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ANALYSIS OF IMPACT CHATTERING

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Abstract. In this paper, mechanical models with Newton's Law of impacts are studied. One of the most interesting properties in some of these models is chattering. This phenomenon is understood as the appearance of an infinite number of impacts occurring in a finite time. Conclusion on the presence of chattering is made exclusively by examination of the right hand side of impact models for the first time. Criteria for the sets of initial data which always lead to chattering are established. The Moon-Holmes model is subject to regular impact perturbations for the chattering generation. Using the chattering solutions, continuous chattering is generated. To depress the chattering, Pyragas control is applied. Illustrative examples are provided to demonstrate the impact chattering.

2010 *Mathematics Subject Classification:* 34A37; 34D10

Keywords: impact mechanism, the Moon-Holmes model, chattering, bouncing bead, Pyragas control

1. INTRODUCTION

The implementation of sliding mode control is often irritated by high frequency oscillations known as "chattering" in system outputs issued by dynamics from actuators and sensors ignored in system modeling [11]. In study [10], chattering is considered as a special type of oscillation characterized by very small amplitudes that are decreasing with time. In impacting systems, it is understood as an infinite number of discontinuity moments occurring in a finite time period, for instance, a ball bouncing to rest on a horizontal surface [7]. It is asserted in [7] that chattering resembles with the inelastic collapse. The balls dissipate their energy through an infinite number of collisions in a finite time interval. Budd and Dux [5] showed that chattering can occur for a periodically forced, single degree of freedom impact oscillator with a restitution law. They demonstrated that chattering can form part of a periodic motion, and this relates to certain types of chaotic behavior. However, they studied through an example. Using the solution, they proved the existence of chattering for a linear system.

Nordmark and Piiroinen [14] considered simulation problems for chattering as well as analysis of stability of the limit cycle, which is chattering by solving the first

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variational equations. Moreover, they used the mappings, which are constructed with the help of a solution, in simulation schemes. Similar to the one in paper [5], it was shown that the existence of chattering for a linear system. Nonetheless, in both papers [5, 14], they do not consider the conditions which guarantee the appearance of chattering. In this study, we consider the chattering as a motion with infinite number of discontinuities in a finite time. This is the first time that sufficient conditions are provided for the chattering based on properties not on maps derived with the help of solutions, but, on conditions for the right-hand side of impulsive systems. Our models essentially are nonlinear (see, for example, Example 1). Since this is the first result in this direction, the models under consideration are respectively simple. Nevertheless, this is a class of mechanical models which can be significantly enlarged in the future investigations by consideration of large ensembles of impact oscillators and weakening conditions of the present paper. We consider models with vibrating surface of impacts as well as analyzed problems of Pyragas controllability and existence of continuous chattering for a model connected unilaterally to a system with an impact chattering. An interesting problem of the regular perturbation of a system with chattering is discussed.

A particular feature of system with impacts is the existence of the chattering. We have two different types of it, namely complete and incomplete chattering [5, 14]. Complete chattering is the phenomenon wherein a system an infinite number of discontinuities in a finite time occurs, where the velocity tends to zero uniformly. Incomplete chattering bears on a sequence of the impacts that initially has the same behavior as complete chattering, but it ends after a large but finite number of impacts [14]. In section 3, we will discuss the transient chattering for systems with small parameter considering the transformation of the incomplete chattering to the complete one when the parameter diminishes to zero.

It was first found by Arnold [4] that the significant characteristic property of chatter vibration is that it is not generated by external periodic forces, but rather it is generated in the dynamic process itself. Therefore, it is important to emphasize that the systems under investigation in this paper are autonomous.

Consider the problem of impact interaction of a body falling in the uniform gravity force field with a fixed horizontal base. After colliding with the base the body bounces back with the velocity whose norm is equal to the norm of the pre-impact velocity multiplied by r, where r is the restitution coefficient, $0 < r < 1$. Then, after some time interval the body will fall on the base again and the norm of its velocity will be equal to the norm of bouncing velocity in the previous collision multiplied by r. The process cannot end in a finite number of collisions. Thus, the considered phenomenon consists in following: after the initial collision a series of repeated collisions of attenuated to zero, which ends in a finite time with establishing a long contact between interacted bodies. Arising this contact results in decreasing number

of degrees of freedom of the system by a unit or more. So, it is reasonable to call this phenomenon the impact chattering.

It is shown by investigations and observations that the impact chattering meets in operating almost every mechanism and machine of impact-oscillating type [13]. Various problems of impact chattering are far from trivial, and their solutions cannot be obtained in closed form for rather general case. As for the use of approximate analytical and numerous methods, it is simplified essentially if one proceeds from the conception about infinite number of impacts inside a finite time range. For example, the existence of impact chattering was investigated in [13]. They simply consider the free falling of a bead on an immobile base and on a vibrating table with constant velocity. In this paper, we consider a more general system and prove the existence of impact chattering.

The chattering phenomena are unwanted in engineering since it is an appearance of infinite discontinuities in a short period of time and this makes theoretical analysis of mechanical models difficult. We have a research plan to consider theoretical and mathematical complexities connected to chattering, and we approach the problem from one of the two possible points of view. The first one is when mechanical models change such that the theoretical chattering disappears [1]. The other point of view, which is considered in this paper, is that we approximate a model with infinite moments of discontinuities with those having a finite number of impacts.

This article is organized as follows. First of all, the impact model is stated. In this model each collision is assumed instantaneous, and it comes to rest after an infinite number of impulse moments in a finite time. The existence of chattering is proved. Asymptotic approximation of solutions with chattering are discussed in section 3. Then, we show that the chattering occurs for a bead bouncing on a sinusoidally vibrating table in section 4. The modified Moon-Holmes model with a small perturbation is discussed in section 5. Using the continuous dependence on parameters and initial value for the impulsive differential equations with non-fixed moments, it is shown that the solution of the modified Moon-Holmes model is chattering. Following that, the appearance of continuous chattering by perturbation method is demonstrated in section 6. Finally, by Pyragas control method the chattering solution is controlled to be periodic.

2. EXISTENCE OF CHATTERING

An impacting system admits a chattering if there is a solution with infinite impulse moments in a finite time. Moreover, we will say that a perturbed system admits a transient chattering, if a number of impacts increases to infinity on a fixed interval as the small parameter tends to zero.

A mechanism with a rigid flat surface of impacts and the constant coefficient of restitution r, $0 < r < 1$, can be modeled by the following impulsive system

$$\begin{aligned} \ddot{x} &= f(x,\dot{x}), \\ \Delta\dot{x}|_{x=\varphi} &= -(1+r)\dot{x}, \end{aligned} \tag{2.1}$$

where $x(t)$ is the coordinate of the bead which is over the impact surface $x = \varphi$, $\dot{x}(t)$ is its velocity, $f(u,v)$ is a continuous function on the domain $H = \{0 < \varphi \leq u \leq h, |v| \leq \bar{h}\}$ for fixed positive numbers $h, \bar{h}$, and it satisfies the local Lipschitz condition in its variables on H. The equality $\Delta\dot{x}(\theta) = \dot{x}(\theta+) - \dot{x}(\theta-)$ denotes the jump operator in which $t = \theta$ is the time when the bead reaches the rigid obstacle, $\dot{x}(\theta-)$ is the pre-impact velocity and $\dot{x}(\theta+)$ is the post-impact velocity.

In system (2.1), we need the following conditions.

(C1) There is a positive number m such that $f(u,v) < -m$ for all $(u,v) \in H$,

(C2) $f(u,v) = f(u,-v)$ for all $(u,v) \in H$.

Conditions on function $f(u,v)$ and compactness of domain H imply that there exists a positive number M such that $f(u,v) \geq -M$ for all $(u,v) \in H$.

Theorem 1. *If conditions (C1), (C2) are satisfied and the following inequality*

$$M\sqrt{\frac{2(h-\varphi)}{m}} < \bar{h} \tag{2.2}$$

is valid, then all solutions with initial value $(x(0),\dot{x}(0)) = (x_0,0)$, $\varphi < x_0 < h$, *of system* (2.1) *are chattering.*

Proof. Consider an initial value $(x_0,0) \in H$, $\varphi < x_0 < h$. Denoting $x_1 = x, x_2 = \dot{x}$ present the system (2.1) as

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= f(x_1,x_2), \\ \Delta x_2|_{x_1=\varphi} &= -(1+r)x_2. \end{aligned} \tag{2.3}$$

The solution of system (2.3) starting at $(x_0,0)$ is

$$x_1(t) = x_0 + \int_0^t (t-s) f(x_1(s),x_2(s))ds, \tag{2.4a}$$

$$x_2(t) = \int_0^t f(x_1(s),x_2(s))ds, \tag{2.4b}$$

while it is continuous. By equation (2.4a) and condition $(C1)$, the coordinate $x_1(t)$ decreases to φ such that there exists a moment θ_1 where $x_1(\theta_1) = \varphi$ and $x_2(\theta_1) < 0$. Moreover, $x_1(\theta_1+) = \varphi$ and $x_2(\theta_1+) = -rx_2(\theta_1) > 0$.

Let us show that the solution is continuable to $+\infty$ and it remains in the domain H. First of all, consider the interval $[0,\theta_1]$. From conditions (C1) and (C2), it implies

that $x_1(t) \le x_0 < h, t \in [0, \theta_1]$. Using (2.4a) and inequality $\varphi < x_0 < h$ we get

$$|h-\varphi| > |\varphi - x_0| = \left| \int_0^{\theta_1} (\theta_1 - s) f(x_1(s), x_2(s)) ds \right| \ge \int_0^{\theta_1} (\theta_1 - s) m ds = m \frac{\theta_1^2}{2},$$

which implies that $\theta_1 < \sqrt{\frac{2(h-\varphi)}{m}}$.

Consequently, from (2.4b) and condition (2.2)

$$|x_2(\theta_1)| = \left| \int_0^{\theta_1} f(x_1(s), x_2(s)) ds \right| \le M\theta_1 < \bar{h}.$$

Thus, we obtain that $\varphi \le x_1(t) < h$ and $|x_2(t)| < \bar{h}$ for $t \in [0, \theta_1]$.

Applying the same arguments as for θ_1 one can show that there is an intersection moment θ_2 such that $x_1(\theta_2) = \varphi$ and $x_1(t) > \varphi, t \in (\theta_1, \theta_2)$. In this interval, we have

$$x_1(t) = \varphi + x_2(\theta_1+)(t-\theta_1) + \int_{\theta_1}^{t} (t-s) f(x_1(s), x_2(s)) ds, \tag{2.5a}$$

$$x_2(t) = x_2(\theta_1+) + \int_{\theta_1}^{t} f(x_1(s), x_2(s)) ds, \tag{2.5b}$$

By condition $(C2)$, $x_2(\theta_1+)$ is the maximum value of $|x_2(t)|$ for $t \in (\theta_1, \theta_2]$. Thus, $|x_2(t)| \le r|x_2(\theta_1)| < r\bar{h} < \bar{h}$. Moreover, from conditions $(C1)$ and $(C2)$, there exists a moment ξ_1, $\theta_1 < \xi_1 < \theta_2$, such that $x_2(\xi_1) = 0$ and $x_1(\xi_1)$ is the maximum value of $x_1(t)$ on $(\theta_1, \theta_2]$. Thus, $x_1(t) \le x_1(\xi_1) < x_0 < h$, and the trajectory of $x(t)$ is in H for $t \in [\theta_1, \theta_2]$. Next, recursively, it can be shown that there exists an increasing sequence $\theta_i, i = 1, 2, \ldots$, such that $x_1(\theta_i) = \varphi, i = 1, 2, \ldots$, and the orbit of $x(t)$ is in H for all $t \ge 0$.

Now, we will show that the sequence θ_i converges. The solution of system (2.3) is defined by

$$x_1(t) = \varphi + x_2(\theta_i+)(t-\theta_i) + \int_{\theta_i}^{t} (t-s) f(x_1(s), x_2(s)) ds, \tag{2.6a}$$

$$x_2(t) = x_2(\theta_i+) + \int_{\theta_i}^{t} f(x_1(s), x_2(s)) ds. \tag{2.6b}$$

on the interval $(\theta_i, \theta_{i+1}], i = 1, 2, \ldots$.

Using condition (C1), it can be shown that there exists a moment ξ_i, $\theta_i < \xi_i < \theta_{i+1}$, such that $x_2(\xi_i) = 0$. Also, utilizing condition (C2), we obtain $\xi_i = \frac{\theta_i + \theta_{i+1}}{2}$.

The solution on the interval $(\theta_{i+1}, \theta_{i+2}]$ is

$$\begin{aligned} x_1(t) &= \varphi + r x_2(\theta_i+)(t-\theta_{i+1}) + \int_{\theta_{i+1}}^{t} (t-s) f(x_1(s), x_2(s)) ds, \\ x_2(t) &= r x_2(\theta_i+) + \int_{\theta_{i+1}}^{t} f(x_1(s), x_2(s)) ds. \end{aligned} \tag{2.7}$$

From $x_2(\xi_i) = 0$ and $x_2(\xi_{i+1}) = 0$, we get

$$x_2(\theta_i+) = -\int_{\theta_i}^{\xi_i} f(x_1(s), x_2(s)) ds, \tag{2.8}$$

$$r x_2(\theta_i+) = -\int_{\theta_{i+1}}^{\xi_{i+1}} f(x_1(s), x_2(s)) ds. \tag{2.9}$$

Let us divide (2.9) by (2.8) in order to get

$$r = \frac{\int_{\theta_{i+1}}^{\xi_{i+1}} f(x_1(s), x_2(s)) ds}{\int_{\theta_i}^{\xi_i} f(x_1(s), x_2(s)) ds}.$$

Using mean value theorem, we have

$$r = \frac{(\xi_{i+1} - \theta_{i+1}) f(x_1(s^*), x_2(s^*))}{(\xi_i - \theta_i) f(x_1(s^{**}), x_2(s^{**}))}, \tag{2.10}$$

for some s^{**} and s^* in (θ_i, θ_{i+1}) and $(\theta_{i+1}, \theta_{i+2})$ respectively.

Then,

$$\frac{\theta_{i+2} - \theta_{i+1}}{\theta_{i+1} - \theta_i} = \frac{\xi_{i+1} - \theta_{i+1}}{\xi_i - \theta_i} < \frac{M_i}{m_i} r, i = 1, 2, 3...., \tag{2.11}$$

where $M_i = \max_{[\theta_i, \theta_{i+1}]} |f(x_1(t), x_2(t))|$ and $m_i = \min_{[\theta_i, \theta_{i+1}]} |f(x_1(t), x_2(t))|$. Since $r < 1$, $\max_{[\theta_i, \theta_{i+1}]} |x_1(t)| \to \varphi$ and $\max_{[\theta_i, \theta_{i+1}]} |x_2(t)| \to 0$ as $i \to \infty$. Moreover, continuity of $f(u, v)$ implies that $\frac{M_i}{m_i} \to 1$ as $i \to \infty$. This and (2.11) prove the convergence. The theorem is proved. □

Example 1. Consider the following non-linear system

$$\begin{aligned} \ddot{x} + \cos(\dot{x}) + x^3 &= 0, \\ \Delta \dot{x}|_{x=2} &= -(1+r)\dot{x}, \end{aligned} \tag{2.12}$$

in the domain $2 \le x \le 2.5, |\dot{x}| < 7$. We have $f(x, \dot{x}) = -\cos(\dot{x}) - x^3 \le -7$, $f(x, \dot{x}) = f(x, -\dot{x})$, $f(x, \dot{x}) \ge -16.625$ in the domain. Condition (2.2) is true since $16.625\sqrt{1/7} \approx 6.28 < 7$. That is, we are in circumstances of Theorem 1 and if we choose $r = 0.8, x(0) = 2.1, \dot{x}(0) = 0$, the solution of system (2.12) is chattering. The simulation of this solution can be seen in Figure 1.

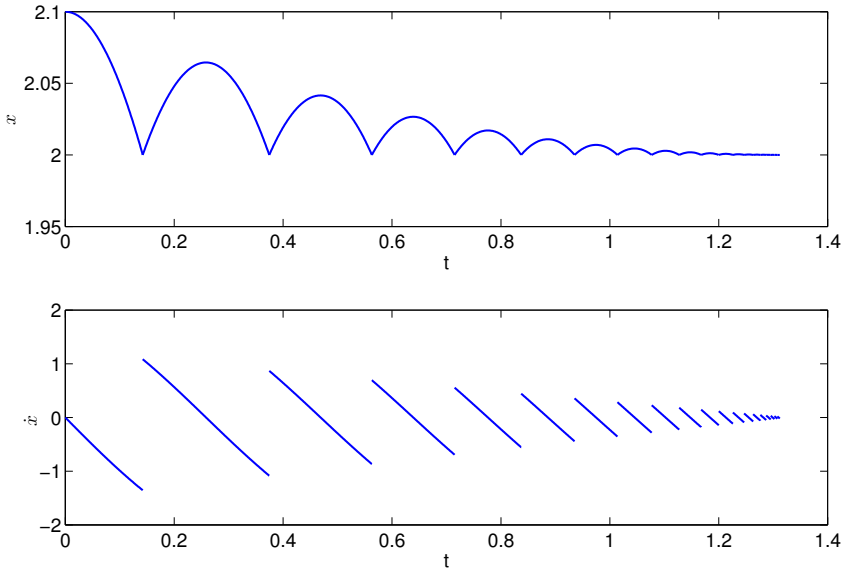

FIGURE 1. The graphs of coordinates $x(t)$ and $\dot{x}(t)$ with initials $x(0) = 2.1$ and $\dot{x}(0) = 0$ of system (2.12) with $r = 0.8$.

3. ASYMPTOTICS

Solutions of the system (2.1) admit infinitely many jumps, and this makes, in general, impossible to find an exact solution or adequately to simulate it. So, in this section we suggest considering degenerate equation to find the perturbed system approximately. In order to increase the precision of approximation we follow the idea of asymptotic approximations. Consider the system

$$\begin{aligned} \ddot{x} &= f(x,\dot{x}), \\ \Delta\dot{x}\,|_{\substack{x=\varphi \\ i<[\frac{1}{r}]}} &= -(1+r)\dot{x}, \end{aligned} \tag{3.1}$$

where i is the index of impacts θ_i, [.] denotes the greatest integer function, with additional condition that the number of impulsive moments has to be not more than $\left[\frac{1}{r}\right]$, i.e., $\theta_i, i = 1,2,...,\left[\frac{1}{r}\right]$. One can guarantee for the fixed value of the parameter r, the incomplete chattering occurs only. The number of impacts increases unboundedly as the parameter tends to zero. For this reason, we say that system (3.1) admits the transient chattering. Assume that this system satisfy all conditions of Theorem 1. For time $t > \theta_{[\frac{1}{r}]}$, the system is only governed by $\ddot{x} = f(x,\dot{x})$. Condition $(C1)$ implies that on the interval $[\theta_{[\frac{1}{r}]}, \theta_\infty]$ the bead stays on the position $x = \varphi$.

For each its solution, system (3.1) has finite number of discontinuity moments. That is why, one can find an exact solution of the problem or at least it is possible to make proper simulations. One can easily see that solutions of the last system and system (2.1) with identical initial data coincide on the interval $[0, \theta_{[\frac{1}{r}]})$. They are different only in the interval $[\theta_{[\frac{1}{r}]}, \theta_\infty]$. The length of the last interval diminishes to 0 as $r \to 0$. Consequently, the solutions of system (3.1) are asymptotic approximations for the solutions of system (2.1).

4. The dynamics of repeated impacts against a sinusoidally vibrating table

In this section, we consider a mechanical model consisting of a bead bouncing on a vibrating table, which is investigated in the papers of Holmes and Guckenheimer [8,9]. It is demonstrated that the model can generate chaos [9]. In this paper, we show that in the mechanism one can observe another type of complex dynamics, namely chattering.

Consider a bouncing bead colliding with a sinusoidally vibrating table. Assume that the table is so massive that it does not react to collisions with the bouncing bead and it moves according to law $X(t) = X_0 \sin \omega t$. The change of the velocity of the bouncing bead at the impact moment is given by the relation $r = \frac{\dot{X}_+ - \dot{x}_+}{\dot{x}_- - \dot{X}_-}$, where r is the restitution coefficient, $0 < r < 1$, $\dot{X}_-, \dot{X}_+, \dot{x}_-, \dot{x}_+$ are the velocities of the table and the bouncing bead before and after impact, respectively. Since the collision does not affect the velocity of the table, we can write $\dot{X}_- = \dot{X}_+$. Then the model will be as follows

$$\begin{aligned} \ddot{x} &= -g, \\ \Delta \dot{x}|_{x=X} &= -(1+r)(\dot{x} - \dot{X}), \\ X(t) &= X_0 \sin(\omega t), \end{aligned} \tag{4.1}$$

where g is the gravitational acceleration ($g \approx 9.8 m/s^2$).

Now, let us consider a general form. Instead of gravitational constant g, take a function $f(u, v)$. Then, the model will be of the form

$$\begin{aligned} \ddot{x} &= f(x, \dot{x}), \\ \Delta \dot{x}|_{x=X} &= -(1+r)(\dot{x} - \dot{X}), \\ X(t) &= X_0 \sin \omega t, \end{aligned} \tag{4.2}$$

where function $f(u, v)$ is a continuous function on the domain $G = \{X_0/10 \le u \le h, |v| \le \bar{h}\}$, for fixed positive numbers $h, \bar{h}$, and it satisfies the local Lipschitz condition in its variables on G. Also, this system has conditions (C1), (C2) defined in the first section for all $(u, v) \in G$. By conditions on function $f(u, v)$ and compactness of the domain G, we have a positive number M such that $f(u, v) \ge -M$ for all $(u, v) \in G$.

Next, consider the graph of the function $X(t) = X_0 \sin \omega t$. The slope of the graph is $\dot{X}(t) = X_0 \omega \cos \omega t$. It is easily seen that if ω is small and t is near $\pi/2\omega$ the graph is close to a horizontal line. Consequently, for sufficiently small ω and for time t near $\pi/2\omega$ if the following inequality

$$M\sqrt{\frac{2(h-X_0/10)}{m}} < \bar{h}, \tag{4.3}$$

is true and conditions (C1), (C2) are satisfied, according to Theorem 1 there is chattering for solutions whose integral curves are near to the point $P(\pi/2\omega, X_0)$. Finally, to demonstrate the result through simulation, we continue with the bouncing bead on the sinusoidally vibrating table.

Example 2. Let us return to the bouncing bead on the sinusoidally vibrating table with the same properties of system (4.2). Then the model will be as follows

$$\begin{aligned} &\ddot{x} = -g, \\ &\Delta\dot{x}|_{x=X} = -(1+r)(\dot{x}-\dot{X}), \\ &X(t) = X_0 \sin(\omega t), \end{aligned} \tag{4.4}$$

where $t \geq 0$. Let us take $\varphi = X_0 = 1$ and consider the domain $0.1 \leq x \leq 2, |\dot{x}| < 7$. Then, we have $f(x,\dot{x}) = -g < 0$, $|f(x,\dot{x})| = |-g| = g = M = m$ and $M\sqrt{\frac{2(h-X_0/10)}{m}} = \sqrt{37.24} < 7$. If we choose the initial conditions $x(2\pi/\omega) = 1.9$, $\dot{x}(2\pi/\omega) = 0$, where $r = 0.9$, $\omega = 0.29$, it can be seen that the conditions of Theorem 1 are satisfied and consequently, this solution is chattering. In Figure 2, one can observe the coordinates of system (4.4) which supports our theoretical result.

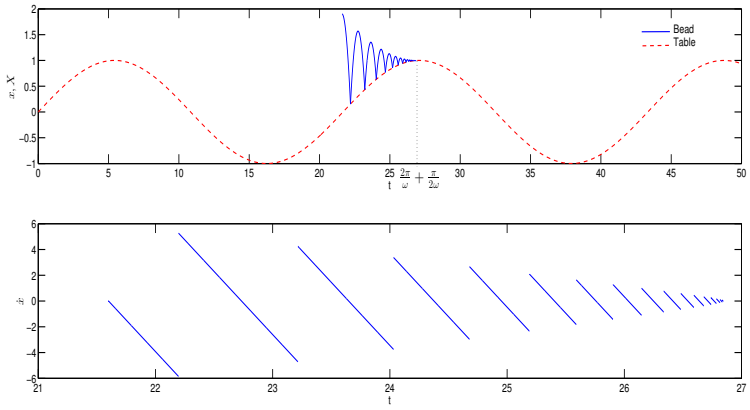

FIGURE 2. The graph of the coordinates of system (4.4).

5. The modified Moon-Holmes Model

The main task of this section is to consider the modified Moon-Holmes Model. Moon and Holmes [12] showed that the Duffing equation in the form

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma \cos wt$$

provides the simplest possible model for the forced vibrations of a cantilever beam in the nonuniform field of two permanent magnets. Such an equation describes the dynamics of a buckled beam or plate when only one mode of vibration is considered. We modify the model as adding a rigid obstacle over the magnet and in front of the beam such that the beam collides the obstacle and from Newton Law of impacts it bounces back. (The system is sketched in Figure 3.) The suggested model has the form of the following impulsive system

$$\begin{aligned} \ddot{x} &= -\delta\dot{x} + x - x^3 + \gamma \cos wt, \\ \Delta\dot{x}|_{x=\varphi} &= -(1+r)\dot{x}, \end{aligned} \tag{5.1}$$

where x is the distance from the wall to the end of the beam, φ is the position of the obstacle, r is the restitution coefficient. Now, if the coefficients γ and δ are equal to zero, one obtains

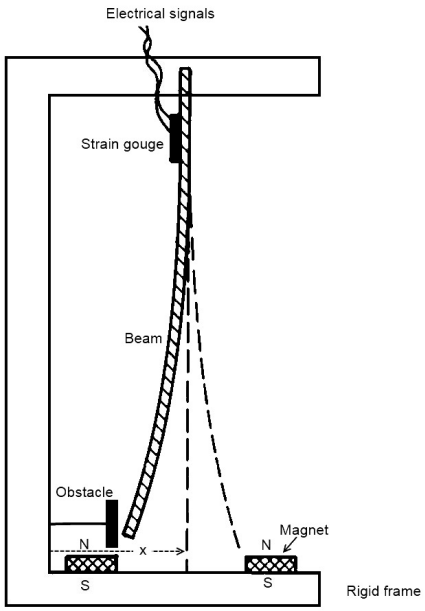

FIGURE 3. The magneto-elastic beam with the obstacle.

$$\begin{aligned} \ddot{x} &= x - x^3, \\ \Delta\dot{x}|_{x=\varphi} &= -(1+r)\dot{x}. \end{aligned} \tag{5.2}$$

For this system, choose $\varphi = 1.1$ and for the domain H let $h = 1.5, \bar{h} = 3$. One can see that function $f(x,\dot{x}) = x - x^3$ satisfies conditions (C1) and (C2), i.e. $-1.875 \leq f(u,v) \leq -0.331$ for all $(u,v) \in H$ and $f(u,v)$ is an even function in v. Moreover, condition (2.2) is valid since $1.875\sqrt{\frac{2(1.5-1.1)}{0.331}} \approx 2,91 < 3$. Therefore, by Theorem 1 all solutions of system (5.2) with initial values $(x(0),\dot{x}(0)) = (x_0,0)$, $\varphi < x_0 < h$, are chattering. Obviously, system (5.1) does not satisfy condition (C2). But, one can easily notice that for sufficiently small δ and γ, by the continuous dependence on parameters and initial value for the impulsive differential equations with non-fixed moments [2], the solutions of (5.1) with the same initial conditions of (5.2) are chattering as well. For the numerical simulation, let $(x(0),\dot{x}(0)) = (1.3,0)$ and $r = 0.9$. Then, one can see that Figure 4 supports our theoretical discussion.

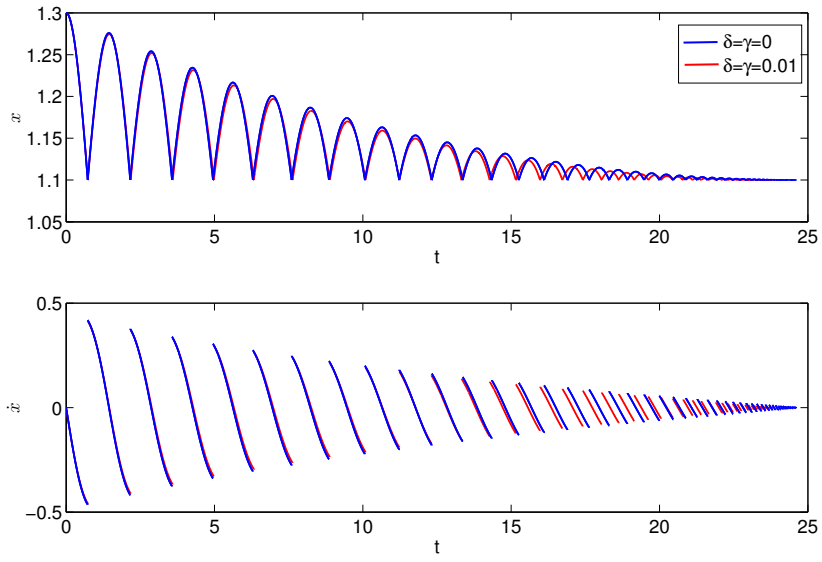

FIGURE 4. The coordinates of systems (5.1) and (5.2) with $w = 0.1$. It can be seen that the solution of perturbed system (5.1) is also chattering.

6. Continuous chattering

In this section, we demonstrate the continuous chattering which is understood as infinitely many oscillations in finite time. Let us observe how continuous chattering appears if a mechanical model is perturbed with a discontinuous one. For this reason, we couple system (2.1) with the following equation of a mass-spring-damper equation

$$m\ddot{y} + c\dot{y} + ky = 0, \tag{6.1}$$

with mass m, spring constant k, and viscous damper of damping coefficient c. If the characteristic equation of system (6.1) has roots with negative real parts, then it admits asymptotically stable equilibrium. By the argument of periodicity theorem for system with stable equilibrium, one can expect that in system (6.1), continuous chattering appears if it is perturbed by a chattering solution of (2.1). Thus, let us write the coupled system taking $f(x,\dot{x}) = -g$ in (2.1), $m = 1, c = 3, k = 2$ in (6.1) and $x = x_1, \dot{x} = x_2, y = x_3, \dot{y} = x_4$, in the form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -g, \\ \Delta|_{x_1=1} &= -(1+r)x_2 \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -2x_3 - 3x_4 + 20x_2^2. \end{aligned} \tag{6.2}$$

with initial conditions $x_1(0) = 6, x_2(0) = 0, x_3(0) = 10, x_4(0) = -1000$ and $r = 0.9$.

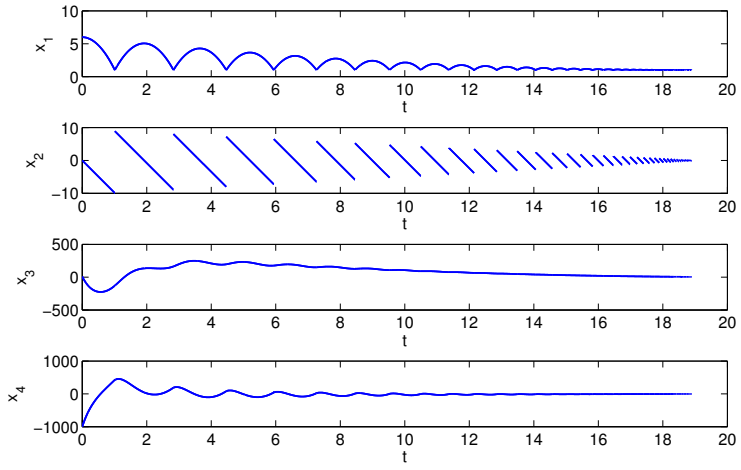

FIGURE 5. The graphs of the coordinates of system (6.2).

Since the first coupling is unilateral, the second equation does not influence the first one. That is why, its dynamics are the same as in Figure 5. But, for the second coupling in Figure 5, we can see the effect of perturbation which we call as continuous chattering.

7. Pyragas control

There are many papers which are searching methods to minimize and control different types of chattering [3, 6]. The problem definitely has to be analyzed for the impact chattering, also. One can accept that the control of impact chattering is a concrete perturbation, which brings the system under control to a regular motion. That is, equilibria or periodic motions. In the circumstances of the present research, it is desired that a family of chattering solutions has to be regularized, if not all of them. We will discuss, in this part of the paper, system (2.12) of Example 1. It was shown that any solution of this system, which starts in a domain, is chattering. Let us apply the control of the form $C[x_1(t-\tau)-x_1(t)]$ to the system. It is applied, for instance, to stabilize periodic motions of chaotic dynamics, and it is called Pyragas control [15]. Now, we will apply the control to depress the chattering in the system. Let us construct the following system denoting $x_1 = x$ and $x_2 = \dot{x}$

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= -x_1^3 - \cos x_2 + C[x_1(t-\tau)-x_1(t)] \\ \Delta x_2|_{x_1=2} &= -(1+r)x_2. \end{aligned} \tag{7.1}$$

We performed a series of simulations of the system with fixed $C = -30$, $\tau = 1$ and $r = 0.6$. Consider $x_2(0)$ as it was requested to prove family of chattering solutions in Theorem 1. For the initial first coordinate $x_1(0)$ we tried values starting from 2.5 to 200.

$x_1(0)$	2.5	3	5	10	100	200
Period T	1.22	1.22	1.22	1.22	1.22	1.22
Amplitude	2.933	2.933	2.933	2.933	2.933	2.933

TABLE 1. Periods and amplitudes of the first coordinate $x_1(t)$ of system (7.1) for different values of $x_1(0)$.

For all these solutions the ultimate periodicity has been approved with period $T = 1.22$. Observe that the period is different from the delay term $\tau = 1$. One can see from the table that the amplitudes are equal to 2.933 for all values of $x_1(0)$ as well. At the same time, the chattering has not been decaying for the solution with $x_1(0) = 2.1$. These all demonstrate that the control problem can be solved for the chattering, but certain conditions have to be determined to specify the controllable domains and

conditions for the stability of the arranged periodic motions. We suppose that these problems will be researched in next papers.

For $x_1(0) = 3$, the periodic orbit $x_1(t)$ can be seen in Figure 6, which shows the effectiveness of the control.

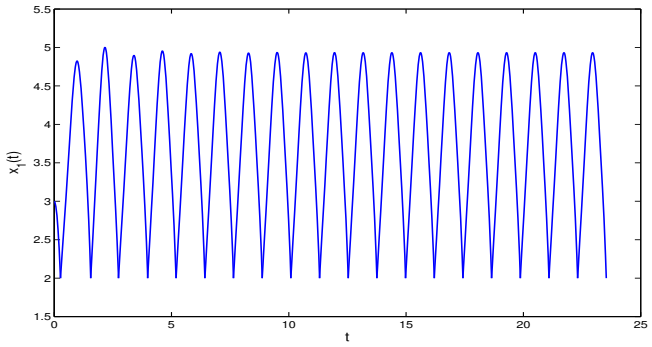

FIGURE 6. Simulation of the first coordinate $x_1(t)$ of controlled system (7.1) with initial conditions $x_1(0) = 3, x_2(0) = 0$.

8. CONCLUSION

In this article, we have considered the mechanical models with impacts. For these models, the chattering phenomenon, which is defined as a motion with infinitely many discontinuities in a finite time, is studied. The sufficient conditions are determined for the existence of the chattering. Asymptotics are discussed to find an approximation solution and to simulate the chattering solution. We study the famous example: the bouncing bead on a sinusoidally vibrating table which generates chaos [9]. It is shown that this mechanism has chattering solutions. Furthermore, we modify the Moon-Holmes model [12], which yields chaos also, with an obstacle to obtain an impacting model. We demonstrate that this model provides chattering. Perturbing a continuous mechanical process by a discontinuous one having chattering solutions, continuous chattering, which is defined as the appearance of infinitely many oscillations in a finite time, is constructed.

The application of results of paper [14] is to prove the existence of a unique chattering solution of the bouncing ball, see Section 3.5. At the same time, by simulation it is proven that a double pendulum admits chattering. Our method, in some sense, is more wider than the one in paper [14]. For example, by Theorem 1 in this paper, we have verified that there are infinitely many chattering motions with initial values in an interval. Thus, the present result is complement to that one accomplished in [14]. However, our approach does not work for the double pendulum, since condition $(C2)$ is not valid for the model. Nevertheless, in our next investigation, we plan to extend the method without condition $(C2)$.

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CLOSURES OF PROPER CLASSES

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Abstract. For an integral domain R we consider the closures $\widehat{M}$ ($\widehat{M}_r, r \in R$) of a submodule M of an R-module N consisting of elements n of N with $tn \in M$ ($r^m n \in M$) for some nonzero $t \in R$ ($m \in \mathbb{Z}^+$) and its connections with usual closure $\overline{M}$ of M in N. Using these closures we study the closures $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_r$ of a proper class $\mathcal{P}$ of short exact sequences and give a decomposition for the class of quasi-splitting short exact sequences of abelian groups into the direct sum of "p-closures" of the class $\mathcal{S}plit$ of splitting short exact sequences and description of closures of some classes. In the general case of an arbitrary ring we generalize these closures of a proper class $\mathcal{P}$ by means of homomorphism classes $\mathcal{F}$ and $\mathcal{G}$ and prove that under some conditions this closure $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is a proper classes.

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Keywords: proper class of short exact sequences, closure of a proper class, sum of proper classes, closure of a module

1. INTRODUCTION

Throughout R is an associative ring with identity and all modules are unital left R-modules; restrictions on the ring R concerned, if there are any, are mentioned at the beginning of each section.

In this paper, we study closures of proper classes and their relation to closures of splitting short exact sequences. Over an integral domain R, for a class $\mathcal{P}$ of short exact sequences of R-modules, we denote by $\hat{\mathcal{P}}$ the class of short exact sequences E such that $kE \in \mathcal{P}$ for some $0 \neq k \in R$, where k also denotes the multiplication homomorphism by k. Thus

$$\hat{\mathcal{P}} = \{E \mid kE \in \mathcal{P} \text{ for some } 0 \neq k \in R\}.$$

In case of abelian groups, the class $\hat{\mathcal{P}}$ firstly was studied in [14] for $\mathcal{P} = \mathcal{S}plit$, the class of splitting short exact sequences, and it was denoted by Text (since $\mathrm{Ext}_{\mathcal{S}\hat{plit}}(C, A) = T(\mathrm{Ext}(C, A))$. In [8] $\hat{\mathcal{P}}$ was studied for $\mathcal{P} = \mathcal{P}ure$, the class of pure exact sequences and $\mathcal{P} = \mathcal{D}$, torsion splitting short exact sequences. For an arbitrary proper class $\mathcal{P}$ in the category of abelian groups the class $\hat{\mathcal{P}}$ was studied in [3].

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For an integral domain R we consider the closures $\widehat{M}$ ($\widehat{M}_r, r \in R$) of a submodule M of an R-module N consisting of elements n of N with $tn \in M$ ($r^m n \in M$) for some nonzero $t \in R$ ($m \in \mathbb{Z}^+$). We study its connections with usual closure of M in N (see [7]) and show that a submodule M of a torsion-free module N over an integral domain have unique closure which coincides with $\widehat{M}$.

For an element r of an integral domain R we define *r-closures* of the class $\mathcal{P}$ of short exact sequences contained in $\hat{\mathcal{P}}$ and give a direct sum decomposition for the class $\mathcal{S}\hat{p}lit$ of quasi-splitting short exact sequences of abelian groups into the direct sum of p-closures of the class $\mathcal{S}plit$ (for sum of proper classes see [1, 12]).

The classes $\mathcal{S}mall$, $\mathcal{S}$, $\mathcal{SB}$ and $\mathcal{WS}$ determined by small submodules, submodules that have supplements, submodules that have supplements with bounded intersection and weak supplement submodules respectively, are studied in [4]. Over a hereditary ring, it is shown in the same work that the least proper class $\overline{\mathcal{WS}}$ can be defined using the class $\mathcal{WS}$. In section 3, we give the relation between $\mathcal{SB}$, $\overline{\mathcal{WS}}$ and Ivanov Classes (see [3, 9]).

For investigation of $\hat{\mathcal{P}}$ over an integral domain R we have used the fact that multiplication by an element of R is an endomorphism of a module if R is commutative which need not be true for arbitrary associative ring with identity. For this reason, the last section is devoted to the study of closures of proper classes over an associative ring with identity. Instead of multiplication by an element of R we take classes $\mathcal{F}$ and $\mathcal{G}$ of homomorphisms and define closure $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ for a class $\mathcal{P}$ of short exact sequences with respect to these classes of homomorphisms of R-modules. We define a *compatible* pair $(\mathcal{F}, \mathcal{G})$ for a class $\mathcal{P}$ and use its properties to prove that for a compatible pair $(\mathcal{F}, \mathcal{G})$ for a proper class $\mathcal{P}$, the class $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ is proper. We give some restrictions on the compatible pair $(\mathcal{F}, \mathcal{G})$ to ensure that $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ contains $\mathcal{P}$. Furthermore, under some conditions on the compatible pair $(\mathcal{F}, \mathcal{G})$ for the proper class $\mathcal{P}$, we obtain a relation between the closure $\mathcal{S}\hat{p}lit^{\mathcal{G}}_{\mathcal{F}}$ of splitting short exact sequences and the class $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ at the end of this section.

2. Preliminaries

Let R be an associative ring with identity and $\mathcal{P}$ be a class of short exact sequences of left R-modules and R-module homomorphisms. If a short exact sequence

$E : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ belongs to $\mathcal{P}$, then f is said to be a *$\mathcal{P}$-monomorphism* and g is said to be a *$\mathcal{P}$-epimorphism*. E is determined by the monomorphism f and the epimorphism g uniquely up to isomorphism.

Definition 1. The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see [5, 10, 13]):

P-1) If a short exact sequence E is in $\mathcal{P}$, then every short exact sequence isomorphic to E is contained in $\mathcal{P}$.

P-2) $\mathcal{P}$ contains all splitting short exact sequences.
P-3) The composition of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if it is defined.
P-3') The composition of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if it is defined.
P-4) If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.
P-4') If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

Another and frequently used way to check if a given class $\mathcal{P}$ is proper is given by R. J. Nunke. A class $\mathcal{P}$ of short exact sequences is a proper class if $\operatorname{Ext}^1_{\mathcal{P}}(C, A)$ is a subfunctor of $\operatorname{Ext}^1_R(C, A)$, $\operatorname{Ext}^1_{\mathcal{P}}(C, A)$ is a subgroup of $\operatorname{Ext}^1_R(C, A)$ for all R-modules A, C and the composition of two $\mathcal{P}$-monomorphisms (or $\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism (a $\mathcal{P}$-epimorphism respectively) (see Theorem 1.1 in [11]).

For any class $\mathcal{E}$ of short exact sequences the proper class generated by $\mathcal{E}$ is the least proper class that contains $\mathcal{E}$ and denoted by $\langle \mathcal{E} \rangle$. It is the intersection of all proper classes containing $\mathcal{E}$ (see [12]).

A module M is $\mathcal{P}$-coprojective ($\mathcal{P}$-coinjective) if every short exact sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$ (of the form $0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$) is in $\mathcal{P}$. For classes $\mathcal{M}$ and $\mathcal{J}$ of modules, the smallest proper class $\overline{k}(\mathcal{M})$ (resp. $\underline{k}(\mathcal{J})$) for which all modules in $\mathcal{M}$ (resp. $\mathcal{J}$) are $\overline{k}(\mathcal{M})$-coprojective (resp. $\underline{k}(\mathcal{J})$-coinjective) is said to be coprojectively (resp. coinjectively) generated by $\mathcal{M}$ (resp. $\mathcal{J}$).

For classes $\mathcal{R}$ and $\mathcal{L}$ of short exact sequences, the class $\mathcal{R} + \mathcal{L}$ is defined as the least proper class that contains $\mathcal{R}$ and $\mathcal{L}$, i.e. $\mathcal{R} + \mathcal{L} = \langle \mathcal{R} \cup \mathcal{L} \rangle$, and it is called the *sum* of the classes $\mathcal{R}$ and $\mathcal{L}$ (see [12]).

We will use Ext instead of Ext^1 and for a homomorphism f, we will denote $\operatorname{Ext}^1(1, f)$ by f_* and $\operatorname{Ext}^1(f, 1)$ by f^*.

3. The classes $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_r$

Let R be an integral domain throughout this section unless otherwise stated. For a class $\mathcal{P}$ of short exact sequences of R-modules, we denote by $\hat{\mathcal{P}}$ the class of short exact sequences $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules such that $kE \in \mathcal{P}$ for some $0 \neq k \in R$ where k also denotes the multiplication homomorphism by $k \in R$. Thus

$$\hat{\mathcal{P}} = \{E \mid kE \in \mathcal{P} \text{ for some } 0 \neq k \in R\}.$$

For $E \in \mathcal{P}$, we have $1 \cdot E = E \in \mathcal{P}$, therefore $\mathcal{P} \subseteq \hat{\mathcal{P}}$ for every class $\mathcal{P}$ of short exact sequences.

In case of abelian groups, the class $\hat{\mathcal{P}}$ was studied in [14] and in [3] for $\mathcal{P} = \mathcal{S}plit$,the class of splitting short exact sequences. In [8] $\hat{\mathcal{P}}$ was studied for $\mathcal{P} =$

$\mathcal{P}ure$, the class of pure exact sequences and $\mathcal{P} = \mathcal{D}$, torsion splitting short exact sequences. In [3] $\hat{\mathcal{P}}$ was studied for $\mathcal{P} = \mathcal{S}plit$, where it was denoted by *Text* (since $\mathrm{Ext}_{\hat{\mathcal{S}plit}}(C, A) = T(\mathrm{Ext}(C, A))$, the torsion part of $\mathrm{Ext}(C, A)$), and for every proper class $\mathcal{P}$. The following result gives a general answer when R is an integral domain.

Theorem 1 ([2], Theorem 3.1)**.** *For every proper class $\mathcal{P}$ of short exact sequences of R-modules, the class $\hat{\mathcal{P}}$ is proper.*

We can also consider $\hat{M}$ for R-modules $M \leq N$ with the definition

$$\hat{M} = \{x \in N \mid kx \in M \text{ for some } 0 \neq k \in R\}.$$

Some results involving this definition are obtained in [12] for the case of abelian groups (see [12, Lemma 4.1]).

The submodule $\hat{M}$ of the R-module N can be used in finding closures. Let us remind the definition of the closure of an R-module.

Definition 2. (see [7]) A submodule X of an R-module Z is closed in Z if $X \leq Y \leq Z$ and $X \trianglelefteq Y$ implies $Y = X$. For R-modules $M \leq X \leq N$, X is called a closure of M in N if $M \trianglelefteq X$ and X is closed in N.

Closure of a module need not be unique in general as the following example shows.

Example 1. Let R be the ring of integers and $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Consider the subgroups $N = (1+2\mathbb{Z}, 2+8\mathbb{Z})\mathbb{Z}$, $L = (0+2\mathbb{Z}, 1+8\mathbb{Z})\mathbb{Z}$ and $X = (0+2\mathbb{Z}, 4+8\mathbb{Z})\mathbb{Z}$ of M. Then N and L are both closures of the subgroup X in M.

Proposition 1. *Let $M \leq N$ be R-modules, then $\hat{M}$ is the unique closure of $M + T(N)$ in N, where $T(N)$ is the torsion part of N.*

Proof. For $x \in \hat{M}$, $rx \in M \leq M + T(N)$ for some $0 \neq r \in R$, therefore $M + T(N) \trianglelefteq \hat{M}$. If $\hat{M} \trianglelefteq M'$ for some $M' \leq N$, then for $y \in M'$, $ky \in \hat{M}$ for some $0 \neq k \in R$ which implies $l(ky) \in M$ for some $0 \neq l \in R$. Since $(lk)y = l(ky) \in M$ for $0 \neq lk \in R$, $y \in \hat{M}$. Therefore, $\hat{M}$ is closed.

Let N' be another closure of $M + T(N)$ in N. If $z \in N'$, then $sz \in M + T(N)$ for some $0 \neq s \in R$ which shows that $N' \leq \hat{M}$. Since $M + T(N) \trianglelefteq N' \leq \hat{M}$ and N' is closed in N, $N' = \hat{M}$. □

Corollary 1. *Let N be a torsion-free R-module. Then for every submodule M of N, $\hat{M}$ is the unique closure of M in N.*

It is possible to define closures of the class $\mathcal{P}$ that happen to be in between $\mathcal{P}$ and $\hat{\mathcal{P}}$. For a class $\mathcal{P}$ of short exact sequences of R-modules and $0 \neq r \in R$, we denote by $\hat{\mathcal{P}}_r$ (r-*closure* of $\mathcal{P}$) the class of short exact sequences
$E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules such that $r^t E \in \mathcal{P}$ for some

for some nonnegative integer t, where r^t also denotes the multiplication homomorphism by $r^t \in R$. Thus

$$\hat{\mathcal{P}}_r = \{E \mid r^t E \in \mathcal{P} \text{ for some nonnegative integer } t\}.$$

For $E \in \mathcal{P}$, we have $r^0 \cdot E = 1 \cdot E = E \in \mathcal{P}$ and clearly $\hat{\mathcal{P}}_r \subseteq \hat{\mathcal{P}}$, therefore $\mathcal{P} \subseteq \hat{\mathcal{P}}_r \subseteq \hat{\mathcal{P}}$ for every class $\mathcal{P}$ of short exact sequences.

For a given proper class $\mathcal{P}$, the following result gives us proper classes that are contained in $\hat{\mathcal{P}}$. We omit its proof since it uses similar ideas used in the proof of [3, Theorem 1].

Proposition 2. *$\hat{\mathcal{P}}_r$ is a proper class for every proper class $\mathcal{P}$ and every $0 \neq r \in R$.*

Direct sum of two proper classes are defined in [1]. We say that the sum $\sum_{i \in I} \mathcal{L}_i$ of proper classes is direct if $\mathcal{L}_j \bigcap \sum_{\substack{i \neq j \\ i \in F}} \mathcal{L}_i = \mathcal{S}plit$ for every $j \in I$ and for every finite index set $F \subseteq I$.

Over the ring of integers, for every group A we have a decomposition of $T(A)$ into primary parts as $T(A) = \bigoplus_{p \text{ prime}} T_p(A)$. The following result holds when $R = \mathbb{Z}$, and it gives a direct sum decomposition for the class $\mathcal{S}plit$ of splitting short exact sequences in terms of proper classes.

Proposition 3. *Over the ring of integers, $\hat{\mathcal{S}plit} = \bigoplus_p \hat{\mathcal{S}plit}_p$, where p ranges over all prime numbers.*

A relation between quasi-splitting short exact sequences and $\hat{\mathcal{P}}$ for a proper class $\mathcal{P}$ is given by [2, Theorem 3.2]. A similar result for the closures we have defined also holds. Its proof uses similar ideas used in [2, Theorem 3.2], therefore it is omitted.

Theorem 2. *Let $\mathcal{P}$ be a proper class. Then*

$$\mathcal{P} + \hat{\mathcal{S}plit}_r = \hat{\mathcal{P}}_r$$

for every $0 \neq r \in R$.

In order to show the relation of the classes defined so far with the proper classes related to supplements, we use Ivanov classes and give some results over the ring of integers (see [3, 9]).

Definition 3. For the classes $\mathcal{M}$ and $\mathcal{J}$ of modules, the class $i(\mathcal{M}, \mathcal{J})$ of short exact sequences is the least proper class for which every module from $\mathcal{M}$ is coprojective and every module from $\mathcal{J}$ is coinjective.

These classes are called Ivanov classes.

Theorem 3 ([3], Theorem 2)**.** *For all classes $\mathcal{M}$ and $\mathcal{J}$ of abelian groups*

$$\widehat{i(\mathcal{M},\mathcal{J})} = i(\mathcal{M}\cup\mathcal{B},\mathcal{J}\cup\mathcal{B})$$

where $\mathcal{B}$ denotes the class of bounded abelian groups.

This result is also true for modules over an integral domains since its proof can easily be modified. It is also possible to use the proof of this result and obtain another closure of the class $i(\mathcal{M},\mathcal{J})$.

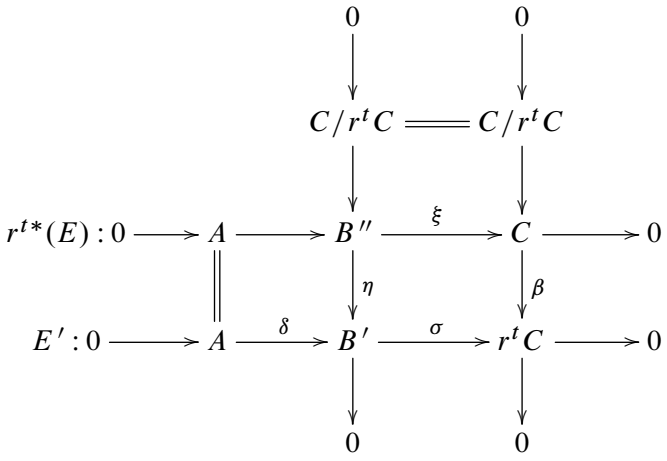
Theorem 4. *Let R be an integral domain, $\mathcal{M}$ and $\mathcal{J}$ classes of R-modules. Then for every $0 \neq r \in R$*

$$\widehat{i(\mathcal{M},\mathcal{J})}_r = i(\mathcal{M}\cup\mathcal{B}_r,\mathcal{J}\cup\mathcal{B}_r)$$

where $\mathcal{B}_r$ denotes the class of R-modules bounded by a power of r.

Proof. Let $i(\mathcal{M}\cup\mathcal{B}_r,\mathcal{J}\cup\mathcal{B}_r) = \mathcal{L}$. It is clear that $i(\mathcal{M},\mathcal{J}) \subseteq \mathcal{L}$ and $\mathcal{L} \subseteq \widehat{i(\mathcal{M},\mathcal{J})}_r$ since every module from $\mathcal{B}_r$ is $\widehat{i(\mathcal{M},\mathcal{J})}_r$-coinjective and $\widehat{i(\mathcal{M},\mathcal{J})}_r$-coprojective. To show that $\widehat{i(\mathcal{M},\mathcal{J})}_r \subseteq \mathcal{L}$, let
$E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \ \in \widehat{i(\mathcal{M},\mathcal{J})}_r$. Then there is a nonnegative integer t such that $r^t E \in i(\mathcal{M},\mathcal{J})$. We can write the homomorphism $r^t : C \longrightarrow C$ as $r^t = \alpha\circ\beta$, where $\alpha : r^tC \longrightarrow C$ is the inclusion and $\beta : C \longrightarrow r^tC$ is the standard epimorphism. Applying these homomorphisms we obtain the following commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
E': 0 & \longrightarrow & A & \xrightarrow{\delta} & B' & \xrightarrow{\sigma} & r^tC & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\alpha} & & \\
E: 0 & \longrightarrow & A & \xrightarrow{\gamma} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C/r^tC & = & C/r^tC & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$



In the second diagram, β is an $\mathcal{L}$-epimorphism since $C[r^t] \in \mathcal{B}_r$ and ξ is also an $\mathcal{L}$-epimorphism. Then $\sigma \circ \eta = \beta \circ \xi$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class and σ is an $\mathcal{L}$-epimorphism since $\mathrm{Ext}_{\mathcal{L}}$ is a subfunctor of Ext. Therefore, $E' \in \mathcal{L}$ and δ is an $\mathcal{L}$-monomorphism.

In the first diagram, θ is an $\mathcal{L}$-monomorphism since $C/r^tC \in \mathcal{B}_r$. Since $\mathcal{L}$ is a proper class, $\alpha = \theta \circ \sigma$ is an $\mathcal{L}$-monomorphism and $E \in \mathcal{L}$. □

Corollary 2. *Let R be a Noetherian integral domain of Krull dimension 1, and $0 \neq r \in R$. Then*

$$\hat{\mathcal{S}}\mathcal{B} = i(\mathcal{B}, \mathcal{B})$$

$$\hat{\mathcal{S}}\mathcal{B}_r = i(\mathcal{B}_r, \mathcal{B})$$

where $\mathcal{B}$ is the class of bounded R-modules and $\mathcal{B}_r$ is the class of modules bounded by a power of r.

Proof. By [6, Proposition 4.3], $\mathcal{S}\mathcal{B} = \underline{k}(\mathcal{B}) = i(\varnothing, \mathcal{B})$. Theorem 3 and Theorem 4 complete the proof. □

We need the following result to see another application of these closures.

Proposition 4 ([4], Proposition 4.13)**.** *Over a hereditary ring $\overline{\mathcal{W}\mathcal{S}} = \underline{k}(\mathcal{S}m)$, where $\mathcal{S}m$ is the class of all small modules.*

Corollary 3. *Let R be a Dedekind domain and $0 \neq r \in R$. Then*

$$\hat{\overline{\mathcal{W}\mathcal{S}}} = i(\mathcal{B}, \mathcal{S}m)$$

$$\hat{\overline{\mathcal{W}\mathcal{S}}}_r = i(\mathcal{B}_r, \mathcal{S}m)$$

where $\mathcal{B}$ is the class of bounded R-modules, $\mathcal{B}_r$ is the class of modules bounded by a power of r and $\mathcal{S}m$ is the class of small modules.

Proof. By Proposition 4, $\overline{\mathcal{WS}} = \underline{k}(\mathcal{S}m) = i(\varnothing, \mathcal{S}m)$. Since $\mathcal{B} \subseteq \mathcal{S}m$, Theorem 3 and Theorem 4 completes the proof. □

Until now, we have dealt with the classes that include the given class $\mathcal{P}$. In [2], the existence of a class that is contained in the given class $\mathcal{P}$ is given using the class $\hat{\mathcal{P}}$ over a principal ideal domain. It is also possible to obtain a similar result using the class $\hat{\mathcal{P}}_r$. Note also that by $r\mathcal{P}$ we mean the class $r\mathcal{P} = \{E : E = rE' \text{ for some } E' \in \mathcal{P}\}$ defined in [2].

Theorem 5 ([2], Theorem 4.4)**.** *Let R be a principal ideal domain. Then for every proper class $\mathcal{P}$ with $\hat{\mathcal{P}} = \mathcal{P}$ and for every $k \in R$, the class $k\mathcal{P}$ is proper.*

With a slight change on the condition given, we can obtain a result for the class $\hat{\mathcal{P}}_r$.

Theorem 6. *Let R be a principal ideal domain. Then for every $0 \neq r \in R$ and every proper class $\mathcal{P}$ with $\hat{\mathcal{P}}_r = \mathcal{P}$, the class $r^t\mathcal{P}$ is proper for every nonnegative integer t.*

4. The class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$

In the previous section, the classes $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_r$ for the given class $\mathcal{P}$ of short exact sequences are defined over an integral domain using the homomorphisms that are multiplication by elements of R since multiplication by elements of R give homomorphisms when R is an integral domain. We cannot use elements of R to obtain homomorphisms in order to find a closure for the class $\mathcal{P}$ when R is an associative ring with an identity element since the multiplication by an element of R need not give a homomorphism in this case. Therefore, we turn our attention to the classes of R-module homomorphisms. Throughout this section we will study modules over an associative ring R with an identity element.

Definition 4. Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules. We say that $\mathcal{F}$ is closed under pushout diagrams if $f : A \longrightarrow B$, $f \in \mathcal{F}$ and $\alpha : A \longrightarrow A'$ is a homomorphism, then in the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ A' & \xrightarrow[f']{} & P \end{array}$$

we have $f' \in \mathcal{F}$. We say that $\mathcal{G}$ is closed under pullback diagrams if $g : C \longrightarrow D$, $g \in \mathcal{G}$ and $\beta : D' \longrightarrow D$, then in the pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ \uparrow & & \uparrow \beta \\ P' & \xrightarrow[g']{} & D' \end{array}$$

we have $g' \in \mathcal{G}$.

Lemma 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules. Let $\mathcal{F}$ be closed under pushout diagrams and $\mathcal{G}$ closed under pullback diagrams. Then the following hold.*

(*i*) *If $f : A \longrightarrow A_1$ belongs to $\mathcal{F}$, then the homomorphisms $f \oplus 1_A : A \oplus A \longrightarrow A_1 \oplus A$ and $1_A \oplus f : A \oplus A \longrightarrow A \oplus A_1$ are also in $\mathcal{F}$.*

(*ii*) *If $f : A \longrightarrow B$ belongs to $\mathcal{F}$, then the inclusion $i : \operatorname{Im} f \longrightarrow B$ (or the monomorphism $\bar{i} : A/\operatorname{Ker} f \longrightarrow B$ induced by f) is also in $\mathcal{F}$.*

(*iii*) *If $g : C_1 \longrightarrow C$ belongs to $\mathcal{G}$, then the homomorphisms $g \oplus 1_C : C_1 \oplus C \longrightarrow C \oplus C$ and $1_C \oplus g : C \oplus C_1 \longrightarrow C \oplus C$ are also in $\mathcal{G}$.*

(*iv*) *If $g : C \longrightarrow D$ belong to $\mathcal{G}$, then the canonical epimorphism $\pi : C \longrightarrow C/\operatorname{Ker} g$ induced by g (or the epimorphism $\bar{\pi} : C \longrightarrow \operatorname{Im} g$ induced by g) is also in $\mathcal{G}$.*

Let R be an associative ring with identity. Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules, and $\mathcal{P}$ a class of short exact sequences. We say that the pair $(\mathcal{F}, \mathcal{G})$ is "*compatible*" for the class $\mathcal{P}$ if for every short exact sequence E, there is $f \in \mathcal{F}$ such that $f_*(E) \in \mathcal{P}$ if and only if there is $g \in \mathcal{G}$ such that $g^*(E) \in \mathcal{P}$ with one (or both) of the following conditions satisfied:

(*i*) $\mathcal{F}$ is closed under compositions and pushout diagrams,

(*ii*) $\mathcal{G}$ is closed under compositions and pullback diagrams.

For a class $\mathcal{P}$ of short exact sequences and a "compatible" pair $(\mathcal{F}, \mathcal{G})$ for $\mathcal{P}$, we define the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ as

$$\begin{aligned} \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} &= \{E \mid f_*(E) \in \mathcal{P} \text{ for some } f \in \mathcal{F}\} \\ &= \{E \mid g^*(E) \in \mathcal{P} \text{ for some } g \in \mathcal{G}\}. \end{aligned}$$

Theorem 7. *For every proper class $\mathcal{P}$ of short exact sequences and every compatible pair $(\mathcal{F}, \mathcal{G})$ for $\mathcal{P}$, the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is proper.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ with $f_{1*}(E), g_1^*(E) \in \mathcal{P}$ for $f_1 : A \longrightarrow A_1$, $f_1 \in \mathcal{F}$ and $g_1 : C_1 \longrightarrow C$, $g_1 \in \mathcal{G}$.

If $f : A \longrightarrow A'$, then $g_1^*(f_*(E)) = (g_1^* \circ f_*)(E) = (f_* \circ g_1^*)(E) = f_*(g_1^*(E)) \in \mathcal{P}$ since $\operatorname{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $f_*(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $g_1 \in \mathcal{G}$.

If $g : C' \longrightarrow C$, then $f_{1*}(g^*(E)) = (f_{1*} \circ g^*)(E) = (g^* \circ f_{1*})(E) = g^*(f_{1*}(E)) \in \mathcal{P}$ since $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $g^*(E) \in \hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ since $f_1 \in \mathcal{F}$.

These arguments show that $\mathrm{Ext}_{\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}}$ is a subfunctor of Ext.

At this point, we separate the proof into two parts and give the proof when $\mathcal{F}$ is closed under compositions and pushout diagrams. Proof for the other case can be done using similar arguments and duality.

Let $\mathcal{F}$ be closed under compositions and pushout diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $E' : 0 \longrightarrow A \longrightarrow B' \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ with $f_{1*}(E) \in \mathcal{P}$ for $f_1 : A \longrightarrow A_1$, $f_1 \in \mathcal{F}$ and $f_{2*}(E') \in \mathcal{P}$ for $f_2 : A \longrightarrow A_2$, $f_2 \in \mathcal{F}$, say $f_{1*}(E) : 0 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow C \longrightarrow 0$ and $f_{2*}(E') : 0 \longrightarrow A_2 \longrightarrow B_2 \longrightarrow C \longrightarrow 0$. We have the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\ & & {\scriptstyle f_1 \oplus 1_A}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A_1 \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \end{array}$$

and

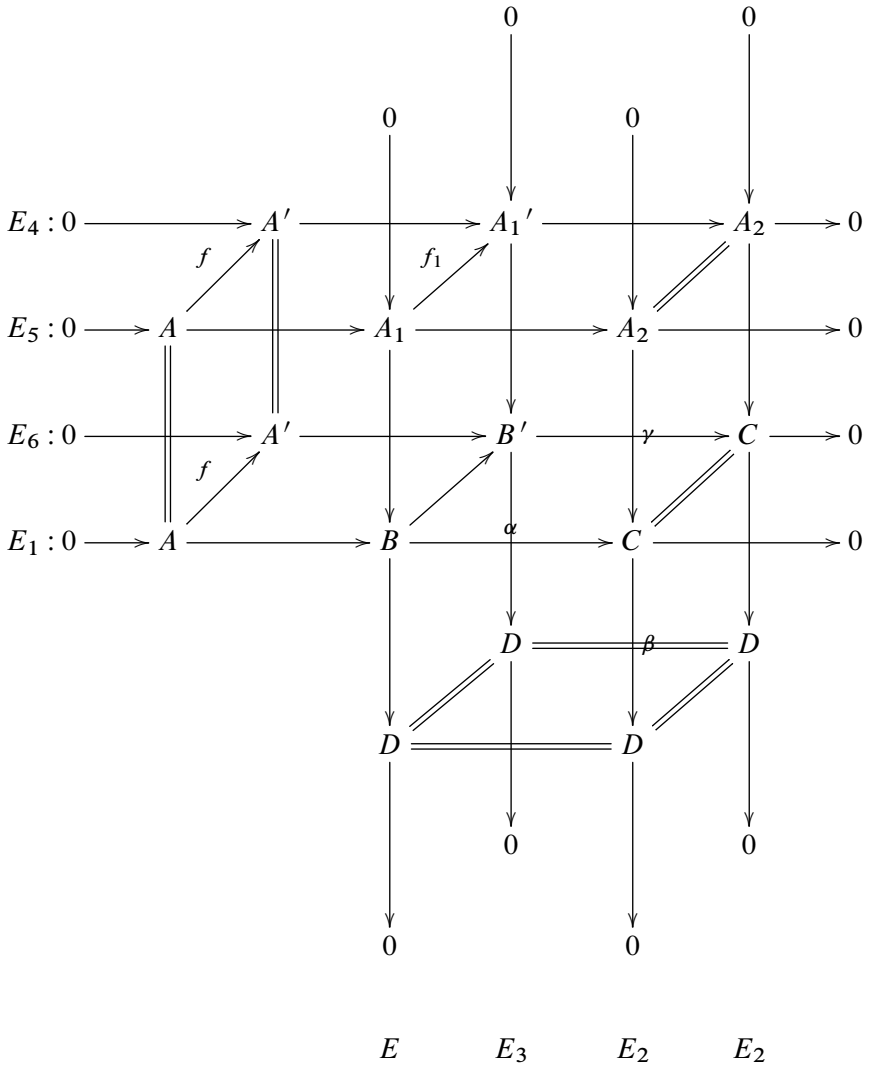
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\ & & {\scriptstyle 1_{A_1} \oplus f_2}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & C \oplus C & \longrightarrow & 0 \end{array}$$

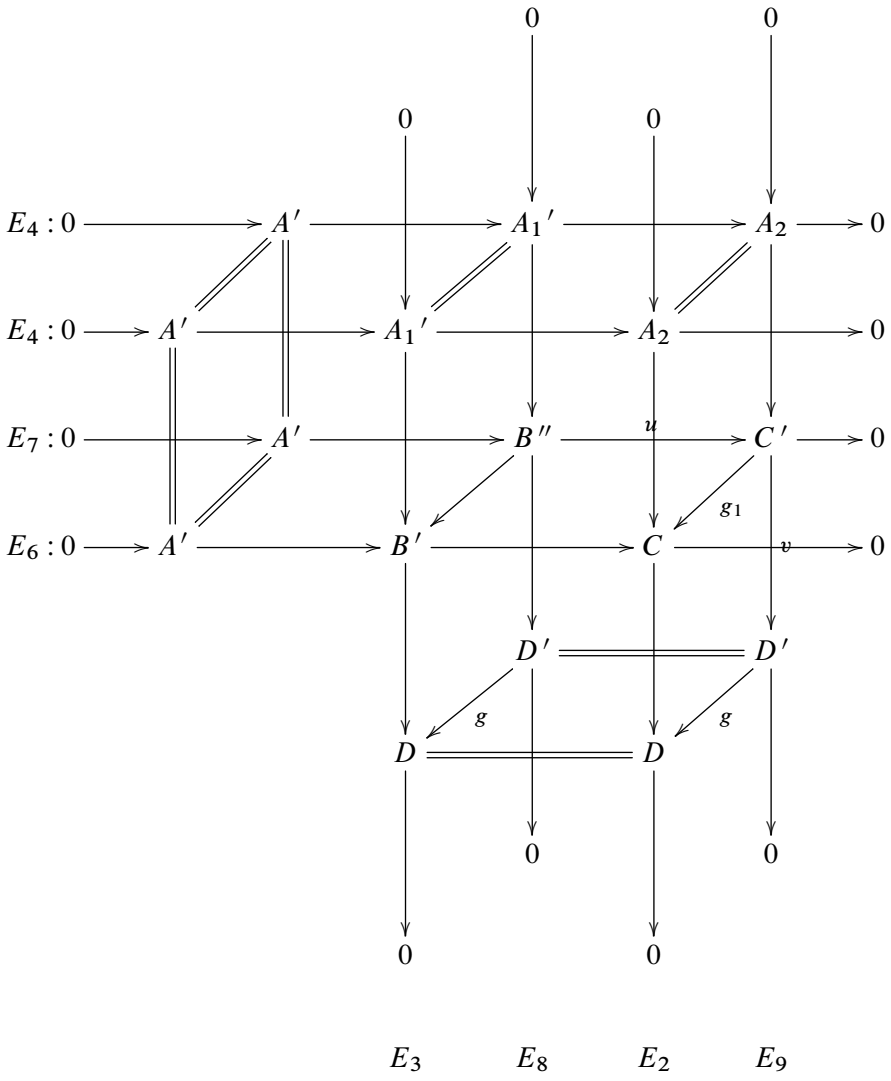
with exact rows.

By Lemma 1, $f_1 \oplus 1_A$ and $1_{A_1} \oplus f_2$ are in $\mathcal{F}$. Since $\mathcal{F}$ is closed under compositions, we have $(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A) \in \mathcal{F}$. Since $\mathcal{P}$ is a proper class, $[(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A)]_*(E \oplus E') \in \mathcal{P}$ with $(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A) \in \mathcal{F}$. Therefore, $(E \oplus E') \in \hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$.

These arguments show that $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ is an e-functor. Using [11, Theorem 1.1], in order to show that $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ is a proper class, it is enough to show that the composition of two $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$-epimorphisms is a $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$-epimorphism.

Let $E_1 : 0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} C \longrightarrow 0$ and $E_2 : 0 \longrightarrow A_2 \xrightarrow{\gamma} C \xrightarrow{\beta} D \longrightarrow 0 \in \hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$. Then there are homomorphisms $f : A \longrightarrow A'$, $f \in \mathcal{F}$ and $g : D' \longrightarrow D$, $g \in \mathcal{G}$ such that $f_*(E_1) \in \mathcal{P}$ and $g^*(E_2) \in \mathcal{P}$. We have the following commutative diagrams with exact rows and columns:


0
0
0
0
$E_4 : 0$
A'
A_1'
A_2
0
f
f_1
$E_5 : 0$
A
A_1
A_2
0
$E_6 : 0$
A'
B'
γ
C
0
f
$E_1 : 0$
A
B
α
C
0
D
β
D
D
D
0
0
0
0
E
E_3
E_2
E_2



In the first diagram, $E_6 = f_*(E_1) \in \mathcal{P}$, $E_4 = f_*(E_5) = f_*(\gamma^*(E_1)) = \gamma^*(f_*(E_1))$ since $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext, and $f_1 \in \mathcal{F}$ since f_1 is constructed using pushout.

In the second diagram, $E_9 = g^*(E_2)$, $E_7 = g_1^*(E_6) \in \mathcal{P}$. Then $u : B'' \longrightarrow C'$ and $v : C' \longrightarrow D'$ are $\mathcal{P}$-epimorphisms. Since $\mathcal{P}$ is a proper class, $v \circ u$ is a $\mathcal{P}$-epimorphism and $E_8 \in \mathcal{P}$. $g^*(E_3) = E_8$ and $g \in \mathcal{G}$ implies $E_3 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$, so there is $f_2 : A_1{}' \longrightarrow A_1{}''$, $f_2 \in \mathcal{F}$ for some R-module $A_1{}''$ such that $f_{2*}(E_3) \in \mathcal{P}$.

Then $(f_2 \circ f_1)_*(E) = f_{2*}(f_{1*}(E)) = f_{2*}(E_3) \in \mathcal{P}$ and $(f_2 \circ f_1) \in \mathcal{F}$ since $f_1, f_2 \in \mathcal{F}$, and $\mathcal{F}$ is closed under compositions. Hence $E \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. □

Remark 1. For the compatible pair $(\mathcal{F}, \mathcal{G})$ for the proper class $\mathcal{P}$, if there is a homomorphism $f \in \mathcal{F}$ from every R-module A or equivalently there is a homomorphism $g \in \mathcal{G}$ to every R-module C, then $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

In case the condition $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is satisfied, a similar result obtained for the classes $\mathcal{P}$ and $\hat{\mathcal{P}}_r$ in the previous section also holds for the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ under some extra conditions on the classes $\mathcal{F}$ and $\mathcal{G}$.

Theorem 8. *Let $\mathcal{P}$ be a proper class, $(\mathcal{F}, \mathcal{G})$ a compatible pair for $\mathcal{P}$, and one of the classes $\mathcal{F}$ and $\mathcal{G}$ contain identity homomorphisms. Moreover, let the inclusions $i_f : \operatorname{Ker} f \longrightarrow A$ belong to $\mathcal{F}$ for all $f : A \longrightarrow A' \in \mathcal{F}$ for R-modules A, A' if $\mathcal{F}$ is closed under pushout diagrams, and the epimorphisms $\pi_g : C \longrightarrow C/\operatorname{Im} g$ belong to $\mathcal{G}$ for all $g : C' \longrightarrow C \in \mathcal{G}$ for R-modules C, C' if $\mathcal{G}$ is closed under pullback diagrams. Then we have*

$$\mathcal{P} + \mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}} = \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}.$$

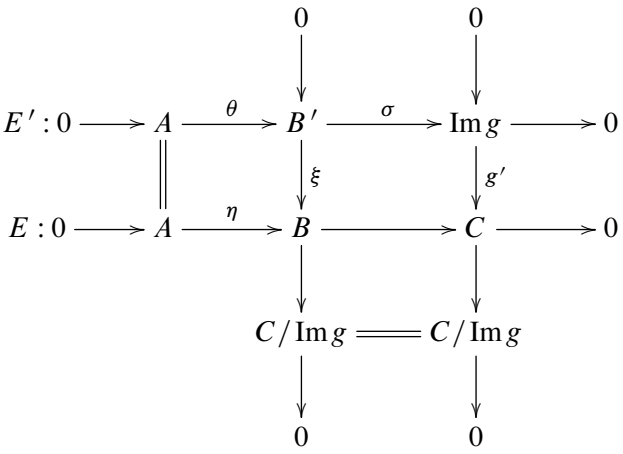
Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \mathcal{P}$. If $\mathcal{F}$ contains identity homomorphisms, then $E = 1 \cdot E = 1_{A*}(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $E \in \mathcal{P}$. If $\mathcal{G}$ contains identity homomorphisms, then $E = 1 \cdot E = 1_C^*(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $E \in \mathcal{P}$. Since $\mathcal{S}plit \subseteq \mathcal{P}$ for every proper class $\mathcal{P}$, $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Combining these inclusions we obtain $\mathcal{P} + \mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

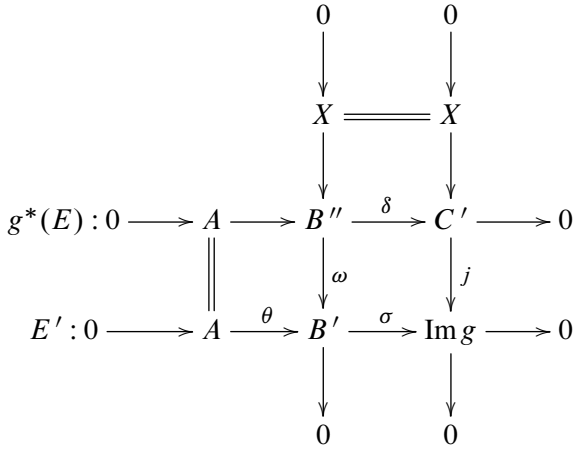
Let us write $\mathcal{P} + \mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}} = \mathcal{L}$. We will show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$.

At this point, we separate the proof into two parts and give the proof when $\mathcal{G}$ is closed under compositions and pullback diagrams. Proof for the other case can be done using similar arguments and duality.

Let $\mathcal{G}$ be closed under compositions and pullback diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Then there is a homomorphism $g : C' \longrightarrow C$, $g \in \mathcal{G}$ for some R-module C' such that $g^*(E) \in \mathcal{P}$. We can write the homomorphism g as $g = g' \circ j$, where $g' : \operatorname{Im} g \longrightarrow C$ is the inclusion and $j : C' \longrightarrow \operatorname{Im} g$ is the epimorphism induced by g.

Using the homomorphisms j and g', we obtain the following commutative diagrams with exact rows and columns:





E_2

We have $g'^*(E') = g^*(E) \in \mathcal{P}$, then δ is a $\mathcal{P}$-epimorphism. Since $\mathcal{P} \subseteq \mathcal{L}$, δ is an $\mathcal{L}$-epimorphism. Since $j \in \mathcal{G}$ by Lemma 1 and $j^*(E_2) \in \mathcal{S}plit$ by [10, Ch. 3, Proposition 1.7], j is a $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}}$-epimorphism. Since $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$, j is an $\mathcal{L}$-epimorphism. Then $\sigma \circ \omega = j \circ \delta$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class by definition, and σ is an $\mathcal{L}$-epimorphism since $\operatorname{Ext}_{\mathcal{L}}$ is a subfunctor of Ext. Therefore θ is an $\mathcal{L}$-monomorphism.

Since $\pi_g : C \longrightarrow C/\operatorname{Im} g$ and $g : C' \longrightarrow C$ belong to $\mathcal{G}$, $\pi_g \circ g$ is in $\mathcal{G}$ since $\mathcal{G}$ is closed under compositions. Therefore, $C/\operatorname{Im} g$ is $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}}$-coprojective and ξ is a

$\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}}$-monomorphism, and ξ is an $\mathcal{L}$-monomorphism since $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$. Then $\eta = \xi \circ \theta$ is an $\mathcal{L}$-monomorphism since $\mathcal{L}$ is a proper class by definition. Hence $E \in \mathcal{L}$ and $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$. □

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ESTIMATE FOR INITIAL MACLAURIN COEFFICIENTS OF CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, estimates for second and third MacLaurin coefficients of certain subclasses of bi-univalent functions in the open unit disk defined by convolution are determined, and certain special cases are also indicated. The main result extends and improve a recent one obtained by Srivastava et al.

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1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Koebe one-quarter theorem [3] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains the disk with the center in the origin and the radius $1/4$. Thus, every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\mathbb{D}) \to \mathbb{D}$, satisfying $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$, and

$$f\left(f^{-1}(w)\right) = w, \ |w| < r_0(f), \ r_0(f) \geq \frac{1}{4}.$$

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots, \ w \in f(\mathbb{D}). \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent*, if both f and f^{-1} are univalent in $\mathbb{D}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{D}$, and we denote by σ this class of bi-univalent functions.

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In [8] the authors defined the classes of functions $\mathcal{P}_m(\beta)$ as follows: let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions P, normalized with $P(0) = 1$, and satisfying

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} P(z) - \beta}{1-\beta} \right| \mathrm{d}\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{D}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions p analytic in $\mathbb{D}$, normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} \mathrm{d}\mu(t), \tag{1.2}$$

where μ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m,\ m \geq 2. \tag{1.3}$$

Clearly, $\mathcal{P} := \mathcal{P}_2$ is the well-known class of *Carathéodory functions*, i.e. the normalized functions with positive real part in the open unit disk $\mathbb{D}$.

Lewin [6] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the concept of bi-starlike functions and the bi-convex functions, and obtained estimates for the initial coefficients. Recently, Ali et al. [1], Srivastava et al. [9], Frasin and Aouf [4], Goyal and Goswami [5] and many others have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Motivated by work of Srivastava et al. [9], we introduce a new subclass of bi-univalent functions, as follows.

For the functions $f,\ h \in \mathcal{A}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad h(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ z \in \mathbb{D},$$

we recall the *Hadamard (or convolution) product* of f and h, defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ z \in \mathbb{D}.$$

Definition 1. For a given function $k \in \sigma$, a function $f \in \sigma$ is said to be in the class $\mathcal{BR}^k(m;\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied

$$\frac{(f * k)(z)}{z} \in \mathcal{P}_m(\beta),$$

$$\frac{(g*k)(w)}{w} \in \mathcal{P}_m(\beta),$$

where $g = f^{-1}$ and $z, w \in \mathbb{D}$.

Remark 1. Taking $k(z) = z/(1-z)^2$ and $m = 2$ in the Definition 1 we obtain the class $\mathcal{B}(\beta) := \mathcal{BR}^{z/(1-z)^2}(2;\beta)$ studied by Srivastava et al. [9, Definition 2].

Definition 2. For a given function $k \in \sigma$ and a number $\alpha \in \mathbb{C}$, a function $f \in \sigma$ is said to be in the class $\mathcal{BV}^k(m;\alpha,\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied

$$(1-\alpha)\frac{z(f*k)'(z)}{(f*k)(z)} + \alpha\left(1 + \frac{z(f*k)''(z)}{(f*k)'(z)}\right) \in \mathcal{P}_m(\beta),$$
$$(1-\alpha)\frac{w(g*k)'(w)}{(g*k)(w)} + \alpha\left(1 + \frac{w(g*k)''(w)}{(g*k)'(w)}\right) \in \mathcal{P}_m(\beta),$$

where $g = f^{-1}$.

Remark 2. (*i*) Taking $\alpha = 0$ and $\alpha = 1$ in the above class $\mathcal{BV}^k(m;\alpha,\beta)$ we obtain the classes $\mathcal{S}_m^k(\beta) := \mathcal{BV}^k(m;0,\beta)$ and $\mathcal{C}_m^k(\beta) := \mathcal{BV}^k(m;1,\beta)$, respectively.

(*ii*) Moreover, if we take $k(z) = z/(1-z)$ and $m = 2$, the classes $\mathcal{S}_m^k(\beta)$ and $\mathcal{C}_m^k(\beta)$ reduces to the well-known classes of *bi-starlike* and *bi-convex functions*, respectively (see also [2]).

The object of the paper is to find estimates for the coefficients a_2 and a_3 for functions in the subclass $\mathcal{BR}^k(m;\beta)$ and $\mathcal{BV}^k(m;\alpha,\beta)$, and these bounds are obtained by employing the techniques used earlier by Srivastava et al. [9].

2. Main results

In order to prove our main result for the functions $f \in \mathcal{BR}^k(m;\beta)$, first we will prove the following lemma:

Lemma 1. *Let the function* $\Phi(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$, $z \in \mathbb{D}$, *such that* $\Phi \in \mathcal{P}_m(\beta)$. *Then,*

$$|h_n| \leq m(1-\beta),\ n \geq 1.$$

Proof. From (1.2) and (1.3), like in [8] and [7], we can see that if $p \in \mathcal{P}_m$, then

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z), \tag{2.1}$$

where $p_1, p_2 \in \mathcal{P}$.

Further, if $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in \mathbb{D}$, where $p_1(z) = 1 + \sum_{n=1}^{\infty} p_n^{(1)} z^n$ and $p_2(z) = 1 + \sum_{n=1}^{\infty} p_n^{(2)} z^n$ for all $z \in \mathbb{D}$, comparing the coefficients of both sides of (2.1) we get

$$p_n = \left(\frac{m}{4} + \frac{1}{2}\right) p_n^{(1)} - \left(\frac{m}{4} - \frac{1}{2}\right) p_n^{(2)}, \; n \geq 1.$$

Since $p_1, p_2 \in \mathcal{P}$, where $\mathcal{P}$ is the class of Carathéodory functions, it is well-known that $|p_n^{(1)}| \leq 2$ and $|p_n^{(2)}| \leq 2$ for all $n \geq 1$, and thus

$$|p_n| \leq \left(\frac{m}{4} + \frac{1}{2}\right)\left|p_n^{(1)}\right| + \left(\frac{m}{4} - \frac{1}{2}\right)\left|p_n^{(2)}\right| \leq$$
$$2\left(\frac{m}{4} + \frac{1}{2}\right) + 2\left(\frac{m}{4} - \frac{1}{2}\right) = m, \; n \geq 1. \tag{2.2}$$

Now, the proof of this lemma is straight forward, if we write

$$\Phi(z) = (1-\beta)\, p(z) + \beta, \quad \text{where} \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}_m.$$

Then,

$$\Phi(z) = 1 + (1-\beta) \sum_{n=1}^{\infty} p_n z^n, \; z \in \mathbb{D},$$

which gives

$$h_n = (1-\beta)\, p_n, \; n \geq 1,$$

and using the inequality (2.2) we obtain the desired result. □

Theorem 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{BR}^k(m;\beta)$, where $k \in \sigma$ has the form $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. If $k_2, k_3 \neq 0$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{|k_3|}}; \frac{m(1-\beta)}{|k_2|}\right\}, \quad |a_3| \leq \frac{m(1-\beta)}{|k_3|},$$

and

$$|2a_2^2 - a_3| \leq \frac{m(1-\beta)}{|k_3|}.$$

Proof. Since $f \in \mathcal{BR}^k(m;\beta)$, from the Definition 1 we have

$$\frac{(f * k)(z)}{z} = p(z) \tag{2.3}$$

and

$$\frac{(g*k)(w)}{w} = q(w), \tag{2.4}$$

where p, $q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Using the fact that the functions p and q have the following Taylor expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots, \; z \in \mathbb{D}, \tag{2.5}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots, \; w \in \mathbb{D}, \tag{2.6}$$

and equating the coefficients in (2.3) and (2.4), from (1.1) we get

$$k_2 a_2 = p_1, \tag{2.7}$$

$$k_3 a_3 = p_2, \tag{2.8}$$

and

$$-k_2 a_2 = q_1,$$

$$k_3 \left(2a_2^2 - a_3\right) = q_2. \tag{2.9}$$

Since p, $q \in \mathcal{P}_m(\beta)$, according to Lemma 1, the next inequalities hold:

$$|p_k| \leq m(1-\beta), \; k \geq 1, \tag{2.10}$$

$$|q_k| \leq m(1-\beta), \; k \geq 1, \tag{2.11}$$

and thus, from (2.8) and (2.9), by using the inequalities (2.10) and (2.11), we obtain

$$|a_2|^2 \leq \frac{|q_2| + |p_2|}{2|k_3|} \leq \frac{m(1-\beta)}{|k_3|},$$

which gives

$$|a_2| \leq \sqrt{\frac{m(1-\beta)}{|k_3|}}. \tag{2.12}$$

From (2.7), by using (2.10) we obtain immediately that

$$|a_2| = \left|\frac{p_1}{k_2}\right| \leq \frac{m(1-\beta)}{|k_2|},$$

and combining this with the inequality (2.12), the first inequality of the conclusion is proved.

According to (2.8), from (2.10) we easily obtain

$$|a_3| = \left|\frac{p_2}{k_3}\right| \leq \frac{m(1-\beta)}{|k_3|},$$

and from (2.9), by using (2.10) and (2.11) we finally deduce

$$\left|2a_2^2 - a_3\right| = \left|\frac{q_2}{k_3}\right| \leq \frac{m(1-\beta)}{|k_3|},$$

which completes our proof. □

Setting $\beta = 0$ in Theorem 1 we get the following special case:

Corollary 1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{BR}^k(m;0)$, *where* $k \in \sigma$ *has the form* $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. *If* $k_2, k_3 \neq 0$, *then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m}{|k_3|}}; \frac{m}{|k_2|}\right\}, \quad |a_3| \leq \frac{m}{|k_3|}, \quad \text{and} \quad \left|2a_2^2 - a_3\right| \leq \frac{m}{|k_3|}.$$

For $k(z) = z/(1-z)^2$ the above corollary reduces to the next result:

Example 1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{BR}^{z/(1-z)^2}(m;0)$, then

$$|a_2| \leq \sqrt{\frac{m}{3}}, \quad |a_3| \leq \frac{m}{3}, \quad \text{and} \quad \left|2a_2^2 - a_3\right| \leq \frac{m}{3}.$$

Taking $k(z) = z/(1-z)$ in Corollary 1, we get:

Example 2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{BR}^{z/(1-z)}(m;0)$, then

$$|a_2| \leq \sqrt{m}, \quad |a_3| \leq m, \quad \text{and} \quad \left|2a_2^2 - a_3\right| \leq m.$$

If we put $k(z) = z/(1-z)^2$ in Theorem 1, we deduce the next corollary:

Corollary 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is in the class* $\mathcal{B}(\beta)$, *then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & \text{if} \quad 0 \leq \beta \leq \frac{1}{3}, \\ 1-\beta, & \text{if} \quad \frac{1}{3} < \beta < 1, \end{cases} \qquad |a_3| \leq \frac{2(1-\beta)}{3},$$

and

$$\left|2a_2^2 - a_3\right| \leq \frac{2(1-\beta)}{3}.$$

Remark 3. For the special case $\frac{1}{3} < \beta < 1$, the above first inequality, and the second one for all $0 \leq \beta < 1$, improve the estimates given by Srivastava et al. in [9, Theorem 2].

Theorem 2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{BV}^k(m;\alpha,\beta)$, *with* $\alpha \in \mathbb{C} \setminus \{-1\}$, *where* $k \in \sigma$ *has the form* $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. *If* $k_2, k_3 \neq 0$ *and*

$$2(1+2\alpha)k_3 - (1+3\alpha)k_2^2 \neq 0,$$

then

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{\left|2(1+2\alpha)k_3-(1+3\alpha)k_2^2\right|}}; \frac{m(1-\beta)}{|1+\alpha|\,|k_2|}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{m(1-\beta)}{\left|2(1+2\alpha)k_3-(1+3\alpha)k_2^2\right|}+\frac{m(1-\beta)}{2|1+2\alpha|\,|k_3|}; \frac{m(1-\beta)}{2|1+2\alpha|\,|k_3|}\left(1+\frac{m(1-\beta)|1+3\alpha|}{|1+\alpha|^2}\right); \frac{m(1-\beta)}{2|1+2\alpha|\,|k_3|}\left(1+\frac{m(1-\beta)\left|4(1+2\alpha)k_3-(1+3\alpha)k_2^2\right|}{|k_2|^2\,|1+\alpha|^2}\right)\right\},$$

whenever $\alpha \in \mathbb{C}\setminus\left\{-\frac{1}{2}\right\}$.

Proof. If $f \in \mathcal{BV}^k(m;\alpha,\beta)$, according to the Definition 2 we have

$$(1-\alpha)\frac{z(f*k)'(z)}{(f*k)(z)}+\alpha\left(1+\frac{z(f*k)''(z)}{(f*k)'(z)}\right)=p(z)$$

and

$$(1-\alpha)\frac{w(g*k)'(w)}{(g*k)(w)}+\alpha\left(1+\frac{w(g*k)''(w)}{(g*k)'(w)}\right)=q(w),$$

where $p,\ q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Since

$$(1-\alpha)\frac{z(f*k)'(z)}{(f*k)(z)}+\alpha\left(1+\frac{z(f*k)''(z)}{(f*k)'(z)}\right)=$$
$$1+(1+\alpha)a_2k_2z+\left[2(1+2\alpha)a_3k_3-(1+3\alpha)a_2^2k_2^2\right]z^2+\ldots\ z\in\mathbb{D},$$

and according to (1.1)

$$(1-\alpha)\frac{z(g*k)'(w)}{(g*k)(w)}+\alpha\left(1+\frac{z(g*k)''(w)}{(g*k)'(w)}\right)=1-(1+\alpha)a_2k_2w+$$
$$\left\{\left[4(1+2\alpha)k_3-(1+3\alpha)k_2^2\right]a_2^2-2(1+2\alpha)a_3k_3\right\}w^2+\ldots,\ w\in\mathbb{D},$$

from (2.5) and (2.6) combined with the above two expansion formulas, it follows that

$$(1+\alpha)a_2k_2=p_1, \tag{2.13}$$
$$2(1+2\alpha)a_3k_3-(1+3\alpha)a_2^2k_2^2=p_2, \tag{2.14}$$

and

$$-(1+\alpha)a_2k_2=q_1,$$
$$\left[4(1+2\alpha)k_3-(1+3\alpha)k_2^2\right]a_2^2-2(1+2\alpha)a_3k_3=q_2. \tag{2.15}$$

Now, from (2.14) and (2.15) we deduce that

$$a_2^2 = \frac{p_2 + q_2}{4(1+2\alpha)k_3 - 2(1+3\alpha)k_2^2}, \tag{2.16}$$

whenever $2(1+2\alpha)k_3 - (1+3\alpha)k_2^2 \neq 0$, and

$$4(1+2\alpha)k_3\left(a_3 - a_2^2\right) = p_2 - q_2.$$

Using (2.16) in the above relation, we obtain

$$a_3 = \frac{p_2 + q_2}{4(1+2\alpha)k_3 - 2(1+3\alpha)k_2^2} + \frac{p_2 - q_2}{4(1+2\alpha)k_3}, \tag{2.17}$$

$$\text{whenever} \quad 2(1+2\alpha)k_3 - (1+3\alpha)k_2^2 \neq 0,\ \alpha \in \mathbb{C} \setminus \left\{-\frac{1}{2}\right\}.$$

From (2.13) and (2.14) we get

$$a_3 = \frac{1}{2(1+2\alpha)k_3}\left[p_2 + \frac{1+3\alpha}{(1+\alpha)^2}p_1^2\right], \quad \text{for} \quad \alpha \in \mathbb{C} \setminus \left\{-1; -\frac{1}{2}\right\}, \tag{2.18}$$

while from (2.13) and (2.15) we deduce that

$$a_3 = \frac{1}{2(1+2\alpha)k_3}\left[-q_2 + \frac{4(1+2\alpha)k_3 - (1+3\alpha)k_2^2}{k_2^2(1+\alpha)^2}p_1^2\right], \tag{2.19}$$

for $\alpha \in \mathbb{C} \setminus \left\{-1; -\frac{1}{2}\right\}$.

Combining (2.13) and (2.16) for the computation of the upper-bound of $|a_2|$, and (2.17), (2.18) and (2.19) for the computation of $|a_3|$, by using Lemma 1 we easily find the estimates of our theorem. □

Taking $\alpha = 0$ and $\alpha = 1$ in Theorem 2 we obtain the following two special cases, respectively:

Corollary 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{S}_m^k(\beta)$, *where* $k \in \sigma$ *has the form* $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. *If* $k_2, k_3 \neq 0$ *and*

$$2k_3 - k_2^2 \neq 0,$$

then

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{|2k_3 - k_2^2|}}; \frac{m(1-\beta)}{|k_2|}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{m(1-\beta)}{|2k_3 - k_2^2|} + \frac{m(1-\beta)}{2|k_3|}; \frac{m(1-\beta)\left(1 + m(1-\beta)\right)}{2|k_3|};\right.$$

$$\frac{m(1-\beta)}{2|k_3|}\left(1+\frac{m(1-\beta)\left|4k_3-k_2^2\right|}{|k_2|^2}\right)\Bigg\}.$$

Corollary 4. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *be in the class* $\mathcal{C}_m^k(\beta)$*, where* $k\in\sigma$ *has the form* $k(z)=z+\sum_{n=2}^{\infty}k_nz^n$*. If* $k_2,k_3\neq 0$ *and*

$$3k_3-2k_2^2\neq 0,$$

then

$$|a_2|\leq\min\left\{\sqrt{\frac{m(1-\beta)}{\left|6k_3-4k_2^2\right|}};\frac{m(1-\beta)}{2|k_2|}\right\},$$

and

$$|a_3|\leq\min\left\{\frac{m(1-\beta)}{\left|6k_3-4k_2^2\right|}+\frac{m(1-\beta)}{6|k_3|};\frac{m(1-\beta)\left(1+m(1-\beta)\right)}{6|k_3|};\right.$$
$$\left.\frac{m(1-\beta)}{6|k_3|}\left(1+\frac{m(1-\beta)\left|3k_3-k_2^2\right|}{|k_2|^2}\right)\right\}.$$

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GLOBAL RAINBOW DOMINATION IN GRAPHS

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Abstract. For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \varnothing$, the condition $\bigcup_{u\in N(v)} f(u) = \{1,2,\ldots,k\}$ is fulfilled, where $N(v)$ is the neighborhood of v. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. A kRDF f is called a *global k-rainbow dominating function* (GkRDF) if f is also a kRDF of the complement $\overline{G}$ of G. The *global k-rainbow domination number* of G, denoted by $\gamma_{grk}(G)$, is the minimum weight of a GkRDF on G. In this paper, we initiate the study of the global k-rainbow domination number and we establish some sharp bounds for it.

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1. INTRODUCTION

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E). The order $|V|$ of G is denoted by $n = n(G)$. Denote by K_n the *complete graph*, by C_n the *cycle* and by P_n the *path* of order n, respectively. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *open neighborhood* of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N[S] = N(S) \cup S$. The *minimum* and *maximum* degrees of G are respectively denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$. A *leaf* of a graph is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v. Let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We use [12, 19] for terminology and notation which are not defined here.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S of

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G is *global dominating set* of G if S is a dominating set both of G and $\overline{G}$. The *global domination number* $\gamma_g(G)$ of G is the minimum cardinality of a global dominating set. The global domination number was introduced independently by Brigham and Dutton [7] (the term factor domination number was used) and Sampathkumar [15] and has been studied by several authors (see for example [3, 20]). Since then some variants of the global domination parameter, such as connected (total) global domination, global minus domination, and global Roman domination, have been studied [4, 5, 10, 13].

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \varnothing$, the condition $\bigcup_{u\in N(v)} f(u) = \{1,2,\ldots,k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [6] and has been studied by several authors (see for example [1, 2, 8, 9, 11, 14, 16–18]). A 2-rainbow dominating function (briefly, rainbow dominating function) $f : V \longrightarrow \mathcal{P}(\{1,2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V_0^f, V_1^f, V_2^f, V_{1,2}^f)$ to refer f) of V, where $V_0 = \{v \in V \mid f(v) = \varnothing\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$ and $V_{1,2} = \{v \in V \mid f(v) = \{1,2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$.

A kRDF f is called a *global k-rainbow dominating function* (GkRDF) if f is also a kRDF of the complement $\overline{G}$ of G. The *global k-rainbow domination number* of G, denoted by $\gamma_{grk}(G)$, is the minimum weight of a GkRDF on G. A $\gamma_{grk}(G)$-function is a GkRDF of G with weight $\gamma_{grk}(G)$. Since every global k-rainbow dominating function f of G is a kRDF of G and $\overline{G}$, and assigning 1 to the vertices with nonempty label under f is a global dominating set of G, and since assigning $\{1,2,\ldots,k\}$ to the vertices of a global dominating set yields a GkRDF, we deduce that

$$\max\{\gamma_g(G), \gamma_{rk}(G), \gamma_{rk}(\overline{G})\} \le \gamma_{grk}(G) \le k\gamma_g(G). \tag{1.1}$$

We note that the global k-rainbow domination number can differ significantly from the k-rainbow domination number. For example, for $n \ge k+1$, $\gamma_{rk}(K_n) = k$ and $\gamma_{grk}(K_n) = n$.

