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ACTA PHYSICA

ACADEMIAE SCIENTIARUM HUNGARICAE

TOMUS XIX

AKADÉMIAI KIADÓ, BUDAPEST
1965

QUANTUM THEORY OF SYSTEMS WITH MANY DEGREES OF FREEDOM

SYMPOSIUM AT KESZTHELY
September 1964

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Presented by *K. F. Novobátzky*

Introduction

The quantum theory of systems with low and bounded number of degrees of freedom has been one of the closed and successfully applied chapters of physics for decades. Today the great open question of theoretical physics is the quantum theory of systems with many and unbounded degrees of freedom. The study of problems with an infinite number of degrees of freedom has turned out to be as fundamental as the classical two-body problem. On the one hand, it has become clear, that the low energy systems built up by very many strongly interacting particles (fluid and solid states of matter, heavy nuclei) can be approximated more successfully starting from systems with an infinite number of degrees of freedom, than by starting from the two-body problem. On the other hand it has become clear, too, that any small piece of matter possesses essentially an infitite number of degrees of freedom, which are mostly frozen in at a lower temperature, but become active at high energies.

We have learned in recent years, that the quantum theory with an infinite number of degrees of freedom shows several qualitatively new characteristics, which could not be foreseen by studying only the quantum mechanics of a few particles. The first and very spectacular indication of this was the understanding of superconductivity, starting with the pioneering work of BARDEEN, COOPER and SCHRIEFFER, and culminating with BOGOLJUBOV and VALATIN. After this first success the special mathematical properties of material systems with an infinite number of degrees of freedom were investigated by the theoreticians interested in solid state physics and by those interested in high energy physics. Having recognized the role of the nonseparable Hilbert space and the inequivalent representations in quantum theory, it was suggested that the wider mathematical framework offered new physical consequences of primary importance. HEISENBERG was the first to suggest that the degeneracy of the vacuum state and the spontaneous breakdown of symmetries could explain many strange phenomena in particle physics.

The most exciting and the most discussed property of present day particle physics is that there are more broken symmetries than perfect ones. The original hypothesis that the different approximate symmetries are associated with

different interactions, led to a very complicated picture. (SU_3 is violated by the medium strong interactions, SU_2 is violated by the electromagnetic interactions, P is violated by the weak interactions, and CP is violated by the very weak interactions, discovered only in 1964.) According to the opinion of many prominent physicists, the many approximate symmetries may offer an experimental indication of the spontaneous symmetry breakdown in the state vector space, while the Lagrangean of the world and the field equations still possess the maximum symmetry. This possibility is offered in the case of infinite number of degrees of freedom.

The quantum theory of systems with an infinite number of degrees of freedom has not yet been developed into an exact mathematical framework. The starting point was the mathematical study of simple models, originating from the nonrelativistic many-body problem or from relativistic field theory. The solution of the realistic problems has been tried with non-unitary canonical transformations (the self-consistent method), or with functional integrals. Another very promising approach is the axiomatic one. But the "convergence" of the different methods into an exact and practically useful theory, comparable to the quantum mechanics of the atoms, is only a hope today.

Having seen this many-sided interest, the Institute for Theoretical Physics of the Roland Eötvös University in Budapest proposed to the Hungarian Physical Society the organization of a symposium on the problems mentioned above. Our suggestion was accepted by the Presidium of the Society. The symposium was held in September, 1964, in the small town of Keszthely, West-Hungary, at Lake Balaton. With the generous help of the Hungarian Academy of Sciences and of the Union of Technical and Scientific Societies, we had the opportunity to invite prominent physicists working in this field, mainly from various countries of Europe to deliver lectures on the different aspects of the theory. The short reports and the informal discussions among the hundred participants of the conference contributed to a better understanding of these problems. One example is the so-called Goldstone theorem, which was at the centre of the most animated discussions.

The participants suggested the publication of the Conference proceedings. The organizing committee is indebted to all those who placed the text of their lectures at our disposal, and is indebted to Prof. P. GOMBÁS, Chief Editor of Acta Physica Hungarica, for having made possible the printing of the proceedings as a separate issue of this periodical.

GEORGE MARX

MORNING SESSION

K. F. Novobátzky's Opening Address

Ladies and Gentlemen; our honoured guests!

May I, as the doyen of Hungarian physicists, welcome you to our country. Allow me now, to propose a motto for our colloquium, by quoting Lessing who said: "When God offers me in his right hand the full knowledge of the Universe, in his left hand the toil and struggle towards obtaining the same knowledge, I without hesitation choose the left hand." This struggle is the most cherished activity of all those, who have made a lifelong treaty with science.

In my long life I have had the opportunity to follow up the development of quantum theory from its birth to the present hour. I feel that for quantum mechanics, too, the poetical words of Duhem, describing the struggle of science towards more general and advanced concepts apply in full, without reservation: "Through the theories, which arise to become later forgotten, through the hypotheses, acclaimed for a decade as unravelling the hidden mechanisms of the Universe to be regarded in the next decade as childrens' folly, proceeds the slow but continuous advance of theoretical physics. When the tide surges on the coast, a wave develops first climbing high on the hitherto dry shore; but the wave with its impetus is soon lost in the following wave. This continuous struggle of the individual waves, arising only to be submerged later, looks like a hopeless task of sea to occupy a height. But after a couple of hours the shore where men had walked is deep under water. In the incessant to and fro of the waves, in their relentless advance and retreat, creation and decay, has the vast ocean moved ahead."

You know very well indeed, my honoured colleagues, that a host of mathematical methods has been developed around the idea of canonically conjugate operators, due originally to Heisenberg. These methods have been developed, mainly because of particular difficulties inherent to quantum mechanics. Quantum mechanics has to cope with singularities, ghosts, with an insufficient number of conservation laws, with the prodigal number of elementary particles, starting from a modest 30 and now around 185. Confronted with these problems the scientists themselves assume two attitudes. The optimists, such as Pauli was, pin their hopes on a future quantum theory,

to be capable of theoretically deducing a few evidently quantized physical properties, e.g. the elementary charge; the pessimists, however, think that present day mathematical methods and tools no longer have any connection with physics.

This present colloquium deals exclusively with problems of quantum mechanics when the number of the degrees of freedom is very high, and even tends to infinity. The central problems are therefore the quantum field theory and the case of a large number of particles. Problems, such as vacuum degeneration, leading to asymmetry of the vacuum, strongly deviating from the common symmetry properties of the Hamiltonian, will be discussed as well.

I can compare this colloquium perhaps to a single wave of DUHEM, which is but a component of the tide. I express the hope that your discussions will produce gratifying results. Wishing you a pleasant stay in our country, I hereby declare this colloquium opened.

ON THE SELF-CONSISTENT METHOD

By

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This article gives a brief account of a formulaion of the self-consistent method which has been proposed by DELL'ANTONIO and myself [1]. The present report covers the following subjects: a formulation of the self-consistent method, the V-limit, the asymptotic condition in field theory, broken symmetry and local conservation laws and the GOLDSTONE theorem.

1. Introduction

The quantized field theory with which we will be concerned is a canonical one: we start from a given field equation in the canonical form:

$$\dot{\psi} = i[H, \psi] , \tag{1}$$

where the Hamiltonian H is given explicitly in terms of canonical operators satisfying the commutation relations

$$\begin{aligned} [a_k, a_l^+] &= \delta(\vec{k} - \vec{l}), \\ [a_k, a_l] &= 0. \end{aligned} \tag{2}$$

As is well known the first relation in (2) should be interpreted in terms of distribution theory. Its precise meaning is the following: Let $\{f_i(k)\}$ be an orthonormal set of continuous functions with compact support, complete in the L^2 norm. Defining

$$a_i^+ = \int d^3k \, f_i(k) \, a_k^+ ,$$

the commutation relation is

$$[a_i, a_j^+] = \delta_{ij} . \tag{3}$$

The right hand side of the field equation (1) can be expressed explicitly using only the commutation relations in (2), without specifying their representation. However, when we are concerned with physical interpretations of the theory, we come to the question of the representation in which the observed particles are described. As we want to introduce particles in the theory, we

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shall assume that the representation should be of the FOCK type. However, we find an infinite set of FOCK representations and in many cases certain properties of the Hamiltonian depend on the choice of the representation. This situation remains true when the volume is infinite, even when the interaction Hamiltonian contains a cut-off factor to eliminate the ultraviolet divergencies.

In the recent formulation of field theory, the observed particles (which will be called the physical particles in the following) are described by the asymptotic fields (ψ^{in}), which are the weak limits of certain interpolating fields (ψ):

$$w - \lim_{t \to -\infty} \psi = \psi^{\text{in}}. \tag{4}$$

When the ψ^{in} form an irreducible set of our algebra, the Hamiltonian $H(\psi)$ can be written in terms of ψ^{in}. It is usually anticipated that

$$H(\psi) = H_0(\psi^{\text{in}}), \tag{5}$$

where $H_0(\psi^{\text{in}})$ is the free Hamiltonian of ψ^{in} the mass of which is the physical mass.

In certain many-body problems relation (5) is the condition which the quasi particle ψ^{in} should satisfy.

Let us now come back to our starting point, i.e. equations (1) and (2). Since the operator a_k is the SCHROEDINGER operator at a certain time (say $t = 0$), the HEISENBERG operator $a_k(t)$ is given by

$$a_k(t) = e^{iHt} a_k e^{-iHt}. \tag{6}$$

However, we do not know whether or not $a_k(t)$ in (6) refers to the interpolating field. Given the field equation (1), we should be able to find out the interpolating fields together with the asymptotic fields. Furthermore, since we want to formulate the theory in the FOCK space of the physical particles, it is desirable to find out the latter representation at any given time (say $t = 0$) without having recourse to the weak limit ($t = -\infty$).

Thus, we first look for the FOCK representations of the physical particles satisfying the condition (5) and afterwards we examine the possibility of the existence of interpolating fields, the asymptotic limits of which coincide with the fields of the physical particles. This is precisely the self-consistent method. In the following we shall illustrate the self-consistent method for a scalar field. We shall thus define the annihilation operator α_k of the physical representation (the FOCK representation for the physical particles) by the relations

$$H = H_0(\alpha), \tag{7}$$

$$H_0(\alpha) = \int d^3k \, E_k \alpha_k^+ \alpha_k + W_0, \tag{8}$$

$$[\alpha_k, \alpha_l^+] = \delta(\vec{k} - \vec{l}),$$

where E_k and W_0 are c-numbers. When (a_k, a_k^+) is an irreducible set in our algebra, we can write the operator a_k in the following form:

$$a_k = c\,\delta(\vec{k}) + d_k\,a_k + e_k\,a_{-k}^+ + \ldots . \tag{9}$$

Here c, d_k and e_k are c-numbers and the dots represent normal products of higher degrees. The invariance of the theory under *three* dimensional translations and rotations was taken into account in (9). Rotational invariance also requires that d_k and e_k depend only on the magnitude of $\vec{k}$. Our problem is to find out the necessary and sufficient conditions for the existence of the canonical transformation (9), which transforms a into a such that (7) is satisfied.

To do this, it is convenient to introduce the operators $\bar{a}_k$ by a canonical transformation of the form

$$a_k = \cos\theta_k\,\bar{a}_k - e^{i\varphi_k}\sin\theta_k\,\bar{a}_{-k}^+ + \chi\delta(\vec{k})\,, \tag{10}$$

where θ_k, φ_k and χ are so chosen that $\bar{a}_k$ relates itself to the physical operator a_k as follows:

$$\bar{a}_k = Z^{1/2}\,a_k + \ldots , \tag{11}$$

where Z is a c-number and the dots denote normal products of higher degrees.*

The self-consistent method can be formulated as follows: when we are given a Hamiltonian written in terms of $\{a_k\}$, we first prepare a continuous set of transformations (10) parametrized by (θ, φ, χ) and then determine these parameters by requiring the existence of the canonical transformation (11) under the condition (7). Corresponding to each solution for (θ, φ, χ) under these conditions there exist the operators $\{\bar{a}_k\}$ and $\{a_k\}$, respectively.

The operators $\{\bar{a}_k\}$ have not only a conventional meaning but also the following important physical content. The HEISENBERG operator $a_k(t)$ defined by (6) can be written as

$$a_k(t) = c\delta(\vec{k}) + d_k\,a_k\,e^{-iE_kt} + e_k\,a_k^+\,e^{+iE_kt} + \ldots . \tag{12}$$

Here the relation (8) is taken into account. The dots denote the normal products of higher degrees. It is evident that the weak limit of $a(t)$ cannot coincide with a unless $c = 0$ and $e_k = 0$. On the other hand, defining

$$\bar{a}_k(t) = e^{iHt}\,\bar{a}_k e^{-iHt}\,, \tag{13}$$

its weak limit can be $Z^{1/2}\,a_k$ in the case when the higher degree terms vanish in the limit as happens in many cases. In such cases, $\bar{a}(t)$ is just the interpolating field.

* We assume that $|d_k| > |e_k|$. For fermion fields we do not need this assumption.

2. The V-limit

As will be shown later, the canonical transformation (9) is in general not unitary. To see this, we shall begin with the theory of fields confined in a finite volume V and afterwards proceed to the limit $V \to \infty$ in a suitable way. Even when V is finite the canonical transformation (9) is in general not unitary unless e_k tends to zero with increasing $|\vec{k}|$. Therefore, if necessary, we let e_k tend to zero rapidly in the limit $|\vec{k}| \to \infty$ by introducing cut-off factors in the interactions. We do this because the ultraviolet divergences ought to be eliminated in some way, whereas the infinite extension of the world may be expected to remain even in future theories. Thus we are mainly concerned with the problem of the limit $V \to \infty$.

We shall now briefly discuss the problem of determining the algebraic form of some observables, in particular of the Hamiltonian, associated with a given representation of the canonical commutation relations [1]. Let $\{f_i(k)\}$ be an orthonormal set of continuous functions with compact support, complete in L^2 norm. We shall also introduce the quantities $f_i(k)$ which are defined on the set Γ_V

$$\left\{k \in \Gamma_V : \vec{k} = \left(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L}\right),\ L = V^{1/3},\ n_i = 1, 2, \ldots\right\}$$

and which satisfy

$$f_i^V(k) = f_i(k) \quad \text{for} \quad k \in \Gamma_V . \tag{14}$$

We can then show that

$$\lim_{V\to\infty} \frac{(2\pi)^3}{V} \sum_{k\in\Gamma_V} f_i^V(k) = \int d^3k\, f_i(k) \tag{15}$$

and that $\{f_i^V(k)\}$ is complete in the Hilbert space of functions defined on Γ_V with the scalar product

$$(f_i^V, f_j^V) = \frac{1}{V} \sum_{k\in\Gamma_V} f_i^{V*}(k) f_j^V(k) .$$

Although the set $\{f_i^V\}$ is in general not orthonormal, we shall orthonormalize it by SCHMIDT's procedure and denote by $\{\hat{f}_i^V\}$ any such resulting set. Then we can prove [1] that*

$$\lim_{V\to\infty} |f_i^V - \hat{f}_i^V| = 0 \tag{16}$$

* According to SCHMIDT's procedure, the functions $\{\hat{f}_i^V\}$ are defined by

$$f_i^V = \sum_{i\geq j}{}' C_{ij} \hat{f}_j^V \qquad \text{with} \qquad C_{ij} = (f_i^V, \hat{f}_j^V).$$

Let us now consider a system in the box of volume V and an operator B^V which can be written as a superposition of terms of type

$$\sum_{k_1 \ldots k_n} \sum_{l_1 \ldots l_m} b(k_1 \ldots k_n, l_1 \ldots l_m, V)\, a_{k_1}^+ \ldots a_{k_n}^+ a_{l_1} \ldots a_{l_m}\,. \tag{17}$$

Here the commutation relations are

$$\begin{aligned} [a_k, a_l^+] &= \frac{V}{(2\pi)^3}\,\delta_{kl}\,, \\ [a_k, a_l] &= [a_k^+, a_l^+] = 0. \end{aligned} \tag{18}$$

Let us denote by $| i_1 \ldots i_n \rangle$ the vector $a_{i_1}^+ \ldots a_{i_n}^+ | 0 \rangle$ where $| 0 \rangle$ is the vacuum of the FOCK representation for the $a_k's$ and

$$a_i^+ = \frac{(2\pi)^3}{V} \sum_{k \varepsilon \Gamma_V} \hat{f}_i(k)\, a_k^+\,, \tag{19}$$

the sum being extended over all $\vec{k} \in \Gamma_V$. Let Σ be the set of vectors of the form

$$|\Phi\rangle = \sum_{n=1}^{\infty} \sum_{i_1 \ldots i_n} \Phi(i_1 \ldots i_n)\, |i_1 \ldots i_n\rangle \tag{20}$$

with

$$\Phi(i_1 \ldots i_n) = 0 \quad \text{if} \quad \max\ \{i_k\} > N_0 \text{ for some } N_0\,. \tag{21}$$

Then Σ is dense in the Hilbert space under consideration. We shall say that

$$V_a - \lim B^V = B \tag{22}$$

if there exists an operator B such that

$$\lim_{V \to \infty} \langle \Phi \mid B^V \mid \Psi \rangle = \langle \Phi \mid B \mid \Psi \rangle \quad \text{for all} \quad |\Phi\rangle, |\Psi\rangle \in \Sigma\,, \tag{23}$$

where, on the right hand side the state vectors are defined by (20) with

$$a_i^+ = \int d^3k\, f_i(k)\, a^+(k). \tag{24}$$

As we shall presently see, B may depend on the choice of the FOCK representation. We have therefore used the notation V_a-lim in (22) in order to show explicitly that the limit is performed in the $\{a\}$-representation.

Eq. (16) implies that, when B^V assumes the form

$$B^V = \left(\frac{(2\pi)^3}{V}\right)^{n+m} \sum_{k_1 \ldots k_n} \sum_{l_1 \ldots l_m} b(k_1 \ldots k_n, l_1 \ldots l_m) \tag{25}$$
$$a^+_{k_1} \ldots a^+_{k_n} a_{l_1} \ldots a_{l_m},$$

where $b(k_1 \ldots, l_1 \ldots)$ is a continuous function of $k_1 \ldots$ and $l_1 \ldots$, then

$$B = V_a - \lim B^V,$$
$$= \int \prod_i d^2 k_i \cdot \prod_j d^3 l_j \, b(k_1 \ldots, l_1 \ldots) \, a^+_{k_1} \ldots a_{l_1} \ldots . \tag{26}$$

Let us now consider the inhomogeneous linear canonical transformation

$$a_k \to a_k = \cosh \theta_k \, a_k - e^{i\varphi_k} \sinh \theta_k \, a^+_{-k} + \frac{V}{(2\pi)^3} \chi \, \delta_{k0}, \tag{27}$$

where $\vec{k} \in \Gamma_V$ and θ_k is a continuous function. In terms of $a_i's$, (27) reads as follows:

$$a_i \to a_i = \sum_j c^V_{ij} a_j + s^V_{ij} a^+_j + \chi_i, \tag{28}$$

where

$$c^V_{ij} = \frac{(2\pi)^3}{V} \sum_{k \, \Gamma_V} \hat{f}_i(k) \hat{f}^*_j(k) \cosh \theta_k,$$

$$s^V_{ij} = \frac{(2\pi)^3}{V} \sum_{k \in \Gamma_V} \hat{f}_i(k) \hat{f}^*_j(k) \sinh \theta_k, \tag{29}$$

$$\chi_i = \chi \hat{f}_i(0).$$

Then

$$V_a - \lim a_i = \sum_j c_{ij} a_j + s_{ij} a^+_j + \chi_i, \tag{30}$$

where

$$c_{ij} = \int d^3 k \, f_i(k) f^*_j(k) \cos \theta_k, \text{ etc.} \tag{31}$$

In (30) the operator a_j is defined by (24). It is clear that (30) is a canonical transformation. Although the transformation (27) is unitary when

$$\sum_{k \in \Gamma_V} \log \cosh \theta_k < \infty, \tag{32}$$

the transformation (30) is not [2]. To see this, we shall first consider the case $\chi = 0$. Then (27) becomes

$$a_k = G a_k G^{-1},$$

where

$$G = \exp\left[\frac{1}{2}\frac{(2\pi)^3}{V}\sum_{k\in\Gamma_V}\theta_k\,(e^{-i\varphi_k}\,a_k\,a_{-k} - e^{i\varphi_k}\,a_k^+\,a_{-k}^+)\right]. \tag{33}$$

This is unitary when (32) is satisfied. However, we have

$$\lim_{V\to\infty}\sum_{k\in\Gamma_V}\log\cosh\theta_k = \infty \tag{34}$$

and thus the canonical transformation is not unitary in the limit $V \to \infty$. When $\theta_k = 0$ and $\chi \neq 0$, (27) is given by

$$a_k = G a_k G^{-1}$$

with

$$G = \exp\left[-(\chi^* a_0 - \chi a_0^+)\right]. \tag{35}$$

Then we can show [2] that

$$\langle 0\,|G|\,0\rangle = \exp\left[-\frac{V}{2(2\pi)^3}\,|\chi|^2\right] \tag{36}$$

and therefore the canonical transformation is not unitary in the limit $V \to \infty$.

We shall denote V_α-lim a_i simply by a_i. Then (30) can be written as

$$a_k = \cosh\theta_k\,a_k - e^{i\varphi_k}\sinh\theta_k\,a_{-k}^+ + \chi\,\delta(\vec{k}). \tag{37}$$

Although the operators $\{a_k\}$ satisfy the commutation relations

$$\begin{aligned}[a_k, a_l^+] &= \delta(\vec{k} - \vec{l}),\\ [a_k, a_l] &= [a_k^+, a_l^+] = 0,\end{aligned} \tag{38}$$

the operator a_k cannot be called an annihilation operator in the $\{\alpha\}$-representation, because no state vector Φ satisfying $a_k\Phi = 0$ exists in this representation.

When we are given two operators A and B (in the case of finite volume), V-lim (AB) is not necessarily equal to (V-lim A) (V-lim B). However, when A and B take the form (17) with

$$b(k\ldots l\ldots, V) \approx V^p b'(k\ldots l\ldots) \tag{39}$$

for large V (where the function $b'(k \ldots l \ldots)$ does not depend explicitly on V) and when V-lim A and V-lim B are well defined, then we have

$$V\text{-lim}\,(AB) = (V\text{-lim}\,A)\,(V\text{-lim}\,B)\,. \tag{40}$$

To illustrate our way of determining the V-limit, we shall consider the pair model for the fermions a and b:

$$H_V = \frac{(2\pi)^3}{V} \sum_{k\varepsilon\Gamma_V} \omega_k\,(a_k^+\,a_k + b_k^+\,b_k) + \tag{41}$$

$$+ \left(\frac{(2\pi)^3}{V}\right)^3 \sum_{k,l\varepsilon\Gamma_V} \nu(k,\,l)\,a_k^+\,b_{-k}^+\,b_{-l}\,a_l\,,$$

where $\nu(k, l)$ is real and symmetric

$$\nu(k,\,l) = \nu^*(k,\,l) = \nu(l,\,k)\,.$$

The commutation relations are

$$[a_k,\,a_l^+]_+ = [b_k,\,b_l^+]_+ = \frac{V}{(2\pi)^3}\,\delta_{kl}\,, \quad \text{etc.} \tag{42}$$

We shall introduce the operators $\{\alpha_k,\,\beta_k\}$ by the transformations

$$\begin{aligned} a_k &= \cos\theta_k\,\alpha_k - \sin\theta_k\,\beta_{-k}^+\,, \\ b_k &= \cos\theta_k\,\beta_k + \sin\theta_k\,\alpha_{-k}^+\,. \end{aligned} \tag{43}$$

Let us define

$$\nu(k) = -\frac{1}{2}\int d^3\,l\,\nu(k,\,l)\sin 2\theta_l \tag{44}$$

and choose θ_l to be the solution of the following equation:

$$\cos 2\theta_k = \frac{\omega_k}{\sqrt{\omega_k^2 + (\nu(k))^2}}\,. \tag{45}$$

Then, we can show that

$$V_a - \lim H_V = \int d^3\,k\,\omega_k\,(a_k^+\,a_k + b_k^+\,b_k) \tag{46}$$

and

$$V_\alpha - \lim H_V = \int d^3\,k\,\sqrt{\omega_k^2 + (\nu(k))^2}\,(\alpha_k^+\,\alpha_k + \beta_k^+\,\beta_k) + c\text{-number}. \tag{47}$$

This example clearly shows that our procedure for determining the V-limit sometimes leads to different free Hamiltonians in different representations.

3. A method of successive approximations

We shall begin with a system in a finite volume V and assume a Hamiltonian of the following form*

$$H = h_0(a) + \lambda h(a) , \tag{48}$$

$$h_0(a) = \frac{(2\pi)^3}{V} \sum_k{}' \omega_k a_k^+ a_k . \tag{49}$$

In (48), λ denotes the coupling constant. The commutation relations are

$$\begin{aligned} [a_k, a_l^+] &= \frac{V}{(2\pi)^3} \delta_{kl} , \\ [a_k, a_l] &= [a_k^+, a_l^+] = 0 . \end{aligned} \tag{50}$$

Our problem is to find the transformation

$$a_k = T_V^{-1} \alpha_k T_V , \tag{51}$$

where α_k should satisfy the following condition:

$$H_V = H_0(\alpha) + Q_V , \tag{52}$$

$$H_0(\alpha) = \frac{(2\pi)^3}{V} \sum_k{}' E_k \alpha_k^+ \alpha_k + W_0 , \tag{53}$$

$$V_\alpha\text{-lim } Q_V = 0 . \tag{54}$$

Here E_k and W_0 are certain c-numbers. Following the argument in § 1, we introduce the operators $\{\bar{\alpha}_k\}$ by

$$\alpha_k = G^{-1} \bar{\alpha}_k G , \tag{55}$$

$$a_k = \cosh \theta_k \bar{\alpha}_k - a^{i\varphi_k} \sinh \theta_k \bar{\alpha}_{-k}^+ + \frac{V}{(2\pi)^3} \chi \delta_{k0} . \tag{56}$$

* In the following $\sum_k{}'$ means the summation extended over the domain Γ_V.

Let us write H_V in terms of $\bar{a}$. Then we obtain

$$H_V = H_0(\bar{a}) + \lambda \bar{h}(\bar{a}) + \delta h(\bar{a}), \tag{57}$$

where

$$H_0(\bar{a}) = \frac{(2\pi)^3}{V} \sum_k E_k \, a_k^+ \, a_k + W_0, \tag{58}$$

$$\bar{h}(\bar{a}) = h(a), \tag{59}$$

$$\begin{aligned} \delta h(\bar{a}) &= h_0(a) - H_0(\bar{a}) = \\ &= \frac{(2\pi)^3}{V} \sum_k (v_k \, \bar{a}_k^+ \, \bar{a}_k + u_k \, \bar{a}_k^+ \, \bar{a}_{-k}^+ + u_k^* \, \bar{a}_{-k} \, \bar{a}_k) + \\ &+ (w \bar{a}_0^+ + w^* \, \bar{a}_0) + \delta W_0, \end{aligned} \tag{60}$$

with

$$\left.\begin{aligned} v_k &= -E_k + \omega_k \cosh 2\theta_k, \\ u_k &= -\frac{1}{2} \omega_k \, e^{i\varphi_k} \sinh 2\theta_k, \\ w &= \omega_0 (\chi \cosh \theta_0 - \chi^* \, e^{i\varphi_0} \sinh \theta_0), \\ \delta W_0 &= \frac{V}{(2\pi)^3} |\chi|^2 \, \omega_0 + \sum_k \omega_k \sinh^2 \theta_k - W_0. \end{aligned}\right\} \tag{61}$$

Let us now define S_V by

$$T_V = S_V \, G. \tag{62}$$

Then, we obtain

$$\bar{a}_k = S_V^{-1} \, a_k \, S_V \tag{63}$$

which, together with (57), gives

$$H_V = S_V^{-1} [H_0(a) + \lambda \bar{h}(a) + \delta h(a)] \, S_V. \tag{64}$$

Taking account of (52), (53) and (54), we see that S_V must satisfy the following conditions:

$$S_V^{-1} [H_0(a) + \lambda \bar{h}(a) + \delta h(a)] \, S_V = H_0(a) + Q_V, \tag{65}$$

$$V_a\text{-lim } Q_V = 0. \tag{66}$$

However, these conditions are not sufficient, because, when we change the parameters (θ, φ, χ), we can still keep T_V invariant by modifying S_V.

To have a unique separation of T_V into S_V and G, we follow the argument in § 1 and introduce the condition (11), that is

$$\bar{a}_k = S_V^{-1} a_k S_V = Z^{1/2} a_k + \ldots , \tag{67}$$

where Z is a c-number and the dots represent normal products of higher degrees. In the self-consistent method we attempt to determine the parameters (θ, φ, χ) by requiring the existence of S_V satisfying the relations (65), (66) and (67). We try to express this requirement by a set of equations for (θ, φ, χ), which, in the following, will be called the self-consistency equations. To each solution of the self-consistency equations there corresponds a T_V and therefore also a physical representation.

However, it is not easy in general to derive the exact form of the self-consistency equations. We shall therefore develop a method of successive approximations. What we have in mind is not an ordinary perturbative expansion such as

$$f(\lambda) = f(0) + \lambda f'(0) + \ldots$$

but rather, an expansion of the following type:

$$f(\lambda) = f_0(\lambda) + \lambda f_1(\lambda) + \lambda^2 f_2(\lambda) + \ldots .$$

We shall now expand the functions u_k, v_k, w and δW_0 in (60):

$$\left.\begin{aligned} v_k &= v_k^{(1)} + v_k^{(2)} + \ldots , \\ u_k &= u_k^{(1)} + u_k^{(2)} + \ldots , \\ w &= w^{(1)} + w^{(2)} + \ldots , \\ \delta W_0 &= \delta W_0^{(1)} + \delta W_0^{(2)} + \ldots . \end{aligned}\right\} \tag{68}$$

The operator $\delta h(a)$ is expanded as follows:

$$\delta h(a) = \delta h^{(1)}(a) + \delta h^{(2)}(a) + \ldots , \tag{69}$$

where

$$\begin{aligned} \delta h^{(i)}(a) = \frac{(2\pi)^3}{V} \sum_k (v_k^{(i)} a_k^+ a_k + u_k^{(i)} a_k^+ a_{-k}^+ + u_k^{(i)*} a_{-k} a_k) + \\ + (w^{(i)} a_0^+ + w^{(i)*} a_0) + \delta W_0^{(i)} . \end{aligned} \tag{70}$$

Let us now introduce a real parameter s and define

$$H_I(s) = e^{-\varepsilon|s|}\,\lambda\bar{h}(s) + \sum_{n=1}^{\infty} e^{-n\varepsilon|s|}\,\delta h^{(n)}(s)\,, \tag{71}$$

where

$$h(s) = e^{iH_0(\alpha)s}\,\bar{h}(\alpha)\,e^{-iH_0(\alpha)s} = \bar{h}(\alpha e^{-iEs})\,, \tag{72}$$

$$\delta\bar{h}^{(n)}(s) = e^{iH_0(\alpha)s}\,\delta\bar{h}^{(n)}(\alpha)\,e^{-iH_0(\alpha)s} = \delta\bar{h}^{(n)}(\alpha e^{-iEs})\,, \tag{73}$$

and ε is a function of V satisfying the following condition:

$$\varepsilon \sim V^p \qquad \text{with} \qquad 0 > p > -\frac{1}{3}\,, \quad \text{for large } V\,. \tag{74}$$

We have thus

$$\begin{aligned} &\lim_{V\to\infty} \varepsilon = 0\,, \\ &\lim_{V\to\infty} V^{-1/3}\varepsilon^{-1} = 0\,. \end{aligned} \tag{75}$$

We shall further define $S_V(s)$ by

$$S_V(s) = 1 + (-i)\int_{-\infty}^{s} ds'\,H_I(s')\,S_V(s')\,. \tag{76}$$

Writing

$$S_V = S_V(0) \tag{77}$$

we can prove [1] that

$$S_V^{-1}\,[H_0(\alpha) + \lambda\bar{h}(\alpha) + \delta h(\alpha)]S_V = H_0(\alpha) + Q_V\,, \tag{78}$$

where Q_V has the following V_α-limit:

$$V_\alpha\text{-lim}\; Q_V = \int d^3k q_k\,a_k^+\,a_k + q^0\,. \tag{79}$$

In (79), q_k and q^0 are the self-energies of the one-particle and vacuum states, respectively. We can choose v_k and δW_0 in $\delta h(\alpha)$ in such a way that $q_k = 0$ and $q^0 = 0$, so that the condition (54) is satisfied. On the other hand the condition (67) determines u_k and w in $\delta h(\alpha)$. According to (61), u_k and w thus obtained determine the parameters (θ, φ, χ) and then E_k and W_0 are given by v_k and δW_0.

Instead of presenting our general argument [1], we shall consider the first-order approximation:

$$S_V \approx 1 + (-i) \int_{-\infty}^{0} ds\, H_I(s)\,. \tag{80}$$

Here the symbol $\approx$ means equality in the first order. We now have

$$H_V = S_V^{-1}\,[H_0(a) + \lambda\bar{h}(a) + \delta h(a)]\,S_V \approx H_0(a) + \lambda\bar{h}(a) + \delta h(a) + \\ + (-i) \int_{-\infty}^{0} ds[H_0(a), H_I(s)]\,, \tag{81}$$

with

$$H_I(s) \approx e^{-\varepsilon|s|}\,[\lambda\bar{h}(s) + \delta h^{(1)}(s)]\,. \tag{82}$$

We obtain from (72) and (73) the relation

$$i \int_{-\infty}^{0} ds[H_0(a), H_I(s)] = \int_{-\infty}^{0} ds\, e^{\varepsilon s} \frac{d}{ds}\,[\lambda\bar{h}(s) + \delta h^{(1)}(s)] = \\ = \lambda\bar{h}(a) + \delta h^{(1)}(a) - \varepsilon \int_{-\infty}^{0} ds\, e^{\varepsilon s}\,[\lambda\bar{h}(s) + \delta h^{(1)}(s)]. \tag{83}$$

Thus,

$$H_V \approx H_0(a) + \varepsilon \int_{-\infty}^{0} ds\, e^{\varepsilon s}\,[\lambda\bar{h}(s) + \delta h^{(1)}(s)]\,, \tag{84}$$

or

$$Q_V \approx \varepsilon \int_{-\infty}^{0} ds\, e^{\varepsilon s}\,\big[\lambda\bar{h}(s) + \delta h^{(1)}(s)\big]. \tag{85}$$

This shows that

$$\langle i|Q_V|j\rangle \approx -i\,\frac{\varepsilon}{(E_i - E_j) - i\varepsilon}\,\langle i|\,\lambda\bar{h}(a) + \delta h^{(1)}(a)\,|j\rangle\,, \tag{86}$$

where $|i\rangle$ and $|j\rangle$ are eigenstates of $H_0(a)$:

$$H_0(a)\,|i\rangle = E_i\,|i\rangle\,. \tag{87}$$

Let us first assume that each of the states $|i\rangle$ and $|j\rangle$ contains more than one particle. Then E_i does not necessarily coincide with E_j, while $|i\rangle$ and $|j\rangle$ have the same total momentum. Since ε tends to zero in the limit $V \to \infty$, $\langle i\,|\,Q_V\,|\,j\rangle$ is zero unless $E_i = E_j$. Since, when $V = \infty$, the coin-

cident energies $E_i = E_j$ represent just one point in the continuous energy spectrum, we see that $\langle i \mid Q_V \mid j\rangle = 0$ when both $|i\rangle$ and $|j\rangle$ are the many-particle states. When $|j\rangle$ is a one-particle state, say $|\vec{k}\rangle$, momentum conservation leads us to the conclusion that $\langle i \mid Q_V \mid j\rangle$ is zero in the limit $V \to \infty$ ($\varepsilon \to 0$) unless $|i\rangle$ is a one-particle state. When $|i\rangle$ is a one-particle state, say $|\vec{l}\rangle$, (86) leads to

$$\langle \vec{l}| Q_V |\vec{k}\rangle \approx \langle \vec{k}| \lambda \bar{h}(\alpha) + \delta h^{(1)}(\alpha) |\vec{k}\rangle \delta_{kl} . \tag{88}$$

In a similar way, we can show that

$$\lim_{V\to\infty} \langle i| Q_V |0\rangle \approx 0 , \text{ when } |i\rangle \neq |0\rangle . \tag{89}$$

We have thus proved (79) with

$$\left.\begin{aligned} q_k &= \langle \vec{k}| \lambda \bar{h}(\alpha) + \delta h^{(1)}(\alpha) |\vec{k}\rangle , \\ q^0 &= \langle 0| \lambda \bar{h}(\alpha) + \delta h^{(1)}(\alpha) |0\rangle \end{aligned}\right\} \tag{90}$$

in the first order approximation. The condition (66) requires that $q_k = 0$ and $q^0 = 0$. This means that $(\lambda \bar{h}(\alpha) + \delta h^{(1)}(\alpha))$ should not contain any $\alpha^+\alpha$- and c-number terms. Since $\lambda \bar{h}(\alpha)$ depends explicitly on the parameters (θ, φ, χ), the above-mentioned condition determines $v_k^{(1)}$ and $\delta W_0^{(1)}$ in $\delta h_{(\alpha)}^{(1)}(\alpha)$ as certain functions of (θ, φ, χ):

$$v_k^{(1)} = v_k^{(1)}(\theta, \varphi, \chi) , \quad \delta W_0^{(1)} = \delta W_0^{(1)}(\theta, \varphi, \chi) .$$

Since $v_k \approx v_k^{(1)}$ and $\delta W_0 \approx \delta W_0^{(1)}$, (61) leads us to the equations

$$\left.\begin{aligned} E_k &= -v_k^{(1)}(\theta, \varphi, \chi) + \omega_k \cosh 2\theta_k , \\ W_0 &= \frac{V}{(2\pi)^3} |\chi|^2 \omega_0 + \sum_k \omega_k \sinh^2 \theta_k - \delta W_0^{(1)}(\theta, \varphi, \chi) . \end{aligned}\right\} \tag{91}$$

To determine the parameters (θ, φ, χ), we make use of the condition (67). We have

$$\bar{a}_k \approx a_k + (-i) \int_{-\infty}^{0} ds\, e^{\varepsilon s} [a_k, \lambda \bar{h}(s) + \delta h^{(1)}(s)] . \tag{92}$$

The condition (67) is satisfied if and only if $(\lambda \bar{h}(\alpha) + \delta h^{(1)}(\alpha)$ contains neither $a_k^+ a_{-k}^+$- nor a_0^+-terms. This condition determines $u_k^{(1)}$ and $w^{(1)}$ in $\delta h_{(\alpha)}^{(1)}$ as certain functions of (θ, φ, χ):

$$u_k^{(1)} = u_k^{(1)}(\theta, \varphi, \chi) , \quad w^{(1)} = w^{(1)}(\theta, \varphi, \chi) . \tag{93}$$

Since $u_k \approx u_k^{(1)}$ and $w \approx w^{(1)}$ (61) leads us to

$$\left.\begin{aligned} u_k^{(1)}(\theta, \varphi, \chi) &= -\frac{1}{2}\omega_k e^{i\varphi_k}\sinh 2\theta_k, \\ w^{(1)}(\theta, \varphi, \chi) &= \omega_0(\chi\cosh\theta_0 - \chi^* e^{i\varphi_0}\sinh\theta_0). \end{aligned}\right\} \tag{94}$$

These are the self-consistency equations which determine the parameters (θ, φ, χ). Then E_k and W_0 are determined by (91).

In this way we can proceed [1] with our calculation up to any order of the approximation and determine $\{u_k^{(i)}\}$, $\{w^{(i)}\}$, $\{v_k^{(i)}\}$ and $\{\delta W_0^{(i)}\}$ as functions of (θ, φ, χ):

$$u_k^{(i)} = u_k^{(i)}(\theta, \varphi, \chi), \text{ etc.} \tag{95}$$

The n-th order approximation means that the relations in (68) are approximated as follows:

$$\left.\begin{aligned} v_k &= v_k^{(1)}(\theta, \varphi, \chi) + \ldots + v_k^{(n)}(\theta, \varphi, \chi), \\ u_k &= u_k^{(1)}(\theta, \varphi, \chi) + \ldots + u_k^{(n)}(\theta, \varphi, \chi), \\ w &= w^{(1)}(\theta, \varphi, \chi) + \ldots + w^{(n)}(\theta, \varphi, \chi), \\ \delta W_0 &= \delta W_0^{(1)}(\theta, \varphi, \chi) + \ldots + \delta W_0^{(n)}(\theta, \varphi, \chi). \end{aligned}\right\} \tag{96}$$

Then, (61) leads to

$$\left.\begin{aligned} E_k &= \omega_k\cosh 2\theta_k - \sum_{i=1}^{n} v_k^{(i)}(\theta, \varphi, \chi), \\ W_0 &= \frac{V}{(2\pi)^3}|\chi|^2\omega_0 + \sum_k \omega_k\sinh^2\theta_k - \sum_{i=1}^{n}\delta W_0^{(i)}(\theta, \varphi, \chi) \end{aligned}\right\} \tag{97}$$

and to

$$\left.\begin{aligned} \frac{1}{2}\omega_k e^{i\varphi_k}\sinh 2\theta_k &= -\sum_{i=1}^{n} u_k^{(i)}(\theta, \varphi, \chi), \\ \omega_0(\chi\cosh\theta_0 - \chi^* e^{i\varphi_0}\sinh\theta_0) &= \sum_{i=1}^{n}\delta W_0^{(i)}(\theta, \varphi, \chi). \end{aligned}\right\} \tag{98}$$

The self-consistency equations in (98) determine the parameters (θ, φ, χ) which specify the representations for $\bar{a}$, following which the equations in (97) give E_k and W_0. It can be proved [1] that the condition $p > -1/3$ in (74) is necessary in order to avoid the singularities at $\varepsilon = 0$ due to the coincidence of the virtual state energy and the initial energy in $\lim (V \to \infty) \langle i| S_V |j\rangle$.

It has been stated in some articles that the Hamiltonian in the lowest order approximation is $(H_0(a) + \lambda\bar{h}(a) + \delta h^{(1)}(a))$ and that the diagonalization of the bilinear and linear terms (a^+a, a^+a^+, aa, a^+ and a) of this Hamiltonian determines the parameters (θ, φ, χ) together with (E_k, W_0). Such an argument, however, is not correct, as was shown by (81). The last term in (81) naturally eliminates the off-diagonal bilinear terms (a^+a^+ and aa) and the linear terms (a^+ and a). On the other hand the parameters (θ, φ, χ) are determined by (67). As was shown in the previous Section, when T_V exists, so does S_V for any choice of (θ, φ, χ), i.e. $S_V = T_V G^{-1}$ (cf. (62)), unless (67) is not taken into account. This is the reason why the parameters (θ, φ, χ) are determined, not by the diagonalization of the Hamiltonian, but by (67).

It may be worth noting that the calculation in the higher order approximation should not be carried out using the parameters (θ, φ, χ) determined by the self-consistency equations of the lower order approximation. When we want to perform any calculation in the n-th order approximation, we should first determine (θ, φ, χ) by solving the self-consistency equations of this approximation (i.e. (98)) and then proceed with the calculations using the values of the parameters (θ, φ, χ) thus determined.*

It can sometimes happen that the physical operators $\{a_k\}$ and their hermitian conjugates $\{a_k^+\}$ do not form an irreducible set even when the operators $\{a_k, a_k^+\}$ do so. In such a case, assume that the physical operators form an irreducible set upon adjunction of other creation-annihilation operators (say β_k, β_k^+). Then *we may call β_k the annihilation operator of the bound (or composite) particle.*

4. The interpolating and asymptotic fields

In this Section we shall consider the asymptotic condition by making use of the successive approximation method presented in the last section. We define

$$\bar{a}_k(t) = e^{iHt}\,\bar{a}_k\,e^{-iHt} = e^{iH_0(a)t}\,\bar{a}_k\,e^{-iH_0(a)t}. \tag{99}$$

Then we can prove [1] that

$$w\text{-}\lim_{t\to-\infty}\bar{a}_k(t) = Z^{1/2}\,a_k. \tag{100}$$

Here $Z^{1/2}$ is the c-number which has appeared in (67). The relation (100) shows that $\bar{a}_k(t)$ and a_k refer to the interpolating and incoming fields. A remarkable feature of the present formulation is that the asymptotic field can be determined at any time (say $t = 0$) by the self-consistent method, without having *recourse* to the limit ($t \to -\infty$).

* A similar remark is made also by P. SZÉPFALUSY in his report [3] in this volume.

5. The local conservation law broken symmetry

When the field equation (1) has a certain symmetry, it frequently leads to a local conservation law:

$$\partial_\mu j_\mu(x) = 0 . \tag{101}$$

Since, in the self-consistent method, we do not modify the field equation at all and since the equation (101) can be derived immediately from the field equation *the local conservation law* (101) *can never be violated.* Still in many cases we find transition matrices where the symmetry under consideration is violated. To show how such a situation can arise, we shall consider the NAMBU—HEISENBERG model [4] for a fermion field ψ. The field equation is

$$\gamma_\mu \partial_\mu \psi = \lambda\{\bar{\psi}(\bar{\psi}\psi) - \gamma_5\psi(\bar{\psi}\gamma_5\psi)\} . \tag{102}$$

This is invariant under the transformation

$$\psi \to \psi' = e^{ic\gamma_\cdot}\psi \tag{103}$$

and leads to the following conservation law:

$$\partial_\mu j_\mu = 0 , \quad j_\mu = \bar{\psi}\gamma_\mu \gamma_5 \psi . \tag{104}$$

Now, in the first order approximation we have the following interaction Hamiltonian (cf. (82)):

$$\lambda\bar{h}(t) + \delta h^{(1)}(t) = \frac{\lambda}{2}\int d^3x : \left[(\bar{\varphi}\varphi)^2 - (\bar{\varphi}\gamma_5\varphi)^2\right] : \tag{105}$$

where φ is the physical operator

$$\left.\begin{aligned} &\varphi(x) = S\psi(x)S^{-1} \text{ at } t = 0 , \\ &(\gamma_\mu \partial_\mu + M)\varphi(x) = 0 . \end{aligned}\right\} \tag{106}$$

In (106) M denotes the observed mass and S is the V-limit of S_V.

Defining

$$S(t) = e^{iH_0(a)t}\, S e^{-iH_0(a)t} \tag{107}$$

we can prove [1] that $S(t)$ is given by (76) (s being replaced by t). Thus we have

$$i\frac{d}{dt}S(t) = e^{-\varepsilon|t|}(\lambda\bar{h}(t) + \delta h^{(1)}(t) + \ldots)\, S(t) , \tag{108}$$

where the dots stand for the higher order interactions ($\delta h^{(n)}$-terms with $n < 1$). Since

$$j_\mu(x) = S^{-1}(t)\bar{\varphi}(x)\gamma_\mu\gamma_5\varphi(x)S(t)\,, \tag{109}$$

we obtain

$$\partial_\mu j_\mu(x) = S^{-1}(t)\left\{2M\bar{\varphi}(x)\,\gamma_5\varphi(x) - \right.$$
$$\left. - [\bar{\varphi}(x)\,\gamma_4\gamma_5\varphi(x),\ \lambda\bar{h}(t) + \delta h^{(1)}(t) + \ldots]\right\} S(t). \tag{110}$$

We shall ignore the terms represented by the dots and thus consider the first order approximation.

We can show that

$$[\bar{\varphi}(x)\gamma_4\gamma_5\varphi(x),\ :\{(\bar{\varphi}(x')\varphi(x')^2 - (\bar{\varphi}(x')\gamma_5\varphi(x')^2\}:]_{t=t'} =$$
$$= 4C\bar{\varphi}(x)\gamma_5\varphi(x)\delta(\vec{x} - \vec{x}')\,, \tag{111}$$

where

$$C = \langle 0|\,\bar{\varphi}\,\varphi\,|0\rangle\,. \tag{112}$$

It is a crucial point in our considerations that (111) is not zero, although

$$[\bar{\varphi}(x)\gamma_4\gamma_5\varphi(x),\ \{(\bar{\varphi}(x')\varphi(x')^2) - (\bar{\varphi}(x')\gamma_5\varphi(x'))^2\}]_{t=t'} = 0\,.$$

The relation (110) now leads to

$$\partial_\mu j_\mu(x) = 2(M - \lambda C)S^{-1}(t)\bar{\varphi}(x)\,\gamma_5\varphi(x)\,S(t)\,, \tag{113}$$

which vanishes owing to the self-consistency equation

$$M = \lambda C\,. \tag{114}$$

It is usually stated that the local conservation law (104) leads to the macroscopic conservation law

$$\frac{d}{dt}\,N = 0,$$

where

$$N = \int d^3x\, j_4\,. \tag{115}$$

The operator N would be the generator of the transformation (103):

$$\psi' = \exp\,[icN]\psi\,\exp\,[-icN] \tag{116}$$

if this was a unitary transformation. However, when $M \neq 0$, we can show that V_a-lim exp $[-icN] = 0$ (for $0 < c < 2\pi$). This means that N is not well-defined in the $\{\alpha\}$-representation and therefore it is meaningless to speak about eigenstates of N in this representation. However, the helicity operator of the physical particles, i.e.

$$N_\varphi = \int d^3 x \; \bar{\varphi}\gamma_4 \gamma_5 \varphi \tag{117}$$

is well defined, but is not conserved in time.

It is a remarkable fact that the local conservation law (101) holds in the theory of spontaneous breakdown of symmetry. In this sense it is an urgent problem to study if there would be any way to check local conservation laws by experiments.

6. The Goldstone theorem

It has been stated several times that the spontaneous breakdown of symmetries must be associated with the existence of massless particles. This is known as the GOLDSTONE theorem [5].

A general proof of the GOLDSTONE theorem has been offered by GOLDSTONE, SALAM and WEINBERG [6]. Their proof is based on the perturbative expansion in which all internal lines of the FEYNMAN diagrams carry the same mass. This means that the perturbative expansion is associated with the FOCK representation corresponding to the symmetric mass spectrum, although the original aim was to find the asymmetric mass spectrum. This means that the above-mentioned solution corresponds, not to an asymmetric case, but to the case of *symmetric massless* physical particles. Such a situation was clearly illustrated [2] in the first order calculation of the GOLDSTONE model. There, we are concerned with a symmetric Hamiltonian for two scalar fields ψ_1 and ψ_2. By making use of the asymmetric canonical transformation

$$\varphi_1 = \Phi_1 + \chi \,, \quad \varphi_2 = \Phi_2 \qquad (\chi \neq 0) \tag{118}$$

we obtain nonzero unequal masses ($M_1 \neq M_2$), in contradiction with the GOLDSTONE theorem. When we proceed to the symmetric limit $M_1 \to M_2$ in the self-consistency equations, we obtain the result $\chi = M_1 = M_2 = 0$. GOLDSTONE, SALAM and WEINBERG have presented another proof of the GOLDSTONE theorem [6], without using the perturbative expansion. They in effect used the conservation law together with the canonical commutation relations and energy spectrum. However, it has been known that such an argument frequently leads to inconsistent conclusions [8], [2].

Summarizing, there exists no reliable proof of the GOLDSTONE theorem, while there is an example (the GOLDSTONE model), where the first order approximation of the self-consistent method leads to a solution which contradicts the GOLDSTONE theorem.

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О МЕТОДЕ САМОСОГЛАСОВАНИЯ

Г. УМЕЗАВА

Резюме

Статья даёт краткий обзор развития метода самосогласования, предложенного Делл'Антонио и мною. В настоящей работе рассматриваются следующие вопросы: формирование метода самосогласования, предела V, асимптотического условия в теории поля, нарушенной симметрии, законы локального сохранения и теоремы Голдстоуна.

ON THE UNITARILY INEQUIVALENT REPRESENTATIONS IN THE QUANTUM FIELD THEORY AND THE MANY BODY PROBLEM

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1. Introduction

This lecture is devoted to the quantum systems of infinitely many degrees of freedom e.g. to the relativistic and non-relativistic field theory. We shall pay special attention to the unitarily inequivalent representations of the commutation relations which appear only for systems of infinitely many degrees of freedom.

For systems with finite number of degrees of freedom e.g. for systems considered in quantum mechanics all irreducible representations of the commutation relations

$$[p_i, q_k] = i\delta_{ik} \qquad i, k = 1, 2, \ldots n \ (n \text{ finite})$$
$$[p_i, p_k] = [q_i, q_k] = 0\,, \tag{1}$$

i.e. all such representations where the operator set (q, p) forms a complete set in the Hilbert space, are equivalent up to a unitary transformation [1].

For systems with infinitely many degrees of freedom, however, not all irreducible representations of the commutation relations are unitarily equivalent [2]. We have to distinguish between the class of canonical and the class of unitary transformations, the last constituting a small subclass of the former.

The appearance of the inequivalent representations in the theory is by no means a pathological or alarming feature and is rather an indispensable characteristic of a properly formulated theory of infinitely many degrees of freedom. As shown recently [3], see also [15], different unitarily inequivalent representations may be equivalent physically, i.e. they differ from each other only as far as the behaviour of the states with respect to observations made infinitely far away is concerned; this is of little physical relevance.

2. The case of a scalar real field

We shall explain the main idea of the theory of inequivalent representations by a simple example of a scalar real field discussed by HAAG. We start with a system of finite number of degrees of freedom, say n. To get the Hilbert space we use the Fock construction. Assume the system is determined by the $2n$ Hermitian operators

$$q_\alpha, p_\alpha \qquad (\alpha = 1, \ldots n)$$

satisfying relations (1). We introduce new operators

$$a_\alpha = \sqrt{\frac{\omega_\alpha}{2}}\, q_\alpha + i \sqrt{\frac{1}{2\omega_\alpha}}\, p_\alpha \qquad (\alpha = 1, \ldots n)\,, \tag{2}$$

ω_α being arbitrary, real, positive numbers. We get from (1)

$$\begin{aligned} &[a_\alpha, a_\beta^+] = \delta_{\alpha\beta}\,, \\ &[a_\alpha, a_\beta] = [a_\alpha^+, a_\beta^+] = 0, \quad (\alpha = 1, \ldots n)\,. \end{aligned} \tag{3}$$

Assume further that the state, the so-called Fock vacuum state, ψ_0, defined by

$$a_\alpha \psi_0 = 0 \tag{4}$$

exists; then we get the Fock space by applying a_α^+ to ψ_0, viz.,

$$\begin{aligned} &\psi_0 \\ &a_\alpha^+ \psi_0 = \psi_\alpha \qquad (\alpha, \beta = 1, \ldots n) \\ &a_\alpha^+ a_\beta^+ \psi_0 = \psi_{\alpha\beta} \\ &\cdot \quad \cdot \quad \cdot \end{aligned} \tag{5}$$

the closure of this linear manifold gives a separable Hilbert space. In case n is finite, VON NEUMANN [1] proved that the Fock representation (5) is the only irreducible one up to unitary equivalence. In other words, given two sets of canonical operators (q, p) and

$$Q = Q(q, p), \quad P = P(q, p)$$

one can always find a unitary operator U which yields

$$\begin{aligned} &Q_\alpha = U q_\alpha U^{-1}\,, \\ &P_\alpha = U p_\alpha U^{-1}\,. \qquad (\alpha = 1, \ldots n)\,. \end{aligned} \tag{6}$$

The Fock vacuum for the new set (Q, P), say Ψ_0, is linked to the Fock vacuum ψ_0 as follows

$$\Psi_0 = U\psi_0. \tag{7}$$

Let us now consider the case when n becomes infinite, e.g., a real scalar field

$$\begin{aligned} &\varphi(\vec{x}, x_0 = 0) = \varphi(\vec{x}) , \\ &\pi(\vec{x} , x_0 = 0) = \pi(\vec{x}) = \left| \frac{\partial \varphi(\vec{x}, x_0)}{\partial x_0} \right|_{x_0=0} , \end{aligned} \tag{8}$$

which satisfies the conventional commutation relations. Assume the fields (φ, π) to form an irreducible set and to be operator valued 3-dimensional distributions, then by means of an orthogonal real set $f_\alpha(\vec{x})$ $(\alpha = 1, 2, \ldots)$ we may introduce new operators

$$\begin{aligned} q_\alpha &= \int d^3x\, \varphi(\vec{x})\, f_\alpha(\vec{x}) = q_\alpha^+ \quad (\alpha = 1, 2, \ldots), \\ p_\alpha &= \int d^3x\, \pi(\vec{x})\, f_\alpha(\vec{x}) = p_\alpha^+ \end{aligned} \tag{9}$$

as well as a_α in a similar way as in (2); the operators (a, a^+) will satisfy the commutation relations (3) (notice: now n is infinite).

We are able to construct formally the Fock space in the same way as formerly, provided the vacuum state ψ_0 exists,* and obtain a separable Hilbert space.

The situation can change, however, as compared with the case of finite number of degrees of freedom if we take another representation of the commutation relations. The theorem of VON NEUMANN does not work any more (conf. [2]). To exhibit this clearly let us consider, following HAAG [4], a new set of canonical variables

$$\begin{aligned} b_\alpha &= \cos h\varphi_\alpha \cdot a_\alpha - \sin h\varphi_\alpha \cdot a_\alpha^+ , \qquad (\alpha = 1,\ 2 \ldots), \\ b_\alpha^+ &= -\sin h\varphi_\alpha \cdot a_\alpha + \cos h\varphi_\alpha \cdot a_\alpha^+ , \\ e^{2\varphi} &= \omega . \end{aligned} \tag{10}$$

If (a, a^+) is complete so is (b, b^+). In terms of $(q, p \to Q, P)$ this reads

$$\begin{aligned} Q_\alpha &= \sqrt{\frac{1}{\omega_\alpha}}\, q_\alpha, \qquad (\alpha = 1, 2, \ldots), \\ P_\alpha &= \sqrt{\omega_\alpha}\, p_\alpha . \end{aligned} \tag{11}$$

* If the Hamiltonian has the form $\int \omega(\vec{p}^2) a^+(\vec{p}) a(\vec{p}) d^3p$ with $\omega(\vec{p}^2) \geqslant \omega(\vec{p}^2 = 0) > 0$ then the Fock vacuum exists (see H. J. BORCHERS, R. HAAG, B. SCHROER, "The vacuum state in Quantum Field Theory").

Notice how simple this transformation is. Notice also that the operators (b, b^+) operate in the same Hilbert space as the operators (a, a^+); the Fock representation for the operators (a, a^+) defines also the representation for the (b, b^+) in the same Hilbert space in virtue of (10). We show, however, that it is not possible in general to carry out a Fock construction for the operators (b, b^+), i.e. the Fock vacuum for (b, b^+) does not in general exist; the same is true for the unitary operator U which interlocks the operators (b, b^+) with the operators (a, a^+)

$$b_\alpha = U a^\alpha U^{-1} \qquad (\alpha = 1, 2, \ldots) \tag{12}$$

in spite of the fact that both, (a, a^+) and (b, b^+), satisfy the same commutation relations.

To prove or disprove the existence of U we do not need to find the explicit expression for U. The only property of U which will prove useful for our purpose is

$$U = \prod_{\alpha=1}^{\infty} U_\alpha , \tag{13}$$

where U_α depends only on a_α and a_α^+, due to the special form of the canonical transformation (10)*.

We are going now to compute the matrix elements

$$\begin{aligned} &\left(\psi(\alpha, m), U\psi(\beta, n)\right), \\ &\psi_{(\alpha,\ m)} = a_{\alpha_1}^+ \ldots a_{\alpha_m}^+ \psi_0, \\ &\psi_{(\beta,\ n)} = a_{\beta_1}^+ \ldots a_{\beta_n}^+ \psi_0 \end{aligned} \tag{14}$$

and to show that they are zero. To begin with, we consider

$$(\psi_0, U\psi_0) = (\psi_0, \Psi_0) \tag{15}$$

(see (4) and (7), n infinite). A straightforward formal procedure (see Appendix I) leads us to

$$\Psi_0 = c \cdot \exp\left\{\frac{1}{2} \sum_{\alpha=1}^{\infty} \operatorname{tgh} \varphi_\alpha \cdot a_\alpha^+ a_\alpha\right\} \psi_0 . \tag{16}$$

This state should be normalizable; the normalization, say, to unity yields

$$|c|^2 = \prod_{\alpha=1}^{\infty} (\cos h\varphi_\alpha)^{-1} . \tag{17}$$

* I am grateful to Dr. A. PAWLIKOWSKI for a hint.

A necessary and sufficient condition for the product (17) to converge is

$$\sum_{\alpha=1}^{\infty} |(\cos \varphi_\alpha - 1)| < \infty \,. \tag{18}$$

If (18) is not fulfilled then c vanishes and so does the matrix element (15).

Let us consider (14). Taking into account that a_α, a_α^+ commute with a_β, a_β^+ for $\beta \neq \alpha$ we get (conf. Appendix I)

$$(\psi_{(\alpha,m)}, U\psi_{(\beta,n)}) = \prod_\gamma F_\gamma \,, \tag{19}$$

where $F_\gamma \neq (\cos h\varphi_\gamma)^{-\frac{1}{2}}$ at most for $(n + m)$ factors F_γ. In case (17) diverges, the change of a finite number of factors cannot improve the convergence. Thus (19) vanishes for every finite m and n. The states $\psi_{(\alpha,\ m)}$ form a complete set, consequently U exists only if (18) holds, otherwise it vanishes.

For $n = 0$ we get from (19) in case of divergence

$$(\psi_{(\alpha,m)} U\psi_0) = (\psi_{(\alpha,m)}, \Psi_0) = 0, \tag{20}$$

i.e. Ψ_0 does not exist.

To each unitary transformation there belongs a canonical one; the reverse is not true.

3. Haag's theorem

In Section 2 we showed that two representations of the commutation relations can, but need not, be equivalent. We are now going to show that in a relativistic theory of a scalar real field $A(x)$ which satisfies the usually stated axioms* and is governed by the Hamiltonian

$$H = H_0(t) + H_i(t) \,,$$

(HEISENBERG picture !) where $H_0(t)$ is the Hamiltonian of the mathematical free particles and $H_i(t)$ the interaction Hamiltonian, the eigenstates of $H_0(t)$ cannot belong to the same Hilbert space $\mathscr{H}$ as the eigenstates of H, until $A(x)$ is a free field. This is due to the rather trivial fact that the 4-dimensional

* We include into the axioms the spectral condition in the sharper form, i.e., we assume the existence of a discrete eigenstate of energy and momentum zero cyclic with respect to $A(x)$. The very interesting case of a continuous spectrum (particles of mass zero) giving rise to infra-red inequivalent representations is not considered here. See e.g., B. SCHROER, "Infra-teilchen in der Quantenfeldtheorie", Fortschritte der Physik, **11**, 1, 1963, or J. TARSKI "Representations of Fields in Two Dimensional Model Theory" (preprint).

volume of the MINKOWSKI space is infinite* and has little to do with the renormalization problem (ultraviolet divergences).

HAAG's theorem [5] can be expressed barring rigour as: a unitary transformation interconnecting the eigenstates of $H_0(t)$ and H does not exist. A rigorous proof of this theorem was given by GREENBERG [6]. We shall outline another proof (see e.g. [7]) which although somewhat simpler needs an additional assumption about the existence of a complete asymptotic field.

We are going to prove the following: Under conditions stated below a real, scalar field $A(x)$ becomes a free field. The conditions are: $A(x)$ satisfies the usually stated axioms, possesses an asymptotic incoming field $A_{in}(x)$ with a mass $m \neq 0$; the state $\Psi_0(t)$ defined by

$$a_\alpha(t)\,\Psi_0(t) = 0, \qquad (\alpha = 1, 2, \ldots)$$

$$a_\alpha(t) = i \int\limits_{t=x_0} d^3x \left\{ f_\alpha^*(x) \frac{\partial A(x)}{\partial x_0} - A(x) \frac{\partial f_\alpha^*(x)}{\partial x_0} \right\}, \tag{21}$$

(where $f_\alpha(x)$ are the normalized $K - G$ solutions with positive frequencies satisfying the orthogonality and completeness conditions) exists for each finite time t in the Fock space of the incoming field and is invariant with respect to 3-dimensional translations.

Our proof makes use of the theorem stated by RUELLE [8], BORCHERS [9] and REEH and SCHLIEDER [10] and [11] that if an eigenstate of the energy and momentum belonging to the discrete eigenvalue zero exists and is cyclic with respect to the field operator ring the necessary and sufficient condition for the uniqueness of the physical vacuum is the irreducibility of the operator ring. In our case — according to the axioms — $A(x)$ is irreducible and at least one (discrete) vacuum state exists. Thus the vacuum must be unique. On the other hand if the vacuum state is unique, then it is the only state invariant with respect to the 3-dimensional translations, (see Appendix II). Thus

$$\Psi_0(t) = c(t)\,\Psi_0\,, \tag{22}$$

where $c(t)$ is a number $\neq 0$ and Ψ_0 is the physical vacuum. From (21) we get

$$a_\alpha(t)\Psi_0 = 0 \tag{23}$$

* This becomes clear if one looks at the nonrelativistic case, e.g., at the connection between the ground state and the mathematical vacuum; there appears an exponential factor e^{-VL}, $L\,(>0)$ being the linked diagram integral independent of volume; V is the volume appearing due to one integration which is left over in the diagram. Thus $\lim\limits_{V\to\infty} e^{-VL} = 0$.

and from the Yang-Feldman equation — after some simple manipulations —

$$KA(x)\Psi_0 = -j(x)\Psi_0 = 0\,. \tag{24}$$

According to a theorem due to REEH and SCHLIEDER [12] and [13] for a complete field $A(x)$ the ring of operators $R(\beta)$ with an arbitrary small open 4-dimensional support β operating on Ψ_0 gives a dense set in the Hilbert space. Because of locality, we are always able to find an open region β spacelike to y so that

$$[j(y), R(\beta)] = 0 \tag{25}$$

holds. Formula (25) together with (24) yields

$$\begin{aligned} j(y)R(\beta)\psi_0 &= 0, \qquad \text{for } \beta \text{ spacelike to } y \text{ or}^*, \\ j(y) &= 0, \end{aligned} \tag{26}$$

which accomplishes the proof.

4. The Bogoliubov — B. C. S. model; irreducible representation

Let us consider an infinite system satisfying the periodicity conditions on the boundary of the volume V described by the spinor field

$$\psi_r(\vec{x}), \psi_r^+(\vec{x}) \qquad r = 1, 2$$

satisfying the conventional anticommutation relations

$$\begin{aligned} \{\psi_r(\vec{x}), \psi_s(\vec{y})\} &= 0, \\ \{\psi_r(\vec{x}), \psi_s^+(\vec{y})\} &= \delta_{rs}\delta(\vec{x} - \vec{y})\,. \end{aligned} \tag{27}$$

The bounded operators

$$\psi(f) = \int \psi(\vec{x})\, f(\vec{x})\, d^3x\,, \tag{28}$$

where $f(\vec{x})$ is a square integrable test function, form a VON NEUMANN ring.

* Notice that $j(y)$ is an unbounded operator, nevertheless it is symmetric and we assume that this extension is unique so that it is self-adjoint.

We start with the BOGOLIUBOV — B. C. S. Hamiltonian [14]

$$K(V) = \int_V d^3x \left\{ \psi_r^+(\vec{x}) \left(-\frac{1}{2m}\vec{\nabla}^2 - \mu \right) \psi_r(\vec{x}) - c \right\} = H_i(V), \tag{29}$$

$$Hi(V) = \frac{1}{V} \int_V \psi_1^+(\vec{x})\, \psi_2^+(\vec{x}+\vec{z})\, \psi_2(\vec{x}^{\mathrm{I}}+\vec{z}^{\mathrm{I}}) \psi_1(\vec{x}^{\mathrm{I}})\, v(\vec{z},\vec{z}^{\mathrm{I}})\, d^3x d^3x^{\mathrm{I}}\, d^3z d^3z^{\mathrm{I}},$$

where $v(\vec{z},\vec{z}^{\mathrm{I}}) \xrightarrow[|\vec{z}-\vec{z}^{\mathrm{I}}|\to\infty]{} 0$ sufficiently rapidly so that

$$\lim_{V\to\infty} \int_V \int_V v(\vec{z},\vec{z}^{\mathrm{I}})\, d^3z d^3z^{\mathrm{I}} < \infty\,; \tag{30}$$

μ is the chemical potential, c is a constant adjusted to make the ground state expectation value of K equal to zero.

Our goal [15] is to construct the Hilbert space where the expression

$$\lim_{V\to\infty} (\psi, H_i(V)\psi^1)$$

makes sense for vectors forming a dense set in $\mathscr{H}$.

To begin with we compute the commutator

$$[H_i(V), \psi_1(y)] = -\int d^3z^{\mathrm{I}}\, \omega_2(\vec{z}^{\mathrm{I}}, \vec{y})\, \varphi_V(\vec{z}^{\mathrm{I}}), \tag{31}$$

where

$$\omega_2(\vec{z},^{\mathrm{I}}\vec{y}) = \int v(\vec{z},\vec{z}^{\mathrm{I}})\, \psi_2^+(\vec{y}+\vec{z})\, d^3z, \tag{32}$$

$$\varphi_V(\vec{z}^{\mathrm{I}}) = \frac{1}{V}\int \psi_2(\vec{x}^{\mathrm{I}}+\vec{z}^{\mathrm{I}})\, \psi_1(\vec{x}^{\mathrm{I}})\, d^3x^{\mathrm{I}}. \tag{33}$$

One can show by inspection that

$$\varphi(\vec{z}^{\mathrm{I}}) = \lim_{V\to\infty} \varphi_V(\vec{z}^{\mathrm{I}}) \tag{34}$$

commutes with all $\psi_a(f)$ and $\psi_a^+(f)$ for each finite $\vec{z}^{\mathrm{I}}$. As the operator $\omega_2(\vec{z},\vec{y})$ is bounded and its norm tends to zero sufficiently fast as $|\vec{z}^{\mathrm{I}}| \to \infty$ we may replace $\varphi_V(\vec{z}^{\mathrm{I}})$ in (31) in the limit $V\to\infty$ by a number, provided that the representation of the VON NEUMANN ring is irreducible. The sufficient condition for that is (30) [16]*.

Thus we arrive at the conclusion that for an irreducible representation the limit $V\to\infty$ of the right hand side of (31) (and similar relations with $\psi_2(y)$, $\psi_1^+(y)$, $\psi_2^+(y)$ on the left hand side) is a linear expression in $\psi_2^+(\vec{y}+\vec{z})$.

* I am indebted to Dr. UMEZAWA for calling my attention to this paper.

If we take this into account as well as the commutation relations (27) we may reconstruct a new Hamiltonian K^{I} bilinear in the field operators, viz.

$$K^{\mathrm{I}} = \lim_{V\to\infty} \int_V \left\{ \psi_r^+(\vec{x}) \left(-\frac{1}{2m}\vec{\nabla}^2 - \mu \right) \psi_r(\vec{x}) - c \right\} d^3x + H_i^{\mathrm{I}}, \tag{35}$$

$$\begin{aligned} H_i^{\mathrm{I}} &= \int \left\{ \Delta(\vec{z})\, \psi_1^+(\vec{x})\, \psi_2^+(\vec{x}+\vec{z}) + \Delta^*(\vec{z})\, \psi_2(\vec{x}+\vec{z})\, \psi_1(\vec{x}) \right\} d^3x\, d^3z + C^{\mathrm{I}}, \\ \Delta(\vec{z}) &= \int v(\vec{z}, \vec{z}^{\mathrm{I}})\, \varphi(\vec{z}^{\mathrm{I}})\, d^3 z^{\mathrm{I}}. \end{aligned} \tag{36}$$

So far $\Delta(\vec{z}^{\mathrm{I}})$ is not determined.

We are going now to diagonalize K^{I}. It is convenient to use the Fourier transform of the field operators as well as the BOGOLIUBOV transformation

$$\begin{aligned} \psi_1(\vec{p}) &= u(\vec{p})\, \gamma_1(\vec{p}) - v(\vec{p})\, \gamma_2^+(-\vec{p}), \\ \psi_2(\vec{p}) &= v(-\vec{p})\, \gamma_1^+(-\vec{p}) + u(-\vec{p})\, \gamma_2(\vec{p}), \end{aligned} \tag{37}$$

$$\begin{aligned} &\left.\begin{array}{ll} \text{for} & |\vec{p}| < P_F\, u \\ \text{for} & |\vec{p}| > P_F\, v \end{array}\right\} = \tilde{\Delta}(-\vec{p}) \left\{ (E(\vec{p}^2) - \varepsilon(\vec{p}^2))^2 + |\tilde{\Delta}(-\vec{p})|^2 \right\}^{-\frac{1}{2}}, \\ &\left.\begin{array}{ll} \text{for} & |\vec{p}| < P_F\, v \\ \text{for} & |\vec{p}| > P_F\, u \end{array}\right\} = (E(\vec{p}^2) - \varepsilon(\vec{p}^2)) \left\{ (E(\vec{p}^2) - (\vec{p}^2))^2 + |\tilde{\Delta}(-\vec{p}|^2 \right\}^{-\frac{1}{2}}, \end{aligned} \tag{38}$$

with

$$p_{\mathrm{F}} = \sqrt{2m\mu}$$

and

$$E(\vec{p}^2) = (\varepsilon(\vec{p}^2)^2 + |\tilde{\Delta}(-\vec{p})|^2)^{\frac{1}{2}};\ \varepsilon(\vec{p}^2) = \frac{p^2}{2m} - \mu.$$

We get

$$K^{\mathrm{I}} = \int E(\vec{p}^2) \left\{ \gamma_1^+(\vec{p})\, \gamma_1(\vec{p}) + \gamma_2^+(\vec{p})\, \gamma_2(\vec{p}) \right\} d^3 p. \tag{39}$$

The ground state Ω is defined by

$$\gamma_r(\vec{p})\Omega = 0\ \ (\Omega, \Omega) = 1 \text{ for every } \vec{p} \text{ and } r. \tag{40}$$

One can show using the definition of the momentum operator and the commutation relations (27) that Ω is invariant with respect to 3-dimensional translations. If the VON NEUMANN ring is irreducible one can also show that there exists only one state invariant under 3-dimensional translations (see Appendix III). Thus Ω is that very state.

We have still to determine $\tilde{\Delta}(-\vec{p})$ or — which turns out to the be same (see (36)) — $\varphi(\vec{z})$. In the irreducible case we have

$$\varphi(\vec{z}) = (\Omega, \varphi(\vec{z})\,\Omega) =$$

$$= \lim_{V\to\infty} \frac{1}{V} \int (\Omega, \psi_2(\vec{x}+\vec{z})\,\psi_1(\vec{x})\,\Omega)\, d^3x =$$

$$= (\Omega, \psi_z(\vec{z})\,\psi_1(\vec{o})\,\Omega)\,,$$

or, in the momentum representation substituting the operator γ for ψ,

$$\varphi(\vec{p}) = -\,u(\vec{p})\,v(\vec{p})\,. \tag{41}$$

Taking into account (36) and (38) we get from (41) an integral equation for $\tilde{\Delta}(\vec{p})$

$$\Delta(\vec{p}) = -\frac{1}{(2\pi)^3} \int v(\vec{p},\vec{q})\, \frac{\tilde{\Delta}(\vec{q})\,(E(\vec{q}^2) - \varepsilon(\vec{q}^2))}{\{E(\vec{q}^2) - \varepsilon(\vec{q}^2)\}^2 + [\tilde{\Delta}(\vec{q})]^2}\,. \tag{42}$$

This equation has a trivial solution $\tilde{\Delta}(\vec{p}) = 0$ corresponding to no interaction in (35). If one non-trivial solution of (42) exists (see [17]) then there exist many other non-trivial solutions. From a real solution

$$\tilde{\Delta}_0(\vec{p}) = \tilde{\Delta}_0^*(\vec{p}) \tag{43a}$$

we can generate complex solutions

$$\tilde{\Delta}_\alpha(\vec{p}) = e^{i\alpha}\,\tilde{\Delta}_0(\vec{p})\,, \qquad (0 \leqslant \alpha < 2\pi)\,. \tag{43b}$$

This in turn implies

$$\varphi_\alpha(\vec{z}) = e^{i\alpha}\,\varphi_0(\vec{z})\,.$$

Because of (33) different solutions (43b) correspond to different gauges of the first kind of the operators ψ

$$\psi(\vec{x}) \to \psi(\vec{x})\, e^{\,i\frac{\alpha}{2}}. \tag{44}$$

It is worthwhile to emphasize that K is invariant with respect to the gauge transformation and consequently conserves the number of mathematical "particles" while K^I is neither invariant nor does it conserve the number of mathematical "particles". This can be expressed in a different way: The assumption of irreducibility of the operator ring is equivalent to a fixed choice of the gauge or with the breaking of one of the symmetries of the model (gauge invariance) of the original Hamiltonian.

The question arises: Can the BOGOLIUBOV transformation (37) be expressed by means of a unitary operator, viz.,

$$\psi_r(\vec{p}) = U(\alpha)\,\gamma_r(\vec{p};\alpha)\,U^{-1}(\alpha)\,. \qquad (r = 1, 2;\ 0 \leq \alpha < 2\pi)\,.$$

The states $\Omega(\alpha)$ as well as the mathematical vacuum Φ defined by

$$\psi_r(\vec{p})\,\Phi = 0 \qquad (r = 1, 2, \text{ all } \vec{p}) \qquad (45)$$

are all translationally invariant. If the operator ring $(\gamma_r(\vec{p};\alpha), \gamma_\alpha^+(\vec{p};\alpha))$ is irreducible then neither Φ nor $\Omega(\beta)$, $\beta \neq \alpha$ belongs to the space $\mathscr{H}_\alpha$ which space arises by Fock construction from $\Omega(\alpha)$. Consequently a unitary transformation does not exist and the representations for different α^Is are unitarily inequivalent to each other and to the Fock space of the mathematical "particles".

So we come to the conclusion that the so-called degeneracy of the ground state means for a system of infinitely many degrees of freedom that the Hamiltonian admits several inequivalent representations of the basic operator algebra. This stays in a close connection with some symmetry properties.

5. The Bogoliubov — B. C. S. model; reducible representation

In the Fock space $\mathscr{H}_\alpha$ representing the field operator ring R a state with a fixed a number of mathematical "particles" does not exist. We recall the expression $\varphi_\alpha(\vec{z})$ given by (33) and (34) which, applied to a normalized state of exactly N particles, would give on the one hand the same state with a numerical factor ($\varphi_\alpha(\vec{z})$ is considered to be a number); on the other hand a state of $(N-2)$ particles ($\varphi_\alpha(\vec{z})$ contains two destruction operators). This is, of course, meaningless. So the eigenvectors of the original Hamiltonian K belonging to a fixed number of "particles" do not belong to $\mathscr{H}_\alpha$.

The only escape to get these states seems to be to give up the irreducibility of the operator ring R. Then the expression

$$\frac{1}{\varphi_0^*(\vec{z})} \lim_{V\to\infty} \frac{1}{V} \int \psi_2(\vec{x}+\vec{z})\,\psi_1(\vec{x})\,d^3x = X^+ \qquad (46)$$

no longer has to be considered as a number, although it commutes with all the elements of the operator ring. A detailed investigation shows that X is unitary and does not depend on $\vec{z}$.

In case of a reducible ring there can be several ground states which all are translationally invariant. To find these ground states which belong to a definite number of "particles" one can adopt the procedure of EZAWA [16]

who in turn followed the idea of ARAKI et al [18]. The method consists of finding a suitable representation for a given Hamiltonian. In the quantum mechanics (finite number of degrees of freedom) the diagonalization of a given Hamiltonian can be carried out starting with an arbitrary representation. The principal axis transformation does always exist as all the representations of the canonical commutation relations are equivalent up to a unitary transformation. In the quantum field theory the situation does not look so nice, since the principal axis transformation does not necessarily exist in an arbitrarily chosen representation. Thus we have to guess the proper representation at the very beginning. This looks rather hopeless. In some cases one can find the suitable representation by a limiting procedure. In our case we can start with a theory with a finite volume. The ground state for

$$K^{\mathrm{I}}(V) = \sum_{\vec{p}} E(\vec{p}^2)\left\{\gamma_1^+(\gamma_1(\vec{p}) + \gamma_2^+(\vec{p})\,\gamma_2(\vec{p})\right\}, \tag{47}$$

$$\vec{p} = \frac{\vec{n}\,\pi}{V^{1/3}}\,\vec{n} = n_1, n_2, n_3 \text{ (integers)}$$

(cf. (39)) is*

$$\begin{aligned}\Omega_V(a) &= \prod_{\vec{p}} \left(u_a^*(\vec{p}) - v_a(\vec{p})\,\psi_1^+(\vec{p})\,\psi_2^+(-\vec{p})\right)\Phi = \\ &= \prod_{\vec{p}}^{|\vec{p}|>p_F} \left(u_0(\vec{p})e^{ia} - v_0(\vec{p})\,\psi_1^+(\vec{p})\,\psi_2^+(-\vec{p})\right)\times \\ &\times \prod_{\vec{p}}^{|\vec{p}|>p_F} \left(u_0 - v_0(\vec{p})\,e^{-ia}\,\psi_1^+(\vec{p}\,\psi_2^+(-\vec{p})\right)\Phi = \\ &= \prod_{\vec{p}} u_a^+(\vec{p})\exp\left\{-\frac{v_a(\vec{p})}{u_a^*(\vec{p})}\,\psi_1^+(\vec{p})\,\psi_2^+(-\vec{p})\right\}\Phi.\end{aligned} \tag{48}$$

We may construct the Wightman functions

$$(\Omega_V(a), \ldots \psi_2(\vec{x}_2) \ldots \psi_1(\vec{x}_1)\,\psi_1^+(y_1) \ldots \psi_2^+(y_2) \ldots \Omega_V(a))$$

and take the limit $V \to \infty$. We get in this way a right representation required by the Hamiltonian. The limit representation is reducible i.e., the Wightman functions do not determine the state uniquely.

As $\lim_{V\to\infty} K^{\mathrm{I}}(V)$ yields a good approximation to $\lim_{V\to\infty} K(V)$, (see (29)) which is manifestly gauge invariant, we expect that the linear combination of $\Omega_V(a)$ will give a good approximation to an eigenstate of $K(V)$. This is

* We get this state in the same formal way we got the Fock vacuum for the operators b, b^+) in Appendix I, (cf. (Al. 7)) but in a somewhat more involved manner.

indeed so and we arrive in the limit $V \to \infty$ at the Ansatz of HAAG [15]. The ground state Ω with a definite number of particles is defined by

$$(\Omega, \psi^+(\vec{x}_1) \ldots \psi^+(\vec{x}_m)\, \psi(\vec{y}_1) \ldots \psi(\vec{y}_n)\, \Omega) =$$
$$= \begin{cases} (\Omega(\alpha), \psi^+(\vec{x}_1) \ldots \psi^+(\vec{x}_m)\, \psi(\vec{y}_1) \ldots \psi(\vec{y}_n)\, \Omega(\alpha) & \text{for } n = m. \\ 0 \end{cases} \tag{49}$$

The expectation values of gauge dependent quantities with respect to states with a given number of "particles" vanish. The right hand side of (49) does not depend on α. The state Ω is not uniquely determined by (49) as the representation is reducible. Indeed, the Wightman functions for Ω as well as for $X^n\ \Omega$ (where X is given by (46)) are the same due to the commutability of X with the ring R. The states defined by (49) are all translationally invariant because the states $\Omega(\alpha)$ are translationally invariant. The positive-definiteness of the Wightman functions results from the definiteness of the Wightman functions for finite V in the limit.

The unitary operator X applied to Ω gives a state Ω_r orthogonal to Ω. Repeated application of X to Ω gives a whole string of states orthogonal to each other, translationally invariant, having the properties of a ground state with a fixed number of particles. As mentioned before all these states have the same expectation values, i.e., they all describe the same physical state.

From these states we may generate the Hilbert space $\mathscr{H}_\Omega$ by applying the operator ring R.

Let us diagonalize the operator X whose eigenvalues are

$$e^{-i\alpha} \quad 0 \leqslant \alpha < 2\,\pi$$

(a continuous spectrum). Then the differential subspaces belonging to the eigenvalue $e^{-i\alpha}$ are the before-mentioned spaces $\mathscr{H}_\alpha \cdot \mathscr{H}_\Omega$ is a direct integral of $\mathscr{H}_\alpha$ over α. Each such differential subspace gives an irreducible representation of R.

Appendix I*

We want to find a state Ψ_0 which satisfies

$$b_\alpha\, \Psi_0 = 0 \text{ for each } b_\alpha\ (\alpha = 1, 2, \ldots)\,, \tag{A1.1}$$

i.e. according to (10)

$$(a_\alpha - \operatorname{tgh} \varphi_\alpha \cdot a_\alpha^+)\Psi_0 = 0\,. \tag{A1.2}$$

* I am grateful to Dr. A. PAWLIKOWSKI for many suggestions exploited in this Appendix.

We look for such a non-singular operator M which yields simultaneously

$$M a_\alpha M^{-1} = a_\alpha + \operatorname{tgh} \varphi_\alpha \cdot a_\alpha^+ , \tag{A1.3}$$

$$M a_\alpha^+ M^{-1} = a_\alpha^+ . \tag{A1.4}$$

The operator

$$M = \exp\left\{ -\frac{1}{2} \sum_\alpha \operatorname{tgh} \varphi_\alpha \cdot a_\alpha^+ a_\alpha \right\} \tag{A1.5}$$

will do; indeed, from (A1.2) we get, using (Al. 3—4),

$$M(a_\alpha - \operatorname{tgh} \varphi_\alpha a_\alpha^+) M^{-1} M \Psi_0 = a_\alpha M \Psi_0 = 0 , \tag{Al. 6}$$

consequently

$$M \Psi_0 = c \psi_0 \ (c \text{ is a number}) , \tag{Al. 7}$$

and we arrive at the formula (16).

To get c we have to normalize the vector Ψ_0. Because a_α, a_α^+ commute with a_β, a_β^+ for $\alpha \neq \beta$ and the states with different number of "particles" are orthogonal to each other, the product

$$\left|\left| \prod_\alpha \exp\left\{ \frac{1}{2} \operatorname{tgh} \varphi_\alpha a_\alpha^+ a_\alpha^+ \right\} \psi_0 \right|\right|^2 = |c|^{-2} \tag{A1.8}$$

can be factorized, and we get

$$\prod_\alpha \cosh \varphi_\alpha = |c|^{-2} ,$$

which coincides with (17).

From formula (16) one sees that

$$(\psi_0, \Psi_0) = c \left(\psi_0, \exp\left\{ \frac{1}{2} \sum_\alpha \operatorname{tgh} \varphi_\alpha a_\alpha^+ a_\alpha^+ \right\} \psi_0 \right) = c .$$

Appendix II*

One of the Lie—Cartan equations for the Lorentz group reads

$$[M_{10}, P_1] = -iP_0 . \tag{A2.1}$$

* I am grateful to Dr. A. UHLMANN for showing this proof to me.

Because $P_1 = P_1^+$ (we assume the extension of P_1 is unique) we get for the state $\Psi_0(t)$ since it is 3-dimensionally invariant

$$(\Psi_0(t),\ P_0\Psi_0(t)) = 0\,, \tag{A2.2}$$

where we assumed in addition that the state $M_{10}\psi_0(t)$ is bounded.

Because of the spectral condition the spectrum of P_0 is non-negative; thus

$$P_0\Psi_0(t) = 0\,. \tag{A2.3}$$

We know (by assumption) that the physical vacuum is the only state in $\mathscr{H}$ satisfying (A2.3), consequently

$$\Psi_0(t) = \text{const.}\ \Psi_0\,, \tag{A2.4}$$

which is just the result we need (cf. (22)).

Appendix III

If the VON NEUMANN ring R is irreducible, then there exists only one state invariant with respect to 3-dimensional translations. To prove it let us assume the contrary that there are two vacuum states $\Omega^{(1)}$ and $\Omega^{(2)}$. We can always make them orthogonal to each other. Both of them are cyclic because the ring R is supposed to be irreducible. Thus some of the expressions of type

$$(\Omega^{(1)},\ \psi^+(x_1)\ldots\psi^+(x_n)\,\psi(y_1)\ldots\psi(y_m)\,\Omega^{(2)})$$

will be different from zero. In particular, taking into account the superselection rules, we are able to find an operator W with $(n+m)$ even, quasilocal, for which

$$(\Omega^{(1)},\ W\Omega^{(2)}) \neq 0\,. \tag{A3.1}$$

Now, for such a quasilocal operator of the form

$$\int\ldots\int \mathscr{F}(x-z_1,\ldots x-z_n;\, x-z_1^{\mathrm{I}},\ldots x-z_m^{\mathrm{I}})^*\times$$
$$\times\,\psi^+(z_1)\ldots\psi^+(z_n)\,\psi(z_1^{\mathrm{I}})\ldots\psi(z_m^{\mathrm{I}})d^3z_1\ldots d^3z_m\,,$$

where $\mathscr{F}$ vanishes sufficiently rapidly for $z_1,\ldots,z_m^{\mathrm{I}}\to\infty$, the following theorem due to Haag is valid: The operator $\overline{W}$

$$\overline{W} = \lim_{V\to\infty}\frac{1}{V}\int W(\vec{x})\,d^3x\,,\qquad W(\vec{x}) = e^{-i\vec{\mathrm{P}}\vec{x}}\,W e^{i\vec{\mathrm{P}}\vec{x}} \tag{A3.2}$$

commutes with all operators of the ring, consequently it is a number. We have

$$(\Omega^{(1)}, W\Omega^{(2)}) = \lim_{V\to\infty} \frac{1}{V} \int (\Omega^{(1)}, W(\vec{x})\,\Omega^{(2)})\, d^3x =$$

$$= (\Omega^{(1)}, \vec{W}\Omega^{(2)}) = \vec{W}(\Omega^{(1)}, \Omega^{(2)}) = 0,$$

which contradicts (A3.1).

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ОБ УНИТАРНО-НЕЭКВИВАЛЕНТНЫХ ПРЕДСТАВЛЕНИЯХ В КВАНТОВОЙ ТЕОРИИ ПОЛЯ И В ТЕОРИИ МНОГИХ ТЕЛ

Я. Т. ЛОПУШИНСКИЙ

Резюме

1. Введение. 2. Случай вещественного скалярного поля. 3. Теорема Хаага. 4. Модель Боголюбова BCS; неприводимые представления. 5. Модель Боголюбова BCS; приводимые представления. Приложение I, Приложение II, Приложение III.

GAUGE INVARIANCE AND STRUCTURE OF THE CORRELATION FUNCTIONS OF AN IMPERFECT BOSE GAS

By

G. JONA—LASINIO

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Summary

This report refers to some work done in collaboration with F. DE PASQUALE of the University of Rome and E. TABET of the Laboratorio Fisico dell'Istituto Superiore di Sanità — Rome.*

We have analysed the structure of the correlation functions of an imperfect BOSE gas with condensation in the zero momentum state, with the aim of developing methods to calculate the effect of the residual interaction of quasiparticles. The main tool of our analysis is a generalized Ward identity which follows the invariance of the theory under infinitesimal gauge transformations. The usefulness of the approach has been illustrated by constructing explicitely density correlation functions which satisfy the longitudinal sum rule and the continuity equation in the case of the BOGOLIUBOV approximation and in the case of a non linear self-consistent approximation. The role of the theorem of HUGENHOLTZ and PINES in this context has also been discussed.

* This work has now appeared in Annals of Physics (N. Y) **33**, 381, 1965.

КАЛИБРОВОЧНАЯ ИНВАРИАНТНОСТЬ И СТРУКТУРА КОРРЕЛЯЦИОННЫХ ФУНКЦИЙ НЕСОВЕРШЕННОГО ГАЗА БОЗЕ

Г. ЙОНА—ЛАЗИНИО

Резюме

В основе доклада лежит работа, сделанная вместе с Ф. де Пасквале (Римский Университет) и Э. Табэ (LABORATORIO FISICO DELL' STITUTO SUPERIORE DI SANITA — РИМ).*

Мы проанализировали структуру функций корреляции несовершенного газа Бозе с конденсацией в состоянии с нулевым импульсом с целью развить методы для вычисления эффекта остаточного взаимодействия квазичастиц. Главным методом нашего анализа является использование обобщенного тождества Уорда, которое следует из инвариантности теории относительно бесконечно малых калибровочных преобразований. Полезность метода иллюстрируется явным построением корреляционных функций плотности, которые удовлетворяют продольному правилу сумм и уравнению непрерывности в случае приближения Боголюбова и в случае нелинейного самосогласованного приближения. Также обсуждается роль теоремы Гугенгольца и Пайнса в этом отношении.

* Опубликовано в Annals of Physics (N. Y) **33**, 381, 1965.

STRANGE PARTICLES AND THE UNSYMMETRICAL VACUUM

By

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In HEISENBERG's nonlinear theory some new results are presented concerning mainly the properties of strange elementary particles.

In the nonlinear theory of HEISENBERG and coworkers it was suggested that the occurrence of the approximate conservation laws in nature may result from a rather complicated structure of the ground state, the vacuum state, rather than an unsymmetrical dynamical law. This conjecture has been very successful in various fields, and, in fact, constitutes the main topic of this conference. Its importance in elementary particle physics, however, has still to be demonstrated.

In our theory, it was particularly assumed that the deviations from the isotopic spin symmetry in the interactions of elementary particles arise from isotopic spin properties of the vacuum, which otherwise has the common features.

Such an asymmetrical vacuum state may be pictured as a state filled with an infinite number of certain "bosons". These bosons do not have any Lorentz properties, i.e. mass and spin, but do carry isotopic spin. We have called these entities "spurions" because of their similarity to the particles introduced by WENTZEL and others in connection with weak interactions.

Such a "vacuum" has two different effects on the particles:

1. As a result of the interaction of the particles with the infinite spurion sea as a whole, which eventually will be assumed to have an infinite net isospin polarization, the mass degeneracy of the states with respect to isospin rotations, in particular the mass degeneracy of all particle isomultiplets, will be removed. E.g. the proton and neutron states which differ by the direction of their isotopic spin, will split up as a consequence of their different residual interaction with the unsymmetrical ground state; their isospin will be parallel or antiparallel to the vacuum isospin polarization.

A vacuum state which carries an infinite isotopic spin is necessarily infinitely degenerate in an isotopic spin symmetrical theory. The degenerate vacuum states differ by the direction of their infinite polarization in isospin space. They are transformed into each other under isospin rotations. Because of the infinite size of the polarization, however, such a rotation, as was frequently emphasized, cannot be generated by a unitary transformation in Hilbert

space: vacuum states which are polarized in different directions do not belong to the same Hilbert space; they are orthogonal to each other in the sense of VAN HOVE and HAAG. Hence we have only to consider one vacuum isospin polarization which we identify with the z-direction. We also prefer to talk about an "unsymmetrical vacuum" instead of a "degenerate vacuum" which are group theoretically synonymous.

2. The particles do not interact so to say only with the sea as a whole but also with its members, the spurions. This intercation will be present even if the net isospin polarization of the vacuum would be assumed to be zero. In particular, "particles" may be strongly bound to a finite number of spurions. In this way we may create from every "particle" a whole family of particles with indentical Lorentz properties but different isospin properties. Such "anormal" states may be identified with the strange particles. It should be emphasized that the operation of rotating the isospin of any finite number of spurions can be represented by a unitary transformation. It does not lead out of the Hilbert space. (We are studying, at present, certain models in which the existence of such "anormal" states can be more clearly established. In ferromagnets it would correspond to an electron riding on a spin wave.)

In my report today I will solely concern myself with this second aspect of an unsymmetrical ground state. I will try to demonstrate that strange particles and their properties may be understood on such a basis.

I will subdivide my talk into two parts:

A) I will discuss the symmetry properties of the spurion in particular, in contrast to earlier assumptions, and consider the general form of their interaction terms. Subsequently we investigate the form of the interaction terms of particles constructed from the fundamental fields and spurions.

B) I will outline the procedure by which we advance to calculate the masses of the simplest fermions and bosons and their coupling constants.

A. Symmetry properties of the spurions and the form of their interaction terms

Spurions are considered to carry an isospin $^1/_2$. However, in addition to isospin further properties have to be attributed to the spurions to be consistent with the requirement of Lorentz- and CPT-invariance of the vacuum. According to earlier investigations three such possibilities present themselves: The first and simplest possibility from a group theoretical point of view adds a parity property. About this possibility I have reported earlier. Here the spurions transform according to the simplest, nontrivial representation of $P \times SU_2$ (discrete reflection group $\times$ isospin group). This leads to a close connec-

tion between parity and isotopic spin. Some immediate implications with respect to the mass spectrum of the baryons and their properties, e.g. odd $\Lambda\Sigma$-parity, however, are in contradiction to present experimental evidence. Therefore this possibility is ruled out. This possibility also has the disadvantage that strangeness, or hypercharge, cannot be understood without enlarging the group space of the spurions to include a gauge group.

Hence the second possibility was investigated which is the subject of the present report. (The third possibility is a combination of this possibility with the first one, which was actually used in the former papers after the inclusion of hypercharge). In this second possibility a "spurion number" (gauge group) is defined in addition to the isospin which then is identified with the hypercharge Y. The spurions in this case transform according to the simplest, nontrivial representation of $U_2 = g \times \mathrm{SU}_2$ in which four different states (spurions and antispurions with $I_3 = \pm 1/2$) can be distinguished. Spurions and antispurions are transformed into each other under $\mathcal{C}$-conjugation. Because of the absence of spin they obey Bose statistics.

As a first step we consider a general system composed of n_s spurions and n_a antispurions, and construct the most general 2-spurion correlation operator, i.e. the operator which for ordinary particles corresponds to the pair interaction operator. We find that

$$O(n_s, n_a) = \left[\sum_{s=1}^{n_s} \vec{\tau}_s - \sum_{a=1}^{n_a} \vec{\tau}_a\right]^2 - f(n_s + n_a) = 4(\vec{I}_s - \vec{I}_a)^2 - f(n_s + n_a) =$$

$$= 2 \sum_{i<j}^{n_s+n_a} \varepsilon_{ij} \vec{\tau}_i \vec{\tau}_j + [3n - f(n)],$$

$$n = n_s + n_a,$$

where $\vec{\tau}_s$, $\vec{\tau}_a$ are the isospin matrices acting on the spurions and antispurions, respectively; $f(n)$ is some arbitrary function of the total number of spurions and antispurions $n = n_s + n_a$; ε_{ij} is a sign function ± 1 depending on whether the indices refer to like or unlike "particles". The operator is determined by the requirements of isospin-, gauge- and $\mathcal{C}$-invariance, and the condition that $O(n_s, n_a) \to O(n_s \pm 1, n_a \mp 1)$ under $\mathcal{C}$-conjunction of a single spurion or antispurion, respectively, which can be deduced from the fact, that the annihilation of a spurion is equivalent to a creation of an antispurion.

