ACTA PHYSICA

ACADEMIAE SCIENTIARUM HUNGARICAE

TOMUS XXVI FASCICULI 1—2

AKADÉMIAI KIADÓ, BUDAPEST
1969

ACTA PHYS. HUNG.

ACTA PHYSICA

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ACTA PHYSICA

ACADEMIAE SCIENTIARUM HUNGARICAE

REDIGIT
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TOMUS XXVI

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1969

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RECENSIONES

PROCEEDINGS

of the

SYMPOSIUM ON HADRON SPECTROSCOPY

Keszthely, Hungary
6—11 September, 1968

The final manuscript was put
into production on 30th October
1968.

HADRON SPECTROSCOPY

In the jungle of particles and resonances the great break-through started with the discovery of the Eightfold Way. The zoological systematization of hadrons was made possible with the help of the compact groups SU(3) and SU(6). This first success made the theoreticians conceited. We started thinking in terms of quarks, and in our mind we followed the classical trail of atomic physics: we wanted to understand the hadrons as molecule-like composite systems. (See e.g. the Proceedings of our last Balaton meeting, Balatonvilágos, 1966, Acta Physica Hungarica Vol. 22). But the hopes turned out to be premature. The hadronic jungle is more entangled.

Real progress was made possible only by more sophisticated methods, which grew out of the compact description. The union of the SU(6) with the Lorentz group led us to non-compact algebras. The interplay of the analytic Regge description and the Lorentz symmetry resulted in a conspiracy of poles; thus, we arrived also from another direction at the infinite representations of non-compact algebras. Trying to build up a relativistic quark theory, expressing the higher symmetries in terms of local currents we got a coupled system of current commutators. This current algebraic approach enabled us to understand quite a number of relations among the different measurable properties of hadronic matter. Thus, we have learned that it is not possible to understand a single resonance or a single decay mode in itself, but only the complex system of all the hadronic energy levels, the correlated transitions among them, offer us an insight into the fundamental nature of hadronic matter.

The increasing importance of this *hadron spectroscopy* suggested to us the organization of a symposium on this subject. In the series of Balaton meetings of the Hungarian Physical Society this conference was held in the small town of Keszthely, at the west end of the Lake, about 150 km from Vienna. The Symposium on Hadron Spectroscopy joined with the International Conference on High Energy Physics in Vienna. This provided an excellent opportunity for theoreticians coming from overseas to meet young physicists from Europe. Most of the review talks and short reports are included in this volume of the Acta Physica Hungarica.

The Organizing Committee is highly indebted to all the speakers for the rapid preparation of manuscripts, to the Hungarian Physical Society and to the Hungarian Academy of Sciences for their financial support, and to the Editor-in-Chief of the Acta Physica for publishing these Proceedings.

G. MARX

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 5—19 (1969)

ANALYTICITY, REGGE POLES AND CONSPIRACY

IMPLICATIONS OF THE O(4.2) MODEL OF STRONG INTERACTIONS FOR REGGE POLE AND HIGH-ENERGY PHENOMENOLOGY

By

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Linear Regge trajectories, secondary trajectories, form factors and their relation to high-energy hadron scattering are consequences of the O(4, 2) model of strong interactions. The O(4), O(2, 1) and the two distinct SL(2, C) classifications of trajectories used at present, as well as the exchange degenerate trajectories, are all given by the subalgebras of the O(4, 2).

I. Introduction

The purpose of this paper is to present the predictions of a model of strong interactions [1] in terms of Regge poles and high-energy phenomenology. The model is a "soluble" relativistic theory of a composite system in the sense that many properties of the system can be completely and exactly evaluated. The results are in remarkable agreement with experiment. However, because the theory has been formulated in a somewhat abstract language of noncompact groups, it is important to show its equivalence to other approaches to strong interactions. In many instances the results obtained go much beyond those of other approaches so that one can make predictions concerning Regge poles and high-energy phenomenology. The salient features of the theory are: (1) systematic consideration of higher spin resonances, analytic expressions as a function of spin J, mass m, and the introduction of the new principal quantum number n, (2) description of both the high energy and the low energy aspects of form factors and scattering phenomena, and (3) a definite prescription to take mass differences between the members of internal symmetry groups into account, that is, a description of broken symmetry.

In Section II I briefly review the basic hypotheses underlying the calculations and explain the factors and ingredients which make the complete relativistic calculations possible. Then, in Section III I discuss the results in the language of high-energy phenomenology and compare them with other results and experiments. Finally, in the last section, a physical interpretation of the mathematical model is presented together with conclusions and the outstanding problems.

* On leave of absence from the University of Colorado, Boulder, Colo., USA.

II. Basic hypotheses

There are two sets of basic hypotheses:

(A) The rest frame states of a composite system belong to the Hilbert spcace of an irreducible representation of a dynamical group G that contains the homogeneous Lorentz group SO(3.1). The physical rest states are, in general, some suitable combinations of the basic states labelled by quantum numbers. The states with momentum p_μ are obtained by pure Lorentz transformations ("boosts") from the rest states.

(B) The current operator for a vertex of the system with external scalar, vector, tensor, ... interactions is made up of the generators of the group G and of the momenta p_μ and p'_μ (Fig. 1) and acts on the physical states defined in (A). The matrix elements of the scalar, vector, ... currents are directly related to observables (see Section II).

These assumptions are quite general and allow families of possible theories. I immediately give the simplest, realistic, specific model that is used in the following.

(A') The bases for rest frame states are labelled by $|\, njm, \pi, II_3Y)$, where π is the parity of the state, n the new principal quantum number, (jm) spin and (II_3Y) the internal quantum numbers. The range of n is 1, 2, ... for bosons and 3/2, 5/2, 7/2, ... for fermions. For each n, j ranges from 0 to $(n-1)$ of from 1/2 to $(n-1)$, respectively. For fixed internal quantum numbers the states $|\, njm)$ form an irreducible representation of the group SO(4.1) and also SO(4.2) extended by parity (doubling).*

Let J and M denote the generators of the Lorentz subgroup, i.e., the angular momentum in the rest frame and generators of pure Lorentz transformations, respectively. The additional generators in O(4.2) are the following: A three-vector A (which together with J, generates the compact O(4) subgroup), a four-vector Γ_μ, a Lorentz scalar operator S and a rotational scalar operator T; altogether 15 generators.

The following representations of O(4.2) are of interest:

(1) An irreducible unitary boson representation (with or without parity doubling), D^B.

(2) A unitary irreducible fermion representation (with parity doubling), D^F.

(3) A non-unitary representation combining (1) and the four-dimensional Dirac representation, $D^B \otimes D^{\text{Dirac}}$.

For fermions, another doubling will be introduced to describe the antiparticles.

* We are dealing here with the simplest (so-called most degenerate) representation of O(4, 2): of the three Casimir operators, two vanish so that the quantum numbers (njm) are sufficient to label the states. In general one needs 3 Casimir operators and 6 quantum numbers.

Thus, when I speak of the O(4.2) algebra for short, I mean to include also the extensions involving parity and antiparticles.

In general, it turns out that the physical states are not the basis states $|\, njm \ldots)$, but the so-called "tilted" states defined by

$$|\overline{njm}) = e^{i\theta_n T}\,|\,njm)\,. \tag{2.1}$$

It is convenient to introduce these new states, because then the current operator j_μ (or the interaction operator in general) simply transforms as a group generator.

We define the spinorial wave functions of momentum $p_\mu = (mch\xi, m\hat{\xi}sh\xi)$ for our system by pure Lorentz transformations

$$|\,\bar{n}jm, p) \equiv e^{i\vec{\xi}\cdot\vec{M}}\,|\,\bar{n}jm) \tag{2.2}$$

and denote them with a round ket, whereas the complete state vector is given by

$$|\,\bar{n}jm\,;\ \vec{p}\rangle = |\,\bar{n}jm\,;\ \vec{p})\ e^{ipx} \tag{2.3}$$

with the usual invariant normalization*

$$\langle \bar{n}'\, j'\, m'\,;\ \vec{p}'\,|\,\bar{n}jm\,;\ p\rangle = (2\pi)^3\,\delta(\vec{p}' - \vec{p})\,\frac{P_0}{m}\,\delta_{n'n}\,\delta_{j'j}\,\delta_{m'm}\,. \tag{2.4}$$

(B') The second assumption is that the quantum mechanical conserved probability current operator for the system evaluated between the wave functions (2.3) is the most general vector operator, linear in the generators of the rest frame group O(4.2) and linear in the momenta of the external lines:

$$j_\mu = \alpha_1\,\Gamma_\mu + \alpha_2\,P_\mu + \alpha_3\,P_\mu\,S + i\alpha_4\,L_{\mu\nu}\,q^\nu\,. \tag{2. 5}$$

Here α_i are *tensor operators* with respect to the internal symmetry group. I believe that a current operator of this kind describes the properties of the system after all the relevant diagrams have been summed up. It is, in field

* The place of the Poincaré group is as follows: The $L_{\mu\nu}$-part of the O(4, 2) algebra (i.e., $\vec{J}$ and $\vec{M}$ defined above) refer to the rest frame. The generators of the Poincaré group are P_μ and $J_{\mu\nu}=L_{\mu\nu} + O_{\mu\nu}$, where the orbital part is $O_{\mu\nu} = i(p_\mu\,\partial/\partial p_\nu - p_\nu\,\partial/\partial p_\mu)$. The states $|njm\,;\vec{p}>$ are then the Poincaré states, but belonging to one of its reducible representations The operators P_μ are defined only implicitly by eqs. (2.2) and (2.3). The McGlinn-O'Raifeartaigh type of impossibility theorems (see e.g., F. J. Dyson (Edit.) Symmetry groups, W. A. Benjamin, N. Y., 1967) are not relevant, because it is not required here that the commutation relations of P_μ with O(4, 2) close to a bigger Lie algebra. Yet a discrete mass spectrum will be obtained[2].

theory language, the effective current operator for the "dressed" particle,* and it is quite striking that the properties of these final physical "dressed" particles can be described in such a simple form.

The framework of non-compact groups predicts an infinite number of states. At the time of the introduction of these concepts, in 1964, this was unusual and startling, because one does not want to introduce an infinite number of particles or fields into a theory. However, if the nucleons are composite, then it is natural to do so. In fact, the infinite number of states for nucleons becomes more and more accepted; also from the point of view of Regge phenomenology or quark model, or current algebra, one arrives at a description with an infinite number of nucleon states of given internal quantum numbers.

The form of the current has been abstracted from the theory of composite particles, in particular the H atom. The structure of the system is embodied in the use of higher representation of O(4.2) (in contrast to the four-dimensional representation of O(4.2) describing a Dirac particle) and also by the presence of the convective terms.

Why the group O(4.2)?

Without going into detailed historical developments [1] I can answer this question as follows:

The smallest rest frame group containing the Lorentz group is the Lorentz group itself, SO(3.1). However, the spectrum, decay rates and form factors predicted by this model do not agree with experiment. There are, in fact, more than one state of the same spin, parity and internal quantum numbers (I, I_3 Y), therefore one needs a *new* quantum number. This has led one to choose SO(4.2) which so far agrees with all the available data. Actually the complete *rest* frame group that is used is SO(4.2) $\otimes$ SU(3). It will be indicated that this type of group is at present preferable to higher groups such as SU(6.6), involving a more intimate linking of SU(3) with the space time group O(4.2). Let me also remark that the group SO(4.2) also occurs in Dirac theory and in H-atom whose properties show a remarkable resemblance to those of nucleons [1].

Why is the theory capable of making complete calculations?

I want to make four remarks concerning this:

(i) The relativistic wave functions of the type (2.3) carry information about the structure of the composite system, i.e., about all its excited states.

* In other words the theory discussed here would correspond to *non-perturbative solutions* of some complicated field equations.

It would be possible to consider each excited state $| n >$ as an elementary particle and then use either field theoretical perturbation theory or analyticity and unitarity of the S-matrix theory, but these methods constitute poor substitutes to the use of wave functions for the system as a whole [1].

(ii) From an algebraic point of view, the formalism specifies completely the *one-particle states* as well as *the algebra of current operators.* Moreover, the matrix elements of all group operators are determined. Thus, one obtains not only sum rules, but complete individual matrix elements of currents. It is important to stress that the algebra which gives particle multiplets is quite distinct from the algebra of currents or the algebra of invariance of the S-matrix. The fact that for the isospin SU(2) group the multiplets are assigned to irreducible representations of SU(2) and the S-matrix is invariant under the same SU(2), leads to the belief that this would always be so. I have tried for many years to emphasise in the examples of composite systems that this is not so [3]. In the present model, all the particles are assigned to a representation of O(4.1) subgroup with O(4) multiplets, but the current is, for example, a four-vector Γ_μ outside the O(4.1) subgroup.

(iii) The internal symmetry group SU(3) is assumed to hold for the rest frame states. The states boosted to their actual momentum values automatically break this symmetry because of the mass differences. But there is a definite prescription to take mass differences into account in applying SU(3) to vertices and to reactions.

(iv) The current operator (2.5) seems to enter in a universal way into strong, weak and electromagnetic interactions of the hadrons.

III. Discussion of results

1. *Mass spectrum, Regge trajectories and secondary trajectories*

The requirement that the current j_μ in (2.5) is conserved between the wave functions (2.3)* (Fig. 1)

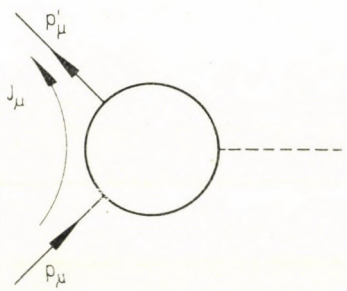

Fig. 1. Vertex kinematics

* Let $|n)$ denote the collection of all quantum numbers $|njm\pi II_3 Y \ldots)$ unless explicitly stated.

$$(\bar{n}',p'|\, j_\mu (p'-p)^\mu \,|\bar{n},p) = 0, \tag{3.1}$$

implies a mass spectrum. This can easily be seen if one rewrites (3.1) in the form

$$m_{n'}(\bar{n}'\,|\,j_0\, e^{i\vec{\xi}\cdot\vec{M}}\,|\,\bar{n}) = m_n(\bar{n}'\,|e^{i\xi\cdot M} j_0\,|\,\bar{n}), \tag{3.2}$$

nserts j_0 and evaluates $j_0 \mid \bar{n})$ or $(\bar{n}' \mid j_0$, respectively.

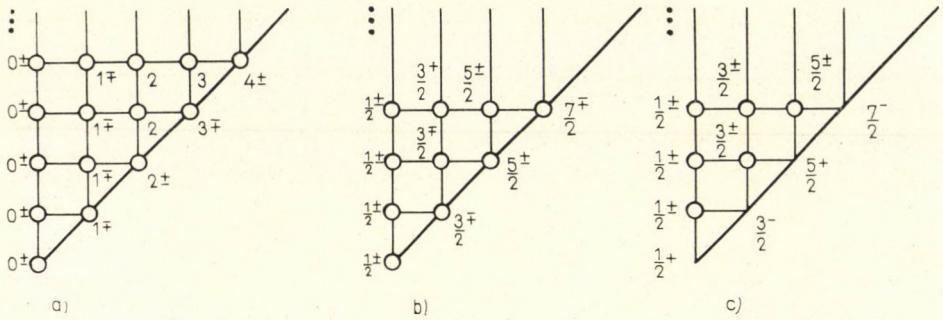

Fig. 2. Range of the quantum numbers n and j shown in weight diagram of the O(4.2) representations. Circles are the doubled states.
a) Boson tower D^B (unitary representation). *b)* Fermion tower D^F (unitary representation). *c)* Boson tower combined with Dirac matrices (non-unitary representation $n > 0$)

For the current (2.5) the result is [4]

$$\mathrm{M}_\eta^2 = \left[2\left(\alpha_3^2 + \frac{\alpha_2^2}{n^2}\right)\right]^{-1}\left\{\alpha_1^2 + 2\beta\alpha_3 + \frac{2\gamma\alpha_2}{n^2} + \right.$$

$$\left. + \left[\left(\alpha_1^2 + 2\beta\alpha_3 + \frac{2\gamma\alpha_2}{n^2}\right)^2 - 4\left(\beta^2 + \frac{\gamma^2}{n^2}\right)\left(\alpha_3^2 + \frac{\alpha_2^2}{n^2}\right)\right]^{1/2}\right\}, \tag{3.3}$$

where β and γ are two new constants.* The requirement (3.1) also determines the tilting angles θ_n in eq. (2.1) in terms of the constants α_i, β and γ. Because there are no terms in j_μ, for simplicity, depending on spin, the mass spectrum (3.3) depends only on the principal quantum number n. "Spin orbit coupling" terms (i.e., j-dependence for a given n) have been neglected. Fig. 2 shows the predicted range of quantum numbers and Fig. 3 the trajectory in the n-plane. Because for each n, $j = 0(1/2), \ldots, (n-1)$, we obtain, in the j-plane, the set of trajectories as shown in Fig. 4. [5].

* There is also the normalization condition for the wave functions

$$(\bar{n}|j_0|\bar{n}) = 1, \text{ for all } n. \tag{3.4}$$

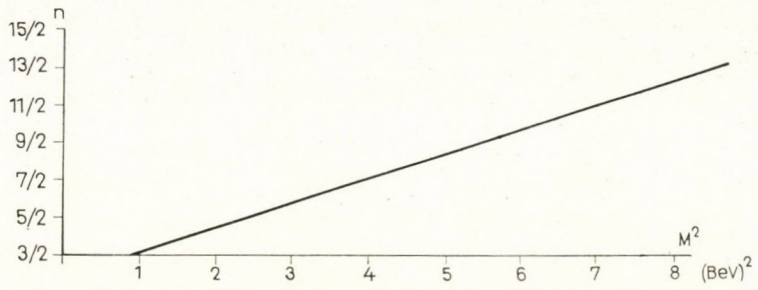

Fig. 3. Calculated trajectory in the n-plane (for proton tower)

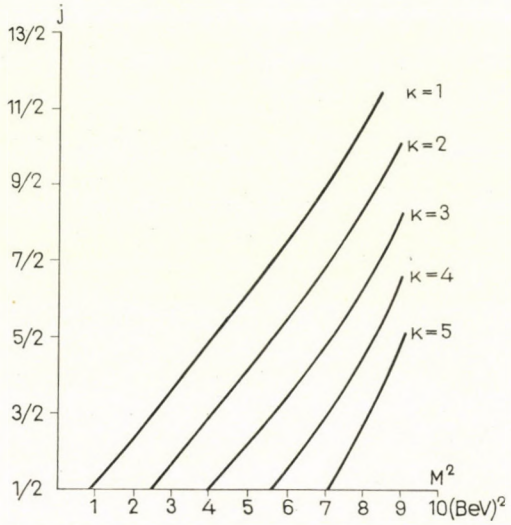

Fig. 4. The trajectories in the j-plane derived from Fig. 4 (spin-orbit terms neglected)

Conclusions and predictions

(1) Approximately linear trajectories** up to about 3 $(\text{BeV})^2$, then possible saturation (i.e., turning *upward* of the trajectory).

(2) Parallel "daughter" or secondary trajectories.***

(3) Parity doubling (depending on the representation shown in Fig.2, (a), (b), (c)).

(4) The classification of families of Regge trajectories of DOMOKOS and SURÁNYI et al. [7], according to the representations of SL(2, C), corresponds

** Actually each trajectory in O(4. 2) contains *two* Regge trajectories of opposite signature (e.g., N_α and N_γ, or ϱ and A_2: "*exchange degeneracy*").

*** The "daughter trajectories" have been derived in a specific way from the singularity structure of the amplitude [6] and refer to a *subset* of our secondary trajectories.

to a subalgebra of the enveloping algebra, whose weight diagrams are lines (in Fig. 4) parallel to the diagonal. Each such line represents an irreducible representation of the Lorentz group SL(2, C) and is characterized by the lowest spin

$$j_0 = \frac{1}{2} \text{(or 0, for boson case)} \tag{3.5}$$

and by

$$n - j = k, \; k = 1, \; 2, \; 3, \; \ldots, \tag{3.6}$$

that is $(n - j)$ is constant along each line.

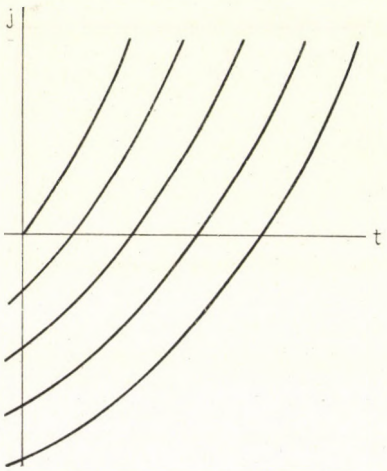

Fig. 5. Analytic continuation of O(4) states [vertical lines in Fig. 4] into O(3.1) states by the extrapolation of the trajectories. Note the particular O(3.1) tower at $M^2 = t = 0$.

(5) Two other subgroups on the O(4.2) weight diagram (Fig. 2) are of interest:

(a) States with the same spin j, but increasing n (vertical lines in Fig. 2), form representations of the group O(2.1) (degeneracy group of spin [3]). In the trajectory picture of Fig. 4, these are given by the horizontal lines.

(b) States with constant n ($j = 1/2$ to $n - 1$) form representations of the algebra of O(4); these representation spaces are given by horizontal lines in Fig. 2 and by vertical lines in Fig. 4. In the simplest most degenerate representations that we use, the second Casimir operator of O(4) vanishes. The analytic continuation of this O(4), i.e., the extrapolation of the trajectories in Fig. 4 to $t = 0$ as shown in Fig. 5, gives the O(3.1) or SL(2, C) classification of trajectories [8]. Again, in our case, the second Casimir operator, or the so-called M-quantum number [8] vanishes. $M \neq 0$ would imply an entirely new representation of O(4.2) and the form factors would depend on M.

(6) The existence of convective terms in the effective current is essential in obtaining *raising trajectories*. With these terms present, the composite system (or the "dressed" particle) has properties distinct from a local behaviour concerning *position operators* and zitterbewegung [9].

2. *Decay rates of higher spin states*

For the extensive numerical results I refer to the published literature [10]. The important point to stress here, is that the decay amplitudes of the form

$$A = (n'j'm', \ldots; p' \mid \mathscr{J} \mid njm, \ldots p), \tag{3.7}$$

where $\mathscr{J}$ is the interaction (scalar, pseudoscalar, ... operator in the theory), is covariant and is an *analytic* function in the quantum numbers n, spin j, ... so that the decay rates

$$\Gamma \sim \mid A \mid^2 \varrho, \quad \varrho = \textit{invariant} \text{ phase space} \tag{3.8}$$

are unambiguously calculated, without the use of barrier penetration factors.

3. *Magnetic moments and form factors*

The electromagnetic current for charged particles is assumed to be proportional to the matter current (2.5). The electromagnetic vertex is then proportional to

$$F_\mu = (n'p' \mid j_\mu \mid np), \tag{3.9}$$

from which we obtain the form factors. For the baryon ground states ($n = 3/2$, $j = 1/2$) we have, for example,

$$\begin{aligned} F_0 &= ch\ \xi/2\ G_E(t), \\ F_1 &= sh\ \xi/2\ G_M(t), \end{aligned} \tag{3.10}$$

($ch\ \xi = E/m$, when one of the particles is at rest),

where G_E and G_M are the electric and magnetic form factors of the proton. The current (2.5) gives [4]

$$G_M^P(t) = \frac{\mu}{(1 - ch^2\ \theta\ t/4m^2)^2},$$

$$G_E^P(t) = \frac{q}{(1 - ch^2\theta t/4m^2)^2} + \frac{b_1 t + b_2 t^2}{(1 - ch^2\theta t/4m^2)^3} . \tag{3.11}$$

Thus, the following conclusions can be made:

(i) The important gross double pole behaviour of the magnetic form factor is predicted. One has to expect small deviations from this behaviour and, indeed, there seem to be, empirically, because the spin orbit terms have not been included in the current (2.5), also, there must be the contribution of the cut in the unphysical time-like region of t.

(ii) Small deviations from the so-called scaling law $G_E(t)/q = G_M(t)/\mu$ are predicted, which now seem to be measured [11]. We would also have

$$G_E(t) \xrightarrow[t\to\infty]{} 1/t \quad \text{but} \quad G_M(t) \xrightarrow[t\to\infty]{} 1/t^2$$

so that both

$$F_1(t) \to 1/t^2 \text{ and } F_2(t) \to 1/t^2 .$$

(iii) The essence of the double pole behaviour comes from the composite structure as described by the O(4.2) representation. Various terms in the current (2.5) serve to fix the absolute values of the magnetic moment. A smaller group like O(3.1) gives a form factor of the type $(1 - at)^{-3/2}$ and larger groups like SU(4.4) of the type $(1 - at)^{-4}$, both disagreeing with experiment at present.

(iv) For the neutral tower, the electromagnetic current is not proportional to j_μ of eq. (2.5), but it must be a conserved term and not contributing to charge, like $i\alpha_4 L_{\mu\nu}q^\nu$, which gives

$$G_M(t) = \frac{-m\alpha_4}{(1 - at)^2}$$

and

$$G_E(t) = \frac{1}{(1 - at)^2}(-t/4m^2) - \frac{4\mu sh^2\theta}{(1 - at)^3}(-t/4m^2) ,$$

or a more general term of the form $i\alpha_5/2\ \varepsilon^{\mu\nu\lambda\varrho}L_{\nu\lambda}q_\varrho\pi$, where π is a pseudoscalar operator in O(4.2). Such a term gives [12]

$$G_M(t) = \frac{-m\alpha_5 sh\theta}{(1 - at)^2} \quad \text{and} \quad G_E(t) = (t/4m^2)\, G_M(t) .$$

(v) It is interesting that, in the case of the H atom, the singularity of the form factor as calculated in O(4.2) coincides exactly with that of the triangular diagram (Fig. 6) representing a virtual dissociation of the composite system. Thus, for our model, we can also interpret the form factor singularity as an "anomalous threshold" [1].

It has been pointed out recently, on the basis of the triangular diagram, that the asymptotic behaviour of the form factor goes as $1/t^2$ for target t [13]. This is in agreement with and a special aspect of our theory.

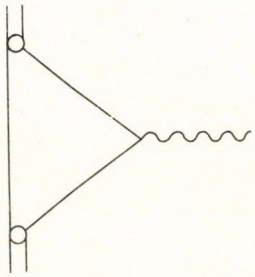

Fig. 6. Diagram whose singularity coincides with that of the form factor calculated group-theoretically

4. *Transition form factors*

Once the parameters of the theory (α_1, α_2, α_3, β, γ) are fixed by the ground state properties of the system* (ground state masses, magnetic moments and one point on the form factors), the transition vertex amplitudes for fixed internal quantum numbers (i.e., within each O(4.2) tower)

$$(n'j'm', p' \mid j_\mu \mid njm, p),\ j' \neq j,$$

are predicted and are again analytic functions of n, j and m. They have the correct kinematical threshold behaviour. Details have been given elsewhere [12].

5. *Weak form factors*

The hadron structure can be probed by electromagnetic, weak and strong interactions. Thus, the same basic hadron structure should reveal itself in all these processes. With the weak interactions, however, one probes transitions not accessible with the other two interactions, namely the transitions with $\Delta Q \neq 0$ and/or with $\Delta S \neq 0$. Thus, here, the SU(3) properties of the current (2.5), more precisely of the coefficients α_i, come into play. Consequently, according to our assumptions, the vector part of the weak interaction current gives

$$\sum_s (\bar{n} \mid F_\mu^s \mid \bar{n}'\, p) \otimes (\overline{II_3}\, Y \mid \alpha_s \mid \overline{I'\, I_3'}\, Y'),$$

* These parameters are functions of the masses of the constituents in specific composite models. For the moment they are not tied to a specific model, but are determined from experiment.

where we have written the current (2.5) as $j_\mu^{\text{weak}} = \sum_s \alpha_s F_\mu^s$. Here $|\bar{n})$ are the tilted O(4.2) states and $|II_3Y)$ are the tilted (by the *Cabibbo angle)* SU(3) states. We immediately obtain form factors in weak interactions. In fact we predict the same double pole behaviour for form factors as in eq. (3.11), with the same slope which seems to be observed in high-energy neutrino reactions. Applications to non-leptonic decays [14] and to K_{l3} form factors [15] have been recently discussed.

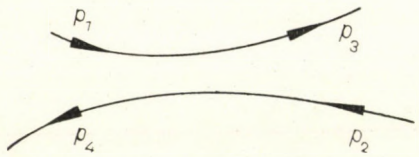

Fig. 7. Kinematics of the elastic scattering and the direct "hard sphere" interaction

6. *Diffraction scattering*

First I want to show that when the nucleons are represented by their relativistic wave functions, then a simple contact interaction (i.e., the bouncing off of the two nucleons from each other — "*hard sphere*" interaction*)* gives the essential features of the diffraction scattering (scattering without the exchange of quantum numbers). Let us first consider a single diagram of direct scattering shown in Fig. 7. Then the transition amplitude is given by

$$A = g(\bar{n}_3 p_3 \mid \mathcal{J} \mid \bar{n}_1 p_1)\ (\bar{n}_4 p_4 \mid \mathcal{J} \mid \bar{n}_2 p_2).$$

Again the important part is the use of wave functions rather than the detailed form of the interaction operator $\mathcal{J}$. Taking $\mathcal{J} = I$, the identity operator in O(4.2), and using the momenta

$$p_1 = m(ch\xi, 00, sh\xi),\ p_2 = m(ch\xi, 00 - sh\xi)$$
$$p_3 = m(ch\xi, \sin\theta\ sh\xi, 0, \cos\theta\ sh\xi)$$
$$p_4 = m(ch\xi, -\sin\theta\ sh\xi, 0, -\cos\theta\ sh\xi),$$

we obtain, for the ground states (e.g., $p - p$ scattering),

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol.}} = g_s^2 \frac{1}{s} \frac{(1 - t/4m^2)^2}{(1 - at)^8},$$

where $a = \cos h^2\theta/4m^2 = 1.4\ (\text{BeV/c})^{-2}$ is the same parameter as in the form

factors (3.11). With vector etc., interactions one gets more complicated terms in the numerator, but essentially the same $(1 - at)^8$ dependence in the denominator.

This simple result explains the characteristic (almost exponential) behaviour of the diffraction peak. It also explains, when applied to $p - p$ scattering [16], the apparent proportionality of $d\sigma/dt$ to $[G_M^P(t)]^4$, the fourth power of the electromagnetic form factor [17, 18]. The point is that both scalar (strong) and electromagnetic (vector) form factors contain the factor $(1 - at)^8$. It is also seen that it would be appropriate to introduce the notion of "*strong form factors*" which one can empirically isolate in diffraction scattering.

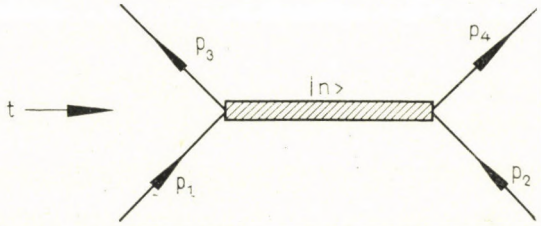

Fig. 8. Exchange of an O(4.2) tower

In addition to the direct scattering via a contact interaction, there are contributions of meson exchanges in the t-channel* (one may neglect the exchange of $N = 2$ states in the s and u-channels). Now the exchange of an infinite multiplet (Fig. 8) is equivalent to a Regge pole term, as can be seen from a Sommerfeld—Watson transformation [20], symbolically,

$$A = \sum_{nj} \frac{g_{nJ}^* D^j g_{nj}}{M_n^2 - t} \to \frac{g_\alpha^*(t)\, D^j(s,t)\, g_\alpha(t)}{\sin \pi\alpha(t)}\, e^{i\pi n},$$
$$\alpha(M_n^2) = n\,.$$

Thus, the composite model provides a theoretical basis for both of the empirical ingredients of the model given in [18] and generalizes this model to arbitrary resonances [21].

In general, the process of obtaining a Regge amplitude by the exchange of an infinite multiplet is precisely the opposite to the procedure of factorizing a Regge (or a Lorentz) pole contribution into vertex parts.

IV. Interpretation of the model and conclusions

From the calculations discussed in the previous section one can deduce a qualitative model for proton not unlike an atomic model [19].

* One may invoke here the analogy with the H—H scattering. There is the direct interaction when the two atoms bounce of from each other; then there is the electron exchange term.

(1) With respect to external interactions (up to the energies available at present) the proton behaves like a two-part system, such that one of the parts interacts locally with the external field. The form factors are in complete agreement with this interpretation.

(2) The excited states can be interpreted as those of a two-body system.

(3) From the slope of the mass spectrum as a function of the principal quantum number, n, dM/dn and from the slope of the form factor dG/dt, one can deduce operationally an analogue of Bohr radius for the proton of $5M_p^{-1}$ and an analogue of the fine structure constant of the order of 3 [19].

(4) During the collisions the "meson clouds" of the two protons can exchange (in addition to direct momentum transfer), which gives rise to exchange contributions of the Regge pole type.

(5) In meson—baryon scattering, the mesons can be captured in excited "Bohr-orbits" giving rise to direct channel resonances — as in atomic Compton effect. Thus, more structure is expected in $\pi - p$ scattering compared with $p - p$ scattering.

Finally, there is the question: Is there a local field theory of strong interactions along the lines discussed here? It is true that some interactions, e.g., the electromagnetic vertex, can be written as a local interaction using infinite-component wave equations. However, there are many differences as compared with the characteristics of the usual local field theories. I should like to take the point of view that, because the theory presented here describes a composite or rather a "dressed" particle (with all renormalization and radiation effects included), it need not be local, in fact, almost certainly it will not be completely, although the constituents (or the underlying "bare" fields) may be governed by a local field theory. The "non-locality" that one is talking about here, however, is of a different nature than the conventional non-local theories.

Acknowledgments

The author is grateful to Professors ABDUS SALAM and P. BUDINI and the IAEA for hospitality at the International Centre for Theoretical Physics, Trieste.

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ПРИМЕНЕНИЕ МОДЕЛИ O(4,2) СИЛЬНОГО ВЗАИМОДЕЙСТВИЯ ДЛЯ ПОЛЮСОВ РЕДЖЕ И ЯВЛЕНИЙ ВЫСОКОЙ ЭНЕРГИИ

А. О. БАРУТ

Р е з ю м е

Линейные траектории Редже, вторичные траектории, форм факторы и их отношение к рассеянию гадронов высокой энергии являются следствиями O(4,2) модели сильных взаимодействий O(4), O(2,1) и два различных классификаций SL (2,C) траекторий, применяемых в настоящее время, так и обменные выраженные траектории даются субалгеброй O(4,2).

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26(1—2) pp. 21—24 (1969)

ON THE INFINITE COMPONENT WAVE EQUATIONS

SOME PROBLEMS OF INTERPRETATION-OUTLINE OF AN APPLICATION

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The form factors obtained in the study of the infinite components wave functions are compared with the ones obtained in a much more conventional way through the use of the Schrödinger equation. The classical problem like the elastic scattering of photons by a hydrogen atom is treated in detail.

In the study of the infinite components wave functions the two first results one tries to get are the mass spectrum and the form factors. The existence and the structure of the form factors are in general considered as characteristic manifestations of the composite nature of the systems studied. Therefore it is of some interest to compare the form factors obtained in this way with the ones obtained in a much more conventional way, viz. through the use of the Schrödinger equation.

This comparison can be done in the study of a classical problem like the elastic scattering of photons by a hydrogen atom. In fact the hydrogen atom has been one of the first systems treated by means of the infinite component wave equations [1, 2, 3] and is therefore a good test for the formalism. Following [1] we can say that the introduction of the infinite component wave function is achieved by representing the relative variables $\mathbf{r}$, $\mathbf{q}$ by means of generators of O(4, 2)

$$r_i = (S_{i0} - S_{i4})/a, \qquad q_i = a(\Gamma_0 - \Gamma_4)^{-1}\Gamma_i,$$
$$r = (\Gamma_0 - \Gamma_4)/a, \qquad q^2 = a^2(\Gamma_0 - \Gamma_4)^{-1}(\Gamma_0 + \Gamma_4).$$

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In this way we can rewrite the Schrödinger equation of the relative motion

$$(q^2/2\mu - e^2/r)\varphi(\mathbf{q}) = E\varphi(\mathbf{q}) \tag{1}$$

in an algebraic form

$$(Q_b \Gamma_b + e^2 a)\psi(0) = 0, \qquad b = 0, \ldots, 4 \tag{2}$$

with the correspondence in the normalization

$$\int \varphi'^* \varphi d^3 q \to \Psi'^+ (\Gamma_0 - \Gamma_4) \Psi .$$

The five vector is

$$Q^b \equiv (E - a^2/2\mu,\ 0,\ E + a^2/2\mu); \qquad \mu = m_e m_p/(m_e + m_p).$$

Now, in the form factor we have the product of two wave functions, viz.

$$F_e(\mathbf{k}) = \int \varphi^*(\mathbf{q})\, \varphi(\mathbf{q} - c_2 \mathbf{k})\, d^3 q, \text{ where } c_2 = m_p/(m_e + m_p),$$

so that the form factor is the inner product between two solutions of Eq. (2), the second one with another five vector

$$Q'^b \equiv (E - a^2/2\mu + c_2^2 k^2/2\mu,\ -ac_2 \mathbf{k}/\mu,\ E + \partial^2/2\mu + c_2^2 k^2/2\mu).$$

In the case in which we have both particles charged the above considered form factor is not complete; we have also a term like

$$F_p(k) = \int \varphi^*(\mathbf{q})\, \varphi(\mathbf{q} + c_1 \mathbf{k})\, d^3 q, \text{ where } c_1 = m_e/(m_e + m_p).$$

Performing the same substitution we obtain in this case another expression for the five vector (simply substituting c_2 with $-c_1$. In this simple way we see that the form factor is not simply the matrix element of the boost connected with the total momentum transfer, because of the presence of the coefficients c_1, c_2. But in the actual case $m_p \gg m_e$, $c_2 \simeq 1$, then the final state in formally obtained by applying a boost on the relative wave function $\varphi(q)$. Since the problem is a non relativistic one the boosts are Galilei transformations. In conclusion the requirement that the form factors are completely given by the "external" Lorentz or Galilei transformation properties seems to be a particular statement on the constitution of the system, namely that the external interaction involves only one of the constituents, which is nearly the case of the electromagnetic interaction of the hydrogen, at low energy.

The scattering problem initially stated can be completely worked out within the formalism. The elastic photon scattering by hydrogen has been

recently studied by GAVRILA [4] which used, to sum over the intermediate states, the Schwinger form of the Green function for the Coulomb problem [5]. We simply here state the problems and describe briefly how they are solved; we have the vertex problem and the propagator problem. For the vertex we do not have any difficulty in taking the complete expression, i.e., with all the recoil effects taken into account, because this implies simply the calculation of the action of a Γ-matrix on the state, calculation which is in any case required, in fact the vertex

$$\frac{1}{m_e}\int \varphi^*(\mathbf{q})\,\varphi(\mathbf{q}-c_2\,\mathbf{k})\,[c_1\,\mathbf{p}-\mathbf{q}]\cdot\mathbf{e}(\mathbf{k})\,d^3q$$

becomes simply

$$\Psi^+(0)\,\eta^b\,\Gamma_b\,\Psi(-c_2\,\mathbf{k})$$

with η^b a numerical five vector.

We have now to sum over all intermediate states ν with the denominators* $E^T - E^T_\nu$. The intermediate states are all solutions of Eq. (2) with different Q, and since these solutions are in one-to-one correspondence with the solutions of the Schrödinger equation they include the continuum. The procedure is to sum over ν writing

$$\Sigma_\nu \frac{|\nu\rangle\langle\nu|}{E^T - E^T_\nu} = \frac{1}{E^T - H}$$

and then transform the denominator, multiplied by r again in a linear form in the Γ-matrices; like in the transformation from Eq. (1) to Eq. (2); at this point we see that, for sufficiently small photon energy, $Q^b\Gamma_b$ has only a discrete spectrum. The eigenstates of $Q^b\Gamma_b$ are not the eigenstates of the Hamiltonian, in fact we work at fixed Q while solutions of Eq. (2) exist only for particular Q vectors and to every different solution (principal quantum number) it corresponds a different five vector. We have a discrete sum of states which are not physical states, but which allow a simpler expression of completeness. The discrete sum can be carried out, at least for elastic scattering on the fundamental state and the complete amplitude is given in terms of hypergeometric functions (as in [4]).

We can remark that there is a strong connection between this procedure and the method of SCHWINGER. SCHWINGER's propagator was also obtained, through an analytic continuation, from a discrete sum and the sum was over a

* With E^T we mean the total non relativistic energy, including the centre-of-mass kinetic energy.

class of representations of the O(4). Here we sum over states which make diagonal $Q^b\Gamma_b$, and as long as Q is timelike these states build up representations of O(4) because this is the stability group of Q. Even here we can have to perform an analytic continuation. Once the sum over the discrete states is carried out we can perform an analytic continuation in the total energy to the situations in which Q would be space-like.

Note added in proof. More detailed calculations will appear in two separate papers by C. FRONSDAL and by G. BISIACCHI and G. CALUCCI, in the Physical Review I.

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О БЕСКОНЕЧНОМ КОМПОНЕНТЕ ВОЛНОВЫХ УРАВНЕНИЙ НЕКОТОРЫЕ ПРОБЛЕМЫ ИНТЕРПРЕТАЦИОННОГО ОЧЕРТАНИЯ ОДНОГО ПРИМЕНЕНИЯ

ДЖ. БИЗИАККИ, Г. КАЛУЧЧИ и К. ФРОНСДАЛ

Р е з ю м е

Форм факторы, полученные при изучении бесконечных компонентов волновых уравнений, сравниваются с таковыми, полученными более конвенциональным путем применения уравнения Шредингера. Классическая проблема упругого рассеяния фотонов на атоме водорода подробно истолкуется.

Acta Physica Academiae Scientiarum Hungaricae, Tomus, 26 (1—2), pp. 25—34 (1969)

GROUP THEORETICAL DESCRIPTION OF RELATIVISTIC SCATTERING NEAR ZERO MOMENTUM TRANSFER

By

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The behaviour of scattering amplitudes is investigated near zero momentum transfer. The contribution of Lorentz poles to high energy scattering is studied. The theoretical considerations are compared with experimental results on resonance data.

1. Introduction

Recent investigations of the structure of relativistic scattering [1—9] show that Lorentz invariance implies a set of non-trivial conditions on the contribution of Regge poles, which probably play an essential role in high energy scattering and low energy pole structure. In some cases (unequal mass scattering, scattering of particles with non-zero spin) unwanted singularities appear, if we follow the usual reggeization procedure. Using the general concepts of group theory and analyticity, expansions of the scattering amplitude have been given [5—8] which are free of kinematic singulartities and constraints at zero momentum transfer. The structure of symmetry-breaking was investigated [5—7] off the point $t = 0$, where t is defined by $t = (p_2 - p_3)^2 = (p_2 - p_4)^2$, t is the invariant momentum transfer in a two particle process with momenta p_1, p_2 and p_3, p_4 in the initial and final states, respectively. A knowledge of the structure of symmetry breaking allowed a mass formula to be constructed for daughter trajectories. At the end we shall describe some attempts [10, 11] to connect internal symmetries and current algebras with SL(2, C) the trajectory generating algebra (TGA) of scattering amplitudes at $t = 0$.

2. Equal mass scattering at t = 0

The simplest problem is to consider the scattering of equal mass non-zero spin particles at $t = 0$. The most elegant way of treating this problem is that of TOLLER's group theoretical approach [2], in which the scattering amplitude is regarded as a function over the little group belonging to the momentum transfer four-vector Q. Following from Lorentz invariance we can find a spe-

cial coordinate system in which the three-moneta p_1 and p_3 point in the z direction. Following from momentum conservation $p_1 - p_3 = p_4 - p_2 = Q$ is of the form $Q = (Q_0, 0, 0, Q_z)$. For the elements of the little group, H: $\mathbf{L}(h)Q = Q$ if $h \in H$, by definition.

For physically admissible p_2 and p_4 vectors it is always possible to find a $h \in H$, which satisfies $p_2 = \mathbf{L}(h)p_2$ and $p_4 = \mathbf{L}(h)p_4$, where p_2 and p_4 are of the form $p_2 = (p_{20}, 0, p, p_{2z})$ and $p_4 = (p_{40}, 0, 0, p_{4z})$. Then the scattering amplitude can be written as

$$T_{m_1 m_2 m_3 m_4}(p_1, L(h)\, p_2, p_3, L(h)\, p_4).$$

a function over the little group. It is essential to remark that while $t = (p_1 - p_3)^2$ is independent of h, $s = (p_1 + p_2)^2 = (p_1 + \mathbf{L}(h)p_2)^2$ is a function of h.

If $Q = 0$, the little group is SL(2, C). Under general conditions the amplitude can be expanded using representations of SL(2, C). Here we do not consider otherwise essential complication connected with reflections. The contribution of a Lorentz-pole to the scattering amplitude is:

$$\begin{aligned} T_{s\lambda s'\lambda'}(h) &= \sum (S\lambda\,|S_1\lambda_1 S_3 - \lambda_3)(S'\lambda'|\, S_4\lambda_4 S_2 - \lambda_2) \\ (-1)^{S_3-\lambda_2+S_2-\lambda_2}\; T_{\lambda_1\lambda_2\lambda_3\lambda_4}(h) &\approx \left[(\sigma+1)^2 - j_0^2\right]\gamma_{1s}\,\gamma_{2s'}\cdot \\ &\cdot \left[D^{\sigma j_0}_{s\lambda s'\lambda'}(h) + D^{\sigma - j_0}_{s\lambda s'\lambda'}(h)\right], \end{aligned} \tag{1}$$

where σ and j_0 label the representations of SL(2, C), γ_{1s} and $\gamma_{2s'}$ are the Lorentz-pole vertices and $D^{\sigma j_0}_{s\lambda s'\lambda'}(h)$ is the matrix element of Lorentz transformation h in the representation labelled by quantum numbers $\{\sigma, j_0\}$. The contribution of a Lorentz pole can be expanded in an infinite sum of Regge poles, which are displaced by unity in the l plane and have alternating signs.

3. Unequal mass scattering near t = 0

For unequal masses $Q \neq 0$ for $Q^2 = t = 0$, the little group, H_t at $t = 0$: $H_0 = E(2) \in \mathrm{SL}(2, C)$. Expanding the scattering amplitude e.g. at $t > 0$ in little group (SU(2)) representations and continuing in t, we arrive at a singularity at $t = 0$. This singularity has a twofold origin. At first, with the usual choices the momentum transfer, Q has a singularity at $t = 0$ (e.g. if $Q = L(h)\bar{Q}$, $\bar{Q} = (\sqrt{t}, 0, 0, 0)$). This singularity appears in the expansion functions as well. However, it is easy to build up a Q vector as has been pointed but by Domokos and Tindle [5] which has an analytic behaviour near $t = 0$, $Q = (\sqrt{t + p^2}, 0, 0, p)$. Then singularities may appear in the t plane, near $t = 0$ if the expansion functions have any. This happens at $t = 0$ as a result of a phenomenon called contraction. As can be shown, the subalgebra

of SL(2, C) commuting with Q is spanned by the operators S_+, S_- and S_0, satisfying commutation relations

$$[S_0, S_\pm] = \pm S_\pm,$$
$$[S_+, S_-] = 2(1 - v^2)S_0, \tag{2}$$

where $v = p/\sqrt{t + p^2}$. At $t = 0$ the structure constants of the algebra are essentially changed; we say that the algebra contracts. As a result, the expansions in representations of H_+ or H_- (the little groups for time or spacelike momentum transfer) will not be valid at $t = 0$. This fact is reflected in a singularity of the expansion functions at $t = 0$.

Domokos and Tindle [5] proposed a modified expansion of the scattering amplitude which avoids the difficulty connected with the singularity at $t = 0$. Their argument goes as follows: If we do not want to put a singularity into the amplitude, we have to expand it in terms of representations of a group, which does not contract at $t = 0$. This group must contain the subgroups H_+, H_- and H_0. They have found that the minimal group satisfying their general requirements is exactly SL(2, C). So all difficulties at $t = 0$ can be avoided by a conspiracy of little group representations giving a "representation" of SL(2, C).

Cosenza, Sciarrino and Toller [8] (CST) placed the above ideas on a slightly more rigorous basis. First they noticed that since we are dealing with an analyticity problem at $t = 0$, we have to regard complex values of t as well, which leads to the necessity for considering complex Lorentz transformations. In addition we have to extend the scattering amplitude as a function of the complex little group to the total complex Lorentz group, or more exactly to the covering group of it: SL(2, C) $\otimes$ SL(2, C). This extension is obviously not unique, but we hope to find one extension, which satisfies the physical conditions given by CST [8]. Then we expand T by means of representations of SL(2, C) $\otimes$ SL(2, C), which can be obtained from those of the real Lorentz group by appropriate continuation. The actual procedure is much more involved because the expanded function is not arbitrary, but satisfies covariance conditions under transformations of the so-called left and right complex covariance groups, defined by

$$K_L^C = \{ k \,|\, k \in SL(2, C) \otimes SL(2, C); L(k)\, p_1 = p_1, L(k)\, p_3 = p_3 \},$$
$$K_R^C = \{ k \,|\, k \in SL(2, C) \otimes SL(2, C), L(k)\, p_2 = p_2, L(k)\, p_4 = p_4 \}.$$

CST shows that kinematic singularities at thresholds and pseudo-thresholds appear because the covariance groups change their structure at these points.

At the end CST obtain idealized solutions of the conspiracy problem, with parallel daughter trajectories and factorized residues. The contribution

of a Lorentz-pole satisfies the kinematic constraints and has no singularity at $t = 0$. The Lorentz-pole contribution can be decomposed into Regge-pole contributions. The behaviour of Regge vertex functions for unequal masses at $t = 0$ is given by

$$\beta^{\sigma, j_0, \varkappa}_{\lambda, \lambda_3} \sim t^{\frac{1}{2}[-\sigma(t)+1+|j_0-|\lambda_1-\lambda_3||]}. \tag{3}$$

This behaviour is the same as obtained using an O(4) expansion of the amplitude, by decomposing the representation function of the Lorentz group into Regge-pole contributions:

$$D^{\sigma j_0}_{s\lambda s'\lambda'}(h\,h'^{-1}) = \sum_{\varkappa=0}^{\infty} d^{\sigma j_0}_{s,\lambda,\sigma-\varkappa}(\alpha)\, D^{\sigma-\varkappa}_{\lambda\lambda'}(\vartheta,\varphi)\, d^{\sigma j_0}_{s,\lambda,\sigma-\varkappa}(\alpha'), \tag{4}$$

where $d^{\sigma j_0}_{s\lambda\sigma-\varkappa}(\alpha)$ is the matrix element of the boost operator along the z axis with velocity. $ch\alpha$, α and α' are given by

$$ch\alpha\, W \sqrt{\frac{m_1^2+m_3^2}{2} - \frac{t}{4}} = m_1^2 - m_3^2,$$

$$ch\alpha'\, W \sqrt{\frac{m_2^2+m_4^2}{2} - \frac{t}{4}} = m_4^2 - m_2^2.$$

$D^{\sigma-\varkappa}_{\lambda\lambda'}(\vartheta,\varphi)$ is the usual rotation matrix continued to complex values of angular momentum. The vertex functions are proportional to the functions $d^{\sigma j_0}_{s\lambda\sigma-\varkappa}(\alpha)$, which have exactly the behaviour indicated in Eq.(3), if the masses are different.

4. Breaking SL(2, C) symmetry

A general off-mass shell approach [4, 6] and model calculations[12], show that the SL(2, C) symmetry is broken at $t \neq 0$. The nature of deviations from the symmetry limit can be understood very simply if we consider an off-shell amplitude obtained, e.g. by the usual introduction of interpolating fields in field theory. The scattering amplitude $T(p_1, p_2; p_3, p_4) \equiv T(Q, p, p')$, where $p = p_1 + p_3/2$, $p' = p_2 + p_4/2$, can be regarded as a two variable function over the SL(2, C) group (or SL(2, C) $\otimes$ SL(2, C), respectively

$$T(Q, p, p') = T(Q, L(h)\bar{p}, L(h)\bar{p}').$$

This amplitude is assumed to be a regular function of Q at the point $Q = 0$. In addition we assume the possibility of extending transformations

h and h' to the complex Lorentz group and so, e.g. to the O(4) region of p and p' vectors.

The scattering amplitude as a function of Q, h and h' satisfies the covariance condition $T(Q, h, h') = T(L(k)Q, kh, kh')$, where $k \in \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ (the compact counterpart of SL(2, C)).

If we expand T in power series of Q we obtain

$$T(Q, h, h') = T(0, h, h') + Q_\mu \frac{\partial}{\partial Q_\mu} T(0, h, h') + \ldots . \tag{5}$$

The covariance condition shows that $T(0, h, h')$ is a scalar operator, while $\partial/\partial Q_\mu T(0, h, h')$ is a tensor operator of representation $\{\sigma, j_0\} = \{1, 0\}$. The expansion of the first term of Eq. (5) is similar to Eq.(1) writing hh'^{-1} into the argument of the D function. The expansion of the following terms leads to a double series of representations of SU(2) $\otimes$ SU(2)

$$\frac{\partial T_{s\lambda s'\lambda'}(0, h\bar{p}, h'\bar{p}')}{\partial Q_0} = \sum_{\sigma,\sigma',j_0,j_0',j,m} \bar{T}^{\sigma,\sigma',j_0,j_0'}_{j,ss'} D^{\sigma j_0}_{s\lambda jm}(h) D^{\sigma' j_0'}_{s'\lambda' jm}(h') . \tag{6}$$

As $\partial T/\partial Q_0$ is a tensor operator of representation $\{1, 0\}$, T is proportional to the CGC of the Lorentz group

$$\bar{T}^{\sigma,\sigma',j_0,j_0'}_{j,ss'} \sim \langle \{\sigma, j_0\}, j, m \,|\, \{1, 0\}\, 0, 0; \{\sigma', j_0'\}, j, m \rangle \bar{\bar{T}}^{\sigma,\sigma',j_0,j_0'}_{ss'} . \tag{7}$$

The j dependence of $\bar{T}$ is factored out in the CGC.

Similarly, higher order derivatives of T transform according to reducible representations of SU(2) $\otimes$ SU(2), symbolically

$$\left(\frac{\partial}{\partial Q_k}\right)^k \sim \{k, 0\} \otimes \{k-2, 0\} \otimes \ldots .$$

Eq.(6) is essentially a simple expansion because of selection rules following from Eq.(7). After a continuation in σ we separate the contribution of a Lorentz pole, which will have the form

$$\frac{\partial T(0, h\bar{p}, h'\bar{p}')}{\partial Q_0} = \sum_{\sigma,\sigma',j_0,j_0',\varkappa} \bar{T}^{\sigma',\sigma'',j_0,j_0'}_{\sigma-\varkappa,ss'} d^{\sigma' j_0}_{s\lambda\sigma-\varkappa}(h)\, D^{\sigma-\varkappa}_{\lambda\lambda'}(\vartheta, \varphi)\, d^{\sigma'' j_0'}_{s'\lambda'\sigma-\varkappa}(h') . \tag{8}$$

The index $\varkappa$ enumerates the daughter trajectories.

The derivatives of the amplitude can be connected easily with the Regge pole parameters thus obtaining the $\varkappa$ dependence of Regge trajectories

and residues. After a tedious calculation we obtain for the Regge trajectories the four (or three) parameter "mass formula" in $W^2 = t$ order

$$\alpha_\varkappa(W) = \alpha_0 - \varkappa + \left(\varkappa + \frac{1}{2}\right) W\beta + \\ + \left[\left(\varkappa + \frac{1}{2}\right)^2 \gamma + \left(\varkappa + \frac{1}{2}\right)\beta^2 + \delta\right] W^2 + O(W^3), \qquad (9)$$

where

$$\beta = 0, \quad \text{if} \quad |j_0| \neq \frac{1}{2}.$$

Similarly, we obtain in the equal mass case a simple decomposition of the derivatives of the vertex functions in terms of a finite number of different $d^{\sigma' j_0'}_{s\lambda\sigma-\varkappa}$ functions, with coefficients of a simple $\varkappa$ dependence.

Eq.(9) was compared with data on resonances [13]. It seems that most baryon resonances can be put into one (or two for Σ and Ξ particles) family of Regge poles. The most interesting feature of the fit is the identification of the B_γ trajectories

$$\left(J = \frac{3}{2} = 2n\,, \; P = (-1)^{\frac{1}{2}+J}\right)$$

as the first daughter of B_α trajectories

$$\left(J = \frac{1}{2} + 2n, \; P = (-1)^{\frac{1}{2}+J}\right).$$

The fits for $I = 1/2$, $Y = 1$ and $I = 3/2$, $Y = 1$ families are shown in Figs. 1 and 2. In the symmetry limit (which is a highly idealized situation due to unequal masses) the ratio of coupling constants of different resonances at a given σ can be expressed by the ratio of $d^{\sigma j_0}_{s\lambda\sigma-\varkappa}(\alpha)$ functions. Using the identifications of the above described fit the elastic widths of $I = 1/2$, $Y = 1$, $\sigma = 5/2$ resonances were calculated [14]. Here we present the results in the form of a Table. The absolute scale of coupling constants was the only free parameter fitted.

Γ elastic	P_{11} (1470)	D_{13} (1520)	F_{15} (1690)	S_{11} (1550)	P_{13} (?)	D_{15} (1580)
Γ_{exp} (MeV)	68	76	85	39		68
Γ_{theor} (MeV)	81	105	82	156	105	81

If the second daughter of D_{15} (1580) is S_{11} (1770), which cannot be excluded, then the fit is much better, since then $\Gamma_{\text{exp}} = 240$ MeV/c and $\Gamma_{\text{theor}} = 180$ MeV/c for the S_{11} member of the family.

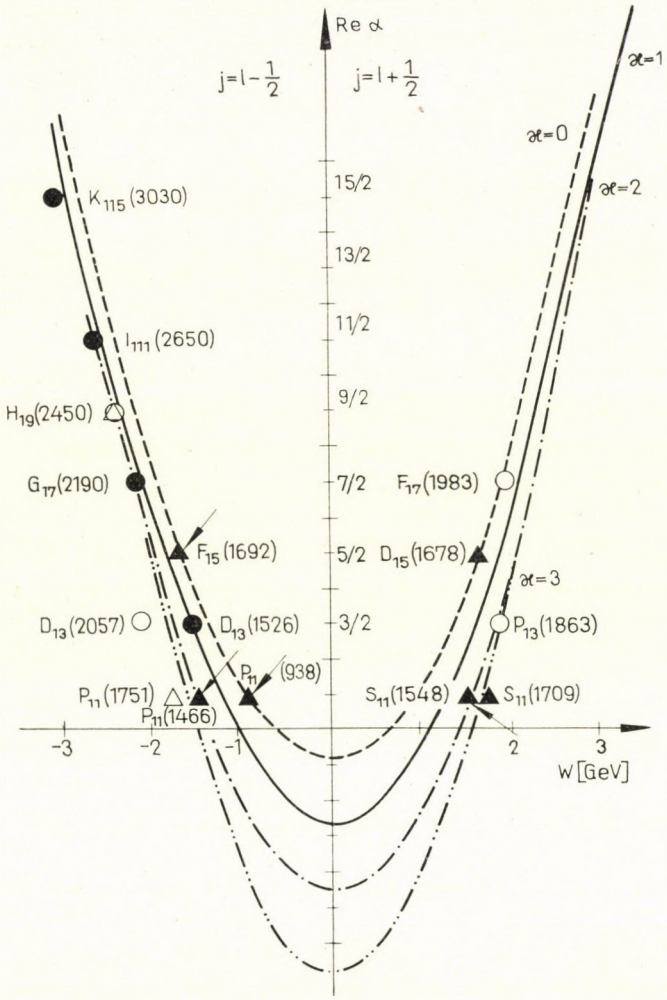

Fig. 1. Regge trajectories with the observed $I = 1/2$ nucleon resonances. Resonances of positive sign are marked by triangles, those of negative sign by circles. Well-established resonances are drawn with full triangles and circles. Points classified as "resonance interpretation in doubt" in the paper of LOVELACE[15] are distinguished by a question mark. Resonances indicated by arrows were used for the fit

5. Extension of SL(2, C) symmetry

If baryon resonances belong to families of Regge trajectories, it is comparatively easy to connect those resonances which belong to the same SU(3) multiplet through a trivial SL(2, C) $\otimes$ SU(3) extension of SL(2, C) and we can obtain mass formulas by breaking SU(3) at $t = 0$ and SL(2, C)

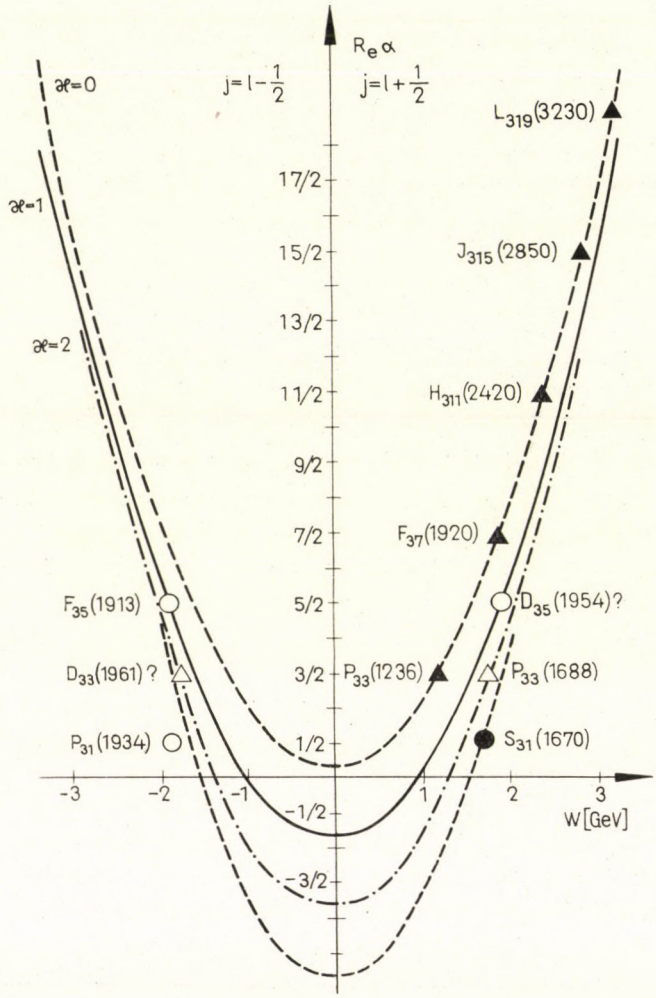

Fig. 2. Regge trajectories with the observed $I = 3/2$ resonances. The notation is the same as in Fig. 1

at $t \neq 0$. It is much harder to connect SL(2, C) with SU(6). If we believe that on the one hand some form of SU(6) is relevant in hadron spectroscopy and on the other that baryon resonances are members of Regge trajectory families we need some drastic extension of the TGA SL(2, C).

It has been proposed [10, 11] to decompose the angular momentum operator, $M_{\mu\nu}$ into orbital and spin parts $M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$.

The spin part of the angular momentum tensor can be connected with the internal symmetry group SU(3) to give a $SU(6)_L \otimes SU(6)_R$ or $SL(6, C)_S$ extension.

As DOMOKOS and KÖVESI—DOMOKOS show in their paper, the generators of this group can be connected with the vector and axial-vector current operators of the quark model. If the scattering operator is invariant under the transformation of the group $SL(2, C)_L \otimes SL(6, C)_S$, Lorentz poles are grouped in "tribes". The members of these tribes are obtained by decomposing $SL(6, C)_S$ into $SU(3) \otimes SL(2, C)_S$ and combining $SL(2, C)_S$ and $SL(2, C)_L$ into the usual SL(2, C) representations. E.g. if we accept the $(1,56) \otimes (56,1)$ representation for baryons and combine it with a $\{\sigma_0, 0\}$ representation fo $SL(2, C)_L$ we obtain six families of Regge trajectories, the leading families being a $\{\sigma_0 + 3/2, \pm 3/2\}$ decuplet and a $\{\sigma_0 + 1/2, \pm 1/2\}$ octet families, which can be identified as known decuplet and octet families. At the first recurrence all families but the above mentioned ones decouple in the SL(2, C) symmetry limit. DOMOKOS and KÖVESI—DOMOKOS predict the masses of the members of the decuplet using mass values of the members of the octet assuming only SU(3) symmetry breaking at $t = 0$. The mass values obtained, $M_\Delta = 1120$ $M_{\Sigma^*} = 1290$, $M_{\Xi^*} = 1460$, $M_\Omega = 1590$ are in rough agreement with experimental data.

It was shown [11] that Regge vertices may not be invariant under $SL(2, C) \otimes SL(6, C)_S$ but only under the maximal compact subgroup of it $SU(6)_S \otimes SU(2)_L$.

If we put mesons into a $(6, \bar{6}) \otimes (\bar{6}, 6)$ representation of $SL(6, C)_S$ and decompose this representation into Regge-pole families we obtain vector, pseudo-vector and conspiring scalar and pseudo-scalar meson families (they have $j_0 = 0$ and 1, respectively). The conspiring axial vector and pseudo-scalar families (II. and III. class conspiracies of FREEDMAN and WANG) appear necessarily together. The vector meson family is coupled to the octet baryon-antibaryon family with a pure F coupling, while $D/F = 3/2$ for the axial vector and pseudo-scalar families. The symmetry scheme gives definite relations among couplings of the above-mentioned families to members of given SU(6) multiplets.

6. Discussion

The applications of group theory to relativistic scattering described above are only the first steps of a broad programme. Even the contours of this programme are not known at present, but probably it is connected with most of the known parts of hadron physics. The most interesting open questions in this field are to find a "complete set of general solutions" of the conspiracy problem as defined by COSENZA, SCIARRINO and TOLLER [3], to find further consequences of the higher symmetries, and to clarify the connections with current algebras.

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ГРУППОВОЕ ТЕОРЕТИЧЕСКОЕ ОПИСАНИЕ РЕЛЯТИВИСТИЧЕСКОГО РАССЕЯНИЯ ВБЛИЗИ НУЛЕВОЙ ПЕРЕДАЧИ МОМЕНТА

П. ШУРАНЬИ

Резюме

Исследуется поведение амплитуд рассеяния вблизи нулевой передачи момента. Изучается вклад полюсов Лорентца в рассеяние высокой энергии. Результаты теоретических рассуждений сравниваются с экспериментальными данными по разонансам.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 35—46 (1969)

ANALYTICITY, FACTORIZATION, AND LORENTZ SYMMETRY*

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The properties of the scattering amplitude near $t = 0$ have been studied on the basis of analyticity, factorization and Lorentz symmetry.

During the last few years, the properties of the scattering amplitude near $t = 0$ have been extensively studied in the framework of the Regge pole model. The origin of the interest to this particular point, which at high energy in the general mass configuration, is very close to the physical region, has to be found in the existence at $t = 0$ of two kinds of complications. In fact, the analyticity properties of the scattering amplitude and the crossing symmetry require the existence of some kinematical constraints between different helicity amplitudes. These constraints in the framework of the Regge pole model give rise to the concept of "conspiracy" between different Regge trajectories.

Additional interest in the point $t = 0$ arises in connection with the Regge expansion of the scattering amplitude, which is singular at $t = 0$ in the unequal—unequal (UU) and equal—unequal (EU) mass configuration. The concepts of daughter trajectory and of Regge pole family come into play in order to avoid these unwanted singularities and to restore the pure Regge behavior.

The properties of the scattering amplitude near $t = 0$ have been studied in the following two different ways: the group theoretical approach and the analytic approach.

The first approach is based on the invariance of the scattering amplitude under the Lorentz group O(3, 1) at $t = 0$ in the pairwise equal mass configuration. In fact, Toller [1—3] Reggeized expansions of amplitudes

* Work supported in part by the Air Force Office of Scientific Research through the European Office of Aerospace Research, OAR, United States Air Force, under contract F61052 67 C 0084; and by the U.S. Atomic Energy Commission under contract AT(11—1)—68 of the San Francisco Operations Office, U.S. Atomic Energy Commission.

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in terms of the representation of the group O(3, 1); the simpler compact group O(4) was used later by FREEDMAN and WANG [4] and by DOMOKOS [5].

The results of this approach are that a TOLLER pole leads to an infinite family of Regge poles with defined relations between the trajectories and the residue functions at $t = 0$. These families are characterized, apart from the internal quantum numbers, by the TOLLER quantum number M which, for boson trajectories, can assume all the integer values.

Since these formalisms apply rigorously only at the point $t = 0$ and for equal mass scattering, more general formalisms have been developed recently in order to overcome these limitations [6, 7]. The most general work in this direction has been done by COSENZA, SCIARRINO, and TOLLER [8, 9] by a generalization of the Lorentz group formalism and the introduction of quite strong assumptions.

The analytic approach on the other hand is based on the usual assumptions of the S-matrix theory, which adapted to the Regge pole theory permits us to clarify the properties of the scattering amplitude near $t = 0$ for any mass configuration [10—15]. The assumptions made in the analytic approach are the following: (a) analyticity; (b) simplicity; (c) crossing symmetry; and (d) factorization.

Assumption (a) is the fundamental one of the S-matrix theory and claims that the scattering amplitude once its kinematical singularities have been removed, should have only the singularities required by the Mandelstam representation. In the UU and EU mass configurations, the contribution of a single Regge pole is not an analytic function at $t = 0$; assumption (b) requires that the analyticity of the scattering amplitude has to be realized by the introduction of the minimum number of daughter trajectories with residue sufficiently singular. Moreover, the requirements of analyticity and crossing symmetry impose some constraints on the helicity amplitudes that must be satisfied by the Regge pole families.

The assumption (d) is a consequence of unitarity and is necessary in order to connect the various mass configurations. Following the above assumptions, it is possible to define a quantum number M which has been found to be the TOLLER quantum number, and to classify the Regge poles in families with well-defined quantum numbers. This classification is, in some sense, equivalent to the group theoretical one; moreover, the $t = 0$ behavior of the factorized Regge pole residue functions, satisfying all the kinematical constraints, is the same as that derived in [8] and [9] by the group theoretical methods. It is possible then to reconstruct the scattering amplitude at $t = 0$ for the process $N + N \to N + N$ due to the exchange of a Regge pole family with a defined value of M and to show the complete equivalence between the group-theoretical and the analytic approach at $t = 0$ in the equal mass configuration. This equivalence eliminates the possibility that an "analy-

ticity family" represents a string of integer spaced TOLLER poles rather than a single TOLLER pole. As a consequence of the analytic approach, it is possible to go away from $t = 0$ and evaluate the "mass formulas" for the Regge trajectories. FREEDMAN and WANG [10] showed that in the spinless case the introduction of daughter trajectories is necessary in order to ensure the analyticity of the full amplitude. In the scattering between particles with spin, the daughter trajectories have necessarily to come into play but the singularity structure of their residue functions is, however, somewhat different than in the spinless case [12]. Therefore, in any spin configuration, the analyticity requires, for every parent trajectory exchanged, the exchange of an infinite family of Regge poles with well-defined quantum numbers with respect to the parent pole. In order to classify the Regge poles into families with well-defined quantum numbers, we first study the "minimal" solutions of the factorization conditions and the constraint equations. In this paper we will limit ourselves to the reactions $S + N \to J + N$ and the others related to these through factorization. Here N is a nucleon and $J(S)$ is a spin $J(S)$ and mass m_J (m_S) particle with $m_J \neq m_S \neq$ nucleon mass. Obviously these reactions are the most interesting from the physical point of view.

We give now the kinematical constraints at $t = 0$ for the various cases in our discussion:

(1) EU case (i.e., $S + N \to J + N$). The constraints turn out to be [12, 14, 16]

$$i\tilde{f}^{(+)}_{cd;1/2-1/2} - \tilde{f}^{-}_{cd;1/2,1/2} = 0(t) \tag{1}$$

for any c and d satisfying the inequality $c \neq d$, and

$$i\tilde{f}^{(-)}_{cc;1/2-1/2} - \tilde{f}^{-}_{cc;1/2,1/2} = 0(t) \tag{2}$$

for any c.

(2) UU case (i.e., $S + S \to J + J$). The constraints are

$$\tilde{f}^{+}_{cd;ab} + \tilde{f}^{-}_{cd;ab} = 0(t^m)\,, \tag{3}$$

if $|\lambda - \mu| < |\lambda + \mu|$; and

$$\tilde{f}^{+}_{cd;ab} - \tilde{f}^{-}_{cd;ab} = 0(t^m)\,, \tag{4}$$

if $|\lambda - \mu| > |\lambda + \mu|$, where $m =$ Minimum $(|\lambda|, |\mu|)$.

(3) EE case. In the simplified treatment given here, we have to consider only the nucleon—nucleon scattering. In this case the constraint is well-known [17]:

$$\tilde{f}^{-}_{1/2,1|2;1/2,1/2} - z\tilde{f}^{+}_{1/2-1/2;1/2-1/2} - \tilde{f}^{-}_{1/2-1/2;1/2-1/2} = 0(t)\,. \tag{5}$$

It is well-known that in the Regge pole model these constraints can be satisfied in three different ways; by evasion, by conspiracy between different poles and by a daughter-like conspiracy. In order to discuss this problem, we first study the minimal solutions of the factorization conditions. The objects which are assumed to factorize are the residues of the individual poles in $F^{J\pm}_{cd;ab}$; therefore, in our case, the residue is

$$K^{\pm}_{cd;ab}(t)\,\gamma^{\pm}_{cd;ab}(t)\left(\frac{q_i\,q_0}{s_0}\right)^{\alpha^{\pm}(t)-N}, \tag{6}$$

where $K^{\pm}_{cd;ab}(t)$ is the WANG kinematical factor, $(q_i q_0/s_0)^{\alpha^{\pm}(t)-N}$ is the usual threshold factor and $\gamma^{\pm}_{cd;ab}(t)$ in the reduced residue free from kinematical singularities at $t = 0$.

The factorization conditions in all the three channels considered are consistent and, in general, from a given solution, one can obtain other solutions by increasing the number of t powers of some of the Regge pole residues in the original solution. If a given solution cannot be obtained from another in this way, we shall call it "minimal". Once the factorization conditions are written down, it is not too difficult to find the "minimal" solutions. For the reactions with unequal masses of the type $S + S \to J + J$ or $S + S' \to J + J'$, one finds [12, 18, 19]:

$$\gamma^{\sigma}_{cd;cd} \sim t^{|\,|\mu|-M|-n}, \tag{7}$$

where $\mu = c - d$ and σ is the Regge natural parity.

When an equal mass vertex is involved, the selection rules due to parity and G-parity invariance must be taken into account. This implies the identical vanishing of the residue if the following conditions are not satisfied:

$$\begin{aligned} &\sigma\xi\,\tau(-1)^{S+1+N} = 1\,, \\ &S = 1\,, \quad \text{if} \quad \sigma = +1\,. \end{aligned} \tag{8}$$

τ is the signature of the parent Regge trajectory, the integer n is the order of the daughter, and S is the total spin of the $N\bar{N}$ system. The quantity ξ is defined in terms of the internal quantum numbers I (isospin) and G (G-parity) of the exchanged family by

$$\xi = G(-1)^T\,.$$

In the equal mass reaction $N + N \to N + N$ for the residues which do not vanish identically, we have [12]

$$\gamma^{\sigma}_{ab;ab} \sim \begin{cases} t^M & \text{if } (-1)^{\lambda+n} = \sigma, \\ t^{|M-1|} & \text{if } (-1)^{\lambda+n} = -\sigma, \end{cases}$$

where $\lambda = a - b$.

For the families with $\sigma = +1$ and $\tau = -\xi$, whose parent (and even daughter) trajectories do not couple to the $N\bar{N}$ system, we have for the odd daughters [12]:

$$\gamma^{+}_{ab;ab} \sim \begin{cases} t^{M+1} & \text{if } \lambda = 0 \\ t^{|M-1|+1} & \text{if } |\lambda| = 1 \end{cases} . \tag{10}$$

The solutions for the equal—unequal mass case can be obtained through the factorization and can be found explicitly in [18].

M is a number that we introduce in order to label the minimal solutions and can assume all the integer values between zero and infinity. An interesting feature of our results is that, for all the values of the masses, a family with a given value of M contributes asymptotically only to the forward s-channel helicity amplitudes with helicity flip equal to $\pm M$. M is therefore the TOLLER quantum number introduced in the group theoretical approach.

The Regge pole families can be then classified [12,20] according to the values of M, σ, ξ. These families and the residue behavior near $t = 0$ are coincident with those found in the general group theoretical approach.

Class I: $M = 0$, $\sigma = +1$, $\tau = \xi$. Only the trajectories with n even can couple to the $N\bar{N}$ system. Poles of this class verify the constraints 1, 2, 3, 4, 5 by evasion.

Class Ia: $M = 0$, $\sigma = +1$, $\tau = -\xi$. The parent and the even daughters of this class do not couple to the $N\bar{N}$ system. This explains why this class is not contained in the FREEDMAN and WANG classification.

Class II: $M = 0$, $\sigma = -1$, $\tau = -\xi$. The poles of this class satisfy the constraints 2 and 5 by a daughter-like conspiracy. All the other constraints are satisfied by evasion.

Class IIa: $M = 0$, $\sigma = -1$, $\tau = \xi$. The parent trajectory of this class, which is decoupled from the $N\bar{N}$ system at $t = 0$, satisfies the constraints by evasion. Conspiracy between daughters is allowed.

Class III: $M = 1$, $\tau = \xi$. In this class we find the well-known parity doubling phenomenon, which not only allows us to satisfy the constraints 1, 3, 4, 5 by conspiracy, but it is also imposed by other general analyticity requirements. The members of the doublet have the following quantum numbers:

$$\sigma = +1 \begin{cases} n \text{ even}: & P_n = \xi = \tau , \\ n \text{ odd}: & P_n = -\xi = -\tau , \end{cases}$$

$$\sigma = -1 \begin{cases} n \text{ even}: & -P_n = \xi = \tau\,, \\ n \text{ odd}: & -P_n = -\xi = -\tau, \end{cases}$$

where P means parity.

Class IIIa: $M = 1$, $\tau = -\xi$. The poles of this class have the following quantum numbers:

$$\sigma = +\begin{cases} n \text{ even}: & P_n = -\xi = \tau\,, \\ n \text{ odd}: & P_n = \xi = -\tau\,, \end{cases}$$

$$\sigma = -\begin{cases} n \text{ even}: & -P_n = -\xi = \tau\,, \\ n \text{ odd}: & -P_n = \xi = -\tau\,. \end{cases}$$

The parent trajectories satisfy the constraints by evasion. Conspiracy between daughter trajectories is allowed. Poles with $M > 1$ are decoupled, at $t = 0$, from the $N\bar{N}$ system, in agreement with the group theoretical results.

Using the assumptions (a)—(d), we were able to classify the Regge families according to the values of M, σ, ξ and we showed that this classification is equivalent to that obtained in the group theoretical approach. However, the previous discussion does not eliminate the possibility that an "analyticity family" represents a string of integer spaced TOLLER poles rather than a single TOLLER pole.

In the following, we will reconstruct the scattering amplitude at $t = 0$ for the process $N + N \to N + N$ due to the exchange of a Regge pole family with a definite value of M and we will show that the scattering amplitude obtained in the various cases is the same as that deduced using the group theoretical approach [11, 14, 15]. We will reconstruct only the classes I, II, III because the reconstruction of the other classes is quite similar to the previous ones.

Class I: $M = 0$, $\sigma = +1$, $\tau = \xi$. In order to study the Class I family in the UU case, we have to consider the amplitude $\tilde{f}^{+}_{0;0}$. The contribution of a family of Regge poles to this amplitude is given by:

$$\tilde{f}^{+}_{0;0}(s,t) = \sum_{n=0}^{\infty} \frac{2\alpha_n + 1}{\sin \pi\, \alpha_n} [1 + \tau_n\, e^{-i\pi\alpha_n}]\, \mathscr{P}_{\alpha_n}(-\cos \Theta_t)\, \beta_n(t)\,, \tag{11}$$

where

$$\mathscr{P}_{\alpha}(z) = -\frac{\tan \pi\alpha}{\pi} Q_{-\alpha-1}(z) = \frac{2^{\alpha}\, \Gamma(\alpha + 1/2)}{\Gamma(\alpha + 1)\, \pi^{1/2}}\, z^{\alpha} \cdot$$

$$\cdot F\left(-\frac{\alpha}{2}\,,\ \frac{1}{2} - \frac{\alpha}{2}\,;\ \frac{1}{2} - \alpha;\ \frac{1}{z^2}\right) =$$

$$= \frac{\Gamma(\alpha + 1/2)\, 2^{\alpha}}{\pi^{1/2}\, \Gamma(\alpha + 1)}\, z^{\alpha} \sum_{k=0}^{\infty} a_k(\alpha)\, z^{-2k}\,,$$

$$z = \cos \Theta_t = \frac{s}{2q_i q_0} + \frac{t\left(t - \sum_{i=1}^{4} m_i^2\right) + (m_1^2 - m_\infty)(m_2^2 - m_4^2)}{4tq_i q_0}, \tag{12}$$

$$a_k(\alpha) = \frac{\Gamma\left(-\frac{\alpha}{2} + k\right)\Gamma\left(-\frac{\alpha}{2} + \frac{1}{2} + k\right)\Gamma\left(\frac{1}{2} - \alpha\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2} + \frac{1}{2}\right)\Gamma\left(\frac{1}{2} - \alpha + k\right)k!}$$

and q_i and q_0 are the initial and final moments in the c.m. frame of the t channel.

Expanding the right-hand side of Eq. (11) in a power series, after some rearrangements, we get

$$\tilde{f}^{+}_{0;0}(s,t) = \sum_{m=0}^{\infty} [B(t)/t]^m (s/s_0)^{-m} \sum_{k=0}^{N(m)} \sum_{n=0}^{m-2k} d^m_{n;k}(t)\,(s/s_0)^{\alpha_n(t)+n}, \tag{13}$$

where s_0 is a scale factor

$$B(t) = \frac{t\left(t - \sum_{i=1}^{4} m_i^2\right) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{2s_0},$$

$$N(m) = \begin{cases} \dfrac{m}{2} & \text{if} \quad (-1)^m = 1, \\ \dfrac{m-1}{2} & \text{if} \quad (-1)^m = -1, \end{cases} \tag{14}$$

$$d^m_{n;k}(t) = -\frac{2\alpha_n + 1}{\sin \pi \alpha_n}\left[1 + \tau_n e^{-i\pi a_n}\right] \frac{\Gamma(\alpha_n + 1/2)\,2^{a_n}}{\pi^{1/2}\Gamma(\alpha_n + 1)} \gamma_n(t)\, a_k(\alpha_n) \cdot$$

$$\cdot \left[D(t)/B(t)\right]^{2k} [B(t)/t_0]^{-n} \frac{\Gamma(\alpha_n - 2k + 1)}{(m - n - 2k)!\,\Gamma(\alpha_n + 1 - m + n)},$$

$$D(t) = \left[\frac{[t - (m_1 + m_3)^2][t - (m_1 - m_3)^2][t - (m_2 + m_4)^2][t - (m_2 - m_4)^2]}{4s_0^2}\right]^{1/2}.$$

Since the function $\tilde{f}^{+}_{0;0}$ has to be analytic at $t = 0$, for any s, we must require that:

$$\sum_{k=0}^{N(m)} \sum_{n=0}^{m-2k} d^m_{n;k}(t)\ (s/s_0)^{\alpha_n(t)+n} = 0(t^m) \text{ for } m \geq 1. \tag{15}$$

These are the fundamental relations of our approach; from these equations in fact not only the quantities $\gamma_n(0)$ can be expressed in terms of $\gamma_0(0)$ but the behavior of the family for $t \neq 0$ can be studied. If we evaluate the expression (15) at $t = 0$, we get a system for the residue functions of the daughter trajectories at $t = 0$ in function of $\gamma_0(0)$. The solution of this system has been shown to be [11, 14]:

$$\gamma_n(0) = \frac{(-1)^n}{n!} \frac{\Gamma(n-1-2\alpha)}{\Gamma(-1-2\alpha)} \gamma_0(0). \tag{16}$$

Once these relations are known we can deduce the mass formula. In fact, from the system (15), differentiating with respect to t, we obtain for $m \geqslant 2$:

$$\sum_{k=0}^{N(m)} \sum_{n=0}^{m-2k} d^m_{n;k}(0)\, \alpha'_n(0) = 0 . \tag{17}$$

The solution of this system is the DOMOKOS—SURÁNYI [21] mass formula

$$\alpha'_n(0) = a_1 + a_2(\alpha - n)(\alpha - n + 1) . \tag{18}$$

We then study the Class I in the EU mass configuration; this case is somewhat simpler than the UU one, due to the presence of weaker singularities. At the equal mass vertex, the selection rules due to parity and G-parity invariance must be taken into account. In the present case one finds that the parent and the even daughter trajectories couple to the amplitude $\tilde{f}^+_{cc;1/2,1/2} \equiv \tilde{f}^+_{0;0}$ while the odd daughters are decoupled from the $N\bar{N}$ system.

Using then essentially the same methods described for the UU case, we get

$$\gamma_{2n}(0) = \frac{(-1)^n}{n!} \frac{\Gamma\left(n - \frac{1}{2} - \alpha\right)}{\Gamma\left(-\frac{1}{2} - \alpha\right)} \gamma_0(0) \tag{19}$$

and

$$\alpha'_{2n}(0) = a_1 + a_2(\alpha - 2n)(\alpha - 2n + 1) \tag{20}$$

which, as expected, is again of the form of Eq. (18). The factorization theorem provides now the bridge necessary to study the contribution of the analyticity family with $M = 0$ and $\sigma = +1$ to the nucleon—nucleon scattering.

Using therefore the factorization theorem, we get:

$$\beta^{(\pm)2n}_{1/2,1/2;1/2,1/2} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\Gamma(\alpha+1-n)}{\Gamma(\alpha+1)} \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}-n\right)} \beta^{(+)n=0}_{1/2,1/2;1/2,1/2.} \tag{21}$$

The same expression can be found using the group theoretical approach. This proves that the Class I "analyticity families" are indeed the same as the Class I "group theoretical families".

Class II: $M = 0$, $\sigma = -1$, $\tau = -\xi$. In order to study the Class II family in the UU case, we have to consider the amplitude $\tilde{f}_{0;0}$, whose discussion goes on exactly the same way as that for the Class I UU mass case. The results, therefore, are exactly the same as reported above.

In the EU case, however, there is a new complication due to the spin. Due to the selection rules at the nucleon vertex, one finds that the parent and the even daughters contribute to the amplitude $\tilde{f}_{cc;1/2,1/2} \equiv \tilde{f}_{0;1}$, while the odd daughters contribute to the amplitude $\tilde{f}_{cc;1/2,1/2} \equiv \tilde{f}_{0;0}$. Moreover, there is the constraint 2 which is satisfied by conspiracy. If we impose that the amplitudes $\tilde{f}_{0;0}$ and $\tilde{f}_{0;1}$ are analytic at $t = 0$ and satisfy by conspiracy the constraint equation, we get the following relations for the residue functions:

$$\gamma_{0;1}^{2n}(0) = \frac{(-1)^n}{n!}\left[\frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)}\right]^{1/2}\frac{\Gamma\left(n-\frac{1}{2}-\alpha\right)}{\Gamma\left(-\frac{1}{2}-\alpha\right)},$$

$$\gamma_{0;}^{2n+1}(0) = \frac{(-1)^n}{n!}\frac{\Gamma\left(n+\frac{1}{2}-\alpha\right)}{\Gamma\left(-\frac{1}{2}-\alpha\right)}\gamma_{0;0}^{n=1}(0), \tag{22}$$

$$\gamma_{0;0}^{n=1}(0) = i\frac{2\alpha+1}{\sqrt{\alpha(\alpha+1)}}\gamma_{0;}^{n=0}(0).$$

Using the factorization theorem, we get for the complete residue functions in nucleon–nucleon scattering:

$$\beta_{1/2-1/2;1/2-1/2}^{2n}(0) = \frac{(2n)!}{2^{2n}(n!)^2}\frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}\times$$

$$\times\frac{\Gamma\left(n-\frac{1}{2}-\alpha\right)}{\Gamma\left(-\frac{1}{2}-\alpha\right)}\beta_{1/2-1/2;1/2-1/2}^{n=0}(0), \tag{23}$$

$$\beta_{1/2,1/2;1/2,1/2}^{2n+1}(0) = \frac{(2n+1)!}{2^{2n}(n!)^2}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}\frac{\Gamma\left(n+\frac{1}{2}-\alpha\right)}{\Gamma\left(\frac{1}{2}-\alpha\right)}\beta_{1/2,1/2;1/2,\,1/2}^{n=1}(0),$$

$$\beta^{n=1}_{1/2,1/2;1/2,1/2}(0) = \frac{2\alpha+1}{\alpha(\alpha+1)}\,\beta^{n=0}_{1/2-1/2;1/2;1/2-1/2}(0).$$

These results are coincident with those obtained using the group theoretical approach. It is easily seen that in this case the analytic and the group theoretical families are the same. The mass formula for poles belonging to the Class II is the same that has been obtained for the Class I.

Class III: $M = 1$, $\tau = \xi$. The discussion of this family is much more involved, essentially due to the fact that σ is no longer diagonal with M. Here, for the first time, the parity doubling phenomenon appears. In order to study the Class III family in the UU case, we must consider the amplitudes $\tilde{f}^{\pm}_{1;1}(s,t)$. In fact, because of the existence of the parity doubling phenomenon, we cannot restrict consideration of only one amplitude like in the first two classes. If we require that the contribution to the amplitudes $\tilde{f}^{\pm}_{1;1}$ of the two parent Regge trajectories and of their daughters is analytic at $t = 0$ and satisfies the constraint 3 by conspiracy, we get the following relations for the residue functions:

$$\begin{aligned} \gamma^{(\pm)n}_{1;1}(0) &= \frac{(-1)^n}{n!}\,\frac{\Gamma(n-1-2\alpha)}{\Gamma(-1-2\alpha)}\,\gamma^{\pm n=0}_{1;1}(0)\,, \\ \gamma^{(+)n=0}_{1;1}(0) &= -\gamma^{-n=0}_{1;1}(0)\,, \end{aligned} \tag{24}$$

and the mass formula for $M = 1$ Regge families:

$$\alpha^{\pm}_n(0) = c_1 + [c_2 \pm c_3]\,(\alpha - n)\,(\alpha - n + 1)\,. \tag{25}$$

Owing to the parity and G-parity selection rules, in the EU case the parent and even daughters of the sub-family with $\sigma = -1$ couple to the amplitude $F^{J-}_{1;0}$ and the odd daughters to the amplitude $F^{J-}_{1;1}$. The parent and even daughters of the sub-family with $\sigma = +1$ couple to the amplitude $F^{J+}_{1;1}$ while the $\sigma = +1$ odd daughters are decoupled from the $N\bar{N}$ system.

In this case we have to impose the analyticity at $t = 0$ to the amplitudes $\tilde{f}^{-}_{1;0}$ and $\tilde{f}^{+}_{1;1}$ and the constraint 1. In this way, we get

$$\begin{aligned} \gamma^{(+)2n}_{1;1}(0) &= \frac{(-1)^n}{n!}\,\frac{\Gamma\left(n-\frac{1}{2}-\alpha\right)}{\Gamma\left(-\frac{1}{2}-\alpha\right)}\,\gamma^{(+)n=0}_{1;1}(0)\,, \\ \gamma^{(-)n=1}_{1;1}(0) &= -\frac{2\alpha+1}{\alpha+1}\,\gamma^{(+)n=0}_{1;1}\,, \end{aligned} \tag{26}$$

$$\gamma_{1;1}^{(-)2n+1}(0) = \frac{(-1)^n}{n!} \frac{\Gamma\left(n+\frac{1}{2}-\alpha\right)}{\Gamma\left(+\frac{1}{2}-\alpha\right)} \gamma_{1;1}^{(-)n=1}(0),$$

$$\gamma_{1;0}^{(-)n=0}(0) = i\sqrt{\frac{\alpha+1}{\alpha}}\,\gamma_{1;1}^{(+)n=0}(0),$$

and a mass formula which is consistent with Eq. (25). The factorization theorem permits one to get the following results for the nucleon—nucleon scattering:

$$\beta_{1/2-1/2;1/2-1/2}^{(+)2n} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)} \frac{\Gamma\left(n-\frac{1}{2}-\alpha\right)}{\Gamma\left(-\frac{1}{2}-\alpha\right)} \beta_{1/2-1/2;1/2-1/2}^{(+)n=0},$$

$$\beta_{1/2,1/2;1/2,1/2}^{(-)2n} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)} \frac{\Gamma\left(n-\frac{1}{2}-\alpha\right)}{\Gamma\left(-\frac{1}{2}-\alpha\right)} \times$$

$$\times \frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)} \beta_{1/2,1/2;1/2,1/2}^{(-)n=0}(0),$$

$$\beta_{1/2-1/2;1/2-1/2}^{(-)2n\pm1} = \frac{(2n+1)!}{2^{2n}(n!)^2} \frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)} \frac{\Gamma\left(n+\frac{1}{2}-\alpha\right)}{\Gamma\left(\frac{1}{2}-\alpha\right)} \beta_{1/2-1/2;1/2-1/2}^{(-)n=1},$$

$$\beta_{1/2,1/2;1/2,1/2}^{(-)n=0}(0) = \frac{\alpha}{\alpha+1} \beta_{1/2-1/2;1/2-1/2}^{(+)n=0}(0),$$

$$\beta_{1/2-1/2;1/2-1\backslash2}^{(-)n=1} = \frac{2\alpha+1}{(\alpha+1)^2} \beta_{1/2-1/2;1/2-1/2}^{(+)n=0},$$

which are coincident with those derived from the group theoretical approach.

The proof that the $M = 1$ Regge pole families are the same as the Regge pole families derived from $M = 1$ TOLLER poles in EE mass scattering is therefore completed.

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АНАЛИТИЧНОСТЬ, ФАКТОРИЗАЦИЯ И СИММЕТРИЯ ЛОРЕНЦА

П. ДИ ВЕЧЧИА и Ф. ДРАГО

Резюме

На базе аналитичности, факторизации и симметрии Лоренца изучаются свойства амплитуды рассеяния вблизи $t = 0$.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 47—55 (1969)

REGGE FAMILIES AND LORENTZ SYMMETRY

By

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Some of the fundamental ideas of the conspiracy problem are discussed in a simple way. It is shown why we are forced to consider families of Regge poles. A model in which Regge families are characterized by representations of the homogeneous Lorentz group is also examined.

The papers on Regge families and constraints are widely scattered and tend to contain complicated mathematics which obscures the basic ideas involved. For this reason, I should like to discuss some of the fundamental ideas in a simple way. This heuristic presentation will necessarily contain incomplete statements, which may appear to be incorrect when left unqualified; so for more careful explanations, and greater detail I refer you to the Berkeley preprint [1] "Daughters, Conspiracies and Lorentz Symmetry" which KHALIL BITAR and I wrote in April. This contains the formalism underlying the material I shall present today.

First of all, I shall show why we are forced to consider families of Regge poles, and explain what I mean by "daughters" and "conspirators". We shall then examine some hypotheses which lead to a model where Regge families are characterised by representations of the homogeneous Lorentz group. Our approach will differ from the approaches of DOMOKOS and SURÁNYI [2] and SCIARRINO and TOLLER [3] and collaborators, and lead to slightly different results. However, at $t = 0$, in the case of elastic scattering, the different theories coincide. In conclusion, we shall see how our representation may be written in a form which resembles the contribution of pole terms in conventional Field Theory.

Let us start by looking at the unequal mass kinematics [4]. We shall be concerned with the s-channel scattering of particles (a) and (b) into particles (c) and (d), (Fig. 1). We take a particular mass configuration

$$m_a \geqslant m_c \geqslant m_d \geqslant m_b\,, \qquad (1$$

so that the line $t = 0$ lies partly in the s-channel physical region (Fig. 2)

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We take p_i to denote the t-channel c.m. four momentum of particle (i). The c.m. energy parameters s, t and u are then defined by

$$s = (p_a - p_b)^2,\ t = (p_a + p_c)^2 \text{ and } u = (p_a - p_d)^2. \tag{2}$$

Now the s-channel helicity amplitude $\mathscr{S}_{cd;ab}$ can be expressed as a linear combination of t-channel amplitudes,

$$\mathscr{S}_{cd,ab} = \sum_{a'b'c'd'} d^{s_a}_{a'a}(\chi_a)\, d^{s_b}_{b'b}(\chi_b)\, d^{s_c}_{c'c}(\chi_c)\, d^{s_d}_{d'd}(\chi_d)\, \mathscr{T}_{b'd',a'c'}\,. \tag{3}$$

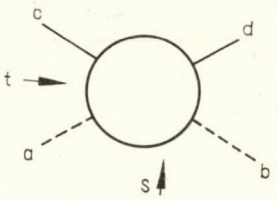

Fig. 1. Two-body scattering

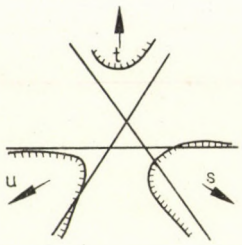

Fig. 2. Physical region boundaries

The function $d(\chi)$ is the usual representation of a y-axis rotation through angle χ. In the physical region the crossing angles χ_i are always real. The dependence on s and t is given by expressions of the form

$$\sin \chi_a = \frac{2\, m_a\, \Phi(s,t)}{\Delta(s;a,b)\,\Delta(t;a,c)}, \tag{4}$$

where $\Delta(s;\ a,\ b)$ is a product of threshold factors,

$$\Delta(s;a,b) = \left[\left(s - (m_a + m_b)^2\right)\left(s - (m_a - m_b)^2\right)\right]^{1/2}, \tag{5}$$

and $\Phi\ (s,t)$ is a function which vanishes on the physical region boundary.

We are interested in looking at the large s behaviour of the s-channel amplitude generated by poles in the t channel amplitude. Before doing this I should like to sketch a way of obtaining an expansion of the $\mathscr{T}$ amplitude in terms of rotation group representation functions, because we shall refer to it later when we consider a "Lorentz pole" expansion. We first of all indicate the full momentum dependence of the t channel amplitude in the form

$$\mathscr{T} \approx \langle R_{p'}\,\hat{z}\,|T(P)|R_p\,\hat{z}\rangle, \tag{6}$$

with

$$P = p_a + p_c\,, \tag{7}$$

where we have taken energy momentum conservation into account. Transformations R_p and $R_{p'}$ rotate the unit vector in the z direction $\hat{z}$ into the direc-

tions of $p = p_a - p_c$ and $p' = p_b - p_d$ respectively. Note that in general the mass shell conditions determine the *magnitudes* of vectors p and p' once P is known. Now Lorentz invariance of the scattering operator T, and rotational invariance of P in the t-channel c.m. system, allow us to express $\mathscr{T}$ in the form:

$$\mathscr{T} \approx \langle R_p^{-1} R_{p'} \hat{z} \,|T(P)|\, \hat{z} \rangle = \mathscr{T}(P; R_p^{-1} R_{p'}) . \tag{8}$$

Group theory then tells us that we may expand $\mathscr{T}$ in terms of rotation group representation functions,

$$\mathscr{T} \approx \sum_j D^j (R_p^{-1} R_{p'}) \, \mathscr{T}^j(P) . \tag{9}$$

Note that P is a function of t alone. If we take p to lie along the z axis with p' inclined at an angle Θ_t in the $x - z$ plane we have the usual partial wave expansion

$$\mathscr{T} \approx \sum_j D^j (\Theta_t) \, \mathscr{T}^j(t) . \tag{10}$$

Regge theory then tells us that the high energy behaviour in the s-channel is dominated by single t-channel pole contributions of the form

$$\mathscr{T} \approx D^\alpha(\Theta_t) \, \beta^\alpha(t) , \tag{11}$$

where α is the position of the pole in the complex j plane, and $\beta^\alpha(t)$ the residue. For simplicity I am suppressing helicity indices, and neglecting effects of signature and parity.

Let us now examine the behaviour of this contribution (11) for large s. In terms of s and t the c.m. scattering angle is given by

$$z = \cos \Theta_t = \frac{t\,(s - u) + (m_a^2 - m_c^2)\,(m_b^2 - m_d^2)}{\Delta(t; a, c)\, \Delta(t; d, b)} . \tag{12}$$

For finite t, $\cos \Theta_t$ grows like s as $s \to \infty$ and the property $D^\alpha = z^\alpha$ for large z means that a single Regge pole (11) generates s^α behaviour of the scattering amplitude $\mathscr{T}$. In the special case where t is zero, unless $m_a = m_c$ and $m_b = m_d$ we loose the dependence of $\cos \Theta_t$ on s and cannot have Regge asymptotic behaviour. Moreover, s-matrix analyticity implies that $[\Delta(t; a, c)\Delta(t; b, d)/t]^\alpha D^\alpha(z)$ should be analytic at $t = 0$ and we observe that it is not. However, in the exceptional equal mass case, which corresponds to elastic scattering, square root of t factors in the threshold functions ensure Regge asymptotic behaviour s^α for all t.

The question now arises, "How can we retain the s^{α} power behaviour in the case of inelastic scattering, even when the momentum transfer vanishes?" A possible answer was given by MANDELSTAM who suggested that corresponding to each single Regge pole there exists an infinite family of *daughter* Regge poles. These poles are spaced by integers in the j plane with singular reduced residues $\beta(t)$.

To see how this may work consider the function st which vanishes when t is zero. We should expect all well behaved functions of st to be independent of s for zero t. However, the non uniformly convergent series

$$\sum_{k=0}^{\infty} t^{-k}(st)^k = (1-s)^{-1} \tag{13}$$

is s dependent for all values of t.

With these ideas in mind D. Z. FREEDMAN and J. M. WANG [5] have examined the zero spin scattering problem. They have justified a prescription which tells us how the daughter residues $\beta^{\alpha-\varkappa}(t)$ must be related near $t=0$ to preserve the s^{α} dependence of $\mathscr{S}$ for large s. KHALIL BITAR and I have considered the case where particles have non zero spin [1] and find that for certain solutions of the daughter problem one must have parity doublet Regge poles with intersecting trajectories when $t=0$. More recently F. ARBAB and J. D. JACKSON [6] have investigated the more complicated constraints implied by daughter and parent residue factorisation. This enables us to express a residue $\beta_{bd;ac}$ as a product of partial residues,

$$\beta_{bd,ac}(t) = \beta_{bd}(t)\,\beta_{ac}(t)\,. \tag{14}$$

However, the essential thing to note is that *whenever a Regge pole couples to unequal mass particles there must exist a family of integrally spaced daughters.* When the parent residue vanishes at $t=0$ it is said to be evasive, and we have a slightly different formalism which I do not wish to discuss here.

Leaving the daughters which arise from analyticity troubles when Regge poles couple to *unequal* masses I should like to discuss constraints which arise when Regge poles couple to *equal* mass particles. At the boundaries of physical regions where we have forward or backward scattering, helicity flip amplitudes must vanish. In this case when the masses are all unequal and the crossing angles (4) are either zero or π, the relation (3) takes the form

$$\mathscr{S}_{cd;ab} = \mathscr{T}_{bd;ac} = 0\,,\quad b-d \neq a-c\,. \tag{15}$$

We have several less trivial conditions with equal external mass configurations. For example if we consider elastic scattering, where the crossing angles are $\pm\pi/2$, one has the relation

$$\mathscr{S}_{cd;ab} = \sum d^{s_a}_{a'a}\left(-\frac{\pi}{2}\right) d^{s_b}_{b'b}\left(-\frac{\pi}{2}\right) d^{s_c}_{c'c}\left(\frac{\pi}{2}\right) d^{s_d}_{d'd}\left(\frac{\pi}{2}\right) \mathscr{T}_{b'd';a'c'} \equiv 0, \overset{a-d\neq}{b-c}. \quad (16)$$

This constraint on helicity amplitudes leads to corresponding restrictions on Regge pole residues, and is known as an equal mass conspiracy relation. If we put a Regge pole in one helicity amplitude, we are forced to introduce a number of conspiring poles or "conspirators", into other amplitudes to satisfy the boundary constraints (16). The general result may be summarised by saying that *when a Regge pole couples to equal mass particles with spin, there may exist a family of conspirators.*

Two questions now arise. Do there exist simple group theoretical ways of grouping together daughter trajectories, to preserve analyticity and asymptotic behaviour at $t = 0$, and conspirator trajectories, to satisfy the physical boundary constraints? May we identify conspirator and daughter trajectories? A partial answer to the first question is given by the work of DOMOKOS and SURÁNYI who investigated bound state classification in a BETHE—SALPETER framework [7], and SCIARRINO and TOLLER and collaborators [8] who investigated the crossed channel Lorentz decomposition of the backward elastic scattering amplitude. To see how this comes about we follow a procedure similar to that we used to obtain the partial wave expansion (9). In the case of elastic scattering when the crossed channel four momentum vanishes the mass shell conditions no longer determine the magnitudes of the relative three momenta p and p'. We may then, by analogy with equation (6) write the amplitude $\mathscr{T}$ in the form

$$\mathscr{T} \approx \langle \Lambda_{p'} \hat{z} \,|T(P)|\, \Lambda_p \hat{z}\rangle\,, \quad (17)$$

where Λ_p and $\Lambda_{p'}$ boost and rotate the unit vector in the t direction $\hat{z}$ so that $p = |p|\,\Lambda_p\hat{z}$ and $p' = |p'|\,\Lambda_{p'}\hat{z}$. Lorentz invariance of the scattering operator T and four momentum P allows us to express $\mathscr{T}$ in the form

$$\mathscr{T} \approx \langle \Lambda_p^{-1}\Lambda_{p'}\hat{z} \,|T(P)|\, \hat{z}\rangle = \mathscr{T}(P; \Lambda_p^{-1}\Lambda_{p'}). \quad (18)$$

Group theory tells us that we may expand in terms of homogeneous Lorentz group representation functions

$$\mathscr{T} \approx \sum_{j_0\sigma} D^{j_0\sigma}(\Lambda_p^{-1}\Lambda_{p'})\,\mathscr{T}^{j_0\sigma}(P). \quad (19)$$

The parameter j_0, which takes integral or half integral values, and the continuous parameter σ are eigenvalues of the Casimir operators of the homogeneous Lorentz group, analogous to the parameter j which labels representations of the rotation group. Arguments similar to those of Regge theory then lead to Lorentz poles in the complex σ plane. A single pole at α with Lorentz parameter M gives a contribution to the $\mathscr{T}$ matrix of the form

$$\mathscr{T}(s,t) \approx D^{M\alpha}(\Lambda_p^{-1}\Lambda_{p'})\,\beta^{M\alpha}. \tag{20}$$

In a standard $|jm>$ basis the functions $D^{j_0\sigma}$ have the properties

$$D^{j_0\sigma}_{jm,j'm'}(R) = \delta_{jj'}\,D^{j}_{mm'}(R)\,, \tag{21}$$

and

$$D^{j_0\sigma}_{jm,j'm'}(Z) = \delta_{mm'}\,D^{j_0\sigma}_{jj'm}(Z)\,, \tag{22}$$

where R is a rotation and Z is a pure boost in the z direction.

After decomposing the Lorentz transformations Λ_p and $\Lambda_{p'}$ in the form $\Lambda_p = Z_p$, $\Lambda_{p'} = R_{p'}Z_{p'}$ detailed analysis shows that we may write equation (20) in the form

$$\mathscr{T}_{bd;ac} \approx \beta^{M\alpha} \sum_k C^{s_a s_c\, s}_{a-c\lambda}\, C^{s_b s_d\, s'}_{b-d\lambda'}\, D^{M\alpha+1}_{s\alpha-k\,\lambda}(Z_p^{-1})\, D^{M\alpha+1}_{s'\alpha-k\lambda'}(Z_{p'})\, D^{\alpha-k}_{\lambda\lambda'}(\Theta_t). \tag{23}$$

On comparing this expression with equation (11) we see immediately that a Lorentz pole at α in the σ plane corresponds to a family of Regge poles at $\alpha - k$ in the j plane with correlated residues. The family of poles is such that the conspiracy conditions (16) are automatically satisfied.

The solution to the problem of the group theoretical classification of *daughter* poles has been given in terms of group contraction theory [9]. We have already seen examples in which the little group of the Poincaré group which leaves the crossed channel for momentum invariant can be the three dimensional rotation group or full homogeneous Lorentz group. Contraction theory suggests that when the group structure changes at $t = 0$ we should expect to find singularities when trying to continue little group representations as functions of t from positive t to $t = 0$. We may thus solve the singularity problem by finding a classification group for Regge poles which does *not* have contracting representations when $t = 0$. Groups with this property are such that the root vectors of the Lie algebra with respect to boosts span the algebra, and one group with such an algebra is the homogeneous Lorentz group. Thus if we classify Regge poles at $t = 0$ in the case of *inelastic* scattering according to representation of the homogeneous Lorentz group, all daughter conditions will be satisfied automatically.

We now examine this hypothesis in a simple model. We take Regge daughter trajectories to run parallel to the parent trajectories, although this is not necessary. We correlate residues in such a way that they give contributions to the scattering amplitude of the form

$$\mathscr{T} \approx \beta^{j_0\sigma}\, D^{j_0}(\Lambda^{-1}_{p_{ac}}\Lambda_{p_{bd}})\,,$$

where the momenta p_{bd} and p_{ac} are to be determined. The most general candidates which lead to a Regge pole expansion are in fact of the form

$$p_{ac} = p_a + \xi p_c = 1/2\,(1+\xi)\,P + 1/2\,(1-\xi)\,p$$

and

$$p_{bd} = p_b + \xi' p_d = 1/2\,(1+\xi')\,P + 1/2\,(1-\xi')\,p'. \tag{24}$$

It is important to note that only when the Lorentz pole expansion is valid with $P=0$ is there no ambiguity in the definitions of P_{ac} and P_{bd}. They effectively reduce to p and p' for all ξ and ξ'. In order to proceed further we shall need an identity connecting homogeneous Lorentz group representations [1]

$$\sum_{\varkappa=0}^{\infty} D^{j_0\sigma+1}_{s\sigma-\varkappa\lambda}(\delta)\, D^{j_0\sigma+1}_{s'\sigma-\varkappa\lambda'}(\delta')\, D^{\sigma-\varkappa}_{\lambda\lambda'}(\Theta) = \sum_{\mu} D^{s}_{\lambda\mu}(\psi)\, D^{s'}_{\mu\lambda'}(\psi')\, D^{j_0\sigma+1}_{ss'\mu}(\gamma), \tag{25}$$

where

$$\sin\psi = \frac{\sinh\delta'\sin\Theta}{\sinh\gamma}, \quad \sin\psi' = \frac{\sinh\delta\sin\Theta}{\sinh\gamma}, \tag{26}$$

and

$$\cosh\gamma = \cosh\delta'\cosh\delta + \sinh\delta'\sinh\delta\cos\Theta.$$

We set $\Lambda_{p_{ac}} = Z_{p_{ac}} = e^{-ik_3\delta}$ and $\Lambda_{p_{bd}} = R(\Theta_t)Z_{p_{bd}}$ and identify the L.H.S. of equation (25) with the contribution of a single Lorentz family. If we proceed to compute the angle γ we shall find that it is such that the expression on the R.H.S. has simple Regge asymptotic behaviour *for all values of t.* This result is quite independent of the choice of parameters ξ and ξ' entering to the definitions of P_{bd} and P_{ac}. In the case where $p_{bd} = p'$ and $p_{ac} = p$ we find that at $t = 0$

$$\cosh\gamma = \frac{2s - \sum m^2}{2\left[(m_a^2 + m_c^2)\,(m_b^2 + m_d^2)\right]^{1/2}}$$

and

$$\sinh\gamma = \left[\frac{(2s - \sum m^2)^2}{4\,(m_0^2 + m_c^2)(m_b^2 + m_d^2)} - 1\right]^{1/2}.$$

This form of Lorentz family sum has been used in the off mass shell expansions of DOMOKOS and SURÁNYI [2]. On close examination we find that the function $\sinh\gamma$ has square root branch points which have no simple physical interpretation. We have eliminated the daughter problem at the expense of introducing new unphysical kinematic branch points into the model. By choosing the parameters ξ and ξ' carefully it is possible to explicitly eliminate these spurious branch points altogether. The condition is $\xi = 0$ or ∞ and we identify p_{ac} with p_a or p_c and p_{bd} with p_b or p_d. In such circumstances with $p_{ac} = p_a$ and $p_{bd} = p_b$ the angle γ is given by

$$\sinh\gamma = \frac{\Delta(s;a,b)}{2\,m_a\,m_b}, \quad \cosh\gamma = \frac{-s + m_a^2 + m_b^2}{2\,m_a\,m_b}, \tag{27}$$

where the kinematic threshold factor $\Delta(s; a, b)$ is defined in equation (5). Moreover the angles ψ and ψ' are simply related to the crossing angles (4). A careful analysis indicates that the phases are such that *all* equal mass conspiracy relations and physical boundary constraints are automatically satisfied. I should perhaps remark here that we are principally interested in the behaviour near $t = 0$, but observe that our model has interesting properties for all values of t.

If we now take the Lorentz pole expansion (23) as a boundary condition on the form of our Lorentz family sum in the case where masses become equal, we find that our expansion also gives the correct threshold behaviour to Regge residue functions. Moreover we may identify the Lorentz family parameter j_0 with the Lorentz pole parameter M.

We have shown that if we group Regge poles to form Lorentz families according to this particular model, all daughter relations and conspiracy conditions are automatically satisfied. We have also been able to identify daughters with conspirators.

In conclusion, I should like to show that our model is *not uniquely determined* by the Lorentz pole boundary condition. One unsatisfactory feature as it stands is the manifestly unsymmetric appearance of bound state poles and external particles. This may be overcome by writing the contribution of a single Lorentz family in the form

$$\mathscr{T} \approx V^{M_a\sigma_a\ M_c\sigma_c\ M\sigma}_{s_a a\quad s_c c\quad j\lambda} D^{j}_{\lambda\lambda'}(\Theta_t)\, V^{M_b\sigma_b\ M_d\sigma_d\ M\sigma}_{s_b b\quad s_d d\quad j\lambda'}, \tag{28}$$

where M and σ denote Lorentz parameters associated with each external particle and bound state. The function $V^{M_a\sigma_a\ M_c\sigma_c\ M\sigma}_{s_a a\quad s_c c\quad j\lambda}$ is given by

$$\begin{aligned} V^{M_a\sigma_a\ M_c\sigma_c\ M\sigma}_{s_a a\quad s_c c\quad j\lambda} &= D^{M_a\sigma_a}_{s_a j_a a}(Z_{p_a})\, D^{M_c\sigma_c}_{s_c j_c - c}(Z_{p_c})\, C^{F_a G_a j_a}_{f_a g_a a}\, C^{F_c G_c j_c}_{s_c \xi_c - c}\, C^{F_a\ F_c\ F}_{f_a\ f_c\ f}\, C^{G_a\ G_c\ G}_{g_a\ g_c\ g}\, C^{F\ G\ j}_{f\ g\ \lambda} \\ &= D^{M_c\sigma_c}_{s_c j_c - c}(Z_{p_c} Z_{p_a}^{-1})\, C^{F_a\ G_a\ s_a}_{f_a\ g_a\ a}\, C^{F_c\ G_c\ j_c}_{f_c\ g_c\ -c}\, [C^{F_a\ F_c\ F}_{f_a\ f_c\ f}\, C^{G_a\ G_c\ G}_{g_a\ g_c\ g}]\, C^{F\ G\ S}_{f\ g\ \lambda}\, D^{M\sigma}_{sj\lambda}(Z_{p_a}), \end{aligned} \tag{29}$$

where the functions C are analytic continuations of ordinary Clebsch Gordan coefficients and

$$M = F - G, \quad \sigma = F + G. \tag{30}$$

The term in square brackets may be regarded as a vertex function coupling two particles to a Regge pole and is by no means unique. We have effectively written the contribution of a Lorentz family in the form of a pole term in a conventional Field Theory with non derivative coupling.

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СЕМЕЙСТВА РЕДЖЕ И СИММЕТРИЯ ЛОРЕНЦА

Г. Л. ТИНДЛ

Резюме

Простым методом дискутируются некоторые основные идеи заговорщицких проблем. Раскрывается причина, побуждающая нас сравнивать семейства полюсов Редже. Исследуются также некоторые гипотезы, ведущие к модели, в которой семейства Редже характеризуются в представлении однородных групп Лоренца.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 57—63 (1969)

π-N RESONANCE WIDTHS IN THE BROKEN SL(2, C) MODEL

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The elastic decay width of some πN resonances is evaluated in the SL(2, C) model of Regge poles. Two families of resonances are examined in the first order of symmetry breaking, one of them has isotopic spin $I = 1/2$, the other $I = 3/2$. The width of other resonances along the trajectories is calculated in symmetry limit and the differential cross-section is examined for πN backward scattering. The results are in good agreement with experiments.

I

The analyticity problems of the Regge theory at $u = 0$ for unequal mass scattering led physicists to the discovery of a higher symmetry of the Regge poles, the SL(2, C) one [1]. This symmetry manifests itself in grouping the poles into families at $u = 0$. Near $u = 0$ the SL(2, C) symmetry is broken. The breaking mechanism has been elaborated by DOMOKOS and SURÁNYI [2]. They have applied it successfully to classify the πN resonances [3]. The aim of this paper is to calculate the elastic widths of those resonances.

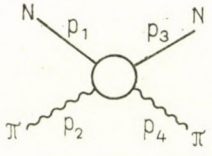
II

The elastic decay width of $\pi - N$ resonances is given by the formula

$$\Gamma_{\text{el}} = \frac{1}{2\pi}\int dp\, dq\, \delta^4(P - p - q)\, |\langle N^*(P)|\, T\, |\pi(q)\, N(p)\rangle|^2 . \tag{1}$$

In what follows we shall calculate the transition matrix elements making use of the SL(2, C) symmetry of the Regge poles. $\langle N^* \mid T \mid \pi N\rangle = g_J(P^2)$ can be continued analytically not only in P^2 but in J, the spin of N^* as well (the kinematical singularities are separable), and it is evident that this quantity is solely the vertex function of a N^* type Regge pole at $J = \alpha(P^2)$. The fact that the Regge poles are grouped into families makes it possible to connect the residua of the daughters.

