Acta Mathematica Hungarica

VOLUME 50, NUMBERS 1–2, 1987

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0236—5294

ACTA MATHEMATICA
HUNGARICA

Acta Mathematica publishes papers on mathematics in English, German, French and Russian.

Acta Mathematica is published in two volumes of four issues a year by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to

Acta Mathematica, H-1364 Budapest, P.O.B. 127

Orders may be placed with

KULTURA Foreign Trading Company
H-1389 Budapest P.O. Box 149

or with its representatives abroad.

Acta Mathematica Hungarica is abstracted/indexed in Current Contents — Physical, Chemical and Earth Sciences, Mathematical Reviews.

ACTA
MATHEMATICA
HUNGARICA

VOLUME 50

AKADÉMIAI KIADÓ, BUDAPEST
1987

CONTENTS

VOLUME 50

87-403—Szegedi Nyomda — Felelős vezető: Surányi Tibor igazgató

CONTENTS

Acta Math. Hung.
50 (1—2) (1987), 3—16.

ON SOME CLASSES OF ARITHMETICAL FUNCTIONS ON A SEMIGROUP G_K

A. GRYTCZUK (Zielona Góra)

1. Introduction

Let K be an algebraic number field of degree n over the rational number field Q. Denote by G_K the multiplicative semigroup of all non-zero integral ideals of K. In this paper we consider some general classes $\mathscr{H}$ of arithmetical functions.

Let

$$C_1(K) = \frac{1}{C_1 n!|\Delta|^{1/n}} = \frac{1}{K_1}, \tag{1.1}$$

where $C_1>0$ is some numerical constant and n, Δ denote the degree and the discriminant of K, respectively. Moreover let

$$D_1 = \left\{s = \sigma+it\colon \sigma \geqq 1-\frac{C_1(K)}{\log|\Delta|(|t|+2)^n} > \frac{1}{2},\quad |t| \geqq 2\right\}, \tag{1.2}$$

$$D_2 = \left\{s = \sigma+it\colon \sigma \geqq 1-\eta > \frac{1}{2},\quad |t| < 2\right\}, \tag{1.3}$$

where $\eta = \dfrac{C_1(K)}{\log|\Delta|4^n}$, and

$$D = D_1 \cup D_2. \tag{1.4}$$

We will say that the function h belongs to $\mathscr{H}$ iff it satisfies the following conditions:

(i) $h\colon G_K\times\{z\in C, |z| \leqq 1\} \to C$,

(ii) there exists a constant $R=R(K)>0$ such that

$$\sum_{x<N(I)\leqq y} |h(I, z)| \leqq R(y-x)+O_K(y^\beta)$$

for some $0 \leqq \beta < 1$,

(iii) $\displaystyle\sum_{I\in G_K} \frac{h(I, z)}{N(I)^s} = g(s, z)\zeta_K^z(s)$, for $\operatorname{Re} s > 1, |z| \leqq 1$,

(iv) $g(s, z)$ is holomorphic with respect to $s\in D$ for every fixed $|z|\leqq 1$ and is holomorphic with respect to z in the circle $|z|<1$ for every fixed $s\in D$, moreover $g(s, z)$ is bounded for $(s, z)\in D\times E$ where $E=\{z\colon |z|\leqq 1\}$.

We prove the following theorems:

THEOREM 1. *Let* $h\in\mathscr{H}$, *then there exists a sequence of functions* $A_j(z)$ *defined and continuous in the circle* $|z|\leqq 1$ *and holomorphic in the circle* $|z|<1$ *such that for every integer* $q\geqq 0$ *we have*

$$\sum_{\substack{I\in G_K\\ N(I)\leqq x}} h(I,z) = \sum_{j-0}^{q} xA_j(z)(\log x)^{z-j-1}+O_K\big(x(\log x)^{\mathrm{Re}\, z-q-2}\big) \tag{1.5}$$

uniformly with respect to $|z|\leqq 1$ *as* $x\to\infty$, *where*

$$A_j(z)=\frac{B_j(z)}{\Gamma(z-j)},\quad B_j(z)=\frac{1}{2\pi i}\int\limits_{|s-1|=\delta<\eta}\frac{H(w,z)}{w(w-1)^{j+1}}\,dw,\quad j=0,1,2,\ldots,q \tag{1.6}$$

and $\delta>0$ *is an arbitrary real number such that* $\delta<\eta$,

$$H(s,z)=g(s,z)\exp\big(z\log(s-1)\zeta_K(s)\big). \tag{1.7}$$

Throughout this paper for complex w, $\log w$ denotes its principal branch.

THEOREM 2. *Let* σ_1 *be some fixed real number such that* $\frac{1}{2}<\sigma_1\leqq 1-\eta$ *and the function* h *satisfies the following conditions:*

(a) $|h(I,z)|\leqq R_1$ *for* $I\in G_K$ *and* $z\in E$, *where* $R_1=R_1(K)\geqq 1$,

(b) $\displaystyle\sum_{\mathfrak{p}}\frac{|h(\mathfrak{p},z)-z|}{N(\mathfrak{p})^{\sigma}}<\infty$ *for* $\sigma\geqq\sigma_1$,

(c) $h(I\circ J,z)=h(I,z)h(J,z)$ *for* $I,J\in G_K$ *such that* $(I,J)=1$,

(d) $h(I,z)$ *is a holomorphic function with respect to* z *in the circle* $|z|<1$; *then* $h\in\mathscr{H}$.

COROLLARY. *Let* $h=h(I,z)=f(I)z^{F(I)}$ *for* $|z|\leqq 1$ *and* $f\colon G_K\to C$, $|f(I)|\leqq R_1$ *for* $I\in G_K$, *where* $R_1=R_1(K)\geqq 1$, $F\colon G_K\to N\cup\{0\}$, $f(I\circ J)=f(I)f(J)$, $F(I\circ J)=F(I)+F(J)$ *for* $I,J\in G_K$ *such that* $(I,J)=1$ *and* $f(\mathfrak{p})=F(\mathfrak{p})=1$ *for every prime ideal* $\mathfrak{p}\in G_K$. *Then* $h\in\mathscr{H}$.

REMARK. From this Corollary and Theorem 1 we get in the case $n=[K:Q]=1$ some results which have been proved by A. Selberg in [4] and by H. Delange in [1]. The proof of our theorems is based on a method due to H. Delange [1].

In a forthcoming paper we shall prove explicit asymptotic formula for the function $h=f(I)z^{F(I)}$, where f, F satisfy the assumptions of the Corollary.

2. Basic lemmas

The following lemma follows easily from results of K. Wiertelak [6], Lemma 7 and H. M. Stark [5], Theorem 1.

LEMMA 1. *Let* $\zeta_K(s)$ *denote the Dedekind zeta function of the field K. There exists a numerical constant* $c>0$ *such that* $\zeta_K(s)\neq 0$ *in the region*

$$\sigma>1-\frac{1}{cn!|\Delta|^{1/n}\log|\Delta|(|t|+2)^n},\qquad -\infty<t<+\infty. \tag{2.1}$$

LEMMA 2. *Let* C *denote the complex plane and let* $A\subset C$, $u_n\colon A\to C$, $v_n\colon A\to C$. *Let* $B\subset A$ *and suppose that there are sequences* $\{U_n\}$, $\{V_n\}$ *where* $U_n\geqq 0$, $V_n\geqq 0$ *such that*

$$|u_n(z)|\leqq U_n,\quad |u_n(z)-v_n(z)|\leqq V_n,\quad \sum_n U_n^2<\infty,\quad \sum_n V_n<\infty.$$

Then the product

$$\prod_{n-1}^{\infty}(1+u_n(z))\exp(-v_n(z))$$

is absolutely and uniformly convergent and bounded for $z\in B$.

For the proof see [1], pp. 108—109.

LEMMA 3. *Let* $K_1=C_1 n!|\Delta|^{1/n}$ *and* $C_1>c>0$, *where* C_1 *is a numerical constant and* c *is as in Lemma* 1. *Then*

$$|\log\zeta_K(s)|\leqq C_3+n[\log 3K_1+\log\log(|\Delta|(|t|+2)^n)] \tag{2.2}$$

for $s\in D_1$, *where* $C_3>0$ *is a numerical constant.*

PROOF. We have

$$\log\zeta_K(s)=\int_a^s\frac{\zeta_K'}{\zeta_K}(z)\,dz+\log\zeta_K(a), \tag{2.3}$$

where $a\in D_1$. We remark that for $\sigma\geqq 1+\dfrac{1}{2K_1\log|\Delta|(|t|+2)^n}$ we obtain

(2.4)

$$|\log\zeta_K(\sigma+it)|=\left|\log\prod_{\mathfrak{p}}\left(1-\frac{1}{N(\mathfrak{p})^{\sigma+it}}\right)^{-1}\right|\leqq n\log\prod_{\mathfrak{p}}\left(1-\frac{1}{p^{\sigma}}\right)^{-1}\leqq n\log\zeta_Q(\sigma)\leqq$$

$$\leqq n\log\left(1+\frac{1}{\sigma-1}\right)\leqq n[\log 3K_1+\log\log(|\Delta|(|t|+2)^n)].$$

For $\sigma<\operatorname{Re}a=1+\dfrac{1}{2K_1\log|\Delta|(|t|+2)^n}$ we integrate over the segment parallel to the real axis. Then from (2.3) we obtain

$$|\log\zeta_K(s)|\leqq\max_{\overrightarrow{a,s}}\left|\frac{\zeta_K'}{\zeta_K}(x+iy)\right||a-s|+|\log\zeta_K(a)|. \tag{2.5}$$

For $s\in D_1$ we have

$$|a-s|\leqq\frac{3}{2K_1\log|\Delta|(|t|+2)^n}, \tag{2.6}$$

and from [6]

$$-\frac{\zeta_K'}{\zeta_K}(s)=\frac{1}{s-1}+O(\log(|\Delta|(|t|+2)^n)). \tag{2.7}$$

Hence $\left|\frac{1}{s-1}\right| \leqq \frac{1}{2}$ for $|t| \geqq 2$, therefore from (2.7) we get

$$\left|\frac{\zeta_K'}{\zeta_K}(\sigma+it)\right| \leqq C_2 \log\left(|\Delta|(|t|+2)^n\right). \tag{2.8}$$

From (2.8) and (2.6) we obtain

$$|\log \zeta_K(s)| \leqq \frac{3C_2}{2K_1} + |\log \zeta_K(a)|. \tag{2.9}$$

By (2.9) and (2.4) the Lemma follows.

In a similar way we get the following lemma.

LEMMA 4. *Let* K_1 *denote the constant from Lemma* 3. *Then we have*

$$|\log (s-1)\zeta_K(s)| \leqq C_4 + n\left(\log 3K_1 + \log\log(|\Delta| 4^n)\right) \tag{2.10}$$

for $s \in D_2$, *where* $C_4 > 0$ *is a numerical constant.*

Finaly we remark that by using Landau's theorem [2], Satz 210, pp. 131—139 we get the following lemma:

LEMMA 5. *Let* $n = [K: Q] \geqq 1$, *then*

$$\Big|\sum_{\substack{I \in G_K \\ N(I) \leqq x}} 1 - \lambda h x\Big| \leqq C_2(K) x^{1-\frac{2}{n+1}},$$

where

$$C_2(K) = n^{C_5 n} |\Delta|^{\frac{1}{n+1}} \log(|\Delta| e),$$

and $\lambda h = \operatorname*{res}_{s=1} \zeta_K(s)$.

3. Proof of Theorem 1

Let $P_z(x) = \sum_{\substack{I \in G_K \\ N(I) \leqq x}} h(I, z)$ for $|z| \leqq 1$, then from (iii) and the classical Perron's formula we get

$$\int_0^x P_z(t)\,dt = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^{s+1}}{s(s+1)} g(s,z) \zeta_K^z(s)\,ds, \tag{3.1}$$

where $c > 1$ and $x > 0$.

Putting $g(s,z)\zeta_K^z(s) = H(s,z)(s-1)^{-z}$ and

$$\Phi(s,z,x) = \frac{x^{s+1}}{s(s+1)} H(s,z)(s-1)^{-z} \tag{3.2}$$

by (3.1), (3.2) we get

$$\int_0^x P_z(t)\,dt = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \Phi(s,z,x)\,ds. \tag{3.3}$$

Since $\eta=\frac{C_1(K)}{\log|\Delta|4^n}$ and $1-\eta>\frac{1}{2}$ thus $0<\eta<\frac{1}{2}$. Let $0<r<\eta$ and $0<\varepsilon<$ $<\operatorname{Arc}\operatorname{tg}\frac{2}{\eta}$. Then by Cauchy's theorem the integral on the right-hand side of (3.3) may be replaced by the integrals $I_1, \dots, I_9$ over the paths $\Gamma_1, \dots, \Gamma_9$ which are defined as follows:

Γ_1 is the segment $\langle c-iT, 1-\frac{C_1(K)}{\log|\Delta|(|T|+2)^n}-iT\rangle$;

Γ_2 is the curve described by $1-\frac{C_1(K)}{\log|\Delta|(|t|+2)^n}+it$ as t increases from $-T$ to -2;

Γ_3 is the segment $\langle 1-\eta-2i,\ 1-\eta-i\eta\operatorname{tg}\varepsilon\rangle$;
Γ_4 is the segment $\langle 1-\eta-i\eta\operatorname{tg}\varepsilon,\ 1-re^{i\varepsilon}\rangle$;
Γ_5 is the arc of the circle $1+re^{i\theta}$ described as θ increases from $-\pi+\varepsilon$ to $\pi-\varepsilon$;
Γ_6 is the segment $\langle 1-re^{-i\varepsilon},\ 1-\eta+i\eta\operatorname{tg}\varepsilon\rangle$;
Γ_7 is the segment $\langle 1-\eta+i\eta\operatorname{tg}\varepsilon,\ 1-\eta+2i\rangle$;

Γ_8 is the curve described by $1-\frac{C_1(K)}{\log|\Delta|(|t|+2)^n}+it$ as t increases from 2 to T;

Γ_9 is the segment $\langle 1-\frac{C_1(K)}{\log|\Delta|(T+2)^n}+iT, c+iT\rangle$.

We note that Γ_1, Γ_2, Γ_8 and Γ_9 depend only on T and do not depend on r or ε. For fixed T and r and for $\varepsilon\to 0$, Γ_3 and Γ_7 become the segments

$$\Gamma_3' = \langle 1-\eta-2i, 1-\eta) \quad\text{and}\quad \Gamma_7' = (1-\eta, 1-\eta+2i\rangle.$$

For I_4 and I_6 we have

$$\lim_{\varepsilon\to 0} I_4 = \lim_{\varepsilon\to 0}\int_{\Gamma_2}\Phi(s,z,x)\,ds = \int_{1-\eta}^{1-r} H(\sigma,z)(1-\sigma)^{-z}(e^{-i\pi})^{-z}\frac{x^{\sigma+1}}{\sigma(\sigma+1)}\,d\sigma,$$

$$\lim_{\varepsilon\to 0} I_6 = \lim_{\varepsilon\to 0}\int_{\Gamma_6}\Phi(s,z,x)\,ds = \int_{1-r}^{1-\eta} H(\sigma,z)\ (1-\sigma)^{-z}(e^{i\pi})^{-z}\frac{x^{\sigma+1}}{\sigma(\sigma+1)}\,d\sigma.$$

If $\gamma_r=\{s\colon |s-1|=r\}$ excluding the point $1-r$, then

$$\lim_{\varepsilon\to 0}(I_4+I_5+I_6) = \int_{\gamma_r}\Phi(s,z,x)\,ds+2i\sin\pi z\int_r^{\eta}\frac{H(1-u,z)}{(1-u)(2-u)}u^{-z}x^{2-u}\,du$$

which does not depend on the choice of r.

If $T\to\infty$ then $I_1\to 0$ and $I_9\to 0$ so that

$$\int_0^x P_z(t) = \varphi_z(x)+\omega(x,z), \tag{3.4}$$

where

$$\text{(3.5)} \qquad \varphi_z(x) = \frac{\sin \pi z}{\pi} \int_r^{\eta} \frac{H(1-u, z)}{(1-u)(2-u)} x^{2-u}\, du + \frac{1}{2\pi i} \int_{\gamma_r} \Phi(s, z, x)\, ds$$

and

$$\text{(3.6)} \qquad \omega(x, z) = J_2 + J_3 + J_7 + J_8,$$

(3.7)

$$J_2 = \frac{1}{2\pi i} \int_{-\infty}^{-2} \Phi\left(1 - \frac{C_1(K)}{\log |\Delta| (|t|+2)^n} + it, z, x\right)\left(i + \frac{C_1(K)n}{(|t|+2) \log^2 |\Delta| (|t|+2)^n}\right) dt,$$

$$\text{(3.8)} \qquad J_3 = \frac{1}{2\pi i} \int_{\Gamma_3'} \Phi(s, z, x)\, ds, \quad J_7 = \frac{1}{2\pi i} \int_{\Gamma_7'} \Phi(s, z, x)\, ds,$$

(3.9)

$$J_8 = \frac{1}{2\pi i} \int_{2}^{+\infty} \Phi\left(1 - \frac{C_1(K)}{\log |\Delta| (|t|+2)^n} + it, z, x\right)\left(i + \frac{C_1(K)n}{(t+2) \log^2 |\Delta| (t+2)^n}\right) dt.$$

By Lemma 3 and assumption (iv) for $s \in D_1$ we have

$$\text{(3.10)} \qquad |H(s,z)(s-1)^{-z}| \leqq |g(s, z)| e^{|z|\,|\log \zeta_K(s)|} \leqq R_2(K)(\log |\Delta| (|t|+2)^n)^n,$$

where $|z| \leqq 1$. Since $|t| \geqq 2$ and $n < 2 + \log |\Delta|$ thus from (3.10) we get

$$\text{3.11)} \qquad |H(s, z)(s-1)^{-z}| \leqq R_3(K)(\log |t|)^n,$$

where

$$R_3(K) = R_2(K) 4^n \log^n (|\Delta| e).$$

For $s \in \Gamma_2$ or $s \in \Gamma_8$ from (3.2) and (3.11) we have

$$\text{(3.12)} \qquad |\Phi(s, z, x)| \leqq R_3(K) x^{2 - \frac{C_1(K)}{\log |\Delta| (|t|+2)^n}} |t|^{-2} (\log |t|)^n.$$

Since $|\Delta| \geqq 1$, $n \geqq 1$, $|t| \geqq 2$ we obtain

$$\text{(3.13)} \qquad \left| i + \frac{C_1(K)\, n}{(|t|+2) \log^2 |\Delta| (|t|+2)^n} \right| < \sqrt{1 + \frac{1}{16 C_1^2}} = C_4,$$

where $C_4 > 0$ is a numerical constant. From (3.13), (3.12), (3.7) and (3.8) we get

$$\text{(3.14)} \qquad J_2 + J_8 \ll_K \int_2^{+\infty} x^{2 - \frac{C_1(K)}{\log |\Delta| (|t|+2)^n}} t^{-2} \log^n t\, dt.$$

Let $0 < \varepsilon_0 < 1$, then we have

(3.15)

$$x^{2 - \frac{C_1(K)}{\log |\Delta| (|t|+2)^n}} t^{-2} = x^2 t^{-2+\varepsilon_0} \exp\left(-\varepsilon_0 \log t - C_1(K) \frac{\log x}{\log |\Delta| (|t|+2)^n}\right).$$

Putting $\alpha(K)=\frac{C_1(K)}{4\log(|\Delta|e)}$ and remarking that for $|t|\geqq 2$ and $n<2+\log|\Delta|$ we get

(3.16)

$$\varepsilon_0\log|t|+\frac{C_1(K)}{\log|\Delta|(|t|+2)^n}\log x\geqq\varepsilon_0\log|t|+\alpha(K)\frac{\log x}{\log|t|}\geqq 2\sqrt{\alpha(K)\varepsilon_0}\sqrt{\log x}.$$

Let $\alpha=\sqrt{\alpha(K)\varepsilon_0}$ then from (3.16), (3.15), (3.14) we get

$$J_2+J_8\underset{K}{\ll}x^2\int_2^{+\infty}t^{-2+\varepsilon_0}\exp\left(-2\alpha\sqrt{\log x}\right)\log^n t\,dt= \tag{3.17}$$

$$=x^2\exp\left(-2\alpha\sqrt{\log x}\right)\int_2^{+\infty}t^{-2+\varepsilon_0}\log^n t\,dt.$$

Denoting by

$$I_n=\int_2^{+\infty}\frac{(\log t)^n}{t^{2-\varepsilon_0}}dt$$

then we obtain

$$I_n\leqq\frac{n!}{(1-\varepsilon_0)^n}(e^{\log 2}-1)=\frac{n!}{(1-\varepsilon_0)^n}. \tag{3.18}$$

By (3.18) and (3.17) it follows that

$$J_2+J_8\underset{K}{\ll}x^2\exp\left(-2\alpha\sqrt{\log x}\right). \tag{3.19}$$

Since

$$H(s,z)=g(s,z)\exp z\left(\log(s-1)\zeta_K(s)\right)$$

thus by Lemma 4 and assumption (iv) for $s\in D_2$, $|z|\leqq 1$ we obtain

$$|H(s,z)|\leqq R_5(K). \tag{3.21}$$

For $s\in\Gamma_3'$ or $s\in\Gamma_7'$ and $|z|\leqq 1$ from (3.21) follows that

$$|\Phi(s,z,x)|\ll x^{2-\eta},\quad\text{where}\quad\eta=\frac{C_1(K)}{\log|\Delta|4^n}. \tag{3.22}$$

Since $x^{2-\eta}=x^2\exp(-\eta\log x)$ and

$$\eta\log x+\frac{\alpha^2}{\eta}=\left(\sqrt{\eta\log x}-\frac{\alpha}{\sqrt{\eta}}\right)^2+2\alpha\sqrt{\log x}\geqq 2\alpha\sqrt{\log x},$$

where $\alpha=\sqrt{\alpha(K)\varepsilon_0}$, we have

$$-\eta\log x\leqq -2\alpha\sqrt{\log x}+\frac{\alpha^2}{\eta}. \tag{3.23}$$

From (3.23) we obtain

$$x^{2-\eta}\leqq x^2\exp\left(\frac{\alpha^2}{\eta}\right)\exp\left(-2\alpha\sqrt{\log x}\right)=R_6(K)x^2\exp\left(-2\alpha\sqrt{\log x}\right). \tag{3.24}$$

From (3.24), (3.22) we get

$$J_3+J_7 \underset{K}{\ll} x^2 \exp\left(-2\alpha \sqrt{\log x}\right). \tag{3.25}$$

From (3.25), (3.20) and (3.6) we have

$$\omega(x, z) \underset{K}{\ll} x^2 \exp\left(-2\alpha \sqrt{\log x}\right). \tag{3.26}$$

It is easy to see that the function $\varphi_z(x)$ in (3.5) is infinitely differentiable with respect to x for $|z| \leqq 1$. From (3.5) swe get

(3.27)

$$\varphi_z'(x) = \frac{\sin \pi z}{\pi} \int_r^\eta \frac{H(1-u, z)}{1-u} u^{-z} x^{1-u}\, du + \frac{1}{2\pi i} \int_{\gamma_r} \frac{H(s, z)}{s} (s-1)^{-z} x^s\, ds$$

(3.28)

$$\varphi_z''(x) = \frac{\sin \pi z}{\pi} \int_r^\eta H(1-u, z) u^{-z} x^{-u}\, du + \frac{1}{2\pi i} \int_{\gamma_r} H(s, z)(s-1)^{-z} x^{s-1}\, ds.$$

We remark that in the region $|s-1| \leqq \eta$, $|z| \leqq 1$ the function $H(s, z)$ is bounded, since the condition $|s-1| \leqq \eta$ implies $\operatorname{Re} s \geqq 1-\eta > \frac{1}{2}$ and by assumption (iv) and Lemma 4 we get

$$|H(s, z)| = |g(s, z)|\,|\exp\left(z \log (s-1)\zeta_K(s)\right)| \leqq R_7(K) \tag{3.29}$$

for $|s-1| \leqq \eta$ and $|z| \leqq 1$. From (3.29) we get

$$\int_r^\eta H(1-u, z) u^{-z} x^{-u}\, du \underset{K}{\ll} \int_r^\eta u^{-1} x^{-u}\, du. \tag{3.30}$$

Putting $u = \frac{v}{\log x}$ we obtain

$$\int_r^\eta u^{-1} x^{-u}\, du = \int_{r \log x}^{\eta \log x} v^{-1} e^{-v}\, dv. \tag{3.31}$$

For $|s-1| = r < \eta$ and $|z| \leqq 1$ we have

$$(s-1)^{-z} x^{s-1} \ll x^r r^{-1}$$

and therefore

$$\int_{\gamma_r} H(s, z)(s-1)^{-z} x^{s-1}\, ds \underset{K}{\ll} x^r. \tag{3.32}$$

Putting $r = \frac{1}{\log x}$ for sufficiently large x by (3.27), (3.30)—(3.32) follows that

$$\varphi_z''(x) = O_K(1). \tag{3.33}$$

Let $0<\xi<\dfrac{x}{2}$ and $t\in\langle x, x+\xi\rangle$ then

$$\left|\frac{1}{\xi}\int\limits_{x}^{x+\xi} P_z(t)\,dt-P_z(x)\right| \leqq \frac{1}{\xi}\int\limits_{x}^{x+\xi} |P_z(t)-P_z(x)|\,dt. \tag{3.34}$$

By assumption (ii) we have

$$|P_z(t)-P_z(x)| \leqq \sum_{x<N(I)\leqq t} |h(I, z)| \leqq R(K)(t-x)+O_K(t^{\beta}). \tag{3.35}$$

Since $x\leqq t\leqq x+\xi$ thus $0\leqq t-x\leqq\xi$ and therefore for $0<\xi<\dfrac{x}{2}$ and (3.34), (3.35) we obtain

$$\left|\frac{1}{\xi}\int\limits_{x}^{x+\xi} P_z(t)\,dt-P_z(x)\right| \leqq R(K)\xi+O_K(x^{\beta}). \tag{3.36}$$

We have

$$|P_z(x)-\varphi_z'(x)| \leqq \left|\frac{1}{\xi}\int\limits_{x}^{x+\xi} P_z(t)\,dt-\varphi_z'(x)\right|+\left|\frac{1}{\xi}\int\limits_{x}^{x+\xi} P_z(t)\,dt-P_z(x)\right|. \tag{3.37}$$

Since $\int\limits_{x}^{x+\xi} P_z(t)dt=\int\limits_{0}^{x+\xi} P_z(t)dt-\int\limits_{0}^{x} P_z(t)dt$ thus by (3.4) we get

$$\int\limits_{x}^{x+\xi} P_z(t)\,dt = \varphi_z(x+\xi)-\varphi_z(x)+\omega(x+\xi, z)-\omega(x, z). \tag{3.38}$$

For the function $\varphi_z(x)$ we have

$$\varphi_z(x+\xi)-\varphi_z(x) = \xi\varphi_z'(x)+\xi^2\int\limits_{0}^{1}(1-u)\varphi_z''(x+u\xi)\,du. \tag{3.39}$$

From (3.38) and (3.39) we get

(3.40)

$$\frac{1}{\xi}\int\limits_{x}^{x+\xi} P_z(t)\,dt = \varphi_z'(x)+\xi\int\limits_{0}^{1}(1-u)\varphi_z''(x+u\xi)du+\frac{\omega(x+\xi, z)-\omega(x-z)}{\xi}.$$

From (3.26) for $0<\xi<\dfrac{x}{2}$ we get

$$\omega(x+\xi, z) \underset{K}{\ll} x^2\exp\left(-2\alpha\sqrt{\log x}\right). \tag{3.41}$$

By (3.33) follows that

$$\varphi_z''(x+u\xi) = O_K(1). \tag{3.42}$$

From (3.40)—(3.42) we have

$$\left|\frac{1}{\xi}\int\limits_{x}^{x+\xi} P_z(t)\,dt-\varphi_z'(x)\right| \underset{K}{\ll} \xi+\frac{1}{\xi}x^2\exp\left(-2\alpha\sqrt{\log x}\right). \tag{3.43}$$

By (3.43), (3.37) and (3.36) and (3.36) it follows that

$$|P_z(x)-\varphi_z'(x)| \underset{K}{\ll} \xi+x^\beta+\frac{1}{\xi}x^2\exp\left(-2\alpha\sqrt{\log x}\right). \tag{3.44}$$

Putting $\xi=x\exp(-\alpha\sqrt{\log x})$ and remarking that

$$x^\beta < x\exp\left(-\alpha\sqrt{\log x}\right) \quad \text{for} \quad x>\exp\left(\frac{\alpha}{1-\beta}\right)^2 \quad \text{and} \quad 0\leqq\beta<1$$

we get

$$|P_z(x)-\varphi_z'(x)| \underset{K}{\ll} x\exp\left(-\alpha\sqrt{\log x}\right). \tag{3.45}$$

To finish the proof it remains to evaluate $\varphi_z'(x)$. Let $L_\eta=\{s\colon |s-1|<\eta\}$ and $L_1=\{z\colon |z|<1\}$. By (iv), $g(s,z)$ is holomorphic with respect to $s\in L_\eta$ for every fixed $|z|\leqq 1$ and it is holomorphic with respect to $z\in L_1$ for every fixed $s\in L_\eta$. Since $H(s,z)=g(s,z)\exp\left(z\log(s-1)\zeta_K(s)\right)$ thus $\dfrac{H(s,z)}{s}$ has this property. Let $L=L_\eta\times L_1$ then by a well-known theorem of Hartogs, $\dfrac{H(s,z)}{s}$ is holomorphic with respect to both variables $(s,z)\in L$. Therefore $\dfrac{H(s,z)}{s}$ can be represented in the form

$$\frac{H(s,z)}{s}=\sum_{j=0}^{\infty}B_j(z)(s-1)^j \tag{3.46}$$

where

$$B_j(z)=\frac{1}{2\pi i}\int\limits_{L_\delta}\frac{H(w,z)}{w(w-1)^{j+1}}\,dw, \quad j=0,1,\ldots \tag{3.47}$$

are holomorphic functions in the circle $|z|<1$. By Cauchy's inequality we have

$$|B_j(z)|\leqq\frac{M}{\eta^j}, \quad \text{for} \quad j=0,1,\ldots,|z|<1 \tag{3.48}$$

where

$$M=\max_{\substack{z\in L_1\\ |s-1|<\eta}}\left|\frac{H(s,z)}{s}\right|.$$

We note that $H(s, z)$ is bounded in the region $\sigma \geqq 1-\eta$, $|t|<2$, $|z| \leqq 1$, thus we have also

$$|H(s, z)| \leqq R_7(K) \quad \text{for} \quad |s-1| \leqq \eta, \quad |t|<2,\ |z| \leqq 1.$$

Since $|s-1| \leqq \eta$ implies $\operatorname{Re} s \geqq 1-\eta > \frac{1}{2}$ thus we have

$$\frac{|H(s, z)|}{|s|} \leqq \frac{R_7(K)}{\operatorname{Re} s} < \frac{R_7(K)}{1-\eta} < 2R_7(K) = M(K). \tag{3.49}$$

Let $|s-1| = \delta < \eta$, then putting in (3.46)

$$W_{q+1}(s, z) = \sum_{j=q+1}^{\infty} B_j(z)(s-1)^{j-q-1} \tag{3.50}$$

we have

$$|W_{q+1}(s, z)| \leqq M(K) \sum_{j=q+1}^{\infty} \frac{(\delta)^{j-q-1}}{\eta^j} = M(K) \frac{1}{\eta^q(\eta-\delta)} = M_1(K, q, \delta). \tag{3.51}$$

Substituting

$$\frac{H(s, z)}{s} = \sum_{j=0}^{q} = B_j(z)(s-1)^j + W_{q+1}(s, z)(s-1)^{q+1}$$

into (3.27) we get

(3.52)

$$\varphi_z'(x) = \sum_{j=0}^{q} xB_j(z) \left[\frac{\sin \pi(z-j)}{\pi} \int_r^{+\infty} u^{j-z} x^{-u}\, du + \frac{1}{2\pi i}(s-1)^{j-z} x^{s-1}\, ds + \right] + W(x, z),$$

where

$$\begin{aligned} W(x, z) = &-\sum_{j=0}^{q} xB_j(z) \frac{\sin \pi\ (z-j)}{\pi} \int_\eta^{+\infty} u^{j-z} x^{-u}\, du + \\ &+\frac{x}{2\pi i} \int_{\gamma_r} W_{q+1}(s, z)(s-1)^{q+1-z} x^{s-1}\, ds + \\ &+x \frac{\sin \pi(z-q-1)}{\pi} \int_r^{\eta} W_{q+1}(1-u, z) u^{q+1-z} x^{-u}\, du. \end{aligned} \tag{3.53}$$

Since for $0 \leqq j \leqq q$ we have

$$\frac{\sin \pi(z-j)}{\pi} \int_r^{+\infty} u^{j-z} x^{-u}\, du + \frac{1}{2\pi i} \int_{\gamma_r} x^{s-1}(s-1)^{j-z}\, ds = \frac{(\log x)^{z-j-1}}{\Gamma(z-j)}$$

thus from (3.52) we get

$$\varphi_z'(x) = \sum_{j=0}^{q} xB_j(z) \frac{(\log x)^{z-j-1}}{\Gamma(z-j)} + W(x, z). \tag{3.54}$$

For estimating $W(x, z)$ we remark that

$$\int_{\eta}^{+\infty} u^{j-z} x^{-u}\, du \underset{K}{\ll} (\log x)^{-q-3}, \quad 0 \leqq j \leqq q \tag{3.55}$$

$$\int_{\gamma_r} W_{q+1}(s, z)(s-1)^{q+1-z} x^{s-1}\, du \underset{K}{\ll} (\log x)^{\operatorname{Re} z-q-2}. \tag{3.56}$$

$$\int_{r}^{\eta} W_{q+1}(1-u, z) u^{q+1-z} x^{-u}\, du \underset{K}{\ll} (\log x)^{\operatorname{Re} z-q-2}. \tag{3.57}$$

From (3.55)—(3.57) and (3.53) we obtain

$$W(x, z) \underset{K}{\ll} x(\log x)^{\operatorname{Re} z-q-2}. \tag{3.58}$$

By (3.58), (3.54) follows that

$$\varphi_z'(x) = \sum_{j=0}^{q} x B_j(z) \frac{(\log x)^{z-j-1}}{\Gamma(z-j)} + O_K\big(x(\log x)^{\operatorname{Re} z-q-2}\big). \tag{3.59}$$

Finally we remark that for sufficiently large x we have

$$P_x(z) = \sum_{\substack{I \in G_K \\ N(I) \leqq x}} h(I, z) = \sum_{j=0}^{q} x \frac{B_j(z)}{\Gamma(z-j)} (\log x)^{z-j-1} + O_K\big(x(\log x)^{\operatorname{Re} z-q-2}\big)$$

and the proof is complete.

4. Proof of Theorem 2

We remark that by (a) we have

$$\left|\sum_{I \in G_K} \frac{h(I, z)}{N(I)^s}\right| \leqq R_1 \sum_{I \in G_K} \frac{1}{N(I)^\sigma} < \infty, \quad \text{for} \quad \sigma > 1, |z| \leqq 1. \tag{4.1}$$

By (c) it follows that $h(I, z)$ is multiplicative with respect to $I \in G_K$. From (4.1) and Lemma 7.1, [3] follows that

$$\sum_{I \in G_K} \frac{h(I, z)}{N(I)^s} = \prod_{\mathfrak{p}} \left(1 + \sum_{k=1}^{\infty} \frac{h(\mathfrak{p}^k, z)}{N(\mathfrak{p})^{ks}}\right) \tag{4.2}$$

for $\operatorname{Re} s > 1$ and $|z| \leqq 1$.

Consider the product

$$\prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^z \left(1 + \sum_{k=1}^{\infty} \frac{h(\mathfrak{p}^k, z)}{N(\mathfrak{p})^{ks}}\right). \tag{4.3}$$

It is easy to see that (4.3) can be represented in the following form:

$$\prod_{\mathfrak{p}} \big(1 + u_{\mathfrak{p}}(s, z)\big) \exp\big(-v_{\mathfrak{p}}(s, z)\big) \tag{4.4}$$

where

$$u_{\mathfrak{p}}(s,z)=\sum_{k=1}^{\infty}\frac{h(\mathfrak{p}^k,z)}{N(\mathfrak{p})^{ks}}, \tag{4.5}$$

$$v_{\mathfrak{p}}(s,z)=z\sum_{k=1}^{\infty}\frac{1}{kN(\mathfrak{p})^{ks}}. \tag{4.6}$$

Since $|h(I,z)|\leqq R_1$ for $I\in G_K$ and $|z|\leqq 1$ thus

$$|u_{\mathfrak{p}}(s,z)|\leqq R_1\sum_{k=0}^{\infty}\frac{1}{N(\mathfrak{p})^{k\sigma}}=R_1\frac{1}{N(\mathfrak{p})^{\sigma}-1}=U_{\mathfrak{p}}. \tag{4.7}$$

From (4.7) for $\sigma\geqq\sigma_1$ we have

$$\sum_{\mathfrak{p}}U_{\mathfrak{p}}^2=R_1^2\sum_{\mathfrak{p}}\frac{1}{(N(\mathfrak{p})^{\sigma}-1)^2}\leqq R_2\sum_{\mathfrak{p}}\frac{1}{N(\mathfrak{p})^{2\sigma_1}}<\infty \tag{4.8}$$

for $\sigma_1>\frac{1}{2}$. Similarly we get for $\sigma\geqq\sigma_1$

$$|u_{\mathfrak{p}}(s,z)-v_{\mathfrak{p}}(s,z)|\leqq\frac{|h(\mathfrak{p},z)-z|}{N(\mathfrak{p})^{\sigma}}+\sum_{k=2}^{\infty}\left|\frac{h(\mathfrak{p}^k,z)}{N(\mathfrak{p})^{ks}}-\frac{z}{kN(\mathfrak{p})^{ks}}\right| \tag{4.9}$$

where

$$\sum_{k=1}^{\infty}\left|\frac{h(\mathfrak{p}^k,z)}{N(\mathfrak{p})^{ks}}-\frac{z}{kN(\mathfrak{p})^{ks}}\right|\leqq R_3\frac{1}{N(\mathfrak{p})^{2\sigma_1}}=\overline{V}_{\mathfrak{p}}. \tag{4.10}$$

Because $\sum_{\mathfrak{p}}\overline{V}_{\mathfrak{p}}<\infty$ for $\sigma_1>\frac{1}{2}$ and by (b) we have

$$\sum_{\mathfrak{p}}\frac{|h(\mathfrak{p},z)-z|}{N(\mathfrak{p})^{\sigma}}<\infty\quad\text{for}\quad\sigma\geqq\sigma_1,\quad\text{where}\quad\frac{1}{2}<\sigma_1\leqq 1-\eta.$$

Thus we obtain from (4.9) and (4.10)

$$|u_{\mathfrak{p}}(s,z)-v_{\mathfrak{p}}(s,z)|\leqq\frac{|h(\mathfrak{p},z)-z|}{N(\mathfrak{p})^{\sigma}}+\overline{V}_{\mathfrak{p}}=V_{\mathfrak{p}}$$

that is

$$\sum_{\mathfrak{p}}V_{\mathfrak{p}}<\infty\quad\text{for}\quad\sigma\geqq\sigma_1>\frac{1}{2}. \tag{4.11}$$

From (4.8) and (4.11) and Lemma 2 we get that the product (4.4) is absolutely and uniformly convergent and bounded in the region $\sigma\geqq\sigma_1>\frac{1}{2}$, $|z|\leqq 1$, so we have

$$g(s,z)=\prod_{\mathfrak{p}}\left(1+u_{\mathfrak{p}}(s,z)\right)\exp\left(-v_{\mathfrak{p}}(s,z)\right),\quad |g(s,z)|\leqq N(K) \tag{4.12}$$

for $\sigma\geqq\sigma_1>\frac{1}{2}$, $|z|\leqq 1$. By (d), $h(I,z)$ is holomorphic with respect to z in the circle $|z|<1$ therefore the functions given by formulas (4.5) and (4.6) are holomorphic with respect to z in the circle $|z|<1$ for every fixed s such that

$$\operatorname{Re} s = \sigma \geqq \sigma_1 > \frac{1}{2}.$$

From (4.5) follows that $u_p(s, z)$ and $v_p(s, z)$ are holomorphic with respect to s in the region $\sigma \geqq \sigma_1 > \frac{1}{2}$ for every fixed z, $|z| \leqq 1$. Therefore $g(s, z)$ safitsies (iv).

From (4.12) and (4.3) and (4.2) we obtain

$$\sum_{I \in G_K} \frac{h(I, z)}{N(I)^s} = g(s, z)\zeta_K^z(s)$$

so that the assumption (iii) is fullfied. To finish the proof it remains to verify (ii). From (a) we have $|h(I, z)| \leqq R_1$ and therefore we get

$$\sum_{x<N(I)\leqq y} |h(I, z)| \leqq R_1 \sum_{x<N(I)\leqq y} 1. \tag{4.13}$$

By Lemma 5 it follows that

$$\sum_{\substack{I \in G_K \\ x<N(I)\leqq y}} 1 = \lambda h(y-x) + O_K(y^{1-\frac{2}{n+1}}). \tag{4.14}$$

Putting $\beta = 1 - \frac{2}{n+1}$ we have $0 \leqq \beta < 1$ for $n \geqq 1$ and from (4.14) and (4.13) we get

$$\sum_{x<N(I)\leqq y} |h(I, z)| \leqq R_1 \lambda h(y-x) + O_K(y^\beta) = R(K)(y-x) + O_K(y^\beta)$$

and the proof is complete.

References

[1] H. Delange, Sur des formules de Atle Selberg, *Acta Arith.*, **19** (1971), 105—146.
[2] E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig—Berlin, 1927).
[3] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*. PWN (Warszawa, 1974).
[4] A. Selberg, Note on a paper by L. G. Sathe, *J. Indian Math. Soc.*, **18** (1954), 83—87.
[5] H. M. Sark, Some effective cases of the Brauer—Siegel theorem, *Inven. Math.*, **23** (1974), 135—152.
[6] K. Wiertelak, On the density of some sets of primes, II. *Acta Arith.*, **34** (1978), 197—210.

(Received June 18, 1984)

DEPARTMENT OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
ZIELONA GÓRA
PLAC SŁOWIAŃSKI 8. B.
POLAND

Acta Math. Hung.
50 (1—2) (1987), 17—20.

HIGH ORDER SMOOTHNESS

M. J. EVANS (Raleigh)

All functions considered in this paper will be assumed to be Lebesgue measurable and defined on all of the real line R. In a recent paper [3], T. K. Dutta examined the notion of a generalized smooth function and obtained several interesting results for continuous generalized smooth functions analogous to those obtained for continuous smooth functions in [5] and [8]. The purpose of the present paper is to sharpen the hypotheses of Dutta's results by taking advantage of known results for smooth functions and symmetrically differentiable functions along with a form of a lemma due to Auerbach [1], which is slightly stronger in appearance than that quoted by Dutta [3]. These observations used in conjunction with Dutta's methods generate stronger results than those obtained in [3].

We begin by defining the necessary terms. Let f denote a function and let $x_0\in R$ with $f(x_0)=\alpha_0$. If there exist real numbers $\alpha_2, \alpha_4, \dots, \alpha_{2k}$ such that

$$\{f(x_0+h)+f(x_0-h)\}/2 = \sum_{r=0}^{k} \frac{h^{2r}}{(2r)!}\alpha_{2r}+o(h^{2k}),$$

then α_{2k} is called the symmetric de la Vallée Poussin (d.l.V.P.) derivative of f at x_0 of order $2k$ and is denoted by $D^{2k}f(x_0)$. Similarly, if there are numbers $\beta_1, \beta_3, \dots, \dots, \beta_{2k+1}$ such that

$$\{f(x_0+h)-f(x_0-h)\}/2 = \sum_{r=0}^{k} \frac{h^{2r+1}}{(2r+1)!}\beta_{2r+1}+o(h^{2k+1}),$$

then β_{2k+1} is called the symmetric d.l.V.P. derivative of f at x_0 of order $2k+1$ and is denoted by $D^{2k+1}f(x_0)$.

Let m be a natural number and assume that $D^{2m-2}f(x_0)$ exists. Let

$$\theta_{2m}(f;\, x_0, h) = \frac{(2m)!}{h^{2m}}\left[\{f(x_0+h)+f(x_0-h)\}/2-\sum_{r=0}^{m-1}\frac{h^{2r}}{(2r)!}D^{2r}f(x_0)\right].$$

Then f is said to be smooth of order $2m$ at x_0 (or $2m$-smooth at x_0) provided $\lim_{h\to 0} h\theta_{2m}(f;\, x_0, h)=0$. If f is $2m$-smooth at each $x\in R$, we say that f is $2m$-smooth.

Smoothness of order $2m+1$ $(m=1, 2, \dots)$ is then defined similarly.

Next, if there are numbers $\gamma_0, \gamma_1, \dots, \gamma_m$ such that $f(x_0)=\gamma_0$ and

$$f(x_0+h) = \sum_{r=0}^{m}\frac{h^r}{r!}\gamma_r+o(h^m),$$

then γ_m is called the unsymmetric d.l.V.P. derivative (or the Peano derivative) of f at x_0 of order m and is denoted by $f_m(x_0)$. It is well known that if the ordinary m^{th} derivative of f at x_0, $f^{(m)}(x_0)$, exists then so does $f_m(x_0)$ and $f_m(x_0)=f^{(m)}(x_0)$. Similarly, if $f_m(x_0)$ exists, then so does $D^mf(x_0)$ and they are equal.

A set A in R is said to be scattered (separated or *clairseme*) provided that A has no dense in itself subset. A function f is said to be a Baire* one function provided that for each closed set P there are numbers $a<b$ such that $(a, b)\cap P\neq\emptyset$ and the restriction of f to $(a, b)\cap P$ is continuous. (Dutta [3] referred to this property of f as property $\mathscr{R}$.) This notion was introduced by O'Malley [7] and is important in what follows. It is easily seen that if the points at which a function f fails to be continuous form a scattered set, then f is a Baire* one function. A useful tool in showing that a function is a Baire* one function is the following lemma due to Auerbach [1].

LEMMA A. *Let P be a nonempty closed set and $\{f_n\}$ a sequence of functions such that the restriction of each f_n to P is continuous. Further, suppose that Σa_n is a convergent series of positive numbers and that for each $x\in P$ there is a natural number $N(x)$ such that $|f_n(x)|\leqq a_n$ whenever $n>N(x)$. Then there exist numbers $a<b$ such that $(a, b)\cap P\neq\emptyset$ and Σf_n converges uniformly on $(a, b)\cap P$.*

Although Auerbach's original lemma was stated for functions f_n continuous on a closed interval instead of a closed set, only the obvious changes in his proof need be made to prove Lemma A.

The next two results are the ones that allow us to sharpen the hypotheses in the results of Dutta [3].

THEOREM 1. *If f is continuous and m-smooth, then $D^{m-2}f$ is a Baire* one function.*

PROOF. First, consider the case where m is even; say $m=2k$. For $k=1$ there is nothing to prove. Assume the theorem holds for $k=1, 2, \dots, r$. We shall show that it is true $k=r+1$ as well. Let P be a nonempty closed set. By our assumption, there exist numbers $a<b$ such that $(a, b)\cap P\neq\emptyset$ and the restrictions to $(a, b)\cap P$ of $D^2f, D^4f, \dots, D^{2r-2}f$ are continuous. Choose an interval $[a^*, b^*]\subseteqq(a, b)$ such that $(a^*, b^*)\cap P\neq\emptyset$. Let $\{h_n\}$ be a strictly decreasing sequence of positive terms such that $\sum h_n$ converges and $a^*-h_n>a$, $b^*+h_n<b$ for all n. For each natural number n let $\Psi_n^{2r}(x)=\theta_{2r}(f; x, h_n)$ and note, in light of the continuity of $f, D^2f, D^4f, \dots, D^{2r-2}f$, that the restriction of Ψ_n^{2r} to $[a^*, b^*]\cap P$ is continuous. Furthermore, since f is $(2r+2)$-smooth, we have that for each x, $\lim\limits_{h\to 0} h\theta_{2r+2}(f; x, h)=0$, and hence

$$\lim_{n\to\infty} [\Psi_n^{2r}(x)-D^{2r}f(x)]\, h_n^{-1} = 0.$$

Thus, there is a natural number $N(x)$ such that $|\Psi_n^{2r}(x)-D^{2r}f(x)|<h_n$ for all $n>N(x)$. For $n>N(x)$ we have

$$|\Psi_{n+1}^{2r}(x)-\Psi_n^{2r}(x)| \leqq |\Psi_{n+1}^{2r}(x)-D^{2r}f(x)|+|\Psi_n^{2r}(x)-D^{2r}f(x)| < h_{n+1}+h_n < 2h_n.$$

So the series $\sum\limits_n (\Psi_{n+1}^{2r}-\Psi_n^{2r})$ satisfies all the conditions of Lemma A on the closed set $[a^*, b^*]\cap P$. Hence there are numbers $a'<b'$ such that $[a', b']\subseteqq[a^*, b^*]$, $(a', b')\cap P\neq\emptyset$, and $\sum_n (\psi^{2r}_{n+1}-\Psi^{2r}_n)$ converges uniformly on $[a', b']\cap P$. Hence $\Psi^{2r}_1+\sum_{n=1}^{\infty}(\Psi^{2r}_{n+1}-\Psi^{2r}_n)$, i.e. $\{\Psi^{2r}_n\}$ converges uniformly on $[a', b']\cap P$ and so the limit function $D^{2r}f$ restricted to $[a', b']\cap P$ is continuous. Therefore, $D^{2r}f$ is a Baire* 1 function, and the theorem holds for all even natural numbers m.

For m odd, say $m=2k+1$, the proof is very similar. The initial case $k=1$ can be handled by setting $\Psi^1_n(x)=(f(x+h_n))-(f(x-h_n))/2h_n$, noting that $\lim_{n\to\infty}(\Psi^1_n(x)-D^1f(x))/h_n=0$, and utilizing Lemma A as above. Assuming the theorem true for $k=1, 2, \ldots, r$, the inductive stage is handled as above using $\Psi^{2r+1}_n(x)==\theta_{2r+1}(f; x, h_n)$.

THEOREM 2. *If f is m-smooth, then the set of points at which f is discontinuous is scattered.*

PROOF. If m is even, then f is 2-smooth; i.e., f is smooth. According to Theorem 2.1 in [4], the set of points of discontinuity is scattered. On the other hand, if m is odd, then f is 3-smooth. Consequently, f has a finite symmetric derivative D^1f everywhere. According to Theorem 1 in [2], the set of points of discontinuity of f is scattered.

THEOREM 3. *If f is m-smooth, then*

i) $f^{(m-2)}$ exists and is continuous on a dense open set,

ii) the set of points where $f_{m-1}(x)$ exists and is finite of the power of the continuum in every interval.

PROOF. To prove i), let (a, b) be any interval. Since the set of discontinuities of f is scattered, there is a subinterval $(a', b')\subseteqq(a, b)$ on which f is continuous. From Theorem 1 we have that $D^{m-2}f$ is continuous on a subinterval $(a'', b'')\subseteqq(a', b')$. Applyng Lemmas 2 and 3 from [3] we see that $f^{(m-2)}$ exists on (a'', b'') and is, of course, continuous there since $f^{(m-2)}=D^{m-2}f$.

To prove ii), again let (a, b) be any interval, utilize Theorem 2 to obtain a subinterval on which f is continuous, and apply Theorem 3 in [3] to f on that subinterval.

We now turn to those results of Dutta [3] wherein in addition to assuming that f is m-smooth, we assume that f_{m-2} exists.

THEOREM 4. *Let f be an m-smooth function which has the Darboux property and for which f_{m-2} exists everywhere. Then if we let $E=\{x: f_{m-1}(x)$ exists$\}$, we have*

i) f_{m-1} has the Darboux property on E,

ii) if $f_{m-1}(x)\geqq 0$ for all $x\in E$, then f_{m-2} is continuous and nondecreasing.

PROOF. First consider the case $m=2$. O'Malley showed that i) holds in [7]. To obtain ii), we first suppose $f'(x)>0$ for all $x\in E=\{x: f'(x)$ exists$\}$. If f is continuous then f is nondecreasing according to the lemma on page 27 of [5]. If f is not continuous, then because it has the Darboux property it cannot be monotone. Hence there are two points, $a<b$, where $f(a)=f(b)$. Then according to Theorem

3 in [7] f has a local extremum at some point $x_0 \in (a, b)$. Because of the smoothness condition, it is easily seen that $f'(x_0)$ exists and equals 0. This contradiction shows that f must be continuous and monotone. The general case, $f'(x) \geqq 0$ on E, is handled in the usual manner by considering $g(x) = f(x) + \varepsilon x$ for arbitrary positive numbers ε.

Finally, as Dutta observed at the end of his paper [3], if $m > 2$, f must be continuous since it is in fact differentiable under the stated hypotheses. Consequently, the validity of i) and ii) is an immediate consequence of Theorem 1 in this paper and Theorems 4 and 5 in [3].

The reader will note that analogous sharpening of the hypotheses can be made in Theorems 6, 7 and 8 in [3].

It should be noted that Neugebauer has provided an example [5, p. 27], showing that neither i) nor ii) is valid in Theorem 4 if the Darboux hypothesis is dropped in the $m=2$ case.

The function constructed by Oliver in Theorem 5 of [6] shows that the conclusion of part i) of Theorem 3 is the best possible, in that given any open dense set G, there is an m-smooth function for which $f^{(m-2)}(x)$ exists if and only if $x \in G$. The same example shows that the conslusion of Theorem 1 cannot be replaced by the stronger statement that the discontinuities of $D^{m-2}f$ form a scattered set, except, of course, in the $m=2$ case.

References

[1] H. Auerbach, Sur les dérivées généralisées, *Fund. Math.*, **8** (1926), 49—55.
[2] Z. Charzynski, Sur les fonctions dont la derivée symétrique est partout finite, *Fund. Math.*, **21** (1933), 214—225.
[3] T. K. Dutta, Generalised smooth functions, *Acta Math. Acad. Sci. Hungar.*, **40** (1982), 29—37.
[4] M. J. Evans and L. Larson, The continuity of symmetric and smooth functions, *Acta Math. Hung.*, **43** (1984), 251—257.
[5] C. J. Neugebauer, Symmetric, continuous and smooth functions, *Duke Math. J.*, **31** (1964), 23—32.
[6] H. W. Oliver, The exact Peano derivative, *Trans. Amer. Math. Soc.*, **76** (1954), 444—456.
[7] R. J. O'Malley, Baire* l, Darboux functions, *Proc. Amer. Math. Soc.*, **60** (1976), 187—192.
[8] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

(Received July 27, 1984)

DEPARTMENT OF MATHEMATICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA 27695-8205
U.S.A.

Acta Math. Hung.
50 (1—2) (1987), 21—31.

STRONG SUMMABILITY AND CONVERGENCE OF SUBSEQUENCES OF ORTHOGONAL SERIES

H. SCHWINN (Berlin)

1. Introduction

We take as a basis general real orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x), \quad \sum_{n=0}^{\infty} c_n^2 < \infty, \tag{1}$$

where $\{\varphi_n(x)\}$ is an arbitrary orthonormal system on [0, 1], i.e. $\int\limits_0^1 \varphi_i(x)\varphi_j(x)dx=\delta_{ij}$ $(i,j=0,1,\dots)$. We consider a summability method $A=(a_{nk})$ and the means

$$t_n(x) = \sum_{k=0}^{\infty} a_{nk} s_k(x),$$

where $s_k(x)$ denote the partial sums of the series (1). We always assume A being regular, i.e. A transforms a general convergent sequence into a convergent one with the same limit. It is well-known that A is regular if and only if the following conditions are fulfilled:

(a) $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = 1$

(b) $\sum\limits_{k=0}^{\infty} |a_{nk}| \leqq K \quad (n=0,1,\dots)$[1]

(c) $\lim\limits_{n\to\infty} a_{nk} = 0 \quad (k=0,1,\dots)$

(cf. K. Zeller; W. Beekmann [15], p. 57).

The series (1) is called *A-summable* if $t_n(x)\to f(x)$ a.e. on [0, 1], and *strongly A-summable with order* $\gamma>0$ ($[A]^\gamma$-summable) if the strong means

$$\tau_n^{(\gamma)}(x) = \sum_{k=0}^{\infty} |a_{nk}||s_k(x)-f(x)|^\gamma \to 0 \quad \text{a.e.}$$

on [0, 1] with a suitable function $f(x)$. In the following "convergence" always means "convergence a.e.". It is obvious by Hölder's inequality that with regular methods A, $[A]^\gamma$-summability implies both $[A]^{\gamma'}$-summability, if $\gamma>\gamma'>0$ and A-summability if $\gamma\geqq 1$.

---

[1] $K, L, \dots$ denote constants.

In the first part we want to prove direct theorems for strong summability. The convergence of the sequence $\{s_{n_i}(x)\}$ may imply $[A]^\gamma$-summability with arbitrary exponents $\gamma>0$, if $\{n_i\}$ is chosen in a suitable way depending on A alone. Strong $(C, 1)$-summability of orthogonal series with exponents $\gamma\leqq 2$ were first investigated by A. Zygmund [18] and S. Borgen [4]. The exponent $\gamma=2$ may be considered with respect to $\varphi_n(x)\in L^2[0, 1]$ as a natural borderline. Exponents $\gamma>2$ were first considered for Cesàro methods (C, α), $\alpha>0$, by G. Sunouchi [14], later on by O. A. Ziza [16], [17] for series-to-sequence methods (φ, λ) and recently by L. Leindler [9] for generalized Abel methods (A, p). Our interest is directed to general matrix methods.

The second part is devoted to inverse theorems. From A-summability resp. $[A]^\gamma$-summability of an orthogonal series one may deduce convergence of a subsequence $\{s_{n_i}(x)\}$, $\{n_i\}$ depending on A alone. The classical result goes back to A. N. Kolmogoroff [8] and S. Kaczmarz [7] who proved that the series (1) is $(C, 1)$-summable if and only if $\{s_{2^n}(x)\}$ is convergent. We want to stress here the relations to general gap theorems and gap methods. Finally we want to point out the equivalence of summability and strong summability for orthogonal series when applying some special methods.

2. Strong summability

For a given regular method $A=(a_{nk})$ we consider the column majorants

$$\alpha_k = \sup_n |a_{nk}| \quad (k = 0, 1, \ldots), \tag{2}$$

and prove the following theorems.

THEOREM 1. *For a given regular method* $A=(a_{nk})$ *and an increasing sequence* $\{n_i\}$ *of natural numbers let*

$$\sum_{k=n_i+1}^{n_{i+1}-1} \alpha_k \leqq M \quad (i = 0, 1, \ldots) \tag{3}$$

be fulfilled. Then the convergence of $\{s_{n_i}(x)\}$ *implies* $[A]^\gamma$*-summability of* (1) *for* $0<\gamma\leqq 2$.

THEOREM 2. *For a given regular method* $A=(a_{nk})$ *let the sequences* $\{\alpha_k\}$ *and* $\{n_i\}$ *satisfy condition* (3) *and let*

$$\alpha_k \leqq \frac{M^*}{n_{i+1}-n_i} \quad (n_i \leqq k \leqq n_{i+1}) \tag{4}$$

be fulfilled. Then the convergence of $\{s_{n_i}(x)\}$ *implies* $[A]^\gamma$*-summability of* (1) *for any* $\gamma>0$.

To prove these theorems we need some lemmas. At first we state a result concerning Cesàro methods $(C, \alpha)=\left(A_{n-k}^{\alpha-1}/A_n^\alpha\right)$.

LEMMA 1 (G. Sunouchi [14]). *Let* $\alpha>0$, $\gamma>0$; *then*

$$\int_0^1 \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} |s_k(x)-\sigma_k(x)|^\gamma \right)^{2/\gamma} \right\} dx \leqq C(\alpha;\gamma) \sum_{n=0}^{\infty} c_n^2$$

with $\sigma_k(x)=\frac{1}{k+1}\sum_{\mu=0}^{k} s_\mu(x)$ *and* $A_n^\alpha=\binom{n+\alpha}{n}$.

Next we construct the modified arithmetic means for (1) with an increasing $\{n_i\}$:

$$\sigma_n^*(x) = \sigma_n^*(x;\{n_i\}) = \frac{1}{n+n_{i+1}-2n_i+1} \sum_{k=n_i}^{n} (s_k(x)-s_{n_i}(x)) \tag{5}$$

$$(n_i \leqq n < n_{i+1};\ i=0,1,\ldots),$$

and prove

LEMMA 2. *Let* $\{n_i\}$ *be an increasing sequence. Then* $\sigma_n^*(x)\to 0$ *a.e.*

PROOF. Let us consider

$$\delta_i(x) = \sup_{n_i\leqq n<n_{i+1}} |\sigma_n^*(x)|^2,$$

then by $\sigma_{n_i}^*(x)=0$, using Schwarz inequality, we get

$$\delta_i(x) \leqq \Big\{ \sum_{n=n_i+1}^{n_{i+1}-1} |\sigma_n^*(x)-\sigma_{n-1}^*(x)| \Big\}^2 \leqq (n_{i+1}-n_i) \sum_{n=n_i+1}^{n_{i+1}-1} (\sigma_n^*(x)-\sigma_{n-1}^*(x))^2.$$

With the aid of the representation

$$\sigma_n^*(x) = \sum_{k=n_i+1}^{n} \left(1-\frac{k+n_{i+1}-2n_i}{n+n_{i+1}-2n_i+1}\right) c_k\varphi_k(x)$$

his leads us for $n_i\leqq n<n_{i+1}$, resp. $n_{i+1}-n_i\leqq n+n_{i+1}-2n_i<2(n_{i+1}-n_i)$ to

$$\int_0^1 \delta_i(x)\,dx \leqq (n_{i+1}-n_i) \sum_{n=n_i+1}^{n_{i+1}-1} \frac{1}{(n+n_{i+1}-2n_i)^4} \sum_{k=n_i+1}^{n} (k+n_{i+1}-2n_i)^2 c_k^2.$$

Obviously $\sum_{i=0}^{\infty}\int_0^1 \delta_i(x)dx<\infty$ and thus by B. Levi's theorem $\sum_{i=0}^{\infty}\delta_i(x)<\infty$ a.e. This proves the assertion.

LEMMA 3. *Let* $\{n_i\}$ *be an increasing sequence and let* $\gamma>0$. *Then*

$$\int_0^1 \left\{ \frac{1}{2(n_{i+1}-n_i)} \sum_{k=n_i}^{n_{i+1}-1} |s_k(x)-s_{n_i}(x)-\sigma_k^*(x)|^\gamma \right\}^{2/\gamma} dx \leqq L \sum_{n=n_i+1}^{n_{i+1}-1} c_n^2 \quad (i=0,1,\ldots).$$

PROOF. We construct the first partial sums of a new orthogonal series $\Sigma\bar{c}_n\tilde{\Phi}_n(x)$ in the following way:

$$\tilde{s}_n(x) = \begin{cases} 0 & (n=0,1,\ldots,n_{i+1}-n_i), \\ \sum_{k=n_i+1}^{n-(n_{i+1}-2n_i)} c_k\varphi_k(x) & (n=n_{i+1}-n_i+1,\ldots,2(n_{i+1}-n_i)). \end{cases}$$

The corresponding arithmetic means $\tilde{\sigma}_n(x)$ of this sequence are

$$\tilde{\sigma}_n(x) = \begin{cases} 0 & (n = 0, 1, \ldots, n_{i+1}-n_i) \\ \sigma^*_{n-(n_{i+1}-2n_i)}(x) & (n = n_{i+1}-n_i+1, \ldots, 2(n_{i+1}-n_i)). \end{cases}$$

Now by Lemma 1 the estimation is proved.

LEMMA 4. *Let* $A=(a_{nk})$ *be a regular summability method and let* $s_k \geqq 0$. *If with the majorants* (2) $\sum_{k=0}^{\infty} \alpha_k s_k < \infty$ *is fulfilled, then*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} s_k = 0.$$

Applying the regularity conditions the statement of this lemma is easily proved.

PROOF OF THEOREM 2. As already mentioned strong summability with higher order implies the one with a smaller order. Therefore we may assume $\gamma \geqq 2$. We pick up again the means $\sigma^*_n(x)$ (cf. (5)) and the function $f(x)$ determined uniquely a.e. by the Riesz—Fischer theorem. Now with $l(k)=i$ if $n_i \leqq k < n_{i+1}$

$$\tau_n^{(\gamma)}(x) = \sum_{k=0}^{\infty} |a_{nk}| \cdot |s_k(x) - f(x)|^{\gamma} =$$

$$= O(1)\Big\{\sum_{k=0}^{\infty} |a_{nk}| \cdot |s_k(x) - s_{n_{l(k)}}(x) - \sigma^*_k(x)|^{\gamma} + \sum_{k=0}^{\infty} |a_{nk}| \cdot |\sigma^*_k(x)|^{\gamma} +$$

$$+ \sum_{k=0}^{\infty} |a_{nk}| \cdot |s_{n_{l(k)}}(x) - f(x)|^{\gamma}\Big\} =: O(1)\{\tau_n^{(\mathrm{I})}(x) + \tau_n^{(\mathrm{II})}(x) + \tau_n^{(\mathrm{III})}(x)\}.$$

Since the method $(|a_{nk}|)$ transforms the space of nullsequences c_0 into c_0, so $\tau_n^{(\mathrm{III})}(x) \to 0$ a.e., and by Lemma 2 we also get $\tau_n^{(\mathrm{II})}(x) \to 0$ a.e. Then with the majorants α_k (cf. (2)) we get

$$\tau_n^{(\mathrm{I})}(x) \leqq \sum_{i=0}^{\infty} \sum_{k=n_i}^{n_{i+1}-1} \alpha_k |s_k(x) - s_{n_i}(x) - \sigma^*_k(x)|^{\gamma}. \tag{6}$$

We prove that the series on the right is finite a.e. For (3), (4) and Lemma 3 with consideration to the estimation $\{\Sigma a_i\}^p \leqq \Sigma a_i^p$ $(a_i \geqq 0;\ 0<p<1)$ yields

$$\int_0^1 \{\sum_{i=0}^{\infty} \sum_{k=n_i}^{n_{i+1}-1} \alpha_k |s_k(x) - s_{n_i}(x) - \sigma^*_k(x)|^{\gamma}\}^{2/\gamma}\, dx =$$

$$= O(1) \sum_{i=0}^{\infty} \left\{\frac{1}{2(n_{i+1}-n_i)} \sum_{k=n_i}^{n_{i+1}-1} |s_k(x) - s_{n_i}(x) - \sigma^*_k(x)|^{\gamma}\right\}^{2/\gamma} dx =$$

$$= O(1) \sum_{i=0}^{\infty} \sum_{n=n_i+1}^{n_{i+1}-1} c_k^2 < \infty.$$

This shows the finiteness of the series in (6). Lemma 4 immediately furnishes the assertion.

The proof of Theorem 1 may be carried out in a similiar way for the exponent $\gamma=2$ without using the means $\sigma_n^*(x)$. We only use the estimation

$$\int_0^1 \sum_{k=n_i}^{n_{i+1}-1} \alpha_k |s_k(x)-s_{n_i}(x)|^2\,dx \leqq M \sum_{k=n_i}^{n_{i+1}-1} c_k^2.$$

We omit the proof.

The relations of equivalence between convergence of subsequences and (strong) summability of orthogonal series show that condition (3) may not be weakened in general. On the other hand we state

Remark 1. For a regular method $A=(a_{nk})$ and an increasing sequence $\{n_i\}$ let

$$\lim_{i\to\infty} \sum_{k=n_i}^{n_{i+1}-1} \alpha_k = 0.$$

Then there exists an $[A]^\gamma$-summable orthogonal series (1) with divergent partial sums $\{s_{n_i}(x)\}$ for any $0<\gamma\leqq 2$.

For in this case there exists an increasing sequence $\{\mu_i\}$ with

$$\sum_{k=\mu_i+1}^{\mu_{i+1}} \alpha_k \leqq M^* \quad (i=0,1,\ldots)$$

exhibiting the following property: the number of members out of $\{n_i\}$ between two adjacent μ_j and μ_{j+1} $(j=0,1,\ldots)$ is not bounded. Then, by [13], there exists a series (1) with divergent partial sums $\{s_{n_i}(x)\}$ and convergent partial sums $\{s_{\mu_i}(x)\}$.

Theorem 1 applied to $\{\mu_i\}$ shows the $[A]^\gamma$-summability of the series (1) for $0<\gamma\leqq 2$.

Condition $\lim_{k\to\infty} \alpha_k=0$, which corresponds to the particular case $n_i=i$ $(i=0,1,\ldots)$, plays a certain role in the field of summability of general divergent sequences. It is well-known that such a method sums at least a divergent sequence (cf. K. Zeller; W. Beekmann [15], p. 46 f). By Remark 1 it immediately follows that this result may be stated for the restriction to orthogonal series, too:

Remark 2. For a regular method $A=(a_{nk})$ let $\lim_{k\to\infty} \alpha_k=0$. Then there exists a divergent series (1) which is $[A]^\gamma$-summable for $0<\gamma\leqq 2$. However, in general, condition $\lim_{k\to\infty} \alpha_k=0$ may not be replaced by $\liminf_{k\to\infty} \alpha_k=0$.

3. Summability and convergence

In the following we first want to turn to aspects of the field of general gap theorems restricting the methods to regular triangular matrices $A=(a_{nk})$, $a_{nk}=0$ if $k>n$. For a gap series $\sum_{n=0}^{\infty} u_n$ with $u_n=0$ if $n\neq n_1, n_2, \ldots,$ the transforms of its partial sums $\{s_n\}$ for $n_i\leqq n<n_{i+1}$ are

$$t_n = \sum_{k=0}^{n} a_{nk} s_k = \sum_{j=0}^{i} a_{nj}^* s_{n_j},$$

where

$$a_{nj}^* = \begin{cases} \sum_{k=n_j}^{n_{j+1}-1} a_{nk} & (j < i) \\ \sum_{k=n_i}^{n} a_{nk} & (j = i) \\ 0 & (j > i). \end{cases}$$

The method $A^*=(a_{nj}^*)$ is called a gap method (cf. K. Zeller; W. Beekmann [15]; p. 79). In the context of gap theorems one asks for equivalence of A^* to convergence. Possibly the convergence of a subsequence of $\{t_n\}$ may already imply convergence of $\{s_{n_i}\}$ or of the series, respectively. To this end, with $S_j=s_{n_j}$ we consider the following subsequence of $\{t_n\}$:

$$t_{n_{i+1}-1} = \sum_{j=0}^{i} a_{n_{i+1}-1,j}^* s_{n_j} = \sum_{j=0}^{i} A_{ij} S_j,$$

where $A_{ij}=a_{n_{i+1}-1,j}^*$. The method $\tilde{A}=\tilde{A}(\{n_i\})=(A_{ij})$ has the advantage of having a triangular form. These gap submethods play an important role in our investigations of inverse theorems in summability of orthogonal series. In this connection we introduce the modified means

$$T_i(x) = \sum_{j=0}^{i} A_{ij} s_{n_j}(x)$$

of the series (1). With the majorants

$$\alpha_k^* = \sup_i |a_{n_i-1,k}| \tag{7}$$

similar to (2), we prove

LEMMA 5. *For an increasing sequence $\{n_i\}$ let*

$$\sum_{k=n_i}^{n_{i+1}-1} \alpha_k^* \leqq \tilde{M} \tag{8}$$

be fulfilled. Then for the series (1), $\{T_i(x)\}$ *is convergent if and only if* $\{t_{n_i-1}(x)\}$ *is convergent.*

PROOF. Without loss of generality, $n_0=0$ may be assumed. By the regularity of A (cf. condition (a)) and by Hölder's inequality we get

$$|t_{n_{i+1}-1}(x)-T_i(x)|^2 = \Big|\sum_{j=0}^{i}\sum_{k=n_j}^{n_{j+1}-1} a_{n_{i+1}-1,k}\big(s_k(x)-s_{n_j}(x)\big)\Big|^2 =$$

$$= O(1)\sum_{j=0}^{i}\sum_{k=n_j}^{n_{j+1}-1} |a_{n_{i+1}-1,k}|\,|s_k(x)-s_{n_j}(x)|^2 = O(1)\sum_{j=0}^{i}\sum_{k=n_j}^{n_{j+1}-1} \alpha_k^*\big(s_k(x)-s_{n_j}(x)\big)^2.$$

This yields

$$\int_0^1 \sup_i \big(t_{n_{i+1}-1}(x)-T_i(x)\big)^2\,dx = O(1)\sum_{n=0}^{\infty} c_n^2,$$

or $\sup_i |t_{n_{i+1}-1}(x)-T_i(x)|<\infty$ a.e. With the regularity condition (c) it follows $t_{n_{i+1}-1}(x)-T_i(x)\to 0$ a.e. which completes the proof.

REMARK 3. If the columns of A are decreasing ($a_{nk}\geqq a_{n+1,k}$; $n\geqq k$), the statement of Lemma 5 holds for every increasing sequence $\{n_i\}$.

The sequence $\{s_{n_i}(x)\}$ considered so far may be represented as a (direct) sequence of partial sums of a new series $\sum_{i=0}^{\infty} C_i \Phi_i(x)$ which belongs to the class of series (1), namely with

$$C_i = \{\sum_{n=n_i}^{n_{i+j}-1} c_n^2\}^{1/2}; \quad \Phi_i(x) = \begin{cases} \varphi_{n_i}(x) & (C_i = 0) \\ \dfrac{1}{C_i} \sum\limits_{n=n_i}^{n_{i+1}-1} c_n \varphi_n(x) & (C_i \neq 0). \end{cases}$$

Following Lemma 5 it has to be investigated what are the conditions for a given method not to sum a divergent orthogonal series (1). For general sequences we mention the result obtained by R. P. Agnew.

LEMMA 6 (R. P. Agnew [1]). *Let* $B=(b_{nk})$ *be a regular triangular method with*

$$|b_{nn}| - \sum_{k=0}^{n-1} |b_{nk}| \geqq \varepsilon > 0.$$

Then B *is equivalent to convergence.*

The application of the last lemma to 'blocked' matrices is due to J. A. Fridy [5] in connection with Tauberian theorems. Another condition for a method to be equivalent to convergence is

LEMMA 7. *For a regular triangular method* $B=(b_{nk})$, *let*

$$b_{n+1,k} \leqq q b_{nk} \quad (0<q<1;\ k=0,1,\dots;\ n\geqq k).$$

Then B *is equivalent to convergence.*

PROOF. The means $t_n=\sum_{k=0}^{n} b_{nk}s_k$ associated with B are first transformed into the means

$$t_n^* = \frac{-q^*}{1-q^*}\,t_{n-1} + \frac{1}{1-q^*}\,t_n,$$

where $q\leqq q^*<1$. By Lemma 6, $\{t_n\}$ is convergent if and only if $\{t_n^*\}$ is convergent. Now

$$t_n^* = \frac{1}{1-q^*}\{\sum_{k=0}^{n-1}(b_{nk}-q^* b_{n-1,k})\,s_k + b_{nn}s_n\} =: \sum_{k=0}^{n} c_{nk}s_k.$$

We want to prove that $C=(c_{nk})$ is equivalent to convergence; for $c_{nk}\leqq 0$ if $k=0,1,\dots,n-1$ and therefore

$$c_{nn} - \sum_{k=0}^{n-1} |c_{nk}| = \sum_{k=0}^{n} c_{nk} \to 1.$$

Then Lemma 6 shows the equivalence of C to convergence. Thus the convergence of $\{t_n\}$ implies that $\{s_n\}$ is convergent, too.

With respect to orthogonal series we prove

LEMMA 8. *Let* $B=(b_{nk})$ *be a regular triangular method with* $\sum_{k=0}^{n} b_{nk}=1$ *and*

$$|b_{nk}| \leqq K^* q^{n-k} \quad (n = 0, 1, \ldots;\ n \geqq k).$$

Then B *is equivalent to convergence with respect to the orthogonal series* (1).

PROOF. Because

$$s_n(x)-t_n(x) = \sum_{\mu=1}^{n} c_\mu \varphi_\mu(x) \sum_{k=0}^{\mu-1} b_{nk},$$

the assertion follows by

$$\sum_{n=1}^{\infty} \int_0^1 (s_n(x)-t_n(x))^2\, dx = O(1) \sum_{n=1}^{\infty} \sum_{\mu=1}^{n} c_\mu^2 \Big(\sum_{k=0}^{\mu-1} q^{n-k}\Big)^2 = O(1) \sum_{\mu=0}^{\infty} c_\mu^2 < \infty.$$

We want to mention that with respect to the orthogonal series (1), a necessary condition for a method to be equivalent to convergence is deducible from Remark 3. With respect to orthogonal series we summarize

THEOREM 3. *Let* $\{n_i\}$ *be an increasing sequence. If condition* (8) *is fulfilled for the majorants* $\{\alpha_k^*\}$ *and if the gap submethod* $\tilde{A}=\tilde{A}(\{n_i\})$ *is equivalent to convergence with respect to the orthogonal series* (1), *then* $\{t_{n_i-1}(x)\}$ *and* $\{s_{n_i}(x)\}$ *are both convergent or both divergent.*

REMARK 4. If a gap submethod $\tilde{A}=\tilde{A}(\{n_i\})$ sums a divergent orthogonal series (1), then there exists a series (1) with divergent partial sums $\{s_{n_i}(x)\}$ for which the means $\{t_{n_i}(x)\}$ are convergent.

To prove this remark we take a divergent series $\sum_{i=1}^{\infty} C_i \Phi_i(x)$. Let $\{\mu_k\}$ be chosen such that $\sum_{k=1}^{\infty} C_{\mu_k} \Phi_{\mu_k}(x)$ is convergent which holds, for example, if $\sum_{k=2}^{\infty} C_{\mu_k}^2 (\log k)^2 < \infty$ (cf. G. Alexits [2], p. 76). We then consider the series $\sum_{i=1}^{\infty} C_i^* \Phi_i(x)$ with partial sums $\{S_j^*(x)\}$ defined by $C_i^*=C_i$ if $i \neq \mu_k$ $(k=1, 2, \ldots)$ and otherwise $C_{\mu_k}^*=0$, $k=1, 2, \ldots$. This series is divergent and its transforms $\{T_i^*(x)\}=\tilde{A}\{S_j^*(x)\}$ converge. Now with respect to $\{n_i\}$ we define the requested series $\sum_{n=0}^{\infty} c_n \varphi_n(x)$ by

$$c_n = \begin{cases} C_i & (n = n_i;\ i \neq \mu_k,\ k = 1, 2, \ldots) \\ 0 & \text{(otherwise)}, \end{cases}$$

and we put $\varphi_{n_i}(x)=\Phi_i(x)$ if $i \neq \mu_k$, $k=1, 2, \ldots$. For the remaining n (with the coefficients $c_n=0$), where $n \neq n_i$, $i=1, 2, \ldots$, or $n=n_i$ and $i=\mu_k$, $k=1, 2, \ldots$, we take for $\varphi_n(x)$ successively those functions $\Phi_{\mu_k}(x)$ which have not been used so far. It is obvious that $\{s_{n_i}(x)\}=\{S_i^*(x)\}$ and that for the A-means of the partial sums $\{s_n(x)\}$ of the constructed series $t_{n_{i+1}}(x)=T_i^*(x)$, $i=1, 2, \ldots$. Finally $\sum_{n=0}^{\infty} c_n \varphi_n(x)$ belongs to the class of series (1).

When applying gap submethods in summability of general sequences, slight modifications of well-known gap theorems can be stated by Lemma 7 in the following way:

Remark 5. (a) If $n_{i+1}/n_i \geqq q > 1$ $(i=0, 1, \ldots)$ then for a gap series $\sum_{n=0}^{\infty} u_n$, $u_n=0$, if $n \neq n_1, n_2, \ldots$, the arithmetic means $\{\sigma_{n_i}\}$ are convergent if and only if the partial sums $\{s_{n_i}\}$ (i.e. the series itself) are convergent.

(b) For the discontinuous Riesz means $(R^*, \lambda, 1)$, $\lambda=\{\lambda_n\}$ increasing to infinity, let $\lambda_{n_{i+1}}/\lambda_{n_i} \geqq q > 1$ $(i=1, 2, \ldots)$. Then with a gap series $\sum_{n=0}^{\infty} u_n$, $u_n=0$ if $n \neq n_1, n_2, \ldots$ the $(R^*, \lambda, 1)$-means $\{t_{n_i}\}$ are convergent if and only if $\{s_{n_i}\}$ (resp. $\sum_{n=0}^{\infty} u_n$) converges.

Theorem 3 allows to conclude from summability (and strong summability with order $\gamma \geqq 1$) to convergence of the partial sums $\{s_{n_i}(x)\}$. For an order $\gamma < 1$ we can also prove

Theorem 4. *Let $\{n_i\}$ be an increasing sequence, and let together with* (8) *for the gap submethod $\hat{A}(\{n_i\})$ concerning the method $(|a_{nk}|)$ in dependence of $\{n_i\}$ at least one of the following conditions be fulfilled:*

(a) *$\hat{A}=\hat{A}(\{n_i\})$ is equivalent to convergence (with respect to general sequences),*

(b) *$\hat{A}=\hat{A}(\{n_i\})$ satisfies the assumption of Lemma* 8.

Then for a series (1) *the convergence of the strong means $\{\tau_{n_i-1}^{(\gamma)}(x)\}$ $(\gamma>0)$ implies convergence of $\{s_{n_i}(x)\}$.*

Proof. It suffices to carry out the proof for the exponents $\gamma \leqq 2$; otherwise Theorem 3 yields the assertion. Now with $\hat{A}=(\hat{A}_{ij})$

$$\sum_{j=0}^{i} \hat{A}_{ij} |s_{n_j}(x)-f(x)|^\gamma = O(1)\Big\{\sum_{j=0}^{i} \sum_{k=n_j}^{n_{j+1}-1} |a_{n_{i+1}-1,k}| \cdot |s_k(x)-s_{n_j}(x)|^\gamma +$$

$$+\sum_{k=0}^{n_{i+1}-1} |a_{n_{i+1}-1,k}|\, |s_k(x)-f(x)|^\gamma\Big\} = O(1)\{\tau_i^*(x)+\tau_i^{**}(x)\}.$$

The second term is identical with $\tau_{n_{i+1}-1}^{(\gamma)}(x)$, and therefore $\tau_i^{**}(x) \to 0$ $(i \to \infty)$. The estimation

$$\tau_i^*(x) = O(1)\Big\{\sum_{j=0}^{i} \sum_{k=n_j}^{n_{j+1}-1} |a_{n_{i+1}-1,k}|\, |s_k(x)-s_{n_j}(x)|^2\Big\}$$

holds on the basis of the regularity of A. In the proof of Lemma 5 we have shown that, if (8) is assumed, the right side tends to 0 $(i \to \infty)$ for every series (1), i.e. we have

$$\sum_{j=0}^{i} \hat{A}_{ij} |s_{n_j}(x)-f(x)|^\gamma \to 0 \quad (i \to \infty).$$

With (a) the statement now follows immediately. On the other hand, in case (b) there exists a natural number K, so that with the sequence $\{m_i\}=\{n_{i\cdot K}\}$ the gap submethod $\tilde{A}(\{m_i\})$ satisfies the conditions of Lemma 6. Therefore by (a) $\{s_{m_i}(x)\}$ converges and it is easy to see that so does $\{s_{n_i}(x)\}$, too.

4. Applications

We represent the sequence of partial sums $\{s_{m_i}(x)\}$ as a transformation of $\{s_n(x)\}$ obtained with the aid of the (simple) summability method $T(\{m_i\})$. If two summability methods are equivalent with respect to the orthogonal series (1), we use the symbol $\sim^*$.

(a) For the class A^p of triangular methods $A=(a_{nk})$ with

$$(n+1)^{1-1/p}\Big\{\sum_{k=0}^{n}|a_{nk}|^p\Big\}^{1/p}\leqq M \quad (n=0,1,\ldots),$$

which were first considered in connection with orthogonal series by V. A. Bolgov [3], we have

$$T(\{2^i\})\sim^* A\sim^*[A]^\lambda \quad (\gamma>0).$$

Theorems 2 and 4 prove this result in the first instance for the special method $(C,1)$. In general

$$\sum_{k=0}^{n}a_{nk}|s_k(x)-f(x)|^\gamma\leqq M\Big\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-f(x)|^{\gamma p}\Big\};$$

with $n_i=2^i$, $i=0,1,\ldots$. Lemma 7 may be applied and Theorem 4 helps to prove the stated relation. We notice that this class contains a subclass of Hausdorff-methods $H=(h_{nk})$ with (cf. G. H. Hardy [6], Theorem 215)

$$h_{nk}=\binom{n}{k}\int_0^1 t^k(1-t)^{n-k}\psi(t)\,dt, \quad \psi(t)\in L_p[0,1] \quad (p>1)$$

and, in particular, the regular Cesàro methods (C,α), $\alpha>0$. The classical result on strong summability with large exponents $(C,\alpha)\Rightarrow^*[(C,\alpha)]^\gamma$, $\alpha>0$, $\gamma>0$ was obtained by G. Sunouchi [14], and V. A. Bolgov [3] proved $A\sim^*T(\{2^i\})$.