Our purpose in this paper is to initiate the study of the global k-rainbow domination number in graphs. We study basic properties of the global k-rainbow domination number and we establish some bounds for it.

We make use of the following results in this paper.

Theorem A ([14]). *For any graph G of order n and maximum degree $\Delta(G) \ge 1$, $\gamma_{rk}(G) \ge \dfrac{kn}{\Delta(G)+k}$.*

Theorem B ([6]). *For $n \geq 1$,*

$$\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Theorem C ([6]). *For $n \geq 3$,*

$$\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

Theorem D ([1]). *If G is a graph of order n, then $\gamma_{rk}(G) \leq n - \Delta(G) + k - 1$.*

Theorem E ([9]). *Let G be a connected graph. If there is a path $v_3v_2v_1$ in G with $\deg(v_2) = 2$ and $\deg(v_1) = 1$, then G has a $\gamma_{r2}(G)$-function f such that $f(v_1) = \{1\}$, and $2 \in f(v_3)$.*

Since the function f defined by $f(v) = \{1\}$ for each $v \in V(G)$ is a GkRDF of a graph G, we have the first part of the following observation. The second part is easy to see and therefore its proof is omitted.

Observation 1. If G is a graph of order n, then $\gamma_{grk}(G) \leq n$. Furthermore, if $1 \leq n \leq 4$, then $\gamma_{grk}(G) = n$.

2. Graphs with $\gamma_{rk}(G) = \gamma_{grk}(G)$

In this section we provide some sufficient conditions for a graph to satisfy $\gamma_{rk}(G) = \gamma_{grk}(G)$.

Proposition 1. If G is a disconnected graph with at least two components of order at least k, then

$$\gamma_{grk}(G) = \gamma_{rk}(G).$$

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of G. Assume, without loss of generality, that $|V(G_i)| \geq k$ for $i = 1, 2$. Let f be a $\gamma_{rk}(G)$-function. Obviously, $\sum_{v \in V(G_i)} |f(v)| \geq k$ for $i = 1, 2$. If $f(x) = \varnothing$ for some $x \in V(G_i)$, then clearly $\bigcup_{v \in V(G_i)} f(v) = \{1, 2, \ldots, k\}$, otherwise we may assume $\bigcup_{v \in V(G_i)} f(v) = \{1, 2, \ldots, k\}$ for $i = 1, 2$ because $|V(G_1)| \geq k$ and $|V(G_2)| \geq k$. Then f is a GkRDF of G and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G)$. Now the result follows from (1.1). □

According to Proposition 1, if G is the disjoint union of two copies of the complete graph K_n $(n \geq k)$, then $\gamma_{grk}(G) = \gamma_{rk}(G)$.

Proposition 2. If G is a disconnected graph with $r \geq 2$ components $G_1, G_2, \ldots, G_r$ of order at most $k - 1$ such that $\sum_{i=1}^{r} |V(G_i)| \geq k$, then

$$\gamma_{grk}(G) = \gamma_{rk}(G).$$

Proof. Assume that $\bigcup_{i=1}^{r} V(G_i) = \{v_1, v_2, \ldots, v_s\}$, and let f be a $\gamma_{rk}(G)$-function. Then clearly $f(v_i) \neq \varnothing$ for each i. Define $g : V(G) \longrightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ by $g(v_i) = \{k - i - 1\}$ for $1 \leq i \leq k - 1$, $g(v_i) = \{1\}$ for $i = k, k + 1, \ldots, s$ and $g(x) = f(x)$

for $x \in V(G) - \{v_1, v_2, \ldots, v_s\}$. Then obviously g is a GkRDF of G of weight $\omega(g) = \gamma_{rk}(G)$ and the proof is complete. □

According to Proposition 2, if G is the disjoint union of k copies of K_1 and a copy of the complete graph K_n $(n \geq k)$, then $\gamma_{grk}(G) = \gamma_{rk}(G)$.

Theorem 1. For any connected graph G with radius $\mathrm{rad}(G) \geq 4$, $\gamma_{gr2}(G) = \gamma_{r2}(G)$.

Proof. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(G)$-function such that $|V_{1,2}|$ is maximum. We show that f is a G2RDF of G. Suppose to the contrary that f is not a 2RDF of $\overline{G}$. Then there exists a vertex $v \in V_0$ such that $V_{1,2} \subseteq N(v)$ and either $V_1 \subseteq N(v)$ or $V_2 \subseteq N(v)$. Assume, without loss of generality, that $V_1 \subseteq N(v)$. Let u be an arbitrary vertex in $V(G)$. If $u \in V_1 \cup V_{1,2}$, then $d(u,v) = 1$. If $u \in V_0$, then u and v have a common neighbor in V_1 or $V_{1,2}$ implying that $d(u,v) \leq 2$. Let $u \in V_2$. If u has a neighbor in $V_1 \cup V_{1,2}$, then $d(u,v) \leq 2$ as above. If u has a neighbor w in V_0, then $d(u,v) \leq d(u,w) + d(w,v) \leq 3$. Otherwise, since G is connected, u has a neighbor x in V_2. Then the function g defined by $g(u) = \varnothing, g(x) = \{1,2\}$ and $g(y) = f(y)$ for $y \in V(G) - \{u,x\}$, is a $\gamma_{r2}(G)$-function which contradicts the choice of f. Thus f is a G2RDF of G and the proof is complete. □

Corollary 1. Let G be a connected graph of diameter $\mathrm{diam}(G) \geq 7$. Then

$$\gamma_{gr2}(G) = \gamma_{r2}(G).$$

The next results is an immediate consequence of Theorems B, C and 1.

Corollary 2. For $n \geq 8$,

$$\gamma_{gr2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Corollary 3. For $n \geq 8$,

$$\gamma_{gr2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

3. Bounds on the global k-rainbow domination number

In this section we present some sharp lower and upper bounds on $\gamma_{grk}(G)$.

Proposition 3. For any integer $k \geq 2$ and any graph G of order $n \geq 2k$,

$$\gamma_{grk}(G) \geq 2k.$$

Proof. Let f be a $\gamma_{grk}(G)$-function, and let $V_0 = \{v \in V(G) \,|\, f(v) = \varnothing\}$. If $V_0 = \varnothing$, then $\gamma_{grk}(G) = n \geq 2k$. Let $V_0 \neq \varnothing$ and $v \in V_0$. Then $\bigcup_{x \in N_G(v)} f(x) = \{1,2,\ldots,k\}$ and $\bigcup_{x \in N_{\overline{G}}(v)} f(x) = \{1,2,\ldots,k\}$. Since $N_G(v) \cap N_{\overline{G}}(v) = \varnothing$, we obtain $\gamma_{grk}(G) = \omega(f) \geq 2k$, as desired. □

This bound is sharp for the disjoint union of two copies of the complete graph K_n $(n \geq k+1)$.

Proposition 4. For any graph G of order $n \geq 4$, $\gamma_{gr2}(G) = 4$ if and only if of G satisfies one of the following properties.
(i) $n = 4$,
(ii) there exist two vertices u and v in G such that $N(u) \cap N(v) = \varnothing$ and $N[u] \cup N[v] = V$,
(iii) there exist three distinct vertices u, v, w in G such that $N(u) \cap (N(v) \cup N(w)) = \varnothing$ and $N(u) \cup (N(v) \cap N(w)) = V - \{u, v, w\}$,
(iv) there exist four distinct vertices u, v, w, x in G such that $(N(u) \cap N(v)) \setminus \{w, x\} = \varnothing, (N(w) \cap N(x)) \setminus \{u, v\} = \varnothing$, $(N[u] \cup N[v]) \setminus \{w, x\} = V - \{w, x\}$ and $(N[w] \cup N[x]) \setminus \{u, v\} = V - \{u, v\}$.

Proof. If $n = 4$, then it is clear that $\gamma_{gr2}(G) = 4$. Let $n \geq 5$. If (ii) holds, then the function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $f(u) = f(v) = \{1, 2\}$ and $f(z) = \varnothing$ for $z \in V(G) - \{u, v\}$, is a 2RDF of G and $\overline{G}$ which yields $\gamma_{gr2}(G) = 4$ by Proposition 3. If (iii) holds, then the function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $f(u) = \{1, 2\}$, $f(v) = \{1\}$, $f(w) = \{2\}$ and $f(z) = \varnothing$ for $z \in V(G) - \{u, v, w\}$, is a 2RDF of G and $\overline{G}$ which yields $\gamma_{gr2}(G) = 4$ again. Let (iv) hold. Then the function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $f(u) = f(v) = \{1\}$, $f(w) = f(x) = \{2\}$ and $f(z) = \varnothing$ for $z \in V(G) - \{u, v, x, w\}$, is a 2RDF of G and $\overline{G}$. This implies that $\gamma_{gr2}(G) = 4$.

Conversely, Let $\gamma_{gr2}(G) = 4$ and let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{gr2}(G)$-function such that $|V_{1,2}|$ is maximum. We consider three cases.

Case 1. $|V_{1,2}| = 2$.
Let $V_{1,2} = \{u, v\}$. Then $V_0 = V(G) - \{u, v\}$. Since f is a G2RDF, each vertex in $w \in V(G) - \{u, v\}$ must be adjacent to a vertex in $\{u, v\}$ in both G and $\overline{G}$. It follows that $N[u] \cup N[v] = V$ and $N(u) \cap N(v) = \varnothing$, i.e. G satisfies (ii).

Case 2. $|V_{1,2}| = 1$.
Then $|V_1| = |V_2| = 1$. Let $V_{1,2} = \{u\}, V_1 = \{v\}$ and $V_2 = \{w\}$. Hence $V_0 = V(G) - \{u, v, w\}$. Every vertex of $w \in V(G) - \{u, v, w\}$ must be adjacent to u or both of v, w in G and $\overline{G}$ because f is a 2RDF of G and $\overline{G}$. This yields $N(u) \cap (N(v) \cup N(w)) = \varnothing$ and $N(u) \cup (N(v) \cap N(w)) = V - \{u, v, w\}$. Thus G satisfies (iii) in this case.

Case 3. $|V_{1,2}| = 0$.
If $V_0 = \varnothing$, then $V_1 \cup V_2 = V(G)$ which implies that $4 = \gamma_{gr2}(G) = |V_1 \cup V_2| = n$, i.e. G satisfies (i). Now assume that $V_0 \neq \varnothing$ and let $z \in V_0$. Since f is a 2RDF of G and $\overline{G}$, $\bigcup_{v \in N_G(z)} f(v) = \{1, 2\}$ and $\bigcup_{v \in N_{\overline{G}}(z)} f(v) = \{1, 2\}$. Assume that $u, w \in N_G(z)$ and $x, v \in N_{\overline{G}}(z)$ such that $f(u) = f(v) = \{1\}$ and $f(w) = f(x) = \{2\}$. Since f is a G2RDF, each vertex in $V(G) - \{u, v, w, x\}$ must be adjacent to a vertex in $\{u, v\}$ and a vertex in $\{w, x\}$ in G and $\overline{G}$. It follows that $(N(u) \cap N(v)) \setminus \{w, x\} = \varnothing, (N(w) \cap N(x)) \setminus \{u, v\} = \varnothing$, $(N[u] \cup N[v]) \setminus \{w, x\} = V - \{w, x\}$ and $(N[w] \cup N[x]) \setminus \{u, v\} = V - \{u, v\}$. Thus G satisfies (iv). This completes the proof. □

Proposition 5. Let $k \geq 2$ be an integer. If the graph G has $r \geq 1$ components $G_1, G_2, \ldots, G_r$ with $\sum_{i=1}^{r} |V(G_i)| \leq k-1$ then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + k - \sum_{i=1}^{r} |V(G_i)|.$$

Proof. Let $\bigcup_{i=1}^{r} V(G_i) = \{v_1, v_2, \ldots, v_s\}$, and let f be a $\gamma_{rk}(G)$-function. Clearly, $f(v_i) \neq \varnothing$ for each i. Define $g : V(G) \longrightarrow \mathcal{P}(\{1,2,\ldots,k\})$ by $g(v_s) = \{s, s+1, \ldots, k\}, g(v_i) = \{i\}$ for $i = 1,2,\ldots,s-1$ and $g(x) = f(x)$ for $x \in V(G) - \{v_1, v_2, \ldots, v_s\}$. Then obviously g is a GkRDF of G with weight $\omega(g) = \gamma_{rk}(G) + k - s$ and so $\gamma_{grk}(G) \leq \gamma_{rk}(G) + k - \sum_{i=1}^{r} |V(G_i)|$. □

Let H be the disjoint union of $r \leq k-1$ isolated vertices and a star $K_{1,s}$ with $s \geq k$. Then $\gamma_{rk}(H) = r + k$ and $\gamma_{grk}(H) = 2k$. This example demonstrates that Proposition 5 is tight.

Proposition 6. Let G be a graph of order $n \geq 4$ and $u, v \in V(G)$. If $uv \notin E(G)$, then

$$\gamma_{grk}(G) \leq n - \deg(u) - \deg(v) + 2|N(u) \cap N(v)| + 2k - 2,$$

and if $uv \in E(G)$, then

$$\gamma_{grk}(G) \leq n - \deg(u) - \deg(v) + 2|N(u) \cap N(v)| + 2k.$$

Proof. Define $f : V(G) \longrightarrow \mathcal{P}(\{1,2,\ldots,k\})$ as follows

$$f(z) = \begin{cases} \{1,2,\ldots,k\} & \text{if } z \in \{u,v\} \\ \varnothing & \text{if } z \in ((N(u) \cup N(v)) - \{u,v\}) \setminus (N(u) \cap N(v)) \\ \{1\} & \text{otherwise.} \end{cases}$$

It is easy to see that f is a GkRDF of G which attains the bound. This completes the proof. □

Corollary 4. If G is a connected triangle-free graph of order $n \geq 4$, then

$$\gamma_{grk}(G) \leq \min\{n - \Delta(G) - \delta(G) + 2k, \gamma_{rk}(G) + 2k - 1\}.$$

Proof. By considering a vertex of maximum degree and one of its neighbors, it follows from Proposition 6 that $\gamma_{grk}(G) \leq n - \Delta(G) - \delta(G) + 2k$. Hence, it is sufficient to show that $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 1$. If $n \leq \gamma_{rk}(G) + 2k - 1$, the result is immediate. Let $n > \gamma_{rk}(G) + 2k - 1$ and let f be a $\gamma_{rk}(G)$-function. Then there exists a vertex u such that $f(u) = \varnothing$. Then u has a neighbor v such that $|f(v)| \geq 1$. Define $g : V(G) \longrightarrow \mathcal{P}(\{1,2,\ldots,k\})$ by $g(u) = g(v) = \{1,2,\ldots,k\}$ and $g(x) = f(x)$ otherwise. Clearly, g is a GkRDF of G and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 1$. This completes the proof. □

Proposition 7. Let $k \geq 2$ be an integer, and let G be a graph of diameter $\text{diam}(G) \geq 5$. Then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 2.$$

Proof. If G is disconnected, then the result follows from Propositions 1 and 5. Henceforth, we assume that G is connected. Let f be a $\gamma_{rk}(G)$-function. Let $v_1 v_2 \ldots v_d$ be a diametral path in G. If $f(v_1) = f(v_d) = \varnothing$, then we have $\bigcup_{x \in N(v_1)} f(x) = \{1, 2, \ldots, k\}$ and $\bigcup_{x \in N(v_d)} f(x) = \{1, 2, \ldots, k\}$. Since $\text{diam}(G) \geq 5$, we have $N(v_1) \cap N(v_d) = \varnothing$. It follows that f is a GkRDF of G and hence $\gamma_{grk}(G) = \gamma_{rk}(G)$. If $f(v_1) \neq \varnothing$ and $f(v_d) \neq \varnothing$, then the function $g : V \longrightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $g(v_1) = g(v_d) = \{1, 2, \ldots, k\}$ and $g(x) = f(x)$ for $x \in V(G) - \{v_1, v_d\}$, is a GkRDF of G of weight at most $\omega(f) + 2k - 2$ and so $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 2$. Now let $f(v_1) = \varnothing$ and $f(v_d) \neq \varnothing$ (the case $f(v_1) \neq \varnothing$ and $f(v_d) = \varnothing$ is similar). Define $g : V \longrightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ by $g(v_d) = \{1, 2, \ldots, k\}$ and $g(x) = f(x)$ for $x \in V(G) - \{v_d\}$. Obviously, g is a GkRDF of G of weight at most $\omega(f) + k - 1$ and so $\gamma_{grk}(G) \leq \gamma_{rk}(G) + k - 1$. This completes the proof. □

Proposition 8. If G is a graph of diameter 3 or 4, then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k.$$

Proof. Let f be a $\gamma_{rk}(G)$-function, and let u and v be two vertices of G such that $d(u, v) = \text{diam}(G)$. Then the function $g : V \longrightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $g(u) = g(v) = \{1, 2, \ldots, k\}$ and $g(x) = f(x)$ for $x \in V(G) - \{u, v\}$, is a GkRDF of G and therefore $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k$. □

Theorem 2. If G is a graph of order $n \geq 4$ with minimum degree $\delta(G)$, then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1.$$

This bound is sharp for stars $K_{1,t}$ ($t \geq 2k - 1$) by Proposition 3.

Proof. If G is disconnected, then the result follows from Propositions 1 and 5. Therefore we assume that G is connected. Let u be a vertex of minimum degree $\delta(G)$, f be a $\gamma_{rk}(G)$-function and $B = \{x \in N(u) \mid f(x) = \varnothing\}$.

If $f(u) = \varnothing$, then $\bigcup_{v \in N(u) - B} f(v) = \{1, 2, \ldots, k\}$. Then obviously the function $g : V(G) \longrightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $g(u) = \{1, 2, \ldots, k\}, g(x) = \{1\}$ if $x \in B$ and $g(z) = f(z)$ otherwise, is a GkRDF of G with weight at most $\gamma_{rk}(G) + \delta(G) + k - 1$ and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1$.

Let $|f(u)| \geq 1$. Define $g : V(G) \longrightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ by $g(u) = \{1, 2, \ldots, k\}$, $g(v) = \{1\}$ if $v \in B$ and $g(z) = f(z)$ for each $z \in V(G) - (B \cup \{u\})$. It is clear that g is a GkRDF of G with weight at most $\gamma_{rk}(G) + \delta(G) + k - 1$ and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1$. This completes the proof. □

4. Global rainbow domination numbers of trees

According to Theorem 2, for any tree T of order $n \geq 4$ we have

$$\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 2. \tag{4.1}$$

In this section we characterize all extremal trees attaining equality in (4.1). We begin with some lemmas giving some sufficient conditions for a tree to have global 2-rainbow domination number less than $\gamma_{r2}(T) + 2$. As a special case, Corollary 1 and Proposition 3 imply the next results.

Corollary 5. For any tree T with $\operatorname{diam}(T) \geq 7$, $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

Corollary 6. If T is a star of order $n \geq 4$, then $\gamma_{gr2}(T) = \gamma_{r2}(T) + 2$.

Lemma 1. Let T be a tree. If T has two strong support vertices, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$.

Proof. Let u and v be two strong support vertices of T and let f be a $\gamma_{r2}(T)$-function. Obviously we may assume that $f(u) = f(v) = \{1,2\}$. Since T is a tree, u and v have at most one common neighbor. If u and v have no common neighbor, then clearly f is a G2RDF of T and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. If u and v has a common neighbor, say w, then the function g defined by $g(w) = f(w) \cup \{1\}$ and $g(x) = f(x)$ otherwise, is a G2RDF of T of weight at most $\gamma_{r2}(T) + 1$ and the result follows. □

Lemma 2. Let T be a tree. If $\operatorname{diam}(T) = 6$, then $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

Proof. Let $P = v_1 v_2 \ldots v_7$ be a diametral path of T and let f be a $\gamma_{r2}(T)$-function. Root T at v_1. If v_2 and v_6 are strong support vertices, then f is a $\gamma_{gr2}(T)$-function since v_2 and v_6 have no common neighbor. Hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Assume, without loss of generality, that $\deg(v_2) = 2$. By Theorem E, we may assume $f(v_1) = \{1\}$ and $2 \in f(v_3)$. If v_6 is a strong support vertex, then we can assume $f(v_6) = \{1,2\}$ and clearly f is a G2RDF of T implying that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Henceforth, we assume $\deg(v_6) = 2$. By Theorem E, we may assume $f(v_7) = \{1\}$ and $2 \in f(v_5)$. Define the function g by $g(v) = \{1\}$ if $v \in V(T_{v_5})$ and $f(v) = \{2\}$, $g(v) = \{2\}$ if $v \in V(T_{v_5})$ and $f(v) = \{1\}$ and $g(x) = f(x)$ otherwise. Clearly, g is a G2RDF of T of weight $\gamma_{r2}(T)$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. This completes the proof. □

Lemma 3. Let T be a tree. If $\operatorname{diam}(T) = 5$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$.

Proof. Let $P = v_1 v_2 \ldots v_6$ be a diametral path of T, and let f be a $\gamma_{r2}(T)$-function. If v_2 and v_5 are strong support vertices, then f is a $\gamma_{gr2}(T)$-function and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Assume, without loss of generality, that all support vertices adjacent to v_4 have degree 2. By Theorem E, we may assume $f(v_6) = \{1\}$ and $2 \in f(v_4)$. Then the function g defined by $g(v_3) = f(v_3) \cup \{1\}$ and $g(x) = f(x)$ otherwise, is a G2RDF of T of weight at most $\gamma_{r2}(T) + 1$ that implies $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$. □

A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider*. A *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. The *center of a spider*, is the center of the star whose subdivision produced the spider.

Definition 1. For $1 \leq i \leq 2$, let $\mathcal{B}_i$ be the family of trees T defined as follows and let $\mathcal{B} = \bigcup_{i=1}^{2} \mathcal{B}_i$.
$\mathcal{B}_1$: T is a spider S_t for some $t \geq 2$ with exception of stars, wounded spiders S_t ($t \geq 3$) with exactly one wounded leg or wounded spiders S_t ($t \geq 3$) with at least four healthy legs.
$\mathcal{B}_2$: T is obtained from stars $K_{1,r_1}, K_{1,r_2}, \ldots, K_{1,r_j}$ where $r_k \geq 3$ for $1 \leq k \leq j$, with centers $y_1, y_2, \ldots, y_j$ ($j \geq 2$) by adding a new vertex x and joining x to all vertices y_j and adding at most one pendant edge at x.

Lemma 4. Let T be a tree. If $\operatorname{diam}(T) = 4$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$ and equality holds if and only if $T \in \mathcal{B}$.

Proof. Let $\operatorname{diam}(T) = 4$ and let $P = v_1v_2v_3v_4v_5$ be a diametral path of T. Let f be a $\gamma_{r2}(T)$-function. Consider the following cases.
Case 1. $\deg(v_2) = 3$.
Suppose u, v_1 are the leaves adjacent to v_2. Then we can assume that $f(v_2) = \{1,2\}$. If $\deg(v_4) \geq 3$, then we may assume $f(v_4) = \{1,2\}$ and if $\deg(v_4) = 2$ then by Theorem E we can assume $f(v_5) = \{1\}$ and $2 \in f(v_3)$. Define $g : V(T) \to \mathcal{P}(\{1,2\})$ by $g(v_1) = \{1\}, g(u) = \{2\}, g(v_2) = \varnothing$ and $g(x) = f(x)$ otherwise. Obviously g is a G2RDF of T of weight $\gamma_{r2}(T)$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

By Case 1, we may assume that all support vertices adjacent to v_3 have degree different from 3.

Case 2. $\deg(v_2) > 3$.
Then $f(v_2) = \{1,2\}$. If $\deg(v_4) = 2$, then by Theorem E we may assume $f(v_5) = \{1\}$ and $2 \in f(v_3)$, and clearly f is a G2RDF of T and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. So we assume that each support vertex adjacent to v_3 has degree at least 4. If v_3 is a strong support vertex, then $f(v_3) = \{1,2\}$ and clearly f is a G2RDF of T and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Let v_3 be not a strong support vertex. Then $T \in \mathcal{B}_2$ and T has at most two $\gamma_{r2}(T)$-functions which none of them is G2RDF of T and hence $\gamma_{gr2}(T) \geq \gamma_{r2}(T) + 1$. On the other hand, the function g defined by $g(v_3) = \{1\}$ and $g(x) = f(x)$ otherwise is a G2RDF of T of weight $\gamma_{r2}(T) + 1$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T) + 1$.

By Cases 1 and 2, we may assume that all support vertices adjacent to v_3 have degree 2. Thus T is a spider of diameter 4. If T is a wounded spiders S_t ($t \geq 3$) with exactly one wounded leg, then the function g that assigns $\varnothing$ to all support vertices

of T with exception of the center of spider, $\{1\}$ to the center of spider and the leaf adjacent to the center of spider, and $\{2\}$ to the other leaves, is a G2RDF of T of weight $\gamma_{r2}(T)$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Now T is a wounded spider S_t $(t \geq 3)$ with at least four healthy legs. Suppose x is the center of T and u_1, u_2, u_3, u_4 are leaves at distance two from x. Then the function g that assigns $\{1,2\}$ to x, $\varnothing$ to all support vertices of T, $\{1\}$ to u_1, u_2, and $\{2\}$ to the other leaves, is a G2RDF of T of weight $\gamma_{r2}(T)$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Finally let T be a spider that is not a wounded spider S_t $(t \geq 3)$ with exactly one wounded leg or a wounded spider S_t $(t \geq 3)$ with at least four healthy legs, that is $T \in \mathcal{B}_1$. It is easy to see that in this case $\gamma_{gr2}(T) = \gamma_{r2}(T) + 1$ and the proof is complete. □

For $p, q \geq 1$, a double star $DS(p,q)$ is a tree with exactly two vertices that are not leaves, with one adjacent to p leaves and the other to q leaves.

Lemma 5. Let T be a tree. If $\operatorname{diam}(T) = 3$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$ and equality holds if and only if $T = DS(p,q)$ with $q \geq p = 1$.

Proof. Let $\operatorname{diam}(T) = 3$. Then T is a double star $DS(p,q)$ with $q \geq p \geq 1$. Let u, v be the vertices of T of degree p and q, respectively. If $p \geq 2$, then u, v are strong support vertices with no common neighbor and it follows from the proof of Lemma 1 that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Henceforth, assume $p = 1$. If $q = 1$, then $T = P_4$ and clearly $\gamma_{gr2}(T) = \gamma_{r2}(T) + 1$. Let $q \geq 2$ and u' be the leaf adjacent to u. Then T has exactly two $\gamma_{r2}(T)$-functions f_i $(i = 1,2)$ defined by $f_i(v) = \{1,2\}$, $f_i(u') = \{i\}$ and $f_i(x) = \varnothing$ otherwise. Obviously, none of f_1 or f_2 is not a G2RDF of T and also the function g defined by $g(u) = \{1\}$ and $g(x) = f_1(x)$ for $x \in V(T) - \{u\}$ is a G2RDF of T that yields $\gamma_{gr2}(T) \geq \gamma_{r2}(T) + 1$. □

The next theorem is an immediate consequence of (4.1), Corollaries 5, 6 and Lemmas 2, 3, 4, 5.

Theorem 3. Let T be a tree of order $n \geq 4$. Then $\gamma_{gr2}(T) = \gamma_{r2}(T) + 2$ if and only if T is the star $K_{1,t}$ for some $t \geq 3$.

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SOME VARIANTS OF THE SZEGED INDEX UNDER ROOTED PRODUCT OF GRAPHS

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Abstract. The Szeged index $Sz(G)$ of a connected graph G is defined as the sum of the terms $n_u(e|G)n_v(e|G)$ over all edges $e = uv$ of G, where $n_u(e|G)$ is the number of vertices of G lying closer to u than to v and $n_v(e|G)$ is the number of vertices of G lying closer to v than to u. In this paper, some variants of the Szeged index such as the edge PI index, edge Szeged index, edge-vertex Szeged index, vertex-edge Szeged index, and revised edge Szeged index are studied under rooted product of graphs. Results are applied to compute these graph invariants for some chemical graphs by specializing components in rooted products.

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1. INTRODUCTION

In this paper, we consider connected finite graphs without loops or multiple edges. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $d(u,v|G)$ the distance between the vertices u and v in G which is the length of any shortest path in G connecting u and v. Let $e = uv$ be the edge of G connecting the vertices u and v. The quantities $n_0(e|G)$, $n_u(e|G)$, and $n_v(e|G)$ are defined to be the number of vertices of G equidistant from u and v, the number of vertices of G whose distance to u is smaller than the distance to v, and the number of vertices of G whose distance to v is smaller than the distance to u, respectively, i.e.,

$$n_0(e|G) = |\{z \in V(G) : d(z,u|G) = d(z,v|G)\}|,$$
$$n_u(e|G) = |\{z \in V(G) : d(z,u|G) < d(z,v|G)\}|,$$
$$n_v(e|G) = |\{z \in V(G) : d(z,v|G) < d(z,u|G)\}|.$$

For an edge $e = uv \in E(G)$ and a vertex $z \in V(G)$, the distance between z and e is defined as $d(z,e|G) = \min\{d(z,u|G), d(z,v|G)\}$. The quantities $m_0(e|G)$, $m_u(e|G)$, and $m_v(e|G)$ are defined to be the number of edges of G equidistant from u and v, the number of edges of G whose distance to u is smaller than the distance

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to v, and the number of edges of G whose distance to v is smaller than the distance to u, respectively, i.e.,

$$m_0(e|G) = |\{f \in E(G) : d(u,f|G) = d(v,f|G)\}|,$$

$$m_u(e|G) = |\{f \in E(G) : d(u,f|G) < d(v,f|G)\}|,$$

$$m_v(e|G) = |\{f \in E(G) : d(v,f|G) < d(u,f|G)\}|.$$

For the vertex $z \in V(G)$, we define

$$m_z(G) = |\{e = uv \in E(G) : d(u,z|G) \neq d(v,z|G)\}|.$$

Chemical graphs, particularly *molecular graphs*, are graph-based descriptions of molecules, with vertices representing the atoms and edges representing the bonds. A numerical invariant associated with a chemical graph is called *topological index* or *graph invariant*. Topological indices are used in theoretical chemistry for the design of chemical compounds with given physicochemical properties or given pharmacologic and biological activities [6, 20]. The *Wiener index* [21], defined as the sum of distances between all pairs of vertices in a chemical graph, is the oldest and the most thoroughly studied topological index from both theoretical and practical point of view. Motivated by the original definition of the Wiener index, the *Szeged index* [11] was introduced in 1994 which coincides with the Wiener index for a tree. It found applications in quantitative structure-property-activity-toxicity modeling [16]. The Szeged index of a graph G is defined as

$$Sz(G) = \sum_{e=uv \in E(G)} n_u(e|G) n_v(e|G).$$

In recent years, some variants of the Szeged index such as the *vertex PI index* [17], *edge PI index* [15], *edge Szeged index* [12], *edge-vertex Szeged index* [18], *vertex-edge Szeged index* [9], *revised Szeged index* [19], and *revised edge Szeged index* [7] have attracted much attention in both chemistry and mathematics. These indices are defined for a graph G as follows:

$$PI_v(G) = \sum_{e=uv \in E(G)} \big[n_u(e|G) + n_v(e|G)\big],$$

$$PI_e(G) = \sum_{e=uv \in E(G)} \big[m_u(e|G) + m_v(e|G)\big],$$

$$Sz_e(G) = \sum_{e=uv \in E(G)} m_u(e|G) m_v(e|G),$$

$$Sz_{ev}(G) = \frac{1}{2} \sum_{e=uv \in E(G)} \big[n_u(e|G) m_v(e|G) + n_v(e|G) m_u(e|G)\big],$$

$$Sz_{ve}(G) = \frac{1}{2} \sum_{e=uv\in E(G)} \big[n_u(e|G)m_u(e|G) + n_v(e|G)m_v(e|G)\big],$$

$$Sz^*(G) = \sum_{e=uv\in E(G)} \big[n_u(e|G) + \frac{n_0(e|G)}{2}\big]\big[n_v(e|G) + \frac{n_0(e|G)}{2}\big],$$

$$Sz^*_e(G) = \sum_{e=uv\in E(G)} \big[m_u(e|G) + \frac{m_0(e|G)}{2}\big]\big[m_v(e|G) + \frac{m_0(e|G)}{2}\big].$$

We refer the reader to [1, 10, 14] for more information on these indices.

The *rooted product* $G_1\{G_2\}$ of a graph G_1 and a rooted graph G_2 is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and by identifying the root vertex of the i-th copy of G_2 with the i-th vertex of G_1, for $i = 1, 2, ..., |V(G_1)|$.

In this paper, we study the edge PI index, edge Szeged index, edge-vertex Szeged index, vertex-edge Szeged index, and revised edge Szeged index under rooted product of graphs. Results are applied to compute these invariants for some chemical graphs by specializing components in rooted products. For more information on computing topological indices of rooted product see [2–5, 8, 13, 22].

2. Results and discussion

Let G_1 and G_2 be two connected graphs with vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively. In this section, we compute some variants of the Szeged index for rooted product of G_1 and G_2. Throughout this section, the graph G_2 is assumed to be rooted on the vertex $x \in V(G_2)$ and the degree of x in G_2 is denoted by δ. Also, we denote by n_i and m_i, the order and size of the graph G_i, respectively, where $i \in \{1, 2\}$. In addition, for notational convenience, we define

$$\overline{N} = \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} n_v(e|G_2), \quad \underline{N} = \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} n_u(e|G_2),$$

$$\overline{M} = \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} m_v(e|G_2), \quad \underline{M} = \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} m_u(e|G_2).$$

Theorem 1. *The edge PI index of the rooted product $G_1\{G_2\}$ is given by*

$$PI_e(G_1\{G_2\}) = PI_e(G_1) + m_2 PI_v(G_1) + n_1 PI_e(G_2) + n_1\big(m_1 + (n_1 - 1)m_2\big)m_x(G_2). \tag{2.1}$$

Proof. From the definition of the edge PI index, we have

$$PI_e(G_1\{G_2\}) = \sum_{e=uv\in E(G_1\{G_2\})} \big[m_u(e|G_1\{G_2\}) + m_v(e|G_1\{G_2\})\big].$$

We partition the above sum into two sums as follows:

The first sum S_1 consists of contributions to $PI_e(G_1\{G_2\})$ of edges from G_1,

$$S_1 = \sum_{e=uv\in E(G_1)} \big[m_u(e|G_1\{G_2\}) + m_v(e|G_1\{G_2\})\big].$$

By definition of rooted product, we have

$$\begin{aligned}
S_1 &= \sum_{e=uv\in E(G_1)} \big[\big(m_u(e|G_1) + m_2 n_u(e|G_1)\big) + \big(m_v(e|G_1) + m_2 n_v(e|G_1)\big)\big] \\
&= \sum_{e=uv\in E(G_1)} \big[m_u(e|G_1) + m_v(e|G_1)\big] \\
&\quad + m_2 \sum_{e=uv\in E(G_1)} \big[n_u(e|G_1) + n_v(e|G_1)\big] \\
&= PI_e(G_1) + m_2 PI_v(G_1).
\end{aligned}$$

The second sum S_2 consists of contributions to $PI_e(G_1\{G_2\})$ of edges from n_1 copies of G_2,

$$S_2 = n_1 \sum_{e=uv\in E(G_2)} \big[m_u(e|G_1\{G_2\}) + m_v(e|G_1\{G_2\})\big].$$

By definition of rooted product, we have

$$\begin{aligned}
S_2 &= n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)\neq d(v,x|G_2)}} \big[m_1 + (n_1-1)m_2 + m_u(e|G_2) + m_v(e|G_2)\big] \\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)= d(v,x|G_2)}} \big[m_u(e|G_2) + m_v(e|G_2)\big] \\
&= n_1 \Big[\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)\neq d(v,x|G_2)}} \big[m_u(e|G_2) + m_v(e|G_2)\big] \\
&\quad + \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)= d(v,x|G_2)}} \big[m_u(e|G_2) + m_v(e|G_2)\big]\Big] \\
&\quad + n_1\big(m_1 + (n_1-1)m_2\big)\,|\{e = uv \in E(G_2) : d(u,x|G_2) \neq d(v,x|G_2)\}| \\
&= n_1 PI_e(G_2) + n_1\big(m_1 + (n_1-1)m_2\big)m_x(G_2).
\end{aligned}$$

Eq. (2.1) is obtained by adding the quantities S_1 and S_2. □

Theorem 2. *The edge Szeged index of the rooted product $G_1\{G_2\}$ is given by*

$$\begin{aligned}
Sz_e(G_1\{G_2\}) &= Sz_e(G_1) + m_2^2 Sz(G_1) + 2m_2 Sz_{ev}(G_1) + n_1 Sz_e(G_2) \\
&\quad + n_1\big(m_1 + (n_1-1)m_2\big)\overline{M}.
\end{aligned} \tag{2.2}$$

Proof. From the definition of the edge Szeged index, we have

$$Sz_e(G_1\{G_2\}) = \sum_{e=uv\in E(G_1\{G_2\})} m_u(e|G_1\{G_2\})m_v(e|G_1\{G_2\}).$$

We partition the above sum into two sums as follows:
The first sum S_1 consists of contributions to $Sz_e(G_1\{G_2\})$ of edges from G_1,

$$S_1 = \sum_{e=uv\in E(G_1)} m_u(e|G_1\{G_2\})m_v(e|G_1\{G_2\}).$$

By definition of rooted product, we have

$$\begin{aligned}
S_1 &= \sum_{e=uv\in E(G_1)} \big[m_u(e|G_1) + m_2 n_u(e|G_1)\big]\big[m_v(e|G_1) + m_2 n_v(e|G_1)\big] \\
&= \sum_{e=uv\in E(G_1)} m_u(e|G_1)m_v(e|G_1) + m_2^2 \sum_{e=uv\in E(G_1)} n_u(e|G_1)n_v(e|G_1) \\
&\quad + m_2 \sum_{e=uv\in E(G_1)} \big[n_u(e|G_1)m_v(e|G_1) + n_v(e|G_1)m_u(e|G_1)\big] \\
&= Sz_e(G_1) + m_2^2 Sz(G_1) + 2m_2 Sz_{ev}(G_1).
\end{aligned}$$

The second sum S_2 consists of contributions to $Sz_e(G_1\{G_2\})$ of edges from n_1 copies of G_2,

$$S_2 = n_1 \sum_{e=uv\in E(G_2)} m_u(e|G_1\{G_2\})m_v(e|G_1\{G_2\}).$$

By definition of rooted product, we have

$$\begin{aligned}
S_2 &= n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} \big[m_1 + (n_1-1)m_2 + m_u(e|G_2)\big]m_v(e|G_2) \\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}} m_u(e|G_2)m_v(e|G_2) \\
&= n_1\Bigg[\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} m_u(e|G_2)m_v(e|G_2) \\
&\quad + \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}} m_u(e|G_2)m_v(e|G_2)\Bigg] \\
&\quad + n_1\big(m_1 + (n_1-1)m_2\big) \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} m_v(e|G_2) \\
&= n_1 Sz_e(G_2) + n_1\big(m_1 + (n_1-1)m_2\big)\overline{M}.
\end{aligned}$$

Eq. (2.2) is obtained by adding the quantities S_1 and S_2. □

Theorem 3. *The edge-vertex Szeged index of the rooted product $G_1\{G_2\}$ is given by*

$$Sz_{ev}(G_1\{G_2\}) = n_2 Sz_{ev}(G_1) + n_2 m_2 Sz(G_1) + n_1 Sz_{ev}(G_2) + \frac{1}{2}n_1 n_2 (n_1-1)\overline{M} + \frac{1}{2}n_1\big(m_1 + m_2(n_1-1)\big)\overline{N}. \tag{2.3}$$

Proof. From the definition of the edge-vertex Szeged index, we have

$$Sz_{ev}(G_1\{G_2\}) = \frac{1}{2}\sum_{e=uv\in E(G_1\{G_2\})} \big[n_u(e|G_1\{G_2\})m_v(e|G_1\{G_2\}) + n_v(e|G_1\{G_2\})m_u(e|G_1\{G_2\})\big].$$

We partition the above sum into two sums as follows:
The first sum S_1 consists of contributions to $Sz_{ev}(G_1\{G_2\})$ of edges from G_1,

$$S_1 = \frac{1}{2}\sum_{e=uv\in E(G_1)} \big[n_u(e|G_1\{G_2\})m_v(e|G_1\{G_2\}) + n_v(e|G_1\{G_2\})m_u(e|G_1\{G_2\})\big].$$

By definition of rooted product, we have

$$\begin{aligned} S_1 &= \frac{1}{2}\sum_{e=uv\in E(G_1)} \Big[n_2 n_u(e|G_1)\big[m_v(e|G_1) + m_2 n_v(e|G_1)\big] \\ &\quad + n_2 n_v(e|G_1)\big[m_u(e|G_1) + m_2 n_u(e|G_1)\big]\Big] \\ &= \frac{1}{2}n_2 \sum_{e=uv\in E(G_1)} \big[n_u(e|G_1)m_v(e|G_1) + n_v(e|G_1)m_u(e|G_1)\big] \\ &\quad + n_2 m_2 \sum_{e=uv\in E(G_1)} n_u(e|G_1)n_v(e|G_1) \\ &= n_2 Sz_{ev}(G_1) + n_2 m_2 Sz(G_1). \end{aligned}$$

The second sum S_2 consists of contributions to $Sz_{ev}(G_1\{G_2\})$ of edges from n_1 copies of G_2,

$$S_2 = \frac{1}{2}n_1 \sum_{e=uv\in E(G_2)} \big[n_u(e|G_1\{G_2\})m_v(e|G_1\{G_2\}) + n_v(e|G_1\{G_2\})m_u(e|G_1\{G_2\})\big].$$

By definition of rooted product, we have

$$S_2 = \frac{1}{2}n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} \Big[\big[n_u(e|G_2) + n_2(n_1-1)\big]m_v(e|G_2)$$

$$
\begin{aligned}
&+ n_v(e|G_2)\big[m_1 + m_2(n_1-1) + m_u(e|G_2)\big]\Big] \\
&+ \frac{1}{2}n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}} \big[n_u(e|G_2)m_v(e|G_2) + n_v(e|G_2)m_u(e|G_2)\big] \\
&= \frac{1}{2}n_1\Big[\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} \big[n_u(e|G_2)m_v(e|G_2) + n_v(e|G_2)m_u(e|G_2)\big] \\
&+ \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}} \big[n_u(e|G_2)m_v(e|G_2) + n_v(e|G_2)m_u(e|G_2)\big]\Big] \\
&+ \frac{1}{2}n_1n_2(n_1-1) \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} m_v(e|G_2) \\
&+ \frac{1}{2}n_1\big(m_1 + m_2(n_1-1)\big) \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} n_v(e|G_2) \\
&= n_1Sz_{ev}(G_2) + \frac{1}{2}n_1n_2(n_1-1)\overline{M} + \frac{1}{2}n_1\big(m_1 + m_2(n_1-1)\big)\overline{N}.
\end{aligned}
$$

Eq. (2.3) is obtained by adding the quantities S_1 and S_2. □

Let G be a graph with edge set $E(G)$. We define the *second vertex PI index* of G as

$$PI_v^{(2)}(G) = \sum_{e=uv\in E(G)} \big[n_u(e|G)^2 + n_v(e|G)^2\big].$$

Theorem 4 ([9]). *Let G be a graph of order n and size m. Then*

$$Sz^*(G) = \frac{mn^2}{4} - \frac{1}{4}PI_v^{(2)}(G) + \frac{1}{2}Sz(G). \tag{2.4}$$

Theorem 5. *The vertex-edge Szeged index of the rooted product $G_1\{G_2\}$ is given by*

$$
\begin{aligned}
Sz_{ve}(G_1\{G_2\}) = {} & n_2Sz_{ve}(G_1) + \frac{1}{2}n_2m_2PI_v^{(2)}(G_1) + n_1Sz_{ve}(G_2) \\
& + \frac{1}{2}n_1n_2(n_1-1)\big(m_1 + m_2(n_1-1)\big)m_x(G_2) \\
& + \frac{1}{2}n_1\big(m_1 + m_2(n_1-1)\big)\underline{N} + \frac{1}{2}n_1n_2(n_1-1)\underline{M}.
\end{aligned} \tag{2.5}
$$

Proof. From the definition of the vertex-edge Szeged index, we have

$$Sz_{ve}(G_1\{G_2\}) = \frac{1}{2}\sum_{e=uv\in E(G_1\{G_2\})}\Big[n_u(e|G_1\{G_2\})m_u(e|G_1\{G_2\}) + n_v(e|G_1\{G_2\})m_v(e|G_1\{G_2\})\Big].$$

We partition the above sum into two sums as follows:
The first sum S_1 consists of contributions to $Sz_{ve}(G_1\{G_2\})$ of edges from G_1,

$$S_1 = \frac{1}{2}\sum_{e=uv\in E(G_1)}\Big[n_u(e|G_1\{G_2\})m_u(e|G_1\{G_2\}) + n_v(e|G_1\{G_2\})m_v(e|G_1\{G_2\})\Big].$$

By definition of rooted product, we have

$$\begin{aligned}
S_1 &= \frac{1}{2}\sum_{e=uv\in E(G_1)}\Big[n_2n_u(e|G_1)\big[m_u(e|G_1) + m_2n_u(e|G_1)\big] \\
&\quad + n_2n_v(e|G_1)\big[m_v(e|G_1) + m_2n_v(e|G_1)\big]\Big] \\
&= \frac{1}{2}n_2\sum_{e=uv\in E(G_1)}\big[n_u(e|G_1)m_u(e|G_1) + n_v(e|G_1)m_v(e|G_1)\big] \\
&\quad + \frac{1}{2}n_2m_2\sum_{e=uv\in E(G_1)}\big[n_u(e|G_1)^2 + n_v(e|G_1)^2\big] \\
&= n_2Sz_{ve}(G_1) + \frac{1}{2}n_2m_2PI_v^{(2)}(G_1).
\end{aligned}$$

The second sum S_2 consists of contributions to $Sz_{ve}(G_1\{G_2\})$ of edges from n_1 copies of G_2,

$$S_2 = \frac{1}{2}n_1\sum_{e=uv\in E(G_2)}\Big[n_u(e|G_1\{G_2\})m_u(e|G_1\{G_2\}) + n_v(e|G_1\{G_2\})m_v(e|G_1\{G_2\})\Big].$$

By definition of rooted product, we have

$$\begin{aligned}
S_2 &= \frac{1}{2}n_1\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}}\Big[\big[n_u(e|G_2) + n_2(n_1-1)\big]\big[m_1 + m_2(n_1-1) \\
&\quad + m_u(e|G_2)\big] + n_v(e|G_2)m_v(e|G_2)\Big] \\
&\quad + \frac{1}{2}n_1\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}}\big[n_u(e|G_2)m_u(e|G_2) + n_v(e|G_2)m_v(e|G_2)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}n_1\Bigg[\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} \big[n_u(e|G_2)m_u(e|G_2)+n_v(e|G_2)m_v(e|G_2)\big] \\
&\quad + \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}} \big[n_u(e|G_2)m_u(e|G_2)+n_v(e|G_2)m_v(e|G_2)\big]\Bigg] \\
&\quad + \frac{1}{2}n_1n_2(n_1-1)\big(m_1+m_2(n_1-1)\big)\Big|\{e=uv\in E(G_2): d(u,x|G_2) \\
&< d(v,x|G_2)\}\Big| + \frac{1}{2}n_1\big(m_1+m_2(n_1-1)\big)\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} n_u(e|G_2) \\
&\quad + \frac{1}{2}n_1n_2(n_1-1)\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} m_u(e|G_2) \\
&= n_1Sz_{ve}(G_2) + \frac{1}{2}n_1n_2(n_1-1)\big(m_1+m_2(n_1-1)\big)m_x(G_2) \\
&\quad + \frac{1}{2}n_1\big(m_1+m_2(n_1-1)\big)\underline{N} + \frac{1}{2}n_1n_2(n_1-1)\underline{M}.
\end{aligned}
$$

Eq. (2.5) is obtained by adding the quantities S_1 and S_2. □

Using Eq. (2.4), we can get an alternative formula for the vertex-edge Szeged index of the rooted product of G_1 and G_2.

Corollary 1. *The vertex-edge Szeged index of the rooted product $G_1\{G_2\}$ is given by*

$$
\begin{aligned}
Sz_{ve}(G_1\{G_2\}) &= n_2Sz_{ve}(G_1)+n_2m_2Sz(G_1)-2n_2m_2Sz^*(G_1)+n_1Sz_{ve}(G_2) \\
&\quad + \frac{1}{2}n_1^2n_2m_1m_2 + \frac{1}{2}n_1n_2(n_1-1)\big(m_1+m_2(n_1-1)\big)m_x(G_2) \\
&\quad + \frac{1}{2}n_1n_2(n_1-1)\underline{M} + \frac{1}{2}n_1\big(m_1+m_2(n_1-1)\big)\underline{N}. \qquad (2.6)
\end{aligned}
$$

Proof. From Eq. (2.4), we get

$$PI_v^{(2)}(G_1) = m_1n_1^2 - 4Sz^*(G_1) + 2Sz(G_1).$$

Eq. (2.6) is obtained by applying the above equation in Eq. (2.5) and simplifying the resulting expression. □

Let G be a graph with edge set $E(G)$. We define the *second edge PI index* of G as

$$PI_e^{(2)}(G) = \sum_{e=uv\in E(G)} \big[m_u(e|G)^2 + m_v(e|G)^2\big].$$

Theorem 6 ([9]). *Let G be a graph of size m. Then*

$$Sz_e^*(G) = \frac{m^3}{4} - \frac{1}{4}PI_e^{(2)}(G) + \frac{1}{2}Sz_e(G). \tag{2.7}$$

Lemma 1. *The second edge PI index of $G_1\{G_2\}$ is given by*

$$\begin{aligned} PI_e^{(2)}(G_1\{G_2\}) &= PI_e^{(2)}(G_1) + m_2^2 PI_v^{(2)}(G_1) + 4m_2 Sz_{ve}(G_1) \\ &\quad + n_1 PI_e^{(2)}(G_2) + n_1\big(m_1 + (n_1-1)m_2\big)^2 m_x(G_2) \\ &\quad + 2n_1\big(m_1 + (n_1-1)m_2\big)\underline{M}. \end{aligned} \tag{2.8}$$

Proof. From the definition of the second edge PI index, we have

$$PI_e^{(2)}(G_1\{G_2\}) = \sum_{e=uv\in E(G_1\{G_2\})} \big[m_u(e|G_1\{G_2\})^2 + m_v(e|G_1\{G_2\})^2\big].$$

We partition the above sum into two sums as follows:
The first sum S_1 consists of contributions to $PI_e^{(2)}(G_1\{G_2\})$ of edges from G_1,

$$S_1 = \sum_{e=uv\in E(G_1)} \big[m_u(e|G_1\{G_2\})^2 + m_v(e|G_1\{G_2\})^2\big].$$

By definition of rooted product, we have

$$\begin{aligned} S_1 &= \sum_{e=uv\in E(G_1)} \big[\big(m_u(e|G_1) + m_2 n_u(e|G_1)\big)^2 + \big(m_v(e|G_1) + m_2 n_v(e|G_1)\big)^2\big] \\ &= \sum_{e=uv\in E(G_1)} \big[m_u(e|G_1)^2 + m_v(e|G_1)^2\big] \\ &\quad + m_2^2 \sum_{e=uv\in E(G_1)} \big[n_u(e|G_1)^2 + n_v(e|G_1)^2\big] \\ &\quad + 2m_2 \sum_{e=uv\in E(G_1)} \big[n_u(e|G_1)m_u(e|G_1) + n_v(e|G_1)m_v(e|G_1)\big] \\ &= PI_e^{(2)}(G_1) + m_2^2 PI_v^{(2)}(G_1) + 4m_2 Sz_{ve}(G_1). \end{aligned}$$

The second sum S_2 consists of contributions to $PI_e^{(2)}(G_1\{G_2\})$ of edges from n_1 copies of G_2,

$$S_2 = n_1 \sum_{e=uv\in E(G_2)} \big[m_u(e|G_1\{G_2\})^2 + m_v(e|G_1\{G_2\})^2\big].$$

By definition of rooted product, we have

$$S_2 = n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} \big[\big(m_1 + (n_1-1)m_2 + m_u(e|G_2)\big)^2 + m_v(e|G_2)^2\big]$$

$$
\begin{aligned}
&+n_1 \sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}} \big[m_u(e|G_2)^2+m_v(e|G_2)^2\big]\\
&=n_1\Big[\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} \big[m_u(e|G_2)^2+m_v(e|G_2)^2\big]\\
&+\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)=d(v,x|G_2)}} \big[m_u(e|G_2)^2+m_v(e|G_2)^2\big]\Big]\\
&+n_1\big(m_1+(n_1-1)m_2\big)^2\,|\{e=uv\in E(G_2): d(u,x|G_2)<d(v,x|G_2)\}|\\
&+2n_1\big(m_1+(n_1-1)m_2\big)\sum_{\substack{e=uv\in E(G_2),\\ d(u,x|G_2)<d(v,x|G_2)}} m_u(e|G_2)\\
&=n_1PI_e^{(2)}(G_2)+n_1\big(m_1+(n_1-1)m_2\big)^2m_x(G_2)+2n_1\big(m_1+(n_1-1)m_2\big)\underline{M}.
\end{aligned}
$$

Eq. (2.8) is obtained by adding the quantities S_1 and S_2. □

Theorem 7. *The revised edge Szeged index of the rooted product $G_1\{G_2\}$ is given by*

$$
\begin{aligned}
Sz_e^*(G_1\{G_2\}) &= Sz_e^*(G_1)+m_2^2Sz^*(G_1)+m_2\big(Sz_{ev}(G_1)-Sz_{ve}(G_1)\big)\\
&\quad+n_1Sz_e^*(G_2)+\frac{1}{2}n_1(m_1+(n_1-1)m_2)(\overline{M}-\underline{M})\\
&\quad-\frac{1}{4}n_1\big(m_1+(n_1-1)m_2\big)^2m_x(G_2)\\
&\quad+\frac{1}{4}n_1m_2\big[m_1(3m_1+2n_1m_2)+m_2^2(n_1^2-1)\big]. \qquad (2.9)
\end{aligned}
$$

Proof. By Eq. (2.7),

$$
Sz_e^*(G_1\{G_2\})=\frac{1}{4}(m_1+n_1m_2)^3-\frac{1}{4}PI_e^{(2)}(G_1\{G_2\})+\frac{1}{2}Sz_e(G_1\{G_2\}).
$$

Now using Eq. (2.2) and Eq. (2.8), we get

$$
\begin{aligned}
Sz_e^*(G_1\{G_2\}) &= \frac{1}{4}\big[m_1^3+3m_1^2n_1m_2+3m_1n_1^2m_2^2+n_1^3m_2^3\big]-\frac{1}{4}\big[PI_e^{(2)}(G_1)\\
&\quad+m_2^2PI_v^{(2)}(G_1)+4m_2Sz_{ve}(G_1)+n_1PI_e^{(2)}(G_2)\\
&\quad+n_1\big(m_1+(n_1-1)m_2\big)^2m_x(G_2)+2n_1\big(m_1+(n_1-1)m_2\big)\underline{M}\big]\\
&\quad+\frac{1}{2}\big[Sz_e(G_1)+m_2^2Sz(G_1)+2m_2Sz_{ev}(G_1)+n_1Sz_e(G_2)\\
&\quad+n_1\big(m_1+(n_1-1)m_2\big)\overline{M}\big].
\end{aligned}
$$

Eq. (2.9) is obtained by simplifying the above expression. □

3. Examples and corollaries

In this section, we apply the results of the previous section to compute the edge PI index, edge Szeged index, edge-vertex Szeged index, vertex-edge Szeged index, and revised edge Szeged index of some graphs by specializing components in rooted product.

Let P_n, S_n, and C_n denote the n-vertex path, star, and cycle, respectively. Some Szeged-related topological indices of these graphs have been given in Table 1.

$Graph$	P_n	S_n	C_n, n is even	C_n, n is odd
PI_v	$n(n-1)$	$n(n-1)$	n^2	$n(n-1)$
PI_e	$(n-1)(n-2)$	$(n-1)(n-2)$	$n(n-2)$	$n(n-1)$
Sz	$\binom{n+1}{3}$	$(n-1)^2$	$\frac{n^3}{4}$	$\frac{n(n-1)^2}{4}$
Sz_e	$\binom{n-1}{3}$	0	$\frac{n(n-2)^2}{4}$	$\frac{n(n-1)^2}{4}$
Sz_{ev}	$\binom{n}{3}$	$\frac{(n-1)(n-2)}{2}$	$\frac{n^2(n-2)}{2}$	$\frac{n(n-1)^2}{2}$
Sz_{ve}	$2\binom{n}{3}$	$\frac{(n-1)^2(n-2)}{2}$	$\frac{n^2(n-2)}{2}$	$\frac{n(n-1)^2}{2}$
Sz^*	$\binom{n+1}{3}$	$(n-1)^2$	$\frac{n^3}{4}$	$\frac{n^3}{4}$
Sz_e^*	$\frac{(n-1)(2n^2-4n+3)}{12}$	$\frac{(n-1)(2n-3)}{4}$	$\frac{n^3}{4}$	$\frac{n^3}{4}$

TABLE 1. Some topological indices of path, star, and cycle.

As the first example, consider the rooted product of P_n and P_m, where the root vertex of P_m is assumed to be on one of its pendant vertices (vertices of degree one). This molecular graph is called the *comb lattice graph*. Using Eqs. (2.1)–(2.3), (2.6), (2.9), and Table 1, we easily arrive at:

Corollary 2. *Let $G = P_n\{P_m\}$, where the root vertex of P_m is assumed to be on one of its pendant vertices. Then*

(i) $PI_e(G) = (nm-1)(nm-2)$,

(ii) $Sz_e(G) = \frac{nm^3}{6}(3n-2) + \frac{nm^2}{6}(n^2-9n+2) + \frac{11nm}{6} - 1$,

(iii) $Sz_{ev}(G) = \frac{nm^3}{6}(3n-2) + \frac{nm^2}{6}(n^2-6n+2) + \frac{nm}{3}$,

(iv) $Sz_{ve}(G) = \frac{nm^3}{6}(3n^2-3n+2) - \frac{nm^2}{6}(n+1)(n+2) + \frac{2nm}{3}$,

(v) $Sz_e^*(G) = \frac{nm^3}{6}(3n-2) + \frac{nm^2}{6}(n^2-6n+2) + \frac{7nm}{12} - \frac{1}{4}$.

Let $P_n^*(m)$ denote the *m–thorn path* which is the graph obtained by attaching m pendant vertices to each vertex of the path P_n. This graph can be viewed as the rooted

product of P_n and the star graph on $m+1$ vertices, where the root vertex of S_{m+1} is assumed to be on its central vertex (vertex of degree m). Using Eqs. (2.1)–(2.3), (2.6), (2.9), and Table 1, we easily arrive at:

Corollary 3. *The following equalities hold:*

(i) $PI_e(P_n^*(m)) = n^2m^2 + nm(2n-3) + (n-1)(n-2)$,

(ii) $Sz_e(P_n^*(m)) = \frac{nm^2}{6}(n-1)(n+1) + \frac{nm}{3}(n-1)(n-2) + \binom{n-1}{3}$,

(iii) $Sz_{ev}(P_n^*(m)) = \frac{nm^2}{6}(n^2+3n-1) + \frac{nm}{6}(2n^2-5) + \binom{n}{3}$,

(iv) $Sz_{ve}(P_n^*(m)) = \frac{n^3m^3}{2} + \frac{nm^2}{6}(8n^2-12n+1) + \frac{nm}{6}(n-1)(7n-11) + 2\binom{n}{3}$,

(v) $Sz_e^*(P_n^*(m)) = \frac{nm^2}{6}(n^2+3n-1) + \frac{nm}{12}(4n^2-7) + \frac{n}{12}(2n^2-6n+7) - \frac{1}{4}$.

Let $C_n^*(m)$ denote the *m−thorn cycle* which is the graph obtained by attaching m pendant vertices to each vertex of the cycle C_n. This graph can be seen as the rooted product of C_n and the star graph on $m+1$ vertices, where S_{m+1} is assumed to be rooted on its central vertex. Using Eqs. (2.1)–(2.3), (2.6), (2.9), and Table 1, we easily arrive at:

Corollary 4. *The following equalities hold:*

(i) $PI_e(C_n^*(m)) = \begin{cases} n^2m^2 + nm(2n-1) + n(n-2) & n\ is\ even, \\ n^2m^2 + 2nm(n-1) + n(n-1) & n\ is\ odd, \end{cases}$

(ii) $Sz_e(C_n^*(m)) = \begin{cases} \frac{n(nm+n-2)^2}{4} & n\ is\ even, \\ \frac{n(n-1)^2(m+1)^2}{4} & n\ is\ odd, \end{cases}$

(iii) $Sz_{ev}(C_n^*(m)) = \begin{cases} \frac{n^2m^2}{4}(n+2) + \frac{nm}{2}(n^2-1) + \frac{n^2}{4}(n-2) & n\ is\ even, \\ \frac{nm^2}{4}(n^2+1) + \frac{n^2m}{2}(n-1) + \frac{n(n-1)^2}{4} & n\ is\ odd, \end{cases}$

(iv) $Sz_{ve}(C_n^*(m)) = \begin{cases} \frac{n^3m^3}{2} + \frac{n^2m^2}{4}(5n-4) & \\ + \frac{nm}{2}(2n^2-3n+1) + \frac{n^2(n-2)}{4}, & n\ is\ even \\ \frac{n^3m^3}{2} + \frac{nm^2}{4}(5n^2-6n+1) & \\ + nm(n-1)^2 + \frac{n(n-1)^2}{4}, & n\ is\ odd \end{cases}$

(v) $Sz_e^*(C_n^*(m)) = \frac{n^2m^2}{4}(n+2) + \frac{nm}{4}(2n^2+2n-1) + \frac{n^3}{4}$.

Finally, consider the rooted product of P_n and C_m. Note that because of the symmetry of C_m any vertex of this graph can be considered at its root vertex. Using Eqs. (2.1)–(2.3), (2.6), (2.9), and Table 1, we easily arrive at:

Corollary 5. *Let $G = P_n\{C_m\}$. Then*

$$\textit{(i)}\ \ PI_e(G) = \begin{cases} n^2m^2 + 2nm(n-2) + (n-1)(n-2) & m\ is\ even, \\ n^2m^2 + nm(n-2) - 2(n-1) & m\ is\ odd, \end{cases}$$

$$\textit{(ii)}\ Sz_e(G) = \begin{cases} \frac{nm^3}{4}(2n-1) + \frac{nm^2}{6}(n+1)(n-4) & m\ is\ even \\ +\frac{nm}{3}(n-2)(n-4) + \binom{n-1}{3}, & \\ \frac{nm^3}{4}(2n-1) + \frac{nm^2}{6}(n^2-3n-1) & m\ is\ odd \\ +\frac{nm}{12}(4n^2-18n+17) + \frac{n-1}{6}(n^2-2n+6), & \end{cases}$$

$$\textit{(iii)}\ \ Sz_{ev}(G) = \begin{cases} \frac{n^2m^3}{2} + \frac{nm^2}{12}(2n^2-3n-11) + m\binom{n}{3} & m\ is\ even, \\ \frac{n^2m^3}{2} + \frac{nm^2}{12}(2n^2-9n-5) & m\ is\ odd \\ +\frac{nm}{6}(n^2-3n+5) + \frac{n(n-1)}{4}, & \end{cases}$$

$$\textit{(iv)}\ Sz_{ve}(G) = \begin{cases} \frac{nm^3}{2}(n^2-n+1) + \frac{nm^2}{12}(10n^2-21n-1) & m\ is\ even \\ +2m\binom{n}{3}, & \\ \frac{nm^3}{2}(n^2-n+1) + \frac{nm^2}{12}(4n^2-15n-1) & m\ is\ odd \\ -\frac{nm}{6}(n^2-4) + \frac{n(n-1)}{4}, & \end{cases}$$

$$\textit{(v)}\ \ Sz_e^*(G) = \begin{cases} \frac{nm^3}{4}(2n-1) + \frac{nm^2}{6}(n-1)(n+4) & m\ is\ even \\ +\frac{nm}{6}(2n^2-3n+1) + \frac{n-1}{12}(2n^2-4n+3), & \\ \frac{nm^3}{4}(2n-1) + \frac{5m^2}{2}\binom{n+1}{3} + \frac{nm}{6}(5n^2-9n+4) & m\ is\ odd \\ +\frac{1}{12}(5n^3-12n^2+10n-3). & \end{cases}$$

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COMPUTING GRÖBNER BASES AND INVARIANTS OF THE SYMMETRIC ALGEBRA

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Abstract. We study algebraic invariants of the symmetric algebra $Sym_R(L)$ of the square-free monomial ideal $L = I_{n-1} + J_{n-1}$ in the polynomial ring $R = K[X_1, \dots, X_n; Y_1, \dots, Y_n]$, where I_{n-1} (resp. J_{n-1}) is generated by all square-free monomials of degree $n-1$ in the variables $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$). In particular, the dimension and the depth of $Sym_R(L)$ are investigated by techniques of Gröbner bases and theory of s-sequences.

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INTRODUCTION

Let R be a commutative noetherian ring and M be a finitely generated R-module $M = Rf_1 + \cdots + Rf_q$. If (a_{ij}), $i = 1, \dots, q$, $j = 1, \dots, p$, is the matrix associated to a free presentation of M, then $Sym_R(M) = R[T_1, \dots, T_q]/J$, where J is generated by the linear forms $g_j = \sum_{i=1}^{q} a_{ij} T_i$ for $j = 1, \dots, p$.

In [3], in order to study the symmetric algebra $Sym_R(M)$, it is introduced the concept of s-sequence for the generators $f_1, \dots, f_q$ of M. We say that $f_1, \dots, f_q$ is an s-sequence for M if there exists a monomial order $\prec$ for the monomials in the variables T_i with $T_1 \prec T_2 \prec \cdots \prec T_q$ such that $in_\prec(J) = (\mathfrak{I}_1 T_1, \dots, \mathfrak{I}_q T_q)$, with $\mathfrak{I}_i = (f_1, \dots, f_{i-1}) :_R f_i$ ideals of R.

The ideals I_i are called the annihilator ideals of the sequence $f_1, \dots, f_q$. If M is generated by an s-sequence, the standard algebraic invariants of M can be expressed only by the ideals $\mathfrak{I}_i$ and in more cases the dimension can be computed in terms of the annihilators ideal of the sequence. The crucial point is that we can easily calculate the invariants, starting by the structure of the initial ideal $in_\prec(J)$ of J, stated that it is $\underline{T}$-linear, being $\underline{T} = \{T_1, \dots, T_q\}$ the variables that correspond in the presentation of symmetric algebra of M to the generators of M. A natural question arises: if $in_\prec(J)$ is not linearly generated in the variables $T_1, \dots, T_q$ and we write $in_\prec(J) = J_L + J^*$, where J^* is the not linear part of $in_\prec(J)$, what is the maximum degree (with respect the variables $T_1, \dots, T_q$) of the monomial generators of J^*?

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We are interested to check when J^* is generated by monomials of minimum degree (quadratic). Moreover, we are interested to describe explicitly a Gröbner basis of J, given the importance of the initial ideal in more open problems about the invariants of $Sym_R(M)$ ([2], [4], [5], [8]).

Now, let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_m]$ be the polynomial ring on a field K in two sets of variables and L be a mixed products ideal, as defined in [7]. In [6] the authors selected the mixed products ideals that are generated by an s-sequence in order to compute the value or a bound for standard invariants of the symmetric algebra $Sym_R(L)$. The problem is open for ideals L that are not generated by an s-sequence. In this paper we examine a first class of mixed products ideals not generated by an s-sequence. More precisely, we consider the ideal $L = I_{n-1} + J_{n-1}$ of $K[X_1,\dots,X_n;Y_1,\dots,Y_n]$, with I_{n-1} (resp. J_{n-1}) the monomial ideal of R generated by all square-free monomials of degree $n-1$ in the variables $X_1,\dots,X_n$ (resp. $Y_1,\dots,Y_n$).

The aim is to compute a Gröbner basis of the relations ideal J of the symmetric algebra $Sym_R(L)$ and to study some invariants of $Sym_R(L)$. More precisely, in section 1 we give the structure of a Gröbner basis of J with respect to the lexicographic order $\prec$. It should be noted that we are able to compute the generators of $in_\prec(J)$ not linear in the variables $T_1\dots,T_{2n}$ and to establish the degree. For $n = m = 3$, we obtain the only case in which the not linear part of $in_\prec(J)$ is of degree two in the variables T_i. In section 2, we compute the dimension and the depth of $Sym_R(L)$. For the computation of the dimension, we inspire to the techniques used in the theory of the s-sequences. More precisely, we consider the linear part J_L of $in_\prec(J)$ and we apply the results given in [3] for computing dimension in terms of the annihilator ideals of the monomial sequence generating L. Then we obtain that $Sym_R(L)$ is a Cohen-Macaulay algebra.

1. Gröbner bases of relation ideals

In [3] the notion of s-sequence is introduced for finitely generated modules in a noetherian ring R.

For every $i = 1,\dots,q$, we set $M_{i-1} = Rf_1 + \cdots + Rf_{i-1}$ and $\mathfrak{I}_i = M_{i-1} :_R f_i$ be the colon ideal. We set $\mathfrak{I}_0 = (0)$. Since $M_i/M_{i-1} \simeq R/\mathfrak{I}_i$, so $\mathfrak{I}_i$ is the annihilator of the cyclic module $R/\mathfrak{I}_i$. $\mathfrak{I}_i$ is called *annihilator ideal* of the sequence $f_1,\dots,f_q$.

Let (a_{ij}), for $i = 1,\dots,q$, $j = 1,\dots,p$, be the relation matrix of M. The symmetric algebra $Sym_R(M)$ has a presentation $R[T_1,\dots,T_q]/J$, with $J = (g_1,\dots,g_p)$ where $g_j = \sum_{i=1}^q a_{ij}T_i$ for $j = 1,\dots,p$.

We consider $S = R[T_1,\dots,T_q]$ as a graded ring by assigning to each variable T_i degree 1 and to the elements of R degree 0.

Let $\prec$ be a monomial order on the monomials of S in the variables T_i such that $T_1 \prec T_2 \prec \cdots \prec T_q$. With respect to this term order, if $f = \sum a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha =$

$T_1^{\alpha_1}\cdots T_q^{\alpha_q}$ and $\alpha=(\alpha_1,\ldots,\alpha_q)\in\mathbb{N}^q$, we put $in_\prec(f)=a_\alpha\underline{T}^\alpha$, where $\underline{T}^\alpha$ is the largest monomial in f such that $a_\alpha\neq 0$.

So we define the monomial ideal $in_\prec(J)=(\{in_\prec(f)|\, f\in J\})$. In general we have $(\mathcal{I}_1T_1,\mathcal{I}_2T_2,\ldots,\mathcal{I}_qT_q)\subseteq in_\prec(J)$ and the two ideals coincide in degree 1.

The sequence $f_1,\ldots,f_q$ is an *s-sequence* for M if

$$(\mathcal{I}_1T_1,\mathcal{I}_2T_2,\ldots,\mathcal{I}_qT_q)=in_\prec(J).$$

If R is a polynomial ring over a field and $f_1,\ldots,f_q$ are monomials of R, then we have a criterion to be s-sequences. Set $f_{ij}=\frac{f_i}{[f_i,f_j]}$ for $i\neq j$, where $[f_i,f_j]$ is the greatest common divisor of the monomials f_i and f_j. J is generated by $g_{ij}=f_{ij}T_j-f_{ji}T_i$ for $1\leq i<j\leq q$. The monomial sequence $f_1,\ldots,f_q$ is an s-sequence if and only if g_{ij}, for $1\leq i<j\leq q$, is a Gröbner basis for J for a term order that agrees with the order of the variables in $S=R[T_1,\ldots,T_q]$. Note that the annihilator ideals of the monomial sequence $f_1,\ldots,f_q$ are the ideals $\mathcal{I}_i=(f_{1i},f_{2i},\ldots,f_{i-1,i})$ for $i=1,\ldots,q$ ([3]).

Now we consider the polynomial ring $R=K[X_1,\ldots,X_n;Y_1,\ldots,Y_m]$ over a field K in two sets of variables and the class of monomial *ideals of mixed products* of $R=K[X_1,\ldots,X_n;Y_1,\ldots,Y_m]$:

$$L=I_kJ_r+I_sJ_t,$$

where I_k (resp. J_r) is the monomial ideal of R generated by all the square-free monomials of degree k (resp. r) in the variables $X_1,\ldots,X_n$ (resp. $Y_1,\ldots,Y_m$).

In [6] the authors investigate in which cases these monomial ideals are generated by an s-sequence. The system of generators of L is an s-sequence only in the following cases:

1) $L=I_{n-1}J_m$, 2) $L=I_1J_m$, 3) $L=I_{n-1}J_m+I_nJ_{m-1}$, 4) $L=J_m+I_nJ_1$.

Set $I_0=J_0=R$, then we have the following case for $r=0$ and $s=0$ $L=I_k+J_k$ with $1\leq k\leq\inf\{n,m\}$. We study this class of square-free monomial ideals for $k=n-1$ and $n=m$, then $R=K[X_1,\ldots,X_n;Y_1,\ldots,Y_n]$ and $L=I_{n-1}+J_{n-1}$.

Since the property to be an s-sequence may depend on the order of the sequence, in the sequel we will suppose $L=(f_1,f_2,\ldots,f_{2n})$ where $f_1\prec f_2\prec\cdots\prec f_{2n}$ with respect to the monomial order $\prec_{lex}$ on $X_1,\ldots,X_n,Y_1,\ldots,Y_n$ and $X_1\prec X_2\prec\cdots\prec X_n\prec Y_1\prec Y_2\prec\cdots\prec Y_n$.

J is generated by $g_{ij}=f_{ij}T_j-f_{ji}T_i$ for $1\leq i<j\leq 2n$. The monomial sequence $f_1,\ldots,f_{2n}$ is an s-sequence if and only if g_{ij} for $1\leq i<j\leq 2n$ is a Gröbner basis for J in $K[X_1,\ldots,X_n;Y_1,\ldots,Y_n;T_1,\ldots,T_{2n}]$, with $X_1\prec X_2\prec\cdots\prec X_n\prec Y_1\prec Y_2\prec\cdots\prec Y_n\prec T_1\prec T_2\prec\cdots\prec T_{2n}$.

Theorem 1 ([6]). *Let $R=K[X_1,\ldots,X_n]$ be the polynomial ring over a field K and I_k with $2\leq k\leq n$. The ideal I_k is generated by an s-sequence if and only if $k=n-1$.*

Theorem 2. *Let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$. L is not generated by an s-sequence for any $n \neq 2$.*

Proof. For $n = 2$, $L = I_1 + J_1 = (X_1,\dots,X_n,Y_1,\dots,Y_n)$ is generated by an s-sequence for any admissible term order, since L is generated by a regular sequence ([3]).

For $n > 2$, let $L = I_{n-1} + J_{n-1} = (f_1, f_2,\dots, f_n) + (f_{n+1}, f_{n+2},\dots, f_{2n})$, where $f_1 \prec f_2 \prec \cdots \prec f_n$ and $f_{n+1} \prec f_{n+2} \prec \cdots \prec f_{2n}$ with respect to the monomial order $\prec_{lex}$. L is generated by an s-sequence $\Leftrightarrow$ $G = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid\ 1 \le i < j \le 2n\}$ is a Gröbner basis for J $\Leftrightarrow$ $S(g_{ij}, g_{hl}) \xrightarrow{G} 0$ for all $i,j,h,l \in \{1,\dots,2n\}$ and $g_{ij} \neq g_{hl}$. We consider a lexicographic Gröbner basis for J with respect to the order on the variables $T_1 \prec T_2 \prec \cdots \prec T_{2n}$.

The generators of L are the following: $f_1 = X_1\cdots X_{n-1}$, $f_2 = X_1\cdots X_{n-2}X_n$, $f_n = X_2\cdots X_n$, $f_{n+1} = Y_1\cdots Y_{n-1}$, $f_{n+2} = Y_1\cdots Y_{n-2}Y_n$, …, $f_{2n} = Y_2\cdots Y_n$.

One has $S(g_{1n}, g_{2,n+1}) = \frac{f_{1n}f_{n+1,2}}{[f_{1n},f_{2,n+1}]}T_2T_n - \frac{f_{2,n+1}f_{n1}}{[f_{1n},f_{2,n+1}]}T_1T_{n+1} = Y_1\cdots Y_{n-1}T_2T_n - X_2\cdots X_{n-2}X_n^2T_1T_{n+1}$. By the structure of the generators of L there is no $g_{st} \in G$ whose initial term with respect to the admissible order on the variables $T_1 \prec T_2 \prec \cdots \prec T_{2n}$ divides the terms of $S(g_{1n}, g_{2,n+1})$. It follows that it is not possible to get a standard expression of $S(g_{1n}, g_{2,n+1})$ with respect G with remainder 0. Hence G is not a Gröbner basis for J. It follows that L can not be generated by an s-sequence ([3], Lemma 1.2). In fact, from the theory of Gröbner bases, suppose that $f_1,\dots,f_{2n}$ is a monomial s-sequence with respect to some admissible term order $\prec$, then $f_1,\dots,f_{2n}$ is an s-sequence for any other admissible term order, and as a consequence it is an s-sequence for the lexicographic order, that is a contradiction. □

The main result of this section gives the structure of the Gröbner basis of the relation ideal J of the symmetric algebra $Sym_R(L)$.

Theorem 3. *Let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$. A Gröbner basis for J is the set*

$$BG(J) = \{g_{ij} = f_{ij}T_j - f_{ji}T_i, 1 \le i < j \le 2n\} \cup \mathcal{S},$$

where $\mathcal{S} = \bigcup_{t=1}^{n-2} H_t$ with $H_1 = \bigcup_{k_1=3}^{n} H_{1k_1}$ where $H_{1k_1} = \{S(g_{1k_1}, g_{ij}) | i < i < k_1,\ \ j = n+1,\dots,2n\}$ and $H_t = \bigcup_{k_t=t+2}^{n} H_{tk_t}$ where $H_{tk_t} = \{S(g_{1k_t}, h) | h \in H_{t-1k_{t-1}},\ \ k_{t-1} < k_t\}$.

Proof. Let $L = I_{n-1} + J_{n-1} = (f_1,\dots,f_n) + (f_{n+1}\dots,f_{2n})$ and $G = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid\ 1 \le i < j \le 2n\}$, where $f_{ij} = \frac{f_i}{[f_i,f_j]}$ for $i \neq j$ and $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j. By the Theorem 2 L is not generated by an s-sequence, then $G \subsetneq BG(J)$ and there are S-polynomials $S(g_{ij}, g_{kl})$ that have not a standard expression with respect G with remainder 0. Let $g_{ij}, g_{kl} \in G$,

it is known that $S(g_{ij},g_{kl}) = \frac{f_{ij}f_{lk}}{[f_{ij},f_{kl}]}T_jT_k - \frac{f_{kl}f_{ji}}{[f_{ij},f_{kl}]}T_iT_l$. Knowing the structure of the monomials $f_1,\dots,f_{2n}$, we are able to compute the S-polynomials of G that do not reduce to 0 modulo G and using Buchberger algorithm we construct the elements of $BG(J)\setminus G$. Then for $j = n+1,\dots,2n$ we compute $S(g_{1k_1},g_{ij})$ with $i < k_1$ and $k_1 = 3,\dots,n$:

$$S(g_{13},g_{2j}) = f_jT_2T_3 - \delta_1X_n^2T_1T_j$$

$$S(g_{14},g_{2j}) = f_jT_2T_4 - \delta_2X_n^2T_1T_j$$

$$S(g_{15},g_{2j}) = f_jT_2T_5 - \delta_4X_n^2T_1T_j$$

.....

$$S(g_{1n},g_{n-1,n+1}) = f_jT_{n-1}T_n - \delta_{\binom{n-1}{n-3}}X_n^2T_1T_j,$$

where $\delta_1,\delta_2,\dots,\delta_{\binom{n-1}{n-3}}$ are the generators of the Veronese ideal of degree $n-3$ in the variables $X_1,X_2,\dots,X_{n-1}$. Being $f_1 = X_1\cdots X_{n-1}$, $f_2 = X_1\cdots X_{n-2}X_n$, $f_n = X_2\cdots X_n$, $f_{n+1} = Y_1\cdots Y_{n-1}$, $f_{n+2} = Y_1\cdots Y_{n-2}Y_n$, …, $f_{2n} = Y_2\cdots Y_n$, it follows that there is no $g_{st}\in G$ whose initial term divides the terms of these S-polynomials. Then $S(g_{1k_1},g_{ij})\in BG(J)$ for $j = n+1,\dots,2n$, $i < k_1$ and $k_1 = 3,\dots,n$. Set $H_{1k_1} = \{S(g_{1k_1},g_{ij}) \mid j = n+1,\dots,2n,\ i < k_1\}$ for $k_1 = 3,\dots,n$. Now we continue to compute the S-polynomials $S(g_{1k_2},S(g_{1k_1},g_{ij}))$ with $i < k_1 < k_2$ and $k_2 = 4,\dots,n$, $j = n+1,\dots,2n$:

$$S(g_{14},S(g_{13},g_{2,n+1})) = f_{n+1}T_2T_3T_4 - \gamma_1X_n^3T_1^2T_{n+1}$$

$$S(g_{14},S(g_{13},g_{2,n+2})) = f_{n+2}T_2T_3T_4 - \gamma_2X_n^3T_1^2T_{n+2}$$

.....

$$S(g_{14},S(g_{13},g_{2,2n})) = f_{2n}T_2T_3T_4 - \gamma_{\binom{n-1}{n-4}}X_n^3T_1^2T_{2n}$$

and so on up to $k_2 = n$:

$$S(g_{1n},S(g_{1,n-1},g_{n-2,n+1})) = f_{n+1}T_{n-2}T_{n-1}T_n - \gamma_1X_n^3T_1^2T_{n+1}$$

$$S(g_{1n},S(g_{1,n-1},g_{n-2,n+2})) = f_{n+2}T_{n-2}T_{n-1}T_n - \gamma_2X_n^3T_1^2T_{n+2}$$

.....

$$S(g_{1n}, S(g_{1,n-1}, g_{n-2,2n})) = f_{2n}T_{n-2}T_{n-1}T_n - \gamma_{\binom{n-1}{n-4}}X_n^3T_1^2T_{2n},$$

where $\gamma_1, \gamma_2, \ldots, \gamma_{\binom{n-1}{n-4}}$ are the generators of the Veronese ideal of degree $n-4$ in the variables $X_1, X_2, \ldots, X_{n-1}$. The terms of these S-polynomials are not divided by the initial term of any $g_{st} \in G$, then $S(g_{1k_2}, S(g_{1k_1}, g_{ij})) \in BG(J)$ for $i < k_1 < k_2$ and $k_2 = 4, \ldots, n$, $j = n+1, \ldots, 2n$. Set $H_{2k_2} = \{S(g_{1k_2}, h_{k_1}) | h_{k_1} \in H_{1k_1}\}$. Continuing the computation of the S-polynomials one obtains:

$$H_{3k_3} = \{S(g_{1k_3}, h_{k_2}) | h_{k_2} \in H_{2k_2}\} \text{ for } k_2 < k_3, k_3 = 5, \ldots, n$$

.....

$$H_{n-3k_{n-3}} = \{S(g_{1k_{n-3}}, h_{k_{n-4}}) | h_{k_{n-4}} \in H_{n-4k_{n-4}}\} \text{ for } k_{n-4} < k_{n-3},\ k_{n-3} = n-1, n$$

$$H_{n-2k_{n-2}} = \{S(g_{1k_{n-2}}, h_{k_{n-3}}) | h_{k_{n-3}} \in H_{n-3k_{n-3}}\} \text{ for } k_{n-3} < k_{n-2}, k_{n-2} = n.$$

Set $\mathcal{S} = \bigcup_{t=1}^{n-2} H_t$ with $H_1 = \bigcup_{k_1=3}^{n} H_{1k_1}$ where $H_{1k_1} = \{S(g_{1k_1}, g_{ij}) | i < i < k_1, \quad j = n+1, \ldots, 2n\}$ and $H_t = \bigcup_{k_t=t+2}^{n} H_{tk_t}$ where $H_{tk_t} = \{S(g_{1k_t}, h) | h \in H_{t-1k_{t-1}},\ k_{t-1} < k_t\}$. Then elements of $\mathcal{S}$ do not reduce to 0 modulo G. Moreover by construction no term of an element h of $\mathcal{S}$ is divisible by the initial term of an element of $\mathcal{S} \setminus \{h\}$. Set $B = G \cup \mathcal{S}$, it follows that $BG(J) \supseteq B$.

In order to show that $BG(J) = B$ we must prove that for all $g, h \in B$ the S-polynomial $S(g,h)$ reduces to 0 modulo B.

Let's start to prove that all the S-polynomials $S(g_{ij}, g_{hl})$, with $i, j, h, l \in \{1, \ldots, 2n\}$, has a standard expression with respect to B with remainder 0. We have:

$$S(g_{ij}, g_{hl}) = \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}T_jT_h - \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}T_iT_l \ . \quad (*)$$

Let's find a standard expression of $S(g_{ij}, g_{hl})$, for all $i, j, h, l \in \{1, \ldots, n-1\}$.

If $[in_{\prec}(g_{ij}), in_{\prec}(g_{hl})] = 1$, then $S(g_{ij}, g_{hl}) = f_{lh}g_{ij}T_h - f_{ji}g_{hl}T_i$.

If $[in_{\prec}(g_{ij}), in_{\prec}(g_{hl})] \neq 1$, we apply $(*)$ to obtain a standard expression for the S-polynomials $S(g_{ij}, g_{hl})$. It results:

$$S(g_{ij}, g_{il}) = -[f_{ji}, f_{li}]g_{jl}T_i$$
$$S(g_{ij}, g_{lj}) = [f_{ji}, f_{jl}]g_{il}T_j$$
$$S(g_{ij}, g_{lk}) = [f_{ji}, f_{kl}](\frac{f_{kl}}{[f_{ji}, f_{kl}]}g_{il}T_j - \frac{f_{lk}}{[f_{ij}, f_{lk}]}g_{jk}T_i) \text{ or}$$
$$S(g_{ij}, g_{lk}) = [f_{ji}, f_{kl}](\frac{f_{ji}}{[f_{ji}, f_{kl}]}g_{il}T_k - \frac{f_{ij}}{[f_{ij}, f_{lk}]}g_{jk}T_l).$$

Hence all the S-polynomials $S(g_{ij}, g_{hl})$ reduce to 0 with respect to B. It remains to prove that the elements of $B \setminus G$ reduce to 0 with respect to B. If the elements

of B have initial terms coprime, then they reduce to 0 with respect to B. Otherwise we observe that by the structure of the elements of $\mathcal{S}$ it follows that the initial terms of the elements of $\mathcal{S}$ are $\gamma_{i_k} X_n^t T_1^s T_j$ with γ_{i_k} a generator of the Veronese ideal of degree k in the variables $X_1, X_2, \ldots, X_{n-1}$ for $n+1 \leq j \leq 2n, 0 \leq k \leq n-3, 0 \leq t \leq n-1, 0 \leq s \leq n-3, k+t = n-1, s = t-1$.

Let $f, g \in B$, let d be the greater common divisor of $\text{in}_\prec(f)$ and $\text{in}_\prec(g)$, c be the greater common divisor of the no initial terms of f and g. In order to prove that $S(f,g)$ reduces to 0 with respect to B for all $f, g \in B$, we start to consider the elements of B of the form $g_{ij} \in G$ and $g_{1m,st} = S(g_{1m}, g_{st}) \in H_{1k_1}$. We have the following cases:

– if T_j is a variable of d, then $S(g_{ij}, g_{1m,st})$ reduces to 0 by the elements of B g_{is} and g_{1m} for $i < s$;

– if T_j is a variable of d and T_i is a variable of c, then $S(g_{ij}, g_{1j,si})$ reduces to 0 by the element $g_{1s} \in B$;

– if no variable $T_1, \ldots, T_{2n}$ is in d and c, then $S(g_{ij}, g_{1m,st})$ reduces to 0 by the elements of B g_{1i} and $g_{11m,stj} = S(g_{1j}, S(g_{1t}, g_{sm})) \in H_{2k_2}$ for $s < t < j$.

For the elements of the form $g_{1j,lk} = S(g_{1j}, g_{lk}) \in G$ and $g_{1m,st} = S(g_{1m}, g_{st}) \in H_{1k_1}$, we have the following cases:

– if T_1, T_j are variables of d and T_l is a variable of c, then $S(g_{1j,lk}, g_{1j,tl})$ reduces to 0 by $g_{kt} \in B$ for $k < t$;

– if T_1 is a variable of d and T_l, T_k are variables of c, then $S(g_{1j,lk}, g_{1j,lk})$ reduces to 0 by $g_{jt} \in B$ for $j < t$;

– if T_k is a variable of c then $S(g_{1j,lk}, g_{1t,mk})$ reduces to 0 by $g_{jt}, g_{lm} \in B$ for $j < t, l < m$.

The same argument is applied for the S-polynomials of the elements of all H_{jk_j} for $j > 1$. The assertion follows. □

Corollary 1. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$.*
The presentation ideal of $Sym_R(L)$ admits a lexicographic Gröbner basis of degree $\leq n-1$ in the variables $T_1, \ldots, T_{2n}$.

Corollary 2. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K and $L = I_{n-1} + J_{n-1}$. Then we have:*

1) The presentation ideals of $Sym(I_{n-1})$ and $Sym(J_{n-1})$ have a linear Gröbner basis respectively in the variables $U_1,\dots,U_n$ and $V_1,\dots,V_n$, which correspond to the generators of I_{n-1} and J_{n-1} respectively.
2) The presentation ideal of $Sym(I_{n-1}+J_{n-1})$ has a Gröbner basis not linear in the variables $T_1,\dots,T_{2n}$, which correspond to the generators of $L=I_{n-1}+J_{n-1}$.

Proof. 1) Since I_{n-1} and J_{n-1} are generated by an s-sequence, there exist a monomial order $\prec_1$ in the variables $U_1,\dots,U_n$ with $U_1\prec_1\cdots\prec_1 U_n$ and a monomial order $\prec_2$ in the variables $V_1,\dots,V_n$ with $V_1\prec_2\cdots\prec_2 V_n$ such that the presentation ideals of $Sym(I_{n-1})$ and $Sym(J_{n-1})$ have a linear Gröbner basis respectively in the variables $U_1,\dots,U_n$ and $V_1,\dots,V_n$.

2) It follows by Corollary 1. □

2. Studying standard invariants

In this section, we shall compute the dimension and the depth of the symmetric algebra $Sym_R(L)$, with L the mixed product ideal $L=I_{n-1}+J_{n-1}$.

In order to apply the theory of s-sequences, at the beginning we prove the main result concerning the annihilator ideals of the monomial sequence that generates L.

Proposition 1. *Let $R=K[X_1,\dots,X_n;Y_1,\dots,Y_n]$ be the polynomial ring over a field K and $L=I_{n-1}+J_{n-1}=(f_1,\dots,f_{2n})$. Then the annihilator ideals of the sequence $f_1,\dots,f_{2n}$ are:*

$$\mathcal{I}_1=(0),\mathcal{I}_i=\begin{cases}(X_{n-i+1}) & \text{for } i=2,\dots,n\\ I_{n-1} & \text{for } i=n+1\\ (I_{n-1},Y_{2n-i+1}) & \text{for } i=n+2,\dots,2n\end{cases}$$

Proof. Set $f_{ij}=\frac{f_i}{[f_i,f_j]}$ for $i<j$ and $i,j=1,\dots,2n$. Then the annihilator ideals of the monomial sequence $f_1,\dots,f_{2n}$ are $\mathcal{I}_i=(f_{1i},f_{2i},\dots,f_{i-1,i})$ for $i=1,\dots,2n$. For $i=1$ we have $\mathcal{I}_1=(0)$ and by the structure of these monomials it follows $\mathcal{I}_2=(f_{12})=(X_{n-1})$, $\mathcal{I}_3=(f_{13},f_{23})=(X_{n-2})$, … , $\mathcal{I}_{n-1}=(f_{1,n-1},\dots,f_{n-2,n-1})=(X_2)$, $\mathcal{I}_n=(f_{1n},f_{2n},\dots,f_{n-1,n})=(X_1)$, $\mathcal{I}_{n+1}=(f_{1,n+1},f_{2,n+1},\dots,f_{n,n+1})=(f_1,\dots,f_n)=I_{n-1}$, $\mathcal{I}_{n+2}=(f_{1,n+2},\dots,f_{n+1,n+2})=(f_1,\dots,f_n,Y_{n-1})=(I_{n-1},Y_{n-1})$, …, $\mathcal{I}_{2n}=(f_{1,2n},\dots,f_{2n-1,2n})=(f_1,\dots,f_n,Y_1)=(I_{n-1},Y_1)$. Hence the assertion follows. □

Proposition 2. *Let $R=K[X_1,\dots,X_n;Y_1,\dots,Y_n]$ be the polynomial ring over a field K and $L=I_{n-1}+J_{n-1}$ for $n\geq 3$. Then*

$$in_{\prec}(J)=(\mathcal{I}_2T_2,\dots,\mathcal{I}_{2n}T_{2n})+(I'_kX_n^tT_1^sT_j)$$

where $n+1 \le j \le 2n$, $0 \le k \le n-3$, $0 \le t \le n-1$, $0 \le s \le n-2$ *such that* $k+t = n-1$, $s = t-1$, *and* I'_k *is the square-free Veronese ideal generated by all the monomials of degree* k *in the variables* $X_1, \ldots, X_{n-1}$.

Proof. It is known that the initial ideal of J is given by $\mathrm{in}_\prec(J) = (\{\mathrm{in}_\prec(f) | f \in BG(J)\})$. One has $BG(J) = \{g_{ij} = f_{ij}T_j - f_{ji}T_i, 1 \le i < j \le 2n\} \cup \mathcal{S}$ as in Theorem 3. By the structure of the monomials $f_1, \ldots, f_{2n}$ that generate L we deduce the linear forms $g_{ij} = f_{ij}T_j - f_{ji}T_i, 1 \le i < j \le 2n$ and then we compute $\mathrm{in}_\prec(g_{ij}) = f_{ij}T_j$ for $1 \le i < j \le 2n$:

$\{X_{n-1}T_2, \ldots, X_2T_{n-1}, X_1T_n, Y_{n-1}T_{n+2}, \ldots, Y_1T_{2n}, I_{n-1}T_j,\ j = n+1, \ldots, 2n\}$.

It remains to compute the initial terms of the element of $\mathcal{S}$. By the structure of the S-polynomials of $\mathcal{S}$ (see Theorem 3) we obtain that the initial term of these S-polynomials are the elements of the set $\{I'_k X_n^t T_1^s T_j\}$, where I'_k is the ideal generated by the square-free monomials of degree k in the variables $X_1, \ldots, X_{n-1}$ and $j = n+1, \ldots, 2n$, $k+t = n-1$, $s = t-1$, $k = 0, \ldots, n-3$, $t = 2, \ldots, n-1$, $s = 1, \ldots, n-2$. Then the initial ideal of J is:

$$\mathrm{in}_\prec(J) = (X_{n-1}T_2, \ldots, X_1T_n, I_{n-1}T_{n+1}, (I_{n-1}, Y_{n-1})T_{n+2}, \ldots, (I_{n-1}, Y_1)T_{2n}) + (I'_k X_n^t T_1^s T_j).$$

Using Proposition 1 we can write $\mathrm{in}_\prec(J) = (\mathcal{I}_2 T_2, \ldots, \mathcal{I}_{2n} T_{2n}) + (\{I'_k X_n^t T_1^s T_j |$ $n+1 \le j \le 2n, 0 \le k \le n-3, 0 \le t \le n-1, 0 \le s \le n-2, k+t = n-1, s = t-1\})$. □

Now, we recall some general results about an ideal $I = (H_1T_1, \ldots, H_tT_t) \subset R[T_1, \ldots, T_t]$, where R is a noetherian ring and $H_1, \ldots, H_t$ are ideals of R. We say that I is linear in the variables $T_1, \ldots, T_t$, or $\underline{T}$-linear. For $\underline{T}$-linear ideals we have:

Proposition 3 ([3], Lemma 2.3)**.** *Let* $I = (H_1T_1, \ldots, H_tT_t)$ *be a* $\underline{T}$*-linear ideal of* $R[T_1, \ldots, T_t]$. *Then*

$$I = \bigcap_{1 \le r \le t} (H_{i_1} + \ldots + H_{i_r}, T_1, \ldots, \widehat{T}_{i_1} \ldots, \widehat{T}_{i_r}, \ldots, T_t),$$

with $1 \le i_1 \le \ldots \le i_r \le t$.

Proposition 4 ([3], Prop. 2.4)**.** *Let* $I = (H_1T_1, \ldots, H_tT_t)$ *be a* $\underline{T}$*-linear ideal of* $R[T_1, \ldots, T_t]$. *Then*

$$d = dim(R[T_1, \ldots, T_t]/I) = max_{1 \le r \le t}\{dim(R/(H_{i_1} + \ldots + H_{i_r})) + r\}$$

with $1 \le i_1 \le \ldots \le i_r \le t$.

In order to apply the previous results, let $\mathrm{in}_\prec(J) = (\mathcal{I}_2 T_2, \ldots, \mathcal{I}_{2n} T_{2n}) + (I'_k X_n^t T_1^s T_j)$ as in Proposition 2. We set $J_L = (\mathcal{I}_2 T_2, \ldots, \mathcal{I}_{2n} T_{2n})$ the *linear part* of $\mathrm{in}_\prec(J)$ and $J^* = (I'_k X_n^t T_1^s T_j)$, where $n+1 \le j \le 2n$, $0 \le k \le n-3$, $0 \le t \le n-1$,

$0 \leq s \leq n-2$ such that $k+t = n-1$, $s = t-1$, and I'_k is the square-free Veronese ideal generated by all the monomials of degree k in the variables $X_1, \ldots, X_{n-1}$.

Proposition 5. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K, $L = I_{n-1} + J_{n-1}$ and $J_L \subset R[T_1, \ldots, T_{2n}]$. Then:*

$$dim(R[T_1, \ldots, T_{2n}]/J_L) = 2n+2.$$

Proof. Let $J_L = (\mathcal{I}_1 T_1, \ldots, \mathcal{I}_{2n} T_{2n})$, $\dim(R[T_1, \ldots, T_{2n}]/(\mathcal{I}_1 T_1, \ldots, \mathcal{I}_{2n} T_{2n})) = \max_{1 \leq r \leq 2n}\{\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r,\ \ 1 \leq i_1 \leq \ldots \leq i_r \leq 2n\}$ by Proposition 3. For $r = 1, \ldots, 2n$ one computes:

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r \leq 2n+1$ for $r < n$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(n-1)] + n = 2n+1$ for $r = n$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(n-1)] + n + 1 = 2n+2$ for $r = n+1$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = (2n-n) + n + 2 = 2n+2$ for $r = n+2$,

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(n+1)] + n + 3 = 2n+2$ for $r = n+3$,

.....

$\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r = [2n-(2n-2)] + 2n = 2n+2$ for $r = 2n$.