The eigenvalue of the operator for a system with isospin I and hypercharge $Y = 2(n_s - n_a)$ is given by

$$O(n_s, n_a) = -4\left[I(I+1) - \frac{1}{4} Y^2\right] + [n(n+4) - f(n)],$$

i.e. the I, Y combination is exactly of the form which occurs in connection with broken SU_3 theories. However, it should be emphasized that the above formula does not contain any of the feicatures partular to SU_3, which — depending on the representation — express themselves in certain limitations on the possible values of I and Y, leading to the characteristic multiplets. I and Y in the above formula are only limited by $I \geq \frac{1}{2} Y$. The characteristic combination of I and Y in the above formula is brought about by the special form of the interaction (opposite sign for particle-particle and particle-anti-particle systems; this, in fact, is common to many other theories, e.g. interaction via π-mesons) and the permutation symmetry of the spurions (only symmetrical combinations are admitted).

Up to now we have dealt only with the spurions and general spurion systems. To connect these results with our actual problem, i.e. the determination of the isospin-hypercharge dependence of the mass operators of single particle states, we have to consider systems which are constructedfrom spurions and the spinor-isospinor field operator $\psi(x)$ which occurs in the differential equation ofour theory:

$$i \sigma^{\nu} \frac{\partial}{\partial x_{\nu}} \psi(x) = l^2 \sigma_{\mu} : \psi(x) \left[\psi^* (x) \sigma^{\mu} \psi(x)\right] : .$$

In the field operator the isospin-hypercharge properties seem inseparably connected with the Lorentz properties. In particular, the isospins of two field operators will not be symmetrical, in general, as required for the spurions, since only the space-spin-isospin-dependence is required to be antisymmetrical.

However, the situation is different if only field operators at the same spacetime point are considered. Due to our particular interaction $\sigma_{\mu} \sigma^{\mu} = = - I I + \vec{\sigma}\vec{\sigma}$, the interaction will vanish whenever the spins in $\psi(x)\, \psi(x)$ are symmetrical ($\vec{\sigma}\vec{\sigma} = +1$). Since in this case the space-dependence is symmetrical this immediately implies that there is no interaction if the isospins are antisymmetrical, i.e. whenever they do not behave like spurions. Hence the above spurion considerations can be equally applied for states which are constructed from spurions and field operators at the same space-time point. Particles which can be constructed in this way will be called "primary particles". Therefore we expect "primary particles" to behave very much like a many-spurion system. On the other hand "primary particles" are also distinguished dynamically since they can take immediate advantage of the original contact interaction.

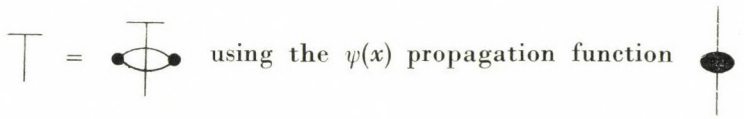
If we deal with systems of non-zero baryon number also the operator BY can occur besides $O(n_s, n_a)$, which also appears in the GELL-MANN—OKUBO mass formula.

I will now proceed to the second part of my talk and give a rough description how the above considerations can be utilized to calculate the masses and coupling constants of the simplest fermions and mesons.

B. Procedure to calculate the masses and coupling constants

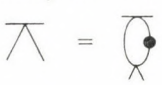
The actual calculations of the masses of the particles proceed according to the following program:

1. We assume the existence of a dominating baryon pole at some mass in the propagation function of the field operator $\psi(x)$. The eigenvalue equation for the baryon wave function $\varphi(x) = \langle 0 \mid \psi(x) \mid p \rangle$ is then derived in the lowest Tamm—Dancoff approximation as represented by the equation



using the $\psi(x)$ propagation function with only the (regularized) baryon pole, as an approximation for the contraction function, and is examined with respect to a discrete baryon solution. Such a solution exists. The mass of the baryon is fixed by the selfconsistency requirement. This calculation is the same as carried out in 1958. We may call it a baryon boot-strap calculation. The calculation does not change if we include spurions in the matrix elements, i.e. there is no splitting of the baryon levels at this point.

2. We proceed to consider the eigenvalue equation for the "primary" strangeness zero bosons with the eigenfunctions $\varphi_B(x) = \langle 0 \mid \psi(x)\, \psi^*(x) \mid p \rangle$. They are derived again in lowest NTD-approximation which is of the form



using the ψ-contraction function with the selfconsistent baryon pole. The bosons appear as "S-states" of the baryon-antibaryon system bound by the contact interaction represented by the nonlinear term. The discrete solutions belong to the η, π, ω, ϱ mesons. However, only the spin 0 states η, π were considered. In case of the ω, ϱ also D-states have to be included. This is more difficult and is, at present, carried out by STUMPF and YAMAMOTO.

For the η, π the eigenvalue equation is of the form

$$1 + [3 - I(I+1)]\, 2\, Q\left(\frac{\mu_I^2}{\varkappa^2}\right) = 0\,,$$

where $Q\left(\frac{\mu_I^2}{\varkappa^2}\right)$, is a known function which depends on the mass ratio of meson to baryon mass. This again is nothing new (1960).

3. As a new step we now consider primary boson wave functions which involve spurions. E. g. the K-meson wave function is related to the matrix element $\varphi_K(x) = \langle 0| s_1 \psi_2(x) \psi_3^*(x) |p\rangle$. The isospin wave-function of the spurion s and of ψ (which behaves like a spurion) must be symmetrical. Because of the well-known relationship (change of coupling transformation)

$$(\overline{123}) = \sqrt{\frac{3}{4}}\,(1\tilde{\tilde{2}}\tilde{\tilde{3}}) + \sqrt{\frac{1}{4}}\,(1\overline{23}),$$

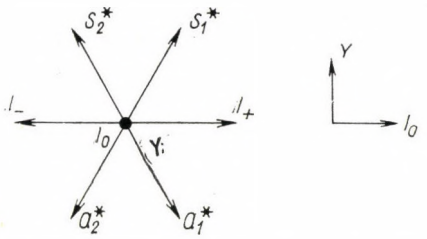
where — designates a symmetrical, $\sim$ an antisymmetrical combination, one immediately obtains the result that the K, because of the spurion permutation symmetry, behaves like 75% $(s\eta)$ and 25% $(s\pi)$. Since the spurion does not participate in the interaction (only ψ occur in the nonlinear term), the K-eigenvalue operator directly reflects this "mixture", which is familiar to us from broken SU_3. Hence the boson eigenvalue equation can be generalized to

$$1 + \left[3 - \left[I(I+1) - \frac{1}{4}Y^2\right]\right] 2Q\left(\frac{\mu_{I,Y}^2}{\varkappa^2}\right) = 0$$

to include the K. According to the above derivation this formula is not to be used for other than η, π, K states. E.g. a K-quartet state $\left(I = \frac{3}{2},\ Y = \frac{1}{2}\right)$ can only be derived if also a $I = \pm 2$, $Y = 0$ state is contained from the beginning. Roughly speaking the $K_{3/2}$ would be a "mixture" of such a state and a π. The $I = 2$, $Y = 0$, however, in our framework cannot be constructed from field operators at the same space-time point. They are not "primary" particles and hence are dynamically quite different. If we consider $K_{3/2}$ as a pure $(s\pi)$ state then the $K_{3/2}$ eigenvalue equation is identical with the equation for the π. The spurion in this case is not "bound" nor localized but only formally attached.

Hence we conclude that there exist only 8 "primary" boson states of spin 0. Their eigenvalue operator contains as a factor the familiar $I(I+1) - \frac{1}{4}Y^2$. Since the function $Q^{-1}(x^2)$ behaves essentially like $\ln^{-1} x^2$ with $x^2 = \mu^2/\varkappa^2$, it can be very well approximated in the relevant region by $a + bx^2$ which then leads to the well-known boson mass formulas. The same procedure would lead to a spin 1 "octet" if the spin 1 states were predominantly 3s-states, which does not, however, seem likely from our calculations.

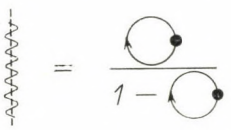
To emphasize a certain similarity with the adjoint representation of SU_3 we may draw the following diagram of the operators: I_+, I_-, I_0, Y are the familiar generators of the U_2 group. In addition we now have the 4 spurion



creation operators (or corresponding antispurion annihilation operators) s_1^*, s_2^*, a_1^*, a_2^* which simulate to some extent the operators V_-, U_+, U_-, V_+ of SU_3.

4. As the next step we now repeat the baryon eigenvalue calculation with the inclusion of not only the baryon contraction functions but also the boson contraction functions. This means a generalisation of the Tamm—Dancoff approximation in the sense that we now not only explicitly pull out the 2 point contractions but also a certain part which contains the meson poles. Our integral equation for the baryons is now graphically represented by

where the first graph corresponds to the dressing of the baryons by themselves, the second graph, however, corresponds to the boson corrections, as calculated in the third step. Or more precisely the boson propagator is essentially represented by



where

is the kernel of the boson eigenvalue integral equation as calculated before. The calculation of the residium of the poles which occur at the boson masses leads to a determination of the coupling constants.

From the above improved eigenvalue equation for the baryons in lowest approximations we therefore get corrections due to the virtual exchange of η, π and K mesons. However, now due to the K-exchange the eigenvalue operators

of baryons with zero, one- and two spurions will differ from each other. One can show that the eigenvalue equation is of the form:

$$\sigma_\nu p^\nu \left[1 + \frac{3}{2} \left(\frac{\varkappa l}{2\pi} \right)^4 L\left(-\frac{p^2}{\varkappa^2} \right) - \frac{3}{2} \sum_B \frac{C_{F,B}\, r\left(-\frac{p^2}{\mu_B^2} \right)}{Z_B^2 q'\,(\mu_B^2/\varkappa^2)} \right] \varphi(p) = 0 ,$$

where L, r, q' are all known functions. The L-term already occurs in the original approximation, the $\sum_B$ stems from the boson contributions. There the K-contribution is of principal importance, since it depends on the isospin and the hypercharge of the baryon. In fact we find:

$$C_{F,K} = c_0 + c_1 BY + c_2 \left[I(I+1) - \frac{1}{4} Y^2 \right],$$

i.e. we denote that the eigenvalue operator is multiplied by a factor which has the GELL-MANN—OKUBO form. To the extent that the dependence of the integral operators on the masses is smooth this will also reproduce the GELL-MANN—OKUBO formula for the masses. In our case L is not sufficiently smooth and hence does not give too good an agreement with the experimental masses, but these functions can scarcely be considered reliable to such an extent.

To summarize we may state that the inclusion of the boson contraction terms improve firstly the mass ratios between baryons and boson (the baryons become relatively heavier, which is in the right direction), and secondly it leads to a *splitting* of the baryons which agrees with the empirical mass sequence. However, it turns out that the zero-spurion system (the original system) has to be identified with Ξ, rather than the N. The one-spurion system corresponds to the Λ, Σ. The numerical values of c_1 and c_2 are still somewhat too small, but one can show that the inclusion of the spin 1 mesons will enlarge the effects.

The baryon calculation is not necessarily limited to the "octet" states but may also include other states, e.g. isoquartet states. To exclude these unwanted states the hypothesis was used that spurions can only be "bound" to a single field operator $\psi(x)$ such that the resulting electric charge never exceeds the value one. Such an assumption seems necessary in view of the local conservation of charge. In this case there would be only 8 "primary" baryons. However, this point has to be further investigated.

5. As a last step one may consider all elementary particles which cannot be constructed from field operators at the same space-time point. These particles we may call "secondary particles". These "secondary" particles cannot take immediate advantage of the contact interaction. For interactions

at a distance, however, the finite range interactions via exchange of the "primary" bosons may be the most important. E.g. the $^3/_2$, $^3/_2$ resonance state N^* may be considered in a good approximation as a bound state of an N and a π by virtue of an interaction produced by virtual mesons. For the calculation of these "secondary" particles an approximation scheme which assumes the "primary" particles as really "elementary" particles as a starting point may be better than an NTD-approximation. This seems rather obvious in the case of the deuteron and higher nuclei. Since the "primary" particles roughly allow a broken SU_3 terminology, also the secondary particles may allow a similar description. Of course, this has not to be the case since nothing comparable to the SU_3-Clebsch—Gordan algebra can be provided in our case. On the other hand bootstrap calculations like the ones carried out by ZEMACH and ZACHARIASEN and by CUTKOVSKY seem to indicate that the bootstrap mechanism may produce the higher group if the dimension of the adjoint representation (here 8) is somehow provided. On the basis of our present calculation we do not expect the coupling constants to follow very closely the SU_3 predictions.

СТРАННЫЕ ЧАСТИЦЫ И НЕСИММЕТРИЧНЫЙ ВАКУУМ

Г. П. ДЮРР

Р е з ю м е

Показываются некоторые новые результаты в нелинейной теории Гейзенберга главным образом по отношению свойств странных элементарных частиц.

VACUUM DEGENERACY AND BROKEN SYMMETRIES IN NONLINEAR SPINOR THEORIES

By

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The possibility of a spontaneous breakdown of some invariances will be discussed by considering three different nonlinear spinor equations. The breakdown of the γ^5-symmetry is considered by using the parity-symmetrized HEISENBERG equation. A possible mechanism is suggested for the spontaneous electromagnetic type breakdown of the G-invariance by introducing a simple G-invariant (but not isoinvariant) nonlinear field equation. The third field equation is invariant under the group Sp_6 of unitary symplectic transformations in six dimensions. It is assumed that the medium strong mass splitting operator is due to a spontaneous mixing of the SU_3 singlet (representation 21), the isosinglet member of the SU_3 octet (representation 21), and the Sp_6 singlet (representation 1).

This report contains some remarks about a possibility of the breakdown of some invariances in nonlinear spinor theories. Three different nonlinear spinor equations will be considered briefly in the following. The first one is the parity-symmetrized HEISENBERG equation in a form given by DÜRR [1—3]

$$-i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi + \lambda T\gamma^\mu \psi(\overline{\psi}\gamma_\mu\psi) + \lambda T i\gamma^5\gamma^\mu\psi(\overline{\psi}i\gamma^5\gamma_\mu\psi) = 0, \tag{1}$$

where the 2×4 component DIRAC isospinor ψ is the field operator of matter and T is WICK's chronological operator. Eq. (1) is invariant under the isogroup. The second (nonisoinvariant) field equation is the following

$$-i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi + \lambda \sum_{j=1}^{2}\left[T\gamma^\mu P_j^+ \psi(\overline{\psi}\gamma_\mu P_j^- \psi) + T i\gamma^5\gamma^\mu P_j^+\psi(\overline{\psi}\, i\gamma^5\gamma_\mu P_j^-\psi)\right] = 0, \tag{2}$$

where

$$P_1^{\pm} = \frac{1}{2}(1 \pm \tau^3), \qquad P_2^{\pm} = \frac{1}{2}(1 \mp \tau^3),$$

and the matrices τ^j $(j = 1, 2, 3)$ are the usual isospin matrices. Eq. (2) is SO_2-invariant. The third equation is invariant under the group Sp_6 of unitary symplectic transformations in six dimensions. The group Sp_6 and its applications in the theory of strongly interacting particles have been discussed by BACRY, NUYTS and VAN HOVE [4]. In this report the following Sp_6 invariant nonlinear

field equation will be considered

$$-i\gamma^\mu \frac{\partial}{\partial x^\mu} t_m + \sum_{m',n,n'} h_{mm'} h_{nn'} \left[G_V T \gamma^\mu \bar{t}_n (\bar{t}_{m'} \gamma^\mu t_{n'}) + \right.$$
$$\left. + G_A T i\gamma^5 \gamma^\mu t_n (\bar{t}_{m'} i\gamma^5 \gamma_\mu t_{n'}) \right] = 0 , \tag{3}$$

where $t_m(x\alpha)$ is a 6×4 component spinor field. The components of t_m are the trions introduced by BACRY and al.,

$$t_1 = T^+, \qquad t_2 = T^0, \qquad t_3 = T'^0 ,$$
$$t_4 = \theta^0, \qquad t_5 = \theta^+, \qquad t_6 = \theta'^+ ,$$

and the matrix h is defined by

$$h = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} . \tag{4}$$

The invariance of eq. (3) under Sp_6 can be easily proved by taking into account that the group Sp_6 can be defined as the group of six dimensional unitary transformations which leave invariant the following bilinear form

$$f = \sum_{m,n} t_m (x\alpha) h_{mn} t_n (y\beta) . \tag{5}$$

According to HEISENBERG's suggestions, the basis of our considerations is a generalization of the usual postulates of quantum mechanics. We assume that the states of the systems are represented by vectors in a space of states with indefinite metric and the observables of the systems can be represented by pseudo-hermitian operators. The observables are connected with the generators of the symmetry transformations in the usual way and the action principle [5] is not valid in general. In this case the usual canonical commutation relations are not required for the field operators ψ and t.

Next, the possibility of a spontaneous breakdown of symmetries will be discussed along the lines suggested by HEISENBERG [1], NAMBU and JONA—LASINIO [6], BAKER and GLASHOW [7], SAKURAI [8] and many others. The basic idea may be formulated in the following way. It is well known that some invariance properties of eqs. (1) — (3) are violated in nature by mass terms and medium strong, electromagnetic and weak interactions. These violated

symmetries and the exact symmetries of the field equations will be called "bad" and "good" symmetries, respectively. We reject to destroy the "bad" symmetries by introducing mass terms or medium strong, electromagnetic and weak interactions with empirical coupling constants from a "philosophical" point of view. It will be assumed that the breakdown of the "bad" invariances is spontaneous and very similar to the breakdown of the gauge-invariance in the theory of superconductivity.

We next turn to a brief discussion of the breakdown of the γ^5-invariance by considering the HEISENBERG equation (1) [6, 9]. In a "first approximation" the isogroup is a "good" symmetry. The basis of our considerations is an appropriate generalization of the relativistic FOCK method to calculate the simplest many-point amplitudes of the type

$$\tau(x_1\,\alpha_1 \ldots x_m\,\alpha_m \mid y_1\,\beta_1 \ldots y_n\,\beta_n) =$$

$$= <0 \mid T\psi(x_1\,\alpha_1) \ldots \psi(x_m\,\alpha_m)\,\overline{\psi}(y_1\,\beta_1) \ldots \overline{\psi}(y_n\,\beta_n)|\Phi>.$$

The solutions of the FOCK equations for the τ amplitudes, the masses of the particles and the coupling constants may be obtained, in principle, by a self-consistent method, which is, to a certain extent, similar to the bootstrap mechanism of S-matrix theory [10]. According to NAMBU and JONA—LASINIO [6] we expect that the physical vacuum state and its elementary excitations are not eigenstates of the chirality χ. Assuming the invariance of the vacuum state under the full Lorentz group and the isogroup we obtain for the vacuum expectation value F

$$F(x\alpha \mid y\,\beta) = \langle 0 \mid T\,\psi(x\alpha)\,\overline{\psi}(y\,\beta) \mid 0\rangle$$

$$= i\gamma^{\mu}_{\alpha\beta}\,z_{\mu}\,H(z^2) + \delta_{\alpha\beta}\,M(z^2)\,, \tag{6}$$

where

$$z = x - y, \quad z^2 = z^{\mu}\,z_{\mu}\,.$$

We next turn to a brief discussion of the two point amplitudes

$$\tau^{(SI)}\,(x\alpha \mid y\beta) = \langle 0 \mid T\,\psi(x\alpha)\,\overline{\psi}(y\beta) \mid \varphi^{(SI)}\rangle\,, \tag{7}$$

where $\mid \varphi^{(SI)} >$ is a state vector characterized by isospin assignments $I = 0$ or 1, intrinsic angular momentum assignments $S = 0$ or 1, intrinsic parity $P = -1$ and G parity $G = (-1)^{S+I}$.

We emphasize that $\mid \varphi^{(SI)} >$ may contain contributions from various bound and scattering states. The general form of the amplitude (7) for $I = 1$,

$S = 0$ has been given by BAUMANN, FREUND and THIRRING [11]

$$\tau^{(01)}(x\alpha|y\beta) = \frac{1}{(2\pi)^2}\int d^4 q \exp\left[-iq\frac{1}{2}(x+y)\right]\varphi^{(01)j}_{\alpha\beta}(q; x-y)\,\tau(q,j), \quad (8)$$

$$\varphi^{(01)j}_{\alpha\beta}(q; z) = \Big\{\gamma^5\left[f_1^{(01)}(z^2, (qz)^2) + (\gamma q) f_2^{(01)}(z^2, qz)^2\right] +$$

$$+ (\gamma z)(qz) f_3^{(01)}(z^2, (qz)^2) + \left((\gamma z)(\gamma q) - (\gamma q)(\gamma z)\right) f_4^{(01)}(z^2, (qz)^2)\Big]\tau^j\Big\}_{\alpha\beta}. \quad (9)$$

The corresponding equations for $S = 1$ states can be obtained in a similar way

$$\tau^{(11)}(x\alpha|x\beta) = \frac{1}{(2\pi)^2} f_2^{(11)}(0, 0)\int d^4 q \exp(-iqx)\,(\gamma^\mu \tau^j)_{\alpha\beta}\,\tau(q, \mu, j), \quad (10)$$

$$\mathrm{q}^\mu \tau(\mathrm{q}, \mu, \mathrm{j}) = 0. \quad (11)$$

For the isoscalar amplitudes we have the unit matrix instead of τ^j.

Let us consider a bound state with self-energy μ_{SI}. In this case the amplitudes $\tau(q, j)$ etc. satisfy the KLEIN—GORDON equation, e.g.

$$(q^2 - \mu_{01}^2)\,\tau(q, j) = 0. \quad (12)$$

The eigenvalues of the four-momentum P^μ and of the other observables may be obtained from the covariant systems of FOCK equations without making use of the explicit form of the observables as functionals of the field operator ψ.

Consider first the FOCK equations for the one-point amplitude $\tau(x\alpha'|)$. It follows from eq. (1) that

$$-i\gamma^\mu_{\alpha\alpha'}\frac{\partial}{\partial x^\mu}\tau(x\alpha'|) = \lambda\Gamma_{\alpha\alpha'\gamma\delta}\,\tau(x\alpha'\, x\delta|x\gamma), \quad (13)$$

where

$$\Gamma_{\alpha\alpha'\gamma\delta} = (\gamma^\nu)_{\alpha\alpha'}(\gamma_\nu)_{\gamma\delta} + (i\gamma^5\gamma^\nu)_{\alpha\alpha'}(i\gamma^5\gamma_\nu)_{\gamma\delta}. \quad (14)$$

Let us consider the free one fermion states $|\varphi^{(f)}>$ with mass m. For the corresponding amplitudes we obtain

$$-\lambda\,\Gamma_{\alpha\alpha'\gamma\delta}\,\tau^{(f)}(x\alpha'\, x\delta\,|\,x\,\gamma) = m\tau^{(f)}(x\alpha\,|\,). \quad (15)$$

The general three point amplitude τ will be decomposed in the following way [9]

$$\tau(x\alpha_1\, x\alpha_2|x\beta_1) = \left\{\frac{1}{\sqrt{2}\,(2\pi)^4}\iint d^4p\, d^4q \exp(-ipx_1)\exp\left[-iq\,\frac{1}{2}\,(x_2+y_1)\right]\cdot\right.$$
$$\cdot\left[\varphi^{(01)j}_{\alpha_2\beta_1}(q;\, x_2-y_1)\,\tau(p\alpha_1\|qj)+\ldots\right]-$$
$$-\frac{1}{\sqrt{2}\,(2\pi)^4}\iint d^4p\, d^4q \exp(-ipx_2)\exp\left[-iq\,\frac{1}{2}\,(x_1+y_1)\right]\cdot$$
$$\left.\cdot\left[\varphi^{(01)j}_{\alpha_1\beta_1}(q;\, x_1-y_1)\,\tau(p\alpha_2\|qj)+\ldots\right]\right\}_{x_1=x_2=y_1=x}. \tag{16}$$

This amplitude is antisymmetrical in the "coordinates" x_1, α_1 and x_2, α_2. The first term contains the contributions from a system of a fermion + fermion-antifermion pair in $S=0$, $I=1$ state. The dots denote similar terms containing contributions from fermion-antifermion pairs with $S=0$, $I=0$ and $S=1$, $I=0$ or $I=1$. The functions $\tau(p\alpha\,|\,|\,qj)$, etc., are the Fourier amplitudes of the decomposition. Substituting (16) into the one-point FOCK equation (13) we obtain

$$-i\gamma^\mu\frac{\partial}{\partial x^\mu}\tau(x\alpha'|) = \frac{g_{01}}{\mu_{01}}(i\gamma^5\gamma^\mu\tau^j)_{\alpha\alpha'}\left[\frac{\partial}{\partial x'^\mu}\tau(x\alpha'\|x'\,j)\right]_{x=x'}+$$
$$+\frac{g_{00}}{\mu_{00}}(i\gamma^5\gamma^\mu)_{\alpha\alpha'}\left[\frac{\partial}{\partial x'^\mu}\tau(x\alpha'\|x'\,0)\right]_{x=x'}+ \tag{17}$$
$$+g_{11}(\gamma_\mu\tau^j)_{\alpha\alpha'}\tau(x\alpha'\|x\mu j)+g_{10}(\gamma^\mu)_{\alpha\alpha'}\tau(x\alpha'\|x\mu 0),$$

where

$$\tau(x\alpha'\|x'\,j) = \frac{1}{(2\pi)^4}\iint d^4p\, d^4q \exp[-ipx-iqx']\,\tau(p\alpha'\|qj), \tag{18}$$

etc., further

$$\frac{g_{01}}{\mu_{01}} = -\frac{4}{\sqrt{2}}f_2^{(01)}(0,0)\lambda, \quad \frac{g_{00}}{\mu_{00}} = -\frac{12}{\sqrt{2}}f_2^{(00)}(0,0)\lambda, \tag{19}$$

$$g_{11} = \frac{4}{\sqrt{2}}f_2^{(11)}(0,0)\lambda, \qquad g_{10} = \frac{12}{\sqrt{2}}f_2^{(10)}(0,0)\lambda. \tag{20}$$

In this paper the constants g_{SI} and masses μ_{SI} will be fixed conveniently by corresponding nucleon "bound pair" coupling constants and "bound pair" self-energies, respectively, to be determined later. This only means an appropriate normalization of the functions φ.

Eq. (17) is very similar to the one-point FOCK equation derived from a conventional field theory, in which the fermions are coupled with pseudoscalar and vector bosons with isospin 0 and 1. We have derivative coupling for pseudoscalar bosons. The fundamental difference is that the stable bosons are bound fermion-antifermion pairs, and the forces generated by bound fermion-antifermion pairs guarantee the existence of these bound pairs self-consistently. This situation is due to the nonlinear character of the underlying field equation. Thus, as a next step, we shall investigate the FOCK equations for the fermion-antifermion amplitude and the following approximation will be suggested

$$\begin{aligned}
&\left(-i\gamma^{\mu}_{\alpha\alpha'}\frac{\partial}{\partial x^{\mu}}+m\delta_{\alpha\alpha'}\right)\tau(x\alpha'\,|y\beta')\left(i\gamma^{\nu}_{\beta'\beta}\frac{\overleftarrow{\partial}}{\partial y^{\nu}}+m\delta_{\beta'\beta}\right)= \\
&=\frac{g^2_{01}}{\mu^2_{01}}(i\gamma^5\gamma^{\mu}\tau^k)_{\alpha\alpha'}\tau(x\alpha'|y\beta')(i\gamma^5\gamma^{\nu}\tau^k)_{\beta'\beta}\frac{\partial^2}{\partial x^{\mu}\partial y^{\nu}}\Delta^c(x-y;01)+ \\
&+\frac{g^2_{00}}{\mu^2_{00}}(i\gamma^5\gamma^{\mu})_{\alpha\alpha'}\tau(x\alpha'|(y\beta')(i\gamma^5\gamma^{\nu})_{\beta'\beta}\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\Delta^c(x-y;00)+ \qquad (21) \\
&+g^2_{11}(\gamma^{\mu}\tau^k)_{\alpha\alpha'}\tau(x\alpha'|y\beta')(\gamma^{\nu}\tau^k)_{\beta'\beta}\Delta^c_{\mu\nu}(x-y;11)+ \\
&+g^2_{10}(\gamma^{\mu})_{\alpha\alpha'}\tau(x\alpha'|y\beta')(\gamma^{\nu})_{\beta'\beta}\Delta^c_{\mu\nu}(x-y;10)+ \\
&+\lambda^2\Gamma_{\beta\beta'\varepsilon\varphi}\Gamma_{\alpha\alpha'\gamma\delta}\tau^{(W)}(x\alpha'\,x\delta y\varphi|y\varepsilon x\gamma y\beta')\,,
\end{aligned}$$

where m is the fermion mass, the functions $\Delta^c(z, SI)$ and $\Delta^c_{\mu\nu}(z; SI)$ are the free pseudoscalar and vector boson propagators, respectively, with mass μ_{SI}, and $\tau^{(W)}$ contains the WICK terms with two contractions in the WICK decomposition of the six-point function. The contraction is defined in the following way

$$\underline{\psi(x\alpha)\,\bar{\psi}(y\beta)} = \langle 0 \mid T\,\psi(x\alpha)\,\bar{\psi}(y\beta) \mid 0 \rangle\,. \qquad (22)$$

The terms containing the functions Δ^c are the ladder contributions from the bound pairs. In this approximation the π, η, ϱ and ω mesons are stable bound pairs, the instabilities result from higher approximations to be considered later. The masses and coupling constants are fixed, in principle, by self-consistency. The last term in (21) contains, physically speaking, contributions from highly "dissociated" pairs. The mass m is the contribution of the fermion self-energy. To obtain the fermion mass the calculation of the three point function $\tau(x_1x_2\,|\,y_1)$ is necessary by using a generalization of the methods discussed previously (c. f. eq. (15)). Thus, we have to discuss the binding problem of a two fermion + antifermion system. The mass is fixed by self-consistency. The whole self-consistent calculation is very complicated. The first steps of the calculations, however, may lead to useful semiempirical

methods. E.g. as a first step the two-point vacuum expectation value (6) (fermion propagator function) and various bound and scattering amplitudes may be calculated by using the experimental masses and coupling constants. The dangerous singularities due to the vector boson propagators are cancelled by the corresponding contributions of the pseudoscalar bosons due to the pseudovector coupling between fermions and pseudoscalar bosons. We obtain the following relation for the coupling constants

$$\frac{g_{0I}^2}{\mu_{0I}^2} = \frac{g_{1I}^2}{\mu_{1I}^2} . \tag{23}$$

We next turn to a brief discussion of the fermion propagator solution. We assume that the physical propagator function (6) may be approximated by an appropriate solution of the homogeneous eq. (21). Thus, according to HEISENBERG's suggestions, we assume that the equal-time anticommutator of the field operators has no $\delta^3(x - y)$ type singularities on the light cone. This assumption must be justified self-consistently by considering the propagator solution of (21). By substituting (6) into (21) we obtain a system of coupled nonlinear differential equations for H and M. The nonlinearity is due to the last term of (21). We have obtained a finite solution near the light cone. The asymptotic behaviour in infinity is given by a free fermion propagator. The complete solution can be obtained by well known numerical methods. Our fermion propagator function exhibits an invariant cut-off near the light cone. This cut-off, however, is not arbitrary but prescribed by the dynamics of the theory.

Next we turn to the discussion of the spontaneous breakdown of the isosymmetry. We expect that this breakdown is due to a spontaneous mixing of neutral isovector and isoscalar states. The $\omega - \varrho$ mixing is an illustrative example. The nonisoinvariant interaction generated by these mixed states may be obtained by an appropriate generalization of the methods discussed in the first part of this report [12].

Let us consider the electromagnetic interactions. We expect that a bound fermion-antifermion state exists with zero rest mass and $S = 1$. This state is the photon in this theory. As a first step in a self-consistent calculation we assume that the photon exhibits a maximal charge asymmetry, i.e. that the photon amplitude has the following form for $x = y$

$$\tau^{(\gamma)}(x\alpha|x\beta) \sim \int d^4 q \exp(-iqx) \left| \gamma^\mu \frac{1}{2}(1 - \tau^3) \right|_{\alpha\beta} \tau^\gamma(q, \mu) . \tag{24}$$

If we calculate the interaction generated by this state by using the field equation (1) we obtain the surprising result, that the electric charge of the neutron does not vanish if the charge of proton is different from zero.

It is interesting that by using the SO_2 invariant field equation (2) the interaction generated by photons with maximal charge asymmetry leads to an electromagnetic violation of the charge symmetry i.e. the electric charge of neutrons is zero.

Eq. (2) is nonisoinvariant. We may, however, expect that the self-consistency incorporated in our approximations leads to approximately isosymmetrical solutions for masses and coupling constants, along the lines suggested e.g. by ABERS, ZACHARIASEN and ZEMACH [13].

Finally the Sp_6 invariant nonlinear field equation (3) will be discussed. Let us consider the bosons belonging to representations 1,14 and 21 of Sp_6. The interactions generated by these bosons and the covariant FOCK equations can be obtained by an appropriate generalization of the methods suggested in the first part of this report [14]. By a generalization of the ideas of SAKURAI [8] we assume that the medium strong symmetry breaking interactions are due to a spontaneous mixing of the SU_3 singlet and the isosinglet member of the SU_3 octet in representation 21 and the Sp_6 singlet (representation 1). This mixing leads to a mass splitting operator Δm with the following transformation properties

$$\Delta m = \Delta m\,(21,1) + \Delta m\,(21,8) + \Delta m\,(14,8)\,, \tag{25}$$

where $\Delta(a, b)$ belongs to representation a of Sp_6 and b of SU_3. The effect of the mass splitting operator of the type (25) has been discussed by BACRY, NUYTS and VAN HOVE [4]. The last term $\Delta m\,(14,8)$ contributes to mass splittings within any SU_3 multiplet and the GELL-MANN—OKUBO type mass formulae can be obtained in the usual way. The second order effect of the first term $\Delta m\,(21,1)$ produces a mass shift of all mesons with quantum number $Z \neq 0$, retaining nevertheless valid predictions for the masses of the $Z = 0$ mesons.

Let us denote the 21 linearly independent infinitesimal generators of the basic representation of Sp_6 by $\lambda_1, \lambda_2 \ldots \lambda_{21}$. The "photon amplitude" with maximal charge asymmetry

$$\tau^{(\gamma)}(x\alpha|x\beta) \sim \int d^4 q \exp(-iqx)\left[\gamma^\mu \frac{1}{2}\left(1 - \lambda_3 - \frac{1}{\sqrt{3}}\lambda_8 + \frac{1}{\sqrt{6}}\lambda_{12}\right)\right]_{\alpha\beta} \tau^{(\gamma)}(q, \mu)\,, \tag{26}$$

generates a correct electromagnetic type interaction i.e. the electric charge of the neutral particles is zero.

It is interesting to speculate about the possibility of the dynamical origin of the internal symmetries of the strong interactions supported by recent "bootstrap" calculations [13, 15]. We may believe that the internal symmetries cor-

responding to quantum numbers of isospin, hypercharge etc. are not necessarily arbitrary constraints to be imposed at the beginning of the calculation. The self-consistency, incorporated in our approximations discussed in the first part of this report, may lead to symmetric solutions for masses and coupling constants. Thus we may believe that the T and θ trions are created by the 2×4 component ψ field, which satisfies the very simple field equation (2), and a field equation with higher internal symmetry is only an approximate semiphenomenological basis to calculate some important amplitudes in a relatively simple way.

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ВЫРОЖДЕНИЕ ВАКУУМА И НАРУШЕНИЕ СИММЕТРИЙ В НЕЛИНЕЙНЫХ СПИНОРНЫХ ТЕОРИЯХ

К. ЛАДАНЬИ

Резюме

Рассматриваются три разных нелинейных спинорных уравнений и обсуждается возможность спонтанного нарушения некоторых симметрий. Рассматривается нарушение J^5 — инвариантности в случае уравнения Гейзенберга, симметризованного по четности. Вводя простое G-инвариантное (но не изо-инвариантное) нелинейное уравнение поля, предлагается возможный механизм для спонтанного нарушения электромагнитного типа G-инвариантности. Третье уравнение поля инвариантно относительно группы Sp_6, т. е. относительно группы унитарных симплектических преобразований в шестимерном пространстве. Предполагается, что оператор расщеплений масс связан со спонтанным смешиванием Sp_6-синглета (представление 1), изосинглетного члена SU_3-октета (представление 21) и SU_3-синглета (представление 21).

BROKEN SYMMETRIES IN THE TWO GOLDSTONE MODELS

By

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The energy spectra of the two GOLDSTONE models are investigated using the RITZ variational method. More quantum mechanical ground states are obtained, but they do not belong to the same Hilbert space. One can solve the eigenvalue problem using only one Hilbert space but in this case the ground state does not show all the symmetries possessed by the Hamiltonian.

§ 1. For the understanding of spontaneous symmetry breakdown the GOLDSTONE model seems to be the most simple example [1]. Here the ground state possesses a broken symmetry already in the classical theory. It was this model in which GOLDSTONE recognised the so-called GOLDSTONE theorem, according to which spontaneous breaking of a continuous symmetry is necessarily connected with the appearance of zero mass bosons. For the understanding of the physical consequences of the GOLDSTONE model the SCHIFF representation seems to be the most convenient one i.e. working in a representation in which not the particle number but the field potential is diagonal [2]. In the following approximate solutions of the two GOLDSTONE models will be given and the physical causes and consequences of the symmetry breaking will be discussed.

§ 2. Let us consider the first GOLDSTONE model: a self-coupled real scalar field described by the field equation

$$\Box \varphi + \frac{1}{2} \mu_0^2 \varphi - \lambda_0^2 \varphi^3 = 0 .$$

The Hamiltonian of the field has the following form:

$$H = \int \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{4} \mu_0^2 \varphi^2 + \frac{1}{4} \lambda_0^2 \varphi^4 \right] d^3 r . \tag{1}$$

The field equation and the Hamiltonian are invariant with respect to the substitution

$$\varphi(x) \to -\varphi(x) , \quad \pi(x) \to -\pi(x) . \tag{2}$$

In the framework of the classical theory the field has two ground states, corresponding to the homogeneous solutions

$$\varphi = \frac{\mu_0}{\lambda_0 \sqrt{2}} \qquad \text{and} \qquad \varphi = -\frac{\mu_0}{\lambda_0 \sqrt{2}}, \tag{3}$$

in which the energy density possesses the same minimum value:

$$\frac{W_0^{\text{class}}}{\Omega} = \min \frac{H}{\Omega} = -\frac{\mu_0^4}{16\,\lambda_0^2}. \tag{4}$$

(Ω is the normalisation volume.) The third homogeneous solution $\varphi = 0$ corresponds to an unstable excited state with energy $H = 0$. The excitations with lower energies than this are vibrations around one of the two ground states (3).

§ 3. Let us go over to the quantum theory of the first GOLDSTONE model [3]. The commutation relations in Schrödinger picture are of the usual form:

$$[\varphi(\mathbf{r}), \pi(\mathbf{r}')] = i\delta(\mathbf{r} - \mathbf{r}'). \tag{5}$$

The commutators, the Hamiltonian (1) and the field momentum

$$\mathbf{P} = -\frac{1}{2}\int (\pi \cdot \underline{\nabla}\varphi - \underline{\nabla}\varphi \cdot \pi)\, d^3\mathbf{r} \tag{6}$$

are invariant with respect to the transformation

$$U\varphi(\mathbf{r})U^{-1} = -\varphi(\mathbf{r}), \qquad U\pi(\mathbf{r})U^{-1} = -\pi(\mathbf{r}). \tag{7}$$

U is a formally unitary operator. Our main task is to evaluate the simultaneous eigenvalues of H, $\mathbf{P}$ and U.

Let us carry out the calculation in the SCHIFF representation, in which $\varphi(\mathbf{r})$ is diagonal. Starting with a finite normalisation volume Ω the field operator can be expanded into Fourier series:

$$\varphi(\mathbf{r}) = \frac{x_0}{\sqrt{\Omega}} + \sqrt{\frac{2}{\Omega}}\, \Sigma\, x_{\mathbf{k}} \cos(\mathbf{kr} + \vartheta_{\mathbf{k}}). \tag{8}$$

(Summation is over the half $\mathbf{k}$ space, $\mathbf{k} = 0$ excluded.) If x_0, $x_{\mathbf{k}}$ and $\vartheta_{\mathbf{k}}$ are simple multiplicative parameters, π will be a combination of the differential

operators $\partial/\partial x_0$, $\partial/\partial x_{\mathbf{k}}$, $\partial/\partial \vartheta_{\mathbf{k}}$. With these parameters H and $\mathbf{P}$ take the following form:

$$H^{(1)} = -\frac{1}{2}\frac{\partial^2}{\partial x_0^2} - \frac{1}{4}\mu_0^2 x_0^2 - \frac{1}{2}\sum\left[\frac{1}{x_{\mathbf{k}}}\cdot\frac{\partial}{\partial x_{\mathbf{k}}}\left(x_{\mathbf{k}}\cdot\frac{\partial}{\partial x_{\mathbf{k}}}\right) + \frac{1}{x_{\mathbf{k}}^2}\cdot\frac{\partial^2}{\partial\vartheta_{\mathbf{k}}^2} + \right. \tag{9}$$

$$\left. + \left(\frac{1}{2}\mu_0^2 - k^2\right)x_{\mathbf{k}}^2\right] + \frac{\lambda_0^2}{\Omega^2}\int_{\Omega}\left[\frac{x_0}{\sqrt{2}} + \sum x_{\mathbf{k}}\cos\left(\mathbf{k}\cdot\mathbf{r} + \vartheta_{\mathbf{k}}\right)\right]^4 d^3\mathbf{r}\,,$$

$$\mathbf{P}^{(1)} = \sum i\mathbf{k}\cdot\frac{\partial}{\partial\vartheta_{\mathbf{k}}}\,. \tag{10}$$

In order to determine the approximate eigenfunctions of H, $\mathbf{P}$ and U a version of the variational method of RITZ will be applied, which was adapted by L. I. SCHIFF to quantum field theory [2]. Let our trial function be

$$\psi = \psi_0(x_0)\,\Pi\psi_{\mathbf{k}}(x_{\mathbf{k}}, \vartheta_{\mathbf{k}})\,. \tag{11}$$

(This is a reasonable supposition if one wishes to remain inside the practical solubility of the problem.) The partial functions ψ_0, $\psi_{\mathbf{k}}$ are normalised to unity and (11) is an exact eigenfunction of $\mathbf{P}$ and U if

$$\psi_0(-x_0) = \pm\psi_0(x_0), \qquad \psi_{\mathbf{k}}(x_{\mathbf{k}}, \vartheta_{\mathbf{k}}) = f_{\mathbf{k}}(x_{\mathbf{k}})e^{-in_{\mathbf{k}}\vartheta_{\mathbf{k}}}\,, \tag{12}$$

$$(n_{\mathbf{k}} = 0, \pm 1, \pm 2, \ldots),$$

To make (11) an approximate eigenfunction of H, let us minimalise $\langle H\rangle$ with respect to ψ_0 and $\psi_{\mathbf{k}}$. Using (12), one arrives at the expression

$$\langle\Psi|H|\Psi\rangle = \langle\Psi_0|-\frac{1}{2}\frac{d^2}{dx_0^2} - \frac{1}{4}\mu_0^2 x_0^2 + \frac{\lambda_0^2}{4\Omega}x_0^4|\Psi_0\rangle +$$

$$+ \sum\langle f_{\mathbf{k}}|-\frac{1}{2}\left[\frac{1}{x_{\mathbf{k}}}\cdot\frac{\partial}{\partial x_{\mathbf{k}}}\left(x_{\mathbf{k}}\cdot\frac{\partial}{\partial x_{\mathbf{k}}}\right) - \frac{1}{x_{\mathbf{k}}^2}n_{\mathbf{k}}^2\right] + \frac{1}{2}\left(-\frac{1}{2}\mu_0^2 + \mathbf{k}^2\right)x_{\mathbf{k}}^2 + \tag{13}$$

$$+ \frac{3}{8}\frac{\lambda_0^2}{\Omega}x_{\mathbf{k}}^4|f_{\mathbf{k}}\rangle + \frac{3\lambda_0^2}{4\Omega}\left[\sum_{\mathbf{k}\neq\mathbf{k}'}\sum\langle f_{\mathbf{k}}|x_{\mathbf{k}}^2|f_{\mathbf{k}}\rangle\langle f_{\mathbf{k}'}|x_{\mathbf{k}'}^2|f_{\mathbf{k}'}\rangle + \right.$$

$$\left. + 2\langle\psi_0|x_0^2|\psi_0\rangle\sum\langle f_{\mathbf{k}}|x_{\mathbf{k}}^2|f_{\mathbf{k}}\rangle\right].$$

The minimum conditions for this expression are the following equations:

$$\left[-\frac{1}{2}\frac{d^2}{dx_0^2}+V(x_0)\right]\psi_0(x_0)=E_0\,\psi_0(x_0)\,, \tag{14}$$

$$\left\{-\frac{1}{2x_{\mathbf{k}}}\cdot\frac{d}{dx_{\mathbf{k}}}\left(x_{\mathbf{k}}\cdot\frac{d}{dx_{\mathbf{k}}}\right)+\frac{n_{\mathbf{k}}^2}{2x_{\mathbf{k}}^2}+\right.$$
$$\left.+\frac{1}{2}\,\omega_{\mathbf{k}}^2\,x_{\mathbf{k}}^2+\frac{3\lambda_0^2}{2\Omega}\left[\frac{1}{4}\,x_{\mathbf{k}}^4-<f_{\mathbf{k}}\,|x_{\mathbf{k}}^2|\,f_{\mathbf{k}}>x_{\mathbf{k}}^2\right]\right\}\cdot f_{\mathbf{k}}(x_{\mathbf{k}})=E_{\mathbf{k}}\cdot f_{\mathbf{k}}(x_{\mathbf{k}})\,, \tag{15}$$

where

$$V(x_0)=-\frac{1}{4}\,(\mu_0^2-6\lambda_0^2\,B)\,x_0^2+\frac{\lambda_0^2}{4\Omega}\,x_0^4\,, \tag{16}$$

$$\omega_{\mathbf{k}}^2=\mathbf{k}^2-\frac{1}{2}\,\mu_0^2+3\lambda_0^2\left[B+\frac{1}{\Omega}<\psi_0\,|x_0^2|\,\psi_0>\right], \tag{17}$$

$$B=\frac{1}{\Omega}\,\Sigma<f_{\mathbf{k}}\,|x_{\mathbf{k}}^2|\,f_{\mathbf{k}}>\,. \tag{18}$$

E_0 and $E_{\mathbf{k}}$ are the Lagrange multiplicators of the normalisation conditions.

In the limiting case $B=0$ the potential curve (16) of the field oscillator would have two minima. If

$$\mu^2=\mu_0^2-6\lambda_0^2\,B \tag{19}$$

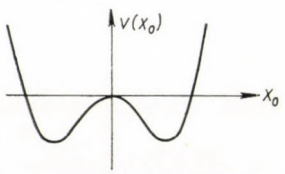

Fig. 1

turns out to be negative, the potential energy of the oscillator $\mathbf{k}=0$ will have only one minimum at $x_0=0$, i.e. only one quantum mechanical ground state appears. If, however, the influence of the other field oscillators (through B) is smaller, i.e. (19) is of positive value, then the original two minima survive, but their separation

$$2a=2\,\frac{\mu}{\lambda_0}\sqrt{\frac{\Omega}{2}} \tag{20}$$

is modified (Fig. 1). The potential wall separating the two wells has the height

$$V(0) - V(\pm a) = \frac{\mu^4 \Omega}{16 \lambda_0^2} .$$

Performing a transformation $\xi = x_0 + a$ (or $\eta = x_0 - a$), the potential energy $V(x_0)$ will have the form

$$V = V(a) + \frac{1}{2} \mu^2 \left(\xi^2 - a^{-1} \xi^3 + \frac{1}{4} a^{-2} \xi^4 \right) =$$

$$= V(a) + \frac{1}{2} \mu^2 \left(\eta^2 + a^{-1} \eta^3 + \frac{1}{4} a^{-2} \eta^4 \right) . \tag{21}$$

Thus one has two ground state solutions essentially of the form

$$\psi_0^{(0)} \sim e^{-\frac{\mu}{2}(x_0+a)^2} \pm e^{-\frac{\mu}{3}(x_0-a)^2} , \tag{22}$$

modified only slightly by the anharmonicity, i.e. by terms proportional to $\Omega^{-1/2}$, Ω^{-1} etc. The mean elongation is

$$\langle \psi_0^{(0)} | x_0^2 | \psi_0^{(0)} \rangle = a^2 + \frac{1}{2\mu} \left[1 + 0 \left(\frac{\lambda_0^2}{\Omega \mu^3} \right) \right] . \tag{23}$$

Taking this into account, the proper frequencies of the other oscillators ($\mathbf{k} \neq 0$) turn out to be

$$\omega_{\mathbf{k}} = \mathbf{k}^2 + \mu^2 \left[1 + 0 \left(\frac{\lambda_0^2}{\Omega \mu^3} \right) \right] . \tag{24}$$

Using this, the ground state solution of the equ. (15) will have essentially the form

$$\psi^{(0)} = \left(\frac{\omega_{\mathbf{k}}}{\pi} \right)^{1/2} e^{-\frac{1}{2} \omega_{\mathbf{k}} x_{\mathbf{k}}^2} , \quad (n_{\mathbf{k}} = 0) , \tag{25}$$

modified again only slightly by the anharmonicity. The eigenvalues of the two ground states, built up from (22) and (25), can be obtained easily by substituting these into (13)

$$\psi^{(0)} = \psi_0^{(0)} \Pi \psi_{\mathbf{k}}^{(0)} , \tag{26}$$

$$W^{(0)} = \min \langle \psi | H | \psi \rangle = -\frac{\mu^4}{16 \lambda_0^2} \Omega + \frac{1}{2} \mu + \Sigma \omega_{\mathbf{k}} - \frac{3}{4} \mu^2 \Sigma \frac{1}{\omega_{\mathbf{k}}} -$$

$$- \frac{3}{4} \frac{\lambda_0^2}{\Omega} \left(\Sigma \Sigma' \frac{1}{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}} + \frac{1}{\mu} \Sigma \frac{1}{\omega_{\mathbf{k}}} \right) \pm \frac{\Delta}{2} . \tag{27}$$

It is — as in the case of other fields — a divergent expression, but it exhibits a level splitting; the amount of the splitting is of the order

$$\triangle \sim \mu e^{-\mu^3 \lambda_0^{-4} \Omega}. \tag{28}$$

§ 4. The eigenfunctions of the excited states are determined also by the system of differential equations (14) and (15). These equations are formally separated, but the parameters μ, $\omega_{\mathbf{k}}$ occurring in them depend on the wave functions of the other oscillators, thus in reality they describe a rather strong coupling among the oscillators. This causes serious complications in the course of the evaluation of the higher excitations: it is difficult to assure their orthogonality to the wave functions of the lower states. (Exciting one partial oscillator the proper frequencies of all the other oscillators change!) This problem does not produce a great difficulty in the case of the first excitation. The eigenfunctions of the first excited states are essentially

$$\psi^{(1)} \sim \eta e^{-\frac{\mu}{2}\eta^2} \pm \xi e^{-\frac{\mu}{2}\xi^2} \quad \text{or} \quad \psi^{(1)} \sim x_{\mathbf{k}} \cdot e^{-i\vartheta_{\mathbf{k}}} \psi^{(0)}, \tag{29}$$

with the momentum eigenvalue

$$\mathbf{P} = 0 \text{ and } \mathbf{P} = \mathbf{k}. \tag{30}$$

The energy eigenvalue can be calculated again with the help of (13). To make comparison with the ground state possible, let us denote by μ the ground state value of the parameter defined by (19):

$$\langle \psi_{\mathbf{k}}^{(0)} | x_{\mathbf{k}}^2 | \psi_{\mathbf{k}}^{(0)} \rangle = \int_0^\infty x_{\mathbf{k}}^3 \psi^{(0)}(x_{\mathbf{k}})^2 \, dx_{\mathbf{k}} = \frac{1}{\omega_{\mathbf{k}}} \cdot \left[1 + 0\left(\frac{\lambda_0^4}{\Omega^2 \mu^6}\right)\right],$$

$$\mu^2 = \mu_0^2 - 6\frac{\lambda_0^2}{\Omega} \cdot \Sigma \frac{1}{\omega_{\mathbf{k}}} \left[1 + 0\left(\frac{\lambda_0^4}{\Omega^2 \mu^6}\right)\right]. \tag{31}$$

Here $0(\lambda_0^4 \Omega^{-2} \mu^{-6})$ is finite and proportional to Ω^{-2}. On the other hand, using (17) for the ground state eigenfunctions we have

$$\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + \mu^2 \left[1 + 0\left(\frac{\lambda_0^2}{\mu^3 \Omega}\right)\right].$$

Thus one arrives at the following value of the energy of the first excited states:

$$W^{(1)} = W^{(0)} + \mu + 0\left(\frac{\lambda_0^2}{\mu^3 \Omega}\right) \pm \frac{\triangle^{(1)}}{2}$$

or

$$W^{(1)} = W^{(0)} + \omega_{\mathbf{k}} + 0\left(\frac{\lambda_0^2}{\mu^3\,\Omega}\right) \pm \frac{\triangle^{(1)}}{2}\,. \tag{32}$$

The splitting $\triangle^{(1)}$ is again of the order of (28). One sees that the ground state is a discrete doublet level, but at the energy $W_0 + \mu$ an energy continuum starts (Fig. 2). The energy quantum of an excitation with momentum $\mathbf{k}$ is

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2} + 0\left(\frac{\lambda_0^2}{\mu^3\,\Omega}\right) \tag{33}$$

and it is finite, if (31) is of a finite value.

The sum is diverging quadratically but with a proper choice of the "bare mass" parameter μ_0 the "physical mass" μ can be made finite (mass renormalisation). This approximation corresponds to the summation of the

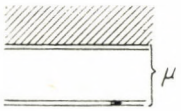

Fig. 2

Fig. 3

diagrams shown in Fig. 3. For very large values of the normalisation volume Ω the neglected terms will be very small, and the excitation will have a particle character.

§ 5. To get some information about the interaction of quanta, and to perform the charge renormalisation, it is necessary to discuss the two-quantum excitations, too. Investigating the two-particle scattering it is expedient to work in C. M. system. The function

$$\psi_{\mathbf{k}}^{(2)} = \left(\frac{\omega_{\mathbf{k}}}{\pi}\right)^{1/2} (1 - \omega_{\mathbf{k}}\, x_{\mathbf{k}}^2)\, e^{-\frac{1}{2}\omega_{\mathbf{k}} x_{\mathbf{k}}^2}$$

describes the oscillator of wave number $\mathbf{k}$ in the second excited state with momentum $\mathbf{P} = 0$ i.e. with two particles of opposite momenta. Let us suppose that the other oscillators are in ground state. To get the scattered state in C. M. system, we have to compose a wave packet trial form

$$\psi^{\text{scat}} = \Sigma\, a_{\mathbf{k}}\, \psi_{\mathbf{k}}^{(2)} \cdot \prod_{\mathbf{k}' \neq \mathbf{k}} \cdot \psi_{\mathbf{k}'}^{(0)}\,. \tag{34}$$

The scattering amplitude is represented by $a_{\mathbf{k}}$. The normalisation of the wave function (34) is assured by the requirement

$$\Sigma a_{\mathbf{k}}^*\, a_{\mathbf{k}} = 1\,. \tag{35}$$

We look for the optimal amplitude $a_{\mathbf{k}}$ by minimizing the expression $\langle \psi^{scat} | H | \psi^{scat} \rangle$ under the subsidiary condition (35). The condition of the minimum is the following

$$2\omega_{\mathbf{k}} a_{\mathbf{k}} + \frac{3}{2} \frac{\lambda_0^2}{\Omega} \cdot \frac{1}{\omega_{\mathbf{k}}} \cdot \Sigma' \frac{a_{\mathbf{k}'}}{\omega_{\mathbf{k}'}} = \alpha \, a_{\mathbf{k}} \, ,$$

up to the orders $0(\lambda_0^4/\mu^6\Omega^2)$. This equation is entirely similar to the equation obtained and discussed in [2], and we can give immediately the peculiarities of the scattering. In C. M. system there is only S-wave scattering. The scattering cross section turns out to be finite, if

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_0^2} + \frac{3\pi}{\Omega} \Sigma \frac{1}{\omega_{\mathbf{k}}^3} \tag{36}$$

is a finite expression. λ is the renormalised coupling constant. (This approximation corresponds to the summation of the bouble diagrams of Fig. 4.) Thus the whole theory is renormalisable in SCHIFF's approximation.

Fig. 4

§ 6. The renormalisability remains when Ω goes to infinity. In this case the separation (20) of the two potential wells of (16) grows in an unlimited way and the potential barrier (12) between them tends to infinity. In (21) and in (15) the anharmonicity ceases and one arrives at a simple harmonic oscillator, vibrating in the potential well $\xi = 0$, with the exact ground state eigenfunction

$$\psi_0^{(-)} = A \cdot e^{-\frac{1}{2}\mu\xi^2} \Pi e^{-\frac{1}{2}\omega_{\mathbf{k}} x_{\mathbf{k}}^2} \, . \tag{37}$$

The Ω-limit of the Hamiltonian describes particles with momentum $\mathbf{k}$, energy $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2}$ and rest mass μ

$$\lim_{\Omega \to \infty} H = \lim_{\Omega \to \infty} \Sigma \sqrt{\mu^2 + \mathbf{k}^2} \cdot N(\mathbf{k}) \quad \text{with} \quad N(\mathbf{k}) = 0, 1, \ldots \, . \tag{38}$$

This result is exact in the framework of the function class (11). This Hamiltonian can be represented and diagonalised in a separable Fock space built up above the ground state (37). In this Hilbert space no unitary operator describing the transformation (2) exists. The symmetry (2) of the classical theory is broken because all the states occurring in this Fock space are characterised by

$$\langle \psi \mid \varphi(x) \mid \psi \rangle > 0 \, . \tag{39}$$

To find an irreducible representation of lim H, $\mathbf{P}$ and U, one has to introduce two Fock spaces, one $\mathscr{H}_{(-)}$ built up above the ground state (37), the second one $\mathscr{H}_{(+)}$ built up above

$$\psi^{(+)} = A \cdot e^{-\frac{1}{2}\mu\eta^2} \Pi e^{-\frac{1}{2}\omega_{\mathbf{k}} x^2_{\mathbf{k}}},$$

where η was defined in (21). The two Hilbert spaces $\mathscr{H}_{(-)}$ and $\mathscr{H}_{(+)}$ have the property

$$U\mathscr{H}_{(-)} = \mathscr{H}_{(+)},$$
$$\langle \psi^{(\pm)} \mid \varphi(x) \mid \psi^{(\pm)} \rangle \gtrless 0, \quad \text{if} \quad \psi^{(\pm)} \in \mathscr{H}_{(\pm)}.$$

No physical quantity (built up from finite products of creation and destruction operators) can produce a transition between $\mathscr{H}_{(+)}$ and $\mathscr{H}_{(-)}$. Thus one may conclude by saying, that our scalar field has two ground states in the classical theory, but it possesses only one ground state in the Hilbert space of the physically connected state vectors. This quantum theoretical ground state does not show all the symmetries of the Hamiltonian.

The "topology" of the state vector space can be visualised in the following way. Let $g_\alpha(\mathbf{r})$ be a complete set of functions in the infinite space volume with the properties

$$\int g_\alpha(\mathbf{r}) g_\beta(\mathbf{r}) d^3\mathbf{r} = \delta_{\alpha\beta} \quad \text{and} \quad \int g_\alpha(\mathbf{r}) \quad d^3\mathbf{r} < \infty.$$

Introduce

$$\varphi(\mathbf{r}) = \frac{1}{\sqrt{2}} \cdot \sum_\alpha (a_\alpha + i a^+_\alpha) g_\alpha(\mathbf{r}) - \frac{\mu_0}{\lambda_0\sqrt{2}},$$

$$\pi(\mathbf{r}) = \frac{1}{\sqrt{2}} \cdot \sum_\alpha (a_\alpha - i a^+_\alpha) g_\alpha(\mathbf{r}),$$

where a^+_α and a_α are creation and destruction operators. The transformation (2) corresponds now to the substitution

$$a_\alpha \to b_\alpha \equiv -a_\alpha + c_\alpha, \quad c_\alpha = \frac{\mu_0}{\lambda_0} \cdot \int g_\alpha(\mathbf{r}) d^3\mathbf{r}.$$

This is formally a canonical transformation, with the formally unitary operator

$$U = e^{-\Sigma c_\alpha (a_\alpha - a^+_\alpha)} P \quad \text{with} \quad P a_\alpha P^{-1} = -a_\alpha.$$

Now let us investigate the representation of these operators. Let us build up a separable Fock space $\mathscr{H}_{(-)}$ above the vacuum state $\mid 0_- \rangle$ characterised by

$$a_\alpha \mid 0_- \rangle = 0 \qquad \text{(for every } \alpha).$$

The operators $\varphi(x)$, a_α, a_α^+ can be represented in $\mathscr{H}_{(-)}$, but U cannot because

$$\langle \psi^{(-)} \mid U \mid \psi'^{(-)} \rangle = 0 \quad \text{for} \quad \psi^{(-)},\ \psi'^{(-)} \in \mathscr{H}_{(-)} .$$

Let us define a second vacuum state $\mid 0_+ \rangle$ through

$$b_\alpha \mid 0_+ \rangle = 0 \qquad \text{(for every } \alpha) .$$

This state is *not* contained in $\mathscr{H}_{(-)}$. The separable Hilbert space $\mathscr{H}_{(+)}$ built up above $\mid 0_+ \rangle$ with the help of the b_α-s offers also a representation of the operators $\varphi(x)$, a_α, a_α^+, but the two representations are inequivalent. Both separable spaces $\mathscr{H}_{(+)}$ and $\mathscr{H}_{(-)}$ are subspaces of the nonseparable $\tilde{\mathscr{H}}$ obtained through a nonlimited application of the creation operators on either of the two vacuum states, and U can be represented only in $\tilde{\mathscr{H}}$ with the property

$$U\mathscr{H}_{(\pm)} = \mathscr{H}_{(\mp)} .$$

The occurrence of the broken symmetry is very similar also in the first GOLDSTONE model with hermitian $\varphi(x)$, discussed above: $\mathscr{H}_{(-)}$ and $\mathscr{H}_{(+)}$ offer two inequivalent irreducible representations of the Hamiltonian, but the symmetry (2) is lost in any of them. The SCHIFF representation is very convenient to visualise the breaking of the symmetry: the main effect of the transformation (2) is concentrated on the variable x_0. For $\Omega \to \infty$ the state $U\psi^{(-)}$ contains an unlimited number of particles, concentrated on the field oscillator with wave number $\mathbf{k} = 0$, thus it lays outside the Hilbert space $\mathscr{H}_{(-)}$. Our calculations are very similar to those of UMEZAWA [4], obtained recently with a different method. This indicates that our approximation is of similar accuracy as that of UMEZAWA and the BCS approximation for fermion fields.

§ 7. Let us use SCHIFF's method in the second GOLDSTONE model in which the potential operator $\varphi(x)$ is not Hermitian. The Hamiltonian

$$H = \int \left[\frac{1}{2} \pi^+ \pi + \frac{1}{2} \nabla\varphi^+ \nabla\varphi - \frac{1}{4} \mu_0^2 \varphi^+ \varphi + \frac{1}{4} \lambda_0^2 (\varphi^+ \varphi)^2 \right] d^3\,\mathbf{r} \tag{40}$$

possesses a continuous symmetry

$$\varphi \to e^{ic}\varphi , \quad \pi \to e^{-ic}\pi . \tag{41}$$

The classical ground state is given by the homogeneous solution

$$\varphi = \frac{\mu_0}{\lambda_0 \sqrt{2}} e^{i\alpha}$$

for any real a, and is not invariant with respect to the transformation (41). Let us see the behaviour of this broken symmetry in the quantum theory. The SCHIFF representation will be used again, with the real variables x_0, y_0, $x_\mathbf{k}$, $y_\mathbf{k}$, $\vartheta_\mathbf{k}$, $\Theta_\mathbf{k}$. Let us expand the potential operator $\varphi(x)$ (supposed to be diagonal) into a Fourier series in the finite normalisation volume Ω

$$\varphi = \frac{1}{\sqrt{\Omega}}\left[\frac{1}{\sqrt{2}}\cdot(x_0 + iy_0) + \Sigma\left[x_\mathbf{k}\cos(\mathbf{kr} + \vartheta_\mathbf{k}) + iy_\mathbf{k}\sin(\mathbf{kr} + \Theta_\mathbf{k})\right]\right]. \quad (42)$$

With the help of this field energy H and field momentum $\mathbf{P}$ can be represented by differential operators. To evaluate their approximative values, the trial function

$$\psi = \psi_0(x_0, y_0)\,\Pi\psi_\mathbf{k}(x_\mathbf{k}, y_\mathbf{k}, \vartheta_\mathbf{k}, \Theta_\mathbf{k}) \quad (43)$$

is introduced. With the help of (43) let us form $\langle\psi \mid H \mid \psi\rangle$.

$$\begin{aligned}
\langle\psi|H|\psi\rangle = \Big\langle\psi_0 - \Big| & \frac{1}{2}\frac{\partial^2}{\partial x_0^2} - \frac{1}{2}\frac{\partial^2}{\partial y_0^2} - \frac{1}{4}\mu_0^2(x_0^2 + y_0^2) + \\
& + \frac{\lambda_0^2}{4\Omega}(x_0^4 + y_0^4)\Big|\psi_0\Big\rangle + \Sigma\Big\langle\psi_\mathbf{k}\Big| - \frac{1}{2}\left[\frac{1}{x_\mathbf{k}}\cdot\frac{\partial}{\partial x_\mathbf{k}}\left(x_\mathbf{k}\cdot\frac{\partial}{\partial x_\mathbf{k}}\right) + \right. \\
& \left. + \frac{1}{x_\mathbf{k}^2}\frac{\partial^2}{\partial\vartheta_\mathbf{k}^2} + \frac{1}{y_\mathbf{k}}\frac{\partial}{\partial y}\left(y_\mathbf{K}\cdot\frac{\partial}{\partial y_\mathbf{k}}\right) + \frac{1}{y_\mathbf{k}^2}\frac{\partial^2}{\partial\Theta_\mathbf{k}^2}\right] + \\
& + \left(-\frac{1}{4}\mu_0^2 + \frac{1}{2}\mathbf{k}^2\right)(x_\mathbf{k}^2 + y_\mathbf{k}^2) + \frac{3}{8}\frac{\lambda_0^2}{\Omega}(x_\mathbf{k}^4 + y_\mathbf{k}^4)\Big|\psi_\mathbf{k}\Big\rangle + \\
& + \frac{3}{4}\frac{\lambda_0^2}{\Omega}\sum_{\mathbf{k}'\neq\mathbf{k}}{}'\left[\langle\psi_\mathbf{k}|x_\mathbf{k}^2|\psi_\mathbf{k}\rangle\langle\psi_{\mathbf{k}'}|x_{\mathbf{k}'}^2|\psi_{\mathbf{k}'}\rangle + \langle\psi_\mathbf{k}|y_\mathbf{k}^2|\psi_\mathbf{k}\rangle\langle\psi_{\mathbf{k}'}|\psi_{\mathbf{k}'}^2|\psi_{\mathbf{k}'}\rangle\right] + \\
& + \frac{3}{2}\frac{\lambda_0^2}{\Omega}\left[\langle\psi_0|x_0^2|\psi_0\rangle\cdot\Sigma\langle\psi_\mathbf{k}|x_\mathbf{k}^2|\psi_\mathbf{k}\rangle + \langle\psi_0|y_0^2|\psi_0\rangle\Sigma\langle\psi_\mathbf{k}|y_\mathbf{k}^2|\psi_\mathbf{k}\rangle\right] + \\
& + \frac{1}{2}\frac{\lambda_0^2}{\Omega}\cdot\Sigma\langle\psi_\mathbf{k}|x_\mathbf{k}^2\cdot y_\mathbf{k}^2|\psi_\mathbf{k}\rangle + \frac{1}{2}\frac{\lambda_0^2}{\Omega}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum\langle\psi_\mathbf{k}|x_\mathbf{k}^2|\psi_\mathbf{k}\rangle\langle\psi_{\mathbf{k}'}|y_{\mathbf{k}'}^2|\psi_{\mathbf{k}'}\rangle + \\
& + \frac{1}{2}\frac{\lambda_0^2}{\Omega}\langle\psi_0|x_0^2\cdot y_0^2|\psi_0\rangle + \\
& + \frac{1}{2}\frac{\lambda_0^2}{\Omega}\Sigma\left[\langle\psi_0|x_0^2|\psi_0\rangle\langle\psi_\mathbf{k}|y_\mathbf{k}^2|\psi_\mathbf{k}\rangle + \langle\psi_0|y_0^2|\psi_0\rangle\langle\psi_\mathbf{k}|x_\mathbf{k}^2|\psi_\mathbf{k}\rangle\right]. \quad (44)
\end{aligned}$$

The condition for the minimum of $\langle H \rangle$ is given by the following differential equations:

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial y_0^2}\right)+V_0(x_0,\,y_0)-E_0\right]\psi_0(x_0,\,y_0)=0\,, \tag{45}$$

$$\left\{-\frac{1}{2}\left[\frac{1}{x_{\mathbf{k}}}\cdot\frac{\partial}{\partial x_{\mathbf{k}}}\left(x_{\mathbf{k}}\cdot\frac{\partial}{\partial x_{\mathbf{k}}}\right)+\frac{1}{x_{\mathbf{k}}^2}\cdot\frac{\partial^2}{\partial\vartheta_{\mathbf{k}}^2}+\frac{1}{y_{\mathbf{k}}}\cdot\frac{\partial}{\partial y_{\mathbf{k}}}\left(y_{\mathbf{k}}\cdot\frac{\partial}{\partial y_{\mathbf{k}}}\right)+\right.\right.$$

$$\left.\left.+\frac{1}{y_{\mathbf{k}}^2}\cdot\frac{\partial^2}{\partial\Theta_{\mathbf{k}}^2}\right]+V_{\mathbf{k}}(x_{\mathbf{k}},\,y_{\mathbf{k}})\;-\;E_{\mathbf{k}}\right\}\cdot\psi_{\mathbf{k}}(x_{\mathbf{k}},\,y_{\mathbf{k}},\,\vartheta_{\mathbf{k}},\,\Theta_{\mathbf{k}})=0\,. \tag{46}$$

Here E_0, $E_{\mathbf{k}}$ are the Lagrange multiplicators, coming from the normalisation conditions, and

$$V_0(x_0,y_0)=\left[-\frac{1}{4}\,\mu_0^2+\frac{\lambda_0^2}{2\Omega}\,\Sigma\,\left(3\,\langle\psi_{\mathbf{k}}\,|x_{\mathbf{k}}^2|\,\psi_{\mathbf{k}}\rangle+\langle\psi_{\mathbf{k}}\,|y_{\mathbf{k}}^2|\,\psi_{\mathbf{k}}\rangle\right)\right]x_0^2+$$

$$+\left[-\frac{1}{4}\,\mu_0^2+\frac{\lambda_0^2}{2\Omega}\cdot\Sigma\left(\langle\psi_{\mathbf{k}}\,|x_{\mathbf{k}}^2|\,\psi_{\mathbf{k}}\rangle\;+\;3\,\langle\psi_{\mathbf{k}}\,|y_{\mathbf{k}}^2|\,\psi_{\mathbf{k}}\rangle\right]y_0^2+ \tag{47}$$

$$+\frac{\lambda_0^2}{4\Omega}\,(x_0^2+y_0^2)^2\,,$$

$$V_{\mathbf{k}}(x_{\mathbf{k}},y_{\mathbf{k}})=\frac{1}{2}\left(-\frac{1}{2}\,\mu_0^2+\mathbf{k}^2\right)(x_{\mathbf{k}}^2+y_{\mathbf{k}}^2)+\frac{3}{8}\,\frac{\lambda_0^2}{\Omega}\left(\;x_{\mathbf{k}}^4+y_{\mathbf{k}}^4+\frac{4}{3}\,x_{\mathbf{k}}^2\,y_{\mathbf{k}}^2\right)+$$

$$+\frac{1}{2}\,\frac{\lambda_0^2}{\Omega}\sum_{\substack{\mathbf{k}'\neq\mathbf{k}\\ \mathbf{k}'=0}}\left[\langle\psi_{\mathbf{k}'}\,|3x_{\mathbf{k}'}^2+y_{\mathbf{k}'}^2|\,\psi_{\mathbf{k}'}\rangle\,x_{\mathbf{k}}^2+\langle\psi_{\mathbf{k}'}\,|x_{\mathbf{k}'}^2+3y_{\mathbf{k}'}^2|\,\psi_{\mathbf{k}'}\rangle\,y_{\mathbf{k}}^2\right]. \tag{48}$$

The solution of these equations goes along the lines of § 3, using perturbation theory with respect to $\lambda_0^2\,\mu^{-3}\Omega^{-1}$, making use of the fact that finally $\Omega\to\infty$. The actual work is made, however, more complicated because of the more numerous degrees of freedom of the field oscillators. (The "potential energies" are defined in two and in four dimensional spaces !)

§ 8. Let us try to find the ground state solution by supposing that

$$\langle\psi_{\mathbf{k}}\,|x_{\mathbf{k}}^2|\,\psi_{\mathbf{k}}\rangle=\langle\psi_{\mathbf{k}}\,|y_{\mathbf{k}}^2|\,\psi_{\mathbf{k}}\rangle\,. \tag{49}$$

In this case let μ^2 be

$$\mu^2={}^1/_4\,\mu_0^2-\frac{\lambda_0^2}{\Omega}\,\Sigma\langle\psi_{\mathbf{k}}\,|x_{\mathbf{k}}^2+y_{\mathbf{k}}^2|\,\psi_{\mathbf{k}}\rangle\,, \tag{50}$$

thus (47) takes the following simple form:

$$V_0(x_0, y_0) = -\mu^2 \cdot \varrho^2 + \frac{\lambda_0^2}{4\Omega} \varrho^4 , \tag{51}$$

where

$$\varrho^2 = x_0^2 + y_0^2 . \tag{52}$$

If $\mu^2 < 0$ the minimum of (51) is at $\varrho = a = \mu \lambda_0^{-1} \sqrt{2\Omega}$. Introducing the coordinates

$$r = \varrho - a , \quad \alpha = \text{arc tg}\, y_0/x_0 ,$$

equ. (45) is satisfied by the following expression

$$\psi_0(x_0, y_0) = R(r) \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{im\alpha} \quad (m = 0, \pm 1, \pm 2 \ldots)$$

$$\left[-\frac{1}{2} \frac{d^2 R}{dr^2} - \frac{1}{2(a+r)} \cdot \frac{dR}{dr} + \left(\frac{m^2}{2(a+r)^2} + 2\mu^2 r^2 + 2 \frac{\mu \cdot \lambda_0}{\sqrt{2\Omega}} r^3 + \right. \right.$$

$$\left. \left. + \frac{\lambda_0^2}{4\Omega} r^4 \right) R \right] = \left[E_0 + \frac{\Omega \mu^4}{\lambda_0^2} \right] R .$$

In the limit $\Omega \to \infty$ this will be an equation describing a harmonic vibration:

$$\left[-\frac{1}{2} \frac{d^2 R}{dr^2} + 2\mu^2 r^2 R \right] = \lim_{\Omega \to \infty} \left[E_0 + \frac{\Omega \mu^4}{\lambda_0^2} \right] R \cdot$$

Thus we see that

$$\psi_0 \sim e^{-\mu r^2} \cdot e^{im\alpha} \tag{53}$$

up to terms of the order $\Omega^{-1/2}$, Ω^{-1} and

$$E_0 = \lim_{\Omega \to \infty} \left(-\frac{\Omega \mu^4}{\lambda_0^2} \right) + \mu + \varepsilon(m) ,$$

where $\varepsilon(m)$ depends on the angular quantum number m and

$$\lim_{\Omega = \infty} \varepsilon(m) = 0.$$

The ground state has many discrete levels for finite Ω, characterised by the quantum number m, but the level spacing disappears in the Ω-limit. For $\Omega = \infty$ the ground state turns out to be completely degenerate, described by the eigenfunctions (53), or by any superposition of them. The last state-

ment would only be strictly acceptable if the eigenfunctions obtained were exact ones. In this case a wave packet concentrated in the neighbourhood of a given angle α would be again an eigenfunction. The reason for this is clear: (51) is a circle-like chanel. With radius a, with channel width μ and with length $2\pi a$ (53) describes standing waves in this channel, and any wave packet is a superposition of them. In the Ω-limit $a \to \infty$ and the channel will be infinitely long in one direction. The field oscillators with wave number $\mathbf{k}$ will also be simplified if (49) is valid. Making use of

$$\langle \psi_0 | x_0^2 | \psi_0 \rangle = \langle \psi_0 | y_0^2 | \psi_0 \rangle = \frac{3}{8\mu} + \frac{1}{2} a^2 ,$$

we arrive at

$$V_{\mathbf{k}}(x_{\mathbf{k}}, y_{\mathbf{k}}) = \frac{1}{2}\left[\mathbf{k}^2 + \mu^2 + \frac{\lambda_0^2}{\Omega}\left(\frac{3}{2\mu} - 2\langle \psi_{\mathbf{k}} | x_{\mathbf{k}}^2 + y_{\mathbf{k}}^2 | \psi_{\mathbf{k}} \rangle\right)\right](x_{\mathbf{k}}^2 + y_{\mathbf{k}}^2) +$$

$$+ \frac{3}{8}\frac{\lambda_0^2}{\Omega}\left(x_{\mathbf{k}}^4 + y_{\mathbf{k}}^4 + \frac{4}{3} x_{\mathbf{k}}^2 \cdot y_{\mathbf{k}}^2\right),$$

which is a simple four dimensional anharmonic oscillator with center at $x_{\mathbf{k}} = y_{\mathbf{k}} = 0$, and its anharmonicity disappears for $\Omega \to \infty$. Its frequency turns out to be

$$\omega_{\mathbf{K}} = \sqrt{\mu^2 + \mathbf{k}^2}, \tag{54}$$

isotropically with μ defined in (50). The eigenfunction is

$$\psi_{\mathbf{K}} \sim e^{-\frac{1}{2}\omega_{\mathbf{k}}(x_{\mathbf{k}}^2 + y_{\mathbf{k}}^2)} \tag{55}$$

up to terms of the order Ω^{-1}.

The ground state energy can be calculated by substituting (53) and (55) into (44). The mass renormalisation can be performed by (50), postulating μ to be finite. The continuum spectrum begins with the energy value μ above the ground state energy. It is worth to mention that in our approximation there is no "direction" in which the "curvature" (the second derivative) of the potential energy would be $|\mathbf{k}|^2$, characteristic for bosons with vanishing rest mass. The calculation quoted above is, however, valid only in the case of "isotropic excitations" given by (49).

§ 9. Let us consider the possibility of an "anisotropic" ground state. We define

$$\mu_1^2 = \mu_0^2 - 2\frac{\lambda_0^2}{\Omega} \Sigma \langle \psi_{\mathbf{k}} | 3x_{\mathbf{k}} + y_{\mathbf{k}}^2 | \psi_{\mathbf{k}} \rangle ,$$

$$\mu_2^2 = \mu_0^2 - 2\frac{\lambda_0^2}{\Omega} \Sigma \langle \psi_{\mathbf{k}} | x_{\mathbf{k}}^2 + 3y_{\mathbf{k}}^2 | \psi_{\mathbf{k}} \rangle \tag{56}$$

and we suppose

$$\mu_1^2 > \mu_2^2 > 0\,. \tag{57}$$

In this case

$$V_0(x, y_0) = -\frac{1}{4}(\mu_1^2 x_0^2 + \mu_2^2 y_0^2) + \frac{\lambda_0^2}{4\Omega}(x_0^2 + y_0^2)^2\,.$$

This has a maximum at $x_0 = y_0 = 0$, two saddle points at $x_0 = 0$, $y_0 = \pm\frac{\mu^2}{\lambda_0}\sqrt{\frac{\Omega}{2}}$ and two real minima at $x_0 = \pm\frac{\mu_1}{\lambda_0}\sqrt{\frac{\Omega}{2}}$, $y_0 = 0$. The "curvature" (second derivative) in the minimum is in the y_0 direction $m^2 = \mu_1^2 - \mu_2^2$, in the x_0 direction $\mu_2^2 = M^2$. Visually the channel, representing the "potential energy" of the $\mathbf{k} = 0$ oscillator is deformed: its deepest line is elliptical and it is not a *flat* curve, but it runs on a *bended* surface.

Corresponding to the two minima, the field oscillator $\mathbf{k} = 0$ will have two ground states:

$$\psi_0^0(x_0, y_0) = \frac{mM}{\pi^2}^{1/4} \cdot e^{-\frac{1}{2}M\left(x_0 \pm \frac{\mu_1}{\lambda_0}\sqrt{\frac{\Omega}{2}}\right)^2 - \frac{1}{2}my^2}\,,$$

thus

$$\langle\psi_0^0|x_0|\psi_0^0\rangle = \pm\frac{M^2}{\lambda_0^2}\sqrt{\frac{\Omega}{2}}, \quad \langle\psi_0^0|y_0|\psi_0^0\rangle = 0\,,$$

$$\langle\psi_0^0|x_0^2|\psi_0^0\rangle = \frac{M^2}{\lambda_0^2}\frac{\Omega}{2} + \frac{1}{2M}, \quad \langle\psi_0^0|y_0^2|\psi_0^0\rangle = \frac{1}{2m}\,.$$

The potential energy (48) of the other oscillators have the form:

$$V_{\mathbf{k}}(x_{\mathbf{k}}, y_{\mathbf{k}}) = \frac{1}{2}\left[\mathbf{k}^2 + M^2 + \frac{\lambda_0^2}{\Omega}\left(\frac{3}{2M} + \frac{1}{2m} - \langle\psi_{\mathbf{k}}|3x_{\mathbf{k}}^2 + y_{\mathbf{k}}^2|\psi_{\mathbf{k}}\rangle\right)\right]x_{\mathbf{k}}^2 +$$

$$+ \frac{1}{2}\left[\mathbf{k}^2 + m^2 + \frac{\lambda_0^2}{\Omega}\left(\frac{1}{2M} - \frac{3}{2m} - \langle\psi_{\mathbf{k}}|x_{\mathbf{k}}^2 + 3y_{\mathbf{k}}|\psi_{\mathbf{k}}\rangle\right)\right]y_{\mathbf{k}}^2 +$$

$$+ \frac{3}{8}\frac{\lambda_0^2}{\Omega}\left(x_{\mathbf{k}}^4 + y_{\mathbf{k}} + \frac{4}{3}x_{\mathbf{k}}^2 \cdot y\right).$$

The corresponding eigenfunctions in the ground state are

$$\psi_{\mathbf{k}}^{(0)} = \left(\frac{\omega_{\mathbf{k}}\Omega_{\mathbf{k}}}{\pi^2}\right)^{1/2} \cdot e^{-\frac{1}{2}(\Omega_{\mathbf{k}}x_{\mathbf{k}}^2 + \omega_{\mathbf{k}}y_{\mathbf{k}}^2)}$$

up to terms of order Ω^{-1}, with

$$\Omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M^2}\,, \qquad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}\,.$$

Let us resubstitute these into the mass equations (56). We get in the Ω-limit

$$M^2 = \mu_0^2 - \frac{\lambda_0^2}{8\pi^3}\int\left[3(M^2 + \mathbf{k}^2)^{-1/2} + (m^2 + \mathbf{k}^2)^{-1/2}\right] d^3\mathbf{k}\,,$$

$$m^2 = \frac{\lambda_0^2}{8\pi^3}\int\left[(m^2 + k^2)^{-1/2} - (M^2 + \mathbf{k}^2)^{-1/2}\right] d^3\mathbf{k}\,.$$

This is a system of equations which has two nonvanishing finite solutions m and M, if μ_0^2 is a quadratically divergent constant, and if λ_0 is renormalised by a logarithmically divergent factor (making the scattering cross sections finite at the same time).

Now the field energy (44) can be evaluated also for the "anisotropic ground state". We may compare it with the energy of the "isotropic ground state" obtained in § 8. For suitable values of the renormalised coupling constant λ the anisotropic ground state will have lower energy than the isotropic ground state. In this case the field will have two types of quanta: those with rest mass M and those with rest mass m. The values of the two masses M and m and their coupling constant λ are not independent of each other, their connection is the following: $M^2 = (1 + \lambda^{-2})m^2$.

The difference in the masses of the two vibrations are produced by the coupling of the field oscillators through nonlinearity. The formation of the asymmetric ground state vibrations, the possibility of the spontaneous concentration of the energy in one "polarisation" state is the cause of the breakdown of the gauge invariance (41). The Hamiltonian (40) is gauge invariant, so we can define a "charge" operator which is conserved:

$$Q = \int (\pi \cdot \varphi - \varphi^+ \pi^+) d^3\mathbf{r}\,,$$

but the vacuum is no eigenstate of this Q.

It should be mentioned that in our approximation no bosons with vanishing rest mass were found. To find the essence of the theorem of GOLDSTONE — if it is valid at all — we have to go deeper in the case of the GOLDSTONE model than it was supposed earlier.

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НАРУШЕНИЕ СИММЕТРИЙ В ДВУХ МОДЕЛЯХ ГОЛДСТОУНА

Д. КУТИ и Г. МАРКС

Р е з ю м е

Исследованы энергетические спектры двух моделей Голдстоуна с помощью вариационного метода Рица. Получено больше чем одно квантовомеханическое основное состояние, но эти состояния принадлежат к разным гильбертовым пространствам. Проблема собственных значений может быть решена, используя только одно из гильбертовых пространств, но в этом случае основное состояние не обладает всеми симметриями гамильтониана.

RELATIVISTIC CALCULATIONS OF THE GREEN'S FUNCTIONS BY MEANS OF THE CONTINUOUS INTEGRATION IN THE CASE OF STRONG COUPLING

(APPLICATION TO THE GOLDSTONE MODEL)*

By

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A strong coupling theory is studied for self coupled meson field with vanishing bare mass by making use of continuous integration. Particularly, the one-particle propagator is discussed. Some remarks are made on the GOLDSTONE model.

The aim of our work is to develop a method for the calculation of Green's functions from a given Lagrangean in a Lorentz-invariant manner and in the case of strong interactions. For renormalizable cases the result should also be renormalizable. So it should be a true alternative to perturbation theory, if coupling is strong. It seems to us very important to have such a method; it would be very useful for giving an answer to many problems of quantum field theory and the theory of elementary particles.

We develop our method here on the simple model with the Lagrangean

$$L = \frac{1}{2}\varphi \Box \varphi - \lambda\varphi^4 .$$

The next simple generalization is

$$L = \frac{1}{2}\varphi(\Box - m^2)\varphi - \lambda\varphi^4$$

about which we shall also make some remarks. But the method can also be applied to some other cases.

2. As is well-known, the propagator $\Delta_F(x_1 - x_2)$ of the real scalar field φ can be written as a continuous integral in the form

$$\Delta_F(x_1 - x_2) = \frac{\int \varphi(x_1)\varphi(x_2)\exp\left(i\int L dx\right) D(\varphi)}{\int \exp\left(i\int L dx\right) D(\varphi)} = \frac{T_2}{T_0} .$$

Here we wish to mention first that there is no difficulty with integration-measure $D(\varphi)$ in function space, as this quantity appears in the nominator and

* The work, which is the subject of this lecture, was done in collaboration with Dr. KAISER and Dr. KÜHNEL at Dubna.

the denominator of Δ_F as well. — If one expands T_2 and T_0 in powers of $i \int L_W dx = -i\lambda \int \varphi^4 dx$ one gets the usual perturbation series for Δ_F. We however expand in powers of $i\int L_0 dx = \frac{i}{2}\int \varphi \,\Box\, \varphi dx$. This gives for T_2 after some manipulations the series

$$T_2(x_1, x_2) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{2}{\sqrt{\lambda}}\right)^n \int \ldots \int G(x_3, x_4) \ldots$$

$$\ldots G(x_{2n+1}, x_{2n+2})\, \varphi_{n+1}(x_1 \ldots x_{2n+2})\, dx_3 \ldots dx_{2m+2}{}^*,$$

with

$$\varphi_n = \int \bar{\varphi}(x_1) \ldots \bar{\varphi}(x_{2n}) \exp\left(-i \int \bar{\varphi}^4 dx\right) D(\varphi),$$

$$G(x_i, x_j) = \delta(x_i - x_j) \frac{\Box}{2} \qquad \text{and} \qquad \bar{\varphi} = \sqrt[4]{\lambda}\, \varphi .$$

For T_0 one gets a similar expression.

3. How to calculate the functional integrals φ_m? A possible method is to do it in lattice-space, i.e. write

$$\varphi_n = \lim \prod_l \int_{-\infty}^{+\infty} \ldots \int \bar{\varphi}(x_1) \ldots \bar{\varphi}(x_{2n}) \exp\left(i\varepsilon\, \bar{\varphi}^4(x_e)\right) d\bar{\varphi}(x_e) .$$

Here ε is the volume of the primitive cell in lattice-space and the lim means going back to continuum. We see that the functional integral is represented as an infinite product of simple integrals of the form:

$$\int_{-\infty}^{+\infty} y^{2k} e^{-i\varepsilon y^4} dy = M_k \cdot M_0 ;$$

$$M_0 = \int_{-\infty}^{\infty} e^{-i\varepsilon y^4} dy .$$

The M_k are essentially Γ-functions.** Of course, all the products φ_n are divergent, so we have for Δ_F essentially an expression of the form $\frac{\infty}{\infty}$, (as was, in a similar context, noted by SYMANZIK). But we see that an infinite number of M_0 cancel, because of their appearance in T_2 and in T_0. For finite n there remains only a finite number of M_k. Going over to the limit of the continuum, the φ_n-functions come out as certain products of DIRAC's Delta-

* This expansion was by CAIANIELLO et al. in their preprint of June 1964 denoted as a good one for this model.

** The integrals M_k are only conditionally convergent; to give them a good meaning, a convergence factor is needed, as usual.

functions. This is not very nice, because we have to differentiate them (according to our formula for T_2) several times. It is, however, also good because the following integrations are to be performed simply for such functions. — In any case, we were cautious and treated the φ_n's first as non-differentiable functions, going back to lattice-space and replacing the differential-operators $\Box$ by difference-operators and the integrals by sums. It turned out that one gets the same results by treating the δ-functions as usual functions interpreting, however, expressions of the form $\delta(0)$, $\Box\ \delta(0)$ and so on by going back to lattice space. Performing the necessary differentiations and integrations, one gets for T_2 an expression of the form

$$T_2(x_1 - x_2) = \sum_{k=0}^{\infty} a_k \Box^k \delta(x_1 - x_2),$$

or after a Fourier transformation

$$T_2(p) = \sum_{k=0}^{\infty} a_k P^{2k}.$$

Here the a'_ks are infinite series in decreasing powers of $\lambda^{1/2}$. It is important to note that the series for T_2 must not be cut off for finite k: This would give no propagation at all. But we assume that the series representing each a_k can be cut off, if λ is large enough. This will be our approximation. A closer inspection of the different contributions to the quantities T_2 and T_0 shows, that there are also terms proportional to the volume of the considered space-time region. One can show, that such contributions are simple vacuum-terms and they can be cancelled with corresponding terms from T_0. Renormalization can be done in the following manner: After omission of the vacuum-terms, the remaining contributions contain certain ε-powers, steming from the above-mentioned M_k and from expressions of the form $\delta(0)$, $\Box\ \delta(0)$ and so on. These ε-powers are collected. In our (renormalizable) example we find that each a_k contains ε in the form $a_k = \varepsilon^{k/2}\bar{a}_k$, $\bar{a}_k$ being independent of ε. So we are able to make the substitution $p^2\varepsilon^{1/2} \to \bar{p}^2$ and to introduce $\bar{p}_\mu$ as the new variable in momentum space. This has the meaning of a mass-renormalization. After this renormalization is performed, T_2 and T_0 do not contain ε any longer (or some other not-finite quantity). We wish to stress that the ε-dependence of T_2 and T_0 cannot be transformed off so simply in nonrenormalizable or super-renormalizable cases. The quantum-mechanical case, which can be treated in principle also by our method, belongs to the super-renormalizable cases. Therefore this method seems to be not very useful for quantum mechanics (as was also the case for SYMANZIK's considerations on similar problems).

4. We have represented the quantity $T_2(p)$ (which is essentially the fourier-transform of Δ_F) as a Taylor-series. This series, of course, has a radius of convergence equal to the distance between the point $p^2 = 0$ and the first

singularity of the function $T_2(p)$. The singularities of $T_2(p)$, however, have physical meaning. The poles of $T_2(p)$ give the masses of single-particle states; branching points and cuts are connected with more-particle states. So we can, from inspection of the convergence of the Taylor-series for T_2, find the smallest mass of the considered system. To determine the other singularities of $T_2(p)$, we have considered 4 possibilities:

a) Analytical completion, which is very difficult.

b) Subtraction of the first singularity from $T_2(p)$, which is not very useful, as we can determine the position and the residuum of the first singularity only approximately (for big λ).

c) Shift of the point about which $T_2(p)$ is expanded. That is, writing $T_2(p) = \Sigma a_k(m)\,(p^2 - m^2)^k$. This can be done simply by writing $L = \frac{1}{2}\,\varphi \cdot (\Box - m^2)\varphi - \lambda\varphi^4 + \frac{1}{2}\,m^2\varphi^2$, considering $L_0 = 1/2\varphi(\Box - m^2)\varphi$ and the rest of L as L_W. Then, however, the integrals M_k are much more difficult and the further calculations are more involved numerically. Therefore we have for the moment not tried to look more closely into this possibility; but it may be very well that this is the most effective method.

d) A closer inspection of the structure of the a_k shows the possibility of splitting from each a_k a certain part ϱ_k, which essentially contain splitting graphs. Then one can perform a formal summation of the Taylor-series: the result is an expression of the form:

$$T_2(p) = \frac{a_0 + \sum_{k=1}^{\infty} g_k p^{2k}}{1 - p^2\left(a_0 + \sum_{k=1}^{\infty} g_k p^{2k}\right)}\,.$$

So we have to look for the zeros of the quantity $1 - p^2(a_0 + \sum_{k=1}^{\infty} g_k p^{2k})$ in order to find the singularities of T_2. This would be very well, if one could sum up also the series $\sum_{1}^{\infty} g_k p^{2k}$ exactly. But this can be done only in certain approximations (for big λ).

5. In the last-mentioned manner we have determined approximate expressions for the location of the first two poles of T_2. They are

$$K_0^2 = \frac{i\sqrt{i}\,\sqrt{\lambda}}{0{,}338} + c_0 - c_1\,\lambda^{-1/2} + \ldots$$

$$K_1^2 = d_{-3}\,\lambda^{3/2} + d_{-2}\,\lambda + d_{-1}\,\lambda^{1/2} + d_0 + \ldots$$

with c_i, d_i numerical factors.

You see, that these poles in general do not lie on the real p^2-axis, but they lie anywhere in the complex p^2-plane. The lowest pole of a propagator, however, must lie on the positive-real p^2-axis, otherwise the propagator will not have the right physical behaviour for large $(x_1 - x_2)^2$. We have tried to pull K_0^2 to the real p^2-axis by determining that value of λ for which the imaginary part of K_0^2 vanishes. Using only the first 3 terms from the series for K_0^2, we get from this condition $\lambda \approx 0{,}01$. But for such small λ it is surely not enough to take only the first 3 terms of the K_0^2-series. It is even not sure, that our series in decreasing powers of $\sqrt{\lambda}$ hold for such small λ.

It seems to us plausible and we suppose that for example the series for a_0 is convergent for some $\lambda > \lambda_0$; divergent for $\lambda < \lambda_0$. That would mean that for $\lambda = \lambda_0$ there appears a pole at $p^2 = 0$. We hope to determine this critical λ_0 later. But what happens for $\lambda < \lambda_0$ really, we cannot say at the moment.

So our conclusion is that for large λ the considered model gives a propagator with unphysical properties. For small λ the propagator may give physically correct properties, but we are presently not able to tell anything definite about this region of smaller λ.* — We suppose, that this unphysical behaviour for large λ is connected with the breakdown of unitarity for large λ.**

6. Now we wish to make some remarks on the application of our method to the GOLDSTONE Model with the Lagrangian $L = \frac{1}{2}\varphi\Box\varphi - \mu^2\varphi^2 - \lambda\varphi^4$, where $\lambda > 0$ and $\mu^2 > 0$, so that this model has (as discussed by MARX et al.) degenerate vacuum-state.

The first question would be, if our method (more exactly: the representation of the Green's functions by continuous integrals) is valid in the case of degenerate vacuum-state. The author thinks that there is no difficulty because of translation invariance of all continuous integrals in function space; the new vacuum-states differ from the usual one by the property $\langle 0 \mid \varphi(x) \mid 0\rangle \neq 0$ whereas in conventional theory $\langle 0 \mid \varphi \mid 0\rangle = 0$. A rigorous proof of this fact is, however, lacking. Let us apply our method to this model! The new model differs from the one considered formerly by the term $\mu^2\varphi^2$ in L. We have two possibilities: Either treating this new term together with L_W or together with L_0. The last-mentioned possibility is, however, not a good one, because renormalization is done in such a way that all unrenormalized masses must go to infinity. Since μ plays the role of a mass (the opposite sign does not alter this situation), so also μ must go to infinity. Then it is, of course, not any longer justified to treat L_0 as a "Perturbation" (relativ zu L_W) if $\mu^2\varphi^2$ is included in L_0. — So we have to write $L_W = \mu^2\varphi^2 - \lambda\varphi^4$. The only difference is then that

* Perturbation theory is not applicable to this model, according to the results of CAIANIELLO et al.