(b) *Discontinuous Riesz methods* $(R^*,\lambda,1)$, $\lambda=\{\lambda_n\}$ *increasing to infinity*. If $\{n_i\}$ satisfies

$$\lambda_{n_{i+1}}/\lambda_{n_i}\geqq q_1>1 \quad\text{and}\quad \lambda_{n_{i+1}-1}/\lambda_{n_i}\leqq q_2,$$

we get by Remark 5, Theorems 1 and 4

$$T(\{n_i\})\sim^*(R^*,\lambda,1)\sim^*[(R^*,\lambda,1)]^\gamma \quad (0<\gamma\leqq 2).$$

The first relation is due to A. Zygmund [19], and the second to J. Meder [10].

If, in addition,

$$\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}=O(1)\cdot\frac{1}{n_{i+1}-n_i} \quad (n_i\leqq n<n_{i+1})$$

we get with Theorem 2 even

$$(R^*, \lambda, 1) \sim^* [(R^*, \lambda, 1)]^\gamma \quad (\gamma > 0).$$

We finally mention that in the case of general summability methods the equivalence of the summability processes with respect to orthogonal series is not given as was shown by D. E. Menchoff [11] in the case of summability and convergence of subsequences or by F. Móricz [12] in the case of strong summability and summability.

References

[1] R. P. Agnew, Equivalence of methods for evaluation of sequences, *Proc. Amer. Math. Soc.*, **3** (1932), 550—556.

[2] G. Alexits, *Konvergenzprobleme der Orthogonalreihen* (Berlin, 1960).

[3] V. A. Bolgov, The summation of orthogonal series by linear methods, *Math. Notes,* **4** (1968), 907—911.

[4] S. Borgen, Über die (*C*, 1)-Summierbarkeit von Reihen orthogonaler Funktionen, *Math. Ann.*, **98** (1928), 125—150.

[5] J. A. Fridy, Tauberian theorems via block dominated matrices, *Pacific, J. Math.*, **81** (1979), 81—91.

[6] G. H. Hardy, *Divergent series* (Oxford, 1947).

[7] S. Kaczmarz, Über Reihen von allgemeinen Orthogonalfunktionen, *Math. Ann.*, **96** (1927), 148—151.

[8] A. Kolmogoroff, Une contribution à l'étude de la convergence des séries de Fourier, *Fund. Math.*, **5** (1924), 96—97.

[9] L. Leindler, On the strong and very strong summability and approximation of orthogonal series by generalized Abel method, *Studia Sci. Math. Hungar.*, **16** (1981), 35—43.

[10] J. Meder, On the very strong Riesz-summability of orthogonal series, *Studia Math.*, **20** (1961), 285—300.

[11] D. E. Menchoff, On the summability of orthogonal series, *Trudy Moscow Math. Obs.*, **10** (1961), 351—418 (in Russian).

[12] F. Móricz, A note on the strong *T*-summation of orthogonal series, *Acta Sci. Math. (Szeged)*, **30** (1969), 69—76.

[13] H. Schwinn, Über die Konvergenz und Summierbarkeit von Teilfolgen allgemeiner Orthogonalreihen, *Acta Math. Acad. Sci. Hung.*, **37** (1981), 373—380.

[14] G. Sunouchi, On the strong summability of orthogonal series, *Acta Sci. Math. (Szeged),* **27** (1966), 71—76.

[15] K. Zeller, W. Beekmann, *Theorie der Limitierungsverfahren* (Berlin—Heidelberg—New York, 1970).

[16] O. A. Ziza, Summation of orthogonal series by the methods (φ, λ), *Math. Notes,* **25** (1979), 91—102.

[17] O. A. Ziza, On the application of summability methods of class (φ, λ) to orthogonal series, *Soviet Math. Dokl.*, **20** (1979), 14—18.

[18] A. Zygmund, Sur l'application de la première moyenne arithmétique dans la théorie des séries de fonctions orthogonales, *Fund. Math.*, **10** (1927), 356—362.

[19] A. Zygmund, Sur la sommation des séries de fonctions orthogonales, *Bull. Intern. Acad. Polon. Sci. Lettres (Cracovie)*, série A (1927), 295—312.

(Received July 31, 1984)

MATHEMATISCHES INSTITUT
TECHNISCHE UNIVERSITÄT BERLIN
STRASSE DES 17. JUNI 137
1000 BERLIN 12

Acta Math. Hung.
50 (1—2) (1987), 33—37.

NOTES ON LACUNARY INTERPOLATION WITH SPLINES. III

(0, 2)-INTERPOLATION WITH QUINTIC G-SPLINES

TH. FAWZY (Ismailia)

1. Introduction

Recently A. Meir and A. Sharma [1], B. K. Swartz and R. S. Varga [2] and A. K. Varma [3] have studied the use of splines to solve the (0, 2) lacunary interpolation problem. All of these methods are global and require the solution of a large system of equations.

Recently, Th. Fawzy [5], [6], [7] presented several local methods for solving lacunary interpolation problems using piecewise polynomials with certain continuity properties.

In this paper we study the following (0, 2)-lacunary interpolation problem:

Problem 1. Given $\Delta\colon \{x_i=ih\}_{i=0}^{n}$ and real numbers $\{f_i,\ f_i''\}_{i=0}^{n}$, find S such that

$$S(x_i)=f_i \quad\text{and}\quad S''(x_i)=f_i'', \quad i=0,\ldots,n. \tag{1.1}$$

The prupose of this paper is to construct a spline method for solving Problem 1 using piecewise quintic polynomials, such that for all functions $f\in C^5$, the order of approximation is the same as the best order of approximation with quintic splines.

2. Construction of the spline interpolant

We shall construct a solution S of Problem 1 in the form

$$S_\Delta(x)=S_k(x)=\sum_{j=0}^{5}\frac{S_K^{(j)}}{j!}(x-x_k)^j, \quad x_k\leqq x\leqq x_{k+1}, \tag{2.1}$$

where $k=0,\ldots,n-1$.

We shall define each of the $S_k^{(j)}$ explicitly in terms of the data. In particular we choose

$$S_k^{(0)}=f_k=f(x_k), \quad S_k^{(2)}=f_k^{(2)}=f^{(2)}(x_k), \quad k=0,\ldots,n-1. \tag{2.2}$$

For $k=1,2,\ldots,n-2$ we take

$$S_k^{(5)}=\frac{1}{h^3}\{f_{k+2}''-3f_{k+1}''+3f_k''-f_{k-1}''\}, \tag{2.3}$$

$$S_k^{(4)}=\frac{1}{h^2}\{f_{k+1}''-2f_k''+f_{k-1}''\}, \tag{2.4}$$

$$S_k^{(3)}=\frac{1}{h}\left\{f_{k+1}''-f_k''-\frac{h^2}{2}S_k^{(4)}-\frac{h^3}{3!}S_k^{(5)}\right\} \tag{2.5}$$

and

$$(2.6)\qquad S_k^{(1)} = \frac{1}{h}\left\{f_{k+1}-f_k-\frac{h^2}{2}f_k''-\sum_{r=3}^{5}\frac{h^r}{r!}S_k^{(r)}\right\}.$$

For $k=0$ we take

$$(2.7)\qquad S_0^{(5)} = S_1^{(5)},$$

$$(2.8)\qquad S_0^{(4)} = S_1^{(4)}-hS_0^{(5)},$$

$$(2.9)\qquad S_0^{(3)} = \frac{1}{h}\left\{f_1''-f_0''-\frac{h^2}{2}S_0^{(4)}-\frac{h^3}{3!}S_0^{(5)}\right\},$$

$$(2.10)\qquad S_0^{(1)} = \frac{1}{h}\left\{f_1-f_0-\frac{h^2}{2}f_0''-\sum_{r=3}^{5}\frac{h^r}{r!}S_0^{(r)}\right\},$$

and finally for $k=n-1$ we take

$$(2.11)\qquad S_k^{(j)} = S_{k-1}^{(j)}(x_k),\quad j=1,3,4 \text{ and } 5.$$

Clearly, the function S defined in (2.1)—(2.11) solves the (0, 2)-interpolation problem 1. Moreover, by the construction it is clear that S is a piecewise quintic polynomial.

The $S_k^{(3)}$ have been chosen to make S_Δ'' right continuous, i.e.

$$D_L^2 S_k(x_{k+1}) = D_R^2 S_{k+1}(x_{k+1}),$$

while the $S_k^{(1)}$ have been chosen to make S_Δ continuous. Thus

$$(2.12)\qquad S\in C^{(0,2)}[x_0,x_n] = \{f\in C[x_0,x_n]\colon D_R^2 f\in C[x_0,x_n]\}$$

where D_R is the right derivative.

Indeed, S is the unique piecewise quintic polynomial in

$$C^{(0,2)}[x_0,x_n]\cap C^5[x_{n-2},x_n]$$

satisfying the interpolation condition (1.1).

S is a special kind of g-spline, we refer to it as lacunary g-spline.

3. Error bounds for the interpolation method

Suppose $f\in C^5[x_0,x_n]$. Then using the Taylor and dual Taylor expansions it is easy to establish the following lemma showing how well the $S_k^{(j)}$ approximate $f^{(j)}(x_k)$ in terms of the modulus of continuity $\omega(D^5f;h)$ of $f^{(5)}(x)$.

LEMMA 3.1. *For* $0 \leqq k \leqq n-2$ *and* $j=1, 3, 4$ *and* 5,

$$|S_k^{(j)} - f^{(j)}(x_k)| \leqq c_{kj} h^{5-j} \omega(D^5 f;\ h) \tag{3.1}$$

where the constants c_{kj} *are given in the following table:*

	c_{k1}	c_{k3}	c_{k4}	c_{k5}
$k=0$	$\frac{169}{720}$	$\frac{91}{36}$	4	$\frac{19}{6}$
$1 \leqq k \leqq n-2$	$\frac{61}{540}$	$\frac{7}{9}$	$\frac{5}{6}$	$\frac{13}{6}$

THEOREM 3.1. *Let* $f \in C^5[x_0, x_n]$ *and let* S_Δ *be the lacunary g-spline constructed in* (2.1)—(2.11). *Then for all* $0 \leqq j \leqq 5$ *and all* $0 \leqq k \leqq n-2$, *the inequality*

$$\|D^{(j)}(f - S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \leqq c_{kj}^* h^{5-j} \omega(D^5 f;\ h)$$

o lds true where the contants c_{kj}^* *are given in the following table:*

	c_{k0}^*	c_{k1}^*	c_{k2}^*	c_{k3}^*	c_{k4}^*	c_{k5}^*
$k=0$	$\frac{917}{1080}$	$\frac{827}{360}$	$\frac{91}{18}$	$\frac{73}{9}$	$\frac{43}{6}$	$\frac{19}{6}$
$1 \leqq k \leqq n-2$	$\frac{319}{1080}$	$\frac{1579}{2160}$	$\frac{14}{9}$	$\frac{97}{36}$	3	$\frac{13}{6}$

PROOF. Suppose $1 \leqq k \leqq n-2$ and let $x_k \leqq x \leqq x_{k+1}$. Then using the Taylor expansion of $f(x)$ we have

$$|f(x) - S_\Delta(x)| = |f(x) - S_k(x)| \leqq \sum_{j=0}^{4} \frac{|f^{(j)}(x_k) - S_k^{(j)}|}{j!} h^j + \frac{|f^{(5)}(\xi_k) - S_k^{(5)}|}{5!} h^5 \leqq$$

$$\leqq \sum_{j=0}^{5} \frac{|f^{(j)}(x_k) - S_k^{(j)}|}{j!} h^j + \frac{|f^{(5)}(\xi_k) - f^{(5)}(x_k)|}{5!} h^5$$

where $x_k < \xi_k < x_{k+1}$.

Now, using the above lemma, it is easy to get the required result.

Similar procedures for the derivatives with the help of Lemma 3.1 will easily complete the proof for $1 \leqq k \leqq n-2$.

For $k=0$ and $x_0 \leqq x \leqq x_1$ we repeat the same technique as above and the results could be easily obtained.

LEMMA 3.2. *For* $k=n-1$ *and* $j=1, 3, 4$ *and* 5 *we have*

$$|S_k^{(j)} - f^{(j)}(x_k)| \leqq c_{kj} h^{5-j} \omega(D^5 f;\ h)$$

where

$$c_{k1}=\frac{1579}{2160},\quad c_{k3}=\frac{97}{36},\quad c_{k4}=3,\quad c_{k5}=\frac{13}{6}.$$

PROOF. This lemma is a direct consequence of Theorem 3.1, using (2.11).

THEOREM 3.2. *Let* $f\in C^5[x_0, x_n]$ *and let* S_Δ *be the lacunary g-spline constructed in* (2.1)—(2.11). *Then for* $k=n-1$ *we have*

$$\|D^{(j)}(f-S_\Delta)\|_{L_\infty[x_k, x_{k+1}]}\leqq c^*_{kj}h^{5-j}\omega(D^5f;\, h)$$

where $j=1, 3, 4$ *and* 5, *and*

$$c^*_{k0}=\frac{719}{540},\quad c^*_{k1}=\frac{2927}{1080},\quad c^*_{k2}=\frac{170}{36},\quad c^*_{k3}=\frac{131}{18},\quad c^*_{k4}=\frac{37}{6},\quad c^*_{k5}=\frac{19}{6}.$$

PROOF. Using Lemma 3.2, the Taylor expansion of $f(x)$ for $x\in[x_{n-1}, x_n]$ and the construction of $S_{n-1}(x)$, it will be easy to prove this theorem.

4. Numerical example

The method is tested for the following example:

$$f(x) = 1+xe^x,\quad 0\leqq x\leqq 1.$$

We carried out the calculation at $x=0.55$ and for $h=0.1$. The following results are obtained:

	Exact values	Numerical values	Absolute error
f:	1.953 289 160	1.953 289 187	$2.7\cdot 10^{-8}$
f':	2.686 542 178	2.686 542 177	$1\cdot 10^{-9}$
f'':	4.419 795 196	4.419 768 559	$2.6637\cdot 10^{-5}$
$f^{(3)}$:	6.153 048 214	6.153 042 055	$6.159\cdot 10^{-6}$
$f^{(4)}$:	7.886 301 232	7.909 976 2	$2.367\,496\,8\cdot 10^{-2}$
$f^{(5)}$:	9.619 554 25	9.635 924	$1.636\,975\cdot 10^{-2}$

5. Remarks

1. The method defined here, in contrast to the other methods, does not require any end condition to be imposed.
2. The method defined here converges faster than any other known method.
3. A similar method for solving the (0, 2)-interpolation problem using splines of degree 6 will be presented elsewhere.
4. The constants presented here are not guaranteed to be the best.

References

[1] A. Meir and A. Sharma, Lacunary interpolation by splines, *Siam J. Num. Anal.*, **10** (1973).
[2] B. K. Swartz and R. S. Varga, A note on lacunary interpolation by splines, *Siam J. Num. Anal.*, **10** (1973), 433—447.
[3] A. K. Varma, Lacunary interpolation by splines. I, *Acta Math. Acad. Sci. Hungar.*, **31** (1978).
[4] L. Schumaker, *Spline Functions: Basic Theory*, John Wiley and Sons (New York, Chichester, 1981).
[5] Th. Fawzy, Notes on lacunary interpolation by splines. I, *Annales Univ. Sci. Budapest, Sectio Math.*, **28** (1985).
[6] Th. Fawzy, Notes on lacunary interpolation by splines. II, *Annales Univ. Sci. Budapest, Sectio Comp.*
[7] Th. Fawzy and F. Holail, (0, 2) lacunary interpolation with splines, *Annales Univ. Sci. Budapest, Sectio Comp.*

(Received August 3, 1984)

MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE
SUEZ-CANAL UNIVERSITY
ISMAILIA
EGYPT

Acta Math. Hung.
50 (1—2) (1987), 39—48.

KLASSEN VON MATRIZENABBILDUNGEN IN FK-RÄUMEN

E. MALKOWSKY (Gießen)

1. Einleitung und Bezeichnungen

In [1] und [2] ist es gelungen, für Folgenräume X und Y in bestimmten Fällen die Klasse (X, Y) aller unendlichen komplexen Matrizen A zu bestimmen, die X in Y abbilden: (X, l_∞), (X, c) und (l_∞, Y). Hier wollen wir ähnliche Ergebnisse im Zusammenhang mit den Räumen l_1, γ und bv herleiten. In allen Fällen wird die Charakterisierung der Matrizenklassen auf die Bestimmung der Dualräume der auftretenden Folgenräume zurückgeführt. Aus unseren Sätzen erhält man durch Spezialisierung viele der Ergebnisse aus der Tabelle in [3].

Zunächst benötigen wir einige Bezeichnungen. Wir setzen die Begriffe „r-normierter Raum“, „Schauder-Basis“ oder kurz „Basis“ und „FK-Raum“ als bekannt voraus (s. [4], S. 94, S. 84 und [5], S. 202).

Mit A bezeichnen wir unendliche Matrizen $(a_{nk})_{n,k}$ komplexer Zahlen.

Mit s bezeichnen wir die Menge aller komplexen Folgen $x=(x_k)_k$, und wir benutzen die üblichen Bezeichnungen für die Folgenräume l_p $(0<p<\infty)$, l_∞, c_0 und c und die zugehörigen natürlichen p-Normen bzw. Normen $\|\dots\|_p$ und $\|\dots\|_\infty$. Weiter betrachten wir die Folgenräume

$$\gamma := \Big\{x\in s \Big| \Big(\sum_{k=1}^{n} x_k\Big)_n \in c\Big\}, \quad \gamma_0 := \Big\{x\in s \Big| \Big(\sum_{k=1}^{n} x_k\Big)_n \in c_0\Big\},$$

$$\gamma_\infty := \Big\{x\in s \Big| \Big(\sum_{k=1}^{n} x_k\Big)_n \in l_\infty\Big\} \quad \text{und} \quad \text{bv} := \Big\{x\in s \Big| \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty\Big\};$$

γ, γ_0 und γ_∞ sind mit

$$\|x\|_{\gamma_\infty} := \sup_{n\in\mathbf{N}} \Big|\sum_{k=1}^{n} x_k\Big|$$

FK-Räume, und bv ist ein FK-Raum mit

$$\|x\|_{\text{bv}} := \sum_{k=1}^{\infty} |x_k - x_{k-1}| \quad (\text{hier ist } x_0:=0 \text{ gesetzt}).$$

Sind X und Y Teilmengen von s, so schreiben wir (X, Y) für die Klasse aller Matrizen A, für die $A_n(x) := \sum_{k=1}^{\infty} a_{nk} x_k$ für alle $x\in X$ und für alle $n\in\mathbf{N}$ existiert und $A(x) := \big(A_n(x)\big)_n \in Y$ für alle $x\in X$. Ist X ein r-normierter Raum, so setzen wir

$$S_X := \{x\in X \,|\, \|x\| = 1\}.$$

Für Teilmengen X von s definieren wir die folgenden Dualräume von X:

$$X^{\dagger} := \{a \in s \mid \sum_{k=1}^{\infty} a_k x_k \text{ konvergiert für alle } x \in X\},$$

den Köthe—Toeplitz-Dualraum von X,

$$X^{|\dagger|} := \{a \in s \mid \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ für alle } x \in X\},$$

falls X ein r-normierter Teilraum von s ist

$$X^{||\dagger||} := \{a \in s \mid \sup_{x \in S_X} \sum_{k=1}^{\infty} |a_k x_k| < \infty\},$$

und wir setzen

$$\|a\|^{\dagger} := \sup_{x \in S_X} \Big|\sum_{k=1}^{\infty} a_k x_k\Big| \quad \text{sowie} \quad \|a\|^{||\dagger||} := \sup_{x \in S_X} \Big(\sum_{k=1}^{\infty} |a_k x_k|\Big)$$

für alle $a \in s$, für die die Ausdrücke rechts existieren; mit X^* bezeichnen wir den Raum der stetigen linearen Funktionale auf X. Sind X eine Teilmenge von s und $a \in X^{\dagger}$, so wird durch

$$f_a(x) := \sum_{k=1}^{\infty} a_k x_k \quad \text{für alle} \quad x \in X$$

ein lineares Funktional auf X definiert. Wir schreiben $X^{\dagger} \subset X^*$, wenn aus $a \in X^{\dagger}$ folgt $f_a \in X^*$.

Zum Schluß dieses Kapitels machen wir einige Bemerkungen über Beziehungen zwischen den einzelnen Dualräumen:

Aus den Definitionen von $X^{\dagger}$, $X^{|\dagger|}$ und $X^{||\dagger||}$ ergibt sich sofort:

1. Für alle Folgenräume X gilt $X^{|\dagger|} \subset X^{\dagger}$; für alle r-normierten Folgenräume gilt $X^{||\dagger||} \subset X^{|\dagger|}$.

Dabei ist die Inklusion in beiden Fällen echt: Für $X := \gamma$ ist $\gamma^{|\dagger|} = l_1 \subsetneqq \mathrm{bv} = \gamma^{\dagger}$ (s. [4], S. 117). Wenn wir mit φ den Raum aller Folgen bezeichnen, die nur endlich viele von Null verschiedene Glieder haben (s. [6], S. 273), dann wird $X := \varphi$ mit $\|\ldots\|_{\infty}$ zu einem normierten Raum, und es gilt $\varphi^{||\dagger||} \subsetneqq s = \varphi^{|\dagger|}$. Bezeichnen wir nämlich für alle $n \in \mathbf{N}$ mit $e^{(n)}$ die Folge, für die $e_k^{(n)} := 0$ für $k \neq n$ und $e_n^{(n)} := 1$, und definieren wir die Folge $a \in s$ durch $a_k := k$ $(k = 1, 2, \ldots)$, so gilt

$$\sum_{k=1}^{\infty} |a_k e_k^{(n)}| = n,$$

$\|a\|^{||\dagger||}$ existiert nicht, und somit ist $a \notin \varphi^{||\dagger||}$. Wir erhalten die folgenden Charakterisierungen für FK-Räume:

2. Ist $X \subset s$ ein vollständiger linearer metrischer Raum, so ist X genau dann ein FK-Raum, wenn $X^{\dagger} \subset X^*$.

3. Ist $X \subset s$ ein vollständiger r-normierter Raum, so ist X genau dann ein FK-Raum, wenn $X^{|\dagger|} \subset X^{||\dagger||}$.

(2. folgt leicht. 3.: Ist X ein FK-Raum und $a \in X^{|\dagger|}$, so folgt $a \in X^{\|\dagger\|}$ mit einer Anwendung eines bekannten Satzes (s. [4], Satz 11, S. 114). Die Umkehrung ist klar.)

Bekannt ist:

4. Ist X ein normaler Folgenraum,* so gilt $(X, Y) = |X, Y|$ (s. [7], S. 374), wobei $|X, Y| := \{A \in (X, Y) \mid \sum_{k=1}^{\infty} |a_{nk} x_k| < \infty$ für alle $n \in \mathbf{N}$ und für alle $x \in X\}$.

Daraus folgt insbesondere:

5. Für normale Folgenräume ist $X^{\dagger} = X^{|\dagger|}$.

Aus $X^{\dagger} = X^{|\dagger|}$ folgt jedoch im allgemeinen nicht, daß X ein normaler Folgenraum ist, wie das Beispiel $X := c$ zeigt.

2. Matrizenklassen (X, Y) bei FK-Räumen

Wir wollen nun einige Matrizenklassen (X, Y) bestimmen, in denen jeweils einer der beiden Räume X oder Y ein beliebiger r-normierter FK-Raum ist und der andere l_1, bv, γ_∞ oder γ. Zunächst beweisen wir ein Ergebnis, das Satz 1.2 in [1] mit l_1 anstelle von l_∞ entspricht.

Satz 1. *Ist X ein r-normierter FK-Raum, so gilt*

$$A \in (X, l_1) \Leftrightarrow \sup \left\{ \left\| \left(\sum_{n \in N} a_{nk} \right)_k \right\|^{\dagger} \,\middle|\, N \subset \mathbf{N} \textit{ endlich} \right\} < \infty.$$

Beweis. Wir setzen

$$M := \sup \left\{ \left\| \left(\sum_{n \in N} a_{nk} \right)_k \right\|^{\dagger} \,\middle|\, N \subset \mathbf{N} \text{ endlich} \right\}.$$

Es gelte $A \in (X, l_1)$. Da X und l_1 FK-Räume sind, ist die Abbildung $A: X \to l_1$ linear und stetig. Es gibt daher eine Konstante K, so daß für alle $x \in X$

$$\|A(x)\|_1 \leqq K \|x\|^{1/r}.$$

Für alle endlichen Teilmengen N von $\mathbf{N}$ und für alle $x \in X$ folgt daraus

$$\left| \sum_{k=1}^{\infty} \sum_{n \in N} a_{nk} x_k \right| \leqq \sum_{n \in N} \left| \sum_{k=1}^{\infty} a_{nk} x_k \right| \leqq K \|x\|^{1/r}$$

und daher

$$\sup_{x \in S_X} \left| \sum_{k=1}^{\infty} x_k \left(\sum_{n \in N} a_{nk} \right) \right| = \left\| \left(\sum_{n \in N} a_{nk} \right)_k \right\|^{\dagger} \leqq K.$$

Da die letzte Ungleichung für alle endlichen Teilmengen N von $\mathbf{N}$ gilt, erhalten wir $M < \infty$.

Es gelte umgekehrt $M < \infty$. Dann folgt für die einelementigen Mengen $\{n\}$ $(n = 1, 2, \ldots)$ die Existenz von $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ für alle $x \in X$. Für alle $m \in \mathbf{N}$

---

* Ein Folgenraum X heißt normal, wenn aus $x \in X$ und $|y_k| \leqq |x_k|$ für alle $k = 1, 2, \ldots$ folgt $y \in X$ (s. [6], S. 273).

gilt mit einer bekannten Ungleichung (s. [8], S. 33)

$$\sum_{n=1}^{m} |A_n(x)| = \sum_{n=1}^{m} \Big|\sum_{k=1}^{\infty} a_{nk}x_k\Big| \leqq 4\cdot \max_{N_m\subset\{1,\dots,m\}} \Big|\sum_{k=1}^{\infty} x_k \sum_{n\in N_m} a_{nk}\Big| \leqq$$

$$\leqq 4\cdot \max_{N_m\subset\{1,\dots,m\}} \Big\|\Big(\sum_{n\in N_m} a_{nk}\Big)_k\Big\|^{\dagger} \|x\|^{1/r} \leqq 4M\|x\|^{1/r}.$$

Da $m\in\mathbf{N}$ beliebig war, folgt

$$\sum_{n=1}^{\infty} |A_n(x)| \leqq 4M\|x\|^{1/r} \quad \text{für alle} \quad x\in X$$

und daher $A(x)=\big(A_n(x)\big)_n\in l_1$.

Wir beweisen nun ein Ergebnis, das Satz 1.3 in [1] mit l_1 anstelle von l_∞ entspricht.

SATZ 2. *Es sei* Y *ein Teilraum von* s *und* $(Y^\dagger, \|\dots\|_\dagger)$ *ein normierter Raum. Gilt* $Y^{\dagger\|\dagger\|}=Y$, *so folgt*

$$A\in(l_1, Y) \Leftrightarrow \sup_{k\in\mathbf{N}} \|(a_{nk})_n\|_{\dagger}^{\|\dagger\|} < \infty.\text{*}$$

BEWEIS. Wir setzen $M:=\sup_{k\in\mathbf{N}} \|(a_{nk})_n\|_{\dagger}^{\|\dagger\|}$. Es gelte $A\in(l_1, Y)$. Dann ist $\big(A_n(x)\big)_n\in Y$ für alle $x\in l_1$, und daher existiert $\big\|\big(A_n(x)\big)_n\big\|_{\dagger}^{\|\dagger\|}$ für alle $x\in l_1$ wegen $Y^{\dagger\|\dagger\|}=Y$. Da l_1 und Y (mit $\|\dots\|_{\dagger}^{\|\dagger\|}$) FK-Räume sind, ist $A: l_1\to Y$ linear und stetig, also gibt es eine Konstante K, so daß für alle $x\in l_1$

$$\big\|\big(A_n(x)\big)_n\big\|_{\dagger}^{\|\dagger\|} \leqq K\|x\|_1.$$

Speziell für $x=e^{(k)}$ $(k=1, 2, \dots)$ folgt daraus

$$\|(a_{nk})_n\|_{\dagger}^{\|\dagger\|} \leqq K \quad \text{für alle} \quad k=1,2,\dots$$

also $M<\infty$.

Es gelte umgekehrt $M<\infty$. Daraus folgt für alle $y\in S_{Y^\dagger}$

$$\sum_{k=1}^{\infty} |y_n a_{nk}| \leqq M \quad (k=1,2,\dots)$$

und speziell für $e^{(n)}\in Y^\dagger$ $(n=1, 2, \dots)$

$$|a_{nk}| \leqq M\|e^{(n)}\|_\dagger =: M_n \quad \text{für alle} \quad k, n=1,2,\dots$$

also

$$\sup_{k\in\mathbf{N}} |a_{nk}| \leqq M_n \quad \text{für alle} \quad n=1,2,\dots$$

Somit gilt $\sum_{k=1}^{\infty} |a_{nk}x_k| \leqq M_n\|x\|_1$ für alle $x\in l_1$ und für alle $n\in\mathbf{N}$. Daher existiert $A_n(x)$ für alle $x\in l_1$ und für alle $n\in\mathbf{N}$.

---

* Hierbei ist $\|a\|_{\dagger}^{\|\dagger\|} := \sup\{\sum_{k=1}^{\infty} |a_k y_k| \mid y\in S_{Y^\dagger}\}$.

Es sei nun $y \in S_{Y^\dagger}$ beliebig. Dann gilt für alle $x \in l_1$

$$\sum_{n=1}^{\infty} |y_n||A_n(x)| \leqq \sum_{n=1}^{\infty} |y_n| \sum_{k=1}^{\infty} |a_{nk}x_k| = \sum_{k=1}^{\infty} |x_k| \sum_{n=1}^{\infty} |y_n||a_{nk}| \leqq$$

$$\leqq \sup_{k \in \mathbf{N}} \Big(\sum_{n=1}^{\infty} |y_n||a_{nk}| \Big) \|x\|_1 \leqq M \|x\|_1 < \infty.$$

Aus

$$\|A(x)\|_{\dagger}^{\|\dagger\|} = \|(A_n(x))_n\|_{\dagger}^{\|\dagger\|} \leqq M \|x\|_1 < \infty$$

folgt daher $A(x) \in Y^{\dagger\|\dagger\|} = Y$ für alle $x \in l_1$.

Wir beweisen nun zwei Ergebnisse, die den Sätzen 1 und 2 mit bv anstelle von l_1 entsprechen.

SATZ 3. *Ist X ein r-normierter FK-Raum, so gilt*

$$A \in (X, \mathrm{bv}) \Leftrightarrow \sup \Big\{ \Big\| \Big(\sum_{n \in N} (a_{nk} - a_{n-1,k}) \Big)_k \Big\|^{\dagger} \Big| N \subset \mathbf{N} \text{ endlich} \Big\} < \infty.$$

SATZ 4. *Es sei Y ein Teilraum von s und $(Y^\dagger, \|\ldots\|_\dagger)$ ein normierter Raum. Gilt $Y^{\dagger\|\dagger\|} = Y$, so folgt*

$$A \in (\mathrm{bv}, Y) \Leftrightarrow \begin{cases} \text{(i)} \quad \sup_{m \in \mathbf{N}} \Big\| \Big(\sum_{k=1}^{m} a_{nk} \Big)_n \Big\|_{\dagger}^{\|\dagger\|} < \infty, \\ \text{(ii)} \quad \Big\| \Big(\sum_{k=1}^{\infty} a_{nk} \Big)_n \Big\|_{\dagger}^{\|\dagger\|} \text{ existiert.} \end{cases}$$

BEWEIS VON SATZ 3. Wir setzen

$$M := \sup \Big\{ \Big\| \Big(\sum_{n \in N} (a_{nk} - a_{n-1,k}) \Big)_k \Big\|^{\dagger} \Big| N \subset \mathbf{N} \quad \text{endlich} \Big\}.$$

Es gelte $A \in (X, \mathrm{bv})$. Da X und bv FK-Räume sind, ist $A\colon X \to \mathrm{bv}$ linear und stetig, also gibt es eine Konstante K, so daß für alle $x \in X$

$$\|A(x)\|_{\mathrm{bv}} \leqq K \|x\|^{1/r}.$$

Daraus folgt $M < \infty$ wie im Beweis von Satz 1 mit $A_n(x) - A_{n-1}(x)$ anstelle von $A_n(x)$ bzw. mit $a_{nk} - a_{n-1,k}$ anstelle von a_{nk}.

Es gelte umgekehrt $M < \infty$. Für die Menge $\{1\}$ folgt die Existenz von $A_1(x) = \sum_{k=1}^{\infty} a_{1k}x_k$ für alle $x \in X$ und dann nacheinander für die Mengen $\{n\}$ $(n = 2, 3, \ldots)$ die Existenz von $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ für alle $x \in X$.

Wie im Beweis von Satz 1 mit $A_n(x) - A_{n-1}(x)$ anstelle von $A_n(x)$ bzw. mit $a_{nk} - a_{n-1,k}$ anstelle von a_{nk} folgt $(A_n(x))_n \in \mathrm{bv}$ für alle $x \in X$.

BEWEIS VON SATZ 4. Wir setzen

$$M := \sup_{m \in \mathbf{N}} \Big\| \Big(\sum_{k=1}^{m} a_{nk} \Big)_n \Big\|_{\dagger}^{\|\dagger\|} \quad \text{und} \quad M_\infty := \Big\| \Big(\sum_{k=1}^{\infty} a_{nk} \Big)_n \Big\|_{\dagger}^{\|\dagger\|}.$$

Es gelte $A \in (\mathrm{bv}, Y)$. Dann ist $(A_n(x))_n \in Y$ für alle $x \in \mathrm{bv}$, und wegen $Y = Y^{\dagger\|\dagger\|}$ existiert

$$\|(A_n(x))_n\|_{\dagger}^{\|\dagger\|} \quad \text{für alle} \quad x \in \mathrm{bv}.$$

Da bv und Y (mit $\|\cdots\|_{\dagger}^{\|\dagger\|}$) FK-Räume sind, ist $A\colon \mathrm{bv} \to Y$ linear und stetig, also gibt es eine Konstante K, so daß für alle $x \in \mathrm{bv}$

$$\|(A_n(x))_n\|_{\dagger}^{\|\dagger\|} \leqq K \|x\|_{\mathrm{bv}}.$$

Es sei $m \in \mathbf{N}$ beliebig. Dann gilt für die Folge $x^{(m)}$ mit

$$x_k^{(m)} := \begin{cases} 1 & \text{für} \quad 1 \leqq k \leqq m \\ 0 & \text{für} \quad k > m \end{cases}$$

$x^{(m)} \in \mathrm{bv}$ und

$$\|(A_n(x^{(m)}))_n\|_{\dagger}^{\|\dagger\|} = \|(\sum_{k=1}^{m} a_{nk})_n\|_{\dagger}^{\|\dagger\|} \leqq K \|x^{(m)}\|_{\mathrm{bv}} = 2K.$$

Da $m \in \mathbf{N}$ beliebig war, folgt (i). Da $e := (1, 1, \ldots) \in \mathrm{bv}$ ist, existiert

$$M_\infty = \|(A_n(e))_n\|_{\dagger}^{\|\dagger\|}$$

wegen $(A_n(e))_n \in Y = Y^{\dagger\|\dagger\|}$.

Umgekehrt seien (i) und (ii) erfüllt. Aus (ii) folgt, daß die Reihen $\sum_{k=1}^{\infty} a_{nk}$ für alle $n \in \mathbf{N}$ existieren. Daher ist $(a_{nk})_k \in c_0$ für alle $n \in \mathbf{N}$. Aus (i) folgt, daß es zu jedem $n \in \mathbf{N}$ eine Konstante K_n gibt, so daß

$$\sup_{m \in \mathbf{N}} \left|\sum_{k=1}^{m} a_{nk}\right| \leqq K_n.$$

Also ist $(a_{nk})_k \in \gamma_\infty$ für alle $n \in \mathbf{N}$. Insgesamt erhalten wir

$$(a_{nk})_k \in \gamma_\infty \cap c_0 = \mathrm{bv}^\dagger \quad \text{für alle} \quad n \in \mathbf{N}.$$

Das bedeutet aber, daß $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ für alle $n \in \mathbf{N}$ und für alle $x \in \mathrm{bv}$ existiert. Es seien $m \in \mathbf{N}$ beliebig, $y \in S_{Y^\dagger}$ und $x \in \mathrm{bv}$. Für alle $l \in \mathbf{N}$ und für alle Teilmengen N_m von $\{1, 2, \ldots, m\}$ folgt mit

$$A_k^{(N_m)}(y) := \sum_{j=1}^{k} \Big(\sum_{n \in N_m} a_{nj} y_n\Big) \quad (k = 1, 2, \ldots)$$

und abelscher partieller Summation

$$\Big|\sum_{k=1}^{l} x_k \Big(\sum_{n \in N_m} a_{nk} y_n\Big)\Big| \leqq \sum_{k=1}^{l-1} |x_k - x_{k+1}|\,|A_k^{(N_m)}(y)| + |x_l|\,|A_l^{(N_m)}(y)| \leqq$$

$$\leqq 2\|x\|_{\mathrm{bv}} \cdot \sup_{l \in \mathbf{N}} |A_l^{(N_m)}(y)|;$$

da $x \in \mathrm{bv}$ und $\bigl(\sum_{n \in N_m} a_{nk} y_n\bigr)_k \in \mathrm{bv}^\dagger$ ist, existieren die Reihen $\sum_{k=1}^{\infty} x_k\bigl(\sum_{n \in N_m} a_{nk} y_n\bigr)$, und es gilt

$$\Bigl|\sum_{k=1}^{\infty} x_k\bigl(\sum_{n \in N_m} a_{nk} y_n\bigr)\Bigr| \leqq 2\|x\|_{\mathrm{bv}} \cdot \sup_{l \in \mathbf{N}} |A_l^{(N_m)}(y)|.$$

Damit folgt für alle $y \in S_{Y^\dagger}$ und für alle $x \in \mathrm{bv}$

$$\sum_{n=1}^{m} |y_n||A_n(x)| \leqq 4 \cdot \max_{N_m \subset \{1, \ldots, m\}} \Bigl|\sum_{k=1}^{\infty} x_k\bigl(\sum_{n \in N_m} a_{nk} y_n\bigr)\Bigr| \leqq$$

$$\leqq 8\|x\|_{\mathrm{bv}} \cdot \max_{N_m \subset \{1, \ldots, m\}} \Bigl[\sup_{l \in \mathbf{N}} \Bigl|\sum_{k=1}^{l}\bigl(\sum_{n \in N_m} a_{nk} y_n\bigr)\Bigr|\Bigr] \leqq$$

$$\leqq 8\|x\|_{\mathrm{bv}} \cdot \sup_{l \in \mathbf{N}} \Bigl[\sum_{n=1}^{\infty} |y_n| \Bigl|\sum_{k=1}^{l} a_{nk}\Bigr|\Bigr] \leqq 8\|x\|_{\mathrm{bv}} \cdot \sup_{l \in \mathbf{N}} \Bigl\|\bigl(\sum_{k=1}^{l} a_{nk}\bigr)_n\Bigr\|_{\dagger}^{\|\dagger\|} < \infty$$

wegen (i). Da $m \in \mathbf{N}$ beliebig war, folgt für alle $y \in S_{Y^\dagger}$ und für alle $x \in \mathrm{bv}$

$$\sum_{n=1}^{\infty} |y_n A_n(x)| \leqq 8\|x\|_{\mathrm{bv}} \cdot \sup_{l \in \mathbf{N}} \Bigl\|\bigl(\sum_{k=1}^{l} a_{nk}\bigr)_n\Bigr\|_{\dagger}^{\|\dagger\|}$$

also $(A_n(x))_n \in Y^{\dagger\|\dagger\|} = Y$ für alle $x \in \mathrm{bv}$.

In Verbindung mit den Räumen γ_∞ können wir die folgenden Ergebnisse beweisen:

SATZ 5. *Ist X ein r-normierter FK-Raum, so gilt*

$$A \in (X, \gamma_\infty) \Leftrightarrow \sup_{m \in \mathbf{N}} \Bigl\|\bigl(\sum_{n=1}^{m} a_{nk}\bigr)_k\Bigr\|^\dagger < \infty.$$

SATZ 6. *Es sei Y ein Teilraum von s und $(Y^\dagger, \|\cdots\|_\dagger)$ ein normierter Raum. Gilt $Y^{\dagger\|\dagger\|} = Y$, so folgt*

$$A \in (\gamma_\infty, Y) \Leftrightarrow \begin{cases} \text{(i)}\ \sup_{K \subset \mathbf{N}} \Bigl\|\bigl(\sum_{k \in K} (a_{nk} - a_{n,k+1})\bigr)_n\Bigr\|_{\dagger}^{\|\dagger\|} \\ \text{(ii)}\ (a_{nk})_k \in c_0 \quad \textit{für alle} \quad n \in \mathbf{N}. \end{cases}$$

(Satz 5 ließe sich zwar auf Satz 1.2 in [1] zurückführen. Die Arbeitsersparnis wäre allerdings nicht erheblich; zudem geben wir hier eine Modifikation gegenüber dem Beweis von Satz 1.2 in [1].)

BEWEIS VON SATZ 5. Wir setzen $M := \sup_{m \in \mathbf{N}} \Bigl\|\bigl(\sum_{n=1}^{m} a_{nk}\bigr)_k\Bigr\|^\dagger$. Es gelte $A \in (X, \gamma_\infty)$. Da X und γ_∞ FK-Räume sind, ist die Abbildung $A\colon X \to \gamma_\infty$ linear und stetig, also gibt es eine Konstante C, so daß für alle $x \in X$

$$\|A(x)\|_{\gamma_\infty} \leqq C\|x\|^{1/r}.$$

Also ist für alle $m \in \mathbf{N}$

$$\Big|\sum_{n=1}^{m} A_n(x)\Big| = \Big|\sum_{k=1}^{\infty} x_k \sum_{n=1}^{m} a_{nk}\Big| \leqq C\|x\|^{1/r}$$

und

$$\Big\|\Big(\sum_{n=1}^{m} a_{nk}\Big)_k\Big\|^{\dagger} = \sup_{x \in S_X} \Big|\sum_{k=1}^{\infty} x_k\Big(\sum_{n=1}^{m} a_{nk}\Big)\Big| \leqq C \quad \text{für alle} \quad m \in \mathbf{N}.$$

Daraus folgt $M < \infty$.

Es gelte umgekehrt $M < \infty$. Es sei $x \in X$ beliebig. Dann folgt mit $m := 1$ die Existenz von

$$A_1(x) = \sum_{k=1}^{\infty} a_{1k} x_k$$

und daraus nacheinander für $n = 2, 3, \ldots$ die Existenz von

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k \quad \text{für alle} \quad n \in \mathbf{N} \quad \text{und für alle} \quad x \in X.$$

Für alle $m \in \mathbf{N}$ und für alle $x \in X$ ist

$$\Big|\sum_{k=1}^{m} A_n(x)\Big| = \Big|\sum_{k=1}^{\infty} x_k\Big(\sum_{n=1}^{m} a_{nk}\Big)\Big| \leqq \Big\|\Big(\sum_{n=1}^{m} a_{nk}\Big)_k\Big\|^{\dagger} \|x\|^{1/r} \leqq M\|x\|^{1/r}$$

und daher

$$\sup_{m \in \mathbf{N}} \Big|\sum_{k=1}^{m} A_n(x)\Big| \leqq M\|x\|^{1/r} \quad \text{für alle} \quad x \in X.$$

Das bedeutet aber $(A_n(x))_n \in \gamma_\infty$ für alle $x \in X$.

Beweis von Satz 6. Wir setzen $M := \sup_{K \subset \mathbf{N}} \Big\|\Big(\sum_{k \in K} (a_{nk} - a_{n,k+1})\Big)_n\Big\|_{\dagger}^{||\dagger||}$. Es gelte $A \in (\gamma_\infty, Y)$. Dann existiert $A_n(x)$ für alle $x \in \gamma_\infty$ und für alle $n \in \mathbf{N}$; das bedeutet

$$(a_{nk})_k \in \gamma_\infty^{\dagger} = \mathrm{bv} \cap c_0 \quad \text{für alle} \quad n \in \mathbf{N}. \tag{1}$$

Daraus folgt (ii). Wir definieren die Folge y durch

$$y_m := \sum_{k=1}^{m} x_k \quad (m = 1, 2, \ldots) \tag{2}$$

und die Matrix $B = (b_{nk})_{n,k}$ durch

$$b_{nk} := a_{nk} - a_{n,k+1} \quad (n, k = 1, 2, \ldots). \tag{3}$$

Ist $m \in \mathbf{N}$ beliebig, so folgt mit abelscher partieller Summation für alle $x \in \gamma_\infty$ und für alle $n \in \mathbf{N}$

$$\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m-1} b_{nk} y_k + a_{nm} y_m$$

und daher wegen (ii), (1) und $y \in l_\infty$

$$\sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} b_{nk} y_k. \tag{4}$$

Also ist $B \in (l_\infty, Y)$, und es folgt (i) mit Satz 1.3 aus [1].

Es seien umgekehrt (i) und (ii) erfüllt. Ist $m \in \mathbf{N}$ beliebig, so folgt aus (i) für alle $n \in \mathbf{N}$ mit b_{nk} aus (3)

$$\sum_{k=1}^{m} |b_{nk}| \leqq 4 \cdot \max_{N_m \subset \{1, \dots, m\}} \Big|\sum_{k \in N_m} b_{nk}\Big| = 4 \cdot \max_{N_m \subset \{1, \dots, m\}} \Big(\sum_{l=1}^{\infty} |e_l^{(n)}| \Big|\sum_{k \in N_m} b_{nk}\Big|\Big) \leqq$$
$$\leqq 4 \cdot M \|e^{(n)}\|_\dagger < \infty.$$

Da $m \in \mathbf{N}$ beliebig war, ist $(a_{nk})_k \in \mathrm{bv}$ für alle $n \in \mathbf{N}$.

Wie im ersten Teil des Beweises gilt dann mit y aus (2) und B aus (3) unter Beachtung von (ii) die Beziehung (4), und mit Satz 1.3 aus [1] folgt $\big(B_n(y)\big)_n \in Y$ für alle $y \in l_\infty$. Das bedeutet aber $\big(A_n(x)\big)_n \in Y$ für alle $x \in \gamma_\infty$.

Zum Abschluß dieses Kapitels beweisen wir noch ein Ergebnis, das Satz 1.1 aus [2] mit γ anstelle von c entspricht.

SATZ 7. *Es sei X ein vollständiger r-normierter Teilraum von s mit Basis $(e^{(k)})_k$. Dann gilt*

$$A \in (X, \gamma) \Leftrightarrow \begin{cases} \text{(i)} \ \sup_{m \in \mathbf{N}} \Big\|\Big(\sum_{n=1}^{m} a_{nk}\Big)_k\Big\|^\dagger < \infty, \\ \text{(ii)} \ (a_{nk})_n \in \gamma \quad \text{für alle} \quad k = 1, 2, \dots. \end{cases}$$

BEWEIS. Ist $A \in (X, \gamma)$, so folgen die Bedingungen (i) und (ii) wie im Beweis von Satz 1.1 in [2].

Es seien umgekehrt (i) und (ii) erfüllt. Aus (i) folgt die Existenz von $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ für alle $x \in X$ und für alle $n \in \mathbf{N}$ wie im Beweis von Satz 5. Wenn wir für alle $k \in \mathbf{N}$

$$b_{mk} := \sum_{n=1}^{m} a_{nk} \quad (m = 1, 2, \dots) \text{ und } b_k := \lim_{m \to \infty} b_{mk}$$

setzen, so folgt $\big(A_n(x)\big)_n \in \gamma$ für alle $x \in X$ wie im Beweis von Satz 1.1 in [2] mit b_{mk} anstelle von a_{nk} und b_k anstelle von a_k.

Aus Satz 7 folgt sofort

KOROLLAR 1. *Es sei X ein vollständiger r-normierter Teilraum von s mit Basis $(e^{(k)})_k$. Dann gilt*

$$A \in (X, \gamma_0) \Leftrightarrow \begin{cases} \text{(i)} \ \sup_{m \in \mathbf{N}} \Big\|\Big(\sum_{n=1}^{m} a_{nk}\Big)_k\Big\|^\dagger < \infty, \\ \text{(ii)} \ (a_{nk})_n \in \gamma_0 \quad \textit{für alle} \quad k = 1, 2, \dots. \end{cases}$$

Wir schließen mit zwei Anmerkungen:

6. Ist $X \supset \varphi$ ein normaler r-normierter FK-Raum, so ist unter Beachtung eines bekannten Satzes von Allen (s. [7], Satz 1) offensichtlich, daß Satz 1.3 in [1] ein Spezialfall von 1 Satz und Satz 2 ein Spezialfall von Satz 1.2 in [1] ist.

7. Als ein Anwendungsbeispiel für unsere Sätze betrachten wir lediglich einige Spezialfälle von Satz 1. Sind die natürliche Norm für $X^\dagger$ und die Norm $\|\cdots\|^\dagger$ äquivalent, wie das unter anderem für die Räume l_p $(0<p\leqq\infty)$, c und c_0 der Fall ist, so ist die Bestimmung von (X, l_1) auf die Bestimmung von $X^\dagger$ zurückgeführt. Wenn wir für alle p mit $0<p\leqq\infty$ die zu p konjugierte Zahl bezeichnen, d.h. q so bestimmen, daß $l_p^\dagger=l_q$, mit $\|\cdots\|_q$ die übliche Norm für l_q bezeichnen und

$$M_q := \sup\left\{\left\|\left(\sum_{n\in \mathbf{N}} a_{nk}\right)_k\right\|_q \,\middle|\, N\subset \mathbf{N} \quad \text{endlich}\right\}$$

setzen, so erhalten wir aus Satz 1:

$$A\in(l_p, l_1) \Leftrightarrow M_q < \infty \quad (0 < p \leqq \infty)$$

(s. [3], Nr. 76 für $1<p<\infty$, Nr. 72, (72.2) für $p=\infty$, Nr. 77 für $p=1$, man überlegt sich leicht, daß $M_\infty<\infty$ gleichbedeutend mit (77.1) ist);

$$A\in(c, l_1) = (c_0, l_1) \Leftrightarrow M_1 < \infty \quad \text{(s. [3], Nr. 72, (72.2)).}$$

Literaturverzeichnis

[1] E. Malkowsky, Toeplitz-Kriterien für Matrizenklassen bei Räumen absolut limitierbarer Folgen, *Mitt. Math. Sem. Gießen,* **158** (1983), 1—30.

[2] E. Malkowsky, Toeplitz-Kriterien für Matrizenklassen bei Räumen stark limitierbarer Folgen, *Acta Sci. Math. (Szeged),* **48** (1985), 297—313.

[3] M. Stieglitz, H. Tietz, Matrixtransformationen von Folgenräumen, *Math. Z.,* **154** (1977), 1—16.

[4] I. J. Maddox, *Elements of Functional Analysis,* Cambridge University Press (1970).

[5] A. Wilansky, *Functional Analysis,* Blaisdell Publishing Company (New York, 1964).

[6] R. G. Cooke, *Infinite Matrices and Sequence Spaces,* MacMillan (1950).

[7] H. S. Allen, Transformations of sequence spaces, *J. London Math. Soc.,* **31** (1956).

[8] A. Peyerimhoff, Über ein Lemma von Herrn H. C. Chow, *J. London Math. Soc.,* **32** (1957), 33—36.

(Eingegangen am 27. August 1984.)

MATHEMATISCHES INSTITUT
JUSTUS-LIEBIG-UNIVERSITÄT
GIESSEN, ARNDTSTR. 2
BRD

Acta Math. Hung.
50 (1—2) (1987), 49—61.

НАИЛУЧШЕЕ ПРИБЛИЖЕНИЕ УГЛОМ И ПРИБЛИЖЕНИЕ УГЛОМ ИЗ СИНГУЛЯРНЫХ ИНТЕГРАЛОВ ФУНКЦИЙ $f \in L_p(R_n)$, $2<p<\infty$

М. ТОМИЧ (Сараево)

В этой работе приближение углом из некоторых сингулярных интегралов оценивается через наилучшие приближения углом из целых функций. Полученный результат пользуется для сравнения классов функций определенных с помощью указанных приближений.

1. Введение и вспомогательные результаты

Приближение углом является хорошим методом для исследования некоторых классов функций (см. [4], [8], [9]). В этой работе мы результат работы [8] относящийся на функции $f \in L_p(R_n)$, $1<p\leqq 2$, распространяем и на случай $2<p<\infty$. Для 2π периодических функций соответствующий результат дан в [9].

Пусть даны ядра $\mathscr{H}^j_{l_j}$, $j=1, 2, \dots, l_j=1, 2, \dots$ для которых выполнено

$$\int_{-\infty}^{\infty} \mathscr{H}^j_{l_j}(t)\,dt = \sqrt{2\pi}, \quad \int_{-\infty}^{\infty} |\mathscr{H}^j_{l_j}(t)|\,dt \leqq M, \quad \lim_{l_j\to\infty} \int_{\delta\leqq|t|} |\mathscr{H}^j_{l_j}(t)|\,dt = 0$$

причем константа M не зависит от l_j.

От функции f и этих ядер образуем интегралы

$$\mathscr{H}^j_{l_j} * f = I^j_{l_j} f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_1, \dots, x_{j-1}, x_j - t_j, x_{j+1}, \dots, x_n)\, \mathscr{H}^j_{l_j}(t_j)\,dt_j,$$

$$I_{l_i l_j} f = I^i_{l_i} I^i_{l_j} f, \;\dots,\; I_{l_1 \dots l_n} f = I^1_{l_1} \dots I^n_{l_n} f.$$

Для любого набора индексов $i_1, \dots, i_m$, причем $1\leqq i_j \leqq n$, $1\leqq j\leqq m\leqq n$, из этих интегралов образуем m-мерный угол

$$y_{l_{i_1} \dots l_{i_m}} f = I^{i_1}_{l_{i_1}} f + \dots + I^{i_m}_{l_{i_m}} f - I_{l_{i_1} l_{i_2}} f - \dots - I_{l_{i_{m-1}} l_{i_m}} f + \dots + (-1)^{m-1} I_{l_{i_1} \dots l_{l_m}} f.$$

Норму $\|f - y_{l_{i_1} \dots l_{i_m}} f\|_p$ мы будем оценивать через наилучшие приближения углом $Y_{k_{i_1} \dots k_{i_m}}(f)_p$. При том

$$Y_{k_{i_1} \dots k_{i_m}}(f)_p = \inf_{g_{k_{i_j}}} \Big\| f - \sum_{j=1}^{m} g_{k_{i_j}} \Big\|_p \quad (k_{i_j} \geqq 0),$$

$g_{k_{i_j}}$ целая функция экспоненциального типа k_{i_j} по переменной x_{i_j} а по остальным переменным во общем то произвольная функция.

В получении результата работы мы будем пользоватся преобразованием Фурье. Поскольку преобразование Фурье функции $f \in L_p$, $2 < p < \infty$, является обобщенной функцией, то мы приведем понятия и результаты которые обеспечивают применение преобразования Фурье.

Буквой S обозначается пространство на R_n бесконечно дифференцируемых функций $\varphi(x)$, $x=(x_1, \dots, x_n)$, $|x|=\sqrt{x_1^2+\dots+x_n^2}$ для которых

$$\sup_x (1+|x|^l)\,|\varphi^{(k)}(x)| \leqq C(l, k, \varphi) < \infty$$

где C постоянная, $l=1, 2, \dots$, $k=(k_1, \dots, k_n)$, $k_i=1, 2, \dots$.

Буквой S' обозначается пространство линейных непрерывных функционалов на S. Значение функционала $f \in S'$ на функции $\varphi \in S$ будем означать символом $\langle f, \varphi \rangle$. Функционал $f \in S'$ тоже называется обобщенной функцией.

По определению $f_1 = f_2$ в S' если $\langle f_1, \varphi \rangle = \langle f_2, \varphi \rangle$ для каждой функции $\varphi \in S$.

Каждая функция $f(x) \in L_p(R_n)$, $1 \leqq p < \infty$, определяет функционал с помощью равенства

$$\langle f, \varphi \rangle = \int f(x)\,\varphi(x)\,dx, \quad \int = \int_{R_n}. \tag{1.1}$$

Каждая на R_n локально интегрируемая функция f дла которой

$$\int f(x)(1+x^2)^{-\alpha/2}\,dx < \infty \quad (\alpha \geqq 0),$$

определяет функционал $f \in S'$ с помощью равенства (1.1).

Обобщенные функции, определяемые локально интегрируемыми в R_n функциями по формуле (1.1), называюутся регулярными обобщенными функциями.

Лемма 1 ([10], гл. II, §5, [11], гл 1, §1), (дю Буа-Реймонд). *Для того чтобы локально интегрируемая в R_n функция $f(x)$ обращалась в нуль в R_n в смысле обобщенных функций, необходимо и достаточно, чтобы $f(x)=0$ почти везде в R_n.*

Преобразование Фурье $\tilde{\varphi}(x)$ функции $\varphi \in S$ дается равенством

$$\tilde{\varphi}(x) = \frac{1}{(\sqrt{2\pi})^n} \int \varphi(t)\,e^{-ixt}\,dt, \quad \left(tx = \sum_{j=1}^{n} t_j x_j\right).$$

Преобразование Фурье $\tilde{f}$ для $f \in S'$ определяется равенством

$$\langle \tilde{f}, \varphi \rangle = \langle f, \tilde{\varphi} \rangle.$$

Обратное преобразование Фурье $\hat{f}$ дается равенством $(\hat{\tilde{f}})=f$.

Преобразование Фурье преобразует S на S и S' на S' и притом взаимно однозначно и взаимно непрерывно ([10], гл. II, §9).

Кроме того, если обозначим $y=(x_1, \ldots, x_m)$, $z=(x_{m+1}, \ldots, x_n)$, $1 \leqq m \leqq n$, тогда $\tilde{f}^y$, $(\hat{f}^y)$ обозначает, что преобразование $\sim$, $(\hat{\ })$ относится к величине y.

Если $a(x)$ бесконечно дифференцируемая функция для которой

$$|a^{(k)}(x)| \leqq C(k)(1+|x|^{l(k)})$$

($l(k)$ — натуральное число, $C(k)$ — константа), то для $f \in S'$ произведение $af \in S'$ определяется ([10], [11]) равенством

$$\langle af, \varphi\rangle = \langle f, a\varphi\rangle.$$

Произведение af можно определить чтобы принадлежать к S' и тогда когда функция a не обладает предидущими свойствами, при предположении, что f принадлежит какому-то подпространству от S'. Так произведение $a\tilde{f}$ для $f \in L_p$, $1 \leqq p < \infty$, можно определить с помощью мультипликатора в L_p, т.е. всегда когда $a(x)$ является мультипликатором в L_p.

Определение 1 ([2], 1.5.1.1). Функция $\lambda(x)$ называется мультипликатором в L_p, $1 \leqq p < \infty$, если она измерима и ограничена на R_n и для любой функции $f \in S$ выполняется неравенство

$$\|(\lambda\tilde{f})^\wedge\|_p \leqq C_p\|f\|_p \tag{1.2}$$

где константа C_p не зависит от f.

Для любой функции $f(x) \in L_p(R_n)$ существует последовательность функций $f_l \in S$ так, что ([2])

$$\|f-f_l\|_p \to 0, \quad l \to \infty. \tag{1.3}$$

Если функция λ мультипликатор в L_p, тогда неравенство (1.2) имеет место и для $f \in L_p$, причем функция $F=(\lambda\tilde{f})^\wedge$ определена соотношением

$$\|F-(\lambda\tilde{f}_l)^\wedge\|_p \to 0, \quad l \to \infty \tag{1.4}$$

и для функции $f \in L_p$, $f_l \in S$ справедливо (1.3), ([2], 1.5.1.1).

Определение 2. Пусть λ мультипликатор в L_p и $f \in L_p$. Произведение $\lambda\tilde{f}$ это $\tilde{F}$ причем функция $F \in L_p$ определена с помощью соотношения (1.4).

Таким образом определенное произведение $\lambda\tilde{f}$ принадлежит к S' так как $F \in L_p$, т.е. по этому определению $\lambda\tilde{f}$ понимается как преобразование Фурье функций $F \in L_p$.

Справедливы следующие леммы.

Лемма 2. *Пусть* μ, λ_k, $k=1, \ldots, m$ *мультипликаторы в* L_p *и пусть* $f, g \in L_p$, $1 \leqq p < \infty$. *Тогда*

a) $$\big(\mu(\tilde{f}+\tilde{g})\big)^\wedge = (\mu\tilde{f})^\wedge + (\mu\tilde{g})^\wedge \quad \textit{п.в.}$$

b) $$\Big(\Big(\sum_{k=1}^{m} \lambda_k\Big)\tilde{f}\Big)^\wedge = \sum_{k=1}^{m} (\lambda_k\tilde{f})^\wedge \quad \textit{п.в.}$$

и тем самым

a′) $$\mu(\tilde{f}+\tilde{g}) = \mu\tilde{f}+\mu\tilde{g} \quad \text{в } S'$$

b′) $$\Big(\sum_{k=1}^{m} \lambda_k\Big)\tilde{f} = \sum_{k=1}^{m} \lambda_k\tilde{f} \quad \text{в } S'.$$

Эту лемму легко утверждить.

Лемма 3 ([2], 1.5.1.1, (9)). *Пусть* λ, μ *мультипликаторы в* L_p *и* $f\in L_p$, $1\leqq p<\infty$. *Тогда произведение* $\lambda\mu$ *мультипликатор в* L_p *и имеет место*

$$(\lambda(\mu\tilde{f}))\hat{} = ((\lambda\mu)\tilde{f})\hat{} \quad \text{п.в.}$$

и тем самым

$$\lambda(\eta\tilde{f}) = (\lambda\eta)\tilde{f} \quad \text{в } S'.$$

Для $f\in L_p(R)$, $1\leqq p\leqq 2$, и $\mathscr{H}\in L_1(R)$ равенство $\widetilde{\mathscr{H}*f}=\tilde{\mathscr{H}}\cdot\tilde{f}$ имеет место причем произведение понимаем в обичном смысле. Для $2<p<\infty$ это равенство справедливо если произведение $\tilde{\mathscr{H}}\cdot\tilde{f}$ понимаем в смысле определения 2. Именно имеет место

Лемма 4. *Если* $\mathscr{H}\in L_1$ *и* $\tilde{\mathscr{H}}$ *мультипликатор в* L_p, *то для* $f\in L_p(R)$, $1\leqq p<\infty$, *имеет место* $\mathscr{H}* f=F$ *п.в. и тем самым* $\widetilde{\mathscr{H}*f}=\tilde{F}=\tilde{\mathscr{H}}\cdot\tilde{f}$ *в* S' *причем функция* $F\in L_p$ *дана соотношением*

$$\|F-(\tilde{\mathscr{H}}\cdot\tilde{f}_l)\hat{}\,\|_p \to 0, \quad l \leftarrow \infty$$

и для f, f_l *имеет место* (1.3).

Доказательство. Для $\mathscr{H}\in L_1$ и $f_l\in S$ имеет место

$$\mathscr{H}*f_l = (\tilde{\mathscr{H}}\cdot\tilde{f}_l)\hat{}, \quad \|\mathscr{H}*f_l-\mathscr{H}*f\| \to 0, \quad l\to\infty.$$

Имеем

$$\|\mathscr{H}*f-F\| \leqq \|\mathscr{H}*f-\mathscr{H}*f_l\|+\|(\tilde{\mathscr{H}}\cdot\tilde{f}_l)\hat{}-F\| \to 0, \quad l\to\infty,$$

откуда следует утверждение леммы.

Теперь введем функции которые являются свертками функции f с ядрами Дирихле. Мы даем рассмотрение для случая $n=2$ т.е. для $f(x_1, x_2)$, так как для $n>2$ делается аналогично.

Будем обозначать

$$S_{n_1}f = D_{n_1}*f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_1-t_1, x_2)\, D_{n_1}(t_1)\, dt_1 \quad (n_1>0),$$

$$S_{n_2}f = D_{n_2}*f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_1, x_2-t_2)\, D_{n_2}(t_2)\, dt_2 \quad (n_2>0),$$

$$S_{n_1n_2}f = S_{n_1}S_{n_2}f, \quad D_{n_i}(t_i) = \frac{\sin n_i t_i}{t_i}.$$

Лемма 5. *Для функции* $f \in L_p(R)$ $(n=1)$, $1<p<\infty$, *имеет место* $D_k * f = \varphi$ *п.в. и тем самым* $\widehat{D_k * f} = \tilde{\varphi} = \tilde{D}_k \cdot \tilde{f}$ *в* S', *причем функция* $\varphi \in L_p(R)$ *определена соотношением*

$$\|\varphi - (\tilde{D}_k \cdot \tilde{f}_l)\hat{}\,\|_p \to 0, \quad l \to \infty,$$

и для $f_l \in S$ *справедливо* (1.3).

Доказательство. Функция $\tilde{D}_k$ $(\tilde{D}_k = 1$ для $|x| \leqq k$ и $\tilde{D}_k = 0$ для $|x| > k)$, является мультипликатором в L_p. Поэтому существует функция $\varphi \in L_p$ для которой

$$\|\varphi - (\tilde{D}_k \cdot \tilde{f}_l)\hat{}\,\|_p \to 0, \quad l \to \infty.$$

Кроме того имеем $(f_l \in S,\ D_k \in L_q,\ 1<q\leqq\infty)$,

$$D_k * f_l = (\tilde{D}_k \cdot \tilde{f}_l)\hat{}, \quad \|D_k * f - D_k * f_l\| \leqq c_p \|f - f_l\|.$$

Теперь имеем

$$\|D_k * f - \varphi\| \leqq \|D_k * f - D_k * f_l\| + \|(\tilde{D}_k \cdot \tilde{f}_l)\hat{} - \varphi\| \to 0, \quad l \to \infty,$$

откуда следует утверждение леммы.

Лемма 6 ([8]). *Если для функции* $f(x_1, x_2) \in L_p(R_2)$, $1<p<\infty$, *обозначим*

$$S^*_{n_1 n_2} f = S_{n_1} f + S_{n_2} f - S_{n_1 n_2} f$$

то имеет место

$$\|f - S^*_{n_1 n_2} f\|_p \leqq C Y_{n_1 n_2}(f)_p, \quad \|f - S_{n_i} f\|_p \leqq C Y_{n_i}(f)_p \quad (i = 1, 2),$$

причем константа C *не зависит от* f.

Лемма 7 ([2], 1.5.4). *Пусть функция* $\lambda(x)$ *непрерывна разве что за исключением конечного числа точек и пусть* $|\lambda(x)| \leqq M$,

$$\sum_{j=\pm 2^{|k|-1}}^{\pm 2^{|k|}-1} \left| \lambda\left[(j+1)\frac{\pi}{s}\right] - \lambda\left(j\frac{\pi}{s}\right) \right| \leqq M,$$

причем константа M *не зависит от* x, $k = \pm 2, \pm 3, \dots$ *и* $s > 0$; *ставится* $+$ *или* $-$ *зависимо от того, будет ли* $k > 0$ *или* $k < 0$.

Тогда существует константа C_p *независящая от* M *и* f *так, что*

$$\|(\lambda \tilde{f})\hat{}\,\|_p \leqq C_p M \|f\|_p, \quad 1 < p < \infty,$$

для всех $f \in L_p$.

Эта лемма содержится в [2], 1.5.4.

Обозначим

$$\gamma_k(f) = \left((1)_{A_k} \tilde{f}\right)\hat{}, \quad k = 0, 1, 2, \dots$$

где

$$A_0 = \{x\colon |x| \leqq 1\}, \quad A_k = \{x\colon 2^{k-1} < |x| \leqq 2^k\}, \quad (1)_A = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Лемма 8 ([2], 1.5.6.1). *Для функции* $f \in L_p(R)$, $1 < p < \infty$, *справедливы неравенства*

$$\|f\|_p \leqq C_1 \left\| \left\{ \sum_{k=0}^{\infty} \gamma_k^2(f) \right\}^{1/2} \right\|_p \leqq C_2 \|f\|_p,$$

причем константы C_1, C_2 *не зависят от* f.

Лемма 9 ([8]). *Пусть функция* $\lambda(x_1)=0$ *для* $|x_1|>l_1$ *и пусть она удовлетворяет условиям леммы* 7. *Если оператор* $P^1_{l_1} f$ *определим равенством*

$$\widetilde{P^1_{l_1} f}^1 (x_1, x_2) = \lambda(x_1) \tilde{f}^1 (x_1, x_2)$$

причем символ ~ 1 *значит, что преобразование* $\sim$ *относится к переменной* x_1, *то справедливо равенство*

$$P^1_{l_1} S_{k_2} f(x_1, x_2) = S_{k_2} P^1_{l_1} f(x_1, x_2) \quad \text{п.в.}$$

для любой функции $f(x_1, x_2) \in L_p(R_2)$, $1 < p < \infty$.

Лемма 10 ([8]). *Для функции* $\lambda_i(x_i)$ *предположим, что удовлетворяет условиям леммы* 7 *и что* $\lambda_i(x_i)=0$ *для* $|x_i|>l_i$. *Оператор* $P^i_{l_i} f$ *определим равенством*

$$\widetilde{P^i_{l_i} f}^i (x_1, x_2) = \lambda_i(x_i) \tilde{f}^i (x_1, x_2)$$

причем символ $\sim i$ *значит, что преовразование* $\sim$ *относится к переменной* x_i. *Тогда*

$$P^1_{l_1} P^2_{l_2} f(x_1, x_2) = P^2_{l_2} P^1_{l_1} f(x_1, x_2) \quad \text{п.в.}$$

для любой функции $f(x_1, x_2) \in L_p(R_2)$, $1 < p < \infty$.

Символом $a \ll b$ обозначается неравенство $a < Cb$ для некоторой константы C.

2. Оценка приближения

Теорема 1. *Предположим, что функции* $\tilde{\mathscr{H}}^j_{l_j}$ *мультипликаторы в* L_p *и, что*

$$1 - \tilde{\mathscr{H}}^j_{l_j}(t_j) = \omega_j(l_j)\, \psi^j_{l_j}(t_j), \quad l_j = 1, 2, \dots \quad (j = 1, \dots, n),$$

причем $\omega_j(l_j) > 0$ *и функции* $\psi^j_{l_j}(t_j)$ *четные. Для фиксированного числа* l_j *подберем число* s_j *так, что* $2^{s_j} \leqq l_j < 2^{s_j+1}$ *и предположим, что выполнены условия:*

(α) *функция* $|\psi^j_{l_j}(t_j)|$ *не убывет на* $[0, \infty)$ *и* $|\psi^j_{l_j}(t_j)| \leqq C_1$, $t_j \in [0, 1]$,

(β) $|\psi^j_{l_j}(2k_j)| \leqq C_2 |\psi^j_{l_j}(k_j)|$, $\quad 2k_j \leqq 2^{s_j}$,

(δ) $0 < C_3 \leqq \omega_j(l_j) |\psi^j_{l_j}(2^{s_j})|$,

где константы C_1, C_2, C_3 *не зависят от параметров* t_j, k_j, l_j.

Тогда для каждого набора индексов $i_1, \ldots, i_m$ $(1 \leqq i_j \leqq n,\ \ 1 \leqq j \leqq m \leqq n)$ *справедливы неравенства*

$$\|f - y_{l_{i_1} \ldots l_{i_m}} f\|_p \leqq \tag{2.1}$$

$$\leqq C \prod_{j=1}^{m} \omega_j(l_j) \Bigg\{ \sum_{k_{i_1}=0}^{l_{i_1}} \ldots \sum_{k_{i_m}=0}^{l_{i_m}} \sum_{j=1}^{m} \frac{|\psi_{l_{i_j}}^{i_j}(k_{i_j}+1)|^2}{k_{i_j}+1} Y_{k_{i_1} \ldots k_{i_m}}^2 (f)_p \Bigg\}^{1/2},$$

$2 < p < \infty$, *где константа* C *не зависит от* f *и* $l_j = 1, 2, \ldots$.

Доказательство дадим для случая $n=2$. Сначала рассмотрим одномерный угол. Имеем

$$\|f - I_{l_1}^1 f\| \leqq \|f - S_{2^{s_1}} f\| + \|I_{l_1}^1 S_{2^{s_1}} f - I_{l_1}^1 f\| + \tag{2.2}$$

$$+ \|S_{2^{s_1}} f - I_{l_1}^1 S_{2^{s_1}} f\| \ll Y_{2^{s_1}}(f) + \|S_{2^{s_1}} f - S_{2^{s_1}} I_{l_1}^1 f\|.$$

Обозначим $B_{s_1} = \{x_1 : |x_1| \leqq 2^{s_1}\}$. Считая $f(x_1, x_2)$ функцией (одной) переменной x_1 на основании лемм 3 и 5 получаем справедливость в S' равенства

$$\tilde{\gamma}_{k_1}[S_{2^{s_1}}(f - I_{l_1}^1 f)] = (1)_{A_{k_1}} \tilde{S}_{2^{s_1}}(f - I_{l_1}^1 f) =$$

$$= (1)_{A_{k_1}}[(1)_{B_{s_1}}(\tilde{f} - \widetilde{I_{l_1}^1 f})] = [(1)_{A_{k_1}}(1)_{B_{s_1}}](\tilde{f} - \widetilde{I_{l_1}^1 f}) =$$

$$= (1)_{A_{k_1}}(\tilde{f} - \widetilde{I_{l_1}^1 f}) = \tilde{\gamma}_{k_1}(f - I_{l_1}^1 f), \quad k_1 = 0, 1, \ldots, s_1.$$

Из этого равенства следует

$$\gamma_{k_1}[S_{2^{s_1}}(f - I_{l_1}^1 f)] = \gamma_{k_1}(f - I_{l_1}^1 f) \quad \text{в } S'$$

откуда на основании леммы 1 получаем

$$\gamma_{k_1}[S_{2^{s_1}}(f - I_{l_1}^1 f)] = \gamma_{k_1}(f - I_{l_1}^1 f) \quad \text{п.в.} \quad (k_1 = 0, 1, \ldots, s_1). \tag{2.3}$$

Пользуясь леммой 8 имеем

$$\int |S_{2^{s_1}}(f - I_{l_1}^1 f)|^p \, dx_1 \ll \int \Big\{ \sum_{k_1=0}^{\infty} \gamma_{k_1}^2 [S_{2^{s_1}}(f - I_{l_1}^1 f)] \Big\}^{p/2} dx_1$$

откуда интегрированием по x_2 следует

$$\Big(\int\!\!\int |S_{2^{s_1}}(f - I_{l_1}^1 f)|^p \, dx_1 \, dx_2 \Big)^{2/p} \ll$$

$$\ll \Big(\int\!\!\int \Big\{ \sum_{k_1=0}^{\infty} \gamma_{k_1}^2 [S_{2^{s_1}}(f - I_{l_1}^1 f)] \Big\}^{p/2} dx_1 \, dx_2 \Big)^{2/p}.$$

Так как $1 < p/2 < \infty$ то применением неравенства Минковского получаем

$$\|S_{2^{s_1}}(f - I_{l_1}^1 f)\|_p^2 \ll \sum_{k_1=0}^{\infty} \|\gamma_{k_1}[S_{2^{s_1}}(f - I_{l_1}^1 f)]\|_p^2.$$

Учитывая, что

$$\gamma_{k_1}[S_{2^{s_1}}(f - I_{l_1}^1 f)] = 0 \quad \text{для} \quad k_1 > s_1$$

то

$$(2.4)\qquad \|S_{2^{s_1}}(f-I_{l_1}^1 f)\|_p^2 \ll \sum_{k_1=0}^{s_1} \|\gamma_{k_1}[S_{2^{s_1}}(f-I_{l_1}^1 f)]\|_p^2 .$$

Пусть оператор $P_{l_1}^1 f$ дан равенством

$$\widetilde{P_{l_1}^1 f}^{\,1} = [(1)_{B_{s_1}} \psi_{l_1}^1(x_1)]\, \tilde{f}^1(x_1, x_2) \quad \text{в} \quad S'$$

(функция $(1)_{B_{s_1}} \psi$ является мультипликатором в L_p), и пусть функции $f_r(x_1, x_2) \in \in S(R_2)$ такие, что

$$\|f-f_r\|_p \to 0, \quad r \to \infty.$$

Имеет место (в обычном смысле) равенство

$$\tilde{\gamma}_{k_1}^1(f_r-I_{l_1}^1 f_r) = (1)_{A_{k_1}}(1-\tilde{\mathscr{H}}_{l_1}^1)\, \tilde{f}_r^1 = \omega_1(l_1)\,[(1)_{A_{k_1}} \psi_{l_1}^1(t_1)]\, \tilde{f}_r^1 =$$

$$= \omega_1(l_1)(1)_{A_{k_1}}[(1)_{B_{s_1}} \psi_{l_1}^1(t_1)]\, \tilde{f}_r^1, \quad k_1 = 0, 1, \ldots, s_1,$$

откуда следует

$$(2.5)\qquad \gamma_{k_1}(f_r-I_{l_1}^1 f_r) = \omega_1(l_1)\gamma_{k_1}(P_{l_1}^1 f_r) \quad \text{п.в.}$$

На основании леммы 7 имеем

$$\|\gamma_{k_1}(f-I_{l_1}^1 f)-\gamma_{k_1}(f_r-I_{l_1}^1 f_r)\|_{p(R_1)}^p =$$

$$= \|\gamma_{k_1}\{f-I_{l_1}^1 f-(f_r-I_{l_1}^1 f_r)\}\|_{p(R_1)}^p \leqq \|f-I_{l_1}^1 f-(f_r-I_{l_1}^1 f_r)\|_{p(R_1)}^p$$

откуда интегрированием по x_2 получаем

$$\|\gamma_{k_1}(f-I_{l_1}^1 f)-\gamma_{k_1}(f_r-I_{l_1}^1 f_r)\|_{p(R_2)} \leqq \|f-I_{l_1}^1 f-(f_r-I_{l_1}^1 f_r)\|_{p(R_2)}^p .$$

Отсюда когда $r \to \infty$ следует

$$(2.6)\qquad \|\gamma_{k_1}(f-I_{l_1}^1 f)-\gamma_{k_1}(f_r-I_{l_1}^1 f_r)\|_{p(R_2)} \to 0, \quad r \to \infty.$$

Кроме того имеем

$$\|\gamma_{k_1}(P_{l_1}^1 f)-\gamma_{k_1}(P_{l_1}^1 f_r)\| = \|\gamma_{k_1}(P_{l_1}^1 f-P_{l_1}^1 f_r)\| \leqq \|P_{l_1}^1 f-P_{l_1}^1 f_r\| \ll \|f-f_r\|,$$

т.е.

$$(2.7)\qquad \|\gamma_{k_1}(P_{l_1}^1 f)-\gamma_{k_1}(P_{l_1}^1 f_r)\|_p \to 0, \quad r \to \infty.$$

Теперь пользуясь неравенством

$$\|\gamma_{k_1}(f-I_{l_1}^1 f)-\omega_1(l_1)\,\gamma_{k_1}(P_{l_1}^1 f)\|_{p(R_2)} \leqq \|\gamma_{k_1}(f-I_{l_1}^1 f)-\gamma_{k_1}(f_r-I_{l_1}^1 f_r)\| +$$

$$+\|\gamma_{k_1}(f_r-I_{l_1}^1 f_r)-\omega_1(l_1)\,\gamma_{k_1}(P_{l_1}^1 f_r)\| + \|\omega_1(l_1)\,\gamma_{k_1}(P_{l_1}^1 f_r)-\omega_1(l_1)\,\gamma_{k_1}(P_{l_1}^1 f)\|$$

получаем

$$(2.8)\qquad \gamma_{k_1}(f-I_{l_1}^1 f) = \omega_1(l_1)\,\gamma_{k_1}(P_{l_1}^1 f) \quad \text{п.в.}$$

Из (2.4) на основании (2.3) и (2.8) следует

$$\|S_{2^{s_1}}(f-I^1_{l_1}f)\|^2_p \ll \omega^2_1(l_1) \sum_{k_1=0}^{s_1} \|\gamma_{k_1}(P^1_{l_1}f)\|^2_p. \tag{2.9}$$

Функции $P^{i*}_{l_i} f$ мы определим с помощью (в S') равенства

$$\widetilde{P^{i*}_{l_i}}^{\,i} = \eta^i_{l_i}\tilde{f}^i, \quad \eta = \eta^i_{l_i}(t_i) = \begin{cases} 1, & |t_i| \leqq 1 \\ \psi^i_{l_i}(2^{k_i}), & t_i \in A_{k_i} \\ 0, & |t_i| > 2^{s_i}. \end{cases}$$

Имеет место (см. (2.8) в [8])

$$\|\gamma_{k_1}(P^1_{l_1}f)\|_{p(R_2)} \leqq C_p \|\gamma_{k_1}(P^{1*}_{l_1}f)\|_{p(R_2)}, \quad k_1 = 0, 1, \ldots, s_1, \tag{2.10}$$

где C_p не зависит от $\gamma_{k_1}(P^{1*}_{l_1}f)$.

Будем пользоваться равенством

$$\gamma_{k_1}(P^{1*}_{l_1}f) = S_{2^{k_1}}(P^{1*}_{l_1}f) - S_{[2^{k_1-1}]}(P^{1*}_{l_1}f) \tag{2.11}$$

причем $[2^{k-1}]=2^{k-1}$ для $k \geqq 1$, $[2^{k-1}]=0$ для $k=0$. Теперь неравенство (29) дает

$$\|S_{2^{s_1}}(f-I^1_{l_1}f)\|^2_p \ll \omega^2_1(l_1) \sum_{k_1=0}^{s_1} \|S_{2^{k_1}}(P^{1*}_{l_1}f) - S_{[2^{k_1-1}]}(P^{1*}_{l_1}f)\|^2_p. \tag{2.12}$$

В работе [8] констатировано, что для функции $f(x_1, x_2) \in L_p(R_2)$, $1 < p \leqq 2$, а это значит и для функции $f_r \in S$, справедливо равенство

$$\begin{aligned} S_{2^{k_1}}(P^{1*}_{l_1}f) - S_{[2^{k_1-1}]}(P^{1*}_{l_1}f) = \sum_{v_1=[2^{k_1-1}]}^{2^{k_1-1}} (S_{\gamma_1}f - f)\,\Delta\eta^1_{l_1}(v_1) + \\ + (S_{2^{k_1}}f - f)\,\eta^1_{l_1}(2^{k_1}) - (S_{[2^{k_1-1}]}f - f)\,\eta^1_{l_1}([2^{k_1-1}]) \quad \text{п.в.} \end{aligned} \tag{2.13}$$

Обозначая буквой $G(f)$ выражение на левой стороны и буквой $H(f)$ выражение на правой стороны равенства (2.13) и учитывая, что $G(f_r)=H(f_r)$ имеем

$$\|G(f)-H(f)\| \leqq \|G(f)-G(f_r)\| + \|H(f_r)-H(f)\|.$$

Так как

$$\|G(f)-G(f_r)\| \to 0, \quad \|H(f)-H(f_r)\| \to 0, \quad r \to \infty,$$

при предположении, что $\|f-f_r\| \to 0$, то утверждаем справедливость равенства (2.13) и для $f \in L_p$, $2 < p < \infty$.

Пользуясь условиями (α), (β) и леммой 6 из (2.12) на основании (2.13) получается

$$\|S_{2^{s_1}}(f-I^1_{l_1}f)\|^2_p \ll \omega^2_1(l_1) \sum_{k_1=0}^{s_1} |\psi^1_{l_1}([2^{k_1-1}])|^2\, Y^2_{[2^{k_1}-1]}(f)_p. \tag{2.14}$$

Из (2.2), (2.14) пользуясь условием (δ) следует утверждение теоремы для одномерного угла.