Let us now consider the πN backward-scattering amplitude at $u = 0$, $s \gg 0$.



$$s = (p_1 + p_2)^2 , \qquad u = (p_1 + p_4)^2 = P^2,$$

$$2q = p_1 - p_4 , \qquad 2q' = p_2 - p_3 ,$$

The Lorentz-pole terms, giving the main contribution to the amplitude can be written as [4]

$$F_{\lambda\mu}(s, u=0) \sim \sum_{i, j_0} \frac{1}{8\pi} \frac{(\sigma_i^2 - j_0^2)}{\cos \pi \sigma_i} \xi_i \cdot \mathcal{D}^{j_0 \sigma_i}_{\frac{1}{2}\lambda \frac{1}{2}\mu}(L_q L_{q'}^{-1}) T^{\sigma_i j_0} \tag{2}$$

$$\xi_i = \frac{1}{2}\left(1 + \tau \exp\left(i\pi\left(\sigma - \frac{1}{2}\right)\right)\right),$$

τ is the signature factor, $\Gamma^{j_0\sigma}$, $\Gamma^{*j_0\sigma}$ are the factorized residua of the Lorentz poles, $T^{j_0\sigma} = \Gamma^{j_0\sigma}\,\Gamma^{*j_0\sigma}$. Further, we can write

$$\begin{aligned} T^{j_0\sigma}\,\mathcal{D}^{j_0\sigma}_{s\lambda s'\lambda'}(L_q L_{q'}^{-1}) &= \sum_{jm} \langle j_0\,\sigma jm\,|T|\,j_0\,\sigma jm\rangle\,\mathcal{D}^{j_0\sigma}_{s\lambda jm}(L_q)\,\mathcal{D}^{j_0\sigma}_{jms'\lambda'}(L_{q'}^{-1}) = \\ &= \sum_{jm} \langle j_0\,\sigma\,jm\,|T|\,j_0\,\sigma jm\rangle\, d^{j_0\sigma}_{s\lambda j}\left(\frac{q_0}{q}\right) \cdot d^{j}_{\lambda\lambda'}\left(\frac{qq'}{|q|\cdot|q'|}\right) d^{j_0\sigma}_{j\lambda' s'}\left(\frac{q_0'}{q'}\right). \end{aligned} \tag{3}$$

Here $(j_0\sigma\,jm)$ are the Lorentz quantum numbers of the Regge poles which "intermediate" between the initial and final states. The parity is a good quantum number of the poles, so we diagonalize in it. The parity operator P acts on a state $|\,j_0\sigma\;jm\rangle$ as $P|\,j_0\sigma jm\rangle = \eta(-1)^{j-s}\,|\,j_0\sigma jm\rangle$. So we introduce the parity eigenstates as

$$j_0\,\sigma\,jm, s, \pm > \frac{1}{\sqrt{2}}\left\{|\,j_0\,\sigma\,jm, s > \pm\,(-1)^{j-s}\,\eta|-j_0\,\sigma\,jm, s\rangle\right\}.$$

We need not label the reduced matrix element of T with the parity quantum number because $\langle j_0\sigma jm, +\,|\,T\,|\,j_0\sigma jm, +\rangle = \langle j_0\sigma jm, -\,|\,T\,|\,j_0\sigma jm, -\rangle = \eta(-1)^{j-s}\langle -\,j_0\,\sigma jm, +\,|\,T\,|\,j_0\,\sigma jm, +\rangle = -\,\eta(-1)^{j-s}\,\langle j_0\,\sigma jm, -\,|\,T\,|\,-\,j_0\sigma jm, -\rangle$.

There is nothing extraordinary in this; the III. class conspiracy means the same: the residua of the parity doublets are equal at $u = 0$. Introducing the parity quantum number into Eq. (3) we can write:

$$\begin{aligned} &T^{j_0\,\sigma}\,\mathcal{D}^{j_0\sigma}_{s\lambda s'\lambda'} + T^{-j_0\sigma}\;\mathcal{D}^{-j_0\sigma}_{s\lambda s'\lambda'} = \\ &= \sum_{j,m} \frac{1}{2}\left\{(d^{j_0\sigma}_{s\lambda j} + \eta(-1)^{j-s}\,d^{-j_0\sigma}_{s\lambda j})\,\langle j_0\,\sigma jm, s, +\,|T|\,j_0\,\sigma\,jm, s', +\rangle \cdot\right. \\ &\cdot (d^{j_0\sigma}_{j\lambda' s'} + \eta'(-1)^{j'-s'}\,d^{-j_0\sigma}_{j\lambda' s'}) + (d^{j_0\sigma}_{s\lambda j} - \eta\,(-1)^{j-s}\,d^{-j_0}_{s\lambda j}) \cdot \\ &\left.\cdot\,\langle j_0\,\sigma jm, s, -\,|T|\,j_0\,\sigma jm, s', -\rangle \cdot (d^{j_0\sigma}_{j\lambda' s'} - \eta(-1)^{j'-s'}\,d^{-j_0\sigma}_{j\lambda' s'})\right\} d^{j}_{\lambda\lambda}\,. \end{aligned} \tag{4}$$

We suppose that the Lorentz residuum $T^{j_0\sigma}_{s,s'}$ is factorizable: $T^{j_0\sigma}_{s,s'} = \Gamma^{j_0\sigma}_{s}\,\Gamma^{*j_0\sigma}_{s'}$. Hence, if we compare Eqs. (2) and (4) with the ordinary Regge decomposition

we find that the residuum of a pole of parity P, being the $\varkappa$-th member of a family, labelled by (j_0, σ) is:

$$\beta^{\pm}_{\sigma-1-\varkappa} = \Gamma_s^{j_0\sigma}\left(d^{j_0\sigma}_{s\lambda\sigma-1-\varkappa} \pm \eta(-1)^{\sigma-1-\varkappa-s}\, d^{-j_0\sigma}_{s\lambda\sigma-1-\varkappa}\right), \tag{5}$$

where s is the total spin and λ the total helicity of the in (out) going state with which the pole is coupled.

Up to now we have adhered to the point $u = 0$: we apply the SL(2, C) symmetry breaking method [2] to go to the region of resonances. We shall work in the first order of the symmetry breaking. So we write the scattering amplitude as is done in Eq. 14 of [2a] and separate the residue in the same way as we did in the symmetry limit. The result is:

$$\begin{aligned}\beta^{\pm}_{\mu\lambda}(N^* \to N\pi) = \frac{1}{\sqrt{2}} C_I(N^*, N\pi)\{&A\left(d^{j_0\sigma}_{j\lambda 1/2}(x) \pm (-1)^{j+1/2} d^{-j_0\sigma}_{j\lambda 1/2}(x)\right) + \\ &+ W\left[BC^{j_0\sigma+1}_{SL(2,C)}\left(d^{j_0\sigma+1}_{j\lambda 1/2}(x) \pm (-1)^{j+11/2} d^{-j_0\sigma+1}_{j\lambda 1/2}(x)\right) + \right. \\ &+ C\cdot C^{j_0\sigma-1}_{SL(2,C)}\left(d^{j_0\sigma-1}_{j\lambda 1/2}(x) \pm (-1)^{j+1/2} d^{-j_0\sigma-1}_{j\lambda' 2}(x)\right) + \\ &\left.+ D\cdot C^{j_0-1\sigma}_{SL(2,C)}\left(d^{j_0-1\sigma}_{j\lambda 1/2}(x) \pm (-1)^{j_0+1/2} d^{-j_0+1\sigma}_{j\lambda 1/2}(x)\right)\right]\} d^{j}_{\mu\lambda}(\vartheta)\end{aligned} \tag{6}$$

and

$$\Gamma(N^* \to N\pi) = \frac{1}{2\pi}\,\frac{1}{2j+1}\,\frac{m_r}{M}\,p\sum_{\mu\lambda}|\beta_{\mu\lambda}|^2.$$

In the case of a πN system we have only three breaking terms because of the constraint for the symmetry limit: $|j_0| = 1/2$. In Eq. (16) j is the spin-parity of the resonance, μ is its spin-projection quantized along the z-axis of a coordinate system in which the three-momentum of the N^* is zero, λ is the helicity of the nucleon. The index σ is a half integer denoting the actual family to which the resonance belongs. W stands for the mass of the resonance, $W = M$ for the resonances of natural parity, and $W = -M$ for those of unnatural parity. As can easily be seen [5]:

$$\begin{aligned}x &= \frac{1}{4uq^2}\left[m_N^2 - m_\pi^2 - \sqrt{-4uq^2 + (m_N^2 - m_\pi^2)^2}\right] = \\ &= \frac{(m_N^2 - m_\pi^2)}{4uq^2}\left[1 - \frac{2Wp}{m_N^2 - m_\pi^2}\right]^2,\end{aligned}$$

where $u = W^2 = M^2$, $4q^2 = 2(m_N^2 + m_\pi^2) - u$ and p is the magnitude of the three-momentum of the pion and nucleon in the final state.

Now we have to say a few words about the "reduced matrix elements" A, B, C, D. As an example we take A. It consists of a $\sqrt{\sigma^2 - j_0^2}$ factor and

a function $A'(u)$. For compensating the singularity of the $d^{J_0\sigma}$ functions at the point $u = 2(m_N^2 + m_\pi^2)$ we write $A'(u)$ as

$$A'(u) = \left(\frac{4q^2}{u_0}\right)^{1/2(\sigma-1)} g(u) \tag{7}$$

and suppose $g(u)$ to be a smooth function of u.

Finally, we notice the factor $1 \pm (-1)^{J+1/2}\sqrt{x}$ in $\beta_{\mu\lambda}$ coming from the combination $d^{1/2\sigma} \pm (-1)^{J+1/2} d^{-1/2\sigma}$. To have the well known threshold behaviour we define the physical sheet by the prescription:

$$\sqrt{x} = \frac{m_N^2 - m_\pi^2}{2W\sqrt{q^2}} \left(1 - \sqrt{-4uq^2 + (m_N^2 - m_\pi^2)^2}\right)$$

for the resonances of natural parity ($W = M$), and

$$\sqrt{x} = \frac{m_N^2 - m_\pi^2}{-2W\sqrt{q^2}} \left(1 - \sqrt{-4uq^2 + (m_N^2 - m_\pi^2)^2}\right)$$

for the resonances of unnatural parity ($-W = M$). C_I and $C^{j_0\sigma}_{SL(2,C)}$ in Eq. (6) are isospin and SL(2, C), Clebsch—Gordan coefficients,

$$C^{j_0'\sigma'}_{SL(2,C)} = \langle \sigma' j_0' jm; 1000 \,|\, \sigma j_0 jm \rangle.$$

The following interpretation differs from that of Eq. (6) in [2]. However, it was indicated for us by the authors of [2]. In the original form of Eq. (6) every quantity is to be taken at $u = 0$. But this is not necessary, as can be seen from a consideration of the following.

A $F_{\lambda\mu\lambda'\mu'}$ scattering amplitude is the function of the six invariants P^2, Pq, Pq', qq', q^2, q'^2. When introducing $F_{\lambda\mu\lambda'\mu'}$ over a group, we sought a group G so that if $g \in G$, $gP = P$, but $gq \neq q$. If $P = 0$, this group is the SL(2, C). If $P \neq 0$, only the Pq, Pq, type quantities break the invariance but p^2 does not. In this way, we expand $F_{\lambda\mu\lambda'\mu'}$ into Taylor series in Pq, Pq', but not in P^2. That is to say, in Eq. (6) every quantity has a P^2-dependence. The further steps are the same as in the previous case, so the final form remains the same.

The five unknown functions what would be in the general case in Eq. (6) can be chosen to be real: at $u = 0$ where only the symmetric term is not zero, the trajectory is real so the residue is real as well. As we neglect the imaginary part of the trajectory throughout our calculation, it is consistent to take the residua to be real. (There is another argument, leading to the same result. The first derivative the of residue function that gives the first order symmetry breaking term transforms as a vector. But only two types of vectors

can be composed out of the operators we have: q_μ and γ_μ; each yields a complex parameter, so the total number of the parameters is four.)

III

After summarizing the main point we apply the method for getting the elastic decay width of πN resonances. For numerical calculations we have chosen the $I = 3/2$, $\sigma = 9/2$ and $I = 1/2$, $\sigma = 7/2$ families classified in [3]. To reduce the work we have taken degenerate masses in the families, except for calculating the phase spaces. The central masses were obtained from the symmetry limit of the trajectory formula fitted in [3].

$$a) \quad I = \frac{3}{2} .$$

The central mass is $M_0 = 1.94$ GeV. From a least squares fit we obtained the following values for the parameters defined as $a = 1/\sqrt{2\pi}\, x^{1/2(\sigma-1)} A$, $b = 1/\sqrt{2\pi}\, x^{1/2\sigma} B$, etc.: $a = 2.15$, $b = -0.40$, $c = 0.00$, $d = -0.45$. In the symmetry limit $a = 2.29$. The results for the widths, summarized in Table I, are in good agreement with experiment. The width of the missing G_{37} resonance is predicted with the same mass value as that of F_{37}.

$$b) \quad I = \frac{1}{2} .$$

The central mass is $M_0 = 1.66$ GeV. Parameters are: $a = 2.60$, $b = 1.25$, $c = -0.48$, $d = 0.53$. In the symmetry limit $a = 2.56$. As can be seen from Table II, there is a problem connected with the S_{11} resonance. Either the $\Gamma_{el} = 186$ MeV is right for the $N^*(1550)$, or the resonance $N^*(1700)$ belongs to the $\sigma = 7/2$ family.

To get information on the u-dependence of the function $g(u)$ in (7), we evaluated some other elastic widths in symmetry limit supposing u_0 to be constant. We obtained:

$$c) \quad I = \frac{3}{2} .$$

	theor (MeV)		exp (MeV) [7]
$P_{33}(1236)$	6.29	$g^2 u_0^2 = 120$	120
$H_{311}(2420)$	$2.30 \cdot 10^5$	$g^2 u_0^{-2} = 30$	34
$J_{315}(2850)$	$1.28 \cdot 10^8$	$g^2 u_0^{-4} = 13$	13
$L_{319}(3230)$	$9.79 \cdot 10^{10}$	$g^2 u_0^{-6} = 300$	2

For the neighbours P_{33} and H_{311} of the fitted family taking the same value of $g(u)$ as it is at $u = 1.94^2$ GeV^2, $\sqrt{20}g(1.94^2) = 0.22$ we obtained nearly the right widths if $u_0 = 20$ GeV^2. If we desire a qualitatively nice picture in the symmetry limit, $g(u)$ must decrease when u is increasing. To obtain the cross-section of $N\pi$ backward scattering the $g(u)$ function has to decrease again at small u values.

d) $I = \frac{1}{2}$.

	theor(MeV)[7]		exp(MeV)[7]
$G_{17}(2190)$	$16.3 \cdot 10^3$	$\tilde{g}^2 u_0^{-2} = 68$	75
$I_{11}(2650)$	$61.2 \cdot 10^5$	$\tilde{g}^2 u_0^{-4} = 170$	27
$K_{115}(3030)$	$12.4 \cdot 10^8$	$\tilde{g}^2 u_0^{-6} = 240$	2.5

For the πN coupling constant we have taken: $\tilde{g}^2/4\pi = 15$. Again taking $\tilde{g}(u)$ at $u = 1.66^2$ GeV^2, $u_0 = 12$ GeV^2. We evaluated the width of $S_{11}(1550)$ supposing to be the Mac Dowell pair of the nucleon, with $u_0 = 12$, $g(1.55^2) = g(1.66^2)$. The result is wrong (3 GeV). However, the results are wonderful for $\pi^+ p \to \pi^+ p$ backward scattering if $\tilde{g}(0) = \tilde{g}(1.66^2)$. (We left out the small contribution of the Δ-trajectory.) [6].

p_{lab}(GeV/c)			5.9	9.9	13.7	17.1
$\frac{d\sigma}{du}(\pi^+ p \to \pi^+ p)\|_{u=0}$	theor.	μ barn / GeV	16	4	1 75	1
	exp.		21 ± 1	6 ± 0.5	3 ± 0.5	2 ± 1

Acknowledgement

We are grateful to Drs. G. Domokos and P. Surányi for valuable discussions. The help of F. Telbisz and G. Vesztergombi in the numerical work is highly appreciated.

Table I

$I = 3/2$ resonance widths (elastic width in MeV)

	exp. [7]	in symmetry limit	with first order symmetry breaking
$F_{37}(1920)$	85	53.5	84.5
$D_{35}(1954)$	47	46	35
$P_{33}(1688)$	28	64	53
$S_{31}(1670)$	50	66	41
$G_{37}(1920)$	—	53	20
$F_{35}(1913)$	57	45	46
$D_{33}(1690)$	37	64	60
$P_{31}(1934)$	101	74	91

Table II

$I = 1/2$ resonance widths (elastic width in MeV)

	exp. [7]	in symmetry limit	with first order symmetry breaking
$D_{15}(1680)$	68	81	77
$P_{13}(1530)$	—	105	130
$S_{11}(1550)$ ROSENFELD	39	156	182
$S_{11}(1550)$ LOVELACE	186		
$S_{11}(1710)$	240	180	205
$F_{15}(1690)$	85	81	92
$D_{13}(1530)$	76	105	76
$P_{11}(1466)$	138	144	133

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РЕЗОНАНСНЫЕ ШИРИНЫ $\pi - N$ В МОДЕЛИ НАРУШЕННОЙ SL (2, C)

К. СЕГЭ и К. ТОТ

Р е з ю м е

Исследуется ширина упругого распада некоторых резонансов πN в модели *SL* (2, C) полюсов Редже. Два семейства резонансов рассматриваются в первой степени наружения симметрии, одно из которых имеет изотопический спин $I = 1/2$, другое $I = 3/2$. Ширина других резонансов вдоль траекторий вычислялась в лимите симметрии, дифференциальное поперечное сечение рассматривается для πN рассеяния назад. Результаты хорошо согласуются с экспериментальными данными.

Acta Physica Academiae Scientiarum Hungaricae 26 (1—2), Tomus, pp. 65—69 (1969)

THE MULTIPERIPHERAL MODEL

By

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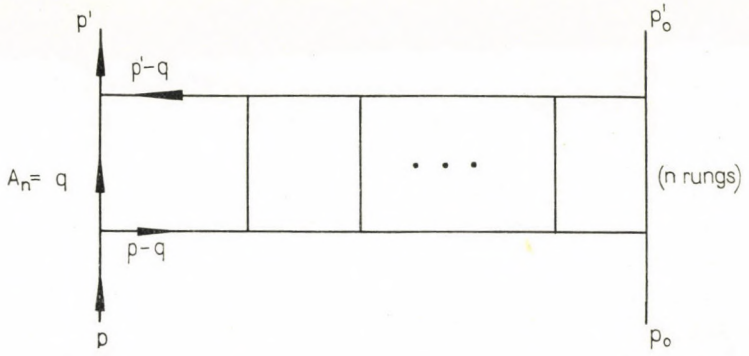
The multiperipheral Regge model attempts to substitute a more plausible model for the n-particle production amplitude which is significant in unitarity-type equations.

Let me start with a review of the method of AMATI, FUBINI and STANGHELLINI. They calculate the imaginary part of the scattering amplitude by adding multiperipheral contributions to the unitary equation, thus if A_n is the contribution from n mesons (in a hypothetical scalar meson model):



or, in terms of a recursion relation,

$$A_n(p, p'; p_0, p_0') = g^2 \int \frac{d^4q\,\delta(q^2 + m^2)}{[(p-q)^2 + m^2]\,[(p'-q)^2 + m^2]} A_{n-1}(p-q, p'-q; p_0, p_0'),$$

$$= \int k(p, p', q)\, A_{n-1}\, dq. \qquad (1)$$

The physical absorptive part is given by $A = \sum_{n=2}^{\infty} A_n$, which thus satisfies the integral equation

$$A(p, p'; p_0, p_0') = A_2 + \int k(p, p', q)\, A(p-q, p'-q; p_0, p_0'). \qquad (2)$$

The invariance of the kernel of this equation with respect to Lorentz transformation of p, p' and q (but not p_0 or p_0') leads to an asymptotic power law behavior for the solution of the homogeneous equation, A_h:

$$A_h \sim s^{\alpha(t)}\, a(t, p^2, p'^2)\,, \tag{3}$$

where $t = -(p - p')^2$ and $s = -(p + p_0)^2$ are the usual momentum transfer and energy variables.

Since the inhomogeneity A_2 goes like $1/s$, any $\alpha(t)$ greater than -1 will dominate the solution.

We know, in fact, that the term A_2 is completely extinguished. To see this, we write

$$A(s, t) \cong \int_C d\tau\, A(\tau, t)\, s^\tau\, d^\tau \tag{4}$$

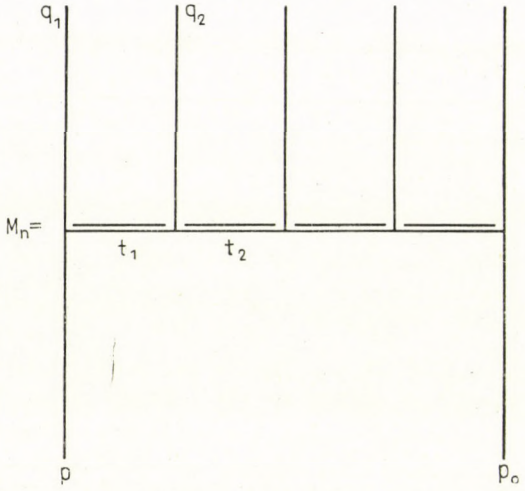
The solution is then, roughly,

$$A(\tau) \sim \frac{A_2(\tau)}{D(\tau)}\,, \tag{5}$$

where $A_2(\tau)$ has a fixed pole at $J = -1$ corresponding to the $1/s$ behavior of $A_2(s)$. However, $D(J)$ also has a fixed pole at $J = -1$, so that in the ratio there is no pole at $J = -1$, and hence a pure power. (Note that this argument does not apply to the extinction of cuts, since branch points do not cancel like poles.)

The extension I will now discuss grew out of conversation with M. L. GOLDBERGER and G. F. CHEW, and was particularly stimulated by the work of CHEW and PIGNIOTTI.

The multiperipheral Regge model attempts to substitute a more plausible model for the n particle production amplitude, to wit:

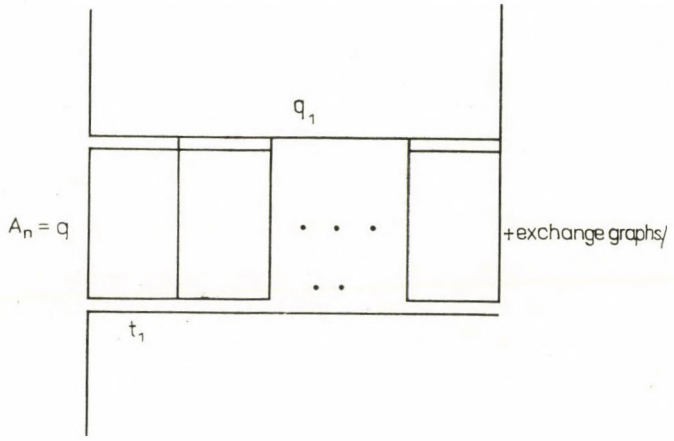


(6)

where the heavy lines indicate Regge rather than particle exchange (we assume one Regge input pole α_0. for simplicity). The corresponding formula has

$$\prod_{i=1}^{n-1} (s_{i,\,i+1})^{\alpha_0(t_i)+\alpha_0(t_{i'})}\, \beta(t_i, t_{ir'}, \varphi_i)\, \beta^*(t_{i'}, t_{ir'}, \varphi_i)\,, \tag{7}$$

(where $s_{i,i+1} = -(q_i + q_{i+1})^2$ and φ_i is a Toller angle) in place of the simple Feynman denominators of the multiperipheral model. The diagram for A_n becomes



Clearly, it does not satisfy a simple recursion relation since a correlation s_{12} has to be introduced.

We may, however, back off one integration, and define a new function B_n which leaves out the last q integral. Thus:

$$A_n = \int d^4q B_n(p, p', q; p_0 p_0')\, g(q^2 + m^2)\, \beta_0(t_i)\, \beta_0^+\, (t_i') \tag{8}$$

and B_n satisfies the equation:

$$\begin{aligned} B_n(p, p', q_i p_0 p_0') = \int d^4 q_1\, \delta(q_1^2 + m^2)\, [-(q + q_1)^2]^{\alpha_0(p-q)+\alpha_0(p'-s)} \\ \beta(p - q, p - q - q_1)\, \beta^*(p' - q,\ p' - q - q_1) \\ B_{n-1}(p - q, p' - q, q_1\,;\, p_0, p_0')\,, \end{aligned} \tag{9}$$

for which again an integral equation for $B = \sum_{n=2}^{\infty} B_n$ can be derived. Again, the Lorentz invariance of the kernel leads to an asymptotic $s_1^{\alpha(t)}$ dependence of B (where $s_1 = -(p - q + p_0)^2$) and a corresponding $s^{\alpha(t)}$ behavior of A.

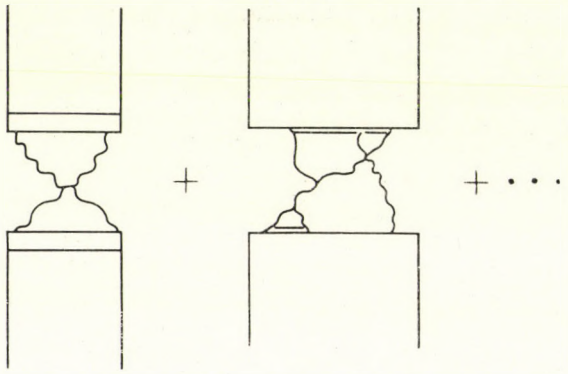
The following points are significant:

i. The term A_2 is not now given by a fixed J pole at $J = -1$, but by a cut running up to

$$\tau = 2\alpha(0) - 1 \qquad (\text{for } t = 0)\,.$$

Thus, as I remarked before, there is no extinction, but, if the cut is sufficiently concentrated to resemble a pole, a considerable suppression is possible.

ii. The method evidently extends to an arbitrary (but finitely) correlated approximation: if instead of q and q_1 correlations one has q, q_1 and q_2 correlations one must back off two q integrals, etc. Also, clearly, exchange terms will modify the $\delta(q^2 + m^2)$ into the absorptive part of the crossed diagrams



Finally, the correlation need not have the simple form suggested in (7), but can be any Lorentz invariant function of p, p', q_1, q_i ... q_N, where N is the number of correlated exchanges.

iii. The FINKELSTEIN—KAJANTIS difficulty with $\alpha_0(0) = 1$ emerges in a very simple way. One has, in the most naive possible approximation, the following kind of eigenvalue equation for $\alpha(0)$:

$$1 = \int_{-\infty}^{0} \frac{|\psi(t)|^2\, dt}{\alpha(0) - (2\alpha_0(t) - 1)}\,, \tag{10}$$

where ψ is related to the residue functions β.

Now, if $\alpha_0(0) = 1$, then we obviously have $\alpha > 1$, violating Froissart's unitarity bound. Thus, if $\alpha_0(0) = 1$, $\psi(0)$ must equal zero, *and* $\alpha(0)$ must, for consistency, come out equal to one. This requires an apparent double effect of unitarity (through Froissart), and appears fairly unlikely to me at the moment. More likely is the result $\alpha_0(0) < 1$ and thus, consistently,

$$\alpha(0) = \alpha_0(0) > 2\alpha_0(0) - 1\,.$$

One can only decide which choice is made when two particle unitarity has been used to calculate A_2 in terms of $\int M^* M d\Omega$, where M is the elastic amplitude which has been consistently calculated from A itself. This is a horribly non-linear functional integral equation, but it may represent the first genuine boot-strap possibility. So far, I have no results to report on it.

МУЛЬТИПЕРИФЕРИЧЕСКАЯ МОДЕЛЬ

Ф. Е. ЛАУ

Р е з ю м е

Мультипериферическая модель Регге пытается дать более правдоподобную модель амплитуды образования n-частиц, которая играет заметную роль в уравнениях типа унитарности.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 71—76 (1969)

THE VENEZIANO FORM

By

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The VENEZIANO form is a model scattering amplitude which satisfies all finite energy sum rules. Its properties, some simple consequences, and the technique of constructing from it amplitudes with specified crossing symmetry are briefly described. Also the relation of the 'Schmid circles' computed from it to its resonances is discussed.

The VENEZIANO form [1] is a model scattering amplitude which has Regge trajectories of resonances, and Regge asymptotic behaviour, in two channels; it thus satisfies all superconvergence relations in *both* channels. It is real (the resonance poles are on the real axes) and so doesn't satisfy unitarity, but it is the solution to what MANDELSTAM [2] called the 'zero order problem' in which unitarity and hence $Im\alpha$ is neglected, in his scheme of bootstrapping Regge trajectories; the V-form contains all the information that has been gotten piecemeal using finite energy sum rules in the zero width approximation. It is not yet clear how unique the V-form is, aside from trivial modifications (see below for some) such as taking linear combinations of different V-forms. It *looks* unique in the sense that it is hard to see what another would look like, but of course that was true even for this one.

The V-form is fundamentally a product representation,

$$V(\sigma,\tau)=\frac{1\cdot 2\cdot 3\ldots(-\sigma-\tau)(1-\sigma-\tau)\ldots}{(-\sigma)(1-\sigma)\ldots(-\tau)(1-\tau)\ldots}, \tag{1}$$

where $\sigma = a_s s + b_s$, $\tau = a_t t + b_t$ are the leading trajectories in the s and t channels, respectively. The σ-factors in the denominator give resonance poles at integers $\sigma = 0, 1, 2, \ldots$; similarly for τ (henceforth, we shall not continually draw attention to the symmetry between σ and τ). The factors $1\cdot 2\cdot\ldots$ in the numerator simply make the infinite product finite. The numerator factors $(-\sigma-\tau)(1-\sigma-\tau)$ cancel against the τ-factors of the denominator when σ approaches an integer, in such a way that

$$V(\sigma,\tau)\xrightarrow{\sigma\to n}\frac{a_0+a_1\tau+\ldots+a_n\tau^n}{n-\sigma}+\text{finite}. \tag{2}$$

Since the trajectory $\tau(t)$ was *assumed linear* in t, the residue of the $\sigma = n$ pole term is a polynomial in z, and so (2) implies that the highest spin resonance at $\sigma = n$ has $J = n$, which was required by "σ is the leading s-channel trajectory". In general, the polynomial in t is a linear combination of *all* $P_l(z)$, $0 \leq l \leq n$. The 'secondary resonances', $l < n$, lie on trajectories σ_m which are integrally spaced from the leading trajectory: $\sigma_m = \sigma - m$, $m = 1, 2, \ldots$ [They are not "daughters" in the technical sense of FREEDMAN—WANG, TOLLER, et al.] Their occurrence in this model is inescapable, but may not be too unrealistic: One sees in the $\pi^+\pi^-$ channel a 0^+ (730) which is nearly degenerate with the $1^-\varrho$ (765), and a $1^-\varrho'$ ($\approx$ 1100) which is somewhat degenerate with the 2^+ $f_0(1260)$. The 0^+ which should be degenerate with the last two may be too broad and/or inelastic to be easily seen; the ϱ' ($\approx$ 1100) is quite inelastic. Also, the P_{11} "Roper" πN resonance, $N(1470)$, might lie on a first secondary trajectory of the N.

Asymptotically, $V(\sigma, \tau)$ has Regge behaviour. This can be exhibited directly, but perhaps it is better to first express $V(\sigma, \tau)$ in terms of the factorial function

$$\begin{aligned} V(\sigma,\tau) &= \lim_{N\to\infty} \frac{1\cdot 2\ldots N(-\sigma-\tau)(1-\sigma-\tau)\ldots(N-\sigma-\tau)}{(-\sigma)(1-\sigma)\ldots(N-\sigma)(-\tau)(1-\tau)\ldots(N-\tau)} \\ &= \lim_{N\to\infty} \frac{N!(N-\sigma-\tau)!/(-1-\sigma-\tau)!}{[(N-\sigma)!/(-1-\sigma)!]\,[(N-\tau)!/(-1-\tau)!]} \\ &= \frac{(-1-\sigma)!\,(-1-\tau)!}{(-1-\sigma-\tau)!}, \end{aligned} \tag{3}$$

where we used $(A+x)! \xrightarrow{|A|\to\infty} A!\,A^*$, A not negative real. Using this last again,

$$V(\sigma,\tau) \xrightarrow{|\sigma|\to\infty} (-1-\tau)!\,(-\sigma)^\tau, \quad \sigma \text{ not positive real}, \tag{4}$$

which is Regge behaviour, because σ is linear in s. For the case that σ is positive real, it is best to transform V again, using $(-x)!\,x! = \pi x/\sin \pi x$.

$$V(\sigma,\tau) = -\pi \frac{(\sigma+\tau)!}{\sigma!\;\tau!}(\cot \pi\sigma + \cot \pi\tau). \tag{5}$$

Here the resonance poles at $\sigma = 0, 1, \ldots$ are exhibited as poles of $\cot \pi\sigma$. Comparing with (4), we see that V would have Regge behaviour as $\sigma \to +\infty$ if $\cot \pi\sigma$ in (5) were replaced by $-i$. The average of $\cot \pi\sigma$ for fixed positive (but arbitrarily small) $Im\sigma$ is $-i$, so in this sense $V(\sigma, \tau)$ does have Regge behaviour 'on the average' for positive σ. In other words, since $V(\sigma, \tau)$ is real, it can have the correct (Regge) imaginary part for real σ (= real s) only in this average sense [cut approximated by sequence of poles.] Thus V necessarily has the HORN—SCHMID [3] duality: Its imaginary part is entirely given by resonances.

We digress briefly to remark on another aspect of duality: SCHMID [4] has noted that if one takes the partial wave projections R_l of a Regge amplitude R in which the exchanged trajectories are rising ones, one finds that each R_l as a function of energy performs circles, which are qualitatively similar to resonance circles. In a realistic case, namely πN scattering, COLLINS, JOHNSON and SQUIRES [5] found considerable agreement between the "SCHMID circles" and the actual resonance circles of the πN partial wave amplitudes. This raises the question: Is this coincidental, or is there a good reason for the agreement? [We shall ignore the alternative suggestion of SQUIRES et al. that the observed circles are *only* SCHMID circles and do not correspond to resonances at all; the resonance character of at least half a dozen is experimentally demonstrated by now. However, it may well be true that if the true width of a resonance (as defined by the position of its pole) is large compared to the 'width' of the corresponding SCHMID circle, the latter behaviour will dominate the energy dependence of the amplitude.] *The V-form is a model in which this agreement occurs* (subject to one specializing assumption): The Regge form corresponding to V is obtained, as we said above, by replacing $\cot \pi\sigma$, in (5) by $-i$, thus

$$R(\sigma, \tau) = \pi \frac{(\sigma + \tau)!}{\sigma!\ \tau!} (i - \cot \pi\tau) \equiv \bar{R} + i\hat{R}. \tag{6}$$

As functions of z, the zeros of $\bar{R}$ and $\hat{R}$ interlace, since they are at the zeros and poles, respectively, of $\cot \pi\sigma$. Thus the partial wave projections $\bar{R}_l$ and $\hat{R}_l$ tend to be out of phase, which implies circles in the $R_l = \bar{R}_l + i\hat{R}_l$ plane; a little more detailed consideration shows that the circles will be counterclockwise for increasing energy. So far, this is generally true for any Regge amplitude with rising trajectories. But now take the case of equal external masses (more specifically, equal masses in either the initial or final s-channel states) and trajectory parameters a and b (intercepts and slopes) such that the odd secondary trajectories are uncoupled; this condition is

$$a_s = a_t, \quad 2b_t + b_s + a_s \sum + 1 = 0, \text{ where } \sum \equiv \sum_{i=1}^{4} m_i^2 = s + t + u. \tag{7}$$

The reason why the odd trajectories are then uncoupled, is that the residue of the pole [$= \hat{R}$] at $\sigma = n$ is then an even (odd) function of z when n is even (odd). Just the opposite is true for $\bar{R}$, because of the factor $\cot \pi\tau$. Hence

$$\begin{aligned} \hat{R}_l &= 0, \qquad l = 0, 1, 2, \ldots, l - 1, l + 1, l + 3, \ldots \\ \bar{R}_l &= 0, \qquad l = \ldots, l - 2, l, l + 2, \ldots \end{aligned} \tag{8}$$

Explicit calculation shows, as might be expected, that $\hat{R}_l$ reaches maxima roughly at $\sigma = l,\ l+2,\ \ldots$, which is also where $\bar{R}_l$ vanishes. Hence the circles of R_l "go over the top", i.e. appear to resonate, at $\sigma = l,\ l+2,\ \ldots$, which are just the resonance energies of the original complete V-form. But if the trajectory parameters are not such that the odd secondary trajectories are uncoupled, the number of resonances doubles, but the SCHMID circles remain qualitatively the same. Hence in this case the SCHMID circles do not have a close, one-to-one correspondence with resonances.

We now briefly discuss generalizations of the V-form. One may remove the ground state s-channel resonance by multiplying by σ. (Note that cancellation by *subtracting* a $1/\sigma$ term would introduce a non-Regge "elementary particle" term.) This promotes by 1 the degrees of the polynomials in σ which are the residues of the t-channel poles; thus we must replace τ by $\tau - 1$ in order that τ continues to signify the leading trajectory in the t-channel. Thus the modified V-form is

$$V_1(\sigma, \tau) = \frac{1 \cdot 2 \cdot 3(1-\sigma-\tau)\,(2-\sigma-\tau)\ldots}{(1-\sigma)\,(2-\sigma)\ldots(1-\tau)\,(2-\tau)\ldots} \,. \tag{1'}$$

Notice that the V-form remained symmetric in σ and τ; removal of the ground state in the s-channel implied removal of the ground state in the t-channel as well. We can repeat the process and define a V_p in which the resonances at $\sigma = 0,\ 1,\ \ldots,\ p-1$ have been removed; but we shall find immediately that only the cases $p = 0$ and $p = 1$ can occur.

So far, our discussion has been limited to spinless external particles. The starting point for treating spin is the remark that the V-form

$$V(\sigma - \lambda,\ \tau - \mu)$$

is a model for an invariant (singularity free) amplitude for a process in which the minimum channel helicities are λ, μ respectively (by channel helicity, we mean the larger of the absolute values of the spin projection on the relative momentum in the initial and final states.)

We now point out the existence of constraints on the trajectory parameters. It almost goes without saying that the slopes a_s, a_t must be real and positive: real so that $V(\sigma, \tau)$ is real below threshold, and positive so that spins increase with energy and there is a ground state. The intercepts b_s, b_t again must be real. Now, if $b_t > p$ the first t-channel resonances, at $\tau = p$, would occur at $t < 0$; this is bad. Likewise, $b_t > p - 1$ is bad because this would mean that the highest zeros of the residues of the s-channel pole residues, at $\tau = p - 1$ $[V_p \xrightarrow{\sigma \to n} (\tau + n - p)\ldots(\tau + 1 - p)/(n - \sigma),\ n \geqslant p]$ would occur at $t > 0$; but for elastic scattering, $Im\, f = \sum_l (2l+1)\, Im f_l P_l(z) > 0$ for $z > 1$ and hence for $t > 0$, for large s. Hence we find

$$p - 1 < b_t < p. \tag{10}$$

Invoking FROISSART, i.e. $b_t \leqslant 1$, we see that $p \leqslant 1$, i.e. only $p = 0$ and $p = 1$ are possible.

Stronger constraints will result if the resonance residues (i.e. partial widths) are required to be non-negative. For instance, this condition applied to the first secondary trajectory, in both the s and t channels, results in

$$a_s = a_t,\ 2b_t + b_s + a\,\Sigma + 1 > 0,\ 2b_s + b_t + a\,\Sigma + 1 > 0. \tag{11}$$

The first result is the most interesting, namely the equality of slopes in the two channels. Perhaps this is the simplest way to explain the observed fact that the slopes of all hadron trajectories are equal.

As a final topic, we briefly discuss the construction of model scattering amplitudes with specified crossing symmetries as linear combinations of V forms. The simple $V(\sigma, \tau)$ represents a scattering amplitude, for spinless particles, with no resonances in the u channel (it correspondingly exhibits exchange degeneracy, i.e. the s and t channel trajectories do not have pure signature, but instead have resonances with l both even and odd.). For instance, let us construct the $\pi - \pi$ scattering amplitudes, A_{I_t}. On the hypothesis that there are no $I = 2$ $\pi - \pi$ resonances, we have simply $A_2 = V_1(\sigma, v)$ up to a constant factor, where v is the u channel trajectory. (We had to choose $p = 1$ because it is known that the intercept of the leading trajectory lies above zero.) Permuting, s, t, and u and using the well known crossing relations, one arrives at

$$A_0(s, t, u) = 3/2\,V_1(\tau, \sigma) + 3/2\,V_1(\tau, v) - 1/2\,V_1(\sigma, v),$$
$$A_1(s, t, u) = V_1(\tau, \sigma) - V_1(\tau, v),$$
$$A_2(s, t, u) =$$

We have omitted to write an overall constant factor; $V_1(\sigma, v)$. It is fixed by, say, the width of the ϱ. This now predicts the partial widths into two pions of the resonances on the $\varrho - f$ trajectory and its secondary trajectories.

A more elaborate example is provided by the process $\pi\pi \to \pi A_2$. It is found that the six amplitudes ($6 = 3 \times 2$; 3 i-spin states and 2 spin states) can be written as linear combinations of the six V-forms $V(\sigma - 2, \tau - 1)$, with s, t, and u permuted. The proper amplitudes to represent this way are the invariant amplitudes, which RUBINSTEIN et al. [6] have noted to have simple crossing symmetry.

A glaring omission from (12) is the POMERON. The POMERON does not fit into the VENEZIANO form, and hence presumably into the 'zero order approximation', for at least three reasons: It has a definite signature, it does

not have the same slope as other hadron trajectories, and it has a strongly energy dependent residue. Probably it is best regarded as something that will appear when, but not before, unitarity is taken into account. In the meantime, it possibly could be simply added to amplitudes such as (12), if desired; HARARI [7] has expressed this view of the approximate independence of the POMERON and the resonance-trajectories.

I would like to acknowledge the benefit I have had of conversations with B. SAKITA and M. VIRASORO on these matters.

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ФОРМА ВЕНЕЗИАНО

Ч. ГЭБЕЛ

Р е з ю м е

Форма Венезиано — модельная амплитуда рассеяния, удовлетворяющая всем правилам сумм конечных энергий. Коротко описываются ее свойства, некоторые простые следствия и техника конструкции амплитуд со специфической симметрией скрещения. Также дискутируется связь кругов Шмидта, вычисленных по модели, к ее резонансам.

Acta Physica Academice Scientiarum Hungaricae, Tomus 26 (1—2), pp. 77—79 (1969)

UNITARITY AND NON-REGGE SINGULARITIES

By

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Continuing a relativistic amplitude into the complex angular momentum plane, new singularities are forced to appear from the crossed channel unitarity requirement.

As well known, Regge-pole theory, though rigorously proved only in the framework of potential scattering, succeeded in explaining many important features of the elementary particle phenomenology. It was, however, realized that in continuing a relativistic amplitude into the complex angular momentum plane, new singularities are forced to appear from the crossed channel unitarity requirement.

Since 1962, GRIBOV and POMERANCHUK [1] noticed that the presence of the third double spectral function ϱ_{tu} entails fixed poles at wrong signature non-sense points in the left-hand discontinuity of the partial-wave amplitude that are turned into essential singularities for the amplitude in the elastic approximation. At the same time FUBINI and his coworkers [2] found that a Reggeon diagram, when iterated in the crossed channel, gives rise to moving branch points in the angular momentum plane.

As shown later by MANDELSTAM [3], these branch points can appear on the physical sheet only if in the Reggeon production amplitude effects coming from ϱ_{tu} are taken into account. On the other side these branches provide the only way in order to avoid the GRIBOV—POMERANCHUK essential singularity, reconciling the presence of fixed poles with the complete unitarity constraint. In this scheme one recognizes that inelastic contributions are responsible for the appearance of both kinds of non-Regge singularities; the underlying dynamical connection, however, in our opinion, is not yet fully expounded.

We should like in this note to reobtain from a different point of view and to generalize some known results as well as to discuss the subject on a general basis. The most natural way of handling inelastic processes in the N/D method is by considering the FRYE—WARNOCK integral equation [4], that in the equal-mass spinless case is:

$$\varphi(l,\nu) = Z(l,\nu) + \frac{1}{\pi}\int_0^\infty \frac{(\nu''-\nu_0)Z(l,\nu'') - (\nu-\nu_0)\,Z(l,\nu)}{(\nu''-\nu)\,(\nu''-\nu_0)\,\eta\,(l,\nu'')}\,\varrho(\nu'')\nu''^{l}\,\varphi(l,\nu'')\,d\nu'' \quad (1)$$

with:

$$Z(l,\nu) = \frac{1}{\pi}\int_L \frac{\Delta A(l,\nu')}{\nu'-\nu}\,d\nu' + \frac{P}{2\pi}\int_0^\infty \frac{1-\eta(l,\nu')}{\varrho(\nu')\,\nu'^{l}}\,\frac{d\nu'}{\nu'-\nu} \quad (2)$$

and

$$\varphi(l,\nu) = -\frac{\eta(l,\nu)\,ImD(l,\nu)}{\varrho(\nu)\,\nu^{l}}\,. \quad (3)$$

As usual, $A(l,\nu) = N(l,\nu)/(D(l,\nu)$ is the partial wave amplitude and $\eta(l,\nu)$ the inelasticity factor. Moreover for real l and ν above threshold:

$$Im\,A(\,,\nu) = \frac{1}{2\varrho(\nu)\,\nu^{l}}\left[1 - \frac{1-[Im\,D(l,\nu)/Re\,D(l,\nu)]^2}{[\,1+Im\,D(l,\nu)/Re\,D(l,\nu)]^2}\,\eta(l,\nu)\right]. \quad (4)$$

Now the left-hand discontinuity of the amplitude $\Delta A(l,\nu)$ has a fixed pole at $l=-1$ if ϱ_{tu} is present, and this pole, through the function $Z(l,\nu)$, enters the kernel of eq. (1) and makes it singular unless the function η in the denominator has a pole at the same position. This mechanism avoids the generation of the GRIBOV—POMERANCHUK singularity, leaving a fixed pole in the amplitude coming from the known term. On the other side eq. (4) shows that this choice is a consistent one. We notice that, for physical values of l, η has to be one below the inelastic threshold and can develop a fixed pole at $l=-1$, only if a branch point of the amplitude emerges from the inelastic threshold in the ν-plane and reaches the origin for $l=-1$.

All this picture [5] agrees in its conclusions with the one of [6]. Moreover we have generalized it to the case where a coupled-channel process is considered with particles having different masses and spins. The WARNOCK procedure [7] enabled us to perform a matrix N/D calculation with inelastic contribution in every channel. The formalism is rather involved and will be reported elsewhere [8]; here we summarize the main results we have obtained. We found that only the poles present in the "forces" are reproduced in the amplitude in a way consistent with unitarity; this is not a trivial fact since "a priori" unitarity mixes the singularities of different channels.

The MANDELSTAM cuts are still needed in the proof and this time they will affect the whole matrix amplitude. Again this can be easily understood on the basis of unitarity arguments.

The singularities in the inelasticity function have been so far requested by a general consistency condition. Now, η is related from eq. (4) to $Im\,A(l,\nu)$, that in turn receives the main contribution from the double spectral function ϱ_{st}. On the other hand η has to contain some information coming from ϱ_{tu}, which is the only responsible of the appearance of the fixed poles as well as of

the fact that the cuts do not cancel on the physical sheet. The dynamical pependence of ϱ_{st} on ϱ_{tu}, due again to unitarity, plays an essential role in clarifying this point and in explaining why the simplest diagram that has both fixed poles and moving cuts is the one firstly considered by MANDELSTAM [3].

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УНИТАРНОСТЬ И НЕ-РЕДЖЕ СИНГУЛЯРНОСТИ

А. БАЗЕТТО и Ф. ПАККАНОНИ

Резюме

Продолжая релятивистическую амплитуду в комплексную плоскость углового момента, появляются новые сингулярности из требования унитарности скрещенных каналов.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 81—88 (1969)

MULTI-REGGEON BEHAVIOUR OF PRODUCTION AMPLITUDES

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GRIBOV has developed a calculus based on perturbation theory which permits a meaning to be attached to an arbitrary Reggeon diagram in two-body scattering. The rules that GRIBOV gives are for calculating the t-channel partial wave amplitude $a(j, t)$. It is indicated how GRIBOV's discussion can be taken over to the case of production amplitude for the process $A + B \to 1 + 2 + 3$.

Recently there has been much interest both phenomenological [1] and theoretical in multi-Reggeon exchange models of production amplitudes. These models are straight-forward generalisations of Regge exchange [2] in two-body scattering, namely

$$A(s, t) = g_1(t)s^{\alpha(t)}g_2(t), \tag{1}$$

where A is the amplitude for the process

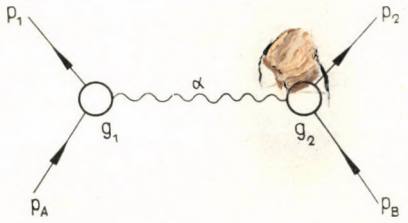

Fig. 1

and $t = (p_A - p_1)^2$, $s = (p_A + p_B)^2$. The corresponding amplitude for a production process $A + B \to 1 + 2 + 3$ which proceeds through double Reggeon exchange,

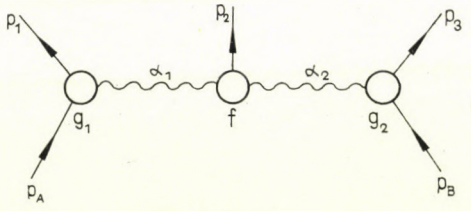

Fig. 2

is

$$T(s, s_1, s_2, t_1, t_2) = g_1(t_1)\, s_1^{\alpha_1(t_1)} f_{\alpha_1\alpha_2}\, s_2^{\alpha_2(t_2)}\, g_2(t_2), \tag{2}$$

where

$$t_1 = (p_A - p_1)^2,\ t_2 = (p_B - p_3)^2,\ s_1 = (p_1 + p_2)^2,\ s_2 = (p_2 + p_3)^2.$$

A novel feature of this case is the amplitude $f_{\alpha_1\alpha_2}$ which describes the coupling of the trajectories α_1 and α_2 to particle 2. This function depends on t_1, t_2 and $\eta = s_1 s_2/s$. In the limit of high energies

$$\frac{s_1 s_2}{s} = m_2^2 - q'^2, \tag{3}$$

where q' is that part of p_2 transverse to the incident beam in the lab or centre of mass frame. As one would intuitively expect $f_{\alpha_1\alpha_2}$ really only depends on the momenta of the three objects whose coupling it describes.

In two-body scattering it is well known that there are contributions to the asymptotic behaviour which arise from cuts in the angular momentum plane of the t-channel reaction $A + \bar{1} \to \bar{B} + 2$. One of these contributions can be associated with the two Reggeon exchange diagram.

Fig. 3

GRIBOV [3] has developed a calculus based on perturbation theory which permits a meaning to be attached to an arbitrary Reggeon diagram in two-body scattering. The rules that GRIBOV [3] gives are for calculating the t-channel partial wave amplitude $a(j, t)$. They are:

1) Associated with the n^{th} line of the diagram, are a space-like two dimensional momentum k_n, an angular momentum l_n, and a Reggeon propagator $(l_n - \alpha_n(k_n^2))^{-1}$.

2) At each internal vertex

$$\Sigma k_i = \Sigma(l_i - 1) = 0, \tag{4}$$

where the Reggeon lines are suitably directed. A similar law holds at external vertices. This allows all momenta to be expressed in terms of loop variables and the external momentum q (chosen to be two-dimensional and space-like).

The same is true of the angular momenta.

3) Certain amplitudes are associated with the vertices of the diagram.

4) The contribution to the partial wave amplitude is given by

$$a(j, q^2) = \int \prod_{\text{loops}} d^2k\, dl \frac{(\text{numerator})}{\prod_n [l_n - \alpha_n(k_n^2)]}, \tag{5}$$

where the numerator is computed from the amplitudes mentioned in 3).

For example the simple two-Reggeon bubble (Fig. 3) is given by

$$a(j, q^2) = \int d^2k_1\, dl_1 \frac{(\text{numerator})}{[l_1 - \alpha_1(k_1^2)][l_2 - \alpha_2(k_2^2)\ ]}, \tag{6}$$

where the l_1 — integration runs in an imaginary direction and lies to the right of the pole at $l_1 = \alpha_1$. If we use

$$\begin{aligned} k_2 &= q - k_1, \\ l_2 &= j - l_1 + 1, \end{aligned} \tag{7}$$

and move the l_1 contour to the left keeping only the pole contribution then we have

$$a(j, q^2) = \int \frac{d^2k_1(\text{numerator})}{[j - \alpha_1(k_1^2) - \alpha_2((q - k_1)^2) + 1]}. \tag{8}$$

It is easy to see that the denominator gives rise to the cuts pointed out by MANDELSTAM.

The aim of this talk is to indicate how GRIBOV's discussion can be taken over to the case of production amplitudes. Related work has been undertaken by ANSELM and DYATLOV [4].

In deducing his results from Feynman graphs GRIBOV uses a certain approximation technique. When this is applied to the graph

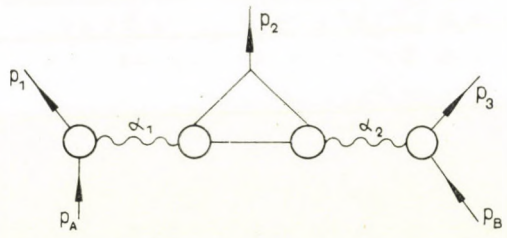

Fig. 4

the result has just the form of the double Regge exchange amplitude. Moreover one can deduce something about the dependence of $f_{\alpha_1\alpha_2}$ on η, namely that near $\eta = 0$,

$$f_{\alpha_1\alpha_2} \cong A_1 \eta^{-\alpha_1} + A_2 \eta^{-\alpha_2}. \tag{9}$$

Assuming that A_1 and A_2 are slowly varying functions of η this formula suggests a possible parametrisation of $f_{\alpha_1\alpha_2}$. This result is consistent with that of BLANKENBECLER and SUGAR [5] and ZAKRZEWSKI [6]. Actually A_1 and A_2 each have a pole when $\alpha_1 = \alpha_2$ but with opposite residues. When $\alpha_1 = \alpha_2 = \alpha$ the result becomes

$$f_{\alpha\alpha} \cong \eta^{-\alpha}(A - B\ln\eta). \tag{10}$$

The asymptotic behaviour exhibited by these double Reggeon exchange amplitudes is of the same type as already encountered in two-body scattering. The same is true of amplitudes associated with graphs such as

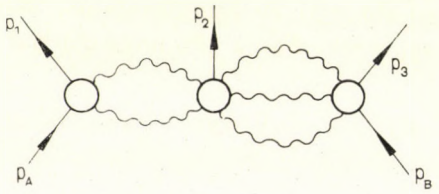

Fig. 5

The resulting asymptotic behaviour is simply ordinary cut-type behaviour in each sub-energy s_1 and s_2 (at fixed t_1, t_2, η). In order to encounter a new phenomenon it is necessary to examine a graph with no counter-part in two-body scattering. The simplest such graph is the Reggeon triangle graph.

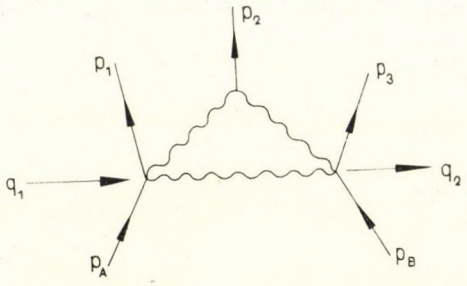

Fig. 6

Such a graph determines the asymptotic behaviour of the Feynman amplitude associated with

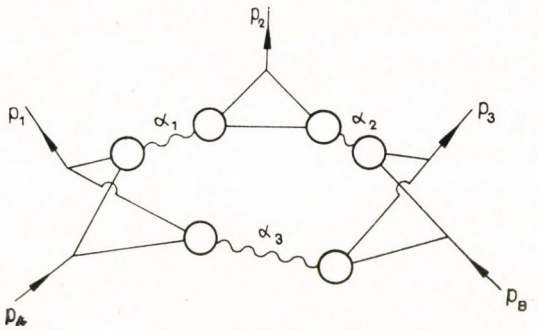

Fig. 7

Just as in the two-body case discussed by MANDELSTAM it is necessary to have a crossed box on each end of the diagram.

In order to apply GRIBOV's ideas to this situation it is necessary to define a double partial wave of the production amplitude. $T_{j_1 j_2}$ which depends on *two* angular momenta (j_1, j_2). BALI, CHEW and PIGNOTTI [7] have shown how an extension of TOLLER's work allows this problem to be solved in a manner consistent with group theoretical considerations. We shall follow GRIBOV and adopt a cruder approach. For a two-body partial wave we use

$$a(j,t) = \frac{2}{\pi}\int_{s_0}^{\infty} ds'\, s'^{-j-1}\, Im\, A(s',t) \tag{11}$$

and inverse formula

$$A(s,t) = -\frac{1}{4i}\int dj\, \xi_j\, s^j\, a(j,t). \tag{12}$$

In the case of the production amplitude, we regard it as a function of s_1 and s_2 at fixed t_1, t_2 and η and compute a double absorbtive part $\Delta_2 T$ in s_1, and s_2. The double partial wave is then

$$T_{j_1 j_2} = \left(\frac{2}{\pi}\right)^2 \int_{s_0}^{\infty} ds_1\, s_1^{-j_1-1} \int_{s_0}^{\infty} ds_2\, s_2^{-j_2-1}\, \Delta_2\, T \tag{13}$$

and the inverse formula is

$$T = \left(\frac{-1}{4i}\right)^2 \int dj_1\, dj_2\, \xi_{j_1}\, \xi_{j_2}\, s_1^{j_1}\, s_2^{j_2}\, T_{j_1 j_2}\,. \tag{14}$$

If $T_{j_1 j_2}$ has a pole in each variable as right most singularities

$$T_{j_1 j_2} \cong \frac{R}{(j_1 - \alpha_1)\,(j_2 - \alpha_2)} \tag{15}$$

then the leading contribution to T is

$$T = R\, s_1^{\alpha_1}\, s_2^{\alpha_2}. \tag{16}$$

In order that this calculation of a double partial wave make sense it is necessary at least in the asymptotic region of s_1 and s_2 that T have simple cut plane analyticity. This property does seem to emerge when GRIBOV's approach is applied to the Reggeon triangle graph.

The rules suggested by this study are the same as GRIBOV proposed for the two-body case except for some modifications to the numerator. The result is that for the triangle graph in Fig. 6

$$T_{j_1 j_2} = \int d^2 k_3\, dl_3\, \frac{(\text{numerator})}{[l_1 - \alpha_1(k_1^2)]\,[l_2 - \alpha_2(k_2^2)]\,[l_3 - \alpha_3(k_3^2)]} \tag{17}$$

where

$$\begin{aligned} k_1 &= q_1 - k_3, \\ k_2 &= q_2 - k_3, \\ l_1 &= j_1 - l_3 + 1, \\ l_2 &= j_2 - l_3 + 1. \end{aligned} \tag{18}$$

Using these results and keeping only the pole contribution to the l_3 — integration it follows that

$$T_{j_1 j_2} = \int \frac{d^2 k_3 (\text{numerator})}{[j_1 - \alpha_1((q_1 - k_3)^2) - \alpha_3(k_3^2) + 1]\,[j_2 - \alpha_2((q_2 - k_3)^2) - \alpha_3(k_3^2) + 1]}. \tag{19}$$

By using standard pinching techniques it is easy to analyse the analytic structure of $T_{j_1 j_2}$ which originates in the denominator of the integrand. The results are

1) Branch points at $j_1 = \alpha_{13}(t_1)$ and $j_2 = \alpha_{23}(t_2)$, where α_{13} and α_{23} are the two-Reggeon branch points produced, in the usual way by the Reggeon pairs (α_1, α_3) and (α_2, α_2). Each factor in the denominator gives rise separately to one of these singularities.

2) A leading curve which arises from a pinch involving both factors in the denominator. When the trajectories are linear this curve is a parabola and lies as shown in Fig. 8. It touches the two lines of singularities mentioned in 1).

The precise disposition of P depends on η as well as t_1 and t_2. In order to see what this structure implies for the asymptotic behaviour it is con-

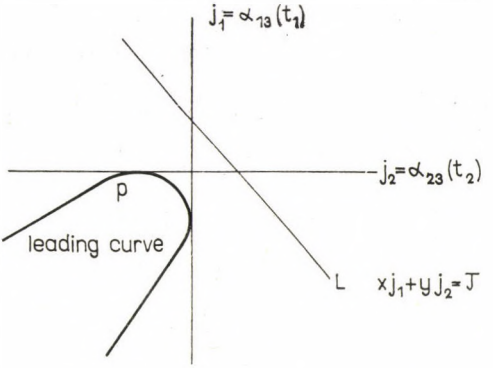

Fig. 8

venient to parametrise as follows,

$$s_1 = \lambda(\eta s)^x, \qquad s_2 = \lambda^{-1}(\eta s)^y, \qquad x + y = 1. \tag{20}$$

If we put

$$\tau = xj_1 + yj_2 \qquad j = j_1 + j_2 \tag{21}$$

then it follows from eq. (14) that

$$T = \int d\tau (\eta s)^\tau \varphi(\tau), \tag{22}$$

where

$$\varphi(\tau) = \int dj \, \xi_{j_1} \xi_{j_2} \lambda^j T_{j_1 j_2}. \tag{23}$$

The leading asymptotic behaviour of T is given by the right most singularity of $\varphi(J)$. This arises from a pinch of the j — integration contour which lies on the (complex part of) the line L in Fig. 8. Pinches occur when J is such that L passes through the intersection of the two-Reggeon branch lines or L touches the leading curve P. Only this latter pinch is effective in this case with the result that the asymptotic behaviour of T is determined by the leading curve. That is $T \sim s^J$, where J depends on η as well as on t_1 and t_2. (In fact the behaviour is $T \sim s^J/\ln s$).

While it may be beyond the scope of present experiments it would be of great interest to detect such a dependence on η in the asymptotic behaviour of a production amplitude. Of theoretical interest is the possibility of developing a Landau theory of singularities in the space of angular momenta which encompasses production amplitudes as well as elastic scattering.

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МУЛЬТИ-РЕДЖЕ ПОВЕДЕНИЕ АМПЛИТУД ОБРАЗОВАНИЯ

И. Т. ДРУМОНД

Резюме

На основе теории возмущений Грыбов разработал вычисление, в рамках которого выдвигается идея, использованная в случае произвольной диаграммы Редже при рассеянии двух частиц. Правила, сформулированные Грыбовым, служат для вычисления амплитуды a (j, t) парциальной волны t-канала. Описывается возможность применения обсуждения Грыбова для случая амплитуды образования для процесса $A + B \to 1 + 2 + 3$.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 89—90 (1969)

FINITE ENERGY SUM RULES FOR MESON–MESON SCATTERING

By

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Finite energy sum rules, based on analyticity properties of scattering amplitudes, are applied to meson—meson scattering.

Finite energy sum rules (FESR) [1], based on analyticity properties of scattering amplitudes, provide a useful tool in relating the high energy and the low energy phenomenology. Assuming that the scattering amplitude at low energy is dominated by a few resonances, and that the high energy behaviour is determined by a few Regge poles, one can evaluate the Regge parameters in terms of the low energy resonances.

We consider here the scattering amplitudes for the $\pi - \pi$, $\pi - K$, $\pi - \varrho$ and $\pi - K^*$ systems, corresponding to the exchange of the ϱ-trajectory in the t-channel. We write FESR's for forward amplitudes at $t = 0$:

$$R_{\pi x}(n) = \beta_{\pi x} \frac{\nu_L^{\alpha_\varrho + n - 1}}{\pi(\alpha_\varrho + n + 1)} = \frac{1}{\pi} \int_{\nu_0}^{\nu_L} \nu^n \, Im F_{\pi x}^{(1)}(\nu, t = 0) \, d\nu . \tag{1}$$

The zero moment ($n = 0$) sum rules are used to determine the residues $\beta_{\pi x}$, assuming for the intercept $\alpha_\varrho(0)$ the value obtained from high energy fits [2]. A determination of $\alpha_0(0)$ itself is provided, making use also of the $n = 2$ sum rules, by the relation [3]

$$\frac{\alpha_\varrho(0) + 3}{\alpha_\varrho(0) + 1} = \frac{\nu_L^2 \, R_{\pi x}(n = 0)}{R_{\pi x}(n = 2)} . \tag{2}$$

The saturation of the sum rules has been carried out in the narrow width resonance approximation, using the available experimental information [4]. The data used in the FESR's are consistent with the ADLER—WEISBERGER sum rules [5] and the generalized superconvergent relations [6, 7] applied to the different processes [8] discussed in this communication. We have taken values of (ν_L ($\nu = 1/2$ ($s - u$)) corresponding to cutoff C. M. energy in the range 1.6—1.8 GeV; the corresponding variations in the l.h.s. of eq. (1) are

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within the experimental errors. Assuming $\alpha_\varrho(0) = 0.57 \pm 0.03$ from [2], our results are summarized by:

$$\begin{aligned} \beta_{\pi\pi} &= 1.5 \pm 0.5\,, & \beta_{\pi\varrho} &= 1.5 \pm 0.5\,, \\ \beta_{\pi\varkappa} &= 0.9 \pm 0.2\,, & \beta_{\pi\varkappa^*} &= 0.75 \pm 0.25\,, \end{aligned} \tag{3}$$

where all quantities have been evaluated in pion mass units.

The corresponding results, assuming factorization of the Regge residues and universality of the couplings of the ϱ-trajectory, are:

$$\frac{1}{2}\beta_{\pi\pi} = \frac{1}{2}\beta_{\pi\varrho} = \beta_{\pi\varkappa} = \beta_{\pi\varkappa^*} = \beta_{\pi N}\,. \tag{4}$$

We see that these ratios are well satisfied by our results, although the errors involved are rather large. Moreover, from [2] we get $\beta_{\pi N} = 0.825$ and 0.872 for solution I and III, respectively, so that the above values have also the right absolute magnitude. This is also consistent with the results obtained for the $\pi\Sigma$ system [7].

We apply now eq. (2) to the $\pi - \pi$ system, which is relatively better known than the others. Using the same cut-off ν_L, we obtain:

$$\frac{\alpha_\varrho(0) + 3}{\alpha_\varrho(0) + 1} = 2.17 \pm 0.45\,, \tag{5}$$

to be compared with the value 2.27 ± 0.03, corresponding to the quoted $\alpha_\varrho(0)$. We think that the two values are in rather good agreement, although the determination of $\alpha_\varrho(0)$ from eq. (5) is sensitive to the large error.

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ПРАВИЛА СУММ КОНЕЧНЫХ ЭНЕРГИЙ ДЛЯ РАССЕНИЯ МЕЗОНОВ МЕЗОНАМИ

Г. КОСТА и К. А. САВОА

Резюме

Основанные на аналитических свойствах амплитуд рассеяния правила сумм конечных энергий применяются в случае рассеяния мезонов мезонами.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 91—93 (1969)

ON THE DECOUPLET SUPERCONVERGENCE RELATIONS FOR MESON–BARYON SCATTERING

By

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The possible superconvergence relations which correspond to the decouplet exchange in the t-channel were considered for meson—baryon scattering. It was found that superconvergence relations hold for non-spin-flip but fail for the spin-flip amplitudes. It is suggested that the failure of the latter may be due to the j-plane branch point singularity which protects the fixed singularity at the wrong signature nonsense point $j = 0$.

The superconvergence relations [1], namely the sum rules of the following form:

$$\int_{-\infty}^{+\infty} ImA(\nu\,;t)\,d\nu = 0 \tag{1}$$

are the rigorous consequence of dispersion relations and (assumed) asymptotic behaviour of scattering amplitudes, the latter being determined by the j-plane singularities in the crossed channel. In particular, the absence of meson resonances which could be assigned to the 27-plet representation of SU(3) has led to the conjecture that the corresponding j-plane singularity $\alpha_{27}(0)$ has negative intercept:

$$\alpha_{27}(0) < 0 \tag{2}$$

and, as a consequence, we receive the familar superconvergence relation for the corresponding meson—baryon spin-flip amplitude B_{27}:

$$\int_{0}^{\infty} ImB_{27}(\nu\,;0)\,d\nu = 0 \tag{3}$$

which has been discussed by several authors [2].

In this report we wish to discuss the possible superconvergence relations for that part of the meson—baryon scattering amplitude which corresponds to the decouplet exchange in the t-channel.

The lack of experimental evidence for meson resonances which could be assigned to the decouplet representation of SU(3) suggests that the following condition may be true:

$$\alpha_{10}(0) < -1\,, \tag{4}$$

where $\alpha_{10}(0)$ denotes the intercept of the possible decouplet meson trajectory.

The condition (4) then leads to the following independent superconvergence relations:

$$a_0 = \int_0^\infty Im\, A'_{10}(\nu\,;0)\, d\nu = 0\,, \tag{5}$$

$$b_1 = \int_0^\infty Im\, \nu\, B_{10}(\nu\,;0)\, d\nu = 0\,, \tag{6}$$

where

$$A'_{10} = A_{10} + \nu\, B_{10}\,,$$

$$\nu = \frac{s-u}{4M}\,.$$

A_{10} and B_{10} are the conventional invariant amplitudes.

The sum rules (5) and (6) can be most easily analysed if we use the following representation of the T_{10} amplitude:

$$T_{10} = T_{\pi^- p} - T_{\pi^+ p} + T_{K^+ p} - T_{K^- p} + T_{\bar{K}^\circ p} - T_{K^\circ p}. \tag{7}$$

The sum rule (5) can then be converted into the sum rule over respective total cross-sections and in this form it has recently been analysed [3] and found to be well satisfied.

We have analysed the sum rule (6) for the spin-flip amplitude B_{10} in the narrow-width resonance approximation taking the representation (7) for numerical estimates.

It appears [4] that contrary to the sum rule (5) the possible superconvergence relation (6) exhibits a remarkable tendency to fail. This is due to the fact that predominant baryonic resonances (α, γ octet recurrences and δ decouplet recurrences) contribute with the same sign to the integral b_1. (The same situation exists if we analyse this sum rule within SU(3) symmetry).

The failure of this sum rule can be compatible with the fact that the sum rule (5) holds only if we accept the t-channel j-plane singularity, located near $j = -1$, which is present only in $f_{-;10}(j\,;t)$ partial wave amplitude (in SINGH notation [5]) since the asymptotic behaviour of the $A'_{10}(\nu\,;t)$ amplitude is controlled by j-plane singularities of $f_{+;10}(j\,;t)$ alone. A possible candidate for this singularity is the moving branch point [6] which protects the fixed pole of $f_{-;10}(j\,;t)$ amplitude at $j = 0$. It can be shown [4] that this branch point singularity must be present at the amplitude $f_{-;10}(j\,;t)$ and may decouple from $f_{+;10}(j\,;t)$.

The more detailed discussion of the problems presented here is given in [4].

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ДЕКУПЛЕТНЫЕ СВЕРХСХОДЯЩИЕСЯ СООТНОШЕНИЯ ДЛЯ МЕЗОН-БАРИОННОГО РАССЕЯНИЯ

Я. КВИЕЦИНСКИ

Резюме

Рассматриваются возможные соотношения сверхсходимости, относящиеся к декуплетному обмену в t-канале при мезон-барионном рассеянии. Найденно, что соотношения сверхсходимости имеют место в случае не спин-флиповых, но отсутствуют в случае спин-флиповых амплитуд. Оказывается, что отсутствие последних возможно обусловлено существованием разреза в j-плоскости, который связан с неподвижной особенностью в абсурдной точке $j = 0$.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 95—99 (1969)

S-MATRIX THEORY OF PION–PION SCATTERING

By

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A one parameter model of the pion—pion scattering amplitude is given, which satisfies crossing symmetry and unitarity approximately. The results are qualitatively correct.

There have been several formulations of the solution to the problem of pion—pion scattering in S-matrix theory [1]. One needs a real analytic function of the Mandelstam variables s and t which satisfies crossing symmetry and unitarity. Instead of beginning with a dispersion relation which necessarily gives the correct analytic properties, an amplitude will be defined by its Taylor expansion. It proves convenient to expand about the point $s = 4$, $t = u = 0$ and the variables of the expansion are $\sqrt{4 - s}$, t, u. The mass unit is the pion mass. It is possible to re-expand the amplitude about the unphysical point $s = t = u = 0$, and to require the amplitude to be crossing symmetric in the s and t variables to given order. This leads to a set of linear equations amongst the coefficients of the Taylor expansion. Unitarity is imposed to given order on the partial waves of the original form, leading to a set of quadratic equations. The problem is then the problem of analysing and solving these equations. The mathematical model is explained most clearly by considering the scattering of uncharged pions which have no u-channel forces. The expansion of the amplitude to first order in the variables $\sqrt{4 - s}$, t is

$$A(s, t) = a + b\sqrt{4 - s} + ct.$$

a, b and c are constants, $s < 4$. To impose crossing symmetry, re-expand about $s = t = 0$,

$$A(s, t) \sim a + b(2 - s/4) + ct,$$

therefore

$$c = -b/4.$$

Continue the amplitude above the branch point $s = 4$, and impose unitarity,

$$A(s, t) = a - 2ibk + ct,$$

$$\operatorname{Im} A_l = (k/E)\,|A_l|^2,$$

$$A_l = \frac{1}{2}\int_{-1}^{1} A(s,t)\, P_l(x)\, dx,$$

where k is the c.o.m. momentum, E the c.o.m. energy of a pion, and x is the cosine of the c.o.m. scattering angle. Working to first order, one can only impose s-wave unitarity

$$b = -a^2/2,$$

so the complete amplitude to first order is

$$A(s,t) = a + ika^2 + ta^2/8.$$

The amplitude is expressed in terms of one parameter a, the s-wave scattering length, and the p-wave scattering length is implicitly deduced.

The realistic model has three amplitudes $A^\alpha(s,t,u)$, $\alpha = 0, 1, 2$ and two symmetries:

$$A^\alpha(s,t,u) = (-)^\alpha A^\alpha(s,u,t),$$

$$A^\alpha(s,t,u) = S^{\alpha\beta} A^\beta(t,s,u),$$

where S is the pion–pion crossing matrix. To impose crossing symmetry, use a matrix X such that $SX = X\Lambda$, where Λ is the diagonal matrix of eigenvalues. $\Lambda^{00} = \Lambda^{11} = 1$, $\Lambda^{22} = -1$. Then new amplitudes

$$G^\alpha(s,t,u) = (X^{-1})^{\alpha\beta} F^\beta(s,t,u)$$

have the symmetries that G^0, G^1 are symmetric and G^2 is antisymmetric under interchange of s and t.

Use trial amplitudes

$$A^0 = a_0 + b_0\sqrt{4-s} + c_0(t+u),$$

$$A^1 = c_1(t-u),$$

$$A^2 = a_2 + b_2\sqrt{4-s} + c_2(t+u).$$

Expand about $s = t = u = 0$, form the G amplitudes and symmetrise or antisymmetrise: one finds that

$$\begin{pmatrix} -4 & -5 & -16 & -12 & -20 \\ -2 & -7 & -8 & 12 & -28 \\ 2 & -5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & -5 \\ 2 & -5 & -8 & 12 & 20 \end{pmatrix} \begin{pmatrix} b_0 \\ b_2 \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -a_0 + \dfrac{5a_2}{2} \\ 0 \\ 0 \end{pmatrix}. \qquad (1)$$

The unitarity equations are

$$b_0 = -a_0^2/2\,, \quad b_2 = -a_2^2/2\,. \tag{2}$$

One might hope that seven equations would determine seven parameters, but one crossing symmetry equation is consistent with the other four and so is redundant. Working on the assumption that $a_2 = 0$, strongly suggested by the experimental data [2] one finds that

$$a_0 = 1, \quad b_0 = {}^{-1}/_{2}, \quad c_0 = c_1 = c_2 = {}^{1}/_{24}, \quad a_2 = b_2 = 0\,.$$

An estimate of the errors involved in using this procedure is given by evaluating A^0/A^2 at the symmetry point. Its value is 2.651 instead of the theoretical 2.5, an error of 6%.

It seems impossible to form any estimate of the convergence domain of the series. There should be crossed channel normal thresholds, and there are second type singularities in the two particle phase space function. It would seem reasonable to expect convergence in $|k| \lesssim 1$ or total energy in the range $0 \to 400$ MeV. To make best use of the results, form the K-matrix

$$\frac{\tan \delta_0^0}{k} = \frac{1}{k} \frac{\operatorname{Im} A_0^0}{\operatorname{Re} A_0^0} \tag{3}$$

which is meromorphic in $k > 0$, but below inelastic thresholds. This particular procedure predicts that

$$\tan \delta_0^0 = k/(1 - k^2/6)$$

and that there is an s-wave, $I = 0$ resonance at 741 MeV.

The second order calculation is tedious but interesting.

$$\begin{aligned}
A^0 &= a_0 + b_0 \sqrt{4 - s} + c_0(t + u) + d_0(4 - s) \\
&\quad + f_0(t^2 + u^2) + h_0 \sqrt{4 - s}(t + u) + k_0\, tu\,, \\
A^1 &= c_1(t - u) + h_1 \sqrt{4 - s}(t - u) + f_1(t^2 - u^2)\,, \\
A^2 &= a_2 + b_2 \sqrt{4 - s} + c_2(t + u) + d_2(4 - s) \\
&\quad + f_2(t^2 + u^2) + h_2 \sqrt{4 - s}(t + u) + k_2\, tu\,.
\end{aligned} \tag{4}$$

There are 13 crossing symmetry equations, of which three are compatible and redundant. There are five unitarity equations, two of which are equations (2) and one of which is $h_1 = 0$. The remaining two unitarity equations and the 10 independent crossing symmetry equations may be written in the form

$$R\,\mathbf{x} = \mathbf{b} + d_2\,\mathbf{d}\,,$$

where $x^T = (k_2\, h_2 f_2,\ f_1\, k_0\, h_0,\ f_0\, d_0\, c_2,\ c_1\, c_0\, b_2)$. R is a 12×12 matrix, **b** and **d** twelve dimensional vectors. R, **b** and **d** both depend on a_0 and a_2 and so, from equations (2), on b_0, b_2. Inversion of the matrix R reveals the remarkable coincidence that the vector $R^{-1}\,\mathbf{d}$ is zero, except for $(R^{-1}\,\mathbf{d})_8 = -(R^{-1}\,\mathbf{d})_{11} = 2.5$ and $(R^{-1}\,\mathbf{d})_9 = -1$. This means that the combinations $c_0 + d_0$, $c_2 + d_2$ and the elements k_2, h_2, f_2, $f_1\, k_0\, h_0$, f_0, b_2 are determined uniquely in terms of a_0, b_0, a_2 and b_2. The equations are solved by giving a_0 a fixed value and a_2 trial values so that b_2 obtained by matrix inversion eventually equals b_2 obtained from unitarity in equation (2). Further the scattering amplitude is uniquely determined in terms of a_0, because $4 - s = t + u$ in the physical

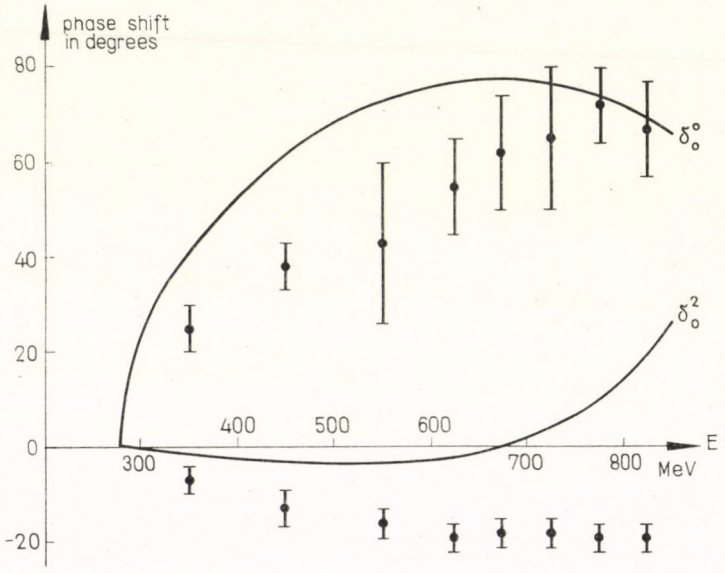

Fig. 1. The phase shift as deduced by methods described in the text. The experimental points are taken from [2]

region, and so the combinations $c_0 + d_0$, $c_2 + d_2$ fix the amplitudes of equations (4) in the physical region. I can see no simple reason for this accident.

There does not appear to be a unique solution to these equations for given a_0, but the solution given is the one of the two found which is closest to the experimental data. The error at the symmetry point in the case $a_0 = 1$ is. 8%.

The $I = 1$ amplitude is purely real and so non-unitary in the second order model. Therefore one expects the $I = 1$ results to be imperfect. But the real part of the amplitude vanishes at 580 MeV, indicating the presence of a low energy rho resonance. Secondly, the K-matrix parametrisation (3) is inappropriate for δ_0^2. This is because Im A_0^2 should be positive, but is not

in this model — it is small and of oscillating sign. So the phase shift was determined from the magnitude of the real part, Re f_0^2. The value 1 for a_0 was chosen (a) because it is in accord with the previous calculation, (b) because for a_0 larger or smaller than 1 the zero of Re f_1^1 is at lower energies and (c) there is a qualitative fit to δ_0^0. It did not seem worthwhile to do a least squares fit to the data and ignore criterion (b). Small changes in a_0 do not produce any substantial changes in the results. The results are shown in Fig. 1 for the $I = 0$ and $I = 2$ s-wave phase shifts given experimentally by WALKER's analysis [2]. The predicted scattering lengths are

$$a_0 = 1, \quad a_2 = \cdot 04115.$$

Further work on the third order calculation appears feasible, and in retrospect one can see arguments in favour of re-expanding about the symmetry point.

Acknowledgement

I am grateful to Professor E. LEADER, Dr. D. LYTH and Dr. A. K. COMMON for their comments.

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ТЕОРИЯ S-МАТРИЦ ПРИ ПИОН-ПИОННОМ РАССЕЯНИИ

П. Р. ГРЕЙВС-МОРРИС

Р е з ю м е

Дается модель с одним параметром для амплитуды пион-пионного рассеяния, которая приближенно удовлетворяет требованиям симметрии скрещения и унитарности. Результаты качественно правильны.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 101—105 (1969)

UNITARITY STRUCTURE OF SCATTERING AMPLITUDES

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As a first step towards constructing scattering amplitudes satisfying unitarity, analyticity and crossing symmetry, we derive a linear non-singular integral equation for the total scattering amplitude which is equivalent to the unitarity condition. For this purpose we use the partial-wave N/D representation (with inelasticity) and the convolution theorem for Legendre transforms. We also discuss briefly the choice of two functions $N(s, \cos\Theta)$, $C(s, \cos\Theta)$ which determine the unitary scattering amplitude through the integral equation. These functions may hopefully be chosen so that the analyticity and crossing symmetry requirements are satisfied.

1. Introduction

This note is to a large extent inspired by a paper by A. W. MARTIN [1], dealing with the formidable problem of combining unitarity and crossing symmetry in a single formalism. MARTIN uses the general structure of the partial-wave N/D representation to develop a Mandelstam-like representation for the full scattering amplitude that satisfies elastic unitarity in the s-channel. The key points in this development consist of a fairly general ansatz for the partial wave numerator function and a technique for evaluating infinite sums over Legendre polynomials and Legendre functions of the second kind. As a result, one obtains a series expansion for the unitary scattering amplitude, each term in the series having analytic properties corresponding approximately to the Mandelstam double dispersion relation. However, due to the complicated features of his expansion, MARTIN did not prove that the unitary scattering amplitude actually had the correct analytic properties. Secondly, inelastic unitarity was not incorporated in the formalism. However, inelastic unitarity is a necessary ingredient in the problem, as it is known that non-trivial scattering amplitudes cannot satisfy exactly both *elastic* unitarity and crossing symmetry.

Now, we will show, that by using the convolution theorem for Legendre transforms [2, 3] one can circumvent some of the difficulties connected with the series expansion of the unitary scattering amplitude. Also, by repeated use of the convolution theorem, we can, at least formally, include inelastic unitarity in the formalism. Instead of a series expansion for the unitary scattering amplitude we obtain a linear non-singular integral equation for the full

scattering amplitude, the kernel and inhomogeneous term being determined by the ansatz made for the partial-wave numerator function. The next Section is devoted to a discussion of the unitarity integral equations, while the concluding Section 3 contains some comments on MARTIN's ansatz for the partial-wave numerator function and a discussion of further developments.

2. The unitarity integral equation

For simplicity we consider the scattering of distinguishable spinless particles of equal mass. For notations and conventions we refer e.g. to MARTIN's paper [1].

As is known, a partial-wave amplitude $A_l(s)$, which is represented in the form N_l/D_l with D_l given by

$$D_l(s) = 1 - \frac{1}{\pi}\int_1^\infty \frac{ds'\, p(s')\, N_l(s')}{s' - s}\,, \quad \varrho^2(s) = \frac{s-1}{s}\,, \tag{1}$$

will satisfy elastic unitarity for rather arbitrary functions $N_l(s)$.

From Eq. (1) then follows,

$$A_l(s) = N_l(s) + \frac{1}{\pi}\int_1^\infty \frac{ds'\, \varrho(s')\, A_l(s)\, N_l(s')}{s' - s}\,. \tag{2}$$

Let us then perform the inverse Legendre transform in Eq. (2). On the right hand side of Eq. (2) we have to transform the product $A_l(s)N_l(s')$, which according to the convolution theorem [2, 3] can be expressed as an integral involving the functions $A(s, \cos\Theta)$ and $N(s, \cos\lambda)$ defined below. We obtain

$$A(s,z) = N(s,z) + \frac{1}{\pi}\int_{-1}^{+1} dz'\, K(s,z,z')\, A(s,z'), \tag{3}$$

with $(z = \cos\Theta)$,

$$\begin{aligned} A(s,z) &= \Sigma\,(2l+1)\, A_l(s)\, P_l(z), \\ N(s,z) &= \Sigma\,(2l+1)\, N_l(s)\, P_l(z), \end{aligned} \tag{4}$$

and

$$\begin{aligned} K(s,z,z') &= \frac{1}{\pi}\int_1^\infty \frac{ds'\, p(s')}{s' - s}\int_0^\pi d\varphi\, N(s', \cos\lambda), \\ \cos\lambda &= zz' + \left((1-z^2)\,(1-z'^2)\right)^{1/2}\cos\varphi, \end{aligned} \tag{5}$$

The Eq. (3) which, for a given function $N(s,z)$, constitutes a nonsingular linear integral equation for $A(s,z)$, is equivalent to the elastic unitarity con-

dition. It is amusing to note that the Fredholm determinant of Eq. (3) is given by the product of the partial-wave denominator functions,

$$\mathcal{D}(s) = \prod_l D_l(s). \tag{6}$$

Now the difficulty in Eq. (3) lies in the finding of a "correct" ansatz for $N(s, z)$. The ansatz proposed by MARTIN corresponds to the choice,

$$N(s, \cos\Theta) = \frac{1}{\pi}\int_1^\infty \frac{dt' f_t(s,t')}{t'-t} + \frac{1}{\pi}\int_1^\infty \frac{du' f_u(s,u')}{u'-u}, \tag{7}$$

with

$$\begin{aligned} t &= -\frac{s-1}{2}(1-\cos\Theta), \\ u &= -\frac{s-1}{2}(1+\cos\Theta). \end{aligned} \tag{8}$$

Inserting Eq. (7) in the integral equation (3), it is easily checked that the first few terms in the Neumann—Liouville series solution of Eq. (3) agree with the first few terms in the series expansion obtained by MARTIN. It is clear that it is easier to investigate the analytic properties of the solution $A(s, \cos\Theta)$, by examining the integral equation than by examining the series expansion for $A(s, \cos\Theta)$. However, as mentioned in the Introduction, it is necessary to include inelastic unitarity in the formalism before considering the constraints given by analyticity and crossing symmetry requirements. We devote the remainder of this Section to the problem of inelastic unitarity.

As is known, one way of introducing inelastic unitarity in the N/D scheme consists of introducing an inelasticity factor $R_l(s)$ in the integral defining the denominator function $D_l(s)$,

$$D_l(s) = 1 - \frac{1}{\pi}\int_{s_1}^\infty \frac{ds'\,\varrho(s')\,R_l(s')\,N_l(s')}{s'-s}, \tag{9}$$

where

$$R_l(s) = 1 + \frac{\sigma^{(l)}_{\text{inel}}}{\sigma^{(l)}_{\text{el}}} \equiv 1 + C_l(s). \tag{10}$$

The unknown function $C_l(s)$ in Eq. (10) is non-negative above the inelastic threshold s_1. Let us now make the assumption that the function $C_l(s)$ is the Legendre transform of some function $C(s, \cos\Theta)$. (This assumption is not necessarily very brave, in view of existing upper and lower bounds on σ_{inel} and σ_{el}, respectively). We can then apply the inverse Legendre transform

just as in the previous case, and obtain again the integral equation (3), with the only difference that the kernel $K(s, z, z')$ now contains a double convolution integral,

$$K(s,z,z') = \frac{1}{2\pi}\int_1^\infty \frac{ds'\,\varrho(s')}{s'-s}\int_0^\pi d\varphi\, N(s', \cos\lambda) + \\ + \frac{1}{4\pi^2}\int_{s_1}^\infty \frac{ds'\,\varrho(s')}{s'-s}\int_0^\pi d\varphi\int_0^\pi d\varphi'\int_{-1}^{+1} dn\, C(s',\eta)\, N(s', \cos\lambda'), \tag{11}$$

with

$$\zeta \equiv \cos\lambda = zz' + \left((1-z^2)\,(1-z'^2)\right)^{1/2}\cos\varphi, \\ \cos\lambda' = \zeta n + \left((1-\zeta^2)\,(1-\eta^2)\right)^{1/2}\cos\varphi'. \tag{12}$$

The integral equation (3), with the kernel $K(s, z, z'')$ given by Eq. (11) is thus equivalent to the requirement of inelastic unitarity expressed by Eq. (9). In order to satisfy the analyticity and crossing symmetry requirements we have the two (fairly arbitrary) functions $N(s, \cos\Theta)$ and $C(s, \cos\Theta)$ at our disposal. It is of course clear that the "correct" choice of these functions will require a lot of inspired guesswork.

3. Summary and discussion

We have shown that the unitarity condition for the full scattering amplitude $A(s, \cos\Theta)$ can be introduced by means of a linear non-singular integral equation whose inhomogeneous term equals the inverse Legendre transform $N(s, \cos\Theta)$ of the partial-wave numerator function (of the N/D method), and whose kernel is given in terms of the function $N(s, \cos\Theta)$ and an additional function $C(s, \cos\Theta)$ which describes the inelasticity.

These two functions may hopefully be chosen so that the scattering amplitude also has the correct analytic properties and satisfies crossing symmetry. The integral equation obtained replaces the cumbersome summation techniques and series expansions introduced by MARTIN [1] for the purpose of combining the analyticity-, unitarity- and crossing symmetry requirements. The formalism presented here is also more general, as we incorporate inelastic unitarity and are able to deal with general functions $N(s, \cos\Theta)$ and $C(s, \cos\Theta)$ respectively. Admittedly, our analysis has been heuristic, as we have not found it worthwhile to state conditions for the validity of the interchange of various limiting processes, which occur frequently in the paper. Also, we have disregarded complications such as the appearance of bound states, subtractions, etc. Apart from such questions one may of course criticize the

use of the N/D representation to impose unitarity, in view of the ambiguous nature of this representation.

The achievements made in this paper are mainly technical, as the heart of the problem lies in the determination of the functions $N(s, \cos\Theta)$ and $C(s, \cos\Theta)$. Here the results obtained in potential scattering for Yukawian potentials provide a useful hint. In fact, from the papers by DE ALFARO et al. [4] and especially by PETRÁŠ [5], we learn that the function $N(s, \cos\Theta)$ has, apart from minor modifications, the representation given by Eq. (7), provided the relevant spectral functions satisfy certain non-linear dynamical equations. There is thus some ground for optimism although the fully relativistic problem is more complicated. We hope to be able to discuss the construction of the functions $N(s, \cos\Theta)$ and $C(s, \cos\Theta)$ in more detail in the near future.

Acknowledgement

The author is indebted to Oskar Öflunds Stiftelse for financial support.

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УНИТАРНАЯ СТРУКТУРА АМПЛИТУД РАССЕЯНИЯ

К. КРОНСТРЁМ

Резюме

Первым шагом в направлении составления амплитуд рассеяния, удовлетворяющих унитарности, аналитичности и симметрии скрещения, выводится линейное несингулярное интегральное уравнение для полной амплитуды рассеяния, которая эквивалентна унитарному условию. Для достижения этой цели использованы N/D представление парциальных волн (с неупругостью) и теорема свертки функций для преобразований Лежандра. Коротко истолкуется также выбор двух функций $N(s, \cos\Theta)$, $C(s, \cos\Theta)$, которые определяют амплитуду унитарного рассеяния через интегральное уравнение. Эти функции можно выбирать с надежностью так, что требования симметрии скрещивания и аналитичности удовлетворились.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 107—120 (1969)

$\pi^{\pm} - p$ ELASTIC SCATTERING ON POLARIZED PROTONS AT LARGE MOMENTUM TRANSFERS

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Recent results of the measurements of $\pi^{\pm} - p$ elastic scattering on polarized protons at large momentum transfers are reported and compared with Regge pole phenomenology.

O dulces comitum valete coetus
Catullus, Carmina

I. Introduction

A whole new class of polarization measurements has recently become possible in elementary particle physics by the development of polarized proton targets, which have reduced polarization determinations in reactions taking place on protons to asymmetry measurements [1].

In $\pi - p$ scattering, for energies around or above 1 GeV, several polarization measurements have been already carried out.

They can be divided, roughly speaking, into two groups.

On the one hand, extensive polarization measurements exist [2] for energies up to about 2 GeV. In this energy region total cross sections show a considerable structure as function of energy and several resonances have been established. The knowledge of polarization and of differential cross sections at closely spaced energy intervals has made possible the determinations of spins and parities of several resonances.

On the other hand, polarization determinations in the forward direction have been obtained for energies above 6 GeV [3]. At these energies cross sections change slowly with energy, showing no evidence for resonances and differences between particle and antiparticle cross sections become small. Many hypotheses on the asymptotic behavior of scattering amplitudes have been formulated in recent years [4]. The polarization data have played a very important role in increasing our understanding of the general features of high-energy two-body collisions.

It is the purpose of this paper to describe the features and present some results of an experiment [5] presently set up at the Argonne ZGS to measure polarizations in $\pi^{\pm} - p$, $K^{+} - p$, and $p - p$ elastic scattering in the energy region between 2.5 and 5 GeV.

Some results on polarizations in $p - p$ scattering have been already published [6].

The $\pi - p$ part of this experiment has two objectives:

i) Supplement the existing data on polarizations in $\pi^{\pm} - p$ elastic scattering at high energies and small momentum transfers with similar measurements at lower energies but larger momentum transfers. The extended range of $-t$ is essential for testing the various Regge-pole models that have been proposed.

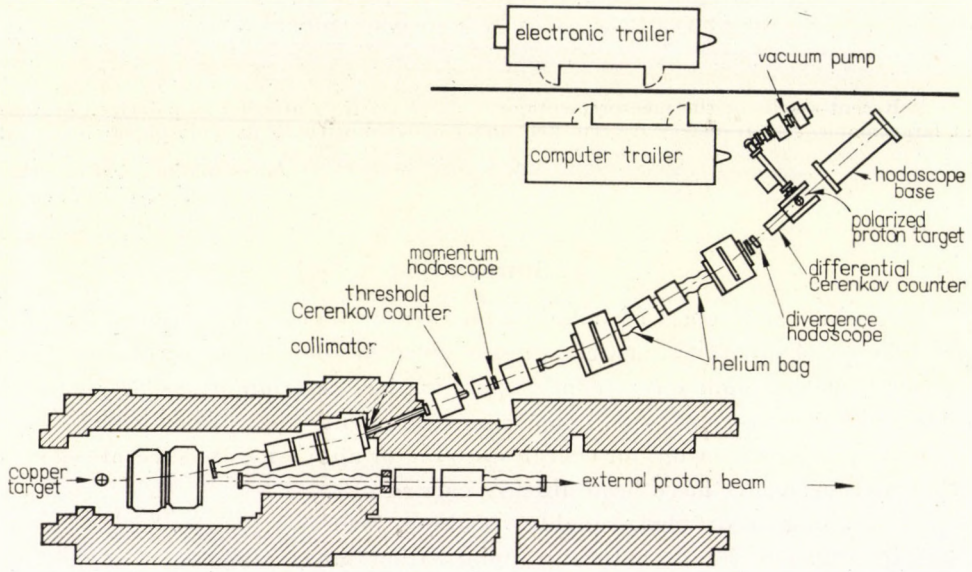

Fig. 1. Experimental layout

ii) For energies between 2.5 and 3.75 GeV and in steps of 0.25 GeV, measure polarizations in backward $\pi^{+} - p$ elastic scattering. This information is very important for determining the relative roles and the characteristics of the resonance, nucleon-exchange and Regge-exchange amplitudes which can be used to describe this process [7].

II. Experimental method

Fig. 1 shows the beam layout. The slow extracted proton beam of the Argonne National Laboratory Zero Gradient Synchrotron interacts in a copper target. Secondary particles produced at $\sim 0°$ are deflected outward by the first two magnets and are transported to the Argonne horizontally polarized proton target by the various beam elements.

The momentum acceptance of the beam is $\pm$ 3.5%. The intensity at the target is typically 10^6 particles per pulse. The composition of the positive

beam is about 75% protons and 25% pions for momenta around 5 GeV/c and it is about 50% protons and 50% pions for momenta around 3 GeV/c. The negative beam is composed almost exclusively by pions.

A seven-counter hodoscope ("momentum hodoscope") placed at an intermediate focus is used for defining the incoming particle momentum to $\pm$ 0.5%. A threshold gas Cerenkov counter in anticoincidence rejects the electrons present in the beam. A second threshold gas Cerenkov counter placed next to the first one, allows to distinguish between pions and heavier particles.

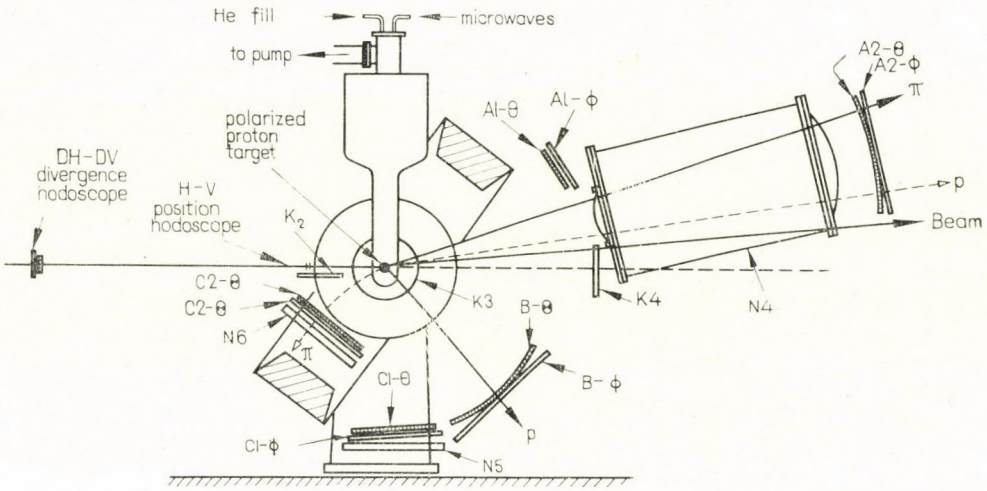

Fig. 2. Schematic of the $\pi - p$ scattering experiment

At momenta for which the amount of K^+'s in the beam is appreciable, a further distinction between K^+'s and protons is obtained by using a differential gas Cerenkov counter, placed further downstream on the beam.

A schematic of the counter arrangement around the target is shown in Fig. 2. The two counter matrices DH$\times$DV (6×6, "divergence hodoscope") and H$\times$V (6×8, "position hodoscope") placed on the beam are used for the definition of the incoming particle angle and position.

A "beam particle" is then defined as a coincidence among the momentum, divergence and position hodoscopes, not accompanied by a count in the first threshold Cerenkov counter.

Particles produced in the target are detected by the two matrices A1$\Theta\times$A1φ (17×19), A2$\Theta\times$A2φ (45×19) placed above the beam line, and by the three matrices B$\Theta\times$Bφ (41×12), C1$\Theta\times$C1φ (10×13), C2$\Theta\times$C2φ (15×13), placed below the beam line.

Two typical configurations occurring in $\pi - p$ scattering are represented in Fig. 2.

The first one (full line) refers to forward $\pi - p$ scattering. In this case any particle detected by the B hodoscope is defined to be a "proton" and any particle detected by the A hodoscopes is defined to be a "pion".

The second one (dashed line) refers to backward $\pi - p$ scattering. In this case, much more rare due to the smaller cross section, the counters N4, N5, and N6 are also used. N4, a threshold gas Cerenkov counter in anticoincidence, covering the whole solid angle subtended by the A2 hodoscope, imposes the condition that the forward going particle must be relatively slow and

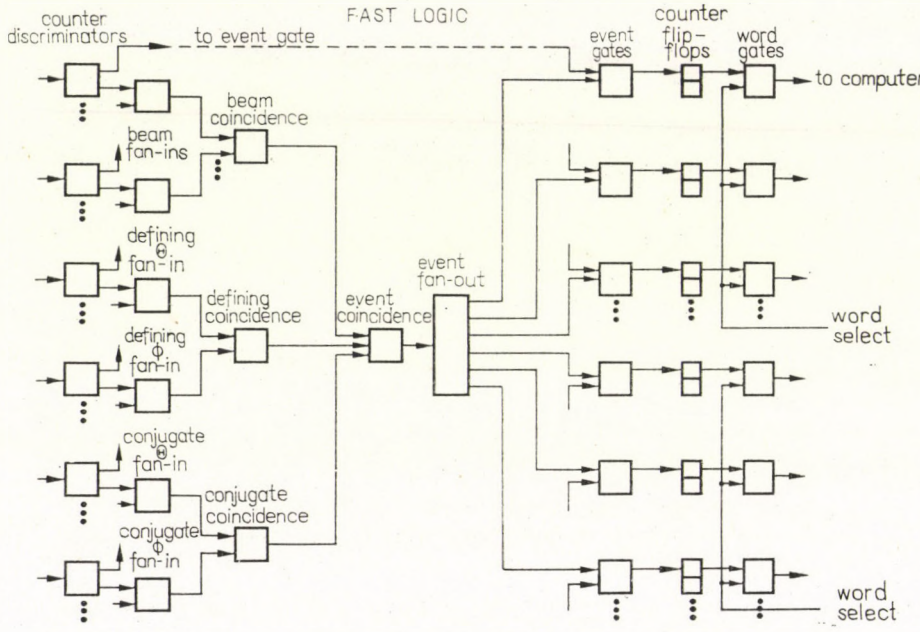

Fig. 3. Simplified block diagram of the fast electronics

a "proton" is defined as particle detected by the A2 hodoscope but not by N4. N5, and N6, two lucite Cerenkov counters in coincidence, covering the whole solid angle subtended by the C hodoscopes, impose the condition that the backward going particle must be relatively fast and a "pion" is defined as a particle that is simultaneously detected by C1 and N5 or by C2 and N6.

The counters K2, K3, and K4 are in anticoincidence. K2, in the horizontal plane, shields the C hodoscopes from products of interactions taking place in the position hodoscope. K3, consisting of two discs in the vertical plane, rejects all events in which a particle goes into or comes from the pole tips of the polarized target magnet. K4, perpendicular to the beam, vetoes non-interacting beam particles.

A simplified block diagram of the fast electronics is shown in Fig. 3. Whenever

— one "beam particle"
— one "proton" in the B hodoscope
— one "pion" in the A hodoscopes

or

— one "beam particle"
— one "proton" in the A2 hodoscope
— one "pion" in the C hodoscopes

are detected simultaneously and no signal from K2, K3, or K4 is present, the information of which counters have been triggered is sent to an on-line computer.

The apparatus is thus capable of taking forward $\pi^+ - p$, $K^+ - p$, $p - p$ and backward $\pi^+ - p$ scattering data simultaneously [8].

The one-line computer [9]

— rejects those events with more than one particle in one of the beam hodoscopes or more than two particles in the final state
— determines nature, momentum, angle and position of the incoming particle
— calculates the angles Θ and φ of the two outgoing particles
— sorts and stores the parameters of the analyzed events to form various distributions
— reads in miscellaneous information such as counting rates at various points in the electronics, target polarization data, etc.
— writes all the raw and calculated information on a magnetic tape for subsequent analysis.

A cathode ray tube is used for an on-line display of the distributions formed in the computer memory, thus allowing a continuous monitoring of the experiment. As an example, Figs. 4 and 5 show two pictures of the computer cathode ray tube displaying the count distributions in the A2Θ and BΘ hodoscopes, respectively, for the events read in by the computer.

Up to about 300 events per pulse can be read in by the on-line computer.

The material of the polarized target is a crystal of LMN [10]. A polarization of about 55% is obtained for the free protons contained in the crystal, which amount, however, to only about 3% of the total number of protons present.

Events from $\pi - p$ elastic scattering off the free protons of the LMN target are separated from other events on the basis of coplanarity and angular correlation in the scattering plane. Figs. 6 and 7 show two pictures of the computer cathode ray tube displaying the "proton" distribution in the BΘ hodoscope for events in which the "pion" is detected in two particular bins in the center (bin 24, Fig. 6) and at the top (bin 8, Fig. 7) of the A2 hodoscope. The full line is for "coplanar" events, the dotted line for "non-coplanar" events [11]. The vertical dotted lines indicate the region where protons from elastic $\pi - p$ scattering on protons at rest are expected to appear.

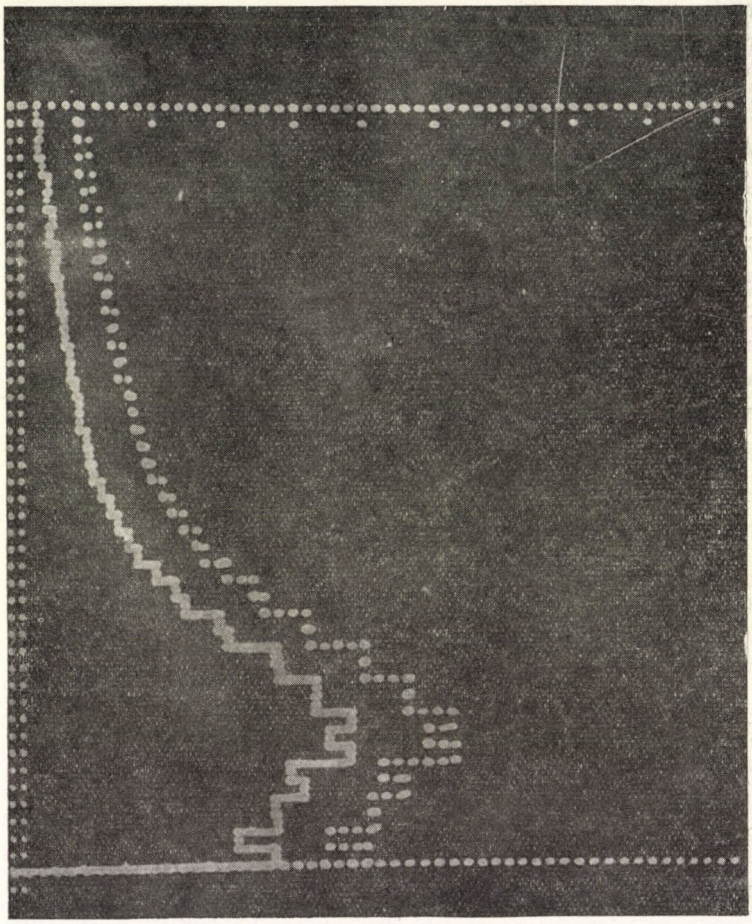

Fig. 4. Pictures of the on-line computer cathode ray tube displaying the count distributions in the $A2\Theta$ (Fig. 4) and $B\Theta$ (Fig. 5) hodoscopes for the events read in by the computer (dotted line) and the same distribution after subtraction of events with more than one particle in one of the beam hodoscopes or more than two particles in the final state (full line)

Protons of events due to elastic scattering on free protons show up as peaks in top of a smooth background, mainly due to events taking place on the complex nuclei of the LMN target. The background under the peaks is subtracted in two independent ways. The first method consists in using the "non-coplanar" events in order to establish the shape of the Θ distribution for the background events. The second method consists simply in interpolating the background under the peaks by fitting the Θ distribution outside the peaks with a simple function. Both methods gave the same results well within errors.

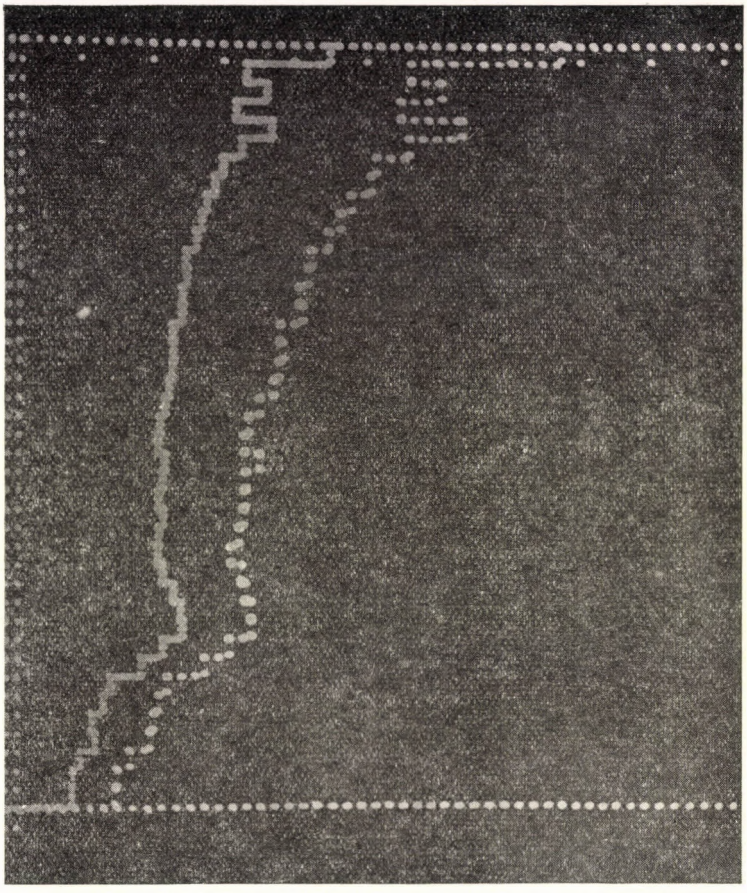

Fig. 5. (For caption see Fig. 4 on p. 112)

Fig. 8 shows the "proton" distribution in the A2 hodoscope for events in which the "pion" is detected in the region of the C hodoscope corresponding to $\cos \Theta_{cm} = -0.99$ and illustrates the effect of the Cerenkov counter N4. These data have been obtained for π^+'s of 2.75 GeV/c in 8 hours of running time.

The measurement of the asymmetry from which the polarization of the proton in $\pi - p$ scattering on an unpolarized target can be derived is performed by determining, at each angle, the number of elastic scattering events off the free protons of the LMN target for the two signs of the target polarization.

III. Results

The results for the polarization in $\pi^{\pm} - p$ elastic scattering at 5.15 GeV/c in the forward direction are shown in Fig. 9 [12].

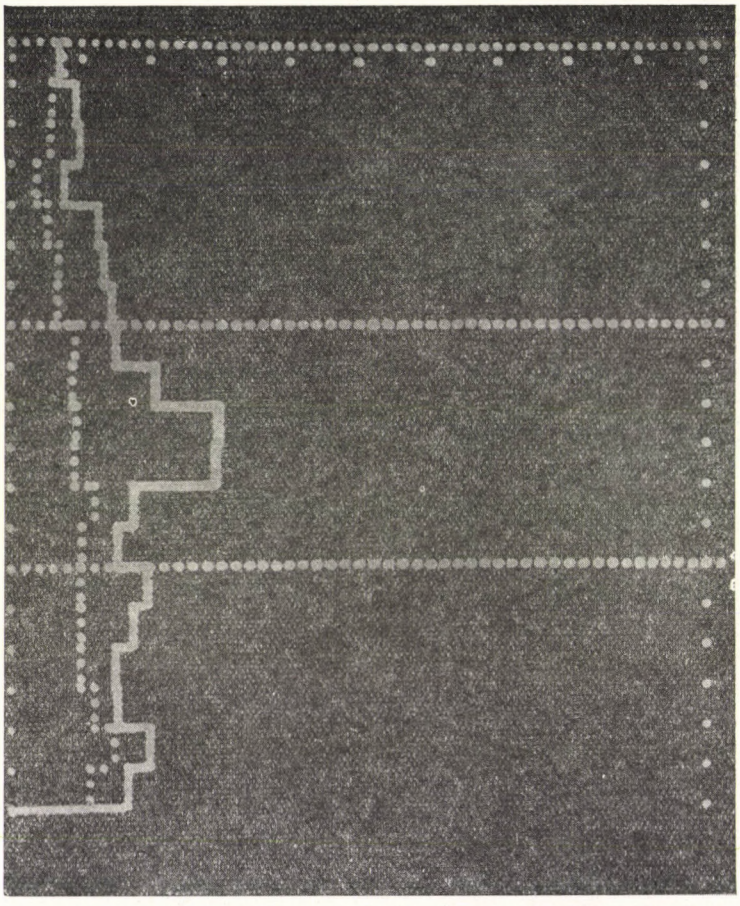

Fig. 6. Pictures of the on-line computer cathode ray tube displaying the "proton" distributions in the $B\Theta$ hodoscope for events in which the "pion" is detected in two particular bins in the center (bin 24, Fig. 6) and at the top, (bin 8 Fig. 7) of the $A2$ hodoscope. The full line is for "coplanar" events, the dotted line for "non coplanar" events. The vertical dotted lines indicate the region where protons from elastic π-scattering on free protons are expected to appear

The data are summed over the seven bins of the momentum hodoscope since, within the accepted $\pm$ 3.5%, there is no significant variation of the polarization with momentum. The errors are statistical and include the uncertainty in background subtraction. There is, in addition, a $\pm$ 10% normalization error due to the uncertainty in the target polarization.

At small momentum transfers, $P_+(t)$ (the polarization in $\pi^+ - p$ scattering) is positive while $P_-(t)$ (the polarization in $\pi^- - p$ scattering) is negative [13] and both become small near $-t = 0.6$. In the region $0.5 \lesssim -t \lesssim 0.8$, $P_-(t)$ becomes positive and very closely equal to $P_+(t)$, which remains positive. For $-t \gtrsim 0.8$, $P_+(t)$ and $P_-(t)$ become large again, with $P_+(t)$ remaining positive and $P_-(t)$ becoming again negative.

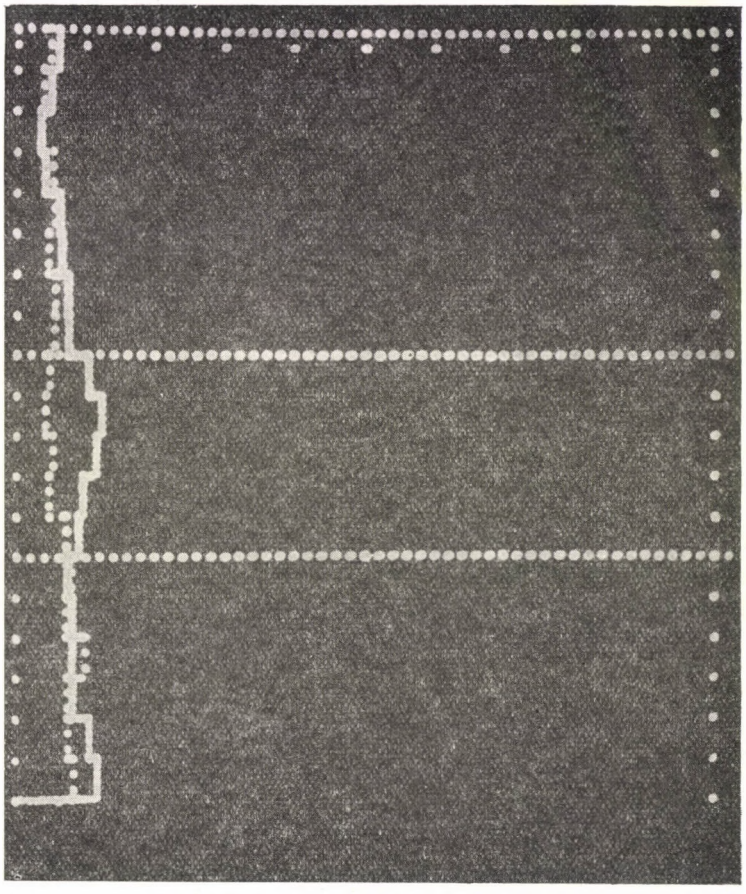

Fig. 7. (For caption see Fig. 6 on p. 114)

Experimental data on $\pi - p$ scattering at high energies are generally analyzed in the framework of the $P + P' + \varrho$ Regge-pole model [14].