Hence $\max_{1 \leq r \leq 2n}\{\dim(R/(\mathcal{I}_{i_1} + \ldots + \mathcal{I}_{i_r})) + r\} = 2n+2$. □

Using the Proposition 5 we state the following

Theorem 4. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be the polynomial ring over a field K, $L = I_{n-1} + J_{n-1}$ and $J \subset R[T_1, \ldots, T_{2n}]$ be the relation ideal of $Sym_R(L)$. Then:*
1) $dim(R[T_1, \ldots, T_{2n}]/in_{\prec}(J)) = 2n+1$
2) $depth(R[T_1, \ldots, T_{2n}]/in_{\prec}(J)) = 2n+1$
3) $pd(R[T_1, \ldots, T_{2n}]/in_{\prec}(J)) = 2n-1$.

Proof. 1) Let $\mathrm{in}_{\prec}(J) = J_L + J^*$ as in Proposition 2. By the structure of the ideals, $\mathrm{ht}(J_L + J^*) = \mathrm{ht}(J_L) + 1$. Hence one has

$\dim(R[T_1, \ldots, T_{2n}]/\mathrm{in}_{\prec}(J)) = \dim(R[T_1, \ldots, T_{2n}]/(J_L + J^*)) = \dim(R[T_1, \ldots, T_{2n}]) - \mathrm{ht}(J_L) - 1 = \dim(R[T_1, \ldots, T_{2n}]/J_L) - 1 = (2n+2) - 1 = 2n+1$ (by Proposition 5).

2) $\mathrm{ht}(\mathrm{in}_\prec(J)) = \mathrm{ht}(J_L) + 1 = \dim(R[T_1,\dots,T_{2n}]) - \dim(R[T_1,\dots,T_{2n}]/J_L) + 1 = 4n-(2n+2)+1 = 2n-1$. Hence $\mathrm{depth}(\mathrm{in}_\prec(J)) \le \mathrm{ht}(\mathrm{in}_\prec(J)) = 2n-1$.

By Proposition 2 one has $\mathrm{depth}(\mathrm{in}_\prec(J)) \ge 2n-1$ being $\{X_{n-1}T_2,\dots,X_1T_n, Y_{n-1}T_{n+2},\dots,Y_1T_{2n}, I'_kX_n^tT_1^sT_j\}$ a regular sequence of $2n-1$ elements of $\mathrm{in}_\prec(J)$, where $k+t=n-1$, $s=t-1$ and $n+1\le j\le 2n$. One has $2n-1\le\mathrm{depth}(\mathrm{in}_\prec(J))\le 2n-1$, hence the equality holds. If follows $\mathrm{depth}(\mathrm{in}_\prec(J)) = \mathrm{ht}(\mathrm{in}_\prec(J))$, then $\mathrm{in}_\prec(J)$ is Cohen-Macaulay that is equivalent to say that $R[T_1,\dots,T_{2n}]/\mathrm{in}_\prec(J)$ is Cohen-Macaulay.

Hence $\mathrm{depth}(R[T_1,\dots,T_{2n}]/\mathrm{in}_\prec(J)) = \dim(R[T_1,\dots,T_{2n}]/\mathrm{in}_\prec(J)) = 2n+1$.

3) $\mathrm{pd}(R[T_1,\dots,T_{2n}]/\mathrm{in}_\prec(J)) = \dim(R[T_1,\dots,T_{2n}]) - \mathrm{depth}(R[T_1,\dots,T_{2n}]/\mathrm{in}_\prec(J)) = 4n-(2n+1) = 2n-1$. □

The following result holds for the the symmetric algebra $Sym_R(L)$.

Theorem 5. *Let* $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$, $L = I_{n-1}+J_{n-1}\subset R$, $Sym_R(L) = S/J$ *with* $S = R[T_1,\dots,T_{2n}]$.

Then:

1) $dim(Sym_R(L)) = 2n+1$

2) $depth(Sym_R(L)) = 2n+1$.

3) $pd(Sym_R(L)) = 2n-1$.

Proof. 1) $\dim(Sym_R(L)) = \dim(S/\mathrm{in}_\prec(J)) = 2n+1$.

2) One has $\mathrm{depth}(Sym_R(L)) \ge \mathrm{depth}(\mathrm{in}_\prec(J)) = 2n+1$. On the other hand $\mathrm{depth}(Sym_R(L)) \le \dim(Sym_R(L)) = 2n+1$. The thesis follows.

3) $\mathrm{pd}(Sym_R(L)) = \mathrm{pd}(S/J) = \dim(S) - \mathrm{depth}(S/J) = 4n-(2n+1) = 2n-1$. □

Corollary 3. *Let* $R = K[X_1,\dots,X_n;Y_1,\dots,Y_n]$, $L = I_{n-1}+J_{n-1}\subset R$. *Then* $Sym_R(L)$ *is Cohen-Macaulay.*

Example 1. $R = K[X_1,X_2,X_3;Y_1,Y_2,Y_3]$

$$L = I_2 + J_2 = (X_1X_2, X_1X_3, X_2X_3, Y_1Y_2, Y_1Y_3, Y_2Y_3)$$

Set $f_1 = X_1X_2$, $f_2 = X_1X_3$, $f_3 = X_2X_3$, $f_4 = Y_1Y_2$, $f_5 = Y_1Y_3$, $f_6 = Y_2Y_3$, where $f_1 \prec \cdots \prec f_6$ with respect to the lex order and $X_1 \prec X_2 \prec X_3 \prec Y_1 \prec Y_2 \prec Y_3$.

$G = \{X_2T_2 - X_3T_1, X_1T_3 - X_3T_1, X_1X_2T_4 - Y_1Y_2T_1, X_1X_2T_5 - Y_1Y_3T_1, X_1X_2T_6 - Y_2Y_3T_1, X_1T_3 - X_2T_2, X_1X_3T_4 - Y_1Y_2T_2, X_1X_3T_5 - Y_1Y_3T_2, X_1X_3T_6 - Y_2Y_3T_2, X_2X_3T_4 - Y_1Y_2T_3, X_2X_3T_5 - Y_1Y_3T_3, X_2X_3T_6 - Y_2Y_3T_3, Y_2T_5 - Y_3T_4, Y_1T_6 - Y_3T_4, Y_1T_6 - Y_2T_5\}$ is a set of generators for J.

$$BG(J) = G \cup \{S(g_{13}, g_{24}), S(g_{13}, g_{25}), S(g_{13}, g_{26})\},$$

where

$$S(g_{13}, g_{24}) = Y_1 Y_2 T_2 T_3 - X_3^2 T_1 T_4;$$

$$S(g_{13}, g_{25}) = Y_1 Y_2 T_2 T_3 - X_3^2 T_1 T_5;$$

$$S(g_{13}, g_{26}) = Y_1 Y_2 T_2 T_3 - X_3^2 T_1 T_6.$$

The annihilator ideals are

$$\mathcal{I}_2 = (X_2);$$

$$\mathcal{I}_3 = (X_1);$$

$$\mathcal{I}_4 = (X_1X_2, X_1X_3, X_2X_3) = I_2;$$

$$\mathcal{I}_5 = (X_1X_2, X_1X_3, X_2X_3, Y_2) = (I_2, Y_2);$$

$$\mathcal{I}_6 = (X_1X_2, X_1X_3, X_2X_3, Y_1) = (I_2, Y_1).$$

Moreover $\text{in}_{\prec}(J) = J_L + J^*$, with
$J_L = ((X_2)T_2, (X_1)T_3, (X_1X_2, X_1X_3, X_2X_3)T_4, (X_1X_2, X_1X_3, X_2X_3, Y_2)T_5,$
$(X_1X_2, X_1X_3, X_2X_3, Y_1)T_6)$ and $J^* = (X_3^2T_1T_4, X_3^2T_1T_5, X_3^2T_1T_6)$.

By direct calculations, one obtains:

$$\dim(Sym_R(L)) = \text{depth}(Sym_R(L)) = 7$$

$$\text{pd}(Sym_R(L)) = 5.$$

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NECESSARY AND SUFFICIENT CONDITIONS FOR GEOMETRIC MEANS OF SEQUENCES IN MULTIPLICATIVE CALCULUS

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Abstract. In this paper we have introduced the concept of geometric convergence of a sequence and determined the necessary and sufficient condition under which convergence follows from geometric convergence of a sequence in multiplicative sense. Corollaries allow this condition to be replaced by multiplicative analogues of Schmidt type slow oscillation condition or Landau type two-sided condition.

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1. INTRODUCTION AND PRELIMINARIES

During the 17th century many scholars, as Galileo, discussed the following problem:
" Two estimates, 10 and 1000, are proposed as the value of a horse, which estimates, if any, deviates more from the true value of 100?"

The scholars who maintained that deviations should be measured by differences concluded that the estimate of 10 was closer to the true value. However, Galileo eventually maintained that the deviations should be measured by ratios, and he concluded that two estimates deviated equally from the true value (see [8]).

In this situation, the question "If we measure by ratios, what kind of a calculus do we have?" appears. The answer is the main idea of multiplicative calculus. Multiplicative calculus is alternative to the classical calculus. It provides differentiation and integration tools based on multiplication instead of addition. Every fact in classical calculus has an analogue in multiplicative calculus.

In recent years, multiplicative calculus has attained noticeable importance and popularity, due mainly to its applications in different areas such as biomedical image analysis [7], physics [3], dynamical systems [2], numerical analysis [9], complex analysis [10], sequence spaces [5] and economy [6].

Now, we give some basic properties of multiplicative calculus.

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Let $x \in \mathbb{R}^+$. The multiplicative absolute value of x (see [4]) is defined as

$$|x|^* = \begin{cases} x, & \text{if } x \geq 1, \\ \frac{1}{x}, & \text{if } x < 1. \end{cases}$$

Using the above definition we have the following properties of the multiplicative absolute value function.

For any $x, y \in \mathbb{R}^+$, the followings are valid (see [1]):

i) $|x|^* \geq 1$,
ii) $|\frac{1}{x}|^* = |x|^*$,
iii) $|x|^* \leq y$ if and only if $\frac{1}{y} \leq x \leq y$,
iv) $|xy|^* \leq |x|^*|y|^*$.

By means of the multiplicative absolute value, Bashirov et al. [4] introduced the multiplicative distance between positive real numbers x and y as

$$d^*(x, y) = \left|\frac{x}{y}\right|^*$$

which satisfies the following properties:

i) $d^*(x, y) \geq 1$ for all $x, y \in \mathbb{R}^+$,
ii) $d^*(x, y) = 1$ if and only if $x = y$,
iii) $d^*(x, y) = d^*(y, x)$ for all $x, y \in \mathbb{R}^+$,
iv) $d^*(x, z) \leq d^*(x, y)d^*(y, z)$ for all $x, y, z \in \mathbb{R}^+$.

As a consequence, $\mathbb{R}^+$ is a multiplicative metric space and a sequence (x_n) in $\mathbb{R}^+$ converges to the limit $L \in \mathbb{R}^+$ in the multiplicative sense if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d^*(x_n, L) = \left|\frac{x_n}{L}\right|^* < \epsilon$ for all $n > N$, or equivalently,

$$d^*(x_n, L) \xrightarrow{*} 1 \text{ as } n \to \infty.$$

It is well known that $\mathbb{R}^+$ is not complete according to the Euclidean metric. To emphasize the importance of this work, we should first note that $(\mathbb{R}^+, |.|^*)$ is a complete multiplicative metric space [1].

Also, it should be noted that throughout this paper the concept of convergence of a sequence is understood in multiplicative sense.

Let (x_n) be a sequence of positive real numbers. The n-th geometric mean g_n of (x_n) is defined by

$$g_n := \left(\prod_{k=0}^{n} x_k\right)^{\frac{1}{n+1}}, \quad n = 0, 1, 2, \ldots.$$

We say that (x_n) converges to L by the geometric mean method, briefly: G-convergent to $L \in \mathbb{R}^+$ if

$$\lim_{n\to\infty} g_n = L \tag{1.1}$$

and we write $x_n \overset{*}{\to} L\ (G)$.

If the sequence (x_n) diverges, then we may assign a limit value to (x_n) by a different limitation method. In this case we still say that (x_n) diverges but, even so, we are able to find a finite number that can be considered as its limit value.

We can replace $\lim_{n\to\infty} x_n$, which possibly does not exist, by

$$\lim_{n\to\infty}\left(\prod_{k=0}^{n} x_k\right)^{\frac{1}{n+1}} = L \in \mathbb{R}^+$$

if this multiplicative limit exists and we use this value for the limit of the sequence (x_n). This method is completely natural because it just takes geometric means and settles for computing a kind of "mean" limit where an actual limit fails to exist.

Our concern while studying a new kind of convergence is to determine whether it assigns the "correct" value to a sequence which is already convergent.
That is, does

$$x_n \overset{*}{\to} L \quad \Longrightarrow \quad x_n \overset{*}{\to} L\ (G)?$$

Any method assigning a limit value to a sequence is said to be regular if this is the case. In Lemma 1, we prove that if the limit

$$\lim_{n\to\infty} x_n = L \tag{1.2}$$

exists, then the limit (1.1) also exists. That the converse of Lemma 1 is not true in general is provided by the following example.

Example 1. The sequence $(x_n) = \left(\prod_{k=0}^{n} e^{(-1)^k}\right)$ is divergent, but it is G-convergent to $\sqrt{e}$.

Here, we define the concept of slowly oscillation which is more general than the convergence of a sequence in multiplicative calculus.

Definition 1. A sequence (x_n) of positive real numbers is said to be $*$-slowly oscillating if

$$\liminf_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n<m\le\lambda_n}\left|\frac{x_m}{x_n}\right|^* = 1, \tag{1.3}$$

where by λ_n we denote the integral part of the product λn, in symbol $\lambda_n := [\lambda n]$, or equivalently, if for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ and $\lambda_0 = \lambda_0(\epsilon) > 1$, as close to 1 as we wish, such that

$$\left|\frac{x_m}{x_n}\right|^* < \epsilon \tag{1.4}$$

whenever $n_0 < n < m \le \lambda_n$.

It is also easy to show that (1.3) is satisfied if and only if $\lim \left|\frac{x_m}{x_n}\right|^* = 1$ whenever $1 < \frac{m}{n} \overset{*}{\to} 1$, $m,n \to \infty$.

Example 2. The sequence $(x_n) = (n)$ is $*$-slowly oscillating, but not convergent. In fact,

$$\lim \left|\frac{x_m}{x_n}\right|^* = \lim \left|\frac{m}{n}\right|^* = 1,$$

whenever $1 < \frac{m}{n} \overset{*}{\to} 1$, $m,n \to \infty$.

Our main goal in this paper is to obtain a necessary and sufficient condition under which convergence of a sequence follows from convergence of its geometric means, and vice versa.

2. Main result

We prove the following theorem. In main theorem, some relations with convergence and G-convergence of a sequence are given. Also, we present some corollaries which include the multiplicative analogues of conditions of Schmidt type and Landau type.

Theorem 1. *Let (x_n) be a sequence of positive real numbers. If (x_n) is G-convergent to a finite limit L, then (x_n) is convergent to L if and only if one of the following two conditions hold:*

$$\liminf_{\lambda\to 1^+} \limsup_{n\to\infty} \left(\left| \prod_{i=n+1}^{\lambda_n} \frac{x_i}{x_n} \right|^* \right)^{\frac{1}{\lambda_n - n}} = 1, \tag{2.1}$$

or

$$\liminf_{\lambda\to 1^-} \limsup_{n\to\infty} \left(\left| \prod_{i=\lambda_n+1}^{n} \frac{x_n}{x_i} \right|^* \right)^{\frac{1}{n-\lambda_n}} = 1, \tag{2.2}$$

where by λ_n we denote the integral part of the product λn, in symbol $\lambda_n := [\lambda n]$.

The proof of Theorem 1 relies on the representations (3.4) and (3.5).

The next corollaries of Theorem 1 follows immediately.

Corollary 1. *If the sequence (x_n) is G-convergent to L and $*$-slowly oscillating, then it is convergent to L.*

Corollary 2. *If (x_n) is G-convergent to L and $((\Delta^* x_n)^n)$ is bounded, where*

$$\Delta^* x_n = \frac{x_n}{x_{n-1}} \text{ for } n \geq 1 \text{ and } \Delta^* x_0 = x_0,$$

then (x_n) is convergent to L.

Corollary 3. *If (x_n) is G-convergent to L and*

$$(\Delta^* x_n)^n \overset{*}{\to} 1, \text{ as } n \to \infty,$$

then (x_n) is convergent to L.

3. Auxiliary results

We need following lemmas for the proof of our main theorem.

Lemma 1. *If the sequence (x_n) is convergent to L, then the sequence (g_n) of its geometric means is convergent to L.*

Proof. Let $\lim_{n\to\infty} x_n = L$. Then for all $\epsilon > 1$, there exists $N_0 \in \mathbb{N}$ such that $\left|\frac{x_n}{L}\right|^* < \epsilon^{1/2}$ whenever $n > N_0$, and $\left|\frac{x_n}{L}\right|^* < M$ whenever $n \le N_0$. So, we have

$$\begin{aligned}
\left|\frac{g_n}{L}\right|^* &= \left|\left(\prod_{k=0}^{n} x_k\right)^{\frac{1}{n+1}} \Big/ \left(\prod_{k=0}^{n} L\right)^{\frac{1}{n+1}}\right|^* \\
&= \left|\left(\prod_{k=0}^{n} \frac{x_k}{L}\right)^{\frac{1}{n+1}}\right|^* \\
&\le \left(\prod_{k=0}^{n} \left|\frac{x_k}{L}\right|^*\right)^{\frac{1}{n+1}} \\
&= \left(\prod_{k=0}^{N_0} \left|\frac{x_k}{L}\right|^*\right)^{\frac{1}{n+1}} \left(\prod_{k=N_0+1}^{n} \left|\frac{x_k}{L}\right|^*\right)^{\frac{1}{n+1}} \\
&\le M^{\frac{N_0+1}{n+1}} (\epsilon^{1/2})^{\frac{n-N_0}{n+1}} \\
&\le M^{\frac{N_0+1}{n+1}} \epsilon^{1/2}
\end{aligned}$$

Since $\lim_{n\to\infty} M^{\frac{N_0+1}{n+1}} = 1$, there exists $N_1 \in \mathbb{N}$ such that $\left|M^{\frac{N_0+1}{n+1}}\right|^* < \epsilon^{1/2}$ whenever $n > N_1$. Therefore, there exists $N = \max\{N_0, N_1\}$ such that $\left|\frac{g_n}{L}\right|^* < \epsilon$ for all $n > N$. This completes the proof. □

Next, we prove that if a sequence is G-convergent to a finite limit L, then the so-called moving geometric means converge to the same limit. More precisely, the following lemma is valid.

Lemma 2. *If the sequence (x_n) is G-convergent to a finite limit L, then for each $\lambda > 1$*

$$\tau^{>}_{n,\lambda_n}(x) := \left(\prod_{m=n+1}^{\lambda_n} x_m\right)^{\frac{1}{\lambda_n - n}} \xrightarrow{*} L, \tag{3.1}$$

and for each $0 < \lambda < 1$

$$\tau^{<}_{n,\lambda_n}(x) := \left(\prod_{m=\lambda_n+1}^{n} x_m\right)^{\frac{1}{n-\lambda_n}} \xrightarrow{*} L, \tag{3.2}$$

where by λ_n we denote the integral part of the product λn, in symbol $\lambda_n := [\lambda n]$.

Proof. If $\lambda > 1$ and n is large enough in the sense that $\lambda_n > n$, then

$$\begin{aligned}
\left(\prod_{m=n+1}^{\lambda_n} x_m\right)^{\frac{1}{\lambda_n - n}} &= \left(\prod_{m=0}^{\lambda_n} x_m \Big/ \prod_{m=0}^{n} x_m\right)^{\frac{1}{\lambda_n - n}} \\
&= \left[\left(\left(\prod_{m=0}^{\lambda_n} x_m\right)^{\frac{1}{\lambda_n+1}}\right)^{\frac{\lambda_n+1}{\lambda_n-n}} \Big/ \left(\left(\prod_{m=0}^{n} x_m\right)^{\frac{1}{n+1}}\right)^{\frac{n+1}{\lambda_n-n}}\right] \\
&= \left(\left(g_{\lambda_n}\right)^{\frac{\lambda_n+1}{\lambda_n-n}} \Big/ \left(g_n\right)^{\frac{n+1}{\lambda_n-n}}\right) \\
&= g_n \left(\frac{g_{\lambda_n}}{g_n}\right)^{\frac{\lambda_n+1}{\lambda_n-n}}
\end{aligned}$$

Since

$$\left|\frac{\tau^{>}_{n,\lambda_n}(x)}{L}\right|^{*} = \left|\frac{g_n}{L}\right|^{*} \left(\left|\frac{g_{\lambda_n}}{L}\right|^{*}\right)^{\frac{\lambda_n+1}{\lambda_n-n}} \left(\left|\frac{g_n}{L}\right|^{*}\right)^{\frac{\lambda_n+1}{\lambda_n-n}}, \tag{3.3}$$

(3.1) follows from (1.1). The proof of (3.2) can be similarly obtained. □

The following two representations of the ratio $\dfrac{x_n}{g_n}$ will be used in the proof of the main theorem.

Lemma 3. *(i) For all $\lambda > 1$ and large enough n, that is, when $\lambda_n > n$,*

$$\frac{x_n}{g_n} = \left(\frac{g_{\lambda_n}}{g_n}\right)^{\frac{\lambda_n+1}{\lambda_n-n}} \left[\left(\prod_{i=n+1}^{\lambda_n} \frac{x_i}{x_n}\right)^{\frac{1}{\lambda_n-n}}\right]^{-1}. \tag{3.4}$$

(ii) For all $0 < \lambda < 1$ *and large enough* n*, that is, when* $n > \lambda_n$*,*

$$\frac{x_n}{g_n} = \left(\frac{g_n}{g_{\lambda_n}}\right)^{\frac{\lambda_n+1}{n-\lambda_n}} \left[\left(\prod_{i=\lambda_n+1}^{n} \frac{x_n}{x_i}\right)^{\frac{1}{n-\lambda_n}}\right]^{-1}. \tag{3.5}$$

Proof. By definition, for every $\lambda > 1$, we have

$$\begin{aligned}\frac{g_{\lambda_n}}{g_n} &= \frac{\left(\prod_{i=0}^{\lambda_n} x_i\right)^{\frac{1}{\lambda_n+1}}}{\left(\prod_{i=0}^{n} x_i\right)^{\frac{1}{n+1}}} \\ &= \left(\prod_{i=0}^{n} x_i\right)^{\frac{n-\lambda_n}{(n+1)(\lambda_n+1)}} \left(\prod_{i=n+1}^{\lambda_n} x_i\right)^{\frac{1}{\lambda_n+1}}\end{aligned}$$

Hence,

$$\begin{aligned}\frac{g_{\lambda_n}}{g_n} &= g_n{}^{\frac{n-\lambda_n}{\lambda_n+1}} \left(\prod_{i=n+1}^{\lambda_n} \frac{x_i}{x_n} x_n\right)^{\frac{1}{\lambda_n+1}} \\ &= g_n{}^{\frac{n-\lambda_n}{\lambda_n+1}} \left(\prod_{i=n+1}^{\lambda_n} \frac{x_i}{x_n}\right)^{\frac{1}{\lambda_n+1}} x_n{}^{\frac{\lambda_n-n}{\lambda_n+1}}\end{aligned}$$

and

$$\left(\frac{g_{\lambda_n}}{g_n}\right)^{\frac{\lambda_n+1}{\lambda_n-n}} = \frac{x_n}{g_n} \left(\prod_{i=n+1}^{\lambda_n} \frac{x_i}{x_n}\right)^{\frac{1}{\lambda_n-n}},$$

which is equivalent to (3.4).

(ii) The proof of (3.5) is similar. □

4. Proof

Proof of Theorem 1. Necessity. Assume that (1.1) holds. Since (x_n) converges to L, we have

$$\lim_{n\to\infty} \left|\frac{x_n}{g_n}\right|^* = 1. \tag{4.1}$$

It follows by (1.1) that for all $\lambda > 1$,

$$\lim_{n\to\infty} \left(\left|\frac{g_{\lambda_n}}{g_n}\right|^*\right)^{\frac{\lambda_n+1}{\lambda_n-n}} = 1. \tag{4.2}$$

Since

$$\left(\left|\prod_{i=n+1}^{\lambda_n}\frac{x_i}{x_n}\right|^*\right)^{\frac{1}{\lambda_n-n}} \le \left|\frac{x_n}{g_n}\right|^* \left(\left|\frac{g_{\lambda_n}}{g_n}\right|^*\right)^{\frac{\lambda_n+1}{\lambda_n-n}}, \tag{4.3}$$

(2.1) follows from (3.4), (4.1), (4.2), and (4.3).

It follows by (1.1) that for all $0 < \lambda < 1$,

$$\lim_{n\to\infty}\left(\left|\frac{g_n}{g_{\lambda_n}}\right|^*\right)^{\frac{\lambda_n+1}{n-\lambda_n}} = 1. \tag{4.4}$$

Since

$$\left(\left|\prod_{i=\lambda_n+1}^{n}\frac{x_n}{x_i}\right|^*\right)^{\frac{1}{n-\lambda_n}} \le \left|\frac{x_n}{g_n}\right|^* \left(\left|\frac{g_n}{g_{\lambda_n}}\right|^*\right)^{\frac{\lambda_n+1}{n-\lambda_n}}, \tag{4.5}$$

(2.2) follows from (3.5), (4.1), (4.4), and (4.5).

Sufficiency. Assume the fulfilment of (1.1) and (2.1). It follows from (2.1) that there exists a decreasing sequence (λ_j) converging to 1 such that

$$\lim_{j\to\infty}\limsup_{n\to\infty}\left(\left|\prod_{i=n+1}^{\lambda_{jn}}\frac{x_i}{x_n}\right|^*\right)^{\frac{1}{\lambda_{jn}-n}} = 1, \tag{4.6}$$

where $\lambda_{jn} = [\lambda_j n]$.

By (3.4), we have

$$\limsup_{n\to\infty}\left|\frac{x_n}{g_n}\right|^* \le \lim_{j\to\infty}\limsup_{n\to\infty}\left(\left|\frac{g_{\lambda_{jn}}}{g_n}\right|^*\right)^{\frac{\lambda_{jn}+1}{\lambda_{jn}-n}} \lim_{j\to\infty}\limsup_{n\to\infty}\left(\left|\prod_{i=n+1}^{\lambda_{jn}}\frac{x_i}{x_n}\right|^*\right)^{\frac{1}{\lambda_{jn}-n}} \tag{4.7}$$

Taking (1.1) and (4.6) into account, we obtain

$$\lim_{n\to\infty}\left|\frac{x_n}{g_n}\right|^* = 1 \tag{4.8}$$

Assume the fulfilment of (1.1) and (2.2). It follows from (2.2) that there exists an increasing sequence (λ_j) converging to 1 such that

$$\lim_{j\to\infty}\limsup_{n\to\infty}\left(\left|\prod_{i=\lambda_{jn}+1}^{n}\frac{x_n}{x_i}\right|^*\right)^{\frac{1}{n-\lambda_{jn}}} = 1, \tag{4.9}$$

where $\lambda_{jn} = [\lambda_j n]$.

By (3.5), we have

$$\limsup_{n\to\infty}\left|\frac{x_n}{g_n}\right|^* \leq \lim_{j\to\infty}\limsup_{n\to\infty}\left(\left|\frac{g_n}{g_{\lambda_{jn}}}\right|^*\right)^{\frac{\lambda_{jn}+1}{n-\lambda_{jn}}} \lim_{j\to\infty}\limsup_{n\to\infty}\left(\left|\prod_{i=\lambda_{jn}+1}^{n}\frac{x_n}{x_i}\right|^*\right)^{\frac{1}{n-\lambda_{jn}}} \tag{4.10}$$

Taking (1.1) and (4.9) into account, we obtain

$$\lim_{n\to\infty}\left|\frac{x_n}{g_n}\right|^* = 1 \tag{4.11}$$

Combining (1.1) and (4.8) or (4.11), in either case we conclude that (x_n) converges to L. □

Since $*$-slow oscillation implies both (2.1) and (2.2), the proof of Corollary 1 follows from Theorem 1. However, the proof of Corollary 1 seems to be fundamental, we prove it.

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NEAR SEMIRINGS AND SEMIRINGS WITH INVOLUTION

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Abstract. We study so-called near semirings endowed with an antitone involution. Such a near semiring is in fact a bounded lattice which has one more binary operation, the multiplication. We classify several families of bounded lattices which can be organized in such near semirings, e.g. chains or orthomodular lattices. A particular case are the so-called balanced near semirings which form a variety which is congruence distributive, permutable and regular.

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Keywords: Near semiring, antitone involution, orthomodular lattice, congruence distributive, congruence permutable, congruence regular, balanced near semiring

1. INTRODUCTION

The concept of a near semiring was introduced by the authors in [3]. It seems to be useful in order to axiomatize certain so-called quantum structures, e.g. basic algebras, MV-algebras and orthomodular lattices. With respect to addition, near semirings used in these representations are in fact join semilattices with an additional operation, which is multiplication. Hence, in every such a near semiring there can be introduced a semilattice order and these near semirings can be considered to be ordered. Among other things this means that if an antitone involution is introduced, such a near semiring becomes a lattice with multiplication. The question arises which lattices can be equipped with a suitable multiplication in order to become a near semiring with antitone involution. In the present paper we will study this problem and provide partial solutions. Adding some natural identities to the axioms of a near semiring with involution we obtain so-called balanced near semirings. They form a variety having very strong congruence properties. Namely, this variety turns out to be congruence distributive, permutable and regular.

We start with the following definition:

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Definition 1. A *near semiring* is an algebra $\mathbf{R} = (R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that $(R, +, 0)$ is a commutative monoid and $\mathbf{R}$ satisfies the following identities:

- $(x + y)z \approx xz + yz$ (so-called *right distributivity*),
- $x0 \approx 0x \approx 0$,
- $x1 \approx 1x \approx x$.

$\mathbf{R}$ is called

- *commutative* if it satisfies $xy \approx yx$,
- *idempotent* if it satisfies $x + x \approx x$,
- *multiplicatively idempotent* if it satisfies $xx \approx x$,
- a *semiring* if it satisfies $(xy)z \approx x(yz)$ and $z(x + y) \approx zx + zy$.

If $\mathbf{R}$ is idempotent then we define a partial order relation $\leq$ on R by

$$x \leq y \text{ if and only if } x + y = y.$$

Then $(R, \leq)$ is a poset with smallest element 0 since it corresponds to the join-semilattice $(R, +)$. In the following we will call $\leq$ the *induced order* of the near semiring $\mathbf{R}$.

If $\mathbf{R}$ is a commutative semiring which is multiplicatively idempotent then we can define a partial order relation $\leq_1$ on R by

$$x \leq_1 y \text{ if and only if } xy = x$$

$(x, y \in R)$. Of course, the partial order relations $\leq$ and $\leq_1$ may be different as we will see later.

Now we define our main concept.

Definition 2. A *near semiring with involution* is an algebra $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(R, +, \cdot, 0, 1)$ is an idempotent near semiring and the following conditions hold for all $x, y \in R$:

(a) If $x \leq y$ then $y' \leq x'$.
(b) $(x')' = x$.

Let $\mathcal{N}$ denote the class of near semirings with involution. Directly by definition, $\mathcal{N}$ is a variety. $\mathbf{R} \in \mathcal{N}$ is called a *semiring with involution* if $(R, +, \cdot, 0, 1)$ is a semiring. Let $\mathcal{S}$ denote the class of semirings with involution. Clearly, also $\mathcal{S}$ is a variety.

Lemma 1. *Every* $\mathbf{R} = (R, +, \cdot, ', 0, 1) \in \mathcal{N}$ *satisfies* $(x + y)' + x' \approx x'$.

Proof. For $a, b \in R$ we have $a \leq a + b$ and hence $(a + b)' \leq a'$ which shows $(a + b)' + a' = a'$. □

It is almost evident that condition (a) of Definition 2 can be replaced by the identity from Lemma 1.

2. Duals of near semirings with involution

In what follows, we use the antitone involution in order to show that the concept of a near semiring with involution can be dualized. Although $(R,+,\cdot)$ need not be a lattice, we can define new operations as follows by using the De Morgan laws.

Definition 3. For every $\mathbf{R} = (R,+,\cdot,',0,1) \in \mathcal{N}$ define an algebra $\mathbf{R}' := (R,+',\cdot',',0',1')$ by $x +' y := (x' + y')'$ and $x \cdot' y := (x'y')'$ for all $x, y \in R$. The algebra $\mathbf{R}'$ will be called the *dual* of $\mathbf{R}$.

Of course, if $\mathbf{R} = (R,+,\cdot,',0,1) \in \mathcal{N}$ then the induced algebra $\mathbb{L}(\mathbf{R}) := (R,+,+',',0,0')$ is a bounded lattice with an antitone involution and hence it satisfies the De Morgan laws.

We can prove the following statement concerning the dual of $\mathbf{R}$.

Theorem 1. *If* $\mathbf{R} = (R,+,\cdot,',0,1) \in \mathcal{N}$ *then* $\mathbf{R}' := (R,+',\cdot',',0',1') \in \mathcal{N}$ *and* $(\mathbf{R}')' = \mathbf{R}$. *The mapping* $h(x) = x'$ *is an isomorphism from* $\mathbf{R}$ *onto* $\mathbf{R}'$. *Moreover,* $\mathbf{R}' \in \mathcal{S}$ *if and only if* $\mathbf{R} \in \mathcal{S}$.

Proof. It is easy to check that $(R,+',\cdot',0',1')$ is an idempotent near semiring whose induced order is $\geq$. Hence $\mathbf{R}'$ satisfies the corresponding conditions mentioned in Definition 2. Now $(\mathbf{R}')' = \mathbf{R}$ follows by a straightforward calculation. The remaining assertions are immediate. □

Let us recall that a ternary term $m(x,y,z)$ is called a *majority term* in a variety $\mathcal{V}$ if it satisfies the identities

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x.$$

It is well-known that if $\mathcal{V}$ has a majority term then it is congruence distributive, see e.g. [2].

The following important result follows directly from the fact that $(R,+,+')$ is a lattice as mentioned above.

Theorem 2. *The variety* $\mathcal{N}$ *has the majority term*

$$m(x,y,z) = (x+y) +' (y+z) +' (z+x)$$

and hence is congruence distributive.

3. Ortho near semirings

Up to now, we did not assume the involution to have some more important properties. This will be done now. In what follows we will ask the involution to be a complementation in the induced bounded lattice $(R,+,+',0,0')$. Hence we define

Definition 4. A member of $\mathcal{N}$ is called an *ortho near semiring* if it satisfies $x + x' \approx 0'$. If, in addition, it belongs to $\mathcal{S}$ then it is called an *orthosemiring*.

It is evident that in this case also the identity $x +' x' \approx 0$ holds and that the dual of an ortho near semiring is an ortho near semiring again. It is clear that the class of ortho near semirings forms a variety.

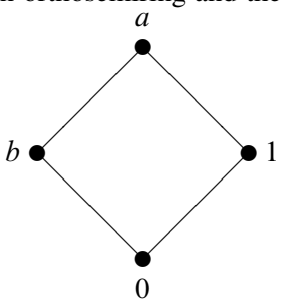
Example 1. Let $R = \{0, a, b, 1\}$ and define binary operations $+$ and $\cdot$ and a unary operation $'$ on R by

$+$	0	a	b	1
0	0	a	b	1
a	a	a	a	a
b	b	a	b	a
1	1	a	a	1

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	b	b	b
1	0	a	b	1

x	x'
0	a
a	0
b	1
1	b

Then $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is an orthosemiring and the Hasse diagram of its induced order looks as follows:



Hence $(R, +, +')$ is in fact a distributive lattice which is not a chain. Since $\mathbf{R}$ is, moreover, multiplicatively idempotent, commutative and associative, we can introduce the order $\leq_1$ induced by multiplication. For this order we have $0 <_1 b <_1 a <_1 1$ which is a chain.

An example of an ortho near semiring whose induced lattice is not distributive is shown in the following

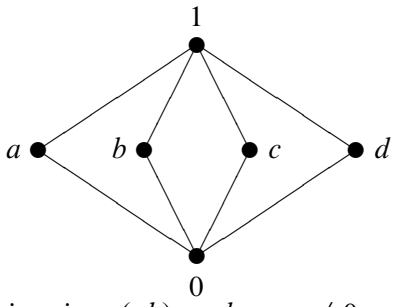
Example 2. Let $R = \{0, a, b, c, d, 1\}$ and define binary operations $+$ and $\cdot$ and a unary operation $'$ on R by

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	1	1	1
b	b	1	b	1	1	1
c	c	1	1	c	1	1
d	d	1	1	1	d	1
1	1	1	1	1	1	1

$\cdot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	0	d	a
b	0	a	b	c	0	b
c	0	0	b	c	d	c
d	0	a	0	c	d	d
1	0	a	b	c	d	1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Then $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is an ortho near semiring the Hasse diagram of its induced order looks as follows:



R is neither an orthosemiring since $(ab)c = bc = c \neq 0 = ac = a(bc)$ nor commutative since $cd = d \neq c = dc$.

4. Constructions of near semirings with involution

Now we investigate the question which bounded lattices $(L, \vee, \wedge, ', 0, 1)$ can be equipped with a binary operation of multiplication in such a way that $(L, \vee, \cdot, ', 0, 1)$ becomes a near semiring with involution and $(L, \vee, \cdot)$ is not a lattice.

If $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $a \in L$ we call the interval $[0, a]$ a *section*. We say that the mapping $x \mapsto x^a$ from $[0, a]$ to $[0, a]$ is a *sectional antitone involution* on $([0, a], \leq)$ if $(x^a)^a = x$ and $x \leq y$ implies $y^a \leq x^a$ for each $x, y \in [0, a]$.

Theorem 3. *Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice where for each $a \in L$ there exists a sectional antitone involution ^{a} on the section $[0, a]$. Define $xy := (x^1 \wedge y)^y$ for all $x, y \in L$. Then $(L, \vee, \cdot, ^1, 0, 1)$ is a near semiring with involution.*

Proof. It is evident that $(L, \vee, 0)$ is an idempotent commutative monoid. Moreover,

$$\begin{aligned}
(x \vee y)z &= ((x \vee y)^1 \wedge z)^z = (x^1 \wedge y^1 \wedge z)^z = ((x^1 \wedge z) \wedge (y^1 \wedge z))^z \\
&= (x^1 \wedge z)^z \vee (y^1 \wedge z)^z = (xz) \vee (yz), \\
x0 &= (x^1 \wedge 0)^0 = 0^0 = 0, \\
0x &= (0^1 \wedge x)^x = x^x = 0, \\
x1 &= (x^1 \wedge 1)^1 = (x^1)^1 = x \text{ and} \\
1x &= (1^1 \wedge x)^x = 0^x = x
\end{aligned}$$

for all $x, y, z \in L$. □

For the reader's convenience let us recall that an *orthocomplemented lattice* is a bounded lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ with an antitone involution satisfying the identities $x \vee x' \approx 1$ and $x \wedge x' \approx 0$.

Not for every orthocomplemented lattice $(L, \vee, \wedge, ', 0, 1)$ there exists a multiplication $\cdot$ such that $(L, \vee, \cdot, ', 0, 1)$ becomes an ortho near semiring. A certain sufficient condition for this will be given in the next corollary.

Corollary 1. *If $(L,\vee,\wedge,',0,1)$ is an orthocomplemented lattice such that for every $a\in L$ there exists a sectional antitone involution ^{a} on $([0,a],\leq)$ such that ${}^1={}'$ and $xy:=(x'\wedge y)^y$ for all $x,y\in L$ then $(L,\vee,\cdot,',0,1)$ is an ortho near semiring.*

For finite chains the antitone involution is determined uniquely and we can prove that there exists a multiplication such that the induced near semiring is even a semiring.

Theorem 4. *Let $\mathbf{L}=(L,\vee,\wedge,',0,1)$ be a finite chain with antitone involution containing more than two elements. Then there exists a binary operation $\cdot$ on L such that $\mathbf{R}=(L,\vee,\cdot,',0,1)\in\mathcal{S}$ and $(L,\vee,\cdot)$ is not a lattice.*

Proof. Without loss of generality we can assume $L=\{1,\ldots,n\}$ with an integer $n>2$ and we define an order on L as follows: $1<3<\cdots<n-1<n<\cdots<4<2$ if n is even and $1<3<\cdots<n<n-1<\cdots<4<2$ if n is odd and denote by 'the unique antitone involution on $(L,\leq)$. As proved in [5], if $\cdot$ denotes the minimum operation on L with respect to the natural ordering $\leq_1$ of integers then $(L,\vee,\cdot,0,1)$ is a commutative idempotent semiring and hence $\mathbf{R}=(L,\vee,\cdot,',0,1)\in\mathcal{S}$. Because of $\leq\neq\leq_1$, $(L,\vee,\cdot)$ is not a lattice. □

Theorem 5. *For every infinite cardinal k there exists a bounded chain $\mathbf{L}=(L,\vee,\wedge,',0,1)$ of cardinality k with an antitone involution and a binary operation $\cdot$ on L such that $\mathbf{R}=(L,\vee,\cdot,',0,1)\in\mathcal{S}$ and $(L,\vee,\cdot)$ is not a lattice.*

Proof. Let $(C,\leq_2,0,1)$ be a bounded chain of infinite cardinality k, put $L:=C\times\{1,2\}$ and define binary relations $\leq$ and $\leq_1$ and a unary operation $'$ on L as follows:

- $(x,i)\leq(y,j)$ if either $((i,j)=(1,1)$ and $x\leq_2 y)$ or (i,j)=(1,2) or $((i,j)=(2,2)$ and $x\geq_2 y)$,
- $(x,i)\leq_1(y,j)$ if either $x<_2 y$ or $(x=y$ and $i\leq j)$,
- $(x,i)':=(x,3-i)$

for $(x,i),(y,j)\in L$. Then $(L,\leq,(0,1),(0,2))$ and $(L,\leq_1,(0,1),(1,2))$ are bounded chains of cardinality k and $'$ is an antitone involution on $(L,\leq)$. Let $\mathbf{L}=(L,\vee,\wedge,',(0,1),(0,2))$ denote the bounded lattice with antitone involution corresponding to $(L,\leq)$. Then $\mathbf{L}$ is a chain of cardinality k with an antitone involution as proved in [5]. If $\cdot$ denotes the minimum operation on L with respect to $\leq_1$ then $(L,\vee,\cdot,0,1)$ is a commutative idempotent semiring and hence $\mathbf{R}=(L,\vee,\cdot,',0,1)\in\mathcal{S}$. Because of $\leq\neq\leq_1$, $(L,\vee,\cdot)$ is not a lattice. □

For the reader's convenience let us recall that an *orthomodular lattice* is an orthocomplemented lattice $\mathbf{L}=(L,\vee,\wedge,',0,1)$ satisfying the identity $x\vee y\approx x\vee((x\vee y)\wedge x')$.

Now we can show that also every orthomodular lattice $\mathbf{L}=(L,\vee,\wedge,',0,1)$ can be equipped with a multiplication operation $\cdot$ in such a way that $(L,\vee,\cdot,',0,1)$ becomes

an ortho near semiring such that $(L,\vee,\cdot)$ is not a lattice provided $\mathbf{L}$ is not a Boolean algebra.

As it was defined in [6] and [1], two elements a,b of an orthomodular lattice $\mathbf{L}=(L,\vee,\wedge,',0,1)$ are said to *commute with each other*, in symbols $a\,\mathrm{C}\,b$ if $a=(a\wedge b)\vee(a\wedge b')$. The following result can be found in [6] and [1]:

Proposition 1. *If* $\mathbf{L}=(L,\vee,\wedge,',0,1)$ *is an orthomodular lattice and* $a,b,c\in L$ *then the following hold:*

(1) *If* $a\,\mathrm{C}\,b$ *then* $b\,\mathrm{C}\,a$ *and* $a\,\mathrm{C}\,b'$.
(2) *If* $a\leq b$ *then* $a\,\mathrm{C}\,b$.
(3) *If one of the three elements* a,b,c *commutes with the remaining two then the distributive laws hold for* a,b,c.
(4) $\mathbf{L}$ *is a Boolean algebra if and only if any two elements commute with each other.*

Now we are ready to get a positive answer to the previous question in the case of orthomodular lattices.

Theorem 6. *Let* $\mathbf{L}=(L,\vee,\wedge,',0,1)$ *be an orthomodular lattice which is not a Boolean algebra. Then there exists a binary operation* $\cdot$ *on* L *such that* $\mathbf{R}=(L,\vee,\cdot,',0,1)$ *is an ortho near semiring and* $(L,\vee,\cdot)$ *is not a lattice.*

Proof. Let $a,b,c\in L$. Define a binary operation $\cdot$ on L by $ab:=(a\vee b')\wedge b$. Since $c'\leq a\vee c'$ and $c'\leq b\vee c'$ we have by Proposition 1(2) that $c'\,\mathrm{C}\,a\vee c'$ and $c'\,\mathrm{C}\,b\vee c'$. Again according to Proposition 1(1) we have $c\,\mathrm{C}\,a\vee c'$ and $c\,\mathrm{C}\,b\vee c'$ and we can use distributivity for these elements (by Proposition 1(3)) and hence compute

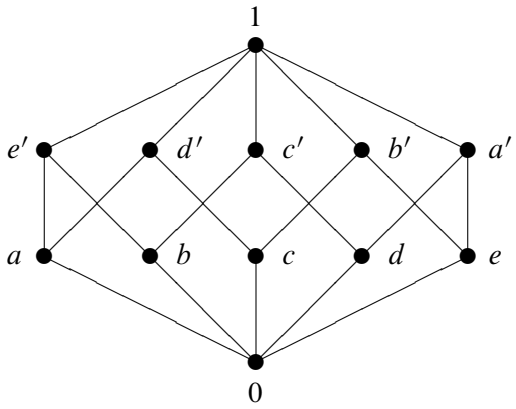
$$\begin{aligned}(a\vee b)c&=((a\vee b)\vee c')\wedge c=((a\vee c')\vee(b\vee c'))\wedge c\\&=((a\vee c')\wedge c)\vee((b\vee c')\wedge c)=ac\vee bc.\end{aligned}$$

By straightforward calculations it follows that $\mathbf{R}=(L,\vee,\cdot,',0,1)$ is a ortho near semiring. If $(L,\vee,\cdot)$ would be a lattice then we would have

$$\begin{aligned}a&=(a')'=(a'b'\vee a')'=(((a'\vee b)\wedge b')\vee a')'=(((a'\vee b)\vee a')\wedge(b'\vee a'))'\\&=((a'\vee b')\wedge(a'\vee b))'=(a\wedge b)\vee(a\wedge b'),\end{aligned}$$

i.e. $a\,\mathrm{C}\,b$ and hence $\mathbf{L}$ would be a Boolean algebra (by Proposition 1(4)) contradicting our assumption. This shows that $(L,\vee,\cdot)$ is not a lattice. □

The following example shows that there exist orthocomplemented lattices $(L,\vee,\wedge,',0,1)$ which are not orthomodular but have an antitone involution on every section $([0,a],\leq)$ and hence according to Corollary 1 can be converted into an ortho near semiring:



5. VARIETIES OF NEAR SEMIRINGS WITH INVOLUTION

In the remaining part of our paper we investigate some varieties of near semirings with involution. For this purpose we define the following:

Let $R = \{0, a, 1\}$ and define binary operations $+$ and $\cdot$ and a unary operation $'$ on R by

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

x	x'
0	a
a	0
1	1

Then $(R, +, \cdot, 0, 1)$ is a commutative idempotent semiring which was denoted by $\mathbf{S}_3$ in [5] and $\mathbf{S}_3' := (R, +, \cdot, ', 0, 1) \in \mathcal{S}$. The induced order of $\mathbf{S}_3$ is $0 < 1 < a$. However, the order induced by multiplication is as follows: $0 <_1 a <_1 1$. Thus $\leq \neq \leq_1$ and hence $(R, +, \cdot)$ is not a lattice.

As pointed out in [5], a prominent role plays the variety $\mathcal{V}(\mathbf{S}_3)$ generated by the near semiring $\mathbf{S}_3$. If we endow $\mathbf{S}_3$ with an antitone involution, we can ask about the properties of the variety $\mathcal{V}(\mathbf{S}_3')$ generated by $\mathbf{S}_3'$. It is rather surprising that contrary to $\mathcal{V}(\mathbf{S}_3)$, which is residually large, $\mathcal{V}(\mathbf{S}_3')$ has completely different properties.

It was shown in [5] that $\mathbf{S}_3$ is a subdirectly irreducible semiring and in [4] that $\mathcal{V}(\mathbf{S}_3)$ is the variety of commutative idempotent and multiplicatively idempotent semirings satisfying the identity $x + y + xy \approx x + y$ and that it has a proper class of subdirectly irreducible members. Contrary to this we can prove:

Theorem 7. *The only subdirectly irreducible member of* $\mathcal{V}(\mathbf{S}_3')$ *is* $\mathbf{S}_3'$.

Proof. Apparently, $\mathcal{V}(\mathbf{S}_3')$ is a subvariety of $\mathcal{S}$ which is in turn a subvariety of $\mathcal{N}$. By Theorem 2, $\mathcal{V}(\mathbf{S}_3')$ is congruence distributive. Since $\mathcal{V}(\mathbf{S}_3')$ is congruence distributive and $\mathbf{S}_3'$ is finite, every subdirectly irreducible member of $\mathcal{V}(\mathbf{S}_3')$ belongs to $\mathrm{HS}(\mathbf{S}_3')$ according to Jónsson's Lemma. However, as can be easily verified, $\mathbf{S}_3'$ has no proper subalgebras and it is simple and hence it has no non-trivial homomorphic image. This shows that $\mathbf{S}_3'$ is the only subdirectly irreducible member of $\mathcal{V}(\mathbf{S}_3')$. □

Very strong congruence properties have varieties of ortho near semirings satisfying a certain more or less natural property. Hence we define:

Definition 5. A *balanced near semiring* is an ortho near semiring $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ satisfying $((x + y)' + y)' + x \approx x$. Let $\mathcal{B}$ denote the class of balanced near semirings.

This identity is surely satisfied in an ortho near semiring provided the elements x and y are comparable. Namely, if $x \leq y$ then $x + y = y$ and thus $((x + y)' + y)' + x = (y' + y)' + x = (0')' + x = 0 + x = x$ and, if $y \leq x$ then $((x + y)' + y)' + x = (x' + y)' + x = (x +' y') + x = x$ according to the absorption law.

Remark 1.

(1) Of course, $\mathcal{B}$ is a variety.
(2) Every bounded distributive lattice with an antitone involution belongs to $\mathcal{B}$.
(3) We have $\mathcal{B} \subseteq \mathcal{N}$ and hence $\mathcal{B}$ is congruence distributive according to Theorem 2.

Theorem 8. *The variety $\mathcal{B}$ is congruence permutable and hence arithmetical.*

Proof. If $\mathbf{R} = (R, +, \cdot, ', 0, 1) \in \mathcal{B}$, $a, b, c \in R$ and

$$p(x, y, z) := (((x' + y)' + z)' + ((z' + y)' + x)')'$$

for all $x, y, z \in R$ then

$$\begin{aligned} p(a, a, b) &= (((a' + a)' + b)' + ((b' + a)' + a)')' = (b' + ((b' + a)' + a)')' \\ &= (((b' + a)' + a)' + b')' = (b')' = b \text{ and} \\ p(x, y, z) &= p(z, y, x) \text{ for all } x, y, z \in R \text{ and hence} \\ p(a, b, b) &= a \end{aligned}$$

which shows that p is a Malcev term proving congruence permutability of $\mathcal{B}$. According to Remark 1(3), $\mathcal{B}$ is arithmetical. □

Recall that a *variety* $\mathcal{V}$ is called *congruence regular* if for every $\mathbf{A} = (A, F) \in \mathcal{V}$ and each $a \in A$ and for every $\Theta, \Phi \in \operatorname{Con}\mathbf{A}$ the equality $[a]\Theta = [a]\Phi$ implies $\Theta = \Phi$. It is well-known (cf. [2]) that a variety $\mathcal{V}$ is congruence regular if and only if there exists a positive integer n and ternary terms $t_1, \ldots, t_n$ such that $t_1(x, y, z) = \cdots = t_n(x, y, z) = z$ is equivalent to $x = y$.

Theorem 9. *The variety $\mathcal{B}$ is congruence regular.*

Proof. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1) \in \mathcal{B}$ and $a, b, c \in R$ and put

$$\begin{aligned} t_1(x, y, z) &:= (x' + y)' + (x + y')' + z \text{ and} \\ t_2(x, y, z) &:= ((x + y')' + (x' + y)' + z')' \end{aligned}$$

for all $x, y, z \in R$. A straightforward calculation shows $t_1(a,a,b) = t_2(a,a,b) = b$. Conversely, assume $t_1(a,b,c) = t_2(a,b,c) = c$. Then $(a'+b)'+(a+b')' \leq c$ and $(a+b')'+(a'+b)' \leq c'$ whence $((a+b')'+(a'+b)')' \geq (c')' = c$ and $(a'+b)'+(a+b')' \leq ((a+b')'+(a'+b)')'$. Hence $(a'+b)' \leq ((a'+b)')' = a'+b$ and $(a+b')' \leq ((a+b')')' = a+b'$. From this we obtain $a+b' = a'+b = 0'$. Now we have

$$\begin{aligned} a &= (a')' = (((a'+b)'+b)'+a')' = ((0+b)'+a')' = (b'+a')' \leq (b')' \\ &= b \leq b+a = (0+b')'+a = ((a+b')'+b')'+a = a \end{aligned}$$

whence $a = b$. This shows that t_1 and t_2 are Csákány terms proving congruence regularity of $\mathcal{B}$ (cf. [2]). □

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A NEW SUMS AND ITS RECIPROCITY THEOREM

GUOHUI CHEN AND DI HAN

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Abstract. The main purpose of this paper is introduced a new sums analogous to Dedekind sums, then using the analytic method and the properties of Dirichlet L-functions to study the arithmetical properties of this sums, and give an interesting reciprocity theorem for it.

2010 *Mathematics Subject Classification:* 11M20

Keywords: a new sums, Dedekind sums, reciprocity theorem, identity, analytic method

1. Introduction

For a positive integer k and an arbitrary integer h, the classical Dedekind sums $S(h,k)$ is defined by

$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x))=\begin{cases}x-[x]-\frac{1}{2} & \text{if } x \text{ is not an integer}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

The various arithmetical properties of $S(h,k)$ were investigated by many authors, one of the most important results (see Tom M. Apostol [2] or L. Carlitz [3]) is its reciprocity theorem. That is, for all positive integers h and q with $(h,q)=1$, we have the identity

$$S(h,q)+S(q,h)=\frac{h^2+q^2+1}{12hq}-\frac{1}{4}. \tag{1.1}$$

The other properties of $S(h,q)$ can also be found in [2], [3], [4], [6], [7] [8] and [9]. But we think that the formula (1.1) is very important and interesting, because from it we can compute the value of $S(h,q)$ by $S(q,h)$, providing $(h,q)=1$.

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In this paper, we introduce a new sums $C(h,q)$ as follows:

$$C(h,q)=\sum_{a=1}^{q}{}'\cot\left(\frac{\pi ha}{q}\right)\cot\left(\frac{\pi a}{q}\right),$$

where $\sum'$ denotes the summation over all a such that $(a,q)=1$, $(h,q)=1$ and $\cot(x)=\cos(x)/\sin(x)$. We also provide $C(h,q)=0$, if $q|h$.

This sums looks very similar to Dedekind sums, so we think that it must have some similar properties with Dedekind sums. Based on this reason, we use the analytic method and the properties of Dirichlet L-functions to study the reciprocity properties of $C(h,k)$, and obtain an interesting reciprocity theorem. That is, we shall prove the following:

Theorem 1. *For any integers $h>1$ and $q>1$ with $(h,q)=1$, we have the reciprocity formula*

$$\frac{1}{q}\sum_{d|q}C(h,d)+\frac{1}{h}\sum_{d|h}C(q,d)=\frac{q^2+h^2+1}{3qh}-1,$$

where $\sum_{d|q}$ denotes the summation over all divisors d of q.

Theorem 2. *For any square-full number q, we have the identity*

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'K(a,1;q)K(b,1;q)\,C(a\overline{b},q)=\frac{1}{3}\cdot q^2\cdot\phi^2(q)\cdot\prod_{p|q}\left(1+\frac{1}{p}\right),$$

where $K(m,n;q)=\sum_{u=1}^{q}{}'e\left(\frac{mu+n\overline{u}}{q}\right)$ is the Kloostermann sums, $e(y)=e^{2\pi iy}$, $\prod_{p|q}$ denotes the product over all distinct prime divisors p of q, and $u\cdot\overline{u}\equiv 1 \bmod q$.

Note that $C(h,1)=C(q,1)=0$, so from our Theorem 1 we may immediately deduce the following:

Corollary 1. *For any odd primes p and q with $p\neq q$, we have the reciprocity formula*

$$\frac{C(p,q)}{q}+\frac{C(q,p)}{p}=\frac{p^2+q^2+1}{3pq}-1.$$

2. Some Lemmas

In this section, we shall give some lemmas which are necessary in the proof of our theorems. First we have the following:

Lemma 1. *Let $q > 2$ be an integer, and let χ be any Dirichlet character* mod q *with $\chi(-1) = -1$. Then we have the identity*

$$L(1,\chi) = \frac{\pi}{2q}\sum_{r=1}^{q}\chi(r)\cot\left(\frac{\pi r}{q}\right),$$

where $L(1,\chi)$ denotes Dirichlet L-function corresponding to χ mod q.

Lemma 2. *Let $q \geq 3$ be an integer, then for any integer h with $(h,q) = 1$, we have the identity*

$$S(h,q) = \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\overline{\chi}(h)|L(1,\chi)|^2,$$

where χ runs through the Dirichlet characters mod d *with $\chi(-1) = -1$.*

Proof. The proofs of Lemma 1 and Lemma 2 can be found in [8]. □

Lemma 3. *Let $q \geq 3$ be an integer, then for any integer h with $(h,q) = 1$, we have the identity*

$$C(h,q) = \frac{4q^2}{\pi^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2.$$

Proof. For any integer a with $(a,q) = 1$, note that $\sum_{b=1}^{q}\chi(b)\cot\left(\frac{\pi b}{q}\right) = 0$ if $\chi(-1) = 1$, from Lemma 1 and the orthogonality of characters mod q we have

$$\begin{aligned}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(a)L(1,\chi) &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(a)\left(\frac{\pi}{2q}\sum_{r=1}^{q}\chi(r)\cot\left(\frac{\pi r}{q}\right)\right)\\ &= \frac{\pi}{2q}\sum_{\chi \bmod q}\overline{\chi}(a)\left(\sum_{r=1}^{q}\chi(r)\cot\left(\frac{\pi r}{q}\right)\right) = \frac{\pi\phi(q)}{2q}\cdot\cot\left(\frac{\pi a}{q}\right).\end{aligned}$$

or

$$\cot\left(\frac{\pi a}{q}\right) = \frac{2q}{\pi\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(a)L(1,\chi). \tag{2.1}$$

Then from the definition of $C(h,q)$, (2.1) and the orthogonality of characters mod q we have the identity

$$C(h,q) = \sum_{a=1}^{q}{}'\cot\left(\frac{\pi ha}{q}\right)\cot\left(\frac{\pi a}{q}\right)$$

$$= \frac{4q^2}{\pi^2\phi^2(q)} \sum_{\substack{\chi_1 \bmod q \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2(-1)=-1}} \sum_{a=1}^{q} \overline{\chi}_1(ha)\overline{\chi}_2(a)L(1,\chi_1)L(1,\chi_2)$$

$$= \frac{4q^2}{\pi^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h)|L(1,\chi)|^2 = \frac{4q^2}{\pi^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(h)|L(1,\chi)|^2.$$

This proves Lemma 3. □

Lemma 4. *Let $q \geq 3$ be a square-full number. Then we have the identity*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right),$$

where $\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*}$ *denotes the summation over all primitive odd characters* mod q.

Proof. The proof of Lemma 4 can be found in [8]. □

Lemma 5. *Let q be a square-full number. Then for any non-primitive character χ mod q, we have the identity*

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a)\, e\left(\frac{a}{q}\right) = 0.$$

Proof. It is clear that Gauss sums $|\tau(\chi)|$ is a multiplicative function of mod q (see references [1] and [5]), so without loss of generality, we can assume $q = p^\alpha$, where p be a prime and $\alpha \geq 2$. Now if χ is a non-primitive character mod p^α, then it is also a character mod $p^{\alpha-1}$. So from the properties of the trigonometric sums we know that for any positive integers $q \geq 2$ and integer n with $(n,q) = 1$, we have the identity

$$\sum_{u=0}^{q-1} e\left(\frac{un}{q}\right) = 0. \tag{2.2}$$

From (2.2) and the definition of the reduce residue system modulo p^α we have

$$\tau(\chi) = \sum_{a=1}^{p^\alpha} \chi(a)\, e\left(\frac{a}{p^\alpha}\right) = \sum_{u=0}^{p-1}\sum_{v=1}^{p^{\alpha-1}}{}' \chi\left(up^{\alpha-1}+v\right)\, e\left(\frac{up^{\alpha-1}+v}{p^\alpha}\right)$$

$$= \sum_{u=0}^{p-1}\sum_{v=1}^{p^{\alpha-1}}{}' \chi(v)\, e\left(\frac{u}{p}+\frac{v}{p^\alpha}\right) = \sum_{v=1}^{p^{\alpha-1}} \chi(v)\, e\left(\frac{v}{p^\alpha}\right)\sum_{u=0}^{p-1} e\left(\frac{u}{p}\right) = 0.$$

This proves Lemma 5. □

3. Proof of Theorems

In this section, we shall complete the proof of our theorems.

Proof of Theorem 1. For any positive integers $h>1$ and $q>1$ with $(h,q)=1$, note that $C(h,1)=0$ and $(h,d)=1$ for all $d\,|\,q$. So from Lemma 3 and Lemma 2 we have

$$\frac{1}{q}\sum_{d|q}C(h,d)=\frac{4}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2=4S(h,q). \tag{3.1}$$

Similarly, using the method of proving (3.1) we can also deduce that

$$\frac{1}{h}\sum_{d|h}C(q,d)=\frac{4}{\pi^2 h}\sum_{d|h}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(q)|L(1,\chi)|^2=4S(q,h). \tag{3.2}$$

Combining (1.1), (3.1) and (3.2) we may immediately deduce the reciprocity formula

$$\frac{1}{q}\sum_{d|q}C(h,d)+\frac{1}{h}\sum_{d|h}C(q,d)=S(h,q)+S(q,h)=\frac{q^2+h^2+1}{3qh}-1.$$

This proves Theorem 1. □

Proof of Theorem 2. If q is a square-full number and χ is not a primitive character mod q, then from Lemma 5 we know that $\tau(\chi)=0$. If χ is a primitive character mod q, then $|\tau(\chi)|^2=q$. Note that the identities $\tau^2(\overline{\chi})=\overline{\tau(\chi)}^2$ and

$$\sum_{a=1}^{q}{}'\chi(a)K(a,1;q)=\sum_{u=1}^{q}{}'\sum_{a=1}^{q}{}'\chi(a)\,e\left(\frac{au+\overline{u}}{q}\right)=\tau(\chi)\cdot\sum_{u=1}^{q}{}'\overline{\chi}(u)e\left(\frac{\overline{u}}{q}\right)=\tau^2(\chi),$$

from Lemma 3 and Lemma 4 we have

$$\begin{aligned}&\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'K(a,1;q)K(b,1;q)\;C(a\overline{b},q)\\&=\frac{4q^2}{\pi^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\sum_{a=1}^{q}{}'\chi(a)K(a,1;q)\sum_{b=1}^{q}{}'\overline{\chi}(b)K(b,1;q)|L(1,\chi)|^2\\&=\frac{4q^2}{\pi^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\tau^2(\chi)\cdot\tau^2(\overline{\chi})\cdot|L(1,\chi)|^2\end{aligned}$$

$$= \frac{4q^2}{\pi^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} |\tau(\chi)|^4 \cdot |L(1,\chi)|^2$$

$$= \frac{4q^4}{\pi^2\phi(q)} \cdot \frac{\pi^2}{12} \frac{\phi^3(q)}{q^2} \prod_{p|q} \left(1+\frac{1}{p}\right) = \frac{1}{3} \cdot q^2 \cdot \phi^2(q) \cdot \prod_{p|q} \left(1+\frac{1}{p}\right).$$

This completes the proof of Theorem 2. □

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CONTINUOUS SPECTRUM FOR SOME CLASSES OF $(p,2)$-EQUATIONS WITH LINEAR OR SUBLINEAR GROWTH

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Abstract. We are concerned with two classes of nonlinear eigenvalue problems involving equations driven by the sum of the p-Laplace ($p > 2$) and Laplace operators. The main results of this paper establish the existence of a continuous spectrum consisting in an unbounded interval, which is described by using the principal eigenvalue of the Laplace operator.

2010 *Mathematics Subject Classification:* 35P30; 49R05; 58C40

Keywords: nonlinear eigenvalue problem; $(p,2)$-equation; continuous spectrum

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be an open bounded set with smooth boundary. A central result in elementary functional analysis and in the linear theory of partial differential equations asserts that the spectrum of the Laplace operator $(-\Delta)$ in $H_0^1(\Omega)$ is *discrete*. More precisely, the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.1}$$

admits a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$. The proof of this result relies on the Riesz-Fredholm theory for compact self-adjoint operators (see, e.g., H. Brezis [6, Ch. VI]).

The anisotropic linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda V(x)u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.2}$$

was studied starting with the pioneering papers of M. Bocher [5] and P. Hess and T. Kato [11]. We also refer to S. Minakshisundaram and A. Pleijel [14] who proved that problem (1.2) admits an unbounded sequence (λ_n) of eigenvalues, provided that

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V is nonnegative, $V \in L^\infty(\Omega)$ and $V > 0$ in $\omega \subset \Omega$ with $|\omega| > 0$. The case where the weight function V may change sign (that is, V is indefinite) and may have singular points was studied by A. Szulkin and M. Willem [20] who established sufficient conditions for the existence of an unbounded sequence of eigenvalues.

Fix $p \in (1, \infty)$. The quasilinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.3}$$

was studied by several mathematicians (see, e.g., A. Anane [1], J. Garcia Azorero and I. Peral Alonso [9], P. Lindqvist [12], A. Szulkin and M. Willem [20]). For instance, A. Anane [1] and P. Lindqvist [12] proved that the first eigenvalue $\lambda = \lambda_1$ of problem (1.3) is *simple* and *isolated* in any bounded domain Ω. By combining topological and variational arguments, A. Szulkin and M. Willem [20] established the existence of a countable family of eigenvalues for a class of quasilinear eigenvalue problems with indefinite weight.

The analysis developed in these papers can be extended to homogeneous eigenvalue problems of the type

$$\begin{cases} -\operatorname{div} A(x, \nabla u) u = \lambda V(x) |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases}$$

where $A(x, \xi) \simeq |\xi|^{p-2} \xi$ fulfills restrictive structural conditions and $V \geq 0$, $V \not\equiv 0$.

In the present paper, we are concerned with the spectral analysis of two classes of $(p, 2)$-*equations*, that is, equations driven by the sum of the p-Laplace ($p > 2$) and Laplace operators. These equations describe phenomena arising in mathematical physics. We refer to V. Benci, P. D'Avenia, D. Fortunato and L. Pisani [4] (quantum physics) and L. Cherfils and Y. Ilyasov [7] (plasma physics). Problems involving Laplace operators with different homogeneity have been studied recently by S. Barile and G. Figueiredo [2], D. Motreanu and M. Tanaka [15], N. Papageorgiou and V. Rădulescu [17], N. Papageorgiou, V. Rădulescu and D. Repovš [16], etc.

In comparison with the results described in the first part of this section, the properties established in the present paper deal with a *continuous spectrum* that concentrates at infinity.

2. Main results

Consider the eigenvalue problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \tag{2.1}$$

where a, b are positive real numbers and $p > 2$.

We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a solution of problem (2.1) if

$$a \int_\Omega \nabla u \nabla v dx + b \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_\Omega uv dx$$

whenever $v \in W_0^{1,p}(\Omega)$.

In such a case, the corresponding λ is called an *eigenvalue* of problem (2.1). Since a and b are positive real numbers, it follows that any eigenvalue λ is positive, too.

Let λ_1 be the *first eigenvalue* (or the *principal frequency*) of the Laplace operator in $H_0^1(\Omega)$, namely the smallest eigenvalue of problem (1.1). The first result of this paper establishes the striking property that the spectrum of problem (2.1) is *continuous*. This description will be performed in terms of λ_1 and does not take into account any contribution of the p-Laplace operator that arises in problem (2.1). More precisely, we prove the following property.

Theorem 1. *Assume that a, b are positive real numbers and $p > 2$.*
Then λ is an eigenvalue of problem (2.1) if and only if $\lambda > a\lambda_1$.

This result shows that the eigenvalues of the nonlinear operator $-a\Delta u - b\Delta_p u$ depend *only* on a and λ_1. The spectrum is continuous even for $b \to 0^+$, which corresponds to the case when this operator is "close" to the Laplace operator (hence, with a *discrete* spectrum).

The right-hand side of problem (2.1) is linear. We establish a related continuity property of the spectrum in the case of a suitable linear or sublinear perturbation. In such a case it is not possible to describe the whole spectrum (as done in Theorem 1) but we can assert two facts:
(i) any $\lambda < a\lambda_1$ cannot be an eigenvalue;
(ii) all λ sufficiently large is an eigenvalue.
We refer to [13] and [18] for related concentration properties of the spectrum.

Consider the nonlinear problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \tag{2.2}$$

where a, b are positive real numbers and $p > 2$.

We assume that $f : \Omega \to \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and we set $F(x,t) := \int_0^t f(x,s)ds$.

We suppose that the following hypotheses are fulfilled:

(f1) we have $|f(x,t)| \le |t|$ for a.a. $x \in \Omega$, all $t \in \mathbb{R}$;

(f2) there exists $t_0 \in \mathbb{R}$ such that $F(x,t_0) > 0$ for all $x \in \Omega$;

(f3) we have $f(x,t) = o(t)$ as $|t| \to \infty$ uniformly for a.a. $x \in \Omega$.

The following functions satisfy the above assumptions:
(i) $f(x,t) = V(x)\sin(\alpha t)$, $\alpha > 0$, $V \in L^\infty(\Omega)$, $V > 0$, $||V||_{L^\infty} \le 1$;
(ii) $f(x,t) = V(x)\log(1+|t|)$, $V \in L^\infty(\Omega)$, $V > 0$, $||V||_{L^\infty} \le 1$;
(iii) $f(x,t) = V(x)(|t|^r - |t|^q)$, $0 < q < r < 1$, $V \in L^\infty(\Omega)$, $V > 0$, $||V||_{L^\infty} \le 1$.

We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a solution of problem (2.1) if

$$a\int_\Omega \nabla u \nabla v dx + b\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega f(x,u)v dx \tag{2.3}$$

whenever $v \in W_0^{1,p}(\Omega)$.

In such a case, the corresponding λ is called an *eigenvalue* of problem (2.2).

Theorem 2. *Assume that a, b are positive real numbers, $p > 2$ and hypotheses (f1)-(f3) are fulfilled.*

Then any $0 < \lambda \leq a\lambda_1$ is not an eigenvalue of problem (2.2). Moreover, there exists $\lambda^* > 0$ such that all $\lambda > \lambda^*$ is an eigenvalue of problem (2.2).

We do not have any estimate on the value of λ^*. We consider that this is an interesting subject, which should be considered in accordance with the behavior of the nonlinear term f.

The methods developed in this paper allow to consider several classes of differential operators in the left-hand side of problem (2.1), for instance

$$-a\Delta u - b\,\mathrm{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{(p-2)/2}}\right)$$

or

$$-a\Delta u - b\Delta_p u - b\,\mathrm{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{(p-2)/2}}\right).$$

We refer for more details to S. Barile and G. Figueiredo [2].

The approach used in this paper can be applied to the abstract framework developed by Mingione *et al.* [3, 8] and corresponding to differential operators of the form

$$-a\Delta_p u - b\,\mathrm{div}\,(a(x)|\nabla u|^{q-2}\nabla u) \quad \text{with } 1 < p < q,$$

where $0 \leq a(\cdot) \in C^{0,\alpha}(\overline{\Omega})$.

Notation: for all $u \in W_0^{1,p}(\Omega)$ we denote

$$u_\pm(x) := \max\{\pm u(x), 0\}, \quad \text{for } x \in \Omega.$$

By [10, Theorem 7.6] we have $u_\pm \in W_0^{1,p}(\Omega)$ and

$$\nabla u_+ = \begin{cases} \nabla u & \text{on } [u > 0] \\ 0 & \text{on } [u \leq 0] \end{cases} \qquad \nabla u_- = \begin{cases} \nabla u & \text{on } [u < 0] \\ 0 & \text{on } [u \geq 0]. \end{cases}$$

3. Proof of Theorem 1

We first argue that any $\lambda \leq a\lambda_1$ is not an eigenvalue of problem (2.1). Arguing by contradiction, let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ denote the eigenfunction corresponding to the eigenvalue $\lambda \leq a\lambda_1$. Then

$$a\int_\Omega |\nabla u|^2 dx + b\int_\Omega |\nabla u|^p dx = \lambda \int_\Omega u^2 dx \leq a\lambda_1 \int_\Omega u^2 dx. \tag{3.1}$$

Since $p > 2$, it follows that $u \in H_0^1(\Omega) \setminus \{0\}$, hence the variational characterization of λ_1 yields

$$\lambda_1 \int_\Omega u^2 dx \leq \int_\Omega |\nabla u|^2 dx. \tag{3.2}$$

Combining relations (3.1) and (3.2) we deduce that

$$a\lambda_1 \int_\Omega u^2 dx + ab \int_\Omega |\nabla u|^p dx \le a\lambda_1 \int_\Omega u^2 dx,$$

a contradiction.

It remains to show that any $\lambda > a\lambda_1$ is an eigenvalue of problem (2.1).

The energy functional $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated to problem (2.1) is defined by

$$\mathcal{E}(u) = \frac{a}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{2} \int_\Omega u^2 dx.$$

We have

$$\mathcal{E}(u) \ge \frac{b}{p} ||u||^p_{W_0^{1,p}(\Omega)} - \frac{\lambda - a\lambda_1}{2\lambda_1} ||u||^2_{H_0^1(\Omega)}.$$

Our assumption $p > 2$ implies that

$$\lim_{||u||_{W_0^{1,p}(\Omega)} \to \infty} \mathcal{E}(u) = +\infty,$$

hence $\mathcal{E}$ is coercive.

Consider the minimization problem

$$\inf\{\mathcal{E}(u);\ u \in W_0^{1,p}(\Omega)\} \tag{3.3}$$

and let (u_n) be a minimizing sequence of (3.3). Since $\mathcal{E}$ is coercive, it follows that (u_n) is bounded. Thus, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \subset H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, we can also assume that

$$u_n \to u \quad \text{in } L^2(\Omega).$$

Next, using the weakly lower semicontinuity of $\mathcal{E}$, we deduce that $u \in W_0^{1,p}(\Omega)$ minimizes $\mathcal{E}$. In order to show that u is nontrivial (hence, an eigenvalue of problem (2.1)), we argue by contradiction and assume that $u = 0$. This implies that $\mathcal{E}$ takes only nonnegative values, so it is enough to prove that

$$\inf\{\mathcal{E}(v);\ v \in W_0^{1,p}(\Omega)\} < 0.$$

For this purpose we first choose $w \in C_0^\infty(\Omega)$ such that

$$\lambda_1 < \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega w^2 dx} < \frac{\lambda}{a}. \tag{3.4}$$

This choice is possible due to the hypothesis $\lambda > a\lambda_1$ combined with the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$. We also observe that we have $w \in W_0^{1,p}(\Omega) \setminus \{0\}$. So, for all $t > 0$, we have

$$\mathcal{E}(tw) = \frac{at^2}{2} \int_\Omega |\nabla w|^2 dx + \frac{bt^p}{p} \int_\Omega |\nabla w|^p dx - \frac{\lambda t^2}{2} \int_\Omega w^2 dx$$

$$= \frac{bt^p}{p}\int_\Omega |\nabla w|^p dx + \frac{t^2}{2}\left(a\int_\Omega |\nabla w|^2 dx - \lambda\int_\Omega w^2 dx\right)$$
$$= A\frac{t^2}{2} + B\frac{bt^p}{p},$$

where

$$A := a\int_\Omega |\nabla w|^2 dx - \lambda\int_\Omega w^2 dx < 0$$

and

$$B := \int_\Omega |\nabla w|^p dx > 0.$$

Moreover, by the choice of w, cf. (3.4), we have $A < 0$.

In order to obtain $\mathcal{E}(tw) < 0$ it is enough to choose

$$0 < t < \left(-\frac{pA}{2B}\right)^{1/(p-2)}.$$

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

We first establish that all positive eigenvalues of problem (2.2) are bigger than $a\lambda_1$. Let us observe that relation (2.3) can be rewritten as

$$\begin{aligned} &a\int_\Omega (\nabla u_+ - \nabla u_-)\nabla v dx + b\int_\Omega |\nabla u|^{p-2}(\nabla u_+ - \nabla u_-)\nabla v dx = \\ &\lambda\int_\Omega (f(x,u_+) + f(x,-u_-))v dx \end{aligned} \tag{4.1}$$

whenever $v \in W_0^{1,p}(\Omega)$.

In particular, relation (4.1) shows that $u = e_1$ (namely, the first eigenfunction of the Laplace operator in $H_0^1(\Omega)$) cannot be an eigenvalue of problem (2.2), provided that $\lambda \le a\lambda_1$.

Taking $v = u_+$ in (4.1) we obtain

$$a\int_\Omega |\nabla u_+|^2 dx + b\int_\Omega |\nabla u|^{p-2}|\nabla u_+|^2 dx = \lambda\int_\Omega f(u_+)u_+ dx. \tag{4.2}$$

Taking $v = u_-$ in (4.1) we obtain

$$a\int_\Omega |\nabla u_-|^2 dx + b\int_\Omega |\nabla u|^{p-2}|\nabla u_-|^2 dx = -\lambda\int_\Omega f(u_-)u_- dx. \tag{4.3}$$

Relations (4.2) and (4.3) in combination with hypothesis (f1) yield, respectively,

$$a\lambda_1\int_\Omega u_+^2 dx \le a\int_\Omega |\nabla u_+|^2 dx \le \lambda\int_\Omega f(u_+)u_+ dx \le \lambda\int_\Omega u_+^2 dx$$

and

$$a\lambda_1 \int_\Omega u_-^2 dx \le a \int_\Omega |\nabla u_-|^2 dx \le -\lambda \int_\Omega f(u_+)u_+ dx \le \lambda \int_\Omega u_-^2 dx.$$

Since u is nontrivial, at least one of u_+ or u_- is nontrivial. Thus, the above relations imply that $\lambda \ge a\lambda_1$. Moreover, as we have already observed, $\lambda = a\lambda_1$ cannot be an eigenvalue of problem (2.2), since this would imply that $u = e_1$ is an eigenfunction of problem (2.2), which is impossible. In conclusion, if problem (2.2) admits a solution then $\lambda > a\lambda_1$.

It remains to show that problem (2.2) has a solution for all λ large enough.

The energy functional associated to problem (2.2) is $\mathcal{J} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \frac{a}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{p} \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega F(x,u) dx.$$

Fix $\lambda > a\lambda_1$ (which is a necessary condition for the existence of solutions to problem (2.2)).

Hypothesis (f3) implies that there is a positive constant $C = C(\lambda)$ such that

$$\lambda F(x,u) \le \frac{a\lambda_1}{2} u^2 + C \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}.$$

It follows that

$$\begin{aligned} \mathcal{J}(u) &\ge \frac{a}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{p} \int_\Omega |\nabla u|^p dx - \frac{a\lambda_1}{2} \int_\Omega u^2 dx - C|\Omega| \\ &\ge \frac{b}{p} ||u||^p_{W_0^{1,p}} - C|\Omega|, \end{aligned}$$

hence $\mathcal{J}$ is coercive.

Next, we show that there exists $\lambda^* > 0$ such that

$$\inf\{\mathcal{J}(u);\ u \in W_0^{1,p}(\Omega)\} < 0.$$

For this purpose we use our assumption (f2) and fix $t_0 \in \mathbb{R}$ such that

$$F(x,t_0) > 0 \quad \text{for all } x \in \Omega.$$

Fix arbitrarily a compact set $K \subset \Omega$ and let $w \in W_0^{1,p}(\Omega)$ such that $w = t_0$ in K and $0 \le w \le t_0$ in Ω.

Using hypotheses (f1) it follows that

$$\begin{aligned} \int_\Omega F(x,w) dx &= \int_K F(x,w) dx + \int_{\Omega \setminus K} F(x,w) dx \\ &\ge \int_K F(x,t_0) dx - \frac{1}{2} \int_{\Omega \setminus K} w^2 dx \\ &\ge \int_K F(x,t_0) dx - \frac{t_0^2}{2} |\Omega \setminus K|. \end{aligned} \tag{4.4}$$

Relation (4.4) shows that increasing eventually the size of K (in order to have $|\Omega \setminus K|$ small enough) we can assume that

$$\int_\Omega F(x,w)dx > 0.$$

We deduce that

$$\mathcal{J}(w) = \frac{a}{2}\int_\Omega |\nabla w|^2 dx + \frac{b}{p}\int_\Omega |\nabla w|^p dx - \lambda \int_\Omega F(x,w)dx < 0,$$

provided that $\lambda > 0$ is large enough. For these values of λ, the energy functional $\mathcal{J}$ has a negative global minimum, hence problem (2.2) admits a solution. This completes the proof. □

The proof of Theorem 2 shows that we can assume the growth imposed by hypothesis (f3) only on one side, say at $+\infty$:

$$f(x,t) = o(t) \quad \text{as } t \to +\infty \text{ uniformly for a.a. } x \in \Omega.$$

In such a case, the final part of the proof of Theorem 2 (the existence of λ^*) follows by considering the auxiliary problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda f(x,u_+) & \text{in } \Omega \\ u = 0 & \text{on } \Omega. \end{cases} \tag{4.5}$$

Let u be a solution of problem (4.5). By taking $v = u_-$ as test function we deduce that $u_- = 0$, hence $u \geq 0$. This implies that any solution of (4.5) is also a solution of problem (2.2).

From now on, we follow the same arguments as those developed in the second part of the proof of Theorem 2 by replacing the energy functional $\mathcal{J}$ with $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{p}\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega F(x,u_+)dx.$$

The main result of this paper can be extended in the framework of differential operators with variable exponent; we refer to Rădulescu and Repovš [19] for related results.

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FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY USING SALAGEAN OPERATOR

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Abstract. In this paper, we obtain some coefficients bounds (namely, Fekete-Szegö inequalities) for functions belonging to certain subclass of analytic functions defined by using Salagean operator. Moreover, connections of the results presented here with those obtained in earlier works are pointed out.

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1. INTRODUCTION

Let $\mathcal{A}$ denotes the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ (z \in U), \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further let S denote the family of functions of the form (1.1) which are univalent in U. A classical theorem of Fekete-Szegö [6] states that, for $f(z) \in S$ given by (1.1) that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu, & if \ \mu \leq 0, \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & if \ 0 \leq \mu \leq 1, \\ 4\mu - 3, & if \ \mu \geq 1. \end{cases} \tag{1.2}$$

The result is sharp.

Given two functions $f(z)$ and $g(z)$, which are analytic in U with $f(0) = g(0)$, the function $f(z)$ is said to be subordinate to $g(z)$ in U if there exists a function $w(z)$, analytic in U, such that $w(0) = 0$ and $|w(z)| < 1 \, (z \in U)$ and $f(z) = g(w(z)) \ (z \in U)$. We denote this subordination by $f(z) \prec g(z)$ in U (see [11]).

Let $\varphi(z)$ be an analytic function with positive real part on U, which satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$, and which maps the unit disc U onto a region starlike with respect

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to 1 and symmetric with respect to the real axis. Let $S^*(\varphi)$ be the class of functions $f(z) \in S$ for which

$$\frac{z f'(z)}{f(z)} \prec \varphi(z) \ (z \in U), \tag{1.3}$$

and $C(\varphi)$ be the class of functions $f(z) \in S$ for which

$$1 + \frac{z f''(z)}{f'(z)} \prec \varphi(z) \ (z \in U). \tag{1.4}$$

The classes of $S^*(\varphi)$ and $C(\varphi)$ were introduced and studied by Ma and Minda [10]. The familier class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ are the special cases of $S^*(\varphi)$ and $C(\varphi)$, respectively, when $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$.
Ma and Minda [10] have obtained the Fekete-Szegö problem for the functions in the class $C(\varphi)$.
For a function $f(z) \in \mathcal{S}$, Salagean [18] introduced the following differential operator as following:

$$D^0 f(z) = f(z),\ D^1 f(z) = z f'(z), \ldots,\ D^n f(z) = D(D^{n-1} f(z)) (n \in \mathbb{N} = \{1, 2, \ldots\}).$$

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.5}$$

Using Salagean operator we introduce a new class of analytic functions as following:

Definition 1. For $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the class $M_{\alpha,n}^b(\varphi)$ consists of all functions $f(z) \in \mathcal{A}$ satisfying the following subordination:

$$1 + \frac{1}{b}\left(\frac{z(D^n f(z))' + \alpha z^2 (D^n f(z))''}{(1-\alpha) D^n f(z) + \alpha z (D^n f(z))'} - 1\right) \prec \varphi(z), (\alpha \geq 0; n \in \mathbb{N}_0; z \in U). \tag{1.6}$$

Specializing the parameters α, b, n, and $\varphi(z)$, we obtain the following subclasses studied by various authors:
(i) $M_{\alpha,n}^1(\varphi) = M_{\alpha,n}(\varphi)$ (see Orhan and Gunes [16]);

(ii) $M_{0,n}^b(\varphi) = H_{n,b}(\varphi)$ (see Aouf and Silverman [3]);

(iii) $M_{\alpha,0}^1(\varphi) = M_\alpha(\varphi)$ (see Shanmugam and Sivasubramanian [19]);

(iv) $M_{0,n}^b\left(\frac{1+Az}{1+Bz}\right) = H_n^b(A, B)$ $(-1 \leq B < A \leq 1)$ (see Attiya [4]);

(v) $M_{0,0}^b(\varphi) = S_b^*(\varphi)$ and $G_{0,1}^b(\varphi) = C_b(\varphi)$ (see Ravichandran et al. [17]);

(vi) $M_{0,n}^b\left(\frac{1+z}{1-z}\right) = S^n(b)$ (see Aouf et al. [1]);

(vii) $M_{0,0}^{b}\left(\frac{1+z}{1-z}\right)=S\left(b\right)$ (see Nasr and Aouf [14, 15] see also Aouf et al. [2]);

(viii) $M_{0,1}^{b}\left(\frac{1+z}{1-z}\right)=C\left(b\right)$ (see Nasr and Aouf [12, 13] see also Aouf et al. [2]);

(ix) $M_{0,0}^{(1-\rho)\cos\eta e^{-i\eta}}\left(\frac{1+z}{1-z}\right)=S^{\eta}\left(\rho\right)$ $\left(|\eta|<\frac{\pi}{2},0\leq\rho<1\right)$ (see Libera [9] see also Keogh and Merkes [8]);

(x) $M_{0,1}^{(1-\rho)\cos\eta e^{-i\eta}}\left(\frac{1+z}{1-z}\right)=C^{\eta}\left(\rho\right)$ $\left(|\eta|<\frac{\pi}{2},0\leq\rho<1\right)$ (see Chichra [5]).

Also we note that for additional choices of parameters we have the following new subclasses of $\mathcal{A}$:

(i) $M_{1,n}^{b}\left(\varphi\right)=N_{n}^{b}\left(\varphi\right)=\left\{f(z)\in\mathcal{A}:1+\frac{1}{b}\frac{z(D^{n}f(z))''}{(D^{n}f(z))'}\prec\varphi(z)\left(n\in\mathbb{N}_{0};b\in\mathbb{C}^{*};z\in U\right)\right\}$;

(ii) $M_{\alpha,n}^{(1-\rho)\cos\eta e^{-i\eta}}\left(\varphi\right)=M_{\alpha,n}^{\rho,\eta}\left(\varphi\right)$

$$=\left\{f(z)\in\mathcal{A}:\frac{e^{i\eta}\frac{z(D^{n}f(z))'+\alpha z^{2}(D^{n}f(z))''}{(1-\alpha)D^{n}f(z)+\alpha z(D^{n}f(z))'}-\rho\cos\eta-i\sin\eta}{(1-\rho)\cos\eta}\prec\varphi(z)\right.$$
$$\left.\left(|\eta|<\tfrac{\pi}{2};\alpha\geq0;0\leq\rho<1;n\in\mathbb{N}_{0};z\in U\right)\right\}.$$

In this paper, we obtain the Fekete-Szegö inequalities for functions in the class $M_{\alpha,n}^{b}\left(\varphi\right)$.

2. Fekete-Szegö problem

Unless otherwise mentioned, we assume in the reminder of this paper that $\alpha\geq0$, $b\in\mathbb{C}^{*}$, $n\in\mathbb{N}_{0}$ and $z\in U$.

To prove our results, we shall need the following lemmas (see [10]):

Lemma 1. *If $p(z)=1+c_{1}z+c_{2}z^{2}+...$ is a function with positive real part in U and μ is a complex number, then*

$$\left|c_{2}-\mu c_{1}^{2}\right|\leq2\max\{1;|2\mu-1|\}. \tag{2.1}$$

The result is sharp for the functions given by

$$p(z)=\frac{1+z^{2}}{1-z^{2}}\ \text{and}\ p(z)=\frac{1+z}{1-z}\ \left(z\in U\right). \tag{2.2}$$

Lemma 2. *If $p_{1}(z)=1+c_{1}z+c_{2}z^{2}+...$ is a function with positive real part in U, then*

$$\left|c_{2}-\nu c_{1}^{2}\right|\leq\begin{cases}-4\nu+2, & if\ \nu\leq0,\\ 2, & if\ 0\leq\nu\leq1,\\ 4\nu-2, & if\ \nu\geq1.\end{cases}$$

When $\nu < 0$ *or* $\nu > 1$, *the equality holds if and only if* $p_1(z) = \dfrac{1+z}{1-z}$ *or one of its rotations. If* $0 < \nu < 1$, *then the equality holds if and only if* $p_1(z) = \dfrac{1+z^2}{1-z^2}$ *or one of its rotations. If* $\nu = 0$, *the equality holds if and only if*

$$p_1(z) = (\frac{1}{2} + \frac{1}{2}\gamma)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\gamma)\frac{1-z}{1+z} \ (0 \le \gamma \le 1),$$

or one of its rotations. If $\nu = 1$, *the equality holds if and only if*

$$\frac{1}{p_1(z)} = (\frac{1}{2} + \frac{1}{2}\gamma)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\gamma)\frac{1-z}{1+z} \ (0 \le \gamma \le 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \le 2 \ (0 < \nu < \frac{1}{2}),$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \le 2 \ (\frac{1}{2} < \nu < 1).$$

Using Lemma 1, we have the following theorem:

Theorem 1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, *where* $\varphi(z) \in \mathcal{A}$ *and* $\varphi'(0) > 0$. *If* $f(z)$ *given by (1.1) belongs to the class* $M_{\alpha,n}^{b}(\varphi)$ *and if* μ *is a complex number, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b| B_1}{2.3^n (1+2\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{3^n (1+2\alpha)}{2^{2n-1}(1+\alpha)^2}\mu\right) b B_1\right|\right\}. \quad (2.3)$$

The result is sharp.

Proof. If $f(z) \in M_{\alpha,n}^{b}(\varphi)$, then there exists a Schwarz function $w(z)$ which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b}\left(\frac{z(D^n f(z))' + \alpha z^2 (D^n f(z))''}{(1-\alpha) D^n f(z) + \alpha z (D^n f(z))'} - 1\right) = \varphi(w(z)). \quad (2.4)$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots . \quad (2.5)$$

Since $w(z)$ is a Schwarz function, we see that $\Re\{p_1(z)\} > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by:

$$p(z) = 1 + \frac{1}{b}\left(\frac{z(D^n f(z))' + \alpha z^2 (D^n f(z))''}{(1-\alpha) D^n f(z) + \alpha z (D^n f(z))'} - 1\right) = 1 + b_1 z + b_2 z^2 + \ldots . \quad (2.6)$$

In view of the equations (2.4), (2.5) and (2.6), we have

$$\begin{aligned} p(z) &= \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \varphi\left(\frac{c_1 z + c_2 z^2 + \ldots}{2 + c_1 z + c_2 z^2 + \ldots}\right) \\ &= \varphi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \ldots\right) \\ &= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \ldots. \end{aligned} \tag{2.7}$$

Thus

$$b_1 = \frac{1}{2}B_1 c_1 \qquad \text{and} \qquad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2. \tag{2.8}$$

Since

$$1+\frac{1}{b}\left(\frac{z\,(D^n f(z))' + \alpha z^2\,(D^n f(z))''}{(1-\alpha)\,D^n f(z) + \alpha z\,(D^n f(z))'} - 1\right) = 1 + \left\{\frac{1}{b}2^n\,(1+\alpha)\,a_2\right\} z + \left\{\frac{2.3^n(1+2\alpha)a_3 - 2^{2n}(1+\alpha)^2 a_2^2}{b}\right\} z^2 + \ldots .$$

Then from 2.6 and 2.8, we obtain

$$a_2 = \frac{bB_1 c_1}{2^{n+1}\,(1+\alpha)}, \tag{2.9}$$

and

$$a_3 = \frac{bB_1 c_2}{4.3^n\,(1+2\alpha)} + \frac{c_1^2}{8.3^n\,(1+2\alpha)}\left[b^2 B_1^2 - b\,(B_1 - B_2)\right]. \tag{2.10}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4.3^n\,(1+2\alpha)}\left[c_2 - \nu c_1^2\right], \tag{2.11}$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{3^n\,(1+2\alpha)}{2^{2n-1}\,(1+\alpha)^2}\mu - 1\right)bB_1\right]. \tag{2.12}$$

Our result now follows by an application of Lemma 1. The result is sharp for the functions

$$1 + \frac{1}{b}\left(\frac{z\,(D^n f(z))' + \alpha z^2\,(D^n f(z))''}{(1-\alpha)\,D^n f(z) + \alpha z\,(D^n f(z))'} - 1\right) = \varphi(z^2), \tag{2.13}$$

and

$$1 + \frac{1}{b}\left(\frac{z\,(D^n f(z))' + \alpha z^2\,(D^n f(z))''}{(1-\alpha)\,D^n f(z) + \alpha z\,(D^n f(z))'} - 1\right) = \varphi(z). \tag{2.14}$$

This completes the proof of Theorem 1. □

Remark 1. (i) Taking $\alpha = n = 0$ in Theorem 1, we improve the result obtained by Ravichandran et al. [17, Theorem 4.1];
(ii) Taking $\alpha = 0$, $b = (1-\rho)\cos\eta e^{-i\eta}\left(|\eta| < \frac{\pi}{2}, 0 \leq \rho < 1\right)$ and $\varphi(z) = \frac{1+z}{1-z}$ (equivalently $B_1 = B_2 = 2$) in Theorem 1, we obtain the result obtained by Goyal and Kumar [7, Corollary 2.10];
(iii) Taking $\alpha = n = 0$, $b = (1-\rho)\cos\eta e^{-i\eta}\left(|\eta| < \frac{\pi}{2}, 0 \leq \rho < 1\right)$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result obtained by Keogh and Merkes [8, Theorem 1];
(iv) Taking $\alpha = 0$ in Theorem 1, we obtain the result obtained by Aouf and Silverman [3, Theorem 1].

Using Lemma 2, we obtain the following theorem:

Theorem 2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, (b > 0; B_i > 0; i \in \mathbb{N})$ *.Also let*

$$\sigma_1 = \frac{2^{2n-1}(1+\alpha)^2\left(bB_1^2 + B_2 - B_1\right)}{3^n(1+2\alpha)bB_1^2},$$

and

$$\sigma_2 = \frac{2^{2n-1}(1+\alpha)^2\left(bB_1^2 + B_2 + B_1\right)}{3^n(1+2\alpha)bB_1^2}.$$

If $f(z)$ *is given by (1.1) belongs to the class* $M_{\alpha,n}^b(\varphi)$, *then we have the following sharp results:*
(i) If $\mu \leq \sigma_1$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{b}{2.3^n(1+2\alpha)}\left[B_2 - \left(\frac{3^n(1+2\alpha)}{2^{2n-1}(1+\alpha)^2}\mu - 1\right)bB_1^2\right]; \tag{2.15}$$

(ii) If $\sigma_1 \leq \mu \leq \sigma_2$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{b}{2.3^n(1+2\alpha)}; \tag{2.16}$$

(iii) If $\mu \geq \sigma_2$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{b}{2.3^n(1+2\alpha)}\left[-B_2 + \left(\frac{3^n(1+2\alpha)}{2^{2n-1}(1+\alpha)^2}\mu - 1\right)bB_1^2\right]. \tag{2.17}$$

Proof. For $f(z) \in M_{\alpha,n}^b(\varphi)$, $p(z)$ given by (2.6) and $p_1(z)$ given by (2.5), then a_2 and a_3 are given as same as in Theorem 1. Also

$$a_3 - \mu a_2^2 = \frac{bB_1}{4.3^n(1+2\alpha)}\left[c_2 - \nu c_1^2\right], \tag{2.18}$$

where ν is given by (2.12).
First, if $\mu \leq \sigma_1$, then we have $\nu \leq 0$, then by applying Lemma 2 to equality (2.18), we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{b}{2.3^n(1+2\alpha)}\left[B_2 - \left(\frac{3^n(1+2\alpha)}{2^{2n-1}(1+\alpha)^2}\mu - 1\right)bB_1^2\right],$$

which is evidently inequality (2.15) of Theorem 2.
If $\mu = \sigma_1$, then we have $\nu = 0$, therefore equality holds if and only if

$$p_1(z) = (\tfrac{1+\gamma}{2})\frac{1+z}{1-z} + (\tfrac{1-\gamma}{2})\frac{1-z}{1+z}\ (0 \le \gamma \le 1; z \in U).$$

Next, if $\sigma_1 \le \mu \le \sigma_2$, we note that

$$\max\left\{\frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{3^n(1+2\alpha)}{2^{2n-1}(1+\alpha)^2}\mu - 1\right)bB_1\right]\right\} \le 1, \tag{2.19}$$

then applying Lemma 2 to equality (2.18), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{2.3^n(1+2\alpha)},$$

which is evidently inequality (2.16) of Theorem 2.
If $\sigma_1 < \mu < \sigma_2$, then we have

$$p_1(z) = \frac{1+z^2}{1-z^2}.$$

Finally, If $\mu \ge \sigma_2$, then we have $\nu \ge 1$, therefore by applying Lemma 2 to (2.18), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{2.3^n(1+2\alpha)}\left[\left(\frac{3^n(1+2\alpha)}{2^{2n-1}(1+\alpha)^2}\mu - 1\right)bB_1^2 - B_2\right],$$

which is evidently inequality (2.17) of the theorem. If $\mu = \sigma_2$, then we have $\nu = 1$, therefore equality holds if and only if

$$\frac{1}{p_1(z)} = (\frac{1+\gamma}{2})\frac{1+z}{1-z} + (\frac{1-\gamma}{2})\frac{1-z}{1+z}\ (0 \le \gamma \le 1; z \in U).$$

To show that the bounds are sharp, we define the functions $K_\varphi^s (s \ge 2)$ by

$$1+\frac{1}{b}\left(\frac{z\left(D^n K_\varphi^s(z)\right)' + \alpha z^2\left(D^n K_\varphi^s(z)\right)''}{(1-\alpha)D^n K_\varphi^s(z) + \alpha z\left(D^n K_\varphi^s(z)\right)'} - 1\right) = \varphi(z^{s-1}), K_\varphi^s(0) = 0 = K_\varphi^{'s}(0) - 1, \tag{2.20}$$

and the functions F_t and G_t $(0 \le t \le 1)$ by

$$1+\frac{1}{b}\left(\frac{z\left(D^n F_t(z)\right)' + \alpha z^2\left(D^n F_t(z)\right)''}{(1-\alpha)D^n F_t(z) + \alpha z\left(D^n F_t(z)\right)'} - 1\right) = \varphi\left(\frac{z(z+t)}{1+tz}\right), F_t(0) = 0 = F_t'(0) - 1, \tag{2.21}$$

and

$$1+\frac{1}{b}\left(\frac{z\left(D^n G_t(z)\right)' + \alpha z^2\left(D^n G_t(z)\right)''}{(1-\alpha)D^n G_t(z) + \alpha z\left(D^n G_t(z)\right)'} - 1\right) = \varphi\left(-\tfrac{z(z+t)}{1+tz}\right), G_t(0) = 0 = G_t'(0) - 1. \tag{2.22}$$

Cleary, the functions K_φ^s, F_t and $G_t \in M_{\alpha,n}^b(\varphi)$. Also we write $K_\varphi = K_\varphi^2$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is K_φ^3 or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_t or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_t or one of its rotations. □

Remark 2. (i) Taking $b = 1$ and $n = 0$ in Theorem 2, we improve the result obtained by Shanmugam and Sivasubramanian [19, Theorem 2.1];
(ii) Taking $b = 1$ in Theorem 2, we obtain the result obtained by Orhan and Gunes [16, Theorem 2.1];
(iii) Taking $\alpha = 0$ in Theorem 2, we obtain the result obtained by Aouf and Silverman [3, Theorem 2].