Теперь докажем теорему для приближения двухмерным углом. На основании теоремы Фубини утверждаем справедливость разложения

$$f-\mathscr{Y}_{l_1l_2}f = \sum_{i=1}^{9} B_i \tag{2.15}$$

причем

$$B_1 = f-S^*_{2^{s_1}2^{s_2}}f, \quad B_2 = -I^1_{l_1}B_1,$$

$$B_3 = -I^2_{l_2}B_1, \quad B_4 = I_{l_1l_2}B_1,$$

$$B_5 = S_{2^{s_1}}(f-I^1_{l_1}f-S_{2^{s_2}}f+I^1_{l_1}S_{2^{s_2}}f), \quad B_6 = -I^2_{l_2}B_5,$$

$$B_7 = S_{2^{s_2}}(f-S_{2^{s_1}}f-I^2_{l_2}f+I^2_{l_2}S_{2^{s_1}}f), \quad B_8 = -I^1_{l_1}B_7,$$

$$B_9 = S_{2^{s_1}2^{s_2}}(f-\mathscr{Y}_{l_1l_2}f).$$

Имеет место

$$\sum_{i=1}^{4} \|B_i\|_p^2 \ll Y^2_{2^{s_1}2^{s_2}}(f)_p. \tag{2.16}$$

Обозначим $F=f-S_{2^{s_2}}f$. Тогда

$$B_5 = S_{2^{s_1}}(F-I^1_{l_1}F) = S_{2^{s_1}}F-I^1_{l_1}S_{2^{s_1}}F.$$

Для оценки нормы $\|B_5\|$ пользуемся методом одномерного случая, так, что, в (2.13) надо писать F вместо f (см. (2.7)—(2.11) в [8]). Таким образом учитывая, что

$$S_{v_1}F-F = -(f-S^*_{v_12^{s_2}}f)$$

получаем

$$\|B_5\|_p^2 \ll \omega_1^2(l_1)\sum_{k_1=0}^{s_1}|\psi^1_{l_1}([2^{k_1-1}])|^2 Y^2_{[2^{k_1-1}]2^{s_2}}(f)_p$$

откуда пользуясь условием (δ) следует

$$\|B_5\|_p^2 \ll \omega_1^2(l_1)\,\omega_2^2(l_2)|\,\psi^2_{l_2}(2^{s_2})|^2\sum_{k_1=0}^{s_1}|\psi^1_{l_1}([2^{k_1-1}])|^2 Y^2_{[2^{k_1-1}]2^{s_2}}(f)_p. \tag{2.17}$$

Тем же способом получается

$$\|B_7\|_p^2 \ll \omega_1^2(l_1)\,\omega_2^2(l_2)|\,\psi^1_{l_1}(2^{s_1})|^2\sum_{k_2=0}^{s_2}|\psi^2_{l_2}([2^{k_2-1}])|^2 Y^2_{2^{s_1}[2^{k_2-1}]}(f)_p. \tag{2.18}$$

Имеем

$$\|B_6\| \ll \|B_5\|, \quad \|B_8\| \ll \|B_7\|. \tag{2.19}$$

Чтобы оценить $\|B_9\|$ напишем

$$B_9 = S_{2^{s_1}}(\varphi-I^1_{l_1}\varphi), \quad \varphi = S_{2^{s_2}}f-I^2_{l_2}S_{2^{s_2}}f.$$

Для функции φ справедливы равенства (2.8) и это значит, что

$$(1)_{A_{k_1}}(\tilde{\varphi}-\widetilde{I^1_{l_1}\varphi}) = \omega_1(l_1)\,\tilde{\gamma}_{k_1}(P^1_{l_1}\varphi) \quad \text{в} \;\; S'.$$

Суммированием по k_1 получаем

$$\sum_{k_1=0}^{s_1}(1)_{A_{k_1}}(\tilde{\varphi}-\widetilde{I^1_{l_1}\varphi}) = \omega_1(l_1)\sum_{k_1=0}^{s_1}\tilde{\gamma}_{k_1}(P^1_{l_1}\varphi),$$

$$(1)_{B_{s_1}}(\tilde{\varphi}-\widetilde{I^1_{l_1}\varphi}) = \omega_1(l_1)\sum_{k_1=0}^{s_1}\tilde{\gamma}_{k_1}(P^1_{l_1}\varphi),$$

откуда следует

$$B_9 = S_{2^{s_1}}(\varphi - I^1_{l_1}\varphi) = \omega_1(l_1)\sum_{k_1=0}^{s_1}\gamma_{k_1}(P^1_{l_1}\varphi) \quad \text{п.в.} \tag{2.20}$$

Также считая функцию $f(x_1, x_2)$ функцией (одной) переменной x_2 имеем

$$\varphi = \omega_2(l_2)\sum_{k_2=0}^{s_2}\gamma_{k_2}(P^2_{l_2}f) \quad \text{п.в.} \tag{2.21}$$

Дальнейшие выкладки как в работе [8]. Именно, аналогично неравенству (2.12) имеем неравенство

$$\|B_9\|_p^2 \ll \omega_1^2(l_1)\sum_{k_1=0}^{s_1}\|S_{2^{s_1}}(P^{1*}_{l_1}\varphi) - S_{[2^{k_1-1}]}(P^{1*}_{l_1}\varphi)\|^2_{p(R_2)}. \tag{2.22}$$

Пользуясь равенствами (2.21) и

$$\gamma_{k_2}(P^2_{l_2}f) = S_{2^{k_2}}(P^2_{l_2}f) - S_{[2^{k_2-1}]}(P^2_{l_2}f)$$

из (2.22) получается

$$\begin{aligned}\|B_9\|_p^2 \ll \omega_1^2(l_1)\,\omega_2^2(l_2)\sum_{k_1=0}^{s_1}\|S_{2^{k_1}}[P^{1*}_{l_1}\sum_{k_2=0}^{s_2}\{S^{2k_2}(P^2_{l_2}f) - S_{[2^{k_2-1}]}P^2_{l_2}f)\}] - \\ - S_{[2^{k_1-1}]}[P^{1*}_{l_1}\sum_{k_2=0}^{s_2}\{S_{2^{k_2}}(P^2_{l_2}f) - S_{[2^{k_2-1}]}(P^2_{l_2}f)\}]\|_{p(R_2)}.\end{aligned} \tag{2.23}$$

Если операции $P^{1*}_{l_1}$ и $S_{2^{k_1}}$ внести под суммы и применить леммы 9 и 10 получаем

$$\|B_9\|_p^2 \ll \omega_1^2(l_1)\,\omega_2^2(l_2)\sum_{k_1=0}^{s_1}\Big\|\sum_{k_2=0}^{s_2}\gamma_{k_2}(P^2_{l_2}\xi_{k_1})\Big\|_p^2 \tag{2.24}$$

где

$$\xi_{k_1} = S_{2^{k_1}}(P^{1*}_{l_1}f) - S_{[2^{k_1-1}]}(P^{1*}_{l_1}f).$$

Из (2.24) пользуя лемму 8 и неравенство Минковского получается

$$\begin{aligned}\|B_9\|_p^2 \ll \omega_1^2(l_1)\,\omega_2^2(l_2)\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}\|\gamma_{k_2}(P^2_{l_2}\xi_{k_1})\|_p^2 \ll \\ \ll \omega_1^2(l_1)\,\omega_2^2(l_2)\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}\|\gamma_{k_2}(P^{2*}_{l_2}\xi_{k_1})\|_p^2.\end{aligned} \tag{2.25}$$

Аналогично равенству (2.13) имеет место равенство

$$(2.26)\qquad \gamma_{k_2}(P_{l_2}^{2*}\xi_{k_1}) = S_{2^{k_2}}(P_{l_2}^{2*}\xi_{k_1}) - S_{[2^{k_2-1}]}(P_{l_2}^{2*}\xi_{k_1}) =$$

$$= \sum_{\nu_2=[2^{k_2-1}]}^{2^{k_2}-1} (S_{\nu_2}\xi_{k_1}-\xi_{k_1})\,\Delta\eta_{l_2}^2(\nu_2) + (S_{2^{k_2}}\xi_{k_1}-\xi_{k_1})\,\eta_{l_2}^2(2^{k_2}) -$$

$$-(S_{[2^{k_2-1}]}\xi_{k_1}-\xi_{k_1})\,\eta_{l_2}^2([2^{k_2-1}]).$$

В равенстве (2.26) подставим для ξ_{k_1} выражение дано равенством (2.13), и потом применим лемму 6. Тогда пользуясь условиями (α) и (β) и фактом, что

$$S_{\nu_2}(S_{\nu_1}f-f)-(S_{\nu_1}f-f) = f-S_{\nu_1\nu_2}^{*}f$$

получаем

$$(2.27)\qquad \|\gamma_{k_2}(P_{l_2}^{2*}\xi_{k_1})\|_p \ll |\psi_{l_1}^1([2^{k_1-1}])|\cdot|\psi_{l_2}^2([2^{k_2-1}])|\,Y_{[2^{k_1-1}][2^{k_2-1}]}(f)_p.$$

Из (2.25) на основании (2.27) получается оценка для нормы $\|B_9\|_p$. На конец, из равенства (2.15) на основании оценки для $\|B_i\|_p$ $(i=1, \ldots, 9)$ следует неравенство (2.1).

Теорема доказана.

3. Иллюстрация

Пусть сингулярный интеграл Рисса дан ядром $\mathscr{H}_{l_j}^j(t_j)$ $(j=1, \ldots, n)$, для которого

$$\widetilde{\mathscr{H}_{l_j}^j}(tj) = \begin{cases} 1-\dfrac{|t_j|^{r_j}}{l_j^{r_j}}, & |t_j| \leqq l_j \\ 0, & |t_j| > l_j, \end{cases}$$

$r_j>0$. Обозначим

$$S_p^r F(R_n) = \{f\in L_p(R_n)\colon\ \|f-\mathscr{Y}_{l_{i_1}\ldots l_{i_m}}f\|_p = O\big(\prod_{j=1}^m l_{i_j}^{-r_{i_j}}\big),$$

$l_j=1, 2, \ldots,$ для всех наборов индексов $i_1, \ldots, i_m$ таких, что $1\leqq i_j\leqq n$, $1\leqq j\leqq$ $\leqq m\leqq n\}$.

На основании доказанной теоремы имеем неравенство

$$\|f-y_{l_{i_1}\ldots l_{i_m}}f\|_p \ll \prod_{j=1}^m l_{i_j}^{-r_{i_j}}\Big\{\sum_{k_{i_1}=0}^{l_{i_1}}\cdots\sum_{k_{i_m}=0}^{l_{i_m}}\prod_{j=1}^m k_{i_j}^{2r_{i_j}-1}Y_{k_{i_1}\ldots k_{i_m}}^2(f)_p\Big\}^{1/2},\quad 2<p<\infty.$$

Пусть $S_p^r H(R_n)$, $r=(r_1, \ldots, r_n)$, классы Никольского которые данные с помощью наилучших приближений углом, [6]

$$S_p^r H = \{f\in L_p\colon\ Y_{l_{i_1}\ldots l_{i_m}}(f)_p = O\big(\prod_{j=1}^m l_{i_j}^{-r_{i_j}}\big),\ 1\leqq i_j\leqq n,\ 1\leqq j\leqq m\leqq n\}.$$

Следствием теоремы является вложение

$$S_p^{\mathbf{r}^{(1)}} H(R_n) \subset S_p^{\mathbf{r}} F(R_n) \subset S_p^{\mathbf{r}} H(R_n), \quad 2 < p < \infty,$$

если $r_j^{(1)} > r_j$ $(j=1, \ldots, n)$.

Если учесть результат работы [8], то мы утверждили, что это вложение справедливо для $1 < p < \infty$.

Литература

[1] Т. И. Аманов, *Пространства дифференцируемых функций с доминирующей смешанной производной,* АН Каз ССР (Алма-Ата, 1976).

[2] С. М. Никольский, *Приближение функций многих переменных и теоремы вложеиня* (Москва, 1969).

[3] М. К. Потапов, О приближении углом, *Труды конференции по конструктивной теории функций* (Будапешт, 1972), 371—399.

[4] М. К. Потапов, Изучение некоторых классов функций при помощи приближения углом, *Труды МИАНСССР,* **117** (1972), 256—300.

[5] М. Томич, О приближении углом функций с доминирующим смешанным модулем гладкости, *Публ. де л'инст. мат. Београд,* **23 (37)** (1978), 193—206.

[6] М. Томич, Изучение пространств функций типа Никольского $S_p^\beta H(R_n)$, *Радови АНБИХ,* кн. **66 (19)** (1980), 111—128.

[7] М. Томич, Наилучшее приближение функции и приближение некоторыми сингулярными интегралами, *Гласник математички,* **15 (35)** (1980), 71—78.

[8] М. Томич, Наилучшее приближение углом и приближение углом из сингулярных интегралов, *Мат. весник,* **3, 35** (1983), 289—304.

[9] М. Томич, Приближение 2π периодических функций углом из сингулярных интегралов, *Публ. де л'инст. мат.,* **26 (40)** (1979), 297—305.

[10] В. С. Бладимиров, *Уравнения математической физики* (Москва, 1971).

[11] В. С. Владимиров, *Обобщенные функции в математической физике* Москва, 1976).

(Поступило 19. 9. 1984.)

УЛ. ДЖ. НЕХРУА 10
71000 САРАЕВО
ЮГОСЛАВИЯ

Acta Math. Hung.
50 (1—2) (1987), 63—69.

(0, 2, 3) AND (0, 1, 3) INTERPOLATION THROUGH SPLINES

R. B. SAXENA and H. C. TRIPATHI (Lucknow)

1. Introduction

Let

$$\Delta:\ 0 = x_0 < x_1 < \ldots < x_{m-1} < x_m = 1$$

be the uniform partition of the interval $I=[0, 1]$ with $x_k=\dfrac{k}{m}$, $k=0, \ldots, m$ and $h=\dfrac{1}{m}$. We define the class of spline functions $S^{*(3)}_{m,6}$ as follows:

Any element $S_\Delta \in S^{*(3)}_{m,6}$ if the following conditions are fulfilled:

(i) $S_\Delta \in C^3(I)$,

(ii) in each interval $[x_k, x_{k+1}]$, $k=0, \ldots, m-1$ $S_\Delta \in \Pi_6$ except in one of the end intervals, say in $[x_0, x_1]$, where $S_\Delta \in \Pi_7$.

Here, as usual, Π_n denotes the set of polynomials of degree at most n.

Let there be given two sets of real numbers $f(x_k)$, $f^{(q)}(x_k)$, $q=2, 3$, and $q=1, 3$ with $k=0, \ldots m$, which we shall denote in the sequel by y_k, $y_k^{(q)}$. In this paper, we solve two lacunary interpolation problems, viz. the (0, 2, 3) and (0, 1, 3) described in (1.1) and (1.2), respectively, by the elements of $S^{*(3)}_{m,6}$. The first one is formulated as follows:

$$\text{(1.1)}\qquad \begin{cases} \tilde{S}_\Delta(x_k) = y_k, & \tilde{S}^{(q)}_\Delta(x_k) = y_k^{(q)}, \quad q = 2, 3;\ k = 0, \ldots, m, \\ \tilde{S}'_\Delta(x_0) = y'_0, & \tilde{S}'_\Delta(x_m) = y'_m; \end{cases}$$

and the second problem as

$$\text{(1.2)}\qquad \begin{cases} \hat{S}_\Delta(x_k) = y_k, & \hat{S}^{(q)}_\Delta(x_k) = y_k^{(q)}, \quad q = 1, 3;\ k = 0, \ldots, m, \\ \hat{S}''_\Delta(x_0) = y''_0, & \hat{S}''_\Delta(x_m) = y''_m. \end{cases}$$

Here y'_0, y'_m, y''_0, y''_m are given reals.

In Section 2, two theorems on the existence and uniqueness of solutions of the above problems are given. In [1], J. Győrvári has considered the (0, 2, 3) problem by different spline functions of class $C(I)$. The essential difference here is in the continuity class and the nature of the spline function. In Section 3, we obtain error bounds for the error $|\tilde{S}^{(q)}_\Delta - f^{(q)}|$ and $|\hat{S}^{(q)}_\Delta - f^{(q)}|$ in the case when $f \in C^6(I)$ and $q=0, \ldots, 5$.

2. Existence and uniqueness

We first consider the interpolation problem (1.1). Let

$$\tilde{S}_{\Delta}(x)=\begin{cases}\tilde{S}_0(x), & x_0\leqq x\leqq x_1\\ \tilde{S}_k(x), & x_k\leqq x\leqq x_{k+1}, \quad k=1,\dots,m-2\\ \tilde{S}_{m-1}(x), & x_{m-1}\leqq x\leqq x_m.\end{cases}$$

Then owing to the conditions (1.1), we can write

$$\tilde{S}_0(x)=y_0+\sum_{r=1}^{3}\frac{(x-x_0)^r}{r!}y_0^{(r)}+\sum_{r=4}^{7}\frac{(x-x_0)^r}{r!}a_{0,r}, \tag{2.1}$$

$$\tilde{S}_k(x)=y_k+(x-x_k)\,a_{k,1}+\frac{(x-x_k)^2}{2!}y_k''+\frac{(x-x_k)^3}{3!}y_k'''+\sum_{r=4}^{6}\frac{(x-x_k)^r}{r!}a_{k,r}, \tag{2.2}$$

$$k=1,\dots,m-1$$

where the coefficients of these polynomials are to be determined by the following conditions:

$$\begin{cases}\tilde{S}_k(x_{k+1})=\tilde{S}_{k+1}(x_{k+1})=y_{k+1},\\ \tilde{S}_k^{(q)}(x_{k+1})=\tilde{S}_{k+1}^{(q)}(x_{k+1})=y_{k+1}^{(q)}, & q=2,3\\ \tilde{S}_k'(x_{k+1})=\tilde{S}_{k+1}'(x_{k+1}), & k=0,\dots,m-2\end{cases} \tag{2.3}$$

and

$$\tilde{S}_{m-1}(x_m)=y_m,\quad \tilde{S}_{m-1}^{(q)}(x_m)=y_m^{(q)},\quad q=1,2,3. \tag{2.4}$$

For brevity we set

$$\alpha_k:=y_{k+1}-y_k-hy_k'-\frac{h^2}{2}y_k''-\frac{h^3}{6}y_k''',$$

$$\beta_k:=y_{k+1}'-y_k'-hy_k''-\frac{h^2}{2}y_k''',$$

$$\gamma_k:=y_{k+1}''-y_k''-hy_k''',$$

$$\delta_k:=y_{k+1}'''-y_k''',\quad k=0,\dots,m-1.$$

To obtain the coefficients in $\tilde{S}_{m-1}(x)$, we use (2.2) for $k=m-1$ and apply the conditions (2.4). We get the following equations:

$$h(a_{m-1,1}-y_{m-1}')+\sum_{r=4}^{6}\frac{h^r}{r!}a_{m-1,r}=\alpha_{m-1},$$

$$(a_{m-1,1}-y_{m-1}')+\sum_{r=4}^{6}\frac{h^{r-1}}{(r-1)!}a_{m-1,r}=\beta_{m-1},$$

$$\sum_{r=4}^{6}\frac{h^{r-2}}{(r-2)!}a_{m-1,r}=y_{m-1},\quad \sum_{r=4}^{6}\frac{h^{r-3}}{(r-3)!}a_{m-1,r}=\delta_{m-1}.$$

Solving these equations we get

$$a_{m-1,1}-y'_{m-1}=\frac{2}{h}\alpha_{m-1}-\beta_{m-1}+\frac{h}{5}\gamma_{m-1}-\frac{h^2}{60}\delta_{m-1}, \tag{2.5}$$

$$a_{m-1,4}=-\frac{120}{h^4}\alpha_{m-1}+\frac{120}{h^3}\beta_{m-1}-\frac{36}{h^2}\gamma_{m-1}+\frac{4}{h}\delta_{m-1}, \tag{2.6}$$

$$a_{m-1,5}=\frac{720}{h^5}\alpha_{m-1}-\frac{720}{h^4}\beta_{m-1}+\frac{240}{h^3}\gamma_{m-1}-\frac{30}{h^2}\delta_{m-1}, \tag{2.7}$$

$$a_{m-1,6}=-\frac{1440}{h^6}\alpha_{m-1}+\frac{1440}{h^5}\beta_{m-1}-\frac{504}{h^4}\gamma_{m-1}+\frac{72}{h^3}\delta_{m-1}. \tag{2.8}$$

We shall now determine the coefficients of $\tilde{S}_k(x)$, $k=1, \ldots, m-2$. Here we have

$$h(a_{k,1}-y'_k)+\sum_{r=4}^{6}\frac{h^r}{r!}a_{k,r}=\alpha_k,$$

$$\sum_{r=4}^{6}\frac{h^{r-2}}{(r-2)!}a_{k,r}=\gamma_k, \quad \sum_{r=4}^{6}\frac{h^{r-3}}{(r-3)!}a_{k,r}=\delta_k,$$

and

$$(a_{k,1}-y'_k)-(a_{k+1,1}-y'_{k+1})+\sum_{r=4}^{6}\frac{h^{r-1}}{(r-1)!}a_{k,r}=\beta_k.$$

Solving the first three equations we obtain

$$a_{k,4}=-\frac{120}{h^3}(a_{k,1}-y'_k)+\frac{120}{h^4}\alpha_k-\frac{12}{h^2}\gamma_k+\frac{2}{h}\delta_k, \tag{2.9}$$

$$a_{k,5}=\frac{720}{h^4}(a_{k,1}-y'_k)-\frac{720}{h^5}\alpha_k+\frac{96}{h^3}\gamma_k-\frac{18}{h^2}\delta_k, \tag{2.10}$$

$$a_{k,6}=-\frac{1440}{h^5}(a_{k,1}-y'_k)+\frac{1440}{h^6}\alpha_k-\frac{216}{h^4}\gamma_k+\frac{48}{h^3}\delta_k. \tag{2.11}$$

Substituting the values of $a_{k,4}$, $a_{k,5}$ and $a_{k,6}$ in the last equation we get the following relation between $(a_{k,1}-y'_k)$ and $(a_{k+1,1}-y'_{k+1})$, $k=1, \ldots, m-2$:

$$(a_{k,1}-y'_k)+(a_{k+1,1}-y'_{k+1})=\frac{2}{h}\alpha_k-\beta_k+\frac{h}{5}\gamma_k-\frac{h^2}{60}\delta_k. \tag{2.12}$$

The coefficient matrix of the system of equations (2.5) and (2.12) in the unknowns $(a_{k,1}-y'_k)$, $k=1, \ldots, m-1$ is a non-singular matrix and hence the coefficients $a_{k,1}$, $k=1, \ldots, m-2$ are uniquely determined and so are, therefore, the coefficients $a_{k,4}$, $a_{k,5}$ and $a_{k,6}$.

Lastly, for the coefficients of $\tilde{S}_0(x)$, we have

$$\sum_{r=4}^{7}\frac{h^r}{r!}a_{0,r}=\alpha_0,\qquad \sum_{r=4}^{7}\frac{h^{r-2}}{(r-2)!}a_{0,r}=\gamma_0,$$

$$\sum_{r=4}^{7}\frac{h^{r-3}}{(r-3)!}a_{0,r}=\delta_0,\qquad \sum_{r=4}^{7}\frac{h^{r-1}}{(r-1)!}a_{0,r}=(a_{1,1}-y_1')+\beta_0.$$

Solving these equations we get

$$a_{0,4}=-\frac{360}{h^3}(a_{1,1}-y_1')-\frac{360}{h^3}\beta_0+\frac{840}{h^4}\alpha_0+\frac{60}{h^2}\gamma_0-\frac{4}{h}\delta_0, \tag{2.13}$$

$$a_{0,5}=\frac{4680}{h^4}(a_{1,1}-y_1')+\frac{4680}{h^4}\beta_0-\frac{10\,080}{h^5}\alpha_0-\frac{840}{h^3}\gamma_0+\frac{60}{h^2}\delta_0, \tag{2.14}$$

$$a_{0,6}=-\frac{24\,480}{h^5}(a_{1,1}-y_1')-\frac{24\,480}{h^5}\beta_0+\frac{50\,400}{h^6}\alpha_0+\frac{4680}{h^4}\gamma_0-\frac{360}{h^3}\delta_0, \tag{2.15}$$

$$a_{0,7}=\frac{50\,400}{h^6}(a_{1,1}-y_1')+\frac{50\,400}{h^6}\beta_0-\frac{100\,800}{h^7}\alpha_0-\frac{10\,080}{h^5}\gamma_0+\frac{840}{h^4}\delta_0. \tag{2.16}$$

Since $a_{1,1}$ is already determined, the coefficients $a_{0,i}$, $i=4, \ldots, 7$ are uniquely determined. Hence we obtain

THEOREM 2.1. *For a uniform partition Δ of the interval I, there exists exactly one spline function $\tilde{S}_\Delta(x)\in S_{m,6}^{*(3)}$ which is a solution of the interpolation problem* (1.1).

We now give a theorem which asserts that the interpolation problem (1.2) has a unique solution in the class $S_{m,6}^{*(3)}$.

THEOREM 2.2. *For a uniform partition of the interval I, there exists a unique spline function $\hat{S}_\Delta(x)\in S_{m,6}^{*(3)}$ which is a solution of the interpolation problem* (1.2).

The proof of this theorem is similar to the above theorem and so we omit it.

3. Error bounds

We shall first prove

THEOREM 3.1. *Let $f\in C^6(I)$ and $\tilde{S}_\Delta(x)\in S_{m,6}^{*(3)}$ be the solution of the problem* (1.1). *Then*

$$|\tilde{S}^{(q)}(x)-f^{(q)}(x)|\leqq k_1h^{5-q}w_6(h),\quad q=0,\ldots,5$$

where

$$k_1=\begin{cases}604, & \text{when}\quad x\in[x_0,x_1]\\ 24, & \text{when}\quad x\in[x_k,x_{k+1}],\quad k=1,\ldots,m-2\\ 35h, & \text{when}\quad x\in[x_{m-1},x_m]\end{cases}$$

and $w_6(\cdot)$ is the modulus of continuity of $f^{(6)}$.

For the proof of this theorem we shall need

LEMMA 1. *Let* $f \in C^6(I)$. *Then*

$$|e_{k,1}| \leqq (m-k)\frac{h^5}{90}w_6(h), \quad k = 1, \dots, m-1$$

where

$$e_{k,1} := a_{k,1} - y'_k. \tag{3.1}$$

PROOF. If $f \in C^6(I)$ then on using Taylor's formula, we can write

$$\begin{cases} \alpha_k = \dfrac{h^4}{4!}y_k^{(4)} + \dfrac{h^5}{5!}y_k^{(5)} + \dfrac{h^6}{6!}f^{(6)}(\xi_k), \\ \beta_k = \dfrac{h^3}{3!}y_k^{(4)} + \dfrac{h^4}{4!}y_k^{(5)} + \dfrac{h^5}{5!}f^{(6)}(\eta_k), \\ \gamma_k = \dfrac{h^2}{2!}y_k^{(4)} + \dfrac{h^3}{3!}y_k^{(5)} + \dfrac{h^4}{4!}f^{(6)}(\zeta_k), \\ \delta_k = hy_k^{(4)} + \dfrac{h^2}{2!}y_k^{(5)} + \dfrac{h^3}{3!}f^{(6)}(\sigma_k), \end{cases} \tag{3.2}$$

where $x_k < \xi_k, \eta_k, \zeta_k, \sigma_k < x$.

From (2.12) and (3.2) in the notation of (3.1) we have

(3.3)

$$e_{k,1} + e_{k+1,1} = \frac{h^5}{360}[f^{(6)}(\xi_k) - f^{(6)}(\sigma_k)] + \frac{h^5}{120}[f^{(6)}(\zeta_k) - f^{(6)}(\eta_k)], \quad k = 1, \dots, m-2$$

and

$$e_{m-1,1} = \frac{h^5}{360}[f^{(6)}(\xi_{m-1}) - f^{(6)}(\sigma_{m-1})] + \frac{h^5}{120}[f^{(6)}(\zeta_{m-1}) - f^{(6)}(\eta_{m-1})]. \tag{3.4}$$

We easily see that the system of equations (3.3) and (3.4) in the unknowns $e_{k,1}$, $k=1, \dots, m-1$ have a unique solution

$$e_{k,1} = d_k - d_{k+1} + d_{k+2} - \dots + (-1)^{m-1-k}d_{m-1},$$

where

$$d_k = \frac{h^5}{360}[f^{(6)}(\xi_k) - f^{(6)}(\sigma_k)] + \frac{h^5}{120}[f^{(6)}(\zeta_k) - f^{(6)}(\eta_k)].$$

It is clear that

$$|d_k| \leqq \frac{h^5}{90}w_6(h).$$

Hence

$$|e_{k,1}| \leqq (m-k)\frac{h^5}{90}w_6(h)$$

and this completes the proof of the lemma.

PROOF OF THE THEOREM. Let $x \in [x_k, x_{k+1}]$, $k=1, \dots, m-1$. From (2.2) we have

$$\tilde{S}_k^{(5)}(x) = a_{k,5} + (x-x_k)\, a_{k,6}$$

and

$$f^{(5)}(x) = y_k^{(5)} + (x-x_k) f^{(6)}(\lambda_k), \quad x_k < \lambda_k < x$$

so that

$$|\tilde{S}_k^{(5)}(x) - f^{(5)}(x)| \leqq |a_{k,5} - y_k^{(5)}| + h|a_{k,6} - f^{(6)}(\lambda_k)|.$$

We estimate the quantities $|a_{k,5} - y_k^{(5)}|$ and $h|a_{k,6} - f^{(6)}(\lambda_k)|$. From (2.10) and (2.11) on using (3.2) we have

$$a_{k,5} - y_k^{(5)} = \frac{720}{h^4} e_{k,1} - h[f^{(6)}(\xi_k) - 4f^{(6)}(\xi_k) + 3f^{(6)}(\sigma_k)]$$

and

$$a_{k,6} - f^{(6)}(\lambda_k) = -\frac{1440}{h^5} e_{k,1} + 2f^{(6)}(\xi_k) - 9f^{(6)}(\zeta_k) + 8f^{(6)}(\sigma_k) - f^{(6)}(\lambda_k)$$

from which owing to Lemma 1 we get

$$|a_{k,5} - y_k^{(5)}| \leqq 4[2m-(2k-1)]\, h w_6(h) \leqq 8 w_6(h)$$

and

$$h|a_{k,6} - f^{(6)}(\lambda_k)| \leqq [16m - (16k-10)]\, h w_6(h) \leqq 16 w_6(h).$$

Thus

$$|\tilde{S}_k^{(5)}(x) - f^{(5)}(x)| \leqq 24 w_6(h).$$

Set

$$g(x) := \tilde{S}_k'''(x) - f'''(x).$$

Then by (1.1) $g(x_k) = g(x_{k+1}) = 0$ and so by Rolle's theorem there exists μ_k $(x_k < < \mu_k < x_{k+1})$ such that

$$g'(\mu_k) = \tilde{S}_k^{(4)}(\mu_k) - f^{(4)}(\mu_k) = 0,$$

from which we obtain

$$|\tilde{S}_k^{(4)}(x) - f^{(4)}(x)| = \left| \int_{\mu_k}^{x} \{\tilde{S}_k^{(5)}(t) - f^{(5)}(t)\}\, dt \right| \leqq$$

$$\leqq \int_{\mu_k}^{x} |\tilde{S}_k^{(5)}(t) - f^{(5)}(t)|\, dt \leqq 24 h w_6(h).$$

Carrying on similar arguments we easily find that

$$|\tilde{S}_k^{(q)}(x) - f^{(q)}(x)| \leqq 24 h^{5-q} w_6(h)$$

is true for $q=1, \dots, 5$.

For $x_0 \leqq x \leqq x_1$ and $x_{m-1} \leqq x \leqq x_m$, we have from (2.1) and (2.2),

$$|\tilde{S}_0^{(5)}(x) - f^{(5)}(x)| \leqq |a_{0,5} - y_0^{(5)}| + h|a_{0,6} - f^{(6)}(\lambda_0)| + \frac{h^2}{2}|a_{0,7}|$$

and

$$|\tilde{S}_{m-1}^{(5)}(x) - f^{(5)}(x)| \leqq |a_{m-1,5} - y_{m-1}^{(5)}| + h|a_{m-1,6} - f^{(6)}(\lambda_{m-1})|.$$

From (2.7) and (2.8) we can show that

$$|a_{m-1,5}-y_{m-1}^{(5)}| \leqq 11hw_6(h)$$

and

$$h|a_{m-1,6}-f^{(6)}(\lambda_{m-1})| \leqq 24hw_6(h)$$

so that

$$|\tilde{S}_{m-1}^{(5)}(x)-f^{(5)}(x)| \leqq 35hw_6(h).$$

Similarly by (2.14), (2.15) and (2.16), we have

$$|a_{0,5}-y_0^{(5)}| \leqq [52(m-1)+49]\, hw_6(h) \leqq 52w_6(h),$$

$$h|a_{0,6}-f^{(6)}(\lambda_0)| \leqq h[272(m-1)+265]\, w_6(h) \leqq 272w_6(h)$$

and

$$\frac{h^2}{2}\,|a_{0,7}| \leqq \frac{h}{2}[560(m-1)+560]\, w_6(h) \leqq 280w_6(h).$$

Hence

$$|\tilde{S}_0^{(5)}(x)-f^{(5)}(x)| \leqq (52+272+280)\, w_6(h) = 604w_6(h).$$

By the method of successive integration we find that the inequalities

$$|\tilde{S}_0^{(q)}(x)-f^{(q)}(x)| \leqq 604h^{5-q}w_6(h)$$

and

$$|\tilde{S}_{m-1}^{(q)}(x)-f^{(q)}(x)| \leqq 35h^{6-q}w_6(h)$$

are valid for $q=1, ..., 5$.

Regarding the error bounds of the spline function $\hat{S}_\Delta(x)$ which is the solution of the interpolation problem (1.2), we simply state the theorem and omit the proof

THEOREM 3.2. *Let* $f\in C^6(I)$ *and* $\hat{S}_\Delta(x)\in S_{m,6}^{*(3)}$ *be the solution of the problem* (1.2). *Then*

$$|\hat{S}^{(q)}(x)-f^{(q)}(x)| \leqq k_2h^{5-q}w_6(h), \quad q = 0, ..., 5,$$

where

$$K_2 = \begin{cases} 587+8h, & \text{when} \quad x\in[x_0, x_1] \\ 60, & \text{when} \quad x\in[x_k, x_{k+1}], \quad k = 1, ..., m-2 \\ 54h, & \text{when} \quad x\in[x_{m-1}, x_m] \end{cases}$$

and as usual $w_6(\cdot)$ *denotes the modulus of continuity of* $f^{(6)}$.

Reference

[1] J. Győrvári, Lakunäre Interpolation mit Spline-Funktionen, *Acta Math. Hung.*, **42** (1983), 25—33.

(Received October 2, 1984)

DEPARTMENT OF MATHEMATICS AND ASTRONOMY
LUCKNOW UNIVERSITY
LUCKNOW, INDIA

Acta Math. Hung.
50 (1—2) (1987), 71—78.

ON JACOBSON TYPE RADICALS OF NEAR-RINGS

K. KAARLI (Tartu)

The main result of this paper is a negative answer to the problem concerning the radicalness in the sense of Kurosh—Amitsur of the near-ring radical J_0. We shall give negative answers to some hereditariness problems of near-ring radicals too.

1. Introduction

For fundamental concepts and notations of near-ring theory we refer to the book of Pilz [7]. The only difference is that we accept the left distributive law $x(y+z)=xy+xz$, not the right as Pilz. All near-rings will be zero-symmetric, i.e. the identity $0x=0$ is satisfied in all near-rings. We shall use the standard terminology of radical theory, see for example [2]. A class of near-rings is called hereditary if it is closed under taking of ideals.

In 1963, Betsch in his thesis [1] defined three Jacobson type radicals for near-rings: J_0, J_1 and J_2. He defined $J_i(N)$ to be the intersection of kernels of all N-groups of type i, $i=0, 1, 2$. Recall here the definitions of these N-groups.

An N-group G with $GN\neq 0$ is said to be

— of type 0 if it is simple and monogenic,

— of type 1 if it is simple and strongly monogenic, i.e. for any $g\in G$ either $gN=G$ or $gN=0$,

— of type 2 if it has only two N-subgroups.

Since then these radicals have been objects of extensive study. In 1976 the problem arose whether they are radicals in the sense of Kurosh—Amitsur [5]. The same problem was also discussed by Pilz [7]. Let S_i stand for the class of all J_i-semisimple near-rings and R_i for the class of all J_i-radical near-rings, $i=0, 1, 2$. It was proved in [5] that any non-zero ideal of a near-ring $N\in S_i$ has a non-zero homomorphic image belonging to S_i, $i=0, 1, 2$. Hence S_i determines the upper Kurosh—Amitsur radical K_i and it was proved that the class of all K_i-radical near-rings coincides with R_i, $i=0, 1, 2$. Also we showed that J_i is Kurosh—Amitsur iff it is idempotent (i.e. $J_i(N)=J_i(J_i(N))$ for any near-ring N) and in this case $J_i=K_i$, $i=0, 1, 2$.

We succeeded in proving the relation $J_2(I)=J_2(N)\cap I$ for any ideal I of any near-ring N [5]. This implies immediately that J_2 is Kurosh—Amitsur and both R_2 and S_2 are hereditary.

It is very easy to construct near-rings N with $J_1(N)\neq 0$ and $J_1(J_1(N))=0$, so J_1 is not Kurosh—Amitsur. For example, let G be a finite group having a proper non-zero subgroup H. Then the near-ring of all mappings from G into G which preserve 0 and H is just we need.

The case of J_0 turned out to be the most difficult one. It was known that $J_0(N)$ is nilpotent provided N satisfies the DCC on right N-subgroups (Ramakotaiah [8]). Hence, J_0 is idempotent on these near-rings. In [5] this result was generalized in two directions. We showed that J_0 is idempotent on so-called semiprimary near-rings and also on near-rings satisfying the weak DCC on right N-subgroups. The latter means that for any element a of a near-ring N there exists a natural number k such that $a^kN=a^{k+1}N$. In [6] we proved that J_0 is idempotent on the least class of near-rings closed with respect of taking ideals and homomorphic images and containing the class of distributively generated near-rings.

In the present paper we shall construct a countable near-ring N with Abelian additive group, whose J_0-radical has a finite 1-primitive homomorphic image. Hence, $J_0(N)\neq J_1(J_0(N))$ and, of course, J_0 is not Kurosh—Amitsur.

We shall also show that K_0 and K_1 have non-hereditary radical classes and non-hereditary semisimple classes as well.

We shall use the characterization of $J_0(N)$ via quasiregularity given by Ramakotaiah in [8]. Denote by $(S)_r$ the right ideal of a near-ring N generated by the subset $S\subseteq N$. The element $a\in N$ is called quasiregular if

$$a\in(\{n-an|n\in N\})_r.$$

An ideal of N is called quasiregular if all its elements are quasiregular. Ramakotaiah has proven that $J_0(N)$ is the largest quasiregular ideal in any near-ring N.

2. J_0 is not Kurosh—Amitsur

Let A be a cyclic group of order 4 with a generator a_0 and let B be a countable elementary Abelian group of exponent 2. We represent the elements of B as infinite 0, 1-sequences having only a finite number of non-zero entries. Denote by C the subgroup of B consisting of all sequences having an even number of non-zero entries. Consider a mapping $\varphi: B\to B$ defined as follows:

$$\varphi(0) = (1, 0, 0, \ldots),$$

$$\varphi(\underbrace{0, \ldots, 0,}_{k}\ 1, x_1, x_2, \ldots) = (\underbrace{0, \ldots, 0,}_{k+1} 1, x_1, x_2, \ldots).$$

Obviously, φ acts homomorphically on $B\setminus\{0\}$, i.e. if $b_1\neq 0$, $b_2\neq 0$ and $b_1+b_2\neq 0$ then $\varphi(b_1+b_2)=\varphi(b_1)+\varphi(b_2)$.

Let $G=A\dotplus B$ and let us identify A and B with their canonical images in G. Define $s_0: G\to G$ as follows

$$(a+b)s_0 = \begin{cases} 0 & \text{if} \quad a\neq a_0 \\ a_0+\varphi(b) & \text{if} \quad a=a_0. \end{cases}$$

PROPOSITION 2.1. *Let* S *be the near-ring of transformations on* G *generated by the single transformation* s_0. *Then*

(i) S *consists of polynomials* $n_1s_0+n_2s_0^2+\ldots+n_ks_0^k$ *where* k *is any natural number and* n_i *are arbitrary integers;*

(ii) $S^2=s_0S$;

(iii) $GS\cap B=C$;

(iv) *if* $s\in S$, $s\neq 0$ *and* $a_0 s\in B$ *then* $(a_0+b)s\neq 0$ *for any* $b\in B$;
(v) $B \underset{S}{\lhd} G$;
(vi) G/B *is an S-group of type* 1;
(vii) $S/(B\colon G)_S$ *is a finite* 1*-primitive near-ring*;
(viii) $(B\colon G)_S^2=0$.

PROOF. (i) Denote by S_0 the set of all polynomials in s_0, i.e.

$$S_0 = \{n_1 s_0+n_2 s_0^2+\ldots+n_k s_0^k \mid k \text{ is a natural number, } n_i\in Z\}.$$

Obviously, S_0 is closed under addition and to prove that it is a sub-near-ring it suffices to show that $S_0 s_0 \subseteqq S_0$. Take an arbitrary element $s=n_1 s_0+\ldots+n_k s_0^k$ from S_0 and consider the product ss_0. Evidently, $(a+b)ss_0=0$ for $a\neq a_0$, so we have to consider the action of ss_0 only on elements a_0+b. By easy induction argument it follows

$$(a_0+b)ss_0 = \big((n_1+\ldots+n_k)a_0+(n_1\varphi+\ldots+n_k\varphi^k)(b)\big)s_0.$$

Now we have two possibilites. If $n_1+\ldots+n_k\not\equiv 1 \pmod 4$ then $(a_0+b)ss_0=0$ for any $b\in B$ and $ss_0=0\in S_0$. Otherwise

$$(a_0+b)ss_0 = a_0+\varphi(n_1\varphi+\ldots+n_k\varphi^k)(b).$$

Now observe that $\varphi(b)\neq 0$ for any $b\in B$ and, moreover, given an arbitrary $b\in B$, the set $\varphi(b), \varphi^2(b), \ldots, \varphi^k(b)$ is linearly independent over Z_2. Hence, if $i_1, \ldots, i_m$ are all the indices i for which n_i is odd, we get

$$a_0+\varphi(n_1\varphi+\ldots+n_k\varphi^k)(b) = a_0+\varphi\big(\varphi^{i_1}(b)+\ldots+\varphi^{i_m}(b)\big) =$$
$$= a_0+\varphi^{i_1+1}(b)+\ldots+\varphi^{i_m+1}(b) = a_0+(n_1\varphi^2+\ldots+n_k\varphi^{k+1})(b) =$$
$$= (a_0+b)(n_1 s_0^2+\ldots+n_k s_0^{k+1}).$$

Thus, the property (i) is proved, moreover, we obtained the following multiplication rule:

$$(1)\qquad (n_1 s_0+\ldots+n_k s_0^k)s_0 = \begin{cases} 0 & \text{if } \ n_1+\ldots+n_k\not\equiv 1 \pmod 4 \\ n_1 s_0^2+\ldots+n_k s_0^{k+1} & \text{otherwise.} \end{cases}$$

(ii) This property is a straightforward consequence of the left distributive law and the multiplication rule (1).

(iii) Let $s=n_1 s_0+\ldots+n_k s_0^k$ and let $(a+b)s\in B$ for some $a\in A$ and $b\in B$. If $a\neq a_0$ then $(a+b)s=0\in C$. Let now $a=a_0$. Then

$$(a+b)s = (n_1+\ldots+n_k)a_0+n_1\varphi(b)+\ldots+n_k\varphi^k(b)$$

and the condition $(a+b)s\in B$ yields $n_1+\ldots+n_k\equiv 0 \pmod 4$. Since $\varphi(b), \ldots, \varphi^k(b)$ have the same number of non-zero entries, the latter implies $(a+b)s\in C$. Thus, we proved the inclusion $GS\cap B\subseteqq C$. Conversely, for any $b=(n_1, \ldots, n_k, 0, 0, \ldots)$ with $n_1+\ldots+n_k\equiv 0 \pmod 2$ we have

$$b = \begin{cases} a_0(n_1 s_0+\ldots+n_k s_0^k) & \text{if } \ n_1+\ldots+n_k\equiv 0 \pmod 4 \\ a_0\big((n_1+2)s_0+\ldots+n_k s_0^k\big) & \text{if } \ n_1+\ldots+n_k\not\equiv 0 \pmod 4. \end{cases}$$

(iv) Let s be a non-zero element of S such that $a_0s\in B$. Then $s=n_1s_0+\ldots+n_ks_0^k$ where some of the coefficients n_i is odd and $n_1+\ldots+n_k\equiv 0 \pmod 4$. Now, given $b\in B$, we get

$$(a_0+b)s = n_1\varphi(b)+\ldots+n_k\varphi(b)$$

and since $\varphi(b), \ldots, \varphi^k(b)$ are linearly independent, we are done.

(v) Since s_0 acts on G compatibly with respect to B, so do all elements from the near-ring generated by it.

(vi) For $ka_0\in A$ we have $ka_0-a_0(ks_0)\in B$, so a_0+B is a generator of the S-group G/B. If G/B were not a simple S-group then $2A+B$ would be an S-ideal of G. However, this conjecture yields a contradiction since

$$(2a_0+a_0)s_0-a_0s_0 = 0-a_0-\varphi(0)\notin 2A+B.$$

(vii) Since G/B is an S-group of type 1, the near-ring $S/(B: G)_S$ is 1-primitive. It is an easy exercise to verify that it is isomorphic to the near-ring $(A, +, \cdot)$ where the multiplication is defined as follows:

$$aa' = \begin{cases} a' & \text{if } a = a_0 \\ 0 & \text{otherwise.} \end{cases}$$

(viii) By the definition of S, $a\neq a_0$ implies $(a+b)S=0$. This yields immediately $(B: G)_S^2=0$.

The Proposition is proven.

Now consider another near-ring of transformations on the same group G. Let T be the set of all transformations on G satisfying the following conditions:

1) $gt\neq 0$ implies $g\in B\setminus\{0\}$;

2) $(B\setminus C)t$ equals the fixed element (depending on t) of $2A+B$;

3) $(C\setminus\{0\})t$ equals either 0 or $2a_0$.

It is easy to check that the set T is closed under addition and multiplication, so it is indeed a near-ring. Clearly G and $2A+B$ as well can be regarded as T-groups. The main property of T (for us) will be proved in the next Proposition.

PROPOSITION 2.2. *The group $H=2A+B$ is a faithful T-group of type 0, so T is a 0-primitive near-ring.*

PROOF. The second condition from the definition of T implies that all elements from $B\setminus C$ are generators of the T-group H. Now suppose there exists a non-zero proper T-ideal F in H. If $F\subseteq B$ then, evidently, $F\subseteq C$, but this yields a contradiction since $cT\nsubseteq C$ for any non-zero $c\in C$ by the third condition. Hence, there exists $b\in B$ such that $2a_0+b\in F$. Take now arbitrary $b'\in B\setminus C$ and $t\in T$ such that $b't\notin F$. Then

$$b't-(b'+2a_0+b)t = b't\notin F,$$

a contradiction. The Proposition is proven.

Now we are in the position to finish our construction.

THEOREM 2.3. *Let N be the near-ring of transformations on G generated by S and T. Then*

(i) $S\lhd N$, $T\underset{r}{\lhd} N$ *and* $N=S+T$;

(ii) $S = J_0(N)$.

PROOF. Since the subsets in G on which S and T take non-zero values are disjoint, the additive group generated by S and T is their direct sum. Let us show that the subgroup $S+T$ is closed under multiplication, too. To do this take $s, s'\in S$, $t, t'\in T$ and prove that

(I) $(s+t)s'\in S+T$;

(II) $(s+t)t'\in S+T$.

(I) Consider the action of $(s+t)s'$ on $a+b$. If $a\neq a_0$ then

$$(a+b)(s+t)s' = (a+b)ts'\in(2A+B)S = 0 = (a+b)ss'.$$

For $a=a_0$ we have

$$(a+b)(s+t)s' = \big((a_0+b)s+(a_0+b)t\big)s' = (a_0+b)ss'.$$

Hence, $(s+t)s'=ss'\in S+T$.

(II) Suppose $(C\setminus\{0\})t'=ma_0$ $(m=0, 2)$ and consider the action of $(s+t)t'$ on $a+b$. For $a\neq a_0$ we have

$$(a+b)(s+t)t' = (a+b)tt' = (a+b)(ms_0+tt')$$

and for $a=a_0$

$$(a+b)(s+t)t' = \big((a_0+b)s+(a_0+b)t\big)t' = (a_0+b)st'.$$

Now we have two possibilities.

(II_1) $a_0s\notin B$. Then, by the definition of S, $(a_0+b)s\notin B$ for any $b\in B$. Hence $(a_0+b)st'=0=(a_0+b)tt'$. Therefore, in this case $(s+t)t'=tt'$.

(II_2) $a_0s\in B$. If $s=0$ then, obviously, $(s+t)t'=tt'$. If $s\neq 0$ then, by Proposition 2.1, $(a_0+B)s\subseteq C\setminus\{0\}$ and so $(a_0+b)st'=ma_0=(a_0+b)(ms_0)$ for any $b\in B$. Hence, in this case $(s+t)t'=ms_0+tt'\in S+T$.

Thus, we proved that $S+T$ is a near-ring of transformations on G and, moreover, we obtained the following multiplication rule:

$$(2)\qquad (s+t)(s'+t') = \begin{cases} ss'+tt' & \text{if} \quad a_0s\notin B \quad \text{or} \quad s=0 \\ ss'+ms_0+tt' & \text{if} \quad a_0s\in B, \quad s\neq 0 \quad \text{and} \quad (C\setminus\{0\})t' = ma_0. \end{cases}$$

It is easy to see that $S=(0\colon H)_N$ and $T=(0\colon a_0)_N$, hence both S and T are right ideals of N. Moreover, since $HN\subseteq H$, $S\lhd N$.

(ii) By Proposition 2.2, $N/S=T$ is a 0-primitive near-ring and so $J_0(N)\subseteq S$. Hence, to prove the Theorem we need to show that S is a quasiregular ideal of N. This will be the crucial point of our construction.

Take an arbitrary element $s\in S$ and set

$$A_s = \big(\{n-sn|n\in N\}\big)_r.$$

We have to prove that $s\in A_s$. If $a_0s\in a+B$, $a\neq a_0$, then $s^2=0$ and we are done. Let now $a_0s\in a_0+B$, i.e. $s=n_1s_0+\ldots+n_ks_0^k$ where n_i are integers with $n_1+\ldots+n_k\equiv$ $\equiv$ (mod 4). Let u be the first index for which $n_u\not\equiv 0 \pmod 4$ and let $s^2=m_1s_0+\ldots$ $\ldots+m_vs_0^v$. By making use of the multiplication rule (1) we get immediately $m_1=$ $=\ldots=m_u\equiv 0 \pmod 4$. Hence, $s^2\neq s$. Since $a_0s\in a_0+B$, $a_0s^2\in a_0+B$, too, and $G(s-s^2)\in B$. Denote $s_1=s-s^2$. Obviously, $s_1\in A_s$ and by Proposition 2.1 $Gs_1\subseteq$ $\subseteq C\setminus\{0\}$. Take an element $t\in T$ such that $(C\setminus\{0\})t=2a_0$. Then, using the multiplication rule (2) we get $2s_0=s_1t\in A_s$. Further, by the definition of right ideal, $(2s_0+3s_0)s'-(3s_0)s'\in A_s$ for any $s'\in S$. Since $5s_0=s_0$ and $(3s_0)s'=0$, this yields

$s_0 S \subseteq A_s$. Applying once more Proposition 2.1, we get $S^2 \subseteq A_s$. Hence, $s^2 \in A_s$ and $s \in A_s$. The Theorem is proven.

Summarizing, we state the main result of the present paper.

THEOREM 2.4. *There exists a countable near-ring N with Abelian additive group whose J_0-radical can be mapped homomorphically onto* 4*-element* 1*-primitive near-ring.*

All but maybe the countability is clear. But S is countable because it is single-generated and there is a $1-1$ correspondence between T and the set of all pairs (m, h) where $m \in \{0, 2\}$ and $h \in H$ (the pair (m, h) corresponds to the element $t \in T$ if and only if $(B \setminus C)t = h$ and $(C \setminus \{0\})t = ma_0$), hence the countability is also clear.

It is well known that given any ideal I of any ring R, any irreducible I-module may be regarded as an R-module. In [6] the following generalization of this result was obtained. If N is a near-ring belonging to the ideally and homomorphically closed class generated by the class of distributively generated near-rings and $I \lhd N$ then any I-group of type 0 can be regarded as an N-group. In general, this result fails even for finite near-rings. Nevertheless, for a semi-primary near-ring N we proved in [4] that any I-group of type 0, while I being an ideal of N, is an I-homomorphic image of some N-group of type 0. Now we are able to show that this weaker property also fails in the class of all near-rings.

COROLLARY 2.5. *There exists a near-ring N having an ideal S for which there exists an S-group of type* 0 *which cannot be represented as an S-homomorphic image of some N-group of type* 0.

PROOF. Let $S = J_0(N)$ and let G be an S-group of type 0. Suppose G is an S-homomorphic image of some N-group G' of type 0. Then, by the definition of J_0, $G'S = 0$ which implies $GS = 0$, a contradiction.

3. On hereditariness of K_0 and K_1

As we mentioned in the introduction, the class of K_i-radical near-rings coincides with R_i, $i = 0, 1$. Denote by SK_i the class of K_i-semisimple near-rings, $i = 0, 1$. So there are four classes of near-rings: R_0, R_1, SK_0 and SK_1 and for each of these classes we have a problem of hereditariness. We solve all these problems negatively.

THEOREM 3.1. *There exists a countable J_0-radical near-ring N with Abelian additive group which has an ideal S of index* 2 *having a* 4*-element* 1*-primitive homomorphic image.*

PROOF. The construction is nearly the same as that in Section 2. We only change the right ideal T by $T' = \{0, t_0'\}$ where

$$(a+b)t_0' = \begin{cases} 2a_0 & \text{if} \quad a = 0 \quad \text{and} \quad b \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the near-ring N generated by S and T' equals again $S \dotplus T'$ and $S \lhd N$, $T' \underset{r}{\lhd} N$. Now the multiplication rule is the following:

$$(s+t)(s'+t') = \begin{cases} ss' & \text{if} \quad a_0 s \notin B \quad \text{or} \quad t' = 0 \\ ss' + 2s_0 & \text{if} \quad a_0 s \in B \quad \text{and} \quad t' \neq 0. \end{cases}$$

When proving quasiregularity in Theorem 2.3 we used T only once. We needed an element $t\in T$ such that $(C\setminus\{0\})t=2a_0$. Since t_0' is good for this, S is a quasiregular ideal in our new context, too. Suppose $N\notin R_0$. Then there exists an N-group G of type 0. Since S is a quasiregular ideal, $GS=0$ and G is an N/S-group of type 0. But $N/S=T'$, $T'^2=0$, a contradiction. The Theorem is proven.

COROLLARY 3.2. *The class* R_0 *is not hereditary, hence the radical* K_0 *is not hereditary.*

In [4] we introduced the class $\mathfrak{A}$ of semiprimary near-rings containing all near-rings with DCC on right N-subgroups and intersecting with the class of rings by semiprimary ones. We do not want to give here the exact definition of a semi-primary near-ring but for our purposes it is enough if we recall three of their properties.

(i) $\mathfrak{A}$ contains all nilpotent near-rings.

(ii) An extension of a nilpotent near-ring by a finite 1-primitive near-ring is semiprimary (see [4], Corollary 5 and Theorem 6).

(iii) J_0-radical semiprimary near-rings are nilpotent ([5], Corollary 10).

It was proved in [4] that $\mathfrak{A}$ is closed under ideals and homomorphic images. Now we get from Theorem 3.1 the immediate

COROLLARY 3.3. *The class* $\mathfrak{A}$ *is not closed under extensions.*

Next we shall present a simple example to show that the class R_1 is not hereditary.

PROPOSITION 3.4. *There exists a finite* J_1*-radical near-ring with Abelian additive group having an ideal which is a* 1*-primitive near-ring.*

PROOF. Let A be a cyclic group of order 8. Consider the set N of all transformations n on A such that $(4A)n=0$ and $(2A)n\subseteq 4A$. Clearly, the set N is a near-ring and A is a monogenic N-group. A straightforward computation will show that $2A$ and $4A$ as well are not N-ideals in A, so A is a faithful N-group of type 0. Now, applying Lemma 3.2 from [3], we conclude that any other N-group of type 0 must have a form B/C where B is a proper N-subgroup of A and $C\underset{k}{\lhd}B$.[1] Since $2A$ is the largest proper N-subgroup of A and $(2A)N^2=0$, we obtain that A is the only N-group of type 0. Obviously, A is not of type 1, hence $N\in R_1$. However, one can easily check that A is an S-group of type 1 where $S=(0\colon 2A)_N$. This proves the proposition.

To conclude this paper we construct an example of a near-ring which will show that both SK_0 and SK_1 are not hereditary. Note that this example was first published in [3] to show that a minimal ideal S wiht $S^2\neq 0$ of a finite near-ring need not be a simple near-ring.

PROPOSITION 3.5. *There exists a finite* K_1*-semisimple near-ring* N *with Abelian additive group having an ideal* S *which has a non-zero nilpotent ideal.*

PROOF. Let A be again a cyclic group of order 8 and let Γ be its automorphism group consisting of the unit automorphism and the multiplication by 5. Let N be a

---

[1] Note that in [3] N-groups of type 0 were called irreducible N-groups and the notion of the type of an irreducible N-group had a different meaning.

so called centralizer near-ring, i.e. the near-ring of all zero-preserving transformations on A which permute with elements of Γ. To understand the structure of N first consider the acting of Γ on A. It is easy to see that the elements of $2A$ are fixed points of Γ and Γ acts fixed-point-freely on the complement of $2A$. Now an arbitrary element $n\in N$ is uniquely determined by its action on $\{1, 2, 3, 4, 6\}$ (there are two 2-element orbits: $\{1, 5\}$ and $\{3, 7\}$). Obviously, $1n$ and $3n$ range independently over A but $2n$, $4n$ and $6n$ over $2A$. From this it follows that N is a direct sum of an ideal $S=(0\colon 2A)_N$ and a right ideal $T=(0\colon A\setminus 2A)_N$. Evidently, T is isomorphic to the near-ring of all zero-preserving transformations on $2A$, so it is simple and 2-primitive. On the other hand, the ideal S consists of all transformations on A which permute with Γ and annihilate $2A$. Since Γ acts fixed-point-freely on $A\setminus 2A$, we can apply results of [4]. We observe that A is an N-group of type 0 and conclude by making use of Proposition 7 from [4] that S is a minimal ideal of N. Further, since A is a faithful N-group of type 0, N is 0-primitive, hence it is a prime near-ring. This yields that S is a unique minimal ideal of N. The latter together with the simplicity of N/S gives that N has only one proper non-zero ideal: S. Therefore, to show $N\in SK_1$ we have only to prove that S has a 1-primitive homomorphic image. But this is obvious because A is a strongly monogenic S-group.

Now the theorem will be proved if we shall show that S has a non-zero nilpotent ideal. To do this first observe that $4A \underset{S}{\triangleleft} A$. If $a\in A\setminus 2A$ then $a+4=5a$. Hence $(a+4)s-as=(5a)s-as=5(as)-as=4(as)\in 4A$ for any $s\in S$. If $a\in 2A$ then obviously

$$(a+4)s-as = 0\in 4A.$$

Hence, $U=(4A\colon A)_S$ is an ideal of S. Clearly, $U\neq 0$, but $U^2=0$ because of $AU^2\subseteqq (4A)S=0$. The proposition is proven.

COROLLARY 3.6. *The class* SK_0 *is not hereditary.*

COROLLARY 3.7. *The class* SK_1 *is not hereditary.*

REMARK. It is interesting to point out that the class S_1 is hereditary (see [5], Theorem 11).

References

[1] G. Betsch, *Struktursätze für Fastringe*, Diss. Univ. Tübingen, 1963.
[2] N. J. Divinsky, *Rings and radicals* (London, 1965).
[3] K. Kaarli, The classification of irreducible R-groups over a semiprimary near-ring (in Russian), *Tartu Riikliku Ülikooli Toimetised*, **556** (1981), 47—63.
[4] K. Kaarli, Minimal ideals in near-rings (in Russian), *Tartu Riikliku Ülikooli Toimetised*, **336** (1975), 105—142.
[5] K. Kaarli, Radicals in near-rings (in Russian), *Tartu Riikliku Ülikooli Toimetised*, **390** (1976), 134—171.
[6] K. Kaarli, Special radicals of near-rings (in Russian), *Tartu Riikliku Ülikooli Toimetised*, **610** (1982), 53—68.
[7] G. Pilz, *Near-rings* (North-Holland, 1977).
[8] D. Ramakotaiah, Radicals for near-rings, *Math. Z.*, **97** (1967), 45—56.

(Received October 4, 1984)

DEPARTMENT OF MATHEMATICS
TARTU STATE UNIVERSITY
TARTU, ESTONIA, USSR

Acta Math. Hung.
50 (1—2) (1987), 79—84.

NON-EXISTENCE OF CERTAIN TYPES OF NP-FINSLER SPACES

P. N. PANDEY (Allahabad)

1. Introduction

K. Yano [12] defined normal projective connection coefficients Π^i_{kh} by $\Pi^i_{kh}=G^i_{kh}-\dot{x}^i G^r_{khr}/(n+1)$, where G^i_{kh} are connection coefficients of Berwald, $G^i_{jkh}=\dot{\partial}_j G^i_{kh}$ and $\dot{\partial}_j$ denotes the operator for partial differentiation with respect to $\dot{x}^j$. R. B. Misra and F. M. Meher [3] considered a space equipped with normal projective connection coefficients Π^i_{kh} whose curvature tensor N^i_{jkh} is recurrent with respect to Π^i_{kh}, and called it an RNP—Finsler space. They also studied the projective motion in such space. R. B. Misra, N. Kishore and the present author [4] studied an SNP—Finsler space characterized by the vanishing of covariant derivative of the curvature tensor N^i_{jkh} with respect to Π^i_{kh}. These spaces are also studied by U. P. Singh and A. K. Singh [9], [10], S. B. Misra and A. K. Misra [2]. It seems that the authors [9], [10] were not aware of the papers [3], [4], this is why they used different nomenclature, viz. NP—RF$_n$ and NP—SF$_n$ for RNP-Finsler space and SNP-Finsler space respectively. The aim of the present paper is to show the non-existence of non-trivial RNP-Finsler spaces and SNP-Finsler spaces. The notation of this paper is based on [3, 4, 6] and differs from that of [2, 8, 9, 10, 12].

2. Preliminaries

Let us consider an n-dimensional normal projective space equipped with the normal projective connection Π^i_{kh} given by

$$\Pi^i_{kh} = G^i_{kh}-\dot{x}^i G^r_{khr}/(n+1), \tag{2.1}$$

where G^i_{kh} are connection coefficients[1] of Berwald, $G^i_{jkh}=\dot{\partial}_j G^i_{kh}$ and $\dot{\partial}_j\equiv\partial/\partial\dot{x}^j$. The connection coefficients G^i_{kh} and the tensor G^i_{jkh} are symmetric in their lower indices and are positively homogeneous of degree 0 and -1 respectively. The tensor G^i_{jkh} satisfies

$$G^i_{jhk}\dot{x}^h = G^i_{jkh}\dot{x}^h = G^i_{hjk}\dot{x}^h = 0. \tag{2.2}$$

Due to symmetry of G^i_{kh} and G^i_{jkh} in their lower indices and their homogeneity in $\dot{x}^i$ the derivatives $\dot{\partial}_j \Pi^i_{kh}$, denoted by Π^i_{jkh}, satisfy the following:

$$\left\{\begin{array}{l} \text{a) } \Pi^i_{jkh} = \Pi^i_{jhk}, \quad \text{b) } \Pi^i_{khi} = G^i_{khi}, \quad \text{c) } \dot{x}^j\Pi^i_{jkh} = 0, \\ \text{d) } \dot{x}^h\Pi^i_{jkh} = \dot{x}^i G^r_{jkr}/(n+1), \quad \text{e) } \Pi^i_{ikh} = 2G^i_{ikh}/(n+1). \end{array}\right. \tag{2.3}$$

---

[1] Unless stated otherwise, all the entities are considered as functions of the line-elements $(x^i, \dot{x}^i)$. The indices $i, j, k, \ldots$ assume positive integral values $1, 2, 3, \ldots, n$.

Π^i_{jkh} are the same as U^i_{jkh} of K. Yano [12, p. 197]. The normal projective covariant derivative $\nabla_k T^i_j$ of an arbitrary tensor T^i_j, defined by

$$\nabla_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)\Pi^r_{kh}\dot{x}^h + T^r_j \Pi^i_{kr} - T^i_r \Pi^r_{kj}, \quad \partial_k = \partial/\partial x^k, \tag{2.4}$$

gives rise to the commutation formula[2]

$$2\nabla_{[j}\nabla_{k]}T^i_h = N^i_{jkr}T^r_h - N^r_{jkh}T^i_r - N^r_{jks}\dot{x}^s\dot{\partial}_r T^i_h \tag{2.5}$$

where N^i_{jkh} are components of a normal projective curvature tensor. The present author [6] established the relation between this tensor and Berwald's curvature tensor H^i_{jkh} which is given by

$$N^i_{jkh} = H^i_{jkh} - \dot{x}^i \dot{\partial}_h H^r_{jkr}/(n+1). \tag{2.6}$$

The Weyl projective curvature tensor[3] W^i_{jkh} and the normal projective curvature tensor N^i_{jkh} are connected [12, p. 197] by

$$W^i_{jkh} = N^i_{jkh} + 2\{\delta^i_{[j}M_{k]h} - \delta^i_h M_{[jk]}\}, \tag{2.7}$$

where $M_{jk} = -(nN_{jk} + N_{kj})/(n^2-1)$ and $N_{jk} = N^r_{rjk}$. For the tensors H^i_{jkh} and W^i_{jkh} we have the following:

$$\begin{cases} \text{a) } H^i_{jkh}\dot{x}^h = H^i_{jk}, \quad \text{b) } H^i_{jk}\dot{x}^k = H^i_j, \quad \text{c) } \dot{\partial}_h H^i_{jk} = H^i_{jkh}, \\ \text{d) } H^i_{ikh} = H_{kh}, \quad \text{e) } H^i_{ik} = H_k, \quad \text{f) } H^i_i = (n-1)H, \\ \text{g) } H_{kh}\dot{x}^h = H_k, \quad \text{h) } \dot{\partial}_h H_k = H_{kh}, \quad \text{i) } H_h\dot{x}^h = (n-1)H, \end{cases} \tag{2.8}$$

and

$$\begin{cases} \text{a) } W^i_{jkh}\dot{x}^h = W^i_{jk}, \quad \text{b) } \dot{\partial}_h W^i_{jk} = W^i_{jkh}, \quad \text{c) } W^i_{jk}\dot{x}^k = W^i_j, \\ \text{d) } W^i_r\dot{x}^r = 0, \quad \text{e) } W^i_{ikh} = 0, \quad \text{f) } W^i_{jki} = 0. \end{cases} \tag{2.9}$$

It should be noted that if the Berwald connection satisfies $G^r_{jkr}=0$, the normal projective connection coefficients and Berwald's connection coefficients coincide. Thus, a Finsler space whose connection coefficients G^i_{jk} satisfy $G^r_{jkr}=0$ is a trivial normal projective space.

3. An RNP-Finsler space

Let us consider an RNP-Finsler space [2, 3, 9] characterized by

$$\nabla_m N^i_{jkh} = \lambda_m N^i_{jkh}, \tag{3.1}$$

$N^i_{jkh} \neq 0$ and λ_m are components of a non-null covariant vector field positively homogeneous of degree zero in $\dot{x}^i$. Transvecting (2.6) by $\dot{x}^h$, using (2.8a) and taking

---

[2] The square brackets denote the skew-symmetric part of the object with respect to the indices enclosed within them.

[3] The tensor W^i_{jkh} is the same as P^i_{jkh} of K. Yano [12, p. 197].

care of the degree of homogeneity of H^r_{jkr} in $\dot{x}^i$, we find

$$N^i_{jkh}\dot{x}^h = H^i_{jk}. \tag{3.2}$$

Transvecting (3.1) by $\dot{x}^h$ and using (3.2), we get

$$\nabla_m H^i_{jk} = \lambda_m H^i_{jk}. \tag{3.3}$$

Further transvection of (3.3) by $\dot{x}^k$, in view of (2.8b), gives

$$\nabla_m H^i_j = \lambda_m H^i_j. \tag{3.4}$$

Contraction of the indices i and h in (2.6) and the fact that H^r_{jkr} are positively homogeneous of degree zero in $\dot{x}^i$'s imply

$$N^r_{jkr} = H^r_{jkr}, \tag{3.5}$$

in view of which contraction of the indices i and h in (3.1) gives

$$\nabla_m H^r_{jkr} = \lambda_m H^r_{jkr}. \tag{3.6}$$

By virtue of (2.8e), contraction of the indices i and j in (3.3) gives

$$\nabla_m H_k = \lambda_m H_k. \tag{3.7}$$

Suppose F and g_{ij} are the fundamental metric function and components of the metric tensor from which G^i_{jk} are derived. The present author [5] proved that the tensors H^i_{jk} and H^i_j satisfy the following:

$$\text{a) } y_i H^i_{jk} = 0, \quad \text{b) } y_i H^i_j = 0, \quad \text{c) } g_{ij} H^i_k = g_{ik} H^i_j, \tag{3.8}$$

where $y_i = g_{ij}\dot{x}^j$. Transvecting (3.4) by y_i and using (3.8b), we get

$$H^i_j \nabla_m y_i = 0. \tag{3.9}$$

Writing the expression for $\nabla_m y_i$ with the help of (2.4) and using (2.1), we have

$$\nabla_m y_i = \partial_m y_i - (\dot{\partial}_r y_i) G^r_{mh}\dot{x}^h - y_r G^r_{mi} + F^2 G^r_{mir}/(n+1), \tag{3.10}$$

since $\Pi^r_{mh}\dot{x}^h = G^r_{mh}\dot{x}^h$ and $y_r\dot{x}^r = F^2$. Since the covariant derivative of y_i in the sense of Berwald vanishes identically, we have

$$\mathscr{B}_m y_i \overset{\text{def}}{=} \partial_m y_i - (\dot{\partial}_r y_i) G^r_{mh}\dot{x}^h - y_r G^r_{mi} = 0 \tag{3.11}$$

where $\mathscr{B}_m$ is the operator for Berwald's covariant differentiation. From (3.10) and (3.11) we have

$$\nabla_m y_i = F^2 G^r_{mir}/(n+1), \tag{3.12}$$

by virtue of which (3.9) gives

$$H^i_j G^r_{mir} = 0. \tag{3.13}$$

Differentiating (3.7) partially with respect to $\dot{x}^h$, utilizing the commutation formula [12, p. 196]

$$\dot{\partial}_h \nabla_m T^i_j - \nabla_m \dot{\partial}_h T^i_j = T^r_j \Pi^i_{hmr} - T^i_r \Pi^r_{hmj} - (\dot{\partial}_r T^i_j) \Pi^r_{hms}\dot{x}^s \tag{3.14}$$

and using (2.8h), we have

$$\nabla_m H_{kh} - H_r \Pi^r_{hmk} - H_{kr} \Pi^r_{hms} \dot{x}^s = (\dot{\partial}_h \lambda_m) H_k + \lambda_m H_{kh}. \tag{3.15}$$

We know that the tensor H^i_{jkh} satisfies the Bianchi identity [8, p. 127]

$$H^i_{jkh} + H^i_{khj} + H^i_{hjk} = 0.$$

Contracting the indices i and j in the above identity, using (2.8d) and the skew-symmetry of the tensor H^i_{jkh} in the first two lower indices, we have

$$H^r_{khr} = H_{hk} - H_{kh}. \tag{3.16}$$

Taking skew-symmetric part of (3.15) with respect to the indices k and h, and using (3.16), (3.6) and (2.3d), we have

$$-2H_r \Pi^r_{[h|m|k]} - 2\dot{x}^r H_{[k|r|} G^s_{h]ms}/(n+1) = 2H_{[k} \dot{\partial}_{h]} \lambda_m. \tag{3.17}$$

Differentiating (2.1) partially with respect to $\dot{x}^j$, taking skew-symmetric part with respect to the indices j and h, and using the symmetric property of the tensors G^i_{jkh} and $\dot{\partial}_j G^i_{khm}$ in their lower indices, we have

$$2\Pi^i_{[j|k|h]} = -\frac{2}{n+1} \delta^i_{[j} G^r_{h]kr}. \tag{3.18}$$

Using (2.8g) and (3.18) in (3.17), we have

$$4H_{[h} G^s_{k]ms}/(n+1) = 2H_{[k} \dot{\partial}_{h]} \lambda_m. \tag{3.19}$$

According to the authors [3, 9], the Weyl's projective curvature tensor W^i_{jkh} is recurrent in an RNP-Finsler space of dimension greater than 2, i.e.,

$$\nabla_m W^i_{jkh} = \lambda_m W^i_{jkh}. \tag{3.20}$$

Transvecting (3.20) by $\dot{x}^h$ and using (2.9a), we get

$$\nabla_m W^i_{jk} = \lambda_m W^i_{jk}. \tag{3.21}$$

Differentiating (3.21) partially with respect to $\dot{x}^h$, using the commutation formula (3.14) and taking care of the equations (2.3d), (2.9a), (2.9b), (3.20) and (3.21), we have

$$(\dot{\partial}_h \lambda_m) W^i_{jk} = \Pi^i_{hmr} W^r_{jk} - \Pi^r_{hmj} W^i_{rk} - \Pi^r_{hmk} W^i_{jr} - \frac{1}{n+1} W^i_{jk} G^s_{hms}. \tag{3.22}$$

Transvecting this equation by $\dot{x}^k$, and using (2.3d) and (2.9c), we have

$$(\dot{\partial}_h \lambda_m) W^i_j = \Pi^i_{hmr} W^r_j - \Pi^r_{hmj} W^i_r - \frac{2}{n+1} W^i_j G^s_{hms}. \tag{3.23}$$

Transvecting (3.23) by $\dot{x}^m$ and using (2.2), (2.3d), (2.9d) and the symmetry of Π^i_{jkh} in its last two lower indices, we get

$$(\dot{x}^m \dot{\partial}_h \lambda_m) W^i_j = \frac{\dot{x}^i}{n+1} G^s_{hrs} W^r_j. \tag{3.24}$$

The projective deviation tensor W^i_j and the deviation tensor H^i_j are related by [8, p. 140]

$$W^i_j = H^i_j - H\delta^i_j - \frac{\dot{x}^i}{n+1}(\dot{\partial}_r H^r_j - \dot{\partial}_j H). \tag{3.25}$$

In view of (3.25), (3.13) and (2.2), the equation (3.24) reduces to

$$(\dot{x}^m \dot{\partial}_h \lambda_m) W^i_j = -\frac{\dot{x}^i}{n+1} H G^r_{jhr}. \tag{3.26}$$

Thus we conclude that an RNP-Finsler space F_n $(n>2)$ admits (3.26). Transvecting (3.19) by $\dot{x}^h$, using (2.2), (2.8i) and the fact that the vector λ_m is positively homogeneous of degree zero in $\dot{x}^i$, we get

$$2\left(\frac{n-1}{n+1}\right) H G^r_{kmr} = -(n-1) H \dot{\partial}_k \lambda_m. \tag{3.27}$$

Since $n>1$, (3.27) is equivalent to

$$H\left(\dot{\partial}_k \lambda_m + \frac{2}{n+1} G^r_{kmr}\right) = 0. \tag{3.28}$$

This equation gives at least one of the following:

$$\text{a) } H = 0, \quad \text{b) } \dot{\partial}_k \lambda_m + \frac{2}{n+1} G^r_{kmr} = 0. \tag{3.29}$$

If (3.29a) holds, (3.26) gives at least one of the following:

$$\text{a) } W^i_j = 0, \quad \text{b) } \dot{x}^m \dot{\partial}_h \lambda_m = 0. \tag{3.30}$$

If $W^i_j=0$, the space is of scalar curvature [1, 7, 11], and hence we have

$$H^i_j = H(\delta^i_j - l^i l_j) \tag{3.31}$$

where $l^i=\dot{x}^i/F$ and $l_j=g_{ij}l^i$. But (3.31) together with (3.29a) implies $H^i_j=0$ which leads to $H^i_{jkh}=0$. In view of this fact the relation (2.6) gives $N^i_{jkh}=0$, a contradiction. Therefore condition (3.30a) can not hold. Condition (3.30b) is equivalent to $\lambda_h=\dot{\partial}_h\lambda$, where $\lambda=\lambda_m\dot{x}^m$. Hence the skew-symmetric part of (3.23) with respect to the indices h and m, in view of symmetry of G^s_{hms} and Π^i_{jmh} in the indices m and h and the equation (3.18), is given by

$$\delta^i_{[h} G^s_{m]rs} W^r_j - W^i_{[h} G^s_{m]js} = 0. \tag{3.32}$$

Transvecting (3.32) by $\dot{x}^h$, and using (2.2) and (2.9d), we get $W^r_j G^s_{mrs}=0$; in view of which the equation (3.32) gives $W^i_{[h}G^s_{m]js}=0$. Using (3.25) and (3.29a) in $W^i_{[h}G^s_{m]js}=$ $=0$, we get

$$H^i_{[h} G^s_{m]js} - \dot{x}^i (\dot{\partial}_r H^r_{[h}) G^s_{m]js}/(n+1) = 0. \tag{3.33}$$

Transvecting (3.33) by y_i, and using (3.8b) and $y_i\dot{x}^i=F^2$, we have $(\dot{\partial}_r H^r_{[h})G^s_{m]js}=0$. Using this in (3.33) we find $H^i_{[h}G^s_{m]js}=0$. If the tensor G^s_{mjs} is non-zero, we may choose a vector Y^i such that $G^s_{mjs}Y^j\neq 0$. Multiplying $H^i_{[h}G^s_{m]js}=0$ by Y^j and putting $G^s_{mjs}Y^j=\varphi_m$, we get $H^i_{[h}\varphi_{m]}=0$; which implies $H^i_h=X^i\varphi_h$, where X^i is a contravariant vector. Multiplying $H^i_h=X^i\varphi_h$ by g_{im} and using (3.8c), we have $\varphi_h=\psi g_{ih}X^i$, where ψ is a non-zero scalar. Hence we have $H^i_h=\psi X^i g_{rh}X^r$. Contracting the indices i and h in this equation, and using (2.8f) and (3.29a), we get $g_{rs}X^rX^s=$ $=0$; which implies $X^i=0$ because the metric of the space considered is positive definite. Substituting $X^i=0$ in $H^i_h=X^i\varphi_h$, we find $H^i_h=0$. This will lead to $N^i_{jkh}=0$, a contradiction. Therefore the supposition $G^s_{mjs}\neq 0$ is wrong. Hence (3.29a) implies $G^s_{mjs}=0$. Now, we consider (3.29b). Transvection of (3.29b) by $\dot{x}^m$ and utilizing (2.2), we have $\dot{x}^m\dot{\partial}_h\lambda_m=0$; in view of which (3.26) becomes $HG^r_{jhr}=0$. This means at least one of the scalar H and the tensor G^r_{jhr} is zero. We have seen that $H=0$ implies $G^r_{jhr}=0$. Therefore we certainly have $G^r_{jhr}=0$. In view of this fact, (2.1) shows that the normal projective connection Π^i_{kh} coincides with Berwald's connection G^i_{kh}. Thus, our normal projective space is a trivial one. Therefore, an RNP-Finsler space F_n $(n>2)$ is a trivial one. This leads to:

THEOREM 3.1. *A non-trivial RNP-Finsler space* F_n $(n>2)$ *does not exist.*

Adopting the procedure similar to Theorem 3.1, we may prove that if an SNP-Finsler space F_n $(n>2)$ characterized by $\nabla_m N^i_{jkh}=0$ exists, the tensor G^r_{jkr} vanishes, and hence the normal projective space becomes trivial. This gives:

THEOREM 3.2. *A non-trivial SNP-Finsler space* F_n $(n>2)$ *does not exist.*

References

[1] Makoto Matsumoto, Projective changes of Finsler metrics and projectively flatFinsler spaces, *Tensor (N.S.)*, **34** (1980), 303—315.

[2] S. B. Misra and A. K. Misra, Affine motion in RNP-Finsler space, *Indian J. Pure Appl. Math.*, **15** (1984), 463—472.

[3] R. B. Misra and F. M. Meher, Projective motion in an RNP-Finsler space, *Tensor (N.S.)*, **22** (1971), 117—120.

[4] R. B. Misra, N. Kishore and P. N. Pandey, Projective motion in an SNP—F_n, *Boll. Un. Mat. Ital.*, **14—A** (1977), 513—519.

[5] P. N. Pandey, On decomposability of curvature tensor of a Finsler manifold, *Acta Math. Acad. Sci. Hungar.*, **38** (1981), 109—116.

[6] P. N. Pandey, On NPR-Finsler manifolds, *Annales Fac. Sci. Kinshasa*, **6** (1980), 65—77.

[7] P. N. Pandey, On a Finsler space of zero projective curvature, *Acta Math. Acad. Sci. Hungar.*, **39** (1982), 387—388.

[8] H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag (1959).

[9] U. P. Singh and A. K. Singh, On the N-curvature collineation in Finsler spaces, *Annales Soc. Sci. Bruxelles*, **95** (1981), 69—77.

[10] U. P. Singh and A. K. Singh, On the N-curvature collineation in Finsler space II, *Indian J. Pure Appl. Math.*, **12** (1981), 1208—1212.

[11] Z. I. Szabó, Ein Finslerscher Raum ist gerade dann von scalarer Krümmung wenn seine Weylsche Projektivkrümmung verschwindet, *Acta Sci. Math. (Szeged)*, **39** (1977), 163—168.

[12] K. Yano, *The theory of Lie derivatives and its applications*, North Holland Publisching Co. (Amsterdam, 1957).

(Received October 15, 1984; revised January 3, 1985)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALLAHABAD
ALLAHABAD—211002, INDIA

Acta Math. Hung.
50 (1—2) (1987), 85—95.

ABELIAN GROUPS LIKE MODULES

S. SINGH (Kuwait)

Consider the following two conditions on a module M_R:

(I) Every finitely generated submodule of a homomorphic image of M is a direct sum of uniserial modules.

(II) Given two uniserial submodules of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f: W \to V$ can be extended to a homomorphism $g: U \to V$ provided, composition length $d(U/W) \leqq d(V/f(W))$.

The study of modules satisfying (I) and (II) was initiated by Singh [12]. It has been seen through a number of papers, some of which are [1, 5, 6, 12, 13, 15], that the structure theory of these modules is similar to that of torsion abelian groups; keeping this in view these were called torsion abelian groups like modules (in short TAG-modules) in [1]. A uniserial module which is not quasi-injective trivially satisfies (I), but not (II). In this paper we study modules only satisfying condition (I), and call them QTAG-modules. In Section 2, Lemmas (2.2) and (2.3) show that a certain weaker version of condition (II) is implied by (I). Further (2.4) shows that a certain weaker version of the dual of (II) is also implied by (I). In Section 3, the structure theory of QTAG-modules is developed. Using the results in Section 2, first of all it is seen that the concepts of exponent and height of an element in a TAG-module can be defined verbatim for an element in a QTAG-module (see (3.1), (3.2), and (3.3)). In (3.5) [11, Lemma 7] is strongly improved. After this it is discussed in brief that almost all the concepts and results for TAG-modules given in [12, 13] can be defined, stated and proved for QTAG-modules. In particular it is shown that any QTAG-module M admits a basic submodule and that any two basic submodules of M are isomorphic. In Section 4, those rings R for which R_R is a QTAG-module, are studied. The main results are given in (4.5) and (4.6).

§ 1. Preliminaries

All rings considered in this paper are with unity $1 \neq 0$ and all modules are unital right modules, unless otherwise stated. For any ring R, $J(R)$ or simply J denotes its Jacobson radical. Consider a module M_R. The symbols $J(M)$, $E_R(M)$ (or simply $E(M)$) will denote its Jacobson radical and injective hull respectively; $N \subset' M$ denotes that N is an essential submodule of M. $\operatorname{soc}(M)$ will denote the socle of M. Put $\operatorname{soc}^0(M)=0$. For any $k \geqq 0$, $\operatorname{soc}^k(M)$ is defined inductively by $\operatorname{soc}(M/\operatorname{soc}^k(M)) = \operatorname{soc}^{k+1}(M)/\operatorname{soc}^k(M)$. Similarly $J^k(M)$ is defined inductively by putting $J^{k+1}(M) = J(J^k(M))$ and $J^0(M) = M$. M is said to be serial if the lattice of its submodules is linearly ordered under inclusion [18]; however if M is serial and of finite composition length it is said to be uniserial. Following Warfield [18] a ring R is said to be right (left) serial if R_R (respectively ${}_RR$) is a direct sum of serial modules. A ring R which is right as well as left, serial and artinian, is called a generalized uniserial ring. For most of basic concepts on rings and modules, we refer to Faith [2]. For the definitions and properties of push out and pull back diagrams we refer to Stenstrom 17]. For concepts and results on abelian groups, we refer to Fuchs [3], [4]. An element x in M is said to be uniform, if xR is a non-zero uniform submodule of M. If S and T are two simple R-modules, then T is called a successor of S, and S is called a predecessor of T if $\operatorname{Ext}(S, T) \neq 0$ [18, Definition 5.2]. If x is a uniform element of a TAG-module M_R, following [11], [12], $e(x)$ and $H_M(x)$ (or simply $H(x)$) will denote the exponent of x and the height of x in M respectively. For all the basic concepts for TAG-modules, we refer to [12], [13].

§ 2. Some general results for condition (I)

In this section we show that any QTAG-module satisfies a certain weaker version of condition (II) and its dual. We further establish some general results needed in subsequent sections. We start with the observation that the class of QTAG is closed under submodules and homomorphic images.

LEMMA 2.1. *Let* $A_1, A_2, \dots, A_n$ *be any (non-zero) uniserial submodules of a module* M_R *and let* $A_1+A_2+\dots+A_n=B_1\oplus B_2\oplus\dots\oplus B_m$ *for some uniserial submodules* B_i, $1\leqq i\leqq m$. *Then*

(i) $m\leqq n$.

(ii) *Each* B_i *is a homomorphic image of some* A_j.

(iii) *Each* A_j *embeds in some* B_i *under the projection* $p_i\colon \oplus \sum_k B_k \to B_i$.