In this model, the Regge-pole contributions to the invariant amplitudes A' and B [15] are generally parametrized as follows:

$$A' = C_0 \exp(C_1 t)\, \alpha(\alpha + 1)\, \xi\, (E_L/E_0)^{\alpha} \text{ for } P \text{ and } P'$$

$$= C_0 [(1 + C_2) \exp(C_1 t) - C_2]\, (\alpha + 1)\, \xi\, (E_L/E_0)^{\alpha} \text{ for } \varrho$$

and

$$B = D_0 \exp(D_1 t)\, \alpha^2 (\alpha + 1)\, \xi\, (E_L/E_0)^{\alpha - 1} \text{ for } P \text{ and } P'$$

$$= D_0 \exp(D_1 t)\, \alpha(\alpha + 1)\, \xi\, (E_L/E_0)^{\alpha - 1} \text{ for } \varrho,$$

where

$$\xi(t) = [\exp(-i\pi\alpha) \pm 1]/\sin \alpha$$

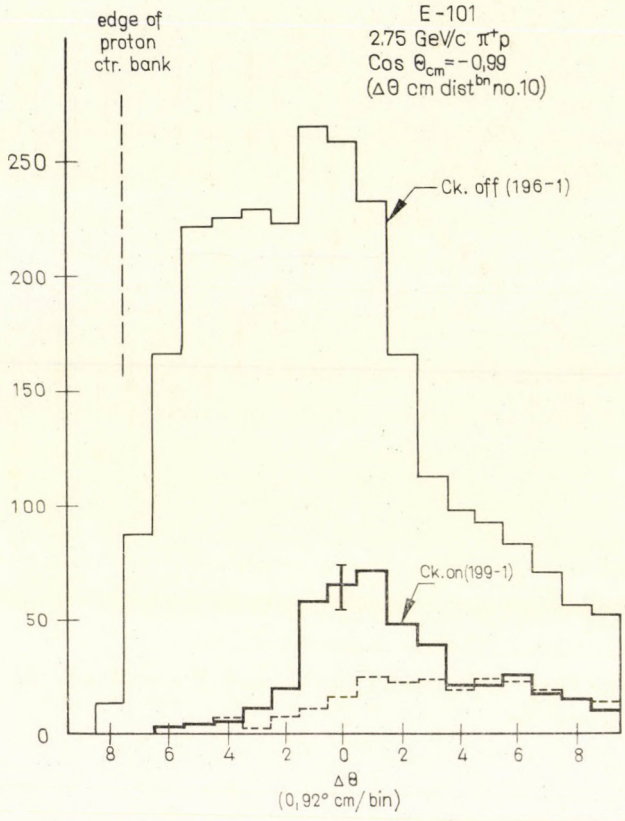

Fig. 8. The "proton" distribution in the $A2$ hodoscope for events in which the "pion" is detected in the region of the C hodoscope corresponding to cos $\Theta_{cm} = -0.99$, illustrating the effect of the Cerenkov counter $N4$. The full line histogram is for "coplanar" events, the dotted line histogram for "non-coplanar" events. Data have been obtained for π^+'s of 2.75 GeV/c in 8 hours of running time

and

$$\alpha(t) = \alpha(0) + t\alpha'.$$

$\alpha(t)$ is the trajectory, $\xi(t)$ the signature factor (with signature $+$ for P and P' and $-$ for ϱ), E_L the total pion lab energy and E_0 a scale factor. C_0, C_1, C_2, D_0, D_1, $\alpha(0)$ and α' are adjustable parameters.

Another parametrization for the P' amplitudes, the so-called no-compensation mechanism,

$$A' = C_0 \exp(C_1 t)\, \alpha^2 (\alpha + 1)^2\, \xi\, (E_L/E_0)^{\alpha},$$

$$B = D_0 \exp(D_1 t)\, \alpha^2 (\alpha + 1)\, \xi\, (E_L/E_0)^{\alpha-1}$$

has also been proposed [16].

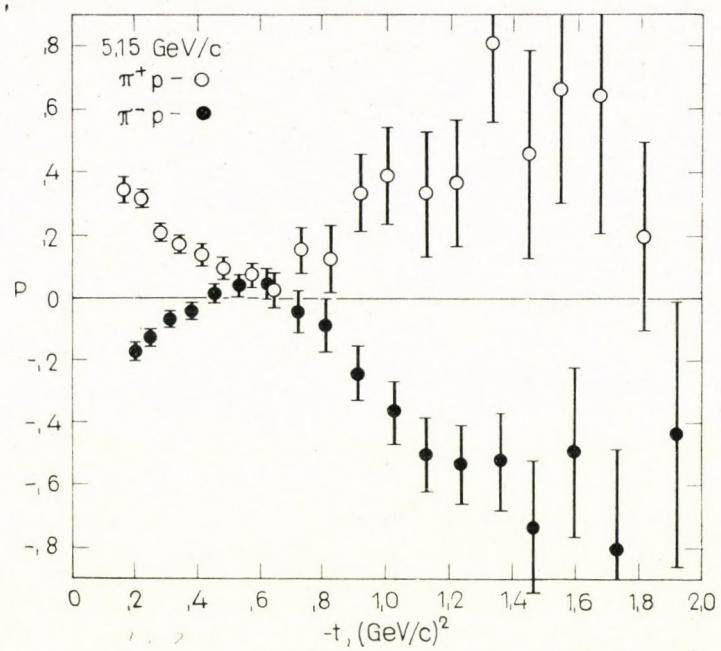

Fig. 9. Polarization in $\pi^+ - p$ elastic scattering in the forward direction at 5.15 GeV/c

In the three pole model the A' and B amplitudes for $\pi^\pm - p$ scattering are then given by

$$A' = A'_p + A'_{p'} \mp A'_\varrho ,$$

$$B = B_p + B_{p'} \mp B_\varrho .$$

In terms of these amplitudes, one has [13]

$$P_\pm (t) = \pm \frac{\sin \Theta_{\mathrm{cm}}}{16\pi \sqrt{s}} \cdot \frac{\mathrm{Im}\,(A'\,B^*)}{(d\sigma/dt)_\pm} .$$

The A'_ϱ amplitude is known to be small and is normally neglected. Since the A' and B amplitudes have the same phase for a given trajectory, the terms that can contribute to the polarization are $(A'_p + A'_{p'}) \times B_p$, $A'_p \times B_{p'}$ and $A'_{p'} \times B_p$.

The first term, which is known to dominate at small $-t$, has opposite sign for $\pi^- - p$ and $\pi^+ - p$ scattering, whereas the last two terms give equal contribution to $P_+(t)$ and to $P_-(t)$. Thus, in first approximation, $P_+(t) = -P^-(t)$ and both vanish at $\alpha_\varrho(t) = 0$, whish occurs at $-t \approx 0.6$ [14].

The data of Fig. 9 show that $P_+ (t) \approx -P_- (t)$ at large $-t$ as well, thus indicating that the term $(A'_p + A'_{p'}) \times B_\varrho$ still dominates the polarization at values of the momentum transfer well beyond $-t = 0.6$.

The amplitude A'_p is always mainly imaginary. In the vicinity of $-t = 0.6$

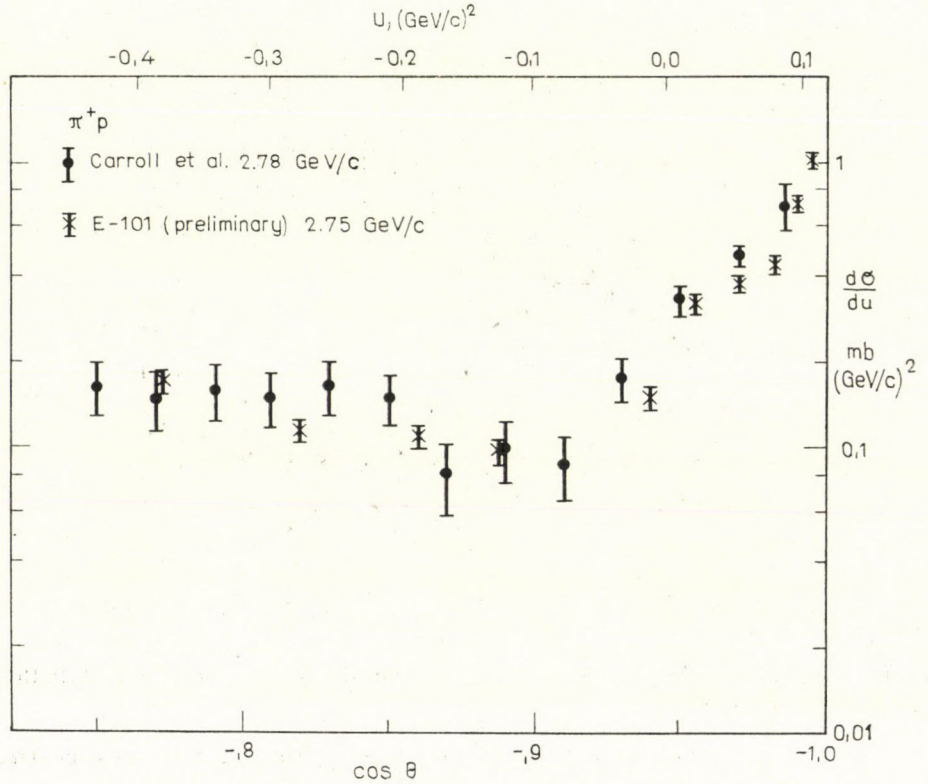

Fig. 10. Preliminary backward $\pi^+ - p$ cross section data at 2.75 GeV/c

the amplitude B_ϱ is also mainly imaginary, and the polarization is dominated by the term $\mathrm{Im}B_\varrho$ Re $A'_{p'}$.

Now $\mathrm{Im}B_\varrho$ changes sign when $\alpha_\varrho(t)$ passes through zero. Therefore, $P_+(t)$ stays positive and $P_-(t)$ negative, as required by the experimental data, only if $\mathrm{Re}A'_{p'}$ also changes sign somewhere near $-t = 0.6$.

On the other hand, $\alpha_{p'}(t)$ is known to pass through zero near $-t = 0.5$ [16]. Thus, only the parametrization of the $A'_{p'}$ amplitude according to the no-compensation mechanism has the wanted t dependence.

However, the no-compensation mechanism makes both $A'_{p'}$ and $B_{p'}$ vanish when $\alpha_{p'}(t) = 0$. This implies that, in the region around $-t = 0.5$, $P_+(t)\,(\mathrm{d}\sigma/\mathrm{d}t)_+ = -P_-(t)\,(\mathrm{d}\sigma/\mathrm{d}t)_-$. This conclusion is clearly in disagreement with the data of Fig. 9.

This may very well be an indication of an inadequacy of the model to explain the data [17].

As yet, only few data for backward $\pi^+ - p$ scattering have been taken. As an example, backward $\pi^+ - p$ cross section and polarization data at 2.75 GeV/c are shown in Figs. 10 and 11, respectively.

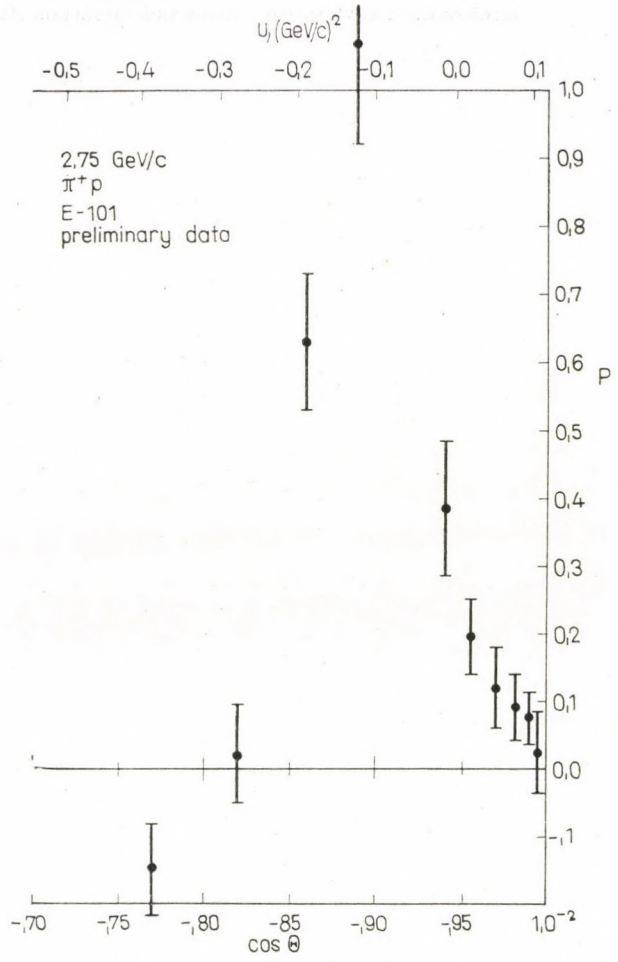

Fig. 11. Preliminary backward $\pi^+ - p$ polarization data at 2.75 GeV/c

The backward $\pi^+ - p$ scattering results are still preliminary and are presented here to illustrate the capabilities of the experiment. The analysis of their implications will have to wait until these data and the forthcoming ones at different energies will be in their final shape.

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9. An ASI 6020. Its main characteristics are a 16K word (24 bit plus parity bit) core memory and a 1.9 microseconds memory cycle time.
10. LMN is a conventional abbreviation for $La_2Mg_3(NO_3)_{12} \cdot 24H_2O$.
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УПРУГОЕ РАССЕЯНИЕ $\pi^\pm - p$ НА ПОЛЯРИЗОВАННЫХ ПРОТОНАХ ПРИ БОЛЬШОМ ПЕРЕНОСЕ ИМПУЛЬСА

Г. КОНФОРТО

Резюме

Описываются последние результаты измерений упругого рассеяния $\pi^+ - p$ на поляризационных протонах при большом переносе импульса. Результаты сравниваются с явлением полюсов Редже.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 121—125 (1969

CHIRAL SYMMETRY CURRENTS AND CURRENT ALGEBRAS

INTRODUCTION TO CHIRAL SYMMETRY AND NON-LINEAR LAGRANGIANS

By

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The non-linear realization of the approximate chiral symmetry of the strong interactions is presented.

In recent years the approximate chiral symmetry of the strong interactions has been firmly established. The first indication of the existence of this symmetry appeared a decade ago after the discovery of parity non-conservation in the weak interactions and the subsequent unraveling of its V—A structure. The occurrence of the hadronic axial current A in close correspondence to the vector current V associated with SU(2) symmetry, suggested the existence of a higher symmetry, the chiral $SU(2)\times SU(2)$. In particular, the absence of a large renormalization of the axial coupling constant by the strong interactions led FEYMANN and GELL-MANN [1] to propose that the axial current be approximately conserved in analogy to the conservation of the vector current. The renormalizable Kemmer Lagrangian, which was then the popular pion—nucleon Lagrangian, does not possess a symmetry which would imply this property. POLKINGHORNE [2] added to it a scalar meson coupled to the nucleon with the same strength as the pion thus obtaining a chiral symmetric Lagrangian. Another solution was to consider the non-renormalizable pseudovector pion—nucleon interaction which is invariant under the transformation of adding a constant to the pion field. This transformation generates an axial current which is conserved in the limit of zero mass for the pion. This idea, known as PCAC [3, 4] was stimulated by the success of the GOLDBERGER—TREIMAN relation [5] and played an important role in further developments of the theory, particularly since BLIN-STOYLE[6] pointed out that axial current conservation does not imply non-renormalization of the axial charge (because of the existence of a massless pion) as had been expected originally.

Lagrangian models possessing chiral $SU(2)\times SU(2)$ symmetry were constructed by GÜRSEY [7], GELL-MANN and LEVY [8] and NAMBU and JONA-LASINIO [9]. The commutation relations for the generators of this group were

* Work supported by the National Science Foundation.

specially emphasized by GELL-MANN and LEVY [8] and constituted the beginnings of GELL-MANN's formulation of current algebra [10]. NAMBU and his collaborators [9, 11, 12] deduced further physical consequences in the form of soft pion relations, but otherwise interest in chiral Lagrangians waned until after the success of current algebra methods. WEINBERG [13] then pointed out that naive perturbation calculations with chiral symmetric Lagrangians lead to the results of current algebra.* Further extensions of chiral Lagrangians have been made to include SU(3) symmetry [13], the vector mesons [14, 15] and a relativistic extension of SU(6) which Professor GÜRSEY presented here [17, 18].

To illustrate an elementary method for constructing chiral invariant Lagrangians let us restrict this presentation to the pion-nucleon system. Consider the familiar Kemmer Lagrangian

$$\mathscr{L} = i\bar{\psi}\gamma_\mu \partial_\mu \psi - m\bar{\psi}\psi - ig\bar{\psi}\gamma_5 \tau\cdot\varphi\,\psi + \\ + \frac{1}{2}\partial_\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\mu^2\varphi^2, \tag{1}$$

where ψ and φ are the conventional nucleon and pion fields. This Lagrangian is invariant under infinitesimal SU(2) transformations

$$\delta\psi = i\frac{\tau}{2}a\,\psi, \tag{2}$$

$$\delta\varphi = \varphi \times a, \tag{3}$$

which leads to the conserved vector isospin current

$$V_\mu = \overline{\psi}\,\gamma_\mu\frac{\tau}{2} + \varphi\times\partial_\mu\varphi. \tag{4}$$

The charge component of this current is associated with the vector part of the weak interaction current. The question naturally arises what current plays the role of the corresponding weak axial current. Insofar as conserved currents are generated by symmetry operations it is suggestive that we consider the axial analog of the isospin transformation eq. (2)

$$\delta\psi = i\gamma_5\frac{\tau}{2}b\psi. \tag{5}$$

* It is interesting to speculate what would have happened if such calculations had been attempted seven years earlier!

We shall leave open the corresponding transformation for the pion field φ required to obtain invariance. The first point to notice is that although the nucleon kinetic energy term is invariant under this transformation, the nucleon mass term becomes

$$m\delta\,\overline{\psi}\psi = im\,\overline{\psi}\,\gamma_5\tau\cdot b\psi. \tag{6}$$

This variation corresponds to the pion—nucleon interaction term with φ replaced by $mg^{-1}\,b$, and suggests that we attempt to cancel it by defining the chiral transformation of the pion field to be

$$\delta\varphi = -\,mg^{-1}\,b\,. \tag{7}$$

Notice that in this sense the existence of a pseudoscalar isovector pion coupled to the nucleon is demanded by the requirement of chiral symmetry. As yet, however, we do not have chiral invariance, because we must also consider the contribution due to the variation of the nucleon field in the pion—nucleon interaction term. Combining the net variation of the nucleon mass term with the pion—nucleon interaction we have

$$\delta\overline{\psi}\,(1 + igm^{-1}\,\gamma_5\tau\cdot\varphi)\,\psi = -\,gm^{-1}\,\overline{\psi}\psi\varphi\cdot b\,. \tag{8}$$

To cancel this variation we introduce a term quadratic in φ to the Lagrangian

$$c\overline{\psi}\psi\varphi\,. \tag{9}$$

The constant c is chosen by demanding that the chiral variation of Φ, eq. (7), gives a term opposite in sign to that in eq. (8)

$$c = -g^2/2m \tag{10}$$

leading to chiral invariance up to second order in g.

We are left now with the contribution from the chiral variation of the nucleon field ψ. To compensate for it we might try adding a term cubic in φ

$$x\overline{\psi}\,i\gamma_5\tau\cdot\varphi\varphi^2\,\psi, \tag{11}$$

where x is a constant. Two new features now appear. First, in order to achieve the desired cancellation we must modify the chiral variation $\delta\varphi$, eq. (7), to include quadratic terms in φ

$$\delta\varphi = b\,(-1 + y\varphi^2) + z\varphi b\varphi, \tag{12}$$

where we have redefined $gm^{-1}\varphi \to \varphi$ as a dimensionless field. Second, we find that three constants x, y and z are not uniquely determined, but are constrained by the two equations,

$$y - x = 1/2, \tag{13}$$

$$z - 2x = 0. \tag{14}$$

The chiral Lagrangian is therefore not unique; nevertheless it turns out that this lack of uniqueness is related to redefinitions of the pion field which are physically equivalent. It is worthwhile to note at this point that we can introduce other chiral invariant pion—nucleon interactions by allowing gradients in the pion field. An example is the familiar pseudovector coupling

$$f\bar{\psi}\gamma_\mu \gamma_5 \tau \cdot \partial_\mu \varphi \psi \tag{15}$$

which is chiral invariant in lowest order. It is required to obtain renormalization of the weak axial coupling constant as well as the correct s-wave pion-nucleon scattering lengths.

Proceeding further in this manner we can achieve a chiral invariant Lagrangian to any desired order in the coupling constant g. It should be clear that to obtain exact chiral invariance we require an infinite power series. The properties of the general solution were given by GÜRSEY in 1961 and have been summarized in his lecture, so we shall not reproduce them here. Note, for example, that the exponential model corresponds to the choice $x = 1/3$ while the square root model is $x = 0$.

We turn now to the free pion part of the Lagrangian, eq. (1). The pion kinetic energy is chiral invariant under the lowest order variation of φ, eq. (7), but not in higher orders. For example, to maintain invariance under the chiral variation $\delta\varphi$ to second order in g, eq. (13), we add the fourth order pion—pion interaction term

$$s\varphi^2 (\partial_\mu \varphi)^2 + t (\varphi \partial_\mu \varphi)^2, \tag{16}$$

where the constants s and t are determined to be

$$s = x, \tag{17}$$

$$t = \frac{1}{2} + 2x, \tag{18}$$

where x is the third order pion—nucleon coupling constant, eq. (11). On the other hand, the pion mass term is not invariant under chiral transformation,

and there exist no counter terms either; i.e., to obtain exact chiral invariance the pion mass must vanish. Again, GÜRSEY [7] has given the pion invariant chiral Lagrangian to all orders in the coupling constant g.

For practical purposes our perturbative approach is quite useful, particularly in dealing with higher symmetries; i.e., $SU(3)\times SU(3)$. Since for applications the exact Lagrangian must be expanded into its power series, it is simpler to do this from the outset.

Acknowledgement

I would like to thank Professor GEORGE MARX for his hospitality at the Lake Balaton.

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ВВЕДЕНИЕ В КИРАЛЬНУЮ СИММЕТРИЮ И НЕЛИНЕЙНЫЕ ЛАГРАНЖИАНЫ

М. НАЙЕНБЕРГ

Р е з ю м е

Показывается нелинейная реализация приближенной киральной симметрии сильных взаимодействий.

Acta Physica Academice Scientiarum Hungaricae, Tomus 26 (1—2), pp. 127—137 (1969)

GENERALIZED CHIRAL SYMMETRY GROUPS AND THE CLASSIFICATION OF HADRONS

By

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The existence of Lagrangian models partially invariant under groups that generalize chiral symmetry is shown. The implications of these models with regard to be classification of hadrons is discussed. The currents defined from the Lagrangian satisfy a current algebra which includes tensor currents and provides a generalization of the usual SU(3) ⊗ SU(3) current algebra with the extension of the PCAC condition to the more general currents.

1. Introduction

In this talk I would first like to show the existence of Lagrangian models partially invariant under groups that generalize chiral symmetry. These groups include both the chiral $SU(3)\otimes SU(3)$ and the SU(6) groups as subgroups. In the second part I shall briefly discuss the implications of these models with regard to the classification of hadrons. The relevant groups are $SL(6, C)\otimes SL(6, C)$ and the larger group SL(12, C). The currents defined from the Lagrangian satisfy a current algebra which includes tensor currents and provides a generalization of the usual $SU(3)\otimes SU(3)$ current algebra. The Lagrangian also shows an unambiguous way of extending the PCAC condition to these more general currents. One thus obtains larger meson multiplets associated with the compact parts of the groups already mentioned. The Lagrangian models can then be used for a new classification of hadrons. At the end I will discuss a possible identification of the members of the multiplets associated with SL(12, C) and a possibility of extending the Cabibbo rotation within this multiplet.

This work is based on an extension of previous work [1] on non linear Lagrangians invariant under $SU(2)\otimes SU(2)$ and more recent work in collaboration with P. CHANG [2, 3, 4].

2. The partially chiral invariant model with PCAC

Lagrangian models partially invariant under the $SU(2)\otimes SU(2)$ chiral group have been constructed during the years 1959 and 1960. These models have several characteristic features:

a) The Lagrangians are highly non-linear and they are non renormalizable unless new scalar mesons are introduced.

b) The pions are associated with non-linear representations of the chiral group.

c) The symmetry becomes exact in the limit of vanishing π mass. WEINBERG has recently shown that such Lagrangians, when used phenomenologically reproduce the results of current algebra for soft pions [5]. An example of a partially chiral invariant Lagrangian for the pion—nucleon system is [6]

$$L = -\bar{\psi}\gamma_\mu \partial_\mu \psi + \frac{1}{2} g' \bar{\psi} U \gamma_\mu (\partial_\mu U)\psi - m\bar{\psi} U\psi - \frac{1}{16 f^2} Tr(\partial_\mu U)(\partial_\mu U) + L^{S.B.},$$

where U is a matrix function of $\gamma_5 \vec{\tau}\cdot\vec{\Phi}$, ($\vec{\Phi}$ being the pion field) with the property

$$UU^+ = 1 \qquad \text{and} \qquad U(0) = 1.$$

U has the general form

$$U = \sigma(\alpha) + 2if\gamma_5 \varrho(\alpha)\vec{\tau}\cdot\vec{\Phi} = 1 - 2f^2\varphi^2 + 2if\gamma_5 \vec{\tau}\cdot\vec{\Phi} + O(f^3)\ldots,$$

where

$$\alpha = f^2 \vec{\varphi}\cdot\vec{\varphi}$$

and

$$\sigma^2 + 4f^2\varrho^2\varphi^2 = 1.$$

f is a universal length and g' is dimensionless. To first order in f we obtain

$$L' = -\bar{\psi}\gamma_\mu \partial_\mu \psi - m\bar{\psi}\psi - \frac{1}{2}\partial_\mu \vec{\varphi}\cdot\partial_\mu \vec{\varphi} - 2mf\bar{\psi} i\gamma_5 \vec{\tau}\cdot\vec{\varphi}\psi -$$

$$- g' f \bar{\psi} i\gamma_5 \gamma_\mu \vec{\tau}\psi \partial_\mu \vec{\varphi} + O(f^2) + \ldots.$$

This shows that $2mf$ and $g'f$ are respectively the pseudoscalar and the pseudovector pion—nucleon coupling constants. If this Lagrangian is used as a phenomenological Lagrangian to lowest order in f, the Dyson—Foldy equivalence theorem allows us to calculate the effective $\pi - N$ pseudoscalar interaction constant $g_{\pi NN} = g$ to be $g = 2mf(1 + g')$.

The corresponding dimensionless pseudovector coupling constant is $F = \mu f(1 + g')$. Experimentally $F^2/4\pi = 0.08$ and μ is the pion mass, provided L is understood as a phenomenological Lagrangian in which the constants f, m, g', μ are all renormalized quantities.

The model is invariant under isospin and the remaining three transformations of SU(2)⊗SU(2), namely

$$\psi \to e^{i/2\,\gamma_5 \tau\cdot\vec{a}}\psi,\ U \to e^{-i/2\,\gamma_5\vec{\tau}\cdot\vec{a}} U e^{-i/2\,\gamma_5\vec{\tau}\cdot\vec{a}}.$$

The transformation law shows that the pion belongs to a non-linear representation of the chiral group. The operators that transform linearly are $\sigma(f^2\varphi^2)$ and $\varrho(f^2\varphi^2)\varphi^i$. Using the usual technique we can construct vector and axial vector currents that satisfy GELL-MANN's current algebra.

There are two principles that can guide us in choosing the symmetry breaking term.

a) *Exact PCAC*

$L^{S.B.}$ is determined so that the pion fieldis proportional to the divergence of the axial vector current. This principle gives

$$L^{S.B.}(\alpha) = \frac{\mu^2}{2f^2}\int_0^{f^2\varphi^2} \frac{\sqrt{\alpha}\,\sigma'(\alpha)}{\sqrt{1-\sigma^2(\alpha)}}\,d\alpha = -\frac{1}{2}\mu^2\vec{\varphi}\cdot\vec{\varphi} + O(f^2).$$

The Lagrangian leads to the PCAC relation $\partial_\mu J^i_{5\mu} = \mu^2/2f\varphi^i = \mu^2 C_\pi \varphi^i$ for an arbitrary $\sigma(\alpha)$. For the choice $\sigma = \cos 2\sqrt{\alpha}$ associated with the exponential model, we have exactly $L^{S.B.} = -1/2\mu^2\vec{\varphi}\cdot\vec{\varphi}$ which is the ordinary mass term.

b) *Simple behaviour under* SU(2)⊗SU(2).

The simplest choice would be for $L^{S.B.}$ to be a mixture of a SU(2)⊗SU(2) scalar and the 4th component of the vector associated with the (1/2, 1/2) representation.

These two principles are satisfied simultaneously if we choose $\sigma(\alpha) = \sqrt{1-4\alpha}$ (the square root model). Then we have $\varrho = 1$ and we find

$$L^{S.B.} = \frac{\mu^2}{4f^2}(\sigma(\alpha) - 1).$$

Since in this special case $(\sigma, \vec{\varphi})$ forms a 4-vector, the second condition is satisfied. The model is now unique and any redefinition of the pion field does not change the physical predictions [7]. The parameters g' and f are fixed by the pion decay constant and $G_A/G_V = -g_A$ through

$$g_A = 1 + g', \quad F_\pi = C_\pi\sqrt{2} = \frac{1}{f\sqrt{2}}.$$

Another popular form of the effective chiral invariant Lagrangian is obtained by making a unitary transformation on ψ and $\vec{\varphi}$ to the fields

$$\vec{\pi} = \vec{\varphi} \text{ and } N = U^{1/2}\psi.$$

This is the form used by SCHWINGER [8], WEINBERG [5, 9], ZUMINO and

WESS [10]. The chiral transformation is the same for $\vec{\pi}$, but for N it assumes the form

$$N \to \Lambda(\vec{a}, \vec{\pi})\, N\,,$$

where

$$\Lambda = 1 + \frac{i}{2} f \vec{\tau}\,(\vec{a} \times \vec{\pi}) + O(f^2)\,.$$

The square root model gives WEINBERG'S values [5, 3]

$$a_0 = 0.20\mu^{-1}, \quad a_2 = -\frac{7}{2} a_0$$

for the S-wave $\pi - \pi$ scattering lengths and $\pi - N$ scattering lengths consistent with the ϱ-dominance model

$$a_1 + 2a_3 = 0 \quad \text{and} \quad a_1 - a_3 = f_\mu^2 \left(1 + \frac{\mu}{m}\right)^{-1}\,.$$

3. Extension of the chiral model to include SU(6) symmetry

The successes of the phenomenological SU(6) symmetry [11] for vertices and the classification of hadrons is well-known. To combine this kind of symmetry with the chiral symmetry in a covariant way we consider the group generated by λ_i, $\gamma_5 \lambda_i$, $\sigma_{\mu\nu}$ and $\lambda_i \sigma_{\mu\nu}$. The group is SL(6, C)$\otimes$SL(6, C) with parameters

$$\frac{1 \pm \gamma_5}{2} (\lambda_i, \sigma_{\mu\nu}, \lambda_i \sigma_{\mu\nu})\,.$$

Let H be a unimodular matrix. We define [3, 4]

$$\bar{H} = \gamma_4 H^\dagger \gamma_4.$$

The generators of the group SL(6, C)$\otimes$SL(6, C) all commute with γ_5. Impose the condition

$$\gamma_5 H = H \gamma_5.$$

Then H can be put in the canonical form

$$H = e^{-i\Phi'} e^{\Phi},$$

where

$$Tr\, \Phi' = Tr\, \Phi = 0$$

and $$\Phi = \bar{\Phi},\ \Phi' = \bar{\Phi}',$$

Φ and Φ' commuting with γ_0. The most general form is given by

$$\Phi = f\left(\lambda_i \varphi_i + i\gamma_5 \varphi_5 + i\gamma_5 \lambda_i \varphi_{5i} + \frac{1}{2}\sigma_{\mu\nu}\varphi_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}\lambda_i \varphi_{\mu\nu i}\right),$$

$$\Phi' = f\left(\lambda_i \varphi'_i + i\gamma_5 \varphi'_5 + i\gamma_5 \lambda_i \varphi'_{5i} + \frac{1}{2}\sigma_{\mu\nu}\varphi'_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}\lambda_i \varphi'_{\mu\nu i}\right).$$

In the free quark Lagrangian

$$-\bar{\psi}\gamma_\mu \partial_\mu \psi - \bar{\psi} m \psi$$

we make the substitutions

$$\gamma_\mu \to M_\mu = \bar{H}\gamma_\mu H, \quad m \to m\bar{H}H.$$

Then, if B is a C-number matrix with the same structure as H, namely

$$B = e^{-i\Omega'} e^{\Omega},$$

such matrices B form a finite dimensional representations of the group $SL(6, C) \otimes SL(6, C)$ and the new Lagrangian is invariant under this group, since we have

$$H \to H' = HB^{-1},$$

$$\psi \to \psi' = B\psi$$

under the group.

We can also add an invariant kinetic meson term of the form

$$L^{\text{meson}} = \frac{1}{16 f^2} Tr\left\{(\partial_\lambda H)(\partial_\lambda H^{-1}) + (\partial_\lambda \bar{H}^{-1})(\partial_\lambda \bar{H}) - \right.$$

$$\left. - \frac{1}{2}\partial_\mu (\bar{H}\gamma_\mu H)(\partial_\nu \bar{H}\gamma_\nu H)\right\} + L^{\text{m}}.$$

For $f \to 0$, this Lagrangian leads to a positive definite Hamiltonian.

From the equations of motion $\varphi^i_{\mu\nu}$ and $\varphi'^i_{\mu\nu}$ satisfy the subsidiary conditions

$$\partial_\nu \tilde{\varphi}_{\nu\lambda i} = 0, \qquad \partial_\mu \varphi_{\mu\nu i} = 0.$$

This shows that $\varphi_{\mu\nu i}$ and $\varphi'_{\mu\nu i}$ represent respectively 1^- and 1^+ mesons. If we study the C-conjugation property of their neutral member we find that these

are 1^{--} and 1^{++} mesons while φ_i, φ_{5i}, φ'_i and φ'_{5i} are respectively 0^{++}, 0^{-+}, and 0^{--}. Now the linear couplings of φ'_i and φ'_{5i} to ψ vanish, so that the only meson, left are 0^{++}, 0^{-+}, 1^{--} and 1^{++} associated with a scalar octet, a pseudoscalar nonet containing π, a vector nonet containing σ and an axial vector nonet containing A_1. These mesons correspond to the representations

$$(1,1) \qquad \text{and} \qquad (1,35)\otimes(35,1)$$

of $SU(6)\otimes SU(6)$, which is the compact part of $SL(2, C)\otimes SL(2, C)$.

It should be noted here that the 0^{++} fields play the role of the function σ in the simple chiral theory, so that they can be expressed in terms of the remaining mesons π, ϱ and A.

4. Generalization to SL(12, C)

If we lift the restriction that H commutes with γ_5, then we still have

$$H = e^{-i\Phi'} e^{\Phi}$$

with

$$\Phi = \bar{\Phi} = f\Big(\lambda_i \Phi_i + i\gamma_5 \varphi_5 + \frac{1}{2}\cdot \sigma_{\mu\nu}\varphi_{\mu\nu} + i\gamma_5\gamma_\nu\varphi_{5\nu} + i\gamma_\nu\varphi_\nu +$$

$$+ i\lambda_i\gamma_5\varphi_{5i} + \frac{1}{2}\lambda_i\sigma_{\mu\nu}\varphi_{\mu\nu i} + i\lambda_i\gamma_5\gamma_\nu\varphi_{5\nu i} + i\lambda_i\gamma_\nu\varphi_{\nu i}\Big)$$

and similarly for Φ'. The Lagrangian is formally the same. For $f \to 0$ with a meson mass term it leads to the subsidiary conditions

$$\partial_\mu\varphi_{5\nu i} - \partial_\nu\varphi_{5\mu i} = 0,\ \partial_\nu\varphi_{\nu i} = 0,\ \partial_\nu\tilde{\varphi}_{\nu\lambda} = 0,$$
$$\partial_\mu\varphi'_{5\mu i} = 0,\ \partial_\mu\varphi'_{\mu\nu i} = 0,\ \partial_\mu\varphi'_{\nu i} - \partial_\nu\varphi'_{\mu i} = 0.$$

These leave 143 field components filling the adjoint representation of SU(12) which is the compact part of SL(12, C). They can be classified in the following I^{PC} SU(3) multiplets, C showing the charge conjugation parity of the neutral members of the multiplets:

a)	One	0^{++}	octet	φ_i	that are functions of the remaining mesons,
b)	One	0^{++}	nonet	$(\varphi'_\nu,\ \varphi'_{\nu i})$	contains σ (700) and δ,
c)	One	0^{-+}	nonet	$(\varphi_5,\ \varphi_{5i})$	contains η' and π,
d)	One	0^{-+}	nonet	$(\varphi_{5\nu},\ \varphi_{5\nu i})$	?

e)	One	1^{--}	nonet	$(\varphi_{\mu\nu},\ \varphi_{\mu\nu i})$	contains	ϱ.
f)	One	1^{--}	nonet	$(\varphi_{\nu},\ \varphi_{\nu i})$	contains	G (1650),
g)	One	1^{++}	nonet	$(\varphi'_{\mu\nu},\ \varphi'_{\mu\nu i})$	contains	A_1,
h)	One	1^{+-}	nonet	$(\varphi'_{5\nu},\ \varphi'_{5\nu i})$	contains	B.

The nonets c and e may be thought of forming (1 + 35) SU(6) multiplets: they contain respectively π and ϱ. b and g have positive parity. They may contain δ and A_1, respectively. b, c, e and h form a representation of the chiral SU(6)$\otimes$SU(6). f may contain G (1650) which has the same quantum numbers as ϱ. h contains B. d is as yet unidentified. Finally the octet a) plays the role of the $\sigma(\alpha)$ field in the simple theory and does not correspond to physical mesons. The symmetry is only exact in the limit of zero mass for all these mesons. However, in that limit the subsidiary conditions do not follow from the equations of motion and have to be imposed.

5. Currents and generalized PCAC

Associated with the transformation parameters Ω are the currents

$$j^i_{\mu,A} = \frac{i}{4}\,\overline{\psi}\,[M_\mu, \Gamma_A]\,\lambda^i\,\psi - \frac{i}{16f^2}\,Tr\left\{\left[(\partial_\mu H^{-1})\,H + \overline{H}\,(\partial_\mu \overline{H}^{-1})\right]\Gamma_A\,\lambda^i\right\}$$

$$+\frac{i}{64f^2}\,Tr\left\{\left[\left\{M_\mu,\partial_\lambda M_\lambda^{-1}\right\} - \left\{M_\mu^{-1}\partial_\lambda M_\lambda\right\}\right]\Gamma_A\,\lambda^i\right\},$$

where i is an SU(3) index that can also take the value of zero, μ is the covariant index ($\mu = 1, \ldots 4$) and A is the Dirac index ($A = 1, \ldots, 15$). Similarly, to the parameters Ω' correspond the currents

$$j'^i_{\mu,B} = \frac{1}{4}\,\overline{\psi}\left\{M_\mu, \Gamma_B\right\}\lambda^i\,\psi - \frac{1}{16f^2}\,Tr\left\{\left[(\partial_\mu H^{-1})\,H - \overline{H}(\partial_\mu \overline{H}^{-1})\right]\Gamma_B\,\lambda^i\right\}$$

$$-\frac{1}{64f^2}\,Tr\left\{\left([M_\mu,\partial_\lambda M_\lambda^{-1}] + [M_\mu^{-1},\partial_\lambda M_\lambda]\right)\Gamma_B\,\lambda^i\right\}.$$

We see that these currents include vector and axial vector currents as well as tensor currents of order 2 and 3.

It is useful to introduce the complex currents

$$g^i_{\mu,A} = j^i_{\mu,A} + i j^i_{\mu,A}.$$

The complex charges

$$G^i_A = -\,i\int g^i_{4,A}\,d^3x$$

are conserved for zero meson masses. The hermitian part of G^i_A obeys the SU(12) commutation relations even when the currents are not conserved. This leads to the plausibility of the current algebra

$$[H g^i_{0A}(x),\ H g^j_{0B}(y)]\,\delta(x_0 - y_0) = i\,C^{ijk}_{ABC}\,H g^k_{0c}(x)\,\delta(x-y),$$

where C^{ijk}_{ABC} are the structure constants of SU(12).

The hermitian and antihermitian parts of G^i_A together generate SL(12, C). It should be noted that the operation of taking the hermitian part of G^i_A, namely

$$H\ G^i_A = \frac{1}{2}(G^i_A + G^{i\dagger}_A)$$

is not a covariant operation except when $A = 5$.

For simplicity let us take a common mass μ for all the mesons. Then it can be shown that PCAC can be generalized to

$$\partial_\mu g^i_{\mu,A}(x) = \frac{\mu^2}{2f}\eta_A^{-1}(\varphi^i_A + i\,\varphi^{i\prime}_A) + O(f),$$

where the factor η_A defined by

$$\eta_A = 1 \quad \text{for} \quad A = 5 \quad \text{and} \quad A = \mu,$$

$$\eta_A = -i \quad \text{for} \quad A = 5\mu, \quad \eta_A = i \quad \text{for} \quad A = \mu\nu$$

is necessary to preserve hermiticity.

With each field φ^i_A are associated the covariant canonical variables $\Pi^i_{A\mu}$. The canonical conjugate of φ^i_A is just Π^i_{A0}. The current—meson correspondence is now expressed by the relation

$$g^i_{\mu,A} = \frac{\eta_A^{-1}}{2f}(\Pi^i_{\mu,A} + i\,\Pi^{i\prime}_{\mu,A}) + 0(f).$$

It might be possible to construct models in which these equalities hold to all orders in f. Then we would have a new generalization of PCAC and field algebra.

Once such general results are extracted from the model we could postulate the SL(12, C) algebra and the generalized PCAC for these general currents and derive low energy theorems for soft mesons. In this case the symmetry would be postulated as being exact. Extrapolation of matrix elements and vertex functions to the physical values of the meson masses would then be postulated to be smooth in virtue of the generalized PCAC relations just as

in the case of chiral symmetry. SU(6) relations in this model are exact only when mesons are extrapolated to zero mass.

6. Formulation in terms of new fields

Let us introduce the new quark field

$$\xi = (\bar{H}H)^{1/2}\psi\, e^{\varphi}\psi .$$

The Lagrangian takes the form

$$L = -\frac{1}{2}\bar{\xi}\, e^{i\varphi'}\gamma_\mu e^{-i\varphi'}\partial_\mu \xi + \frac{1}{2}(\partial_\mu \bar{\xi})\, e^{i\varphi'}\gamma_\mu e^{-i\varphi'}\xi -$$

$$-\frac{1}{2}\bar{\xi}\, e^{i\varphi'}\gamma_\mu\, e^{-i\varphi'}(e^{\varphi}\partial_\mu e^{-\varphi})\,\xi + \frac{1}{2}\bar{\xi}(\partial_\mu e^{-\varphi})e^{\varphi}\, e^{i\varphi'}\gamma_\mu e^{-i\varphi'}\xi - m\,\bar{\xi}\xi + L^{\text{meson}} .$$

The field ξ no longer transforms linearly under SL(12, C). Instead it obeys a law of the form

$$\xi \to \xi' = \Lambda(\Omega, \Omega', \Phi)\,\xi ,$$

where Λ is a certain matrix depending on the parameters Ω, Ω' and the meson fields Φ. Consequently, under such relativistic transformations, a quark will transform into a quark by a spin rotation plus a quark state together with soft pseudoscalar and vector mesons. This provides a physical interpretation of relativistic SU(6) transformations as well as chiral transformations in Hilbert space. In the non relativistic limit we get only the spin rotations.

In our treatment chiral symmetry and SU(6) symmetry synthesized in such a way that soft pions and soft rho's are treated on the same footing.

7. Possibility of a generalized Cabibbo rotation

In the theory of weak interactions it is assumed that the weak current consists of the first and second components of the generators of an SU(2) group obtained from the isotopic spin current plus the chiral current by a unitary transformation within the multiplet to which the electric current belongs. The Cabibbo theory results if an SU(3) octet is taken for this multiplet.

In the SL(12, C) theory we have a larger multiplet for currents, namely an SU(12) multiplet with the quantum numbers discussed above. We now have the possibility of making a more general unitary transformation on the charged isotopic plus chiral current in the SU(12) space. In particular consider the unitary transformation matrix

$$U(\theta, \varphi)\, e^{-i\lambda_7(\theta+\varphi\gamma_4)}.$$

If $\varphi = 0$ we obtain the Cabibbo rotation with the angle θ. In general the charged current takes the form

$$\begin{aligned} J_\lambda^{(+)} = \cos\varphi\,[\cos\theta\,\{(\varrho_\lambda^1 + i\varrho_\lambda^2) + A_\lambda^1 + iA_\lambda^2\} + \sin\theta\,\{(\varrho_\lambda^4 + i\varrho_\lambda^5) + A_\lambda^4 + iA_\lambda^5\}] - \\ - \sin\varphi\sin\theta\,\{i(\varrho_\lambda'^1 + i\varrho_\lambda'^2) + (B_\lambda^1 + iB_\lambda^2)\} - \\ - \cos\theta\,\{i(\varrho_\lambda'^4 + i\varrho_\lambda'^5) + (B_\lambda^4 + iB_\lambda^5)\}], \end{aligned}$$

where ϱ stands for the isospin current, A stands for the axial current with the same quantum number as A_1 while ϱ' and B are associated respectively with currents with same quantum numbers as the G meson and the B meson.

Now if leptons are coupled to $J_\lambda^{(+)}$ and $J_\lambda^{(+)}$ is coupled to itself, we see that both the vector and axial vector current become mixtures of operators with opposite CP properties and hence CP is violated. The angle φ characterizes the strength of this violation.

8. Concluding remarks

I have tried to show the existence of non-linear Lagrangian models that are partially invariant under SL(12, C) or SL(6, C)$\otimes$SL(6, C) which are groups generalizing both the chiral and the SU(6) symmetries. This leads to a classification of mesons under SU(12) which is not inconsistent with the new table of resonances. A current—meson correspondence still exists and the weak current is now allowed to be a mixture of current of 1st and 2nd class, leading to the possibility of CP violation.

The strong symmetry is exact at the unphysical point where the meson masses are all extrapolated to zero. The group being non compact its unitary representations would be associated with infinite towers of resonances that can also be classified as charged Toller poles since SL(6, C) and SL(6, C) are subgroups. This raises an intriguing question of the relation of the present scheme to the Regge classification. In the latter Regge trajectories are extrapolated to zero mass where the corresponding mesons no longer have integer spin. It is at this point that the SL(2, C) classification holds. In the present treatment mesons retain their spin but their masses are made to vanish at the symmetry point. Since both classifications seem to be successful there exists enough motivation to attempt a synthesis of the two approaches.

Another promising line is indicated by the possibility of writing phenomenological Lagrangians where mesons are coupled to baryons that should be represented as third rank SL(12, C) tensors. Then the vertex symmetries

can be read off the Lagrangian which should embody the results of chiral symmetry as well as SU(6) symmetry.

In conclusion I would like to record my indebtedness to Prof. G. MARX for hospitality at the Balaton meeting.

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ОБОБЩЕННЫЕ ГРУППЫ КИРАЛЬНОЙ СИММЕТРИИ И КЛАССИФИКАЦИЯ ГАДРОНОВ

[Ф. ГЮРСЕЙ

Резюме

Показывается существование частично инвариантных относительно групп моделей Лагранжианов, которые обобщают киральную симметрию. Истолкуется применение этих моделей по отношению классификации гадронов. Токи, определенные из Лагранжианов, удовлетворяют алгебре токов, которая включает в себя тензорные токи и дает возможность для обобщения обычной алгебры токов SU (3) ⊗ SU (3) с расширением PCAC-условия на более общие токи.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 139—146 (1969)

NON-LINEAR LAGRANGIANS AND RELATIVISTIC SU(6)

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A method for constructing a phenomenological nonlinear Lagrangian of the hadrons within the relativistic framework is presented. A particular SL(6, C)-invariant series of meson-meson and meson—baryon interactions are introduced and it is shown that in a first-order calculation a relation between the various couplings is possible if PCAC is maintained. Using the coupling constant for $\varrho\pi\pi$ as input the following results are noted: $\Gamma(X^0 \to \eta\pi\pi) = 2$ MeV, $\Gamma(\omega \to 3\pi) = 8.9$ MeV, $g^2_{\bar{N}N\varrho/4\pi} = 0.54$.

Non-linear Lagrangians and relativistic SU(6)

There has been considerable interest lately [1] in the calculation of strong interaction processes by means of non-linear Lagrangians which arise as a consequence of invariance or partial invariance (PCAC) under gauge transformations which generate particular algebras (SU(2)⊗SU(2); SU(3)⊗ ⊗SU(3)). The purpose of this discussion is to review some previous work [2] which describes the extension of the non-linear models to the relativistic formulation of the SU(6) symmetry. The physical particles then possess the same transformation properties as tensors formed from the fundamental quark and anti-quark states. The twelve component bispinor representation of SL(6, C) provides the basis for these quarks and the physical states transform like tensors of SL(6, C). The physical fields are required to satisfy wave equations [3] which are not invariant under SL(6, C) but yield conditions on the physical fields so that their multiplet structure is the same as in the nonrelativistic SU(6) theory.

The total phenomenological hadronic Lagrangian is constructed by imposing the following conditions:

a) The linear gauge transformations for the physical particles are constructed by means of their transformations as tensors formed from fundamental quarks. The quarks undergo the gauge transformations which generate the SU(3)⊗SU(3) algebra.

b) The effective meson—meson and meson—baryon interactions in the lowest order perturbation are obtained by requiring the partially conserved axial-vector current condition (PCAC)

$$\partial_\mu J^{(k)}_{\mu,5} \sim P^{(k)} ,$$

where $P^{(k)}$ is the pseudo-scalar nonet.

The last requirement assumes that PCAC would hold only in a fully SL (6, C)-invariant theory and since the part of the Lagrangian which breaks SL(6, C) is the kinematic term, the divergence of the axial-vector current will contain then kinematic dependent terms. If we assume the canonical momentum is unaltered in the presence of interaction then it is only in the lowest order, where the free-field solutions are utilized, that the kinematic terms may be eliminated by cancellation with the contributions from the interactions.

The non-linear Lagrangian

In this particular model the physical fields will be considered as irreducible tensors of SL(6, C) formed by products of the quark fields. The meson field $\Phi^{j\beta}_{i\alpha} \equiv \Phi^B_A$ is assumed to transform like the 144-component mixed tensor $Q_A \bar{Q}^B$; a totally symmetric 364-component third-rank tensor Ψ_{ABC} which transforms like the symmetric product of three quarks $Q_A Q_B Q_C$ is assumed to describe the baryon field. The notation followed will be the same as SAKITA and WALI (SW) [4]. The gauge transformations will then be derived by assuming that the quarks obey a Dirac equation and by defining the gauge transformations which produce the nonet vector current $1/2\bar{Q}\gamma_\mu \lambda^{(k)} Q$ and the nonet axial-vector current $1/2\bar{Q}\gamma_\mu\gamma_5\lambda^{(k)}\ Q$.

Consider the second-rank tensor $\Phi^{j\beta}_{i\alpha}$ which transforms like $Q_{i\alpha}\bar{Q}^{j\beta}$. Using the infinitesimal transformations for the quark vector and axial-vector currents, respectively

$$Q_{i,\alpha} \to Q_{i,\alpha} + \frac{1}{2} i\varepsilon'^{(k)} \lambda^{(k)\alpha'}_{\alpha} Q_{i,\alpha'} \ (k = 0, \ldots, 8) \tag{1}$$

and

$$Q_{i,\alpha} \to Q_{i,\alpha} + \frac{1}{2} i\varepsilon^{(k)} \lambda^{(k)\alpha'}_{\alpha} (\gamma_5)^{i'}_{i} Q_{i'\alpha'} . \tag{2}$$

We obtain the transformations which give rise to the vector and axial-vector meson currents, respectively

$$\Phi \to \Phi + \frac{1}{2} i\varepsilon'^{(k)}[\lambda^{(k)}, \Phi] \tag{3}$$

and

$$\Phi \to \Phi + \frac{1}{2} i\varepsilon^{(k)}\{\gamma_5 \lambda^{(k)}, \Phi\} . \tag{4}$$

It will be sufficient for illustration to consider here only the boson and quark fields and to note that the baryon fields are formed from the symmetrie

364-component tensor $\psi_{i\alpha,j\beta,k\gamma} = Q_{i\alpha} Q_{j\beta} Q_{k\gamma} +$ (sym.) The considerations of the gauge transformations for this tensor are straightforward but somewhat tedious and therefore the reader is referred to the literature [2].

The 144-complex-component object Φ will appropriately describe the meson field if we impose the reality condition

$$\gamma_4 \Phi \gamma_4 = -\Phi^+. \tag{5}$$

This insures that the antiparticles are contained in the same multiplet as the particles. We then expand in terms of the 16 independent Dirac matrices:

$$\Phi_{i\alpha}^{j\beta} = \delta_i^j S_\alpha^\beta + (\gamma_\mu)_i^j V_{\mu,\alpha}^{\ \beta} + \frac{1}{2}(\sigma_{\mu\nu})_i^j T_{\mu\nu,\alpha}^{\ \beta} + (\gamma_\mu \gamma_5)_i^j A_{\mu,\alpha}^{\ \beta} + (\gamma_5)_i^j P_\alpha^\beta, \tag{6}$$

where, S, V, ..., P represent nonets of scalar, vector, tensor, axial-vector, and pseudoscalar fields, respectively.

The meson field Φ obeys the DUFFIN—KEMMER (DK) equation

$$[\gamma_\mu, \partial_\mu \Phi] + 2 m_0 \Phi = 0, \tag{7}$$

and the BARGMANN—WIGNER (BW) equations,

$$\gamma_\mu \partial_\mu \Phi + m_0 \Phi = 0, \tag{8}$$

$$\partial_\mu \Phi \gamma_\mu - m_0 \Phi = 0. \tag{9}$$

Using the equation of motion (7), the free-field solutions for Φ are subject to the following conditions:

$$\begin{aligned} S &= 0, \\ A_\mu &= -(1/m_0)\,\partial_\mu P, \\ T_{\mu\nu} &= (1/m_0)(\partial_\nu V_\mu - \partial_\mu V_\nu). \end{aligned} \tag{10}$$

The Lagrangian for Φ which produces the DK equations (7) may be written in the form [5]

$$\mathcal{L}(\Phi) = -\frac{1}{8} m_0 \, Tr\, \Phi([\gamma_\mu, \partial_\mu \Phi] + 2 m_0) \Phi. \tag{11}$$

Introducing the infinitesimal transformation (3) and (4) into the above, we obtain the vector and axial-vector meson currents and their derivatives.

For the vector current and its derivative we obtain

$$J_{\mu}^{(k)}(\Phi) = \frac{1}{8} m_0 \, Tr \Phi \gamma_{\mu} [\lambda^{(k)}, \Phi] , \tag{12}$$

and

$$\partial_{\mu} J_{\mu}^{(k)}(\Phi) = 0 . \tag{13}$$

For the axial-vector current and its derivative we have

$$J_{\mu,5}^{(k)}(\Phi) = \frac{1}{8} m_0 \, Tr \Phi \gamma_{\mu} \{\gamma_5 \lambda^{(k)}, \Phi\} \tag{14}$$

and

$$\partial_{\mu} A_{\mu}^{(k)}(\Phi) = \frac{1}{4} m_0 \, Tr \gamma_5 \lambda^{(k)} \Phi (\gamma_{\mu} \partial_{\mu} \Phi + 2 m_0 \Phi) . \tag{15}$$

For the free field Φ we may utilize the BW equation (8) and convert the source (9) into the following form:

$$\begin{aligned} \partial_{\mu} J_{\mu,5}^{(k)}(\Phi_{\text{free}}) &= \frac{1}{4} m_0^2 \, Tr \gamma_5 \lambda^{(k)} (\Phi\Phi)_{\text{free}} = \\ &= \frac{1}{2} Tr(m_0 [\partial_{\mu} P, V_{\mu}] + \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \{\partial_{\mu} V_{\nu}, \partial_{\alpha} V_{\beta}\}) \lambda^{(k)} . \end{aligned} \tag{16}$$

To obtain the PCAC condition we now introduce into the infinitesimal meson transformation an inhomogeneous term which will project out, in the divergence of the current, a linear term in pseudoscalar mesons. To this end we consider the modification of (4)

$$\Phi \to \Phi + \frac{1}{2} i \varepsilon^{(k)} \{\gamma_5 \lambda^{(k)}, \Phi\} + i K \gamma_5 \lambda^{(k)} \varepsilon^{(k)} , \tag{17}$$

where K is a constant to be determined. With this transformation the divergence of the axial-vector current for the mesons becomes

$$\begin{aligned} \partial_{\mu} J_{\mu,5}^{(k)}(\Phi) = &\frac{1}{4} m_0 \, Tr \lambda^{(k)} \gamma_5 (\gamma_{\mu} \partial_{\mu} \Phi + 2 m_0 \Phi) + \\ &+ \frac{1}{4} m_0 K Tr \lambda^{(k)} \gamma_5 (\gamma_{\mu} \partial_{\mu} \Phi + 2 m_0 \Phi) . \end{aligned} \tag{18}$$

If we use the equation of motion (8) for the free field Φ the following form is obtained:

$$\partial_{\mu} J_{\mu,5}^{(k)}(\Phi_{\text{free}}) = \frac{1}{4} m_0^2 \, Tr(\lambda^{(k)} \gamma_5 \Phi\Phi + K \lambda^{(k)} \gamma_5 \Phi) . \tag{19}$$

The last term produces the desired PCAC term $2m_0^2 K P^{(k)}$.

In view of our initial assumptions we consider the following $U(6.6)$ invariant Hermitian series of interactions:

$$\mathscr{L}_{\text{int}}(\Phi, Q) = \sum_{n=1}^{\infty} \frac{i^{n^2} g_n}{m_0^{n-1}} (\bar{Q}\Phi^n Q) + \sum_{n=3}^{\infty} \frac{i^{n^2} a_n}{m_0^{n-4}} Tr(\Phi^n), \tag{20}$$

where the g_n and a_n are dimensionless coupling constants. The contribution to the divergence of the axial-vector current from (20) is then

$$\begin{aligned} \frac{i\partial \mathscr{L}_{\text{int}}(\Phi, Q)}{\partial \varepsilon^{(k)}} = &- \sum_{n+1}^{\infty} \frac{i^{n^2} g_n}{m_0^{n-1}} (\bar{Q}\gamma_5 \lambda^{(k)} \Phi^n Q + \bar{Q}\Phi\gamma_5 \lambda^{(k)} \Phi^{n-1} Q + \ldots + \\ &+ \bar{Q}\Phi^n \gamma_5 \lambda^{(k)} Q) - \sum_{n=1}^{\infty} \frac{i^{n^2} g_n K}{m_0^{n-1}} (\bar{Q}\gamma_5 \lambda^{(k)} \Phi^{n-1} Q + \\ &+ \bar{Q}\Phi\gamma_5 \lambda^{(k)} \Phi^{n-2} Q + \ldots + \bar{Q}\Phi^{n-1} \gamma_5 \lambda^{(k)} Q) - \\ &- \sum_{n=3}^{\infty} \frac{i^{n^2} n a_n}{m_0^{n-4}} [Tr(\gamma_5 \lambda^{(k)} \Phi^n) + K Tr(\gamma_5 \lambda^{(k)} \Phi^{n-1})]. \end{aligned} \tag{21}$$

Adding (21) to (19) and $m_Q \bar{Q}\gamma_5 \lambda^{(k)} Q$, the divergence of the axial vector quark current, we obtain the total divergence of the axial-vector current for our quark-meson model. To effect the cancellation of the interaction contributions and the terms which arise from the free Lagrangian so as to obtain the PCAC condition, we derive a relationship between the effective meson—meson and meson-quark interactions in first order. Indeed, using (8), (19) and (20) we obtain the following identities:

$$K = - i m_Q / g_1, \tag{22}$$

where we have eliminated the pure quark term $m_Q \bar{Q}\gamma_5 \lambda^{(k)} Q$. Cancellation to higher powers in the quark—meson interaction series yields the relation

$$i^{n^2} g_n / m_0^{n-1} = - i^{(n+1)^2} g_{n+1} K / m_0^n \tag{23}$$

or

$$g_{n+1}/g_n = i^{-2(n-1)} m_0 g_1 / m_Q, \tag{24}$$

where $n \geq 1$.

Likewise, we obtain by cancellation of the lowest power meson term in 21) the relation

$$a_3 = - i m_0 / 12 K = m_0 g_1 / 12 m_Q. \tag{25}$$

Cancellation of the higher power terms in the meson—meson interaction series then yields the recursion relation

$$\frac{a_{n+1}}{a_n} = \frac{nm_0}{(n+1)K} i^{-2n+1}, \tag{26}$$

where $n \geq 3$.

If we use (1) and (3) and compute the divergence of the vector current, it is straightforward to show that it vanishes identically for each term in the series (20). Thus, our interaction terms guarantee a conserved vector current.

If we consider the 364-component field ψ in place of the quark in the discussion above then the Lagrangian becomes

$$\begin{aligned} \mathscr{L} = \mathscr{L}(\Psi) + \mathscr{L}(\Phi) + \sum_{n=3}^{\infty} \frac{i^{n^2} a_n}{m_0^{n-4}} Tr\Phi^n + \\ + \sum_{n,m,l=0}^{\infty} \frac{i^{(n^2+m^2+l^2)}}{m_0^{n+m+l-1}} g_{nml} \times Tr\overline{\Psi}(\Phi^{(n)})_1 (\Phi^{(m)})_2 (\Phi^{(l)})_3 \Psi, \end{aligned} \tag{27}$$

where $g_{000} \equiv 0$. The $U(6.6)$-invariant series have been added to produce the desired PCAC form analogous to the meson—quark model. It is straightforward to show that the coupling condition and recursion relations become

$$K = iM/g_{100}, \tag{28}$$

where M is the baryon mass,

$$a_3 = -im_0/12K, \tag{29}$$

and

$$\frac{a_{n+1}}{a_n} = \frac{n}{(n+1)} \frac{m_0}{K} i^{-2n+1} \tag{30}$$

and

$$\frac{g_{(n+1)ml}}{g_{nml}} = \frac{m_0}{K} i^{-2n+1}, \tag{31}$$

or all n, m, l but $n = m = l = 0$.

To make comparison with some of the results of the relativistic SU(6) theories we again emphasize that we are considering our Lagrangian to be the effective model for hadronic processes. Thus we regard the coupling constants to be completely renormalized. Furthermore, the matrix elements used to identify the coupling constants have been derived in first order (free field solutions have been used), only one experimental input needs be employed to calculate the coupling constants for all effective vertices. We shall use the experimental value for the coupling of $\varrho \to 2\pi$, $g^2_{\pi\pi\varrho}/4\pi = 1/2$.

If we consider the trilinear meson vertex $ia_3 m_0 \times Tr(\Phi\Phi\Phi)$ and pick out the matrix element $\vec{\varrho}_\mu \cdot (\pi \times \overleftrightarrow{\partial}_\mu \pi)$ we find that

$$a_3 = (2\sqrt{2}/36)\, g_{\varrho\pi\pi}. \tag{32}$$

We may now compute the meson processes if we use (32) and the recursion relation (30). It should be noted that the relations for the mesons, where we dealt only with quarks rather than the baryons, are the same as those derived here. Thus the various meson processes could just as well have been considered in the simpler quark—meson model.

In particular we shall consider the quadrilinear processes $X_0 \to \eta\pi\pi$ and $\omega \to 3\pi$. The latter will be thought of as a correction to the trilinear description of that width [6] using the GELL-MANN, SHARP and WAGNER (GSW) model [7]. We then use the effective interaction $a_4 Tr\Phi\Phi\Phi\Phi$, (32), and the recursion relation (30) and obtain the identity

$$a_4 = 9\, a_3^2 = g_{\varrho\pi\pi}^2/18\,. \tag{33}$$

If we use this value for a_4 and compute the width for $X_0 \to \eta\pi\pi$, we find $\Gamma(X_0 \to \eta\pi\pi) = 2.0$ MeV.

We now evaluate the width for $\omega \to 3\pi$, where we employ both the quadrilinear and trilinear vertices. We use (33), the GSW model for $\omega \to 3\pi$, and the relation $g_{\omega\varrho\pi}^2/g_{\varrho\pi\pi}^2 = 4/m_\varrho^2$ from relativistic SU(6) to obtain for the combined trilinear and quadrilinear terms the width $\Gamma(\omega \to 3\pi) = 8.9$ MeV compared to the experimental value of 9.4 MeV.

Within our baryon—meson model we may also derive relations between the meson and the various meson—baryon couplings given by (27). Using (28), (29), and (32) we find that

$$G = (2\sqrt{2}\, M/m_0)\, g_{\varrho\pi\pi}\,, \tag{34}$$

where G is defined by the baryon—meson interaction $iG\overline{\Psi}\Phi\Psi$ in lowest order and $3g_{100} \equiv G$. Thus it follows that $G^2/4\pi = 7.9$, where we have used the mass values $M = 1065$ MeV and $m_0 = m_\varrho = 756$ MeV. If we use this value for G and select as an example the coupling $\overline{N}\gamma_\mu\vec{\tau}N\vec{\varrho}_\mu$ from $\overline{\Psi}\Phi\Psi$ we obtain $g_{\overline{N}N\varrho/4\pi}^2 = 0.54$, which is consistent with universality assumptions. We should caution, however, that if one believes universality and then employs relativistic SU(6) to relate this to pseudoscalar coupling, the result is in very poor agreement with the experimental value for $g_{\overline{N}N\pi}$ [8].

It is proposed that the difficulty lies in allowing only the parity doublet 0^-, 1^-. If one includes the less known 0^+ and 1^+ particles then it is hoped that the known effective coupling will be improved. It is interesting to note that, in the formulation of this model, the 0^+, 1^+ pair are introduced by means of the 144-component field Φ' which satisfies the equation

$$\{\partial_\mu \Phi', \gamma_\mu\} + 2m_0' \Phi' = 0 \tag{35}$$

and which is derivable from the Lagrangian

$$\mathscr{L}(\Phi') = Tr\gamma_5 \Phi' \gamma_5(\gamma \cdot \partial + m_0') \Phi' . \tag{36}$$

Thus the mass term is invariant only under SL(6, C) and not U(6.6), as was the mass term for the 0^-, 1^-. The interacting series then required by our philosophy of PCAC will involve terms which are invariant only under SL(6, C). If the gauge transformation then also allows mixtures between 0^+, 1^+ and 0^-, 1^- the effective interacting Lagrangian will no longer be invariant under U(6.6) but only under the smaller group SL(6, C).

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НЕЛИНЕЙНЫЙ ЛАГРАНЖИАН И РЕЛЯТИВИСТИЧЕСКАЯ SU(6)

Д. УЭЛИНГ

Резюме

Разрабатывается метод для составления феноменологического нелинейного Лагранжиана адронов в рамках релявистических соображений. Вводятся особо SL (6, C) — инвентарные серии мезон-мезонных и мезон-барионных взаимодействий и показывается, что в первом приближении соотношение между различными связями возможно в случае сохранения PCAC. Используя константу связи как вводную величину для $\varrho\pi\pi$ получены следующие результаты: $\Gamma(X^0 \to \eta\pi\pi) = 2$ Мэв, $\Gamma(\omega \to 3\pi) = 8{,}9$ Мэв, $g^2_{\bar{N}N\varrho/4\pi} = 0{,}54$.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 147—152 (1969)

CURRENT DIVERGENCES IN THE $SU_3 \times SU_3$ ALGEBRA

By

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The scheme of the charge-current equal time commutators for an octet of vector currents and axial-vector currents implying an $SU_3 \times SU_3$ algebra generated by the charges, is extended to commutators of charges with current divergences.

In 1962, M. GELL-MANN [1] proposed the following equal time commutators for an octet of vector currents $V_\mu^i(x)$ and axial-vector currents $A_\mu^i(x)$

$$\begin{aligned}
[\int V_0^i(x)\,(dx)^3, \quad V_\mu^j(y)] &= if^{ijk}\,V_\mu^k(y)\,,\\
[\int V_0^i(x)\,(dx)^3, \quad A_\mu^j(y)] &= if^{ijk}\,A_\mu^k(y)\,,\\
[\int A_0^i(x)\,(dx)^3, \quad V_\mu^j(y)] &= if^{ijk}\,A_\mu^k(y)\,,\\
[\int A_0^i(x)\,(dx)^3, \quad A_\mu^j(y)] &= if^{ijk}\,V_\mu^k(y)\,,
\end{aligned} \tag{1}$$

implying an $SU_3 \times SU_3$ algebra generated by the charges. An extension of this scheme to commutators of charges with current divergences will be the subject of this contribution. Its essential ideas are to some extent already contained in M. GELL-MANN's pioneering paper [1], the details have been worked out recently by GELL-MANN, OAKES and the author [2]. We acknowledge that commutators of charges and current divergences have been used by several authors [3] before our recent work, yet we find that a systematic approach has led to results which to the best of our knowledge have not yet been reported elsewhere.

The algebraic behaviour of current divergences is closely connected with the properties of the energy operator H under the $SU_3 \times SU_3$ algebra, through Heisenberg's equation of motion

$$\int \partial^\mu j_\mu(x)\,(dx)^3 = \frac{d}{dt}\int j_0(x)\,(dx)^3 = i\,[H, \int j_0(x)\,(dx)^3]\,. \tag{2}$$

Apart from operators which vanish when being integrated over space, the current divergences belong to the same $SU_3 \times SU_3$ multiplets as the symmetry breaking part of the energy density $\vartheta_{00}(x)$. For simplicity we assume that only a scalar part $g_{00}\,U(x)$ in $\vartheta_{00}(x)$ breaks the symmetry and the proper

tensor part (like the kinetic term in the quark model) commutes with the charges [4]. This assumption allows us to propose

$$\partial^\mu j_\mu(x) = i\,[U(x)\,,\ \int j_0(y)\,(dy)^3]\,. \tag{3}$$

The operator $U(x)$ will in gerenal consist of an SU_3 invariant part $U_0(x)$ and a term responsible for SU_3 breaking. In accordance with the success of the Gell-Mann—Okubo mass formula, we confine ourselves to an octet term $U_8(x)$. Neither of these two terms can be expected to commute with the axial charges.

To specify the commutators of $U_i(x)$ with the axial charges we speculate on a principle of minimality: we look for a multiplet of operators under $SU_3\times$ $\times SU_3$ which contains all the observable current divergences and the smallest possible number of other less readily observable operators with them. In particular we assume that no operators of isospin or hypercharge two occur; this implies the vanishing of certain commutators, for example

$$\begin{aligned}
&\left[\int \left(A_0^1(x) + iA_0^2(x)\right)(dx)^3,\quad \partial^\mu A_\mu^1(y) + i\partial^\mu A_\mu^2(y)\right] = 0\,,\\
&\left[\int \left(A_0^4(x) + iA_0^5(x)\right)(dx)^3,\quad \partial^\mu A_\mu^4(y) + i\partial^\mu A_\mu^5(y)\right] = 0\,,
\end{aligned} \tag{4}$$

which have been tested in generalized Adler-Weisberger relations and have been found consistent with zero.

This restriction on the algebraic behaviour of the current divergences leaves only two types of $SU_3\times SU_3$ multiplets for $U(x)$ to have components in, these are (8.1) + (1.8) and (3.3*) + (3*.3). Only the latter one can accommodate both an SU_3 singlet and an SU_3 octet operator, it is the minimal multiplet we wanted to find.

The representation (3.3*) + (3*.3) contains nine scalar and nine pseudoscalar operators, an SU_3 octet and an SU_3 singlet of either parity, to be denoted by $u^i(x)$, $v^i(x)$, $i = 0 \ldots 8$. The commutators with the generators of the algebra are given by

$$\begin{aligned}
&[\int V_0^i(x)\,(dx)^3,\quad u^j(y)] = if^{ijk}u^k(y)\,,\\
&[\int V_0^i(x)\,(dx)^3,\quad v^j(y)] = if^{ijk}v^k(y)\,,\\
&[\int A_0^i(x)\,(dx)^3,\quad u^j(y)] = -\,id^{ijk}v^k(y)\,,\\
&[\int A_0^i(x)\,(dx)^3,\quad v^j(y)] = id^{ijk}u^k(y)\,.
\end{aligned} \tag{5}$$

In terms of these operators the symmetry-breaking part of the energy density is assumed to be

$$U(x) = u_0(x) + cu_8(x). \tag{6}$$

The expressions for the current divergences follow from Eqs. (5) and (6)

$$\partial^\mu V_\mu^i = cf^{i8k}u^k, \tag{7}$$

$$\partial^\mu A^i_\mu = -\, d^{i0k} v^k - c d^{i8k} v^k = -\, W^i(c)\, \delta^{ik} v^k - \sqrt{\frac{2}{3}}\, c \delta^{i8} v^0 \tag{8}$$

with $i = 1 \ldots 8$ and

$$\begin{aligned} W^i(c) &= (\sqrt{2} + c)/\sqrt{3} \quad \text{for} \quad i = 1, 2, 3\,, \\ &= (\sqrt{2} - c/2)/\sqrt{3} \quad \text{for} \quad i = 4, 5, 6, 7\,, \\ &= (\sqrt{2} - c)/\sqrt{3} \quad \text{for} \quad i = 8 \end{aligned} \tag{9}$$

commutators of charges with current divergences can be read off by combining Eqs. (5)—(9).

The particular characteristics of this simplest proposal are the strong relations between violations of chiral symmetry and of SU_3 symmetry, specified in terms of a single and universal constant $c \cdot c = 0$ would correspond to SU_3 symmetry, $c = -\sqrt{2}$ to $SU_2 \times SU_2$ symmetry (see Eq. (9)). The main part of this contribution will be devoted to an estimate of c.

We would expect c to come out small as compared to unity, if a chiral symmetry limit would imply zero baryon (or quark) masses; if on the other hand a chiral symmetry limit with zero pseudoscalar meson masses is favoured by the data, we would expect c in the neighbourhood of $(-\sqrt{2})$. If determinations from different data give different values for c, then Eq. (6) would be wrong and our search for a minimal multiplet would be an over-simplification.

Estimates for c are not easily available, because no approximate $SU_3 \times \times SU_3$ multiplets of states have yet been identified, and the usual application of the Wigner-Eckart theorem is not possible. In particular we have no general method of separating $U(x)$ from the $SU_3 \times SU_3$ invariant part in $\vartheta_{00}(x)$ and of obtaining independently matrix elements of $c u_8(x)$ and $u_8(x)$.

In principle an estimation of the σ-terms in low energy scattering of pseudoscalar mesons would help here. Combining Eqs. (5)—(9) with the usual current algebra treatment of low energy scattering of π and K mesons, we obtain

$$\langle \pi p \,|\, T \,|\, \pi p \rangle\, |_{q_\pi = 0} = -\frac{1}{3} (\sqrt{2} + c) \frac{1}{F_\pi^2} \langle p \,|\, \sqrt{2}\, u_0 + u_8 \,|\, p \rangle\,, \tag{10}$$

$$\langle K^+ p \,|\, T \,|\, K^+ p \rangle\, |_{q_K = 0} = -\frac{1}{3} \left(\sqrt{2} - \frac{c}{2} \right) \frac{1}{F_K^2} \left\langle p \,\middle|\, \sqrt{2}\, u_0 + \frac{\sqrt{3}}{2} u_3 - \frac{1}{2} u_8 \,\middle|\, p \right\rangle. \tag{11}$$

Since it is not known a priori how much of the nucleon mass should be attributed to chiral symmetry violation, we have to leave $< p \,|u_0|\, p >$ as an undetermined parameter, and to determine c, we would need low energy estimates of at least two scattering processes. It appears questionable, whether this is possible with the present data [5]. (See Note 1 added in proof.)

In the pseudoscalar system additional information is provided through the application of meson pole dominance (PCAC). PCAC for π-mesons is hardly disputed any more; for K-mesons its status is in doubt, but it may well be valid [6], and for η-mesons it should be of similar accuracy as in the K-meson case, since (η η') mixing is small and may be neglected in our accuracy. Assuming PCAC for πK and η mesons (see Note 2 added in proof) we consider the low energy limit of the following vertex of pseudoscalar mesons.

$$\lim_{p\to 0}\langle P_i(p)\,|\,u_0+cu_8\,|\,P_i(p')\rangle=\frac{(-i)}{F_i}\langle 0\,|\,[\int A_0^i(x)\,(dx)^3,\,u_0+cu_8]\,|\,P_i(p')\rangle=$$

$$=\frac{1}{F_i}\langle 0\,|\,\partial^\mu A_\mu^i\,|\,P_i(p')\rangle=m_i^2\,. \tag{12}$$

We see that a chiral symmetry limit ($u_0+cu_8)\to 0$ would imply the vanishing of the pseudoscalar meson masses. There is no need left for parity-doubling of single-particle states; degenerate states of both parities are created by the addition of massless Goldstone-type bosons. This explains why we may be near to a world with chiral symmetry without being able to classify singe-particle like states in approximate $SU_3\times SU_3$ multiplets. This result coincides with the view of chiral symmetry taken in the contemporary Lagrangian models, and it is probably inherent in the PCAC-approximation.

Now we proceed to an estimate of c. We start with the GELL-MANN—OKUBO mass formula

$$\langle P_i(p)\,|\,u_0+cu_8\,|\,P_i(p)\rangle=m_i^2\approx\bar{m}^2+d^{ii8}\Delta m^2 \tag{13}$$

generalized to

$$\langle P_i(p)\,|\,u_j\,|\,P_k(p')\rangle\approx\alpha(t)\,\delta_{j0}\,\delta_{ik}+\beta(t)\,d_{ijk}\,;\begin{pmatrix}i,k=1\ldots 8\\ j=0\ldots 8\end{pmatrix}, \tag{14}$$

$$\alpha(0)=\bar{m}^2-\sqrt{\frac{2}{3}}\,\Delta m^2/c\,;\quad \beta(0)=\Delta m^2/c\,. \tag{15}$$

Assuming that possible scalar mesons have a substantially larger mass than the pseudoscalar mesons, we neglect the dependence of α and β on t and we examine the consistency of Eq. (14) with its low energy values at $p\to 0$ and at $p'\to 0$

$$\alpha\delta_{j0}\,\delta_{ik}+\beta d_{ijk}\approx\frac{d_{ijl}}{F_i}\langle 0\,|\,v_l\,|\,P_k\rangle\approx\frac{d_{ljk}}{F_k}\langle P_i\,|\,v_l\,|\,0\rangle\,. \tag{16}$$

Using all possible values for (i, j, k) we find consistency in Eq. (16) only if

$$F_\pi \approx F_k \approx F_\eta, \tag{17}$$

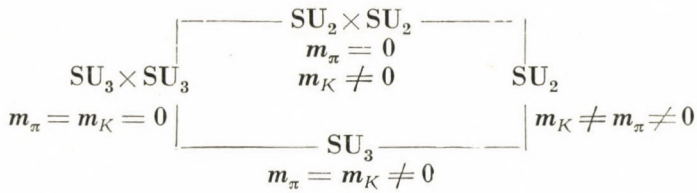
$$\langle 0 | v_i | P_i \rangle \approx F \cdot \beta, \qquad \text{independent of } i \tag{18}$$

$$\langle 0 | v_0 | \eta \rangle \approx 0, \tag{19}$$

$$\alpha \approx 0; \quad \overline{m}^2 \approx \sqrt{\frac{2}{3}}\, \Delta m^2/c; \quad c \approx -1.25. \tag{20}$$

Equation (17) gives the approximate equality of the $K_{\mu 2}$ and $\pi_{\mu 2}$ decay constants, which is observed experimentally. (Possible discrepancies in the order of 25% may be attributed to the neglect of scalar mesons in the t-dependence of α and β). Eq. (18), when combined with Eqs. (7), (8), (9) and (17) reproduces the GELL-MANN—OKUBO mass formula for the squares of the pseudoscalar meson masses (Eq. (13)). Eq. (18) confirms our neglect of $(\eta\, \eta')$ mixing. If there were substantial mixing we would have to abandon the octet mass formula and η-PCAC.

Our most important result is Eq. (20) with the estimate of $c \approx -1.25$. Its closeness to the $SU_2 \times SU_2$ symmetric value $c = -\sqrt{2}$ confirms in our language the successes of chirally symmetric Lagrangians. One may summarize our view of the situation by considering two chains of symmetry breaking



It depends entirely on the process under consideration which chain of symmetry breaking leads to a better understanding of the data. For the classification of states the SU_3 chain is obviously more useful, for pion reactions it is the $SU_2 \times SU_2$ chain. The vacuum is clearly not invariant under chiral transformations, yet it is probably not far from SU_3 symmetric. A low energy limit to Eq. (18) shows that

$$\langle 0 | u_0 | 0 \rangle \approx -\frac{3}{2} F^2 \overline{m}^2, \quad \langle 0 | u_8 | 0 \rangle \approx 0.$$

The estimate of c, as given here, is clearly not sufficient to establish its universality. Better data or new ideas will be needed to make independent estimates outside the pseudoscalar meson system. It should not be an accident that SU_3 mass-splittings within multiplets are always of the same order as pseudoscalar meson masses.

Apart from these problems of principle, we hope that our specification of new commutators will allow new applications [7] of current algebras, although presently only on a tentative level.

Note added in proof. 1. F. VON HIPPEL and J. K. KIM have recently confirmed the predictions of this theory for low-energy meson baryon scattering, as given in Eqs. (10) and (11). F. VON HIPPEL and J. K. KIM: A Systematic Test of the Soft Meson Theorems and of a Theory of $SU_3 \times SU_3$ Breaking. (Stanford preprint)

2. J. ELLIS has recently shown that the constant c can be determined through an argument similar to the one given here, without assuming K or PCAC, J. ELLIS: Current Divergence Commutators and Scalar Mesons (Cambridge University preprint.)

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РАСХОДИМОСТИ ТОКОВ В АЛГЕБРЕ $SU_3 \times SU_3$

Б. РЕННЭР

Резюме

Схема коммутаторов зарядных токов при равных временах для октета векторных токов и аксиально векторных токов, включающих в себе алгебру $SU_3 \times SU_3$ обусловленную зарядами, распространяется на коммутаторы зарядов с расходимостью токов.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 153—158 (1969)

ELECTROMAGNETIC MASS DIFFERENCES, EQUAL-TIME COMMUTATORS, AND OSCILLATING SPECTRAL FUNCTIONS*

By

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A connection is suggested between the ambiguities in the definition of the equal-time commutator $[J_\mu^{em}(0, \vec{x}), J_\nu^{em}(0)]$ and the experimental indications of an exponentially falling electromagnetic form factor for the nucleon. On this basis, a resolution is proposed of the difficulty of current algebra models which appear to give divergent expressions for the electromagnetic mass shifts of hadrons.

I. Introduction

I realize that, like Dr. NAUENBERG yesterday, I am handicapped by the fact that lunch time is almost upon us. In addition, Dr. RENNER has, in his talk, enunciated the principle that wrong equations should not be written and/or left on the blackboard. Nevertheless, to save time I have already written a number of equations on the board, to which I will refer as needed. I will erase them at the end to avoid possible conflict with RENNER's principle.

Let me begin by reminding you of a disturbing feature of the current-algebra approach, pointed out last year by a number of authors [1, 2], following the successful calculation [3] of the $\pi^+ - \pi^0$ mass difference for zero-mass pions: Use of techniques introduced by BJORKEN [4] leads to the conclusion that the familiar models of current algebra appear to give divergent answers for the electromagnetic mass-shifts of massive hadrons. The work I would like to discuss today concerns an attempt, made in collaboration with RICHARD BRANDT [5, 6], to avoid this conclusion.

Let us recall the argument that, say, $\delta m^2 = m_{\pi^+}^2 - m_{\pi^0}^2$ is divergent. Let $M_{\mu\nu}^{(\delta)}(q\,;\,p)$ denote the forward amplitude for a virtual photon of momentum q to scatter from a pion of momentum p and charge δ. Then

$$\delta m^2 = \text{const.} \int d^4q\, q^{-2}\, F(-q^2, \nu) \tag{1}$$

with $\nu \equiv q \cdot p$ and

$$F(-q^2, \nu) \equiv M_{\mu\nu}^{(+)}(q; p) - M_{\mu\nu}^{(0)}(q; p). \tag{2}$$

* Supported in part by the U. S. Air Force Office of Scientific Research under Grant AFOSR 68—1453.

** John Simon Guggenheim Fellow.

Now

$$M_{\mu\nu}^{(\delta)}(q;p) = T_{\mu\nu}^{(\delta)}(q;p) + S_{\mu\nu}^{(\delta)}(q;p), \tag{3}$$

where

$$T_{\mu\nu}^{(\delta)}(q;p) = -i\int d^4x e^{-iq\cdot x}\langle p,\delta\,|\,TJ_\mu(x)\,J_\nu(0)\,|\,p,\delta\rangle$$

and $S_{\mu\nu}^{(\delta)}$ is the "Schwinger term", defined as the difference, if any, between M and T. Following BJORKEN [4], one assumes that T may be written in the unsubtracted form (we drop the charge index)

$$T_{\mu\nu}(q;p) = \int_0^\infty \frac{dq_0'}{2\pi}\left[\frac{\varrho_{\mu\nu}(q_0',\mathbf{q};p)}{q_0 - q_0'} - \frac{\varrho_{\mu\nu}^-(q_0',-\mathbf{q};p)}{q_0 + q_0'}\right]$$

and further expanded in the form,

$$T_{\mu\nu}(q;p) = \langle p|\int d\mathbf{x}e^{-i\mathbf{q}\cdot\mathbf{x}}\{q_0^{-1}\left[J_\mu(0,\mathbf{x}),J_\nu(0)\right] + iq_0^{-2}\left[\dot{J}_\mu(0,\mathbf{x}),J_\nu(0)\right]\}|\,p\rangle + \mathcal{O}(q_0^{-3}), \tag{4}$$

obtained by expansion of the denominators $(q_0 \pm q_0')$. The first term does not contribute in the usual models but the second term does not vanish and so appears to give rise to a logarithmic divergence in δm^2.

More explicitly, we have from (2), (3), and (4), ignoring Schwinger terms, for $|q_0| \to \infty$

$$F(-q^2,\nu) \sim I_1/\pi q_0^2 + \mathcal{O}(q_0^{-3}), \tag{5}$$

where

$$I_1 = \frac{1}{2}\int dq_0'\,(\Delta\varrho_{\mu\mu} + \Delta\bar{\varrho}_{\mu\mu}) = \pi\int dx e^{-i\mathbf{q}\cdot\mathbf{x}}E(0,\mathbf{x};p) \tag{6}$$

with $\Delta\varrho = \varrho^{(+)} - \varrho^{(0)}$, and

$$E(x;p) \equiv \langle p,+|\left[\dot{J}_\mu(x),J_\mu(0)\right]|p,+\rangle - \langle p,0|\ldots|p,0\rangle. \tag{7}$$

Since, in the usual models, $E(o,\mathbf{x};p)$ does not vanish (unless one is in a world with $m_\pi = o$) neither does I_1 and using (1) and (5) we see the origin of the logarithmic divergence of δm^2. Inclusion of the Schwinger term, a polynominal in q_0, cannot help matters [6].

II. Evasion

The basic idea for rescuing current algebra from this difficulty is the recognition that the expansion (4) and hence the above argument breaks down *if the* ETC *matrix element* $E(0,\mathbf{x};p)$ *is not well-defined*, i.e., if lim $E(x;p)$

as $x^0 \to 0$ *is ambiguous, and that this corresponds to the non-existence of the moment of the spectral function*, in a sense to be made precise below.

For simplicity, let us work at $p = 0$ and ignore Schwinger terms. Then

$$\delta m^2 \propto \int \frac{d^4q}{q^2} F(-q^2) \propto \int_{-\infty}^{0} dz\, F(z)$$

on rotation of the q_0 — contour (with $z = -q^2$), where

$$F(z) = F(-q^2, 0) = \frac{1}{\pi} \int_{a_0}^{\infty} da \frac{\sigma(a)}{a - z} \tag{8}$$

with $\sigma(a) = \sigma(q_0^2 - \mathbf{q}^2) \propto \Delta\varrho(q_0, \pm \mathbf{q})$. We will refer to (8) as a U.S.S.R. (unsubtracted spectral representation) for $F(z)$. We now ask: What are necessary conditions on $\sigma(a)$ for δm^2 to be finite? Let us define a moment function

$$I(x) = \int_{a_0}^{x} da\, \sigma(a)$$

and let $I = \lim_{x \to \infty} I(x)$. Then we have the

Theorem [6] If $F(z)$ satisfies a U.S.S.R. and δm^2 is finite, then $I(x)$ cannot be asymptotically bounded away from zero. In particular, (a) I cannot be $\pm \infty$ and if I exists it must be zero, (b) if I *does not exist* $\sigma(x)$ *must oscillate at infinity*. More generally, if $F(z)$ admits a S.R. with one or more *needed* subtractions, [and perhaps an additive polynomial] then $\sigma(x)$ *must* oscillate at infinity if δm^2 is finite.

Now the usual models do not give $E = 0$ and hence imply $I \neq 0$. However, the theorem suggests that we explore the *remaining possibility* that I *does not exist* (not $I = \pm\infty$). What does this mean in terms of the commutator? One can infer from the preceding equations that

$$E(x) \equiv E(x; 0) = \int_{a_0}^{\infty} da\, \sigma(a)\, \Delta(x, a).$$

Now, the orthodox definition of the equal time limit is

$$E(0, \mathbf{x}) = \lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(t)\, E(x)\, dt,$$

where $[f_n(t)]$ is a sequence of testing functions with $f_n(t) \to \delta(t)$ as $n \to \infty$. If $\sigma(a)$ does not grow too rapidly (the general analysis has been given by BRANDT [7]) one finds

$$E(0, \mathbf{x}) = -\left(\lim_{n \to \infty} K_n\right) \delta(\mathbf{x}),$$

where

$$K_n = \int da\, \sigma(a)\, \tilde{f}_n(\sqrt{a})$$

with $\tilde{f}_n(\varkappa)$ (the Fourier-transform of $f_n(t)$) $\to 1$ as $n \to \infty$. Thus, the K_n are simply *regularizations* of the moment I and I is well-defined (regularization independent) if and only if the ET limit is unique (sequence-independent). Clearly, $\lim_{n\to\infty} K_n$ can be sequence dependent only if $\sigma(a)$ oscillates at infinity and the connection between oscillations and ambiguous ETC's is established.

As an example, consider

$$F(z) = iz^{-1/2}\, e^{iz^{1/2}} = \pi^{-1} \int_0^\infty da\, \frac{a^{-1/2}\, \cos\, a^{1/2}}{a - z}\,,$$

so that

$$I = \int_0^\infty da\, a^{-1/2}\, \cos\, a^{1/2} = 2 \int_0^\infty d\varkappa\, \cos\varkappa,$$

which does not exist. It can be shown [6] that for a suitable choice of regulator functions $\tilde{f}_n(\varkappa\,;\, c)$ one can have, for any real number c,

$$\lim_{n\to\infty} K_n = c\,.$$

Of course for no value of c is it true that $F(z) \sim c/z$ for $|z| \to \infty$. Many other examples can be constructed, illustrating the fact that $\sigma(a)$ must oscillate at infinity if $\delta m^2 < \infty$, unless $I = 0$; the non-existence of I is, as we see, directly related to the breakdown of the BJORKEN expansion of $F(z)$ in inverse powers of z, which is analogous to the expansion of T in inverse powers of q_0.

III. Plausibility

Oscillating spectral functions may seem to be strange beasts at first sight, but their existence may be made plausible in a number of ways. Indeed, for the spectral function $\varrho(q^2)$ associated with the electromagnetic form factor $G(q^2)$ of a hadron, i.e., with the amplitude for *one* photon emission, there is experimental evidence which is very suggestive of oscillations in $\varrho(q^2)$. Thus, a very good fit to the data for large momentum transfer $e - p$ scattering [8] is obtained with a form factor $G(q^2) \sim e^{-b(q^2)^{1/4}}$, for which the corresponding ϱ oscillates. (More generally, if $G(q^2)$ falls off more rapidly than any inverse power of q^2, the cut-plane analyticity of G *requires* that ϱ oscillate, at least if G is polynomially bounded in the complex q^2-plane.) In the case of the proton form factor there is also a good dipole fit, $G \sim (1/q^2)^2$. However, it is interesting to note that recent experiments [9] in electro-pion production are consistent with an $N^*N\gamma$ form factor which falls exponentially and here there is *no* satisfactory dipole fit.

Since, apart from seagull terms, the two-photon amplitude $M_{\mu\nu}(q, p)$ can be expressed as a sum of integrals involving a product of (off-shell) one-photon amplitudes, $\Gamma_\mu \Gamma_\nu$, and a kernel involving strong interactions only, it can be argued [5] that if the Γ_μ fall off exponentially as $q^2 \to \infty$, so will $M_{\mu\nu}(q, p)$ (In any case, it is sufficient that $M_{\mu\nu}$ have an exponential piece, e.g. $F \sim a/q^4 + b\, e^{-\sqrt{q^2}}$ has a spectral function which oscillates at infinity.)

On the theoretical side, support for the idea of exponentially-falling form factors and hence for oscillating spectral functions comes from a number of places, including work on high-energy, large-angle $p - p$ scattering [10], the concept of minimal interaction [11], strictly local field theory [12], and work on composite models of the nucleon in a bootstrap approach [13]. Also, it should be noted that the Regge asymptotic behaviour for hadronic scattering amplitudes itself leads to spectral functions which are likely to oscillate. However, there is no time to enter into detail on these matters.

IV. Concluding comments

In conclusion, let me add two comments. The first, in the nature of a side remark, is that the vanishing of the moment is *not* a sufficient condition for the finiteness of the mass-shift. A counter example is provided by defining $\sigma(x) = x - a$, for $0 < x < b$, and $\sigma(x) = \sigma(b)\,(b/x)\,(\ln b/\ln x)^2$ with $a = b(1 + 2\ln b)/(2 + 2\ln b)$. Then, for $x \geq b$, $I(x) = \text{const.}/\ln x$, so that $I(\infty) = 0$. However, by reversing some orders of integration one can show that

$$\delta m^2 \sim \int_0^\infty \frac{I(x)}{x}\, dx,$$

so that in this case δm^2 diverges like $\ln \ln \Lambda$. Thus, even when $[\dot{J}_\mu, J_\mu]$ is a c-number, as is the case in recent theory of T. D. LEE [14], one must check explicitly that δm^2 is finite.

The second more important comment, relates to the question of whether the present ideas can be extended to the case of the divergent radiative correction to weak decays. CLAUDIO ORZALESI and I looked into this possibility during the summer and came to the conclusion that in order to create a breakdown of the BJORKEN method in this problem, and to have finite radiative corrections to the matrix element of the vector current alone (without relying on cancellations from the axial-vector matrix element) it seems to be necessary that ETC's of the type

$$C_{0i}^{ab} = \left[J_0^a(\mathbf{x}, t), J_i^b(\mathbf{x}', t)\right]$$

be ambiguous. One might have hoped that, for example, ambiguities in space—space commutators would suffice, or that other ambiguities connected with

the definition of the T-product might help. Ambiguities in C_{0i}^{ab} are not as palatable from the viewpoint of other applications of current algebra as ambiguities in C_{ij}^{ab} would be. If one thinks that the ("observed") finiteness of radiative corrections to both masses and weak-decay amplitudes should have a common explanation within a current-algebra framework [15], one's enthusiasm, if any, for the possibilities outlined, may be somewhat reduced. In any case, the proposal that the ETC $[\dot{J}_\mu, J_\mu]$ is ambiguous seems to be consistent and the possible connection with the experimental evidence on the rapidly falling form factors seems very suggestive. Even if these ideas do not survive, at the minimum they serve to indicate how on occasion relatively sophisticated aspects of quantum field theory may make contact with physics.

There may be still more things twixt quantum field theory and experiment than are normally dreamt of in our philosophy.

Note added in proof: Recently several authors [H. EPSTEIN and R. JACKIW (unpublished); D. G. BOULWARE and S. DESER, Phys. Rev., **172,** 1912, 1968 (note added in proof)] have pointed out that, for spectral functions of the type we considered, $K_n \to 0$ for $n \to \infty$ for large classes of $f_n(t)$. Our essential point, however, is that as long as there exists *one* sequence for which $K_n \to c \neq 0$, the commutator *is* ambiguous in the sense we have defined. If, furthermore, this sequence is the one "chosen by nature", then the commutator will effectively be non-vanishing for low energy considerations.

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15. Such a point of view is of course debatable, in view of our relatively incomplete understanding of the weak interactions.

РАЗНОСТИ ЭЛЕКТРОМАГНИТНЫХ МАСС, КОММУТАТОРЫ ПРИ РАВНЫХ ВРЕМЕНАХ И ОСЦИЛЛИРУЮЩИЕ СПЕКТРАЛЬНЫЕ ФУНКЦИИ

Д. СУЧЕР

Резюме

Рассматривается соотношение между двусмысленностью в определении коммутатора при равных временах $[\dot{J}_\mu^{em}(0, \vec{x}), J_\nu^{em}(0)]$ и экспериментальными индикациями экспоненциально падающего электромагнитного форм-фактора для нуклона. На базе этого предлагается разрешение трудности модели алгебры токов, результирующей расходимые выражения для сдвигов электромагнитных масс гадронов.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 159—168 (1969)

SUM RULES AND COVARIANCE

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A representation of commutator matrix elements is derived in Lehmann—Symanzik—Zimmermann formalism and applied to current density algebra sum rules in an arbitrary frame of reference. The sum rules take a remarkably simple form if the current correlation functions obey multiple integral representations in the current masses. The reference frame dependence of the sum rules becomes explicit.

I. Introduction

In a previous paper [1] we investigated the sum rules derived from the current density commutators [2, 3] in different infinite momentum limits and showed that the sum rules collapse into a single sum rule if the current correlation functions are given by multiple integral representations. The purpose of the present paper is to investigate in more detail the dependence of sum rules on the reference frame. As a result we get the explicit frame dependence of equal-time commutators consistent with multiple integral representations.

It was pointed out in a paper by AMATI, JENGO, and REMIDDI [4] that disconnected intermediate states play an essential role in sum rules. They showed that the singularities in current masses automatically cancel if one takes into account disconnected contributions. Assuming multiple i tegral representations this means that the double (in the case of four-point functions also the triple) spectral functions cancel and the equal-time commutator is given entirely by the single spectral functions. In other words, current algebra gives constraints for the subtraction constants.

In Section II we derive a general representation of commutator matrix elements, taking into account the splitting of intermediate state contributions into connected and disconnected parts. Using this representation we investigate the covariance properties of retarded product and equal-time commutator of currents in Section III. The assumption of multiple integral representations is introduced in Section IV, where the sum rules are expressed in terms of the spectral functions. Section V contains some concluding remarks.

II. A representation of commutator matrix elements

In [1] a representation of commutator matrix elements proved very useful in dealing with the frame dependence of sum rules. Here we give a proof of this using the Lehmann—Symanzik—Zimmermann reduction technique and the TCP theorem. Let us consider, for simplicity, the commutator of scalar currents $A(x)$, $B(x)$ taken between scalar states of momentum p_1, p_2:

$$\tau(q^2, k^2, S^2, U^2, t) = \frac{1}{2}\int d^4x e^{ikx} < p^2 |[A(x), B(0)]| p_1 > . \tag{1}$$

The extension of the following proof to particles and currents with spin is immediate. One has only to project out the appropriate Lorentz-invariant functions. The kinematic variables we use are the following:

$$\begin{aligned} p_1 + q &= p_2 + k, & p_1^2 &= p_2^2 = m^2, \\ S &= k + p_2, & t &= (p_1 - p_2)^2, \\ U &= k - p_1, & S^2 + U^2 + t &= 2m^2 + k^2 + q^2. \end{aligned} \tag{2}$$

First, let us insert a complete set of intermediate states between the two currents in Eq. (1). For later purposes it will be convenient to choose a complete set of in-states (out-states) in the first (second) term of the commutator:

$$\begin{aligned} \tau = \frac{1}{2}\int d^4x e^{ikx} \sum_n \{ &< p_2 |A(x)| n >_{\text{in in}} < n |B(0)| p_1 > - \\ &- < p_2 |B(0)| n >_{\text{out out}} < n |A(x)| p_1 > \}. \end{aligned} \tag{3}$$

The contribution of intermediate states can be split up into connected and disconnected parts. If $a^+_{\text{in out}}(p)$ denotes the creation operator of an in (out) state with momentum p, then we have for every local operator $A(y)$

$$\begin{aligned} A(y)a^+_{\text{in}}(p) &= \frac{i}{\sqrt{2(2\pi)^3}}\int d^4x e^{-ipx}(\Box_x + m^2)\,\theta(y_0 - x_0)\,[A(y), \varphi(x)] + \\ &\quad + a^+_{\text{in}}(p)\,A(y), \\ A(y)a^+_{\text{out}}(p) &= \frac{i}{\sqrt{2(2\pi)^3}}\int d^4x e^{-ipx}(\Box_x + m^2)\,\theta(x_0 - y_0)\,[\varphi(x), A(y)] + \\ &\quad + a^+_{\text{out}}(p)\,A(y). \end{aligned} \tag{4}$$

(Here $\varphi(x)$ is the interpolating field to $a^+_{\text{in out}}(p)$. The first term on the right hand side gives the "connected contribution" while in the second the particle $a^+(p)$ "goes through" and gives rise to the "disconnected parts". Having

in mind the stability of one particle states $|p\rangle_{\text{in}} = |p\rangle_{\text{out}}$, and using Eq. (4) we get 8 terms in the commutator (3), which can be visualized by the graphs in Fig. 1. The graphs A, B, C, D ($\overline{\text{A}}$, $\overline{\text{B}}$, $\overline{\text{C}}$, $\overline{\text{D}}$) come from the first (second) term of the commutator. The dashed lines cross the intermediate states. Thus, e.g. in A. the intermediate state consists of n and in both matrix elements the connected part is taken. In graph B. the intermediate state contains $n +$ the particle $|p_1\rangle$, and $|p_1\rangle$ goes through in the matrix element $\langle n + p_1 | B(0) | p_1\rangle$ but we have to take the connected part of $\langle p_2|A(x) | n + p_1\rangle$, etc

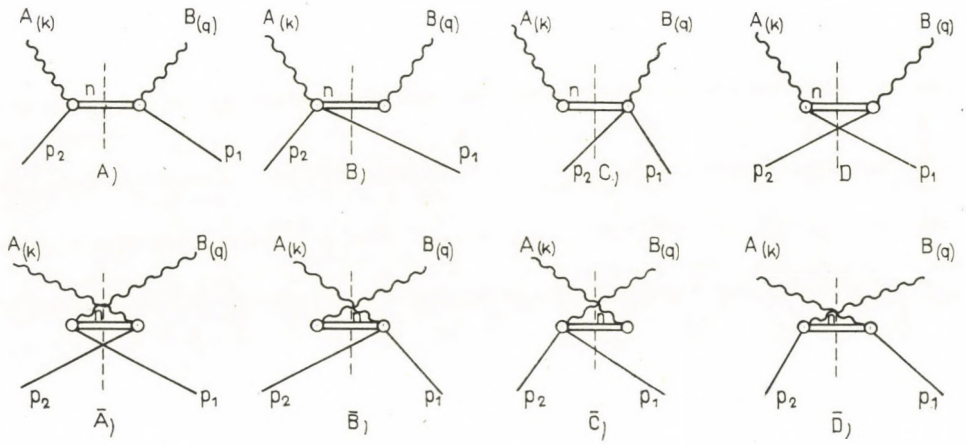

Fig. 1

It can be shown that the contributions of the graphs A and $\overline{\text{A}}$, B and $\overline{\text{B}}$, C and $\overline{\text{C}}$, D and $\overline{\text{D}}$ summed over n are very similar to each other: apart from an overall minus sign (due to the negative sign of the second term in the commutator) the only difference is that every momentum has its sign reversed in the corresponding pairs. Let us show this on the example of graphs A and $\overline{\text{A}}$. Considering the states $|p_{1,2}\rangle$ as in-states, reducing them out and summing over n, the contribution of the graphs A to τ is the following:

$$\frac{1}{2}\int d^4x\, e^{ikx} \sum_n \langle 0 | a_{\text{in}}(p_2)\, A(x) | x\rangle^{(c)}_{\text{in}}\ {}^{(c)}_{ix}\langle n | B(0)\, a^{+}_{\text{in}}(p_1) | 0\rangle =$$

$$= -\frac{1}{4(2\pi)^3}\int d^4x_1\, d^4x_2\, d^4x\, e^{ikx - ip_1x_1 + ip_2x_2}\,(\Box_{x_1} + m^2)(\Box_{x_2} + m^2) \qquad (5)$$

$$\theta(-x_{10})\,\theta(x_0 - x_{20})\,\langle 0 | \big[A(x), \varphi(x_2)\big]\big[B(0), \varphi(x_1)\big] | 0\rangle .$$

(The suffix (c) means "connected part".) Calculating $\overline{\text{A}}$ we consider $|p_{1,2}\rangle$ as out-states and get

$$-\frac{1}{2}\int d^4x e^{ikx} \sum_n \langle 0 | B(0)\, a^+_{\text{out}}(p_1) | n \rangle_{\text{out}}\ {}_{\text{out}}\langle n | a_{\text{out}}(p_2)\, A(x) | 0 \rangle =$$

$$= \frac{1}{4(2\pi)^3}\int d^4x_1\, d^4x_2\, d^4x e^{ikx - ip_1x_1 + ip_2x_2} (\Box_{x_1} + m^2)(\Box_{x_2} + m^2) \tag{6}$$

$$\theta(x_{10})\,\theta(x_{20} - x_0) \langle 0 | [\varphi(x_1), B(0)][\varphi(x_2), A(x)] | 0 \rangle .$$

The TCP theorem states that

$$\langle 0 | [(\varphi(x_1), B(0)][\varphi(x_2), A(x)] | 0 \rangle =$$
$$= \langle 0 | [A(-x), \varphi(-x_2)][B(0), \varphi(-x_1)] | 0 \rangle, \tag{7}$$

which immediately proves the above statement. Introducing

$$S(S^2, q^2, k^2, U^2, t) = (2\pi)^4 \sum_{\alpha_s} \langle p_2 | A(0) | S\alpha_s \rangle^{(c)}_{\text{in}}\ {}^{(c)}_{\text{in}}\langle S\alpha_s | B(0) | p_1 \rangle, \tag{8}$$

where α_s includes all the quantum numbers besides mass and momentum, the contribution of the graphs A and $\overline{\text{A}}$ together can be expressed as

$$\int dM_s^2\, \delta(M_s^2 - S^2)\, \varepsilon(S_0)\, S(M_s^2, q^2, k^2, U^2, t) . \tag{9}$$

The integration $\int dM_s^2$ has to be extended to the masses of states contributing to the graph A, and, of course, also includes the sum over pole terms. It is remarkable that graph A represents contributions of type, connected $\otimes$ connected, whereas $\overline{\text{A}}$ is disconnected $\otimes$ disconnected. Their contributions to the commutator τ, however, are in fact very similar to each other.

The graphs B, ..., $\overline{\text{D}}$ can be treated in a similar way resulting in the following representation for τ:

$$\begin{aligned}\tau(q^2, k^2, S^2, U^2, t) = &\int dM_q^2\, \delta(M_q^2 - q^2)\, \delta(q_0)\, Q(M_q^2, k^2, S^2\, U^2, t) + \\ &+ \int dM_k^2\, \delta(M_k^2 - k^2)\, \varepsilon(k_0)\, K(M_k^2, q^2, s^2, U^2, t) + \\ &+ \int dM_s^2\, \delta(M_s^2 - S^2)\, \varepsilon(S_0)\, S(M_s^2, q^2, k^2, U^2, t) + \\ &+ \int dM_u^2\, \delta(M_u^2 - U^2)\, \varepsilon(U_0)\, U(M_u^2, q^2, k^2, S^2, t) .\end{aligned} \tag{10}$$

We remark that Eq. (10) resembles the JOST—LEHMANN—DYSON (JLD) representation [5], which is of the form

$$\begin{aligned}\tau(Q) &= \int d^4x e^{iQx} \langle p_2 \left| \left[A\left(\frac{x}{2}\right), B\left(-\frac{x}{2}\right) \right] \right| p_1 \rangle = \\ &= \int d^4u \int_0^\infty d\sigma\, \varepsilon(Q_0 - U_0)\, \delta[\sigma - (Q-u)^2]\, \Phi(u, \sigma) .\end{aligned} \tag{11}$$

As is well known the JLD representation expresses local commutativity. Such a property was not used explicitly in our calculation, but we used the TCP theorem which is known to be equivalent to local commutativity (more exactly to "weak local commutativity" [6]).

III. Retarded product and equal time commutator

In the remaining part of this paper we shall deal with three-point functions only, the extension to four-point functions being trivial. In particular, results of the next section can be easily extended to the four-point function case if a triple integral representation holds. (See, in this respect [1]). For definiteness let us consider the matrix element of the commutator of an axial-vector current A_μ with a vector V_ν taken between the vacuum and one pion state $\pi(p)$ of momentum p:

$$t_{\mu\nu}(k,p) = \frac{1}{2}\int d^4x e^{ikx} \langle 0 | [A_\mu(x), V_\nu(0)] | \pi(p) \rangle = \\ = a_{MN}(\nu_K) P_{M\mu} P_{N\nu} + a(\nu_K) g_{\mu\nu}, \tag{12}$$

where

$$P_1 = \frac{1}{2}(k+q), \qquad P_2 = \frac{1}{2}(k-q), \qquad q = k - p,$$

$$\nu_1 = \frac{1}{2}(k^2+q^2), \qquad \nu_2 = \frac{1}{2}(k^2-q^2), \qquad \nu'_K = \nu_K + 2P_{K_0} r_0 + \delta_{K_1} r_0^2, \tag{13}$$

$$(K = 1, 2).$$

The retarded function corresponding to $t_{\mu\nu}$ in Eq. (12) is defined as in [7] by

$$R_{\mu\nu}(k,p) = i\int d^4x\, e^{ikx}\, \theta(x_0) \langle 0 | [A_\mu(x), V_\nu(0)] | \pi(p) \rangle = \\ = \frac{1}{\pi}\int \frac{dr_0}{r_0 - i\varepsilon} t_{\mu\nu}(k_0 + r_0, \underline{k}, p). \tag{14}$$

Writing down this expression we supposed that the commutator is not too singular near the origin. The extension to more singular cases is a difficult task and needs further investigations. The retarded commutator as defined in Eq. (14) is generally a noncovariant quantity. In any case, from rotation invariance we have

$$R_{\mu\nu} = R_{MN} P_{M\mu} P_{N\nu} + R g_{\mu\nu} + R_M P_{M\mu} g_{0\nu} + R'_N g_{0\mu} P_{N\nu} + R' g_{0\mu} g_{0\nu}, \tag{15}$$

where R_{MN} , ..., R' are rotation invariant functions. Comparing Eqs. (15), (14), (12) we have:

$$R_{MN} = \frac{1}{\pi}\int \frac{dr_0}{r_0 + i\varepsilon} a_{MN}(\nu'_k), \qquad R = \frac{1}{\pi}\int \frac{dr_0}{r_0 - i\varepsilon} a(\nu'_k),$$
$$R_M = \frac{1}{\pi}\int dr_0\, a_{M1}(\nu'_k), \qquad R'_N = \frac{1}{\pi}\int dr_0\, a_{1N}(\nu'_k), \tag{16}$$
$$R' = \frac{1}{\pi}\int dr_0\, a_{11}(\nu'_k)\, r_0 .$$

The equal-time commutator can be defined in a similar way:

$$E_{\mu\nu} = \int d^4x\, e^{ikx}\, \delta(x_0)\, \langle 0 \,|\, [A_\mu(x), V_\nu(0)] \,|\, \pi(p)\rangle = \frac{1}{\pi}\int dr_0\, t_{\mu\nu}(k_0 + r_0, \underline{k}, p) =$$
$$= E_{MN} P_{M\mu} P_{N\nu} + E g_{\mu\nu} + E_M P_{M\mu} g_{0\nu} + E'_N P_{N\nu} g_{0\mu} + E' g_{0\mu} g_{0\nu}. \tag{17}$$

Here the rotation invariant functions are

$$E_{MN} = \frac{1}{\pi}\int dr_0\, a_{MN}(\nu'_k), \qquad E = \frac{1}{\pi}\int dr_0\, a(\nu'_k),$$
$$E_M = \frac{1}{\pi}\int dr_0\, a_{M1}(\nu'_k)\, r_0, \qquad E'_N = \frac{1}{\pi}\int dr_0\, a_{1N}(\nu'_k)\, r_0, \tag{18}$$
$$E' = \frac{1}{\pi}\int dr_0\, a_{11}(\nu'_k)\, r_0^2 .$$

Let us now use the representation of the commutator derived in the preceding Section. For the invariants in the commutator (12) it reads:

$$a_{MN}(\nu_k) = \int dM_q^2\, Q_{MN}(M_q^2, k^2)\, \delta(q^2 - M_q^2)\, \varepsilon(q_0) +$$
$$+ \int dM_k^2\, K_{MN}(M_k^2, q^2)\, \delta(k^2 - M_k^2)\, \varepsilon(k_0). \tag{19}$$

The advantage of this form is that with its help the noncovariant integration $\int dr_0$ in Eqs. (16), (18) (which goes along a parabola in the ν_1, ν_2 - plane) can be performed leaving us with the covariant integrations on the masses of intermediate states. All the expressions in Eqs. (16) and (18) can be rewritten in terms of Q_{MN} and K_{MN} with the help of the following formula ($\alpha = 0, 1, 2, 3$):

$$\frac{1}{\pi}\int \frac{dr_0}{r_0 - i\varepsilon}\, a(\nu'_k)\, r_0^\alpha =$$

$$= \frac{1}{\pi}\int \frac{dM_q^2}{r_{0+a} - r_{0-q}} \left\{ \frac{r_{0+q}^\alpha}{r_{0+a} - i\varepsilon} Q(M_q^2, k^2 + 2r_{0+q}(P_{10} + P_{20} + r_{0+q}^2) - \right. \tag{20}$$

$$\left. - \frac{r_{0-q}^\alpha}{r_{0-q} - i\varepsilon} Q(M_q^2, k^2 + 2r_{0-q}(P_{10} + P_{20}) + r_{0-q}^2) \right\} + \left[P_{a_0} \to \overset{q \leftrightarrow k}{P}_{a_0} (-1)^{\alpha+1} \right]$$

$$r_{0\pm q} = -(P_{10} - P_{20}) \pm \sqrt{M_q^2 - q^2 + (P_{10} - P_{20})^2}\,.$$

The case $\alpha = 0$ corresponds to a noncovariant dispersion relation. Taking a limit to infinity with some linear combination of P_{10} and P_{20} we arrive at a covariant dispersion relation in the same combination of ν_1, and ν_0. In particular, from $q_0 \to \infty$ we conclude that Q is the discontinuity of

$$A(\nu_K) = \frac{1}{\pi}\int \frac{dr_0}{r_0 - i\varepsilon}\, a(\nu'_K) = R(\nu_K) \tag{21}$$

in the variable q^2. Analogously, K turns out to be the k^2-discontinuity of A. Here we assumed that the function A defined in Eq. (21) depends only on ν_1, ν_2 which are kept fixed during the limit to infinite momentum. In other words we supposed that the R-product is covariant in the scalar (both currents and particles) case and its noncovariance is due only to the spins. In this way the "covariant part" of $R_{\mu\nu}$ can be defined as

$$T_{\mu\nu} = A_{MN} P_{M\mu} P_{N\nu} + A g_{\mu\nu} := R_{MN} P_{M\mu} P_{N\nu} + R g_{\mu\nu}. \tag{22}$$

This can be considered as the definition of the physical current correlation function relevant in weak or electromagnetic processes.

From Eqs. (20) and (18) we can express the equal-time commutator with the help of the discontinuities Q and K. The sum rules derived in this way from the commutator of a time component of the current with the other components were given in [1] in the infinite momentum limit.

IV. Double integral representations and sum rules

It was shown in [1] that in the infinite momentum sum rules a cancellation of singularities occurs if the current correlation functions (in the case of three-point functions) are given by double integral representations. In fact, in the infinite momentum limit the double spectral functions do not contribute to the equal-time commutators which are given by the single spectral functions alone.

Let us now investigate the equal-time commutators in an arbitrary frame of reference. We assume a double integral representation of the form

$$A_{MN}(q^2, k^2) = \frac{1}{\pi}\int \frac{dM_q^2}{M_q^2 - q^2}\varrho_q^{MN}(M_q^2) + \frac{1}{\pi}\int \frac{dM_k^2}{M_k^2 - k^2}\varrho_k^{MN}(M_k^2) + \\ + \frac{1}{\pi^2}\int \frac{dM_q^2\, dM_k^2}{(M_q^2 - q^2)(M_k^2 - k^2)}\varrho_{qk}^{MN}(M_q^2, M_k^2). \tag{23}$$

In this case the discontinuities Q_{MN} and K_{MN} are:

$$Q_{MN}(M_q^2, k^2) = \varrho_q^{MN}(M_q^2) + \frac{1}{\pi}\int \frac{dM_k^2}{M_k^2 - k^2}\varrho_{qk}^{MN}(M_q^2, M_k^2),$$
$$K_{MN}(M_k^2, q^2) = \varrho_k^{MN}(M_k^2) + \frac{1}{\pi}\int \frac{dM_q^2}{M_q^2 - q^2}\varrho_{qk}^{MN}(M_q^2, M_k^2). \tag{24}$$

Using these expressions from Eqs. (20) and (16) one can easily express the retarded commutator in terms of the spectral functions:

$$R_{MN} = A_{MN}, \quad R = A, \quad R_M = R'_N = 0,$$
$$R' = \frac{1}{\pi}\int dM_q^2\, \varrho_q^{11} + \frac{1}{\pi}\int dM_k^2\, \varrho_{qk}^{11}. \tag{25}$$

The first two equalities in Eq. (25) show the consistency of our assumptions about the covariance of retarded products in the scalar case. The last equality shows an explicitly noncovariant part of the R-product in Eq. (14). The analogous expressions for the equal-time commutators are

$$E_{MN} = E = 0,$$
$$E_M = \frac{1}{\pi}\int dM_q^2\, \varrho_q^{M1} + \frac{1}{\pi}\int dM_k^2\, \varrho_k^{M1},$$
$$E'_N = \frac{1}{\pi}\int dM_q^2\, \varrho_q^{1N} + \frac{1}{\pi}\int dM_k^2\, \varrho_k^{1N}, \tag{26}$$
$$E' = -2(P_{10} - P_{20})\frac{1}{\pi}\int dM_q^2\, \varrho_q^{11} - 2(P_{10} + P_{20})\frac{1}{\pi}\int dM_k^2\, \varrho_k^{11}.$$

Eqs. (26) contains the sum rules we looked for.