** *Note added in proof:* Very recent calculations of Dr. KÜHNEL and the present author show that one can give a certain recipe of calculation such, that the lowest pole gets real.

the quantities M_k (resulting from the splitting of the continuous integrals φ_n in lattice-space) become more complicated. They contain now, besides the T-function, the functions of the parabolic cylinder D in the form

$$D - \frac{2n+1}{2}\left(\frac{\sqrt{i}\,\mu_{\text{ren}}^2}{2\sqrt{\lambda}}\right).$$

Therefore numerical results cannot be obtained as easily as in the case with $\mu^2 = 0$ and we have not yet such results. There exists the possibility that the situation is quite different if we go from $\mu^2 = 0$ to $\mu^2 \neq 0$. Especially our conclusion regarding the non-physical behaviour of the propagator for large λ must not necessarily be true in that case. But it seems to us that we can make some statements without detailed calculations:

a) It is very probable that this model also will contain not only one but different masses, and that there may be also some resonances or even ghost-poles.

b) We do not need in the course of our calculations any renormalization of the coupling constant λ. Such a renormalization was found necessary in the calculations of SCHIFF, MARX and KUTI and others, to obtain a finite 4-point function. We have also calculated the 4-point function by our method in the lowest two approximations* and did not find any necessity for λ-renormalization. Mass-renormalization is enough also for the 4-point function according to our calculations. Presently we do not know what is responsible for this discrepancy. It may be that it has to do with the divergency of the perturbation series in that case as mentioned by CAIANIELLO et al.

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РЕЛЯТИВИСТСКИЕ ВЫЧИСЛЕНИЯ ФУНКЦИЙ ГРИНА С ПОМОЩЬЮ НЕПРЕРЫВНОГО ИНТЕГРИРОВАНИЯ В СЛУЧАЕ СИЛЬНОЙ СВЯЗИ

(Применение к модели Голдстоуна)

Г. ХЕБЕР

Р е з ю м е

С помощью непрерывного интегрирования изучается теория сильной связи для мезонного поля, исчезающей незначительной массы, взаимодействующего с самим собой. В частности, обсуждается одночастный пропагатор. Сделаны несколько замечаний относительно модели Голдстоуна.

* These calculations will be described elsewhere.

APPROXIMATE SYMMETRIES IN FIELD THEORY

By

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Some ideas concerning the broken and approximate symmetries are discussed based on the simplest relativistic field theories satisfying the asymptotic condition.

Introduction

The aim of this paper is to discuss some ideas concerning the approximate symmetries occurring in nature. Our treatment here, when explicitely referred to, deals with one of the simplest exactly soluble and well known relativistic models, namely, the linear coupling of two fields with the interaction Lagrangian $\mathscr{L}_1 = g\varphi_1\varphi_2$. As for the notations and treatment we refer to DEO's work [1].

We were led to this choice by the exact solubility of these models including also the existence of asymptotic states which we feel to be highly essential and still it shows, essentially, the symmetry and approximate symmetry properties we wish to discuss.

The main point is the same as it was in [2]; the symmetry properties of the real fields or particles might be and are — at least in [2] and the models below — quite different from the symmetries of the fields of bare particles. Therefore, exact symmetries in bare fields might yield approximate symmetries of the same type in real fields an vice versa. We do not claim, of course, that the real world shows the same symmetry behaviour as the models treated here, but one might feel that these calculations give some insight into what really happens.

As from the point of technical details: We have learnt that in field theory, where the number of degrees of freedom is infinite, there exist unitarily inequivalent representations of the algebra of field variables. Starting from this, a series of examples have been constructed (see e.g. [3], all of them by means of some kind of approximate methods) in such a way that the Hamiltonian does, but the symmetry does not have a representation by hermitian and unitary operators.

We would like to mention here that even in these cases, other representations can also be constructed (see e.g. FANO's paper [4]) in which the symmetry survives. The same happens also in NAMBU's theory [3] where an infinity of physically equivalent but not intersecting worlds described by $\Omega_\alpha^{(m)}$, $0 \leq \alpha \leq 2\pi$ exists. These worlds (in which the masses of particles are functions of α) are not invariant under γ_5-gauge transformations. One can construct,

however, ground states which are eigenstates of the generator of the γ_5-gauge group, the chirality, considering suitable integrals of $\Omega_a^{(m)}$ over a.

At present, neither the role of the "non-symmetrical" representations in a future theory of elementary particles (see e.g. [5]) nor the physical properties of the extra "symmetrical" representations constructed by direct integrals is clear. In our opinion, when discussing the problem of the proper choice of the representations describing the physical world, the asymptotic condition plays a decisive role.

In what follows we show that

(i) formal substitutional invariances of several exactly soluble models remain symmetries also in the concrete representations, but the form of the exact symmetry expressed in terms of bare or real particles is quite different,

(ii) under appropriate circumstances depending on the dynamics, the transformation which led to a substitutional invariance in bare fields, produces an approximate symmetry, if — keeping its form — we apply it to the real particles. In our picture the existence of approximate symmetries are quite independent of the fact whether or not the substitutional invariances in bare fields can be represented by unitary operators over the physical state vector space.

The models

Let us take as a first example the case of two scalar boson fields of equal bare masses with the Lagrangian

$$L(\Phi_1, \Phi_2) = -\frac{1}{2}\sum_{i=1}^{2}\left[(\partial_\mu \Phi_i)^2 + m^2 \Phi_i^2\right] - g\Phi_1 \Phi_2 . \tag{1}$$

This Lagrangian is invariant under the transformation (P)

$$(P) \quad \varphi_1 \to \varphi_2 , \ \varphi_2 \to \varphi_1 . \tag{2}$$

(P) can be represented by a unitary operator on the bare particle Fock's space (charge conjugation).

As it is well-known and can easily be seen, this model can be brought to the form of the sum of two free field Lagrangeans

$$L(\psi_1, \psi_2) = -\frac{1}{2}\sum_{i=1}^{2}\left[(\partial_\mu \psi_i)^2 + M_i^2 \psi_i^2\right] \tag{3}$$

by means of the canonical transformations

$$\psi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2)\,,$$
$$\psi_2 = \frac{1}{\sqrt{2}}(\varphi_2 - \varphi_1)\,, \tag{4}$$

where in (3)

$$M_{\frac{1}{2}}^2 = m^2 \pm g\,. \tag{5}$$

In order to get real particles with positive mass square and no ghosts, we take g as real and $m^2 > |g|$.

As we know e.g. from WEIDLICH's paper [6], the transformation (4) is only canonical; it is described by a so-called pseudo-unitary operator.

The problem has been solved and the theory can be represented over the Fock space of the real particles corresponding to ψ_1 and ψ_2. (Asymptotic condition is stated in the next part.)

The transformation (P) expressed in terms of real particles is

$$(P) \quad \psi_1 \to \psi_1\,,\ \psi_2 \to -\psi_2\,, \tag{6}$$

which is obviously an exact symmetry and can be described by the well-known operator $\exp(-i\pi \cdot$ number of "2" particles) unitary on the real particle Fock space (parity). The vacuum state of the Fock space is clearly invariant under (6). What is essential, (P) is an exact symmetry of the model but possesses quite a different form in terms of bare and real fields. If we ask, led by the form of (P) in (2), whether or not the transformation (P') exchanging the real fields

$$(P') \quad \psi_1 \to \psi_2\,,\ \psi_2 \to \psi_1 \tag{7}$$

is a symmetry, we see that it is not, but for the case of a weak coupling, $|g| \ll m^2$, (P') is an approximate symmetry.

This example already shows what we wished to express, but in order to explain some other points, let us treat now the case of fermions. Starting with the Lagrangean

$$L(\varphi_1, \varphi_2) = -\sum_{i=1}^{2} \overline{\varphi}_i (\gamma_\mu \partial_\mu + m_i)\,\varphi_i + g(\overline{\varphi}_2 \varphi_1 + \overline{\varphi}_1 \varphi_2)\,, \tag{8}$$

by means of the transformations

$$\psi_1 = (\varphi_1 + \lambda\varphi_2)/(1 + \lambda^2)^{1/2},\ \psi_2 = (\varphi_2 - \lambda\varphi_1)/(1 + \lambda^2)^{1/2},$$

where

$$\lambda = (M_1 - m_1)g\,, \quad M_2^1 = \frac{1}{2}\left[m_1 + m_2 \pm((m_1 - m_2)^2 + 4g^2)^{1/2}\right],$$

(8) can be brought to the free form

$$L(\psi_1, \psi_2) = -\sum_{i=1}^{2} \overline{\psi}_i\,(\gamma_\mu \partial_\mu + M_i)\psi_i\,. \tag{9}$$

Now, we can treat several symmetries. If $m_1 = m_2 = m$, we find the same situation as before with (P). If $m_1 = m_2 = 0$, (8) is Touschek-invariant

$$\varphi_1' = e^{i\gamma_5\alpha}\,\varphi_1\,, \quad \varphi_2' = e^{-i\gamma_5\alpha}\,\varphi_2\,,$$
$$(T) \quad \varphi_i' = U\varphi_i\,U^{-1}\,, \tag{10}$$
$$U = \exp\,(i\alpha \int d^3x\,(\varphi_2^+\,\gamma_5\,\varphi_2 - \varphi_1^+\,\gamma_5\,\varphi_1))\,.$$

This transformation for ψ-s gives

$$(T) \quad \psi_1' = \psi_1 \cos\alpha + i\gamma_5\,\psi_2 \sin\alpha\,,$$
$$\psi_2' = i\gamma_5\,\psi_1 \sin\alpha + \psi_2 \cos\alpha\,, \tag{11}$$
$$U = \exp\,(-i\alpha \int d^3x(\psi_2^+\,\gamma_5\,\psi_1 + \psi_1^+\,\gamma_5\,\psi_2))\,,$$

which is indeed an exact symmetry. Thus, Touschek-invariance of the bare fields does not prevent us in obtaining a theory with massive particles. If we wish to get rid of negative masses (since in this case $M_2 = -g$), by introducing $\varphi_1 = \psi_1$ and $\varphi_2 = \gamma_5\psi_2$, (9) gives

$$L(\varphi_1, \varphi_2) = \sum_{i=1}^{2} \overline{\varphi}_i\,(\gamma_\mu \partial_\mu + g)\,\varphi_i\,. \tag{12}$$

(T) in terms of φ_i-s

$$\varphi_1' = \varphi_1 \cos\alpha + i\varphi_2 \sin\alpha\,,$$
$$(T) \qquad \varphi_2' = i\varphi_1 \sin\alpha + \varphi_2 \cos\alpha\,,$$
$$U = \exp\,(-i\alpha \int d^3x\,(\varphi_2^+\,\varphi_1 + \varphi_1^+\,\varphi_2))\,,$$

is quite different from the form of (10) and can be represented by a simple unitary operator over the Fock's space of the φ-particles. Again, if g is small

$$(T') \quad \varphi_j' = e^{i\gamma_5 a_j}\varphi_j, \quad j = 1, 2$$

are approximate symmetries.

As another example, from (9) we see that the theory is invariant also under the independent gauge transformations

$$(G) \qquad \begin{aligned} \psi_1' &= e^{i\alpha}\psi_1, \\ \psi_2' &= e^{i\beta}\psi_2. \end{aligned} \tag{13}$$

For the bare fields this transformation gives

$$(G) \qquad \begin{aligned} \varphi_1' &= [(e^{i\alpha} + \lambda^2 e^{i\beta})\varphi_1 + \lambda(e^{i\alpha} - e^{i\beta})\varphi_2]/(1 + \lambda^2), \\ \varphi_2' &= [\lambda(e^{i\alpha} - e^{i\beta})\varphi_1 + (e^{i\beta} + \lambda^2 e^{i\alpha})\varphi_2]/(1 + \lambda^2). \end{aligned} \tag{14}$$

These two forms are again quite different.

Asymptotic condition

Next we turn to discuss the reason why these types of breaking of the "forms of the symmetries" occur. Simply saying the reason is that real particles are certain combinations of bare ones (c.f. (4) or (8)). By making such combinations the symmetries change their form. If the relation

$$\psi_{\text{r al}} = \text{a single renormalization constant-times } \varphi \text{ bare} \tag{15}$$

were valid, such change would not occur. This, expressed in a much more accurate and modern language, means that both of the bare fields of the examples are simultaneously and separately interpolating fields of each real field. Indeed, an exact calculation gives e.g. for the case of the first scalar model

$$\begin{aligned} \lim_{t\to\pm\infty} \langle 0|\, \varphi_1(\alpha, M_{1,2}, t)\,|0\cdot p\cdot s\rangle &= \pm\frac{1}{\sqrt{2}}\langle 0|\, \psi_{1,2}(\alpha)\,|0\cdot p\cdot s\rangle, \\ \lim_{t\to\pm\infty} \langle 0|\, \varphi_2(\alpha, M_{1,2}, t|0\cdot p\cdot s\rangle &= \frac{1}{\sqrt{2}}\langle 0|\, \psi_{1,2}(\alpha)\,|0\cdot p\cdot s\rangle, \end{aligned} \tag{16}$$

where

$$\varphi_i(a, M_j, t) = i \int_t \varphi_i \overleftrightarrow{\partial} f_a(M_j)\, d^3x \tag{17}$$

and $f_a(M_j)$ is a normalized positive frequency solution of the free Klein-Gordon equation with the mass $M_j \cdot \psi_i(a)$-s are smeared out with functions belonging always to their own masses $M_i \cdot |0\rangle$ is the vacuum and $|$ o. p. s.$\rangle$-s are the different one-particle states of the representing Fock space of real particles. (The most important point in prooving (16) was that, when looking at the inverse of (4)

$$\varphi_1 = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2), \quad \varphi_2 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

and performing (17), $f_a(M_i)$ cancelled the contributions of the corresponding field with the other mass.)

By the way, we would like to mention that the above models present also the simplest illustrations to part 18. § 8 of UMEZAWA's textbook where the case with different stable mass states is discussed [8].

Conclusions

A possible mechanism for the explanation of the approximate symmetries were presented. In the above models everything is exact, even if it is looked at from the point of view of representations.

We are fully aware of the fact that these models (as it can also be seen from the asymptotic condition) do not give any scattering among real particles. Therefore, these models are not real. Nevertheless, we belive that if we want to handle approximate symmetries by introducing an exactly invariant bare Lagrangian, the program has to be understood in the sense of our examples. As they show, exact symmetries take quite different forms in terms of bare or real particles and exact symmetries in one of them being written in the same form expressed by the other variables, turn out to be approximate symmetries when the coupling is weak. The condition for this is that fields describing real particles should be formed from (linear or non-linear) combinations (functions or functionals) of one or more bare fields.

It is tempting to assume that in reality not always the simplest situation (15) occurs.

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ПРИБЛИЖЕННЫЕ СИММЕТРИИ В ТЕОРИИ ПОЛЯ

К. Л. НАДЬ, Т. НАДЬ и Д. ПОЧИК

Р е з ю м е

Обсуждаются некоторые идеи относительно нарушенных и приближенных симметрий в простейших релятивистских теориях поля, удовлетворяющих асимптотическому условию.

ON THE ABSTRACT APPROACH TO THE MANY-BODY PROBLEM WITH INFINITE VOLUME

By

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After discussing the classification of the possible representations of the canonical commutation and anticommutation relations, the problem of finding the suitable representation for a given Hamiltonian is investigated. The consequences of the Galilean invariance of the theory are considered.

In the many body problem, the idealization N (number of particles) $= s$, V (volume) $= \infty$ and $\varrho = \lim_{V\to\infty} \frac{N}{V} =$ constant is used, more or less explicitly, in a large number of works. The physically correct point of view is the following: the only systems which are observable are those with a finite number of particles; therefore the formalism must first be developed keeping N finite, and only at the end of the calculations, and for practical reasons, one is allowed to approximate the formulae performing the $V \to \infty$ limit. On the other hand, like many other idealizations of physics (reversible processes in thermodynamics, point charge in electromagnetic theory, or a plane wave in quantum mechanics) it is interesting to consider the limiting situation, i.e., the case corresponding to a strictly infinite system, as if it were observable.

There is some advantage in considering an infinite system:

1. Some nice simplification appears in the calculations when we take the $V \to \infty$ limit. For instance, as proved first by BOGOLIUBOV, the B. C. S. model is exactly soluble only in this limit.

2. With the exception of the case of a solid, the theory is invariant under translations and rotations.

3. The concepts of thermodynamics are easily discussed in this framework.

4. It is mathematically more rigorous to study the existence problem and the properties of the Hilbert space and of the representation of the operators corresponding to infinite volume, than to perform the V limit at some intermediate stage of the calculations without justifying this procedure.

It is well known, and it has been repeated many times at this Conference, that the representation of the algebra of the field operators becomes inequivalent to the Fock representation when the number of particles becomes infinite.

The first problem to be faced is the classification of the possible representations of the canonical commutation (or anticommutation) relations (C. C. R. or C. A. R.). A very convenient way of studying this classification is given by GELFAND's construction of a representation, which also provides a very good method for describing states of a Hilbert space representing an infinite system. This method was used for the first time by WIGHTMAN in field theory.

Suppose we have an algebra R with an adjoint operation. Let Q be an element of the algebra and Q^+ its adjoint, and let ψ_Q be a symmetric representation of the algebra (ψ_Q is therefore a linear operator in a Hilbert space H, and $\psi_{Q^+} = \psi_Q^+$).

A vector $\xi_0 \in H$ is said cyclic if the set $\psi_Q \xi_0$ is dense in H. The mathematical concept of "cyclic vector" is very similar to DIRAC's idea of "standard ket". The definition means that by applying to ξ_0 the operators of the algebra, we get essentially all the states of physical interest.

Let us now consider the functional over the algebra defined by:

$$f(Q) = \langle \xi_0, \psi_Q \xi_0 \rangle \,, \tag{1}$$

where ξ_0 is a cyclic vector of H.

Of course it is a positive functional, i.e.

$$f(Q^+Q) < 0 \,. \tag{2}$$

Now the GELFAND theorem states that the expectation values $\langle \xi_0, \psi_Q \xi_0 \rangle$ determine, to within equivalence, the vectors ψ_Q, ξ_0, and the operators ψ_{Q^+}.

In other words to any positive functional f over the algebra, there corresponds a Hilbert space H, a cyclic vector ξ_0 and a representation ψ_Q.

Let us consider for instance a Fermi system. The basic quantities of the algebra are the "smeared out" fields:

$$\psi(b) = \int \psi(x) f(x) dx \quad \psi^*(f) = \int \psi^+(x) f(x) dx \,, \tag{3}$$

where $f(x)$ is some kind of test function (x is a point of the three-dimensional space).

Due to the anticommutation relations it is easy to prove that $\psi(f)$, $\psi^*(f)$ are bounded operators.

Therefore if we make linear combinations of products of a finite number of $\psi(b)$, $\psi^*(b)$, we get well defined expressions. These "polynomials in the field operators" are the elements of an algebra, and we want to obtain a representation of this algebra or, more precisely, of the weak closure of this algebra.

Without insisting on these more technical points, it appears quite obvious that the following expressions that I will call "WIGHTMAN functions"

$$W(x_1 \ldots x_n; y_1 \ldots y_m) = \langle \xi_0 \mid \psi^*(x_1) \ldots \psi^*(x_n)\, \psi(y_1) \ldots \psi(y_m) \mid \xi_0 \rangle$$

determine, to within equivalence, the state ξ_0, the Hilbert space and the representation of the operators $\psi(f)$, $\psi^*(f)$. For the case of a BOSE system, the analogous operators are unbounded, and therefore equations like

$$[\psi(f), \psi^*(g)]_- = \int f(x)\, g(x)\, dx = (f, g) \tag{4}$$

are not mathematically correct.

In fact on the right we have an operator (a multiple of the identity) whose domain is the entire Hilbert space, while nothing is a priori known about the domain of the operator on the left hand side; it may even consist of the zero vector.

For this reason one uses the WEIL form of the C. C. R. by introducing the unitary operators (f, g are real):

$$U(f) = e^{i\varphi(f)} \quad V(g) = e^{i\pi(g)} \tag{5}$$

where

$$\varphi(f) = \frac{\psi(f) + \psi^*(f)}{\sqrt{2}}, \qquad \pi(g) = \frac{\psi(g) - \psi^*(g)}{\sqrt{2}\, i}, \tag{6}$$

WEIL form of the C. C. R.
$$\begin{cases} U(f)V(g) = V(g)U(f)e^{-i(f,g)}, \\ U(f)U(g) = U(f+g), \\ V(f)V(g) = V(f+g). \end{cases} \tag{7}$$

$U(f)$, $V(g)$ are bounded operators, and the C. C. R. are now correctly written down.

Similar to the previous case, one considers the algebra generated by $U(f)$, $V(g)$ (for all f and g); then a straightforward application of the GELFAND theorem gives the following result:

Necessary and sufficient conditions for a functional $E(f, g)$ to define a representation of the C. C. R. with a cyclic vector ξ_0 such that

$$E(f, g) = \langle \xi_0,\, U(f)V(g)\xi_0 \rangle \tag{8}$$

are:

$$\left.\begin{aligned} E(f, g)^* &= E(-f, -g)e^{-i(f\,g)}, \\ E(0, 0) &= 1, \\ \sum_{ij=1}^{n} c_i\, c_j^*\, E(f_i - f_j,\ g_i - g_j)\, e^{i(g_j, f_j) - i(g_i, f_i)} &\geq 0. \end{aligned}\right\} \tag{9}$$

The first two equations guarantee the unitarity of the operators $U(f)$, $V(g)$, the last equation guarantees that the functional over the algebra is positive.

Even a stronger result holds, i.e. the representation of the C. C. R. is determined simply by the functional defined by:

$$E(f) = E(f, 0) = \langle \xi_0 \, e^{i\varphi(f)} \xi_0 \rangle \, . \tag{10}$$

The following formula holds (ARAKI [1], LEW [2])

$$E(f) = \int e^{i(x, f)} d\mu_x \, , \tag{11}$$

where $\mu(x)$ is a positive measure on the infinite dimensional space S' of the linear functionals on the test functions. $\mu(x)$ must fulfil only one condition, the so called "quasi invariance". (If a subset of S' has zero measure, also the translated subset must have zero measure).

Therefore the problem of the classification of the C. C. R. can be reduced to the problem of the classification of these quasi-invariant measures. There are no manageable methods of evaluating the functional integration, and therefore I will not insist on this point. I will only point out that there is considerable work to be done in this area. In the case of the Fock representation, it is well known that $E(f) = e^{\frac{1}{4}\int f^2 dx}$ and the restriction to any finite dimensional space of the corresponding measure is the Gaussian measure.

Let us now go back to our infinite body problem. The main question is then: for a given Hamiltonian which is the suitable representation of the C. C. R. (or of the C. A. R.)?

I want to mention that in field theory the following result holds (see reference [1]):

Let φ be a neutral scalar field, π the conjugate momenta, and H a Hamiltonian invariant under time reversal which verifies the equation:

$$[H, \varphi(f)]_- = -i\pi(f) \tag{12}$$

(this equation can be written in a rigorous form using the $U(f)$, etc).

Under these hypotheses it can be proved that the functional $E(f)$ "almost" determines the Hamiltonian. (The term "almost" refers to a fine mathematical point about the domain of the operators).

Unfortunately, for a non-relativistic infinite BOSE or FERMI system the Hamiltonian does not verify eq. (12), and except for few and simple cases it is an open problem to find rigorous relations between the form of the Hamiltonian and the representations of the C. C. R. (or C. A. R.).

Let us show some examples of application of these concepts. First, we want to add some comment on the very beautiful discussion made by R. HAAG on the B. C. S. model, (see reference [3]) and the lecture of Prof. LOPUSZANSKI).

HAAG considers the usual reduced Hamiltonian of a FERMI gas with a nonlocal potential corresponding to interaction between pairs of total momentumzero. He looks for irreducible representations of the closed algebra R generated by the operators $\psi(f)$, $\psi^*(f)$ (let us forget the spin indices).These irreducible representations can be labelled by a continuous parameter α $(0 \leq \alpha \leq 2\pi)$. Let us recall briefly how this happens. Due to the gauge invariance of the first kind of the Hamiltonian, i.e. invariance under the transformation $\psi \to \psi e^{i\beta}$, the BOGOLIUBOV—VALATIN canonical transformation, which maps the vacuum of the Fock representation $|\, \psi_{0F}\rangle$ into the vacuum of the quasi particles $|\, \xi_{BCS}\rangle$ depens on a "phase" α. So we get the famous "infinite degeneracy of the vacuum", i.e. we get an infinite number of ground states $|\, \xi^{(\alpha)}_{BCS}\rangle$, labelled by the parameter α.

Applying the operators of the algebra to they cyclic state $|\, \xi^{(\alpha)}_{BCS}\rangle$, and taking the completion, we get a Hilbert space H_α in which the representation of the algebra is irreducible. As is well known, the representations in H_α $(0 \leq \alpha \leq 2\pi)$ are all inequivalent to each other, and the state $|\, \xi^{(\alpha)}_{BCS}\rangle$ is not a state with sharp particle number. Later we will give a more precise meaning to the last statement. ANDERSON argued that by taking an infinitesimal contribution in each space H_α one can obtain a ground state with "sharp particle number". His idea is essentially correct, but care is needed in the definition of such a state.

First of all, are we allowed to say that a state corresponds to a "sharp particle number", notwithstanding that only the number operator for a finite region of space

$$N_V = \int_V \psi^*(x)\, \psi(x) = \sum_n \cdot \psi^*(f_n)\, \psi(f_n)$$

is a sensible operator? ($\{f_n\}$ is a complete orthonormal set of functions.)

The answer is in the affirmative. First we notice that the "WIGHTMAN functions"

$$W_\alpha(x_1 \ldots x_n\, y_1 \ldots y_m) = \langle \xi^{(\alpha)}_{BCS} |\psi^*(x_1) \ldots \psi^*(x_n)\, \psi(y_1) \ldots \psi(y_m)|\, \xi^{(\alpha)}_{BCS}\rangle$$

do not vanish, in general, for $n \neq m$. This fact gives an exact meaning to the proposition "$|\, \xi^{(\alpha)}_{BCS}\rangle$ is not a state with sharp particle number"; then, following HAAG, we define a new set of "WIGHTMAN functions":

$$W_\Omega(x_1 \ldots x_n\, y_1 \ldots y_m) = \begin{cases} W_\alpha(x_1 \ldots x_n\, y_1 \ldots y_m) & \text{for } n = m\,, \\ 0 \text{ for } n \neq m\,. \end{cases}$$

It is easy to see that these functions W_Ω verify the positivity condition; therefore we can apply the GELFAND theorem and introduce a new "cyclic state" $|\Omega\rangle$ such that:

$$\langle\Omega\,|\psi^*(x_1)\ldots\psi^*(x_n)\,\psi(y_1)\ldots\psi(y_m)\,|\,\Omega\rangle = W_\Omega\,(x_1\ldots x_n, y_1\ldots y_m)\,.$$

We know that this is a correct way of defining a state, and we are allowed to consider it to be a state of sharp particle number, because it verifies the same property of an eigenstate of the number of particles in the case of finite volume; i.e., it gives rise to a vanishing expectation value for any non-gauge invariant operator.

It is easy to prove that the space H_Ω, obtained by applying to $|\Omega\rangle$ the operators of the algebra and by taking the completion, is a space in which the representation is reducible; roughly speaking, if we want to split H_Ω into differential subspaces in which the representation is irreducible, we find again the spaces H_α. More rigorously, as α is a continuous parameter, we must use the mathematical tool provided by the direct integral of Hilbert spaces:

$$H_\Omega = \int^{\oplus} H_\alpha \frac{d\alpha}{2\pi}\,.$$

(An element ξ of H_Ω is a vector-valued function $\xi = \{\xi_\alpha\}$, where $\xi_\alpha \in H_\alpha$, satisfying the condition $\int \frac{d\alpha}{2\pi}\langle\xi_\alpha\,\xi_\alpha\rangle < \infty$; of course the scalar product of two vectors $\xi = \{\xi_\alpha\}$, $\eta = \{\eta_\alpha\}$ of H_Ω, is defined as:

$$(\eta, \xi) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\,\langle\eta_\alpha\,\xi_\alpha\rangle\,.)$$

The states with "sharp particle number" have been defined by ANDERSON in the following way:

$$|\,\Omega'_n\rangle = \int e^{in\alpha}\,|\,\xi^{(\alpha)}_{BCS}\rangle\,d\alpha \qquad n = 0, \pm 1, \pm 2\ldots$$

and have vanishing norm in the limit $V \to \infty$. Instead, with the formalism of the direct integral we can define the following states:

$$|\,\Omega_n\rangle = \{e^{in\alpha}\,\xi^{(\alpha)}_{BCS}\} \qquad n = 0, \pm 1, \pm 2\ldots$$

which have a non-vanishing norm; we will prove that the vectors $|\,\Omega_n\rangle$ verify the property of giving rise to vanishing expectation value for all non-gauge

invariant quantities. It will follow that $|\ \Omega_n\rangle$ can be identified with the vector $|\ \Omega\rangle$ previously defined by means of the Gelfand theorem.

Let A_α be an operator representing an element of the algebra in H_α, and let A be the corresponding operator in H_Ω. As the algebra is irreducible in H_α, if $\xi_\alpha \in H_\alpha$, also $A_\alpha \xi_\alpha \in H_\alpha$. Therefore the definition of the operator A is given by the formula:

$$A\{\xi_\alpha\} = \{A_\alpha \xi_\alpha\}\ .$$

Let us assume that A is the product of n operators ψ^* and m operator ψ. The expectation value of A in the state $|\ \Omega_{\dot{n}}\rangle$ is given by:

$$\langle \Omega_{\dot{n}}\, A \Omega_{\dot{n}}\rangle = \int_0^{2\pi} \frac{d\alpha}{2\pi} \langle \xi_{BCS}^{(\alpha)}\, A_\alpha\, \xi_{BCS}^{(\alpha)}\rangle$$

and it contains the factor $\int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{i(m-n)} = \delta^{m,n}$ Q. E. D.

So $|\ \Omega_{\dot{n}}\rangle$ is a state with "sharp particle number". We notice that the Hilbert space H is separable.

We can understand the meaning of the formalism used, noticing that H_α is not a subspace of H_Ω, in the same way as a bounded function differing from zero only in one point of the interval $0 \leq \alpha < 2\pi$, is not a vector (different from the nul vector) of the Hilbert space of the the square-integrable function $f(\alpha)$. In other words in H_Ω only "wave packets" of spaces H_α are allowed to constitute a sensible vector. [1] Finally, I want to point out that the vector $|\ \Omega_{n+p}\rangle$ can be interpreted as corresponding to a physical situation with p more "COOPER pairs" than the vector $|\ \Omega_n\rangle$. Of course they both correspond to an infinite number of "COOPER pairs".

In the case of a free BOSE gas, with $\varrho \neq 0$ and $V = \infty$, the situation is completely analogous. Let ψ_0 be the ground state. It is very easy to find the appropriate functional

$$E(f, g) = \langle \psi_0,\ U(f) V(g) \psi_0\rangle$$

[1] This procedure of restoring a "broken symmetry" using the concept of the direct integral of Hilbert spaces can be generalized to the case of an arbitrary compact group G of symmetry. In the general case, in the place of the integration $\int \frac{d\alpha}{2\pi}$, we must perform the invariant integration over the group G. States which are analogous to the "states with sharp particle number" of the B. C. S. case are obtained by taking into the subspace H_0 of the translationally invariant states a basis adapted to the decomposition of H_0 into irreducible representations of G. (These simple results came to light during discussions with Dr. S. DOPLICHER.)

It would be very interesting to study the physical consequences of this mathematical procedure, especially in connection with the GOLDSTONE theorem.

Note added in proof: see also on the subject of the "broken symmetries": G. FANO, Nuovo Cimento, **38**, 597, 1965.

by taking the V limit of the analogous quantity computed in the FOCK representation:

$$E_F(f, g) = \langle \psi_{OF}, U_F(f) V_F(g) \psi_{OF} \rangle ,$$

where $| \psi_{OF} \rangle$ is a state corresponding to n particles with wave function $V^{-\frac{1}{2}}$ in a box of volume V.

As in the B. C. S. case, the Hilbert space H corresponding to the functional E can be decomposed into invariant "differential" subspaces H_Θ, $0 \leq \theta < 2\pi$, in which the representation is irreducible. In each H_Θ the field operators are related to the FOCK operators by the well known canonical transformation

$$\psi_\Theta(x) = \psi_F(x) + \sqrt{\varrho}\, e^{i\Theta} .$$

For the excited states of an infinite BOSE system, which corresponds to systems of particles with a continuous distribution $\varrho(k)$ in momentum space, the calculation of the functional E proceeds in the same way, and one gets new representations of the C. C. R. The corresponding Hilbert space is given simply by a tensor product of two Hilbert spaces corresponding to the FOCK representation. This corresponds physically to the particle-hole description. All these calculations (performed by ARAKI and WOOD [4]) are also useful for finding the representations suitable at a given temperature.

Let me now say a few words about the consequences of the Galilean invariance of the theory. First of all, I want to point out that not all the infinite systems are translationally and rotationally invariant. For instance, a good charaterization of an equilibrium state to represent a solid, is not to be invariant under rotations.

If ξ_0 is the only state invariant under the Euclidean group (let $\vec{a}$ be the vector of the translation, R the rotation, and $U(a, R)$ the representation of the Euclidean group in the Hilbert space), i.e. if

$$U(a, R)\, \xi_0 = \xi_0 ,$$

then a very elegant argument (used by BORCHERS in field theory for the case of the Lorentz group) can be used in order to prove that:

$$\text{weak} \lim_{|a| \to \infty} U(\vec{a}, I) = P_{\xi_0} ,$$

where P_{ξ_0} is the projection operator on the state ξ_0.

An important consequence of this result is the so called "linked cluster property". Let $T_{\vec{a}}$ be the operator defined by

$$T_{\vec{a}} f)(\vec{x}) = f(\vec{x} + \vec{a}) .$$

Then we have:

$$\lim_{|a|\to\infty} E(f_1 + T_a f_2) = \lim_{|a|\to\infty} \langle \xi_0, U(f_1)\, U(\vec{a}, I)^{-1}\, U(f_2)\, U(\vec{a}, I)\, \xi_0 \rangle =$$

$$= \langle \xi_0\, U(f_1)\, \xi_0 \rangle \langle \xi_0\, U(f_2)\, \xi_0 \rangle = E(f_1)\, E(f_2)\,.$$

The physical meaning of this result is that regions far separated in space must not interfere.

This property must be taken into account if one wants to devise some approximation method. For this purpose, it is better to work with the "truncated" expectation values,[1] defined by: ($W^{n,m}$ is the W-function with n fields ψ^* and m fields ψ)

$$W^{(1\,0)}(x_1) = W_T^{(1\,0)}(x_1)\,,$$

$$W^{(1,1)}(x_1, y_1) = W_T^{(1\,0)}(x_1)\, W_T^{(1\,0)}(y_1)^* + W_T^{(1\,1)}(x_1, y_1)\,,$$

$$W^{(2\,0)}(x_1, x_2) = W_T^{(1\,0)}(x_1) W_T^{(1\,0)}(x_2) + W_T^{(2,0)}(x_1, x_2)\,,$$

etc.

Because of the translation invariance, the truncated functions depend only on the difference of the arguments ($W_T^{(1,0)}(x)$ is then a constant,[2] which can be different from zero because the representation of the operator is not the FOCK representation) and the linked cluster property is equivalent to the condition that the W_T vanish whenever one of these differences becomes very large. Therefore the following approximation procedure can be devised:

One takes only a finite number of W_T different from zero, i.e.:

$$W_T^{(n,m)} = 0 \text{ for } n + m > s$$

and one writes down the positivity condition. The corresponding $E(f)$ defines, by the GELFAND theorem, a state that can be considered as a trial state for the particular Hamiltonian of the problem. In other words one imposes the energy density to be minimal at a given particle density. The advantage of using the truncated functions is that one cannot take a finite number of WIGHTMAN functions different from zero without violating the linked cluster property. There is some unpublished result by HAAG and collaborators in this direction. For instance, considering only the truncated functions written above to be different from zero ($s = 2$), they have found, in a very simple way, the same results as BOGOLIUBOV, VALATIN and BUTLER etc. for the BOSE gas. It is not possible to take $s = 3$, because an unpublished theorem by D. W. ROBINSON states that the case in which $s = 3$ does not differ from the

[1] We assume that the field operators have a common dense domain.

[2] This constant, which corresponds to a delta function in momentum space, is proportional to the square root of the particle density of zero momentum.

case in which $s = 2$; this theorem can be derived using the positivity condition only. It seems therefore very likely that, as in field theory, if one wants the truncated WIGHTMAN function with three arguments to be non-vanishing, one is forced to have all the truncated WIGHTMAN functions non-vanishing.[1]

It is an open and interesting problem to find approximation methods which go beyond the case $s = 2$, and to find the relation between this approach and the usual time-dependent Green's function method widely used in many-body problems.

Another unpublished result is given by a theorem of ARAKI, which, roughly speaking, states that if the representation of the algebra of the "quasi local" observables is a factor, the cluster property holds.

(Let R be a set of bounded operators in a Hilbert space, and R' be the set of all bounded operators commuting with every operator in R; a closed algebra R, containing all the adjoints of its elements, is said to be a factor if $R \cap R'$ consists only of multiples of the identity).

Concluding this talk, I would like to remark that the consideration of systems with an infinite number of particles certainly forces us to use new and more powerful mathematical tools, but this fact makes our physical problems richer and more interesting.

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АБСТРАКТНЫЙ МЕТОД В ТЕОРИИ МНОГИХ ТЕЛ С БЕСКОНЕЧНЫМ ОБЪЕМОМ

Г. ФАНО

Р е з ю м е

Обсуждается классификация возможных представлений канонических соотношений коммутации и антикоммутации, и исследована проблема нахождения подходящего представления заданного гамильтониана. Рассмотрены следствия габилеевой инвариантности теории.

[1] *Note added in proof*: this result has been now proved by D. W. ROBINSON: Commun. math. Phys., **1**, 89, 1965.

ON THE PERTURBATION-THEORETIC CALCULATION OF THE ONE-PARTICLE EXCITATION SPECTRUM IN A LARGE BOSE SYSTEM

By

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Perturbation theoretic calculation of the one-particle Green's functions is studied for a large system of bosons at zero temperature. We start from a splitting of the Hamiltonian in which the unperturbed part is chosen to be diagonal in the BOGOLYUBOV quasi particle operators. The self-consistent form of the perturbation expansion required by the appearance of inequivalent representations of canonical commutation relations is formulated and discussed.

1. Introduction

In this paper a large Bose system of particles interacting via a two-body potential is considered at the temperature of absolute zero. In a system of particles obeying Bose statistics the large number of particles of momentum zero plays a particular role. In the case of weakly interacting bosons, the excitation spectrum was calculated by BOGOLYUBOV [1] in 1947. It has been shown by BELIAEV [2], how the Green's function method of the field theoretic perturbation theory can be formulated for Bose systems and equations were derived for the one particle Green's functions. Similar result was obtained by HUGENHOLTZ and PINES [3] who eliminated the condensed particles from the problem by adopting BOGOLYUBOV's prescription which consists in treating the quantum amplitude referring to the zero momentum state as a c-number. Moreover they proved that to any order in the perturbation theory the one-particle excitation spectrum does not exhibit an energy gap. In the perturbation theory of BELIAEV, and HUGENHOLTZ and PINES the random phase approximation is equivalent to the BOGOLYUBOV canonical transformation carried out on the BOGOLYUBOV's approximate Hamiltonian.

There are, of course, other possibilities for the way of generating the perturbation expansion. In this paper we start out from a splitting of the Hamiltonian in which the unperturbed part is chosen to be diagonal in terms of the BOGOLYUBOV quasi particle operators. The splitting used by BELIAEV, HUGENHOLTZ and PINES is obtainable as a special case. The self-consistent form of the perturbation expansion required by the appearance of inequivalent representations of canonical commutation relations is formulated and discussed.

2. The one-particle Green's functions

We consider a Bose system of N particles, contained in a box of volume Ω with periodic boundary conditions. The Hamiltonian of the system is as follows

$$H = \sum_{\mathbf{k}} \varepsilon_k \, a_{\mathbf{k}}^+ \, a_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}_i} \langle \mathbf{k}_1 \, \mathbf{k}_2 \, | \mathrm{v} | \, \mathbf{k}_3 \, \mathbf{k}_4 \rangle \, a_{\mathbf{k}_1}^+ \, a_{\mathbf{k}_2}^+ \, a_{\mathbf{k}_3} \, a_{\mathbf{k}_4} \,, \tag{1}$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^+$ satisfying the usual commutation relations for bosons, destroy and create, respectively, free particle states of momentum $\mathbf{k}$. Furthermore v denotes the interaction between the particles and $\varepsilon_k = k^2/2m - \mu$ (we choose units such that $\hbar = 1$). μ standing for the chemical potential has been introduced because states with an undetermined number of particles will be considered. μ is determined by the requirement that the average number of particles should be N.

From the possibilities to take into account the condensation of the particles into the zero momentum state [1], [2], [3], [4] for our purposes the most suitable one is to introduce the canonical transformation as follows [4]:

$$b_{\mathbf{k}} = a_{\mathbf{k}} - \sqrt{N_0} \, \delta_{k,0} \,, \tag{2}$$

where $\delta_{k,0}$ is the Kronecker symbol and N_0 is the average number of particles in the zero momentum state. After substituting (2) into (1) we have a Hamiltonian which can be treated by the usual methods of field theoretic perturbation theory. Taking $\sum_{\mathbf{k}} \varepsilon_k b_{\mathbf{k}}^+ b_{\mathbf{k}}$ as unperturbed Hamiltonian and determining N_0 in a self-consistent manner the results of BELIAEV, and HUGENHOLTZ and PINES can be rederived. Instead of doing that we start out from a more general splitting of the Hamiltonian

$$H = H_0 + H_1 \,, \tag{3}$$

where

$$H_0 = \sum_{\mathbf{k}} (\varepsilon_k + A_k) \, a_{\mathbf{k}}^+ \, a_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}} B_k \, (a_{\mathbf{k}}^+ \, a_{-\mathbf{k}}^+ + a_{\mathbf{k}} \, a_{-\mathbf{k}}) + C_0 \, (a_0^+ + a_0)$$

and

$$\begin{aligned} H_1 = {} & \frac{1}{2} \sum_{\mathbf{k}_i} \langle \mathbf{k}_1 \, \mathbf{k}_2 \, | \mathrm{v} | \, \mathbf{k}_3 \, \mathbf{k}_4 \rangle \, a_{\mathbf{k}_1}^+ \, a_{\mathbf{k}_2}^+ \, a_{\mathbf{k}_3} \, a_{\mathbf{k}_4} - \sum_{\mathbf{k}} A_k \, a_{\mathbf{k}}^+ \, a_{\mathbf{k}} - \\ & - \frac{1}{2} \sum_{\mathbf{k}} B_k \, (a_{\mathbf{k}}^+ \, a_{-\mathbf{k}}^+ + a_{\mathbf{k}} \, a_{-\mathbf{k}}) - C_0 \, (a_0^+ + a_0) \,. \end{aligned} \tag{4}$$

H_0 can be diagonalized by performing the canonical transformation (2) together with BOGOLYUBOV's canonical transformation [1].

$$\beta_{\mathbf{k}} = u_k \, b_{\mathbf{k}} - v_k \, b_{-\mathbf{k}}^+ \,, \quad u_k^2 - v_k^2 = 1 \,, \tag{5}$$

where u_k and v_k are now considered to be real. In this way it is obtained that

$$H_0 = U + \sum_{\mathbf{k}} E_k \beta_{\mathbf{k}}^+ \beta_{\mathbf{k}} , \tag{6}$$

where U is a constant,

$$E_k = \sqrt{(\varepsilon_k + A_k)^2 - B_k^2} \tag{7}$$

and the parameters of the canonical transformations are determined by

$$\sqrt{N_0} = \frac{C_0}{\mu - A_0 - B_0} , \tag{8}$$

$$\left.\begin{matrix} u_k^2 \\ v_k^2 \end{matrix}\right\} = \frac{1}{2}\left[\pm 1 + \frac{\varepsilon_k + A_k}{E_k}\right], \tag{9}$$

$$u_k v_k = -\frac{B_k}{2E_k} . \tag{10}$$

Let us introduce now the one-particle Green's functions

$$\left.\begin{aligned} G(\mathbf{k}, t - t') &= -i \langle \Psi_0 \mid T b_{\mathbf{k}}(t)\, b_{\mathbf{k}}^+(t') \mid \Psi_0 \rangle , \\ \hat{G}(\mathbf{k}, t - t') &= -i \langle \Psi_0 \mid T b_{\mathbf{k}}(t)\, b_{-\mathbf{k}}(t') \mid \Psi_0 \rangle , \\ \check{G}(\mathbf{k}, t - t') &= -i \langle \Psi_0 \mid T b_{-\mathbf{k}}^+(t)\, b_{\mathbf{k}}^+(t') \mid \Psi_0 \rangle , \end{aligned}\right\} \tag{11}$$

where the operators are in the Heisenberg representation and Ψ_0 denotes the exact ground state wave function of the system. If we take H_0 as unperturbated Hamiltonian and go over to the interaction representation, it is possible to perform a perturbation expansion of the one-particle Green's functions in powers of H_1. We have the following internal lines

$$\overrightarrow{G_0(k, k_0)} \quad , \qquad \overrightarrow{\check{G}_0(k, k_0)} \quad , \qquad \overrightarrow{\hat{G}_0(k, k_0)} \quad ,$$

where

$$G_0(k, k_0) = \frac{u_k^2}{k_0 - E_k + i\delta} - \frac{v_k^2}{k_0 + E_k - i\delta} \tag{12}$$

and

$$\hat{G}_0(k, k_0) = \check{G}_0(k, k_0) = u_k v_k \left[\frac{1}{k_0 - E_k + i\delta} - \frac{1}{k_0 + E_k - i\delta}\right]. \tag{13}$$

Substituting eq. (2) into (4) we obtain the vertices shown in Fig. 1.
We call those v (A and B) vertices incomplete which have less than four (two) lines joining to them. Those v and B incomplete vertices which are obtainable from the given ones by interchanging the roles of the opposite ends of the

v and B interactions are not shown in Fig. 1. The places of the missing lines are indicated by open circles for each of which is a factor $\sqrt{N_0}$.

Before writing down the perturbation expansion of the one-particle Green's functions it will be expedient to deal with the condition determining

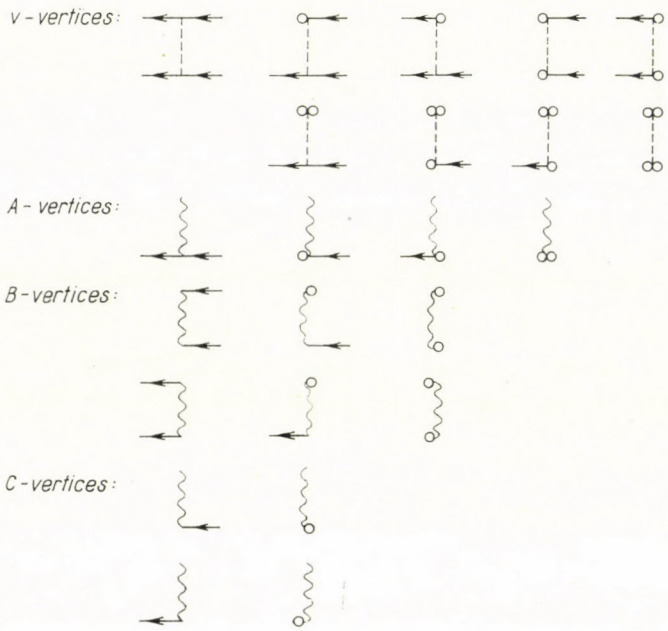


Fig. 1

N_0. In the present treatment N_0 is to be determined from the requirement

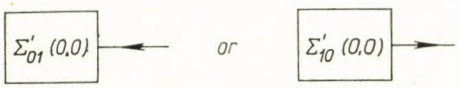
$$\langle \Psi_0 \mid b_0 \mid \Psi_0 \rangle = \langle \Psi_0 \mid b_0^+ \mid \Psi_0 \rangle = 0 \,, \tag{14}$$

which implies

$$\langle \Psi_0 \mid a_0 \mid \Psi_0 \rangle = \langle \Psi_0 \mid a_0^+ \mid \Psi_0 \rangle = \langle \Phi_0 \mid a_0 \mid \Phi_0 \rangle = \langle \Phi_0 \mid a_0^+ \mid \Phi_0 \rangle = \sqrt{N_0} \,,$$

where Φ_0 is the ground state wave function of the unperturbed Hamiltonian H_0.

The graphs contributing to $\langle \Psi_0 \mid b_0^+ \mid \Psi_0 \rangle$ are of the types shown in Fig. 2 and contain, of course, at least one irreducible part* like



which cannot be divided into two parts by cutting a single solid line.

* Only those segments of the lines are indicated here which are joining to Σ'_{01} and Σ'_{10}.

It can be seen easily that $\Sigma'_{10}(0, 0) = \Sigma'_{01}(0,0)$ and eq. (14) is satisfied by

$$\Sigma'_{01}(0,0) = 0 . \tag{15}$$

Let us separate out from Σ'_{01} the terms of first order in A_0, B_0 and C_0:

$$\Sigma'_{01}(0, 0) = -\sqrt{N_0}(A_0 + B_0) - C_0 + \Sigma_{01}(0,0) .$$

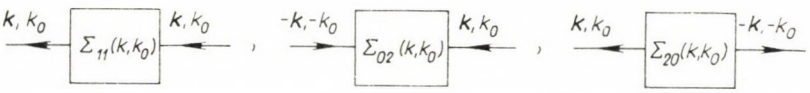
Using equs. (15) and (8) we get

$$\sqrt{N_0}\,\mu = \Sigma_{01}(0, 0) . \tag{16}$$

Fig. 2

From now on we shall always consider N_0 as a quantity satisfying this equation. As a consequence graph parts as in Fig. 2 will not occur in the perturbation expansion.*

Let us introduce the irreducible self-energy graphs



By definition we mean by Σ_{11} and Σ_{02}, Σ_{20} the sums of all the above types of self-energy graphs except the ones of first order in A_k and B_k respectively. Furthermore, for the beforementioned reasons, they do not contain insertions like those shown in Fig. 2. It can be seen easily, that $\Sigma_{20}(k, k_0) = \Sigma_{02}(k, k_0)$ and that they are even functions of k_0. All the Σ's depend only on the absolute value of **k**.

For the Fourier transforms of the one particle Green's functions (11) the Dyson's equations shown in Fig. 3 can be obtained**, where

$$G_{00} = \frac{1}{k_0 - \varepsilon_k + i\delta} .$$

* This implies, especially, that C-vertices are not to be drawn any more.

** If condition (14) had not been taken into account further terms would have appeared in the Green's functions e.g.

The appearance of the G_{00} line here may seem strange because we should expect to appear only lines coming from the pairings of the operators b, b^+, i.e. the lines which occur in Σ_{11} and Σ_{02}. However, this form of the Dyson's equations follows directly if we start from the equation of motion of the

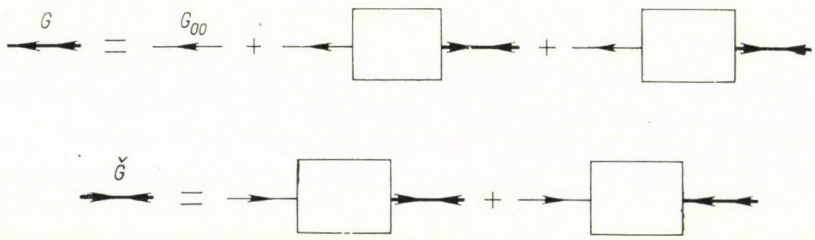


Fig. 3

operator $b_k(t)$. Instead of proceeding in this way, we derive in the Appendix that form of the Dyson's equations which results from analysing the perturbation expansion of the one particle Green's functions in the usual way and show the identity of the two forms.

The algebraic equations corresponding to those shown in Fig. 3 can be solved to yield

$$G(k, k_0) = \frac{k_0 + \varepsilon_k + \Sigma_{11}(k, -k_0)}{F}, \tag{17}$$

$$\hat{G}(k, k_0) = \check{G}(k, k_0) = \frac{-\Sigma_{02}(k, k_0)}{F}, \tag{18}$$

where

$$F = \left[k_0 - \frac{1}{2}\left(\Sigma_{11}(k, k_0) - \Sigma_{11}(k, -k_0)\right)\right]^2 -$$

$$- \left[\varepsilon_k + \frac{1}{2}\left(\Sigma_{11}(k, k_0) + \Sigma_{11}(k, -k_0)\right)\right]^2 + \Sigma_{02}^2(k, k_0)$$

which are just BELIAEV's equations. There is a difference, however, which lies in the fact that the graphs giving the Σ's are defined here in terms of internal lines generally different from those used by BELIAEV and HUGENHOLTZ and PINES. Their perturbation expansion can be obtained from ours as a special case, namely by choosing $A_k = B_k = 0$. In that case $u_k = 1$, $v_k = 0$, $\check{G}_0 = \hat{G}_0 = 0$, $G_0 = G_{00}$ and $C_0 = \sqrt{N_0}\,\mu$.

It is to be noted that C_0 appears in the formalism because of our treatment of the zero momentum state which differs from both of the treatments followed by BELIAEV and HUGENHOLTZ and PINES. All of the three treatments are, however, essentially equivalent. HUGENHOLTZ and PINES, adopting BOGOLYUBOV's prescription, replaced a_0, a_0^+ by a c-number $\sqrt{N_0}$ and determined the value of N_0 by minimizing the ground state energy. This requirement is equivalent to our eq. (14).

3. Relation among the Σ's

It has been proved by HUGENHOLTZ and PINES that to every order in their perturbation expansion

$$\mu = \Sigma_{11}(0, 0) - \Sigma_{02}(0, 0) , \tag{19}$$

which is, as can be seen easily, the necessary condition for the existence of a pole of the one particle Green's function (17) at $k = 0$, $k_0 = 0$, which means that the one particle excitation spectrum does not exhibit an energy gap, if Σ_{11} and Σ_{02} are not too badly behaved for small k and k_0.

We can derive a relation among the Σ's in the framework of the present perturbation expansion in a similar manner, by proceeding as follows.

The diagrams contributing to $\Sigma_{01}(0, 0)$, $\Sigma_{11}(0, 0)$ and $\Sigma_{02}(0, 0)$ are obtainable from those vacuum-vacuum diagrams which cannot be divided into two parts by cutting a single solid line and which are not of first order in A, B or C. Let us group these vacuum-vacuum diagrams according to the total number of the open circles of their incomplete vertices. Namely, let us denote by D_{ij} the contribution of the sum of such diagrams which have open circles i and j in number at the places of ingoing and outgoing lines, respectively. All the diagrams of $\Sigma_{11}(0, 0)$ can be obtained by setting back in all possible ways one ingoing and one outgoing missing line with momentum and energy equal to zero. To obtain $\Sigma_{02}(0, 0)$ and $\Sigma_{01}(0, 0)$ one attaches two ingoing lines and one ingoing line, respectively, in a similar fashion. Then we get

$$\Sigma_{11}(0, 0) = \frac{1}{N_0} \sum_{i,j} ij\, D_{ij} ,$$

$$\Sigma_{02}(0, 0) = \frac{1}{N_0} \sum_{i,j} i(i-1)\, D_{ij} ,$$

$$\Sigma_{01}(0, 0) = \frac{1}{\sqrt{N_0}} \sum_{i,j} i\, D_{ij} ,$$

whence we obtain

$$\frac{1}{\sqrt{N_0}} \Sigma_{01}(0, 0) + \frac{1}{N_0} D = \Sigma_{11}(0, 0) - \Sigma_{02}(0, 0) , \tag{20}$$

where D is defined by

$$D = -\sum_{i>j}(i-j)^2 D_{ij}\,.$$

Here we have made use of the fact that $D_{ij} = D_{ji}$.

According to eqs. (20) and (16) relation (19) is fulfilled if $D = 0$. It is a characteristic feature of the graphs contributing to D that $i \neq j$ i.e. all of them must have at least one $\hat{G}_0$ or $\check{G}_0$ line or B_k vertex. Choosing $B_k = 0$ such elements do not come into the perturbation expansion and we get HUGENHOLTZ and PINES's theorem.

4. The self-consistent perturbation expansion

In connection with the application of perturbation theory it is important that, as is well known, the representations of the canonical commutation relations corresponding to different values of the parameters u_k, v_k become inequivalent to each other at the limit $\Omega \to \infty$. This implies that the quasi particle vacuum states belonging to different values of the parameters u_k, v_k are orthogonal at the limit $\Omega \to \infty$. Furthermore, as to the canonical transformation (2), the same is true for those representations which belong to different values of N_0/Ω as not only the density of particles but also the density of particles of momentum zero remains finite when $\Omega \to \infty$. This fact suggests that u_k, v_k and N_0 should be determined self-consistently.*

Eq. (14) is just a self-consistent condition for the determination of N_0. For the self-consistent determination of u_k and v_k let us proceed as follows.** Introduce the one quasi-particle Green's functions

$$\begin{aligned} \mathcal{G}_{\mathbf{k}}(k, t-t') &= -i\langle \Psi_0 \mid T\beta_{\mathbf{k}}(t)\,\beta^{+}_{\mathbf{k}}(t') \mid \Psi_0\rangle\,, \\ \hat{\mathcal{G}}_{\mathbf{k}}(k, t-t') &= -i\langle \Psi_0 \mid T\beta_{\mathbf{k}}(t)\,\beta_{-\mathbf{k}}(t') \mid \Psi_0\rangle\,, \end{aligned} \tag{21}$$

which can be expressed in terms of the one-particle Green's functions (11) by inserting the BOGOLYUBOV transformation (5) for the operators in the Heisenberg representation. Consequently the Fourier transforms of $\mathcal{G}_{\mathbf{k}}(k, t-t')$ and $\hat{\mathcal{G}}_{\mathbf{k}}(k, t-t')$ can be obtained as the linear combinations of $G(k, \pm k_0)$ and $\hat{G}(k, k_0)$. Moreover the poles of $\mathcal{G}_{\mathbf{k}}(k, k_0)$ and $\hat{\mathcal{G}}_{\mathbf{k}}(k, k_0)$ are the same as those of $G(k, k_0)$ and $\hat{G}(k, k_0)$.

We require that the E'_k s occurring in H_0 be the renormalized one-particle energies, i.e. the real parts of the poles be equal to the E'_ks. Restricting ourselves

* The similar problem in the field theory was discussed extensively by H. UMEZAWA in his lecture at this Conference [5].

** In the case of fermion systems a similar procedure was used by N. MENYHÁRD [6].

to cases, for which the imaginary parts of the poles are negligible, we obtain according to eq. (17)

$$E_k = \sqrt{\left[\varepsilon_k + \frac{1}{2}\left(\Sigma_{11}(k, E_k) + \Sigma_{11}(k, -E_k)\right)\right]^2 - \Sigma_{02}^2(k, E_k)} + \\ + \frac{1}{2}\left(\Sigma_{11}(k, E_k) - \Sigma_{11}(k, -E_k)\right).$$

The Σ's depend on u_k and v_k which has not been determined yet. According to eqs. (9) and (10) all the parameters appearing in the perturbation expansion can be obtained if E_k and $u_k v_k$ are known. In accordance with the self-consistency requirement, $u_k v_k$ will be chosen in such a way that $\hat{\mathcal{Q}}_{\mathfrak{f}}(k, E_k)$ be equal to zero, as we have $\hat{\mathcal{Q}}_{\mathfrak{f}0} = -i \langle \Phi_0 \mid T\beta^i_{\mathbf{k}}(t)\beta^i_{-\mathbf{k}}(t') \mid \Phi_0 \rangle = 0$. This condition is equivalent to the BOGOLYUBOV's principle of compensation of dangerous diagrams in the theory of superconductivity [7]. Expressing $\hat{\mathcal{Q}}_{\mathfrak{f}}(k, k_0)$ in terms of $G(k, \pm k_0)$ and $\hat{G}(k, k_0)$ and setting the numerator of the resulting expression for $k_0 = E_k$ equal to zero, we obtain

$$(u_k^2 + v_k^2)\,\Sigma_{02}(k, E_k) + 2u_k v_k \left(\varepsilon_k + \frac{\Sigma_{11}(k, E_k) + E_{11}(k, -E_k)}{2}\right) = 0,$$

whence using $u_k^2 - v_k^2 = 1$,

$$u_k v_k = -\frac{\Sigma_{02}(k, E_k)}{2\sqrt{\left[\varepsilon_k + \frac{1}{2}\left(\Sigma_{11}(k, E_k) + \Sigma_{11}(k, -E_k)\right)\right]^2 - \Sigma_{02}^2(k, E_k)}}$$

follows, which determines B_k according to eq. (10).

Calculating to the first order approximation we get the results (obtained by different methods) of VALATIN and BUTLER [8] and GIRARDEAU and ARNOWITT [9]. In this approximation D is not equal to zero.* The general behaviour of the self-consistent expansion needs further investigations.**

* It is to be noted that VALATIN and BUTLER eliminated the energy gap by determining the number of the particles in the zero momentum state just from the condition that the excitation energy should be zero for zero momentum.

** *Note added in proof:* It can turn out to be the case that in the self-consistent perturbation expansion the HUGENHOLTZ—PINES theorem does not hold if we calculate up to any given order. By summing up, however, the whole perturbation expansion the HUGENHOLTZ—PINES theorem reappears. To prove it let us calculate the Σ's by taking into account only skeleton graphs i.e. graphs without any self-energy insertion and replace G_0, $\hat{G}_0$ and $\check{G}_0$ by G, $\hat{G}$ and $\check{G}$, respectively. (Using skeleton graphs with thick lines means, of course, a rearrangement of the perturbation expansion, but starting from the self-consistent perturbation expansion i.e. having determined E_k and $u_k v_k$ self-consistently it is reasonable to assume that this rearrangement does not alter the final results.) The skeleton graphs contributing to Σ_{02} have been analysed by GAVORET and NOZIÈRES in the Appendix B of their paper (Annals of Phys. **28**, 349, 1964),

(continued on page 118)

Appendix

In order to get the diagrams contributing to the Green's functions defined by eqs. (11) let us write them down in the interaction representation

$$G(k, t-t') = -i \langle \Phi_0 \mid T b^i_{\mathbf{k}}(t)\, b^{i+}_{\mathbf{k}}(t')\, S \mid \Phi_0 \rangle / \langle \Phi_0 \mid S \mid \Phi_0 \rangle ,$$

$$\hat{G}(k, t-t') = -i \langle \Phi_0 \mid T b^i_{\mathbf{k}}(t)\, b^{i}_{-\mathbf{k}}(t') S \mid \Phi_0 \rangle / \langle \Phi_0 \mid S \mid \Phi_0 \rangle ,$$

$$\check{G}(k, t-t') = -i \langle \Phi_0 \mid T b^{i+}_{-\mathbf{k}}(t)\, b^{i+}_{\mathbf{k}}(t') S \mid \Phi_0 \rangle / \langle \Phi_0 \mid S \mid \Phi_0 \rangle ,$$

where the S-matrix is given by the well-known expression

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \ldots dt_n\, T\left[H^i_1(t_1) \ldots H^i_1(t_n)\right] .$$

Applying Wick's theorem we are led to the pairings

$$iG_0(k, t-t') = \langle \Phi_0 \mid T b^i_{\mathbf{k}}(t)\, b^{i+}_{\mathbf{k}}(t') \mid \Phi_0 \rangle ,$$

$$i\hat{G}_0(k, t-t') = \langle \Phi_0 \mid T b^i_{\mathbf{k}}(t)\, b^{i}_{-\mathbf{k}}(t') \mid \Phi_0 \rangle ,$$

and

$$i\check{G}_0(k, t-t') = \langle \Phi_0 \mid T b^{i+}_{-\mathbf{k}}(t)\, b^{i+}_{\mathbf{k}}(t') \mid \Phi_0 \rangle ,$$

the Fourier transforms of which as well as the possible vertices have been given in Section 2. The complete vertices give the respective contributions $-\frac{i}{2} \langle \mathbf{k}_1 \mathbf{k}_2 \mid \mathrm{v} \mid \mathbf{k}_3 \mathbf{k}_4 \rangle$, iA_k, iB_k and $iC_0\, \delta_{k,0}$ from which the contributions of the

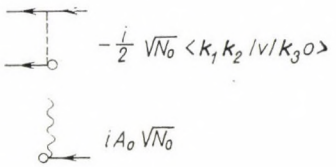
footnote continued)
which has led to the inhomogeneous BETHE—SALPETER equation (B.8). It is easy to see, that applying their method to the skeleton graphs contributing to Σ_{01} we get the following relation

$$\frac{1}{\sqrt{N_0}} \Sigma_{01}(0, 0) = \tilde{\Sigma}_{11}(0, 0) - \tilde{\Sigma}_{02}(0,0) + \frac{2i}{\sqrt{N_0}} \sum_{\mathbf{p}} \int dp_0 \left[J^{++}_{+}(\mathbf{p}, p_0; 0, 0) - \right.$$

$$\left. - J^{--}_{+}(\mathbf{p}\, p_0; 0, 0) \right] \check{G}(\mathbf{p}, p_0).$$

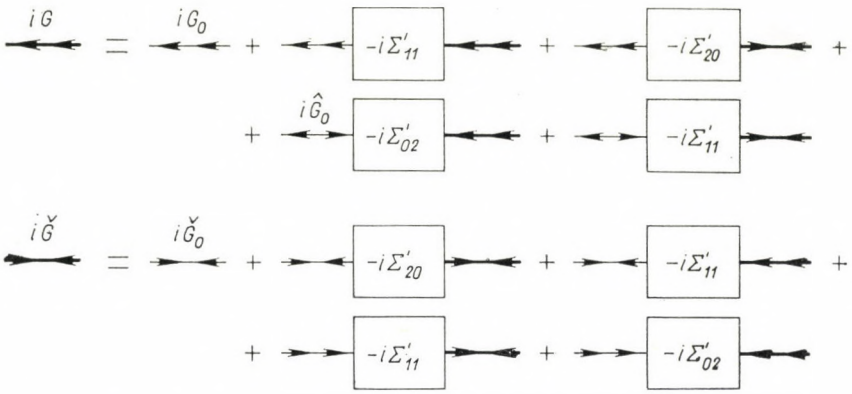
Here the $\tilde{\Sigma}$'s are the sums of those skeleton Σ graphs, which do not contain parts connected by only two propagators; $J^{++}_{+}(\mathbf{p}, p_0; \mathbf{k}, k_0)$ and $J^{--}_{+}(\mathbf{p}p_0; \mathbf{k}, k_0)$ are irreducible three-point kernels with one ingoing line $(\mathbf{k}, k_0)$, two outgoing lines $(\mathbf{k} \pm \mathbf{p}, k_0 \pm p_0)$ and three ingoing lines $(\mathbf{k}, k_0; -\mathbf{k} \pm \mathbf{p}, -k_0 \pm p_0)$, respectively. The left hand side of this equation is equal to μ according to our equation (16), the right hand side is equal to $\Sigma_{11}(0,0) - \Sigma_{02}(0,0)$ according to eq. (3,13) of GAVORET and NOZIÈRES's paper. It is important that the latter equation makes use of the expression of Σ_{02} which is not a result of comparing skeleton graphs of given orders, but is the solution of GAVORET and NOZIÈRES's eq. (B.8).

incomplete vertices can be obtained by substituting zero for the momenta of the missing lines and attaching the factor $\sqrt{N_0}$ to each of the open circles e.g.



Due to condition (14) graphs containing graph-parts as in Fig. 2 are not to be taken into account. The momentum conserving δ-functions are included in the contributions of the vertices. The integrations $\frac{1}{2\pi}\int \mathrm{d}k'_0 \ldots$ over the energies are to be performed after ensuring energy conservation at each vertex.

By analysing, in the usual way, the structure of the diagrams we arrive at the Dyson equations as follows



Here Σ'_{11}, Σ'_{02} and Σ'_{20} are the sum of all the irreducible self-energy parts having one ingoing and one outgoing line, two ingoing lines and two outgoing lines, respectively. Let us separate out from them the terms of first order in A_k and B_k.

$$\Sigma'_{11}(k, k_0) = -A_k + \Sigma_{11}(k, k_0) ,$$

$$\Sigma'_{02}(k, k_0) = -B_k + \Sigma_{02}(k, k_0) ,$$

$$\Sigma'_{20}(k, k_0) = -B_k + \Sigma_{20}(k, k_0) .$$

Their substitution into the Dyson equations obtained and taking into account eqs. (7), (9), (10), (12), (13), a straightforward rearrangement leads to the equations represented in Fig. 3.

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О ПЕРТУРБАТИВНОМ ВЫЧИСЛЕНИИ ОДНОЧАСТИЧНОГО СПЕКТРА ВОЗБУЖДЕНИЯ В БОЛЬШОЙ БОЗЕ-СИСТЕМЕ

П. СЕПФАЛУШИ

Р е з ю м е

Изучается пертурбативное вычисление одночастичной функции Грина большой системы бозонов при нулевой температуре. Мы исходим из расщепления Гамильтониана в котором невозмущенная часть выбирается диагональной в квазичастичных операторах Боголюбова. Сформулируется и исследуется самосогласованная форма ряда теории возмущения, обусловленная появлением инэквивалентных представлений канонических коммутационных соотношений.

ON THE RENORMALIZATION OF THE ENERGIES OF THE ONE-PARTICLE EXCITATIONS IN A SUPERFLUID FERMI SYSTEM

By

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Summary

Fully renormalized energy gap equation is derived for an infinite system of fermions interacting through a two-body potential, at the absolute zero of temperature. After introducing an unperturbed Hamiltonian with one-particle energies to be determined by renormalization, the one quasi-particle Green's functions are calculated by perturbation theory. The self-energy parts are given in terms of generalized Feynman diagrams. The renormalization conditions lead to the equations determining the energy gap and single-particle energies. Ladder approximation to these equations is given and discussed.

Published in Nuovo Cimento, **37**, 831, 1965.

О ПЕРЕНОРМИРОВКЕ ЭНЕРГИЙ ОДНОЧАСТИЧНЫХ ВОЗБУЖДЕНИЙ В СВЕРХТЕКУЧЕЙ СИСТЕМЕ ФЕРМИОНОВ

Н. МЕНЬХАРД

Резюме

Получено полностью перенормированное уравнение энергетической щели для бесконечной системы взаимодействующих фермионов с двухчастичным потенциалом при температуре абсолютного нуля. Введется Гамильтониан квазичастиц одночастичные энергетические уровни которого определяются перенормировкой, а функции Грина квазичастицы методом итерации. Собственно-энергетические части описываются обобщенными диаграммами Фейнмана. Перенормировочные соотношения ведут к уравнениям определяющим энергетическую щель и одночастичные энергетические уровни. Обсуждается лестничное приближение к этим уравнениям.

Опубликовано в Nuovo Cimento, **37**, 831, 1965.

ON SYMPLECTIC-INVARIANT MANY-FERMION PROBLEMS

By

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A class of symplectic-invariant many-fermion Hamiltonians is investigated. A special model Hamiltonian is found for which a complete set of "generalized BCS eigenstates" (i.e. eigenstates with definite values of the quasi-particle number) can be constructed.

A particularly instructive and simple special case of the BCS Hamiltonian has been treated by Professor THIRRING in his lecture delivered at the last Balaton Conference on Theoretical Physics (Balatonföldvár, 1961), and in his paper written in collaboration with BAUMANN, EDER and SEXL [1]. This is what has been called by workers in superconductivity "the strong coupling Hamiltonian" (see e.g. [2]), while nuclear theoreticians often refer to it as "the degenerate model" [3]. This model is exactly soluble, and possesses a non-degenerate ground state. Besides this, as shown in [1], the exact ground state becomes degenerate in the limit $\Omega \to \infty$, where we have denoted the number of single particle states by $2\,\Omega$.

Naturally, it is gratifying that in an exactly soluble special case an approximation method immediately yields the exact result. With the above-mentioned "degenerate model" and the BCS approximation this is only the case in the limit $\Omega \to \infty$. It might be desirable to consider a modification of this model which possesses a degenerate exact ground state, and in the case of which the exact and approximate results coincide for all values of Ω. (In applications in nuclear physics the number of single particle states involved is not very large.) Although in the modified model to be discussed the "ordinary" BCS states are still in general not exact eigenstates, nevertheless the concept of a BCS state can be generalized in such a way that among these more general BCS states one can find a complete set of exact eigenstates of the problem.

Consider the simple pairing Hamiltonian of the degenerate model

$$H = -gA^{+}A\,, \tag{1}$$

where

$$\left.\begin{aligned} A &= \sum_{K>0} a_{-K}\, a_K \equiv \frac{1}{2} \sum_{K\gtrless 0} \varepsilon_K\, a_{-K}\, a_K\,, \\ A^+ &= \frac{1}{2} \sum_{K\gtrless 0} \varepsilon_K\, a_K^+\, a_{-K}^+\,, \end{aligned}\right\} \tag{2}$$

($\varepsilon_K = K/|\,K\,|$). In this model the BCS ground state is a superposition of exact eigenstates corresponding to the particle numbers 0, 2, 4, . . . The BCS method does not yield an exact ground state energy because the eigenvalue depends (and not only linearly) on the particle number. In order to eliminate effects of fluctuation of the number of particles, LIPKIN [4] has suggested the introduction of a modified Hamiltonian

$$H' = H - f(N)\,, \tag{3}$$

with $f(N)$ chosen in such a way that H' should possess a spectrum degenerate in the particle number. NOGAMI [5], on the other hand, has found that for the degenerate model

$$f(N) = \lambda_1 N + \lambda_2 N^2\,, \tag{4}$$

with

$$\left.\begin{aligned} &\lambda_1 = \frac{1}{2}\, g(\Omega + 1)\,, \qquad \lambda_2 = \frac{1}{4}\, g\,, \\ &N = \sum_{K\lesssim 0} a_K^+\, a_K\,, \end{aligned}\right\} \tag{5}$$

has the desired property.

In the following we present a general discussion of symplectic-invariant many-fermion Hamiltonians. Consider many-body problems of identical spin-one-half particles. The space of single particle states is assumed to be finite in dimension. Time reversal invariance requires the number of dimensions to be even: $2\,\Omega$ (Ω = integer). If one transforms the single particle creation operators a_K^+ ($K = \Omega, \Omega - 1, \ldots, -\Omega$) according to the unitary transformations of the $U_{2\Omega}$ group, the n-particle states $a_{K_1}^+\, a_{K_2}^+ \ldots a_{K_n}^+ \mid 0 <$ will be transformed among themselves according to an irreducible representation of $U_{2\Omega}$ labelled by

$$[\overset{1}{\breve{1}}\ \overset{2}{\breve{1}} \ldots \overset{n}{\breve{1}}] \tag{6}$$

as components of a fully antisymmetric tensor of rank n. Such an irreducible subspace is of course transformed into itself by any Hamiltonian conserving the number of particles. The classification of states can be made more complete by using appropriate subgroups of $U_{2\Omega}$. Our choice will be the symplectic subgroup $USp_{2\Omega}$ of $U_{2\Omega}$ consisting of those unitary transformations of the

single particle creation operators which leave invariant the antisymmetric bilinear form A^+ defined by (2). Thus, $USp_{2\Omega}$ is a symmetry group of both the pairing Hamiltonian (1), and NOGAMI's modified Hamiltonian defined under (3—5). Matrices of the irreducible representation (6), which correspond to the elements of $USp_{2\Omega}$, form in general a reducible representation of the latter group. The reduction process yields the following irreducible representations of $USp_{2\Omega}$:

$$\sum_s (\overbrace{1}^{1}\ \overbrace{1}^{2}\ldots\overbrace{1}^{s}\ \overbrace{0}^{s+1}\ldots\overbrace{0}^{\Omega}) ; \tag{7}$$

here s, the seniority, runs through the values

$$s = \begin{Bmatrix} 0, & 2, & \ldots, & n \\ 1, & 3, & \ldots, & n \end{Bmatrix} \quad \text{for } n \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix} . \tag{8}$$

The reduction process can be carried out by repeated trace operations, using the fact that A^+, a second rank tensor under $U_{2\Omega}$, becomes a scalar under $USp_{2\Omega}$. The typical term in (7) represents a traceless antisymmetric tensor of rank s. It is of importance that in the decomposition (7) no irreducible representation of $USp_{2\Omega}$ occurs more than once. As a consequence, a basis in which the representation in question of $USp_{2\Omega}$, assumed to be unitary, takes a reduced form, automatically diagonalizes any Hamiltonian invariant under $USp_{2\Omega}$, which conserves the particle number. Such Hamiltonians can be constructed using Casimir operators. Consider first the group $U_{2\Omega}$ in which $USp_{2\Omega}$ is contained as a subgroup. This is not semisimple but it can be written as a direct product of two simple groups: $U_{2\Omega} = U_1 \times SU_{2\Omega}$. The Casimir operators of the two direct factors, as well as that of $USp_{2\Omega}$, are given in a convenient normalization by

$$G(U_1) = \frac{1}{4} N^2 , \qquad G(SU_{2\Omega}) = \frac{1}{4} N(N - 2\Omega) ; \tag{9}$$

$$G(USp_{2\Omega}) = \frac{1}{4} N(N - 2\Omega - 2) + A^+ A . \tag{10}$$

An alternative form of the latter makes use of the concept of the so-called quasi-spin [6—8]:

$$\left.\begin{aligned} S_1 &= \frac{1}{2}(A + A^+), \\ S_2 &= \frac{1}{2i}(A - A^+), \\ S_3 &= \frac{1}{2}(\Omega - N) . \end{aligned}\right\} \quad \mathrm{S}(S_1, S_2 S_3) ; \tag{11}$$

These components are the three independent bilinear unitary-symplectic invariants. They form a basis of a simple Lie algebra of rank 1. We have

$$G(USp_{2\Omega}) = \mathbf{S}^2 - \frac{1}{2}\,\Omega\left(\frac{1}{2}\,\Omega + 1\right). \tag{12}$$

The basis described above diagonalizes any linear combination of $G(U_1)$, $G(SU_{2\Omega})$ and $G(USp_{2\Omega})$:

$$\mathscr{H} = aG(U_1) + bG(SU_{2\Omega}) + cG(USp_{2\Omega})\,. \tag{13}$$

If $a = -g$, $b = g\,\dfrac{\Omega+1}{\Omega}$, $c = -\dfrac{g}{\Omega}$, the pairing Hamiltonian (1) is obtained. If a Hamiltonian is needed the eigenvalues of which are determined solely by the representation label s of $USp_{2\Omega}$, i.e. degenerate in n, one has to supress $G(U_1)$ and $G(SU_{2\Omega})$. What remains is a coupling constant times $G(USp_{2\Omega})$, i.e. NOGAMI's modified Hamiltonian (3—5). The form (12) shows immediately that this Hamiltonian admits a new symmetry group generated by the components of the quasi-spin. The "rotations in quasi-spin space" represented by the unitary operator $U = \exp i\alpha\,\mathbf{nS}$ are complex BOGOLIUBOV—VALATIN transformations:

$$\begin{aligned} a'_K &\equiv Ua_K\,U^{-1} = ua_K - va^{+}_{-K}\,, \\ a'_{-K} &\equiv Ua_{-K}\,U^{-1} = ua_{-K} + va^{+}_{K}\,, \end{aligned} \qquad (K > 0) \tag{14}$$

with $u = \cos\dfrac{1}{2}\,\alpha - \mathrm{in}_3 \sin\dfrac{1}{2}\,\alpha$, $v = i(n_1 - \mathrm{in}_2)\sin\dfrac{1}{2}\,\alpha$. From this new symmetry property it follows that particles and quasi-particles (corresponding to any fixed $\alpha = \alpha n$) play a symmetrical rôle, as already observed by NOGAMI [5]. In particular, one can construct a complete set of "generalized BCS states" (more precisely: of states with definite values of the quasi-particle number) which diagonalizes the Hamiltonian. One has to proceed exactly as sketched above, only the word "quasi particle" must be substituted for "particle". To each eigenvalue there corresponds an irreducible representation space of the group obtained by forming the direct product of $USp_{2\Omega}$ with the group of rotations in quasi-spin space.

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О СИМПЛЕКТИЧЕСКИ-ИНВАРИАНТНЫХ МНОГОФЕРМИОННЫХ ПРОБЛЕМАХ

Г. ДЬЁРДЬИ и Й. РЕВАИ

Р е з ю м е

Исследован класс симплектически — инвариантных многофермионных гамильтонианов. Найден специальный модельный гамильтониан, для которого можно построить полный набор «обобщенных собственных состояний BCS» (т. е. собственных состояний с определенными значениями числа квазичастиц).

A SUPERCONDUCTIVE MODEL WITH TWO KINETIC ENERGIES FOR EVEN AND ODD SYSTEMS

By

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The reader is referred to [1] where the greater part of this lecture has appeared. Here the determination of the energy of an odd system is added.

The energy of an odd system

The determination of the energy of an odd system is somewhat difficult with the help of the exact wave function method, but it is very simple with the quasi-spin formalism. The energy of an odd system is (for the notations see [1])

$$H_0 = \bar{\varepsilon}(\Omega - 2S_3) - V[\bar{S}^2 - S_3(S_3 + 1)], \tag{1}$$

$$H_1 = -(\varepsilon_2 - \varepsilon_1)\,[(S_2)_3 - (S_1)_3]\,. \tag{2}$$

Let us take into account that now

$$j = \Omega/2 - 1/2\,, \qquad m = \frac{\Omega - N}{2}$$

and

$$m_1 = \frac{\Omega - 2}{4}\,, \qquad m_1 = \frac{\Omega}{4}\,,$$

or

$$m_2 = \frac{\Omega}{4}\,, \qquad m_2 = \frac{\Omega - 2}{4}\,. \tag{3}$$

With the help of (3) and (5) the energy in zero order turns out to be

$$E_0 = \bar{\varepsilon}N - \frac{V}{2}(N-1)\left(\Omega - \frac{N}{2} + \frac{1}{2}\right) \tag{4}$$

and with the degenerate perturbation theory the wave function in zero approximation is

$$(\psi^{j_1 j_2}_{jm})^0 = \Sigma(j_1\, m_1\, j_2\, m_2 \,|\, jm)\,\psi_{j_1 m_1}\psi_{j_2 m_2}\,. \tag{5}$$

The energy in first and second order approximation is the following

$$E_1 = -(\varepsilon_2 - \varepsilon_1)\frac{\Omega - N}{2(\Omega - 1)}, \tag{6}$$

$$E_2 = -\frac{(\varepsilon_2 - \varepsilon_1)^2}{V}\frac{\Omega}{4(\Omega - 1)^3}(N - 1)(2\Omega - N + 1). \tag{7}$$

So the total energy is

$$E = \bar{\varepsilon}E - \frac{V}{2}(N - 1)\left(\Omega - \frac{N}{2} + \frac{1}{2}\right) - (\varepsilon_2 - \varepsilon_1)\frac{(\Omega - N)}{2(\Omega - 1)} - \\ - \frac{(\varepsilon_2 - \varepsilon_1)^2}{V}\frac{\Omega}{4(\Omega - 1)^3}(N - 1)(2\Omega - N - 1). \tag{8}$$

Let us determine now the energy with the help of the corrected BCS theory. The corrected energy in the BCS approximation [2] is:

$$E' = E_{BCS} - \left(\frac{V}{4} + \frac{(\varepsilon_2 - \varepsilon_1)^2}{V}\frac{\Omega}{4(\Omega - 1)^3}\right)(\delta N)^2 = \\ = \Sigma 2\varepsilon_k v_k^2(1 - r_k^2) + \Sigma \varepsilon_k r_k^2 - V \Sigma\Sigma u_k v_k u_{k'} v_{k'}(1 - r_k^2 - r_{k'}^2) + \\ + V \Sigma\Sigma v_k^2 v_{k'}^2 r_k^2 r_{k'}^2 - V \Sigma v_k^2 - \frac{(\varepsilon_2 - \varepsilon_1)^2}{V} \cdot \frac{\Omega}{4(\Omega - 1)^3}(\delta N)^2, \tag{9}$$

where $r_\varkappa$ is the probability that the state K is occupied by an odd particle and

$$r_1^2 + r_2^2 = \frac{2}{\Omega}, \\ \Sigma 2v_k^2(1 - r_k^2) = n - 1. \tag{10}$$

Minimalising the energy as function of v_1^2, v_2^2, r_1^2 and r_2^2 and taking into account (10) we get

$$r_1^2 = \frac{2}{\Omega}, \qquad r_2^2 = 0,$$

$$v_1^2 = \frac{n - 1}{2(\Omega - 1)} + \frac{\Omega}{\Omega - 1}a,$$

$$v_2^2 = \frac{n - 1}{2(\Omega - 1)} - a,$$

where

$$a = \frac{\varepsilon_2 - \varepsilon_1}{V} \frac{(\Omega - 2)}{2(\Omega - 1)^2} (n - 1)\left[1 - \frac{n - 1}{2(\Omega - 1)}\right],$$

and the energy becomes

$$E = \bar{\varepsilon} N - (\varepsilon_2 - \varepsilon_1) \frac{\Omega - N}{2(\Omega - 1)} - \frac{(\varepsilon_2 - \varepsilon_1)^2}{V} \frac{\Omega}{4(\Omega - 1)^3} (N-1)(2\Omega - N - 1), \tag{11}$$

which is just equivalent to (8).

With the help of this model it is easy to see that the correction to the BCS energy proves to be very useful and that we may get more correct results with its help.

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СВЕРХПРОВОДЯЩАЯ МОДЕЛЬ С ДВУМЯ КИНЕТИЧЕСКИМИ ЭНЕРГИЯМИ ДЛЯ ЧЕТНЫХ И НЕЧЁТНЫХ СИСТЕМ

Ю. НЭМЕТ

Р е з ю м е

Читателю предлагается просмотреть статью [1], где напечатана главная часть данной лекции. В данной работе лекция дополняется определением энергии одной нечетной системы.

GAUGE TRANSFORMATIONS AND INEQUIVALENT REPRESENTATIONS*

By

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The connection between gauge transformation and inequivalent representations is investigated in a self-consistent way.

It is well known now that in the case of systems of infinite volume the BOGOLIUBOV transformations of the type

$$a_k \to \alpha_k = \cos\theta_k\, a_k + \sin\theta_k\, a^+_{-k} \tag{1}$$

lead to inequivalent representations and that the FOCK representation for the physical particles (or the quasi particles) is singled out of all the representations by the condition that the Hamiltonian takes the form of the free Hamiltonian:

$$H = \int d^3k\, E_k\, a^+_k\, a_k + c\text{-number}\,. \tag{2}$$

In electrodynamics the field equations are invariant under the gauge transformations:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\,\chi(x)\,, \tag{3}$$

$$\psi(x) \to \psi'(x) = e^{ie\chi(x)}\,\psi(x)\,. \tag{4}$$

This gauge invariance leads to the current conservation law

$$\partial_\mu j_\mu = 0\,. \tag{5}$$

The transformation (4) can formally be written as

$$\psi(x) \to \psi'(x) = U\psi(x)\, U^{-1}\,, \tag{6}$$

where

$$U = \exp\,[i \int d^3x\; \varrho(x)\,\chi(x)]\,, \tag{7}$$

* This is a report of a work done in collaboration with H. UMEZAWA, Istituto di Fisica Teorica dell'Universitá di Napoli, and J. P. VIGIER, of the Institut Henri Poincaré (Paris, France).

$\varrho(x)$ being the charge density. The gauge invariance means that the vector potential $\vec{A}$ in the field equation is changed into $(\vec{A} + \vec{\nabla}\chi(x))$ by means of the transformation (6). Suppose a representation where the Hamiltonian takes the form of the free Hamiltonian (2). Then, in the new representation obtained by the transformation (6) the Hamiltonian is no more of the form (2). We thus want to clarify the meaning of the gauge transformation in the self-consistent method.

1. Inequivalent representations

We shall first consider a non-relativistic electron gas in a box of volume V. Then $\varrho(x) = \psi^+(x)\,\psi(x)$ and therefore

$$\int d^3x\, \varrho(x)\,\chi(x) = \frac{(2\pi)^3}{V} \sum_{p,q} a_p^+\, a_{p+q}\,\chi(q)\,, \tag{8}$$

where the commutation relations are

$$[a_p\,, a_q^+] = \frac{V}{(2\pi)^3}\,\delta_{pq}\,.$$

Let us now see how the creation and destruction operators transform under the transformation (4)*.

$$\alpha_k = U^{-1}\,a_k\,U =$$
$$= a_k + i\,[a_k, X] + \frac{i^2}{2!}\,[X, [X, a_k]] + \ldots\,,$$

where $X = e\,\Sigma a_p^+\, a_{p+q}\,\chi(g)$, we get

$$\alpha_k = a_k + ie \sum_q a_{\vec{k}+\vec{q}}\chi(q) - \frac{e^2}{2} \sum_{q,l} a_{\vec{k}+\vec{q}+\vec{l}}\,\chi(q)\,\chi(l) + \ldots\,.$$

If we examine the second term of this expression we see that a certain number of $\vec{k} + \vec{q}$ will be above the FERMI surface and a certain number under it.

We can thus rewrite α_k

$$\alpha_k = a_k + ie \sum_{|\vec{k}+\vec{q}|>k_F} a_{\vec{k}+\vec{q}}\,\chi(q) + ie \sum_{|\vec{k}+\vec{q}|<k_F} b^+_{-\vec{k}-\vec{q}}\,\chi(q) + \ldots\,, \tag{9}$$

* We might regard the transformation (4) as

$$a_k \to e^{ie\chi(x)}\,a_k\,.$$

We have no interest in this transformation because this transformation has nothing to do with the transformation of the FOCK representation.

where we have made use of the well known fact that

$$\text{for } |\vec{k}| < k_F \qquad a_k = b^+_{-k}\,, \tag{10}$$

where b^+_{-k} is the creation operator of the corresponding hole in the FERMI sea.

Thus we see that a_k is now a mixture of creation and annihilation operators and such transformation is known to lead to inequivalent representations in the limit $V \to \infty$.

This also means that

$$\lim_{V\to\infty} \langle 0|U|0\rangle = 0\,, \tag{11}$$

where $|\,0>$ is the ground state in the $\{a\}$-representation:

$$a_k\,|\,0> = 0\,.$$

In derivating (11) we have taken into account the fact that $U^+\,|\,0>$ is the ground state in the $\{a\}$-representation and that the two ground states in $\{a\}$-representation and $\{a\}$-representation are orthogonal. Thus, the operator X is no more well defined in the $\{a\}$ representation and the ground state $|\,0>$ looses the property of eigenstate of X, while the ground state in the $\{a\}$-representation is an eigenstate of X.

It may be worth noting that the calculation with different choice of the gauge cannot violate the electric current conservation law. This is because, by changing the gauge we are solving the electron equation in different representations and the conservation law (5) is a conclusion straightly derived from the electron equation.

2. Meissner current

It is interesting to apply this result to the case of the MEISSNER current, and see if the conclusions are consistent with the results obtained by several other authors.

Let us begin with the H_{red} used in the BCS theory without electromagnetic field. Then the transformation (6) changes $\vec{\nabla}\,\psi$ into $(\vec{\nabla} - \vec{\nabla}\chi)\psi$. However, the BOGOLIUBOV transformation (1) can transform the Hamiltonian into the free Hamiltonian only after all the gauge field χ is eliminated. In this consideration we have two kinds of parameters $(\chi(q),\theta_k)$ which specify the representation. By choosing $\chi = 0$ and θ_k being specified by the famous gap equation we obtain the Hamiltonian of the form (2). This shows the importance of the parameter χ.

Let us now consider the superconductive current in a magnetic field. When we specify θ_k in (1) by the gap equation and calculate the second order induced current, we obtain

$$\vec{j}(k) = \frac{ne^2}{m} \left(\vec{A}(k) + k\chi(\vec{k})\right) \quad \text{for} \quad k \approx 0 \,. \tag{12}$$

Here the fact is taken into account that the vector potential is $(\vec{A} + \vec{\nabla}\chi)$ when the gauge is shifted by χ. The relation (12) shows that the ground state electromagnetic energy is

$$W_0 = \frac{1}{2} \frac{ne^2}{m} \left(\vec{A}(k) + \chi(\vec{k})\, k\right) \cdot \left(\vec{A}(k) + \chi(k)\, \vec{k}\right) . \tag{13}$$

It is very hard to see which choice of χ leads us to the free Hamiltonian. However, we know that in the lowest order approximation, the result of the self-consistent method agrees with that given by the variation principle, applied to W_0. The variation principle in our case reads as follows:

$$\frac{\partial W_0}{\partial \theta_k} = 0 \,, \tag{14}$$

$$\frac{\partial W_0}{\partial \chi(k)} = 0 \,. \tag{15}$$

It is well known that the first equation leads to the gap equation. There is a slight doubt about the applicability of (15), because here we are concerned with the second order approximation. However, we shall ask what could be the answer if (15) is correct. The relation (15) together with (13) leads to

$$\chi(k) = -\frac{1}{|k|^2} \left(\vec{k} \cdot \vec{A}(k)\right) , \tag{16}$$

which leads to

$$\vec{A}(k) + \vec{k}\,\chi\,(k) = \vec{A}^T(k) \,,$$

where $\vec{A}^T$ means the transverse component of the vector potential A. It is now clear that

$$\operatorname{div} \vec{j} = 0 \,.$$

In other words the current conservation law holds as it should be.

This result does not contradict the arguments which have been presented by other authors. It has been shown that in order to recover the gauge invariance of the theory, we have to take into account the higher order effects due to the non-Cooper pairs interactions (i.e. pairs having a total momentum $q \neq 0$): the latter effects cancel the longitudinal component of the right hand side in (12). To take into account the higher order effects is equivalent to modifying the representation from the beginning. What we have shown is that the self-consistent method seems to single out the representation where the higher order effect of the above-mentioned kind is unnecessary. It is needless to add that A^T is gauge invariant by its definition.

In the relativistic theory, the charge density $\varrho(q)$ is not as simple as $\Sigma a^+_{k+q} a_k$ but contains terms like $\Sigma a^+_{k+q} b^+_{-k}$, where b^+ denote the creation operators of the antiparticles. Due to these terms the gauge transformation (6) again leads to inequivalent representation in the limit $V \to \infty$. In the gauge invariant vector meson theory (13) may correspond to the mass term of vector meson. On the other hand, it is well-known that the massive vector field u_μ has the four-dimensional transverse gauge:

$$\partial_\mu u_\mu = 0 .$$

This suggests that the self-consistent method applied to the gauge invariant vector meson theory might lead to a non zero physical mass of the vector meson.*

КАЛИБРОВОЧНЫЕ ПРЕОБРАЗОВАНИЯ И НЕЭКВИВАЛЕНТНЫЕ ПРЕДСТАВЛЕНИЯ

Л. ЛЕПЛЭ

Р е з ю м е

Самосогласованным образом исследована связь между калибровочным преобразованием и неэквивалентными представлениями.

* An attempt to derive a non zero physical mass in the gauge invariant vector meson by making use of the self-consistent method was presented by S. KAMEFUCHI and H. UMEZAWA: Nuovo Cimento, **32**, 448, 1964.

RELATIVISTIC FIELD THEORIES WITH SYMMETRY BREAKING SOLUTIONS

By

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Following some recent developments in many-body theory, we suggest a functional approach to relativistic field theories particularly suitable to treat the case of symmetry breaking solutions. The method is characterized by the introduction of a functional of the vacuum expectation value of the field variables which has the property of being stationary around the values produced by the solutions of the theory. To illustrate the formalism we discuss the GOLDSTONE theorem, whose formal derivation becomes very simple in this language. Finally, we discuss a stability or degeneracy property of the solutions of a field theory expressed by the persistence of the stationary conditions if one performs certain infinitesimal variations of the vacuum expectation values of the field. When one has symmetry breaking solutions, the admissible variations correspond to the superposition of zero mass classical waves of infinitesimal amplitude.

The possibility that a relativistic field theory defined by a Lagrangian invariant under a certain transformation group might possess unsymmetrical solutions has been widely discussed in recent years[1]. It is generally believed that, if this is the case, the "anomalous" solutions must be degenerate in some sense and, furthermore, if the symmetry group of the Lagrangian is continuous, massless particles of dynamical origin should appear[2]. The latter statement is actually only a possible interpretation of a result which follows quite generally once the existence of the unsymmetrical solution has been assumed. Stated in its most general form, this result indicates that a Green's function suitably defined and which has the character of a boson propagator must be singular at $p = 0$ when transformed into momentum space. This is the content of the so-called GOLDSTONE theorem and there are indeed many examples of field theoretic models in which the singular behaviour described above is associated with massless particles. This happens at least with the approximate solutions that one is able to construct. However, the general validity of this rule has been questioned[3]. One is forced to recognize that, in spite of the progress made by various authors, the structure of asymmetrical solutions has not been yet satisfactorily clarified. This is not very surprising considering the general status of relativistic field theories. However, one feels that an additional difficulty is caused by the lack of a sufficiently general and flexible formulation of the problem. In fact, all the results obtained so far are largely model dependent. Substantial progress, on the other hand, has been made towards a much better treatment of this type of questions in the non-relativistic many-body

problem. We refer specifically to the work of MARTIN, HOHENBERG and DE DOMINICIS [4]. These authors have been concerned with the construction of a suitable formalism allowing, in principle, a unified treatment in terms of Green's functions of both normal and asymmetrical problems, e.g., liquid ^{4}He above and below the critical temperature, superconductors, etc. While nobody expects that the classical difficulties of field theories might be overcome by simply using a different language, one may still hope to obtain some insight into the problems by giving them the most appropriate formulation. The new insight can take, for example, the form of general theorems that would be difficult to state in a different language.

The work of the above-mentioned authors is in the line of classical and quantum statistical thermodynamics and makes an extensive use of the functional calculus which allows to reduce the relevant problems to variational problems. In this note, which is mainly pedagogical in character, we shall make a preliminary attempt to extend these techniques to relativistic field theories with the aim of discussing from a more general point of view and possibly generalizing the results so far obtained in the study of relativistic field theories with symmetry breaking solutions. However, as it will be realized soon, the techniques proposed appear to have a rather general applicability.

The starting point of our discussion will be the analogy known since a long time between the partition function in statistical mechanics and the vacuum expectation value of the S matrix in relativistic field theories[5]. The essential point is that they both allow to construct functionals which generate the time ordered Green's functions. In the first part of the paper, we shall reduce to a variational problem the conditions for the existence of solutions in a relativistic field theory possessing a certain basic invariance property. To this end, an appropriate functional of the vacuum expectation values of the field variables is introduced. An invariance property of this functional which is easily established makes it possible to prove and give a precise meaning to the degeneracy of the anomalous solutions.

A very simple formal derivation of the GOLDSTONE theorem will be then presented. It can be compared with the proof of the HUGENHOLTZ—PINES theorem given by HOHENBERG[6] and is even more direct.