(iv) *Any* A_j *of maximal length among* $A_1, A_2, \dots, A_n$ *is a summand of* $\sum_{i=l}^{n} A_i$.

PROOF. (i) Consider the external direct sum $K=A_1\oplus A_2\oplus\dots\oplus A_n$. Then $K/J(K)$ is a direct sum of n simple modules. Now

$$(\oplus \sum_{j=1}^{m} B_i)/J(\oplus \sum_{i=l}^{m} B_i)$$

is a direct sum of m simple modules, and is a homomorphic image of $K/J(K)$. Consequently $m\leqq n$. This proves (i). Now (ii) and (iii) follow as [11, Lemma 3]. (iv) follows from (iii).

LEMMA 2.2. *Let* A *and* B *be two uniserial submodules of a QTAG-module* M *such that* $A\cap B=0$. *Let* σ *be any homomorphism from a submodule* W *of* A *into* B *such that* $d(A/W)\leqq d(B/\sigma(W))$. *Then* σ *can be extended to a homomorphism* $\bar{\sigma}\colon A\to B$.

PROOF. Consider the push out diagram

$$\begin{tikzcd}
W \arrow[r, "i"] \arrow[d, "\sigma"'] & A \arrow[d, "\eta"] \\
B \arrow[r, "j"'] & K
\end{tikzcd}$$

where i is the inclusion map. Since i is a monomorphism, j is a monomorphism [17, p. 92]. Consider the injections $i_1\colon A\to A\oplus B$, $i_2\colon B\to A\oplus B$. Using the fact that

$$K=\operatorname{coker}(i_1 i-i_2\sigma\colon W\to A\oplus B)$$

it follows that $\ker\eta=\ker\sigma$. Thus by hypothesis

$$d(\eta(A))=d(A)-d(\ker\sigma)\leqq d(B)=d(j(B)).$$

Since K is a homomorphic image of $A+B$, it is a QTAG-module. As $K=j(B)+\eta(A)$, by (2.1) $K=j(B)\oplus C$. Consider the projection $p\colon j(B)\oplus C\to j(B)$. We have $j^{-1}\colon j(B)\to B$. Then $\bar\sigma=j^{-1}p\eta\colon A\to B$ extends σ.

LEMMA 2.3. *Let* A *and* B *be any two unisersal submodules of a QTAG-module* M *such that* $A\cap B\neq 0$ *and* $d(A)\leqq d(B)$. *Then there exists a monomorphism* $\sigma\colon A\to B$, *which is identity on* $A\cap B$.

PROOF. As $d(A)\leqq d(B)$, by (2.1) $A+B=B\oplus C$. σ, the restriction of the projection $p\colon B\oplus C\to B$ to A, is a desired map.

LEMMA 2.4. *Let* A *and* B *be two uniserial submodules of a QTAG-module* M *such that* $A\cap B=0$. *Let* W *be any submodule of* B *and* $\sigma\colon A\to B/W$ *be any homomorphism such that* $d(W)\leqq d(\ker\sigma)$. *Then there exists a homomorphism* $\bar\sigma\colon A\to B$ *lifting* σ.

PROOF. Without loss of generality we take $\sigma\neq 0$. Consider the pull back diagram

$$\begin{tikzcd}
K \arrow[r, "\bar\pi"] \arrow[d, "\lambda"'] & A \arrow[d, "\sigma"] \\
B \arrow[r, "\pi"'] & B/W
\end{tikzcd}$$

where π is natural homomorphism. Since π is an epimorphism, $\bar\pi$ is also an epimorphism. Further as $\sigma\neq 0$, $\lambda(K)\not\subset W$. Since B is uniserial, $W\subset\lambda(K)$. Then $\lambda(K)/W\approx\sigma(A)\approx A/\ker\sigma$. Thus $d(\lambda(K))\leqq d(A)$, as $d(W)\leqq d(\ker\sigma)$. Since K is a subdirect sum of $\lambda(K)\subseteqq B$ and $\bar\pi(K)=A$, it is a QTAG-module and it cannot have a uniserial submodule of length greater than

$$\max(d(\lambda(K)),d(A))=d(A).$$

As $\bar\pi$ is epimorphism, we get $K=A'\oplus C$ for some uniserial submodule A' such that $\bar\pi$ maps A' isomorphically onto A. So we have a monomorphism $\eta\colon A\to K$ such that $\bar\pi\eta=i_A$. Then $\bar\sigma=\lambda\eta\colon A\to B$ lifts σ and the result follows.

It follows from the above lemmas that if two uniserial submodules A and B of a QTAG-module M have zero intersection and same composition lengths, they are isomorphic whenever either they have isomorphic socles or $A/J(A)\approx B/J(B)$; further in that case A and B are quasi-injective as well as quasi-projective. If $M\oplus M$ is a QTAG-module, then M is a TAG-module.

PROPOSITION 2.5. *Let* N *and* K *be any two submodules of a QTAG-module* M. *Let* $x+N$ *be a uniform element of* $(K+N)/N$. *Then*

(a) *For some uniform element* $y\in K$, $x+N=y+N$.

(b) *If* u, v *are two uniform elements in* $x+N$ *such that* $uR\cap vR=0$ *and* $e(u)\leqq e(v)$, *then there exists an epimorphism* $\sigma: vR\to uR$ *such that* $v-\sigma(v)\in N$.

Proof. (a) Now $\bar{x}=x+N=u+N=\bar{u}$ for some non-zero $u\in K$. By condition (I)

$$uR=\sum_{i=1}^{k} u_i R$$

for some uniform elements u_i. Then

$$\bar{u}R=\sum_{i} \bar{u}_i R.$$

However $\bar{u}R$ is uniserial. So for some i, say for $i=1$, $\bar{u}R=\bar{u}_1 R$. Consequently $\bar{u}=\bar{u}_1 r$, for some $r\in R$. Then $y=u_1 r\in K$ is uniform. Further $\bar{x}=\bar{y}$. This proves (a).

(b) Let $uR\cap vR=0$. Now $u+N=v+N$. So we have the epimorhism

$$\eta: vR\to uR/(uR\cap N)$$

such that $\eta(v)=u+(uR\cap N)$. As $\ker\eta=vR\cap N$ and $d(vR\cap N)\geqq d(uR\cap N)$, by (2.4) η can be lifted to an epimorphism $\sigma: vR\to uR$. Clearly $v-\sigma(v)\in N$. This proves the result.

Proposition 2.6. *Let* N *be any submodule of a QTAG-module* M. *Let* $\bar{x}=x+N$ *be a uniform element in* $\operatorname{soc}(M/N)$. *Let* $x_0\in x+N$ *be a uniform element of smallest exponent. Let* y *be any uniform element in* M *such that* $\bar{x}R=\bar{y}R$. *Then either* $x_0R\cap yR=0$ *or* $e(x_0)=e(y)$.

Proof. Let $x_0R\cap yR\neq 0$. By hypothesis $e(x_0)\leqq e(y)$. By (2.3) there exists a monomorphism $\sigma: x_0R\to yR$, which is identity on $x_0R\cap yR$. On the contrary let $e(x_0)<e(y)$. Then $\sigma(x_0)\in N$. Now

$$\eta: x_0R\to \big(x_0-\sigma(x_0)\big)R$$

given by

$$\eta(x_0 r)=\big(x_0-\sigma(x_0)\big)r,\quad r\in R$$

is an epimorphism with $x_0R\cap yR\subset\ker\eta$. Thus $x_1=x_0-\sigma(x_0)$ is a uniform element in $x+N$ such that $e(x_1)<e(x_0)$. This contradicts the choice of x_0. Hence $e(x_0)=e(y)$. This proves the result.

§ 3. Some decomposition theorems

For any uniform element x in a QTAG-module, the concept of height of x in M, denoted by $H_M(x)$ (or simply by $H(x)$) and of exponent of x, denoted by $e(x)$, is verbatim same as in TAG-module [12]. The proof of the following lemma is verbatim same as of [11, Lemma 4], except that we use (2.2) instead of Lemma 2(b) in [11].

Lemma 3.1. *Let* $x_1, x_2, \ldots, x_n$ *be any finitely many uniform elements in a QTAG-module* M_R *such that for some non-negative integer* k, $H(x_i)\geqq k$ *for* $1\leqq i\leqq n$. *Then for any uniform element* x *of* M *in* $\sum_i x_i R$, $H(x)\geqq k$.

Let x be any non-zero element in a QTAG-module M_R. By definition $xR= = \sum_{i=l}^{n} x_i R$, for some uniform elements $x_i \in M$. Define the height $H_M(x)$ (or simply $H(x)$) of x, by $H(x)=\min(H(x_i))$. This is well defined. Put $H(0)=\infty>n$, n any integer, and $\infty+n=\infty$. As in [11] we define $H_n(M)$, for any $n \geqq 0$. Then

$$H_n(M) = \{x \in M \colon H(x) \geqq n\}$$

and it is a submodule of M. The following lemma has similar proof as of [11, Lemma 6].

LEMMA 3.2. *Let A and B be two submodules of a QTAG-module M such that $M=A+B$. Then for any $k \geqq 0$*

$$H_k(M) = H_k(A)+H_k(B).$$

One can easily see that for a QTAG-module M, $H_k(M)=J^k(M)$ and that $H_{k+1}(M)$ is the smallest submodule of $H_k(M)$ such that $H_k(M)/H_{k+1}(M)$ is completely reducible. For any uniform element $y \in M$, a uniform element x is called a predecessor of y if $y \in xR$ and $d(xR/yR)=1$, in this case also y is called a successor of x. Let N be a submodule of M. A uniform element $y \in M$ is in $H_1(N)$ if and only if y admits a predecessor in N. We list some properties of height.

LEMMA 3.3. *Let M be any QTAG-module. Let $x, y \in M$.*

(i) *$H(x) \geqq n$ if and only if $x \in H_n(M)$.*

(ii) *$H(x+y) \geqq \min(H(x), H(y))$; equality holds whenever $H(x) \neq H(y)$.*

(iii) *If $y \in H_1(xR)$, then $H(y) \geqq H(x)+1$.*

(iv) *If $x-y \in \operatorname{soc}(M)$, then $H_1(xR)=H_1(yR)$.*

(v) *If $xR=yR$, then $H(x)=H(y)$.*

LEMMA 3.4. *Let N and K be two submodules of a QTAG-module M. Then, for any $k \geqq 0$*

$$H_k[(K+N)/N] = (H_k(K)+N)/N$$

PROOF. Let $\bar{x}=x+N$ be a uniform element of $H_k[(K+N)/N]$. There exists a uniform element $y \in K+N$ such that $\bar{x} \in \bar{y}R$ and $d(\bar{y}R/\bar{x}R)=k$. So $\bar{x}=\bar{y}r$ for some $r \in R$. Now $z=yr$ is uniform and $yr \in H_k(K+N)=H_k(K)+H_k(N)$. Thus $\bar{x}=\bar{y}r=\bar{u}$ for some $u \in H_k(K)$. Obviously $[H_k(K)+N]/N \subset H_k[(K+N)/N]$. This proves the lemma.

THEOREM 3.5. *Let M be a QTAG-module over a ring R, T be any submodule of M and K be any complement of T in M. Then the following hold:*

(a) *For any simple submodule S of $M/(T+K)$ and any uniform element $x \in M$, generating S modulo $T+K$, $L_S=[xR \cap T+H_1(T)]/H_1(T)$ is a simple submodule of $[(H_1(M)+K) \cap T]/H_1(T)$, and is independent of the choice of x.*

(b) *$S \to L_S$ gives a mapping of the family of all simple submodules of $M/(T+K)$ onto the family of all simple submodules of $[(H_1(M)+K) \cap T]/H_1(T)$.*

PROOF. Now $(T\oplus K)/K\subset' M/K$ and $M/(T+K)\approx[M/K]/[(T+K)/K]$. By (3.4)

$$\frac{H_1(M/K)\cap[(T+K)/K]}{H_1[(T+K)/K]}\approx\frac{(H_1(M)+K)\cap(T+K)}{H_1(T)+K}\approx$$

$$\approx\frac{(H_1(M)+K)\cap T+K}{H_1(T)+K}\approx\frac{(H_1(M)+K)\cap T}{H_1(T)}.$$

Thus without loss of generality we take $K=0$ and hence $T\subset' M$.

Consider a simple submodule $S=\bar{x}R$ of M/T. By (2.5) we take x to be uniform. Clearly $e(x)>1$. Let $yR=xR\cap T$. If $y\in H_1(T)$ it has a predecessor $z\in T$. By (2.3) we can choose z such that $xr\leftrightarrow zr$, $r\in R$ is an isomorphism between xR and zR, which is identity on yR. Then $x-z\in\operatorname{soc}(M)\subset T$. This in turn gives $x\in T$. This is a contradiction. Hence $y\notin H_1(T)$. Hence $\bar{y}R$, where $\bar{y}=y+H_1(T)$ is a simple submodule of $[H_1(M)\cap T]/H_1(T)$. yR is uniquely determined by xR. Let if possible $S=\bar{x}R=\bar{x}'R$ for another uniform element x', but let $xR\neq x'R$. Choose x to be of smallest exponent such that $S=\bar{x}R$. By (2.6) either $e(x)=e(x')$ or $xR\cap x'R=0$. Let $xR\cap x'R=0$. By (2.5) (b) we get an epimorphism $\eta\colon x'R\to xR$ such that $x'-\eta(x')\in T+K$. We can choose $x=\eta(x')$. Let $y'R=x'R\cap T$ and $y=\eta(y')$. Then $yR=xR\cap T$. Further $y'-y\in H_1((x'-x)R)\subset H_1(T)$. This gives $\bar{y}R=\bar{y}'R$ in $T/H_1(T)$. Let $xR\cap x'R\neq 0$. By (2.6) $e(x)=e(x')$. As $xR\neq x'R$, $xR+x'R=$ $=x'R\oplus uR$ for some uniform element u with $e(u)<e(x')$. In M/T, $\bar{x}R=\bar{x}R+$ $+\bar{x}'R=\bar{x}'R+\bar{u}R$. By the minimality of $e(x)$, $\bar{x}R\neq\bar{u}R$. Hence $\bar{u}R=0$ and $u\in T$. Consider the projection $p\colon x'R\oplus uR\to x'R$. Write $x=p(x)+ur$. Then $x'R=p(x)R$. For $yR=xR\cap T$, $x'R\cap T=p(y)R$. Now $x-p(x)\in T$ yields $y-p(y)\in H_1(T)$. Hence $\bar{y}R=\bar{y}'R$ in $T/H_1(T)$. This show that $L_S=\bar{y}R$ in $[H_1(M)\cap T]/H_1(T)$ is uniquely determined by S. Now consider any simple submodule $\bar{y}R$ in $[H_1(M)\cap T]/H_1(T)$ and take y to be uniform. Since $y\in H_1(M)$ it has a predecessor x in M. Then $S=\bar{x}R$ in M/T is a simple submodule to which yR corresponds in the above given correspondence. This proves the theorem.

REMARK 3.6. Consider any QTAG-module M. M is said to be h-divisible if $H_1(M)=M$, equivalently if every element in M has infinite height (see also [15,. p. 2034]). Let N be an h-divisible submodule of M. Let K be any complement of N. As $N=H_1(N)$, $[(H_1(M)+K)\cap N]/H_1(N)=0$. So by the above theorem $M/(N+K)=$ $=0$. Consequently $M=N\oplus K$ and N is an absolute summand of M. In view of (2.2), (2.3), the proof of [16, Proposition (2.1)] shows that a submodule N of M is a complement submodule of M if and only if $H_1(M)\cap N=H_1(N)$. (See also [5, Theorem 3].)

Kulilov's theorem for the decomposition of abelian p-groups was generalized to modules over bounded (hnp)-rings in [11, Theorem 3] and to TAG-modules in [12]. We now extend it to QTAG-modules. Since the proof is being adapted from [11, Theorem 3], we shall only outline the proof, indicating the necessary changes needed.

THEOREM 3.7. *Let* M_R *be any QTAG-module. Then* M_R *is direct sum of uniserial modules if and only if* M *is a union of an ascending sequence* M_n $(n=1, 2, 3, \ldots)$ *of submodules of* M *such that for each* n, *there exists a positive integer* k_n *with the property* $H_M(x)\leqq k_n$ *for all uniform elements* x *of* M_n.

PROOF. *Sufficiency*. For each n, let $P_n = \operatorname{soc}(M_n)$. Then $P = \bigcup_n P_n$ is the socle of M and $P \subset' M$. Follow the construction of a basis S_n of P_n for every n, as in [11, Theorem 3]. Then $S_n \subset S_{n+1}$ and $S = \bigcup_n S_n$ is a basis of P_n. Let $S = \{c_\lambda : \lambda \in \Lambda\}$. Since each c_λ is uniform and is in some M_n it has finite height. So we can find a uniform element $a_\lambda \in M$ such that $c_\lambda \in a_\lambda R$ and $d(a_\lambda R / c_\lambda R) = H(c_\lambda)$. Then $\sum_\lambda a_\lambda R = \oplus \Sigma a_\lambda R$. Let

$$M' = \oplus \sum_\lambda a_\lambda R.$$

If we show that $M = M'$, the result follows. Let $M \neq M'$. We can find a uniform element $g \in M$ of smallest exponent such that $g \notin M'$. Then $e(g) > 1$. Let $yR = \operatorname{soc}(gR)$. Now

$$y = c_1 r_1 + c_2 r_2 + \ldots + c_t r_t$$

for some $c_i \in S$, $r_i \in R$ such that $c_i r_i \neq 0$. If $t = 1$, then $y = c_1 r_1$. So $H(y) = H(c_1)$. Consequently $e(g) \leqq e(a_1)$. By (2.3) there exists a monomorphism $\sigma : gR \to a_1 R$, which is identity on yR. Then $g - \sigma(g)$ is a uniform element such that $e(g - \sigma(g)) < e(g)$ and $g - \sigma(g) \notin M'$. This contradicts the choice of g. Hence $t > 1$. Consequently

$$gR \cap c_i R = gR \cap a_i R = 0, \quad 1 \leqq i \leqq t.$$

Using (2.2) and following the arguments in [11, Theorem 3] the sufficiency follows.
Necessity is obvious.

REMARK 3.8. Call a QTAG-module M to be decomposable if it is a direct sum of uniserial modules. It follows from the above theorem that any submodule of a decomposable QTAG-module is decomposable. As for TAG-modules [12] we can define a bounded QTAG-module. It immediately follows from (3.7) that any bounded QTAG-module is decomposable. Let M be any QTAG-module. As in [12, p. 185] any submodule N of M is said to be h-pure in M if $H_k(M) \cap N = H_k(N)$ for all k. Similar to [12, Theorem 2], any bounded h-pure submodule of M is a summand of M. In particular if a uniform element $u \in \operatorname{soc}(M)$ has finite height k, and x is any uniform element in M such that $u \in xR$ and $d(xR/uR) = H(u)$, then xR is h-pure in M and hence is a summand of M. Union of a chain of h-pure submodules of M is h-pure in M.

LEMMA 3.9. *Let N be a submodule of a QTAG-module M. Then N is an h-pure submodule of M if and only if for every uniform element $\bar{x} = x + N$ of M/N, there exists a uniform element $x' \in M$ such that $x + N = x' + N$, $e(x') = (\bar{x})$.*

PROOF. Necessity follows as in [12, Lemma 2], using (2.3) instead of condition (II). Conversely let the given condition hold, but N be not h-pure in M. Let k be the smallest positive integer such that $H_k(M) \cap N \neq H_k(N)$. We can find a uniform element $x \in H_k(M) \cap N$ such that $x \notin H_k(N)$. Clearly $x \in H_{k-1}(N)$. Now there exists a uniform element $y \in M$ such that $x \in yR$ and $d(yR/xR) = k$. Then $yR \cap N = xR$. By hypothesis there exists a uniform element $y' \in M$ such that $\bar{y} = y + N = y' + N$ and $e(y') = e(\bar{y})$. Thus $y'R \cap N = 0$ and we have the epimorphism $\sigma : yR \to y'R$ satisfying $\sigma(y) = y'$. As $y - y' \in N$,

$$H_k((y - y')R) \subset H_k(N).$$

We have the epimorphism $\eta\colon yR\to(y-y')R$ given by $\eta(yr)=(y-y')r=yr-\sigma(yr)$. Since $e(y')=d(yR/xR)=k$, we get $\eta(x)=x$ and $H_k((y-y')R)=xR$. Hence $x\in H_k(M)$. This proves the result.

The following is easy to establish.

LEMMA 3.10. *Let N and K be any two submodules of a QTAG-module M. Then*

(i) *If K is h-pure in M and $N\subset K$, then K/N is h-pure in M/N.*

(ii) *If $N\subset K$, such that K/N is h-pure in M/N and N is h-pure in M, then K is h-pure in M.*

For this see also [12, Lemma 2].

After this one can easily see that Lemma 3 and Theorem 4 in [12] hold for QTAG-modules. If there exists a uniform element u of finite height in the socle of a QTAG-module M, as seen in (3.7), M admits a uniserial summand containing u.

This all gives, as in [12, Theorem 4 and 5] the following.

THEOREM 3.11. (a) *If every element in* soc (M) *is of infinite height, then M is a direct sum of serial modules, each of infinite length.*

(b) *Any QTAG-module M admits a uniform summand, which can be chosen to be of finite length in case not all uniform elements in* soc (M) *are of infinite height.*

REMARK 3.12. The concept of a basic submodule of a TAG-module in [13, Definition 2.5] can be verbatim defined for QTAG-modules. (3.8), (3.10) and (3.11) yield that any QTAG-module M admits a basic submodule. An easy adaption of the proof of [1, Lemma 1.1] shows that any complement of $H_k(M)$ is h-pure and hence a summand of M. One can prove every result in [13, Section 2] for QTAG-modules on similar lines. Consequently any QTAG-module M admits a basic submodule, and any two basic submodules of M are isomorphic.

§ 4. Right artinian serial rings

In this section we determine the structure of a ring R such that R_R is a QTAG-module.

LEMMA 4.1. *Let R be any ring such that R_R is a QTAG-module. Then any simple right R-module admits at most one predecessor.*

PROOF. Let S_R be simple. Let K and T be two predecessors of S. There exist two indecomposable idempotents e, f of R such that $eR/eJ\approx K$, $eJ/eJ^2\approx S$, $fR/fJ\approx\approx K$ and $fJ/fJ^2\approx S$. Let $K\neq T$, then we can choose e, f to be orthogonal. But by (2.2), there exists an isomorphism of eR/eJ^2 onto fR/fJ^2. Hence $K\approx T$. This is a contradiction. Hence S has at most one predecessor.

LEMMA 4.2. *Let R be any ring such that R_R is a QTAG-module. Let E be an indecomposable injective R-module. Then either E is uniserial or there exists an indecomposable idempotent $e\in R$ and an integer $k\geqq 1$ such that eR/eJ^2 is not quasi-injective,* $\operatorname{soc}^k(E)$ *is uniserial and eR/eJ^2 embeds in* $E/\operatorname{soc}^{k-1}(E)$.

PROOF. Let E be not uniserial. As R is a right artinian right serial ring, for some simple R-module S, $E=E(S)$, and for some t, $E=\mathrm{soc}^t(E)$. As E is not uniserial, we can find $k<t$ such that $\mathrm{soc}^k(E)$ is uniserial, but $\mathrm{soc}^{k+1}(E)$ is not uniserial. Clearly $k\geqq 1$, and we can find two uniserial submodules A and B of E, each of length $k+1$ such that $A\not\subset B\not\subset A$. Now $A\cap B=\mathrm{soc}^k(E)$. Further

$$K=[A/\mathrm{soc}^{k-1}(E)]\cap[B/\mathrm{soc}^{k-1}(E)]$$

is a simple module. By (4.1) K admits only one predecessor. So

$$A/\mathrm{soc}^k(E)\approx B/\mathrm{soc}^k(E).$$

Consequently there exists an indecomposable idempotent $e\in R$ such that

$$A/\mathrm{soc}^{k-1}(E)\approx eR/eJ^2\approx B/\mathrm{soc}^{k-1}(E).$$

We have an isomorphism σ of $A/\mathrm{soc}^{k-1}(E)$ onto $B/\mathrm{soc}^{k-1}(E)$. Now the injective hull E' of $E/\mathrm{soc}^{k-1}(E)$ is uniform and σ can be extended to an endomorphism of E'. If eR/eJ^2 were quasi injective, we get $\eta\big(A/\mathrm{soc}^{k-1}(E)\big)=A/\mathrm{soc}^{k-1}(E)$. So $A=B$. This is a contradiction. Hence the result follows.

Firstly we give another proof of [14, Theorem 4.1].

THEOREM 4.3. *Let R be any ring such that R_R is a TAG-module. Then R is a generalized uniserial ring.*

PROOF. Now R/J^2 is also a TAG-module. So $\bar{R}=R/J^2$ is a TAG-module as a right $\bar{R}$-module. In a TAG-module every uniserial submodule is quasi-injective. In particular for any indecomposable idempotent $\bar{e}\in\bar{R}$, $\bar{e}\bar{R}$ is quasi-injective. Now $\bar{e}\bar{J}^2=0$. By (4.2) every indecomposable injective $\bar{R}$-module is uniserial. So by [9, Theorem 1], every finitely generated $\bar{R}$-module is a finite direct sum of uniserial modules. Then by [8, Theorem 13], $\bar{R}$ is a generalized uniserial ring. Hence by [10, Theorem 10] R is generalized uniserial.

We now prove a theorem analogous to that of Kupisch for generalized uniserial rings [7].

THEOREM 4.4. *Let R be an indecomposable right serial right artinian ring, over which any simple right R-module admits not more than one predecessor. Let $e_1, e_2, \ldots, e_k$ be a maximal set of non-isomorphic orthogonal indecomposable idempotents of R. Then $e_1, e_2, \ldots, e_k$ can be so arranged that for $i<k$, $e_iJ\neq 0$, and there exists an epimorphism of $e_{i+1}R$ onto e_iJ. Further if $e_kJ\neq 0$, there exists an epimorphism of e_1R onto e_kJ.*

PROOF. Since R is right serial, any simple right R-module admits at most one successor. By hypothesis a simple right R-module admits at most one predecessor. Let $S_1, S_2, \ldots, S_l$ be longest length sequence of non-isomorphic simple R-modules, such that for $i<l$, S_{i+1} is the successor of S_i. Let $l<k$. By renumbering we can take

$$S_i\approx e_iR/e_iJ,\quad 1\leqq i\leqq l.$$

It is clear that if an e_jR has a composition factor among S_i $(1\leqq i\leqq l)$, then all the composition factors of e_jR are among S_i's. So there exists a simple right R-module S which is not isomorphic to any S_i. For some e_j, $S\approx e_jR/e_jJ$. No composition factor of e_jR can be among S_i's. Thus $e_jRe_iR=0=e_iRe_jR$ for $1\leqq i\leqq l$. This in turn gives that R is decomposable; which is a contradiction. Hence $l=k$. Thus for $i<k$, as S_{i+1} is a successor of S_i, $S_{i+1}\approx e_iJ/e_iJ^2\approx e_{i+1}R/e_{i+1}J$. Consequently $e_iJ\neq 0$ and there exists an epimorphism of $e_{i+1}R$ onto e_iJ, for $i<k$. In addition let $e_kJ\neq 0$. Then e_kJ/e_kJ^2 is the successor of S_k and we must have

$$e_k\,J/e_k\,J^2\approx e_1\,R/e_1\,J.$$

This proves the theorem.

THEOREM 4.5. *Let R be any indecomposable ring such that R_R is a QTAG-module and for any indecomposable idempotent $e\in R$, $eJ^2\neq 0$. Then either R is a local ring or R is generalized uniserial.*

PROOF. If R is local, there is nothing to prove. Let R be not local. We can write

$$R=e_1\,R\oplus e_2\,R\oplus\ldots\oplus e_n\,R$$

for some orthogonal indecomposable idempotents e_i, $1\leqq i\leqq n$, such that for some $t\leqq n$, $e_1R, e_2R, \ldots, e_tR$ is the largest set of non-isomorphic indecomposable summands of R_R. Now $n>1$. Let $t=1$ then $e_iR\approx e_1R$ for every i. Now, e_1R is quasi-injective. Consequently R_R being a direct sum of n copies of e_1R, is quasi-injective. Hence R_R is self injective. So R is quasi Frobenius. Then duality between right and left ideals of R, gives R is also left serial. Hence R is a generalized uniserial ring. Let $t>1$. By (4.4) we can take $e_1R, e_2R, \ldots, e_tR$ such that $e_iJ\approx e_{i+1}R/e_{i+1}J^{k(i)}$ for $i<t$ and $e_tJ\approx e_1R/e_1J^{k(t)}$, for some integers $k(i)\geqq 1$. Then $e_iJ/e_iJ^3\approx$ $\approx e_{i+1}R/e_{i+1}J^2$. By the remark following (2.4), $e_{i+1}R/e_{i+1}J^2$ is quasi-injective. Similarly also e_1R/e_1J^2 is quasi-injective. So as in (4.3), using (4.2) we get R/J^2 is generalized uniserial. Hence by [10, Theorem 10], R is generalized uniserial. This proves the result.

In the above theorem, the condition that $eJ^2\neq 0$, for any indecomposable idempotent e of R, is used to show that $\bar{e}\bar{R}$ is quasi-injective for any indecomposable idempotent $\bar{e}$ of $\bar{R}=R/J^2$. In fact we can prove the following.

THEOREM 4.6. *Let R be any non-local indecomposable ring such that R_R is a QTAG-module. Then either R is a generalized uniserial ring or it has an indecomposable idempotent e such that $eJ^2=0$ and eR/eJ^2 is not quasi-injective.*

There exist indecomposable non-local rings R, such that R is not a generalized uniserial ring, but R_R is a QTAG-module.

EXAMPLE. Let D be any division ring admitting a bi-vector space ${}_DV_D$ such that $\dim_D V>1$, and $\dim V_D=1$. Then the matrix ring

$$R=\begin{bmatrix} D & V\\ O & D\end{bmatrix}$$

is indecomposable, non-local, it is not a generalized uniserial ring, but R_R is a QTAG-module. Consider the ring

$$T=\begin{bmatrix} D & V\\ V & D\end{bmatrix}$$

in which the multiplication is defined by

$$\begin{bmatrix} a_{11} & v_{12} \\ v_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & w_{12} \\ w_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}w_{12}+v_{12}b_{22} \\ v_{21}b_{11}+a_{22}w_{21} & a_{22}b_{22} \end{bmatrix}.$$

$\mathrm{Soc}(T_T)$ is a direct sum of two non-isomorphic simple modules. Any right ideal A of T is of the form $B\oplus C$, where B is a summand of T_T and $C\subset \mathrm{soc}(T_T)$. Using this and that $d(T_T)=4$, it can be easily seen that T_T is a QTAG-module, T is indecomposable, non-local and is not generalized uniserial.

Acknowledgement. The author is extremely thankful to the referee for his various suggestions.

References

[1] K. Benabdallah and S. Singh, *On torsion abelian groups like modules,* Lectures Notes in Mathematics, Springer Verlag, 1006 (1983), 639—653.

[2] C. Faith, *Algebra II, Ring Theory,* Die Grundlehren der Math. Wissenchaften 191, Springer Verlag, 1976.

[3] L. Fuchs, *Infinite Abelian Groups,* Vol. I, Pure and Applied Math., Series 36, Academic Press (1970).

[4] L. Fuchs, *Infinite Abelian Groups,* Vol. II, Pure and Applied Math., Series 36-II, Academic Press (1973).

[5] M. Z. Khan, Modules behaving like torsion abelian groups, *Tamkang J. Math.,* **9** (1978), 15—20.

[6] M. Z. Khan, Modules behaving like abelian groups, *Can. Math. Bull.,* **22** (1979), 449—457.

[7] H. Kupisch, Beiträge zur Theorie nichthalbeinfacher Ringe mit Minimalbedingung, *J. Reine Angew. Math.,* **201** (1959), 100—112.

[8] G. Ivanov, Left generalized uniserial rings, *J. Alg.,* **31** (1974), 166—181.

[9] G. Ivanov, Decomposition of modules over serial rings, *Comm. Alg.,* **3** (1975), 1031—1036.

[10] I. Murase, On the structure of generalized uniserial rings I, *Sci. papers Coll. Gen. Edu.,* **13** (1963), 1—22.

[11] S. Singh, Modules over hereditary noetherian prime rings, *Can. J. Math.,* **27** (1975), 867—883.

[12] S. Singh, Some decomposition theorems in abelian groups and their generalizations, In: *Ring Theory,* Proc. Ohio Univ. Conf., Lecture Notes in Pure and Applied Math., Marcel Dekker **25** (1976), 183—189.

[13] S. Singh, Some decomposition theorems in abelian groups and their generalizations II, *Osaka J. Math.,* **16** (1979), 45—55.

[14] S. Singh, Serial right noetherian rings, *Can. J. Math.,* **36** (1984), 22—37.

[15] S. Singh and W. A. Ansari, On Ulm's theorem, *Comm. Alg.,* **10** (1982), 2031—2042.

[16] S. Singh and S. Talwar, Pure submodules of modules over bounded (hnp)-rings, *Arch. Math:,* **30** (1978), 570—577.

[17] B. Stenström, *Rings of Quotients,* Die Grundlehren der Math. Wissenschaften 217, Springer Verlag (1975).

[18] R. B. Warfield Jr., Serial rings and finitely presented modules, *J. Alg.,* **37** (1975), 187—222.

(Received October 17, 1984; Revised April 24, 1985)

DEPARTMENT OF MATHEMATICS
KUWAIT UNIVERSITY
KUWAIT

Acta Math. Hung.
50 (1–2) (1987), 97–109.

PROXIMITIES, SCREENS, MEROTOPIES, UNIFORMITIES. II

Á. CSÁSZÁR (Budapest), member of the Academy

7. Screens and extensions. RE-proximities are defined with the help of extensions of topological spaces. For this purpose, let us say that, for a TP-proximity δ on X, an extension (Y, c') of the topological space (X, c_δ) is *compatible* with δ iff, for $A, B\subset X$,

$$A\delta B \Leftrightarrow c'(A)\cap c'(B) \neq \emptyset.$$

Then we can say that δ is an RE-proximity iff there exists a regular extension compatible with δ.

A similar characterization is possible concerning LO-proximities. For this purpose, let us recall that, if (X, c) is a topological space, $Y\supset X$, $\mathfrak{s}(y)$ is a c-open filter in X for $y\in Y$, in particular, $\mathfrak{s}(x)$ is the c-neighbourhood filter of x if $x\in X$, and

$$s(G) = \{y\in Y:\ G\in\mathfrak{s}(y)\}$$

for a c-open set $G\subset X$, then the sets $s(G)$ constitute an open base for a topology c' on Y; the space (Y, c') is said to be a *strict extension* of (X, c) corresponding to the *trace filters* $\mathfrak{s}(y)$ (in fact, $\mathfrak{s}(y)$ is the trace in X of the c'-neighbourhood filter of $y\in Y$) ([4], (6.1.2)).

An extension (Y, c') of the topological space (X, c) is said to be T_1*-reduced* iff $x, y\in Y$, $x\neq y$, $y\in Y-X$ implies that each of the points x, y has a c'-neighbourhood that does not contain the other. If (Y, c') is a T_1-reduced extension of (X, c) and the latter is an S_1-space or a T_1-space, then the same holds for (Y, c').

An *extension* (Y, c') of a topological space (X, c) will be said to be *free* iff the trace in X of the c'-neighbourhood filter of any point $y\in Y-X$ is free.

(7.1) LEMMA. *If δ is a TP-proximity on X, and (Y, c') is an extension of (X, c_δ) compatible with δ, then the collection of the traces in X of the c'-neighbourhood filters of the points $y\in Y$ is an open screen on X compatible with δ so that δ is an LO-proximity.*

PROOF. The traces in question are c_δ-open filters in X, in particular, the trace of the c'-neighbourhood filter of $x\in X$ is the c_δ-neighbourhood filter of x, hence fixed at x. Therefore we obtain a screen $\mathfrak{S}$ on X the elements of which are c_δ-open filters.

For $A, B\subset X$, $A\delta B$ iff there is some $y\in Y$ whose c'-neighbourhood filter has a trace $\mathfrak{s}$ in X satisfying $A, B\in\sec\mathfrak{s}$. Thus $\delta=\delta(\mathfrak{S})$, and $\mathfrak{S}$ is an open screen. By (6.14) δ is LO. □

Conversely, we can say:

(7.2) LEMMA. *Let* $\mathfrak{S}$ *be an open screen on* X, $\delta=\delta(\mathfrak{S})$. *Then there exists an extension* (Y, c') *of the space* (X, c_δ) *such that the collection of all traces in* X *of the* c'*-neighbourhood filters of the points* $y\in Y$ *coincides with* $\mathfrak{S}$. *Such an extension is compatible with* δ.

PROOF. By (6.14), c_δ is a topology, and the filters $\mathfrak{s}\in\mathfrak{S}$ are c_δ-open, in particular, an $\mathfrak{s}\in\mathfrak{S}$ fixed at $x\in X$ coincides with the c_δ-neighbourhood filter of x. Hence there exist extensions satisfying the hypotheses. By $\delta=\delta(\mathfrak{S})$, $A\delta B$ holds iff there is an $\mathfrak{s}\in\mathfrak{S}$ such that $A, B\in\sec\mathfrak{s}$, hence iff there is $y\in Y$ such that $y\in c'(A)\cap c'(B)$. □

Now we can prove:

(7.3) THEOREM. *For a TP-proximity* δ *on* X, *the following statements are equivalent:*

(a) δ *is an LO-proximity,*
(b) *There is a* T_1*-reduced, compact extension compatible with* δ,
(c) *There is an extension that is an* S_1*-space and is compatible with* δ,
(d) *There is a free extension compatible with* δ,
(e) *There is an extension* (Y, c') *compatible with* δ.

PROOF. (a)⇒(b): Let $\mathfrak{S}_0$ denote the collection of all minimal δ-compressed filters. By (6.13) and (6.16), $\mathfrak{S}_0$ is an open screen such that $\delta=\delta(\mathfrak{S}_0)$. Define, for $x\in X$, $\mathfrak{s}(x)$ to be the c_δ-neighbourhood filter of x, and choose a set $Y\supset X$ such that, to the elements $y\in Y-X$, we make correspond bijectively the free elements of $\mathfrak{S}_0$; let $\mathfrak{s}(y)\in\mathfrak{S}_0$ be the filter corresponding to y. Consider the strict extension (Y, c') of (X, c_δ) obtained from the trace filters $\mathfrak{s}(y)$.

If $x, y\in Y$, $x\neq y$, $y\in Y-X$, then $\mathfrak{s}(x)$ and $\mathfrak{s}(y)$ are distinct elements of $\mathfrak{S}_0$ (cf. (6.14)) so that neither $\mathfrak{s}(x)\subset\mathfrak{s}(y)$ nor $\mathfrak{s}(x)\supset\mathfrak{s}(y)$ can hold. Hence (Y, c') is a T_1-reduced extension of (X, c_δ). By (7.2), (Y, c') is compatible with δ.

We show that (Y, c') is compact. Let $\mathfrak{u}$ be an ultrafilter in Y, $\mathfrak{s}'$ the c'-neighbourhood filter of $\mathfrak{u}$, and $\mathfrak{s}=\mathfrak{s}'|X$. Then $\mathfrak{s}$ is δ-compressed. In fact, if $A\bar{\delta}B$, then $c'(A)\cap c'(B)=\emptyset$, hence either $Y-c'(A)$ or $Y-c'(B)$ belongs to $\mathfrak{u}$ and to $\mathfrak{s}'$, consequently either $X-A\supset X-c_\delta(A)$ or $X-B\supset X-c_\delta(B)$ belongs to $\mathfrak{s}$. By (6.12) there is a point $y\in Y$ satisfying $\mathfrak{s}(y)\subset\mathfrak{s}$. If G is c_δ-open and $y\in\mathfrak{s}(G)$, then $G\in\mathfrak{s}(y)\subset\mathfrak{s}$, hence there is a c'-open set G' such that $G'\in\mathfrak{s}'$ and $G'\cap X\subset G$. Clearly $G'\subset s(G)$ and $G'\in\mathfrak{u}$, hence $s(G)\in\mathfrak{u}$, and $\mathfrak{u}\to y$.

(b)⇒(c) is obvious because (X, c_δ) is an S_1-space.

(b)⇒(d) follows from the fact that a T_1-reduced extension is free.

(c)⇒(e) and (d)⇒(e): obvious.

(e)⇒(a): (7.1). □

The implication (a)⇒(b) is contained in [22] for separated proximities, while (a)⇒(c) and (a)⇒(d), for separated proximities and so for T_1-extensions again, are contained in [17] and [18]. The technique of the proof is, however, essentially different from ours; bunches, clusters, and clans are considered instead of filters.

The implications (c)⇒(a) and (d)⇒(a) can be formulated in a slightly more general manner:

(7.4) LEMMA. *Let* (X, c) *be an* S_1*-space,* (Y, c') *an extension of* (X, c) *that is either* S_1 *or free, and define, for* $A, B\subset X$,

$$A\delta B \text{ iff } c'(A)\cap c'(B)\neq\emptyset.$$

Then δ *is an LO-priximity compatible with* c.

PROOF. It is easy to check that δ is a proximity on X. If $x\in X$, $A\subset X$, $x\in c(A)$, then $c'(\{x\})\cap c'(A)\neq\emptyset$. If $x\notin c(A)$ then $x\notin c'(A)$ and, if (Y, c') is S_1, also $c'(\{x\})\cap\cap c'(A)=\emptyset$; the same is true if (Y, c') is free because then $c'(\{x\})=c(\{x\})$ and $c(\{x\})\cap c(A)=\emptyset$. Hence $c=c_\delta$, and (7.1) applies by (4.1). □

Our next purpose is to examine, for LO-proximities, an extension somewhat similar to the one constructed in the proof of (a)⇒(b) in (7.3).

(7.5) LEMMA. *For any LO-proximity* δ *on* X, *let* $\mathfrak{S}_1$ *denote the collection of all* δ*-open,* δ*-compressed filters. Then* $\mathfrak{S}_1$ *is a screen such that* $\delta=\delta(\mathfrak{S}_1)$.

PROOF. If $\mathfrak{S}$ and $\mathfrak{S}_0$ denote the sets of all δ-compressed and all minimal δ-compressed filters, respectively, then $\mathfrak{S}_0\subset\mathfrak{S}_1\subset\mathfrak{S}$ by (6.16). From (6.11) and (6.13), it follows easily that $\mathfrak{S}_1$ is a screen and $\delta(\mathfrak{S}_0)=\delta(\mathfrak{S}_1)=\delta(\mathfrak{S})=\delta$. □

(7.6.) LEMMA. *Let* δ *be an LO-proximity on* X, *and* (Y_1, c_1') *be a strict extension of* (X, c_δ) *such that the collection of the traces in* X *of the* c_1'*-neighbourhood filters of the points* $y\in Y_1-X$ *coincides with the collection of all free elements of* $\mathfrak{S}_1$, *the set of all* δ*-open,* δ*-compressed filters. Then* (Y_1, c_1') *is a compact, free extension compatible with* δ.

PROOF. By (7.5) and (7.2) (Y_1, c_1') is an extension compatible with δ. Clearly it is a free extension. It is compact, too. In fact, we can argue in the same way as in the proof of (a)⇒(b) in (7.3), replacing (Y, c') by (Y_1, c_1'), and with the modification that $\mathfrak{s}$ is a δ-open, δ-compressed filter, hence $\mathfrak{s}=\mathfrak{s}(y)$ for some $y\in Y_1$. □

The extension described in (7.6) plays a role in a theorem on extensions of maps. Let us first observe:

(7.7) LEMMA. *Let* Z, X *be topological spaces,* Z_1 *an extension of* Z, Y_1 *a strict extension of* X, $f: Z\to X$ *continuous, and suppose that, for* $z\in Z_1-Z$ *and the neighbourhood filter* $\mathfrak{v}(z)$ *in* Z_1 *of* z, *the neighbourhood filter in* X *of* $f(\mathfrak{v}(z)|Z)$ *coincides with* $\mathfrak{v}'(y)|X$ *where* $\mathfrak{v}'(y)$ *is the neighbourhood filter of some point* $y\in Y_1$. *Then there exists a continuous extension* $g: Z_1\to Y_1$ *of* f.

PROOF. The condition assumed for $z\in Z_1-Z$ is fulfilled for $z\in Z$ as well; in fact, by the continuity of f, $f(\mathfrak{v}(z)|Z)\to f(z)$ so that the neighbourhood filter in X of $f(\mathfrak{v}(z)|Z)$ coincides with $\mathfrak{v}'(f(z))|X$. Define $g(z)=y$ if the neighbourhood filter of $f(\mathfrak{v}(z)|Z)$ is $\mathfrak{v}'(y)|X$, $y\in Y_1$, in particular $y=f(z)$ for $z\in Z$. Then $g: Z_1\to Y_1$ is an extension of f.

g is continuous at any point $z_0\in Z_1$. In fact, let $G_1\subset X$ be open in X, $g(z_0)\in\in s_1(G_1)=\{y\in Y_1: G_1\in\mathfrak{v}'(y)|X\}$. Then $s_1(G_1)\cap X=G_1$ contains a subset of the form $f(G)$ where $G=U\cap Z$ and U is an open neighbourhood of z_0 in Z_1. Hence $z\in U$ implies $G\in\mathfrak{v}(z)|Z$, $G_1\in\mathfrak{v}'(g(z))|X$, $g(z)\in s_1(G_1)$. Since the sets $s_1(G_1)$ constitute a neighbourhood base of $g(z_0)$, g is continuous at z_0. □

A statement similar to the following one can be found in [9], (3.7):

(7.8) THEOREM. *Let δ_1, δ be LO-proximities on Z and X, respectively, (Z_1, c_1) an extension compatible with δ_1, (Y_1, c_1') the extension described in* (7.6), *and $f\colon Z\to X$ (δ_1, δ)-continuous. Then there exists a (c_1, c_1')-continuous extension of f.*

PROOF. For $z\in Z_1-Z$, the trace $\mathfrak{s}$ in Z of the c_1-neighbourhood filter of z is δ_1-compressed by (7.1) and (6.9). Hence $f(\mathfrak{s})$ is δ-compressed, and the c_δ-neighbourhood filter of $f(\mathfrak{s})$ is still δ-compressed by (6.15); it is also c_δ-open, hence it coincides with the trace in X of the c_1'-neighbourhood filter of some $y\in Y_1$. Thus (7.7) applies. □

We add some easy remarks on separation properties of compatible extensions. Let us say that a screen $\mathfrak{S}$ is *independent* if $\mathfrak{s}_1, \mathfrak{s}_2\in\mathfrak{S}$, $\mathfrak{s}_1\subset\mathfrak{s}_2$ implies $\mathfrak{s}_1=\mathfrak{s}_2$. A topology is S_1 iff the neighbourhood filters constitute an independent screen.

(7.9) LEMMA. *Let (Y, c') be a strict extension of the topological space (X, c). Then the traces in X of the c'-neighbourhood filters constitute an independent screen on X iff c' is an S_1-topology.*

PROOF. For $p\in Y$, let $\mathfrak{v}(p)$ denote the c'-neighbourhood filter of p. For $p, q\in Y$, $\mathfrak{v}(p)|X\subset\mathfrak{v}(q)|X$ iff $\mathfrak{v}(p)\subset\mathfrak{v}(q)$ because (Y, c') is a strict extension. □

Let us say that a screen $\mathfrak{S}$ on X is *disjoint* iff $\mathfrak{s}_1, \mathfrak{s}_2\in\mathfrak{S}$, $\mathfrak{s}_1\neq\mathfrak{s}_2$ implies the existence of $S_i\in\mathfrak{s}_i$ such that $S_1\cap S_2=\emptyset$. A topology is S_2 iff the neighbourhood filters constitute a disjoint screen.

(7.10) LEMMA. *Let (Y, c') be a strict extension of the topological space (X, c). Then the traces of the c'-neighbourhood filters constitute a disjoint screen iff c' is an S_2-topology.*

PROOF. Since the extension is strict, with the above notations, $\mathfrak{v}(p)$ and $\mathfrak{v}(q)$ are distinct iff $\mathfrak{v}(p)|X\neq\mathfrak{v}(q)|X$, and they contain disjoint elements iff $\mathfrak{v}(p)|X$ and $\mathfrak{v}(q)|X$ do so. □

In spite of the fact that, for strict extensions, the properties of the extension and those of the corresponding screen are strongly related, it can happen that a topological space (X, c) has two strict extensions (Y_i, c_i) $(i=1, 2)$, for the screens $\mathfrak{S}_i$ composed of the traces in X of the c_i-neighbourhood filters, $\mathfrak{S}_1$ is coarser than $\mathfrak{S}_2$, but (Y_1, c_1) is not coarser as an extension than (Y_2, c_2) (i.e. id_X does not possess a (c_2, c_1)-continuous extension $g\colon Y_2\to Y_1$).

(7.11) EXAMPLE. Let $X=\mathbf{R}-\{0\}$ be equipped with the subspace topology c of the usual topology of $\mathbf{R}$, $Y_1=\mathbf{R}$, $Y_2=\mathbf{R}\cup M$ where $\mathbf{R}\cap M=\emptyset$, $M=\{p_n\colon n\in\mathbf{N}\}$, $\mathbf{N}=\{1, 2, \dots\}$. Let $\mathfrak{S}_1$ be composed of the c-neighbourhood filters and a filter $\mathfrak{s}_0$ generated by the filter base $\mathfrak{r}=\{U_\varepsilon\colon \varepsilon>0\}$, $U_\varepsilon=(-\varepsilon, \varepsilon)-(\{0\}\cup N)$, $N=\left\{\frac{1}{n}\colon n\in\mathbf{N}\right\}$. Let $\mathfrak{s}_0$ be the trace of the c_1-neighbourhood filter of $0\in Y_1$. Let $\mathfrak{S}_2$ be composed of the c-neighbourhood filters, of $\mathfrak{s}_0$, and of the filters $\mathfrak{s}_n$ where $\mathfrak{s}_n$ is generated by the filter base $\mathfrak{r}_n$ composed of the sets $V-\left\{\frac{1}{n}\right\}$ where V is a c-neighbourhood of $\frac{1}{n}$.

Let $\mathfrak{s}_0$ be the trace in X of the c_2-neighbourhood filter of $0\in Y_2$, $\mathfrak{s}_n$ be the similar trace for $p_n\in Y_2$. Both (Y_1, c_1) and (Y_2, c_2) are supposed to be strict extensions of (X, c). Clearly, $\mathfrak{S}_1$ is coarser than $\mathfrak{S}_2$ (more precisely, $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are equivalent screens).

Now suppose that id_X has a (c_2, c_1)-continuous extension $f: Y_2\to Y_1$. Since $f(\mathfrak{s}_0)\to 0$, $f(\mathfrak{s}_n)\to\frac{1}{n}$ and Y_1 is a T_2-space, necessarily $f(0)=0$, $f(p_n)=\frac{1}{n}$. However, f cannot be continuous at $0\in Y_2$; in fact, $U_\varepsilon\in\mathfrak{s}_n$ if $\frac{1}{n}<\varepsilon$, hence, for these n, $p_n\in\{y\in Y_2: U_\varepsilon\in\mathfrak{v}_2(y)|X\}$, where $\mathfrak{v}_2(y)$ is the c_2-neighbourhood filter of y, but $f(p_n)$ does not belong to the c_1-neighbourhood $U_1\cup\{0\}$ of $0\in Y_1$. □

8. Subcategories of Scr. We have already defined open, independent, disjoint, and round screens. The following definition, too, is plausible: a screen $\mathfrak{S}$ on X is said to be *minimal* iff every filter $\mathfrak{s}\in\mathfrak{S}$ is minimal $\delta(\mathfrak{S})$-compressed (see (6.9)).

(8.1) LEMMA. *If δ is a proximity on X, then a δ-round, δ-compressed filter is minimal δ-compressed. Hence a round screen is minimal.*

PROOF. Let $\mathfrak{S}$ be a δ-round, δ-compressed filter, $\mathfrak{s}'\subset\mathfrak{s}$ a δ-compressed filter. For $S\in\mathfrak{s}$, there is $S_1\in\mathfrak{S}$ such that $S_1\bar{\delta}X-S$. Since $S_1\in\sec\mathfrak{s}'$, necessarily $S\in\mathfrak{s}'$, and $\mathfrak{s}'=\mathfrak{s}$. □

The converse of the first statement in (8.1) is true if δ is an EF-proximity: then, for an arbitrary filter $\mathfrak{s}$ is X, the sets $S'\subset X$ such that $S\bar{\delta}X-S'$ for some $S\in\mathfrak{s}$ constitute a δ-round filter ([4], (6.3.9)) coarser than $\mathfrak{s}$, and $\mathfrak{s}'$ is δ-compressed provided so is $\mathfrak{s}$ ([4], (6.3.10)), hence $\mathfrak{s}'=\mathfrak{s}$ if $\mathfrak{s}$ is minimal δ-compressed.

This is certainly not true if δ is not an RE-proximity. In fact, by (6.13), the screen $\mathfrak{S}_0$ composed of all minimal δ-compressed filters is compatible with δ while δ does not admit any compatible screen composed of δ-round filters ([5], (5.12)).

There are RE-proximities, too, such that a minimal δ-compressed filter need not be δ-round:

(8.2) EXAMPLE. Let $X=\omega+1$, $Y=\omega_1+1$, equipped with the order topology, $Z=X\times Y$, $T=Z-\{(\omega, \omega_1)\}$ be the Tikhonov plank with the subspace topology c, and, for $A, B\subset T$, $A\delta B$ iff $c(A)\cap c(B)=\emptyset$. Then δ is an RE-proximity because T is regular. Let $\mathfrak{s}$ denote the filter in T generated by the filter base $\mathfrak{r}=\{R(n,\alpha): n\in\omega,\ \alpha\in\omega_1\}$, where

$$R(n,\alpha)=\{(m,\beta):\ n<m<\omega,\ \alpha<\beta<\omega_1\}.$$

The filter $\mathfrak{s}$ is δ-compressed. In fact, if $A\bar{\delta}B$, then $c(A)\cap c(B)=\emptyset$, hence there is $\alpha\in\omega_1$ such that, say, $(\omega,\beta)\notin c(A)$ for $\alpha\leqq\beta<\omega_1$. For every such β, there are $m_\beta\in\omega$ and an open subset V_β of Y such that $(n,\gamma)\notin A$ for $m_\beta<n<\omega$, $\gamma\in V_\beta$. Define $U_m=\bigcup\{V_\beta: m_\beta=m\}$. Since $\{\beta\in\omega_1: \alpha\leqq\beta<\omega_1\}$ is countably compact, finitely many sets U_{m_i} cover it, and $R(n,\alpha)\cap A=\emptyset$ if $n\geqq m_i$ for every i.

By (6.12) there is a minimal δ-compressed filter $\mathfrak{s}_0\subset\mathfrak{s}$. $\mathfrak{s}_0$ cannot be δ-round. In fact, by [5], (8.1), (8.4), and (3.9), the δ-round, δ-compressed filters coincide with the c-neighbourhood filters, and $\mathfrak{s}$ does not converge to any point of T. □

It is easy to show now that, if δ is an LO-proximity on X, then a δ-open, δ-compressed filter need not be minimal δ-compressed (i.e. the converse of (6.16) is not true): let $X=\mathbf{R}$, $\delta=\delta_c$ for the usual topology c of $\mathbf{R}$, $G=(0,1)\subset X$, $\mathfrak{r}=\{G-F\colon F\subset G,\ F=c(F)\}$. Then $\mathfrak{r}$ is a filter base; let $\mathfrak{u}$ be an ultrafilter finer than $\mathfrak{r}$, and $\mathfrak{s}$ the c-neighbourhood filter of $\mathfrak{u}$. Clearly $\mathfrak{u}$ is δ-compressed, by (6.15) so is $\mathfrak{s}$, moreover, $\mathfrak{s}$ is c_δ-open. However, $\mathfrak{s}$ is not δ-round, because $G\in\mathfrak{s}$, and $S\bar{\delta}X-G$ implies $c(S)\subset G$, hence $G-c(S)\in\mathfrak{s}$, $S\notin\mathfrak{s}$. Since δ is an EF-proximity, $\mathfrak{s}$ is not minimal δ-compressed.

In order to introduce a further type of screens, let us say that a screen $\mathfrak{S}$ on X is *ascending* iff $\mathfrak{s}\in\mathfrak{S}$ implies $\mathfrak{s}'\in\mathfrak{S}$ for any filter $\mathfrak{s}'$ in X such that $\mathfrak{s}\subset\mathfrak{s}'$.

(8.3) LEMMA. *If $\mathfrak{S}$ is an arbitrary screen on X, then the collection of all filters $\mathfrak{s}'$ in X that contain some $\mathfrak{s}\in\mathfrak{S}$, is an ascending screen $\mathfrak{S}^a$ on X equivalent to $\mathfrak{S}$. $\mathfrak{S}$ is ascending iff $\mathfrak{S}=\mathfrak{S}^a$. A screen $\mathfrak{S}'$ is finer than $\mathfrak{S}$ iff $\mathfrak{S}'\subset\mathfrak{S}^a$, $\mathfrak{S}$ and $\mathfrak{S}'$ are equivalent iff $\mathfrak{S}^a=\mathfrak{S}'^a$.*

PROOF. Obviously $\mathfrak{S}\subset\mathfrak{S}^a$, hence $\mathfrak{S}^a$ is a screen on X, and it is ascending by definition. By $\mathfrak{S}\subset\mathfrak{S}^a$, $\mathfrak{S}$ is finer than $\mathfrak{S}^a$, and clearly $\mathfrak{S}^a$ is finer than $\mathfrak{S}$. The remaining part is obvious. □

Let us denote by **Oscr, Iscr, Dscr, Rscr, Mscr, Ascr** the full subcategory of **Scr** the objects of which are the open, independent, disjoint, round, minimal, ascending screens, respectively. Since some of these types of screens are defined with the help of properties of the induced proximity, it is useful to study the behaviour of the functor $F(\mathfrak{S})=\delta(\mathfrak{S})$ with respect to the operations g^{-1} and sup.

(8.4) LEMMA. *If $\mathfrak{S}'$ is a screen on X, $g\colon Z\to X$, $\mathfrak{S}=g^{-1}(\mathfrak{S}')$, $\delta'=\delta(\mathfrak{S}')$, $\delta=\delta(\mathfrak{S})$, then $\delta=g^{-1}(\delta')$, consequently $c_\delta=g^{-1}(c_{\delta'})$.*

PROOF. For $A,B\subset Z$, $A\delta B$ holds iff $A,B\in\sec g^{-1}(\mathfrak{s}')$ for some $\mathfrak{s}'\in\mathfrak{S}'$ ((6.7), (6.2)), hence iff $g(A)$, $g(B)\in\sec\mathfrak{s}'$ for some $\mathfrak{s}'\in\mathfrak{S}'$, i.e. iff $Ag^{-1}(\delta')B$ by (3.4). The second part follows from (3.10). □

Unfortunately, if $\mathfrak{S}_i$ is a screen on X for $i\in I$, $\mathfrak{S}=\sup\{\mathfrak{S}_i\colon i\in I\}$, then, in general,

$$\delta(\mathfrak{S})\neq\sup_{\mathbf{Prox}}\{\delta(\mathfrak{S}_i)\colon i\in I\}.$$

(8.5) EXAMPLE. Let $X=Y=\mathbf{R}$, $Z=X\times Y$, let p_1 and p_2 denote the projections from Z onto X and Y, respectively, let $\mathfrak{S}'$ be the screen on $X=Y$ composed of all neighbourhood filters with respect to the usual topology c of $\mathbf{R}$, $\mathfrak{S}_k=p_k^{-1}(\mathfrak{S}')$, $\mathfrak{S}=\sup\{\mathfrak{S}_1,\mathfrak{S}_2\}$. Clearly $P\delta(\mathfrak{S}')Q$ iff $c(P)\cap c(Q)\neq\emptyset$.

Now let

$$A=\{(x,x)\colon x\in\mathbf{R}\},\quad B=\{(x,y)\colon x,y\in\mathbf{R},\ |x-y|\geqq 1\}.$$

Then $A\bar{\delta}B$ for $\delta=\delta(\mathfrak{S})$. In fact, $A\delta B$ would mean, by (6.7), (6.2), (6.4), that there are $\mathfrak{v}(x_1)$ and $\mathfrak{v}(x_2)$ (where $\mathfrak{v}(x)$ denotes the c-neighbourhood filter of $x\in\mathbf{R}$) such that every finite intersection of the elements of $p_1^{-1}(\mathfrak{v}(x_1))\cup p_2^{-1}(\mathfrak{v}(x_2))$ meets both A and B. However, this would imply that every neighbourhood of (x_1,x_2) with respect to the Euclidean plane topology of Z meets both A and B, a contradiction.

At the same time, $A\delta' B$ for $\delta'=\sup_{\mathrm{Prox}}\{\delta_1, \delta_2\}$, $\delta_k=\delta(\mathfrak{S}_k)$. In fact, if $A=\bigcup_1^m A_i$, $B=\bigcup_1^n B_j$, then at least one of the sets A_i is unbounded; choose $z_1, z_2\in A_i$, $|x_1-x_2|>1$ for $z_k=(x_k, x_k)$. Then $(x_1, x_2)\in B$, say, $(x_1, x_2)\in B_j$, so that $A_i\delta_k B_j$ for $k=1, 2$; indeed, we have $A_i, B_j\in \sec p_k^{-1}(\mathfrak{v}(x_k))$. □

The validity of a weaker statement is the subject of the following problem: let $\mathfrak{S}_i$ be a screen on X for $i\in I$, $\mathfrak{S}=\sup\{\mathfrak{S}_i: i\in I\}$, $\delta_i=\delta(\mathfrak{S}_i)$, $\delta=\delta(\mathfrak{S})$, $c_i=c_{\delta_i}$, $c=c_\delta$, and $c'=\sup_{\mathrm{Cl}}\{c_i: i\in I\}$; is it always true that $c'=c$? The following example shows that the answer is negative in general.

(8.6) Example. Let $X=\mathbf{R}^2$, q be the point $(0, 0)\in X$, and let K denote the circle with radius 1 and centre q, i.e. $K=\{x\in X: |x|=1\}$ where we use vectorial notation. For $x\in X$, $u\in K$, $\varepsilon>0$, let

$$V(x, u, \varepsilon) = \{x+ru:\ 0 \leqq r < \varepsilon\},$$

and let $\mathfrak{s}(x, u)$ denote the filter in X generated by the filter base $\{V(x, u, \varepsilon): \varepsilon>0\}$. Define $\mathfrak{S}'$ to be the screen on X composed of the filters $\mathfrak{s}(x, u)$ $(x\in X,\ u\in K)$.

Now consider $Y=X\times X$ with the projections p_1 and p_2, and define $\mathfrak{S}_k=$ $=p_k^{-1}(\mathfrak{S}')$ for $k=1, 2$, $\mathfrak{S}=\sup\{\mathfrak{S}_1, \mathfrak{S}_2\}$, $\delta_k=\delta(\mathfrak{S}_k)$, $\delta=\delta(\mathfrak{S})$, $c_k=c_{\delta_k}$, $c=c_\delta$, $c'=\sup_{\mathrm{Cl}}\{c_1, c_2\}$. We show $c'\neq c$.

Define, for this purpose,

$$A = \{(q+|u-v|u,\ q+|u-v|v)\in Y:\ \ u, v\in K,\ u\neq v\}.$$

Then $(q, q)\notin c(A)$. In fact, a filter $\mathfrak{s}\in\mathfrak{S}$ is generated by the finite intersections of the elements of $\mathfrak{s}_1\cup\mathfrak{s}_2$ where $\mathfrak{s}_k\in\mathfrak{S}_k$. If $(q, q)\in c(A)$ were true, then there would exist $u, v\in K$ such that

$$p_1^{-1}(V(q, u, \varepsilon))\cap p_2^{-1}(V(q, v, \varepsilon))\cap A \neq \emptyset$$

for every $\varepsilon>0$. However, this is not true if $u=v$ or $\varepsilon<|u-v|$.

On the other hand, $(q, q)\in c'(A)$. In order to see this, we have to show that, whenever $A=\bigcup_1^n A_i$, $n\in\mathbf{N}$, there is an i such that $(q, q)\in c_1(A_i)\cap c_2(A_i)$, i.e. (cf. (8.4)) there are $u, v\in K$ such that

$$p_1(A_i)\in \sec\mathfrak{s}(q, u), \quad p_2(A_i)\in\sec\mathfrak{s}(q, v).$$

Assume this is not true for some representation $A=\bigcup_1^n A_i$. Then, for every i, either

$$p_1(A_i)\notin \sec\mathfrak{s}(q, u) \quad \text{for each} \quad u\in K,$$

or

$$p_2(A_i)\notin \sec\mathfrak{s}(q, v) \quad \text{for each} \quad v\in K.$$

Let B and C denote the union of the sets A_i of the first and second type, respectively. Then $A=B\cup C$, and

$$p_1(B)\notin \sec\mathfrak{s}(q, u), \quad p_2(C)\notin \sec\mathfrak{s}(q, v)$$

for $u, v\in K$. Hence there is a positive valued function $f: K\to(0, 1]$ such that

$$q+ru\notin p_1(B) \quad \text{for} \quad 0\leqq r<f(u),$$

$$q+rv\notin p_2(C) \quad \text{for} \quad 0\leqq r<f(v).$$

Now if $u, v\in K$, $0<r=|u-v|$, and $f(u)>r$, $f(v)>r$, then $(q+ru, q+rv)\in A$, but this point can belong neither to B nor to C: a contradiction. Hence, for a given $u\in K$, $v\in K$ and $|v-u|<f(u)$ imply $f(v)\leqq|v-u|$ i.e. $\lim\limits_{v\to u} f(v)=0$. From the compactness of K, we easily see that $f(u)=0$ with an exception of countably many values of u, a contradiction again. □

A positive result can be obtained by restricting the class of the screens involved:

(8.7) LEMMA. *Let* $\mathfrak{S}_i$ *be a screen on* X *for* $i\in I$ *such that, for each* $x\in X$, *there is a single filter* $\mathfrak{s}\in\mathfrak{S}_i$ *fixed at* x, $\mathfrak{S}=\sup\{\mathfrak{S}_i: i\in I\}$, $\delta_i=\delta(\mathfrak{S}_i)$, $\delta=\delta(\mathfrak{S})$, $c_i=c_{\delta_i}$, $c=c_\delta$, $c'=\sup_{\mathrm{Cl}}\{c_i: i\in I\}$. *Then* $c'=c$.

PROOF. The case $I=\emptyset$ is obvious. Assume $I\neq\emptyset$. Since $\mathfrak{S}_i$ is coarser than $\mathfrak{S}$ for every $i\in I$, δ_i is coarser than δ, c_i is coarser than c, and c' is coarser than c as well ((6.8), (3.13)). Hence $x\in c(A)$ implies $x\in c'(A)$.

Suppose $x\notin c(A)$. Then a filter $\mathfrak{s}\in\mathfrak{S}$ fixed at x satisfies $A\notin\sec\mathfrak{s}$. Now it is clear that there is a single filter $\mathfrak{s}\in\mathfrak{S}$ fixed at x, namely the one generated by the finite intersections of the elements of $\bigcup\limits_{i\in I}\mathfrak{s}_i$ where $\mathfrak{s}_i\in\mathfrak{S}_i$ is the unique element of $\mathfrak{S}_i$ fixed at x. Hence $A\cap\bigcap\limits_1^n S_k=\emptyset$ for suitable sets $S_k\in\mathfrak{s}_{i_k}$, $i_k\in I$. Define $A_k=A-S_k$; then $A=\bigcup\limits_1^n A_k$, and $x\notin c_{i_k}(A_k)$ because $\mathfrak{s}_{i_k}$ is the unique element of $\mathfrak{S}_{i_k}$ fixed at x. Thus $x\notin c'(A)$. □

Let us say that a screen $\mathfrak{S}$ on X is *unipunctual* iff, for $x\in X$, there is a single filter $\mathfrak{s}\in\mathfrak{S}$ fixed at x.

In order to obtain a positive result concerning the problem discussed in (8.5), we need another class of screens; observe that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are unipunctual in (8.5). We say that a screen $\mathfrak{S}$ on X is *saturated* iff it is composed of all $\delta(\mathfrak{S})$-compressed filters; such a filter is ascending.

For ascending, in particular, for saturated screens, the construction of sup is very simple:

(8.8) LEMMA. *If* $\mathfrak{S}_i$ *is an ascending screen on* X *for* $i\in I\neq\emptyset$, *then*

$$\sup\{\mathfrak{S}_i\colon\ i\in I\}=\bigcap_{i\in I}\mathfrak{S}_i.$$

PROOF. Denote by $\mathfrak{S}$ the sup at the left hand side. Then $\mathfrak{s}\in\mathfrak{S}$ implies, for $i\in I$, the existence of $\mathfrak{s}_i\in\mathfrak{S}_i$ such that $\mathfrak{s}_i\subset\mathfrak{s}$, so that $\mathfrak{s}\in\mathfrak{S}_i$ for every i. Conversely, $\mathfrak{s}\in\mathfrak{S}_i$ for each i implies that $\mathfrak{s}$ is generated by the finite intersections of $\bigcup\limits_{i\in I}\mathfrak{s}_i$ provided $\mathfrak{s}_i=\mathfrak{s}$ for every i. □

(8.9) COROLLARY. *If* $\mathfrak{S}_i$ *is an ascending screen on* X *for* $i\in I\neq\emptyset$, *then* $\sup\{\mathfrak{S}_i\colon i\in I\}$ *is ascending.* □

(8.10) LEMMA. *Let* $\mathfrak{S}_i$ *be a saturated screen on* X *for* $i\in I\neq\emptyset$, $\mathfrak{S}=\sup\{\mathfrak{S}_i: i\in I\}$, $\delta_i=\delta(\mathfrak{S}_i)$, $\delta=\delta(\mathfrak{S})$. *Then* $\delta=\sup_{\mathbf{Prox}}\{\delta_i: i\in I\}$, *and* $\mathfrak{S}$ *is saturated as well.*

PROOF. Denoting by δ' the sup at the right hand side, clearly δ' is coarser than δ by (6.8). A δ-compressed filter is δ_i-compressed for every i, hence it belongs to $\mathfrak{S}_i$ by hypothesis, and also to $\mathfrak{S}$ by (8.8). Hence $\mathfrak{S}$ is saturated.

A δ-compressed filter is δ'-compressed. Conversely, a δ'-compressed filter is δ_i-compressed for every i, hence it belongs to $\mathfrak{S}$ and it is δ-compressed. Therefore, δ-compressed and δ'-compressed filters coincide. By (6.11), $\delta=\delta'$. □

Let us denote by **Uscr** and **Sscr** the full subcategories of **Scr** the objects of which are all unipunctual and saturated screens, respectively.

(8.11) LEMMA. *If* $\mathfrak{S}'$ *is an open, or disjoint, or round, or unipunctual screen on* X, $g: Z\to X$, *then so is* $\mathfrak{S}=g^{-1}(\mathfrak{S}')$.

PROOF. For $\mathfrak{s}'\in\mathfrak{S}'$, let $\mathfrak{s}$ be the filter in Z generated by $g^{-1}(\mathfrak{s}')$ (provided the latter is a filter base).

If $\mathfrak{s}'$ is $\delta(\mathfrak{S}')$-open, $S\in\mathfrak{s}$, then $S\supset g^{-1}(S')$ for some $S'\in\mathfrak{s}'$, and $S'\supset S_1'\in\mathfrak{s}'$ for some $\delta(\mathfrak{S}')$-open S_1'. By (8.4) $g^{-1}(S_1')$ is $\delta(\mathfrak{S})$-open, $S\supset g^{-1}(S_1')\in g^{-1}(\mathfrak{s}')$.

If $\mathfrak{s}_i$ is generated by $g^{-1}(\mathfrak{s}_i')$, $\mathfrak{s}_i'\in\mathfrak{S}'$ for $i=1, 2$, and $\mathfrak{s}_1\neq\mathfrak{s}_2$, then $\mathfrak{s}_1'\neq\mathfrak{s}_2'$. If $\mathfrak{S}'$ is disjoint, there are $S_i'\in\mathfrak{s}_i'$ such that $S_1'\cap S_2'=\emptyset$. Clearly $g^{-1}(S_1')\cap g^{-1}(S_2')=$ $=\emptyset$ so that $\mathfrak{S}$ is disjoint.

If $\mathfrak{s}'$ is $\delta(\mathfrak{S}')$-round, $S\in\mathfrak{s}$, then $S\supset g^{-1}(S')$ for some $S'\in\mathfrak{s}'$, and $S_1'\overline{\delta(\mathfrak{S}')}X-S'$ for some $S_1'\in\mathfrak{s}'$. By (8.4) again, $g^{-1}(S_1')\overline{\delta(\mathfrak{S})}Z-g^{-1}(S')\supset Z-S$ so that $\mathfrak{s}$ is $\delta(\mathfrak{S})$-round.

If $\mathfrak{s}$ is fixed at $z\in Z$, then clearly $\mathfrak{s}'$ is fixed at $g(z)\in X$, hence both $\mathfrak{s}'$ and $\mathfrak{s}$ are uniquely determined. □

(8.12) LEMMA. *If every* $\mathfrak{S}_i$ *is an open, or disjoint, or round, or unipunctual screen on* X *for* $i\in I$, *then so is* $\mathfrak{S}=\sup\{\mathfrak{S}_i: i\in I\}$.

PROOF. The case $I=\emptyset$ is obvious. Assume $I\neq\emptyset$.

If $\mathfrak{s}\in\mathfrak{S}$, let $\mathfrak{s}_i\in\mathfrak{S}_i$ be chosen such that the finite intersections of the elements of $\bigcup_{i\in I}\mathfrak{s}_i$ generate $\mathfrak{s}$.

If each $\mathfrak{S}_i$ is open, $S\in\mathfrak{s}$, $S\supset\bigcap_1^n S_k$, $S_k\in\mathfrak{s}_{i_k}$, $i_k\in I$, then we can choose $\delta(\mathfrak{S}_{i_k})$-open sets $S_k'\in\mathfrak{s}_{i_k}$ such that $S_k\supset S_k'$. Clearly S_k' is $\delta(\mathfrak{S})$-open since $\delta(\mathfrak{S}_{i_k})$ is coarser than $\delta(\mathfrak{S})$, hence $\bigcap_1^n S_k'\subset S$ is $\delta(\mathfrak{S})$-open as well and it belongs to $\mathfrak{s}$.

If each $\mathfrak{S}_i$ is disjoint, let $\mathfrak{s}'$ be generated by the finite intersections of $\bigcup_{i\in I}\mathfrak{s}_i'$, $\mathfrak{s}_i'\in\mathfrak{S}_i$, and $\mathfrak{s}\neq\mathfrak{s}'$. Then $\mathfrak{s}_i\neq\mathfrak{s}_i'$ for at least one i, hence there are $S\in\mathfrak{s}_i$, $S'\in\mathfrak{s}_i'$ such that $S\cap S'=\emptyset$. Clearly $S\in\mathfrak{s}$, $S'\in\mathfrak{s}'$ so that $\mathfrak{S}$ is disjoint.

The case of round screens is discussed in [5], (7.10). The argument for the case of unipunctual screens is contained in the proof of (8.7). □

(8.13) THEOREM. *The categories* **Oscr**, **Dscr**, **Rscr**, **Uscr** *are bireflective subcategories of* **Scr** *and of every larger full subcategory of* **Scr**, *and they are strongly topological.*

Proof. (0.2), (8.11), (8.12), (6.6). □

Instead of (8.11), a weaker statement can be proved for the subcategories **Iscr** and **Mscr**:

(8.14) Lemma. *If* $\mathfrak{S}'$ *is an independent or minimal screen on* X, $g\colon Z\to X$ *is surjective, then* $\mathfrak{S}=g^{-1}(\mathfrak{S}')$ *is independent or minimal, respectively.*

Proof. Let $\mathfrak{s}'_i\in\mathfrak{S}'$, and $g^{-1}(\mathfrak{s}'_i)$ generate $\mathfrak{s}_i$, $i=1,2$. If $\mathfrak{s}_1\subset\mathfrak{s}_2$, then $g(g^{-1}(\mathfrak{s}'_1))<g(g^{-1}(\mathfrak{s}'_2))$. Now $g(g^{-1}(S'))=S'$ $(S'\subset X)$ implies $g(g^{-1}(\mathfrak{s}'_i))=\mathfrak{s}'_i$, hence $\mathfrak{s}'_1\subset\mathfrak{s}'_2$. If $\mathfrak{S}'$ is independent, we have $\mathfrak{s}'_1=\mathfrak{s}'_2$, $\mathfrak{s}_1=\mathfrak{s}_2$.

Now assume that $\mathfrak{S}'$ is minimal, $\mathfrak{s}'\in\mathfrak{S}'$, let $\mathfrak{s}$ be generated by $g^{-1}(\mathfrak{s}')$, and let $\mathfrak{s}_1$ be a $\delta(\mathfrak{S})$-compressed filter, $\mathfrak{s}_1\subset\mathfrak{s}$. Then $g(\mathfrak{s}_1)$ generates a $\delta(\mathfrak{S}')$-compressed filter $\mathfrak{s}'_1$, and $\mathfrak{s}'_1<g(g^{-1}(\mathfrak{s}'))=\mathfrak{s}'$. Hence $\mathfrak{s}'_1=\mathfrak{s}'$, and $S'\in\mathfrak{s}'$ implies $S'\supset g(S_1)$ for some $S_1\in\mathfrak{s}_1$, hence $g^{-1}(S')\supset g^{-1}(g(S_1))\supset S_1$ so that $g^{-1}(S')\in\mathfrak{s}_1$, $\mathfrak{s}\subset\mathfrak{s}_1$, finally $\mathfrak{s}_1=\mathfrak{s}$. □

(8.15) Lemma. *If* $\mathfrak{S}'$ *is an ascending screen on* X, $g\colon Z\to X$ *is injective, then* $\mathfrak{S}=g^{-1}(\mathfrak{S}')$ *is ascending as well.*

Proof. Let $\mathfrak{s}'\in\mathfrak{S}'$ and let $\mathfrak{s}_1$ be a filter in Z such that $g^{-1}(\mathfrak{s}')<\mathfrak{s}_1$. Then $\mathfrak{s}'<g(g^{-1}(\mathfrak{s}'))<g(\mathfrak{s}_1)$ so that the filter generated in X by $g(\mathfrak{s}_1)$ belongs to $\mathfrak{S}'$. By $g^{-1}(g(S))=S$ $(S\subset Z)$ we get $g^{-1}(g(\mathfrak{s}_1))=\mathfrak{s}_1$, hence $\mathfrak{s}_1\in\mathfrak{S}$. □

(8.16) Lemma. *If* $\mathfrak{S}'$ *is a saturated screen on* X, $g\colon Z\to X$, *then, for* $\mathfrak{S}=g^{-1}(\mathfrak{S}')$, $\mathfrak{S}^a$ *is saturated. In particular, if* g *is injective,* $\mathfrak{S}$ *is saturated itself.*

Proof. If $\mathfrak{s}$ is a $\delta(\mathfrak{S})$-compressed filter, then $g(\mathfrak{s})$ generates a $\delta(\mathfrak{S}')$-compressed filter $\mathfrak{s}'$ by (8.4), hence $\mathfrak{s}'\in\mathfrak{S}'$, and $g^{-1}(\mathfrak{s}')<g^{-1}(g(\mathfrak{s}))<\mathfrak{s}$ furnishes $\mathfrak{s}\in\mathfrak{S}^a$. By (8.3), $\delta(\mathfrak{S})=\delta(\mathfrak{S}^a)$ so that $\mathfrak{S}^a$ is saturated. The second statement follows from (8.15). □

(8.15) and (8.16) are not sufficient for applying (0.2) or (0.3). However, we can prove directly the following theorem.

(8.17) Theorem. **Ascr** *is a bi(co)reflective subcategory of* **Scr**; *the (co)reflection of* $\mathfrak{S}$ *is* $\mathfrak{S}^a$ *with the (co)reflector* id.

Proof. By (8.3) $\mathfrak{S}^a$ is an object of **Ascr**. If $\mathfrak{S}'$ is an ascending screen on Z, $g\colon Z\to X$, and g is $(\mathfrak{S}', \mathfrak{S})$-continuous, then it is $(\mathfrak{S}', \mathfrak{S}^a)$-continuous because $\mathfrak{S}$ and $\mathfrak{S}^a$ are equivalent; if $g\colon X\to Z$ and g is $(\mathfrak{S}, \mathfrak{S}')$-continuous, then it is $(\mathfrak{S}^a, \mathfrak{S}')$-continuous. □

(8.18) Theorem. **Ascr** *is a strongly topological category with the operatoins*

$$f^{-1}_{\mathrm{Ascr}}(\mathfrak{S}_0)=f^{-1}_{\mathrm{Scr}}(\mathfrak{S}_0)^a$$

for an ascending screen $\mathfrak{S}_0$,

$$\sup\nolimits_{\mathrm{Ascr}}\{\mathfrak{S}_i\colon\ i\in I\}=(\sup\nolimits_{\mathrm{Scr}}\{\mathfrak{S}_i\colon\ i\in I\})^a$$

for ascending screens $\mathfrak{S}_i$.

PROOF. If $f: X\to Y$, $\mathfrak{S}_0$ is an ascending screen on Y, then $\mathfrak{S}=f^{-1}(\mathfrak{S}_0)^a$ is an ascending screen on X, f is $(\mathfrak{S}, \mathfrak{S}_0)$-continuous, and if it is $(\mathfrak{S}', \mathfrak{S}_0)$-continuous, then it is $(f^{-1}(\mathfrak{S}_0), \mathfrak{S}_0)$-continuous, so $(\mathfrak{S}, \mathfrak{S}_0)$-continuous, always because $\mathfrak{S}$ and $f^{-1}(\mathfrak{S}_0)$ are equivalent. (We write f^{-1} instead of $f_{\mathbf{Scr}}^{-1}$ and sup for $\sup_{\mathbf{Scr}}$.)

Similarly, if $\mathfrak{S}_i$ is an ascending screen on X for $i\in I$, then $\mathfrak{S}^a$ is an ascending screen on X for $\mathfrak{S}=\sup\{\mathfrak{S}_i: i\in I\}$, finer than every $\mathfrak{S}_i$ and, if $\mathfrak{S}'$ is ascending and finer than every $\mathfrak{S}_i$, then $\mathfrak{S}'$ is finer than $\mathfrak{S}$ and than $\mathfrak{S}^a$. Observe that, by (8.9), $\mathfrak{S}=\mathfrak{S}^a$ if $I\neq\emptyset$. If $I=\emptyset$, $\mathfrak{S}^a$ is the collection of all filters in X.

Since, if $g: Z\to X$, $f: X\to Y$, and $\mathfrak{S}_0$ is an ascending screen on Y, $f^{-1}(\mathfrak{S}_0)$ and $f^{-1}(\mathfrak{S}_0)^a$ are equivalent, so are $(f\circ g)^{-1}(\mathfrak{S}_0)=g^{-1}(f^{-1}(\mathfrak{S}_0))$ and $g^{-1}(f^{-1}(\mathfrak{S}_0)^a)$, hence by (8.3)

$$(f\circ g)^{-1}(\mathfrak{S}_0)^a = g^{-1}(f^{-1}(\mathfrak{S}_0)^a)^a$$

(both sides are equivalent and ascending), i.e.

$$(f\circ g)_{\mathbf{Ascr}}^{-1}(\mathfrak{S}_0) = g_{\mathbf{Ascr}}^{-1}(f_{\mathbf{Ascr}}^{-1}(\mathfrak{S}_0)).$$

Finally if $\mathfrak{S}_i$ is ascending on X for $i\in I$, $g: Z\to X$ then it is easy to see again that both sides of the equality

$$g^{-1}((\sup\{\mathfrak{S}_i\})^a)^a = (\sup\{g^{-1}(\mathfrak{S}_i)^a\})^a$$

are equivalent and ascending. □

(8.19) THEOREM. **Sscr** *is a bireflective subcategory of* **Scr**; *the reflection of a screen* $\mathfrak{S}$ *on* X *is given by* $\mathfrak{S}^s$, *the collection of all* $\delta(\mathfrak{S})$*-compressed filters, with the reflector* id_X.

PROOF. By (6.11), $\delta(\mathfrak{S}^s)=\delta(\mathfrak{S})$ so that $\mathfrak{S}^s$ is saturated. $\mathfrak{S}\subset\mathfrak{S}^s$ by (6.9), hence $\mathfrak{S}^s$ is coarser than $\mathfrak{S}$, and if $\mathfrak{S}'$ is a saturated screen on Y, $f: X\to Y$ is $(\mathfrak{S}, \mathfrak{S}')$-continuous, then, for a filter $\mathfrak{s}\in\mathfrak{S}^s$, $\mathfrak{s}$ is $\delta(\mathfrak{S})$-compressed, hence $f(\mathfrak{s})$ generates a $\delta(\mathfrak{S}')$-compressed filter $\mathfrak{s}'$, so $\mathfrak{s}'\in\mathfrak{S}'$ and f is $(\mathfrak{S}^s, \mathfrak{S}')$-continuous. □

The strongly topological character of the category **Sscr** is a consequence of the following:

(8.20) THEOREM. *For a saturated screen* $\mathfrak{S}$, *define* $F(\mathfrak{S})=\delta(\mathfrak{S})$. *Then* F *induces an isomorphism from* **Sscr** *onto* **Prox**.