V. Concluding remarks

We can see from Eq. (26) that the double spectral functions cancel in the equal-time commutator in any frame of reference. Calculating the values of E_{MN}, ..., E' from Eq. (17) in some model and substituting them into Eq. (26) we get the sum rules expressing the constraints imposed by the current-density commutators on the current correlation functions. Taking infinite momentum limit is not essential in deriving sum rules as Eq. (26) holds in any frame of reference.

In our calculations besides the assumption of double integral representations, assumptions about asymptotic behaviour also play an essential role. It can be seen from Eq. (24) that we supposed the discontinuities to behave like constants in infinity. An extension of our results to a more divergent asymptotic behaviour does not seem impossible. The sum rules in Eq. (26) show explicitly the dependence of the equal-time commutator matrix element on the reference frame. It is remarkable that the results $E_{MN} = E = 0$ correspond to the field algebra [3] (and contradict the free quark model [2]). This shows that the quark commutators are incompatible with the asymptotic behaviour we assumed.

Finally, we remark that the assumption of multiple integral representations may be, in general, a severe restriction. In practical calculations, however, pole dominance is almost always assumed and the contribution of a one particle pole is always of the form (23). Hence assuming pole dominance the sum rules can always be evaluated from Eq. (26).

In our approach a particle (in Eq. (12) $\pi(p)$) is on the mass shell. Thus, if we want to collect all the restrictions of current commutators on a given three-point function, then we have to put all particles subsequently on the mass shell and consider the sum rules (26) in each case. As an example, we can calculate the $A_1\,\varrho\pi$-system. Using pole dominance in once subtracted dispersion relations, PCAC and the sum rules (26) we get the hard pion results [8—11]. Another example is the system containing the K_{l3} form factors. We shall deal with this in a separate paper [12], where the sum rules (26) are given in the case of linear behaviour at infinity (in the "truncation approximation").

The author thanks Dr. F. Csikor for valuable comments and discussions.

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ПРАВИЛА СУММ И КОВАРИАНТНОСТЬ

И. МОНТВАИ

Р е з ю м е

Представление коммутаторных матричных элементов производится от формализма Лемана—Симанцика—Циммермана и применяется к правилам сумм алгебры токов в любой системе отчета. Правила сумм принимают особенно простую форму в случае, если корреляционные функции тока подчиняются интегральным представлениям в массах токов. Зависимость правил сумм от системы отчета становится явной.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 169—177 (1969)

K_{l3} FORM FACTORS, CURRENT ALGEBRA AND DOUBLE INTEGRAL REPRESENTATIONS

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Predictions of current algebra for the K_{l3} form factors are considered using double integral representations of the current correlation functions. The K_{l3} form factor parameters ξ, $\lambda_{\pm}$ and the mass and width of the $\varkappa = K(0^+)$ meson are calculated in good agreement with currently known experimental values.

I. Introduction and summary of results

The K_{l3} decay form factors have received much interest in recent years in connection with current algebra. Since the first calculations [1—3] much work has been done in order to avoid too many restrictive assumptions. The main problem in current algebraic calculations is always the extrapolation from the non-physical point (where current algebra gives definite statements) to the physically measurable region. This can be done most simply by pole dominance approximation, but pole dominance based on unsubtracted dispersion relations is known to require corrections in the K_{l3} case [4—5]. Recently [6—7], once subtracted dispersion relations have been proposed instead, which have proved to be successful, for example, in the $A_1\,\varrho\pi$ system [8—10].

In the present paper we apply a method previously worked out for the exploitation of current density commutators [11—12] to the case of the $K_A\,(K)\,K^*\,(\varkappa)\,A1\,(\pi)$ system. Our assumptions correspond to once subtracted dispersion relations for the form factors. The SU(3) breaking is taken into account by assuming a model where the divergence of the strangeness-carrying vector current is proportional to the strange scalar meson ($\varkappa$) field (see, for instance, in [13—14]). From the sum rules we obtain the values $\lambda_+ = 0.024$; $\xi = -0.067$; $\lambda_- = 0.042$ for the K_{l3} parameters. The mass of the $\varkappa$ meson comes out to $m_\varkappa = 1020$ MeV and its width to $\Gamma_\varkappa = 275$ MeV. These $\varkappa$ meson data agree with the recently reported observation [15] of the broad S-wave $K\pi$ resonance K_N (1100—1200), but differ from those derived in [14]. The reason for this is that we do not assume the two WEINBERG sum rules. Moreover we need not specify the commutators of currents with current divergences which are model dependent and essentially use only the commutator of time-component of the current with space components (in this context see also [16]).

Our results also imply that the coupling of $\varkappa$ to leptons is relatively weak. This justifies to some extent the assumption made in [7] that the omission of scalar mesons does not alter very much the results for the K_{l3} form factors.

II. K_{l3} form factors

Let us first consider the commutator

$$t_{\mu\nu} = \frac{1}{2}\int d^4x\, e^{ikx} \langle 0 \,|\, [A_\mu(x)_R, V_\nu(0)_S] \,|\, \pi(p)_T \rangle\,, \tag{1}$$

where $A_\mu(x)_R$ $\left(V_\nu(x)_S\right)$ denotes a member of the octet of axialvector (vector) currents and $\pi(p)_T$ is chosen from the pseudoscalar meson octet (with four momentum p). More specifically, we take for the moment $R = ''3'' = \pi^\circ$; $s = 1/\sqrt{2}\,(''4'' - i''5'') = K^-$, $T = 1/\sqrt{2}\,(''4'' + i''5'') = K^+$. Let us define the following kinematic variables:

$$q = k - p\,; \quad P_1 = \frac{1}{2}(k+q)\,; \quad P_2 = \frac{1}{2}(k-q)\,; \tag{2}$$

$$\nu_1 = \frac{1}{2}(k^2+q^2)\,; \quad \nu_2 = \frac{1}{2}(k^2-q^2)\,; \quad \nu'_K(r_0) = \nu_K + 2P_{K0}\,r_0 + \delta_{K1}\,r_0^2\,;$$

then from Lorentz-invariance we have

$$t_{\mu\nu} = a_{MN}(\nu_K)\,P_{M\mu}\,P_{N\nu} + a(\nu_K)\,g_{\mu\nu}. \tag{3}$$

If the commutator is not too singular near the origin (which will be assumed in the following) then the invariants a_{MN} define the corresponding retarded functions like

$$A_{MN}(\nu_K) = \frac{1}{\pi}\int \frac{dr_0}{r_0 - i\varepsilon}\, a_{MN}\big(\nu'_K(r_0)\big). \tag{4}$$

Following [11—12] we assume that these functions are invariant and define the physical current correlation functions by

$$T_{\mu\nu} = A_{MN}(\nu_K)\,P_{M\mu}\,P_{N\nu} + A(\nu_K)\,g_{\mu\nu}. \tag{5}$$

Furthermore, we suppose the invariants A_{MN}, A to be given by double integral representations of the following form

$$A_{MN}(q^2, k^2) = \frac{1}{\pi}\int \frac{dM_q^2}{M_q^2 - q^2}\,[\varrho_q^{MN}(M_q^2) + \varrho_q^{MN}(M_q^2)k^2] +$$

$$+\frac{1}{\pi}\int\frac{dM_k^2}{M_k^2-k^2}\left[\varrho_k^{MN}(M_k^2)+\varrho_{\bar{k}}^{MN}(M_k^2)\,q^2\right]+ \tag{6}$$

$$+\frac{1}{\pi^2}\iint\frac{dM_q^2\,dM_k^2}{(M_q^2-q^2)\,(M_k^2-k^2)}\,\varrho_{qk}(M_q^2,M_k^2)\,.$$

Note that here we allowed linear terms also in single spectral functions. As we shall see below this corresponds to once subtracted dispersion relations for the form factors. With one variable fixed the above representation corresponds to a twice subtracted dispersion relation in the "truncated form". (Such representations are also used in a recent preprint of MATHUR [17] treating the $A1\varrho\pi$ system.)

The sum rules derived in [11—12] are expressed in terms of the single spectral functions in Eq. (6). The equal-time commutator is now

$$E_{\mu\nu}=\int d^4x\,e^{ikx}\,\delta(x_0)\,\langle 0\,|\,[A_\mu(x)_R,V_\nu(0)_S]\,|\,\pi(p)_T\rangle= \tag{7}$$
$$=E_{MN}\,P_{M\mu}\,P_{N\nu}+Eg_{\mu\nu}+P_{M\mu}\,g_{0\nu}\,E_M^{(1)}+P_{N\nu}\,g_{0\mu}\,E_N^{(2)}+g_{0\mu}\,g_{0\nu}\,E^{(3)},$$

and the sum rules including the linear terms $\varrho_{\bar{q}}$, $\varrho_{\bar{k}}$ are the following:

$$E_{MN}=4P_{20}\left\{\frac{1}{\pi}\int dM_q^2\,\varrho_{\bar{q}}^{MN}-\frac{1}{\pi}\int dM_k^2\varrho_{\bar{k}}^{MN}\right\};$$

$$E=4P_{20}\left\{\frac{1}{\pi}\int dM_q^2\,\varrho_{\bar{q}}-\frac{1}{\pi}\int dM_k^2\,\varrho_{\bar{k}}\right\};$$

$$E_M^{(1)}=\frac{1}{\pi}\int dM_q^2\,\varrho_q^{M1}+\frac{1}{\pi}\int dM_k^2\,\varrho_k^{M1}+\frac{1}{\pi}\int dM_q^2\,\varrho_{\bar{q}}^{M1}[k^2+M_q^2-q^2-8P_{20}(P_{10}-P_{20})]+$$

$$+\frac{1}{\pi}\int dM_k^2\,\varrho_{\bar{k}}^{M1}\,[q^2+M_k^2-k^2+8P_{20}(P_{10}+P_{20})];$$

$$E_N^{(2)}=\frac{1}{\pi}\int dM_q^2\,\varrho_q^{1N}+\frac{1}{\pi}\int dM_k^2\,\varrho_k^{1N}+\frac{1}{\pi}\int dM_q^2\,\varrho_{\bar{q}}^{1N}\,[k^2+M_q^2-q^2-8P_{20}(P_{10}-P_{20})]$$

$$+\frac{1}{\pi}\int dM_k^2\,\varrho_{\bar{k}}^{1N}\,[q^2+M_k^2-k^2+8P_{20}(P_{10}+P_{20})]; \tag{8}$$

$$E^{(3)}=2(P_{20}-P_{10})\frac{1}{\pi}\int dM_q^2\,\varrho_q^{11}-(2P_{20}+P_{10})\frac{1}{\pi}\int dM_k^2\,\varrho_k^{11}+$$

$$+\frac{1}{\pi}\int dM_q^2\,\varrho_{\bar{q}}^{11}\,[2(P_{20}-P_{10})\,k^2+(M_q^2-q^2)\,(6P_{20}-2P_{10})+16P_{20}(P_{10}-P_{20})^2]+$$

$$+\frac{1}{\pi}\int dM_k^2\,\varrho_{\bar{k}}^{11}\,[-2(P_{20}+P_{10})\,q^2-(M_k^2-k^2)\,(6P_{20}+2P_{10})-16P_{20}(P_{10}+P_{20})^2].$$

Our next task is to calculate the spectral functions in pole dominance approximation. We shall consider two cases: (1) the one mentioned above: $R = \pi^0$, $S = K^-$, $T = K^+$ and (2) $R = K^-$, $S = K^+$, $T = \pi^0$. Let us define the relevant form factors (M_{KA,A,K^*} denote the KA, $A1$, K^* masses; the ε's are the polarization vectors of the spin one particles):

$$\langle 0|A_\mu(0)_{K\pm}|K_A^{\mp}(p)\rangle = -\frac{M_{KA}^2}{G_{KA}}\frac{\varepsilon_\mu^{KA}(p)}{\sqrt{2(2\pi)^3}}\,;\quad \langle 0|A_\mu(0)_{K\mp}|K^{\pm}(p)\rangle = \frac{ip_\mu}{\sqrt{2(2\pi)^3}}F_K\,;$$

$$\langle 0|A_\mu(0)_{\pi^0}|A1^0(p)\rangle = -\frac{M_A^2}{G_A}\frac{\varepsilon_\mu^{A}(p)}{\sqrt{2(2\pi)^3}}\,;\quad \langle 0|A_\mu(0)_{\pi^0}|\pi^0(p)\rangle = \frac{ip_\mu}{\sqrt{2(2\pi)^3}}F_\pi\,;$$

$$\langle 0|V_\mu(0)_{K\mp}|K^{*\pm}(p)\rangle = -\frac{M_{K^*}^2}{G_{K^*}}\frac{\varepsilon_\mu^{K^*}(p)}{\sqrt{2(2\pi)^3}}\,;\quad \langle 0|V_\mu(0)_{K\mp}|\varkappa^{\pm}(p)\rangle = \frac{ip_\mu}{\sqrt{2(2\pi)^3}}F_\varkappa\,;$$

$$-\frac{M_A^2}{G_A}F_K\langle A1^0(k)|V_\nu(0)_{K^-}|K^+(p)\rangle = \frac{i}{2(2\pi)^3}\{\varepsilon_\nu^A G^{(1)}(q^2) + \varepsilon^A p\, P_{N\nu}\, G_N^{(1)}(q^2)\}\,;$$

$$-\frac{M_{KA}^2}{G_{KA}}F_\pi\langle K_A^+(k)|V_\nu(0)_{K^\pm}|\pi^0(p)\rangle = \frac{i}{2(2\pi)^3}\{\varepsilon_\nu^{KA} G^{(2)}(q^2) + \varepsilon^{KA} p\, P_{N\nu}\, G_N^{(2)}(q^2)\}$$

$$F_\pi F_K\langle\pi^0(k)|V_\nu(0)_{K^-}|K^+(p)\rangle = \frac{-1}{2(2\pi)^3}g_N^{(1)}(q^2)\,P_{N\nu}\,; \tag{9}$$

$$F_K F_\pi\langle K^+(k)|V_\nu(0)_{K^+}|\pi^0(p)\rangle = \frac{-1}{2(2\pi)^3}g_N^{(2)}(q^2)\,P_{N\nu}\,;$$

$$-\frac{M_{K^*}^2}{G_{K^*}}F_K\langle K^{*+}(p-k)|A_\mu(0)_{\pi^0}|K^+(p)\rangle = \frac{i}{2(2\pi)^3}\{\varepsilon_\mu^{K^*}F^{(1)}(k^2) + \varepsilon^{K^*}p\,P_{M\mu}F_M^{(1)}(k^2)\}\,;$$

$$-\frac{M_{K^*}^2}{G_{K^*}}F_\pi\langle K^{*-}(p-k)|A_\mu(0)_{K^-}|\pi^0(p)\rangle = \frac{i}{2(2\pi)^3}\{\varepsilon_\mu^{K^*}F^{(2)}(k^2) + \varepsilon^{K^*}p\,P_{M\mu}F_M^{(2)}(k^2)\}\,;$$

$$F_\varkappa F_K\langle\varkappa^+(p-k)|A_\mu(0)_{\pi^0}|K^+(p)\rangle = \frac{-1}{2(2\pi)^3}P_{M\mu}f_M^{(1)}(k^2)\,;$$

$$F_\varkappa F_\pi\langle\varkappa^-(p-k)|A_\mu(0)_{K^-}|\pi^0(p)\rangle = \frac{-1}{2(2\pi)^3}P_{M\mu}f_M^{(2)}(k^2)\,.$$

All the form factors will be assumed to obey a once subtracted dispersion relation, dominated by the one particle poles. Therefore, we may write, e.g. in the case (1):

$$G^{(1)}(q^2) = \frac{G^{(1)}}{M_{K^*}^2 - q^2} + \bar{G}^{(1)}\,,\quad G_M^{(1)}(q^2) = \frac{G_M^{(1)}}{M_{K^*}^2 - q^2} + \frac{(-1)^M A^{(1)}}{m_\varkappa^2 - q^2} + \bar{G}_M^{(1)}\,,$$

$$g_M^{(1)}(q^2) = \frac{g_M^{(1)}}{M_{K^*}^2 - q^2} + \frac{(-1)^M a^{(1)}}{m_\varkappa^2 - q^2} + \bar{g}_M^{(1)}\,, \tag{10}$$

$$F^{(1)}(k^2) = \frac{F^{(1)}}{M_A^2 - k^2} + \bar{F}^{(1)}, \qquad F_M^{(1)}(k^2) = \frac{F_M^{(1)}}{M_A^2 - k^2} + \frac{B^{(1)}}{m_\pi^2 - k^2} + \bar{F}_M^{(1)},$$

$$f_M^{(1)}(k^2) = \frac{f_M^{(1)}}{M_A^2 - k^2} + \frac{b^{(1)}}{m_\pi^2 - k^2} + \bar{f}_M^{(1)}.$$

The case (2) can be obtained from this by making the changes: $A \to KA$, $\pi \to K$. It is now an easy matter to calculate $t_{\mu\nu}$ in Eq. (1) inserting the one particle intermediate states (and taking into account the corresponding disconnected parts too). Projecting out the invariants a_{MN} and a, we can determine the spectral functions from the representation [11—12]:

$$\begin{aligned} a_{MN}(q^2, k^2) = &\int dM_q^2 \, Q_{MN}(M_q^2, k^2) \, \delta(q^2 - M_q^2) \, \varepsilon(q_0) + \\ &+ \int dM_k^2 \, K_{MN}(M_k^2, q^2) \, \delta(k^2 - M_k^2) \, \varepsilon(k_0). \end{aligned} \tag{11}$$

(In the case of pole dominance the integrals $\int dM_q^2$, $\int dM_k^2$ are substituted of course by finite sums.) Taking an appropriate infinite momentum limit [11] of Eq. (4) one can easily show that the functions Q_{MN} and K_{MN} are the discontinuities of A_{MN} in q^2 and k^2, respectively. Hence, we have from Eq. (6):

$$\begin{aligned} Q_{MN} &= \varrho_q^{MN} + \varrho_{\bar{q}}^{MN} \, k^2 + \frac{1}{\pi} \int \frac{dM_k^2}{M_k^2 - k^2} \, \varrho_{qk}, \\ K_{MN} &= \varrho_k^{MN} + \varrho_{\bar{k}}^{MN} \, q^2 + \frac{1}{\pi} \int \frac{dM_q^2}{M_q^2 - q^2} \, \varrho_{qk}. \end{aligned} \tag{12}$$

Comparing the expression of a_{MN} with Eqs. (11, 12) the spectral functions $\varrho_q^{MN}, \ldots, \varrho_{qk}^{MN}$ can be computed. Substituting back into Eq. (8) we get the sum rules expressing the constraints imposed by the knowledge of the equal time commutator in Eq. (7).

Besides the equations so obtained we also require the fulfilment of the divergence equations which specify the SU(3)⊗SU(3) symmetry breaking in strong interactions:

$$\begin{aligned} \partial^\mu A_\mu(x)_{K^\pm} &= m_K^2 \, F_K \, \varphi_{K^\pm}(x), \\ \partial^\mu A_\mu(x)_{\pi^\circ} &= m_\pi^2 \, F_\pi \, \varphi_{\pi^\circ}(x), \\ \partial^\mu V_\mu(x)_{K^\pm} &= \pm i m_\varkappa^2 \, F_\varkappa \, \varphi_{\varkappa^\pm}(x). \end{aligned} \tag{13}$$

Here $\varphi_{K^\pm}$, π°, $\varkappa^\pm(x)$ denote the $K^\pm$, π°, $\varkappa^\pm$ fields with masses $m_{K,\pi,\varkappa}$ respectively.

Considering the cases (1) and (2) separately, the divergence equations (13) and the sum rules (8) determine all the coupling constants and subtraction constants in Eq. (10) in terms of six parameters, say $\bar{g}_1^{(1,2)}$, $\bar{G}^{(1,2)}$, $f^{(1,2)}$. Inspect-

ing the sum rules we see that in the field-algebra [18] only the commutator of the time-component with space components gives a non-trivial equation, while the other commutators do not contain independent information. In the quark model [19], however, the equations are contradictory. This situation is analogous to the one encountered in the $A_1 \varrho\pi$ system [20]. Thus, in the following we shall consider only the field-algebra. We write out here only the resulting expressions for the coupling constants relevant in the K_{l3} form factors:

$$g_1^{(1)} = (M_{K^*}^2 + m_K^2 - m_\pi^2)\left[F_K^2 \frac{m_K^2 - m_\varkappa^2}{m_\pi^2 - m_\varkappa^2} - \bar{g}_1^{(1)}\right], \quad a^{(1)} = F_K^2(m_K^2 - m_\pi^2)\frac{m_K^2 - m_\varkappa^2}{m_\pi^2 - m_\varkappa^2},$$

$$g_2^{(1)} = (3M_{K^*}^2 - m_K^2 + m_\pi^2)\left[F_K^2 \frac{m_K^2 - m_\varkappa^2}{m_\pi^2 - m_\varkappa^2} - \bar{g}_1^{(1)}\right], \tag{14}$$

$$g_1^{(2)} = (M_{K^*}^2 + m_\pi^2 - m_K^2)\left[F_\pi^2 \frac{m_\pi^2 - m_\varkappa^2}{m_K^2 - m_\varkappa^2} - \bar{g}_1^{(2)}\right], \quad a^{(2)} = F_\pi^2(m_\pi^2 - m_K^2)\frac{m_\pi^2 - m_\varkappa^2}{m_K^2 - m_\varkappa^2},$$

$$g_2^{(2)} = (3M_{K^*}^2 - m_\pi^2 + m_K^2)\left[F_\pi^2 \frac{m_\pi^2 - m_\varkappa^2}{m_K^2 - m_\varkappa^2} - \bar{g}_1^{(2)}\right].$$

Fortunately, these quantities depend only on $\bar{g}_1^{(1,2)}$. Moreover, as can be seen from Eq. (9) we have

$$g_1^{(2)}(q^2) = \frac{1}{2}[g_2^{(1)}(q^2) - g_1^{(1)}(q^2)],$$
$$g_2^{(2)}(q^2) = \frac{1}{2}[g_2^{(1)}(q^2) + 3g_1^{(1)}(q^2)]. \tag{15}$$

This, together with Eq. (14), gives

$$\left(\frac{F_K}{F_\pi}\right)^2 = \left(\frac{m_\varkappa^2 - m_\pi^2}{m_\varkappa^2 - m_K^2}\right)^2. \tag{16}$$

Experimentally $|F_K/F_\pi| = 1.28$ (and SU(3) suggests F_K and F_π to be of the same sign). Thus, from Eq. (16) we get $m_\varkappa = 1020$ MeV. The other root is $m_\varkappa = 380$ MeV, but such a particle does not exist. Let us introduce the usual form factors

$$F_+^{(1)}(q^2) \equiv F_+(q^2) = \frac{1}{4F_K F_\pi}[g_1^{(1)}(q^2) + g_2^{(1)}(q^2)],$$
$$F_-^{(1)}(q^2) \equiv F_-(q^2) = \frac{1}{4F_K F_\pi}[g_2^{(1)}(q^2) - 3g_1^{(1)}(q^2)]. \tag{17}$$

Then from Eqs. (10, 14) we obtain a modified version of the CALLAN—TREIMAN relations [1]:

$$F_+^{(1)}(m_K^2)+F_-^{(1)}(m_K^2)=\frac{F_K}{F_\pi}+\frac{m_\pi^2}{M_{K^*}^2-m_K^2}\left[\frac{F_K}{F_\pi}\frac{m_K^2-m_\varkappa^2}{m_\pi^2-m_\varkappa^2}-\frac{\bar g_1^{(1)}}{F_K F_\pi}\right],$$

$$F_+^{(2)}(m_\pi^2)+F_-^{(2)}(m_\pi^2)=F_+^{(1)}(m_\pi^2)-F_-^{(1)}(m_\pi^2)= \tag{18}$$

$$=\frac{F_\pi}{F_K}+\frac{m_K^2}{M_{K^*}^2-m_\pi^2}\left[\frac{F_\pi}{F_K}\frac{m_\pi^2-m_\varkappa^2}{m_K^2-m_\varkappa^2}-\frac{\bar g_1^{(1)}}{F_K F_\pi}\right].$$

The corrections are of order $m_\pi^2/M_{K^*}^2$ and $m_K^2/M_{K^*}^2$ in the two cases, respectively, and are due to the fact that we treated both pions and kaons as "hard" (i.e. we did not assume that form factors vary little from $q^2=0$ to $q^2=m_\pi^2$ or m_K^2).

Let us now write down the expressions resulting from Eqs. (14, 15) for the usual parameters ξ and $\lambda_\pm$:

$$\xi=\frac{F_-(0)}{F_+(0)};\quad F_\pm(q^2)=F_\pm(0)\left[1+\lambda_\pm\frac{q^2}{m_\pi^2}+\dots\right];$$

$$\lambda_+=\frac{m_\pi^2}{M_{K^*}^2}\left[1-\gamma\frac{m_K^2-m_\varkappa^2}{m_\pi^2-m_\varkappa^2}\right]=0{,}024\,,$$

$$F_+(0)=1,\quad F_-(0)=\xi=\frac{m_\pi^2-m_K^2}{m_\pi^2}\lambda_+ +\frac{m_K^2-m_\pi^2}{m_\varkappa^2}=-0{,}067\,, \tag{19}$$

$$\xi\lambda_-=\frac{m_\pi^2-m_K^2}{M_{K^*}^2}\lambda_+ +\frac{m_\pi^2}{m_\varkappa^2}\frac{m_K^2-m_\pi^2}{m_\varkappa^2}=-0{,}0028\,.$$

The numerical values correspond to $m_\varkappa=1020$ MeV and $\gamma=\bar g_1^{(1)}F_\pi^{-2}=0$ (that is to unsubtractedness of the form factor $g_1^{(1)}(q^2)$). Experimentally, the situation with K_{l3} form factors is rather unclear. Only the value of λ_+ is known to a good accuracy: $\lambda_+=0{,}019\pm0{,}006$ [21], in fair agreement with the above value.

III. The width of $\varkappa$ meson

The sum rules and divergence conditions could be considered also with the other particles (K_A, $A1$, K^*, $\varkappa$) on the mass shell as was done, e.g. in [10], for the $A_1\varrho\pi$ system. The relations obtained, however, are hard to test experimentally as most of the coupling constants and form factors involved are unknown at present. Therefore, we consider only the matrix elements

$$t_{\mu\nu}^{(1)}=\frac{1}{2}\int d^4x e^{ikx}\langle 0\,|\,[A_\mu(x)_{K^-},\quad A_\nu(0)_{\pi^0}]\,|\,\varkappa^+(p)\rangle\,,$$

$$t_{\mu\nu}^{(2)}=\frac{1}{2}\int d^4x e^{ikx}\langle 0\,|\,[A_\mu(x)_{\pi^0},\quad A_\nu(0)_{K^+}]\,|\,\varkappa^-(p)\rangle\,. \tag{20}$$

Here one can obtain additional information for the form factors $f_M^{(1,2)}(k^2)$ defined in the last two of Eq. (9). The derivation and evaluation of sum rules is exactly the same as before and comparing the results with those of the preceding Section we have

$$b^{(1)} = -b^{(2)} = F_\varkappa^2 \frac{(m_\varkappa^2 - m_\pi^2)(m_K^2 - m_\varkappa^2)}{m_\pi^2 - m_K^2} = F_K^2 \frac{(m_K^2 - m_\pi^2)(m_K^2 - m_\varkappa^2)}{m_\pi^2 - m_\varkappa^2} . \qquad (21)$$

Here $b^{(1)}$ is the $\varkappa K\pi$ coupling constant defined in Eq. (10). Eq. (21) gives for $F_\varkappa$:

$$\left| \frac{F_\varkappa}{F_K} \right| = \frac{m_K^2 - m_\pi^2}{m_\varkappa^2 - m_\pi^2} = 0{,}22 , \qquad (22)$$

where we again used the values $F_K/F_\pi = 1{,}28$, and $m_\varkappa = 1020$ MeV. We can see that $F_\varkappa$ is considerably smaller than F_K and F_π. This justifies to some extent the neglect of $\varkappa$ in sum rules (see [7]). From the value (21) of $b^{(1)}$ we can determine the $\varkappa$-width too. For $F_K/F_\pi = 1{,}28$ (1,26) we have $m_\varkappa = 1020$ (1050) MeV and $\Gamma_\varkappa = 275$ (300) MeV. As we have already mentioned, a recent experiment [15] seems to favour data like this for the strange scalar meson $K_N(1100-1200)$.

Note added in proof:

Comparing the results of this paper with the equations coming from the commutators of currents and divergences (B. RENNER, these Proceedings) one can determine the value of the parameter c (in the notations of [13] $c = -2g_0(3a_0)^{-1}$):

$$c = -2\sqrt{2}\, \frac{m_K^2 F_K - m_\pi^2 F_\pi}{m_\pi^2 F_\pi + 2m_K^2 F_K} \sim -1{,}28.$$

This value agrees very well with $c = -1{,}25$ obtained by GELL-MANN, RENNER and OAKES, and coincides with the one derived, if one assumes $\varphi_\varkappa$ and φ_π in Eq. (13) to be members of the same octet. I am indebted to Prof. B. RENNER for a useful conversation on divergence commutators.

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ФОРМ-ФАКТОРЫ K_{l3}, АЛГЕБРА ТОКОВ И ДВОЙНО-ИНТЕГРАЛЬНЫЕ ПРЕДСТАВЛЕНИЯ

И. МОНТВАИ

Р е з ю м е

Предсказания алгебры токов по отношению формфакторов K_{l3} рассматриваются применением двойноинтегральных представлений корреляционных функций тока. Параметры форм-факторов K_{l_3} ξ, $\lambda\pm$, далее масса и ширина мезона $\varkappa = K(0^+)$ вычислялись в хорошем согласии с экспериментальными данными, известными в настоящее время.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 179—180 (1969)

THE $A_1\varrho\pi$ SYSTEM

By

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The current algebra calculations of the $A_1\varrho\pi$ system are reviewed and results of different methods are compared.

Recently, a great deal of work has been devoted to the examination of systems involving only mesons by the method of current algebra. In particular the $A_1\varrho\pi$ system has been studied in detail by various authors [1—9]. While the results of the first papers [1, 2] seemed to disagree with experiment, the improved calculations are compatible with it. In this paper we want to discuss the work done by I. MONTVAY and the author [7] and compare the results with those of other papers.

In [7] current algebra is exploited in the form of FUBINI—HALLIDAY—LANDSHOFF [10] infinite momentum type sum rules for the vertex functions. Matrix elements of current commutators taken between vacuum and one meson states (pseudo-scalar, vector, and axial-vector mesons) are treated. The current commutators are saturated by one particle states (disconnected contributions [11] are also taken into account), and we have supposed dispersion relations with at most one subtraction for the appearing form factors. Of course, pole-dominance of dispersion relations was also used. The treatment of all three cases (pseudo-scalar, vector and axial-vector mesons) yielded consistent results. We obtained all the appearing form factors containing 2 axial-vector (or pseudo-scalar) and 1 vector current (or particle) in terms of only two parameters, as well as a reasonable value for the pion charge radius. In the expressions of experimentally accessible form factors, however, we have only a single parameter. The first WEINBERG sum rule [12] was also obtained. Imposing the condition of PDDAC we got expressions without any free parameter for all the measurable quantities (including the A_1 and ϱ width), which imply however, a vanishing d wave for the $A_1 \to \varrho\pi$ decay. The remaining single parameter appears only in form factors, which are not experimentally accessible.

Our work is nearest in spirit to that of BROWN and WEST [3], who used unsubtracted dispersion relations for linear combinations of the invariants. In fact, our work can be considered to be the $p \to \infty$ formulation of the dispersion theoretical method of BROWN and WEST.

SCHNITZER and WEINBERG [6] derive Ward identities from current algeb-

ras and use the meson dominance approximation in the form of a smoothness condition for the proper vertex functions. The results of this method are identical with those given above.

In [4] current algebra, PCAC and extrapolation in the π mass are used to obtain the subtraction constants for some form factors. (We note that our calculation yields unsubtracted (subtracted) dispersion relations for those form factors where [4] assumes unsubtracted (subtracted) form factors, but we get this as a result from our equations without the necessity for assumptions.) Divergence conditions for the currents and as additional information both WEINBERG sum rules [12] are used to obtain a definite answer for the ϱ and A_1 width. So this calculation uses more assumptions than those mentioned above. However, if only the first WEINBERG sum rule is exploited, the results are again identical with those of the above-mentioned works.

We have reviewed the existing calculations of the system $A_1\sigma\pi$ which are based on current algebras and conclude that the results are in good order of magnitude agreement with experiment. Results of [3, 4, 6, 7], although obtained by different methods, agree exactly with one another.

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СИСТЕМА $A_1\varrho\pi$

Ф. ЧИКОР

Резюме

Осматриваются вычисления системы $A_1\varrho\pi$ методами алгебры токов и сравниваются результаты разных методов.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 181—189 (1969)

EQUAL-TIME BEHAVIOUR OF CURRENT COMMUTATORS IN PERTURBATION THEORY*

By

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The singular equal-time behaviour of the vacuum expectation value of current commutators is determined for a field-theoretic model in second order perturbation theory. Using the well-known properties of the Wightman functions involved and relations for generalized functions studied in detail by GELFAND and SHILOV it is shown in an unambiguous way that a gradient term appears whose coefficient has the singular behaviour P/z_0^2 and $\log|z_0|$ with respect to the relative time (the first term is also found in second order quantum electrodynamics). The relation to the T-product of currents is investigated and defined within a special class of test functions. Furthermore, it is shown that the divergence of the T-product of conserved vector currents is localized in the sense of quasi-local operators involving arbitrary renormalization constants.

I. Introduction

Recently various methods for the calculation of equal-time commutators of currents have been proposed [1—8]. They led to different results caused by questionable limiting processes including the shrinking procedure for the test function with respect to the time coordinate. The present paper is devoted to a mathematically unambiguous determination of the singular equal-time behaviour of the vacuum expectation value of current commutators in a field-theoretic model where the necessary calculations can be carried out in a transparent way. For this we consider the first order currents of a field theory which couples a charged scalar field to a neutral massive vector field. Using the well-known properties of the Wightman functions involved and relations for generalized functions studied in detail by GELFAND and SHILOV [9] we show in an unambiguous way that the singular equal-time behaviour is determined by a gradient term whose coefficient is singular with respect to the relative time in the form P/z_0^2 and $\log|z_0|$. We mention that the first term appears also in second order quantum electrodynamics. The relation to the T-product given by the formal expression of JOHNSON and LOW [1]

* Reported at the International Seminar on Theory of Elementary Particles in Varna (1968); a preliminary version was contributed to the International Conference on Elementary Particles at Heidelberg (1967).

$$\langle 0\,|[j_\mu(x), j_\nu(y)]|\,0\rangle_{x_0=y_0} = \Big\{ \lim_{x_0-y_0\to+0} - \lim_{x_0-y_0\to-0} \Big\} \langle 0\,|Tj_\mu(x)\,j_\nu(y)|\,0\rangle \quad (1)$$

is explored and defined within a special class of test functions. Thus, using the T-product, we arrive at a corresponding result loosing, however, the possibility to identify the singularities at $z_0 = 0$ in the sense of generalized functions. In this connection we discuss also some ambiguities caused by the performances of ill-defined limits as has been proposed in the literature. Finally, we consider the divergence relation for the renormalized T-product

$$\partial_\mu \langle 0\,|Tj_\mu(x)\,j_\nu(y)|\,0\rangle = \delta(x_0 - y_0)\,\langle 0\,|[j_0(x), j_\nu(y)]|\,0\rangle\,. \quad (2)$$

We find that the left-hand side represents a quasilocal part, i.e. it consists of δ-functions and their derivatives with respect to the relative coordinate $z = x - y$ multiplied by arbitrary renormalization constants. Because of the singular equal-time behaviour of the current commutator the right-hand side of eq. (2) needs a redefinition which is determined by the renormalization prescription used on the left-hand side.

II. Equal-time commutator

The field-theoretic model we consider describes the interaction of charged scalar particles with neutral massive vector particles determined by the interaction Lagrangian

$$L_{\text{int}} = g{:}\varphi^x(x)\,\overleftrightarrow{\partial}_\mu\,\varphi(x)\,U_\mu(x){:} + g^2{:}\varphi^x(x)\,\varphi(x)\,U_\mu(x)\,U_\mu(x){:}\,. \quad (3)$$

The conserved current

$$j_\mu(x) = g{:}\varphi^x(x)\overleftrightarrow{\partial}_\mu\,\varphi(x){:} + g^2{:}\varphi^x(x)\,\varphi(x)\,U_\mu(x){:} \quad (4)$$

couples to the vector field $U_\mu(x)$. Second order perturbation theory yields for the vacuum expectation value of the current commutator

$$\begin{aligned} C_{\mu\nu}(z) &= \langle 0\,|[j_\mu(x), j_\nu(y)]|\,0\rangle = \\ &= -\,2\,g^2\{\partial_\mu\,\Delta^+(z)\,\partial_\nu\,\Delta^+(z) - \Delta^+(z)\,\partial_\mu\,\partial_\nu\,\Delta^+(z) - c.\,c.\} \end{aligned} \quad (5)$$

where $z = x - y$ and

$$\Delta^+(z) = \frac{-\,i}{(2\pi)^3} \int d^4\,k e^{-ikz}\,\Theta(k_0)\,\delta(k^2 - m^2)\,. \quad (6)$$

In terms of the Wightman function

$$W^{+}_{\mu\nu}(z) = \\ = \frac{g^2}{12(2\pi)^5} \int d^4q e^{-iqz}\, \Theta(q_0)\, \Theta(q^2 - 4m^2)\, (q_\mu q_\nu - q_{\mu\nu} q^2) \left(1 - \frac{4m^2}{q^2}\right)^{3/2} . \tag{7}$$

$C_{\mu\nu}$ has the structure

$$C_{\mu\nu}(z) = W^{+}_{\mu\nu}(z) - W^{+}_{\mu\nu}(-z) . \tag{8}$$

In the following we consider time and space components only where the spectral function behaves as constant at infinity. The integrable part of the spectral function, i.e. the part

$$\left(1 - \frac{4m^2}{q^2}\right)^{3/2} - \left(1 - \frac{6m^2}{q^2}\right), \tag{9}$$

contributes finite gradient terms to the equal-time commutator. The remaining more singular parts may be evaluated by means of the formulae

$$\begin{aligned} &\int d^4q e^{-iqz}\, \Theta(q_0)\, \Theta(q^2 - 4m^2)\, 1/q^2 = 4\pi\zeta^{-1} K_0(2m\, \zeta^{1/2}) , \\ &\int d^4q e^{-iqz}\, \Theta(q_0)\, \Theta(q^2 - 4m^2) = -\, 4\pi(2m)^2\, \zeta^{-1} K_2(2m\, \zeta^{1/2}) \end{aligned} \tag{10}$$

for $\zeta = -(z_0^2 - \vec{z}^2) > 0$. K_i denote the modified Hankel functions. Analytic continuation to $\zeta < 0$, $z_0 \gtrless 0$ is defined as usually. From eqs. (9) and (10) the singularities on the light cone can be exhibited explicitely so that we get

$$C_{0k} = \frac{g^2}{3(2\pi)^4} \left[\partial_0 \partial_k \left\{ \frac{1}{(-z^2 + i\varepsilon(z_0)\,\eta)^2} + \frac{3m^2}{2} \frac{\log(-m^2 z^2 + i\varepsilon(z_0)\,\eta)}{-z^2 + i\varepsilon(z_0)\eta} + \right. \right. \\ \left. \left. + \frac{d}{-z^2 + i\varepsilon(z_0)\,\eta} \right\} - c.\,c. \right] + r_{0k}(z) . \tag{11}$$

$\varepsilon(z_0) = sgn z_0$, $\eta \to +\,0$. The part (9) contributes to the real constant d whereas the expression $r_{0k}(z)$ denotes the regular part which vanishes for $z_0 = 0$. The singular terms of eq. (11) are considered as functionals on the space $S(R_4)$ of test functions defined on R_4. We may use these test functions in the form $g(\vec{z})l(z_0)$ of the space $S(R_3) \times S(R_1)$. Evaluating the corresponding functionals we arrive at the result

$$C_{0k}(z) = \frac{ig^2}{6(2\pi)^2} [\{P/z_0^2 - \tfrac{1}{2}\Delta - 6m^2 \log(m\,|\,z_0\,|\,) + d'\}\, \partial_k\, \delta(\vec{z}) + \\ + R_{0k}(z_0^2, \vec{z})] + r_{0k}(z) \tag{12}$$

with

$$R_{0k}(z_0^2, \vec{z}) = \sum_{n=1}^{\infty} a_n \frac{z_0^{2n}}{n!} \Delta^{n+1}\, \delta(\vec{z}) , \tag{13}$$

which vanishes for $z_0 = 0$ and is non-local in ordinary space. Up to the regular part $r_{0k}(z)$ the expressions (12) and (13) represent the Laurent series of $C_{0k}(z)$ with respect to z_0^2. The term P/z_0^2 belongs to the type of generalized functions studied in detail by GELFAND and SHILOV [9] and has the explicit meaning in R_1

$$(P/z_0^2, f) = \int_0^\infty dz_0 \frac{f(z_0) + f(-z_0) - 2f(0)}{z_0^2} \tag{14}$$

which defines the principal value for a second order pole.

We see that the singular behaviour of $C_{0k}(z)$ at $z_0 = 0$ is determined by a gradient term $\partial_k \delta(\vec{z})$ whose coefficient is singular with respect to the relative time in the form P/z_0^2 and $\log | z_0 |$. It is worthwile to mention also the term $\Delta \partial_k \delta(\vec{z})$. The relation (12) can be derived in R_3 for fixed $z_0 \neq 0$ and needs then a fixing of the singularity at $z_0 = 0$ according to the principal value (14) as it comes out in R_4. The singular part P/z_0^2 can also be found in second order quantum electrodynamics.*

We add the remark that the derivation of expressions (12) and (13) is based on formulae of the following type. If one considers, for instance, the contribution

$$\frac{1}{2\pi i} \partial_0 \left\{ \frac{1}{[z^2 + i\varepsilon(z_0)\eta]^2} - \frac{1}{[z^2 - i\varepsilon(z_0)\eta]^2} \right\} = \partial_0 \{\varepsilon(z_0)\, \delta'(z_0^2 - r^2)\} \tag{15}$$

one may derive in R_4 the relation [Appendix I]

$$\frac{1}{\pi} \partial_0 \{\varepsilon(z_0)\, \delta'(z_0^2 - r^2)\} = - P/z_0^2\, \delta(\vec{z}) - \frac{1}{z_0^2} \left[\frac{\delta(z_0^2 - r^2)}{2\pi r} - \delta(\vec{z}) \right] +$$

$$+ 4 \frac{\delta'(z_0^2 - r^2)}{2\pi r} + 4 z_0^2 \frac{\delta''(z_0^2 - r^2)}{2\pi r} . \tag{16}$$

We note the expansion

$$\frac{1}{2\pi r} \delta(z_0^2 - r^2) = \delta(\vec{z}) + \sum_{n=1}^{\infty} \frac{z_0^{2n}}{n!} \left(\frac{\Delta}{6} \right)^n \delta(\vec{z}) \tag{17}$$

which we employed in formulae (12) and (13).

* We mention that a logarithmic singularity with respect to the time behaviour was also exhibited for a special model by SCHROER and STICHEL [6] where a shrinking procedure with respect to the test functions of R_1 was used which we avoid completely. Without such a procedure the term P/z_0^2 can also be derived from BRANDT's formulae [5]. Compare also the paper by LUKIERSKI [4].

III. T-product and equal-time commutator

In this Section we investigate within our model the relations (1) and (2) connecting equal-time commutator and T-product of currents. For this we consider first the second order T-product of the vector-scalar theory (we always deal with vacuum expectation values only)

$$T_{\mu\nu}(z) = \langle 0 \,|Tj_\mu(x)\, j_\nu(y)|\, 0\rangle = \\ = \frac{g^2}{2}\{\partial_\mu \Delta_F(z)\,\partial_\nu \Delta_F(z)\,\partial_\mu \partial_\nu \Delta_F(z)\}, - \triangle_F(z), \tag{18}$$

where

$$\Delta_F(z, m) = \frac{2i}{(2\pi)^4}\int d^4k e^{-ikz}\,\frac{1}{k^2 - m^2 + i\varepsilon}\,. \tag{19}$$

Formula (18) is mathematically meaningless unless it is understood by means of regularization which has to be removed finally. We use the Pauli—Villars regularization defined by the substitution $\Delta_F \to \Delta_F^{\text{reg}}$, where

$$\Delta_F^{\text{reg}}(z, m) = \Delta_F(z, m) - \Delta_F(z, M). \tag{20}$$

The regularized T-product has the structure

$$T_{\mu\nu}^{\text{reg}}(z) = \frac{1}{(2\pi)^4}\int d^4p e^{-ipz}\,\Pi_{\mu\nu}^{\text{reg}}(p) \tag{21}$$

with

$$\Pi_{\mu\nu}^{\text{reg}}(p) = \frac{-4g^2}{(2\pi)^4}\int d^4k\left[\frac{k_\mu k_\nu}{\left(k - \frac{p}{2}\right)^2 - m^2} - \frac{k_\mu k_\nu}{\left(k - \frac{p}{2}\right)^2 - M^2}\right]\cdot \\ \cdot\left[\frac{1}{\left(k + \frac{p}{2}\right)^2 - m^2} - \frac{1}{\left(k + \frac{p}{2}\right)^2 - M^2}\right]. \tag{22}$$

For the special case of time and space components Feynman parametrization leads to

$$\Pi_{0k}^{\text{reg}}(p) = \frac{+ig^2}{4(2\pi)^2}\,p_0\,p_k\int_0^1 d\alpha(1-2\alpha)^2\cdot \\ \cdot\left\{\log\frac{\alpha(1-\alpha)\,p^2 - m^2}{\alpha(1-\alpha)\,p^2 - \alpha M^2 - (1-\alpha)\,m^2} + (m \leftrightarrow M)\right\}. \tag{23}$$

We note that analogously to the results of other papers one may derive non-unique results from the regularized T-product on the basis of formal manipulations. Performing the momentum integration in eqs. (21) and (23) for $z_0 \neq 0$ one gets

$$\lim_{z_0 \to 0} T_{0k}^{\mathrm{reg}}(z_0, \vec{z}) = 0. \tag{24}$$

If we regularize only one of the propagators in formula (22) (which means that we have to drop the term $(m \leftrightarrow M)$ in eq. (23)) we arrive at

$$T_{0k}^{\mathrm{reg}}(z) = \frac{ig^2}{2(2\pi)^2}\,\varepsilon(z_0)\left\{\left[M^2 \log(M|z_0|) - m^2 \log(m|z_0|) + \right.\right.$$
$$\left.\left. + (M^2 - m^2)\left(C + \frac{1}{2} + \frac{i\pi}{2}\right)\right]\partial_k\,\delta(\vec{z}) + 0(z_0)\right\} \tag{25}$$

(C denotes Euler's constant). From eq. (24) one may conclude that the T-product vanishes at equal times also in the limit where the regularization is removed. In this way one arrives at the surprising result that no renormalization is necessary to determine the equal-time behaviour of the T-product.

Hamprecht and Polkinghorne [2, 3] first pointed out that the equal-time commutator calculated via the T-product in a regularized field theory vanishes, provided the regularization procedure leads to a sufficiently strong damping to the asymptotic behaviour (as e.g. by appropriate Pauli—Villars regularization).

On the other hand the behaviour of the T-product near $z_0 = 0$ can be studied in such a way that first the regularization is removed. This means that now the T-product occurs in the standard renormalized form. In this way we arrive in momentum space at

$$\pi_{\mu\nu}^{\mathrm{ren}}(p) =$$
$$= \frac{-ig^2}{2(2\pi)^2}\left\{\int_0^1 d\alpha\left[\frac{(1-2\alpha)^2}{2}\,p_\mu p_\nu - \left(\alpha(1-\alpha)\,p^2 - m^2\right)g_{\mu\nu}\right]\right.$$
$$\left.\log\left(p^2 - \frac{m^2}{\alpha(1-\alpha)}\right) + C_1\,p_\mu p_\nu + g_{\mu\nu}\,(C_2\,p^2 + C_3)\right\} \tag{26}$$

and obtain the generalized function

$$T_{0k}^{\mathrm{ren}}(z) =$$
$$= \frac{ig^2}{12(2\pi)^2}\left\{\varepsilon(z_0)\left[\frac{1}{z_0^2} - \frac{1}{2}\Delta - 6\,m^2\left(\log(m|z_0|) + C + \frac{1}{2} + \frac{1\pi}{2}\right)\right]\partial_k\delta(\vec{z}) + \right.$$
$$\left. + \left[6\,c_1 - (2C\,i\pi)\right]\partial_0\,\partial_k\,\delta^4(z) + t_{0k}(z)\right\}, \tag{27}$$

where in eqs. (26) and (27) a polynomial or a quasilocal term involving arbitrary constants C_i is included, respectively [10]. The regular part $t_{0k}(z)$ vanishes for $z_0 = 0$. According to GELFAND and SHILOV [9] the relation

$$\frac{\varepsilon(z_0)}{z_2^0} = \frac{P_+}{z_0^2} - \frac{P_-}{z_0^2} \tag{28}$$

holds, where the principal values, P_+ and P_- are defined as follows

$$(P_\pm/z_0^2, f) = \int_0^\infty dz_0 \frac{f(\pm z_0) - f(0) \mp z_0 f'(0)\,\Theta(1 - z_0)}{z_0^2} . \tag{29}$$

Thus we are confronted with different equal-time results of the T-product. To clarify the situation we remind that actually we are dealing with improper limiting procedures which require the application of test functions and the consideration of the corresponding functionals. Taking this into account we have to study the following relation (compare Section II)

$$\int d^4 z\, T_{0k}^{\mathrm{reg}}(z)\, g(\vec{z})\, f(z_0) = \frac{1}{(2\pi)^4} \int d^4 p\, \Pi_{0k}^{\mathrm{reg}}(p) \tilde{g}(\vec{p}) \tilde{f}(p_0) \tag{30}$$

considered in ordinary and momentum space, respectively. Here, it is obvious that the removing of the regularization leads to the replacement of $\Pi_{0k}^{\mathrm{reg}}(p)$ by $\Pi_{0k}^{\mathrm{reg}}(p)$ in the standard way which is given by (26). The corresponding situation holds in ordinary space where the behaviour near $z_0 = 0$ is given by expression (27). On the other hand the proceeding in formula (24) means that we put in Eq. (30)

$$f(z_0) = \delta(z_0) \quad \text{or} \quad \tilde{f}(p_0) = 1 , \tag{31}$$

respectively. Then, however, the limit $M \to \infty$ in relation (30) becomes ill-defined. Thus we see that the ambiguities discussed follow from mathematically undefined limiting procedures. It should be noted in this connection that a similar objection holds against the argumentation of KALLÉN [8] where a regularization approach for the calculation of the equal-time commutator itself was applied after interchanging limiting processes (compare also the paper by BRANDT [5]).

Until now we considered the T-product itself. Let us now study the relation (1) in more detail. First we note that because of the symmetry relation

$$T_{0k}(z_0, \vec{z}) = - T_{0k}(-z_0, \vec{z}) \tag{32}$$

the proposal (1) leads in our case to the following connection between the equal-time commutator and the T-product

$$C_{0k}(z)|_{z_0=0} = 2\,\varepsilon(z_0)\,T_{0k}(z)|_{z_0=0}. \tag{33}$$

Since the T-product is multiplied by a discontinuous function relation (33) is obviously not well-defined. We stress that such products are not treated within the frame-work of standard renormalization theory.

However, we may still give a meaning to the expression (33) by using a special class of test functions (compare also the paper by BRANDT [5]). We define $\varepsilon(z_0)T_{0k}(z)$ on a space of test functions which vanish identically for $z_0 \leq 0$ or $z_0 \geq 0$, respectively. It is easy to see that this prescription yields for the right-hand side of eq. (33) the expression (27) multiplied by $2\varepsilon(z_0)$ where now the quasi-local operators do not contribute.* The non-local singular parts are in agreement with those of the left-hand side of eq. (33), given by formula (12). However, since all expressions are now considered on the special class of test functions introduced above there is actually no possibility to identify the singularities in the sense of generalized functions, i.e. to determine the local behaviour at $z_0 = 0$.

Finally let us consider the divergence relation (2). We observe that the right-hand side is not defined if one uses formula (12). This is not surprising because the left-hand side is based on a renormalization procedure which obviously requires also a redefinition of the right-hand side. The divergence $\delta_\mu T^{\text{ren}}_{\mu k}(z)$ ean be determined from eq. (26) in the form

$$\partial_\mu T^{\text{ren}}_{\mu k}(z) = \frac{-ig^2}{2(2\pi)^2}\,(C_1' + C_2' -)\,\delta^4(z)\,, \tag{34}$$

where the constants C_1' and C_2' involve the renormalization constants C_1, C_2 and C_3 used in expression (26). The corresponding redefinition of the right-hand side of eq. (2) has to reproduce this result. This prescription includes a definition for the products $\delta(z_0)P/z_0^2$ and $\delta(z_0)\log|z_0|$ in agreement with the procedure of standard renormalization theory.

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* We note for this that the test functions used vanish together with all derivatives at $z_0 = 0$.

ПОВЕДЕНИЕ КОММУТАТОРОВ ТОКА ПРИ РАВНЫХ ВРЕМЕНАХ В ТЕОРИИ ВОЗМУЩЕНИЙ

Ф. КАШЛУН, Е. ВИЦОРЕК и В. ЦЕЛЛНЕР

Резюме

Сингулярное поведение при равных временах вакуумного ожидаемого значения коммутаторов тока определяется для полево-теоретической модели во втором приближении теории возмущений. Используя хорошо известные свойства функций Уитмана и соотношения для обобщенных функций, подробно изученных Гельфандом и Шиловым, однозначно показывается, что появляется один градиентный член, коэффициент которого по отношению относительного времени имеет сингулярность типа P/z_0^2 и $\log(z_0)$ (первый член найден также во втором приближении квантовой электродинамики). Соотношение к T-произведении токов исследуется и определяется на базе специального класса пробных функций, Далее показывается, что расходимость T-произведения сохраненных векторных токов локализирована в смысле квази-локальных операторов, содержащих произвольные ренормализационные постоянные.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 191—199 (1969)

REVIEW OF CURRENT ALGEBRA RESULTS IN THE WEAK DECAYS OF THE METASTABLE HADRONS

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Without claiming completeness the main achievements in the applications of the current algebraic approach to weak decays are surveyed.

I. Introduction

It is generally accepted that the considerations based on the current algebra generated by weak currents and on the hypothesis of a partially conserved axial-vector current (PCAC) are highly fruitful [1]. In particular, the soft-pion method has led to successful non-trivial relations between the matrix elements of different decay processes. (See [2]—[7]).

These relations, however, hold at non-physical points and problems arise how to extrapolate them to the physical region. Recently, many attempts have been made to exploit current algebraic relations without taking the pions as soft (hard pion methods [8], [9]).

The commutation relations of current densities have also been exploited in the evaluation of electromagnetic corrections to the weak hadronic decay amplitudes and some new interesting results have been obtained.

II. Relations based on the soft-pion theorem

In the soft-pion method the current algebra and the PCAC hypothesis are applied to connect the matrix elements $\langle \pi b \mid O \mid a \rangle$ and $\langle b \mid O \mid a \rangle$ of an operator O; reduction of the pion gives

$$\lim_{k \to 0} \sqrt{2k_0} \langle \pi k, b \mid 0 \mid a \rangle = - i \frac{\sqrt{2}}{f\pi} \langle b \mid [A_i^{(0)}, 0] \mid a \rangle + \lim_{k \to 0} \frac{\sqrt{2}}{f\pi} k_\mu T_\mu , \qquad (1)$$

where A_i is the axial charge

$$A_i(t) = - i \int d^3 x A_{i4}(x, t),$$

f_π is defined as follows:

$$H_W = \frac{G}{\sqrt{2}} (J_\mu + j_\mu)(J_\mu + j_\mu)^*, \; J_\mu = A_\mu + V_\mu ,$$

$$\langle 0 \mid A_\mu(0) \mid \alpha k \rangle = \frac{i}{\sqrt{2k_0}} k_\mu f_\alpha k \tag{2}$$

$$f_\pi = f_{\pi^+}$$

and

$$T_\mu = - \int d^4x e^{-ikx} \Theta(x_0) \langle b \mid [A_{i\mu}(x), O] \mid a \rangle .$$

If $\lim_{k \to 0} k_\mu T_\mu \to 0$, then equation (1) connects the weak amplitudes $A(a \to b)$ and $A(a \to b + \pi)$. If the behaviour of T_μ is singular at $k = 0$, special considerations are needed in evaluating the pole contributions to the surface term $k_\mu T_\mu$.

By applying this method to the K_{l_3} decay we get the CTMOP relation [2]:

$$f_+(-m_K^2, m_\pi = 0) + f_-(-m_K^2, m_\pi = 0) = - \frac{f_K}{f_\pi \sqrt{2}} , \tag{3}$$

where f_π, f_K, f_+, f_- are the K_{l_2} and K_{l_3} form factors, respectively:

$$\langle \pi_\beta k_2 \mid V_\mu(0) \mid K_\alpha k_1 \rangle = \frac{1}{\sqrt{4k_{10} k_w}} [f_+(q^2) k_\mu + f_-(q^2) q_\mu] , \tag{4}$$

where

$$q = k_1 - k_2 ; \quad k = k_1 + k_2 .$$

In the case of K_{l_4} decay we have:

(a) $$F_1 = \frac{\sqrt{2}\, m_K}{f\pi} \delta_{i_1 i_2} \delta_{mn} f_+(0, -m_\pi^2) ,$$

(b) $$F_2 = - \frac{\sqrt{2}\, m_K}{f\pi} \varepsilon_{i_1 i_3 i_3} (\tau_{i_3})_n^m f_+(0, -m_\pi^2) , \tag{5}$$

(c) $$F_3 = 0, \quad f_+(0, -m_\pi^2) + f_-(0, -m_\pi^2) = 0, \quad \xi = \frac{f_-}{f_+} = -1 ,$$

where F_i's are the corresponding axial vector form factors in the process K_{l_4}:

$$\langle i_1 k_1 i_2 k_2 \mid A_\mu(0) \mid K_n k \rangle = \frac{1}{\sqrt{8k_0 k_{10} k_{20}}} [F_1(k_1 + k_2)_\mu +$$

$$+ F_2(k_1 - k_2)_\mu + F_3(k - k_1 - k_2)_\mu] .$$

The relations (5) are obtained if only one of the pions is reduced at one time and the F_i's are assumed to be constant.

WEINBERG has shown [3] that if the two pions are treated symmetrically (by reducing both of them simultaneously) and the σ-term is neglected, the K-pole contribution makes F_3 essentially momentum-dependent and that is why one obtains the relations (5, c) on assuming a constant F_3. In a paper, S. BERMAN and PROBIR ROY [11] criticize WEINBERG's result and consider the relations (5, c) to be right. They call attention to the crucial role of the σ-term which, if non-vanishing, makes F_3 to blow up at the double limit

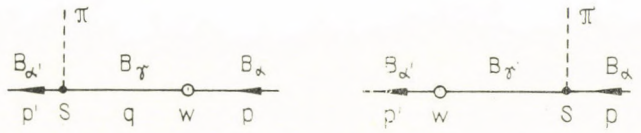

Fig. 1

$k_1 \to 0$, $k_2 \to 0$. At the same time they emphasize that the (5, c) relations can be reconciled with the CTMOP relations if the K_{l_3} form factors have an appropriate m_π-dependence.

As to the non-leptonic decays, the $K_{3\pi} - K_{2\pi}$ relations ([4], [5]) seem to be all right.

The problem of the non-leptonic hyperon decays, however, though it has been attacked by many authors ([6], [7], [12], [13], [15]), cannot be considered as satisfactorily settled up to now. When applying the soft-pion method one must take baryon poles into account [14]. This can be done by assuming that it is the difference of the amplitude.

$$A(k, p', p) = \sqrt{2k_0}\,\langle \pi_i\, k, B_{\alpha' p'} \,|\, H_w \,|\, B_{\alpha\, p}\rangle$$

and the Born term $B(k, p', p)$ given by the diagrams in Fig. 1 — and not $A(k, p', p)$ itself — which can smoothly be extrapolated to $k = 0$. In other words we assume that (c.f. (1))

$$\begin{aligned} A(k, p', p) - B(k, p', p) \approx -\frac{i}{f\pi}\langle p' \,|\, [A_i, H_w] \,|\, p\rangle + \\ + \lim_{k\to 0} [k_\mu T_\mu - B(k, p', p)] \end{aligned} \tag{6}$$

with

$$T_\mu = -\frac{1}{f\pi}\int e^{-ikx}\,\Theta(x_0)\,\langle p' \,|\, [A_{i\mu}(x), H_w(0)] \,|\, p\rangle$$

In order to calculate the Born term and the r.h.s. of Eq. (6) we must evaluate matrix elements of the type $\langle B' \mid Hw \mid B \rangle$. Since these are not known experimentally, the first analyses were made resorting to SU(3) symmetry [6]. From these calculations one gets a two parameter fit, which gives the Lee—Sugawara relation for the s-wave amplitudes; the simultaneous fit of S-and P-waves, however, is far from being in agreement with the experimental data.

It is very likely that the SU(3) breaking effects play an essential role. We get an SU(3) violating contribution from the K-pole diagram of Fig. 2,

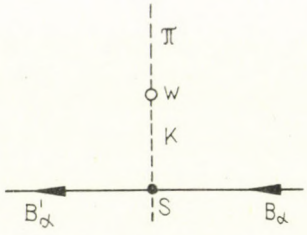

Fig. 2

and taking this into account we arrive at a three parameter fit [7]. Nevertheless, even in the improved fit there are discrepancies as large as 50% between theoretical and experimental values.

A systematic treatment of the SU(3) breaking effects was undertaken in [12] with a relatively good reproduction of the experimental data; it seems however, that the author made errors in the numerical calculations (cf. [13]). In a recent critical study of non-leptonic decays, OKUBO [15] casts doubts on the applicability of the standard current algebra technique to hyperon decay problems.

III. Hard-pion methods

The soft-pion limit can be abandoned if one makes some assumptions about the structure of the propagator and vertex functions occurring in the Ward type relations obtained by the use of the current commutators. Such an assumption is the so-called "smoothness conditions" [8] which implies that the various vertex functions in question depend as smoothly on the momenta as allowed by the Ward type identities. We mention only two applications of this technique. One is the calculation of the K_{l_3} form factor $f_+(0)$ by GLASHOW and WEINBERG [16]. In this work a model is proposed for the violation of the SU(3) $\otimes$ SU(3) symmetry, by assuming a Lagrangian of the form

$$L = L_0 + \varepsilon_i \varphi_i,$$

where L_0 is SU(3) $\otimes$ SU(3) invariant and the Φ_i's are local fields forming a basis of the representation $(3, 3^*) \otimes (3^*, 3)$ (i.e. they form a scalar and a pseudoscalar nonet). By exploiting the smoothness conditions one arrives at the relation

$$F_+(0) = \frac{F_\pi^2 + F_K^2 - F_\varkappa^2}{2F_K F_\pi},$$

where F_+, F_π and F_K are connected with the form factors f_+, f_π and f_k defined by Eqs. (2) and (3), respectively, as follows:

$$\begin{aligned} f_+ &= \frac{1}{\sqrt{2}} F_+ \sin\Theta = \frac{1}{\sqrt{2}} \sin\Theta_V, \\ f_\pi &= F_\pi \cos\Theta = F\cos\Theta_A, \\ f_K &= F_K \sin\Theta = F\sin\Theta_A. \end{aligned} \tag{7}$$

Θ is the Cabibbo angle, $F_\varkappa$ is the amplitude for the scalar kanon ($\varkappa$), analogous to F_K. The relations (7) give

$$\frac{F_K}{F_\pi F_+(0)} = \frac{\operatorname{tg}\Theta_A}{\operatorname{tg}\Theta_V} = 1{,}28.$$

By making use of the first and second sum rules of WEINBERG and saturating them with K, K^*, K^{**} and K_A (1250) the authors obtain

$$\frac{F_K^2}{F_\pi^2} = 1{,}17, \quad F_+(0) = 0{,}85.$$

The interesting thing is that second order SU(3) breaking effects may be significant for $F_+(0)$ according to this result. If one assumes that there is no $\varkappa$ at all, i.e. $F_\varkappa = 0$, one gets $F_+(0) = 1.05$. Such a possibility is in contradiction with an inequality of BJÖRKEN and QUINN [17] which states that $F_+(q^2) \leq 0$ for $q^2 \geq 0$.

The hard-pion treatment of the K_{l_4} form factors was undertaken by SARKER [18]. He got very good agreement with the experimental $K^+ \to \pi^-\pi^+e^+\nu$ width for $\xi = f_-(0)/f_+(0) = 0.6$, but regarding the numerous assumptions he was compelled to make in the various phase of the calculation, it is hard to say, how significant this agreement is.

We think that a general remark is proper at this place. The current algebraic approach is based on several simple ideas, such as well defined equal time commutators, CVC, PCAC. The simple-minded application of these ideas leads to relations which are more or less in agreement with experience

or, if the disagreement is too large the reason for this can be guessed. Now, if — in order to improve numerical coincidence of theoretical and experimental values of certain quantities — one starts to introduce sophisticated ideas and accumulate assumptions a priori approximately satisfied one sooner or later loses control of what is checked and how reliable the various predictions are. One ought to spend at least the same amount of energy on checking the reliability of the assumptions made as that spent in reproducing the experimental data as closely as possible.

Results similar to those mentioned above can also be obtained by a dispersion theoretical approach. In this connection, we refer to the papers of I. MONTVAY [19] and T. NIEH [20].

IV. The electromagnetic corrections

It is well known that the radiative corrections are finite for the muon lifetime, but they turn out to be divergent for the hadronic weak amplitudes, if hadrons are treated as pointlike particles. It might be thought that divergences would disappear if one took strong interactions into account. The framework of current algebra has provided a means to estimate the strong effects; the calculations led to the result that the divergent corrections to the vector part of the amplitudes were independent of the details of the strong interactions ([21], [22], [23]).

We outline briefly the procedure of the exploitation of current algebra on the example of β decay. (The lepton momenta are neglected and we disregard trivial factors.)

The uncorrected amplitude has the form

$$M_0 \sim \bar{u}_e \gamma_\mu (1 + \gamma_5) v \langle p | J_\mu | p \rangle . \tag{8}$$

The corrections are described by the three diagrams given in Fig. 3. Their contribution can be given as follows:

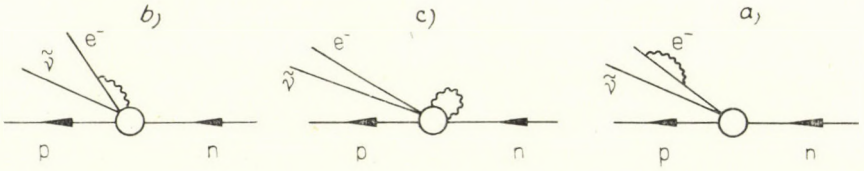

Fig. 3

Diagram a)

$$\delta M_1 \sim \bar{u}_e \int d^4k \, D_{\mu\nu}(k) \gamma_\mu S(k + q) \gamma_\nu S(q) \gamma_\lambda (1 + \gamma_5) v \langle p | J_\lambda | p \rangle .$$

Diagram b)

$$\delta M_2 \sim \bar{u}_e \int d^4k\, D_{\mu\nu}(k)\, \gamma_\mu\, S(k)\, \gamma_\lambda(1+\gamma_5)\, v\, \Gamma_{\nu\lambda}(p,k).$$

Diagram c)

$$\delta M_3 \sim \bar{u}\, \gamma_\lambda(1+\gamma_5)\, v \int d^4k\, \Theta_{\mu\nu}(k)\, \Gamma_{\lambda\mu\nu}(p,k),$$

where

$$D_{\mu\nu}(k) = \delta_{\mu\nu} \frac{1}{k^2 - x^2} + \text{soft photon term}$$

$$S(p) = i \frac{i\hat{p}^1 - m}{p^2 + m^2},$$

$$\Gamma_{\mu\nu}(p,k) = \int d^4x\, e^{ikx} \langle p \,|\, T J_{e\mu}(x)\, J_\nu(0) \,|\, p\rangle,$$

$$\Gamma_{\mu\nu\lambda}(p,k) = \int d^4x\, d^4y\, e^{ik(x-y)} \langle p \,|\, T J_{e\mu}(x)\, J_{e\nu}(y)\, J_\lambda(0) \,|\, p\rangle.$$

The axial vector part of J_μ makes a contribution only to diagram *b*, and its contribution to $\Gamma_{\mu\nu}$ is non-zero only if it is coupled with the isoscalar part of J_e. In the vector part of $\Gamma_{\mu\nu}$ only the isovector electromagnetic current contributes; let us denote this expression by $\Gamma^{3+}_{\mu\nu}$. Defining

by
$$\Gamma_{\mu\nu\varrho}(p,k,q)$$
$$\Gamma_{\mu\nu\varrho}(p,k,q) = \int d^4x\, d^4y\, e^{i(k+q)x} e^{-iky} \langle p \,|\, T J_{e\mu}(x)\, J_{e\nu}(y)\, J_\varrho(0) \,|\, p\rangle$$

and using current algebra, we get the relation

$$q_\lambda\, \Gamma_{\lambda\mu\varrho} = -\, i\, \Gamma^{3+}_{\mu\varrho}(p,-k) - i\, \Gamma^{3+}_{\mu\varrho}(p,k+q).$$

Differentiating with respect to q_μ, in the limit $q \to 0$ we obtain

$$\Gamma_{\mu\nu\varrho}(p,k) = -\, i \frac{\partial}{\partial k_\mu} \Gamma^{3+}_{\nu\varrho}(p,k). \tag{9}$$

Thus, the contribution of diagram c) is reduced to an expression similar to that of diagram b).

Taking only the vector part we are left with the following divergent term in the sum of the contributions of the three diagrams

$$\delta M^V = M_0 \frac{3\alpha}{8\pi} \ln \frac{\Lambda^2}{\lambda^2}, \tag{10}$$

where Λ is an ultraviolet cut-off.

The high energy part of the axial vector contribution can be connected with the equal time commutator of the space components of $V(x)$ and $A(x)$. In a model which contains only particles of isospin 1/2, we get

$$[V_{S,1}(x,t), A_{+2}(0)] = 2\bar{Q}\, V_{+3}(0)\, \delta^3(x)\,, \tag{11}$$

where $\bar{Q}$ is the mean charge of the isodoublet. Using Eqs. (11) one obtains the axial divergent term in the form

$$\delta M^A = 2\bar{Q}\, \delta M^V$$

and combining this with Eq. (10) we have

$$\frac{\delta M}{M_0} = \frac{3\alpha}{8\pi} \ln \frac{\Lambda^2}{\lambda^2} (1 + 2\bar{Q})\,.$$

This result shows that the total correction is model dependent. In the usual quark model $\bar{Q} = 1/6$, but models can be constructed where the vector and axial vector divergent parts cancel each other (see [23]).

Finally, we should like to mention an interesting work by BAILIN [24] who used the field algebraic commutation relations of LEE [25] to estimate electromagnetic corrections. LEE had postulated that the total electromagnetic current and the *total* weak current were proportional to a neutral and a charged vector boson field, respectively, and he had obtained the following commutators:

$$\left[\frac{\partial}{\partial x_0} J_{e\mu}(x)\,,\, J_{e\mu}(y)\right] \delta(x_0 - y_0) = \Omega\, \delta(x - y)\,;$$

$$[J_\mu(x), J_{e\lambda}(y)]\, \delta(x_0 - y_0) = i(\delta_{\lambda 4} J_\mu + \delta_{\mu 4} J_\lambda - \delta_{\lambda 4}\, \delta_{\mu 4}\, J_4)\, \delta(x - y)\,.$$

Having started with these relations, BAILIN showed that the electromagnetic divergent correction is the same even for *weak* process, namely

$$\frac{\delta M^{\mathrm{div}}}{M_0} = \frac{3\alpha}{8\pi} \ln \frac{\Lambda}{m}\,,$$

(m is some mass parameter). If, following LEE, one assumes that the electromagnetic interaction goes through the chain: particle $\leftrightarrow W_0 \leftrightarrow \gamma \leftrightarrow W_0 \leftrightarrow$ particle, the effective photon propagator behaves like k^{-6} as $k^2 \to \infty$ and the general correction will be finite.

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ИЗЛОЖЕНИЕ РЕЗУЛЬТАТОВ АЛГЕБРЫ ТОКОВ В СЛАБОМ РАСПАДЕ МЕТАСТАБИЛЬНЫХ ГАДРОНОВ

З. КУНСТ и Т. НАДЬ

Резюме

Без требования полноты исследуются главные достижения применений приближения алгебры токов в слабых распадах.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 201—204 (1969)

PROBLEMS OF THE THEORY OF WEAK INTERACTIONS

By

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It is not a theorem that no conventional theory of weak interactions exists.

The leptonic decays of hadrons may be (at least approximately) described by an interactions Hamiltonian

$$H_\tau = \frac{G}{\sqrt{2}} \int d\vec{x}\, j_\mu^{(h)}\, j_\mu^{(l)+} + \text{Hermitian Conjugate}\,, \tag{1}$$

where $j_\mu^{(h)}$ is the hadronic and $j_\mu^{(l)}$ the leptonic current. The small constant G is given by

$$G \sim 10^{-5}/M_N^2\,, \tag{2}$$

where M_N is the nucleon mass. We use units where $h = c = 1$.

If one tries to formulate a field theory which will give rise to the phenomenological interaction (1), one is naturally led to one of two theories:

(a) The original Fermi field theory, in which one postulates an interaction Lagrangian density

$$L_1 = -\frac{G}{\sqrt{2}}\, j_\mu j_\mu^+\,, \tag{3}$$

where

$$j_\mu = j_\mu^{(h)} + j_\mu^{(l)}.$$

(b) An intermediate vector boson theory with

$$L_I = -g\,[j_\mu A_\mu^+ + H.c.]\,, \tag{4}$$

where A_μ is a charged vector boson field.

The interaction (4) gives rise, in second order, to an effective Fermi coupling

$$G/\sqrt{2} = g^2/M_V^2\,, \tag{5}$$

(where M_V is the mass of the vector boson) provided the energy and momentum transfer of the reaction are much less than M_V. This is analogous to the

direct Moller interaction between electrons which is generated by the basic coupling of electrons with the electromagnetic field.

Both the interactions (3) and (4), if taken very seriously, give rise to nonrenormalizable theories. Such theories require an infinite number of arbitrary subtraction constants. This in itself is not a compelling argument against them. There is, however, a more physical way of stating the difficulty, to wit: No matter how weak the coupling, at a sufficiently high energy the coupling becomes so strong that its virtual effect in higher order can drastically change the low-energy predictions of the original theory.

We consider as an example the electron—neutrino ($e - \nu$) system. The original Lagrangian is designed to give $e - \nu$ scattering of order G, but no $\nu - \nu$ or $\nu - \bar{\nu}$ scattering. In lowest order, the $e - \nu$ scattering amplitude turns out to be purely s-wave, and equal to (in the Fermi theory)

$$F_S = \sqrt{2}\, Gq/\pi\,, \tag{6}$$

where q is the center-off-mass momentum. Since unitarity requires all partial wave amplitudes to be bounded by $1/q$, we find that the first-order theory breaks down by $q = q_m$, where

$$\sqrt{2}\,\frac{Gq_m}{\pi} = \frac{1}{q_m}\,. \tag{7}$$

This maximum energy is of the order of four or five hundred BeV.

If we now calculate $\nu - \nu$ scattering, we find (if we cut off all integrals at q_m)

$$F_{\nu\nu} \sim 2\,\frac{G^2 q_m}{\pi^2} \tag{8}$$

or

$$F_{\nu\nu} \sim F_{e\nu}/\pi\,. \tag{9}$$

The experimental limit on $\nu - \nu$ scattering (or even elastic $\nu - p$ scattering) is not very good. There do exist weak processes, however, which vanish according to first-order theory, are permitted in second order, but are known to be much smaller than first-order weak. For example, $\Delta S = 2$ transitions, as in the $K_1 - K_2$ mass difference, and neutral $\Delta S = 1$ currents, as in $K^\circ \to \to \mu^+ - \mu^-$. The experimental limits on these processes have been used by IOFFE and SHABALIN to show that the cutoff produced by nature is in fact much less than the q_m of equation (7). The importance of the work of IOFFE and SHABALIN is that their results are independent of the strong interactions except for the assumption of a conventional current algebra for $j_\mu^{(h)}$. Their results are as follows:

(a) For the Fermi theory, Eq. (3)

$$\frac{1/\tau(K_2^\circ \to \mu^+ + \mu^-)}{1/\tau(K^+ \to \mu^+ + \nu)} = \left(\frac{\sqrt{2}\, G\, \Lambda^2}{\pi^2}\right)^2 . \tag{10}$$

(b) For the vector boson theory, Eq. (4),

$$\frac{1/\tau(K_2^\circ \to \mu^+ + \mu^-)}{1/\tau(K^+ \to \mu^+ + \nu)} = \left(\frac{\sqrt{2}\, G\, \Lambda^2}{(4\pi)^2}\right)^2 , \tag{11}$$

where Λ is the cutoff (i.e. the momentum at which the theory becomes less singular). The experimental limit is $\sim 10^{-6}$. This yields, for case (a), $\Lambda < 30$ BeV, and for case (b) $\Lambda < 120$ BeV.

(c) For the vector boson theory (ignoring a highly model dependent Λ^4 term)

$$m_{K_1} - m_{K_2} \cong \frac{1}{2m_k}\left(\frac{G}{\sqrt{2}}\right)^2 \frac{\Lambda^2}{8\pi^2} \langle K^\circ \,|\, j_\mu^{K_0}(0)\, j_\mu^{K_0}(0) |\, \bar{K}^\circ \cdot \rangle , \tag{12}$$

where $j_\mu^{k_0}$ is the neutral isotopic partner of the $\Delta Q = \Delta S = 1$ current. The matrix element on the right of (12) is hard to evaluate; indeed, it is probably divergent (i.e. diverges with Λ^2). IOFFE and SHABALIN therefore prefer to disregard it. However, a very conservative estimate, based on the simplest intermediate states, gives

$$\left(\frac{\Lambda}{m_\mu}\right)^2 = \pi \frac{\tau_{K+\to\mu+\nu}}{\tau_{K_1}} , \tag{12'}$$

where $1/2\ \tau_{K_1}$ is simply a mnemonic for $m_{K_1} - m_{K_2}$, and $1/m_\mu^2 \tau_{K+\to\mu+\nu}$ is our estimate of the controversial matrix element. Here m_μ is the muon mass. The limit given by (12) is $\Lambda \sim 4$ BeV!

The fascinating question is: how does nature produce such a low cutoff? One may even ask: is it a theorem that there cannot be a Lagrangian theory of weak interactions consistent with present experiments?

The answer is, it is not a theorem. There appear to be at least two classes of theories (aside from the statement that since perturbation theory does not apply, one cannot *prove* there *is* a difficulty) which do not lead to experimental contradictions. We may call these

(i) deception (as in conspiracy, evasion, etc.), and
(ii) careful planning.

(i) Deception might, for example, mean that what appears to be a vector interaction is really a scalar, as suggested, for example, by TANIKAWA (who considered a Fierz transformation responsible for the deception), or by

KUMMER and SEGRE (who supposed that the observed weak interaction was a fourth order effect due to an $S + iP$ interaction). It has been shown by NORMAN CHRIST that the KUMMER—SEGRE model, suitably modified, can be made consistent with present experiments, although it does require considerable agility. Of course, the idea of deception is contrary to our usual experience but that is a religious argument.

(ii) A careful planning method is described in a paper to be published in collaboration with GELL-MANN, GOLDBERGER and KROLL. The main idea is the following: the difficulty of the vector theory is contained in the presence of a term in the propagator

$$\underset{q\to\infty}{\Delta^V_{\mu\nu}(q)} \to \frac{q_\mu q_\nu}{q^2} \cdot \frac{1}{m_y^2},$$

which is very singular at high energies. (As shown by OKUBO, this term is present irrespective of the structure of the vector field and is not therefore curved by strong bi-linear interactions of the latter.) Now a derivate coupled scalar also gives rise to an effective propagator

$$\Delta^5_{\mu\nu} \to \frac{q_\mu q_\nu}{q^2}.$$

Question: can they be subtracted? Answer: not for all processes. However, it is possible to construct many models in which the scalars and possible other vectors are coupled in such a way that in all quantum-number violating processes the singular behavior does cancel when the virtual momenta become larger than the scalar and vector masses. Such models involve very careful planning, in that the coupling of the scalars and vectors to the weak current and to each other must be very precisely related; how they acquire this relationship is not stated in any simple or convincing way.

I will not give examples here; they will appear in a publication. The main conclusion at this point should be, I believe, not that one can construct a beautiful theory of weak interactions, but that it is not a theorem that no conventional theory exists.

ПРОБЛЕМЫ ТЕОРИИ СЛАБЫХ ВЗАИМОДЕЙСТВИЙ

Ф. Е. ЛАУ

Р е з ю м е

Нет теоремы, что не существует конвенциональной теории слабых взаимодействий.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 205—215 (1969)

A MODEL OF MAXIMAL CP VIOLATION*

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A model of weak interaction is introduced which shows maximal CP violation. The model contains an octet (or nonet) of intermediate vector meson fields with a strong tri-linear self-coupling. First order terms are forbidden, while second order terms give CP = 1, third order terms CP = —1 effects. The smallness of CP-violating effects is thus easily explained.

§ 1. Introduction

The history of weak interaction is full of surprises. First, it is found to possess a kind of universal coupling between (ev), $(\mu\nu)$ and (pn) pairs, culminating finally to the introduction of the Cabibbo angle. Then, the parity (P) and the charge conjugation (C) invariances were discovered to be violated, leading to two-component neutrino and the successful V—A theory. Soon, the experiment showed existence of two different types of neutrinos which we may call muonic ν_μ and electronic ν_e neutrinos, respectively. More recently, the time reversal or equivalently the product CP by TCP theorem turned out to be also violated. The surprising fact about this CP violation is its smallness by a factor of 10^{-3} in comparison with the CP-conserving part. Note that the violation of P and C is very large in a sense that it enters in the famous combination $(1 + \gamma_5)$.

Since the discovery of the CP violation, many theoretical models have been presented by various authors. They may be roughly classified into the following categories:

(i) We assume [1] rather in an ad-hoc way an existence of small CP-violating weak Hamiltonian in addition to the dominant CP-conserving one.

(ii) The CP violation may be introduced [2] by a distortion due to Cabibbo rotation between CP conserving and violating currents.

(iii) The electromagnetic (or even strong) interaction may violate [3] the charge conjugation invariance of theory. Through the combined effect of both electromagnetic and CP-conserving weak interaction, we may produce an apparent violation of the CP.

(iv) There may be a super-weak interaction [4], whose existence affects the $K_1^0 - K_2^0$ mixing suitably, which leads also to an apparent CP violation

* Work supported in part by the U. S. Atomic Energy Commission.

in $K_2^0 \to 2\pi$ decay. The C violation of the gravitational interaction may produce a similar effect. However, it has been found to lead to experimentally incorrect results.

(v) The CP is maximally violated, although experimentally its observable effects are somehow very small. This may be achieved without introducing intermediate vector meson (hereafter W-meson) as in NISHIJIMA's theory[5] or with the W-meson as in this paper [6] or with other mesons [7].

Each model has its merits and defects, but we shall not go into detailed comparisons of these models. Here, I simply state the motivation for introducing my model. First, we notice a remarkable property of the V—A theory. It violates automatically P and C conservations of the theory in a very definite specific way. However, it is really not P or C per se which is maximally violated in the theory. Now, we know that the weak interaction is intimately related to the algebra of currents and to the chiral W_3 group, $W_3 = \mathrm{SU}_3^{(+)} \otimes \mathrm{SU}_3^{(-)}$. The nature violates the W_3 symmetry by choosing only the combination $(1 + \gamma_5)$ corresponding to $\mathrm{SU}_3^{(+)}$ group, rather than other combinations $(1 - \gamma_5)$. Hence, one may say that it is the chiral W_3 symmetry which is maximally violated in the weak interaction. Since C and P generate only discrete transformations, it seems natural to expect a similar maximal violation of the product CP if it is violated at all. Then the question is why the experimentally observed CP violation is so small. At first glance, this is insurmountable, but in reality it is a very simple matter to explain this fact in our model as we shall see shortly. In this connection, it may be observed that theories based upon a distortion of Cabibbo angle do not seem [2] to preserve the V—A structure. On the other hand, it is possible to preserve the V—A structure in the theory of the category (i) but the smallness of the CP violation may be a bit mysterious.

At any rate, we shall present a rough idea of our scheme. We assume the existence of the intermediate vector meson, and schematically we write

$$H_w = igW_\mu \cdot (J_\mu + l_\mu) + h.c. \tag{1}$$

where W_μ, J_μ and l_μ are W-field, hadronic current and leptonic current, respectively and where g is the semi-weak coupling constant. The factor i in the front of g is added so that H_w have CP $= -1$, i.e. the maximal CP violation. However, to give a definite meaning to the CP transformation property for the W-meson, we have to assume an existence of strong tri-linear self-coupling among W-field. This automatically implies an existence of at least three different W-fields $W^\pm$ and W°, so that the form Eq. (1) is rather shorthanded way of writing a more complicated Hamiltonian. Now, if one can show that the first order effect in g does not give any observable result, then all physically interesting effects should come from the second order g^2 and

the third order g^3. Since by assumption H_w has CP $= -1$, the g^2 and g^3 terms must have CP $= +1$ and CP $= -1$, respectively. Hence, if g is of the order 10^{-3} in some unit, one can explain both CP-conserving and violating weak interactions in a unified way. In my model, the forbiddenness of the first order effect is simply obtainable by the conservation of the charge and hypercharge. We remark that such an idea has been already proposed by NISHIJIMA [5]. However, his theory deals exclusively with non-leptonic part in contrast to the present model.

As we mentioned earlier, we need at least three different W-fields. Hence, the simplest choice is to assume a triplet of W-mesons. However, it so turned out that there appears to be no simple way to make a consistent theory, unless we double its number to six instead of three W-fields. Even in that case, the resulting theory is a bit awkward and here we shall assume that the W-meson is a member of an octet or nonet.

§ 2. A model of maximally CP-violating Hamiltonian

Our fundamental assumptions are as follows:

(i) Weak intermediate vector mesons exist and they are members of a unitary nonet or octet of Yang—Mills type. Also, the weak interaction Hamiltonian H_w has CP $= -1$. Further H_w consists of a sum of unitary octets alone, assigning leptons to unitary singlets.

(ii) Leptonic part of H_w has U-spin 1/2.

(iii) Non-leptonic part of H_w must have U-spin one.

(iv) W-mesons with $Q = Y = 0$ should not be contained in the weak interaction H_w but they may have strong and electromagnetic interactions with other hadrons [8].

(v) V—A theory with the ordinary Cabibbo angle.

Before going into details, we remark that because of our assumption (ii), only charged lepton currents enter in the weak interaction, thus excluding the presence of neutral lepton currents in agreement with the experiment. This is due to the fact that all particles with $U = 1/2$ must be automatically electrically charged. It may also be worthwhile to mention the fact that our motivation for assuming (ii) and (iii) is due to a consideration that the electromagnetic interaction has zero U-spin if we assign the same for the electromagnetic field.

Now, the first part of our assumption (i) implies the existence of a nonet of self-conjugate (hermitian) W-meson field $W_\mu^{(\alpha)}(x)(\alpha = 0,1, \ldots 8)$. The hermiticity condition gives

$$\overline{W}_\mu^{(\alpha)}(x) = W_\mu^{(\alpha)}(x), \tag{2}$$

where the adjoint $\overline{V}_\mu(x)$ of a vector $V_\mu(x)$ is defined as usual by

$$\overline{V}_\mu(x) = \varepsilon_\mu V_\mu^+(x), \qquad \text{(no summation over } \mu) \tag{3}$$

with $\varepsilon_\mu = 1$ $(\mu \neq 4)$ and $\varepsilon_4 = -1$. It is a bit more convenient to introduce a tensor notation by

$$(W_\mu)_b^a = \frac{1}{\sqrt{2}} \cdot \sum_{\alpha=0}^{8} (\lambda_\alpha)_{ba} \cdot W_\mu^{(\alpha)}(x), \qquad (a, b = 1, 2, 3). \tag{4}$$

Conversely $W_\mu^{(\alpha)}(x)$ is given by

$$W_\mu^{(\alpha)}(x) = \frac{1}{\sqrt{2}} \cdot \sum_{a,b=1}^{3} (\lambda_\alpha)_{ab} \cdot (W_\mu)_b^a(x), \qquad (\alpha = 0, 1, \ldots 8). \tag{5}$$

Further, it is convenient to regard $(W_\mu)_b^a(x)$ as a 3×3 matrix with respect to SU_3 indices "a" and "b" by

$$(W_\mu)_{ab}(x) \equiv (W_\mu)_a^b(x). \tag{6}$$

Now, let us introduce 3×3 matrix $F_{\mu\nu}$ by

$$F_{\mu\nu}(x) = \partial_\mu W_\nu(x) - \partial_\nu W_\mu(x) - \frac{i}{2} f_0 [W_\mu(x), W_\nu(x)], \tag{7}$$

where the coupling constant f_0 is supposed to be of the order unity. Then, when we make a local gauge transformation

$$W_\mu(x) \to W'_\mu(x) = S^{-1}(x) \cdot W_\mu(x) \cdot S(x) + \frac{2i}{f_0} \cdot S^{-1}(x) \cdot \frac{\partial S(x)}{\partial x\mu}, \tag{8}$$

$F_{\mu\nu}(x)$ transforms covariantly as

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = S^{-1}(x) \cdot F_{\mu\nu}(x) \cdot S(x), \tag{9}$$

where $S(x)$ is an arbitrary 3×3 matrix. Accordingly, the Yang—Mills Lagrangian is given by

$$\mathscr{L}_0 = -\frac{1}{4} Tr [F_{\mu\nu}(x) F_{\mu\nu}(x)] - \frac{1}{2} Tr [W_\mu(x) m_0^2 W_\mu(x)]. \tag{10}$$

We shall prove in the Appendix I that m_0^2 must be a multiple of a unit matrix if we demand $\partial_\mu W_\mu(x) = 0$ should follow from the Lagrangian Eq.(10). Hence,

we shall assume it hereafter in this note. Then, $\mathscr{L}_0$ given by Eq.(10) is invariant under Eq. (8) if $S(x)$ is independent of the co-ordinate x. This invariance is achieved without any corresponding change of other hadronic quantities. Thus, one can define two independent SU(3) spaces which we may call weak and hadronic SU(3) transformations. As a matter of fact, the weak SU(3) is the SU(3) group of W-meson field while the hadronic SU(3) is defined as that with respect to the ordinary hadrons. Hence, one can define two charges [6] Q_w, and Q_h, two hypercharges Y_w and Y_h, and two charge conjugations C_w and C_h of weak and hadronic spaces. The physical Q, Y and C are given now by

$$Q = Q_w + Q_h, \quad Y = Y_w + Y_h, \quad C = C_w \cdot C_h . \tag{11}$$

Now we remark that the Lagrangian Eq. (10) contains a desired trilinear coupling among W-field as is required. Since f_0 is the strong coupling constant, $\mathscr{L}_0$ must be invariant under both parity (P) and charge conjugation (C) operations. The inspection of Eq. (10) shows that W meson must have $P = -1$, i.e.

$$P: \; W_\mu(x) \to - \varepsilon_\mu \cdot W_\mu(-x) , \text{ (no summation over } \mu) . \tag{12}$$

As for the charge conjugation, it is a bit more complicated. We require that C must reverse its electric charge. Also, C must be an automorphism of the weak SU(3) group. The most general form satisfying these conditions are expressed [9] by

$$C: \; W_\mu(x) \to - S W_\mu^T(x) \, S^{-1}, \tag{13}$$

where S is an arbitrary 3×3 unitary matrix which commutes with the charge matrix

$$Q = 1/2 \left(\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right)$$

and where W_μ^T implies the transpose matrix of W_μ. Since the presence of the arbitrary matrix S can be removed by an SU(3) rotation, one may set $S = 1$. We call this to be a canonical choice. In the Appendix II, we shall prove that without loss of generality, one can always set $S = 1$. This is due to the fact that we may use the new field $\widetilde{W}_\mu(x)$ instead of $W_\mu(x)$, defined by

$$\widetilde{W}_\mu(x) = V^{-1} \cdot W_\mu(x) \, V,$$

if necessary. Choosing V suitably, one can reduce Eq. (13) into the canonical form of $\widetilde{W}_\mu(x)$, i.e.

$$C: \; \widetilde{W}_\mu(x) \to - \widetilde{W}_\mu^T(x) . \tag{13}'$$

Hence, using $\widetilde{W}_\mu(x)$ instead of $W_\mu(x)$, one can set $S = 1$ without the loss of generality. Further, we notice that due to the invariance of theory under the weak SU(3) group, we have

$$\begin{aligned} \langle W_\mu^{(\alpha)}(x) \cdot W_\nu^{(\beta)}(y) \rangle_0 &= \langle \widetilde{W}_\mu^{(\alpha)}(x) \cdot \widetilde{W}_\nu^{(\beta)}(y) \rangle_0 , \\ \langle W_\mu^{(\alpha)}(x) \cdot W_\nu^{(\beta)}(y) \cdot W_\lambda^{(\gamma)}(z) \rangle_0 &= \langle \widetilde{W}_\mu^{(\alpha)}(x) \cdot \widetilde{W}_\nu^{(\beta)}(y) \cdot \widetilde{W}_\lambda^{(\gamma)}(z) \rangle_0 . \end{aligned} \tag{14}$$

As we shall see shortly, these are all that we need to prove the equivalence of final results when we replace $W_\mu(x)$ by $\widetilde{W}_\mu(x)$ everywhere. This conclusion is also valid in the presence of the electromagnetic interaction, provided that it possesses zero U-spin. The reason for this is due to the fact that the matrix S in Eq. (13) is essentially a 2×2 unitary matrix corresponding to U-spin subgroup SU(2) and the electromagnetic interaction is invariant under U-spin transformation.

Now, let us define leptonic and hadronic currents by

$$\begin{aligned} l_\lambda(x) &= i \left[\bar{\nu}_e(x) \gamma_\lambda (1+\gamma_5) \, e(x) + \nu_\mu(x) \gamma_\lambda (1+\gamma_5) \, \mu(x) \right], \\ (j_\lambda)_b^a(x) &= i \bar{q}_a(x) \gamma_\lambda (1+\gamma_5) \, q_b(x) , \end{aligned} \tag{15}$$

where $q_a(x)$, $(a = 1, 2, 3)$ are the quark field and we have assumed the quark model for hadrons for the sake of the illustrations.

Under these preparations, only weak Hamiltonian satisfying our postulates stated in the beginning of this section is given by

$$\begin{aligned} H_w = ig \{ \xi (W_3^2 - W_2^3) \, j_a^a + \left[(W \cdot j)_2^2 - (W \cdot j)_3^3 - (j \cdot W)_2^2 + (j \cdot W)_3^3 \right] + \\ + \left[\cos \Theta W_2^1 + \sin \Theta W_3^1 \right] \bar{l} - \left[\cos \Theta W_1^2 + \sin \Theta W_1^3 \right] l \}, \end{aligned} \tag{16}$$

where we have omitted for simplicity Lorentz indices and ξ is a real number of the order unity. To see the fact that Eq. (16) has CP $= -1$, we notice that under the canonical choice, we have only to observe

$$\begin{aligned} CP: \quad & W_\mu(x) \leftrightarrow \varepsilon_\mu \cdot W_\mu^T(-x) , \\ & j_\mu(x) \leftrightarrow \varepsilon_\mu j_\mu^T(-x), \text{ (no summation over } \mu) \\ & l_\mu(x) \leftrightarrow \varepsilon_\mu \bar{l}_\mu(-x) . \end{aligned} \tag{17}$$

However, we must be cautioned for the CP-property of the leptonic current l_μ since it is not unambiguously defined. To see it, one may for instance change the neutrino field ν into $i \cdot \nu$ without changing any physical consequences, but the CP parity of l_μ changes the sign also, implying that it is really not meaningful to assign the definite CP-parity for the lepton current. This is of course due to the fact that the leptons do not have the strong interaction.

However, again we assign the CP parity of the lepton current as is given by Eq. (17) and we may call it to be a canonical choice. As we see easily, what is important for reactions involving leptons is not its absolute CP parity which is not well defined, but its relative CP parity between g^2 and g^3 terms, which is independent of how we assign the CP parity for l_μ.

Now, we can easily see that the first order effect in g without any real W-meson emission or absorption is zero due to the conservation of Q_w and Y_w quantum numbers of W-mesons. This can be easily seen since for instance

$$\langle W_3^2(x)\rangle_0 = 0\,,$$

because of the conservation of the hypercharge Y_w. This assertion holds valid in all order of both electromagnetic and strong interactions. As a corollary of this theorem, we find that the electric dipole moment of the neutron must be absent in the order $g \cdot e^n$ (n = arbitrary integer) and it must come at least in the order $g^3 \cdot e$. Similarly, the ordinary weak interaction should appear in the order g^2 or higher. For the order g^2, our Hamiltonian Eq. (16) gives the usual CP-conserving leptonic, semi-leptonic and non-leptonic weak interaction which is purely a hadronic octet, provided that the weak SU(3) is exact. We may remark that if the weak SU(3) is not exact and if there is a mass difference among W-mesons, this will bring a tiny $\Delta I = 3/2$ contribution.

A more interesting term is that of the order g^3 since it corresponds to CP $= -1$. Note that we have

$$\langle W_\mu^{(\alpha)}(x) \cdot W_\nu^{(\beta)}(y) \cdot W_\lambda^{(\gamma)}(z)\rangle_0 \neq 0$$

in our theory, because of the existence of the tri-linear self-coupling among W-mesons. It is interesting to note that we have now [10] $K_1^0 \to \mu\bar{\mu}$ (or $\bar{e}e$) or $K_2^0 \to \pi_0\mu\bar{\mu}$ (or, $\pi_0 e\bar{e}$) in the order g^3 in our scheme. We can estimate the rough order of magnitudes for these processes. A typical Feynman diagram for $K_0^2 \to 2\pi$ and $K_0^1 \to \mu\bar{\mu}$ is illustrated in Figs. 1 and 2.

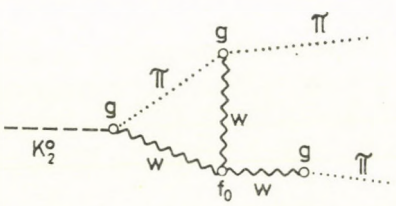

Fig. 1

Both reactions are proportional to $g^3 \cdot f_0$. These diagrams have been computed by MOHAPATRA [11.] By choosing the cut-off value Λ to be $\Lambda \approx$ $\approx M_w \approx 4$ BeV we can reproduce

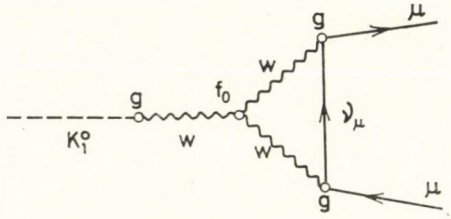

Fig. 2

$$\frac{\Gamma(K_2^0 \to 2\pi)}{\Gamma(K_1^0 \to 2\pi)} \sim 10^{-6}; \qquad \frac{\Gamma(K_1^0 \to \mu\bar{\mu})}{\Gamma(K_1^0 \to 2\pi)} = (10^{-6} \sim 10^{-8}). \tag{18}$$

Appendix I

Here let us consider a Lagrangian

$$\mathscr{L}_0 = -\frac{1}{4} Tr(F_{\mu\nu} F_{\mu\nu}) - \frac{1}{2} Tr(W_\mu m_0^2 W_\mu) - Tr(J_\mu \cdot W_\mu), \tag{A.1}$$

$$F_{\mu\nu} = \frac{\partial}{\partial x_\mu} W_\nu - \frac{\partial}{\partial x_\nu} W_\mu + i\lambda [W_\mu, W_\nu], \tag{A.2}$$

where we added an external current J_μ to the Lagrangian Eq. (10) for the sake of generality. Without loss of generality, the mass matrix m_0^2 can be taken to be symmetric, i.e.

$$(m_0^2)^T = m_0^2. \tag{A.3}$$

Now, the equation of the motion on the basis of Eq. (A.1) is given by

$$\frac{\partial}{\partial x_\nu} F_{\mu\nu} + i\lambda [W_\nu, F_{\mu\nu}] + \frac{1}{2} \{m_0^2, W_\mu\}_+ + F_\mu = 0. \tag{A.4}$$

Taking the derivative ∂_μ of both sides of this equation and noting

$$\partial_\mu \partial_\nu F_{\mu\nu}(x) = 0,$$

one finds

$$\frac{i\lambda}{2} [W_\mu \cdot W_\mu, m_0^2] + i\lambda [W_\mu, F_\mu] + \frac{1}{2} \{\partial_\mu W_\mu, m_0^2\}_+ + \partial_\mu F_\mu = 0, \tag{A.5}$$

where we used Eq. (A.4) together with a Jacobi identity. First, let us consider the case $J_\mu \equiv 0$. Then, if we demand $\partial_\mu W_\mu = 0$, then Eq. (A.5) leads us to $[W_\mu \cdot W_\mu, m_0^2] = 0$ which in turn requires that m_0^2 must be proportional to

the unit matrix. Conversely, if m_0^2 is proportional to the unit matrix and if it is not identically zero, then Eq. (A.5) gives us $\partial_\mu W_\mu = 0$, provided $J_\mu = 0$.

If we have a non-zero external current and if it is generated by the Yang—Mills gauge field, then it should satisfy

$$\frac{\partial}{\partial x_\mu} F_\mu(x) + i\lambda [W_\mu(x), F_\mu(x)] = 0 . \tag{A. 6}$$

Thus, the same argument for the equivalence of $\partial_\mu W_\mu = 0$ with proportionality of m_0^2 to a unit matrix is still valid. Then, Eq. (A.6) can be rewritten also as a conservation law

$$\frac{\partial}{\partial x_\mu} \cdot \{F_\mu(x) + i\lambda [W_\nu(x), F_{\mu\nu}(x)]\} = 0 . \tag{A .7}$$

Appendix II

First of all, the charge conjugation Eq. (13) implies an existence of the second-quantized unitarity operator U_c satisfying

$$U_c W_\mu(x) U_c^{-1} = - S W_\mu^T(x) S^{-1} . \tag{A. 8}$$

Since U_c has no SU(3) matrix indices, Eq. (A.8) also implies

$$U_c W_\mu^T(x) U_c^{-1} = - (S^{-1})^T W_\mu(x) S^T \tag{A. 9}$$

by taking its transpose. Now, if we apply the charge conjugation operation twice on the vector meson field, we should obtain the same value, i.e.

$$U_c \cdot U_c \cdot W_\mu(x) U_c^{-1} \cdot U_c^{-1} = W_\mu(x) .$$

Using Eqs. (A.8) and (A.9), this gives us

$$S \cdot (S^{-1})^T W_\mu(x) \cdot S^T \cdot S^{-1} = W_\mu(x) .$$

Therefore, we must have

$$S^T = \lambda \cdot S ,$$

where λ is a number. Taking the transpose of this equation, we find $\lambda^2 = 1$, i.e. $\lambda = \pm 1$. However, $\lambda = -1$ is not acceptable. The reason is the following. Since S must commute with the charge operator

$$Q = \frac{1}{2}\left[\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8\right],$$

it must have the structure

$$S = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where T is a unitary 2×2 matrix. Hence, $S^T = -S$ is impossible. Thus, we conclude

$$S^T = S,$$

implying that the 2×2 sub-matrix T is also symmetric. But a unitary symmetric matrix S can be expressed as

$$S = V \cdot V^T \tag{A.10}$$

for some non-singular matrix V.

This is because such a matrix S can be diagonalized by a real rotation matrix R as

$$S = R \cdot S_D \cdot R^{-1}, \quad R^T \cdot R = 1,$$

where S_D is diagonal. So by setting

$$V = R \cdot S_D^{1/2} \cdot R^{-1}$$

we obtain $S = V \cdot V^T$. Hence defining $\widetilde{W}_\mu(x)$ by

$$\widetilde{W}_\mu(x) = V^{-1} \cdot W_\mu(x) \cdot V \tag{A.11}$$

one finds

$$U_c \cdot \widetilde{W}_\mu(x)\, U_c^{-1} = -\widetilde{W}_\mu^T(x)\,, \tag{A.12}$$

so that $S = 1$ in Eq. (13) is achieved in this way. Since the transition from $W_\mu(x)$ to $\widetilde{W}_\mu(x)$ is a SU(3) transformation, it must be connected by a second-quantized unitary transformation. This fact is sufficient to prove Eq. (14) in the text.

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МОДЕЛЬ МАКСИМАЛЬНОГО CP НАРУШЕНИЯ

С. ОКУБО

Р е з ю м е

Разрабатывается модель слабых взаимодействий, обнаруживающая максимальное CP нарушение. Модель содержит один октет (или нонет) промежуточных векторных мезонных полей с сильной трехлинейной самосвязью. Члены первого порядка запрещены, в то время как члены второго порядка дают эффекты CP = 1, а члены третьего порядка — CP = — 1. Малость эффектов CP нарушения так легко объяснена.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (1—2), pp. 217—221 (1969)

FIELD OPERATOR FOR AN INTERACTING UNSTABLE ELEMENTARY SYSTEM

By

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The field operator $\varphi_{j;\alpha}(x\,;s)$, describing an interacting unstable elementary system is introduced. Asymptotic fields are represented by free fields with continuous additional parameters. Consistency with the Haag-Ruelle theory of asymptotic states is achieved.

Usually one assumes that every intermediate state occurring in the collisions of elementary particles can be described in terms of free stable particles, with definite spins and masses. In recent years it has been shown, however, that one gets remarkably good fits to experimental data if one introduces the exchange of *unstable* objects, like resonances or Reggeons. It is interesting, therefore, to modify the conventional formalism of quantum field theory in such a way that the unstable objects become primary ones, and can be defined by means of one-particle states.

In this lecture we wish to outline the quantum field theory of interacting resonances. We assume for simplicity that the resonances are characterized by a sharp value of spin, independent from interaction. This restriction should be removed if we pass to the description of Reggeons, a case which we shall discuss in detail in another publication.

Resonances as one-particle states have been discussed already in the frame of Feynmann diagrams method [1], by means of a group-theoretical approach [2], and using the techniques of infinite-component wave equations [3]. All these considerations are, however, insufficient for our purpose because the following two basic problems, concerning unstable systems, remained unsolved*:

a) The relation between the S-matrix** and field-theoretic definition of the elastic resonance.

b) The problem of asymptotic limits for the field operator describing interacting resonance, and the consistency with Haag—Ruelle theory of asymptotic states.

* The difficulties in solving problems *a)* and *b)* are caused by the instability property, and this explains why it is often assumed that the resonances are approximated by stable particles.

** We define the resonance by the property that the scattering phase passes through $\pi/2$. The definition by means of the complex pole on the unphysical sheet seems to be less suitable (see G. CALLUCCI, L. FONDA and G. C. GHIRARDI, Phys. Rev., **166,** 1719, 1968.

We intend to give the answer to these two questions in this lecture.

Let us discuss briefly the notion of resonant scattering. If the scattering develops a resonance this is an effect describing particular correlation of ingoing and outgoing wave packets. The resonance, contrary to the case of appearance of a bound state, is not observed asymptotically as a new object,* but represents a name for a particular type of scattering process. The field operator describing resonance should therefore describe by means of its one-particle wave function the space-time development of the resonant scattering. The main idea is *to relate a single field operator with the resonant scattering channel.*

The interacting multi-particle states, defining scattering channel, are kinematically characterized by total mass s, total angular momentum J, and by some additional quantum numbers α which are called the degeneracy parameters [4]**. We shall consider here only *elementary* channels, defined by means of multiparticle states fully characterized by their total four-momentum and the angular momentum parameters J and α.*** Such multi-particle states represent an unstable elementary system [4] because one cannot measure the observables characterizing its separate components without destroying the system. We attribute to every such elementary system a field operator. It should be stressed that the notion of an unstable elementary system is much wider than the notion of a resonance, and also describes nonresonant ways of scattering.

Our main formal assumptions are the following:

1. The interacting elementary system, characterized kinematically by the choice of the mass spectrum $s \in \Sigma(\Sigma \subset R_+^1)$, spin J and the degeneracy parameters α, is described by the field operator $\varphi_{j;\alpha}(x; s)$ $(j = J, J-1, \ldots -J)$.

2. We assume Lorentz invariance and the locality condition

$$\left[\varphi_{j;\alpha}(x;s), \varphi_{j;\alpha}(x';s')\right] = 0 \qquad \begin{matrix} x, x' \text{ space-like,} \\ s, s' \text{ arbitrary.} \end{matrix}$$

3. The interaction between the multi-particle states goes only *through short range forces acting between the elementary systems* described by the field operators $\varphi_{j;\alpha}(x; s)$ and it is possible to introduce the *free asymptotic fields* $\varphi_{j;\alpha}^{\text{in}}(x; s)$ and $\varphi_{j;\alpha}^{\text{out}}(x; s)$, satisfying the following commutation relations

$$\left[\overset{\text{in}}{\varphi}{}^{\text{out}}_{j;\alpha}(x;s),\ \overset{\text{in}}{\varphi}{}^{\text{out}}_{j';\alpha'}(x';s')\right] = i\delta_{\alpha\alpha'}\delta_{JJ'}\hat{P}_{jj}^{(J)}(\partial_\mu)\,\Delta(x-x';s)\,\delta(s-s')\,, \tag{1}$$

* The bound state is obtained as a limiting case, when the correlation between ingoing and outgoing states becomes so singular that we are forced to enlarge the space of asymptotic states, unless the unitarity condition is violated.

** One can say that the degeneracy parameters determine internal spins of the channel function.

*** In an N-particle elementary channel α denotes fixed set of eigenvalues of $3N-6$ commuting angular momentum operators, forming together with the total four-momentum operator, total angular momentum and its third component the complete set of commuting observables, fully characterizing the N-particle states.

where $P^{(J)}_{jj'}(\partial_\mu)$ represents the projection operator defining commutator funotion for the spin J field [5].

The fields (1) and the asymptotic condition for the simplest case when the indices j and α are not present (S-wave two-particle subsystem) have been introduced by LICHT [6]. It can be shown [7] that introducing suitably modified Wightman axioms for the field $\varphi_{j;\alpha}(x;s)$ one can prove rigorously the asymptotic condition, leading to asymptotic fields (1), following the lines of HEPP's proof of LSZ formulation in the frame of Wightman formalism (see [8]).

Now we introduce the notion of a *free elementary unstable system*. Such systems are defined by means of free field operators $\varphi^{\circ}_{j;\alpha}(x;s)$, having c-number commutators:

$$[\varphi^{\circ}_{j;\alpha}(x;s), \varphi^{\circ}_{j';\alpha'}(x';s')] = i \int d\varkappa^2 \varrho^{(J)}_{\alpha\alpha'}(s,s';\varkappa^2) \cdot \hat{P}^{(J)}_{jj'}(\partial_\mu) \Delta(x-x';\varkappa^2). \tag{2}$$

It is easy to see that *every decoupled elementary channel, and particularly a channel developing elastic resonance, can be described by such a free field with a parameter*. Because it follows from (2) that all truncated VEV of order higher than two vanish; the field $\varphi^{\circ}_{j;\alpha}(x;s)$ can lead only to the scattering of one-particle states. It should be stressed that the presence of the continuous parameter s is responsible for the fact that the scattering of one-particle states is possible. We define the asymptotic one-particle states as follows

$$\left|\vec{p}, s; j; \alpha >_{\substack{\text{in} \\ \text{out}}} = a^{\overset{\text{in}}{\text{out}}}_{;\alpha}(\vec{p}, s)\right| 0 >, \tag{3}$$

where the creation operators, occurring in the definition (3), are obtained from the asymptotic fields (1) by means of the conventional formulae for spin J fields [5], and satisfy the following commutation relations:

$$\left[a^{+\overset{\text{in}}{\text{out}}}_{j;\alpha}(\vec{p}', s')\, a^{\overset{\text{in}}{\text{out}}}_{j',\alpha'}(\vec{p}', s') \right] = \delta_{\alpha\alpha'} \delta_{jj'} \delta(\vec{p} - \vec{p}') \delta(s - s'). \tag{4}$$

Lorentz invariance and the unitarity condition imply that

$$|\vec{p}, s; j; \alpha >_{\text{out}} = e^{2i\eta_{J;\alpha}(s)} |\vec{p}, s; j; \alpha >_{\text{in}}, \tag{5}$$

where the *function* $\eta_{J;\alpha}(s)$ decribes the phase shift in the *decoupled* channel (J, α). We see, therefore that there is one-to-one correspondence between the elastic scattering and the scattering of one-particle states in our formalism. An example of such formulation has been provided by THIRRING [9] in his Lagrangean approach to the ZACHARIASEN model [10].

Now we are prepared to solve our problems. The first answer (see a) follows from the relation (5). Every field operator $\varphi^{\circ}_{j;\alpha}(x;s)$ having such

asymptotic limits that the relation (5) remains valid is a good candidate for the space-time description of an elastic resonance present in the phase shift $\eta_{J;\alpha}(s)$. The consistency with Haag—Ruelle theory of asymptotic states (see b)) follows directly from the physical interpretation of the asymptotic fields (1). The asymptotic states

$$|\vec{p}_1 \dots \vec{p}_n, s_1 \dots s_n; j_1 \dots j_n; \alpha_1 \dots \alpha_n >_{\substack{\text{in} \\ \text{out}}} = \prod_{l=1}^{n} a^{\text{in}}_{\text{out}\, j_i;\alpha_i} (\vec{p}_i, s_i) |0> \qquad (6)$$

represent only another way of description of conventional asymptotic states*. Particularly if $n = 1$ (see (3)) we obtain in the considered elementary channel (J, α) a description of ingoing and outgoing multi-particle states with a given value of total four-momenta $(\vec{p}, p_0 = \sqrt{\vec{p}^2 + s})$ and a given polarization index j. The *physical* in- and outgoing wave packets are obtained by smearing-out with some smooth function $f(s)$, with supp $f(s) \subset \Sigma$. Such asymptotic wave packets can be described by means of generalized free fields [11]. The intuition that the propagators with continuous mass spectrum, substituted in Feynmann graphs, describe in relativistic quantum field theory the exchange of unstable objects is already an old one [12]. In the formalism, presented here, such a conclusion follows naturally.

The existence of our alternative description of an interacting system and its asymptotic states is caused by a particular type of dynamics, which allowed the introduction of the asymptotic fields (1) for the field operators $\varphi_{j;\alpha}(x; s)$. If the interaction is such that it does not require the introduction of separate asymptotic particles, we can forget about the stable multi-particle asymptotic states, and discuss only the scattering of the free elementary unstable objects, described by the asymptotic fields with continuous mass parameter. The scattering matrix operator can be defined as follows:

$$\varphi^{\text{out}}_{j;\alpha}(x; s) = \hat{S}^{-1} \varphi^{in}_{j;\alpha}(x; s) \hat{S} . \qquad (7)$$

Further details are similar to those in the LSZ scattering theory.

The simplest way of expressing the dynamics, consistent with our formulation, is to introduce local Lagrangeans of the field operators $\varphi_{j;\alpha}(x; s)$. It is also possible, however, to formulate the appropriate dynamics, using the dispersion-theoretical approach, or, at least in some cases, by introducing a suitably truncated set of conventional Feynmann graphs.

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ОПЕРАТОР ПОЛЯ ДЛЯ ВЗАИМОДЕЙСТВИЯ НЕСТАБИЛЬНОЙ ЭЛЕМЕНТАРНОЙ СИСТЕМЫ

Й. ЛУКИРСКИ

Резюме

Вводится оператор поля $\varphi_{j;\alpha}(x;s)$, описывающий взаимодействие нестабильной элементарной системы. Ассимптотические поля представляются свободными полями с непрерывными аддитивными параметрами. Достигается совпадение с теорией Хааг—Рюель для асимптотических состояний.

INDEX

I. KOVÁCS

Rotational Structure of the Spectra in Diatomic Molecules

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A kézirat nyomdába érkezett: 1969. I. 7. — Terjedelem: 19.75 (A/5) ív, 43 ábra

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ACADEMIAE SCIENTIARUM HUNGARICAE

TOMUS XXVI FASCICULUS 3

AKADÉMIAI KIADÓ, BUDAPEST
1969

ACTA PHYS. HUNG.

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Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (3), pp. 223—237 (1969)

THE HYDRODYNAMICAL MODEL OF WAVE MECHANICS V.

By

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(Received 1. II. 1968)

The hydrodynamical equations of motion corresponding to the wave equation including spin-orbit coupling are formulated for the hydrodynamical variables introduced in Part IV. It is shown that there is a one-to-one correspondence between the hydrodynamical equations and the wave equation. A physical interpretation of the forces involved in the equations thus obtained is given.

1. Introduction

The construction of the hydrodynamical model that also describes spin-orbit coupling was begun in Part IV. by the introduction of hydrodynamical variables. We succeeded in showing that there exists a one-to-one correspondence between these hydrodynamical variables and the wave function ψ of the Pauli equation including spin-orbit coupling.