Also, using Lemma 2 we have the following theorem:

Theorem 3. *For $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ..., (b > 0; B_i > 0; i \in \mathbb{N})$ and $f(z)$ given by (1.1) belongs to the class $M_{\alpha,n}^b(\varphi)$ and $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 2, Theorem 2 can be improved. Let*

$$\sigma_3 = \frac{2^{2n-1}(1+\alpha)^2\left(bB_1^2 + B_2\right)}{3^n(1+2\alpha)bB_1^2},$$

(i) If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n-1}(1+\alpha)^2}{3^n(1+2\alpha)bB_1^2}\left[B_1 - B_2 + \frac{3^n\mu(1+2\alpha) - 2^{2n-1}(1+\alpha)^2}{2^{2n-1}(1+\alpha)^2}bB_1^2\right]|a_2|^2 \le \frac{bB_1}{2.3^n(1+2\alpha)}; \tag{2.23}$$

(ii) If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n-1}(1+\alpha)^2}{3^n(1+2\alpha)bB_1^2}\left[B_1 + B_2 - \frac{3^n\mu(1+2\alpha) - 2^{2n-1}(1+\alpha)^2}{2^{2n-1}(1+\alpha)^2}bB_1^2\right]|a_2|^2 \le \frac{bB_1}{2.3^n(1+2\alpha)}. \tag{2.24}$$

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_2|^2 &= \frac{bB_1}{4.3^n(1+2\alpha)}\left|c_2 - \nu c_1^2\right| \\ &\quad + \left(\mu - \frac{2^{2n-1}(1+\alpha)^2\left(bB_1^2 + B_2 - B_1\right)}{3^n(1+2\alpha)bB_1^2}\right)\frac{b^2B_1^2}{2^{2n+2}(1+\alpha)^2}|c_1|^2 \\ &= \frac{bB_1}{2.3^n(1+2\alpha)}\left\{\frac{1}{2}\left(\left|c_2 - \nu c_1^2\right| + \nu|c_1|^2\right)\right\}. \end{aligned} \tag{2.25}$$

Now apply Lemma 2 to equality (2.25), then we have

$$\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_2|^2 \le \frac{bB_1}{2.3^n(1+2\alpha)},$$

which is evidently inequality (2.23) of Theorem 3.

Next, for the values of $\sigma_3 \leq \mu \leq \sigma_2$, we have

$$\begin{aligned}\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)\left|a_2\right|^2 &= \frac{bB_1}{4.3^n(1+2\alpha)}\left|c_2-\nu c_1^2\right| \\ &\quad+\left(\frac{2^{2n-1}(1+\alpha)^2\left(bB_1^2+B_2+B_1\right)}{3^n(1+2\alpha)bB_1^2}-\mu\right)\frac{b^2B_1^2}{2^{2n+2}(1+\alpha)^2}\left|c_1\right|^2 \\ &= \frac{bB_1}{2.3^n(1+2\alpha)}\left\{\frac{1}{2}\left(\left|c_2-\nu c_1^2\right|+(1-\nu)\left|c_1\right|^2\right)\right\}. \quad (2.26)\end{aligned}$$

Now apply Lemma 2 to equality (2.26), then we have

$$\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)\left|a_2\right|^2 \leq \frac{bB_1}{2.3^n\,(1+2\alpha)}.$$

Which is inequality (2.24). This completes the proof of Theorem 3. □

Remark 3. (i) Taking $b = 1$ and $n = 0$ in Theorem 3, we improve the result obtained by Shanmugam and Sivasubramanian [19, Remark 2.2];
(ii) taking $\alpha = 0$ in Theorem 3, we improve the result obtained by Goyal and Kumar [7, Remark 2.8].

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ON A MAX TYPE RECURSIVE SEQUENCE OF ORDER THREE

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Abstract. Our aim in this paper is to investigate the behavior of the solution of the following max type difference equation of order three

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, x_{n-2}\right\}, \quad n = 0, 1, ...,$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers and $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of period two.

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Keywords: solution of difference equations, recursive sequences, max-type difference equation, periodic solution, boundedness

1. INTRODUCTION

Our purpose in this paper is to study the behavior of the solution of the following max type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, x_{n-2}\right\}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers and $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of period two.

Nonlinear rational difference equations are of great importance in their own right because diverse nonlinear phenomena occurring in science and engineering can be modeled by such equations. Furthermore, the results about such equations offer prototypes towards the development of the basic theory of nonlinear difference equations. See [8–10, 23, 24, 26].

The study of max-type difference equations attracted a considerable attention recently, see, for example, [1, 5, 7, 13–16] and the references therein. This type of difference equations stem from, for example, certain models in automatic control theory (see [17] and [18]). In the beginning of the study of these equations the experts have been focused on the investigation of the behavior of some particular cases

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of the following difference equation

$$x_n = \max\left\{\frac{A_n^{(1)}}{x_{n-1}}, \frac{A_n^{(2)}}{x_{n-2}}, \ldots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, $A_n^{(i)}$, $i = 1, \ldots, k$, are real sequences (mostly constant or periodic) and the initial values $x_{-1}, \ldots, x_{-k}$ are different from zero (see, e.g., [2, 3, 6, 15] and the references therein).

Cinar et al. [4] dealt with the positive solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_n x_{n-2}^2}\right\}.$$

Recently, in the paper [12] it was showed that every solution of the third-order max-type difference equation (1.1), where the initial conditions are arbitrary nonzero real numbers and A_n =constant$\in R$, is eventually periodic with period three.

Also, in [11] we proved that every positive solution to the same third order non–autonomous max-type difference equation (1.1), with $\{A_n\}$ is a three-periodic sequence of positive numbers, is periodic with period three. The same result was proved for the same equation but with min-type difference equation.

Simsek et al. [19] investigated the solutions of the following difference equation

$$x_{n+1} = \max\left\{x_{n-1}, \frac{1}{x_{n-1}}\right\}.$$

Also, Simsek [20] studied the behavior of the solutions of the following system of difference equations

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{y_n}{x_n}\right\}, \quad y_{n+1} = \max\left\{\frac{A}{y_n}, \frac{x_n}{y_n}\right\}.$$

Stevic [21] studied the boundedness and global attractivity for the positive solutions of the difference equation

$$x_{n+1} = \max\left\{c, \frac{x_n^p}{x_{n-1}^p}\right\}.$$

In [25] Yalcinkaya et al. investigated the periodic nature of the solution of the max-type difference equation

$$x_{n+1} = \max\{x_n, A\}/x_n^2 x_{n-1}.$$

See also [22], [27].

Definition 1. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there is $n_0 \in \{-k, \ldots, -1, 0, 1, \ldots\}$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$, then we say that the sequence $\{x_n\}_{n=-k}^{\infty}$ is periodic with period p.

Remark 1. Note that if $A_n = 0$, then Eq. (1.1) becomes $x_{n+1} = x_{n-2}$, from which it follows that every solution is periodic with period three. Hence, in the sequel we will consider the case $A_n \neq 0$.

Remark 2. Note that if $A_n = A$, then Eq. (1.1) becomes $x_{n+1} = \max\left\{\frac{A}{x_n}, x_{n-2}\right\}$, it was shown that this equation is periodic with period three see [12]. Hence, in the sequel we will consider the case $A_n = \{A_0, A_1, A_0, A_1, ...\}$, $A_1 \neq A_0$.

2. Main results

2.1. *The Case* $A_0 > A_1$

Theorem 1. *Consider the difference equation (1.1) with* $A_0 > A_1$. *Then every solution of Eq. (1.1) is eventually periodic with period three.*

Proof. From Eq.(1.1), we see that

$$x_1 = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\}.$$

We consider two cases

Case (a_1) If $x_{-2}x_0 < A_0$, then $x_1 = \dfrac{A_0}{x_0}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1x_0}{A_0}, x_{-1}\right\}.$$

(a_{11}) If $x_{-1}A_0 < A_1x_0$, then $x_2 = \dfrac{A_1x_0}{A_0}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0^2}{A_1x_0}, x_0\right\}.$$

(a_{111}) If $A_1x_0^2 < A_0^2$, then $x_3 = \dfrac{A_0^2}{A_1x_0}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2x_0}{A_0^2}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$(\text{since } A_1x_0^2 < A_0^2 \quad \Rightarrow \quad A_1^2x_0^2 < A_0A_1x_0^2 < A_0^3)$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{x_0, \frac{A_1x_0}{A_0}\right\} = x_0,$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0^2}{A_1x_0}\right\} = \frac{A_0^2}{A_1x_0},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_1x_0}{A_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1x_0}{A_0}, x_0\right\} = x_0,$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0^2}{A_1x_0}\right\} = \frac{A_0^2}{A_1x_0},$$

and the solution has the following form

$$\left\{\frac{A_0^2}{A_1x_0}, \frac{A_0}{x_0}, x_0, \frac{A_0^2}{A_1x_0}, \frac{A_0}{x_0}, x_0, \frac{A_0^2}{A_1x_0}, \frac{A_0}{x_0}, ...\right\}.$$

(a_{112}) If $A_1x_0^2 > A_0^2$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$
$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{x_0, \frac{A_1x_0}{A_0}\right\} = x_0,$$
$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, x_0\right\} = x_0,$$

(since $A_1x_0^2 > A_0^2 > A_1^2$)

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1x_0}{A_0}, x_0\right\} = x_0,$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, x_0\right\} = x_0,$$

(since $A_0x_0^2 > A_1x_0^2 > A_0^2$)

then the solution is eventually periodic with period three. Moreover, it has the following form

$$\left\{\frac{A_0}{x_0}, x_0, x_0, \frac{A_0}{x_0}, x_0, x_0, \frac{A_0}{x_0}, ...\right\}.$$

(a_{12}) If $x_{-1}A_0 > A_1x_0$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{121}) If $x_0 < \frac{A_0}{x_{-1}}$, then $x_3 = \frac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1x_{-1}}{A_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$\left(\text{since } x_0 < \frac{A_0}{x_{-1}} \;\Rightarrow A_1 x_0 < \frac{A_1 A_0}{x_{-1}} < \frac{A_0^2}{x_{-1}}\right),$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\{x_0, x_{-1}\},$$

(a_{1211}) If $x_0 < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_{-1}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\} = x_{-1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}}.$$

Clearly, the derived solution is eventually periodic with period three and takes the form

$$\left\{\frac{A_0}{x_0}, x_{-1}, \frac{A_0}{x_{-1}}, \frac{A_0}{x_0}, x_{-1}, \frac{A_0}{x_{-1}}, \ldots\right\}.$$

(a_{1212}) If $x_0 > x_{-1}$, then $x_5 = x_0$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$
$$\left(\text{since } x_0 > x_{-1} \;\Rightarrow A_0 x_0 > A_0 x_{-1} > A_1 x_{-1}\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_{-1}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_0\right\} = x_0,$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

we obtain that the solution is eventually periodic with period three. It has the following form

$$\left\{\frac{A_0}{x_{-1}}, \frac{A_0}{x_0}, x_0, \frac{A_0}{x_{-1}}, \frac{A_0}{x_0}, x_0, \ldots\right\}.$$

(a_{122}) If $A_0 < x_{-1} x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\{x_0, x_{-1}\},$$

(a_{1221}) If $x_0 < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\} = x_0,$$

$$\left(\text{since } x_0 > \frac{A_0}{x_{-1}} > \frac{A_1}{x_{-1}}\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\} = x_{-1},$$

$$(\text{since } x_0 < x_{-1} \Rightarrow A_1 x_0 < A_0 x_0 < A_0 x_{-1}),$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

the presented solution has the form

$$\left\{x_{-1}, x_0, \frac{A_0}{x_0}, x_{-1}, x_0, \frac{A_0}{x_0}, \ldots\right\}.$$

(a_{1222}) If $x_0 > x_{-1}$, then $x_5 = x_0$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_0}, x_0\right\} = x_0,$$

$$\left(\text{since } x_0 > \frac{A_0}{x_{-1}} > \frac{A_1}{x_{-1}} \;\&\; x_0 > x_{-1} \Rightarrow x_0^2 > x_0 x_{-1} > A_1\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_0\right\} = x_0,$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_0}, x_0\right\} = x_0,$$

we obtain that the solution is eventually periodic with period three and the developed solution has the following form

$$\left\{x_0, \frac{A_0}{x_0}, x_0, x_0, \frac{A_0}{x_0}, x_0, x_0, \ldots\right\}.$$

Case (a_2) If $x_{-2}x_0 > A_0$, then $x_1 = x_{-2}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\}.$$

(a_{21}) If $x_{-1}x_{-2} < A_1$, then $x_2 = \dfrac{A_1}{x_{-2}}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_0\right\}.$$

(a_{211}) If $A_1 x_0 < A_0 x_{-2}$, then $x_3 = \dfrac{A_0 x_{-2}}{A_1}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2}{A_0 x_{-2}}, x_{-2}\right\} = x_{-2},$$

(since $A_1 x_0 < A_0 x_{-2} \;\Rightarrow \dfrac{1}{x_0} > \dfrac{A_1}{A_0 x_{-2}} \,\&\, x_{-2} > \dfrac{A_0}{x_0}$

$$\Rightarrow\; x_{-2} > \frac{A_0}{x_0} > \frac{A_0 A_1}{A_0 x_{-2}} > \frac{A_1^2}{A_0 x_{-2}}),$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-2}\right\} = x_{-2},$$

(since $A_0 x_{-2}^2 > A_0 A_1$),

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{x_{-2}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

the resulting solution is then eventually periodic with period three in the closed form

$$\left\{\frac{A_0 x_{-2}}{A_1}, x_{-2}, \frac{A_0}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}, x_{-2}, \frac{A_0}{x_{-2}}, \ldots\right\}.$$

(a_{212}) If $A_1 x_0 > A_0 x_{-2}$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\} = x_{-2},$$

(since $x_0 x_{-2} > A_0 > A_1$),

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, x_0\right\} = x_0,$$

(since $A_0x_0 > A_1x_0 > A_0x_{-2} > A_1x_{-2}$),

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\{x_{-2}, x_0\} = x_0,$$

(since $A_1x_0 > A_0x_{-2} > A_1x_{-2}$).

Thus, the solution is

$$\left\{x_{-2}, \frac{A_0}{x_{-2}}, x_0, x_{-2}, \frac{A_0}{x_{-2}}, x_0, x_{-2}, ...\right\}.$$

(a_{22}) If $x_{-1}x_{-2} > A_1$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{221}) If $x_0x_{-1} < A_0$, then $x_3 = \dfrac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1x_{-1}}{A_0}, x_{-2}\right\} = x_{-2},$$
$$\left(\text{since } x_{-2} > \frac{A_0}{x_0} > \frac{x_0x_{-1}}{x_0} \Rightarrow A_0x_{-2} > A_1x_{-2} > A_0x_{-1} > A_1x_{-1}\right),$$
$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\},$$

(a_{2211}) If $\dfrac{A_0}{x_{-2}} < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\{x_{-1}, x_{-2}\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

this provides an eventually periodic with period three, which takes the form

$$\left\{x_{-2}, x_{-1}, \frac{A_0}{x_{-1}}, x_{-2}, x_{-1}, \frac{A_0}{x_{-1}}, \ldots\right\}.$$

(a_{2212}) If $\dfrac{A_0}{x_{-2}} > x_{-1}$, then $x_5 = \dfrac{A_0}{x_{-2}}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$
$$\left(\text{since } x_{-2}x_{-1} < A_0 \Rightarrow A_1 x_{-2} x_{-1} < A_1 A_0 < A_0^2\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\{x_{-1}, x_{-2}\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, \frac{A_0}{x_{-1}}\right\} = \frac{A_0}{x_{-1}},$$

obviously, the solution is eventually periodic with period three, which can be expressed in the form

$$\left\{\frac{A_0}{x_{-1}}, x_{-2}, \frac{A_0}{x_{-2}}, \frac{A_0}{x_{-1}}, x_{-2}, \frac{A_0}{x_{-2}}, \ldots\right\}.$$

(a_{222}) If $A_0 < x_{-1}x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\} = x_{-2},$$
$$(\text{since } x_{-2}x_0 > A_0 > A_1)$$
$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\},$$

(a_{2221}) If $\dfrac{A_0}{x_{-2}} < x_{-1}$, then $x_5 = x_{-1}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\} = x_0,$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

this in turn gives the eventually periodic solution with period three

$$\{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

(a_{2222}) If $\dfrac{A_0}{x_{-2}} > x_{-1}$, then $x_5 = \dfrac{A_0}{x_{-2}}$ and

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1 x_{-2}}{A_0}, x_0\right\} = x_0,$$

$$\left(\text{since } x_0 > \frac{A_0}{x_{-1}} > x_{-2} \Rightarrow A_0 x_0 > A_0 x_{-2} > A_1 x_{-2}\right),$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0}{x_{-2}}\right\} = \frac{A_0}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\{x_{-2}, x_0\} = x_0,$$

this gives the solution

$$\left\{x_0, x_{-2}, \frac{A_0}{x_{-2}}, x_0, x_{-2}, \frac{A_0}{x_{-2}}, ...\right\}.$$

□

2.2. *The Case $A_1 > A_0$*

Theorem 2. *Consider the difference equation (1.1) for $A_1 > A_0$. Then the solution is periodic with period three or the equation has unboundedness solution takes the form:*

$$\{x_{2n-1}\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-2}\left(\frac{A_0}{A_1}\right), x_{-2}\left(\frac{A_0}{A_1}\right)^2, x_{-2}\left(\frac{A_0}{A_1}\right)^3, ...\right\},$$

$$\{x_{2n}\}_{n=1}^{\infty} = \left\{\frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right), \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^2, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^3, ...\right\}.$$

Proof. From Eq.(1.1), we see that

$$x_1 = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\}.$$

We consider the following two cases:

Case (a_1) If $x_{-2}x_0 < A_0$, then $x_1 = \dfrac{A_0}{x_0}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\}.$$

(a_{11}) If $x_{-1}A_0 < A_1x_0$, then $x_2 = \dfrac{A_1x_0}{A_0}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0^2}{A_1x_0}, x_0\right\}.$$

(a_{111}) If $A_1x_0^2 < A_0^2$, then $x_3 = \dfrac{A_0^2}{A_1x_0}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2x_0}{A_0^2}, \frac{A_0}{x_0}\right\},$$

(a_{1111}) If $A_1^2x_0^2 < A_0^3$, then $x_4 = \dfrac{A_0}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{x_0, \frac{A_1x_0}{A_0}\right\} = \frac{A_1x_0}{A_0},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0^2}{A_1x_0}\right\} = \frac{A_0}{x_0},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_0, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

$$\left(\text{since } x_0 < \frac{A_0^2}{x_0A_1} < \frac{A_0A_1}{x_0A_1} < \frac{A_0}{x_0}\right),$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1x_0}{A_0}, \frac{A_1x_0}{A_0}\right\} = \frac{A_1x_0}{A_0},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0^2}{A_1x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

then the solution takes the form

$$\left\{\frac{A_0}{x_0}, \frac{A_1x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_0}{x_0}, \frac{A_1x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_0}{x_0}, \ldots\right\},$$

which is eventually periodic with period three.

(a_{1112}) If $A_1^2x_0^2 > A_0^3$, then $x_4 = \dfrac{A_1^2x_0}{A_0^2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^3}{A_1^2x_0}, \frac{A_1x_0}{A_0}\right\} = \frac{A_1x_0}{A_0},$$

$$\left(\text{since } A_1^2x_0^2 > A_0^3 \Rightarrow A_1^3x_0^2 > A_1A_0^3 > A_0^4\right),$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_0^2}{A_1 x_0}\right\} = \frac{A_0}{x_0},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_0, \frac{A_1^2 x_0}{A_0^2}\right\} = \frac{A_1^2 x_0}{A_0^2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_0^2}{A_1 x_0}, \frac{A_1 x_0}{A_0}\right\} = \frac{A_1 x_0}{A_0},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0^2}{A_1 x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

and the solution has the following form which is period three solution

$$\left\{\frac{A_1^2 x_0}{A_0^2}, \frac{A_1 x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_1^2 x_0}{A_0^2}, \frac{A_1 x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_1^2 x_0}{A_0^2}, \ldots\right\}.$$

(a_{112}) If $A_1 x_0^2 > A_0^2$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0 x_0}{A_1}, \frac{A_1 x_0}{A_0}\right\} = \frac{A_1 x_0}{A_0},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_0}{x_0}, x_0\right\},$$

(a_{1121}) If $x_0^2 > A_0$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{x_0, \frac{A_1 x_0}{A_0}\right\} = \frac{A_1 x_0}{A_0},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0^2}{A_1 x_0}, x_0\right\} = x_0,$$

then the solution is eventually periodic with period three and the solution has the following expression:

$$\left\{\frac{A_1 x_0}{A_0}, x_0, \frac{A_1}{x_0}, \frac{A_1 x_0}{A_0}, x_0, \frac{A_1}{x_0}, \ldots\right\}.$$

(a_{1122}) If $x_0^2 < A_0$, then $x_6 = \dfrac{A_0}{x_0}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{x_0, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{x_0, \frac{A_1 x_0}{A_0}\right\} = \frac{A_1 x_0}{A_0},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0^2}{A_1 x_0}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$

then this solution can be written as

$$\left\{\frac{A_1}{x_0}, \frac{A_1 x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_1}{x_0}, \frac{A_1 x_0}{A_0}, \frac{A_0}{x_0}, \frac{A_1}{x_0}, \ldots\right\},$$

which is eventually periodic with period three.

(a_{12}) If $x_{-1}A_0 > A_1 x_0$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{121}) If $x_0 < \dfrac{A_0}{x_{-1}}$, then $x_3 = \dfrac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1 x_{-1}}{A_0}, \frac{A_0}{x_0}\right\},$$

(a_{1211}) If $\dfrac{A_1 x_{-1}}{A_0} < \dfrac{A_0}{x_0}$, then $x_4 = \dfrac{A_0}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$(\text{since } A_0 x_{-1} > A_1 x_0 > A_0 x_0),$$
$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$
$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_0}{x_0}\right\} = \frac{A_0}{x_0},$$
$$\left(\text{since } x_0 x_{-1} < A_0 < A_1 \;\Rightarrow A_0 x_{-1} < \frac{A_1 A_0}{x_0}\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1 x_0}{A_0}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

we obtain that the solution is eventually periodic with period three. It has the following form

$$\left\{\frac{A_0}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_0}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, ...\right\}.$$

(a_{1212}) If $\dfrac{A_1x_{-1}}{A_0} > \dfrac{A_0}{x_0}$, then $x_4 = \dfrac{A_1x_{-1}}{A_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^2}{A_1x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$\left(\text{since } x_{-1} > \frac{A_1x_0}{A_0} \ \ \& \ \ x_0 > \frac{A_0^2}{A_1x_{-1}} \Rightarrow x_{-1} > \frac{A_1A_0^2}{A_0A_1x_{-1}} > \frac{A_0^2}{A_1x_{-1}}\right)$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0x_{-1}}{A_1}, \frac{A_1x_{-1}}{A_0}\right\} = \frac{A_1x_{-1}}{A_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

$$\left(\text{since } A_1x_{-1}^2 > A_0A_1\right),$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

then the solution is eventually periodic with period three. Moreover, it has the following form:

$$\left\{\frac{A_1x_{-1}}{A_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1x_{-1}}{A_0}, x_{-1}, \frac{A_1}{x_{-1}}, ...\right\}.$$

(a_{122}) If $A_0 < x_{-1}x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, \frac{A_0}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0x_0}{A_1}, x_{-1}\right\} = x_{-1},$$

$$\left(\text{since } x_{-1} > \frac{A_1x_0}{A_0} > \frac{A_0x_0}{A_1}\right),$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\}.$$

(a_{1221}) If $A_1 < x_0x_{-1}$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$\left(\text{since } x_{-1} > \frac{A_1 x_0}{A_0} > \frac{A_0 x_0}{A_0} = x_0\right),$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

the developed solution has the following form

$$\left\{x_{-1}, x_0, \frac{A_1}{x_0}, x_{-1}, x_0, \frac{A_1}{x_0}, \ldots\right\}.$$

(a_{1222}) If $A_1 > x_0 x_{-1}$, then $x_6 = \dfrac{A_1}{x_{-1}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$\left(\text{since } A_1 > x_0 x_{-1} \;\Rightarrow\; A_1^2 > A_1 A_0 > A_0 x_0 x_{-1}\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$\left(\text{since } x_{-1} > \frac{A_1 x_0}{A_0} > \frac{A_0 x_0}{A_0} = x_0\right),$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

we get the solution which is eventually periodic with period three. It has the following form

$$\left\{\frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

Case (a_2) If $x_{-2} x_0 > A_0$, then $x_1 = x_{-2}$ and

$$x_2 = \max\left\{\frac{A_1}{x_1}, x_{-1}\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\}.$$

(a_{21}) If $x_{-1} x_{-2} < A_1$, then $x_2 = \dfrac{A_1}{x_{-2}}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_0\right\}.$$

(a_{211}) If $A_1 x_0 > A_0 x_{-2}$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\}.$$

(a_{2111}) If $A_1 > x_0x_{-2}$, then $x_4 = \dfrac{A_1}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0x_0}{A_1}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

(since $A_1 > x_0x_{-2} \Rightarrow > A_1^2 > A_1x_0x_{-2} > A_0x_0x_{-2}$),

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\{x_{-2}, x_0\}.$$

(a_{21111}) If $x_0 < x_{-2}$, then $x_6 = x_{-2}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

(since $A_1x_{-2} > A_1x_0 > A_0x_0$),

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_0x_0}{A_1}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0x_{-2}}{A_1}, x_{-2}\right\} = x_{-2},$$

the solution of the form

$$\left\{x_{-2}, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_{-2}, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_{-2}, ...\right\},$$

is eventually periodic with period three.

(a_{21112}) If $x_0 > x_{-2}$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{x_0, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0x_{-2}}{A_1}, x_0\right\} = x_0,$$

this shows that the solution

$$\left\{x_0, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_0, \frac{A_1}{x_0}, \frac{A_1}{x_{-2}}, x_0, ...\right\},$$

is just eventually periodic with period three.

(a_{2112}) If $A_1 < x_0x_{-2}$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\{x_{-2}, x_0\}.$$

(a_{21121}) If $x_0 < x_{-2}$, then $x_6 = x_{-2}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-2}\right\} = x_{-2},$$

$$\left(\text{since } x_{-2} > \frac{A_1}{x_0} > \frac{A_1}{x_{-2}} > \frac{A_0}{x_{-2}}\right),$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_{-2}\right\} = x_{-2},$$

this gives the solution

$$\left\{x_{-2}, \frac{A_1}{x_{-2}}, x_{-2}, x_{-2}, \frac{A_1}{x_{-2}}, x_{-2}, x_{-2}, \ldots\right\}.$$

(a_{21122}) If $x_0 > x_{-2}$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\right\} = \frac{A_1}{x_{-2}},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0 x_{-2}}{A_1}, x_0\right\} = x_0,$$

then the solution is eventually periodic with period three, moreover, it has the following form

$$\left\{x_0, x_{-2}, \frac{A_1}{x_{-2}}, x_0, x_{-2}, \frac{A_1}{x_{-2}}, x_0, \ldots\right\}.$$

(a_{212}) If $A_1 x_0 < A_0 x_{-2}$, then $x_3 = \dfrac{A_0 x_{-2}}{A_1}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1^2}{A_0 x_{-2}}, x_{-2}\right\}.$$

(a_{2121}) If $A_1^2 < A_0 x_{-2}^2$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1^2}{A_0 x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\}.$$

(a_{21211}) If $A_1^3 < A_0^2 x_{-2}^2$, then $x_6 = \dfrac{A_0 x_{-2}}{A_1}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-2}\right\} = x_{-2},$$

(since $A_0 x_{-2}^2 > A_1^2 > A_0 A_1$),

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

and the solution has the following expression

$$\left\{x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_0 x_{-2}}{A_1}, x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_0 x_{-2}}{A_1}, x_{-2}, ...\right\}.$$

(a_{21212}) If $A_1^3 > A_0^2 x_{-2}^2$, then $x_6 = \dfrac{A_1^2}{A_0 x_{-2}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0^2}{A_1^2} x_{-2}, x_{-2}\right\} = x_{-2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_0 x_{-2}}{A_1}\right\} = \frac{A_0 x_{-2}}{A_1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_1}{x_{-2}}, \frac{A_1^2}{A_0 x_{-2}}\right\} = \frac{A_1^2}{A_0 x_{-2}},$$

also the the solution has the following period three as follows

$$\left\{x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_1^2}{A_0 x_{-2}}, x_{-2}, \frac{A_0 x_{-2}}{A_1}, \frac{A_1^2}{A_0 x_{-2}}, x_{-2}, ...\right\}.$$

(a_{2122}) If $A_1^2 > A_0 x_{-2}^2$, then $x_4 = \dfrac{A_1^2}{A_0 x_{-2}}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^2 x_{-2}}{A_1^2}, \frac{A_1}{x_{-2}}\right\}.$$

Case (i) If we take $x_5 = \dfrac{A_1}{x_{-2}}$. Then the solution has the following form:

$$\left\{\frac{A_1}{x_{-2}}, x_{-2}, \frac{A_1^2}{A_0 x_{-2}}, \frac{A_1}{x_{-2}}, x_{-2}, \frac{A_1^2}{A_0 x_{-2}}, \frac{A_1}{x_{-2}}, x_{-2}, ...\right\}.$$

Case (ii) If $x_5 = \dfrac{A_0^2 x_{-2}}{A_1^2}$, then the solution will be periodic with period three or being in the following form

$$\{x_{2n-1}\}_{n=1}^{\infty} = \left\{ x_{-2}, x_{-2}\left(\frac{A_0}{A_1}\right), x_{-2}\left(\frac{A_0}{A_1}\right)^2, x_{-2}\left(\frac{A_0}{A_1}\right)^3, \ldots \right\},$$

$$\{x_{2n}\}_{n=1}^{\infty} = \left\{ \frac{A_1}{x_{-2}}, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right), \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^2, \frac{A_1}{x_{-2}}\left(\frac{A_1}{A_0}\right)^3, \ldots \right\}.$$

which is unboundedness solution.

(a_{22}) If $x_{-1}x_{-2} > A_1$, then $x_2 = x_{-1}$ and

$$x_3 = \max\left\{\frac{A_2}{x_2}, x_0\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\}.$$

(a_{221}) If $x_0 x_{-1} < A_0$, then $x_3 = \dfrac{A_0}{x_{-1}}$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1 x_{-1}}{A_0}, x_{-2}\right\}.$$

(a_{2211}) If $A_0 x_{-2} < A_1 x_{-1}$, then $x_4 = \dfrac{A_1 x_{-1}}{A_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0^2}{A_1 x_{-1}}, x_{-1}\right\} = x_{-1},$$

(since $x_{-1} > \dfrac{A_1}{x_{-2}}$ & $A_1 x_{-1} > A_0 x_{-2} \Rightarrow A_1 x_{-1}^2 > A_0 A_1 > A_0^2$),

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_1 x_{-1}}{A_0}\right\} = \frac{A_1 x_{-1}}{A_0},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

(since $A_1 x_{-1}^2 > A_1 A_0$),

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}}.$$

Obviously, the solution is eventually periodic with period three, which can be expressed in the form

$$\left\{ x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1 x_{-1}}{A_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1 x_{-1}}{A_0}, \ldots \right\}.$$

(a_{2212}) If $A_0x_{-2} > A_1x_{-1}$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, \frac{A_0}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0x_{-1}}{A_1}, x_{-2}\right\} = x_{-2},$$

(since $x_{-2} > \dfrac{A_1x_{-1}}{A_0} > \dfrac{A_0x_{-1}}{A_1}$),

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

this provides an eventually periodic with period three, which takes the form

$$\left\{\frac{A_1}{x_{-1}}, x_{-2}, x_{-1}, \frac{A_1}{x_{-1}}, x_{-2}, x_{-1}, ...\right\}.$$

(a_{222}) If $A_0 < x_{-1}x_0$, then $x_3 = x_0$ and

$$x_4 = \max\left\{\frac{A_3}{x_3}, x_1\right\} = \max\left\{\frac{A_1}{x_0}, x_{-2}\right\}.$$

(a_{2221}) If $A_1 < x_{-2}x_0$, then $x_4 = x_{-2}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_0\right\}.$$

(a_{22211}) If $\dfrac{A_1}{x_{-1}} < x_0$, then $x_6 = x_0$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_0}, x_{-2}\right\} = x_{-2},$$

$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$

$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_0\right\} = x_0,$$

and the solution has the following form

$$\{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

(a_{22212}) If $\dfrac{A_1}{x_{-1}} < x_0$, then $x_6 = \dfrac{A_1}{x_{-1}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, x_{-2}\right\} = x_{-2},$$
$$\left(\text{since } x_{-2}x_{-1} > A_1 > x_0 x_{-1} \Rightarrow x_{-2} > x_{-1} \Rightarrow A_1 x_{-2} > A_1 x_{-1} > A_0 x_{-1}\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\left\{\frac{A_1}{x_{-2}}, x_{-1}\right\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}},$$

we obtain that the solution is eventually periodic with period three. It has the following form

$$\left\{x_{-2}, x_{-1}, \frac{A_1}{x_{-1}}, x_{-2}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

(a_{2222}) If $A_1 > x_{-2}x_0$, then $x_4 = \dfrac{A_1}{x_0}$ and

$$x_5 = \max\left\{\frac{A_4}{x_4}, x_2\right\} = \max\left\{\frac{A_0 x_0}{A_1}, x_{-1}\right\} = x_{-1},$$
$$\left(\text{since } x_{-2}x_{-1} > A_1 > x_0 x_{-2} \Rightarrow x_{-1} > x_0 \Rightarrow A_1 x_{-1} > A_1 x_0 > A_0 x_0\right),$$
$$x_6 = \max\left\{\frac{A_5}{x_5}, x_3\right\} = \max\left\{\frac{A_1}{x_{-1}}, x_{-1}\right\}.$$

(a_{22221}) If $\dfrac{A_1}{x_{-1}} > x_{-1}$, then $x_6 = \dfrac{A_1}{x_{-1}}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0 x_{-1}}{A_1}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$\left(\text{since } A_1 > x_{-1}^2 > x_{-1}x_0 \Rightarrow A_1^2 x_{-1} > A_1 x_{-1} x_0 > A_0 x_{-1} x_0\right),$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_{-1}}\right\} = \frac{A_1}{x_{-1}}.$$

Then the solution is eventually periodic with period three and has the following form

$$\left\{\frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \frac{A_1}{x_0}, x_{-1}, \frac{A_1}{x_{-1}}, \ldots\right\}.$$

(a_{22222}) If $\dfrac{A_1}{x_{-1}} < x_{-1}$, then $x_6 = x_{-1}$ and

$$x_7 = \max\left\{\frac{A_6}{x_6}, x_4\right\} = \max\left\{\frac{A_0}{x_{-1}}, \frac{A_1}{x_0}\right\} = \frac{A_1}{x_0},$$
$$x_8 = \max\left\{\frac{A_7}{x_7}, x_5\right\} = \max\{x_0, x_{-1}\} = x_{-1},$$
$$x_9 = \max\left\{\frac{A_8}{x_8}, x_6\right\} = \max\left\{\frac{A_0}{x_{-1}}, x_{-1}\right\} = x_{-1},$$

we obtain that the solution is eventually periodic with period three and takes the form

$$\left\{x_{-1}, x_{-1}, \frac{A_1}{x_0}, x_{-1}, x_{-1}, \frac{A_1}{x_0}, \ldots\right\}.$$

□

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ON THE SIZE OF DIOPHANTINE M-TUPLES FOR LINEAR POLYNOMIALS

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Abstract. In this paper we prove that there does not exist a set with more than 16 nonzero polynomials in $\mathbb{K}[X]$, where $\mathbb{K}$ is any field of characteristic 0, such that the product of any two of them increased by a linear polynomial $n \in \mathbb{K}[X]$ is a square of a polynomial from $\mathbb{K}[X]$.

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1. INTRODUCTION

Diophantus of Alexandria [1] was the first who studied the problem of finding sets with the property that the product of any two of its distinct elements increased by 1 is a perfect square. Such a set consisting of m elements is therefore called a Diophantine m-tuple. Diophantus found the first Diophantine quadruple consisting of rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$, while the first Diophantine quadruple of integers, the set $\{1, 3, 8, 120\}$, was found by Fermat. In the case of rational numbers no upper bound for the size of such sets is known. In integer case which is the most studied, Dujella [4] proved that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples. The folklore conjecture is that there does not exist a Diophantine quintuple over the integers.

Many generalizations of this problem were also considered, for example by adding a fixed integer n instead of 1, looking at kth powers instead of squares, or considering the problem over other domains than $\mathbb{Z}$ or $\mathbb{Q}$.

Definition 1. Let $m \geq 2$, $k \geq 2$ and let R be a commutative ring with 1. Let $n \in R$ be a nonzero element and let $\{a_1, \ldots, a_m\}$ be a set of m distinct nonzero elements from R such that $a_i a_j + n$ is a kth power of an element of R for $1 \leq i < j \leq m$. The set $\{a_1, \ldots, a_m\}$ is called a kth power Diophantine m-tuple with the property $D(n)$ or simply a kth power $D(n)$-m-tuple in R.

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The first such set, a second power $D(256)$-quadruple $\{1,33,68,105\}$, was found by Diophantus [1]. It is interesting to find upper bounds for the number of elements of such sets. Dujella [2,3] found such bounds for the integer case and for $k=2$. For other similar results see [5,6,9,12].

The first polynomial variant of the above problem was studied by Jones [13,14] for $R=\mathbb{Z}[X]$, $k=2$ and $n=1$. In this case, Dujella and Fuchs [6] proved that there does not exist a second power Diophantine quintuple. Dujella and Luca [11] considered the case $n=1$, $k\geq 3$ and $R=\mathbb{K}[X]$, where $\mathbb{K}$ is an algebraically closed field of characteristic 0. Using many results from [11], Dujella and Jurasić [9] proved that there does not exist a second power Diophantine 8-tuple in $\mathbb{K}[X]$ for $n=1$. There were also considered other variants of such a polynomial problem. Dujella and Fuchs, jointly with Tichy [7] and later with Walsh [8], considered the case $R=\mathbb{Z}[X]$, $k=2$ and n is a linear polynomial from $\mathbb{Z}[X]$. They proved that in this case $m\leq 12$. Jurasić [15] proved that $m\leq 98$ for n a quadratic polynomial in $\mathbb{Z}[X]$.

We will consider the case $R=\mathbb{K}[X]$, where $\mathbb{K}$ is any field of characteristic 0, $k=2$ and n is a linear polynomial from $\mathbb{K}[X]$. Without loss of generality we assume that $\mathbb{K}$ is algebraically closed. If we omit the condition that $\mathbb{K}$ is a field of characteristic 0, then we could not obtain some results where the factorisation of a polynomial is considered. For brevity, instead of second power $D(n)$-m-tuple in $\mathbb{K}[X]$, from now on we shall refer to a polynomial $D(n)$-m-tuple. Observe that, at most one polynomial a_i for $i\in\{1,\ldots,m\}$ in such a polynomial $D(n)$-m-tuple is constant. Otherwise, we would have two different constants a and b for which $ab+n=r^2$, where $r\in\mathbb{K}[X]$. This is not possible, because then $\deg(n)=1=2\deg(r)$.

An improper kth power $D(n)$-m-tuple in R is an m-tuple with the property from Definition 1, but with relaxed condition that its elements need not be distinct and need not be nonzero. For linear n, we cannot have 0 in an improper polynomial $D(n)$-m-tuple. Let us assume that there exists a non-constant polynomial a such that $a^2+n=r^2$, for some $r\in\mathbb{K}[X]$. Then $\deg(a)=\deg(r)\geq 1$ and $\deg(n)=\deg(r-a)+\deg(r+a)$. This is not possible if $\deg(a)\geq 2$ but, for example, we have $(X+3)^2-4X-8=(X+1)^2$. So, in an improper polynomial $D(n)$-m-tuple we can have infinitely many equal linear polynomials. In the rest of the paper we consider only proper $D(n)$-m-tuples, described in Definition 1. We have the following theorem.

Theorem 1. *There are at most* 16 *elements in a polynomial* $D(n)$*-*m*-tuple for a linear polynomial* n*, i.e.*

$$m\leq 16.$$

In order to prove Theorem 1, we follow the strategy used in [7,8] for linear n and in [15] for quadratic n. In those papers the ring $\mathbb{Z}[X]$ was considered, using the relation "$<$" between its elements. Instead of that, in $\mathbb{K}[X]$ we have to use the relation "$\leq$" between the degrees of its elements. The paper is organized as follows. In Section 2

we estimate the number of polynomials with given degree k in a polynomial $D(n)$-m-tuple and consider separate cases depending on k. In Section 3, we adapt the gap principle for the degrees of the elements of a polynomial $D(n)$-quadruple, proved in [7] for the ring $\mathbb{Z}[X]$, to $\mathbb{K}[X]$. Using the bounds from Section 2 and combining the gap principle with an upper bound for the degree of the largest element in a polynomial $D(n)$-quadruple, obtained in [8] and also valid in $\mathbb{K}[X]$, in Section 4 we give the proof of Theorem 1.

2. Sets with polynomials of equal degree

Let L_k be the number of polynomials of degree k from $\mathbb{K}[X]$ in a polynomial $D(n)$-m-tuple for linear n. The first step which leads us to the proof of Theorem 1 is to estimate the numbers L_k for $k \geq 0$. We already proved that

$$L_0 \leq 1.$$

The following lemma, which is [7, Lemma 1], plays the key role in our proofs. It is proved for polynomials with integer coefficients, but the proof is obtained using only algebraic manipulations so it holds in $\mathbb{K}[X]$ also.

Lemma 1. *Let $\{a,b,c\}$ be a polynomial $D(n)$-triple in $\mathbb{K}[X]$ and let*

$$ab+n=r^2,\ \ ac+n=s^2,\ \ bc+n=t^2, \tag{2.1}$$

for some $r,s,t \in \mathbb{K}[X]$. Then there exist polynomials $e,u,v,w \in \mathbb{K}[X]$ such that

$$ae+n^2=u^2,\ \ be+n^2=v^2,\ \ ce+n^2=w^2. \tag{2.2}$$

More precisely,

$$e=n(a+b+c)+2abc-2rst, \tag{2.3}$$

$$u=at-rs,\ \ v=bs-rt,\ \ w=cr-st \tag{2.4}$$

and

$$c=a+b+\frac{e}{n}+\frac{2}{n^2}(abe+ruv). \tag{2.5}$$

We also define

$$\overline{e}=n(a+b+c)+2abc+2rst \tag{2.6}$$

and we get

$$e\cdot\overline{e}=n^2(c-a-b-2r)(c-a-b+2r). \tag{2.7}$$

Also, from (2.3), using (2.1) and (2.4), we get

$$e=n(a+b-c)+2rw, \tag{2.8}$$

$e=n(a-b+c)+2sv$ and $e=n(-a+b+c)+2tu$.

Let $\deg(n)=1$ and $\deg(a)=\deg(b)=\deg(c)=k\geq 1$. From (2.1) we obtain $\deg(r)=\deg(s)=\deg(t)=k\geq 1$. Let A,B,C,R,S,T be the leading coefficients

of the polynomials a,b,c,r,s,t, respectively. Then, from (2.1), we have $AB = R^2$, $AC = S^2$ and $BC = T^2$ so $ABC = \pm RST$. Let us consider both cases.
1.) If $ABC = RST$, from (2.6) we have $\deg(\overline{e}) = 3k$ and then, from (2.7),

$$\deg(e) \leq 2-k. \tag{2.9}$$

2.) If $ABC = -RST$, from (2.3) we conclude

$$\deg(e) = 3k, \tag{2.10}$$

and then, from (2.7), $\deg(\overline{e}) \leq 2-k$.

In order to bound the number L_k for $k \geq 1$, we are interested to find the number of possible c-s, for fixed a and b, such that (2.1) holds. The first step is finding all possible e-s from Lemma 1. The trivial situation in **1.)** is $e = 0$. Then, from (2.7), we obtain

$$c_{\pm} = a + b \pm 2r. \tag{2.11}$$

Analogue situation in **2.)** is $\overline{e} = 0$. Beside those trivial situations we suppose that $\deg(e) \geq 0$ and $\deg(\overline{e}) \geq 0$.

Polynomial $D(n)$-triples can be classified as regular or irregular, depending on whether they satisfy the condition given in the next definition (see e.g. [10]).

Definition 2. A $D(n)$-triple $\{a,b,c\}$ is called regular if it satisfies the condition

$$(c-b-a)^2 = 4(ab+n). \tag{2.12}$$

Observe that any permutation of a,b,c leaves equation (2.12) invariant. Also, from (2.12), using (2.1), we get (2.11) and we have

$$ac_{\pm} + n = (a \pm r)^2, \quad bc_{\pm} + n = (b \pm r)^2. \tag{2.13}$$

Moreover, a semi-regular $D(n)$-quadruple is one which contains a regular triple, and a twice semi-regular $D(n)$-quadruple is one that contains two regular triples.

Let us consider a polynomial $D(n)$-triple $\{a,b,c\}$ in the two following lemmas. We will use the divisibility of polynomials in $\mathbb{K}[X]$. A polynomial p divides a polynomial q if there exists a polynomial m from $\mathbb{K}[X]$ such that $q = mp$. This situation will be denoted by $p|q$.

Lemma 2. *At most one element of a polynomial $D(n)$-triple $\{a,b,c\}$ is divisible by n.*

Proof. Without loss of generality, let a and b be divisible by n. Then, from (2.1), it follows that $n|r$. Hence, $n^2|n$, a contradiction. □

In the following lemma, we adapt the important result from [8] for $\mathbb{K}[X]$.

Lemma 3. *Let $\{a,b\}$ be a polynomial $D(n)$-pair with $\deg(a) = \deg(b) = k \geq 1$ and such that $ab+n = r^2$, where $r \in \mathbb{K}[X]$. Let $ae+n^2 = u^2$, $be+n^2 = v^2$, where $u,v,e \in \mathbb{K}[X]$ and $\deg(e) \geq 0$. Then for each such e there is at most one $c \in \mathbb{K}[X]$, with $\deg(c) = k$, such that $\{a,b,c\}$ is a polynomial $D(n)$-triple.*

Proof. Suppose that $\{a,b\}$ is a polynomial $D(n)$-pair such that $ab+n=r^2$, where $r\in\mathbb{K}[X]$. From $ae+n^2=u^2$ and $be+n^2=v^2$, where $u,v\in\mathbb{K}[X]$, and from the fact that $\mathbb{K}$ is an algebraically closed field, we conclude that there are at most two u-s and at most two v-s. Namely, we have $\pm u$ and $\pm v$. Then, from (2.5), we obtain two possible c-s:

$$c_{\pm}=a+b+\frac{e}{n}+\frac{2}{n^2}(abe\pm ruv). \tag{2.14}$$

From this, we get

$$c_+\cdot c_-=b^2+a(a-2b)+\frac{e^2}{n^2}-\frac{2ae}{n}-\frac{2be}{n}-4n. \tag{2.15}$$

If $c_+\cdot c_-\in\mathbb{K}[X]$, then $\frac{e(e-2n(a+b))}{n^2}\in\mathbb{K}[X]$. From this, we conclude that $n|e$. Then, from (2.2), we get that $n|u$, $n|v$, and further $n^2|ae$ and $n^2|be$. If $n^2\nmid e$, then $n|a$ and $n|b$, which is in contradiction with Lemma 2. Hence, $n^2|e$. In that case (2.9) is not possible and we must have (2.10). Then, from (2.15), we obtain that

$$\deg(c_+)+\deg(c_-)=\deg\Big(\frac{e^2}{n^2}\Big)=6k-2.$$

Finally, we conclude that one of the polynomials $c_{\pm}$ has a degree $5k-2$ if the other of them has a degree k. □

Remark 1. From the proof of Lemma 3 we have that if $n|e$, then $n^2|e$. A polynomial $D(16X+9)$-triple

$$\{X,16X+8,36X+20\}, \tag{2.16}$$

from [7], for which $e=33X+18$ is an example for which (2.9) holds. But in this case $c_+\cdot c_-\notin\mathbb{K}[X]$.

2.1. *Linear polynomials*

Let us prove the following proposition.

Proposition 1. $L_1\leq 7$.

For the proof of Proposition 1 we use the results from the previous lemmas and the results stated below. Let us fix $a,b\in\mathbb{K}[X]$ such that $\deg(a)=\deg(b)=1$. We are looking at the extensions of $\{a,b\}$ to a polynomial $D(n)$-triple $\{a,b,c\}$ with $\deg(c)=1$ and then at the corresponding $e\in\mathbb{K}[X]$ defined by (2.3). From (2.9), we have the first possibility that $\deg(e)\leq 1$ and from (2.10) we have the second possibility that $\deg(e)=3$. Therefore, we have to consider the possibilities that $\deg(e)\in\{0,1,3\}$.

Lemma 4. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a)=\deg(b)=1$ there is at most one c with $\deg(c)=1$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}\setminus\{0\}$.*

Proof. By Lemma 1, there is $u \in \mathbb{K}[X]$ such that $ae + n^2 = u^2$ and $\deg(u) = 1$. Since $a = A(X - \phi)$, where $\phi \in \mathbb{K}$ and $A \in \mathbb{K}\backslash\{0\}$, we assume without loss of generality that $u - n = \varepsilon_1(X - \phi)$, $u + n = \varepsilon_2$, where $\varepsilon_1, \varepsilon_2 \in \mathbb{K}\backslash\{0\}$ and $\varepsilon_1\varepsilon_2 = Ae$. This implies that

$$2n = -\varepsilon_1 X + \varepsilon_2 + \varepsilon_1\phi. \tag{2.17}$$

Assume that, for fixed a and b, two distinct e-s exist, namely e and f, where $f \in \mathbb{K}\backslash\{0\}$ such that Lemma 1 holds. Therefore, $af + n^2 = u_1^2$, where $u_1 \in \mathbb{K}[X]$ and $\deg(u_1) = 1$. We conclude that

$$\begin{aligned} u_1 - n &= \varphi_1(X - \phi), \\ u_1 + n &= \varphi_2, \end{aligned} \tag{2.18}$$

or

$$\begin{aligned} u_1 - n &= \varphi_1, \\ u_1 + n &= \varphi_2(X - \phi), \end{aligned} \tag{2.19}$$

where $\varphi_1\varphi_2 = Af$ and $\varphi_1, \varphi_2 \in \mathbb{K}\backslash\{0\}$. Let us first consider the case (2.18). We get $2n = -\varphi_1 X + \varphi_2 + \varphi_1\phi$. Hence, from (2.17), $-\varepsilon_1 = -\varphi_1$ and $\varepsilon_2 + \varepsilon_1\phi = \varphi_2 + \varphi_1\phi$. So, we have $\varepsilon_1 = \varphi_1$ and $\varepsilon_2 - \varphi_2 = \phi(-\varepsilon_1 + \varphi_1)$, from which it follows that and $\varepsilon_2 = \varphi_2$. Therefore, $e = f$.

Assume now that (2.19) holds. Then, $2n = \varphi_2 X - \varphi_2\phi - \varphi_1$. Therefore, $-\varepsilon_1 = \varphi_2$ and $\varepsilon_2 = -\varphi_1$. This yields $e = f$. Hence, for fixed a and b, there is at most one $e \in \mathbb{K}\backslash\{0\}$. For that e, from Lemma 3, there is at most one $c \in \mathbb{K}[X]$, with $\deg(c) = 1$, such that $\{a, b, c\}$ is a polynomial $D(n)$-triple. □

Remark 2. Using the proof of Lemma 4, we get a polynomial $D(-2X + \frac{1}{2})$-triple

$$\left\{X + \frac{\sqrt{2} - 10}{49}, 8X + \frac{64\sqrt{2} + 46}{49}, X(9 - 4\sqrt{2})\right\},$$

for which $e = \frac{36 + 16\sqrt{2}}{49}$. Hence, the situation described in Lemma 4 is possible.

Lemma 5. *For a fixed polynomial $D(n)$-pair $\{a, b\}$ with $\deg(a) = \deg(b) = 1$ there are at most two c-s with $\deg(c) = 1$ such that $\{a, b, c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e) = 1$.*

Proof. From (2.2), we have

$$c = \frac{(w - n)(w + n)}{e}. \tag{2.20}$$

Therefore, $w \pm n = \lambda e$ where $\lambda \in \mathbb{K}\backslash\{0\}$, for at least one of the signs $\pm$. Inserting that into (2.20), we get

$$c = \lambda(\lambda e \mp 2n). \tag{2.21}$$

Then, from (2.8), it follows that $e(1-2r\lambda) = n(a+b-c\mp 2r)$. From Remark 1, we know that if $n|e$ then $n^2|e$, so $n \nmid e$. Therefore,

$$1-2r\lambda = \lambda_1 n, \tag{2.22}$$

$$\lambda_1 e = a+b-c\mp 2r, \tag{2.23}$$

where $\lambda, \lambda_1 \in \mathbb{K}\backslash\{0\}$.

Suppose that for fixed a and b another e exists. We call it f, and we suppose that for such $f \in \mathbb{K}[X]$, where $\deg(f) = 1$, Lemma 1 holds. For a polynomial $D(n)$-triple $\{a,b,c'\}$, where $\deg(c') = 1$, by Lemma 1 there is $w' \in \mathbb{K}[X]$ such that $c'f + n^2 = (w')^2$. Analogously as for e, it holds

$$1-2r\xi = \xi_1 n, \tag{2.24}$$

where $\xi_1 f = a+b-c'\mp 2r$, $w' \pm n = \xi f$ (for at least one of the signs $\pm$) and $\xi, \xi_1 \in \mathbb{K}\backslash\{0\}$.

From (2.22) and (2.24), we get $-2r(\lambda-\xi) = n(\lambda_1 - \xi_1)$. If $n|r$, we obtain a contradiction with (2.22). Therefore, $\lambda = \xi$ and $\lambda_1 = \xi_1$.

By inserting (2.21) into (2.23), we obtain

$$e = \frac{1}{\lambda^2+\lambda_1}(a+b\mp 2r \pm 2n\lambda). \tag{2.25}$$

Analogously, we get

$$f = \frac{1}{\xi^2+\xi_1}(a+b\mp 2r \pm 2n\xi). \tag{2.26}$$

Comparing (2.26) and (2.25), we conclude that there is at most one such $f \neq e$, namely one obtained for different combination of signs in (2.25) and (2.26).

Suppose that there is a third e, namely $h \in \mathbb{K}[X]$, where $\deg(h) = 1$, for which Lemma 1 holds. Analogously we conclude that $h = e$ or $h = f$. For each of polynomials e and f, by Lemma 3, we have at most one different linear polynomial c, namely c and c', such that $\{a,b,c\}$ and $\{a,b,c'\}$ are polynomial $D(n)$-triples. □

Remark 3. The situation from the proof of Lemma 5 is possible. Examples for that are the polynomial $D(16X+9)$-triples (2.16) and $\{X, 16X+8, 100X+44\}$, for which $f = 273X+126$. In this case $\lambda = \frac{2}{3}$ and $\lambda_1 = -\frac{1}{3}$. Also, by iserting X^2 instead of X in those examples, we get examples obtained in [15].

Let us now consider the last possibility, that (2.10) holds, i.e. $\deg(\overline{e}) \leq 1$. Since we assumed that $\deg(\overline{e}) \geq 0$, from (2.7) we conclude that $n|e$.

Lemma 6. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a) = \deg(b) = 1$ there does not exist c with $\deg(c) = 1$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e) = 3$.*

Proof. Assume that such an e exists. Since $n|e$ then $n^2|e$, by Remark 1. If we, for fixed a and b, have such a triple $\{a,b,c\}$, then $ce+n^2=w^2$, for $w\in\mathbb{K}[X]$. We conclude that $\deg(w)=2$ and $n|w$. Therefore, from the last equation, we get $ce_1+1=w_1^2$, where $e_1,w_1\in\mathbb{K}[X]$ and $\deg(e_1)=\deg(w_1)=1$. We actually have two possibilities for the polynomial w_1, namely $\pm w_1$.

By dividing (2.8) by n, we obtain

$$c=a+b-ne_1\pm 2rw_1, \tag{2.27}$$

which is actually $c_\pm$ given with (2.14), and one of the polynomials $c_\pm$ must have a degree equal to 2. Since $\deg(c_\pm)+\deg(c_\mp)=4$, we conclude that neither one of the polynomials c obtained in this way has degree equal to 1. □

Now we can estimate the number L_1.

Proof of Proposition 1. Let $a,b\in\mathbb{K}[X]$ be linear polynomials such that $ab+n=r^2$, with $r\in\mathbb{K}[X]$. We want to find the number of possible $D(n)$-triples $\{a,b,c\}$, where $c\in\mathbb{K}[X]$ is also a linear polynomial.

For $e=0$, from (2.11), we get as candidates for c at most two polynomials

$$c_{1,2}=c_\pm=a+b\pm 2r.$$

By Lemma 4, there can exists another c, which is one of the polynomials

$$c_3=a+b+\frac{e_3}{n}+\frac{2}{n^2}(abe_3\pm ru_3v_3),$$

with $e_3\in\mathbb{K}\setminus\{0\}$, for which $ae_3+n^2=u_3^2$, $be_3+n^2=v_3^2$, where $u_3,v_3\in\mathbb{K}[X]$. By Lemma 5, as a possible c we obtain at most one of the polynomials

$$c_i=a+b+\frac{e_i}{n}+\frac{2}{n^2}(abe_i\pm ru_iv_i)$$

for each $i=4,5$, with linear polynomials $e_i\in\mathbb{K}[X]$ and with $u_i,v_i\in\mathbb{K}[X]$ such that $ae_i+n^2=u_i^2$, $be_i+n^2=v_i^2$.

Since there are no other possibilities for e, we conclude that a polynomial $D(n)$-tuple which contains polynomials a and b, and consists only of linear polynomials, has at most seven elements, namely, $\{a,b,c_1,c_2,c_3,c_4,c_5\}$. □

Remark 4. The example $a=X-\frac{7}{12}$, $b=4X$, $n=X+\frac{1}{9}$, $r=2X-\frac{1}{3}$, $c_1=c_+=9X-\frac{5}{4}$ and $c_2=c_-=X+\frac{1}{12}$ shows that a twice semi-regular $D(n)$-quadruple of the form $\{a,b,c_1,c_2\}$ is possible in the case of linear polynomials so we can not exclude this possibility in the proof of Proposition 1.

2.2. *Quadratic polynomials*

We intend to prove the following proposition.

Proposition 2. $L_2\leq 4$.

The proof of Proposition 2 is based on the constructions from Lemmas 1- 3 and from the following lemmas which deal with a polynomial $D(n)$-triple $\{a,b,c\}$, where $\deg(a)=\deg(b)=\deg(c)=2$. First, we are looking for the possible e-s for fixed a and b. We have the following possible cases: $\deg(e)\leq 0$, which comes from (2.9), and $\deg(e)=6$, from (2.10). Therefore, we consider the possibilities that $\deg(e)=0$ and $\deg(e)=6$.

Lemma 7. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a)=\deg(b)=2$ there is at most one c with $\deg(c)=2$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}\setminus\{0\}$.*

Proof. By Lemma 1, there is $u\in\mathbb{K}[X]$ such that $ae+n^2=u^2$ and $\deg(u)\leq 1$. Since $a=A(X-\phi_1)(X-\phi_2)$, where $\phi_1,\phi_2\in\mathbb{K}$ and $A\in\mathbb{K}\setminus\{0\}$, we assume that

$$\begin{aligned} u-n&=\varepsilon_1(X-\phi_1),\\ u+n&=\varepsilon_2(X-\phi_2), \end{aligned} \tag{2.28}$$

where $\varepsilon_1,\varepsilon_2\in\mathbb{K}\setminus\{0\}$ and $\varepsilon_1\varepsilon_2=Ae$. From that we conclude

$$2n=X(\varepsilon_2-\varepsilon_1)+\varepsilon_1\phi_1-\varepsilon_2\phi_2. \tag{2.29}$$

Let, for fixed a and b, two distinct e-s from $\mathbb{K}\setminus\{0\}$ exist, i.e. there is also $f\in\mathbb{K}\setminus\{0\}$ for which $af+n^2=u_1^2$, where $u_1\in\mathbb{K}[X]$ and $\deg(u_1)\leq 1$. We have

$$\begin{aligned} u_1-n&=\varphi_1(X-\phi_1),\\ u_1+n&=\varphi_2(X-\phi_2), \end{aligned} \tag{2.30}$$

or

$$\begin{aligned} u_1-n&=\varphi_1(X-\phi_2),\\ u_1+n&=\varphi_2(X-\phi_1), \end{aligned} \tag{2.31}$$

where $\varphi_1\varphi_2=Af$ and $\varphi_1,\varphi_2\in\mathbb{K}\setminus\{0\}$.

From (2.30), we get $2n=X(\varphi_2-\varphi_1)+\varphi_1\phi_1-\varphi_2\phi_2$. By comparing that with (2.29), we obtain $\phi_1(\varepsilon_1-\varphi_1)=\phi_2(\varepsilon_2-\varphi_2)=\phi_2(\varepsilon_1-\varphi_1)$, from which it follows that $\phi_1=\phi_2$ or $\varepsilon_1=\varphi_1$. If $\phi_1=\phi_2$ then, from (2.28), we conclude that $n^2|a$. Further, from (2.1), we get that $n|r$, and then, from (2.3), it follows that $n|e$, which is not possible. If $\varepsilon_1=\varphi_1$, then $\varepsilon_2=\varphi_2$, so $e=f$.

Assume now that (2.31) holds. Then, $2n=(\varphi_2-\varphi_1)X-\varphi_2\phi_1+\varphi_1\phi_2$. By comparing that with (2.29), we obtain $\phi_1(\varepsilon_1+\varphi_2)=\phi_2(\varepsilon_2+\varphi_1)=\phi_2(\varepsilon_1+\varphi_2)$, from which $\phi_1=\phi_2$ or $\varepsilon_1=-\varphi_2$. For $\phi_1=\phi_2$ we obtain a contradiction, as in the previous case. If $\varepsilon_1=-\varphi_2$, then $\varepsilon_2=-\varphi_1$, so $e=f$. Hence, for fixed a and b, there is at most one $e\in\mathbb{K}\setminus\{0\}$ for which Lemma 1 holds. For that e, from Lemma 3, there is at most one $c\in\mathbb{K}[X]$ with $\deg(c)=2$, such that $\{a,b,c\}$ is a polynomial $D(n)$-triple. □

We are left with the last possibility, that (2.10) holds, i.e. $\deg(\overline{e}) \leq 0$. Since we assumed that $\deg(\overline{e}) \geq 0$, from (2.7) we conclude that $n^2|e$.

Lemma 8. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a) = \deg(b) = 2$ there does not exist c with $\deg(c) = 2$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e) = 6$.*

Proof. Assume, on the contrary that such an e exists. Then $ce+n^2 = w^2$ for some $w \in \mathbb{K}[X]$. We conclude that $\deg(w) = 4$ and $n|w$. By dividing the last equation by n^2 we get $ce_1 + 1 = w_1^2$, where $e_1, w_1 \in \mathbb{K}[X]$, $\deg(e_1) = 4$ and $\deg(w_1) = 3$. The polynomial w_1 is only determined up to the sign.

Analogously as in the proof of Lemma 6, we obtain (2.27), from which we conclude that one of the polynomials $c_\pm$ has a degree equal to 5. Since $\deg(c_\pm) + \deg(c_\mp) = 10$, we conclude that neither one of polynomials $\pm w_1 c$ obtained this way has a degree equal to 2. □

Before we estimate the number L_2 we will exclude one possibility, which exists in the case of linear polynomials.

Lemma 9. *Let $a,b \in \mathbb{K}[X]$ such that $\deg(a) = \deg(b) = 2$ and $ab + n = r^2$. Then the set of the form $\{a,b,a+b+2r,a+b-2r\}$, which contains only quadratic polynomials, is not a polynomial $D(n)$-quadruple.*

Proof. Assume that such a set is a polynomial $D(n)$-quadruple. Let us consider the triples $\{a,a+b+2r,a+b-2r\}$ and $\{b,a+b+2r,a+b-2r\}$. If the first triple is regular, then, from (2.12), we have $(-4r-a)^2 = 4(a(a+b+2r)+n)$. From that, $-4r-a = \pm 2(a+r)$ so $r|a$. Then, from (2.1), it follows that $r|n$, which is not possible, because $\deg(r) = 2$ and $\deg(n) = 1$. The second case is analogous. Therefore, neither one of those triples is regular.

By Lemma 7, for the pair $\{a+b+2r,a+b-2r\}$, there is at most one $e \in \mathbb{K}\backslash\{0\}$ such that Lemma 1 holds. Since no other e-s exist for that pair, it holds $a = b$ which is not possible. □

Remark 5. The example $a = X^2 + X + \frac{17}{36}$, $b = X^2 + 2X - \frac{1}{9}$, $n = -\frac{2}{3}X + \frac{1}{18}$, $r = X^2 + \frac{3}{2}X + \frac{1}{18}$, $a+b+2r = 4X^2 + 6X + \frac{17}{36}$ and $a+b-2r = \frac{1}{4}$ shows that if we omit the condition in Lemma 9 that all polynomials in a set of the form $\{a,b,a+b+2r,a+b-2r\}$ are quadratic, then such a $D(n)$-quadruple can exist.

Now we determine the upper bound for L_2.

Proof of Proposition 2. Let $a,b \in \mathbb{K}[X]$ be quadratic polynomials such that $ab + n = r^2$, with $r \in \mathbb{K}[X]$. We look for the number of possible $D(n)$-triples $\{a,b,c\}$, where $c \in \mathbb{K}[X]$ and c is also a quadratic polynomial.

By Lemma 7, there is at most one possibility for c, namely c_1, for which $e \in \mathbb{K}\backslash\{0\}$. For $e = 0$ we obtain $c_{2,3} = a+b \pm 2r$. In Lemma 8, we excluded the last option

which comes from Lemma 1, those that $\deg(e) = 6$. Hence, the largest set which we can obtain is of the form $\{a, b, c_1, c_2, c_3\}$.

By Lemma 9, the set $\{a, b, c_2, c_3\}$ is not a polynomial $D(n)$-quadruple. Therefore, the pair $\{a, b\}$ can be extended with at most 2 quadratic polynomials (c_1 and one of the polynomials $c_{2,3}$), so $L_2 \leq 4$. □

2.3. *Polynomials of degree $k \geq 3$*

Now we are looking for the upper bound for the number of polynomials of degree $k \geq 3$ in a polynomial $D(n)$-tuple.

Proposition 3. *$L_k \leq 3$ for $k \geq 3$.*

Let $a, b \in \mathbb{K}[X]$ be polynomials of degree $k \geq 3$ such that $ab + n = r^2$, with $r \in \mathbb{K}[X]$. We look for the number of possible c-s such that $\{a, b, c\}$, where $c \in \mathbb{K}[X]$ and $\deg(c) = k$ is a polynomial $D(n)$-triple. First, we look for possible e-s such that Lemma 1 holds. From (2.9), $\deg(e) \leq -1$, and from (2.10), $\deg(\overline{e}) \leq -1$. Therefore, $e = 0$ or $\overline{e} = 0$. From (2.7), we obtain $c_{1,2} = a + b \pm 2r$. Hence, we have at most four elements $\{a, b, c_1, c_2\}$ in such a polynomial $D(n)$-tuple, but we also have the following lemma.

Lemma 10. *Let $a, b \in \mathbb{K}[X]$ such that $\deg(a) = \deg(b) = k \geq 3$ and $ab + n = r^2$. Then the set of the form $\{a, b, a + b + 2r, a + b - 2r\}$, which contains only polynomials of degree $k \geq 3$, is not a polynomial $D(n)$-quadruple.*

Proof. The first part of the proof is analogue as the proof of Lemma 9, except that here $\deg(r) = k \geq 3$. Since in the case of polynomials of degree $k \geq 3$ the only possible triples are regular ones, we proved the lemma. □

Therefore, the pair $\{a, b\}$ can be extended with at most one polynomial of degree k, where $k \geq 3$, (namely, one of $c_{1,2}$), so we proved Proposition 3.

Example for this case is $D(X)$-triple $\{X^3 - 1, X^3 + 2X^2 + X - 1, 4X^3 + 4X^2 + X - 4\}$, from [8].

3. Gap principle

We will prove a gap principle for the degrees of the elements in a polynomial $D(n)$-quadruple. This result will be used in the proof of Theorem 1, together with the bounds from Section 2 and with the upper bound for the degree of the element in a polynomial $D(n)$-quadruple [8, Lemma 1], given in the following lemma. This bound was obtained for polynomials with integer coefficients but it also holds in $\mathbb{K}[X]$ with slightly different assumption on the degrees of polynomials in quadruple.

Lemma 11. *Let $\{a, b, c, d\}$, where $\deg(a) \leq \deg(b) \leq \deg(c) \leq \deg(d)$, be a polynomial $D(n)$-quadruple with $n \in \mathbb{K}[X]$. Then*

$$\deg(d) \leq 7\deg(a) + 11\deg(b) + 15\deg(c) + 14\deg(n) - 4.$$

The proof of this lemma is based on the theory of function fields, precisely it is obtained by using Mason's inequality [16].

Now we will adjust the result from [7, Lemma 3], very similar as in the classical case for integers, to achieve the needed gap principle. We cannot use the gap principle from [7] and [8], because we do not have the relation " $<$ " between elements of $\mathbb{K}[X]$.

Lemma 12. *Let $\{a,b,c,d\}$, where $3 \leq \deg(a) \leq \deg(b) \leq \deg(c) \leq \deg(d)$, be a polynomial $D(n)$-quadruple for linear $n \in \mathbb{K}[X]$. Then*

$$\deg(d) \geq \deg(b) + \deg(c) - 2.$$

Proof. Applying Lemma 1 to the polynomial $D(n)$-triple $\{a,c,d\}$, we conclude that there exist $e, \overline{e} \in \mathbb{K}[X]$ such that, by (2.7), we have

$$e \cdot \overline{e} = n^2(d-a-c-2s)(d-a-c+2s). \tag{3.1}$$

From (3.1), for $e = 0$ or $\overline{e} = 0$, we get

$$d = a + c \pm 2s. \tag{3.2}$$

In this case $\deg(d) \leq \deg(c)$, so $\deg(d) = \deg(c)$.

Let $e, \overline{e} \in \mathbb{K}[X]$ be nonzero polynomials. Since $\deg(s) = \frac{\deg(a)+\deg(c)}{2} \leq \deg(c)$, from (3.1) we obtain

$$\deg(e) + \deg(\overline{e}) \leq 2 + 2\deg(d). \tag{3.3}$$

By (2.3) and (2.6), we conclude that the degree of one of the polynomials e and $\overline{e}$ is equal to $\deg(a) + \deg(c) + \deg(d)$ and the degree of the other one is ≥ 0. Hence, from (3.3) it follows that

$$\deg(d) \geq \deg(a) + \deg(c) - 2. \tag{3.4}$$

Analogously, applying Lemma 1 to the polynomial $D(n)$-triple $\{b,c,d\}$, we obtain that either

$$d = b + c \pm 2t \tag{3.5}$$

or

$$\deg(d) \geq \deg(b) + \deg(c) - 2. \tag{3.6}$$

Assume that (3.5) holds. As for (3.2), we conclude that $\deg(d) = \deg(c)$. Therefore, (3.4) cannot hold at the same time. Otherwise, we would obtain $\deg(a) \leq 2$, a contradiction. Assume that (3.2) also holds. From (3.2) and (3.5) we obtain $a = b \pm 2s \pm 2t$. Therefore, if $\deg(a) < \deg(c)$ then $\deg(s) > \deg(a)$ and the polynomial on the right hand side of the previous equation has degree $> \deg(a)$ unless $\deg(a) = \deg(b)$ and $S = \pm T$. We conclude that

$$\deg(a) = \deg(b) = \deg(c) = \deg(d) \tag{3.7}$$

or

$$\deg(a) = \deg(b) < \deg(c) = \deg(d) \text{ and } S = \pm T. \tag{3.8}$$

If (3.7) holds, then we have a polynomial $D(n)$-quadruple whose elements have degrees equal to k, where $k \geq 3$. This is not possible, according to Proposition 3. Therefore, it holds (3.8). Since $\{a,c,d\}$ and $\{b,c,d\}$ are both regular triples (i.e. $\{a,b,c,d\}$ is a twice semi-regular $D(n)$-quadruple), by (2.13) we have $cd+n = (c \pm s)^2$ and $cd+n = (c \pm t)^2$, where the signs $\pm$ in last two equations are the same as signs in (3.2) and (3.5), respectively. If $c \pm s = c \pm t$, then we obtain $s^2 = t^2$ so $a = b$, which is not possible. If $c \pm s = -c \mp t$, then $\pm s = -2c \mp t$. Since (3.8) holds, the degree of the polynomial on the left hand side of the previous equation is $< \deg(c)$ and the polynomial on the right hand side of that equation has a degee equal to $\deg(c)$. This is not possible.

Assume that (3.6) holds. Then (3.2) can not hold, because otherwise we would have $\deg(b) \leq 2$, a contradiction. But, from (3.6) we obtain (3.4), which shows that this situation is indeed possible. □

4. Proof of Theorem 1

Let $S = \{a_1, a_2, ..., a_m\}$, where $\deg(a_1) \leq \deg(a_2) \leq ... \leq \deg(a_m)$, be a polynomial $D(n)$-m-tuple with $n \in \mathbb{K}[X]$ a linear polynomial. Since the product of each two elements from S increased by n is a square of a polynomial in $\mathbb{K}[X]$, it follows that if S contains a polynomial of degree ≥ 1, then it contains only polynomials of even or only polynomials of odd degree. We proved that in S we have at most 1 nonzero constant. By Proposition 1, in S there are at most 7 linear polynomials. By Proposition 2, the number of quadratic polynomials in S is at most 4, and, by Proposition 3, in S there are at most 3 polynomials of degree k for every $k \geq 3$.

Assume that in S there is a polynomial of degree ≥ 1. Let us first consider the case where the degrees of all polynomials in S are odd. We will combine the gaps between the degrees of the elements in S with the upper bound on the degree of the element of a $D(n)$-quadruple. We assume that there are smallest possible gaps between the degrees of the elements in S. Then, there are possible the following bounds for the degrees:

$$\deg(a_1) \geq 1,\ \deg(a_2) \geq 1,\ ...,\ \deg(a_7) \geq 1,$$
$$\deg(a_8) \geq 3,\ \deg(a_9) \geq 3,\ \deg(a_{10}) \geq 3.$$

Applying Lemma 12 to the polynomial $D(n)$-quadruple $\{a_8, a_9, a_{10}, a_{11}\}$ gives $\deg(a_{11}) \geq 4$ and, since this degree is odd, we conclude that

$$\deg(a_{11}) \geq 5.$$

If we continue in analogue way, we obtain

$$\begin{array}{llll} \deg(a_{12}) \geq 7, & \deg(a_{13}) \geq 11, & \deg(a_{14}) \geq 17, & \deg(a_{15}) \geq 27, \\ \deg(a_{16}) \geq 43, & \deg(a_{17}) \geq 69, & \deg(a_{18}) \geq 111, & \deg(a_{19}) \geq 179, ... \end{array}$$

We will separate the cases depending on the number of linear polynomials in S. Assume first that in S we have at least three (and ≤ 7) linear polynomials. For

the quadruple $\{a_1, a_2, a_3, a_m\} \subseteq S$, where $\deg(a_1) = \deg(a_2) = \deg(a_3) = 1$ and $\deg(a_m) \geq 1$, applying Lemma 11, we get

$$\deg(a_m) \leq 7 + 11 + 15 + 14 - 4 = 43.$$

Hence, in this case

$$m \leq 16.$$

Assume next that in S we have two linear polynomials. Now we observe the quadruple $\{a_1, a_2, a_3, a_m\} \subseteq S$, where $\deg(a_1) = \deg(a_2) = 1$ and $\deg(a_3) = \mathrm{A}$, for $A \geq 3$ an odd positive integer. As before, we have

$$\deg(a_m) \leq 7 + 11 + 15A + 14 - 4 = 15A + 28$$

and

$$\begin{array}{lll} \deg(a_4) \geq \mathrm{A}, & \deg(a_5) \geq \mathrm{A}, & \deg(a_6) \geq 2A-2, \ldots, \\ \deg(a_{10}) \geq 13A-24, & \deg(a_{11}) \geq 21A-40, & \deg(a_{12}) \geq 34A-66, \\ \deg(a_{13}) \geq 55A-108, & \deg(a_{14}) \geq 89A-176, & \deg(a_{15}) \geq 144A-286, \ldots, \end{array}$$

so we obtain $m \leq 13$.

Let in S we have one linear polynomial, i.e. $\deg(a_1) = 1, \deg(a_2) = A$, $\deg(a_3) = B$, where $3 \leq A \leq B$ and A, B are odd positive integers. We obtain

$$\deg(a_m) \leq 7 + 11A + 15B + 14 - 4 \leq 26B + 17.$$

Further,

$$\begin{array}{lll} \deg(a_4) \geq B, & \deg(a_5) \geq 2B-2, & \deg(a_6) \geq 3B-4, \ldots, \\ \deg(a_{12}) \geq 55B-108, & \deg(a_{13}) \geq 89B-176, & \deg(a_{14}) \geq 144B-286, \ldots, \end{array}$$

so $m \leq 13$.

Finally, suppose that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$ where $3 \leq A \leq B \leq C$ and A, B, C are odd positive integers. We get

$$\deg(a_m) \leq 7A + 11B + 15C + 14 - 4 \leq 33C + 10$$

and

$$\begin{array}{ll} \deg(a_4) \geq B + C - 2, & \deg(a_5) \geq B + 2C - 4, \ldots, \\ \deg(a_{12}) \geq 34B + 55C - 176, & \deg(a_{13}) \geq 55B + 89C - 286, \ldots, \end{array}$$

then $m \leq 12$. Hence, if S contains only polynomials of odd degree then $m \leq 16$.

Let all polynomials in S have even degree. Now we have

$$\deg(a_1) \geq 0,$$

$$\deg(a_2) \geq 2,\ \deg(a_3) \geq 2,\ \deg(a_4) \geq 2,\ \deg(a_5) \geq 2 \text{ and}$$

$$\deg(a_6) \geq 4,\ \deg(a_7) \geq 4,\ \deg(a_8) \geq 4.$$

Applying Lemma 12 to the polynomial $D(n)$-quadruple $\{a_6, a_7, a_8, a_9\}$ we get

$$\deg(a_9) \geq 6.$$

Analogously, it follows

$$\begin{array}{llll} \deg(a_{10}) \geq 8, & \deg(a_{11}) \geq 12, & \deg(a_{12}) \geq 18, & \deg(a_{13}) \geq 28, \\ \deg(a_{14}) \geq 44, & \deg(a_{15}) \geq 70, & \deg(a_{16}) \geq 112, & \deg(a_{17}) \geq 180, \ldots. \end{array}$$

Assume first that $\deg(a_1) = 0, \deg(a_2) = A, \deg(a_3) = B$ where $2 \leq A \leq B$ and A, B are even positive integers. If we apply Lemma 11 to a polynomial $D(n)$-quadruple $\{a_1, a_2, a_3, a_m\}$, it follows that

$$\deg(a_m) \leq 0 + 11A + 15B + 14 - 4 \leq 26B + 10.$$

If $A = B = 2$, then $m \leq 14$. If $B \geq 4$, then

$$\begin{array}{lll} \deg(a_4) \geq B, & \deg(a_5) \geq B, & \deg(a_6) \geq 2B - 2, \ldots, \\ \deg(a_{12}) \geq 34B - 66, & \deg(a_{13}) \geq 55B - 108, & \deg(a_{14}) \geq 89B - 176, \ldots, \end{array}$$

so we obtain that $m \leq 13$.

Suppose finally that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$, where $2 \leq A \leq B \leq C$ and A, B, C are even positive integers. We have

$$\deg(a_m) \leq 7A + 11B + 15C + 14 - 4 \leq 33C + 10.$$

If $A = B = C = 2$ and

$$\begin{array}{lll} \deg(a_4) \geq C, & \deg(a_5) = D, & \deg(a_6) \geq D, \\ \deg(a_7) \geq D, & \deg(a_8) \geq 2D - 2, & \deg(a_9) \geq 3D - 4, \ldots, \\ \deg(a_{14}) \geq 34D - 66, & \deg(a_{15}) \geq 55D - 108, & \deg(a_{16}) \geq 89D - 176, \ldots, \end{array}$$

where $D \geq 4$ and D is even positive integer, then $m \leq 15$. Also, $m \leq 13$ if $C \geq 4$. We conclude that the set S has at most 15 polynomials of even degree. Therefore,

$$L \leq 16.$$

□

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UNIFORM APPROXIMATION BY GENERALIZED Q-BERNSTEIN OPERATORS

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Abstract. For a sequence of integral type operators involving q-integers we study its uniform convergence in $C[0,1]$ and estimate the rate of convergence with the aid of modulus of continuity. As applications we obtain quantitative estimates for old and new q-parametric operators.

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1. INTRODUCTION

The study of q-parametric operators was initiated by Lupaş [8] and Phillips [14]. The so-called q-Bernstein operators were introduced by Phillips in 1997 (see [14]) and they mean another generalization of the well-known Bernstein operators [4] based on q-integers. Nowadays, q-Bernstein operators form an area of an intensive research. A survey of the obtained main results and references in this area during the first decade of study can be found in [12]. Different type of q-integral operators, q-Bernstein type integral operators and q-summation-integral operators were studied in [3].

The goal of the paper is to define a sequence of general q-integral type operators involving q-integers which approximate each continuous function on $[0,1]$ in the uniform norm. The rate of convergence will be estimated by the modulus of continuity. As special cases we recover some q-Bernstein type operators and q-Bernstein type integral operators, respectively. For these operators we also obtain quantitative estimations.

To present our operators we recall some basic definitions and notations of quantum calculus (see [7]). For any $q>0$ and any non-negative integer n, the q-integers $[n]_q$ and the q-factorials $[n]_q!$ are defined by $[0]_q=0$,

$$[n]_q=\begin{cases} 1+q+\ldots+q^{n-1}, & \text{if} \quad q\neq 1 \\ n, & \text{if} \quad q=1 \end{cases}$$

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and

$$[n]_q! = \begin{cases} [1]_q[2]_q \dots [n]_q, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0. \end{cases}$$

For any integers n and k satisfying $0 \leq k \leq n$, the q-binomial coefficients are given by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For $x \in [0,1]$ and m non-negative integer, we set

$$(1-x)_q^m = \begin{cases} (1-x)(1-qx)\dots(1-q^{m-1}x), & \text{if} \quad m \geq 1 \\ 1, & \text{if} \quad m = 0. \end{cases}$$

Let $0 < b$, $0 < q < 1$ and f a real-valued function. The q-Jackson integral of f over the interval $[0,b]$ is defined by

$$\int_0^b f(t)\, d_q t = (1-q)b \sum_{j=0}^{\infty} f(bq^j)q^j. \tag{1.1}$$

On the general interval $[a,b]$ the q-Jackson integral is not appropriate to derive the q-analogues of some well-known integral inequalities. For this reason we consider the Riemann type q-integral defined as follows (see [6, 10]):

$$\int_a^b f(t)\, d_q^R t = (1-q)(b-a) \sum_{j=0}^{\infty} f(a+(b-a)q^j)q^j, \tag{1.2}$$

where $0 \leq a < b$ and $0 < q < 1$. If $a = 0$ in (1.2) then we recover the q-Jackson integral given by (1.1).

For $n \in \{1,2,\dots\}$, $0 < q < 1$, $r, r' \in \{0,1,2,\dots\}$, $f \in C[0,1+r']$ and $x \in [0,1]$, we consider the following q-integral type operators:

$$L_{n,q}^{r,r'}(f;x) = \sum_{k=0}^{n+r'} \left[\begin{array}{c} n+r' \\ k \end{array} \right]_q x^k (1-x)_q^{n+r'-k} \tag{1.3}$$
$$\times \frac{1}{b(n,k,r,q) - a(n,k,r,q)} \int_{a(n,k,r,q)}^{b(n,k,r,q)} f(u(n,k,q) + tv(n,k,q))\, d_q^R t,$$

where $0 \leq a(n,k,r,q) < b(n,k,r,q) \leq 1+r'$, $u(n,k,q) \geq 0$, $v(n,k,q) \geq 0$ and $u(n,k,q) + b(n,k,r,q)v(n,k,q) \leq 1+r'$ for all $k \in \{0,1,\dots,n+r'\}$. With the notation $g_{n,k,r,q}(t) = u(n,k,q) + tv(n,k,q)$, $t \in [a(n,k,r,q), b(n,k,r,q)] \subseteq [0,1+r']$, we have for $f \in C[0,1+r']$ that there exists $M = M(f) > 0$ such that $|f(g_{n,k,r,q}(t))| \leq M$ for $t \in [0,1+r']$. Hence, in view of (1.3), (1.2) and

$0 \le a(n,k,r,q) + (b(n,k,r,q) - a(n,k,r,q))q^j \le 1 + r'$ for $j = 0,1,2,\ldots$, we get

$$\begin{aligned}
|L_{n,q}^{r,r'}(f;x)| &\le \sum_{k=0}^{n+r'} \left[\begin{matrix} n+r' \\ k \end{matrix} \right]_q x^k (1-x)_q^{n+r'-k} \\
&\quad \times \frac{1}{b(n,k,r,q) - a(n,k,r,q)} \left| \int_{a(n,k,r,q)}^{b(n,k,r,q)} f(g_{n,k,r,q}(t))\, d_q^R t \right| \\
&\le \sum_{k=0}^{n+r'} \left[\begin{matrix} n+r' \\ k \end{matrix} \right]_q x^k (1-x)_q^{n+r'-k} \frac{1}{b(n,k,r,q) - a(n,k,r,q)} \\
&\quad \times (1-q)(b(n,k,r,q) - a(n,k,r,q)) \sum_{j=0}^{\infty} \{q^j \\
&\quad \times |f(g_{n,k,r,q}(a(n,k,r,q) + (b(n,k,r,q) - a(n,k,r,q))q^j))|\} \\
&\le M \sum_{k=0}^{n+r'} \left[\begin{matrix} n+r' \\ k \end{matrix} \right]_q x^k (1-x)_q^{n+r'-k} = M
\end{aligned}$$

for all $x \in [0,1]$ (see [14, (13)]). In conclusion the operators (1.3) are well-defined for all continuous functions f on the interval $[0, 1+r']$.

In what follows we study the uniform convergence of $L_{n,q}^{r,r'}(f;x)$ to $f(x)$ in $x \in [0,1]$ as $n \to \infty$. The rate of convergence will be estimated with the aid of the modulus of continuity of $f \in C[0,1+r']$ defined by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1+r'], |x-y| \le \delta\},\ \delta > 0. \tag{1.4}$$

For particular cases of $a(n,k,r,q)$, $b(n,k,r,q)$, $u(n,k,q)$ and $v(n,k,q)$ we recover some known operators (see [1, 2, 5, 9, 11, 13–15]), and we may introduce some new integral type operators. As applications we give quantitative estimations for these operators using (1.4).

2. Main results

Theorem 1. *Let $L_{n,q_n}^{r,r'}(f;x)$ be defined by (1.3), where $n \ge 2$ and $q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$. If there exist $C_1 = C_1(r,r') > 0$ and $C_2 = C_2(r,r') > 0$ such that*

$$\begin{aligned}
(i)\ &\Big| u(n,k,q_n) + v(n,k,q_n)\{a(n,k,r,q_n) + \tfrac{1}{1+q_n}(b(n,k,r,q_n) \\
&\quad - a(n,k,r,q_n))\} - \tfrac{[k]_{q_n}}{[n+r']_{q_n}} \Big| \le \tfrac{C_1}{[n]_{q_n}}, \\
(ii)\ &\Big| u^2(n,k,q_n) + 2u(n,k,q_n)v(n,k,q_n)\{a(n,k,r,q_n) + \tfrac{1}{1+q_n}(b(n,k,r,q_n) \\
&\quad - a(n,k,r,q_n))\} + v^2(n,k,q_n)\{a^2(n,k,r,q_n) + 2a(n,k,r,q_n)(b(n,k,r,q_n)
\end{aligned}$$

$$-a(n,k,r,q_n))\frac{1}{1+q_n}+(b(n,k,r,q_n)-a(n,k,r,q_n))^2\frac{1}{1+q_n+q_n^2}\}$$
$$-\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}\Big|\le\frac{C_2}{[n]_{q_n}}$$

hold true for all $k\in\{0,1,\dots,n+r'\}$, *then* $L_{n,q_n}^{r,r'}(f;x)$ *converges uniformly to* $f(x)$ *with respect* $x\in[0,1]$ *as* $n\to\infty$.

Proof. In view of Korovkin's theorem we have to prove that $L_{n,q_n}^{r,r'}(e_i;x)$ converges uniformly to $e_i(x)$ in $x\in[0,1]$ as $n\to\infty$, where $e_i(x)=x^i$, $x\in[0,1]$ and $i\in\{0,1,2\}$.
We denote the n-th order q-Bernstein polynomial associated to f by

$$B_{n,q}(f;x)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q x^k(1-x)_q^{n-k}f\left(\frac{[k]_q}{[n]_q}\right).$$

Then, (1.3), (1.2) and [14, (13)] imply that

$$\begin{aligned}L_{n,q_n}^{r,r'}(e_0;x)&=\sum_{k=0}^{n+r'}\begin{bmatrix} n+r'\\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\\&=B_{n+r',q_n}(e_0;x)=e_0(x).\end{aligned}\tag{2.1}$$

Further, using (1.2), we have

$$\begin{aligned}&\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}e_1(u(n,k,q_n)+tv(n,k,q_n))\,d_{q_n}^R t\\&=\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)\}\,d_{q_n}^R t=u(n,k,q_n)\{b(n,k,r,q_n)\\&\quad-a(n,k,r,q_n)\}+v(n,k,q_n)a(n,k,r,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}\\&\quad+v(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}^2\frac{1}{1+q_n}.\end{aligned}\tag{2.2}$$

But

$$B_{n+r',q_n}(e_1;x)=\sum_{k=0}^{n+r'}\begin{bmatrix} n+r'\\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\frac{[k]_{q_n}}{[n+r']_{q_n}}=x\tag{2.3}$$

(see [14, (14)]), therefore, by (1.3), (2.2), (i) and (2.1), we have

$$\begin{aligned}&|L_{n,q_n}^{r,r'}(e_1;x)-e_1(x)|\\&\le\sum_{k=0}^{n+r'}\begin{bmatrix} n+r'\\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\Big|u(n,k,q_n)+v(n,k,q_n)\\&\quad\times\left\{a(n,k,r,q_n)+\frac{1}{1+q_n}(b(n,k,r,q_n)-a(n,k,r,q_n))\right\}-\frac{[k]_{q_n}}{[n+r']_{q_n}}\Big|\end{aligned}$$

$$\leq \frac{C_1}{[n]_{q_n}}. \tag{2.4}$$

If $q_n \to 1$ as $n \to \infty$ then $[n]_{q_n} \to \infty$ as $n \to \infty$. Thus, by (2.4), $L^{r,r'}_{n,q_n}(e_1;x)$ converges uniformly to $e_1(x)$ in $x \in [0,1]$ as $n \to \infty$.

Analogously, using (1.2), we have

$$\begin{aligned}
&\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)} e_2(u(n,k,q_n)+tv(n,k,q_n))\, d^R_{q_n}t \\
&= \int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)} \{u^2(n,k,q_n)+2u(n,k,q_n)v(n,k,q_n)t+v^2(n,k,q_n)t^2\}\, d^R_{q_n}t \\
&= u^2(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}+2u(n,k,q_n)v(n,k,q_n)(1-q) \\
&\quad \times \{b(n,k,r,q_n)-a(n,k,r,q_n)\}\sum_{j=0}^{\infty} q^j\{a(n,k,r,q_n)+(b(n,k,r,q_n) \\
&\quad -a(n,k,r,q_n))q^j\}+v^2(n,k,q_n)(1-q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\} \\
&\quad \times \sum_{j=0}^{\infty} q^j\{a(n,k,r,q_n)+2a(n,k,r,q_n)(b(n,k,r,q_n)-a(n,k,r,q_n))q^j \\
&\quad +(b(n,k,r,q_n)-a(n,k,r,q_n))^2q^{2j}\} \\
&= u^2(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}+2u(n,k,q_n)v(n,k,q_n) \\
&\quad \times \{b(n,k,r,q_n)-a(n,k,r,q_n)\}\left\{a(n,k,r,q_n)+\frac{1}{1+q_n}(b(n,k,r,q_n)\right. \\
&\quad \left.-a(n,k,r,q_n))\right\}+v^2(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\} \\
&\quad \times \left\{a^2(n,k,r,q_n)+2a(n,k,r,q_n)\frac{1}{1+q_n}(b(n,k,r,q_n)-a(n,k,r,q_n))\right. \\
&\quad \left.+\frac{1}{1+q_n+q_n^2}(b(n,k,r,q_n)-a(n,k,r,q_n))^2\right\}.
\end{aligned} \tag{2.5}$$

Because

$$\sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_{q_n} x^k(1-x)^{n+r'-k}_{q_n} \frac{[k]_{q_n}}{[n+r']_{q_n}} \frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}} = x^2, \tag{2.6}$$

we get, in view of (1.3), (2.5), (ii) and (2.6), that

$$|L^{r,r'}_{n,q_n}(e_2;x)-e_2(x)| \leq \sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_{q_n} x^k(1-x)^{n+r'-k}_{q_n} \left| u^2(n,k,q_n) \right.$$

$$+2u(n,k,q_n)v(n,k,q_n)\Big\{a(n,k,r,q_n)+\frac{1}{1+q_n}(b(n,k,r,q_n)$$
$$-a(n,k,r,q_n))\}+v^2(n,k,q_n)\{a^2(n,k,r,q_n)+2a(n,k,r,q_n)(b(n,k,r,q_n)$$
$$-a(n,k,r,q_n))\frac{1}{1+q_n}+(b(n,k,r,q_n)-a(n,k,r,q_n))^2\frac{1}{1+q_n+q_n^2}\Big\}$$
$$-\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}\Big|\le\frac{C_2}{[n]_{q_n}}.$$

Hence, because of $[n]_{q_n}\to\infty$ as $n\to\infty$, we find that $L_{n,q_n}^{r,r'}(e_2;x)$ converges uniformly to $e_2(x)$ in $x\in[0,1]$ as $n\to\infty$. This completes the proof of the theorem. □

Theorem 2. *Let $L_{n,q_n}^{r,r'}(f;x)$ and q_n, $n\ge 2$, be defined as in Theorem 1 satisfying the conditions (i) and (ii). Then*

$$|L_{n,q_n}^{r,r'}(f;x)-f(x)|\le\{1+\sqrt{2C_1+C_2}\}\omega(f;[n]_{q_n}^{-1/2})$$

for all $f\in C[0,1+r']$ and $x\in[0,1]$.

Proof. Taking into account (1.4), we have

$$|f(u(n,k,q_n)+tv(n,k,q_n))-f(x)|$$
$$\le\omega(f;|u(n,k,q_n)+tv(n,k,q_n)-x|)$$
$$\le\{1+\delta^{-1}|u(n,k,q_n)+tv(n,k,q_n)-x|\}\omega(f;\delta).$$

Hence, by (2.1) and Hölder's inequality,

$$|L_{n,q_n}^{r,r'}(f;x)-f(x)|$$
$$\le\sum_{k=0}^{n+r'}\begin{bmatrix}n+r'\\k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}$$
$$\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}|f(u(n,k,q_n)+tv(n,k,q_n))-f(x)|\,d_{q_n}^Rt$$
$$\le\omega(f;\delta)\sum_{k=0}^{n+r'}\begin{bmatrix}n+r'\\k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}$$
$$\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{1+\delta^{-1}|u(n,k,q_n)+tv(n,k,q_n)-x|\}\,d_{q_n}^Rt$$
$$\le\omega(f;\delta)\Bigg\{1+\delta^{-1}\sum_{k=0}^{n+r'}\begin{bmatrix}n+r'\\k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}$$

$$\times\left(\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\right.$$
$$\left.\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}(u(n,k,q_n)+tv(n,k,q_n)-x)^2\,d_{q_n}^R t\right)^{1/2}\Bigg\}$$
$$\le\omega(f;\delta)\left\{1+\delta^{-1}\left(\sum_{k=0}^{n+r'}\begin{bmatrix} n+r' \\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\right.\right.$$
$$\times\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}$$
$$\left.\left.\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}(u(n,k,q_n)+tv(n,k,q_n)-x)^2\,d_{q_n}^R t\right)^{1/2}\right\}. \tag{2.7}$$

On the other hand, by (2.2), (i), (2.5) and (ii), we have

$$\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)-x\}^2\,d_{q_n}^R t$$
$$=\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)\}^2\,d_{q_n}^R t$$
$$-\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-\frac{2x}{b(n,k,r,q_n)-a(n,k,r,q_n)}$$
$$\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)\}\,d_{q_n}^R t+2x\frac{[k]_{q_n}}{[n+r']_{q_n}}$$
$$+\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-2x\frac{[k]_{q_n}}{[n+r']_{q_n}}+x^2$$
$$\le\frac{C_2}{[n]_{q_n}}+\frac{2C_1}{[n]_{q_n}}+\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-2x\frac{[k]_{q_n}}{[n+r']_{q_n}}+x^2.$$

Hence, in view of (2.7), (2.3), (2.6) and (2.1),

$$|L_{n,q_n}^{r,r'}(f;x)-f(x)|$$
$$\le\omega(f;\delta)\left\{1+\delta^{-1}\left(\frac{C_2}{[n]_{q_n}}+\frac{2C_1}{[n]_{q_n}}+\sum_{k=0}^{n+r'}\begin{bmatrix} n+r' \\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\right.\right.$$
$$\left.\left.\times\left(\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-2x\frac{[k]_{q_n}}{[n+r']_{q_n}}+x^2\right)\right)^{1/2}\right\}$$

$$= \omega(f;\delta)\left\{1+\delta^{-1}\left(\frac{2C_1+C_2}{[n]_{q_n}}\right)^{1/2}\right\}.$$

Choosing $\delta = [n]_{q_n}^{-1/2}$, we arrive at the required estimate. □

3. Applications

In this section we apply the main results for some q-Bernstein type operators and q-Bernstein type integral operators. This means that we have to verify the conditions of Theorem 1, and apply Theorem 2 to obtain quantitative estimates for these operators.