PROOF. By (6.11), $\delta(\mathfrak{S}(\delta))=\delta$ so that $\mathfrak{S}(\delta)$ is a saturated screen. Conversely, for a saturated screen $\mathfrak{S}$, $\mathfrak{S}(\delta(\mathfrak{S}))=\mathfrak{S}$ by definition. If $f: X\to Y$ is $(\mathfrak{S}, \mathfrak{S}')$-continuous then it is $(\delta(\mathfrak{S}), \delta(\mathfrak{S}'))$-continuous by (6.8). Conversely, if f is (δ, δ')-continuous, and $\mathfrak{s}$ is a δ-compressed filter, then $f(\mathfrak{s})$ generates a δ'-compressed filter, hence f is $(\mathfrak{S}(\delta), \mathfrak{S}(\delta'))$-continuous. □

It is an attractive idea to use the results on the category **Rscr** for examining the category RE-**Prox**. In fact, by [5], (5.1) and (5.12), if $\mathfrak{S}$ is a round screen, then $\delta(\mathfrak{S})$ is an RE-proximity, and conversely, if δ is an RE-proximity, then the collection $\mathfrak{S}^r(\delta)$ of all δ-round, δ-compressed filters is a round screen inducing δ.

However, this method yields modest results only. We have e.g.:

(8.21) LEMMA. *If* δ' *is an RE-proximity on* X, $g: Z\to X$, *then* $g^{-1}(\delta')=\delta$ *is an RE-proximity on* Z.

Proof. (8.11) and (8.4). □

On the other hand, (8.12) (or [5], (7.10)) does not imply similar consequences for $\sup_{\mathrm{Prox}}\{\delta_i\colon i\in I\}=\delta'$ if every δ_i is an RE-proximity because, taking round screens $\mathfrak{S}_i$ such that $\delta_i=\delta(\mathfrak{S}_i)$, $\mathfrak{S}=\sup\{\mathfrak{S}_i\colon i\in I\}$ will be a round screen but $\delta'\neq\delta(\mathfrak{S})$ in general (observe that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in (8.5) are round screens).

If we examine the relation between δ and $\mathfrak{S}^r(\delta)$, for RE-proximities δ, we expect that $F(\delta)=\mathfrak{S}^r(\delta)$ induces an isomorphism from RE-**Prox** onto a subcategory of **Rscr**. This is not true because a map $f\colon X\to Y$ can be (δ, δ')-continuous for RE-proximities δ, δ' on X and Y, respectively, without being $(\mathfrak{S}^r(\delta), \mathfrak{S}^r(\delta'))$-continuous ([5], (8.11)). However, we can obtain again a positive result concerning inverse images.

(8.22) Lemma. *Let* $f\colon Z\to X$, *let* δ *be a semi-proximity on* X, *and let* $\delta'=f^{-1}(\delta)$. *If* $\mathfrak{s}$ *is a* δ*-compressed or* δ*-round filter in* X, *then* $f^{-1}(\mathfrak{s})$ *generates (if it is a filter base) a* δ'*-compressed or* δ'*-round filter in* Z, *respectively. Conversely, let* $\mathfrak{s}'$ *be a filter in* Z, *and let* $\mathfrak{s}$ *be the filter generated in* X *by* $f(\mathfrak{s}')$. *If* $\mathfrak{s}'$ *is* δ'*-round, then* $f^{-1}(\mathfrak{s})$ *generates* $\mathfrak{s}'$; *if* $\mathfrak{s}'$ *is* δ'*-compressed, then* $\mathfrak{s}$ *is* δ*-compressed; if* $\mathfrak{s}'$ *is* δ'*-round and* f *is surjective, then* $\mathfrak{s}$ *is* δ*-round.*

Proof. If $A, B\in\sec\mathfrak{s}'$, $\mathfrak{s}'$ is generated by $f^{-1}(\mathfrak{s})$, then $f(A), f(B)\in\sec\mathfrak{s}$, hence $f(A)\delta f(B)$ provided $\mathfrak{s}$ is δ-compressed, so that $A\delta'B$.

Now let $\mathfrak{s}$ be δ-round, $\mathfrak{s}'$ generated by $f^{-1}(\mathfrak{s})$, and $S'\in\mathfrak{s}'$. Then $S'\supset f^{-1}(S)$ for some $S\in\mathfrak{s}$, and there is $S_1\in\mathfrak{s}$ such that $S_1\bar{\delta}X-S$. Then $f^{-1}(S_1)\in\mathfrak{s}'$, and $f(f^{-1}(S_1))\subset S_1$, $f(Z-S')\subset f(Z-f^{-1}(S))\subset X-S$ imply $f^{-1}(S_1)\bar{\delta}'Z-S'$, so that $\mathfrak{s}'$ is δ'-round.

Conversely, let $\mathfrak{s}'$ be a δ'-round filter and let $\mathfrak{s}$ be generated by $f(\mathfrak{s}')$. Then $S\in\mathfrak{s}$ implies $S\supset f(S')$ for some $S'\in\mathfrak{s}'$, hence $f^{-1}(S)\supset S'$, $f^{-1}(S)\in\mathfrak{s}'$; on the other hand, $S'\in\mathfrak{s}'$ implies the existence of $S_1'\in\mathfrak{s}'$ satisfying $S_1'\bar{\delta}'Z-S'$, hence $f(S_1')\in\mathfrak{s}$, $f(S_1')\bar{\delta}f(Z-S')$, thus $f^{-1}(f(S_1'))\subset S'$ so that $f^{-1}(\mathfrak{s})$ generates $\mathfrak{s}'$. If $\mathfrak{s}'$ is δ'-compressed then $\mathfrak{s}$ is δ-compressed since f is (δ', δ)-continuous. Finally if f is surjective, $S\in\mathfrak{s}$, then $S\supset f(S')$ for some $S'\in\mathfrak{s}'$, and $S_1'\bar{\delta}'Z-S'$ for some $S_1'\in\mathfrak{s}'$; we have again $f(S_1')\in\mathfrak{s}$, $f(S_1')\bar{\delta}f(Z-S')\supset X-f(S')\supset X-S$, so that $\mathfrak{s}$ is δ-round. □

(8.23) Corollary. *Let* $f\colon Z\to X$ *be surjective,* δ *an RE-proximity on* X, $\delta'=f^{-1}(\delta)$. *Then* $\mathfrak{S}^r(\delta')=f^{-1}(\mathfrak{S}^r(\delta))$. □

Concerning the operation sup, the situation is less advantageous. In fact, if δ_i is an RE-proximity on X for $i\in I$, then $\sup\{\mathfrak{S}^r(\delta_i)\colon i\in I\}=\mathfrak{S}$ need not be of the form $\mathfrak{S}^r(\delta(\mathfrak{S}))$. In fact, if T is a set of cardinality ω_1 (e.g. if T denotes the Tikhonov plank), and $\{\delta_i\colon i\in I\}$ denotes the family of all possible RE-proximities on T, then, by [5], (8.10), there is no RE-proximity on T such that $\mathfrak{S}^r(\delta)$ is finer than $\mathfrak{S}^r(\delta_i)$ for every i.

The answer to the following questions is unknown to the author:

Let δ_i be an RE-proximity on X for $i\in I$, $\delta'=\sup_{\mathrm{Prox}}\{\delta_i\colon i\in I\}$.

(a) Is it true that $\delta'=\delta(\mathfrak{S})$ for $\mathfrak{S}=\sup\{\mathfrak{S}^r(\delta_i)\colon i\in I\}$?

(b) Is it true at least that δ' is an RE-proximity?

(c) Is it true at least that there is an RE-proximity δ'' finer than δ' and coarser than every RE-proximity finer than δ'?

A candidate for δ'' can be obtained in the following way. By (5.1), δ' is an R-proximity, and by (8.7) $c_{\delta'}=c_{\delta}$ for $\delta=\delta(\mathfrak{S})$, and, of course, δ is and RE-proximity finer than δ' by (8.12), (6.8), and [5], (5.12). Now [5], (5.16) says that, if $\mathfrak{S}'$ is the collection of all δ'-compressed, δ'-round filters, then $\delta(\mathfrak{S}')$ is an RE-proximity finer than δ'. However, the author cannot prove that $\delta(\mathfrak{S}')$ is coarser than every RE-proximity finer than δ', or, at least, that $\delta(\mathfrak{S}')$ is coarser than δ.

The relation of screens to merotopies will be discussed in the following Part III.

References

[1] E. Čech, *Topological Spaces* (Prague—London—New York—Sydney, 1966).
[2] Á. Császár, *Foundations of General Topology* (Budapest—Oxford—London—New York—Paris, 1963.)
[3] Á. Császár, Syntopogene Gruppen III, *Ann. Univ. Budapest., Sect. Math.*, **14** (1971), 23—52.
[4] Á. Császár, *General Topology* (Budapest—Bristol, 1978).
[5] Á. Császár, RE-proximities, *Acta Math. Hungar.*, (in print).
[6] K. Császár, Separation axioms for proximity and closure spaces (in print).
[7] A. S. Davis, Indexed systems of neighbourhoods for general topological spaces, *Amer. Math. Monthly*, **68** (1961), 886—893.
[8] V. A. Efremovich, The geometry of proximity I, *Mat. Sb.*, **31** (73) (1952), 189—200 (in Russian).
[9] M. S. Gagrat and S. A. Naimpally, Proximity approach to extension problems, *Fundam. Math.*, **71** (1971), 63—76.
[10] D. Harris, Regular-closed spaces and proximities, *Pacific J. Math.*, **34** (1970), 675—685.
[11] H. Herrlich, Topological structures, in Topological Structures, *Math. Centre Tracts*, **52** (1974), 59—122.
[12] H. Herrlich, Categorical topology 1971—1981, *General Topology and its Relations to Modern Analysis and Algebra V*, 1982, 279—383.
[13] J. R. Isbell, *Uniform Spaces* (Providence, 1964).
[14] M. Katětov, On continuity structures and spaces of mappings, *Comm. Math. Univ. Carol.*, **6** (1965), 257—278.
[15] M. Katětov, Convergence structures, *General Topology and its Relations to Modern Analysis and Algebra II*, 1967, 207—216.
[16] M. Katětov, Spaces defined by giving a family of centred systems, *Uspehi Mat. Nauk*, **31** (1976), 95—107 (in Russian).
[17] M. W. Lodato, On topologically induced generalized proximity relations, *Proc. Amer. Math. Soc.*, **15** (1964), 417—422.
[18] M. W. Lodato, On topologically induced generalized proximity relations II, *Pacific J. Math.*, **17** (1966), 131—135.
[19] S. A. Naimpally and B. D. Warrack, *Proximity Spaces* (Cambridge, 1970).
[20] V. Yu. Sandberg, New definition of uniform spaces, *Dokl. Akad. Nauk SSSR*, **135** (1960), 535—537 (in Russian).
[21] Yu. M. Smirnov, On proximity spaces, *Mat. Sb.*, **31** (**73**) (1952), 543—574 (in Russian).
[22] W. J. Thron, On a problem of F. Riesz concerning proximity structures, *Proc. Amer. Math. Soc.*, **40** (1973), 323—326.
[23] J. W. Tukey, Convergence and uniformity in topology, *Annals of Math. Studies* (1940).
[24] A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, *Actual. Sci. Industr.* fasc. 551 (Paris, 1937).

(Received July 19, 1985)

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF MATHEMATICS
BUDAPEST, MÚZEUM KRT. 6—8.
H—1088

Acta Math. Hung.
50 (1—2) (1987), 111—115.

METRIZATION AND LIAPUNOV FUNCTIONS. V

B. M. GARAY (Budapest)

This paper is the continuation of [2a—2d]. The problem we deal with is the possibility of constructing remetrizations of the phase space such that the new metrics describe attraction and repulsion properties of the trajectories.

An obstacle to this approach is the presence of nonequilibrium recurrent trajectories (precluding the existence of global Liapunov functions). This difficulty can be overcome by factorization according to Auslander recurrence classes [4].

Now we concern ourselves with an additional obstacle. Replacing the finiteness condition of a previous result [3, Theorem 2.1] by local finiteness, we prove a theorem on the existence of metrics of Liapunov type. On the other hand, we point out by examples that, in general, local finiteness can not be weakened further. The dynamical system defined in Example 1 has no recurrent trajectories but equilibrium points. Example 2 shows that the local finiteness condition can not be dropped even if compact isolated invariant sets are replaced by asymptotically stable equilibrium points.

Throughout this paper, let (X, d) be a locally compact separable metric space. Given a point $x\in X$ and a set $Y\subset X$, the distance between x and Y is defined as $d(x, Y)=\inf\{d(x, y)|y\in Y\}$. For $c>0$, the set $\{x\in X|d(x, Y)\leqq c\}$ is denoted by $S(Y, c)$.

Let $T:\mathbf{R}\times X\to X$ be a dynamical system. For terminology and notations, see [1], [2a—2d], [3]. For brevity, we say that a continuous function $V: X\to\mathbf{R}^+$ is a Liapunov function for a compact isolated invariant set K if it is a strictly monotonous Liapunov function for K on $A(K)\cup R(K)$ and in addition, it is a hyperbolic Liapunov function for K on some compact neighbourhood N of K.

Let $\emptyset=C_0\subset C_1\subset C_2\subset\dots$ be a nested system of compact sets such that $\cup\{C_n|n\in\mathbf{N}\}=X$ and $C_n\subset\operatorname{int} C_{n+1}$, $n\in\mathbf{N}$.

THEOREM 1. *Let* $K_0, K_1, K_2, \dots$ *be pairwise disjoint compact isolated invariant sets. For each* $n\in\mathbf{N}$ *assume that the set* $\{m\in\mathbf{N}|K_m\cap C_n\neq\emptyset\}$ *is finite. Further, assume that* K_m *is positively resp. negatively relatively asymptotically stable with respect to* $A(K_m)$ *resp.* $R(K_m)$, $m\in\mathbf{N}$. *Then there exists a remetrization* ϱ *of* X *such that the function* $V_m: X\to\mathbf{R}^+$ *defined by* $V_m(x)=\varrho(x, K_m)$ *is a Liapunov function for* K_m, $m\in\mathbf{N}$.

PROOF. We divide the proof into three parts:

A) *A preliminary remetrization.* There is no loss of generality in assuming that
(1) $\sup\{d(x, y)|x, y\in X\}\leqq 3$,
(2) $d(x, y)=3$ whenever $x\in K_p$, $y\in K_q$; $p, q\in\mathbf{N}$, $p\neq q$.

Proof. Let $Y=\bigcup\{K_m|m\in\mathbf{N}\}$ and define a metric δ on Y by letting

$$\delta(x,y)=\begin{cases}\min\{3, d(x,y)\} & \text{if } x,y\in K_p, p\in\mathbf{N}.\\ 3 & \text{if } x\in K_p, y\in K_q;\ p,q\in\mathbf{N}, p\neq q.\end{cases}$$

It is easy to check that Y is a closed subset of X and δ is a metric on Y topologically equivalent to $d|Y$. Applying Hausdorff's extension theorem for metrics [6], δ can be extended to a remetrization Δ of X. $d(x,y)$ can be replaced by $\min\{3, \Delta(x,y)\}$. □

From now on, assume that conditions (1) and (2) are satisfied.

B) *Extension lemma for hyperbolic Liapunov functions (a technical lemma).* Let $K\subset X$ be a compact isolated invariant set. It is clear that $K\subset X\setminus C_{n(K)}$ but $K\cap C_{n(K)+1}\neq\emptyset$ for some $n=n(K)\in\mathbf{N}$. (Recall that $\emptyset=C_0$.) Let $L\subset X$ be a closed set for which $L\cap(A(K)\cup R(K))=\emptyset$, and in addition, $d(x,y)=3$ whenever $x\in K$, $y\in L$. Assume that K is positively resp. negatively relatively asymptotically stable with respect to $A(K)$ resp. $R(K)$. Then there exists a continuous function $V\colon X\to[0,5]$ with the following properties:

(3) V is a Liapunov function for K,

(4) $V^{-1}(0)=K$, $V^{-1}(5)=L\cup(\mathrm{cl}(A(K)\cup R(K))\setminus(A(K)\cup R(K)))$,

(5) for each $x\in X$, there holds $d(x,K)\leqq V(x)$,

(6) for each $x\in X\setminus S(K,1)$, there holds $V(x)\geqq 4$,

(7) for each $x\in S(L,1/n)\setminus S(L,1/(n+1))$, there holds $V(x)\geqq 5-1/(n+1)$, $n=1,2,\dots$,

(8) for $n=0,1,\dots,n(K)$, there holds $\inf\{V(x)|x\in C_n\}\geqq 5-1/(n(K)+1-n)$.

(By definition, $\inf\{F(x)|x\in\emptyset\}=\infty$ for each real function F.)

Proof. Let $\hat{N}$ and $\hat{W}\colon\hat{N}\to\mathbf{R}^+$ be constructed as in the proof of Lemma 2.1 of [3]. $\hat{N}$ is a compact neighbourhood of K and $\hat{W}$ is a hyperbolic Liapunov function for K on $\hat{N}$. Without loss of generality, we may assume that

$$\hat{N}\subset S(K,1)\cap(X\setminus C_{n(K)}).$$

The stability conditions imposed on K imply the existence of a constant $c_0>0$ and of a compact neighbourhood $N\subset\hat{N}$ of K for which $N:=\{x\in\hat{N}|\hat{W}(x)\leqq c_0\}$ and in addition, $\{T_tx|x\in A(K)\cap N,\ t\geqq 0\}\subset\hat{N}$, $\{T_tx|x\in R(K)\cap N,\ t\leqq 0\}\subset\hat{N}$. Since $\hat{W}$ is a hyperbolic Liapunov function for K on $\hat{N}$, the sets

$$S_{A(K)}:=\{x\in A(K)|\hat{W}(x)=c_0\}=A(K)\cap\partial N,$$

$$S_{R(K)}:=\{x\in R(K)|\hat{W}(x)=c_0\}=R(K)\cap\partial N$$

are closed and for each $x\in A(K)\setminus K$ resp. $x\in R(K)\setminus K$ there exists a unique $t_A(x)\in\mathbf{R}$ resp. $t_R(x)\in\mathbf{R}$ for which $T_{t_A(x)}x\in S_{A(K)}$ resp. $T_{t_R(x)}x\in S_{R(K)}$ and in addition, the functions $t_A\colon A(K)\setminus K\to\mathbf{R}$ resp. $t_R\colon R(K)\setminus K\to\mathbf{R}$ are continuous. In other words, $S_{A(K)}$ is a section for $A(K)\setminus K$ and $S_{R(K)}$ is a section for $R(K)\setminus K$; both $A(K)\setminus K$ and $R(K)\setminus K$ are parallelizable.

Let $g\colon[0,c_0]\to[0,4]$ be a strictly increasing continuous function with $g(0)=0$, $g(c_0)=4$. Define a function $V\colon N\to\mathbf{R}^+$ by $V(x)=g(\hat{W}(x))$. It is obvious that

(3)′ V is a hyperbolic Liapunov function for K on N, and

(4)′ $V^{-1}(0)=K$.

By a simple compactness argument, an appropriate choice of g guarantees for V to have the following properties as well:

(5)′ for each $x \in N$, there holds $d(x, K) \leqq V(x)$,

(6)′ for each $x \in \partial N$ (observe that $\partial N \subset S(K, 1)$), there holds $V(x)=4$.

In what follows, we extend V from N onto X.

Observe that the sets $\{T_{-t}x | x \in S_{A(K)},\ t \in [0, \tau]\}$ and $\{T_t x | x \in S_{R(K)},\ t \in [0, \tau]\}$ are compact, for each $\tau > 0$. Consequently, the inclusion $N \subset S(K, 1) \cap (X \setminus C_{n(K)})$ and the conditions imposed on L imply that the sequences

$$s_A^n := \inf\{t_A(x) | x \in (A(K) \setminus K) \cap (S(L, 1/n) \setminus S(L, 1/(n+1)))\},$$
$$s_R^n := \inf\{-t_R(x) | x \in (R(K) \setminus K) \cap (S(L, 1/n) \setminus S(L, 1/(n+1)))\},$$

$n=1, 2, \dots$ consist of positive elements and

$$s_A^n, s_R^n \to \infty \quad \text{as} \quad n \to \infty$$

and that

$$t_A^n, t_R^n > 0 \quad \text{for} \quad n = 0, 1, \dots, n(K)$$

where

$$t_A^n := \inf\{t_A(x) | x \in (A(K) \setminus K) \cap C_n\},$$
$$t_R^n := \inf\{-t_R(x) | x \in (R(K) \setminus K) \cap C_n\}.$$

Choose a strictly increasing continuous function $h\colon \mathbf{R}^+ \to [0, 1)$ for which $h(0)=0$,

$$h(t_A^n), h(t_R^n) \geqq 1 - 1/(n(K)+1-n) \quad \text{for} \quad n = 0, 1, \dots, n(K),$$
$$h(s_A^n), h(s_R^n) \geqq 1 - 1/(n+1) \quad \text{for} \quad n = 1, 2, \dots$$

For $x \in A(K) \setminus N$, let $V(x) = 4 + h(t_A(x))$ and for $x \in R(K) \setminus N$, let $V(x) = 4 + h(-t_R(x))$. Since $A(K) \cap R(K) = K \subset N$, $V(x)$ is well-defined. By (6)′, we have obtained a continuous extension of V onto $N \cup A(K) \cup R(K)$. Since $h(t) \to 1$ as $t \to \infty$, V can be continuously extended onto $Z := N \cup L \cup \mathrm{cl}\,(A(K) \cup R(K))$ by setting $V(x)=5$ for $x \in L \cup (\mathrm{cl}\,(A(K) \cup R(K)) \setminus (A(K) \cup R(K)))$.

Since h is strictly increasing, (3)′ and $t_A(T_t x) = t_A(x) - t$, $t_R(T_t x) = t_R(x) - t$ imply that

(3)″ V is a Liapunov function for K on Z. Further, in virtue of the properties of h, we have that,

(4)″ $V^{-1}(0) = K$, $V^{-1}(5) = L \cup (\mathrm{cl}\,(A(K) \cup R(K)) \setminus (A(K) \cup R(K)))$,

(7)″ for each $x \in Z \cap (S(L, 1/n) \setminus S(L, 1/(n+1)))$, there holds $V(x) \geqq 5 - 1/(n+1)$, $n=1, 2, \dots$,

(8)″ for $n=0, 1, \dots, n(K)$, there holds $\inf\{V(x) | x \in Z \cap C_n\} \geqq 5 - 1/(n(K)+1-n)$.

By (1,) (5)′ and (6)′, we have that

(5)″ for each $x \in Z$, there holds $d(x, K) \leqq V(x)$.

(6)″ for each $x \in Z \setminus S(K, 1)$, there holds $V(x) \geqq 4$.

The extension of V from Z onto X can be completed now by a repeated application of Tietze's extension theorem. Using (1), (4)″—(7)″, it is easy to construct a continuous extension V_L of V satisfying (4)—(7). In fact, V_L can be obtained by a step by step extension of V from $Z \cap \{x \in X | d(x, L) = 1/n\}$ onto $\{x \in X | d(x, L) = 1/n\}$ followed by an other step by step extension onto $Z \cup \mathrm{cl}\,(X \setminus S(L, 1/n))$, $n=1, 2, \dots$. Similarly, it is not hard to extend V continuously so that the extension $V_c\colon X \to [0, 5]$ satisfies (4)—(6), (8). $V(x)$ can be defined as $\max\{V_L(x), V_c(x)\}$. $\square$

Let us remark a useful consequence of (7) in advance:

(9) $V(x) \geqq 5-d(x, L)$ whenever $d(x, L) \leqq 1$.

C) *The final remetrization.* We apply the Lemma for $K=K_m$, $L=Y \setminus K_m$. The resulting Liapunov functions are denoted by V_m, $m \in \mathbf{N}$.

In virtue of (5), we have that

(10) for each $x \in X$, there holds $d(x, K_m) \leqq V_m(x)$, $m \in \mathbf{N}$.

We show that

(11) for each $x \in X$, there holds $V_p(x)+V_q(x) \geqq 5$, $p, q \in \mathbf{N}$, $p \neq q$.

We distinguish two cases according as $x \in S(K_p, 1) \cup S(K_q, 1)$ or not. If $x \in S(K_p, 1)$, then $x \in S(Y \setminus K_q, 1)$, consequently, by (9) and (2), $V_q(x) \geqq 5-d(x, Y \setminus K_q)=$ $=5-d(x, K_p)$ and the desired result follows from inequality $d(x, K_p) \leqq V_p(x)$. The subcase when $x \in S(K_q, 1)$ can be settled similarly. Finally, if $x \in X \setminus (S(K_p, 1) \cup$ $\cup S(K_q, 1))$, (6) implies that $V_p(x)+V_q(x) \geqq 4+4>5$.

Now we are in a position to define ϱ. For $x, y \in X$, let

$$\varrho(x, y) = \max \{d(x, y), \sup \{|V_m(x)-V_m(y)| \mid m \in \mathbf{N}\}\}.$$

We show that

(12) for each $x \in X$, there holds $\varrho(x, K_m)=V_m(x)$, $m \in \mathbf{N}$.

In fact, using (4), $y \in K_m$ implies that

$$\varrho(x, y) = \max \{d(x, y), \sup \{V_m(x), 5-V_p(x) | p \in \mathbf{N}, p \neq m\}.$$

Therefore, by (10) and (11), taking infimum for $y \in K_m$, (12) follows.

It is left to show that d and ϱ are topologically equivalent. It is clear that ϱ is a metric on X and that $d(x_n, x) \to 0$ whenever $\varrho(x_n, x) \to 0$. Therefore, in order to prove topological equivalence, we have to show that $\varrho(x_n, x) \to 0$ whenever $d(x_n, x) \to 0$. In the contrary, suppose there exist a sequence $\{x_n\}_0^\infty \subset X$ and a constant $\varepsilon_0>0$ such that $d(x_n, x) \to 0$ for some $x \in X$ but $\varrho(x_n, x)>\varepsilon_0$, $n \in \mathbf{N}$. Since X is locally compact, we may assume that $x \in C_{n(x)}$, $\{x_n\}_0^\infty \subset C_{n(x)}$ for some $n=n(x)$. Combining (8) with the local finiteness condition, it follows that there is an $m_0=$ $=m_0(n(x), \varepsilon_0) \in \mathbf{N}$ such that $V_m(z) \geqq 5-\varepsilon_0/2$ whenever $z \in C_{n(x)}$, $m \in \mathbf{N}$, $m \geqq m_0$. Consequently, we obtain that $m \geqq m_0$ implies that

$$\begin{aligned} |V_m(x_n)-V_m(x)| &\leqq |V_m(x_n)-5|+|5-V_m(x)| = \\ &= (5-V_m(x_n))+(5-V_m(x)) \leqq \varepsilon_0/2+\varepsilon_0/2 = \varepsilon_0 \end{aligned}$$

for each $n \in \mathbf{N}$. Thus,

$$\varepsilon_0 < \varrho(x_n, x) = \max \{d(x_n, x), \max \{|V_m(x_n)-V_m(x)| \mid m \in \mathbf{N}, m < m_0\}\},$$

which is a contradiction for n sufficiently large. □

We now give two examples showing that the previous theorem is incorrect without the assumption that the sequence of subsets $K_0, K_1, K_2, \ldots$ be locally finite.

EXAMPLE 1. Let $X=(\{0\} \cup \{1/m | m=1, 2, \ldots\}) \times [-1, 1] \subset \mathbf{R}^2$. Set $C_0=(0, 1)$, $D_0=(0, -1)$, $E_0=(0, 0)$ and for $m=1, 2, \ldots$, set $C_m=(1/m, 1)$, $D_m=(1/m, -1)$,

$E_m=(1/m, 0)$. For $m\in\mathbf{N}$, let $K_m=C_{m+1}\cup D_{m+2}$. Let $T\colon \mathbf{R}\times X\to X$ be a dynamical system satisfying the following conditions:

- (C_1') For each $m\in\mathbf{N}$, C_m is a source, D_m is a sink.
- (C_2') There are no other equilibrium points but C_m and D_m, $m\in\mathbf{N}$.

It is clear that all conditions of Theorem 1 are fulfilled with the exception of the one concerning local finiteness. Assume that we are given a metric ϱ on X topologically equivalent to the usual Euclidean metric for which $V_m\colon X\to\mathbf{R}^+$ defined by $V_m(x)=\varrho(x, K_m)$ is a Liapunov function for K_m on X, $m\in\mathbf{N}$. Since $A(K_m)=C_{m+1}\cup\big(\{1/(m+2)\}\times[-1, 1)\big)$, we have that

$$\min\{\varrho(C_{m+2}, C_{m+1}),\ \varrho(C_{m+2}, D_{m+2})\}=\varrho(C_{m+2}, K_m)>$$

$$>\varrho(E_{m+2}, K_m)=\min\{\varrho(E_{m+2}, C_{m+1}),\ \varrho(E_{m+2}, D_{m+2})\}.$$

Consequently, by letting $m\to\infty$, we obtain

$$0=\min\{\varrho(C_0, C_0), \varrho(C_0, D_0)\}\geqq\min\{\varrho(E_0, C_0), \varrho(E_0, D_0)\},$$

a contradiction.

EXAMPLE 2. For each $m\in\mathbf{N}$, set $X_m=[1/(3m+4), 1/(3m+2)]\times[-1, 1]\subset\mathbf{R}^2$, $K_m=\big(1/(3m+3), 0\big)$. Let $X=\big(\{0\}\times[-1, 1]\big)\cup X_0\cup X_1\cup X_2\cup\ldots\subset\mathbf{R}^2$. Let $T\colon\mathbf{R}\times X\to X$ be a dynamical system satisfying the following conditions:

- (C_1'') For each $m\in\mathbf{N}$, K_m is a sink with $A(K_m)=\operatorname{int} X_m$,
- (C_2'') The set $\{0\}\times[-1, 1]$ is filled up with equilibrium points.
- (C_3'') For each $m\in\mathbf{N}$, X_m is a periodic orbit.

Of course the sequence of the asymptotically stable equilibrium points $K_0, K_1, K_2, \ldots$ is not locally finite. (The other conditions of Theorem 1 are fulfilled.) The failure of Theorem 1 can be shown easily. We leave it to the reader.

REMARK. We do not know whether the pairwise disjointness condition of Theorem 1 can be omitted or not. Even the pure metrization aspects of the weakening of this condition seem to be very difficult. Cf. [5].

References

[1] N. P. Bhatia and G. P. Szegő, *Dynamical Systems: Stability Theory and Applications*, Springer (Berlin, 1967).

[2] B. M. Garay, Metrization and Liapunov functions. I—II—III, *Acta Math. Hung.*, **42** (1983), 337—348; **44** (1984), 157—167, 379—387. IV. *Proc. Diff. Eq. Conf.*, 1984 (Szeged, Hungary), 305—328.

[3] B. M. Garay, On dynamical systems in metric spaces: metrics as Liapunov functions, *Nonlin. Anal.*, **8** (1984), 1033—1042.

[4] B. M. Garay, Auslander recurrence and metrization via Liapunov functions, *Funkc. Ekvac.*, **28** (1985), 299—308.

[5] L. F. McAuley, *On the metrizability of decomposition spaces*, Top. Conf. 1979 (Greensboro, N. C.) (Guilford College, Greensboro, N. C., 1980).

[6] S. Willard, *General Topology*, Addison-Wesley (London, 1970).

(Received August 9, 1985)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TECHNOLOGY
H—1521 BUDAPEST
HUNGARY

Acta Math. Hung.
50 (1—2) (1987), 117—122.

ON DIVISIBILITY PROPERTIES OF INTEGERS OF THE FORM $a+a'$

P. ERDŐS, member of the Academy and A. SÁRKÖZY (Budapest)

1. Throughout this paper, we use the following notations:

For any real number x let $[x]$ denote the greatest integer less than or equal to x, and let $\|x\|$ denote the distance from x to the nearest integer: $\|x\|=\min(x-[x], 1+[x]-x)$. We write $e^{2\pi i x}=e(x)$. The cardinality of the set X is denoted by $|X|$. $\Lambda(n)$ denotes the Mangoldt symbol.

In this paper, our goal is to study the following problem: how large can $|\mathcal{A}|$ be if $\mathcal{A}\subset\{1, 2, ..., N\}$ and $a+a'$ is squarefree for all $a\in\mathcal{A}$, $a'\in\mathcal{A}$? (See [1], [2] and [4] for other somewhat related results. In fact, in all these papers arithmetic properties of sums of sequences of integers are studied.)

We will prove the following results:

THEOREM 1. *For* $N>N_0$, *there exists a sequence* $\mathcal{A}\subset\{1, 2, ..., N\}$ *such that*

$$|\mathcal{A}| > \frac{1}{248}\log N \tag{1}$$

and $a+a'$ *is squarefree for all* $a\in\mathcal{A}$, $a'\in\mathcal{A}$.

THEOREM 2. *If* $N>N_1$, $\mathcal{A}\subset\{1, 2, ..., N\}$ *and* $a+a'$ *is squarefree for all* $a\in\mathcal{A}$, $a'\in\mathcal{A}$ *then we have*

$$\mathcal{A} < 3N^{3/4}\log N. \tag{2}$$

There is a considerable gap between the lower and upper bounds above. We guess that the lower bound is nearer to the truth. In fact, we conjecture that the upper bound in (2) can be replaced by N^{ε} (for all $\varepsilon>0$ and $N>N_2(\varepsilon)$) and, perhaps, even by $(\log N)^c$. Unfortunately, we have not been able to prove this.

By similar but slightly more complicated methods we can get analogous results for k-th power free numbers.

Also the following related problem can be considered: Let $1\leqq a_1<a_2<\dots<a_k\leqq N$, $1<b_1<b_2<\dots<b_l\leqq N$ be two sequences of integers. Assume that all the sums

$$a_i+b_j,\quad 1\leqq i\leqq k,\quad 1\leqq j\leqq l$$

are squarefree. Our method gives that

$$kl < N^{3/2+\varepsilon}$$

and we can show that $kl/N\to\infty$ is possible, but we of course have no satisfactory upper bound for kl. Perhaps the following remark is of some interest: there is an absolute constant c so that $k > cN$, $l \to \infty$ is possible. Here perhaps l must be less than $\log N$ or $(\log N)^c$.

2. In this section, we prove Theorem 1. Let p_i denote the i-th prime number. Let N be a large positive integer, define the positive integer K by

$$\prod_{i=1}^{K-1} p_i^2 < N^{1/2} \leqq \prod_{i=1}^{K} p_i^2, \tag{3}$$

and put

$$P = \prod_{i=1}^{K} p_i^2 .$$

Then by the prime number theorem we have

$$\log P = 2 \sum_{i=l}^{K} \log p_i \sim 2 \sum_{n \leqq p_K} \Lambda(n) \sim 2p_K \tag{4}$$

so that for $N \to +\infty$ we obtain from (3) that

$$\frac{4P}{(\log P)^2} \sim \frac{P}{p_K^2} < N^{1/2} \leqq P$$

hence $\log P \sim \frac{1}{2} \log N$, so that, in view of (4), for large N

$$N^{1/2} \leqq P = p_K^2 \sum_{i=1}^{K-1} p_i^2 < 1/3 (\log P)^2 N^{1/2} < \frac{1}{11} N^{1/2} (\log N)^2. \tag{5}$$

Let us take all the integers n satisfying

$$n \equiv 2 \pmod 4 \tag{6}$$

and

$$n \not\equiv 0 \quad (\bmod\, p_i^2) \quad \text{for} \quad i = 2, 3, \ldots K. \tag{7}$$

These integers lie in

$$\prod_{i=2}^{K} (p_i^2 - 1) = \frac{1}{3} \prod_{i=1}^{K} (p_i^2 - 1) = \frac{1}{3} P \prod_{i=1}^{K} \left(1 - \frac{1}{p_i^2}\right) >$$

$$< \frac{1}{3} P \prod_{i=1}^{+\infty} \left(1 - \frac{1}{p_i^2}\right) = \frac{1}{3} P \cdot \frac{1}{\zeta(2)} = \frac{2}{\pi^2} P > \frac{1}{5} P$$

residue classes modulo P. Let us take the intersection of the set $\{1, 2, \ldots, N\}$ with each of these residue classes. In this way, we get $\prod_{i=2}^{K} (p_i^2 - 1)$ arithmetic progressions; let us denote the set of them by $\mathbf{B}$, so that

$$|\mathbf{B}| = \prod_{i=2}^{K} (p_i^2 - 1) > \frac{1}{5} P. \tag{8}$$

Then for $\mathscr{B}\in\mathbf{B}$, clearly we have

$$(9)\qquad [N/P]\leqq|\mathscr{B}|<[N/P]+1 \quad (\text{for}\quad \mathscr{B}\in\mathbf{B}).$$

If $\mathscr{B}\in\mathbf{B}$, $n\in\mathscr{B}$, then n satisfies (6) and (7), so that n is not divisible by $p_1^2, \dots, p_K^2$. Thus if n is not squarefree then $p_i^2|n$ for some $K<i\leqq\pi(N^{1/2})$. In view of (4), the number of the integers n with $n\leqq N$, $p_i^2|n$ $\left(K<i\leqq\pi(N^{1/2})\right)$ is

$$\sum_{i=K+1}^{\pi(N^{1/2})}\left[\frac{N}{p_i^2}\right]<\sum_{i=K+1}^{+\infty}<\frac{N}{p_i^2}<N\sum_{n=p_{K+1}}^{+\infty}\frac{1}{n^2}<N\sum_{n=p_{K+1}}^{+\infty}\frac{1}{(n-1)n}=$$

$$=N\sum_{n=p_{K+1}}^{+\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)=N\frac{1}{p_{K+1}-1}<N\frac{1}{p_K}<\frac{3N}{P\log P}$$

for N large enough. Thus, in view of (8) and (9), there exists an arithmetic progression $\mathscr{B}_0\in\mathbf{B}$ which contains less than

$$\frac{3N(\log P)^{-1}}{|\mathbf{B}|}\leqq\frac{3N(\log P)^{-1}}{P/5}<15\frac{N}{P\log P}$$

integers which are not squarefree (for N large enough). Let $n_1<n_2<\ldots<n_t$ be those integers in $\mathscr{B}_0$ which are not squarefree so that

$$(10)\qquad t<\frac{15\,N}{P\log P}.$$

Put $n_0=0$, $n_{t+1}=N+1$,

$$M=\max_{0\leqq i\leqq t}|\mathscr{B}_0\cap(n_i, n_{i+1})|,$$

and assume that this maximum is assumed at $i=r$:

$$M=|\mathscr{B}_0\subset(n_r, n_{r+1})|.$$

In view of (5), (8) and (10), for large N we have

$$\frac{N}{2P}<\left[\frac{N}{P}\right]\leqq|\mathscr{B}_0|=\sum_{i=0}^{t}\mathscr{B}_0\cap[n_i, n_{i+1})|\leqq\sum_{i=0}^{t}\left(1+|\mathscr{B}_0\cap(n_i, n_{i+1})|\right)\leqq$$

$$\leqq\sum_{i=0}^{t}(1+M)=(t+1)2M<\left(\frac{15N}{P\log P}+1\right)2M<\frac{31MN}{P\log P}$$

hence, by (3),

$$(11)\qquad M>\frac{1}{62}\log P\geqq\frac{1}{62}\log N^{1/2}=\frac{1}{124}\log N.$$

Let us write

$$\mathscr{B}_0\cap(n_r, n_{r+1})=\{2b, 2b+P, \dots, 2b+(M-1)P\}.$$

(Note that by (6) and $\mathscr{B}_0\in\mathbf{B}$, all the elements of $\mathscr{B}_0$ are even.) The elements of $\mathscr{B}_0\cap(n_r, n_{r+1})$ are squarefree. In fact, if $n\in\mathscr{B}_0\cap(n_r, n_{r+1})$, then by (6) and (7),

n is not divisible by $p_1^2, p_2^2, \ldots, p_K^2$, and by $n_r < n < n_{r+1}$, it is not divisible by p_{K+1}^2, $p_{K+2}^2, \ldots, p_{\pi(\sqrt{N})}^2$.

Let us put

$$\mathcal{A} = \left\{b, b+P, \ldots, b+\left[\frac{M-1}{2}\right]P\right\}.$$

Then for $a \in \mathcal{A}$, $a' \in \mathcal{A}$ we have $a+a' \in \mathcal{B}_0 \cap (n_r, n_{r+1})$, so that $a+a'$ is squarefree.

Finally, by (11) we have

$$|\mathcal{A}| = \left[\frac{M+1}{2}\right] > \frac{M}{2} > \frac{1}{248} \log N$$

which completes the proof of Theorem 1.

3. The proof of Theorem 2 will be based on the large sieve but we shall sieve by squares of primes. In this section, we derive the sieve result needed in the proof.

LEMMA 1. *If M, N are integers, $N \geqq 1$, $a_{M+1}, a_{M+2}, \ldots, a_{M+N}$ are arbitrary complex numbers, we put*

$$S(x) = \sum_{n=M+1}^{M+N} b_n e(nx).$$

Let $\mathcal{X}$ be a set of real numbers for which

$$\|x-x'\| \geqq \delta > 0 \tag{12}$$

whenever x and x' are distinct members of $\mathcal{X}$. Then

$$\sum_{x \in \mathcal{X}} |S(x)|^2 \leqq (\delta^{-1} + \pi N) \sum_n |b_n|^2.$$

PROOF. This is Corollary 2.2 in [3], p. 12.

LEMMA 2. *Let M, N be integers, $N \geqq 1$, and let $\mathcal{N}$ be a set of Z integers in the interval $[M+1, M+N]$. Put*

$$Z(q, h) = \sum_{\substack{n \in \mathcal{X} \\ n \equiv h \pmod{q}}} 1. \tag{13}$$

Then for $Q > 0$ we have

$$\sum_{p^2 \leqq Q} p^2 \sum_{h=1}^{p^2} \left(Z(p^2, h) - \frac{Z}{p^2}\right)^2 \leqq (Q^2 + \pi N) Z. \tag{14}$$

PROOF. Let us write

$$S(x) = \sum_{n \in \mathcal{N}} e(n\alpha).$$

Then by [3], p. 23, (3.1) we have

$$p \sum_{h=1}^{p} \left(Z(p, h) - \frac{Z}{p^2}\right)^2 = \sum_{a=1}^{p-1} \left|S\left(\frac{a}{p^2}\right)\right|. \tag{15}$$

Furthermore, by [3], p. 24, (3.4) we have

$$q\sum_{h=1}^{q}\left|\sum_{d|q}\frac{\mu(d)}{d}Z\left(\frac{q}{d},h\right)\right|^2=\sum_{\substack{1\leqq a\leqq q\\(a,q)=1}}\left|S\left(\frac{a}{q}\right)\right|^2.$$

Putting $q=p^2$ here, we obtain that

$$p^2\sum_{h=1}^{p^2}\left(Z(p^2,h)-\frac{1}{p}Z(p,h)\right)^2=\sum_{\substack{1\leqq a\leqq p^2\\(a,p)=1}}\left|S\left(\frac{a}{p^2}\right)\right|^2. \tag{16}$$

By (15) and (16), we have

$$\sum_{p^2\leqq Q}p^2\sum_{h=1}^{p^2}\left(Z(p^2,h)-\frac{1}{p^2}\right)^2= \tag{17}$$

$$=\sum_{p^2\leqq Q}\left(p^2\sum_{h=1}^{p^2}\left(Z(p^2,h)-\frac{1}{p}Z(p,h)\right)^2+p\sum_{h=1}^{p}\left(Z(p,h)-\frac{Z}{p}\right)^2\right)=$$

$$=\sum_{p^2\leqq Q}\Biggl(\sum_{\substack{1\leqq a\leqq p^2\\(a,p)=1}}\Bigl|S\Bigl(\frac{a}{p^2}\Bigr)\Bigr)^2+\sum_{a=1}^{p-1}\Bigl|S\Bigl(\frac{a}{p}\Bigr)\Bigr|^2\Biggr).$$

In order to estimate this last sum, we use Lemma 1 where $\mathscr{X}$ is taken as the set of the fractions of the form $\frac{a}{p^2}$ $(p^2\leqq Q,\ 1\leqq a\leqq p^2,\ (a,p)=1)$ and $\frac{a}{p}$ $(p^2\leqq Q,$ $1\leqq a\leqq p-1)$. Then for $x=\frac{a_1}{p_1^{\alpha_1}}\in\mathscr{X}$, $x'=\frac{a_2}{p_2^{\alpha_2}}\in\mathscr{X}$ (where $\alpha_1,\alpha_2=1$ or 2), $x\neq x'$ we have

$$\|x-x'\|=\left\|\frac{a_1}{p_1^{\alpha_1}}-\frac{a_2}{p_2^{\alpha_2}}\right\|=\left\|\frac{a_1p_2^{\alpha_2}-a_2p_1^{\alpha_1}}{p_1^{\alpha_1}p_2^{\alpha_2}}\right\|\geqq\frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}}\geqq\frac{1}{Q^2}$$

so that (12) in Lemma 1 holds with $\delta=Q^{-2}$. Thus by using Lemma 1, we obtain from (17) that

$$\sum_{p^2\leqq Q}p^2\sum_{h=1}^{p^2}\left(Z(p^2,h)-\frac{Z}{p^2}\right)^2\leqq(Q^2+\pi N)\sum_{n\in\mathscr{N}}1=(Q^2+\pi N)Z$$

which completes the proof of Lemma 2.

4. In this section, we derive Theorem 2 from Lemma 2.

Let $\mathscr{A}\subset\{1,2,\dots,N\}$ be a sequence such that for all $a\in\mathscr{A}$, $a'\in\mathscr{A}$ the sum $a+a'$ is squarefree. Then for all p, $a+a'\not\equiv 0 \pmod{p^2}$. Thus $\mathscr{A}$ may lie in at most $\frac{p^2-1}{2}$ residue classes modulo p^2, hence, defining $Z(q,h)$ by (13) (with $\mathscr{A}$ in place of $\mathscr{N}$), we have $Z(p^2,h)=0$ for at least $p^2-\frac{p^2-1}{2}=\frac{p^2+1}{2}$ incongruent values of h. Thus the left hand side of (14) in Lemma 2 can be estimated in the following way:

$$\sum_{p^2 \leqq Q} p^2 \sum_{h=1}^{p^2} \left(Z(p^2, h) - \frac{Z}{p^2} \right)^2 \geqq \sum_{p^2 \leqq Q} p^2 \sum_{\substack{1 \leqq h \leqq p^2 \\ Z(p^2, h)=0}} \frac{Z^2}{p^4} = \tag{18}$$

$$= \sum_{p^2 \leqq Q} \frac{Z^2}{p^2} \sum_{\substack{1 \leqq h \leqq p^2 \\ Z(p^2, h)=0}} 1 \geqq \sum_{p^2 \leqq Q} \frac{Z^2}{p^2} \frac{p^2+1}{2} > \frac{Z^2}{2} \sum_{p^2 \leqq Q} 1 = \frac{Z^2}{2} \pi(Q^{1/2}).$$

Setting $Q=N^{1/2}$, we obtain from (18) and Lemma 2 that

$$\frac{Z^2}{2} \pi(N^{1/4}) < (N+\pi N)Z$$

hence, by the prime number theorem, for large N we have

$$Z < \frac{2(1+\pi)N}{\pi(N^{1/4})} < \frac{9N}{N^{1/4}\left(\frac{1}{4}\log N\right)^{-1}} < 3N^{3/4} \log N$$

which completes the proof of Theorem 2.

References

[1] A. Balog and A. Sárközy, On sums of sequences of integers. II, *Acta Math. Hung.*, **44** (1984), 339—349.

[2] P. Erdős and A. Sárközy, On differences and sums of integers, I, *J. Number Theory*, **10** (1978), 430—450.

[3] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer Verlag (1971).

[4] A. Sárközy and C. L. Stewart, On divisors of sums of integers, II, *J. Reine Angew. Math.*, to appear.

(Received August 21, 1985)

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13—15.
H—1053

Acta Math. Hung.
50 (1—2) (1987), 123—125.

ON THE CHARACTERIZATION OF ADDITIVE FUNCTIONS ON RESIDUE CLASSES

K. KOVÁCS (Budapest)

In 1946 P. Erdős [1] discovered the following theorem:

If a real-valued additive function is monotonic, then it is $c\cdot\log n$.

The following assertion can be proved similarly and seems to be known among number theorists, though I cannot give a specific reference.

($*$) Let $q\geqq 2$ be a fixed natural number. If the additive function $f: N\to R$ is monotonic on the residue class (1) modulo q from a point on $(n\geqq n_0)$, then $f(n)=c\log n$ for all $(n, q)=1$.

In [2] we proved the following generalization of the mentioned theorem of Erdős:

($**$) If a function $f: N\to R^k$ is additive and its Euclidean norm is monotonic from some point on, then $f(n)=\mathbf{c}\log n$ with a constant vector $\mathbf{c}\in R^k$.

Applying ($*$) and ($**$) we prove the following theorem:

THEOREM. *If a function $f: N\to R^k$ is additive and its Euclidean norm is monotonic on an arbitrary fixed reduced residue class a* $(\mathrm{mod}\, q)$ $(q\in N,\ q\geqq 2)$ *from a number n_0 on, then $f(n)=\mathbf{c}\log n$ for all $(n, q)=1$.*

Throughout the paper the following notations are used: P is the set of all primes; $P_1, P_2\subset P$ are infinite sets; $\mathbf{u}$, $\mathbf{v}$ etc. are vectors in R^k; $\langle \mathbf{u}, \mathbf{v}\rangle$ is the scalar product; $T=\{n\in N: (n, q)=1\}$; $(\alpha)=\{n: n\equiv\alpha \ (\mathrm{mod}\, q)\}$.

LEMMA 1. *If $|f|$ is convergent on some (α), $(\alpha, q)=1$, then $f(n)=0$ on T.*

PROOF OF THE LEMMA. There exists a set of primes $p_i\in(\alpha)$, such that $\lim\limits_{i\to\infty} f(p_i)=\mathbf{c}$ with some constant $\mathbf{c}\in R^k$, because the sphere with radius $|\mathbf{c}|$ is compact. For all $a\in(\alpha)$ we have $a^{\varphi(q)+1}\in(\alpha)$, so

$$\mathbf{c}=\lim_{s\to\infty} f\Big(\prod_{i=s}^{s+\varphi(q)} p_i\Big)=\lim_{s\to\infty}\Big(\sum_{i=s}^{s+\varphi(q)} f(p_i)\Big)=(\varphi(q)+1)\mathbf{c},$$

which implies $\mathbf{c}=0$.

For a fixed $a\in(\alpha)$, let $p_j\in(\alpha)$ be primes such that $(p_j, a)=1$. Consider the numbers $t_{ai}=\prod\limits_{j=i}^{i+\varphi(q)-1} p_j$. Thus $at_{ai}\in(\alpha)$ and

$$|f(a)|=|f(at_{ai})-f(t_{ai})|\leqq|f(at_{ai})|+\sum_{j=1}^{i+\varphi(q)-1}|f(p_j)|\to 0,$$

consequently $f\big((\alpha)\big)=0$.

It is easy to see that $f((\alpha))=0$ yields $f(n)=0$ on T.

Consequence: If $|f|$ is monotonic and bounded on a reduced residue class (α) mod q, then $f(n)=0$ on T. So it is enough to examine the case that $|f|$ is monotonic and unbounded on (α).

MAIN LEMMA. *If* $|f|\to\infty$ *on* (α) *and* $|f|$ *is monotonic on* (α), *then* $f(n)=\mathbf{c}\log n+g(n)$ *where* $\langle g(n), \mathbf{c}\rangle=0$ *for all* $n\in T$ *and* $g(n)=o(\log n)$ *on* T.

PROOF. We proceed as in [2], so we note only the different steps. $|f(n)|\to\infty$ implies that there exists a $P_1\subset P$, such that $\lim\limits_{\substack{p\to\infty \\ p\in P_1}}\frac{f(p)}{|f(p)|}=\boldsymbol{\tau}$ with $|\tau|=1$. These primes are all in the reduced residue classes mod q and there exists at least one (μ) containing infinitely many primes of P_1 with the previous property.

Let $b<d$, $b, d\in(1)$. Take an $m\in N$ with $mb\geqq n_0$, $(m, qbd)=1$ and $m\mu\equiv\alpha$ (mod q). Then $bmp, dmp\in(\alpha)$ for all $p\in P_1\cap(\mu)$. By the monotonicity of $|f|$ on (α) we have $|f(bwp)|\leqq|f(dwp)|$, which gives like in [2]

$$\langle f(bw), \tau\rangle \leqq \langle f(dw), \tau\rangle, \quad \text{i.e.} \quad \langle f(b), \tau\rangle \leqq \langle f(d), \tau\rangle.$$

This yields by $(*)$

$$\langle f(n), \tau\rangle = c\cdot\log n \quad \text{on } T,$$

so $f(n)=\mathbf{c}\log n+g(n)$ with $\langle g, \mathbf{c}\rangle=0$ too, where $\mathbf{c}=c\boldsymbol{\tau}$. As in [2] (Lemmas 2 and 3) we get that we have only one $\boldsymbol{\tau}$ and that $g(n)=o(\log n)$ for all primes, further for all squarefree numbers.

We know that g is small on squarefree numbers and we want to estimate it on an arbitrary $n\in T$. First assume $n\in(\alpha)$. Take a prime $m\in[n, 2n]$, $m\in(\alpha)$, which exists for large n. By the monotonicity of $|f|$ we know

$$0 \leqq |f(m)|^2-|f(n)|^2 = c^2(\log^2 m-\log^2 n)+|g(m)|^2-|g(n)|^2,$$

hence

$$|g(n)|^2 \leqq |g(m)|^2+O(\log^2 m-\log^2 n) = o(\log^2 n).$$

Next, let $n\in T$ be arbitrary. Take a prime $p\in[n, 2n]$, $pn\in(\alpha)$. Then

$$g(n) = g(pn)-g(p) = o(\log n),$$

where the first term is small because $pn\in(\alpha)$ and the second because p is a prime.

The proof of the Theorem is continued as in [2]. The proof there is based on four lemmas (A, B, C, D). Lemmas A and B can be applied unchanged. Instead of Lemma C we use

LEMMA C′. *For any large* r, *in the interval* $[n, n+D]$, $D=e^{r^2}$ *there are* r *pairwise coprime elements of* T.

Instead of Lemma D, we work with

LEMMA D′. *If* $n, n+j \in T$ *and* $0<j\leqq q$, *we have*

$$|g(n+j)| \geqq g(n)-\frac{C_5}{\sqrt[3]{n}}.$$

For an n such a j can always be found. The proofs present no difficulties.

So the case $g\not\to 0$ on T contradicts $g(n)=o(\log n)$ as in [2]. Therefore $g\to 0$, which gives $g(n)=0$ on T by Lemma 1.

References

1] P. Erdős, On the distribution function of additive functions, *Ann. of Math.*, **47** (1946), 1—20.
[2] K. Kovács, On the characterization of additive functions with monotonic norm, to appear in *Journal of Number Theory*.

(Received August 29, 1985)

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13—15.
H—1053

Acta Math. Hung.
50 (1—2) (1987), 127—145.

ON THE RESIDUAL PARTS OF COMPLETELY NON-UNITARY CONTRACTIONS

L. KÉRCHY (Szeged)

Introduction

By a celebrated theorem of B. Sz.-Nagy, to every contraction operator T on a Hilbert space there corresponds a unitary operator U, closely related to T, the so-called unitary dilation of T. Starting from this result B. Sz.-Nagy and C. Foiaş have developed an effective theory for contractions (cf. [14]). If T is not of class C_{00} then especially two restrictions of U to certain reducing subspaces play important roles in the study of T. These unitary operators are called the residual and $*$-residual parts of T. By their aid it could be proved e.g. that every C_{11}-contraction has many hyperinvariant subspaces.

In the present paper we intend to give a systematic study of the residual parts and, as a consequence, to prove a reflexivity theorem for a class of contractions.

We shall deal only with completely non-unitary contractions and use their functional model. The model operator and the residual parts are introduced in Section 1. The residual part is well-represented in this model, but the $*$-residual part is not. In Section 2 we give a natural representation for the $*$-residual part too. Section 3 deals with the canonical intertwining operators between the contraction and the residual parts and provides a representation for them as a multiplication by an operator-valued function. Section 4 is devoted to the study of the weak invertibility of the canonical intertwiners. In Section 5 the residual parts together with the canonical intertwining operators are characterized as universal intertwiners corresponding to T. In Section 6 generalizing C_{11}-contractions we define the class of quasi-C_{11} contractions, and using a recent result of Bercovici and Takahashi we prove a reflexivity theorem for this class. This theorem partly extends the validity of the main result in [11]. Finally, in Section 7 our reflexivity theorem is applied for C_{11}-contractions, and the connection between the result obtained and the general reflexivity problem of C_{11}-contractions is discussed.

We shall use the terminology of [14].

1. The residual parts of completely non-unitary contractions

Let $\mathfrak{E}$ and $\mathfrak{E}_*$ be complex, separable Hilbert spaces and let us consider the Hilbert space $L^2(\mathfrak{E}_*)\oplus L^2(\mathfrak{E})\cong L^2(\mathfrak{E}_*\oplus\mathfrak{E})$ of vector-valued functions defined with respect to the normalized Lebesgue measure m on the unit circle $\partial\mathbf{D}$ (the boundary of the open unit disc $\mathbf{D}=\{\lambda\in\mathbf{C}: |\lambda|<1\}$). Let M denote the operator of multiplication by the identical function $f(\lambda)=\lambda$ in every L^2-space. Let us consider a purely contractive, analytic, operator-valued function $\Theta:\mathbf{D}\to\mathscr{L}(\mathfrak{E},\mathfrak{E}_*)$, where $\mathscr{L}(\mathfrak{E},\mathfrak{E}_*)$ stands for the Banach space of bounded, linear transformations mapping $\mathfrak{E}$ into

$\mathfrak{E}_*$. In a.e. point $\zeta\in\partial\mathbf{D}$ the function Θ has a radial limit $\Theta(\zeta)\in\mathscr{L}(\mathfrak{E},\mathfrak{E}_*)$ in the strong operator topology. This limit function, which will be also denoted by $\Theta:\partial\mathbf{D}\to\mathscr{L}(\mathfrak{E},\mathfrak{E}_*)$, is contractive, measurable, and so is its defect function

$$\Delta:\ \partial\mathbf{D}\to\mathscr{L}(\mathfrak{E})(=\mathscr{L}(\mathfrak{E},\mathfrak{E})),\quad \Delta(\zeta)=(I-\Theta(\zeta)^*\Theta(\zeta))^{1/2}$$

too.

The subspace $\mathfrak{K}=\mathfrak{K}(\Theta):=L^2(\mathfrak{E}_*)\oplus\mathfrak{R}$, where $\mathfrak{R}=(\Delta L^2(\mathfrak{E}))^-$, is reducing for $M\in\mathscr{L}(L^2(\mathfrak{E}_*)\oplus L^2(\mathfrak{E}))$, hence $U:=M|\mathfrak{K}$ is a unitary operator. Let $H^2(\mathfrak{E}_*)$ stand for the Hardy space $H^2(\mathfrak{E}_*)=\bigvee_{n\geqq 0}M^n\mathfrak{E}_*$. Then $\mathfrak{K}_+:=H^2(\mathfrak{E}_*)\oplus\mathfrak{R}$ is invariant for U, so the restriction $U_+:=U|\mathfrak{K}_+$ is an isometry. The operator $R=U_+|\mathfrak{R}$ is the unitary part of U_+ according to the Wold decomposition. The isometry V: $L^2(\mathfrak{E})\to\mathfrak{K}$, $Vw=\Theta w\oplus\Delta w$ intertwines the multiplications by the identical functions, consequently the subspace $\mathfrak{M}_+:=VH^2(\mathfrak{E})$ of $\mathfrak{K}_+$ is invariant for U_+. Let $P_\Theta\in\mathscr{L}(\mathfrak{K}_+)$ denote the orthogonal projection onto $\mathfrak{H}=\mathfrak{H}(\Theta):=\mathfrak{K}_+\ominus\mathfrak{M}_+$, and let $T=S(\Theta)\in\mathscr{L}(\mathfrak{H})$ be defined as the compression $T:=P_\Theta U_+|\mathfrak{H}$.

One of the central results of the theory of contractions, having developed by Sz.-Nagy and Foiaş (cf. [14]), states that this operator T can be considered as the model of completely non-unitary contractions acting on separable Hilbert spaces. This means that for every Θ, T is a c.n.u. contraction, and every c.n.u. contraction can be obtained up to unitary equivalence in this way. Furthermore, U_+ and U are the minimal isometric and unitary dilations of T, respectively. The unitary operator $R_T=R$ is called the *residual part of* T.

The subspace $\mathfrak{M}:=VL^2(\mathfrak{E})=\bigvee_{n\geqq 0}U^{-n}\mathfrak{M}_+$ of $\mathfrak{K}$ reduces U and so does its orthogonal complement $\mathfrak{R}_*=\mathfrak{K}\ominus\mathfrak{M}$ too. The unitary operator $R_{*,T}=R_*=U|\mathfrak{R}_*$ is called the $*$*-residual part of* T. Taking into account that the adjoint $U^*\in\mathscr{L}(\mathfrak{K})$ is the minimal unitary dilation of $T^*\in\mathscr{L}(\mathfrak{H})$, it easily follows in view of the geometric structure of U (cf. [14, ch. II]) that

$$R_{*,T}=(R_{T^*})^*\quad\text{and}\quad R_T=(R_{*,T^*})^*. \tag{1}$$

2. The representation of the $*$-residual part

In the model introduced above the residual part of T is well-represented as the range-space of the defect function of Θ. In order to give a similar representation for the $*$-residual part of T let us consider the operator-valued function

$$\begin{cases} W:\ \partial\mathbf{D}\to\mathscr{L}(\mathfrak{E}_*\oplus\mathfrak{E}),\\ W(\zeta)=\begin{bmatrix}-\Delta_*(\zeta) & \Theta(\zeta)\\ \Theta(\zeta)^* & \Delta(\zeta)\end{bmatrix},\end{cases} \tag{2}$$

where $\Delta_*(\zeta)=[I-\Theta(\zeta)\Theta(\zeta)^*]^{1/2}$ is the defect function of the adjoint function $\Theta^*(\zeta):=\Theta(\zeta)^*$. Let $x_*\in\mathfrak{E}_*$ and $x\in\mathfrak{E}$ be arbitrary vectors. Applying the commuting relations

$$\Theta^*\Delta_*=\Delta\Theta^*,\quad \Theta\Delta=\Delta_*\Theta, \tag{3}$$

we get that

$$\begin{aligned}\|W(\zeta)(x_*\oplus x)\|^2_{\mathfrak{E}_*\oplus\mathfrak{E}} &= \|(-\Delta_*(\zeta)x_*+\Theta(\zeta)x)\oplus(\Theta^*(\zeta)x_*+\Delta(\zeta)x)\|^2_{\mathfrak{E}_*\oplus\mathfrak{E}} = \\ &= \|-\Delta_*(\zeta)x_*+\Theta(\zeta)x\|^2_{\mathfrak{E}_*}+\|\Theta^*(\zeta)x_*+\Delta(\zeta)x\|^2_{\mathfrak{E}} = \langle\Delta_*(\zeta)^2x_*, x_*\rangle_{\mathfrak{E}_*}- \\ &\quad -2\mathrm{Re}\,\langle\Theta^*(\zeta)\Delta_*(\zeta)x_*, x\rangle_{\mathfrak{E}}+\langle\Theta^*(\zeta)\Theta(\zeta)x, x\rangle_{\mathfrak{K}}+ \\ &\quad +\langle\Theta(\zeta)\Theta^*(\zeta)x_*, x_*\rangle_{\mathfrak{E}_*}+2\mathrm{Re}\,\langle\Delta(\zeta)\Theta^*(\zeta)x_*, x\rangle_{\mathfrak{E}}+ \\ &\quad +\langle\Delta(\zeta)^2x, x\rangle_{\mathfrak{E}} = \|x_*\|^2_{\mathfrak{E}_*}+\|x\|^2_{\mathfrak{E}} = \|x_*\oplus x\|^2_{\mathfrak{E}_*\oplus\mathfrak{E}}\end{aligned}$$

holds for a.e. $\zeta\in\partial\mathbf{D}$. Since $\mathfrak{E}$ and $\mathfrak{E}_*$ are separable spaces and $W(\zeta)$ is selfadjoint, it follows that $W(\zeta)$ is a selfadjoint, unitary operator a.e. Therefore, the operator $W\in\mathscr{L}(L^2(\mathfrak{E}_*)\oplus L^2(\mathfrak{E}))$ of multiplication by the function W is unitary, selfadjoint, and clearly commutes with $M\in\mathscr{L}(L^2(\mathfrak{E}_*)\oplus L^2(\mathfrak{E}))$.

Let $\mathfrak{K}', \mathfrak{K}'_+, \ldots$ denote the subspaces obtained by applying the transformation W for $\mathfrak{K}, \mathfrak{K}_+, \ldots$, i.e. $\mathfrak{K}'=W\mathfrak{K}$, $\mathfrak{K}'_+=W\mathfrak{K}_+, \ldots$. Then for the operators $U'=M|\mathfrak{K}'$, $U'_+=U'|\mathfrak{K}'_+, \ldots$ we have

$$U'\cong U, \quad U'_+\cong U_+, \quad T'\cong T, \quad R'\cong R, \quad R'_*\cong R_*,$$

and the unitary equivalences are implemented by the operators $W'=W|\mathfrak{K}\in\mathscr{L}(\mathfrak{K}, \mathfrak{K}')$, $W'_+=W|\mathfrak{K}_+\in\mathscr{L}(\mathfrak{K}_+, \mathfrak{K}'_+)$, $W_0=W|\mathfrak{H}\in\mathscr{L}(\mathfrak{H}, \mathfrak{H}')$, $W_r=W|\mathfrak{R}\in\mathscr{L}(\mathfrak{R}, \mathfrak{R}')$, $W_{*,r}=W|\mathfrak{R}_*\in\mathscr{L}(\mathfrak{R}_*, \mathfrak{R}'_*)$, respectively.

PROPOSITION 1. *The subspaces introduced above are of the following form:*

$$\mathfrak{K}'=(\Delta_*L^2(\mathfrak{E}_*))^-\oplus L^2(\mathfrak{E}), \tag{4}$$

$$\mathfrak{M}'_+=H^2(\mathfrak{E}), \quad \mathfrak{M}'=L^2(\mathfrak{E}), \tag{5}$$

and

$$\mathfrak{R}'_*=(\Delta_*L^2(\mathfrak{E}_*))^-. \tag{6}$$

PROOF. On account of (3) we can write for any functions $u\in L^2(\mathfrak{E}_*)$ and $v\in L^2(\mathfrak{E})$ that $W(u\oplus\Delta v)=(-\Delta_*u+\Theta\Delta v)\oplus(\Theta^*u+\Delta^2v)=\Delta_*(-u+\Theta v)\oplus(\Theta^*u+\Delta^2v)$, which yields that

$$\mathfrak{K}'\subset(\Delta_*L^2(\mathfrak{E}_*))^-\oplus L^2(\mathfrak{E}). \tag{7}$$

Since for every $w\in L^2(\mathfrak{E})$

$$WVw=\begin{bmatrix}-\Delta_* & \Theta\\ \Theta^* & \Delta\end{bmatrix}\begin{bmatrix}\Theta\\ \Delta\end{bmatrix}w=\begin{bmatrix}-\Delta_*\Theta+\Theta\Delta\\ \Theta^*\Theta+\Delta^2\end{bmatrix}w=\begin{bmatrix}0\\ I\end{bmatrix}w=0\oplus w \tag{8}$$

holds, the relations of (5) follow.

For the sake of clarity let us now write M_Θ for the operator of multiplication by Θ, i.e. $M_\Theta\in\mathscr{L}(L^2(\mathfrak{E}), L^2(\mathfrak{E}_*))$ is a contraction. Then the adjoint of M_Θ is the multiplication by the adjoint function $(M_\Theta)^*=M_{\Theta^*}$, and its defect operators are the multiplications by the defect functions $D_{M_\Theta}=M_\Delta$ and $D_{M_\Theta^*}=M_{\Delta_*}$. It is well-known (cf. [14, sec. I.3]) that

$$\mathfrak{D}_{M_\Theta^*}=(M_\Theta\mathfrak{D}_{M_\Theta})^-\oplus\ \ker\ M_\Theta^*$$

is true for the defect spaces $\mathfrak{D}_{M_\Theta}$ and $\mathfrak{D}_{M_\Theta^*}$. Hence we obtain

$$(\varDelta_* L^2(\mathfrak{E}_*))^- = (\Theta\varDelta L^2(\mathfrak{E}))^- \oplus \ker \Theta^*. \tag{9}$$

If $u \in \ker \Theta^*$, then $\varDelta_* u = u$ and so

$$\begin{bmatrix} -\varDelta_* & \Theta \\ \Theta^* & \varDelta \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} -\varDelta_* u \\ 0 \end{bmatrix} = - \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

We infer that

$$W|\ker \Theta^* = -I|\ker \Theta^*, \tag{10}$$

hence

$$W(L^2(\mathfrak{E}_*)) \supset W(\ker \Theta^*) = \ker \Theta^*. \tag{11}$$

On the other hand, for any vector $v \in (\varDelta L^2(\mathfrak{E}))^-$ we have

$$P_* W \begin{bmatrix} 0 \\ v \end{bmatrix} = P_* \begin{bmatrix} \Theta v \\ \varDelta v \end{bmatrix} = \begin{bmatrix} \Theta v \\ 0 \end{bmatrix}, \tag{12}$$

which yields

$$(P_* W \mathfrak{R})^- = (\Theta\varDelta L^2(\mathfrak{E}))^- \oplus \{0\}, \tag{13}$$

where $P_* \in \mathscr{L}(L^2(\mathfrak{E}_*), L^2(\mathfrak{E}))$ denotes the orthogonal projection onto the subspace $L^2(\mathfrak{E}_*)$.

Since W is unitary (6) and then (4) follow by (7), (5), (9), (11) and (13).

3. Operators intertwining T and its residual parts

We shall consider the canonical operators intertwining T and its residual parts, introduced by Sz.Nagy and Foiaş, and examine how they can be represented by the aid of the unitary operator W. In the sequel let $P_{\mathfrak{K}_+}, P_{\mathfrak{H}}, P_{\mathfrak{R}}, P_{\mathfrak{R}_*}, \ldots \in \mathscr{L}(\mathfrak{K})$ denote the orthogonal projections onto the subspaces $\mathfrak{K}_+, \mathfrak{H}, \mathfrak{R}, \mathfrak{R}_*, \ldots$, respectively.

Since $P_{\mathfrak{R}_*}$ commutes with U we obtain $R_*(P_{\mathfrak{R}_*}|\mathfrak{H}) = UP_{\mathfrak{R}_*}|\mathfrak{H} = P_{\mathfrak{R}_*}U|\mathfrak{H} = P_{\mathfrak{R}_*}P_{\mathfrak{H}}U|\mathfrak{H} + P_{\mathfrak{R}_*}P_{\mathfrak{M}_+}U|\mathfrak{H} = (P_{\mathfrak{R}_*}|\mathfrak{H})T$. Hence, the operator

$$X := P_{\mathfrak{R}_*}|\mathfrak{H} \in \mathscr{L}(\mathfrak{H}, \mathfrak{R}_*) \tag{14}$$

intertwines T and its $*$-residual part R_*: $XT = R_* X$, i.e. X belongs to the set $\mathscr{I}(T, R_*) := \{Q \in \mathscr{L}(\mathfrak{H}, \mathfrak{R}_*) : QT = R_* Q\}$. Taking into consideration that $P_{\mathfrak{M}_+}\mathfrak{R}_* = \{0\}$, it immediately follows that

$$\mathscr{L}(\mathfrak{R}_*, \mathfrak{H}) \ni X^* = P_{\mathfrak{H}}|\mathfrak{R}_* = P_{\mathfrak{K}_+}|\mathfrak{R}_*, \tag{15}$$

and

$$X^* \in \mathscr{I}(R_*^*, T^*). \tag{16}$$

Now, we infer by (1) that the operator

$$Y := P_{\mathfrak{H}}|R \in \mathscr{L}(\mathfrak{R}, \mathfrak{H}) \tag{17}$$

intertwines R and T:

$$Y \in \mathscr{I}(R, T), \tag{18}$$

and so for its adjoint

$$Y^* = P_{\mathfrak{R}} | \mathfrak{H} \in \mathscr{L}(\mathfrak{H}, \mathfrak{R}) \tag{19}$$

we have

$$Y^* \in \mathscr{I}(T^*, R^*). \tag{20}$$

This adjoint Y^* is of very simple form, namely $Y^* = P | \mathfrak{H}$, where $P \in \mathscr{L}\big(L^2(\mathfrak{E}_*) \oplus \oplus L^2(\mathfrak{E})\big)$ stands for the orthogonal projection onto the subspace $L^2(\mathfrak{E})$.

On the other hand, if X', Y' are defined in an analogous way with respect to the space $\mathfrak{K}'$, then X' has a simple form:

$$\mathscr{L}(\mathfrak{H}', \mathfrak{R}'_*) \ni X' = P_{\mathfrak{R}'_*} | \mathfrak{H}' = P_* | \mathfrak{H}'.$$

Let us consider the operator

$$\tilde{X} := X' W_0 \in \mathscr{L}(\mathfrak{H}, \mathfrak{R}'_*). \tag{21}$$

Then $\tilde{X}$ intertwines T and R'_*, and for any vector $u \oplus v \in \mathfrak{H}$ we have

$$\tilde{X}(u \oplus v) = P_* \begin{bmatrix} -\Delta_* & \Theta \\ \Theta^* & \Delta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = P_* \begin{bmatrix} -\Delta_* u + \Theta v \\ \Theta^* u + \Delta v \end{bmatrix} = (-\Delta_* u + \Theta v) \oplus 0. \tag{22}$$

Now, we examine the operator

$$Z := P_{\mathfrak{R}_*} | \mathfrak{R} \in \mathscr{L}(\mathfrak{R}, \mathfrak{R}_*). \tag{23}$$

Since every vector $v \in \mathfrak{R}$ can be decomposed into the sum $v = Yv + v'$, where $v' \in \mathfrak{M}_+$, it follows that $P_{\mathfrak{R}_*} v = P_{\mathfrak{R}_*} Yv + P_{\mathfrak{R}_*} v' = XYv$, hence

$$Z = XY \in \mathscr{I}(R, R_*). \tag{24}$$

Let us consider the operator

$$\tilde{Z} := W_{r,*} Z \in \mathscr{I}(R, R'_*). \tag{25}$$

It is immediate that

$$\tilde{Z} = W_{r,*} XY = X' W_0 Y = \tilde{X} Y. \tag{26}$$

On the other hand, since $\tilde{Z} = W_{r,*} Z = Z' W_r = P_* W | \mathfrak{R}$, we infer by (12) that

$$\tilde{Z}(0 \oplus v) = \Theta v \oplus 0, \tag{27}$$

for every vector $v \in \mathfrak{R}$. The adjoint of Z is of the form

$$Z^* = P_{\mathfrak{R}} | \mathfrak{R}_* = P | \mathfrak{R}_* \in \mathscr{I}(R_*, R). \tag{28}$$

It follows that $\tilde{Z}^* = (W_{r,*} Z)^* = Z^* W_{r,*}^* = Z^* W^* | \mathfrak{R}'_* = Z^* W | \mathfrak{R}'_* = PW | \mathfrak{R}'_*$. Hence, we conclude

$$\tilde{Z}^*(u \oplus 0) = 0 \oplus \Theta^* u, \tag{29}$$

for every vector $u \in \mathfrak{R}'_*$.

Finally, let us assume that Θ is an inner function, or equivalently that T is of class $C_{\cdot 0}$. Then $\Theta(\zeta)$ is isometric, and so $\Delta(\zeta)=0$ a.e. on $\partial\mathbf{D}$. This yields that $\mathfrak{R}=\{0\}$, $\mathfrak{K}=L^2(\mathfrak{E}_*)$, $\mathfrak{K}_+=H^2(\mathfrak{E}_*)$ and $\mathfrak{H}=H^2(\mathfrak{E}_*)\ominus\Theta H^2(\mathfrak{E})$. Since $\Theta(\zeta)$ is an isometry, it follows that $\Delta_*(\zeta)\in\mathscr{L}(\mathfrak{E}_*)$ is the orthogonal projection onto the subspace $\ker\Theta(\zeta)^*$ a.e. on $\partial\mathbf{D}$. Hence we obtain by (6)

$$\mathfrak{R}'_* = \Delta_* L^2(\mathfrak{E}_*) = \ker\Theta^*. \tag{30}$$

In virtue of (10) and (30) we infer $W_{r,*}^*|\mathfrak{R}'_*=W|\mathfrak{R}'_*=-I|\mathfrak{R}'_*$, which implies that

$$\mathfrak{R}_* = W_{r,*}^*\mathfrak{R}'_* = \mathfrak{R}'_*, \tag{31}$$

and by (22) that

$$Xu = W_{r,*}^*\tilde{X}u = -\tilde{X}u = \Delta_* u \tag{32}$$

holds, for every vector $u\in\mathfrak{H}$.

4. Weak invertibility of the canonical intertwining operators

In the following proposition the characterizations of injectivity and quasi-surjectivity of the operators X and Y, introduced in the previous section, are collected.

PROPOSITION 2. a) *X is injective if and only if T is of class $C_{1\cdot}$, or equivalently if Θ is a $*$-outer function;*

b) *Y is quasi-surjective, i.e. it has a dense range, if and only if T belongs to $C_{\cdot 1}$, or equivalently if Θ is an outer function;*

c) *we have*

$$\ker X^* = \ker\Theta^*\cap\big(L^2(\mathfrak{E}_*)\ominus H^2(\mathfrak{E}_*)\big); \tag{33}$$

hence X is quasi-surjective if and only if

$$\ker\Theta^*\cap\big(L^2(\mathfrak{E}_*)\ominus H^2(\mathfrak{E}_*)\big) = \{0\}; \tag{34}$$

d) *we have*

$$\ker Y = \ker\Theta\cap H^2(\mathfrak{E}); \tag{35}$$

hence Y is injective if and only if

$$\ker\Theta\cap H^2(\mathfrak{E}) = \{0\}. \tag{36}$$

PROOF. The statement a) is an immediate consequence of [14, Proposition II.3.1], which claims among others that $\|Xh\|=\lim\limits_{n\to\infty}\|T^n h\|$ holds, for every $h\in\mathfrak{H}$. The statement b) follows from a) in virtue of the relation (1).

Let us prove (33). Since $\tilde{X}=W_{r,*}X$, we infer, using (15), that

$$\tilde{X}^* = X^*W_{r,*}^* = X^*W|\mathfrak{R}'_* = P_{\mathfrak{K}_+}W|\mathfrak{R}'_*. \tag{37}$$

This implies that

$$\ker X^* = W(\ker\tilde{X}^*). \tag{38}$$

On the other hand, for an arbitrary vector $u\in\mathfrak{R}'_*$ we have $\tilde{X}^*u=P_{\mathfrak{K}_+}(-\Delta_*u\oplus$

$\oplus\Theta^* u)=0$, if and only if $\Theta^* u=0$ and $\Delta_* u\in L^2(\mathfrak{E}_*)\ominus H^2(\mathfrak{E}_*)$. However, taking into account that $\Delta_* u=u$ whenever $\Theta^* u=0$, we obtain that $\tilde{X}^* u=0$ exactly when $u\in\ker\Theta^*\cap\big(L^2(\mathfrak{E}_*)\ominus H^2(\mathfrak{E}_*)\big)$. Therefore,

$$\ker\tilde{X}^* = \ker\Theta^*\cap\big(L^2(\mathfrak{E}_*)\ominus H^2(\mathfrak{E}_*)\big), \tag{39}$$

and (33) follows by (38), (39) and (10).

Finally, let us verify (35). It is clear that $\ker Y=\mathfrak{N}\cap\mathfrak{M}_+$, so we have to show that

$$\mathfrak{N}\cap\mathfrak{M}_+ = \ker\Theta\cap H^2(\mathfrak{E}). \tag{40}$$

Let us assume first that $v\in\ker\Theta\cap H^2(\mathfrak{E})$. Then $\Theta v=0$ and $v\in H^2(\mathfrak{E})$, which implies $\Delta v=v$. Hence $v=\Delta v\in\mathfrak{N}$ and $v=0\oplus v=Vv\in\mathfrak{M}_+$, i.e. $v\in\mathfrak{N}\cap\mathfrak{M}_+$. Let us consider now an arbitrary vector $u\oplus v\in\mathfrak{N}\cap\mathfrak{M}_+$. Since $u\oplus v\in\mathfrak{N}$, it follows that $u=0$. On the other hand, $u\oplus v\in\mathfrak{M}_+$ implies $0\oplus v=\Theta w\oplus\Delta w$ with a vector $w\in H^2(\mathfrak{E})$. But here $\Theta w=0$ results that $v=\Delta w=w\in\ker\Theta\cap H^2(\mathfrak{E})$.

The statements a) and c) of Proposition 2 together with the relations (22) and (27) give a generalization of [12, Theorem 10]. In the following remarks we shall discuss the equation (34), noting that an analogous discussion can be carried out in connection with (36) too.

REMARK 1. First we note that the relation (34) does not imply $\ker\Theta^*=\{0\}$ even if we assume that Θ is an inner and $*$-outer function. Indeed, we can find by [13, Theorem 2] an inner and $*$-outer function Θ such that $\sigma(R_*)\neq\partial\mathbf{D}$ is true for the spectrum $\sigma(R_*)$ of the $*$-residual part R_*. For this function we have $\Delta_*(\zeta)=0$ in a.e. point ζ of the set $\partial\mathbf{D}\setminus\sigma(R_*)$ of positive Lebesgue measure. Hence by (30) we infer that $\ker\Theta^*=\Delta_* L^2(\mathfrak{E}_*)\neq\{0\}$, while (34) is fulfilled.

REMARK 2. There is a close connection between (34) and the point spectrum $\sigma_p(T^*)$ of T^*. In fact, let us assume that there is a non-zero function $u\in\ker\Theta^*\cap\cap(L^2(\mathfrak{E}_*)\ominus H^2(\mathfrak{E}_*))$. Then $\tilde{u}(\zeta):=u(\bar{\zeta})$ $(\zeta\in\partial\mathbf{D})$ will be a non-zero function belonging to $H^2(\mathfrak{E}_*)$. Let us consider the expansion $\tilde{u}=\sum_n h_n e_n$, $h_n\in H^2$, of the function $\tilde{u}$ with respect to an orthonormal basis $\{e_n\}_n$ of $\mathfrak{E}_*$. Let us form the largest common inner divisor $h\in H^\infty$ of the inner parts of the functions h_n, and define $u_1\in H^2(\mathfrak{E}_*)$ as the product $u_1=\bar{h}\tilde{u}$. Since the system of functions $\bar{h}h_n$ is relatively prime, it follows that

$$u_1(\lambda)\neq 0,\quad \text{for every}\quad \lambda\in\mathbf{D}. \tag{41}$$

On the other hand, for the function $\Theta^\sim u_1\in H^2(\mathfrak{E})$ we have

$$\Theta^\sim(\zeta)u_1(\zeta)=\overline{h(\zeta)}\Theta(\bar{\zeta})^* u(\bar{\zeta})=0\ \text{ for a.e. }\zeta\in\partial\mathbf{D},$$

and so

$$\Theta\quad\lambda)u_1(\lambda)=0,\quad \text{for every}\quad \lambda\in\mathbf{D}. \tag{42}$$

We infer by (41) and (42) that $\ker\Theta^\sim(\lambda)\neq\{0\}$ for every $\lambda\in\mathbf{D}$, and this implies by [14, Theorem VI. 4.1] that $\sigma_p(T^*)\supset\mathbf{D}$. Therefore, if $\sigma_p(T^*)$ does not cover the unit disc $\mathbf{D}$:

$$\sigma_p(T^*)\not\supset\mathbf{D}, \tag{43}$$

then (34) holds, and so X is a quasi-surjective operator. So the preceding argumentation performs a new proof for [14, Proposition II. 3.2].