The new variables introduced are as follows:

a) The charge density and mass density of the medium representing the particle given by

$$\varrho_e = e\varrho, \qquad \varrho_m = m\varrho, \tag{1}$$

where

$$\varrho = \psi^+ \psi .$$

b) The velocity distribution of the medium defined as

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{r}, t) = \frac{1}{\varrho} \frac{\hbar}{2mi} (\psi^+ \nabla \psi - (\nabla \psi)^+ \psi) - \frac{e}{mc} \boldsymbol{A} + \\ + \frac{1}{\varrho} \frac{\hbar}{4m} \operatorname{rot} \psi^+ \boldsymbol{\sigma} \psi) - \frac{2\lambda}{\varrho} \psi^+ \boldsymbol{\sigma} \psi \times \boldsymbol{E}, \end{aligned} \tag{2}$$

where

$$\lambda = \frac{e\hbar}{8 m^2 c^2} .$$

c) The spin vector characterizing the magnetic moment distribution of the medium

$$\boldsymbol{s}(\boldsymbol{r}, t) = \psi^+ \boldsymbol{\sigma} \psi \tag{3}$$

or the unit vector pointing in the direction of the magnetic polarization

$$\boldsymbol{T}(\boldsymbol{r},t)=\frac{1}{\varrho}\,\mathbf{s}\,. \tag{4}$$

d) The vector representing the density of the electric polarization, given by

$$\boldsymbol{P}(\boldsymbol{r},t)=\lambda\psi^{+}\left[\left(-i\hbar\nabla-\frac{e}{c}\boldsymbol{A}\right)\times\boldsymbol{\sigma}-\boldsymbol{\sigma}\times\left(i\hbar\overleftarrow{\nabla}-\frac{e}{c}\boldsymbol{A}\right)\right]\psi \tag{5}$$

or expressed as

$$\boldsymbol{R}(\boldsymbol{r},t)=\frac{1}{\varrho}\,\boldsymbol{P}(\boldsymbol{r},t)\,. \tag{6}$$

The symbol $\overleftarrow{\nabla}$ in expression (5) denotes that the operator ∇ acts in reverse, i.e. on the preceding ψ^{+}.

The conditions which have to be satisfied by the new hydrodynamical variables as well as by their equations of motion have been given in [1] as follows:

1) The wave function and the hydrodynamical variables must be calculable from each other unambiguously at any time. In this mapping the wave function and the related hydrodynamical variables are called corresponding quantities.

2) If at time $t=0$ the wave function $(\psi(0))$ and the complete set of the corresponding hydrocynamical variables $(D(0))$ are corresponding quantities, then $\psi(t)$ and $D(t)$ calculated from the wave equation and the hydrodynamical equations respectively, at any time t are also required to be corresponding quantities.

3) The hydrodynamical variables must be invariant under symmetry transformations or, rather, they have to transform like the corresponding classical quantities.

4) a) Each term in the equations of motion of the hydrodynamical variables integrated over the whole space should be interpretable in terms of classical physics.

b) The hydrodynamical variables must be equal to the experimentally obtained values of the quantities represented by them.

In Part IV it has already been shown that the hydrodynamical variables (1)—(6) satisfy condition 1), i.e. ϱ, $\boldsymbol{v}$, $\mathbf{s}$, $\boldsymbol{P}$ are expressed in a form showing a one-to-one correspondence with the wave function ψ.

2. The equations of motion

The wave equation including the spin-orbit coupling term was given in [1] in the form

$$\mathscr{H}\psi = i\hbar\dot{\psi}\,, \tag{7}$$

with

$$\mathscr{H} = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)^2 - \mu(\boldsymbol{\sigma B}) +$$

$$+ \lambda\boldsymbol{\sigma}\left[\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)\times\boldsymbol{E} - \boldsymbol{E}\times\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)\right],$$

where

$$\mu = \frac{e\hbar}{2mc} \qquad \text{and} \qquad \lambda = \frac{e\hbar}{8m^2c^2}\,.$$

From this wave equation we have to derive the equations of motion of the hydrodynamical variables given in (1)—(6). It is to be noted that since in that case the terms of order of magnitude v^4/c^4 are neglected in the Hamiltonian, the terms containing v^4/c^4 or those of higher order in v/c will be neglected in the following calculations.

Let us start with the equation of motion of hydrodynamical variables with vector characteristics. If $\boldsymbol{D}$ represents a vector quantity than as is apparent from equations (2), (3), (4), (5) and (6), it always has the form

$$\varrho\boldsymbol{D} = \psi^+\,\boldsymbol{d}\psi + (\boldsymbol{d}\psi)^+\,\psi\,, \tag{8}$$

where $\boldsymbol{d}$ is a vector operator (e.g. for $\boldsymbol{D} = \boldsymbol{T}$ we have $\boldsymbol{d} = 1/2\,\boldsymbol{\sigma}$). In order to obtain the equation of motion for $\boldsymbol{D}$, the quantity $d\boldsymbol{D}/dt$ must be calculated. The procedure can be considerably simplified by making use of the identity

$$\varrho\frac{d\boldsymbol{D}}{dt} = \frac{\partial(\varrho\boldsymbol{D})}{\partial t} + \mathrm{Div}\,(\varrho\boldsymbol{v}\circ\boldsymbol{D})\,. \tag{9}$$

Substituting (8) into (9) and expressing the time derivatives of the wave function by making use of (7), we get

$$\varrho\frac{d\boldsymbol{D}}{dt} = \frac{1}{i\hbar}\left[\psi^+\,\boldsymbol{d}\mathscr{H}\psi - (\mathscr{H}\psi)^+\,\boldsymbol{d}\psi\right] +$$

$$+ \frac{1}{i\hbar}\left[(\boldsymbol{d}\psi)^+\,\mathscr{H}\psi - (\boldsymbol{d}\mathscr{H}\psi)^+\,\psi\right] + \psi^+\frac{\partial\,\boldsymbol{d}}{\partial\,t}\psi +$$

$$+ \left(\frac{\partial\,\boldsymbol{d}}{\partial\,t}\psi\right)^+\psi + \mathrm{Div}\,(\varrho\boldsymbol{v}\circ\boldsymbol{D})\,.$$

By adding and subtracting the expression

$$\frac{1}{i\hbar}\psi^+ \mathscr{H}\boldsymbol{d}\psi + \frac{1}{i\hbar}(\mathscr{H}\boldsymbol{d}\psi)^+\psi$$

one arrives to the expression

$$\begin{aligned}\varrho\frac{d\boldsymbol{D}}{dt} = {} & \frac{i}{\hbar}\psi^+[\mathscr{H},\boldsymbol{d}]\psi - \frac{i}{\hbar}([\mathscr{H},\boldsymbol{d}]\psi)^+\psi + \\ & + \psi^+\frac{\partial\boldsymbol{d}}{\partial t}\psi + \left(\frac{\partial\boldsymbol{d}}{\partial t}\psi\right)^+\psi + \operatorname{Div}(\varrho\boldsymbol{v}\circ\boldsymbol{D}) + \\ & + \frac{1}{i\hbar}\{\psi^+\mathscr{H}\boldsymbol{d}\psi - (\mathscr{H}\psi)^+\boldsymbol{d}\psi + (\boldsymbol{d}\psi)^+\mathscr{H}\psi - (\mathscr{H}\boldsymbol{d}\psi)^+\psi\}.\end{aligned} \tag{10}$$

In this arrangement of the terms of the right hand side the integral of the expression in brackets extended over the whole space vanished because of the Hermiticity of $\mathscr{H}$.

The equations of motion will be derived for each of the hydrodynamical variables with vector characteristics by the aid of eq. (10).

a) *The continuity equation*

The equation of motion for the only scalar quantity, the mass density ϱ_m, is actually the continuity equation which has the form

$$\frac{\partial\varrho_m}{\partial t} + \operatorname{div}(\varrho_m\boldsymbol{v}) = 0, \tag{11}$$

as was shown in [1]. The expression of $\boldsymbol{v}$ is given in (2). In analogy to the equations of motion of the vector quantities, (11) can be expressed as the total derivative of the mass density in the form

$$\frac{d\varrho_m}{dt} = -\varrho_m\operatorname{div}\boldsymbol{v}. \tag{12}$$

b) *Equation of motion of the velocity distribution*

Making use of (10) and (8) lengthy calculations lead to the equation

$$\begin{aligned}\varrho_m\frac{d\boldsymbol{v}}{dt} = {} & \left[\varrho_e\boldsymbol{E} + \frac{1}{c}\varrho_e\,\boldsymbol{v}\times\boldsymbol{B}\right] + \frac{\mu}{2}[(\mathbf{s}\operatorname{grad})\boldsymbol{B} + \mathbf{s}\times\operatorname{rot}\boldsymbol{B}] + \\ & + [(\boldsymbol{P}\operatorname{grad})\boldsymbol{E} + \boldsymbol{P}\times\operatorname{rot}\boldsymbol{E}] - 2\,m\lambda\frac{\partial}{\partial t}(\mathbf{s}\times\boldsymbol{E}) + \\ & + \operatorname{Div}(\mathfrak{D}^S + \mathfrak{D}^A),\end{aligned} \tag{13}$$

where

$$
\begin{aligned}
(\mathfrak{Q}^S)_{ik} = {} & \frac{\hbar^2}{4m}\varrho\,\frac{\partial^2 \ln \varrho}{\partial x_i\,\partial x_k} + \frac{\mu}{2}\delta_{ik}\,\mathbf{s}\,\mathbf{B} - \frac{\mu}{4}(s_i B_k + s_k B_i) + \\
& + \frac{\hbar}{4}(v_i\,\mathrm{rot}_k\,\mathbf{s} + v_k\,\mathrm{rot}_i\,\mathbf{s}) - \frac{\hbar^2}{16\,m}\,\frac{1}{\varrho}\,\mathrm{rot}_i\,\mathbf{s}\cdot\mathrm{rot}_k\,\mathbf{s} - \\
& - \frac{\hbar^2}{4m}\varrho\,\frac{\partial T_r}{\partial x_i}\cdot\frac{\partial T_r}{\partial x_k} - m\lambda\left[(\mathbf{s}\times\boldsymbol{E})_i\,v_k + (\mathbf{s}\times\boldsymbol{E})_k\,v_i\right] + \\
& + \frac{\hbar\lambda}{4}\,\frac{1}{\varrho}\left[(\mathbf{s}\times\boldsymbol{E})_i\,\mathrm{rot}_k\,\mathbf{s} + (\mathbf{s}\times\boldsymbol{E})_k\,\mathrm{rot}_i\,\mathbf{s}\right] - \\
& - \frac{\hbar\lambda}{2}\,\boldsymbol{E}\mathbf{s}\left(\frac{\partial T_k}{\partial x_i} + \frac{\partial T_i}{\partial x_i}\right) + \frac{\hbar\lambda}{2}\,E_r\left(s_k\,\frac{\partial T_r}{\partial x_i} + s_i\,\frac{\partial T_r}{\partial x_k}\right)
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
(\mathfrak{Q}^A)_{ik} = {} & \frac{\mu}{4}(s_i B_k - s_k B_i) + \frac{1}{2}(P_i E_k - P_k E_i) - \\
& - \frac{\hbar\lambda}{2}\,\frac{\partial}{\partial x_r}(\varrho\delta_{ir}E_k - \varrho\delta_{kr}E_i) - \varepsilon_{ikl}\,\frac{\partial}{\partial x_p}\left\{\frac{\hbar}{4}\,s_l\,v_p - \right. \\
& \left. - \frac{\hbar^2}{16\,m}\,\frac{1}{\varrho}\,s_l\,\mathrm{rot}_p\,\mathbf{s} - \frac{\hbar^2}{8\,m}\,\varrho\varepsilon_{lrq}T_r\,\frac{\partial T_q}{\partial x_p} + \frac{\hbar\lambda}{2\varrho}\,s_l\,(\mathbf{s}\times\boldsymbol{E})_r\right\}.
\end{aligned}
\tag{15}
$$

Here the following notations have been introduced: δ_{ik} is the Kronecker delta. ε_{ikl} is $+1$ if (i, k, l) is an even permutation of $(1, 2, 3)$, -1 if it is an odd permutation of 1, 2, 3 and zero otherwise. For repeated indices summation from 1 to 3 is to be understood. S and A indicate the symmetry and antisymmetry of the tensor $\mathfrak{Q}$, respectively.

It should be noted here that it is only the divergence of the stress tensor $\mathfrak{Q}$ that can be unambiguously determined. Let us use instead of $\mathfrak{Q}$ a tensor with elements

$$
(\bar{\mathfrak{Q}})_{ik} = (\mathfrak{Q})_{ik} + \frac{\partial}{\partial x_l}(\mathfrak{q})_{ikl}(\boldsymbol{r}, t),
$$

where $(\mathfrak{q})_{ikl}$ is an arbitrary tensor antisymmetric in the last two indices. The density of force corresponding to $\bar{\mathfrak{Q}}$ will be the same as that corresponding to $\mathfrak{Q}$, since

$$
(\mathrm{Div}\,\bar{\mathfrak{Q}})_i = \frac{\partial(\bar{\mathfrak{Q}})_{ik}}{\partial x_k} = \frac{\partial(\mathfrak{Q})_{ik}}{\partial x_k} = (\mathrm{Div}\,\mathfrak{Q}).
$$

The physical meaning of this arbitrariness lies in the fact that the force acting on a volume element does not determine unambiguously the surface forces

replacing it, since these can always be completed by surface forces resulting zero when integrated over the closed surface surrounding the volume element. In equation (13), however, $\mathfrak{Q}$ arises in an unambiguously determined form as *Dis* $\mathfrak{Q}$. The determination of the exact form of $\mathfrak{Q}$ would require the measurement of the moment of force produced by the inner forces.

c) *Equation of motion of the spin vector*

With the use of (10) and (7) the equation of motion for the vector $\boldsymbol{T}$ defined by (4) can be formulated as

$$\varrho \frac{d\boldsymbol{T}}{dt} = \frac{2\mu}{\hbar} \varrho \boldsymbol{T} \times \boldsymbol{B} - \frac{1}{2c} \frac{2\mu}{\hbar} \varrho \boldsymbol{T} \times (\boldsymbol{v} \times \boldsymbol{E}) + \tag{16}$$

$$+ 2\lambda \left[\varrho(\boldsymbol{E} \operatorname{rot} \boldsymbol{T}) \boldsymbol{T} - \varrho([\boldsymbol{T} \times \boldsymbol{E}] \operatorname{grad}) \boldsymbol{T} - \frac{1}{2} \boldsymbol{T} \times (\boldsymbol{E} \times \operatorname{rot}(\varrho \boldsymbol{T})) \right] +$$

$$+ \operatorname{Div} \mathfrak{Q}',$$

where

$$(\mathfrak{Q}')_{ik} = \frac{\hbar}{2m} \varepsilon_{irq} \varrho T_r \frac{\partial T_q}{\partial x_k} + \frac{\hbar}{4m} T_i \operatorname{rot}_k \varrho \boldsymbol{T} - 2\lambda \varrho T_i (\boldsymbol{T} \times \boldsymbol{E})_k - 2\lambda \varepsilon_{ikr} \varrho E_r .$$

d) *Equation of motion of the electric polarization vector*

It appears more convenient to formulate the equation of motion for the vector $\boldsymbol{R}$ introduced by (6), than for $\boldsymbol{P}$. Putting (6) into (10) we find

$$\varrho \frac{d\boldsymbol{R}}{dt} = -2e\lambda \varrho (\boldsymbol{T} \times \boldsymbol{E}) + \operatorname{Div} \mathfrak{Q}'', \tag{17}$$

where

$$(\mathfrak{Q}'')_{ik} = \varrho R_i v_k .$$

The equations of motion given by (12), (13), (16) and (17) are equivalent to the Schrödinger equation (7), since the hydrodynamical variables ϱ, $\boldsymbol{v}$, $\boldsymbol{T}$, $\boldsymbol{R}$ and the wave function (ψ_1, ψ_2) can be unambiguously calculated from one another (for details see [1]). Furthermore, the equations of motion derived can be used for calculating the values of the hydrodynamical variables for any time provided their values are known at $t = 0$.

e) *Equation of motion of the angular momentum*

The angular momentum can be expressed in terms of the other variables as

$$\boldsymbol{r} \times \varrho_m \boldsymbol{v} .$$

For the sake of completeness, however, we have also derived the equation of motion of the total angular momentum of the medium describing the electron.

Using equation (13) in the form

$$\varrho_m \frac{d\boldsymbol{v}}{dt} = \varrho \boldsymbol{f} + \operatorname{Div} \mathfrak{Q}$$

we have for the time derivative of the angular momentum of the medium

$$\frac{d}{dt}\int \boldsymbol{r} \times \varrho_m \boldsymbol{v}\, d\tau = \int \boldsymbol{r} \times \varrho \boldsymbol{f}\, d\tau + \int \boldsymbol{r} \times \operatorname{Div} \mathfrak{Q}\, d\tau \,. \tag{18}$$

Integrating by parts we find for the i-th component of the second term, remembering that the tensor $\mathfrak{Q}$ is built up from $\mathfrak{Q}^S$ and $\mathfrak{Q}^A$:

$$\left(\int \boldsymbol{r} \times \operatorname{Div} \mathfrak{Q}\, d\tau\right)_i = \varepsilon_{ikl} \int \mathfrak{Q}^A_{kl}\, d\tau \,. \tag{19}$$

This means that it is only the antisymmetric part of the stress tensor which contributes to the rate of change of the angular momentum. Substituting (19) into (18), we have for the i-th components of the moment of force:

$$\frac{d}{dt}\int (\boldsymbol{r} \times \varrho_m \boldsymbol{v})_i\, d\tau = \int (\boldsymbol{r} \times \varrho \boldsymbol{f})_i\, d\tau + \varepsilon_{ikl} \int \mathfrak{Q}^A_{kl}\, d\tau \,. \tag{20}$$

Taking into account the form of $\boldsymbol{f}$ given in (13) and that of $\mathfrak{Q}^A$ given in (15), the equation of motion for the total angular momentum can be written as

$$\begin{aligned} \frac{d}{dt}\int \boldsymbol{r} \times \varrho_m \boldsymbol{v}\, d\tau &= \int \boldsymbol{r} \times \left[\varrho_e \boldsymbol{E} + \frac{1}{C}\varrho_e \boldsymbol{v} \times \boldsymbol{B}\right] d\tau + \\ &+ \int \boldsymbol{r} \times \frac{\mu}{2}\left[(\boldsymbol{s}\,\mathrm{grad})\,\boldsymbol{B} + \boldsymbol{s} \times \mathrm{rot}\,\boldsymbol{B}\right] d\tau + \\ &+ \int \boldsymbol{r} \times \left[(\boldsymbol{P}\,\mathrm{grad})\,\boldsymbol{E} + \boldsymbol{P} \times \mathrm{rot}\,\boldsymbol{E}\right] d\tau - \int \boldsymbol{r} \times 2\, m\lambda \frac{\partial}{\partial t}(\boldsymbol{s} \times \boldsymbol{E})\, d\tau + \\ &+ \frac{\mu}{2}\int \boldsymbol{s} \times \boldsymbol{B}\, d\tau + \int \boldsymbol{P} \times \boldsymbol{E}\, d\tau \,. \end{aligned} \tag{21}$$

(The total derivatives in (20) when integrated over the whole space obviously become zero.)

Equations (20) and (21) show the physical meaning of the decomposition

of the tensor into symmetric and antisymmetric parts: The symmetric part does not produce any resultant moment of force while the antisymmetric part gives the moment of the inner forces.

3. Equivalence of the equations of motion and the wave equation

It has already been shown that in accordance with condition 1) there is a one-to-one correspondence between the hydrodynamical variables and the wave function. The wave function and the related hydrodynamical variables satisfying the auxiliary condition will be referred to as corresponding quantities.

It still remains to be shown that the hydrodynamical variables and the related equations of motion also satisfy requirement 2).

According to point 2. the hydrodynamical equations of motion can be derived unambiguously from the wave equation. This means that the value of the hydrodynamical variables will be the same at any time t whether calculated by the wave equation or by the hydrodynamical equations of motion. In other words: if the hydrodynamical variables satisfying the auxiliary condition at $t = 0$ are known, their values at time t can be calculated in either of two ways:

a) by using the hydrodynamical equations,

b) by using the wave equation; i.e. at first the value of the wave function will be determined at $t = 0$ from the value of the hydrodynamical variables at $t = 0$. Then, knowing $\psi(0)$, the function $\psi(t)$ can be determined from the wave equation and from that we get the hydrodynamical variables at any time t.

Naturally, these two possibilities also exist for the determination of the wave function.

Similarly to the above, the wave function at time t can be evaluated either directly from the wave equation or from the hydrodynamical equation of motion and the result of the two procedures must again be identical. To prove this we should have had the wave equation derived from the hydrodynamical equations of motion.

This calculation though simple in principle is so cumbersome that we have not carried it out. Considering, however, that in the case of the Pauli equation we succeeded in obtaining the wave equation from the corresponding hydrodynamical equations of motion [2], it appears probable that a similar proof can be obtained here, too.

Finally, it should be noted that the fulfilment of the auxiliary condition needs to be considered only at $t = 0$, since, as already shown [2], it will then hold at any later time t.

4. Rotation and gauge invariance

Let the coordinate vector $\boldsymbol{r}$ transform by rotation R as

$$\boldsymbol{r}' = \mathfrak{D}(R)\,\boldsymbol{r}\,,$$

where $\mathfrak{D}(R)$ is a 3×3 orthogonal matrix. We put the transformed wave function in the form

$$\psi'(\boldsymbol{r}', t) = \mathfrak{U}(R)\,\psi(\boldsymbol{r}, t)\,,$$

where $\mathfrak{U}(R)$ is a 2×2 matrix dependent of the rotation R. The actual form of $\mathfrak{U}(R)$ can be determined from the requirement that the density $\varrho = \psi^+\psi$ should transform as a scalar and the spin vector $\boldsymbol{s} = \psi^+\,\boldsymbol{\sigma}\,\psi$ as a vector. These two conditions provide the following equations for the matrix

$$\mathfrak{U}^+(R)\,\mathfrak{U}(R) = 1\,,$$
$$\mathfrak{U}^+(R)\,\boldsymbol{\sigma}\mathfrak{U}(R) = \mathfrak{D}(R)\,\boldsymbol{\sigma}\,,$$

whence the matrix $\mathfrak{U}(R)$ can be determined up to an irrelevant phase factor. Using this $\mathfrak{U}(R)$ it is easy to show that $\boldsymbol{v}$ and $\boldsymbol{P}$ transform as

$$\boldsymbol{v}'(\boldsymbol{r}', t) = \mathfrak{D}(R)\,\boldsymbol{v}(\boldsymbol{r}, t)\,,$$
$$\boldsymbol{P}'(\boldsymbol{r}', t) = \mathfrak{D}(R)\,\boldsymbol{P}(\boldsymbol{r}, t)\,.$$

This results in all hydrodynamical variables showing the correct transformation properties.

A further possibility for the ambiguity of ψ arises from the fact that the wave equation contains explicitely the scalar (Φ) and the vector ($\boldsymbol{A}$) potentials of the electromagnetic field. As is well known, the electromagnetic field strengths are invariant under gauge-transformation:

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \operatorname{grad} f\,,$$
$$\Phi \to \Phi' = \Phi - \frac{1}{c}\,\frac{\partial f}{\partial t}\,,$$

where f is an arbitrary function of time and coordinates.

Replacing $\boldsymbol{A}$ and Φ by $\boldsymbol{A}'$ and Φ' and requiring the hydrodynamical variables to be gauge invariant, the wave function — as can be seen from Point 6. in Part IV — has to be transformed as

$$\psi \to \psi' = e^{i\frac{e}{\hbar c}f}\,\psi\,.$$

In order to prove this it has to be shown that the hydrodynamical variables are invariant at any time with respect to the above gauge transformation $\psi \to \psi'$, $\boldsymbol{A} \to \boldsymbol{A}'$, $\Phi \to \Phi'$. As an example, this will be illustrated for the polarization vector $\boldsymbol{P}$. Substituting the above expression of $\boldsymbol{A}', \Phi'$ and ψ' into (5) we get for $\boldsymbol{P}'$:

$$\boldsymbol{P}' = \lambda\psi^+ e^{-i\frac{e}{\hbar c}f}\left[\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)\times\boldsymbol{\sigma} - \boldsymbol{\sigma}\times\left(i\hbar\overleftarrow{\nabla} - \frac{e}{c}\boldsymbol{A}\right)\right]\psi e^{i\frac{e}{\hbar c}f} +$$

$$+ 2\lambda\psi^+ e^{-i\frac{e}{\hbar c}f}\left(\boldsymbol{\sigma}\times\frac{e}{c}\nabla f\right)\psi e^{i\frac{e}{\hbar c}f} = \boldsymbol{P}.$$

The same can be shown equally easily and briefly for the other variables.

Consequently, the hydrodynamical equations of motion are also gauge invariant.

5. Interpretation of the equations of motion

a) Let us consider first the force density in (13). By integrating equation (13) over the whole space the resulting force acting on the medium can be obtained. That part of the force density which can be expressed as the divergence of the tensor $\mathfrak{Q}$ gives zero on integration; i.e. Div $\mathfrak{Q}$ describes the density of the inner forces acting in the medium, the tensor itself being the stress tensor of these inner forces.

Let us see now the density of the outer force. The expression in the first brackets in (13) describes the density of the Lorentz force:

$$\boldsymbol{f}_L = \varrho_e \boldsymbol{E} + \frac{1}{c}\varrho_e \boldsymbol{v}\times\boldsymbol{B}, \tag{22}$$

while the force density acting on the permanent magnetic moment of the medium written in the second brackets is expressed as

$$\boldsymbol{f}_\mu = \frac{\mu}{2}\left[(\boldsymbol{s}\,\mathrm{grad})\,\boldsymbol{B} + \boldsymbol{s}\times\mathrm{rot}\,\boldsymbol{B}\right]. \tag{23}$$

The value of the coefficient in (23) (a half Bohr magneton: $\frac{\mu}{2}$) is originated in the coefficient of the rotation term of the expression of the velocity (see IV. eq. (21)). In this interpretation one half of the magnetic moment of the electron is considered as permanent magnetization, while the other half

can be attributed to convection currents arising in the electrically charged medium. The force acting on that latter is included in the Lorentz force (22) and $\boldsymbol{f}_\mu$ corresponds only to that acting on the permanent magnetic moment. Thus, we get the proper value for the gyromagnetic factor of the electron, in good agreement with the Einstein—de Haas experiment (see [2]).

The first term in (23) stands for the translation force acting on the magnetic moment $\frac{\mu}{2}\boldsymbol{s}$ in an inhomogeneous magnetic field of field strength $\boldsymbol{B}$. The second term, however, requires more detailed consideration. The force

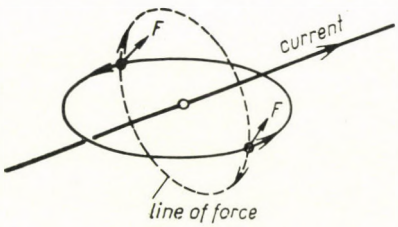

Fig. 1

$\frac{\mu}{2}\boldsymbol{s} \times \operatorname{rot}\boldsymbol{B}$ differs from zero only if conduction (or displacement) current flows at the place of the magnetic moment. Take the simple model represented in Fig. 1. Let the magnetic moment be represented by a small closed circuit. Insert a straight wire through the surface bounded by this circuit. In that case the resultant Lorentz force exerted by the magnetic field of the current flowing through the wire has the actual value $\frac{\mu}{2}\boldsymbol{s} \times \operatorname{rot}\boldsymbol{B}$. Placing the wire outside the circuit, at two points facing each other on the circuit the Lorentz forces act in opposite directions. In this case only the first term from (23) contributes to the expression of the force, i.e. the resulting force can be attributed solely to the difference in the field strengths at those points.

Let us see now how the above model can be formulated mathematically. If the current density of the circuit is $\boldsymbol{j}(\boldsymbol{r})$, the density of the force acting on it will have the form

$$\boldsymbol{f} = \frac{1}{c}\int \boldsymbol{j}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})\, d\tau\,. \tag{24}$$

Suppose the diameter of the circuit is so small that in this range no appreciable variation takes place in the value of $\boldsymbol{B}(\boldsymbol{r})$. Expanding the magnetic field strength in series in the vicinity of the point $\boldsymbol{r}_0$ of the area bounded by

the circuit and introducing the abbreviations

$$\boldsymbol{B}(\boldsymbol{r}_0) = \boldsymbol{B}_0 \qquad \text{and} \qquad \boldsymbol{r} - \boldsymbol{r}_0 = \boldsymbol{R},$$

we get

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_0 + (\boldsymbol{R}\nabla_0)\,\boldsymbol{B}_0. \tag{25}$$

(The index $_0$ refers to derivatives with respect to $\boldsymbol{r}_0$).

Introducing (25) into (24) after some calculations (for details see [3]) the force can be obtained in the form:

$$\boldsymbol{f} = (\boldsymbol{m}\,\mathrm{grad}_0)\,\boldsymbol{B}_0 + \boldsymbol{m}\times\mathrm{rot}_0\boldsymbol{B}_0, \tag{26}$$

where $\boldsymbol{m}$ is the magnetic moment of the circuit, i.e.

$$\boldsymbol{m} = \frac{1}{2c}\int \boldsymbol{R}\times\boldsymbol{j}\,d\tau.$$

Putting the magnetic moment density $\frac{\mu}{2}\boldsymbol{s}$ in (26) instead of $\boldsymbol{m}$ we get for the density of the magnetic force exactly the expression given in (23).

The next term of the force density in (13) is the electric equivalent of $\boldsymbol{f}_\mu$ given as

$$\boldsymbol{f}_p = (\boldsymbol{P}\,\mathrm{grad})\,\boldsymbol{E} + \boldsymbol{P}\times\mathrm{rot}\,\boldsymbol{E}. \tag{27}$$

The electric dipole moment can always be represented — independently of its origin — by a bar-magnet rotating around a perpendicular axis. Thus, expression (27) can be explained in analogy to the considerations described above for the magnetic forces with the only difference that in this case the circular lines of the electric force occur due to induction produced by the variation of the magnetic field.

The following term in expression (13) is the force density containing a time derivative, namely:

$$\boldsymbol{f}_t = -\,2\,m\lambda\,\frac{\partial}{\partial t}(\boldsymbol{s}\times\boldsymbol{E}). \tag{28}$$

Let us again represent the magnetic dipole as a current flowing in a circular wire. The current induced by switching on the electromagnetic field tends to reduce the magnetic moment. f_t represents the density of force needed to keep the magnetic moment constant.

In connection with the force density (28) the question arises whether it would not have been more convenient to substitute $\frac{\partial \boldsymbol{E}}{\partial t}$ by $c\,\mathrm{rot}\,\boldsymbol{B}$ from the Maxwell equations, especially because in this way the second term in (23)

also could be eliminated. However, as is apparant from the above, the terms (28) and (23) have their particular physical meaning. Moreover, if taking into account the anomalous magnetic moment of the electron $[(1 + \varkappa)\mu]$ and repeating the hydrodynamical calculus, the reduction of the two terms in question gives instead of zero

$$2\varkappa \frac{2\varkappa + 1}{4\varkappa + 4} \frac{\mu}{2} \mathbf{s} \times \operatorname{rot} \boldsymbol{B}$$

with $\varkappa = 0.0011453$.

b) In the equation of motion (16) of the magnetic polarization vector $\boldsymbol{T}$ the term Div $\mathfrak{Q}'$ represents the internal moment of force and it vanishes when integrating over the whole space. The terms in the square brackets yield the moment of force exerted by the external electromagnetic field on the medium when inhomogeneously magnetized. The interpretation of the first two terms of the right-hand side of (16) requires some more detailed consideration. Taking into account that the magnetic field strength detected by the electron in its proper system is instead of $\boldsymbol{B}$

$$\boldsymbol{B}' = \boldsymbol{B} - \frac{1}{c} \boldsymbol{v} \times \boldsymbol{E}$$

and that owing to the Thomas precession the actual value of $\boldsymbol{E}$ reduces to $\frac{1}{2}\boldsymbol{E}$, the effective magnetic field acting on the electron will be

$$\boldsymbol{B}_{\text{eff}} = \boldsymbol{B} - \frac{1}{2c} \boldsymbol{v} \times \boldsymbol{E}. \tag{29}$$

Thus the density of the moment of force acting on a magnet of momentum $\boldsymbol{m} = \mu \mathbf{s}$ in the field $\boldsymbol{B}_{eff}$ can be written as

$$\boldsymbol{N}_T = \mu \mathbf{s} \times \boldsymbol{B}_{\text{eff}} = \mu \varrho \boldsymbol{T} \times \boldsymbol{B} - \frac{\mu}{2c} \varrho \boldsymbol{T} \times (\boldsymbol{v} \times \boldsymbol{E}), \tag{30}$$

which is the same as the first two terms of the right-hand side of (16) multiplied by $\hbar/2$.

Let us now see a somewhat more exact deduction of the above expression. The potential energy of a particle of magnetic moment $\boldsymbol{m}$ moving with velocity $\boldsymbol{v}$ in an electromagnetic field $\boldsymbol{B}$ and $\boldsymbol{E}$, as shown in IV. (5), is:

$$\mathscr{E} = -\boldsymbol{m} \left(\boldsymbol{B} - \frac{1}{c} \boldsymbol{v} \times \boldsymbol{E} \right) - \frac{1}{2c} \boldsymbol{m} (\boldsymbol{v} \times \boldsymbol{E}), \tag{31}$$

where the last term is originated from the Thomas precession. Rotating the magnetic dipole by the angular vector $\delta\boldsymbol{\varphi}$, the momentum will change with

$$\delta\boldsymbol{m} = \delta\boldsymbol{\varphi} \times \boldsymbol{m} ,$$

which results in a change of the potential energy:

$$-\delta\mathscr{E} = \left[\boldsymbol{m} \times \left(\boldsymbol{B} - \frac{1}{c}\boldsymbol{v} \times \boldsymbol{E}\right) + \frac{1}{2c}(\boldsymbol{m} \times (\boldsymbol{v} \times \boldsymbol{E}))\right]\delta\boldsymbol{\varphi} . \tag{32}$$

This loss in the potential energy will be compensated for by the work done by the moment of the outer forces during the rotation of the magnetic moment, i.e.

$$-\delta\mathscr{E} = \boldsymbol{N}_m \, \delta\boldsymbol{\varphi} , \tag{33}$$

where $\boldsymbol{N}_m$ is the moment of the outer forces acting on a magnet of momentum $\boldsymbol{m}$.

Substituting (33) into (32) and comparing the two sides of the new equation we get for $\boldsymbol{N}_m$ exactly the expression discussed.

c) For the interpretation of the density of moment of force

$$\boldsymbol{N}_p = -2e\lambda\varrho\,(\boldsymbol{T} \times \boldsymbol{E}) \tag{34}$$

acting on the electric polarization vector, as can be seen in equ. (17), let us write the interaction energy $\mathscr{E}$ in the form:

$$\mathscr{E} = -\boldsymbol{m}\boldsymbol{B} - \boldsymbol{p}\boldsymbol{E} .$$

Comparing this with (30) we obtain for the electric polarization

$$\boldsymbol{p} = -\frac{1}{2c}\boldsymbol{m} \times \boldsymbol{v} .$$

Derivating it with respect to time we get

$$\frac{d\boldsymbol{p}}{dt} = -\frac{1}{2c}\frac{d\boldsymbol{m}}{dt} \times \boldsymbol{v} - \frac{1}{2c}\boldsymbol{m} \times \frac{d\boldsymbol{v}}{dt} . \tag{35}$$

Substituting $\dfrac{d\boldsymbol{v}}{dt}$ from (13) and for $\boldsymbol{m} = \mu\varrho\,\boldsymbol{T}$ the expression $\dfrac{d\boldsymbol{m}}{dt}$ from (16) and taking into account that the terms of second or higher order in v/c are neglected in the course of this calculation, (35) takes the form:

$$\frac{d\boldsymbol{p}}{dt} \equiv \boldsymbol{N}_p = -\frac{e}{2mc}\boldsymbol{m} \times \boldsymbol{E}$$

which agrees with (34).

The moment of force occurring in the equation of motion of the angular momentum (21) can be interpreted in a similar way as was done above in connection with the force terms of the equation of motion of the velocity.

The term in the first bracket is the moment of the Lorentz-force. The meaning of the second integral can be interpreted by using again the circuit model. The moment of the outer magnetic field acting on the circuit is:

$$\boldsymbol{n} = \frac{1}{c}\int \boldsymbol{r}\times\big(\boldsymbol{j}(\boldsymbol{r})\times\boldsymbol{B}(\boldsymbol{r})\big)\,d\tau\,.$$

Using the expansion from (26) and in accordance with (21) we have

$$\boldsymbol{n} = \boldsymbol{m}\times\boldsymbol{B}_0 + \boldsymbol{r}_0\times\big[(\boldsymbol{m}\nabla_J)\,\boldsymbol{B}_0 + \boldsymbol{m}\times\mathrm{rot}_0\,\boldsymbol{B}_0\big].$$

A similar formula can also be derived for electric polarization. In expression (21) the term containing the time derivative is the moment of force given in (28). Thus, finally the moment of force originating from both the inner forces and the moment of the outer forces have been interpreted.

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ГИДРОДИНАМИЧЕСКАЯ МОДЕЛЬ ВОЛНОВОЙ МЕХАНИКИ V.

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Резюме

В работе выводится соответствующее гидродинамическому уравнению движения волновое уравнение, принимающее во внимание спин-орбитальное взаимодействие. Наряду с этим дается физическая интерпретация данного уравнения. После представления уравнений движения, выведенных в предыдущей работе для гидродинамических переменных (1), показывается, что связь между гидродинамическими уравнениями и волновым уравнением взаимно однозначна. Наконец, с физической точки зрения истолкуются выражения для сил, пяовляющиеся в новых уравнениях.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (3), pp. 239—246 (1969)

ON THE SPUR OF THE PRODUCT OF DIRAC MATRICES

By

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(Received 10. VII. 1968)

In this paper several methods for the evaluation of the spur of the product of γ_5 matrix and any number of Dirac matrices have been established. The formulae for the determination of the various types of products of two spurs when γ_i, $\gamma_i\gamma_j$, $\gamma_i\gamma_j\gamma_k$, $\gamma_i\gamma_j\gamma_k\gamma_l$ and these terms associated with γ_5 matrix occur in both of the spurs, have been derived. It has been shown that the product of two spurs each involving γ_5 matrix, can be equated to an expression involving no γ_5 matrix. One useful identity for spurs involving γ_5 matrix has been deduced.

Introduction

It is necessary to evaluate the spurs of products of Dirac matrices when we deal with various types of problems of transition probabilities in electrodynamics, meson theories and weak interaction processes. Spurs involving γ_5 matrix generally occur in weak interaction problems and in the calculation of the polarization of particles. One of the purposes of this paper is to reduce the problem of the calculation of the spur of the product of γ_5 matrix and an arbitrary number of Dirac matrices to one involving γ_5 matrix and a smaller number of Dirac matrices. Continuing this process we shall come to the stage when we have to calculate the spur of the product of γ_5 matrix and four Dirac matrices. We have also derived formulae for the evaluation of various types of products of two spurs when γ_i, $\gamma_i\gamma_j$, $\gamma_i\gamma_j\gamma_k$ and $\gamma_i\gamma_j\gamma_k\gamma_l$ terms occur in both of them. These formulae are also extended to the case in which γ_5 terms occur in any one or in both of the spurs. We have also reduced an expression involving the product of two spurs, with γ_5 occurring in both of them, to one involving no γ_5 matrix. One useful identity for spurs involving γ_5 matrix has been deduced. With the help of these formulae we have given methods for determining the spur of the product of any number of Dirac matrices. Finally, we have obtained a formula which shows that the spur of the product of γ_5 matrix and an arbitrary number of Dirac matrices can be equated to a series of terms each of which involves the spur of the product of γ_5 matrix and any four of the Dirac matrices and the spur of the product of the remaining Dirac matrices. In this connection we may mention that CAIANIELLO and FUBINI [1], and CHISHOLM [2] have investigated various aspects of the problem of the determination of the spur of product of Dirac matrices.

Calculation

All the formulae derived here are based on the following anticommutation property of Dirac matrices,

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}. \tag{1}$$

Let us use the abbreviated notation

$$\text{spur } (A_1A_2A_3 \ \ldots \ A_n) = (A_1A_2A_3 \ \ldots \ A_n) \tag{2}$$

$$A = A_i\gamma_i \qquad \ldots \tag{3}$$

$$A \cdot B = A_i B_i \qquad \ldots \tag{4}$$

$$\gamma_5 = \gamma_1\ \gamma_2\ \gamma_3\ \gamma_4 \qquad \ldots \tag{5}$$

Summation over $i = 1,2,3,4$ is to be done. Throughout this paper summation is implied whenever repeated suffixes occur.

Let us denote $(\gamma_5\ A_1A_2A_3A_4 \ \ldots)$ by S.
We can write

$$S = \sum_{i,j,k} A_{1i}A_{2j}A_{3k}\,(\gamma_5\gamma_i\gamma_j\gamma_k\,A_4A_5A_6 \ \ldots). \tag{6}$$

The summation $\sum_{i,j,k}$ in equation (6) can be split up in the following manner

$$\sum_{i,j,k} = \sum_{i\neq j\neq k} + \sum_{i=j\neq k} + \sum_{j=k\neq i} + \sum_{k=i\neq j} + \sum_{i=j=k} \cdots . \tag{7}$$

Now we have

$$\sum_{i\neq j\neq k} = -\sum_{\substack{i<j<k \\ l\neq i,j,k}} (-1)^l\ (\gamma_5\gamma_l\gamma_5\,A_4A_5A_6\ldots) \quad \sum_P \delta_P\,P[A_{1i}A_{2j}A_{3k}]. \tag{8}$$

In equation (8) and elsewhere, P is any permutation of the suffixes (in this case i, j and k) and $\delta_p = \pm 1$ depending on whether P is an even or odd permutation. The notation $\sum_P$ denotes summation over all the possible permutations.

With the help of equations (6), (7) and (8) we obtain

$$S = (\gamma_5\,[A_1 \cdot A_2A_3 - A_1 \cdot A_3A_2 + A_2 \cdot A_3A_1]\ A_4A_5A_6 \ldots) + \frac{1}{4}\sum_{i\geq 4}(-1)^i\ (\gamma_5A_1A_2A_3A_i)\ (A_4A_5 \ldots A_{i-1}\,A_{i+1} \ldots). \tag{9}$$

We can rewrite equation (8) in the form

$$\sum_{i\neq j\neq k} = \sum_{i\neq j\neq k} A_{1i}A_{2j}A_{3k}A_{4l}\ (\gamma_i\gamma_j\gamma_k\gamma_l\gamma_5\,A_5A_6\ldots)$$

$$= \frac{1}{4}(\gamma_5A_1A_2A_3A_4)\,(A_5A_6\ldots) + \sum_{\substack{i\neq j\neq k \\ l=i}} + \sum_{\substack{i\neq j\neq k \\ l=j}} + \sum_{\substack{i\neq j\neq k \\ l=k}} . \tag{10}$$

Application of equations (7) and (10) provides us with the second formula for S, as given below

$$S = (\gamma_5[A_1 \cdot A_2A_3A_4 - A_1 \cdot A_3A_2A_4 + A_1 \cdot A_4A_2A_3 + A_2 \cdot A_3A_1A_4 - A_2 \cdot A_4A_1A_3 + A_3 \cdot A_4A_1A_2]\, A_5A_6 \ldots) - \frac{1}{4}(A_1A_2A_3A_4)\ (\gamma_5A_5A_6A_7A_8 \ldots) + \frac{1}{4}(\gamma_5A_1A_2A_3A_4)\ (A_5A_6 \ldots). \tag{11}$$

Let us now discuss some relations involving the product of two spurs. We have

$$(\gamma_iA_1A_2A_3 \ldots)\ (\gamma_iA_1'A_2'A_3' \ldots) = \sum_i (-1)^{i+1}\ (A_1A_2 \ldots A_{i-l}\ A_{i+l} \ldots) \cdot (A_iA_1'A_2'A_3' \ldots) \tag{12}$$

With the help of equation (12) we obtain

$$(\gamma_i\gamma_jA_1A_2 \ldots)\ (\gamma_i\gamma_jA_1'A_2' \ldots) = 4(A_1A_2 \ldots)\ (A_1'A_2' \ldots) - 2\sum_{i>j}(-1)^{i+j}\,(A_1A_2 \ldots)''\,[(A_iA_jA_1'A_2' \ldots) - A_i \cdot A_j(A_1'A_2' \ldots)]. \tag{13}$$

In equation (13), the notation of double primes over the spur $(A_1A_2 \ldots)''$ implies that those two unprimed Dirac matrices (in this case A_i and A_j) which are now present in the other term, are now absent from the spur $(A_1A_2 \ldots)$. This notation in the general form with any number of primes over the notation of spur will be widely used in this paper.

We can write,

$$(\gamma_5\gamma_iA_1A_2A_3 \ldots)\ (\gamma_5\gamma_iA_1'A_2'A_3' \ldots) = \sum_{i<j<k}(\gamma_i\gamma_j\gamma_kA_1A_2A_3 \ldots)\ (\gamma_i\gamma_j\gamma_kA_1'A_2'A_3' \ldots) \tag{14}$$

$$= \frac{1}{6}\sum_{i\neq j\neq k}(\gamma_i\gamma_j\gamma_kA_1A_2A_3 \ldots)\ (\gamma_i\gamma_j\gamma_kA_1'A_2'A_3' \ldots). \tag{15}$$

Applying equation (15) we can write

$$(\gamma_5\gamma_i\gamma_jA_1A_2 \ldots)\ (\gamma_5\gamma_i\gamma_jA_1'A_2' \ldots) = 4(\gamma_5A_1A_2A_3A_4 \ldots)\ (\gamma_5A_1'A_2'A_3'A_4' \ldots) + \sum_{i\neq j}(\gamma_i\gamma_jA_1A_2 \ldots)\ (\gamma_i\gamma_jA_1'A_2' \ldots)$$

$$= 4(\gamma_5A_1A_2A_3A_4 \ldots)\ (\gamma_5A_1'A_2'A_3'A_4' \ldots) - 2\sum_{i>j}(-1)^{i+j}\ (A_1A_2 \ldots)''\,[(A_iA_jA_1'A_2' \ldots) - A_i \cdot A_j(A_1'A_2' \ldots)] \tag{16}$$

Let us discuss some particular cases of equations (15) and (16).
We have,

$$(\gamma_i\gamma_jA_1A_2)\ (\gamma_i\gamma_jA_1'A_2' \ldots) = 8\,[(A_2A_1A_1'A_2' \ldots) + A_2 \cdot A_1(A_1'A_2' \ldots)] \tag{17}$$

$$(\gamma_5\gamma_i\gamma_jA_1A_2)\ (\gamma_5\gamma_i\gamma_jA_1'A_2' \ldots) = 8\,[(A_2A_1A_1'A_2' \ldots) - A_2 \cdot A_1(A_1'A_2' \ldots)], \tag{18}$$

$$(\gamma_5\gamma_i\gamma_jA_1A_2)\ (\gamma_5\gamma_i\gamma_jA_1'A_2') = 32\ [A_2 \cdot A_2'A_1 \cdot A_1' - A_2 \cdot A_1'A_1 \cdot A_2']. \tag{19}$$

With the repeated application of equation (12) we arrive at

$$(\gamma_i\gamma_j\gamma_k A_1A_2A_3\ldots)\ (\gamma_i\gamma_j\gamma_k A_1'A_2'A_3'\ldots..)$$
$$=\sum_{i>j>k}(-1)^{i+j+k}(A_1A_2....)'''\sum_P \delta_P P(A_iA_jA_kA_1'A_2'A_3'..)$$
$$-10\sum_i(-1)^i\ (A_1A_2\ldots.A_{i-1}A_{i+1}\ldots)\ (A_iA_1'A_2'A_3'\ldots..). \tag{20}$$

It can be shown that

$$\sum_P \delta_P P(A_iA_jA_kA_1'A_2'A_3'\ldots)$$
$$6=([A_iA_jA_k-A_i\cdot A_jA_k+A_i\cdot A_kA_j-A_j\cdot A_kA_i]A_1'A_2'A_3'..). \tag{21}$$

We can write

$$(\gamma_i\gamma_j\gamma_kA_1A_2A_3\ldots)\ (\gamma_i\gamma_j\gamma_kA_1'A_2'A_3'\ldots)=\sum_{i\neq j\neq k}(\gamma_i\gamma_j\gamma_kA_1A_2A_3\ldots)$$
$$(\gamma_i\gamma_j\gamma_kA_1'A_2'A_3'\ldots)+10(\gamma_iA_1A_2A_3\ldots)\ (\gamma_i\ A_1'A_2'A_3'\ldots). \tag{22}$$

Equation (22) combined with equation (15) enables us to establish

$$(\gamma_i\gamma_j\gamma_kA_1A_2A_3\ldots)\ (\gamma_i\gamma_j\gamma_kA_1'A_2'A_3'\ldots)=6(\gamma_5\gamma_iA_1A_2A_3\ldots)\ (\gamma_5\gamma_iA_1'A_2'A_3'\ldots)$$
$$+10(\gamma_iA_1A_2A_3\ldots)\ (\gamma_iA_1'A_2'A_3'\ldots). \tag{23}$$

Like equation (22) we have the relation

$$(\gamma_5\gamma_i\gamma_j\gamma_kA_1A_2A_3\ldots)\ (\gamma_5\gamma_i\gamma_j\gamma_kA_1'A_2'A_3'\ldots)=6\ (\gamma_iA_1A_2A_3\ldots)\ (\gamma_iA_1'A_2'A_3'\ldots)$$
$$+10(\gamma_5\gamma_iA_1A_2A_3\ldots)\ (\gamma_5\gamma_iA_1'A_2'A_3'\ldots). \tag{24}$$

Comparing equations (20) and (23) we have

$$(\gamma_5\gamma_iA_1A_2A_3\ldots.)\ (\gamma_5\gamma_iA_1'A_2'A_3'\ldots)$$
$$=\frac{1}{6}\sum_{r>s>t}(-1)^{r+s+t}(A_1A_2\ldots)'''\sum_P\delta_P P(A_rA_sA_tA_1'A_2'A_3'\ldots)\ldots \tag{25}$$
$$=\sum_{r>s>t}(-1)^{r+s+t}\ (A_1A_2\ldots)'''\quad\sum_{i>j}(-1)^{i+j+1}(A_r\cdot A_i'A_s\cdot A_j'-$$
$$-A_r\cdot A_j'A_s\cdot A_i')(A_tA_1'A_2'A_3'\ldots)''\ldots \tag{26}$$

A special case of equation (26) yields

$$(\gamma_5\gamma_iA_1A_2A_3)(\gamma_5\gamma_iA_1'A_2'A_3'..)=4\sum_{i>j}(-1)^{i+j+1}(A_3\cdot A_i'A_2\cdot A_j'-A_3\cdot A_j'A_2\cdot A_i')$$
$$(A_tA_1'A_2'A_3'\ldots.)''\ldots \tag{27}$$

and

$$(\gamma_5\gamma_iA_1A_2A_3)\ (\gamma_5\gamma_iA_1'A_2'A_3')=16A_1\cdot A_1'(A_2\cdot A_2'A_3\cdot A_3'-A_2\cdot A_3'A_3\cdot A_2')$$
$$-16A_1\cdot A_2'(A_3\cdot A_3'A_2\cdot A_1'-A_3\cdot A_1'A_2\cdot A_3')+16A_1\cdot A_3'(A_3\cdot A_2'A_2\cdot$$
$$\cdot A_1'-A_3\cdot A_1'A_2\cdot A_2')\ldots \tag{28}$$

With the help of equation (23) we can write

$$(\gamma_i\gamma_j\gamma_k\gamma_l A_1'A_2'\ldots)\,(\gamma_i\gamma_j\gamma_k\gamma_l A_1'A_2'\ldots) = 6\,(\gamma_5\gamma_i\gamma_j A_1A_2\ldots)\,(\gamma_5\gamma_i\gamma_j A_1'A_2'\ldots)$$
$$+ 10\,(\gamma_i\gamma_j A_1A_2\ldots)\,(\gamma_i\gamma_j A_1'A_2'\ldots\ldots) \tag{29}$$
$$= 24\,(\gamma_5 A_1A_2A_3A_4\ldots)\,(\gamma_5 A_1'A_2'A_3'A_4'\ldots) + 40\,(A_1A_2\ldots)\,(A_1'A_2'\ldots)$$
$$-32\sum_{i>j}(-1)^{i+j}\,(A_1A_2\ldots)^{//}\left[(A_iA_jA_1'A_2'\ldots) - A_i\cdot A_j\cdot(A_1'A_2'\ldots)\right. . \tag{30}$$

Equation (30) is deduced with the help of equations (13), (16) and (29).

By repeatedly applying equation (12) as in the deduction of relation (20) we obtain the expression

$$(\gamma_i\gamma_j\gamma_k\gamma_l A_1A_2\ldots)\,(\gamma_i\gamma_j\gamma_k\gamma_l A_1'A_2'\ldots) = 40\,(A_1A_2\ldots)\,(A_1'A_2\ldots)$$
$$-32\sum_{i>j}(-1)^{i+j}\,(A_1A_2\ldots)^{//}\left[(A_iA_jA_1'A_2'\ldots) - A_lA_j(A_1'A_2'\ldots)\right]$$
$$+\sum_{i>j>k>l}(-1)^{i+j+k+l}(A_1A_2\ldots)^{////}\sum_P\delta_P\,P(A_iA_jA_kA_lA_1'A_2'\ldots) \tag{31}$$

We have

$$\sum_P\delta_P\,P(A_iA_jA_kA_lA_1'A_2'\ldots) = 24\left([A_iA_jA_kA_l - A_i\cdot A_jA_kA_l + A_i\cdot A_kA_jA_l -\right.$$
$$-\left. A_i\cdot A_lA_jA_k - A_j\cdot A_kA_iA_l + A_j\cdot A_lA_iA_k - A_k\cdot A_lA_iA_j]\,A_1'A_2'\ldots\right) +$$
$$+ 6(A_iA_jA_kA_l)\,(A_1'A_2'\ldots) \tag{32}$$

Like equation (29) we have the relation

$$(\gamma_5\gamma_i\gamma_j\gamma_k\gamma_l\,A_1A_2\ldots)\,(\gamma_5\gamma_i\gamma_j\gamma_k\gamma_l\,A_1'A_2'\ldots) = 6(\gamma_i\gamma_j\,A_1A_2)\,(\gamma_i\gamma_j\,A_1'A_2'\ldots)$$
$$+ 10(\gamma_5\gamma_i\gamma_j\,A_1A_2\ldots)\,(\gamma_5\gamma_i\gamma_j\,A_1'A_2'\ldots) \tag{33}$$

Equations (12), (13), (23) and (29) hold even when γ_5 is introduced at the beginning of the second spur terms occurring on both the right-hand and left-hand sides of these equations.

From equations (30), (31) and (32) we obtain the relation

$$(\gamma_5\,A_1'A_2'A_3'A_4'\ldots)\,(\gamma_5\,A_1A_2A_3A_4\ldots)$$
$$= \frac{1}{24}\sum_{r>s>t>u}(-1)^{r+s+t+u}\,(A_1'A_2'\ldots)^{////}\sum_P\delta_P\,P(A_r'A_s'A_t'A_u'A_1A_2\ldots)$$
$$= \sum_{r>s>t>u}(-1)^{r+s+t+u}\,(A_1'A_2'\ldots)^{////}\sum_{i>j>k>l}(-1)^{i+j+k+l}(A_1A_2\ldots)^{////}$$
$$\cdot\sum_P\delta_P\,P[A_r'\cdot A_i\,A_s'\cdot A_j\,A_t'\cdot A_k\,A_u'\cdot A_l]\,. \tag{34}$$

The general relation (34) leads to the particular relations

$$(\gamma_5\,A_1'A_2'A_3'A_4')\,(\gamma_5\,A_1A_2A_3A_4\ldots)$$
$$= 4\sum_{i>j>k>l}(-1)^{i+j+k+l}\,(A_1A_2\ldots)^{////}\sum_P\delta_P\,P[A_4'\cdot A_i\,A_3'\cdot A_j\,A_2'\cdot A_k\,A_1'\cdot A_l] \tag{35}$$

and

$$(\gamma_5 \mathrm{ABCD})\,(\gamma_5 A_iA_jA_kA_l) = 16 \sum_P \delta_P P[A\cdot A_i\, B\cdot A_j C\cdot A_k D\cdot A_l]. \qquad (36)$$

Applying equations (11) and (35) we get,

$$\begin{aligned}&(A_1A_2A_3A_4\,..)\;(A_1A_2A_3A_4\ldots\gamma_5\gamma_5)\\ &= ([A_1\cdot A_2A_3A_4 - A_1\cdot A_3A_2A_4 + A_1\cdot A_4A_2A_3 + A_2\cdot A_3A_1A_4 - A_2\cdot\\ &\quad A_4A_1A_3 + A_3\cdot A_4A_1A_2]A_5A_6\ldots) - \frac{1}{4}(A_1A_2A_3A_4)\;(A_5A_6\,..)\\ &+ \sum_{i>j>k>l\geq 5}(-1)^{i+j+k+l}(A_5A_6\ldots)^{////}\sum_P \delta_P P[A_4\cdot A_i A_3\cdot A_j A_2\cdot A_k A_1\cdot A_l]. \qquad (37)\end{aligned}$$

In equations (34), (35), (36) and (37) the permutation P of the variables A_i, A_j, A_k and A_l is implied.

Let us obtain another formula for $(A_1A_2A_3A_4\ldots.)$ with the help of equations (25) and (27)

$$\begin{aligned}&\frac{1}{4}(\gamma_5\gamma_iA_1A_2A_3)\;(\gamma_5\gamma_iA_4A_5A_6\ldots) = ([A_1\cdot A_2A_3 - A_1\cdot A_3A_2 + A_2\cdot A_3A_1 -\\ &A_1\cdot A_2A_3]A_4A_5A_6\ldots) = \frac{1}{4}(\gamma_iA_1A_2A_3)\,(\gamma_iA_4A_5A_6\ldots) - (A_1A_2A_3A_4A_5A_6\ldots).\end{aligned} \qquad (38)$$

Thus, we have

$$\begin{aligned}(A_1A_2A_3A_4\ldots) &= \frac{1}{4}\sum_{i\geq 4}(-1)^i\;(A_1A_2A_3A_i)\;(A_4A_5\ldots A_{i-1}A_{i+1}\ldots.)\\ &\quad - \sum_{i>j\geq 4}(-1)^{i+j}\;(A_1\cdot A_iA_2\cdot A_j - A_1\cdot A_jA_2\cdot A_i)\,(A_3A_4\ldots.)^{//}. \qquad (39)\end{aligned}$$

Equation (37) can be used to get the third formula for S, which is written for $(\gamma_5A_1A_2A_3A_4\ldots.)$

$$S = \frac{1}{4}\sum_{i>j>k>l}(-1)^{i+j+k+l}(\gamma_5A_iA_jA_kA_l)\,(A_1A_2A_3A_4\ldots)^{////}. \qquad (40)$$

We can write

$$\begin{aligned}&(A_1A_2A_3A_4\ldots A_n\gamma_1\gamma_2\gamma_3\gamma_4)\\ &= \sum_{i=2}^{n}(-1)^i A_1\cdot A_i\,(A_2A_3\ldots A_{i-1}A_{i+1}\ldots A_n\gamma_5) + \sum_i A_{1i}(A_2A_3A_4\ldots A_n\gamma_5\gamma_i).\end{aligned}$$

Then we have the identity

$$\sum_{i=2}^{n}(-1)^i A_1\cdot A_i(A_2A_3\ldots A_{i-1}A_{i+1}\ldots A_n\gamma_5) = 0\ldots. \qquad (41)$$

For $n = 8$, the identity (41) can be rewritten with the help of equation (40) in the form

$$\sum_{i<j<k} (-1)^{i+j+k} (A_1 A_i A_j A_k) (A_2 A_3 A_4 \ldots A_8 \gamma_5)^{///} = 0 \ldots . \tag{42}$$

Now we can write

$$T = (\gamma_i A_1 A_2 \ldots . A_n \gamma_5) \ (\gamma_i A_1' A_2' A_3' \ldots .)$$
$$= \sum_{i=1}^{n} (-1)^{i+1} (A_1 A_2 A_3 \ldots A_{i-1} A_{i+1} \ldots A_n \gamma_5) (A_i A_1' A_2' A_3' \ldots) + T \tag{43}$$

and obtain the identity

$$\sum_{i=1}^{n} (-1)^i (A_1 A_2 A_3 \ldots A_{i-1} A_{i+1} \ldots A_n \gamma_5) (A_i A_1' A_2' A_3' \ldots) = 0 \tag{44}$$

Since we have the equality,

$$(A_1 A_2 A_3 \ldots . A_n \gamma_5) = \frac{1}{4} (\gamma_i A_2 A_3 \ldots . A_n \gamma_5) \ (\gamma_i A_1). \tag{45}$$

Equation (41) can be thought of as a special case of equation (44). We can obtain other identities when we take γ_i, $\gamma_i \gamma_j$, $\gamma_i \gamma_j \gamma_k$ and $\gamma_i \gamma_j \gamma_k \gamma_l$ successively instead of γ_i which occurs simultaneously in the two spur terms of equation (43).

We have given three equations (9), (11) and (40) for determining $(\gamma_5 A_1 A_2 A_3 \ldots . A_n)$.

For $n = 6$, we get according to equation (9)

$$S_6 = (\gamma_5 A_1 A_2 A_3 A_4 A_5 A_6)$$
$$= A_1 \cdot A_2 (\gamma_5 A_3 A_4 A_5 A_6) - A_1 \cdot A_3 (\gamma_5 A_2 A_4 A_5 A_6) + A_2 \cdot A_3 (\gamma_5 A_1 A_4 A_5 A_6)$$
$$+ A_5 \cdot A_6 (\gamma_5 A_1 A_2 A_3 A_4) - A_4 \cdot A_6 (\gamma_5 A_1 A_2 A_3 A_5) + A_4 \cdot A_5 (\gamma_5 A_1 A_2 A_3 A_6). \tag{46}$$

Equations (11) and (40) give 7 and 15 terms respectively for S_6 instead of 6 terms which occur on the right hand side of equation (46) when we apply equation (9) to determine S_6. These 7 and 15 terms can be reduced to 6 terms if the identity given by equation (41) is used. For $n = 8$ we get 33, 48 and 210 terms of the type which occur in equation (46) if we apply equations (9), (11) and (40) respectively for evaluating S_8. Again the 48 and 210 terms can be reduced to 33 terms with the help of equations (41) and (42). It is seen that equation (9) is most convenient for determining $(\gamma_5 A_1 A_2 A_3 \ldots A_n)$.

Acknowledgement

The author wishes to thank Professor D. Basu and Dr. N. C. Sil for stimulating discussions.

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О ШПУРЕ ПРОИЗВЕДЕНИЯ МАТРИЦ ДИРАКА

С. САРКАР

Резюме

В данной работе устанавливаются методы для определения шпура произведения матрицы γ_5 и любого числа матриц Дирака. Выводятся формулы для вычисления произведений различного типа двух шпуров в случае, когда в обоих шпурах появляются γ_i, $\gamma_i\gamma_j$, $\gamma_i\gamma_j\gamma_k$, $\gamma_i\gamma_j\gamma_k\gamma_l$ и эти же члены, связанные с матрицей γ_5. Показывается, что произведение двух шпуров, каждый из которых содержит матрицу γ_5, может равняется выражению, не содержащему матрицу γ_5. Для шпуров, содержащих матрицу γ_5, выводится полезное тождество.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (3), pp. 247—259 (1969)

INVESTIGATION OF THE DENSITY DISTRIBUTION OF NUCLEI, TAKING INTO CONSIDERATION THE RADIAL KINETIC SELF-ENERGY CORRECTION

By

M. TISZA*

(Received 25. VI. 1968)

Based on the statistical model of the nucleus developed by P. GOMBÁS the proton and neutron density of nuclei is determined. By the statistical perturbation method the radial kinetic self-energy correction is taken into account and the extent to which the density distribution is altered by this correction is investigated. The density is calculated for three nuclei and the results are used for the determination of the nuclear form factors.

1. The statistical model of the nucleus

The statistical model of the nucleus developed by GOMBÁS [1—5] gives the following expression for the total energy of the nucleus

$$E = E_K + E_s + E_A^{np} + E_A^{nn} + E_A^{pp} + E_C\,, \tag{1}$$

where the first two terms are the kinetic energy, the next three are the interaction energy due to the nuclear interaction of nucleons, and the last is the Coulomb energy of the protons.

The individual energy terms in detailed form are:

a) The Fermi kinetic energy

$$E_K = \varkappa_K \int (\varrho_n^{5/3} + \varrho_p^{5/3})\, dv\,. \tag{2}$$

Here ϱ_n and ϱ_p are the neutron and the proton density, respectively, m is the mass of the nucleon, $\varkappa_K = 3/40\,(3/\pi)^{2/3}\, h^2/m$.

b) The Weizsäcker inhomogeneity correction

$$E_s = \varkappa_s \int \left[\frac{(\operatorname{grad} \varrho_n)^2}{\varrho_n} + \frac{(\operatorname{grad} \varrho_p)^2}{\varrho_p} \right] dv\,, \tag{3}$$

where

$$\varkappa_s = \frac{h^2}{32\pi^2 m}\,.$$

* This article by the late M. TISZA has been arranged for publication by L. URBÁN, Rescarch Group for Theoretical Physics, Hungarian Academy of Sciences, Budapest.

c) The exchange energy between neutrons and protons

$$E_A^{np} = -\frac{4\varepsilon_0}{r_0^3}\int f(\omega_n, \omega_p)\, dv, \tag{4}$$

where ω_n and ω_p are the dimensionless quantities introduced to replace the densities

$$\omega_i = (3\pi^2)^{1/3} r_0 \varrho_i^{1/3}, \quad i = n, p,$$

$r_0 = 1{,}355 \cdot 10^{-13}$ cm. The value of the parameter of energy dimension specifying the strength of the interaction can be determined so that the calculated and experimental energies agree to the greatest possible extent. For Yukawa interaction (the calculations have been carried out for this case) $\varepsilon_0 = 71{,}28$ MeV.

The shape of the function $f(\omega_n, \omega_p)$ in the interaction energy depends on the primary interaction between the nucleons. For Yukawa interaction

$$f(\omega_n, \omega_p) = \frac{1}{24\pi^3}\left[g\left(\frac{\omega_n + \omega_p}{2}\right) - g\left(\frac{\omega_n - \omega_p}{2}\right) + \right.$$

$$\left. + (\omega_n - \omega_p)^2 h\left(\frac{\omega_n + \omega_p}{2}\right) - (\omega_n + \omega_p)^2 h\left(\frac{\omega_n - \omega_p}{2}\right)\right],$$

where $$g(x) = 6x^4 - x^2 + \frac{1}{4}(1 + 12x^2)\ln(1 + 4x^2) - 8x^3 \operatorname{arctg} 2x$$

and $$h(x) = \frac{3}{4}(1 - 4x^2)\ln(1 + 4x^2) - 6x \operatorname{arctg} 2x.$$

d) $$E_A^{nn} = -\frac{\varepsilon_0}{r_0^3}\int f(\omega_n, \omega_n)\, dv, \tag{5}$$

e) $$E_A^{pp} = -\frac{\varepsilon_0}{r_0^3}\int f(\omega_p, \omega_p)\, dv, \tag{6}$$

f) $$E_C = \frac{1}{2} e^2 \iint \frac{\varrho_p(r)\,\varrho_p(r')}{|r - r'|}\, dv\, dv'. \tag{7}$$

The neutron and the proton density are determined by the distribution functions ϱ_n and ϱ_p minimizing the energy expression (1). The distribution functions must also satisfy the auxiliary conditions

$$\int \varrho_n\, dv = N, \qquad \int \varrho_p\, dv = Z.$$

The distribution functions can be calculated e.g. by the Ritz variational method.

This form of the nuclear model does not include the kinetic energy correction. As was shown by GOMBÁS [6] a better approximation is obtained for the kinetic energy of the fermion gas if instead of $E_K + E_J$ the following expression is used in the calculation:

$$E_K^a + E_K^r + E_s, \tag{8}$$

where

$$E_K^a = \varkappa_0 \int \varrho^{5/3} dv \quad \varkappa_0 = \frac{1}{20}\left(\frac{3}{\pi^2}\right)^{2/3} \frac{h^2}{m},$$

$$E_K^r = \int_\Omega \left(\varkappa_1 \varrho^{5/3} - \varkappa_2 \varrho^{4/3} \frac{1}{r} + \varkappa_3 \varrho \frac{1}{r^2} + \varkappa_4 \frac{1}{r^5}\right) dv, \tag{9}$$

$$\varkappa_1 = \frac{\varkappa_0}{2}; \quad \varkappa_2 = \frac{1}{96}\left(\frac{3}{\pi}\right)^{4/3} \frac{h^2}{m}; \quad \varkappa_3 = \frac{1}{96\pi^2} \frac{h^2}{m}; \quad \varkappa_4 = \frac{1}{2^5 \cdot 720\pi^4} \frac{h^2}{m}.$$

In the energy expression (9) the region Ω of the integration is determined by the inequality $p_{\mp} F, \geqslant 1/4\pi r$ the integration is to be carried out for that part of space for which the inequality is satisfied. (The Fermi momentum is space dependent). E_s is the former expression, given by (3).

The kinetic energy correction is

$$v = (E_K^a + E_K^r) - E_K. \tag{10}$$

This is taken as a perturbating effect and is included in the nuclear model by the method of statistical perturbation calculation [7].

2. Perturbation calculation in the statistical model of the nucleus

The approximation $\varrho_n = \varrho_p = \varrho$ is applied, which is permitted for light nuclei and nuclei of medium mass number. In this approximation the total exchange energy can be written as one term.

Let the density ϱ be changed by some perturbation effect to ϱ' where

$$\varrho' = \varrho + \eta,$$

η is the perturbation of the density.

Then the total energy becomes

$$E(\varrho + \eta) = E_K(\varrho + \eta) + E_s(\varrho + \eta) + E_A(\varrho + \eta) + E_C(\varrho + \eta) + v(\varrho + \eta). \tag{11}$$

Here v is the perturbating potential.

Taking into account the perturbation to second order, let us expand each energy term in series according to η to second order

$$E_K(\varrho+\eta) = 2\varkappa_K \int \left(\varrho^{5/3} + \frac{5}{3}\varrho^{2/3}\eta + \frac{5}{9}\varrho^{-\frac{1}{3}}\eta^2\right) dv,$$

$$E_s(\varrho+\eta) = 2\varkappa_s \int \left[\frac{(\operatorname{grad}\varrho)^2}{\varrho} - \frac{(\operatorname{grad}\varrho)^2}{\varrho^2}\eta + \frac{2\operatorname{grad}\varrho\,\operatorname{grad}\eta}{\varrho} + \right.$$

$$\left. + \frac{1}{2}\frac{(\operatorname{grad}\varrho)^2}{\varrho^3}\eta^2 - \frac{2\operatorname{grad}\varrho\,\operatorname{grad}\eta}{\varrho^2}\eta + \frac{(\operatorname{grad}\eta)^2}{\varrho}\right] dv, \tag{12}$$

$$E_A(\varrho+\eta) = \varkappa \int \left[f(\varrho) + f'(\varrho)\eta + \frac{1}{2}f''(\varrho)\eta^2\right] dv,$$

$$E_C(\varrho+\eta) = \frac{1}{2}e^2 \iint \frac{[\varrho(r)+\eta(r)][\varrho(r')+\eta(r')]}{|r-r'|}\, dv\, dv'.$$

The perturbating energy v is small compared with the other terms, so v is expanded in series only to first order according to η

$$v(\varrho+\eta) = \int \left[v(\varrho) + \frac{\delta v}{\delta\varrho}\eta\right] dv.$$

The terms of 0th order in η and v give the original energy. The terms of first order give zero, as we started from the equilibrium state. The contribution of terms of second order should be minimum.

$$\frac{10}{9}\varkappa_K \int \varrho^{-1/3}\eta^2\, dv + \frac{1}{2}\varkappa \int f''(\varrho)\eta^2\, dv +$$

$$+ 2\varkappa_s \int \left[\frac{(\operatorname{grad}\eta)^2}{\varrho} - \frac{2\operatorname{grad}\varrho\,\operatorname{grad}\eta}{\varrho^2}\eta + \frac{1}{2}\frac{(\operatorname{grad}\varrho)^2}{\varrho^3}\eta^2\right] dv + \tag{13}$$

$$+ \frac{1}{2}e^2 \iint \frac{\eta(r)\,\eta'(r)}{|r-r'|}\, dv\, dv' + \int \frac{\delta v}{\delta\varrho}\eta\, dv = \min,$$

i.e. the variation of (13) according to η should vanish.

By carrying out the variation, taking into account the auxiliary condition $\int \eta\, dv = 0$, and performing some identical transformations we obtain

$$\left[\frac{20}{9}\varkappa_K \varrho^{-1/3} + \varkappa f''(\varrho) + 4\varkappa_s \frac{\Delta\varrho}{\varrho^2} - 6\varkappa_s \frac{(\operatorname{grad}\varrho)^2}{\varrho^3}\right]\eta +$$

$$+ \frac{\delta v}{\delta\varrho} - 4\varkappa_s \frac{\Delta\eta}{\varrho} + 4\varkappa_s \frac{\operatorname{grad}\varrho\,\operatorname{grad}\eta}{\varrho^2} + e^2 \int \frac{\eta(r')}{|r-r'|}\, dv' - \lambda = 0, \tag{14}$$

where λ is a Lagrange multiplier introduced on account of the auxiliary condition.

By rearrangement of (14) we obtain

$$\eta = \frac{\lambda - \frac{\delta v}{\delta \varrho} + 4\varkappa_s \frac{\Delta \eta}{\varrho} - 4\varkappa_s \frac{\operatorname{brad} \varrho \operatorname{grad} \eta}{\varrho^2} - e^2 \int \frac{\eta(r')}{|r - r'|} dv'}{\frac{20}{9} \varkappa_K \varrho^{-1/3} + \varkappa f''(\varrho) + 4\varkappa_s \frac{\Delta \varrho}{\varrho^2} - 6\varkappa_s \frac{(\operatorname{grad} \varrho)^2}{\varrho^3}}, \tag{15}$$

from which η can be calculated by iteration.

In first approximation

$$\eta_1 = \frac{\lambda - \frac{\delta v}{\delta \varrho}}{\frac{20}{9} \varkappa_K \varrho^{-1/3} + \varkappa f''(\varrho) + 4\varkappa_s \frac{\Delta \varrho}{\varrho^2} - 6\varkappa_s \frac{(\operatorname{grad} \varrho)^2}{\varrho^3}} \tag{16}$$

λ can be obtained from the auxiliary condition $\int \eta \, dv = 0$.

3. Introduction of radial kinetic self-energy for light and medium-heavy nuclei

The neutron and the proton density of nuclei are calculated in the $\varrho_n = \varrho_p = \varrho$ approximation:

$$\varrho_n = \varrho_p = \varrho = \varrho_0 e^{-\frac{r^2}{a^2}}, \tag{17}$$

where $\varrho_0 = C^3/24\pi^2 r_0^3$, C is the dimensionless variational parameter introduced by the relation $a = 2(\pi)^{1/5} A^{1/3} r_0/C$ instead of the variational parameter a.

From the equality of the neutron and proton density it follows that $\omega_n = \omega_p = \omega$, so $f(\omega_u, \omega_p) = f(\omega_n, \omega_u) = f(\omega_p, \omega_p)$ takes the following form:

$$f(\omega) = \frac{1}{24\pi^3} \left[6\omega^4 - \omega^2 + \frac{1}{4}(1 + 12\omega^2) \ln (1 + 4\omega^2) - 8\omega \operatorname{arc\,tg} 2\omega \right]. \tag{18}$$

The perturbating potential

$$V = \int v dv = (E_K^q + E_K^r) - E_K, \tag{19}$$

whence

$$\frac{\delta v}{\delta \varrho} = \begin{cases} -\frac{5}{9} \varkappa_K \varrho^{2/3}, & 0 \leq r < r_i, \\ -\frac{4}{3} \varkappa_2 \varrho^{1/3} \frac{1}{r} + \varkappa_3 \frac{1}{r^2}, & r_i < r < r_\alpha, \\ -\frac{5}{9} \varkappa_K \varrho^{2/3}, & r_\alpha < r < \infty, \end{cases} \tag{20}$$

the values r_i and r_α can be determined from the relation

$$\varrho(r) = \frac{1}{24\pi^2 r^3} .$$

To determine η the following procedure has been used: The parameter λ is to be determined from the auxiliary condition $\int \eta \, dv = 0$. The integral is calculated by an approximation procedure: the integral

$$\int_0^\infty f(r)\, dr = C_1 f(0) + C_2 f\left(\frac{a}{2}\right) + C_3 f(a) + C_4 f(2a) \tag{21}$$

is calculated for four different functions $f(r)$, with the aid of which the constants C_i can be determined. We have chosen the following functions:

$$f_1(r) = \varrho$$
$$f_2(r) = r^2\varrho,$$
$$f_3(r) = \varrho^2,$$
$$f_4(r) = r^2\varrho^2.$$

The values obtained from the integration constants are:

$$c_1 = 1.7800 \cdot 10^{-1}, \qquad c_3 = 4.2851 \cdot 10^{-1},$$
$$c_2 = 1.9415 \cdot 10^{-1}, \qquad c_4 = 7.4630 \cdot 10^{-1}.$$

If the values c_i are known $\int \eta \, dv$ can be calculated, from which λ and then the first approximation of η can be obtained. Calculations have been carried out for the nuclei of mass numbers $A = 40$ and $A = 100$. The results are shown in Table I.

Table I

	r	$\varrho\left[\frac{1}{r_0^3}\right]$	$\eta\left[\frac{1}{r_0^3}\right]$	η/ϱ [%]	$\varrho + \eta\left[\frac{1}{r_0^3}\right]$
$A = 40$	0	1.399	−0.34673	−24.78	1.0523
	$a/2$	1.035	−0.06753	− 6.52	0.9674
	a	0.4889	−0.00818	− 1.67	0.4807
	$2a$	0.0243	+0.00110	4.56	0.0254
$A = 100$	0	1.5328	−0.52721	−34.39	1.0055
	$a/2$	1.1937	−0.09468	− 7.93	1.0991
	a	0.5638	−0.00114	− 0.20	0.5627
	$2a$	0.0280	+0.00170	6.07	0.0297

Thus, the value of the density perturbation η is known at four points. For the calculation of the second approximation the values grad η and $\Delta\eta$ are also required, so we can fit some function to the four known points from which the derivatives required can be determined.

Let this function be of the form

$$\eta = e^{-r^2/a^2}(b_0 + b_1 r^2 + b_2 r^4 + b_3 r^6).$$

The values calculated for the coefficients are

$$A = 40,\ b_0 = -0.3467,\ b_1[a^{-2}] = 1.3030,\ b_2[a^{-4}] = -1.2049,\ b_3[a^{-6}] = 0.2248,$$

$$A = 100,\ b_0 = -0.5272,\ b_1[a^{-2}] = 2.0372,\ b_2[a^{-4}] = 1.9000,\ b_3[a^{-2}] = 0.3577.$$

For the calculation of the Coulomb term another approximation has been applied, η has been replaced by a step function and the width of the step has been determined from the following integral (the heights of the steps are determined from the respective η values, see Table I).

$$\eta_0 \int_0^x y^2\,dy + \eta_{a/2} \int_x^{kx} y^2\,dy + \eta_a \int_{kx}^{5x} y^2\,dy + \eta_{2a} \int_{5x}^{12x} y^2\,dy = 0 \qquad (22)$$

where $x = a/4$, whence, for the value of k we obtained 2,8 if $A = 40$ and 2,84 if $A = 100$. Thus, the Coulomb integral can be calculated and the second approximation of η can be determined (Table II).

Table IIa

$A = 40$				
r	$\varrho\left[\frac{1}{r_0^3}\right]$	$\eta_2\left[\frac{1}{r_0^3}\right]$	$\eta_2/\varrho[\%]$	$\varrho + \eta_2\left[\frac{1}{r_0^3}\right]$
0	1.399	−0.6419	−45.88	0.768
$a/2$	1.0350	−0.1800	−17.39	0.855
a	0.4889	−0.0165	− 3.38	0.4724
$2a$	0.0243	0.0053	21.90	0.0296

Table IIb

$A = 100$				
r	$\varrho\left[\frac{1}{r_0^3}\right]$	$\eta_2\left[\frac{1}{r_0^3}\right]$	$\eta_2/\varrho[\%]$	$\varrho + \eta_2\left[\frac{1}{r_0^3}\right]$
0	1.5328	−0.8094	−52.80	0.7233
$a/2$	1.1937	−0.2131	−17.85	0.9806
a	0.5638	−0.0222	− 3.95	0.5416
$2a$	0.0280	0.00666	23.73	0.03473

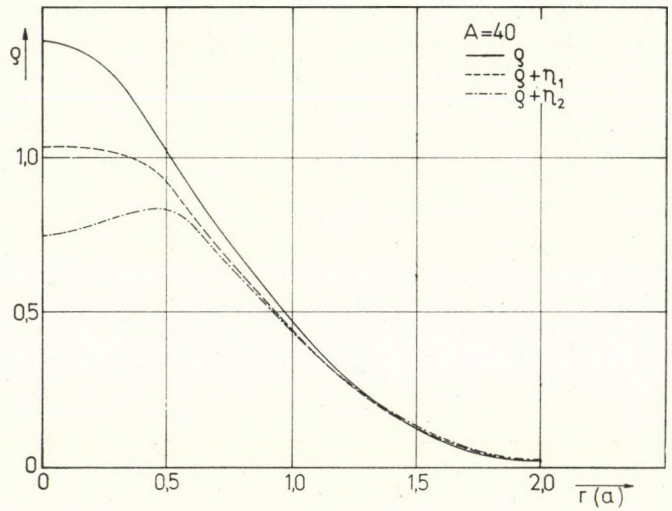

Fig. 1

Figs. 1 and 2 show the density distribution calculated for two nuclei. Besides the Gauss distribution obtained from the original statistical model of the nucleus, the figures show the first and second approximation calculated perturbationally. The values of the nuclear radius R and the surface layer thickness can be read from the figures and it can be seen that in the inner region of the nucleus the density is approximately constant.

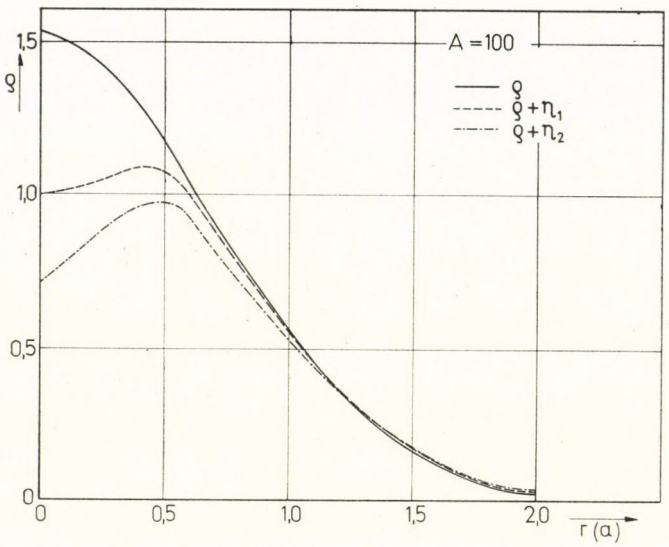

Fig. 2

4. Consideration of the difference between the neutron and the proton density. Investigation of the density distribution of heavy nuclei

We investigate how the perturbation of density develops when the neutron and the proton density are assumed to be different.

Let

$$\varrho = \varrho_n + \varrho_p, \qquad \varrho_n = \frac{N}{A}\varrho, \qquad \varrho_p = \frac{Z}{A}\varrho, \tag{23}$$

where $\varrho = \varrho_0 e^{-r^2/a^2}$. The energy expression can be written on the basis of formulae (1)—(7), and by calculating in the same way as in Section 2, we obtain for the perturbation of density

$$\eta = \frac{\lambda - \frac{\delta v}{\delta \varrho} - 2\varkappa_s \frac{\operatorname{grad}\varrho \operatorname{grad}\eta}{\varrho^2} + 2\varkappa_s \frac{\Delta\eta}{\varrho} - e^2\left(\frac{Z}{A}\right)^2 \int \frac{\eta(r')}{|r-r'|} dv'}{\frac{5}{9}\varkappa_K\left[\left(\frac{Z}{A}\right)^{5/3} + \left(\frac{N}{A}\right)^{5/3}\right]\varrho^{-1/3} + \frac{1}{2}\varkappa f''(\varrho) - \varkappa_s \frac{(\operatorname{grad}\varrho)^2}{\varrho^3} + 2\varkappa_s \frac{\Delta\varrho}{\varrho^2}}. \tag{24}$$

Compared with Part 3 the term $1/2\ \varkappa f''(\varrho)$ has changed essentially; instead of the single function $f(\omega)$ we have to work with three different functions $f(\omega_n, \omega_p)$, $f(\omega_n, \omega_n)$ and $f(\omega_p, \omega_p)$. The constant $\varkappa$ will also differ for each of the three functions.

The calculations have been carried out in first approximation for the nuclei

$$\begin{aligned} A &= 240 \\ Z &= 99 \\ N &= 141 \end{aligned}$$

The results are shown in Table III and Fig. 3, respectively.

Table III

$A = 240$ $Z = 99$				
r	$\varrho\left[\frac{1}{r_0^3}\right]$	$\eta\left[\frac{1}{r_0^3}\right]$	$\eta/\varrho[\%]$	$\varrho + \eta\left[\frac{1}{r_0^3}\right]$
0	2.7484	−1.0443	−37.99	1.7041
$a/2$	2 1404	−0.3074	−14.36	1.8330
a	1.0108	−0.0203	− 2.01	0.9905
$2a$	0.0503	0.0057	11.38	0.0560

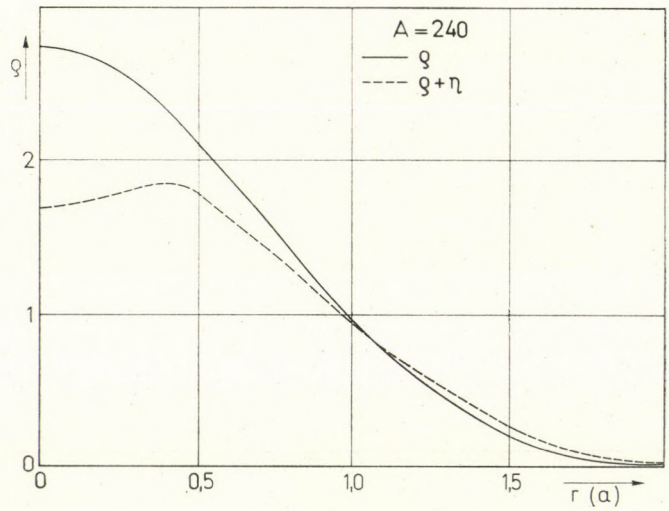

Fig. 3

It can be seen that by assuming the neutron and the proton density to be different, in the first approximation we obtain a greater improvement than for equal densities.

5. Application of the results to the determination of nuclear form factors

Our results obtained so far can be used for the calculation of the form factor of the nucleus, with the aid of which the differential cross-section of electron scattering on the nucleus can be determined.

Let us investigate the electron scattering on the nucleus in Born approximation. If an electron beam describable by a plane wave falls on the nucleus, the electrons scattered by the nucleus leave the scattering field of force in the form of an outgoing spherical wave. The form of the outgoing spherical wave in Born approximation is

$$\psi = f(\vartheta)\frac{e^{ikr}}{r},$$

$$f(\vartheta) = -\frac{m}{2\pi\hbar^2}\int v_c(r')\, e^{iqr'}\, dv', \tag{25}$$

where V_c is the potential acting on the electron

$$V_c(r') = -e^2 \int \frac{\varrho_p(r)}{|r - r'|} dv, \tag{26}$$

q is the change of momentum of the electron in the scattering process $|q| = 2k \sin \vartheta/2$, where ϑ is the angle of scattering. The differential cross-section of scattering is given by

$$\sigma(\vartheta) = |f(\vartheta)|^2 .$$

From (25) and (26) we obtain by carrying out the integration according to r'

$$f(\vartheta) = \frac{2e^2 m}{\hbar^2 q^2} F(q). \tag{27}$$

The expression $F(q) = \int_0^\infty \varrho_p(r) \sin qr/qr \, 4\pi r^2 \, dr$ is the form or shape factor of the nucleus, knowing which the cross-section can be determined in a simple way.

If the density is known, $F(q)$ can be calculated in a simple way. As the density is known at four points only, between these points the function giving the density distribution has been approximated by straight sectors. The calculated values of $F(q)$ are:

Table IV

q	0	$\frac{1}{a}$	$\frac{2}{a}$	$\frac{4}{a}$
$Z = 19$ $A = 40$	20	17.66	9.75	0.541
$Z = 45$ $A = 100$	50	34.13	19.48	1.273
$Z = 99$ $A = 240$	99	95.26	38.45	4.960

For the three nuclei considered here, $F(q)$ is shown in Figs. 4—6.

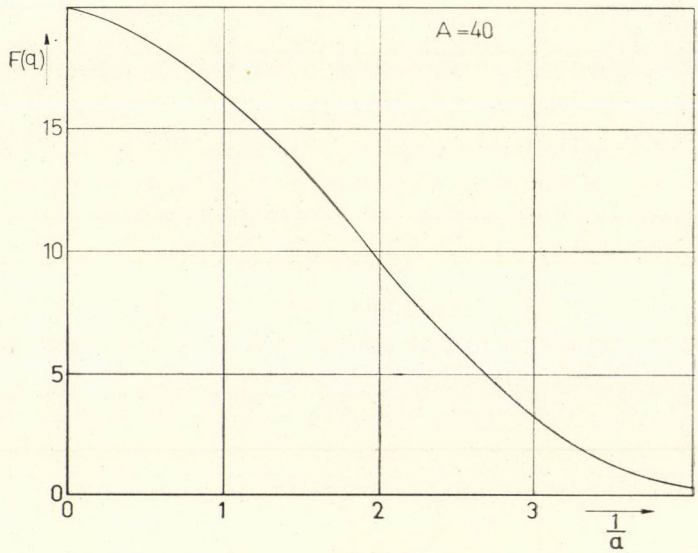

Fig. 4

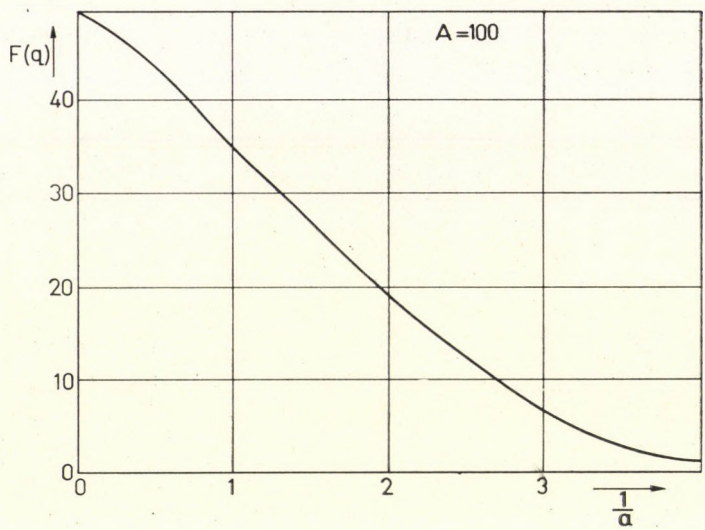

Fig. 5

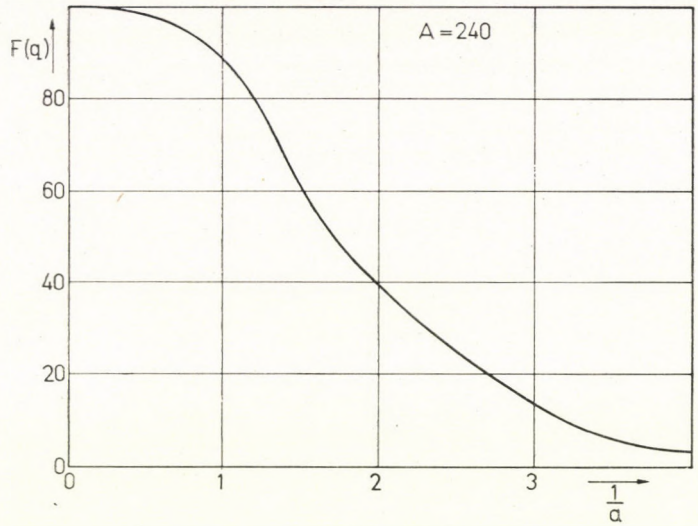

Fig. 6

The author is indebted to Prof. P. Gombás for suggesting the problem and for useful advice, as well as to Dr. D. Kisdi for his help during the course of this work.

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ИССЛЕДОВАНИЕ РАСПРЕДЕЛЕНИЯ ПЛОТНОСТИ ЯДЕР С УЧЕТОМ КОРРЕКЦИИ РАДИАЛЬНОЙ КИНЕТИЧЕСКОЙ СОБСТВЕННОЙ ЭНЕРГИИ

М. ТИСА*

Р е з ю м е

В статье определяются плотности протонов и нейтронов ядер на основе разработанной П. Гомбашом статической модели ядра. Методом расчета статистического возмущения учитывается коррекция радиальной кинетической собственной энерги и исследуется влияние учета этой коррекции на распределение плотности. Расчеты плотности проведены для ядер и полученные результаты используются для определения формфакторов ядер.

* Автор умер, работа подготовлена к печати Л. Урбаном.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (3), pp. 261—267 (1969)

AN EFFECTIVE FIELD APPROACH FOR MANY-BODY FORCES BETWEEN IONS OF A METAL

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(Received 30. VII. 1968)

For calculating the phonon spectrum of a metal lattice a self-consistent field approximation is used. At a given lattice deformation the restoring forces between the ions have a many-body character because of the effects of the periodic field. The condition of self-consistency and the use of first order Bloch functions for correcting the dielectric tensor lead to a physical understanding of a formula proposed earlier for calculating the elements of the dynamical matrix arising from many-body forces.

In calculating the phonon frequency spectrum of a crystal one generally uses the BORN—KÁRMÁN scheme [1]. This model assumes the adiabatic approximation for electronic and ionic motions to be valid, and consists essentially in finding the second derivatives of an effective potential acting between the ions which is responsible for the stable lattice configuration as well as for the lattice vibrations.

In the case of noble-gas crystals or ionic crystals the potential can be calculated in a fairly simple way, since the nature of the relevant forces is well known. For metals, however, the situation is more complicated, because the most important part of the cohesive forces comes from the interaction mediated by the conduction electrons.

Since the pioneer work of TOYA [2], calculations based on computing the electronic part of the potential have been performed by several authors for a number of metals [3—7]. In these works, however, when calculating the response of the conduction electrons to a lattice deformation resulting eventually in a restoring force, the electron gas was treated as free. This corresponds to a lowest order calculation using two-body central forces, neglecting the effect of the periodic crystal field. Only in the paper of BROVMAN and KAGAN [7] can one find the general form of the interaction energy including the effects of the crystal field also, resulting in a potential which represents many-body forces.

For the calculation of the potential energy formal a diagram technique was proposed.

In the present paper an attempt is made to point out, that the third order diagram of BROVMAN and KAGAN is the most relevant one, and calculating this diagram properly is equivalent to a dielectric formulation where in

the dielectric function the effect of the periodic crystal field is taken into account. The treatment results in an understanding of this main diagram in the same sense as the usual self-consistent field approach [8] helps one to understand the underlying idea of summing up bubble diagrams by the Brueckner method.

In the adiabatic approximation the quantity one has to calculate is the change in the electronic energy caused by a given deformation of the equilibrium lattice. The Hamiltonian for the electron gas filling up the ionic lattice has the form

$$H_{el} = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|r_i - r_j|} + \sum_{ils} v(r_i - R_{ls}), \tag{1}$$

where p_i is the momentum, r_i is the coordinate of the i-th electron, R_{ls} is the coordinate of the ion at the l-th cell in the s-th place, while $v(r)$ is the electron-ion pseudopotential [8].

Assuming small lattice deformations one can expand v as

$$v(r - R_{ls}) \approx v(r - R_{ls}^{(0)}) - u_{ls} (\operatorname{grad} v)_{r - R_{ls}^{(0)}}, \tag{2}$$

where $R_{ls}^{(0)}$ is the equilibrium position and u_{ls} is the deviation from it.

Introducing now the absorption and emission operators a_k and a_k^+ for Bloch electrons with momentum k, in the usual second quantized form, one has for H_{el}*:

$$H_{el} = \sum_k \varepsilon_k a_k^+ a_k + \frac{1}{2} \sum_{q \neq 0} \frac{4\pi e^2}{q^2} \varrho_q^+ \varrho_q + \sum \delta v_q^* \varrho_q, \tag{3}$$

where

$$\delta v_q = - iq\, v_q \frac{1}{N} \sum_{ls} e^{-iqR_{ls}^{(0)}} u_{ls}; \quad v_q = \frac{1}{\Omega_0} \int e^{-iqr} v(r)\, dr \tag{4}$$

and

$$\varrho_q = \sum_{kk'} \left(\int \psi_k^* e^{-iqr} \psi_{k'}\, dr \right) a_k^+ a_{k'}. \tag{5}$$

In this formula ε_k is the Bloch energy of the electrons, Ω_0 is the volume of the elementary cell.

We are interested in the second derivative of the ground state energy with respect to u_{ls}. Therefore, we have to determine the energy in second or-

* We do not deal with spin indices, but wherever a sum for k is present, a summing for spins is also understood.
The volume of the crystal is taken to be unity.

der in u. Once having calculated this, the relation

$$\Delta E = \frac{1}{2} \sum_{sl's'} u_{ls} \mathbf{A}_{ls,l's'} u_{l's'} \tag{6}$$

gives us the tensors $\mathbf{A}_{ls,l's'}$ which play the fundamental role in determining the lattice vibration spectrum [9].

Making use of the well known theorem of PAULI [10]:

$$\Delta E = \int_0^1 \frac{d\lambda}{\lambda} \langle \lambda H' \rangle_\lambda , \tag{7}$$

where H' is the perturbation and $< \quad >_\lambda$ means an averaging using the exact eigenfunction with value λ of the coupling constant, one sees that by choosing

$$H' = \sum_q \delta v_q^* \varrho_q ,$$

one has to calculate $<\varrho_q>$ in first order with respect to u_{ls}.

The problem is identical with that of determining the change in density by the action of an external field as treated e.g. in [10].

Standard perturbation theory leads to the result:

$$\langle \varrho_q \rangle^{(1)} = - \sum P_{qq'} \delta v_{q'} \tag{8}$$

$$P_{qq'} = A_{qq'} + A_{-q'-q} , \tag{9}$$

$$A_{qq'} = \sum_m \frac{\langle 0 | \varrho_q | m \rangle \langle m | \varrho_{q'}^+ | 0 \rangle}{E_m - E_0} . \tag{10}$$

Here $| 0>$ and $| m>$ mean the ground state and the excited states of the electron gas, which are the true states of an interacting system of electrons.

The self-consistent field approximation first neglects in the above formula the Coulomb interaction using for $| 0>$ and $| m>$ the free states, but at the same time introduces on the right hand side of the equation the term $(\delta v_{q'} + \Phi_{q'})$ instead of δv_q where

$$\Phi_q = \frac{4\pi e^2}{q^2} \langle \varrho_q \rangle , \tag{11}$$

so that one has to solve the equation

$$\sum_{q'} \left(\delta_{qq'} + \frac{4\pi e^2}{q^2} P_{qq'}^{(B)} \right) \langle \varrho_{q'} \rangle = - \sum_{q'} P_{qq'}^{(B)} \delta v_q , \tag{12}$$

where $P^{(B)}_{qq'}$ denotes $P_{qq'}$ calculated from the non-interacting Bloch states.

For the energy we then have

$$\Delta E = \frac{1}{2} \sum_{qq'q''} \delta v^*_{q'} \, \tilde{\varepsilon}^{-1}_{qq''} P^{(B)}_{q''q'} \, \delta v_{q'}, \tag{13}$$

where the matrix to be inverted is defined as

$$\tilde{\varepsilon}_{qq'} = \delta_{qq'} + \frac{4\pi e^2}{q'^2} P^{(B)}_{qq'}. \tag{14}$$

This quantity with a change of q' by q in the denominator is known as the dielectric constant matrix of a solid [11].

From (10) it is seen that crystal field effects come from the exact form of the Bloch states (which in principle can be expanded into series with coefficients having matrix elements of the periodic potential) and from the energy denominators. Neglecting both one would have the well known free gas approximation for $P_{qq'}$ [10]:

$$P^{(0)}_{qq'} = P^{(0)}_q \delta_{qq'}; \quad P^{(0)}_q = \sum_p \frac{n_p - n_{p+q}}{E^{(0)}_{p+q} - E^{(0)}_p}, \tag{15}$$

thus

$$\Delta E^{(0)} = -\frac{1}{2} \sum_q \frac{|\delta v_q|^2 P^{(0)}_q}{1 + \dfrac{4\pi e^2}{q^2} P^{(0)}_q}.$$

Our purpose is to consider the first order effects of the periodic field. The energy denominators are of second order, therefore one can use the free electron energies. For the eigenfunctions one has:

$$\psi_k = e^{ikr} + \sum_G \alpha_{kG} \, e^{i(k-G)r}, \tag{16}$$

$$\alpha_{kG} = \frac{\int (e^{i(k-G)r})^* \, \bar{v}(r) \, e^{ikr} \, dr}{E^{(0)}_k - E^{(0)}_{k-G}} = \frac{\bar{v}_{-G} \sum_s e^{iGr_s}}{E^{(0)}_k - E^{(0)}_{k-G}},$$

where $\bar{v}(r)$ and $\bar{v}_q$ stand for the screened pseudopotential:

$$\bar{v}_q = v_q/\varepsilon_q; \quad \varepsilon_q = 1 + \frac{4\pi e^2}{q^2} P^{(0)}_q. \tag{17}$$

$E^{(0)}_k = k^2/2m$ and G are the reciprocal lattice vectors while r_s is the equilibrium place of the s-th atom in the elementary cell. Making use of this series and of Eq. (10) one has in first order

$$P^{(B)}_{qq'} = P^{(0)}_q \delta_{qq'} + \frac{v_{-G}}{\varepsilon_G} \left(\sum_s e^{iGrs} \right) P^{(1)}_{qq'},$$

$$P^{(1)}_{qq'} = \sum_k n_k \left\{ \frac{1}{(E^{(0)}_{k+q} - E^{(0)}_k)(E^{(0)}_k - E^{(0)}_{k-G})} + \frac{1}{(E^{(0)}_{k+q+G} - E^{(0)}_k)(E^{(0)}_k - E^{(0)}_{k+G})} + \right.$$

$$\left. + \frac{1}{(E^{(0)}_{k+q} - E^{(0)}_k)(E^{(0)}_k - E^{(0)}_{k+q+G})} \right\} \delta_{q',q+G} \,. \tag{18}$$

The sums in the above expression can be transformed into a simpler form of a one dimensional integral, as treated in the Appendix.

In the language of diagram technique [7] [10] one has the correspondence

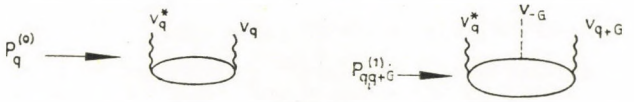

Fig. 1

where the solid lines mean free electron propagators and the external lines indicating the action of the phonon field or the periodic field are excluded from the definition of the P-s.

On inverting Eq. (14) one now gets

$$\tilde{\varepsilon}^{-1}_{qq'} = \frac{1}{\varepsilon_q} \delta_{qq'} - \frac{1}{\varepsilon_q} \frac{4\pi e^2}{q'^2} \frac{v_{-G}}{\varepsilon_G} \sum_s e^{iGr_s} \frac{1}{\varepsilon_{q'}} P^{(1)}_{qq'} + O(v_G^2). \tag{19}$$

Using these values one has for the dynamical matrix in the case of $ls \neq l's'$

$$\mathbf{A}_{ls,l's'} = \mathbf{A}^{(0)}_{ls,l's'} + \mathbf{A}^{(1)}_{ls,l's'}, \qquad \text{a)}$$

$$\mathbf{A}^{(0)}_{\substack{ls,l's' \\ (\alpha,\beta)}} = -\frac{1}{N^2} \sum_q q^\alpha q^\beta |v_q|^2 e^{iq(R^{(0)}_{ls} - R^{(0)}_{l's'})} \frac{P^{(0)}_q}{\varepsilon_q}, \qquad \text{b)} \tag{20}$$

$$\mathbf{A}^{(1)}_{\substack{ls,l's' \\ (\alpha,\beta)}} = -\frac{1}{N^2} \cdot \qquad \text{c)}$$

$$\cdot \sum_q \sum_{G \neq 0} q^\alpha (q^\beta + G^\beta) \frac{v_q}{\varepsilon_q} \frac{v_G}{\varepsilon_G} \frac{v_{q+G}}{\varepsilon_{q+G}} \left(\sum_{s''} e^{iGr_{s''}} \right) e^{iq(R^{(0)}_{ls} - R^{(0)}_{l's'})} e^{-iGr_{s'}} P^{(1)}_{q,q+G} \,.$$

The terms for $ls = l's'$ can be determined from the relation [1]

$$\sum_{l's'} \mathbf{A}_{ls,l's'} = 0 \,.$$

The potential energy between ions leading to the force constants $A^{(1)}_{ls,l's'}$

must have the form

$$V(r_{1\ldots}r_N) = \sum_{i,j} \tilde{v}(r_i, r_j); \quad \tilde{v}(r_i, r_j) = \sum_{G\neq 0} u(r_i - r_j, G)\, e^{-iGr_j} \quad . \tag{21}$$

Here the function $\tilde{v}(r_i, r_j)$ does not represent any real two-body interaction, since $(\mathrm{grad})_i\, \tilde{v} \neq -(\mathrm{grad})_j\, \tilde{v}$, therefore the terms in (21) represent many-body forces.

In metals, however, where the Cauchy relations [1] are usually not satisfied, and especially in substances with complicated elementary cells (e.g. white-tin structure) the role of such forces describing non-central, many-body interactions should be very important.

The formula (20/c) implicitly given in [7] is therefore an effective self-consistent field approximation. As we have seen, evaluating the force constants in such a way is equivalent to

1) taking into account the electronic response self-consistently and correcting the dielectric function with the effect of the periodic field, and

2) using first order wave functions with a screened pseudopotential.

Numerical studies based on these considerations are in progress.

Thanks are due to Prof. Yu. Kagan and Dr. E. Brovman (Kurchatov Institute, Moscow) for proposing this problem and for several discussions.

Appendix

For $P^{(1)}_{qq'}$ one has

$$P^{(1)}_{qq'} = G(q,q') + G(q,q-q') + G(q',q'-q),$$

$$G(q,q') = -\,2\cdot(2\pi)^{-3}\int dk\{[(k+q)^2/2m - k^2/2m]\cdot[(k+q')^2/2m - k^2/2m]\}^{-1},$$

where the factor 2 arises from summing up for the two spin directions, and the integral is taken inside the Fermi-sphere. Simple calculation gives

$$G(a,b) = \frac{m^2}{4\pi^3}\int_{-k_F}^{k_F} \frac{2(k_F^2 - z^2)}{az + \dfrac{a^2}{2}}\, dz \int_{-1}^{1} \frac{\sqrt{1-\xi^2}\, d\xi}{\xi b_\perp \sqrt{k_F^2 - z^2} + b_{//} z + \dfrac{b^2}{2}},$$

where $b_\perp$ and $b_{//}$ are the components of b perpendicular and parallel to a, and the integral is understood in the sense of the principal value. This can be transformed [12] to the form:

$$G(a,b) = -\frac{m^2}{2\pi^2}\int_{-k_F}^{k_F} \frac{b_{//} z + \dfrac{b^2}{2}}{az + \dfrac{a^2}{2}}\, F(b_{//}, b_\perp, z)\, dz,$$

$$F(u,v,z)=\begin{cases}\dfrac{1}{v^2}\ (1-\sqrt{1-p^2}) & |p|<1\,,\\[2ex] \dfrac{1}{v^2} & |p|\geq 1\,,\end{cases}$$

with

$$p=\frac{v^2(k_F^2-z^2)}{uz+\dfrac{u^2+v^2}{2}}\,.$$

For further evaluation of the expression numerical integration is needed.

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ПРИБЛИЖЕНИЕ ЭФФЕКТИВНОГО ПОЛЯ ДЛЯ СИЛ МНОГИХ ТЕЛ ИОНОВ МЕТАЛЛА

Г. ШОЛТ

Р е з ю м е

Для вычисления фононного спектра металлической решетки применяется приближение самосогласованного поля. При данной деформации решетки востанавливающие силы между ионами имеют характер многих тел, что является следствием эффектов периодического поля. Условие самосогласогласованности и применение функций Блоха первого порядка для диэлектрического тензора, ведут к физическому пониманию формулы, предложенный ранее для определения элементов динамической матрицы, обусловленной силами многих тел.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (3), pp. 269—283 (1969)

ANGULAR DISTRIBUTIONS OF THE REACTION $^{12}C(d,p)^{13}C$ BELOW $E_d = 2$ MeV

By

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(Received 6. VIII. 1968)

Excitation functions of the $^{12}C(d, p_0)$ and $^{12}C(d, d)$ reactions were measured at 90° in the range $E_d = 900-2000$ keV in 10 keV steps. Angular distributions of the $^{12}C(d, p_0)^{13}C$ reaction were measured in steps of 25 keV. The experimental data are interpreted in terms of interfering compound levels. Interference phenomena suggest additional new levels in ^{14}N.

1. Introduction

Angular distributions of the $^{12}C(d,p_0)^{13}C$ reaction in the range $E_d = 800-1500$ keV were first measured by PHILLIPS [1] who also suggested spins and parities for the compound levels. SARMA et al. [2] determined angular distributions at 17 different bombarding energies in the same range. On assuming a pure compound nuclear reaction mechanism, these authors succeeded in assigning spins to four of the five known levels of ^{14}N in the energy range concerned. The parity and spin assignments were made on the assumption that the neutron and proton reduced widths of the highly excited states in the $N = Z$ nucleus ^{14}N were of the same order of magnitude. Angular distributions of the (d, d), (d, p_0) and (d, p_1) reactions were measured by KASHY et al. [3] up to 2 MeV. Spins, parities, as well as neutron, proton and deuteron reduced widths of five ^{14}N levels were identified by least square fits of the theoretical angular distribution formula for many-level resonance reactions. In the same energy region, WILLIAMSON [4] studied the energy dependence of the distributions and observed, for the first time, significant Legendre components above $L = 4$. Two tentative spin assignments were made unambiguous by this very detailed experiment. The results of [1—4] are summarized in Table I.

However, every previous attempt had failed to reproduce the angular distributions above $E_d = 1500$ keV in terms of overlapping levels. Some of the authors [3] suspect the existence of several unidentified compound states in this region. — Resonances at $E_d = 1730$ keV and 1950 keV are known from $^{12}C(d, n)^{13}N$ reaction [5, 6] but have not yet been observed in (d, p) processes. — As regards the non-resonant contribution, it seems to play an important

Table I

Levels in ^{14}N as excited in the reaction $^{12}C(d, p)^{13}C$ (g.s).[a]

E_d/keV	$E_{x_{keV}}$ ^{14}N	Ref.[1]	Ref.[2]	Ref.[3]	Ref.[4]	Present work
940	11055	1^+	1^+	1^+	1^+	1^+
1130	11235	—	I	—	—	—
1160	11260	1^-, 2^-	2^-	2^-	2^-	2^-
1225	11315	—	—	—	—	3^+
1310	11390	1^+, 2^+	3^+	1^+	3^+	2^+, 3^+
1390	11450	—	—	—	—	2^-
1440	11500	*	3^+	3^+	3^+	3^+
1550	11595	—	.	—	—	2^-
1630	11655	.	.	1^-	*	2^-
1800	11810	.	.	1^+	*	1^+
1950	11935	.	.	—	—	1^-

[a] Asterisk: no assignment made. Parenthesis: dubious assignment. Dash: no resonance found. Dot: no measurement.

part both below [7, 8] and above [9, 10] the region 900—2000 keV. In addition, ELWYN et al. [11] were able to interpret the neutron angular distributions for the mirror reaction $^{12}C(d, n)^{13}N$ in the region $E_d = 1—2$ MeV by DWBA calculations. Moreover, distorted wave calculations [12] were also successful in reproducing the experimental proton distributions in the $^{12}C+d$ reaction at $E_d = 1200$, 1300 and 1700 keV. Thus, it seemed to be of interest to re-investigate the detailed structure in the energy dependence of the cross-section for the reaction $^{12}C(d, p_0)$ in this energy range.

2. Apparatus

The deuteron beam of about 1 μA intensity and 1—2 keV energy spread from the 2 MeV electrostatic generator of the Institute was focussed onto the target after magnetic analysis. The targets were self-supporting foils of 30—50 μ g/cm^2 thickness obtained by vacuum deposition of carbon of spectroscopic purity.

Target chambers of different types were used for measuring excitation functions and absolute cross-sections, and for investigating the angular distributions (Fig. 1). With both arrangements, the ground state proton groups were detected outside the chamber by a surface barrier Ortec detector with a sensitive area of 7,1 mm^2. When measuring the *excitation function* and the absolute cross-section, both the elastically scattered deuterons and the p_1 protons were

detected within the vacuum by a Toshiba detector with a sensitive area of 17.5 mm^2. When studying *angular distributions*, the p_0 group was monitored at a fixed angle of 120° lab. While measuring the excitation function at 90° lab, the charge collection method had to be used in order to obtain absolute cross-section data.

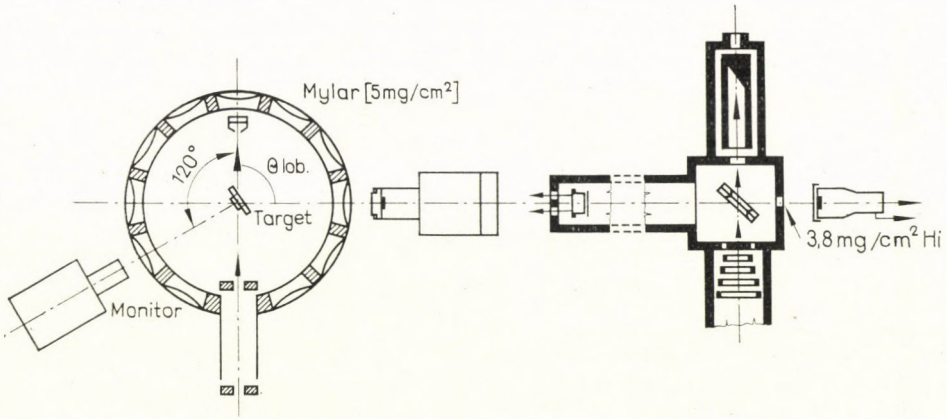

Fig. 1. The scattering chambers (see text)

Signals from the semiconductor detectors were fed into a 128-channel analyser of the type KFKI-NK-103. The pulses of the monitor detector were counted by means of a differential discriminator. The current from the charge collector was integrated in a 0.69 μ F capacitor whose voltage was detected by an electrostatic device.

3. Experimental procedure

The accuracy of our conventional charge measurement was estimated to be about 6%.

The target thickness, 17.6 keV ($\theta = 45°$) was determined by measuring the shift in the bombarding energy of the $E_p = 874$ keV $^{19}F(p, \gamma)$ resonance (Fig. 2).

To check the bombarding energy the field of the deflecting magnet was measured by NMR.