Finally, we shall discuss a result which comes out most clearly in this type of formalism. This is a stability property common to both the normal and the symmetry breaking solutions. This property is exhibited by the invariance of the stationary conditions of the system under infinitesimal transformations which are essentially gauge transformations of the second kind. This invariance property appears to be a characteristic feature of relativistic theories and is strictly connected with the singularities of the propagator.

I. In order to make our discussion definite, we shall consider the model theory already studied by BLUDMAN and KLEIN[2] and defined by a set of

N Hermitian boson fields Φ_i transforming according to the fundamental representation of the orthogonal group and satisfying equations of motion of the form

$$(\Box - \mu^2)\Phi_i = J_i \,. \tag{1}$$

Let us add to the corresponding Lagrangian $\mathscr{L}(x)$ which is invariant under the group, a symmetry breaking term of the form $\mathscr{L}'(x) = \sum_i \lambda_i(x)\Phi_i(x)$, where the λ_i describe external sources, and construct the vacuum functional

$$S_0(\lambda) = \langle 0 \mid T \exp i \int d^4x \,(\mathscr{L}_I(x) + \mathscr{L}'(x)) \mid 0\rangle \,, \tag{2}$$

where $\mid 0>$ is the bare vacuum. We define next the functional $Z(\lambda)$ generating the connected parts of the time ordered functions

$$Z(\lambda) = -i \ln S_0(\lambda) \,. \tag{3}$$

We have in particular

$$\frac{\delta Z}{\delta \lambda_j(x)} = \langle \Phi_i(x)\rangle = \varphi_i(x) \,,$$

$$\frac{\delta^2 Z}{\delta\lambda_i(x)\,\delta\lambda_j(y)} = i\big[\langle T(\Phi_i(x)\,\Phi_j(y))\rangle - \varphi_i(x)\,\varphi_j(y)\big] = \Delta_{ij}(x,y) \,. \tag{4}$$

The "normal" solution of the theory is the one for which

$$\lim_{(\lambda_i)\to 0} \frac{\delta Z}{\delta\lambda_j} = 0 \qquad j = 1, 2, \ldots N \,,$$

while for the asymmetrical solution this limit is different from zero[7]. It is now convenient to perform a Legendre transformation and introduce the functional of the natural variables φ_i

$$W(\varphi) = Z - \sum_i \int d^4x\, \lambda_i(x)\,\varphi_i(x) \,. \tag{5}$$

The relevant property of this functional is that it is stationary when $(\lambda_i) \to 0$. We have in fact

$$\frac{\delta W}{\delta\varphi_i(x)} = -\lambda_i(x) \,. \tag{6}$$

To clarify further the structure of W let us take the functional derivative of the first of the equations (4) with respect to φ'_i

$$\delta_{ij}\,\delta(x-y) = \frac{\delta^2 Z}{\delta\varphi_i(x)\,\delta\lambda_i(y)} = \sum_K \int d^4\xi\, \frac{\delta^2 Z}{\delta\lambda_j(y)\,\delta\lambda_K(\xi)}\,\frac{\delta\lambda_K(\xi)}{\delta\varphi_i(x)} = \tag{7}$$

$$= -\sum_K \int d^4\xi\, \Delta_{jK}(\gamma,\xi)\,\frac{\delta^2 W}{\delta\varphi_K(\xi)\,\delta\varphi_i(x)} \,.$$

From (7) we learn that the second functional derivatives of W generate the inverse propagators. We can now proceed further by taking the functional derivative of (7) and find after some manipulations

$$\frac{\delta^3 W}{\delta\varphi_i(x)\,\delta\varphi_j(y)\,\delta\varphi_K(z)} =$$

$$= -\sum_{l,m,n}\iiint d^4\xi\, d^4\eta\, d^4\varrho\, \frac{\delta^2 W}{\delta\varphi_i(x)\,\delta\varphi_l(\xi)}\;\frac{\delta^3 Z}{\delta\lambda_l(\xi)\,\delta\lambda_m(\eta)\,\delta\lambda_n(\varrho)}\times$$

$$\times\frac{\delta^2 W}{\delta\varphi_m(\eta)\,\delta\varphi_j(y)}\;\frac{\delta^2 W}{\delta\varphi_n(\varrho)\,\delta\varphi_K(z)}\,. \tag{8}$$

This is the ordinary proper Γ vertex. From (8) all the higher derivatives can now be obtained explicitly in terms of time ordered functions by successive functional differentiation. The n-th order derivative gives the connected proper part of the n-point Green's function.

We want now to prove an invariance property of the W functional that will be crucial for the following. Let U be the unitary operator which induces on the fields the transformations of the invariance group according to the rule

$$U\Phi_i(x)\,U^{-1} = \sum_j T_{ij}\,\Phi_j(x)\,, \tag{9}$$

where T is an $N \times N$ orthogonal matrix. It is now very easy to verify, using (3), (4), (9) and the assumed invariance property of the Lagrangian, that the transformation

$$\lambda_i \to \lambda_i' = \sum_j \lambda_j\, T_{ji} \tag{10}$$

is such that

$$Z(\lambda') = Z(\lambda)\,, \tag{11a}$$

$$\varphi_i(x,\lambda') = \sum_j T^{-1}_{ij}\,\varphi_j(x,\lambda)\,. \tag{11b}$$

From (11b) it follows that the substitution

$$\varphi_i \to \sum_j T_{ij}\,\varphi_j \tag{12}$$

induces the transformation

$$\lambda_i \to \sum_j \lambda_j\, T^{-1}_{ij} \tag{13}$$

on the source function considered as a functional of φ.

W is then invariant under the transformation (12). Furthermore, *the unsymmetrical solution, if it exists, is determined up to the transformation* (12). *It is, therefore, degenerate.*

II. We are now in a position to supply a very simple derivation of the GOLDSTONE theorem.

Let us consider an infinitesimal transformation of the group $T_{ij} = \delta_{ij} + t_{ij}$. From (6), (12) and (13) we have

$$\delta\lambda_i(x) = \sum_j \int d^4\xi \frac{\delta\lambda_i(x)}{\delta\varphi_j(\xi)} \delta\varphi_j(\xi) = -\sum_{j,K} \int \frac{\delta^2 W}{\delta\varphi_i(x)\,\delta\varphi_j(\xi)} t_{iK}\varphi_K(\xi)\, d^4\xi\,. \quad (14)$$

We now take the limit $(\lambda_i) \to 0$ and remember that because of translation invariance $\varphi_i(x,\lambda) \to \varphi_i = \text{constant}$[8]. This yields

$$\sum_{j,K} \Delta_{ij}^{-1}(p=0)\, t_{jK}\varphi_K = 0\,. \quad (15)$$

In order that the matrix $\Delta^{-1}(0)$ may have a non zero eigenvector corresponding to a zero eigenvalue the following equation must hold

$$\text{Det}\,(\Delta^{-1}(0)) = 0\,. \quad (16)$$

In the particular case considered here all the φ_is but one (say φ_1) can be set equal to zero and we find for all $j \neq 1 \Delta_{jj}(o) = 0$. However (16) represents the general form of the result.*

We notice that a somewhat weaker result already follows from the invariance property of W which gives

$$\delta W + \delta^2 W + \ldots = \sum_i \int d^4\xi \frac{\delta W}{\delta\varphi_i(\xi)} \delta\varphi_i(\xi) + \\ + \frac{1}{2} \sum_{ij} \iint d^4\xi\, d^4\eta \frac{\delta^2 W}{\delta\varphi_i(\xi)\,\delta\varphi_j(\eta)} \delta\varphi_i(\xi)\,\delta\varphi_j(\eta) + \ldots = 0\,, \quad (17)$$

when $\delta\varphi_i$ corresponds to a transformation of the form (12).

III. In the previous Section we have seen that the eigenvalue problem

$$\sum_j \int d^4\,\xi \frac{\delta^2 W}{\delta\varphi_i(x)\,\delta\varphi_j(\xi)} \delta\varphi_j(\xi) = 0 \quad (18)$$

* Some comments are here in order. The derivation given above depends on an implicit assumption which has been made in the last step. In fact we have freely interchanged the limit $(\lambda_i) \to 0$ with the four-dimensional space-time integration. This step may not be allowed because of singularities arising in the limit $(\lambda_i) \to 0$. We have here a possible reason limiting the validity of the theorem. The difficulty we have mentioned was first pointed out to the author by Prof. E. S. FRADKIN, whose stimulating criticism is gratefully acknowledged.

in the limit $(\lambda)_i \to 0$ may have the solution

$$\delta\varphi_i = \sum_j t_{ij}\varphi_j .$$

We want to investigate now whether there exist other possible eigenvectors of the system (18). It is clear that if we shall be able to find such variations that satisfy (18), the equations $\delta W/\delta\varphi_i = 0$ will hold for the displaced vacuum expectation values $\varphi_i + \delta\varphi_i(x)$. We now admit that the matrix $\Delta(p)$ singular at $p = 0$, is singular also for every other $p^2 = 0$ as it depends on p only through the invariant p^2.

This being the case, a non-trivial eigenvector of (18) will be

$$\delta\varphi_i = \sum_j t_{ij}\varphi_i f(x) , \tag{19}$$

where $f(x)$ is of the form $\int d^4p\tilde{f}(p)\,\delta(p^2)\,e^{ipx}$.

A similar stability property of the solution holds also in the normal case with

$$\delta\varphi_i = f_i(x) \quad (\Box - \mu^2)f_i(x) = 0 ,$$

where μ is the renormalized mass. We can perhaps get a physical picture of this phenomenon already in the free field case. We consider a single a field obeying $(\Box - \mu^2)\Phi = 0$ and the generator F of gauge transformations such that $\Phi \to \Phi + f(x)$ with $(\Box - \mu^2)f = 0$. If we consider the modified vacuum

$$|\,0'\rangle \simeq (1 + iF)\,|\,0\rangle ,$$

we have $\langle 0' \mid \Phi(x) \mid 0'\rangle = f(x)$ but there is no first order change in the vacuum expectation value of the Hamiltonian H_0 i.e., $\langle 0' \mid H_0 \mid 0'\rangle \simeq 0$. The same happens for the expectation value in many particle states as well as for the other observables of the theory.

From the above analysis we conclude that oscillations of the vacuum corresponding to zero mass particles can be associated with symmetry breaking solutions. At this point, however, it may be objected that we have not considered the possibility of "spurious" states such as those mentioned in [3]. As far as we can understand this question, it seems to us that in the pure Green's function approach we have been describing spurious states would manifest themselves with the existence of several solutions for our variational problem. In this case stability conditions should play a role in discussing their meaning[9]. It is clear that a decision on this problem requires a much more elaborate analysis that goes beyond the general ideas considered here.

In conclusion we have presented an alternative formal approach to the problem of symmetry breaking solutions in relativistic field theories and previously known results have been obtained in a simpler way. A number of questions are open. Among others, it is of prominent importance to investigate the nature of the stationary points of the W functional [10]. Furthermore the invariance properties of W should be exploited more systematically to obtain relationships among higher order Green's functions. Also functional equations in which some of the Green's functions play the role of independent variables seem to offer interesting possibilities.

We would like to thank Professor N. CABIBBO for a very useful conversation.

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10. It can actually be shown that the stationary points of W correspond to extrema provided the class of variations is suitably chosen.

РЕЛЯТИВИСТСКИЕ ТЕОРИИ ПОЛЯ С РЕШЕНИЯМИ, НАРУШАЮЩИМИ СИММЕТРИИ

Г. ИОНА—ЛАЗИНИО

Р е з ю м е

Следуя некоторым новейшим достижениям теории многих тел, предлагается функциональный метод для релятивистских теорий поля, особенно подходящий для исследования решений, нарушающих симметрии. Метод характеризуется введением функционала ожидаемых вакуумных значений переменных поля, который обладает свойством стационарности около значений, соответствующих решениям теории. Для иллюстрации формализма обсуждается теорема Голдстоуна, формальное выведение которой становится весьма простым в этом формализме. Наконец, обсуждается свойство стабильности или вырождения решений теории поля, выраженное через устойчивость условий стационарности в случае проведения некоторых инфинитезимальных вариаций ожидаемых вакуумных значений поля. Если имеются решения, нарушающие симметрии, разрешенные вариации соответствуют суперпозиции классических волн нулевой массы с инфинитезимальной амплитудой.

SUM RULES IN THE QUANTUM FIELD THEORY AND IN THE MANY-BODY PROBLEM

By

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In case of interacting fermion systems integral restrictions are derived for the spectral functions of the simplest Green's functions. Some applications are shown.

I

The first sum rules were derived about thirty years ago in the theory of atomic spectra and since then they appear in different topics of physics. The use of the sum rules is that in many cases when the entire physical problem cannot be solved exactly these rules give valuable information about the physical systems because one can deduce them without approximations. We will show in this lecture how sum rules can be derived for interacting fermion fields and how one can use them. We shall treat interacting fields in the frame of nonlinear, self-coupled field theory and superconducting many-fermion system.

II

The first sum rules in the quantum theory of elementary particles were treated by LEHMANN just ten years ago giving integral relations for the spectral functions of the propagators of the fermion-pseudoscalar meson coupling. In the following we shall show that a similar integral relation is valid also for the selfcoupled fermion fields [1].

Let us start with a general field equation

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = \sum_j A^j(x)\, O_j\, \psi(x)\,,$$

where

$$A^j(x) = g_j\, \bar{\psi}(x) O^j \psi(x)\ ;$$

O_j denotes the matrices of the scalar, vector, tensor and the different pseudo-couplings. To avoid divergencies we shall use cut-off. Now, one can write a

similar equation also for the vacuum expectation value containing the anti-commutators of the field operators

$$J(x, y) \equiv (i\gamma^\mu \partial_\mu - m) \langle \{\psi(x), \overline{\Psi}(y)\} \rangle_0 = \sum_j \langle \{A^j(x) O_j \psi(x), \overline{\Psi}(y)\} \rangle_0 .$$

For the different two-point functions we shall use LEHMANN's spectral representation

$$S^{()\prime}(x) = \int_0^\infty d\mu^2 \left[\varrho_1(\mu^2) S^{()}(x, \mu) + \varrho_2(\mu^2) \Delta^{()}(x, \mu^2)\right].$$

The spectral function $\varrho_1(\mu^2)$ satisfies the condition

$$\int_0^\infty \varrho_1(\mu^2) d\mu^2 = 1 ,$$

which is derived by using the equal-time commutation rules. For $x_0 = y_0$, the function $J(x, y)$ can be calculated by making use of the commutation rules and spectral representation

$$J(x, y) = - G\gamma_0 \delta(x - y) \int_0^\infty d\mu^2 (\varrho_2 - \mu\varrho_1) \Delta_F(0, \mu^2) .$$

Here we have used the relation

$$S_F(0, \mu) = - \mu \Delta_F(0, \mu^2)$$

supposing the application of an invariant cut-off. G denotes a combination of the coupling constants,

$$G = \frac{1}{2}(- 3g_s + 4g_v + g_p + 4g_a + 16g_t) .$$

On the other hand, $J(x, y)$ is defined by

$$J(x, y) = \gamma_0 \delta(x - y) \int_0^\infty d\mu^2 \left[\varrho_1(\mu - m) - \varrho_2\right].$$

These two expressions of J give the following integral relation for the spectral functions

$$m = \int_0^\infty d\mu^2 (\mu\varrho_1 - \varrho_2) \left(1 - \frac{G}{8\pi^2}\left[\Lambda^2 - \mu^2 \ln\left(\frac{\Lambda^2}{\mu^2} + 1\right)\right]\right).$$

Here we have substituted the function Δ_F in its cut-off form. It is interesting to remark that for suitable nonvanishing g's the extra term vanishes.

The general form of ϱ_1 with one stable particle of mass $\varkappa$ is

$$\varrho_1(\mu^2) = Z\delta(\mu^2 - \varkappa^2) + \sigma_1(\mu^2) .$$

Provided that the continuous parts of the spectral functions are negligible, we get the renormalized mass equation

$$\delta m = \varkappa - m = \frac{\varkappa G}{8\pi^2}\left[\Lambda^2 - \varkappa^2 \ln\left(\frac{\Lambda^2}{\varkappa^2} + 1\right)\right].$$

This result can also be obtained by using the functional integral method [2]. For vanishing bare mass, we can get two solutions for $\varkappa$, one of them is $\varkappa = 0$; the two different solutions, however, correspond to two different Hilbert spaces. In this case our results are the same as were obtained by NAMBU and JONA—LASINIO in a self-consistent HARTREE—FOCK type approximation [3] and by UMEZAWA et al. in the method of the V-limit. We can see that the HARTREE-FOCK type approximation is equivalent with simple pole-approximation in the spectral function. This will be our result for a superconducting system too.

III

For the sake of simplicity we shall consider the simplest model [5] although our results are valid also in the case of an arbitrary potential [6]. So we want to treat GORKOV's superconducting model. In our considerations we use states only with a definite number of particle and finite volume. The model is given by the equation of motion

$$\left(i\frac{\partial}{\partial x_0} + \frac{\nabla^2}{2m}\right)\psi(x) - g\left(\psi^+(x)\,\psi(x)\right)\psi(x) = 0\,, \qquad g < 0$$

and the equal-time commutation rules. Introduce the following simple correlation functions and the corresponding spectral functions

$$K = \langle N \mid \psi^+(x)\psi(x') \mid N\rangle \to J(p, E)\,,$$

$$K' = \langle N \mid \psi(x')\psi^+(x) \mid N\rangle \to J'(p, E)\,,$$

$$K^{(+)} = \langle N \mid \{\psi^+(x), \psi(x')\} \mid N\rangle \to J^+(p, E)\,.$$

$|\, N\rangle$ denotes the ground state with N particles. By using the commutation rules in the equation of motion of the correlation function $K^{(+)}$ in the term containing four field operators we get

$$\left[\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)K^{(+)}\right]_{t=0}=-\frac{g}{2}\delta(x-x')\varrho, \qquad t=x_0-x_0',$$

where ϱ is the particle density. Now we must take into account also the anomalous correlation functions with the corresponding spectral functions

$$\langle N-2\,|\,\psi(x)\psi(x')\,|\,N\rangle e^{2iE_0x_0}=F(x-x')\to L,$$
$$\langle N-2\,|\,\{\psi(x),\psi(x')\}\,|\,N\rangle e^{2iE_0x_0}=F^{(+)}(x-x')\to L^{(+)},$$

where E_0 denotes the half of the energy difference of the states $|\, N\rangle$ and $|\, N-2\rangle$. For the anomalous quantities it is possible to get equations of motion similar to that for $K^{(+)}$. These equations can be expressed by the spectral function. The commutation relations give simply

$$\int (J(p,E)+J'(p,E))dE=1\,,\quad \int L^{(+)}(p,E)dE=0$$

and the before-mentioned equations are

$$\int E\big(J(p,E)+J'(p,E)\big)\,dE=\frac{p^2}{2m}+\frac{g\varrho}{2},$$
$$\int EL^{(+)}(p,E)\,dE=\frac{g}{(2\pi)^3}\int L(p',E)\,d^3p'\,dE\,.$$

These four equations are exact ones and to get some results we must make use of some approximation. Consider for example the matrix elements

$$\langle N\,|\,a(p,x_0)a^+(p,x_0)\,|\,N\rangle\sim\langle N\,|\,a(p,x_0)\,|\,N+1,p\rangle$$
$$\langle N+1,p\,|\,a^+\,(p,x_0)\,|\,N\rangle\,.$$

Here we have supposed that only one state with particle number $N+1$, momentum p and energy $E_N+P_0+E_p$ contributes to the matrix element. If we introduce the usual notation

$$v_p=\langle N-1,p\,|\,a(-p,0)\,|\,N\rangle\,,$$
$$u_p=\langle N+1,p\,|\,a^+(p,0)\,|\,N\rangle\,,$$
$$\tilde{u}_p=\langle N-1,p\,|\,a^+(p,0)\,|\,N-2\rangle\,,$$

then for a large system $| u_p | = | \tilde{u}_p |$. These are the coefficients of the BOGOLJUBOV transformation but we do not apply it. The spectral function can be expressed by the coefficients

$$J(p) = v_p^2 \delta(E - E_0 + E_p),$$

$$J'(p) = u_p^2 \delta(E - E_0 - E_p),$$

$$L = \tilde{u}_p^* v_p \delta(E - E_0 + E_p)$$

and by a straightforward calculation we get the well-known results

$$E_p = (\Sigma_p^2 + |\Delta|^2)^{1/2}$$

$$1 = -\frac{g}{2(2\pi)^3} \int \frac{d^3 p}{(\Sigma_p^2 + |\Delta|^2)^{1/2}}, \qquad \Sigma_p = \frac{p^2}{2m} + \frac{g\varrho}{2},$$

which are the results of GORKOV.

We have shown that GORKOV's truncating of the Green's functions containing four field operators and the pole approximation of the exact sum rules give the same results. The ground-state energy E_n could be obtained from the one-particle correlation function in the usual simple way. Furthermore, calculation was made for the case of an arbitrary two-particle potential $V(x, x')$ [6] and the results were in agreement with the calculation of VALATIN, who used well-known canonical transformations and gave the coefficients by variational assumption [7].

We conclude that the results of the simple pole approximation are equivalent to those of the usual approximations. The reason of the agreement is that almost all the approximations give simple spectral functions with one pole and just this is our unique approximation. Naturally, an improvement of the present approximation would be very useful.

We have seen that the pole approximation of the exact sum rules, one of the simplest methods, gives good informative results. The application of the method to Bose systems is the subject of Dr. Cs. HARGITAI's lecture [8]. Finally, we should like to emphasise that the sum rules are exact and so they are applicable to the investigation of the correctness of other methods by inserting the spectral functions of other approximations into the sum rules. For example, we think that the sum rules of the Bose system can decide the appearance of the gap in the excitation spectrum of the Bose gas.

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ПРАВИЛА СУММ В КВАНТОВОЙ ТЕОРИИ ПОЛЯ И В ТЕОРИИ МНОГИХ ТЕЛ

А. ЗАВАДОВСКИЙ и Д. ПОЧИК

Резюме

В случае взаимодействующих фермионных систем выведены интегральные ограничения для спектральных функций простейших функций Грина. Показаны некоторые их применения.

A GAUGE-INVARIANT METHOD IN THE THEORY OF INTERACTING BOSE SYSTEMS

By

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The pole-approximation method based on the exact sum rules is given for interacting Bose systems. The relations derived here are the same as the ones known from the Valatin—Butler treatment.

The applied method is based on the equation of motion and on the equal-time commutation rules [1]. Some exact integral relations concerning the spectral functions of the various one-particle correlation functions are derived. To obtain some information about the ground state energy and the spectrum of low lying one-particle excitations we apply the pole-approximation. The results given by these considerations agree with results derived by Valatin and Butler [2].

1. As A. Zawadowski and G. Pócsik [1] have pointed out the sum rules combined with the pole-approximation can be successfully applied in the theory of superconducting fermion systems. It seems to be obvious that this method is, maybe after some modifications, suitable to investigate the zero-temperature properties of a many-boson system, too.

In most of the methods applied to boson systems the creation and annihilation operators of condensed particles are considered to be c-numbers. This leads to a distinction between the operators of the condensed and non-consed particles. In our treatment we are able to avoid this distinction.

Besides our treatment there is another method developed by Valatin and Butler [2] which does not use the mentioned distinction but in their treatment they use a trial ground state vector which is not an eigenvector of the particle number operator, therefore their treatment is not a gauge-invariant one.

We are going to show that using the pole-approximation just the same relations can be derived from the exact sum rules as in the Valatin—Butler treatment, but now in a gauge-invariant manner.

2. We consider here a system of spinless bosons interacting via two-body forces. The system is characterised by the second quantised Hamiltonian

$$H = -\frac{1}{2}\int d^3\vec{x}\,\psi^+(\vec{x}, x_0)\,\Delta\psi(\vec{x}, x_0) + \\ + \frac{1}{2}\int d^3\vec{x}\,d^3\vec{y}\,\psi^+(\vec{x}, x_0)\,\psi^+(\vec{y}, x_0)\,U(\vec{x}-\vec{y})\,\psi(\vec{y}, x_0)\,\psi(\vec{x}, x_0)\,, \qquad (1)$$

where $U(\vec{x}-\vec{y})$ is the two-body interaction potential, which is a function of the distance of two particles only; $\psi(\vec{x}, x_0)$ and $\psi^+(\vec{x}, x_0)$ are the Heisenberg field operators satisfying the following equal-time commutation rules

$$\begin{aligned}[\psi(\vec{x}, x_0), \psi^+(\vec{x}', x_0)] &= \delta(\vec{x}-\vec{x}'),\\ [\psi(\vec{x}, x_0), \psi(\vec{x}', x_0)] &= [\psi^+(\vec{x}, x_0), \psi^+(\vec{x}', x_0)] = 0\,.\end{aligned} \tag{2}$$

The field operator $\psi(x)$ ($x \equiv \{\vec{x}, x_0\}$) satisfies the equation of motion

$$\left(i\frac{\partial}{\partial x_0} + \frac{\Delta}{2}\right)\psi(x) - \int d^3\vec{y}\,\psi^+(\vec{y}, x_0)\,\psi(\vec{y}, x_0)\,U(\vec{x}-\vec{y})\,\psi(x) = 0\,. \tag{3}$$

3. We introduce the following one-particle correlation functions, the normal ones

$$\begin{aligned}K(x-x') &\equiv \langle N \mid \psi(x)\psi^+(x') \mid N\rangle\,,\quad K'(x-x') \equiv \langle N \mid \psi^+(x')\psi(x) \mid N\rangle,\\ K^{(-)}(x-x') &\equiv \langle N \mid [\psi(x), \psi^+(x')] \mid N\rangle = K(x-x') - K'(x-x')\end{aligned} \tag{4}$$

and the anomalous ones

$$\begin{aligned}F^{(-)}(x-x') &\equiv e^{i2\mu x_0}\langle N-2 \mid [\psi(x), \psi(x')] \mid N\rangle\,,\\ F(x-x') &\equiv e^{2i\mu x_0}\langle N-2 \mid \psi(x)\psi(x') \mid N\rangle\,,\end{aligned} \tag{5}$$

where $\mid N\rangle$ and $\mid N-2\rangle$ are the exact ground state eigenvectors of systems with N and $N-2$ particles, respectively, $\mu = \frac{1}{2}\left(E_0(N) - E_0(N-2)\right)$ is the chemical potential at zero temperature ($E_0(N)$ is the exact ground state energy of the N-particle system.)

With the help of equ. (3) the equations of motion for $K^{(-)}(x-x')$ and $F^{(-)}(x-x')$ can easily be derived. From these equations using the equal-time commutation rules it follows that

$$\begin{aligned}\left[\left(i\frac{\partial}{\partial x_0} + \frac{\Delta}{2}\right)K^{(-)}(x-x')\right]_{x_0=x_0'} &= \left[\delta(\vec{x}-\vec{x}')\int d^3\vec{y}\,U(\vec{y}-\vec{x})\,n + \right.\\ &\left. + U(\vec{x}'-\vec{x})\,K'(x-x')\right]_{x_0=x_0'}.\end{aligned} \tag{6}$$

Here n is the density of particles. Similarly,

$$\begin{aligned}\left[\left(i\frac{\partial}{\partial x_0} + \frac{\Delta}{2}\right)F^{(-)}(x-x') + 2\mu\,F^{(-)}(x-x')\right]_{x_0=x_0'} &=\\ = -U(\vec{x}'-\vec{x})\,F(\vec{x}-\vec{x}'\,, x_0-x_0'=0)\,.\end{aligned} \tag{7}$$

It is easy to see that the right hand side of equ. (6) is the mass operator in the HARTREE—FOCK approximation.

Let us denote the Fourier-transform of $K(x-x')$, $K'(x-x')$, $F^{(-)}(x-x')$ and $F(x-x')$ by $J(\vec{p}, E)$, $J'(\vec{p}, E)$, $L^{(-)}(p, E)$ and $L(\vec{p}, E)$, resp., i.e. for example

$$K(x-x') = \frac{1}{(2\pi)^3}\int d^3\vec{p}\, dE\, J(\vec{p}, E)\, e^{i\vec{p}(\vec{x}-\vec{x}')-iE(x_0-x)_0'}.$$

First of all the equal-time commutation rules are identical with the conditions

$$\int dE\big(J(\vec{p}, E) - J(\vec{p}, E)\big) = 1\,, \tag{8a}$$

$$\int dE L^{(-)}(\vec{p}, E) = 0\,. \tag{8b}$$

If we insert the spectral representations into equs. (6) and (7) using equ. (8a) we get

$$\int dE\, E\big(J(\vec{p}, E) - J'(\vec{p}, E)\big) = \Sigma_1(\vec{p}) + \frac{p^2}{2}\,, \tag{9}$$

$$\int dE\,(E+\mu)\, L(\vec{p}, E) = -\frac{1}{2(2\pi)^3}\int dE\, d^4\vec{q}\, U(\vec{p}-\vec{q})\, L(\vec{q}, E), \tag{10}$$

where $U(\vec{q})$ is the Fourier-transform of the two-body interaction potential and

$$\Sigma_1(\vec{p}) = U(0)\, n + \frac{1}{(2\pi)^3}\int d^3\vec{q}\, U(\vec{p}-\vec{q})\int dE\, J'(\vec{q}, E)\,. \tag{11}$$

Equs. (8a), (9) and (10) can be used for the calculation of the low-lying one-particle excitation spectrum of Bose systems. For fermion systems the analogous equations have been derived by A. ZAWADOWSKI and G. PÓCSIK [1].

4. It is possible to derive a simple formula for the ground state energy of the system in terms of the one-particle correlation function K'. To this end we proceed as follows. For the ground state energy $E_0(N)$ we have

$$\begin{aligned} E_0(N) = \langle N|H|N\rangle &= -\frac{1}{2}\int d^3\vec{x}\, \langle N|\psi^+(x)\, \Delta\psi(x)|N\rangle + \\ &+ \frac{1}{2}\int d^3\vec{x}\, d^3\vec{y}\, \langle N|\psi^+(\vec{x})\, \psi^+(\vec{y})\, U(\vec{y}-\vec{x})\, \psi(\vec{y})\, \psi(\vec{x})\, |N\rangle\,. \end{aligned} \tag{12}$$

The second term of the right hand side can be written with the help of the equation of motion for the normal correlation function K' as

$$\int d^3\vec{y}\, U(\vec{x}-\vec{y})\, \langle N|\, \psi^+(x')\, \psi^+(\vec{y}, x_0)\, \psi(\vec{y}, x_0)\, \psi(x)|N\rangle = \left(i\frac{\partial}{\partial x_0} + \frac{\Delta}{2}\right) K'(x-x').$$

Therefore

$$E_0(N) = \frac{1}{2}\int d^3\vec{x}\left[\left(i\frac{\partial}{\partial x_0} - \frac{\Delta}{2}\right)K'(x-x')\right]_{x_0=x_0'} \tag{13a}$$

or in terms of spectral function

$$\frac{E_0}{\Omega} = \frac{1}{2}\int dE\, d^3\vec{p}\,\frac{1}{(2\pi)^3}\left(E + \frac{p^2}{2}\right)J'(\vec{p}, E)\,, \tag{13b}$$

where Ω is the volume of the system. (13b) is a simplified form of the expression for the ground state energy given by BELIAEV [3], HUGENHOLTZ and PINES [4].

5. To determine the spectral functions from the relations derived we must make some assumptions. Let us begin with the energy of the one-particle excitations.

The exact eigenstates of the Hamiltonian can be labelled by the number of particles N, its total momentum $\vec{p}$ and some other observables. We can refer to states $|\, N \pm 1, \vec{p}, \alpha\rangle$ as one-particle excitations. We suppose that its energy $E(\vec{p})$ is uniquely determined by its momentum, that is,

$$H\,|\, N \pm 1, \vec{p}, \alpha\rangle = \left(E_0(N) \pm \mu + E(\vec{p})\right)\,|\, N \pm 1, \vec{p}, \alpha\rangle \tag{14}$$

holds for every α.

This assumption has the following fundamental consequence. As the ground states of the $(N \pm 1)$ particle systems are one-particle excited states of the N-particle system with vanishing momentum

$$E(\vec{p} = 0) = 0 \tag{15}$$

must hold. This means that there is no gap in the one-particle excitation spectrum.

Using the translational invariance the correlation functions can be reexpressed as

$$\begin{aligned}
K(x-x') &= \int d^3\vec{p} \sum_\alpha \langle N|\,\psi(0)\,|N+1, \vec{p}, \alpha\rangle \times \\
&\times \langle N+1, \vec{p}, \alpha|\psi^+(0)|N\rangle\, e^{i\vec{p}(\vec{x}-\vec{x}') - i(E(\vec{p})+\mu)(x_0-x_0')}, \\
K'(x-x') &= \int d^3\vec{p} \sum_\alpha \langle N|\,\psi^+(0)|N-1, \vec{p}, \alpha\rangle \times \\
&\times \langle N-1, \vec{p}, \alpha\,|\psi(0)\,|N\rangle\, e^{-i\vec{p}(\vec{x}-\vec{x}') - i(\mu - E(\vec{p}))(x_0-x_0')}. \\
F(x-x') &= \int d^3\vec{p} \sum_\alpha \langle N-2|\,\psi(0)\,|N-1, \vec{p}, \alpha\rangle \times \\
&\times \langle N-1, \vec{p}, \alpha|\,\psi(0)|\,N\rangle\, e^{i\vec{p}(\vec{x}-\vec{x}') - i(E(\vec{p})-\mu)(x_0-x_0)}\,.
\end{aligned} \tag{16}$$

Now we assume that for a given value of the momentum there is only one excited state, which contributes to the spectral functions. Let us denote this state by $| N \pm 1, \vec{p}\rangle$. Introducing the quantities

$$\begin{aligned} u_{\vec{p}} &= (2\pi)^{3/2}\langle N+1, \vec{p} \mid \psi^{+}(0) \mid N\rangle\,, \qquad |u_{\vec{p}}| = |u_{-\vec{p}}|\,, \\ v_{\vec{p}} &= (2\pi)^{3/2}\langle N-1, \vec{p} \mid \psi(0) \mid N\rangle\,, \qquad |v_{\vec{p}}| = |v_{-\vec{p}}|\,, \end{aligned} \tag{17}$$

we get

$$J(\vec{p}, E) = |u_{\vec{p}}|^2 \delta\big(E - \mu - E(\vec{p})\big)\,,\ J'(\vec{p}, E) = |v_{\vec{p}}|^2 \delta\big(E - \mu + E(\vec{p})\big). \tag{18}$$

By assuming that, if the system is sufficiently large, for the quantities $\tilde{u}_{\vec{p}} \equiv \langle N-2 \mid \psi(0) \mid N-1, \vec{p}\rangle (2\pi)^{3/2}$ the relation

$$|\tilde{u}_{\vec{p}}| = |u_{\vec{p}}|\,.$$

holds, we get for the spectral function

$$L(\vec{p}, E) = \tilde{u}_{\vec{p}} v_{\vec{p}} \delta(E - E(\vec{p}) + \mu)\,. \tag{19}$$

Equs. (8), (9) and (10) enable us to determine the parameters $|u_{\vec{p}}|$, $|v_{\vec{p}}|$ and $E(\vec{p})$. Equ. (8a) gives

$$|u_{\vec{p}}|^2 - |v_{\vec{p}}|^2 = 1\,, \tag{20}$$

while taking into account that $E(0) = 0$, from equs. (9) and (18)

$$\begin{aligned} |u_{\vec{p}}|^2 &= \frac{1}{2}\left(\frac{\Sigma(\vec{p})}{E(\vec{p})} + 1\right) + (2\pi)^3 n_0 \delta(\vec{p})\,, \\ |v_{\vec{p}}|^2 &= \frac{1}{2}\left(\frac{\Sigma(\vec{p})}{E(\vec{p})} - 1\right) + (2\pi)^3 n_0 \delta(\vec{p}) \end{aligned} \tag{21}$$

follows, where

$$\begin{aligned} \Sigma(\vec{p}) &= \frac{p^2}{2} + \Sigma_1(\vec{p}) - \mu = \frac{p^2}{2} - \mu + nU(0) - \\ &- \frac{1}{2(2\pi)^3}\int d^3\vec{q}\, U(\vec{q}) + n_0 U(\vec{p}) + \frac{1}{2(2\pi)^3}\int d^3\vec{q}\, U(\vec{p}-\vec{q})\frac{\Sigma(\vec{q})}{E(\vec{q})}\,. \end{aligned} \tag{22}$$

Here we have used equs. (11) and (21). Moreover, n_0 is the density of the condensed particles determined by the equation

$$n = K'(0) = \frac{1}{(2\pi)^3}\int d^3\vec{p}\, dE\, J'(\vec{p}, E) = n_0 + \frac{1}{2(2\pi)^3}\int d^3\vec{p}\left(\frac{\Sigma(\vec{p})}{E(\vec{p})} - 1\right). \tag{23}$$

On the other hand, equs. (10) and (19) lead to the equation

$$2E(\vec{p})\,\tilde{u}^{*}_{\vec{p}}\,v_{\vec{p}} = -\frac{1}{(2\pi)^3}\int d^3\vec{q}\,U(\vec{p}-\vec{q})\,(\tilde{u}^{*}\,v)_{\vec{q}} \equiv \Sigma_2(\vec{p})\,. \tag{24}$$

From equs. (21) and (24) the one-particle excitation energy can be expressed in terms of $\Sigma(\vec{p})$ and $\Sigma_2(\vec{p})$. Indeed, we get

$$E(\vec{p}) = (\Sigma^2(\vec{p}) - |\Sigma_2(\vec{p})|^2)^{1/2}\,. \tag{25}$$

Hence, using equ. (15), the chemical potential is

$$\mu = \Sigma_1(0) - |\,\Sigma_2(0)\,|\,, \tag{26}$$

which is a special form of HUGENHOLTZ and PINES' result. Using equ. (24) and taking again into account that $E(0) = 0$, we get the following equation for $\Sigma_2(\vec{p})$

$$\Sigma_2(\vec{p}) = -n_0\,U(\vec{p}) - \frac{1}{2(2\pi)^3}\int d^3\vec{q}\,U(\vec{p}-\vec{q})\,\frac{\Sigma_2(\vec{q})}{E(\vec{q})}\,. \tag{27}$$

6. With the help of equs. (13), (22), (23), (25), (26) and (27) the energy spectrum of one-particle excitations, the ground state energy and the chemical potential can be evaluated.

It is easy to see that our equations derived to the very end in a gauge invariant manner are identical to those which have been derived by VALATIN and BUTLER [2]. This means that their variational method and the pole-approximation are in essence equivalent for the case of boson systems. Similar agreement has also been found by PÓCSIK and ZAWADOWSKI between the two approximations in the case of fermion systems,

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КАЛИБРОВОЧНЫЙ ИНВАРИАНТНЫЙ МЕТОД В ТЕОРИИ ВЗАИМОДЕЙСТВИЯ БОЗЕ-СИСТЕМ

Ч. ХАРГИТАИ

Р е з ю м е

Для взаимодействия систем Бозе-частиц даётся метод полюсного приближения, основанный на точном правиле суммирования. Выведённые в работе выражения совпадают с известными из работ Валатина—Батлера соотношениями.

QUASI-PARTICLE OPERATORS FOR THE CONDENSED BOSE SYSTEM

By

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The concept of quasi-particle operators is used for the condensed BOSE system. The operators depend on the relative motion of the superfluid and the normal component.

1. Quasi-particle operators (QPO) are originally defined and extensively used for the FERMI system by NOZIÈRES and LUTTINGER [1, 2]. The QPO, though only approximately defined, are especially useful, because it is possible to construct a quasi-particle distribution function easily by means of them. The definition of such a distribution function is necessary for the discussion of quasi-phenomenological theories from the point of view of the microscopic theory. LUTTINGER and NOZIÈRES [2] have derived in this way the quasi-phenomenological LANDAU theory of the normal FERMI liquid for temperatures $T > 0$.

2. We are here interested in the condensed BOSE system (liquid He^4 II). The definition of the QPO rests primarily on the assumption of narrow peaks of the spectral function $\mathfrak{a}$ for the one-particle GREEN's function $\ll a_{\mathbf{k}}; a_{\mathbf{k}}^+ \gg$. ($a_{\mathbf{k}}$ is the field operator in momentum space.) We use for the system with superfluid and normal component at rest ($\hbar = 1$)

$$\mathfrak{a}(\mathbf{k}, \omega) - \mathfrak{a}^{\mathrm{inc}}(\mathbf{k}, \omega) = \frac{1}{\pi} \frac{z_k^+ \Gamma_k^+}{(\omega - E_k)^2 + \Gamma_k^{+2}} + \frac{1}{\pi} \frac{z_k^- \Gamma_k^-}{(\omega + E_k)^2 + \Gamma_k^{-2}} \approx$$
$$\approx z_k^+ \delta(\omega - E_k) + z_k^- \delta(\omega + E_k) ; \tag{1}$$

$$\mathfrak{a}(\mathbf{k}, \omega) = i \ll a_{\mathbf{k}}; \; a_{\mathbf{k}}^+ \gg_{\omega + i\varepsilon} - i \ll a_{\mathbf{k}}; \; a_{\mathbf{k}}^+ \gg_{\omega - i\varepsilon}, \; \varepsilon \to +0 .$$

$\mathfrak{a}^{\mathrm{inc}}$ is the incoherent part of the spectral function. For the quasi-particle energies E_k we use the LANDAU spectrum; $\Gamma_k^{\pm} \ll E_k$. $z_k^+ > 0$ and $z_k^- < 0$ are the residua of the GREEN's function. In contrast to the FERMI case the absolute values of the residua are here not limited to 1. The use of the above form for the spectral function is only possible for low temperature. We neglect the temperature dependence of the quantities in (1), which means essentially neglecting the temperature dependence of the energy spectrum.

3. We now turn to a system with moving condensate (velocity $\mathbf{v}_s$) and moving normal component ($\mathbf{v}_n$) [3]. In the HAMILTONian operator of the system we use two additional terms: $-\mathbf{v}_n\mathbf{P}$, where $\mathbf{P}$ is the momentum operator of the system, and BOGOLJUBOV's source term [4] in the form

$$H_\nu = -\nu \int e^{-i\mathbf{k}_0\mathbf{r}}\,\psi(\mathbf{r}t)\,d^3\mathbf{r} + c.c. \tag{2}$$

ψ is the field operator, $\nu \to +0$, $\mathbf{k}_0 = m\mathbf{v}_s$ and m is the mass of the He^4 atom. This leads for the wave function of the condensate to

$$\langle\psi\rangle = \sqrt{\varrho_0}\,e^{i\mathbf{k}_0\mathbf{r}}\,.$$

$\langle\ldots\rangle$ means the average over the grand canonical ensemble and ϱ_0 is the density of the particles in the condensate. It is convenient to split off the motion of the condensate: $\psi = e^{i\mathbf{k}_0\mathbf{r}}\hat{\psi}$, or in momentum space $\hat{a}_\mathbf{k} = a_{\mathbf{k}+\mathbf{k}_0}$.

The source term (2) has the consequence that the momentum operator $\mathbf{P}$ does not commute with the Hamiltonian H; instead of this the momentum operator relative to the motion of the condensate

$$\hat{\mathbf{P}} = \mathbf{P} - \mathbf{k}_0 N = \sum_\mathbf{k} \mathbf{k}\,\hat{a}_\mathbf{k}^+\,\hat{a}_\mathbf{k} \tag{3}$$

(N is the particle number operator) commutes with H.

We describe the moving system by the spectral function $\hat{\mathfrak{a}}(\mathbf{k}, \omega, \mathbf{v}_n - \mathbf{v}_s)$ belonging to the GREEN's function $\ll\hat{a}_\mathbf{k};\,\hat{a}_\mathbf{k}^+\gg$. For this spectral function [3] we can write

$$\hat{\mathfrak{a}}(\mathbf{k}, \omega, \mathbf{v}_n - \mathbf{v}_s) = \mathfrak{a}\big(\mathbf{k}, \omega + (\mathbf{v}_n - \mathbf{v}_s)\,\mathbf{k}\big)\,. \tag{4}$$

4. The QPO are now defined in analogy with the case of the FERMI system [1] but generalized to the situation of systems with moving superfluid and normal component

$$\hat{A}_\mathbf{k}^+ = \frac{\alpha}{\sqrt{z_k^+}} \int_{-\infty}^{0} dt\, e^{-iE_\mathbf{k}^+ t + \alpha t}\,\hat{a}_\mathbf{k}^+(t)\,, \tag{5}$$

with $\Gamma_k^\pm \ll \alpha \ll E_k$ and $E_\mathbf{k}^\pm = E_k \mp (\mathbf{v}_n - \mathbf{v}_s)\mathbf{k}$. The physical idea of the QPO concept is to single out by interference the quasi-particle state out of the state $\hat{a}_\mathbf{k}^+\,|0\rangle$, ($|0\rangle$ is the ground state of the system). The operator $\hat{A}_\mathbf{k}^+$ creates a quasi-particle with momentum $\mathbf{k} + \mathbf{k}_0$ and energy $E_\mathbf{k}^+$.

As in the case of the superconducting FERMI system [1] there are two equivalent forms of the QPO.

$$\hat{A}_\mathbf{k}^+ = \frac{\alpha}{\sqrt{-z_k^-}} \int_{-\infty}^{0} dt\, e^{-iE_\mathbf{k}^+ t + \alpha t}\,\hat{a}_{-\mathbf{k}}(t)\,. \tag{5a}$$

This is connected with the existence of the anomalous GREEN's function $\ll \hat{a}^+_{-\mathbf{k}}; \hat{a}^+_{\mathbf{k}} \gg$. For its spectral function $\hat{\mathfrak{b}}(\mathbf{k}, \omega)$

$$\hat{\mathfrak{b}}(\mathbf{k}, \omega) - \hat{\mathfrak{b}}^{\,\mathrm{inc}}(\mathbf{k}, \omega) = \tilde{z}_k \, \delta(\omega - E^+_{\mathbf{k}}) - \tilde{z}_k \, \delta(\omega + E^-_{\mathbf{k}})$$

holds corresponding to (1). The HUGENHOLTZ—PINES expressions [5] for the GREEN's functions in terms of the self energy parts lead then to the relation $z^+_k \, z^-_k = -\tilde{z}^2_k$, from which the equivalence of (5) and (5a) follows.

Furthermore one proves easily, similarly to the case of the FERMI system [1, 2], the BOSE commutation relations for the QPO and gets for the quasi-particle distribution function $\hat{n}_{\mathbf{k}}$ for equilibrium $\left(\beta = \dfrac{1}{\varkappa T}, \ \varkappa \text{ BOLTZMANN's constant}\right)$

$$\hat{n}_{\mathbf{k}} \equiv \langle \hat{A}^+_{\mathbf{k}} \, \hat{A}_{\mathbf{k}} \rangle = \frac{1}{e^{\beta E^+_{\mathbf{k}}} - 1} . \tag{6}$$

5. For applications it is necessary to express the basic physical quantities of the system in terms of the quasi-particle distribution function. Naturally this is possible only approximately, because the QPO themselves are defined only approximately. We are here interested in the energy and in the density of the normal component ϱ_n. For this we start with the relation

$$[\mathcal{O}, \hat{A}^+_{\mathbf{k}}] = \mathcal{O}_{\mathbf{k}} \, \hat{A}^+_{\mathbf{k}} , \tag{7}$$

where $\mathcal{O} = \hat{P}_i , \quad \mathcal{O}_{\mathbf{k}} = k_i \quad \text{for} \quad i = 1, 2, 3 ;$

or $\mathcal{O} = H, \quad \mathcal{O}_{\mathbf{k}} = E^+_{\mathbf{k}} .$

For the momentum operator (7) follows exactly from definitions (3) and (5). To prove this it is essential that $\hat{\mathbf{P}}$ commutes with H because $\hat{a}^+_{\mathbf{k}}$ enters into (5) in the HEISENBERG picture. This is an interesting point; the quantity $\langle \hat{a}^+_{\mathbf{k}} \hat{a}_{\mathbf{k}} \rangle$ for example does not commute with H and has therefore no direct relation with $\langle \hat{A}^+_{\mathbf{k}} \hat{A}_{\mathbf{k}} \rangle$. For the case of the Hamiltonian operator H equ. (7) follows from (6), the BOSE commutation relation and the invariance of the trace under cyclic permutations of the operators $\hat{A}^+_k$, $\hat{A}_k$ and $e^{-\beta H}$.

From relation (7) follows

$$\mathcal{O} = \sum_{\mathbf{k}} \mathcal{O}_{\mathbf{k}} \, \hat{A}^+_{\mathbf{k}} \, \hat{A}_{\mathbf{k}} + \mathcal{O}' , \tag{8}$$

where $\mathcal{O}'$ commutes approximately (because of the approximate validity of the BOSE commutation relations) with each $\hat{A}^+_k$ and $\hat{A}_k$. For low temperatures we assume the states of the system to be described totally by the quasi-particles

included in the spectral function (1) and we neglect $\mathcal{O}'$. There is no necessity to add a c-number part to $\mathcal{O}$, because $\langle \hat{\mathbf{P}} \rangle = 0$ for the ground state ($\varrho_n = 0$) and we can use $\langle H \rangle = 0$ for this state. Then

$$\langle \hat{P} \rangle = \sum_{\mathbf{k}} \mathbf{k} \langle \hat{A}_{\mathbf{k}}^{+} \hat{A}_{\mathbf{k}} \rangle , \qquad \langle H \rangle = \sum_{\mathbf{k}} E_{\mathbf{k}}^{+} \langle \hat{A}_{\mathbf{k}}^{+} \hat{A}_{\mathbf{k}} \rangle . \tag{9}$$

For the density of the normal component then follows e.g. the well known LANDAU formula (Ω is the normalization volume)

$$v_n \varrho_n = \frac{1}{\Omega} \sum_{k} k \frac{1}{e^{\beta(E_k - k v_n)} - 1} \qquad \text{for } v_s = 0 . \tag{10}$$

To conclude we wish to mention in connection with equations (9) and (10) other treatments starting from perturbation theory. BALIAN and DE DOMINICIS [6] have derived ideal-gas-like expressions for the entropy and for ϱ_n (approximately for low temperatures). In their theory the quasi-particle energies appearing in the BOSE distribution functions are real for all temperatures and coincide with the poles of the one-particle GREEN's functions provided these poles lie near the real axis. GÖTZE and WAGNER [7] have shown that the specific heat term proportional to T^3 of the BOSE liquid is rigorously given by the ideal-gas value. These results confirm the approximate validity of the QPO method. But the advantage of the QPO lies in their simple and explicit definition (5). So the QPO seem to be a very useful tool for the discussion of transport processes and quasi-phenomenological theories such as the LANDAU–CHALATNIKOV theory of the BOSE liquid.

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КВАЗИЧАСТИЧНЫЕ ОПЕРАТОРЫ ДЛЯ СКОНДЕНСИРОВАННОЙ СИСТЕМЫ БОЗЕ

В. ВЕЛЛЕР

Р е з ю м е

Понятие квазичастичных операторов применяется к сконденсированной системе Бозе. Операторы зависят от относительного движения сверхтекучей и нормальной компоненты.

ON THE RELAXATION TIME FOR NONEQUILIBRIUM STATE OF SUPERFLUID

By

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The present work deals with the idealized problem of spatially homogeneous flow with different velocity of the normal and superfluid components. The aim of the investigation is the development of the expression for the number of vortex rings which reach the critical size in the unit of time.

1. In the original theory of superfluidity developed by LANDAU [1] it was supposed that the normal and superfluid components can flow at subcritical velocities without any friction between them. Later a number of experiments showed that the critical velocities are very small and strongly dependent on the size of the channel. These phenomena were qualitatively interpreted in terms of the vortex lines, postulated first by ONSAGER and FEYNMAN. These vortex lines are a kind of local excitations, and if they are created some mutual friction force opposes the relative motion of superfluid and normal components [2].

The present work does not pretend to consider the real experimental conditions in which the boundaries of the liquid probably play a very important role, but deals with the idealized problem of spatially homogeneous flow with different velocities of the normal and superfluid components. Under LANDAU's criteria $|v_s - v_n| < \min \frac{\varepsilon(p)}{p} = v_c$, where v_s, v_n are the velocities of the superfluid and normal components respectively, ε is the energy, p is the momentum of the elementary excitation, such a state will be metastable because the state of the same momentum, but with no relative motion will have smaller free energy.

The equilibrium will be established due to the creation of vortex lines and subsequent loss of relative momentum by the mutual friction between the normal and superfluid components.

As the vortex lines are no elementary excitations but complicated macroscopic formations it is very difficult to treat this problem in terms of GREEN's functions. Instead of that it is more appropriate to use a hydrodinamical description.

The vortex lines can be created in the superfluid by the thermal fluctuations. The creation of the closed lines, namely the vortex rings is the most

probable, because they can be created with the lowest work A on the unit of length. Such rings are essentially HIBB's embryons for metastable states. Their critical size R is defined by the equilibrium between the mutual friction force and the hydrodynamical forces acting on the vortex line in the superfluid.

As the thickness of vortex line is of atomic dimensions [2], that is much smaller than the critical size of the vortex ring if $|v_s - v_n| \ll v_c$, it is not difficult to calculate the quantity $A(R)$. The minimum work is obtained in a reversible process, so the friction force must be absent, or (see [2, 3]) the velocity of line must be equal to the normal velocity v_n, which we assume to be zero. In this case the only force acting on unit length of vortex line is the MAGNUS force and $A(R)$ is the energy of the vortex ring in the perfect fluid with velocity v_s at the infinity. Using Galilean transformation we obtain

$$A(R) = \varepsilon(R) + P(R)v_s\,, \tag{1.1}$$

where $\varepsilon(R)$ and $p(R)$ are the energy and momentum of the vortex ring in fluid:

$$\varepsilon(R) = \varrho_s \frac{R\varkappa^2}{2}\left(\ln\frac{8R}{a} - \frac{7}{4}\right) p(R) = \varkappa\pi R^2 \varrho_s\,, \tag{1.2}$$

where $\varkappa = \dfrac{h}{m}$ is the quantum of circulation, a is the length of atomic dimensions.

The critical radius of the ring can be obtained from the condition:

$$F = \frac{\partial A}{\partial R} = \varrho_s \frac{\varkappa^2}{2}\left(\ln\frac{8R}{a} - \frac{7}{4}\right) + \varrho_s \frac{\varkappa^2}{2} - 2\pi\varkappa R\varrho_s v_s = 0 \tag{1.3}$$

or

$$R_1 = \frac{\varkappa}{4\pi v_s}\ln\frac{8R_1}{a} - \frac{3}{4}\frac{\varkappa}{\pi v_s}\,. \tag{1.4}$$

We see from this expression that we must consider only vortex rings with $\varkappa > 0$.

The energy $A(r)$ has the form of a potential barrier with the height

$$A_1 = \frac{\varrho_s R_1}{4}\varkappa^2 \ln\frac{R_1}{a} \tag{1.5}$$

and if the size of the vortex ring reaches R_1, the further expansion of it will be energetically preferable and has almost unit probability.

The aim of the further investigation is the refinement of these elementary considerations and the development of the expression for the number of vortex rings which reach the critical size in the unit of time.

2. We are interested in the initial stage of the process, when there are few vortex rings and we can neglect their interaction. This problem is very close to the problem of boiling of pure liquid considered earlier by ZELDOVICH [4] and YU. KAGAN [5] who used the general method suggested by KRAMERS [6].

The vortex rings can reach critical size by many small random expansions or by single large expansion. This second possibility has a very small probability because it demands the instant organisation of motion in a volume of liquid with critical dimensions. So we shall consider only the first slow diffusion process.

In this case we can describe the growth of a vortex ring as a motion in the presence of external medium (the gas of excitation), which drive the ring in a random way. Essentially it is a special kind of Brownian motion.

Let us introduce the distribution function for vortex rings $f(\{n\})$, which depends on a set of quantum numbers $\{n\}$, which takes into account the oscillations and arbitrary orientations of the ring. We denote by $w_\tau(\{n'\}, \{n' - n\}, t)$ the probability of the transition from the state $\{n\}$ to the state $\{n'\}$ in time τ. Then the number of rings in the state $\{n\}$ at the moment $t + \tau$ will be

$$f(\{n\}, t + \tau) = \sum_{n'} w_\tau(\{n'\}, \{n' - n\}, t) f(\{n'\}, t). \tag{2.1}$$

Here the sum is over the whole set $\{n'\}$, in particular over the state with $\{n'\} = \{n\}$. The equation (2.1) must have such a property that HIBBS' distribution $f(\{n\}) = \text{const.} \exp(-E(n)/kT)$ is its solution.

We shall distinguish quantum numbers n_χ describing oscillations of the ring and quantum numbers N_i describing its motion as a whole.

The energy of oscillation for the ring of radius R is $\left(\frac{1}{2} + n_\chi\right) \hbar\omega(\chi, R)$, where $\omega(\chi, R)$ can be determined from the problem of small oscillations, χ is the wave vector of oscillation or the number of appropriate harmonic, n_χ are the appropriate occupation numbers.

We can write down the whole energy in the form

$$E(N, n_\chi) = \sum_\chi \hbar\omega(\chi, R) n_\chi + E(N_i), \tag{2.2}$$

where $E(N_i)$ is the energy of round ring, including the energy of zero oscillations.

The establishment of equilibrium in quantum numbers n_χ for vortex rings of some radius R is a rather fast process, becasue there are many fluctuations with energy $\hbar\omega$. So the distribution function will take approximately the form

$$f(N_i, n_\chi, t) = f_1(N_i, t) \exp\left\{-\sum_\chi \frac{\hbar\omega_\chi h_\chi}{kT}\right\}. \tag{2.3}$$

The average change of the quantum numbers N_i is relatively small for a short time τ and the equation (2.1) can be transformed into the FOKKER—PLANCK equation for the function $f(N, t)$.

Since the transitions between states with different n_χ do not change the distribution fuction we can leave in the sum over $\{n'\}$ only the summation over N_i' (performing the summation over n_χ)

$$\sum_{n_\chi} f(N_i, n_\chi, t + \tau) = \sum_{n_\chi} \sum_{N_i'} w_\tau(N_i', \{N_i' - N_i\}, n_\chi) f(N_i', n_\chi, t). \tag{2.4}$$

Developing the right hand side in power series of $(N_i' - N)$ in the first argument of w_τ and f, and assuming w_τ and f sufficiently smooth functions of N_i we obtain the following FOKKER—PLANCK equation in the space of quantum numbers N_i

$$\sum_{n_\chi} \frac{\partial f(N_i, n_\chi, t)}{\partial t} = \sum_{n_\chi} \sum_i \frac{\partial}{\partial N_i} \left(- \frac{dN_i}{dt} f + D_{ij} \frac{\partial f}{\partial N_j} \right), \tag{2.5}$$

where the quantity $\dfrac{dN_i}{dt}$ is the average velocity change of N_i and the second term describes the diffusion process connected with fluctuations. The average velocity $\dfrac{dN_i}{dt}$ can be written in the form

$$\frac{dN_i}{dt} = - \sum_j \eta_{ij} \frac{\partial E}{\partial N_j}, \tag{2.6}$$

where η_{ij} are some kinetic coefficients.

As the HIBBS distribution must be a solution of (2.5), we have the following relations

$$D_{ij} = kT \eta_{ij} \tag{2.7}$$

and the equation (2.5) can be written as

$$\sum_{n_\chi} \frac{\partial f}{\partial t} = \sum_{n_\chi} \sum_{i\,j} \frac{\partial}{\partial N_i} \eta_{ij} \left(\frac{\partial E}{\partial N_j} f + kT \frac{\partial f}{\partial N_j} \right). \tag{2.8}$$

The energy of a round vortex depends on its radius R and its orientation defined by the angles ϑ, φ. According to (1.1) and (2.2) the energy of an arbitrary ring is

$$E(N, n_\chi) = E(R) + E'(n_\chi, \vartheta, R), \tag{2.9}$$

where

$$E(R) = \varepsilon(R) - p(R)\, v_s,$$

$$E'(n_\chi, \vartheta, R) = \sum_\chi \hbar\omega(\chi, R)\, n_\chi + p(R)\, v_s (1 - \cos\vartheta) \tag{2.10}$$

and ϑ is the angle between $-v_s$ and the momentum of the ring $P(R)$.

Let us change the variables N_i into new independent variables N_i' which include R, ϑ, φ. It is easy to show that after such transformation equation (2.8) will take the form

$$\sum_{n_\chi} \frac{\partial(fI)}{\partial t} = \sum_{n_\chi} \frac{\partial}{\partial N_l'} \left\{ \eta_{ij} \frac{\partial N_m'}{\partial N_j} \frac{\partial N_l'}{\partial N_i} \left[\left(\frac{\partial E}{\partial N_m'} - kT \frac{\partial \ln I}{\partial N_m'} \right) fI + kT \frac{\partial (fI)}{\partial N_m'} \right] \right\}, \tag{2.11}$$

where

$$I = \frac{\partial(N_1 \ldots N_n)}{\partial(N_1' \ldots N_n')} \tag{2.12}$$

is the Jacobian of our transformation.

The establishment of the equilibrium in the orientations of the rings is a rather fast process, because the energy is almost independent of ϑ for small R, and for large R almost all rings are concentrated near $\vartheta = 0$.

For this reason we can assume that

$$If = If'(R)\, e^{-E'/kT}. \tag{2.13}$$

We are interested in the creation of vortex rings of some radius R only, independently of their other quantum numbers. So we can integrate (2.11) over the angles ϑ and φ and other N_i' excluding $R = N_l'$. Making use of (2.13) and performing the summation and integration we obtain from (2.11)

$$\frac{\partial \sigma}{\partial t} = \frac{\partial}{\partial R} \left\{ \eta_{RR} \left[\left(\frac{\partial E(R)}{\partial R} - kT \frac{\partial \ln \Gamma}{\partial R} \right) \sigma + kT \frac{\partial \sigma}{\partial R} \right] \right\}, \tag{2,14}$$

where $\sigma(R, t)$ is the distribution function of the rings in the one dimensional space of their radius, Γ is the statistical weight of the state with the ring of radius R

$$\ln \Gamma = \ln \sum_{n_\chi} \int I e^{-E'/kT}\, d\vartheta\, d\varphi\, dN'\,. \tag{2.15}$$

We can see from (2.14) in agreement with the thermodynamic theory of fluctuations that the number of vortex rings of radius R is proportional to $\exp(-A_{\min}/kT)$, where $A_{\min} = \delta F = \delta E - kT\,\delta S$, and δE and δS are the changes of the energy and entropy due to the creation of vortex ring.

3. To complete the equation (2.14) it is necessary to calculate Γ and η_{RR}.

The kinetic coefficient η_{RR} can be obtained from the consideration of the macroscopic motion of a vortex ring, because the quantity $-\eta_{RR} \dfrac{\partial E}{\partial R}$ gives us the average radial velocity of the ring in the presence of dissipative

forces. The force acting on the unit length of the vortex line from the side of the normal component is according to [3]:

$$F_1 = -b\frac{\varrho_n \varrho_s}{\varrho}\frac{[\varkappa[\varkappa v_L]]}{\varkappa} + b'\frac{\varrho_n \varrho_{s'}}{\varrho}[\varkappa v_L], \tag{3.1}$$

where v_L is the velocity of the line, the dimensionless coefficients b and b' are of the order of unity and are slightly different from the HALL's and VINEN's coefficients B, B'.

In addition there is a MAGNUS force acting on the unit length of line in the superfluid [7]:

$$F_2 = \varrho_s[\varkappa(v_s + V(R) - v_L)], \tag{3.2}$$

where $V(R)$ is the velocity of the vortex ring in the stationary fluid. Because the vortex line has no mass the whole force acting on the line must be equal to zero and we obtain for the radial velocity

$$\frac{dR}{dt} = -\frac{\varrho \varrho_n b}{\varrho_n^2 b^2 + \varrho^2\left(1 - b'\frac{\varrho_n}{\varrho}\right)^2}\frac{\partial E}{\partial R}\frac{\partial R}{\partial \overline{\overline{P}}}. \tag{3.3}$$

We can calculate from this expression the desired kinetic coefficient

$$\eta_{RR} = \frac{\varrho \varrho_n b}{\varrho_n^2 b^2 + \varrho^2\left(1 - b'\frac{\varrho_n}{\varrho}\right)^2}\frac{\partial R}{\partial \overline{\overline{P}}}. \tag{3.4}$$

In order to calculate the statistical weight Γ it is necessary to know all quantum numbers of the vortex ring. If we consider the vortex ring as an excitation with energy ε and momentum p, the latter must be quantized:

$$p = \left(\frac{2\pi\nu_1\hbar}{L_1}, \frac{2\pi\nu_2\hbar}{L_2}, \frac{2\pi\nu_3\hbar}{L_3}\right),$$

where ν_1, ν_2, ν_3 are integers, L_1, L_2, L_3 are the dimensions of the box. Now we can calculate the Jacobian (2.12)

$$I = \frac{V}{(2\pi\hbar)^3}p^2\frac{dp}{dR}\sin\vartheta. \tag{3.5}$$

Using (2.15) we obtain

$$\ln\Gamma = S_L + S_1, \tag{3.6}$$

where

$$S_1 = \ln\frac{kTV}{v_s(2\pi\hbar)^3}2\pi p\frac{dp}{dR}(1 - e^{-2p\nu_s/kT}) \tag{3.7}$$

and for the entropy of the line, performing the summation over n_χ, we get

$$S_L = -\sum_\chi \ln\left(1 - e^{-\hbar\omega\chi/kT}\right). \tag{3.8}$$

For rings of sufficiently large radius R the quantities χ are distributed practically continuously and for ω we can use the hydrodynamic formula [2]

$$\omega_\chi = \frac{\hbar^2}{2m}\chi^2 \ln\frac{1{,}046}{\chi a_0}. \tag{3.9}$$

We can obtain in that way the following approximate expression

$$S_L \approx 2R\frac{\sqrt{2mkT}}{\hbar}\frac{1}{\sqrt{\ln\dfrac{1{,}046\,\hbar}{a\sqrt{2mkT}}}}\left(\frac{1}{2}\right)!\,\zeta(3/2) \tag{3.10}$$

valid for low temperatures, where ζ is RIEMANN's function.

So we have determined all the quantities in the equation (2.14).

Generally speaking the equation (2.14) must be solved under the initial conditions when there are no rings with radius larger than some small R_0. However, the process is very slow because of the large potential barrier and can be treated approximately as a stationary one ([6] [4]). In this case we have the following boundary conditions; at small distances $R \sim R_0$ the rings are in equilibrium

$$\sigma(R_0) = \Gamma(R_0)\exp\left(-\frac{E(R_0)}{kT}\right) \tag{3.11}$$

and at $R \to \infty$ the rings are absent.

Solving the stationary equation (2.14) we obtain

$$j_R = \frac{kT\,\sigma(R)\,\Gamma^{-1}(R)\exp\left(\dfrac{E(R)}{kT}\right)\Big|_{R_0}^{\infty}}{\displaystyle\int_{R_0}^{\infty}\frac{1}{\eta_{RR}}\exp\left[\frac{E(R)}{kT} - \ln\Gamma(R)\right]dR}, \tag{3.12}$$

where j_R is a constant giving us the flux of the vortex rings along R, or the number of the vortex rings of supercritical size created per unit time.

The integrand has a sharp maximum at $R = R_c$ defined by

$$\frac{\partial E(R)}{\partial R} - kT\frac{\partial \ln\Gamma(R)}{\partial R} = 0 \tag{3.13}$$

or

$$R_c = \frac{\hbar}{m v_s}\left[\ln\frac{8R_c}{a} - \frac{3}{4}\right] - \frac{kT\sqrt{2mkT}}{\hbar\sqrt{\ln\dfrac{1{,}046\,\hbar}{a\sqrt{2mkT}}}}\,\frac{4{,}4m}{4\pi^2\varrho_s v_s \hbar}. \qquad (3.14)$$

It is necessary to point out that this equation in general has no solution. The calculations show that the real root dissappears when $v_s \geqslant 70$ cm/sec, at $T = 1{,}8$ °K, $a = 19$Å (according to [2]). In this case there is no potential barrier, and the vortex rings can be easily created. Unfortunately we do not know the exact value of a, and the expression (3.10) is very crude for $T = 1{,}8$ °K, so the value of velocity, 70 cm/sec, is only an approximate estimation.

When the equation (3.14) has a real root, the main contribution in the integral is given by the neighbourhood of R_c, and using the boundary conditions we obtain:

$$j_R = \frac{2\pi kT\sqrt{kT}}{v_s(2\pi\hbar)^3\sqrt{\pi}}\,\frac{\hbar}{m}\,\varrho_s R_c^2 \exp\left[-\frac{E(R_c) - kT\,S_L(R_c)}{kT}\right] \cdot$$

$$\cdot\,\frac{\varrho_n\varrho b}{\varrho_n^2 b^2 + \left(1 - b'\dfrac{\varrho_n}{\varrho}\right)^2\varrho^2}\sqrt{\frac{\partial^2}{\partial R^2}(E - kTS_L)\,|_R}\,.$$

The calculations according to this formula show that there is an intensive creation of vortex rings at relative velocities larger than 60 cm/sec at 1,8 °K, that is about 100 times smaller than the critical velocity calculated from LANDAU's criteria for elementary excitation. Below 60 cm/sec the probability of the creation of vortex rings abruptly tends to zero, giving us for relaxation time such values as e^{1000} for $v_s = 40$ cm/sec. The results are very sensitive to the temperature.

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О ВРЕМЕНИ РЕЛЯКСАЦИИ НЕРАВНОВЕСНОГО СОСТОЯНИЯ СВЕРХТЕКУЧЕЙ ЖИДКОСТИ

С. В. ИОРДАНСКИЙ

Р е з ю м е

В настоящей работе изучается идеализированная проблема пространственно однородного течения с различными скоростями нормальной и сверхтекучей компонент. Целью исследования является получение выражения для числа вихревых колец, достигающих критического размера в единицу времени.

FUNCTIONAL METHODS AND EXACT SOLUTIONS IN FIELD THEORY

By

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A theory of charged scalar field with fixed source in a nonperturbative functional approach is studied. In this model for the generator of the boson Green's function a closed expression exists.

Functional methods offer a compact formal expression for the sums of all Feynman diagrams, however, there are few cases where all the functional operations can be carried out explicitly. One example is one-dimensional quantum electrodynamics, where the vacuum polarisation is only a quadratic functional in the external field. I would like to give here another example of a theory where a functional operation can be carried out, namely the charged scalar theory with a fixed source. This theory is nonrelativistic but it is nontrivial in the sense that it allows for the production of an arbitrary number of particles. It seems to be an unfortunate feature of all soluble field theories up to date that they allow only for the creation of a few particles if any. Consequently they are not suited for discussing the essential problem of field theory namely the high energy behaviour. In our solution the functional differentiations are carried out but one is still left with an infinite series each term still containing integrals. This sum is of course identical with the sum of all Feynman diagrams for which there exists in this model a comparatively simple closed expression.

The theory is characterized by a Hamiltonian which in standard notation is

$$H = \int d^3k \, \{\dot{\theta}(k)\dot{\theta}^+(K) + \omega_k^2 \, \theta(k) \, \theta^+(k) - \varrho(k) \, (\theta \, (k_1 t)\tau^- + \theta(k_1 t)\tau^+)\} \, . \quad (1)$$

We will concentrate on calculating the generator for the boson Green's function for which we need the motion of the isospin which can be calculated by standard procedures. In particular we need the amplitude (f) for a proton remaining a proton under the influence of an external charged field.

$$f(\varphi, \varphi^+) = T\langle p| \, e^{\,i \int_{-\infty}^{\infty} dt(\tau^- \varphi^-(t) + \tau^+ \varphi^+(t))} \, |p\rangle =$$

$$= \sum_{n=0}^{\infty} (-)^n \int dt_1 \ldots dt_n \, ds_1 \ldots ds_n \, \theta^{(n)} \, (t_1, s) \, \varphi^+ \, (t_1) \ldots \varphi^+ \, (t_n) \, \varphi^- \, (s_1) \ldots \varphi^- \, (s_n) \, , \quad (2)$$

where the kernel θ is given by

$$\theta^{(n)}(t, s) = \theta(t_1 - s_1)\,\theta(s_1 - t_2)\,\theta(t_2 - s_2)\ldots\theta(t_n - s_n)\,, \tag{3}$$

which reflects that positive and negative mesons have to be emitted alternatively. From this expression we find the generator of the boson Green's function using the standard formula [1]

$$G(J) = T\langle p|\, e^{i\int_{-\infty}^{\infty} dt_u{}^3k[J^*(k_it)\phi(k_it)+J(k_it)\phi^+(k_it)]}\,|p\rangle =$$

$$= f\left(\int d^3k\,\varrho(k)\frac{\delta}{\delta i\,J^*(k)}\,,\quad \int d^3k\,\varrho(k)\frac{\delta}{\delta i\,J(k)}\right)\cdot \tag{4}$$

$$\cdot\, e^{i\int d^3k\,J(k_it)\,\Delta(k_i\,t-t')\,J^*(k_it)}\,.$$

The first problem we meet on inserting (2) is to carry out the functional differentiations. Since J and J^* are independent, the problem here is actually a little simpler than for theories with only one kind of boson. Carrying out the necessary combinatorics gives

$$G(J) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\int dt_1\ldots dt_n\,ds_1\ldots ds_n\,d^3k_1\ldots d^3k_r\,d^3q_1\ldots$$

$$\ldots d^3p_r\,d^3p_1\,.d^3p_{n-r}\,d\bar{t}_1\ldots d\bar{t}_r$$

$$d\bar{s}_1\ldots d\bar{s}_r\,\theta^{(n)}(s,t)\sum_{\pi,\varkappa}\frac{1}{(r!)^2\,(n-r)!}\,i\Delta(k_1, s_{\pi_1}-\bar{t}_1)\,J^*(\bar{t}_1, K_1)\,\varrho(K_1)_i\,\Delta(k_2, s_{\pi_2}-\bar{t}_2)\cdot$$

$$\cdot\,J^*(\bar{t}_2, k_2)\,\varrho(k_2)\ldots i\,\Delta(k_r, s_{\pi_r}-\bar{t}_r)\;\;J^*(\bar{t}_r, k_r)\,\varrho(k_r)i\,\Delta(p_1, s_{\pi_{r+1}}-t_{\varkappa 1})$$

$$\varrho^2(p_1)\ldots i\Delta\,(p_{n-r}, s_{\pi_n}-t_{\varkappa n-r})\,\varrho^2(p_{n-r})\;\;J(\bar{s}_1, q_1)i\,\Delta(q_1, \bar{s}_1 - t_{\varkappa n-r+1})$$

$$\varrho(q_1)\ldots J(\bar{s}_r, q_r)i\Delta(\bar{s}_r - t_{\varkappa_n})\,\varrho(q_r)\,e^{i\int J\Delta J}\,, \tag{5}$$

where $\pi_1\ldots\pi_n$ and $\varkappa_1\ldots\varkappa_n$ are two independent permutations of the numbers $1\ldots n$. Relabeling the integration variables this can be rewritten in the form

$$G(J) = \sum_{r=0}^{\infty}\frac{e^{i\int J\Delta J^*}}{(r!)^2}\,i\Delta(k_1\,, s_1 - \bar{t}_1)\,J^*\,(\bar{t}_1\,, k_1)\,\varrho(k_1)\ldots i\Delta(k_r\,, s_r - \bar{t}_r)\,.$$

$$J^*\,(\bar{t}_r\,, k_r)\,\varrho(k_r)\,J(\bar{s}_1\,, q_1)\,i\Delta(q_1\,, \bar{s}_1 - t_1)\,\varrho(q_1)\ldots J(\bar{s}_r,\, q_r)\,i\Delta\,(q_r, \bar{s}_r - t_r)\,.$$

$$\varrho(q_r)\sum_{n=r}^{\infty}\frac{1}{(n-r)!}\,D(s_{r+1} - t_{r+1})\ldots D(s_n - t_n)\sum_{\pi,\varkappa}\theta(s_\pi\,, t_\varkappa)\;. \tag{6}$$

Here we have introduced the boson propagator

$$D(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} D(\omega); D(\omega) = i \int \frac{d^3k \, \varrho(k^2)}{\omega_k^2 - \omega^2 + i\varepsilon}$$

for

$$\varrho^2 = \frac{M^2}{k^2 + M^2}, \quad D(\omega) = i \frac{M^2}{M^2 + \omega^2} \left(\sqrt{1 - \omega^2} - \sqrt{1 + M^2}\right). \tag{7}$$

Furthermore we have understood that variables occurring twice are integrated over. If one is only interested in the lowest order Green's function like the twopoint function which describes the scattering in this theory the expression (6) simplifies considerably. On expanding in J we find

$$\begin{aligned} G(J) = e^{i \int J \Delta J^*} \Bigg\{ & \sum_{n=0}^{\infty} \frac{1}{n!} D(s_1 - t_1) \ldots D(s_n - t_n) \sum_{\pi,\varkappa} \theta^{(n)}(s_\pi, t_\varkappa) + \\ & + i\Delta(k_1, s_1 - \bar{t}_1) J^*(\bar{t}_1, k_1) J(\bar{s}_1, q_1) i\Delta(q_1, \bar{s}_1 - t_1) \varrho(k_1) \varrho(q_1) \\ & \sum_{n=1}^{\infty} \frac{1}{(n-1)!} D(s_2 - t_2) \ldots D(s_n - t_n) \sum_{\pi,\varkappa} \theta^{(n)}(s_\pi, t_\varkappa) + \ldots . \end{aligned} \tag{8}$$

It is to be hoped that this comparatively simple exact expression for the scattering amplitude may be useful for deducing some of its important properties.

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ФУНКЦИОНАЛЬНЫЕ МЕТОДЫ И ТОЧНЫЕ РЕШЕНИЯ В ТЕОРИИ ПОЛЯ

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Р е з ю м е

Изучается теория заряженного скалярного поля с фиксированным источником с помощью непертурбативного функционального приближения. В этой модели существует замкнутое выражение для генератора бозонной функции Грина.

К ЭЛЕКТРОДИНАМИКЕ ЧАСТИЦ СО СПИНОМ НУЛЬ

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В работе рассмотрено функциональным методом взанмодействие векторного нейтрального поля (без массы (фотоны) а также при наличии массы) с бозе-полем спина нуль.

С помощью операторного решения для производящего функционала и операторного решения для функции Грина в произвольном внешнем векторном поле найдено замкнутое выражение для S-матрицы. Развивается модифицированная теория возмущения весьма удобная для нахождения асимптотик функции Грина и сечений.

Рассмотрены асимптотики для одночастичной, двухчастичной и вершинной функций Грина.

Введение

Решение проблемы о внутренней замкнутости теории и нахождение асимптотик сечений при больших энергиях (а в случае электродинамики и в инфракрасной области, даже в теории со слабой связью нуждается в построении методов, существенно отличных от обычной теории возмущений. Дело в том, что при нахождении, к примеру, асимптотики (или связанных состояний) необходимо просуммировать весь бесконечный ряд существенных неприводимых фейнмановских диаграмм. При этом вовсе нет необходимости учесть точно каждую из неприводимых диаграмм в отдельности, а достаточно лишь ухватить часть существенную для асимптотики. Такую программу построения модифицированной теории возмущений, существенно выходящую за рамки обычной теории возмущений, удается построить с помощью функциональных методов. Применительно к электродинамике спинорных и скалярных частиц в инфракрасной области (с помощью операторного решения для производящего функционала [1], такая программа выполнена в работе автора [2]. Более последовательно задача построения модифицированной теории возмущений решается [3] с помощью операторного решения для функции Грина в произвольном внешнем электромагнитном поле. Такое операторное решение удается построить [3] не только в случае взаимодействия электромагнитного поля с бозе-частицами, но и в спинорной электродинамике (где задача усложнена наличием дираковских матриц в уравнении для функции Грина во внешнем поле). Найденное функциональным методом замкнутое решение для S-матрицы позволяет найти и асимптотики для сечений различных процессов (см. также [4]).

В настоящей работе мы с помощью операторного решения [1] построим модифицированную теорию возмущений для различных функций Грина в случае векторного взаимодействия нейтрального векторного поля с бозе-полей со спином нуль.

1. Функциональные уравнения и операторное решение для производящего функционала

Как известно, основной величиной функционального метода является роизводящий функционал Z, который определяется следующим образом [1]

$$Z = \langle T \exp i \int d^4x \{I_\mu(x)\varphi_\mu(x) + \zeta^*(x)\,\psi(x) + \psi^*(x)\zeta(x)\}\,, \qquad (1.1)$$

где I_μ — источник векторного поля (в частности электромагнитное поле). η^* — коммутирующий источник бозе-поля $\psi(x)$. При этом операторы поля φ_μ, ψ, ψ^* и осреднение по вакууму $\langle\ \rangle$ взяты в обычном гейзенберговском представлении (без учета взаимодействия с внешними источниками), т. е. уравнение движения для этих операторов имеет вид:

$$\left.\begin{aligned} &\{-(\partial_\mu - ie\varphi_\mu(x))^2 + \mu^2\}\,\psi(x) = 0 \\ &\{-(\partial_\mu + ie\varphi_\mu(x))^2 + \mu^2\}\,\psi^*(x) = 0 \\ &\{-\partial_\nu^2 + \lambda^2\}\,\varphi_\mu(x) = j_\mu(x) \end{aligned}\right\}, \qquad (1.2)$$

где

$$j_\mu(x) = ie\left(\frac{\partial\psi^*(x)}{\partial x_\mu}\psi(x) - \psi^*(x)\frac{\partial\psi(x)}{\partial x_\mu}\right) - 2e^2\,\varphi_\mu(x)\,\psi^*(x)\,\psi(x)\,. \qquad (1.3)$$

Из (1.1)—(1.3) обычной методикой получим следующую систему функциональных уравнений для Z[1]:

$$\left[-\left(\partial_\mu - e\frac{\delta}{\delta I_\mu(x)}\right)^2 + \mu^2\right]\frac{\delta Z}{\delta\zeta^*(x)} - i\,\zeta(x)\,Z\,,$$

$$\left[-\left(\partial_\mu + e\frac{\delta}{\delta I_\mu(x)}\right)^2 + \mu^2\right]\frac{\delta Z}{\delta\zeta(x)} = i\,\zeta^*(x)\,Z\,, \qquad (1.4)$$

$$\left[-\partial_\nu^2 + \lambda^2\right]\frac{\delta Z}{\partial I_\mu(x)} = i\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)I_\nu(x) +$$

$$+ e\left[\left(\partial_\mu\frac{\delta}{\delta\zeta(x)}\right)\frac{\delta}{\delta\zeta^*(x)} - \frac{\delta}{\delta\zeta(x)}\left(\partial_\mu\frac{\delta}{\delta\zeta^*(x)}\right)\right]Z - 2e^2\,\frac{\delta^3 Z}{\delta I_\mu(x)\,\delta\zeta^*(x)\,\delta\zeta(x)}\,.$$

[1] При получении уравнения $\delta\dfrac{\delta Z}{I_\mu(x)}$ мы воспользуемся тем, что и при наличии взаимодействия выполняется условие Лоренца $\partial_\mu\ \varphi_\mu(x) = 0$, поэтому продольная часть поля φ_μ не вносит вклад в теорию. Можно выбрать, в частности, φ_μ в поперечной калибровке (или, что то же самое, внешний источник φ_μ взять в поперечной калибровке т. е.

$$I_\mu \to \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)I_\nu(x).$$

Операторное решение для Z следуя результатам работы [1] имеет вид:

$$cZ = e^{iA} \exp i \int d^4x\, d^4y \{I_\mu(x)\, D^0_{\mu\nu}(x-y)\, I_\nu(y) + \zeta^*(x)\, G^0(x-y)\, \zeta(y)\}, \quad (1.5)$$

где

$$G^0(x-y) = \frac{1}{(2\pi)^4} \int \frac{d^4p}{p^2 + \mu^{2-i\varepsilon}} e^{ipx},$$

$$D^0_{\mu\nu}(x-y) = \frac{1}{(2\pi)^4} \int \frac{d^4k}{k^2 + \lambda^2 - i\varepsilon} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) e^{ikx} \quad (1.6)$$

$$A = \int d^4x\, L_{\text{in}}\left(\varphi^u_u \to \frac{\delta}{i\delta I_\mu(x)};\ \psi(x) \to \frac{\delta}{i\delta\zeta^*(x)};\ \psi^*(x) \to \frac{\delta}{i\delta\zeta(x)}\right) =$$

$$= e\left[\frac{\delta}{\delta I_\mu(x)} \frac{\delta}{\delta\zeta^*(x)} \left(\partial_\mu \frac{\delta}{\delta\zeta(x)}\right) - \frac{\delta}{\delta I_\mu(x)} \left(\partial_\mu \frac{\delta}{\delta\zeta^*(x)}\right) \frac{\delta}{\delta\zeta(x)}\right] - \quad (1.7)$$

$$- e^2 \frac{\delta^4}{\delta I_\mu(x)\, \delta_\mu I(x)\, \delta\zeta^*(x)\, \delta\zeta(x)},$$

где постоянная «С» выбирается из условия, чтобы $Z = 1$ при $I = \zeta = \zeta^* = 0$. Не представляет труда провести функциональное дифференцирование по одному из полей и получить для Z выражение, содержащее лишь функциональные производные по оставшемуся полю. Для дальнейшего исследования нам будет полезно выражение для Z, в котором остаются лишь функциональные производные по I_μ. Действуя методом работы [1], мы найдем:

$$cZ = \exp i\left[\int \zeta^*(x)\, G\left(x, y \middle| \frac{\delta}{\delta I}\right) \zeta(y)\, d^4x\, d^4y + \int \pi\left(\frac{\delta}{\delta I}\right) d^4x\right] \times$$

$$\times \exp i \int I_\mu(x)\, D^0_{\mu\nu}(x-y)\, I_\nu(y)\, d^4x\, d^4y, \quad (1.8)$$

где поляризационный член $\pi\left(\frac{\delta}{\delta I}\right)$ имеет вид:

$$\pi\left(\frac{\delta}{\delta I}\right) = \int_0^e de \left[\left(\frac{\partial}{\partial x_\mu} G(x, x') - G(x, x') \frac{\partial}{\partial x'_\mu}\right) \frac{\delta}{\delta I_\mu(x)|_{x=x'}} - \right. \quad (1.9)$$

$$\left. - 2e \frac{\delta^2}{\delta I_\mu(x)\, \delta I_\mu(x)} G(x, x)\right] =$$

$$= - i\left\{\ln G\left(x, x \middle| \frac{\delta}{\delta I}\right) - \ln G^0(x, x)\right\}.$$

Функция Грина $G(x,y)$ бозона во внешнем поле φ_μ определяется из следующего уравнения:

$$\{-(\partial_\mu - ie\varphi_\mu(x))^2 + \mu^2\}\, G(x, y \mid i\,\varphi) = \delta(x - y)\,. \tag{1.10}$$

2. Решение уравнения для функции Грина во внешнем поле

Рассмотрим формальное решение уравнения для функции Грина (1.10)

$$G(x, y \mid i\varphi) = \frac{1}{-(\partial_\mu - ie\,\varphi_\mu(x))^2 + \mu^2 - i\varepsilon}\,\delta(x - y) =$$

$$= i\int_0^\infty dv e^{i[(\partial_\mu - ie\varphi_\mu(x))^2 - \mu^2 + i\varepsilon]v}\,\delta(x - y) = \tag{2.1}$$

$$= \frac{i}{(2\pi)^4}\int d^4 p e^{ip(x-y)}\, e^{-i(p^2+\mu^2-i\varepsilon)v}\,\Phi(v)\,dv\,,$$

где $\Phi(v)$ определяется из следующего уравнения

$$-i\frac{\partial\Phi}{\partial v} = \left[\partial_\mu^2 + 2ip\,\partial + 2ep_\mu\varphi_\mu(x) - 2ie\,\varphi_\mu(x)\,\partial_\mu - e^2\varphi^2(x)\right]\Phi. \tag{2.2}$$

Будем искать решение для Φ в виде:

$$\Phi = \exp\, J\,. \tag{2.3}$$

Из (2.2) получим:

$$-i\frac{\partial J}{\partial v} = (\partial_\mu^2 + 2ip\,\partial)\,J - 2ie\varphi_\mu(x)\,\partial_\mu J + (\partial_\mu J)^2 + 2e\partial p_\mu\,\varphi_\mu(x) - e^2\varphi_\mu^2(x)\,. \tag{2.4}$$

Будем искать решение для J в виде ряда теории возмущений по e:

$$J = \sum_{n=1}^{\infty} e^n J_n\,. \tag{2.5}$$

При этом для J_n получаем следующее уравнение:

$$-i\frac{\partial J_n}{\partial v} = (\partial_\mu^2 + 2ip_\mu\partial_\mu)\,J_n - 2i\,\varphi_\mu(x)\,\partial_\mu J_{n-1} + \sum_{m=1}^{n-1}\partial_\mu J_m\,\partial_\mu J_{n-m} +$$
$$+ 2p_\mu\varphi_\mu(x)\,\delta_{n1} - \varphi_\mu^2(x)\,\delta_{n2}\,. \tag{2.6}$$

Из (2.6) имеем решение для J_n:

$$J_n(v) = i\int_0^v e^{i(\partial^2_\mu + 2ip_\mu\partial_\mu)(v-v')}\{2p_\mu\varphi_\mu(x)\,\delta_{n1} - \varphi^2(x)\,\delta_{n2} - \\ - 2i\,\varphi_\mu\partial_\mu J_{n-1}(v') + \sum_{m=1}^{n-1}\partial_\mu J_m(v')\partial_\mu J_{n-m}(v')\}. \tag{2.7}$$

Из (2.7), в частности, имеем:

$$J_1(v) = \frac{2i}{(2\pi)^2}\int p_\mu\varphi_\mu(k)\,d^4k\int_0^v e^{ikx - i(k^2+2pkv')}\,dv', \tag{2.8}$$

$$J_2(v) = -\frac{i}{2}\int d^4k_1\,d^4k_2 f_{\mu\varphi}(k_1, k_2, x, v)\,\varphi_\mu(k_1)\,\varphi_\varphi(k_2)\,, \tag{2.9}$$

где

$$f_{\mu\varphi}(k_1, k_2, x, v) = \frac{1}{(2\pi)^4}e^{i(k_1+k_2)x}\int_0^v dv' e^{-i[(k_1+k_2)^2+2p(k_1+k_2)](v-v')}\left\{\delta_{\mu p} + \right. \\ + \frac{4k_{1p}p_\mu}{(k_1^2+2pk_1)}(e^{-i(k_1^2+2pk_1)v'}-1) + \tag{2.10}$$

$$\left. + \frac{4p_\mu p_\varrho(k_1k_2)}{(k_1^2+2pk_1)(k_2^2+2pk_2)}(e^{-i(k_1^2+2pk_1)v'}-1)(e^{-i(k_2^2+2pk_2)v'}-1)\right\}.$$

Подставив найденное решение в виде ряда по «e» для J в (2.3) мы с помощью (2.1) найдем выражение для функции Грина G во внешнем поле. Легко увидеть, что полученная таким образом модифицированная теория возмущений для G существенно улучшает результаты обычной теории возмущений. Действительно, уже первое приближение для J (т. е. eJ_1) соответствует сумме всех фейнмановских диаграмм для функции Грина в произвольном внешнем поле φ_μ со взаимодействием вида $2ep_\mu\varphi_\mu$ с тем только отличием, что вместо точных фейнмановских знаменателей (типа $(p+\Sigma k_i)^2+\mu^2)$) здесь фигурируют знаменатели (типа $p^2+2p_\mu\Sigma k_i+\Sigma k_i^2+\mu^2$) с опущенными коррелиционными членами k_ik_j $i\neq j$ для внутренних (виртуальных) импульсов. На долю всех остальных J_n и приходится методом последовательных приближений учесть эти опущенные члены (и учесть вклад от оставшегося взаимодействия с φ_μ вида φ_μ^2 и $2\varphi_\mu\partial_\mu$). Сходимость ряда модифицированной теории возмущений для G определяется малостью эффективного* параметра «a». Оценивая этот параметр a, легко убедиться, что он эффективно мал в инфракрасной области и при больших энергиях (кроме узкой области углов). Нашей конечной целью является получение модифицирован-

* Параметр $a \approx \dfrac{2\sum_{i\neq j}' k_ik_j}{p^2+\mu^2+2p_i\Sigma k_i+\Sigma k_i^2|_{k=k\alpha\varphi}}$, где $k_{\alpha\varphi}$ — эффективное значение виртуального импульса в максимуме подинтегрального выражения.

ной теории возмущений для производящего функционала. Для этого целесообразно выразить решение для функции Грина во внешнем поле в такой форме, чтобы функциональное осреднение по внешнему полю можно было провести точно. Для этой цели весьма полезно операторное решение для функции Грина во внешнем поле. Для получения операторного решения для G достаточно найти операторное решение уравнения (2.2) для Φ, для чего введем добавочное взаимодействие с источником (t_μ) обобщенного импульса π_μ. При этом уравнение для Φ приобретает вид [3]:

$$-i\frac{\partial\Phi}{\partial\nu}=\left[\pi_\mu^2(x)+i\pi_\mu t_\mu(\nu)+2p_\mu\varphi_\mu(x)+2ip_\mu\partial_\mu\right]\Phi, \tag{2.11}$$

где

$$\pi_\mu(x)=(\partial_\mu-ie\,\varphi_\mu(x))\,. \tag{2.12}$$

Можно показать, что решение для $\Phi(\nu)$ имеет вид [3]:

$$\Phi=e^{i\int_0^\nu\frac{\delta^2}{\delta t_\mu(\xi)\delta t_\mu(\xi)}d\xi}\Phi_1(t)_{|t=0}, \tag{2.13}$$

где $\Phi_1(t)$ определяется из следующего уравнения:

$$-i\frac{\partial\Phi_1(t)}{\partial\nu}=\left\{2ip\partial+i(\partial_\mu-ie\,\varphi_\mu(x))\,t_\mu(\nu)+2p_\mu\varphi_\mu(x)\right\}\Phi_1(t)\,. \tag{2.14}$$

Из (2.14) имеем

$$\Phi_1(t)=\exp\{i\int_0^\nu d\nu'(2p_\mu+t_\mu(\nu'))\,\varphi_\mu(x-2p(\nu-\nu')-\int_{\nu'}^\nu t\,(\xi)d\xi)\}. \tag{2.15}$$

Итак, окончательно Φ имеет вид [3]:

$$\begin{aligned}\Phi=e^{i\int_0^\nu\frac{\delta^2}{\delta t_\mu(\xi)\delta t_\mu(\xi)}d\xi}\exp i\Big\{e\int_0^\nu d\nu'\,(2p_\mu+t_\mu(\nu'))\,\varphi_\mu(x-2p(\nu-\nu')-\\-\int_{\nu'}^\nu t(\xi)\,d\xi)\Big\}|t=0\,.\end{aligned} \tag{2.16}$$

Выражение (2.16) можно переписать в виде:

$$\varphi(\nu)=\langle\exp 2ie\int_0^\nu P_\mu(\nu')\,\varphi_\mu(x(\nu')d\nu'\rangle,$$

где

$$P_\mu(\nu')=p_\mu+\frac{1}{2}\,\frac{\delta}{\delta t_\mu(\nu')}\,;$$

$$x(\nu')=x-2\int_{\nu'}^\nu P_\mu(\xi)\,d\xi=x-2p(\nu-\nu')-\int_{\nu'}^\nu\frac{\delta}{\delta t(\xi)}\,d\xi\,. \tag{2.17}$$

Причем для любой функции от оператора $\frac{\delta}{\delta t}$ знак $\langle\ \rangle$ определяется следующим соотношением:

$$\left\langle A\left(\frac{\delta}{\delta t}\right)\right\rangle = A\left(\frac{\delta}{\delta t}\right) e^{i\int_{\tau}^{\nu} t^2_{\tau}(\xi)\,d\xi}\,|t=0\,. \tag{2.18}$$

Для функции Грина в произвольном внешнем поле φ_μ мы получаем окончательно следующее операторное решение

$$G(x, y|i\varphi) = \frac{i}{(2\pi)^4}\int e^{ip(x-y)}\,d^4p\int_0^\infty e^{-i(p^2+\mu^2-i\varepsilon)\nu}\,d\nu\left\langle \exp 2ie\int_0^\nu P_\mu(\nu')\,\varphi_\mu(x(\nu'))\,d\nu'\right\rangle. \tag{2.19}$$

Полученное операторное решение обладает тем преимуществом, что внешнее поле φ_μ входит в экспоненту линейно и потому функциональное осреднение по этому полю (при нахождении Z) проводится тривиальным образом. Отметим также, что перейдя к импульсному представлению в функциональном пространстве t_μ, мы получим континуальное представление для G, которое является обобщением лагранжевой формулировки Фейнмана для уравнения Клейна—Годрона на случай функции Грина этого уравнения (применение этой лагранжевой формулировки в инфракрасной области рассмотрено* — в приложении № 2 работы [2]). Нам остается ещё проследить методику получения модифицированной теории возмущений для $G(x, y|i\varphi)$ с помощью операторного решения (2.19). Как видно из (2.3)—(2.5) модифицированная теория возмущений формально соответствует теории возмущений в экспоненте до проведения интегрирования по собственному времени (ν). Следовательно в то время как обычный ряд теории возмущений для G соответствует представлению

$$\langle \exp\left[2ie\int_0^\nu P_\mu(\nu')\,\varphi_\mu(x(\nu')\,d\nu'\right]\rangle$$

ряда теории возмущений по заряду, т. е. в виде

$$\left\langle \exp\left[2ie\int_0^\nu P_\mu(\nu')\varphi_\mu(x(\nu'))\,d\nu'\right]\right\rangle = \sum_{n=0}^{\infty}\frac{e^n a_n}{n!}\,. \tag{2.20}$$

Модифицированная теория возмущений соответствует представлению той же величины в виде ряда теории возмущений в экспоненте, т. е.

$$\langle \exp 2ie\int_0^\nu P_\mu(\nu')\,\varphi_\mu(x(\nu'))\,d\nu'\rangle = \exp\sum_{n=1}^{\infty} e^n b_n\,. \tag{2.21}$$

* Примечание: См. также препринт [5], где фейнмановское решение для функции Грина во внешнем поле также применено для построения инфракрасной асимптотики для одночастичной функции Грина электрона.