However, relation (43) is only a sufficient but not a necessary condition for (34). Indeed, let us consider the domains $\Omega_1=\left\{\lambda\in\mathbf{D}:\operatorname{Re}\lambda>-\frac{1}{2}\right\}$ and $\Omega_2=-\Omega_1$. By the Riemann mapping theorem there is a function $h_1\in H^\infty$ such that $h_1(\mathbf{D})=\Omega_1$. Then $h_2=-h_1$ is also of class H^∞, and $h_2(\mathbf{D})=\Omega_2$. For $j=1, 2$, let us consider the contraction $h_j(S)$ of class C_{10} (cf. [13, Lemma 5]), where S denotes the simple unilateral shift. By the properties of the Sz.-Nagy—Foiaş functional calculus we can easily infer that

$$\Omega_j\subset\sigma_p\big(h_j(S)^*\big)\subset\sigma\big(h_j(S)^*\big)=\Omega_j^- \quad (j=1, 2). \tag{44}$$

Let us choose purely contractive analytic functions $\Theta_j:\mathbf{D}\to\mathscr{L}(\mathfrak{E}_j, \mathfrak{E}_{*,j})$ such that $T_j=S(\Theta_j)$ is unitarily equivalent to $h_j(S)$ for $j=1, 2$, and let us define $\Theta:\mathbf{D}\to\mathscr{L}(\mathfrak{E}=\mathfrak{E}_1\oplus\mathfrak{E}_2, \mathfrak{E}_*=\mathfrak{E}_{*,1}\oplus\mathfrak{E}_{*,2})$ as the orthogonal sum $\Theta=\Theta_1\oplus\Theta_2$. Then for the operator $T=S(\Theta)$ we have by (44) that $\sigma_p(T^*)=\sigma_p(T_1^*)\cup\sigma_p(T_2^*)=$ $=\sigma_p\big(h_1(S)^*\big)\cup\sigma_p\big(h_2(S)^*\big)=\mathbf{D}$, i.e. (43) does not hold. On the other hand, let us assume that $\Theta^*u=0$ for a function $u\in L^2(\mathfrak{E}_*)\ominus H^2(\mathfrak{E}_*)$. Then u is of the form $u=u_1\oplus u_2$, where $u_j\in L^2(\mathfrak{E}_{*,j})\ominus H^2(\mathfrak{E}_{*,j})$ and $\Theta_j^*u_j=0$ for $j=1, 2$. Since in virtue of (44) $\sigma_p(T_j^*)$ does not cover $\mathbf{D}$, it follows by the preceding part of this remark that (34) is fulfilled for Θ_j, hence $u_j=0$ $(j=1, 2)$. This results that $u=u_1\oplus u_2=0$, so we obtain that (34) does hold for Θ.

We conclude this section with a simple but perhaps suprising corollary of Proposition 2.

COROLLARY 1. a) *If X is injective, then so is Y too. Hence $T\overset{i}{\prec}R_*$ implies that $R\overset{i}{\prec}T$, and so $R\overset{i}{\prec}R_*$.*

b) *If Y is quasi-surjective, then so is X too. Hence $R\overset{d}{\prec}T$ implies that $T\overset{d}{\prec}R_*$, and so $R\overset{d}{\prec}R_*$.*

(Here the notation $T_1\overset{i}{\prec}T_2$ $(T_1\overset{d}{\prec}T_2)$ means that $\mathscr{I}(T_1, T_2)$ contains an injective (quasi-surjective, resp.) operator.)

PROOF. By (1) the statements a) and b) of the Corollary are equivalent. Let us prove a), so let us assume that X is injective. Then Proposition 2/a yields that Θ is $*$-outer, which results by [14, Proposition V.2.4] that $\Theta(\zeta)$ is injective for a.e. $\zeta\in\partial\mathbf{D}$. Therefore (36) holds, and Proposition 2/d results that Y is also an injection.

5. Characteristic properties of the residual parts

The following proposition states that the mapping Y is a universal intertwiner between the unitary operator R and T, and that this property together with a minimality condition uniquely determine the pair (R, Y).

PROPOSITION 3. *The ordered pair (R, Y) possesses the following properties:*
(i) *$R|\ker Y$ is of class C_{10};*

(ii) *for every unitary operator* G *and for every mapping* $A\in\mathcal{I}(G,T)$, *there exists a mapping* $B\in\mathcal{I}(G,R)$ *such that*

$$A=YB. \tag{45}$$

Furthermore, if (R_1,Y_1) *is a pair, where* R_1 *is a unitary and* $Y_1\in\mathcal{I}(R_1,T)$, *such that* (i) *and* (ii) *are valid with* (R_1,Y_1) *in place of* (R,Y), *then* R_1 *is unitarily equivalent to* R *and*

$$Y_1=YB \tag{46}$$

with an invertible transformation $B\in\mathcal{I}(R_1,R)$.

PROOF. Let R_0 denote the unitary part of the isometry $R|\ker Y$. Since the subspace $\mathfrak{K}_+\ominus\operatorname{dom}R_0$ is invariant for U_+ and contains $\mathfrak{H}$, taking into account that U_+ is the minimal isometric dilation of T, we infer that $\operatorname{dom}R_0=\{0\}$. Hence, the pair (R,Y) satisfies (i).

Let us consider a unitary operator G and a mapping $A\in\mathcal{I}(G,T)$. By the Lifting theorem (cf. [14, Theorem II.2.3]) we can find an operator $B_0\in\mathcal{I}(G,U_+)$ such that

$$A=P_{\mathfrak{H}}B_0. \tag{47}$$

Since, for every positive integer n, $\operatorname{ran}B_0=\operatorname{ran}(B_0G^n)=\operatorname{ran}(U_+^nB_0)=U_+^n\operatorname{ran}B_0$, it follows that

$$\operatorname{ran}B_0\subset\mathfrak{R}. \tag{48}$$

(47) and (48) show that the operator $B\in\mathcal{I}(G,R)$, obtained from B_0 by restricting its final space to $\mathfrak{R}$, fulfills the relation (45).

Let us now assume that the statements (i) and (ii) are true for the pair (R_1,Y_1). On account of (ii) there exist operators $B\in\mathcal{I}(R_1,R)$ and $B_1\in\mathcal{I}(R,R_1)$ such that

$$Y_1=YB \quad\text{and}\quad Y=Y_1B_1. \tag{49}$$

From (49) it readily follows that $Y(I-BB_1)=0$ and $Y_1(I-B_1B)=0$, i.e. $\operatorname{ran}(I-BB_1)^-\subset\ker Y$ and $\operatorname{ran}(I-B_1B)^-\subset\ker Y_1$. Applying (i) we can infer that

$$R|\operatorname{ran}(I-BB_1)^-\in C_{10} \quad\text{and}\quad R_1|\operatorname{ran}(I-B_1B)^-\in C_{10}. \tag{50}$$

However, since $I-BB_1\in\{R\}'$ (= the commutant of R) and $I-B_1B\in\{R_1\}'$, [9, Lemma 5] yields that

$$R|\operatorname{ran}(I-BB_1)^-\in C_{11} \quad\text{and}\quad R_1|\operatorname{ran}(I-B_1B)^-\in C_{11}. \tag{51}$$

The relations (50) and (51) result that $BB_1=I$ and $B_1B=I$, i.e. the mapping B is invertible. Since similar unitary operators are unitarily equivalent (cf. [14, Proposition II. 3.4]), the proof is complete.

On account of (1) we can obtain from Proposition 3 an analogous characterization for R_* and X.

PROPOSITION 4. *The ordered pair* (R_*,X) *satisfies the properties*

(i) $R_*^*|\ker X^*\in C_{10}$;

(ii) *for every unitary operator* F *and for every mapping* $C\in\mathcal{I}(T, F)$, *there exists a mapping* $D\in\mathcal{I}(R_*, F)$ *such that*

$$C = DX. \tag{52}$$

Moreover, if (R_2, X_2) *is a pair, where* R_2 *is a unitary and* $X_2\in\mathcal{I}(T, R_2)$, *such that* (i) *and* (ii) *hold with* (R_2, X_2) *in place of* (R_*, X), *then* R_2 *is unitarily equivalent to* R_* *and*

$$X_2 = DX \tag{53}$$

with an invertible $D\in\mathcal{I}(R_*, R_2)$.

REMARK 3. Applying the argumentation used in the proof of Proposition 3 it can be shown that the operator B in (45) is unique. Hence the operator D in (52) is also uniquely determined.

As consequences of the preceding proposition we can obtain the following statements.

COROLLARY 2. *The three conditions below are equivalent:*

(i) $T\overset{i}{\prec}F$, *for some unitary operator* F;

(ii) X *is injective*;

(iii) T *is the quasi-affine transform of* R_*: $T\prec R_*$.

($T\prec R_*$ means that $\mathcal{I}(T, R_*)$ contains a quasi-affinity, i.e. an injective operator with dense range.)

PROOF. The condition (i) implies condition (ii) by the statement (ii) of Proposition 4. On the other hand, the statement (i) of Proposition 4 together with the fact that the simple unilateral shift is the quasi-affine transform of the simple bilateral shift (cf. e.g. [9, Example 1]) give the proof of the implication (ii)⇒(iii). Finally, (i) is a trivial consequence of (iii).

COROLLARY 3. *If* F *is unitary and* $T\overset{d}{\prec}F$, *then* F *is unitarily equivalent to an orthogonal summand of* R_*.

PROOF. The statement follows from Proposition 4 (ii) and [5, Lemma 4.1].

From the two previous corollaries we conclude

COROLLARY 4. *If* $T\prec F$, *for a unitary operator* F, *then* $T\prec R_*$ *and* F *is unitarily equivalent to an orthogonal summand of* R_*.

REMARK 4. If T is of class C_{11}, then the unitary operator F in Corollary 4 must be unitarily equivalent to R_* (cf. Proposition 2 and [14, Proposition II. 3.4]). However, in general F is not unique. For example, let us consider the contraction $T=S(\Theta)$, where $\dim\mathfrak{E}=0$, $\dim\mathfrak{E}_*=1$ and $\Theta:\mathbf{D}\to\mathcal{L}(\mathfrak{E}, \mathfrak{E}_*)$ is identically zero. Then $\mathfrak{H}=H^2(\mathfrak{E}_*)=H^2$, i.e. T is the simple unilateral shift, and $\mathfrak{K}=\mathfrak{R}_*=L^2(\mathfrak{E}_*)=L^2$, i.e. R_* is the simple bilateral shift. For any Borel subset $\alpha\subset\partial\mathbf{D}$ of positive Lebesgue measure, T is a quasi-affine transform of $R_*|\chi_\alpha L^2$. Hence, in this case F can be chosen for any non-zero orthogonal summand of R_*.

By the aid of (1) we can obtain the duals of Corollaries 2—4.

COROLLARY 5. *The following three conditions are equivalent:*

(i) $G \overset{d}{\prec} T$, *for some unitary operator* G;

(ii) Y *is quasi-surjective;*

(iii) $R \prec T$.

COROLLARY 6. *If* G *is unitary and* $G \overset{i}{\prec} T$, *then* G *is unitarily equivalent to an orthogonal summand of* R.

COROLLARY 7. *If* $G \prec T$ *for a unitary operator* G, *then* $R \prec T$ *and* G *is unitarily equivalent to an orthogonal summand of* R.

6. Quasi-C_{11} contractions

By a recent result of Bercovici and Takahashi there is a close connection between the injectivity of the canonical interwining operators X, Y and the reflexivity of $T=S(\Theta)$. Namely, they proved in [4] that if $\mathscr{I}(T, Q)$ contains a quasi-surjective operator for a non-zero c.n.u. isometry Q, then T is reflexive (cf. [7, ch. 9]), even more the set $\operatorname{Alg}\operatorname{Lat} T \subset \mathscr{L}(\mathfrak{H})$ of operators leaving invariant every invariant subspace of T coincides with the set $H^\infty(T)=\{u(T): u \in H^\infty\}$ of functions of T:

$$\operatorname{Alg}\operatorname{Lat} T = H^\infty(T). \tag{54}$$

The assumption above is clearly equivalent to the one that $\mathscr{I}(T, S) \neq \{0\}$ where S stands for the simple unilateral shift.[1] By Proposition 4 this happens exactly when X has no dense range. Considering also adjoints, the previous result can be stated in the following form:

If

$$\ker X^* \neq \{0\} \quad or \quad \ker Y \neq \{0\}, \tag{55}$$

then (54) *holds.*

Now we are going to deal with the case when (55) is not fulfilled. In particular, by Proposition 2 and [14, Proposition VI.3.5] this happens if T is of class C_{11}. Here we shall consider a more general class of contractions.

DEFINITION. The contraction $T \cong S(\Theta)$ is called *quasi-*C_{11}, if

$$T \text{ is of class } C_{.1}, \text{ i.e. } \Theta \text{ is outer,} \tag{56}$$

and

$$\Theta(\zeta) \text{ is a quasi-affinity for a.e. } \zeta \in \partial\mathbf{D}. \tag{57}$$

The class of quasi-C_{11} contractions will be denoted by QC_{11}.

This definition can be contrasted with the notion of contraction weakly similar to unitary. Namely, by [10, Theorem 4] T is of the latter type exactly when $T \in C_{11}$ and $\Theta(\zeta)$ is invertible for a.e. $\zeta \in \partial\mathbf{D}$.

---

[1] When I was writing this paper I knew only a preliminary version of [4] written by Takahashi. In the final version the assumption already is that $\mathscr{I}(T, S) \neq \{0\}$.

It is clear that C_{11} is a subclass of QC_{11}. However, there exist quasi-C_{11} contractions which are not of class C_{11}. The following example is a modification of the operator constructed in the proof of [12, Proposition 7]. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in the Hilbert space $\mathfrak{H}_1$, and let $T_1 \in \mathcal{L}(\mathfrak{H}_1)$ be the unilateral, weighted shift defined by $T_1 e_n = w_n e_{n+1}$, where the weight sequence $\{w_n\}_{n=1}^{\infty}$ consists of non-zero numbers, with $|w_n| \leqq 1$, and tends to zero. Let $G \in \mathcal{L}(\mathfrak{H}_2)$ be a bilateral shift of infinite multiplicity, and let us consider a decomposition $\mathfrak{H}_2 = \bigoplus\limits_{n=-\infty}^{\infty} G^n \mathfrak{M}$, where $\dim \mathfrak{M} = \aleph_0$. Let us choose a non-invertible c.n.u. C_{11}-contraction $A \in \mathcal{L}(\mathfrak{M})$ (cf. [3]), and let $T_2 \in \mathcal{L}(\mathfrak{H}_2)$ be the product $T_2 = GD$, where $D \in \mathcal{L}(\mathfrak{H}_2)$ is the diagonal operator defined by $D|G^n\mathfrak{M} = G^n A G^{-n}|G^n\mathfrak{M}$ $(n=0, \pm 1, \pm 2, \ldots)$. By [2, Lemma 3.2] we can find a vector $f \in \mathfrak{M}$ such that

$$\begin{bmatrix} 0 & 0 \\ f\otimes e_1 & A \end{bmatrix} \in \mathcal{L}(\mathfrak{H}_{1,1} \oplus \mathfrak{M})$$

is an injective contraction. Here $\mathfrak{H}_{1,1}$ denotes the subspace spanned by e_1 in $\mathfrak{H}_1$, and $f\otimes e_1 \in \mathcal{L}(\mathfrak{H}_{1,1}, \mathfrak{M})$ is the operator defined by $(f\otimes e_1)h = \langle h, e_1 \rangle f$, for every $h \in \mathfrak{H}_{1,1}$. Finally, let $T \in \mathcal{L}(\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2)$ be the operator whose matrix is

$$T = \begin{bmatrix} T_1 & e_1\otimes f \\ 0 & T_2^* \end{bmatrix} \tag{58}$$

in the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$.

It can be easily verified that T is a c.n.u. contraction belonging to $C_{\cdot 1} \setminus C_{11}$. Let us consider the regular factorization $\Theta_T = \Theta_2 \Theta_1$ of the characteristic function of T corresponding to the invariant subspace $\mathfrak{H}_1$ (cf. [14, ch. VII]). Since T_1 and T_2 are of class C_{00} and C_{11}, resp., we infer by [14, Proposition VI. 3.5 and Section V. 3] that $\Theta_1(\zeta)$ and $\Theta_2(\zeta)$ are quasi-affinities in a.e. point $\zeta \in \partial \mathbf{D}$. Hence $\Theta_T(\zeta)$ is also a quasi-affinity for a.e. $\zeta \in \partial \mathbf{D}$. Therefore, T is a quasi-C_{11} contraction which is not of class C_{11}.

An argumentation similar to the previous one results that the operators T_1 and T_2 occurring in the $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$-type decomposition $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ of a quasi-C_{11} contraction T are of class C_{00} and C_{11}, respectively.

On account of (27) and (29) we can see that (57) is true for the operator-valued function Θ whenever the operator $Z \in \mathcal{I}(R, R_*)$ introduced in (23) is a quasi-affinity. Hence, in this case the residual and $*$-residual parts are unitarily equivalent:

$$R \cong R_*. \tag{59}$$

Furthermore, an application of Proposition 2 yields that if T is a quasi-C_{11} contraction, then X is quasi-surjective and Y is a quasi-affinity, i.e.

$$R \prec T \overset{d}{\prec} R_*. \tag{60}$$

Corollary 4 shows that T is quasi-similar to a unitary operator, i.e. $T \in C_{11}$, if and only if X is a quasi-affinity.

We have proved in [11] that if $T=S(\Theta)$ is a contraction weakly similar to unitary, and

$$\int_{\partial \mathbf{D}} \log \|\Theta v\|_{\mathfrak{E}_*}\, dm > -\infty, \tag{61}$$

for a function $v \in \mathfrak{N}$, then (54) holds validity. Takahashi noticed that [11, Lemma 7] is true in general for every c.n.u. C_{11}-contraction, so our proof for [11, Theorem 5] results (54) whenever $T=S(\Theta)$ is an arbitrary C_{11}-contraction satisfying (61). Now we further generalize this statement by taking T from QC_{11} instead of C_{11} and replacing the condition (61) by a weaker one.

THEOREM 1. *Let* $T=S(\Theta)$ *be a quasi-*C_{11} *contraction. If there exist functions* $u \in H^2(\mathfrak{E}_*)$ *and* $v \in (\Delta L^2(\mathfrak{E}))^-$ *such that*

$$\int_{\partial \mathbf{D}} \log \|-\Delta_* u + \Theta v\|_{\mathfrak{E}_*}\, dm > -\infty, \tag{62}$$

then

$$\text{Alg Lat } T \cap \{T\}'' = H^\infty(T). \tag{63}$$

Since the bicommutant $\{T\}''$ of every C_{11}-contraction T is a reflexive algebra (cf. [15]), this theorem is indeed a generalization of [11, Theorem 5 (iii)].

PROOF. Let $u \in H^2(\mathfrak{E}_*)$ and $v \in (\Delta L^2(\mathfrak{E}))^-$ be functions satisfying (62). Let us consider the decomposition $u \oplus v = h + Vw$ of the vector $u \oplus v \in \mathfrak{K}_+$, where $h \in \mathfrak{H}$ and $Vw \in \mathfrak{M}_+$ $(w \in H^2(\mathfrak{E}))$. On account of (3) and (22) we obtain that $WXh = \tilde{X}h = -\Delta_* u + \Theta v$. Since W is a unitary-valued function, it follows that $\|(Xh)(\zeta)\|_{\mathfrak{E}_* \oplus \mathfrak{E}} = \|(-\Delta_* u + \Theta v)(\zeta)\|_{\mathfrak{E}_*}$ for a.e. $\zeta \in \partial \mathbf{D}$. Hence, by the assumption we infer that

$$\int_{\partial \mathbf{D}} \log \|Xh\|_{\mathfrak{E}_* \oplus \mathfrak{E}}\, dm > -\infty. \tag{64}$$

Let us form the non-zero cyclic subspaces $\mathfrak{M} = \bigvee_{n \geq 0} T^n h$ and $\mathfrak{N} = \bigvee_{n \geq 0} R_*^n Xh$. In virtue of [11, Lemma 9], (64) implies that $R_*|\mathfrak{N}$ is of class C_{10}, i.e. it is a unilateral shift. Since the operator $X|\mathfrak{M} \in \mathcal{I}(T|\mathfrak{M}, R_*|\mathfrak{N})$ has dense range we conclude by [4] that

$$\text{Alg Lat}(T|\mathfrak{M}) = H^\infty(T|\mathfrak{M}). \tag{65}$$

Let us choose now an arbitrary operator $A \in \text{Alg Lat } T \cap \{T\}''$. The restriction $A|\mathfrak{M}$ clearly belongs to $\text{Alg Lat}(T|\mathfrak{M})$, hence by the relation (65) $A|\mathfrak{M}$ is of the form $A|\mathfrak{M} = w(T|\mathfrak{M}) = w(T)|\mathfrak{M}$, with a suitable function $w \in H^\infty$. Let us consider the difference $B = A - w(T)$, for which we have

$$B \in \{T\}'' \quad \text{and} \quad B\mathfrak{M} = \{0\}. \tag{66}$$

It is enough to show that $B=0$, because in this case $A = w(T) \in H^\infty(T)$.

Since $R_*|\mathfrak{N}$ is a unilateral shift, it follows that

$$\bigvee_{D \in \{R_*\}'} D\mathfrak{N} = \mathfrak{R}_*. \tag{67}$$

On the other hand, $YZ^*DX\in\{T\}'$ whenever $D\in\{R_*\}'$, and Y and Z^* are quasi-affinities, as T is a quasi-C_{11} contraction. So we obtain from (67) that

$$\bigvee_{C\in\{T\}'} C\mathfrak{M}=\mathfrak{H}. \tag{68}$$

Now, the relations (66) and (68) result $B\mathfrak{H}=B(\bigvee_{C\in\{T\}'} C\mathfrak{M})=\bigvee_{C\in\{T\}'} BC\mathfrak{M}=$ $=\bigvee_{C\in\{T\}'} CB\mathfrak{M}=\{0\}$, i.e. $B=0$ and the proof is complete.

7. Reflexivity of C_{11}-contractions

As an application of Theorem 1 we prove a reflexivity theorem for a class of C_{11}-contractions.

THEOREM 2. *Let* $\{\alpha_k\}_{k=1}^{\infty}$ *be a decomposition of the unit circle* $\partial\mathbf{D}$ *into the sequence of Borel sets of positive Lebesgue measure. For every* k, *let us given non-zero, complex, separable Hilbert spaces* $\mathfrak{E}_k$ *and* $\mathfrak{E}_{*k}$ *of equal dimension such that* $n_k=\dim\mathfrak{E}_k=\dim\mathfrak{E}_{*k}$ *is finite if* $k\geqq 2$. *For every* k, *let us consider a purely contractive, analytic function* $\Theta_k\colon \mathbf{D}\to\mathcal{L}(\mathfrak{E}_k,\mathfrak{E}_{*k})$, *supposed to be outer from both sides. Let* Δ_k *denote its defect function* $\Delta_k(\zeta)=(I-\Theta_k(\zeta)^*\Theta_k(\zeta))^{1/2}$ $(\zeta\in\partial\mathbf{D})$, *and let* μ_k *stand for its lower bound function, i.e.*

$$\mu_k(\zeta)=\inf\{\|\Theta_k(\zeta)x\|\colon x\in\mathfrak{E}_k,\ \|x\|=1\},\quad \zeta\in\partial\mathbf{D}.$$

We assume that

$$\operatorname{rank}\Delta_k=\chi_\alpha \quad \text{a.e.,} \tag{69}$$

and

$$\int_{\alpha_1}\log\mu_k\,dm+\sum_{k=2}^{\infty}\int_{\alpha_k}\log\frac{1-\mu_k^2}{n_k}\,dm>-\infty, \tag{70}$$

and form the orthogonal sum $\Theta=\bigoplus_{k=1}^{\infty}\Theta_k\colon\partial\mathbf{D}\to\mathcal{L}\big(\mathfrak{E}=\bigoplus_{k=1}^{\infty}\mathfrak{E}_k,\ \mathfrak{E}_*=\bigoplus_{k=1}^{\infty}\mathfrak{E}_{*k}\big)$.

Then for the C_{11}*-contraction* $T=S(\Theta)$ *we have*

$$H^\infty(T)=\operatorname{Alg\,Lat} T\neq\{T\}''=\{T\}'. \tag{71}$$

As a corollary we infer that (71) holds if the sequence $\{n_k\}_{k=2}^{\infty}$ is bounded and, for every k, $\mu_k=c_k\chi_{\alpha_k}+\chi_{\partial\mathbf{D}\setminus\alpha_k}$ a.e. on $\partial\mathbf{D}$, where $0<c_k<1$ is an arbitrary constant. Indeed, after attaching the set $\cup\left\{\alpha_k\colon k\geqq 2,\ c_k\geqq\frac{1}{2}\right\}$ to α_1 the assumptions of Theorem 2 will be fulfilled. So we obtain a positive answer for the problem raised at the end of [11], i.e. the operator constructed in [11, Example 12] always possesses the properties of [11, Theorem 5].

PROOF. On account of (69) rank $\Delta=1$ a.e. on $\partial\mathbf{D}$, hence the residual part R is a cyclic operator (cf. [8, Lemma 1]). It follows that T is cyclic too, and so [10]

Theorem 15 and Corollary 12] yield that $\{T\}''=\{T\}'\neq H^\infty(T)$. On the other hand, Theorem 1 together with [15] will result the coincidence $H^\infty(T)=\mathrm{Alg\,Lat}\,T$, if we find functions $u\in H^2(\mathfrak{E}_*)$ and $v\in(\Delta L^2(\mathfrak{E}))^-$ satisfying (62).

Let k be an arbitrary positive integer. In view of (69) there exists a function $h_k\in L^2(\mathfrak{E}_k)$ such that

$$\|h_k(\zeta)\|_{\mathfrak{E}_k} = \chi_{\alpha_k}(\zeta) \tag{72}$$

and

$$\Delta_k(\zeta) = (1-\mu_k(\zeta)^2)^{1/2}\, h_k(\zeta)\otimes h_k(\zeta) \tag{73}$$

hold for a.e. $\zeta\in\partial\mathbf{D}$. Since $h_k\in(\Delta_k L^2(\mathfrak{E}_k))^-$, it follows that $\tilde h_{*k}=\Theta_k h_k\in(\Delta_{*k}L^2(\mathfrak{E}_{*k}))^-$, moreover in virtue of (72) and (73) we have $\|\tilde h_{*k}(\zeta)\|=\mu_k(\zeta)\chi_{\alpha_k}(\zeta)$ a.e. Θ_k being $*$-outer we know that $0<\mu_k(\zeta)<1$ a.e. in α_k, while $\mu_k(\zeta)=1$ for a.e. $\zeta\in\partial\mathbf{D}\setminus\alpha_k$. Hence the function $h_{*k}=\mu_k^{-1}\tilde h_{*k}\in(\Delta_{*k}L^2(\mathfrak{E}_{*k}))^-$ will possess the following properties:

$$\|h_{*k}(\zeta)\|_{\mathfrak{E}_{*k}} = \chi_{\alpha_k}(\zeta), \tag{74}$$

$$\Theta_k(\zeta)\, h_k(\zeta) = \mu_k(\zeta)\, h_{*k}(\zeta), \tag{75}$$

and

$$\Delta_{*k}(\zeta) = (1-\mu_k(\zeta)^2)^{1/2}\, h_{*k}(\zeta)\otimes h_{*k}(\zeta) \tag{76}$$

in a.e. point $\zeta\in\partial\mathbf{D}$.

Let us now assume that $k=1$. In this case we define the functions $u_1\in H^2(\mathfrak{E}_{*1})$ and $v_1\in(\Delta_1 L^2(\mathfrak{E}_1))^-$ as

$$u_1 = 0 \quad\text{and}\quad v_1 = h_1. \tag{77}$$

It follows that

$$\|u_1\oplus v_1\|_{L^2(\mathfrak{E}_{*1})\oplus L^2(\mathfrak{E}_1)} = m(\alpha_1), \tag{78}$$

and, in view of (74) and (75),

$$\|(-\Delta_{*1}u_1+\Theta_1 v_1)(\zeta)\|_{\mathfrak{E}_{*1}} = \mu_1(\zeta)\chi_{\alpha_1}(\zeta), \tag{79}$$

for a.e. $\zeta\in\partial\mathbf{D}$.

Let us consider now the case $k\geqq 2$, and take the expansion $h_{*k}=\sum_{j=1}^{n_k} h_j^{(k)}e_j^{(k)}$, $h_j^{(k)}\in L^2$, of h_{*k} with respect to an orthonormal basis $\{e_j^{(k)}\}_{j=1}^{n_k}$ of $\mathfrak{E}_{*k}$. The relation (74) shows that $\sum_{j=1}^{n_k}|h_j^{(k)}(\zeta)|^2=1$ in a.e. point ζ of α_k. Hence, we can decompose α_k into the union $\alpha_k=\bigcup_{j=1}^{n_k}\gamma_j^{(k)}$ of pairwise disjoint Borel sets such that

$$|h_j^{(k)}(\zeta)| \geqq (2n_k)^{-1/2} \tag{80}$$

holds in a.e. $\zeta \in \gamma_j^{(k)}$, for $j=1, 2, \ldots, n_k$. For every $1 \leqq j \leqq n_k$, let us choose a function $u_j^{(k)} \in H^2$ such that

$$|u_j^{(k)}| = \chi_{\gamma_j^{(k)}} + \varrho_j^{(k)} \chi_{\partial \mathbf{D} \setminus \gamma_j^{(k)}} \quad \text{a.e.,} \tag{81}$$

where $0 < \varrho_j^{(k)} < 1$ is a constant, and define the functions $u_k \in H^2(\mathfrak{E}_{*k})$ and $v_k \in \in (\Delta_k L^2(\mathfrak{E}_k))^-$ as

$$u_k = \sum_{j=1}^{n_k} u_j^{(k)} e_j^{(k)} \quad \text{and} \quad v_k = 0. \tag{82}$$

Then for the norm of the vector $u_k \oplus v_k$ we have

$$\|u_k \oplus v_k\|^2_{L^2(\mathfrak{E}_{*k}) \oplus L^2(\mathfrak{E}_k)} = \|u_k\|^2_{L^2(\mathfrak{E}_{*k})} = \sum_{j=1}^{n_k} \|u_j^{(k)}\|^2_{L^2} = \tag{83}$$

$$= \sum_{j=1}^{n_k} \left(m(\gamma_j^{(k)}) + \varrho_j^{(k)2}(1 - m(\gamma_j^{(k)}))\right) \leqq m(\alpha_k) + \sum_{j=1}^{n_k} \varrho_j^{(k)2} < 2m(\alpha_k),$$

provided

$$\sum_{j=1}^{n_k} \varrho_j^{(k)2} < m(\alpha_k). \tag{84}$$

On the other hand, on account of (76), (81) and (80) the following estimate is true in a.e. point $\zeta \in \gamma_j^{(k)}$ $(1 \leqq j \leqq n_k)$:

$$\|(-\Delta_{*k} u_k + \Theta_k v_k)(\zeta)\|_{\mathfrak{E}_{*k}} = \|(\Delta_{*k} u_k)(\zeta)\|_{\mathfrak{E}_{*k}} =$$

$$= (1 - \mu_k(\zeta)^2)^{1/2} |\langle u_k(\zeta), h_{*k}(\zeta)\rangle_{\mathfrak{E}_{*k}} = (1 - \mu_k(\zeta)^2)^{1/2} \left| \sum_{l=1}^{n_k} u_l^{(k)}(\zeta) \overline{h_l^{(k)}(\zeta)} \right| \geqq$$

$$\geqq (1 - \mu_k(\zeta)^2)^{1/2} \Big[|u_j^{(k)}(\zeta)|\, |h_j^{(k)}(\zeta)| - \sum_{\substack{l=1 \\ l \neq j}}^{n_k} |u_l^{(k)}(\zeta)|\, |h_l^{(k)}(\zeta)| \Big] \geqq$$

$$\geqq (1 - \mu_k(\zeta)^2)^{1/2} \Big[(2n_k)^{-1/2} - \sum_{l=1}^{n_k} \varrho_l^{(k)} \Big].$$

Taking into consideration that $\|(-\Delta_{*k} u_k + \Theta_k v_k)(\zeta)\| = 0 = 1 - \mu_k(\zeta)$ a.e. on $\partial \mathbf{D} \setminus \alpha_k$, we infer that

$$\|(-\Delta_{*k} u_k + \Theta_k v_k)(\zeta)\|_{\mathfrak{E}_{*k}} \geqq (4n_k)^{-1/2} (1 - \mu_k(\zeta)^2)^{1/2} \quad \text{a.e.,} \tag{85}$$

if

$$\sum_{l=1}^{n_k} \varrho_l^{(k)} < \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right) n_k^{-1/2}. \tag{86}$$

So if we choose $\varrho_l^{(k)}$' s satisfying (84) and (86), then (83) and (85) are valid.

The relations (78) and (83) result that

$$\sum_{k=1}^{\infty} \|u_k \oplus v_k\|^2_{L^2(\mathfrak{E}_{*k}) \oplus L^2(\mathfrak{E}_k)} \leqq m(\alpha_1)^2 + \sum_{k=2}^{\infty} 2m(\alpha_k) \leqq \sum_{k=1}^{\infty} 2m(\alpha_k) = 2, \tag{87}$$

hence we obtain vectors $u\in H^2(\mathfrak{E}_*)$ and $v\in(\Delta L^2(\mathfrak{E}))^-$ by the definitions

$$u=\bigoplus_{k=1}^{\infty} u_k \quad\text{and}\quad v=\bigoplus_{k=1}^{\infty} v_k. \tag{88}$$

In virtue of (79) and (85) it follows that

$$\|(-\Delta_* u+\Theta v)(\zeta)\|^2_{\mathfrak{E}_*}=\sum_{k=1}^{\infty}\|(-\Delta_{*k}u_k+\Theta_k v_k)(\zeta)\|^2_{\mathfrak{E}_{*k}}\geqq \tag{89}$$

$$\geqq \mu_1(\zeta)^2\chi_{\alpha_1}(\zeta)+\sum_{k=2}^{\infty}(4n_k)^{-1}(1-\mu_k(\zeta)^2)\quad\text{a.e.}$$

Finally, by the inequalities (89) and (70) we conclude that the functions u and v satisfy (62), so the proof is complete.

REMARK 5. As far as we know it is an open question whether the relation (54) holds for every C_{11}-contraction $T=S(\Theta)$ with the property:

(90) $\Theta(\zeta)$ is not an isometry, i.e. rank $\Delta(\zeta)\geqq 1$, for a.e. $\zeta\in\partial\mathbf{D}$.

This problem can be reduced to the case when T is cyclic. In fact, let us assume that the answer is affirmative for cyclic contractions, and let us consider an arbitrary C_{11}-contraction $T=S(\Theta)$ possessing the property (90). We shall show that (54) will be true for this T too.

By the functional model of unitary operators (cf. [6]) we can see that the commutant $\{R\}'$ of the residual part R is a cyclic operator algebra, i.e. there is a vector $v_0\in\mathfrak{R}$ such that $\bigvee_{C\in\{R\}'} Cv_0=\mathfrak{R}$. More precisely, taking into account (90) and [8, Lemma 1], a function $v_0\in\mathfrak{R}$ is cyclic for $\{R\}'$ if and only if

$$v_0(\zeta)\neq 0\quad\text{for a.e.}\quad \zeta\in\partial\mathbf{D}. \tag{91}$$

Let us consider a cyclic vector v_0 of $\{R\}'$ such that

$$\int_{\partial\mathbf{D}}\log\|v_0\|_{\mathfrak{E}}\,dm=-\infty. \tag{92}$$

Then, in virtue of [11, Lemma 9] $R|\bigvee_{n\geqq 0} R^n v_0\in C_{11}$, i.e. by (90) this operator is a simple bilateral shift. Since by (25), (27) and (29) Z^*Z is the operator of multiplication by $\Theta^*\Theta$, and $\Theta(\zeta)$ is an injective contraction a.e., it follows that the function $Z^*Zv_0\in\mathfrak{R}$ satisfies the condition (91) too. Hence Z^*Zv_0 is also cyclic for $\{R\}'$. Taking into account (24) and that Y is quasi-surjective (cf. Proposition 2) we infer

$$\bigvee_{C\in\{R\}'}(YCZ^*X)(Yv_0)=\bigvee_{C\in\{R\}'}YC(Z^*Z)v_0=\mathfrak{H}.$$

As YCZ^*X clearly belongs to $\{T\}'$ whenever $C\in\{R\}'$, this implies that the vector $x_0=Yv_0$ is cyclic for $\{T\}'$. Let $\mathfrak{M}$ denote the subspace $\mathfrak{M}=\bigvee_{n\geqq 0}T^n x_0$, cyclic for T. Since the bilateral shift $R|\bigvee_{n\geqq 0}R^n v_0$ is a quasi-affine transform of $T|\mathfrak{M}$, applying

[8, Corollary 1] we obtain that $T|\mathfrak{M}\in C_{11}$ and (90) holds for its characteristic function.

Let A be an arbitrary operator from Alg Lat T. On account of [15] $A\in\{T\}''$. On the other hand, $A|\mathfrak{M}$ belonging to Alg Lat $(T|\mathfrak{M})$ by the assumption it can be found a function $w\in H^\infty$ such that $A|\mathfrak{M}=w(T|\mathfrak{M})=w(T)|\mathfrak{M}$. Since the operator $A-w(T)\in\{T\}''$ annihilates the cyclic vector x_0 of $\{T\}'$, we conclude that $A=w(T)$.

Therefore (54) is true for the contraction T.

Finally, let us assume that $T=S(\Theta)$ is a cyclic C_{11}-contraction with the property (90). Then we have $0<\mu(\zeta)<1$ a.e., for the lower bound function $\mu(\zeta)=\inf\{\|\Theta(\zeta)x\|: x\in\mathfrak{E}, \|x\|=1\}$ of Θ. If $\int_{\partial\mathbf{D}}\log\mu\, dm>-\infty$, then Θ has a scalar multiple (cf. [14, Propositions V.7.1 and V. 4.1]) and we infer by [11, Corollary 6] and [10, Theorem 15] that T possesses (71). So it can be supposed that

$$\int_{\partial\mathbf{D}}\log\mu\, dm=-\infty. \tag{93}$$

Let $\{c_k\}_{k=1}^{\infty}$ be a strictly decreasing sequence converging to zero such that $c_1=1$ and the Borel set

$$\alpha_k=\{\zeta\in\partial\mathbf{D}:\ c_{k+1}<\mu(\zeta)\leqq c_k\}$$

is of positive Lebesgue measure, for every k. On account of [10, Theorem 4] there (uniquely) exists a basic system (cf. [1]) $\{\mathfrak{H}_k\}_{k=1}^{\infty}$ consisting of hyperinvariant subspaces of T such that rank $\Delta_k=\chi_{\alpha_k}$ and $\mu_k=\mu\chi_{\alpha_k}+\chi_{\partial\mathbf{D}\setminus\alpha_k}$ a.e., where Δ_k is the defect function and μ_k is the lower bound function of the characteristic function Θ_k of $T|\mathfrak{H}_k$.

Now we can conclude by Theorem 2 that T *has the property* (71) *if*

$$\sum_{k=k_0}^{\infty} m(\alpha_k)\log\frac{1}{\dim\mathfrak{H}_k}>-\infty \tag{94}$$

is true with a $k_0>1$, *and if the basic system* $\{\mathfrak{H}_k\}_k$ *can be obtained from an orthogonal decomposition of* $\mathfrak{H}$ *by the aid of an affine transformation.*

Added in proof (March 13, 1987). In a subsequent paper, appearing in *Bull. London Math. Soc.* **19** (1987), resting on the results of this work I succeeded in proving that every C_{11}-contraction $T=S(\Theta)$ with the property (90) is reflexive.

References

[1] C. Apostol, Operators quasi-similar to a normal operator, *Proc. Amer. Math. Soc.*, **53** (1975), 104—106.

[2] H. Bercovici and L. Kérchy, Quasisimilarity and properties of the commutant of C_{11} contractions, *Acta Sci. Math. (Szeged)*, **45** (1983), 67—74.

[3] H. Bercovici and L. Kérchy, On the spectra of C_{11} contractions, *Proc. Amer. Math. Soc.*, **95** (3) (1985), 412—418.

[4] H. Bercovici and K. Takahashi, On the reflexivity of contractions on Hilbert space, *J. London Math. Soc.*, **32** (1985), 149—156.

[5] R. G. Douglas, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math. (Szeged)*, **30** (1960), 19—32.
[6] N. Dunford and J. Schwartz, *Linear Operators*, Part II, John Wiley (New York, 1963).
[7] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887—933.
[8] L. Kérchy, On the commutant of C_{11}-contractions, *Acta Sci. Math. (Szeged)*, **43** (1981), 15—26.
[9] L. Kérchy, On invariant subspace lattices of C_{11}-contractions, *Acta Sci. Math. (Szeged)*, **43** (1981), 281—293.
[10] L. Kérchy, Contractions being weakly similar to unitaries, *Operator Theory: Advances and Applications*, Vol. **17** (1986, Birkhauser Verlag Basel), 187—200.
[11] L. Kérchy, Contractions weakly similar to unitaries. II, to appear in *Acta Sci. Math. (Szeged)*.
[12] L. Kérchy, A description of invariant subspaces of C_{11}-contractions, *J. Operator Theory*, **15** (1986), 327—344.
[13] L. Kérchy, On the spectra of contractions belonging to special classes, *J. Funct. Analysis*, **67** (1986), 153—166.
[14] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).
[15] K. Takahashi, Double commutants of operators quasi-similar to normal operators, *Proc. Amer. Math. Soc.*, **92** (1984), 404—406.

(Received October 1, 1985)

BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

Acta Math. Hung.
50 (1—2) (1987), 147—153.

INHOMOGENEOUS NORM FORM EQUATIONS IN TWO DOMINATING VARIABLES OVER FUNCTION FIELDS

B. BRINDZA and I. GAÁL (Debrecen)

1. Introduction

The purpose of the present paper is to give effective bounds for the solutions of certain inhomogeneous norm form equations over function fields.

The first effective bounds for the integral solutions of Thue's equations over number fields were given by Baker [1], using his deep effective method concerning linear forms in the logarithms of algebraic numbers (see [2]). Baker's result was later generalized and improved by many authors, see e.g. Vinogradov and Sprindžuk [35], Coates [4], Feldman [5], Kotov [18] Győry and Papp [16] and Győry [11]—[15]. Effective bounds for the solutions of certain general classes of norm form equations in several variables (over number fields) were obtained by Győry and Papp [16], [17], Győry [8]—[13] and Kotov [19]—[22].

Sprindžuk [31] (see also [32]) gave a so called inhomogeneous generalization of Baker's result [1]. Let α be an algebraic integer of degree $\geqq 3$, let $K=\mathbf{Q}(\alpha)$ and $0\neq m\in\mathbf{Z}$. Sprindžuk gave effective bounds for the solutions of the equation

$$N_{K/\mathbf{Q}}(x+\alpha y+\lambda)=m \tag{1}$$

in $x, y\in\mathbf{Z}$ and[1] $\lambda\in\mathbf{Z}_K$, where x, y are dominating variables and λ is a non-dominating variable such that[2] $\overline{|\lambda|}<(\max(|x|,|y|))^{1-\zeta}$ $(0<\zeta<1$ is a given constant). In the special case $\lambda=0$ this result implies (apart from the form of the bounds) Baker's theorem on Thue's equation.

As a common generalization of the above mentioned result of Sprindžuk on equation (1) and of a theorem of Győry and Papp [17] concerning norm form equations in several variables, Gaál [6], [7] obtained effective bounds for certain inhomogeneous norm form equations in several dominating variables.

On the other hand, Osgood [26], [27], Schmidt [28]—[30], Stepanov [33], Mason [23], [24], Győry [14] and Brindza [3] obtained effective results on Thue's equations over function fields. Mason [23] (see also [24]) established an algorithm to determine all solutions of Thue's equations over function fields. Recently Győry [14] and Mason [25] established effective results on norm form equations in several variables over function fields.

We remark that Győry [14], [15] derived effective results concerning some gen-

---

[1] $\mathbf{Z}_K$ denotes the ring of integers of an algebraic number field K.

[2] For an algebraic number α $\overline{|\alpha|}$, denotes the size of α, that is the maximum absolute value of its conjugates.

eral classes of norm form equations also in the case when the ground ring is an arbitrary integral domain finitely generated over **Z** or over a field of characteristic 0.

Our purpose is to derive an effective theorem for an analogue of equation (1) over function fields. Our result provides an effective bound for the solutions of Thue's equations over function fields.

2. Results

In order to formulate our results we need the following notation. Let k be an algebraically closed field of characteristic 0 and denote by $k(z)$ the field of rational functions over k. Further if K is any finite extension field of $k(z)$, denote by Ω_K the set of all (additive) valuations on K with value group **Z**. For any non-zero $\alpha \in K$ let

$$H_K(\alpha) = -\sum_{v \in \Omega_K} \min\{0, v(\alpha)\}$$

be the additive height of α. Obviously $H_K(\alpha)=0$ if and only if $\alpha \in k$. (For $\alpha=0$ put $H_K(\alpha)=0$.) By the additive form

$$\sum_{v \in \Omega_K} v(\alpha) = 0$$

of the well-known product formula one can easily see that

$$H_K(\alpha^m) = |m| H_K(\alpha),$$

$$H_K(\alpha\beta) \leqq H_K(\alpha)+H_K(\beta), \quad H_K(\alpha+\beta) \leqq H_K(\alpha)+H_K(\beta)$$

for any non-zero $\alpha, \beta \in K$ and $m \in \mathbf{Z}$. We remark that if L is an other finite extension of $k(z)$ and $L \subset K$ then

$$H_K(\alpha) = [K:L]\, H_L(\alpha)$$

for any $\alpha \in L$ (see e.g. [29], [14] or [24]). We shall only use these general properties of valuations of function fields and of the height function. For further properties see e.g. Mason [24].

In our Theorem, L, M will denote finite extension fields of $k(z)$ such that $L \subset M$ and $[M:L]=n \geqq 3$. Further, K will denote the smallest normal extension of L containing M. Denote by $\sigma_1, \ldots, \sigma_n$ the L-isomorphisms of M in K. For any $\gamma \in M$ denote by $\gamma_i=\sigma_i(\gamma)$, $i=1, \ldots, n$ the conjugates of γ over L. We shall denote by g the genus of K and put $G=\max(0, 2g-2)$.

Let O_K be the set of those elements of K which are integral over $k[z]$ (that is $\gamma \in O_K$ if and only if $v(\gamma) \geqq 0$ for all finite valuation[3] v in Ω_K). Denote by S a finite subset of Ω_K which contains the infinite valuations. Suppose that if $v \in S$ then all the conjugate valuations of v over L (that is the valuations of the form $v(\sigma(\cdot))$, where σ is any L-isomorphism of K into itself) are also in S. Let $O_{K,S}$ be the ring of S-integers of K, that is the set of those elements γ of K for which $v(\gamma) \geqq 0$ for all $v \in \Omega_K \setminus S$. We remark that $O_{K,S}$ is a ring and $k[z] \subseteqq O_K \subseteqq O_{K,S}$. An element

---

[3] We adopt the usual definition of additive valuations (see e.g. [24]). Those valuations of K for which $v(z) \geqq 0$ are called finite, others are called infinite.

$\gamma\in O_{K,S}$ is called S-unit if $\gamma^{-1}\in O_{K,S}$ also holds. Finally, let $O_{M,S}=O_{K,S}\cap M$, $O_{L,S}=O_{K,S}\cap L$ and let $|S|$ denote the cardinality of S. Let $\alpha\in O_{K,S}$ be a primitive element of M over L with $H_K(\alpha)\leqq A$ and let $0\neq\mu\in L$.

As a function field analogue of equation (1), let us consider the equation

$$N_{M/L}(x+\alpha y+\lambda)=\mu \tag{2}$$

where $x, y\in O_{L,S}$ are dominating variables and $\lambda\in O_{M,S}$ is a non-dominating variable which is in a certain sense "small" compared with the dominating variables.

THEOREM. *If* $x, y\in O_{L,S}$ *and* $\lambda\in O_{M,S}$ *are solutions of equation* (2) *and*

$$H_K(\lambda)\leqq c_0\max\{H_L(x), H_L(y)\}$$

then we have

$$\max\{H_L(x), H_L(y)\}<\frac{64}{[K:L]}(|S|+G+2H_K(\mu)+25A) \tag{3}$$

where $c_0=\dfrac{[K:L]}{400}$.

In the special case $\lambda=0$ our theorem gives an effective bound for the solutions $x, y\in O_{L,S}$ of the Thue's equation

$$N_{M/L}(x+\alpha y)=\mu.$$

3. Proofs

The proof of our Theorem is based on the following results of Mason[4]:

LEMMA 1 (Mason [25]). *Let* S *denote a finite set of valuations on* K *and suppose that* $u_1, \dots, u_m$ *are* S*-units in* K *with* $u_1+\dots+u_m=1$ *but with no proper subset of* $1, u_1, \dots, u_m$ *is linearly dependent over* k. *Then*

$$\max_{1\leqq i\leqq m} H_K(u_i)\leqq 4^{m-1}(|S|+G).$$

In the case $m=2$ a corresponding theorem is as follows:

LEMMA 2 (Mason [24]). *Let* S *denote a finite set of valuations on* K *and suppose that* $\gamma_1, \gamma_2, \gamma_3$ *are non-zero* S*-units in* K *such that* $\gamma_1+\gamma_2+\gamma_3=0$. *Then we have*

$$H_K\left(\frac{\gamma_1}{\gamma_2}\right)\leqq|S|+G.$$

PROOF OF THE THEOREM. Let $x, y\in O_{L,S}$ and $\lambda\in O_{M,S}$ be a fixed solution of equation (2) satisfying the assumptions of our theorem. Let $X=\max\{H_K(x), H_K(y)\}$.

---

[4] We remark that the assertions of Lemmas 1 and 2 hold for any finite extension field K of $k(z)$.

Then $H_K(\lambda)<\bar{c}X$ with $\bar{c}=\frac{1}{400}$ and equation (2) may be written in the form

$$\prod_{i=1}^{n}(x+\alpha_i y+\lambda_i) = \mu. \tag{4}$$

We remark that $\alpha_1, \ldots, \alpha_n$ are distinct, since M/L is a separable extension. (Our method could be used to equation (4) also under the weaker assumption that there exist at least three distinct elements among $\alpha_1, \ldots, \alpha_n$ if we supposed that $\alpha_i=\alpha_j$ implies $\lambda_i=\lambda_j$ $(i\neq j)$.) For brevity, put $\beta_i=x+\alpha_i y+\lambda_i$ $(i=1, \ldots, n)$.

Let r, s, t be pairwise distinct indices from $\{1, \ldots, n\}$. Then we have

$$(\alpha_r-\alpha_s)(x+\alpha_t y)+(\alpha_s-\alpha_t)(x+\alpha_r y)+(\alpha_t-\alpha_r)(x+\alpha_s y) = 0,$$

whence

$$\beta'_r+\beta'_s+\beta'_t+\Lambda = 0, \tag{5}$$

where $\beta'_r = (\alpha_s-\alpha_t)\beta_r, \beta'_s = (\alpha_t-\alpha_r)\beta_s, \beta'_t = (\alpha_r-\alpha_s)\beta_t$ and

$$\Lambda = (\alpha_t-\alpha_s)\,\lambda_r+(\alpha_r-\alpha_t)\,\lambda_s+(\alpha_s-\alpha_r)\,\lambda_t.$$

In the following we shall see that equation (5) may yield one of the three cases below (A, B1, B2) for the indices r, s, t.

A) First suppose that any three of $\beta'_r, \beta'_s, \beta'_t$ and Λ are linearly independent over k. Dividing (5) by $(-\Lambda)$ we obtain

$$u_r+u_s+u_t = 1 \tag{6}$$

where $u_i=-\beta'_i/\Lambda$, $i=r, s, t$.

We shall use the following notation. For any non-zero $\gamma\in K$ let

$$\mathscr{H}(\gamma) = \{v\in\Omega_K | v(\gamma)\neq 0\}.$$

We remark that for the cardinality of $\mathscr{H}(\gamma)$ we have

$$|\mathscr{H}(\gamma)| \leqq 2H_K(\gamma)$$

(see e.g. Brindza [3]).

Let

$$S_1 = S\cup\mathscr{H}(\mu)\cup\mathscr{H}(\Lambda)\cup\mathscr{H}(\alpha_s-\alpha_r)\cup\mathscr{H}(\alpha_r-\alpha_t)\cup\mathscr{H}(\alpha_t-\alpha_s).$$

Then in equation (6) u_r, u_s, u_t are all S_1-units. For the cardinality of S_1 we have

$$|S_1| \leqq |S|+2H_K(\mu)+24A+6\bar{c}X.$$

Further, any three of $1, u_r, u_s, u_t$ are linearly independent over k, since this property holds for $\beta'_r, \beta'_s, \beta'_t, \Lambda$. Thus we may apply Lemma 1 to equation (6) and we get

$$H_K(u_i) \leqq 16(|S_1|+G), \quad i = r, s, t$$

that is

$$\begin{aligned} H_K(\beta_i) &\leqq H_K(u_i)+H_K(\Lambda)+2A \leqq \\ &\leqq 16\,|S|+32H_K(\mu)+392A+99\bar{c}X+16G = C_1(X), \quad i = r, s, t. \end{aligned} \tag{7}$$

B) Consider now the case when there exist three of $\beta'_r, \beta'_s, \beta'_t, \Lambda$ which are linearly dependent over k. In this case we have two subcases.

B1) First assume that β_r', β_s' and β_t' are linearly dependent over k, that is, there exist non-zero elements k_i in k such that if $\beta_i''=k_i\beta_i'$ $(i=r, s, t)$ then

$$\beta_r''+\beta_s''+\beta_t''=0. \tag{8}$$

Now let

$$S_2=S\cup\mathscr{H}(\mu)\cup\mathscr{H}(\alpha_r-\alpha_s)\cup\mathscr{H}(\alpha_s-\alpha_t)\cup\mathscr{H}(\alpha_t-\alpha_s)$$

then β_r'', β_s'', β_t'' are S_2-units, and applying Lemma 2 to (8) we get

$$\max\left\{H_K\left(\frac{\beta_r''}{\beta_t''}\right),\ H_K\left(\frac{\beta_s''}{\beta_t''}\right)\right\}\leqq|S_2|+G. \tag{9}$$

Put $\sigma=\beta_t$, $\varrho_t=1$, $\varrho_r=\dfrac{\beta_r}{\beta_t}$, $\varrho_s=\dfrac{\beta_s}{\beta_t}$ then we have

$$\beta_i=\sigma\varrho_i,\quad i=r, s, t \tag{10}$$

where σ, ϱ_r, ϱ_s, ϱ_t are non-zero elements in K and by (9)

$$\max_{i=r,s,t} H_K(\varrho_i)\leqq|S|+2H_K(\mu)+16A+G=C_2. \tag{11}$$

B2) Secondly, suppose that two of β_r', β_s', β_t', say β_r' and β_s' together with Λ form a linearly dependent system over k. We may assume that $\Lambda\neq0$, because otherwise we have case B1. Thus, there are non-zero elements k_r, k_s, k_Λ in k such that if $\beta_i''=k_i\beta_i'$ $(i=r, s)$ and $\Lambda'=k_\Lambda\Lambda$ then

$$\beta_r''+\beta_s''+\Lambda'=0. \tag{12}$$

Let

$$S_3=S\cup\mathscr{H}(\mu)\cup\mathscr{H}(\Lambda')\cup\mathscr{H}(\alpha_s-\alpha_t)\cup\mathscr{H}(\alpha_t-\alpha_r).$$

Then β_r'', β_s'' and Λ' are S_3-units and applying Lemma 2 to equation (12) we get

$$\max\left\{H_K\left(\frac{\beta_r''}{\Lambda'}\right),\ H_K\left(\frac{\beta_s''}{\Lambda'}\right)\right\}\leqq|S_3|+G$$

that is

$$\begin{aligned} H_K(\beta_i)&\leqq|S_3|+G+H_K(\Lambda)+2A\leqq \\ &\leqq|S|+2H_K(\mu)+28A+9\bar{c}X+G\quad(i=r, s). \end{aligned} \tag{13}$$

Let us consider now the system of all possible choices of the indices r, s, t.

First assume that for all choices of the indices we have case B1. Let us choose the indices 1, 2, 3, then (10) and (11) imply that there exist non-zero σ, ϱ_1, ϱ_2, ϱ_3 in K such that

$$\beta_i=\sigma\varrho_i,\quad i=1, 2, 3 \tag{14}$$

where $H_K(\varrho_i)\leqq C_2$ $(i=1, 2, 3)$. Let r be any index with $r>3$ (if $n>3$), and consider the choice 1, 2, r. Again by (10) and (11) we can see that there exist non-zero σ_r, ϱ_1', ϱ_2', ϱ_r' in K such that

$$\beta_i=\sigma_r\varrho_i',\ i=1, 2, r \tag{15}$$

where $H_K(\varrho_i')\leqq C_2$ $(i=1, 2, r)$. Comparing the expression for β_1 in (14) and (15) we get $\sigma_r=\sigma\frac{\varrho_1}{\varrho_1'}$, that is

$$\beta_r = \sigma\varrho_r, \tag{16}$$

where $H_K(\varrho_r)\leqq 3C_2$. Since r was an arbitrary index with $r>3$, hence (16) holds for any $r>3$ (if $n>3$). Substituting the expressions (14) and (16) into our equation (4) we have $\sigma^n\varrho_1\ldots\varrho_n=\mu$ whence

$$H_K(\sigma) \leqq \frac{1}{n}(H_K(\mu)+(n-2)3C_2)\leqq H_K(\mu)+3C_2,.$$

Thus, from (14) and the above inequality we obtain that there exist indices r, s (say $r=1$, $s=2$) such that

$$H_K(\beta_i) \leqq H_K(\mu)+4C_2 \leqq 4|S|+9H_K(\mu)+64A+4G, \quad i = r, s. \tag{17}$$

In the opposite case if we can choose r, s, t so that we have case A or case B2, then there are indices r, s such that (7) or (13) holds. Combining this with (17) we obtain that in any situation there exist indices r, s such that

$$\max_{i=r,\,s} H_K(\beta_i) \leqq C_1(X).$$

This estimate together with the expressions

$$x = \frac{\alpha_r(\beta_s-\lambda_s)-\alpha_s(\beta_r-\lambda_r)}{\alpha_r-\alpha_s}, \quad y = \frac{(\beta_r-\lambda_r)-(\beta_s-\lambda_s)}{\alpha_r-\alpha_s}$$

imply

$$X < 2C_1(X)+4A+2\bar{c}X,$$

whence by the definition of $C_1(X)$ and $\bar{c}$, (3) follows.

References

[1] A. Baker, Contributions to the theory of Diophantine equations, *Phil. Trans. Roy. Soc. London*, **A 263** (1968), 173—208.

[2] A. Baker, *Transcendental number theory* (2nd ed.) (Cambridge, 1979).

[3] B. Brindza, On the equation $F(x, y)=z^m$ over function fields, *Acta Math. Hungar.*, **49** (1987), 267—275.

[4] J. Coates, An effective p-adic analogue of a theorem of Thue, *Acta Arith.*, **15** (1969), 279—305.

[5] N. I. Feldman, An effective refinement of the exponent in Liouville's theorem (in Russian), *Izv. Akad. Nauk SSSR*, **35** (1971), 973—990.

[6] I. Gaál, Norm form equations with several dominating variables and explicit lower bounds for inhomogeneous linear forms with algebraic coefficients, *Studia Sci. Math. Hungar.*, **19** (1984), 399—411.

[7] I. Gaál, Norm form equations with several dominating variables and explicit lower bounds for inhomogeneous linear forms with algebraic coefficients, II. *Studia Sci. Math. Hungar.*, to appear.

[8] K. Győry, On the greatest prime factors of decomposable forms at integer points, *Ann. Acad. Sci. Fenn. Ser. A. I.*, **4** (1978/1979), 341—355.

[9] K. Győry, Explicit upper bounds for the solutions of some diophantine equations, *Ann. Acad. Sci. Fenn. Ser. A. I.*, **5** (1980), 3—12.

[10] K. Győry, Résultats effectifs sur la représentation des entiers par des formes décomposables, *Queen's Papers in Pure and Applied Math.*, No. 56 (Kingston, Canada, 1980).

[11] K. Győry, Sur certaines généralisations de l'équation de Thue-Mahler, *Enseignement Math.*, **26** (1980), 247—255.

[12] K. Győry, On the representation of integers by decomposable forms in several variables, *Publ. Math. (Debrecen)*, **28** (1981), 89—98.

[13] K. Győry, On S-integral solutions of norm form, discriminant form and index form equations, *Studia Sci. Math. Hungar.*, **16** (1981), 149—161.

[14] K. Győry, Bounds for the solutions of norm form, discriminant form and index form equations in finitely generated integral domains, *Acta Math. Hungar.*, **42** (1983), 45—80.

[15] K. Győry, On norm form, discriminant form and index form equations, Coll. Math. Soc. János Bolyai 34, *Topics in Classical Number Theory* (Budapest, 1981), pp. 617—676, (1984).

[16] K. Győry and Z. Z. Papp, Effective estimates for the integer solutions of norm form and discriminant form equations, *Publ. Math. (Debrecen)*, **25** (1978), 311—325.

[17] K. Győry and Z. Z. Papp, Norm form equations and explicit lower bounds for linear forms with algebraic coefficients, *Studies in Pure Mathematics* (To the Memory of Paul Turán), pp. 245—267, Akadémiai Kiadó (Budapest, 1983).

[18] S. V. Kotov, The Thue—Mahler equation in relative fields (in Russian), *Acta Arith.*, **27** (1975), 293—315.

[19] S. V. Kotov, On diophantine equations of norm form type I (in Russian), *Inst. Mat. Akad. Nauk BSSR*, Preprint No. 9 (Minsk, 1980).

[20] S. V. Kotov, On diophantine equations of norm form type II (In Russian), *Inst. Mat. Akad. Nauk BSSR*, Preprint No. 10 (Minsk, 1980).

[21] S. V. Kotov, Effective bound for a linear form with algebraic coefficients in archimedean and p-adic metrics (in Russian), *Inst. Mat. Akad. Nauk BSSR*, Preprint No. 24 (Minsk, 1981).

[22] S. V. Kotov, Effective bound for the values of the solutions of a class of diophantine equations of norm form type (in Russian), *Mat. Zametki*, **33** (1983), 801—806.

[23] R. C. Mason, On Thue's equation over function fields, *J. London Math. Soc. Ser.* 2, **24** (1981), 414—426.

[24] R. C. Mason, Diophantine equations over function fields, *London Math. Soc. Lecture Note Series*, No. 96 (Cambridge, 1984).

[25] R. C. Mason, Norm form equations I, *J. Number Theory*, **22** (1986), 190—207.

[26] C. F. Osgood, An effective lower bound on the diophantine approximation of algebraic functions by rational functions, *Mathematika*, **20** (1973), 4—15.

[27] C. F. Osgood, Effective bounds on the diophantine approximation of algebraic functions over fields of arbitrary characteristic and applications to differential equations, *Indag. Math.*, **37** (1975), 105—119.

[28] W. M. Schmidt, On Osgood's effective Thue theorem for algebraic functions, *Commun. on Pure and Applied Math.*, **29** (1976), 759—773.

[29] W. M. Schmidt, Thue's equation over function fields, *J. Austral Math. Soc. Ser. A*, **25** (1978), 385—422.

[30] W. M. Schmidt, Polynomial solutions of $F(x, y)=z^n$, Proc. Queen's Number Theory Conf. (1979), pp. 33—65. *Queen's Papers in Pure and Applied Math.*, No. 54 (Kingston, Canada, 1980).

[31] V. G. Sprindžuk, Representation of numbers by norm forms with two dominating variables, *J. Number Theory*, **6** (1974), 481—486.

[32] V. G. Sprindžuk, *Classical diophantine equations in two variables* (in Russian), Izd. Nauka (Moscow, 1982).

[33] S. A. Stepanov, Diophantine equations over function fields (in Russian), *Mat. Sbornik*, **112** (1980), 86—93.

[34] A. Thue, Über Annäherungswerte algebraischer Zahlen, *J. Reine Angew. Math.*, **135** (1909), 284—305.

[35] A. I. Vinogradov and V. G. Sprindžuk, The representation of numbers by binary forms (in Russian), *Mat. Zametki*, **3** (1968), 369—376.

(Received October 10, 1985)

MATHEMATICAL INSTITUTE
KOSSUTH LAJOS UNIVERSITY
4010 DEBRECEN
HUNGARY

Acta Math. Hung·
50 (1—2) (1987), 155—163.

A SECOND ORDER NONLINEAR DIFFERENTIAL INEQUALITY

I. BIHARI (Budapest)

Consider the differential inequality

$$(1) \qquad (px'^{\overset{*}{n}})' - qf(x) > 0, \;\; ' = \frac{d}{dt}, \quad t\in I = [t_0, \infty], \quad t_0 \in \mathbf{R},$$

where $n > 0, p, q > 0 \,\&\, \in C(I), f(x)\in C(R), xf > 0, x \neq 0, f(0) = 0,$ f is increasing and $u^{\overset{*}{n}} = |u|^n \operatorname{sgn} u,\ u\in \mathbf{R}$.

By a solution of inequality (1) a function $x(t)$ is meant with the property $x\in C_1(I)$, $px'^{\overset{*}{n}} \in C_1(I)$ and satisfying (1). We take into account the class of solutions existing in the *whole* interval I. It will be proved that K is not empty.

The present paper aims at the extension of N. Parhi's results [1]. The following definitions from [1] will be recalled here. A solution $x(t)$ is *non-oscillatory* if $x(t)\neq 0$ for $t\geqq t_1$ with some $t_1\geqq t_0$ and is *oscillatory* if it has zeros τ_k, $k=1, 2, \ldots,$ arbitrarily large and changes sign at τ_k, finally $x(t)$ is of *type Z* if it has arbitrarily large zeros and is ultimately non-negative or non-positive, respectively. The zeros of $x(t)$ cannot have a finite cluster point $\hat{\tau}$, since then $x(\hat{\tau}) = x'(\tau) = f\big(x(\hat{\tau})\big) = 0$ and x' vanishes by Rolle's theorem at some $\tau_{k-1} < t_k < \tau_k$, therefore $px'^{\overset{*}{n}}|_{t_k} = 0$ and $(px'^{\overset{*}{n}})'|_{\hat{\tau}} = 0$ involving $0 > 0$ in (1).

THEOREM 1. *Every solution of* (1) *is non-oscillatory.*

PROOF. In the opposite case (1) would possess oscillatory or Z type solution.

(i) First suppose the existence of an oscillatory or non-negative Z type solution having two consecutive zeros $t_0 \leqq a < b$ between which $x(t) > 0$, i.e.

$$x(a) = x(b) = 0, \quad x'(a) \geqq 0, \quad x'(b) \leqq 0.$$

The zeros of $x'(t)$ cluster nowhere (at a finite point). In such point $\hat{t}$ it would hold $x'(\hat{t}) = px'^{\overset{*}{n}}|_{\hat{t}} = (px'^{\overset{*}{n}})'|_{\hat{t}} = 0$ excluded by (1). Therefore there is a first point c $(a < c < b)$ where $x'(c) = 0$ and integrating (1) over (a, c) we have

$$-px'^{\overset{*}{n}}|_a - \int_a^c qf(x)\,dt > 0$$

which involves contradiction, because the terms on the left hand side are $\leqq 0$ and < 0, respectively.

(ii) If a non-positive Z type solution $x(t)$ existed with successive zeros $a<b$, then $(px'^{\overset{*}{n}})'|_b>0$, so $px'^{\overset{*}{n}}$ is increasing at $t=b$, but $px'^{\overset{*}{n}}|_b=0$, thus in a small enough neighbourhood of b, $px'^{\overset{*}{n}}<0$ $(t<b)$ and $px'^{\overset{*}{n}}>0$ $(t>b)$, i.e. $x'>0$ and $x>0$ for $t-b>0$ which is incompatible with the negative Z type character of $x(t)$.

In the sequel the following simple Lemma will be necessary.

LEMMA. *If* $x(t)>0$, $t\geqq t_1$, *then* $x'(t)$ *can have one zero at most.*

Namely if it had two zeros $t_1<\tau_1<\tau_2$, then from (1) $0-\int\limits_{\tau_1}^{\tau_2} qf(x)dt>0$ which is impossible.

THEOREM 2. *The class K is not empty. There are both ultimately positive increasing and decreasing solutions belonging to K i.e. existing on the whole I provided* $F(\infty)=\infty$, *where* $F(x)=\int\limits_{x_1}^{x} f^{-\nu}(\xi)d\xi$, $\nu=\frac{1}{n}$.

PROOF. 1° Suppose $\hat{x}(t)$ is a positive solution of the *equation*

$$(p\hat{x}'^{\overset{*}{n}})'-kqf(\hat{x})=0, \quad k=\text{const}>1 \tag{2}$$

with $\hat{x}(t_0)=\hat{x}_0>0, \hat{x}'(t_0)=0$. This surely exists in a right hand neighbourhood of t_0, and it can be continued to the whole I. Namely, by the Lemma — which is valid concerning (2), too — $\hat{x}'$ has no more zeros. Besides, $\hat{x}(t)$ is increasing, because by the transformation $\tau=\int\limits_{t_0}^{t} p^{-\nu}(s)ds$, the equation (2) turns to

$$\frac{d}{d\tau}\left[\left(\frac{d\tilde{x}}{d\tau}\right)^{\overset{*}{n}}\right]-\tilde{p}^{1/n}k\tilde{q}f(\tilde{x})=0, \tag{2'}$$

or

$$\left|\frac{d\tilde{x}}{d\tau}\right|^{n-1}\frac{d^2\tilde{x}}{d\tau^2}-\frac{1}{n}\tilde{p}^{1/n}k\tilde{q}f(\tilde{x})=0. \tag{2''}$$

(Here $\tilde{x}(\tau)=\hat{x}(t), \tilde{p}(\tau)=p(t), \tilde{q}(\tau)=q(t)$.) Hence the convexity of $\tilde{x}(\tau)$ follows which involves that $\tilde{x}(\tau)$ and $\hat{x}(t)$ increase, since $p\hat{x}'^{\overset{*}{n}}=\left(\frac{d\tilde{x}}{d\tau}\right)^{\overset{*}{n}}$, i.e. $\hat{x}'=\frac{d\hat{x}}{dt}$ and $\frac{d\tilde{x}}{d\tau}$ have the same sign. From (2) we have (the star $*$ can be omitted since $\hat{x}'>0$)

$$(p\hat{x})'^n)'\cdot f^{-1}(\hat{x})=kq$$

hence by integration by parts

$$[(p\hat{x}'^n)f^{-1}(\hat{x})]_{t_0}^{t}+\int\limits_{t_0}^{t} p\hat{x}'^{n+1}f^{-2}(\hat{x})f'(\hat{x})\,dt=k\int\limits_{t}^{t} q\,dt.$$

The second term on the left is positive, thus

$$\hat{x}' f^{-\nu}(\hat{x}) < \Bigl(cp^{-1}+kp^{-1}\int_{t_0}^{t} q\,dt\Bigr)^{\nu} = G(t),\ \ c = (p\hat{x}'^{n})f^{-1}(\hat{x})\big|_{t_0}$$

where $G(t)$ is a function continuous on I. Hence

$$F(\hat{x}) = \int_{\hat{x}_0}^{\hat{x}} f^{-\nu}(\xi)\,d\xi < \int_{t_0}^{t} G(s)\,ds$$

which exhibits together with $F(\infty)=\infty$ that $\hat{x}(t)$ exists on the whole I (namely it is not becoming infinity at a finite point and neither the zeros of $\hat{x}$, nor those of $\hat{x}'$, cluster at a finite point). Furthermore $\hat{x}(t)$ is a solution of (1), too.

2° Equation (2) has ultimately positive decreasing solution existing on the *whole I.*

Take the solution $\tilde{x}(\tau)$ of (2′)—(2″) satisfying $\tilde{x}(\tau_1)=0$, $\tilde{x}'(\tau_1)=\mu<0$ and continue $\tilde{x}(\tau)$ backwards to intersect the ordinate $\tau=\tau_0<\tau_1$ at the point (τ_0, x_0), $x_0>0$, what is possible if $\tau_1-\tau_0$ is small enough. Now fixing the point (τ_0, x_0) let the slope $m=\tilde{x}'(\tau_0)<0$ increase. By convexity, the graph of $\tilde{x}(\tau)$ is above its tangent line at (τ_0, x_0), therefore it is intuitively clear that there is an $m=m_l<0$ at which $\lim\limits_{m\to m_l}\tau_1=\infty$ i.e. the point $A=(\tau_1, 0)$ ceases to exist. More exactly this can be seen as follows. First of all a Sturmian theorem (see Theorem 7) holds concerning (2) or (2′)—(2″) saying that if $\tilde{x}_i(\tau)$ $(i=1, 2)$ are solutions of (2′)—(2″) with $\tilde{x}_1(\tau_0)=x_2(\tau_0)$ and $m_1=\tilde{x}_1'(\tau_0)<m_2=\tilde{x}_2'(\tau_0)<0$, then $\tilde{x}_2>\tilde{x}_1$ as long as $\tilde{x}_1\geqq 0$, therefore they intersect the τ axis at the points $\tau_{11}<\tau_{12}$. In consequence the numbers $m\in(-\infty, 0]$ can be ranged in two classes α and β, respectively: the numbers m belonging to α have the property that the corresponding solutions $\tilde{x}_m(\tau)$ intersects the τ axis (for them the point A exists), while for the m's of β, A does not exist. If $m\in\alpha$ and $m'<m$, then $m'\in\alpha$ and if $m\in\beta$ and $m''>m$, then $m''\in\beta$. In this way a Dedekind's section is obtained for $m\in(-\infty, 0]$. Observe that β is not empty, since $m=0$ is in β. The value $m_l=\{\alpha, \beta\}$ determines a solution $\tilde{x}_l(\tau)=\hat{x}(t)$ which is positive decreasing and exists on the whole I. In the sequel only solutions belonging to the class K will be taken into account. An exception is Theorem 7.

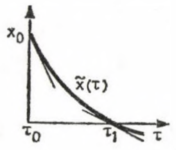

Fig. 1

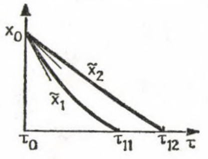

Fig. 2

THEOREM 3. *Each of the following three sets of conditions involves that the ultimately positive solutions of* (1) *tend either to infinity or to zero:*

(i) $\int^{\infty} p^{-\nu} = \int^{\infty} q = \infty$,

(ii) $I(\infty)=J(\infty)=\infty$, *where* $I(t)=\int\limits_0^t p^{-\nu}(s)\big(\int\limits_0^S q(r)\,dr\big)^\nu ds,\ J(t)=\int\limits_0^t q(s)\,ds,$

(iii) $\int\limits^\infty p^{-\nu}=\infty,\ \int\limits^\infty q<\infty,\ I_1(\infty)=\infty$, *where* $I_1(t)=\int\limits_0^t p^{-\nu}(s)\big(\int\limits_s^\infty q(r)\,dr\big)^\nu ds.$

PROOF. 1° Assuming (i) (by the Lemma) x' is of constant sign for $t\geqq t_1$ with some $t_1\geqq t_0$. If $x'>0$, $t\geqq t_1$, then by (1) $(px'^n)'>0$, $t\geqq t_1$ and in turn $px'^n>px'^n|_{t_1}=c_1$, $x'>c_2p^{-\nu}$ $(c_i=\text{const}>0,\ i=1,2)$, whence

$$x>x(t_1)+c_2\int\limits_{t_1}^t p^{-\nu}\,ds,\quad \lim_{t\to\infty}x=\infty.$$

If $x'<0$, $t\geqq t_1$, then $k=\lim\limits_{t\to\infty}x$ exists and $\geqq 0$. If it were $k>0$ then by (1)

$$px'^{\overset{*}{n}}-px'^{\overset{*}{n}}|t_1>\int\limits_{t_1}^t qf(x)\,dt>f(k)\int\limits_{t_1}^t q(s)\,ds.$$

Therefore $px'^{\overset{*}{n}}$ and x' too, would be positive for t large enough. Consequently $k=0$ what was to be proved.

2° Assume now (ii). Then if $x'<0$, $t\geqq t_1$ we argue as in 1°. If $x'>0$, $t\geqq t_1$, then $x>M$, $t\geqq t_1$ with some $M=\text{const}>0$. Thus from (1)

$$px'^n>px'^n|_{t_1}+f(M)\int\limits_{t_1}^t q\,ds>f(M)\int\limits_{t_1}^t q\,ds,$$

whence

$$x'>f^\nu(M)\,p^{-\nu}\big(\int\limits_{t_1}^t q\,ds\big)^\nu,$$

$$x(t)>x(t_1)+f^\nu(M)\int\limits_{t_1}^t p^{-\nu}(s)\big(\int\limits_{t_1}^s q(r)\,dr\big)^\nu ds$$

thus $\lim\limits_{t\to\infty}x=\infty$.

3° Assume now (iii). If $x'>0$, $t\geqq t_1$ we argue as in 1°. If $x'<0$, $t\geqq t_1$ then $px'^{\overset{*}{n}}$ is negative and increasing by (1), so $\lim\limits_{t\to\infty}(px'^{\overset{*}{n}})=c$ exists and is $\leqq 0$. But c cannot be negative what can be seen as follows. With the transformation $\tau=\int\limits_{t_0}^t p^{-\nu}(s)ds$ used before inequality (1) assumes the form

$$\frac{d}{d\tau}\left[\left(\frac{d\bar{x}}{d\tau}\right)^{\overset{*}{n}}\right]-\bar{p}^\nu\bar{q}f(\bar{x})>0,$$

where

$$\bar{x}(\tau)=x(t),\ \text{etc.},\ \lim\tau=\infty,\quad c=\lim_{\tau\to\infty}\left(\frac{d\bar{x}}{d\tau}\right)^{\overset{*}{n}}.$$

If c were negative, $\bar{x}(\tau)$ would have an asymptote with negative slope $c^{\overset{*}{\nu}}$ involving the vanishing of $\bar{x}$ at a finite point (cutting axis τ). Therefore $c=0$ and by (1) in turn

$$-px'^{\overset{*}{n}} > \int_t^\infty qf(x), \quad -x' > p^{-\nu}\Big(\int_t^\infty qf(x)\Big)^\nu,$$

$$x_1-x > \int_{t_1}^t p^{-\nu}(s)\Big(\int_s^\infty q(r)f(x(r))\,dr\Big)^\nu ds, \quad x_1 = x(t_1).$$

Since x is decreasing,

$$x_1-x > f^\nu(x)\,I_1(t), \quad f(x) = f\big(x(t)\big),$$

whence $x_1 > x+f^\nu(x)I_1(t)$. Being $I_1(\infty)=\infty$ we get $\lim\limits_{t\to\infty} x=0$. Let us observe that if $f(x)=x^{\overset{*}{n}}$, then $x<x_1/\big(1+I_1(t)\big)$ which is an explicit estimate.

THEOREM 4. *If* $\int^\infty p^{-\nu}=\infty$, $\int^\infty q<\infty$, $I_1(\infty)<\infty$, $k^\nu I_1(\infty)f^\nu(x_1)<x_1$ $(x_1\in\mathbf{R}^+)$ *where* $I_1(t)$ *is as above in Theorem* 3 *and* $k=\text{const}>1$, *then the ultimately positive increasing solutions tend to infinity and there are ultimately decreasing positive solutions tending to a positive limit as* $t\to\infty$.

PROOF. The first assertion can be proved as in Theorem 3, Part 3°. To prove the second assertion observe that for a solution $\hat{x}(t)$ of *equation* (2) with $\hat{x}'<0$, $t\geqq t_1$ we had (see Theorem 3, Part 3°) $c=\lim\limits_{t\to\infty} p\hat{x}'^{\overset{*}{n}}=0$ involving

$$-p\hat{x}'^{\overset{*}{n}} = k\int_t^\infty qf(\hat{x}), \quad -\hat{x}' = p^{-\nu}\Big(k\int_t^\infty qf(\hat{x})\Big)^\nu,$$

whence

$$\hat{x}_1-\hat{x} = \int_{t_1}^t p^{-\nu}(s)\Big(\int_s^\infty kq(r)f(x(r))\,dr\Big)^\nu ds, \quad \hat{x}_1 = \hat{x}(t_1) = x_1,$$

hence

$$\hat{x}_1-\hat{x} < f^\nu(\hat{x}_1)\,k^\nu I_1(t) < f^\nu(\hat{x}_1)\,k^\nu I_1(\infty)$$

finally

$$\hat{x} > \hat{x}_1-k^\nu f^\nu(\hat{x}_1)\,I_1(\infty) > 0.$$

i.e. $\lim\limits_{t\to\infty}\hat{x}(t)>0$. Being $\hat{x}(t)$ a solution of (1) too, the proof is complete.

REMARK 1. Condition $k^\nu f^\nu(x_1)I_1(\infty)<x_1$ can always be satisfied — at fixed $\hat{x}_1$ — by choosing t_1 large enough. If $f(x)=x^{\overset{*}{n}}$ then this condition reduces to $k^\nu I_1(\infty)<1$.

REMARK 2. Of course, Theorem 3 has a negative wording, too, namely: if (1) has ultimately positive increasing *bounded* solution, then $I<\infty$ and if it has a positive decreasing one not tending to zero, then $J<\infty$, etc.

REMARK 3. Every solution $\hat{x}$ of (2) has at most one zero, since $\tilde{x}(\tau)$ is convex where it is positive and concave where it is negative. Thus there are solutions of (1) with the same property.

REMARK 4. If $f(x)=x^{\overset{*}{n}}$, then with the substitution

$$u=\frac{p\hat{x}'^{\overset{*}{n}}}{\hat{x}^{n}}\quad(\hat{x}>0),$$

equation (2) turns into the Riccati-like equation

$$u'+np^{-\nu}|u|^{1+\nu}-kq=0$$

whence

$$u'<kq\quad\text{or}\quad\left(\frac{p\hat{x}'^{\overset{*}{n}}}{\hat{x}^{n}}\right)'<kq.$$

Under the conditions of Theorem 4 and $\hat{x}'<0$ we have $c=\lim\limits_{t\to\infty}(p\hat{x}^{\overset{*}{n}})=0$, $\lim\limits_{t\to\infty}\hat{x}>0$, thus

$$-\frac{p\hat{x}'^{\overset{*}{n}}}{\hat{x}^{n}}<k\int_{t}^{\infty}q\,ds,\quad -\frac{\hat{x}'}{\hat{x}}<p^{-\nu}\Big(k\int_{t}^{\infty}q\,ds\Big)^{\nu},$$

whence

$$\log\hat{x}-\log\hat{x}_{\infty}<\int_{t}^{\infty}p^{-\nu}(s)\Big(k\int_{s}^{\infty}q(r)\,dr\Big)^{\nu}ds=k^{\nu}I_2(t)$$

where

$$I_2(t)=\int_{t}^{\infty}p^{-\nu}(s)\Big(\int_{s}^{\infty}q(r)\,dr\Big)^{\nu}ds$$

or

$$\hat{x}<\hat{x}_{\infty}\exp\big(k^{\nu}I_2(t)\big),\quad \hat{x}_{\infty}=\lim_{t\to\infty}\hat{x}>0.$$

Since $I_2(t_1)=I_1(\infty)$ and $k^{\nu}I_1(\infty)<1$ we have $\hat{x}_1=x_1<\hat{x}_{\infty}e=x_{\infty}e$, whence $x_1>x_{\infty}>x_1e^{-1}$.

THEOREM 5. *If for an ultimately positive increasing solution $x(t)$ of (1) the function px'^{n} — essentially the derivative — is bounded, then*

$$\int_{t_0}^{\infty}q(t)f\Big(c\int_{t_0}^{s}p^{-\nu}(s)ds\Big)\,dt<\infty, \tag{3}$$

where c is a constant defined below.

PROOF. Let $x>0$, $x'>0$, $t \geqq t_1$. Then by (1) px'^n increases and so in turn

$$x' > cp^{-\nu}(t), \quad c = p^{\nu}(t_1)\, x'(t_1) > 0,$$

$$x > x(t_1) + c \int_{t_1}^{t} p^{-\nu}\, ds > c \int_{t_1}^{t} p^{-\nu}\, ds,$$

$$(px'^n)' > qf(x) > qf\Bigl(c \int_{t_1}^{t} p^{-\nu}\, ds\Bigr),$$

$$px'^n > px'^n|_{t_1} + \int_{t_1}^{t} q(s) f\Bigl(c \int_{t_1}^{s} p^{-\nu}(r)\, dr\Bigr)\, ds.$$

Since px'^n is bounded, so is the integral in question, too. If $f(x) = x^{\overset{*}{m}}$, $m>0$, then (3) assumes the form

$$\int_{t_0}^{\infty} q(t) \Bigl(\int_{t_0}^{s} p^{-\nu}(s)\, ds\Bigr)^m dt < \infty.$$

The converse of Theorem 5 is perhaps more interesting: if the above integral (3) is infinite then px'^n is not bounded.

THEOREM 6. *If* $F(-\infty) = \infty$ *where again* $F(x) = \int_{x_1}^{x} f^{-\nu}(\xi) d\xi$ *and* $I_2(\infty) < \infty$ *where*

$$I_2(t) = \int_{t_0}^{t} p^{-\nu}(s) \Bigl(\int_{t_0}^{s} q(r)\, dr\Bigr)^{\nu} ds, \quad n \leqq 1 \ (\nu \geqq 1),$$

$x_1 = x(t_1) < 0$, *then every ultimately negative solution of* (1) *is bounded below.*

PROOF. The proof itself will show the existence of $x(t)$ for $t \geqq t_1$. First of all we show that the convergence of $I_2(t)$ involves that of $\int^{\infty} p^{-\nu} dt$. Namely, if t_1 is a *fixed* number with $t_0 \leqq t_1 < t$ then

$$I_2(t) = I_2(t_1) + \int_{t_1}^{t} p^{-\nu}(s) \Bigl(\int_{t_0}^{S} q(r)\, dr\Bigr)^{\nu} ds,$$

$$K > I_2(t) > I_2(t_1) + \Bigl(\int_{t_0}^{t_1} q(r)\, dr\Bigr)^{\nu} \int_{t_1}^{t} p^{-\nu}(s)\, ds,$$

where K is a positive constant. So $\int_{t_1}^{t} p^{-\nu}(s)\, ds$ is bounded. Multiplying (1) by $f^{-1}(x)$ we have (being $f(x)<0$, $t \geqq t_1$)

$$(px'^{\overset{*}{n}})' f^{-1}(x) - q < 0,$$

whence following the lines of the proof of Theorem 2 we have

$$x'f^{-1}(x) < \big(cp^{-1}+p^{-1}\int_{t_1}^{t} q\,ds\big)^{\nu}, \quad c = px'^{\overset{*}{n}}f^{-1}(x)\big|_{t_1}. \tag{4}$$

Now two cases must be distinguished:

(i) $x'(t_1)>0$, i.e. $c<0$, then from (4)

$$F(x) < \int_{t_1}^{t} p^{-\nu}(s)\,\big(\int_{t_1}^{s} q(r)\,dr\big)^{\nu}\,ds. \tag{5}$$

(ii) $x'(t_1)<0$, $c>0$, then by the use of the inequality

$$\left(\frac{A+B}{2}\right)^{\nu} < \frac{A^{\nu}+B^{\nu}}{2} \quad \text{or} \quad (A+B)^{\nu} < 2^{\nu-1}(A^{\nu}+B^{\nu})$$

we get from (4)

$$2^{1-\nu}F(x) < c^{\nu}\int_{t_1}^{t} p^{-\nu}(s)\,ds+\int_{t_1}^{t} p^{-\nu}(s)\,\big(\int_{t_1}^{t} q(r)\,dr\big)^{\nu}\,ds. \tag{6}$$

Inequalities (5)—(6) involve the result stated above.