The excitation functions for both p_0 and p_1 protons and the deuterons were taken at 90° lab in 10 keV steps throughout the region 900—2000 keV. During this 24-h measurement with the same target considerable carbon deposition due to oil vapour was observed. To determine the rate of carbon deposition the proton yield was remeasured rapidly in 100 keV intervals immediately after the first run. A typical energy spectrum is shown in Fig. 3. Ex-

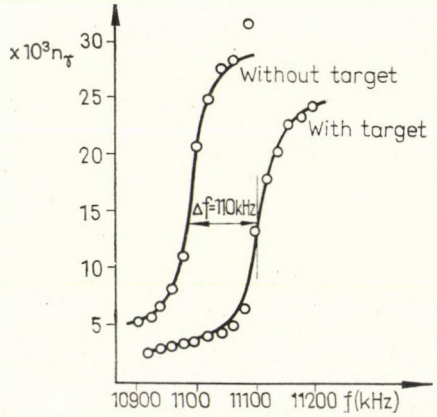

Fig. 2. The energy calibration and target thickness measurement (see text)

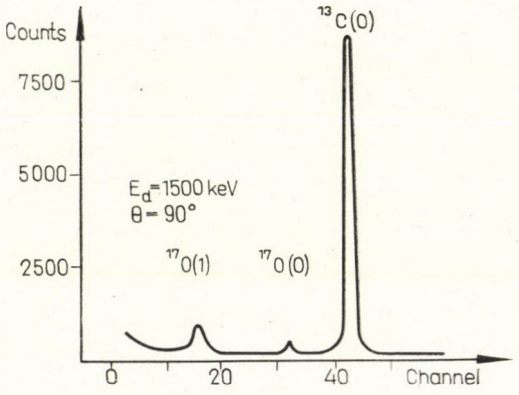

Fig. 3. Typical energy spectrum of the reaction $^{12}C + d$ at $E_d = 1500$ keV

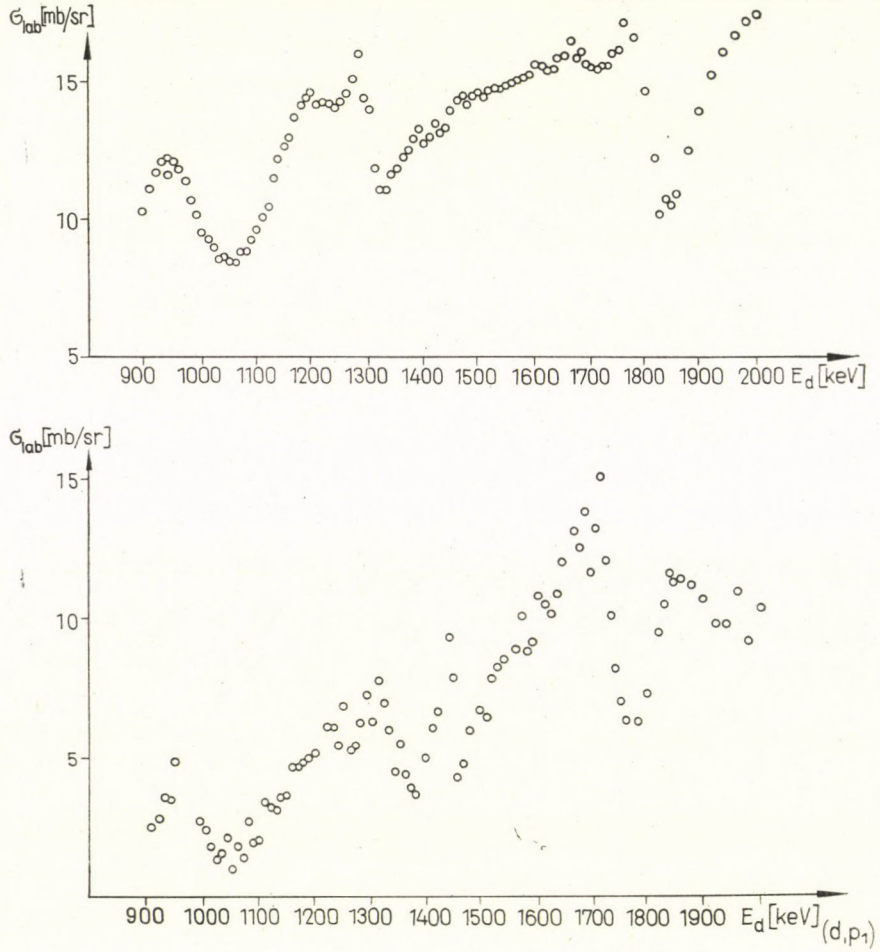

Fig. 4. Excitation functions of the reaction $^{12}C + d$ at 90° lab for (a) the p_0 proton group; (b) the p_1 proton group

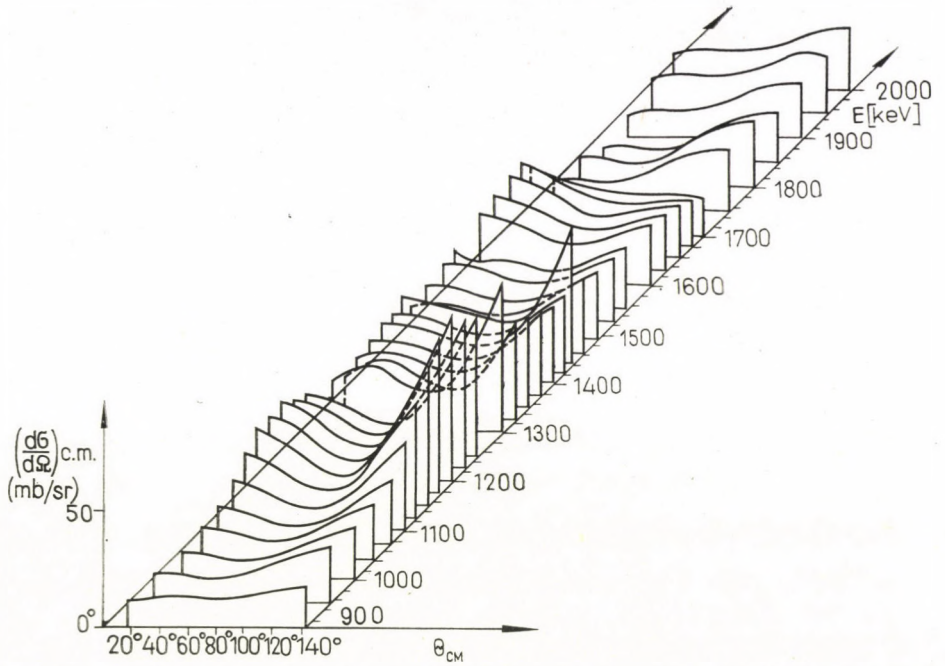

Fig. 5. The energy dependence of the angular distribution of the ground state proton group of the reaction $^{12}C + d$

citation functions of the (d, p_0) and (d, p_1) processes at 90° lab are shown in Fig. 4.

The energy dependence of the p_0 angular distribution was studied at 34 deuteron energies in steps of about 25 keV over the angular range 20°—140° (150°) lab angle (see Fig. 5).

4. Evaluation of data

The coefficients of the expansion in terms of Legendre polynomials of the c.m. differential cross-sections are given in Fig. 6 and Table II. The analysis was restricted to $L \leq 6$.

The error in the cross-section is estimated to be $\pm$ 15% for the p_0 protons and $\pm$ 25% for the p_1 protons and deuterons owing to uncertainties in the background. The absolute cross-section data for protons are systematically 20% lower than those of KASHY et al. [3] and MCELLISTRAM et al. [10] probably because of the uncertainty of our thickness measurement. The differential cross-section for the (d, d) process at 90° lab angle and $E_d = 1100$ keV is found to be 15% *below* that of the Rutherford scattering, while the data of Kashy et al. for the same energy at 80.5° lab angle lie 8% *above* it.

Table II

Legendre coefficients (in mb) with relative errors in the angular distribution of the reaction $^{12}C(d, p)^{13}C$ (g.s.)

E_{lab}(keV)	B_0	B_1	B_2	B_3	B_4	B_5	B_6
902	11.37	— 3.33	1.52	—0.96	—0.39	—0.34	—8.02
	0.007	0.046	0.16	0.29	0.91	1.00	4.68
950	11.64	— 5.99	3.07	0.38	0.25	0.37	0.18
	0.008	0.029	0.09	0.82	1.61	1.01	2.10
1000	11.37	— 6.99	4.62	0.71	—0.22	—0.83	—0.51
	0.008	0.026	0.06	0.46	1.90	0.45	0.73
1040	11.46	— 7.11	6.63	0.99	—2.08	—0.69	8.11
	0.008	0.027	0.04	0.35	0.00	0.54	4.73
1070	13.42	— 8.88	10.06	0.24	—0.49	—0.72	—0.14
	0.008	0.027	0.03	1.74	1.09	0.62	3.20
1100	17.26	—12.18	15.99	—1.31	—0.29	—0.95	8.49
	0.008	0.027	0.03	0.44	2.51	0.60	6.77
1125	21.52	—14.71	21.34	—1.44	—1.31	—0.51	—0.27
	0.008	0.028	0.03	0.50	0.68	1.40	2.67
1150	26.52	—19.01	27.49	—0.37	—2.96	0.66	0.64
	0.008	0.027	0.03	2.49	0.38	1.33	1.38
1175	29.57	—21.69	29.85	—1.88	—4.32	1.14	7.03
	0.008	— 0.026	0.03	—0.54	—0.29	0.00	0.00
1200	30.28	—22.60	30.29	—1.79	—4.14	0.50	0.21
	0.008	— 0.026	0.03	—0.58	—0.31	2.02	4.71
1225	28.24	—23.34	31.11	—6.06	1.81	—4.86	4.23
	0.017	— 0.054	0.06	—0.35	0.00	—0.31	0.28
1250	26.90	—21.01	25.22	—3.88	—2.17	—1.57	0.31
	0.018	— 0.058	0.07	—0.52	—0.99	—0.96	3.78
1300	26.39	—18.73	23.15	—13.36	—6.11	—0.88	—1.79
	0.017	— 0.063	0.079	— 0.143	—0.328	1.631	—0.643
1325	21.95	— 9.85	17.59	—15.43	—3.44	—0.66	—1.74
	0.016	— 0.090	0.078	— 0.095	—0.437	—1.630	—0.548
1350	19.71	— 6.85	12.23	— 6.34	—4.65	0.84	—1.33
	0.016	— 0.116	0.102	— 0.211	—0.304	1.228	—0.684
1375	19.59	— 7.13	13.26	— 5.87	—0.54	—1.16	0.22
	0.015	— 0.106	0.089	— 0.216	—2.539	—0.854	4.017
1400	20.40	— 5.77	13.67	— 4.99	—0.71	—1.27	1.11
	0.015	— 0.133	0.089	— 0.260	—1.978	—0.802	0.843
1420	19.03	— 4.53	10.34	— 1.35	—2.42	0.61	—0.17
	0.015	— 0.159	0.111	— 0.908	—0.554	1.601	5.157

Table II (continued)

E_{lab}(keV)	B_0	B_1	B_2	B_3	B_4	B_5	B_6
1440	24.49	−14.30	23.32	−11.16	2.18	2.64	−1.37
	0.016	−0.072	0.069	−0.152	0.811	0.471	−0.776
1460	18.41	− 5.69	7.16	− 1.87	−3.63	0.34	−1.47
	0.015	− 0.125	0.157	− 0.646	−0.361	2.858	−0.591
1490	18.49	− 2.93	6.78	− 0.23	−3.01	−4.00	−1.28
	0.015	− 0.230	0.159	− 5.016	−0.426	−0.002	−0.692
1525	18.40	− 1.72	6.81	− 7.31	−2.65	−0.33	−0.86
	0.014	− 0.377	0.153	− 0.004	−0.471	−2.820	−1.021
1550	19.13	− 2.81	8.39	− 1.45	1.60	0.62	3.31
	0.014	− 0.230	0.126	0.762	0.780	1.500	0.267
1600	19.94	1.53	7.85	0.85	−1.78	−0.22	−0.10
	0.014	0.424	0.135	1.299	−0.721	−4.266	− 9.347
1630	20.84	3.14	8.74	0.88	−0.48	−0.15	0.99
	0.013	0.206	0.122	1.241	−2.729	−6.136	0.995
1658	20.39	5.39	7.92	1.63	−1.36	−0.20	−0.30
	0.013	0.112	0.129	0.660	−0.918	−4.718	−3.296
1680	19.16	7.42	6.09	2.94	−0.99	0.67	0.19
	0.014	0.085	0.186	0.406	−1.537	1.623	5.830
1700	17.93	5.95	4.61	1.17	0.28	−0.31	0.81
	0.012	0.082	0.181	0.491	3.687	−2.422	1.044
1750	16.52	−7.93	1.94	3.68	−1.34	−0.67	−0.20
	0.015	−0.080	0.516	0.295	−0.901	−1.422	−3.944
1800	14.99	−9.10	2.23	6.38	−4.27	0.30	−2.50
	0.017	0.070	0.446	0.171	−0.283	3.092	−0.285
1850	10.52	−8.50	1.24	4.17	−2.89	−1.20	−1.01
	0.017	−0.053	0.563	0.189	−0.295	−0.543	−0.468
1900	13.82	−6.94	−0.32	2.48	−3.53	−0.58	−1.46
	0.016	−0.078	−2.626	0.374	−0.288	−1.391	−0.443
1950	16.09	−3.75	−2.71	5.03	−6.83	1.81	−3.01
	0.017	−0.178	−0.393	0.229	−0.189	0.565	−0.297
1995	17.69	−5.10	0.55	1.86	−3.62	5.06	−1.38
	0.016	−0.138	2.051	0.649	−0.372	0.003	−0.677

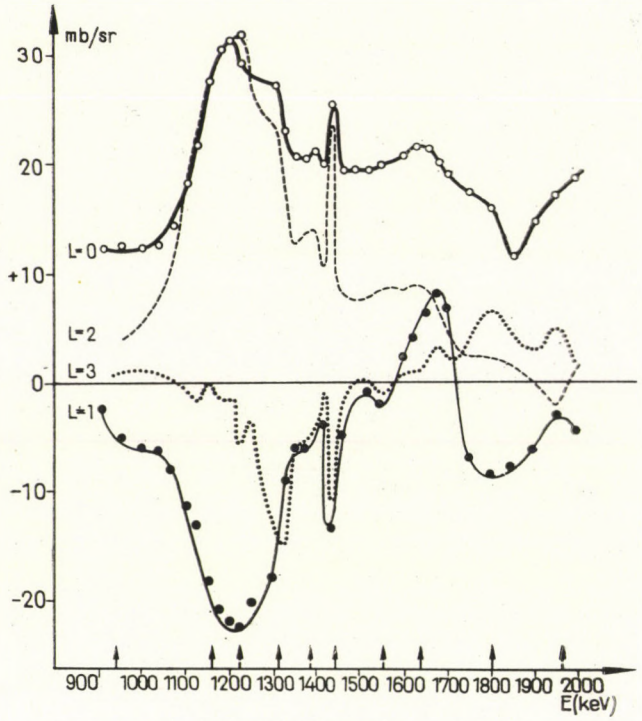

Fig. 6. The energy dependence of the first Legendre coefficients in the differential cross-section of the reaction $^{12}C(d, p)^{13}C$ (g.s.)

5. Discussion

The measured angular distributions are compared in terms of the Legendre coefficients with the earlier experimental results [2, 4] in Fig. 7. An attempt is made to interpret the measured distributions in terms of the energy dependence of these coefficients.

The Legendre coefficients of the *odd*-order L are given at energy E in the region where two resonances of different parities overlap by the expression

$$B_L(E) = \sum_{s_a} \sum_{s_b} (-1)^{s_a - s_b}$$

$$Z(l_{a1} I_1 \, l_{a2} \, I_2; \, s_a \, L) \;\; Z(l_{b1} \, I_1 \, l_{b2} \, I_2; \, s_b \, L)$$

$$\cos \varphi \; (l_{a1} \, l_{a2} \, I_1 \, I_2 \, l_{b1} \, l_{b2}; \; E)$$

$$P_{la1}(E)^{1/2} \;\; P_{lb1}(E)^{1/2} \;\; P_{la2}(E)^{1/2} \;\; P_{lb2}(E)^{1/2}$$

$$\frac{g_a(s_a l_{a1}\,; I_1 \pi_1)\, g_b(s_b l_{b1}\,; I_1 \pi_1)\, g_a(s_a l_{a2}\,; I_2 \pi_2)\, g_b(s_b l_{b2}\,; I_2 \pi_2)}{\left\{(E - \mathscr{E}_1)^2 + \frac{1}{4} \Gamma_1^2\right\}^{1/2} \qquad \left\{(E - \mathscr{E}_2)^2 + \frac{1}{4} \Gamma_2^2\right\}^{1/2}} \,. \tag{1}$$

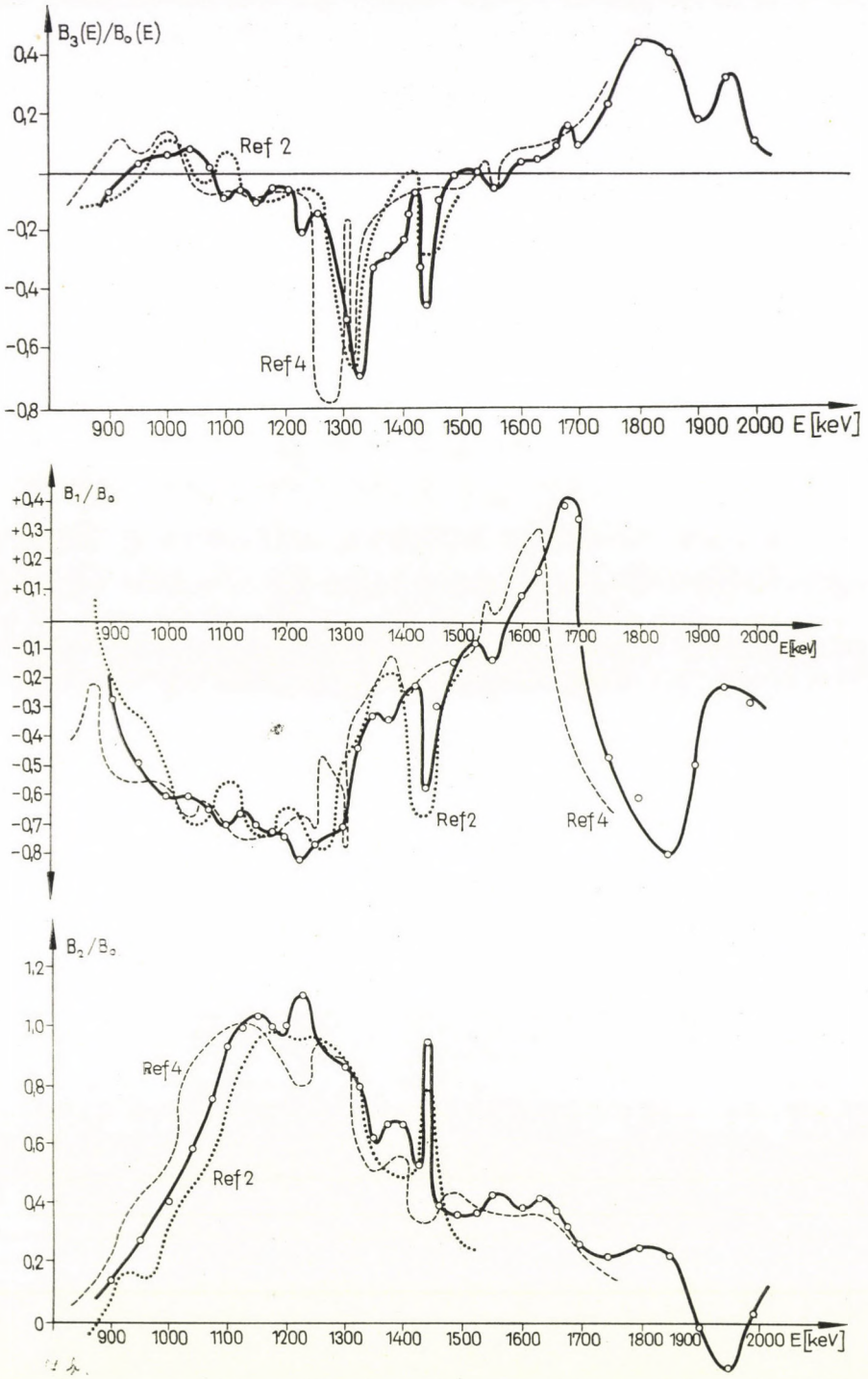

Fig. 7. The energy dependence of the relative Legendre coefficients of the reaction $^{12}C(d, p)^{13}C$ (g.s.) as compared with the results of [2, 4]

Indices 1 and 2 denote the interfering resonances. Each resonance is identified by its spin und parity I, π, position $\mathscr{E}$, total width Γ and the set of g's, the reduced width amplitudes. Subscripts a and b represent the input and the output channels, s the channel spin, and l the minimum orbital angular momentum permitted by the conservation laws and selection rules for the s, I, π values involved in the channel concerned. The Z stands for the coefficients of BLATT and BIEDENHARN [13] and HUBY [14]. The argument φ of the

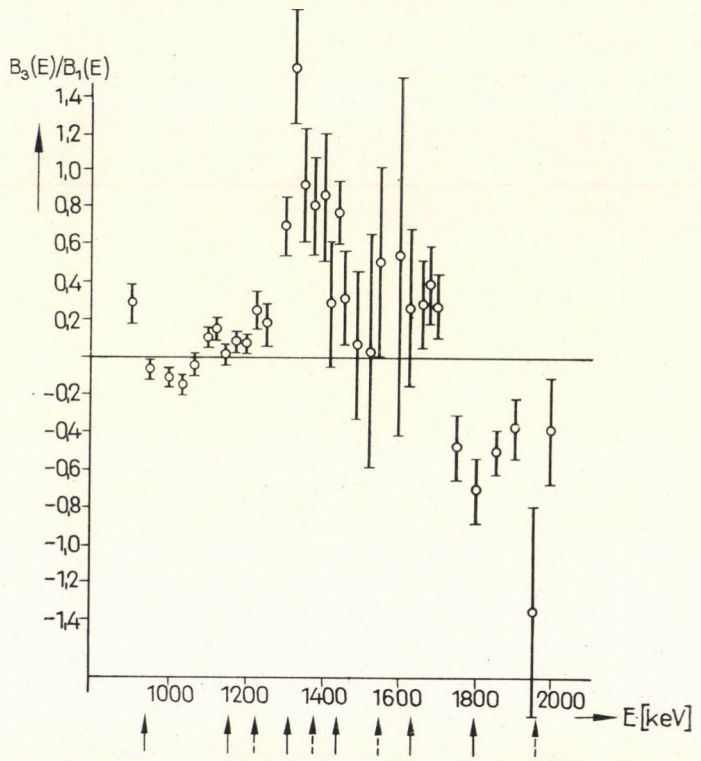

Fig. 8. Energy dependence of the ratio of the first odd Legendre coefficients in the angular distribution of the reaction $^{12}C(d, p)^{13}C$ (g.s.)

cosine factor is the total phase shift including hard sphere scattering, Coulomb scattering and resonance scattering contributions while $P_l(E)$ is a penetration factor. Each term of the sum in Eq. (1) is the contribution from two interfering 'channel pairs'; either of them consists of an input and an output channel both matching the same resonance in spin and parity. Eq. (1) holds only under the conditions: (i) the non-resonant contribution to the scattering matrix is negligible; (ii) the cut-off in l by the penetration factors of the higher orbital angular momenta is not compensated for by other l-dependent factors;

(iii) the interference with any third level can be ignored. If, in spite of the factorization of the L- and E- dependence in Eq. (1), the experimental ratios of the *odd* coefficients are not energy independent one or more of the above assumptions must be invalid. The variation of the ratio B_3/B_1 with energy in Fig. 8 shows the limitations of the method of assigning spins and parities to the interfering levels by comparing the experimental and theoretical values of B_3/B_1. It was found, however, that in the region in question the successive levels of ^{14}N have alternating parities (see Table I). For this reason, the even Legendre coefficients are expected to be virtually of wholly incoherent origin and may be approximated by a simple expression like Eq. (1). The values of the ratio B_2/B_0 can then serve as a check on the spin and parity assignments inferred from the coherent contributions, and vice versa.

For simplicity, the channel spin mixing $s_b = 0, 1$ in the output channel was first disregarded, and the calculated ratios of the Legendre coefficients were tabulated for the interference of levels of different parities from the expression

$$\frac{B_{2\lambda+1}}{B_1} = \frac{Z(l_{a1} I_1 l_{a2} I_2 ; s_a\, 2\lambda+1)\, Z(l_{b1} I_1 l_{b2} I_2 ; s_b\, 2\lambda+1)}{Z(l_{a1} I_1 l_{a2} I_2 ; s_a\, 1)\, Z(l_{b1} I_1 l_{b2} I_2 ; s_b\, 1)} \qquad (\lambda = 1, 2) \tag{2}$$

and for each 'parity isolated' resonance from

$$\frac{B_{2\lambda}}{B_0} = \frac{Z(l_a I l_a I ; s_a 2\lambda)\, Z(l_b I l_b I ; s_b\, 2\lambda)}{Z(l_a I l_a I ; s_a 0)\, Z(l_b I l_b I ; s_b 0)}, \qquad (\lambda = 1, 2, 3) \tag{3}$$

for all the possible combinations of the angular momenta and compared with the measured data in order to determine the discrete resonance parameters where possible.

Wherever a reasonable spin and parity assignment could be made, the channel spin mixing ratio was approximated by the simple formula valid for the particular case of the parity isolated resonances in the $^{12}C(d, p)$ process

$$\begin{aligned} \frac{g^2(0)}{g^2(1)} \equiv \frac{g_b^2(0\, l_b ; I\pi)}{g_b^2(1\, l_b ; I\pi)} = \\ = \{ - [B_2 Z(l_a I l_a I ; 10)\, Z(l_b I l_b I ; 10) \\ - B_0 Z(l_a I l_a I ; 12)\, Z(l_b I l_b I ; 12)] \\ [B_2 Z(l_a I l_a I ; 10)\, Z(l_b I l_b I ; 00) \\ - B_0 Z(l_a I l_a I ; 12)\, Z(l_b I l_b I ; 02)]^{-1} \} \end{aligned} \tag{4}$$

the use of which was justified by the fact that in all practical cases the channels with $s_b = 0$ and those with $s_b' = 1$ had the same lowest partial waves l_b and l'_b. The incoherent contribution is expected to involve a single channel-pair owing to the cut-off of the higher partial waves by the penetration factors and to yield reasonable ratios of reduced widths. The channel spin mixing ratios calculated from Eq. (4) for $L = 2$ are listed in Table III. A similar calculation of the coherent contribution did not yield consistent results. This may be due to the participation of more than two channel-pairs in the interference.

Table III

Channel spin mixing ratios of the ^{14}N levels excited in the reaction $C^{12} + d$

Energy (keV)	I^π	(B_2/B) calc.[a]		(B_2/B_0) exp.	$g^2(1)/g^2(0)$
		$s_b = 1$	$s_b = 0$		
940	1^+	0	0	0.13—0.26	1
1160	2^-	0.5	1	1.04	0.26
1225	3^+	0.85	1.14	1.10	0.41
1310	2^+	0.5	no decay	0.88	0.0
	3^+	0.85	1.14	0.88	3.6
1390	2^-	0.5	1	0.67	1.4
1440	3^+	0.85	1.14	0.95	1.41
1550	2^-	0.5	1	0.44	3.05
1630	2^-	0.5	1	0.42	2.68
1800	1^+	0	0	0.15	1.0
1950	1^-	0	no decay	—0.17	0.0

[a] The calculated B_2/B_0 data refer to single channel-pair transitions with the lowe s possible input and output partial waves.

Significant negative B_4 measured at the anomaly about $E_d = 1160$ keV suggests unequivocally $I = 2$ for this level. The almost equal experimental values of B_2 and B_0 exclude positive parity which would involve $B_2/B_0 = 0.50$ or 0.57 for $s_b = 0$ or 1. The decay of the 2^- resonance favours the $s_b = 0$ output, see Table III. Also the experimental $B_4/B_0 = -0.11$ supports the assumption that there is an admixture from the $s_b = 1$ transition ($l_a = 3$, $B_4/B_0 = -0.56$) to the dominant one with $s_b = 0$ ($l_a = 1$, $B_4/B_0 = 0$).

The large values of B_1 in the interval $E_d = 920—1160$ keV suggest even parity for the resonance at 920 keV. The rapid decrease in B_2 to almost zero shows that $L = 2$ is forbidden and the only possible assignment is 1^+. The interference of 1^+ and 2^- resonances through the dominant partial waves implies $B_3 = 0$ for both output channel spins, which is in agreement with the experimental data.

The striking bulges on the curves for B_0 and B_2 indicate the presence of a narrow level at $E_d = 1300$ keV the even parity of which is uniquely deter-

mined by the large values of B_1. The assumed 'parity isolation' of this level, however, seems to be inconsistent with the experimental values of the even Legendre coefficients. The measured ratio $B_4/B_0 = -0.23$ suggests for this level 2^+, but only through an admixture from the higher order $l_b = 3$ channel. The predicted value of B_2/B_0 for the 2^+ assignment is 0.50 ($l_b = 1$) or 0.57 ($l_b = 3$). The experimental $B_2/B_0 = 0.88$, on the other hand, would be in excellent agreement with the $B_2/B_0 = 0.86$ predicted for the other alternative, $I^\pi = 3^+$ in the dominant partial waves and $s_b = 1$ (see Table III). While the absence of significant B_6 does not contradict this channel pair selection, the measured value of B_4/B_0 is inconsistent with the prediction $B_4/B_0 = 0.85$ ($s_b = 0$) or 0.14 ($s_b = 1$).

The above inconsistency in the pattern could be perhaps explained if one assumes the existence in this region of an unidentified positive parity level as indicated by the rather sharp peaks at $E_d = 1225$ keV on both the B_1 and B_3 curves as well as by the smaller but pronounced extra anomaly in B_0 superimposed on the broad 2^- resonance. If the existence of this level is admitted and the measured value of B_6 at $E_d = 1225$ keV is considered significant its spin and parity must be 3^+. The coherent contribution strongly supports this assignment. The experimental value $B_3/B_1 = 0.26$ is in excellent agreement with the 0.25 predicted for this ratio, assuming the channel spin conserving ($s_b = 1$) interference of a 3^+ and a 2^- (1160 keV) level through the dominant channel pairs. The vanishing of B_5 fits quite well into the picture. But the measured values of the even coefficients giving $B_2/B_0 = 1.10$ favour a transition with $l_a = 0$, $l_b = 3$, $s_b = 0$ for which the prediction is $B_2/B_0 = 1.14$ (see Table III).

The centre of the characteristic narrow resonance at $E_d = 1440$ keV is almost symmetrically surrounded by large minima on the B_1 and the B_3 curves which may be explained by the positive parity of this level and its interference with the distant, broad 2^- resonance at $E_d = 1160$ keV. The measured ratio $B_2/B_0 = 0.95$ restricts the possible spins to 3^+ or 4^+. The non-observation of B_6 does not exclude the latter since the theoretical coefficient is very small. What eventually decides in favour of 3^+ is the measured ratio $B_4/B_0 = 0.09$ as compared with the predictions for $B_4/B_0 = 0.14$ (for $I^\pi = 3^+$, $s_b = 1$); 0.54 ($I^\pi = 4^+$, $s_b = 0$); 0.58 ($I^\pi = 4^+$, $s_b = 1$) (see also Table III).

If the small, but significant anomaly on the B_0 and B_2 curves at $E_d = 1380$ keV is not due to some experimental error one has to assume here a negative parity level since 1^+ is ruled out by the observed coherent contribution from the 1440 keV level. The 2^- assignment seems to be the only one that is compatible with the predictions if one compares the experimental value of $B_2/B_0 = 0.62$ with the calculated ones, 0.5 ($s_b = 1$) and 1.0 ($s_b = 0$) for 2^- as well as 0.9 for 3^- (see also Table III). The odd coefficients do not contradict the assumed interference of two narrow but closely packed 2^- and 3^+ levels.

The predictions $B_3/B_1 = 0.66$ and 0.25 for $s_b = 0$ and 1, respectively, are comparable with the observed values, namely $B_3/B_1 = 0.86$ (both 1375 and 1400 keV); 0.30 (1420 keV) and 0.77 (1440 keV).

The structure observed at $E_d = 1550$ keV in the energy dependence of the first five Legendre coefficients may perhaps be regarded as significant. In this case, the experimental value of B_2/B_0 would indicate the presence of a 2^- state (see Table III). The higher order coefficients are, however, too small to permit any definite inferences to be made about this hypothetical level.

At $E_d = 1800$ keV, all the measured coefficients of the angular distributions exhibit some anomaly. In single channel pair transitions with allowed $L = 2$, the lowest positive value predicted for B_2/B_0 is 0.5. Thus, the observed $B_2/B_0 = 0.14$ cannot be explained unless one admits the admixture of a transition forbidding $L = 2$. Such an admixture would suggest 1^+ for this level (see also Table III). This assignment is, however, inconsistent with the significance of B_4 and B_6 (!); this can be explained barring experimental errors only by postulating the proximity of a non-identified positive parity level.

With the assumption of even parity for this last level, the strongly varying odd coefficients within the preceding interval suggest the existence of a negative-parity resonance localized at $E_d = 1630$ keV as apparent from the structure on the first two even Legendre coefficients. The experimental value $B_2/B_0 = 0.42$ suggests 2^- (see also Table III). Also the measured insignificant $B_4 = -0.02$ shows that the dominant channel again conserves the channel spin. The odd coefficients vanish in the interval $E_d = 1550-1630$ keV which is in agreement with the assumption of two 2^- levels. The drastic change in the angular distribution pattern at about 1680 keV is revealed by the sudden change in B_1 that is not accompanied by a corresponding change in B_3. Possibly because of the above mentioned, non-identified level, there is no correlation between the odd coefficients in the second part of this interval, and the present analysis fails to work here. It is of interest to note also the shift from 1625 to 1675 keV in the peaks of the even and the odd coefficients, respectively.

All the coefficients $L = 1-6$ at $E_d = 1950$ keV exhibit systematic anomalies but they could be hardly taken as indications of a parity-isolated resonance level. The spin and parity assignment would be restricted by the experimental ratio $B_2/B_0 = 0.17$ to the alternatives 1^+ and 1^-, and the incoherent contribution below this level favours the latter (see also Table III). On the other hand, such a low spin value is obviously in contradiction with the observed value of $B_4/B_0 = -0.42$ even if one ignores the significant B_6 at this energy. Either the direct background or some non-identified compound state with higher spin, or both, must be held responsible for the inconsistent behaviour of the coefficients.

The information on the ^{14}N levels obtained from the present work is

in good agrement with the results of other authors; see Table I. The known negative-parity level at $E_d = 1620$ keV has been assigned the spin $I = 2$ in the present analysis. The states suggested at $E_d = 1225$ keV (3^+), 1380 keV (2^-), 1550 keV (1^-) have still to be checked in further detailed studies of the competing channels.

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УГЛОВОЕ РАСПРЕДЕЛЕНИЕ РЕАКЦИИ $^{12}C(d, p)^{13}C$ НИЖЕ $E_d = 2$ Мэв

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Р е з ю м е

Измерялись функции возбуждения реакций $^{12}C(d, p)$ и $^{12}C(d, d)$ под углом 90°, в интервале $E_d = 900 - 2000$ Кэв шагом 10 Кэв. Угловые распределения реакции $^{12}C(d, p_0)^{13}C$ измерялись шагом 25 Кэв. Экспериментальные данные обсуждаются с точки зрения перекрывающихся составных уровней. Явление интерференции внушает добавочные новые уровни в ^{14}N.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (3), pp. 285—289 (1969)

THE YOUNG'S AND SHEAR MODULI OF NaCl WHISKERS

By

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(Received 1. X. 1968)

The static and dynamic Young's moduli, and the dynamic shear modulus of sodium chloride whiskers grown by various methods were measured at room temperature in the ⟨100⟩ direction. In the thickness range of 2—18 μ no dependence on whisker dimension was detected. According to the experimental results the values of the elastic moduli measured correspond to values obtained with large single crystals.

The physical properties of whiskers differ in many instances from the properties of large single (bulk) crystals. Thus, e.g. certain mechanical properties (tensile strength, yield point) of the whiskers are better than those of single crystals [1]. However, as the elastic constants of the crystals are mainly determined by their crystal structure and the forces acting between the crystal components, one would expect the elastic constants of the whiskers not to deviate very much from the values for large single crystals. Measurements carried out on various whiskers supported this expectation in some cases [2, 3], whereas, in other cases, Young's modulus values were obtained which were less [4], or many times more [5, 6] than those measured with large single crystals. The absolute value of the shear modulus has until now been determined only for iron whiskers; the values obtained were smaller than for large single crystals [7]. Considering the contradictory results [3] it was decided to measure the Young's and shear moduli of NaCl whiskers grown by various methods.

The samples were grown by methods described earlier, either on cellophane or from solutions containing polyvinyl alcohol additions [8]. With the latter method whiskers were grown either from "pure" solutions or from solutions which were contaminated with $CaCl_2$. The crystals grew in the ⟨100⟩ directions and had rectangular cross sections.

The Young's modulus of the whiskers was measured in two different ways: dynamically and with a statical method. Applying the dynamic method the Young's modulus of the samples was determined by measuring the bending vibrations. One end of the whiskers was glued to a fixed support with diphenylcarbazide and was made to vibrate by on alternating electric field in a vacuum of 10^{-2} torr. The possibility for having the whiskers vibrated in this way was the fact that, according to our observations, the whiskers were

electrically charged. The rectangular shaped whiskers vibrated in one set of the experiments along their short sides, and in another set along their long sides. The vibration of the whiskers was observed microscopically. As with negligibly small friction, the resonance frequency corresponds to the natural or free frequency of the whiskers, Young's modulus (E) could have been computed by the following formula:

$$E = \frac{48\pi^2}{m^4} \cdot \frac{l^4 \nu^2 \varrho}{h^2},$$

where l is the length of the whisker, ν = free frequency, ϱ = whisker density, h = side length of the whisker cross-section in the direction of the vibration

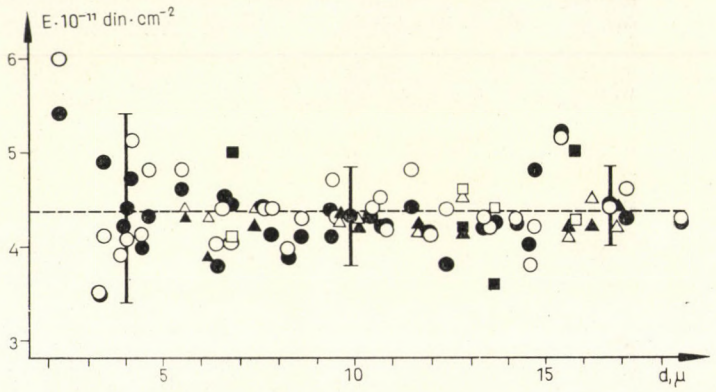

Fig. 1. The thickness dependence of the Young's modulus of NaCl whiskers measured in the ⟨100⟩ direction. ○ ● whiskers grown from solutions containing polyvinyl alcohol; △ ▲ whiskers grown from solutions containing polyvinyl alcohol and doped with 0.1 mol% $CaCl_2$ □ ■ whiskers grown on cellophane. The dark marks indicate the values obtained from vibrations along the direction of the longer cross-sectional side

and m is 1,875 for rods fixed at one end in the case of ground vibration. The resonance frequency of the crystals (lying in the range of 70—150 c/s) was measured with a DAWE 1205 type stroboscope. The dimensions of the whiskers cross-section were determined microscopically with an accuracy of 0,3 μ. The cross-section was measured at several points along the length of the whiskers. The longitudinal dimensions of the samples varied between 5 and 15 millimetres. On a few occasions repeated measurements were carried out on one and the same whisker at various lengths with various vibrational amplitudes and in a varying vacuum. These experiments always yielded the same Young's modulus values. The results of the dynamical Young's modulus measurements carried out on various whiskers are shown in Fig. 1, which represents the Young's modulus as a function of the whisker thickness (the square root of the sample cross-section). The black dots correspond to the vibration along the longer side of the cross section and the white dots represent

the results obtained by vibrating along its shorter side. For three measurements the value of the probable error resulting mainly from incorrect measurements of the cross section is also indicated. The probable error is greater for thinner crystals. The dotted line in Fig. 1 corresponds to the Young's modulus of large single NaCl crystals ($E = 4{,}37 \cdot 10^{11}$ dyn $\cdot$ cm^{-2} [10]). As can be seen from Fig. 1, the E values do not depend upon the dimensions of the samples and the results scatter about the value obtained for large single crystals within the error of measurements.

The long NaCl whiskers grown from solutions containing polyvinyl alcohol allowed, the Young's modulus of the whiskers to be determined by a simple static method. The samples were freely supported at their ends. The deflection caused by the dead weight of the samples was determined microscopically. Employing the relation concerning the bending deflection caused by the dead weight of horizontal beams [11] the Young's modulus of the samples can be expressed with the maximum deflection (f), the distance between the two wedges supporting the sample (l), the length of the vertical side of the whisker and the specific weight of NaCl by applying the formula

$$E = \frac{60}{384} \cdot \frac{\gamma l^4}{f b^2} .$$

Generally, whiskers 5—6 cm long and 10—20 μ thick were measured. From eleven experiments an average value of $E = 4{,}26 \cdot 10^{11}$ dyn. cm^{-2} was obtained with an error of 18%.

The shear modulus of the NaCl whiskers was measured by torsional vibrations. Reversed T-shaped quartz rods were glued to the lower end of the whiskers which were fixed vertically to the support with diphenylcarbazide. In this way a torsional pendulum was obtained. The period of the oscillation of the pendulum was determined in a vacuum of 10^{-2} torr. The oscillation was induced mechanically by twisting the support. If the period of the oscillation (T) is known, the shear modulus (G) can be computed by the relation

$$G = \frac{4\pi^2 \Theta l}{\alpha T^2 b^4} .$$

In this formula Θ is the moment of inertia of the torsional pendulum, l is the length of the whiskers, b represents the length of the shorter side of the rectangular cross-section of the whisker, α is a constant depending upon the ratio of the two sides of the cross section of the samples. The value of this constant was taken from [11].

The period of oscillation of the torsional pendulum was generally 1—5 sec, the length of the whiskers varied between 5 and 15 mm. The moment of

inertia of the torsional pendulum was computed by the dimensions and the mass (0.3—3 mg) of the cross rod. The results of the measurements of the shear modulus are represented in Fig. 2. As can be seen no thickness dependence can be detected. The experimental data scatter about the value of large single crystals $G = 1{,}26 \cdot 10^{11}$ dyn $\cdot$ cm^{-2} [10] within the experimental error.

The results show unambiguously that the values of the elastic moduli for NaCl are within the experimental error the same for whiskers as for large

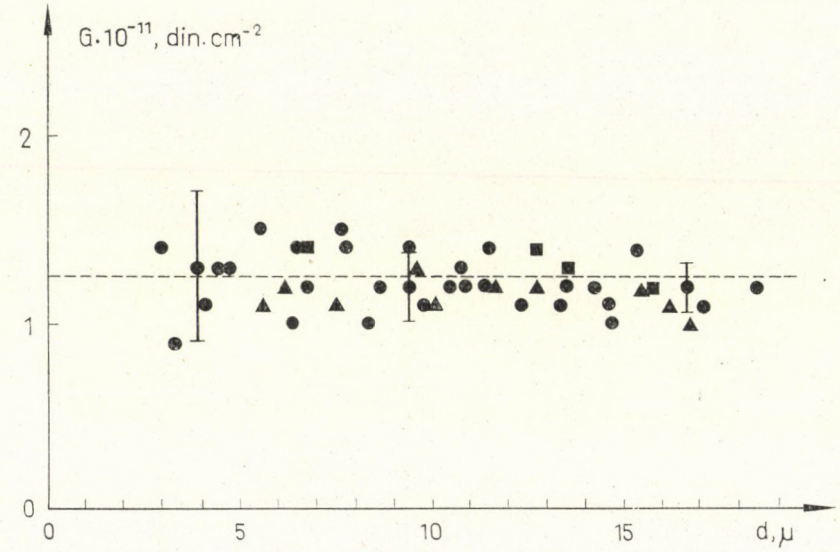

Fig. 2. The thickness dependence of the shear modulus of NaCl whiskers measured in the ⟨100⟩ direction. ● whiskers grown from solutions containing polyvinyl alcohol; ▲ whiskers grown from solutions containing polyvinyl alcohol and doped with 0.1 mol% $CaCl_2$; ■ whiskers grown on cellophane

single crystals. The fact that the elastic moduli of whiskers of various degrees of purity and grown with various methods are identical within the experimental error seems to show that the contradictions in the experimental results as found in the literature are due neither to the various methods of growth nor to the various degree of purity of the samples.

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МОДУЛЬ ЮНГА И МОДУЛЬ СДВИГА НИТЕВИДНЫХ КРИСТАЛЛОВ NaCl

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Р е з ю м е

Определены величины статического и динамического модулей Юнга и величина динамического модуля сдвига при комнатной температуре в направлении <100> для интевидных кристаллов NaCl, выращенных разными методами. Толщина интевидных кристаллов изменялась от 2 мк до 18 мк. Не обнаружена зависимость модулей от толщины образцов. По экспериментальным данным величины модулей согласуются с величинами модулей монокристаллов NaCl большого размера.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (3), pp. 291—299 (1969)

DAMPING AND PHASE SHIFT OF MAGNETOHYDRODYNAMIC WAVES IN A PLASMA HAVING FINITE ELECTRICAL AND THERMAL CONDUCTIVITY

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(Received 30. VII. 1968)

The formulae describing the propagation of small amplitude magnetohydrodynamic waves in any direction relative to the external magnetic field in a plasma of infinite extension and of finite electrical and thermal conductivity have been derived from the fundamental equations of magnetohydrodynamics. The waves generated in the medium which is non-ideal in this sense are damped and phase differences appear between the perturbances of the magnetohydrodynamic quantities. These properties of the waves can be described by a complex refractive index.

I. Introduction

The properties of magnetohydrodynamic waves generated in electrically conducting fluids or gases differ from those of electromagnetic and acoustic waves as a result of coupled electromagnetic and hydrodynamic phenomena. A conducting medium acquires a special anisotropy in a magnetic field. This means that the velocity of the wave propagation depends on the angle included between the directions of wave propagation and the external magnetic field. On the other hand, not only longitudinal waves but also transverse waves can be generated in an ideal plasma. In an ideal medium of infinite extension, the electrical conductivity of which is infinitely large but whose thermal conductivity and viscosity are negligible, there is no dispersion; that is, the waves are undamped and their phase and group velocities are equal [1], [2].

The fundamental equations of magnetohydrodynamics serve as a basis for studying the propagation of magnetohydrodynamic waves:

$$\frac{\partial \vec{H}}{\partial t} = \mathrm{rot}\left[\vec{v}x\vec{H}\right] + \nu_m \Delta \vec{H}, \tag{1}$$

$$\mathrm{div}\,\vec{H} = 0, \tag{2}$$

* This work was performed at the Roland Eötvös University.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{1}{\varrho}\nabla p - \frac{1}{4\pi\varrho}[\vec{H} x \operatorname{rot}\vec{H}] + \\ + \frac{\eta}{\varrho}\Delta\vec{v} + \frac{1}{\varrho}\left(\zeta + \frac{\eta}{3}\right)\nabla \operatorname{div}\vec{v}, \tag{3}$$

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}\varrho\vec{v} = 0, \tag{4}$$

$$\varrho T\left[\frac{\partial s}{\partial t} + (\vec{v}\nabla)s\right] = \sigma'_{ik}\frac{\partial v_i}{\partial x_k} + \operatorname{div}(\chi\nabla T) + \frac{c^2}{16\pi^2\sigma}(\operatorname{rot}\vec{H})^2, \tag{5}$$

$$p = p(\varrho, T), \tag{6}$$

where

$$\nu_m = \frac{c^2}{4\pi\sigma}, \tag{7}$$

$$\sigma'_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}. \tag{8}$$

The notation of these equations:

ϱ	— density of the medium,
$\vec{v}$	— flow velocity,
p	— pressure (assumed to be scalar)
η, ζ	— kinematic viscosity coefficients,
ν_m	— magnetic viscosity,
σ'_{ik}	— viscosity tensor,
s	— entropy per unit mass,
T	— absolute temperature,
σ	— electrical conductivity,
χ	— thermal conductivity,
$\vec{H}$	— magnetic field strength,
c	— velocity of light in free space.

II. Propagation of small amplitude magnetohydrodynamic waves in a medium having finite electrical conductivity

First of all, we discuss the propagation of waves in a medium whose thermal conductivity and kinematic viscosity can be neglected and whose electrical conductivity (and for this reason its magnetic viscosity, too) has finite values. The terms of Eqs. (3) and (5) containing the coefficients χ, η and ζ can be omitted because of this restriction. We also assume that the behaviour of the medium is governed by the equation of state of ideal gases. For this

reason, Eq. (6) has the form:

$$\frac{p}{\varrho T} = R = c_p - c_v,$$

where c_p and c_v are the specific heats under constant pressure and constant volume, respectively. If we introduce the entropy density instead of temperature as a state variable, we obtain the equation of state in the following form:

$$\frac{\partial p}{\partial t} + (\vec{v}\nabla)\, p - \varkappa \frac{p}{\varrho}\left[\frac{\partial \varrho}{\partial t} + (\vec{v}\nabla)\,\varrho\right] - \frac{p}{c_v}\left[\frac{\partial s}{\partial t} + (\vec{v}\nabla)\, s\right] = 0, \qquad (9)$$

$$\varkappa = \frac{c_p}{c_v}.$$

Arbitrary constant values of $\vec{H}_0$, $\vec{v}_0$, p_0, ϱ_0 and s_0 are solutions of the system of equations (1)—(9). We superimpose small disturbances on this steady state:

$$\vec{H} = \vec{H}_0 + \vec{H}', \quad \vec{v} = \vec{v}_0 + \vec{v}', \quad p = p_0 + p', \quad \varrho = \varrho_0 + \varrho', \quad s = s_0 + s'. \qquad (10)$$

If we substitute these quantities into Eqs. (1)—(9), neglecting the terms involving squares and products of the small perturbances and of their derivatives and neglecting the terms involving the material coefficients mentioned above, we obtain the following linear equations:

$$\frac{\partial \vec{H}'}{\partial t} = (\vec{H}_0\nabla)\,\vec{v}' - (\vec{v}_0\nabla)\,\vec{H}' - \vec{H}_0 \operatorname{div} \vec{v}' + \nu_m \Delta \vec{H}', \qquad (11)$$

$$\operatorname{div} \vec{H}' = 0, \qquad (12)$$

$$\frac{\partial \vec{v}'}{\partial t} + (\vec{v}_0\nabla)\,\vec{v}' = -\frac{1}{\varrho_0}\nabla p' - \frac{1}{4\pi\varrho_0}\nabla(\vec{H}_0\,\vec{H}') + \frac{1}{4\pi\varrho_0}(\vec{H}_0\,\nabla)\,\vec{H}', \qquad (13)$$

$$\frac{\partial \varrho'}{\partial t} + (\vec{v}_0\nabla)\,\varrho' + \varrho_0 \operatorname{div} \vec{v}' = 0, \qquad (14)$$

$$\frac{\partial s'}{\partial t} + (\vec{v}_0\nabla)\, s' = 0, \qquad (15)$$

$$\frac{\partial p'}{\partial t} + (\vec{v}_0\nabla)\, p' - c_s^2\left[\frac{\partial \varrho'}{\partial t} + (\vec{v}_0\nabla)\,\varrho'\right] - \frac{p_0}{c_v}\left[\frac{\partial s'}{\partial t} + (\vec{v}_0\nabla)\, s'\right] = 0, \qquad (16)$$

$$c_s = \sqrt{\varkappa \frac{p_0}{\varrho_0}},$$

where c_s is the velocity of sound in the medium in the absence of a magnetic field. Since Eqs. (11)—(16) are linear their solutions can be given as superpositions of plane waves, which are the following functions of space coordinates and time, e.g. in the case of magnetic field disturbance:

$$\vec{H}' = \vec{H}_1 \exp\left\{i\omega\left[t - \frac{n^*(\vec{k}\vec{r})}{c_0}\right]\right\}. \tag{17}$$

The small perturbances of the other magnetohydrodynamic quantities can be written in the same form. In Formula (17) ω represents the frequency; $\vec{k}$ the unit vector pointing in the direction of wave propagation; c_0 is the velocity of the wave propagation in an ideal medium, relative to the coordinate system; n^* is the complex refractive index, the determination of which is one of the purposes of our calculations. Substituting the plane wave formulae of the small disturbances like (17) into the system of Eqs. (11)—(16), we obtain the following algebraic equations for the amplitudes of the waves:

$$\left[1 - \frac{n^*(\vec{k}\vec{v}_0)}{c_0}\,\frac{i\omega\nu_m\, n^{*2}\vec{k}^2}{c_0^2}\right]\vec{H}_1 + \frac{n^*(\vec{k}\vec{H}_0)}{c_0}\,\vec{v}_1 - \frac{n^*(\vec{k}\vec{v}_1)}{c_0}\,\vec{H}_0 = 0, \tag{18}$$

$$(\vec{k}\vec{H}_1) = 0, \tag{19}$$

$$\left[1 - \frac{n^*(\vec{k}\vec{v}_0)}{c_0}\right]\vec{v}_1 + \frac{n^*(\vec{k}\vec{H}_0)}{4\pi\varrho_0 c_0}\vec{H}_1 - \frac{p_1 n^*}{\varrho_0 c_0}\vec{k} - \frac{(\vec{H}_0\vec{H}_1)\,n^*}{4\pi\varrho_0 c_0}\vec{k} = 0, \tag{20}$$

$$\left[1 - \frac{n^*(\vec{k}\vec{v}_0)}{c_0}\right]\varrho_1 - \frac{\varrho_0 n^*}{c_0}(\vec{k}\vec{v}_1) = 0, \tag{21}$$

$$\left[1 - \frac{n^*(\vec{k}\vec{v}_0)}{c_0}\right]s_1 = 0, \tag{22}$$

$$\left[1 - \frac{n^*(\vec{k}\vec{v}_0)}{c_0}\right]\left[p_1 - c_s^2\varrho_1 - \frac{p_0}{c_v}s_1\right] = 0. \tag{23}$$

Non-vanishing solutions of the homogeneous linear system of Equations (18)—(23) for the amplitudes of the small perturbances exist only in the case when the determinant of the system of equations is equal to zero:

$$\Theta\left[\Theta\Theta^* - \frac{(\vec{k}\vec{H}_0)^2}{4\pi\varrho_0}\right]\left[\Theta^4 - \vec{k}^2\left(c_s^2 + \frac{\Theta}{\Theta^*}\,\frac{\vec{H}_0^2}{4\pi\varrho_0}\right)\Theta^2 + \vec{k}^2 c_s^2\,\frac{\Theta}{\Theta^*}\,\frac{(\vec{k}\vec{H}_0)^2}{4\pi\varrho_0}\right]\Theta^{*2} = 0, \tag{24}$$

where

$$\Theta = \frac{c_0}{n^*} - (\vec{k}\vec{v}_0), \tag{25}$$

$$\Theta^* = \frac{c_0}{n^*} - (\vec{k}\vec{v}_0) - \frac{i\omega\nu_m n^* \vec{k}^2}{c_0}. \tag{26}$$

We remark that since $\vec{k}$ denotes a unit vector we could perform the substitution $\vec{k}^2 = 1$ in all formulae. Still, we always indicate $\vec{k}^2$ for the sake of completeness, so that our formulae can be compared easily with the results of those papers in which $\vec{k}$ denotes the wave-number vector.

Three types of waves can be distinguished according to which of the left-side factors of Eq. (24) vanishes.

a) The root $\Theta = 0$ of Eq. (24) means a perturbance which moves together with the medium. The solutions of Eqs. (18)—(23) are, in this case:

$$\varrho_1 = -\frac{p_0}{c_s^2 c_v} s_1, \quad \vec{v}_1 = 0, \quad \vec{H}_1 = 0, \quad p_1 = 0. \tag{27}$$

This is the *entropy wave* which can also be generated in an ideal medium. Since, in this case, the refractive index is a real number and its value is unity, this wave is undamped.

b) It is also possible that the second factor of Eq. (24) vanishes:

$$\sqrt{\Theta\Theta^*} = \pm \frac{(\vec{k}\vec{H}_0)}{\sqrt{4\pi\varrho_0}}. \tag{28}$$

Taking into account Eq. (28) we get the following solutions of Eqs. (18)—(23) for the amplitudes of the small perturbances:

$$\left.\begin{array}{lll} \varrho_1 = 0, & p_1 = 0, & s_1 = 0, \\ \vec{v}_1 = \mp\sqrt{\dfrac{\Theta^*}{\Theta}}\dfrac{\vec{H}_1}{\sqrt{4\pi\varrho_0}}, & (\vec{k}\vec{H}_1) = 0, & (\vec{H}_0\vec{H}_1) = 0 \end{array}\right\} \tag{29}$$

Eqs. (28) and (29) describe the **properties of the *Alfvén* waves.** We suppose that $(\vec{k}\vec{v}_0) = 0$. In this case, the complex **refractive** index can easily be determined from (28).

$$\frac{n^*}{c_0} = \frac{1}{c_A}(n_A - i\varkappa_A), \tag{30}$$

where

$$c_A = \pm \frac{(\vec{k}\vec{H}_0)}{\sqrt{4\pi\varrho_0}} . \tag{31}$$

c_A is the velocity of the propagation of the Alfvén wave in an ideal medium.

$$\varkappa_A = \sqrt{\frac{c_A^2(\sqrt{c_A^4+\alpha^2}-c_A^2)}{2(c_A^4+\alpha^2)}}, \quad n_A = \sqrt{\frac{c_A^2(\sqrt{c_A^4+\alpha^2}+c_A^2)}{2(c_A^4+\alpha^2)}}, \tag{32}$$

$$\alpha = \omega\nu_m = \frac{\omega c^2}{4\pi\sigma} .$$

Substituting (30) into (17) the perturbance of the magnetic field strength has the form:

$$\vec{H}' = \vec{H}_1 \exp\left\{i\omega\left[t - \frac{n_A}{c_A}(\vec{k}\vec{r})\right] - \frac{\omega\varkappa_A}{c_A}(\vec{k}\vec{r})\right\} . \tag{33}$$

It can be seen that the wave of the magnetic field strength is damped; the damping is in direct proportion to the frequency and the quantity $\varkappa_A$. The velocity of wave propagation in a medium having finite electrical conductivity is:

$$c_\sigma = \frac{c_A}{n_A} ,$$

while its penetration depth is:

$$d = \frac{c_A}{\omega\varkappa_A} .$$

On the other hand, it can be shown that

$$\mp\sqrt{\frac{\Theta^*}{\Theta}} = \mp(n_A - i\varkappa_A) = \mp\sqrt{n_A^2+\varkappa_A^2}\exp(-i\gamma_A), \tag{34}$$

where

$$\operatorname{tg}\gamma_A = \frac{\varkappa_A}{n_A} .$$

Making use of this expression and of (29) the flow velocity wave is

$$\vec{v}' = \mp\sqrt{\frac{n_A^2+\varkappa_A^2}{4\pi\varrho_0}}\vec{H}'\exp(-i\gamma_A) =$$

$$= \mp\sqrt{\frac{n_A^2+\varkappa_A^2}{4\pi\varrho_0}}\vec{H}_1\exp\left\{i\omega\left[t - \frac{n_A}{c_A}(\vec{k}\vec{r})\right] - \frac{\omega\varkappa_A}{c_A}(\vec{k}\vec{r}) - i\gamma_A\right\} . \tag{35}$$

It can be seen that *a phase difference γ_A appears between the disturbances of the flow velocity and the magnetic field strength.* n^* determines the phase difference between $\vec{v}$ and $\vec{H}$ (see Eq. (35)) in the same way as the complex refractive index determines the phase difference between $\vec{E}$ and $\vec{H}$ [3] in the case of electromagnetic wave propagation in metals.

c) The *magnetoacoustic waves* are generated if the third factor of (24) vanishes:

$$\Theta^4 - \vec{k}^2 \left(c_s^2 + \frac{\Theta}{\Theta^*} \frac{\vec{H}_0^2}{4\pi\varrho_0} \right) \Theta^2 + \vec{k}^2 c_s^2 \frac{\Theta}{\Theta^*} \frac{(\vec{k}\vec{H}_0)^2}{4\pi\varrho_0} = 0 . \tag{36}$$

From this equation n^*, which is again a complex quantity, can be expressed. This means that these waves are damped, too. The expressions of the amplitudes of the waves are:

$$\vec{v}_1 = -\frac{\Theta}{\varrho_0 \vec{k}^2} \frac{\frac{(\vec{k}\vec{H}_0)}{4\pi\varrho_0} \vec{k}^2 \vec{H}_0 - \Theta\Theta^* \vec{k}}{\Theta\Theta^* - \frac{(\vec{k}\vec{H}_0)^2}{4\pi\varrho_0}} \varrho_1 , \tag{37}$$

$$\vec{H}_1 = \frac{\Theta^2}{\varrho_0 \vec{k}^2} \frac{\vec{k}^2 \vec{H}_0 - (\vec{k}\vec{H}_0)\vec{k}}{\Theta\Theta^* - \frac{(\vec{k}\vec{H}_0)^2}{4\pi\varrho_0}} \varrho_1 , \tag{38}$$

$$p_1 = c_s^2 \varrho_1 , \qquad s_1 = 0 . \tag{39}$$

It is evident from these expressions that phase differences appear between the perturbances of the flow velocity, the magnetic field strength and the density. The phase differences can be determined if the complex factors in the formulae of $\vec{v}_1$ and $\vec{H}_1$ are written in trigonometric forms.

In all three cases discussed above the formulae for the thermodynamical state variables of the medium (pressure, mass and entropy density) are the same as in the case of the ideal medium. The reason for this fact is that we neglected the Joule heat as a second order small quantity when we linearized the energy equation (5).

It must also be pointed out that all the roots of the system of Eqs. (18)—(23) are equal to zero, if the last factor of (24) vanishes ($\Theta^* = 0$).

III. Magnetohydrodynamic waves in a nonviscous medium having finite electrical and thermal conductivity

The above results can be readily extended for the case when the thermal conductivity of the medium is a scalar quantity which does not depend on the

space coordinates and has a finite value. In this case, energy equation (5) can be written in the following form:

$$\varrho T\left[\frac{\partial s}{\partial t}+(\vec{v}\,\nabla)\,s\right]=\chi\Delta T. \tag{40}$$

If we use the entropy density instead of temperature as a state variable and linearize the equation it has the form:

$$\varrho_0 c_v\left[\frac{\partial s'}{\partial t}+(\vec{v}_0\,\nabla)\,s'\right]=\chi\left(\Delta s'+\frac{R}{\varrho_0}\Delta\varrho'\right). \tag{41}$$

Substituting the plane wave formulae of s' and ϱ' like (17) into (41) we get the following equation:

$$Zs_1-\frac{R}{\varrho_0}\varrho_1=0, \tag{42}$$

where

$$Z=\frac{c_0 c_v \varrho_0 \Theta}{i\omega\chi\, n^*\vec{k}^2}-1. \tag{43}$$

The amplitudes of the small perturbances can be determined from the system of equations (11)—(16), but Eq. (42) has to be considered instead of (15). This system of equations has to be solved when Eq. (24) is valid with the modification that its first factor Θ is replaced by Z. If $Z=0$, all the roots of the system of equations are equal to zero. This means that *no entropy wave exists in this case.* The second factor of (24) and all the formulae describing the properties of Alfvén waves remain unchanged. Only the expressions (36) and (39) of the formulae describing the magnetoacoustic waves are altered; namely, c_s^2 is replaced by c_s^{*2} in both expressions.

$$c_s^{*2}=c_s^2+(\varkappa-1)\frac{p_0}{\varrho_0}\,\frac{i\omega\chi\, n^*\vec{k}^2}{c_0 c_v \varrho_0\Theta-i\omega\chi\, n^*\vec{k}^2}, \tag{44}$$

for this reason

$$p_1=c_s^{*2}\varrho_1, \tag{45}$$

and the perturbance of the entropy has a finite value:

$$s_1=\frac{R}{\varrho_0}\,\frac{i\omega\chi\, n^*\vec{k}^2}{c_0 c_v \varrho_0\Theta-i\omega\chi\, n^*\vec{k}^2}\varrho_1. \tag{46}$$

Expressions (44) and (46) involve complex quantities, which means that phase differences appear between the perturbances of the thermodynamic quantities p', ϱ' and s'. This fact and the disappearance of the entropy wave are connected with energy transport through heat conduction.

IV. Acknowledgement

The author wishes to express sincere thanks to Dr. I. Abonyi of the Roland Eötvös University for his critical remarks and helpful suggestions.

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ЗАТУХАНИЕ И СДВИГ ФАЗЫ МАГНИТОГИДРОДИНАМИЧЕСКИХ ВОЛН В ПЛАЗМЕ С КОНЕЧНЫМИ ЭЛЕКТРИЧЕСКОЙ И ТЕПЛОВОЙ ПРОВОДИМОСТЯМИ

ДЬ. ШИМОНИЧ

Р е з ю м е

Исходя из основных уравнений магнитогидродинамики определяются главнейшие зависимости распространения магнитогидродинамических волн с малой амплитудой, распространяющихся в любое направление по отношению внешнего магнитного поля в неограниченной плазме с конечными электрической и тепловой проводимостьями. Возникающие в таком смысле неидеальной среде волны затухаются и появляется сдвиг фазы между возмущениями некоторых магнитогидродинамических величин. Данные свойства волн можно описывать комплексным показателем преломления.

RECENSIONES

Les centres colorés dans les cristaux ioniques

Colloque de la Société Française de Physique. Presses Universitaires de France. Exposés et Communications présentés à SACLAY, les 16, 17 et 18 mars 1967.

D'entre les exposés tenus par des physiciens en renom nous désirons mentionner les suivants: J. FRIEDEL rappelle les types de défaut et la structure électronique de quelques centres colorés simples et il analyse certains des problèmes qu'ils posent. W. D. COMPTON s'occupe des problèmes concernant l'état excité relaxé des centres colorés à excès d'electron dans les halogenures alcalins. Une brève revue des propriétés des centres neutres est donnée ainsi que celles des états une fois ionisés et celles des centres complexes qui ont piégé un électron supplémentaire. G. SPINOLO décrit l'état relâché du centre F, en rappelant sa relation et différence avec celui pas relâché. Il donne une attention particulière au temps de vie moyenne radiative de l'état excité et relâché. A. A. KAPLYANSKIJ discute les propriétés des centres non cubiques dans les cristaux cubiques et leur spectre sous l'action du champ électrique et magnétique extérieur. Il est rendu compte de l'information sur les propriétés des centres et des transitions optiques, obtenue en examinant les spectres sous l'influence des effets extérieurs. A. E. HUGHES donne une courte introduction théorique aux raisons physiques qui font apparaître la structure vibrationelle, puis l'auteur décrit les résultats de deux types principaux d'experiences et leur interprétation. Ces résultats servent à discuter les déplacements isotopiques des transitions à zéro phonon. J. MARGERIE s'occupe du dichroïsme circulaire magnétique et de l'effet Faraday de la bande F de KCl, puis des résultats expérimentaux obtenus sur les centres colorés par des méthodes magneto-optiques. F. LÜTY fait connaître les centres OH^- dans les halogénures alcalins. Le dopage contrôlé permet l'étude des systèmes de dipoles sans interaction ou réagissants, par différentes techniques expérimentales. Les expériences donnent les valeurs des moments dipolaires électriques et élastiques efficaces, et l'anisotropie optique des centres. A partir de l'effet électrocalorique, et faisant varier les temps de croissance et décroissance du champ électrique appliqué, on mesure directement le temps de relaxation dipole-réseau. On peut déduire, des relations de ce temps avec le champ et la température, des conclusions relatives au mécanisme de couplage entre dipole et phonons.

Parmi les communications nous trouvons un compte rendu commun des physiciens hongrois, I. TARJÁN, R. VOSZKA, Á. SIEGLER, L. BERKES sous le titre suivant: Sur quelques propriétés des cristaux de NaCl dopés au calcium. Les auteurs ont observé, que le courant d'obscurité des cristaux NaCl (Ca) diminue et s'approche d'une valeur constante au cours de la coloration par rayons X à la température ambiante. Le rapport du courant du cristal non coloré à la valeur finale du courant est 2,5. Le photocourant des cristaux NaCl(Ca) diminue, si le contenu de Ca augmente dans le cristal. Il s'ensuit de ces résultats qu'au cours de la coloration se produit dans le voisinage des ions de Ca^{++} un centre qui contribue au courant à l'obscurité. Les cristaux dopés au calcium étaient préparés par une méthode mise au point dans le laboratoire de l'Institut de Physique Médicale, Budapest. Une phase importante de la méthode était l'élimination des ions OH^- et l'exclusion du $Ca(OH)_2$.

ZS. CSOMA

WILLIAM A. RENSE: **Physical Science**

Blaisdell Publishing Co., Waltham. Mass. 1966, pp. 429

The author of the book is Professor of Physics at Colorado University. According to him the book has been written for students who may not wish to use physical sciences for later professional work. Under the title "physical sciences" the author summarizes physics, chemis-

try, astronomy and geology, pointing out how closely they are correlated. These sciences keeping close pace with each other's results form very important foundations for present civilization, to an extent much greater than ever before. Thus it is an important requirement that the basic results of physical sciences become known as extensively as possible.

The material of the book is divided into fourteen Chapters and two Appendices. Chapter 1 presents the molecular theory of matter as well as the basic and most important quantities occurring in physics. Chapter 2 describes the properties of gases, liquids and solids. This is followed by two Chapters, one devoted to astronomy and the other to the foundations of the atomic theory of matter. The next Chapters deal with the atomic theory of chemical structure and geological problems. After geology spectra are treated, and in connection with stellar spectra the problems of astronomy are again considered. The book also treats of problems of astronomy in connection with the properties and cosmic distribution of electric charges. In the Chapter on "Space, Time and Matter" the fundamental results of the theory of relativity are summarized. In the following Chapter the results achieved by quantum mechanics are described, the applications of which are also presented in connection with the collection of particles: solid bodies. Here the reader gets acquainted with semiconductors so important in electronics and luminescent materials. Naturally, a Chapter is devoted to nuclear physics, together with the physics of elementary particles. Then the Chapter entitled "The Expanding Universe" again reverts to problems of astronomy. The closing Chapter of the book sets forth the ultimate aims of physical science.

At the end of each Chapter the book contains "Questions" relating to the subject as well as mathematical "Problems". At the end of the book the solutions of all mathematical problems are given.

Appendix 1 contains the brief history of physics, including biographical sketches of famous physicists. The work of 19 physicists is described in a few lines and some others are mentioned in some words. In Appendix 2 the numerical values of some important physical constants are given.

To understand the book only a minimum mathematical knowledge is required. Thus it can be understood by anybody who has graduated from a secondary school. As regards the future it would be important for everybody having a university degree to study it. Without the knowledge contained in this book our present world and civilization can hardly be understood.

J. BOROS

S. G. BRUSH: **Kinetic Theory, Vol. 2. Irreversible Processes**

Pergamon Press, pp. 249, 1966

The interesting series "Selected Readings in Physics" edited by D. TER HAAR has been launched by Pergamon Press. The purpose of the series is to publish the collection of fundamental original papers which have appeared in connection with the development of various fields of research and in most cases are hardly accessible today. The second volume of the collection on kinetic theory contains one paper by MAXWELL, four by BOLTZMANN, two by POINCARÉ, one by W. THOMSON and two by ZERMELO.

The Introduction and Bibliography by S. G. BRUSH are followed by a paper of MAXWELL, and one by BOLTZMANN. These contain MAXWELL'S fundamental equation of the transport processes in gases and the first formulation of the H-theorem, respectively. The Introduction and these two papers occupy three-quarters of the book, while the remaining part contains several shorter papers by the above authors on the second law of thermodynamics and irreversible processes.

The author of the Volume has written summaries for each paper and the papers originally written in German have been translated into English. A subject and author index are also provided.

The book is published in an excellent layout characteristic of Pergamon Press.

J. ANTAL

FRANÇOIS CANAC: L'acoustique des théâtres antiques. Ses enseignements

Éditions du Centre National de la Recherche Scientifique, Paris
184 pages, 12 planches, 35.— Frs.

FRANÇOIS CANAC est très connu par les acousticiens hongrois. Nous nous rappellons encore vivement de sa première visite quand les jeunes chercheurs hongrois en ultrason ont reçu leurs premières connaissances de sa conférence sur les recherches acoustiques françaises développées. A partir de ce temps-là il nous a frequenté plusieurs fois. Son film en couleur, présenté à la IV-ème Conférence Acoustique à Budapest en 1967 a illustré justement le livre, qui se trouve devant nous.

Les deux domaines d'activité principaux du M. CANAC, la recherche d'ultrason et l'acoustique des théâtres antiques sont strictement attachés l'un à l'autre dans son travail de recherche. Il a examiné les maquettes des théâtres avec des ondes d'ultrason, et ainsi il a modélé les circonstances acoustiques originales en plein air. Le livre est presque le résumé de telle tendence de travail du savant méritant.

Après avoir esquissé les bases physiques et physiologiques, il fait connaître les données architecturales de 14 théâtres antiques. Puis il s'occupe des questions théoriques, en détaillant l'ensemble des problèmes des réflexions. C'est le moment où le travail expérimental y entre: la vérification des réflexions comptées par des expériences d'ultrason. A la base de ces examens l'auteur a formulé les équations canoniques des théâtres antiques. Enfin, dans l'un des chapîtres les plus émouvents il rend compte de ses expériences sur place, lesquelles il a acquis à Épidaure, Argos, Aspendos, Delphes et à Pergame, dans les deux théâtres d'Athènes et même à Orange et Vaison. Par conséquent il a tiré des conclusions instructives pour les points de vue de l'architecture du théâtre moderne.

Cette monographie, contenant beaucoup de données précieuses intéressera vivement non seulement les acousticiens, mais aussi les architectes et les historiens de la civilisation.

T. TARNÓCZY

R. LAMORAL: Problèmes d'acoustique des salles et des studios

Éditions Chiron, Paris 1967, 189 pages

ROGER LAMORAL est un acousticien pratique, au nom de qui se rattache — entre autres — l'élaboration des projets des studios pour le nouveau Centre de la Radio Paris. Dans ce livre il décrit surtout ses expériences gagnées sur place. C'est pourquoi — en particulier pour les acousticiens, s'occupant de faire des projets — la matière travaillée est instructive et la documentation photographique constituant 24 grandes planches est aussi de haute valeur.

On ne peut pas dire les mêmes louanges pour le passage didactique. Dans l'introduction même M. LAMORAL trouve des difficultés en ce que ces paroles s'adressent à la fois aux architectes et aux ingénieurs. C'est le développement que les premiers verraient plus volontiers sans mathématiques, mais que les derniers voudraient faire fond sur celles-ci. Le compromis né n'est pas fort heureux. A côté du caractère de recommandations pratiques surgissent soudainement des parties, constituées à la base mathématique comme par exemple le dixième chapitre qui s'occupe avec la protection contre les vibrations. D'autre part les dérivations mathématiques sont extrêmement détaillées, presque minutieusement précises desquelles aucune sorte des lecteurs n'a besoin. Ce sont deux pages de justification perpendiculaire de 20,5 cm que l'auteur utilise pour vérifier par développement en série et avec exemples les divergences entre les équations de MILLINGTON et EYRING.

La division du livre ne suit pas les traditions classiques. La première partie correspond à l'acoustique interne, après vient la partie de l'acoustique physiologique et enfin les problèmes concernant la protection contre les vibrations finissent les développements. Il y a beaucoup de valeurs dans les chapitres qui contiennent des données des matériaux utilisés et de structure, même dans l'instruction de quelques standards s'y rattachant.

Les figures sont surdimensionnées et quelques fois superflues. La famille de courbes d'isosonie établi par CHURCHER-KING et de FLETCHER-MUNSON est déjà dépassée dans nos jours, ainsi celle de ROBINSON et DADSON étant normalisée occupent une page totale. Puis, sur une autre demi-page on peut contempler à nouveau les courbes de FLETCHER-MUNSON.

Mais reçoivent des pages totales tels diagrammes aussi qui figurent des relations connues approchement, comme celle de la sonie de masque, ou celle des bandes critiques, etc. La dernière n'est pas faite sur la base des résultats les plus nouveaux.

La présentation du livre est somptueuse: le papier correspond à une qualité supérieure, la typographie se lit bien, les photographies sont très belles. Auprès des figures trop grandes il apparaît que le livre est fait avec beaucoup de tâches blanches, voire avec des pages blanches. Par une rédaction plus économique on pourrait bien diminuer l'étendue du livre presque avec 15%.

T. TARNÓCZY

Les systèmes sonars animaux

Biologie et Bionique Tome I. et II. Édité par R.-G. Busnel, Laboratoire de Physiologie Acoustique, Jouy-en-Josas, 1966

Dans deux volumes de grandes dimensions, en forme polycopiée a été publiée la matière du Colloque des «Systèmes Sonars Animaux» qui a été tenu à Frascati entre le 26 septembre et le 3 octobre 1966. Le mot «sonar» correspond à l'abréviation pour les appareillages acoustiques de localisation, il s'est acclimaté en particulier dans la littérature anglo-saxonne. Les appareillages acoustiques de localisation, employés dans la technique sous-marine se sont généralisés dans la deuxième guerre mondiale. Plus tard on a reconnu que dans le règne animal cette méthode de l'orientation est utilisée presqu'en général. D'abord en 1938 on a justifié l'aptitude de cette sorte de la chauve-souris, bien qu'à la fin du XVIII^{e} siècle, SPALLANZANI s'est douté quelque chose de cela, avec une reconnaissance géniale. C'est pourquoi pour honneurs de la postérité, dans la matière du colloque on a publié en fac-similé, en manière d'introduction l'une partie convénable de la correspondance, venante de l'an 1794 des oeuvres de SPALLANZANI.

Le colloque, lui-même n'était pas une réunion de travail nombreuse (45 rapporteurs et 30 observateurs dont une partie était compagnon), mais où, une discussion approfondie a suivi les rapports détaillés. Les volumes contiennent également la matière des discussions. A la réunion ont été également représentés des experts de la physiologie, de la zoologie, de l'acoustique, de la physique technique et de la théorie d'information. Dans la plupart des rapports, ces étaient les chauves-souris qui ont figuré comme exemples classiques des systèmes de localisation animaux. Mais en même temps beaucoup de thèmes de la biologie générale et des théories d'information se sont aussi posés. Parmi les autres animaux on a traité surtout l'aéptitude de localisation des balaines, des phoques et des dauphins, mais on a fait connaitre aussi nombreux résultats expérimentaux, en ce qui concerne les aptitudes de cette sorte de l'homme.

Il est presqu'incroyable que non seulement les aveugles, mais même l'homme avec les yeux bandés peut être sensitif à la présence d'un obstacle de 15 centimètres, distant de 1 mètre d'un écho, observé au milieu inéchoique de sa voix claquante ou sifflante. Par une petite pratique est réalisable ce que nous recevions l'information non seulement de la présence et de la position (de la direction et de la distance) de l'objet, mais encore de sa forme. L'auteur de ce compte rendu, à l'occasion de sa visite faite en 1965 chez le groupe de recherches de M. TH. POULTER (Stanford Research Institute, Menlo Park, California) s'est convaincu par ses propres yeux de la possibilité de la répétition des effets y examinés.

Les sujets exposés et débattus du colloque intéressant étaient d'ailleurs les suivants: I. Communications libres, II. Discrimination et identification par les sonars des animaux, III. Les caracteristiques générales des signaux acoustiques d'orientation et les performances des sonars dans le règne animal, IV. Résistance aux signaux interferents, V. Actions réciproques d'autres systèmes sensoriels et du système sonar, VI. Le sonar et l'aveugle, VII. Contenu social de certains impulsions, autres que celles utilisées dans l'écholocation animale, VIII. Structures néurologiques impliquées dans les systèmes sonars biologiques, IX. Les théories des systèmes sonars et leur applications aux organismes biologiques, X. Types et protocoles d'expériences à effectuer pour obtenir des résultats comparatifs.

L'organisation du colloque et la publication de la collection des rapports est devenue possible par l'appui commun des différents organismes internationaux et nationaux. Les matières ont été classées et ordonnées aux fins de publication par l'aide de M. R.-G. BUSNEL, chef du Laboratoire de Physiologie Acoustique, Jouy-en-Josas et de ses collaborateurs. Dans le travail de recherche d'acoustique biologique la matière des volumes s'emploie très utilement.

T. TARNÓCZY

R. J. STRUTT: Life of J. W Strutt, third Baron Rayleigh

Augmented Edition. University of Wisconsin Press. Madison — Milwaukee — London
1968. pp 439. 10 $

Lord RAYLEIGH is one of the interesting personalities of physics. He does not belong to its brightest stars, at least, not in scientific public opinion. This may be due to RAYLEIGH's having remained a real classical physicist in the dividing era between classical and modern physics. Yet, by his extraordinary abilities: his wonderful knowledge in mathematics and at the same time his recognitions anticipating applied physics, he could have been able to achieve whatever results in the field of modern physics.

But even as a classical physicist, many methods and results are due to him which are pointing ahead today's scientific research. His classical formula on black body radiation (RAYLEIGH-JEANS law) generated the order of ideas which led to the quantum theory by PLANCK. In the theory of quantum mechanics there are a number of mathematical methods (RAYLEIGH-RITZ approximation, SCHRÖDINGER-RAYLEIGH method, etc.) which were elaborated by RAYLEIGH. The formulation of the scattering of fourth power by RAYLEIGH also leads to modern scattering theory. At that time RAYLEIGH only needed the terms of lowest order and did not investigate those of higher order. In this respect we must not forget that in his time neither high speed computers nor function tables were available for researchers. For the graphic representation of one single mathematical function a calculation of some 50 pages was needed.

A lot of discussions arose especially concerning the dimensional analysis method used by him intensively which led him, among others, to the discovery of his scattering law. This method is flourishing anew in our days. Up-to-date works in this field refer over and over again to RAYLEIGH's papers even as modern articles on gravitation theory refer to the results of EÖTVÖS. But there is no up-to-date acoustical or optical problem either, the mathematical origin of which could not be found in his epoch-marking work "The Theory of Sound" published in 1877/78.

Hence, RAYLEIGH is a modern "classical" physicist, indeed. Therefore, the new edition of his biography really is timely. To all this it must be added that to the activities of RAYLEIGH there belong such items as the discovery of argon, the even now useful theoretical discussion of the whole of acoustics in a voluminous book, more than 500 scientific articles, and the highest scientific appreciations including the Nobel prize. All these facts result in an estimation which — though mosaic-like — had been alive in most of us which, however, becomes more distinct by the help of this biography. According to this appreciation RAYLEIGH is to be considered as one of the greatest physicists; though not the brightest star, at any rate one of the giant stars.

We have to be grateful to the long since dead author — RAYLEIGH's son who had been a physicist, too — for having preserved for posterity this valuable and instructive documentary material illustrated by original letters and period photographs. Thanks must be given for the new augmented edition to the writer of the preface J. N. HOWARD and to the University of Wisconsin Press. Even we ourselves who have long been devoted to RAYLEIGH have learned much from this book, and according to our opinion those who till now hardly knew or estimated this great physicist will also join his admirers.

T. TARNÓCZY

Proceedings of the International Conference on Luminescense, Budapest, 1966

Editor: G. Szigeti
Publishing House of the Hungarian Academy of Sciences
Budapest, 1968, pp. 2165

The two volumes of the Proceedings contain the English text of all the papers presented at the Conference giving a good survey of current research in the field of luminescence. The three Introductory Lectures summing up the recent accounts of progress in various fields of luminescence — mainly of inorganic luminescence and the trends in solid state physics in general — have been published both in English and Russian.

The book follows the arrangement of subjects of the Conference. According to this the Section of Introductory Lectures is followed by a Section of General Problems of Luminescense,

dealing again with inorganic phosphors mainly, whereas the next Section devoted to "Organic and Amorphous Luminescent Materials" contains a further chapter on "General Problems". Actually the 518 pages on organic materials deal with a group of questions having little in common with problems discussed in other parts of the book. (The only exception is perhaps the section on "Rare-Earth-Activated Luminophores" where some organic problems are discussed). The Section on organic luminescence is divided into chapters according to the different processes and phenomena (general problems, long duration processes, energy transfer, special problems). The next Section, that on "The Luminescence of Inorganic Materials" has been subdivided according to another conception: separate chapters are devoted to the most important luminescent compounds, and so to halogenides, sulphides, oxyphosphors and to the miscellaneous inorganic luminophores. Here again some exceptions have been made: rare-earth activators, phenomena like electroluminescence, the application of luminescence and the question of using resonance methods in luminescence research are discussed in a special section. Therefore, the material dealing with these problems can be found under separate chapters. Although this might seem to be an inconsistency in the grouping of the material, it proves to be quite reasonable for many purposes. Especially in the chapter on rare-earth-activated luminophores, material from quite different fields had been collected and discussed from a uniform view-point.

Each chapter is headed by Invited Papers dealing with new scientific results not published so far. Also the contributed papers contain new results. These papers were not read in full at the Conference, but reported by the referees. Each group of contributed papers concludes with these reports. They proved to be very useful because of summing up not only the latest results of the field in question but serving as excellent abstracts for the preceding contributed papers. A great deal of effort has been put into making these reports a useful summary of the Conference. Each chapter ends with the discussion material on the presented papers. These discussions prove that the luminescence of sulphides and the electroluminescence are still the most intensively investigated fields of luminescence.

The Publishing House has made great efforts in transforming a heterogeneous conference material into a homogeneous unilingual book. The lucid style in which the Proceedings contained in 2165 pages is presented will appeal to all those who wish to gain some insight into the rapid development of luminescence.

It is a great pity that these Proceedings became available only two years after the Conference, but most of the papers collected in it are of sufficient general interest to make it a standard volume for some years.

J. SCHANDA

I. FÉNYES: Thermostatics and Thermodynamics

(In Hungarian) Műszaki Könyvkiadó, Budapest

As is well known, the structure of thermodynamics differs essentially from that of other fields of physics. Thermodynamics generally deals with the existence of functions — whose physical interpretation causes difficulties — and with their relationships instead of the solution of the usual differential equation systems.

Because of this, a fundamental task of books on thermodynamics is to give its foundation by which the system of concepts and the mathematical apparatus can be well and uniformly handled. It is most probably due to the above-mentioned difficulties that the classical work of C. CARATHÉODORY did not become popular enough.

The excellent book of I. FÉNYES, written in a clear style, gives a common basis for both classical and irreversible thermodynamics, classifying the physical interactions and consistently using the notions of intensive and extensive quantities described in his earlier paper (Z. Phys. *134*, 95, 1952). The "reconciliation" of classical and irreversible thermodynamics was only possible after the elimination of the fiction of reversible processes. (Real processes are always irreversible!) On the other hand, heat and entropy are no longer basic notions of thermodynamics.

At first sight it might be surprising that heat is not a basic notion in heat theory but this might be understood by taking into consideration the corresponding analogy in mechanics. Indeed, both mechanical work and heat are functionals of mechanical and thermodynamical states, resp., and "work" is not a basic notion of mechanics, whereas in the usual treatment of thermodynamics heat is considered as a basic notion.

As for entropy, it is introduced here as an extensive quantity conjugated with thermal interaction. This kind of approach is entirely new in the literature. Nevertheless, owing to its lucid style and completeness the book is appealing both to beginners and experts. We are convinced that this book facilitates the understanding of thermodynamics and its application to the solution of given physical problems.

Besides the principal problems of thermodynamics — and in this respect the book may be considered as a monograph — it brings together many particulars, some of them presenting the author's results not published so far.

The titles of the chapters listed below give a clear survey of the book: 1. Interactions. Basic static and dynamical laws. 2. The role of entropy in characterizing the state. 3. Characteristic (potential) functions. 4. The entropy principle of thermostatics. Extremal properties. 5. Special physical systems. 6. Phase and chemical equilibrium. 7. Processes of indefinite rate (These differ from the usual quasi-static processes). 8. Thermodynamic processes near the equilibrium. 9. Thermodynamics of continua. 10. Utilization of energy sources.

It is worth while to point out that the Le Chatelier-Braun principle is treated both by thermostatic and thermodynamical methods. The book presents an original solution of the Gibbs paradox. It would be desirable to complete the book with a part on statistical mechanics treated similarly. The book would be more convenient if the equations were numbered and if it contained a subject index. The production of the book, published in Hungarian by the Műszaki Könyvkiadó is very good.

G. PATAKI

M. A. FILYAND and E. I. SEMENOVA: **Handbook of the Rare Elements**

Vol. 1. (an English translation by M. E. Alferieff) Macdonald Technical and Scientific, London

The book was originally published in Russian as a single volume work of 913 pages. Its publication was dictated by the lack of collected information on such important elements as, for example, Ga, In, Tl. The same necessity motivated its translation into English, which, unlike the original edition, appears in three volumes. The present book represents the first of these covering information of theoretical and experimental interest on the trace elements Ga, In, Tl, Ge, Se, Te, Rh and the light elements Li, Be, Rb and Cs.

The data are largely in tabular form collected in 385 tables of the 1300 total in the whole work. The tables are provided with bibliographical references, the complete index of which is indicated in each volume, amounting to about 500 sources up to 1962.

The material consists of data on isotopes, degree of purity, crystal lattice constants, density, coefficients of linear and volumetric expansion, heat capacity, heat of fusion and vaporization, melting and boiling points, surface tension, specific resistivity, Hall coefficient, temperature at which the substance passes into a superconductive state, magnetic susceptibility, normal electrode potential, indices of absorption, reflection and refraction, compressibility factor, hardness, tensile strength, interaction with other metals, maximum temperatures of chemical stability and finally the melting points and other properties of some alloys. The data refer only to elements, although a few compounds are also dealt with.

The chief areas of use and the principal chemical properties of the elements concerned are also summarized in descriptive form.

The book is the first on this subject attempting to gather together the vast amount of data on these elements. Taking into consideration that the most authentic source of chemical information on inorganic chemistry, Gmelin's "Handbuch der Chemie" has no up-to-date volume on many of these elements, one can really appreciate the importance of this work. We believe that the properties presented in the book should provide solid state physicists and chemists, metallurgists and electrical engineers with a convenient reference source.

I. TARJÁN

Printed in Hungary

A kiadásért felel az Akadémiai Kiadó igazgatója — Műszaki szerkesztő: Farkas Sándor

A kézirat nyomdába érkezett: 1969. II. 28. — Terjedelem: 7,50 (A/5) ív, 21 ábra

69.67225 Akadémiai Nyomda, Budapest — Felelős vezető: Bernát György

CORRIGENDA

Acta Physica Academiae Scientiarum Hungaricae, Tomus 25 (3)
pp. 245—250 (1968)

QUENCHING OF FLUORESCENCE OF EOSIN IN SOLUTIONS

By

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Acta Phys. Hung. Tom. XXVI. Fasc. 3. Budapest, 8. VIII. 1969 Index: 26.022

ACTA PHYSICA

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TOMUS XXVI FASCICULUS 4

AKADÉMIAI KIADÓ, BUDAPEST
1969

ACTA PHYS. HUNG.

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Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 311—318 (1969)

COUPLING OF THREE ANGULAR MOMENTA

By

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(Received 4. XII. 1968)

Some earlier results on the coupling of two angular momenta are extended in the present paper to three angular momenta. As in the previous case, the eigenfunctions with sharp total angular momentum are obtained in closed forms which do not contain the Clebsch—Gordan coefficients explicitly. The occurrence of the hypergeometric function in the present formulation renders the treatment highly flexible and makes it possible to derive a number of alternative expressions for the $6-j$ symbol. Moreover, recursion relations between contiguous $6-j$ symbols are derived more easily from Gauss's relations between contiguous hypergeometric functions.

1. Introduction

In a previous paper [1] certain simplifications were introduced into the theory of coupling of two angular momenta by using the one-variable operators [2]

$$J_{\pm} = \varkappa^{\pm 1}(j \mp \varkappa \partial_{\varkappa}), \quad J_z = \varkappa \partial_{\varkappa} . \tag{1}$$

This led to some convenient forms for the eigenfunctions of the coupled states which could be utilized for calculating the CG (Clebsch—Gordan) coefficients. But, as the meaning of the variable $\varkappa$ remained obscure no connection could be established between these eigenfunctions and the CG series. The defect has now been removed by noting that $\varkappa$ is simply the ratio u/v of the variables which transform according to the fundamental doubled-valued representation of the three-dimensional rotation group O_3. With this interpretation of $\varkappa$ the CG series is found to take the form

$$|{}^{j_1 j_2 j}_{m}\rangle = [(j+m)!\ (j-m)!]^{-1/2} C_{j_1 j_2 j}\, u_1^{j+m} v_1^{j_1-j_2-m} v_2^{-j_1+j_2+j} \times$$

$$\times (u_1 v_2 - u_2 v_1)^{j_1+j_2-j} \cdot F\left(-j-m,\ j_1-j_2-j;\ -2j;\ 1-\frac{u_2 v_1}{u_1 v_2}\right), \tag{2}$$

$$C_{j_1 j_2 j} = \left[\frac{(2j)!\,(2j+1)!}{(j_1+j_2-j)!\,(j_1-j_2+j)!\,(-j_1+j_2+j)!\,(j_1+j_2+j+1)!}\right]^{1/2}. \tag{3}$$

Other forms of the series can be obtained by using Kummer's relations for the hypergeometric function (HGF) occurring in Eq. (2).

The above results were recently derived by the author [2] for studying the representations of the unitary unimodular group SU(3). It turns out, however, that the results can be easily extended to the case of coupling of three angular momenta. The eigenfunctions of the coupled states are, thus, obtained in closed forms analogous to (2) which *do not contain the CG coefficients explicitly*. The new eigenfunctions contain only three independent variables, one for each angular momentum. But, since one of them occurs in its m-th power merely as a multiplier, the number of variables is effectively reduced to two. The situation is similar to that arising in the theory of coupling of two angular momenta.

Of all the eigenfunctions for a given j that for the lowest value of m is the simplest. This function is used in Section 3 for calculating the $6-j$ symbol (RACAH coefficient). Although this problem was solved satisfactorily by RACAH in 1942, the use of the present technique renders the treatment highly flexible and makes it possible to derive a number of alternative expressions for these quantities.

The occurrence of the HGF in the present formulation has another interesting consequence. Recursion relations connecting $6-j$ symbols differing in their j-values by $0, 1/2, 1$ can now be obtained more easily from the relations between contiguous HGFs. The method is, however, found to work only when some restriction is placed on the parameters of the HGFs occurring in a particular relation.

2. Eigenfunctions with sharp total angular momentum

An eigenfunction ψ_{jm} with sharp angular momenta arising from the coupling of j_1, j_2, j_3 can be written as a double sum

$$\psi_{jm} = \sum_{m_2, m_3} A_{m_1 m_2 m_3} \varkappa_1^{m_1} \varkappa_2^{m_2} \varkappa_3^{m_3} = \xi^m \sum_{m_2, m_3} A_{m_1 m_2 m_3} \eta^{m_2} \zeta^{m_3} \tag{4}$$

with $\xi = \varkappa_1$, $\eta = \varkappa_2/\varkappa_1$, $\zeta = \varkappa_3/\varkappa_1$. For the coupling scheme $(j_1 + j_2 = j_{12}, j_{12} + j_3 = j)$,

$$A_{jm} = [(j+m)!\ (j-m)!]^{1/2} \psi_{jm} =$$

$$\xi^m \sum_{m_{12}} \left[\sum_{m_2} (j_1 m_1 j_2 m_2 / j_{12} m_{12}) \eta^{m_2} \right] (j_{12} m_{12} j_3 m_3 / jm) \zeta^{m_3},$$

where, $(j_1 m_1 j_2 m_2 / j_{12} m_{12})$ is the 'CG coefficient of the second kind' as defined in Section 2 of [1]. By using the results of [1] this expression can be brought into the form

$$\begin{aligned} \Lambda_{jm} = {} & \mathrm{C}\,\sigma!\,(j+m)!\,(2j)!^{-1}\,\xi^m\,\eta^{-j_2}\,(1-\eta)^{j_1+j_2-j_{12}}\,\zeta^{m+j_{12}}\cdot \\ & \cdot \sum_t \frac{(2j-t)!}{t!(j+m-t)!\,(\sigma-t)!} \times \\ & \times \left(\frac{1}{\zeta}-1\right)^{\mu+t} F\left(-\mu-t,\ j_1-j_2-j_{12};\quad -2j_{12};\ \frac{1-\eta}{1-\zeta}\right), \end{aligned} \tag{5}$$

where $C = C_{j_1 j_2 j_{12}}\, C_{j_{12} j_3 j}$, $\mu = j_{12} + j_3 - j$, $\sigma = j_{12} - j_3 + j$.

The correctness of the above expression for Λ_{jm} can be tested by applying the raising operator J_+ to the lowest eigenfunction

$$\begin{aligned} \Lambda_{j-j} = {} & \mathrm{C}\,\xi^{-j}\,\eta^{-j_2}\,\zeta^{-j_3}\,(1-\eta)^{j_1+j_2-j_{12}}\,(1-\zeta)^{j_{12}+j_3-j}\cdot \\ & F\left(-j_{12}-j_3+j,\ j_1-j_2-j_{12};\quad -2j_{12};\ \frac{1-\eta}{1-\zeta}\right). \end{aligned} \tag{6}$$

In terms of the new variables the operators J_+, J_-, J_z are given by

$$\begin{aligned} J_+ &= \xi\left[j_1 + j_2\,\eta + j_3\,\zeta - \xi\,\partial_\xi + \eta(1-\eta)\,\partial_\eta + \zeta(1-\zeta)\,\partial_\zeta\right], \\ J_- &= \xi^{-1}\left[j_1 + \frac{j_2}{\eta} + \frac{j_3}{\zeta} + \xi\,\partial_\xi + (1-\eta)\,\partial_\eta + (1-\zeta)\,\partial_\zeta\right], \\ J_z &= \xi\,\partial_\xi\,. \end{aligned} \tag{7}$$

It is easily seen that the operator J_+ applied to Λ_{jm} gives $(j-m)\,\Lambda_{jm+1}$

3. The 6 — j symbol

The $6-j$ symbols [6—9] (or RACAH coefficients) are closely related to the coefficients of transformation from one scheme of coupling of j_1, j_2, j_3 to another leading to the same j, m. For a definite scheme of coupling there are, in general, several values of intermediate angular momentum which give a particular final j. Since the corresponding eigenfunctions form a complete set it is possible to expand a function $\Lambda(j_1 j_2 (j_{12}), j_3; jm)$ for a given j_{12} in terms of the functions $\Lambda(j_1 j_3 (j_{13}), j_2; jm)$ for different j_{13}. The $6-j$ symbols are related to these expansion coefficients by the equation

$$\begin{aligned} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix} = {} & (-)^{j_2+j_3+j_{12}+j_{13}}\,(2j_{12}+1)^{-1/2}\,(2j_{13}+1)^{-1/2}\cdot \\ & \cdot \langle j_1 j_2\,(j_{12}),\ j_3;\ j / j_1 j_3\,(j_{13}),\ j_2;\ j \rangle . \end{aligned} \tag{8}$$

In evaluating these coefficients we are at liberty to choose a value of m for which the eigenfunctions take a particularly simple form. We have seen that the value of m most suitable for the purpose is $-j$. The corresponding eigenfunctions in the coupling scheme $(j_1 + j_2 = j_{12},\ j_{12} + j_3 = j)$ are, by Eq. (6)

$$\begin{aligned} \Lambda(j_1 j_2 (j_{12}), j_3; j - j) = C_{j_1 j_2 j_{12}} C_{j_{12} j_3 j} \xi^{-j} \eta^{-j_2} \zeta^{-j_3} (1 - \zeta)^{j_1 + j_2 + j_3 - j} \times \\ \times z^{j_1 + j_2 - j_{12}} F(-j_{12} - j_3 + j,\ j_1 - j_2 - j_{12};\ -2j_{12};\ z), \\ z = (1 - \eta)/(1 - \zeta)\,. \end{aligned} \tag{9}$$

Interchanging 2, 3 and η, ζ we get the eigenfunctions in the coupling scheme $j_1 + j_3 = j_{13},\ j_{13} + j_2 = j$. These are

$$\begin{aligned} \Lambda(j_1 j_3 (j_{13}), j_2; j - j) = C_{j_1 j_3 j_{13}} C_{j_{13} j_2 j} \xi^{-j} \eta^{-j_2} \zeta^{-j_3} (1 - \zeta)^{j_1 + j_2 + j_3 - j} \times \\ \times z^{j_{13} + j_2 - j} F\left(-j_{13} - j_2 + j,\ j_1 - j_3 - j_{13};\ -2j_{13};\ \frac{1}{z}\right). \end{aligned} \tag{10}$$

For the evaluation of the $6-j$ symbol it is, therefore, sufficient to expand

$$f(z) \equiv z^{j_1 + j_2 - j_{12}} F(-j_{12} - j_3 + j,\ j_1 - j_2 - j_{12};\ -2j_{12};\ z)$$

in a series of the form

$$\begin{aligned} f(z) = \sum_{j_{13}} (j_1 j_2 (j_{12}),\ j_3;\ j / j_1 j_3 (j_{13}),\ j_2;\ j) \cdot z^{j_{13} + j_2 - j} \cdot \\ \cdot F\left(-j_{13} - j_2 + j,\ j_1 - j_3 - j_{13};\ -2j_{13};\ \frac{1}{z}\right). \end{aligned} \tag{11}$$

The coefficients in this expansion are connected with the $6-j$ symbols by the relation

$$\begin{aligned} (j_1 j_2 (j_{12}),\ j_3;\ j / j_1 j_3 (j_{13},)\, j_2;\ j) = \\ = \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix} (-)^{j_2 + j_3 + j_{12} + j_{13}} (2j_{12} + 1)^{1/2} (2j_{13} + 1)^{1/2} \cdot \frac{C_{j_1 j_3 j_{13}} C_{j_{13} j_2 j}}{C_{j_1 j_2 j_{12}} C_{j_{12} j_3 j}} \end{aligned} \tag{12}$$

These coefficients can be determined by writing the function $f(z)$ as a polynomial in z and then developing each power of z in a series of the type (11). This is easily accomplished by transforming the functions on the r.h.s. side of (11) into Jacobi polynomials and making use of the well-known results on the latter.

The connection of the functions on the r.h.s. of Eq. (11) with the Jacobi polynomials is easily seen by reversing the hypergeometric series. Writing $a = -j_{13} - j_2 + j$, $b = j_1 - j_3 - j_{13}$, $c = -2j_{13}$, we have, for $a - b > 0$,

$$z^{-a} F(a, b; c; 1/z) = (-)^a \cdot \frac{(-b)!\,(a-c)!}{(-c)!\,(a-b)!} \cdot F(a, 1-c+a; 1-b+a; z) \tag{13}$$

and this is a multiple of the Jacobi polynomial

$$P_n^{\alpha,\beta}(1-2z) = \binom{n+\alpha}{n} \times F(-n, n+\alpha+\beta+1; \alpha+1; z)$$

with

$$n = -a = j_{13} + j_2 - j, \quad \alpha = a - b = -j_1 - j_2 + j_3 + j,$$
$$\beta = a + b - c = j_1 - j_2 - j_3 + j. \tag{14}$$

From the well-known formula [5]

$$z^\varrho = \sum_n \frac{(\alpha+\varrho)!\,(2n+\alpha+\beta+1)\,(n+\alpha+\beta)!\,(-\varrho)_n}{(n+\alpha+\beta+\varrho+1)!\,(n+\alpha)!} \cdot P_n^{\alpha,\beta}(1-2z) \tag{15}$$

it is now easy to obtain the expansion

$$z^\varrho = \sum_{j_{13}} \frac{(1-c)!\,(a-b+\varrho)!\,\varrho!}{(-a)!\,(-b)!\,(1-c+a+\varrho)!\,(a+\varrho)!} \cdot z^{-a} F\left(a, b; c; \frac{1}{z}\right). \tag{16}$$

All the formulae needed for the calculation of the $6-j$ symbol are now at our disposal. Expanding the function $f(z)$ in powers of z and using Eqs. (12), (14) and (16) we now have

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \frac{(b, df)\,(e, af)}{(b, ac)\,(e, cd)} \cdot \sum (-)^{a+c+d+f+r} \cdot \frac{(2c-r)!\,(-c+d+e+r)!}{r!\,(c+d-e-r)!} \cdot \frac{(a+b-c+r)!}{(a-b+c-r)!\,(1+b-c+e+f+r)!\,(b-c+e-f+r)!},$$

where

$$(b, ac) = [(b+a-c)!\,(b-a+c)!\,(b+a+c+1)!/(-b+a+c)!]^{1/2} \tag{17}$$

This is a generalized hypergeometric series ${}_4F_3$ (1) of the Saalschützian type. From this a large number of expressions for the $6-j$ symbol can be derived by using BAILEY's transformation [10, 11].

4. Relations between contiguous 6 — j symbols

From the way the HGF occurs in the present formulation it appears that the relations between contiguous HGFs should lead to corresponding relations between $6-j$ symbols differing in their j-values by 0, 1/2, 1. This is found to be true provided a consistency condition is satisfied by the parameters of the HGFs. The known relations between contiguous HGFs can all be put into the form

$$z^{\delta} F(\alpha, \beta; \gamma; z) + A z^{\delta+\varrho} F(\alpha+\lambda, \beta+\mu; \gamma+\nu; z) + \\ + A' z^{\delta+\varrho'} F(\alpha+\lambda', \beta+\mu'; \gamma+\nu'; z) + \ldots = 0. \tag{18}$$

For $\alpha = -j_{12}-j_3+j$, $\beta = j_1-j_2-j_{12}$, $\gamma = -2j_{12}$, $\delta = j_1+j_2-j_{12}$, the function $f(z) = z^{\delta} F(\alpha, \beta; \gamma; z)$ can be expanded in a series of Jacobi polynomials of the type (11). Let us now see if the other functions occurring in the above relation can also be expanded in a series of the same type by changing the j-values from j_i to $j_i + h_i$. For such an expansion to be possible the increments h_i must satisfy the equations

$$h_1 = h_3, h_2 = h, \\ -h_3 + h - h_{12} = \lambda, \quad h_1 - h_2 - h_{12} = \mu, \quad -2h_{12} = \nu, \quad h_1 + h_2 - h_{12} = \varrho. \tag{19}$$

These equations are consistent if $\lambda + \mu = \nu$, and the solution, subject to this restriction, is

$$h_1 = h_3 = \frac{1}{2}(\varrho - \lambda), \quad h_2 = h = \frac{1}{2}(\varrho - \mu), \quad h_{12} = -\frac{1}{2}\nu. \tag{20}$$

Using these values one can write

$$z^{\delta+\varrho} F(\alpha+\lambda, \ \beta+\mu; \ \gamma+\nu; \ z) = \\ = \sum_{j_{13}} (j_1 + h_1 j_2 + h_2(j_{12}+h_{12}), \ j_3+h_3; \ j+h/j_1+h_1 j_3 + \\ + h_3(j_{13}), \ j_2+h_2; j+h) z^{-a} F\left(a, b, c; \frac{1}{z}\right). \tag{21}$$

Relations between $6-j$ symbols are now obtained by expanding the l.h.s. of (18) in a series of the above type and equating each expansion coefficient to zero.

As an illustration let us consider the relation

$$F - F(\alpha+1, \ \gamma+1) + \frac{\beta(\gamma-\alpha)}{\gamma(\gamma+1)} \cdot z F(\alpha+1, \ \beta+1; \ \gamma+2) = 0. \tag{22}$$

Multiplying this by z^{δ} and proceeding in the manner explained above we get a relation between $6-j$ symbols which is obtained by EDMONDS [6] and BIEDENHARN, BLATT and ROSE [7] from considerations of recoupling of four angular moments. The present method of deriving the relations appears to be slightly quicker.

Appendix

The formula (16) has been derived under the restriction, $a-b>0$. It is, therefore, necessary to show that it remains valid when this restriction is removed. The preceding formula (15) from which it is derived is valid if $\alpha \geqslant 0$, $k=\alpha+\beta \geqslant 0$. To see this we write $2n+k+1$ occurring in the numerator of the r.h.s. of (15) as a sum of two parts $n+k+\varrho+1$ and $-\varrho+n$ and collect the coefficients of the various powers of z. The coefficient of z^{ϱ} is found to be unity, and the coefficient of z^{r} for $r<\varrho$ can be put into the form

$$\frac{\varrho!\,(\alpha+\varrho)!\,(K+2r)!}{r!\,(\alpha+r)!\;(\varrho-r)!\;(k+r+\varrho+1)!}\big[F(r-\varrho,\ k+2r+1;\ k+r+ \\ +\varrho+1;\ 1)\cdot(k+r+\varrho+1)-F(r-\varrho+1,\ k+2r+ \\ +1;\ k+r+\varrho+2;\ 1)\,(\varrho-r)\big].$$

This expression is seen to vanish identically when the HGFs are replaced by their values given by Gauss's formula $F(a, b; c; 1)=\Gamma(c)\,\Gamma(c-a-b)/\Gamma(c-a)\,\Gamma(c-b)$. Thus, the series on the r.h.s. of (15) does indeed sum up to z^{ϱ} when the parameters of the Jacobi polynomials satisfy the inequalities $\alpha \geqslant 0$, $k \geqslant 0$. From this we conclude that formula (16) is certainly valid if $a-b>0$ and $2a-c \geqslant 0$. To demonstrate its validity for the case, $b-a>0$, $2b-c \geqslant 0$, we use the symbol $C(a, b, \varrho)$ for the expansion coefficient and write Eq. (16) as

$$z^{\varrho}=\sum_{a=0}^{-\varrho} C(a,\ b,\ \varrho)\ z^{-a}\,F(a,\ b;\ c;\ z^{-1}).$$

From symmetry considerations the corresponding formula for $b-a>0$, $2b-c \geqslant 0$ can be written as

$$z^{\varrho}=\sum_{b=0}^{-\varrho} C(b, a, \varrho)\,z^{-b}\,F(a, b; c; z^{-1}).$$

Multiplying this by z^{b-a} and putting $\varrho'=\varrho+b-a$ we have

$$z^{\varrho'}=\sum_{b=0}^{-\varrho'+b-a} C(b, a, \varrho'+a-b)\,z^{-a}\,F(a, b; c; z^{-1}).$$

It turns out, however, that $C(b, a, \varrho' + a - b) = C(a, b, \varrho')$. Thus, the same expansion holds in both cases, although the ranges of summation are different. In the first case, the minimum value of ϱ for which an expansion is possible is 0, and, in the second case, the minimum value of ϱ' is $b - a$.

We have shown that the same expansion (16) can be used for z^{ϱ} irrespective of whether a is greater or less than b provided at least one of the quantities $2a - c$, $2b - c$ is non-negative. It seems doubtful that this will be true when both the quantities are negative. The derivation of the expression (17) for the $6-j$ symbol can, therefore, be justified only when this condition is satisfied. Its validity for all physical values of the parameters is a consequence of the invariance of the $6-j$ symbol and *of the expression itself* under the well-known symmetry operations. The invariance can be established by using BAILEY's [10] transformations. As already mentioned, this also gives a variety of expressions for the $6-j$ symbol.

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СВЯЗЬ ТРЕХ УГЛОВЫХ МОМЕНТОВ

С. Д. МАЮМДАР

Резюме

В данной работе некоторые ранние результаты по отношению связи двух угловых моментов распространяются к трем угловым моментам. Как и в предыдущем случае собственные функции с острым полным угловым моментом получены в закрытых формах, в которые явно не входят коэффициенты Клебша—Гордона. Появление гипергеометрической функции в данной формулировке сделает дискуссию высоко гибкой и дает возможность для вывода некоторого числа альтернативных выражений для символов $6-j$. Далее, рекурсионные соотношения между зависящими друг от друга символами $6-j$ выводятся очень легко из соотношения Гаусса между связанными друг с другом гипергеометрическими функциями.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 319—333 (1969)

QUANTUM PROBABILITY WEIGHTED PATHS

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(Received 4. XII. 1968)

The paper is concerned with the generalization of FEYNMAN's method. A real "probability" measure is introduced for weighting trajectories and this is expressed as the boundary case of a measure containing one real parameter. For the finite values of the real parameter an equation of motion of the Fokker—Planck type is used. In the zero boundary case a Schrödinger equation is derived. A simple deduction is given for the WIGNER phase space distribution by using the differentiability of quantum trajectories, which is also proved. It is suggested that the equations of motion obtained for the finite values of the real parameter included in the theory describe real processes.

Introduction

This paper is concerned with the clarification and improvement of an idea put forward by the present author in an earlier paper [1]. There it was proposed that the concept of "The path integral" in quantum mechanics could be made mathematically more satisfactory by using a probability measure defined on a space of quantum paths. Although that work showed interesting possibilities, the probability measure used was not as precisely defined as one would wish. More recent work [2] has contained another related way of considering quantum probabilities which essentially contains the solution to this problem of defining a satisfactory probability measure for use in quantum studies. We shall thus redefine our measure. We shall then examine some of its properties, consider its relation to Wiener measure and, on the way, discuss the relation of our formalism to some closely related work by WIGNER [3], MOYAL [4], BARTLETT [5] and others. Other work which probably has similar or related motivations to that which we are about to discuss here can be found in [6], [7] and [8].

The probability measure

In [2] equation (14) it was shown that the quantum transition probability for the movement of density from position u to position x in a limitingly small time can be put into the form

$$P(x|u) = \lim_{\tau \to 0} \widetilde{P}(x, t + \tau \,|\, u, t), \tag{1}$$

where

$$\widetilde{P}(x, t+\tau \mid u, t) = \\ = \left(\frac{m}{\pi \hbar \tau}\right) \int_{-\infty}^{+\infty} \left\{ \exp \frac{i}{\hbar} \left[\frac{2m\,(x-u)\,v}{\tau} \right] \right\} \frac{\psi(u-v, t)\,\psi^*(u+v, t)}{\psi(u, t)\,\psi^*(u, t)}\, dv \quad (2)$$

and $\psi(u, t)$ is a solution to Schrödinger's equation at time t. We have introduced the tilde into our notation in order to distinguish between the limiting exact quantum $P(x|u)$ and the $\widetilde{P}(x, t+\tau|u, t)$ which is in general an approximation to it.

We have also shown in [2] that the condition,

$$\int_{-\infty}^{+\infty} \widetilde{P}(x, t+\tau \mid u, t)\, dx = 1 \quad (3)$$

holds for the approximate transition probability $\widetilde{P}$. A second condition on $\widetilde{P}$ is,

$$\text{Im. } \widetilde{P} \equiv 0\,. \quad (4)$$

That $\widetilde{P}$ has this last property of always being real can be seen by taking its complex conjugate and making the replacement $v \to -v$ in formula (2). However, $\widetilde{P}$ as defined by (2) need not be positive everywhere. In view of (3) and (4), $\widetilde{P}$ is a suitable function to use for weighting those paths which pass through u at time t and x at time $t+\tau$. We shall use $\widetilde{P}$ for this purpose, but because $\widetilde{P}$ can assume negative values we should remark that our measure is not strictly pure probability. Such regions of negative 'probability' have been encountered and discussed earlier by WIGNER [3], MOYAL [4] and BARTLETT [9]. It turns out that one can still work consistently with and get correct results from this apparently pathological measure. The regions of negative 'probability' generally have no adverse influence on calculations and in particular the work in this paper does not require that our weighting be pure probability. A more detailed examination and discussion of the reasons for the appearance of regions of negative 'probability' will be given by the present author in another paper [10].

That we have decided to use the approximate quantum transition probability is of no particular significance at this stage. We do not claim to be doing quantum mechanics before the limit $\tau \to 0$ is taken. Further, because we must weight all conceivable paths even the discontinuous and non-differentiable ones, $\widetilde{P}(x, t+\tau|u, t)$ will be taken to be the weighting for all paths which pass through u at t and x at $t+\tau$ whatever happens to them between time t and time $t+\tau$. If it should turn out that a certain subclass of this very large class of paths is particularly important for our choice of weighting,

then this must be deduced from the definition of our weighting. These conceptions are completely in line with WIENER measure [11] where it can be shown that a certain class of paths is very important in that the members of this class carry all the probability. We shall return to this point and further model our weighting on WIENER measure.

When N. WIENER defined his functional integral, he divided the total time of interest into n equal segments of length τ, weighted each segment with the Brownian motion transition probability and then took the repeated integral with appropriate limits to define the weight to be given to the class of paths restricted only by the integral limits. We shall do the same here except that we shall use the approximate quantum transition probability $\tilde{P}(x, t + \tau | u, t)$. Thus we shall take the value of the $n - 1$ fold integral

$$\tilde{\mu}_a^b(n^n, n^0) = \int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} \tilde{P}(x^n, T \mid x^{n-1}, T - \tau)\, \tilde{P}(x^{n-1}, T - \tau \mid x^{n-2}, T - 2\tau) \dots$$
$$\dots \tilde{P}(x^1, \tau \mid x^0, 0)\, dx^{n-1} \dots dx^1, \tag{6}$$

where $n\,\tau = T$, to be the total measure (or 'probability') to be associated with those paths $\{x(t)\}$ which start at $x(0) = x^0$ at time zero and terminate at $x(T) = x^n$ at time $T = n\,\tau$, and are only restricted by the conditions,

$$\text{a)} \qquad x^n = x(T)\,,$$
$$\text{b)} \qquad a_i \leq x(i\tau) \leq b_i\,, \quad i = 1, 2 \dots, n - 1\,, \tag{7}$$
$$\text{c)} \qquad x^0 = x(0)\,.$$

The a_i and b_i here are all real numbers. The expression (6) is essentially the same as was used in [1] equation (I. 20) but the notation is now such that the expressions for the individual $\tilde{P}$ contain in their structure part of the instruction to integrate which appeared explicitly in [I. 20] (the v integration in (2)). Thus here our measure is probability (except in the respect discussed earlier) defined on a line and is much nearer to WIENER measure which is genuine probability defined on a one dimensional configuration space. If we now let the $b_i \to +\infty$ and the $a_i \to -\infty$ and then form the integral $\int_{-\infty}^{+\infty} \tilde{\mu}_\infty^\infty (x^n, x^0) dx^n$, we see by successive application of formula (1) that

$$\int_{-\infty}^{+\infty} \tilde{\mu}_\infty^\infty (x^n, n^0)\, dx^n = 1. \tag{8}$$

Thus, as one would wish, the total probability for all paths only restricted by (7b) and (7c) is unity.

As mentioned earlier, we do not claim at this stage that our path integral has much to do with conventional quantum mechanics, except that we employ a wave function in its definition. The connection with quantum mechanics ultimately depends on taking the limit $\tau \to 0$. Before this limit is taken, the equation which governs the behaviour of our formalism as a function of time is of the Fokker—Planck type. This will be discussed later but firstly we shall examine a connection of this formalism with some related work by WIGNER [3], MOYAL [4] and BARTLETT [5].

Relation with phase space distributions

It is interesting to use our expression for the quantum transition probability to derive a phase space distribution. Our transition probability $\widetilde{P}(x, t + \tau | u, t)$ is the probability that a density element situated at u at time t will be found at x at time $t + \tau$, Thus we can say that provided τ is small we can ignore the existence of any potential between the times t and $t + \tau$ and conclude that $\widetilde{P}(x, t + \tau | u, t)$ gives, alternatively, the probability density that an element at u has the average velocity $x—u/\tau$ for the interval τ. If we then take a simple classical view, we can then infer that $\widetilde{P}$ also gives the probability density for average momentum $\tilde{p} = m(x - u)/\tau$ at u. Now surely the phase space distribution for our system is the joint probability for density at u and momentum

$$p = \lim_{\tau \to 0} \tilde{p} = \lim_{\tau \to 0} \frac{m(x - u)}{\tau} \tag{9}$$

at u all at time t. Thus we conclude that if a phase space distribution comes out of our scheme, it should be the product

$$\lim_{\tau \to 0} \widetilde{P}(x, t + \tau \,|\, u, t)\, \varrho(u, t)\,, \tag{10}$$

since $\varrho(u, t) = \psi^*(u, t)\, \psi(u, t)$ is the density at u. If we now substitute for $\widetilde{P}$ by (2), we get that our phase space distribution (or probability for momentum p at u together with density $\varrho(u, t)$) is

$$mF(p, u) = \lim_{\tau \to 0} \frac{m}{\pi h \tau} \int_{-\infty}^{+\infty} \left\{ \exp \frac{i}{\hbar} 2\tilde{p}v \right\} \psi(u - v, t)\, \psi^*(u + v, t)\, dv\,. \tag{11}$$

To compare (11) with the expression used by MOYAL ([4] equation (3.7)) we need only make the change of variables,

$$a = -2v/\hbar\,, \tag{12}$$

when we get

$$F(p, u) = \frac{1}{2\pi} \psi^* \left(u - \frac{1}{2}\hbar\alpha\right) e^{-i\alpha P} \psi \left(u + \frac{1}{2}\hbar\alpha\right) d\alpha , \tag{13}$$

which is identical with MOYAL's expression for $F(p, u)$. Now this is undoubtedly an elegant and remarkably simple derivation of WIGNER's phase space distribution, but the reader may justifiably have doubts about the validity of the steps. Particularly, because the substitution (9) involves the assumption that the limit of $(x - u)/\tau$ is finite as $\tau \to 0$. Such doubts are naturally strengthened by the knowledge of a theorem proved by N. WIENER [11] (see also DOOB [12]) some years ago concerning the nature of Brownian motion paths. The steps we have made in deducing the phase space distribution would certainly not be correct if this theorem were applicable to our measure, because in that case the limit of $(x - u)/\tau$ would be infinite, as $\tau \to 0$, for almost all paths. A simplified statement of the theorem to which we refer is as follows.