Из (2.20) имеем для коэффициентов a_n следующее выражение:

$$a_n = \langle (2i \int_0^\nu P_\mu(\nu')\ \varphi_\mu(x(\nu'))\, d\nu')^n \rangle\,. \tag{2.22}$$

В частности,

$$a_1 = \frac{2i}{(2\pi)^2} \int d^4k\, p_\mu\, \varphi_\mu(k)\, e^{ikx} \int_0^\nu e^{-i(k^2+2pk)\nu'}\, d\nu'\,, \tag{2.23}$$

$$a_2 = -\frac{2}{(2\pi)^4} \int d^4k\, d^4k_1 \int_0^\nu d\nu_1 \int_0^\nu d\nu_2\, \varphi_\mu(k_1)\, \varphi_\varrho(k_2)\, \exp i \{(k_1+k_2)\, x -$$

$$- \sum_{n=1}^{2} 2pk_n\,(\nu-\nu_n) - \sum_{n,m=1}^{2} k_n\, k_m \left(\nu - \frac{\nu_n+\nu_m}{2} - \frac{|\nu_n-\nu_m|}{2}\right)\Big\} \times \tag{2.24}$$

$$\times \left[2p_\mu p_\varrho + 2k_{2\varrho} p_\mu\, \Theta(\nu_2-\nu_1) + 2p_\varrho\, k_{2\mu}\, \Theta(\nu_1-\nu_2) + i\delta_{\mu\varrho}\, \delta(\nu_1-\nu_2)\right].$$

Легко убедиться, что коэффициенты b_n в (2.21) связаны с найденными коэффициентами a_n следующим соотношением:

$$b_j = \sum_n (-1)^{(\sum_i n_i - 1)} \Big(\sum_i n_i - 1\Big)!\, \prod_i \left[\frac{1}{n_i!} \left(\frac{a_i}{i!}\right)^{n_i}\right], \tag{2.25}$$

где первое суммирование производится по всем совокупностям положительных целых чисел (или нулей) $n = n_1, n_2 \ldots$ таких, для которых $\sum_i i n_i = j$

$$b_1 = a_1\,; \qquad b_2 = \frac{a_2 - a_1^2}{2}\,;$$

$$b_3 = \frac{1}{3!} \{a_3 - 3a_1\, a_2 + 2a_1^3\}\,; \tag{2.26}$$

$$b_4 = \frac{1}{4!} \{a_4 - 4a_1\, a_3 - 3a_2^2 + 12a_1^2\, a_2 - 6a_1^4\}\,.$$

Из приведенных формул легко убедиться, что $b_n = J_n$ и, следовательно, указанной процедурой вычисление b_n мы также получаем модифицированную теорию возмущений для $G(x, y|i\varphi)$.

3. Замкнутое выражение для Z

Подставив найденное операторное решение (2.19) для функции Грина во внешнем поле в операторное решение (1.18) для Z мы получим

$$cZ = \exp - \left\{ \frac{1}{(2\pi)^2} \int\int d^4x\, d^4p\, \zeta(x)\, e^{ipx}\, \zeta(p) \int_0^\infty dv e^{-i(p^2+\mu^2-i\varepsilon)v} \times \right.$$

$$\left. \times \left\langle \exp 2e \int_0^v dv'\, P_\mu(v') \frac{\delta}{\delta I_\mu(x(v'))} \right\rangle \right\} \times$$

$$\times \exp \frac{2}{(2\pi)^4} \left\{ \int\int d^4x\, d^4p \int_{s_0}^\infty \frac{dv}{v} e^{-i(p^2+\mu^2-i\varepsilon)v} \times \right. \tag{3.1}$$

$$\left. \times \left[\left\langle \exp 2e \int_0^v dv'\, P_\mu(v') \frac{\delta}{\delta I_\mu(x(v'))} \right\rangle - 1 \right] \right\} \times$$

$$\times \exp \frac{i}{2} \int\int d^4x\, d^4y\, I_\mu(x)\, \bar{D}_{\mu\varrho}(x-y)\, I_\varrho(y)\,,$$

$$\bar{D}_{\mu\varrho}(k) = \left(\delta_{\mu\varrho} - \frac{k_\mu k_\varrho}{k^2} \right) i \int_{a_0}^\infty e^{-ia(k^2+\lambda^2-i\varepsilon)}\, da\,. \tag{3.2}$$

По поводу полученному и последующих выражений для Z сделаем общее замечание. Как известно, выражение для Z до проведения программы перенормировок содержит расходимости. Практически бывает удобнее провести программу перенормировок в конце вычислений, но для этого необходимо как-то регуляризовать выражение для Z так, чтобы при вычислении бесконечности отсутствовали. При этом существенно, чтобы регуляризационная процедура не нарушила релятивистскую и калибровочную инвариантности теории. В рамках предложенного выражения для Z это удается сделать путем введения двух малых параметров (типа собственного времени) s_0 и a_0, где s_0 введен как нижний предел (вместо нуля) интегрирования по «v» в поляризационном члене, а a_0 как нижний предел (вместо нуля) в интеграле (3.2) по «a». Легко видеть, что пока a_0, s_0 не равны нулю, в теории бесконечности отсутствуют. Это дает нам возможность провести все вычисления, а также программу перенормировок. После проведения программы перенормировок мы можем перейти к пределу $s_0 = a_0 = 0$. Все бесконечности теории возмущений, которые естественно появляются в пределе $s_0 = a_0 = 0$ входят лишь в постоянные перенормировки. При этом перенормированные величины не зависят от них. Вернемся к выражению (3.1). Не представляет труда

провести явно оставшиеся функциональное дифференцирование по источникам I_μ векторного поля φ_μ. Действительно, разлагая выражение (3.1) в ряд по $G\left(x, y\left|\frac{\delta}{\delta I}\right.\right)$ мы легко можем провести оставшееся функциональное дифференцирование по $\frac{\delta}{\delta I}$. При этом получаем следующее выражение для Z;

$$cZ = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2\pi)^{2n}} \prod_{s=1}^{n} \int d^4p^{(s)}\, d^4x_s\, \zeta^*(x_s)\, e^{ip^{(s)}x_s}\, \zeta(p^{(s)}) \int_0^{\infty} dv_s\, e^{-i((p^{(s)})^2+\mu^2-i\varepsilon)v_s} \times$$

$$\times \left\{1 + \sum_{z=1}^{\infty} \frac{i^r}{r!\,(2\pi)^4} \prod_{s=1}^{r}\right] d^4x'_s\, d^4p_1^{(s)} \int_{s_0\to 0}^{\infty} \frac{dv'_s}{v'_s}\, e^{-i((p_1^{(s)})^2+\mu^2-i\varepsilon)v'_s}\Bigg\} \times \tag{3.3}$$

$$\times \exp i\, A_{nr}\left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t_1}\right) \cdot \exp i\left\{\sum_{s=1}^{n} \int_0^{v_s} t_s^2(\xi)\, d\xi + \sum_{s=1}^{r} \left\{\int_0^{v'_s} d\xi t_{1s}^2(\xi)\right\}\right|_{t_s=t_{1s}=0},$$

где

$$A_{nr}\left(\frac{\delta}{\delta t};\ \frac{\delta}{\delta t_1}\right) = \frac{1}{2}\int I_\mu(x)\, \bar{D}_{\mu\varrho}(x-y)\, I_\varrho(y)\, d^4x\, d^4y +$$

$$+ 2e \sum_{s=1}^{n} \int_0^{v_s} d\zeta \int d^4y\, I_\mu(y)\, \bar{D}_{\mu\varrho}(y - x_s(\zeta))\, P_\varrho^{(s)}(\zeta) +$$

$$+ 2e \sum_{s=1}^{r} \int_0^{v'_s} d\xi' \int d^4y\, I_\mu(y)\, \bar{D}_{\mu\varrho}(y - x'_s(\xi'))\, P_{1\varrho}^{(s)}(\xi) + \tag{3.4}$$

$$+ 2e^2 \sum_{s,s'=1}^{n} \int_0^{v_s} d\xi \int_0^{v'_s} d\xi'\, P_\mu^{(s)}(\xi)\, \bar{D}_{\mu\varrho}(x_{s'}(\xi) - x_{s'})\,\xi')\, P_\varrho^{(s')}(\xi') +$$

$$+ 2e^2 \sum_{s,s'=1}^{r} \int_0^{v'_s} d\xi \int_0^{v'_s} d\xi'\, P_{1\mu}^{(s)}(\xi)\, \bar{D}_{\mu\varrho}(x'_s(\xi) - x'_{s'}(\xi'))\, P_{1\varrho}^{(s')}(\xi') +$$

$$+ 4e^2 \sum_{s=1}^{n} \sum_{s'=1}^{r} \int_0^{v_s} d\xi \int_0^{v'_{s'}} d\xi'\, P_\mu^{(s)}(\xi)\, \bar{D}_{\mu\varrho}(x_s(\xi) - z'_{s'}(\xi'))\, P_{1\varrho}^{(s')}(\xi').$$

Не представляет труда получить выражение для Z, в котором поляризационные эффекты представлены не в виде ряда по степеням функции Грина G (в (3.3) ряд по «r»). Перегруппировав члены ряда по «r» и проведя суммирование можно представить этот ряд в виде другого ряда в exp. Наиболее

простой вид выражения для Z приобретает в том случае, когда поляризационными эффектами можно пренебречь. В этом случае Z имеет вид:

$$Z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2\pi)^{2n}} \prod_{s=1}^{n} \int d^4x_s\, d^4p^{(s)}\, \zeta^*(x_s)\, e^{ip^{(s)}x_s}\, \zeta(p)^{(s)} \int_0^{\infty} dv_s\, e^{-i((p^{(s)})^2+\mu^2-i\varepsilon)v_s} \tag{3.5}$$

$$\langle \exp i\, A_n \rangle$$

$$A_n = \frac{1}{2} I_\mu(x)\, \bar{D}_{\mu\varrho}(x-y) I_\varrho(y)\, d^4x\, d^4y + 2e \times$$

$$\times \sum_{s=1}^{n} \int d^4y \int_0^{v_s} d\xi\, P_\mu^{(s)}(\xi)\, \bar{D}_{\mu\varrho}(x_s(\xi) - y)\, I_\varrho(y) + \tag{3.6}$$

$$+ 2e^2 \sum_{s,s'=1}^{n} \int_0^{v_s} d\zeta \int_0^{v'_s} d\xi'\, P_\mu^{(s)}(\xi)\, \bar{D}_{\mu\varrho}(x_s(\xi) - x_{s'}(\xi'))\, P_\varrho^{(s')}(\xi'),$$

где

$$\left\langle I_n\left(\frac{\delta}{\delta t_1} \dots \frac{\delta}{\delta t_n}\right)\right\rangle = I_n\left(\frac{\delta}{\delta t_1} \dots \frac{\delta}{\delta t_n}\right) e^{i \sum_{s=1}^{n} \int_0^{v_s} d\xi\, t_s^2(\xi)}. \tag{3.7}$$

Из (3.5) легко получить методом § 2 модифицированную теорию возмущений для интересующих нас функций Грина. В частности, в инфракрасной области, в том случае, когда все векторные частицы являются мягкими, достаточно ограничиться первым приближением модифицированной теории возмущений. В этом случае $\langle \exp\, i\, A_n \rangle = \exp i \langle A_n \rangle$ выражение Z принимает вид:

$$Z = \Sigma \frac{(-1)^n}{n!\,(1\pi)_n^2} \prod_{s=1}^{n} \int d^4z_s\, d^4p^{(s)}\, \zeta^*(x_s)\, e^{ip^{(s)}x_s}\, \zeta(p^{(s)}) \times$$

$$\times \int_0^{\infty} dv_s \exp^{-i} i \left\{(p^{(s)})^2 + \mu^2 - i\varepsilon)\, v_s\right\} \exp i \langle A_n \rangle; \tag{3.8}$$

$$\langle A_n \rangle = \frac{1}{2} \int I_\mu(x)\, \bar{D}_{\mu\varrho}(x-y)\, I_\varrho(y)\, d^4x\, d^4y +$$

$$+ \sum_{s=1}^{n} \frac{2ie}{(2\pi)^2} \int p_\mu^{(s)}\, \bar{D}_{\mu\varrho}(k)\, I_\varrho(k)\, e^{ikx_s}\, \frac{e^{-i(k^2+2p^{(s)}k)v_s} - 1}{k^2 + 2p^{(s)}k}\, d^4k - \tag{3.9}$$

$$- \sum_{s=1}^{n} \frac{(2\ )^2}{(2\pi)^4} \int d^4k \left[p_\mu^{(s)}\, \bar{D}_{\mu\varrho}(k)\, p_\varrho^{(s)} \left(\frac{e^{-i(k^2+2p^{(s)}k)v_s} - 1}{(k^2 + 2p^{(s)}k)^2} + \right.\right.$$

$$\left.\left. + \frac{iv_s}{k^2 + 2p^{(s)}k} \right) - \frac{iv_s}{2^2}\, \bar{D}_{\mu\mu}(k) \right] -$$

$$- \sum_{s \neq s'=1}^{n} \frac{2e^2}{(2\pi)^2} \int d^4k \, p_\mu^{(s)} \, \bar{D}_{\mu\varrho}(k) \, p_\varrho^{(s')} \times$$

$$\times \frac{(e^{-i(k^2+2p^{(s)}k)\nu_s} - 1)\,(e^{-i(k^2-2p^{(s')}k)\nu_s} - 1)}{(k^2 + 2p^{(s)}k)\,(k^2 - 2p^{(s')}k)} .$$

В частности из (3.8) обычной процедурой получим в этом приближении следующее выражение для одночастичной функции Грина бозе-частицы с массой μ^2 (см. также приложение № 2 работы автора [2]):

$$G^{(1)}(p) = i \int_0^\infty d\nu e^{-i(p^2+\mu^2-i\varepsilon)\nu + e^2 a_1(\nu)} , \tag{3.10}$$

где

$$a_1(\nu) = -\frac{2^2 i}{(2\pi)^4} \int d^4k \left[p_\mu \bar{D}_{\mu\varrho}(k) \, p_\varrho \left\{ \frac{e^{-i(k^2+2pk)\nu} - 1}{(k^2+2pk)^2} + \right. \right.$$

$$\left. \left. + \frac{i\nu}{k^2 + p2k} \right\} - \frac{i\nu}{4} D_{\mu\mu}(k) \right] . \tag{3.11}$$

Из (3.10) в случае взаимодействия с электромагнитным полем ($\lambda = 0$) получаем [2] известную инфракрасную асимптотику для перенормированной функции Грина $G(p)$ (ответ в поперечной калибровке!) ($p^2 + \mu^2 \geqslant 0$)

$$G^{(1)}(p)_{|p^2 \to -\mu^2} = \frac{1}{p^2+\mu^2} \frac{f(p^2)}{\left(1 + \frac{p^2}{\mu^2}\right)^\alpha} , \quad \text{где} \quad \alpha = \frac{3e^2}{8\pi^2} ; \tag{3.12}$$

$$f(p^2) = i \int_0^\infty d\nu \, \nu^\sigma \exp\left\{ -i\nu - \varepsilon\nu - \frac{i\alpha}{2\nu}\left(1 + \frac{p^2}{\mu^2}\right) \right\} \approx 1 + \frac{1}{2}\left(1 + \frac{p^2}{\mu^2}\right).$$

4. Модифицированная теория возмущений для одночастичной функции Грина

Обычной методикой получим следующее выражение для одночастичной функции Грина бозона:

$$G(p) = i \int_0^\infty d\nu_n \, {}^{-i(p^2+\mu^2-i\varepsilon)\nu} \langle \exp 2i\varepsilon^2 \int_0^\nu d\nu' \int_0^\nu d\nu'' \, P_\mu(\nu') \, \bar{D}_{\mu\varrho}\left(2 \int_{\nu'}^\nu P(\xi)\, d\xi\right) P_\varrho(\nu'') \rangle , \tag{4.1}$$

$$P_\mu(\nu') = P_\mu + \frac{\delta}{2\delta t_\mu(\nu')} .$$

С помощью (4.1) легко, в частности, развить модифицированную теорию возмущений для $G(p)$.

В самом деле, из (4.1) имеем

$$G(p) = i \int_0^\infty d\nu \exp\{-i(p^2 + \mu^2 - i\varepsilon)\nu + \sum_{n=1}^{\infty} e^{2n} b_n\}. \tag{4.2}$$

При этом коэффициенты b_n связаны соотношениями (2.25—2.26) с более простыми по структуре коэффициентами a_n. Последние в данном случае равны соответственно

$$a_n = \langle (2i \int_0^\nu d\nu' \int_0^\nu d\nu'' P_\mu(\nu') \bar{D}_{\mu\varrho}(2 \int_{\nu'}^{\nu''} P(\xi)\, d\xi) P_\varrho(\nu'')^n \rangle =$$

$$= \frac{(2i)^n}{(2\pi)^{4n}} \prod_{s=1}^{n} \int_0^\nu d^4k_s \int_0^\nu d\nu_s' \int_0^\nu d\nu_s'' \left\{ \left[p_\mu + \sum_{s_1=1}^{n} k_{s,\mu} [\Theta(\nu_s' - \nu_{s,}'') - \Theta(\nu_s' - \nu_{s_1}')] + \right.\right. \tag{4.3}$$

$$+ \frac{1}{2} \frac{\delta}{\delta t_\mu(\nu_s')} \bar{D}_{\mu\varrho}(k_s) \left[p_\varrho + \sum_{s,=1}^{n} [\Theta(\nu_s'' - \nu_{s,}'') - \Theta(\nu_s'' - \nu_{s,}')] k_{s,\varrho} + \frac{1}{2} \frac{\delta}{\delta t_\varrho(\nu_s'')} \right]$$

$$\exp\left[2ipk_s(\nu_s'' - \nu_s') + \frac{i}{2} \sum_{s,s_1=1}^{n} k_s k_{s_1} (|\nu_s' - \nu_{s_1}'| + |\nu_s'' - \nu_{s_1}''| - \right.$$

$$\left.\left. - |\nu_s' - \nu_{s_1}''| - |\nu_{s_1}'' - \nu_s''|) \right] e^{i \int_0^\nu t^2(\xi) d\xi} \Big| t = 0 \right\}.$$

В частности из (4.3) имеем (в поперечной калибровке):

$$a_1 = \frac{2i}{(2\pi)^4} \int d^4k \int_0^\nu d\nu' \int_0^\nu d\nu'' \left[e^{2ipk(\nu''-\nu') - ik^2|\nu''-\nu'|} p_\mu p_\varrho + \right.$$

$$\left. + \frac{1}{2} i\delta(\nu' - \nu'') \delta_{\mu\varrho} \right] D_{\mu\varrho}(k), \tag{4.4}$$

$$a_2 = -\frac{4}{(2\pi)^8} \int d^4k \int d^4k' \int_0^\nu d\nu' \int_0^\nu d\nu'' \int_0^\nu d\nu_1' \int_0^\nu d\nu_1'' D_{\mu\varrho}(k) D_{\mu_1\varrho_1}(k') \times$$

$$\times \left\{ \left[(p_\mu + k_\mu'(\Theta(\nu' - \nu_1'') - \Theta(\nu' - \nu_1'))(p_\varrho + k_\varrho'(\Theta(\nu'' - \nu_1'') - \Theta(\nu'' - \nu_1'))) + \right.\right.$$

$$+ \frac{1}{2} i\delta(\nu' - \nu'') \delta_{\mu\varrho} \left] \left[(p_{\mu_1} + k_{\mu_1}(\Theta(\nu_1' - \nu'') - \Theta(\nu_1' - \nu')(p_\varrho + \right. \tag{4.5}$$

$$\left.\left. + k_{\varrho_1}(\Theta(\nu_1'' - \nu'') - \Theta(\nu_1'' - \nu')) + \frac{1}{2} i\delta(\nu_1' - \nu_1'') \delta_{\mu_1\varrho_1} \right] + \right.$$

$$+\frac{i}{2}\left[c^{\nu'\nu_1''\nu''\nu_1''}_{\mu\mu_1\varrho\varrho_1}+c^{\nu'\nu_1''\nu''\nu'}_{\mu\varrho_1\varrho\mu_1}+c^{\nu''\nu_1'\nu'\nu_1''}_{\varrho\mu_1\mu_2}+c^{\nu''\nu_1''\nu'\nu_1'}_{\varrho\varrho_1\mu\mu_1}\right]-$$

$$=\frac{1}{4}\delta(\nu'-\nu_1')\,\delta(\nu''-\nu_1'')\,\delta_{\mu\mu_1}\delta_{\varrho\varrho_1}-\frac{1}{4}\delta(\nu'-\nu_1'')\,(\nu''-\nu,')\,\delta_{\mu\varrho_1}\delta_{\varrho\mu_1}\Bigg\}$$

$$\exp\left\{2ipk(\nu''-\nu')+2ipk'\,(\nu_1''-\nu_1')-ik^2\,|\,\nu''-\nu'|-\right.$$

$$\left.-\,i(k')^2|\nu_1''-\nu_1'|+ikk'\left(|\nu'-\nu_1'|+|\nu''-\nu_1''|-|\nu'-\nu_1''|-|\nu''-\nu_1'|\right)\right\}.$$

В первом приближении модифицированной теории возмущений (оставляя в (4.2) лишь коэффициент b_1) мы как и следовало ожидать получаем для $G(p)$ выражение (3.10)–(3.11). Легко убедиться (разложив решение (3.10) в ряд обычной теории возмущений в проведя интегрирование по «ν») что первое приближение модифицированной теории возмущений соответствует сумме всех фейнмановских диаграмм для $G(p)$ с тем только отличием, что вместо характерного знаменателя для фейнмановских диаграмм для $G(p)$ вида $(p+\Sigma k_i)^2+\mu^2$ здесь появляются знаменатели типа $(p^2+2p\,\Sigma\,k_i+\Sigma\,k_i^2)+\mu^2$. На долю остальных приближений модифицированной теории возмущений и приходится учесть вклад опущенных корреляционных членов k_ik_j $i\neq j$. Очевидно, что в инфракрасной области эти члены малосущественны и потому формула (3.10) уже передает инфракрасную асимптотику для $G(p)$. Оставляя в решении (4.2) первые два члена b_1 и b_2 мы получаем следующее* выражение для $G(p)$:

$$G^{(2)}(p)=i\int\limits_0^\infty d\nu e^{-(p^2+\mu^2-i\varepsilon)i\nu+\epsilon^2b_1+\epsilon^4b_2} \tag{4.2}$$

$$b_1=i\nu\,M_1(p^2)-\frac{2^2\,i}{(2\pi)^4}\int d^4k\,p_\mu\,D_{\mu\varrho}(k)\,p_\varrho\,\frac{e^{-i(k^2+2pk)\nu}-1}{(k^2+2pk-i\varepsilon)^2},$$

$$b_2=i\nu M_2(p^2)+b_2'-i\nu M_2'(p^2), \tag{4.6}$$

где

$$M_1(p^2)=-\frac{2i}{(2\pi)^4}\int d^4k\,p_\mu\,\bar{D}_{\mu\varrho}(k)\,p_\varrho\,\frac{2}{k^2+2pk-i\varepsilon}, \tag{4.7}$$

$$M_2(p^2)=-\frac{4^2\,i}{(2\pi)^8}\int D_{\mu\varrho}(k)\,D_{\mu_1\varrho_1}(k_1)\,\frac{d^4k}{(p+k)^2-p^2-i\varepsilon}\times$$

$$\times\frac{d^4k_1}{(p+k+k_1)^2-p^2-i\varepsilon}, \tag{4.8}$$

* В выражениях для $G(p)$ опущены некоторые члены вида ν const поскольку они приводят лишь к перенормировке массы, в формуле (4.5)

$$c^{\nu'\nu_1'\nu''\nu_1''}_{\mu\mu_1\varrho\varrho_1}=\delta_{\mu\mu_1}\delta(\nu'-\nu_1')[(p_\varrho+k_\varrho'\,(\Theta(\nu''-\nu_1')-\Theta(\nu''-\nu_1''))]\;[p_{\varrho_1}+k_{\varrho_1}(\Theta(\nu_1''-\nu')-\Theta(\nu_1''-\nu''))]$$

$$\left\{\frac{p_\mu\left(p_\varrho+\frac{1}{2}k_{1\varrho}\right)\left(p_{\mu 1}+\frac{1}{2}k_{\mu 1}\right)p_{\varrho 1}-\frac{1}{2}\delta_{\mu\mu 1}c_{\varrho\varrho 1}(x,x_1)}{(p+k_1)^2p^2-i\varepsilon}+\right.$$

$$\left.+\frac{p_\mu p_\varrho\left(p_{\mu 1}+\frac{1}{2}k_{\mu 1}\right)\left(p_{\varrho 1}+\frac{1}{2}k_{\varrho 1}\right)-\frac{1}{2}\delta_{\mu\mu 1}c_{\varrho\varrho 1}(k,k)}{}\right\}$$

$$c_{\varrho\varrho 1}(k,k_1)=\left[\frac{1}{2^3}((p+k_1)^2-p^2-i\varepsilon)\,\delta_{\varrho\varrho 1}-\right.$$

$$\left.-\left(p_\varrho+\frac{1}{2}k_{1\varrho}\right)p_{\varrho 1}\right]\left[(p+k)^2-p^2-i\varepsilon\right] \tag{4.9}$$

$$M_2'(p^2)=-\frac{4i}{(2\pi)^8}\int p_\mu D_{\mu\varrho}(x)\,p_\varrho p_{\varrho 1}D_{\varrho 1}(k_1)\,p_{\mu 1}\times \tag{4.10}$$

$$\times\frac{d^4k\,d^4k_1}{[(p+k)^2-p^2-i\varepsilon]](p+k_1)^2-p^2-i\varepsilon][(p+k+k_1)^2-p^2-2k_1k-i\varepsilon]}$$

Необходимо отметить, что программа перенормировок и исключения бесконечности (если работать сразу с $D_{\mu\nu}$, а не $\bar{D}_{\mu\nu}$) может быть проведено в выражении для функции Грина ещё до окончательного интегрирования по собственному времени ν. Действительно перенормировочные члены содержатся лишь в членах линейных по ν типа $i\nu M_n(p^2)e^{2n}$ в экспоненте (4.2) причем $M_n(p^2)$ совпадает с массовым оператором обычной теории возмущений, (с тем только отличием, что в массовом операторе теории возмущений нужно формально заменить μ^2 через $-p^2$) т. е. $M_n(p^2)=\bar{M}_n(p^2,\ \mu^2\to -p^2)$. Перенормировка и сводится к замене $M_n(p^2)\to M_n^r(p^2)=\int\limits_{-\mu^2}^{p_2}\frac{d^2M_n(z)}{d^2z}\,dz$, ($\bar{M}_n$ — перенормированное выражение для массового оператора по теории возмущений.)

5. Вершинная часть

Инфракрасная асимптотика для вершинной части в случае малых импульсов векторного поля легко получить непосредственно из приближенного выражения (3.8) для Z. Однако можно получить выражение для вершинной части, справедливое не только для малых но и при больших импульсах излученного (поглощенного) кванта векторного поля. Это нетрудно сделать методом построения модифицированной теории возмущений для вершинной части непосредственно с помощью точного выражения (3.5) для Z. Здесь мы дадим несколько другой вывод эквивалентного выражения

для вершинной части весьма удобное для перехода на массовую оболочку для импульсов и для построения приближенных методов расчета. Рассмотрим

$$\frac{\delta G(x, x')}{\delta I_\mu(z)|_{I=0}} = ie\int G(x-y)\,\Gamma_\varrho(x-y, z_1-y)\,G(y-x')\,D_{\varrho\mu}(z_1-z)\,d^4y\,d^4z\,.$$

С помощью операторного решения легко получить следующее выражение для $\Gamma_\mu(p, p+k, k)$

$$G(p)\,\Gamma_\mu(p, p+k, k)\,G(p+k) = \int d^4x\,d^4x'\,d^4k_1\,e^{-ipx+i(p+k)x'}\,A_\mu(x, x', k_1)\,,$$

$$A_\mu(x, x', k_1) = \frac{\delta G(x, x'|i\varphi)}{\delta ie\,\varphi_\mu(k_1)}_{i\varphi_\mu=\frac{\delta}{\delta I_\mu}} \exp\frac{i}{2}\int I_\mu(x)\,D_{\mu\nu}(x-y)\,I_\nu(y)\,d^4x\,d^4y|_{I=0}\,, \tag{5.1}$$

где $G(p)$ — одночастичная функция Грина бозона. Из уравнения для $G(x, x'/i\varphi)$ (см. 1.10) легко убедиться, что

$$\frac{\delta G(x, x'|i\varphi)}{\delta ie\varphi_\mu(k)} = \int d^4y e^{-iky}\,G(x, y|i\varphi)\left\{i\frac{\overleftarrow{\partial}}{\partial y} - i\frac{\vec{\partial}}{\partial y} - 2e\varphi_\mu(y)\right\}G(y, x'|i\varphi)\,. \tag{5.2}$$

Подставив для $G(x, y, /i\varphi)$ и $G(y, x'/i\varphi)$ в операторное решение для (2.19) (и учитывая, что функция Грина во внешнем поле симметрично относительно координат x и y точнее $G(xy) = G(yx)$)* мы получим из (5.2) после проведения функционального дифференцирования по I следующее выражение:

$$G(p)\,\Gamma_\mu(p, p+s, s)\,G(p) = -\int_0^\infty dv_1\int_0^\infty dv_2\,e^{-i(p^2+\mu^2-i\varepsilon)v_1-i((p+s)^2+\mu^2-i\varepsilon)v_2}\times$$

$$\times\left\{(2p_\mu+s_\mu)\langle\exp i\,A\rangle - 4e^2\Big\langle\sum_{n=1}^{2}\int_0^{v_n}d\xi\,\bar{D}_{\mu\varrho}\Big(2\int_\xi^{v_n}d\xi'\,P_\varrho^{(n)}(\xi')\Big)\,P^{(n)}(\xi)\exp i\,A\Big\rangle\right\}, \tag{5.3}$$

где

$$A = \sum_{n=1}^{2}2e^2\int_0^{v_n}d\xi\int_0^{v_n}d\xi'\,P_\mu^{(n)}(\xi)\,\bar{D}_{\mu\varrho}\Big(2\int_\xi^{\xi'}d\xi_1\,P^{(n)}(\xi_1)\Big)\,P_\varrho^{(n)}(\xi') +$$

$$+ 4e^2\int_0^{v_1}d\xi\int_0^{v_2}d\xi'\,P_\mu^{(1)}(\xi)\,\bar{D}_{\mu\varrho}\Big(2\int_{\xi'}^{v_2}P^{(2)}(\xi_1)\,d\xi_1 + 2\int_\xi^{v_1}P^{(1)}(\xi_1)\,d\xi_1\Big)\,P_\varrho^{(2)}(\xi')\,,$$

где

$$P^{(1)}(\xi) = p + \frac{\delta}{2\delta t_1(\xi)}\,; \qquad P^{(2)}(\xi) = p + s + \frac{\delta}{2\delta t_2(\xi)}\,. \tag{5.4}$$

Здесь и в дальнейшем $\left\langle J\left(\frac{\delta}{\delta t_1}, \ldots \frac{\delta}{\delta t_n}\right)\right\rangle$ означает

$$\left\langle J\left(\frac{\delta}{\delta t_1}, \ldots \frac{\delta}{\delta t_n}\right)\right\rangle = J\left(\frac{\delta}{\delta t_1}, \ldots \frac{\delta}{\delta t_n}\right)e^{i\sum_{m=0}^{n}\int_0^{v_m}t_m(\xi)\,\xi}\Big|_{t_m=0}\,. \tag{5.5}$$

В первом приближении модифицированной теории возмущений мы получим из (5.4) следующее выражение для $\Gamma_\mu(p, s)$*:

$$G(p)\,\Gamma_\mu(p, p+s, s)\,G(p+s) \approx -(2p_\mu + s_\mu) \times$$
$$\times \int_0^\infty dv_1 \int_0^\infty dv_2\, e^{-i(p^2+\mu^2-i\varepsilon)v_1 - i((p+s)^2+\mu^2-i\varepsilon)v_2} \exp -iA, \tag{5.6}$$

где

$$A = \frac{(2e)^2}{(2\pi)^4}\int d^4k\, p_\mu \bar{D}_{\mu\varrho}(k)\, p_\varrho \left[\frac{e^{-i(k^2+2pk)v'}-1}{(k^2+2pk)^2} + \frac{iv_1}{k^2+2pk}\right] +$$
$$+ \frac{(2e)^2}{(2\pi)^4}\int d^4(p_\mu+s_\mu)\,\bar{D}_{\mu\varrho}(k)(p+s)_\varrho \left[\frac{e^{-i(k^2+2(p+s)k)v_2}-1}{(k^2+2pk+2ks)^2} + \right.$$
$$\left. + \frac{iv_2}{k^2+2pk+2sk}\right] + \frac{4e^2}{(2\pi)^4}\int d^4k\, p_\mu \bar{D}_{\mu\varrho}(k)(p_\varrho + s_\varrho) \times \tag{5.7}$$
$$\times \frac{(e^{-i(k^2+2pk)v_1}-1)\,(e^{-i(k^2+2pk+2sk)v_2}-1)}{(k^2+2pk)\,(k^2+2pk+2sk)}.$$

В частности из (5.6—5.7) следует следующая асимптотика для перенормированной $\Gamma_\mu(p, p+s, s)$ при $p^2 \to -\mu_e^2$, $(p+s)^2 \to -\mu_e^2$ (μ_e — экспериментальное значение для масс) при $\lambda_e \neq 0$

$$\Gamma_\mu(p, p+s, s) \approx (2p_\mu + s_\mu)\exp\left\{F(p^2, (p+s)^2, s^2) - F(-\mu_e^2; -\mu_e^2; -\lambda_e^2)\right\}$$

$$F(p^2, (p+s)^2, s^2) = -\frac{4ie^2}{(2\pi)^2}\int d^4k\, \frac{p_\mu D_{\mu\varrho}(k)\,(p+s)_\varrho}{(k^2+2pk)(k^2+2pk+2sk)}. \tag{5.8}$$

В частности из (5.8) имеем при $s^2 \gg p^2$; $(p+s)^2$

$$F(p^2, (p(p+s)^2, s^2) \approx -\frac{e^2}{16\pi^2}\,\frac{p(p+s)}{\sqrt{(p(p+s))^2 - p^2(p+s)^2}}\ln \times$$
$$\times \frac{-p(p+s) + \sqrt{(p(p+s))^2 - p^2(p+s)^2}}{(p^2(p+s)^2)^{1/2}} \times \tag{5.9}$$
$$\times \ln \frac{\left[-p(p+s) + \sqrt{p^2(p+s)^2}\right]\mu^2}{\lambda^4} \approx -\frac{e^2}{4\pi^2}\ln\frac{\sqrt{s^2}}{\mu}\ln\frac{\sqrt{s^2}\mu}{\lambda^2}.$$

* Отметим, что поскольку мы работаем в поперечной калибровке, то член пропорциональный s_μ в выражении для $\Gamma_\mu(p, p+s, s)$ можно и не выписывать, поскольку будучи умноженный на $D_{\mu\varrho}(s)$, он выпадает.

Не представляет труда построить с помощью (5.3—5.4) модифицированную теорию возмущений для Γ_μ. При этом для $\Gamma_\mu(p, s)$ получаем следующее выражение.*

$$G(p)\,\Gamma_\mu(p, p+s, s)\,G(p+s) = -\int_0^\infty dv_1 \int_0^\infty dv_2\, e^{-i(p^2+\mu^2-i\varepsilon)v_1 - i((p+s)^2+\mu^2-i\varepsilon)v_2} \times \tag{5.10}$$

$$\times\, 2p_\mu \left\{ e^{\Sigma e^{2n} b_n} + \frac{4e^2}{p^2}\, c(p, v_1)\, e^{\Sigma e^{2n} b'_n(p, p+s, v_1)}\, \frac{4e^2}{(p+s)^2}\, c(p+s, v_2)\, e^{\Sigma e^{2n} b_n(p+s, p, v_2)} \right\},$$

где

$$\frac{(2\pi)^4}{i}\, c(p, v_1) = -\int d^4k\, \frac{e^{-i(k^2-2pk)v_1} - 1}{k^2 - 2pk}\, p_\mu D_{\mu\varrho}(k)\, p_\varrho; \quad a_n = \frac{i^n}{e^{2n}} \langle A^n \rangle, \tag{5.11}$$

$$a'_n(p, p+s, v_1) = -\frac{i^n}{e^{2n}\, c(p)} \left\langle \int_0^{v_1} d\xi p_\mu \bar{D}_{\mu\varrho} \left(\int_\xi^{v_1} P^{(1)}(\xi_1)\, d\xi_1 \right) P^{(1)}_\varrho(\xi)\, A^n \right\rangle, \tag{5.12}$$

$$a'_n(p+s, p, v_2) = -\frac{i^n}{e^{2n}\, c(p+s, v_2)} \times \tag{5.13}$$

$$\times \left\langle \int_0^{v^2} d\xi (p_\mu + s_\mu)\, \bar{D}_{\mu\varrho} \left(\int_\xi^{v^2} P^{(2)}(\xi_1)\, d\xi_1 \right) P^{(2)}_\varrho(\xi)\, A^n \right\rangle.$$

Коэффициенты b_n и b'_n связаны соответственно с a_n и a'_n формулами (2.25)—(2.26).

6. Двухчастичная функция Грина (Рассеяние векторной частицы k на бозе частицу p)

Не представляет труда с помощью (3.5) получить выражение для двухчастичной функции Грина

$$G^{\mu\nu}_{12}(x, y|x', y') = \frac{\delta^4 Z}{\delta I_\mu(x)\, \delta\bar{\xi}(y)\, \delta I_\nu(x')\, \delta\xi(y')} \Big| \xi = I = \bar{\xi} = 0 .$$

Поскольку для вычисления соответствующего сечения рассеяния нам придется перейти на массовую оболочку для импульсов участвующих в процессе частиц, то целесообразно иметь дело с таким выражением для G_{12}, в, котором этот переход наиболее просто совершить. Для этой цели весьма удобна методика получения выражения для G_{12}, аналогичная уже рассмот-

* В случае $p^2 = -\mu_e^2 (p+s)^2 = -\mu_e^2$ $\Gamma_\mu = 2p_\mu B(s^2)$, где B-совпадает с перенормированным значением фигурной скобки в (5.10) после подстановки $v_1 = v_2 = \infty$.

ренной в предыдущем параграфе. Опуская подробные выкладки аналогичные проведенным при нахождении вершинной функции Грина, приведем окончательное выражение для $G_{12}^{\mu\nu}$ (поляризационными эффектами мы и здесь пренебрегаем)

$$D^{-1}(k_1)\,D^{-1}(k_2)\,G_{12}^{\mu\nu}(p_1, k_1 \mid p_2, k_2) = g_{\mu\nu}(p_1, k_1 \mid p_2, k_2) + g_{\nu\mu}(p_1, k_2 \mid p_2, k_1)\,. \quad (6.1)$$

При этом p_1 и (p_2) — импульсы начальной (конечной) бозе частицы, $k_1(k_2)$ — импульс начальной (конечной) векторной частицы.

Импульсы связаны законом сохранения

$$p_1 + k_1 = p_2 + k_2\,. \quad (6.2)$$

Для $g_{\mu\varrho}(p_1k_1/p_2k_2)$ имеем: (здесь и ниже наложено* $\mu \neq \varrho$)

$$g_{\mu\varrho}(p_1, k_1 \mid p_2, k_2) = -4ie^2 \int d^4x e^{i(q_1 - p_3)x} \int_0^\infty dv_1 \int_0^\infty dv_3 \int_0^\infty dv_2\, e^{-i\sum_{n=1}^{3}(p_n^2+\mu^2-i\varepsilon)v_n} \times$$
$$\times \left\langle \left(q_1 + \frac{\delta}{\delta t_1(v_1-\varepsilon)}\right)_\mu \left(q_1 + \frac{\delta}{\delta t_2(v_1-\varepsilon)}\right)_\varrho \exp ie^2 A_{12}' \right\rangle, \quad (6.3)$$

где

$$q_1 = p_1 + k_1 = p_2 + k_2\,,$$

$$A_{12}' = 2\sum_{n=1}^{3} \int_0^{v_n} d\xi \int_0^{v_n} d\xi'\, P_\mu^{(n)}(\xi)\, \bar{D}_{\mu\varrho}\Big(2\int_{\xi'}^{\xi} d\xi_1\, P^{(n)}(\xi_1)\Big) P_\varrho^{(n)}(\xi') + \quad (6.4)$$
$$+ 4\int_0^{v_1} d\xi \int_0^{v_3} d\xi'\, P_\mu^{(1)}(\xi)\, \bar{D}_{\mu\varrho}\Big(x + 2\int_{\xi'}^{v_3} P^{(3)}(\xi_1)\, d\xi_1 -$$
$$- 2\int_{\xi}^{v_1} P^{(1)}(\xi)\, d\xi_1\Big) P^{(3)}(\xi') + 4\int_0^{v_2} d\xi \int_0^{v_3} d\xi'\, P^{(2)}(\xi)\, \bar{D}_{\mu\varrho} \times$$
$$\times \Big(2\int_{\xi'}^{v_3} P^{(3)}(\xi_1)\, d\xi_1 + 2\int_{\xi}^{v_2} P^{(2)}(\xi_1)\, d\xi_1\Big) P^{(3)}(\xi') +$$
$$+ 4\int_0^{v_1} d\xi \int_0^{v_2} d\xi'\, P_\mu^{(1)}(\xi)\, \bar{D}_{\mu\varrho}\Big(x - 2\int_{\xi'}^{v_2} P^{(2)}(\xi_1)\, d\xi -$$
$$- 2\int_{\xi}^{v_1} P^{(1)}(\xi_1)\, d\xi_1\Big) P_\varrho^{(2)}(\xi')\,,$$

$$P^{(n)}(\xi) = P_n + \frac{\delta}{2\delta t_n(\xi)}\,. \quad (6.5)$$

* В случае $\mu = \varrho$ в (6.3) появляется добавочный член $\delta_{\mu\varrho}\, e^1 \int_0^\infty dv_1 \int_0^\infty dv_2 \langle \exp i \times \times \Big\{ -\sum_{n=1}^{2}(p_n^2 + \mu^2 - i\varepsilon)\, v_n + A \Big\} \rangle$, где A — определяется формулой (5.4) при замене $p \to p_1$, $p + s \to p_2$ и $s \to k_1 - k_2$

Нетрудно увидеть из сравнения с теорией возмущений, что первые два члена в выражении для A описывают радиационные поправки к одночастичным функциям Грина, третий и второй члены описывают радиационные добавки к вершинным частям («обрастание» узлов взаимодействия Γ_μ), оставшееся выражение в A возникает в результате обмена частиц p и p' векторными квантами при рассеянии (так называемые радиационные добавки типа $\frac{\delta \Gamma_\nu}{\delta \varphi_\mu}$).

Нетрудно убедиться, что интегрирование по x в выражении (6.2) может быть проведено в общем виде. Опуская необходимые выкладки приведем получающееся при этом выражение для

$$g_{\mu\varrho}(p_1, k_1 \mid p_2, k_2) = -4ie^2 \int_0^\infty dv_1 \int_0^\infty dv_2 \int_0^\infty dv_3\, e^{-i\sum_{n=1}^{2}(p_n^2+\mu^2-i\varepsilon)v_n - i(q^2+\mu^2-i\varepsilon)v_3} \times$$

$$\times \left\langle \left(q_1 + \frac{\delta}{\delta t_1 (v_1 - \varepsilon)}\right)_\mu \left(q_1 + \frac{\delta}{\delta t_2 (v_2 - \varepsilon)}\right)_\varrho \exp ie^2 A_{12} \right\rangle \tag{6.6}$$

где

$$\begin{aligned} A_{12} = {} & 2\sum_{n=1}^{3} \int_0^{v_n} d\xi \int_0^{v_n} d\xi'\, P_\mu^{(n)}(\xi)\, \bar{D}_{\mu\varrho}\Big(2\int_\xi^{\xi'} d\xi_1\, P^{(n)}(\xi_1)\Big) P_\varrho^{(n)}(\xi') + \\ & + 4\int_0^{v_1} d\xi \int_0^{v_3} d\xi'\, P_\mu^{(1)}(\xi)\, \bar{D}_{\mu\varrho}\Big(2\int_0^{\xi'} d\xi_1\, P^{(3)}(\xi_1') + \\ & + 2\int_\xi^{v_1} d\xi_1\, P^{(1)}(\xi_1)\Big) P_\varrho^{(3)}(\xi') + \\ & + 4\int_0^{v_2} d\xi \int_0^{v_3} d\xi'\, P_\mu^{(2)}(\xi)\, \bar{D}_{\mu\varrho}\Big(2\cdot\int_{\xi'}^{v_3} P^{(3)}(\xi_1)\, d\xi_1 + \\ & + 2\int_\xi^{v_2} d\xi_1\, P^{(2)}(\xi_1)\Big) P_\varrho^{(3)}(\xi') + \\ & + 4\int_0^{v_1} d\xi \int_0^{v_2} d\xi'\, P_\mu^{(1)}(\xi)\, \bar{D}_{\mu\varrho}\Big(2\int_0^{v_3} d\xi_1\, P^{(3)}(\xi_1) + \\ & + 2\int_{\xi_1}^{v_2} P^{(2)}(\xi_1)\, d\xi_1 + 2\int_\xi^{v_1} P^{(1)}(\xi_1)\, d\xi_1\Big) P^{(2)}(\xi'), \end{aligned} \tag{6.7}$$

здсеь

$$P^{(3)}(\xi) = q_1 + \frac{\delta}{\delta t_3(\xi)}\ .$$

В первом приближении модифицированной теории возмущений мы получаем следующие выражения для $g_{\mu\varrho}$:

$$g_{\mu\varrho}(p_1, k_1 | p_2, k_2) = -4ie^2 \int_0^\infty dv_1 \int_0^\infty dv_2 \int_0^\infty dv_3 \exp\Big(-i \sum_{n=1}^{2} (p_n^2 + \\ + \mu^2 - i\varepsilon) v_n - i(q^2 + \mu^2 - i\varepsilon) v_2\Big) \times q_{1\mu} q_{1\varrho} \exp ie^2 \langle A_{12} \rangle , \tag{6.8}$$

где

$$\langle A_{12} \rangle = -\frac{4}{(2\pi)^4} \sum_{n=1}^{2} \int d^4k \, p_{n\mu} \bar{D}_{\mu\varrho}(k) \, p_{n\varrho} \left\{ \frac{e^{-i(k^2+2p_nk)v_n} - 1}{(k^2 + 2p_n k)^2} + \right.$$

$$\left. + \frac{iv_n}{(k^2 + 2p_n k)} \right\} - \frac{4}{(2\pi)^4} \int d^4k \, q_{1\mu} \bar{D}_{\mu\varrho}(k) \, q_{1\varrho} \times$$

$$\times \left\{ \frac{e^{-i(k^2+2q_1k)v_3} - 1}{(k^2 + 2q_1 k)^2} + \frac{iv_3}{(k^2 + 2q_n k)} \right\} -$$

$$- \frac{4}{(2\pi)^4} \int d^4k \, p_{1\mu} \bar{D}_{\mu\varrho}(k) \, q_{1\varrho} \times$$

$$\times \frac{(e^{-i(k^2-2qk)v_3} - 1)(e^{-i(k^2-2p_1k)v_1} - 1)}{(k^2 - 2q_1 k)(k^2 - 2p_1 k)} - \tag{6.9}$$

$$- \frac{4}{(2\pi)^4} \int d^4k \, p_{2\mu} \bar{D}_{\mu\varrho}(k) p_{1\varrho} \times$$

$$\times \frac{(e^{-i(k^2-2q_1k)v_3} - 1)(e^{-i(k^2-2p_2k)v_2} - 1)}{(k^2 - 2q_1 k)(k^2 - 2p_2 k)} -$$

$$- \frac{4}{(2\pi)^4} \int d^4k \, p_{1\mu} D_{\mu\varrho}(k) \, p_{2\varrho} \times$$

$$\times \frac{(e^{-i(k^2-2p_1k)v_1} - 1)(e^{-i(k^2-2p_2k)v_2} - 1) e^{-i(k^2-2q_1k)v_3}}{(k^2 - 2p_1 k)(k^2 - 2p_2 k)} .$$

При этом для амплитуды рассеяния $f_{\mu\varrho}$ на массовой оболочке получаем следующее выражение:

$$f_{\mu\varrho}(p_1, k_1 | p_2, k_2) = (p_1^2 + M_e^2) \, G_{12}^{\mu\varrho}(p_1, k_1 | p_2, k_2) (p_2^2 + \mu_e^2)|_{p_1^2 = -\mu_e^2, p_2^2 = -\mu_e^2} = \\ = ie^2 \sum_{n=1}^{2} \int_0^\infty dv_n \, e^{-i(q_n^2 + \mu^2 - i\varepsilon)v_n} 4q_{n\mu} q_{n\varrho} \exp ie^2 A_2(q_n) , \tag{6.10}$$

где

$$q_1 = p_1 + k_1\,; \qquad q_2 = p_1 - k_2$$

$$A_2(q_n) = \frac{4}{(2\pi)^4}\int P_{1\mu}\,D_{\mu\varrho}(k)\,q_{n\varrho}\,\frac{d^4k\,(e^{-i(k^2-2q_nk)\nu_n}-1)}{(k^2-2q_n\,k)\,(k^2-2p_1\,k-i\varepsilon)} -$$

$$-\frac{4}{(2\pi)^4}\int P_{2\mu}\,D_{\mu\varrho}(k)\;q_{n\varrho}\,\frac{d^4k\,(e^{-i(k^2-2q_nk)\nu_n}-1)}{(k^2-2q_n\,k)\,(k^2-2p_2\,k-i\varepsilon)} - \qquad (6.11)$$

$$-\frac{4}{(2\pi)^4}\int P_{1\mu}\,D_{\mu\varrho}(k)\,P_{2\varrho}\times$$

$$\times\,\frac{d^4k}{(k^2-2p_1k-i\varepsilon)(k^2-2p_2k-i\varepsilon)}\,e^{-i(k^2-2q_1k)\nu_n} -$$

$$-\frac{4}{(2\pi)^4}\int q_{n\mu}\,D_{\mu\varrho}(k)\,q_{n\varrho}\,\frac{d^4k}{(k^2-2q_n\,k)}\left\{i\nu_n+\frac{e^{i(k^2-2q_nk)\nu_n}-1}{(k^2-2q_n\,k)}\right\}.$$

Обычной процедурой мы можем с помощью $f_{\mu\varrho}$ найти сечение процесса. При этом в том случае, когда масса векторного поля равна нулю (фотонное поле) амплитуда чисто упругого взаимодействия обращается в нуль («инфракрасная» катастрофа). Физический смысл имеет однако «упругое» рассеяние с излучением мягких фотонов, суммарная энергия которых не превосходит Δ (разрешающая способность прибора).

С помощью операторного решения легко получить и в этом случае суммарное выражение для сечения (см. также [4]). Мы не станем проводить здесь необходимые выкладки, ограничимся лишь приведением выражения для асимптотики сечения рассеяния фотона на бозе-частицу (с излучением мягких квантов) суммарная энергия которых не превосходит Δ) при больших энергиях в основной области углов (исключая углы в системе ц. и. близкие к «0» и «π»)

$$\frac{d\sigma}{d\Omega}\approx\frac{d\sigma_0}{d\Omega}\int_{-\infty}^{+\infty}\frac{e^{i\Delta\tau}-1}{2\pi i\tau}\,e^{2B(\tau)}\,d\tau\approx\frac{d\sigma_0}{d\Omega}\left(\frac{\Delta}{E}\right)^{2\beta}\frac{1}{\Gamma(1+2\beta)}\,, \qquad (6.12)$$

где $d\sigma_0$ — борновское сечение,

$$B(\tau)=\frac{e^2}{\pi^4}\int d^4k\sum_{n\geq m=1}^{3}\left\{A_{\mu\varrho}^{(n,m)}\,e^{-ik_0\tau}\,Im\,D_{\mu\varrho}^{+}(k)-Im\left(A_{\mu\varrho}^{(n,m)}\,D_{\mu\varrho}(k)\right)\right\}, \qquad (6.13)$$

$$A_{\mu\varrho}^{(n,m)}\approx\frac{(-1)^{(n-m)}\,P_{n\mu}\,P_{m\varrho}}{(k^2-2p_n\,k-i\varepsilon)(k^2-2p_m\,k-i\varepsilon)'}\,; \qquad P_3=\min\{q_1;\,q_2\}\,,$$

$$\beta\approx\frac{e^2}{4\pi^2}\,\frac{p_1p_2}{\sqrt{(p_1\,p_2)^2-(p_1^2p_2^2)}}\ln\frac{p_1\,p_2+\sqrt{(p_1p_2)^2-p_1^2p_2^2}}{m^2}\,.$$

Хотелось бы отметить особо существенное отличие в поведении амплитуды рассеяния при больших энергиях в случае рассеяния назад, когда масса векторного поля равна нулю (фотонное поле) или отлично от нуля. Это отличие ещё более усугубляется в случае взаимодействия со спинорным полем, где сечение рассеяния назад только в случае $\lambda^2 \neq 0$ имеет реджевский вид, причем на реджевской траектории лежит сама ферми-частица. (более подробно этот вопрос будет рассмотрен в другом месте)

Отметим, что с помощью выражения (3.5) для производящего функционала можно получить модифицированную теорию возмущений и найти соответствующие асимптотики сечений различных процессов также при наличии внешнего векторного поля (например, во внешнем электромагнитном поле). Для этого необходимо считать, что $I_\mu \neq 0$ и φ определяется равенством:

$$\varphi_\mu^{(bn)}(x) = \int D^r_{\mu\nu}(x - y)\, I_\nu(y)\, d^4y\,. \tag{6.14}$$

В (3.4) необходимо формально заменить $I_\mu(k)$ через $I'_\mu(k) + D'_{\mu\nu}(k)\,\varphi_\nu^{(bn)}(k)$, где $D^r_{\mu\nu}$ — запаздывающая функция Грина векторного поля.

Отметим также, что все полученные результаты легко обобщаются на случаи, когда процессы рассеяния быстрых частиц происходят в среде, а последнюю можно учитывать феноменологически. Единственное изменение в этом случае заключается в том, что всюду вместо $D_{\mu\nu}$ необходимо брать функцию распространения $D^{(cp)}_{\mu\nu}$ в среде.

В заключение приведем выражение для амплитуды одной ферми частицы и n — бозе частиц с произвольными импульсами на массовой оболочке. Это выражение следует непосредственно из (3.5), причем уже первое приближение модифицированной теории возмущений для него приводит к правильной асимптотике для сечений в основной области углов.

$$f^{(n)}(p, p', k_1 \ldots k_n) = \prod_{s=1}^{n} \int_{-\nu_1}^{\nu-\nu_1} d\nu'_s \langle \tilde{P}\mu_s(\nu'_s)\, e^{F(\tilde{P})}\rangle \langle e^{-F(P)}\rangle\, \delta(\nu'_1) \times$$

$$\times \exp i \sum_{s=1}^{n} \left\{ \frac{1}{2} \sum_{s_1=1}^{n} k_s k_{s_1} (\nu'_s + \nu'_{s_1} + |\nu'_s - \nu'_{s_1}|) - 2pk_s \nu'_s \right\} \Bigg|_{\substack{\nu_1 \to \infty \\ \nu-\nu_1 \to \infty}};$$

здесь

$$F(P) = 2ie^2 \int_{-\nu_1}^{\nu-\nu_1} d\xi \int_{\nu_1}^{\nu-\nu_1} d\xi'\, P_\mu(\xi)\, D_{\mu\varrho}\Big(2 \int_{\xi'}^{\xi} P(\eta)\, d\eta\Big)\, P_\varrho(\xi''),$$

$$P_\mu(\xi) = p_\mu + \frac{\delta}{2\delta t(\xi + \nu_1)}; \qquad \tilde{P}(\xi) = P(\xi) - \sum_{s=1}^{n} k_s\, \theta(\xi - \nu'_s);$$

$$p = p' + \sum_{s=1}^{n} k_s\,.$$

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ON ELECTRODYNAMICS OF PARTICLES WITH ZERO SPIN

By

E. S. Fradkin

Abstract

The interaction of a vector neutral field (without the mass (photons) and with the mass available as well) with a field of a zero spin is considered with the help of a functional method.

A closed expression for S-matrix is found with the aid of an operator solution for a generating functional and an operator solution for the Green function in an arbitrary external field. A modified perturbation theory is developed which seems to be fairly convenient for finding out asymptotics of the Green function and cross sections.

Asymptotics for one-particle, two-particle and vertex Green functions are discussed.

ON SOME RECENT ACHIEVEMENTS IN THE AXIOMATIC APPROACH TO QUANTUM FIELD THEORY

By

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Some recent achievements in the general trend of axiomatics are treated. Emphasis is given to asymptotic relations for scattering amplitudes.

Introduction

I was asked by our amiable hosts to review recent results in the axiomatic approach to relativistic quantum field theory. LOGUNOV's report at the Dubna conference [1] was confined to the same topic and, not being acquainted with any significant development in the few weeks after it was given, I shall be concerned essentially with the same problems. Naturally, my report is by no means intended to exhaust the matter.

As an introduction I shall start with the following diagram of different approaches in axiomatic quantum field theory.

The characteristic feature of all rectangles in this diagram is that no equation of motion or specific Lagrangian is presupposed. Only some general properties of the "solutions" of such unknown (or even nonexisting) equations are postulated. Unfortunately, the different approaches have still another feature in common: no nontrivial model (i.e. a model in which $S \neq 1$) is known. The usual interpretation of this quite distressing fact is that all the schemes allow only extremely realistic nontrivial models (if any).

The program of direct investigation of the S-matrix was first set by HEISENBERG in the forties. An intensive development of the axiomatic approach began about 10 years ago when schemes A, B I and B II arose. Rectangle C regarded as an independent approach with its own primary notions and axioms (and not merely as an application of B) appeared later — some 4 years ago. The basic postulates of this approach, namely the "maximal analiticity" and "saturated unitarity" were set by G. CHEW in a rather metaphysical manner and as far as I know, they have not been formulated until now in a precise mathematical way. Nor any real success seems to have been obtained on this vague basis, in spite of the "hopes" and the "strong belief" professed by CHEW. Further on C will refer to the consequences and the applications of B only.

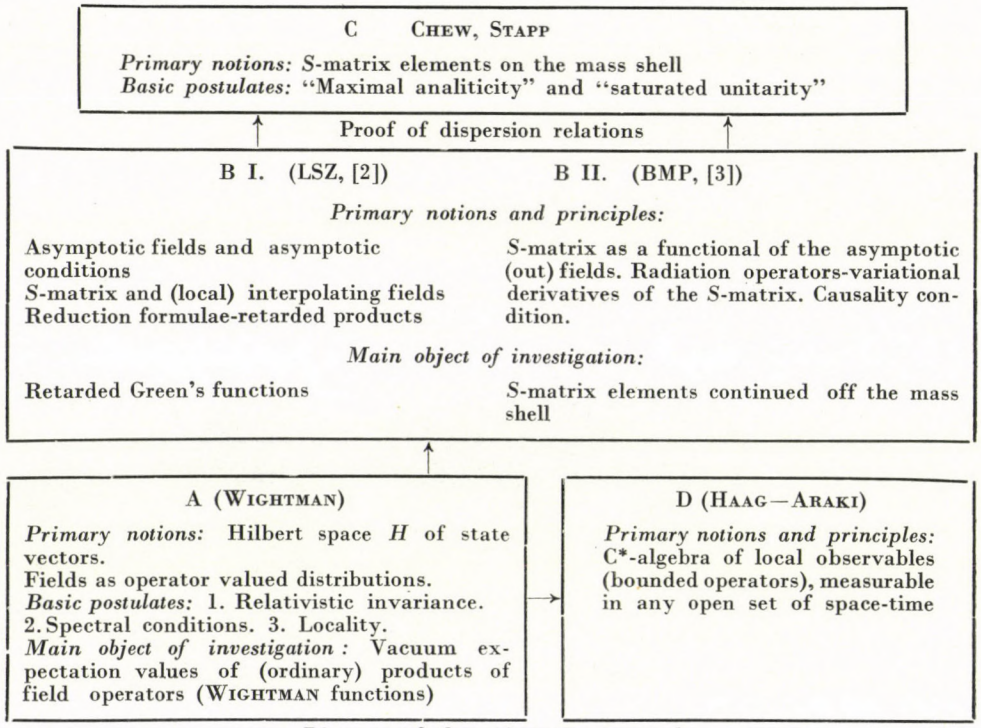

Diagram of the axiomatic approach

(An analogous diagram was proposed by R. JOST at the Sienna Conference (1963)).

Still more recent is the approach of HAAG and ARAKI (*D*) based on the notion of the VON NEUMANN-algebra of bounded operators (or more generally — *C** algebra) associated with any open set in space-time. The work in this direction is at a preliminary stage so I think it is early to judge.

I go on to explain the role of the arrows in the above diagram. It is natural to expect that the different schemes in axiomatic field theory are closely related (or even equivalent) to each other. Nonetheless, the exact connection between the different approaches is not yet completely clear.

Now, it is generally accepted that WIGHTMAN's formulation, based on primary quantum theoretical notions (states and field operators), is mathematically the most elaborated and logically consistent. On the other hand, the construction of asymptotic free-particle states and of the scattering matrix in the WIGHTMAN framework given fairly recently in an important work by RUELLE [4] is quite involved and not explicit. The difficulty to pass from the vacuum expectation values of ordinary products of field operators (WIGHTMAN functions) to the matrix elements of LSZ retarded commutators is twofold.

First, the multiplication of (tempered) distributions with a discontinuous function like $\theta(x)$ is not a well defined operation. Second, the connection between HAAG—RUELLE and LSZ asymptotic conditions is not completely clear. Some light on this question was thrown by the paper of K. HEPP, presented at the recent Dubna Conference on High Energy Physics [5].

A connection between A and D was established recently by BORCHERS and ZIMMERMANN [6]. They add a new postulate to the usual WIGHTMAN-axioms of the theory of a neutral scalar field A, namely: the vacuum state $|0\rangle$ is an analytic vector with respect to the (smeared-out) field operators $A(f)$. This means that for any test function f there exists a positive constant M (depending on f) such that

$$|| [A(f)]^h | 0 \rangle || \leq M^n n !$$

(The notion of an analytic vector was introduced in a mathematical paper by NELSON (1959)). As a consequence it is proved that for real $f(x)$ with compact support, the operator $A(f)$ is essentially selfadjoint i.e. $[A(f)]^{+\,+} = [A(f)]^+$. Further it is shown that the spectral projections of $A(f)$ (for different f with compact supports) give rise to a local VON NEUMANN algebra of bounded operators, satisfying all the requirements of HAAG and ARAKI.

A fundamental connection between B and C, the proof of dispersion relations for a number of processes, was established in the early stage of the development of the axiomatic approach by BOGOLIUBOV et al. [3] and in subsequent papers. The last word in this direction (concerning the analyticity of the two-particle scattering amplitude) was said in the work of BROS, EPSTEIN and GLASER [7].

In what follows I want to discuss in some more detail two questions.

The first one concerns the arrow $A \to B$, that is HAAG—RUELLE's collision theory and the contribution of HEPP to the proof of dispersion relations in the WIGHTMAN framework.

The second one deals with an application of rigorously proved analytic properties of matrix elements to obtain some asymptotic relations between the amplitudes of different high energy processes.

§ 1. The Haag—Ruelle collision theory and analyticity properties of the scattering amplitude

For the sake of simplicity we shall consider only the case of a neutral scalar field A.

I will not formulate here the basic postulates of the WIGHTMAN approach. I want to mention only the restricted form of the spectrality requirements

used by RUELLE. It includes the uniqueness (nondegeneratedness) of the vacuum state and the existence of a minimal positive mass m in the theory corresponding to a discrete one-particle state (the photons being exluded from consideration). This leads to the following type of KÄLLÉN—LEHMANN representation for the two-point WIGHTMAN function

$$\langle A(x)\, A(y)\rangle_0 = iA_m^+(y-x) + i\int_M^\infty A_\mu^+(y-x)\, d\varrho(\mu)\,,\quad M > m > 0 \qquad (1.1)$$

($i\,A_\mu^+$ is the free field two point function).

Briefly, in a simple special case HAAG and RUELLE's results about the asymptotic conditions may be stated as follows.

Let $D_M \equiv \{p : p^2 < M^2\}$ and let $\tilde{f}(p)$ be an infinitely differentiable fast decreasing function with support in D_M (i.e., $\tilde{f}(p) \in S(D_M)$). Define the function $f(x, t)$ of five variables, $x = (x_0, \mathbf{x})$ and t by

$$f(x,t) = \frac{1}{(2\pi)^{5/2}}\int d^4p\, \tilde{f}^*(p)\, \frac{p_0 + w_p}{2w_p}\, e^{i(w_p - p_0)t}\, e^{ipx}\,, \qquad (1.2)$$

where

$$w_p = \sqrt{m^2 + p^2}$$

and let

$$A(f,t) = \int d^4x\, A(x) f(x,t) = \frac{1}{\sqrt{2\pi}}\int d^4p\, \frac{p_0 + w_p}{2w_p}\, e^{i(w_p - p_0)t}\tilde{f}^*(p)\,\tilde{A}(p)\,. \qquad (1.3)$$

In the case when $A(x)$ is a free field of mass m, $\tilde{A}(p)$ contains a factor $\delta(p^2 - m^2)$ so that $A(f, t)$ does not depend in fact on t, and represents an annihilation operator. $A(f, t)$ possesses some properties of an annihilation operator in the general case too. Actually, using the spectral conditions in the form (1.1) and taking tinto account the support properties of $\tilde{f}(p)$ it is easy to see that

$$A(f,t)\mid 0\rangle = 0\,,\qquad [A(f,t)]^+\mid 0\rangle = |\tilde{f}>\,, \qquad (1.4)$$

where $|\tilde{f}\rangle$ is a one particle state of mass m and spin 0 with wave function

$$\tilde{f}(p),\ \ (p_0 = \sqrt{m^2 + p^2})\,.$$

The main result of RUELLE [4] is the rigorous proof of HAAG's conjecture that the strong limits

$$s - \lim_{t\to\pm\infty} A^+(f_1,t)\ldots A^+(f_n,t)\mid 0\rangle = |\tilde{f}_1\ldots\tilde{f}_n^{\,\text{in}}{}^{\text{out}}\rangle \qquad (1.5)$$

exist, where $|\tilde{f}_1 \ldots \tilde{f}_n^{\text{in}}\rangle$ are asymptotic states consisting of n free particles.

In order to obtain the conventional collision theory we have to add to the WIGHTMAN axioms the postulate of asymptotic completeness

$$H_{\text{in}} = H = H_{\text{out}} \tag{1.6}$$

(in fact it is sufficient to assume only one of these equalities, the other follows from the TCP-theorem). Then a unitary operator S exists whose matrix elements are determined by

$$S(f_1 \ldots f_n \mid g_1 \ldots g_k) = \langle f_1 \ldots f_n^{\text{out}} \mid g_1 \ldots g_k^{\text{in}} \rangle . \tag{1.7}$$

The right hand side of (1.7) may be expressed of course in terms of some limits of integrals of WIGHTMAN functions, but it is clear that this expression is too complicated to deal with.

HEPP [5] solves the problem of transition from the distribution

$$\langle \tilde{f}_1, \tilde{f}_2^{\text{out}} \mid \tilde{g}_1 \tilde{g}_2^{\text{in}} \rangle$$

to the boundary value of an analytic function $T((p_1 + p_2)^2 (p_1 - q_1)^2)$(on the mass shell $p_i^2 = m = q_i^2$) finding by the way reduction formulae of the LSZ type for the two-particle amplitude. T is related to the 2-particle S-matrix elements by the conventional formula

$$iT_{22}(p_1 p_2 \mid -q_1 - q_2) \equiv \langle p_1 p_2^{\text{out}} \mid q_1 q_2^{\text{in}} \rangle - \langle p_1 p_2^{\text{out}} \mid q_1 q_2^{\text{out}} \rangle =$$

$$= \delta(p_1 + p_2 - q_1 - q_2)\delta_{\text{m}}^{+}(p_1)\, \delta_{\text{m}}^{+}(p_2)\, \delta_{\text{m}}^{+}(q_1)\delta_{\text{m}}^{+}(q_2)\, T((p_1 + p_2)^2, (p_1 - q_1)^2)$$

$$\delta_{\text{m}}^{+}(p) \equiv \Theta(p_0)\, \delta(p_2 - m_2) \tag{1.8}$$

(to which HEPP gives a precise meaning).

I shall mention the principal steps of HEPP's construction of T. The problem of analyticity is reduced to BOGOLIUBOV's proof of ordinary dispersion relations making extensive use of the JOST—LEHMANN—DYSON representation [8] of the causal commutator.

Define a smooth and bounded approximation of $\delta_{\text{m}}^{+}(p)$ by

$$\delta_{ab}^{m}(p) = \frac{e^{i(p_0 - w_p)a} - e^{-i(p_0 - w_p)b}}{4\pi i\,(p_0 - w_p)\, w_p}\,; \quad (W \lim_{a,b \to +\infty} \delta_{ab}^{m}(p) = \delta_m^{+}(p)). \tag{1.9}$$

Let further $R(x, y)$ be the smooth retarded product

$$R(x, y) = i\Theta_c(x - y)\,[A(x), A(y)]\,, \tag{1.10}$$

where $\Theta_c(x)$ is a C^∞ (i.e. infinitely differentiable) function such that $\Theta - \Theta$ has a compact support.

The first step in HEPP's considerations consists in the following proposition.

Proposition 1. If WIGHTMAN's axioms are assumed, then for $p_1^2 < M^2$ and $q_1^2 < M^2$ one has*

$$T_{22}(p_1 p_2 \mid -q_1 - q_2) = 2\pi \lim_{a,b\to\infty} \lim_{c,d\to\infty} \delta_{ab}^{m*}(p_1)\, \delta_{cd}^{m}(q_1)\,(p_1^2 - m^2)\,(q_1^2 - m^2) \times \times \langle p_2 \mid \tilde{R}(p_1, -q_1) \mid q_2 \rangle , \tag{1.11}$$

where $\tilde{R}$ is the Fourier transform of the smooth retarded product (1.10). If (1.6) is also true, then (1.11) holds for all p_1 and q_1.

In the proof of this proposition the HAAG—RUELLE theorem (eq. (1.5)) is used. In the same way also the following is proved.

Proposition 2. Under the same assumption, for $p_i^2 < M^2, q_i^2 < M^2$

$$T_{22}(p_1 p_2 \mid - q_1 q_2) = 2\pi \lim_{a,b\to\infty} \lim_{c,d\to\infty} \delta_{ab}^{m*}(p_1\, \delta_{cb}^{m*}(p_2)\,(p_1^2 - m^2) \times \times (p_2^2 - m^2) \langle 0 \mid \tilde{R}(p_1 p_2) \mid q_1 q_2^{\text{in}} \rangle \tag{1.12}$$

and

$$(p_1^2 - m^2)\,(q_1^2 - m^2) \langle p_2 \mid \tilde{A}(p_1)\, \tilde{A}(-q_1) \mid q_2 \rangle =$$

$$= 2\pi \lim_{a,b\to\infty} \lim_{c,d\to\infty} \delta_{ab}^{m*}(p_2)\, \delta_{cd}^{m}(q_2)\,(p_1^2 - m^2)\,(q_1^2 - m^2)\,(p_2^2 - m^2)\,(q_2^2 - m^2) \times \times \langle 0 \mid \tilde{R}(p_1, p_2)\, \tilde{R}(-q_1, -q_2) \mid 0 \rangle . \tag{1.13}$$

These reduction formulae give a large class of "off-shell" extrapolations for the 2-particle scattering amplitude T_{22} and for the matrix element $\langle p_2 \mid \mid \bar{j}(p_1)\, \bar{j}(-q_1) \mid q_2 \rangle$ of the currents $\bar{j}(p) = (p^2 - m^2)\, \tilde{A}(p)$. All these "retarded" expressions derived from the 4-point function $\langle 0 \mid A(x_1)\, A(x_2)\, A(x_3)\, A(x_4) \mid 0 \rangle$ are equivalent on the mass shell.

As a next step this class is reduced to a set of "sharp" admissible extrapolations, i.e. extrapolations using discontinuous θ-functions instead of θ_c. In the proof the JOST—LEHMANN—DYSON integral representation is used and invariance properties are explored. The assumption is made that the continuous mass-spectrum starts at $M_2 = 4\ m^2$. Then admissible "sharp" extrapolations are obtained as tempered distributions invariant under the proper POINCARÉ** group P_+ with the support properties in x and p spaces

* We mention that $A^+(f, t) \mid 0 \rangle$ is a time independent vector so that the one-particle states $\mid \tilde{f}^{\text{in}} \rangle$ and $\mid \tilde{f}^{\text{out}} \rangle$ coincide: therefore the superscript $(^{\text{in}}_{\text{out}})$ for these states may be omitted without confusion.

** I.e. inhomogeneous LORENTZ group.

necessary for the proof of the analyticity properties of T_{22}.

The last step in HEPP's analysis consists of an accurate transition to the mass shell in order to prove usual dispersion relations and analyticity of the physical scattering amplitude in the LEHMANN ellipse [9]. In this way all the implicit "technical" assumptions needed in the classical proof of dispersion relations (as mentioned by OMNÉS [10] and FROISSART) are made unnecessary.

The difficulty of extending HEPP's result for many particle amplitudes is due to the lack of a substitute of the JOST—LEHMANN—DYSON representation for this case.

Another nice result of HEPP reported in Dubna (unpublished) is the rigorous proof of LSZ asymptotic conditions for the case when different particles have wave functions with compact nonoverlapping supports.

§ 2. Asymptotic relations between the amplitudes of elastic and inelastic processes

I shall start with some historical remarks.

Besides the elegant proof of the TCP-theorem and the connection between spin and statistics, for a long time dispersion relations were the only physical outcome in axiomatic field theory. They seemed to be particularly useful at low-energy, below the inelastic threshold, because in this region an approximate system of equations may be obtained from dispersion relations using elastic unitarity condition.

In 1958 POMERANCHUK [11] showed that dispersion relations can be used to prove asymptotic equality between the total cross sections of particle and antiparticle interactions for high energy. POMERANCHUK's argument was improved and generalised in several subsequent papers. I want to mention particularly the paper of SUGAWARA and KANAZAWA [12] in which essentially the PHRAGMÉN—LINDELÖF theorem was rediscovered and proved. Later MEIMAN [13] mentioned that this is a classical theorem in the theory of analytic functions. On the basis of this theorem both SUGAWARA and MEIMAN gave a simple and natural proof of POMERANCHUK's theorem under quite general assumptions.

Using the PHRAGMÉN—LINDELÖF theorem the Dubna theoreticians [14—17] obtained asymptotic relations between differential cross sections and polarizations for a large class of elastic and inelastic pairs of processes. Some of these results were obtained (under more restrictive requirements) independently by VAN HOVE [18] who extended the original POMERANCHUK method (using dispersion relations).

In contradistinction to the REGGE pole model (and other semiphenomenological treatments of high energy phenomena) only a very small number

of basic physical assumptions are needed to obtain the above mentioned results. It is worthwhile to list these assumptions in terms of the properties of the scattering amplitude in p-space.

1. Analyticity of $T(s, t)$ for sufficiently large (in absolute value) s in the upper half plane. — This analyticity requirement always follows from the causality condition which in the BMP scheme is stated as follows

$$\frac{\delta j(x)}{\delta \varphi(y)} = 0 \quad \text{for} \quad y \lesssim x .$$

2. Crossing symmetry (or more precisely — substitution law) permitting to express the amplitude for negative energy through the physical antiparticle scattering amplitude.

3. Limited growth at high energy. — The combination of the assumption that S-matrix elements are tempered distributions with BOGOLIUBOV's causality condition leads to at most polynomical growth in momentum space (even for complex momenta in the domain of analyticity of the amplitude [19]). What is needed in the following is the still weaker assumption that the amplitude is less than any linear exponent

$$| T(s, t) | < A_\varepsilon(t) e^{\varepsilon |s|} \quad \text{for each} \quad \varepsilon > 0 .$$

This latter assumption is in fact necessary for the causality condition to have a meaning [20]. Such a generalization might be useful for incorporating unrenormalizable theories too.

4. A less general assumption is made about the absence of (strong) oscillations in the amplitude at high energy (it will be soon formulated in a more precise way). This additional assumption may be given up for the case of forward scattering when only the high energy behaviour of the observed differential cross section is relevant.

Let me now discuss in some more detail the simplest case of scalar particle scattering in order to illustrate the idea of the proof.

Consider the pair of related processes

$$a_1 + b_1 \to a_2 + b_2 \tag{I}$$

and

$$\bar{a}_2 + b_1 \to \bar{a}_1 + b_2 , \tag{II}$$

where the bar stands for the transition to the antiparticle.

Let the amplitudes of these processes $Ty(s, t)$ $(y = \text{I}, \text{II})$ be analytic outside some bounded region of the cut plane s. For fixed t and sufficiently large s (or $| u |$) we shall make use of the substitution law

$$T_{\mathrm{II}}(s, t; M_1^2, m_2^2, M_2^2, m_1^2) = T_{\mathrm{I}}^*(u, t; M_1^2, m_1^2, M_2^2, m_2^2) , \tag{2.2}$$

where M_i and m_i are the masses of particles b_i and a_i, respectively $(i = 1, 2)$.

In order to formulate assumption 4 in a more precise way we introduce the following auxiliary notion. Call the function $\varphi(s, t)$ admissible if at fixed physical t the function $\dfrac{1}{\varphi(s, t)}$ is *a)* analytic outside some finite region in the upper half plane *s*,*b)* less than any exponent $e^{\varepsilon|s|}$, $(\varepsilon > 0)$ for $s \to \infty \dfrac{Im\, s}{|s|} > 0$, *c)* continuous for sufficiently large $|s|$ on the real axis, *d)* such that the limit

$$\lim_{s \to +\infty} \frac{\varphi(s, t)}{\varphi(-s, t)} = e^{-i\pi\alpha(t)} \tag{2.3}$$

exists with real $\alpha(t)$. An example of admissible function is given by

$$\varphi(s, t) = s^{\alpha(t)} (\ln s)^{\beta(t)} (\ln \ln s)^{\gamma(t)} \ldots ,$$

where $\alpha, \beta, \gamma \ldots$ are real.

Let $Ty\,(s, t)$ satisfy the requirements 1—3 stated above. Then the following theorem holds.

Theorem. Let us suppose that for fixed t and for some choice of the admissible function $\varphi(s, t)$ the limits

$$\lim_{s \to +\infty} \frac{T_{\mathrm{I}}(s, t)}{\varphi(s, t)} = V_{\mathrm{I}}(t) , \qquad \lim_{s \to +\infty} \frac{T_{\mathrm{II}}^*(t, t)}{\varphi(-s, t)} = V_{\mathrm{II}}(t) \tag{2.4}$$

exist. Then these limits coincide

$$V_{\mathrm{I}}(t) = V_{\mathrm{II}}(t) \tag{2.5}$$

and hence, for nonvanishing $V_{\mathrm{I}}(t)$ the differential cross sections of processes I and II are asymptotically equal:

$$\lim_{t \to \infty} \left(\frac{d\sigma_{\mathrm{I}}}{dt}\right) \left(\frac{d\sigma_{\mathrm{II}}}{dt}\right)^{-1} = 1 . \tag{2.6}$$

Proof. Under our assumption the function

$$V(s, t) = \frac{T_{\mathrm{I}}(s, t)}{\varphi(s, t)} \tag{2.7}$$

is analytic and does not exceed any exponent $e^{\varepsilon|s|}$ outside some region of the upper half plane s and is bounded along the real axis. Furthermore, it follows from (2.2) and (2.4) that

$$\lim_{s\to\pm\infty} V(s,t) = V_{\mathrm{I,II}}(t)\,. \tag{2.8}$$

Therefore, one can apply the PHRAGMÉN—LINDELÖF theorem [21] to V which may be stated as follows.

Phragmén—Lindelöf theorem. Let $f(z)$ be an analytic function of $z = x + iy$, regular outside some bounded region of the upper half plane and continuous on the real axis for sufficiently large (x). Let further the limit $\lim_{x\to\pm\infty} f(x) = a_{\pm}$ exist.

Then if $a_+ \neq a_-$ a sequence $z_n \to \infty$ exists such that $\lim \frac{y_n}{|z_n|} > 0$ and $|f(z_n)| \geq e^{\nu|z_n|}$ for some choice of $\nu > 0$.

Now $V(s,t)$ satisfies all the conditions of this theorem and, moreover it is less than any linear exponent (in view of (2.1) and the definition of the admissible function). Consequently the limits V_{I} and V_{II} cannot be different, hence equality (2.5) is proved.

In the special case of the forward elastic scattering, under the additional assumption $\alpha(0) = 1$ (or more generally $\alpha(0) = 2n + 1$) $\alpha(t)$ being defined by (2.3), we obtain from (2.5) besides the equality of differential cross sections, asymptotic equality between total cross sections of particle and antiparticle interactions (POMERANCHUK theorem).

If "a" is a neutral scalar (or pseudoscalar) particle, coinciding with its antiparticle (*as* π^0, $\mathrm{K}^0_{1,2}$) then the amplitudes T_{I} and T_{II} coincide $T_{\mathrm{I}} = T_{\mathrm{II}} \equiv T$ and in the case of forward scattering (with $\alpha(0) = 1$) we can conclude that the amplitude is asymptotically purely imaginary, i.e.

$$\lim_{s\to\infty} \frac{Re\,T(s,0)}{Im\,T(s,0)} = 0. \tag{2.9}$$

Combining (2.9) with the optical theorem we immediately arrive at the relation

$$\left.\frac{d\sigma(s,t)}{dt}\right|_{t=0} \sim \frac{1}{16\pi}\left[\sigma_{\mathrm{tot}}(s)\right]^2 \quad \text{at} \quad s\to\infty\,. \tag{2.10}$$

These results are generalized in a straightforward manner [14, 16] for processes involving spinor and charged particles, where the amplitude is a matrix characterized by several scalar functions. I shall not go into any detail and only want to quote some typical physical results.

1. The differential cross sections $\frac{d\sigma(s,t)}{dt}$ for particle and antiparticle scattering are equal to each other. This is true for instance for the following pairs of processes:

$$\begin{array}{ll} \pi^+ + p \to \pi^+ + p\,, & \pi^- + p \to \pi^- + p; \\ K^+ + p \to K^+ + p\,, & K^- + p \to K^- + p; \end{array} \tag{2.11}$$

$$\begin{array}{ll} \pi^+ + p \to K^+ + \Sigma^+\,, & K^- + p \to \pi^- + \Sigma^+; \\ K^- + p \to K^\circ + \Xi^\circ\,, & \bar{K}_0 + p \to K^- + \Xi^\circ; \end{array} \tag{2.12}$$

$$\begin{array}{ll} p + p \to p + p\,, & \bar{p} + p \to \bar{p} + p\,; \\ \Sigma^- + p \to n + \Lambda\,, & \bar{n} + p \to \bar{\Sigma}^- + \Lambda\,. \end{array} \tag{2.13}$$

2. The recoil fermion polarizations in processes (2.11), (2.12) are equal in absolute value and opposite in sign at $s \to \infty$.

3. For forward scattering, assuming that elastic scattering amplitude behaves like e.g. $s(\ln s)^\beta$ and taking into account the isotopic invariance we get

$$\left[\frac{d\sigma(\pi^+ + p \to \pi^+ + p)}{dt} - \frac{1}{2}\,\frac{d\sigma(\pi^- + p \to \pi^- + p)}{dt}\right]_{t=0} \sim$$

$$\sim \frac{1}{16\pi}\left[\sigma_{\text{tot}}(\pi^+ p)\right]^2, \tag{2.14}$$

$$\sigma_{\text{tot}}(\pi^+ p) = \sigma_{\text{tot}}(\pi^0 p)\,. \tag{2.15}$$

4. The limits of electromagnetic form-factors at $t \to \pm\infty$ are equal. It follows that in e^2 approximation the differential cross sections of the processes

$$e^- + p \to e^- + p \quad \text{and} \quad p^- + p \to e^- + e^+ \tag{2.16}$$

must coincide at high energy.

Let me say finally some words about the asymptotic relations between the amplitudes of production processes.

The difficulty in this case is that the five point function (as suggested by perturbation theoretical consideration) has complex singularities in momentum space so that it is not clear from the beginning whether the PHRAGMÉN—LINDELÖF theorem may be applied. This problem can be solved by using the so-called asymptotic amplitude.

The amplitude of both elastic and inelastic scattering processes may be written in the form

$$T^{\text{ret}}\left(w, \sqrt{w^2 - a^2}\right) = \int F^{\text{ret}}(x)\, e^{i(wx_0 - \sqrt{w^2-a^2}\mathbf{ex})}\, d^4x\,, \tag{2.17}$$

where $F^{\text{ret}}(x) = 0$ for $x \lesssim 0$. In the case of elastic scattering (considered in [20]) $a^2 = \mu^2 + p^2$ while in the case of the production amplitude [17] it is a more complicated (rational) function of the four fixed invariants. Define $T_\infty^{\text{ret}}(w)$ as

$$T_\infty^{\text{ret}}(w) = T^{\text{ret}}(w, w) = \int F^{\text{ret}}(x)\, e^{iw(x_3 - ex)}\, d^4x\,. \tag{2.18}$$

Under quite general assumptions it may be proved that

$$\lim_{w \to +\infty} \frac{T^{\text{ret}}(w, \sqrt{w^2 - a^2})}{T_\infty^{\text{ret}}(w)} = 1\,. \tag{2.19}$$

On the other hand, it is easily seen from (2.18) that $T_\infty^{\text{ret}}(w)$ is analytic in the upper half plane w, so that the PHRAGMÉN—LINDELÖF theorem may be applied to it in order to show that for some admissible $\varphi(w)$

$$\lim_{w \to +\infty} \frac{T_\infty^{\text{ret}}(w)}{\varphi(w)} - \lim_{w \to +\infty} \frac{T_\infty^{\text{ret}}(w)}{\varphi(w)}\,. \tag{2.20}$$

Then, taking (2.19) into account, we conclude that the same asymptotic equality holds for the physical amplitude T^{ret}. In this way asymptotic equality between the differential cross sections of the processes

$$\pi + N \to \pi' + \pi'' + N' \quad \text{and} \quad \bar{\pi} + N' \to \bar{\pi}' + \bar{\pi}'' + \text{N} \tag{2.21}$$

is proved.

In conclusion, I want to stress once more the importance of the experimental check of the obtained relations. As it was argued, the disproof of some of them would mean essentially a breakdown of the causality condition (or in any case — of some of the general physical requirements imposed on the scattering amplitude).

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О НЕКОТОРЫХ НЕДАВНИХ ДОСТИЖЕНИЯХ В АКСИОМАТИЧЕСКОМ ПОДХОДЕ В КВАНТОВОЙТЕОРИИ ПОЛЯ

И. Т. ТОДОРОВ

Р е з ю м е

Обсуждаются некоторые недавние достижения и общие тенденции аксиоматического подхода. Особое внимание уделено рассмотрению асимптотических соотношений для амплитуд рассеяния.

THEORY OF FERMI INTERACTIONS

By

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A field theoretical model of Fermi interactions is developed. Explicit expressions are given for lepton scattering amplitudes at high energies. The problem of locality and consistency of a non-renormalizable interaction is discussed.

I. Introduction

The problem of the higher approximations to weak interactions (and unrenormalizable interactions in general) has been a long-standing and till now unresolved one. The following features are interesting both from theoretical and practical point of view.

1. As unrenormalizable interactions always contain a parameter of the dimension of length (say, l), one expects that they can play an essential role in explaining the mass spectrum of elementary particles. However, the perturbation expansion of Green's functions contains powers of lE (E is some energy) so one expects an essential singularity in the energy and/ or momentum transfer variables; at the same time, the perturbation series are expected to diverge even in the simplest cases.

2. i) In view of what has been said above, one begins to doubt the validity of the usual approximations ("for a β — decay process it is sufficient to calculate matrix elements in lowest order, because of the smallness of one coupling constant").

   ii) Further, can higher approximations cure the "unitarity catastrophe"?

   iii) How do "weak scattering amplitudes" actually behave as s and/or t tend to infinity? In particular, can one write down dispersion relations for them? In what follows, we try to answer some of these questions in the framework of a simple, probably oversimplified, model.

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II. Theory

We consider processes where leptons only participate; for the sake of simplicity we neglect their masses; this does not affect the asymptotic behaviour of the amplitude in the kinematic variables if we adopt the philosophy that the latter is determined by the "leading singularities" of the diagrams.

The four-point Green's function G obeys the Bethe—Salpeter equation:

$$G = G_0 + G_0 KG\,, \tag{1}$$

where G_0 is the two-particle Green function, K the Bethe—Salpeter kernel. Our approximation consists in replacing K by the contribution of one single diagram (Fig. 1).

Fig. 1

Remarks

1. If in spite of neglecting masses we distinguish electrons and muons with the corresponding neutrinos we arrive at a two-channel problem. This has been treated fully elsewhere[1]; here we content ourselves by illustrating the main points on a single-channel problem.

2. If we replace the fermion loop in Fig. 1 by a W-meson, we arrive at the model of FEINBERG and PAIS[2]; our method applies equally well to it.

We now assume that the elementary four-fermion interaction is of the usual $V - A$ form; we approximate G by the following expression:

$$G = \gamma_i(1 + \gamma_5)\,\gamma(1 + \gamma_5)\,\mathcal{A}_f,$$

where $\mathcal{A}_f$ is a scalar function. We expand $\mathcal{A}_f$ with respect to four dimensional spherical harmonics. Then in Euclidean coordinate space, the n-th projection of the BS wave function obeys the equation[3]

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{n^2 - 1/4}{r^2} - \frac{\lambda^2}{r^6}\right)\psi_n(r) = 0\,,$$

$$\Psi_n(0) = 0; \quad \psi_n(r) \sim e^{-ikr} + S_n\, e^{ikr}\,, \qquad (r \to \infty) \tag{2}$$

$$(k^2 = 1/4\, E^2)\,.$$

Here n is the eigenvalue of the four dimensional angular momenta, $n = 1, 2 \ldots$; S_n the four dimensional projection of the S-matrix, $4k^2$ is the total CM energy squared, $\lambda^2 = 32\, f^2/\pi$, where f is the Fermi coupling constant.

Usual methods of the theory of singular differential equations can be applied to 2).

In particular, one verifies that the solution which satisfies the boundary condition at $r = 0$ behaves as

$$\psi(r) \sim \exp - \lambda/2r^2 \tag{3}$$

independently of n. Therefore, we immediately see that the divergent part of the diagram on Fig. 1 gives a vanishing contribution. To illustrate this point, we observe that the divergent part of the diagram corresponds to a contribution to the local interaction with "divergent coupling constant". However, in the chain diagrams of Fig. 2 the two particle propagator is the full, rather than

Fig. 2

the free one. The full propagator, however, is strongly damped, as indicated by eq. (3); therefore, it kills the contribution of the chain-diagram altogether.

How can then we reproduce the low energy behaviour of the matrix element? One can show, actually as in elementary quantum mechanics, that an effective range expansion of the transition matrix diverges; nevertheless, it works as an asymptotic expansion for the lowest partial wave[1] (this gives the leading contribution at threshold). In particular, the scattering length turns out to be

$$a = \frac{\lambda}{2} = \left(\frac{8}{\pi}\right)^{1/2} f, \tag{4}$$

so at threshold the higher orders renormalize the coupling constant only. (Let us remark in passing that our solution cannot be reproduced by perturbation theory; in fact, the partial wave amplitudes have a branch point $\sim \lambda^{n/2}$ at $\lambda = 0$).

The W-meson exchange can be treated along the same lines, as the Fermi theory. There is, however, an essential difference. In the low-energy approximation the interaction term in eq. (2) behaves as $g^2 r^{-2}$; therefore, there is no branch point at $g^2 = 0$. For sufficiently small values of the coupling constant an iterative solution can be obtained (although the higher terms do not decrease) and summed explicitly. In this way, one arrives exactly at the F—P result[3].

III. High energy lepton physics

Here "high energies" mean probably some hundreds of GeV's in the CMS. We consider a "multiperipheral model" of lepton scattering (Fig. 3).

Angular distribution

The total scattering amplitude in the t-channel is given by:

$$T(t, \cos \Theta_t) = \sum_n \frac{\sin n\Theta_t}{\sin \Theta_t} T_n(t) ,$$

$$t = 4k^2 , \qquad \cos \Theta_t = 1 + \frac{2s}{t} . \tag{5}$$

We take $T_n(t)$ as obtained from eq. (2) in an effective range approximation. The latter converges for $Re\, n < 1$ i.e. for unphysical values of n only.

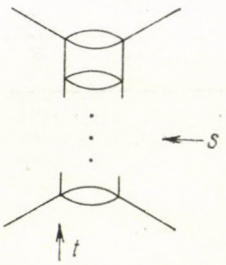

Fig. 3

One verifies that $S_n(t)$ is meromorphic in the whole n-plane and satisfies the relation, characteristic of a hard core[4]

$$S_{-n}(t) = e^{-2\pi i n} S_n(t) .$$

In spite of the nice analytic properties of $S_n(t)$, the concept of Regge poles is not a useful one in evaluating eq. (5) for $t < 0$, $s \to \infty$. One finds namely that there are infinitely many poles in the n-plane and for $t < 0$ all of them lie on the imaginary axis. However, if one transforms $\sum_n$ in eq. (5) into a contour integral:

$$\sum_n = \int dn \operatorname{ctg} n ,$$

and deforms the contour, the integral can be evaluated by means of the saddle point method. The resulting asymptotic expression is very simple:

$$T(s, t) \sim i \frac{16f}{s \log s} M(x) ,$$

where

$$x = -\log s / \log(-t) ,$$

$$M(x) = x^{3/2} \exp\left(-\frac{\log x}{x}\right) ;$$

(s, t are measured in units of $1/f$). Thus one obtains:

a) an essential singularity both in s and t;

b) the angular distribution shows a sharp diffraction peak;

c) the differential cross section decreases with the energy, in particular there is no "unitarity catastrophe".

IV. Problems of locality and zero charge

In the previous sections we have found that the Euclidean Bethe—Salpeter wave function behaves as $\exp -\frac{\lambda}{2r^2}$ near the origin. Such a behaviour has the property that expressions of the type

$$\int_0 dr\, R(r)\, \delta^{(n)}(r)\psi(r) \tag{6}$$

vanish, if $R(r)$ is any rational function of r and $\delta^{(n)}(r)$ is a derivative of any finite order (including zero) of the delta function. In treating the chain diagrams of Fig. 2 we had expressions exactly of the above type (that is the mathematical reason, why the contribution of the chain vanishes).

The above property of the wave function has far-reaching consequences.

Let us observe that the local part of the four-fermion interaction such as the term $f(\bar{\psi}\gamma_n(1+\gamma_5)\psi)^2$, say, in the Lagrangian, when corrected for interactions in the final state, gives rise just to integrals of the type (6). Now we see that such terms vanish, at least in our approximation. Thus the "effective interaction" is a non-local one. This is the same phenomenon as noticed some time ago by BOGOLIUBOV and SHIRKOV [5] and more recently by SCHROER [6]. Although none of the arguments (including ours) is rigorous, there seems to be thus an increasing evidence that a local, nonrenormalizable interaction does not lead to a consistent theory.

This immediately gives rise to the question of the consistency of our approximation. In fact, we summed up the "ladder-diagrams" of Fig. 3. However, by construction, these diagrams contain local vertices in the Bethe—Salpeter kernel (Fig. 1).

In principle, they should be replaced by the full vertex everywhere, so the "local bubble" in Fig. 1 vanishes. At this point there are two possibilities. Either the whole theory as it stands is inconsistent (we obtain e.g. identically zero for every Green's function), or we are not allowed to use a finite number of diagrams even for the Bethe—Salpeter kernel. In other words this means that if a non-renormalizable theory is to exist at all, there should be a singularity in the coupling constant in at least two of the Green's functions.

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ТЕОРИЯ ФЕРМИ-ВЗАИМОДЕЙСТВИЙ

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Резюме

Развивается теоретико-полевая модель взаимодействий Ферми. Даются явные выражения для амплитуды рассеяния лептонов при высоких энергиях. Обсуждается проблема локальности и состоятельности неперенормируемого взаимодействия.

REMARKS ON THE CONVERGENCE OF THE PERATIZATION

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The FEINBERG—PAIS equation for zero incoming momenta is investigated. In this case it turns out to be possible to show the mathematical accuracy and convergence of the peratization method.

In the preceding lecture of Mr. G. DOMOKOS you heard about the greatest defect of the field theory of weak interactions, namely that there exists no appropriate method of calculating the probability amplitudes in higher order. There are methods however which we can consider as the first attempts to solve this problem. These methods are: the one proposed by DOMOKOS, SURÁNYI and VANČURA, and the corresponding one in the intermediate vector meson theory, namely the peratization invented by G. FEINBERG and A. PAIS.

The peratization method consists of the following steps: one first writes down the equation which formally sums up the ladder graphs, that is the formal iteration of this integral equation (1) corresponds to the series of the contributions of the ladder graphs. After this one considers the graphs only as the graphical representation of the non-existent iteration terms and one tries to solve the equation obtained, hoping that the solution is a better approximation for the probability amplitude than the contribution of the first order graph only.

FEINBERG and PAIS gave a procedure for the solution of this equation, but without proving the mathematical accuracy and convergence of the method. They proceeded as follows: instead of the equation (1) they considered the so called regularized equation. The regularization corresponds to an invariant cut off. Denote M the regulator mass e.g. the cut off parameter.

The next step consists of the solution of the regularized equation with a special iteration scheme. Naturally the terms in the obtained iteration series depend on M (the cut off parameter).

After this, renouncing the summation of the series one let M tend to infinity in each term separately. At this point one sees the so-called self-damping of divergences. Loosely speaking the sum of the divergences in the graphs all together give zero.

With this procedure we get the peratization series, which gives the amplitude in power series of the coupling constant.

The mathematical problems which we have to prove are the following: 1. It is allowed to let M tend to infinity in the terms of the series. 2. The peratization series obtained in this way is convergent. 3. The sum of the peratization series satisfies the original integral equation (1).

We dealt with these problems in the simplest case: the trace equation for zero incoming momenta.[1] We answered first the second problem. It is possible to show.[1] that this series is identical with the NEUMANN series of an iterable equation (P), e.g. the convergence is assured for sufficiently small values of the coupling constant. (Probably also for the physical value.)

It is already more difficult to answer the first and third problems. The main mathematical difficulties lie in the fact that the space of variables in the integral equation (the four momentum space) is not Euclidean. The situation is much more favourable if the solution has suitable analytic properties for turning over the integration path in the complex p_0 plane to the imaginary axis. This reduces the difficulties for we can work in a Euclidean space. We assumed these analytic properties for the solution and so we could derive equation (P) from equation (1).[2] Thus the third problem is solved and the first is justified. Although it is possible to derive eq. (P) from eq. (1) in the original pseudo-euclidean metric too, in this case contrary to the Euclidean case we could not prove the existence of solutions of equation (P).