If $f(x)=x^{\overset{*}{m}}$ $0<m<n<1$, then $F(x)=\dfrac{|x|^{1-r}}{1-r}$, $r=\dfrac{m}{n}$ and if $m=n$ then $F(x)=\log|x|$.

THEOREM 7. *Consider the inequalities*

$$\begin{cases} (x'^{\overset{*}{n}})'-q_1(t)f(x) \geqq 0 \\ (x'^{\overset{*}{n}})'-q_2(t)f(x) \leqq 0 \end{cases} \quad (q_1 \geqq q_2 > 0,\ t \geqq t_0), \tag{7}$$

admitting the solutions x_1 *and* x_2, *respectively, satisfying the conditions*

$$x_1(t_1) = x_2(t_1) \geqq 0, \tag{8_1}$$

$$x_1'(t_1) > x_2'(t_1). \tag{8_2}$$

Then $x_1>x_2$, $x_1'>x_2'$, $t>t_1$ *as long as* $x_2\geqq 0$ *or* x_1 *ceases to exist.*

PROOF. From (7) we obtain

$$(x_1'^{\overset{*}{n}}-x_2'^{\overset{*}{n}})' \geqq (q_1-q_2)f(x_1)+q_2\big(f(x_1)-f(x_2)\big). \tag{9}$$

At $t=t_1$ the second term on the right hand side is zero. Then at $t=t_1$ $(x_1'^{\overset{*}{n}}-x_2'^{\overset{*}{n}})'>0$ and it remains positive for $t-t_1>0$ and small enough, $x_1'^{\overset{*}{n}}-x_2'^{\overset{*}{n}}$ increases there, but with respect to (8_2)

$$x_1'^{\overset{*}{n}} > x_2'^{\overset{*}{n}},\ x_1' > x_2',\ x_1 > x_2 > 0$$

for $t-t_1>0$ and small enough. Thus for small $t-t_1$ the assertion is true. It must be shown that it remains valid as long as indicated. If it were not so then a first (smallest) $t=c>t_1$ would exist where $x_1'=x_2'$ (namely at first the derivatives can get equal while $x_1>x_2$ remains valid for a while). But then we get from (9)

$$0=(x_1'^{\overset{*}{n}}-x_2'^{\overset{*}{n}})|_c \geqq (x_1'^{\overset{*}{n}}-x_2'^{\overset{*}{n}})|_{t_1}+\int_{t_1}^{c}[(q_1-q_2)f(x_1)+q_2(f(x_1)-f(x_2))]\,dt>0$$

which involves contradiction.

Remark 5. If in (1) $f(-x)=f(x)$ and $x(t)$ satisfies (1) then $-x(t)$ satisfies

$$(px'^{\overset{*}{n}})'-qf(x)<0. \tag{10}$$

Therefore Theorems 1—6 remain valid concerning (10) provided in their wording the adjectives positive and negative are interchanged. Theorem 7 must be modified in an obvious way.

I express my deep gratitude to Á. Elbert for his valuable remarks and criticism.

References

[1] N. Parhi, On nonoscillatory solutions of second order differential inequalities, *Acta Math. Hung.*, **42** (1983), 101—109.

[2] Ph. Hartman, *Ordinary differential equations* (London, 1964), pp. 357—358; Corollary 6.4 and Exercice 6.7.

[3] Ph. Hartman, Differential equations with nonoscillatory eigenfunctions, *Duke Math. J.*, **15** (1948), 697—709.

(Received October 10, 1985)

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13—15.
H—1053

Acta Math. Hung.
50 (1—2) (1987), 165—175.

POLYNOMIAL APPROXIMATION WITH EXPONENTIAL WEIGHTS[1]

Z. DITZIAN (Edmonton), D. S. LUBINSKY (Pretoria), P. NEVAI (Columbus)
and V. TOTIK (Szeged)

1. Introduction

As many other problems in approximation theory, the problem of weighted polynomial approximation was initiated by S. N. Bernstein. His question about characterization of weights w for which polynomials are dense in $C_0(w)$ stimulated a great deal of interest. The final solution came some forty years later, and though the general description is fairly delicate, for sufficiently regular weights it is equivalent to

$$\int_{\mathbf{R}} [\log w(x)][1+x^2]^{-1}\,dx = -\infty$$

(cf. [1]). The related qualitative results seem to have appeared first in the works of M. M. Dzarbasyan and his collaborators [6, 7, 8] (cf. [35]). In the early seventies G. Freud started a systematic study of the rate of polynomial approximation with weight

$$w_\lambda(x) = \exp(-|x|^\lambda), \quad x\in\mathbf{R}, \lambda>0.$$

Freud and his associates (cf. [9—28]) developed a powerful method which runs parallel with harmonic approximation and is based on orthogonal polynomials that are today commonly known as Freud polynomials (cf. [4], [29—34], [35], [36], [38—40], [41—45], [47—48] and the references therein). The rate of best approximation was characterized in terms of a K-functional by G. Freud and H. N. Mhaskar in [27] where a possible modulus of smoothness of orders one and two was proposed as well. The characterization of the relevant K-functional by a fairly simple modulus of smoothness was given in [4—5].

For technical reasons Freud's method restricts the parameter λ in w_λ to the range of $\lambda\geqq 2$. Freud's theory has recently been extended to $1<\lambda<2$ by A. L. Levin and D. S. Lubinsky in [29—30].

Let $n=1, 2, \ldots$ and $1\leqq p\leqq\infty$. For a function f and a positive weight w such that $fw\in L_p(\mathbf{R})$, let $E_n(f, w)_p$ be defined by

$$E_n(f, w)_p = \inf_{P_n} \|w(f-P_n)\|_r \tag{1.1}$$

---

[1] This material is based upon work supported by the Canadian National Research Council (first author), by National Science Foundation under Grant No. DMS 84—19525 (third author), by the United States Information Agency under Senior Research Fulbright Grant No. 85—41612 (third author) and by the Hungarian Ministry of Education (third author). The work was started while the first, second and fourth author visited The Ohio State University on various occasions between 1983 and 1985, and it was completed during the third author's visit to Hungary in 1985.

where the infimum is taken for all polynomials P_n of degree at most n. This is what we call the rate of best weighted approximation of the function f by polynomials. For $r \in \mathbf{N}$ we define the r-th symmetric difference $\Delta_h^r f$ of the function f with step h by

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^{r+k} C(r, k) f\big(x+(k-r/2)h\big)$$

where $C(r, k)$ is the binomial coefficient.

The following theorem is a summary of the results mentioned so far (cf. [4—5]).

THEOREM A. *Let* $\lambda > 1$, $1 \leqq p \leqq \infty$ *and* $fw_\lambda \in L_p(\mathbf{R})$. *Let* r *be a positive integer. Then for* $0 < \alpha < r$

$$E_n(f, w_\lambda)_p = O(n^{-\alpha(\lambda-1)/\lambda}), \quad n = 1, 2, \ldots,$$

is equivalent to

$$\|\chi_h w_\lambda \Delta_h^r f\|_p = O(h^\alpha), \quad h \to 0+$$

where χ_h *denotes the characteristic function of the interval* $(-h^{1/(1-\lambda)}, h^{1/(1-\lambda)})$.

Now let us turn to the question what happens when $0 < \lambda \leqq 1$. For $0 < \lambda < 1$ the question is vague since the polynomials are not dense in the corresponding weighted L_p spaces (cf. [33] and [41]). In the singular case $\lambda = 1$, G. Freud, A. Giroux and Q. I. Rahman [25] proved some convergence theorems in L_1 with weight w_λ but the characterization of best approximation in the sense of Theorem A was left unresolved. In Section 2 we give this characterization, and in Section 3 we apply Theorem A to a constructive description of weighted Lipschitz classes. The result we prove is complete for all $\lambda > 1$.

2. Polynomial approximation with weight $\exp(-|x|)$

Here we prove the following

THEOREM 1. *Let* r *be a positive integer,* $0 < \alpha < r$, *and let* $fw_1 \in L_1(\mathbf{R})$. *Then*

$$E_n(f, w_1)_1 = O\big((\log n)^{-\alpha}\big), \quad n = 1, 2, \ldots, \tag{2.1}$$

holds if and only if

$$\|w_1 \Delta_h^r f\|_1 = O(h^\alpha), \quad h \to 0+, \tag{2.2}$$

and

$$\|\zeta_h w_1 f\|_1 = O(h^\alpha), \quad h \to 0+, \tag{2.3}$$

where ζ_h *is the characteristic function of the set* $\mathbf{R} \setminus (-\exp(1/h), \exp(1/h))$.

Clearly, (2.1) is equivalent to

$$E_{2^n}(f, w_1)_p = O(n^{-\alpha}), \quad n = 1, 2, \ldots.$$

REMARK 1. One would naturally like to find out whether Theorem 1 remains true for $1 < p \leqq \infty$. The necessity of (2.2) and (2.3) can be proved by the method applied in the proof of Theorem 1 below for every $1 < p \leqq \infty$. To prove their sufficiency, the most natural approach would be the one used by Freud (cf. [13—15, 19, 21—23]). However, Freud's method does not seem to lend itself to treating L_p spaces for $p>1$, the main obstacle being that we do not have appropriate information about the L_∞ boundedness of the de la Vallée—Poussin (delayed arithmetic) means of the generalized Fourier series expansion in the orthogonal polynomial system associated with w_1. Interestingly, the weight w_1 is almost classical in the sense that it behaves like $2(\cosh x)^{-1}$ and the orthogonal polynomials corresponding to the latter weight function are a special case of the Pollaczek polynomials (cf. [46] and [40]).

Proof of Theorem 1. (i): (2.1)⇒(2.2). Define the sequence $\{n(m)\}$ by $n_{-1}=0$ and $\log_2(n(m))=2^m$, $m=0, 1, 2, \ldots$. Let $0<h<1/2$. Then

$$[\log_2(n(m))]^{-1} \leqq h < [\log_2(n(m-1))]^{-1}$$

for some $m \geqq 1$. In [41, Theorem 3] it was proved that the Markov—Bernstein-type inequality

$$\|P'w_1\|_1 \leqq K \log k \, \|Pw_1\|_1, \quad k > 1,$$

holds for every polynomial P of degree at most k with an appropriate absolute constant $K>0$. Using this inequality the proof of (2.2) becomes a fairly routine exercise. Let P_n be a polynomial of degree at most n such that

$$\|(f-P_n)\,w_1\|_1 \leqq 2E_n(f, w_1)_1 \leqq K\big(\log_2(n)\big)^{-\alpha}.$$

Then

$$\|w_1[\Delta_h^r f]\|_1 \leqq \|w_1[\Delta_h^r(f-P_{n(m)})]\|_1 + \|w_1[\Delta_h^r P_{n(m)}]\|_1.$$

Here

$$\|w_1[\Delta_h^r(f-P_{n(m)})]\|_1 \leqq K\big(\log_2(n(m))^{-\alpha} \leqq Kh^\alpha$$

and

$$\|w_1[\Delta_h^r P_{n(m)}]\|_1 \leqq Kh^r \|P_{n(m)}^{(r)} w_1\|_1 \leqq Kh^r \sum_{k=1}^{m} \|[P_{n(k)}-P_{n(k-1)}]^{(r)} w_1\|_1 \leqq$$

$$\leqq Kh^r \sum_{k=0}^{m} (\log_2(n(k)))^r E_{n(k-1)}(f, w_1)_1 \leqq Kh^\alpha.$$

Now (2.2) follows directly from the latter three estimates.

(ii): (2.1)⇒(2.3). We start with choosing a constant $C>0$ such that the infinite-finite range inequality

$$\|\xi_n Q_n w_1\|_1 \leqq n^{-1}\|Q_n w_1\|_1$$

holds for every polynomial Q of degree at most n where ξ_n denotes the characteristic function of $\mathbf{R}\setminus(-Cn, Cn)$ (cf. [2], [15], [31—33] and [37—40]). Let $\{n(m)\}$ and $\{P_{n(m)}\}$ be defined as in (i), and given $h>0$ let M be such that

$$Cn(M) < \exp(1/h) \leqq Cn(M+1).$$

Then

$$\|\zeta_h w_1 f\|_1 \leqq \|\zeta_h w_1(f-P_{n(M)})\|_1 + \|\zeta_h w_1 P_{n(M)}\|_1 \leqq$$

$$\leqq K\big(\log_2(n(M))\big)^{-1} + n(M)^{-1}\|w_1 P_{n(M)}\|_1 \leqq Kh^\alpha + 2n(M)^{-1}\|w_1 f\|_1 \leqq Kh^\alpha$$

(for ζ_h cf. (2.3)), and this was to be proved.

(iii): (2.2) & (2.3)$\Rightarrow$(2.1). By (2.2) for every $n\in\mathbf{N}$ there is a function g_n such that $g_n\in\mathbf{C}^{(r-1)}$, $g_n^{(r-1)}$ is locally absolutely continuous and

$$\|w_1(f-g_n)\|_1+(\log_2 n)^{-r}\|w_1 g_n(r)\|_1 \leqq K(\log_2 n)^{-\alpha} \tag{2.4}$$

is satisfied with K independent of n (cf. [4, Remark 1] and [5]). By Theorem 2.3 in [25] we obtain

$$E_n(g_n, w_1)_1 \leqq K(\log_2 n)^{-r}\|w_1 g_n^{(r)}\|_1+$$

$$+K\exp(-n^2/3)\sum_{k=0}^{m}(\log_2 n)^{-k+1}\|w_1 g_n^{(k)}\|_1+K\|\psi_n w_1 g_n\|_1$$

where ψ_n is the characteristic function of $\mathbf{R}\diagdown(-n^{1/2}, n^{1/2})$. By (2.4) the first term on the right-hand side is $O((\log n)^{-\alpha})$. The second term there can be estimated by

$$K\exp(-n^2/3)(\|w_1 g_n\|_1+\|w_1 g_n^{(r)}\|_1) = O((\log n)^{-\alpha})$$

(cf. [3, Lemma 2.1] and [5]). Finally, by (2.3) and (2.4) we have

$$\|\psi_n w_1 g_n)\|_1 \leqq \|\psi_n w_1(f-g_n)\|_1+\|\psi_n w_1 f\|_1 = O((\log n)^{-\alpha}).$$

Consequently, we obtain

$$E_n(g_n, w_1)_1 = O((\log n)^{-\alpha}), \tag{2.5}$$

and since

$$E_n(f, w_1)_1 \leqq E_n(g_n, w_1)_1+\|w_1(f-g_n)\|_1,$$

(2.1) follows from (2.4) and (2.5).

Theorem 1 has been proved. □

3. Constructive characterization of weighted Lipschitz classes

Our next goal is characterizing

$$\|w_\lambda \Delta_h^r f\|_p = O(h^\alpha), \quad h\to 0+,$$

for the weights w_λ $(\lambda>1)$ in a constructive way as follows.

THEOREM 2. *Let* $r\in\mathbf{N}$, $1\leqq p\leqq\infty$, $0<\alpha<r$ *and* $\lambda>1$. *Assume that* f *is an almost everywhere finite valued Lebesgue measurable function. Then the following assertions are pairwise equivalent.*

(i) *The r-th symmetric difference of the function* f *satisfies*

$$\|w_\lambda \Delta_h^r f\|_p = O(h^\alpha), \quad h\to 0+. \tag{3.1}$$

(ii) *The rate of best weighted approximation of* f *satisfies*

$$E_n(f, w_{\lambda,n})_p = O(n^{-\alpha(\lambda-1)/\lambda}), \quad n\to\infty, \tag{3.2}$$

where $w_{\lambda,n}(x)=w_\lambda(|x|-1/n)$.

(iii) *We have* $w_\lambda f(\cdot\pm\delta)\in L_p(R)$ *for some* $\delta>0$ *and*

$$E_n(f, w_\lambda)_p = O(n^{-\alpha(\lambda-1)/\lambda}), \quad n\to\infty. \tag{3.3}$$

REMARK 2. Applying Fatou's Lemma it is possible to show that for $\alpha=r$ condition (3.1) is equivalent to the following: $w_\lambda f(\cdot\pm\delta)\in L_p(R)$ for some $\delta>0$, $f^{(r-1)}$ is locally absolutely continuous and $wf^{(r)}\in L_p(R)$. Moreover, if $\alpha>r$ in (3.1) then f almost everywhere coincides with a polynomial of degree at most $r-1$. We wish we could characterize all functions f satisfying (3.1) when $0<\lambda\leqq 1$. Alas, we cannot do this at the present time.

REMARK 3. Let us observe that for $h>0$ and $|x|<Kh^{1/(1-\lambda)}$ we have $w_\lambda(x)\sim\sim w_\lambda(x\pm h)$, and this is the reason for the appearance of the characteristic function of the interval $(-h^{1/(1-\lambda)}, h^{1/(1-\lambda)})$ in Theorem A.

The proof of Theorem 2 is based on Theorem A and the following three lemmas of technical nature. The next proposition is probably known though we could not find a reference to it.

LEMMA 1. *Let f be an almost everywhere finite valued Lebesgue measurable function on the real line, and let $D_h f$, $h\in\mathbf{R}$, be defined by*

$$D_h f(x) = \sum_{i=0}^{r} c_i f(x+d_i h)$$

where $\{c_i\}$ and $\{d_i\}$ are real numbers such that $d_i\neq d_j$ $(i\neq j)$ and not every c_i equals 0. If for some $0<p\leqq\infty$ and $\delta_0>0$ we have $D_h f\in L_p[-1,1]$ for $0<h<\delta_0$ then $f\in L_p[a,b]$ for every subinterval $[a,b]\subset(-1,1)$.

REMARK 4. It would have been sufficient to assume $D_h f\in L_p[-1,1]$ on a measurable set for which $h=0$ is a density point. However, even the latter seems to be superfluous, and we think it is an interesting problem to find thin sets $\mathbf{H}$ with the property that $D_h f\in L_p[-1,1]$ for $h\in\mathbf{H}$ would be sufficient for the conclusion of the lemma.

PROOF OF LEMMA 1. Let $[a,b]\subset(-1,1)$. Since

$$\|D_h f\|_{L^p[-1,1]} \geqq \|D_h f(\cdot\,-dh)\|_{L^p[-1+dh,1-dh]}$$

for $h>0$, we may assume without loss of generality that $d_0=0$ and $c_0=1$. Let $\delta<\min\{\min(1+a, 1-b)/|d_i|;\ i=1,2,\dots,r\}$, $\delta<\delta_0$, and choose $\eta>0$ sufficiently small, for instance $\eta<2^{-1}\delta(1+\Sigma|d_i|^{-1})^{-1}$ is appropriate. Fix a constant $C>0$ such that the sets

$$H = \{h\in(0,\delta) \mid \|D_h f\|_p < C\} \quad\text{and}\quad F = \{x \mid |f(x)| < C\}$$

satisfy

$$m(H) > \delta-\eta \quad and \quad m(F\cap[-1,1]) > 2-\eta. \tag{3.4}$$

Let the sets E_i, E and E^x be defined by

$$E_i = \{(x,h) \mid x\in[a,b],\ h\in(0,\delta),\ x+d_i h\in F\},\quad i=1,2,\dots,r,$$

$$E = \bigcap_{i=1}^{r} E_i \quad\text{and}\quad E^x = \{h \mid (x,h)\in E\}.$$

Then each E_i is a measurable subset of the plane and thus so is E. Moreover, E^x is measurable for almost every x. Thus if g is a nonnegative measurable function then the function defined by

$$x \to \int\limits_{h\in H\cap E^x} g(h)\,dh$$

is also measurable. First let $0<p<\infty$, and let

$$M = 2C\sum_{i=1}^{r}(|c_i|+1).$$

By the choice of H we have

$$\int\limits_{H}\int\limits_{-1}^{1}|D_h f(x)|^p\,dx\,dh < \infty$$

which, by Fubini's theorem, implies

$$\int\limits_{\substack{[a,b]\\ |f(x)|>M}}\;\int\limits_{h\in H\cap E^x}|D_h f(x)|^p\,dh\,dx < \infty.$$

If $|f(x)|>M$ and $h\in E^x$ then by the definition of M, E^x, E and F we obtain $|f(x)|\leqq 2|D_h f(x)|$. Thus

$$\int\limits_{\substack{[a,b]\\ |f(x)|>M}}|f(x)|^p\int\limits_{h\in H\cap E^x}dh\,dx < \infty, \tag{3.5}$$

and if we can show that the inequality

$$\int\limits_{h\in H\cap E^x} dh = m(H\cap E^x) > \delta/2$$

holds for every $x\in[a,b]$, then (3.5) proves the assertion of the lemma. But

$$m(H\cap E^x) \geqq \delta - m([0,\delta]\setminus H) - m([0,\delta]\setminus E^x),$$

and here by (3.4) $m([0,\delta]\setminus H)<\eta$. Furthermore

$$m([0,\delta]\setminus E^x) \leqq \sum_{i=1}^{r} m([0,\delta]\setminus(E_i)^x)$$

where

$$[0,\delta]\setminus(E_j)^x = \{h|\,h\in(0,\delta),\, x+d_i hF\},\quad i=1,2,\dots,r.$$

Therefore we obtain

$$m([0,\delta]\setminus(E_i)^x) \leqq m(\{t|\,t\in[-1,1],\, t\notin F\})/|d_i| \leqq \eta/|d_i|.$$

Collecting the above estimates we can conclude

$$\int\limits_{h\in H\cap E^x} dh \geqq \delta - \eta - \sum_{i=1}^{r}[\eta/|d_i|] > \delta/2$$

provided η is sufficiently small. This completes the proof of the lemma for $0<p<\infty$. If $p=\infty$ then $|D_h f(x)| \leqq C$ for almost every $(h, x) \in H \times [a, b]$. The argument above yields that for every $x \in [a, b]$ the set of those h for which $x+d_i h \in F$, $i=1, 2, \ldots, r$, has measure at least $\delta/2$. Thus for almost every $x \in [a, b]$ there is $h \in H$ such that $|D_h f(x)| \leqq C$ and $x+d_i h \in F$, $i=1, 2, \ldots, r$, which implies

$$|f(x)| \leqq |D_h f(x)| + \Big|\sum_{i=1}^{r} c_i f(x+d_i h)\Big| \leqq C\Big(1+\sum_{i=0}^{r} |c_i|\Big).$$

Hence we proved the lemma for $p=\infty$ as well. □

LEMMA 2. *Let* $\lambda>1$, $1 \leqq p \leqq \infty$, *and let* $f \in L_p[a, b]$ *for every finite interval* $[a, b]$. *If* $w_\lambda \Delta_h^r f \in L_p[0, \infty)$ *for some* $0<h<1/4$ *then*

$$w_\lambda(\cdot - h/2)[\Delta_h^{r-1} f] \in L_p[0, \infty),$$

and if $w_\lambda \Delta_h^r f \in L_p(-\infty, 0]$ *for some* $0<h<1/4$ *then*

$$w_\lambda(\cdot + h/2)[\Delta_h^{r-1} f] \in L_p(-\infty, 0].$$

PROOF. Let a_n and b_n be defined by

$$a_n = \|w_\lambda(\cdot - h/2)[\Delta_h^r f(\cdot - h/2)]\|_{L^p[n/4, (n+1)/4]}$$

and

$$c_n = \|w_\lambda(\cdot - h/2)[\Delta_h^{r-1} f]\|_{L^p[n/4, (n+1)/4]}.$$

By the assumption of the lemma

$$\sum_{n=1}^{\infty} a_n < \infty \tag{3.6}$$

and we want to prove

$$\sum_{n=1}^{\infty} c_n < \infty. \tag{3.7}$$

Applying the identity

$$w_\lambda(x-h/2)[\Delta_h^{r-1} f(x)] = w_\lambda(x-h/2)[\Delta_h^r f(x-h/2)] + w_\lambda(x-h/2)[\Delta_h^{r-1} f(x-h)]$$

and the relation

$$\lim_{x\to\infty} w_\lambda(x-h/2)/w_\lambda(x-3h/2) = 0$$

we can easily obtain

$$c_n \leqq a_n + (c_{n-1}+c_{n-2})/4 \tag{3.8}$$

for sufficiently large n. Since $c_1+c_2<\infty$, (3.7) follows from (3.6) and (3.8) by standard estimates. The case of the negative real line can be treated similarly. □

LEMMA 3. *Let* $\lambda>1$, $1 \leqq p \leqq \infty$ *and* $\delta>0$. *Suppose* $w_\lambda f(|\cdot| \pm \delta) \in L_p(\mathbf{R})$. *Then for every* $M>0$ *and* $0 \leqq h<\delta/2$

$$\|w_\lambda(|\cdot|-h) f\|_{L^p(\mathbf{R}/[-M, M])} \leqq K \exp(-cM^{\lambda-1}) \tag{3.9}$$

holds with positive constants K *and* c *which are independent of* h *and* M. *In addition, given* $r\in\mathbf{N}$, *there is* $h_0>0$ *such that*

$$\|w_\lambda[\Delta_h^r f\|_{L^p(\mathbf{R}\setminus[-M,M])} \leqq K_1 \exp(-c2^{-1}M^{\lambda-1}) \tag{3.10}$$

for every $M>0$ *and* $0\leqq h<h_0$ *where the constant* $K_1>0$ *is independent of* h *and* M.

PROOF. By the assumptions $w_\lambda(|\cdot|\pm\delta)f\in L_p(\mathbf{R})$. Since

$$w_\lambda(|x|-h)/w_\lambda(|x|-\delta) \leqq K\exp(-c(|x|)^{\lambda-1}) \leqq K\exp(-cM^{\lambda-1})$$

for $0\leqq h<\delta/2$ and $|x|\geqq M$, inequality (3.9) follows immediately. Inequality (3.10) is a straightforward consequence of (3.9). □

After these preliminary results, let us turn our attention to the

PROOF OF THEOREM 2. We will prove the implications (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii). By Lemma 1, $f\in L_p[a, b]$ on every finite interval $[a, b]$. Thus Lemma 2 may be applied to obtain $\|w_\lambda\Delta_h^{r-1}f\|_p<\infty$ for every sufficiently small $h>0$. Repeating this argument we get $\|w_\lambda\Delta_h^1 f\|_p<\infty$, and a final application of Lemma 2 yields

$$\|w_\lambda f(\cdot\pm\delta)\|_p < \infty \tag{3.11}$$

for some $\delta>0$. By Theorem A there exists a sequence of polynomials $\{P_n\}$ ($\deg P_n\leqq n$) such that

$$\|w_\lambda(f-P_n)\|_p = O(n^{-\alpha(\lambda-1)/\lambda}). \tag{3.12}$$

Since for $|x|\leqq Kn^{1/(\lambda-1)}$ we have $w_\lambda(x)\sim w_\lambda(|x|\pm 1/n)$, this implies

$$\|w_{\lambda,n}(f-P_n)\|_{L^p(-n^s,n^s)} = O(n^{-\alpha(\lambda-1)/\lambda}), \quad s = 1/(\lambda-1). \tag{3.13}$$

It follows from (3.11) and Lemma 3 that

$$\|w_{\lambda,n}f\|_{L^p(\mathbf{R}\setminus[-n^s,n^s])} = O(\exp(-cn)), \quad s = 1/(\lambda-1), \tag{3.14}$$

holds with some $c>0$. Finally, the inequality

$$\|w_{\lambda,n}P_n\|_{L^p(\mathbf{R}\setminus[-n^s,n^s])} = O(\exp(-c_1 n)), \quad s = 1/(\lambda-1), \tag{3.15}$$

$c_1>0$, can be obtained from (3.12) and from the existence of a constant $C>0$ such that

$$\|w_{\lambda,n}Q_n\|_{L^p(\mathbf{R}\setminus[-Cn^{1/\lambda},Cn^{1/\lambda}])} = O(\exp(-c_1 n))\|w_{\lambda,n}Q_n\|_p \tag{3.16}$$

holds for every polynomial Q_n of degree at most n (cf. [2], [15], [31—33] and [37—40]). Indeed, since $n^{1/\lambda}\leqq n^{1/(\lambda-1)}$, we can use (3.16) to establish

$$\|w_{\lambda,n}P_n\|_p \leqq K\|w_{\lambda,n}P_n\|_{L^p[-Cn^{1/\lambda},Cn^{1/\lambda}]} \leqq K\|w_\lambda P_n\|_{L^p[-Cn^{1/\lambda},Cn^{1/\lambda}]},$$

and thus by (3.12) $\|w_{\lambda,n}P_n\|_p$ is uniformly bounded in $n\in\mathbf{N}$. Now we obtain (3.15) directly from (3.16). Part (ii) of the theorem (cf. (3.2)) follows from (3.13)—(3.15).

(ii)⇒(iii). This is obvious.

(iii)⇒(i). By Theorem **A**

$$\|\chi_h w_\lambda \Delta_h^r f\|_p = O(h^\alpha), \quad h \to 0+$$

where χ_h denotes again the characteristic function of the interval $(-h^{1/(1-\lambda)}, h^{1/(1-\lambda)})$. The relation

$$\|(1-\chi_h)w_\lambda \Delta_h^r f\|_p = O(h^\alpha), \quad h \to 0+$$

follows from $w_\lambda f(\cdot\pm\delta)\in L_p(\mathbf{R})$ for some $\delta>0$ and Lemma 3. Thus we have proved Theorem 2. □

References

[1] N. I. Akhiezer, On the weighted approximation of continuous functions by polynomials on the entire real axis, in *Amer. Math. Soc. Translations*, Ser. 2, Vo˙. 22 (Providence, Rhode Island, 1962), 95—137.

[2] W. C. Bauldry, Estimates of Christoffel functions of generalized Freud type weights, *J. Approx. Theory*, **46** (1986), 217—229.

[3] Z. Ditzian, On interpolation of $L_p[a, b]$ and weighted Sobolev spaces, *Pacific J. Math.*, **90** (1980), 307—323.

[4] Z. Ditzian and V. Totik, *K* functionals and best polynomial approximation in weighted $L_p(\mathbf{R})$, *J. Approx. Theory*, **46** (1986), 38—41.

[5] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer Series for Computational Mathematics, Springer Verlag (New York, to appear).

[6] M. M. Dzarbasyan, On weighted best polynomial approximation on the whole real axis (in Russian), *Dokl. Akad. Nauk SSSR*, **84** (1952), 1123—1126.

[7] M. M. Dzarbasyan, Some questions of the theory of weighted polinomial approximations in a complex domain (in Russian), *Matem. Sbornik*, **36 (78)** (1955), 353—440.

[8] M. M. Dzarbasyan and A. B. Tavadyan, On weighted uniform approximation by polynomials of functions of several variables (in Russian), *Matem. Sbornik*, **43 (85)** (1957), 227—256.

[9] G. Freud, On weighted polynomial approximation on the whole real axis, *Acta Math. Acad. Sci. Hungar.*, **20** (1969), 223—225.

[10] G. Freud, On weighted approximation by polynomials on the real axis, *Soviet Math. Dokl.*, **11** (1970), 370—371.

[11] G. Freud, On an inequality of Markov type, *Soviet Math. Dokl.*, **12** (1971), 570—573.

[12] G. Freud, On approximation with weight $\exp\{-x^2/2\}$ by polynomials, *Soviet Math. Dokl.*, **12** (1971), 1837—1840.

[13] G. Freud, A contribution to the problem of weighted polynomial approximation, in *Linear Operators and Approximation*, ed. P. L. Butzer et al., ISNM, vol. 20, Birkhäuser Verlag (Basel, 1972), 431—447.

[14] G. Freud, On direct and converse theorems in the theory of weighted polynomial approximation, *Math. Zeitschr.*, **126** (1972), 123—134.

[15] G. Freud, On weighted L_1-approximation by polynomials, *Studia Math.*, **46** (1973), 125—133.

[16] G. Freud, Investigations on weighted approximation by polynomials, *Studia Sci. Math. Hungar.*, **8** (1973), 285—305.

[17] G. Freud, On polynomial approximation with the weight $\exp\{-x^{2k}/2\}$, *Acta Math. Acad. Sci. Hungar.*, **24** (1973), 363—371.

[18] G. Freud, On the converse theorems of weighted polynomial approximation, *Acta Math. Acad. Sci. Hungar.*, **24** (1973), 389—397.

[19] G. Freud, On polynomial approximation with respect to general weights, in *Functional Analysis and its Applications*, ed. H. G. Garnir et al., Lecture Notes in Math., vol. 399, Springer Verlag (Berlin, 1973), 149—179.

[20] G. Freud, On the theory of one sided weighted L_1-approximation by polynomials, in *Linear Operators and Approximation, II*, ed. P. L. Butzer et al., ISNM, vol. 25, Birkhäuser Verlag (Basel, 1974), 285—303.

[21] G. Freud, Markov—Bernstein type inequalities in $L_p(-\infty, \infty)$, in *Approximation Theory, II*, ed. G. G. Lorentz et al., Academic Press (New York, 1976), 369—377.

[22] G. Freud, Weighted polynomial approximation and K-functionals, in *Theory of Approximation, with Applications*, ed. A. G. Law et al., Academic Press (New York, 1976), 9—23.

[23] G. Freud, On Markov—Berstein-type inequalities and their applications, *J. Approx. Theory*, **19** (1977), 22—37.

[24] G. Freud, A. Giroux and Q. I. Rahman, Sur l'approximation polynomiale sur tout l'axe reel, *Canad. J. Math.*, **28** (1976), 961—967.

[25] G. Freud, A. Giroux and Q. I. Rahman, Sur l'approximation polynomiale avec poids $\exp\{-|x|\}$ *Canad. J. Math.*, **30** (1978), 358—372.

[26] G. Freud and H. N. Mhaskar, Weighted polynomial approximation in rearrangement invariant Banach function spaces on the whole real line (with H. N. Mhaskar), *Indian J. Math.*, **22** (1980), 209—224.

[27] G. Freud and H. N. Mhaskar, K-functionals and moduli of continuity in weighted polynomial approximation, *Arkiv. för Mat.*, **21** (1983), 145—161.

[28] G. Freud and P. Nevai, Weighted L_1 and one-sided weighted L_1 polynomial approximation on the real line (in Hungarian), *Magyar Tud. Akad. III. Oszt. Közl.*, **21** (1973), 485—502.

[29] A. L. Levin and D. S. Lubinsky, Canonical products and the weights $\exp(-|x|^\alpha)$, $\alpha>1$, with applications, *J. Approx. Theory*, **49** (1987), 149—169.

[30] A. L. Levin and D. S. Lubinsky, Weights on the real line that admit good relative polynomial approximation, with applications, *J. Approx. Theory*, **49** (1987), 170—195.

[31] D. S. Lubinsky, A weighted polynomial inequality, *Proc. Amer. Math. Soc.*, **92** (1984), 263—267.

[32] D. S. Lubinsky, Gaussian quadrature, weights on the whole real line, and even entire functions with nonnegative even order derivatives, *J. Approx. Theory*, **46** (1986), 296—313.

[33] D. S. Lubinsky, Estimates of Freud—Christoffel functions for some weights with the whole real line as support, *J. Approx. Theory*, **44** (1985), 343—379.

[34] D. S. Lubinsky, Even entire functions absolutely monotone in $[0, \infty)$ and weights on the whole real line, in *Orthogonal Polynomials and Their Applications*, ed. C. Brezinski et al., Lecture Notes in Mathematics, Springer Verlag (Berlin, 1985).

[35] H. N. Mhaskar, Weighted polynomial approximation, *J. Approx. Theory*, **46** (1986), 100—110.

[36] H. N. Mhaskar and E. B. Saff, Extremal problems for polynomials with exponential weights, *Transactions Amer. Math. Soc.*, **285** (1984), 203—234.

[37] P. Nevai, Orthogonal polynomials on the real line associated with the weight $|x|^\alpha \exp(-|x|^\beta)$. I (in Russian), *Acta Math. Acad. Sci. Hungar.*, **24** (1973), 335—342.

[38] P. Nevai, Two of my favorite ways of obtaining asymptotics for orthogonal polynomials, in *Functional Analysis and Approximation*, ed. P. L. Butzer, R. L. Stens and B. Sz.-Nagy, ISNM 65, Birkhäuser Verlag (Basel, 1984), 417—436.

[39] P. Nevai, Orthogonal polynomials on infinite intervals, *Rend. Sem. Mat. Univ. Polit. Torino*, to appear.

[40] P. Nevai, Géza Freud, Orthogonal Polynomials and Christoffel Functions (A Case Study), *J. Approx. Theory*, **48** (1986), 3—167.

[41] P. Nevai and V. Totik, Weighted polynomial inequalities, *Constructive Approximation*, **2** (1986), 113—127.

[42] P. Nevai and V. Totik, Sharp Nikolskii inequalities with exponential weights, *Analysis Mathematica*, **13** (1987) (to appear).

[43] E. A. Rahmanov, On asymptotic properties of polynomials orthogonal on the real axis, *Math USSR Sbornik*, **47** (1984), 155—193.

[44] E. B. Saff, Incomplete and orthogonal polynomials, in *Approximation Theory, IV*, ed. C. K. Chui et al., Academic Press (New York, 1983), 219—256.

[45] R. C. Sheen, Asymptotics for orthogonal polynomials associated with $\exp(-x^6/6)$, *J. Approx. Theory*, **49** (1986), to appear.

[46] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ., vol. 23, Amer. Math. Soc. (Providence, Rhode Island, 1939), 4th ed., 1975.
[47] J. L. Ullman, Orthogonal polynomials associated with an infinite interval, *Michigan Math. J.*, **27** (1980), 353—363.
[48] J. L. Ullman, On orthogonal polynomials associated with an infinite interval, in *Approximation Theory III*, ed. by E. W. Cheney, Academic Press (New York, 1980), 889—895.

(Received October 22, 1985)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA
CANADA TEG 2G1

NATIONAL RESEARCH INSTITUTE
FOR MATHEMATICAL SCIENCES
CSIR, P.O. BOX 395
PRETORIA, REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210
USA

BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
SZEGED, ARADI VÉRTANÚK TERE 1
H—6720

PRINTED IN HUNGARY
Szegedi Nyomda, Szeged

Instructions for authors. Manuscripts should be typed on standard size paper (25 rows; 50 characters in each row). Mathematical symbols (except sin, log, etc.) will be set in italics, so *use handwritten letters, or indicate them by simple underlining*. The text of theorems, lemmas, corollaries are to be printed also in italics. The words THEOREM, LEMMA. COROLLARY, PROOF, REMARK, PROBLEM, etc. (or their German and French equivalents) should be typed in capital letters. Sub-headings will be set in boldface lower case; indicate them by red underlining. Other special types of symbols (such as German, Greek, script, etc.) should also be clearly identified. Mark footnotes by consecutive superscript numbers. When listing references, please follow the following pattern:

[1] G. Szegő, *Orthogonal polynomials*, AMS Coll, Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

For abbreviation of names of journals follow the Mathematical Reviews. After the references give the author's affiliation.

Do not send abstract and second copy of the manuscript. Authors will receive only galley-proofs (one copy). Manuscripts will not be sent back to authors (neither for the purpose of proof-reading nor when rejecting a paper);

Authors obtain 100 reprints free of charge. Additional copies may be ordered from the publisher.

Periodicals of the Hungarian Academy of Sciences are sold at the following addresses:

AUSTRALIA
C.B.D. LIBRARY AND SUBSCRIPTION SERVICE
Box 4886, G.P.O., *Sydney N.S.W. 2001*
COSMOS BOOKSHOP, 145 Ackland Street
St. Kilda Melbourne , Victoria 3182

AUSTRIA
GLOBUS, Höchstädtplatz 3, *1206 Wien XX*

BELGIUM
OFFICE INTERNATIONAL DE LIBRAIRIF
30 Avenue Marnix, *1050 Bruxelles*
LIBRAIRIE DE MONDE ENTIER
162 rue du Midi, *1000 Bruxelles*

BULGARIA
HEMUS, Bulvar Ruszki 6, *Sofla*

CANADA
PANNONIA BOOKS, P.O. Box 1017
Postal Station „B", *Toronto, Ontario M5T 2T8*

CHINA
CNPICOR, Periodical Department, P.O. Box 50 *Peking*

CZECHOSLOVAKIA
MAD'ARSKÁ KULTURA, Národní třida 22
115 66 Praha
PNS DOVOZ TISKU, Vinohradská 46, *Prha 2*
PNS DOVOZ TLAČE, *Bratislava 2*

DENMARK
EJNAR MUNKSGAARD, Norregade 6
1165 Copenhagen K

FEDERAL REPUBLIC OF GERMANY
KUNST UND WISSEN ERICH BIEBER
Postfach 46, *7000 Stuttgart 1*

FINLAND
AKATEEMINEN KIRJAKAUPPA, P.O. Box 128
SF-00101 Helsinki 10

FRANCE
DAWSON-FRANCE S. A., B. P. 40, *91121 Palaiseau*
EUROPÉRIODIQUES S. A., 31 Avenue de Versailles, *78170 La Celle St. Cloud*
OFFICE INTERNATIONAL DE DOCUMENTATION ET LIBRAIRIE, 48 rue Gay-Lussac *75240 Paris Cedex 05*

GERMAN DEMOCRATIC REPUBLIC
HAUS DER UNGARISCHEN KULTUR
Karl Liebknecht-Straße 9, *DDR-102 Berlin*
DEUTSCHE POST ZEITUNGSVERTRIEBSAMT
Straße der Pariser Kommune 3-4, *DDR-104 Berlin*

GREAT BRITAIN
BLACKWELLS'S PERIODICALS DIVISION
Hythe Bridge Street, *Oxford OXI 2ET*
BUMPUS, HALDANE AND MAXWELL LTD.
Cowper Works, *Olney, Bucks MK46 4BN*
COLLET'S HOLDINGS LTH. Denington Estate *Wellingborough, Northants NN8 2QT*
WM. DAWSON AND SONS LTD., Cannon House *Folkstone, Kent CT19 5EE*
H. K. LEWIS AND CO., 136 Gower Street
London WC1E 6BS

GREECE
KOSTARAKIS BROTHERS INTERNATIONAL
BROOKSELLERS, 2 Hippokratous Street, *Athens-143*

HOLLAND
MEULENHOFF-BRUNA B.V, Beulingstraat 2, Amsterdam
MARTINUS NIJHOFF B.V.
Lange Voorhout 9-11, *Den Haag*
SWETS SUBSCRIPTION SERVICE
347b Heereweg, *Lisse*

INDIA
ALLIED PUBLISHING PRIVATE LTD., 13/14
Asaf Ali Road, *New Delhi 110001*
150 B-6 Mount Road, *Madras 600002*
INTERNATIONAL BOOK HOUSE PVT. LTD.
Madame Cama Road, *Bombay 400039*
THE STATE TRADING CORPORATION OF INDIA LTD,.
Books Import Division, Chandralok
36 Janpath, *New Delhi 110001*

ITALY
INTERSCIENTIA, Via Mazzé 28, *10149 Torino*
LIBRERIA COMMISSIONARIA SANSONI, Via Lamarmora 45, *50121 Firenze*
SANTO VANASIA, Via M. Macchi 58 *20124 Milano*
D. E. A., Via Lima 28, *00198 Roma*

JAPAN
KINOKUNIYA BOOK-STORE CO. LTD.
17-7 Shinjuku 3 chome, Shinjuku-ku, *Tokyo 160-91*
MARUZEN COMPANY LTD., Book Department,
P.O. Boy 5050 Tokyo International, *Tokyo 100-31*
NAUKA LTD. IMPORT DEPARTMENT
2-30-19 Minami Ikebukuro, Toshima-ku, *Tokyo 171*

KOREA
CHULPNMUL, *Phenjan*

NORWAY
TANUM-TIDSKRIFT-SENTRALEN A.S.,
Karl Johansgatan 41-43, *1000 Oslo*

POLAND
WEGIERSKI INSTYTUT KULTURY,
Marszalkowska 80, *00-517 Warszawa*
CKP W, ul. Towarowa 28, *00-958 Warszawa*

ROUMANIA
D. E. P., *Buçuresti*
ILEXIM, Calea Grivitei 64-66, *Bucuresti*

SOVIET UNION
SOJUZPECHAT—IMPORT, *Moscow*
and the post offices in each town
MEZHDUNARODNAYA KNIGA, *Moscow G-200*

SPAIN
DIAZ DE SANTOS, Lagasca 95, *Madrid 6*

SWEDEN
GUMPERTS UNIVERSITETSBOKHANDEL AB
Box 346, *401 25 Göteborg 1*

SWITZERLAND
KARGER LIBRI AG., Petersgraben 31, *4011 Basel*

USA
EBSCO SUBSCRIPTION SERVICES
P.O. Box 1943, *Birmingham, Alabama 35201*
F.W. FAXON COMPANY, INC.
15 Southwest Park, *Westwood Mass. 02090*
READ-MORE PUBLICATIONS, INC.
140 Cedar Street, *New York, N.Y. 10006*

YUGOSLAVIA
JUGOSLAVENSKA KNJIGA, Terazije 27, *Beograd*
FORUM, Vojvode Mišića 1, *21000 Novi Sad*

Index: 26.014

Acta Mathematica Hungarica

VOLUME 50, NUMBERS 3–4, 1987

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0236—5294

ACTA MATHEMATICA
HUNGARICA

Acta Mathematica publishes papers on mathematics in English, German, French and Russian.

Acta Mathematica is published in two volumes of four issues a year by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to
Acta Mathematica, H-1364 Budapest, P.O.B. 127

Orders may be placed with

KULTURA Foreign Trading Company
H-1389 Budapest P.O. Box 149

or with its representatives abroad.

Acta Mathematica Hungrica is abstracted/indexed in Current Contents — Physical, Chemical and Earth Sciences, Mathematical Reviews.

CONTENTS

Acta Math. Hung.
50 (3—4) (1987), 177—201.

BASIC TOOLS AND MILD CONTINUITIES IN RELATOR SPACES

Á. SZÁZ (Debrecen)

Introduction

Starting with this paper, we offer a simple, unified foundation to general topology and abstract analysis by using a straightforward generalization of uniform spaces which leads us to more general structures than the ordinary topologies.

As the extensive references show uniform spaces have been defined and generalized in terms of various objects. The most widely used ones are certain metrics, relations and covers. Covers are versatile tools in topology, while metrics are better adapted to arguments in analysis. However, almost everything can be formulated more simply in terms of relations. Therefore, we adhere to relations.

Before describing the main features of our present approach, it seems necessary to make some historical remarks. Uniform spaces in terms of relations were first introduced by Weil [97], and later standardized by Bourbaki [10] with some adjustments. A uniform space in the Weil—Bourbaki sense is an ordered pair $X(\mathcal{U})=(X, \mathcal{U})$ consisting of a set X and a nonvoid family $\mathcal{U}$ of relations $U\subset X\times X$ such that

(I) $U\in\mathcal{U} \Rightarrow \Delta_X \subset U;$

(II) $U\in\mathcal{U}\Rightarrow\exists V\in\mathcal{U}\colon V\subset U^{-1};$

(III) $U\in\mathcal{U} \Rightarrow\exists V, W\in\mathcal{U}\colon W\circ V \subset U;$

(IV) $U, V\in\mathcal{U} \Rightarrow\exists W\in\mathcal{U}\colon\ W \subset U\cap V;$

(V) $U\in\mathcal{U},\ U\subset V\subset X\times X \Rightarrow V\in\mathcal{U}.$

Each uniform space $X(\mathcal{U})$ gives rise to a completely regular topological space $X(\mathcal{T}_{\mathcal{U}})$ such that the family $\mathcal{U}(x)=\{U(x)\colon U\in\mathcal{U}\}$ is the complete neighbourhood system of each point x in $X(\mathcal{T}_{\mathcal{U}})$. Moreover, each completely regular space can be obtained in this manner, and thus precisely those spaces are uniformizable.

Generalizations of uniform spaces, by omitting or weakening some of the axioms (I)—(V), were introduced by Appert [3], Nachbin [71], Krishnan [54], Alfsen—Njåstad [1], Čech [13], Hušek [41], Mordkovič [66], Williams [98], Thampuran [93], Nakano—Nakano [73] and Mozzochi—Gagrat—Naimpally [67]. Similar generalizations were also given by Cohen—Goffman [14], Konishi [51] and Davis [20] by using certain neighbourhood-valued relations which were later called neighbournets by Junnila [44]. The interested reader can get a rapid overview on the subject by consulting a recent book of Page [76].

The most widely used generalizations of uniform spaces are the quasi-uniform ones. (See Császár [16, p. 66], Pervin [79, p. 174], Murdeshwar—Naimpally [70] and Fletcher—Lindgren [28].) These were first introduced in 1948 by Nachbin [71, p. 104] by omitting the axiom (II) of symmetry. Nachbin originally called them semi-uniform spaces, and he only used them to define and study uniform preordered spaces [71, p. 58]. The importance of quasi-uniform spaces might become apparent only after the striking discoveries of Krishnan [53], Császár [16, (13.53)] and Pervin [78] that each topological space is quasi-uniformizable. This suggests that topology and analysis should be rather based on generalized uniform spaces than on the topological ones in which several useful notions of metric spaces become meaningless.

Adopting this new point of view, in this paper we aim to initiate a simple, unified foundation to topology and analysis. In order that all the reasonable generalizations of uniform spaces may be included, and we at once have a more workable concept without defining bases or subbases, following the ideas of Konishi [51], Krishnan [54] and Nakano—Nakano [73], we drop all the axioms of a uniform space except (I). Thus, we consider spaces $X(\mathcal{R})=(X, \mathcal{R})$ consisting of a set X and a nonvoid family $\mathcal{R}$ of reflexive relations R on X. We call these spaces relator spaces and show how naturally and easily the fundamental notions and statements of topology and analysis can be extended to such spaces. The novelty of our treatment is largely due to an extensive and systematic use of nets and relations whose knowledge is the only prerequisite for reading this paper.

In Sections 1, 2 and 3, we define and study limits $\operatorname{Lim}_{\mathcal{R}}$, $\lim_{\mathcal{R}}$ and adherences $\operatorname{Adh}_{\mathcal{R}}$, $\operatorname{adh}_{\mathcal{R}}$ of nets, and closures $\operatorname{Cl}_{\mathcal{R}}$, $\operatorname{cl}_{\mathcal{R}}$ and interiors $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ of sets in a relator space $X(\mathcal{R})$. The results obtained partly reveal the relationship of relator spaces to convergence, proximity and closure spaces. Moreover, they show that our most important basic tool is the big limit relation $\operatorname{Lim}_{\mathcal{R}}$ which expresses convergence of nets to nets in the relator space $X(\mathcal{R})$. In this respect, it is very surprising that the importance of such a relation between nets seems to have formerly been recognized only by Efremovič—Švarc [26], Mamuzič [63, p. 119] and Hušek [41]. (Similar relations for sequences have been considered by Mrówka [68], Goetz [35], Poljakov [80], Naimpally—Warrack [72, p. 100] and Frič [29].)

In Sections 4, 5 and 6, we introduce and investigate mild continuities of relations. We define a relation f from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$ to be mildly $(\mathcal{R}, \mathcal{S})$-continuous if $f^{-1}\circ S\circ f\in\mathcal{R}$ for all $S\in\mathcal{S}$. By introducing a straightforward notion of hyperspace of a relator space, we show that mild continuities of relations can be reduced to continuities of the induced set-valued functions. Moreover, by defining appropriate operations $*$, $\#$ and $\hat{}$ on relators, we prove that a function f from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$ is

(i) $(\mathcal{R}^*, \mathcal{S})$-continuous if and only if $y_\alpha\in\underset{\alpha}{\operatorname{Lim}_{\mathcal{R}}}\, x_\alpha$ implies $f(y_\alpha)\in\underset{\alpha}{\operatorname{Lim}_{\mathcal{S}}}\, f(x_\alpha)$;

(ii) $(\mathcal{R}^\#, \mathcal{S})$-continuous if and only if $B\in\operatorname{Cl}_{\mathcal{R}}(A)$ implies $f(B)\in\operatorname{Cl}_{\mathcal{S}}(f(A))$;

(iii) $(\hat{\mathcal{R}}, \mathcal{S})$-continuous if and only if $x\in\underset{\alpha}{\lim_{\mathcal{R}}}\, x_\alpha$ implies $f(x)\in\underset{\alpha}{\lim_{\mathcal{S}}}\, f(x_\alpha)$, or equivalently $x\in\operatorname{cl}_{\mathcal{R}}(A)$ implies $f(x)\in\operatorname{cl}_{\mathcal{S}}(f(A))$.

Thus, the most important continuity properties (i.e., uniform, proximal and topological continuity) of a function from one relator space into another can be obtained as particular relator continuities. This remarkable fact has already been suggested by Nakano—Nakano [73] who also used the useful expression $f^{-1}\circ S\circ f$ instead of $(f\times f)^{-1}(S)$.

Our terminology and notation in this paper will mainly follow Kelley [49]. The only essential difference is that, following an idea of Čech [13], functions defined merely on nonvoid preordered sets will also be called nets here. This generalization is necessary since relators cannot at once be required to be directed with respect to the reverse set inclusion $\supset$ which will usually serve as a preorder when a family of sets is concerned. Moreover, for a net x defined on A, we shall rather use the more convenient notations $(x_\alpha)_{\alpha\in A}=x$ and $\{x_\alpha\}_{\alpha\in A}=x(A)$, where $x_\alpha=x(\alpha)$. And, when confusion seems unlikely, we shall simply write (x_α) and $\{x_\alpha\}$ instead of $(x_\alpha)_{\alpha\in A}$ and $\{x_\alpha\}_{\alpha\in A}$, respectively.

In connection with relations, we shall also use some particular terminology and notation. Adopting the functional point of view, a relation with domain X and range contained in Y (being equal to Y) will be called a relation from X into (onto) Y. Moreover, if in particular $X=Y$, then we shall simply speak of a relation on X. Furthermore, for families $\mathscr{R}$ and $\mathscr{S}$ of relations, we shall also use the following straightforward notations:

$$\mathscr{R}(A) = \{R(A)\colon R\in\mathscr{R}\} \quad \text{whenever } A \text{ is a set,}$$

$$\mathscr{R}^{-1} = \{R^{-1}\colon R\in\mathscr{R}\}, \quad \mathscr{R}\circ\mathscr{S} = \{R\circ S\colon R\in\mathscr{R},\ S\in\mathscr{S}\},$$

$$\mathscr{R}\wedge\mathscr{S} = \{R\cap S\colon R\in\mathscr{R},\ S\in\mathscr{S}\}, \quad \mathscr{R}\vee\mathscr{S} = \{R\cup S\colon R\in\mathscr{R},\ S\in\mathscr{S}\}.$$

And, when confusion seems unlikely, we shall identify singletons with their elements. Thus, for instance, we shall write $\mathscr{R}(x)$ instead of $\mathscr{R}(\{x\})$.

For their help in the effort leading to the present paper, I wish to express my gratitude and admiration to Á. Császár and S. Gacsályi who suggested improvements and provided encouragement. Moreover, I am also indebted to several further mathematicians, especially to P. Fletcher, W. N. Hunsaker and N. Levine who took the trouble to send me several relevant reprints.

1. Limits and adherences

DEFINITION 1.1. A nonvoid family $\mathscr{R}$ of reflexive relations R on a set X will be called a relator on X.

An ordered pair $X(\mathscr{R})=(X, \mathscr{R})$ consisting of a set X and a relator $\mathscr{R}$ on X will be called a relator space.

REMARK 1.2. Relators appear to be the ultimate reasonable generalizations of the various uniformities. (See, for instance, [76] and [67].)

They have formerly been studied only by Konishi [51], Krishnan [54] and Nakano—Nakano [73] in greater detail who called them generalized uniformities and connector systems.

For a preliminary illustration of the forthcoming concepts, it seems appropriate to use the following

EXAMPLE 1.3. Let X be a set and $\mathscr{D}$ be a nonvoid family of nonnegative functions d on $X\times X$ such that $d(x, x)=0$ for all $x\in X$. For each $d\in\mathscr{D}$ and $\varepsilon>0$, define the relation R_d^ε on X by

$$R_d^\varepsilon(x) = \{y:\ d(x, y) < \varepsilon\}.$$

Then

$$\mathscr{R}_{\mathscr{D}} = \{R_d^\varepsilon:\ d \in \mathscr{D},\ \varepsilon > 0\}$$

is a relator on X, and thus $X(\mathscr{R}_{\mathscr{D}})$ is a relator space.

REMARK 1.4. If $\mathscr{R}$ is an arbitrary relator on X, then by defining $\mathscr{D}_{\mathscr{R}} = \{d_R:\ R \in \mathscr{R}\}$ such that $d_R(x, y) = 0$ if $(x, y) \in R$ and $d_R(x, y) = +\infty$ if $(x, y) \in X \times X \setminus R$, we clearly have $\mathscr{R}_{\mathscr{D}_{\mathscr{R}}} = \mathscr{R}$.

However, this simple fact means by no means that relators are superfluous since almost everything can be expressed more simply in terms of the "surroundings" R_d^ε than in that of the "metrics" d.

The next definition has mainly been suggested to us by Efremovič—Švarc [26] and Hušek [41] who showed that uniformities and their generalizations can also be described in terms of nets.

DEFINITION 1.5. If $\mathscr{R}$ is a relator on X and $\mathscr{N} = \mathscr{N}(X)$ is the class of all nets in X, then the relations

$$\mathrm{Lim}_{\mathscr{R}} \subset \mathscr{N} \times \mathscr{N} \quad \text{and} \quad \mathrm{Adh}_{\mathscr{R}} \subset \mathscr{N} \times \mathscr{N}$$

defined such that for any $(x_\alpha) \in \mathscr{N}$

$$\mathrm{Lim}_{\mathscr{R}}\big((x_\alpha)\big) = \big\{(y_\alpha) \in \mathscr{N}:\ \big((y_\alpha, x_\alpha)\big) \text{ is eventually in each } R \in \mathscr{R}\big\}$$

and

$$\mathrm{Adh}_{\mathscr{R}}\big((x_\alpha)\big) = \big\{(y_\alpha) \in \mathscr{N}:\ \big((y_\alpha, x_\alpha)\big) \text{ is frequently in each } R \in \mathscr{R}\big\}$$

will be called the big limit and the big adherence on X induced by $\mathscr{R}$, respectively.

REMARK 1.6. In the sequel, trusting to the reader's good sense to avoid confusion, we shall simply write

$$y_\alpha \in \underset{\alpha}{\mathrm{Lim}_{\mathscr{R}}}\, x_\alpha \quad \text{and} \quad y_\alpha \in \underset{\alpha}{\mathrm{Adh}_{\mathscr{R}}}\, x_\alpha$$

instead of

$$(y_\alpha) \in \mathrm{Lim}_{\mathscr{R}}\big((x_\alpha)\big) \quad \text{and} \quad (y_\alpha) \in \mathrm{Adh}_{\mathscr{R}}\big((x_\alpha)\big),$$

respectively.

EXAMPLE 1.7. If $\mathscr{R}_{\mathscr{D}}$ is as in Example 1.3, and (x_α) and (y_α) are nets in X, then

$$y_\alpha \in \underset{\alpha}{\mathrm{Lim}_{\mathscr{R}}}\, x_\alpha \quad \text{iff} \quad \varlimsup_{\alpha} d(y_\alpha, x_\alpha) = 0 \quad \text{for all} \quad d \in \mathscr{D},$$

and

$$y_\alpha \in \underset{\alpha}{\mathrm{Adh}_{\mathscr{R}}}\, x_\alpha \quad \text{iff} \quad \varliminf_{\alpha} d(y_\alpha, x_\alpha) = 0 \quad \text{for all} \quad d \in \mathscr{D}.$$

To check this, recall that

$$\varliminf_{\alpha} r_\alpha = \sup_{\alpha} \inf_{\beta \geqq \alpha} r_\beta \quad \text{and} \quad \varlimsup_{\alpha} r_\alpha = \inf_{\alpha} \sup_{\beta \geqq \alpha} r_\beta$$

for any net (r_α) of extended real numbers.

THEOREM 1.8. *If $\mathscr{R}$ is a relator on X, then the relation* $\mathrm{Lim}_{\mathscr{R}}$ *has the following properties:*

(i) *If (x_α) is a net in X, then* $x_\alpha \in \underset{\alpha}{\mathrm{Lim}}_{\mathscr{R}}\, x_\alpha$.

(ii) *If* $y_\alpha \in \underset{\alpha}{\mathrm{Lim}}_{\mathscr{R}}\, x_\alpha$ *and* $((z_\beta, w_\beta))$ *is a subnet of* $((x_\alpha, y_\alpha))$, *then* $w_\beta \in \underset{\beta}{\mathrm{Lim}}_{\mathscr{R}}\, z_\beta$.

(iii) *If (x_α) and (y_α) are nets in X such that for any cofinal subnet* $((z_\beta, w_\beta))$ *of* $((x_\alpha, y_\alpha))$ *there exists a net* $((u_\gamma, v_\gamma))$ *being frequently in* $\{(z_\beta, w_\beta)\}$ *such that* $v_\gamma \in \underset{\gamma}{\mathrm{Lim}}_{\mathscr{R}}\, u_\gamma$, *then* $y_\alpha \in \underset{\alpha}{\mathrm{Lim}}_{\mathscr{R}}\, x_\alpha$.

PROOF. The properties (i) and (ii) are quite obvious. To check (iii), note that if $(x_\alpha)_{\alpha \in A}$ and $(y_\alpha)_{\alpha \in A}$ are nets in X such that $y_\alpha \notin \underset{\alpha}{\mathrm{Lim}}_{\mathscr{R}}\, x_\alpha$, then there exists $R \in \mathscr{R}$ such that the set

$$B = \{\alpha \in A \colon (y_\alpha, x_\alpha) \notin R\}$$

is cofinal in A. Thus, $((x_\beta, y_\beta))_{\beta \in B}$ is a cofinal subnet of $((x_\alpha, y_\alpha))_{\alpha \in A}$ such that $v_\gamma \notin \underset{\gamma}{\mathrm{Lim}}_{\mathscr{R}}\, u_\gamma$ for any net $((u_\gamma, v_\gamma))$ being frequently in $\{(x_\beta, y_\beta)\}_{\beta \in B}$.

COROLLARY 1.9. *If (x_α) and (y_α) are nets in a relator space $X(\mathscr{R})$, then the following assertions are equivalent:*

(i) $y_\alpha \in \underset{\alpha}{\mathrm{Lim}}_{\mathscr{R}}\, x_\alpha$;

(ii) *each subnet* $((z_\beta, w_\beta))$ *of* $((x_\alpha, y_\alpha))$ *has a subnet* $((u_\gamma, v_\gamma))$ *such that* $v_\gamma \in \underset{\gamma}{\mathrm{Lim}}_{\mathscr{R}}\, u_\gamma$.

REMARK 1.10. Necessary or sufficient conditions in order that the relation $\mathrm{Lim}_{\mathscr{R}}$ may have various useful additional properties will be given later.

Instead of establishing the basic properties of the relation $\mathrm{Adh}_{\mathscr{R}}$, we shall now briefly discuss its relationship to $\mathrm{Lim}_{\mathscr{R}}$.

THEOREM 1.11. *If (x_α) and (y_α) are nets in a relator space $X(\mathscr{R})$, then the following assertions hold:*

(i) *If* $y_\alpha \in \underset{\alpha}{\mathrm{Adh}}_{\mathscr{R}}\, x_\alpha$, *then* $w_\beta \in \underset{\beta}{\mathrm{Lim}}_{\mathscr{R}}\, z_\beta$ *for some subnet* $((z_\beta, w_\beta))$ *of* $((x_\alpha, y_\alpha))$.

(ii) *If* $w_\beta \in \underset{\beta}{\mathrm{Adh}}_{\mathscr{R}}\, z_\beta$ *for any subnet* $((z_\beta, w_\beta))$ *of* $((x_\alpha, y_\alpha))$, *then* $y_\alpha \in \underset{\alpha}{\mathrm{Lim}}_{\mathscr{R}}\, x_\alpha$.

PROOF. Suppose that $y_\alpha \in \underset{\alpha \in A}{\mathrm{Adh}}_{\mathscr{R}}\, x_\alpha$, and define

$$B = \{(\alpha, R) \in A \times \mathscr{R} \colon (y_\alpha, x_\alpha) \in R\},$$

and $z_{(\alpha, R)} = x_\alpha$ and $w_{(\alpha, R)} = y_\alpha$ for all $(\alpha, R) \in B$. Then, by preordering $\mathscr{R}$ with the reverse set inclusion and B by the restriction of the product preorder, one can easily check that $((z_\beta, w_\beta))_{\beta \in B}$ is a subnet of $((x_\alpha, y_\alpha))_{\alpha \in A}$ such that $w_\beta \in \underset{\beta}{\mathrm{Lim}}_{\mathscr{R}}\, z_\beta$. This proves (i).

To prove (ii), note that if $w_\beta \in \underset{\beta}{\mathrm{Adh}}_{\mathscr{R}}\, z_\beta$ for any subnet $((z_\beta, w_\beta))$ of $((x_\alpha, y_\alpha))$, then by (i), any subnet $((z_\beta, w_\beta))$ of $((x_\alpha, y_\alpha))$ has a subnet $((u_\gamma, v_\gamma))$ such that $v_\gamma \in \underset{\gamma}{\mathrm{Lim}}_{\mathscr{R}}\, u_\gamma$. Thus by Corollary 1.9, $y_\alpha \in \underset{\alpha}{\mathrm{Lim}}_{\mathscr{R}}\, x_\alpha$.

REMARK 1.12. Unfortunately, the converses of the assertions (i) and (ii) in Theorem 1.11 are not, in general, true.

This follows at once from the next example which reveals a serious disadvantage of nondirected nets.

EXAMPLE 1.13. Let $X(\mathcal{R})$ be a relator space such that $(b, a)\notin R_0$ for some $a, b\in X$ and $R_0\in\mathcal{R}$. Moreover, let A be a preordered set such that for some $\alpha_1, \alpha_2\in A$, the set $\{\alpha_1, \alpha_2\}$ has no upper bound in A. Define $x_\alpha=a$ for all $\alpha\in A$ with $\alpha\not\geqq\alpha_1$ and $x_\alpha=b$ for all $\alpha\geqq\alpha_1$, and $y_\alpha=b$ for all $\alpha\in A$. Then, it is clear that $(y_\alpha, x_\alpha)\in R$ for all $R\in\mathcal{R}$ and $\alpha\geqq\alpha_1$, but $(y_\alpha, x_\alpha)\notin R_0$ for all $\alpha\geqq\alpha_2$. Consequently, $y_\alpha\in\underset{\alpha}{\mathrm{Lim}}_{\mathcal{R}}\, x_\alpha$, but $y_\alpha\notin\underset{\alpha}{\mathrm{Adh}}_{\mathcal{R}}\, x_\alpha$.

REMARK 1.14. A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$ will be called uniformly directed if $\mathcal{R}$ is directed with respect to the reverse set inclusion.

In uniformly directed relator spaces, we may restrict ourselves to directed nets, thus the above inconveniences can be avoided there.

DEFINITION 1.15. If $\mathcal{R}$ is a relator on X and $\mathcal{N}=\mathcal{N}(X)$, then the relations

$$\lim_{\mathcal{R}}\subset\mathcal{N}\times X \quad\text{and}\quad \mathrm{adh}_{\mathcal{R}}\subset\mathcal{N}\times X$$

defined such that for any $(x_\alpha)\in\mathcal{N}$

$$\lim_{\mathcal{R}}((x_\alpha)) = \{x\in X\colon (x)\in\mathrm{Lim}_{\mathcal{R}}((x_\alpha))\}$$

and

$$\mathrm{adh}_{\mathcal{R}}((x_\alpha)) = \{x\in X\colon (x)\in\mathrm{Adh}_{\mathcal{R}}((x_\alpha))\}$$

will be called the little limit and the little adherence on X induced by $\mathcal{R}$, respectively.

REMARK 1.16. Again, we shall simply write

$$x\in\underset{\alpha}{\lim}_{\mathcal{R}}\, x_\alpha \quad\text{and}\quad x\in\underset{\alpha}{\mathrm{adh}}_{\mathcal{R}}\, x_\alpha$$

instead of

$$x\in\lim_{\mathcal{R}}((x_\alpha)) \quad\text{and}\quad x\in\mathrm{adh}_{\mathcal{R}}((x_\alpha)),$$

respectively.

The following theorem is an immediate consequence of the corresponding definitions.

THEOREM 1.17. *If (x_α) is a net in a relator space $X(\mathcal{R})$, then*

$$\underset{\alpha}{\lim}_{\mathcal{R}}\, x_\alpha = \bigcap_{R\in\mathcal{R}}\underset{\alpha}{\underline{\lim}}\, R^{-1}(x_\alpha)$$

and

$$\underset{\alpha}{\mathrm{adh}}_{\mathcal{R}}\, x_\alpha = \bigcap_{R\in\mathcal{R}}\underset{\alpha}{\overline{\lim}}\, R^{-1}(x_\alpha).$$

REMARK 1.18. Recall that

$$\underset{\alpha}{\underline{\lim}}\, A_\alpha = \bigcup_{\alpha}\bigcap_{\beta\geqq\alpha} A_\beta \quad\text{and}\quad \underset{\alpha}{\overline{\lim}}\, A_\alpha = \bigcap_{\alpha}\bigcup_{\beta\geqq\alpha} A_\beta$$

for any net (A_α) of sets.

The next two theorems follow at once from Theorems 1.8 and 1.11.

THEOREM 1.19. *If $\mathscr{R}$ is a relator on X, then the relation* $\lim_{\mathscr{R}}$ *has the following properties:*

(i) *If* $x\in X$ *and A is a nonvoid preordered set, then* $x\in\lim\limits_{\alpha\in A} x$.

(ii) *If* (x_α) *is a net in X and* (y_β) *is a subnet of* (x_α), *then* $\lim\limits_{\alpha}{}_{\mathscr{R}}\, x_\alpha\subset\lim\limits_{\beta}{}_{\mathscr{R}}\, y_\beta$.

(iii) *If* (x_α) *is a net in X and* $x\in X$ *such that for each cofinal subnet* (y_β) *of* (x_α) *there exists a net* (z_γ) *being frequently in* $\{y_\beta\}$ *such that* $x\in\lim\limits_{\gamma}{}_{\mathscr{R}}\, z_\gamma$, *then* $x\in\lim\limits_{\alpha}{}_{\mathscr{R}}\, x_\alpha$.

COROLLARY 1.20. *If* (x_α) *is a net and x is a point in a relator space* $X(\mathscr{R})$, *then the following assertions are equivalent:*

(i) $x\in\lim\limits_{\alpha}{}_{\mathscr{R}}\, x_\alpha$;

(ii) *each subnet* (y_β) *of* (x_α) *has a subnet* (z_γ) *such that* $x\in\lim\limits_{\gamma}{}_{\mathscr{R}}\, z_\gamma$.

REMARK 1.21. A further important property which the relation $\lim_{\mathscr{R}}$ may have is the following iterated limit property:

If $(x_{\alpha\beta})_{\beta\in B_\alpha}$ is a net in X for each α in a nonvoid preordered set A, $y_\alpha\in\lim\limits_{\beta}{}_{\mathscr{R}}\, x_{\alpha\beta}$ for each $\alpha\in A$, and $z\in\lim\limits_{\alpha}{}_{\mathscr{R}}\, y_\alpha$, then $z\in\lim\limits_{(\alpha,\varphi)}{}_{\mathscr{R}}\, x_{\alpha\varphi(\alpha)}$, where $(\alpha,\varphi)\in A\times\bigtimes\limits_{\alpha\in A} B_\alpha$.

THEOREM 1.22. *If* (x_α) *is a net and x is a point in a relator space* $X(\mathscr{R})$, *then the following assertions hold:*

(i) *If* $x\in\operatorname*{adh}\limits_{\alpha}{}_{\mathscr{R}}\, x_\alpha$, *then* $x\in\lim\limits_{\beta}{}_{\mathscr{R}}\, y_\beta$ *for some subnet* (y_β) *of* (x_α).

(ii) *If* $x\in\operatorname*{adh}\limits_{\beta}{}_{\mathscr{R}}\, y_\beta$ *for any subnet* (y_β) *of* (x_α), *then* $x\in\lim\limits_{\alpha}{}_{\mathscr{R}}\, x_\alpha$.

REMARK 1.23. Also by Example 1.13, it is clear that the converses of the assertions (i) and (ii) in Theorem 1.22 are not, in general, true.

This strongly suggests that in a relator space $X(\mathscr{R})$, even the relations $\lim_{\mathscr{R}}$ and $\operatorname{adh}_{\mathscr{R}}$ cannot, in general, be equivalent tools. However, as we shall soon see, this is, fortunately, not the case.

2. Closures and interiors

DEFINITION 2.1. If $\mathscr{R}$ is a relator on X and $\mathscr{P}=\mathscr{P}(X)$ is the family of all subsets of X, then the relations

$$\mathrm{Cl}_{\mathscr{R}}\subset\mathscr{P}\times\mathscr{P} \quad\text{and}\quad \mathrm{Int}_{\mathscr{R}}\subset\mathscr{P}\times\mathscr{P}$$

defined such that for any $A\in\mathscr{P}$

$$\mathrm{Cl}_{\mathscr{R}}(A)=\{B\in\mathscr{P}\colon\ \forall R\in\mathscr{R}\colon\ A\cap R(B)\neq\emptyset\}$$

and

$$\mathrm{Int}_{\mathscr{R}}(A)=\{B\in\mathscr{P}\colon\ \exists R\in\mathscr{R}\colon\ R(B)\subset A\}$$

will be called the big closure and the big interior on X induced by $\mathscr{R}$, respectively.

EXAMPLE 2.2. If $\mathcal{R}_{\mathcal{D}}$ is as in Example 1.3 and $A, B \subset X$, then

$$B \in \mathrm{Cl}_{\mathcal{R}_{\mathcal{D}}}(A) \quad \text{iff} \quad d(B,A) = 0 \quad \text{for all} \quad d \in \mathcal{D},$$

where $d(B, A) = \inf \{d(b, a)\colon b \in B,\ a \in A\}$.

REMARK 2.3. Note also that if $\mathcal{R}$ is in particular a quasi-uniformity, then $\mathrm{Cl}_{\mathcal{R}}$ is precisely the inverse of the induced quasi-proximity $\delta_{\mathcal{R}}$ [28, p. 12].

In this respect, it is also worth mentioning that we have $\mathrm{Cl}_{\mathcal{R}}^{-1} = \mathrm{Cl}_{\mathcal{R}^{-1}}$ for any relator $\mathcal{R}$. (Note that the same assertions hold also for the relations $\mathrm{Lim}_{\mathcal{R}}$ and $\mathrm{Adh}_{\mathcal{R}}$.)

The following theorem is an immediate consequence of the definition of $\mathrm{Cl}_{\mathcal{R}}$ and the equality $\mathrm{Cl}_{\mathcal{R}}^{-1} = \mathrm{Cl}_{\mathcal{R}^{-1}}$.

THEOREM 2.4. *If $\mathcal{R}$ is a relator on X, then the relation $\mathrm{Cl}_{\mathcal{R}}$ has the following properties:*

(i) $\mathrm{Cl}_{\mathcal{R}}(\emptyset) = \emptyset$;
(ii) $A \in \mathrm{Cl}_{\mathcal{R}}(A)$ *if* $\emptyset \neq A \subset X$;
(iii) $\mathrm{Cl}_{\mathcal{R}}(A) \subset \mathrm{Cl}_{\mathcal{R}}(B)$ *and* $\mathrm{Cl}_{\mathcal{R}}^{-1}(A) \subset \mathrm{Cl}_{\mathcal{R}}^{-1}(B)$ *if* $A \subset B \subset X$.

REMARK 2.5. An equivalent reformulation of the first part of (iii) says that

$$\mathrm{Cl}_{\mathcal{R}}(A) \cup \mathrm{Cl}_{\mathcal{R}}(B) \subset \mathrm{Cl}_{\mathcal{R}}(A \cup B)$$

for all $A, B \subset X$.

Later, we shall see that the converse inclusion can hold for all $A, B \subset X$ if and only if $\mathcal{R}$ is proximally directed in the sense that the family $\mathcal{R}(A)$ is directed with respect to the reverse set inclusion for all $A \subset X$.

THEOREM 2.6. *If A is a set in a relator space $X(\mathcal{R})$, then*

$$\mathrm{Int}_{\mathcal{R}}(A) = \mathcal{P}(X) \setminus \mathrm{Cl}_{\mathcal{R}}(X \setminus A)$$

and

$$\mathrm{Cl}_{\mathcal{R}}(A) = \mathcal{P}(X) \setminus \mathrm{Int}_{\mathcal{R}}(X \setminus A).$$

PROOF. For $B \subset X$, we have $B \in \mathrm{Int}_{\mathcal{R}}(A)$ iff $R(B) \subset A$ for some $R \in \mathcal{R}$ iff $(X \setminus A) \cap R(B) = \emptyset$ for some $R \in \mathcal{R}$ iff $B \notin \mathrm{Cl}_{\mathcal{R}}(X \setminus A)$. This proves the first assertion.

The second assertion can be at once derived from the first one by writing $X \setminus A$ instead of A.

REMARK 2.7. Using this theorem, the properties of the relation $\mathrm{Int}_{\mathcal{R}}$ can be easily derived from that of $\mathrm{Cl}_{\mathcal{R}}$.

Moreover, this theorem shows that in a relator space $X(\mathcal{R})$, the relations $\mathrm{Cl}_{\mathcal{R}}$ and $\mathrm{Int}_{\mathcal{R}}$ are equivalent tools.

DEFINITION 2.8. If $\mathcal{R}$ is a relator on X and $\mathcal{P} = \mathcal{P}(X)$, then the relations

$$\mathrm{cl}_{\mathcal{R}} \subset \mathcal{P} \times X \quad \text{and} \quad \mathrm{int}_{\mathcal{R}} \subset \mathcal{P} \times X$$

defined such that for any $A \in \mathcal{P}$

$$\mathrm{cl}_{\mathcal{R}}(A) = \{x \in X\colon \{x\} \in \mathrm{Cl}_{\mathcal{R}}(A)\}$$

and

$$\mathrm{int}_{\mathcal{R}}(A) = \{x \in X\colon \{x\} \in \mathrm{Int}_{\mathcal{R}}(A)\}$$

will be called the little closure and the little interior on X induced by $\mathcal{R}$, respectively.

REMARK 2.9. When it seems convenient, we shall also use the notations $\bar{A} = \mathrm{cl}_{\mathcal{R}}(A)$ and $\mathring{A} = \mathrm{int}_{\mathcal{R}}(A)$.

The following theorem is an immediate consequence of the corresponding definitions.

THEOREM 2.10. *If A is a set in a relator space $X(\mathcal{R})$ then* $\mathrm{cl}_{\mathcal{R}}(A) = \bigcap \mathcal{R}^{-1}(A)$.

The next two theorems follow at once from Theorems 2.4 and 2.6.

THEOREM 2.11. *If $\mathcal{R}$ is a relator on X, then the relation $\mathrm{cl}_{\mathcal{R}}$ has the following properties:*

(i) $\mathrm{cl}_{\mathcal{R}}(\emptyset) = \emptyset$;

(ii) $A \subset \mathrm{cl}_{\mathcal{R}}(A)$ *if* $A \subset X$;

(iii) $\mathrm{cl}_{\mathcal{R}}(A) \subset \mathrm{cl}_{\mathcal{R}}(B)$ *if* $A \subset B \subset X$.

REMARK 2.12. The property (iii) can again be reformulated by stating that

$$\mathrm{cl}_{\mathcal{R}}(A) \cup \mathrm{cl}_{\mathcal{R}}(B) \subset \mathrm{cl}_{\mathcal{R}}(A \cup B)$$

for all $A, B \subset X$.

But, now the converse inclusion can hold for all $A, B \subset X$ if and only if $\mathcal{R}$ is topologically directed in the sense that $\mathcal{R}(x)$ is directed with respect to the reverse set inclusion for all $x \in X$.

THEOREM 2.13. *If A is a set in a relator space $X(\mathcal{R})$, then*

$$\mathrm{int}_{\mathcal{R}}(A) = X \setminus \mathrm{cl}_{\mathcal{R}}(X \setminus A) \quad \textit{and} \quad \mathrm{cl}_{\mathcal{R}}(A) = X \setminus \mathrm{int}_{\mathcal{R}}(X \setminus A).$$

DEFINITION 2.14. If $\mathcal{R}$ is a relator on X, then the members of the families

$$\mathcal{F}_{\mathcal{R}} = \{A \subset X\colon \mathrm{cl}_{\mathcal{R}}(A) = A\} \quad \textit{and} \quad \mathcal{T}_{\mathcal{R}} = \{A \subset X\colon \mathrm{int}_{\mathcal{R}}(A) = A\}$$

will be called the $\mathcal{R}$-closed and the $\mathcal{R}$-open subsets of X, respectively.

REMARK 2.15. Note that for $A \subset X$, we have $A \in \mathcal{F}_{\mathcal{R}}$ $(A \in \mathcal{T}_{\mathcal{R}})$ if and only if $\mathrm{cl}_{\mathcal{R}}(A) \subset A$ $(A \subset \mathrm{int}_{\mathcal{R}}(A))$.

The following theorem can be easily derived from Theorem 2.11.

THEOREM 2.16. *If $\mathcal{R}$ is a relator on X, then the family $\mathcal{F}_{\mathcal{R}}$ has the following properties:*

(i) $\emptyset \in \mathcal{F}_{\mathcal{R}}$ *and* $X \in \mathcal{F}_{\mathcal{R}}$;
(ii) $\bigcap \mathcal{A} \in \mathcal{F}_{\mathcal{R}}$ *if* $\emptyset \neq \mathcal{A} \subset \mathcal{F}_{\mathcal{R}}$.

REMARK 2.17. If $\mathcal{R}$ is a topologically directed relator on X, then by Remark 2.12, we also have $A \cup B \in \mathcal{F}_{\mathcal{R}}$ for all $A, B \in \mathcal{F}_{\mathcal{R}}$.

Corollary 2.18. *If $\mathscr{R}$ is a relator on X, then the following assertions are equivalent:*

(i) $\operatorname{cl}_{\mathscr{R}}(A)\in\mathscr{F}_{\mathscr{R}}$ *for all* $A\subset X$;
(ii) $\operatorname{cl}_{\mathscr{R}}(A)=\bigcap\{F\in\mathscr{F}_{\mathscr{R}}\colon A\subset F\}$ *for all* $A\subset X$.

Remark 2.19. The property (i) can, somewhat inprecisely, be expressed by saying that the relation $\operatorname{cl}_{\mathscr{R}}$ is idempotent.

Later, we shall call a relator $\mathscr{R}$ on X topological if $x\in R(x)^{00}$ for all $x\in X$ and $R\in\mathscr{R}$, and show that: $\mathscr{R}$ is topological iff $\operatorname{cl}_{\mathscr{R}}$ is idempotent iff $\lim_{\mathscr{R}}$ has the iterated limit property.

The next theorem is an immediate consequence of Theorem 2.13.

Theorem 2.20. *If $\mathscr{R}$ is a relator on X, then*

$$\mathscr{T}_{\mathscr{R}}=\{X\setminus A\colon A\in\mathscr{F}_{\mathscr{R}}\}.$$

Definition 2.21. If $\mathscr{R}$ is a relator on X, then the relation $\varrho_{\mathscr{R}}$ defined on X by $\varrho_{\mathscr{R}}(x)=\operatorname{cl}_{\mathscr{R}}(\{x\})$ will be called the point-closure on X induced by $\mathscr{R}$.

Theorem 2.22. *If $\mathscr{R}$ is a relator on X, then*

$$\varrho_{\mathscr{R}}=\bigcap\mathscr{R}^{-1}=\left(\bigcap\mathscr{R}\right)^{-1}.$$

Proof. By Theorem 2.10, we clearly have

$$\varrho_{\mathscr{R}}(x)=\bigcap\mathscr{R}^{-1}(x)=\left(\bigcap\mathscr{R}^{-1}\right)(x)=\left(\bigcap\mathscr{R}\right)^{-1}(x)$$

for all $x\in X$, whence the assertion immediately follows.

Remark 2.23. The point-closure will play an important role in the descriptions of the so-called weak properties of relators.

For instance, a relator $\mathscr{R}$ will be called weakly topological and weakly symmetric if $\varrho_{\mathscr{R}}$ is closed-valued and symmetric, resp.