Theorem

"The WIENER probability for paths other than the set of equi-continuous paths is zero. The probability that a Brownian particle follows an equi-continuous path is unity."

In order to handle the mathematics of these ideas and also to study the question of whether or not such a theorem applies to our quantum probability measure, we shall need to introduce some further notation and also state WIENER's theorem more fully.

Path properties

We shall need to consider the set of all conceivable functions $x(t)$ (including the discontinuous ones) which can be defined for $0 \leq t \leq T$, and such that $x(0) = x_0$, since in both WIENER measure and our measure all such paths are weighted. The number T is a fixed value representing the total time of interest. Further, we shall also need the symbol

$$S(t, \tau, \lambda) = \frac{|x(t + \tau) - x(t)|}{\tau^\lambda} . \tag{14}$$

τ, here, is as usual a positive time increment. We can now express WIENER's theorem as follows

"Diffusing particles are certain to follow paths, $x(t)$, with the following three properties"

$$\text{a)} \qquad \lim_{\tau \to 0} S(t, \tau, \lambda) = 0, \quad \lambda < \frac{1}{2},$$

$$\text{b)} \qquad \lim_{\tau \to 0} S(t, \tau, \lambda) < \infty, \lambda = \frac{1}{2}, \tag{15}$$

$$\text{c)} \qquad \lim_{\tau \to 0} S(t, \tau, \lambda) = \infty, \lambda > \frac{1}{2}.$$

Hence, if this should be true for quantum paths, the condition (15c) with $\lambda = 1$ would render our derivation of the phase space distribution quite meaningless. We must therefore consider the properties of quantum paths further.

Quantum paths

We only need to consider what the quantum paths do in the neighbourhood of one typical instant of time t. Thus we do not need to employ the full probability measure for the whole interval $0 \to T$. What we prove holds at t will be true for all $0 \leq t \leq T$. Hence let us consider the total probability for all paths $\{x(t)\}$ which pass through $x(t) = u$ at time t and which lie outside the interval $(u - S_0 \tau^\lambda, u + S_0 \tau^\lambda)$ at time $t + \tau$, where S_0 is some specific value of S. This probability is

$$\tilde{\mu}(\lambda) = \int_{-\infty}^{u - S_0 \tau^\lambda} \tilde{P}(x, t + \tau \mid u, t)\, dx + \int_{u + S_0 \tau^\lambda}^{+\infty} \tilde{P}(x, t + \tau \mid u, t)\, dx \tag{16}$$

$$= \int_{-\infty}^{+\infty} \tilde{P}(x, t + \tau \mid u, t)\, dx - \int_{u - S_0 \tau^\lambda}^{u + S_0 \tau^\lambda} \tilde{P}(x, t + \tau \mid u, t)\, dx .$$

The first integral in (16) is unity by (3) and the second integral can be performed after substituting for $\tilde{P}$ by (2). It follows that the total probability for such paths is

$$\tilde{\mu}(\lambda) = 1 - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{1}{v} \left[\exp \frac{i}{\hbar} \left\{ \frac{2m(x - u)v}{\tau} \right\} \right]_{u - S_0 \tau^\lambda}^{u + S_0 \tau^\lambda} x(u, v)\, dv , \tag{17}$$

where

$$x(u, v) = \frac{\psi(u - v, t)\, \psi^*(u + v, t)}{\psi(u, t)\, \psi^*(u, t)} . \tag{18}$$

After substituting the limits into the square brackets in (17), we get the function $\sin(2 m S_0 \tau^{\lambda - 1} v/\hbar)$ inside the integral. Strictly it is the limiting measure, $\mu = \lim_{\tau \to 0} \tilde{\mu}$ which concerns us and this is

$$\mu(\lambda) = 1 - \lim_{\tau \to 0} \int_{-\infty}^{+\infty} \frac{1}{\pi v} \left\{ \sin \left(\frac{2 m S_0 \tau^{\lambda - 1} v}{\hbar} \right) \right\} x(u, v)\, dv . \tag{19}$$

From formula (19), we see that the value $\lambda = 1$ is critical in determining this probability. Let us now consider the three cases (a), $\lambda < 1$; (b), $\lambda = 1$; and (c), $\lambda > 1$ analogous to (15, a, b and c).

(a) When $\lambda < 1$, we can put $\lambda - 1 = -|\eta|$ and use the delta function representation

$$\delta(v) = \frac{1}{\pi} \lim_{\tau \to 0} \left(\frac{1}{\tau |\eta|} \frac{2mS_0 v}{\hbar} \right) \tag{20}$$

to infer that

$$\mu(\lambda) = 1 - \int_{-\infty}^{+\infty} \delta(v)\, x(u, v)\, dv$$

$$- 1 - x(u, 0) = 1 - 1 = 0. \tag{21}$$

Now (21) will be true for arbitrarily small S_0 and so we can let $S_0 \to 0$. Thus we can conclude that for $\lambda < 1$ the probability is concentrated on paths for which

$$\lim_{\tau \to 0} S = \lim_{\tau \to 0} \frac{|x - u|}{\tau^{\lambda}} < S_0 \to 0. \tag{22}$$

(b) When $\lambda = 1$, we get from (19)

$$\mu(1) = 1 - \int_{-\infty}^{+\infty} \frac{1}{\pi v} \left\{ \sin \left(\frac{2mS_0 v}{\hbar} \right) \right\} x(u, v)\, dv \tag{23}$$

Thus in this case we can use the delta function representation

$$\delta(v) = \frac{1}{\pi} \lim_{S_0 \to \infty} \frac{\sin}{v} \left(\frac{2mS_0 v}{\hbar} \right) \tag{24}$$

to infer that when $S_0 \to \infty$

$$\mu(1) = 1 - \int_{-\infty}^{+\infty} \delta(v)\, x\,(u, v)\, dv = 0\,. \tag{25}$$

Thus in this case we see that all the probability is concentrated on paths for which

$$\lim_{\tau \to 0} S = \lim_{\tau \to 0} \frac{|x - u|}{\tau} < S_0 = \infty\,. \tag{26}$$

(c) In the last case when $\lambda > 1$, we see from (19) that the integral vanishes when $\tau \to 0$. Thus

$$\mu(\lambda) = 1, \ \lambda > 1 \tag{27}$$

and all the probability is concentrated on paths for which

$$\lim_{\tau \to 0} S = \lim_{\tau \to 0} \frac{|x - u|}{\tau^{\lambda}} \geq S_0 \to \infty\,, \tag{28}$$

since S_0 can be made arbitrarily large without affecting the argument. In cases (a) and (c) two distinct limits are involved. They do not mutually interfere if in (a) we keep $S_0/\tau|\eta|$ of order $1/\tau|\xi|$ with $|\beta| > |\xi|$ and in (c) we keep $S_0\,\tau|\eta|$ of order $\tau|\xi|$ with $|\xi| < |\eta|$.

From this analysis we conclude that the paths of importance for our probability measure have finite derivatives, unlike the paths which play a central rôle in WIENER's measure. This conclusion is mostly based on (26) which admittedly is not a very strong statement but taken with formula (19) of [2], from which we can infer that the average value of the velocity functional is finite where the quantum current $Q_1'(u)/m$ is finite, we have a strong case.

Hence taking all these points into account, we conclude that our derivation of the phase space distribution is sound. This really being a consequence of $\lambda = 1$ being the critical value for our measure, whereas $\lambda = {}^1/_2$ is critical for WIENER measure. It would not be correct to assume, because the paths of importance for our measure have finite derivatives, that they are classical paths. They merely have a common characteristic with such paths.

Application of the quantum probability functional integral

The motivation for defining a functional integral using the Schrödinger wave function is in the first instance to obtain an alternative and possibly illuminating language to describe the quantum process. We shall now very briefly show how our functional integral gives such an alternative. We have first to consider how the energy functional should be defined.

Definition of local energy

In classical dynamics when one is concerned with the trajectory traced by a single particle of definite mass, the definition of energy is straightforward. There one takes the total energy, H, to be simply the sum of kinetic and potential energies. In the case of the movement of a distribution of either particle density or a probability, local energy can assume a more complicated character. In particular, one can talk of local energy on the one hand and global energy on the other, and the relation between these two energies involves some subtleties. It is our intention to work with a local energy density, $E(x)$. We shall define this to be the energy per unit probability for finding a particle at x. Now the probability density at x is $\varrho(x)$ and the probability in the range dx is $\varrho(x)\,dx$. Thus the energy in dx will be $E(x)\,\varrho(x)\,dx$, and the total energy, H, for this system in the distribution state $\varrho(x)$ will be

$$H = \int_{-\infty}^{+\infty} E(x)\,\varrho(x)\,dx\,. \tag{29}$$

Further, when $E(x) = E =$ constant and $\int_{-\infty}^{+\infty} \varrho(x) = 1$, the total energy H is identically equal to E. Thus in defining the local energy density, $E(x)$, in general, there is some freedom provided it is the value of H which characterises the state of the system. In other words we can always add to $E(x)$ some function $G(x)$, say, such that

$$H = \int_{-\infty}^{+\infty} [E(x) + G(x)]\, \varrho(x)\, dx = \int_{-\infty}^{+\infty} E(x)\, \varrho(x)\, dx\,.$$

Thus suitable functions $G(x)$ will have the property

$$\int_{-\infty}^{+\infty} G(x)\, \varrho(x)\, dx = 0. \tag{30}$$

Now (30) defines a class of functionals $G(x)$ which could be said to be of configuration probability measure zero. We shall use the term configuration probability measure to distinguish the weighting $\varrho(x)\, dx$ from our paths integral weighting $\tilde{P}dx$. The form the functions $G(x)$ can take will clearly depend on the properties we care to assign to our probabilities at $\pm\infty$. We have previously (see [2] equations (18) $\rightarrow$ (19)) restricted our probability densities by the two conditions that $\varrho(x)$ and $\partial\varrho(x)/\partial x$ are both to approach zero at $x \rightarrow \pm\infty$. The second of these conditions is conveniently translated into the integral form

$$\int_{-\infty}^{+\infty} \lambda \frac{\partial^2 \varrho}{\partial x^2}\, dx = \lambda \left[\frac{\partial \varrho}{\partial x} \right]_{-\infty}^{+\infty} = 0\,, \tag{31}$$

where λ is an arbitrary constant. Thus we have a naturally occurring class of functionals of configuration probability measure zero. These are the functions $G(x)$ such that

$$\int_{-\infty}^{+\infty} \varrho(x)\, G(x)\, dx = \int_{-\infty}^{+\infty} \lambda \frac{\partial^2 \varrho}{\partial x^2}\, dx = 0 \tag{32}$$

or

$$G(x) = \frac{\lambda}{\varrho} \frac{\partial^2 \varrho}{\partial x^2}\,. \tag{33}$$

These G are clearly not the most general solutions to (30), but they do turn out to be particularly important in the following work. Thus in defining the local energy functional for our probabilistic system we shall employ the expression

$$H(x, u) = \frac{1}{2} m \left(\frac{x - u}{\tau} \right)^2 + W(u) + \frac{\lambda}{\varrho(u)} \frac{\partial^2 \varrho(u)}{\partial u^2}\,, \tag{34}$$

where $W(u)$ is a prescribed external potential energy. In view of the preceding discussion, this is seen to be a minimal generalisation of what would occur classically. Functionals such as (34) can be regarded as a framework which is only meaningful when supporting some probabilistic weighting. Actual numerical values or functions are obtained by taking some appropriate average For the functional $H(x, u)$ this is achieved by forming either or both of

$$H(u) = \int_{-\infty}^{+\infty} H(x, u)\, \widetilde{P}(x, t + \tau \,|\, u, t)\, dx\,, \tag{35}$$

and

$$H = \int_{-\infty}^{+\infty} H(x, u)\, \widetilde{P}(x, t + \tau \,|\, u, t)\, \varrho\,(u)\, dx\, du \tag{36}$$

according to the information required.

$H(u)$ is the total energy density at u all possible momentum at u being taken into account, and H is the total energy for the whole system. We note that (36), when expressed in terms of MOYAL's function

$$F(\widetilde{p}, u) = \widetilde{P}(x, t + \tau \,|\, u, t)\, \varrho(u)$$

becomes the integral over all phase space

$$H = \iint H(p, u)\, F(p, u)\, dp\, du\,. \tag{37}$$

The energy averages

For the work which follows we shall adopt a different standpoint from that taken in the earlier parts of this paper. We shall no longer assume that the function, $\psi(u, t)$, which appears in the definition of $\widetilde{P}$ is a solution to the Schrödinger equation. We shall accept that $\widetilde{P}$ has the form (2) in terms of some function $\psi(u, t)$, but take the view that $\psi(u, t)$ is now to be determined by probabilistic considerations. These considerations will depend on taking averages of various functionals. In particular the average value of the generalisation of the classical energy functional (34) is particularly important. Thus the basic elements in our new approach are a probability measure of known form and a generalisation of the classical energy equation.

We observe that the significant generalisation in (34) consists in allowing the local energy to be coupled to the probability density ϱ. The constant λ can be regarded as the coupling constant. The various formulas we shall employ ([2] equations (19) and (20)) do not depend on ψ being a solution to the Schrödinger equation. Thus they are all valid from our new standpoint.

Let us now calculate, $H(u)$, the total energy at u due to the momentum distribution at u. From (34), (35) and (3) we have

$$H(u) = \int_{-\infty}^{+\infty} H(x, u)\, \tilde{P}(x, t+\tau \mid u, t)\, dx =$$

$$= \frac{m}{2} \int_{-\infty}^{+\infty} \left(\frac{x-u}{\tau}\right)^2 \tilde{P}(x, t+\tau \mid u, t)\, dx + W(u) + \frac{\lambda \partial^2 \varrho(u)}{\varrho(u)\, \partial u^2}\,. \tag{38}$$

Using ([2] (20)), (38) becomes

$$H(u) = \frac{1}{2m}\left(\frac{\partial \Phi_1(u)}{\partial u}\right)^2 + \frac{h}{4m}\,\frac{\partial^2 \Phi_2(u)}{\partial u^2} + W(u) + \frac{\lambda \partial^2 \varrho(u)}{\varrho(u)\, \partial\, u^2}\,. \tag{39}$$

We now recall [2] equations (7) and (21) from which follow that

$$\Phi_2 = -\frac{\hbar}{2} \log \varrho\,. \tag{40}$$

Using (40), (39) now becomes

$$H(u) = \frac{1}{2m}\left(\frac{\partial\, \Phi_1(u)}{\partial u}\right)^2 - \frac{h^2}{8m}\left[-\frac{1}{\varrho^2}\left(\frac{\partial \varrho}{\partial u}\right)^2 + \frac{1}{\varrho}\,\frac{\partial^2 \varrho}{\partial u^2}\right] +$$

$$+ W(u)\,\frac{\lambda}{\varrho}\,\frac{\partial^2 \varrho}{\partial u^2}\,. \tag{41}$$

From (41) we see that the arbitrary constant λ can now be exploited in order to bring our energy functional into a form which is much nearer to having a classical appearance. This is achieved by taking $\lambda = \hbar^2/8m$, and if we further define

$$p_1 = \frac{\partial\, \Phi_1}{\partial u} \tag{42}$$

and

$$p_2 = -\frac{\hbar}{2\varrho}\,\frac{\partial \varrho}{\partial u} = -\frac{\hbar}{2}\,\frac{\partial \log \varrho}{\partial u}\,, \tag{43}$$

we get

$$H(u) = \frac{1}{2m}\,[p_1^2 + p_2^2] + W(u)\,. \tag{44}$$

Thus we are able to bring the local energy functional into a form in which the kinetic energy term is positive. The oddity here is that now we seem to be working on a plane with momentum having the second component p_2.

Surprisingly, (44) is not at all outrageous because it can be obtained by simply taking the expectation value of the conventional quantum Hamiltonian, $\mathscr{H}$, when one gets

$$\int_{-\infty}^{+\infty} \mathscr{H}(u)\, \varrho(u)\, du\,,$$

$$= \int_{-\infty}^{+\infty} \psi^*(u) \left[-\frac{h^2}{2m} \frac{\partial^2}{\partial u^2} + W(u) \right] \psi(u)\, du\,,$$

$$= \int_{-\infty}^{+\infty} \left[\frac{1}{2m} (p_1^2 + p_2^2) + W(u) \right] \varrho(u)\, du\,. \tag{45}$$

The last form being obtained by integration by parts and using (42) and (43).

Recovering the Schrödinger equation

To recover the Schrödinger equation, we need to make a different choice for λ at stage (41). If we take $\lambda = -\hbar^2/8\,m$, with a little elementary algebra, we get

$$H(u) = \frac{1}{2m} [p_1^2 - p_2^2] + \frac{\hbar}{2m} \frac{\partial p_2}{\partial u} + W\,. \tag{46}$$

(46), which is in essence an energy equation, will be one of the equations needed to determine $\psi(x, t)$. Our system is probabilistic so that in addition to (46) we shall need the equation which describes the movement of our probabilities. This equation is implicit in our formalism and is obtained by standard stochastic arguments [13, 14] applied to the integral

$$\frac{1}{\tau} \int_{-\infty}^{+\infty} [\varrho(x, t) - \varrho(x, t-\tau)]\, F(x)\, dx, \tag{47}$$

where $F(x)$ is an arbitrary function.

From (47) and using ([2], (19) and (20)) we obtain, to the first order in τ, the discrete time Fokker—Planck equation,

$$\frac{\varrho(x, t) - \varrho(x, t-\tau)}{\tau} = -\frac{\partial}{\partial x} \left(\frac{\Phi_1'(x)}{m} \varrho(x, t) \right) + \frac{\tau \partial^2}{\partial x^2} (\alpha(x)\, \varrho(a))\,, \tag{48}$$

where

$$\alpha(x) = \frac{\hbar}{4m} \Phi_2''(x) + \frac{1}{2} \left(\frac{\Phi_1'(x)}{m} \right)^2. \tag{49}$$

Eqs. (46) and (48) together constitute a generalisation of the Schrödinger equation involving the discrete time increment τ. The Schrödinger equation arises from the limit $\tau \to 0$ when (48) becomes the equation of continuity

$$\frac{\partial \varrho}{\partial t} = - \nabla \cdot \left(\frac{p_1}{m} \varrho \right) \tag{50}$$

To see how the Schrödinger equation arises in this limit, it is convenient to re-express (50) in terms of $\log \varrho$ when we get

$$\frac{\hbar}{2} \frac{\partial \log \varrho}{\partial t} = - \frac{\hbar}{2m} \frac{\partial p_1}{\partial x} + \frac{p_1 p_2}{m}, \tag{51}$$

on using (42) and (43). The two equations (46) and (51) now invite making the complex combination

$$p = p_1 + i p_2 \tag{52}$$

and using the notation

$$E_2 = \frac{\hbar}{2} \frac{\partial \log \varrho}{\partial t} . \tag{53}$$

Adding (46) to $\sqrt{-1}$ multiplied into (51) and using (52) and (53), we get the combined equation

$$E = \frac{p^2}{2m} - \frac{i\hbar}{2m} \frac{\partial p}{\partial x} + W. \tag{54}$$

This is the Schrödinger equation for the wave function ψ when

$$p = - i\hbar \frac{\partial \log \psi}{\partial x} , \tag{55}$$

which indeed is a consequence of (40) and (43) as far as the imaginary part is concerned and of ([2] (21)) and (42) as far as its real part is concerned. Thus we have recovered the Schrödinger equation. It is, in fact, implicit in the form we assume for the path weighting and the generalization of the classical expression for energy. Thus Schrödinger quantum mechanics is a limiting case of a more general stochastic theory which involves a discrete time τ. This limit stochastic theory should perhaps be regarded as being degenerate in comparison with stochastic descriptions in general because, as we saw in [2] and as is

also obvious from Eq. (48) here, in the limit $\tau \to 0$ there is no diffusion. On the other hand this limit theory is probably more complicated than the stochastic structures which have been well studied [12]. This appears to be the case in view of the rather involved form of the transition probability. It is highly likely that there are actual physical situations where the generalisation of conventional quantum mechanics represented by (46) and (48) taken together will constitute the correct physical description. Such a situation would pertain in a system of quantum size which also possesses diffusive characteristics.

Concluding discussion

It is clear that this type of analysis based on the idea of a probability measure and functional kernels does contain the basic ingredients of a new and illuminating way of looking at quantum processes. This new viewpoint besides having aesthetic qualities does help to add the visualisation advantages of classical dynamics and probabilistic thinking to the quantum area. The preceding work also illustrates the power of the basic FEYNMAN formula for the time development of a wave function. All our analysis has been built up on that formula and contrary to what MOYAL says ([4] page 102) it would seem that the phase space distributions are indeed part of the basic constructs of quantum mechanics (if not of the basic postulates), since we have derived these distributions from the FEYNMAN integral [15]. We further observe that our statement of quantum mechanics, and its possible generalisation for finite τ, in stochastic terms, is essentially simpler than MOYAL's scheme since our transition probability corresponds to his phase space distribution. Thus MOYAL's scheme is more complicated in that to follow the time development of his phase space distributions he also needs the transition element kernel $K(p, q|p_0, q_0, t - t_0)$ which presumably can be complex. Thus our scheme is as close as perhaps one could hope to the standard stochastic description in configuration space. The generalisation of conventional quantum theory which we have before taking the limit $\tau \to 0$ is close in spirit to the works of E. NELSON [6] and D. KERSHAW [7] but is very different in at least two respects. We only get quantum mechanics in the limit $\tau \to 0$ and even before this limit is taken our basic transition probability $\tilde{P}$ is essentially more complicated than these authors propose. Their papers, though, have had considerable influence on the preceding work.

Acknowledgment

I would like to thank Professor M. S. BARTLETT F.R.S. for a number of very useful discussions and in particular for drawing my attention to the possibility of the remarkable derivation of the phase space distributions from the transition probability.

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КВАНТОВАЯ ВЕРОЯТНОСТЬ ВЗВЕШЕННЫХ ТРАЕКТОРИЙ

Й. Г. ГИЛЬСОН

Резюме

В работе обобщается метод Фейнмана. Вводится действительная «вероятностная» мера для взвешивания траекторий, что выражается пределом меры, содержащей действительный параметр. Для конечных значений действительного параметра выводится уравнение движения типа Фоккера—Планка, а для нулевого предельного случая — уравнение Шредингера. Дается простой вывод для распределения фазового пространства Вигнера применением дифференцируемости квантовых траекторий, что также доказывается. Выдвигается вопрос, что уравнения движения, полученные для конечных значений фигурирующих в теории действительных параметров, могут отражать пожалуй реальные процессы.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 335—338 (1969)

ON THE SHELL EFFECT IN $(n,2n)$ REACTION CROSS-SECTIONS

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(Received 17. XII. 1968)

Shell effects are investigated by comparing measured $(n,2n)$ cross-sections with predictions from a phenomenological equation which is insensitive to threshold energy. The results suggest that only the threshold energy of $(n,2n)$ reaction can be held responsible for any shell effect in the value of the cross-section.

The existence of shell effects has been suggested by several authors [1—5] to describe the cross-sections for (n,p), (n,α) and $(n,2n)$ reactions induced by 14 MeV neutrons. These effects were attributed to irregularities in the level density parameter a, the reaction threshold energy Q at magic neutron and proton numbers [1—5] and the proton pairing energy δ_p [2].

It was shown recently [6] that there is no evidence of shell effects in the measured (n,p) reaction cross-sections which can be well described in a broad region of mass number by the LEVKOVSKI equation [7] of the form

$$\sigma(n,p) = c_1 \cdot f)A) \cdot \exp\left(-c_2 \cdot (N-Z)/A\right), \qquad f(A) = (A^{1/3}+1)^2 . \tag{1}$$

According to LEVKOVSKI's arguments the proton emission probability increases with increasing relative proton concentration. The same relation is expected to hold for neutron emission which should therefore increase as $N—Z/A$ increases. A strong $N—Z$ dependence of the $(n,2n)$ cross-section has already been shown by several authors [8—9].

An attempt was made to describe the compiled experimental values by a cross section formula relating the $(n,2n)$ cross section to $N—Z/A$ in the form

$$\sigma_{\mathrm{emp}}(n,2n) = \left[1 - c_1 \cdot f(A) \cdot \exp\left(-c_2 \cdot (N-U)/A\right)\right] \cdot c_3 , \qquad f(A) = (A^{1/3}+1)^2 . \tag{2}$$

It would have been difficult to account in the formula for the value of Q varying with the different nuclei excited by 14 MeV neutrons. Considering

that near the threshold the $(n, 2n)$ reaction strongly depends on $E_{exc} = E_n - Q$, where E_n is the bombarding energy, E_{exc} was introduced with a constant value, thus eliminating the shell effect of Q to identify those of δ_p and a.

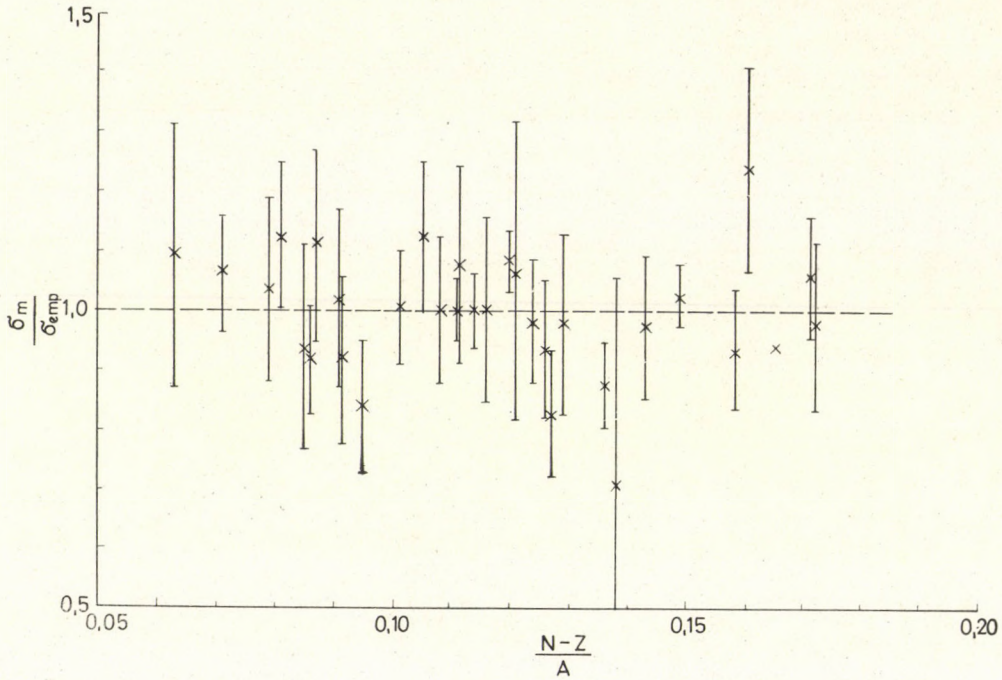

Fig. 1. Ratio of measured to predicted values of $(n, 2n)$ cross-sections

Most of the reported data were measured at $E_n = Q + 3$ MeV, if not, the cross-section was extrapolated from the values measured at 14 MeV using the WEISSKOPF formula [10]:

$$\sigma(n, 2n) = \sigma_c \cdot \left[1 - \left(1 + \frac{E_{exc}}{T}\right) \exp\left(-E_{exc}/T\right)\right],$$
$$T = \left(\frac{E_n}{0{,}115 \cdot A}\right)^{1/2}. \tag{3}$$

σ_c was evaluated from the fit of (3) to the excitation functions. The values of Q were taken from Tables [11]. The fit of Eq. (2) to the measured cross-sections yields the parameters of interest as $c_1 = 0.061$, $c_2 = 8.6$, $c_3 = 2050$ mbarn. The results of the calculations are listed along with the reported data in Table I and the measured and predicted cross-sections are compared graphically in Fig. 1.

Table I

Measured (σ_m) and predicted (σ_{emp}) values of $(n, 2n)$ reaction cross-sections. The error in the extrapolated values is that of the measurement at 14 MeV

target nucleus	$\frac{N-Z}{A}$	σ_m	σ_{emp}	Ref.
1.	2.	3.	4.	5.
^{55}Mn	0.091	750 ± 112	731	8 M*
^{56}Fe	0.071	500 ± 40	470	2 M
^{59}Co	0.085	570 ± 105	609	13 M
^{63}Cu	0.079	495 ± 74	478	8 M
^{65}Cu	0.108	810 ± 120	805	8 M
^{64}Zn	0.063	254 ± 50	232	2 M
^{66}Zn	0.091	550 ± 83	598	8 M
^{70}Zn	0.143	1065 ± 130	1092	8 E*
^{69}Ga	0.101	690 ± 65	685	13 M
^{71}Ga	0.127	780 ± 100	941	8 M
^{70}Ge	0.086	447 ± 45	486	14 M
^{76}Ge	0.158	1095 ± 120	1169	14 E
^{75}As	0.120	910 ± 40	838	13 M
^{74}Se	0.081	415 ± 44	368	15 E
^{76}Se	0.105	745 ± 81	661	15 E
^{79}Br	0.114	740 ± 45	738	16 E
^{81}Br	0.136	835 ± 65	949	16 E
^{85}Rb	0.129	830 ± 125	849	8 M
^{87}Rb	0.149	1056 ± 53	1026	13 M
^{84}Sr	0.095	380 ± 50	452	1 M
^{86}Sr	0.116	701 ± 110	699	2 M
^{88}Sr	0.136	215 ± 24**	898	1 M
^{89}Y	0.124	751 ± 80	764	2, 8, 20 M
^{90}Zr	0.111	608 ± 30	605	2 M
^{93}Nb	0.118	384 ± 60**	664	2 E
^{92}Mo	0.087	280 ± 40	251	8 M
^{100}Mo	0.160	1295 ± 180	1045	17 E
^{103}Rh	0.126	642 ± 80	682	2 E
^{107}Ag	0.121	630 ± 141	592	18 E
^{109}Ag	0.138	553 ± 262	777	18 E
^{108}Cd	0.111	490 ± 75	453	19 E
^{116}Cd	0.172	1044 ± 147	1066	19 E
^{123}Sb	0.171	1090 ± 100	1025	13 M
^{127}J	0.165	900	951	13 M

* M measured at $E_n = Q + 3$ MeV
E extrapolated to $E_n = Q + 3$ MeV
** Cross-section for reaction leading to the ground state only.

The agreement between measured and predicted values is very good for nuclei with $4 \leq N - Z \leq 21$. Since the agreement is good whether the number of neutrons is magic or not, it can be considered as a definite proof of the non-existence of shell effects produced by the pairing energy or the level density. Thus, the shell effect appearing in the cross-section for $(n, 2n)$ reactions seems to be due to the value of Q only.

The results show the usefulness of the empirical formula (2) for the calculation of the $(n, 2n)$ cross-sections at a given bombarding energy. By Eq. (3) it is possible to evaluate the cross-sections at different excitation energies.

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ОБОЛОЧЕЧНЫЙ ЭФФЕКТ В ПОПЕРЕЧНЫХ СЕЧЕНИЯХ РЕАКЦИИ $(n, 2n)$

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Резюме

Исследуются оболочечные эффекты сравнением измеренных поперечных сечений $(n, 2n)$ с предсказаниями на основе феноменологического уравнения, которое нечувствительно к порогой энергии. Результаты внушают, что любой оболочечный эффект в значении поперечного сечения может быть обусловлен только пороговой энергией реакции $(n, 2n)$.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 339—344 (1969)

EXPANSION OF A RADIATING METRIC

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(Received 19. XII. 1968)

The expansion of Vaidya's radiating Schwarzschild metric is obtained by using the vectors formed by Δ-tensor.

1. Introduction

In a previous paper [1] the vectors formed by Δ-tensor are evaluated for some non-static cosmological line elements. In the present paper the vectors are evaluated for the VAIDYA's radiating Schwarzschild metric. Then the components of each vector are explained more clearly. We follow the concepts of [1].

2. Values of R_α and S_i in Vaidya's radiating metric

Let us consider the metric [2]

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 + \frac{\dot{m}^2}{f^2}\left(1 - \frac{2m}{r}\right) dt^2\,, \quad (2.1)$$

where

$$m = m(r, t), \quad (2.2)$$

$$f(m) = m'\left(1 - \frac{2m}{r}\right). \quad (2.3)$$

Here dash and dot indicate the differentiation with respect to r and t, respectively. The corresponding flat space metric is

$$ds^2 = -dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 + dt^2\,. \quad (2.4)$$

Then we find that

$$g_{\mu\nu} = \text{diag}\left[-\left(1-\frac{2m}{r}\right)^{-1}, \; -r^2, \; -r^2\sin\theta, \frac{\dot{m}^2}{f^2}\left(1-\frac{2m}{r}\right)\right], \tag{2.5}$$

$$g^{\mu\nu} = \text{diag}\left[-\left(1-\frac{2m}{r}\right), \; -r^{-2}, \; -r^{-2}(\sin\theta)^{-2}, \frac{f^2}{\dot{m}^2}\left(1-\frac{2m}{r}\right)^{-1}\right], \tag{2.6}$$

$$\nu_{\mu\nu} = \text{diag}\,[-1, -r^2, \; -r^2\sin^2\theta, \; 1,] \tag{2.7}$$

$$g = -r^4\sin^2\theta\frac{\sin^2}{f^2}, \tag{2.8}$$

$$\nu = -r^4\sin^2\theta, \tag{2.9}$$

$$k = \frac{\dot{m}}{f} = \frac{\dot{m}}{m'\left(1-\frac{2m}{r}\right)}, \tag{2.10}$$

$$R_\alpha = \left[\frac{\dot{m}'}{\dot{m}} - \frac{m''}{m'} + \frac{2(m'r-m)}{r^2\left(1-\frac{2m}{r}\right)}, \; 0, \; 0, \; \frac{\ddot{m}}{\dot{m}} - \frac{\dot{m}'}{m'} + \frac{2\dot{m}}{r\left(1-\frac{2m}{r}\right)}\right], \tag{2.11}$$

$$s_i = \left[\frac{m''}{m'} - \frac{\dot{m}'}{\dot{m}}, \; 0, \; 0, \; \frac{\ddot{m}}{\dot{m}} - \frac{\dot{m}'}{m'}\right]. \tag{2.12}$$

We now study the non-zero components of R_α and S_i of this model in some detail.

The component R_1

The component R_1 is given by

$$R_1 = \frac{\dot{m}'}{\dot{m}} - \frac{m''}{m'} + \frac{2(m'r-m)}{r^2\left(1-\frac{2m}{r}\right)}. \tag{2.13}$$

The value of $(\dot{m}'/\dot{m} - m''/m')$ can be obtained [2] as

$$\frac{\dot{m}'}{\dot{m}} - \frac{m''}{m'} = \frac{2m}{r^2\left(1-\frac{2m}{r}\right)}. \tag{2.14}$$

Eq. (2.13) can then be written as

$$R_1 = \frac{2m'}{r\left(1 - \frac{2m}{r}\right)} . \tag{2.15}$$

Eq. (2.15) and the value of g^{11} from (2.6) give

$$R^1 = -\frac{2m'}{r} . \tag{2.16}$$

To the first order approximation, Eq. (2.15) gives

$$R_1 = \frac{2m'}{r} \tag{2.17}$$

The apparent luminosity, L, may be defined [3] as

$$L = 4\pi r^2 q , \tag{2.18}$$

where

$$q = v^\mu v^\nu T_{\mu\nu} . \tag{2.19}$$

Here v^μ are the four components of velocity of the observer, for a stationary observer

$$v^1 = v^2 = v^3 = 0, \quad v^4 = \sqrt{g^{44}} \tag{2.20}$$

and, therefore,

$$q = T_4^4 . \tag{2.21}$$

The value of T_4^4 is given [2] by

$$T_4^4 = \frac{m'}{4\pi r^2} . \tag{2.22}$$

Eq. (2.18) with Eqs. (2.21) and (2.22) gives

$$L = m' . \tag{2.23}$$

Eqs. (2.16) and (2.17) with Eq. (2.23) give

$$R^1 = -\frac{2L}{r} . \tag{2.24}$$

$$R_1 = \frac{2L}{r} . \tag{2.25}$$

The component R^1 and the first approximation of the component R_1 both correspond to the corresponding component of radiating force. The force L/r is the non-Newtonian gravitational force which is produced by the flowing radiation towards the centre of the force. LINDQUIST, SCHWARTZ and MISNER [3] call it an "induction field" associated with the changing Newtonian m/r^2 field.

The component S_1

The value of the component S_1 with Eq. (2.14) can be expressed as

$$S_1 = -\frac{2m}{r^2\left(1-\frac{2m}{r}\right)} . \tag{2.26}$$

To the first order of approximation

$$S_1 = -\frac{2m}{r^2} . \tag{2.27}$$

Eq. (2.26) with the value of g^{11} gives

$$S^1 = \frac{2m}{r^2} . \tag{2.28}$$

The component S^1 and the first order approximation of the component S_1 both correspond to the corresponding component of the gravitational force. This gravitational force is due to the form of the line element like the Schwarzschild line element. NARLIKAR and SINGH [4] have already shown that the component S_1 for the Schwarzschild line element is given by Eq. (2.26).

The component S_4

The component S_4 is given by

$$S_4 = \frac{\partial}{\partial t}\log\left(\frac{\dot{m}}{m'}\right) . \tag{2.29}$$

We know [2] that

$$\frac{\dot{m}}{m'} = \frac{g_{11}}{g_{44}} . \tag{2.30}$$

For the light path ($g_{\mu\nu}w^\mu w^\nu = 0$), we have

$$\frac{g_{11}}{g_{44}} = \frac{w_4}{w_1} . \tag{2.31}$$

From Eqs. (2.29), (2.30) and (2.31) we find that

$$S_4 = \frac{\partial}{\partial t} \log \left(\frac{w_4}{w_1} \right) . \tag{2.32}$$

The substitution

$$R = \frac{w_4}{w_1} \tag{2.33}$$

in Eq. (2.32) gives

$$S_4 = \frac{1}{R} \left(\frac{\partial R}{\partial t} \right) . \tag{2.34}$$

The component S_4, therefore, corresponds to the rate of expansion of the wave of radiation.

The component R_4

The component R_4 is given by

$$R_4 = \left(\frac{\ddot{m}}{\dot{m}} - \frac{\dot{m}'}{m'} \right) + \frac{2\dot{m}}{r \left(1 - \dfrac{2m}{r} \right)} . \tag{2.35}$$

The first term on the right hand side of equation (2.35) corresponds to the rate of expansion of the wave front of flowing radiation while the second term corresponds to the fourth (time) component of radiation force.

3. Discussion

In VAIDYA's radiating Schwarzschild metric the component S_4 corresponds to the rate of expansion of the wave front of radiation because the other term which could correspond to the fourth component of field force is zero, while in the component R_4 both terms exist.

The components R_1, R_2, R_3 correspond to the corresponding components of radiation force while the components S_1, S_2, S_3 correspond to the corresponding components of the gravitational force which exists due to the form of the line element like Schwarzschild metric.

In other non-static models [1], the fourth components of the field forces are always zero, so the components R_4 and S_4 correspond to the Hubble's expansions of the models only and the components R_1, R_2, R_3 and S_1, S_2, S_3 correspond to the corresponding components of the gravitational forces.

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РАЗЛОЖЕНИЕ ИЗЛУЧАЮЩЕЙ МЕТРИКИ

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Резюме

Разложение излучающей Шварцшильд-метрики Вейдя получено применением векторов, образованных тензором.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 345—352 (1969)

GENERAL SUM RULES FROM CURRENT DENSITY ALGEBRA

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(Received 10. I. 1969)

Current density algebra sum rules for three and four point functions are investigated under the assumption of multiple integral representations of current correlation functions in the currents' masses. It is proved that the sum rules corresponding to a limit to infinite momentum in an arbitrary linear combination of momenta ($P_0 \to \infty$, $\Delta_0 \to \infty$ etc.) collapse into a single sum rule.

I. Introduction

It is known that different families of sum rules can be derived from the equal-time commutation relations of current densities [1] by the $P_0 \to \infty$ [2], the $\Delta_0 \to \infty$ [3] and $K_0 \to \infty$ [4] methods. The same sum rules also follow from the FUBINI identity [5] if one disperses in the appropriate variable [5, 6]. In the case of equal time commutation relations taken between the vacuum and one particle states (three-point functions) besides the $P_0 \to \infty$ sum rules [7], more general sum rules have also been derived [8] corresponding to a limit to infinity in an arbitrary linear combination of the momenta. The $A1 \to \varrho\pi$ problem was solved (in one alternative way) exactly by this latter method [9, 10].

Our purpose now is to write down the sum rules corresponding to the most general infinite momentum limit in the case when the current correlation functions are given by multiple integral representations. It turns out that all these sum rules collapse; there are no separate $P_0 \to \infty$, $\Delta_0 \to \infty$ etc. sum rules.

II. Three-point function sum rules

First we consider three-point functions. For definiteness we take the case treated in [10]:

$$t_{\mu\nu}(k, p) = \frac{1}{2} \int d^4x e^{ikx} \langle 0 | \left[A_\mu(x), V_\nu(0) \right] | \pi(p) \rangle . \tag{1}$$

(A_μ is an axial-vector, V_ν a vector current, π is a pion state of momentum p.

We have omitted the isospin indices.) Let us introduce the kinematic variables:

$$q = k - p, \quad P_1 = \frac{1}{2}(k+q), \quad P_2 = \frac{1}{2}p, \tag{2}$$

$$r_1 = \frac{1}{2}(k^2 + q^2), \quad r_2 = \frac{1}{2}(k^2 - q^2), \quad p^2 = m^2.$$

Then, from Lorentz invariance we have:

$$t_{\mu\nu}(k,p) = a_{MN}(r_k)\, P_{M\mu}\, P_{N\nu} + a(r_k)\, g_{\mu\nu}, \tag{3}$$
$$k, M, N = 1, 2.$$

Taking into account the connected and disconnected parts of intermediate states [6] in Eq. (1), one can easily prove* the following representation for the invariant functions in the commutator (1):

$$\begin{aligned} a_{MN}(\nu_k) = &\int dM_q^2\, Q_{MN}(M_q^2,\ k^2)\, \delta(q^2 - M_q^2)\, \varepsilon(q_0) + \\ &+ \int dM_k^2\, K_{MN}(M_k^2, q^2)\, \delta(k^2 - M_k^2)\, \varepsilon(k_0). \end{aligned} \tag{4}$$

The retarded functions corresponding to $a_{MN}(\nu_k)$ can be defined as usual by

$$A_{MN}(\nu_k) = \frac{1}{\pi}\int \frac{dr_0}{r_0 - i\varepsilon}\, a_{MN}(\nu_k'), \tag{5}$$

$$\nu_k' = \nu_k + 2P_{k0}\, r_0 + \delta_{k1}\, r_0^2.$$

Using Eq. (4) this can be written as

$$\begin{aligned} A_{MN}(\nu_k) = &\frac{1}{\pi}\int \frac{dM_q^2}{r_{0+q} - r_{0-q}} \{(r_{0+q} - i\varepsilon)^{-1}\, Q_{MN}(M_q^2, k^2 + 2r_{0+q}\, k_0 + r_{0+q}^2) - \\ &- (r_{0-q} - i\varepsilon)^{-1}\, Q_{MN}(M_q^2, k^2 + 2r_{0-q}\, k_0 + r_{0-q}^2)\} + \\ &+ \frac{1}{\pi}\int \frac{dM_k^2}{r_{0+k} - r_{0-k}} \{(r_{0+k} - i\varepsilon)^{-1} \cdot \\ &K_{MN}(M_k^2,\ q^2 + 2r_{0+k}\, q_0 + r_{0+k}^2) - \\ &- (r_{0-k} - i\varepsilon)^{-1}\, K_{MN}(M_k^2,\ q^2 + 2r_{0-k} + r_{0-k}^2)\}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} r_{0\pm q} &= -q_0 \pm \sqrt{M_q^2 - q^2 + q_0^2}, \\ r_{0\pm k} &= -k_0 \pm \sqrt{M_k^2 - k^2 + k_0^2}. \end{aligned} \tag{7}$$

* A simple proof can be given using Lehmann—Symanzik—Zimmermann reduction technique and TCP theorem. See I. MONTVAY [11].

In the physical current correlation function $T_{\mu\nu}$ the invariant functions are those defined in Eq. (5). Thus, we have

$$T_{\mu\nu} = A_{MN}(\nu_K) P_{M\mu} P_{N\nu} + A(\nu_K) g_{\mu\nu}. \tag{8}$$

It is well known that in the case of currents with spin the physical current correlation function does not coincide with the retarded product of currents. In fact, we have

$$\begin{aligned} R_{\mu\nu} &= i \int d^4x e^{ikx} \theta(x_0) \langle 0 | \left[A_\mu(x), V_\nu(0) \right] | \pi(p) \rangle = \\ &= \frac{1}{\pi} \int \frac{dr_0}{r_0 - i\varepsilon} t_{\mu\nu}(k_0 + r_0, \underset{\sim}{k}, p) = T_{\mu\nu} + P_{M\mu} g_{0\nu} \frac{1}{\pi} \int dr_0 a_{M1}(\nu'_k) + \\ &+ g_{0\mu} P_{N\nu} \frac{1}{\pi} \int dr_0 a_{1N}(\nu'_k) + g_{0\mu} g_{0\nu} \frac{1}{\pi} \int dr_0 a_{11}(\nu'_k) r_0 . \end{aligned} \tag{9}$$

The presence of noncovariant terms in currents' T-product (or R-product) was first shown by K. JOHNSON in quantum electrodynamics [12]. In his proof an essential role is played by the spin of currents. In the case of scalar currents such noncovariant terms do not appear.**

The equal-time commutator can be defined analogously to the retarded product in Eq. (9):

$$\begin{aligned} E_{\mu\nu} &= \int d^4x e^{ikx} \delta(x_0) \langle 0 | \left[A_\mu(x), V_\nu(0) \right] | \pi(p) \rangle = \\ &= \frac{1}{\pi} \int dr_0 t_{\mu\nu}(k_0 + r_0, \underset{\sim}{k}, p) = P_{M\mu} P_{N\nu} \frac{1}{\pi} \int dr_0 a_{MN}(\nu'_k) + \\ &+ g_{\mu\nu} \frac{1}{\pi} \int dr_0 a(\nu'_k) + P_{M\mu} g_{0\nu} \frac{1}{\pi} \int dr_0 a_{M1}(\nu'_k) r_0 + \\ &+ g_{0\mu} P_{N\nu} \frac{1}{\pi} \int dr_0 a_{1N}(\nu'_k) r_0 + g_{0\mu} g_{0\nu} \frac{1}{\pi} \int dr_0 a_{11}(\nu'_k) r_0^2 . \end{aligned} \tag{10}$$

Let us now recapitulate briefly how one derives the infinite momentum sum rules from Eq. (10). One starts with the FUBINI identity [5]:

$$k^\mu R_{\mu\nu} = R_\nu - E_{0\nu} . \tag{11}$$

Here R_ν is the retarded product involving the divergence of the axial-vector current:

$$R_\nu = - \int d^4x e^{ikx} \theta(x_0) \langle 0 | \left[\partial^\mu A_\mu(x), V_\nu(0) \right] | \pi(p) \rangle . \tag{12}$$

** In fact, Eq. (5) means that the retarded product is covariant in the spinless case.

The covariant analogue of Eq. (11) defines the "covariant commutator" of TAHA [4]:

$$C_\nu = T_\nu - k^\mu T_{\mu\nu} \equiv \alpha_N P_{N\nu} = E_{0\nu} - N_\nu + k^\mu N_{\mu\nu} . \tag{13}$$

Here T_ν is the physical amplitude containing the axial-vector divergence, $T_{\mu\nu}$ is given by Eq. (8) and N_ν, $N_{\mu\nu}$ denote the noncovariant pieces of the retarded products R_ν and $R_{\mu\nu}$, respectively. From Eqs. (13), (12) and (9) we easily derive

$$\alpha_N = P_{M0} \frac{1}{\pi} \int dr_0 \, a_{MN}(r'_k) + \frac{1}{\pi} \int dr_0 \, a_{1N}(r'_k) \, r_0 + k_0 \frac{1}{\pi} \int dr_0 \, a_{1N}(r'_k). \tag{14}$$

Let us now take the infinite momentum limit, the Lorentz invariants remaining fixed as in [4]. In order to exploit the full freedom in the limit to infinite momentum we define arbitrary (orthogonal) linear combinations of the momenta by

$$Q_M = P_K A_{KM} , \quad P_K = A_{KM} Q_M \quad (AA^T = 1) . \tag{15}$$

Then we perform the limit $Q_{10} \to \infty$, Q_{20} finite in Eq. (14). Using Eqs. (4), (6), (7) we obtain under the assumption of constant asymptotic behaviour of $Q_{MN}(M_q^2, k^2)$ and $K_{MN}(M_k^2, q^2)$ when $k^2 \to \infty$ and $q^2 \to \infty$, respectively:***

$$\begin{aligned} \alpha_N = \lim_{Q_{10}\to\infty} \alpha_N = {} & \frac{1}{\pi} \int dM_q^2 \, Q_{1N}(M_q^2, \infty) + \frac{1}{\pi} \int dM_k^2 \, K_{1N}(M_k^2, \infty) + \\ & + \frac{A_{M1} + \delta_{M1}(A_{11} + A_{21})}{2\pi} \left\{ \int \frac{dM_q^2}{A_{11} - A_{21}} \left[Q_{MN}\left(M_q^2, k^2 + \right.\right.\right. \\ & \left. + \frac{A_{11} + A_{21}}{A_{11} - A_{21}} (M_q^2 - q^2) \right) - Q_{MN}(M_q^2, \infty) \Big] + \\ & + \int \frac{dM_k^2}{A_{11} + A_{21}} \left[K_{MN}\left(M_k^2, q^2 + \frac{A_{11} - A_{21}}{A_{11} + A_{21}} (M_k^2 - k^2) \right) - \right. \\ & \left. \left. - K_{MN}(M_k^2, \infty) \right] \right\} . \end{aligned} \tag{16}$$

The special cases $A_{11} + A_{21} = 0$ and $A_{11} - A_{21} = 0$ in Eq. (16) need some comments. (Properly speaking they must be considered separately.) Here we suppose, following [6], that if $q^2 \to \infty$

$$K_{MN}(M_k^2, q^2) - K_{MN}(M_q^2, \infty) \to \frac{K_{MN}(M_k^2)}{q^2} , \tag{17}$$

*** The assumption of a constant asymptotic behaviour simplifies the results to a large extent, but extension to the more divergent cases seems possible.

and if $k^2 \to \infty$

$$Q_{MN}(M_q^2, k^2) - Q_{MN}(M_q^2, \infty) \to \frac{Q_{MN}(M_q^2)}{k^2} . \tag{18}$$

Then we get in the two cases, respectively

$$\alpha_N = \lim_{q_0 \to \infty} \alpha_N = \frac{1}{\pi} \int dM_q^2 Q_{1N}(M_q^2, \infty) + \frac{1}{\pi} \int dM_k^2 K_{1N}(M_k^2, \infty) +$$
$$+ \frac{1}{4\pi} \left\{ \int dM_q^2 [Q_{1N}(M_q^2, k^2) - Q_{1N}(M_q^2, \infty)] + \int dM_k^2 \frac{K_{1N}(M_k^2)}{M_k^2 - k^2} - \right.$$
$$\left. - \int dM_q^2 [Q_{2N}(M_q^2, k^2) - Q_{2N}(M_q^2, \infty)] - \int dM_k^2 \frac{K_{2N}(M_k^2)}{M_k^2 - k^2} \right\}; \tag{19}$$

$$\alpha_N = \lim_{k_0 \to \infty} \alpha_N = \frac{1}{\pi} \int dM_q^2 Q_{1N}(M_q^2, \infty) + \frac{1}{\pi} \int dM_k^2 K_{1N}(M_k^2, \infty) +$$
$$+ \frac{1}{4\pi} \left\{ 3 \int dM_k^2 [K_{1N}(M_k^2, q^2) - K_{1N}(M_k^2, \infty)] + \right.$$
$$+ 3 \int dM_q^2 \frac{Q_{1N}(M_q^2)}{M_q^2 - q^2} + \int dM_k^2 [K_{2N}(M_k^2, q^2) - K_{2N}(M_k^2, \infty)] +$$
$$\left. + \int dM_q^2 \frac{Q_{2N}(M_q^2)}{M_q^2 - q^2} \right\} . \tag{20}$$

Eqs. (16), (19) and (20) are essentially the sum rules derived by HALLIDAY and LANDSHOFF [8] for the three-point functions, which correspond to the AMATI, JENGO, REMIDDI [6] sum rules valid in the four-point function case.

Let us now see what happens when the physical current correlation function in Eq. (8) is given by a double integral representation of the form

$$A_{MN}(q^2, k^2) = \frac{1}{\pi} \int dM_q^2 \frac{\varrho_q^{MN}(M_q^2)}{M_q^2 - q^2} + \frac{1}{\pi} \int dM_k^2 \frac{\varrho_k^{MN}(M_k^2)}{M_k^2 - k^2} +$$
$$+ \frac{1}{\pi^2} \iint dM_q^2 dM_k^2 \frac{\varrho_{qk}^{MN}(M_q^2, M_k^2)}{(M_q^2 - q^2)(M_k^2 - k^2)} . \tag{21}$$

Here the single spectral functions correspond to our philosophy of constant behaviour in the limits $q^2 \to \infty$ and $k^2 \to \infty$. Taking the $q_0 \to \infty$ and $k_0 \to \infty$ limits of Eq. (6) one can easily see that Q_{MN} and K_{MN} are the discontinuities of A_{MN} in the variables q^2 and k^2, respectively. Hence

$$Q_{MN} = \varrho_q^{MN} + \frac{1}{\pi} \int dM_k^2 \frac{\varrho_{qk}^{MN}}{M_k^2 - k^2} ,$$
$$K_{MN} = \varrho_k^{MN} + \frac{1}{\pi} \int dM_q^2 \frac{\varrho_{qk}^{MN}}{M_q^2 - q^2} . \tag{22}$$

Substituting this expression back into Eqs. (16), (19) or (20) we always get

$$\alpha_N = \frac{1}{\pi}\int dM_q^2\, \varrho_q^{1N} + \frac{1}{\pi}\int dM_k^2\, \varrho_k^{1N}\,. \tag{23}$$

This shows that all the infinite momentum sum rules collapse into a single sum rule and that TAHA's equivalence conditions [4] are trivially satisfied if Eq. (21) holds. (The equivalence conditions can be obtained by equating the forms (19) and (20) of α_N.)

A remarkable property of the covariant commutator in Eq. (23) is that the double spectral functions ϱ_{qk}^{MN} cancel in it. The covariant commutator is given entirely by the single spectral functions. To get really sum rules from Eq. (23) one must of course give the equal time commutators explicitly and compute α_N. In the quark model [1] we have, e.g. in the case of Eq. (1) treated in [10]:

$$\alpha_N = -\frac{4F_\pi}{\sqrt{2(2\pi)^3}}\,\delta_{N2}\,, \quad \text{if} \quad \langle 0|\, A_\mu(0)|\pi(p)\rangle = \frac{iF_\pi p_\mu}{\sqrt{2(2\pi)^3}}\,. \tag{24}$$

It can be shown that Eqs. (23) and (24) give precisely the "nontrivial" equations of [10] and thus give the hard pion values for the $A1$- and ϱ-widths which agree well with the experiment.****

III. Four-point function sum rules

Four-point functions can be treated analogously. Therefore, we give only the definitions of the different quantities and the results. The integral representation for the invariant functions is here more involved (there is one more variable) and certainly hard to justify. We remark only that in making pole dominance approximation one always makes such an assumption, since the contribution of, say, an s-pole is of the form

$$\frac{g(t)}{(M_k^2 - k^2)\,(M_q^2 - q^2)\,(M_s^2 - s)} + \ldots\,,$$

where the dots contain terms with less factors in the denominator, owing to the possible subtractions.

For definiteness let us now consider the commutator of vector currents between pion states (for simplicity we omit isospin indices here also)

$$t_{\mu\nu}(k, p_i) = \frac{1}{2}\int d^4x e^{ikx}\,\langle\pi(p_1)|\,[V_\mu(x), V_\nu(0)]\,|\pi(p_2)\rangle\,. \tag{25}$$

**** After this paper was completed we received a Rochester preprint by V. S. MATHUR, UR—875—235, May 1968, where the $A1/\varrho\pi$ system is also treated with the use of double integral representations.

The kinematics is now defined by

$$p_1 + q = p_2 + k, \quad P_1 = K = \frac{k+q}{2}, \quad P_2 = \Delta = \frac{p_1 - p_2}{2}, \tag{26}$$

$$P_3 = P = \frac{p_1 + p_2}{2}, \quad \nu_1 = \frac{1}{2}(k^2 + q^2), \quad \nu_2 = \frac{1}{2}(k^2 - q^2), \quad \nu_3 = \nu = 2PK,$$

$$t = (p_1 - p_2)^2, \quad s = (P+K)^2 + \frac{t}{2} - m^2, \quad u = (P-K)^2 + \frac{t}{2} - m^2$$
$$s + u = k^2 + q^2, \quad p_{1,2}^2 = m^2. \tag{27}$$

The Eqs. (3), (5), (8), (13) are also valid here, if we take into account that now $K, M, N = 1, 2, 3$. (The variable t is fixed throughout, hence we do not write it out explicitly.) The intermediate states can now be put in the "channels" s, u, q^2 and k^2. Therefore, we have

$$\begin{aligned} a_{MN}(\nu_K) = &\int dM_q^2\, Q_{MN}(M_q^2, k^2, s, u)\, \delta(q^2 - M_q^2)\, \varepsilon(q_0) + \\ &+ \int dM_k^2\, K_{MN}(M_k^2, q^2, s, u)\, \delta(k^2 - M_k^2)\, \varepsilon(k_0) + \\ &+ \int dM_s^2\, S_{MN}(M_s^2, q^2, k^2, u)\, \delta(s - M_s^2)\, \varepsilon(k_0 + p_{20}) + \\ &+ \int dM_u^2\, U_{MN}(M_u^2, q^2, k^2, s)\, \delta(u - M_u^2)\, \varepsilon(k_0 - p_{10}). \end{aligned} \tag{28}$$

The infinite momentum sum rule corresponding to Eq. (16) takes the from ($N = 1, 2, 3$):

$$\begin{aligned} \alpha_N = &\frac{1}{\pi}\int dM_q^2\, Q_{1N}(M_q^2, \infty) + \frac{1}{\pi}\int dM_k^2\, K_{1N}(M_k^2, \infty) + \\ &+ \frac{1}{\pi}\int dM_s^2\, S_{1N}(M_s^2, \infty) + \frac{1}{\pi}\int dM_u^2\, U_{1N}(M_u^2, \infty) + \\ &+ \frac{A_{M1} + \delta_{M1}(A_{11} + A_{21})}{2\pi} \cdot \left\{ \int \frac{dM_q^2}{A_{11} - A_{21}} \cdot \right. \\ &\cdot \left[Q_{MN}\left(M_q^2, k^2 + \frac{A_{11} + A_{21}}{A_{11} - A_{21}}(M_q^2 - q^2), s + \right.\right. \\ &+ \left.\frac{A_{11} + A_{31}}{A_{11} - A_{21}}(M_q^2 - q^2),\ u + \frac{A_{11} - A_{31}}{A_{11} - A_{21}}(M_q^2 - q^2)\right) - \\ &\left. - Q_{MN}(M_q^2, \infty, \infty, \infty)\right] + \text{similar terms containing } K_{MN}, S_{MN}, U_{MN} \Big\}. \end{aligned} \tag{29}$$

The sum rules $P_0 \to \infty$, $\Delta_0 \to \infty$ as given in [6] correspond to such special cases as Eqs. (19), (20).

Now let us assume a triple integral representation in the variables q^2, k^2, s, u subjected to the constraint $s + u = q^2 + k^2$ and express the covariant commutator α_N in terms of the spectral functions. In this case we have single, double and triple spectral functions like ϱ_q, ϱ_k, ..., ϱ_{qk}, ϱ_{ks}, ..., ϱ_{qks}, ... It can be shown that the double and triple spectral functions cancel from the infinite momentum sum rules (29), and the general sum rules to which all the infinite momentum sum rules collapse are now ($N = 1, 2, 3$):

$$\alpha_N(t) = \frac{1}{\pi}\int dM_q^2\,\varrho_q^{1N}(t,\ M_q^2) + \frac{1}{\pi}\int dM_k^2\,\varrho_k^{1N}(t,\ M_k^2) + \\ + \frac{1}{\pi}\int dM_s^2\,\varrho_s^{1N}(t,\ M_s^2) + \frac{1}{\pi}\int dM_u^2\,\varrho_u^{1N}(t, M_u^2)\,. \tag{30}$$

Here we have written out the dependence of the different terms on t.

The author is grateful to Drs. F. Csikor and G. Pócsik for valuable discussions.

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ОБЩИЕ ПРАВИЛА СУММ ИЗ АЛГЕБРЫ ПОЛНОСТИ ТОКА

И. МОНТВАИ

Р е з ю м е

Исследуются правила сумм алгебры плотности тока для трех- и четырех-точечных функций при предположении представления многократных интегралов корреляционных функций тока в массах токов. Доказывается, что правила сумм, соответствующие пределу бесконечного импульса в любой линейной комбинации моментов ($p_0 \to \infty$, $\Delta_0 \to \infty$ и так далее), переходят в простое правило суммы.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 353—369 (1969)

THE LORENTZ PRINCIPLE AND THE GENERAL THEORY OF RELATIVITY

PART VI

By

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(Received 23. I. 1969)

In a number of publications dealing with the Lorentz principle applied to general relativity we have introduced step by step a notation which is particularly useful to reflect our own physical ideas. In the present paper we give a summary of this notation and the involved technique. We have freed the formalism of a number of minor inconsistencies to be found in the earlier works and show how with the help of this formalism simple expressions can be derived — among others — for the Christoffel symbols and the Riemann—Christoffel tensor.

More-dimensional quantities

§ 1. In our previous articles on the subject [1—5] we have introduced a notation for the formalism of the theory of relativity which notation differs from the usual one. We think that this notation reflects the physical concepts we have developed and therefore the question of notation is not an unimportant feature of our physical considerations. We give here a summary of the notation and show how a number of known relations can be obtained in a straightforward manner making use of the technique proposed by us.

§ 2. We shall write $\overset{(k)}{\mathbf{A}}$ for a k-dimensional matrix, i.e. for a matrix with k suffices, with elements

$$(\overset{(k)}{\mathbf{A}})_{\nu_1\nu_2\ldots\nu_k} = \overset{(k)}{A}_{\nu_1\nu_2\ldots\nu_k}, \quad \nu_1, \nu_2, \ldots, \nu_k = 1, 2, 3, 4.$$

The product of matrices will be defined as follows. The direct product of two matrices will be written:

$$\overset{(k)}{\mathbf{A}} \bigcirc \overset{(l)}{\mathbf{B}} = \overset{(k+l)}{\mathbf{C}}.$$

Expressed in elements we have

$$\overset{(k+l)}{C}_{\nu_1\nu_2\ldots\nu_k\mu_1\mu_2\ldots\mu_l} = \overset{(k)}{A}_{\nu_1\nu_2\ldots\nu_k} \overset{(l)}{B}_{\mu_1\mu_2\ldots\mu_l}.$$

Further we write

$$\left(\overset{(k+m)}{\mathbf{A}} \; \overset{(l+m)}{\mathbf{B}}\right)^{(m)} = \overset{(k+l)}{\mathbf{C}}.$$

where the product is to be taken by summing successively over m pairs of adjacent suffices, i.e.

$$\overset{(k+l)}{C}_{\nu_1\nu_2\ldots\nu_k\,\mu_1\mu_2\ldots\mu_l} = \sum_{\sigma_1\sigma_2\ldots\sigma_m} \overset{(k+m)}{A}_{\nu_1\nu_2\ldots\nu_k\,\sigma_1\sigma_2\ldots\sigma_m} \overset{(l+m)}{B}_{\sigma_m\ldots\sigma_2\sigma_1\,\mu_1\mu_2\ldots\mu_l}$$

In particular in case of $m = 1$ or 2 we write short

$$\overset{(k+1)}{\mathbf{A}}\ \overset{(l+1)}{\mathbf{B}} = \overset{(k+l)}{\mathbf{C}}\,,$$

$$\left(\left(\overset{(k+2)}{\mathbf{A}}\ \overset{(l+2)}{\mathbf{B}}\right)\right) = \overset{(k+l)}{\mathbf{C}}\,.$$

It is also useful to introduce the *invariant* products. We write

$$\overset{(k+1)}{\mathbf{A}} \cdot \overset{(l+1)}{\mathbf{B}} = \overset{(k+1)}{\mathbf{A}}\ \mathbf{g}^{-1}\ \overset{(l+1)}{\mathbf{B}}\,,$$

thus the fat $\cdot$ denotes that a factor $\mathbf{g}^{-1}$ has to be taken between the factors, where $\mathbf{g}^{-1}$ is the inverse of the propagation tensor.

Similarly we introduce

$$\left(\overset{(k+m)}{\mathbf{A}} \cdot \overset{(l+m)}{\mathbf{B}}\right)^{(m)} = \overset{(k+l)}{\mathbf{C}}\,,$$

the above expression stands short for

$$\overset{(k+l)}{C}_{\nu_1\nu_2\ldots\nu_k\,\mu_1\mu_2\ldots\mu_l} = \sum_{\substack{\sigma_1\ldots\sigma_m \\ \tau_1\ldots\tau_m}} \overset{(k+m)}{A}_{\nu_1\ldots\nu_k\,\sigma_1\ldots\sigma_m} \overset{(l+m)}{B}_{\tau_m\ldots\tau_1\,\mu_1\ldots\mu_l} g^{+}_{\sigma_1\tau_1}\ldots g^{+}_{\sigma_m\tau_m}\,.$$

where $g^{+}_{\sigma\tau}$ are the elements of the matrix $\mathbf{g}^{-1}$.

Finally we use the reduction of the dimension of a matrix by the following process

$$\left(\overset{(k+2m)}{\mathbf{A}}\ \mathbf{g}^{(-m)}\right)^{(2m)} = \overset{(k)}{\mathbf{A}}\,,$$

where we have written

$$\mathbf{g}^{(-m)} = \underbrace{\mathbf{g}^{-1} \bigcirc \mathbf{g}^{-1} \ldots \bigcirc \mathbf{g}^{-1}}_{m \text{ factors}}\,.$$

Permutation operations

§ 3. Denoting by P a permutation operator, we write

$$P\ \overset{(k)}{\mathbf{A}} = \overset{(k)}{\mathbf{B}}\,,$$

where $\overset{(k)}{\mathbf{B}}$ is the matrix with elements

$$\overset{(k)}{B}_{\nu_1\nu_2\ldots\nu_k} = \overset{(k)}{A}_{P(\nu_1\nu_2\ldots\nu_k)}.$$

In particular we make use of operators

$$c_l = (1, 2, 3, \ldots l),$$

thus c_l gives the cyclic permutation of the first l suffices of a k-dimensional matrix. Apart from c_l we also use the powers c_l^m of c_l which operators are defined in the usual way.

Particularly useful appears the operator

$$\pi_n = \mathbf{1} - c_n^{-1} + c_n^{-2} - \ldots + (-1)^{n-1} c_n^{-(n-1)}.$$

It can be seen easily

$$\frac{1}{2}(\mathbf{1} + c_n^{-1})\,\pi_n = \begin{cases} 1 \text{ if } l \text{ is odd,} \\ 0 \text{ if } l \text{ is even.} \end{cases}$$

With the help of c_k we can generalize the definition of the transposed matrix, we introduce

$$\overset{(\widetilde{k})}{\mathbf{A}} = c_k \overset{(k)}{\mathbf{A}}.$$

The rule of the transposed of a product can be generalized so as to read

$$\overset{(k\widetilde{+}1)}{\mathbf{A}}\;\overset{(l+1)}{\mathbf{B}} = c_{k+l}^{l}\left(\overset{(l\widetilde{+}1)}{\mathbf{B}}\;\overset{(k+1)}{\mathbf{A}}\right). \tag{1}$$

The above rule, which can be extended in the same manner to the invariant product, includes as a particular case the well-known rule valid for $k = l = 1$.

If a more-dimensional quantity is such that for a given permutation P we have

$$P\overset{(k)}{\mathbf{A}} = \overset{(k)}{\mathbf{A}},$$

then we can take that $\overset{(k)}{\mathbf{A}}$ is symmetric with respect to P. We may also write

$$P \equiv \mathbf{1} \quad \left(\mathrm{mod}\,\overset{(k)}{\mathbf{A}}\right),$$

where we denote by 1 the identical permutation operator.

The $\mathfrak{N}$ operator

§ 4. We introduce the $\mathfrak{N}$ operator the components of which in a representation K are given as

$$K(\mathfrak{N}) = \Box = \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial x_4}.$$

Applying $\Box$ to a k-dimensional matrix field $\overset{(k)}{\mathbf{A}}(\mathbf{x})$ we obtain a $k+1$-dimensional one; we shall write

$$\overset{(k)}{\mathbf{A}}(\mathbf{x}) \circ \overleftarrow{\Box} = \overset{(k+1)}{\mathbf{A}}(\mathbf{x}),$$

where the arrow $\leftarrow$ denotes that the differentiation has to be applied to the factor on the left of $\overleftarrow{\Box}$. We take that $\overleftarrow{\Box}$ produces the $k+1$-st suffix of $\overset{(k+1)}{\mathbf{A}}(\mathbf{x})$.

The differentiation of a product can also be written down in a concise form using the permutation operators. One finds

$$\left(\overset{(k)}{\mathbf{A}}\ \overset{(l)}{\mathbf{B}}\right) \circ \overleftarrow{\Box} = \overset{(k)}{\mathbf{A}}\ \overset{(l+1)}{\mathbf{B}} + c_{k+l-1}\left(\left[c_{k+1}^{-1}\ \overset{(k+1)}{\mathbf{A}}\right]\overset{(l)}{\mathbf{B}}\right), \tag{2}$$

where

$$\overset{(k+1)}{\mathbf{A}} = \overset{(k)}{\mathbf{A}} \circ \overleftarrow{\Box} \quad \text{and} \quad \overset{(l+1)}{\mathbf{B}} = \overset{(l)}{\mathbf{B}} \circ \overleftarrow{\Box}.$$

The above rule can also be extended to m-fold summation, one finds

$$\left(\overset{(k+m)}{\mathbf{A}}\ \overset{(l+m)}{\mathbf{B}}\right)^{(m)} \circ \overleftarrow{\Box} = \left(\overset{(k+m)}{\mathbf{A}}\ \overset{(l+m+1)}{\mathbf{B}}\right)^{(m)} + c_{k+l-1}\left(\left[c_{k+m+1}^{-1}\ \overset{(k+m+1)}{\mathbf{A}}\right]\overset{(l+m)}{\mathbf{B}}\right)^{(m)}.$$

Applying the $\overleftarrow{\Box}$ operator to invariant products, we find e.g.

$$\left(\overset{(k)}{\mathbf{A}} \cdot \overset{(l)}{\mathbf{B}}\right) \circ \overleftarrow{\Box} = \overset{(k)}{\mathbf{A}} \cdot \overset{(l+1)}{\mathbf{B}} + c_{k+l-1}\left(\left[c_{k+1}^{-1}\left(\overset{(k+1)}{\mathbf{A}} - \overset{(k)}{\mathbf{A}} \cdot \overset{(3)}{\mathbf{g}}\right)\right] \cdot \overset{(l)}{\mathbf{B}}\right), \tag{3}$$

where we wrote

$$\overset{(3)}{\mathbf{g}} = \mathbf{g} \circ \overleftarrow{\Box}.$$

Tensor quantities

§ 5. A transition from a representation K to a K' can be obtained by a reversible four-function such that

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{4}$$

The transformation (4) defines a matrix field, i.e.

$$\mathbf{S}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \circ \overleftarrow{\square} .$$

A k-dimensional field $\overset{(k)}{\mathbf{A}}(\mathbf{x})$ will be denoted a tensor field if the connection between the representations $\overset{(k)}{\mathbf{A}}(\mathbf{x})$ in K and $\overset{(k)}{\mathbf{A}}'(\mathbf{x}')$ in K' is of the following form

$$\begin{aligned} \overset{(k)}{\mathbf{A}}(\mathbf{x}) &= \left(\overset{(\mathfrak{k})}{\mathbf{S}}(\mathbf{x})\, \overset{(k)}{\mathbf{A}}'(\mathbf{x}')\right)^{(k)} \qquad \text{a)} \\ \text{or} \qquad & \qquad (5) \\ \overset{(k)}{\mathbf{A}}'(\mathbf{x}') &= \left(\overset{(\mathfrak{k})}{\mathbf{S}^{+}}(\mathbf{x})\, \overset{(k)}{\mathbf{A}}(\mathbf{x})\right)^{(k)} \qquad \text{b)} \end{aligned}$$

where $\overset{(\mathfrak{k})}{\mathbf{S}}(\mathbf{x})$ and $\overset{(\mathfrak{k})}{\mathbf{S}^{+}}(\mathbf{x})$ are 2 k-dimensional matrix fields; we write down the elements explicitly, we omit, however, the variable $\mathbf{x}$ for brevity

$$\left(\overset{(\mathfrak{k})}{\mathbf{S}}\right)_{\nu_1\nu_2\ldots\nu_k\,\mu_k\mu_{k-1}\ldots\mu_1} = S_{\mu_1\nu_1} S_{\mu_2\nu_2} \ldots S_{\mu_k\nu_k} .$$

and

$$\overset{(\mathfrak{k})}{S^{+}}_{\nu_1\nu_2\ldots\nu_k\,\mu_k\mu_{k-1}\ldots\mu_1} = S^{+}_{\mu_1\nu_1} S^{+}_{\mu_2\nu_2} \ldots S^{+}_{\mu_k\nu_k}.$$

In particular for $k = 1, 2$ (5b) reduces to the well-known expressions

$$\mathbf{A}' = \widetilde{\mathbf{S}}^{-1}\,\mathbf{A}, \quad \overset{(2)}{\mathbf{A}}' = \widetilde{\mathbf{S}}^{-1}\,\overset{(2)}{\mathbf{A}}\,\mathbf{S}^{-1} .$$

We note that the above relations are valid for *covariant* tensor representations. The formulae for *contravariant* tensor representations will not be made use of.

§ 6. There are cases where we are given explicitly the representations of a matrix $\mathfrak{A}$ in various systems of reference:

$$K(\mathfrak{A}) = \overset{(k)}{\mathbf{A}}, \quad K'(\mathfrak{A}) = \overset{(k)}{\mathbf{A}}', \ldots .$$

We can form from the representation $\overset{(k)}{\mathbf{A}}'$ a quantity

$$\overset{\overline{(k)}}{\mathbf{A}} = \left(\overset{(\mathfrak{k})}{\mathbf{S}}\, \overset{(k)}{\mathbf{A}}'\right)^{(k)} . \tag{6}$$

We define that $\mathfrak{A}$ is a tensor provided we find for any pair of representations K and K' that

$$\overset{\overline{(k)}}{\mathbf{A}} = \overset{(k)}{\mathbf{A}} .$$

§ 7. We apply the $\overleftarrow{\square}$ operator to both sides of (6). We note that the right hand expression of (6) represents a sum each of the terms of which being a product containing $k+1$ factors. The differentiation gives thus $k+1$ terms.

One of the terms contains the derivative of $\overset{(k)}{\mathbf{A}}{}'(\mathbf{x}')$ into the variable $\mathbf{x}$. Making use of the rule

$$\overleftarrow{\square} = \overleftarrow{\square}' \mathbf{S}$$

we can write

$$\overset{(k)}{\mathbf{A}}{}' \circ \overleftarrow{\square} = \overset{(k+1)}{\mathbf{A}}{}' \mathbf{S},$$

therefore we find for the term obtained by differentiating $\overset{(k)}{\mathbf{A}}{}'$

$$\left(\overset{(\mathfrak{k})}{\mathbf{S}} \left(\overset{(k)}{\mathbf{A}}{}' \circ \overleftarrow{\square}\right)\right)^{(k)} = \left(\overset{(\mathfrak{k})}{\mathbf{S}} \left(\overset{(k+1)}{\mathbf{A}}{}' \mathbf{S}\right)\right)^{(k)} = \left(\overset{(\mathfrak{k}+1)}{\mathbf{S}} \overset{(k+1)}{\mathbf{A}}{}'\right)^{(k+1)} = \overset{(\overline{k+1})}{\mathbf{A}} .$$

The terms obtained from differentiating the factors of the elements of $\overset{(\mathfrak{k})}{\mathbf{S}}$ give contributions of the form

$$c_k^{-l} \left[\left(c_k^l \overset{(\overline{k})}{\mathbf{A}}\right) \mathbf{S}^{-1} \overset{(3)}{\mathbf{S}}\right].$$

Taking the $k+1$ terms together we find.

$$\overset{(\overline{k})}{\mathbf{A}} \circ \overleftarrow{\square} - \overset{(\overline{k+1})}{\mathbf{A}} + \sum_{l=1}^{k} c_k^{-l} \left\{\left(c_k^l \overset{(\overline{k})}{\mathbf{A}}\right) \mathbf{S}^{-1} \overset{(3)}{\mathbf{S}}\right\}. \tag{7}$$

In place of (7) we can also write

$$\overset{(k+1)}{\mathbf{A}} - \overset{(\overline{k+1})}{\mathbf{A}} = \left(\overset{(k)}{\mathbf{A}} - \overset{(\overline{k})}{\mathbf{A}}\right) \circ \overleftarrow{\square} + \sum_{l=1}^{k} c_k^{-l} \left\{\left(c_k^l \overset{(\overline{k})}{\mathbf{A}}\right) \mathbf{S}^{-1} \overset{(3)}{\mathbf{S}}\right\}. \tag{8}$$

We note that provided $\overset{(k)}{\mathbf{A}}$ is a tensor, we have $\overset{(k)}{\mathbf{A}} = \overset{(\overline{k})}{\mathbf{A}}$ and thus the right hand expression becomes simpler, however,

$$\overset{(k+1)}{\mathbf{A}} = \overset{(k)}{\mathbf{A}} \circ \overleftarrow{\square}$$

is in general not a tensor, since the sum on the right of (8), as a rule, does not vanish.

The Christoffel brackets

§ 8. Applying (8) to the tensor $\mathbf{g}$ we find remembering that $\mathbf{g} = \overline{\mathbf{g}} = (12)\,\mathbf{g}$

$$\overset{(3)}{\mathbf{g}} - \overline{\overset{(3)}{\mathbf{g}}} = (\mathbf{1} + (12))\,\mathbf{g}\,\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}}. \tag{9}$$

$\overset{(3)}{\mathbf{S}}$ being a second derivative is symmetric in the second and third suffix, thus

$$(23)\overset{(3)}{\mathbf{S}} = \overset{(3)}{\mathbf{S}},$$

we have therefore also

$$(23)\left(\mathbf{g}\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}}\right) = \mathbf{g}\,\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}}, \tag{10}$$

as the operator (23) acts only on the last two suffices of $\overset{(3)}{\mathbf{S}}$ while the multiplication with $\mathbf{g}\mathbf{S}^{-1}$ gives only a summation over the first suffix. Because of (10) and remembering that $(12)\,(23) = (132) = c_3^{-1}$ we can write in the place of (9) also

$$\overset{(3)}{\mathbf{g}} - \overline{\overset{(3)}{\mathbf{g}}} = (\mathbf{1} + c_3^{-1})\,\mathbf{g}\,\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}}. \tag{11}$$

Applying the operation $1/2\,\pi_3 = 1/2\,(1 - c_3^{-1} + c_3^{-2})$ to both sides of (11) we find

$$\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}} = \mathbf{g}\,\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}} \tag{12}$$

and multiplying Eq. (12) with $\mathbf{S}\mathbf{g}^{-1}$ we have

$$\overset{(3)}{\mathbf{S}} = \mathbf{S}\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right), \tag{13}$$

where

$$\overset{(3)}{\mathbf{C}} = \frac{1}{2}\pi_3\overset{(3)}{\mathbf{g}} \quad \text{and} \quad \overline{\overset{(3)}{\mathbf{C}}} = \frac{1}{2}\pi_3\overline{\overset{(3)}{\mathbf{g}}}. \tag{14}$$

$\overset{(3)}{\mathbf{C}}$ is the Christoffel bracket symbol — comparing it with the usual notation we find

$$\overset{(3)}{C}_{\lambda\mu\nu} = \begin{bmatrix} \mu\nu \\ \lambda \end{bmatrix}.$$

In the literature one finds quantities $\mathbf{g}^{-1}\overset{(3)}{\mathbf{C}} = \overset{(3)}{\mathcal{C}}$ where

$$\overset{(3)}{\mathcal{C}}_{\lambda\mu\nu} = \begin{Bmatrix} \mu\nu \\ \lambda \end{Bmatrix}.$$

Reversing Eq. (14) we can also write

$$\overset{(3)}{\mathbf{g}} = (\mathbf{1} + c_3^{-1})\overset{(3)}{\mathbf{C}}, \tag{15}$$

in place of c_3^{-1} we can also write the operator (12).

The covariant differentiation

§ 9. Introducing Eq. (12) into (8) we can write

$$\overset{(k+1)}{\mathbf{A}} - \overset{\overline{(k+1)}}{\mathbf{A}} = \left(\overset{(k)}{\mathbf{A}} - \overset{\overline{(k)}}{\mathbf{A}}\right) \circ \overleftarrow{\square} + \sum_{l=1}^{k} c_k^{-l} \left\{\left(c_k^l \overset{\overline{(k)}}{\mathbf{A}}\right) \cdot \left(\overset{(3)}{\mathbf{C}} - \overset{\overline{(3)}}{\mathbf{C}}\right)\right\}. \tag{16}$$

Supposing $\overset{(k)}{\mathbf{A}}$ to be a tensor, i.e. supposing $\overset{(k)}{\mathbf{A}} = \overset{\overline{(k)}}{\mathbf{A}}$ we can also write in place of (16)

$$\overset{(k+1)}{\mathbf{B}} = \overset{\overline{(k+1)}}{\mathbf{B}},$$

where

$$\overset{(k+1)}{\mathbf{B}} = \operatorname{Grad} \overset{(k)}{\mathbf{A}} = \overset{(k)}{\mathbf{A}} \circ \overleftarrow{\square} - \sum_{l=1}^{k} c_k^{-l} \left\{\left(c_k^l \overset{(k)}{\mathbf{A}}\right) \cdot \overset{(3)}{\mathbf{C}}\right\}. \tag{17}$$

Eq. (17) defines thus the $k+1$-dimensional Grad of a k-dimensional tensor in a covariant manner.

We can also introduce

$$\operatorname{Div} \overset{(k)}{\mathbf{A}} = \left(\left(\left[\operatorname{Grad} \overset{(k)}{\mathbf{A}}\right] \mathbf{g}^{-1}\right)\right), \tag{18}$$

as the covariant Div operation.

Applying (17) to the tensor $\mathbf{g}$ we find

$$\operatorname{Grad} \mathbf{g} = 0, \tag{19}$$

and thus with the help of (18) we have

$$\operatorname{Div} \mathbf{g} = 0.$$

Homogeneous regions

§ 10. A region $\mathfrak{R}$ is taken to be homogeneous if it admits of representations of $\mathfrak{g}$

$$\mathbf{g}' = K'(\mathfrak{g}) = \text{independent of } \mathbf{x}'. \tag{20}$$

Suppose

$$\mathbf{g}(\mathbf{x}) = K(\mathfrak{g})$$

to be the representation of $\mathfrak{g}$ in an arbitrary system of reference K. If region $\mathfrak{R}$ is a homogeneous one, then it exists a coordinate transformation

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \tag{21}$$

so that

$$\tilde{\mathbf{S}}(\mathbf{x})\, \mathbf{g}'\, \mathbf{S}(\mathbf{x}) = g(\mathbf{x}), \tag{22}$$

with

$$\mathbf{S}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \circ \overleftarrow{\square}. \tag{23}$$

From (20) it follows that $\overset{(3)}{\mathbf{C}}{}'(\mathbf{x}') = 0$ and therefore also $\overline{\overset{(3)}{\mathbf{C}}}(\mathbf{x}) = 0$ and we may write using relation (13),

$$\overset{(3)}{\mathbf{S}}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \cdot \overset{(3)}{\mathbf{C}}(\mathbf{x}). \tag{24}$$

Relation (24) is a differential equation with the help of which $\mathbf{S}(\mathbf{x})$ and also $\mathbf{f}(\mathbf{x})$ can be determined if the region $\mathfrak{R}$ is homogeneous indeed.

Whether or not (24) possesses solutions can be ascertained by differentiating (24) into $\mathbf{x}$. We write

$$\overset{(3)}{\mathbf{S}} \circ \overleftarrow{\square} = \overset{(4}{\mathbf{S}},$$

where we omit to write down the variable $\mathbf{x}$ explicitly. We find with the help of (3)

$$\overset{(4)}{\mathbf{S}} = \mathbf{S} \cdot \overset{(4)}{\mathbf{C}} + c_4 \left(c_3^{-1} \left(\overset{(3)}{\mathbf{S}} - \mathbf{S} \cdot \overset{(3)}{\mathbf{g}} \right) \cdot \overset{(3)}{\mathbf{C}} \right).$$

Inserting $\overset{(3)}{\mathbf{S}}$ from (24) and $\overset{(3)}{\mathbf{g}}$ from (15) we find

$$\overset{(4)}{\mathbf{S}} = \mathbf{S} \cdot \overset{(4)}{\mathbf{C}} - c_4 \left[c_3^{-1} \left(\mathbf{S} \cdot c_3^{-1} \overset{(3)}{\mathbf{C}} \right) \cdot \overset{(3)}{\mathbf{C}} \right].$$

Using the symmetry of $\overset{(3)}{\mathbf{C}}$ we can simplify the second term and find as the result of a short calculation

$$\overset{(4)}{\mathbf{S}} = \mathbf{S} \cdot \left(\overset{(4)}{\mathbf{C}} - (24) \left(\overset{(3)}{\mathbf{C}} \cdot \overset{(3)}{\mathbf{C}} \right) \right). \tag{25}$$

If $\overset{(4)}{\mathbf{S}}$ is to be the third derivative of the transformation function $\mathbf{f}(\mathbf{x})$ then it has to be symmetric in the last three suffices, we expect thus e.g.

$$(\mathbf{1} - (24)) \overset{(4)}{\mathbf{S}} = 0. \tag{26}$$

Since it follows from Eq. (24) that $\overset{(4)}{\mathbf{S}}$ is automatically symmetric in the operator (23) it follows also that provided Eq. (26) is fulfilled, it is also symmetric in (34) since we have

$$(34) = (23)\,(24)\,(23)\,.$$

Thus provided (26) is fulfilled, then $\overset{(4)}{\mathbf{S}}$ is symmetrical in the three last suffices. Inserting (25) into (26) we find

$$(\mathbf{1} - (24))\left(\overset{(4)}{\mathbf{C}} + \overset{\sim}{\overset{(3)}{\mathbf{C}}} \cdot \overset{(3)}{\mathbf{C}}\right) = 0\,.$$

We can also write

$$(\mathbf{1} - (24))\overset{(4)}{\mathbf{C}} = \frac{1}{2}(\mathbf{1} - (24))\,\pi_3 \overset{(4)}{\mathbf{g}} = \frac{1}{2}\pi_4 \overset{(4)}{\mathbf{g}}\,.$$

Defining thus a four dimensional quantity

$$\overset{(4)}{\mathbf{R}}(\mathbf{x}) = \frac{1}{2}\pi_4 \overset{(4)}{\mathbf{g}}(\mathbf{x}) + (\mathbf{1} - (24))\left(\overset{\sim}{\overset{(3)}{\mathbf{C}}}(\mathbf{x}) \cdot \overset{(3)}{\mathbf{C}}(\mathbf{x})\right) \tag{27}$$

we see that a necessary condition for a region $\mathfrak{R}$ to be homogeneous is that

$$\overset{(4)}{\mathbf{R}}(\mathbf{x}) = 0 \quad \text{inside} \quad \mathfrak{R}\,. \tag{28}$$

Indeed unless (28) is fulfilled the differential equation (24) does not admit of solutions.

§ 11. The argument can also be reversed. Differentiating (24) successively into $\mathbf{x}$ we obtain a recursion formula for the derivatives $\overset{(2+l)}{\mathbf{S}}(\mathbf{x})$ of $\mathbf{S}(\mathbf{x})$. Expanding $\mathbf{S}(\mathbf{x})$ or $\mathbf{f}(\mathbf{x})$ into powers of $\mathbf{x}$ using the derivatives thus obtained we obtain a solution of (22), (23) provided (28) is fulfilled

We see therefore that *(28) is a necessary and also sufficient condition for a region $\mathfrak{R}$ to be homogeneous.*

Almost straight representations

§ 12. In an inhomogeneous region the system (22), (23) admits of no solutions; in such a region one can try to introduce *almost straight* coordinates, i.e. one may try to find a representation K' such that $\mathbf{g}'(\mathbf{x}')$ is almost constant.

For any value of $\mathbf{g}(\mathbf{x})$ we are led to (24) which gives a differential equation which, if it admits of solution leads to the transformation (21) defining straight coordinates. Differentiating (24), one is led to (25); if $\overset{(4)}{\mathbf{S}}$ thus obtained is symmetric in the last three suffices, then (22) and (23) admits of solution.

The matrix $\overset{(4)}{\mathbf{S}}$ as obtained from (25) is in any case symmetric with respect to the operator (23); we can introduce a symmetric matrix

$$\overset{(4)}{\langle \mathbf{S} \rangle} = \frac{1}{3}(\mathbf{1} + (24) + (34))\overset{(4)}{\mathbf{S}}.$$

The latter is symmetric in all the three suffices. Since $\overset{(4)}{\langle \mathbf{S} \rangle}$ is equal to $\overset{(4)}{\mathbf{S}}$ in a homogeneous region we can take $\overset{(4)}{\langle \mathbf{S} \rangle}$ to define an approximate solution of (22), (23).

The difference between the two solutions can be written

$$\delta\overset{(4)}{\mathbf{S}} = \overset{(4)}{\langle \mathbf{S} \rangle} - \overset{(4)}{\mathbf{S}} = -\frac{1}{3}[(\mathbf{1} - (24)) + (\mathbf{1} - (34))]\overset{(4)}{\mathbf{S}},$$

however,

$$(\mathbf{1} - (34)) = (23)(\mathbf{1} - (24))(23) \equiv (23)(\mathbf{1} - (24))$$

therefore we can also write

$$\delta\overset{(4)}{\mathbf{S}} = -\frac{1}{3}(\mathbf{1} + (23))(\mathbf{S} \cdot \overset{(4)}{\mathbf{R}}). \tag{29}$$

The function $\langle \mathbf{S}(\mathbf{x}) \rangle$ giving the approximate solution can be obtained by integration; we find

$$\langle \mathbf{S}(\mathbf{x}) \rangle = \mathbf{S}(0) + \overset{3)}{\mathbf{S}}(0)\,\mathbf{x} + \int_0^{\mathbf{x}} \left(\overset{(4)}{\langle \mathbf{S}(\mathbf{x}') \rangle}((\mathbf{x} - \mathbf{x}') \circ d\mathbf{x}') \right)^{(2)} \tag{30}$$

(the integration can be taken on any path from 0 to $\mathbf{x}$) the above transformation function satisfies thus approximately (22) and (23) in the vicinity of $\mathbf{x} = 0$.

§ 13. We may write

$$\mathbf{g}'(\mathbf{x}') = \mathbf{g}' + \delta\mathbf{g}'(\mathbf{x}')$$

for the representation of $\mathfrak{g}$ in the almost straight systems of reference. We have thus

$$(\langle \tilde{\mathbf{S}}(\mathbf{x}) \rangle (\mathbf{g}' + \delta\mathbf{g}'(\mathbf{x}'))\langle \mathbf{S}(\mathbf{x}) \rangle) = \mathbf{g}(\mathbf{x}). \tag{31}$$

Differentiating (31) into $\mathbf{x}$ we find for $\mathbf{x} = 0$

$$\delta\overset{(3)}{\mathbf{g}'}(0) = 0. \tag{32}$$

Differentiating twice with the help of (30) and (32) we find for $\mathbf{x} = 0$

$$(\mathbf{1} + (12))\,(\mathbf{g}\mathbf{S}^{-1}\,\delta\overset{(4)}{\mathbf{S}}) = -\,\delta\overset{\overline{(4)}}{\mathbf{g}}\,,$$

therefore with the help of (31) remembering that

$$(\mathbf{1} + (12))\,(23) \equiv 0 \mod \overset{(4)}{\mathbf{R}}\,,$$

we find

$$\delta\overset{\overline{(4)}}{\mathbf{g}} = \frac{1}{3}\,(\mathbf{1} + (12))\overset{(4)}{\mathbf{R}}$$

and transforming to K'

$$\delta\overset{(4)}{\mathbf{g}'} = \frac{1}{3}\,(\mathbf{1} + (12))\overset{(4)}{\mathbf{R}'},$$

thus we have

$$\mathbf{g}'(\mathbf{x}') = \mathbf{g}' + \frac{1}{6}\,(\mathbf{1} + (12))\overset{(4)}{\mathbf{R}'}\,\mathbf{x}'^2 + \ldots .$$

The above representation is as straight as possible since the second derivative of $\mathbf{g}'(\mathbf{x}')$ must be at last of the order of the elements of $\overset{(4)}{\mathbf{R}'}$.

Tensor character of the Riemann—Christoffel tensor

§ 14. The quantity $\overset{(4)}{\mathbf{R}}(\mathbf{x})$ defined by (27) is a tensor; apart from a change in convention it is equal to the Riemann—Christoffel tensor. One finds

$$\overset{(4)}{\mathfrak{R}}(\mathbf{x}) = -\,(23)\overset{(4)}{\mathbf{R}}(\mathbf{x})\,,$$

where $\overset{(4)}{\mathfrak{R}}(\mathbf{x})$ is the usual form of the Riemann—Christoffel tensor. We prove the tensor character of $\overset{(4)}{\mathbf{R}}$ using our formalism. We start from a relation which is obtained applying the operator $\overleftarrow{\square}$ to $\mathbf{g}\mathbf{S}^{-1}\mathbf{S} = \mathbf{g}$ with the help of (2) one finds thus

$$c_3^{-1}(\mathbf{g}\,\mathbf{S}^{-1} \circ \overleftarrow{\square}) = \left[c_3^{-1}\left(\overset{(3)}{\mathbf{g}} - \mathbf{g}\,\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}}\right)\right]\mathbf{S}^{-1}\,. \tag{33}$$

Applying $\overleftarrow{\square}$ to the relation (12) one finds with the help of (2) and (33)

$$\left(\overset{(3)}{\mathbf{C}} - \overset{\overline{(3)}}{\mathbf{C}}\right) \circ \overleftarrow{\square} = \mathbf{g}\,\mathbf{S}^{-1}\overset{(4)}{\mathbf{S}} + c_4\left\{\left[c_3^{-1}\left(\overset{(3)}{\mathbf{g}} - \mathbf{g}\,\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}}\right)\right]\mathbf{S}^{-1}\overset{(3)}{\mathbf{S}}\right\}\,.$$

With the help of (15) and (12) one can rewrite the second term on the right band side and find

$$c_4\{\ldots\} = c_4(12)\left\{\left(\overset{(3)}{\mathbf{C}} + \overline{\overset{(3)}{\mathbf{C}}}\right)\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right)\right\}.$$

As the $\overset{(3)}{\mathbf{C}}$ are symmetric in the operator (23) the expression in the $\{\ldots\}$ bracket is symmetric in the operator (34) and one can write for the operator before the bracket

$$c_4(12) \equiv (1234)(12)(34) = (24).$$

We have thus

$$\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right)\circ\overleftarrow{\square} = \mathbf{g}\,\mathbf{S}^{-1}\overset{(4)}{\mathbf{S}} + (24)\left\{\left(\overset{(3)}{\mathbf{C}} + \overline{\overset{(3)}{\mathbf{C}}}\right)\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right)\right\}. \tag{34}$$

Applying Eq. (16) to $\overset{(3)}{\mathbf{C}}$ one finds

$$\begin{aligned}\overset{(4)}{\mathbf{C}} - \overline{\overset{(4)}{\mathbf{C}}} &= \left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right)\circ\overleftarrow{\square} + \overline{\overset{(3)}{\mathbf{C}}}\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right) + \\ &+ c_3^{-1}\left[\widetilde{\overset{(3)}{\mathbf{C}}}\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right)\right] + c_3\left[\left(c_3^{-1}\overline{\overset{(3)}{\mathbf{C}}}\right)\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right)\right].\end{aligned} \tag{35}$$

With the help of (1) we find further

$$\widetilde{\overset{(3)}{\mathbf{C}}}\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right) = c_4^2\left[\left(\overset{(3)}{\mathbf{C}} - \widetilde{\overset{(3)}{\mathbf{C}}}\right)\cdot\overline{\overset{(3)}{\mathbf{C}}}\right] \tag{36}$$

and

$$\left(c_3^{-1}\overline{\overset{(3)}{\mathbf{C}}}\right)\cdot\left(\overset{(3)}{\mathbf{C}} - \overline{\overset{(3)}{\mathbf{C}}}\right) = c_4^2\left[\left(\overset{(3)}{\mathbf{C}} - \widetilde{\overset{(3)}{\mathbf{C}}}\right)\cdot\widetilde{\overset{(3)}{\mathbf{C}}}\right]. \tag{37}$$

The expression in the [ ]-bracket of (36) is symmetric with respect to (34) therefore we have

$$c_3^{-1}c_4^2[\ ] \equiv (24)\left[\left(\overset{(3)}{\mathbf{C}} - \widetilde{\overset{(3)}{\mathbf{C}}}\right)\cdot\overline{\overset{(3)}{\mathbf{C}}}\right]. \tag{38}$$

Similarly the expression in the [ ]-bracket of Eq. (37) is symmetric with respect to the operator (12) thus we find

$$c_3 c_4^2 \equiv (14).$$

We note further

$$(\mathbf{1} - (24))(14)\left[\left(\overset{(3)}{\mathbf{C}} - \widetilde{\overset{(3)}{\mathbf{C}}}\right)\cdot\widetilde{\overset{(3)}{\mathbf{C}}}\right] = 0. \tag{39}$$

So as to obtain expressions of the form of $\overset{(4)}{\mathbf{R}}$ we note that

$$(\mathbf{1}-(24))\overset{(4)}{\mathbf{C}}=\frac{1}{2}\pi_4\overset{(4)}{\mathbf{g}}. \tag{40}$$

Thus applying (1 — (24)) to both sides of (35) we obtain with the help of (26), (34), (38), (39) and (40) the following relation

$$\frac{1}{2}\pi_4\left(\overset{(4)}{\mathbf{g}}-\overline{\overset{(4)}{\mathbf{g}}}\right)=(\mathbf{1}-(24))\left\{-\left(\overset{\sim}{\overset{(3)}{\mathbf{C}}}+\overline{\overset{(3)}{\mathbf{C}}}\right)\cdot\left(\overset{(3)}{\mathbf{C}}-\overline{\overset{(3)}{\mathbf{C}}}\right)+\right.$$

$$\left.+\overline{\overset{(3)}{\mathbf{C}}}\cdot\left(\overset{(3)}{\mathbf{C}}-\overline{\overset{(3)}{\mathbf{C}}}\right)-\left(\overset{\sim}{\overset{(3)}{\mathbf{C}}}-\overline{\overset{(3)}{\mathbf{C}}}\right)\cdot\overline{\overset{(3)}{\mathbf{C}}}\right\}.$$

The terms in the { }-bracket reduce to

$$\{\ \}=-\overset{\sim}{\overset{(3)}{\mathbf{C}}}\cdot\overset{(3)}{\mathbf{C}}+\overline{\overset{\sim}{\overset{(3)}{\mathbf{C}}}}\cdot\overline{\overset{(3)}{\mathbf{C}}}$$

and therefore we find

$$\frac{1}{2}\pi_4\overset{(4)}{\mathbf{g}}+(\mathbf{1}-(24))\left(\overset{\sim}{\overset{(3)}{\mathbf{C}}}\cdot\overset{(3)}{\mathbf{C}}\right)=\frac{1}{2}\pi_4\overline{\overset{(4)}{\mathbf{g}}}+(\mathbf{1}-(24))\overset{\sim}{\overset{(3)}{\mathbf{C}}}\cdot\overline{\overset{(3)}{\mathbf{C}}},$$

or using the notation (27)

$$\overset{(4)}{\mathbf{R}}=\overline{\overset{(4)}{\mathbf{R}}}.$$

Thus in accord with § 6 $\overset{(4)}{\mathbf{R}}$ is a tensor.

Symmetries of the $\overset{(4)}{\mathbf{R}}$ tensor

§ 15. We note that

$$(13)(24)=\mathbf{1}\mod\overset{\sim}{\overset{(3)}{\mathbf{C}}}\cdot\overset{(3)}{\mathbf{C}},$$

thus

$$(\mathbf{1}-(24))=\frac{1}{2}(\mathbf{1}-(24))(\mathbf{1}+(13)(24))=\frac{1}{2}(\mathbf{1}-(24))(\mathbf{1}-(13))=\pi_4 \mod\overset{\sim}{\overset{(3)}{\mathbf{C}}}\cdot\overset{(3)}{\mathbf{C}}$$

and we find in place of (27) also

$$\overset{(4)}{\mathbf{R}}=\frac{1}{2}(\mathbf{1}-(24))(\mathbf{1}-(13))\left(\overset{(4)}{\mathbf{g}}+\overset{\sim}{\overset{(3)}{\mathbf{C}}}\cdot\overset{(3)}{\mathbf{C}}\right) \tag{41}$$

or

$$\overset{(4)}{\mathbf{R}}=\frac{1}{2}\pi_4\left(\overset{(4)}{\mathbf{g}}+\overset{\sim}{\overset{(3)}{\mathbf{C}}}\cdot\overset{(3)}{\mathbf{C}}\right). \tag{42}$$

(41) and (42) are of course identic with (27). From the above relations the symmetries of $\overset{(4)}{\mathbf{R}}$ can be obtained in a simple manner. From (41) we see immediately that

$$(24)\overset{(4)}{\mathbf{R}} = (13)\overset{(4)}{\mathbf{R}} = -\overset{(4)}{\mathbf{R}}, \tag{43a}$$

so

$$(13)(24)\overset{(4)}{\mathbf{R}} = \overset{(4)}{\mathbf{R}}. \tag{43b}$$

Further remembering that according to the definitions

$$c_4\,\pi_4 = -\pi_4,\ (1234)\overset{(4)}{\mathbf{R}} = -\overset{(4)}{\mathbf{R}}. \tag{43c}$$

Multiplying (42) from the left by $(12)(34)(1234) = (13)$ and remembering (43a) and (43c) we find

$$(12)(34)\overset{(4)}{\mathbf{R}} = \overset{(4)}{\mathbf{R}}, \tag{43d}$$

thus

$$(12) \equiv (34) \mod \overset{(4)}{\mathbf{R}}.$$

Furthermore one finds as the result of a simple calculation that

$$(\mathbf{1} + c_3 + c_3^2)\,\pi_4 = 0 \quad \left(\mathrm{mod}\overset{(4)}{\mathbf{g}}\right)$$

therefore

$$(\mathbf{1} + c_3 + c_3^2)\overset{(4)}{\mathbf{R}} = 0.$$

Relations (43a) and (43d) give all the symmetries of $\overset{(4)}{\mathbf{R}}$.

The reduced form of the $\overset{(4)}{\mathbf{R}}$ tensor

§ 16. The Riemann—Christoffel tensor $\overset{(4)}{\mathbf{R}}$ can be contracted and thus we obtain the following tensor quantities

$$\overset{(2)}{\mathbf{R}} = \left(\left(\overset{(4)}{\mathbf{R}}\,\mathbf{g}^{-1}\right)\right) = \left(\left(\mathbf{g}^{-1}\,\overset{(4)}{\mathbf{R}}\right)\right). \tag{44}$$

The contraction leads to the same tensor $\overset{(2)}{\mathbf{R}}$ whether we multiply from the left or from the right by $\mathbf{g}^{-1}$, this can be verified with the help of (43b). Further we have

$$R = \left(\left(\mathbf{g}^{-1}\,\overset{(2)}{\mathbf{R}}\right)\right) = \left(\mathbf{g}^{(-2)}\,\overset{(4)}{\mathbf{R}}\right)^{(4)},$$

where we write $\mathbf{g}^{(-2)} = \mathbf{g}^{-1} \circ \mathbf{g}^{-1}$.

From (44) we find with the help of (17) and (18)

$$\operatorname{Div} \overset{(2)}{\mathbf{R}} = \left(\left(\mathbf{g}^{-1} \operatorname{Grad} \overset{(4)}{\mathbf{R}}\right)^{(2)} \mathbf{g}^{-1}\right)^{(2)} = \left(\mathbf{g}^{(-2)} \left((35) \operatorname{Grad} \overset{(4)}{\mathbf{R}}\right)\right)^{(4)}. \tag{45}$$

We find also

$$\operatorname{Div}(\mathbf{g} R) = \operatorname{Grad} R = \left(\mathbf{g}^{(-2)} \operatorname{Grad} \overset{(4)}{\mathbf{R}}\right)^{(4)}. \tag{46}$$

We can reduce (46) to a simple form. With the help of (24) we can write

$$\operatorname{Grad} \overset{(4)}{\mathbf{R}} = \frac{1}{2} \left(\mathbf{1} - (24) - (13) + (13)(24)\right) \overset{(5)}{\boldsymbol{\gamma}}, \tag{47}$$

where

$$\overset{(5)}{\boldsymbol{\gamma}} = \left(1 + (34) + (35)\right) \left\{ \frac{1}{3} \overset{(5)}{\mathbf{g}} + \overset{(3)}{\tilde{\mathbf{C}}} \cdot \overset{(4)}{\mathbf{g}} - (13) \overset{(4)}{\tilde{\mathbf{g}}} \cdot \overset{(3)}{\mathbf{C}} - \overset{(4)}{\mathbf{g}} \cdot \overset{(3)}{\mathbf{C}} - 2 \overset{(3)}{\tilde{\mathbf{C}}} \cdot \overset{(3)}{\mathbf{C}} \cdot \overset{(3)}{\mathbf{C}} \right\}$$

(the calculation of $\overset{(5)}{\gamma}$ is published elsewhere [6]). It is verified easily that

$$(12) \equiv (34) \equiv (35) \equiv 1 \mod \overset{(5)}{\boldsymbol{\gamma}}. \tag{48}$$

Introducing (47) into (46) we can simplify the expression thus obtained remembering that $\mathbf{g}^{(-2)}$ has the following symmetries

$$(12) \equiv (34) \equiv (13)(24) \equiv 1 \mod \mathbf{g}^{(-2)}. \tag{49}$$

Therefore we can apply any of the above operators to $\operatorname{Grad} \overset{(4)}{\mathbf{R}}$ — or to any terms of $\operatorname{Grad} \overset{(4)}{\mathbf{R}}$ without changing the value of the sum. Introducing thus (47) into (46) we obtain four terms; we find that the first and fourth are equal since

$$\left(\mathbf{g}^{-(2)} \overset{(5)}{\boldsymbol{\gamma}}\right)^{(4)} = \left(\mathbf{g}^{(-2)} \left((13)(23) \overset{(5)}{\boldsymbol{\gamma}}\right)\right)^{(4)}. \tag{50}$$

We find also that the second and third terms are equal. Indeed,

$$\left(\mathbf{g}^{-(2)} \left((13) \overset{(5)}{\boldsymbol{\gamma}}\right)\right)^{(4)} = \left(\mathbf{g}^{-(2)} \left((24) \overset{(5)}{\boldsymbol{\gamma}}\right)\right)^{(4)},$$

thus with the help of (46), (47) and (50) we find

$$\operatorname{Grad} R = 2 \left(\mathbf{g}^{(-2)} \left(\left(\mathbf{1} - (24)\right) \overset{(5)}{\boldsymbol{\gamma}}\right)\right)^{(4)}. \tag{51}$$

Similarly introducing (47) into (45) we find, making use of (48) also

$$\left(\mathbf{g}^{(-2)}\left((35)\,\mathrm{Grad}\overset{(4)}{\mathbf{R}}\right)\right)^{(4)} = \left(\mathbf{g}^{(-2)}\left(\left(\mathbf{1}-(24)\right)\overset{(5)}{\boldsymbol{\gamma}}\right)\right)^{(4)} - \\ - \left(\mathbf{g}^{(-2)}\left((35)\left(\mathbf{1}-(24)\right)(13)\overset{(5)}{\boldsymbol{\gamma}}\right)\right)^{(4)}. \tag{52}$$

The second term on the right vanishes as can be seen in the following manner. The operator inside the bracket can also be written

$$(35)\left(\mathbf{1}-(24)\right)(13) \equiv (35)\left(\mathbf{1}-(24)\right)(13)(35) = (15) - (24)(15) \mod \overset{(5)}{\boldsymbol{\gamma}}.$$

However, under the summation we can replace (24) (15) by (13) (15) because of Eq. (49), further

$$(13)(15) \equiv (13)(15)(35) = (15).$$

Thus the second term under the sum cancels the first; thus the second term on the right of (52) can be omitted. We find therefore comparing (51) and (52)

$$\mathrm{Div}\overset{(2)}{\mathbf{R}} = \frac{1}{2}\,\mathrm{Grad}\,R,$$

and also

$$\mathrm{Div}\left(\overset{(2)}{\mathbf{R}} - \frac{1}{2}\,\mathbf{g}\,R\right) = 0.$$

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ПРИНЦИП ЛОРЕНЦА И ОБЩАЯ ТЕОРИЯ ОТНОСИТЕЛЬНОСТИ
Часть VI.

Л. ЯНОШИ и А. ВЕРНЕР

Резюме

В публикациях, занимающихся принципом Лоренца, примененным к общей теории относительности, мы шаг за шагом ввели одно замечание, которое особенно полезно при отражении физического смысла. В настоящей работе дается обзор данного замечания и использованной техники. Формализм освобожден от нескольких незначительных непоследовательностей, имеющих место в предыдущих работах. На основе данного формализма показывается вывод простых выражений для символов Кристофела, тензора Римана—Кристофела и др.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 371—385 (1969)

ON THE AZIMUTHAL EFFECTS OF TWO PRONG πN AND THREE PRONG DD EVENTS PRODUCED BY 17.2 GeV/c π^- MESONS

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(Received 13. II. 1969)

A deviation from isotropy in the azimuthal angular distribution of secondary particles of two and three prong pion-nucleon interactions is observed in emulsion at 17.2 GeV/c. Attempts are made using Monte Carlo calculations to interpret these azimuthal effects on the basis of (i) the kinematics of reactions $\pi^- p \to p \pi^-$ ($k \pi^\circ$) and $\pi^- p \to n \pi^+ \pi^-$ ($k' \pi^\circ$) and (ii) the diffraction dissociation of pions on nuclei. A new method is given for the selection of the contribution of DD interactions from three prong events. Using this method $\lambda_{DD} = (1065 \pm 288)\, m$ and $\lambda_{DD} = (113 \pm 13)\, m$ are obtained for the mean free path of the diffraction dissociation in emulsion at $\sim$7 and 17.2 GeV/c, respectively. The possible influence of the DECK effect in the case of three prong events is also investigated.

1. Introduction

In recent years several attempts have been made to detect azimuthal effects in the angular distribution of secondary particles generated in high energy interactions at accelerator energies (e.g. [1]) as well as in the cosmic ray energy region (e.g. [2]). The importance of these studies, as was pointed out in [3—7], is given by the following: If, in the distribution of azimuthal angles of secondary particles, a deviation from isotropy were to be found this fact would provide important information about the production mechanism — especially about the production of fireballs or other resonant states associated with large angular momenta.

The aim of the present analysis is to investigate the possible azimuthal effects in pion-nucleon (πN) interactions at 17.2 GeV/c. The work can be taken as a continuation of a similar investigation, carried out earlier on πN interactions at a primary momentum of $\sim$7 GeV/c [8]. In that paper an azimuthal anisotropy in the two and three prong events was reported, while at higher multiplicities such an effect could not be detected.

Since the results of different authors [1], [6—11] concerning the question of azimuthal effects are rather contradictory or inconclusive it seemed to be worth while to check our earlier results and carry out a new measurement.

2. Experimental

For the present analysis a total of 928 inelastic πN interactions found in Ilford G5 plates by scanning along the track (for details of the scanning and measurement see [12]—[13]) was used.* The interactions satisfied appropriate selection criteria summarised in [14].** This sample of events contains mostly the inelastic interactions of the primary pion on free and quasi-free nucleons and to a certain degree the so-called diffraction dissociation (DD) events which also satisfy the above selection criteria. Thus, apart from the DD events, our sample is almost free from the more complex pion-nucleus ($\pi\mathfrak{N}$) interactions, which can mask the possible azimuthal effects.

It is remarkable that, in general, the complex nuclear interactions are not separated in the majority of the samples used for investigating azimuthal anisotropy.

3. Results

In order to detect a possible azimuthal anisotropy the frequently applied "method of consecutive angles" [5], [15] was used. According to this method, the observed distribution of consecutive azimuthal angles (i.e. of the separation angle $\Phi = \Phi_{i+1} - \Phi_i$ between the azimuthal angles of the successive prongs) for events of a given multiplicity has to be compared with the corresponding random distribution.

In the case of azimuthal isotropy the probability distribution of Φ for n-prong events is given by the following expression:

$$N(\Phi)\, d\Phi = (n-1)\left(1 - \frac{\Phi}{2\pi}\right)^{n-2} \frac{d\Phi}{2\pi} . \tag{1}$$

The observed distributions together with the random ones obtained from expression (1) for events having different numbers of prongs ($n = 2, 3, 4, 5, 6, 7$) are shown in Fig. 1. The curve for $n \geq 7$ was obtained by calculating the weighted average of the curves for $n = 7, 8, 9, 10, 11, 12$, where the weighting factors were the number of events of the corresponding multiplicities.

A χ^2-test shows (see Table I) that in cases of the two and three prong events, a significant deviation from the azimuthal isotropy exists, whereas at higher multiplicities such deviations could not be detected.

These results are in agreement with those obtained earlier at $\sim$7 GeV/c. In neither case do our samples contain events lying within 50 μ, (in processed emulsion) from either surface of the emulsion, avoiding thus the biases due to the omission of dipping shower tracks. The fact that these unbiased samples

* We are indebted to the Alma-Ata group for kindly providing us with their data concerning the 2-, 3- and 4-prong events, which are included in the above sample.

** These selection criteria were the same as those applied for the $\sim$7 GeV/c material.

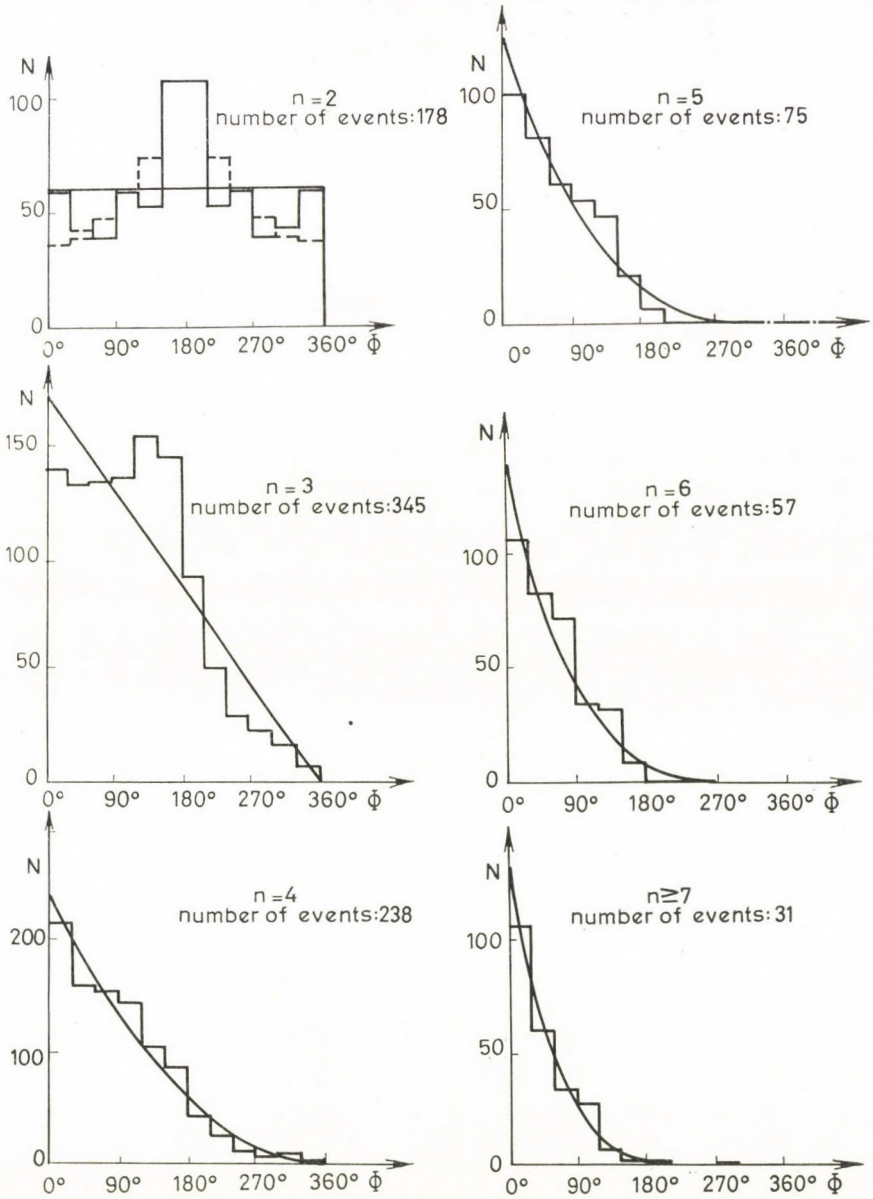


Fig. 1. Observed and calculated from Eq. (1) distributions of consecutive azimuthal angles for events having different numbers ($n \geq 2$) of prongs. The dotted line for $n = 2$ corresponds to the Monte Carlo distribution

show a significant deviation from azimuthal isotropy is against the conclusion of the authors of [1].