The situation can be explained on a model equation. Let us consider the integral equation

$$\psi(x) = (\varphi(x) + \gamma)\,\chi(x) + \lambda \int_0^\infty [\Sigma(xx') + \chi(x)\,\sigma(x')]\,\psi(x')\,dx' . \tag{1}$$

Here $\Sigma(xx')$ and $\varphi(x)$ are square integrable functions of their variables, γ is a constant, $\chi(x)$ is bounded and the integral $\int_0^\infty \chi(x')\sigma(x')dx'$ is divergent. From equation (1) it is clear, that the solution is of the form

$$\psi(x) = (\psi_1(x) + \psi_0)\chi(x) , \tag{2}$$

where ψ_0 does not depend on x. So the integral on the right hand side of (1) can exist only if $\psi_0 = 0$. Thus we get from (1) two equations:

$$\begin{aligned} 0 &= \gamma + \lambda \int_0^\infty \sigma(x')\,\psi(x')\,dx' , \\ \psi(x) &= \varphi(x)\,\chi(x) + \lambda \int_0^\infty \Sigma(xx')\,\psi(x')\,dx' . \end{aligned} \tag{3}$$

[1] The details of the calculation are given in the paper I. MONTVAY, Acta Phys. Hung., **18,** 119, 1965. For references see this paper too.

[2] See the Appendix of the article quoted in Footnote 1.

One can show that the FEINBERG—PAIS equation in suitable new variables is of the type (1). This means that "splitting up" like (3) happens with this equation too. The equation corresponding to the second part of (3) is eq. (*P*). It is a well behaved, iterable equation and its NEUMANN series is identical with the peratization series.

The first equation in (3) shows the self damping of divergences. Namely if we substitute the iteration series of the second equation into the first one and integrate the terms of the series we get more and more divergent results. The formal sum of the "divergences" gives zero on the left hand side. From these considerations we can understand why the perturbation series diverges. There we get the divergent terms together with the "real" ones.

There is a question, however, which arises immediately: whether the two equations in (3) contradict each other or not. To answer this question in the case of the FEINBERG—PAIS equation seems rather difficult in the pseudo-euclidean metric. But if we go over to the Euclidean space it is possible to show that there is no contradiction and thus the solution of (3) and so that of equation (1) exists.

ЗАМЕЧАНИЯ О СХОДИМОСТИ ПЕРАТИЗАЦИИ

И. МОНТВАИ

Р е з ю м е

Исследуется уравнение Файнверга и Пайса для нулевых начальных импульсов. Оказывается, что в этом случае можно показать метаметическую состоятельность и сходимость ператизационного метода.

NONPERTURBATIVE FIELD-THEORETICAL MODEL OF LEPTONIC WEAK INTERACTIONS

By

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In this work the problem of the four-leptonic weak processes is studied in the theory with an intermediate vector W-meson. Neglecting 1. $m_{\text{lept.}}$, 2. the part of amplitude containing induced neutral currents, an integral relation between the four-leptonic amplitude and the weak W-meson form factor is found. On the basis of this relation and the unitarity condition the four-leptonic amplitude is evaluated. The solution obtained contains a resonance in the energy channel with the total width about 1 MeV, and decreases as $\frac{1}{s^2}$ for $s \to \infty$. Some consequences of the model are discussed, first of all with regard to the production of an intermediate boson by neutrinos.

I. Introduction

Since the time of formulation of the FERMI theory of β-decay it is known that this theory (based on the lowest order of the perturbation expansion) is in contradiction with the unitarity of S-matrix [1]. During the development of this theory it was believed that this so called "unitarity catastrophe" is related only to the behaviour of the weak interactions at very high energies ($> 10^3$ BeV) and that the higher approximations do not change the form of the amplitude of the low-energy processes like μ-decay or β-decay. On the basis of such a theory the possibility was often discussed that weak interactions become strong at high energies [2], i.e. that the cross-sections of processes like

$$\nu + e^+ \to \nu + e^+ \tag{I.1}$$

increase as E^2(E is the c.m. total energy). The requirement of unitarity shows, however, that this is clearly not possible for the whole range of energy. Therefore one can conclude that there exists some critical energy, from which the cross-section decreases again.

This situation is without any essential change the same in the theory of weak interactions mediated by vector W-meson. Here too the lowest perturbative order is in contradiction with the unitarity [3], however, the cross-sections of processes like (I.1) do not increase with E^2.

These difficulties of the theory of weak interactions are related to the fact that neither in the case of FERMI interaction nor in the theory with W-meson have we any selfconsistent method for the field theoretical calculation of the S-matrix. Because of the unrenormalizability of both theories we cannot in fact apriori exclude the possibility that higher approximations of the perturbation expansion could essentially change the form of the lowest order amplitude. This statement has nothing to do with the smallness of the weak coupling constant and could appear even at small energies.

There are some examples demonstrating this situation: The theory of leptonic weak interactions with W-meson of G. FEINBERG and A. PAIS (based on the so-called peratization method) [4] and the theory of G. DOMOKOS, P. SURÁNYI and A. VANCURA [5] with direct FERMI interaction is of this kind. The starting point of both theories is the BETHE—SALPETER equation, i.e. one works here with some infinite class of FEYNMANN diagrams (ladder diagrams in the FP theory.) One obtains some relations between the coupling constant appearing in the Hamiltonian and the effective constant of the four leptonic processes like (I.1), which differs from the usual lowest order relation (in the case of W-theory)

$$\left(\frac{f}{m}\right)^2 = \frac{G}{\sqrt{2}} = \frac{10^{-5}}{\sqrt{2}} \frac{1}{m_p^2} . \tag{I.2}$$

We must conclude therefore, that the situation is probably such as indicated before: The role of higher order diagrams cannot be neglected in the whole energy range, beginning at low energies (μ-decay, β-decay etc.).

There are two fundamental questions related to the theories of FP or DSV type: what is the reason for selection of any class of diagrams, which generate the BETHE—SALPETER equation? Of course, the peratization method of FP can be probably extended to another than the uncrossed ladders type of diagrams, but it is not clear how to take into account in this way all possible diagrams or how to prove that taking only some class of diagrams forms a good approximation. The second question concerns the unitarity of amplitudes of FP or DSV, respectively. It is clear that these amplitudes cannot satisfy the unitarity condition at least in the two particle approximation [6].

Because of these two reasons it is possible that the numerical results of FP or DSV are wrong and that these theories show only the qualitative possibility of the important role of higher order diagrams.

However, the basic ideas of FEINBERG and PAIS, which form the starting point of the peratization programme, have a general meaning, which is independent of the selection of some class of diagram or of the renormalisation property of the theory. If one believes that the leading singularities of some diagrams damp themselves we can regard the perturbative expansion as

purely formal. By this we mean that it is only the relations between some (divergent) diagrams what is essential. (For example a relation between $n+1$-th and n-th ladders or between some diagrams which contribute to different processes). Each of the diagrams which appear in such a relation is — in general — meaningless but their sum has well-defined meaning: it represents the amplitude of the corresponding process.

It is possible that we can generalise this statement even to the case, when the perturbation expansion is divergent (as series) and therefore do not represent the solution of any field equations.

We shall demonstrate this point of view in a symbolic way: Let us denote by $M_n^G(A)$ a diagram of the n-th order, contributing to the process A, which belongs to some class G. $M_n^G(A)$ has of course only symbolic meaning, because it represents an infinite (and unmeasurable !) quantity. We assume that it is possible to introduce in a relativistic way the cut off Λ, which transforms the expressions $M_n^G(A)$ into some well-defined functions $M_n^G(\Lambda, A)$ for each n, G and A. In terms of cut off Λ we can write the formal expression $M_n^G(A)$ in a more detailed form

$$M_n^G(A) = \lim_{\Lambda \to \infty} M_n^G(\Lambda, A) . \tag{I.3}$$

We now define $\Lambda \to \infty$ by

I. the equality of $M_n^G(A)$ and $M_{n'}^{G'}(A)$ by the relation $M_n^G(\Lambda, A) = M_{n'}^{G'}(\Lambda, A)$;

II. the action of some (integral) linear operator $\hat{K}$ on $M_n^G(A)$ by

$$\hat{K} M_n^G(A) = \lim_{\Lambda \to \infty} \hat{K} M_n^G(\Lambda, A) ; \tag{I.4}$$

III. the expression

$$\overset{\infty}{\underset{n=1}{S}} M_n^G(A) = M^G(A) , \tag{I.5}$$

where the operation $\overset{\infty}{\underset{n=1}{S}}$ has the following properties:

a.) $M^G(A)$ is a finite function, if $M_n^G(A)$ contain (for different n) all diagrams contributing to the same measurable quantity (i.e. to the amplitude of corresponding process.)

b.) $$\underset{n}{S} M_n^G(A) = \underset{n}{S} M_n^{G'}(B) \quad \text{if} \quad M_n^G(A) = M_n^{G'}(B) ,$$

c.) $$\underset{n}{S} \hat{K} M_n^G(A) = \hat{K} \underset{n}{S} M_n^G(A) \hat{K} M_n^G(A) ,$$

d.) $$\overset{\infty}{\underset{n=1}{S}} M_{n+1}^G(A) = M^G(A) - M_1^G(A) . \tag{I.6}$$

It is possible to identify the expression (I.5) with the "sum of infinite set of divergent diagrams". The relations (I.6a—d) define the meaning of such a "sum", which is in fact independent of the covergence of the series $\sum_n M_n^G(A)$ and does not explicitly contain any cutoff.

It is clear that the point of view just described does not depend on the renormalisation properties of the theory. It defines the general meaning of the perturbation expansion. Using the standard FEYNMAN rules we can find some relations of the type

$$M_{n+1}^G(A) = \hat{K}^G(A) M_n^G(A) \tag{I.7}$$

and with the help of (I.6a—d) derive an integral equation for $M^G(A)$ which represents in the case of (I.7) the equation of BETHE—SALPETER type. However, there are some other relations like

$$M_{n+1}^G(A) = \hat{K}_{AB} M_n^G(B), \tag{I.8}$$

where A, B denote different processes. From such an expression it is possible to deduce some integral relations between the amplitudes of different processes without the assumption that only some class of diagrams contributes [7].

From the point of view of the renormalisability of the theory there are two possibilities: 1. All divergences can be excluded by renormalisation and the renormalized perturbation series converges. Then the conditions (I.5a—d) are valid for $S \equiv \Sigma$. In this case the perturbation series contains probably more information than integral equations, which can be derived from expressions like (I.7,8). 2. The theory is unrenormalizable. In this case the perturbation expansion is meaningless as series and the only physical information which the perturbation expansion contains is given by the FEYNMAN rules and expressions like (I.5,6).

We believe that in both of these cases the equations, which one obtains by application of the operation S on expressions like (I.7,8), together with the unitarity condition define the S-matrix and are equivalent to the basic principles of the quantum theory of fields. We are unable today to give a detailed proof of this equivalence but we believe that it is possible [8].

We shall show in this work, how it is possible to evaluate the amplitudes of four-leptonic processes like (I.1) in the theory with unstable W-meson, in the way just described. We shall not start from the equation of the type (I.7), which generates the BETHE—SALPETER equation, but rather from the expression like (I.8), which leads to some integral equations between amplitudes

of different processes. This starting point permits to take into account all possible diagrams contributing to the amplitudes we shall consider.

We can solve this problem only by using some additional assumptions, which however have nothing to do with the perturbation expansion and could be verified experimentally.

II. The general form of the four-leptonic amplitude

We shall start our investigation of the four-leptonic processes like (I.1) by specifying the basic interactions between vector W-field and leptonic currents. We define this interaction by the following requirements:

1. It conserves the electric charge, muon charge and lepton number. All these quantities we consider to be the additive quantum numbers [9].

2. It is invariant under the transformation $\psi \to -i\gamma_5\psi$, where ψ is an arbitrary lepton field.

3. Only single charged vector meson exists (together with its antiparticle), which has zero lepton number and muon charge.

4. The interaction is invariant under the transformation $\mu \leftrightarrow e$ together with $\nu_\mu \leftrightarrow \nu_e$ i.e. it is invariant under muon charge conjugation.

5. The interaction is local and contains only the simple vertex in which the single W-meson and single lepton pair can be annihilated or created in agreement with [1—4].

None of these assumptions is in contradiction with the experiment and it seems that they represent the simplest system of requirements, on which the theory of leptonic weak interactions can be based. Let us remark that the usual Hamiltonian

$$H_{\text{int}} = g[(\bar{e}O_a\nu_e) + (\bar{\mu}O_a\nu_\mu)]W^a \tag{II.1}$$

with

$$O_a = \gamma_a(1 - i\gamma_5)$$

is consistent with assumptions 1—5 [10]. The symbols e, ν_e, μ, ν_μ, W_a are the field operators of the corresponding particles and g is the coupling constant (dimensionless in $h = c = 1$ units as well as the electric charge). We can say a priori nothing about its numerical value, because we do not know the correct connection between g and the effective constants of four-leptonic processes, which only are directly measurable. Such a connection must be obtained from the theory.

We introduce as usual the invariant amplitude T by the relation

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta^4(p_i - p_f)\langle f \mid T \mid i\rangle. \tag{II.2}$$

Denoting by p_1, p_2, p_3, p_4 the momenta of the four leptons (see Fig. 1, and using the assumption 1) we can write $T(p_1, p_2, p_3, p_4)$ in the form

$$T(p_1, p_2, p_3, p_4) = M_{\alpha\beta}(p_1, p_2, p_3, p_4) J^{\alpha}(p_1, p_2) J^{\beta+}(p_3, p_4) + \\ + N_{\alpha\beta}(p_1, p_2, p_3, p_4) J^{\alpha}_{(0)}(p_1, p_3) J^{\beta}_{(0)}(p_2, p_4), \qquad \text{(II.3)}$$

where J^{α} is a charged and $J^{\alpha}_{(0)}$ is an (induced) neutral current. Both of these currents conserve lepton number and muon charge. The first part of the amplitude (II.3) is called the allowed amplitude, the other one, containing neutral currents, forbidden amplitude.

It is useful to introduce the following classification of all four-leptonic processes allowed by conservation of electric and muon charge together with lepton number:

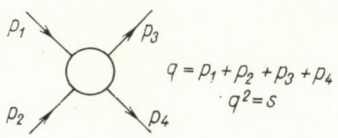

Fig. 1. The orientation of the external lepton lines: we consider p_1, p_2 and p_3, p_4 to be the momenta of leptons forming together some charged current (see II. 3)

I. *Processes of first kind*: These are those processes which are described by the allowed amplitude only. They are μ-decay and all corresponding inverse processes, or processes which can be obtained by the CP-conjugation.

II. *Processes of second kind* — i.e. processes which can be described by both allowed and forbidden amplitude. These are elastic scattering of ν_e on e^+ or of ν_μ on μ^+ and corresponding inverse processes.

III. *Processes of third kind* are those, which are described by the forbidden amplitude only and contain neutrino (i.e. weak interactions are dominant.)

IV. We shall call the *processes of fourth kind* all processes which do not contain any neutrino, i.e. which are electromagnetic: the forbidden amplitude represents the small corrections to them only.

Feinberg and Pais (in the uncrossed ladder approximation) have distinguished the processes, which occur via an exchange of odd or even number of intermediate bosons and called these processes allowed or forbidden, respectively. Our division of amplitude (II.3) into allowed and forbidden parts is in fact a generalisation of such a point of view. One of the most interesting result of FP is the statement that the forbidden amplitude is negligible in comparison with the allowed one [11]. In our model we shall suppose that this is the general property of leptonic weak interactions, which is true independently of the ladder diagrams approximation. This assumption will essen-

tially simplify our investigations, however, it means that we cannot obtain in our model any information about the processes of third kind. Both processes of the first and second kind will be now described by the unique amplitude, which can be obtained from (II.3) by neglecting the term containing neutral currents $J^{\alpha}_{(0)}$.

Let us now discuss the form of charged current J^{α}. From the experiment we know that this current must be of the $V-A$ type at small energies. We shall suppose that this is true in the whole energy range. Such a supposition could be probably simply verified in future $\nu - l$ scattering experiments. To maintain the $V - A$ character of the current J^{α} in all steps of our calculations we put in all intermediate states $m_{\text{lept}} = 0$. This means that the theory becomes γ_5 invariant including the lepton propagators. This reflects itself in the fact that the current J^{α} must be $V - A$.

The Hamiltonian chosen is invariant under muon charge conjugation. As we suppose $m_{\text{lept}} = 0$ this invariance must take place in the amplitude as well. From this we can conclude that the current must have the general form

$$J_{\alpha} = (\bar{U}_e O_{\alpha} U_{\nu e}) + (\bar{U}_{\mu} O_{\alpha} U_{\nu\mu}) \tag{II.4}$$

(the terms $\bar{U}_e U_{\nu\mu}$ cannot in (II.4) arise because they violate conservation of muon charge.)

The general form of the four-leptonic amplitude based on the Hamiltonian (II.1) and on the additional assumptions *a)* the smallness of the forbidden amplitude and *b)* $m_{\text{lept}} = 0$ in intermediate states, can be written as

$$T(p_1, p_2, p_3, p_4) = M_{\alpha\beta}(p_1, p_2, p_3, p_4)\, J^{\alpha}(p_1, p_2)\, J^{\beta+}(p_3, p_4)\,, \tag{II.5}$$

where J^{α} is given by (II.4).

In the next section we shall show how $M_{\alpha\beta}$ can be found.

III. The weak formfactor of W-meson

The W-meson, which mediates the leptonic weak interactions, is unstable and can decay in the following two ways:

$$W \to e + \nu_e\,, \quad W \to \mu + \nu_{\mu}\,. \tag{III.1}$$

(We do not speak about other nonleptonic decay modes of W, because we consider the leptonic interactions only.) The amplitude, which describes the two decays (III.1) can be written (for the reasons which we have discussed before) in the general form

$$\langle p_1, p_2, \mid T \mid k\rangle = U_{\alpha\beta}(p_1, p_2, k) J^{\alpha}(p_1, p_2) W^{\beta}(k)\,, \tag{III.2}$$

where the function $U_{\alpha\beta}(p_1, p_2, k)$ represents the weak formfactor of W.

In this section we shall derive an integral relation between the formfactor $U_{\alpha\beta}$ and the amplitude $M_{\alpha\beta}$ introduced in Section II in the way, which was described in the Introduction.

Let us consider some irreducible diagram, contributing to the allowed four-leptonic amplitude. (By irreducible diagram we mean here a diagram which cannot be divided into two parts by a single cut crossing a single lepton line only — see Fig. 2a).

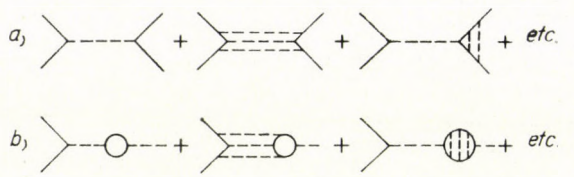

Fig. 2. a) Examples of some irreducible contributions to the four-leptonic allowed amplitude (the "sum" is $M_{\alpha\beta} j^{\alpha} j^{\beta+}$).
b) Corresponding terms $\Gamma^{(1,n)}_{\alpha\beta}$ (their "sum"is $G^{(1)}_{\alpha\beta}\ j^{\alpha}\ W^{\beta}$)

Assuming that it is possible to arrange all these diagrams in some sequence, we can write the n-th diagram as [12]

$$M^{(n)}_{\alpha\beta}(p_1, p_2, p_3, p_4) j^{\alpha}(p_1, p_2) j^{\beta+}(p_3, p_4) \,. \tag{III.3}$$

The meaning of the symbol $M^{(n)}_{\alpha\beta}$, which is in fact a divergent integral, was defined in the Introduction. Using the standard FEYNMAN rules we can form the diagram, which represents (III.3) (contracting the free lepton lines in current $j^{\beta+}$), obtain some other diagrams which represent an irreducible contribution to the W-meson formfactor. Denoting the last as

$$\Gamma^{(1,n)}_{\alpha\beta}(p_1, p_2, k)$$

(see Fig. 2b) we can write

$$j^{\alpha} W^{\beta} \Gamma^{(1,n)}_{\alpha\beta}(p_1, p_2, k) = -ij^{\alpha} W^{\beta} \int M^{(n)}_{\alpha\beta}(p_1, p_2, p_3, p_4) \Phi^{(1)\lambda}_{\beta}(p_3, p_4) \cdot$$
$$\cdot (2\pi)^4 \delta^4(p_3 + p_4 - k) d^4p_3 d^4p_4, \tag{III.4}$$

where

$$\Phi^{(1)}_{\alpha\beta} = 2gSp[O_{\alpha} S^{(1)}_F(\hat{p}_3) O_{\beta} S^{(1)}_F(\hat{p}_4)] \,. \tag{III.5}$$

In (III.5) $S^{(1)}_F$ is the propagator of a free lepton.

We have here an example of a situation which was discussed in Section I. Relation (III.4) is of the general type (I.8). According to (I.5,6) we can define the operation $\underset{n}{S}$, which transforms the sequencies $M^{(n)}_{\alpha\beta}$ and $\Gamma^{(1\,n)}_{\alpha\beta}$ into

$$\underset{n}{S}\, M^{(n)}_{\alpha\beta} = M_{\alpha\beta} \,, \qquad \underset{n}{S}\, \Gamma^{(1n)}_{\alpha\beta} = G^{(1)}_{\alpha\beta} \,. \tag{III.6}$$

Using now the properties of operation S (13) (which define it) we can obtain from (III.4)

$$j^{\alpha} W^{\beta} G^{(1)}_{\alpha\beta}(p_1, p_2, k) = -ij^{\alpha} W^{\beta} \int M_{\alpha\beta}(p_1, p_2, p_3, p_4) \cdot$$
$$\cdot \Phi^{(1)\lambda}_{\beta}(p_3, p_4)(2\pi)^4 \delta^4(p_3 + p_4 - k)\delta^4 p_3 \delta^4 p_4 \,. \qquad \text{(III.7)}$$

However, $M_{\alpha\beta} j^{\alpha} j^{\beta+}$ does not represent the whole amplitude of allowed four-leptonic processes and $G^{(1)}_{\alpha\beta} j^{\alpha} W^{\beta}$ is not the whole amplitude (III.2). Let us first consider the contributions to $M_{\alpha\beta} j^{\alpha} j^{\beta+}$ which give rise (in the renormalisation terminology) to the renormalisation of leptonic wave functions, contained in the current $j^{\beta+}$ (see Fig. 3a). We shall denote these reducible cor-

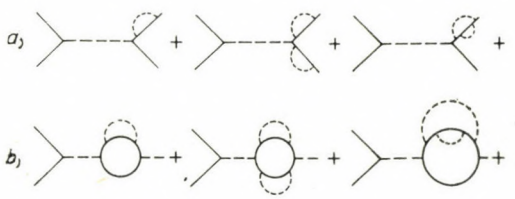

Fig. 3. a) Examples of some reducible ν-corrections to the expression. $M^{(n)}_{\alpha\beta} j^{\alpha} j^{\beta+}$ for some fixed n.
b) The ν-corrections corresponding to the $\Gamma^{(1,n)}_{\alpha\beta} j^{\alpha} W^{\beta}$

rections by index ν. We can again repeat the procedure (based on the standard FEYNMAN rules) which leads to the equation (III.4) and obtain (see Fig. 3b)

$$j^{\alpha} W^{\beta} G^{(1,n)}_{\alpha\beta}(p_1, p_2, k) = -ij^{\alpha} W^{\beta} \int M^{(n)}_{\alpha\lambda}(p_1, p_2, p_3, p_4) \cdot$$
$$\cdot \Phi^{(\nu)\lambda}_{\beta}(p_3, p_4)\,(2\pi)^4 \delta^4(p_3 + p_4 - k) d^4 p_3 d^4 p_4 \,. \qquad \text{(III.8)}$$

The application of the operation $\underset{n}{S}$ to this expression gives

$$j^{\alpha} W^{\beta} G^{(\nu)}_{\alpha\beta}(p_1, p_2, k) = -ij^{\alpha} W^{\beta} \int M_{\alpha\lambda}(p_1, p_2, p_3, p_4) \cdot$$
$$\cdot \Phi^{(\nu)\lambda}_{\beta}(p_3, p_4)\,(2\pi)^4 \delta^4(p_3 + p_4 - k) d^4 p_3 d^4 p_4 \,, \qquad \text{(III.9)}$$

where $\Phi^{(\nu)}_{\alpha\beta}(p_3, p_4)$ has the same meaning as (III.5) but contains the ν-th correction to the free lepton propagator $S^{(1)}_F$. The expression

$$\underset{\gamma=1}{\overset{\infty}{S}} \Phi^{(\nu)}_{\alpha\beta} = \Phi_{\alpha\beta} \qquad \text{(III.10)}$$

has then the form (III.5) with the physical lepton propagators S_F.

To get the whole amplitude of the allowed four-fermion processes it remains to take into account the diagrams contributing to the "renormalisation" of the lepton wave functions, which are contained in the current j^{α}. This can

be simply done by substituting into (III.9) J^α instead of j^α, where J^α is a current constructed from the physical lepton wave functions. The allowed four-leptonic amplitude can now be written as

$$M_{\alpha\beta}(p_1, p_2, p_3, p_4)J^\alpha(p_1, p_2)J^{\beta+}(p_3, p_4)\,, \tag{III.11}$$

where $M_{\alpha\beta}$ is given by (III.6). The whole W-meson form factor is defined by the relation

$$U_{\alpha\beta} = \overset{\infty}{\underset{\nu=0}{S}} G^{(\nu)}_{\alpha\beta} = \overset{\infty}{\underset{\nu=1}{S}} G^{(\nu)}_{\alpha\beta} + G^{(0)}_{\alpha\beta}\,, \tag{III.12}$$

where we have denoted

$$G^{(0)}_{\alpha\beta} = gg_{\alpha\beta}\,. \tag{III.13}$$

Applying the operation $\overset{\infty}{\underset{\nu=1}{S}}$ to (III.9) with J^α instead of j^α we obtain the integral relation between the whole four-leptonic amplitude and the W-meson form factor

$$\begin{aligned} U_{\alpha\beta}(p_1, p_2, k) = g \cdot g_{\alpha\beta} - i \int M_{\alpha\lambda}(p_1, p_2, p_3, p_4) \cdot \\ \cdot \Phi^\lambda_\beta(p_3, p_4)\,(2\pi)^4\delta^4(p_3 + p_4 - k)d^4p_3 d^4p_4\,. \end{aligned} \tag{III.14}$$

It is clear that $U_{\alpha\beta}J^\alpha W^\beta$, where $U_{\alpha\beta}$ is given by (III.14), represents the whole amplitude (III.2). This follows from the fact that (III.11), which is the "sum" of all irreducible diagrams with all possible reducible corrections, is the total four-leptonic amplitude. Each diagram contributing to the amplitude (III.2) must begin by virtual dissociation of W into a lepton pair. Two (virtual) leptons forming this pair can either interact in all possible ways, which gives rise to the second term of the right hand side of (III.14), or can represent free physical particles. The first term of (III.14) corresponds to this second possibility (see Fig. 4).

The relation (III.14) represents one of the basic equations we shall start from in our model.

Let us now consider the $W-W$ amplitude, which must have the general form

$$\langle k \mid T \mid k \rangle = A_{\alpha\beta}(k)W^\alpha(k)W^{\beta+}(k). \tag{III.15}$$

With the help of this amplitude at the point $k^2 = m^2$ we will be able to evaluate the self energy of the W-meson. In a similar way as before we can find for $A_{\alpha\beta}$ the following representation

$$A_{\alpha\beta}(k) = -i \int \Phi^\lambda_\alpha(p_1, p_2)U_{\lambda\beta}(p_1, p_2, k)\,(2\pi)^4\delta^4(p_1 + p_2 - k)d^4p_1 d^4p_2, \tag{III.16}$$

where Φ^λ_α has the same meaning as in equation (III.14) (see Fig. 5).

The equations (III.14,16) together with the unitarity condition are the basic equations of our model. It was pointed out in the Introduction that we believe that such a system of equations is sufficient to obtain a finite expression for the amplitude, and that it is equivalent to the basic principles of the field theory.

We shall now assume that the amplitude $M_{\alpha\beta}(p_1, p_2, p_3, p_4)$ depends on $q = p_1 + p_2$ ($q^2 = s$) only. In the next Section it will become clear that such a choice represents correctly some solution of our equations. From the orientation of the external lines we use (see Fig. 1) it follows that this situation is

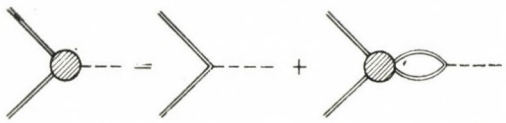

Fig. 4. The graphical representation of equation (III. 14). The doubled lines represent the physical lepton wave functions or propagators

Fig. 5. The graphical representation of equation (III. 16)

very similar to that obtained by FEINBERG and PAIS. They found that some approximate solution of their BETHE—SALPETER equation exists, which depends (just as in the case we consider here) on the variable q only.

Writing

$$M_{\alpha\beta}(p_1, p_2, p_3, p_4) = M_{\alpha\beta}(q)\,, \quad U_{\alpha\beta}(p_1, p_2, k) = U_{\alpha\beta}(q) \tag{III.17}$$

we can obtain from (III.14,16) a simplified system of equations

$$U_{\alpha\beta}(q) = g \cdot g_{\alpha\beta} - iM_{\alpha\lambda}(q)F^{\lambda}_{\beta}(q)\,, \tag{III.18a}$$

$$A_{\alpha\beta}(q) = -iF^{\lambda}_{\alpha}(q)U_{\lambda\beta}(q)\,, \tag{III.18b}$$

where

$$F_{\alpha\beta}(q) = 2g \int Sp\,[O_{\alpha}S_F(\hat{q}_1)O_{\beta}S_F(\hat{q}_2)](2\pi)^4\delta^4(q_1 + q_2 - q)d^4q_1d^4q_2 \tag{III.19}$$

is a real function of q.

Each tensor function of the single variable q, which appears in (III.18a,b) can be generally written as

$$\begin{aligned} M_{\alpha\beta}(q) &= M(s)g_{\alpha\beta} + \bar{M}(s)q_{\alpha}q_{\beta}\,, \\ U_{\alpha\beta}(q) &= U(s)g_{\alpha\beta} + \bar{U}(s)q_{\alpha}q_{\beta}\,, \\ A_{\alpha\beta}(q) &= A(s)g_{\alpha\beta} + \bar{A}(s)q_{\alpha}q_{\beta}\,, \\ F_{\alpha\beta}(q) &= F(s)g_{\alpha\beta} + \bar{F}(s)q_{\alpha}q_{\beta}\,. \end{aligned} \tag{III.20}$$

In Section II we supposed $m_{\text{lept}} = 0$ in all intermediate states. This assumption is necessary to obtain in general the current J^a of the $V-A$ type and also to obtain a closed system of equations in the way we consider. We can now simplify our calculations by neglecting m_{lept} in the initial and final states as well. This is a purely mathematical approximation in general not required by our model and we shall consider the effect of finite lepton masses in real states later.

Assuming $m_{\text{lpt}} = 0$ everywhere (14) we must have

$$q^a J_a = q^a J_a^+ = 0 \,, \tag{III.21}$$

i.e. only the parts of (III. 20) which contain $g_{\alpha\beta}$ contribute to the corresponding matrix elements. We can therefore project out of the system of equations (III.18a,b) those parts only, which do not contain the terms $q_\alpha q_\beta$. For the scattering processes the remaining parts are not necessary. In this approximation we have instead of (III.18a,b)

$$\begin{aligned} U(s) &= 1 - iM(s)F(s) \,, \\ A(s) &= -iU(s)F(s) \,. \end{aligned} \tag{III.22}$$

The whole amplitudes we consider are now given by the following simple expressions

$$\begin{aligned} \langle p_1, p_2 \mid T \mid p_3, p_4 \rangle &= M(s) J^a J_a^+ \,, \\ \langle p_1, p_2 \mid T \mid k \rangle &= U(s) J^a W_a \,, \\ \langle k \mid T \mid k \rangle &= A(s) W^a W_a^+ \,. \end{aligned} \tag{III.23}$$

The last expression in (III.23) is true, because for the free W-meson we have $k^a W_a(k) = 0$.

IV. The unitarity condition

Taking into account the definition of the invariant amplitude (II.2), we can write the unitarity condition in the form

$$Im \langle f|T|i\rangle = -\frac{(2\pi)^4}{2} \sum_n \langle f|T^+|n\rangle \langle n|T|i\rangle . \tag{IV.1}$$

Since we work with weak interactions only, we cannot consider the unitarity condition (IV.1) in the channels where processes of fourth kind contribute, because there the electromagnetic interactions prevail. We must consider such channels in which $|i\rangle$, $|f\rangle$, $|n\rangle$ contain some charged weakly interacting systems — for example $e^{\mp} \nu_e$, $\mu^+ \nu_\mu$, W^+ etc. In these channels the electromagnetic interactions represent some corrections only, we hope, that suffi-

ciently small. We shall choose as $|i\rangle$, $|f\rangle$ either $e^+\nu_e$, $\mu^+\nu_\mu$ or W^+ (together with CP-conjugated states.). In this case the variable s represents the total c.m. energy squared.

We are of course unable to take into account all possible intermediate states $|n\rangle$. We shall consider here the same states as in $|i\rangle$, $|f\rangle$ i.e. $e^+\nu_e$, $\mu^+\nu_\mu$, W^+. In this case the unitarity condition together with expressions (III.22) represents a closed system of equations for the amplitudes $M(s)$, $U(s)$, $A(s)$. We can say apriori nothing about the correctness of such an approximation. We hope only that the effective constants of processes where more particles than we consider are present are small in comparison with processes we take into account. Because of the zero value of lepton masses (or in general

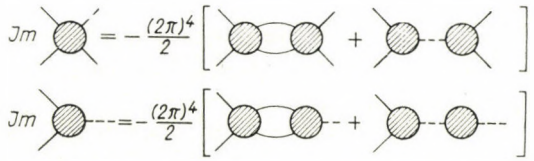

Fig. 6. The approximation used in the unitarity condition

because of their smallness) we assume that there are no "kinematic" arguments for the approximation we are obliged to do. However, it is natural to suppose that this approximation is equally good (if it is good) in the whole energy range.

The unitarity condition can be written now (see Fig. 6) as

$$Im\ M(s) = \frac{s}{3\pi} |M(s)|^2 - \pi |U(m^2)|^2 \delta(s - m^2)\,, \qquad \text{(IV. 2a)}$$

$$Im\ U(s) = \frac{s}{2\pi} U(s)\, M^*(s) - \pi A(m^2)\, U^*(m^2)\, \delta(s - m^2)\,. \qquad \text{(IV. 2b)}$$

Deriving the equations (IV.2a,b) we must take into account that the current J^a contains two parts $(\bar{e}O_a\nu_e)$ and $\bar{\mu}O_a\nu_\mu$, which can independently contribute to the intermediate state $|n\rangle$. The contributions of these two parts are the same because $m_{\text{lept}} = 0$.

The single W-meson intermediate state leads to the δ-singular terms in (IV.2a,b). The presence of such a singularity is related to the fact that we have worked with the W-meson as a stable particle. In this case it would be natural to expect the pole of $M(s)$ at $s = m^2$. However, it was pointed out before that one of the most important features of the theory of weak interactions with intermediate boson is its unstability. The probability of the decay of W is proportional to $|U(m^2)|^2$ and so, to have a meaningful theory, we must

exclude possibilities like $U(m^2) = 0$ or $U(m^2) \to \infty$. If $U(m^2) \neq 0$ the δ-singular terms in (IV.2a,b) cannot vanish, and we must expect the pole of $M(s)$ in the point $s = m^2$. But in this case it follows from the equation (III.22) that $U(m^2) \to \infty$ unless $F(m^2) = 0$. But this latter possibility is unsatisfactory as well, because it leads to zero selfenergy of W. These qualitative considerations show that we must take into account the unstability of the W-meson in the unitarity condition [15]. This can be done in the following phenomenological way: Let us write in the δ-function which appears in the conditions (IV.2a,b) the complex mass

$$m' = m + i\frac{\Gamma}{2} \tag{IV.3}$$

instead of m. In (IV.3) Γ represents the full width of the "W-resonance", i.e. Γ is the total probability of W-decay. Instead of $\delta(s - m^2)$ we shall then use the function $\Delta(s - m'^2)$, where $\Delta(z)$, for any complex $z = x + iy$, is defined in the way very similar to the usual definition of δ-function for $Imz \leqq 0$,

$$\Delta(z) = \frac{1}{2\pi}\int_0^\infty dq[e^{\pm iqz} + e^{\mp iqz^*}] = \frac{1}{\pi}\frac{|Imz|}{|z|^2}. \tag{IV.4}$$

It can be easily seen, that the function $\Delta(z)$ defined in such a way has the following properties:

1.) $$\lim_{Imz\to 0} \Delta(z)\,\delta(Rez)\,,$$

2.) $$\lim_{y\to 0} \int_{-\infty}^{\infty} \Delta(x + iy) f(x)\, dx = f(0)\,.$$

Using in the unitarity conditions (IV.2a,b) $\Delta(z)$ instead of δ and substituting $A(m^2)$ from the second equation (III.22) we obtain

$$\begin{aligned} ImM(s) &= \lambda s \mid M(s) \mid^2 - \xi D(s)\,, \\ ImU(s) &= \lambda s U(s) M^*(s) + i\xi\eta D(s)\,, \end{aligned} \tag{IV.5}$$

where $\lambda = \dfrac{1}{3\pi}$, $\xi = \mid U(m^2) \mid^2$, $\eta = F(m^2)$ and the function $D(s)$ is given by the expression

$$D(s) = \frac{m\Gamma}{\left(s - m^2 + \dfrac{\Gamma^2}{4}\right) + m^2\Gamma^2}\,. \tag{IV.6}$$

In general we must expect $\Gamma^2 \ll m^2$. From (IV.6) it is clear that in the limit $\frac{\Gamma}{m} \to 0$, which corresponds to the stable W-meson, the relations (IV.5) go over into (IV.2a, b).

V. The resultant amplitude

To obtain a final expression for the allowed four-leptonic amplitude we have to solve the system of equations (III.22) and (IV.5). Let us denote by indices 1 and 2 the real and imaginary parts of $M(s)$ and $U(s)$, respectively. (We note, that $F(s)$ is a real function — see (III.19).) Then it can be easily shown that our system of equations requires

$$M_1(s) = U_2(s) = 0 \,. \tag{V.1}$$

The remaining system of equations does not have a unique solution. There are two solutions — one which is finite at $s = 0$ and the other, which is not. To maintain the possibility of defining the effective coupling constant as the zero-energy value of the amplitude $M(s)$, we chose as the physical solution the finite one at $s = 0$. This solution has the form

$$M_2(s) = -2\xi D(s) \frac{1}{1 + \sqrt{1 + 4\lambda\xi D(s)\, s}} \,, \tag{V.2}$$

where $D(s)$ is given by (IV. 6). The corresponding form of the W-meson form-factor is

$$U_1(s) = -\frac{\eta}{2\lambda s}\left[1 + \sqrt{1 + 4\lambda\, \xi s\, Ds}\right]. \tag{V.3}$$

The solutions (V.2,3) contain the constant Γ, which represents the total W-meson decay rate. This quantity can be exactly calculated with the help of the general form of the W-decay amplitude (see III.23)

$$\langle p_1, p_2 \mid T \mid k\rangle\big|_{k^2=m^2} = U(m^2) J_\alpha W^\alpha \,. \tag{V.4}$$

The calculation leads to a very simple expression

$$\Gamma = \lambda\xi m \,, \tag{V.5}$$

where λ, ξ have the same meaning as before.

It remains to determine the constants ξ, η. Using the form of $F(s)$, which follows from our equations too,

$$F(s) = 2\lambda s \frac{-g\left[1 - \sqrt{1 + 4\lambda s\, \xi D(s)}\right] + 2\xi\eta\, D(s)}{\left[1 - \sqrt{1 + 4\lambda s\, \xi D(s)}\right]^2} \tag{V.6}$$

and the approximate relation

$$D(m^2) = \frac{1}{m\Gamma} \tag{V.7}$$

which holds, because $\Gamma^2 \ll m^2$ must be valid, one can obtain

$$U(m^2) = g\frac{1 + \sqrt{5}}{2} \tag{V.8}$$

together with

$$\eta = -g\lambda m^2 . \tag{V.9}$$

Inserting (V.8,9) into (V.2,3,5) and using (IV.6) we obtain the final expression for the allowed four-leptonic amplitude

$$M(s) = -6\pi i \frac{\Gamma^2}{\left(s - m^2 + \frac{\Gamma^2}{4}\right)^2 + m^2\Gamma_2} \times \tag{V.10}$$

$$\times \frac{1}{1 + \left[1 + \frac{4s\Gamma^2}{\left(s - m^2 + \frac{\Gamma^2}{4}\right)^2 + m^2\Gamma^2}\right]^{1/2}},$$

where Γ is given by the formula

$$\Gamma = \frac{g^2}{3\pi}\left(\frac{1 + \sqrt{5}}{2}\right)^2 m . \tag{V.11}$$

The solution just obtained has the sharp resonant character. The resonance energy is given by $s = m^2 - \frac{\Gamma^2}{4} \approx m^2$ and Γ represents the width of the resonance. This resonance can occur in the energy channel, i.e. in the processes of the second or first kind like

$$\begin{aligned} &\nu_e + e^+ \to \nu_e + e^+ , \quad \nu_e + e^+ \to \nu_\mu + \mu^+ , \\ &\nu_\mu + \mu^+ \to \nu_\mu + \mu^+ , \quad \nu_\mu + \mu^+ \to \nu_e + e^+ . \end{aligned} \tag{V.12}$$

The value of the amplitude at resonance is independent of Γ (i.e. of the coupling constant) and is given by

$$M_{\text{res}} = -\frac{6\pi i}{1+\sqrt{5}}\,\frac{1}{m^2}\,. \tag{V.13}$$

The amplitude also increases slowly to the resonant point and then decrease as $\frac{1}{s^2}$. This picture is qualitatively the same as in the case of W-theory in lowest order perturbation calculation, except for the asymptotic behaviour. (In the lowest order theory the asymptotic behaviour is $\frac{1}{s}$).

The presence of the resonance in the processes like (V.12) is a usual conclusion of the W-theory of weak interactions and can be obtained from the lowest perturbation as well [16]. What is, however, a surprising result of our model is the value of the width of such a resonance.

Let us consider the effective constant of interactions like (V.12), which can be defined as the zero-energy value of the amplitude (V.10). Using $\Gamma^2 \ll m^2$ we obtain

$$M(0) = -3\pi i\,\frac{\Gamma^2}{m^4}\,. \tag{V.14}$$

As $\Gamma \sim g^2$ we must conclude that the effective constant (V.14) is proportional to g^4. This result is incomprehensible from the perturbation point of view. Using the perturbation theory language we would be obliged to suppose, that the contribution of all higher order diagrams cancels the lowest perturbation order (which is the only perturbative term proportional to g^2). From this we see the difference of our result from that of FEINBERG and PAIS.

For getting some estimate of the numerical value of Γ we put the effective constant (V.14) equal to the Fermi constant $\frac{10^{-5}}{\sqrt{2}}\,\frac{1}{m^2}$. (In such a way we can of course obtain only an approximate information about the magnitude of Γ and g. To obtain the precise value we must calculate the lifetime of μ. For this we must take into account the effect of nonzero lepton masses in the initial and final states. These calculations are in progress now.)

Using (V.14) and the magnitude of the Fermi constant we obtain with m of the order of nucleon mass the estimate

$$\Gamma \approx 1 \text{ MeV}\,, \tag{V.15}$$

which is about three orders greater than the value of Γ predicted by the lowest order perturbation calculation.

VI. Conclusion

We would like to discuss now some possible consequences of the results derived in the preceding sections.

Let us first say some words about the production of W-meson by dissociation of the neutrino in the Coulomb field of the nucleus, i.e. about the process

$$\nu_\mu + Z \to \mu^- + W^+ + Z\,. \tag{VI.1}$$

This process was experimentally studied at CERN, where the analysis of experimental data was based on the lowest order perturbation calculations of the corresponding crossection [19]. These calculations lead in general to a value proportional to $(Z\alpha)^2\Gamma$ $\left(\text{where } \alpha = \frac{e^2}{\hbar c}\right)$. It seems also, that according to our model the production could be copious, because $\frac{\Gamma}{m}$ is of the same order as α. However, this conclusion is generally wrong. Let us consider the weak formfactor of the W-meson, which in our model (see V.3) is

$$U(s) = g\,\frac{m^2}{2s}\left\{1 + \left[1 + \frac{4s\,\Gamma^-}{\left(s - m^2 + \frac{\Gamma^2}{4}\right)^2 + m^2\,\Gamma^2}\right]^{1/2}\right\}. \tag{VI.2}$$

In lowest order calculations one uses instead of $U(s)$ the coupling constant f, related to the Fermi constant by the relation (I.2). However, from (VI.2) we see that in our model the situation is quite different. We have here the constant $g \sim f^{1/2}$ and $U(s)$ decreases as $\frac{1}{s}$ with s. For $s \sim \frac{m^2}{g}$ U has the same vale as f, and for $s > \frac{m^2}{g}$ it is even smaller. Therefore without calculations (which will be made in the near future) we cannot say anything about the magnitude of the crossection for production (VI.1). It seems that its form will be in our model quite different — one can for example expect that it will decrease with the energy of incident neutrino (at least in the very high energy region.) It can even happen that our cross-section of productions process (VI.1) would be in our model smaller than in the lowest order calculations in perturbation theory.

At the end we have to mention some ideas, which could lead to a further development of our model.

Recently TAKEDA [17] discussed the possibility of identification of some resonance in the $\pi-K$ system (K, mass 725 MeV, $\Gamma_K < 12$ MeV) with the weak intermediate meson W. If our model could be extended to the non-

leptonic interactions (such as $(\pi K)W$) then the high value of our $\Gamma/\Gamma \sim 1$ MeV could be used for demonstrating TAKEDA's point of view.

There is another point we would like to emphasize, i.e. the approximate equality of the dimensionless coupling constant g^2 and the coupling constant α in electrodynamics. It seems reasonable to speculate on this basis about the possibility of constructing a universal theory of electromagnetic and weak interactions, where the W-meson together with the photon would form an "isotopic" multiplet in the sense discussed by SALAM and WARD [18] some time ago. The problem with the smallness of the effective four-fermion coupling constant, which arose in their paper and is the reason for an extremely high value of the mass of intermediate boson (60 proton masses), does not cause any difficulties in our case. From our estimate of Γ it follows that g^2 is of the same order as α, if m is of the order of nucleon mass.

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6. The problem of the unitarity of the theory of FEINBERG and PAIS was discussed by G. V. EFIMOV at JIRN-Dubna. We are indebted to P. SURÁNYI for the private communication concerning this problem.
7. It is clear that the operation S can be extended whithout any complications to the summation over all classes G, i.e. (together with the summation over orders n) over all diagrams contributing to the process A.
8. A similar problem, i.e. whether the field theory can be defined with help of a system of relations between amplitudes of different processes derived from the perturbation expansion considered as purely formal, was discussed by E. CAIANIELLO about ten years ago. E. CAIANIELLO, Nuovo Cimento, **10,** 1634, 1953; **11,** 492, 1954.
9. We consider e^-, ν_e and μ^-, ν_μ to be leptons, having muon charge $+1$; e^-, ν_e — 1, respectively.
10. From the point of view of our model each Hamiltonian, which is consistent with requirements 1—5, leads to the same amplitude. However, we did not find any Hamiltonian different from (II. 1) being in agreement with 1—5.
11. This is not true in the case of direct Fermi interaction. For example in the DSV theory the "forbidden" amplitude is of the same order as the allowed one. (A. VANČURA to be published in Czech. Journ. of Phys.)
12. This is possible because we have $m_{\text{lept}} = 0$ in all intermediate states and (III. 3) must therefore have the same form as (II. 5).
13. None of the reducible corrections will change the form of $M_{\alpha\beta}$. Therefore, according to (I. 6a) $M_{\alpha\beta}$ is a finite function. However, $G^{(1)}_{\rho\alpha}$ has only symbolic meaning because $\Gamma^{(1,n)}_{\alpha\beta}$ (for different n) does not contain all diagrams contributing to the W-meson formfactor and does not represent a measurable quantity either.

14. Both bare and physical m_{lept} are supposed to be zero. In this case we can take (on the mass shell) as physical lepton wave function the solution of the free Dirac equation. This follows from the fact that in the γ_5-invariant theory the proper self-energy part of the lepton can be written as $\Sigma(\hat{p}) = \int_{-\infty}^{+\infty} \frac{\varrho(a)\, da}{a - \beta}$, where $\varrho(-a) = \varrho(a)$. (THIRRING, Phys. Let., **4**, 167, 1963). If $m_{\text{lept}} = 0$, $\Sigma(\hat{p})\, u(p) \equiv 0$ i.e. $u(p)$ is really the wave function of a physical lepton.
15. We note that although the W-meson is not stable, the contribution of a single W-meson intermediate state must be taken into account. Our model is based on the field theory and the quantized W-field arises in the Hamiltonian. This situation is quite different from the case of dispersion theory, where the possibility arises that the single unstable vector meson intermediate state does not contribute to any absorptive part of the amplitude. (See M. GELL-MANN and F. ZACHARIASEN, Phys. Rev., **124**, 953, 1961.
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НЕПЕРТУРБАТИВНАЯ ТЕОРЕТИКО-ПОЛЕВАЯ МОДЕЛЬ СЛАБЫХ ЛЕПТОННЫХ ВЗАИМОДЕЙСТВИЙ

Й. ШТЕРН

Р е з ю м е

В работе изучается проблема четырехлептонных слабых процессов в теории с промежуточным векторным W-мезоном. Пренебрегая 1, m_{lept}. 2, частью амплитуды, содержащей индуцированные нейтральные токи, получено интегральное соотношение между четырехлептонной амплитудой и слабым формфактором W-мезона. На основании этого соотношения и условия унитарности оценивается четырехлептонная амплитуда. Полученное решение содержит резонанс в энергетическом канале с полной шириной около 1 Мэв и убывает как $1/s^2$ при $S \to \infty$. Обсуждаются некоторые следствия модели, прежде всего с точки зрения образования промежуточного бозона посредством нейтрино.

A MULTI-CHANNEL MODEL OF WEAK SCATTERING OF LEPTONS

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A multi-channel model of weak scattering is developed, based on the four-fermion theory of weak interactions using BETHE—SALPETER equation. It leads to several relations among the cross-sections of different weak scattering processes. In low energy limit the cross-sections of all included reactions are expressed with the help of a single constant.

I. Introduction

Recently much attention has been paid to the study of higher order corrections in the theory of weak interactions. In theories with intermediate boson a series of papers by FEINBERG and PAIS [1] was followed by many others, developing their ideas. Higher order processes in four-fermion interaction were studied in [2]. Here the $V-A$ vertex function G for four leptons is investigated in ladder approximation with the help of the BETHE—SALPETER equation. It is shown that the final expression does not contain any subtraction constants, and the Fermi coupling constant is renormalised by a finite factor.

In this paper we shall investigate lepton-lepton (l-l) and lepton-antilepton (l-a) scattering in the framework of the theory of weak interactions developed in [2]. We get relations among cross-sections of various (l-l) and (l-a) scattering processes, which do not depend on the detailed form of the scattering amplitudes.

II. Lepton-lepton scattering

We consider the following reactions

$$\nu_e + e^- \to \nu_e + e^- \qquad (AF)\,, \tag{1a}$$

$$\nu_\mu + \mu^- \to \nu_\mu + \mu^- \qquad (AF)\,, \tag{1b}$$

$$\nu_\mu + e^- \to \nu_\mu + e^- \qquad (F)\,, \tag{1c}$$

$$\nu_\mu + e^- \to \mu^- + \nu_e \qquad (A)\,, \tag{1d}$$

$$\nu_e + \mu^- \to \nu_e + \mu^- \qquad (F)\,, \tag{1e}$$

$$\nu_e + \mu^- \to \nu_\mu + e^- \qquad (A)\,. \tag{1f}$$

Here (A) and (F) denote the allowed and forbidden reactions in the terminology of FEINBERG and PAIS [1]. We chose the basic Hamiltonian in the form

$$\frac{G}{\sqrt{2}}(J_\alpha J^{+\alpha} + h.c.), \tag{2}$$

where

$$\begin{aligned} J_\alpha &= \bar{\mu} O_\alpha \nu_\mu + \bar{e} O_\alpha \nu_e , \\ O_\alpha &= \gamma_\alpha (1 - i\gamma_5) . \end{aligned} \tag{3}$$

We do not consider reactions with antileptons now. We do not consider reaction either in which the total charge in the initial and final state is zero.

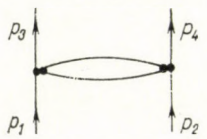

Fig. 1

These are caused (in the first order) by neutral currents, which we do not introduce explicitly. The conservation of muonic and electronic numbers is understood.

We proceed in the same way as in [2], or [3]. The only difference is that we are dealing now with a four-channel problem, the channels being $(\nu_e\, e)$, $(\nu_e\, \mu)$, $(\nu_\mu\, e)$ and $(\nu_e\, \mu)$. We enumerate them as 1, 2, 3, 4, respectively. The BETHE—SALPETER equation reads in this case

$$G = G_0 + G_0 K G , \tag{4}$$

where G, G_0, K are 4×4 matrices in the channel space. As in [2] we put all lepton masses equal to zero. It means that the matrix elements of K are the same for all four channels and given by the graph in Fig. 1.
Eq. (4) written explicitly has the form

$$\begin{pmatrix} G_{11} & & & \\ & G_{22} & & \\ & & G_{33} & G_{34} \\ & & G_{43} & G_{44} \end{pmatrix} = G_0 I + \begin{pmatrix} G_0 K & & & \\ & G_0 K & & \\ & & G_0 K & \\ & & G_0 K & \end{pmatrix} \begin{pmatrix} G_{11} & & & \\ & G_{22} & & \\ & & G_{33} & G_{34} \\ & & G_{43} & G_{44} \end{pmatrix}, \tag{5}$$

where I is a 4×4 unit matrix. Because of T-invariance we may write $G_{34} = G_{43}$; our model allows to put further $G_{11} = G_{22}$, and $G_{33} = G_{44}$ (it is not possible to put all G_{nn} equal to each other !).

With the help of the orthogonal transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & & & \\ & \sqrt{2} & & \\ & & 1 & 1 \\ & & 1 & 1 \end{pmatrix}$$

it is possible to diagonalize eq. (5). We define

$$G^+ = G_{33} + G_{43}\,, \tag{6a}$$

$$G^- = G_{33} - G_{43}\,, \tag{6b}$$

$$G = G_{11} \tag{6c}$$

and after some manipulation we get

$$G = G_0 + G_0 K G\,, \tag{7a}$$

$$G^+ = G_0 + G_0 K G^+\,, \tag{7b}$$

$$G^- = G_0 - G_0 K G^-\,. \tag{7c}$$

From (7a) and (7b) it is seen that $G = G^+$. We can now express the three vertex functions G_{33}, G_{43} and G_{11} with the help of only two functions G^+ and G^-

$$G_{11} = G_{22} = G^+\,, \tag{8a}$$

$$G_{33} = G_{44} = \frac{1}{2}(G^+ + G^-)\,, \tag{8b}$$

$$G_{34} = G_{43} = \frac{1}{2}(G^+ - G^-)\,. \tag{8c}$$

They correspond to the reactions [1a, b], [1c, e] and [1d, f], respectively. The kinematics of the reactions (1) is the same. The differential cross-sections for any of these reactions can therefore be expressed in the form

$$\sigma = f_e(p_i)\,|G|^2\,, \tag{9}$$

where $f_e\ (p_i)$ is a kinematical factor common for all reactions (1). From (8) we get the relations

$$\sigma_{11} = \sigma_{22} = f_e|G^+|^2\,, \tag{10a}$$

$$\sigma_{33} = \sigma_{44} = \frac{1}{4} f_e\,|G^+ + G^-|^2\,, \tag{10b}$$

$$\sigma_{34} = \sigma_{43} = \frac{1}{4} f_e\,|G^+ - G^-|^2\,. \tag{10c}$$

The notation is self-explaining. These cross-sections fulfill the inequality

$$2(\sigma_{33} + \sigma_{34}) > \sigma_{11} \,. \tag{11}$$

This is true both for differential and total cross-sections, and is here the only general relation which holds among the cross-section of reactions (1).

III. Lepton-antilepton scattering

It is possible to generalize the described scheme to the scattering of antileptons and leptons.

The spin structure of the B—S equation (4) is of the from

$$\left(\frac{1}{2}\hat{E}^{(1)} + \hat{p}^{(1)}\right)\left(\frac{1}{2}\hat{E}^{(2)} - \hat{p}^{(2)}\right)\gamma_\mu^{(1)}(1 - i\gamma_5^{(1)})\,\gamma_\nu^{(2)}(1 - i\gamma_5^{(2)})\,G^{\mu\nu}(p, p') =$$

$$= \delta(p - p') + \frac{1}{i(2\pi)^4}\int d^4q\,\gamma_\varrho^{(1)}(1 + i\gamma_5^{(1)})\,\gamma_\sigma^{(2)}(1 + i\gamma_5^{(1)}) \cdot \tag{4}$$

$$\cdot K^{\varrho\sigma}(p, q)\,\gamma_\alpha^{(1)}(1 - i\gamma_5^{(1)})\,\gamma_\beta^{(2)}(1 - i\gamma_5^{(2)})\,G^{\alpha\beta}(q, p') \,.$$

With the assumption [2, 3]

$$G_{\alpha\beta} = g_{\alpha\beta}\,G \tag{12}$$

it is possible to eliminate the γ-matrices by taking the trace of (4). The resulting equation reads

$$\left(\frac{1}{4}E^2 - p^2\right)G(p, p') = \delta(p - p') + \frac{1}{i(2\pi)^4}\int K(p, q)\,G(q, p')\,d^4q \,, \tag{13}$$

where E is the total energy four-vector, p, p' are the relative four-momenta, and $K(p, q) = K_a^a(p, q)$.

This procedure is useful together with the graph in Fig. 1, concerning the $l - l$ scattering.

In this case the kernel has the form [4]

$$Im\,K_{\mu\nu}^{(ll)} = \frac{2}{3}\pi^3 G^2\left(q_{\mu\nu} - \frac{P_\mu P_\nu}{p^2}\right)p^2 \,, \tag{14}$$

which gives

$$Im\,K^{(ll)} = 2\,\pi^3 G^2 p^2$$

$$p^2 = (p_1 = p_3)^2 \,. \tag{15}$$

For $(l - a)$ scattering the basic graph is as in Fig. 2. With the assumption (12) one can easily prove, that the B—S equation has the same form as in the previous case, and the new kernel, $K^{(al)}$, is again given by (15)

$$Im\ K^{(al)} = 2\ \pi^3 G^2 p^2 .$$

The $(l - l)$ and $(l - a)$ scattering is therefore described by the same B—S equation. This allows us to include in the study of multi-channel lepton scattering processes with antileptons too, and to draw some conclusions about the cross-sections.

The relevant scattering reactions are

$$\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^- , \tag{16a}$$

$$\bar{\nu}_e + \mu^- \to \bar{\nu}_e + \mu^- , \tag{16b}$$

$$\bar{\nu}_\mu + \mu^- \to \bar{\nu}_\mu + \mu^- . \tag{16c}$$

$$\bar{\nu}_\mu + \mu^- \to \bar{\nu}_e + e^- , \tag{16d}$$

$$\bar{\nu}_e + e^- \to \bar{\nu}_e + e^- , \tag{16e}$$

$$\bar{\nu}_e + e^- \to \bar{\nu}_\mu + \mu^- . \tag{16f}$$

We do not consider in our discussion other leptonic processes. They can be obtained either by charge-conjugation, or include neutral currents (like $\bar{\nu}_e + \nu_e \to \bar{\nu}_\mu + \bar{\nu}_\mu$), or are governed by electromagnetic interaction (for example $e^+ + e^- \to \mu^+ + \mu^-$, etc.).

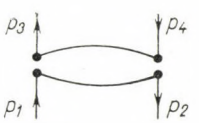

Fig. 2

The $(l - a)$ pairs of 16a, b, c, d, e, f can be grouped into four channels again: $(\bar{\nu}_\mu\, e)$, $(\bar{\nu}_e\, \mu)$, $(\bar{\nu}_\mu\, \mu)$ and $(\bar{\nu}_e\, e)$. We call them channel 5, 6, 7, 8, respectively. Together with the four channels of the $l - l$ scattering, we deal now with an eight-channel problem.

The channels 1, 2, 3, 4 and 5, 6, 7, 8 do not mix (because of muonic and electronic numbers conservation). The eqs. (7a, b, c) are valid even in this generalized case. In the four $(l - a)$ channels we can write

$$\begin{pmatrix} G_{55} & & & \\ & G_{66} & & \\ & & G_{77} & G_{78} \\ & & G_{87} & G_{88} \end{pmatrix} = G_0 I + \begin{pmatrix} G_0K & & & \\ & G_0K & & \\ & & G_0K & G_0K \\ & & G_0K & G_0K \end{pmatrix} \begin{pmatrix} G_{55} & & & \\ & G_{66} & & \\ & & G_{77} & G_{78} \\ & & G_{87} & G_{88} \end{pmatrix} . \tag{17}$$

Using the orthogonal transformation

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & & & \\ & \sqrt{2} & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix}$$

eq. (17) can be diagonalized; the equalities $G_{78} = G_{87}$ (T-invariance) and $G_{55} = G_{66}$, $G_{77} = G_{88}$ (our model) have to be employed. The diagonalization leads to the following equations

$$\bar{G} = G_0 + G_0 K \bar{G} \,. \tag{18a}$$

$$\bar{G}^+ = G_0 + 2\, G_0 K \bar{G}^+ \,, \tag{18b}$$

$$\bar{G}^- = G_0 \,, \tag{18c}$$

where we defined the functions

$$\bar{G} = G_{55} = G_{66} \,, \tag{19a}$$

$$\bar{G}^+ = G_{77} + G_{87} \,, \tag{19b}$$

$$\bar{G}^- = G_{77} - G_{87} \,. \tag{19c}$$

The general function G is a 8×8 matrix in the channel-space with elements G_{kl} ($k, l = 1, 2, \ldots 8$); the equation for G is a direct product of eqs. (5) and (17), which is diagonalized by a matrix ($U \times V$). The result of this procedure are the eqs. (7a, b, c) and (18a, b, c). It is important to note that the kernels of these equations are the same.

From 19a, b, c we get

$$G_{55} = G_{66} = \bar{G} \,, \tag{20a}$$

$$G_{77} = G_{87} = \frac{1}{2} (\bar{G}^+ + \bar{G}^-) \,, \tag{20b}$$

$$G_{78} = G_{87} = \frac{1}{2} (\bar{G}^+ - \bar{G}^-) \,. \tag{20c}$$

Comparing eqs. (7a, b, c) and (18a, b, c) we see that there are only three independent solutions, namely G^+, G^- and $\bar{G}^+$; other functions G can be expressed with the help of these and G_0 ($G = \bar{G} = G^+$, $\bar{G}^- = G_0$).

Collecting now these results, we get

$$G_{11} = G_{22} = G_{55} = G_{66} = G^+ , \tag{21a}$$

$$G_{33} = G_{44} = \frac{1}{2}(G^+ = G^-) , \tag{21b}$$

$$G_{34} = G_{43} = \frac{1}{2}(G^+ - G^-) , \tag{21c}$$

$$G_{77} = G_{88} = \frac{1}{2}(\bar{G}^+ + G_0) , \tag{21d}$$

$$G_{78} = G_{87} = \frac{1}{2}(\bar{G}^- - G_0) . \tag{21e}$$

In the scattering problem the G_0's do not contribute to the scattering amplitude (the initial and final states are free particle states). Therefore we can write instead of (21d) and (21e)

$$G_{77} = G_{88} = G_{78} = G_{87} = \frac{1}{2}\bar{G}^+ . \tag{21f}$$

The cross-sections for reactions (1) and (16) are now given by the formulas

$$\sigma_{AF(l)} \equiv \sigma_{11} = \sigma_{22} = f_l(\bar{G}^+)^2, \quad \sigma_{AF(a)} \equiv \sigma_{55} = \sigma_{66} = f_a(G^+)^2 , \tag{22a}$$

$$\sigma_A \equiv \sigma_{34} = \sigma_{43} = \frac{1}{4} f_l (G^+ - G^-)^2 , \tag{22b}$$

$$\sigma_F \equiv \sigma_{33} = \sigma_{44} = \frac{1}{4} f_l (G^+ + G^-)^2 , \tag{22c}$$

$$\sigma \equiv \sigma_{77} = \sigma_{88} = \sigma_{78} = \sigma_{87} = \frac{1}{4} f_a (\bar{G}^+)^2 . \tag{22d}$$

Here again f_l and f_a are common kinematical factors for $l - l$ and $l - a$ scatterings, respectively. For high energies (higher than any real lepton mass) $f_l/f_a = 3$, in the lowenergy limit ($k^2 \to 0$) $f_l/f_a = 1$ (in the laboratory frame system of reference).

In the low-energy limit ($1/4\, E^2 = k^2 \to 0$) it is possible to find other relations among the cross-sections σ_{kl}. It was shown in [3], that in this limit one can define "scattering lengths" a^+, a^-, and $\bar{a}^+$, corresponding to the vertex-functions G^+, G^-, and $\bar{G}^+$, respectively. If we realize that the eqs. (18a) and (18b) are essentially the same and differ only in the definition of the

coupling constant (one gets eqs. (18b) from (18a) by changing $\lambda^2 \to 2\,\lambda^2$, λ is the coupling constant; see [3]), we can immediately write [3]

$$a^+ = \frac{\lambda}{2}, \tag{23a}$$

$$a^- = i\,\frac{\lambda}{2}, \tag{23b}$$

$$\bar{a}^+ = \frac{\sqrt{2}\lambda}{2} = \sqrt{2}\,a^+ . \tag{23c}$$

The cross-section is obtained from the general formula

$$\Sigma = 4\,\pi k^2 (a)^2;$$

in our case it gives

$$\sigma_{AF(a,l)} = \pi\,k^2\,\lambda^2\,, \tag{24a}$$

$$\sigma_A = \frac{1}{2}\,\pi k^2\,\lambda^2\,, \tag{24b}$$

$$\sigma_F = \frac{1}{2}\,\pi k^2\,\lambda^2\,, \tag{24c}$$

$$\sigma = \frac{1}{2}\,\pi k^2\,\lambda^2\,. \tag{24d}$$

It is interesting to note that these formulas lead to the relations

$$\sigma_{AF} = 2\,\sigma_A = 2\,\sigma_F = 2\,\sigma\,. \tag{25}$$

IV. Conclusions

In (25) the indices A and F denote cross-sections for allowed and forbidden processes of FEINBERG and PAIS. In our model there is no difference between these two categories of weak lepton processes. Moreover, there exist "super allowed" reactions (1a, b) and (16a, b) with cross-sections twice as high as the allowed or forbidden ones. Reactions (16c, d, e, f) are included which do not belong to any category of FEINBERG and PAIS, with the same cross-sections. From (22a, b, c) a relation identical to (11) follows:

$$2(\sigma_A + \sigma_F) > \sigma_{AF(l)} = (f_e/f_a)\,\sigma_{AF(a)}. \tag{11'}$$

This is true for any value of the energy.

The main result of this paper is contained in the relations (11), (11′), and (25), which make direct comparison with experiments possible. The equality of σ_A and σ_F allows to distinguish between our model and the FEINBERG and PAIS theory. Unfortunately, the experimental situation today in lepton physics prevents us from drawing any definite conclusion as regards the validity of these results.

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4. Note: in formulas (2), (14) and (15) G is the weak Fermi coupling constant!

МНОГОКАНАЛЬНАЯ МОДЕЛЬ СЛАБОГО РАССЕЯНИЯ ЛЕПТОНОВ

А. ВАНЧУРА

Р е з ю м е

С помощью уравнения Бете—Солпитера развивается модель слабого рассеяния на основе четырехфермионной теории слабых взаимодействий. Она приводит к некоторым соотношениям между эффективными сечениями разных процессов слабого рассеяния. В низкоэнергетическом пределе эффективные сечения всех рассматриваемых реакций выражаются через единственное постоянное.

LEPTONIC DECAYS WITH $\Delta I = 3/2$ AND $\Delta S = 2$ IN THE UNITARY SYMMETRY SCHEME

By

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A possible generalization of CABIBBO's formalism is investigated in the leptonic decays of the strange particles.

As it seems the leptonic decays of the meson and baryon octets can sufficiently well be described by CABIBBO's scheme [1] which amalgamates the unitary symmetry considerations [2] and the conserved vector current hypothesis [3]. If the $\Delta S = \Delta Q$ and $\Delta I = 1/2$ rules are violated, the violation has to be very small. The question arises, how we should extend CABIBBO's formalism in the case such small violation exists.

Let us suppose that weak current is of the following form:

$$J_\mu = S_\mu + S_{\mu 5} + j_\mu \,. \tag{1}$$

where

$$\begin{aligned} S_\mu &= \sum_{n\alpha} g_{n\alpha} T_\mu^{n\alpha} \,, \\ S_{\mu 5} &= \sum_{n\alpha} g_{n\alpha} T_{\mu 5}^{n\alpha} \,, \end{aligned} \tag{2}$$

and j_μ is the current of leptons. $T^{n\alpha}$ is a tensor operator under SU_3, where n denotes the irreducible representation according to which the tensor operator transforms, and α represents the set of quantum numbers (I, I_3, Y). Assuming that there are no neutral weak currents, $T^{n\alpha}$ is a tensor operator for which $\Delta Q = +1$. If the $\Delta I = 1/2$ and $\Delta Q = \Delta S$ rules are valid, we may restrict ourselves to the currents $T^{8\,110}$ and $T^{8;1/2\,1/2\,1}$; in the general case we have to consider higher representations as well. The simplest candidates are the T^{10}, T^{10^*} and T^{27} currents.

If we want to reduce the number of the unknown parameters, we have to make some assumptions about the relevant combination of the different currents. For simplicity, let us suppose that S_μ can be obtained from $T_\mu^{8\,110} + T_\mu^{10,110} + T_\mu^{10^*,110}$ by an L_2-rotation, where L_2 is one of the infinitesimal

operators of the SU_3-group, as given by DE SWART [4]. More exactly, we require that

$$S_\mu = \sqrt{2}\, e^{i\theta L_2}(T_\mu^{8;110} + T_\mu^{10;110} + T^{10^*;110})\, e^{-i\theta L_2}, \tag{3}$$

that is

$$\begin{aligned} S_\mu = \sqrt{2}\,(&aT^{8;110} + bT^{8;1/2\,1/2\,1} + aT^{10;110} - \\ &- bT^{10;3/2\,1/2\,1} + aT^{10^*;110} + \beta T^{10^*;1/2\,1/2\,1} + \\ &+ \gamma T^{10^*;3/2\,3/2-1} + \delta T^{10^*;002})\,, \end{aligned}$$

where

$$\begin{aligned} a &= \cos\theta/2\,, \\ b &= -\sin\theta/2\,, \\ \alpha &= 1/4(\cos\theta/2 + 3\cos 3\theta/2)\,, \\ \beta &= 1/4(\sin\theta/2 - 3\sin 3\theta/2)\,, \\ \gamma &= \sqrt{3}/4(\sin\theta/2 + \sin 3\theta/2)\,, \\ \delta &= \sqrt{3}/4(\cos\theta/2 - \cos 3\theta/2)\,. \end{aligned}$$

We assume a similar form for $S_{\mu 5}$ with $T_{\mu 5}^8$ and $T_{\mu 5}^{10}$, $T_{\mu 5}^{10^*}$. In addition, we suppose that the T_μ^8 operators are the members of the same octet as the electromagnetic current.

The matrix elements of the leptonic decays will have the following form:

$$\begin{aligned} \langle \gamma_2 n_2\, \alpha_2;\, l_1\, \bar{l}_2 \mid H \mid \gamma,\, n_1\, \alpha_1 \rangle = \frac{f}{\sqrt{2}} \sum_{n\alpha\varrho} g_{n\alpha} \langle n_1\, \alpha_1\, n\, \alpha \mid n_2\, \alpha_2\, \varrho \rangle \cdot \\ \cdot \{\langle \gamma_2\, n_2\, \varrho \,||\, T_\mu^n \,||\, \gamma_1 n_1 \rangle + \langle \gamma_2\, n_2\, \varrho \,||\, T_{\mu 5}^n \,||\, \gamma_1\, n_1 \rangle\} \cdot \langle l_1\, \bar{l}_2 \mid j_\mu \mid 0 \rangle\,, \end{aligned} \tag{4}$$

where γ_1 and γ_2 characterize the state of the strongly interacting particles, l_1 and l_2 stand for the leptons, $\langle n_1\, \alpha_1\, n\alpha \mid n_2\, \alpha_2\, \varrho \rangle$ is the generalized CG-coefficent [4], and the expressions $\langle \| T \| \rangle$ are the reduced matrix elements, which, by taking into account the space-time symmetries, can be further reduced to give the Lorentz-invariant reduced matrix elements.

From the ratio $(\pi^- \to \mu^- \tilde{\nu}_\mu)/(K^- \to \mu^- \tilde{\nu}_\mu)$ one obtains:

$$\theta = 30^\circ\,. \tag{5}$$

Then the coefficients a and b will have just the values, conjectured by MATTHEWS and SALAM [5]. The current $T^{10^*,002}$ appears with reduced weight as compared with the other currents ($\delta = 0{,}11$ while $a = 0{,}97$ and $b = -0{,}26$). Since the reduced matrix elements of the T_μ^8 currents can be taken on the basis

of the CVC-hypothesis, we are left with two parameters in the case of the leptonic decays of the pseudoscalar mesons: they are h_{10} and h_{10^*}, the reduced matrix elements of the T_μ^{10} and $T_\mu^{10^*}$ currents. Assuming that $h_{10} = h_{10^*}$, from the rate of the $K^+ \to \pi^\circ + e^+ + \nu$ decay we get:

$$(\sqrt{30}/15)h_{10} = -0{,}023\,. \tag{6}$$

Some results, calculated for K-mesons, are given in Table I. The rate of the decay $\pi^- \to \pi^\circ e^- \tilde{\nu}$ is not very much affected by the T^{10} and T^{10^*} currents.

Table I

Decay rates for pseudoscalar mesons (in 10^6 sec^{-1})

	Calculated		Experimental	
	$h_{10} \neq 0$	$h_{10} = 0$		
$R(K^+ \to \pi^\circ e^+ \nu)$	3,79	4,9	$3{,}79 \pm 0{,}35$	[6]
$R(K_2^0 \to \pi\, e \nu)$	8	10	$7{,}7 \pm 1{,}2$	[6]
X	0,1	0	$0{,}07 \pm \begin{matrix} 0{,}15 \\ 0{,}2 \end{matrix}$	[7]

$$X = \frac{(K^\circ \to \pi^+ e^- \bar{\nu}) \text{ amplitude}}{(\bar{K}^\circ \to \pi^+ e^- \bar{\nu}) \text{ amplitude}}$$

Assuming again the equality of the T^{10} and T^{10^*} reduced matrix elements, in the case of the baryon octet we remain with four unknown parameters: two reduced matrix elements from $T_{\mu 5}^8$, d_{8_1} and d_{8_2}, and the reduced matrix elements of T_μ^{10} and $T_{\mu 5}^{10}$, which we denote by a_{10} and d_{10}, respectively. $d_1 = (2\sqrt{15}/10)d_{8_1}$ and $d_2 = (1/\sqrt{3})d_{8_2}$ correspond to H^E and H^0, respectively, in the notation of CABIBBO [1]. (The weak magnetism and the induced pseudoscalar term are neglected.)

Since the experimental data are very poor, we shall determine the values of the parameters by making some simple assumptions. For the matrix element of the β-decay we obtain:

$$(a - \sqrt{2}(a + \alpha)A)\gamma_\mu + (a(d_1 + d_2) - \sqrt{2}(a + \alpha)D)\,\gamma_\mu \gamma_\nu\,, \tag{7}$$

where

$$A = (\sqrt{30}/15)a_{10}\,,\quad D = (\sqrt{30}/15)d_{10}\,.$$

Since according to experiments the coefficient of γ_μ is nearly equal to 1, A will be small in any case. If we choose A to make the vector coupling constant equal to that obtained from the O_{14}-decay, we get:

$$A = -5{,}75 \cdot 10^{-3}\,. \tag{8}$$

It is known, that the coefficient of $\gamma_\mu\gamma_5$, divided by that of γ_μ, is about 1,25. Let us suppose, for simplicity, that $D = 1.25\ A$; then for d_1 and d_2 we get 0,6 and 0,65, respectively, as in CABIBBO's paper [1]. The decay rates obtained for the baryons are given in Table II.

Table II

Decay rates for baryons

$\Lambda \to pe^- \tilde{\nu}$	$1{,}1 \cdot 10^{-3}$
$\Sigma^- \to ne^- \tilde{\nu}$	$0{,}9 \cdot 10^{-3}$
$\Xi^- \to \Lambda e^- \tilde{\nu}$	$0{,}9 \cdot 10^{-3}$
$\Xi^- \to \Sigma^\circ e^- \tilde{\nu}$	$0{,}1 \cdot 10^{-3}$
$\Xi^\circ \to \Sigma^+ e^- \tilde{\nu}$	$0{,}3 \cdot 10^{-3}$
$\Sigma^+ \to ne^+ \tilde{\nu}$	$1{,}0 \cdot 10^{-6}$
$\Xi^- \to ne^- \tilde{\nu}$	$1{,}3 \cdot 10^{-6}$
$\Xi^\circ \to pe^- \tilde{\nu}$	$1{,}9 \cdot 10^{-6}$

The non-leptonic decays have not been investigated. However, one may quite well imagine, that the $K^\circ \to \overline{K}^\circ$ amplitude will be considerably reduced as compared with the case when all currents are supposed to appear with the same weight.

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ЛЕПТОННЫЙ РАСПАД С $\Delta I = {}^3/_2$ И $\Delta S = 2$ В СХЕМЕ УНИТАРНОЙ СИММЕТРИИ

Т. НАДЬ

Резюме

Исследуется возможное обобщение формализма Кабиббо в лептонном распаде странных частиц.

THE RENORMALIZABLE VECTOR BOSON THEORY OF WEAK INTERACTION

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Experimental consequences and some theoretical aspects of the use of the vector boson propagator $\frac{-i}{(2\pi)^4}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2-M_v^2}$ are investigated.

In the intermediate vector boson theory of weak interaction one generally takes the boson propagator in the form $\frac{-i}{(2\pi)^4}\left(g_{\mu\varrho}-\frac{k_\mu k_\varrho}{M_v^2}\right)\frac{1}{k^2-M_v^2}$. If $M_v \gg |k_\mu|$, this theory leads to the four-fermion FERMI theory, but as is well known, both of these theories are non-renormalizable. A certain theoretical possibility of the construction of a renormalizable vector boson theory (r.v.b. theory) with the new propagator $\frac{-i}{(2\pi)^4}\left(g_{\mu\varrho}-\frac{k_\mu k_\varrho}{k^2}\right)\frac{1}{k^2-M_v^2}$ has been suggested by BIALYNICKI—BIRULA [1]. With such a propagator, however, serious departures from the FERMI theory may be supposed to occur even if $M_v \gg |k_\mu|$. Since the FERMI theory is in excellent agreement with the overwhelming majority of the experimental data, it seems at first sight that the r.v.b. theory should be in strong disagreement with them. Surprisingly enough, the study of the general behaviour of the energy distributions of the β-particle and of the recoiled ion in allowed nuclear β-decay shows that the r.v.b. theory fits the measured data at least as well as the FERMI theory does [2].

Several methods for the construction of the r. v. b.theory may be conceived. The functional-derivative approach of I. BIALYNICKI—BIRULA has already been mentioned. Another possibility, pointed out to the authors by G. MARX may consist in the introduction of a massless scalar boson with indefinite metric. A third way of treating the problem may be based on the hypothesis that a vector-boson has to be coupled to the conserved part of the current only. If this current itself is not strictly conserved, we are led to a non-local field theory. (The hypothesis when applied to electrodynamics, does not alter the theory, the electric current being strictly conserved.) For second order graphs with an internal boson line this approach leads simply to the replacement of the conventional vector boson propagator with that used in the present note. For other cases the situation is, however, more complicated [3].

The detailed investigation of the general graph rules of the theory is in progress.

Note added in proof. While this paper was in press, it has been found that the r. v. b. theory is in agreement with the experimental data in muon decay [4] and in Λ^0 decay [5], but strongly contradicts the $\frac{v}{c}$ rule for the lepton polarization in neutron decay. Thus the r. v. b. theory would be inadequate to describe weak interaction processes even if the use of the transversal propagator could be theoretically justified.

In a forthcoming paper [6] a new approach to the description of unstable particles (or resonances) will be given, with a possible application to the intermediate vector boson theory of weak interactions.

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ПЕРЕНОРМИРУЕМАЯ ВЕКТОРНАЯ БОЗОННАЯ ТЕОРИЯ СЛАБОГО ВЗАИМОДЕЙСТВИЯ

А. ФРЕНКЕЛ и П. ХРАШКО

Р е з ю м е

Исследуются экспериментальные следствия и некоторые теоретические положения векторного бозонного пропагатора $-\frac{i}{(2\pi)^4}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2-M_v^2}$.

AN APPLICATION OF THE QUASIPOTENTIAL METHOD TO PION NUCLEON SCATTERING

By

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The quasipotential method developed in Dubna Laboratory was applied to pion-nucleon scattering processes up to the energies 500 MeV, for obtaining the resonance behaviour of te P_{33}-phase shift. Although numerical calculations have not yet been completed, the existence of a resonance seems very probable.

In the strong interaction theory use has been recently made of dispersion relations which together with the unitarity condition lead to the system of nonlinear equations. The solution of such is connected with great technical dificulties and, besides, is not unique.

Another approach is the use of the BETHE—SALPETER equations. In that case mathematical difficulties are connected with the solution of integral equations containing four-dimensional integrals.

However, on the basis of the BETHE—SALPETER equations one can derive for each partial wave uncoupled integral equations containing the three-dimensional integration. This approach has been recently developed by LOGUNOV and TAVKHELIDZE [1]. In the present paper it is used for solving the problem of pion-nucleon scattering at energies up to 500 MeV to obtain the resonance behaviour of the P_{33}-phase shift in the considered energy range. We determine the T-matrix in the following way:

$$S = 1 + i(2\pi)^4 \delta(p_1 + q_1 - p_2 - q_2) \frac{M}{\sqrt{4\omega_1 \omega_2 E_1 E_2}} T , \tag{1}$$

$$\bar{u} T u = \bar{u} \left[A(s,t) + B(s,t) \frac{\hat{q}_1 + \hat{q}_2}{2} \right] u , \qquad \bar{u} u = 1 . \tag{2}$$

In deriving a linear integral equation for the T-amplitude we make use of the following definition of the two-time Green function and the potential[1]

$$\tilde{G}(p_{30}, \vec{p}_3, \vec{p}_4; p_{10}, \vec{p}_1, \vec{p}_2) = \int d\varepsilon \int d\varepsilon' \, G(\vec{p}_3, p_{30} - \varepsilon, \vec{p}_4, \varepsilon, \vec{p}_1, p_{10} - \varepsilon', \vec{p}_2, \varepsilon') \tag{3}$$

$$\tilde{G}^{-1} = \tilde{G}_0^{-1} + V , \tag{4}$$

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where $\tilde{G}_0^{-1}$ is the inverse two-time Green function of the system of non-interacting π meson and nucleon, and V is the potential. Using the ordinary technique we write the equation for the amplitude $\tilde{T}(p, q)$ in the form[1].

$$\tilde{T}(p, q) = V(p, q) + \frac{C}{\pi} \int V(p, p') \, \tilde{G}_0(p') \, \tilde{T}(p' q) \, d^3p' . \tag{5}$$

$C = \dfrac{1}{8(2\pi)^2}$ is a constant which can be find from the unitarity condition. The function $\tilde{G}_0(p')$ for the pion-nucleon scattering is determined in the following way:

$$\tilde{G}_0(\vec{p}_1, \vec{p}_2, p_{10}) = \pi i \frac{\sqrt{\vec{p}^2 + M^2} + \sqrt{\vec{p}^2 + \mu^2}}{\sqrt{(\vec{p}^2 + M^2)(\vec{p}^2 + \mu^2)}} \cdot \cdot \frac{u(\vec{p}) \, \bar{u}(\vec{p})}{W^2 - (\sqrt{p^2 + \mu^2} + \sqrt{p^2 + M^2})} , \tag{6}$$

where μ, M — are the pion and nucleon masses, respectively, $\vec{p}$ is a momentum in the centre-of-mass system and $W = \omega + E$, $\omega = \sqrt{p^2 + \mu^2}$, $E = \sqrt{p^2 + M^2}$. The expression (6) and the potential $V(p, p')$ by definition[1], are taken on the mass shell. This means that in eq. (5) the integration over the intermediate state of p' may be considered as a special form of the analytic continuation into the unphysical region.

It should be emphasized that eq. (5) is derived in the framework of field theory.

The equations for the partial wave amplitudes are derived from eq. (5) with the aid of the projection operators[2]

$$\bar{u} \tilde{T} u = \sum_e (\tilde{f}_{e+1/2, e} L_e^{(+)} + \tilde{f}_{e-1/2} L_e^{(-)}) , \tag{7}$$

$$L_e^{(+)} = (l + 1) P_e(\cos\theta) = \frac{i\vec{\sigma} \cdot [\vec{p} \times \vec{q}]}{|\vec{p}| \, |\vec{q}|} P'_e(\cos\theta) , \tag{8}$$

$$L_e^{(-)} = l' P_e(\cos\theta) + \frac{i\vec{\sigma} [\vec{p} \times \vec{q}]}{|\vec{p}| \, |\vec{q}|} P'_e(\cos\theta) .$$

These operators possess the following properties:

$$\int L_e^{\pm}(\vec{p}, \vec{p}') \, L_m^{\pm}(\vec{p}', \vec{q}) \, d\Omega_{\vec{p}'} = 4\pi \cdot \delta_{em} \cdot L_e^{(\pm)}(\vec{p}, \vec{q}) , \tag{9}$$

$$\int L_e^{\pm}(\vec{p}, \vec{p}') \, L_m^{\mp}(\vec{p}', \vec{q}) \, d\Omega_{\vec{p}'} = 0 .$$

From eqs. (5)—(9) we find the equations for the partial wave amplitudes (in the variables W):

$$\tilde{f}^I_{e\pm}(W) = V^I_{e\pm}(W) - \frac{1}{(2\pi)^2} \cdot \int \frac{V^I_{e\pm}(W, W')\tilde{f}^I_{e\pm}(W', W)\sqrt{W'^4 - 2W'^2(M^2+\mu^2) + (M^2-\mu^2)^2}}{W'(W'^2 - W^2)} dx'. \tag{10}$$

There are two possibilities for determinanting the potentials $V^I_{e\pm}(W)$ either from eq. (4), or solving eq. (10) by the method of iterations. For the second case, in the BORN approximation, it is sufficient to take two pole FEYNMANN graphs. It is, however, known that the BORN approximation explains the 33-resonance only qualitatively[3]. Therefore we include additional graphs which are due to the isobar state of the nucleon in the intermediate state. We do not use the contribution from the pion-pion interaction since it is essential only for small s and p waves of πN-scattering as it follows from the dispersion relation method [4].

A partial contribution to the 33-potential from the pole graphs symmetrized in the initial and final states is of the form:

$$V^{3/2}_{p^{3/2}} = -\frac{g^2}{2|b|}\left[\frac{\sqrt{(E'+M)(E+M)}}{2M}(W' - W - 2M)\, Q_1\left(-\left|\frac{a}{b}\right|\right) + \right.$$
$$\left. + \frac{|b|}{2}\,\frac{W' + W + 2M}{2M\sqrt{(E'+M)(E+M)}}\, Q_2\left(-\left|\frac{a}{b}\right|\right)\right], \tag{11}$$

where

$$Q_e\left(-\left|\frac{a}{b}\right|\right) = \frac{1}{2}\int_{-1}^{+1} \frac{P_e(z)\,dz}{-\left|\frac{a}{b}\right| - z},$$

$$|a| = \frac{1}{2}\left[EE' + (E'-E)^2 + (E'+E)(\omega'+\omega) - M^2 - \mu^2 - \omega\omega'\right],$$

$$|b| = 2\,|\vec{p}|\,|\vec{p}'|,$$

g^2 is the coupling constant of πN interaction.

The expressions for the partial contributions of other waves as well as for the contributions to the potential from isobar state of the nucleon are not written down here.

Eq. (10) is solved by the FREDHOLD determinant method. In the general case the integrals will diverge. However, if one subtraction is made then the integrals will converge. Such a subtraction looks as a renormalization of the theory.

The position of the resonance in various states and its width are determined in a usual way.

The numerical calculations have not yet been completed. However, the preliminary calculations point to the existence of resonance solutions in the considered approach.

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ПРИМЕНЕНИЕ КВАЗИПОТЕНЦИАЛЬНОГО МЕТОДА К РАССЕЯНИЮ ПИОНА НА НУКЛОНЕ

П. С. ИСАЕВ и ДЖ. СМИТ

Р е з ю м е

Квазипотенциальный метод, разработанный в Дубне применяется к процессам рассеяния пиона на нуклоне вплоть до энергий 500 Мэв для получения резонансного поведения фазы P_{33}. Несмотря на то, что численные расчеты еще не закончены, наличие резонанса кажется весьма вероятным.

A PROOF OF THE COMPLETENESS OF THE SOLUTIONS OF THE SCHRÖDINGER EQUATION IN THE λ-PLANE

By

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It is proved that the system of solutions of the SCHRÖDINGER equation in the complex λ-plane is complete.

Of great mathematical interest is the problem of determining the potential $V(r)$ from a given distribution of REGGE poles, that is to say the establishment of equations in the λ-plane of the same type as those of GEL'FAND and LEVITAN [1] or MARCHENKO [2], which are valid in the k-plane.

The first step to be overcome is the establishment of a completeness relation for the solutions of the SCHRÖDINGER equation in the complex angular momentum plane, and this is the aim of this paper.

We use the same method as JOST and KOHN [3] have used to prove the completeness relation in the k-plane.

Let us consider the integral:

$$I(k, r) = -\int_{\Gamma} \lambda d\lambda \int_0^{\infty} \frac{h(r')}{rr'} G(\lambda, k, r, r')\, dr' , \tag{1}$$

where $h(r')$ is an arbitrary function which does not make the integral diverge,

$G(\lambda, h, r, r')$ is the GREEN's function which allows us to express a solution $\psi(\lambda, \mathrm{k}, r)$ of the SCHRÖDINGER equation in terms of $\psi_0(\lambda, k, r)$, the solution of the free-SCHRÖDINGER equation (i.e. with zero potential) following the relation

$$\psi(\lambda, k, r) = \psi_0(\lambda, k, r) + \int_0^{\infty} G(\lambda, k, r, r')\ V(r')\ \psi_0(\lambda, k, r^2)\, dr' . \tag{2}$$

It can be shown that $G(\lambda, k, r, r')$ can be expressed in terms of a physical solution φ which is defined in the origin by:

$$\lim_{r \to 0} \varphi(\lambda, k, r) \cdot r^{-\lambda - \frac{1}{2}} = 1 \tag{3}$$

and by a JOST solution defined at infinity by:

$$\lim_{r\to\infty} f(\lambda, -k, r)\cdot e^{-ikr} = 1\,, \tag{4}$$

so that G can be written as

$$G(\lambda, k, r, r') = -\frac{\varphi(\lambda, k, r_>)\, f(\lambda, -k, r_>)}{f(\lambda, -k)}\,, \tag{5}$$

where $f(\lambda - k)$ is the JOST function defined by

$$f(\lambda, -k) = W[f(\lambda, -k, r), \varphi(\lambda, k, r)]. \tag{6}$$

The contour Γ is the imaginary axis of the λ-plane completed by a semi-circle at infinity in the $\mathrm{Re}\lambda \geqslant 0$ half-plane. The potential is assumed to be regular, i.e. its first and second absolute moments exist, which implies (4) that the functions $\varphi(\lambda, k, r)$ and $f(\lambda, -k, k)$ are holomorphic in the half-plane $\mathrm{Re}\lambda \geqslant 0$, such that the only poles are the zeros α_i of the JOST function $f(\lambda - k)$.

Under these assumptions we can evaluate the integral by the residue method

$$I(k, r) = 2\pi i \sum_j \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{\varphi(a_j, k, r_<) f(a_j, -k, r_>)}{\left.\dfrac{\partial f(\lambda, -k)}{\partial\lambda}\right|_{\lambda = dj}}\, dr'\,. \tag{7}$$

Taking into account

$$\left.\frac{\partial f(\lambda, -k)}{\partial\lambda}\right|_{\lambda = d_j} = -i\frac{aj}{k} f(a_j, k)\, M^2(a_j, k), \tag{8}$$

where

$$M^2(\alpha_j, k) = \int_0^\infty \frac{f^2(\alpha_j, -k, r)}{r^2}\, dr \tag{9}$$

and

$$\varphi(\lambda, k, r) = \frac{1}{2ik}[f(\lambda, k) f(\lambda, -k, r) - f(\lambda, -k) f(\lambda, k, r)] \tag{10}$$

which for $\lambda = \alpha_i$ reduces to

$$\varphi(\alpha_j, k, r) = \frac{1}{2ik} f(\alpha_j, k) f(a_j, -k, r)\,. \tag{11}$$

We obtain

$$I(k, r) = i\pi \sum_j \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{f(a_j, -k, r) f(a_j - k, r')}{M^2(a_j, k)}\, dr'\,. \tag{12}$$

On the other hand we can evaluate the integral along the contour, along the imaginary λ axis, taking into account that

$$f(\lambda, -k, r) = f(-\lambda, -k, r) \,. \tag{13}$$

We can show that

$$\int_{i\infty}^{-i\infty} \ldots = \frac{1}{2} \int_{i\infty}^{-i\infty} \lambda d\lambda \int_0^\infty \frac{h(r')}{rr'} \, dr' \left\{ f(\lambda, -k, r)\, \Theta(r - r') \left[\frac{\varphi(\lambda, k, r')}{f(\lambda, -k)} - \right.\right.$$
$$\left.\left. - \frac{\varphi(-\lambda, k, r')}{f(-\lambda, -k)} \right] + f(\lambda, -k, r')\, \Theta(r' - r) \left[\frac{\varphi(\lambda, k, r)}{f(\lambda, -k)} - \frac{\varphi(-\lambda, k, r)}{f(-\lambda, -k)} \right] \right\} . \tag{14}$$

If now we use the relation between the JOST and the φ solutions

$$f(\lambda, -k, r) = \frac{1}{2\lambda} \left[f(\lambda, -k)\, \varphi(-\lambda, k, r) - f(-\lambda, -k)\, \varphi(\lambda, k, r) \right] \tag{15}$$

we have

$$\int_{i\infty}^{-i\infty} \ldots = -2 \int_0^{i\infty} \lambda^2 \, d\lambda \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{f(\lambda, -k, r) f(\lambda, -k, r')}{f(\lambda, -k) f(-\lambda, -k)} \, dr' \,. \tag{16}$$

To evaluate the integral along the curved portion we need the asymptotic forms of the solutions of the SCHRÖDINGER equation and of the JOST function for $|\lambda| \to \infty$.

We know (5) that for regular potentials the functions $\varphi(\lambda, k, r)$ and $f(\lambda, -k)$ tend to the correspondent quantities of the free SCHRÖDINGER equation as $|\lambda| \to \infty$. Hence we have

$$\lim_{|\lambda| \to \infty} \varphi(\lambda, k, r) = r^{\lambda + \frac{1}{2}} ,$$

$$\lim_{|\lambda| \to \infty} \varphi(-\lambda, k, r) = r^{-\lambda + \frac{1}{2}} , \tag{17}$$

$$\lim_{|\lambda| \to \infty} f(\lambda, -k) = 2 \cdot e^{-i\frac{\pi}{4}} \sqrt{\lambda k} \left(\frac{2\lambda}{k} \right)^{\lambda} e^{-\lambda \left(1 - i\frac{\pi}{2}\right)}$$

valid for $\qquad |\arg \lambda| < \pi - \varepsilon .$

Taking into account (15) and (17) we deduce $\lim_{|\lambda|\to\infty} f(\lambda, -k, r)$, and hence we can write the asymptotic form of the integral as

$$\int_c \ldots = \frac{1}{2}\int_0^\infty \frac{h(r')}{\sqrt{rr'}}\,dr'\left[-i\int_c \left(\frac{ek\sqrt{rr'}}{2\lambda}\right)^{2\lambda} d\lambda - \right.$$
$$\left. - \int_c \left[\left(\frac{r'}{r}\right)^\lambda \Theta(r-r') + \left(\frac{r}{r'}\right)^\lambda \Theta(r'-r)\right] d\lambda\right]. \tag{18}$$

The first integral in the bracket can be neglected and we are left with

$$\int_c \ldots = i\int_0 \frac{h(r')}{\sqrt{rr'}}\,dr' \cdot \lim_{\eta\to\infty} \frac{\sin \eta R}{R} \quad \text{with} \quad R = \log\frac{r'}{r} \tag{19}$$
$$= i\pi h(r)$$

in virtue of the properties of the δ-function.