1) In [2] the following operators are introduced:

$$S_{n,p}^{(\alpha,\beta)}(f;q,x) = \sum_{k=0}^{n+p} \left[\begin{array}{c} n+p \\ k \end{array}\right]_q x^k (1-x)_q^{n+p-k} f\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right),$$

where $f \in C[0,1+p]$, $0<q<1$, $0 \le \alpha \le \beta$, $x \in [0,1]$. Using the notations $r' = p$, $a(n,k,r,q) = 0$, $b(n,k,r,q) = 1$, $u(n,k,q) = \frac{[k]_q+\alpha}{[n]_q+\beta}$, $v(n,k,q) = 0$ for $k = 0,1,\dots,n+p$, we obtain $L_{n,q}^{r,r'}(f;x) = S_{n,p}^{(\alpha,\beta)}(f;q,x)$. Moreover, if $\alpha = \beta = 0$ then we recover the q-Bernstein-Schurer operators

$$\tilde{B}_{n,p}(f;q,x) = \sum_{k=0}^{n+p} \left[\begin{array}{c} n+p \\ k \end{array}\right]_q x^k (1-x)_q^{n+p-k} f\left(\frac{[k]_q}{[n]_q}\right)$$

(see [11]), while for $\alpha = \beta = p = 0$ we recover the q-Bernstein operators introduced by Phillips [14].

Now, we verify the conditions of Theorem 1:

(i) for $k = 0,1,\dots,n+p$ we have

$$\begin{aligned}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} - \frac{[k]_{q_n}}{[n+p]_{q_n}}\right| &= \frac{|[k]_{q_n}([n+p]_{q_n}-[n]_{q_n})+\alpha[n+p]_{q_n}-\beta[k]_{q_n}|}{[n+p]_{q_n}([n]_{q_n}+\beta)} \\ &\le \frac{[n+p]_{q_n}q_n^n[p]_{q_n}+(\alpha+\beta)[n+p]_{q_n}}{[n+p]_{q_n}([n]_{q_n}+\beta)} \\ &\le \frac{\alpha+\beta+p}{[n]_{q_n}};\end{aligned}$$

(ii) by (i) and

$$\frac{[n]_{q_n}}{[n-1]_{q_n}} = \frac{1+q_n[n-1]_{q_n}}{[n-1]_{q_n}} = \frac{1}{[n-1]_{q_n}} + q_n \le 2 \tag{3.1}$$

for $n \ge 2$, we get for all $k = 0,1,\dots,n+p$ that

$$\left|\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}\right)^2 - \frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}}\right|$$

$$
\begin{aligned}
&\leq \left| \left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} \right)^2 - \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 \right| \\
&\quad + \left| \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 - \frac{[k]_{q_n}}{[n+p]_{q_n}} \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| \\
&\leq \left\{ \frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} + \frac{[k]_{q_n}}{[n+p]_{q_n}} \right\} \left| \frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \right| \\
&\quad + \left| \frac{[k]_{q_n}+\alpha}{[n+p]_{q_n}+\beta} - \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| \\
&\leq \left\{ \frac{[n+p]_{q_n}+\alpha}{[n]_{q_n}+\beta} + 1 \right\} \frac{\alpha+\beta+p}{[n]_{q_n}} + \frac{q_n^{k-1}[n+p-k]_{q_n}}{[n+p]_{q_n}[n+p-1]_{q_n}} \\
&\leq \left\{ \frac{[n]_{q_n}+q_n^n[p]_{q_n}+\alpha}{[n]_{q_n}+\beta} + 1 \right\} \frac{\alpha+\beta+p}{[n]_{q_n}} + \frac{1}{[n+p]_{q_n}} \\
&\leq \frac{(\alpha+p+2)(\alpha+\beta+p)}{[n]_{q_n}} + \frac{1}{[n]_{q_n}} \frac{[n]_{q_n}}{[n+p-1]_{q_n}} \\
&\leq \frac{(\alpha+p+2)(\alpha+\beta+p)}{[n]_{q_n}} + \frac{1}{[n]_{q_n}} \frac{[n]_{q_n}}{[n-1]_{q_n}} \\
&\leq \{(\alpha+p+2)(\alpha+\beta+p)+2\} \frac{1}{[n]_{q_n}}.
\end{aligned}
$$

Then, by Theorem 2 with $C_1 = \alpha+\beta+p$ and $C_2 = (\alpha+p+2)(\alpha+\beta+p)+2$, we have

$$
|S_{n,p}^{(\alpha,\beta)}(f;q,x) - f(x)| \leq \{1+\sqrt{(\alpha+\beta+p)(\alpha+p+4)+2}\}\,\omega(f;[n]_{q_n}^{-1/2}).
$$

2) The following q-Kantorovich type operators are introduced in [13]:

$$
\begin{aligned}
&K_n^p(f;q,x) \\
&= \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k} \int_0^1 f\left(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t \right) d_q t,
\end{aligned}
$$

where $f \in C[0,1+p]$, $0<q<1$, $x \in [0,1]$. Using the notations $r' = p$, $a(n,k,r,q) = 0$, $b(n,k,r,q) = 1$, $u(n,k,q) = \frac{[k]_q}{[n+1]_q}$, $v(n,k,q) = \frac{1+(q-1)[k]_q}{[n+1]_q}$ for $k = 0,1,\ldots,n+p$, we obtain $L_{n,q}^{r,r'}(f;x) = K_n^p(f;q,x)$. For $p=0$ we recover the operators $B_{n,q}^*(f,x)$ of [9].

We verify the conditions of Theorem 1:

(i) for $k=0,1,\dots,n+p$ we have

$$\begin{aligned}
&\left|\frac{[k]_{q_n}}{[n+1]_{q_n}}+\frac{1+(q_n-1)[k]_{q_n}}{[n+1]_{q_n}}\frac{1}{1+q_n}-\frac{[k]_{q_n}}{[n+p]_{q_n}}\right| \\
&\le \frac{[k]_{q_n}|[n+p]_{q_n}-[n+1]_{q_n}|}{[n+p]_{q_n}[n+1]_{q_n}}+\frac{1}{[n]_{q_n}} \\
&\le \frac{|[n]_{q_n}+q_n^n[p]_{q_n}-[n]_{q_n}-q_n^n|}{[n+1]_{q_n}}+\frac{1}{[n]_{q_n}}\le\frac{p+2}{[n]_{q_n}};
\end{aligned}$$

(ii) as in (ii) of case **1),** we may write for $k=0,1,\dots,n+p$ that

$$\begin{aligned}
&\left|\left(\frac{[k]_{q_n}}{[n+1]_{q_n}}\right)^2+2\frac{[k]_{q_n}}{[n+1]_{q_n}}\frac{1+(q_n-1)[k]_{q_n}}{[n+1]_{q_n}}\frac{1}{1+q_n}+\left(\frac{1+(q_n-1)[k]_{q_n}}{[n+1]_{q_n}}\right)^2\right. \\
&\quad\left.\times\frac{1}{1+q_n+q_n^2}-\frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}}\right| \\
&\le\left|\left(\frac{[k]_{q_n}}{[n+1]_{q_n}}\right)^2-\left(\frac{[k]_{q_n}}{[n+p]_{q_n}}\right)^2+\left(\frac{[k]_{q_n}}{[n+p]_{q_n}}\right)^2-\frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}}\right| \\
&\quad+\frac{2}{[n]_{q_n}}\frac{[n+p]_{q_n}}{[n]_{q_n}}+\frac{1}{[n]_{q_n}} \\
&\le\left\{\frac{[k]_{q_n}}{[n+1]_{q_n}}+\frac{[k]_{q_n}}{[n+p]_{q_n}}\right\}\frac{[k]_{q_n}|[n+p]_{q_n}-[n+1]_{q_n}|}{[n+p]_{q_n}[n+1]_{q_n}} \\
&\quad+\left|\frac{[k]_{q_n}}{[n+p]_{q_n}}-\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}}\right|+\frac{2}{[n]_{q_n}}\frac{[n]_{q_n}+q_n^n[p]_{q_n}}{[n]_{q_n}}+\frac{1}{[n]_{q_n}} \\
&\le\left\{\frac{[n+p]_{q_n}}{[n+1]_{q_n}}+1\right\}\frac{p+1}{[n]_{q_n}}+\frac{2}{[n]_{q_n}}+\frac{2(p+1)}{[n]_{q_n}}+\frac{1}{[n]_{q_n}} \\
&\le\left\{\frac{[n]_{q_n}+q_n^n[p]_{q_n}}{[n]_{q_n}}+1\right\}\frac{p+1}{[n]_{q_n}}+\frac{2p+5}{[n]_{q_n}} \\
&\le\frac{(p+2)(p+1)}{[n]_{q_n}}+\frac{2p+5}{[n]_{q_n}}=\frac{p^2+5p+7}{[n]_{q_n}}.
\end{aligned}$$

For $C_1=p+2$ and $C_2=p^2+5p+7$, we have, by Theorem 2,

$$|K_n^p(f;q,x)-f(x)|\le\{1+\sqrt{p^2+7p+11}\}\omega(f;[n]_{q_n}^{-1/2}).$$

3) Recently, in [1] the following operators were introduced:

$$K_{n,p}(f;q,x) = [n+1]_q \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k} q^{-k} \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t)\, d_q^R t,$$

where $f \in C[0,1+p]$, $0<q<1$, $x \in [0,1]$. When $p=0$ we recover the operators $B_n^*(f;q,x)$ studied in [5]. Using the notations $r'=p$, $a(n,k,r,q)=\frac{[k]_q}{[n+1]_q}$, $b(n,k,r,q)=\frac{[k+1]_q}{[n+1]_q}$, $u(n,k,q)=0$, $v(n,k,q)=1$ for $k=0,1,\ldots,n+p$, we obtain $L_{n,q}^{r,r'}(f;x)=K_{n,p}(f;q,x)$.

Analogously to the previous two cases, and taking into account Theorem 2, we find that

$$|K_{n,p}(f;q,x)-f(x)| \le \{1+\sqrt{p^2+7p+11}\}\,\omega(f;[n]_{q_n}^{-1/2}).$$

In what follows we introduce some new q-Bernstein type integral operators.

4) For $f \in C[0,1]$, $0<q<1$, $x \in [0,1]$, $p \in \{0,1,2,\ldots\}$, we consider the operators

$$U_n^p(f;q,x) = [n+p+1]_q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k} q^{-p-k} \int_{[k+p]_q/[n+p+1]_q}^{[k+p+1]_q/[n+p+1]_q} f(t)\, d_q^R t,$$

where $f \in C[0,1+p]$, $0<q<1$, $x \in [0,1]$. Using the notations $r=p$, $r'=0$, $a(n,k,r,q)=\frac{[k+p]_q}{[n+p+1]_q}$, $b(n,k,r,q)=\frac{[k+p+1]_q}{[n+p+1]_q}$, $u(n,k,q)=0$, $v(n,k,q)=1$ for $k=0,1,\ldots,n$, we have $L_{n,q}^{r,r'}(f;x)=U_n^p(f;q,x)$.

We verify the conditions of Theorem 1 as follows:

(i) for $k=0,1,\ldots,n$ we have

$$\begin{aligned}
&\left| \frac{[k+p]_{q_n}}{[n+p+1]_{q_n}} + \frac{1}{1+q_n}\frac{[k+p+1]_{q_n}-[k+p]_{q_n}}{[n+p+1]_{q_n}} - \frac{[k]_{q_n}}{[n]_{q_n}} \right| \\
&\le \left| \frac{[k+p]_{q_n}}{[n+p+1]_{q_n}} - \frac{[k]_{q_n}}{[n]_{q_n}} \right| + \frac{1}{[n]_{q_n}} = \left| \frac{[k]_{q_n}+q_n^k[p]_{q_n}}{[n]_{q_n}+q_n^n[p+1]_{q_n}} - \frac{[k]_{q_n}}{[n]_{q_n}} \right| + \frac{1}{[n]_{q_n}} \\
&= \frac{|q_n^k[p]_{q_n}[n]_{q_n} - q_n^n[p+1]_{q_n}[k]_{q_n}|}{[n]_{q_n}[n+p+1]_{q_n}} + \frac{1}{[n]_{q_n}} \\
&\le \frac{p[n]_{q_n}+(p+1)[n]_{q_n}}{[n]_{q_n}[n+p+1]_{q_n}} + \frac{1}{[n]_{q_n}} \le \frac{2p+2}{[n]_{q_n}};
\end{aligned}$$

$(i\,i)$ using (3.1), we have for $k=0,1,\dots,n$ that

$$\begin{aligned}
&\left|\left(\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}\right)^2+2\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}\frac{[k+p+1]_{q_n}-[k+p]_{q_n}}{[n+p+1]_{q_n}}\frac{1}{1+q_n}\right.\\
&\quad\left.+\left(\frac{[k+p+1]_{q_n}-[k+p]_{q_n}}{[n+p+1]_{q_n}}\right)^2\frac{1}{1+q_n+q_n^2}-\frac{[k]_{q_n}}{[n]_{q_n}}\frac{[k-1]_{q_n}}{[n-1]_{q_n}}\right|\\
&\le\left|\left(\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}\right)^2-\left(\frac{[k]_{q_n}}{[n]_{q_n}}\right)^2\right|+\left|\left(\frac{[k]_{q_n}}{[n]_{q_n}}\right)^2-\frac{[k]_{q_n}}{[n]_{q_n}}\frac{[k-1]_{q_n}}{[n-1]_{q_n}}\right|\\
&\quad+\frac{2}{[n]_{q_n}}+\frac{1}{[n]_{q_n}}\\
&\le\left\{\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}+\frac{[k]_{q_n}}{[n]_{q_n}}\right\}+\left|\frac{[k+p]_{q_n}}{[n]_{q_n}}-\frac{[k-1]_{q_n}}{[n-1]_{q_n}}\right|+\frac{3}{[n]_{q_n}}\\
&\le 2\frac{2p+1}{[n]_{q_n}}+\frac{q_n^{k-1}[n-k]_{q_n}}{[n]_{q_n}[n-1]_{q_n}}+\frac{3}{[n]_{q_n}}\le\frac{4p+2}{[n]_{q_n}}+\frac{[n]_{q_n}}{[n-1]_{q_n}}\frac{1}{[n]_{q_n}}+\frac{3}{[n]_{q_n}}\\
&\le\frac{4p+7}{[n]_{q_n}}.
\end{aligned}$$

Hence, by Theorem 2,

$$|U_n^p(f;q,x)-f(x)|\le\{1+\sqrt{8p+11}\}\,\omega(f;[n]_{q_n}^{-1/2}).$$

5) For $f\in C[0,1+p]$, $p\in\{0,1,2,\dots\}$, $0<q<1$, $x\in[0,1]$, we introduce the operators

$$V_n^p(f;q,x)=(1-x)_q^{n+p}f(0)+\frac{1}{2}[n]_q^2\sum_{k=1}^{n+p-1}\begin{bmatrix}n+p\\k\end{bmatrix}_q x^k(1-x)_q^{n+p-k}$$
$$\times\int_{\frac{[k]_q}{[n]_q}-\frac{1}{[n]_q^2}}^{\frac{[k]_q}{[n]_q}+\frac{1}{[n]_q^2}}f(t)\,d_q^R t+x^{n+p}f\left(\frac{[n+p]_q}{[n]_q}\right).$$

We set $r'=p$,

$$\begin{aligned}
&a(n,0,r,q)=0,\quad b(n,0,r,q)=1,\quad u(n,0,q)=0,\quad v(n,0,q)=0;\\
&a(n,k,r,q)=\frac{[k]_q}{[n]_q}-\frac{1}{[n]_q^2},\quad b(n,k,r,q)=\frac{[k]_q}{[n]_q}+\frac{1}{[n]_q^2},\\
&u(n,k,q)=0,\quad v(n,k,q)=1,\text{for } k=1,2,\dots,n+p-1;\\
&a(n,n+p,r,q)=0,\quad b(n,n+p,r,q)=1,\\
&u(n,n+p,q)=\frac{[n+p]_q}{[n]_q},\quad v(n,n+p,q)=0.
\end{aligned}$$

Then $L_{n,q}^{r,r'}(f;x) = V_n^p(f;q,x)$. In what follows, we verify the conditions of Theorem 1.

(i) for $k=0$ it is obvious; for $k=1,2,\ldots,n+p-1$ we have

$$\left| \frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} + \frac{2}{[n]_q^2}\frac{1}{1+q_n} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \right| \leq \frac{[k]_{q_n}([n+p]_{q_n}-[n]_{q_n})}{[n]_{q_n}[n+p]_{q_n}} + \frac{3}{[n]_{q_n}} \leq \frac{p+3}{[n]_{q_n}},$$

and for $k=n+p$ we have

$$\left| \frac{[n+p]_{q_n}}{[n]_{q_n}} - \frac{[n+p]_{q_n}}{[n+p]_{q_n}} \right| = \frac{[n+p]_{q_n}-[n]_{q_n}}{[n]_{q_n}} = \frac{q_n^n[p]_{q_n}}{[n]_{q_n}} \leq \frac{p}{[n]_{q_n}};$$

(ii) for $k=0$ it is obvious; for $k=1,2,\ldots,n+p-1$ we have

$$\begin{aligned}
&\left| \left(\frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} \right)^2 + 2\left(\frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} \right)\frac{2}{[n]_q^2}\frac{1}{1+q_n} \right. \\
&\quad \left. + \frac{4}{[n]_q^2}\frac{1}{1+q_n+q_n^2} - \frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| \\
&\leq \left| \left(\frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} \right)^2 - \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 \right| \\
&\quad + \left| \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 - \frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| \\
&\quad + \frac{4}{[n]_{q_n}^2}\frac{[n+p]_{q_n}}{[n]_{q_n}} + \frac{4}{[n]_{q_n}^2} \left| \frac{1}{1+q_n+q_n^2} - \frac{1}{1+q_n} \right| \\
&\leq \left| \frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} + \frac{[k]_{q_n}}{[n+p]_{q_n}} \right| \left| \frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \right| \\
&\quad + \left| \frac{[k]_{q_n}}{[n+p]_{q_n}} - \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| + \frac{4(p+1)}{[n]_{q_n}} + \frac{4}{[n]_{q_n}} \\
&\leq \left\{ \frac{[n+p]_{q_n}}{[n]_{q_n}} + 1 + 1 \right\} \left\{ \frac{[k]_{q_n}([n+p]_{q_n}-[n]_{q_n})}{[n+p]_{q_n}[n]_{q_n}} + \frac{1}{[n]_{q_n}} \right\} \\
&\quad + \frac{q_n^{k-1}[n+p-k]_{q_n}}{[n+p]_{q_n}[n+p-1]_{q_n}} + \frac{4(p+2)}{[n]_{q_n}} \\
&\leq (p+2)\frac{p+1}{[n]_{q_n}} + \frac{[n]_{q_n}}{[n-1]_{q_n}}\frac{1}{[n]_{q_n}} + \frac{4(p+2)}{[n]_{q_n}} \leq \frac{p^2+7p+12}{[n]_{q_n}}
\end{aligned}$$

(see (3.1)); for $k = n+p$ we have

$$\left| \left(\frac{[n+p]_{q_n}}{[n]_{q_n}} \right)^2 - \frac{[n+p]_{q_n}}{[n+p]_{q_n}} \frac{[n+p-1]_{q_n}}{[n+p-1]_{q_n}} \right|$$
$$= \frac{[n+p]_{q_n} + [n]_{q_n}}{[n]_{q_n}} \frac{[n+p]_{q_n} - [n]_{q_n}}{[n]_{q_n}}$$
$$= \frac{2[n]_{q_n} + q_n^n [p]_{q_n}}{[n]_{q_n}} \frac{q_n^n [p]_{q_n}}{[n]_{q_n}} \leq (p+2) \frac{p}{[n]_{q_n}} = \frac{p^2+2p}{[n]_{q_n}}.$$

Choosing $C_1 = p+3$ and $C_2 = p^2+7p+12$, we have, in view of Theorem 2, that

$$|V_n^p(f;q,x) - f(x)| \leq \{1 + \sqrt{p^2+9p+18}\}\, \omega(f;[n]_{q_n}^{-1/2}).$$

6) Analogously, for

$$W_n^p(f;q,x)$$
$$= ([n]_q + \beta) \sum_{k=0}^{n+p} \left[\begin{matrix} n+p \\ k \end{matrix} \right]_q x^k (1-x)_q^{n+p-k} q^{-k} \int_{\frac{[k]_q+\alpha}{[n]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n]_q+\beta}} f(t)\, d_q^R t,$$

where $f \in C[0,1+p]$, $p \in \{0,1,2,\ldots\}$, $0 < q < 1$, $x \in [0,1]$, $0 \leq \alpha \leq \beta$, we have

$$|W_n^p(f;q,x) - f(x)| \leq \{1 + \sqrt{c(p,\alpha,\beta)}\}\, \omega(f;[n]_{q_n}^{-1/2}),$$

where $c(p,\alpha,\beta) = (\alpha+p+1)(\alpha+1) + (\alpha+\beta+p+4)(\beta+1)(p+1) + 3\alpha + 2$.

7) Finally, we consider the operators

$$\bar{K}_n^p(f;q,x) = [n+1]_q \sum_{k=0}^{n+p} \left[\begin{matrix} n+p \\ k \end{matrix} \right]_q x^k (1-x)_q^{n+p-k} \int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)\, d_q^R t,$$

where $f \in C[0,1]$, $0 < q < 1$, $x \in [0,1]$. For $p = 0$ we recover the operators studied in [15]. Then, by Theorem 2, we have

$$|\bar{K}_n^p(f;q,x) - f(x)| \leq \{1 + \sqrt{p^2+7p+11}\}\, \omega(f;[n]_{q_n}^{-1/2}).$$

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ON GENERALIZED STATISTICAL CONVERGENCE OF SEQUENCES OF SETS OF ORDER α

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Abstract. In this paper we define generalized statistical convergence for sequences of sets of order α, $0 < \alpha \leq 1$ in sense of Wijsman and study some basic properties of this concept.

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Keywords: statistical convergence, λ-sequence, Wijsman convergence, Orlicz function

1. INTRODUCTION

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. We shall define Wijsman λ-statistical convergence for sequences of sets of order α, $(0 < \alpha \leq 1)$ and establish some basic results regarding the notions Wijsman λ-statistical convergence and Wijsman statistical convergence for sequences of sets of α, $(0 < \alpha \leq 1)$.

The idea of statistical convergence first appeared, under the name of "almost convergence" in the first edition Zygmund [25] of celebrated monograph of Zygmund [26]. Later, this idea was introduced by Fast [8] and Steinhaus [22] and later was introduced by Schoenberg [21], also independently by Buck [1] and studied various authors (see [5, 9, 20]). Mursaleen [17], introduced the notion λ-statistical convergence for real sequences. For more details on λ-statistical convergence we refer to [2] and many others. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers.

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A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.
The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be $(V,\lambda)-$summable to number L (see [15]) if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then $(V,\lambda)-$summability reduces to (C,1)-summability.

Mursaleen [17] defined $\lambda-$statistically convergent sequence as follows. A sequence $x = (x_k)$ is said to be $\lambda-$ statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0.$$

Let S_λ denotes the set of all $\lambda-$statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S.
Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y \in A} \rho(x, y).$$

Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman convergent (see [23, 24]) to A if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case we write $W - \lim A_k = A$. The concepts of Wijsman statistical convergence and boundedness for the sequence (A_k) were given by Nuray and Rhoades [18] as follows. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman statistical convergent to A if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim_k A_k = A$ or $A_k \to A\left(S^W\right)$. The sequence (A_k) is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets denoted by L_∞. For details on Wijsman statistical convergent we refer to [11–14] and many others.

2. WIJSMAN $\lambda-$STATISTICAL CONVERGENCE

In this section, we will define Wijsman $\lambda-$statistical convergence of sequences of sets and will give the relations between Wijsman statistical and Wijsman $\lambda-$statistical convergence of sequences of sets.

Definition 1. Let (X,ρ) be a metric space and $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1}\leq\lambda_n+1, \lambda_1=1, \lambda_n\to\infty$ as $n\to\infty$ and $I_n=[n-\lambda_n+1,n]$. For any non-empty closed subsets $A, A_k\subset X$ $(k\in\mathbb{N})$, we say that the sequence (A_k) is Wijsman $\lambda-$statistical convergent to A if the sequence $(d(x,A_k))$ is $\lambda-$statistically convergent to $d(x,A)$, i.e., for $\varepsilon>0$ and for each $x\in X$

$$\lim_n \frac{1}{\lambda_n}\left|\{k\in I_n:\ |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|=0.$$

In this case, we write $S_\lambda^W-\lim_k A_k=A$ or $A_k\to A\left(B_\lambda^W\right)$.

Example 1. Let $X=\mathbb{R}^2$ and the sequence (A_k) is defined as follows:

$$A_k=\begin{cases}\left\{(x,y):\ x^2+(y-1)^2=k^{-1}\right\}, & \text{if } n-\left[\sqrt{\lambda_n}\right]+1\leq k\leq n,\ k \\ & \text{is square integer} \\ \{(0,0)\}, & \text{otherwise}\end{cases}.$$

Then the sequence (A_k) is Wijsman $\lambda-$statistical convergent to $A=\{(0,0)\}$ since

$$\lim_n \frac{1}{\lambda_n}\left|\{k\in I_n:\ |d(x,A_k)-d(x,\{(0,0)\})|\geq\varepsilon\}\right|=0.$$

But it is not Wijsman convergent.

Definition 2. Let (X,ρ) be a metric space. Then for any non-empty closed subsets $A, A_k\subset X$ $(k\in\mathbb{N})$, we say that the sequence (A_k) is Wijsman $\lambda-$Cesáro summable to A if

$$\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|=0.$$

In this case, we write $w_\lambda^W-\lim_k A_k=A$ or $A_k\to A\left(w_\lambda^W\right)$ and

$$w_\lambda^W=\left\{(A_k):\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|=0\right\}.$$

If $\lambda_n=n$, then Wijsman λ-Cesáro summable becomes Wijsman-Cesáro summable, i.e.

$$w^W=\left\{(A_k):\lim_n \frac{1}{n}\sum_{k\in\mathbb{N}}|d(x,A_k)-d(x,A)|=0\right\}.$$

Theorem 1. *Let (X,ρ) be a metric space and $A, A_k \subset X$ $(k \in \mathbb{N})$ be non-empty closed subsets of X. Then*

(a) $w_\lambda^W \subset S_\lambda^W$ *and the inclusion is proper.*
(b) *Let* $(A_k) \in L_\infty$, *then* $S_\lambda^W \subset w_\lambda^W$.
(c) $S_\lambda^W \cap L_\infty = w_\lambda^W \cap L_\infty$.

Proof. (a) Let $\varepsilon > 0$ and $(A_k) \in w_\lambda^W$. Then we can write

$$\sum_{k\in I_n} |d(x,A_k)-d(x,A)| \geq \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}} |d(x,A_k)-d(x,A)|$$
$$\geq \varepsilon\,|\{k \in I_n :\ |d(x,A_k)-d(x,A)| \geq \varepsilon\}|$$

which gives the result. To show that the inclusion is strict, we define the sequence (A_k) as follows:

$$A_k = \begin{cases} \{k\}, & \text{if } n-\left[\sqrt{\lambda_n}\right]+1 \leq k \leq n; \\ \{0\}, & \text{otherwise} \end{cases}$$

It is clear that $(A_k) \notin L_\infty$ and for $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n :\ |d(x,A_k)-d(x,\{0\})| \geq \varepsilon\}| = \lim_n \frac{1}{\lambda_n}\left[\sqrt{\lambda_n}\right] = 0.$$

So $(A_k) \in S_\lambda^W$, but

$$\lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} |d(x,A_k)-d(x,\{0\})| = \lim_n \frac{1}{\lambda_n} \frac{\left(\left[\sqrt{\lambda_n}\right]\left(\left[\sqrt{\lambda_n}\right]+1\right)\right)}{2} = \frac{1}{2} \neq 0.$$

Therefore $(A_k) \notin w_\lambda^W$. This completes the proof of (a).

(b) Suppose that $(A_k) \in S_\lambda^W$ and $(A_k) \in L_\infty$, say $|d(x,A_k)-d(x,A)| \leq M$ for each $x \in X$ and for all $k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\frac{1}{\lambda_n} \sum_{k\in I_n} |d(x,A_k)-d(x,A)|$$
$$= \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}} |d(x,A_k)-d(x,A)|$$
$$+ \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|<\varepsilon}} |d(x,A_k)-d(x,A)|$$
$$\leq \frac{M}{\lambda_n} |\{k \in I_n :\ |d(x,A_k)-d(x,A)| \geq \varepsilon\}| + \varepsilon$$

from which the result follows.

(c) It follows from (a) and (b). □

If we let $\lambda_n = n$ in Theorem 1, then we have the following corollary.

Corollary 1. *Let (X,ρ) be a metric space and $A, A_k \subset X$ $(k \in \mathbb{N})$ be non-empty closed subsets of X. Then*

(a) $w^W \subset S^W$ *and the inclusion is proper.*
(b) *Let $(A_k) \in L_\infty$, then $S^W \subset w^W$.*
(c) $S^W \cap L_\infty = w^W \cap L_\infty$.

Theorem 2. $S^W \subset S_\lambda^W$ *if and only if* $\liminf \frac{\lambda_n}{n} > 0$.

Proof. Suppose that $\liminf \frac{\lambda_n}{n} > 0$. For given $\varepsilon > 0$, we have

$$\{k \le n : \ |d(x,A_k) - d(x,A)| \ge \varepsilon\} \supset \{k \in I_n : \ |d(x,A_k) - d(x,A)| \ge \varepsilon\}.$$

Therefore

$$\frac{1}{n}|\{k \le n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}| \ge \frac{1}{n}|\{k \in I_n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}|$$
$$\ge \frac{\lambda_n}{n}.\frac{1}{\lambda_n}|\{k \in I_n : \ |d(x,A_k) - d(x,A)| \ge \varepsilon\}|.$$

Taking the limit as $n \to \infty$ and using $\liminf \frac{\lambda_n}{n} > 0$, we get the desired result.

Conversely, suppose that $\liminf_n \frac{\lambda_n}{n} = 0$. Then we can select a subsequence $(n(i))_{i=1}^\infty$ such that

$$\frac{\lambda_{n(i)}}{n(i)} < \frac{1}{i}.$$

We define a sequence (A_k) as follows:

$$A_k = \begin{cases} \{1\}, & \text{if } n(i) - \left[\sqrt{\lambda_{n(i)}}\right] + 1 \le k \le n(i), i = 1,2,3,\ldots; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then (A_k) is Wijsman-statistically convergent, so $(A_k) \in S^W$. But $(A_k) \notin w_\lambda^W$. Therefore Theorem 1 (b) implies that $(A_k) \notin S_\lambda^W$. This completes the proof. □

Theorem 3. $S_\lambda^W \subset s^W$ *if* $\liminf \frac{\lambda_n}{n} = 1$.

Proof. Since $\lim_n \frac{\lambda_n}{n} = 1$, then for $\varepsilon > 0$, we observe that

$$\frac{1}{n}|\{k \le n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}|$$
$$\le \frac{1}{n}|\{k \le n - \lambda_n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}|$$
$$+ \frac{1}{n}|\{k \in I_n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}|$$
$$\le \frac{n - \lambda_n}{n} + \frac{1}{n}|\{k \in I_n : |d(x,A_k - d(x,A)| \ge \varepsilon\}|$$

$$= \frac{n-\lambda_n}{n} + \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : |d(x, A_k - d(x, A)| \geq \varepsilon\}|.$$

This implies that (A_k) Wijsman-statistically convergent, if (A_k) is Wijsman λ-statistically convergent. Thus $S_\lambda^W \subset S^W$. □

Remark 1. Since $\lim_n \frac{\lambda_n}{n} = 1$, this implies that $\liminf_n \frac{\lambda_n}{n} > 0$, then from Theorem 3, we have $S^W \subset S_\lambda^W$. Hence $S_\lambda^W = S^W$.

3. Wijsman λ-statistical Convergence of order α

The notion of order statistical convergence was introduced by Gadjiev and Orhan [10] and after that statistical convergence of order α studied by Çolak [3], λ-statistical convergence of order α studied by Çolak and Bektaş [4], λ-statistical convergence of order α of sequence of functions studied by Et et al., [6, 7] and many authors.

In this section, we define the concept of $S_{\lambda,\alpha}^W$-statistical convergence and establish the relationship of S_λ^W with w_λ^W. Also we introduce the notion of S_λ^W−statistical convergence of order α of real number sequences and obtain some inclusion relations between the set of S^W−statistical convergence of order α.

Definition 3. Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. We denote the family of such sequences $\lambda = (\lambda_n)$ by Λ. For any non-empty closed subsets A, $A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman λ−statistical convergent to A of order α or $S_{\lambda,\alpha}^W$−convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $S_{\lambda,\alpha}^W - \lim A_k = A$ or $A_k \to A(S_{\lambda,\alpha}^W)$. If $\lambda_n = n$ for all $n \in \mathbb{N}$, we obtain the sequence space S_α^W such that

$$S_\alpha^W = \left\{(A_k) : \ \lim_n \frac{1}{n^\alpha} |\{k \in I_n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0\right\}$$

and if $\lambda_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$ we obtain the sequence space S^W such that

$$S^W = \left\{(A_k) : \ \lim_n \frac{1}{n} |\{k \in I_n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0\right\}.$$

Definition 4. Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and $\lambda = (\lambda_n) \in \Lambda$. For any non-empty closed subsets A, $A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman λ−Cesaro summable to A of order α or $w_{\lambda,\alpha}^W$−convergent to A, if for each $x \in X$

$$\lim_n \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $w^W_{\lambda,\alpha} - \lim A_k = A$ or $A_k \to A(w^W_{\lambda,\alpha})$. If $\lambda_n = n$ for all $n \in \mathbb{N}$, we obtain the sequence space w^W_α such that

$$w^W_\alpha = \left\{ (A_k) : \lim_n \frac{1}{n^\alpha} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| = 0 \right\}$$

and if $\lambda_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$ we obtain the sequence space w^W such that

$$w^W = \left\{ (A_k) : \lim_n \frac{1}{n} \sum_k |d(x, A_k) - d(x, A)| = 0 = 0 \right\}.$$

Theorem 4. *Let (X, ρ) be a metric space and $\alpha \in (0, 1]$. If $S^W_{\lambda,\alpha} - \lim A_k = A$ then A is unique.*

Proof. The proof is easy, so omitted. □

Theorem 5. *Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and non-empty closed subsets $A, B,\ A_k, B_k \subset X\ (k \in \mathbb{N})$, then*

(a) *If $S^W_{\lambda,\alpha} - \lim A_k = A$ and $c \in \mathbb{C}$, then $S^W_{\lambda,\alpha} - \lim cA_k = cA$;*

(b) *If $S^W_{\lambda,\alpha} - \lim A_k = A$ and $S^W_{\lambda,\alpha} - \lim B_k = B$, then $S^W_{\lambda,\alpha} - \lim (A_k + B_k) = A + B$.*

Proof. (a) For $c = 0$, the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows form the following inequality

$$\frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(cx, cA_k) - d(cx, cA)| \geq \varepsilon\}| = \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : |d(x, A_k) - d(x, A)| \geq \frac{\varepsilon}{|c|} \right\} \right|.$$

(b) For every $\varepsilon > 0$. The result follows from the following inequality.

$$\begin{aligned} &\frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k + B_k) - d(x, A + B)| \geq \varepsilon\}| \\ &\leq \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : |d(x, A_k) - d(x, A)| \geq \frac{\varepsilon}{2} \right\} \right| \\ &\quad + \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : |d(x, B_k) - d(x, B)| \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

□

If we take $\lambda_n = n$ in the above theorem, then we have

Corollary 2. *Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and non-empty closed subsets $A, B,\ A_k, B_k \subset X\ (k \in \mathbb{N})$, then*

(a) *If* $S_\alpha^W - \lim A_k = A$ *and* $c \in \mathbb{C}$, *then* $S_\alpha^W - \lim cA_k = cA$;
(b) *If* $S_\alpha^W - \lim A_k = A$ *and* $S_\alpha^W - \lim B_k = B$, *then* $S_\alpha^W - \lim (A_k + B_k) = A + B$.

Theorem 6. *If* $0 < \alpha < \beta \leq 1$, *then* $S_{\lambda,\alpha}^W \subset S_{\lambda,\beta}^W$ *and the inclusion is strict.*

Proof. The proof of the result follows form the following inequality.

$$\frac{1}{\lambda_n^\beta} |\{k \in I_n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}|$$
$$\geq \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}|.$$

To prove the inclusion is strict, let λ be given and we consider a sequence (A_k) of non-empty closed subsets of (X,ρ) metric space defined by

$$A_k = \begin{cases} \{(k,k)\}, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n; \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$
$$= \frac{1}{\lambda_n^\beta} |\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\}| \leq \frac{\sqrt{\lambda_n}}{\lambda_n^\beta}$$

Then we have $(A_k) \in S_{\lambda,\beta}^W$ for $\frac{1}{2} < \beta \leq 1$ but $(A_k) \notin S_{\lambda,\alpha}^W$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 7. *If a sequence* (A_k) *of non-empty closed subsets of* (X, ρ) *metric space is* $S_{\lambda,\alpha}^W$*−convergent to A, then it is* S_λ^W*−convergent to A for* $0 < \alpha \leq 1$.

Proof. It is easy, so omitted. □

Theorem 8. *Let* (X, ρ) *be a metric space and* $\alpha \in (0, 1]$. *Then* $S_\alpha^W \subset S_{\lambda,\alpha}^W$ *if*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{n^\alpha} > 0.$$

Proof. If $A_k \to A\left(S_\alpha^W\right)$ then for every $\varepsilon > 0$ and for sufficiently large n we have

$$\frac{1}{n^\alpha} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|$$
$$\geq \frac{1}{n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|$$
$$\geq \frac{\lambda_n^\alpha}{n^\alpha} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|.$$

Taking the limit as $n \to \infty$ and using the given condition, we get $A_k \to A\left(S_{\lambda,\alpha}^W\right)$. This completes the proof of the theorem. □

Corollary 3. *Let* (X, ρ) *be a metric space,* $\lambda = (\lambda_n) \in \Lambda$ *and* $\alpha \in (0, 1]$. *Then* $S_{\lambda,\alpha}^W \subset S^W$.

Theorem 9. *Let $\lambda = (\lambda_n) \in \Lambda$ and $\alpha \in (0,1]$. Then $S^W \subset S^W_{\lambda,\alpha}$ if and only if*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{n} > 0. \tag{3.1}$$

Proof. Let the condition (3.1) holds and $(A_k) \in S^W$. For a given $\varepsilon > 0$ we have

$$\{k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \supset \{k \in I_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}.$$

Then we have

$$\begin{aligned}\frac{1}{n}|\{k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}| &\ge \frac{1}{n}|\{k \in I_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}| \\ &= \frac{\lambda_n^\alpha}{n}\frac{1}{\lambda_n^\alpha}|\{k \in I_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}|.\end{aligned}$$

By taking limit as $n \to \infty$ and from relation (3.1) we have

$$A_k \to A\left(S^W\right) \Rightarrow A_k \to A\left(S^W_{\lambda,\alpha}\right).$$

Next we suppose that

$$\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{n} = 0.$$

Then we can choose a subsequence (n_i) such that $\frac{\lambda_{n_i}^\alpha}{n_i} < \frac{1}{i}$. Define a sequence (A_k) as follows:

$$A_k = \begin{cases} \{(1,1)\}, & \text{if } k \in I_{n_i}; \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$

Then clearly $(A_k) \in S^W$ but $(A_k) \notin S^W_{\lambda,\alpha}$. Since $S^W_{\lambda,\alpha} \subset S^W_\lambda$, we have $(A_k) \notin S^W_{\lambda,\alpha}$, which is a contradiction. Hence the relation (3.1) holds. □

Theorem 10. *Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \le \beta \le 1$. If*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{\mu_n^\beta}, \tag{3.2}$$

then $S^W_{\lambda,\beta} \subseteq S^W_{\lambda,\alpha}$.

Proof. Suppose that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}$ and the condition (3.2) satisfied. Then $I_n \subset J_n$ and so that for $\varepsilon > 0$ we can write

$$\{k \in J_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \supset \{k \in I_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}.$$

Then we have

$$\frac{1}{\mu_n^\beta}|\{k \in J_n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}|$$

$$\geq \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|,$$

for all $n \in \mathbb{N}$, where $J_n = [n - \mu_n + 1, n]$. Taking limit $n \to \infty$ in the last inequality and using (3.2), we have $S^W_{\lambda,\beta} \subseteq S^W_{\lambda,\alpha}$. □

Corollary 4. *Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$. If (3.2) holds, then*

(a) $S^W_{\mu,\alpha} \subseteq S^W_{\lambda,\alpha}$ *for* $0 < \alpha \leq 1$,
(b) $S^W_{\mu} \subseteq S^W_{\lambda,\alpha}$ *for* $0 < \alpha \leq 1$,
(c) $S^W_{\mu} \subseteq S^W_{\lambda}$.

Theorem 11. *Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If*

$$\lim_{n \to \infty} \frac{\mu_n}{\lambda_n^{\beta}} = 1, \tag{3.3}$$

then $S^W_{\lambda,\alpha} \subseteq S^W_{\mu,\beta}$.

Proof. Let $S^W_{\lambda,\alpha} - \lim A_k = A$ and (3.3) be satisfied. Since $I_n \subset J_n$, for $\varepsilon > 0$ we can write

$$\begin{aligned}
&\frac{1}{\mu_n^{\beta}} |\{k \in J_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\
&= \frac{1}{\mu_n^{\beta}} |\{n - \mu_n + 1 \leq k \leq n - \lambda_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\
&\quad + \frac{1}{\mu_n^{\beta}} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\
&\leq \frac{\mu_n - \lambda_n}{\mu_n^{\beta}} + \frac{1}{\mu_n^{\beta}} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\
&\leq \frac{\mu_n - \lambda_n^{\beta}}{\lambda_n^{\beta}} + \frac{1}{\mu_n^{\beta}} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\
&\leq \left(\frac{\mu_n}{\lambda_n^{\beta}} - 1 \right) + \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|.
\end{aligned}$$

Using the relation (3.3) and $S^W_{\lambda,\alpha} - \lim A_k = A$ the right-hand side of the above inequality tends to zero as $n \to \infty$. This implies that $S^W_{\lambda,\alpha} \subseteq S^W_{\mu,\beta}$. □

Corollary 5. *Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$. If (3.3) holds, then*

(a) $S^W_{\lambda,\alpha} \subseteq S^W_{\mu,\alpha}$ *for* $0 < \alpha \leq 1$,

(b) $S_{\lambda}^{W} \subseteq S_{\mu,\alpha}^{W}$ *for* $0 < \alpha \leq 1$,
(c) $S_{\lambda}^{W} \subseteq S_{\mu}^{W}$.

An *Orlicz function* M is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called the *modulus function* and characterized by Ruckle [19]. An Orlicz function M is said to satisfy Δ_2*−condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1. *An Orlicz function satisfies the inequality* $M(\lambda x) \leq \lambda M(x)$ *for all* λ *with* $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

Definition 5. Let (X,ρ) be a metric space. Let M be an Orlicz function, $p = (p_k)$ be a sequence of strictly positive real numbers, $\alpha \in (0,1]$, $\lambda = (\lambda_n)$ be a sequence of positive reals, and for $\rho > 0$, now we define

$$w_{\lambda,\alpha}^{W}[M,p] = \left\{ (A_k) \in X : \lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} \sum_{k\in I_n} \left[M\left(\frac{|d(x,A_k) - d(x,A)|}{\rho} \right) \right]^{p_k} = 0, \right.$$
$$\left. \text{for some } A \text{ and for } x \in X \right\}.$$

If $M(x) = x$ and $p_k = p$ for all $k \in \mathbb{N}$ then we shall write $w_{\lambda,\alpha}^{W}[M,p] = w_{\lambda,\alpha}^{W}(p)$ and if $M(x) = x$ then we shall write $w_{\lambda,\alpha}^{W}[M,p] = w_{\lambda,\alpha}^{W}[p]$.

Theorem 12. *Let* (p_k) *be a bounded and* $0 < \inf_k p_k \leq p_k \leq sup_k p_k = H < \infty$. *Let* $0 < \alpha \leq \beta \leq 1$, M *be an Orlicz function and* $\lambda = (\lambda_n)$ *be a sequence of positive reals, then* $w_{\lambda,\alpha}^{W}[M,p] \subset S_{\lambda,\beta}^{W}$.

Proof. Let $(A_k) \in w^W_{\lambda,\alpha}[M,p]$. Let $\varepsilon > 0$ be given. As $\lambda_n^\alpha \leq \lambda_n^\beta$ for each n we can write

$$\begin{aligned}
&\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\rho}\right)\right]^{p_k}\\
&=\frac{1}{\lambda_n^\alpha}\left[\sum_{\substack{k\in I_n\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\rho}\right)\right]^{p_k}\right.\\
&\quad\left.+\sum_{\substack{k\in I_n\\ |d(x,A_k)-d(x,A)|<\varepsilon}}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\rho}\right)\right]^{p_k}\right]\\
&\geq\frac{1}{\lambda_n^\beta}\left[\sum_{\substack{k\in I_n\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\rho}\right)\right]^{p_k}\right.\\
&\quad\left.+\sum_{\substack{k\in I_n\\ |d(x,A_k)-d(x,A)|<\varepsilon}}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\rho}\right)\right]^{p_k}\right]\\
&\geq\frac{1}{\lambda_n^\beta}\sum_{\substack{k\in I_n\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}\\
&\geq\frac{1}{\lambda_n^\beta}\sum_{\substack{k\in I_n\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}\min\left([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H\right)\\
&\geq\frac{1}{\lambda_n^\beta}|\{k\in I_n:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|\min\left([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H\right),
\end{aligned}$$

where $\varepsilon_1 = \frac{\varepsilon}{\rho}$. From the above inequality we have $(A_k) \in S^W_{\lambda,\alpha}$.

□

Corollary 6. *Let $0 < \alpha \leq 1$, M be an Orlicz function and $\lambda = (\lambda_n)$ be an element of Λ, then $w^W_{\lambda,\alpha}[M,p] \subset S^W_{\lambda,\alpha}$.*

Theorem 13. *Let M be an Orlicz function, (A_k) be a sequence in L_∞, and $\lambda = (\lambda_n)$ be an element of Λ. If $\lim_{n\to\infty}\frac{\lambda_n}{\lambda_n^\alpha} = 1$, then $S^W_{\lambda,\alpha} \subset w^W_{\lambda,\alpha}[M,p]$.*

Proof. Suppose that (A_k) is in L_∞ and $S^W_{\lambda,\alpha} - \lim_k A_k = A$. As $(A_k) \in L_\infty$ there exists $K > 0$ such that $|d(x, A_k) - d(x, A)| \leq K$ for all k. For given $\varepsilon > 0$ we have

$$\begin{aligned}
&\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \\
&= \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x,A_k) - d(x,A)| \geq \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \\
&\quad + \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x,A_k) - d(x,A)| < \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \\
&\leq \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x,A_k) - d(x,A)| \geq \varepsilon}} \max \left\{ \left[M \left(\frac{K}{\rho} \right) \right]^h, \left[M \left(\frac{K}{\rho} \right) \right]^H \right\} \\
&\quad + \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x,A_k) - d(x,A)| < \varepsilon}} \left[M \left(\frac{\varepsilon}{\rho} \right) \right]^{p_k} \\
&\leq \max \left\{ \left[M \left(\frac{K}{\rho} \right) \right]^h, \left[M \left(\frac{K}{\rho} \right) \right]^H \right\} \frac{1}{\lambda_n^\alpha} |d(x, A_k) - d(x, A)| \geq \varepsilon| \\
&\quad + \frac{\lambda_n}{\lambda_n^\alpha} \max \left\{ \left[M \left(\frac{\varepsilon}{\rho} \right) \right]^h, \left[M \left(\frac{\varepsilon}{\rho} \right) \right]^H \right\}.
\end{aligned}$$

Therefore we have $(A_k) \in w^W_{\lambda,\alpha}[M, p]$. □

Theorem 14. *Let $\lambda = (\lambda_n) \in \Lambda, 0 < \alpha \leq \beta \leq 1$, p be a positive real number, then $w^W_{\lambda,\alpha}(p) \subseteq w^W_{\beta,\alpha}(p)$.*

Proof. The proof is easy, so omitted. □

Corollary 7. *Let $\lambda = (\lambda_n) \in \Lambda$ and p be a positive real number, then $w^W_{\lambda,\alpha}(p) \subseteq w^W_\lambda(p)$.*

Theorem 15. *Let $\lambda = (\lambda_n) \in \Lambda, 0 < \alpha \leq \beta \leq 1$ and p be a positive real number, then $w^W_{\lambda,\alpha}(p) \subseteq S^W_{\lambda,\beta}$.*

Proof. Let $(A_k) \in w^W_{\lambda,\alpha}(p)$ and for $\varepsilon > 0$ we have

$$\sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p = \sum_{\substack{k \in I_n \\ |d(x,A_k) - d(x,A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)|^p$$

$$+ \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|<\varepsilon}} |d(x,A_k)-d(x,A)|^p$$

$$\geq \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}} |d(x,A_k)-d(x,A)|^p$$

$$\geq |\{k \in I_n : |d(x,A_k)-d(x,A)| \geq \varepsilon\}|.\varepsilon^p.$$

Therefore we have

$$\frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} |d(x,A_k)-d(x,A)|^p \geq \frac{1}{\lambda_n^\beta} |\{k \in I_n : |d(x,A_k)-d(x,A)| \geq \varepsilon\}|.\varepsilon^p.$$

The last inequality implies that $(A_k) \in S^W_{\lambda,\beta}$ if $(A_k) \in w^W_{\lambda,\alpha}(p)$. This compltes the proof of the theorem. □

Theorem 16. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If (3.3) holds, then $w^W_{\mu,\beta}(p) \subseteq w^W_{\lambda,\alpha}(p)$.*

Proof. The proof is easy, so omitted. □

Corollary 8. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$. If (3.3) holds, then*

(a) $w^W_{\mu,\alpha}(p) \subseteq w^W_{\lambda,\alpha}(p)$ *for* $0 < \alpha \leq 1$,
(b) $w^W_{\mu}(p)) \subseteq w^W_{\lambda}(p)$ *for* $0 < \alpha \leq 1$,
(c) $w^W_{\mu}(p) \subseteq w^W_{\lambda}(p)$.

Theorem 17. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If (3.3) holds, then $w^W_{\mu,\beta}(p) \subseteq S^W_{\lambda,\alpha}$.*

Proof. Let $(A_k) \in w^W_{\mu,\beta}(p)$ and for $\varepsilon > 0$. Then we have

$$\sum_{k\in I_n} |d(x,A_k)-d(x,A)|^p = \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}} |d(x,A_k)-d(x,A)|^p$$

$$+ \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|<\varepsilon}} |d(x,A_k)-d(x,A)|^p$$

$$\geq \sum_{\substack{k\in I_n \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}} |d(x,A_k)-d(x,A)|^p$$

$$\geq |\{k \in I_n : |d(x,A_k)-d(x,A)| \geq \varepsilon\}|.\varepsilon^p.$$

Therefore we have

$$\frac{1}{\mu_n^\beta}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|^p \geq \frac{\lambda_n^\alpha}{\mu_n^\beta}\frac{1}{\lambda_n^\alpha}|\{k\in I_n : |d(x,A_k)-d(x,A)|\geq\varepsilon\}|.\varepsilon^p.$$

Since (3.3) holds and $(A_k)\in w^W_{\mu,\beta}(p)$. The last inequality implies that $(A_k)\in S^W_{\lambda,\alpha}$. This completes the proof of the theorem. □

Corollary 9. *Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n$ for all $n\in\mathbb{N}$ and $0<\alpha\leq 1$. If (3.3) holds, then*

(a) $w^W_{\mu,\alpha}(p)\subseteq S^W_{\lambda,\alpha}$,
(b) $w^W_{\mu}(p)\subseteq S^W_{\lambda,\alpha}$,
(c) $w^W_{\mu}(p)\subseteq S^W_{\lambda}$.

Theorem 18. *Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n$ for all $n\in\mathbb{N}$ and $0<\alpha\leq\beta\leq 1$. If (3.3) holds, then $L_\infty\cap w^W_{\mu,\alpha}(p)\subseteq w^W_{\lambda,\beta}(p)$.*

Proof. Let $(A_k)\in L_\infty\cap w^W_{\mu,\alpha}(p)$ and suppose that (3.3) holds. Since $(A_k)\in L_\infty$, there esists $K>0$ such that $|d(x,A_k)-d(x,A)|\leq K$ for all k. Since $\lambda_n\leq\mu_n$ and $I_n\subset J_n$ for all $n\in\mathbb{N}$ we can write

$$\begin{aligned}
&\frac{1}{\mu_n^\beta}\sum_{k\in J_n}|d(x,A_k)-d(x,A)|^p\\
&=\frac{1}{\mu_n^\beta}\sum_{k\in J_n-I_n}|d(x,A_k)-d(x,A)|^p+\frac{1}{\mu_n^\beta}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|^p\\
&\leq\left(\frac{\mu_n-\lambda_n}{\mu_n^\beta}\right)K^p+\frac{1}{\mu_n^\beta}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|^p\\
&\leq\left(\frac{\mu_n-\lambda_n^\beta}{\mu_n^\beta}\right)K^p+\frac{1}{\mu_n^\beta}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|^p\\
&\leq\left(\frac{\mu_n-\lambda_n^\beta}{\lambda_n^\beta}\right)K^p+\frac{\lambda_n^\alpha}{\mu_n^\beta}\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|^p\\
&\leq\left(\frac{\mu_n}{\lambda_n^\beta}-1\right)K^p+\frac{\lambda_n^\alpha}{\mu_n^\beta}\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}|d(x,A_k)-d(x,A)|^p.
\end{aligned}$$

This imples that $(A_k)\in w^W_{\mu,\beta}(p)$. Hence $L_\infty\cap w^W_{\mu,\alpha}(p)\subseteq w^W_{\lambda,\beta}(p)$. □

Corollary 10. *Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n$ for all $n\in\mathbb{N}$. If (3.3) holds, then*

(a) $L_\infty\cap w^W_{\mu,\alpha}(p)\subseteq w^W_{\lambda,\beta}(p)$ *for* $0<\alpha\leq 1$,

(b) $L_\infty \cap w^W_{\lambda,\alpha}(p) \subseteq w^W_\mu(p)$ *for* $0 < \alpha \leq 1$,

(c) $L_\infty \cap w^W_\lambda(p) \subseteq w^W_\mu(p)$.

Theorem 19. *Let M be an Orlicz function and if* $\inf_k p_k > 0$, *then limit of any sequence (A_k) in $w^W_{\lambda,\alpha}[M,p]$ is unique.*

Proof. Let $\lim_k p_k = s > 0$. Suppose that $A_k \to A_1 \left(w^W_{\lambda,\alpha}[M,p]\right)$ and $A_k \to A_2 \left(w^W_{\lambda,\alpha}[M,p]\right)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} \left[M\left(\frac{|d(x,A_k) - d(x,A_1)|}{\rho} \right) \right]^{p_k} = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} \left[M\left(\frac{|d(x,A_k) - d(x,A_2)|}{\rho} \right) \right]^{p_k} = 0.$$

Let $\rho = \max\{2\rho_1, 2\rho_2\}$. As M is nondecreasing and convex, we have

$$\begin{aligned} &\frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} \left[M\left(\frac{|d(x,A_1) - d(x,A_2)|}{\rho} \right) \right]^{p_k} \\ &\leq \frac{D}{\lambda_n^\alpha} \sum_{k\in I_n} \frac{1}{2^{p_k}} \left(\left[M\left(\frac{|d(x,A_k) - d(x,A_1)|}{\rho} \right) \right]^{p_k p_k} \right. \\ &\quad \left. + \left[M\left(\frac{|d(x,A_k) - d(x,A_2)|}{\rho} \right) \right] \right) \end{aligned}$$

$$\begin{aligned} &\frac{D}{\lambda_n^\alpha} \sum_{k\in I_n} \left(\left[M\left(\frac{|d(x,A_k) - d(x,A_1)|}{\rho} \right) \right]^{p_k} \right. \\ &\qquad \left. + \frac{D}{\lambda_n^\alpha} \sum_{k\in I_n} \left[M\left(\frac{|d(x,A_k) - d(x,A_2)|}{\rho} \right) \right]^{p_k} \right) \to 0 \text{ as } n \to \infty, \end{aligned}$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. Therefore we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} \left[M\left(\frac{|d(x,A_1) - d(x,A_2)|}{\rho} \right) \right]^{p_k} = 0.$$

As $\lim_k p_k = s$, we have

$$\lim_{k\to\infty} \left[M\left(\frac{|d(x,A_1) - d(x,A_2)|}{\rho} \right) \right]^{p_k} = \left[M\left(\frac{|d(x,A_1) - d(x,A_2)|}{\rho} \right) \right]^{s}$$

and so $A_1 = A_2$. Hence the limit is unique. □

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STURM-LIOUVILLE PROBLEMS WITH FINITELY MANY POINT $\delta-$INTERACTIONS AND EIGEN-PARAMETER IN BOUNDARY CONDITION

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Abstract. This paper deals with the Sturm-Liouville equation with a finite number of point $\delta-$interactions and eigenvalue parameter contained in the boundary condition. Sturm–Liouville problem with discontinuities at one or two points and its different variants have already been investigated. In this study we extend these results to a finite number of point $\delta-$interactions case. The crucial part of this study is the using graph demonstration to obtain asymptotic representation of solutions.

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1. Introduction

We consider the boundary problem (BVP) for the differential equation

$$\ell y := -y'' + q(x)y = \lambda y \tag{1.1}$$

on $[a, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_{n-1}, x_n) \cup (x_n, b]$ and boundary condition at $x = a$

$$L_1(y) := \alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{1.2}$$

with transmission conditions at discontinuous points x_i, $i = \overline{1,n}$

$$U_i(y) := y(x_i - 0) = y(x_i + 0) = y(x_i), \tag{1.3}$$

$$V_i(y) := y'(x_i + 0) - y'(x_i - 0) = \gamma_i y(x_i) \tag{1.4}$$

and the eigenparameter-dependent boundary condition at $x = b$

$$L_2(y) := \lambda[\beta_1' y(b) + \beta_2' y'(b)] + [\beta_1 y(b) - \beta_2 y'(b)] = 0, \tag{1.5}$$

where $q(x)$ is real-valued function and continuous in $L_1[a, b]$. We assume that α_i, γ_i, β_i, β_i', $i = 1, 2$ are real numbers, satisfying $|\alpha_1| + |\alpha_2| \neq 0$, λ is a complex spectral parameter. Throughout this paper, we assume that

$$r := \beta_1'\beta_2 - \beta_1\beta_2' > 0. \tag{1.6}$$

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Notice that, we can understand problem (1.1), (1.3) and (1.4) as one of the treatments of the equation

$$-y''+(p(x)+q(x))\,y=\lambda y,\qquad x\in(a,b), \tag{1.7}$$

when $p(x)=\sum_{i=1}^{n}\gamma_i\delta(x-x_i)$, where $\delta(x)$ is the Dirac function, (see [2]).

Sturmian theory is one of the most extensively developing fields in theoretical and applied mathematics. Particularly, there has been increasing interest in the spectral analysis of BVPs with eigenvalue-dependent boundary conditions. There are quite substantial literatures on such problems. Here we mention the results of [5], [6], [7], [8], [10], [14], [18] and the corresponding references cited therein.

BVPs with discontinuities inside the interval and eigenvalue contained in the boundary conditions often appear in many branches of natural sciences. We note that Sturm-Liouville problems with eigen-dependent boundary conditions and with transmission conditions have been investigated in [1], [3], [9], [13], [11], [12]. Furthermore Green's formula for impulsive differential equation has been studied in [16] and [17].

In this paper we consider Sturm-Liouville problem with eigenparameter depend boundary condition and $\delta-$interactions at finite number of interior points. The aim of this article is to carry results in [9] to the case of finitely many $\delta-$interactions and the main difference of this study is the using graph theory (see [4]) for the complicated asymptotic formulas.

2. The Hilbert space construction and some properties of the spectrum

In the Hilbert space $\mathcal{H}:=L_2[a,b]\oplus\mathbb{C}$ of two component vectors, we define an inner product by

$$\langle f,g\rangle_{\mathcal{H}}:=\int_a^b f_1(x)\bar{g}_1(x)dx+\frac{1}{r}f_2\overline{g}_2 \tag{2.1}$$

for

$$f=\begin{pmatrix} f_1(x)\\ f_2\end{pmatrix},\quad g=\begin{pmatrix} g_1(x)\\ g_2\end{pmatrix}$$

where $f_1(x),\,g_1(x)\in L_2(a,b),\ f_2,\ g_2\in\mathbb{C}$, the constant r is defined in (1.6). For convenience we introduce

$$R_b(y):=\beta_1 y(b)-\beta_2 y'(b),$$

$$R_b'(y):=\beta_1' y(b)-\beta_2' y'(b).$$

In the Hilbert space $\mathcal{H}$ we define the operator $\mathbf{L}$

$$\mathbf{L}:\mathcal{H}\to\mathcal{H}$$

with domain

$$D(\mathbf{L}):\left\{ f\in\mathcal{H}\;\middle|\;\begin{array}{c} f_1,\ f_1'\in AC\left([a,x_1)\cup\bigcup_{i=1}^{n-1}(x_i,x_{i+1})\cup(x_n,b]\right),\\ \ell f_1\in L_2[a,b],\ L_1(f_1)=0,\\ U_i(f_1)=V_i(f_1)=0\ for\ i=\overline{1,n},\ f_2=R_b'(f_1)\end{array}\right\}$$

and operator rule

$$\mathbf{L}(f):=\begin{pmatrix}\ell f_1\\ -R_b(f_1)\end{pmatrix}.$$

Here AC (.) denotes the set of all absolutely continuous functions on related interval. In particular, those functions will have limits at the boundary points x_i.

It is clear that the eigenvalues of the operator $\mathbf{L}$ and the boundary value problem (1.1)-(1.5) are same and the eigenfunctions of (1.1)-(1.5) coincide with the first components of corresponding eigenelements of the operator $\mathbf{L}$.

Theorem 1. *The operator* $\mathbf{L}$ *is symmetric.*

Proof. Let $f,g\in D(\mathbf{L})$. From the inner product defined in (2.1), we obtain

$$\begin{aligned}\langle Lf,g\rangle_{\mathcal{H}}-\langle f,Lg\rangle_{\mathcal{H}}&=[W(f,\overline{g};x_1-0)-W(f,\overline{g};a)]\\&+[W(f,\overline{g};x_2-0)-W(f,\overline{g};x_1+0)]\\&+\sum_{p=2}^{n-1}\left[W(f,\overline{g};x_{p+1}-0)-W(f,\overline{g};x_p+0)\right]\\&+[W(f,\overline{g};b)-W(f,\overline{g};x_n+0)]\\&+\frac{1}{r}\left[R_b'(f)R_b(\overline{g})-R_b(f)R_b'(\overline{g})\right]\end{aligned}\tag{2.2}$$

where $W(f,g;x)=f(x)g'(x)-f'(x)g(x)$ is the wronkskian of the functions f and g. Since f and $\overline{g}$ satisfy the same boundary condition (1.2) and from the transmission conditions (1.3) and (1.4), we get

$$W(f,\overline{g};a)=0,\tag{2.3}$$

$$W(f,\overline{g};x_i-0)=W(f,\overline{g};x_i+0),\quad(i=\overline{1,n}),\tag{2.4}$$

$$R_b'(f)R_b(\overline{g})-R_b(f)R_b'(\overline{g})=rW(f,\overline{g};b).\tag{2.5}$$

Substituting (2.3)-(2.5) in (2.2) we obtain $\langle\mathbf{L}f,g\rangle_{\mathcal{H}}=\langle f,\mathbf{L}g\rangle_{\mathcal{H}}$ for $f,g\in D(\mathbf{L})$. So $\mathbf{L}$ is symmetric. □

Corollary 1. *All eigenvalues of the problem (1.1)-(1.5) are real and if* λ_1 *and* λ_2 *are two different eigenvalues of the problem (1.1)-(1.5) , then the corresponding eigenfunctions* y_1 *and* y_2 *are orthogonal in the sense of*

$$\int_a^b y_1(x)y_2(x)dx+\frac{1}{r}R_b'(y_1)R_b'(y_2)=0.$$

To define a solution of (1.1)-(1.5), we need the following lemma. The proof of this assertion reproduces that of Theorem 1.1 in [15, p. 14] or [12].

Lemma 1. *Let $q(x)$ be a real-valued, continuous function and let $f(\lambda)$ and $g(\lambda)$ are given entire functions. Then for any $\lambda \in \mathbb{C}$, the equation*

$$-y'' + q(x)y = \lambda y, \quad x \in [a,b]$$

has a unique solution $y = y(x,\lambda)$ such that

$$y(a) = f(\lambda), \;\; y'(a) = g(\lambda) \quad (\text{or } y(b) = f(\lambda), \;\; y'(b) = g(\lambda))$$

and for each $x \in [a,b]$, $y(x,\lambda)$ is an entire function of λ.

Now we define two solutions of the equation (1.1) as follows:

$$\phi_\lambda(x) = \begin{cases} \phi_{1\lambda}(x), & x \in [a,x_1) \\ \phi_{2\lambda}(x), & x \in (x_1,x_2) \\ \quad\vdots & \quad\vdots \\ \phi_{n\lambda}(x), & x \in (x_{n-1},x_n) \\ \phi_{(n+1)\lambda}(x), & x \in (x_n,b] \end{cases}, \quad \chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), & x \in [a,x_1) \\ \chi_{2\lambda}(x), & x \in (x_1,x_2) \\ \quad\vdots & \quad\vdots \\ \chi_{n\lambda}(x), & x \in (x_{n-1},x_n) \\ \chi_{(n+1)\lambda}(x), & x \in (x_n,b] \end{cases}$$

Let $\phi_{1\lambda}(x) = \phi_1(x,\lambda)$ be a solution of the equation (1.1) on $[a,x_1)$ which satisfies the initial conditions at the point a

$$y(a) = \alpha_2, \quad y'(a) = -\alpha_1 \tag{2.6}$$

From Lemma 1, we can define a solution $\phi_{i+1}(x,\lambda)$ of the equation (1.1) on $[x_i,x_{i+1}]$, $(i = \overline{1,n-1})$ by means of the solution $\phi_i(x,\lambda)$ by the nonstandard initial conditions

$$y(x_i+0) = \phi_{i\lambda}(x_i-0), \quad y'(x_i+0) = \phi'_{i\lambda}(x_i-0) + \gamma_i\phi_{i\lambda}(x_i). \tag{2.7}$$

In the same manner, we define a solution $\phi_{n+1}(x,\lambda)$ of the equation (1.1) on $[x_n,b]$ by the nonstandard initial conditions

$$y(x_n+0) = \phi_{n\lambda}(x_n-0), \quad y'(x_n+0) = \phi'_{n\lambda}(x_n-0) + \gamma_i\phi_{n\lambda}(x_n). \tag{2.8}$$

Therefore $\phi(x,\lambda)$ satisfies the equation (1.1) on the interval $[a,x_1) \cup (x_1,x_2) \cup \cdots \cup (x_{n-1},x_n) \cup (x_n,b]$ and the boundary condition (1.2) and the transmission conditions (1.3), (1.4).

Similarly let $\chi_{(n+1)\lambda}(x) = \chi_{n+1}(x,\lambda)$ be a solution of (1.1) on $(x_n,b]$ which satisfies the initial conditions

$$y(b) = \beta'_2\lambda + \beta_2, \quad y'(b) = \beta'_1\lambda + \beta_1. \tag{2.9}$$

Likewise let define the solution $\chi_i(x,\lambda)$, on $[x_{i-1},x_i]$, $(i = \overline{n,2})$ satisfies the conditions

$$y(x_i-0) = \chi_{(i+1)\lambda}(x_i+0), \;\; y'(x_i-0) = \chi'_{(i+1)\lambda}(x_i+0) - \gamma_i\chi_{(i+1)\lambda}(x_i). \tag{2.10}$$

Finally we define a solution $\chi_1(x,\lambda)$ on $[a,x_1)$ satisfies the conditions

$$y(x_1-0)=\chi_{2\lambda}(x_1+0), \quad y'(x_1-0)=\chi'_{2\lambda}(x_1+0)-\gamma_i\chi_{2\lambda}(x_1). \tag{2.11}$$

Hence $\chi(x,\lambda)$ satisfies the equation (1.1) on $[a,x_1)\cup(x_1,x_2)\cup\cdots\cup(x_{n-1},x_n)\cup(x_n,b]$, the boundary condition (1.5) and the transmission conditions (1.3), (1.4).

Let define the wronskians of ϕ_i and χ_i for x in the intervals $[a,x_1)$, (x_1,x_2), ..., (x_{n-1},x_n), $(x_n,b]$ respectively:

$$\begin{aligned} w_i(\lambda) &= W_\lambda(\phi_i,\chi_i,x) \\ &= \phi_i(x,\lambda)\chi'_i(x,\lambda)-\phi'_i(x,\lambda)\chi_i(x,\lambda), \quad i=\overline{1,n+1}. \end{aligned} \tag{2.12}$$

Lemma 2. *For each* $\lambda\in\mathbb{C}$ $w(\lambda):=w_1(\lambda)=w_2(\lambda)=w_3(\lambda)=\cdots=w_{n+1}(\lambda)$.

Proof. From (2.7), (2.10), (1.3) and (1.4)

$$\begin{aligned} W_\lambda(\phi_1,\chi_1;x_1-0) &= \phi_1(x_1-0,\lambda)\chi'_1(x_1-0,\lambda)-\phi'_1(x_1-0,\lambda)\chi_1(x_1-0,\lambda) \\ &= \phi_2(x_1+0,\lambda)[\chi'_2(x_1+0,\lambda)-\gamma_1\chi_2(x_1,\lambda)] \\ &\quad -[\phi'_2(x_1+0,\lambda)-\gamma_1\phi_2(x_1,\lambda)]\chi_2(x_1+0,\lambda) \\ &= W_\lambda(\phi_2,\chi_2;x_1+0)-\gamma_1[\phi_2(x_1+0,\lambda)\chi_2(x_1,\lambda) \\ &\quad -\phi_2(x_1,\lambda)\chi_2(x_1+0,\lambda)] \\ &= W_\lambda(\phi_2,\chi_2;x_1+0) \end{aligned}$$

Since the wronskians are independent of x (see [15, pp.7]), we get

$$W_\lambda(\phi_2,\chi_2;x_1+0)=W_\lambda(\phi_2,\chi_2;x_2-0)$$

and similar calculation gives

$$W_\lambda(\phi_2,\chi_2;x_2-0)=W_\lambda(\phi_3,\chi_3;x_2+0).$$

Consequently we arrive at

$$\begin{aligned} W_\lambda(\phi_1,\chi_1;x_1-0) &= W_\lambda(\phi_2,\chi_2;x_1+0)=W_\lambda(\phi_3,\chi_3;x_2+0)=\cdots \\ &= W_\lambda(\phi_{n+1},\chi_{n+1};x_n+0) \end{aligned}$$

□

Theorem 2. *The eigenvalues of the problem (1.1)-(1.5) consist the zeros of the function* $w(\lambda)$.

Proof. For the proof, we will follow the technique in [9]. Suppose that μ is the zero of $w(\lambda)$. Then the wronskian of $\phi_1(x,\mu)$ and $\chi_1(x,\mu)$ is zero, so that $\chi_1(x,\mu)$ is a constant multiple of $\phi_1(x,\mu)$, say

$$\chi_1(x,\mu)=k\phi_1(x,\mu), \quad x\in[a,x_1]$$

for some $k\neq 0$. Therefore $\chi_1(x,\mu)$ satisfies the boundary condition (1.2) and this means that $\chi_1(x,\mu)$ is an eigenfunction for the eigenvalue μ.

For the converse, let $\Phi(x)$ be any eigenfunction corresponding to eigenvalue μ. Then the function $\Phi(x)$ may be represented in the form

$$\Phi(x)=\begin{cases} k_1\phi_1(x,\mu)+k_2\chi_1(x,\mu), & x\in[a,x_1)\\ k_3\phi_2(x,\mu)+k_4\chi_2(x,\mu), & x\in(x_1,x_2)\\ \vdots & \vdots\\ k_{2n-1}\phi_n(x,\mu)+k_{2n}\chi_n(x,\mu), & x\in(x_{n-1},x_n)\\ k_{2n+1}\phi_{n+1}(x,\mu)+k_{2n+2}\chi_{n+1}(x,\mu), & x\in(x_n,b] \end{cases}$$

where at least one of the constants k_j, $(j=\overline{1,2n+2})$ is not zero. By using the initial conditions (2.6)-(2.11), the equations

$$L_1(\Phi)=0;\ \ U_i(\Phi)=0,\ V_i(\Phi)=0,\ i=\overline{1,n};\ \ L_2(\Phi)=0 \tag{2.13}$$

give a system of linear equations in the variables k_j, $(j=\overline{1,2n+2})$ and the coefficient matrix of this system is

$$\begin{bmatrix} 0 & w_1(\mu) & 0 & 0 & \cdots & & & 0 & 0\\ & N_1 & & M_1 & & & & 0 & 0\\ & & & & & & & 0 & 0\\ 0 & 0 & & N_2 & M_2 & & & & \\ 0 & 0 & & & & & & & \\ & & & & N_3 & \ddots & & \vdots & \vdots\\ \vdots & \vdots & & & & & & & \\ & & & & & \ddots & M_{n-1} & 0 & 0\\ & & & & & & & 0 & 0\\ 0 & 0 & & & & & N_n & M_n & \\ 0 & 0 & & & & & & & \\ 0 & 0 & & & \cdots & & 0 \quad 0 & w_{n+1}(\mu) & 0 \end{bmatrix}_{(2n+2)\times(2n+2)} \tag{2.14}$$

where M_i and N_i are 2×2 matrices defined as follows

$$M_i=\begin{bmatrix} -\phi_{(i+1)\mu}(x_i+0) & -\chi_{(i+1)\mu}(x_i+0)\\ -\phi'_{(i+1)\mu}(x_i+0) & -\chi'_{(i+1)\mu}(x_i+0) \end{bmatrix},\quad i=\overline{1,n},$$

$$N_i=\begin{bmatrix} \phi_{i\mu}(x_i-0) & \chi_{i\mu}(x_i-0)\\ \phi'_{i\mu}(x_i-0) & \chi'_{i\mu}(x_i-0) \end{bmatrix},\quad i=\overline{1,n}.$$

The determinant of the matrix (2.14) is $-\prod_{i=1}^{n} w_i(\mu).w^{n+1}(\mu)$ which must be zero in order to the system (2.13) has a nontrivial solution and hence $w(\mu)=0$. □

3. Asymptotic form of solutions and graph representation

In this section, we shall derive the asymptotic formulas for the characteristic function $w(\lambda)$ of (1.1), (1.5) in four different cases. The main difficulty is derivation of asymptotic formulas for the solutions of (1.1), (1.5). Because these formulas fastly lead to a very complicated equations for large values of n. A convenient way to manage the resulting asymptotic representation of solutions is by a graph analogy. For further reading for graph based theory, see, for example, [4]. We start with some lemmas.

Lemma 3. *Let $\lambda=\rho^2$ and $\phi(x,\lambda)$ be solution of the equation (1.1). Then the following integral equations hold for $k=0,1$*

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x)=\alpha_2\frac{d^k}{dx^k}(\cos\rho(x-a))-\alpha_1\frac{1}{\rho}\frac{d^k}{dx^k}(\sin\rho(x-a)) \tag{3.1}$$
$$+\frac{1}{\rho}\int_a^{x_1}\frac{d^k}{dx^k}(\sin\rho(x-t))q(t)\phi_{1\lambda}(t)dt,$$

$$\frac{d^k}{dx^k}\phi_{(i+1)\lambda}(x)=\phi_{i\lambda}(x_i-0)\frac{d^k}{dx^k}(\cos\rho(x-x_i)) \tag{3.2}$$
$$+\frac{1}{\rho}\phi'_{i\lambda}(x_i-0)\frac{d^k}{dx^k}(\sin\rho(x-x_i))$$
$$+\frac{1}{\rho}\int_{x_i}^{x}\frac{d^k}{dx^k}(\sin\rho(x-t))q(t)\phi_{(i+1)\lambda}(t)dt,\ \ i=\overline{1,n}.$$

Proof. The last terms in (3.1) and (3.2) are equal to

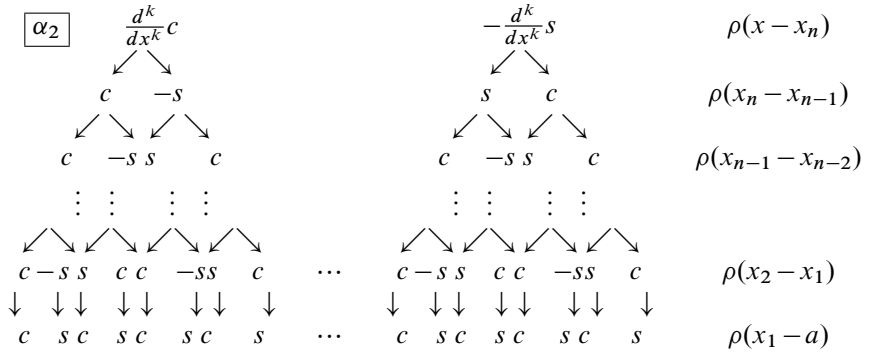
$$\frac{1}{\rho}\int_a^{x_1}\frac{d^k}{dx^k}(\sin\rho(x-t))\{\rho^2\phi_{1\lambda}(t)+\phi''_{1\lambda}(t)\}dt,$$
$$\frac{1}{\rho}\int_{x_i}^{x}\frac{d^k}{dx^k}(\sin\rho(x-t))\{\rho^2\phi_{(i+1)\lambda}(t)+\phi''_{(i+1)\lambda}(t)\}dt,\ \ i=\overline{1,n}$$

by the equation (1.1), respectively. On integrating by parts twice we obtain (3.1) and (3.2). □

Lemma 4. *Let $\tau := Im\rho$ and $\alpha_2 \neq 0$. As $|\lambda| \to \infty$, the asymptotic formula for $\phi_{1\lambda}(x)$ is*

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \alpha_2 \frac{d^k}{dx^k}(\cos\rho(x-a)) + O\left(|\rho|^{k-1} e^{|\tau|(x-a)}\right) \tag{3.3}$$

and the asymptotic formula for $\phi_{(i+1)\lambda}(x)$, $i = \overline{1,n}$, is obtained from the following tree:

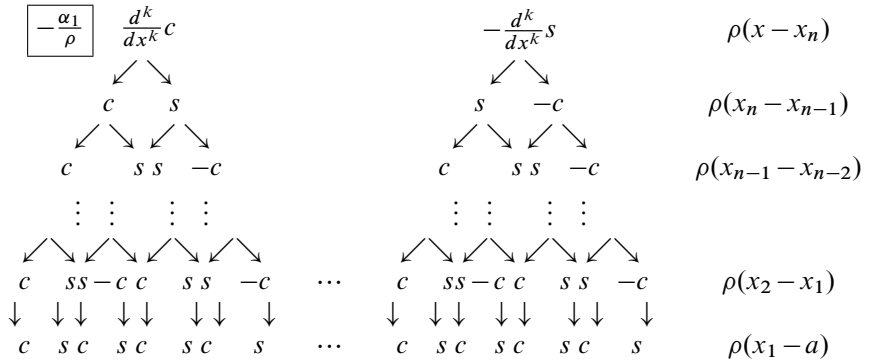


(3.4)

where c and s denote cosine and sine functions respectively. The roots consist of two nodes $\frac{d^k}{dx^k}c$ and $-\frac{d^k}{dx^k}s$. The children of roots are written by repeating the processing $s\ -s\ s\ c$. The last children are written by repeating the processing $c\ s$. After constructing trees for any $\frac{d^k}{dx^k}\phi_{i\lambda}(x)$, $i = \overline{2,n+1}$ using rules above to get its formula, first we write each terms by multiplying all notes on the branch from root to least children and then sum these terms. Then, while if $\alpha_2 = 0$

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = -\frac{\alpha_1}{\rho}\frac{d^k}{dx^k}(\sin\rho(x-a)) + O\left(|\rho|^{k-2} e^{|\tau|(x-a)}\right) \tag{3.5}$$

and



(3.6)

Each result obtained from trees holds uniformly for the intervals $a \le x \le x_1$, $x_i \le x \le x_{i+1}$, $i = \overline{1,n-1}$ and $x_n \le x \le b$ respectively.

Proof. The proof for $i=1,2$ and 3 without graph representation were given in [9] and [15]. Here, we propose an alternative proof in general sense by using graph demonstration. As can be seen in studies [9], [15] the continuation of this process, although theoretically clear, soon leads to very complicated formulas. It is therefore advantageous to use a graphical representation. Indeed, we can construct a link about the array of the functions. Below, we exhibit the trees corresponding to each $\frac{d^k}{dx^k}\phi_{i\lambda}(x)$, $i=\overline{1,n+1}$ and give a systematic pattern. We will write $\frac{d^k}{dx^k}\phi_{i\lambda}(x)$ for $i=1,2,3$ from [9, Lemma 3.2]. For $\alpha_2\neq 0$ and $i=1$ we have

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x)=\alpha_2\frac{d^k}{dx^k}(\cos\rho(x-a))+O\left(|\rho|^{k-1}e^{|\tau|(x-a)}\right).$$

The tree for this formula will consist of only one node will be named by root:

$$\boxed{\alpha_2}\qquad \frac{d^k}{dx^k}c\qquad \rho(x-a)$$

In order to work with trees we introduce the following notations: $\boxed{\alpha_2}$ is common factor of all terms, $\frac{d^k}{dx^k}$ shows the derivative of terms on root and $\rho(x-a)$ is at values of functions which are on the same line with it. For $i=2$,

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{2\lambda}(x)=\alpha_2\Bigg\{&\frac{d^k}{dx^k}(\cos\rho(x-x_1))\cos\rho(x_1-a)\\&-\frac{d^k}{dx^k}(\sin\rho(x-x_1))\sin\rho(x_1-a)\\&+O\left(|\rho|^{k-1}e^{|\tau|[(x-x_1)+(x_1-a)]}\right)\Bigg\}.\end{aligned}\tag{3.7}$$

For this formula the tree will consist of two roots and their one child:

$$\begin{array}{llll}\boxed{-\frac{\alpha_1}{\rho}} & \frac{d^k}{dx^k}c & -\frac{d^k}{dx^k}s & \rho(x-x_1)\\ & \downarrow & \downarrow & \\ & c & s & \rho(x_1-a)\end{array}$$

In this tree, the multiplication of terms on the vertical branch from root to its child gives the first term of (3.7) and similarly, second branch gives the second term of (3.7). Then the sum of these branches gives the formula (3.7). Finally, for $i=3$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{3\lambda}(x)=\alpha_2\Bigg\{&\frac{d^k}{dx^k}(\cos\rho(x-x_2))\cos\rho(x_2-x_1)\cos\rho(x_1-a)\\&-\frac{d^k}{dx^k}(\cos\rho(x-x_2))\sin\rho(x_2-x_1)\sin\rho(x_1-a)\\&-\frac{d^k}{dx^k}(\sin\rho(x-x_2))\sin\rho(x_2-x_1)\cos\rho(x_1-a)\end{aligned}$$

$$-\frac{d^k}{dx^k}(\sin\rho(x-x_2))\cos\rho(x_2-x_1)\sin\rho(x_1-a)$$
$$+O\left(|\rho|^{k-1}\,e^{|\tau|[(x-x_2)+(x_2-x_1)+(x_1-a)]}\right)\Big\}$$

and corresponds to:

$$\boxed{\alpha_2}\qquad \begin{array}{cc} \frac{d^k}{dx^k}c & \\ \swarrow\ \searrow & \\ c \quad -s & \\ \downarrow\quad\downarrow & \\ c \quad s & \end{array} \qquad\qquad \begin{array}{cc} -\frac{d^k}{dx^k}s & \\ \swarrow\ \searrow & \\ s \quad c & \\ \downarrow\quad\downarrow & \\ c \quad s & \end{array} \qquad \begin{array}{l} \rho(x-x_2) \\ \\ \rho(x_2-x_1) \\ \\ \rho(x_1-a) \end{array}$$

The above trees help in giving a procedure to construct trees for $i \geq 4$. From using the rules above we generalize a tree for $\frac{d^k}{dx^k}\phi_{(n+1)\lambda}(x)$ as in (3.4) and by using the same technique we obtain a tree for $\frac{d^k}{dx^k}\phi_{(n+1)\lambda}(x)$ when $\alpha_2 = 0$ as in (3.6). □

Theorem 3. *Let $\lambda = \rho^2$ and $\tau := Im\rho$. Then the estimates obtained from the following four trees for $w(\lambda)$ are valid.*

For $\beta_2' \neq 0$, $\alpha_2 \neq 0$, the estimate obtained from a tree which is the same as in (3.4) except common factor and root changed by

$$\boxed{-\beta_2'\alpha_2\rho^3} \quad s \quad c \quad \rho(b-x_n)$$

For $\beta_2' \neq 0$, $\alpha_2 = 0$, the estimate obtained from a tree which is the same as in (3.6) except common factor and root changed by

$$\boxed{-\beta_2'\alpha_1\rho^2} \quad s \quad c \quad \rho(b-x_n)$$

For $\beta_2' = 0$, $\alpha_2 \neq 0$, the estimate obtained from a tree which is the same as in (3.4) except common factor and root changed by

$$\boxed{-\beta_1'\alpha_2\rho^2} \quad c \quad -s \quad \rho(b-x_n)$$

For $\beta_2' = 0$, $\alpha_2 = 0$, the estimate obtained from a tree which is the same as in (3.6) except common factor and root changed by

$$\boxed{-\beta_1'\alpha_1\rho} \quad c \quad s \quad \rho(b-x_n)$$

Proof. The case $\beta_2' \neq 0$, $\alpha_2 \neq 0$ will be considered; similar proof works for other three cases. Since $w_{n+1}(\lambda)$ in (2.12) is independent of $x \in [a,b]$, from (2.9) we get

$$w_{n+1}(\lambda) = (\beta_1'\lambda + \beta_1)\phi_{(n+1)\lambda}(b) - (\beta_2'\lambda + \beta_2)\phi'_{(n+1)\lambda}(b) \tag{3.8}$$

Putting $x = b$ in estimates obtained from trees (3.4) and (3.6) for the asymptotic behavior of $\frac{d^k}{dx^k}\phi_{(n+1)\lambda}(x)$ and then substituting in (3.8) we obtain four different

cases for the asymptotic behavior of $w_{n+1}(\lambda)$ and their trees as $|\lambda| \to \infty$. For $\beta_2' \neq 0$, $\alpha_2 \neq 0$, we briefly write

$$w_{n+1}(\lambda) = \beta_2'\alpha_2\rho^3 \{\sin\rho(b-x_n)\,[\ldots]_1 + \cos\rho(b-x_n)\,[\ldots]_2\} + O\left(|\rho|^2\, e^{|\tau|[(b-x_n)+(x_n-x_{n-1})+\cdots+(x_2-x_1)+(x_1-a)]}\right) \tag{3.9}$$

where the terms in square brackets $[\ldots]_1$ and $[\ldots]_2$ are the same as the terms in estimate obtained from (3.4). Therefore the tree for (3.9) is similar to the tree (3.4) except common factor and root. The common factor and root are replaced by

$$\boxed{-\beta_2'\alpha_2\rho^3} \qquad s \qquad c \qquad \rho(b-x_n)$$

□

Putting $\rho = i\tau$ $(\tau > 0)$ in these formula it follows that $w(\lambda) \neq 0$ for λ negative and sufficiently large. The eigenvalues of the problem (1.1)-(1.5) are bounded below in all cases.

4. Asymptotic behavior of the eigenvalues and eigenfunctions

Our object in this section is to obtain asymptotic estimates for eigenvalues of problem (1.1)-(1.5). We know from Lemma 2 and Theorem 2 that the eigenvalues coincide with the zeros of the entire function $w_{n+1}(\lambda)$. Since the eigenvalues are real and bounded below we can denote the eigenvalues by λ_n $(n = 0, 1, \ldots)$, where

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

Let $\beta_2' \neq 0$, $\alpha_2 \neq 0$. In the formula (3.9) and so in the corresponding tree, after successive trigonometric operations from last children to the root we arrive at

$$w_{n+1}(\lambda) = \beta_2'\alpha_2\rho^3 \prod_{k=1}^{n} \sin\rho(b-a) + O\left(|\rho|^2\, e^{|\tau|(b-a)}\right). \tag{4.1}$$

By applying Rouché's theorem on a sufficiently large contour it follows that $w_{n+1}(\lambda)$ has the same number of zeros inside the contour as

$$\beta_2'\alpha_2\rho^3 \prod_{k=1}^{n} \sin\rho(b-a).$$

Hence, if $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, are the zeros of $w_{n+1}(\lambda)$ and $\rho_m^2 = \lambda$, we have for sufficiently large m

$$\rho_m = \frac{(m-1)\pi}{b-a} + O\left(\frac{1}{m}\right). \tag{4.2}$$

With the same idea, we obtain: For $\beta_2' \neq 0$, $\alpha_2 = 0$ and for $\beta_2' = 0$, $\alpha_2 \neq 0$,

$$\rho_m = \frac{(m-1/2)\pi}{b-a} + O\left(\frac{1}{m}\right). \tag{4.3}$$

For $\beta_2' = 0,\ \alpha_2 = 0,$

$$\rho_m = \frac{m\pi}{b-a} + O\left(\frac{1}{m}\right). \tag{4.4}$$

Let $\phi(x,\lambda_m)$ is an eigenfunction corresponding to eigenvalue λ_m and $\beta_2' \neq 0,\ \alpha_2 \neq 0$. After inserting (4.2) into (3.3) and (3.4) and taking into account the definition of solutions defined in §2 we get

$$\phi(x,\lambda_m) = \alpha_2 \cos\left(\frac{(m-1)\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m}\right), \quad x \in [a,b],$$

and for $\beta_2' \neq 0,\ \alpha_2 = 0$

$$\phi(x,\lambda_m) = -\alpha_1 \frac{(b-a)}{\pi(m-1/2)} \sin\left(\frac{(m-1/2)\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m^2}\right), \quad x \in [a,b],$$

for $\beta_2' = 0,\ \alpha_2 \neq 0$

$$\phi(x,\lambda_m) = \alpha_2 \cos\left(\frac{(m-1/2)\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m}\right), \quad x \in [a,b],$$

for $\beta_2' = 0,\ \alpha_2 = 0$

$$\phi(x,\lambda_m) = -\alpha_1 \frac{(b-a)}{\pi m} \sin\left(\frac{m\pi}{b-a}(x-a)\right) + O\left(\frac{1}{m^2}\right), \quad x \in [a,b].$$

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ON A GENERALIZED RUBEL'S OPERATOR EQUATION

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Abstract. We describe all pairs of linear operators that act in spaces of analytic functions in domains and satisfy the generalized Rubel's operator equation.

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1. INTRODUCTION

Let G be an arbitrary domain of the complex plane. Let $\mathcal{H}(G)$ denote the space of all analytic functions in G equipped with the topology of compact convergence [6].

In [14], L.A. Rubel posed and solved the problem of finding all pairs of linear continuous functionals L and M on the space $\mathcal{H}(G)$ that satisfy the relation

$$L(fg) = L(f)M(g) + L(g)M(f) \tag{1.1}$$

for arbitrary functions f and g of $\mathcal{H}(G)$. Later, Nandakumar [10] and Zalcman [15] solved Rubel's problem in the class of linear functionals on the space $\mathcal{H}(G)$ in different ways. Further investigations related to the description of pairs of linear functionals on the space $\mathcal{H}(G)$ that satisfy similar relations were carried out by Nandakumar and Kannappan in [5, 13]. These results and their generalizations were systematized by Kannappan in [4]. Further studies in this regard has been devoted to some operator modifications of (1.1) in classes of operators acting in $\mathcal{H}(G)$ [1,2,11].

In the light of the above-mentioned results there naturally arises the problem of finding all pairs of linear operators A, B on the space $\mathcal{H}(G)$ such that

$$(A(fg))(z) = (Af)(z)(Bg)(z) + (Ag)(z)(Bf)(z) \tag{1.2}$$

for arbitrary functions f and g of $\mathcal{H}(G)$ and for $z \in G$. In [3] this problem was solved only for some special classes of linear operators on $\mathcal{H}(G)$. In [7] Rubel's operator equation (1.2) was completely solved in the class of linear operators that act in spaces of analytic functions in domains. Note that all solutions of Rubel's equation in the class of linear continuous operators that act in spaces of analytic functions in arbitrary simply connected domains were described in [8]. The purpose

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of this paper is to determine all pairs of linear operators A and B on $\mathcal{H}(G)$ satisfying the generalized Rubel's operator equation

$$(A(fg))(z) = \alpha(z)(Af)(z)(Bg)(z) + \beta(z)(Ag)(z)(Bf)(z) \tag{1.3}$$

for arbitrary functions f and g of $\mathcal{H}(G)$ for $z \in G$; here α, β are some fixed functions of $\mathcal{H}(G)$.

Note that in the case $B = 0$ we have $A = 0$. If $A = 0$ then (1.3) implies that B is an arbitrary linear operator on the space $\mathcal{H}(G)$. Obviously, we get in these cases that the set of solutions of (1.3) in the class of linear operators on $\mathcal{H}(G)$ coincides with the set of such operators: $A = 0$, B, where B is an arbitrary linear operator on the space $\mathcal{H}(G)$. We will assume that $A \neq 0$ and $B \neq 0$. We will call the pairs of such operators nontrivial solutions of (1.3).

The method of solving of equation (1.3) depends strongly on the relation between functions α and β. To solve our problem we will need auxiliary facts and they are also of independent interest.

2. Auxiliary results

Lemma 1. *In order that the linear operators A and B on $\mathcal{H}(G)$ satisfy the equality*

$$AfBg = AgBf \tag{2.1}$$

for any $f, g \in \mathcal{H}(G)$ it is necessary and sufficient that

$$Af = a\,Cf,\ Bf = b\,Cf, \tag{2.2}$$

where a, b are functions of $\mathcal{H}(G)$, C is a linear operator on $\mathcal{H}(G)$.

Proof. Necessity. Suppose that linear operators A and B on $\mathcal{H}(G)$ satisfy (2.1). If $A = 0$ or $B = 0$ then A and B are of the form (2.2). Therefore, we assume that $A \neq 0$ and $B \neq 0$.

Since $B \neq 0$, there exists $g_0 \in \mathcal{H}(G)$ such that $(Bg_0)(z) \not\equiv 0$ in G. Let $Ag_0 = a_0$, $Bg_0 = b_0$. Substituting $g = g_0$ in (2.1) we get

$$b_0(z)(Af)(z) = a_0(z)(Bf)(z) \tag{2.3}$$

for any $f \in \mathcal{H}(G)$, $z \in G$. Since $b_0(z) \not\equiv 0$ in G and $A \neq 0$, (2.3) implies that $a_0(z) \not\equiv 0$ in G.

Let $S = \{z \in G : a_0(z)b_0(z) = 0\}$. Then S is an at most countable set. If $z_0 \in G$ and $a_0(z_0) = 0$ then by $k(z_0)$ we denote the multiplicity of the zero z_0 of $a_0(z)$. Similarly, if $z_0 \in G$ and $b_0(z_0) = 0$, then let $l(z_0)$ stand for the multiplicity of the zero z_0 of $b_0(z)$. For any $z_0 \in S$ we put

$$m(z_0) = \begin{cases} k(z_0), \text{ if } a_0(z_0) = 0 \text{ and } b_0(z_0) \neq 0; \\ l(z_0), \text{ if } a_0(z_0) \neq 0 \text{ and } b_0(z_0) = 0; \\ \max\{k(z_0), l(z_0)\}, \text{ if } a_0(z_0) = b_0(z_0) = 0. \end{cases}$$

Let $d(z)$ be a function of $\mathcal{H}(G)$ such that the zero set of d coincides with S and herewith the multiplicity of the zero $z_0 \in S$ of d is equal to $m(z_0)$. The existence of such a d follows from the generalized Weierstrass theorem (see. [9], p. 301). If $z \in G\setminus S$ then equality (2.3) implies that

$$\frac{b_0(z)}{d(z)}(Af)(z) = \frac{a_0(z)}{d(z)}(Bf)(z) \tag{2.4}$$

for any $f \in \mathcal{H}(G)$. Using (2.4) the definition of $d(z)$ implies that for any $z_0 \in S$ the following equality holds

$$\lim_{z\to z_0}\left(\frac{b_0(z)}{d(z)}(Af)(z)\right) = \lim_{z\to z_0}\left(\frac{a_0(z)}{d(z)}(Bf)(z)\right),$$

for an arbitrary $f \in \mathcal{H}(G)$ and notice that the above limits are finite. Therefore, each point of S is a removable singularity of the functions $\frac{b_0(z)}{d(z)}(Af)(z)$ and $\frac{a_0(z)}{d(z)}(Bf)(z)$. For any $f \in \mathcal{H}(G)$ we put

$$(Cf)(z) = \begin{cases} \frac{b_0(z)}{d(z)}(Af)(z) \text{ if } z \in G\setminus S; \\ \lim\limits_{t\to z}\left(\frac{b_0(t)}{d(t)}(Af)(t)\right), \text{ if } z \in S. \end{cases}$$

It follows from the preceding arguments that C acts in the space $\mathcal{H}(G)$. The linearity of C is obvious. Let $a(z) = \frac{d(z)}{b_0(z)}$, $b(z) = \frac{d(z)}{a_0(z)}$. The functions $a(z)$ and $b(z)$ are analytic on the set $G\setminus S$ and each point of S is a removable singularity of these functions. Therefore, $a(z)$ and $b(z)$ are analytic in G. The definition of C implies that

$$(Af)(z) = a(z)(Cf)(z), \tag{2.5}$$

$$(Bf)(z) = b(z)(Cf)(z) \tag{2.6}$$

for any $f \in \mathcal{H}(G)$, $z \in G\setminus S$.

Since C acts in $\mathcal{H}(G)$ and $a, b \in \mathcal{H}(G)$, the equalities (2.5) and (2.6) hold for all $z \in G$. Thus, A and B are of the form (2.2). The necessity part is proved. The sufficiency of conditions of Lemma is obvious. □

3. The main result

We begin with the case when $\alpha(z) \not\equiv \beta(z)$ in G.

Theorem 1. *Let G be an arbitrary domain of the complex plane. Let α, β be arbitrary functions of $\mathcal{H}(G)$ and $\alpha(z) \not\equiv \beta(z)$ in G. In order that (1.3) has a non-trivial solution in the class of linear operators on $\mathcal{H}(G)$ it is necessary and sufficient that the following condition holds:*

$$\alpha(z) + \beta(z) \neq 0, \ \forall z \in G. \tag{3.1}$$

Under condition (3.1) a pair of linear operators A, B of $\mathcal{H}(G)$ satisfies (1.3) if and only if A, B are defined by one of the following conditions:

1) $A = 0$, *B is an arbitrary linear operator on the space* $\mathcal{H}(G)$;
2) $Af = \frac{a}{(\alpha+\beta)b}(f \circ \psi)$, $Bf = \frac{1}{\alpha+\beta}(f \circ \psi)$, *where* $a, b, \psi \in \mathcal{H}(G)$, $b(z) \neq 0$ *if* $z \in G$, $\psi(G) \subset G$.