In order to get information about the possible causes of the observed azimuthal anisotropy of two and three prong events, the following analyses were carried out.

Table I

Number of prongs n	$P(\chi^2)$
2	0.01%
3	0.01%
4	3%*
5	5%*
6	6%*
7	55%

* Calculating $P(\chi^2)$, cells containing only a few particles were grouped together, which procedure, though a little arbitrary, is generally used in χ^2-tests.

4. Analysis of two-prong events

The distributions of the consecutive azimuthal angles are plotted in Fig. 2 for those events in which both the charged secondaries were identified and were found to be $p\,\pi^-$ (Fig. 2a) or $\pi^+\,\pi^-$ (Fig. 2b) particles, respectively;* i.e. for reactions

$$\pi^- p \to p\,\pi^-(k\,\pi^\circ) \qquad k = 1, 2, \ldots \tag{2a}$$

and

$$\pi^- p \to n\,\pi^+\,\pi^-(k'\,\pi^\circ) \qquad k' = 0, 1, \ldots. \tag{2b}$$

The isotropic distributions and distributions calculated by a Monte Carlo method (the matrix element of the interactions was chosen to be constant, and appropriate weighting factors were taken into account for the different channels, see Appendix I) are also drawn in the Figure.

One can see:

(1) In case of reaction (2a) there is, both in the experimental and Monte Carlo distributions, a pronounced and comparatively narrow peak around $\Phi = 180°$ (the interval $150° < \Phi < 210°$ contains 46% of all the measured events). This peak is much less pronounced and narrow for reaction (2b) as can be seen from both the experimental and Monte Carlo distributions (the same interval contains only 20% of all the measured events).

(2) The Monte Carlo curves agree quite well with the experimental distributions except at the very edges for the reaction (2b). This is probably due to peripheral collisions where the π^+ and π^- particles are emitted from the upper vertex at small angles with the primary directions in the laboratory system, which process was not taken into account in our Monte Carlo calculation.

* For these distributions appropriate geometrical factors were used (see [12], [13]).

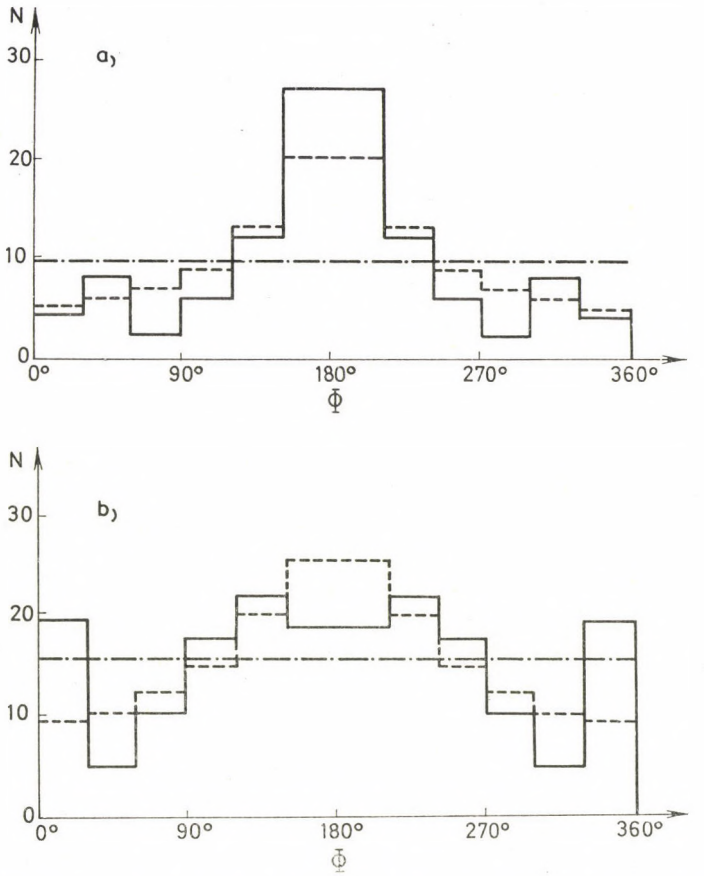


Fig. 2. Distributions of consecutive azimuthal angles observed a) in reaction (2a) and b) in reaction (2b). The dotted (– – – – – –) and the broken (·—·—·—·—) lines correspond to Monte Carlo and random distributions, respectively

The result of a χ^2-test for the isotropic and Monte Carlo distributions can be seen in Table II. In the last column of the table, p'_{MC} χ^2 means the Pearson probabilities if we do not take into account the first and last cells.

Thus our analysis shows that the observed deviation from isotropy can be explained only partly by the kinematics of the processes.

Table II

reaction	$p_I(\chi^2)$	$p_{MC}(\chi^2)$	$p'_{MC}(\chi^2)$
(2a)	0.01%	14%	
(2b)	0.2%	0.03%	25%
total with $n = 2$	0.01%	3%	50%

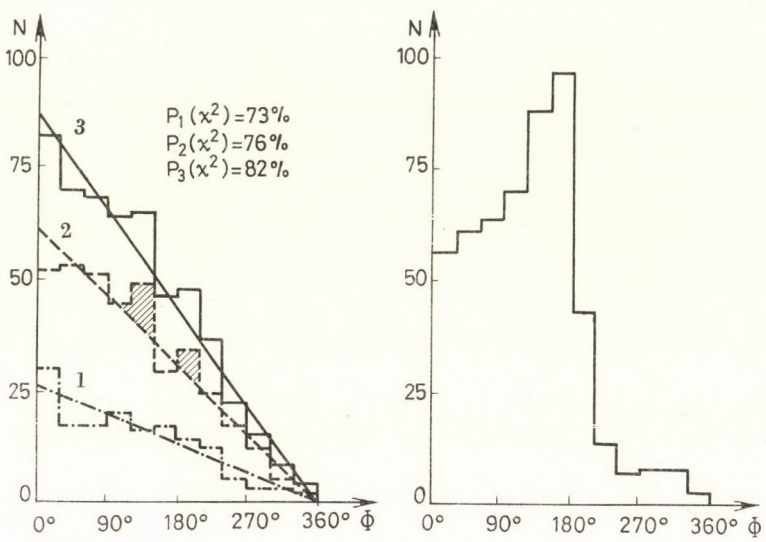
5. Analysis of three-prong events

As has already been mentioned, the three prong events have a certain contamination of DD interactions [13], [16—18].

Therefore, our sample is subdivided into three parts:

(a) Events containing an identified proton.

(b) Events without an identified proton, for which $\sum_{i=1}^{3} \sin \Theta_i \geq 0.44$ (where Θ_i is the emission space angle of the i-th track).



3a 3b

Fig. 3. Distributions of consecutive azimuthal angles for three prong events belonging a) to subsamples (a) (distribution 1), (b) (distribution 2) and to their sum (distribution 3) and b) to subsample (c). The straight lines correspond to the random distributions

(c) Events without an identified proton, for which $\sum_{i=1}^{3} \sin \Theta_i < 0{,}44$; i.e. the DD candidates.

The azimuthal angular distribution of events belonging to subsamples (a), (b) and their sum can be seen in Fig. 3a (distributions quoted by 1, 2 and 3). The corresponding random distributions are also indicated. As can be seen from the Figure and from the values of $P(\chi^2)$ (indicated in the Figure) there is *no* azimuthal effect in either case.

The distribution of subsample (c), however, has a shape quite different from that of the random one. It has a peak at around $\Phi = 180°$, and after a rapid fall, a tail at Φ values greater that $\Phi = 210°$ (see Fig. 3b).

This characteristic shape automatically suggests the assumption that this subsample consists of two types of events; (i) events which yield the isotropic part of the distribution, and (ii) events which cause the deviation from isotropy.

In order to separate these two types of events a straight line was fitted to the tail of the distribution and it was extrapolated down to $\Phi = 0°$. Then, subtracting the area under this line from the distribution of events belonging to subsample (c) a new distribution corresponding to the *events causing the deviation from isotropy* can be obtained.

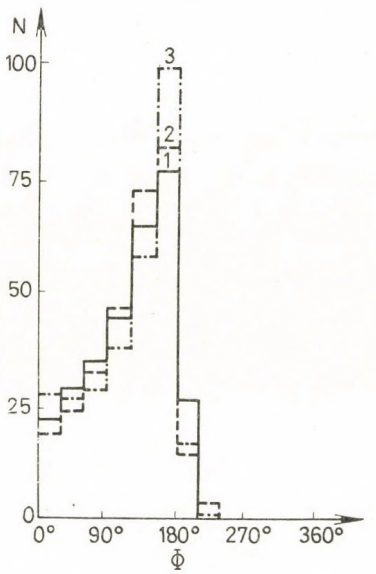


Fig. 4. Distributions of consecutive azimuthal angles obtained by the subtraction method (full line). The distributions drawn with dotted (– – – – – –) and broken (·—·—·—·—) lines correspond to Monte Carlo distributions calculated for DD-interactions of types (3a) and (3b)

A comparison of this new distribution (distribution 1 in Fig. 4) with that calculated by Monte Carlo method for DD interactions (for details see Appendix II) of types

$$\pi^- \mathscr{N} \to \mathscr{N} X^-_{M^*} \to \mathscr{N} p^\circ \pi^- \tag{3a}$$

and

$$\pi^- \mathscr{N} \to \mathscr{N} X^-_{M^*} \to \mathscr{N} \pi^+ \pi^- \pi^-, \tag{3b}$$

where $X^-_{M^*}$ is the coherently produced system having an effective mass M^*, shows (distributions 2 and 3 in Fig. 4) that there is an excellent agreement with the distribution calculated for reaction (3a).* Thus we can conclude

* In this calculation the spin parity assignment of the $X^-_{M^*}$-system was not taken into account since it is shown in Appendix II that the shape of the distribution of consecutive azimuthal angles is not sensitive to the possible spin parity assignments.

that the distribution obtained by the above procedure is due only to DD events (presumably of type (3a)). This means that *this "subtraction" procedure makes it possible to select DD events* from a sample of three prong events.

Knowing this result, we can make the subtraction more precisely, eliminating the arbitrariness in the extrapolation. Denote by y_i the number of angles in the i-th cell in the consecutive azimuthal angular distribution, and by m_i and h_i the same quantities obtained by the Monte Carlo calculation for reaction (3a), and from Eq. (1), respectively. Then, the expected value of y_i will be $\langle y_i \rangle = Am_i + Bh_i$, where A and B are the constants to be determined, and the probability that in the i-th cell y_i is observed instead of $\langle y_i \rangle$ is

$$P_i = \frac{\langle y_i \rangle^{y_i}}{y_i!} e^{-\langle y_i \rangle} .$$

Applying the maximum likelihood method, A and B were determined.

Using the obtained values of the constants, the number of DD events (N_{DD}) among the events belonging to subsample (c), ($N_{(c)}$), and thus the mean free path of the DD process in question in emulsion can be obtained.* The calculation was carried out for the ~7 GeV/c material as well.** The results can be seen in Table III.

Table III

	$N_{(c)}$	$A \cdot \Sigma m_i$	$B \cdot \Sigma h_i$	N_{DD}	λ_{DD}
17.2 GeV/c	173	255 ± 29	260 ± 29	83.1 ± 9.5	(113 ± 13) m
~7 GeV/c	20	28.0 ± 7.6	31.0 ± 7.8	9.3 ± 2.5	(1065 ± 288) m

The above values of λ_{DD} are within the interval obtained earlier [13] from the same material***

$$(61 \pm 5)\, m \leq \lambda_{DD}^{17{,}2\,\mathrm{GeV/c}} \leqslant (300 \pm 83)\, m ,$$

$$(555 \pm 131)\, m \leq \lambda_{DD}^{\sim 7\,\mathrm{GeV/c}} \leqslant (1600 \pm 800)\, m .$$

* It is worth mentioning, that if this subtraction procedure is applied to all three prong events the result for λ_{DD} agrees excellently with the result shown in Table III, which shows the reliability of the method.

** In the case of the ~7 GeV/c material the result of a separate Monte Carlo calculation corresponding to this primary energy was applied.

*** The lower limit was obtained using selection criteria

1. $n = 3$, $N_g = N_h = 0$,
2. $\sum_{i=1}^{3} \sin \Theta_i < 0.44$,

while for the upper limit three more criteria were used:

3. all the three particles have to be identified as pions,
4. $q_\perp \leq 200$ MeV/c, where q is the transverse momentum transferred to the nucleus,
5. $\sum_{i=1}^{3} E > 0.7\, E_0$, where E_0 and E_i are the total energy of the primary and of the i-th pion.

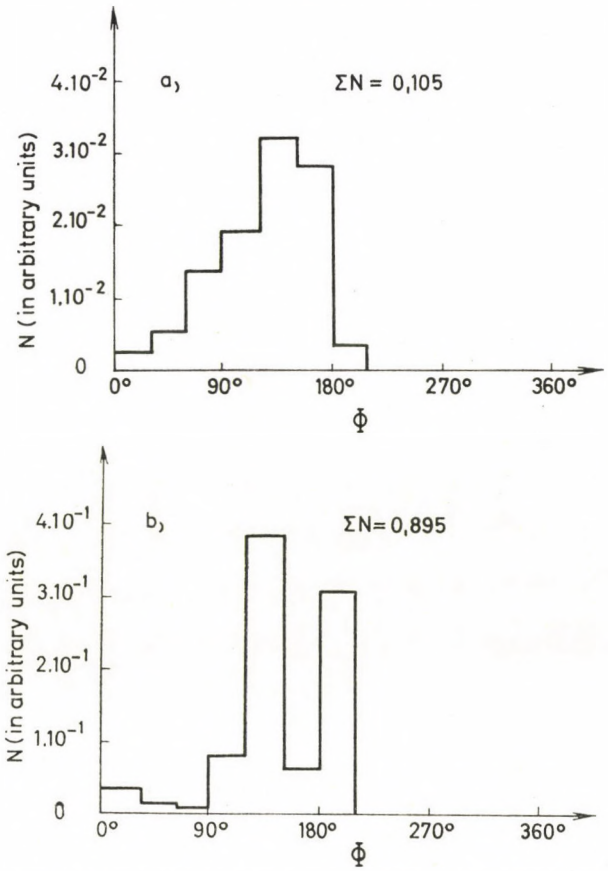


Fig. 5. Monte Carlo distributions of consecutive azimuthal angles of three prong events corresponding to Deck mechanism and a) satisfying the $\sum_{i=1}^{3} \sin \Theta < 0.44$ criterion; b) not satisfying the above criterion

The possible influence of a $\pi^- n \to n\varrho^\circ \pi^-$ DECK mechanism [19] on the shape of the distribution on consecutive angles was also investigated by a separate Monte Carlo calculation (see Appendix III). The calculation yielded the following results:

a) Only 10% of the generated events fulfils the $\sum_{i=1}^{3} \sin \Theta < 0.44$ criterion.

b) The maximum of the consecutive azimuthal angular distribution of events fulfilling the $\sum_{i=1}^{3} \sin \Theta < 0.44$ criterion is slightly shifted towards lower Φ values compared with the distribution of DD process (see Fig. 5a).

c) The shape of the distribution of events having $\sum_{i=1}^{3} \sin \Theta_i \geq 0.44$ is very different from that of experimentally observed (see Fig. 5b).

If the small enhancements (hatched areas in Fig. 3a) in the distribution of events belonging to subsample (b) in the regions $120° \leq \Phi \leq 150°$ and $180° \leq \leq \Phi \leq 210°$ are supposed to be due to DECK mechanism, then the contribution of the DECK mechanism (in the case of subsample (b)) can be estimated as $\lesssim 10\%$ (which corresponds to $\lesssim 17$ events) and it is negligible for subsample (c).

6. Discussion and conclusions

1. Analysing the consecutive azimuthal angular distribution of $\pi^- N$ interactions at 17.2 GeV/c, azimuthal anisotropy was found for the two and three prong events. At higher multiplicities azimuthal effects were not detected.

2. As concerns two prong events it was pointed out that the anisotropy found can be partly attributed to kinematic effects.

3. As concerns three prong events the observed anisotropy was found to be due to DD processes.

4. A new "subtraction" method was found for the selection of DD events from a sample of three prong events. Using this method

$$\lambda_{DD}^{\sim 7 \text{GeV/c}} = (1065 \pm 288)\, m \quad \text{and} \quad \lambda_{DD}^{17.2 \text{GeV/c}} = (113 \pm 13)\, m$$

were obtained for the mean free path of DD processes at ~ 7 and 17.2 GeV/c primary momenta.

5. Since in DD events there is an azimuthal anisotropy, one can assume that the azimuthal effects found in cosmic ray jets (mainly in secondary ones) having few heavy prongs and shower particles (e.g. [20], [21]) are partly due to DD processes. Thus, it is desirable to study separately the azimuthal effects of DD and non-DD events in future cosmic ray work.

6. The absence of azimuthal effect at charged multiplicities $n \geq 3$ (apart from the DD events) can be attributed to the following cause:

Inelastic channels of different multiplicity are superimposed at a given number of charged prongs and this superposition can yield a nearly "random" distribution in spite of the possible azimuthal effects in the separate channels.

Therefore the possibility of the production of fireballs at this energy is not necessarily ruled out. (As a matter of fact, fireball production was detected by e.g. C. W. AKERLOF et al. [22] in $p - p$ collisions.)

To get more information in this respect, it seems to be worth while to investigate azimuthal effects using the "method of consecutive angles" (or an improved, more sensitive method) in the separate inelastic channels, and to study the azimuthal correlations between likely and unlikely charged secondaries.

The authors are grateful to Dr. P. KIRÁLY for his critical comments.

Appendix I

The distributions of consecutive azimuthal angles were calculated for reaction channels

$$\pi^- p \to p\,\pi^-\,\pi^\circ$$
$$p\,\pi^-\,2\,\pi^\circ$$
$$p\,\pi^-\,3\,\pi^\circ$$
$$n\,\pi^+\,\pi^-$$
$$n\,\pi^+\,\pi^-\,\pi^\circ$$
$$n\,\pi^+\,\pi^-\,2\,\pi^\circ$$
$$n\,\pi^+\,\pi^-\,3\,\pi^\circ *$$

separately by generating these types of many-body final states supposing a uniform distribution in the phase space; i.e. supposing a constant transition

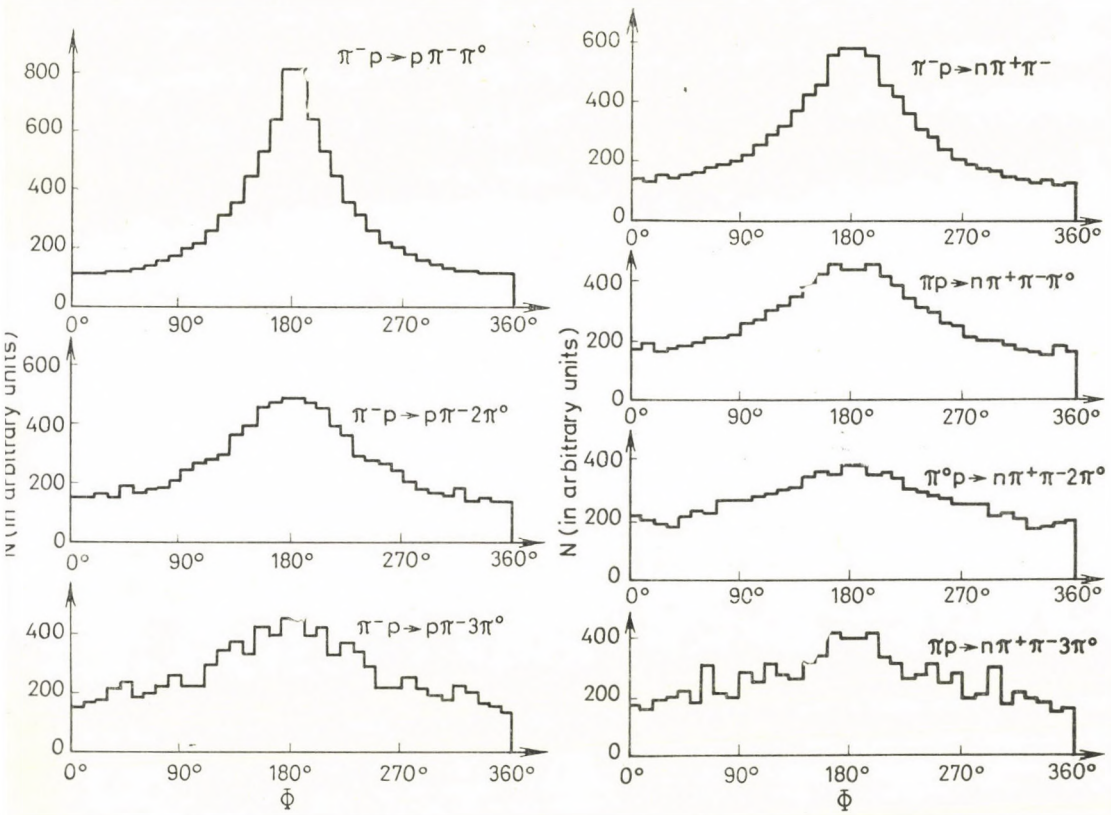


Fig. A1. Monte Carlo distributions of consecutive azimuthal angles calculated for reactions $\pi^- p \to p\,\pi^-\,(k\,\pi^\circ)$ $k = 1, 2, 3$ and $\pi^- p \to n\,\pi^+\,\pi^-\,(k'\,\pi^\circ)$ $k' = 0, 1, 2, 3$

* Reaction channels with more then 3 π° were not calculated since their relative occurrence is very small [24].

matrix element (for the details of the programming of this calculation see [23]). The result of these Monte Carlo distributions can be seen in Fig. A1.

These distributions were summed up using the measured proportions [24] of the above reaction channels as weighting factors. Distributions thus obtained are compared with the experimental ones in Figs. 1a, 2a and 2b.

Appendix II

A distinct Monte Carlo programme was written for the generation of coherent events using the kinematics of coherent production. The method of generation was the following. A value for the effective mass (M^*) of the coherently produced system was randomly chosen from the experimentally measured distribution given in [25]. The momentum, transferred to the target nucleus (in our case to a carbon nucleus) was calculated by choosing random transverse momentum from the distribution

$$f(q_\perp)\,dq_\perp = \begin{cases} q_\perp e^{-\left(\frac{q_\perp}{2} m_\pi^{-1} A^{1/3}\right)^2} dq_\perp, & \text{if } q_\perp \leq 200\ \text{MeV/c}, \\ 0, & \text{if } q_\perp > 200\ \text{MeV/c}. \end{cases}$$

The longitudinal component of this momentum was determined by the use of the formula

$$q_{||} = q_\perp^2 \frac{M_A + P_0}{2M_A P_0} + \frac{M^{*2} - m_\pi^2}{2P_0}.$$

where M_A is the mass of the nucleus and P_0 is the primary momentum in the laboratory system. (They were set equal with 12 and 17.2 GeV/c, respectively.)

Then the momenta and the azimuthal angles of the three pions were calculated in the laboratory system assuming that a coherently produced X_{M^*} system decays in its rest system as

$$X^-_{M^*} \to \varrho^\circ \pi^- \to \pi^+ \pi^- \pi^-, \tag{i}$$

$$X^-_{M^*} \to \pi^+ \pi^- \pi^-. \tag{ii}$$

To take into account the possible spin effects, a further calculation was made for reaction (i), in which the experimentally measured distribution [26] of the direction of the normal of the decay plane in the X_{M^*} rest system was used. (These distributions can be attributed to possible 1^+ or 2^- spin-parity assignments for the produced system.)

The results of the above calculations can be seen in Fig. A2.

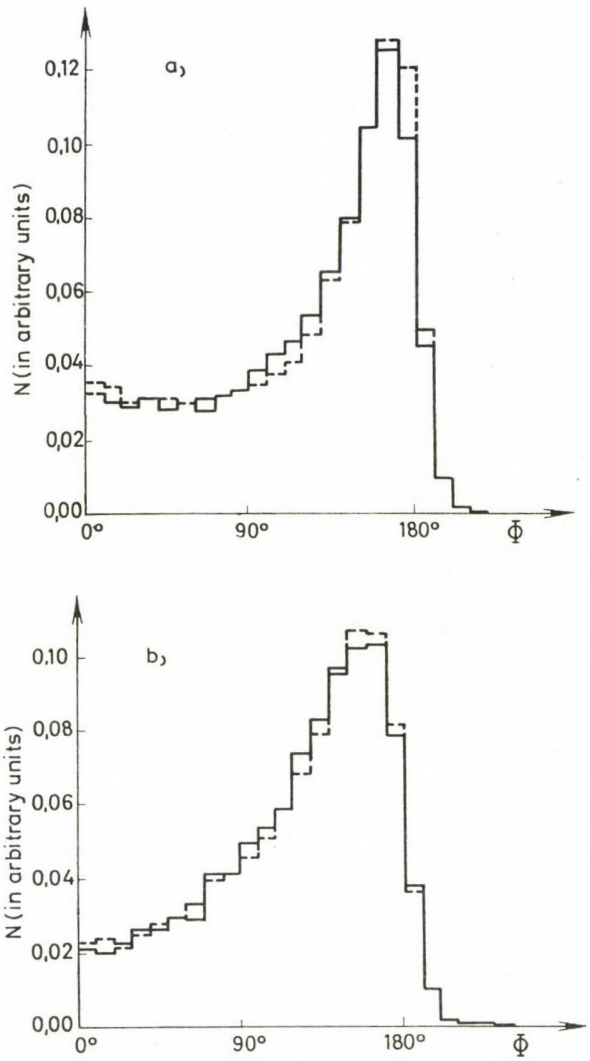


Fig. A2. Monte Carlo distributions of consecutive azimuthal angles of DD events of types a) $\pi^- \mathcal{N} \to \mathcal{N} \pi^+ \pi^- \pi^-$ and b) $\pi^- \mathcal{N} \to \mathcal{N} \varrho^\circ \pi^-$. Distributions drawn with full and dotted lines correspond to calculations, including and neglecting spin effects, respectively

Appendix III

In order to see the effect due to a $\pi^- n \to n\varrho^\circ \pi^-$ DECK mechanism [19], the distribution of consecutive azimuthal angles was calculated by Monte Carlo method. The calculation was carried out by integrating the DECK matrix element on the $n\varrho^\circ \pi$ phase space with a value of the diffraction peak

parameter $k = 9$ GeV^{-2}. The matrix element corresponding to the graph in Fig. A3 can be written as:

$$|M|^2 = \frac{\omega^2 e^{-kt}}{(\Delta + m_\pi^2)^2},$$

where $\omega^2 = (q_1 + q_2)^2$, $t = (q_1 - p_1)^2$, $\Delta = (q_3 - p_2)^2$ and the symbols q, t, Δ correspond to the notations of Fig. A3.

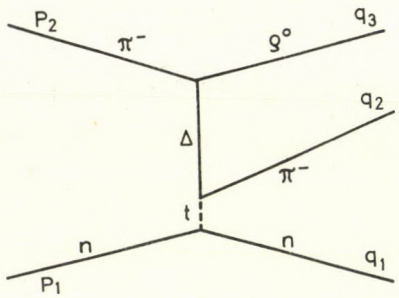


Fig. A3. Graph corresponding to the DECK mechanism in the $\pi^- n \to n\varrho^\circ \pi^-$ reaction. p_1, p_2, q_1, q_2, q_3, Δ and t are the four momenta involved in the interaction

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ОБ АЗИМУТАЛЬНЫХ ЭФФЕКТАХ ДВУХ ЛУЧЕВЫХ πN И ТРЕХ ЛУЧЕВЫХ DD СОБЫТИЯХ ОБРАЗОВАННЫХ π^- МЕЗОНАМИ ПРИ 17,2 Гев/ц.

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Р е з ю м е

Обнаружено отклонение от изотропии в азимутальном угловом распределении вторычных частиц двух- и трех лучевых πN взаимодействий в эмульсии при 17,2 Гев/ц. Сделаны попытки с помощью рассчетов Монте-Карло для интерпретации этих азимутальных эффектов на основе (i) кинематики реакций: $\pi^- p \to p \pi^-$ ($k \pi^\circ$) и $\pi^- p \to n \pi^+ \pi^-$ ($k' \pi^\circ$), и (ii) механизма диффракционной диссоциации пионов на ядрах. Сделан новый метод для избирания вкладов DD взаимодействиях среди троек. С помощью этого метода $\lambda_{DD} = (1065 \pm 288)$ м и $\lambda_{DD} = (113 \pm 13)$ м были получены для эффективного пробега диффракционного взаимодействия в эмульсии при $\sim$7 и 17,2 Гев/ц. Возможное влияние эффекта Дэка в случае трех лучевых событий было тоже исследовано.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 387—400 (1969)

THEORY OF ONE-DIMENSIONAL LATTICE IN PSEUDOHARMONIC APPROXIMATION

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(Received 21. II. 1969)

The properties of the monoatomic linear chain are considered in pseudoharmonic approximation at a fixed length and at constant external tension. It is shown that in the latter case the chain becomes unstable at a sufficiently high temperature or at a sufficiently high zero-point energy.

1. Introduction

In the usual theory of lattice dynamics the harmonic approximation is taken as the starting point for considering the anharmonic terms in the potential energy of the crystal as a small perturbation [1]. But it is well known that such an approach cannot be justified in certain cases: near the phase-transition points, e.g. at melting point; for quantum crystals with large zero-point energy; for a light impurity with small binding energy; etc. when the anharmonic effects are not small. In this connection the generalization of the theory of lattice dynamics has been proposed in a series of works [2—9] which allows the properties of strongly anharmonic crystals to be investigated.

In our previous work [8] we formulated the theory of anharmonic crystals which takes into account all the higher order anharmonic terms in certain approximation. A self-consistent system of equations was obtained for determining the frequency and the damping of lattice vibrations. The generalization of this theory and its application to the linear chain were considered in [9]. In certain conditions the damping of the lattice vibrations is sufficiently small and the most simple pseudoharmonic approximation [8] can be used. In this approximation only the renormalization of the energy of phonons in the phonon self-consistent field is taken into account. This is essentially the self-consistent phonon approximation which was obtained using varying techniques in [7]. Therefore, it is of interest to compare the results of the pseudoharmonic approximation and the approximation which takes into account the damping of phonons [9].

In this work we report the results of the investigation of the linear chain in the pseudoharmonic approximation. Some preliminary results were reported earlier in [10]. In Section 2 using the double-time Green's func-

tions we obtain a self-consistent system of equation for the determination of the equilibrium separation between atoms, the frequency of the vibrations and the internal energy. In Section 3 the properties of the linear chain with fixed ends are investigated. In Section 4 the properties of the linear chain at constant external tension are investigated.

2. Self-consistent system of equations in pseudoharmonic approximation

Let us consider a linear chain of length L which consists of $N+1$ identical atoms with the mass M. Taking into account only the nearest neighbour interaction, the Hamiltonian reads:

$$\mathcal{H} = H + H_1 = \sum_{n=0}^{N} \frac{P_n^2}{2M} + \frac{1}{2} \sum_{n=1}^{N} \Phi(R_n - R_{n-1}) + H_1 , \tag{1}$$

where P_n, R_n are the momentum and position operators for the n-th atom. The interaction potential between the neighbouring atoms is denoted by $\Phi(R_n - R_{n-1})$. In the case of a one-dimensional chain the effect of the external forces can be described by the external tension τ which acts on the end of the chain:

$$H_1 = \tau (R_N - R_0) = \tau \sum_{n=1}^{N} (R_n - R_{n-1}) . \tag{2}$$

It is convenient to introduce the equilibrium separation l between the neighbouring atoms and the relative displacement separators by the following definition:

$$R_n - R_{n-1} \langle R_n - R_{n-1} \rangle + u_n - u_{n-1} \equiv l + u_n - u_{n-1} , \tag{3}$$

where the statistical average $\langle \ldots \rangle$ is calculated for the equilibrium state of the system described by the Hamiltonian (1):

$$\langle \ldots \rangle = Sp\,(e^{-\mathcal{H}/\Theta} \ldots)/Sp(e^{-\mathcal{H}/\Theta}) \qquad (\Theta = kT) . \tag{4}$$

The equilibrium separation l in the one-dimensional case can be obtained from the equation [1]:

$$\tau = -\frac{1}{2} \left\langle \frac{\partial}{\partial R_n} \Phi(R_n - R_{n-1}) \right\rangle , \tag{5}$$

which shows that the average force acting on the arbitrary atom in the euqilibrium position is equal to zero.

It is convenient to introduce explicitly the displacement operators in the Hamiltonian (1) by the Fourier transformation:

$$\Phi(R) = \sum_q \Phi(q)\, e^{iqR}\,; \qquad \Phi(q) = \frac{1}{L}\int_{-L/2}^{L/2} dR\, \Phi(R)\, e^{-iqR}\,. \tag{6}$$

In this representation the Hamiltonian of the lattice (1) takes the form:

$$H = \sum_n \frac{P_n^2}{2M} + \frac{1}{2}\sum_n \sum_q \Phi(q)\, e^{iql}\, e^{iq(u_n - u_{n-1})}\,. \tag{7}$$

For the calculation of the frequency of lattice vibration we apply the method of double-time Green's functions [11]. We use the following one-phonon Green's function:

$$G_{nn'}(t-t') = \ll u_n(t)\,;\ \ u_{n'}(t') \gg = -\,i\,\Theta(t-t') < \left[u_n(t), u_{n'}(t')\right] > . \tag{8}$$

To obtain the equation of motion for the Green's function (8) we differentiate it twice with respect to time t and employ the equations of motion for the Heisenberg operators $u_n(t)$ and $P_n(t)$. In this manner we get:

$$\begin{aligned} M i^2 \frac{d^2}{dt^2} G_{nn'}(t-t') &= \delta(t-t')\,\delta_{nn'} + \\ &+ \frac{1}{2}\sum_q \Phi(q)\, e^{iql}\, iq \ll \left\{e^{iq(u_n - u_{n-1})} - e^{iq(u_{n+1} - u_n)}\right\};\ \ u_{n'}(t') \gg . \end{aligned} \tag{9}$$

The multiphonon Green's function on the r.h.s. of Eq. (9) describes the interaction of two, three or more phonons. We use the pseudoharmonic approximation [8] here, in which the processes connected with the damping of phonons are not considered, but the renormalization of the energy of phonons in the phonon self-consistent field is taken into account. This approximation is equivalent to the first order of the perturbation theory for the self-energy operator, which takes into account all the even anharmonic terms. In this approximation the multiphonon Green's function can be written in the form:

$$\begin{aligned} &\ll e^{iq(u_n - u_{n-1})}\,;\ u_{n'} \gg = \sum_{s=1}^{\infty} \frac{1}{s!} \ll \left\{iq\,(u_n - u_{n-1})\right\}^s ; u_{n'} \gg \approx \\ &\approx \sum_{s=1}^{\infty} \frac{1}{s!} < \left\{iq(u_n - u_{n-1})\right\}^{s-1} > s \ll iq\,(u_n - u_{n-1});\ u_{n'} \gg = \\ &= < e^{iq(u_n - u_{n-1})} > iq \ll (u_n - u_{n-1})\,;\ \ u_{n'} \gg . \end{aligned} \tag{10}$$

For the calculation of the correlation function on the r.h.s. of Eq. (10) we use the same approximation. We introduce the following function:

$$F(\lambda) = < e^{\lambda q(u_n - u_{n-1})} > ; \quad F(0) = 1 .$$

Differentiating it on λ and using the similar approximations as in Eq. (10) we get:

$$\frac{\partial F(\lambda)}{\partial \lambda} = < q\,(u_n - u_{n-1})\, e^{\lambda q(u_n - u_{n-1})} > =$$

$$= < q(u_n - u_{n-1}) \sum_{s=1}^{\infty} \frac{(q\lambda)^s}{s!} (u_n - u_{n-1})^s > \approx$$

$$\approx \lambda\, q^2 < (u_n - u_{n-1})^2 > F(\lambda) .$$

The integration of this equation over λ from $\lambda = 0$ to $\lambda = i$ gives us:

$$< e^{iq(u_n - u_{n-1})} > = e^{-1/2\, q^2 \langle (u_n - u_{n-1})^2 \rangle} = e^{-1/2\, q^2\, \overline{u}^2} , \tag{11}$$

where we take into account that the correlation function of the nearest neighbouring atoms does not depend on n:

$$\overline{u}^2 = < (u_{n+1} - u_n)^2 > = < (u_n - u_{n-1})^2 > .$$

Now we introduce the Fourier transformation for the Green's functions (8):

$$G_{nn'}(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\, G_{nn'}(\omega) \tag{12a}$$

and we take into account that this depends only on the difference of lattice sites $(n - n')$:

$$G_{nn'}(\omega) = \frac{1}{MN} \sum_k e^{ikl(n-n')}\, G_k(\omega) . \tag{12b}$$

Then Eq. (9) takes the form:

$$\omega^2\, G_k(\omega) = 1 +$$

$$+ \frac{1}{2M} \sum_q \Phi(q)\, e^{iql}\, (iq)^2\, e^{-1/2\, q^2\, \overline{u}^2}\, 2(1 - \cos kl)\, G_k(\omega) , \tag{13}$$

where Eqs. (10), (11) have been used. The solution of Eq. (13) reads

$$G_k(\omega) = \frac{1}{\omega^2 - \omega_k^2} \tag{14}$$

as in the harmonic approximation except the renormalization of the strength constant

$$\omega_k^2 = \frac{4f(\Theta, l)}{M} \sin^2 \frac{kl}{2} = \frac{f(\Theta, l)}{f} \omega_{0k}^2 = \alpha^2 \omega_{0k}^2 , \tag{15}$$

where ω_{0k} is the harmonic frequency of vibration and f stands for the harmonic strength constant. The pseudoharmonic strength constant $f(\Theta, l)$ according to Eq. (13) can be written as:

$$f(\Theta, l) = \frac{1}{2} \sum_q \Phi(q)\, e^{iql}\, (iq)^2\, e^{-1/2\, q^2 \overline{u}^2} = \frac{1}{2} \tilde{\Phi}''(l) , \tag{16}$$

where we introduce the self-consistent potential

$$\tilde{\Phi}(l) = < \Phi(R_n - R_{n-1}) > = \sum_q \Phi(q)\, e^{iql}\, e^{-1/2\, q^2 \overline{u}^2} = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{\overline{u}^2}{2} \right)^s \Phi^{(2s)}(l) . \tag{17}$$

In obtaining Eq. (17) we decompose the function exp $(-1/2\, q^2 \overline{u}^2)$ into the series of $\overline{u}^2$ and integrate it over q. The self-consistent potential can also be written in the form:

$$\tilde{\Phi}(l) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2}\, \Phi(l + x\sqrt{\overline{u}^2}) , \tag{17a}$$

where $x = R/\sqrt{\overline{u}^2}$. It is easy to see that in Eq. (17a) the renormalization of the potential due to the vibrations of the atoms is taken into account by averaging it over the small region $R \sim \sqrt{\overline{u}^2} \ll l$ with the Gaussian function exp $(-x^2/2)$ which describes the effect of the phonon self-consistent field. Owing to this function only the shape of the potential $\Phi(R)$ at the bottom of the potential well is of importance. In the case of the potential with hard core the formula (17) in the form of infinite series can be used.

The correlation function of the nearest neighbours in Eqs. (17), (17a) can be obtained from the spectral theorem (11) for the Green's function

$$< u_{n'}\, u_n > = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{e^{\omega/\Theta} - 1} \left\{ -2 \operatorname{Im} G_{nn'}(\omega + i\varepsilon) \right\} .$$

From Eq. (14) we get:

$$\begin{aligned} \overline{u}^2 = <(u_n - u_{n-1})^2> &= \frac{1}{Nf} \sum_k \frac{\omega_{0k}^2}{2\omega_k} \coth \frac{\omega_k}{2\Theta} = \\ &= \frac{\omega_{0L}}{\pi f} \frac{1}{\alpha} \int_0^{\pi/2} d\varphi \sin\varphi \coth \left\{ \frac{\omega_{0L}}{2\Theta} \alpha \sin\varphi \right\} , \end{aligned} \tag{18}$$

where we have replaced the sum over k by integral over $\varphi = kl/2$. The maximal value of the vibrational frequency of the chain in the harmonic approximation is denoted by $\omega_{0L} = (4f/M)^{1/2}$. In the high temperature ($\Theta \gg \omega_{0L}$) and low temperature ($\Theta \ll \omega_{0L}$) limit the integral in Eq. (18) becomes:

$$\alpha^2 f \overline{u}^2 \approx \Theta \left\{1 + \frac{1}{24}\left(\frac{\alpha\omega_{0L}}{\Theta}\right)^2\right\} + O(\Theta^{-3}) \quad (\Theta \gg \omega_{0L}), \tag{18a}$$

$$\alpha f \overline{u}^2 \approx \frac{\omega_{0L}}{\pi}\left\{1 + \frac{\pi^2}{3}\left(\frac{\Theta}{\alpha\omega_{0L}}\right)^2\right\} + O(\Theta^4) \quad (\Theta \ll \omega_{0L}). \tag{18b}$$

In addition to the temperature Θ the properties of the linear chain are determined also by the length of the chain $L = Nl$ or by the external tension τ. According to Eqs. (5), (17) these parameters satisfy the following equation:

$$\tau = -\frac{1}{2} < \Phi'(R_n - R_{n-1}) > = -\frac{1}{2}\tilde{\Phi}'(l), \tag{19}$$

which is the thermal equation of state for the linear chain.

The caloric equation of state is obtained from the internal energy which is given in our approximation by the equation:

$$E = <H> = N < \frac{P_n^2}{2M} > + N\frac{1}{2} < \Phi(R_n - R_{n-1}) > =$$

$$= N\frac{1}{2N}\sum_k \frac{\omega_k}{2} \coth \frac{\omega_k}{2\Theta} + \frac{N}{2}\tilde{\Phi}(l).$$

Taking into account Eqs. (15), (19) this expression for the internal energy can be rewritten in the more convenient form

$$\frac{1}{N}E = \frac{1}{2}\tilde{\Phi}(l) + \frac{1}{2}f(\Phi, l)\,\overline{u}^2. \tag{20}$$

In this way we have a closed system of self-consistent equations (15), (16), (17) or (17a), (18), (19), (20) which determine the properties of the anharmonic linear chain in the pseudoharmonic approximation.

3. Properties of the linear chain with fixed ends

The self-consistent system of equations in the previous Section is determined by the self-consistent potential (17), (18) which can be obtained easily if the form of the potential in the Hamiltonian (1) is known. Let us take the Morse potential as a model potential:

$$\Phi(R) = D\left[(e^{-a(R-r_0)} - 1)^2 - 1\right], \tag{21}$$

where r_0 is the average distance between the neighbouring atoms in the harmonic approximation: $\Phi'(r_0) = 0$ and D is the depth of the potential: $\Phi(r_0) = -D$. The strength constant in the harmonic approximation is given by $f = 1/2\,\Phi''(r_0) = Da^2$.

Applying the expansion of Eq. (17) or taking the integral of Eq. (17a) we get the following expression for the self-consistent potential:

$$\tilde{\Phi}(l) = D\left\{e^{-2a(l-r_0)}\,e^{2y} - 2e^{-a(l-r_0)}\,e^{y/2}\right\}, \tag{22}$$

where $y = a^2\overline{u}^2$.

In this Section we shall consider the case in which the length of the chain is fixed, for example, $l = r_0 = \text{const}$. We note that in general this case has been considered in the framework of the perturbation theory [1, 12, 13]. Taking the derivative of Eq. (22) and substituting $l = r_0$ we get:

$$f(\Theta, l) = \frac{1}{2}\,\tilde{\Phi}''(l) = f(2e^{2y} - e^{y/2}), \tag{23}$$

$$\tau(\Theta, l) = \frac{1}{a}\,\tilde{\Phi}'(l) = \frac{f}{a}\,(e^{2y} - e^{y/2}). \tag{24}$$

Let us investigate the case of high and low temperature separately.

3a. High temperature $\Theta > \omega_{0L}$

Substituting (23) into Eq. (18a) for y we get the following equation

$$\lambda_1\,(y - \lambda_1\,\eta)\,(2e^{2y} - e^{y/2}) = 1\,,$$

where $\lambda_1 = D/\Theta$; $\eta = \omega_{0L}^2/24\,D^2 \ll 1$. It is easy to see that this equation always has a real solution for $y > 0$ at all temperatures. If the temperature is not too high y is small, e.g. $y \sim 0.3$ at $\Theta \sim D$ and $y \sim 1$ at $\Theta \sim 13\,D$. The results of the usual perturbation theory correspond to the limit $\Theta \ll D$. In this case $y \ll 1$ and

$$y \approx \frac{\Theta}{D}\left\{1 - \frac{7}{2}\,\frac{\Theta}{D} + \frac{1}{24}\,\frac{\omega_{0L}^2}{\Theta^2}\left(1 - \frac{7}{2}\,\frac{\Theta}{D}\right)\right\}. \tag{25a}$$

3b. Low temperature $\Theta \ll \omega_{0L}$

Substituting (23) into Eq. (18b) for y we get the following equation:

$$\lambda^2 y^2\,(2e^{2y} - e^{y/2}) = 1 + 2\gamma\,(2e^{2y} - e^{y/2})^{-1}\,,$$

where $\lambda = \pi D/\omega_{0L}$; $\gamma = \pi^2\Theta^2/3\,\omega_{0L}^2 \ll 1$. As in the case of high temperature, this equation has a real solution for $y > 0$ at all temperatures. The results of the usual perturbation theory correspond to the limit $\lambda \gg 1$, when $y \ll 1$ and

$$y \approx \frac{1}{\lambda}\left\{1 - \frac{7}{4\lambda} + \gamma\left(1 - \frac{7}{\lambda}\right)\right\}. \tag{25b}$$

Using solutions (25a), (25b) we can obtain expressions for the renormalized frequency (15), the internal energy (20) and the tension (24) which agree with the results of the ordinary theory of perturbation [1, 12, 13] if te omit cubic anharmonicity, because it is not taken into account in the pseudoharmonic approximation [10]. In the theory [9] which takes into account the damping of phonons these expressions turned out to be identical with the low order terms of the perturbation expansion. But the main result of the pseudoharmonic theory, the stability of the linear chain at all temperature, s not altered when we take the damping of phonons into account.

4. The properties of the linear chain at constant tension

The length of the chain depends on the temperature of the system if we keep the external tension fixed: $\tau = \text{const}$. We consider the special case when the tension is sufficiently small, that is, when the distance l between the atoms does not differ appreciably from the equilibrium separation l_0 at $\tau = 0$. According to Eq. (13) the value of l_0 defined by the condition $\Phi'(l_0) = 0$ is given by

$$l_0 = r_0 + \frac{3}{2a}\,y\,. \tag{26}$$

In this case the pseudoharmonic strength constant can be written as follows:

$$f(\Theta, l) \approx fe^{-y}(1 + pe^{y}) \tag{27}$$

and the length of the chain at given τ is:

$$L = Nl \approx N\left(l_0 - \frac{\tau}{f}\,e^{y}\right), \tag{28}$$

where $p = 3a\,\tau/f$ is the dimensionless measure of the tension which is small: $p \ll 1$. Let us investigate the case of high and low temperatures separately.

4a. High temperature $\Theta > \omega_{0L}$

A self-consistent equation for y can be obtained from Eq. (18a) if we take into account Eqs. (15) and (27):

$$\lambda_1(y - \lambda_1\eta)(1 + pe^{y}) = e^{y}\,, \tag{29a}$$

where $\lambda_1 = D/\Theta$; $\eta = \omega_{0L}^2/24\,D^2 \ll 1$. This equation has a real solution only if $\Theta \leq \Theta_c$ where Θ_c is the critical temperature. The critical solution $y_c = y(\Theta_c)$ is obtained as the simultaneous solution of Eq. (29a) and its derivative. The calculation gives:

$$\Theta_c \approx \frac{D}{e}\left[1 + e(p - \eta)\right], \tag{30a}$$

$$y(\Theta \lesssim \Theta_c) \approx 1 + e(p + \eta) - \sqrt{2(1 - \Theta/\Theta_c)}\,. \tag{31a}$$

The vibrational frequency ω_k at $\Theta \lesssim \Theta_c$ is given by

$$\omega_k^2 \approx \frac{\omega_{0k}^2}{e}\left\{1 - e\eta + \sqrt{2(1 - \Theta/\Theta_c)} + (1 - \Theta/\Theta_c)\right\}. \tag{32a}$$

It becomes complex if $\Theta > \Theta_c$, which shows the instability of the system.

The length of the chain which can be obtained from Eqs. (26), (28) and also the internal energy Eq. (20), which is given by the formula at

$$\frac{1}{N}E \approx -\frac{D}{2} + \Theta_c\left\{1{,}4 - p + \frac{\sqrt{2}}{2}\,\frac{1}{24}\,\frac{\omega_{0L}^2}{\Theta_c^2} - \frac{1}{2}\sqrt{(1 - \Theta/\Theta_c)} - (1 - \Theta/\Theta_c)\right\} \tag{33a}$$

remain finite. The coefficient of the linear thermal expansion

$$\alpha_T = \frac{k}{L}\,\frac{\partial L}{\partial \Theta} \approx \frac{k}{l}\,\frac{3}{2aD}\,\frac{e}{\sqrt{2(1 - \Theta/\Theta_c)}} \tag{34a}$$

and the specific heat at constant pressure

$$c_p = \frac{k}{N}\left[\frac{\partial}{\partial \Theta}(E + \tau L)\right]_\tau \approx k\left\{1 + \frac{0{,}3}{\sqrt{1 - \Theta/\Theta_c}}\right\} \tag{35a}$$

tend to infinity if $\Theta \to \Theta_c$.

In the range of temperatures where $\Theta \ll D$ the solution of Eq. (29a) has the following form:

$$y \approx \frac{\Theta}{D}\left\{1 + \frac{\Theta}{D} + \frac{1}{24}\,\frac{\omega_{0L}^2}{\Theta^2}\left(1 + \frac{\Theta}{D}\right) - p\right\}. \tag{36a}$$

In this case the renormalised frequency, the internal energy, the specific heat and the coefficient of the linear thermal expansion are given by the following formulae:

$$\omega_k^2 \approx \omega_{0k}^2\left\{1 + p - \frac{\Theta}{D}\left(1 + \frac{1}{2}\,\frac{\Theta}{D} + \frac{1}{24}\,\frac{\omega_{0L}^2}{\Theta^2} - p\right)\right\} \tag{37a}$$

$$\frac{1}{N}E \approx -\frac{D}{2} + \Theta\left\{1 + \frac{1}{4}\frac{\Theta}{D} + \frac{1}{24}\frac{\omega_{0L}^2}{\Theta^2} - \frac{1}{2}p\left(1 + 2\frac{\Theta}{D}\right)\right\}, \quad (38a)$$

$$c_p \approx k\left\{1 - \frac{1}{24}\frac{\omega_{0L}^2}{\Theta^2} + \frac{1}{2}\frac{\Theta}{D} + p\frac{\Theta}{D}\right\}, \quad (39a)$$

$$\alpha_T = \frac{k}{l}\frac{3}{2aD}\left\{1 + 2\frac{\Theta}{D} - \frac{1}{24}\frac{\omega_{0L}^2}{\Theta^2} - \frac{11}{9}p\right\}. \quad (40a)$$

4b. Low temperature $\Theta \ll \omega_{0L}$

The self-consistent equation for y obtained from Eq. (18b) taking into account Eqs. (15) and (27) is the following:

$$\lambda^2 y^2 = e^y\left\{1 + \lambda^2 y^2 (2\gamma - p)\right\}, \quad (29b)$$

where $\lambda = \pi D/\omega_{0L}$ is the dimensionless parameter which describes the coupling of the atoms in the chain; $\gamma = \pi^2\Theta^2/3\,\omega_{0L}^2 \ll 1$. This equation has a real solution for $y > 0$ in the range where $\lambda > \lambda_0$ and $\Theta < \Theta_c$. The critical parameters defined as previously in the case of high temperatures are:

$$\Theta_c \approx \frac{\omega_{0L}}{\pi e}\sqrt{\frac{6}{e}(\lambda - \lambda_0)}\,; \qquad \lambda_0 \approx \frac{e}{2}\left(1 - \frac{e^2}{2}p\right). \quad (30b)$$

The solution at $\Theta \leq \Theta_c$ is given by

$$y \approx 2\left\{1 + e^2 p - \frac{4}{e}(\lambda - \lambda_0) - \right.$$
$$\left. - \sqrt{\frac{4}{e}}\sqrt{(\lambda - \lambda_0)(1 - \Theta^2/\Theta_c^2)}\left[1 - \frac{8}{e}(\lambda - \lambda_0)\right] + \frac{4}{e}(\lambda - \lambda_0)(1 - \Theta^2/\Theta_c^2)\right\}. \quad (31b)$$

The vibration frequency ω_k near to the critical point can be expressed as

$$\omega_k^2 \approx \left(\frac{\omega_{0k}}{e}\right)^2\left\{1 - pe^2 + \frac{8}{e}(\lambda - \lambda_0) + \frac{4}{\sqrt{e}}\sqrt{(\lambda - \lambda_0)(1 - \Theta^2/\Theta_c^2)}\right\}. \quad (32b)$$

The frequency becomes complex if $\lambda < \lambda_0$ or $\Theta > \Theta_c$ which shows the instability of the system. It is worth-while to emphasize, that the chain becomes unstable even at zero temperature: $T = 0$ °K if the zero-point energy is sufficiently high $\omega_{0L}/2\pi > D/e$. Such a situation can occur for a chain of high atoms with small binding energy (quantum crystals).

In the vicinity of the critical point $\Theta \lesssim \Theta_c$ the internal energy which is given by

$$\frac{1}{N} E \approx -\frac{D}{2} + \frac{\omega_{0L}}{\pi} \left\{0{,}8 - 2{,}5\, p + 0{,}6\,(\lambda = \lambda_0) - \right.$$
$$\left. - 0{,}6(\lambda - \lambda_0)^{3/2} (1 - \Theta^2/\Theta_c^2)^{1/2} - 0{,}3(\lambda - \lambda_0)\,(1 - \Theta^2/\Theta_c^2)\right\} \tag{33b}$$

and the length of the chain, Eq. (28), remain finite. The coefficient of the linear thermal expansion

$$\alpha_T \approx \frac{k}{al} \frac{\Theta}{\omega_{0L}^2} \frac{12\,\pi^2}{\sqrt{(\lambda - \lambda_0)\,(1 - \Theta^2/\Theta_c^2)}} \tag{34b}$$

and the specific heat at constant pressure

$$c_p \approx k \frac{2\pi}{3} \frac{3\Theta}{\omega_{0L}} \left\{1 + 2 \sqrt{\frac{\lambda - \lambda_0}{1 - \Theta^2/\Theta_c^2}}\right\} \tag{35b}$$

tend to infinity if $\Theta \to \Theta_c$.

For $\lambda \gg 1$ the solution of Eq. (29b) has the following form:

$$y \approx \frac{1}{\lambda} \left\{1 + \frac{1}{2\lambda} + \gamma \left(1 + \frac{2}{\lambda}\right) - \frac{p}{2} \left(1 + \frac{2}{\lambda}\right)\right\} \tag{36b}$$

from which we see, that $y \ll 1$.

Using this solution the renormalised frequency, the internal energy, the specific heat and the coefficient of the linear thermal expansion can be written as:

$$\omega_k^2 \approx \omega_{0k}^2 \left\{1 + p - \frac{1}{\lambda} \left(1 + \gamma - \frac{p}{2}\right)\right\}, \tag{37b}$$

$$\frac{1}{N} E \approx -\frac{D}{2} + \frac{\omega_{0L}}{\pi} \left\{1 - \frac{1}{4\lambda} + \gamma \left(1 + \frac{2}{\lambda}\right)\right\}, \tag{38b}$$

$$c_p \approx k \frac{2\pi}{3} \frac{\Theta}{\omega_{0L}} \left\{1 + \frac{1}{2\lambda}\right\}, \tag{39b}$$

$$\alpha_T \approx \frac{k}{al} \frac{\Theta}{\omega_{0L}} \frac{\pi}{D} \left\{1 + \frac{2}{\lambda}\right\}. \tag{40b}$$

It should be emphasized that the relative displacement of the atoms at the critical point is rather small:

$$\frac{\sqrt{u_c^2}}{l_c} = \frac{\sqrt{y_c}}{a l_c} \approx \frac{1}{4}$$

(at $ar_0 \approx 2.5$) and this is true for both the low and the high temperature cases.

All the results obtained in this Section coincide with those of the theory in which the damping of phonons is taken into account [9] exept for the numerical coefficients, which are somewhat different. The coefficient of the linear thermal expansion, Eq. (40a), and also the average of the quadratic displacement $\overline{u}^2 = y/a^2$, Eq. (36a) coincide with the results obtained in [14]. The phenomenon of nonstability of the linear chain of constant tension at high temperature was investigated also in [4].

4c. The critical temperature

In previous Sections we obtained the expressions for the critical temperature in the high ($\Theta \gg \omega_{0L}$) and low ($\Theta \ll \omega_{0L}$) temperature limit. It is of interest to investigate the critical temperature as a function of the dimen-

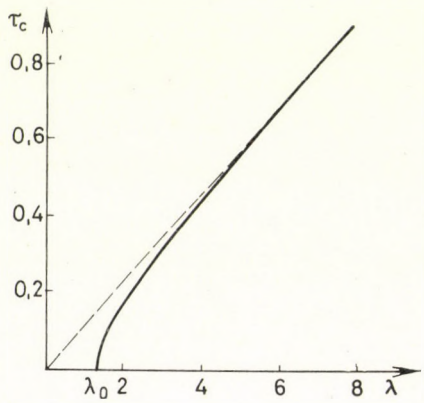

Fig. 1. The critical temperature τ_c as a function of the coupling constant λ

sionless coupling constant $\lambda = \pi D/\omega_{0L}$ of the atoms in the linear chain. Taking into account Eq. (27), Eq. (18) for $\overline{u}^2$ can be written as follow:

$$\lambda\alpha \ln \frac{1}{\alpha^2 - p} = \int_0^{\pi/2} d\varphi \sin\varphi \coth \frac{\alpha \sin\varphi}{2\tau}, \tag{41}$$

where $\tau = \Theta/\omega_{0L}$, $\alpha^2 \equiv f(\Theta, l)/f = e^{-y} + p$. Derivating Eq. (41) on α we obtain also:

$$\lambda \left\{ \frac{2\alpha^2}{\alpha^2 - p} - \ln \frac{1}{\alpha^2 - p} \right\} = \frac{1}{2\tau} \int_0^{\pi/2} d\varphi \sin\varphi \frac{1}{\mathrm{sh}^2 \dfrac{\alpha \sin\varphi}{2\tau}}. \tag{42}$$

The simultaneous solution of Eqs. (41), (42) gives us the critical temperature $\tau_c(\lambda)$ and the renormalization of the strength constant $\alpha_c(\lambda)$ as functions of λ.

The results of numerical solution of Eqs. (41), (42) for $p = 0$ are given in Fig. 1 and Fig. 2. They agree quite well with the asymptotic expressions, Eqs. (30a), (32a) and Eqs. (30b), (32b).

Thus in the one-dimensional case we can obtain explicitly the whole stability region for an anharmonic crystal: it lies below the curve $\tau_c(\lambda)$ in Fig. 1.

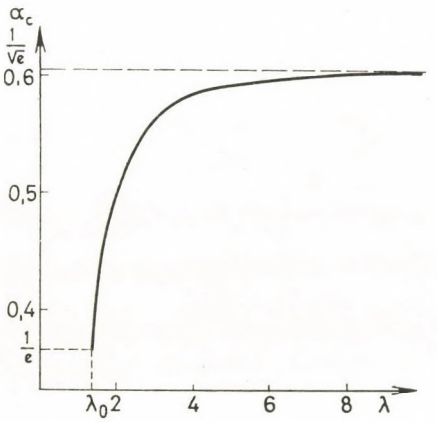

Fig. 1. The renormalization of the strength constant α as a function of the coupling constant λ at the critical temperature τ_c

5. Conclusions

The theory of anharmonic crystals in the pseudoharmonic approximation developed in [8] permit us to investigate the linear chain in a wide range of temperature and to take into account the effect of the external tension. The results obtained here agree with those of the theory in which the damping of the vibration is taken into account [9]. In the case of small anharmonicities the results agree with those of the usual perturbation theory [11, 12, 13].

The results of [15] show that in the three-dimensional case the pseudoharmonic approximation leads to the same results as in the case of one-dimensional linear chain.

Acknowledgments

We should like to point out that the theme of this investigation was proposed by Prof. S.V. Tyablikov and was helpfully discussed with him many times. We should like also to thank Dr. H. Konwent for helpful discussions and Dr. V. Priezzhev for numerical calculations.

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ТЕОРИЯ ОДНОМЕРНОЙ РЕШЕТКИ В ПСЕВДОГАРМОНИЧЕСКОМ ПРИБЛИЖЕНИИ

Н. М. ПЛАКИДА и Т. ШИКЛОШ

Р е з ю м е

Исследуются свойства одноатомной линейной цепочки в псевдогармоническом приближении при фиксированной длине и при постоянном внешнем натяжении. Показано, что цепочка неустойчива во втором случае при достаточно высокой температуре или достаточно большой энергии нулевых колебаний.

Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 401—406 (1969)

COMMUNICATIONES BREVES

CHERENKOV RADIATION DUE TO THE PASSAGE OF AN OSCILLATING DIPOLE

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(Received 13. VIII. 1968)

A simple expression is deduced for Cherenkov radiation caused by the motion of an oscillating dipole. The region of radiation is much wider than that of the usual Cherenkov radiation. Two cones of radiation are produced and energy loss due to radiation can be traced even when the velocity of the oscillator is less than the phase velocity of light in the medium.

1. Introduction

Classical treatments of the problem of Cherenkov radiation for a static dipole moving with a constant velocity in a dielectric medium have been given by FRANK [1], EIDMAN [2], BALAZS [3] and others. We are interested in the case of an electric dipole which is vibrating with a fixed frequency along its axis and also moving with a constant velocity parallel to its axis. The energy expression obtained in our consideration is in full agreement with that of a static dipole as a limiting case. The dipole can be considered as consisting of two oppositely charged particles separated by a small distance where the product of the charge and this distance is constant — known as the moment of the dipole. Vibration can be introduced either by variation of the charges with time or by a change of the length of the dipole with time. It will be found that both cases lead to identical results. One may object that the continuity equation of charge and current densities will be jeopardized owing to charges as a function of time. In our case the continuity equation is preserved when the distance between the charges or the length of the dipole is made to tend to zero. Effects of vibration on radiation have physical and practical importance and it is interesting to note that the Cherenkov radiation exists in a usual non Cherenkov region.

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2. Preliminary equations

Let us consider the dipole consisting of charges q and $-q$ separated by a small distance b. They are vibrating with a fixed frequency ω_0 along the axis and also moving with a constant velocity v in the positive direction of the z-axis inside a medium of dielectric constant ε. Due to Lorentz contraction the length of separation will be $b\sqrt{1-\beta^2} = b_0$ (say), where $\beta = v/c$ and $c =$ velocity of light in vacuum.

Maxwell's equations for electro-magnetic intensities $\vec{E}$ and $\vec{H}$ with Fourier transform are

$$\begin{aligned} \operatorname{rot} H_\omega &= \frac{i\omega\varepsilon}{c} E_\omega + \frac{4\pi}{c} j_\omega , \\ \operatorname{rot} E_\omega &= -\frac{i\omega}{c} H_\omega , \\ \operatorname{div} E_\omega &= \frac{4\pi}{\varepsilon} \varrho_\omega , \\ \operatorname{div} H_\omega &= 0 , \end{aligned} \tag{1}$$

where j is the current density, ϱ the density of free charges. Here $j_x = 0 = j_y$ and

$$j_z = \frac{qv}{2} e^{\pm\omega_0 t}\, \delta(x)\, \delta(y) \left[\delta(z - vt) - \delta(z + b_0 - vt)\right],$$

$$j_z(\omega) = -\frac{v}{4\pi} \delta(x)\, \delta(y) \left[e^{-i(\omega-\omega_0)\frac{z}{v}} + e^{-i(\omega+\omega_0)\frac{z}{v}} - e^{-i(\omega-\omega_0)\frac{z+b_0}{v}} - e^{-i(\omega+\omega_0)\frac{z+b_0}{v}}\right].$$

Since b_0 is very small,

$$j_z(\omega) = -\frac{ivb_0}{4\pi v} \left[(\omega - \omega_0)\, e^{-i(\omega-\omega_0)\frac{z}{v}} + (\omega + \omega_0)^{-i(\omega+\omega_0)\frac{z}{v}}\right] \delta(x)\, \delta(y). \tag{2}$$

If we consider the electric dipole of moment qb and the vibration is imposed by the variation of its length, then it can be written as

$$j_z = qv\, \delta(x)\, \delta(y) \left[\delta)z - vt) - \delta(z - b_0 \cos\omega_0 t - vt)\right],$$

$$j_z(\omega) = -\frac{q}{2\pi} \delta(x)\, \delta(y) \left[e^{-i\frac{\omega z}{v}} + v\int_{\infty}^{-\infty} e^{-i\omega t}\, \delta(z + b_0 \cos_0 t - vt)\, dt\right].$$

As b_0 is very small,

$$j_z(\omega) = -\frac{q}{2\pi}\,\delta(x)\,\delta(y)\,e^{-\frac{i\omega z}{v}}\left[1 - v\,\frac{e^{-i\frac{\omega\, b_0 \cos\frac{\omega_0 z}{v}}{v}}}{b_0\,\omega_0 \sin\dfrac{\omega_0 z}{v} + v}\right] =$$

$$= -\frac{iqb_0}{4\pi\, v}\left[(\omega - \omega_0)\,e^{-i(\omega-\omega_0)\frac{z}{v}} + (\omega + \omega_0)\,e^{-i(\omega+\omega_0)\frac{z}{v}}\right]\delta(x)\,\delta(y)\,. \qquad (3)$$

(2) and (3) are identical.

Now one can easily verify that div $j + \partial\varrho/\partial t = 0$ in each of the above cases when $b_0 \to 0$. Introducing vector and scalar potentials $\vec{A}$ and Φ we have

$$H_\omega = \mathrm{rot}\, A_\omega\,,$$

$$E_\omega = -\frac{i\omega}{c}\,A_\omega - \mathrm{grad}\,\Phi_\omega\,, \qquad (4)$$

$$\nabla^2 A_\omega + \frac{\varepsilon\omega^2}{c^2}\,A_\omega = -\frac{4\pi}{c}\,j_\omega\,,$$

with the condition

$$\mathrm{div}\, A_\omega + \frac{i\omega\varepsilon}{c}\,\Phi_\omega = 0\,.$$

Taking $A_x = 0 = A_y$,

$$\nabla^2 A_z(\omega) + \frac{\varepsilon\omega^2}{c^2}\,A_z(\omega) = \frac{iq\,b_0}{cv}\,\delta(x)\,\delta(y)\left[(\omega - \omega_0)\,e^{-i(\omega-\omega_0)\frac{z}{v}} + \right.$$

$$\left. + (\omega + \omega_0)\,e^{-i(\omega=\omega_0)\frac{z}{v}}\right], \qquad (5)$$

By the help of equation (5)

$$A_z(\omega) = -\frac{vb_0}{4cv}\left[(\omega - \omega_0)\,H_0^{(2)}(s_1 r)\,e^{i\omega\left(t - \frac{\omega-\omega_0}{\omega}\frac{z}{v}\right)} + \right.$$

$$\left. + (\omega + \omega_0)\,H_0^{(2)}(s_2 r)\,e^{i\omega\left(t - \frac{\omega+\omega_0}{\omega}\frac{z}{v}\right)}\right]. \qquad (6)$$

Here

$$s_1^2 = \frac{\varepsilon\omega^2}{c^2} - \frac{(\omega - \omega_0)^2}{v^2}\,, \qquad s_2^2 = \frac{\varepsilon\omega^2}{c^2} - \frac{(\omega + \omega_0}{v^2} \qquad (7)$$

and

$$r = \sqrt{x^2 + y^2}\,.$$

By the condition of (4)

$$\Phi(\omega) = -\frac{vb_0}{4\varepsilon\omega v^2}\left[(\omega-\omega_0)^2 H_0^{(2)}(s_1 r)\, e^{i\omega\left(t-\frac{\omega-\omega_0}{\omega}\frac{z}{v}\right)}\right.$$
$$\left. + (\omega+\omega_0)^2 H_0^{(2)}(s_2 r)\, e^{i\omega\left(t-\frac{\omega+\omega_0}{\omega}\frac{z}{v}\right)}\right]. \tag{8}$$

With the help of (4), (6) and (8) field components in cylindrical co-ordinates (r, Θ, z) can be calculated. The real parts of the relevant field components for radiation are given as

$$E_z = -\frac{qb_0}{2v}\sqrt{\frac{2}{\pi r}}\int_0^\infty \frac{1}{\varepsilon\omega}\left[(\omega-\omega_0)\, s_1 V s_1 \sin\chi_1 + (\omega+\omega_0)\, s_2 V s_2 \sin\chi_2\right] d\omega\,,$$

$$Re\, H_\theta = \frac{qb_0}{2cv}\sqrt{\frac{2}{\pi r}}\int_0^\infty \left[(\omega-\omega_0)\, V s_1 \sin\chi_1 + (\omega+\omega_0)\, V s_2 \sin\chi_2\right] d\omega, \tag{9}$$

when $s_1 r \gg 1$ and $s_2 r \gg 1$.

Here

$$\chi_1 = \omega\left(t - \frac{s_1}{\omega} r - \frac{\omega-\omega_0}{\omega}\frac{z}{v} + \frac{\pi}{4\omega}\right)$$

and

$$\chi_2 = \omega\left(t - \frac{s_2 r}{\omega} - \frac{\omega+\omega_0}{\omega}\frac{z}{v} + \frac{\pi}{4\omega}\right).$$

3. Amount of energy loss

Energy loss due to radiation through the surface of a cylinder of large radius r per unit length is

$$\frac{dW}{dl} = -\frac{cr}{2}\int_{-\infty}^{\infty} (Re\, E_z \cdot Re\, H_\theta)\, dt =$$

$$= \frac{q^2 b_0^2}{4v^2}\int_0^\infty \frac{1}{\varepsilon\omega}\left[(\omega-\omega_0)^2 s_1^2 + (\omega+\omega_0)^2 s_2^2 + (\omega^2+\omega_0^2)\sqrt{s_1 s_2}\, s_1 + s_2)\right] d\omega\,. \tag{10}$$

If the dipole moves without oscillation, then $\omega_0 = 0$. In this case

$$\frac{dW}{dl} = \frac{q^2 b_0^3}{c^2 v^2}\int_0^\infty \omega^3\left(1 - \frac{1}{\varepsilon\beta^2}\right) d\omega\,. \tag{11}$$

The result (11) was deduced by BALAZS and others for the static dipole. The expressions (10) and (11) are divergent since we have assumed no dispersion, which makes the radiation of arbitrary high frequencies possible. A cut-off may be introduced by using the Compton wave length of the particle.

4. Cone of radiation

According to FRANK and TAMM's [4] theory the semi-vertical angle of the cone of Cherenkov radiation is $\cos^{-1} 1/V\varepsilon\beta$. From the expression of (6) with the asymptotic value of $H_0^{(2)}$ for $s_1 r \gg 1$ and $s_2 r \gg 1$ we obtain two plane wave surfaces whose normals make angles Θ_1 and Θ_2 with the z-axis, where

$$\cos \Theta_1 = \frac{1}{V\varepsilon\beta}\left(1 - \frac{\omega_0}{\omega}\right), \tag{12}$$

$$\cos \Theta_2 = \frac{1}{V\varepsilon\beta}\left(1 + \frac{\omega_0}{\omega}\right). \tag{13}$$

Let us denote the cones corresponding to Θ_1 and Θ_2 as the first cone and the second cone, respectively. Eq. (12) is more valuable for studying the effect of vibration. It shows that radiation will be obtained even when $V\varepsilon\beta < 1$, i.e. radiation takes place when the velocity of the dipole is less than the phase velocity of light. Therefore, radiation exists not only in the visible or micro-wave region but also in other regions corresponding to low and high frequencies. This is rarely found in usual Cherenkov radiation.

When $\sqrt{\varepsilon}\beta < 1$ then $s_2^2 < 0$. Thus, electro-magnetic waves corresponding to this part are attenuated and consequently the second cone is not produced. If $1 < \sqrt{\varepsilon}\beta < 2$, the first cone appears at first for low frequencies with a greater value of Θ_1, and Θ_2 gradually decreases with the increase of ω. The second cone will not be in the picture until $\omega > \omega_0$. In the frequency range $\omega > \omega_0$ two cones will be produced simultaneously and the shift of the cones, i.e. $\Theta_1 - \Theta_2$ will decrease with the increase of ω. When $\omega \gg \omega_0$, both cones will coincide with the normal Cherenkov cone. If $\omega < \omega_0$, the second cone will rarely be found and it will appear only when $\sqrt{\varepsilon}\beta > 2$. In this case the first cone is exhibited in a peculiar manner with an obtuse value of Θ_1 whether $\sqrt{\varepsilon}\beta > 1$ or < 1. This may be due to a cumulative Doppler effect. The Doppler effect on Cherenkov radiation was discussed by FRANK [5], GINSBURG and FRANK [6] and AKHIEZER et al. [7]. If $\omega = \omega_0$, $\Theta_1 = 90°$, i.e. a plane of radiation is obtained. Moreover the first wavefront is not attenuated when $\varepsilon = 1$. This indicates that there is a possibility of the existence of radiation in empty space (!). This is perhaps due to non-relativistic calculations.

Acknowledgement

The author thanks Dr. T. C. ROY of Jadavpur University for a great deal of help in the preparation of this paper.

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Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 407—415 (1969)

RADIATION IN AN ANISOTROPIC ELECTRON PLASMA POLARIZED UNDER A STRONG MAGNETIC FIELD

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It is assumed that an unbounded anisotropic electron plasma is linearly polarized under an applied strong magnetic field and the linear distribution of electrons moves with uniform velocity. Electro-magnetic intensities and also radiation are slowly damped owing to collisions of electrons but there is no damping in the plane of the external magnetic field and the velocity of the linear distribution of electrons. As a limiting case the expression of Cherenkov radiation for the motion of a line charge in an infinite homogeneous isotropic medium is obtained.

1. Introduction

There are many papers on wave propagation in plasma. COHEN [1] has considered Cherenkov type radiation in plasma, MAJUMDER [2] has investigated the electro-dynamics of a charged particle moving with a constant velocity in plasma, TUAN and SESHADRI [3] have published a paper on radiation due to the uniform motion of a line charge in compressible plasma. In these treatments plasma is idealized to be homogeneous and free from collisions and stationary ions neutralize the electrons on the average.

In this paper an infinite electron plasma which is incompressible but anisotropic in dielectric property is studied. It is acted on by a strong magnetic field and the electrons are linearly polarized. The linear distribution of the electrons moves with a uniform velocity greater than a velocity like the phase velocity of light in the medium. Collisions of electrons are not ignored and this consideration exhibits the damping nature of electro-magnetic intensities and radiation. Though radiation is attenuated at a large distance it is interesting to note that in a certain plane there is no attenuation. Moreover, radiation is confined between two planes perpendicular to planes of wave propagation and in the limiting case it is identical with the expression of Cherenkov radiation for the motion of line charge considered by the author [4] previously.

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2. Phenomenological equations

We assume that (i) ions are stationary and they neutralize the electrons on average, (ii) an external magnetic field $\vec{B}_0$ is acting in the direction of the y-axis, (iii) the electrons are linearly distributed and they move like line charge with uniform velocity "u" in the direction of the z-axis, (iv) collisions of electrons are not negligible.

Maxwell's equations for field variables $\vec{E}$ and $\vec{H}$ (electric and magnetic vectors) are

$$\text{rot}\,\vec{E} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t}\,, \tag{1}$$

$$\text{rot}\,\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\gamma r} - \frac{4\pi}{c} N e \vec{v} + \frac{4\pi}{c}\vec{J}\,. \tag{2}$$

Force equation is

$$Nm\frac{\partial \vec{V}}{\partial t} = -e[\vec{E} + \vec{V}\times\vec{B}_0] - q\vec{v}\,. \tag{3}$$

Here N, e, m, u, g, V and c are the average electron number density, electron charge, mass of an electron, magnetic permeability in vacuum, the collision factor, the velocity of electrons and the velocity of light in free space, respectively.

$\vec{D} = \varepsilon\,\vec{E}$, where ε is the dielectric tensor described as

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}. \tag{4}$$

$\vec{J}$ is the current density and

$$J_x = 0 = J_y,\ J_z = qu\,\delta(x)\,\delta(z - ut)\,, \tag{5}$$

where q is the line charge density.

With a little manipulation one can say that the problem is two dimensional and all the vector quantities are independent of y. Under Fourier transform of the form

$$f(z, x, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z, x, t)\, e^{-i\omega t}\, dt\,,$$

$$f(z, x, t) = \int_{-\infty}^{\infty} f(z, x, \omega)\, e^{i\omega t}\, d\omega$$

eqs. (1), (2) and (3) transform to

$$\text{rot}\,\vec{E}(z,x,\omega) = -\frac{i\mu\omega}{c}\vec{H}(z,x,\omega)\,, \tag{6}$$

$$\text{rot}\,\vec{H}(z,x,\omega) = \frac{i\omega}{c}\vec{D}(z,x,\omega) - \frac{4\pi}{c}Ne\,\vec{v}(z,x,\omega) + \frac{4\pi}{c}\vec{J}(z,x,\omega)\,, \tag{7}$$

$$i\omega\, Nm\,\vec{v}(z,x,\omega) = -e\left[\vec{E}(z,x,\omega) + \vec{v}(z,x,\omega)\times\vec{\beta}_0\right] - g\,v(z,x,\omega)\,. \tag{8}$$

By the rule of Fourier transform

$$J_z(z,x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} qu\,\delta(x)\,\delta(z-ut)\,e^{-i\omega t}\,dt = -\frac{q}{2\pi}\,\delta(x)\,e^{-\frac{i\omega z}{u}}\,. \tag{9}$$

Without any inconsistency in Eqs. (6), (7) and (8) we can assume that $E_y = 0 = H_x = H_y = V_z$.

From (8)

$$i\,\omega\, Nm\,V_x = -Ne\,E_x + Ne\,B_0\,V_z - gV_x\,, \tag{10}$$

$$i\omega\, Nm\,V_z = -Ne\,E_z - Ne\,\beta_0\,V_x - gV_z\,. \tag{11}$$

By these two equations

$$V_x = -Ne\,\frac{(g+i\omega\, Nm)\,E_x + Ne\,\beta_0\,E_z}{(g+i\omega\, Nm)^2 + N^2e^2\beta_0^2}\,, \tag{12}$$

$$V_z = Ne\,\frac{Ne\,\beta_0\,E_x - (g+i\omega\, Nm)\,E_z}{(g+i\omega\, Nm)^2 + N^2e^2\beta_0^2}\,. \tag{13}$$

Putting these values of V_x and V_z in (6) and (7) and solving for E_x and E_z we have

$$E_x = \frac{-\left(L+\frac{i\omega}{e}\varepsilon_3\right)\frac{\partial H_y}{e} - M\,\frac{\partial H_y}{\partial x} + \frac{4\pi}{c}MJ_z}{\left(L+\frac{i\omega}{c}\varepsilon_1\right)\left(L+\frac{i\omega}{c}\varepsilon_3\right) + M^2}\,, \tag{14}$$

$$E_z = \frac{-M\frac{\partial H_y}{\partial z} + \left(L+\frac{i\omega}{c}\varepsilon_1\right)\frac{\partial H_y}{\partial x} - \frac{4\pi}{c}\left(L+\frac{i\omega}{c}\varepsilon_1\right)J_z}{\left(L+\frac{i\omega}{c}\varepsilon_1\right)\left(L+\frac{i\omega}{c}\varepsilon_3\right) + M^2}\,. \tag{15}$$

Here

$$L = \frac{4\pi}{c}\,\frac{N^2e^2(g+i\omega\, Nm)}{(g+i\omega\, Nm)^2 + N^2e^2\beta_0^2}\,, \quad M = \frac{4\pi}{c}\,\frac{N^2e^2\beta_0}{g+i\omega\, Nm)^2\,N^2e^2\beta_0^2}\,. \tag{16}$$

By (6), (14) and (15),

$$\left(L+\frac{i\omega}{c}\varepsilon_1\right)\frac{\partial^2 H_y}{\partial x^2}+\left(L+\frac{i\omega}{c}\varepsilon_3\right)\frac{\partial^2 H_y}{\partial z^2}-$$
$$-\frac{i\mu\omega}{c}\left\{\left(L+\frac{i\omega}{c}\varepsilon_1\right)\left(\left(L+\frac{i\omega}{c}\varepsilon_3\right)+M^2\right\}H_y=\right.$$
$$=\frac{4\pi}{c}\left\{M\frac{\partial J_z}{\partial z}+\left(L+\frac{i\omega}{c}\varepsilon_1\right)\frac{\partial J_z}{\partial z}\right\}. \tag{17}$$

3. Formal solution of the equation (17)

Considering Fourier transform with respect to x in the form

$$f(z,x,w)=\int_{-\infty}^{\infty}f(z,k,\omega)\,e^{ikx}\,dk$$

and in view of dependence of field components on "z" through the phase factor $e^{-\frac{i\omega z}{u}}$ the equation (17) gives the following result

$$H_y(z,k,\omega)=\frac{iq}{\pi c}\times$$
$$\times\frac{\left\{k\left(L+\frac{i\omega}{c}\varepsilon_1\right)-\frac{M\omega}{u}\right\}e^{-\frac{i\omega z}{u}}}{k^2\left(L+\frac{i\omega}{c}\varepsilon_1\right)+\frac{\omega^2}{u^2}\left(L+\frac{i\omega}{c}\varepsilon_3\right)\frac{i\mu\omega}{c}\left\{\left(L+\frac{i\omega}{c}\varepsilon_1\right)\left(L+\frac{i\omega}{c}\varepsilon_3\right)+M^2\right\}},$$

$$\therefore H_y(z,x,\omega)=\frac{i\omega}{\pi c}\int_{-\infty}^{\infty}\times$$
$$\times\frac{\left\{k-\frac{M\omega}{u\left(L+\frac{i\omega}{c}\varepsilon_1\right)}\right\}e^{-\frac{i\omega z}{u}+ikx}\,dk}{k^2+\frac{\omega^2}{u^2}\frac{L+\frac{i\omega}{c}\varepsilon_3}{L+\frac{i\omega}{c}\varepsilon_1}+\frac{i\mu\omega}{c}\left\{\left(L+\frac{i\omega}{c}\varepsilon_3\right)+\frac{M^2}{L+\frac{i\omega}{c}\varepsilon_1}\right\}}. \tag{18}$$

To perform the integration (18) we assume that B_0 is very large and powers of $1/B_0$ higher than 2 are negligibly small. With this assumption and for a particular value of "ω" the equation (18) reduces to

$$H_y(z,x,\omega)=\frac{iq}{\pi c}\int_{-\infty}^{\infty}\frac{k-\frac{i4\pi Ne}{\varepsilon_1 u\beta_0}}{k^2-(a-ib)}e^{-\frac{i\omega z}{u}+ikx}\,dk, \tag{19}$$

where

$$a = \omega^2 \mu\varepsilon_3 \eta \left\{1 + \frac{4\pi N}{\beta_0^2}\left(\frac{m}{\varepsilon_3} + \frac{m}{\varepsilon_1^2 u^2 \mu\eta} - \frac{4\pi N e^2}{\varepsilon_1 \varepsilon_3 \eta c^2 \omega^2}\right)\right\}$$
$$b = \frac{4\pi g\omega}{\eta_0^2}\left(\eta\mu + \frac{\varepsilon_3}{\varepsilon_1^2 u^2}\right), \qquad \eta = \frac{1}{c^2} - \frac{1}{\varepsilon_1 \mu u^2}. \tag{20}$$

On taking $\eta > 0$, the integration of (19) can be performed by the residue method. The poles are $\pm J(a - ib) = \pm(\alpha - i\beta)$ (say).

Then

$$H_y(z, x, \omega) = -\frac{q}{c}\left[1 + \frac{i\,4\pi Ne}{\varepsilon_1 u \beta_0 (\alpha - i\beta)}\right] e^{-\frac{i\omega z}{u} - i\alpha x - \beta x} \quad \text{when } x > 0,$$
$$H_y(z, x, \omega) = -\frac{q}{c}\left[1 - \frac{i\,4\pi Ne}{\varepsilon_1 u \beta_0 (\alpha - i\beta)}\right] e^{-\frac{i\omega z}{u} + i\alpha x + \beta x} \quad \text{when } x < 0, \tag{21}$$
$$\alpha = \frac{1}{V2} V\left(\sqrt{a^2 + b^2} + a\right), \qquad \beta = \frac{1}{V2} V\left(\sqrt{a^2 + b^2} - a\right).$$

Again

$$\alpha \approx \omega \sqrt{\mu\varepsilon_3 \eta}\left[1 + \frac{2\pi N}{\beta_0^2}\left(\frac{m}{\varepsilon_3} + \frac{m}{\mu\eta\varepsilon_1^2 u^2} - \frac{4\pi Ne^2}{\varepsilon_1 \varepsilon_3 \eta c^2 \omega^2}\right)\right],$$
$$\beta \approx \frac{2\pi g}{\sqrt{\mu\varepsilon_3 \eta}\,\beta_0^2}\left(\mu\eta + \frac{\varepsilon_3}{\varepsilon_1^2 u^2}\right).$$

4. Expressions of E, H and V

By the above approximations the following results are obtained with the help of (21), (12), (13), (14) and (15).

When $x > 0$,

$$H_y(z, x, \omega) = -\frac{q}{c}\left[1 + \frac{i\,4\pi Ne}{\varepsilon_1 u \beta_0 \omega \sqrt{\mu\varepsilon_2 \eta}}\right] e^{-\frac{i\omega z}{u} - i\alpha x - \beta x},$$
$$Ex(z, x, \omega) = -\frac{q}{\varepsilon_1 u}\left[1 + \frac{4\pi N}{\beta_0^2}\left(\frac{8\pi Ne^2}{\varepsilon_1 \varepsilon_3 \omega^2} - \frac{m}{\varepsilon_1}\right) - \right.$$
$$\left. - \frac{i\,4\pi N}{\beta_0}\left(\frac{Ne\,u\sqrt{\mu\varepsilon_3 \eta}}{\varepsilon_3 \omega} - \frac{Ne}{\varepsilon_1 u \omega \sqrt{\mu\varepsilon_3 \eta}} - \frac{g}{\varepsilon_1 \omega \beta_0}\right)\right] e^{-\frac{i\omega z}{u} - i\alpha x - \beta x},$$
$$E_z(z, x, \omega) = \frac{q\sqrt{\mu\varepsilon_3 \eta}}{\varepsilon_3}\left[1 + \frac{2\pi N}{\beta_0^2}\left(\frac{m}{\mu\eta\varepsilon_1^2 u^2} + \frac{8\pi Ne^2}{\varepsilon_1 \varepsilon_3 \omega^2} - \frac{m}{\varepsilon_3} - \right.\right.$$
$$\left. - \frac{4\pi N^2}{\varepsilon_1 \varepsilon_3 \eta c^2 \omega^2} - \frac{8\pi Ne^2}{\varepsilon_1^2 u^2 \omega^2 \mu\varepsilon_3 \eta}\right) + \frac{i\,2\pi}{\beta_0}\left(\frac{4\,Ne}{\varepsilon_1 u \omega \sqrt{\mu\varepsilon_3 \eta}} + \right.$$
$$\left.\left. + \frac{g}{\varepsilon_3 \omega \beta_0} - \frac{g}{\mu\eta\omega\varepsilon_1^2 u^2 \beta_0}\right)\right] e^{-\frac{i\omega z}{u} - i\alpha x - \beta x}, \tag{22}$$

$$V_x(z,x,\omega) = -q\left[\left(\frac{\sqrt{\mu\varepsilon_3\eta}}{\varepsilon_3\beta_9} - \frac{g}{Ne\,\varepsilon_1\,u\,\beta_0^2}\right) + \right.$$

$$\left. + i\left(\frac{8\pi\,Ne}{\varepsilon_1\,\varepsilon_3\,\omega\,\beta_0^2} + \frac{m\omega}{e\,\varepsilon_1\,u\,\beta_0^2}\right)\right] e^{-\frac{i\omega z}{u} - i\alpha x - \beta x},$$

$$V_z(z,x,\omega) = +q\left[\frac{1}{\beta_0\,Ne}\left(\frac{1}{\varepsilon_1\,u} + \frac{\sqrt{\mu\,\varepsilon_3\,\eta}\,g}{\varepsilon_3\beta_0}\right) + \frac{i}{\beta_0^2}\left(\frac{\sqrt{\mu\,\varepsilon_3\,\eta}\,m\,\omega}{e\,\varepsilon_3} - \right.\right.$$

$$\left.\left. - \frac{4\pi\,Ne\sqrt{\mu\,\varepsilon_3\,\eta}}{\varepsilon_1\,\varepsilon_3\,\omega} + \frac{4\pi\,Ne}{\varepsilon_1^2\,u^2\,\omega\sqrt{\mu\,\varepsilon_3\,\eta}}\right)\right] e^{-\frac{i\omega z}{u} - i\alpha x - \beta x}.$$

When $x < 0$,

$$H_y(z,x,\omega) = -\frac{q}{c}\left[1 - \frac{i\,4\pi\,Ne}{\varepsilon_1\,u\,\omega\,\beta_0\sqrt{\mu\,\varepsilon_4\,\eta}}\right] e^{-\frac{i\omega z}{u} + i\alpha x + \beta x},$$

$$E_x(z,x,\omega) = -\frac{q}{\varepsilon_1\,u}\left[1 + \frac{2\pi\,N}{\beta_0^2}\left(\frac{\varepsilon_1\,\varepsilon_3\,\omega^2}{16\,\pi\,Ne^2} - \frac{2m}{\varepsilon_1}\right) + \right.$$

$$\left. + \frac{i4\pi}{\beta_0}\left(\frac{Ne\,u\sqrt{\mu\varepsilon_3\,\eta}}{\varepsilon_3\,\omega} - \frac{Ne}{\varepsilon_1\,u\,\omega\sqrt{\mu\varepsilon_3\,\eta}} + \frac{g}{\varepsilon_1\,\omega\,\beta_0}\right)\right] e^{-\frac{i\omega z}{u} + i\alpha x + \beta x}, \qquad (23)$$

$$E_z(z,x,\omega) = -\frac{q\sqrt{\mu\varepsilon_3\,\eta}}{\varepsilon_3}\left[1 + \frac{2\pi\,N}{\beta_0^2}\left(\frac{m}{\mu\eta\,\varepsilon_1^2\,u^2} + \frac{8\pi\,Ne^2}{\varepsilon_1\,\varepsilon_3\,\omega^2} - \frac{m}{\varepsilon_3} - \right.\right.$$

$$\left. - \frac{4\pi\,Ne^2}{\varepsilon_1\,\varepsilon_3\,\eta\,c^2\,\omega^2} - \frac{8\pi\,Ne^2}{\varepsilon_1^2\,u^2\,\omega^2\,\mu\varepsilon_3\,\eta}\right) - \frac{i\,2\pi}{\beta_0}\left(\frac{4\,Ne}{\varepsilon_1\,u\,\omega\sqrt{\mu\varepsilon_3\,\eta}} + \frac{g}{\mu\eta\omega\varepsilon_1^2\,u^2\,\beta_0} - \right.$$

$$\left.\left. - \frac{g}{\varepsilon_3\,\omega\,\beta_0}\right)\right] e^{-\frac{i\omega z}{u} + i\alpha x + \beta x}.$$

For $x > 0$ and $\omega > 0$ to $\omega_{\max}$,

$$Re\ \ H_y = -\frac{2q}{c}\,e^{-\beta x}\left[\cos\chi + \frac{4\pi\,Ne}{\varepsilon_1\,u\,\omega\,\beta_0\sqrt{\mu\varepsilon_3\,\eta}}\sin\chi\right],$$

$$Re\ \ E_x = -\frac{2q}{\varepsilon_1\,u}\,e^{-\beta x}\left[\left(1 + \frac{32\,\pi^2\,N^2\,e^2}{\varepsilon_1\,\varepsilon_3\,\omega^2\,\beta_0^2} - \frac{4\pi\,Nm}{\varepsilon_1\,\beta_0^2}\right)\cos\chi - \right.$$

$$\left. - \frac{4\pi}{\beta_0}\left(\frac{Ne\,u\sqrt{\mu\varepsilon_3\,\eta}}{\varepsilon_3\,\omega} - \frac{Ne}{\varepsilon_1\,u\,\omega\sqrt{\mu\varepsilon_3\,\eta}} - \frac{g}{\varepsilon_1\,\omega\,\beta_0}\right)\sin\chi\right], \qquad (24)$$

$$Re\, E_z = \frac{2q\sqrt{\mu\varepsilon_3\,\eta}\,e^{-\beta x}}{\varepsilon_3}\left[\left\{1+\frac{2\pi N}{\beta_0^2}\left(\frac{m}{\mu\eta\varepsilon_1^2 u^2}+\frac{8\pi Ne^2}{\varepsilon_1\varepsilon_3\omega^2}-\frac{m}{\varepsilon_3}-\right.\right.\right.$$

$$\left.\left.\left.-\frac{4\pi Ne^2}{\varepsilon_1\varepsilon_3\eta\omega^2c^2}-\frac{8\pi Ne^2}{\varepsilon_1^2u^2\omega^2\mu\varepsilon_3\eta}\right)\right\}\cos\chi+\frac{2\pi}{\beta_0}\left(\frac{4Ne}{\varepsilon_1 u\omega\sqrt{\mu\varepsilon_3\eta}}+\right.$$

$$\left.+\frac{g}{\varepsilon_3\omega\beta_0}-\frac{g}{\mu\eta\omega\varepsilon_1^2u^2\beta_0}\right)\sin\chi\Bigg],$$

where

$$\chi=\frac{\omega z}{u}+\alpha x+\omega t.$$

5. Calculation of radiation

The power radiated per unit time per unit frequency interval is

$$\vec{S}=\frac{c}{4\pi}\int_{-\infty}^{\infty}(Re\,\vec{E}\times Re\,\vec{H})\,dz\,.$$

The energy radiated in the positive direction of the x-axis is

$$S_x=\frac{c}{4\pi}\int_{-\infty}^{\infty}(-\,Re\,E_z\cdot Re\,H_y)\,dz$$

$$=\frac{q^2e^{-2\beta x}\sqrt{\mu\varepsilon_3\,\eta}\,u}{\varepsilon_3}\left[1+\frac{2\pi N}{\beta_0^2}\left(\frac{m}{\mu\eta\,\varepsilon_1^2u^2}+\frac{8\pi Ne^2}{\varepsilon_1\varepsilon_3\omega^2}+\right.\right.$$

$$\left.\left.+\frac{8\pi Ne^2}{\varepsilon_1^2u^2\omega^2\mu\varepsilon_3\eta}-\frac{m}{\varepsilon_3}-\frac{4\pi Ne^2}{\varepsilon_1\varepsilon_3\eta\,c^2\omega^2}\right)\right] \tag{25}$$

with the help of (24).

The energy radiated in the negative direction of the x-axis is

$$S_{-x}=-\frac{c}{4\pi}\int_{-\infty}^{\infty}(-\,Re\,E_z\,,\;Re\,H_y)\,dz\,.$$

By the help of (23) it is found that $S_x=S-x$.

The z-component of $\vec{S}$ is obtained from the equations of (24) and

$$S_z=\frac{q^2e^{-2\beta x}}{\varepsilon_1}\left[1+\frac{4\pi N}{\beta_0^2}\left(\frac{4\pi Ne^2}{\varepsilon_1\varepsilon_3\omega^2}-\frac{m}{\varepsilon_1}+\frac{4\pi Ne^2}{\varepsilon_1^2u^2\omega^2\mu\,\varepsilon_3\eta}\right)\right]. \tag{26}$$

It is of interest to examine the angular distribution of the radiated energy. The angle Θ between the direction of motion of linear distribution and the Cherenkov ray is given by

$$\tan\theta = \frac{S_x}{S_z} = \sqrt{\frac{\varepsilon_1}{\varepsilon_3}\left(\mu\varepsilon_2\frac{u^2}{c^2} - 1\right)}\left[1 - \frac{2\pi N}{\beta_0^2}\left(\frac{4\pi Ne^2}{\varepsilon_1\varepsilon_3\eta\omega^2 c^2} + \right.\right.$$
$$\left.\left. + \frac{m}{\varepsilon_3} - \frac{2m}{\varepsilon_1} - \frac{m}{\mu\eta\varepsilon_1^2 u^2}\right)\right]. \tag{27}$$

6. Conclusion

Eqs. (22), (23), (25) and (26) reveal that the electromagnetic intensities and the radiated energy have exponential damping with x. Since g is comparatively small and B_0 is very large, the penetration depth $1/\beta$ is fairly large and consequently attenuation takes place very slowly. If g is zero, there will be no damping. Therefore collisions of electrons are solely responsible for damping when $\eta > 0$.

Eqs. (22) and (23) show that outgoing plane waves are propagating when $\eta > 0$. This is a Cherenkov like condition and it implies a cut-off in frequency. At a particular frequency, planes are parallel to $dx + \omega/u\, z = 0$ and $\alpha x - \omega/u\, z = 0$. Radiation is confined between two planes perpendicular to the above planes. It is comparable with a Cherenkov cone. The angle between the planes is 2Θ which is obtained by (27). Maximum value of Θ is given by

$$\tan^{-1}\sqrt{\left\{\frac{\varepsilon_1}{\varepsilon_3}\left(\mu\varepsilon_1\frac{u^2}{c^2} - 1\right)\right\}}.$$

Thus, the strong magnetic field polarizes the incompressible and anisotropic electron plasma in such a way that the whole system becomes Cherenkov type when the velocity of the linearly polarized electrons exceeds a certain limit.

An interesting result is obtained if we put $N = 0$, $\varepsilon_1 = \varepsilon_3 = \varepsilon$ and $g = 0$ in (25) and (27). In this case

$$S_x = \frac{q^2}{\varepsilon}\sqrt{\left(\eta\varepsilon\frac{u^2}{c^2} - 1\right)} \quad \text{and} \quad \tan\Theta = \sqrt{\left(\mu\varepsilon\frac{u^2}{c^2} - 1\right)}.$$

For unit value of μ these results have been published by the author [4] on consideration of Cherenkov radiation by line charge.

If $\mu < 0$, the damping depends on α and in this case attenuation takes place more rapidly in comparison with the case of a positive value of η. In the plane $x = 0$, i.e. in the plane of the external magnetic field and the direction of the velocity of linear distribution of electrons, the field vectors and radiation are free from a damping factor. Thus, there is a plane where attenuation does not take place whether $\eta > 0$ or < 0.

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Acta Physica Academiae Scientiarum Hungaricae, Tomus 26 (4), pp. 417—419 (1969)

STATISTICAL MODEL CALCULATION OF THE BRANCHING RATIOS OF N^* (1470)

By

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(Received 6. II. 1969)

Our aim when writing the present paper was to obtain the branching ratios of different decay modes of the nucleon isobar N^* (1470) having the isospin 1/2, and also some of the proposed [1] $T = 5/2$ isobar with the same mass. To do this we used the following formula for the probability of a final state $|f>$ starting from the state $|i>$ [2]:

$$w(i \to f) = |C_{T\vartheta}(T_1 \ldots T_n, \vartheta_1 \ldots \vartheta_n)|^2 \frac{\prod_i (2S_i + 1)^{\sigma_i}}{\prod_j Nj!} \left| \frac{\Omega}{8\pi^3} \right|^{n-1} \varrho_n (E; m_1 \ldots m_n) \tag{A}$$

obtained by means of a statistical model. In the above formula, $C_{T\vartheta}$ is the isospin coefficient projecting the final state, consisting of n particles, on the initial one, having the total isospin T and its third component ϑ. S_i is the i-th kind of spin, appearing in the final state, and σ_i is the number of particles with this spin. N_j is the number of equal particles of the kind j (all the members of an isomultiplet have to be considered as equal). Ω is a volume of the size of a pion: $\Omega/8\,\pi^3 = 0.615 \times 10^{-8}\ \mathrm{MeV}^{-3}$ (we are working with $c = h/2\,\pi = 1$, the mass being measured in MeV). In the following, we shall use the shorthand notation $\Omega/8\,\pi^3 = \mu$. ϱ_n is the relativistic non-invariant phase space density for n particles of masses $m_1 \ldots m_n$ and total centre of mass energy E:

$$\varrho_n = \int d\vec{p}_1 \ldots d\vec{p}_n\, \delta \left(E - \sum_{i=1}^{n} E_i \right) \delta \left(\sum_{i=1}^{n} \vec{P}_i \right) \tag{B}$$

(calculated in the overall c.m. system).

HAGEDORN's formula (A) which we used for calculating the branching ratios is appropriate, by construction, only for central collisions. Therefore, we may expect that this formula works well for decays, which can be considered, in fact, the most central "collisions". Finally, we note that the formula (A) gives the results in a non-normalised form, because of an undetermined constant, not explicitly written.

1. Now, we calculate the probabilities of the different decay modes of the $T = 1/2$ N^* isobar:

$$N^{*+} \to p\,\pi^\circ, \tag{1}$$
$$N^{*+} \to n\,\pi^+, \tag{2}$$
$$N^{*+} \to p\,\pi^+\,\pi^-, \tag{3}$$
$$N^{*+} \to p\,\pi^\circ\,\pi^\circ, \tag{4}$$
$$N^{*+} \to n\,\pi^+\,\pi^\circ, \tag{5}$$
$$N^{*+} \to p\,\sigma \to p\,\pi^+\,\pi^- \text{ or } p\,\pi^\circ\,\pi^\circ, \tag{6}$$
$$N^{*+} \to \Delta^{++}\pi^- \to p\,\pi^+\,\pi^-, \tag{7}$$
$$N^{*+} \to \Delta^+\pi^\circ \leq p\,\pi^\circ\,\pi^\circ \text{ or } n\,\pi^+\,\pi^\circ, \tag{8}$$
$$N^{*+} \to \Delta^\circ\pi^+ \to p\,\pi^-\,\pi^+ \text{ or } n\,\pi^\circ\,\pi^+. \tag{9}$$

σ is a hypothetical dipion resonance, having $T = 0$ and a mass of 410 MeV (see, for instance, the compilation [3]).

Formula (A) yields:

$$W_1 = 1.087 \times 10^6\,\mu, \quad W_2 = 2.175 \times 10^6\,\mu, \quad W_3 = 1.569 \times 10^{13}\,\mu^2,$$
$$W_4 = 1.046 \times 10^{13}\,\mu^2, \quad W_5 = 1.046 \times 10^{13}\,\mu^2, \quad W_6 = 2.254 \times 10^6\,\mu,$$
$$W_7 = 8.164 \times 10^5\,\mu, \quad W_8 = 5.443 \times 10^5\,\mu, \quad W_9 = 2.721 \times 10^5\,\mu.$$

For the reactions (7—9), we took into account that the radiative width of Δ is negligible compared with the total width of Δ(for instance, $\Gamma = 120$ MeV, but $\Gamma_\gamma = 0.65$ MeV [4] for Δ^+).

As can be seen, the dominant inelastic decay mode, predicted by formula (A) is N^* (1470) $\to N\,\sigma$, in agreement with the results obtained using other methods:

— Analysis of the reaction $\pi^- p \to \pi^-\,\pi^+\,n$ using a relativistic S-matrix formalism and assuming a N^* (1480) and N^* (1512) dominance [5].

— A relativistic formalism for analysing 3-body angular momentum states (6).

— Partial wave analysis combined with the isobar model [7]. However, we got a branching fraction for the elastic mode of 0.442, in disagreement with other results found by partial wave analysis (0.68 in [8] and 0.658 in [9]). Now, the dipion resonance σ (410) is a very controversial one (see, for example, the compilation [3]). Assuming that it does not exist, we found for the elastic mode a branching ratio of 0.637, in agreement with [8] and [9].

2. Supposing, as in [1], the existence of a $T = 5/2$ isobar at 1470 MeV, we looked for the probabilities of its decay modes:

$$N^{*+++} \to \Delta^{++}\pi^+, \tag{1}$$
$$N^{*+++} \to p\,\pi^+\,\pi^+. \tag{2}$$

We found:

$$W_1 = 16.328 \times 10^5 \mu, \qquad W_2 = 3.138 \times 10^{13} \mu^2$$

yielding $W_1/W_2 = 8.46$. If this isobar actually exists, this ratio could explain why a corresponding peak is not seen in the effective mass distribution for events with the $p\,\pi^+$ effective mass outside the Δ^{++} band (10).

Conclusions. We tried to use a statistical model to obtain the branching ratios of N^* (1470). The ratios found agree with those obtained by other methods.

Considering that the σ (410) dipion resonance does not exist, we found for the elastic channel a branching ratio of 0.637, very near to that obtained in other papers. This supports the non-existence of σ (410). Further, these calculations show that $N^{*+++}(1470)$ with $T = 5/2$ decays mainly via Δ^{++}.

Acknowledgement. We are grateful to Mr. E. HEGEDŰS for his continuous encouragement and also for helpful discussions.

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RECENSIONES

D. TER HAAR: **Collected Papers of P. L. Kapitza**

Vol. II. Pergamon Press

The second volume of the "Collected Papers of P. L. Kapitza" contains his works written between 1938 and 1964. This means 25 papers, nearly all of them published in Soviet periodicals and a few published in English journals, too.

The papers can be divided into three groups according to their topics: the first comprises papers on low-temperature physics and on the problems of gas-liquefaction. The papers of the second group have miscellaneous subjects. The papers of the third group can be labelled shortly: electrical oscillations, electronics.

One of the works on low-temperature physics deals with the viscosity of Helium II, with the so-called superfluidity. Another discusses the phenomena of heat-transfer in Helium II, attributing its high heat-conductivity to superfluidity. A result of these studies which are remarkable from a technical aspect is that it provides a new method for producing very low temperatures and approaching absolute zero temperature. In one paper KAPITZA works out a new method for air-liquefaction. He gives theoretical foundations for the method, and describes his apparatus called an "expansion turbine". The apparatus works with a high efficiency: 83 per cent. This liquefier works at pressures of about 5—6 atm. producing 30 kg of liquid air per hour. The description of the apparatus was received with interest in the literature and was the subject of considerable discussion.

In 3 more papers a further development of the helium liquefying technique is given. He aims at simplifying this technique and increasing the efficienty of liquefaction.

The group of papers on miscellaneous topics consists of 10 papers. One studies the Zeeman and Paschen-Back effects in strong magnetic fields up to 320 kg. The work on the stability of high-speed rotors deals with problems of considerable importance from technical aspects. The paper on heat-transfer in a two-dimensional turbulent flow is also of technical significance. KAPITZA carried out experiments on the flow of thin viscous fluid layers, too. In the next paper it is shown both theoretically and experimentally that the flow of thin viscous fluid layers is wave-like. In a further paper KAPITZA discusses the formation of sea waves by the wind.

Two papers deal with the motion of a pendulum: with the motion of a pendulum with a vibrating suspension point and with the dynamic stability of such a pendulum. One paper gives a method for the calculation of the sum of negative powers of roots of Bessel functions involved in many problems of mathematical physics.

The increase of heat-transfer and diffusion in a fluid in an undulatory flow is the subject of a further paper.

The following paper giving the hydrodynamical theory of lubrication in the presence of rolling is of practical importance. The series of papers belonging to this group ends with a work on the nature of the widely investigated ball-lightning.

Two of the 5 papers comprising the third group deal with the electronic problems of a hollow cylinder of finite length. In one of them KAPITZA determines the potential and in the other he considers the symmetrical electric oscillations of a perfectly conducting cylinder. The following 3 papers reproduce practically the book published under the title "High-Power Microwave Electronics" in the USSR and in 1964 by Pergamon Press. In the first paper the author deals with the motion of charged particles, then discusses the planotron in detail. Further on he reviews experimental results followed by a study of the magnetron. In the last chapter he deals with gridded cavity resonators: with rectangular and cylindrical cavities. The third paper discusses the electronic processes in a high-power generator of the magnetron type. The device introduced here theoretically and experimentally is called "nigotron". Two types are described: "single-set" and "double-set" nigotrons.

Even a brief survey of these two volumes gives an idea about the man who is an extraordinary figure not only of Soviet science but is recognized all over the world. KAPITZA enriched several fields of physics with achievements of outstanding importance. With his life's work he sets an example for young physicists to follow all over the world.

J. BOROS

D. TER HAAR: **Collected Papers of P. L. Kapitza**

Vol. III. Pergamon Press

Volume III completes the publication of the "Collected Papers of P. L. Kapitza". It contains a few semi-popular scientific papers as well as some papers of more general interest.

The volume contains 31 papers. They can be grouped according to their topics in the following way: magnetism, low temperature physics, organization of science, biographies and obituaries and, in conclusion, papers on miscellaneous topics. The papers were published between 1913 and 1966, some in English journals, but most in Russian ones and in the volume they appear in chronological order of publication.

Two papers deal with magnetism. One of these discusses the future of magnetism, the other, strong magnetic fields. 4 papers provide a policy survey of the elements of low-temperature physics: liquid air and its production, liquid helium and its production and superfluidity.

In one of his papers on the politics of science Kapitza describes the Institute for Physical Problems of the Academy of Sciences of the USSR founded in 1934. A further paper reports on the organization of scientific work in the Institute. The paper published under the title "The unity of science and engineering" is of great interest. As a commemoration of the fortieth anniversary of the Soviet state he reviews the development of physical science during that period. The paper entitled "Theory, Experiment, Practice" is of great value as also are his reflections on the future of science and on the efficacy of scientific work.

The life and work of great scientists particularly physicists is a favourite topic of KAPITZA. It is interesting for everybody to see how these scientists are pictured by this prominent representative of Soviet science. Amongst the biographies and memorial papers are to be found: RUTHERFORD, the great teacher of KAPITZA. Four of the papers deal with him. In one, KAPITZA surveys RUTHERFORD's historically significant scientific achievements. The last paper of the volume is a longer memorial lecture on RUTHERFORD given by KAPITZA in England on the invitation of the President of the Royal Society.

There is a paper on NEWTON for the tricentennial anniversary of his birth, one on Benjamin FRANKLIN for the 250th anniversary of his birth and one written on the occasion of the 200th anniversary of LOMONOSOV's death. In a long paper he describes the activity of Paul LANGEVIN, the progressive French physicist, an honorary member of the Academy of Sciences of the USSR.

The group of miscellaneous papers contains the following: The production of cod-liver oil. Science and war. On stereoscopic films. On the 25th anniversary of the establishment of the Soviet state. We fight for freedom. How is atomic war to be prevented? Problems of intensification of technological processes by oxygen.

J. BOROS

P. KRATOCHVIL: **Crystals**

Iliffe Books Ltd. London, 1967, 112 p.

This little monograph was published as the seventh volume of the series "Introduction to Physics". The monographs in the series have been written by distinguished Czech physists and their publication was supported by English specialists. The book — like the other volumes of the series — provides an introduction to this field for students at universities and technical colleges.

In the Introduction the author states the importance of crystal physics in research of solids, and this fact is also specially technically important. He points out here that the results of basic research are frequently not used in applications, but in this field the results of physical research have provided a basis for rapid technical progress.

As an application of the results of crystal physics the widely known examples of crystal rectifiers and transistors are mentioned.

The book deals with its subject-matter in four chapters. Chapter 1 supplies the knowledge of atomic physics necessary to understand crystal physics. It describes the four types of bonds in crystals. Then, crystal systems and the main lattice types are summarized. The reader is introduced to the concept of ideal and real crystals, followed by a description of various imperfections to be found in real crystals. The determination of crystal structure and the detection of lattice defects are briefly discussed.

Chapter 3 deals with the problem of crystal growth. First, the present accepted theory of crystal growth is outlined. Then, the questions of the formation of crystallization nuclei, growth from the melt, crystallization of impure materials, and segregation of impurities are dealt with. There follows a description of the theory and technique of zone-melting and zone-purification. The chapter ends with a survey of the important methods of growing crystals from the melt, from solution and from the vapour phase.

Chapter 4 ends with a description of some crystal properties and the use of crystals. It discusses the electrical, mechanical, optical and magnetic properties of solids.

J. Boros

M. Françon: Diffraction. Coherence in Optics

Translated from the original French by Barbara Jeffrey, Pergamon Press, 1966, 139 p.

This book presents an account of Fraunhofer diffraction phenomena and is essentially the subject matter of a course given by the author at the Faculty of Science in Paris. As the author states in the Preface, the book is intended to give those basic concepts which are needed by all engineers and researchers working in this field. The book is divided into seven chapters.

As an introduction, vibrations, waves, the Huygens—Fresnel principle, Fraunhofer and Fresnel diffraction phenomena and the general integral giving the luminous intensity at a point, are treated. After this, diffraction phenomena arising from slits of the simplest form are treated: Circular, rectangular aperture, narrow slit. It also gives the distribution of luminous flux in the Airy spot. In the following chapter Fourier integrals and transforms are dealt with. These are indispensable mathematical tools in the understanding of diffraction phenomena.

After this, diffraction by several apertures identical in shape and orientation, for example diffraction by two, three, or more narrow slits, is treated. Diffraction by complementary screens, and Babinet's theorem are also discussed here.

The next chapter includes diffraction by extended luminous sources, and a detailed analysis of spatial and time coherence is presented.

Chapter VI studies diffraction phenomena in perfect optical instruments. It gives the resolving power for astronomical telescopes, microscopes and prism spectroscopes. Diffraction gratings, phase contrast instruments and the Clark ground method are also discussed in detail.

The final chapter studies diffraction phenomena in real optical instruments.

The interested reader can find an extensive bibliography at the end of the book.

J. Boros

Nuclear structure and electromagnetic interactions

Edited by N. Macdonald, Oliver & Boyd Ltd., Edinburgh, 1965

The book is based on lectures given at the 5th Scottish Universities' Summer School in Physics in 1964. The lecturers were recognised experts in this subject. They have written separate parts of this book. The main purpose of the book is to give a particular study of experimental methods and results in the field of electromagnetic and photonuclear processes and to consider the interpretation of these phenomena in terms of modern theoretical concepts of nuclear models and the electromagnetic properties of nuclei.

The first two chapters give an introductory background to nuclear structure theory. Prof. A. DE-SHALIT writes about nuclear models and the electromagnetic properties of nuclei. He makes some remarks on the role of nuclear models, about the information we can get from wave-functions and how we can test wave-functions by comparing the calculated matrix elements with the measured ones. Prof. B. F. BAYMAN writes about the general properties of some nuclear models. He deals in detail with the independent particle model or shell model for a spherical nucleus and for a strongly deformed nucleus, then with the quasi-particle method and the boson model. The third chapter deals with radiative transitions following nuclear reactions and is written by A. E. LITHERLAND. He describes several methods including the Doppler shift attenuation method for measuring nuclear lifetimes down to about $3 \cdot 10^{-14}$ seconds, and the gamma-ray angular distribution and correlation methods. As illustration, he decentres many experimental results from measurements made at the Chalk River Nuclear Laboratories. The following chapter written by E. HAYWARD deals with photonuclear reactions. After a short summary of the theoretical results he describes the various experimental possibilities and deals in more detail with photonuclear cross-section measurements on heavy nuclei and the hydrodynamic model, with light nuclei and the independent particle model and with the decay of the dipole state.

In the next chapter Prof. G. R. BISHOP gives an account of all we know about electron scattering by nuclei, the virtual photon picture, the Born approximation, radiative corrections, phase-shift analysis of elastic scattering, and the scattering form factors for nuclear models, especially the form factors for the independent particle model.

Coulomb excitation and nuclear fission, both theoretical and experimental, are briefly summarized by J. O. NEWTON and J. R. HUIZENGA. The last three chapters of the book deal with nuclear radiation detectors written by A. T. G. FERGUSON; with fast electronics in nuclear physics, by P. R. ORMAN; and with data processing by J. V. KANE. Each of these is a very interesting summary of his subject.

The book is recommended to research workers who are interested in nuclear structure investigations.

D. KISDI

H. A. BETHE u. R. JACKIW: **Intermediate Quantum Mechanics**

Second Edition, XVI + 393 S., W. A. Benjamin, Inc., New York, Amsterdam, 1968.

Die zweite Auflage dieses Buches, das schon oftmals gewürdigt wurde, weist dieselben Vorzüge auf wie die erste. Erfreulicherweise ist die zweite Auflage wesentlich erweitert. Beide Auflagen sind in erster Linie für Studierende der theoretischen Physik geschrieben und dienen als eine Einführung in dieses Gebiet. Das Buch bringt Vieles, das man im allgemeinen in Büchern über Quantenmechanik nicht findet. Die Auswahl des Stoffes ist sehr glücklich und entspricht durchaus den modernsten Bedürfnissen auf diesem Gebiet.

Das Buch gliedert sich in 4 Hauptteile, diese sind: Theorie der Atomstruktur, Halbklassische Theorie der Strahlung, Zusammenstösse von Atomen und schliesslich Relativistische Gleichungen.

Wie die Autoren hervorheben, soll dieses Buch kein Text-Buch der Quantenmechanik sein, sondern eher als ein Ergänzungswerk zu einem solchen dienen, dadurch wird auch die auf die schon weiter oben hingewiesene Auswahl des Stoffes begründet.

Im Verhältnis zur ersten Auflage erweitert wurde das Buch hauptsächlich durch eine ziemlich ausführliche Theorie der Zusammenstösse von Atomen, während einige Kapitel über Feldtheorie der ersten Auflage weggelassen wurden.

Die am Ende der Kapitel angeführten und gut ausgewählten Probleme sollen zur Mitarbeit des Lesers beitragen.

Etwas mangelhaft sind an manchen Stellen die Literaturhinweise. Diese scheinen mir an einigen Stellen etwas rhapsodisch ausgewählt zu sein.

Das Buch kann allen die sich für Quantenmechanik interessieren, insbesondere den Studierenden auf diesem Gebiet, wärmstens empfohlen werden.

P. GOMBÁS

Printed in Hungary

A kiadásért felel az Akadémiai Kiadó igazgatója Műszaki szerkesztő: Farkas Sándor

A kézirat nyomdába érkezett: 1969. VII. 17. — Terjedelem: 10 (A/5) ív, 11 ábra

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