We have finally

$$I(k, r) = i\pi h(r) - 2\int_0^{i\infty} \lambda^2\,d\lambda \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{f(\lambda, -k, r) f(\lambda, -k, r')}{f(\lambda, -k) f(-\lambda, -k)}\,dr' . \tag{20}$$

If we compare the result (20) with that obtained by the residue method (16), we see that

$$h(r) = \int_0^\infty h(r')\,dr' \left[\sum_d M^{-2}(\alpha_j, k) \frac{f(\alpha_j, -k, r)}{r} \cdot \frac{f(\alpha_j, -k, r')}{r'} + \right.$$
$$\left. + \frac{2i}{\pi}\int_0^{i\infty} \frac{\lambda^2}{f(\lambda, -k) f(-\lambda, -k)} \cdot \frac{f(\lambda, -k, r)}{r} \cdot \frac{f(\lambda, -k, r')}{r'}\,dh'\right]. \tag{21}$$

This relation (21) can only be satisfied for all and every square integrable $h(r)$ if

$$\sum_d M^{-2}(\alpha_j, k) \frac{f(\alpha_j, -k, r)}{r} \cdot \frac{f(\alpha_j, -k, r')}{r'} +$$
$$+ \frac{2i}{\pi}\int_0^{i\infty} \frac{\lambda^2\,d\lambda}{f(\lambda, -k) f(-\lambda, -k)} \cdot \frac{f(\lambda, -k, r)}{r} \cdot \frac{f(\lambda, -k, r')}{r'} = \delta(r - r') . \tag{22}$$

This relation (22) can be written as a STIELTJES integral if we define a spectral function $p_k(\lambda)$ in the following sense

$$\frac{dp_k(\lambda)}{d\lambda} = \begin{cases} \dfrac{2i}{\pi} \dfrac{\lambda^2}{f(\lambda, -k) f(-\lambda, -k)} & \text{for } \lambda \varepsilon [0, i\infty] \\ \sum\limits_d \dfrac{\delta(\lambda - \alpha_j)}{M^2(\alpha_j, k)} & \text{for the zeros of } f(\lambda - k) \end{cases} \tag{23}$$

and we obtain

$$\int \frac{f(\lambda, -k, r)}{r} \cdot \frac{f(\lambda, -k, r')}{r'} dp_k(\lambda) = \delta(r - r') . \tag{24}$$

(22) or (24) prove the completeness of the set of the JOST solution of the SCHRÖDINGER equation modified by the factor r^{-1}.

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ДОКАЗАТЕЛЬСТВО ПОЛНОТЫ РЕШЕНИЙ УРАВНЕНИЯ ШРЕДИНГЕРА В λ — ПЛОСКОСТИ

Г. БУРДЭ и М. ЖИФОН

Р е з ю м е

Доказано, что система решений уравнения Шредингера является полной в комплексной λ — плоскости.

ON THE CONNECTION BETWEEN VACUUM-LIKE AND LIGHT-LIKE SOLUTIONS OF THE BETHE-SALPETER EQUATION

By

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It is shown that a vacuum-like solution of the BETHE—SALPETER equation in ladder approximation does imply a light-like solution with the same eigenvalue.

The question whether a vacuum-like solution of the B. S. equation does necessarily imply light-like solutions corresponding to the same eigenvalue has been recently raised in some papers[1].

For vacuum-like solutions and light-like solutions we mean wave functions describing bound states with four momentum $P_\mu = 0$ and $P_\mu \neq 0$, $P^2 = 0$, respectively.

We think the problem rather interesting also in order to throw light on the existence of massless bosons associated with non-conventional solutions of Quantum Field Theory which has often been discussed at this Conference.

A very intuitive although not completely rigorous way of looking at symmetry breaking solutions is the following: suppose you have two particles A and B with different quantum numbers and consider a bound state made up of them with four momentum $P_\mu = 0$ (Fig. 1)

Now, if you look at the graph from below to above you may consider it as a spontaneous coupling between the particles A and B, which breaks the original symmetry.

If a vacuum-like solution of the B. S. equation does imply light-like solutions corresponding to the same coupling constant, then these massless particles so generated correspond to GOLDSTONE's bosons. So non-symmetrical solutions without associated bosons could be obtained from a B. S. equation with a vacuum like solution but no light-like solutions corresponding to the same eigenvalue.

We apply this analysis to a ladder B. S. equation for scalar particles of equal masses m. When $P^2 = 0$ and $P_\mu \neq 0$ the WICK's rotation is feasible but the resulting equation:

$$\left[(p^2 + m^2)^2 - \varepsilon^2 (n \cdot p)^2\right] \varphi(pP) = \lambda^2 \int \frac{\varphi(kP)}{(k-p)^2 + \mu^2} d^4 k \qquad (1)$$

with $P_\mu = \varepsilon\, n_\mu$, $n_\mu = (0\ 0\ 1 - i)$, has a complex Kernel. In fact P_μ is fixed and after the rotation its fourth component becomes imaginary. No choice of Lorentz frame can avoid this difficulty.

Now we suppose we know the eigenvalues λ_a^2 and the eigenfunctions $\{\Phi_0^a\}$ of the equation:

$$(p^2 + m^2)^2\, \varphi(p) = \lambda_a^2 \int \frac{\varphi(k)}{(k-p)^2 + \mu^2}\, d^4 k \tag{2}$$

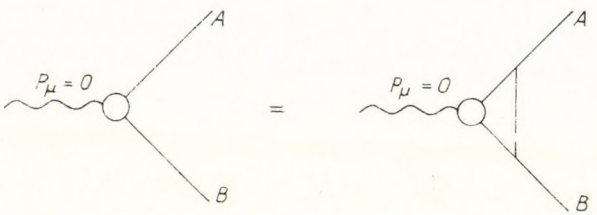

Fig. 1

and recall that usually, when $\mu^2 \neq 0$, they are not degenerate with respect to the four dimensional angular momentum. With the Ansatz:

$$\varphi(pP) = \sum_0^\infty{}_h\, \varepsilon^{2h}\, \varphi_h\,(p, n)\,, \tag{3}$$

and by putting

$$\Phi_h\,(p, n) = (p^2 + m^2)\, \varphi_h\,(p, n) \tag{4}$$

by collecting equal powers of ε^2, we transform the equation (1) with coupling constant λ_a^2 into the following system:

$$\begin{aligned}
\Phi_0(p) &= \lambda_a^2 \int K^-(p, k)\, \Phi_0(k)\, d^4 k, \\
\Phi_1(n, p) &= N(n, p)\ \Phi_0(p) + \lambda_a^2 \int K^-(p, k)\, \Phi_1(n, k)\, d^4 k, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Phi_h(n, p) &= N(n, p)\, \Phi_{h-1}(n, p) + \lambda_a^2 \int K^-(p, k)\, \Phi_h(n, k)\, d^4 k, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{5}$$

with the known term $N(n,p) = \dfrac{(n \cdot p)^2}{(p^2 + m^2)^2}$ complex, but the FREDHOLM's Kernel

$$K(p, k) = -\frac{1}{\left[(k-p)^2 + \mu^2\right](p^2 + m^2)\,(k^2 + m^2)} \quad \text{real and symmetric.}$$

The first equation is just the same as in the case $P_\mu = 0$. Now FREDHOLM's theorems state that the inhomogeneous h-th equation can have solution if and only if the known term is orthogonal to all the eigenfunctions correspond-

ing to the same eigenvalue of the homogeneous associated equation, the first one in our case, that is

$$\int N(n, p)\, \Phi_{h-1}(n, p)\, \{\Phi_0^q(p)\}^*\, d^4 p = 0\,. \tag{6}$$

This condition is verified when no degeneracy is present with respect to the four dimensional angular momentum. We consider here the case in which the eigenfunction is an S wave solution. Then:

$$\int \frac{(n\,p)^2}{(p^2 + m^2)^2}\, \Phi_{h-1}(n, p)\, \Phi_0^q(p^2)\, d^4 p = n_\mu n_\nu \{g_{\mu\nu} F_1 + n_\mu n_\nu F_2\} = 0 \tag{7}$$

because $n^2 = 0$. This argument can be easily generalized to the higher waves. So when there is no degeneracy, FREDHOLM's requirement is satisfied. Therefore we are able to solve the system (5); it is easy to prove that the series (3) so obtained is totally convergent.

Suppose now that there is degeneracy and we have, for instance, a four dimensional "S wave" and a "D wave" solution of the equation (2) corresponding to the same eigenvalue λ_a^2. Then if we start with the "D wave" solution in the known term, the orthogonality conditions:

$$\int \frac{(n\,p)^2}{(p^2 + m^2)^2}\, f_2(p^2)\, Y_2(n \cdot p) f_0^*(p^2)\, d^4 p = 0, \tag{8}$$

$$\int \frac{(n\,p)^2}{(p^2 + m^2)^2}\, f_2(p^2)\, Y_2(n \cdot p) f_2^*(p^2)\, Y_{2lm}(p)\, d^4 p = 0 \tag{9}$$

are clearly satisfied.

In conclusion, vacuum-like solutions always imply lightlike solutions corresponding to the same coupling constants and analytic in the origin $p \cdot P = 0$.

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СВЯЗЬ МЕЖДУ ВАКУУМНО-ПОДОБНЫМИ И СВЕТО-ПОДОБНЫМИ РЕШЕНИЯМИ УРАВНЕНИЯ БЕТЕ—СОЛПЕТЕРА

А. БАССЕТТО, К. ЧИККАРИЕЛЛО и М. ТОНИН

Резюме

Показывается что вакуумноподобное решение уравнения Бете—Солпетера в цепочном приближении содержит светоподобное решение, с тем же собственным значением.

FINITE AND DISCONNECTED SUBGROUPS AND ELEMENTARY PARTICLE SYMMETRIES

By

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The subgroups of the SU_3 group are investigated in connection with elementary particle physics. It is found that the finite and disconnected subgroups of SU_3 cannot account for the elementary particle spectrum.

Introduction

The three dimensional unitary unimodular group [1, 2] (SU_3) has been particularly successful in accounting for many aspects of the spectrum of elementary particles and their strong, electromagnetic and weak interactions. This success has motivated a search for more fundamental symmetries, based at times on dynamical models, at times on attempts to connect SU_3 with other symmetries (e.g. arguments of surrent conservation).

The researh I am to report on, the details of which will be published shortly [3], is less fundamental in its approach than the work referred to above. My co-workes and I have attempted to proceed in the opposite direction. We asked the question: "Are there groups representing symmetries lower than the one corresponding to SU_3 which still retain those characteristics of SU_3 which have been found useful?" We were thus naturally led to restrict ourselves to the subgroups of SU_3, both finite and infinite. Two obvious types of subgroups of SU_3 which are not acceptable in the present context are direct products of Abelian phase groups and SU_2. They are rank one groups and hence can have only one independent quantum number associated with them. Our interest is in subgroups in which there exist operators corresponding to two quantum numbers (e.g. charge, Q, and hypercharge, Y), that is, groups which have a structure similar to that of a rank two group.

One may object that a limitation of our search for groups of lower symmetry to subgroups of SU_3 is too restrictive. Such a narrower view has distinct advantages, however. In she first place, all of these subgroups have

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been listed [4, 5, 6], thus enabling one to completely cover this more limited area. Secondly, one may well hope that some of the more useful aspects of SU_3 may be retained in its subgroups. Finally, the structure of SU_3 can be utilized in applying the subgroups to the elementary particle spectrum (for instance in the assignment of charge and hypercharge operators).

CASE, KARPLUS and YANG (hereafter referred to as C. K. Y.) [7] have considered the analogous problem in relation to isotopic spin and charge independence some years ago. They examined the consequences of replacing SU_2 by the crystal groups in the description of isotopic multiplets. Some of the results of their work provided us with a motivation for making our own attempt. They find it possible to maintain charge independence up to a maximum value of the isotopic spin, I, which is a function of the particular crystal group. An aspect of this result is the well known fact that there exists an upper limit to the dimensions of the irreducible representations, which is again a function of the particular crystal group. They also find that charge is not conserved, except modulo an integer determined by the group.

The first of the results of C. K. Y. quoted above can actually be looked on as a possible advantage of finite groups. It is attractive to consider the possibility of an upper limit to the dimension of irreducible representations into which particles or resonances are placed. There is strong evidence for charge independence only for $I \leq 3/2$. (Evidence for charge independence at higher I exists only for nuclei and is made less precise by the effects of the Coulomb interaction).

The lack of exact charge conservation is of course an unacceptable result of C. K. Y. As I will briefly indicate later, if the tetrahedral group were used to describe isotopic spin (charge conservation modulo three), the reaction $\pi^+ + p \rightarrow \pi^- + n$ would be allowed to take place. It is necessary therefore to make charge conservation a separately imposed postulate. This postulate, however, represents an additional symmetry condition, and therefore serves to generate a new group. In fact, as we shall briefly indicate later, and as can be shown more rigorously, charge cons ervation, when imposed as an additional condition on the finite subgroups of SU_2, generates the full SU_2 group.

Since we consider only those subgroups of SU_3 for which two quantum numbers can be defined, one may be led to expect that the separate imposition of charge conservation is less restrictive than for subgroups of SU_2 and does not lead back to the full SU_3 group. For example, one may hope that, although charge may be made to be conserved exactly in all reactions, hypercharge could still be conserved only modulo an integer. This would necessarily imply, through the GELL—MANN—NISHIJIMA relation, the conservation of I_3 modulo twice that integer. Our subsequent investigations have not led to these originally hoped for results.

II. Subgroups of SU_2 and elementary particles

It is useful at this stage to review briefly, and to extend somewhat, the work of C. K. Y., since it serves to illustrate in a compactway the approach which was followed in the case of subgroups of SU_3.

One can classify subgroups of SU_2 into three types:

a) Abelian phase groups. These correspond to rotations about one axis and are trivial.

b) Finite subgroups. They correspond to restricted finite angles of rotation about a specified and limited number of axes. These groups are the finite dihedral, the tetrahedral, the octahedral and the icosahedral group.

c) Disconnected group. This is the infinite dihedral group, composed of continuous rotations about one axis, and rotations of π radians about the other two orthogonal axes.

We concentrate our attention on the groups listed under *b)* above, and for the sake of specificity, on the tetrahedral group.

Following the analysis of C. K. Y., we consider the double tetrahedral group, T', rather than the tetrahedral group, T. The group T' bears the same relation to T as SU_2 does to the three dimensional rotation group R_3 (i.e. it allows, speaking crudely, for double valuedness). There are twenty-four elements in T' rather than twelve, as in T. It is not necessary in this brief and semi-qualitative review to give the character table of T'. Briefly T' has element types E, C_2 and C_3 and their powers, corresponding to unity and rotations by π and $\frac{2\pi}{3}$ and their multiples, and seven inequivalent irreducible representations, which we shall denote by Γ_1, Γ_1, $\bar{\Gamma}_1$, Γ_2, $\underline{\Gamma}_2$, $\bar{\Gamma}_2$, Γ_3. The subscript denotes the dimension of the representation. Representations without bars are real and $\underline{\Gamma}_n$ $\bar{\Gamma}_n$ are complex conjugates of each other.

Charge multiplets can now be assigned by subducing elements of T' from elements of SU_2. Arbitrarily choosing the z axis as an axis of highest symmetry in T' (a threefold symmetry axis corresponding to element types C_3), elements $e^{iI_3\varphi}$ of SU_2, with φ a continuous variable, go over into elements of T':

$$C_3 = e^{iI_3\psi_j}; \qquad \psi_j = 0\,, \quad \frac{2\pi}{3}\,, \quad \frac{4\pi}{3}\,.$$

We *define* the operator I_3 in T' to be such as to give the correct values of the characters. Not surprisingly, for the irreducible representations Γ_1,

Γ_2 and Γ_3, I_3 turns out to be exactly the same as the corresponding I_3 for SU_2. For example, for Γ_3, the character is correctly given by the choice

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

with

$$\chi(C_3) \equiv T^r C_3 = 0 .$$

Thus the π mesons can be assigned to Γ_3, and, in a similar fashion, the nucleons can be associated with Γ_2 (Γ_1 corresponds to an $I = 0$ state).

The irreducible representations Γ_j, $\bar{\Gamma}_j$ correspond to only parts of irreducible representations of SU_2, as will be indicated later. One can define I_3 for these too. For example

$$C_3(\Gamma_1) \equiv e^{\frac{-2\pi i}{3}} \equiv e^{\frac{2\pi i}{3} I_3} .$$

This shows that the operator I_3 for this representation is the one by one matrix

$$I_3 = -1 .$$

In other words this representation would accommodate a singlet boson of charge -1.

The problem of non-conservation of charge can be illustrated qualitatively by operating in states of $3\pi^+$ and $3\pi^0$ mesons, viewed first as states in representations of SU_2 and then of T'. Consider the product state, operated on by the group element $e^{iI_3\psi}$. Define

$$O_{123}(\psi) = e^{iI_3^{(1)}\psi} e^{iI_3^{(2)}\psi} e^{iI_3^{(3)}\psi}.$$

For SU_2,

$$O_{123}(\psi) \mid \pi^+_{(1)} \pi^+_{(2)} \pi^+_{(3)} \rangle = e^{3i\psi} \mid \pi^+_{(1)} \pi^+_{(2)} \pi^+_{(3)} \rangle ,$$

and

$$O_{123}(\psi) \mid \pi^0_{(1)} \pi^0_{(2)} \pi^0_{(3)} \rangle = 1 \cdot \mid \pi^0_{(1)} \pi^0_{(2)} \pi^0_{(3)} \rangle .$$

The same operation in T' gives

$$O_{123}(\psi_j) \mid \pi^+_{(1)} \pi^+_{(2)} \pi^+_{(3)} \rangle = 1 \cdot \mid \pi^+_{(1)} \pi^+_{(2)} \pi^+_{(3)} \rangle ,$$

and

$$O_{123}(\psi_j) \mid \pi^0_{(1)} \pi^0_{(2)} \pi^0_{(3)} \rangle = 1 \cdot \mid \pi^0_{(1)} \pi^0_{(2)} \pi^0_{(3)} \rangle .$$

Thus, since ψ is arbitrary in SU_2, the two states of π mesons, differing by a charge of three, are distinguished, while in T' they are not. If ψ_j were restricted to the values 0 or π, states differing by two units of charge would not be distinguished.

A more detailed look at the nature of this difficulty can be had if one considers a scattering problem, say $\pi-N$ scattering. We then have to deal with product representations and their reduction to irreducible representations. Thus, for $\pi-N$ scattering (denoting the irreducible representations in SU_2 by their dimension), we have

$$2 \otimes 3 = 2 \oplus 4$$

for SU_2 (i.e. the usual $I = 1/2$ and $I = 3/2$ representations), while for T'

$$\Gamma_2 \otimes \Gamma_3 = \Gamma_2 \oplus \underline{\Gamma}_2 \oplus \bar{\Gamma}_2 .$$

There is no general method for treating the Clebsch—Gordan problem for finite groups. One must actually operate on the states with enough group elements to ascertain that one has obtained the correct irreducible representations. For the case of $\pi-N$ scattering, the states comprising the irreducible representations are:

$$\begin{cases} \Gamma_2\left(\frac{1}{2}\right) = \sqrt{\frac{2}{3}}\,|\pi^+ n\rangle - \frac{1}{\sqrt{3}}\,|\pi^0 p\rangle ; \\ \Gamma_2\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{3}}\,|\pi^0 n\rangle - \sqrt{\frac{2}{3}}\,|\pi^- p\rangle ; \end{cases}$$

$$\begin{cases} \bar{\Gamma}_2\left(\frac{3}{2}\right) = \frac{1}{\sqrt{3}}\,|\pi^+ p\rangle + \sqrt{\frac{2}{3}}\,|\pi^- n\rangle ; \\ \bar{\Gamma}_2\left(\frac{1}{2}\right) = e^{\frac{2\pi i}{3}}\left[\frac{1}{\sqrt{3}}\,|\pi^+ n\rangle + \sqrt{\frac{2}{3}}\,|\pi^0 p\rangle\right]; \end{cases}$$

$$\begin{cases} \underline{\Gamma}_2\left(-\frac{1}{2}\right) = \sqrt{\frac{2}{3}}\,|\pi^0 n\rangle + \frac{1}{\sqrt{3}}\,|\pi^- p\rangle ; \\ \underline{\Gamma}_2\left(-\frac{3}{2}\right) = e^{\frac{-2\pi i}{3}}\left[\sqrt{\frac{2}{3}}\,|\pi^+ p\rangle - \frac{1}{\sqrt{3}}\,|\pi^- n\rangle\right]. \end{cases}$$

As expected, the T' group is adequate to represent isotopic multiplets for which $I \leq 1$ and so the Clebsch—Gordan coefficients for the Γ_2 representation are the usual ones for $\vec{1} + \frac{\vec{1}}{2} = \frac{\vec{1}}{2}$. The representations $\bar{\Gamma}_2$ and $\underline{\Gamma}_2$ each contain pieces of the $I = \frac{3}{2}$ representation of SU_2. If we associate an inde-

pendent scattering amplitude A with each of the representations, we can see immediately that the reaction $\pi^+ p \to \pi^- n$ is allowed, and its amplitude is given by

$$A(\pi^+ p \to \pi^- n) = \frac{\sqrt{2}}{3} [A(\underline{\Gamma}_2) - A(\bar{\Gamma}_2)].$$

Since this reaction violates charge conservation, we must forbid it. This implies

$$A(\underline{\Gamma}_2) = A(\bar{\Gamma}_2),$$

which can occur in general only if the representations $\underline{\Gamma}_2$ and $\bar{\Gamma}_2$ coalesce into a single irreducible representation. A unitary transformation will then take such a representation into the equivalent standard representation for $I = 3/2$. Thus, charge conservation has served to regenerate the four-dimensional representation of SU_2 itself.

III. Subgroups of SU_3

Surprisingly, all the detailed discussions of these subgroups [4, 6] are not very recent and are restricted to giving the orders of each group and the generators. It was therefore necessary for us to first derive the group theoretical properties, such as class structure, character tables and irreducible representations, before carrying out an analysis similar in detail but much more lengthy and complicated than the above one for subgroups of SU_2.

The finite groups possessing "rank two-like" structure are of two types:

a) Analogues of the crystal groups. We call them $\Sigma(n)$, where n is the order of the group, and is 36φ, 72φ, 216φ, 60, 168, 360φ, with $\varphi = 1$ or 3. The last fact is the reflection of the triple valuedness of rotations which is possible in SU_3 (the analogue of the double valuedness in SU_2).

b) Analogues of the dihedral groups. We call these $\Delta(3n^2)$ and $\Delta(6n^2)$, where the order again appears inside the parentheses and n is integer. The disconnected continuous groups arise from the Δ groups in the limit $n \to \infty$.

A number of the Σ groups have eight and even ten dimensional representations. If we further require some reflection of charge independence, we have to demand that one of the crystal groups be a subgroup of the Σ we consider. This restricts us to $\Sigma(216)$, which has T' as a subgroup. We are clearly already in difficulty, since T' accounts for charge independence only for $I \leq 1$, rather than $I \leq 3/2$, as we would wish. What is worse, T' is embedded in Σ (216) the wrong way, so that the wrong charge multiplet is associated with a given hypercharge. The simultaneous assignment of the correct Q and Y is thus quite a severe condition. It turns out that we can still correctly make

such an assignment in the octet representation of $\Sigma(216)$ but at the price of giving up the hoped-for close connection with T'.

We can now again turn to the Clebsch—Gordan problem by looking at the scattering of an octet of mesons by an octet of baryons. Considering the reduction of product representations, we have

$$8 \otimes 8 = 1^S \oplus 8^S \oplus 27^S \oplus 8^A \oplus 10^A \oplus \overline{10}^A$$

for SU_3, and

$$\Sigma_8 \otimes \Sigma_8 = \Sigma_1^s \oplus \Sigma_8^s \oplus (\underline{\Sigma}_8^s \oplus \bar{\Sigma}_8^s \oplus \Sigma_8^s \oplus \Sigma_3^s) \oplus \Sigma_8^A \oplus (\underline{\Sigma}_8^A + \bar{\Sigma}_2^A) \oplus (\bar{\Sigma}_8^A \oplus \underline{\Sigma}_2^A)$$

for $\Sigma(216)$. The superscripts A and S stand for antisymmetric and symmetric combinations of the constituent states, respectively. The highest dimension of an irreducible representation for $\Sigma(216)$ is eight and we can once more see, as we did for SU_2 and T', how higher dimensional irreducible representations of SU_3 become reducible into the lower dimensional representations of $\Sigma(216)$. The detailed Clebsch—Gordan analysis leads to states, as it does in T', in which there are components of different charge present. This time, there are also components of different hypercharge as well, both hypercharge and charge differing from each other by three units. The requirement of charge conservation alone leads also to hypercharge conservation, makes the representations coalesce and causes $\Sigma(216)$ to go over into SU_3.

As far as the disconnected groups associated with the Δ-s are conserved, Q and Y are assignable and are exactly conserved in reactions. However, the maximum dimensions of irreducible representations are three for $\Delta(3\infty^2)$ and six for $\Delta(6\infty^2)$, so that only Sakata-like, rather than octet models can be accommodated. Invariance under these Δ groups implies only charge symmetry, rather than charge independence.

In summary, our failure to apply finite and disconnected subgroups of SU_3 to the elementary particle spectrum serves to strengthen our belief in the tightness of the SU_3 scheme. At the same time, the detailed analysis of the hitherto little known subgroups of SU_3 which we have carried out, may have as yet unforseen physical applications.

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КОНЕЧНЫЕ И РАЗДЕЛЁННЫЕ ПОДГРУППЫ И СИММЕТРИИ ЭЛЕМЕНТАРНЫХ ЧАСТИЦ

Т. ФУЛТОН

Резюме

Рассматриваются подгруппы группы SU_3 в отношении физики элементарных частиц. Найдено, что конечные и разделённые подгруппы SU_3 не могут употребляться для спектра элементарных частиц.

BETHE-SALPETER EQUATION AND CONSERVATION LAWS IN NUCLEAR PHYSICS

By

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Finite nuclei are considered using the method of interacting quasi particles. The justification of this description is based on the application of the BETHE—SALPETER technique. The number of free parameters is reduced with the help of invariance properties.

1. Introduction

Many of the low energy properties of systems of strongly interacting particles can be described very well, if one considers them as systems of weakly interacting quasi particles. This description has been very fruitful in the theory of solids as well as liquid He^3 and He^4.

Recently great progress has been made in generalizing the theory of Fermi liquids to cover the theory of finite nuclei.[1]

Although the method of interacting quasi particles in many cases does not lead to new results as far as applications are concerned it sheds a new light on many older calculations or gives a theoretical justification of certain models or methods.

Histotically one may distinguish three different stages concerning the justification of the quasi particle description:

a) The HARTREE—FOCK approximation. This is essentially a weak interaction approximation assuming that $v_0/(k_f^2/2m \ll 1$, where v_0 is a measure of the strength of the two-body interaction and $k_f^2/2m$ a measure of the average kinetic energy of the particles, k_f being the Fermi momentum.

b) T-matrix and random phase approximation. These are essentially low density or high density approximations assuming $ak_f \ll 1$ or $me^2/k_f \ll 1$ respectively, where a is an effective two particle scattering length.

c) Application of BETHE—SALPETER techniques. In this approach one considers a sufficiently narrow energy range $\Delta\varepsilon$ and singles out those diagrams in a perturbation expansion which vary strongly within $\Delta\varepsilon$ from the ones which can be treated as constants. Usually no attempt is made to calculate

[1] A. B. MIGDAL, Nucl. Phys., **57**, 29, 1964.

these constants. They are considered as microscopic parameters of the system. Since $\Delta\varepsilon/\varepsilon$ can be made arbitrarily small by confining oneself to a sufficiently narrow range of excitation energies this approach is virtually exact.

Usually the number of free parameters in such an approach is rather large, however. The aim of this lecture is to show that invariances such as gauge invariance or Galilean invariance or the corresponding conservation laws following from them lead to many relations between these parameters, thus reducing the number of independent parameters considerably.

2. The response function

There is a large class of experiments which can be described by means of the so-called response function. Let us consider some examples

a) Dipole γ-absorption.

The cross section for the absorption of a dipole γ-ray can be written as

$$\sigma(\omega) = 4\pi e^2 Im R(\omega) \; ; \qquad \omega \geq 0 \, , \tag{1}$$

where $R(\omega)$ is the dipole response function

$$R(\omega) = \left\langle D \frac{1}{H - E_0 - \omega - i\gamma} D + D \frac{1}{H - E_0 + \omega - i\gamma} D \right\rangle , \tag{2}$$

D being the dipole operator.

b) Elastic γ-ray scattering.

The cross-section in this case is given by

$$\sigma_{el}(\omega) = \frac{8\pi}{3} \left| \frac{e^2}{m} Z - \frac{1}{2\pi^2} \int_0^\infty \sigma d\omega - e^2 \omega^2 R(\omega) \right|^2 . \tag{3}$$

c) Inelastic electron scattering.

As an example consider the longitudinal dipole formfactor $F_1(k)$

$$| F_1(k) |^2 = i2\pi \langle j_1(kr) \delta(H - E_0 - \omega) j_1(kr) \rangle \, . \tag{4}$$

Further examples may be found in MIGDAL's paper (see footnote 1) The quantities considered in our examples can all be obtained as special "expectation values" of a general matrix response function

$$\underset{\alpha\lambda,\mu\beta}{R(\omega)} = \left\langle \varrho_{\alpha\mu} \frac{1}{H - E_0 - \omega - i\gamma} \varrho^+_{\beta\lambda} + \varrho^+_{\beta\lambda} \frac{1}{H - E_0 + \omega - i\gamma} \varrho_{\alpha\mu} \right\rangle , \tag{5}$$

where

$$\varrho^{+}_{a\mu} = \psi^{+}_{a}\psi_{\mu} , \tag{6}$$

and ψ^{+}_{a}, ψ_{α} are the creation and annihilation operators of fermions in the state characterized by the quantum number(s) α. For instance the dipole response is obtained by

$$R(\omega) = \Sigma D^{*}_{a\mu} R_{a\lambda\,\mu\beta} D_{\beta\lambda} , \tag{7}$$

where $D_{a\mu}$ are the matrix elements of the dipole operator i.e.

$$D = \Sigma D_{a\mu}\varrho^{+}_{a\mu} . \tag{8}$$

We therefore consider the general response function in more detail.

i) HARTREE—FOCK approximation.

The shell model may serve as a first approximation for the determination of the response function. If the states with the quantum numbers α, μ are chosen so as to coincide with the single particle states of the shell model, the response function takes a particularly simple form

$$R_{\alpha\lambda,\mu\beta} = R^{s0}_{a\varkappa}\delta_{\alpha\beta}\delta_{\mu\lambda} , \tag{9}$$

where

$$R^{s0}_{ak}(\omega) = \frac{(1 - N_{\alpha})\,N_{k}}{\varepsilon^{0}_{\alpha} - \varepsilon^{0}_{k} - \omega - i\gamma} + \frac{(1 - N_{k})\,N_{\alpha}}{\varepsilon^{0}_{k} - \varepsilon^{0}_{o} + \omega - i\gamma} , \tag{10}$$

ε^{0}_{a}, ε^{0}_{k} are the Hartree—Fock single particle energies of the shell model states α, k and N_{α}, K_{k} their occupation numbers in the ground state

$$N_{\alpha} = \begin{cases} 1; \text{ if } \alpha \text{ occupied} , \\ \\ 0; \text{ if } \alpha \text{ unoccupied} . \end{cases} \tag{11}$$

(10) describes the fact, that the particle-hole pair created by ϱ^{+}_{ak} remains in its state and has the excitation energy $\varepsilon^{0}_{a} - \varepsilon^{0}_{k}$.

ii) Random phase approximation.

Usually there are many particle-hole states in the shell model having approximately the same energy. Because of this approximate degeneracy the residual interaction will lead to a strong mixing of the shell model excitations. The effect of this mixing may be treated in the so called random phase approximation. In this approximation one can derive the following equation for the response function

$$R_{\alpha\lambda,\mu\beta} = R^{s0}_{a\mu}(\delta_{\alpha\beta}\,\delta_{\mu\lambda} - \Sigma V^{0}_{ak,\mu\gamma} R_{\gamma\lambda,k\lambda}) , \tag{12}$$

iii) Higher order corrections.

To take into account all higher order corrections it is convenient to use Feynman diagrams. The iterative solution of (12) may be written in matrix form as

$$R = R^{s0}(1 - V^0 R) = R^{s0} - R^{s0} V^0 R^{s0} + R^{s0} V^0 R^{s0} V^0 R^{s0} + \dots \qquad (13)$$

and illustrated by diagrams in the following manner

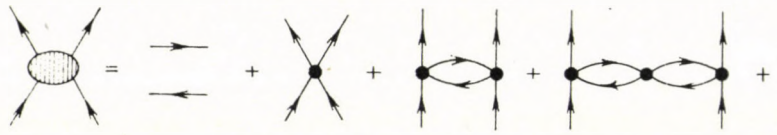

Fig. 1

All higher order diagrams now may be classified as either "single particle dressing" or "vertex renormalization" diagrams. Typical single particle dressing diagrams are

Fig. 2

All these diagrams are summed up if one replaces the single particle propagator of the shell model G^0 (indicated by an open arrow.)

$$G_\alpha^0(\varepsilon) = \frac{1 - N_\alpha}{\varepsilon - \varepsilon_\alpha^0 + i\gamma} + \frac{N_\alpha}{\varepsilon - \varepsilon_\alpha^0 - i\gamma} \qquad (14)$$

by the dressed propagator G indicated by a full arrow

$$G_{\alpha\beta}(\varepsilon) = Z_\alpha g_\alpha(\varepsilon)\,\delta_{\alpha\beta} + G^r_{\alpha\beta}(\varepsilon)\,, \qquad (15)$$

where $g(\varepsilon)$ has the same form as G^0, ε_α^0 being replaced by ε_α and G^r is non-singular at the renormalized single particle energies ε_α.

Typical vertex renormalization diagrams are

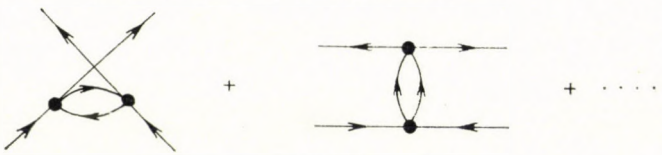

Fig. 3

All these diagrams are summed up if one replaces the bare vertex V^0 by the renormalized vertex V. The equation of the response function for the external field Q

$$\begin{cases} QRQ = QR^s(1 - VR)Q = QR^sT\,, \\ T = Q - VR^sT \end{cases} \tag{16}$$

with

$$R^s_{\alpha\lambda,\mu\beta}(\varepsilon, \omega) = G_{\alpha\beta}\left(\varepsilon + \frac{\omega}{2}\right) G_{\lambda\mu}\left(\varepsilon - \frac{\omega}{2}\right) \tag{17}$$

and the energies of the external lines are chosen according to the diagram

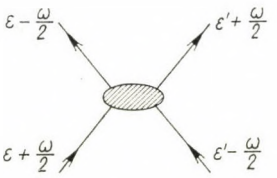

Fig. 4

R therefore depends on ε, ε' and ω. The $R(\omega)$ of eq. (5) is obtained from $R(\omega, \varepsilon, \varepsilon')$ of eq. (16) after integrating over ε and ε'.

3. Transformation to quasi particles

The crucial point in the further treatment of (16) is the fact that R^s contains a part, which is singular at the particle-hole excitation energies $\varepsilon_\alpha - \varepsilon_k$. If one uses (15) one obtains

$$R^s_{\alpha\lambda,\mu\beta} = Z_\alpha Z_\beta r^s_{\alpha\mu}(\omega)\,\delta\left(\frac{\varepsilon_\alpha + \varepsilon_\mu}{2} - \varepsilon\right) + R^r_{\alpha\lambda,\mu\beta}\,, \tag{18}$$

where r^s has the same form as R^{0s} but the ε^0_a's are replaced by the ε_a's. R contains no more poles and δ-functions and V similarly is nonsingular at the particle-hole excitation energies.

One can now eliminate the renormalization factors z as well as R^r. After some algebra one finds

$$\begin{cases} ImQRQ = Imqr^s\ T \\ T = q - vr^sT\,, \end{cases} \qquad \text{with} \tag{19}$$

where we have introduced the quasi particle quantities, r^s, q, v, T instead of the renormalized ones R^s, Q, V, T. In matrix notation they are related by

$$\begin{aligned} r^s &= Z^{-2}R^s - R^r\,, \\ \tau &= ZT\,, \\ q &= Z(1 + VR^r)^{-1}Q\,, \\ v &= Z^2(1 + VR^r)^{-1}V \end{aligned} \tag{20}$$

with the aid of (19) the response function is expressed in terms of the quasi particle quantities. The parameters contained in them are: The matrix elements q of the external field, the renormalized single particle energies ε_a and the effective vertex v. The question then arises how they can be determined.

First of all one sees from the spectral decomposition of G, that the single particle energies of the A particle system can be read off from the energy levels of the $A + 1$ and $A - 1$ particle systems. They may be taken from experiment therefore.

We shall now see that in many cases of interest the requirement of gauge invariance leads to $Q = q$ and a combination of gauge invariance and Galilean invariance leads to further relations between the quantities q, v and the ε_a. In some cases these relations are sufficient to determine all free parameters in terms of the experimentally observable ε_a 's.

4. Conservation laws

To explore the consequences of gauge invariance it is convenient to assume, that the particles have an infinitesimal charge $\delta\varepsilon$ and investigate the response to a purely longitudinal vector potential i.e.

$$Q = \delta e\,(\vec{P}\cdot\vec{k} - \omega)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}. \tag{21}$$

The only effect of this field is to multiply the field operators $\psi(x)$ by the phase factors $\exp(i\delta e \cdot e^{ikx})$. From this after some algebra (compare for instance MIGDAL, loc. cit.) one can deduce an identity for the corresponding τ_Q in matrix notation

$$T_Q = e^{ik\cdot r}\,g^{-1}_{\left(\varepsilon+\frac{\omega}{2}\right)} - g^{-1}_{\left(\varepsilon-\frac{\omega}{2}\right)}\,e^{ik\cdot r}\,. \tag{22}$$

This identity is valid for all $\vec{k}$ and ω. Expansion into powers of $\vec{k}$ after comparison of the corresponding powers on both sides of (22) yields a set of identities for the various multiples. In particular for the dipole term (putting $\vec{r} = D$)

$$(T_P - \omega T_D)_{\mu a} = Z^{-1}(\varepsilon_a - \varepsilon_\mu + \omega)iD_{\mu a}\,. \tag{23}$$

The r. h. s. of this equation may now be inserted into the equation (16) for τ. After applying the procedure which transforms (16) into the quasi particle language (19) one obtains finally

$$(\varepsilon_\mu - \varepsilon_\alpha + \omega) D_{\alpha\mu} = \left(\frac{i}{m} p_{\alpha\mu} + \omega d_{\alpha\mu}\right) + \sum_{\beta,\lambda} v_{\alpha\lambda,\mu\beta} (N_\beta - N_\lambda) D_{\beta\lambda} . \quad (24)$$

Since this is valid for all ω one has the two equations

$$D_{\alpha\mu} = d_{\alpha\mu} \quad (25)$$

and

$$(\varepsilon_\mu - \varepsilon_\alpha) D_{\alpha\mu} = \frac{i}{m} P_{\alpha\mu} + \sum_{\beta\lambda} v_{\alpha\lambda,\mu\beta} (N_\beta - N_\lambda) D_{\beta\lambda} . \quad (26)$$

In an analogous way one may derive from Galilean invariance the two corresponding equations

$$P_{\alpha\mu} = p_{\alpha\mu} \quad (27)$$

and

$$(\varepsilon_\mu - \varepsilon_\alpha) P_{\alpha\mu} = \sum_{\beta\lambda} v_{\alpha\lambda,\mu\beta} (N_\beta - N_\lambda) P_{\beta\lambda} . \quad (28)$$

This means that in the case of n approximately degenerate levels $(\varepsilon_\alpha - \varepsilon_\mu)$ contributing to the response function one has $2n$ relations among the matrix elements of the vertex operator v.

The above considerations apply to excitations with isospin zero. In the case of isospin one Galilean invariance has no effect and the consequences of gauge invariance are modified because of the presence of exchange forces. If one uses the approximation of "effective charges" i.e. $eD = e'd$ one can relate e'/e to the exchange correction of the dipole sum rule.[2]

УРАВНЕНИЕ БЕТЕ—СОЛПЕТЕРА И ЗАКОНЫ СОХРАНЕНИЯ В ЯДЕРНОЙ ФИЗИКЕ

В. БРЕНИГ

Р е з ю м е

Рассматриваются конечные ядра методом взаимодействия квази-частиц. Доказательство данного описания ядер основывается на применении техники Бете—Солпетера. Число свободных параметров уменьшается с помощью инвариантных свойств.

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MANY-BODY FOUNDATIONS OF THE NUCLEAR SHELL MODEL

By

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We wish to show how the general features of the shell model emerge when one extends the BRUECKNER—BETHE—GOLDSTONE theory of nuclear matter to the case of finite nuclei. The basic idea is to use perturbation theory, together with suitable partial summations. The starting point is $H_0 = T + V_{SM}$, where V_{SM} is a one-body operator chosen to optimize convergence of the expansion. After some comments on perturbation theory, we shall indicate how the BRUECKNER reaction matrix can be calculated within the finite geometry defined by the above H_0. Finally, we shall discuss the form of V_{SM} which leads to a rapid convergence of the linked-cluster expansion.

Bloch—Horowitz expansion

For most nuclei, the shell-model wavefunction consists of a superposition of several Slater determinants (H_0 eigenstates), thus a degenerate version of the GOLDSTONE expansion is required. This was discovered by BLOCH and HOROWITZ [1]. Their important and elegant result is not very familiar, and a brief discussion would seem worthwhile.

Let us expand the total nuclear wavefunction in terms of H_0 eigenstates

$$\Psi = \sum_i a_i \Phi_i \,. \tag{1}$$

We break up the sum Σ_i into "degenerate" and "nondegenerate" parts, $P\Psi = \Sigma_{i\varepsilon D}$, $Q\Psi = \Sigma_{i\notin D}$, where D is a finite-dimensional quasi-degenerate subspace. (Exact degeneracy is not necessary). Then the Schroedinger equation can be written in the form [2]

$$[H_0 + V - EI]_D A = 0 \,, \tag{2}$$

A = column vector $(\text{- - - } a_i \text{ - -})$, $i\varepsilon D$.

This looks just like the familiar form of degenerate perturbation theory in lowest order, except that the effective interaction,

$$V = V + V \frac{Q}{E - H_0} V \tag{3}$$

now includes all effects due to the non-degenerate Φ_i 's.

No "many-body" features have been introduced yet. (The Φ_i 's are, of course, Slater determinants.) It is convenient to choose the degenerate Φ_i 's by the following prescription: The single-particle states of the closed-shell "core" must all be occupied, and all the remaining particles (the "valence" particles) must be distributed among certain low-lying single-particle states called valence states. The interaction energy is defined by

$$\Delta E = E - A^+ H_0 A / A^+ A = A^+ V A / A^+ A \,. \tag{4}$$

The important result of BLOCH and HOROWITZ is that ΔE can be expressed as a sum of "core" and "valence" terms, $\Delta E = \Delta E_c + \Delta E_V$. The "core" term ΔE_c is given by the familiar GOLDSTONE expansion for the closed-shell nucleus one would obtain by physically removing the valence particles. Equation (2) reduces to the much simpler form

$$[H_{0V} + V_V - E_V I]_D A = 0 \,, \tag{5}$$

where H_{0V} is the valence-state part of H_0, and $E_V = E - E_c$, $E_c = E_{0c} + \Delta E_c$ being the total energy of the above closed-shell nucleus. The valence interaction V_V is given by a modification of (3) in which (a) the propagator is now $(E - \Delta E_c - H_0)^{-1} Q$, and (b) in the perturbation expansion of a particular matrix element, $(V_V)_{ij}$, there are no diagrams containing unlinked core parts, i.e. parts which look like the conventional vacuum to vacuum GOLDSTONE diagrams for the core particles.

In lowest order this agrees with an old argument by BETHE [3], that one should use the Brueckner reaction matrix as the effective 2-body interaction for S M calculations. The BLOCH—HOROWITZ formalism also provides a systematic way to examine corrections to this simple picture. One such correction, which may be significant, comes from the exchange of a "virtual phonon" between two valence particles, corresponding to a state-dependent polarization of the core.

Effective 2-body interaction

For a simple closed-shell nucleus, the lowest term in the GOLDSTONE expansion is

$$\frac{1}{2} \sum_{mn}^{A} \langle \Phi_{mn} |v| \Psi_{mn} \rangle = \\ = \frac{1}{2} \sum_{mn}^{A} \iint d^3 \vec{r}_1 \, d^3 \vec{r}_2 \, |\Phi_{mn}(\vec{r}_1, \vec{r}_2)|^2 \, g_{mn}(\vec{r}_1, \vec{r}_2) \,. \tag{6}$$

Here we have introduced an effective 2-body interaction,

$$g_{mn} \sim \frac{v\Psi_{mn}}{\Phi_{mn}}, \tag{7}$$

where $\Phi_{mn}(\vec{r}_1, \vec{r}_2) = \varphi_m(\vec{r}_1)\varphi_n(\vec{r}_2)$. The correlated 2-body wavefunction Ψ_{mn} is determined by the BETHE—GOLDSTONE equation,

$$\zeta_{mn}(\vec{r}', \vec{r}_2) \equiv \Phi_{mn} - \Psi_{mn} = \sum_{a,b>A} \frac{\varphi_a(\vec{r}_1)\,\varphi_b(\vec{r}_2)\,\langle ab\,|v|\,\Psi_{mn}\rangle}{E_a + E_b - E_m - E_n}. \tag{8}$$

It turns out that V_{SM} is quite weak for the intermediate states a, b, thus a form of WKB approximation should be quite satisfactory. Furthermore, we know from nuclear matter studies that ζ_{mn} is negligible unless $\vec{r}_1$ and $\vec{r}_2$ are rather close together. This leads us to focus our attention on a particular center-of-mass point $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, and then to assume

$$E_a \approx A(\vec{R}) - [\hbar^2/2Mm^*(\vec{R})]\nabla_1^2. \tag{9}$$

Ignoring (for now) the exclusion-principle restriction that $a, b > A$, we obtain

$$\left[\nabla_r^2 + \frac{1}{4}\nabla_R^2 - \gamma'^2_{mn}(\vec{R})\right]\zeta_{mn}(\vec{r}, \vec{R}) = \\ = -\left[Mm^*(\vec{R})/\hbar^2\right]v(\vec{r})\,\Psi_{mn}(\vec{r}, \vec{R}), \tag{10}$$

where

$$\gamma'^2_{mn}(\vec{R}) = [Mm^*(\vec{R})/\hbar^2]\,[2A(\vec{R}) - E_m - E_n]. \tag{11}$$

(Corrections to this "reference-spectrum" approximation [4] are too large to ignore, but are small enough so that rather simple estimates should suffice.) The remainder of the program [5] involves an approximate separation of relative and center-of-mass motions to convert $\gamma'^2_{mn}(\vec{R}) - \frac{1}{4}\nabla_R^2$ into $\gamma^2_{mn}(\vec{R})$, followed by a partial-wave separation of (10). To calculate the partial-wave effective interaction

$$g_L = v_L\Psi_L/\Phi_L, \tag{12}$$

we may approximate the uncorrelated relative wavefunction by a Bessel function, $\Phi_L(r, \vec{R}) \approx j_L[rk_{mn}(\vec{R})]$, again using something like a WKB approximation. This should be satisfactory because most of the resulting error in Ψ_L will be cancelled when one divides by the corresponding Φ_L. At this stage

the equations for the Ψ_L's are formally the same as in nuclear matter, and can be solved by standard methods [4, 5]. The actual partial-wave expansion of Φ_{mn} can be avoided by a projection-operator technique, replacing the local form (7) of g_{mn} by a nonlocal one:

$$\langle \hat{r}| \, g_{mn}(r, \bar{R})| \, \hat{r}' \rangle = \sum_{L,M} Y_L^M(\hat{r}) \, g_{mn,L}(r, \bar{R}) \, Y_L^{M*}(\hat{r}') = \\ = (4\pi)^{-1} \sum_L (2L+1) \, g_{mn,L}(r, \bar{R}) \, P_L(\hat{r} \cdot \hat{r}') \, . \tag{13}$$

Compared to the corresponding expression of BRUECKNER, GAMMEL and WEITZNER [6], this form of g_{mn} has the virtue of treating the "starting energy" $(Em + En)$ in an exact manner. This should lead to a significant improvement in the treatment of the low-density surface region. We believe that modified forms of "local-density" and WKB approximations, as outlined here, will be satisfactory for the remaining features of the BETHE—GOLDSTONE equation (8).

Shell-model potential

HUGENHOLTZ [7] has shown that the GOLDSTONE wavefunction (for a closed-shell nucleus) can be written in the form

$$\Psi = \exp\{\sum_r W_r\} \Phi_0 \, , \tag{14}$$

where the W's are linked, open diagrams. The first few terms are shown in Fig. 1, where the black box stands for all possible "self-energy" insertions. The third diagram arises from the $-V_{SM}$ term in $V = v - V_{SM}$. We argue that one obtains "optimum" convergence [5] by choosing V_{SM} to make the second and third diagrams of Fig. 1 cancel each other. In other words, $V_{SM} =$ =sum of all self-energy insertions. The first few of these are shown in Fig. 2.

The first diagram, (a), is just the usual Hartree—Fock expression, V_{HF}, but with the potential v replaced by the effective interaction (13). Diagram (b) represents a "renormalization" of V_{HF} due to the fact that the "normally-occupied" states n are actually occupied less than 100% of the time. We estimate that, at the density of nuclear matter (the centre of a nucleus), diagram (b) $\approx -15\%$ times diagram (a). Diagram (c) is the first member of a class of 3-body cluster diagrams which BETHE [8] has recently shown to have a very small net effect. The fourth diagram, (d), is also quite small, of order $+2$ to $+3$ MeV at the nuclear matter density.

The first of these diagrams, V_{HF}, is attractive and (very roughly) proportional to the particle density, ϱ, while the latter 3 are repulsive and nearly proportional to ϱ^2. Clearly the latter diagrams (which represent corrections to V_{HF} arising from correlations within the sea of background particles n, n') are responsible for the saturation phenomenon. For this reason we shall call the sum of the latter (ϱ^2) terms the "saturation potential", V_{sat}, thus $V_{SM} = V_{HF} + V_{\text{sat}}$.

Notice that we have not used any variational argument [9] to obtain V_{SM}. The present method is both simpler and less ambiguous than a *consistent* application of variational methods within the framework of the GOLDSTONE expansion. Furthermore, the present method also applies to the BLOCH—HOROWITZ expansion for open-shell nuclei.

Fig. 1. The first few diagrams in the exponent of equation (14)

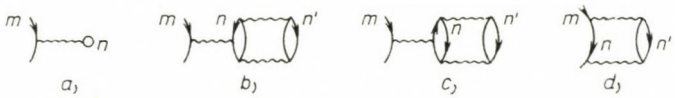

Fig. 2. The first few diagrams in the expansion of V_{SM} for normally-occupied states φ_m

An interesting result of this analysis is that the single-particle eigenvalues (of our $H_0 = T + V_{SM}$) are *not* equal to the "removal energies" which are observed, for example, in "pickup" $(p \to d)$, $(d \to t)$ and "knockout" $(p \to 2p)$ reactions. The same diagrams (Fig. 2) also occur in the calculation of the removal energy [10], but there is an important difference in their numerical evaluation. These terms are actually insertions within the diagrams of Fig. 1, when one is calculating V_{SM}, thus certain of their energy denominators must be evaluated "off the energy shell" [4]. This distinction appears for the first time in the denominator associated with the middle of term $2d$. In V_{SM} this denominator is always quite large, with the result that the net contribution of this term is quite small. For the removal energy, however, this denominator can be very small, or even zero. The net contribution is therefore large ($\sim +15$ MεV) [11] and also complex, corresponding to the finite lifetime of a physical hole. Furthermore, this term becomes strongly velocity-dependent [11], in such a way that the "effective mass" for the "removal potential" (whose expectation values lead to the observed removal energies) is expected to be considerably larger (and perhaps > 1) than the corresponding effective mass describing the velocity-dependence of V_{SM}. There is some experimental evidence for this [12, 13]. (Notice that we are referring here to the single-particle eigenvalues of the $H_0 = T + V_{SM}$ which

optimizes convergence for a *closed-shell* nucleus. We are not yet ready to comment on the relation between the observed removal energies and the valence-state energies which are most appropriate for the H_{0V} of (5).)

For a given two-body potential, v, we believe that the present V_{sat} terms will be a little more repulsive than the corresponding V_R term used by BRUECKNER's group [9, 14]. But in view of several recent developments in the theory of nuclear matter [8], it is not possible to say yet whether the present form of V_{SM} would lead to reasonable density distributions for actual nuclei.

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ОБОСНОВАНИЕ ЯДЕРНОЙ ОБОЛОЧЕЧНОЙ МОДЕЛИ В РАМКАХ ЗАДАЧИ МНОГИХ ТЕЛ

Б. Г. БРЭНДАУ

Р е з ю м е

Работа ставит своей целью показать, как проявляются общие свойства оболочечной модели при распространении теории Брюкнера—Бете—Голдстона на случай конечных ядер. Основная идея метода заключается в применении теории возмущений с соответствующей парциальной суммацией. Исходное выражение определяется, как $H_0 = T + V_{SM}$, где V_{SM} одночастичный оператор, обеспечивающий оптимальную сходимость выражения. После некоторых замечаний по отношению применения теории возмущений показывается метод вычисления матрицы реакции Брюкнера в пределах конечной геометрии, определённой выше упомянутым H_0. Наконец, дискутируется форма V_{SM}, обеспечивающая быструю сходимость связанного разложения по корреляционным функциям.

ON A SIMPLE METHOD OF CALCULATING ATOMIC ONE-ELECTRON EIGENFUNCTIONS

By

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Starting from a statistical atom model in which electrons are grouped in shells according to the principal quantum number, a method is developed to calculate atomic one-electron eigenfunctions in a simple way.

As is well known the HARTREE self-consistent field method is an iteration procedure which makes possible the determination of eigenfunctions and energies of atomic electrons. Since the application of the method involves the solution by iteration of a complicated system of differential equations, the solution can be obtained in a reasonably short time only by computer. Hence the development of a method with the help of which we can obtain Hartree eigenfunctions in a simple way is interesting from the practical as well as theoretical point of view. In recent years a number of papers appeared dealing with such problems. In what follows we would like to talk of a procedure which is completely different from previous ones and which gives good Hartree eigenfunctions and energies.

The procedure is intimately tied to a statistical atom model, in which electrons are grouped in shells according to the principal quantum number [1]. The shells are the K-, L-, M-, . . . shells which correspond to the successive values $n = 1, 2, 3, \ldots$ of the principal quantum number. The atom is then constructed, starting with the K-shell, by the superposition of the successive L-, M-, N,. . . shells. The two electrons of the K-shell have been treated according to HYLLERAAS, whereas the electrons of the other shells — which are assumed to be separated from each other — globally i.e. statistically. The kinetic energy has been calculated by means of the WEIZSÄCKER energy expression. The PAULI exclusion principle, namely the fact that the electrons of one particular shell cannot occupy the completely filled levels of the inner shells, has been taken into account by a repulsive potential deduced earlier from the statistical theory of an electron gas [2]. The potential energy of the electrons of a shell consists of the following contributions: first, the electron—nucleus interaction, second, the interaction of electrons of a particular shell with the electrons of other shells and finally the electron—electron interaction within the one shell. This latter has been corrected according to FERMI and

AMALDI in order to eliminate the electrostatic self interaction of electrons. We are now working on the problem of correlation and exchange between electrons [3].

The radial electron density of the n-th shell is taken to be

$$D_n = 4\pi r^2 \varrho_n = A_n \; r^{\varkappa_n} e^{-\lambda_n r} , \tag{1}$$

where ϱ_n is the electron density of the n-th shell, r is the distance from the nucleus, A_n is a normalising factor, $\varkappa_n$ and λ_n are variational parameters to be determined from the minimization of energy. The distributions of the total radial electron density for atoms obtained in this way are good even in the case of heavy atoms (e.g. Hg^{++}). The calculated radial density maxima, characteristic of different shells, are in very good agreement with those of the Hartree calculations.

The radial density, as a function of r, describes correctly only the outermost, so called principal maximum, but not the inner ones which originate from the orthogonalization of the electron eigenfunctions. Our calculated radial density distribution gives only the average of these inner maxima. This is connected with the fact that the orthogonality conditions for the same values of l, which are equivalent to the Pauli exclusion principle, have been taken into account by means of a repulsive potential. It is to be noted, however, that the averaging of the secondary maxima is hardly noticeable in the total radial electron density of the atom.

In one shell the average values of the squares of radial eigenfunctions can be obtained from the radial density of the particular electron shell by dividing the radial density of the shell by the electron number. In the case of the n-th shell

$$f_n^2 = \frac{D_n}{N_n} ,$$

where f_n is the radial eigenfunction and N_n the number of electrons in the n-th shell.In our approximation f_n turns out to be the same for all the electrons of the n-th shell independently of l. This is not the case in the HARTREE method, where states with different l of the n-th shell are different in that the position and number of inner maxima of the squares of radial eigenfunctions corresponding to different values of l are different. The outermost principal maximum of a particular shell, however, is practically at the same place independently of l. That spatially well separated shells are formed in atoms is due precisely to this fact. The result that our calculated f_n^2 function is independent of l, is reasonable in a first approximation since our f_n^2 corresponds to the outermost (principal) maximum, which also nearly coincide in the Hartree method for different states of the same shell.

If we orthogonalize our approximate f_n radial eigenfunctions with the method of SCHMIDT to the radial wave functions of the inner shells with the same l, then we get a surprisingly good approximation to the Hartree eigenfunctions. The squares of the orthogonalized eigenfunctions describe correctly also the secondary maxima. These maxima and the roots between the maxima are practically the same as the HARTREE values. The orthogonalization has been performed with the method of SCHMIDT as follows. In the K-shell there are two 1s states. Our radial eigenfunction f_1 of the K-shell electrons is therefore identical with f_{1s}, accordingly we have

$$f_{1s} = f_1 \,.$$

Of the 8 electrons of the L-shell two are in the $2s$ and six in the $2p$ states. Our f_2 can be identified with f_{2p}, accordingly we have

$$f_{2p} = f_2 \,.$$

The radial eigenfunction f_{2s} is then obtained, if we put

$$f_{2s} = \alpha f_{1s} + f_2 \,,$$

where the constant α will be determined from the orthogonalization of f_{2s} to f_{1s}. For the next shell M we can proceed in a similar way, and we have

$$\begin{aligned} f_{3d} &= f_3 \,, \\ f_{3p} &= \beta f_{2p} + f_3 \,, \\ f_{3s} &= \gamma f_{1s} + \mu f_{2s} + f_3 \,, \end{aligned}$$

where the constant β is determined from the orthogonalization of f_{3p} to f_{2p} and the constants γ and μ from the orthogonalization of f_{3s} to f_{1s} and f_{3s} to f_{2s}. This procedure can be continued for the next shells.

Apart from the outermost electrons of the atom the obtained eigenfunctions approximate very well the Hartree functions, even for heavy atoms (e.g. Hg^{++}) as it can be seen in the figures (in the figures r is in units of a_0, and f_{nl} in units of $1/\sqrt{a_0}$, where a_0 is the first Bohr radius) [4]. The agreement is not so good for the outer electrons of the atoms. This is because, on the one hand, for the outer electrons even the starting eigenfunction is not as good as for the inner electrons, since on the periphery of the atom the electron density is less accurate than in the inner shells, on the other hand, because of the large number of orthogonalizations for the outer electrons errors might accumulate in this case.

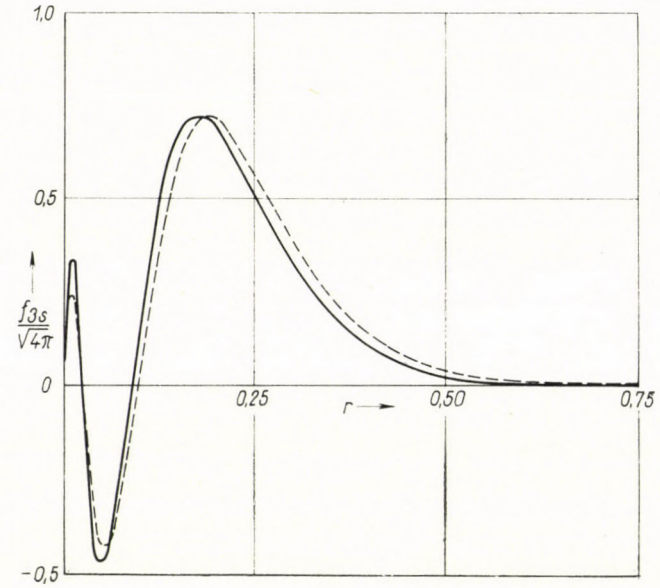

Fig. 1. The radial eigenfunction of the 3*s* state of Hg^{++} as a function of *r*.
——— our eigenfunction,
— — — the HARTREE eigenfunction

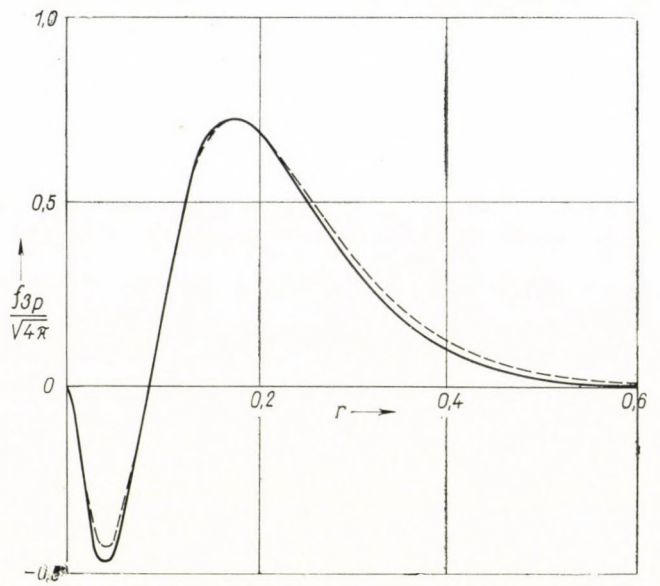

Fig. 2. The radial eigenfunction of the 3*p* state of Hg^{++} as a function of *r*.
——— our eigenfunction,
— — — the HARTREE eigenfunction

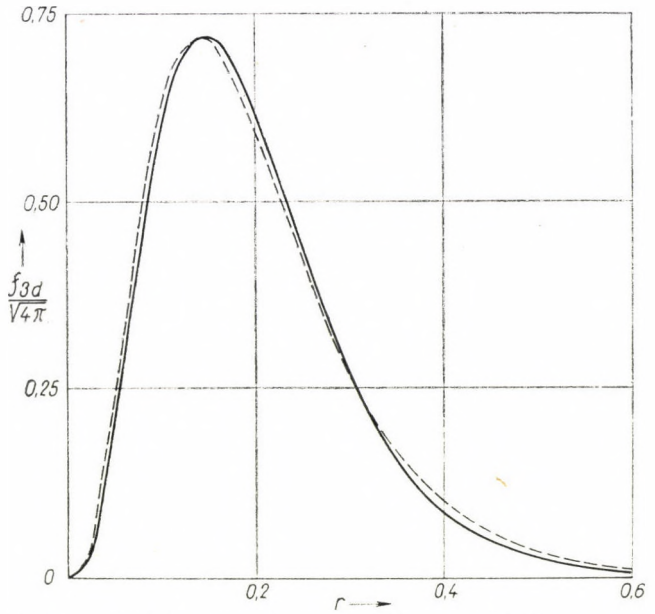

Fig. 3. The radial eigenfunction of the 3*d* state of Hg^{++} as a function of *r*.
——— our eigenfunction,
– – – the HARTREE eigenfunction

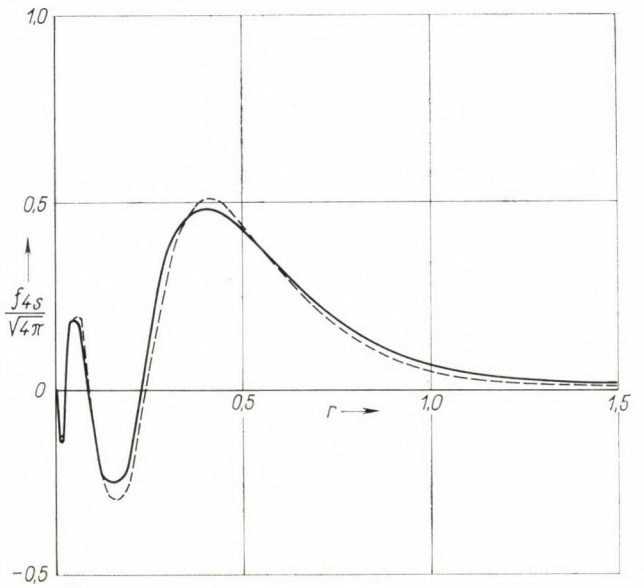

Fig. 4. The radial eigenfunction of the 4*s* state of Hg^{++} as a function of *r*.
——— our eigenfunction,
– – – the HARTREE eigenfunction

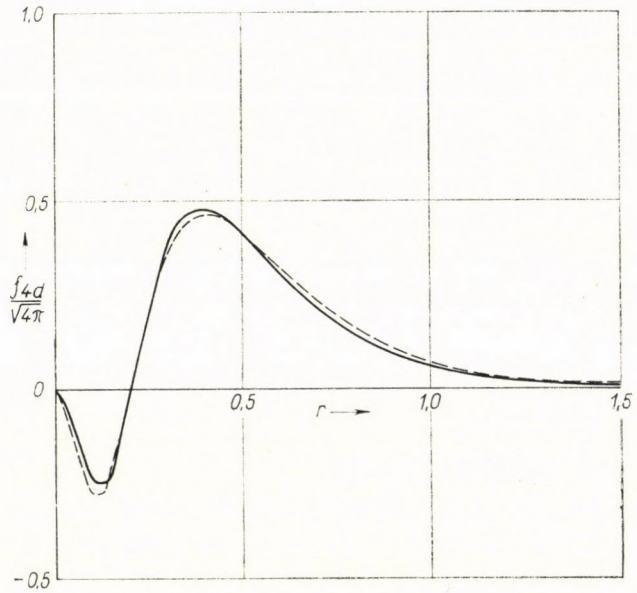

Fig. 5. The radial eigenfunction of the $4d$ state of Hg^{++} as a function of r.
——— our eigenfunction,
– – – the HARTREE eigenfunction

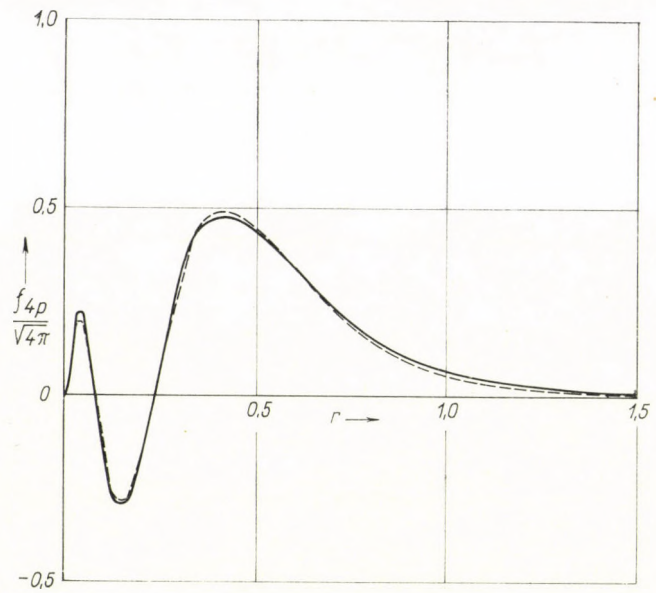

Fig. 6. The radial eigenfunction of the $4p$ state of Hg^{++} as a function of r.
——— our eigenfunction,
– – – the HARTREE eigenfunction

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ОБ ОДНОМ ПРОСТОМ МЕТОДЕ ОПРЕДЕЛЕНИЯ АТОМНЫХ ОДНОЭЛЕКТРОННЫХ СОБСТВЕННЫХ ФУНКЦИЙ

П. ГОМБАШ

Р е з ю м е

Исходя из статистической модели атома, в которой электроны сгруппированы в оболочках по главным квантовым числам, разработан метод для вычисления простым методом атомных одноэлектронных собственных функций.

THE METHOD OF EFFECTIVE INTERACTION IN NUCLEAR PHYSICS

By

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In the nuclear two-body problem the GAMMEL—THALER hard core potential can be replaced by any one of an infinite number of equivalent non local potentials, which are regular everywhere. Such potentials can easily be constructed. It is investigated, to what degree equivalent two-body potentials are also equivalent for the nuclear many body problem, and can therefore be used as effective interaction in the many body Hamiltonian.

Introduction

The nuclear many body problem is characterized by the special nature of the nucleon-nucleon interaction. Our knowledge of these forces comes from an analysis of the deuteron data and of the nucleon-nucleon scattering data.

If one fits the two nucleon data by a local two body potential (GAMMEL—THALER, HAMADA—JOHNSTON [1]) this potential is found to be strongly repulsive for short distances. I will assume here that this repulsion is well approximated by a hard core. (A recent analysis of scattering data seems to show that also a "soft" core would be sufficient. I will not discuss this experimental question here, but those problems which arise from the existence of a hard core).

The nuclear forces are sufficiently attractive to form a superfluid ground state. This superfluidity is well known for finite systems from the energy spectra and the moments of inertia. For nuclear matter the question is still open whether the nuclear forces are strong enough to yield superfluidity. I will assume here that the nuclear matter is also superfluid and discuss the consequences of this assumption.

If the two body interaction v has a hard core the matrix elements $\langle \alpha\beta \mid v \mid \delta\gamma \rangle$ are infinite. In this case neither perturbation theory nor the BCS-BOGOLJUBOV method can be applied to the nuclear many body problem. However, partial summations of the perturbation series can still give finite results. The normal ground state energy is probably well approximated by the BRUECKNER particle-particle ladder approximation, but this is not relevant to our problem since we will assume that the nuclear matter is in a superfluid state.

For the calculation of the superfluid state the BRUECKNER approach is not convenient for the following reasons: If the interaction has a sufficiently

attractive part the BRUECKNER matrix t_B which is defined by the equation

$$t_B = v - v\frac{P}{e}t_B$$

has a singularity. This case is of particular interest since a singularity in t_B is a sufficient (but not necessary) condition for the superfluidity of the ground state of an infinite system* [2]. This difficulty can, however, be removed by principal value integrals [3] or by a convenient renormalization of the energy denominators, which makes the singularity vanish. (Reference Spectrum Method [4]).

A second point is that the BCS state is essentially symmetric in particles and holes, but t_B is a particle-particle ladder only. This problem can be solved by using instead of t_B the GALITZKI matrix t_G which is symmetric in particles and holes. The two-particle operator t_G is defined by the equation

$$t_G = v - v\frac{P}{e}t_G + v\frac{(1-P)}{e}t_G\,.$$

Here the superfluidity of the ground state is necessary and sufficient for the existence of a singularity in t_G. Again this singularity can be treated by principle value methods (LOMON and MCMILLAN [5]).

A more serious difficulty is that a partial summation of the perturbation series like t_B or t_G is a very dubious approach to a superfluid system. This can easily be seen if one calculates the ground state energy E_0 of the reduced BCS-Hamiltonian in ladder approximation. E_0 is then not proportional to the volume of the system but simply a constant. For more realistic forces which interact not only in $P = 0$ states, the break-down of the ladder approximation will be less drastic but still present. The underlying reason is that the superfluid state depends non-analytically on the interaction v and can therefore not be obtained by a power series in v. — The BCS-BOGOLJUBOV method which solves just this problem is, however, not applicable on account of the singular hard core.

These difficulties would disappear if one could combine the BCS-BOGOLJUBOV method with the BRUECKNER method in such a way that one uses the non-singular t-operator (t_B or t_G) as an effective interaction in the many body Hamiltonian, that means one starts with the Hamiltonian

$$H = \sum_i \frac{P_i^2}{2m} a_i^+ a_i + \frac{1}{2}\sum_{\substack{i,k\\ l\,m}} \langle ik\,|t|\,lm\rangle\, a_i^+ a_k^+ a_m a_e\,,$$

* This means that a somewhat weaker interaction is sufficient for superfluidity than for the existence of a singularity in t_B. The difference is, however, very small in the nuclear case. Therefore we will neglect it here.

where v is simply replaced by t. Perturbation theory or the BCS-BOGOLJUBOV method could then be applied to this Hamiltonian.

However, there is no obvious justification for such a procedure. Moreover, it is quite clear that this method cannot be exact, since the t-operator is not hermitian. — Nevertheless, this method has been applied to the calculation of the normal state by BRUECKNER and GOLDMAN [6] and very recently by BRUNNER [7] and to the calculation of the superfluid nuclear matter by BRUECKNER et al. [8]. The results are surprisingly good.

A more convincing method of constructing an effective potential is to replace the singular hard core potential by a non-local pseudo-potential which reproduces the wave function for large distances. The HUANG, YANG, LEE [9] potential and the SCOTT—MOSZKOWSKI [10] separation method potential is of this kind. However in both cases the effective potentials are not hermitian and variational techniques like the HARTREE—FOCK or the HARTREE—BOGOLJUBOV method could become inconsistent.

There is, however, no difficulty in replacing the local hard core potential by a soft hermitian but non-local potential which fits all the two body data. We know that only a local potential v is uniquely determined by the two body scattering data, — or at least up to a constant, if there is one bound state as in the proton-neutron problem — but an infinite number of non-local potentials gives the same two body data [11].

Such potentials have been obtained for example by GREEN[12] and TABAKIN [14] from the experimental two body data. Using these potentials as effective interactions in the many body problem these authors obtained very encouraging results [12] [13] [14].

But even for these hermitian potentials which fit all the two body data the question is still open, how one can justify the use of these potentials as effective interactions in the many body problem. The equivalence of these potentials for the two body problem does not imply their equivalence for the many body problem. This will be discussed now.

The canonical transformation

The Schrödiger equation of the many body problem is

$$H \mid \psi_n \rangle = E_n \mid \psi_n \rangle ,$$

where

$$H = \Sigma T_i + \frac{1}{2} \sum_{i,k} v_{ik} = H_0 + H_v$$

and v_{ik} is the GAMMEL—THALER potential. If we apply a canonical trans-

formation e^{iS} (S hermitian) to this equation we obtain the new problem

$$e^{-iS}He^{iS}e^{-iS} \mid \psi_n\rangle = E_n e^{-iS} \mid \psi_n\rangle ,$$

$$\tilde{H} \mid \tilde{\psi}_n\rangle = E_n \mid \tilde{\psi}_n\rangle$$

with the same spectrum E_n.

We will try to choose e^{iS} such that the interaction in $\tilde{H}$ is not singular [15], [16]. For the operator S we make the ansatz

$$S = \int \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \mathrm{d}\mathbf{r}_1' \mathrm{d}\mathbf{r}_2' \langle \mathbf{r}_1\mathbf{r}_2 \mid \Omega \mid \mathbf{r}_1'\mathbf{r}_2'\rangle \psi^+(\mathbf{r}_1)\psi^+(\mathbf{r}_2)\psi(\mathbf{r}_2')\psi(\mathbf{r}_1') ,$$

where Ω is a two body operator.

The transformed Hamiltonian $\tilde{H}$ is then given by

$$\tilde{H} = H + i[H, S] + \frac{i^2}{2}[[H, S], S] + \ldots .$$

After ordering the $\psi(\mathbf{r})$-operators, $\tilde{H}$ is seen to be of the form

$$\tilde{H} = H_0 + H_w + \sum_{n\geq 3} W(n) .$$

Here H_0 is the free Hamiltonian, H_w is the contribution of the two body forces

$$H_w = \frac{1}{2} \sum_{\substack{i,k \\ l\,m}} \langle ik \,|w|\, lm\rangle \, a_i^+ \, a_k^+ \, a_m \, a_e$$

and $W(n)$ is the contribution of the n-body forces. A canonical transformation e^{iS}, applied to the many particle Hamiltonian H will therefore generally lead to many body forces. This is the price we have to pay for the possibility of making the two body force w regular.

To obtain the two body interaction w, it is sufficient to consider the two body system. In this case, the two particle Hamiltonian

$$h \equiv h_0 + v \quad \text{with} \quad (h_0 + v) \mid \psi_i^v\rangle = \varepsilon_i \mid \psi_i^v\rangle$$

is transformed into

$$\tilde{h} \equiv h_0 + w = e^{-i\Omega} h e^{i\Omega} \quad \text{with} \quad \tilde{h} \mid \psi_i^w\rangle = \varepsilon_i \mid \psi_i^w\rangle$$

where

$$w = e^{-i\Omega} v e^{i\Omega} + e^{-i\Omega} [h_0, e^{i\Omega}] .$$

Our goal is to find a transformation $e^{i\Omega}$, which yields an interaction w, which is regular everywhere and which makes the many body contribution in $\widetilde{H}$ small. The first conditions can easily be satisfied, as will be shown in the next section. The second condition is very difficult to formulate and I will assume in this lecture that the many body terms are sufficiently small to be neglected.

The GAMMEL—THALER potential v is obtained from the deuteron data and from the two body scattering phase shifts $\delta_l(k)$. The transformation $e^{i\Omega}$ leads generally to a Hamiltonian $\tilde{h} = h_0 + w$ with the same eigenvalues

$$(h_0 + w) \mid \psi_i^w \rangle = \varepsilon_i \mid \psi_i^w \rangle$$

but with different eigenfunctions $\mid \psi_i^w \rangle = e^{-i\Omega} \mid \psi_i^v \rangle$ and therefore to different phase shifts $\tilde{\delta}_l(k)$.

The nuclear potential v is not really known. Any other potential w, which yields the same two body data has the same empirical justification. Therefore we will restrict here the transformation $e^{i\Omega}$ so that the new potential w yields the same two body data as v. The potential w will then generally be non-local, since only a local potential is uniquely determined by the experimental two body data. We will call here two potentials "equivalent" if they correspond to the same two body data.

It follows from the transformation properties of H and $\widetilde{H}$ under a canonical transformation e^{is}, that equivalent two body interactions obviously correspond to the same many body spectrum E_n. But $H = H_0 + H_v$ and the two body contribution $H_2 = H_0 + H_w$ of $\widetilde{H}$ does not yield the same energy spectrum. The difference comes from the many body forces in $\widetilde{H}$ mentioned above.

The condition, that the transformation $e^{i\Omega}$ leads to a potential w, which is equivalent to v can easily be formulated in terms of the scattering t-matrix. Let t_v and t_w be the t-matrices of the scattering process which correspond the potentials v and w respectively. The matrices are then defined by

$$t_v |\mathbf{p}\rangle = v|\mathbf{p}\rangle - v \frac{1}{h_0 - \varepsilon(\mathbf{p}) - i\gamma} t_v |\mathbf{p}\rangle ,$$

$$t_w |\mathbf{p}\rangle = w|\mathbf{p}\rangle - w \frac{1}{h_0 - \varepsilon(\mathbf{p}) - i\gamma} t_w |\mathbf{p}\rangle ,$$

where $\varepsilon(p) = \dfrac{p^2}{2m}$. The potentials v and w are equivalent if

$$\langle p \mid t_v \mid p \rangle = \langle p \mid t_w \mid p \rangle$$

i.e.

$$\langle \mathbf{p}| v \left(1 + \frac{1}{e} v\right)^{-1} |\mathbf{p}\rangle = \langle \mathbf{p}| w \left(1 + \frac{1}{e} w\right)^{-1} |\mathbf{p}\rangle ,$$

where e is the energy denominator and

$$w = e^{-i\Omega} v e^{i\Omega} + e^{-i\Omega} [h_0, e^{i\Omega}] .$$

This condition on the operator $e^{i\Omega}$ is difficult to elucidate. Therefore I will discuss here another method of constructing a phase shift conserving transformation $e^{i\Omega}$.

Let $h_A = h_0 + v_A$ and $h_B = h_0 + v_B$ be two arbitrary two particle Hamiltonians but with the same energy spectrum ε_γ. The eigenstates $| \psi_\gamma^A \rangle$ and $| \psi_\gamma^B \rangle$ can then be related to each other by a unitary operator

$$|\psi_\gamma^A\rangle = e^{i\Omega} |\psi_\gamma^B\rangle .$$

Since the states $| \psi_\gamma^A \rangle$ and $| \psi_\gamma^B \rangle$ form a complete orthonormal set, we get

$$e^{i\Omega} = \sum_\nu |\psi_\nu^A\rangle \langle \psi_\nu^B|$$

or

$$\langle \mathbf{r}| e^{i\Omega} |\mathbf{r}'\rangle = \sum_\nu \psi_\gamma^A (\mathbf{r}) \psi_\gamma^{B*} (\mathbf{r}') ,$$

where $\mathbf{r}=\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}' = \mathbf{r}_1' - \mathbf{r}_2'$. The potentials v_A and v_B are equivalent, if $\psi_\gamma^A(\mathbf{r})$ and $\psi_\gamma^B(\mathbf{r})$ are asymptotically equal. The transformation $e^{i\Omega}$ will then not change the asymptotic behaviour of a given wave function. This can easily be seen: An arbitrary wave function $\chi(\mathbf{r})$ is transformed by $e^{i\Omega}$ into the function

$$\zeta(\mathbf{r}) = \int \langle \mathbf{r}| e^{i\Omega} |\mathbf{r}'\rangle \chi(\mathbf{r}') \rangle d\mathbf{r}' ,$$
$$\zeta(\mathbf{r}) = \sum_\nu \psi_\nu^A (\mathbf{r}) \int d\mathbf{r}' \psi_\nu^{B*} (\mathbf{r}') \chi(\mathbf{r}') .$$

For large r we have $\psi_\gamma^A(\mathbf{r}) = \psi_\gamma^B(\mathbf{r})$ and therefore

$$\zeta(\mathbf{r}) = \sum_\nu \psi_\nu^B (\mathbf{r}) \int d\mathbf{r}' \psi_\nu^{B*} (\mathbf{r}') \chi(\mathbf{r}') = \chi(\mathbf{r}) .$$

The construction of a phase shift conserving operator $e^{i\Omega}$ is now straightforward: From an arbitrary pair of equivalent two body Hamiltonians h_A and h_B with eigenstates $| \psi_\gamma^A \rangle$ and $| \psi_\gamma^B \rangle$ respectively, we obtain the operator

$$e^{i\Omega} = \sum_\nu |\psi_\nu^A\rangle \langle \psi_\nu^B| .$$

This operator, applied to the potential v gives an equivalent potential

$$w = e^{-i\Omega} v e^{i\Omega} + e^{-i\Omega} [h_0, e^{i\Omega}] .$$

Explicit examples for $e^{i\Omega}$

a) *The separation method*

To illustrate qualitatively the effect of a canonical transformation on the GAMMEL—THALER potential v, which does not change the asymptotic phase of the wave function and which makes the potential regular for small distances, I will discuss here the SCOTT—MOSZKOWSKI separation method [10] to estimate a regular equivalent potential w. This potential is not exactly hermitian and does not fit the deuteron data precisely, but the method is very instructive.

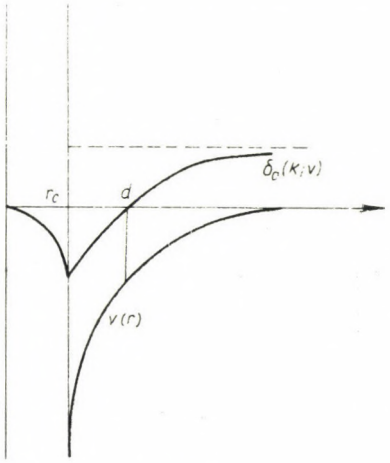

Fig. 1

The potential $v(r)$ changes its sign at $r = r_c$. Therefore the phase function $\delta_0(k; r)$, for the s-wave, which is defined by the differential equation

$$\delta_0'(k; r) = -\frac{1}{k} v(r) \sin^2 (kr + \delta_0 (k; r)),$$

$$\delta_0(k; 0) = 0$$

has a zero at $r = d(k)$. (Fig. 1) Now the potential is separated into two parts: The short range potential

$$v_S(r) = \begin{cases} v(r) & r \leq d \\ 0 & r > d \end{cases}$$

gives no phase shift and the long range part

$$v_{SM}(r) = \begin{cases} 0 & r \leq d \\ v(r) & r > d \end{cases}$$

yields the same phase shift as $v(v)$. If we consider w as approximately obtained by a canonical transformation $e^{i\Omega}$ from v, this transformation has the effect that

1. The short range part of v is projected out;
2. The long range part of v is not changed;
3. The new potential w is momentum dependent, since the separation distance $d = d(k)$ is momentum dependent.

In terms of wave functions the eigenstates $| \psi_i^v >$ of $(h_0 + v)$ are related to the eigenstates $| \psi_i^w >$ of $(h_0 + w)$ by

$$e^{-i\Omega} | \psi_i^v \rangle = | \psi_i^w \rangle .$$

The effect of $e^{i\Omega}$ on the wave function is therefore (Fig. 2).

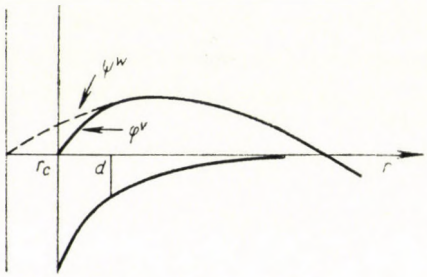

Fig. 2

1. To project out the parts with $r \leq r_c$;
2. Not to change the phase shift i.e. the asymptotic behaviour.

An operator, which transforms the wave function $\psi^w(\mathbf{r})$ into $\psi^v(\mathbf{r})$ and the potential v into the regular potential w can easily be constructed: If we start again from two sets of eigenstates $| \psi_\gamma^A \rangle$ and $| \psi_\gamma^B \rangle$, the transformation

$$\langle \mathbf{r}| e^{i\Omega} |\mathbf{r}' \rangle = \sum_\nu \psi_\nu^A (\mathbf{r}) \psi_\nu^{B*} (\mathbf{r}')$$

which transforms $\psi^w(\mathbf{r})$ into

$$\psi^v (\mathbf{r}) = \sum_\nu \psi_\nu^A (\mathbf{r}) \int \psi_\nu^{*B} (\mathbf{r}') \psi^w (\mathbf{r}') \, d\mathbf{v}'$$

has the desired effect, if $\psi_\gamma^A(r) \cdot r = 0$ for $r \leq r_c$. That part of w, which comes from the transformation of v, i.e.

$$\langle \mathbf{r}|^{-i\Omega} v e^{i\Omega} |\mathbf{r}' \rangle = \sum_{\mu,\nu} \psi_\nu^B (\mathbf{r}) \psi_\mu^{B*} (\mathbf{r}') \int d\mathbf{r}'' \, \psi_\nu^{A*} (\mathbf{r}'') v(\mathbf{r}'') \psi_\mu^A (\mathbf{r}'') ,$$

is then regular everywhere, even if $v(r)$ has a singular hard core.

b) *The Villars transformation*

VILLARS has given an explicit example for a phase shift conserving transformation $e^{i\Omega}$ which transforms the interaction v into a regular one [15]*. The eigenstates of

1. The free two particle equation

$$h_0 \mid \varphi_{\mu\nu}\rangle = \varepsilon_{\mu\nu} \mid \varphi_{\mu\nu} \rangle ,$$

2. The BETHE—GOLDSTONE equation

$$(h_0 + Pv) \mid \tilde{\psi}_{\mu\nu}\rangle = \tilde{\varepsilon}_{\mu\nu} \mid \tilde{\psi}_{\mu\nu} \rangle$$

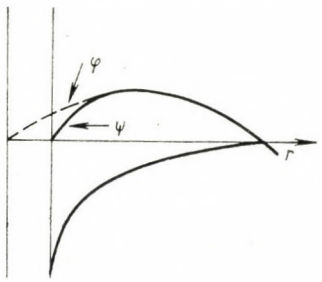

Fig. 3

re asymptotically equal, i.e. they have the same phase and the wave function $_{\mu\nu}(r) \cdot r$ vanishes for $r \leq r_c$ (Fig. 3). The $\mid \tilde{\psi}_{\mu\nu} \rangle$ are not orthogonal, but an rthogonal set $\mid \psi_{\mu\nu} \rangle$ can be constructed. If we assume that $(h_0 + Pv)$ has o bound state the canonical transformation

$$e^{i\Omega} = \sum_{\mu,\nu} |\varphi_{\mu\nu}\rangle \langle\psi_{\mu\nu}|$$

will approximately satisfy the relation

$$e^{-i\Omega}(h_0 + Pv)\, e^{i\Omega} = h_0$$

and will preserve the phase of an arbitrary wave function.

If one applies this transformation to the two body hamiltonian $h = h_0 + + v$, one obtains an equivalent potential

$$w_V = e^{-i\Omega}\, v\, e^{i\Omega} + e^{-i\Omega}[h_0, e^{i\Omega}] = e^{-i\Omega}(v - Pv)\, e^{i\Omega} .$$

* Another interesting example has been given by BELL [16].

The properties of this new potential can easily be seen: For small distances, $v - Pv$ almost vanishes, and therefore w will be very small in that region. For large distances, $Pv \approx 0$, and $e^{i\Omega} \approx 1$ so that $w_V \approx v$ for long distances. The transformed potential w_V is therefore qualitatively similar to the SCOTT—MOSZKOWSKI potential w_{SM}.

VILLARS has shown that w_V can approximately be expressed in terms of BRUECKNER's t_B-matrix. One obtains

$$w_V = \frac{t_B + t_B^+}{2}$$

i.e. the hermitian part of the t_B-matrix. These investigations show that the use of t_B as an effective interaction has the same justification as the use of any other equivalent potential as effective interaction.

c) *Hermitian Bethe—Goldstone equation*

To avoid the non-hermitian operator $(h_0 + Pv)$ and the corresponding nonorthogonal eigenstates $|\tilde{\psi}_{\mu\nu}\rangle$ it has been proposed[17] to use the hermitian BETHE—GOLDSTONE equation

$$(h_0 + PvP)\,|\,\chi_{\mu\nu}\rangle = \eta_{\mu\nu}\,|\,\chi_{\mu\nu}\rangle\,.$$

The eigenstates of this equation belong to two classes:

1. If $P\,|\,\chi_{\mu\nu}\rangle = |\,\chi_{\mu\nu}\rangle$ we have $|\,\chi_{\mu\nu}\rangle = |\,\tilde{\psi}_{\mu\nu}\rangle$,

where $|\,\tilde{\psi}_{\mu\nu}\rangle$ is the solution of the ordinary BG-equation.

2. If $P\,|\,\chi_{\mu\nu}\rangle = 0$ we have $|\chi_{\mu\nu}\rangle = |\,\varphi_{\mu\nu}\rangle$,

where $|\,\varphi_{\mu\nu}\rangle$ is the free two-particle wave function.

Therefore the eigenstates $|\,\chi_{\mu\nu}\rangle$ seem to be very convenient for our problem:

a) The states $|\,\chi_{\mu\nu}\rangle$ are asymptotically equal to the free two-particle states $|\,\varphi_{\mu\nu}\rangle$

b) They form a complete orthonormal set of states.

The operator $e^{i\Omega}$ is then given by

$$e^{i\Omega} = (1 - P) + \sum_{\substack{\alpha,\beta \\ >k_F}} |\tilde{\psi}_{\alpha\beta}\rangle\langle\varphi_{\alpha\beta}|\,.$$

Here the transformed potential is exactly given by

$$w = e^{-i\Omega}(v - PvP)e^{i\Omega}\,,$$

since

$$e^{-i\Omega}\,(h_0 + PvP)\,e^{i\Omega} = h_0\,.$$

However, it is easily seen that for occupied states $|\varphi_{\mu\nu}\rangle$ with $P\,|\varphi_{\mu\nu}\rangle = 0$ we have

$$\langle\varphi_{\mu\nu}\,|\,w\,|\varphi_{\mu'\nu'}\rangle = \langle\varphi_{\mu\nu}\,|\,v\,|\,\varphi_{\mu'\nu'}\rangle$$

and we do not get a new potential. The reason is, that the states $|\,\chi_{\mu\nu}\rangle$ below the Fermi surface are free states and therefore the wavefunction $\chi_{\mu\nu}(r)$ does not vanish inside the hard core.

d) *Construction of equivalent potentials from the two body data*

Instead of searching for a convenient transformation $e^{i\Omega}$ and then transforming the potential v into an equivalent potential w one can simply try to fit the empirical two body data by a nonlocal potential. From the JOST—KOHN theorem we know that only a local potential is uniquely determined (up to a constant) by the nuclear two body data. But an infinite set of nonlocal potentials would give the same results.

Non-local potentials $w(\mathbf{r}, \mathbf{r}')$ which fit the experimental two body data have been obtained for example by GREEN [12], and TABAKIN [14]. Using these potentials, perturbation theory and the BCS-BOGOLJUBOV method can immediately be applied. In the spirit of the canonical transformation philosophy these non local potentials may be considered as obtained from v by the transformation

$$w_\Omega\,(\mathbf{r}, \mathbf{r}') = \langle\mathbf{r}\,|\,e^{-i\Omega}\,v\,e^{i\Omega} + e^{-i\Omega}\,[h_0, e^{i\Omega}]\,|\mathbf{r}'\rangle\,.$$

Each of these equivalent $w_\Omega(\mathbf{r}, \mathbf{r}')$ then belongs to another Hamilton operator $\widetilde{H} = e^{-is}H\,e^{is}$ which has the same eigenvalues E_n as the original H. However, the two body parts of these operators $\widetilde{H}\,[\Omega]$ do not have the same spectrum. The use of an equivalent potential w as an effective interaction can therefore only be justified if the many body terms in H can be neglected. We do not know very much about these terms.

In addition to this problem, we are in the difficult situation, that we do not know, which one of the infinite number of equivalent two body potentials $w_\Omega(\mathbf{r}, \mathbf{r}')$ is the realistic one. Consequently we do not know, when we have to take into account many body forces and when not.

In conclusion, we see that for a justification of the method of effective interaction the following problems must be solved: One has to

1. reduce the great ambiguity in the empirical potential w by new experimental data
2. give at least a rough estimation of the many body forces in $\widetilde{H}$.

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МЕТОД ЭФФЕКТИВНОГО ВЗАИМОДЕЙСТВИЯ В ЯДЕРНОЙ ФИЗИКЕ

П. МИТТЕЛЬШТАДТ

Р е з ю м е

В задаче двух тел ядерной физики потенциал твёрдой сердцевины Гаммела—Талера может быть заменён любым из эквивалентных нелокальных потенциалов бесконечного числа, везде регулярных. Такие потенциалы сконструировать нетрудно. Далее доказывается, что в какой степени эквивалентны потенциалы в случае двух тел, в такой же степени эквивалентны для проблемы многих тел ядерной физики, и, таким образом, могут использоваться для выражения эффективного взаимодействия в гамильтониане многих тел.

A NEW METHOD FOR THE CALCULATION OF TEMPERATURE AVERAGES

By

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A new method will be presented for the calculation of the temperature averages. The method is in close connection with the GREEN's function method, as it is based upon the so called projection relations, which can be regarded as the solutions of the GREEN's function problem. From the projection relations we get the exact formulas and series expansions with the help of the principle of perturbation and renormalization.

Most problems in statistical physics are connected with the calculation of temperature averages. These quantities are usually one-time averages, however, in recent years two-time averages have become of interest too.

Several methods are known for evaluating such averages. One of them is the well known GREEN's function method, by which the two-time averages can be calculated from which the one-time averages can be obtained.

Besides its definite advantages, the GREEN's function method has some drawbacks. It is difficult, for instance, to obtain with this method the results of the high temperature expansions.

A method will now be presented, which in principle is similar to the GREEN's function method, but is markedly different in its form. It is based upon the so-called projection relations. These projection relations can be obtained as follows.

Let A and B be two operators on a physical system. The two-time averages of these operators are defined as

$$\begin{aligned} \langle BA(t)\rangle &= Tr\{BA(t)e^{-\beta H}\}/Tr\{e^{-\beta H}\}\,, \\ \langle A(t)B\rangle &= Tr\{A(t)Be^{-\beta H}\}/Tr\{e^{-\beta H}\}\,. \end{aligned} \tag{1}$$

The latter can be reduced to the former by making use of the general relations

$$\begin{aligned} Tr\{abc\} &= Tr\{bca\}\,, \\ A(t+s) &= e^{-iE(s)t}\,A(s)\,, \end{aligned}$$

where we have put $E(s) = i\dfrac{\partial}{\partial s}$. In fact,

$$\langle A(t)B\rangle = Tr\{Be^{-\beta H}A(t)\}/Tr\langle e^{-\beta H}\} = Tr\{BA(t+i\beta)e^{-\beta H}\}/Tr\{e^{-\beta H}\} =$$

$$= \langle BA(t+i\beta)\rangle = e^{\beta E(t)}\langle BA(t)\rangle .$$

Let us write $\langle A(t)B - \eta BA(t)\rangle$ $(\eta = \pm 1)$ in the shorter form,

$$\langle A(t) \mid B\rangle_\eta = \langle A(t)B - \eta BA(t)\rangle .$$

Using the above relation we get

$$\langle A(t) \mid B\rangle_\eta = [e^{\beta E(t)} - \eta]\langle BA(t)\rangle ,$$

hence the projection relations are obtained in the form

$$\begin{aligned} \langle BA(t)\rangle &= \frac{1}{e^{\beta E(t)} - \eta}\langle A(t)|B\rangle_\eta \\ \langle A(t)B\rangle &= \frac{1}{1 - \eta e^{-\beta E(t)}}\langle A(t)|B\rangle_\eta . \end{aligned} \tag{2}$$

It has been assumed in the last step that $\langle A\rangle = \langle B\rangle = 0$.

Equivalent forms of the projection functions to be used in the present treatment are

$$\begin{aligned} \langle BA(t)\rangle &= \frac{1}{e^{\beta E(t)} - \eta}\langle e^{iHt}Ae^{-iHt}|B\rangle_\eta , \\ \langle A(t)B\rangle &= \frac{1}{1 - \eta e^{-\beta E(t)}}\langle e^{iHt}Ae^{-iHt}|B\langle_\eta , \end{aligned} \tag{3}$$

$$\begin{aligned} \langle BA(t)\rangle &= \frac{1}{e^{\beta E(t)} - \eta}\langle e^{-iEt}A|B\rangle_\eta , \\ \langle A(t)B\rangle &= \frac{1}{1 - \eta e^{-\beta E(t)}}\langle e^{-\beta Et}A|B\rangle_\beta , \end{aligned} \tag{4}$$

$$\begin{aligned} \langle BA(t)\rangle &= \left\langle \frac{e^{-iEt}}{e^{\beta E} - \eta}A|B\right\rangle_\eta , \\ \langle A(t)B\rangle &= \left\langle \frac{e^{-iEt}}{1 - \eta e^{-\beta E}}A|B\right\rangle_\eta . \end{aligned} \tag{5}$$

The set of basic relations (2) and (3—5) can be used for constructing various types of formulae corresponding to the tasks to be performed. In simpler cases, expressions in closed form, in more complicated cases, however, expressions containing an infinite series are obtained.