Proof. Assume that a pair of non-zero linear operators A, B of $\mathcal{H}(G)$ satisfy relation (1.3). Interchanging f and g in (1.3) we get

$$(A(fg))(z) = \alpha(z)(Ag)(z)(Bf)(z) + \beta(z)(Af)(z)(Bg)(z) \tag{3.2}$$

for any $f, g \in \mathcal{H}(G)$. Equalities (1.3) and (3.2) imply that

$$(\alpha(z) - \beta(z))((Af)(z)(Bg)(z) - (Ag)(z)(Bf)(z)) = 0 \tag{3.3}$$

for all $f, g \in \mathcal{H}(G)$, $z \in G$. Since $\alpha(z) \not\equiv \beta(z)$ in G, (3.3) implies that the operators A, B satisfy (2.1). By Lemma 1, these operators are of the form (2.2), where a, b are some functions of $\mathcal{H}(G)$, C is a linear operator on $\mathcal{H}(G)$.

Using (2.2) and (1.3) we get

$$a(z)(C(fg))(z) = (\alpha(z) + \beta(z))a(z)b(z)(Cf)(z)(Cg)(z).$$

Since $A \neq 0$ and $B \neq 0$, $a(z) \not\equiv 0$, $b(z) \not\equiv 0$ in G and $C \neq 0$. Therefore,

$$(C(fg))(z) = (\alpha(z) + \beta(z))b(z)(Cf)(z)(Cg)(z) \tag{3.4}$$

for all $f, g \in \mathcal{H}(G)$, $z \in G$. Since $C \neq 0$, $\alpha(z) + \beta(z) \not\equiv 0$ on G. Using (3.4) we obtain that $(C_1 f)(z) = (\alpha(z) + \beta(z))b(z)(Cf)(z)$ is a multiplicative operator on $\mathcal{H}(G)$ and $C_1 \neq 0$. Therefore, there exists a function $\psi \in \mathcal{H}(G)$ such that $\psi(G) \subset G$ and $C_1 f = f \circ \psi$ [12]. Thus, $(\alpha(z) + \beta(z))b(z)(Cf)(z) = (f \circ \psi)(z)$ for all $f \in \mathcal{H}(G)$, $z \in G$. Setting in the above equality $f \equiv 1$, we get that (3.1) holds and $b(z) \neq 0$ for all $z \in G$. We have that C is of the following form:

$$Cf = \frac{1}{(\alpha + \beta)b}(f \circ \psi). \tag{3.5}$$

Substituting (3.5) in (2.2)we get that the operators A and B are of the form 2). The necessity part is proved.

By a direct calculation we obtain that the pair of operators A and B that is defined by 1) or 2), satisfies the generalized Rubel's operator equation (1.3). □

Let us now assume that $\alpha(z) \equiv \beta(z)$ in G. Consider the operator equation

$$(A(fg))(z) = \alpha(z)((Af)(z)(Bg)(z) + (Ag)(z)(Bf)(z)). \tag{3.6}$$

Theorem 2. *Let G be an arbitrary domain of the complex plane. Let α be an arbitrary function of $\mathcal{H}(G)$. In order that operator equation (3.6) has a non-trivial solution in the class of linear operators on $\mathcal{H}(G)$ it is necessary and sufficient that the following equality holds:*

$$\alpha(z) \neq 0, \ \forall z \in G. \tag{3.7}$$

Under condition (3.7) *a pair of linear operators* A, B *of* $\mathcal{H}(G)$ *satisfies* (3.6) *if and only if either* $A = 0$ *and* B *is an arbitrary linear operator on the space* $\mathcal{H}(G)$, *or the pair of operators* A, B *is defined by one of the following conditions:*

1) $A(f) = \varphi \cdot (f \circ \psi)$; $B(f) = \frac{1}{2\alpha}(f \circ \psi)$, *where* $\varphi, \psi \in \mathcal{H}(G)$, *with* $\psi(G) \subset G$;

2) $A(f) = \varphi \cdot (f' \circ u)$, $B(f) = \frac{1}{\alpha}(f \circ u)$, *where* $\varphi, u \in \mathcal{H}(G)$, *with* $u(G) \subset G$;

3) *the operators* A *and* B *are defined by the following formulas*

$$(Af)(z) = \begin{cases} \varphi(z)\frac{f(u(z)+\sqrt{v(z)})-f(u(z)-\sqrt{v(z)})}{2\sqrt{v(z)}}, & \text{if } z \in V, \\ \varphi(z) f'(u(z)), & \text{if } z \in G \setminus V; \end{cases}$$

$$(Bf)(z) = \begin{cases} \frac{f(u(z)+\sqrt{v(z)})+f(u(z)-\sqrt{v(z)})}{2\alpha(z)}, & \text{if } z \in V, \\ \frac{1}{\alpha} f(u(z)), & \text{if } z \in G \setminus V, \end{cases}$$

where $\varphi, u, v \in \mathcal{H}(G)$ *and* $v \not\equiv 0$, *the set* $G \setminus V$ *coincides with the zero set of the function* $v(z)$; *the functions* $u(z)$ *and* $v(z)$ *are such that the functions* $\chi_1(z) = u(z) + \sqrt{v(z)}$ *and* $\chi_2(z) = u(z) - \sqrt{v(z)}$ *satisfy the following conditions:* $\chi_1(G) \subset G$, $\chi_2(G) \subset G$.

Proof. Assume that the nonzero linear operators A and B of $\mathcal{H}(G)$ satisfy (3.6). Let us show that (3.7) holds. To obtain a contradiction we suppose that there exists a point $z_0 \in G$, such that $\alpha(z_0) = 0$. Substituting $g \equiv 1$ in (3.6) we obtain that $(Af)(z_0) = 0$ holds for any $f \in \mathcal{H}(G)$. Let $n(f)$ stands for the multiplicity of the zero z_0 of the function Af (in the case when $(Af)(z) \equiv 0$ in G we put $n(f) = +\infty$). Let $n_0 = \min\{n(f) : f \in \mathcal{H}(G)\}$. By the assumption the operator A is nonzero. Then for an arbitrary $f \in \mathcal{H}(G)$ the point z_0 is a removable singularity for the function $\frac{(Af)(z)}{(z-z_0)^{n_0}}$. Define $(A_1 f)(z) = \frac{(Af)(z)}{(z-z_0)^{n_0}}$ if $z \neq z_0$. Then A_1 can be extended to an operator which linearly acts in $\mathcal{H}(G)$. The definition of n_0 implies that there exists the function $f_0 \in \mathcal{H}(G)$ such that $(A_1 f_0)(z_0) \neq 0$. For any function $f \in \mathcal{H}(G)$ the following equality holds:

$$(Af)(z) = (z - z_0)^n (A_1 f)(z). \tag{3.8}$$

Substituting (3.8) in (3.6) we obtain that

$$(A_1(fg))(z) = \alpha(z)((A_1 f)(z)(Bg)(z) + (A_1 g)(z)(Bf)(z)) \tag{3.9}$$

for all $f, g \in \mathcal{H}(G)$, $z \in G$. Setting $g \equiv 1$, $f = f_0$ and $z = z_0$ in (3.9) we get a contradiction. Thus, (3.7) holds.

Under condition (3.7) a pair of linear operators A, B on $\mathcal{H}(G)$ satisfies (3.6) if and only if the pair of operators A, αB satisfy Rubel's equation (1.2). Therefore, the assertion of the theorem follows from the main theorem of [7]. □

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A PEROV TYPE THEOREM FOR CYCLIC CONTRACTIONS AND APPLICATIONS TO SYSTEMS OF INTEGRAL EQUATIONS

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Abstract. In this paper we will prove a fixed point theorem of Perov type for cyclic contractions on complete generalized metric spaces. Then, as an application, we will study the existence, uniqueness and approximation of the solution for a system of integral equations.

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1. PRELIMINARIES

We begin the considerations with some notions and results which will be useful further in this paper.

Let (X,d) be a metric space. We denote:

$$P(X) := \{Y \subset X \mid Y \text{ is nonempty}\};$$
$$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\};$$

If $T : Y \subseteq X \to X$ is a single-valued operator, then the symbol

$$\mathrm{F_T} := \{\mathrm{x} \in \mathrm{Y} \mid \mathrm{x} \in \mathrm{Tx}\}$$

denotes the fixed point set of T.

Definition 1. A matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ is called a matrix convergent to zero if $S^k \to 0$ as $k \to +\infty$.

Theorem 1 ([5], [4])**.** *Let $S \in \mathcal{M}_p(\mathbb{R}_+)$. The following statements are equivalent:*

(i) *S is a matrix convergent to zero;*
(ii) *$S^k x \to 0$ as $k \to +\infty$, $\forall\, x \in \mathbb{R}^p$;*
(iii) *$I_p - S$ is non-singular and*

$$(I_p - S)^{-1} = I_p + S + S^2 + \ldots \tag{1.1}$$

(iv) *$I_p - S$ is non-singular and $(I_p - S)^{-1}$ has nonnegative elements;*
(v) *$\lambda \in \mathbb{C}$, $\det(S - \lambda I_p) = 0$ imply $|\lambda| < 1$.*

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The matrices convergent to zero were used by A.I. Perov [2] to generalize the contraction principle in the case of metric spaces with a vector-valued distance.

Definition 2 ([5]). Let (X,d) be a metric space with $d : X \times X \to \mathbb{R}_+^p$ a vector-valued distance and $T : X \to X$. The operator T is called an S-contraction if there exists a matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ such that:

(i) S is a matrix convergent to zero;
(ii) $d(T(x),T(y)) \leq Sd(x,y),\ \forall\ x,y \in X$.

Theorem 2 (Perov, [2]). *Let (X,d) be a complete metric space with $d : X \times X \to \mathbb{R}_+^p$ a vector-valued distance and $T : X \to X$ be an S-contraction. Then:*

(i) *T has a unique fixed point $x^* \in X$;*
(ii) *$T^k x \xrightarrow{d} x^*$ as $k \to +\infty$, for all $x \in X$;*
(iii) *$d(T^k x, x^*) \leq S^k (I_p - S)^{-1} d(x, Tx)$, for all $x \in X$ and $k \in \mathbb{N}$;*
(iv) *$d(x,x^*) \leq (I_p - S)^{-1} d(x,Tx)$ for all $x \in X$.*

Another consistent generalization of the contraction principle was given by Kirk, Srinivasan and Veeramani, using the concept of cyclic operator.

Theorem 3 ([1]). *Let $\{A_i\}_{i=1}^m$ be nonempty subsets of a complete metric space, and suppose $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ satisfies the following conditions (where $A_{m+1} = A_1$):*

(1) *$TA_i \subseteq A_{i+1}$ for $1 \leq i \leq m$;*
(2) *$\exists\ k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y),\ \forall\ x \in A_i,\ y \in A_{i+1}$, for $1 \leq i \leq m$.*

Then T has a unique fixed point.

This theorem suggested the introduction of the following

Definition 3 ([3]). Let X be a nonempty set, m a positive integer and $T : X \to X$ an operator. By definition, $\bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T if:

(i) $X = \bigcup_{i=1}^m A_i$, with $A_i \in P(X)$, for $1 \leq i \leq m$;
(ii) $TA_i \subseteq A_{i+1}$, for $1 \leq i \leq m$, where $A_{m+1} = A_1$.

2. Main results

Definition 4. Let (X,d) be a metric space with $d : X \times X \to \mathbb{R}_+^p$ a vector-valued distance, $A_1, \ldots, A_m \in P_{cl}(X)$ and $T : X \to X$ be an operator. If:

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T;

(ii) there exists a matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ convergent to zero such that

$$d(Tx, Ty) \leq S \cdot d(x, y), \text{ for any } x \in A_i,\ y \in A_{i+1}, \text{ where } A_{m+1} = A_1,$$

then, by definition, we say that T is a cyclic S-contraction.

Theorem 4. *Let (X, d) be a complete metric space with $d : X \times X \to \mathbb{R}_+^p$ a vector-valued distance, $A_1, A_2, \ldots, A_m \in P_{cl}(X)$. If $T : X \to X$ is a cyclic S-contraction then the following statements hold:*

(1) *T has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $\{x_n\}_{n\geq 0}$ given by*

$$x_n = Tx_{n-1},\ n \geq 1,$$

converges to x^* for any starting point $x_0 \in X$;

(2) *the following estimates hold:*

$$d(x_n, x^*) \leq S^n (I_p - S)^{-1} d(x_0, x_1),\ n \geq 1; \tag{2.1}$$

$$d(x_n, x^*) \leq (I_p - S)^{-1} d(x_n, x_{n+1}),\ n \geq 1; \tag{2.2}$$

(3) *for any $x \in X$,*

$$d(x, x^*) \leq (I_p - S)^{-1} d(x, Tx). \tag{2.3}$$

Proof. (1)

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq S d(x_{n-1}, x_n) \\ &\leq \ldots \leq S^n d(x_0, x_1) \end{aligned}$$

For $k \geq 1$ we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq S^n d(x_0, x_1) + S^{n+1} d(x_0, x_1) + \ldots + S^{n+k-1} d(x_0, x_1) \\ &= S^n (I_p + S + S^2 + \ldots + S^{k-1}) d(x_0, x_1) \\ &\leq S^n (I_p + S + S^2 + \ldots) d(x_0, x_1) \to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.4}$$

which means that $(x_n)_{n\geq 0}$ is a Cauchy sequence.

(X, d) is a complete metric space, so the sequence $(x_n)_{n\geq 0}$ is convergent to a $q \in X$.

The sequence $(x_n)_{n\geq 0}$ has an infinite number of terms in each A_i, $i = \overline{1, m}$, so from each A_i one we can extract a subsequence of $(x_n)_{n\geq 0}$ which converges to $q = \lim_{n\to\infty} x_n$.

Because A_i are closed, $q \in \bigcap_{i=1}^{m} A_i$, so $\bigcap_{i=1}^{m} A_i \neq \varnothing$.

Let be the restriction $T\Big|_{\bigcap_{i=1}^{m} A_i} : \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i$.

$\bigcap_{i=1}^{m} A_i$ is also complete. Applying Perov's theorem, $T\Big|_{\bigcap_{i=1}^{m} A_i}$ has a unique fixed point, which can be obtained by means of the Picard iteration starting from any initial point. It remains to prove that the Picard iteration converges to x^*, for any initial guess $x \in X$.

$$\begin{aligned} d(x_{n+1},x^*) = d(Tx_n,Tx^*) &\leq Sd(x_n,x^*) \\ &\leq \ldots \leq S^n d(x_0,x^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

$$\begin{aligned} (2)\quad d(x_n,x_{n+k}) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{n+k-1},x_{n+k}) \\ &\leq d(x_n,x_{n+1}) + Sd(x_n,x_{n+1}) + \ldots + S^{k-1}d(x_n,x_{n+1}) \\ &= (I_p + S + \ldots + S^{k-1})d(x_n,x_{n+1}), \text{ for any } n \in \mathbb{N},\ k \geq 1. \end{aligned} \tag{2.5}$$

Using the statement (iii) from Theorem 1, by letting $k \to \infty$ in (2.4) and (2.5) we obtain the estimates (2.1) and (2.2).

(3) Let $x \in X$. For $n = 0$, $x_0 := x$, the a posteriori estimate (2.2) becomes

$$d(x,x^*) \leq (I_p - S)^{-1} d(x,Tx).$$

□

Theorem 5. (Data dependence theorem) *Let $T : X \to X$ be as in Theorem 4 with $F_T = \{x_T^*\}$. Let $U : X \to X$ be an operator such that:*

(i) *U has at least one fixed point x_U^*;*

(ii) *there exists $\eta > 0$ such that*

$$d(Tx,Ux) \leq \eta, \text{ for any } x \in X.$$

Then $d(x_T^*,x_U^*) \leq \eta(I_p - S)^{-1}$.

Proof. By letting $x := x_U^*$ in the inequality (2.3), we have

$$\begin{aligned} d(x_U^*,x_T^*) &\leq (I_p - S)^{-1} d(x_U^*,Tx_U^*) = (I_p - S)^{-1} d(Ux_U^*,Tx_U^*) \\ &\leq (I_p - S)^{-1}\eta. \end{aligned}$$

□

Theorem 6. *Let $T : X \to X$ be as in Theorem 4. Then the fixed point problem for T is well posed, that is, assuming there exist $z_n \in X$, $n \in \mathbb{N}$ such that $d(z_n,Tz_n) \to 0$, as $n \to \infty$, this implies that $z_n \to x^*$, as $n \to \infty$, where $F_T = \{x^*\}$.*

Proof. By letting $x := z_n$ in the inequality (2.3), we have

$$d(z_n, x^*) \leq (I_p - S)^{-1} d(z_n, Tz_n),\ n \in \mathbb{N}$$

and letting $n \to \infty$ we obtain $d(z_n, x^*) \to 0$, $n \to \infty$. □

3. An application to a system of integral equations

We apply the results given by Theorem 2.1 to study the existence and the uniqueness of solutions of the following system of integral equations:

$$\begin{cases} x_1(t) = \displaystyle\int_a^b G_1(t,s) f_1(s, x_1(s), x_2(s)) ds \\ x_2(t) = \displaystyle\int_a^b G_2(t,s) f_2(s, x_1(s), x_2(s)) ds \end{cases}, \ t \in [a,b] \tag{3.1}$$

where $a, b \in \mathbb{R}$, $a < b$,

$$G_1, G_2 \in C([a,b] \times [a,b], [0,\infty)),$$
$$f_1, f_2 \in C([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}).$$

Theorem 7. *We suppose that:*

(i) *there exist* $\alpha_k, \beta_k \in C([a,b], \mathbb{R})$, $m_k, M_k \in \mathbb{R}$ *with* $m_k \leq \alpha_k(t) \leq \beta_k(t) \leq M_k$, *for any* $t \in [a,b]$, *such that*

$$\begin{cases} \alpha_k(t) \leq \displaystyle\int_a^b G_k(t,s) f_k(s, \beta_1(s), \beta_2(s)) ds \\ \beta_k(t) \geq \displaystyle\int_a^b G_k(t,s) f_k(s, \alpha_1(s), \alpha_2(s)) ds \end{cases} \text{for } k \in \{1,2\} \tag{3.2}$$

(ii) *there exist* $a_1, b_1, a_2, b_2 \in \mathbb{R}_+$ *such that*

$$\begin{aligned} |f_1(s,u_1,u_2) - f_1(s,v_1,v_2)| &\leq a_1|u_1 - v_1| + a_2|u_2 - v_2|, \\ |f_2(s,u_1,u_2) - f_2(s,v_1,v_2)| &\leq b_1|u_1 - v_1| + b_2|u_2 - v_2|, \end{aligned} \tag{3.3}$$

for any $s \in [a,b]$ *and* $u_k, v_k \in \mathbb{R}$, *with*

$$\begin{cases} u_k \leq M_k \\ v_k \geq m_k \end{cases} \quad or \quad \begin{cases} u_k \geq m_k \\ v_k \leq M_k \end{cases} \quad \text{for } k \in \{1,2\};$$

(iii) $\sup\limits_{t \in [a,b]} \displaystyle\int_a^b G_k(t,s) ds \leq 1$ *for* $k \in \{1,2\}$;

(iv) f_k *is decreasing in each of the last two variables, that is,*

$$u_1, u_2, v_1, v_2 \in \mathbb{R},\ u_1 \leq v_1,\ u_2 \leq v_2 \Rightarrow f_k(s,u,v) \geq f_k(s,u_2,v_2),$$

for any $s \in [a,b]$, *and* $k \in \{1,2\}$;

(v) *the matrix* $S = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ *converges to zero.*

Then the system (3.1) has a unique solution $x^* = (x_1^*, x_2^*) \in C([a,b], \mathbb{R}^2)$, *with* $\alpha_k \leq x_k^* \leq \beta_k$, *for* $k \in \{1,2\}$.
This solution can be obtained by the successive approximations method, starting at any element $x^0 \in C([a,b], \mathbb{R}^2)$. *Moreover, if* x^n *is the* n^{th} *successive approximation, then we have the following estimation:*

$$\|x^* - x^n\| \leq S^n (I_2 - S)^{-1} \|x^0 - x^1\|,$$

where

$$\|x\| = \begin{pmatrix} |x_1|_\infty \\ |x_2|_\infty \end{pmatrix} \quad \text{and} \quad |x|_\infty = \max_{t \in [a,b]} |x(t)|.$$

Proof. Let us denote

$$X := (C([a,b], \mathbb{R}), |\cdot|_\infty), \ Z = X \times X \tag{3.4}$$

$$\|\cdot\| : Z \to \mathbb{R}^2, \ \|x\| = \|(x_1, x_2)\| = \begin{pmatrix} |x_1|_\infty \\ |x_2|_\infty \end{pmatrix},$$

where $|x_k|_\infty = \max\limits_{t \in [a,b]} |x_k(t)|$ is the Chebyshev norm.

Then $(Z, \|\cdot\|)$ is a generalized Banach space.

We consider the following closed subsets of X:

$$A_1 = \{(x_1, x_2) \in Z \mid x_k \leq \beta_k, \ k \in \{1,2\}\},$$

$$A_2 = \{(x_1, x_2) \in Z \mid x_k \geq \alpha_k, \ k \in \{1,2\}\},$$

and the operator $T : Z \to Z$,

$$(x_1, x_2) = x \mapsto Tx = (T_1 x, T_2 x),$$

$$T_k x(t) := \int_a^b G_k(t,s) f_k(s, x_1(s), x_2(s)) ds, \text{ for } k \in \{1,2\}. \tag{3.5}$$

The system (3.1) is equivalent with the equation $Tx = x$.
We will prove that $A_1 \cup A_2$ is a cyclic representation of Z with respect to T.
Let $x = (x_1, x_2) \in A_1 \Rightarrow x_k(s) \leq \beta_k(s), \ \forall \ s \in [a,b]$, for $k \in \{1,2\}$.
Using the monotonicity of f_k we have

$$G_k(t,s) f_k(s, x_1(s), x_2(s)) \geq G_k(t,s) f_k(s, \beta_1(s), \beta_2(s)), \text{ for } k \in \{1,2\}$$

and from (i), by integration,

$$\int_a^b G_k(t,s) f_k(s, x_1(s), x_2(s)) ds \geq \alpha_k(t),$$

which means that

$$T_k x(t) \geq \alpha_k(t), \ \forall \ t \in [a,b], \text{ for } k \in \{1,2\} \Rightarrow Tx \in A_2.$$

So $TA_1 \subseteq A_2$. In a similar way we have $TA_2 \subseteq A_1$.

Using the conditions (ii) and (iii) we have

$$\begin{aligned}
|T_k x(t) - T_k y(t)| &\leq \int_a^b G_k(t,s)|f_k(s,x_1(s),x_2(s)) - f_k(s,y_1(s),y_2(s))|ds \\
&\leq \int_a^b G_k(t,s)(a_k|x_1(s)-y_1(s)| + b_k|x_2(s)-y_2(s)|)ds \\
&\leq \int_a^b G_k(t,s)(a_k|x_1-y_1|_\infty + b_k|x_2-y_2|_\infty) \\
&\leq a_k|x_1-y_1|_\infty + b_k|x_2-y_2|_\infty, \ \forall\, t \in [a,b] \\
&\Rightarrow |T_k x - T_k y|_\infty \leq a_k|x_1-y_1|_\infty + b_k|x_2-y_2|_\infty \\
&\Rightarrow \begin{pmatrix} |T_1 x - T_1 y|_\infty \\ |T_2 x - T_2 y|_\infty \end{pmatrix} \leq S \begin{pmatrix} |x_1-y_1|_\infty \\ |x_2-y_2|_\infty \end{pmatrix},
\end{aligned}$$

so we have

$$\|Tx - Ty\| \leq S\|x-y\|, \text{ for any } (x,y) \in A_1 \times A_2,$$

and by the condition (v) it results that the operator T is a cyclic S-contraction.

All the conditions of Theorem 4 are satisfied, so T has a unique fixed point

$$x^* = (x_1^*, x_2^*) \in A_1 \cap A_2, \text{ with } \alpha_k \leq x_k^* \leq \beta_k, \text{ for } k \in \{1,2\}.$$

This finishes the proof.

□

Further on, we will study the continuous dependence phenomenon for the system (3.1).

We consider the perturbed system of integral equations

$$\begin{cases} y_1(t) = \displaystyle\int_a^b H_1(t,s) g_1(s, y_1(s), y_2(s))ds \\ y_2(t) = \displaystyle\int_a^b H_2(t,s) g_2(s, y_1(s), y_2(s))ds \end{cases} \tag{3.6}$$

where

$$H_1, H_2 \in C([a,b]\times[a,b],[0,\infty)), \quad g_1, g_2 \in C([a,b]\times\mathbb{R}\times\mathbb{R},\mathbb{R}).$$

Theorem 8. *We suppose that the conditions of Theorem 7 are satisfied and we denote by x^* the unique solution of the system of integral equations (3.1).*

If $y^* \in C([a,b],\mathbb{R}^2)$ is a solution of the perturbed system of integral equations (3.6), and

$$\sup_{t\in[a,b]} \int_a^b H_k(t,s)ds \leq 1,$$

then we have the following estimation:

$$\|x^* - y^*\|_{\mathbb{R}^2} \leq (I_2 - S)^{-1}(\eta + \tau), \tag{3.7}$$

where $\eta = (\eta_1, \eta_2)$, $\tau = (\tau_1, \tau_2)$ *and*

$$\begin{cases} \eta_k = \sup\{|f_k(s,u,v)| \ | \ s \in [a,b], \ u,v \in \mathbb{R}\}, \\ \tau_k = \sup\{|g_k(s,u,v)| \ | \ s \in [a,b], \ u,v \in \mathbb{R}\}, \end{cases} \quad \text{for } k \in \{1,2\}.$$

Proof. We consider the operator $T : Z \to Z$ attached to the system (3.1), defined by the relation (3.5).

Let $U : Z \to Z$ be an operator attached to the perturbed system (3.6) and defined by the relation:

$$(y_1, y_2) = y \mapsto Uy = (U_1 y, U_2 y),$$

$$U_k y(t) := \int_a^b H_k(t,s) g_k(s, y_1(s), y_2(s)) ds, \text{ for } k \in \{1,2\}.$$

We have

$$\begin{aligned} |T_k x(t) - U_k x(t)| &\leq \int_a^b G_k(t,s) |f_k(s, x_1(s), x_2(s))| ds \\ &\quad + \int_a^b H_k(t,s) |g_k(s, x_1(s), x_2(s))| ds \\ &\leq \eta_k \int_a^b G_k(t,s) ds + \tau_k \int_a^b H_k(t,s) ds \\ &\leq \eta_k + \tau_k, \ \forall \ t \in [a,b], \ \text{ for } k \in \{1,2\} \\ &\Rightarrow |T_k x - U_k x|_\infty \leq \eta_k + \tau_k \\ &\Rightarrow \|Tx - Ux\| \leq \eta + \tau, \ \forall \ x \in Z. \end{aligned}$$

The conditions of Theorem 6 are satisfied, so estimation (3.7) is proved.

□

Remark 1. A similar approach can be achieved for a system of Volterra type integral equations using, instead of the supremum norm, the Bielecki type norm approach.

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ON THE SONG RECURRENCE RELATION FOR THE RIEMANN ZETA FUNCTION

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Abstract. In this paper, the author uses the generating function for the Bernoulli numbers in order to obtain a new proof for a known linear recurrence relation of the Riemann zeta function with even integer arguments, $\zeta(2n)$.

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Keywords: Riemann zeta function, Bernoulli numbers, recurrences

1. Introduction

Originally the Riemann zeta function was defined for real arguments by Euler as

$$\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x},\qquad x>1.$$

Euler first started to develop the theory of this function and obtained in 1734 the famous formula for even positive zeta values

$$\zeta(2n)=(-1)^{n+1}\frac{(2\pi)^{2n}}{2\cdot(2n)!}B_{2n}, \tag{1.1}$$

where n is a positive integer and B_n is the n-th Bernoulli number. There are many proofs of this formula, some of them elementary, see, e.g., [1–3, 5, 10, 12–14].

The following recurrence relation for $\zeta(2n)$ has been derived by Song [11, eq. 15] in 1987 using Fourier series expansion of periodic functions. In this work, Song did not make use of the Riemann zeta function and always wrote the sums explicitly.

Theorem 1. *For $n>0$,*

$$(-1)^n\frac{\pi^{2n}\cdot n}{(2n+1)!}+\sum_{k=0}^{n-1}(-1)^k\frac{\pi^{2k}}{(2k+1)!}\zeta(2n-2k)=0.$$

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We see that this linear recurrence relation does not require a priori knowledge of the Bernoulli numbers. If n is small, then it is relatively easy to evaluate $\zeta(2n)$, namely

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{3!},\\
\zeta(4) &= \frac{\pi^2}{3!}\cdot\zeta(2)-\frac{\pi^4}{5!}\cdot 2=\frac{\pi^4}{90},\\
\zeta(6) &= \frac{\pi^2}{3!}\cdot\zeta(4)-\frac{\pi^4}{5!}\cdot\zeta(2)+\frac{\pi^6}{7!}\cdot 3=\frac{\pi^6}{945},\\
\zeta(8) &= \frac{\pi^2}{3!}\cdot\zeta(6)-\frac{\pi^4}{5!}\cdot\zeta(4)+\frac{\pi^6}{7!}\cdot\zeta(2)-\frac{\pi^8}{9!}\cdot 4=\frac{\pi^8}{9450}.
\end{aligned}$$

In 2008, Lettington [7, Lemma 11.1.2, pp. 140–142] rediscovered this result in his doctoral thesis by deriving the Bernoulli recurrence relation

$$2^{2n-1}B_{2n}=\frac{n}{2n+1}-\frac{1}{2n+1}\sum_{k=1}^{n-1}\binom{2n+1}{2k}2^{2k-1}B_{2k}.$$

Moreover, he showed in [8] that this linear recurrence for the Riemann zeta function occurs naturally in a certain variety of odd-side magic square matrix with multiplicative symmetries. These symmetries are discussed more generally in [4].

In this paper, based on the generating function of the Bernoulli numbers and Euler's formula (1.1), we provide a proof of Theorem 1 which is different from Song's and Lettington's proofs. We remark that the proof via generating functions for the Bernoulli numbers is of interest, as can be observed in the recent paper by Coffey [6].

2. A PROOF OF THEOREM 1

We consider the following two series:

$$\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n=\frac{x}{e^x-1},\qquad |x|<2\pi$$

and

$$\sum_{n=0}^{\infty}\frac{1}{n!}x^n=e^x,\qquad \text{for all } x.$$

For $|x|<\pi$, we can write

$$\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}(2x)^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)=\frac{2xe^x}{e^{2x}-1}$$

and

$$2\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n-\sum_{n=0}^{\infty}\frac{B_n}{n!}(2x)^n=\frac{2x}{e^x-1}-\frac{2x}{e^{2x}-1}=\frac{2xe^x}{e^{2x}-1}.$$

Thus we deduce that

$$\left(\sum_{n=0}^{\infty}\frac{2^nB_n}{n!}x^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)=\sum_{n=0}^{\infty}\frac{2-2^n}{n!}B_nx^n,\qquad |x|<\pi. \tag{2.1}$$

Equating coefficients of x^{2n+1} on each side give the relation

$$\sum_{k=0}^{2n+1}\frac{2^kB_k}{k!\cdot(2n+1-k)!}=0. \tag{2.2}$$

Moreover, taking into account that $B_1=-1/2$ and $B_{2n+1}=0$ for $n>0$, we deduce

$$\frac{1}{(2n)!}-\sum_{k=0}^{n}\frac{2^{2k}B_{2k}}{(2k)!\cdot(2n+1-2k)!}=0$$

or

$$\frac{1}{(2n)!}-\sum_{k=0}^{n}\frac{2^{2n-2k}B_{2n-2k}}{(2n-2k)!\cdot(2k+1)!}=0.$$

Using Euler's formula (1.1), we obtain the relation

$$2\sum_{k=0}^{n}\frac{(-1)^{n-k+1}\zeta(2n-2k)}{\pi^{2n-2k}\cdot(2k+1)!}-\frac{1}{(2n)!}=0,$$

that can be rewritten as

$$\sum_{k=0}^{n-1}(-1)^k\frac{\pi^{2k}}{(2k+1)!}\zeta(2n-2k)+(-1)^n\frac{\pi^{2n}}{(2n+1)!}\cdot n=0.$$

This completes the proof.

3. Concluding remarks.

Due to the results of Euler and the more recent works described herein, we know that the linear recurrence relations for $\zeta(2n)$ are intimately related to the properties of the Bernoulli numbers. In this paper, we have obtained a new proof of the linear recurrence for $\zeta(2n)$ considered by Song and Lettington. Moreover, equating coefficients of x^{2n} on each side of the relation (2.1), we obtain the relation

$$\sum_{k=0}^{n}\frac{2^{2k}B_{2k}}{(2k)!(2n-2k)!}=\frac{2-2^{2n}}{(2n)!}B_{2n}+\frac{2n}{(2n)!},$$

that allows us to derive the following recurrence relation for $\zeta(2n)$.

Theorem 2. *For $n > 0$,*

$$\zeta(2n)+\frac{2^{2n-1}}{2^{2n}-1}\left((-1)^n\frac{\pi^{2n}}{(2n)!}\cdot\frac{2n-1}{2}+\sum_{k=1}^{n-1}(-1)^k\frac{\pi^{2k}}{(2k)!}\zeta(2n-2k)\right)=0.$$

According to the last theorem, we can write

$$\begin{aligned}
\zeta(2)&=\frac{2}{3}\cdot\frac{\pi^2}{2!}\cdot\frac{1}{2}=\frac{\pi^2}{6},\\
\zeta(4)&=\frac{8}{15}\cdot\left(\frac{\pi^2}{2!}\cdot\zeta(2)-\frac{\pi^4}{4!}\cdot\frac{3}{2}\right)=\frac{\pi^4}{90},\\
\zeta(6)&=\frac{32}{63}\cdot\left(\frac{\pi^2}{2!}\cdot\zeta(4)-\frac{\pi^4}{4!}\cdot\zeta(2)+\frac{\pi^6}{6!}\cdot\frac{5}{2}\right)=\frac{\pi^6}{945},\\
\zeta(8)&=\frac{128}{255}\cdot\left(\frac{\pi^2}{2!}\cdot\zeta(6)-\frac{\pi^4}{4!}\cdot\zeta(4)+\frac{\pi^6}{6!}\cdot\zeta(2)-\frac{\pi^8}{8!}\cdot\frac{7}{2}\right)=\frac{\pi^8}{9450}.
\end{aligned}$$

A final remark is that as with Lemma 2.1 of [9], these type of recurrence relations often reduce to relations involving Bernoulli numbers, binomial coefficients and powers of 2. Here we find that Theorems 1 and 2 are consequences of the recurrence relation

$$\sum_{k=0}^{n}\binom{n}{k}2^kB_k=(2-2^n)B_n.$$

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THE R-JACOBI-STIRLING NUMBERS OF THE SECOND KIND

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Abstract. In this paper, we study the r-Jacobi-Stirling numbers of the second kind introduced by Gelineau in his Phd thesis. We give, upon using combinatorial and analytic arguments, the ordinary generating function of these numbers, two recurrence relations, their exact expressions and the log-concavity.

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Keywords: the r-Jacobi-Stirling of the second kind, generating function, recurrence relations, log-concavity

1. INTRODUCTION

Let α and β be real numbers such that $\alpha > -1$ and $\beta > -1$. The n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is defined as the unique polynomial solution with degree n satisfies the following Jacobi ordinary differential equation of second order:

$$\left(1-x^2\right) y''(x) + (\beta - \alpha - (\alpha+\beta+2)\, x)\, y'(x) + n\,(n+\alpha+\beta+1)\, y(x) = 0 \tag{1.1}$$

with initial conditions

$$P_n^{(\alpha,\beta)}(1) = \binom{\alpha+1}{n} \text{ and } P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{\beta+1}{n}. \tag{1.2}$$

We introduce Jacobi differential operator $l_{(\alpha,\beta)}[y](x)$ to be

$$l_{(\alpha,\beta)}[y](x) = \frac{\left(-w_{(\alpha+1,\beta+1)}(x)\, y'(x)\right)'}{w_{(\alpha,\beta)}(x)}, \quad w_{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}. \tag{1.3}$$

Then, the polynomial solution $y = P_n^{(\alpha,\beta)}(x)$ of (1.1) satisfies

$$l_{(\alpha,\beta)}[y](x) = n\,(n+\alpha+\beta+1)\, y(x). \tag{1.4}$$

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For the n-th composite powers of the Jacobi operator $l_{(\alpha,\beta)}$ given inductively by $l^{(n)}_{(\alpha,\beta)}[y] = l_{(\alpha,\beta)}\left(l^{(n-1)}_{(\alpha,\beta)}[y]\right)$, Everitt et al. [2, Thm. 4.2] proved that

$$l^{(n)}_{(\alpha,\beta)}[y](x) = \frac{1}{w_{(\alpha,\beta)}(x)} \sum_{k=0}^{n} (-1)^k JS(n,k;\alpha,\beta) \left(w_{(\alpha+k,\beta+k)}(x)\, y^{(k)}(x)\right)^{(k)}, \tag{1.5}$$

where the coefficients $JS(n,k;\alpha,\beta)$ are the Jacobi-Stirling numbers of the second kind. The same authors proved in [2, Eq. 4.4] that the Jacobi-Stirling number $JS(n,k;\alpha,\beta)$ has the following exact expression

$$JS(n,k;\alpha,\beta) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(\alpha+\beta+2j+1)\,(j\,(j+\alpha+\beta+1))^n}{(\alpha+\beta+k+j+1)_{k+1}}, \tag{1.6}$$

where $(x)_n := x\,(x-1)\cdots(x-n+1)$ if $n \geq 1$ and $(x)_0 := 1$.
We remark that the last expression of $JS(n,k;\alpha,\beta)$ depends only on $\alpha+\beta$, so if we set $z := \alpha+\beta+1$ and $JS(n,k;\alpha,\beta) := JS(n,k;z)$, the identity given in (1.6) becomes

$$JS(n,k;z) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{z+2j}{(z+k+j)_{k+1}} (j\,(j+z))^n, \quad z > -1. \tag{1.7}$$

Using different methods, Everitt et al. [2, Thm. 4.1] and Gelineau et al. [4, Sec. 4.2] showed that

$$X^n = \sum_{k=0}^{n} JS(n,k;z)\,(X)_{k,z}, \tag{1.8}$$

where $(X)_{k,z} := \prod_{i=0}^{k-1} (X - i\,(i+z))$ if $k \geq 1$ and $(X)_{0,z} := 1$.
The equation (1.8) shows the the Jacobi-Stirling numbers satisfy the following recurrence relation:

$$\begin{aligned}
&JS(0,0;z) = 1,\\
&JS(n,0;z) = JS(0,k;z) = 0 \ \text{ if } n,k \geq 1,\\
&JS(n,k;z) = JS(n-1,k-1;z) + k\,(k+z)\,JS(n-1,k;z) \ \text{ if } n,k \geq 1.
\end{aligned}$$

To give a combinatorial interpretation of the Jacobi-Stirling number $JS(n,k;2\gamma-1)$, let $[\pm n]$ be the set $\{\pm 1, \pm 2, ..., \pm n\}$ and we will use definition of the Jacobi-Stirling set partition [1, Def. 4.1] given as follows.

Definition 1. For all positive integers n, k, γ, a Jacobi-Stirling set partition of $[\pm n]$ into γ zero blocks $A_1, \ldots, A_\gamma$ and k nonzero blocks $B_1, \ldots, B_k$ is an ordinary set partition of $[\pm n]$ into $k+\gamma$ blocks for which the following conditions hold:

(1) The blocks $A_1,\ldots,A_\gamma$, called the zero blocks, are distinguishable, but the blocks $B_1,\ldots,B_k$ are indistinguishable.
(2) The zero blocks may be empty, but all other blocks are nonempty.
(3) $\forall i \in [n]=\{1,2,\ldots,n\}$, $\{-i,i\} \not\subseteq \bigcup_{j=1}^{\gamma} A_j$,
(4) $\forall j \in [k]$, $\forall i \in [n]$, we have $\{-i,i\} \subset B_j \Leftrightarrow i = \min B_j$.

Andrews et al. showed that the Jacobi-Stirling number $JS(n,k;2\gamma-1)$ is the number of Jacobi-Stirling set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks [1, Thm. 4.1].
In his Phd Thesis, Gelineau [3, Sec. 1.4] has introduced the r-Jacobi-Stirling numbers of the second kind $JS_r(n,k;z)$, $n \geq r \geq 0$, as follows.

Definition 2. The r-Jacobi-Stirling number $JS_r(n,k;2\gamma-1)$ is the number of Jacobi-Stirling set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks such that $\min B_j = j$ for $j = 1,\ldots,r$.

The author gives some interesting properties of these numbers. In this paper, we study the r-Jacobi-Stirling numbers of the second kind by giving, with combinatorial and analytic proofs, their ordinary generating function, two recurrence relations, their exact expressions and the log-concavity.

2. Main results

To start, we give in the following theorem the ordinary generating function.

Theorem 1. *For all positive integers n, k, the r-Jacobi-Stirling number $JS_r(n,k;z)$ has ordinary generating function to be*

$$\sum_{n\geq 0} JS_r(n+k,k;z)\,t^n = \left(\prod_{i=r}^{k}(1-i\,(i+z)\,t)\right)^{-1}.$$

Proof. Let γ be a positive integer and $z = 2\gamma-1$. According to Definition 2, let $m_j = \min B_j$, $j = 1,\ldots,k$ be fixed and ordering the minimal elements of the nonzero blocks to get $m_1 < m_2 < \cdots < m_k$. We have $m_1 = 1,\ldots,m_r = r$ and $m_j \geq j$ for $j = r+1,\ldots,k$. For instance, let the sets $A_1,\ldots,A_\gamma$ be empty and $B_j = \{m_j\}$, $j = 1,\ldots,k$. Hence, we count the number of ways to construct such blocks $A_1,\ldots,A_\gamma$ and $B_1,\ldots,B_k$ by insertion the elements of the set $[\pm n]-\{\pm m_1,\ldots,\pm m_k\}$. Since the elements of the set $[n]-\{m_1,\ldots,m_k\}$ are the integer elements of the set $]m_r,m_{r+1}[\cup\cdots\cup]m_k,n+1[$, we can proceed the proof by insertion the integer elements (and their opposites) of each interval.
Indeed, let $j \in \{r,\ldots,k\}$ and $s \in \left]m_j,m_{j+1}\right[$ with convention $m_{k+1} = n+1$.
Case 1: If $-s \in \bigcup_{i=1}^{\gamma} A_i$ and $s \in \bigcup_{i=1}^{k} B_i$, there are γ ways to insert $-s$ in the zero blocks and since the blocks $B_{j+1},\ldots,B_k$, when they exist, have minimal elements

$\geq m_{j+1} > s > m_j \geq j$, it follows that s can be inserted only in the blocks $B_1, \ldots, B_j$. So, we count in this case γj possibilities.

Case 2: If $s \in \underset{i=1}{\overset{\gamma}{\cup}} A_i$ and $-s \in \underset{i=1}{\overset{k}{\cup}} B_i$, by symmetry, there are γj ways.

Case 3: If $\pm s \in \underset{i=1}{\overset{k}{\cup}} B_i$, since $-s$ and s can be inserted only in the blocks $B_1, \ldots, B_j$ and can't be inserted in the same block, there are $j\,(j-1)$ ways.

So, there are $2\gamma j + j\,(j-1) = j\,(j+z)$ ways to insert $\pm s$ and since the interval $\left]m_j, m_{j+1}\right[$ contains $n_j := m_{j+1} - m_j - 1$ integer elements, then the number of ways to insert the elements (and their opposites) of the interval $\left]m_j, m_{j+1}\right[$ is $(j\,(j+z))^{n_j}$. It follows that the number to insert the elements (and their opposites) of the set $]m_r, m_{r+1}[\cup \cdots \cup]m_k, m_{k+1}[$ is $\prod_{j=r}^{k} (j\,(j+z))^{n_j}$. So, the number $JS_r\,(n,k;z)$ must be equal to

$$\sum_{r=m_r<\cdots<m_k<n+1=m_{k+1}} \prod_{j=r}^{k} (j\,(j+z))^{n_j} = \sum_{n_r+\cdots+n_k=n-k} \prod_{j=r}^{k} (j\,(j+z))^{n_j}$$

which gives the desired result. □

Proposition 1. *For all positive integers n, k, r, the r-Jacobi-Stirling numbers $JS_r\,(n,k;z)$ satisfy*

$$JS_r\,(n,k;z) = JS_{r-1}\,(n,k;z) - (r-1)\,(r-1+z)\,JS_{r-1}\,(n-1,k;z).$$

Proof. The number of the set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks such that $\min B_j = j$ for $j = 1, \ldots, r-1$ but $\min B_r \neq r$ is $JS_{r-1}\,(n,k;2\gamma-1) - JS_r\,(n,k;2\gamma-1)$.

This number is exactly $(r-1)\,(r-2+2\gamma)\,JS_{r-1}\,(n-1,k;2\gamma-1)$ since the number of set partitions of the set $[\pm n] - \{\pm r\}$ is exactly the number of the set partitions of the set $[\pm\,(n-1)]$ and the elements $\pm r$ can be inserted only in the zero blocks or in $B_1 \cup \ldots \cup B_{r-1}$ in $(r-1)\,(r-2+2\gamma)$ ways, i.e.:

Case 1: If r is in one of the zero blocks (there are γ ways to insert it), then the element $-r$ can be inserted in the nonzero blocks in $r-1$ ways, i.e. $-r$ can't be inserted in the blocks $B_r, \ldots, B_k$ since these blocks have minimal elements $\geq r+1$. So this case gives $\gamma\,(r-1)$ possibilities.

Case 2: By symmetry, if $-r$ is in one of the zero blocks, we get $\gamma\,(r-1)$ possibilities.

Case 3: If $-r$ and r are both in the nonzero blocks, we get $(r-1)\,(r-2)$ possibilities since they can't both be in the same block. So $JS_{r-1}\,(n,k;2\gamma-1) - JS_r\,(n,k;2\gamma-1) = (r-1)\,(r-2+2\gamma)\,JS_{r-1}\,(n-1,k;2\gamma-1)$. □

Proposition 2. *For all non-negative integers n, k, r such that $0 \le r \le k \le n$ and $r < n$, the r-Jacobi-Stirling numbers $JS_r(n,k;z)$ satisfy*

$$JS_r(n,k;z) = JS_r(n-1,k-1;z) + k(k+z)\,JS_r(n-1,k;z).$$

Proof. The number of the set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks such that $\min B_j = j$ for $j = 1,\dots,r$ is $JS_r(n,k;2\gamma-1)$. This number can be obtained in another manner by considering n as a minimal element or not of a nonzero block. Indeed, if n is a minimal of such nonzero block, then this block contains only the two elements $\pm n$, in this case we have $JS_r(n-1,k-1;2\gamma-1)$ ways, otherwise, $[\pm(n-1)]$ can be partitioned in $k+\gamma$ blocks in $JS_r(n-1,k;2\gamma-1)$ ways, and, similar to the above proofs, $\pm n$ can be inserted in the $k+\gamma$ blocks in $k(k+2\gamma-1)$ ways. So $JS_r(n,k;z) = JS_r(n-1,k-1;z) + k(k+z)\,JS_r(n,k-1;z)$, where $z = 2\gamma-1$. □

Theorem 2. *For non-negative integers n, k, r, the r-Jacobi-Stirling numbers have the following expression:*

$$JS_r(n+r,k+r;z) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(2j+2r+z)\,((j+r)(j+r+z))^n}{(j+k+2r+z)_{k+1}}.$$

In particular, for $r = 0$ or $r = 1$, we get

$$JS(n,k;z) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{2j+z}{(j+k+z)_{k+1}}(j(j+z))^n.$$

Proof. From Theorem 1, there exist numbers $\lambda_r,\dots,\lambda_k$ such that

$$\sum_{n\ge 0} JS_r(n+k,k;z)\,t^n = \left(\prod_{i=r}^{k}(1-i(i+z)t)\right)^{-1} = \sum_{j=r}^{k}\frac{\lambda_j}{1-j(j+z)t}.$$

One can verify that we have

$$\lambda_j = \frac{1}{\prod_{i=r,i\ne j}^{k}\left(1-\frac{i(i+z)}{j(j+z)}\right)} = \frac{(j(j+z))^{k-r}}{\prod_{i=r,i\ne j}^{k}((j-i)(j+i+z))}$$

which can be simplified as

$$\lambda_j = \frac{(-1)^{k-j}}{(k-r)!}\binom{k-r}{j-r}\frac{2j+z}{(j+k+z)_{k-r+1}}(j(j+z))^{k-r}.$$

In other words, we have

$$\sum_{n\geq 0} JS_r\left(n+k,k;z\right)t^n=\sum_{j=r}^{k}\frac{\lambda_j}{1-j\left(j+z\right)t}=\sum_{j=r}^{k}\lambda_j\sum_{n\geq 0}\left(j\left(j+z\right)\right)^n t^n$$

which shows with the expressions of $\lambda_r,\ldots,\lambda_k$ that

$$\begin{aligned}JS_r\left(n,k;z\right)&=\sum_{j=r}^{k}\lambda_j\left(j\left(j+z\right)\right)^{n-k}\\&=\frac{1}{(k-r)!}\sum_{j=r}^{k}(-1)^{k-j}\binom{k-r}{j-r}\frac{2j+z}{\left(j+k+z\right)_{k-r+1}}\left(j\left(j+z\right)\right)^{n-r}.\end{aligned}$$

and this completes the proof. □

Remark 1. We have $JS_r\left(n+r,n+r;z\right)=1$ which gives with Theorem 2 the following identity

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{2j+2r+z}{\left(j+n+2r+z\right)_{n+1}}\left(\left(j+r\right)\left(j+r+z\right)\right)^n=n!.$$

Corollary 1. *For non-negative integers n, k, r, the r-Jacobi-Stirling numbers can be expressed in terms of the Jacobi-Stirling numbers as follows:*

$$JS_r\left(n+r,k+r;z\right)=\sum_{i=0}^{n-k}\binom{n}{i}\left(r\left(r+z\right)\right)^i JS\left(n-i,k;z+2r\right).$$

Proof. From Theorem 2 we may state:

$$\begin{aligned}&JS_r\left(n+r,k+r;z\right)\\&=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{2j+2r+z}{\left(j+2r+k+z\right)_{k+1}}\left(j\left(j+z+2r\right)+r\left(r+z\right)\right)^n\\&=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}\left(2j+2r+z\right)}{\left(j+2r+k+z\right)_{k+1}}\sum_{i=0}^{n}\binom{n}{i}\left(j\left(j+z+2r\right)\right)^{n-i}\left(r\left(r+z\right)\right)^i\\&=\sum_{i=0}^{n}\binom{n}{i}\left(r\left(r+z\right)\right)^i\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}\left(2j+2r+z\right)}{\left(j+2r+k+z\right)_{k+1}}\left(j\left(j+z+2r\right)\right)^{n-i}\\&=\sum_{i=0}^{n}\binom{n}{i}\left(r\left(r+z\right)\right)^i JS\left(n-i,k;z+2r\right).\end{aligned}$$

□

Remark 2. For $r=1$ and $z=-1$ in Corollary 1 we obtain the following identity

$$JS(n+1,k+1;-1)=JS(n,k;1).$$

Corollary 2. *For non-negative integers n, r, the r-Jacobi-Stirling numbers satisfy*

$$(X+r(r+z))^n=\sum_{k=0}^{n}JS_r(n+r,k+r;z)(X)_{k,z+2r}.$$

Proof. From the first identity of Corollary 1 and identity (1.8) we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}JS_r(n+r,k+r;z)(X)_{k,z+2r}\\
&=\sum_{k=0}^{n}\left(\sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i JS(n-i,k;z+2r)\right)(X)_{k,z+2r}\\
&=\sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i\left(\sum_{k=0}^{n}JS(n-i,k;z+2r)(X)_{k,z+2r}\right)\\
&=\sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i X^{n-i}\\
&=(X+r(r+z))^n.
\end{aligned}$$

□

Remark 3. 1) Since the Jacobi-Stirling number $JS(n,k;z)$ is a polynomial in z of degree $n-k$ whose leading coefficient is the Stirling number of the second kind $S(n,k)$, it follows from Corollary 2 that the r-Jacobi-Stirling number $JS_r(n+r,k+r;z)$ is also a polynomial in z of degree $n-k$ with leader coefficient the r-Stirling number of the second kind $S_r(n,k)$.
2) Corollary 2 can also be obtained by replacing x_i with $(i+r)(i+r+z)$ and X with $X+r(r+z)$ in the Newton interpolation formula,

$$X^n=\sum_{k=0}^{n}\left(\sum_{j=0}^{k}x_j^n\left(\prod_{l=0,l\neq j}^{k}(x_j-x_l)\right)^{-1}\right)\prod_{i=0}^{k-1}(X-x_i).$$

Theorem 3. *For every positive integer n and every real number $z\geq -1$, the polynomial*

$$P_n(x;r)=\sum_{k=0}^{n}JS_r(n+r,k+r;z)x^k$$

has only real simple non-positive zeros, and then, the r-Jacobi-Stirling numbers of the second kind are strictly log-concave (thus unimodal).

Proof. For $r = 0$ the theorem is Theorem 4 given in [6]. For $r \geq 1$, since the coefficient of x^n in $P_n(x)$ is $JS_r(n+r, n+r; z) = 1$, P_n is a polynomial of degree n. For $n = 1$, $P_1(x) = 1 + x$ has a real negative root. Assume $P_n(x)$ has only real negative simple roots, $n \geq 2$. Then, from Proposition 2, we get

$$\begin{aligned}
&P_{n+1}(x;r) \\
&= \sum_{k=0}^{n+1} JS_r(n+1+r, k+r; z)\, x^k \\
&= \sum_{k=0}^{n+1} \left(JS_r(n+r, k+r-1; z) + (k+r)(k+r+z)\, JS_r(n+r, k+r; z)\right) x^k \\
&= xP_n(x;r) + x^{1-r} D\left(x^{1-z} D\left(x^{r+z} \sum_{k=0}^{n} JS_r(n+r, k+r; z)\, x^k\right)\right) \\
&= xP_n(x;r) + x^{1-r} D\left(x^{1-z} D\left(x^{r+z} P_n(x;r)\right)\right),
\end{aligned}$$

and by setting $Q_n(x) = x^r P_n(x;r)$, the last identity becomes

$$Q_{n+1}(x) = x\left(Q_n(x) + D\left(x^{1-z} D\left(x^z Q_n(x)\right)\right)\right), \quad r \geq 1.$$

By induction, the polynomial $Q_n(x)$ satisfies all the assumptions of Lemma 2 given in [6], so $Q_{n+1}(x)$ has only real negative simple zeros and the root $x = 0$ with multiplicity r. This shows that $P_{n+1}(x;r)$ has only real negative simple zeros. On using Newton's inequality [5, p. 52], it follows that the sequence $(JS_r(n+r, k+r; z),\ 0 \leq k \leq n)$ is strongly log-concave. □

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NEW PARAMETRIC SOLUTIONS OF SOME ABEL EQUATIONS

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Abstract. New parametric solutions of some Abel equations of second kind are found. The scheme for construction of these solutions utilizes a combination of differential invariants of a symmetry of an associated second order ODE and the solutions of the associated ODE.

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1. INTRODUCTION

Riccati and Abel equations form an interesting class of first order ODEs due to the difficulties involved in their integration as well as for the reason that the search for the solutions to many applied problems reduces to the question of solving one of these equations. A description of different approaches, in Section 0.1.4 of [8], to deal with Riccati equation

$$\frac{dV}{dt} = f_2(t)V^2 + f_1(t)V + f_0(t) \tag{1.1}$$

includes conversion of (1.1) to a second order linear equation and utilizing its solution to obtain solution of Riccati equation. In *Lie symmetry setting* it can be viewed as follows:

- Consider the symmetry

$$X = 0\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{1.2}$$

  of a second order linear ODE

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0. \tag{1.3}$$

- Utilize the invariants $\frac{1}{y}\frac{dy}{dx}$ and x of symmetry X to get the transformations

$$t = x, \qquad V = \frac{1}{y}\frac{dy}{dx} \tag{1.4}$$

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to convert Riccati equation (1.1) to an ODE of the form (1.3), which may be easier to solve and hence can lead to solution of (1.1) via the transformations (1.4).

The purpose of this note is to investigate if an approach similar to above can be adapted to find new solutions of some Abel equations. The general form of an Abel equation of the second kind is

$$(V+g(t))\frac{dV}{dt} = f_2(t)V^2 + f_1(t)V + f_0(t).$$

There is no general procedure to find solutions of these equations. However some integration strategies, a listing of known integrable cases and a description of integrable classes can be found, for example, in [3,5,6,8]. An approach of integrating Abel equations admitting nonlocal exponential symmetry is presented by Adam-Mahomed in [1]. It should be noted that most of the solutions given in [8, Section 1.3] are in parametric form. Here we focus on a specific type of Abel equations of second kind, not contained in [8, Section 1.3], and develop an integration strategy that leads to parametric solutions. Our approach involves computations using Lie symmetries; the reader is referred to [2,4,7] for an introduction to the subject of symmetry analysis.

2. Integration scheme and examples

We consider Abel equations of the form

$$(V+bt)\frac{dV}{dt} = aV + F(t), \qquad a,b = \text{constant} \tag{2.1}$$

and ask the question:
Is there a symmetry X of a second order ODE

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \tag{2.2}$$

whose similarity variables provide a conversion relation between ODEs (2.1) and (2.2).
If such a symmetry is found and ODE (2.2) can be solved, it can be used to get solution of Abel equation (2.1) by utilizing the solution of second order ODE (2.2) via similarity variables of X.

As an ansatz for symmetry X of ODE (2.2), we take a scaling symmetry of the form

$$X = x\frac{\partial}{\partial x} + Ay\frac{\partial}{\partial y}$$

whose differential invariants are $\frac{y}{x^A}$, $x^{1-A}\frac{dy}{dx}$. The corresponding similarity variables

$$t = \frac{y}{x^A}, \qquad V = x^{1-A}\frac{dy}{dx} \tag{2.3}$$

provide the transformations between ODEs (2.1) and (2.2) and convert ODE (2.1) to

$$\frac{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-Ax\frac{dy}{dx}}{x\frac{dy}{dx}-Ay}=\frac{ax\left(\frac{dy}{dx}\right)+x^AF\left(yx^{-A}\right)}{x\frac{dy}{dx}+by}$$

which, upon choosing $A=-b$, reduces to

$$x^2\frac{d^2y}{dx^2}+(b+1-a)x\frac{dy}{dx}-x^{-b}F\left(x^by\right)=0 \tag{2.4}$$

In general, the values of undetermined parameters a,b for which ODE (2.4) can be solved would lead to corresponding solvable cases of Abel equation (2.1). For the specific case $b=(a-1)$ we have

Lemma 1. *An Abel equation of second kind of the form*

$$(V+(a-1)t)\frac{dV}{dt}=aV+F(t),\qquad a=\text{constant} \tag{2.5}$$

transforms to a second order ODE of the form

$$\frac{d^2y}{dx^2}=x^{-1-a}F\left(x^{a-1}y\right) \tag{2.6}$$

through the similarity variables

$$t=\frac{y}{x^{1-a}},\qquad V=x^a\frac{dy}{dx} \tag{2.7}$$

of the symmetry

$$X=x\frac{\partial}{\partial x}+(1-a)y\frac{\partial}{\partial y}$$

of ODE (2.6).

An effective implementation of Lemma 1 requires a suitable choice of the value of a for which ODE (2.6) is solved which will lead to parametric solutions of Abel equation (2.5) via the transformations (2.7). The scheme is implemented below to produce some new solutions of Abel equations.

Example 1. We consider the class of Abel equations

$$(V+(a-1)t)\frac{dV}{dt}=aV+t^n,\quad n\neq 1. \tag{2.8}$$

By above Lemma, for $a=\frac{n+1}{n-1}$, it reduces to

$$\frac{d^2y}{dx^2}=y^n \tag{2.9}$$

via transformations

$$t = yx^{\frac{2}{n-1}}, \qquad V = x^{\frac{n+1}{n-1}}\frac{dy}{dx}. \tag{2.10}$$

Case 1: For $n = 2$, the ODE (2.9) can be solved to get solution

$$y(x) = 6\,WeierstrassP\,(x + C_1, 0, C_2)$$

where *WeierstrassP* denotes the Weierstrass elliptic function. Hence we obtain, via (2.10), the parametric solution

$$t(x) = 6x^2\,WeierstrassP\,(x + C_1, 0, C_2)$$
$$V(x) = 6x^3\,WeierstrassPPrime\,(x + C_1, 0, C_2)$$

of Abel equation (2.8) for $n = 2$ and $a = 3$.

Case 2: For $n = 3$, the ODE (2.9) can be solved to get solution

$$y(x) = C_2\,JacobiSN\left(\left(1/2\,i\sqrt{2}x + C_1\right)C_2, i\right)$$

where *JacobiSN* denotes the Jacobi elliptic function. Hence we obtain, via (2.10), the parametric solution

$$t(x) = xC_2\,JacobiSN\left(\left(1/2\,i\sqrt{2}x + C_1\right)C_2, i\right),$$
$$V(x) = \frac{i}{2}x^2{C_2}^2\sqrt{2}\,JacobiCN\left(\left(\frac{i}{2}\sqrt{2}x + C_1\right)C_2, i\right)JacobiDN\left(\left(\frac{i}{2}\sqrt{2}x + C_1\right)C_2, i\right)$$

of Abel equation (2.8) for $n = 3$ and $a = 2$.

Case 3: For $n = -3$, the ODE (2.9) can be solved to get solution

$$y(x) = \frac{\sqrt{C_1\left(1 + x^2{C_1}^2 + 2x{C_1}^2C_2 + {C_2}^2{C_1}^2\right)}}{C_1}.$$

Hence we obtain, via (2.10), the parametric solution

$$t(x) = \frac{\sqrt{C_1\left(1 + x^2{C_1}^2 + 2x{C_1}^2C_2 + {C_2}^2{C_1}^2\right)}}{\sqrt{x}C_1},$$
$$V(x) = \frac{1}{2}\frac{\sqrt{x}\left(2x{C_1}^2 + 2{C_1}^2C_2\right)}{\sqrt{C_1\left(1 + x^2{C_1}^2 + 2x{C_1}^2C_2 + {C_2}^2{C_1}^2\right)}}$$

of Abel equation (2.8) for $n = -3$ and $a = \frac{1}{2}$.

Case 4: For $n = -1$, a first integral the ODE (2.9) is obtained as

$$\frac{dy}{dx} = \pm\sqrt{2\ln(y)}$$

which can be integrated to get

$$x = \frac{-i}{\sqrt{2}}\sqrt{\pi}\,\mathrm{erf}\left(i\sqrt{\ln(y)}\right)$$

where erf denotes the error function. Hence we obtain, via (2.10), the parametric solution

$$t(y) = \pm\frac{\sqrt{2}\,iy}{\sqrt{\pi}\,\mathrm{erf}\left(i\sqrt{\ln(y)}\right)},$$
$$V(y) = \pm\sqrt{2}\sqrt{\ln(y)}$$

of Abel equation (2.8) for $n = -1$ and $a = 0$.

Case 5: $n \neq -1$. For general n with $n \neq -1$ a first integral the ODE (2.9) is obtained as

$$\frac{dy}{dx} = \pm\sqrt{\frac{2}{n+1}}\,y^{\frac{n+1}{2}}$$

which can be integrated for $n \neq 1$ to get

$$x(y) = \mp\frac{\sqrt{2(n+1)}}{n-1}y^{-\frac{n-1}{2}} + C$$

Hence the transformations (2.10) lead to the parametric solutions

$$t(y) = \left(\mp\frac{\sqrt{2(n+1)}}{n-1} + Cy^{\frac{n-1}{2}}\right)^{\frac{2}{n-1}},$$
$$V(y) = \pm\frac{\sqrt{2}}{\sqrt{n+1}}\left(\mp\frac{\sqrt{2(n+1)}}{n-1} + Cy^{\frac{n-1}{2}}\right)^{\frac{n+1}{n-1}}$$

of Abel equation (2.8) for $n \neq \pm 1$ and $a = \frac{n+1}{n-1}$.

For Abel equations of the form (2.1) with $F(t) = \lambda t$ we obtain the following solutions.

Example 2. Consider the Abel equation

$$(V + bt)\frac{dV}{dt} = aV + \lambda t, \qquad a, b = \text{constant} \tag{2.11}$$

Then, as in equation (2.4), the similarity transformations

$$t = \frac{y}{x^{-b}}, \qquad V = x^{1+b}\frac{dy}{dx} \tag{2.12}$$

transform (2.11) to

$$x^2\frac{d^2y}{dx^2}+(b+1-a)x\frac{dy}{dx}-\lambda y=0 \tag{2.13}$$

which has the solution

$$y(x)=C_1x^{\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}}+C_2x^{\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}}.$$

This leads to, via (2.12), the parametric solution

$$\begin{aligned}
t(x)&=C_1x^{\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}}+C_2x^{\frac{a}{2}+\frac{b}{2}-\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}},\\
V(x)&=C_1x^{\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}}\left(\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}\right)\\
&\quad+C_2x^{\frac{a}{2}+\frac{b}{2}-\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}}\left(\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\sqrt{4\lambda+a^2-2ab+b^2}\right)
\end{aligned}$$

of Abel equation (2.11). Viewing equation (2.13) as an eigenvalue problem, we determine that the eigenvalues $\lambda=n(n+b-a)$ and the eigenfunctions

$$y(x)=C_1x^n+C_2x^{-n-b+a}$$

satisfy equation (2.13). Hence, in addition to above solutions, the transformations (2.12) provide the parametric solutions

$$\begin{aligned}
t(x)&=C_1x^{n+b}+C_2x^{-n+a},\\
V(x)&=C_1x^{n+b}n+C_2x^{-n+a}(-n-b+a)
\end{aligned}$$

of Abel equation (2.11) for $\lambda=n(n+b-a)$.

Though we have restricted to a specific class of Abel equations of the form (2.1) the strategy may be adapted for finding solutions of other types, as demonstrated in next example.

Example 3. We consider an Abel equation of second kind given by

$$\left(-2t^2V+t\right)\frac{dV}{dt}=V+2t^3 \tag{2.14}$$

and take as ansatz for symmetry X as

$$X=x\frac{\partial}{\partial x}+Ay\frac{\partial}{\partial y}$$

whose differential invariants give similarity transformations

$$t=\frac{x^A}{y},\qquad V=x^{1-A}\frac{dy}{dx}, \tag{2.15}$$

which transform Abel equation (2.14) to second order ODE

$$\frac{y^2\left(-x^{2-2A}\frac{d^2y}{dx^2}-x^{1-2A}\frac{dy}{dx}+Ax^{1-2A}\frac{dy}{dx}\right)}{x\frac{dy}{dx}-Ay}=\frac{x^{1-2A}y^3\frac{dy}{dx}+2x^{2A}}{-2y\left(x\frac{dy}{dx}-\frac{y}{2}\right)}$$

Choosing $A=\frac{1}{2}$ simplifies above equation to

$$\frac{d^2y}{dx^2}=y^{-3} \tag{2.16}$$

which has the solution

$$y(x)=\frac{\sqrt{C_1\left(1+x^2{C_1}^2+2x{C_1}^2C_2+{C_2}^2{C_1}^2\right)}}{C_1}.$$

So the similarity transformations

$$t=\frac{\sqrt{x}}{y},\qquad V=\sqrt{x}\frac{dy}{dx}$$

give the parametric solution

$$t(x)=\frac{\sqrt{x}C_1}{\sqrt{C_1\left(1+x^2{C_1}^2+2x{C_1}^2C_2+{C_2}^2{C_1}^2\right)}},$$

$$V(x)=\frac{1}{2}\frac{\sqrt{x}\left(2x{C_1}^2+2{C_1}^2C_2\right)}{\sqrt{C_1\left(1+x^2{C_1}^2+2x{C_1}^2C_2+{C_2}^2{C_1}^2\right)}}$$

of the Abel equation (2.14).

CONCLUSION

A new integration strategy for solving Abel equations of second kind is presented. Utilizing the differential invariants of Lie symmetries of an associated second order ODE, this strategy leads to parametric solutions of Abel equations. An implementation of the scheme is carried out to construct many new parametric solutions of some Abel equations.

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THE INDUCED SUBGRAPH OF THE UNITARY CAYLEY GRAPH OF A COMMUTATIVE RING OVER REGULAR ELEMENTS

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Abstract. Let R be a finite commutative ring and R^* be the group of units of R. The unitary Cayley graph of R, denoted by $G(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining vertices x and y to be adjacent if and only if $x-y \in R^*$. We denote by $\Gamma(R)$ the induced subgraph of $G(R)$ whose vertex set is R^*. In this paper, the basic properties of $\Gamma(R)$ are investigated and some characterization results regarding connectedness, chromatic number, chromatic index, girth and planarity of $\Gamma(R)$ are given. In addition, by using the concept of spectrum of graphs, it is shown that for two finite commutative rings R and S, if $\Gamma(R)$ is non empty, then $\Gamma(R) \cong \Gamma(S)$ as graphs if and only if $R/J(R) \cong S/J(R)$ as rings, where $J(R)$ and $J(S)$ are the Jacobson radicals of R and S, respectively. As a consequence of this result, we obtain that the structure of a reduced ring R can be determined by $\Gamma(R)$.

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1. Introduction

The study of algebraic structures, using the properties of graph associated to them, becomes an interesting topic in the last years. There are many papers on assigning a graph to a ring, for example see [6], [8], [12] and [28]. For additional historical background the reader is referred to the article [24], which also contains references to recent work in this area. Our main object is to establish connections between ring theoretic properties with the properties of the associated graph.

For the sake of completeness, first we state some definitions and notions used throughout to keep this paper as self contained as possible. The graph in this paper are simple, that is they have no loops or multiple edges. For a graph G, let $V(G)$ denotes the set of vertices, and let $E(G)$ denotes the set of edges. For $x \in V(G)$ we denote by $N(x)$ the set of all vertices of G adjacent to x. Also, the *degree* of x, denoted $d_G(x)$, is the size of $N(x)$. The maximum degree of G denoted by $\Delta(G)$. The *union* of two simple graphs G and H is the graph $G \cup H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(G)$. If G and H are disjoint, we refer

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to their union as a *disjoint union*, and denote it by $G + H$. The disjoint union of n copies of G is denoted by nG.

Let G be a graph. For two vertices x and y of G, a *path* of length n between x and y is an ordered list of distinct vertices $x = x_0, x_1, ..., x_n = y$ such that x_{i-1} is adjacent to x_i for $i = 1, .., n$. The distance between x and y, denoted by $d(x, y)$, is the length of shortest path between x and y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path between x and y). The largest distance among all distances between pairs of the vertices of a graph G is called the *diameter* of G and is denoted by $\mathrm{diam}(G)$. A *cycle* in G is a path that begins and ends at the same vertex. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G has no cycles). A graph G is called *connected* if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*. A graph with no edges is called an *empty graph*.

A graph in which each pair of distinct vertices is joined by an edge is called *complete graph*. We denote the complete graph on n vertices by K_n. Let r be a a positive integer. A graph is called *r-regular*, if all its vertices have degree r. A graph is called *r-partite* if the vertex set admits a partition into r classes such that vertices in the same partition class are not adjacent. A 2-partite graph is also called *bipartite graph*. An r-partite graph is called complete if every two vertices in different parts are adjacent. A complete r-partite graph with part sizes $m_1, m_2, ..., m_r$ is denoted by $K_{m_1,m_2,...,m_r}$.

Throughout this paper, all rings are assumed to be finite commutative with non-zero identity. For a ring R, we denote by R^* and $J(R)$ the group of units of R and the Jacobson radical of R, respectively. For a set X, $|X|$ denotes the cardinal of X. Also, $\mathbb{F}_{p^n}$ denotes the field with p^n elements and $\mathbb{Z}_n$ denotes for the ring of integers modulo n and following the literature, we write

$$D_2(R) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R\}.$$

It is well known that every finite commutative ring can be expressed as a direct product of finite local rings, and this decomposition is unique up to permutations of such local rings (see [7, Theorem 8.7]). Throughout the paper we assume the following:

Assumption 1. $R = R_1 \times \cdots \times R_n$ is a finite commutative ring, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$ of order m_i, $1 \leq i \leq n$. We assume $\pi_i : R_i \longrightarrow R_i/\mathfrak{m}$ is the canonical map and $f_i = |R_i/\mathfrak{m}_i|$ and

$$f_1 \leq f_2 \leq \ldots \leq f_n.$$

For other notations not mentioned in the paper, one can refer to [7] and [30].

The Cayley graph introduced by Arthur Cayley in 1878 is a useful tool for connection between group theory and the theory of algebraic graphs. The unitary Cayley

graph of a ring R, denoted by $G(R)$, is the graph whose vertex set is R, and in which two vertices x and y are adjacent if and only if $x - y \in R^*$. For some other recent papers on unitary Cayley graphs, see [1], [2] and [4]. Let $\Gamma(R)$ be the induced subgraph of $G(R)$ whose vertex set is R^*. This paper studies the graph $\Gamma(R)$ and the main motivation is the fact that a reduced ring R can be determined by $\Gamma(R)$ (see Corollary 6.4). The graphs $\Gamma(\mathbb{Z}_2)$, $\Gamma(\mathbb{Z}_3)$, $\Gamma(\mathbb{Z}_5)$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ are presented in Figure 1.

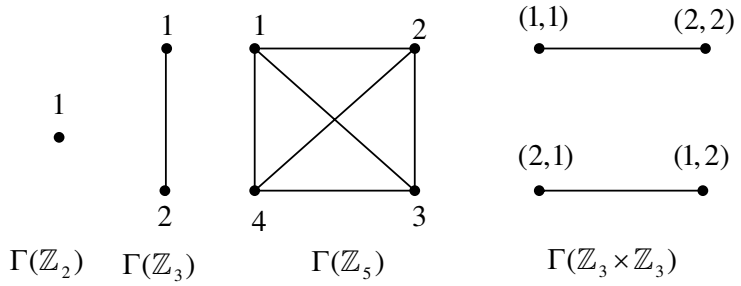

FIGURE 1.

The plan of the paper is as follows: In Section 2, we give some basic properties of $\Gamma(R)$. In Section 3, we determine the number of component of $\Gamma(R)$. Also, we completely characterize the diameter and girth of $\Gamma(R)$. In Section 4, we give a necessary and sufficient condition for the planarity of $\Gamma(R)$. In Section 5, some characterization results regarding chromatic number and chromatic index of $\Gamma(R)$ are given. For two finite rings R and S, the question of when $\Gamma(R) \cong \Gamma(S)$ as graphs, implies that $R \cong S$ as rings is very interesting and this kind of question has been studied extensively in [3], [5] and [18]. In the final section, by using the concept of spectrum of graphs, we show that, if $\Gamma(R)$ is non empty, then $\Gamma(R) \cong \Gamma(S)$ as graphs, if and only if $R/J(R) \cong S/J(R)$ as rings.

2. SOME BASIC PROPERTIES OF $\Gamma(R)$

In this section we study some basic properties of $\Gamma(R)$. The following proposition is well-known, but a simple proof of it is included here for the sake of completeness.

Proposition 1. *Let R be a finite local ring with maximal ideal $\mathfrak{m}$. Then there exists a prime number p such that $|R/\mathfrak{m}|$, $|R|$ and are all power of p.*

Proof. For a group G, we denote the order of an element $x \in G$ by $\mathrm{ord}_G(x)$. Since $F = R/\mathfrak{m}$ is a field, there exists a prime number p such that $|F|$ is a power of p. Let $n = |R|$ and q be a prime divisor of n. Then there exists $x \in R$ such that $\mathrm{ord}_R(x) = q$. Since $\mathrm{ord}_F(x + \mathfrak{m})$ is a power of p and divides q, we must have $q = p$. Therefore $|R|$ is a power of p, which implies that $|\mathfrak{m}|$ is also a power of p □

The *direct product* (sometimes called tensor product or category product) of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (x_1, y_1) adjacent to (x_2, y_2) if and only if x_1 is adjacent to x_2 in G and y_1 is adjacent to y_2 in H.

Theorem 1. *Let R be as in Assumption 1. Then*

(1) *$\Gamma(R)$ is a $\prod_{i=1}^{n}(|R_i|-2|\mathfrak{m}_i|)$-regular graph,*
(2) *$\Gamma(R) \cong \Gamma(R_1) \times \cdots \times \Gamma(R_n)$.*

Proof. (1) Let $1 \leq i \leq n$ and $R_i/\mathfrak{m}_i = \{\mathfrak{m}_i, a_{i1} + \mathfrak{m}_i, ..., a_{i(f_i-1)} + \mathfrak{m}_i\}$. If x_i is an arbitrary element of R_i^*, then there exists $1 \leq k \leq f_i - 1$ such that $x_i \in a_{ik} + \mathfrak{m}_i$. Therefore $N(x_i) = \cup_{j \neq k}(a_{ij} + \mathfrak{m}_i)$ and so $d_{\Gamma(R_i)}(x_i) = (f_i - 2)|\mathfrak{m}_i| = |R_i| - 2|\mathfrak{m}_i|$. It follows that $d_{\Gamma(R)}(x_1, ..., x_n) = \prod_{i=1}^{n}(|R_i| - 2|\mathfrak{m}_i|)$.

(2) The assertion follows easily from the fact that $R^* = R_1^* \times \cdots \times R_n^*$. □

Theorem 2. *Let $(R, \mathfrak{m})$ be a local ring. Then*

(1) *$\Gamma(R)$ is a complete multipartite graph whose partite sets are the set of all cosets of $\mathfrak{m}$ except $\mathfrak{m}$,*
(2) *$\Gamma(R)$ is a complete bipartite graph if and only if $|R/\mathfrak{m}| = 3$.*

Proof. (1) Let $|R/\mathfrak{m}| = f$ and $R/\mathfrak{m} = \{\mathfrak{m}, a_1 + \mathfrak{m}, ..., a_{f-1} + \mathfrak{m}\}$. Then $\Gamma(R)$ is complete bipartite with partites $a_1 + \mathfrak{m}, ..., a_{f-1} + \mathfrak{m}$.

(2) Let $|R/\mathfrak{m}| = 3$. Then it is easy to see that $\Gamma(R)$ is complete bipartite.

If $|R/\mathfrak{m}| = 2$, then $\Gamma(R)$ is non empty and hence $\Gamma(R)$ is not complete bipartite graph. If $|R/\mathfrak{m}| = n$, where $n > 3$, then $\Gamma(R)$ is a complete $n-1$ partite graph and hence $\Gamma(R)$ graph is not complete bipartite graph. Therefore if $\Gamma(R)$ is a complete bipartite graph, then $|R/\mathfrak{m}| = 3$ and the proof is completed. □

Theorem 3. *Let R be a finite ring. Then*

(1) *$\Gamma(R)$ has only one vertex if and only if $R \cong \mathbb{Z}_2^n$ for some natural number n,*
(2) *$\Gamma(R)$ has only two vertices if and only if $R \cong \mathbb{Z}_2^n \times S$, where n is a non negative integer, $S \cong \mathbb{Z}_3$, $S \cong \mathbb{Z}_4$ or $S \cong D_2(\mathbb{Z}_2)$*

Proof. (1) Let R be as in Assumption 1. If $\Gamma(R)$ has only one vertex, then $|R^*| = |R_1^*| \times \cdots \times |R_n^*| = 1$. Therefore $|R_i^*| = |R_i| - |\mathfrak{m}_i| = 1$ for each i. It follows from Proposition 1 that $|\mathfrak{m}_i| = 1$ and hence $|R_i| = 2$. Therefore $R \cong \mathbb{Z}_2^n$ for some natural number n. The converse is easily.

(2) Follows easily from [11, Corollary 4.5]. □

3. Connectivity, diameter and girth

We begin with the following theorem.

Theorem 4. *Let R be as in Assumption 1. Then the following are equivalent:*

(1) *$\Gamma(R)$ is non empty,*
(2) *$f_1 \neq 2$,*
(3) *$R/J(R)$ does not contain a summand isomorphic to $\mathbb{Z}_2$,*
(4) *every element of R can be expressed as the sum of two units.*

Proof. (1) $\Longrightarrow$ (2) Trivial.

(2) $\Longrightarrow$ (1) Let $f_i \neq 2$ for all $1 \leq i \leq n$. Then $f_1 \geq 3$ and hence there exists two elements $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of R^* such that $a_i - b_i \notin \mathfrak{m}_i$ for all $1 \leq i \leq n$. Therefore a and b are two adjacent elements of $\Gamma(R)$, and hence $\Gamma(R)$ is non empty.

(3) $\Longleftrightarrow$ (2) We have $R/J(R) \cong R_1/\mathfrak{m}_1 \times \cdots \times R_n/\mathfrak{m}_n$. By uniqueness of Wedderburn-Artin Theorem ([21, Theorem 3.5]), we have that $f_1 = 2$ if and only if $R/J(R)$ has a summand isomorphic to $\mathbb{Z}_2$.

(3) $\Longleftrightarrow$ (4) Follows from [22, Theorem 1]. □

Theorem 5 ([17, Theorem 5.29])**.** *Let G and H be graphs with at least one edge. Then $G \times H$ is connected if and only if both G and H are connected and at least one of them is non bipartite. Furthermore, if both G and H are connected and bipartite, then $G \times H$ has exactly two components.*

Theorem 6. *Let R be as in Assumption 1. Then $\Gamma(R)$ is disconnected if and only if $f_1 = 2$ or $f_n = 3$.*

Proof. Follows easily from Theorems 2(2), 4 and 5. □

The following theorem determine the number of components of $\Gamma(R)$. For a graph G we write $k(G)$ for the number of components of G.

Theorem 7. *Let R be as in Assumption 1. Then*

$$k(\Gamma(R)) = \begin{cases} |R^*| & \text{if } f_1 = 2, \\ 2^{r-1} & \text{if } f_1 = \cdots = f_r = 3 \text{ and } f_{r+1} > 3, \\ 2^{n-1} & \text{if } f_1 = \cdots = f_n = 3, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. If $f_1 = 2$, then $\Gamma(R)$ is an empty graph and hence $k(\Gamma(R)) = |R^*|$.

If $f_1 = \cdots = f_n = 3$, then we see easily by induction that

$$\Gamma(R) \cong K_{m_1,m_1} \times \cdots \times K_{m_n,m_n} \cong 2^{n-1} K_{m_1 \ldots m_n, m_1 \ldots m_n}.$$

So $k(\Gamma(R)) = 2^{n-1}$.

Now let there exists $1 \leq r < n$ such that $f_1 = \cdots = f_r = 3$ and $f_{r+1} > 3$. For convenience, we will take $S = R_{r+1} \times \cdots \times R_n$. Then

$$\Gamma(R) \cong K_{m_1,m_1} \times \ldots \times K_{m_r,m_r} \times \Gamma(S) \cong 2^{r-1} K_{m_1 \ldots m_r, m_1 \ldots m_r} \times \Gamma(S).$$

Since $\Gamma(S)$ is a connected non bipartite graph, Theorem 5 implies that $k(\Gamma(R)) = 2^{r-1}$.

If $f_1 > 3$, then by Theorem 6, $\Gamma(R)$ is connected and hence $k(\Gamma(R)) = 1$. This completes the proof. □

The following theorem shows that the diameter of $\Gamma(R)$ has five possibilities.

Theorem 8. *Let R be as in Assumption 1 and $|R| > 2$. Then*

$$\operatorname{diam}(\Gamma(R)) = \begin{cases} 1 & \text{if } R \text{ is a field,} \\ 0 & \text{if } f_n = 2, \\ \infty & \text{if } f_n > f_1 = 2 \text{ or } f_1 = f_2 = 3, \\ 3 & \text{if } f_1 = 3, f_2 \geq 4, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two arbitrary distinct vertices of $\Gamma(R)$. We consider the following cases:

(1) R is a field. Since $|R| > 2$, $\Gamma(R)$ is isomorphic to $K_{|R|-1}$ and hence $\operatorname{diam}(\Gamma(R)) = 1$.

(2) $f_n = 2$. Then by Theorem 3(1), $\Gamma(R)$ has only one vertex and hence $\operatorname{diam}(\Gamma(R)) = 0$.

(3) $f_n > f_1 = 2$ or $f_1 = f_2 = 3$. Then by Theorem 2, $\Gamma(R)$ is disconnected and hence $\operatorname{diam}(\Gamma(R)) = \infty$.

(4) Now let $n \geq 2$, $f_1 = 3$ and $f_2 \geq 4$. We consider two cases:

Case 1: $\pi_1(a_1) = \pi_1(b_1)$. In this case, For each $1 \leq i \leq n$, we choose a vertex c_i of $\Gamma(R_i)$ such that $\pi_i(c_i) \notin \{\pi_i(a_i), \pi_i(b_i)\}$. Set $c = (c_1, c_2, ..., c_n)$. Then $a, b \in N(c)$. It follows that $d(a, b) = 2$.

Case 2: $\pi_1(a_1) \neq \pi_1(b_1)$. We consider two subcases:

Subcase 1: If $\pi_i(a_i) \neq \pi_i(b_i)$ for all $2 \leq i \leq n$, then a and b are adjacent. Therefore $d(a, b) = 1$.

Subcase 2: Suppose that $\pi_i(a_i) = \pi_i(b_i)$ for some $2 \leq i \leq n$. For each $1 \leq i \leq n$, if $\pi_i(a_i) \neq \pi_i(b_i)$, we set $c_i = b_i$ and $d_i = a_i$ and if $\pi_i(a_i) = \pi_i(b_i)$, we choose $c_i, d_i \in R_i^*$ such that $|\{\pi_i(a_i) = \pi_i(a_i), \pi_i(c_i), \pi_i(d_i)\}| = 3$. Now set $c = (c_1, c_2, ..., c_n)$ and $d = (d_1, d_2, ..., d_n)$. Then a, c, d, b is path between a and b. Therefore $\operatorname{diam}(\Gamma(R)) \leq 3$. On the other hand, the vertices a and b neither adjacent nor do they have a common neighbor, hence $\operatorname{diam}(\Gamma(R)) \geq 3$. So $\operatorname{diam}(\Gamma(R)) = 3$.

(5) $n = 1$, $f_1 \geq 3$ and R is not a field. Then $\Gamma(R)$ is isomorphic to $K_{m_1, ..., m_{f_1 - 1}}$, where $m_1 = ... = m_{f_1 - 1} = |\mathfrak{m}_1| \geq 2$. It follows that $\operatorname{diam}(\Gamma(R)) = 2$.

(6) $n \geq 2$, $f_1 \geq 4$. Since $f_i \geq 4$ for each $1 \leq i \leq n$, we can choose $c_i \in R_i^*$ such that $\pi_i(c_i) \notin \{\pi_i(a_i), \pi_i(b_i)\}$. Set $c = (c_1, c_2, c_3, ..., c_n)$. Then $a, b \in N(c)$. It follows that $d(a, b) \leq 2$. On the other hand, if $x = (x_1, a_2, ..., a_n)$ is a vertex of $\Gamma(R)$ such that $\pi_1(x_1) \neq \pi_1(a_1)$, then a and x are not adjacent, and so $d(x, a) \geq 2$. Therefore $\operatorname{diam}(\Gamma(R)) = 2$. This completes the proof. □

The following theorem shows that the girth of $\Gamma(R)$ has four possibilities.

Theorem 9. *Let R be as in Assumption 1. Then*

$$\mathrm{gr}(\Gamma(R)) = \begin{cases} \infty & \text{if } f_1 = 2, or R \cong \mathbb{Z}_3^n, \\ 6 & \text{if } f_1 = ... = f_{n-1} = 3, f_n = 4, \\ 4 & \text{if } f_1 = ... = f_{n-1} = 3, f_n \geq 5, \\ 3 & otherwise. \end{cases}$$

Proof. Consider the following cases:

(1) $f_1 = 2$. Then Theorem 4 implies that $\Gamma(R)$ is non empty and hence $\mathrm{gr}(\Gamma(R)) = \infty$.

(2) $f_1 = ... = f_{n-1} = 3$ and $f_n = 4$. In this case, we have

$$\begin{aligned} \mathrm{gr}(\Gamma(R)) &= \mathrm{gr}(K_{m_1,m_1} \times \cdots \times K_{m_{n-1},m_{n-1}} \times K_{m_n,m_n,m_n}) \\ &= \mathrm{gr}(2^{n-2} K_{m_1...m_{n-1},m_1...m_{n-1}} \times K_{m_n,m_n,m_n}) \\ &= \mathrm{gr}(K_{m_1...m_{n-1},m_1...m_{n-1}} \times K_{m_n,m_n,m_n}) \\ &= 6. \end{aligned}$$

(3) $f_1 = ... = f_{n-1} = 3$ and $f_n \geq 5$. In this case, since $\Gamma(R)$ is bipartite, we have $\mathrm{gr}(\Gamma(R)) \geq 4$. For each $1 \leq i \leq n-1$, choose elements a_i, b_i of R_i such that $|\{\pi_i(a_i), \pi_i(b_i)\}| = 2$. Let x_n, y_n, z_n, w_n of R_n such that $|\{\pi_i(x_n), \pi_i(y_n), \pi_i(z_n), \pi_i(w_n)\}| = 4$. Now let

$$\begin{aligned} x &= (a_1, a_2, ..., a_{n-1}, x_n), \\ y &= (b_1, b_2, ..., b_{n-1}, y_n), \\ z &= (a_1, a_2, ..., a_{n-1}, z_n), \\ w &= (b_1, b_2, ..., b_{n-1}, w_n). \end{aligned}$$

Then x, y, z, w, x is a cycle of length 4, and hence $\mathrm{gr}(\Gamma(R)) = 4$.

(4) $f_1 = \cdots = f_n = 3$. Then

$$\Gamma(R) \cong K_{m_1,m_1} \times \cdots \times K_{m_n,m_n} \cong 2^{n-1} K_{m_1...m_n,m_1...m_n}.$$

So $\mathrm{gr}(\Gamma(R)) = 4$ if $m_1...m_n > 1$ and $\mathrm{gr}(\Gamma(R)) = \infty$ if $m_1 = ... = m_n = 1$ (i.e., $\mathrm{gr}(\Gamma(R)) = \infty$ if $R \cong \mathbb{Z}_3^n$).

(5) Finally, suppose that $f_1 \geq 4$. Let a_i, b_i, c_i be elements of R_i such that $|\{\pi_i(a_i), \pi_i(b_i), \pi_i(c_i)\}| = 3$. Let

$$\begin{aligned} a &= (a_1, a_2, ..., a_n), \\ b &= (b_1, b_2, ..., b_n), \\ c &= (c_1, c_2, ..., c_n). \end{aligned}$$

Then a, b, c, a is a cycle of length 3, and hence $\mathrm{gr}(\Gamma(R)) = 3$. This completes the proof. □

4. PLANARITY

A graph is said to be *planar* if it can be drawn in the plane such that its edges intersect only at their ends. A *subdivision* of an edge is obtained by inserting into this edge some new vertices of degree two. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (see [9, Theorem 10.30]). Also if G is a planar graph, then G contains a vertex of degree at most five (see [9, Corollary 10.22]).

The following theorem gives necessary and sufficient condition for the planarity of $\Gamma(R)$.

Theorem 10. *Let R be a finite ring. Then $\Gamma(R)$ is planar if and only if $\Gamma(R)$ is empty or R is isomorphic to one of the following rings:*

$$\mathbb{Z}_3^r \times \mathbb{F}_4, \mathbb{Z}_3^r, \mathbb{Z}_3^r \times \mathbb{F}_4 \times \mathbb{F}_4, \mathbb{Z}_3^r \times \mathbb{Z}_5, \mathbb{Z}_3^r \times \mathbb{Z}_9, \mathbb{Z}_3^r \times D_2(\mathbb{Z}_3) \ \ (r \geq 0).$$

Proof. Let R be as in Assumption 1 and let $1 \leq i \leq n$. By Theorem 1, we may assume $|R_i| = p_i^{\alpha_i}$ and $|\mathfrak{m}_i| = p_i^{\beta_i}$, where p is a prime number and α_i is a natural number and β_i is a non negative integer. Now consider the following cases:

(1) $p_i^{\alpha_i} - 2p_i^{\beta_i} = 0$. In this case, we have $f_i = 2$ and hence $\Gamma(R_i)$ is an empty graph.

(2) $p_i^{\alpha_i} - 2p_i^{\beta_i} = 1$. In this case, we have $p_i = 3$, $\alpha_i = 1$ and $\beta_i = 0$. So $|R_i| = 3$ and hence $R_i \cong \mathbb{Z}_3$.

(3) $p_i^{\alpha_i} - 2p_i^{\beta_i} = 2$ for some i. In this case we have $p_i = 2$, $\alpha_i = 2$ and $\beta_i = 0$. So $|R_i| = 4$ and hence $R_i \cong \mathbb{F}_4$.

(4) $p_i^{\alpha_i} - 2p_i^{\beta_i} = 3$. If $\beta_i = 0$, we have $|R_i| = 5$ and hence $R_i \cong \mathbb{Z}_5$. If $\beta_i \neq 0$, we have $p_i = 3$, $\alpha_i = 2$ and $\beta_i = 1$. So $R_i \cong \mathbb{Z}_9$ or $R_i \cong D_2(\mathbb{Z}_3)$, by [26, Theorem 3(4)].

(5) $p_i^{\alpha_i} - 2p_i^{\beta_i} = 4$. In this case we have $p_i = 2$. If $\beta_i = 0$, then $2^{\alpha_i - 1} = 3$, which is a contradiction. So we must have $\beta_i \neq 0$. Therefore $\alpha_i = 3$ and $\beta_i = 1$, which is again a contradiction by [26, Theorem 3(1)].

(6) $p_i^{\alpha_i} - 2p_i^{\beta_i} = 5$. If $\beta_i \neq 0$, then $p_i = 5$ and hence $5^{\alpha_i - 1} = 3$, which is a contradiction. So, in this case we have $\beta_i = 0$. Therefore $\alpha_i = 1$ and so $R_i \cong \mathbb{Z}_7$. (Note that $\Gamma(\mathbb{Z}_7)$ is not planar).

Since $\Gamma(R)$ is planar, the (minimum) degree of vertices of $\Gamma(R)$ is at most five. So $\prod_{i=1}^{k}(p_i^{\alpha_i} - 2p_i^{\beta_i}) \leq 5$. Therefore either $\Gamma(R)$ is an empty graph or there exits a non negative integer $r \geq 0$ such that R is isomorphic to one of the following rings:

$\mathbb{Z}_3^r \times \mathbb{F}_4$, $\mathbb{Z}_3^r$, $\mathbb{Z}_3^r \times \mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{Z}_3^r \times \mathbb{Z}_5$, $\mathbb{Z}_3^r \times \mathbb{Z}_9$, $\mathbb{Z}_3^r \times D_2(\mathbb{Z}_3)$.

On the other hand, it is easy to see that if R is isomorphic to one of the above rings, then $\Gamma(R)$ is planar. This completes the proof. □

5. CHROMATIC NUMBER AND CHROMATIC INDEX

Let G be a graph. A *k-colouring* of the vertices of G is an assignment of k colours to the vertices of G in such a way that no two adjacent vertices receive the same colour. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number k such that G admits a k-colouring.

Let G and H be two graphs. Since the projections $G\times H\longrightarrow G$ and $G\times H\longrightarrow H$ are homomorphisms, we have $\chi(G\times H)\leq \min\{\chi(G),\chi(H)\}$. One of the most challenging open problems in the context of direct products of graphs is Hedetniemi's Conjecture (see [15, Page 51]). The Hedetniemi's conjecture asserts that for all graphs G and H,

$$\chi(G\times H)=\min\{\chi(G),\chi(H)\}.$$

This conjecture is also known as Lovász-Hedetniemi conjecture. The following theorem shows that this conjecture is true in the special case:

Theorem 11. *Let R and S be two finite rings. Then*

$$\chi(\Gamma(R)\times\Gamma(S))=\min\{\chi(\Gamma(R)),\chi(\Gamma(S))\}.$$

Proof. Let R be as in Assumption 1. First of all, we show that $\chi(\Gamma(R))=f_1-1$. We have $\chi(\Gamma(R))\leq\chi(\Gamma(R_1))=f_1-1$. Now let $a_{i1},...,a_{i(f_1-1)}$ be pairwise adjacent vertices of $\Gamma(R_i)$. Then

$$\begin{aligned}&(a_{11},a_{21},...,a_{n1}),\\&(a_{12},a_{22},...,a_{n2}),\\&\qquad\vdots\\&(a_{1(f_1-1)},a_{2(f_1-1)},...,a_{n(f_1-1)}),\end{aligned}$$

are pairwise adjacent vertices of $\Gamma(R)$. Therefore $\chi(\Gamma(R))\geq f_1-1$ and hence $\chi(\Gamma(R))=f_1-1$. Now let $S=S_1\times...\times S_m$, where S_i is a local ring with maximal ideal $\mathfrak{n}_i$, $g_i=|S_i/\mathfrak{n}_i|$ and $g_1\leq g_2\leq...\leq g_m$. Then

$$\chi(\Gamma(R)\times\Gamma(S))=\min\{f_1-1,g_1-1\})=\min\{\chi(\Gamma(R)),\chi(\Gamma(S))\}.$$

This completes the proof. □

An *k-edge colouring* of a graph G is an assignment of k colours to its edges so that no two edges incident to same vertex the same colour. The chromatic index of G, denoted by $\chi'(G)$, is the least k for which G has an k-edge colouring. Vizing's Theorem ([29]) gives us a basic way of classifying graphs into two classes. A graph G is of *Class 1* if $\chi'(G)=\Delta(G)$ and of *Class 2* if $\chi'(G)=\Delta(G)+1$. The following theorem determines the class of $\Gamma(R)$.

Theorem 12. *Let R be a finite ring. Then*

$$\chi'(\Gamma(R))=\begin{cases}\Delta(\Gamma(R))+1 & \text{if } |R^*| \text{ is odd},\\ \Delta(\Gamma(R)) & \text{if } |R^*| \text{ is even},\end{cases}$$

Proof. If $G = \Gamma(R)$ is a graph of odd order, then $|E(G)| > \Delta(G)\lfloor\frac{V(G)}{2}\rfloor$, that is G is overfull graph. Therefore [10, Corollary 8.18] implies that G is of Class 2.

Now let R be as in Assumption 1. If G is a graph of even order, then there exits $1 \leq i \leq n$ such that $\Gamma(R_i)$ is a graph of even order. By [25, Corollary 2.7], $\Gamma(R_i)$ is of Class 1 and [27, Theorem 4.4] implies that G is of Class 1. □

6. Isomorphisms

The *adjacency matrix* of a graph is the matrix with rows and columns indexed by its vertices such that the (i,j)-entry is equal to 1 if vertices i and j are adjacent and 0 otherwise. The *eigenvalues* of a graph are eigenvalues of its adjacency matrix, and the *spectrum* of a graph is the collection of its eigenvalues together with multiplicities. If $\lambda_1,...,\lambda_k$ are distinct eigenvalues of a graph G and $m_1,...,m_k$ the corresponding multiplicities, then we denote the spectrum of G by

$$\mathrm{Spec}(G) = \begin{pmatrix} \lambda_1 & ... & \lambda_k \\ m_1 & ... & m_k \end{pmatrix},$$

that is, the multiset of eigenvalues of the adjacency matrix of G. For example, for $n > 1$, we have

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

The energy of a graph G with n vertices and with eigenvalues $\lambda_1,...,\lambda_k$ is defined as

$$\varepsilon(G) = \sum_{i=1}^{k} |\lambda_i|.$$

The energy of a graph was introduced in [13] in the context of mathematical chemistry. Since then it has been studied extensively; see for example [14], [16], [19] and [23]. As usual, we use $\mathbb{N}$ for the set of all natural numbers.

Theorem 13. *Let $n(i), m(j) \in \mathbb{N} \setminus \{1\}$, where $1 \leq i \leq r$ and $1 \leq j \leq s$. If $\prod_{i=1}^{r} K_{n(i)} \cong \prod_{j=1}^{s} K_{m(j)}$, then $r = s$ and there is a permutation σ of $\{1,2,...,r\}$ such that $n(i) = m(\sigma(i))$ for all $1 \leq i \leq r$.*

Proof. Let $G = \prod_{i=1}^{r} K_{n(i)}$, $H = \prod_{j=1}^{s} K_{m(j)}$, $a = \prod_{i=1}^{r}(n(i)-1)$ and $b = \prod_{j=1}^{s}(m(j)-1)$. Since G is a-regular and H is b-regular we must have $a = b$. On the other hand, $\varepsilon(G) = 2^r a$ and $\varepsilon(H) = 2^s b$. So $r = s$. Without loss of generality, we can assume that

$$n(1) \leq \cdots \leq n(r),$$
$$m(1) \leq \cdots \leq m(r).$$

We prove that $n(i)=m(i)$ for all $1\leq i\leq r$. Since $(-1)^{r-1}(n(1)-1)\in \mathrm{Spec}(G)$ and $(-1)^{r-1}(m(1)-1)\in \mathrm{Spec}(H)$, we must have $n(1)=m(1)$.

On the contrary let i be the smallest number such that $n(i)\neq m(i)$. Without loss of generality, we can assume that $m(i)<n(i)$. Let $C\subseteq\{1,2,..,i-1\}$. Let X and Y be the multisets of $(-1)^{r-|C|}\prod_{i\in C}(n(i)-1)$ and $(-1)^{r-|C|}\prod_{i\in C}(n(i)-1)$, respectively. By the assumption, $|X|=|Y|$ and hence as multisets we have $\mathrm{Spec}(G)\setminus X=\mathrm{Spec}(H)\setminus Y$. Therefore $(-1)^{r-1}(m(i)-1)\in \mathrm{Spec}(H)\setminus Y$. So there exists a subset $C\subseteq\{1,2,..,r-1\}$ such that $C\nsubseteq\{1,2,..,i-1\}$ and $(-1)^{r-1}(m(i)-1)=(-1)^{r-|C|}\prod_{i\in C}(n(i)-1)$, which is a contradiction. □

We note that the above theorem is proved in [20], when $n(i),m(j)\in\mathbb{N}\setminus\{1,2\}$.

Lemma 1. *Let R be as in Assumption 1. Let $\Gamma(R)$ be a non empty and let x,y be two vertices of $\Gamma(R)$. Then the following are equivalent:*

(1) $x-y\in J(R)$,
(2) $N(x)=N(y)$.

Proof. (1) $\Longrightarrow$ (2) Let $u\in N(x)$. Then $u-x\in U(R)$. We claim that $u\in N(y)$. Suppose on the contrary that $u\notin N(y)$. Then there is a maximal ideal $\mathfrak{m}$ of R such that $u-y\in\mathfrak{m}$. Therefore $u-x=(u-y)+(y-x)\in\mathfrak{m}$, which is a contradiction. It follows that $N(x)\subseteq N(y)$. A similar argument shows that $N(y)\subseteq N(x)$.

(2) $\Longrightarrow$ (1) Suppose that $x-y\notin J(R)$. Let $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$. So there exists $1\leq i\leq n$ such that $x_i-y_i\notin\mathfrak{m}_i$. Since $f_1\neq 2$, there exists $z_j\in R_j$ such that $y_j-z_j\notin\mathfrak{m}_i$ for all $1\leq j\neq i\leq n$. Let

$$z=(z_1,...,z_{i-1},x_i,z_{i+1},...,z_n).$$

Then $z\in N(y)\setminus N(x)$, which is a contradiction. □

We are now in a position to prove the main result of this section.

Theorem 14. *Let R and S be two finite rings such that $\Gamma(R)$ is non empty. If $\Gamma(R)\cong\Gamma(S)$, then the following hold:*

(1) $R/J(R)\cong S/J(S)$,
(2) $|J(R)|=|J(S)|$,
(3) $|R|=|S|$.

Proof. (1) Let $f:\Gamma(R)\longrightarrow\Gamma(S)$ be a graph isomorphism. Define

$$\begin{aligned}\overline{f}:\Gamma(R/J(R))&\longrightarrow\Gamma(S/J(S))\\ x+J(R)&\longmapsto f(x)+J(S)\end{aligned}$$

The above lemma implies that $\overline{f}$ is an isomorphism. By [7, Proposition 1.10] or [21, Theorem 3.5], the rings $R/J(R)$ and $S/J(S)$ are direct product of finite fields. Since two finite fields are isomorphic if and only if they have the same number of elements, Theorem 13 implies that $R/J(R)\cong S/J(S)$.

(2) Let R be as in Assumption 1 and let $S = S_1 \times \cdots \times S_m$, where S_i is a local ring with maximal ideal $\mathfrak{n}_i$ such that $|S_1/\mathfrak{n}_1| \leq ... \leq |S_m/\mathfrak{n}_m|$. By Part (1), we have $R/J(R) \cong S/J(S)$. It follows that

$$\Gamma(R_1/\mathfrak{m}_1) \times \cdots \times \Gamma(R_n/\mathfrak{m}_n) \cong \Gamma(S_1/\mathfrak{n}_1) \times \cdots \times \Gamma(S_m/\mathfrak{n}_m).$$

In view of Part (1), we have $n = m$ and $|R_i/\mathfrak{m}_i| = |S_i/\mathfrak{n}_i|$ for all $1 \leq i \leq n$. Then $(|R_i| - |\mathfrak{m}_i|)/|\mathfrak{m}_i| = (|S_i| - |\mathfrak{n}_i|)/|\mathfrak{n}_i|$ for all $1 \leq i \leq n$. It follows that $|R^*|/|J(R)| = |S^*|/|J(S)|$. Since $|R^*| = |S^*|$, we have $|J(R)| = |J(S)|$.

(3) Follows easily from the Parts (1) and (2). □

We recall that a ring R is called reduced if R has no non-zero nilpotent elements.

Corollary 1. *Let R and S be two finite rings such that R is a reduced ring and $\Gamma(R)$ is non empty. If $\Gamma(R) \cong \Gamma(S)$ as graphs, then $R \cong S$ as rings.*

Proof. Since R is reduced, we have $\mathrm{Nil}(R) = 0$ ($\mathrm{Nil}(R)$ is the intersection of prime ideals of R). On the other hand, since R is finite, every prime ideal of R is maximal. Hence $J(R) = \mathrm{Nil}(R) = 0$. By Theorem 14(2), we have $J(S) = 0$. Now the assertion follows from Theorem 14(1). □

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ON THE WILLMORE ENERGY OF CURVES UNDER SECOND ORDER INFINITESIMAL BENDING

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Abstract. The change of the Willmore energy, as a special case of so-called Helfrich energy, under second order infinitesimal bending of curves in a three-dimensional Euclidean space is studied. The first and the second variation of the Willmore energy are obtained.

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1. INTRODUCTION

Infinitesimal bending of curves and surfaces is a part of the more general bending theory, which presents one of the main consisting parts of the global differential geometry. A concept of infinitesimal bending dealt first with infinitesimal bending of surfaces and then with the same problem in the theory of curves and manifolds.

Infinitesimal bending is determined by the stationarity of arc length s with appropriate precision, which is described with the condition

$$ds_\epsilon^2 - ds^2 = o(\epsilon^m), m \geq 1, \epsilon \geq 0, \epsilon \to 0.$$

Under infinitesimal bending, geometric magnitudes describing a curve and a surface are changing and this change is determined by the variation. The variation is very important in description of the flexibility and rigidity of curves and surfaces under different kinds of deformations.

The Helfrich energy is geometrical magnitude describing the curve. It is given with the equation

$$H_\lambda(C) = \frac{1}{2}\int_I (k - c_0)^2\, ds + \lambda L(C),$$

where k denotes the curvature of a curve C, s is the arc length and $L(C) = \int_I ds$ is the length of C. The map $c_0 : I \mapsto \mathbb{R}, I \subseteq \mathbb{R}$ is called spontaneous curvature.

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The constant $\lambda \in \mathbb{R}$ is taken to be positive, so that the growth in length of a curve is penalized. The above functional is motivated by the elasticity of cell membranes [5].

The special case where $c_0 = 0$ and $\lambda = 0$ is known as *Willmore (bending) energy*

$$W(C) = \frac{1}{2}\int_I k^2\, ds,$$

and it can also be historically motivated by the so-called Euler-Bernoulli model of elastic rods [13]. The most famous model of thin elastic rods is *Euler's elastica*, which is the solution of a variational problem that minimizes the Willmore energy of a thin inextensible wire [11]. The elastica is also the mathematical model of the large-scale configurations of deoxyribonucleic acid (DNA) molecules [12]. In [8] isometric extremals of rotation were studied which are connected with functional $\int_I k\, ds$.

The Willmore energy has been extensively investigated analytically and numerically in recent years and it is very interesting for research yet. In [3], the authors study the long-time evolution of regular open curves in $\mathbb{R}^n$, $n \geq 2$, moving according to the L^2-gradient flow for a generalization of the Helfrich functional. In [2], the authors examine the equilibrium conditions of a curve in space when a local energy penalty is associated with its extrinsic geometrical state characterized by its curvature and torsion. To do this, they tailor the theory of deformations to the Frenet-Serret frame of the curve. In [9], the change of the Willmore energy of a curve under infinitesimal bending of the first order is analyzed. Many papers are dedicated to the Willmore energy of deformed surfaces (see [6, 15, 16]). Some other quantities are also interesting for studying at various deformations. For instance, a linear deformation technique preserving the total curvature of the masterpiece is presented in [1].

This paper deals with the first and the second variation of the Willmore energy under second order infinitesimal bending of curves. It represents a natural extension of our previous paper [10] in which the second order infinitesimal bending of curves was thoroughly studied.

2. Second order infinitesimal bending of a curve in $\mathbb{R}^3$

Let us consider a regular curve

$$C : \mathbf{r} = \mathbf{r}(s) = \mathbf{r}[u(s)], \quad s \in I = [0, L] \subseteq \mathbb{R}, \tag{2.1}$$

of a class C^α, $\alpha \geq 3$, parameterized by the arc length s and included in a family of the curves

$$C_\epsilon : \tilde{\mathbf{r}}(s, \epsilon) = \mathbf{r}_\epsilon(s) = \mathbf{r}(s) + \epsilon \overset{(1)}{\mathbf{z}}(s) + \epsilon^2 \overset{(2)}{\mathbf{z}}(s), \tag{2.2}$$

where $\epsilon \geq 0, \epsilon \to 0$ and we get C for $\epsilon = 0$ ($C = C_0$). The fields $\overset{(j)}{\mathbf{z}}(s) \in C^\alpha$, $\alpha \geq 3$, $j = 1, 2$, are continuous vector functions defined in the points of C.

Definition 1 ([4]). Family of curves C_ϵ is an **infinitesimal bending of the second order of the curve** C if

$$ds_\epsilon^2 - ds^2 = o(\epsilon^2). \tag{2.3}$$

The field $\overset{(j)}{\mathbf{z}} = \overset{(j)}{\mathbf{z}}(s)$ is the **infinitesimal bending field of the order** j, $j = 1,2$ of the curve C.

The previous condition is equivalent to the system of equations ([4]):

$$d\mathbf{r}\cdot d\overset{(1)}{\mathbf{z}} = 0, \quad 2d\mathbf{r}\cdot d\overset{(2)}{\mathbf{z}} + d\overset{(1)}{\mathbf{z}}\cdot d\overset{(1)}{\mathbf{z}} = 0,$$

where $\cdot$ stands for the scalar product in $\mathbb{R}^3$. This system is equivalent to the next system of differential equations:

$$\mathbf{r}'\cdot\overset{(1)}{\mathbf{z}}{}' = 0, \quad 2\mathbf{r}'\cdot\overset{(2)}{\mathbf{z}}{}' + \overset{(1)}{\mathbf{z}}{}'\cdot\overset{(1)}{\mathbf{z}}{}' = 0,$$

where prime denotes a derivative with respect to arc length s.

Let $\{\mathbf{t}, \mathbf{n_1}, \mathbf{n_2}\}$ be an orthonormal basis along the curve C, where $\mathbf{t}$ is the unit tangent, $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit principal normal and binormal vector field of the curve, respectively. We choose an orientation with $\mathbf{n}_2 = \mathbf{t}\times\mathbf{n}_1$.

As the vector fields $\overset{(1)}{\mathbf{z}}$ and $\overset{(2)}{\mathbf{z}}$ are defined in the points of the curve C, they can be presented in the form

$$\overset{(j)}{\mathbf{z}} = \overset{(j)}{z}\,\mathbf{t} + \overset{(j)}{z_1}\mathbf{n}_1 + \overset{(j)}{z_2}\mathbf{n}_2, \quad j = 1,2, \tag{2.4}$$

where $\overset{(j)}{z}, \overset{(j)}{z_1}, \overset{(j)}{z_2}$ are the functions of s.

Theorem 1 ([10]). *Necessary and sufficient conditions for the fields* $\overset{(j)}{\mathbf{z}}$ (2.4), $j = 1,2$, *to be infinitesimal bending fields of the corresponding order of a curve* C *are*

$$\begin{aligned} &\overset{(1)}{z}{}' - k\overset{(1)}{z_1} = 0, \\ &\overset{(2)}{z}{}' - k\overset{(2)}{z_1} = -\frac{1}{2}\{[k\overset{(1)}{z} + \overset{(1)}{z_1}{}' - \tau\overset{(1)}{z_2}]^2 + [\tau\overset{(1)}{z_1} + \overset{(1)}{z_2}{}']^2\}, \end{aligned} \tag{2.5}$$

where k *is the curvature and* τ *is the torsion of* C.

Under infinitesimal bending geometric magnitudes of the curve are changing which is described with variations of these geometric magnitudes.

Definition 2 ([14]). Let $A = A(s)$ be a magnitude which characterizes a geometric property on the curve C and $A_\epsilon = A_\epsilon(s)$ the corresponding magnitude on the curve C_ϵ being infinitesimal bending of the curve C and let the equation

$$\Delta A = A_\epsilon - A = \epsilon\,\delta A + \epsilon^2\delta^2 A + \ldots + \epsilon^n\delta^n A + \ldots \tag{2.6}$$

be a valid one. The coefficients $\delta A, \delta^2 A, \ldots, \delta^n A, \ldots$ are **the first, the second, ..., the n-th variation** of the geometric magnitude A, respectively under the infinitesimal bending C_ϵ of the curve C.

This method was used for infinitesimal F-planar transformations [7].

In this paper we will consider the first and the second variation of a magnitude under infinitesimal bending of the second order. For this reason, we can represent the magnitude A_ϵ by neglecting the terms in ϵ^n, $n \geq 3$.

It is easy to prove that the first and the second variation of a line element ds are equal to zero, i. e. $\delta(ds) = \delta^2(ds) = 0$.

Theorem 2 ([10])**.** *Under a second order infinitesimal bending of a curve C, the first and the second variation of the curvature are respectively*

$$\delta k = k'\overset{(1)}{z} + \overset{(1)}{z_1}'' + (k^2 - \tau^2)\overset{(1)}{z_1} - 2\tau\overset{(1)}{z_2}' - \tau'\overset{(1)}{z_2}, \tag{2.7}$$

$$\begin{aligned}\delta^2 k = &-\frac{k}{2}(\tau\overset{(1)}{z_1} + \overset{(1)}{z_2}')^2 + \frac{1}{2k}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z_1}' + \tau'\overset{(1)}{z_1} + \overset{(1)}{z_2}'' - \tau^2\overset{(1)}{z_2})^2 \\ &+ (k\overset{(2)}{z} + \overset{(2)}{z_1}' - \tau\overset{(2)}{z_2})' - \tau(\overset{(2)}{z_2}' + \tau\overset{(2)}{z_1}),\end{aligned} \tag{2.8}$$

where k is the curvature and τ is the torsion of C.

3. The Willmore energy under infinitesimal bending

Let us consider the Willmore energy of the curve C (2.1):

$$W = \frac{1}{2}\int_I k^2\, ds. \tag{3.1}$$

The Willmore energy of deformed curve is

$$W_\epsilon = \frac{1}{2}\int_I k_\epsilon^2\, ds_\epsilon = \frac{1}{2}\int_I (k + \epsilon\,\delta k + \epsilon^2\delta^2 k)^2(ds + \epsilon\,\delta\, ds + \epsilon^2\delta^2\, ds).$$

As it is $\delta\, ds = \delta^2 ds = 0$, we have

$$W_\epsilon = W + \epsilon\int_I k\,\delta k\, ds + \epsilon^2\int_I (\frac{1}{2}(\delta k)^2 + k\delta^2 k)\, ds,$$

which is obtained after neglecting the terms in ϵ^3 and ϵ^4. Therefore,

$$\delta W = \int_I k\,\delta k\, ds, \quad \delta^2 W = \int_I (\frac{1}{2}(\delta k)^2 + k\delta^2 k)\, ds. \tag{3.2}$$

Theorem 3. *Under a second order infinitesimal bending of a curve C, the first variation of the Willmore energy is*

$$\begin{aligned}\delta W = &\int_I \big[(k'' + \frac{1}{2}k^3 - k\tau^2)\overset{(1)}{z_1} + (2k'\tau + k\tau')\overset{(1)}{z_2}\big]\, ds \\ &+ \int_I \big[\frac{1}{2}k^2\overset{(1)}{z} - k'\overset{(1)}{z_1} + k\overset{(1)}{z_1}' - 2k\tau\overset{(1)}{z_2}\big]'\, ds.\end{aligned} \tag{3.3}$$

Proof. Applying the results (2.7) and (3.2) we get

$$\delta W = \int_I k[k'^{(1)}z + {}^{(1)}z_1'' + (k^2-\tau^2)^{(1)}z_1 - 2\tau\,{}^{(1)}z_2' - \tau'^{(1)}z_2]\,ds. \tag{3.4}$$

Obviously, the equation

$$k\,{}^{(1)}z_1'' = (k\,{}^{(1)}z_1')' - k'\,{}^{(1)}z_1', \tag{3.5}$$

is valid and also

$$-k'\,{}^{(1)}z_1' = k''\,{}^{(1)}z_1 - (k'\,{}^{(1)}z_1)'. \tag{3.6}$$

Further,

$$-2\tau\,{}^{(1)}z_2'k = (-2\tau\,{}^{(1)}z_2k)' + (2\tau k)'\,{}^{(1)}z_2. \tag{3.7}$$

Also, using the first equation in (2.5), we obtain

$$kk'^{(1)}z = (\frac{1}{2}k^2\,{}^{(1)}z)' - \frac{1}{2}k^2\,{}^{(1)}z' = (\frac{1}{2}k^2\,{}^{(1)}z)' - \frac{1}{2}k^3\,{}^{(1)}z_1. \tag{3.8}$$

Substituting the results (3.5)-(3.8) into the equation (3.4) we confirm the validity of the equation (3.3). □

Directly from the previous theorem, in the case of the plane curve ($\tau = 0$), we obtain the next corollary.

Corollary 1. *Under a second order infinitesimal bending of a plane curve, the first variation of the Willmore energy is*

$$\delta W = \int_I (k'' + \frac{1}{2}k^3)\,{}^{(1)}z_1\,ds + \int_I [\frac{1}{2}k^2\,{}^{(1)}z - k'\,{}^{(1)}z_1 + k\,{}^{(1)}z_1']'\,ds.$$

From equation (3.3) we can see that the second summand is an integral of a total derivative. Also, we obtain the Euler-Lagrange equations

$$k'' + \frac{1}{2}k^3 - k\tau^2 = 0, \tag{3.9}$$

$$2k'\tau + k\tau' = 0. \tag{3.10}$$

These equations are the governing ones of the curve that minimize the integral (3.3). After integrating equation (3.10), we obtain

$$\tau k^2 = const, \tag{3.11}$$

which determines τ as a function of k. If we substitute τ into the equation (3.9) we get a second order differential equation for k which is, obviously, integrable.

Theorem 4. *Under a second order infinitesimal bending of a curve C, the second variation of the Willmore energy is*

$$\delta^2 W = I_1 + I_2, \tag{3.12}$$

where

$$
\begin{aligned}
I_1 = \int_I & \frac{1}{2}[k'^{(1)}_z + {}^{(1)}_{z_1}{}'' + (k^2-\tau^2)^{(1)}_{z_1} - 2\tau {}^{(1)}_{z_2}{}' - \tau'^{(1)}_{z_2}]^2 \\
& - \frac{3}{4}(k\tau^{(1)}_{z_1} + k{}^{(1)}_{z_2}{}')^2 + \frac{1}{2}(k\tau^{(1)}_{z} + 2\tau {}^{(1)}_{z_1}{}' + \tau'^{(1)}_{z_1} + {}^{(1)}_{z_2}{}'' - \tau^2 {}^{(1)}_{z_2})^2 \\
& - \frac{1}{4}(k^{2\,(1)}_{z} + k{}^{(1)}_{z_1}{}' - k\tau^{(1)}_{z_2})^2\, ds, \qquad (3.13) \\
I_2 = \int_I & \big[(k'' + \frac{1}{2}k^3 - k\tau^2)^{(2)}_{z_1} + (2k'\tau + k\tau')^{(2)}_{z_2}\big]\, ds \\
& + \int_I \big[\frac{1}{2}k^{2\,(2)}_{z} - k'^{(2)}_{z_1} + k{}^{(2)}_{z_1}{}' - 2k\tau^{(2)}_{z_2}\big]'\, ds
\end{aligned}
$$

Proof. Based on the equations (2.7), (2.8) and (3.2) we get

$$
\begin{aligned}
\delta^2 W = \int_I & \{\frac{1}{2}[k'^{(1)}_z + {}^{(1)}_{z_1}{}'' + (k^2-\tau^2)^{(1)}_{z_1} - 2\tau {}^{(1)}_{z_2}{}' - \tau'^{(1)}_{z_2}]^2 \\
& + k[-\frac{k}{2}(\tau^{(1)}_{z_1} + {}^{(1)}_{z_2}{}')^2 + \frac{1}{2k}(k\tau^{(1)}_{z} + 2\tau {}^{(1)}_{z_1}{}' + \tau'^{(1)}_{z_1} + {}^{(1)}_{z_2}{}'' - \tau^2 {}^{(1)}_{z_2})^2 \\
& + (k^{(2)}_{z} + {}^{(2)}_{z_1}{}' - \tau^{(2)}_{z_2})' - \tau({}^{(2)}_{z_2}{}' + \tau^{(2)}_{z_1})]\}\, ds.
\end{aligned}
$$

We can express the previous equation as $\delta^2 W = I_1 + I_2$ where

$$
\begin{aligned}
I_1 = \int_I & \frac{1}{2}\{[k'^{(1)}_z + {}^{(1)}_{z_1}{}'' + (k^2-\tau^2)^{(1)}_{z_1} - 2\tau {}^{(1)}_{z_2}{}' - \tau'^{(1)}_{z_2}]^2 \\
& - (k\tau^{(1)}_{z_1} + k{}^{(1)}_{z_2}{}')^2 + (k\tau^{(1)}_{z} + 2\tau {}^{(1)}_{z_1}{}' + \tau'^{(1)}_{z_1} + {}^{(1)}_{z_2}{}'' - \tau^2 {}^{(1)}_{z_2})^2\}\, ds, \\
I_2 = \int_I & [k(k^{(2)}_{z} + {}^{(2)}_{z_1}{}' - \tau^{(2)}_{z_2})' - k\tau({}^{(2)}_{z_2}{}' + \tau^{(2)}_{z_1})]\, ds.
\end{aligned}
$$

As it is

$$
k(k^{(2)}_{z} + {}^{(2)}_{z_1}{}' - \tau^{(2)}_{z_2})' = [k(k^{(2)}_{z} + {}^{(2)}_{z_1}{}' - \tau^{(2)}_{z_2})]' - kk'^{(2)}_{z} - (k'^{(2)}_{z_1})' + k''^{(2)}_{z_1} + k'\tau^{(2)}_{z_2},
$$

and also

$$
k\tau {}^{(2)}_{z_2}{}' = (k\tau^{(2)}_{z_2})' - (k\tau)'^{(2)}_{z_2},
$$

we obtain

$$
\begin{aligned}
I_2 = \int_I & [-kk'^{(2)}_{z} + (k'' - k\tau^2)^{(2)}_{z_1} + (2k'\tau + k\tau')^{(2)}_{z_2} \\
& + \int_I [k(k^{(2)}_{z} + {}^{(2)}_{z_1}{}' - \tau^{(2)}_{z_2}) - k'^{(2)}_{z_1} - k\tau^{(2)}_{z_2}]'\, ds.
\end{aligned}
$$

Furthermore, it holds the equation

$$
kk'^{(2)}_{z} = (\frac{1}{2}k^{2\,(2)}_{z})' - \frac{1}{2}k^{2\,(2)}_{z}{}',
$$

which, after using (2.5) gives

$$kk'^{(2)}_z = (\frac{1}{2}k^2{}^{(2)}_z)' - \frac{1}{2}k^3{}^{(2)}_{z_1} + \frac{1}{4}k^2[(k{}^{(1)}_z + {}^{(1)}_{z_1}{}' - \tau{}^{(1)}_{z_2})^2 + (\tau{}^{(1)}_{z_1} + {}^{(1)}_{z_2}{}')^2].$$

If we put the last summand from the previous equation into the expression for I_1, and the rest into I_2, we confirm the validity of the equations (3.12), (3.13) after a bit of calculation. □

Corollary 2. *Under a second order infinitesimal bending of a plane curve, the second variation of the Willmore energy is*

$$\delta^2 W = I_1 + I_2,$$

where

$$I_1 = \int_I \frac{1}{2}[k'^{(1)}_z + {}^{(1)}_{z_1}{}'' + k^2{}^{(1)}_{z_1}]^2 - \frac{3}{4}(k{}^{(1)}_{z_2}{}')^2 + \frac{1}{2}({}^{(1)}_{z_2}{}'')^2 - \frac{1}{4}(k^2{}^{(1)}_z + k{}^{(1)}_{z_1}{}')^2\, ds,$$

$$I_2 = \int_I (k'' + \frac{1}{2}k^3){}^{(2)}_{z_1}\, ds + \int_I [\frac{1}{2}k^2{}^{(2)}_z - k'{}^{(2)}_{z_1} + k{}^{(2)}_{z_1}{}']'\, ds.$$

If we compare the expressions of the first variation and the part I_2 of the second variation of the Willmore energy, we conclude the Euler-Lagrange equations

$$k'' + \frac{1}{2}k^3 - k\tau^2 = 0,$$
$$2k'\tau + k\tau' = 0$$

are the same in both cases. This means that the same governing equations of the curve minimize the integrals δW and I_2.

It is sometimes useful to get a family of closed curves under infinitesimal bending of a closed curve (infinitesimal bending without "tearing"). In that case we must specify the condition $\mathbf{z}(0) = \mathbf{z}(L)$ for the infinitesimal bending field. Also, we suppose that the curve, as well as the infinitesimal bending field are sufficiently smooth. Keeping this in mind we have the next corollary.

Corollary 3. *Under a second order infinitesimal bending of a closed curve C, the first and the second variation of the Willmore energy are:*

$$\delta W = \int_I [(k'' + \frac{1}{2}k^3 - k\tau^2){}^{(1)}_{z_1} + (2k'\tau + k\tau'){}^{(1)}_{z_2}],\, ds$$

$$\begin{aligned}\delta^2 W = \int_I & \frac{1}{2}[k'^{(1)}_z + {}^{(1)}_{z_1}{}'' + (k^2 - \tau^2){}^{(1)}_{z_1} - 2\tau{}^{(1)}_{z_2}{}' - \tau'{}^{(1)}_{z_2}]^2 \\ & - \frac{3}{4}(k\tau{}^{(1)}_{z_1} + k{}^{(1)}_{z_2}{}')^2 + \frac{1}{2}(k\tau{}^{(1)}_z + 2\tau{}^{(1)}_{z_1}{}' + \tau'{}^{(1)}_{z_1} + {}^{(1)}_{z_2}{}'' - \tau^2{}^{(1)}_{z_2})^2 \\ & - \frac{1}{4}(k^2{}^{(1)}_z + k{}^{(1)}_{z_1}{}' - k\tau{}^{(1)}_{z_2})^2 + (k'' + \frac{1}{2}k^3 - k\tau^2){}^{(2)}_{z_1} + (2k'\tau + k\tau'){}^{(2)}_{z_2}]\, ds.\end{aligned}$$

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ALTERNATING SUMS IN HYPERBOLIC PASCAL TRIANGLES

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Abstract. A new generalization of Pascal's triangle, the so-called hyperbolic Pascal triangles were introduced in [1]. The mathematical background goes back to the regular mosaics in the hyperbolic plane. The alternating sum of elements in the rows was given in the special case $\{4,5\}$ of the hyperbolic Pascal triangles. In this article, we determine the alternating sum generally in the hyperbolic Pascal triangle corresponding to $\{4,q\}$ with $q \geq 5$.

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Keywords: Pascal triangle, hyperbolic Pascal triangle, alternating sum

1. INTRODUCTION

In the hyperbolic plane there are infinite types of regular mosaics (see, for example [3]), they are denoted by Schläfli's symbol $\{p,q\}$, where $(p-2)(q-2) > 4$. Each regular mosaic induces a so called hyperbolic Pascal triangle (see [1]), following and generalizing the connection between the classical Pascal's triangle and the Euclidean regular square mosaic $\{4,4\}$. For more details see [1], but here we also collect some necessary information.

There are several approaches to generalize the Pascal's arithmetic triangle (see, for instance [2]). The hyperbolic Pascal triangle based on the mosaic $\{p,q\}$ can be figured as a digraph, where the vertices and the edges are the vertices and the edges of a well defined part of the lattice $\{p,q\}$, respectively, further the vertices possesses a value giving the number of different shortest paths from the base vertex. Figure 1 illustrates the hyperbolic Pascal triangle when $\{p,q\} = \{4,6\}$. Generally, for $\{4,q\}$ the base vertex has two edges, the leftmost and the rightmost vertices have three, the others have q edges. The square shaped cells surrounded by appropriate edges are corresponding to the regular squares in the mosaic. Apart from the winger elements, certain vertices (called "Type A" for convenience) have two ascendants and $q-2$ descendants, the others ("Type B") have one ascendant and $q-1$ descendants. In the figures we denote the vertices type A by red circles and the vertices type B by cyan diamonds, further the wingers by white diamonds. The vertices which are n-edge-long far from the base vertex are in row n.

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The general method of drawing is the following. Going along the vertices of the j^{th} row, according to type of the elements (winger, A, B), we draw appropriate number of edges downwards (2, $q-2$, $q-1$, respectively). Neighbour edges of two neighbour vertices of the j^{th} row meet in the $(j+1)^{th}$ row, constructing a vertex with type A. The other descendants of row j in row $j+1$ have type B. In the sequel, $\binom{n}{k}$ denotes the k^{th} element in row n, which is either the sum of the values of its two ascendants or the value of its unique ascendant. We note, that the hyperbolic Pascal triangle has the property of vertical symmetry.

It is well-known that the alternating sum $\sum_{i=0}^{n}(-1)^i\binom{n}{i}$ of row n in the classical Pascal's triangle is zero ($n \geq 1$). In [1] we showed that the alternating sum $\sum_i (-1)^i \binom{n}{i}$ is either 0 (if $n \equiv 1 \pmod 3$) or 2 (otherwise, with $n \geq 5$) for case $\{4,5\}$. In this paper, we determine an explicit form for the alternating sums generally for $\{4,q\}$ ($q \geq 5$). If one considers the result with $q = 4$, it returns 0 according to the classical Pascal's triangle. The definitions, the signs, the figures, and the method strictly follow the article [1].

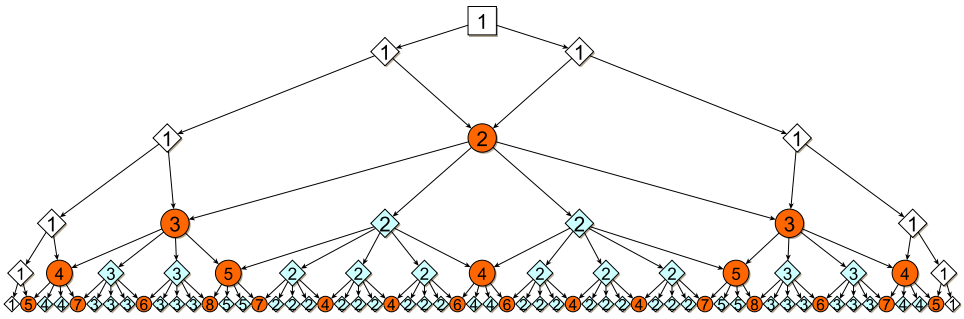

FIGURE 1. Hyperbolic Pascal triangle linked to $\{4,6\}$ up to row 5

2. MAIN THEOREMS

From this point we consider the hyperbolic Pascal triangle based on the mosaic $\{4,q\}$ with $q \geq 5$. Denote by s_n and $\hat{s}_n$ the number of the vertices, and the sum of the elements in row n, respectively.

The sequences $\{s_n\}$ and $\{\hat{s}_n\}$ can be given (see again [1]) by the ternary homogenous recurrence relations

$$s_n = (q-1)s_{n-1} - (q-1)s_{n-2} + s_{n-3} \qquad (n \geq 4), \tag{2.1}$$

(the initial values are $s_1 = 2$, $s_2 = 3$, $s_3 = q$) and

$$\hat{s}_n = q\hat{s}_{n-1} - (q+1)\hat{s}_{n-2} + 2\hat{s}_{n-3} \qquad (n \geq 4),$$

(the initial values are $\hat{s}_1 = 2$, $\hat{s}_2 = 4$, $\hat{s}_3 = 2q$), respectively.

Let $\widetilde{s}_n$ be the alternating sum of elements of the hyperbolic Pascal triangle (starting with positive coefficient) in row n, and we distinguish the even and odd cases.

Theorem 1. *Let q be even. Then*

$$\widetilde{s}_n = \sum_{i=0}^{s_n-1} (-1)^i \left|\binom{n}{i}\right| = \begin{cases} 0, & \text{if} \quad n = 2t+1, \quad n \geq 1, \\ -2(5-q)^{t-1}+2, & \text{if} \quad n = 2t, \quad n \geq 2, \end{cases}$$

hold, further $\widetilde{s}_0 = 1$.

Corollary 1. *In case of $q = 6$ we deduce*

$$\widetilde{s}_n = \begin{cases} 0, & \text{if} \quad n \neq 4t, \quad n \geq 1, \\ 4, & \text{if} \quad n = 4t, \quad n \geq 4. \end{cases}$$

Remark 1. For $q = 4$ Theorem 1 would return with 0, providing the known result $\widetilde{s}_n = 0$ for the original Pascal's triangle.

Theorem 2. *Let $q \geq 5$ be odd. Then $\widetilde{s}_0 = 1$, further*

$$\widetilde{s}_n = \sum_{i=0}^{s_n-1} (-1)^i \left|\binom{n}{i}\right| = \begin{cases} 0, & \text{if} \quad n = 3t+1, \quad n \geq 1, \\ (-2)^t(q-5)^{t-1}+2, & \text{if} \quad n = 3t-1, \quad n \geq n_1, \\ 2(-2)^t(q-5)^{t-1}+2, & \text{if} \quad n = 3t, \quad n \geq n_2, \end{cases}$$

where $(n_1, n_2) = (2,3)$ and $(5,6)$ if $n > 5$ and $n = 5$, respectively. In the latter case $\widetilde{s}_2 = 0$, $\widetilde{s}_3 = -2$.

We note, that by the help of $\hat{s}_n$ and $\widetilde{s}_n$ we can easily determine the alternate sum with the arbitrary weights v and w.

Corollary 2.

$$\begin{aligned} \widetilde{s}_{(v,w),n} &= \sum_{i=0}^{s_n-1} \left(v\delta_{0,i \bmod 2} + w\delta_{1,i \bmod 2}\right) \left|\binom{n}{i}\right| = \frac{\hat{s}_n + \widetilde{s}_n}{2} v + \frac{\hat{s}_n - \widetilde{s}_n}{2} w \\ &= \frac{v+w}{2}\hat{s}_n + \frac{v-w}{2}\widetilde{s}_n, \end{aligned}$$

where $\delta_{j,i}$ is the Kronecker delta.

3. Proofs of Theorems

Since the hyperbolic Pascal triangle has a symmetry axis, if s_n is even then the alternating sum is zero. In the case when s_n is odd (in the sequel, we assume it), the base of both proofs is to consider the vertices type A and B of row n and to observe their influence on $\widetilde{s}_{n+2}$ or $\widetilde{s}_{n+3}$. We separate the contribution of each $\left|\binom{n}{i}\right|$ individually, and then take their superposition.

Let $\widetilde{s}_n^{(A)}$ and $\widetilde{s}_n^{(B)}$ be the subsum of $\widetilde{s}_n$ restricted only to the elements of type A and B, respectively. As $\binom{n}{0} = \binom{n}{s_n-1} = 1$, then

$$\widetilde{s}_n = \widetilde{s}_n^{(A)} + \widetilde{s}_n^{(B)} + 2. \tag{3.1}$$

Using the notations of [1], x_A and x_B denote the value of an element of type A and B, respectively. (In the figures, we indicate them shortly by x.) Their contributions to $\widetilde{s}_{n+k}$ $(k \geq 1)$ are denote by $\mathcal{H}_k(x_A)$ and $\mathcal{H}_k(x_B)$, respectively, and for example, $\mathcal{H}_k^{(A)}(x_A)$ and $\mathcal{H}_k^{(B)}(x_A)$ for the contribution of x_A from the row n to the alternate sum of the row $n+k$ restricted to the elements of type A and B, respectively. Similarly, $\mathcal{H}_k(1)$ shows the contribution of a winger element of row n (having value 1). According to [1] we have

$$\begin{aligned}
\mathcal{H}_k(x_A) &= \mathcal{H}_k^{(A)}(x_A) + \mathcal{H}_k^{(B)}(x_A),\\
\mathcal{H}_k(x_B) &= \mathcal{H}_k^{(A)}(x_B) + \mathcal{H}_k^{(B)}(x_B),\\
\mathcal{H}_k(1) &= \mathcal{H}_k^{(A)}(1) + \mathcal{H}_k^{(B)}(1) + 1.
\end{aligned}$$

Clearly, it is obvious that $\widetilde{s}_0 = 1, \widetilde{s}_2 = 0$ hold for all $q \geq 5$.

3.1. *Proof for even q*

If $n = 2t+1$ $(n \geq 1)$, then in accordance with relation (2.1) s_n is even, otherwise odd. So, we may suppose that n is also even.

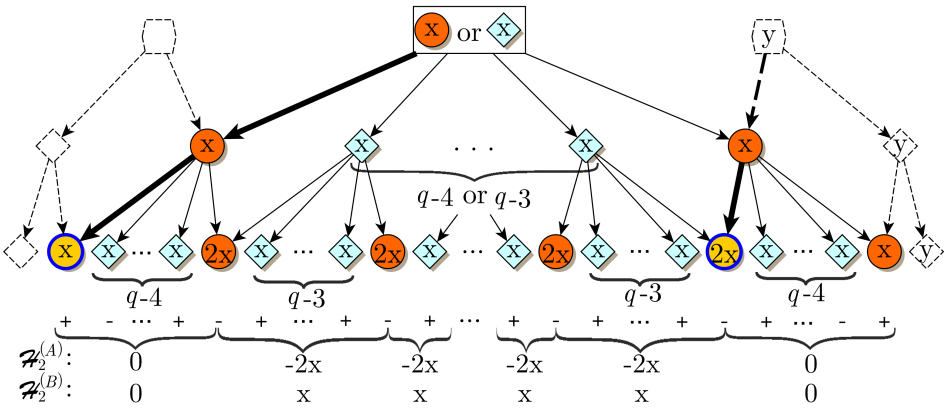

FIGURE 2. The influence $\mathcal{H}_2(x_A)$ and $\mathcal{H}_2(x_B)$

Figure 2 shows the contributions of x_A and x_B from the row n to the alternating sum of the row $n+2$. From the growing method of the hyperbolic Pascal triangle, a vertex type A in row i generates $q-4$ vertices type B in row $i+1$, if the vertex is type B then it has $q-3$ generated vertices type B in row $i+1$. The value of a vertex

is either the value of its ascendant or the sum of values of its two ascendants. Then we consider the value $2x_A$ (and $2x_B$) of vertices type A in rows $n+2$ as $x_A + x_A$ (and $x_B + x_B$). In Figure 2 we drew the values of the alternating sums belonging to x (in row $n+2$) in blocks. The last but one row shows the values $\mathcal{H}_2^{(A)}(x_A)$ (or $\mathcal{H}_2^{(A)}(x_B)$), the last row shows $\mathcal{H}_2^{(B)}(x_A)$ (or $\mathcal{H}_2^{(B)}(x_B)$) in blocks.

Put $\varepsilon = \pm 1$ and $\delta = \pm 1$ in case of the vertex type A and B is being considered, respectively. (In Figure 2, they are $+1$, it is the first sign in row $n+2$.) Now we obtain, by observing Figure 2, that

$$\mathcal{H}_2^{(A)}(x_A) = \varepsilon \cdot (-2x(q-4)) = -2(q-4)\varepsilon x_A,$$
$$\mathcal{H}_2^{(B)}(x_A) = \varepsilon \cdot x(q-4) = (q-4)\varepsilon x_A,$$

and

$$\mathcal{H}_2^{(A)}(x_B) = \delta \cdot (-2x(q-3)) = -2(q-3)\delta x_B,$$
$$\mathcal{H}_2^{(B)}(x_B) = \delta \cdot x(q-3) = (q-3)\delta x_B.$$

Figure 3 shows the contributions of the influence of left winger element of row n to row $n+2$. For the right winger elements the situation is the same, thanks to the vertical symmetry. Thus

$$\mathcal{H}_2^{(A)}(1) = -1,$$
$$\mathcal{H}_2^{(B)}(1) = 0.$$

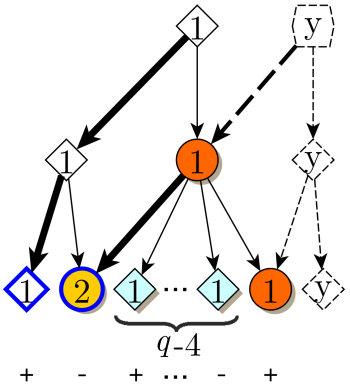

FIGURE 3. The influence $\mathcal{H}_2(1)$

We have given the influence of an element located in row n on row $n+2$. Let suppose that y is the value of the neighbour element of x in row n. Clearly, y has

also influence on row $n+1$. The signs of x and y in the alternating sum in row n are different and the signs of left hand side of their influence structures are also different in row $n+2$. In the figures, the leftmost element of influence structures are highlighted. (The signs of rightmost values of influence structures are also different.) The situation is the same in case of the winger elements. Thus, according to [1] we can give the changing of the alternating sums from row n to row $n+2$.

Summarising the results, we obtain the system of recurrence equations

$$\widetilde{s}^{(A)}_{n+2} = -2(q-4)\widetilde{s}^{(A)}_{n} - 2(q-3)\widetilde{s}^{(B)}_{n} - 2, \qquad (n \geq 0), \tag{3.2}$$

$$\widetilde{s}^{(B)}_{n+2} = \quad (q-4)\widetilde{s}^{(A)}_{n} + \ (q-3)\widetilde{s}^{(B)}_{n}, \qquad (n \geq 0). \tag{3.3}$$

Now we apply the following lemma (see [1]).

Lemma 1. *Let x_0, y_0, further u_i, v_i and w_i $(i = 1,2)$ be complex numbers such that $a_2 b_1 \neq 0$. Assume that the for $n \geq n_0$ terms of the sequences $\{x_n\}$ and $\{y_n\}$ satisfy*

$$x_{n+1} = u_1 x_n + v_1 y_n + w_1,$$
$$y_{n+1} = u_2 x_n + v_2 y_n + w_2.$$

Then for both sequences

$$z_{n+3} = (u_1 + v_2 + 1)z_{n+2} + (-u_1 v_2 + u_2 v_1 - u_1 - v_2)z_{n+1} + (u_1 v_2 - u_2 v_1)z_n$$

holds $(n \geq n_0)$.

Thus we obtain $\widetilde{s}^{(A)}_{n+6} = (6-q)\widetilde{s}^{(A)}_{n+4} + (q-5)\widetilde{s}^{(A)}_{n+2}$ and $\widetilde{s}^{(B)}_{n+6} = (6-q)\widetilde{s}^{(B)}_{n+4} + (q-5)\widetilde{s}^{(B)}_{n+2}$. Obviously, $\widetilde{s}^{(A)}_{2} = -2$ and $\widetilde{s}^{(B)}_{2} = 0$ fulfil. Using (3.2) and (3.3) we gain $\widetilde{s}^{(A)}_{4} = 4q - 18$ and $\widetilde{s}^{(B)}_{4} = -2(q-4)$.

From (3.1) we conclude

$$\widetilde{s}_{n+6} = (6-q)\widetilde{s}_{n+4} + (q-5)\widetilde{s}_{n+2}, \qquad (n \geq 0), \tag{3.4}$$

where $\widetilde{s}_0 = 1$, $\widetilde{s}_2 = 0$ and $\widetilde{s}_4 = 2(q-4)$.

The characteristic equation of (3.4) is

$$\widetilde{p}(x) = x^4 + (q-6)x^2 - (q-5) = \left(x^2 - (5-q)\right)(x^2 - 1).$$

Further, we have $\widetilde{s}_{2t} = \alpha(5-q)^t + \beta$, and from $\widetilde{s}_2$ and $\widetilde{s}_4$ we realize $\alpha = -2/(5-q)$, $\beta = 2$, and then $\widetilde{s}_{2t} = (-2/(5-q))(5-q)^t + 2 = -2(5-q)^{t-1} + 2$.

Remark 2. For all $n \geq 1$, $\widetilde{s}_n = -\widetilde{s}^{(B)}_n$, because $\widetilde{s}_i = -\widetilde{s}^{(B)}_i = 0$ for $i = 1,2,3$ and $\widetilde{s}_4 = -\widetilde{s}^{(B)}_4$.

3.2. *Proof for odd q*

We examine rows $n \neq 3t+1$ ($n \geq 2$), because the number of element in row $n = 3t+1$ is even (see relation (2.1)).

Here, apart from some details, we copy the treatment of the previous case. The first difference is that now we have to examine the influence of the elements from row n on row $n+3$ because the nice property about the signs first appears three rows later. The influence structures $\mathcal{H}_3(x_A)$ and $\mathcal{H}_3(x_B)$ are rather complicated, so we split them into smaller parts. First we draw the structures $\mathcal{H}_2(x_A)$ and $\mathcal{H}_2(x_B)$ when x_A and x_B are in row $n+1$, and we describe the influence of them on row $n+3$. Then we combine the result with the branches of $\mathcal{H}_3(x_A)$ and $\mathcal{H}_3(x_B)$. In Figures 4 and 5 only $\mathcal{H}_2(x_A)$ and $\mathcal{H}_2(x_B)$ can be seen, later in Figure 6 we consider the "skeleton" of $\mathcal{H}_3(x_A)$ and $\mathcal{H}_3(x_B)$.

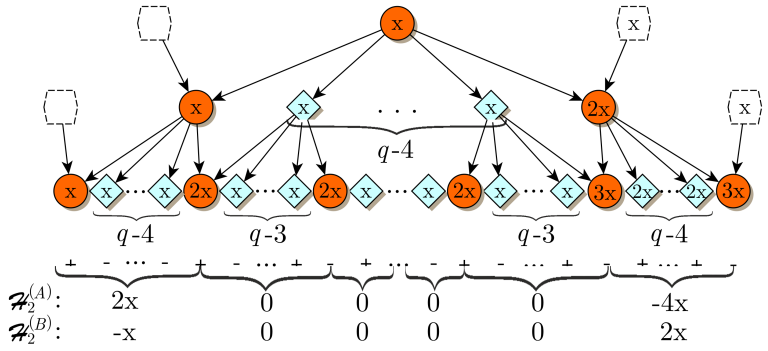

FIGURE 4. The influence $\mathcal{H}_2(x_A)$

From the figures one can derive the observations

$$\mathcal{H}_3^{(A)}(x_A) = \varepsilon \cdot (2x - 4x - 4x(q-4) + 2x) = -4(q-4)\varepsilon x_A,$$
$$\mathcal{H}_3^{(B)}(x_A) = \varepsilon \cdot (-x + 2x + 2x(q-4) - x) = 2(q-4)\varepsilon x_A,$$

and

$$\mathcal{H}_3^{(A)}(x_B) = \delta \cdot (2x - 4x - 4x(q-3) + 2x) = -4(q-3)\delta x_B,$$
$$\mathcal{H}_3^{(B)}(x_B) = \delta \cdot (-x + 2x + 2x(q-3) - x) = 2(q-3)\delta x_B.$$

We also have

$$\mathcal{H}_3^{(A)}(1) = -5 + 2 = -3,$$
$$\mathcal{H}_3^{(B)}(1) = 2 - 1 = 1.$$

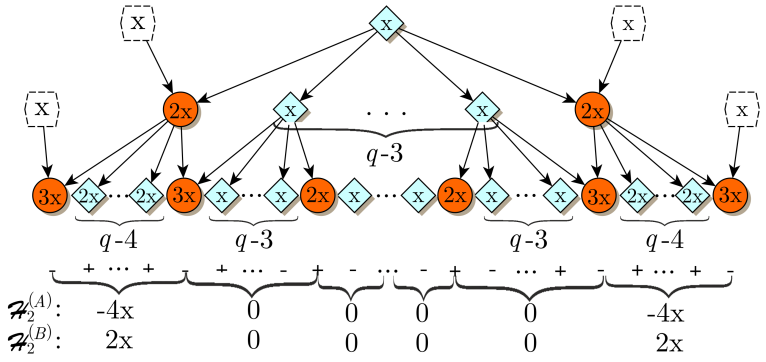

FIGURE 5. The influence $\mathcal{H}_2(x_B)$

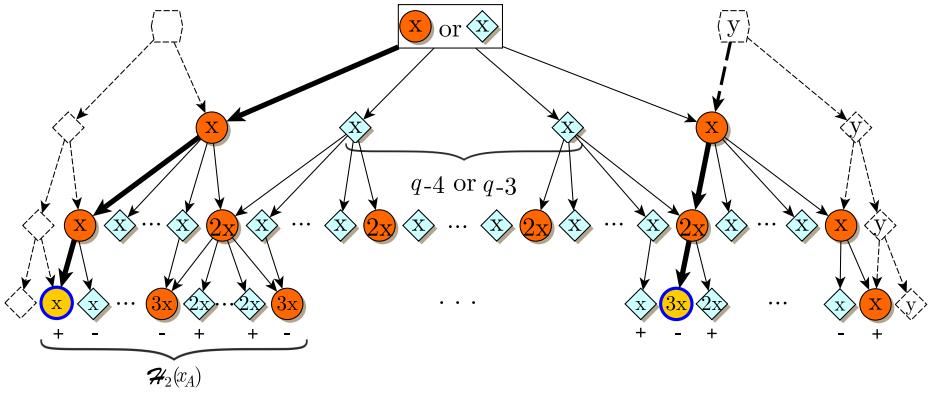

FIGURE 6. The influence $\mathcal{H}_3(x_A)$ and $\mathcal{H}_3(x_B)$

Combining the informations, it results the system of recurrence relations

$$\widetilde{s}^{(A)}_{n+3} = -4(q-4)\widetilde{s}^{(A)}_n - 4(q-3)\widetilde{s}^{(B)}_n + 2\cdot(-3), \qquad (n \geq 0), \tag{3.5}$$

$$\widetilde{s}^{(B)}_{n+3} = \quad 2(q-4)\widetilde{s}^{(A)}_n + 2(q-3)\widetilde{s}^{(B)}_n + 2\cdot 1, \qquad (n \geq 0). \tag{3.6}$$

Lemma 1 yields $\widetilde{s}^{(A)}_{n+9} = (11-2q)\widetilde{s}^{(A)}_{n+6} + (2q-10)\widetilde{s}^{(A)}_{n+3}$ and $\widetilde{s}^{(B)}_{n+9} = (11-2q)\widetilde{s}^{(B)}_{n+6} + (2q-10)\widetilde{s}^{(B)}_{n+3}$. Apparently, $\widetilde{s}^{(A)}_2 = -2$, $\widetilde{s}^{(B)}_2 = 0$, and $\widetilde{s}^{(A)}_3 = -6$, $\widetilde{s}^{(B)}_3 = 2$. From the system (3.5) and (3.6) we gain $\widetilde{s}^{(A)}_5 = 8q-38$, $\widetilde{s}^{(B)}_5 = -4q+18$ and $\widetilde{s}^{(A)}_6 = 16q-78$, $\widetilde{s}^{(B)}_6 = -8q+38$.

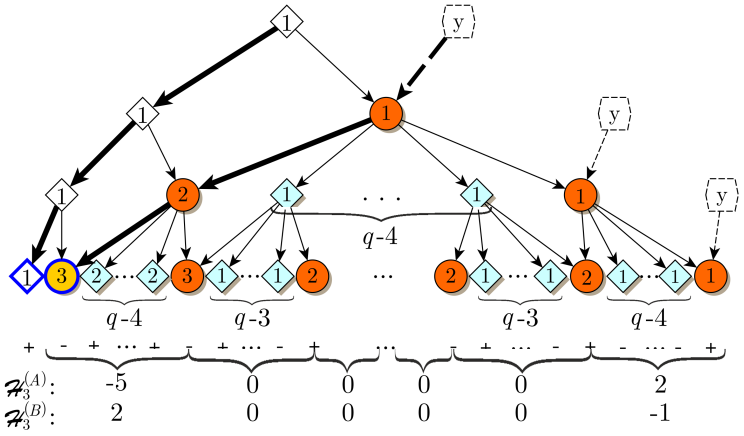

FIGURE 7. The influence $\mathcal{H}_3(1)$

By (3.1) we realize

$$\widetilde{s}_{n+9} = (11-2q)\widetilde{s}_{n+6} + (2q-10)\widetilde{s}_{n+3}, \qquad (n \geq 0), \tag{3.7}$$

where $\widetilde{s}_2 = 0, \widetilde{s}_3 = -2$ and $\widetilde{s}_5 = 4q-18, \widetilde{s}_6 = 8q-38$.

In case of $q = 5$, the equation (3.7) gives $\widetilde{s}_{n+9} = \widetilde{s}_{n+6}$ and $\widetilde{s}_5 = 2, \widetilde{s}_6 = 2$. So $\widetilde{s}_n = 2$ holds if $n \geq 5$. In case $q > 5$, the characteristic equation of (3.7) is

$$\widetilde{p}(x) = x^6 + (2q-11)x^3 - (2q-10) = \left(x^3 - 2(5-q)\right)(x^3-1).$$

Thus $\widetilde{s}_{3t}$ or $\widetilde{s}_{3t-1} = \alpha(2(5-q))^t + \beta$ fulfils for some α and β. Finally, if $n = 3t-1$, then from $\widetilde{s}_2$ and $\widetilde{s}_5$ we obtain $\alpha = 1/(q-5)$ and $\beta = 2$. Hence $\widetilde{s}_{3t-1} = (10-2q)^t/(q-5)+2$. Otherwise, if $n = 3t$, then from $\widetilde{s}_3$ and $\widetilde{s}_6$ we deduce $\alpha = 2/(q-5)$ and $\beta = 2$. Thus $\widetilde{s}_{3t} = 2(10-2q)^t/(q-5)+2$.

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BERTRAND CURVES IN THREE DIMENSIONAL LIE GROUPS

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Abstract. In this paper, we give the definition of harmonic curvature function some special curves such as helix, slant curves, Mannheim curves and Bertrand curves. Then, we recall the characterizations of helices [7], slant curves (see [19]) and Mannheim curves (see [12]) in three dimensional Lie groups using their harmonic curvature function.

Moreover, we define Bertrand curves in a three dimensional Lie group G with a bi-invariant metric and the main result in this paper is given as (Theorem 7): A curve $\alpha : I \subset \mathbb{R} \to G$ with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ is a Bertrand curve if and only if

$$\lambda\kappa + \mu\kappa H = 1$$

where λ, μ are constants and H is the harmonic curvature function of the curve α.

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Keywords: Bertrand curves, Lie groups.

1. INTRODUCTION

The general theory of curves in a Euclidean space (or more generally in a Riemannian manifolds) have been developed a long time ago and we have a deep knowledge of its local geometry as well as its global geometry. In the theory of curves in Euclidean space, one of the important and interesting problem is characterizations of a regular curve. In the solution of the problem, the curvature functions k_1 (or κ) and k_2 (or τ) of a regular curve have an effective role. For example: if $k_1 = 0 = k_2$, then the curve is a geodesic or if k_1 =constant$\neq 0$ and $k_2 = 0$, then the curve is a circle with radius $(1/k_1)$, etc. Thus we can determine the shape and size of a regular curve by using its curvatures. Another way in the solution of the problem is the relationship between the Frenet vectors of the curves (see [15]).

For instance Bertrand curves: In the classical diferential geometry of curves, J. Bertrand studied curves in Euclidean 3-space whose principal normals are the principal normals of another curve. In [3], he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other word, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, then for $\lambda, \mu \in \mathbb{R}$ we have $\lambda k_1 + \mu k_2 = 1$. Since

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the time of Bertrand's paper, pairs of curves of this kind have been called *Conjugate Bertrand Curves*, or more commonly *Bertrand Curves* (see [15]).

In 1888, *C. Bioche* [4] give a new theorem to obtaining Bertrand curves by using the given two curves C_1 and C_2 in Euclidean 3−space. Later, in 1960, *J. F. Burke* [5] give a theorem related with *Bioche's thorem on Bertrand curves.*

The following properties of Bertrand curves are well known: If two curves have the same principal normals, **(i)** corresponding points are a fixed distance apart; **(ii)** the tangents at corresponding points are at a fixed angle. These well known properties of Bertrand curves in Euclidean 3-space was extended by *L. R. Pears* in [21] to Riemannian $n-$space and found general results for Bertrand curves. When we applying these general result to Euclidean n-space, it is easily find that either k_2 or k_3 is zero; in other words, Bertrand curves in ,$\mathbb{E}^n (n > 3)$ are degenerate curves. This result is restated by *Matsuda and Yorozu* [18]. They proved that *there is no special Bertrand curves* in $E^n (n > 3)$ and they define new kind, which is called $(1,3)-$type Bertrand curves in 4−dimensional Euclidean space. Bertrand curves and their characterizations were studied by many authours in Euclidean space as well as in Riemann–Otsuki space, in Minkowski 3- space and Minkowski spacetime (for instance see [1, 2, 10, 14, 17, 22, 23].)

The degenarete semi-Riemannian geometry of Lie group is studied by Çöken and Çiftçi [8]. Moreover, they obtanied a naturally reductive homogeneous semi-Riemannian space using the Lie group. Then Çiftçi [7] defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret's theorem. Also he gave a relation between the geodesics of the so-called cylinders and general helices. Then, Okuyucu et al. [19] defined slant helices in three dimensional Lie groups with a bi-invariant metric and obtained some characterizations using their harmonic curvature function.

Recently, *Izumiya and Takeuchi* [13] have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ was defined by the property that its principal normal vector field makes a constant angle with a fixed direction. Also, Izumiya and Takeuchi showed that α is a slant helix if and only if the geodesic curvature of spherical image of principal normal indicatrix (N) of a space curve α

$$\sigma_N(s) = \left(\frac{\kappa^2}{\left(\kappa^2 + \tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$$

is a constant function .

Harmonic curvature functions were defined by Özdamar and Hacısalihoğlu [20]. Recently, many studies have been reported on generalized helices and slant helices using the harmonic curvatures in Euclidean spaces and Minkowski spaces [6, 11, 16]. Then, Okuyucu et al. [19] defined slant helices in three dimensional Lie groups with a bi-invariant metric and obtained some characterizations using their harmonic curvature function.

In this paper, first of all, we give the definition of harmonic curvature function some special curves such as helix, slant curves. Then, we recall the characterizations of helices [7], slant curves (see [19]) and Mannheim curves (see [12]) in three dimensional Lie groups using their harmonic curvature function. Moreover, we define Bertrand curves in a three dimensional Lie group G with a bi-invariant metric and then the main result to this paper is given as (Theorem 7): A curve $\alpha : I \subset \mathbb{R} \to G$ with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ is a Bertrand curve if and only if

$$\lambda\kappa + \mu\kappa H = 1$$

where λ, μ are constants and H is the harmonic curvature function of the curve α.

Note that three dimensional Lie groups admitting bi-invariant metrics are $SO(3)$, SU^2 and Abelian Lie groups. So we believe that our characterizations about Bertrand curves will be useful for curves theory in Lie groups.

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle\,,\rangle$ and D be the Levi-Civita connection of Lie group G. If $\mathfrak{g}$ denotes the Lie algebra of G then we know that $\mathfrak{g}$ is isomorphic to T_eG where e is neutral element of G. If $\langle\,,\rangle$ is a bi-invariant metric on G then we have

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle \tag{2.1}$$

and

$$D_X Y = \frac{1}{2}[X, Y] \tag{2.2}$$

for all X, Y and $Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted regular curve and $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis of $\mathfrak{g}$. In this case, we write that any two vector fields W and Z along the curve α as $W = \sum_{i=1}^{n} w_i X_i$ and $Z = \sum_{i=1}^{n} z_i X_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Also the Lie bracket of two vector fields W and Z is given

$$[W, Z] = \sum_{i=1}^{n} w_i z_i \left[X_i, X_j\right]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha'}W$ is given as follows

$$D_{\alpha'}W = \dot{W} + \frac{1}{2}[T, W] \tag{2.3}$$

where $T = \alpha'$ and $\dot{W} = \sum_{i=1}^{n} \dot{w}_i X_i$ or $\dot{W} = \sum_{i=1}^{n} \frac{dw}{dt} X_i$. Note that if W is the left-invariant vector field to the curve α then $\dot{W} = 0$ (see for details [9]).

Let G be a three dimensional Lie group and (T, N, B, κ, τ) denote the Frenet apparatus of the curve α. Then the Serret-Frenet formulas of the curve α satisfies:

$$D_T T = \kappa N, \quad D_T N = -\kappa T + \tau B, \quad D_T B = -\tau N$$

where D is Levi-Civita connection of Lie group G and $\kappa = \|\dot{T}\|$.

Definition 1 ([7]). Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve. Then α is called a *general helix* if it makes a constant angle with a left-invariant vector field X. That is,

$$\langle T(s), X \rangle = \cos\theta \text{ for all } s \in I,$$

for the left-invariant vector field $X \in g$ is unit length and θ is a constant angle between X and T, which is the tangent vector field of the curve α.

Proposition 1 ([7]). *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus (T, N, B, κ, τ) then τ_G is defined by*

$$\tau_G = \frac{1}{2} \langle [T, N], B \rangle \tag{2.4}$$

or

$$\tau_G = \frac{1}{2\kappa^2\tau} \langle \ddot{T}, [T, \dot{T}] \rangle + \frac{1}{4\kappa^2\tau} \|[T, \dot{T}]\|^2.$$

Definition 2 ([19]). Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve. Then α is called a *slant helix* if its principal normal vector field makes a constant angle with a left-invariant vector field X which is unit length. That is,

$$\langle N(s), X \rangle = \cos\theta \text{ for all } s \in I,$$

where $\theta \neq \frac{\pi}{2}$ is a constant angle between X and N which is the principal normal vector field of the curve α.

Definition 3 ([19]). Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$. Then the *harmonic curvature function* of the curve α is defined by

$$H = \frac{\tau - \tau_G}{\kappa}$$

where $\tau_G = \frac{1}{2} \langle [T, N], B \rangle$.

Theorem 1 ([7]). *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus (T, N, B, κ, τ). The curve α is a general helix, if and only if*

$$\tau = c\kappa + \tau_G$$

where c is a constant.

Also, the next theorem can be given by using the definition of the harmonic curvature function of the curve α.

Theorem 2. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus (T, N, B, κ, τ). The curve α is a general helix, if and only if the harmonic curvature function of the curve α is a constant function.*

Proof. It is obvious using Definition 3 and Theorem 1. □

Theorem 3 ([19]). *Let $\alpha : I \subset \mathbb{R} \to G$ be a unit speed curve with the Frenet apparatus (T, N, B, κ, τ). Then α is a slant helix if and only if*

$$\sigma_N = \frac{\kappa(1+H^2)^{\frac{3}{2}}}{H'} = \tan\theta$$

is a constant where H is a harmonic curvature function of the curve α and $\theta \neq \frac{\pi}{2}$ is a constant.

Theorem 4 ([12]). *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with arc length parameter s and the Frenet apparatus (T, N, B, κ, τ). Then, α is Mannheim curve if and only if*

$$\lambda\kappa\left(1+H^2\right) = 1, \text{ for all } s \in I \tag{2.5}$$

where λ is constant and H is the harmonic curvature function of the curve α.

Theorem 5. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with arc length parameter s. Then β is the Mannheim partner curve of α if and only if the curvature κ_β and the torsion τ_β of β satisfy the following equation*

$$\frac{d\kappa_\beta H_\beta}{d\bar{s}} = \frac{\kappa_\beta}{\mu}(1+\mu^2\kappa_\beta^2 H_\beta^2)$$

where μ is constant and H_β is the harmonic curvature function of the curve β.

3. Bertrand curves in a three dimensional Lie group

In this section, we define Bertrand curves and their characterizations are given in a three dimensional Lie group G with a bi-invariant metric $\langle\,,\rangle$. Also we give some characterizations of Bertrand curves using the special cases of G.

Definition 4. A curve α in 3-dimensional Lie group G is a *Bertrand curve* if there exists a special curve β in 3-dimensional Lie group G such that principal normal vector field of α is linearly dependent principal normal vector field of β at corresponding point under ψ which is bijection from α to β. In this case β is called the *Bertrand mate curve* of α and (α, β) is called *Bertrand curve couple.*

The curve $\alpha : I \subset \mathbb{R} \to G$ in 3-dimensional Lie group G is parametrized by the arc-length parameter s and from Definition 4 Bertrand mate curve of α is given $\beta : \overline{I} \subset \mathbb{R} \to G$ in 3-dimensional Lie group G with the help of Figure 1 such that

$$\beta(s) = \alpha(s) + \lambda(s) N(s),\ s \in I$$

where λ is a smooth function on I and N is the principal normal vector field of α. We should remark that the parameter s generally is not an arc-length parameter of β.

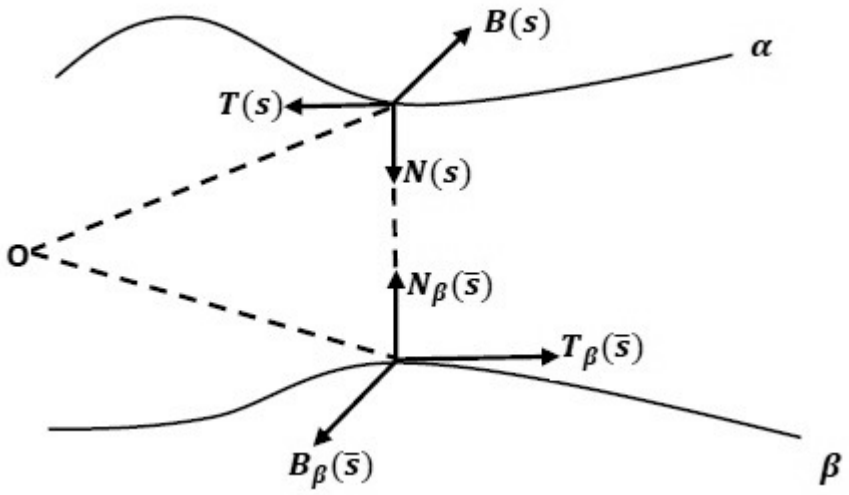

FIGURE 1. Bertrand Partner Curves

So, we define the arc-length parameter of the curve β by

$$\overline{s} = \psi(s) = \int_0^s \left\| \frac{d\beta(s)}{ds} \right\| ds$$

where $\psi : I \longrightarrow \overline{I}$ is a smooth function and holds the following equality

$$\psi'(s) = \kappa H \sqrt{\lambda^2 + \mu^2} \tag{3.1}$$

for $s \in I$.

Proposition 2 ([19]). *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B\}$. Then the following equalities*

$$[T, N] = \langle [T, N], B \rangle B = 2\tau_G B$$
$$[T, B] = \langle [T, B], N \rangle N = -2\tau_G N$$

hold.

Theorem 6. *Let $\alpha : I \subset \mathbb{R} \to G$ and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Bertrand curve couple with arc-length parameter s and $\overline{s}$, respectively. Then corresponding points are a fixed distance apart for all $s \in I$, that is,*

$$d(\alpha(s), \beta(s)) = \text{constant}, \quad \text{for all } s \in I$$

Proof. From Definition 4, we can simply write

$$\beta(s) = \alpha(s) + \lambda(s) N(s) \tag{3.2}$$

Differentiating Eq. (3.2) with respect to s and using Eq. (2.3), we get

$$\frac{d\beta(\bar{s})}{d\bar{s}}\psi'(s) = \frac{d\alpha(s)}{ds} + \lambda'(s)N(s) + \lambda(s)\dot{N}(s)$$
$$= (1-\lambda(s)\kappa(s))T(s) + \lambda'(s)N(s) + \lambda(s)\tau(s)B(s) - \frac{1}{2}[T,N]$$

and with the help of Proposition 2, we obtain

$$\frac{d\beta(\bar{s})}{d\bar{s}}\psi'(s) = (1-\lambda(s)\kappa(s))T(s) + \lambda'(s)N(s) + \lambda(s)((\tau-\tau_G)(s))B(s)$$

or

$$T_\beta(\bar{s}) = \frac{1}{\psi'(s)}\left[(1-\lambda(s)\kappa(s))T(s) + \lambda'(s)N(s) + \lambda(s)((\tau-\tau_G)(s))B(s)\right].$$

And then, we know that $\{N_\beta((\bar{s})), N(s)\}$ is a linearly dependent set, so we have

$$\left\langle T_\beta(\bar{s}), N_\beta(\bar{s})\right\rangle = \frac{1}{\psi'(s)}\left[\begin{array}{c}(1-\lambda(s)\kappa(s))\left\langle T(s), N_\beta(\bar{s})\right\rangle + \lambda'(s)\left\langle N(s), N_\beta(\bar{s})\right\rangle \\ +\lambda(s)\tau(s)\left\langle B(s), N_\beta(\bar{s})\right\rangle\end{array}\right]$$

Since $\left\langle T_\beta(\bar{s}), N_\beta(\bar{s})\right\rangle = 0$, we get $\lambda'(s) = 0$ from the last formula. That is, $\lambda(s)$ is a constant function on I. This completes the proof. □

Theorem 7. *If $\alpha : I \subset \mathbb{R} \to G$ is a parametrized Bertrand curve with arc length parameter s and the Frenet apparatus (T, N, B, κ, τ), then α satisfy the following equality*

$$\lambda\kappa(s) + \mu\kappa(s)H(s) = 1, \text{ for all } s \in I \tag{3.3}$$

where λ, μ are constants and H is the harmonic curvature function of the curve α.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Bertrand curve with arc length parameter s then we can write

$$\beta(s) = \alpha(s) + \lambda N(s)$$

Differentiating the above equality with respect to s and by using the Frenet equations, we get

$$\frac{d\beta(\bar{s})}{d\bar{s}}\psi'(s) = \frac{d\alpha(s)}{ds} + \lambda(s)\dot{N}$$
$$= (1-\lambda(s)\kappa(s))T(s) + \lambda(s)\tau(s)B(s) - \frac{1}{2}[T,N]$$

and with the help of Proposition 2, we obtain

$$T_\beta(\bar{s}) = \frac{(1-\lambda\kappa(s))}{\psi'(s)}T(s) + \frac{\lambda((\tau-\tau_G)(s))}{\psi'(s)}B(s).$$

As $\{N_\beta((\bar{s})), N(s)\}$ is a linearly dependent set, we can write

$$T_\beta(\bar{s}) = \cos\theta(s)T(s) + \sin\theta(s)B(s) \tag{3.4}$$

where

$$\cos\theta(s) = \frac{(1-\lambda\kappa(s))}{\psi'(s)},$$
$$\sin\theta(s) = \frac{\lambda((\tau-\tau_G)(s))}{\psi'(s)}.$$

If we differentiate Eq. (3.4) and consider $\{N_\beta(\bar{s}), N(s)\}$ is a linearly dependent set we can easily see that θ is a constant function. So, we obtain

$$\frac{\cos\theta}{\sin\theta} = \frac{1-\lambda\kappa(s)}{\lambda((\tau-\tau_G)(s))}$$

or taking $c = \dfrac{\cos\theta}{\sin\theta}$, we get

$$\lambda\kappa(s) + c\lambda((\tau-\tau_G)(s)) = 1.$$

Then denoting $\mu = c\lambda =$ constant and using Definition 3, we have

$$\lambda\kappa(s) + \mu\kappa(s)H(s) = 1, \text{ for all } s \in I,$$

which completes the proof. □

Corollary 1. *The measure of the angle between the tangent vector fields of the Bertrand curve couple* (α, β) *is constant.*

Proof. It is obvious from the proof of the above Theorem. □

Remark 1. It is unknown whether the reverse of the above Theorem holds. Because, for the proof of the reverse we must consider a special Frenet curve $\beta(s) = \alpha(s) + \lambda N(s)$ in its proof. So, we give the following Theorem.

Theorem 8. *Let* $\alpha : I \subset \mathbb{R} \to G$ *be a parametrized Bertrand curve whose curvature functions* κ *and harmonic curvature function* H *of the curve* α *satisfy* $\lambda\kappa(s) + \mu\kappa(s)H(s) = 1$, *for all* $s \in I$. *If the curve* β *given by* $\beta(s) = \alpha(s) + \lambda N(s)$ *for all* $s \in I$ *is a special Frenet curve, then* (α, β) *is the Bertrand curve couple.*

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Bertrand curve whose curvature function κ and harmonic curvature function H of the curve α satisfy $\lambda\kappa(s) + \mu\kappa(s)H(s) = 1$ for all $s \in I$. If the curve β given by $\beta(s) = \alpha(s) + \lambda N(s)$ for all $s \in I$ is a special Frenet curve, then differentiating this equality with respect to s and by using Eq. (3.1) with the equation $\lambda\kappa(s) + \mu\kappa(s)H(s) = 1$, we have

$$T_\beta(\bar{s}) = \frac{\mu}{\sqrt{\lambda^2+\mu^2}}T(s) + \frac{\lambda}{\sqrt{\lambda^2+\mu^2}}B(s). \tag{3.5}$$

Then, if we differentiate the last equation with respect to s and by using the Frenet formulas we obtain

$$\kappa_\beta(\bar{s})N_\beta(\bar{s})\psi'(s) = \frac{\kappa(s)}{\sqrt{\lambda^2+\mu^2}}(\mu - \lambda H(s))N(s). \tag{3.6}$$

Thus, for each $s \in I$, the vector field $N_\beta(\bar{s})$ of β is linearly dependent the vector field $N(s)$ of α at corresponding point under the bijection from α to β. This completes the proof. □

Proposition 3. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Bertrand curve with the Frenet vector fields $\{T, N, B\}$ and $\beta : \bar{I} \subset \mathbb{R} \to G$ be a Bertrand mate of α with the Frenet vector fields $\{T_\beta, N_\beta, B_\beta\}$. Then $\tau_{G_\beta} = \tau_G$ for the curves α and β where $\tau_G = \frac{1}{2}\langle [T, N], B\rangle$ and $\tau_{G\beta} = \frac{1}{2}\langle [T_\beta, N_\beta], B_\beta\rangle$.*

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Bertrand curve with the Frenet vector fields $\{T, N, B\}$ and $\beta : \bar{I} \subset \mathbb{R} \to G$ be a Bertrand mate of α with with the Frenet vector fields $\{T_\beta, N_\beta, B_\beta\}$. From Eq. (3.5) and considering $N_\beta = \mp N$ we have

$$B_\beta(\bar{s}) = -\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} T(s) + \frac{\mu}{\sqrt{\lambda^2+\mu^2}} B(s). \tag{3.7}$$

Since $\tau_{G\beta} = \frac{1}{2}\langle [T_\beta, N_\beta], B_\beta\rangle$, using the equalities of the Frenet vector fields T_β, N_β and B_β we obtain $\tau_{G\beta} = \tau_G$, which completes the proof. □

Theorem 9. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Bertrand curve with curvature functions κ, τ and $\beta : \bar{I} \subset \mathbb{R} \to G$ be a Bertrand mate of α with curvatures functions κ_β, τ_β. Then the relations between these curvature functions are*

$$\kappa_\beta(\bar{s}) = \frac{\mu\kappa(s) - \lambda\kappa(s) H(s)}{(\lambda^2+\mu^2) H(s)}, \tag{3.8}$$

$$\tau_\beta(\bar{s}) = \frac{\lambda\kappa(s) + \mu\kappa(s) H(s)}{(\lambda^2+\mu^2) H(s)} + \tau_G \tag{3.9}$$

Proof. If we take the norm of Eq. (3.6) and use Eq. (3.1), we get Eq. (3.8). Then differentiating Eq. (3.7) and using the Frenet formulas, we have

$$\begin{aligned} \dot{B}_\beta(\bar{s})\psi'(s) &= -\frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\dot{T}(s) + \frac{\mu}{\sqrt{\lambda^2+\mu^2}}\dot{B}(s), \\ &= -\frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\kappa(s)N(s) + \frac{\mu}{\sqrt{\lambda^2+\mu^2}}\left(-\tau(s)N(s) - \frac{1}{2}[T, B]\right) \end{aligned}$$

In the above equality, using Eq. (3.1) and Proposition 2, we get

$$(\tau_\beta - \tau_{G\beta}) N_\beta(\bar{s}) = \frac{1}{\kappa H(\lambda^2+\mu^2)} (\lambda\kappa + \mu\kappa H) N(s).$$

If we take the norm of the last equation and use Proposition 3, we get Eq. (3.9), which completes the proof. □

Theorem 10. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\beta : \bar{I} \subset \mathbb{R} \to G$ be a curve with Frenet apparatus*

$\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$. *If* (α, β) *is a Bertrand curve couple then* $\kappa\kappa_\beta HH_\beta$ *is a constant function.*

Proof. We assume that (α, β) is a Bertrand curve couple. Then we can write

$$\alpha(s) = \beta(s) - \lambda(s) N_\beta(\bar{s}). \tag{3.10}$$

If we use the similar method as in the proof of Theorem 7 and consider Eq. (3.10), then we can easily see that $\kappa\kappa_\beta HH_\beta$ is a constant function. □

Theorem 11. *Let* $\alpha : I \subset \mathbb{R} \to G$ *be a parametrized Bertrand curve with Frenet apparatus* $\{T, N, B, \kappa, \tau\}$ *and* $\beta : \bar{I} \subset \mathbb{R} \to G$ *be a Bertrand mate of the curve* α *with Frenet apparatus* $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$. *Then* α *is a slant helix if and only if* β *is a slant helix.*

Proof. Let σ_N and $\sigma_{N\beta}$ be the geodesic curvatures of the principal normal curves of α and β, respectively. Then using Theorem 9 we can easily see that

$$\sigma_{N\beta} = -\frac{\kappa(1+H^2)^{\frac{3}{2}}}{H'} = -\sigma_N.$$

So, with the help of Theorem 3 we complete the proof. □

Theorem 12. *Let* $\alpha : I \subset \mathbb{R} \to G$ *be a parametrized Bertrand curve with curvature functios* κ, τ *and* $\beta : \bar{I} \subset \mathbb{R} \to G$ *be a Bertrand mate of the curve* α *with curvature functions* κ_β, τ_β. *Then* α *is a general helix if and only if* β *is a general helix.*

Proof. Let α be a helix. From Theorem 1, we have that H is a constant function. Then using Theorem 9, we get

$$\frac{\tau_\beta - \tau_{G\beta}}{\kappa_\beta} = \frac{\lambda + \mu H}{\mu - \lambda H}. \tag{3.11}$$

Since H is a constant function, Eq. (3.11) is constant. So, β is a general helix.

Conversely, assume that β be a general helix. So, $\frac{\tau_\beta - \tau_{G\beta}}{\kappa_\beta}$ = constant. From Eq. (3.11) $c = \frac{\lambda+\mu H}{\mu-\lambda H}$ = constant and then $H = \frac{c\mu-\lambda}{\mu+\lambda c}$ =constant. Consequently α is a general helix and this completes the proof. □

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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS OF PARABOLIC KÄHLER MANIFOLDS

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Abstract. In this paper we study fundamental equations of holomorphically projective mappings of parabolic Kähler spaces (which are generalized classical, pseudo- and hyperbolic Kähler spaces) with respect to the smoothness class of metrics. We show that holomorphically projective mappings preserve the smoothness class of metrics.

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1. INTRODUCTION

First we study the general dependence of holomorphically-projective mappings of parabolic Kähler manifolds in dependence on the smoothness class of the metric. We present well known facts, which were proved by M. Shiha, J. Mikeš et al, see [2, 3, 14, 17–21].

I. Hinterleitner [5] has solved the analogically problems for classical, pseudo- and hyperbolic Kähler manifolds. In this paper were clarified results which were proved by Bácsó, Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro, see [4, 10–13, 15, 16, 23, 25]. In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects. This result was inspired in [6, 7] of geodesic mappings.

2. PARABOLIC KÄHLER MANIFOLDS

In the following definition we introduce generalizations of Kähler manifolds [8], see [9, 11, 13]. A basis on this definition see the monography by V.V. Vishnevskii, A.P. Shirokov and V.V. Shurigin [24].

Definition 1. An n-dimensional (pseudo-) Riemannian manifold (M, g) is called an *m-parabolic Kähler manifold* $K_n^{o(m)}$, if beside the metric tensor g, a tensor field

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F of a rank $m \geq 2$ of type $(1,1)$ is given on the manifold M_n, called a *structure* F, such that the following conditions hold:

$$F^2 = 0, \quad g(X, FX) = 0, \quad \nabla F = 0, \tag{2.1}$$

where X is an arbitrary vector of TM_n, and ∇ denotes the covariant derivative in $K_n^{o(m)}$.

We remind, that Kähler spaces, were characterized by conditions $F^2 = -Id$, $g(X, FX) = 0$, $\nabla F = 0$, were first considered by P.A. Shirokov, see [22]. Independently they were studied by E. Kähler [8]. Hyperbolic Kähler space (also *para Kähler space*, see D.V. Alekseevsky [1]) characterized $F^2 = Id$, $g(X, FX) = 0$, $\nabla F = 0$, were considered P.K. Rashevskij, see [9].

3. Holomorphically-projective mapping theory for $K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ of class C^1

Assume the parabolic Kähler manifolds $K_n^{o(m)} = (M, g, F)$ and $\bar{K}_n^{o(\bar{m})} = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here $\bar{K}_n$, $K_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$. Likewise, as in [17, 18] we introduce the following notations, this is an analogy by [15], see [11, p. 240].

Definition 2. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$ $(\neq 0), t \in I$, where t is a parameter is called *analytical planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = Span\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(\lambda) F\lambda,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t. Particularly, in the case $b(t) = 0$, an analytical planar curve is a geodesic.

Definition 3. A diffeomorphism $f\colon K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ is called a *holomorphically-projective mapping* of $K_n^{o(m)}$ onto $\bar{K}_n^{o(\bar{m})}$ if f maps any analytical planar curve in $K_n^{o(m)}$ onto an analytical planar curve in $\bar{K}_n^{o(\bar{m})}$.

Assume a holomorphically-projective mapping $f\colon K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally, $f : K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ maps points onto points with the same coordinates, and $\bar{M} = M$. A manifold $K_n^{o(m)}$ admits a holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ if and only if the following equations [17]:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + \varphi(FX)FY + \varphi(FY)FX \tag{3.1}$$

hold for any tangent fields X, Y and where ψ is a gradient-like form and $\psi(X) = \varphi(FX)$. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically-projective* . Moreover,

structures F and $\bar{F}$ are preserved, i.e. $\bar{F} = F$, and $\bar{m} = m$. This fact implies from the theory of F-planar mappings, see [11, pp. 219-220]. In local form:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i, \quad \psi_i = \varphi_j F^j_i,$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F^h_i are components of ψ, F and δ^h_i is the Kronecker delta,

$$\psi_i = \frac{\partial \Psi}{\partial x^i}, \quad \Psi = \frac{1}{2(n+2)} \ln \left| \frac{\det \bar{g}}{\det g} \right|.$$

Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots \bar{i} \dots} = A_{\dots k \dots} F^k_i.$$

Equations (3.1) are equivalent to the following equations

$$\begin{aligned} \nabla_Z \bar{g}(X,Y) = {} & 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z) \\ & + \varphi(FX)\bar{g}(FY,Z) + \varphi(FY)\bar{g}(FX,Z). \end{aligned} \tag{3.2}$$

In local form:

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} + \varphi_i \bar{g}_{\bar{j}k} + \varphi_j \bar{g}_{\bar{i}k},$$

where "," denotes the covariant derivative on $K^{o(m)}_n$. M. Shiha and J. Mikeš [18] proved that equations (3.1) and (3.2) are equivalent to

$$\nabla_Z a(X,Y) = \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z) + \theta(X)g(FY,Z) + \theta(Y)g(FX,Z). \tag{3.3}$$

In local form:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \theta_i g_{\bar{j}k} + \theta_j g_{\bar{i}k},$$

where

$$\text{(a) } a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \quad \text{(b) } \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha, \quad \text{(c) } \theta_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \varphi_\alpha. \tag{3.4}$$

From (3.3) follows that λ_i is gradient-like vector and it holds

$$\lambda_i = \partial_i \Lambda, \qquad \Lambda = 1/4 a_{\alpha\beta} g^{\alpha\beta}. \tag{3.5}$$

On the other hand [11]:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \| \tilde{g}_{ij} \| = \| g^{i\alpha} g^{j\beta} a_{\alpha\beta} \|^{-1}. \tag{3.6}$$

The above formulas are the criterion for holomorphically-projective mappings $K^{o(m)}_n \to \bar{K}^{o(m)}_n$, globally as well as locally.

Theorem 1. *A diffeomorphism $f : K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ is a holomorphically-projective mapping if and only if there exist a solution of the following linear Cauchy-like system*

$$\begin{aligned} a)\ & a_{ij,k} = \lambda_{(i}g_{j)k} + \theta_{(i}F_{j)k}; \\ b)\ & \theta_{i,j} = \tau F_{ij} + a_{\alpha\beta}M^{\alpha\beta}_{1|ij}; \\ c)\ & \tau_{,i} = \theta_\alpha M^{\alpha}_{2|i} + a_{\alpha\beta}M^{\alpha\beta}_{3|i} \end{aligned} \tag{3.7}$$

on unknown tensor a_{ij} ($a_{ij} = a_{ji}$, $a_{\bar{i}\,j} + a_{i\,\bar{j}} = 0$, $\det a_{ij} \neq 0$), a vector λ_i, and a function τ. Here $M^{\alpha\beta}_{1|ij}$, $M^{\alpha}_{2|i}$, $M^{\alpha\beta}_{3|i}$ are tensors determined from metric and structure tensors g_{ij} and F^h_i of the space $K_n^{o(m)}$.

Remark 1. This theorem was proved with assuming that $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$ belong to C^3 class. We will prove, that the Theorem 1 valides too if $K_n^{o(m)} \in C^3$ and $\bar{K}_n^{o(\bar{m})} \in C^2$.

The system (3.7) has at most one solution for the initial values in a point x_0 : $a_{ij}(x_0)$, $\lambda_i(x_0)$ and $\tau(x_0)$. Hence, the general solution of this system depends on no more than $(n+2)(n+1)/2 - m(n-m+1)$ essential parameters.

The integrability of conditions (3.7) and their differential prolongations are linear algebraic equations on the components of the unknown tensors a_{ij}, λ_{ij} and τ with coefficients from $K_n^{o(m)}$.

4. Holomorphically-projective mapping of parabolic Kähler space of class C^2

The direct substitution of (3.1) implies that Riemannian tensors of spaces $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$, which are holomorphically-projectively corresponding of m-parabolic Kähler space are connected by the following relations:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik} + F^h_k\varphi_{ij} - F^h_j\varphi_{ik} + F^h_i\varphi_{[kj]}, \tag{4.1}$$

where $\bar{R}^h_{ijk}$, R^h_{ijk} are Riemannian tensors of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$,

$$\varphi_{ik} \equiv \varphi_{i,j} - \psi_i\varphi_j - \varphi_i\psi_j; \quad \psi_{ij} \equiv \varphi_{\bar{i}\,j}. \tag{4.2}$$

Tensor $\bar{\varphi}_{ij}$ has this form:

$$\psi_{ij} = \psi_{i,j} - \psi_i\psi_j. \tag{4.3}$$

As ψ_i is a gradient, that this tensor is symmetrical, that is

$$\psi_{ij} = \psi_{ji}. \tag{4.4}$$

Contracting (4.1) with respect to indices h and k, we obtain connection between Ricci tensors

$$\bar{R}_{ij} = R_{ij} + n\psi_{ij}, \tag{4.5}$$

where R_{ij} and $\bar{R}_{ij}$ are Ricci tensors of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$.

Next, we will proceed similarly as in work of M. Shiha [17, 18]. Let we consider the integrability condition of equation (3.3):

$$a_{\alpha(i}R^{\alpha}_{j)kl} = g_{k(i}\lambda_{j),l} - g_{l(}\lambda_{j),k} - F_{k(i}\theta_{j),l} + F_{l(i}\theta_{j),k}. \tag{4.6}$$

Contracting (4.6) with $F^k_{k'}$ and also with $F^l_{l'}$, we obtain two expressions. After removing primes, we sum them up. Since $K_n^{o(m)}$ it holds $R^h_{i\alpha k} + R^h_{ij\alpha}F^{\alpha}_k = 0$, we get $F_{k(i}\Lambda_{j)l} - F_{l(i}\Lambda_{j)k} = 0$, where $\Lambda_{ij} \equiv \lambda_{i,j} + \theta_{i,\alpha}F^{\alpha}_j$. It follows that $\Lambda_{ij} = \Lambda F_{ij}$, i.e. $\lambda_{i,j} + \theta_{i,\alpha}F^{\alpha}_j = \Lambda F_{ij}$ where Λ is a function. Contracting (4.6) with g^{jk}, we obtain

$$n\lambda_{i,j} = \mu g_{ij} + \nu F_{ij} - a_{\alpha i}R^{\alpha}_j - a_{\alpha\beta}R^{\alpha..\beta}_{.ij.}. \tag{4.7}$$

where R^h_{ijk} and R_{ij} are Riemann and Ricci tensors, respectively, the operation of lifting and lowering indices are induced by the metric tensor, and μ, ν are certain functions. After symmetrizing (4.7) we get

$$n\lambda_{i,j} = \mu g_{ij} - \frac{1}{2}a_{\alpha(i}R^{\alpha}_{j)} - a_{\alpha\beta}R^{\alpha..\beta}_{.ij.}. \tag{4.8}$$

Substituting (4.8) to (4.6) we obtain

$$a_{\alpha\beta}M^{\alpha\beta}_{jkl} = F_{li}\theta_{j,k} + F_{lj}\theta_{i,k} - F_{ki}\theta_{j,l} + F_{kj}\theta_{i,l}. \tag{4.9}$$

From this implies that M are tensors determined by g_{ij} and F^h_i on $K_n^{o(m)}$. More precisely

$$M^{\alpha\beta}_{ijkl} \equiv \delta^{\alpha}_{(i}R^{\beta}_{j)kl} + M^{\alpha\beta}_{4|k(i}g_{j)l} - M^{\alpha\beta}_{4|l(i}g_{j)k}; \qquad nM^{\alpha\beta}_{4|li} \equiv \frac{1}{2}\delta^{\alpha}_{(i}R^{\beta}_{j)} - R^{\alpha..\beta}_{.ij.},$$

where δ^h_i is the Kronecker symbol. Let ε^j and ν^k be vectors such that $\varepsilon^j\nu^k F_{jk} = 1$. Denote $M_i \equiv \varepsilon^{\alpha}F_{\alpha i}$. Contracting (4.9) with $\varepsilon^i\varepsilon^j\nu^k$ we get

$$\varepsilon^{\alpha}\theta_{\alpha,l} = \tau M_l + a_{\alpha\beta}M^{\alpha\beta}_{5|l}, \tag{4.10}$$

where $\tau \equiv \lambda_{\alpha,\beta}\varepsilon^{\alpha}\nu^{\beta}$. Contracting (4.9) with $\varepsilon^j\nu^k$ and using (4.10) we have the following formula

$$\theta_{i,j} = \tau F_{ij} + \Lambda_i M_j + a_{\alpha\beta}M^{\alpha\beta}_{6|ij}, \tag{4.11}$$

where λ_i is a vector. Substituting (4.11) into (4.9) we have

$$\Lambda_i(M_kF_{lj} - M_lF_{kj}) + \Lambda_j(M_kF_{li} - M_lF_{ki}) = a_{\alpha\beta}M^{\alpha\beta}_{7|ijkl}. \tag{4.12}$$

So we proved the following lemma:

Lemma 1. *Let $K_n^{o(m)}$ and $\bar{K}_n^{o(\bar{m})}$ belong to class C^2. If $K_n^{o(m)}$ admit a holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ than formulae (3.7a) and (3.7b) hold.*

5. Holomorphically-projective mapping $K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ for $K_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\bar{K}_n^{o(\bar{m})} \in C^2$

Theorem 2. *Let $K_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\bar{K}_n^{o(\bar{m})} \in C^2$. If $K_n^{o(m)}$ admits holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ then $\bar{K}_n^{o(\bar{m})} \in C^r$.*

Proof of Theorem 2 is based on the proof of Theorem 3.

Theorem 3. *Let $K_n^{o(m)} \in C^3$ and $\bar{K}_n^{o(\bar{m})} \in C^2$. If $K_n^{o(m)}$ admits holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ then $\bar{K}_n^{o(\bar{m})} \in C^3$.*

For first, we prove that metric g and structure F have the same differentiation.

Lemma 2. *If $K_n^{o(m)} = (M, g, F) \in C^r$, i.e. $g(x) \in C^r$, then $F(x) \in C^r$, for $r \in \mathbb{N}$ and $r = \infty, \omega$.*

Proof. Let $K_n^{o(m)} \in C^r$, i.e. the components of metric $g_{ij}(x) \in C^r$ in a coordinate chart x. It is a priori valid, that $F_i^h \in C^1$. The formula $\nabla F = 0$ can be written $\partial_k F_i^h = F_\alpha^h \Gamma_{ik}^\alpha - F_i^\alpha \Gamma_{\alpha k}^h$, where $\Gamma_{ijk} = 1/2(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$, $\partial_k = \partial/\partial x^k$, and $\Gamma_{ij}^h = g^{hk}\Gamma_{ijk}$ are the Christoffel symbols of the first and second kind, respectively. It holds, that Γ_{ijk} and $\Gamma_{ij}^h \in C^{r-1}$. From this equation immediately follows $F_i^h(x) \in C^r$, i.e. $F \in C^r$. □

For proving Theorem 3, we need the lemmae:

Lemma 3 ([9])**.** *Let $\lambda^h(x) \in C^1$ be a vector field, $\Phi_i^h = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$ is special form with $I = \delta_b^a$, where $1 \leq a, b \leq r$, $r \geq 2$. If $\partial_i \lambda^h - \rho \Phi_i^h \in C^1$ then $\lambda^h \in C^2$ and $\rho \in C^1$.*

Lemma 4. *Let $\lambda^h(x) \in C^1$ be a vector field, $F_i^h(x) \in C^2$ is a tensor field of* rank $F \geq 2$. *If $\partial_i \lambda^h - \rho F_i^h \in C^1$ then $\lambda^h \in C^2$ and $\rho \in C^1$.*

Proof. Let

$$\partial_i \lambda^h - \rho F_i^h(x) = f_i^h(x) \in C^1.$$

Because rank $F_i^h \geq r$ and $F_i^h(x) \in C^2$, then exist a regular tensor field $\Omega_i^h(x) \in C^2$ that $F_i^h \Omega_j^i = \Phi_i^h$

We put $v^h = \lambda^\alpha \Omega_\alpha^h$. Then we have

$$\partial_i v^h - \rho \Phi_\alpha^h = \partial_i(\lambda^\alpha \Omega_\alpha^h) - \rho \Phi_\alpha^h = f_i^\alpha \Omega_\alpha^h + \lambda^\alpha \partial_i \Omega_\alpha^h.$$

Because $f_i^\alpha \Omega_\alpha^h + \lambda^\alpha \partial_i \Omega_\alpha^h \in C^1$, from Lemma 3 it implies $v^h \in C^2$ and $\rho \in C^1$. Since $\rho \in C^1$, then $\lambda^h(x) \in C^2$. □

It follows prooving of Theorem 3:

Proof. Equations (3.7b) can be written in the following form:

$$\partial_j \theta^h = \tau F_j^h - \theta^\alpha \Gamma_{\alpha j}^h + a_{\alpha\beta} M_{1|j}^{\alpha\beta h},$$

i.e. $\partial_j \theta^h - \tau F_j^h = f_j^h$, where $f_j^h = -\theta^\alpha \Gamma_{\alpha j}^h + a_{\alpha\beta} M_{1|j}^{\alpha\beta h}$. Because $f_j^i \in C^1$ and $F_i^h \in C^3$ then from Lemma 4 follows $\theta^i \in C^2$ and $\tau \in C^1$. Using (3.7a) and (3.7b) we obtain $a_{ij}(x) \in C^3$ and $\Psi \in C^3$. Finally, from (3.6) we have $\bar{g}_{ij}(x) \in C^3$. □

In to this moment, we proved formulae (3.7a) and (3.7b). For complete proof of Theorem 1, we have to prove the latest formula (3.7c).

Proof. Proof of Theorem 3 allows us differentiation of equation (3.7b). Application to this, we have:

$$\theta_{i,jk} = \tau_{,k} F_{ij} + a_{\alpha\beta,k} M_{1|ij}^{\alpha\beta} + a_{\alpha\beta} M_{1|ij,k}^{\alpha\beta}. \tag{5.1}$$

After alternation with respect to indices j and k and using Ricci identity, we have

$$\tau_{,j} F_{ik} - \tau_{,k} F_{ij} = a_{\alpha\beta} M_{8|ijk}^{\alpha\beta} + \theta_\alpha M_{9|ijk}^{\alpha\beta}. \tag{5.2}$$

Contracting (5.2) with $\varepsilon^i \varepsilon^j \nu^k$, we obtain $\tau_{,a} \varepsilon^\alpha = a_{\alpha\beta} M_{9|}^{\alpha\beta} + \lambda_\alpha M_{10|}^{\alpha\beta}$. Finally, contracting (5.2) with $\varepsilon^j \nu^i$, we have (3.7c) and Theorem 1 is proved. □

At the end, we prove Theorem 2.

Proof. If $K_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\bar{K}_n^{o(\bar{m})} \in C^2$, then by Theorem 3, $\bar{K}_n^{o(\bar{m})} \in C^3$ and formulas (3.7) hold. Because the system of equations (3.7) is closed, we can differentiate equation (3.1) $(r-1)$ times. So we can convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ $(\equiv \bar{K}_n^{o(\bar{m})} \in C^r)$. □

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SOME FIXED POINT THEOREMS FOR NONSELF GENERALIZED CONTRACTION

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Abstract. In this paper we give a new proof of a result by S. Reich and A.J. Zaslavski (S. Reich and A.J. Zaslavski, A fixed point theorem for Matkowski contractions, Fixed Point Theory, 8(2007), No. 2, 303–307). Some new fixed point theorems for nonself generalized contractions are also given.