Let us assume that for the operator $A(t)$ the equation of motion is

$$[E(t) - \Omega]\, A(t) = 0\,, \tag{6}$$

in which Ω is time independent,

$$\frac{\partial}{\partial t}\Omega - i[H, \Omega] = 0\,.$$

Then, from (5) we get the expressions

$$\begin{aligned} \langle BA(t)\rangle &= \left\langle \frac{e^{-i\Omega t}}{e^{\beta\Omega} - \eta} A|B \right\rangle_\eta\,, \\ \langle A(t)\, B\rangle &= \left\langle \frac{e^{-i\Omega t}}{1 - \eta e^{-\beta\Omega}} A|B \right\rangle_\eta\,. \end{aligned} \tag{7}$$

In cases to which an equation of the form (6) cannot be applied, expansions must be performed. This can be done by making use of two general principles, namely renormalization and perturbation.

Using renormalization, the expansion becomes simpler, if one assumes the renormalized energy to be temperature dependent only. Let the renormalized energy be ε, then, by expanding (5) in $E - \varepsilon$, we obtain

$$\begin{aligned} \langle BA(t)\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \varepsilon^n} \frac{e^{-i\varepsilon t}}{e^{\beta\varepsilon} - \eta} \langle [E - \varepsilon]^n A|B\rangle_\eta\,, \\ \langle A(t)B\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial \varepsilon^u} \frac{e^{-i\varepsilon t}}{1 - \eta e^{-\beta\varepsilon}} \langle [E - \varepsilon]^n A|B\rangle_\eta\,. \end{aligned} \tag{8}$$

These expressions are useful mainly for the evaluation of one-time averages, like

$$\begin{aligned} \langle BA\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \varepsilon^n} \frac{1}{e^{\beta\varepsilon} - \eta} \langle [E - \varepsilon]^n A|B\rangle_\eta\,, \\ \langle AB\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \delta^n} \frac{1}{1 - \eta e^{-\beta\varepsilon}} \langle [E - \varepsilon]^n A|B\rangle_\eta\,. \end{aligned} \tag{9}$$

ε has not yet been defined. Let us choose it so that in (9) the terms with $n = 1$ be equal to zero, then we have

$$\varepsilon_0 = \frac{\langle EA|B\rangle_\eta}{\langle A|B\rangle_\eta}\,. \tag{10}$$

On the other hand, if the sum of the terms with $n = 1$ and 2 is set equal to zero we get,

$$\varepsilon_1 = \varepsilon_0 - \frac{1 - \sqrt{1 - (\varrho_0^2 - \varepsilon_0^2)\,\beta_0^2}}{\beta_0}\,, \tag{11}$$

where

$$\varrho_0^2 = \frac{\langle E^2\, B|B\rangle_\eta}{\langle A|B\rangle_\eta}\,,$$

$$\beta_0 = \beta\,\frac{e^{\beta\varepsilon_1} + \eta}{e^{\beta\varepsilon_1} - \eta}\,.$$

Let us now renormalize with time dependent energy $\varepsilon(t)$. Expanding (4) in $E - \varepsilon(t)$ we find

$$\begin{aligned}
\langle B\,A(t)\rangle &= \frac{1}{e^{\beta E(t)} - \eta}\left\{\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\,e^{-i\varepsilon(t)t}\,\langle[E - \varepsilon(t)]^n\,A|B\rangle_\eta\right\}, \\
\langle A(t)\,B\rangle &= \frac{1}{1 - \beta e^{-\beta E(t)}}\left\{\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\,e^{-i\varepsilon(t)t}\,\langle[E - \varepsilon(t)]^n\,A|B\rangle_\eta\right\}.
\end{aligned} \tag{12}$$

Taking the terms with $n = 1$ equal to zero reproduces expression (10). The other condition, that is assuming the terms with $n = 1$ and 2 to be zero, leads, however, to

$$\varepsilon_1(t) = \varepsilon_0 - i\gamma_0(t)\,, \qquad \gamma_0 = \frac{\sqrt{1 + (\varrho_0^2 - \varepsilon_0^2)\,t^2} - 1}{t}\,.$$

It is apparent that the complex $\varepsilon_1(t)$ describes damped oscillations.

Let us use now the principle of perturbation. We assume the Hamiltonian to have the form $H = H_0 + gH_1$ and we define

$$e^{iHt} = e^{iH_0 t}\,s(t)\,.$$

The function $s(t)$ can then be expanded in powers of g as

$$\begin{aligned}
s(t) &= \sum_{n=0}^{\infty} g^n\,s_n(t)\,, \qquad H_1(t) = e^{-iH_0 t}\,H_1\,e^{iH_0 t}\,, \\
s_n(t) &= \frac{(i)^n}{n!}\int_0^t \ldots \int_0^t dt_1 \ldots dt_n\,T\{H_1(t^1)\ldots H_1(t_n)\}
\end{aligned} \tag{13}$$

and T is the time ordering Feynman operator. Introducing this formula into (3), we get

$$\langle BA(t)\rangle = \frac{1}{e^{\beta E(t)} - \eta} \sum_{n=0}^{\infty} g^n \langle A_n(t)|B\rangle_n , \tag{14}$$

$$\langle A(t)\ B\rangle = \frac{1}{1 - \eta^{-\beta E(t)}} \sum_{n=0}^{\infty} g^n \langle A_n(t)|B\rangle_\eta \tag{15}$$

as a final result, where

$$A_n(t) = \sum_{m=0}^{n} e^{iH_0 t}\, s_m(t)\, A s^*_{n-m}(t)\, e^{-iH_0 t} .$$

These expressions of closed form as well as the series expansions can be applied to a variety of problems in statistical physics. The method has been applied already to the theory of ferromagnetism and a number of known results could be reproduced.

The advantage of the method seems to be the quick and mechanical way in which the results can be obtained.

НОВЫЙ МЕТОД ДЛЯ ВЫЧИСЛЕНИЯ СРЕДНИХ ТЕМПЕРАТУР

Е. ПРАВЕЦКИ

Р е з ю м е

Работа ознакомливает читателя с новым методом определения средних температур. Метод находится в тесной связи с методом функций Грина, так как он базируется на так называемом проекционном соотношении, которое можно считать решением проблемы функций Грина. На основе теории возмущений и принципа перенормировки, исходя из проекционного соотношения, можно вывести точные формулы и серии выражений.

DER WAHRSCHEINLICHKEITSOPERATOR ϱ BEI VIELTEILCHENPROBLEMEN

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1. Einleitung

Gegenstand dieser Untersuchung ist die Gesamtheit aller physikalischen Fragen hinsichtlich Systemen vom Typ

$$H = \Sigma_i t_i + \Sigma_{i<j} v_{ij} \,. \tag{1}$$

Für die durch derartige HAMILTONoperatoren charakterisierten nichtrelativistischen quantenmechanischen *Vielteilchenprobleme* existierten bereits 1930 alle Voraussetzungen zu deren Behandlung. Trotzdem erschienen in den nachfolgenden Jahren nur vereinzelte Untersuchungen, und erst seit etwa 1955 werden diese Probleme in grossem Umfang und mit wachsendem Erfolg angegriffen. Charakteristisch für diese neuere Entwicklung ist, dass viele Probleme mehrfach gelöst wurden mit scheinbar ganz verschiedenen Methoden, aber gleichen Ergebnissen. Um daher einen Überblick über die vorliegende Problematik und die bisherigen Erfolge zu gewinnen, ist es erforderlich, diese scheinbar so verschiedenen Methoden auf ihren gemeinsamen Ursprung zurückzuführen. Die genauere Untersuchung zeigt nun, dass viele Methoden lediglich durch die Verwendung unterschiedlicher und für jeden speziellen Zweck extra eingeführter charakteristischer Bestimmungsgrössen voneinander abweichen.

Hier soll nunmehr die *gesamte Problematik* so grundsätzlich wie nur irgend möglich angegriffen werden, weil es nur so möglich ist, zu den fundamentalsten charakteristischen Grössen vorzustossen, aus denen sich alle sonst verwendeten Bestimmungsgrössen dann als Spezialfälle ergeben. Ein solches Vorgehen führt automatisch zur Definition eines Wahrscheinlichkeitsoperators ϱ, aus dem die bekannten Dichtematrizen [1] [2] [3] als spezielle Darstellungen hervorgehen.

2. Die physikalische Fragestellung

Die *allgemeinste physikalische Frage*, die in der Theoretischen Physik an ein System gerichtet werden kann, ist folgendermassen formulierbar:

> Eine Messung am System hat zur Zeit 0 die Information »A« ergeben. Welche Information »B« würde eine Messung zur Zeit t ergeben? (1)

Das betrachtete physikalische System wird durch seinen HAMILTONoperator H charakterisiert. Eine durch Messung gewonnene Information besteht in der Angabe der gemessenen Grössen, der erzielten Messergebnisse und einer Angabe über den »Wert« der betreffenden Information. Zur Beantwortung der Frage (1) sind folgende Einzelprobleme zu lösen:

1. Analyse der Anfangsinformation »A«,
2. Beschreibung des Zeitablaufs,
3. Analyse der zur Fragestellung »B« gehörigen Information.

Im Rahmen der Quantentheorie gehören zu einer *Information* »A«, »B«, ... folgende Angaben: Es müssen die zu einer Information gehörigen, gleichzeitig und unabhängig voneinander messbaren Grössen als untereinander vertauschbare HILBERToperatoren

$$A_1, \ldots, A_F \to A \qquad B_1, \ldots, B_f \to B \quad f \leq F \tag{2}$$

bekannt sein. Übersichtshalber werden Indizes, wo immer möglich, fortgelassen. F ist die Zahl der Freiheitsgrade des betrachteten Systems, und im allgemeinen gilt $[A, B]_- \neq 0$. Ferner gehört zur Information die Angabe der erzielten Messergebnisse, die bei optimaler Messgenauigkeit übereinstimmt mit der Angabe der Messwerte

$$a_1, \ldots, a_F \to a \qquad b_1, \ldots, b_f \to b\,, \tag{3}$$

also der Eigenwerte der Operatoren (2). Sie charakterisieren die zum Messergebnis gehörigen Eigenzustände des Systems, die HILBERTvektoren Φ_a bzw. Φ_b. Liegen nur ungefähre Messergebnisse vor, so kann der Zustand nicht eindeutig bestimmt, sondern lediglich eine Wahrscheinlichkeit w_x für das Vorhandensein eines Zustands Φ_x angegeben werden, hier also die Gesamtheit aller w_a bzw. w_b. Dieser Kenntnis äquivalent ist ein Ensemble von M gleichartigen (numerierbaren) Systemen, von denen sich M_x in Zuständen Φ_x befinden. Dann ist $w_x = M_x/M$.

Der Wert »J« *einer* solchen *Information* ist natürlich am grössten bei genauer Kenntnis des Zustands Φ_x und am kleinsten, wenn alle w_x gleichgross

sind. Als Mass für den Unwert (Unkenntnis) kann angesehen werden: die Zahl der Gesamtzustände des Ensembles der numerierten Einzelsysteme, die zur gleichen Verteilung »M_x« gehören (bei vollständiger Kenntnis nur ein einziger!). Diese Zahl stimmt überein mit den möglichen Umnumerierungen der Systeme, die deren Einzelzustände verändert, aber die Gesamtverteilung »M_x« unverändert lässt. Diese Zahl ist bekanntlich

$$R = \frac{M!}{\Pi_x M_x!} \to \left(\prod_x \frac{1}{w_x^{w_x}} \right)^M \text{für } M \to \infty. \tag{4}$$

Somit ist R^{-1} ein Mass für den Informationswert. Das gleiche gilt für eine monotone Funktion von R^{-1}, die so gewählt wird, dass der Informationswert zweier Systeme sich addiert:

$$\bar{I} \equiv \frac{1}{M} \ln R^{-1} = \Sigma_x w_x \ln w_x = -S \leq 0. \tag{5}$$

Diese im Bereich $-\infty \ldots 0$ variierende Funktion stimmt bis auf einen unwesentlichen Dimensionsfaktor k mit der negativen Entropie überein.

Weiterhin sind *Besonderheiten zu beachten*, die im Zusammenhang mit Fragen vom Typ (1) auftreten können. So wird das Problem (2.1) zeitunabhängig, wenn eine der drei Bedingungen

$$t = 0, \qquad [H, A]_- = 0\,, \qquad [H, B]_- = 0 \tag{6}$$

erfüllt ist. Eine gewisse, wenngleich unwesentliche Komplikation tritt auf, wenn der HAMILTONoperator, wie in der Störungstheorie oftmals üblich, zeitabhängig ist:

$$\frac{\partial H}{\partial t} \neq 0\,, \quad \text{etwa} \quad H_t = H^0 + g_t V\,, \tag{7}$$

weil dann die H_t für verschiedene t wegen

$$[H_1, H_2]_- = (g_1 - g_2)\,[V, H^0]_- \neq 0 \tag{8}$$

nicht mehr miteinander vertauschbar sind. Gewisse Vereinfachungen treten auf, wenn in (2) $f < F$ ist, weil dann die gesuchte Information von vornherein von einfacherer Art ist. Dieser Fall wird in Kapitel 4 besprochen.

Nach den im Anschluss an (1) gemachten Ausführungen ist zunächst die *Analyse der Anfangszustände* durchzuführen. Bei vollständiger Kenntnis

des Messergebnisses a besteht sie in der Angabe des zugehörigen Zustands Φ_a, der sich aus dem Eigenwertproblem

$$A\Phi_a = \Phi_a\, a \tag{9}$$

berechnet. Bei ungenauer Messung müssen alle in Frage kommenden Φ_a mit den aus der Messung folgenden Wahrscheinlichkeiten w_a bestimmt werden. Ist nur ein Teil der Grössen A_i von (2) gemessen worden, so müssen die w_a durch geeignete Annahmen bestimmt werden, wie minimale Willkür, minimaler Informationswert, Gleichverteilung, maximal chaotische Verteilung, etc...

In der SCHRÖDINGERdarstellung besteht *der Zeitablauf* darin, dass alle Anfangszustände Φ_a übergehen in

$$\Phi_a \to \Psi_a = {}^t\Pi^0\, e^{-(i/\hbar)H_{t'}dt'}\,\Phi_a \equiv U(t)\,\Phi_a\,. \tag{10}$$

Im besonderen Falle zeitunabhängiger HAMILTONoperatoren vereinfacht sich (10) zu

$$\frac{\partial H}{\partial t} = 0\colon\quad {}^t\Pi^0\, e^{-\frac{i}{\hbar}H dt'} \to e^{-\frac{i}{\hbar}\int_0^t H\,dt'} = e^{-\frac{i}{\hbar}Ht}\,. \tag{11}$$

Diese vereinfachte Zeitabhängigkeit wird im folgenden verwendet. Treffen dagegen die Voraussetzungen (7) zu, so sind in allen nachfolgenden Formeln lediglich die Exponentialfaktoren durch die unendlichen Produkte von (11) zu ersetzen.

Die dritte Aufgabe besteht in der *Analyse der gefragten Information* »B«. Die zugehörigen möglichen Messwerte folgen aus der Eigenwertgleichung

$$B\Phi_b = \Phi_b\, b\,. \tag{12}$$

Im Falle $f < F$ müssen die B_i von (2) evtl. noch ergänzt werden durch Aufsuchen von Operatoren

$$C_{f+1}, \ldots, C_F \to C \qquad c_{f+1}, \ldots, c_F \to c, \tag{13}$$

die untereinander sowie mit den B_i vertauschbar sind und mit ihnen zusammen ein vollständiges System observabler Grössen bilden. Da die Grössen B und C als gleichzeitig und unabhängig voneinander messbar angesehen werden, bilden ihre Eigenzustände einen Produktraum

$$\Phi_{bc} = \Phi_b\,\Phi_c = \Phi_c\,\Phi_b \tag{14}$$

mit den Faktorräumen der Φ_b und der Φ_c.

Die *Antwort auf die Frage* (1) wird nunmehr durch Angabe der Wahrscheinlichkeiten w_b für die Messergebnisse b gegeben. Nach elementaren Regeln der Wahrscheinlichkeitstheorie gelten die Beziehungen

$$w_b = \Sigma_c \, w_{bc} \qquad w_{bc} = \Sigma_a \, w_{bc,a} \, w_a \,, \tag{15}$$

während die hierin auftretenden Übergangswahrscheinlichkeiten in der Quantentheorie durch

$$w_{bc,a} = | \, (\Phi_{bc}, \Psi_a) \, |^2 = (\Phi_{bc}, \Psi_a) \, (\Psi_a, \Phi_{bc}) \tag{16}$$

bestimmt werden. Gleichzeitig mit den w_b aus (15) können die zu den B gehörigen Erwartungswerte in der Form

$$\bar{B} = \Sigma_b \, b w_b = \Sigma_{bc} \, b w_{bc} \tag{17}$$

als bei vielen gleichartigen Messungen beobachtete statistische Mittelwerte bestimmt werden.

3. Der Wahrscheinlichkeitsoperator ϱ

Die in (2.15 und 16) auftretenden Wahrscheinlichkeiten können als Erwartungswerte von Operatoren dargestellt werden. *Bei vollständiger Information* »a« ist die Wahrscheinlichkeit, zur Zeit t die Information »b, c« zu erhalten, nach (2.16):

$$w_{bc\,a} = (\Phi_{bc}, \, \varrho_a \, \Phi_{bc}) \;\text{ mit }\; \varrho_a = \Psi_a) \, (\Psi_a \,. \tag{1}$$

ϱ_a wird allgemein als Wahrscheinlichkeitsoperator bezeichnet und bedeutet hier den zu Ψ_a gehörigen Projektionsoperator mit den Eigenwerten

$$\varrho_a \to 0{,}1 \qquad \varrho_a^2 = \varrho_a \,, \tag{2}$$

denen zufolge die daneben stehende charakteristische Gleichung gilt.

Die *zur unvollständigen Anfangsinformation* »A« *gehörige Wahrscheinlichkeit*, bei t die Information »b, c« zu erhalten, ist

$$w_{bc} = (\Phi_{bc}, \, \varrho \Phi_{bc}) \;\text{ mit }\; \varrho = \Sigma_a \, \varrho_a \, w_a \,. \tag{3}$$

Für den hier auftretenden allgemeineren Wahrscheinlichkeitsoperator gilt

$$\varrho = \Sigma_a \Psi_a) \, w_a (\, \Psi_a = \Sigma_a \, e^{-(i/\hbar)Ht} \, \Phi_a) \, w_a (\Phi_a \, e^{(i/\hbar)Ht} \,. \tag{4}$$

Er besitzt offenbar die Eigenschaften

$$\varrho\, \Psi_a = \Psi_a\, w_a, \quad \text{daher} \quad \Pi_a(\varrho - w_a) = 0\,. \tag{5}$$

Diese Gleichungen lassen seine Bedeutung unmittelbar erkennen: Seine Eigenvektoren sind die zur Information »A« gehörigen Vektoren Ψ_a und seine Eigenwerte die zur gleichen Information gehörigen Wahrscheinlichkeiten w_a.

Die Ausdrücke (1) und (4) lassen erkennen, dass die *Wahrscheinlichkeitsoperatoren* ϱ_a und ϱ hermitisch, normiert und positiv definit sind:

$$\varrho^\dagger = \varrho \qquad Tr\,\varrho = 1 \qquad 0 \leq \bar{\varrho} \leq 1\,. \tag{6}$$

Die mit $Tr\,\varrho$ bezeichnete Spur ist bekanntlich gleich der Summe der Eigenwerte, so dass die Normierung aus $\Sigma_a\, w_a = 1$ folgt. Die letzte Gleichung (6) folgt aus der Wahrscheinlichkeitsbedeutung aller Erwartungswerte. Für den Zeitablauf des Wahrscheinlichkeitsoperators kann die Beziehung

$$\dot{\varrho}(t) + \frac{i}{\hbar}\,[H, \varrho(t)]_- = 0 \qquad \varrho(0) = \text{»}A\text{«} \tag{7}$$

aus (4) entnommen werden. Da $\varrho(0)$ durch die Anfangsinformation »A« vollständig bestimmt ist, was durch die zweite Gleichung (7) symbolisch ausgedrückt werden soll, und die erste Gleichung von erster Ordnung hinsichtlich t ist, wird $\varrho(t)$ durch die Gleichungen (7) vollständig bestimmt.

Erwartungswerte von der Art (2.17) können in der Form

$$\bar{B} = \Sigma_{bc}(\Phi_{bc},\, \varrho B \Phi_{bc}) = Tr\,\varrho B = Tr\,B\varrho \tag{8}$$

durch Spurbildungen mit ϱ dargestellt werden. Innerhalb einer Spur über zwei Operatoren ist deren Reihenfolge bekanntlich beliebig.

Der in (2.5) eingeführte *Informationswert* kann *durch* den *Operator* $\ln\varrho$ *dargestellt* werden, denn sein Erwartungswert (8) stimmt entsprechend

$$\bar{I} = Tr\ \varrho \ln \varrho = \Sigma_a\, w_a \ln w_a \tag{9}$$

mit (2.5) überein. Man beachte, dass bei vollständiger Information

$$\bar{I}_a = Tr\ \varrho_a \ln \varrho_a = 0 \tag{10}$$

gilt. Für die Zeitableitungen der Erwartungswerte (8) und (9) gilt

$$\begin{aligned} \dot{\bar{B}} &= Tr\, B\dot{\varrho} = \frac{i}{\hbar}\, Tr\, B[\varrho, H]_- = \frac{i}{\hbar}\, Tr[H, B]_-\varrho \\ \dot{\bar{I}} &= \frac{i}{\hbar}\, Tr\,[\varrho \ln \varrho, H]_- = 0. \end{aligned} \tag{11}$$

Die erste Gleichung lässt erkennen, dass das Ergebnis das gleiche ist, welches man auch bei HEISENBERGdarstellung der Operatoren zu erwarten hätte. Die Zeitunabhängigkeit des mittleren Informationswerts folgt bereits daraus, dass der Zeitablauf durch eine unitäre Transformation beschrieben wird und die Spurbildung invariant gegen unitäre Transformationen ist.

Zusammenfassend ist festzustellen, dass durch die Einführung von Wahrscheinlichkeitsoperatoren ϱ die zu (2.1) gehörige *Problematik* folgendermassen *aufgeteilt* wird: In $\varrho(t)$ ist die Anfangsinformation und der Zeitablauf enthalten, während die spezielle zur Information »B, C« gehörige Fragestellung durch Bildung geeigneter Erwartungswerte berücksichtigt wird.

4. Reduzierte Wahrscheinlichkeiten ϱ_B

Wir berücksichtigen nunmehr den zu (2.12 und 13) gehörigen Fall $f < F$, in dem die Endinformation aus einem »interessierenden« Anteil B und einem »nicht interessierenden« Anteil C besteht. Dementsprechend enthält auch die Anfangsinformation »A« und der aus ihr konstruierte, in diesem Kapitel mit $\varrho \equiv \varrho_{BC}$ bezeichnete Wahrscheinlichkeitsoperator Anteile, die für die Information B wesentlich sind, und solche, die es nicht sind. Die zur Information B gehörigen Wahrscheinlichkeiten (2.15) können als Erwartungswerte im Faktorraum der Φ_b allein dargestellt werden

$$w_B = (\Phi_b, \varrho_B \Phi_b) , \tag{1}$$

während der hier definierte Operator ϱ_B *der reduzierten Wahrscheinlichkeit* entsprechend (2.15) gemäss

$$\varrho_B = \Sigma_c(\Phi_c, \varrho_{BC} \Phi_c) = Tr_C \varrho_{BC} \tag{2}$$

durch Spurbildung im Faktorraum der Φ_c aus ϱ_{BC} hervorgeht. Dieser Operator ist, je nach der speziellen Fragestellung »B«, oftmals entscheidend einfacher. Auch können entsprechende Erwartungsmerte wegen

$$\bar{B} = TrB\ \varrho \equiv Tr_B\, Tr_C\, B \varrho_{BC} = Tr_B B Tr_C\ \varrho_{BC} = Tr_B B \varrho_B \tag{3}$$

als Spuren über ϱ_B allein dargestellt werden.

Für den *Zeitablauf* allerdings folgt aus (2) und (3.7)

$$\dot{\varrho}_B = Tr_C \dot{\varrho}_{BC} = \frac{i}{\hbar} Tr_C [\varrho_{BC}, H]_- . \tag{4}$$

Den HAMILTONoperator des Systems denken wir uns gemäss

$$H = H_B + H_C + H_{BC} \tag{5}$$

zerlegt in Anteile, die nur von B bzw. C allein abhängen, und einen gemischten Anteil. H_B kann wegen Unabhängigkeit von C aus der Spurbildung (4) herausgezogen werden. Die Vertauschung mit H_C verschwindet unter der Spur, wie im Zusammenhang mit (3.8) erwähnt wurde. Daher geht (4) in

$$\dot{\varrho}_B + \frac{i}{\hbar}[H_B, \varrho_B]_- = \frac{i}{\hbar} Tr_C[\varrho_{BC},\ H_{BC}]_- \tag{6}$$

über. Der Zeitablauf von ϱ_B wird also durch den C-Anteil der Information mitbestimmt, wie anschaulich erwartet werden muss. Die zu ϱ_B gehörige reduzierte Information $J_B = \ln \varrho_B$ wird im Zeitablauf ebenfalls vom C-Anteil her beeinflusst; denn es gilt

$$\bar{I}_B = (i/\hbar)\ Tr_B \ln \varrho_B\ Tr_C[\varrho_{BC}, H_{BC}]_- \tag{7}$$

für die zeitliche Änderung ihres Erwartungswerts.

Enthält der HAMILTONoperator keinen gemischten Anteil H_{BC}, so werden die Teilinformationen B und C unabhängig. Die rechte Seite von (6) verschwindet, und für den hier nicht betrachteten Informationsanteil $\varrho_C = = Tr_B \varrho_{BC}$ lässt sich eine entsprechende, von ϱ_{BC} unabhängige Gleichung mit H_C statt H_B angeben. Der *Operator der* gesamten hier *unkorrelierten Wahrscheinlichkeit* kann als Produkt

$$\varrho_{BC} = \varrho_B\ \varrho_C \tag{8}$$

angesetzt werden. Auch in Fällen $H_{BC} \neq 0$ wird (8) oftmals als Näherungsansatz verwendet, der einer Vernachlässigung der Korrelation entspricht. In diesem Falle geht (6) in

$$\dot{\varrho}_B + \frac{i}{\hbar}[H_B + H_{B\bar{C}},\ \varrho_B] = 0 \text{ mit } H_{B\bar{C}} = Tr_C H_{BC}\, \varrho_C \tag{9}$$

über. $H_{B\bar{C}}$ bedeutet hier den über die Teilinformation C gemittelten Anteil der Wechselwirkungsenergie. Die durch (8) und (9) beschriebene Näherung entspricht im klassischen Falle der VLASOV-gleichung [4] und im quantenmechanischen Fall dem HARTREE—FOCKschen Näherungsverfahren [5]. Ausserdem verschwindet bei Vernachlässigung der Korrelation im Sinne von (8) die rechte Seite von (7), und auch die Teilinformation B bleibt näherungsweise zeitunabhängig.

Bei Anwendung dieser Betrachtungen auf Vielteilchenprobleme mit HAMILTONoperatoren vom Typ (1.1) besitzen alle physikalisch interessierenden Grössen entsprechend

$$B = \Sigma_i\, B_i + \Sigma_{i<j}\, B_{ij} \tag{10}$$

Teilchen- und Paareigenschaften. Dementsprechend sind unter der Information »B« die Eigenschaften »1« bzw. »1,2« von einem bzw. von zwei Teilchen zu verstehen:

$$\begin{aligned} B &= 1 \qquad & C &= 2, \ldots, N \\ B &= 1{,}2 \qquad & C &= 3, \ldots, N\,, \end{aligned} \tag{11}$$

während »C« die übrigen, jeweils nicht interessierenden Teilchen enthält. In diesem Sinne kann eine Hierarchie reduzierter Wahrscheinlichkeitsoperatoren

$$\varrho_B = \varrho_1, \varrho_{12}, \varrho_{123}, \cdots \tag{12}$$

aufgestellt werden, deren Zeitableitungen (6) ein gekoppeltes Gleichunssystem, die sogenannte BBGKY-Hierarchie [6], bilden, während die Erwartungswerte von (10) nur ϱ_1 und ϱ_{12} enthalten:

$$\Sigma_i 1 = N \qquad \Sigma_{i<j} 1 = \binom{N}{2}\,. \tag{13}$$

Die hier auftretenden binomischen Faktoren folgen aus den Summationen von (10) und bedeuten gemäss (13) die Teilchen- bzw. Paarzahl.

5. Untersuchung des Zeitablaufs

Wenn H in den zur Information A gehörigen Messgrössen A_i enthalten ist, so treffen die Vereinfachungen von (2.6) zu, und der *Wahrscheinlichkeitsoperator* wird *zeitunabhägig:*

$$\dot{\varrho} = 0 \qquad [H, \varrho]_- = 0\,. \tag{1}$$

Bei maximalem Wert der Information A besteht hier die Aufgabe in der Berechnung der Eigenwerte von H und den A_i, siehe auch (11 bis 13). Daneben kommen Lösungen von (1) mit geringerem Informationswert in Betracht. Die allgemeinste Lösung von (1) ist eine willkürliche Fuktion

$$\varrho = f(H, A_1, A_2, \ldots) \quad [H, A_i]_- = 0 \tag{2}$$

von H und den mit H und untereinander vertauschbaren Grössen. Liegt nur über H allein eine Anfangsinformation vor, so kommt als Lösung auch nur $\varrho = f(H)$ in Betracht, und es können alle Aussagen der Thermodynamik gewonnen werden. Spezialfälle sind die kanonische und die grosskanonische

Gesamtheit (letztere, wenn neben H auch die Teilchenzahl N in der Anfangsinformation enthalten ist):

$$\varrho = e^{\beta(F-H)} \qquad \varrho = e^{\beta(J-H-\zeta N)}. \tag{3}$$

β, F bzw. J, ζ sind Parameter zur Festlegung der Normierung sowie der mittleren Energie und Teilchenzahl.

Wenn H in den zur Anfangsinformation A gehörigen Grössen nicht enthalten ist, so ist $\dot{\varrho} \neq 0$, und der *Zeitablauf* wird durch (3.7) beschrieben. Eine gewisse *Vereinfachung* dieser Gleichung, die für viele Zwecke nützlich ist, erhalten wir durch den Ansatz

$$\varrho(t) = K(t, t) \quad \text{mit} \quad K^{\dagger}(t', t'') = K(t'', t') \tag{4}$$

mit einem Operator K, der die Gleichungen

$$\left(\frac{\partial}{\partial t'} + \frac{i}{\hbar} H\right) K(t', t'') = 0 = K(t', t'') \left(\frac{\overleftarrow{\partial}}{\partial t''} - \frac{i}{\hbar} H\right) \tag{5}$$

erfüllt. Jeweils eine dieser beiden Gleichungen kann durch die Hermitizitätsforderung (4) ersetzt werden. Analog zu (3.4) ist

$$K(t', t'') = e^{-\frac{i}{\hbar} Ht'} \varrho(0) e^{\frac{i}{\hbar} Ht''} \tag{6}$$

die formale Lösung der Gleichungen (4) und (5).

Eine andere Möglichkeit besteht darin, ϱ entsprechend

$$\varrho(t) = i\hbar G(t + 0,\ t - 0) \qquad G^{\dagger}(t', t'') = G(t'', t') \tag{7}$$

durch einen *Operator* $i\hbar G$ zu beschreiben, *der die inhomogenen Gleichungen*

$$\left(\frac{\partial}{\partial t'} + \frac{i}{\hbar} H\right) i\hbar\, G(t', t'') = \delta(t' - t'') = i\hbar G(t', t'') \left(-\frac{\overleftarrow{\partial}}{\partial t''} + \frac{i}{\hbar} H\right) \tag{8}$$

erfüllt. Der in der Definition (7) hinzugefügte Faktor $i\hbar$ vereinfacht die Hermitizitäts- und Normierungseigenschaften von G. Eine mögliche Lösung von (7) und (8), die für $t' > t''$ mit (6) übereinstimmt, also die Bedingung

$$\theta(t' - t'')\,(i\hbar G(t', t'') - K(t'\, t'')) = 0 \tag{9}$$

erfüllt, ist

$$i\hbar G(t', t'') = K(t', t'') - \theta(t'' - t')\, e^{-\frac{i}{\hbar} H(t'-t'')} \tag{10}$$

(man beachte beim Einsetzen von (10) in (8), dass für die Stufenfunktion $\dot{\vartheta} = \delta$ gilt). Derartige Lösungen inhomogener Gleichungen sind oftmals leichter zu handhaben als die von homogenen Gleichungen. Ein Beispiel hierfür ist die Behandlung stationärer Probleme mit $\dot{\varrho} = 0$ durch die zeitabhängige Theorie. Eine Lösung dieser Art von (8) erhalten wir durch formale Auflösung

$$G(t', t'') = G(t' - t'') = \left| -\frac{\hbar}{i} \frac{\partial}{\partial t'} - H \right|^{-1} \delta(t' - t'') \tag{11}$$

und FOURIER-darstellung der δ-Funktion

$$G(t', t'') = \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi\hbar} \frac{e^{-(i/\hbar)\varepsilon(t'-t'')}}{\varepsilon - H} . \tag{12}$$

Die FOURIERtransformierte von G ist die von HUGENHOLTZ [7] untersuchte Resolvente

$$R(\varepsilon) = \frac{1}{\varepsilon - H} = \sum_n \frac{\Phi_n)(\Phi_n}{\varepsilon - E_n} \tag{13}$$

des Operators H, der zusammen mit den Projektionsoperatoren $P_n = \Phi_n)(\Phi_n$ durch seine Eigenwerte E_n dargestellt werden kann.

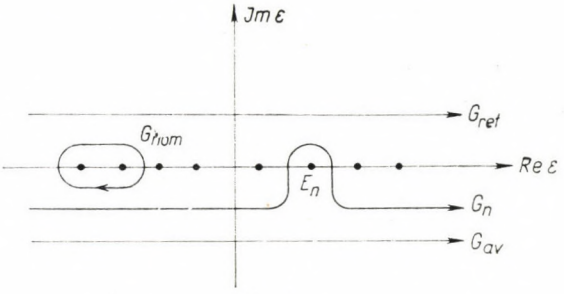

Abb. 1

Die beim Auflösen (11) der Differentialgleichung (8) entstehende Unbestimmtheit entspricht dem Hinzufügen willkürlicher Lösungen der zugehörigen homogenen Gleichung. Über deren Wahl jedoch verfügen wir eindeutig, indem wir in (12) die *Integration* über ε *in einer komplexen Ebene* durchführen und den genauen Weg angeben, siehe Abbildung.

Die Singularitäten des Integranden liegen auf der reellen Achse an den Orten $\varepsilon = E_n$ der Energieeigenwerte. Führen wir die Integration (12) oberhalb der reellen Achse durch, so entsteht die zu $\vartheta(t', t'')$ proportionale, retardierte Lösung G_{ret}. Bei einem Integrationsweg unterhalb der reellen Achse ents-

steht die zu $\theta(t'', t')$ proportionale avancierte Lösung G_{av}. Ein Integrationsweg, der nur bei einer einzigen Singularität E_n oberhalb, bei allen übrigen aber unterhalb verläuft, entspricht einer inhomogenen Lösung G_n, die nur hinsichtlich des Zustands Φ_n retardiert ist. Sie entspricht einem Anfangszustand Φ_n des Systems. Bei geringerem Informationswert, wenn also die Zustände Φ_n mit Wahrscheinlichkeiten w_n besetzt sind, wird ein Integrationsweg benötigt, der sich bei jeder Singularität aufspaltet in einen Teilweg oben herum mit dem Gewicht w_n und einen Anteil unten herum mit dem Anteil $1 - w_n$. G-Funktionen mit geschlossenen Integrationswegen, die bestimmte Singularitäten umfassen, lassen sich als Differenz zweier Lösungen der inhomogenen Gleichung auffassen und sind daher homogene Lösungen G_{hom}.

6. Störungsrechnung

Besteht der HAMILTONoperator $H = H^0 + V$ wie in (2.7) aus zwei Anteilen und sind alle zu H^0 allein gehörigen Aussagen vom Typ (2.1) bekannt, so kann das Problem im Rahmen einer Störungstheorie untersucht werden (*Potenzentwicklung nach V*). Die Erfahrung hat gezeigt, dass die zeitabhängige Form der Störungstheorie der zeitunabhängigen auch in den Fällen (2.6) überlegen ist, in denen sie nicht unbedingt notwendig wäre (der formale Grund besteht letzten Endes darin, dass sich mit Exponentialfunktionen leichter rechnen lässt als mit Partialbrüchen).

Eine störungstheoretische Behandlung kann die *inhomogene Gleichung* (5.8) *als Ausgangspunkt* wählen und neben G einen zu H^0 gehörigen Operator G_0 einführen. Zwischen beiden gilt dann die Integralgleichung

$$G(t, t') = G_0(t, t') + \int_{-\infty}^{+\infty} dt''\, G_0(t, t'')\, V(t'')\, G(t'', t')\,. \tag{1}$$

Ihre Richtigkeit wird durch Differenzieren und Vergleich mit (5.8) und der dazugehörigen ungestörten Gleichung gezeigt. Diese Integralgleichung kann durch eine Potenzreihe gelöst werden, die man dadurch erhält, dass man G auf der rechten Seite von (1) durch den Wert ersetzt, der aus der Gleichung (1) selbst folgt und dieses Verfahren sukzessiv fortsetzt.

Andere Wege der störungstheoretischen Behandlung beginnen damit, alle Operatoren entsprechend

$$Sp\varrho A = Sp\, \tilde{A}\tilde{\varrho}\,, \quad \tilde{A} = e^{(i/\hbar)H^0 t} A e^{-(i/\hbar)H^0 t}, \quad \tilde{\varrho} = e^{(i/\hbar)H^0 t} \varrho e^{-(i/\hbar)H^0 t} \tag{2}$$

von der SCHRÖDINGERdarstellung in die *Wechselwirkungsdarstellung* zu überführen. Sie besitzen dann die durch

$$\dot{\tilde{A}} = \frac{i}{\hbar}[H^0, \tilde{A}]_- \qquad \dot{\tilde{\varrho}} + \frac{i}{\hbar}[\tilde{V}, \tilde{\varrho}]_- = 0 \tag{3}$$

charakterisierten Zeitabhängigkeiten. Für den in (5.4) eingeführten K-Operator entstehen die Beziehungen

$$\tilde{\varrho}(t) = \tilde{K}(t, t) \qquad \tilde{K}^{\dagger}(t', t'') = \tilde{K}(t'', t')$$

$$\left(\frac{\partial}{\partial t'} + \frac{i}{\hbar}\tilde{V}'\right)\tilde{K}(t', t'') = 0 = \tilde{K}(t', t'')\left(\frac{\overleftarrow{\partial}}{\partial t''} - \frac{i}{\hbar}\tilde{V}''\right) \tag{4}$$

$$\tilde{K}(t', t'') = {}^{t'}\Pi^{0}\, e^{-(i/\hbar)\tilde{V}(t)dt}\, \varrho_0\, {}^{0}\Pi^{t''}\, e^{(i/\hbar)\tilde{V}(t)dt} = S(t')\, \varrho_0\, S^{\dagger}(t'')\,.$$

Die letzte Zeile lässt den Zusammenhang mit der bekannten S-Matrix erkennen.

7. Näherungsmethoden

Die zur Lösung des Problemkreises verwendeten Näherungsmethoden sind sehr vielseitig. Sie können hier nur in grossen Zügen dargelegt werden. Eine der Hauptaufgaben besteht darin, die *Gleichung* (4.6) *für den reduzierten Operator* ϱ_B durch eine von ϱ_{BC} unabhängige Näherungsgleichung zu ersetzen. Hinsichtlich der im Zusammenhang mit (4.12) erwähnten BBGKY-Hierarchie bedeutet dies, die Gleichungshierarchie bei irgendeinem gewünschten $\varrho_{12\cdots n}$ abzubrechen. Die einfachste Näherung besteht, wie in (4.8 und 9) bereits ausgeführt wurde, in der Vernachlässigung der Korrelation.

Bei stationären Problemen kann die Extremaleigenschaft der Energie, oder bei geringerem Informationswert der freien Energie, zu einem *Variationsverfahren* ausgebaut werden. Bei der Variation über ϱ müssen dann allerdings die Eigenschaften (3.6) dieses Operators im Ansatz oder durch geeignete Nebenbedingungen erfüllt werden.

Die aus der *Störungstheorie* entspringenden Möglichkeiten für Näherungsverfahren bestehen im Abbrechen der Potenzreihen [8] sowie in der Durchführung von Teilsummationen über einzelne Glieder der Potenzreihen [9]. Schliesslich bleiben noch solche Näherungsmethoden zu erwähnen, die im Rahmen der Operatorendarstellung nicht unmittelbar dargestellt werden können, sondern auf die zu den Operatoren ϱ und ϱ_B gehörigen Matrixdarstellungen aufbauen.

8. Matrixdarstellung von ϱ und ϱ_B

Eine *Matrixdarstellung* der Theorie entsteht durch beiderseitige Projektion aller Operatoren auf die als vollständig, normiert und orthogonal vorausgesetzten Vektoren Φ_x eines Systems von Zuständen x. Insbesondere der Wahrscheinlichkeitsoperator ϱ geht hierbei in die üblicherweise als Dichtematrix bezeichnete Grösse

$$\varrho(x, x') = (\Phi_x, \varrho\, \Phi_{x'}) \quad \text{mit} \quad (\Phi_x, \Phi_{x'}) = \delta_{xx'} \tag{1}$$

über. Alle Gleichungen der bisherigen Theorie bleiben formal gültig als Matrixgleichungen für die entsprechenden Matrixelemente.

Von hier aus erhalten wir durch FOURIERtransformation der Differenzkoordinaten die sogenannte WIGNER-darstellung der *Dichtematrix*[10]

$$\varrho\{p, x\} = \int \frac{dx'}{(2\pi\hbar)^F} e^{\frac{i}{\hbar} px'} \varrho\left(x - \frac{x'}{2}, \quad x + \frac{x'}{2}\right). \tag{2}$$

Damit entsteht eine der klassischen Statistik sehr ähnliche Form der Theorie [11]; denn für viele Erwartungswerte gilt

$$\bar{A} = \int\int dx\, dp\, A\{p, x\}\, \varrho\{p, x\} \quad \text{mit} \quad \int\int dx\, dp\, \varrho\{p, x\} = 1\,. \tag{3}$$

Die Gleichung für den Zeitablauf jedoch ist komplizierter:

$$\dot{\varrho}\{p, x\} = -\frac{2}{\hbar} \sin\left[\frac{\hbar}{2}\left(\frac{\partial}{\partial p}\frac{\partial}{\partial x'} - \frac{\partial}{\partial x}\frac{\partial}{\partial p'}\right)\right] H\{p,x\}\, \varrho\{p, x\}\,|_{p'=p, x'=x}. \tag{4}$$

Sie stimmt nur in erster Ordnung von $\hbar$ mit der klassischen, LIOUVILLEschen Gleichung überein. Ausserdem ist zu erwähnen, dass die Lösungen von (2) bzw. (4) in der Regel nicht positiv definit sind, weswegen auch $\varrho(p, x)dx\, dp$ nicht die einfache Bedeutung der klassischen Statistik haben kann, nämlich die Wahrscheinlichkeit dafür anzugeben, dass das System gewisse Ortseigenschaften x und Impulseigenschaften p besitzt.

Nunmehr betrachten wir den Spezialfall von physikalischen Systemen, bei denen *alle Vektoren* Φ_x sich mittels irgendwelcher Operatoren $a_x^\dagger$ entsprechend

$$\Phi_x = a_x^\dagger \Phi_0 \quad \text{mit} \quad a_x \Phi_0 = 0 \quad H\Phi_0 = 0 \tag{5}$$

durch einen einzigen normierten *Hilbertvektor* Φ_0 mit den Eigenschaften (5) *darstellen* lassen. Φ_0 wird dann als »Vakuum«-Vektor bezeichnet. Die letzte der Eigenschaften (5) kann durch Umnormierung des Hamiltonoperators H (Hinzufügen einer geeigneten Konstante) erreicht werden. Damit die Zustände (5) ein normiertes Orthogonalsystem bilden, müssen die Operatoren a_x die Eigenschaften

$$(\Phi_x, \Phi_{x'}) = (\Phi_0, a_x a_{x'}^\dagger \Phi_0) = \delta_{xx'} \tag{6}$$

besitzen. Etwas weiter als (6) geht die Forderung

$$(a_x a_{x'}^\dagger - \delta_{xx'})\Phi_0 = 0 \tag{7}$$

an die Operatoren a_x. Sie hat zum Beispiel die Gültigkeit der Gleichungen

$$P_0 a_x \Psi_a(t') = a_x \Psi_a(t') \quad H a_x \Psi(t') = 0 \tag{8}$$

mit $P_0 = \Phi_0)(\Phi_0$ als dem Projektionsoperator des Vakuums zur Folge, wenn $\Psi_a(t')$ irgendein durch Linearkombinationen der Φ_x aufgebauter Vektor

$$\Psi_a(t') = \int dx \Phi_x f(x, t') = \int dx a_x^\dagger \Phi_0 f(x, t') \tag{9}$$

ist. Der Beweis erfolgt durch Einsetzen von (9) in (8), Verwendung von (7) und Beachtung von (5).

Bei Systemen mit den Eigenschaften (5) und (7) können die *Matrixelemente des K-Operators* von (5.4) in eine übersichtliche Form gebracht werden. Zunächst gilt infolge (5.6) und analog zu (1)

$$\begin{aligned} K(x't', x''t'') &= \Sigma_a(\Phi_x, \Psi_a(t'))w_a(\Psi_a(t''), \Phi_{x'}) = \\ &= \Sigma_a w_a(\Phi_a, e^{(i/\hbar)Ht''} a_{x'}^\dagger \Phi_0)\ (\Phi_0 a_x e^{-(i/\hbar)Ht'} \Phi_a)\ . \end{aligned} \tag{10}$$

Wegen (8) kann im Innern dieses Ausdrucks ohne Fehler

$$\Phi_0)\ (\Phi_0 \to 1 \to e^{-(i/\hbar)H(t''-t')} \tag{11}$$

ersetzt werden. Bezeichnen wir Erwartungswerte eines Operators O über ein Ensemble von Systemen mit

$$\ll O \gg = \Sigma_a w_a(\Phi_a O \Phi_a) \tag{12}$$

und führen mit

$$a(x, t) = e^{(i/\hbar)Ht} a_x e^{-(i/\hbar)Ht} \quad a^\dagger(x, t) = e^{(i/\hbar)Ht} a_x^+ e^{-(i/\hbar)Ht} \tag{13}$$

die HEISENBERGdarstellung der Operatoren a_x ein, so erhält (10) die übersichtliche Form

$$\begin{aligned} K(x't', x''i'') &= \ll a^\dagger(x'', t'')a(x't') \gg \\ \varrho(x', x'', t) &= \ll a^\dagger(x'', t)a(x', t) \gg\ . \end{aligned} \tag{14}$$

In der zweiten Zeile stehen die aus K mit $t' = t'' = t$ hervorgehenden Matrixelemente von ϱ. Bei vielen Untersuchungsmethoden werden Matrixdarstellungen in der Form (14) benutzt. In anderen Methoden aber werden diese Matrizen durch solche ersetzt, die inhomogenen Gleichungen genügen und nur für $t' > t''$ mit den Grössen (14) übereinstimmen. Dann entstehen die bekannten Greenschen Funktionen.

9. Felddarstellung des Vielteilchenproblems

Ein besonderer Fall, auf den die besprochenen Voraussetzungen (8.5 und 7) zutreffen, ist die *Felddarstellung* des Vielteilchenproblems, beschrieben durch einen HAMILTONoperator

$$H = \int d\mathfrak{r}_1 \, \psi_1^\dagger \; t_1 \, \psi_1 + \int \frac{d\mathfrak{r}_1 \, d\mathfrak{r}_2}{2!} \, \psi_1^\dagger \, \psi_2^\dagger \, v_{12} \, \psi_2 \psi_1 ,$$

$$[\psi_1, \psi_2^\dagger]_\mp = \delta(1,2) \qquad [\psi_1, \psi_2]_\mp = [\psi_1^\dagger, \psi_2^\dagger]_\mp = 0 , \tag{1}$$

dessen Feldoperatoren ψ den angegebenen Minus- oder Plusvertauschungen genügen. Mit der $d\mathfrak{r}_1$, $d\mathfrak{r}_2$, ... sind Summationen über alle Orts- (und gegebenenfalls auch Spin-) Eigenschaften zu verstehen. Die in (8.5 und 7) eingeführten Operatoren sind hier

$$a_x^\dagger = \psi_1^\dagger \ldots \psi_N^\dagger \qquad a_x = \psi_N \ldots \psi_1 , \tag{2}$$

während δ-Funktionen und Summationen über Teilchenzustände im Detail bedeuten:

$$\delta_{x,\,x'} = \sum_P (\pm 1)^P \, \Pi_i \, \delta(i, P_i) \qquad \int dx = \int \frac{d\mathfrak{r}_1 \ldots d\mathfrak{r}_N}{N!} . \tag{3}$$

Der Normierungsfaktor $N!$ rührt daher, dass in Wirklichkeit keine unterscheidbaren Teilchen vorliegen, sondern blosse »Ja—Nein«-Ereignisse, so dass jeweils über die Zahl der Paare, Tripel, ... n-tupel summiert werden muss.

Mit H und den angegebenen Vertauschungen der ψ kann für die Zeitabhängigkeit von ψ_1 (in der HEISENBERGdarstellung) leicht eine *Wellengleichung* abgeleitet werden:

$$-\frac{\hbar}{i} \frac{\partial \psi_1}{\partial t} = [\psi_1, H]_- = (t_1 + \int d\mathfrak{r}_2 \, \psi_2^\dagger \, v_{12} \, \psi_2) \, \psi_1 . \tag{4}$$

Mit (1) und (2) entsteht für die Matrixdarstellung des HAMILTONoperators

$$\langle a_x H a_{x_1}^\dagger \rangle = (\Sigma_i \, t_i + \Sigma_{i<j} \, v_{ij}) \, \delta_{xx'} . \tag{5}$$

Sie stimmt bis auf die unwesentliche δ-Funktion (3) mit dem HAMILTONoperator (1.1) der Teilchentheorie überein. Die Gleichungen (4) und (5) lassen übrigens in einfachster Weise die Äquivalenz von Wellen und (ununterscheidbaren !) Teilchen innerhalb der Quantentheorie erkennen. Die Elemente der Dichtematrix folgen mit (2) aus (8.14):

$$\varrho(x, x', t) = \ll \psi_{1'}^\dagger \ldots \psi_{N'}^\dagger \psi_N \ldots \psi_1 \gg . \tag{6}$$

Zur Darstellung reduzierter Dichten unterteilen wir die Variabeln in zwei Gruppen

$$B = 1, \ldots, n \qquad C = n+1, \ldots, N , \tag{7}$$

die den Teilinformationen »B« und »C« entsprechen. So entstehen durch Spurbildung über C die mit (6) formal fast übereinstimmenden reduzierten Dichten

$$\varrho_n(x, x', t) = \ll\psi_1^\dagger, \ldots \psi_n^\dagger \psi_n \ldots \psi_1 \gg . \tag{8}$$

Allerdings müssen bei der Aufteilung der Spurbildung im Sinne von $Tr = Tr_B Tr_C$ die Normierungsfaktoren beachtet werden. Bildet man nämlich die Teilspuren ebenfalls im Sinne von (3), so bleiben entsprechend

$$\int \frac{d\mathfrak{x}_1 \ldots d\mathfrak{x}_N}{N!} = \int \frac{d\mathfrak{x}_1 \ldots d\mathfrak{x}_n}{n!} \int \frac{d\mathfrak{x}_{n+1} \ldots d\mathfrak{x}_N}{(N-n)!} \frac{(N-n)!\, n!}{N!} \tag{9}$$

noch Binomialkoeffizienten unberücksichtigt, die sich bei der jeweils zweiten Spurbildung von (8) in der Form

$$Tr_n \varrho_n = \frac{N!}{(N-n)!\, n!} \equiv \binom{N}{n} \tag{10}$$

bemerkbar machen. Da (10) die Zahl der unter N Teilchen vorhandenen n-tupel bedeutet, können die Diagonalelemente von (8) vom Teilchenbegriff her gesehen als n-tupeldichten aufgefasst werden.

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ОПЕРАТОР ВЕРОЯТНОСТИ ϱ В ПРОБЛЕМЕ МНОГИХ ЧАСТИЦ

В. МАККЕ

Р е з ю м е

Содержание: 1. Введение, 2. Постановка вопроса. 3. Оператор вероятности ϱ. 4. Приведённые вероятности ϱ_B. 5. Исследование временных процессов. 6. Теория возмущений. 7. Приближённые методы. 8. Матричное представление ϱ и ϱ_B. 9. Полевое представление многих частиц.

ON THE "BEST" APPROXIMATION IN QUANTUM MECHANICS

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In quantum mechanical approximations the difficulty generally arises that the error of calculation cannot be estimated uniquely. By using matrix calculus and the rules of trace calculus the paper facilitates the error of approximation to be defined uniquely for any quantum mechanical calculation (in the framework of matrix formalism). This also makes possible to find a more rapidly convergent method to carry out the calculations. The method is very suitable for use with computers as it contains direct operations only.

The curious title of this report concerns the fairly wide field of approximations in quantum mechanics, where *some quantity* or *some quantities* are to be approximated in the procedure of the approximation in a considerably more exact way than the other ones. Generally some deviation quantities are to be minimized. In *this* sense we consider this problem as the problem of the "best" approximation. To some extent the solution of any variational problem is a "best" approximation too, thus the following general treatment is an extension of the variational procedure in a certain respect.

In the approximating methods of quantum mechanics there is a general drawback, that usually in any approximating stage one does not know the measure of approximation; one does not know, how large is the deviation from the required quantities. In most cases one estimates the measure of the approximation only by comparison of the obtained value with the experimental one. In this case, however, one cannot conclude, whether the mathematical approximation had a certain error, or the physical approximation has neglected some physical facts.

The lack of a measure of the approximations accompanied constantly the calculations in quantum mechanics. Therefore it is worth while to consider the way, how a measure may be constructed for the approximations in general.

If such a measure is obtained one may judge at any step in an approximation the amount of deviation from the exact values, irrespective of the physical approximation, or of the final experimental values. If such a quantity is in our possession, a thorough investigation of this one shows at the same time the best way how to make a most successful next approximation. This is the sense in which we have called this method the "best" approximation method

in quantum mechanics. The detailed description of the method is appearing elsewhere [1] and here we give only a sketch of the features of the procedure.

The question arises how to define the measure of an error. In the variety of real numbers it is very easily defined as for instance

$$d = |b - b_0|, \tag{1}$$

or

$$d = (b - b_0)^2, \tag{2}$$

where b_0 is to be approximated by b. This definition has three essential properties:

1. d is 0, if the approximation is the exact one;
2. d is 0 only in this case;
3. d is always a real non-negative number.

The reality is necessary to make possible the larger-smaller comparison. The non-negativity is necessary in order not to find difficulties near error 0, comparing approximations with positive and negative deviations.

Naturally (1) and (2) are not the only possibilities having properties 1.—3. For instance any

$$d = (b - b_0)^k, \tag{3}$$

where k is an even integer is apt too. However, the choices (1) and (2) are usually used for the sake of simplicity.

Now let us formulate the general exact problem of quantum mechanics in the *density matrix formalism*. The time-independent Schrödinger equation, to which we shall now restrict ourselves, may be written

$$\mathbf{HR} - \mathbf{RH} = \mathbf{0}, \tag{4}$$

where $\mathbf{H}$ is the Hamiltonian matrix and $\mathbf{R}$ is the n-electron density matrix (see e.g. [2], [3]). One can easily show that the n-electron density matrix is *idempotent* (see [3]) i.e.

$$\mathbf{R}^2 - \mathbf{R} = \mathbf{0}. \tag{5}$$

Finally in the case when m separate states of the system must be taken into account in the approximation we have

$$\mathrm{tr}\,\mathbf{R} - m = 0, \tag{6}$$

where $tr\mathbf{R}$ is the trace of matrix $\mathbf{R}$.

One may see that equations (4) and (5) are matrix equations, whereas (6) is a scalar equation. The former ones suggest that we shall look for a measure of the deviation in the matrix form.

A very convenient form of the kind needed is the trace of the square of a matrix (or generally of a product of itself and of its conjugate transposed). Namely, if $\mathbf{D}$ is any squared real matrix (we restrict ourselves for the sake of simplicity of the presentation to squared real commutable matrices, but the extension is obvious, e.g. see [1]), then

$$d = tr(\mathbf{D}^2) \tag{7}$$

has the properties that

1. d is 0, if $\mathbf{D} = \mathbf{0}$;
2. d is 0 only in this case;
3. d is always a real non-negative number.

One may see these properties easily if one takes into account that $tr(\mathbf{D}^2)$ is the sum of the squares of all elements of $\mathbf{D}$.

Now we may define as the measure of the approximation of some matrix quantity the *trace of the square of the deviation*. For example, if we approximate a density matrix $\mathbf{R}_0$ by $\mathbf{R}$, the measure of the approximation will be represented by three non-negative numbers corresponding to the three equations (4)—(6),

$$d_1 = tr\{[(\mathbf{HR} - \mathbf{RH}) - (\mathbf{HR}_0 - \mathbf{R}_0\mathbf{H})]^2\}\,, \tag{8}$$

$$d_2 = tr\{[(\mathbf{R}^2 - \mathbf{R}) - (\mathbf{R}_0^2 - \mathbf{R}_0)]^2\}\,, \tag{9}$$

$$d_3 = [(tr\ \mathbf{R} - m) - (tr\ \mathbf{R}_0 - m)]^2\,. \tag{10}$$

Now, since $\mathbf{R}_0$ satisfies the exact equations (4)—(6), we have

$$d_1 = tr\ [(\mathbf{HR} - \mathbf{RH})^2]\,, \tag{11}$$

$$d_2 = tr\ [(\mathbf{R}^2 - \mathbf{R})^2]\,, \tag{12}$$

$$d_3 = (tr\ \mathbf{R} - m)^2\,. \tag{13}$$

These represent the measures of the approximation. In the case of exactness all three of them are 0.

Now let us make the following change. With a given $\mathbf{R}$ the values d_1, d_2 and d_3 are generally positive numbers, e.g. different from 0. We have to change $\mathbf{R}$ by a matrix $\delta\mathbf{R}$ to obtain a better approximation, i.e. one for which d_1, d_2 and d_3 are smaller positive (or 0) numbers than those before. This may be achieved in the usual way by constructing the variations of d_1, d_2 and d_3. By taking into account only first order terms in $\delta\mathbf{R}$ we obtain

$$\delta d_1 = 2tr\ (\mathbf{B}\delta\mathbf{R})\,, \tag{14}$$

$$\delta d_2 = 2tr\ (\mathbf{b}\delta\mathbf{R})\,, \tag{15}$$

$$\delta d_3 = 2tr\ [(tr\ \mathbf{R} - m)\delta\mathbf{R}]\,, \tag{16}$$

where **B** and **b** are simple matrix expressions of **R** and **H** given by equations

$$\mathbf{A} = \mathbf{HR} - \mathbf{RH}\,, \tag{17}$$

$$\mathbf{B} = \mathbf{AH} - \mathbf{HA}\,, \tag{18}$$

$$\mathbf{a} = \mathbf{R}^2 - \mathbf{R}\,, \tag{19}$$

$$\mathbf{b} = 2\mathbf{Ra} - \mathbf{a}\,. \tag{20}$$

In the evaluation of these quantities we have used several well known rules of the trace forming procedure (see e.g. [1]).

Now we can prove a very important lemma. According to this, if **M** and **N** are two matrices, which have fixed deviations from the matrix **0**, then $tr\,(\mathbf{MN})$ reaches its least value if

$$\mathbf{M} = -\lambda\mathbf{N}\,, \tag{21}$$

where λ is a constant. (For the proof see [1]).

Using this lemma one obtains for the minima of δd_1, δd_2 and δd_3 in (14)—(16) the relations

$$\delta\mathbf{R} = -\lambda\mathbf{B}\,, \tag{22}$$

$$\delta\mathbf{R} = -\mu\mathbf{b}\,, \tag{23}$$

$$\delta\mathbf{R} = -\nu(tr\ \mathbf{R} - m)\mathbf{I}\,, \tag{24}$$

respectively, where λ, μ and ν are constants and **I** is the unity matrix of the same order as **R**.

Since d_1, d_2 and d_3 are non-negative numbers, the best approximation is obtained if δd_1, δd_2 and $d\delta_3$ have the largest possible negative values, i.e. their minima. However, the three choices for $\delta\mathbf{R}$ given in (22)—(24), supplying these minima for the respective quantities, do not generally coincide. Therefore we have chosen in our method instead of them

$$\delta\mathbf{R} = -\lambda\mathbf{B} - \mu\mathbf{b} - \nu(tr\ \mathbf{R} - m)\mathbf{I}\,. \tag{25}$$

Here λ, μ and ν are to be determined from the simultaneous minimization of δd_1, δd_2 and δd_3 *up to the second order* in $\delta\mathbf{R}$ (or in a more exact way up to the fourth order).

The whole procedure consists of direct matrix operations only (matrix multiplication, sum and trace forming) and of the solution of a system of linear equations for the coefficients λ, μ, ν.

Sometimes in the usual approximations some of the equations (e.g. (5)) are taken as fulfilled exactly. However, the analysis has shown that sometimes

even in this case it is more convenient to allow matrices **R**, which do not exactly fulfil (5) but, instead of this, the other equations are fulfilled in a better approximation. The situation is somewhat sketched in the following way.

If we plot the map of the geometric structure given by equations (4), (5) and (6) in the space of the matrix **R**, these supply three hypersurfaces in this space. For the sake of simplicity let us represent **R** in a two-dimensional space and let us investigate only two of equations (4)—(6). The left hand sides

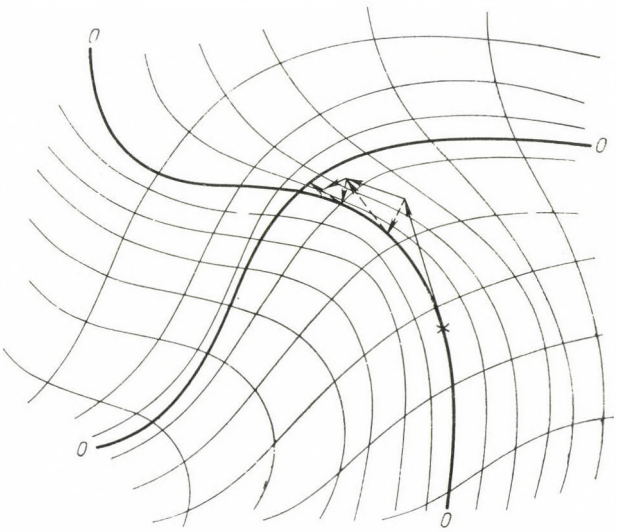

Fig. 1

of the investigated two of equations (4)—(6) give values different from 0 for some **R**. In this way we may construct the level-curves for these equations, i.e. curves connecting points in the **R** space giving fixed values for the investigated left hand sides. One of these level-curves belongs to 0 for each equation. These are represented by the two sets of curves for two equations in Fig. 1. The thick curves are the 0-curves. The exact solution of the problem corresponds to the intersection of the two thick lines of the two 0-curves.

If one of the equations is to be fulfilled exactly in every step, one has to make sure in every step to get back again to the respective 0-curve. This is represented in Fig. 1 by starting from a point of one of the 0-curves and proceeding in the first step along the thin continuous arrow, but afterwards along the thin dashed arrows, in each second step returning to the 0-curve again. However, if we disregard after the first step the exact fulfilment of the respective equation, but consider it as an equation to be approximated stepwise, we see that the approximation of the crosspoint of the two 0-curves

is more smoothly performed than in the former process. (See the continuous arrows in Fig. 1). This approximation removes sometimes one of the approximations already obtained to more distant level curves, i.e. the error of the corresponding equation increases, but at this cost the error of the other one decreases. The question is whether the overall error is thus decreased or not.

This question can be answered only if we define in some way an overall error quantity. Unfortunately, there are many ways to define such a quantity and it depends entirely on personal taste how to choose the decisive overall error quantity. All of the choices are, however, uniform in the fact that they supply 0 in the case, — and only in this case, — when the approximation is an exact one.

In the restricted case given above we may choose for instance

$$d_0 = d_1 + d_2 + d_3 \tag{26}$$

as the overall error quantity (d_1, d_2 and d_3 given by (11)—(13)), or in another way we may choose

$$d_0 = d_1^2 + d_2^2 + d_3^2 . \tag{27}$$

The latter one has the advantage that using this form the weights of the corrections of the errors are proportional to the respective errors in question.

The introduction of the trace calculus for normal, for inner product and for general matrices, helps the manipulations in the analysis a great deal. In the Appendix we summarize some of the rules occurring in this trace calculus.

In [1] the explicit forms are given for a special case, the one electron approximation, for the n-electron density matrices, making full use of this trace calculus.

Let us summarize here the end results of the method given in [1] for a system of n electrons of which n_1 are unpaired, the others paired, having the n-electron Hamiltonian matrix $\mathbf{H}$, taking into account the Pauli principle.

Let us denote the total density matrix (in the three dimensional space) by ϱ and that of the unpaired electrons by ϱ_u. The exact equations for the system are then

$$\varrho_u^2 - \varrho_u = \mathbf{0} , \tag{28}$$

$$\varrho^2 - 2\varrho + \varrho_u = \mathbf{0} , \tag{29}$$

$$\mathbf{H}\varrho_u - \varrho_u\mathbf{H} = \mathbf{0} , \tag{30}$$

$$\mathbf{H}\varrho - \varrho\mathbf{H} = \mathbf{0} , \tag{31}$$

$$tr\, \varrho_u - n_1 = 0 , \tag{32}$$

$$tr\, \varrho - n = 0 . \tag{33}$$

The corresponding deviations $d_1, \ldots, d_6$ can be constructed in a way similar to the generation of equations (11)—(13) from equations (4)—(6). Performing the analysis sketched above, the corrections for the initial approximative matrices ϱ_0 an ϱ_{u0} will be

$$\delta\varrho = -\lambda_1\mathbf{a}_1 - \lambda_2\mathbf{a}_2 - \lambda_3\mathbf{I} \tag{34}$$

and

$$\delta\varrho_u = -\mu_1\mathbf{a}_3 - \mu_2\mathbf{a}_4 - \mu_3\mathbf{a}_5 - \mu_4\mathbf{I}\,, \tag{35}$$

where $\mathbf{I}$ is the unit matrix of the same order as ϱ and ϱ_μ, further

$$\mathbf{a}_3 = (\varrho_{u0}^2 - \varrho_{u0})\,(2\varrho_{u0} - \mathbf{I})\,, \tag{36}$$

$$\mathbf{a}_4 = \varrho_0^2 - 2\varrho_0 + \varrho_{u0}\,, \tag{37}$$

$$\mathbf{a}_5 = \mathbf{H}^2\varrho_{u0} + \varrho_{u0}\mathbf{H}^2 - 2\mathbf{H}\varrho_{u0}\mathbf{H}\,, \tag{38}$$

$$\mathbf{a}_1 = 2\mathbf{a}_4(\varrho_0 - \mathbf{I})\,, \tag{39}$$

$$\mathbf{a}_2 = \mathbf{H}^2\varrho_0 + \varrho_0\mathbf{H}^2 - 2\mathbf{H}\varrho_0\mathbf{H}\,. \tag{40}$$

All these formulas are valid only in the case when the matrices ϱ_0 and ϱ_{u0} are symmetric and commutable, a restriction which can easily be removed in the general case (see [1]).

The seven constants λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 and μ_4 are to be determined by minimizing

$$d_0 = d_1^2 + d_2^2 + \ldots + d_6^2 \tag{41}$$

according to these seven parameters, where ϱ is replaced by $\varrho_0 + \delta\varrho$ and ϱ_u by $\varrho_{u0} + \delta\varrho_u$. Neglecting terms of higher order than the second gives a linear system of equations for them which can easily be solved.

The stepwise approximation in this way is a "best" approximation in the sense that d_0 is tending to reach 0 in the most convenient way. At each step the numerical value of d_0 gives a measure for the approximation.

The process is very adequate to make calculations on automatic computers, since the operations are all direct matrix operations (no inversion operation is necessary), which are very easily programmed on such computers.

Appendix

Some rules of the trace calculus will be given below without proof. The matrices are considered to be real and symmetric ones, except some evident cases using the transposition rules.

1. The trace of a product of two squared matrices is always commutative.

2. The trace of a product of more than two squared matrices does not change its value by cyclic permutation of the matrices, but generally changes by any other permutation.

3. The trace of the sum of matrices is equal to the sum of their traces.

4. The trace of a matrix is equal to the trace of its transposed.

5. If

$$tr\,(\mathbf{A}^2) = tr\,(\mathbf{B}^2)\,,$$

then

$$tr\,(\mathbf{AB}) \geq -tr\,(\mathbf{A}^2) = -tr\,(\mathbf{B}^2)$$

and equality holds only in the case

$$\mathbf{B} = -\mathbf{A}\,.$$

6. The inner product of a matrix $\mathbf{M}$ of size $p \times r$ with a matrix $\mathbf{N}$ of size $q \times s$ is defined as a matrix of the size $pq \times rs$, where each element m_{ij} of matrix $\mathbf{M}$ is replaced by a matrix $m_{ij}\mathbf{N}$. This inner product is denoted by $\mathbf{M} \bigcirc (\mathbf{N})$.

7. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be matrices such that they can be multiplied with each other and $\mathbf{N}_1$ and $\mathbf{N}_2$ similarly. Let us denote the (usual) matrix product of $\mathbf{M}_1$ and $\mathbf{M}_2$ by $\mathbf{M}$ and that of $\mathbf{N}_1$ and $\mathbf{N}_2$ by $\mathbf{N}$, i.e.

$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$$

and

$$\mathbf{N} = \mathbf{N}_1\mathbf{N}_2\,.$$

Then we have

$$\mathbf{M} \bigcirc (\mathbf{N}) = [\mathbf{M}_1 \bigcirc (\mathbf{N}_1)]\,[\mathbf{M}_2 \bigcirc (\mathbf{N}_2)]\,.$$

8. If

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$$

and

$$\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2\,.$$

then

$$\mathbf{M} \bigcirc (\mathbf{N}) = \mathbf{M}_1 \bigcirc (\mathbf{N}_1) + \mathbf{M}_1 \bigcirc (\mathbf{N}_2) + \mathbf{M}_2 \bigcirc (\mathbf{N}_1) + \mathbf{M}_2 \bigcirc (\mathbf{N}_2)\,.$$

9. If both $\mathbf{M}_1$ and $\mathbf{M}_2$, and in addition $\mathbf{N}_1$ and $\mathbf{N}_2$, are commutable the inner product is likewise that in the sense that if

$$\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_1$$

and

$$\mathbf{N}_1\mathbf{N}_2 = \mathbf{N}_2\mathbf{N}_1\,,$$

then e.g.

$$\mathbf{M}_1\mathbf{M}_2\bigcirc(\mathbf{N}_1\mathbf{N}_2) = \mathbf{M}_2\mathbf{M}_1\bigcirc(\mathbf{N}_2\mathbf{N}_1) ,$$

but not in the case if any of the first two equations are not valid.

10. The trace of the inner product is equal to the product of the traces of the factors, i.e.

$$tr\ [\mathbf{M}\bigcirc(\mathbf{N})] = tr\ \mathbf{M} \cdot tr\ \mathbf{N} .$$

11. If none of the matrices $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are $\mathbf{0}$, further

$$\mathbf{M}_1\bigcirc(\mathbf{N}_1) = \mathbf{M}_2\bigcirc(\mathbf{N}_2) ,$$

and the sizes of $\mathbf{M}_1$ and $\mathbf{M}_2$ are the same (this already assures that the sizes of $\mathbf{N}_1$ and $\mathbf{N}_2$ are the same too), then

$$\mathbf{M}_1 = a\,\mathbf{M}_2$$

and

$$a\,\mathbf{N}_1 = \mathbf{N}_2 ,$$

where a is any number.

12. If

$$\mathbf{M}\bigcirc(\mathbf{N}) = \mathbf{0} .$$

then either

$$\mathbf{M} = \mathbf{0} .$$

or

$$\mathbf{N} = \mathbf{0} ,$$

or both. (It is to emphasize that this conclusion is not true in the case of the ordinary matrix multiplication.)

13. The transposition rule for the inner multiplication is

$$[\mathbf{M}\bigcirc(\mathbf{N})]^+ = \mathbf{M}^+\bigcirc(\mathbf{N}^+) .$$

14. Let us take a square matrix $\mathbf{A}$ of the size $k \times k$, where k is a positive integer. Let l be any divisor of k. Then the matrix $\mathbf{A}$ can be divided into small square matrices of the size $l \times l$. Let us take the traces of these small square matrices. In this way the sum of the first l of the diagonal elements should be denoted by A_{11}, the sum of the next l of the diagonal elements by A_{22}, and so on. The sum of the diagonal elements of the second square matrix of size $l \times l$ on the top of the matrix $\mathbf{A}$ be A_{12}, etc. So we obtain a square matrix of the size $\frac{k}{l} \times \frac{k}{l}$ of elements A_{ij}, which will be called the lth *partial trace* of $\mathbf{A}$ denoted by $\mathbf{tr}_l\mathbf{A}$.

15. Evidently by this definition

$$tr\,(\mathbf{tr}_l\,\mathbf{A}) = tr\,\mathbf{A}\,.$$

16. If k has a divisor m, and m has a divisor 1, we have

$$\mathbf{tr}_{\frac{m}{l}}\,(\mathbf{tr}_l\,\mathbf{A}) = \mathbf{tr}_m\,\mathbf{A}\,.$$

17. If the size of matrix $\mathbf{A}$ is $k \times k$

$$\mathbf{tr}_k\,\mathbf{A} = tr\,\mathbf{A}\,.$$

18. The operation of partial trace forming is generally not commutative, i.e.

$$\mathbf{tr}_l\,(\mathbf{AB}) \neq \mathbf{tr}_l\,(\mathbf{BA})\,.$$

19. If $\mathbf{A}$ and $\mathbf{B}$ are commutable matrices so are their partial traces.

20. Let $\mathbf{A}$ be a square matrix of size $k \times k$ and $\mathbf{B}$ a square matrix of size $l \times l$.

Then we have

$$\mathbf{tr}_l\,[\mathbf{A}\bigcirc(\mathbf{B})] = (tr\,\mathbf{B}) \cdot \mathbf{A}\,.$$

21. The partial trace formation is additive, i.e.

$$\mathbf{tr}_l\,(\mathbf{A} + \mathbf{B}) = \mathbf{tr}_l\,\mathbf{A} + \mathbf{tr}_l\,\mathbf{B}\,.$$

22. The partial trace forming corresponds to some kind of integration or summation.

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«НАИЛУЧШЕЕ» ПРИБЛИЖЕНИЕ В КВАНТОВОЙ МЕХАНИКЕ

Т. А. ГОФФМАНН

Р е з ю м е

В квантовомеханических приближениях вообще привело к трудностям то обстоятельство, что ошибка расчетов не могла быть однозначно оценивана. В статье с помощью матричного вычисления и применением правил вычисления следов найден способ однозначного определения ошибки расчета для любого квантовомеханического вычисления. Это позволяет найти более сильно сходящуюся процедуру для проведения расчетов. Процедура весьма пригодна в случае вычислительных машин, так как она содержит только прямые операции.

ВЛИЯНИЕ ОБМЕННОГО ВЗАИМОДЕЙСТВИЯ НА ПАРАМАГНИТНЫЙ РЕЗОНАНС

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В работе исследуется уширение резонансной линии, обусловленное изменением обменного взаимодействия вследствие тепловых колебаний кристаллической решетки.

1. Исследование влияния обменного взаимодействия (о. в.) на резонансные характеристики разведенных парамагнетиков представляет интерес, т. к. это влияние должно существенно зависеть от концентрации парамагнитных атомов. О. в. приводит к появлению дополнительных частот и уширению резонансной линии; последнее обусловлено изменением величины обменного интеграла при тепловых колебаниях атомов решетки.

Резонансные свойства системы описываются обычно с помощью высокочастотной (в. ч.) магнитной восприимчивости, которая при малых амплитудах в. ч. поля может быть представлена через запаздывающие функции Грина [1]

$$\chi_{\alpha\beta} = 2\pi\, i\, \mu^2 \Sigma \ll s_g^\alpha \mid s_f^\beta \gg_{E=\omega}^{\text{ret}}, \tag{1}$$

где μ — магнетомеханическое отношение, S — оператор спина электрона; f, g — координаты узлов решетки; ω — частота в. ч. поля; $(\alpha, \beta) = (x, y\, z)$.

Функции Грина определяются из уравнения [2]

$$i \frac{d}{dt} \ll s_g^\alpha(t) \mid s_f^\beta(t') \gg = i\,\delta(t-t') \langle (s_g^\alpha, s_f^\beta] \rangle + i \ll \frac{ds_g^\alpha(t)}{dt} \mid s_f^\beta(t') \gg . \tag{2}$$

Производные от операторов определяются из уравнений движения

$$i \frac{ds_g^\alpha}{dt} = [s_g^\alpha(t),\ \mathscr{H}], \tag{3}$$

где $\mathscr{H}$ — гамильтониан системы (не включающий взаимодействие системы с в. ч. полем).

В дальнейшем мы будем, для определенности, считать, что постоянное магнитное поле направлено по оси Z, а в. ч. поле — по оси x. Тогда выражение (1) для в. ч. восприимчивости приводится к виду:

$$\chi_{xx} = \frac{\pi i \mu^2}{2} \sum_f \left\{ \ll s_g^+ \mid s_f^- \gg + \ll s_g^- \mid s_f^+ \gg + \ll s_g^+ \mid s_f^+ \gg + \ll s_g^- \mid s_f^- \gg \right\},$$

$$\chi_{yx} = \frac{\pi \mu^2}{2} \sum_f \left\{ \ll s_g^+ \mid s_f^- \gg - \ll s_g^- \mid s_f^+ \gg + \ll s_g^+ \mid s_f^+ \gg - \ll s_g^- \mid s_f^- \gg \right\}. \tag{4}$$

2. Сначала мы рассмотрим влияние обменного взаимодействия на резонансные частоты; при этом гамильтониан системы представим в виде:

$$\mathscr{H}_0 = -\mu \Sigma H s_f^z + \frac{D}{2} \Sigma (s_f^z)^2 - \frac{1}{2} \Sigma I(f_1, f_2)\,(s_{f1}\, s_{f2}), \tag{5}$$

где H — постоянное магнитное поле; D — константа анизотропии; I — обменный интеграл.

Для определения функций Грина мы получим, используя (2) и (3) цепочку уравнений:

$$(E - \mu H) \ll s_g^+ \mid s_f^- \gg + \frac{D}{2} \ll (s_g^+ s_g^z + s_g^z s_g^+) \mid s_f^- \gg -$$

$$- \Sigma I(g, h) \left[\ll s_g^+ s_h^z \mid s_f^- \gg - \gg s_g^z s_h^+ \mid s_f^- \gg \right] = \frac{i}{2\pi} < 2 s_f^z > \Delta(g - f), \tag{6}$$

$$(E - \mu H) \ll (s_g^+ s_g^z + s_g^z s_h^+) \mid s_f^- \gg + \frac{D}{2} \ll s_g^+ \mid s_f^- \gg =$$

$$= \frac{i}{2\pi} < 6 (s_f^z)^2 - 2s(s+1) > + \Sigma I(g, h) \ll (s_g^+ s_g^z + s_g^z s_g^+)\, s_h^z \mid s_f^- \gg + \tag{7}$$

$$+ \Sigma I(g, h) \ll \left[s(s+1) - 3 (s_g^z)^2 \right] s_h^+ \mid s_f^- \gg - \Sigma I(g, h) \ll (s_g^+)^2 s_h^- \mid s_f^- \gg .$$

В последних трех членах уравнения (7) произведем интерполяционное расцепление; при этом последний член исчезает, т. к. $\langle (s_g^+)^2 \rangle = 0$. Средние значения $\langle s_h^z \rangle$ и $\langle (s_g^z)^2 \rangle$, а также $\Sigma I(g, h)$ зависят от координат узлов решетки. Заменим приближенно $\sum_{(h)} I(g, h) \to I$, $\langle s_h^z \rangle \to \langle s^z \rangle$ и $\langle (s_g^z)^2 \rangle \to (\langle s^z \rangle^2)$ и просуммируем уравнения (6) и (7) по индексу g. В результате получим систему уравнений (здесь и далее полагаем $s = 1$)

$$(E - \mu H)\, G_1 + \frac{D}{2} H_2 = \frac{i}{2\pi} \langle 2 s^z \rangle , \tag{8}$$

$$(E - \mu H - \tilde{I} \langle s^z \rangle)\, G_2 + \left[\frac{D}{2} \langle 3 (s^z)^2 - 2 \rangle \right] G_1 = \frac{i}{2\pi} \langle 6 (s^z)^2 - 4 \rangle ,$$

где

$$G_1 = \sum_g \ll s_g^+ \mid s_f^- \gg ; \qquad G_2 = \Sigma \ll (s_g^+ s_g^z + s_g^z s_g^+) \mid s_f^- \gg . \tag{9}$$

Определитель системы (7) имеет следующие конри

$$E_{1,2} = \mu H + \frac{\tilde{I}}{2} \langle s^z \rangle \pm \frac{1}{2} \sqrt{\tilde{I}^2 \langle s^z \rangle^2 + D^2 + 2D\tilde{I} \langle 3(s^z)^2 - 2 \rangle} . \tag{10}$$

Из решения системы (8) получим для G_1 следующее выражение

$$G_1(E) = \frac{i}{2\pi} \left\{ \frac{\langle 2s^z \rangle \left(E_1 - \mu H - \tilde{I} \langle s^z \rangle - E \langle 3(s^z)^2 - 2 \rangle \right)}{E - E_1} - \right.$$

$$\left. - \frac{\langle 2s^z \rangle \left(E_2 - \mu H - \tilde{I} \langle s^z \rangle - D \langle 3(s^z)^2 - 2 \rangle \right)}{E - E_2} \right\}_{E=\omega} . \tag{11}$$

Полюса функции G_1 в точках E_1 и E_2 соответствуют резонансным частотам системы; для больших величин обменного интеграла ($\mathrm{I}\langle S^z \rangle \gg D$ будем иметь приближенно

$$E_1 \simeq \mu H + \tilde{I} \langle s^z \rangle + \frac{D}{2} \frac{\langle 3(s^z)^2 - 2 \rangle}{\langle s^z \rangle} , \tag{12}$$

$$E_2 = \mu H - \frac{D}{2} \frac{\langle 3(s^z)^2 - 2 \rangle}{\langle s^z \rangle} . \tag{13}$$

Таким образом E_2 соответствует частоте обычного парамагнитного резонанса.

Для $D \gg \tilde{J} \langle S^z \rangle$ получим

$$E_1 \simeq \mu H + \tilde{I} \langle s^z \rangle + \frac{D}{2} + \frac{1}{2} \tilde{I} \langle s^z \rangle + \frac{1}{2} \tilde{I} \langle 3(s^z)^2 - 2 \rangle , \tag{14}$$

$$E_2 \simeq \mu H - \frac{D}{2} + \frac{\tilde{I}}{2} \left[\langle s^z \rangle - \langle 3(s^z)^2 - 2 \rangle \right] . \tag{15}$$

3. В работе будет исследоваться уширение резонансной линии, обусловленное изменением о. в. между парамагнитности атомами вследствие тепловых колебаний кристаллической решетки. Так как о. в. является короткодействующим, можно ограничиться учетом его только между ближайшими соседями. Это ограничение является несущественным, так как можно последовательно рассмотреть вклад от о. в. между ближайшими

соседями, вторыми и т. д. При малой концентрации парамагнитных атомов $\left(C = \frac{n}{N}\right.$, c— концентрация, n — число парамагнитных атомов, N — полное число атомов решетки) их можно разбить на группы, из одиночных атомов (не имеющих ближайших парамагнитных соседей), из двух, трех и т. д. атомов. Тогда в. ч. восприимчивость можно представить в виде

$$\chi_{\alpha\beta} = n_1 \chi_{\alpha\beta}^{(2)} + n_2 \chi_{\alpha\beta}^{(2)} + n_3 \chi_{\alpha\beta}^{(3)} + \ldots , \tag{16}$$

где $\chi_{\alpha\beta}^{(i)}$ — в. ч. восприимчивость соответствующей группы атомов, n_i — число таких групп в кристаллической решетке.

Рассматриваемый механизм дает вклад в $\chi_{\alpha\beta}^{(i)}$ для $i \geqslant 2$.

При малых c будем иметь

$$n_2 = \frac{n(n-1)}{2!}\,\frac{1}{N} \simeq \frac{cn}{2}\,; \qquad n_3 = \frac{n(n-1)(n-2)}{3!} \simeq \frac{c^2 n}{6}\,. \tag{17}$$

Мы ограничимся вычислением $\chi_{\alpha\beta}^{(2)} \sim c$, вычисление высших членов $\chi_{\alpha\beta}^{(i)}$ производится аналогично. Представим гамильтониан системы в виде:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}'; \qquad \mathscr{H}' = -\frac{1}{2}\,\Sigma\left[A(k,f,g)b_k + A(k,f,g)b_k^+\right], \tag{18}$$

где

$$A(k,f,g) = N^{-1/2}\,a(k,f,g)\left[e^{i(\vec{k}\vec{f})} - e^{i(\vec{k}\vec{g})}\right], \tag{19}$$

Для определения функций Грина, в том же приближении как и в § 2, получим следующие уравнения.

$$(E - \mu H)\,G_1 + \frac{D}{2}\,G_2 = \frac{i}{2\pi}\,\langle 2s_f^z\rangle, \tag{20}$$

$$\left[E - \mu H - I\,\langle s_2^z\rangle\right]G_2 + \left[\frac{D}{2} + I\langle 3(s_2^z)^2 - 2\rangle\,G_1 + \langle 3(s_2^z)^2\rangle\,\Sigma\,F_1(k,f) - \right.$$

$$- \langle s_2^z\rangle\,\Sigma\left[F_2(k,f)\right] + \langle 3\,(s_2^z)^2 - 2\rangle\,\Sigma\left[F_1(k,f) + F_1^+(k,f)\right] =$$

$$\frac{i}{2\pi}\,\langle 6(s_f^z)^2 - 4\rangle\,, \tag{21}$$

где

$$F_1(k,f) = \Sigma\,A(k,g,h)\ll b_k\,s_g^+ \mid s_f^-\gg;$$

$$F_2(k,f) = \Sigma\,A(k,g,h)\ll(s_g^+\,s_g^z + s_g^z\,s_g^+) \mid s_f^-\gg. \tag{22}$$

$\langle s_2^z \rangle$ и $\langle (s_2^z)^2 \rangle$ средние значения для группы $i = 2$. Для функций $F_j (j = 1, 2)$ используя те же приближения что и для G_j и производя расцепления

$$\ll b_k b_{k'} \varphi(s_g^\beta) \mid s_f^- \gg = \langle b_k b_{k'} \rangle \ll \varphi(s_g^\beta) \mid s_f^- \gg,$$
$$\ll b_k b_{k'}^+ \varphi(s_g^\beta) \mid s_f^- \gg = \langle b_k b_{k'}^+ \rangle \ll \varphi(s_g^\beta) \mid s_f^- \gg, \tag{23}$$

$$\langle b_k b_{k'} \rangle = 0; \quad \langle b_k b_{k'}^+ \rangle = \langle b_k b_k^+ \rangle = \Delta(k - k'); \quad \langle b_k b_k^+ \rangle = \frac{1}{e^{\frac{\hbar\omega_k}{\vartheta}} - 1}$$

получим систему уравнений:

$$(E - \mu H - \hbar\omega_k) F_1(k,f) + \frac{D}{2} F_2(k,f) = 0\,,$$
$$\left[\frac{D}{2} + J\langle 3\,(s_2^z)^2 - 2\rangle\right] F_1(k,f) + [E - \mu H - \hbar\omega_k - J\langle s_2^z\rangle]\, F_2(k,f) = I(k,f)\,, \tag{24}$$

где

$$I(k,f) = \frac{2}{N} a^2(k)\left[1 - \cos(\bar{k}\,\bar{l})\right] \{\langle s_2^z\rangle G_2 + \langle 2 - 3(s_2^z)^2\rangle G_1\} \langle b_k b_k^+ \rangle\,, \tag{25}$$

$\bar{l} = \bar{g} - \bar{h}$ — вектор соединяющий пару атомов. Определяя F_j из (24) и подставляя в (21) мы получим систему уравнений для G_1 и G_2. Из решения последней определим функцию G_1, которая непосредственно входит в выражение для χ.

$$G_1 = \frac{A(E)}{(E - E_1)(E - E_2) - i\pi\, P''(E)}\,, \tag{26}$$

где

$$E_{1,2} = \mu H + \frac{J}{2}\langle S_2^z\rangle \pm R,$$
$$R = \sqrt{\frac{J}{2}\langle s_2^z\rangle^2 + \frac{D}{2}\left(\frac{D}{2} + J\langle 3\,(s_2^z)^2 - 2\rangle\right)}\,, \tag{27}$$

$$A(E) = \frac{i}{2\pi}\left\{\langle 2 s_f^z\rangle (E - \mu H - J\langle s_2^z\rangle) + D\langle 2 - 3(s_f^z)^2\rangle\right\} -$$
$$- \frac{i}{2\pi}\left\{\langle s_2^z\rangle^2 [P_1(E) + P_2(E)] + \frac{D}{2}\langle s_2^z\rangle \langle 3\,(s_2^z)^2 - 2\rangle\, [P_3(E) + P_4(E)]\right\}, \tag{28}$$

$$P(E) = \frac{i}{2\pi}\langle 2 s_f^z\rangle \left\{E - \mu H - J\langle s_2^z\rangle^2 [P_1(E) + P_2(E)] + \right.$$
$$\left. + \frac{D}{2}\langle s_2^z\rangle \langle 2 - 3(s_2^z)^2\rangle\, [P_3(E) + P_4(E)]\right\} - \frac{i}{2\pi} D\langle 3(s_f^z)^2 - 2\rangle\,, \tag{29}$$

$$P_1(E) = \frac{1}{2R} \Sigma\, 2a(k) \left[1 - \cos(\bar{k}, \bar{g} - \bar{h})\right] \langle b_k\, b_k^+ \rangle \times \tag{30}$$

$$\times \left[\frac{\frac{J}{2} \langle s_2^z \rangle + R}{E - \varepsilon_1} - \frac{\frac{J}{2} \langle s_2^z \rangle - R}{E - \varepsilon_2'} \right],$$

$$P_2(E) = \frac{1}{2R} \Sigma\, 2a(k) \left[1 - \cos(\bar{k}, \bar{g} - \bar{h})\right] \langle b_k^+\, b_k \rangle \times \tag{31}$$

$$\times \left[\frac{\frac{J}{2} \langle s_2^z \rangle + R}{E - \varepsilon_1'} - \frac{\frac{J}{2} \langle s_2^z \rangle - R}{E - \varepsilon_2'} \right],$$

$$P_3(E) = \frac{1}{2R} \Sigma\, 2a(k) \left[1 - \cos(\bar{k}, \bar{g} - \bar{h})\right] \langle b_k\, b_k^+ \rangle \left[\frac{1}{E - \varepsilon_1} - \frac{1}{E - \varepsilon_2} \right], \tag{32}$$

$$P_4(E) = \frac{1}{2R} \Sigma\, 2a(k) \left[1 - \cos(\bar{k}, \bar{g} - \bar{h})\right] \langle b_k^+\, b_k \rangle \left[\frac{1}{E - \varepsilon_1'} - \frac{1}{E - \varepsilon_2'} \right], \tag{33}$$

$$P(E) = P'(E) - i\pi\, P''(E), \tag{34}$$

$$\varepsilon_{1,2} = E_{1,2} + \hbar\omega_k\,; \quad \varepsilon_{1,2}' = E_{1,2} - \hbar\omega_k\,. \tag{35}$$

Для других функций, входящих в выражение (4) для χ будем иметь

$$\Sigma \ll s_g^- \mid s_f^- \gg = G_1(-E) = -\frac{A(-E)}{(E + E_1)(E + E_2) + i\pi P''(-E)}, \tag{36}$$

$$\ll s_g^+ \mid s_f^+ \gg = \ll s_g^- \mid s_f^- \gg = 0\,.$$

Резонансные частоты системы определяются уравнением

$$(\omega - E_1)(\omega - E_2) = 0\,. \tag{37}$$

В области $\omega \sim E_2$ будем иметь приближенно:

$$G_1 = \frac{A(E_2)/E_2 - E_1}{\omega - E_2 - \dfrac{i\pi P''(E_2)}{E_2 - E_1}}\,. \tag{38}$$

Ширина линии в окрестности $\omega = E_2$ равна

$$\Gamma = \frac{\pi}{E_2 - E_1} P''(E_2). \tag{39}$$

Выполняя интегрирование по k получим для ширины линии следующее выражение

$$\Gamma_2 = \frac{1}{2\pi R^2} a^2(k^*) \left[1 - 2\frac{\sin k^* l}{k^* l}\right] (k^* \delta)^2 \frac{1}{e^{\frac{\hbar\omega k^*}{\vartheta}} - 1}$$

$$\left[\langle S_2^z\rangle (E_2 - \mu H) + \frac{D}{2}\langle 3(s_2^z)^2 - 2\rangle\right]\left(\frac{J}{2}\langle s_2^z\rangle + R\right), \tag{40}$$

где

$$k^* = \frac{2R}{\hbar c}. \tag{41}$$

Соответственно в окрестности $\omega = E_1$ ширина резонансной линии имеет вид:

$$\Gamma_1 = \Gamma_2 e^{\frac{\hbar\omega k^*}{\vartheta}}. \tag{42}$$

ЛИТЕРАТУРА

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THE EFFECT OF EXCHANGE INTERACTION ON PARAMAGNETIC RESONANCE

By

N. A. POTAPKOV

Abstract

The widening of the resonance line due to the change of the exchange interaction caused by the thermal oscillations of the crystal lattice is investigated.

NUCLEAR MATTER WITH MOMENTUM-DEPENDENT NUCLEAR FORCES

By

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Nuclear matter with momentum-dependent nuclear forces have been investigated with the result that those forces are suitable to explain certain properties of atomic nuclei and nuclear scattering experiments.

Information is obtained on nuclear forces from scattering experiments in the two-nucleon problem and from saturation conditions in the many-nucleon problem. The evaluation of the experiments yields potentials as the GAMMEL—THALER potential [1], the HAMADA—JOHNSTON potential [2] or the YALE potential [3].

These potentials have space-dependent parts containing a repulsive core. Such a core brings some difficulties into the evaluation, causing the perturbation series to diverge. With the method developed by BRUECKNER and others [4] the perturbation calculation may be extended to hard core potentials, but the mathematical evaluation is very complicated — so for finite systems two self consistent fields are needed — and also there are some simplifications whose influence is not known exactly [5].

In the last years momentum-dependent forces [6] attracted more and more attention [7]. There we consider the two equivalent forms

$$V_1 = v_1(r) + \mathfrak{p}\omega_1(r)\mathfrak{p}/2m \,,$$

$$V_2 = v_2(r) + \frac{1}{2}\,\frac{\mathfrak{p}^2}{2m}\,\omega_2(r) + \frac{1}{2}\,\omega_2(r)\,\frac{\mathfrak{p}^2}{2m}\,,$$

with $r = |\mathfrak{r}_1 - \mathfrak{r}_2|$ and $\mathfrak{p} = \mathfrak{p}_1 - \mathfrak{p}_2$. In general we have to add tensor and spin orbital forces and to consider spin- and isospin dependence. The two space functions $v(r)$ and $\omega(r)$ have no singularities.

The pure space-dependent part $v(r)$ is an attractive potential, the momentum-dependent part a repulsive one which increases with energy. It is the counterpart to the hard core. In the potentials [1], [2], [3] for high energies the hard core is the main part and it is repulisve too. There is a straightforward connection between hard core potential and momentum-dependent potential. BELL [8] and BAKER [9] pointed out, that there is a unitary trans-

formation for the SCHRÖDINGER equation of a two nucleon system from hard core potential to momentum-dependent potential. Therefore this potential is only a special case of the more general transformation of the hard core potential, which we know from the lecture of MITTELSTAEDT [10].

The momentum-dependent potential is analysed in some papers [6], [7]. Phase shifts are calculated, mainly by GREEN [11]. He pointed out, that by suitable choice of the space-dependent parts the phase shifts of the momentum-dependent interaction are near to the phase shifts obtained for a hard core potential. But for this potential he evaluates a binding energy which is too low. However, we know other momentum-dependent forces [12] calculated from saturation values.

The binding energy and the saturation density may be determined by a simple THOMAS—FERMI method as opposed to the hard core potential, where this method diverges. The density dependence of the kinetical energy E_K and the part $E_p^{(s)}$ of the potential energy we get from the pure space-dependent part of the interaction is

$$E_k/A \sim \varrho^2/_3, \; E_p^{(s)}/A \sim \varrho$$

(ϱ is the density and A the total number of particles). So without other parts we have the so-called nuclear collapse. But the momentum-dependent part of the interaction energy is proportional to

$$E_p^{(m)}/A \sim \varrho^{5/3} .$$

This part increases faster than $E_p^{(s)}$, and if the momentum dependent part is repulsive we get saturation. So SEYLER and BLANCHARD [12] calculate binding energies for finite nuclei to be near to the BETHE—WEIZSÄCKER mass formula.

So we see that momentum-dependent forces are suitable to explain the properties of nuclear matter and finite nuclei and also of nucleon-nucleon scattering. We know potentials which give a good approximation for saturation values and we know potentials which give a good approximation for phase shifts. But to date we do not know a potential which explains both saturation and scattering in a satisfactory manner. Some calculations are prepared [13] to find such a potential.

I am indebted to Prof. W. MACKE for many helpful discussions.

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ЯДЕРНАЯ МАТЕРИЯ С ЯДЕРНЫМИ СИЛАМИ, ЗАВИСЯЩИМИ ОТ МОМЕНТА КОЛИЧЕСТВА ДВИЖЕНИЯ

П. РЕННЕРТ

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INDEX

AFTERNOON SESSION

Section A

Section B

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