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1. Introduction

There are many techniques in the fixed point theory of nonself operators (see [10], [4], [6], [9], [19], [20], [2], ...). An exotic result is given in [14] (see also, [13] and [15]). This result read as follows:

Theorem 1. *Let (X,d) be a complete metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be a φ-contraction, where φ is a comparison function. We suppose that there exists a bounded sequence $(x_n)_{n\in\mathbb{N}^*}$ such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$. Then f has a unique fixed point x^* and $f^n(x_n) \to x^*$.*

The aim of this paper is to give a new proof of this theorem and to obtain other results of this type.

2. Preliminaries

2.1. *Notations*

$\mathbb{N} = \{0,1,2,\dots\}$, $\mathbb{N}^* = \{1,2,3,\dots\}$.
$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$
Let (X,d) be a metric space. We will use the following symbols:
$\mathcal{P}(X) = \{Y \mid Y \subset X\}$

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$P(X) = \{Y \subset X \mid Y \text{ is nonempty}\}$, $P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}$,
$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$, $P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$.
If $f : X \to X$ is an operator then $F_f := \{x \in X \mid x = f(x)\}$ denotes the fixed point set of the operator f. In the case when the operator f has an unique fixed point $x^* \in X$ then we write $F_f = \{x^*\}$.

The diameter functional $\delta : P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\delta(A) := \sup\{d(a,b) \mid a,b \in A\}.$$

2.2. *Comparison functions*

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. We consider the following conditions relative to φ:

(i_φ) φ is increasing;
(ii_φ) $\varphi(t) < t$, $\forall t > 0$;
(iii_φ) $\varphi(0) = 0$;
(iv_φ) $\varphi^n(t) \to 0$ as $n \to \infty$, $\forall t \in \mathbb{R}_+$;
(v_φ) $t - \varphi(t) \to \infty$ as $t \to \infty$;
(vi_φ) $\sum_{n=0}^{\infty} \varphi^n(t) < +\infty$, $\forall t \in \mathbb{R}_+$.

Definition 1 (I.A. Rus [17])**.** By definition the function φ is a comparison function if it satisfies the conditions (i_φ) and (iv_φ).

Definition 2. A comparison function is:

(a) strict comparison function if it satisfies the condition (v_φ);
(b) strong comparison function if it satisfies the condition (vi_φ).

It is clear that if φ is a comparison function then $\varphi(t) < t$, $\forall t > 0$, and $\varphi(0) = 0$. If φ is a strong comparison function then the functions φ and $\sum_{n=0}^{\infty} \varphi^n$ are continuous in $t = 0$.

For example, if $\varphi(t) := at$, $t \in \mathbb{R}_+$, $a \in [0;1[$, then φ is a strict and strong comparison function and $\varphi(t) := \frac{t}{1+t}$, $t \in \mathbb{R}_+$, is a strict comparison function which is not a strong comparison function.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function. In this case we define the function $\theta_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by,

$$\theta_\varphi(t) = \sup\{s \in \mathbb{R}_+ \mid s - \varphi(s) \leq t\}.$$

We remark that θ_φ is increasing and $\theta_\varphi(t) \to 0$ as $t \to 0$. The function θ_φ appears when we study the data dependence of the fixed points.

For more considerations on comparison functions see [17], [1], [21] and [5].

2.3. *Maximal displacement functional*

Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous nonself operator. By the maximal displacement functional corresponding to f we understand the functional $E_f : P(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$E_f(A) := \sup\{d(x, f(x)) \mid x \in A\}.$$

We have that:

(i) $A, B \in P(Y)$, $A \subset B$ imply $E_f(A) \leq E_f(B)$;
(ii) $E_f(A) = E_f\left(\overline{A}\right)$ for all $A \in P(Y)$.

Definition 3. An operator $f : Y \to X$ is α-graphic contraction if $0 \leq \alpha < 1$ and $x \in Y$, $f(x) \in Y$ imply

$$d\left(f^2(x), f(x)\right) \leq \alpha d(x, f(x)).$$

Example 1. If $f : Y \to X$ is α-contraction then f is α-graphic contraction.

Example 2. If $f : Y \to X$ is α-Kannan operator, i.e., $0 \leq \alpha < \frac{1}{2}$, and

$$d(f(x), f(y)) \leq \alpha [d(x, f(x)) + d(y, f(y))], \ \forall x, y \in Y,$$

then f is $\frac{\alpha}{1-\alpha}$-graphic contraction.

Also, we have that:

Lemma 1. *Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous α-graphic contraction. Then:*

(a) $E_f(f(A)) \leq \alpha E_f(A)$, *for all $A \subset Y$ with $f(A) \subset Y$;*
(b) $E_f(f(A) \cap Y) \leq \alpha E_f(A)$, *for all $A \subset Y$ with $f(A) \cap Y \neq \varnothing$.*

Proof. The proof follows from the definition of E_f. Let, for example, to prove (b). We have

$$\begin{aligned} E_f(f(A) \cap Y) &= \sup\{d(x, f(x)) \mid x \in f(A) \cap Y\} = \\ &= \sup\left\{d\left(f(u), f^2(u)\right) \mid u \in A, \ f(u) \in Y\right\} \leq \\ &\leq \alpha \sup\{d(u, f(u)) \mid u \in A\} = \\ &= \alpha E_f(A) \end{aligned}$$

□

2.4. *Matrices convergent to* 0

Definition 4. A matrix $S \in \mathbb{R}_+^{m \times m}$ is called a matrix convergent to zero iff $S^k \to 0$ as $k \to +\infty$.

Theorem 2 (see [12], [16], [23], [10])**.** *Let $S \in \mathbb{R}_+^{m \times m}$. The following statements are equivalent:*

(i) *S is a matrix convergent to zero;*

(ii) $S^k x \to 0$ *as* $k \to +\infty$, $\forall x \in \mathbb{R}^m$;
(iii) $I_m - S$ *is non-singular and*

$$(I_m - S)^{-1} = I_m + S + S^2 + \cdots$$

(iv) $I_m - S$ *is non-singular and* $(I_m - S)^{-1}$ *has nonnegative elements;*
(v) $\lambda \in \mathbb{C}$, $\det(S - \lambda I_m) = 0$ *imply* $|\lambda| < 1$;
(vi) *there exists at least one subordinate matrix norm such that* $\|S\| < 1$.

The matrices convergent to zero were used by A. I. Perov [11] (see also [10] pp. 432-434) to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of $\mathbb{R}^m$.

3. A NEW PROOF OF THEOREM 1

Now we present a new proof of Theorem 1. Let $A \in P_{b,cl}(Y)$ be such that $x_n \in A$, for all $n \in \mathbb{N}^*$. We consider the following standard construction in the fixed point theory for the nonself operators (see for example [8] and [7]).

Let $A_1 := \overline{f(A)}$, $A_2 := \overline{f(A_1 \cap A)}$, $\ldots$, $A_{n+1} := \overline{f(A_n \cap A)}$, $n \in \mathbb{N}^*$. We remark that:

(a) $A_{n+1} \subset A_n$, $\forall n \in \mathbb{N}^*$;
(b) $f^n(x_n) \in A_n$, $\forall n \in \mathbb{N}^*$, so $A_n \neq \varnothing$, $\forall n \in \mathbb{N}^*$.

Since f is a φ-contraction, i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function such that

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \ \forall x, y \in Y,$$

it follows that

$$\delta(f(B)) \leq \varphi(\delta(B)), \ \forall B \in P_b(Y).$$

From the properties of φ and δ we have

$$\begin{aligned} \delta(A_{n+1}) &= \delta\left(\overline{f(A_n \cap A)}\right) = \delta(f(A_n \cap A)) \leq \delta(f(A_n)) \leq \\ &\leq \varphi(\delta(A_n)) \leq \cdots \leq \varphi^{n+1}(\delta(A)) \to 0 \end{aligned}$$

as $n \to +\infty$. From Cantor intersection lemma we have

$$A_\infty := \bigcap_{n \in \mathbb{N}} A_n \neq \varnothing, \ \delta(A_\infty) = 0 \text{ and } f(A_\infty \cap A) \subset A_\infty.$$

From $A_\infty \neq \varnothing$ and $\delta(A_\infty) = 0$, we have that $A_\infty = \{x^*\}$. On the other hand $f^n(x_n) \in A_n$ and $f^{n-1}(x_n) \in A_{n-1} \cap Y$. This implies that $\{f^n(x_n)\}_{n \in \mathbb{N}}$ and $\left\{f^{n-1}(x_n)\right\}_{n \in \mathbb{N}}$ are fundamental sequences. Since A_n, $n \in \mathbb{N}^*$, are closed, it follows that

$$f^{n-1}(x_n) \to x^* \text{ and } f^n(x_n) \to x^* \text{ as } n \to +\infty.$$

Since f is continuous then $f^n(x_n) \to f(x^*)$, so $f(x^*) = x^*$.

With respect to the data dependence of the fixed point, in Theorem 1, we have the following result:

Theorem 3. *Let $f : Y \to X$ be as in Theorem 1, where φ is a strict comparison function. Then:*

(a) $d\left(f^n(x_n), x^*\right) \leq \varphi\left(d\left(x_n, x^*\right)\right)$, $\forall n \in \mathbb{N}^*$;
(b) $d\left(x, x^*\right) \leq \theta_\varphi\left(d\left(x, f(x)\right)\right)$, $\forall x \in Y$;
(c) *Let $g : Y \to X$ be such that:*
  (1) *there exists $\eta > 0$ such that $d\left(f(x), g(x)\right) \leq \eta$, $\forall x \in Y$;*
  (2) $F_g \neq \varnothing$.

Then

$$d\left(x^*, y^*\right) \leq \theta_\varphi(\eta), \forall y^* \in F_g.$$

Proof. Let us prove (b) and (c).
(b). The conclusion (b) follows from the following estimation

$$d\left(x, x^*\right) \leq d\left(x, f(x)\right) + d\left(f(x), x^*\right) \leq d\left(x, f(x)\right) + \varphi\left(d\left(x, x^*\right)\right), \ \forall x \in Y.$$

So,

$$d\left(x, x^*\right) - \varphi\left(d\left(x, x^*\right)\right) \leq d\left(x, f(x)\right), \ \forall x \in Y.$$

(c). Let $y^* \in F_g$ then from (b) it follows

$$d\left(x^*, y^*\right) \leq \theta_\varphi\left(d\left(y^*, f\left(y^*\right)\right)\right) = \theta_\varphi\left(d\left(g\left(y^*\right), f\left(y^*\right)\right)\right) \leq \theta_\varphi(\eta).$$

□

For more considerations on data dependence of the fixed points for nonself φ-contractions see [3], [18] and [22].

4. A FIXED POINT THEOREM FOR NONSELF KANNAN OPERATORS

We have:

Theorem 4. *Let (X, d) be a complete metric space, $Y \subset X$ a nonempty bounded closed subset and $f : Y \to X$ a continuous operator. We suppose that:*

(i) *f is α-Kannan operator;*
(ii) *there exists a sequence $(x_n)_{n \in \mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$;*
(iii) $E_f(Y) < +\infty$.

Then:

(a) $F_f = \{x^*\}$;
(b) $f^{n-1}(x_n) \to x^*$ *and* $f^n(x_n) \to x^*$ *as* $n \to +\infty$;
(c) $d\left(x, x^*\right) \leq (1+\alpha)\, d\left(x, f(x)\right)$, $\forall x \in Y$;
(d) $d\left(f^{n-1}(x_n), x^*\right) \leq \alpha^{n-1}(1-\alpha)^{1-n}(1+\alpha)\, d\left(x_n, f(x_n)\right)$, $\forall n \in \mathbb{N}^*$;
(e) *Let $g : Y \to X$ be such that:*
  (1) *there exists $\eta > 0$ such that $d\left(f(x), g(x)\right) \leq \eta$, $\forall x \in Y$;*
  (2) $F_g \neq \varnothing$.

Then

$$d\left(x^*, y^*\right) \leq (1+\alpha)\,\eta, \forall y^* \in F_g.$$

Proof. $(a)+(b)$. Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1 \cap Y)}$, $\ldots$, $Y_{n+1} := \overline{f(Y_n \cap Y)}$, $n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$ and $f^n(x_n) \in Y_n$, so $Y_n \neq \varnothing$, $n \in \mathbb{N}^*$. Since f is α-Kannan operator, from Example 2 and Lemma 1, we have that:

$$\begin{aligned}\delta(Y_{n+1}) &= \delta\left(\overline{f(Y_n \cap Y)}\right) = \delta(f(Y_n \cap Y)) \leq 2\alpha \cdot E_f(Y_n \cap Y) = \\ &= 2\alpha \cdot E_f\left(\overline{f(Y_{n-1} \cap Y)} \cap Y\right) = 2\alpha \cdot E_f(f(Y_{n-1} \cap Y) \cap Y) \leq \\ &\leq \frac{2\alpha^2}{1-\alpha} E_f(Y_{n-1} \cap Y) \leq \cdots \leq \frac{2\alpha^{n+1}}{(1-\alpha)^n} E_f(Y) \to 0 \text{ as } n \to +\infty.\end{aligned}$$

Now the proof is similar with the proof of Theorem 1.

(c). Let $x \in Y$. From the definition of the Kannan operator we have:

$$d\left(x, x^*\right) \leq d(x, f(x)) + d\left(f(x), x^*\right) \leq d(x, f(x)) + \alpha d(x, f(x)),\ \forall x \in Y.$$

(d) and (e) follow from (c). □

5. Other nonself generalized contractions

5.1. *Ćirić-Reich-Rus operators*

Let (X, d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a nonself operator. An operator $f : Y \to X$ is a Ćirić-Reich-Rus operator (see [4], [20], [22], $\ldots$) if there exist $a, b \in \mathbb{R}_+$ with $a + 2b < 1$ such that

$$d(f(x), f(y)) \leq a d(x, y) + b[d(x, f(x)) + d(y, f(y))],\ \forall x, y \in Y.$$

Lemma 2. *Let (X, d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ a nonself Ćirić-Reich-Rus operator then f is a nonself α-graphic contraction with $\alpha = \frac{a+b}{1-b}$.*

Proof. Let $x \in Y$ such that $f(x) \in Y$ then

$$d\left(f^2(x), f(x)\right) \leq a d(f(x), x) + b\left[d\left(f(x), f^2(x)\right) + d(x, f(x))\right],$$

so

$$d\left(f^2(x), f(x)\right) \leq \frac{a+b}{1-b} d(x, f(x)).$$

□

Lemma 3. *Let (X, d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ a nonself Ćirić-Reich-Rus operator then:*

(a) $\delta(f(A) \cap Y) \leq a\delta(A) + 2bE_f(A)$, *for all* $A \subset Y$;

(b) $E_f(f(A) \cap Y) \leq \alpha E_f(A)$, *for all* $A \subset Y$, *where* $\alpha = \frac{a+b}{1-b}$.

Proof. (a). Let $A \subset Y$ then

$$\begin{aligned}\delta(f(A)\cap Y) &= \sup\{d(x,y) \mid x,y \in f(A)\cap Y\} = \\ &= \sup\{d(f(u),f(v)) \mid u,v \in A,\ f(u),f(v) \in Y\} \leq \\ &\leq a\sup\{d(u,v) \mid u,v \in A\} + 2b\sup\{d(u,f(u)) \mid u \in A\} = \\ &= a\delta(A) + 2bE_f(A)\end{aligned}$$

(b). The proof follows from Lemma 2 and Lemma 1. □

Also, for the next result we need the following lemma

Lemma 4 (Cauchy Lemma, [21]). *Let a_n, $b_n \in \mathbb{R}_+$, $n \in \mathbb{N}$. We suppose that:*

(i) $\sum_{k=0}^{\infty} a_k < +\infty$;
(ii) $b_n \to 0$ *as* $n \to \infty$.

Then

$$\sum_{k=0}^{n} a_{n-k}b_k \to 0 \text{ as } n \to \infty.$$

Theorem 5. *Let (X,d) be a complete metric space, $Y \subset X$ a nonempty bounded closed subset and $f : Y \to X$ a continuous operator. We suppose that:*

(i) *f is Ćirić-Reich-Rus operator;*
(ii) *there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$;*
(iii) $E_f(Y) < +\infty$.

Then:

(a) $F_f = \{x^*\}$;
(b) $f^{n-1}(x_n) \to x^*$ *and* $f^n(x_n) \to x^*$ *as* $n \to +\infty$;
(c) $d(x,x^*) \leq (1+b)(1-a)^{-1}d(x,f(x))$, $\forall x \in Y$;
(d) $d\left(f^{n-1}(x_n),x^*\right) \leq (1+b)(1-a)^{-1}\alpha^{n-1}d(x_n,f(x_n))$, $\forall n \in \mathbb{N}^*$, *where* $\alpha = \frac{a+b}{1-b}$.
(e) *Let $g : Y \to X$ be such that:*
  (1) *there exists $\eta > 0$ such that $d(f(x),g(x)) \leq \eta$, $\forall x \in Y$;*
  (2) $F_g \neq \varnothing$.
  Then

$$d\left(x^*,y^*\right) \leq (1+b)(1-a)^{-1}\eta, \forall y^* \in F_g.$$

Proof. $(a)+(b)$. Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1\cap Y)}$, …, $Y_{n+1} := \overline{f(Y_n\cap Y)}$, $n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$ and $f^n(x_n) \in Y_n$, so $Y_n \neq \varnothing$, $n \in \mathbb{N}^*$. Since f is Ćirić-Reich-Rus operator, from Lemma 3 (a), we have that:

$$\delta(Y_{n+1}) = \delta\left(\overline{f(Y_n\cap Y)}\right) = \delta(f(Y_n\cap Y)) \leq$$

$$\begin{aligned}
&\leq a\delta\left(Y_n\cap Y\right)+2bE_f\left(Y_n\cap Y\right)\leq\\
&\leq a\delta\left(Y_n\right)+2bE_f\left(Y_n\cap Y\right)\leq\cdots\leq\\
&\leq a^{n+1}\delta\left(Y\right)+a^n2b\cdot E_f\left(Y\right)+\\
&\quad+a^{n-1}2b\cdot E_f\left(Y_1\cap Y\right)+\cdots+2b\cdot E_f\left(Y_n\cap Y\right).
\end{aligned}$$

On the other hand, from Lemma 3 (b) we get

$$\begin{aligned}
E_f\left(Y_k\cap Y\right)&=E_f\left(\overline{f\left(Y_{n-1}\cap Y\right)}\cap Y\right)=E_f\left(f\left(Y_{n-1}\cap Y\right)\cap Y\right)\leq\\
&\leq\alpha E_f\left(Y_{k-1}\cap Y\right)\leq\cdots<\alpha^kE_f\left(Y\right),\ k\in N^*,
\end{aligned}$$

where $\alpha=\frac{a+b}{1-b}$. Applying Lemma 4 for $a_n=a^n$ and $b_n=2b\cdot E_f\left(Y_n\cap Y\right)$ and we get that

$$\delta\left(Y_n\right)\to 0 \text{ as } n\to+\infty$$

and the proof is similar with the proof of Theorem 1.

(c). Let $x\in Y$. From the definition of the Ćirić-Reich-Rus operator we have:

$$\begin{aligned}
d\left(x,x^*\right)&\leq d\left(x,f\left(x\right)\right)+d\left(f\left(x\right),x^*\right)\leq\\
&\leq d\left(x,f\left(x\right)\right)+ad\left(x,x^*\right)+bd\left(x,f\left(x\right)\right),\ \forall x\in Y,
\end{aligned}$$

so

$$d\left(x,x^*\right)\leq\frac{1+b}{1-a}d\left(x,f\left(x\right)\right),\ \forall x\in Y.$$

(d) and (e) follow from (c). □

5.2. *Perov operators*

Let (X,d) be a generalized metric space with $d:X\times X\to\mathbb{R}_+^m$, $Y\in P_{cl}\left(X\right)$ and $f:Y\to X$ be a nonself operator. By definition (see [17], [20]) $f:Y\to X$ is a nonself Perov operator if there exists a matrix convergent to zero $S\in\mathbb{R}_+^{m\times m}$ such that

$$d\left(f\left(x\right),f\left(y\right)\right)\leq S\cdot d\left(x,y\right),\ x,y\in Y.$$

We have the following fixed point results in the case of nonself Perov operators:

Theorem 6. *Let (X,d) be a complete generalized metric space with $d:X\times X\to\mathbb{R}_+^m$, $Y\subset X$ a nonempty bounded closed subset and $f:Y\to X$ an operator. We suppose that:*

(i) *f is a Perov operator;*
(ii) *there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n\left(x_n\right)$ is defined for all $n\in\mathbb{N}^*$.*

Then:

(a) *$F_f=\{x^*\}$;*
(b) *$f^{n-1}\left(x_n\right)\to x^*$ and $f^n\left(x_n\right)\to x^*$ as $n\to+\infty$.*
(c) *$d\left(x,x^*\right)\leq\left(I_m-S\right)^{-1}d\left(x,f\left(x\right)\right)$, $\forall x\in Y$;*

(d) $d\left(f^n(x_n), x^*\right) \leq S^n d\left(x_n, x^*\right)$, $\forall n \in \mathbb{N}^*$;
(e) *Let* $g : Y \to X$ *be such that:*
(1) *there exists* $\eta \in \left(\mathbb{R}_+^*\right)^m$ *such that* $d\left(f(x), g(x)\right) \leq \eta$, $\forall x \in Y$;
(2) $F_g \neq \varnothing$.
Then

$$d\left(x^*, y^*\right) \leq \left(I_m - S\right)^{-1} \eta, \forall y^* \in F_g.$$

Proof. $(a)+(b)$. Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1 \cap Y)}$, ..., $Y_{n+1} := \overline{f(Y_n \cap Y)}$, $n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$ and $f^n(x_n) \in Y_n$, so $Y_n \neq \varnothing$, $n \in \mathbb{N}^*$. Since f is a Perov we have:

$$\delta(Y_{n+1}) = \delta\left(\overline{f(Y_n \cap Y)}\right) = \delta\left(f(Y_n \cap Y)\right) \leq S \cdot \delta(Y_n \cap Y) =\leq$$
$$\leq S \cdot \delta(Y_n) \leq \cdots \leq S^{n+1} \cdot \delta(Y) \to 0 \text{ as } n \to +\infty.$$

Now the proof is similar with the proof of Theorem 1.

(c). Let $x \in Y$ then we have:

$$d\left(x, x^*\right) \leq d\left(x, f(x)\right) + d\left(f(x), x^*\right) \leq d\left(x, f(x)\right) + S d\left(x, x^*\right), \ \forall x \in Y,$$

so

$$d\left(x, x^*\right) \leq \left(I_m - S\right)^{-1} d\left(x, f(x)\right), \ \forall x \in Y.$$

(d) follows from the definition of the Perov operator and (e) is obtained from (c) for $x := y^* \in F_g$. □

6. An open problem

The above considerations give rise to the following problem:

Problem 1. *Let* (X, d) *be a complete metric space,* Y *a nonempty bounded and closed subset of* X *and* $f : Y \to X$ *a nonself operator. We suppose that there exists a sequence* $(x_n)_{n \in \mathbb{N}^*}$ *such that* $f^n(x_n)$ *is defined for all* $n \in \mathbb{N}^*$. *In which additional conditions on* f *we have that:*

(a) $F_f \neq \varnothing$ *?*
(b) $F_f = \{x^*\}$ *?*

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ASYMPTOTIC INTEGRATION OF SINGULARLY PERTURBED LINEAR SYSTEMS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. The paper deals with problem on asymptotic solutions to a system of singular perturbed linear differential-algebraic equations. Case of multiple roots of a characteristic equation is studied. Technique of constructing the asymptotic solutions is developed.

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Keywords: asymptotic solution, differential-algebraic equations, singular perturbed system

1. INTRODUCTION

Linear differential-algebraic equations (DAEs) with constant coefficients

$$A\frac{dx}{dt} = Bx + f(t) \tag{1.1}$$

were well studied in the middle of the past century. In particular, Luzin [12] and Gantmacher [7] have found necessary and sufficient conditions for solvability of DAEs, and they have proposed approaches of constructing their solutions. It should be note that Gantmacher's algorithm [7] for constructing particular solutions of DAEs (1.1) was based on the idea of reduction of a pencil $B-\lambda A$ to Kronecker normal form. At the same time similar technique could not be used to DAEs with variable coefficients

$$A(t)\frac{dx}{dt} = B(t)x + f(t), \tag{1.2}$$

in general case, since its application as a rule causes changing the Kronecker's form of a pencil $B(t)-\lambda A(t)$ of system (1.2).

To solve the problem Campbell has proposed the notion of standard canonical form of system (1.2) that was represented in the following form [3]

$$\begin{pmatrix} I_{n-s} & 0 \\ 0 & N_s(t) \end{pmatrix}\frac{dx}{dt} = \begin{pmatrix} M(t) & 0 \\ 0 & I_s \end{pmatrix}x + h(t), \tag{1.3}$$

where I_s and I_{n-s} are identity matrices of orders s and $n-s$, respectively, and $N_s(t)$ is a nilpotent lower (or upper) triangular matrix.

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Note that in particular case, when matrix $N_s(t)$ is additionally constant, system (1.3) is called the strong standard canonical form of system (1.2).

Later Petzold and Gear have found sufficient conditions of reduction of system (1.2) to its Kronecker's form [8, 13]. It allows to find the general solution of system (1.2) and to study then Cauchy problem, boundary value problems, and others [1, 4, 5, 14].

Another way to deal with DAEs is to decouple them by means of canonical projectors. Using a concept of the tractability index, the effective numerical methods for solving DAEs were developed in the papers by Gear and Petzold [8], Griepentrog and März [9], Brenan, Campbell and Petzold [2].

Since 1980s, there is considered singular perturbed DAEs [1, 14, 19]

$$\varepsilon A(t,\varepsilon)\frac{dx}{dt} = B(t,\varepsilon)x,\ t \in [0;T],\ \varepsilon \in (0;\varepsilon_0], \tag{1.4}$$

where $A(t,\varepsilon)$, $B(t,\varepsilon)$ are $n \times n$–matrices with real or complex-valued elements, possessing uniform asymptotic expansions of the following form

$$A(t,\varepsilon) = \sum_{k\geq 0} \varepsilon^k A_k(t),\ B(t,\varepsilon) = \sum_{k\geq 0} \varepsilon^k B_k(t),\ \varepsilon \to 0,\ t \in [0;T],$$

with infinitely differentiable coefficients, and ε is a small parameter.

In particular, Starun [19] has found as many linearly independent asymptotic solutions of system (1.4) as there are roots of the corresponding characteristic equation

$$det(B_0(t) - \lambda A_0(t)) = 0. \tag{1.5}$$

It follows that if rank$\mathrm{B_0(t)}$ is equal to the degree of the polynomial $det(B_0(t) - \lambda A_0(t))$ then a formal fundamental matrix solution of system (1.4) can be constructed [1, 14]. Generalizing Starun's ideas Yakovets has shown that under certain conditions system (1.4) has two types of formal solutions corresponding to finite or infinite elementary divisors of the pencil $B_0(t) - \lambda A_0(t)$ [14]. Moreover, their linear combination is formal general solution of system (1.4).

It is well known [14], in case of multiple spectrum of the main pencil $B_0(t) - \lambda A_0(t)$ asymptotic expansions of solutions of system (1.4) should be constructed in some fractional powers of small parameter ε. Moreover, value of fractional powers of small parameter depends on the structure of the perturbing matrices $A_k(t)$, $B_k(t)$, $k \in N$. However, in the case of multiple eigenvalues of $B_0(t) - \lambda A_0(t)$ the technique of constructing asymptotic expansions is tedious and quite complicated. That is why we propose in present paper a modified approach for finding formal asymptotic solutions to system (1.4). Namely, our technique is based on transformation of the system with multiple spectrum of the main pencil of matrices into system whose main pencil of matrices has a simple spectrum [17]. On our opinion, the proposed approach is more rational one than others.

2. FORMAL SOLUTIONS

Assume that the following conditions are satisfied:

(i) the pencil of matrices $B_0(t)-\lambda A_0(t)$ is regular for all $t \in [0;T]$;

(ii) the pencil $B_0(t)-\lambda A_0(t)$ has one eigenvalue $\lambda_0(t)$, two finite elementary divisors $(\lambda-\lambda_0(t))^{p_1}$, $(\lambda-\lambda_0(t))^{p_2}$, $2 \le p_1 < p_2$, and two infinite elementary divisors of multiplicity q_1 and q_2, $2 \le q_1 < q_2$; furthermore $p_1+p_2+q_1+q_2=n$.

Then there exist the nonsingular of matrices $P(t,\varepsilon)$, $Q(t,\varepsilon) \in C^{\infty}([0;T]\times[0;\varepsilon_0])$ such that

$$P(t,\varepsilon)A(t,\varepsilon)Q(t,\varepsilon) = H(t,\varepsilon) \equiv diag\{N_q(t,\varepsilon), I_p(t,\varepsilon)\}, \tag{2.1}$$

$$P(t,\varepsilon)B(t,\varepsilon)Q(t,\varepsilon) = \Omega(t,\varepsilon) \equiv diag\{I_q(t,\varepsilon), W_p(t,\varepsilon)\}, \tag{2.2}$$

where $I_q(t,0)=I_q$, $I_p(t,0)=I_p$, I_q and I_p are the identity matrices of orders q $(q=q_1+q_2)$ and p $(p=p_1+p_2)$, respectively;

$N_q(t,0)=N_q \equiv diag\{N_{q_1}, N_{q_2}\}$, $W_p(t,0)=diag\{W_{p_1}(t), W_{p_2}(t)\}$, $p=p_1+p_2$, N_{q_i} is the square matrix of order q_i such that

$$N_{q_i} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \ i=1,2,$$

and $W_{p_i}(t)=\lambda_0(t)I_{p_i}+N_{p_i}$, $i=1,2$ [15, 16, 18].

We set $x(t,\varepsilon)=Q(t,\varepsilon)y(t,\varepsilon)$. Then system (1.4) can be written in the form

$$\varepsilon H(t,\varepsilon)\frac{dy}{dt} = C(t,\varepsilon)y, \tag{2.3}$$

where $C(t,\varepsilon)=\Omega(t,\varepsilon)-\varepsilon H(t,\varepsilon)Q^{-1}(t,\varepsilon)Q'(t,\varepsilon)$, $Q'(t,0)\equiv 0$, $t\in[0;T]$.

The transformation $y(t,\varepsilon)=\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \lambda_0(t)dt\right)z(t,\varepsilon)$ changes system (2.3) into

$$\varepsilon H(t,\varepsilon)\frac{dz}{dt} = (C(t,\varepsilon)-\lambda_0(t)H(t,\varepsilon))z.$$

It should be noted that the pencils $\Omega(t,0)-\lambda_0(t)H(t,0)-\lambda H(t,0)$ and $diag\{I_q, N_p\}-\lambda\, diag\{N_q, I_p\}$ have the same Kronecker normal form [7]. Therefore, without loss of generality, we can assume that

$$\Omega(t,0)=diag\{I_q, N_p\},\ N_p=diag\{N_{p_1}, N_{p_2}\}.$$

Let us define the matrices

$$H(t,\varepsilon)=\sum_{k\ge 0}\varepsilon^k H_k(t),\ C(t,\varepsilon)=\sum_{k\ge 0}\varepsilon^k C_k(t),$$

where $H_0(t) \equiv H_0 = diag\{N_q, I_p\}$, $C_0(t) \equiv C_0 = diag\{I_q, N_p\}$.

We seek an asymptotic solutions of system (2.3) in the following form:

$$y_i(t,\varepsilon) = u_i(t,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int\limits_{a_i}^{t}\lambda_i(t,\varepsilon)dt\right),\ i=\overline{1,n}, \tag{2.4}$$

where $u_i(t,\varepsilon)$ are n-dimensional vectors, $\lambda_i(t,\varepsilon)$ are scalar functions, and

$$u_i(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k\widetilde{u}_i^{(k)}(t,\varepsilon),\ \lambda_i(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k\widetilde{\lambda}_i^{(k)}(t,\varepsilon),\ i=\overline{1,n},$$

[14]. Constants a_i are defined below.

Substituting representation (2.4) into system (2.3), we get

$$(C_0+\varepsilon C_1(t))u_i(t,\varepsilon) - (H_0+\varepsilon H_1(t))u_i(t,\varepsilon)\lambda_i(t,\varepsilon) = \varepsilon H(t,\varepsilon)u_i'(t,\varepsilon) - \\ -\sum_{k\geq 2}\varepsilon^k C_k(t)u_i(t,\varepsilon) + \sum_{k\geq 2}\varepsilon^k H_k(t)u_i(t,\varepsilon)\lambda_i(t,\varepsilon), \tag{2.5}$$

where $C_1(t) = diag\{C_{1q}(t), C_{1p}(t)\}$, $H_1(t) = diag\{H_{1q}(t), H_{1p}(t)\}$; here, $C_{1q}(t)$, $H_{1q}(t)$ are square matrices of order q.

Let $K(t,\varepsilon)$ be defined by $K(t,\varepsilon) = diag\{I_q + \varepsilon C_{1q}(t), I_p + \varepsilon H_{1p}(t)\}$. Then

$$K^{-1}(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k M_k(t) \equiv diag\{I_q + \sum_{k\geq 1}\varepsilon^k M_{kq}(t), I_p + \sum_{k\geq 1}\varepsilon^k M_{kp}(t)\}$$

and $M_1(t) = diag\{-C_{1q}(t), -H_{1p}(t)\}$.

Multiplying both sides of relation (2.5) on the left by $K^{-1}(t,\varepsilon)$, we obtain

$$(C_0+\varepsilon D_1(t))u_i(t,\varepsilon) - (H_0+\varepsilon F_1(t))u_i(t,\varepsilon)\lambda_i(t,\varepsilon) = \varepsilon\sum_{k\geq 0}\varepsilon^k F_k(t)u_i'(t,\varepsilon) - \\ -\sum_{k\geq 2}\varepsilon^k D_k(t)u_i(t,\varepsilon) + \sum_{k\geq 2}\varepsilon^k F_k(t)u_i(t,\varepsilon)\lambda_i(t,\varepsilon),$$

where

$$D_1(t) = diag\{0, D_{1p}(t)\} \equiv diag\{0, C_{1p}(t) - H_{1p}(t)N_p\},$$
$$F_1(t) = diag\{F_{1q}(t), 0\} \equiv diag\{H_{1q}(t) - C_{1q}(t)N_q, 0\},$$
$$D_2(t) = C_2(t) + diag\{0, M_{2p}(t)N_p - H_{1p}(t)C_{1p}(t)\},$$
$$D_k(t) = \sum_{i=0}^{k-2} M_i(t)C_{k-i}(t) + diag\{0, M_{kp}(t)N_p + M_{k-1,p}C_{1p}(t)\},\ k\geq 3,$$
$$F_2(t) = H_2(t) + diag\{M_{2q}(t)N_q - C_{1q}(t)H_{1q}(t), 0\},$$
$$F_k(t) = \sum_{i=0}^{k-2} M_i(t)H_{k-i}(t) + diag\{M_{kq}(t)N_q + M_{k-1,q}H_{1q}(t), 0\},\ k\geq 3.$$

Equating the coefficients at the powers of ε in the following way, we get

$$(C_0+\varepsilon D_1(t)-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)(H_0+\varepsilon F_1(t)))\widetilde{u}_i^{(0)}(t,\varepsilon)=0, \tag{2.6}$$

$$(C_0+\varepsilon D_1(t)-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)(H_0+\varepsilon F_1(t)))\widetilde{u}_i^{(s)}(t,\varepsilon)=d_i^{(s)}(t,\varepsilon),\ s\in N, \tag{2.7}$$

where

$$d_i^{(s)}(t,\varepsilon)=\sum_{k=0}^{s}F_k(t)(\widetilde{u}_i^{(s-k-1)}(t,\varepsilon))'-\sum_{k=2}^{s}D_k(t)\widetilde{u}_i^{(s-k)}(t,\varepsilon)+$$

$$+\sum_{k=2}^{s}\sum_{j=0}^{k}F_k(t)\widetilde{u}_i^{(j)}(t,\varepsilon)\widetilde{\lambda}_i^{(s-k-j)}(t,\varepsilon)+(H_0+\varepsilon F_1(t))\sum_{k=0}^{s-1}\widetilde{u}_i^{(k)}(t,\varepsilon)\widetilde{\lambda}_i^{(s-k)}(t,\varepsilon).$$

According to the form of the matrices C_0 and H_0, we rewrite equation (2.6) as

$$(I_q-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)(N_q+\varepsilon F_{1q}(t)))\widetilde{u}_{i1}^{(0)}(t,\varepsilon)=0, \tag{2.8}$$

$$(N_p+\varepsilon D_{1p}(t)-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)I_p)\widetilde{u}_{i2}^{(0)}(t,\varepsilon)=0, \tag{2.9}$$

where $\widetilde{u}_i^{(0)}(t,\varepsilon)=\begin{pmatrix}\widetilde{u}_{i1}^{(0)}(t,\varepsilon)\\ \widetilde{u}_{i2}^{(0)}(t,\varepsilon)\end{pmatrix}$.

Consider equation (2.8). The characteristic equation of $N_q+\varepsilon F_{1q}(t)$ has the form

$$w^q+\beta_1(t,\varepsilon)w^{q-1}+\ldots+\beta_{q-1}(t,\varepsilon)w+\beta_q(t,\varepsilon)=0, \tag{2.10}$$

where

$$\beta_j(t,\varepsilon)=O(\varepsilon),\ \varepsilon\to 0,\ j=\overline{1,q_2-1},$$

$$\beta_{q_2}(t,\varepsilon)=-h_{q,q_1+1}^{(1)}(t)\varepsilon+O(\varepsilon^2),\ \varepsilon\to 0,$$

$$\beta_j(t,\varepsilon)=O(\varepsilon^2),\ \varepsilon\to 0,\ i=\overline{q_2+1,q-1},$$

$$\beta_q(t,\varepsilon)=h_{q_1,1}^{(1)}(t)h_{q,q_1+1}^{(1)}(t)\varepsilon^2+O(\varepsilon^3),\ \varepsilon\to 0,$$

for all $t\in[0;T]$; here, $h_{ij}^{(1)}(t)$ is the element of the matrix $H_1(t)$.

Let the following condition be satisfied.

(iii) $h_{q_1,1}^{(1)}(t)\neq 0$, $h_{q,q_1+1}^{(1)}(t)\neq 0$, $h(t)\neq 0$, $t\in[0;T]$, where $h(t)=h_{q_1,1}^{(1)}(t)h_{q,q_1+1}^{(1)}(t)-h_{q_1,q_1+1}^{(1)}(t)h_{q1}^{(1)}(t)$.

We construct the characteristic polygon for the equation (2.10) [10,14]. Its vertices are the points $(q;0)$, $(q_1;1)$, and $(0;2)$.

Since $\operatorname{tg}\theta_1=\frac{1}{q_1}$, $\operatorname{tg}\theta_2=\frac{1}{q_2}$, then the solutions of equation (2.10) have the following form:

$$w_j(t,\varepsilon)=\sqrt[q_1]{h_{q_1,1}^{(1)}(t)}\varepsilon^{\frac{1}{q_1}}+O(\varepsilon^{\frac{2}{q_1}}),\ j=\overline{1,q_1},$$

$$w_j(t,\varepsilon)=\sqrt[q_2]{h_{q,q_1+1}^{(1)}(t)}\varepsilon^{\frac{1}{q_2}}+O(\varepsilon^{\frac{2}{q_2}}),\ j=\overline{q_1+1,q}.$$

Let $\varphi_j(t,\varepsilon)$, $j=\overline{1,q}$, be columns of the matrix $T_q(t,\varepsilon)$, where

$$T_q^{-1}(t,\varepsilon)(N_q+\varepsilon F_{1q}(t))T_q(t,\varepsilon)=\widetilde{W}_q(t,\varepsilon)\equiv diag\{w_1(t,\varepsilon),w_2(t,\varepsilon),...,w_q(t,\varepsilon)\}.$$

Then

$$(N_q+\varepsilon F_{1q}(t)-w_j(t,\varepsilon)I_q)\varphi_j(t,\varepsilon)=0,\ j=\overline{1,q}.$$

The components of the vectors $\varphi_j(t,\varepsilon)$, $j=\overline{1,q_1}$, and $\varphi_j(t,\varepsilon)$, $j=\overline{q_1+1,q}$ can be taken as the cofactors of the first and (q_1+1)th rows of the matrix $N_q+\varepsilon F_{1q}(t)-w_j(t,\varepsilon)I_q$, respectively. Hence it follows that

$$\varphi_j(t,\varepsilon)=\begin{pmatrix} a_j^{q_1-1}(t)h^{(1)}_{q,q_1+1}(t)+O(\varepsilon^{\frac{1}{q_1}}) \\ h(t)\varepsilon^{\frac{1}{q_1}}+O(\varepsilon^{\frac{2}{q_1}}) \\ a_j(t)h(t)\varepsilon^{\frac{2}{q_1}}+O(\varepsilon^{\frac{3}{q_1}}) \\ \cdots \\ a_j^{q_1-2}(t)h(t)\varepsilon^{\frac{q_1-1}{q_1}}+O(\varepsilon) \\ -a_j^{q_1-1}(t)h^{(1)}_{q1}(t)+O(\varepsilon^{\frac{1}{q_1}}) \\ -a_j^{q_1}(t)h^{(1)}_{q1}(t)\varepsilon^{\frac{1}{q_1}}+O(\varepsilon^{\frac{2}{q_1}}) \\ \cdots \\ -a_j^{2(q_1-1)}(t)h^{(1)}_{q1}(t)\varepsilon^{\frac{q_1-1}{q_1}}+O(\varepsilon) \\ O(\varepsilon) \\ \cdots \\ O(\varepsilon) \end{pmatrix},\ j=\overline{1,q_1},$$

$$\varphi_j(t,\varepsilon)=\begin{pmatrix} b_{j-q_1}^{q_2-1}(t)h_{q_1,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-q_1}{q_2}}+O(\varepsilon^{\frac{q_2-q_1+1}{q_2}}) \\ b_{j-q_1}^{q_2}(t)h_{q_1,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-q_1+1}{q_2}}+O(\varepsilon^{\frac{q_2-q_1+2}{q_2}}) \\ \cdots \\ b_{j-q_1}^{q_2+q_1-2}(t)h_{q_1,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-1}{q_2}}+O(\varepsilon) \\ b_{j-q_1}^{q_1+q_2-1}(t)+O(\varepsilon^{\frac{1}{q_2}}) \\ b_{j-q_1}^{q_1}(t)h_{q,q_1+1}^{(1)}(t)\varepsilon^{\frac{1}{q_2}}+O(\varepsilon^{\frac{2}{q_2}}) \\ \cdots \\ b_{j-q_1}^{q_1+q_2-2}(t)h_{q,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-1}{q_2}}+O(\varepsilon) \end{pmatrix},\ j=\overline{q_1+1,q},$$

where

$$a_j(t)=\sqrt[q_1]{|h_{q_1,1}^{(1)}(t)|}\left(\cos\frac{\arg h_{q_1,1}^{(1)}(t)+2\pi j}{q_1}+i\sin\frac{\arg h_{q_1,1}^{(1)}(t)+2\pi j}{q_1}\right),\ j=\overline{1,q_1},$$

$$b_j(t)=\sqrt[q_2]{|h_{q,q_1+1}^{(1)}(t)|}\left(\cos\frac{\arg h_{q,q_1+1}^{(1)}(t)+2\pi j}{q_2}+i\sin\frac{\arg h_{q,q_1+1}^{(1)}(t)+2\pi j}{q_2}\right),\ j=\overline{1,q_2}.$$

It is easy to see that the matrix $T_q(t,\varepsilon)$ is nonsingular. Indeed, let us define the matrix

$$T_q(t,\varepsilon)=\begin{pmatrix} T_1(t,\varepsilon) & T_2(t,\varepsilon) \\ T_3(t,\varepsilon) & T_4(t,\varepsilon) \end{pmatrix},$$

where $T_1(t,\varepsilon)$ is a square matrix of order q_1. Then

$$det\,T_q(t,\varepsilon)=det\,T_1(t,\varepsilon)\,det(T_4(t,\varepsilon)-T_3(t,\varepsilon)T_1^{-1}(t,\varepsilon)T_2(t,\varepsilon))=$$

$$=detV_1(t)\,detV_2(t)(h_{q,q_1+1}^{(1)}(t))^{q_2}h^{q_1-1}(t)\prod_{i=1}^{q_2}b_i^{q_1}(t)\varepsilon^{\frac{q-2}{2}}+O(\varepsilon^{\frac{q-2}{2}+\gamma}),\ \gamma>0,$$

where

$$V_1(t)=\begin{pmatrix} a_1^{q_1-1}(t) & a_2^{q_1-1}(t) & \dots & a_{q_1}^{q_1-1}(t) \\ 1 & 1 & \dots & 1 \\ a_1(t) & a_2(t) & \dots & a_{q_1}(t) \\ \dots & \dots & \dots & \dots \\ a_1^{q_1-2}(t) & a_2^{q_1-2}(t) & \dots & a_{q_1}^{q_1-2}(t) \end{pmatrix},$$

$$V_2(t)=\begin{pmatrix} b_1^{q_2-1}(t) & b_2^{q_2-1}(t) & \dots & b_{q_2}^{q_2-1}(t) \\ 1 & 1 & \dots & 1 \\ b_1(t) & b_2(t) & \dots & b_{q_2}(t) \\ \dots & \dots & \dots & \dots \\ b_1^{q_2-2}(t) & b_2^{q_2-2}(t) & \dots & b_{q_2}^{q_2-2}(t) \end{pmatrix},$$

because [7]

$$det\, T_1(t,\varepsilon) = detV_1(t)h^{(1)}_{q,q_1+1}(t)h^{q_1-1}(t)\varepsilon^{\frac{q_1-1}{2}} + O(\varepsilon^{\frac{q_1-1}{2}+\frac{1}{q_1}}).$$

Further, $detV_1(t)$ and $detV_2(t)$ are Vandermonde determinants up to a sign. Thus, $detV_1(t) \neq 0$, $detV_2(t) \neq 0$, $t \in [0;T]$, and $det\, T_q(t,\varepsilon) \neq 0$, $t \in [0;T]$.

Substituting the representation

$$\widetilde{u}^{(0)}_{i1}(t,\varepsilon) = T_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1}(t,\varepsilon), \tag{2.11}$$

into equation (2.8), we get

$$(I_q - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))\widetilde{q}^{(0)}_{i1}(t,\varepsilon) = 0. \tag{2.12}$$

Therefore,

$$\widetilde{\lambda}^{(0)}_i(t,\varepsilon) = \frac{1}{w_i(t,\varepsilon)},$$

$$\{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_i = 1,\ \{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_j = 0,\ t \in [0;T],\ i \neq j,\ i,j = \overline{1,q},$$

where $\{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_j$ is the jth component of $\widetilde{q}^{(0)}_{i1}(t,\varepsilon)$.

Consider now equation (2.9). Suppose the following.

(iv) $c^{(1)}_{q+p_1,q+1}(t) \neq 0$, $c^{(1)}_{n,q+p_1+1}(t) \neq 0$, $c(t) \neq 0$, $t \in [0;T]$, where $c(t) = c^{(1)}_{q+p_1,q+1}(t)c^{(1)}_{n,q+p_1+1}(t) - c^{(1)}_{q+p_1,q+p_1+1}(t)c^{(1)}_{n,q+1}(t)$, $c^{(1)}_{ij}(t)$ is the element of the matrix $C_1(t)$.

Then the solutions of equation

$$det(N_p + \varepsilon D_{1p}(t) - wI_p) = 0 \tag{2.13}$$

have the following form:

$$w_j(t,\varepsilon) = \sqrt[p_1]{c^{(1)}_{q+p_1,q+1}(t)}\varepsilon^{\frac{1}{p_1}} + O(\varepsilon^{\frac{2}{p_1}}),\ j = \overline{q+1,q+p_1},$$

$$w_j(t,\varepsilon) = \sqrt[p_2]{c^{(1)}_{n,q+p_1+1}(t)}\varepsilon^{\frac{1}{p_2}} + O(\varepsilon^{\frac{2}{p_2}}),\ j = \overline{q+p_1+1,n}.$$

Let us $T_p(t,\varepsilon)$ be the square matrix of order p such that

$$T_p^{-1}(t,\varepsilon)(N_p + \varepsilon D_{1p}(t))T_p(t,\varepsilon) = \widetilde{W}_p(t,\varepsilon) \equiv$$
$$\equiv diag\{w_{q+1}(t,\varepsilon), w_{q+2}(t,\varepsilon), ..., w_n(t,\varepsilon)\}.$$

Substituting the representation

$$\widetilde{u}^{(0)}_{i2}(t,\varepsilon) = T_p(t,\varepsilon)\widetilde{q}^{(0)}_{i2}(t,\varepsilon) \tag{2.14}$$

into (2.9), we obtain

$$(\widetilde{W}_p(t,\varepsilon) - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)\widetilde{q}^{(0)}_{i2}(t,\varepsilon) = 0. \tag{2.15}$$

Thus

$$\widetilde{\lambda}^{(0)}_i(t,\varepsilon) = w_i(t,\varepsilon),$$

$$\{\widetilde{q}_{i2}^{(0)}(t,\varepsilon)\}_i = 1,\ \{\widetilde{q}_{i2}^{(0)}(t,\varepsilon)\}_j = 0,\ t \in [0;T],\ i \neq j,\ i,j = \overline{q+1,n},$$

and

$$\widetilde{q}_{i1}^{(0)}(t,\varepsilon) = 0,\ t \in [0;T],\ i = \overline{q+1,n},$$
$$\widetilde{q}_{i2}^{(0)}(t,\varepsilon) = 0,\ t \in [0;T],\ i = \overline{1,q}.$$

We rewrite equation (2.7) for $s = 1$ as follows:

$$(I_q - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)(N_q + \varepsilon F_{1q}(t)))\widetilde{u}_{i1}^{(1)}(t,\varepsilon) = d_{i1}^{(1)}(t,\varepsilon), \tag{2.16}$$

$$(N_p + \varepsilon D_{1p}(t) - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)I_p)\widetilde{u}_{i2}^{(1)}(t,\varepsilon) = d_{i2}^{(1)}(t,\varepsilon), \tag{2.17}$$

where $\widetilde{u}_i^{(1)}(t,\varepsilon) = \begin{pmatrix} \widetilde{u}_{i1}^{(1)}(t,\varepsilon) \\ \widetilde{u}_{i2}^{(1)}(t,\varepsilon) \end{pmatrix}$, and $\widetilde{d}_i^{(1)}(t,\varepsilon) = \begin{pmatrix} \widetilde{d}_{i1}^{(1)}(t,\varepsilon) \\ \widetilde{d}_{i2}^{(1)}(t,\varepsilon) \end{pmatrix}$.

The transformation $\widetilde{u}_{i1}^{(1)}(t,\varepsilon) = T_q(t,\varepsilon)\widetilde{q}_{i1}^{(1)}(t,\varepsilon)$, $\widetilde{u}_{i2}^{(1)}(t,\varepsilon) = T_p(t,\varepsilon)\widetilde{q}_{i2}^{(1)}(t,\varepsilon)$ changes (2.16), (2.17) into the equations

$$(I_q - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))\widetilde{q}_{i1}^{(1)}(t,\varepsilon) = g_{i1}^{(1)}(t,\varepsilon), \tag{2.18}$$

$$(\widetilde{W}_p(t,\varepsilon) - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)I_p)\widetilde{q}_{i2}^{(1)}(t,\varepsilon) = g_{i2}^{(1)}(t,\varepsilon), \tag{2.19}$$

where

$$g_{i1}^{(1)}(t,\varepsilon) = T_q^{-1}(t,\varepsilon)d_{i1}^{(1)}(t,\varepsilon) \equiv T_q^{-1}(t,\varepsilon)N_q T_q'(t,\varepsilon)\widetilde{q}_{i1}^{(0)} + \widetilde{W}_q(t,\varepsilon)\widetilde{q}_{i1}^{(0)}\widetilde{\lambda}_i^{(1)}(t,\varepsilon),$$
$$g_{i2}^{(1)}(t,\varepsilon) = T_p^{-1}(t,\varepsilon)d_{i2}^{(1)}(t,\varepsilon) \equiv T_p^{-1}(t,\varepsilon)T_p'(t,\varepsilon)\widetilde{q}_{i2}^{(0)} + \widetilde{q}_{i2}^{(0)}\widetilde{\lambda}_i^{(1)}(t,\varepsilon).$$

Therefore,

$$\widetilde{\lambda}_i^{(1)}(t,\varepsilon) = -\frac{\{f_{i1}^{(1)}(t,\varepsilon)\}_i}{w_i(t,\varepsilon)},\ \{\widetilde{q}_{i1}^{(1)}(t,\varepsilon)\}_i \equiv 0,\ t \in [0;T],$$

$$\{\widetilde{q}_{i1}^{(1)}(t,\varepsilon)\}_j = \frac{\{g_{i1}^{(1)}(t,\varepsilon)\}_j w_i(t,\varepsilon)}{w_i(t,\varepsilon) - w_j(t,\varepsilon)},\ i \neq j,\ i,j = \overline{1,q},$$

$$\widetilde{q}_{i2}^{(1)}(t,\varepsilon) = (\widetilde{W}_p(t,\varepsilon) - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)I_p)^{-1}g_{i2}^{(1)}(t,\varepsilon),\ i = \overline{1,q},$$

where $f_{i1}^{(1)}(t,\varepsilon) = T_q^{-1}(t,\varepsilon)N_q T_q'(t,\varepsilon)\widetilde{q}_{i1}^{(0)}$, and

$$\widetilde{\lambda}_i^{(1)}(t,\varepsilon) = -\{f_{i2}^{(1)}(t,\varepsilon)\}_i,\ \{\widetilde{q}_{i2}^{(1)}(t,\varepsilon)\}_i \equiv 0,\ t \in [0;T],$$

$$\{\widetilde{q}_{i2}^{(1)}(t,\varepsilon)\}_j = \frac{\{g_{i2}^{(1)}(t,\varepsilon)\}_j}{w_j(t,\varepsilon) - w_i(t,\varepsilon)},\ i \neq j,\ i,j = \overline{q+1,n},$$

$$\widetilde{q}_{i1}^{(1)}(t,\varepsilon) = (I_q - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))^{-1}g_{i1}^{(1)}(t,\varepsilon),\ i = \overline{q+1,n},$$

where $f_{i2}^{(1)}(t,\varepsilon) = T_p^{-1}(t,\varepsilon)T_p'(t,\varepsilon)\widetilde{q}_{i2}^{(0)}$.

Let $T_q^{-1}(t,\varepsilon)$ be defined by

$$T_q^{-1}(t,\varepsilon) = \begin{pmatrix} V_1(t,\varepsilon) & V_2(t,\varepsilon) \\ V_3(t,\varepsilon) & V_4(t,\varepsilon) \end{pmatrix},$$

where $V_1(t,\varepsilon)$ is a square matrix of order q_1.

Using the Frobenius formula for the inverse of the block matrix[7], we find:

$$V_i(t,\varepsilon) = O\left(\varepsilon^{\delta_{2i}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\right)\begin{pmatrix} O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \\ O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \\ \dots & \dots & \dots & \dots \\ O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \end{pmatrix},$$

$t \in [0;T]$, $\varepsilon \to 0$; $\gamma_i = q_{\delta_{1i}} + q_{2\delta_{2i}} + q_{\delta_{3i}} + q_{2\delta_{4i}}$, $i = \overline{1,4}$; δ_{ij} is Kronecker delta.

It should be noted that the matrix $T_p^{-1}(t,\varepsilon)$ has the same form as $T_q^{-1}(t,\varepsilon)$.

Let the following condition be satisfied.

(v) $q_2 < p_1$.

Then

$$||\widetilde{u}_i^{(1)}(t,\varepsilon)|| = O(\varepsilon^{\frac{1}{q_2}}),\ |\widetilde{\lambda}_i^{(1)}(t,\varepsilon)| = O(1),\ i = \overline{1,q},$$

$$||\widetilde{u}_i^{(1)}(t,\varepsilon)|| = O(\varepsilon^{-\frac{1}{p_1}}),\ |\widetilde{\lambda}_i^{(1)}(t,\varepsilon)| = O(1),\ i = \overline{q+1,n},$$

$t \in [0;T]$, $\varepsilon \to 0$. In the same way we define the functions $\widetilde{u}_i^{(s)}(t,\varepsilon)$, $\widetilde{\lambda}_i^{(s)}(t,\varepsilon)$, $i = \overline{1,n}$, $s = 2,3....$ In addition,

$$\begin{aligned} ||\widetilde{u}_i^{(s)}(t,\varepsilon)|| &= O(\varepsilon^{-\left[\frac{s}{2}\right]}),\ i = \overline{1,n}, \\ |\widetilde{\lambda}_i^{(s)}(t,\varepsilon)| &= O(\varepsilon^{-\left[\frac{s}{2}\right]-\frac{1}{q_1}}),\ i = \overline{1,q}, \\ |\widetilde{\lambda}_i^{(s)}(t,\varepsilon)| &= O(\varepsilon^{-\left[\frac{s}{2}\right]-\frac{2}{p_2}\left(\frac{s}{2}-\left[\frac{s}{2}\right]\right)+\frac{1}{p_2}}),\ i = \overline{q+1,n}, \end{aligned} \tag{2.20}$$

$s = 2,3,...$, $t \in [0;T]$, $\varepsilon \to 0$, where $\left[\frac{s}{2}\right]$ is the integer part of $\frac{s}{2}$.

3. Asymptotic behavior of solutions

Assume $\operatorname{Re} w_i(t,\varepsilon) \neq 0$, $t \in [0;T]$, $i = \overline{1,n}$. Then $y_i^{(m)}(t,\varepsilon)$ can be defined by

$$y_i^{(m)}(t,\varepsilon) = u_i^{(m)}(t,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_{a_i}^{t}\lambda_i^{(m)}(t,\varepsilon)dt\right),\ i = \overline{1,n}, \tag{3.1}$$

where $u_i^{(m)}(t,\varepsilon) = \sum\limits_{k=0}^{m}\varepsilon^k\widetilde{u}_i^{(k)}(t,\varepsilon)$, $\lambda_i^{(m)}(t,\varepsilon) = \sum\limits_{k=0}^{m}\varepsilon^k\widetilde{\lambda}_i^{(k)}(t,\varepsilon)$, $i = \overline{1,n}$, and

$$a_i = \begin{cases} 0,\ \operatorname{Re}\widetilde{\lambda}_i^{(0)}(t,\varepsilon) < 0, \\ T,\ \operatorname{Re}\widetilde{\lambda}_i^{(0)}(t,\varepsilon) > 0,\ t \in [0;T],\ \varepsilon \in (0;\varepsilon_0]. \end{cases}$$

The change of variables $y_i(t,\varepsilon) = z_i(t,\varepsilon) + y_i^{(m)}(t,\varepsilon)$ transforms system (2.3) into

$$\varepsilon H(t,\varepsilon)\frac{dz_i}{dt} = C(t,\varepsilon)z_i + f(t,\varepsilon), \tag{3.2}$$

where $f(t,\varepsilon) = C(t,\varepsilon)y_i^{(m)}(t,\varepsilon) - \varepsilon H(t,\varepsilon)\frac{y_i^{(m)}(t,\varepsilon)}{dt}$. It is easy to show that the vector function $f(t,\varepsilon)$ is of order of magnitude $O(\varepsilon^{\alpha(m)})$, $\alpha(m) = m - \left[\frac{m}{2}\right] - \frac{1}{q_1}$.

Let us prove the existence of a solution of system (3.2) such that $z_i(a_i,\varepsilon) = 0$. Multiplying both sides of system (3.2) on the left by $K^{-1}(t,\varepsilon)$, we get

$$\varepsilon F(t,\varepsilon)\frac{dz_i}{dt} = D(t,\varepsilon)z_i + g(t,\varepsilon), \tag{3.3}$$

where

$$F(t,\varepsilon) = H_0 + \sum_{k\geq 1}\varepsilon^k F_k(t),\ D(t,\varepsilon) = C_0 + \sum_{k\geq 1}\varepsilon^k D_k(t),\ g(t,\varepsilon) = K^{-1}(t,\varepsilon)f(t,\varepsilon).$$

Let us define the matrix $T(t,\varepsilon) = diag\{T_q(t,\varepsilon), T_p(t,\varepsilon)\}$. Then

$$T^{-1}(t,\varepsilon)F(t,\varepsilon)T(t,\varepsilon) = diag\{\widetilde{W}_q(t,\varepsilon), I_p\} + \varepsilon L(t,\varepsilon),$$
$$T^{-1}(t,\varepsilon)D(t,\varepsilon)T(t,\varepsilon) = diag\{I_q, \widetilde{W}_p(t,\varepsilon)\} + \varepsilon S(t,\varepsilon),$$

where

$$L(t,\varepsilon) = diag\{L_q(t,\varepsilon), L_p(t,\varepsilon)\} \equiv$$
$$\begin{pmatrix} \varepsilon^{\frac{1}{q_1}}L_{11}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}L_{12}(t,\varepsilon) & 0 & 0 \\ \varepsilon^{\frac{1}{q_2}}L_{21}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}L_{22}(t,\varepsilon) & 0 & 0 \\ 0 & 0 & \varepsilon^{\frac{1}{p_1}}L_{33}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}L_{34}(t,\varepsilon) \\ 0 & 0 & \varepsilon^{\frac{1}{p_2}}L_{43}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}L_{44}(t,\varepsilon) \end{pmatrix},$$

$$S(t,\varepsilon) = \begin{pmatrix} \varepsilon^{\frac{1}{q_1}}S_{11}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}S_{12}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}S_{13}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}S_{14}(t,\varepsilon) \\ \varepsilon^{\frac{1}{q_2}}S_{21}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}S_{22}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}S_{23}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}S_{24}(t,\varepsilon) \\ \varepsilon^{\frac{1}{p_1}}S_{31}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}S_{32}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}S_{33}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}S_{34}(t,\varepsilon) \\ \varepsilon^{\frac{1}{p_2}}S_{41}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}S_{42}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}S_{43}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}S_{44}(t,\varepsilon) \end{pmatrix};$$

the diagonal blocks of $L(t,\varepsilon)$ and $S(t,\varepsilon)$ are square matrices of order q_1, q_2, p_1, p_2, respectively; $||L_{ij}(t,\varepsilon)|| = O(1)$, $||S_{ij}(t,\varepsilon)|| = O(1)$, $t \in [0;T]$, $\varepsilon \to 0$.

The transformation $z_i(t,\varepsilon) = T(t,\varepsilon)r_i(t,\varepsilon)$ changes system (3.3) into

$$\varepsilon\left(\begin{pmatrix}\widetilde{W}_q(t,\varepsilon) & 0\\ 0 & I_p\end{pmatrix} + \varepsilon L(t,\varepsilon)\right)\frac{dr_i}{dt} = \left(\begin{pmatrix} I_q & 0\\ 0 & \widetilde{W}_p(t,\varepsilon)\end{pmatrix} + \varepsilon S(t,\varepsilon) -\right.$$
$$\left. -\varepsilon\left(\begin{pmatrix}\widetilde{W}_q(t,\varepsilon) & 0\\ 0 & I_p\end{pmatrix} + \varepsilon L(t,\varepsilon)\right)T^{-1}(t,\varepsilon)T'(t,\varepsilon)\right)r_i + T^{-1}(t,\varepsilon)g(t,\varepsilon)$$

i.e.

$$\varepsilon\frac{dr_{i1}}{dt} = \widetilde{W}_q^{-1}(t,\varepsilon)r_{i1} + \varepsilon(R_{11}(t,\varepsilon)r_{i1} + R_{12}(t,\varepsilon)r_{i2}) + h_1(t,\varepsilon), \tag{3.4}$$

$$\varepsilon\frac{dr_{i2}}{dt} = \widetilde{W}_p(t,\varepsilon)r_{i2} + \varepsilon(R_{21}(t,\varepsilon)r_{i1} + R_{22}(t,\varepsilon)r_{i2}) + h_2(t,\varepsilon), \tag{3.5}$$

where $r_i(t,\varepsilon) = \begin{pmatrix} r_{i1}(t,\varepsilon) \\ r_{i2}(t,\varepsilon) \end{pmatrix}$, $h(t,\varepsilon) = \begin{pmatrix} h_1(t,\varepsilon) \\ h_2(t,\varepsilon) \end{pmatrix}$,

$$h(t,\varepsilon) = \left(\begin{pmatrix} \widetilde{W}_q(t,\varepsilon) & 0 \\ 0 & I_p \end{pmatrix} + \varepsilon L(t,\varepsilon)\right)^{-1} T^{-1}(t,\varepsilon)g(t,\varepsilon),$$

$$R(t,\varepsilon) = \begin{pmatrix} R_{11}(t,\varepsilon) & R_{12}(t,\varepsilon) \\ R_{21}(t,\varepsilon) & R_{22}(t,\varepsilon) \end{pmatrix} \equiv \frac{1}{\varepsilon}\times$$

$$\times\begin{pmatrix} (\widetilde{W}_q(t,\varepsilon) + \varepsilon L_q(t,\varepsilon))^{-1} - \widetilde{W}_q^{-1}(t,\varepsilon) & 0 \\ 0 & ((I_p + \varepsilon L_p(t,\varepsilon))^{-1} - I_p)\widetilde{W}_p(t,\varepsilon) \end{pmatrix} +$$

$$+\left(\begin{pmatrix} \widetilde{W}_q(t,\varepsilon) & 0 \\ 0 & I_p \end{pmatrix} + \varepsilon L(t,\varepsilon)\right)^{-1} S(t,\varepsilon) - T^{-1}(t,\varepsilon)T'(t,\varepsilon);$$

here, $R_{11}(t,\varepsilon)$ is a square matrix of order q. It is easy to see that $||h(t,\varepsilon)|| = O(\varepsilon^{\alpha(m)-1})$, $t \in [0;T]$, $\varepsilon \to 0$.

Assume that the following condition are satisfied.

(vi) $\operatorname{Re}\frac{1}{w_j(t,\varepsilon)} \le c_1\varepsilon^{l_1} < 0$, $-\frac{1}{q_1} \le l_1 \le 1 - \frac{1}{q_1}$, if $\operatorname{Re} w_j(t,\varepsilon) < 0$,
$\operatorname{Re}\frac{1}{w_j(t,\varepsilon)} \ge c_2\varepsilon^{l_2} > 0$, $-\frac{1}{q_1} \le l_2 \le 1 - \frac{1}{q_1}$, if $\operatorname{Re} w_j(t,\varepsilon) > 0$, $j = \overline{1,q}$.
$\operatorname{Re} w_j(t,\varepsilon) \le c_3\varepsilon^{l_3} < 0$, $\frac{1}{p_2} \le l_3 < 1$, if $\operatorname{Re} w_j(t,\varepsilon) < 0$,
$\operatorname{Re} w_j(t,\varepsilon) \ge c_4\varepsilon^{l_4} > 0$, $\frac{1}{p_2} \le l_4 < 1$, if $\operatorname{Re} w_j(t,\varepsilon) > 0$, $j = \overline{q+1,n}$, $t \in [0;T]$.

Then, without loss of generality, we can assume that

$$\widetilde{W}_q(t,\varepsilon) = diag\{\widetilde{W}_{q-}(t,\varepsilon), \widetilde{W}_{q+}(t,\varepsilon)\},\ \widetilde{W}_p(t,\varepsilon) = diag\{\widetilde{W}_{p-}(t,\varepsilon), \widetilde{W}_{p+}(t,\varepsilon)\}.$$

Here, $\widetilde{W}_{q-}(t,\varepsilon)$ and $\widetilde{W}_{q+}(t,\varepsilon)$ are the diagonal matrices whose eigenvalues are the eigenvalues of $\widetilde{W}_q(t,\varepsilon)$ with negative and positive real parts, respectively, and the matrix $\widetilde{W}_p(t,\varepsilon)$ has the same structure as $\widetilde{W}_q(t,\varepsilon)$.

Thus, systems (3.4), (3.5) can be written as

$$\varepsilon\frac{dr_{i11}}{dt} = \widetilde{W}_{q-}^{-1}(t,\varepsilon)r_{i11} + \varepsilon\left(\sum_{j=1}^{2}(R_{1j}(t,\varepsilon)r_{i1j} + R_{1,j+2}(t,\varepsilon)r_{i2j})\right) + h_{11}(t,\varepsilon), \tag{3.6}$$

$$\varepsilon\frac{dr_{i12}}{dt}=\widetilde{W}_{q+}^{-1}(t,\varepsilon)r_{i12}+\varepsilon\left(\sum_{j=1}^{2}(R_{2j}(t,\varepsilon)r_{i1j}+R_{2,j+2}(t,\varepsilon)r_{i2j})\right)+h_{12}(t,\varepsilon), \tag{3.7}$$

$$\varepsilon\frac{dr_{i21}}{dt}=\widetilde{W}_{p-}(t,\varepsilon)r_{i21}+\varepsilon\left(\sum_{j=1}^{2}(R_{3j}(t,\varepsilon)r_{i1j}+R_{3,j+2}(t,\varepsilon)r_{i2j})\right)+h_{21}(t,\varepsilon), \tag{3.8}$$

$$\varepsilon\frac{dr_{i22}}{dt}=\widetilde{W}_{p+}(t,\varepsilon)r_{i22}+\varepsilon\left(\sum_{j=1}^{2}(R_{4j}(t,\varepsilon)r_{i1j}+R_{4,j+2}(t,\varepsilon)r_{i2j})\right)+h_{22}(t,\varepsilon), \tag{3.9}$$

where

$$R(t,\varepsilon)=\begin{pmatrix}R_{11}(t,\varepsilon) & R_{12}(t,\varepsilon) & R_{13}(t,\varepsilon) & R_{14}(t,\varepsilon)\\ R_{21}(t,\varepsilon) & R_{22}(t,\varepsilon) & R_{23}(t,\varepsilon) & R_{24}(t,\varepsilon)\\ R_{31}(t,\varepsilon) & R_{32}(t,\varepsilon) & R_{33}(t,\varepsilon) & R_{34}(t,\varepsilon)\\ R_{41}(t,\varepsilon) & R_{42}(t,\varepsilon) & R_{43}(t,\varepsilon) & R_{44}(t,\varepsilon)\end{pmatrix};$$

the dimensions of vectors r_{i11}, r_{i12}, r_{i21}, r_{i22} and h_{11}, h_{12}, h_{21}, h_{22} coincide with the orders of the matrices $\widetilde{W}_{q-}(t,\varepsilon)$, $\widetilde{W}_{q+}(t,\varepsilon)$, $\widetilde{W}_{p-}(t,\varepsilon)$, $\widetilde{W}_{p+}(t,\varepsilon)$, respectively.

Let us write a system of integral equations equivalent to system (3.6) – (3.9):

$$r_{i11}(t,\varepsilon)=\int_0^t Z_{11}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{1j}(\tau,\varepsilon)r_{i1j}+R_{1,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{11}(\tau,\varepsilon)\right)d\tau, \tag{3.10}$$

$$r_{i12}(t,\varepsilon)=\int_T^t Z_{12}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{2j}(\tau,\varepsilon)r_{i1j}+R_{2,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{12}(\tau,\varepsilon)\right)d\tau, \tag{3.11}$$

$$r_{i21}(t,\varepsilon)=\int_0^t Z_{21}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{3j}(\tau,\varepsilon)r_{i1j}+R_{3,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{21}(\tau,\varepsilon)\right)d\tau, \tag{3.12}$$

$$r_{i22}(t,\varepsilon)=\int_T^t Z_{22}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{4j}(\tau,\varepsilon)r_{i1j}+R_{4,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{11}(\tau,\varepsilon)\right)d\tau, \tag{3.13}$$

where $Z_{11}(t,\tau,\varepsilon)$ and $Z_{12}(t,\tau,\varepsilon)$ are fundamental matrices solutions of the systems

$$\varepsilon\frac{dr_{i11}}{dt}=\widetilde{W}_{q-}^{-1}(t,\varepsilon)r_{i11},\ Z_{11}(\tau,\tau,\varepsilon)=I_k,$$

and

$$\varepsilon\frac{dr_{i12}}{dt} = \widetilde{W}_{q+}^{-1}(t,\varepsilon)r_{i12},\ Z_{12}(\tau,\tau,\varepsilon) = I_{q-k},$$

respectively; in the same way we define matrices $Z_{21}(t,\tau,\varepsilon)$ and $Z_{22}(t,\tau,\varepsilon)$.

Let us consider the mapping $\psi = Ar$ of the set

$$P = \{r(t,\varepsilon) \in C[0;T] : ||r(t,\varepsilon)|| \le c\varepsilon^{\alpha(m)-2}\},$$

into itself given by system (3.10) – (3.13). The mapping $\psi = Ar$ is a contraction mapping. Thus, the operator equation $r = Ar$ (and consequently system (3.10) – (3.13) also) has one and only one solution.

Thus, the main result of the paper can be formulated as follows.

Theorem 1. *If $A_s(t)$, $B_s(t) \in C^{m+1}[0;T]$, $s \ge 0$, and the assumptions (i) – (vi) are satisfied, then there exist n linearly independent solutions $x_i = x_i(t,\varepsilon)$, $t \in [0;T]$, of system (1.4) such that*

$$||x_i(t,\varepsilon) - x_i^{(m)}(t,\varepsilon)|| = O(\varepsilon^{\alpha(m)-2}),\ \varepsilon \to 0,\ m \ge 5,$$

where $x_i^{(m)}(t,\varepsilon) = Q(t,\varepsilon)y_i^{(m)}(t,\varepsilon)$.

Remark 1. If the pencil $B_0(t) - \lambda A_0(t)$ has more than one distinct eigenvalue, then system (1.4) can be reduced to a set of systems of lower order in each of which the corresponding characteristic equation has only one eigenvalue [6, 10, 11].

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ON HERMITE-HADAMARD TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper, we have established Hermite-Hadamard-type inequalities for fractional integrals and will be given an identity. With the help of this fractional-type integral identity, we give some integral inequalities connected with the left-side of Hermite–Hadamard-type inequalities for Riemann-Liouville fractional integrals.

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Keywords: Hermite-Hadamard's inequalities, Riemann-Liouville fractional integral, integral inequalities.

1. INTRODUCTION

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[20, p.137], [10]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1–4, 10–13, 15–17, 19, 20, 26, 27]) and the references cited therein.

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f:[a,b]\to\mathbb{R}$.

Definition 1. The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

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In [15] in order to prove some inequalities related to Hadamard's inequality Kırmacı used the following lemma:

Lemma 1. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$, be a differentiable mapping on I°, $a,b\in I^\circ$ (I° is the interior of I) with $a<b$. If $f'\in L([a,b])$, then we have*

$$\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)$$
$$=(b-a)\left[\int_0^{\frac{1}{2}}tf'(ta+(1-t)b)dt+\int_{\frac{1}{2}}^1(t-1)f'(ta+(1-t)b)dt\right].$$

Also, in [15], Kırmacı obtained the following inequalities for differentiable mappings which are connected with Hermite-Hadamard's inequality:

Theorem 1. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If the mapping $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.2}$$

Theorem 2. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$, be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and $p>1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le\\&\le\frac{(b-a)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(|f'(a)|^{\frac{p}{p-1}}+3|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right.\\&\quad\left.+\left(3|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right\}\\&\le\frac{b-a}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{1.3}$$

Meanwhile, Sarikaya et al.[25] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha>0$.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]$$

$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(ta+(1-t)b)\,dt.$$

It is remarkable that Sarikaya et al.[25] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha>0$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [14, 18, 21].

Definition 2. Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha>0$ with $a\ge 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x>a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

For some recent results connected with fractional integral inequalities see ([5–9, 22–25, 28]).

The aim of this paper is to establish Hermite-Hadamard's inequalities for Riemann-Liouville fractional integral similar to the method in [25] and we will investigate some integral inequalities connected with the left hand side of the Hermite-Hadamard type inequalities for fractional integrals.

2. Hermite-Hadamard's inequalities for fractional integrals

Hermite-Hadamard's inequalities can be represented in fractional integral forms as follows:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\le\frac{f(a)+f(b)}{2} \tag{2.1}$$

with $\alpha>0$.

Proof. Since f is a convex function on $[a,b]$, we have for $x,y\in[a,b]$ with $\lambda=\frac{1}{2}$

$$f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2} \tag{2.2}$$

i.e., with $x=\frac{t}{2}a+\frac{2-t}{2}b$, $y=\frac{2-t}{2}a+\frac{t}{2}b$,

$$2f\left(\frac{a+b}{2}\right)\le f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right). \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\frac{a+b}{2}\right)\\
&\le\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\alpha-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\
&=\int_b^{\frac{a+b}{2}}\left(\frac{2}{b-a}(b-u)\right)^{\alpha-1}f(u)\frac{2du}{a-b}+\int_a^{\frac{a+b}{2}}\left(\frac{2}{b-a}(v-a)\right)^{\alpha-1}f(v)\frac{2dv}{b-a}\\
&=\frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]
\end{aligned}$$

i.e.

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]$$

and the first inequality is proved.

For the proof of the second inequality in (2.2) we first note that if f is a convex function, then, for $\lambda\in[0,1]$, it yields

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\le\frac{t}{2}f(a)+\frac{2-t}{2}f(b)$$

and

$$f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\le\frac{2-t}{2}f(a)+\frac{t}{2}f(b).$$

By adding these inequalities we have

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq f(a)+f(b). \tag{2.4}$$

Then multiplying both sides of (2.4) by $t^{\alpha-1}$and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\alpha-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt$$
$$\leq [f(a)+f(b)]\int_0^1 t^{\alpha-1}dt$$

i.e.

$$\frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\leq\frac{f(a)+f(b)}{\alpha}.$$

The proof is completed. □

Remark 1. If in Theorem 4, we let $\alpha=1$, then the inequalities (2.1) become the inequalities (1.1).

3. Fractional inequalities for convex functions

We need the following lemma. With the help of this, we give some integral inequalities connected with the left-side of Hermite–Hadamard-type inequalities for Riemann-Liouville fractional integrals.

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2}) \tag{3.1}$$
$$=\frac{b-a}{4}\left\{\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}$$

with $\alpha>0$.

Proof. Integrating by parts

$$\begin{aligned} I_1 &= \int_0^1 t^\alpha f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\ &= t^\alpha \frac{2}{a-b} f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\Big|_0^1 - \int_0^1 \alpha t^{\alpha-1} f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\frac{2}{a-b}dt \\ &= -\frac{2}{b-a} f(\frac{a+b}{2}) - \frac{2\alpha}{a-b}\int_b^{\frac{a+b}{2}} \left(\frac{2}{b-a}(b-x)\right)^{\alpha-1}\frac{2}{a-b} f(x)dx \\ &= -\frac{2}{b-a} f(\frac{a+b}{2}) + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{(\frac{a+b}{2})_-} f(b) \end{aligned} \tag{3.2}$$

and similarly we get,

$$\begin{aligned} I_2 &= \int_0^1 t^\alpha f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\ &= t^\alpha \frac{2}{b-a} f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\Big|_0^1 - \frac{2\alpha}{b-a}\int_0^1 t^{\alpha-1} f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\ &= \frac{2}{b-a} f(\frac{a+b}{2}) - \frac{2\alpha}{b-a}\int_a^{\frac{a+b}{2}} \left(\frac{2}{b-a}(x-a)\right)^{\alpha-1} f(x)\frac{2}{b-a}dx \\ &= \frac{2}{b-a} f(\frac{a+b}{2}) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{(\frac{a+b}{2})_+} f(a). \end{aligned} \tag{3.3}$$

By using (3.2) and (3.3), it follows that

$$I_1 - I_2 = -\frac{4}{b-a} f(\frac{a+b}{2}) + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)_+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)_-} f(a)\right].$$

Thus, by multiplying the both sides by $\frac{b-a}{4}$, we have the conclusion (3.1). □

Corollary 1. *If in Lemma 3, we let $\alpha = 1$, then the equality (3.1) becomes the following equality*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ &= \frac{b-a}{4}\left\{\int_0^1 t f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt - \int_0^1 t f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}. \end{aligned} \tag{3.4}$$

Using this Lemma 3, we can obtain the following fractional integral inequality:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following inequality for fractional*

integrals holds:

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right| \\
&\leq \frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left\{\left((\alpha+1)\left|f'(a)\right|^{q}+(\alpha+3)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left((\alpha+3)\left|f'(a)\right|^{q}+(\alpha+1)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{3.5}$$

Proof. Firstly, we suppose that $q=1$. Using Lemma 3 and the convexity of $|f'|$, we find

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right| \\
&\leq \frac{b-a}{4}\int_0^1 t^{\alpha}\left\{\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right\}dt \\
&\leq \frac{b-a}{4}\left[\left|f'(a))\right|+\left|f'(b))\right|\right]\int_0^1 t^{\alpha}dt \\
&= \frac{b-a}{4(\alpha+1)}\left[\left|f'(a))\right|+\left|f'(b))\right|\right].
\end{aligned}$$

Secondly, we suppose that $q>1$. Using Lemma 3 and the power mean inequality, and the convexity of $|f'|^{q}$, we find

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right| \\
&\leq \frac{b-a}{4}\left\{\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\} \\
&\leq \frac{b-a}{4}\frac{1}{(\alpha+1)^{\frac{1}{p}}}\left\{\left(\int_0^1\left[\frac{t^{\alpha+1}}{2}\left|f'(a)\right|^{q}+\frac{2t^{\alpha}-t^{\alpha+1}}{2}\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\left[\frac{2t^{\alpha}-t^{\alpha+1}}{2}\left|f'(a)\right|^{q}+\frac{t^{\alpha+1}}{2}\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$= \frac{b-a}{4}\frac{1}{(\alpha+1)^{\frac{1}{p}}}\left\{\left(\frac{1}{2(\alpha+2)}\left|f'(a)\right|^q + \frac{(\alpha+3)}{2(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{(\alpha+3)}{2(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q + \frac{1}{2(\alpha+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

The proof of Theorem 5 is complete. □

Remark 2. If we take $\alpha = 1$ and $q = 1$ in Theorem 5, then the inequality (3.5) becomes the inequality (1.2) of Theorem 1.

Theorem 6. *Let $f:[a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for $q > 1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'a)|+3|f'(b))|}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'a)|+|f'(b))|}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \tag{3.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 3, using the well-known Hölder's inequality and the convexity of $|f'|^q$, we find

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left|f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1\left|f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left[\frac{t}{2}\left|f'(a))\right|^q + \frac{2-t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1\left[\frac{2-t}{2}\left|f'(a))\right|^q + \frac{t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a))|+3|f'(b))|}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a))|+|f'(b))|}{4}\right)^{\frac{1}{q}}\right].$$

Let $a_1 = 3|f'(a)|^q, b_1 = |f'(b)|^q, a_2 = |f'(a)|^q, b_2 = 3|f'(b)|^q$. Here, $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}a_k^s + \sum_{k=1}^{n}b_k^s$$

For $(0 \leq s < 1), a_1, a_2, ..., a_n \geq 0, b_1, b_2, ..., b_n \geq 0$, we obtain

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right|$$

$$\leq \frac{b-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(3|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}+\left(|f'(a)|^q+3|f'(b)|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}(3^{\frac{1}{q}}+1)\left[|f'(a)|+|f'(b)|\right]$$

$$\leq \frac{b-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}4\left[|f'(a)|+|f'(b)|\right]$$

which completed proof. □

Remark 3. If we take $\alpha = 1$ in Theorem 6, then the inequality (3.6) becomes the inequality (1.3) of Theorem 2.

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PARTITION PROPERTIES ON COUNTABLE BIPARTITE GRAPHS

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Abstract. We classify all countably infinite (on both sides) bipartite graphs G having a naturally defined version of the pigeonhole principle.

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1. INTRODUCTION

Definition 1. An $(\aleph_0, \aleph_0)$-bigraph is a structure $G = (X, Y, E)$, where $(X \cup Y, E)$ is a digraph such that $E \subseteq \{xy : x \in X, y \in Y\}$ and $|X| = |Y| = \aleph_0$. We call (X, Y) the *bipartition* of G, X the left side, and Y the right side.

The complement $\bar{G}$ of a bigraph $G = (X, Y, E)$ has the same set of vertices with the same bipartition, and $(X \times Y) \setminus E$ for the set of edges.

We say that a countable structure S has property $\mathcal{P}$ (also known as the pigeonhole principle) if for every partition of its domain into finitely many pieces at least one of the induced substructures is isomorphic to S. This property was investigated for various graph-theoretic structures. We list some such results.

Theorem 1 ([2], Proposition 3.4)**.** *The only countable graphs with the property $\mathcal{P}$ up to isomorphism are the empty graph, the complete graph and the Rado graph.*

Theorem 2 ([1], Theorem 1)**.** *The only countable tournaments with the property $\mathcal{P}$ up to isomorphism are the random tournament and tournaments ω^α and $(\omega^\alpha)^*$ for $0 < \alpha < \omega_1$.*

Theorem 3 ([3], Corollary 2.4)**.** *The only countable digraphs with the property $\mathcal{P}$ up to isomorphism are the empty digraph, the random tournament, tournaments ω^α and $(\omega^\alpha)^*$ for $0 < \alpha < \omega_1$, the random digraph, the random acyclic digraph and its inverse.*

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As we shall see, for bigraphs it makes sense only to consider certain altered property that we will call $\mathcal{P}'$. It is our goal to classify all countable bigraphs with property $\mathcal{P}'$.

Since random bigraphs will play an important role in our classification, we will now list some of their properties. More on these structures can be found in [5].

Definition 2. A $(\aleph_0,\aleph_0)$-bigraph (X,Y,E) is $(\aleph_0,\aleph_0)$-*random* if

$$\forall U,W\in[Y]^{<\aleph_0}\ (U\cap W=\varnothing\Rightarrow\exists x\in X(\forall u\in U\ xu\in E\ \wedge\ \forall w\in W\ xw\notin E)).$$

(X,Y,E) is $(\aleph_0,\aleph_0)$-*dense* if

$$\forall U,W\in[X]^{<\aleph_0}\ (U\cap W=\varnothing\Rightarrow\exists y\in Y(\forall u\in U\ uy\in E\ \wedge\ \forall w\in W\ wy\notin E)).$$

If G satisfies both these conditions we will call it $(\aleph_0,\aleph_0)$-random dense.

Proposition 1. *Let $G=(X,Y,E)$ be a $(\aleph_0,\aleph_0)$-random bigraph. Then every $y\in Y$ is of infinite degree in both G and $\bar{G}$.*

Clearly, the same results hold for vertices in X if G is (κ,λ,μ)-dense. In [4] it was shown that there is exactly one $(\aleph_0,\aleph_0)$-random dense bigraph up to isomorphism. This is not true for $(\aleph_0,\aleph_0)$-random bigraphs.

2. The classification

Now we turn to our main concern, a partition property resembling $\mathcal{P}$.

Definition 3. Let $G_1=(X_1,Y_1,E_1)$ and $G_2=(X_2,Y_2,E_2)$ be bigraphs. A bijection $f:X_1\cup Y_1\to X_2\cup Y_2$ is an isomorphism if for all $x\in X_1$, $y\in Y_1$: $xy\in E_1$ iff $f(x)f(y)\in E_2$.

Of course, for $(\aleph_0,\aleph_0)$-bigraphs it makes sense only to consider partitions into $(\aleph_0,\aleph_0)$-bigraphs (otherwise we could split the bigraph in two parts, each having one finite side). So we say that an $(\aleph_0,\aleph_0)$-bigraph G has property $\mathcal{P}'$ if:

$\mathcal{P}'$: for every partition of the set of vertices of G into finitely many pieces that each induce $(\aleph_0,\aleph_0)$-bigraphs at least one of the induced sub-bigraphs is isomorphic to G.

Example 1. $S=(X,Y,E)$ is defined by $X=\{x_n:n\in\omega\}$, $Y=\{y_n:n\in\omega\}$ and $E=\{x_ny_0:n\in\omega\}$. S has the property $\mathcal{P}'$.

It is easy to see that if G has $\mathcal{P}$, then its complement $\bar{G}$ and the bigraph G^* obtained by reversing all the edges also have $\mathcal{P}'$.

Taking into account Propositions 1, 2 and 3 it would be natural to expect the $(\aleph_0,\aleph_0)$-random dense bigraph has the property $\mathcal{P}'$. However, this is not the case.

Lemma 1. *The $(\aleph_0,\aleph_0)$-random dense bigraph does not satisfy $\mathcal{P}'$.*

Proof. Let $y \in Y$ be arbitrary and let X_1 be the set of all its neighbors. By Proposition 1, both X_1 and $X \setminus X_1$ are infinite. Let $x \in X_1$ and let Y_0 be the set of all its neighbors. If we partition vertices of G into $(X \setminus X_1) \cup Y_0$ and $X_1 \cup (Y \setminus Y_0)$, in both induced sub-bigraphs we have an isolated vertex, so none of those parts can be isomorphic to G. □

In order to prove our main result, Theorem 4, we will consider several cases concerning the minimal degree of a vertex in G, and the number of vertices of minimal degree. $d_G(v)$ will denote the degree of a vertex v in G, and we will drop the G if it is clear from the context.

Lemma 2. *If all the vertices in both G and $\bar{G}$ are of infinite degree, then G does not satisfy $\mathcal{P}'$.*

Proof. Let $G = (X,Y,E)$ be an $(\aleph_0,\aleph_0)$-bigraph with property $\mathcal{P}'$ such that all the vertices in both G and $\bar{G}$ are of infinite degree. We prove that G must be $(\aleph_0,\aleph_0)$-random, and in a similar way it is proved that it must be $(\aleph_0,\aleph_0)$-dense. This will be a contradiction with Lemma 1.

Let $u_1,u_2,\ldots,u_m,w_1,w_2,\ldots,w_n \in Y$. Suppose there is no $x \in X$ connected to all $u_1,u_2\ldots,u_m$, and not connected to any of $w_1,w_2\ldots,w_n$. We define the partition $X = X_1 \cup \cdots \cup X_m \cup X'_1 \cup \cdots \cup X'_n$ as follows. First we enumerate the vertices in X: $X = \{x_k : k \in \omega\}$, and begin with $X_1,\ldots,X_m,X'_1,\ldots,X'_n$ empty. We proceed by recursion on k. If there is $i \in \{1,2,\ldots,m\}$ such that $x_k u_i \notin E$, let r_k be such i that X_i is of the smallest cardinality (if there are several such i, choose any of them). On the other hand, if there is $j \in \{1,2,\ldots,n\}$ such that $x_k w_j \in E$, let s_k be such j that X'_j is of the smallest cardinality. Now, if there is no j as above, put x_k in the set X_{r_k}; if there is no i as above then put x_k in the set X'_{s_k}. Otherwise, if $|X_{r_k}| \leq |X'_{s_k}|$, put x_k in the set X_{r_k}, and otherwise in the set X'_{s_k}.

What we get in the end are disjoint sets $X_1,\ldots,X_m,X'_1,\ldots,X'_n$, each X_i containing no neighbors of u_i and each X'_i containing no non-neighbors of w_i. In addition, each X_i is infinite; suppose not and let, for example, X_i be the one with the smallest cardinality (the case when there are several such sets is considered similarly). Let $k_0 \in \omega$ be big enough such that by the k_0-th step of the recursion all the elements of X_i and at least $|X_i|+1$ elements of other sets were added to them. Then for $k > k_0$ we would have no more elements x_k such that $x_k u_i \notin E$, which is impossible because u_i has infinitely many non-neighbors.

Now we partition Y into infinite pieces: $Y = Y_1 \cup \cdots \cup Y_m \cup Y'_1 \cup \cdots \cup Y'_n$ arbitrarily so that $u_i \in Y_i$ and $w_i \in Y'_i$. Finally, we split G into $m+n$ parts induced by $X_i \cup Y_i$ and $X'_i \cup Y'_i$. Clearly, none of the obtained sub-bigraphs can be isomorphic to G, since each of them either has an isolated vertex or a vertex connected to all the vertices from the opposite side, a contradiction. □

Let $G = (X, Y, E)$ be an $(\aleph_0, \aleph_0)$-bigraph. We denote $D_X = \{d(v) : v \in X\}$, $D_Y = \{d(v) : v \in Y\}$, $d_X = \min D_X$ and $d_Y = \min D_Y$.

Lemma 3. *If in an $(\aleph_0, \aleph_0)$-bigraph $G = (X, Y, E)$ that satisfies $\mathcal{P}'$ holds $d_X \in \omega \setminus \{0\}$ or $d_Y \in \omega \setminus \{0\}$, then G is isomorphic either to S or to S^*.*

Proof. Suppose $d_X \in \omega \setminus \{0\}$. We will consider the following cases:

Case 1° There is exactly one vertex in X of degree d_X, say x. Split G arbitrarily into sub-bigraphs G_1 and G_2. Let, for example, $x \in G_1$.

If G_1 is isomorphic to G, we move one of the neighbors of x (call it y) to G_2. If now $G_2 + y$ is isomorphic to G, it has a vertex x_2 of degree d_X, so we also move one of the neighbors of x_2 from G_1 to G_2.

If, on the other hand, G_2 is isomorphic to G then there is $x_2 \in X$ that has degree d_X in G_2. Now by moving one of the neighbors of x_2 (say y) from G_1 to G_2 we get a partition in which G_2 has no vertices of degree d_X. If now the other part $(G_1 - y)$ is isomorphic to G, we move one of the neighbors of x in the same direction.

Either way, we arrive at a partition into two parts, none of which is isomorphic to G.

Case 2° There are at least two (but finitely many) vertices in X of degree d_X, say $x_1, x_2, \ldots, x_k$. Split G into two $(\aleph_0, \aleph_0)$-bigraphs G_1 and G_2 arbitrarily, but in such way that in each of the parts there is at least one of the vertices $x_1, x_2, \ldots, x_k$. Let G_1 be isomorphic to G; then there are exactly k vertices of degree d_X on its left side. Let there be l vertices of degree d_X on the left side of G_2. If $l \geq k$ move one of the vertices $x_1, x_2, \ldots, x_k$ from G_1 to G_2; otherwise move one of them in the other direction. In the partition obtained in this way none of the parts can be isomorphic to G.

Case 3° There are infinitely many vertices in X of degree d_X. If $d_X > 1$ then it is easy to split G so that there are two vertices in different parts such that each of them has in the induced sub-bigraph degree less than d_X. Now let $d_X = 1$. If there is no $y \in Y$ connected to all of the vertices of degree 1, we can split so that in both parts we get isolated vertices on the left side. So let $y \in Y$ be connected to all vertices of degree 1 in X. There may be no other vertices in X, because otherwise we could split into two parts G_1 and G_2 so that in G_1 we get isolated vertices and in G_2 having on the right side more that one vertex connected to vertices of degree 1. It follows that all the other vertices on the right side must be isolated, so the only remaining bigraph is S.

The corresponding cases when $d_Y \in \omega \setminus \{0\}$ are considered analogously and we get the bigraph S^* as another solution. □

This also takes care of the corresponding cases for $\bar{G}$ (i.e. when on one side of $\bar{G}$ there are vertices of finite degree, but no isolated vertices) and gives us two more solutions: $\bar{S}$ and $\bar{S}^*$. For the discussion in the upcoming theorem we will also need

a property stronger than $\mathcal{P}'$:

$\mathcal{P}''$: for every partition of the set of vertices of G into finitely many pieces that each induce $(\aleph_0, \aleph_0)$-bigraphs all of the induced sub-bigraphs are isomorphic to G.

It is easy to see that, when investigating either of the properties $\mathcal{P}'$ and $\mathcal{P}''$, the condition is not weakened when "finitely many pieces" is replaced with "two pieces".

Lemma 4. *The only $(\aleph_0, \aleph_0)$-bigraphs with the property $\mathcal{P}''$ up to isomorphism are the empty $(\aleph_0, \aleph_0)$-bigraph and the complete $(\aleph_0, \aleph_0)$-bigraph.*

Proof. The cases $d_X \in \omega \setminus \{0\}$ and $d_Y \in \omega \setminus \{0\}$ are discussed in Lemma 3 and yield no solutions (S and S^* clearly do not satisfy $\mathcal{P}''$).

Let $d_X = 0$. The case when there are finitely many isolated vertices is easy: partition in an arbitrary way, and if both parts are isomorphic to G, then move one of the vertices that were isolated in G from one part to the other. If infinitely many vertices in X are isolated, we can split so that one of the parts is an empty bigraph, so G itself must be empty.

If in $\bar{G}$ there are vertices of finite degree, the proof is analogous, and gives us the complete bigraph. Finally, if both G and $\bar{G}$ have only vertices of infinite degree, by Lemma 2 we have no further solutions. □

Theorem 4. *The only $(\aleph_0, \aleph_0)$-bigraphs with the property $\mathcal{P}'$ up to isomorphism are the empty $(\aleph_0, \aleph_0)$-bigraph, the complete $(\aleph_0, \aleph_0)$-bigraph, the bigraphs S and S^* and their complements.*

Proof. Let $G = (X, Y, E)$ be an $(\aleph_0, \aleph_0)$-bigraph with property $\mathcal{P}'$. Since some of the cases were analyzed in Lemmas 2 and 3, we consider cases when one of the d_X and d_Y is 0, and the other is not in $\omega \setminus \{0\}$.

Case 1° Either X or Y has exactly one isolated vertex, say $x \in X$. We claim that $G - x$ (the bigraph obtained by removing x from G) has the stronger property $\mathcal{P}''$. To prove it, suppose $G - x$ is partitioned into two parts, thus inducing sub-bigraphs G_1 and G_2. If we add x as an isolated vertex to G_1, we get a partition of G into $G_1 + x$ and G_2. One of the parts must be isomorphic to G. Suppose it is G_2. Then, if we split G into G_1 and $G_2 + x$, G_1 must also be isomorphic to G. Hence each of the bigraphs G_1 and G_2 has exactly one isolated vertex, call them x_1 and x_2. But now, if we split G into $G_1 - x_1$ and $G_2 + x + x_1$, none of the two induced parts is isomorphic to G, a contradiction. So $G_1 + x$ must be isomorphic to G, and it follows that G_1 is isomorphic to $G - x$. In an analogous way we prove that G_2 is also isomorphic to $G - x$.

By Lemma 4 $G - x$ must be either empty or complete. The first possibility would mean that there were more than one isolated vertices in X, a contradiction. So G is isomorphic to $\bar{S}^*$.

If, instead of X, the right side Y has exactly one isolated vertex, we get $\bar{S}$ as a solution.

Case 2° There are at least two (but finitely many) isolated vertices in either X or Y; this is analogous to Case 2° of Lemma 3.

Case 3° There are infinitely many isolated vertices in both X and Y. Then we can split G into two empty sub-bigraphs. None of the parts is isomorphic to G, except if G itself is empty.

Case 4° There are infinitely many isolated vertices in X, and Y contains only vertices of infinite degree (or the other way around). Let $\langle x_k : k < \omega \rangle$ be an enumeration of the vertices from the left side. Again we will construct, by recursion on k, a partition that will be a counterexample for $\mathcal{P}'$. To begin with, let the sets X_0, X_1, Y_0 and Y_1 be empty. If x_k has "unused" neighbors from Y (not yet placed into Y_0 or Y_1), put x_k into X_1 and put one of its neighbors into Y_1; otherwise just place x_k in X_0. When we're finished, we add the rest of the vertices from Y to Y_0. Hence we obtained a partition of G into two parts, one of them containing no edges and the other without isolated vertices, which is a contradiction.

The corresponding cases for $\bar{G}$ (when $\bar{G}$ has isolated vertices) are discussed as above, so we also get the complete $(\aleph_0, \aleph_0)$-bigraph. □

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