Note that weak symmetry corresponds to the famous R_0-property of topological spaces which was mainly studied by Davis [20] and Murdeshwar—Naimpally [70].

3. Interdependence of basic tools

Theorem 3.1. *If A and B are sets in a relator space $X(\mathscr{R})$, then the following assertions are equivalent:*

(i) $B\in\operatorname{Cl}_{\mathscr{R}}(A)$;
(ii) $y_\alpha\in\operatorname*{Lim}_{\alpha}{}_{\mathscr{R}}\,x_\alpha$ *for some net* $((x_\alpha,y_\alpha))$ *in* $A\times B$;
(iii) $y_\alpha\in\operatorname*{Adh}_{\alpha}{}_{\mathscr{R}}\,x_\alpha$ *for some net* $((x_\alpha,y_\alpha))$ *in* $A\times B$.

Proof. If (i) holds, then for each $R\in\mathscr{R}$, there exists $(x_R,y_R)\in A\times B$ such that $(y_R,x_R)\in R$. Hence, by preordering $\mathscr{R}$ with the reverse set inclusion, we can state that $((x_R,y_R))_{R\in\mathscr{R}}$ is a net in $A\times B$ such that $y_R\in\operatorname*{Lim}_{R}{}_{\mathscr{R}}\,x_R$. Consequently, (ii) holds.

On the other hand, by preordering $\mathscr{R}$ with the largest possible preorder, we can also state that $((x_R, y_R))_{R\in\mathscr{R}}$ is a net in $A\times B$ such that $y_R\in\operatorname*{Adh}_{R}{}_{\mathscr{R}}\, x_R$. Consequently (i) implies (iii) too.

The implications (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i) are even more obvious.

Remark 3.2. Note that in the assertion (iii) the net $((x_\alpha, y_\alpha))$ may be required to be directed even if $\mathscr{R}$ is not uniformly directed.

Note also that by using the assertion (i) of Theorem 1.11, the proofs of the implications (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(i) can be spared.

The above theorem shows that in a relator space $X(\mathscr{R})$ the relation $\mathrm{Cl}_{\mathscr{R}}$ cannot be a more powerful tool than $\mathrm{Lim}_{\mathscr{R}}$ or $\mathrm{Adh}_{\mathscr{R}}$.

Theorem 3.3. *If $(x_\alpha)_{\alpha\in A}$ and $(y_\alpha)_{\alpha\in A}$ are nets in a relator space $X(\mathscr{R})$, then among the following assertions the implications* (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii) *hold:*

(i) $y_\alpha\in\operatorname*{Lim}_{\alpha}{}_{\mathscr{R}}\, x_\alpha$;

(ii) $\{y_\beta\}_{\beta\in B}\in\mathrm{Cl}_{\mathscr{R}}(\{x_\beta\}_{\beta\in B})$ *for any cofinal subset B of A;*

(iii) $\{w_\beta\}_{\beta\in B}\in\mathrm{Cl}_{\mathscr{R}}(\{z_\alpha\}_{\beta\in B})$ *for any subnet* $((z_\beta, w_\beta))_{\beta\in B}$ *of* $((x_\alpha, y_\alpha))_{\alpha\in A}$.

Proof. This is an immediate consequence of the corresponding definitions. To check that (ii) implies (iii), note that if $((z_\beta, w_\beta))_{\beta\in B}$ is a subnet of $((x_\alpha, y_\alpha))_{\alpha\in A}$, then there exists a function φ from B into A such that $z_\beta=x_{\varphi(\beta)}$ and $w_\beta=y_{\varphi(\beta)}$ for all $\beta\in B$, and for each $\alpha\in A$ there exists $\beta_\alpha\in B$ such that $\varphi(\beta)\geqq\alpha$ for all $\beta\geqq\beta_\alpha$. Hence, it is clear that $\{z_\beta\}_{\beta\in B}=\{x_\alpha\}_{\alpha\in\varphi(B)}$ and $\{w_\beta\}_{\beta\in B}=\{y_\alpha\}_{\alpha\in\varphi(B)}$, and $\varphi(B)$ is a cofinal subset of A. Thus, (ii) can be used to derive that $\{w_\beta\}_{\beta\in B}\in\mathrm{Cl}_{\mathscr{R}}(\{z_\beta\}_{\beta\in B})$.

Remark 3.4. If in particular A is a linearly ordered set and $\mathscr{R}$ is a uniformity, then (ii) also implies (i).

This follows immediately from a deep result of Ramm—Švarc [82, Lemma 2′] which was later rediscovered by Alfsen—Njåstad [2, Lemma 2].

Theorem 3.5. *If $(x_\alpha)_{\alpha\in A}$ and $(y_\alpha)_{\alpha\in A}$ are nets in a relator space $X(\mathscr{R})$, then any of the following assertions implies the subsequent one:*

(i) $y_\alpha\in\operatorname*{Adh}_{\alpha}{}_{\mathscr{R}}\, x_\alpha$;

(ii) $\{y_\beta\}_{\beta\geqq\alpha}\in\mathrm{Cl}_{\mathscr{R}}(\{x_\beta\}_{\beta\geqq\alpha})$ *for all* $\alpha\in A$;

(iii) $w_\beta\in\operatorname*{Lim}_{\beta}{}_{\mathscr{R}}\, z_\beta$ *for some subnets* $(z_\beta)_{\beta\in B}$ *and* $(w_\beta)_{\beta\in B}$ *of* $(x_\alpha)_{\alpha\in A}$ *and* $(y_\alpha)_{\alpha\in A}$, *respectively.*

Proof. It is clear that (i) implies (ii). To check that (ii) also implies (iii), note that if (ii) holds, then for each $\alpha\in A$ and $R\in\mathscr{R}$, there exist $\varphi_{(\alpha,R)}\geqq\alpha$ and $\psi_{(\alpha,R)}\geqq\alpha$ such that $(y_{\psi_{(\alpha,R)}}, x_{\varphi_{(\alpha,R)}})\in R$. Hence, by preordering $\mathscr{R}$ with the reverse set inclusion and $B=A\times\mathscr{R}$ with the product preorder, it is clear that $(x_{\varphi_{(\alpha,R)}})_{(\alpha,R)\in B}$ and $(y_{\psi_{(\alpha,R)}})_{(\alpha,R)\in B}$ are subnets of $(x_\alpha)_{\alpha\in A}$ and $(y_\alpha)_{\alpha\in A}$, respectively, such that $y_{\psi(\alpha,R)}\in\operatorname*{Lim}_{(\alpha,R)}{}_{\mathscr{R}}\, x_{\varphi(\alpha,R)}$.

Remark 3.6. If in particular B is a directed set in (iii), then (iii) also implies (ii).

But, as the next trivial example shows, the implication (ii) $\Rightarrow$ (i) cannot, even for sequences, be true in general.

Example 3.7. Let X be the set of all real numbers, d the usual metric on X and $\mathcal{R}_d=\mathcal{R}_{\{d\}}$ as in Example 1.3. Define $x_n=n$ and $y_n=n+1$ for each positive integer n. Then, it is clear that (x_n) and (y_n) are sequences in X such that $\{y_m\}_{m\geqq n}\in \mathrm{Cl}_{\mathcal{R}_d}(\{x_m\}_{m\geqq n})$ for all n, but $y_n\notin \underset{n}{\mathrm{Adh}}_{\mathcal{R}}\, x_n$.

By letting B to be a singleton $\{x\}$ in Theorem 3.1, we immediately get

Theorem 3.8. *If A is a set and x is a point in a relator space $X(\mathcal{R})$, then the following assertions are equivalent:*

(i) $x\in \mathrm{cl}_{\mathcal{R}}(A)$;

(ii) $x\in \underset{\alpha}{\lim}_{\mathcal{R}}\, x_\alpha$ *for some net* (x_α) *in* A;

(iii) $x\in \underset{\alpha}{\mathrm{adh}}_{\mathcal{R}}\, x_\alpha$ *for some net* (x_α) *in* A.

Remark 3.9. If $\mathcal{R}$ is, in particular, topologically directed, then the net (x_α) may be required to be directed not only in the assertion (iii), but also in the assertion (ii).

On the other hand, in contrast to Theorems 3.3 and 3.5 and Remarks 3.4 and 3.6, now we simply have

Theorem 3.10. *If $(x_\alpha)_{\alpha\in A}$ is a net in a relator space $X(\mathcal{R})$, then*

$$\underset{\alpha}{\lim}_{\mathcal{R}}\, x_\alpha = \bigcap_{B\in\mathcal{C}(A)} \mathrm{cl}_{\mathcal{R}}(\{x_\beta\}_{\beta\in B}),$$

where $\mathcal{C}(A)$ means the family of all cofinal subsets of A, and

$$\underset{\alpha}{\mathrm{adh}}_{\mathcal{R}}\, x_\alpha = \bigcap_{\alpha\in A} \mathrm{cl}_{\mathcal{R}}(\{x_\beta\}_{\beta\geqq\alpha}).$$

Proof. By letting $y_\alpha=x$ for all $\alpha\in A$ in Theorem 3.3, we immediately get

$$\underset{\alpha}{\lim}_{\mathcal{R}}\, x_\alpha \subset \bigcap_{B\in\mathcal{C}(A)} \mathrm{cl}_{\mathcal{R}}(\{x_\beta\}_{\beta\in B}).$$

To prove the converse inclusion, note that if $x\in X$ such that $x\notin \underset{\alpha}{\lim}_{\mathcal{R}}\, x_\alpha$, then there exists $R\in\mathcal{R}$ such that

$$B=\{\alpha\in A:\ x_\alpha\notin R(x)\}$$

is a cofinal subset of A. Moreover, $x\notin \mathrm{cl}_{\mathcal{R}}(\{x_\beta\}_{\beta\in B})$, since $\{x_\beta\}_{\beta\in B}\cap R(x)=\emptyset$.

This proves the first assertion. The proof of the second one is even more obvious.

Remark 3.11. Theorems 3.8, 3.10 and 2.13 show that in a relator space $X(\mathcal{R})$, the relations $\lim_{\mathcal{R}}$, $\mathrm{adh}_{\mathcal{R}}$, $\mathrm{cl}_{\mathcal{R}}$ and $\mathrm{int}_{\mathcal{R}}$ are equivalent tools.

Moreover, from Theorem 3.8, one can also easily derive the following less surprising

Theorem 3.12. *If A is a set in a relator space $X(\mathcal{R})$, then the following assertions are equivalent:*

(i) $A\in\mathcal{F}_{\mathcal{R}}$;

(ii) $\underset{\alpha}{\lim}_{\mathcal{R}}\, x_\alpha\subset A$ *for any net* (x_α) *in* A;

(iii) $\underset{\alpha}{\mathrm{adh}}_{\mathcal{R}}\, x_\alpha\subset A$ *for any net* (x_α) *in* A.

REMARK 3.13. Hence, we can also state that, for a set A in a relator space $X(\mathcal{R})$, we have $A \in \mathcal{T}_{\mathcal{R}}$ if and only if $A \cap \lim_{\mathcal{R}} x_\alpha \neq \emptyset$ $(A \cap \operatorname{adh}_{\mathcal{R}} x_\alpha \neq \emptyset)$ implies that (x_α) is eventually (frequently) in A.

4. Mild continuities

DEFINITION 4.1. A relation f from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$ will be called mildly continuous, or more precisely mildly $(\mathcal{R}, \mathcal{S})$-continuous, if $f^{-1} \circ S \circ f \in \mathcal{R}$ for all $S \in \mathcal{S}$.

REMARK 4.2. Authors dealing with uniform continuity of a function f usually introduce the auxiliary function $f \times f$ defined by $(f \times f)(x, y) = (f(x), f(y))$ and fail to note that $(f \times f)^{-1}(S) = f^{-1} \circ S \circ f$.

The useful expression $f^{-1} \circ S \circ f$ seems to have formerly been explicitly used only by Konishi [51], Davis [21], Kenyon [50], Doičinov [23], Mathews—Curtis [64] and Nakano—Nakano [73].

The latter two authors have also suggested that by defining appropriate operations on relators all the important continuity properties of a function can be obtained as particular relator continuities.

REMARK 4.3. Mild continuities for relations express in general much weaker continuity properties than upper and lower semicontinuities which are to be defined according to [86].

Namely, the condition of Definition 4.1 can be rephrased by saying that for each $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that the properties $y \in R(x)$ and $f(y) \cap S(f(x)) \neq \emptyset$ are equivalent for any $x, y \in X$.

Fortunately, for functions the above three kinds of continuity properties coincide. Therefore, in that case the term "mildly" may be omitted without any danger of confusion.

The next two theorems contain important, but almost self-evident assertions about mild continuities.

THEOREM 4.4. *If f and g are relations from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$ such that $f \subset g$ and f is mildly continuous, then g is also mildly continuous, provided that $\mathcal{R}$ is a stack on $X \times X$.*

COROLLARY 4.5. *If f is a reflexive relation on a relator space $X(\mathcal{R})$ with $\mathcal{R}$ being a stack, then f is necessarily mildly continuous.*

THEOREM 4.6. *If f is a mildly continuous relation from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$ and g is a mildly continuous relation from $Y(\mathcal{S})$ into a relator space $Z(\mathcal{T})$, then $g \circ f$ is a mildly continuous relation from $X(\mathcal{R})$ into $Z(\mathcal{T})$.*

To partly reduce the study of mild continuities of relations to that of functions, we need a straightforward notion of a hyperspace of a relator space.

DEFINITION 4.7. Let $X(\mathscr{R})$ be a relator space and $\tilde{X}=\mathscr{P}(X)\setminus\{\emptyset\}$. For each $R\in\mathscr{R}$, define the relation $\tilde{R}$ on $\tilde{X}$ by

$$\tilde{R}(A)=\{B\in\tilde{X}:\ B\cap R(A)\neq\emptyset\}.$$

Moreover, let $\tilde{\mathscr{R}}=\{\tilde{R}:\ R\in\mathscr{R}\}$. Then the relator space $\tilde{X}(\tilde{\mathscr{R}})$ will be called the mild hyperspace of $X(\mathscr{R})$.

REMARK 4.8. Note that by identifying singletons with their elements, $\tilde{X}(\tilde{\mathscr{R}})$ may be considered as an extension of $X(\mathscr{R})$.

Thus, by using $\tilde{X}(\tilde{\mathscr{R}})$, the relations defined in Sections 1 and 2 can be naturally extended to nets and families of nonvoid subsets of X, respectively.

Of course, the same assertions hold also for the upper and lower hyperspaces of a relator space which are to be defined according to [8].

However, for the above purposes, mild hyperspaces appear to be more suitable since we have

THEOREM 4.9. *If $\mathscr{R}$ is a relator on X, then*

$$\mathrm{Cl}_{\mathscr{R}}=\varrho_{\tilde{\mathscr{R}}}\quad\text{and}\quad \mathrm{Cl}_{\tilde{\mathscr{R}}}=\mathscr{U}^{-1}\circ\mathrm{Cl}_{\mathscr{R}}\circ\mathscr{U},$$

where $\mathscr{U}$ means the function defined on $\mathscr{P}(\mathscr{P}(X))$ by $\mathscr{U}(\mathscr{A})=\bigcup\mathscr{A}$.

PROOF. The first assertion is quite obvious. To check the second one, note that if $\mathscr{B}\in\mathrm{Cl}_{\tilde{\mathscr{R}}}(\mathscr{A})$, then $\mathscr{A}\cap\tilde{R}(\mathscr{B})\neq\emptyset$ for all $R\in\mathscr{R}$. That is, for each $R\in\mathscr{R}$, there exists $A\in\mathscr{A}$ and $B\in\mathscr{B}$ such that $A\in\tilde{R}(B)$, or equivalently $A\cap R(B)\neq\emptyset$. This means that, for each $R\in\mathscr{R}$, there exist $x\in\bigcup\mathscr{A}$ and $y\in\bigcup\mathscr{B}$ such that $x\in R(y)$. Consequently, $\bigcup\mathscr{A}\cap R(\bigcup\mathscr{B})\neq\emptyset$ for all $R\in\mathscr{R}$, and hence $\bigcup\mathscr{B}\in\mathrm{Cl}_{\mathscr{R}}(\bigcup\mathscr{A})$. That is, $\mathscr{U}(\mathscr{B})\in\mathrm{Cl}_{\mathscr{R}}(\mathscr{U}(\mathscr{A}))$, and hence $\mathscr{B}\in\mathscr{U}^{-1}(\mathrm{Cl}_{\mathscr{R}}(\mathscr{U}(\mathscr{A})))$. This shows that $\mathrm{Cl}_{\tilde{\mathscr{R}}}\subset\mathscr{U}^{-1}\circ\mathrm{Cl}_{\mathscr{R}}\circ\mathscr{U}$. The converse inclusion can be proved quite similarly.

REMARK 4.10. In connection with the mild hyperrelator $\tilde{\mathscr{R}}$, it is also worth mentioning that the relations $\lim_{\tilde{\mathscr{R}}}$ and $\mathrm{adh}_{\tilde{\mathscr{R}}}$ can be used to express the topological lower and upper limits studied by Mrówka [69] and Frolík [30].

DEFINITION 4.11. If f is a relation from a set X into another Y, then the functions φ_f and Φ_f defined on X and $\tilde{X}$ by

$$\varphi_f(x)=f(x)\quad\text{and}\quad\Phi_f(A)=f(A)$$

will be called the little and the big set-valued functions induced by f, respectively.

PROPOSITION 4.12. *If f is a relation from a set X into another Y and S is a reflexive relation on Y, then*

$$\varphi_f^{-1}\circ\tilde{S}\circ\varphi_f=f^{-1}\circ S\circ f\quad\textit{and}\quad\Phi_f^{-1}\circ\tilde{S}\circ\Phi_f=(f^{-1}\circ S\circ f)^{\sim}.$$

PROOF. Straightforward computation. For instance, to check the second assertion, note that for $A,B\in\tilde{X}$ we have $B\in(\Phi_f^{-1}\circ\tilde{S}\circ\Phi_f)(A)$ iff $\Phi_f(B)\in\tilde{S}(\Phi_f(A))$ iff $f(B)\cap S(f(A))\neq\emptyset$ iff $B\cap f^{-1}(S(f(A)))\neq\emptyset$ iff $B\in(f^{-1}\circ S\circ f)^{\sim}(A)$.

THEOREM 4.13. *If f is a relation from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(i) *f is mildly $(\mathcal{R}, \mathcal{S})$-continuous;*
(ii) *φ_f is $(\mathcal{R}, \tilde{\mathcal{S}})$-continuous;*
(iii) *Φ_f is $(\tilde{\mathcal{R}}, \tilde{\mathcal{S}})$-continuous.*

PROOF. By Proposition 4.12, it is clear that (i) and (ii) are equivalent, and (i) implies (iii).

To check that (iii) also implies (i), note that if $S \in \mathcal{S}$ and $T = f^{-1} \circ S \circ f$, then by Proposition 4.12 and the assertion (iii), we have $\tilde{T} = \tilde{R}$ for some $R \in \mathcal{R}$, which implies that $T = R$.

5. Criteria for continuities

DEFINITION 5.1. If $\mathcal{R}$ is a relator on X, then the relator

$$\mathcal{R}^* = \{S \subset X \times X : \exists R \in \mathcal{R} : R \subset S\}$$

will be called the uniform refinement of $\mathcal{R}$.

THEOREM 5.2. *If f is a function from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(i) *f is $(\mathcal{R}^*, \mathcal{S})$-continuous;*
(ii) *$y_\alpha \in \operatorname{Lim}_{\alpha}{}_{\mathcal{R}} x_\alpha$ implies $f(y_\alpha) \in \operatorname{Lim}_{\alpha}{}_{\mathcal{S}} f(x_\alpha)$;*
(iii) *$y_\alpha \in \operatorname{Adh}_{\alpha}{}_{\mathcal{R}} x_\alpha$ implies $f(y_\alpha) \in \operatorname{Adh}_{\alpha}{}_{\mathcal{S}} f(x_\alpha)$.*

PROOF. If (i) does not hold, then there exists $S \in \mathcal{S}$ such that $f^{-1} \circ S \circ f \notin \mathcal{R}^*$. This means that for each R, there exists $(x_R, y_R) \in R$ such that $(x_R, y_R) \notin f^{-1} \circ S \circ f$, i.e., $(f(x_R), f(y_R)) \notin S$. Hence, by preordering $\mathcal{R}$ with the reverse set inclusion, we can state that (x_R) and (y_R) are nets in X such that $x_R \in \operatorname{Lim}_{R}{}_{\mathcal{R}} y_R$, but $f(x_R) \notin \operatorname{Lim}_{R}{}_{\mathcal{S}} f(y_R)$. Consequently, (ii) implies (i).

On the other hand, by preordering $\mathcal{R}$ with the largest possible preorder, we can also state that (x_R) and (y_R) are nets in X such that $x_R \in \operatorname{Adh}_{R}{}_{\mathcal{R}} y_R$, but $f(x_R) \notin \operatorname{Adh}_{R}{}_{\mathcal{S}} f(y_R)$. Consequently, (iii) also implies (i).

The implications (i)⇒(ii) and (i)⇒(iii) are even more straightforward consequences of the corresponding definitions.

REMARK 5.3. Note that any one of the assertions
(iv) $y_\alpha \in \operatorname{Lim}_{\alpha}{}_{\mathcal{R}} x_\alpha$ implies $f(y_\alpha) \in \operatorname{Adh}_{\alpha}{}_{\mathcal{S}} f(x_\alpha)$;
(v) $y_\alpha \in \operatorname{Adh}_{\alpha}{}_{\mathcal{R}} x_\alpha$ implies $f(y_\alpha) \in \operatorname{Lim}_{\alpha}{}_{\mathcal{S}} f(x_\alpha)$;
implies (i), but none of them is implied by (i).

Moreover, note also that in the assertions (iii) and (v) the nets (x_α) and (y_α) may be required to be directed even if $\mathcal{R}$ is not uniformly directed.

COROLLARY 5.4. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X, *then the following assertions are equivalent:*

(i) $\mathcal{S}\subset\mathcal{R}^*$; (ii) $\mathrm{Lim}_{\mathcal{R}}\subset\mathrm{Lim}_{\mathcal{S}}$; (iii) $\mathrm{Adh}_{\mathcal{R}}\subset\mathrm{Adh}_{\mathcal{S}}$.

PROOF. Apply Theorem 5.2 to the identity function of $X(\mathcal{R})$ into $X(\mathcal{S})$.

COROLLARY 5.5. *If* $\mathcal{R}$ *is a relator on* X, *then* $\mathcal{R}^*$ *is the largest relator on* X *such that* $\mathrm{Lim}_{\mathcal{R}^*}=\mathrm{Lim}_{\mathcal{R}}$ $(\mathrm{Adh}_{\mathcal{R}^*}=\mathrm{Adh}_{\mathcal{R}})$.

PROOF. Use Corollary 5.4 and the inclusions $\mathcal{R}^*\subset\mathcal{R}^*$ and $\mathcal{R}\subset(\mathcal{R}^*)^*$.

DEFINITION 5.6. If $\mathcal{R}$ is a relator on X, then the relator

$$\mathcal{R}^{\#}=\{S\subset X\times X:\ \forall A\subset X:\ \exists R\in\mathcal{R}:\ R(A)\subset S(A)\}$$

will be called the proximal refinement of $\mathcal{R}$.

THEOREM 5.7. *If* f *is a function from a relator space* $X(\mathcal{R})$ *into another* $Y(\mathcal{S})$, *then the following assertions are equivalent:*

(i) f *is* $(\mathcal{R}^{\#},\mathcal{S})$*-continuous*;
(ii) $B\in\mathrm{Cl}_{\mathcal{R}}(A)$ *implies* $f(B)\in\mathrm{Cl}_{\mathcal{S}}(f(A))$;
(iii) $f^{-1}(V)\in\mathrm{Cl}_{\mathcal{R}}(f^{-1}(U))$ *implies* $V\in\mathrm{Cl}_{\mathcal{S}}(U)$;
(iv) $V\in\mathrm{Int}_{\mathcal{S}}(U)$ *implies* $f^{-1}(V)\in\mathrm{Int}_{\mathcal{R}}(f^{-1}(U))$.

PROOF. A simple application of the corresponding definitions shows that (i) implies (ii). Moreover, by writing $f^{-1}(U)$ and $f^{-1}(V)$, with $U,V\subset Y$, instead of A and B in (ii), respectively, and $Y\setminus U$ instead of U in (iii), and applying Theorem 2.6, one can easily check that (ii)⇒(iii) and (iii)⇒(iv).

To prove that (iv) also implies (i), suppose that (iv) holds, and let $S\in\mathcal{S}$ and $A\subset X$. Then, we clearly have $f(A)\in\mathrm{Int}_{\mathcal{S}}(S(f(A)))$. Hence, by (iv), it follows that $f^{-1}(f(A))\in\mathrm{Int}_{\mathcal{R}}(f^{-1}(S(f(A))))$. This implies that $A\in\mathrm{Int}_{\mathcal{R}}((f^{-1}\circ S\circ f)(A))$. Thus, there exists $R\in\mathcal{R}$ such that $R(A)\subset(f^{-1}\circ S\circ f)(A)$. Consequently, $f^{-1}\circ S\circ f\in\mathcal{R}^{\#}$.

COROLLARY 5.8. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X, *then the following assertions are equivalent:*

(i) $\mathcal{S}\subset\mathcal{R}^{\#}$; (ii) $\mathrm{Cl}_{\mathcal{R}}\subset\mathrm{Cl}_{\mathcal{S}}$; (iii) $\mathrm{Int}_{\mathcal{S}}\subset\mathrm{Int}_{\mathcal{R}}$.

COROLLARY 5.9. *If* $\mathcal{R}$ *is a relator on* X, *then* $\mathcal{R}^{\#}$ *is the largest relator on* X *such that* $\mathrm{Cl}_{\mathcal{R}^{\#}}=\mathrm{Cl}_{\mathcal{R}}$ $(\mathrm{Int}_{\mathcal{R}^{\#}}=\mathrm{Int}_{\mathcal{R}})$.

DEFINITION 5.10. If $\mathcal{R}$ is a relator on X, then the relator

$$\hat{\mathcal{R}}=\{S\subset X\times X:\ \forall x\in X:\ \exists R\in\mathcal{R}:\ R(x)\subset S(x)\}$$

will be called the topological refinement of $\mathcal{R}$.

The proofs of the next two theorems are quite similar to that of Theorems 5.2 and 5.7.

THEOREM 5.11. *If f is a function from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(i) *f is $(\hat{\mathcal{R}}, \mathcal{S})$-continuous;*
(ii) *$x \in \underset{\alpha}{\operatorname{lim}}_{\mathcal{R}}\, x_\alpha$ implies $f(x) \in \underset{\alpha}{\operatorname{lim}}_{\mathcal{S}}\, f(x_\alpha)$;*
(iii) *$x \in \underset{\alpha}{\operatorname{adh}}_{\mathcal{R}}\, x_\alpha$ implies $f(x) \in \underset{\alpha}{\operatorname{adh}}_{\mathcal{S}}\, f(x_\alpha)$.*

REMARK 5.12. Any one of the assertions

(iv) $x \in \underset{\alpha}{\operatorname{lim}}_{\mathcal{R}}\, x_\alpha$ implies $f(x) \in \underset{\alpha}{\operatorname{adh}}_{\mathcal{S}}\, f(x_\alpha)$,
(v) $x \in \underset{\alpha}{\operatorname{adh}}_{\mathcal{R}}\, x_\alpha$ implies $f(x) \in \underset{\alpha}{\operatorname{lim}}_{\mathcal{S}}\, f(x_\alpha)$

implies (i), but none of them is implied by (i).

If $\mathcal{R}$ is topologically directed, then the net (x_α) can be required to be directed not only in the assertions (iii) and (v), but also in the assertions (ii) and (iv).

COROLLARY 5.13. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(i) $\mathcal{S} \subset \hat{\mathcal{R}}$; (ii) $\operatorname{lim}_{\mathcal{R}} \subset \operatorname{lim}_{\mathcal{S}}$; (iii) $\operatorname{adh}_{\mathcal{R}} \subset \operatorname{adh}_{\mathcal{S}}$.

COROLLARY 5.14. *If $\mathcal{R}$ is a relator on X, then $\hat{\mathcal{R}}$ is the largest relator on X such that* $\operatorname{lim}_{\hat{\mathcal{R}}} = \operatorname{lim}_{\mathcal{R}}$ $(\operatorname{adh}_{\hat{\mathcal{R}}} = \operatorname{adh}_{\mathcal{R}})$.

THEOREM 5.15. *If f is a function from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(i) *f is $(\hat{\mathcal{R}}, \mathcal{S})$-continuous;*
(ii) *$x \in \operatorname{cl}_{\mathcal{R}}(A)$ implies $f(x) \in \operatorname{cl}_{\mathcal{S}}(f(A))$;*
(iii) *$x \in \operatorname{cl}_{\mathcal{R}}(f^{-1}(A))$ implies $f(x) \in \operatorname{cl}_{\mathcal{S}}(A)$;*
(iv) *$f(x) \in \operatorname{int}_{\mathcal{S}}(A)$ implies $x \in \operatorname{int}_{\mathcal{R}}(f^{-1}(A))$.*

COROLLARY 5.16. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(i) $\mathcal{S} \subset \hat{\mathcal{R}}$; (ii) $\operatorname{cl}_{\mathcal{R}} \subset \operatorname{cl}_{\mathcal{S}}$; (iii) $\operatorname{int}_{\mathcal{S}} \subset \operatorname{int}_{\mathcal{R}}$.

COROLLARY 5.17. *If $\mathcal{R}$ is a relator on X, then $\hat{\mathcal{R}}$ is the largest relator on X such that* $\operatorname{cl}_{\hat{\mathcal{R}}} = \operatorname{cl}_{\mathcal{R}}$ $(\operatorname{int}_{\hat{\mathcal{R}}} = \operatorname{int}_{\mathcal{R}})$.

THEOREM 5.18. *If f is a function from an arbitrary relator space $X(\mathcal{R})$ into a topological one $Y(\mathcal{S})$, then the following assertions are equivalent:*

(i) *f is $(\hat{\mathcal{R}}, \mathcal{S})$-continuous;*
(ii) *$A \in \mathcal{F}_{\mathcal{S}}$ implies $f^{-1}(A) \in \mathcal{F}_{\mathcal{R}}$;*
(iii) *$A \in \mathcal{T}_{\mathcal{S}}$ implies $f^{-1}(A) \in \mathcal{T}_{\mathcal{R}}$.*

PROOF. If (i) holds and $A \in \mathcal{F}_{\mathcal{S}}$, then by (iii) in Theorem 5.15, we have

$$\operatorname{cl}_{\mathcal{R}}(f^{-1}(A)) \subset f^{-1}(\operatorname{cl}_{\mathcal{S}}(A)) = f^{-1}(A),$$

whence $f^{-1}(A) \in \mathcal{F}_{\mathcal{R}}$ follows.

Conversely, if (ii) holds and $A \subset Y$, then by Theorem 2.11 and Remark 2.19, we have

$$\mathrm{cl}_{\mathcal{R}}(f^{-1}(A)) \subset \mathrm{cl}_{\mathcal{R}}(f^{-1}(\mathrm{cl}_{\mathcal{S}}(A))) = f^{-1}(\mathrm{cl}_{\mathcal{S}}(A)),$$

whence, again by (iii) in Theorem 5.15, (i) follows.

The equivalence of (ii) and (iii) is an immediate consequence of Theorem 2.20.

COROLLARY 5.19. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on X such that* $\mathcal{S}$ *is topological, then the following assertions are equivalent:*

(i) $\mathcal{S} \subset \hat{\mathcal{R}}$; (ii) $\mathcal{F}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{R}}$; (iii) $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$.

REMARK 5.20. Note that the implications (i)⇒(ii)⇔(iii) in Theorem 8.15 and Corollary 5.19 do not require $\mathcal{S}$ to be topological.

COROLLARY 5.21. *If* $\mathcal{R}$ *is a topological relator on X, then* $\hat{\mathcal{R}}$ *is the largest topological relator on X such that* $\mathcal{F}_{\hat{\mathcal{R}}} = \mathcal{F}_{\mathcal{R}}$ $(\mathcal{T}_{\hat{\mathcal{R}}} = \mathcal{T}_{\mathcal{R}})$.

REMARK 5.22. By Corollary 5.17, it is clear that the equalities $\mathcal{F}_{\hat{\mathcal{R}}} = \mathcal{F}_{\mathcal{R}}$ and $\mathcal{T}_{\hat{\mathcal{R}}} = \mathcal{T}_{\mathcal{R}}$ are always true; and $\hat{\mathcal{R}}$ is topological if and only if $\mathcal{R}$ is topological.

Later, we shall show that a relator $\mathcal{R}$ is topological if and only if $\hat{\mathcal{R}}$ is topologically transitive in the sense that $T(S(x)) \subset R(x)$ for some $S, T \in \hat{\mathcal{R}}$ whenever $x \in X$ and $R \in \hat{\mathcal{R}}$.

The next simple example shows that the condition of topologicalness cannot be omitted from Theorem 5.18 and Corollaries 5.19 and 5.21.

EXAMPLE 5.23. Let $X = \{x, y, z\}$, and define the relators $\mathcal{R} = \{R\}$ and $\mathcal{S} = \{S\}$ on X such that $R(x) = \{x, y\}$ and $S(x) = \{x, z\}$ and $R(t) = S(t) = \{y, z\}$ if $t \in \{y, z\}$. Then, it is clear that

$$\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}} = \{\emptyset, \{y, z\}, X\},$$

but the relators $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ are still incomparable.

REMARK 5.24. Note that the relators $\mathcal{R}$ and $\mathcal{S}$ given in Example 5.23 are not topological since the relations R and S fail to be transitive.

Note also that under the small change $S(z) = \{z\}$ the relator $\mathcal{S}$ becomes topological, and we still have a reasonable example to Theorem 5.18 and Corollary 5.19.

6. Supplements to Section 5

THEOREM 6.1. *If* $\mathcal{R}$ *is a relator on X, then*

$$(\mathcal{R}^*)^{\breve{}} = (\tilde{\mathcal{R}})^* \cap (\{\Delta\}^*)^{\breve{}} \quad \textit{and} \quad (\mathcal{R}^\#)^{\breve{}} = (\tilde{\mathcal{R}})^{\wedge} \cap (\{\Delta\}^*)^{\breve{}},$$

where Δ *is the diagonal of* $X \times X$.

PROOF. We shall only prove the second assertion, since the proof of the first one is similar, but simpler. For this, suppose first that $T \in (\mathcal{R}^\#)^{\breve{}}$. Then, there exists $S\in\mathscr{R}^{\#}$ such that $T=\tilde{S}$. Moreover, for any $A\in\tilde{X}$, there exists $R\in\mathscr{R}$ such that $R(A)\subset S(A)$. This implies that $\tilde{R}(A)\subset\tilde{S}(A)=T(A)$. Consequently, $T\in(\tilde{\mathscr{R}})^{\wedge}\cap\cap(\{\Delta\}^{*})^{\sim}$.

To prove the converse inclusion, suppose now that $T\in(\tilde{\mathscr{R}})^{\wedge}\cap(\{\Delta\}^{*})^{\sim}$. Then, there exists a reflexive relation S on X such that $T=\tilde{S}$. Moreover for any $A\in\tilde{X}$, there exists $R\in\mathscr{R}$ such that $\tilde{R}(A)\subset T(A)=\tilde{S}(A)$. This implies $R(A)\subset S(A)$. Consequently, we have $S\in\mathscr{R}^{\#}$ and hence $T=\tilde{S}\in(\mathscr{R}^{\#})^{\sim}$.

REMARK 6.2. To formulate a similar assertion for the relator $(\hat{\mathscr{R}})^{\sim}$ a weakening of the topological refinement should be introduced.

The importance of the above partial compatibility of the hyperspace operation $\sim$ with the refinement operations $*$, $\#$ and $\hat{}$ lies main lyin the next reduction

THEOREM 6.3. *If f is a relation from a relator space $X(\mathscr{R})$ into another $Y(\mathscr{S})$, then*

(i) *f is mildly $(\mathscr{R}^{*},\mathscr{S})$-continuous if and only if Φ_f is $((\tilde{\mathscr{R}})^{*},\tilde{\mathscr{S}})$-continuous;*
(ii) *f is mildly $(\mathscr{R}^{\#},\mathscr{S})$-continuous if and only if Φ_f is $((\tilde{\mathscr{R}})^{\wedge},\tilde{\mathscr{S}})$-continuous.*

PROOF. This is an immediate consequence of Theorems 4.13 and 6.1 and the second assertion in Proposition 4.12. Namely, for instance, by the above mentioned results, it is clear that f is mildly $(\mathscr{R}^{\#},\mathscr{S})$-continuous iff Φ_f is $((\mathscr{R}^{\#})^{\sim},\tilde{\mathscr{S}})$-continuous iff Φ_f is $((\tilde{\mathscr{R}})^{\wedge}\cap(\{\Delta\}^{*})^{\sim},\tilde{\mathscr{S}})$-continuous iff Φ_f is $((\tilde{\mathscr{R}})^{\wedge},\tilde{\mathscr{S}})$-continuous.

Combining Theorems 4.13 and 6.3 with the results of Section 5, one can formulate several criteria for mild continuities of a relation from one relator space into another.

Among these criteria, it seems appropriate to mention here only the ones contained in the next

THEOREM 6.4. *If f is a relation from a relator space $X(\mathscr{R})$ into another $Y(\mathscr{S})$, then the following assertions are equivalent:*

(i) *f is mildly $(\mathscr{R}^{\#},\mathscr{S})$-continuous;*
(ii) *$B\in\mathrm{Cl}_{\mathscr{R}}(A)$ implies $f(B)\in\mathrm{Cl}_{\mathscr{S}}\big(f(A)\big)$;*
(iii) *$A\in\underset{\alpha}{\lim}{}_{\tilde{\mathscr{R}}}A_{\alpha}$ implies $f(A)\in\underset{\alpha}{\lim}{}_{\tilde{\mathscr{S}}}f(A_{\alpha})$.*

PROOF. By Theorems 4.13, 5.7 and 4.9, it is clear that (i) holds iff φ_f is $(\mathscr{R}^{\#},\tilde{\mathscr{S}})$-continuous iff $B\in\mathrm{Cl}_{\mathscr{R}}(A)$ implies $\varphi_f(B)\in\mathrm{Cl}_{\tilde{\mathscr{S}}}\big(\varphi_f(A)\big)$ iff (ii) holds. Namely, $\varphi_f(A)=\{f(x)\}_{x\in A}$ and $f(A)=\bigcup\limits_{x\in A}f(x)$.

On the other hand, by using Theorems 6.3 and 5.11, one can easily check that (i) holds iff Φ_f is $((\tilde{\mathscr{R}})^{\wedge},\tilde{\mathscr{S}})$-continuous iff $A\in\underset{\alpha}{\lim}{}_{\tilde{\mathscr{R}}}A_{\alpha}$ implies $\Phi_f(A)\in\underset{\alpha}{\lim}{}_{\tilde{\mathscr{S}}}\Phi_f(A_{\alpha})$ iff (iii) holds.

REMARK 6.5. It is a striking fact that the sets A_{α} can be replaced by singletons in (iii).

Namely, if (i) does not hold, then there exist $S\in\mathcal{S}$ and $A\subset X$ such that for each $R\in\mathcal{R}$ there exists $x_R\in R(A)$ such that $x_R\notin(f^{-1}\circ S\circ f)(A)$, i.e., $f(x_R)\cap \cap S(f(A))=\emptyset$. Hence, by preordering $\mathcal{R}$ with the reverse set inclusion, one can easily check that $(\{x_R\})_{R\in\mathcal{R}}$ is a net in $\tilde{X}$ such that $A\in\lim_{R}{}_{\tilde{\mathcal{R}}}\{x_R\}$, but $f(A)\notin\lim_{R}{}_{\tilde{\mathcal{S}}} f(x_R)$.

The above fact was formerly utilized by Hušek [41] in defining convergence classes for generalized proximities.

To derive assertions for various mild continuities of composite relations, the next theorem is also needed.

THEOREM 6.6. *If f is a relation from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$, then the following assertions hold:*

(i) *If f is mildly $(\mathcal{R}^*,\mathcal{S})$-continuous, then f is also mildly $(\mathcal{R}^*,\mathcal{S}^*)$-continuous.*

(ii) *If f is mildly $(\mathcal{R}^\#,\mathcal{S})$-continuous, then f is also mildly $(\mathcal{R}^\#,\mathcal{S}^\#)$-continuous.*

(iii) *If f is mildly $(\hat{\mathcal{R}},\mathcal{S})$-continuous, then f is mildly $(\hat{\mathcal{R}},\mathcal{S}^\#)$-continuous. Moreover, if in addition f is a function, then f is also $(\hat{\mathcal{R}},\hat{\mathcal{S}})$-continuous.*

PROOF. Straightforward computation. For instance, we prove the first part of (iii). For this, suppose that f is mildly $(\hat{\mathcal{R}},\mathcal{S})$-continuous and pick $T\in\mathcal{S}^\#$ and $x\in X$. Then, there exists $S\in\mathcal{S}$ such that $S(f(x))\subset T(f(x))$. Moreover, since $f^{-1}\circ S\circ f\in\hat{\mathcal{R}}$, there exists $R\in\mathcal{R}$ such that $R(x)\subset(f^{-1}\circ S\circ f)(x)$. Hence, it follows that $R(x)\subset(f^{-1}\circ T\circ f)(x)$. Consequently, $f^{-1}\circ T\circ f\in\hat{\mathcal{R}}$ also holds, which proves the mild $(\hat{\mathcal{R}},\mathcal{S}^\#)$-continuity of f.

Finally, we prove an interesting property of the topological refinement $\hat{\mathcal{R}}$ of a relator $\mathcal{R}$.

THEOREM 6.7. *If A is a set in a relator space $X(\mathcal{R})$ then*

$$\mathrm{Cl}_{\hat{\mathcal{R}}}(A)=\{B\subset X\colon\ B\cap\mathrm{cl}_{\mathcal{R}}(A)\neq\emptyset\}$$

and

$$\mathrm{Int}_{\hat{\mathcal{R}}}(A)=\{B\subset X\colon\ B\subset\mathrm{int}_{\mathcal{R}}(A)\}.$$

PROOF. If $B\in\mathrm{Int}_{\hat{\mathcal{R}}}(A)$, then there exists $S\in\hat{\mathcal{R}}$ such that $S(B)\subset A$. Moreover, since $S\in\hat{\mathcal{R}}$, for any $x\in B$, there exists $R\in\mathcal{R}$ such that $R(x)\subset S(x)$. Hence, since $S(x)\subset S(B)$ for any $x\in B$, it is clear that $B\subset\mathrm{int}_{\mathcal{R}}(A)$.

To prove the converse implication, suppose now that $B\subset\mathrm{int}_{\mathcal{R}}(A)$. Then, for each $x\in B$, there exists $R_x\in\mathcal{R}$ such that $R_x(x)\subset A$. Define the relation S on X such that

$$S(x)=R_x(x)\quad\text{if}\quad x\in B\quad\text{and}\quad S(x)=X\quad\text{if}\quad x\in X\setminus B.$$

Then, it is clear that $S\in\hat{\mathcal{R}}$ and

$$S(B)=\bigcup_{x\in B}S(x)=\bigcup_{x\in B}R_x(x)\subset A.$$

Consequently, $B\in\mathrm{Int}_{\hat{\mathcal{R}}}(A)$.

The first assertion of the theorem can be easily derived from the second one by using Theorems 2.6 and 2.13.

REMARK 6.8. The above theorem strongly suggests that the relations $\mathrm{Lim}_{\mathscr{R}^{\#}}$ and $\mathrm{Adh}_{\mathscr{R}^{\#}}$ can also be expressed in terms of the relations $\mathrm{Cl}_{\mathscr{R}}$ and $\mathrm{Int}_{\mathscr{R}}$. However, we could not prove or disprove this.

The results of Sections 5 and 6 make it natural to introduce the following terminology which will be needed in some subsequent papers.

DEFINITION 6.9. A relation f from a relator space $X(\mathscr{R})$ into another $Y(\mathscr{S})$ will be called mildly uniformly, proximally, resp. topologically continuous if it is mildly $(\mathscr{R}^*, \mathscr{S})$-, $(\mathscr{R}^{\#}, \mathscr{S})$-, resp. $(\hat{\mathscr{R}}, \mathscr{S})$-continuous.

REMARK 6.10. The corresponding comparisons of relators $\mathscr{R}$ and $\mathscr{S}$ defined on the same set X is to be defined accordingly.

For instance, we say that $\mathscr{S}$ is uniformly finer than $\mathscr{R}$, resp. $\mathscr{S}$ is uniformly equivalent to $\mathscr{R}$ if $\mathscr{R} \subset \mathscr{S}^*$, resp. $\mathscr{R}^* = \mathscr{S}^*$.

Similarly, we call a relator $\mathscr{R}$ on X, or a relator space $X(\mathscr{R})$, uniformly, proximally, resp. topologically fine if $\mathscr{R}^* = \mathscr{R}$, $\mathscr{R}^{\#} = \mathscr{R}$, resp. $\hat{\mathscr{R}} = \mathscr{R}$.

Some notes

NOTE 1. The basic tools introduced in Sections 1 and 2 make it possible to consider relator spaces as certain generalized convergence, proximity, closure or topological spaces.

To regard a relator space $X(\mathscr{R})$ as a generalized biperfect syntopogenous space (which is to be defined by omitting the axioms (S_1) and (S_2) of Császár [16, p. 58]), we have to consider the finer tool $\{\mathrm{Int}_R\colon R \in \mathscr{R}\}$ instead of $\mathrm{Int}_{\mathscr{R}} = \bigcup\limits_{R \in \mathscr{R}} \mathrm{Int}_R$.

In this respect, it is also worth mentioning that if $\mathscr{R}$ is a relator on X, then $\{\mathrm{Cl}_R\colon R \in \mathscr{R}\}$ is a relator on $\mathscr{P}(X) \setminus \{\emptyset\}$ whose inverse served in Section 4 as the mild hyperrelator induced by $\mathscr{R}$. This suggests that Császár's very general theory could also be based on relators instead of syntopogenous structures.

For this, note that if $<$ is a relation on $\mathscr{P}(X)$ such that

(i) $\emptyset < A < X$ if $A \subset X$,

(ii) $A \not< X \setminus A$ if $\emptyset \neq A \subset X$,

then by defining

$$R_<(A) = \{B \subset X\colon A \not< X \setminus B\}$$

for all $A \subset X$, we get a reflexive relation $R_<$ on $\mathscr{P}(X) \setminus \{\emptyset\}$ such that for any $A, B \subset X$, $A < B$ iff $X \setminus B \notin R_<(A)$. Moreover, note that each reflexive relation R on $\mathscr{P}(X) \setminus \{\emptyset\}$ can be obtained in this manner.

NOTE 2. If $\mathscr{R}$ is a relator on X, then by introducing the relator

$$\mathscr{R}_A = \{\Delta_X \subset S \subset X \times X\colon \exists R \in \mathscr{R}\colon R(A) \subset S(A)\}$$

for any $A \subset X$, one can also prove some localized forms of Theorems 5.7, 5.11 and 5.15. Our conjecture is that a similar thing can also be done in connection with Theorem 5.2 too by using appropriate refinements of $\mathscr{R}$ defined by nets.

More generally, if $\mathscr{R}$ is a relator on X, then one can also introduce the relators

$$\mathscr{R}^*_{\mathscr{A}} = \{\Delta_X \subset S \subset X \times X\colon\ \exists R \in \mathscr{R}\colon\ \forall A \in \mathscr{A}\colon\ R(A) \subset S(A)\}$$

and

$$\mathscr{R}^{\#}_{\mathscr{A}} = \{\Delta_X \subset S \subset X \times X\colon\ \forall A \in \mathscr{A}\colon\ \exists R \in \mathscr{R}\colon\ R(A) \subset S(A)\}$$

for any $\mathscr{A} \subset \mathscr{P}(X)$. Note that all the relators $\mathscr{R}^*$, $\mathscr{R}^{\#}$, $\hat{\mathscr{R}}$ and $\mathscr{R}_A$ are particular instances of some of the relators $\mathscr{R}^*_{\mathscr{A}}$ and $\mathscr{R}^{\#}_{\mathscr{A}}$. Moreover, the necessity of a particular case of $\mathscr{R}^{\#}_{\mathscr{A}}$ has already been indicated in Remark 6.2.

Note 3. Finally, we remark that for any $\mathscr{A} \subset \mathscr{P}(X)$ one can also consider the "Pervin relator" [78]

$$\mathscr{R}_{\mathscr{A}} = \{R_A\colon\ A \in \mathscr{A}\},$$

where R_A means the relation defined on X such that $R_A(x) = A$ if $x \in A$ and $R_A(x) = X$ if $x \in X \setminus A$. In connection with $\mathscr{R}_{\mathscr{A}}$, one can easily prove that $\mathscr{R}_{\mathscr{A}}$ is a "strongly transitive" relator on X such that $\mathscr{A} \subset \mathscr{T}_{\mathscr{R}_{\mathscr{A}}}$. Moreover, $\mathscr{A} = \mathscr{T}_{\mathscr{R}_{\mathscr{A}}}$ if and only if $\mathscr{A}$ is a generalized topology on X in the sense that $\mathscr{A}$ contains X and is closed under arbitrary unions.

Hence, by Corollary 5.21, it is clear that the mapping $\mathscr{R} \to \mathscr{T}_{\mathscr{R}}$ establishes a one-to-one correspondence between topologically fine topological relators on X and generalized topologies on X. This latter statement may be considered as an extension of the "Topology Theorem" of Nakano—Nakano [73, p. 204].

Namely, by Remark 5.22, a topologically fine relator is topological if and only if it is topologically transitive. On the other hand, by Remarks 2.12 and 2.19, for a topological relator $\mathscr{R}$, the family $\mathscr{T}_{\mathscr{R}}$ is an ordinary topology if and only if $\mathscr{R}$ is topologically directed. Moreover, as we shall see later, a topologically fine relator is uniformly directed if and only if it is topologically directed.

References

[1] E. M. Alfsen and O. Njåstad, Proximity and generalized uniformity, *Fund. Math.*, **52** (1963), 235—252.

[2] E. M. Alfsen and O. Njåstad, Totality of uniform structures with linearly ordered base, *Fund. Math.*, **52** (1963), 253—256.

[3] A. Appert, Espaces uniformes généralisés, *Comptes Rendus,* **222** (1946), 986—988.

[4] R. Badard, Fuzzy preuniform structures and the structures they induce, *J. Math. Anal. Appl.*, **100** (1984), 530—560.

[5] B. Banaschewski and J. M. Maranda, Proximity functions, *Math. Nachr.*, **23** (1961), 1—37.

[6] C. Barnhill and P. Fletcher, Topological spaces with a unique compatible quasi-uniform structure, *Arch. Math.*, **21** (1970), 206—209.

[7] C. Berge, *Topological Spaces,* Oliver and Boyd (Edinburgh, 1963).

[8] G. Berthiaume, On quasi-uniformities in hyperspaces, *Proc. Amer. Math. Soc.*, **66** (1977), 335—343.

[9] P. C. Bhakta, Some results on generation of quasi-uniformities, *Riv. Mat. Univ. Parma,* **3** (1977), 89—94.

[10] N. Bourbaki, *Topologie Générale,* Herman (Paris, 1953).

[11] D. Bushaw, *Bibliography on Uniform Topology and its Supplements* I—VI., Washington State University (Pullman, Washington, 1965—1971).

[12] P. Cameron, J. G. Hocking and S. A. Naimpally, Nearness — a better approach to continuity and limits, *Amer. Math. Monthly,* **81** (1974), 739—745.

[13] E. Čech, *Topological Spaces* (Revised edition by Z. Frolik and M. Katetov), Academia (Prague, 1966).

[14] W. Cohen and C. Goffman, A theory of transfinite convergence, *Trans. Amer. Math. Soc.*, **66** (1949), 65—74.

[15] C. H. Cook and H. R. Fischer, Uniform convergence structures, *Math. Ann.*, **173** (1967), 290—306.

[16] Á. Császár, *Foundation of General Topology*, Pergamon (London, 1963).

[17] Á. Császár, *General Topology*, Adam Hilger (Bristol, 1978).

[18] Á. Császár and R. Z. Domiaty, Fine quasi-uniformities, *Ann. Univ. Sci. Budapest*, **22—23** (1970—80), 151—158.

[19] D. W. Curtis and J. C. Mathews, *Generalized uniformities for pairs of spaces*, Topology Conference, Arizona State University, 1967 (Tempe, Arizona, 1968), 212—246.

[20] A. S. Davis, Indexed systems of neighborhoods for general topological spaces, *Amer. Math. Monthly*, **68** (1961), 886—893.

[21] A. S. Davis, The uniform continuity of continuous functions on a topological space, *Proc. Amer. Math. Soc.*, **14** (1963), 977—980.

[22] R. DeMarr and I. Fleischer, Convergence via abstract metrics, *Rev. Roum. Math. Pures Appl.*, **15** (1970), 515—519.

[23] D. Doičinov, A unified theory of topological spaces, proximity spaces and uniform spaces, *Dokl. Akad. Nauk SSSR*, **156** (1964), 21—24 (Russian).

[24] J. Dugundji, *Topology*, Allyn and Bacon (Boston, 1966).

[25] V. A. Efremovič, The geometry of proximity, *Math. Sb.*, **31** (1952), 189—200 (Russian).

[26] V. A. Efremovič and A. S. Švarc, A new definition of uniform spaces. Metrization of proximity spaces, *Dokl. Akad. Nauk SSSR*, **89** (1953), 393—396 (Russian).

[27] R. Engelking, *General Topology*, Polish Scientific Publisher (Warsaw, 1977).

[28] P. Fletcher and W. F. Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker (New York, 1982).

[29] R. Frič, On the sequential uniform structures of A. Goetz, *Proceedings of the International Symposium on Topology and its Applications*, Budva 1972 (Beograd, 1973), 90—93.

[30] Z. Frolík, Concerning topological convergence of sets, *Czechoslovak Math. J.*, **10** (1960), 168—180.

[31] S. Gacsályi, On the syntopogen characterization of continuity structures I, II, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.*, **17** (1967), 161—183, 293—310. (Hungarian).

[32] S. Gacsályi, Remark on a paper of C. J. Mozzochi, *Publ. Math. Debrecen*, **21** (1974), 295—303.

[33] W. Gähler, *Grundstrukturen der Analysis* I, II, Akademie-Verlag (Berlin, 1977—78).

[34] D. B. Gauld, Nearness — a better approach to topology, *Math. Chronicle*, **7** (1978), 84—90.

[35] A. Goetz, *Proximity-like structures in convergence spaces*, Topology Conference, Arizona State University, 1967 (Tempe, Arizona, 1968), 104—137.

[36] G. Grimeisen, Gefilterte Summation von Filtern und iterierte Grenzprozesse I, II, *Math. Ann.*, **141** (1960), 318—342; **144** (1961), 386—417.

[37] P. C. Hammer, Extended topology: The continuity concept, *Math. Mag.*, **36** (1963), 101—105.

[38] D. Harris, *Structures in Topology*, Mem. Amer. Math. Soc. 115 (1971).

[39] H. Herrlich, Topological structures, *Math. Centre Tracts*, **52** (1974), 59—122.

[40] W. N. Hunsaker and W. Lindgren, Construction of quasi-uniformities, *Math. Ann.*, **188** (1970), 39—42.

[41] M. Hušek, Generalized proximity and uniform spaces I, II, *Comment. Math. Univ. Carolinae*, **5** (1964), 247—266; **6** (1965), 119—139.

[42] J. R. Isabell, *Uniform Spaces*, Math. Surveys 12, Amer. Math. Soc. (Providence, 1964).

[43] V. M. Ivanova and A. A. Ivanov, Contiguity spaces and bicompact extensions of topological spaces, *Izv. Akad. Nauk SSSR*, **23** (1959), 613—634 (Russian).

[44] H. J. K. Junnila, Neighbornets, *Pacific J. Math.*, **76** (1978), 83—108.

[45] H. Kamel, Relational algebra and uniform spaces, *J. London Math. Soc.*, **29** (1954), 342—344.

[46] S. Kasahara, Characterizations of convergence structures, *Math. Seminar Notes*, **4** (1976).

[47] M. Katětov, On continuity structures and spaces of mappings, *Comment. Math. Univ. Carolinae*, **6** (1965), 257—278.

[48] A. K. Katsaras and C. G. Petalas, A unified theory of fuzzy topologies, fuzzy proximities and fuzzy uniformities, *Rev. Roumaine Math. Pures. Appl.*, **28** (1983), 845—856.

[49] J. L. Kelley, *General Topology*, Van Nostrand Reinhold (New York, 1955).

[50] H. Kenyon, Two theorems about relations, *Trans. Amer. Math. Soc.*, **107** (1963), 1—9.

[51] I. Konishi, On uniform topologies in general spaces, *J. Math. Soc. Japan*, **4** (1952), 166—188.

[52] R. D. Kopperman, First-order topological axioms, *J. Symbolic Logic*, **46** (1981), 475—489.

[53] V. S. Krishnan, A note on semi-uniform spaces, *J. Madras Univ.*, **25** (1955), 123—124.

[54] V. S. Krishnan, Additivity and symmetry for generalized uniform structures, and characterizations of semi-uniform structures, *J. Madras Univ.*, **25** (1955), 201—212.

[55] W. Kupla, On generalization of uniformities, *Colloq. Math.*, **25** (1972), 227—240.

[56] E. P. Lane, Bitopological Spaces and quasi-uniform spaces, *Proc. London Math. Soc.*, **17** (1967), 241—256.

[57] J. P. Lasalle, Topology based upon the concept of a pseudonorm, *Proc. Nat. Acad. Sci. U. S. A.*, **27** (1941), 448—451.

[58] S. Leader, On pseudometrics for generalized uniform structures, *Proc. Amer. Math. Soc.*, **16** (1965), 493—495.

[59] N. Levine, On Pervin's quasi-uniformity, *Math. J. Okayama Univ.*, **14** (1970), 97—102.

[60] N. Levine, A new uniformity for hyperspaces, *Yokohama Math. J.*, **20** (1972), 69—77.

[61] N. Levine and W. J. Stager, On the hyperspace of a quasi-uniform space, *Math. J. Okayama Univ.*, **15** (1972), 101—106.

[62] M. W. Lodato, On topologically induced generalized proximity relations I, II, *Proc. Amer. Math. Soc.*, **15** (1964), 417—422; *Pacific J. Math.*, **17** (1966), 131—135.

[63] Z. P. Mamuzić, *Introduction to General Topology*, Noordhoff (Groningen, 1963).

[64] J. C. Mathews and D. W. Curtis, Relatively contractive relations in pairs of generalized uniform spaces, *J. London Math. Soc.*, **44** (1969), 100—106.

[65] B. L. McAllister, Hyperspaces and multifunctions. The first half century (1900—1950), *Nieuw Arch. Wisk.*, **26** (1978), 309—329.

[66] A. G. Mordkovič, Test for correctness of uniform spaces, *Dokl. Akad. Nauk SSSR*, **169** (1966), 276—279.

[67] C. J. Mozzochi, M. S. Gagrat and S. A. Naimpally, *Symmetric Generalized Topological Structures*, Exposition Press (New York, 1976).

[68] S. Mrówka, On the uniform convergence in proximity spaces, *Bull. Acad. Polon. Sci. Cl III*, **5** (1957), 255—257.

[69] S. Mrówka, On the convergence of nets of sets, *Fund. Math.*, **45** (1958), 235—246.

[70] M. G. Murdeshwar and S. A. Naimpally, *Quasi-Uniform Topological Spaces*, Noordhoff (Groningen, 1966).

[71] L. Nachbin, *Topology and Order*, D. Van Nostrand (Princeton, 1965).

[72] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press (Cambridge, 1970).

[73] H. Nakano and K. Nakano, Connector theory, *Pacific J. Math.*, **56** (1975), 195—213.

[74] K. Nakano, A notion of completeness of topological structures, *Questiones Math.*, **2** (1977), 235—243.

[75] R. Nielsen and C. Sloyer, Quasi-uniformazibility, *Math. Ann.*, **182** (1969), 273—274.

[76] W. Page, *Topological Uniform Structures*, John Wiley and Sons (New York, 1978).

[77] W. J. Pervin, Uniformization of neighborhood axioms, *Math. Ann.*, **147** (1962), 313—315.

[78] W. J. Pervin, Quasi-uniformization of topological spaces, *Math. Ann.*, **147** (1962), 316—317.

[79] W. J. Pervin, *Foundation of General Topology*, Academic Press (New York, 1964).

[80] V. Z. Poljakov, On uniform convergence spaces, *Colloq. Math.*, **13** (1965), 167—179 (Russian).

[81] A. Pultr, Pointless uniformities I, II, *Comment Math. Univ. Carolinae*, **25** (1984), 91—120.

[82] N. S. Ramm and A. S. Švarc, Geometry of proximity, uniform geometry and topology, *Mat. Sb.*, **33** (1953), 157—180 (Russian).

[83] I. L. Reilly, On generating quasi-uniformities, *Math. Ann.*, **189** (1970), 317—318.

[84] V. J. Sandberg, A new definition of uniform spaces, *Dokl. Akad. Nauk SSSR*, **135** (1960), 535—537 (Russian).

[85] Yu. M. Smirnov, On proximity spaces, *Math. Sb.*, **31** (1952), 543—574 (Russian).

[86] R. E. Smithson, Multifunctions, *Nieuw Arch. Wisk.*, **20** (1972), 31—53.

[87] A. K. Steiner and E. F. Steiner, On semi-uniformities, *Fund. Math.*, **83** (1973), 45—58.

[88] R. A. Stoltenberg, Some properties of quasi-uniform spaces, *Proc. London Math. Soc.*, **17** (1967), 226—240.

[89] Á. Száz, Generalized preseminormed spaces of linear manifolds and bounded linear relations, *Rev. Res. Fac. Sci. Math. Ser. Novi Sad*, **14** (1984), 49—78.

[90] Á. Száz, Coherences instead of convergences, *Proceedings of the Conference on Convergence and Generalized Functions* (Katowice, 1983), Institute of Mathematics, Polish Academy of Sciences (Warsaw, 1984), 141—148.

[91] Á. Száz, Continuities in relator spaces, *Abstracts of the International Conference on Generalized Functions* (Debrecen, Hungary, 1984), 53—54.

[92] D. V. Thampuran, Extended topology: Moore—Smith convergence, *Portugal. Math.*, **24** (1965), 83—98.
[93] D. V. Thampuran, Regularity for bitopological spaces, *Publ. Math. Debrecen*, **20** (1973), 41—44.
[94] W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston (New York, 1966).
[95] J. W. Tukey, *Convergence and Uniformity in Topology*, Ann. Math. Studies 2, Princeton University Press (Princeton, 1940).
[96] C. Votaw, Uniqueness of compatible quasi-uniformities, *Canad. Math. Bull.*, **15** (1972), 575—583.
[97] A. Weil, *Sur les espaces á structure uniforme at sur la topologie générale*, Actual Sci. Ind., 551, Hermann and Cie (Paris, 1937).
[98] J. Williams, Locally uniform spaces, *Trans. Amer. Math. Soc.*, **168** (1972), 435—469.

(Received August 17, 1983; revised June 16, 1986)

MATHEMATICAL INSTITUTE
LAJOS KOSSUTH UNIVERSITY
H—4010 DEBRECEN

Acta Math. Hung.
50 (3—4) (1987), 203—225.

FIXING SYSTEM AND HOMOTHETIC COVERING

S. FUDALI (Szczecin)

Introduction

Fejes Tóth [4] has introduced the notion of *the primitive fixing system* for an open (n-dimensional) convex body K as a set $A \subset \mathrm{bd}\, K$ which stabilizes K with respect to any translation and no subset of A does. In other words, it is a set A of points $a_0, a_1, \dots, a_{p-1}$ not belonging to the interior of K which has the following properties: 1^0 for every translation T there exists a point $a_i \in A$ such that the segment $\langle a_i, T(a_i) \rangle$ and the image $T(K)$ of K has a non-empty intersection, 2^0 for any $p-1$ points of A there exists a translation T' such that $\langle a_j, T'(a_j) \rangle \cap T'(K) = \emptyset$ for each j.

The number of points forming a primitive fixing system for fixed $K \subset E^n$ depends on the form of K and on the position of these points in bd K; e.g. for a circle $C \subset E^2$ there exists a primitive fixing system which consists of three points (Fig. 1a) and there exists one of four points (Fig. 1b), for the regular hexagon $H \subset E^2$ there exist 3-, 4-, 5- or 6-point primitive fixing systems (Fig. 2), for an n-dimensional parallelotope $P \subset E^n$ every one of its primitive fixing systems consists of exactly $2n$ points (Fig. 3). We note [4, p. 382] that in E^n there exist bodies which cannot be stabilized by less than r points, where $r \in \{n+1, n+2, \dots, 2n\}$ (Fig. 4 for $n=3$). In [4] the following is pointed out:

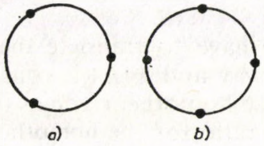

Fig. 1

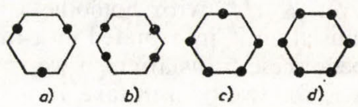

Fig. 2

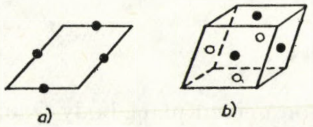

Fig. 3

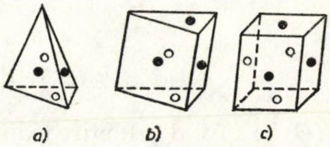

Fig. 4

REMARK FT. For any plane body, different from a parallelogram, there exists a primitive fixing system which consists of three points, while for a parallelogram every such system consists of four points.

The maximal number of points included in a primitive fixing system for fixed $K \subset E^n$ can be greater than $2n$; L. W. Danzer [3] conjectured that this number is equal to $2(2^n - 1)$ — such a primitive fixing system can be constructed for $\operatorname{Conv}(I^n \cup (-I^n))$, where I^n denotes the n-dimensional cube (Fig. 5 for $n=3$).

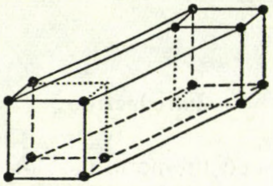

Fig. 5

Since only convex bodies will be considered in the sequel, we agree that the term body always means a convex body. Similarly only primitive fixing systems with minimal number of points for a body $K \subset E^n$ will be considered; each of them will be shortly called a fixing system and denoted by $F(K)$. It is evident that a fixing system for a given body K is determined non-uniquely; for K there exists an uncountable family $\operatorname{Fix}(K)$ of its fixing systems.

A body K' will be called homothetic to a body K if K' is an image of K under some homothety with a ray-ratio $k \in (0, 1)$. Gohberg and Markus [7] posed the problem to find the least number of homothetic bodies K' which form a covering of K; they solved this problem for any plane body.

To cover $K \subset E^n$ with homothetic bodies we have to translate them in E^n. So, if a translation is interpreted in a mechanical way and $F(K)$ is considered, its points impede these translations if we translate these homothetic copies from "outside". This is why we cannot take an arbitrary ray-ratio for the homothetic bodies used to forming of a homothetic covering of K. We must take into account the position of the points belonging to $F(K)$. Therefore, in general, the numbers of covering bodies used to cover K are different depending on whether $F(K)$ is or is not considered. Hence a problem appears: what is the least number of congruent bodies K' homothetic to K which can cover K stabilized by $F(K)$? In the present paper an answer to this question for a plane body Q is given.

1. Preliminaries

Let $F(Q)$ be a fixed primitive fixing system for a given plane body Q and denote by χ_s^k a homothety with centre $s \in Q$ and a ray ratio $k \in (0, 1)$, and by $W_Q(F)$ the set of numbers k with the following property: for every translation T there exists a point $a_i \in F(Q)$ $(i \in \{0, 1, \dots, r-1\})$ such that $\operatorname{int} T(\chi_s^k(Q)) \cap \langle a_i, T(a_i) \rangle \neq \emptyset$. The number $w_Q(F) \overset{\mathrm{df}}{=} \inf W_Q(F)$ will be called *the holding coefficient* of $F(Q)$.

It is evident that $w_Q(F) \notin W_Q(F)$. For $w_Q(F)$ we have

$$\operatorname{int} T'(\chi_s^{w_Q(F)}(Q)) \cap \langle a_i, T'(a_i) \rangle = \emptyset \quad \text{and} \quad T'(\chi_s^{w_Q(F)}(Q)) \cap \langle a_i, T'(a_i) \rangle \neq \emptyset$$

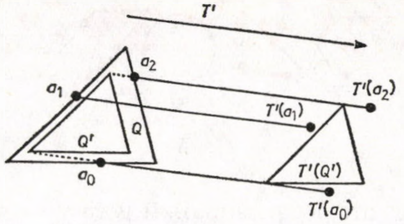

Fig. 6

for some translation T'; the last set may consist of some points only (Fig. 6). Therefore, the holding coefficient can be interpreted geometrically as such a ray ratio of homothety which allows some translation of a copy $w_Q(F)\cdot Q$ among the points of $F(Q)$; a copy $(w_Q(F)+\eta)\cdot Q$, for an arbitrary $\eta>0$, is detained by some points of $F(Q)$. In other words, $w_Q(F)$ is the greatest ray ratio of homothety which admits to translate a copy of Q out of the points of $F(Q)$.

Note that the holding coefficient of any $F(Q)=\{a_0, a_1, \dots, a_{p-1}\}$ $(p\in\{3, 4\})$ for a plane body Q can be interpreted as $\max(w_0(F), w_1(F), \dots, w_{p-1}(F)) \overset{\text{df}}{=} \bar{w}(F)$, where

$$w_i(F)=\frac{\operatorname{dist}(a_i, a_{i+1})}{D(a_i, a_{i+1})} \tag{1}$$

for each $i\in\{0, 1, \dots, p-1\}$; the addition of indices is taken $\operatorname{mod} p$ and $D(a_i, a_{i+1})$ denotes the length of the greatest chord of Q parallel to the segment $\langle a_i, a_{i+1}\rangle$. The number (1) will be called *the passing coefficient* of $F(Q)$ with respect to $\langle a_i, a_{i+1}\rangle$.

DEFINITION. Let $\{F^t(Q)\}_{t\in T}$ be the family of all primitive fixing systems for a body $Q\subset E^n$ and let $W(Q)$ denote the set of holding coefficients of the systems $F^t(Q)$ for all $t\in T$. A system $F^{t_0}(Q)\in\{F^t(Q)\}_{t\in T}$ for which $w_Q(F^{t_0})=\inf W(Q)$ will be called *a pessimal fixing system* for Q and denoted by $F_0(Q)$; the number $\inf W(Q)$ will be shortly denoted by $w(Q)$ and will be called *the coefficient* of Q.

Immediately we have

REMARK 1. For any body $Q\subset E^n$ there exist a pessimal fixing system and the coefficient of Q.

In Fig. 7 we give the pessimal system for a regular hexagon H, a triangle T, a circle C and for a parallelogram P; the coefficient of the respective bodies has the value $w(H)=\frac{3}{4}$, $w(T)=\frac{1}{2}$, $w(C)=\frac{\sqrt{3}}{2}$ and $w(P)=\frac{1}{2}$. The fixing systems in

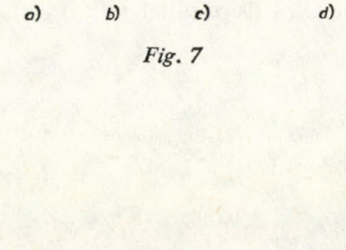

Fig. 7

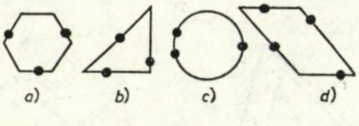

Fig. 8

Fig. 8 for the same bodies are not pessimal. It is easy to see that for T and P the pessimal fixing system is determined uniquely, for H there exist two such systems, and for C we have uncountable many of them.

Let $k^s(Q)$ be the s-th liminal number of Q and let Q' be homothetic to Q with the ray ratio $k^s(Q)\in(0,1)$. (Recall [5] that *the s-th liminal number* of Q is the least upper bound of the set of those real numbers $r\in(0,1)$ such that s congruent bodies, homothetic to Q with the ray ratio r, cannot form a covering of Q.) It is evident that $F(Q)$ is not a fixing system for Q'; it will be called *a holding system* for Q' if $w_Q(F)<k^s(Q)$ and *an impedient system* for Q' if $w_Q(F)\geqq k^s(Q)$. In the first case for every translation T there exists $a_i\in F(Q)$ such that $\operatorname{int} T(Q')\cap\langle a_i, T(a_i)\rangle\neq\emptyset$, and in the second one there exists a translation T' such that for each $a_j\in F(Q)$ we have $\operatorname{int} T'(Q')\cap\langle a_j, T'(a_j)\rangle=\emptyset$. Whether $F(Q)$ is a holding or an impedient system for Q' depends on the ray ratio for Q' and on the choice of $F(Q)$ in $\operatorname{Fix}(Q)$.

2. Some remarks about a fixing system of a plane body

Consider any plane body Q different from a parallelogram. A point $\hat{a}\in\operatorname{bd} Q$ will be called *quasi-antipodal* to a point $a\in\operatorname{bd} Q$ if a supporting straight line of Q passing throught $\hat{a}$ is parallel to one passing through a. (If there is only one such point $\hat{a}$, then it is called *antipodal* to a.) The segment $\langle a, \hat{a}\rangle$ is the greatest chord of Q among all chords of Q parallel to the straight line $(a, \hat{a})$. By Remark FT there exists a fixing system for Q which consists of three points. For such fixing system we have

REMARK 2. In a three-point fixing system for a plane body Q no pair of its points is a pair of quasi-antipodal points.

In other words: no two points of the mentioned system belong to the parallel supporting lines of Q.

Fix any two not quasi-antipodal points $a_{i-1}, a_i\in\operatorname{bd} Q$ $\big(i\in\{0,1,2\}$, addition is taken mod 3$\big)$ such that the curvature of an arc $\overset{\frown}{a_{i-1}a_i}\subset\operatorname{bd} Q$ is positive and less than π. Moreover, let $\overset{\frown}{\hat{a}_{i-1}\hat{a}_i}\subset\operatorname{bd} Q$ always denote the arc disjoint from $\overset{\frown}{a_{i-1}a_i}$ and containing only one point quasi-antipodal to a_j $\big(j\in\{i-1, i\}\big)$, and let $a_{i+1}\in\operatorname{bd} Q$ be an arbitrary point.

REMARK 3. The set $\{a_{i-1}, a_i, a_{i+1}\}\subset\operatorname{bd} Q$ forms a fixing system for a plane body Q if and only if $a_{i+1}\in\operatorname{int}\overset{\frown}{\hat{a}_{i-1}\hat{a}_i}$.

REMARK 4. If a_{i-1}, a_i belong to $F(Q)$ for a plane body Q, then one and only one endpoint of the greatest chord of Q parallel to $\langle a_j, a_{i+1}\rangle$ $\big(j\in\{i-1, i\}, a_{i+1}\in F(Q)\big)$ belongs to $\overset{\frown}{\hat{a}_{i-1}\hat{a}_i}$.

Introduce a parameter $t \in T \subset R$ on the arc $\overset{\smile}{\hat{a}_{i-1}\hat{a}_i} \subset \mathrm{bd}\, Q$ and denote by a^t_{i+1} each point of $\overset{\smile}{\hat{a}_{i-1}\hat{a}_i}$ corresponding to the value of t. By Remark 3 we get a subfamily $\overset{\frown}{\mathrm{Fix}}(Q) \subset \mathrm{Fix}(Q)$ each member of which consists of a_{i-1}, a_i, a^t_{i+1}; i.e. $F^t(Q) = \{a_{i-1}, a_i, a^t_{i+1}\}$. Hence $w_i(F^t) = \dfrac{\mathrm{dist}\,(a_i, a^t_{i+1})}{D(a_i, a^t_{i+1})}$ (cf. (1)) $\left(\text{and } w_{i+1}(F^t) = \dfrac{\mathrm{dist}\,(a^t_{i+1}, a_{i-1})}{D(a^t_{i+1}, a_{i-1})}, \text{ too}\right)$ is some function of t. It is easy to see that this function is continuous on T (i.e. on the arc $\overset{\smile}{\hat{a}_{i-1}\hat{a}_i}$) and assumes a value near to 1 if a^t_{i+1} belongs to an ε-neighbourhood of $\hat{a}_i$; then the value of $w_{i+1}(F^t)$ is near some real number $q_{i+1} < 1$.

Remark 5. If $a^t_{i+1} \in \mathrm{int}\, \overset{\smile}{\hat{a}_{i-1}\hat{a}_i}$ is a point moving from $\hat{a}_{i-1}$ to $\hat{a}_i$, then the value of $w_i(F^t)$ varies from some real number $q_i < 1$ to 1, and the value of $w_{i+1}(F^t)$ from 1 to some number $q_{i+1} < 1$. Moreover, $w_j(F^t)$ $(j \in \{i, i+1\})$, as a continuous function, assumes all values in the interval $(q_j, 1)$.

For shortness, in the sequel, by $\overset{\smile}{pq} \subset \mathrm{bd}\, Q$ we always mean the arc not containing the points $\hat{p}$ and $\hat{q}$.

Lemma 1. *If* $a_{i-1}, a_i \in \mathrm{bd}\, Q$ $(i \in \{0, 1, 2\}$, *addition is taken* $\mathrm{mod}\, 3)$ *are fixed points in each* $F^t(Q) = \{a_{i-1}, a_i, a^t_{i+1}\} \in \mathrm{Fix}\,(Q)$, *then* $w_i(F^t)$ *is a monotonic function of t. If Q is a strictly convex body, then* $w_i(F^t)$ *is strictly monotonic.*

Proof. At first we will prove this for a strictly convex body Q. If $a_{i-1}, a_i \in \mathrm{bd}\, Q$ are fixed, then $a^t_{i+1} \in \mathrm{bd}\, Q$, with a_{i-1}, a_i forming $F^t(Q)$, belongs to $\mathrm{int}\, \overset{\smile}{\hat{a}_{i-1}\hat{a}_i}$ in view of Remark 3. Take such a point and denote by A^t_i, A^t_{i+1} the endpoints of the greatest chord of Q parallel to the segment $\langle a_i, a^t_{i+1} \rangle$. Next choose $t' \in T$ such that $a^{t'}_{i+1} \in \mathrm{int}\, \overset{\smile}{\hat{a}_{i-1}\hat{a}_i}$ and the length of $\overset{\smile}{a_i a^{t'}_{i+1}} \subset \mathrm{bd}\, Q$ is greater than that of $\overset{\smile}{a_i a^t_{i+1}}$ provided both of these arcs contain the point $\hat{a}_i$ (Fig. 9); by this choice we have $A^{t'}_{i+1} \in \overset{\smile}{a^{t'}_{i+1}\hat{a}_i}$ in view of Remark 4, and $A^{t'}_i \in \overset{\smile}{a_i A^t_i}$ because of the strictness of convexity of Q. Hence, again by the strictness of convexity of Q, the straight line $(A^{t'}_{i+1}, a^{t'}_{i+1})$ through $A^{t'}_{i+1}$ and $a^{t'}_{i+1}$ is different from (A^t_{i+1}, a^t_{i+1}) and intersects $(A^{t'}_i, a_i)$ in the point o' which lies on the same side of a_i as $o'' = (A^{t'}_i, a_i) \cap (A^t_{i+1}, a^t_{i+1})$ and

$$o' a_i > o'' a_i. \tag{2}$$

By the same reason $(A^{t'}_i, a_i)$ intersects (A^t_{i+1}, a^t_{i+1}) in o'' which is on the same side of a^t_{i+1} as $o = (A^t_i, a_i) \cap (A^t_{i+1}, a^t_{i+1})$ but at a distance from a^t_{i+1} not greater than the distance of o from a^t_{i+1}.

Consider the trapezoids $A^t_i a_i a^t_{i+1} A^t_{i+1}$ and $A^{t'}_i a_i a^{t'}_{i+1} A^{t'}_{i+1}$, and note that

$$w_i(F^t) = \frac{a_i a^t_{i+1}}{A^t_i A^t_{i+1}} = \frac{o a_i}{o A^t_i} \quad \text{and} \quad w_i(F^{t'}) = \frac{a_i a^{t'}_{i+1}}{A^{t'}_i A^{t'}_{i+1}} = \frac{o' a_i}{o' A^{t'}_i} \tag{3}$$

by the similarity of the triangles $o a_i a^t_{i+1}$, $o A^t_i A^t_{i+1}$ and the triangles $o' a_i a^{t'}_{i+1}$, $o' A^{t'}_i A^{t'}_{i+1}$.

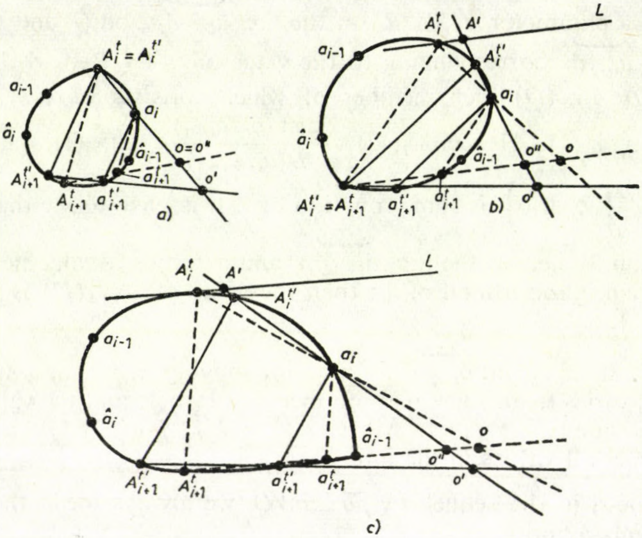

Fig. 9

If $A_i^t = A_i^{t'}$ (Fig. 9a), then both trapezoids have a common lateral side and

$$A_i^t a_i = A_i^{t'} a_i, \quad o = o''. \tag{4}$$

Using (2) and (4) we get

$$\frac{o' A_i^{t'}}{o' a_i} = \frac{o' a_i + a_i A_i^{t'}}{o' a_i} = 1 + \frac{a_i A_i^{t'}}{o' a_i} = 1 + \frac{a_i A_i^t}{o' a_i} < 1 + \frac{a_i A_i^t}{o'' a_i} = 1 + \frac{a_i A_i^t}{o a_i} =$$

$$= \frac{o a_i + a_i A_i^t}{o a_i} = \frac{o A_i^t}{o a_i}$$

which implies that $\dfrac{o' a_i}{o' A_i^{t'}} > \dfrac{o a_i}{o A_i^t}$; as a consequence we have $w_i(F^{t'}) > w_i(F^t)$ in view of (3).

If $A_i^t \neq A_i^{t'}$, then $o \neq o''$ and the considered trapezoids have at most two vertices in common: a_i and A_{i+1}^t (Fig. 9b, c). In this case draw the straight line L passing through A_i^t and parallel to (A_{i+1}^t, a_{i+1}^t) and note that L cannot intersect the interior of the arc $\overset{\frown}{a_i A_i^t}$ because (A_{i+1}^t, a_{i+1}^t) intersects $\overset{\frown}{a_{i-1} a_i}$ and A_i^t, A_{i+1}^t are antipodal (i.e. the supporting straight lines passing through A_i^t, A_{i+1}^t, respectively, are parallel). Therefore, the point $A' = (A_i^{t'}, a_i) \cap L \notin Q$ and

$$A' a_i > A_i^{t'} a_i, \tag{5}$$

and the triangles $A_i^t a_i A'$, $o a_i o''$ are similar. Thus we have

$$\frac{A_i^t a_i}{o a_i} = \frac{A' a_i}{o'' a_i}. \tag{6}$$

Taking into account (3) and using (6), (5), (2) and again (3) we get

$$\frac{1}{w_i(F^t)}=\frac{oA_i^t}{oa_i}=\frac{oa_i+a_iA_i^t}{oa_i}=1+\frac{a_iA_i^t}{oa_i}=1+\frac{A'a_i}{o''a_i}>1+\frac{A_i^{t'}a_i}{o''a_i}>1+\frac{A_i^{t'}a_i}{o'a_i}=$$

$$=\frac{o'a_i+a_iA_i^{t'}}{o'a_i}=\frac{o'A_i^{t'}}{o'a_i}=\frac{A_i^{t'}A_{i+1}^{t'}}{a_ia_{i+1}^{t'}}=\frac{1}{w_i(F^{t'})}$$

which implies $w_i(F^{t'})>w_i(F^t)$, i.e. the same as in case $A_i^t=A_i^{t'}$. This means that $w_i(F^t)$ is a strictly monotonic function.

In the general case, i.e. when $\mathrm{bd}\,Q$ contains some segment of a straight line, note that w_i is constant only in the case when $a_{i+1}^{t'}$ and $A_{i+1}^{t'}$ simultaneously belong to the interior of a segment $B\subset\mathrm{bd}\,Q$ and $A_i^t=A_i^{t'}$ (Fig. 10b). Then each of the ratios $\frac{oa_{i+1}^t}{oA_{i+1}^t}$ and $\frac{oa_{i+1}^{t'}}{oA_{i+1}^{t'}}$ is equal to $\frac{oa_i}{oA_i^t}$ which implies that $w_i(F^t)=w_i(F^{t'})$. If $A_i^t\neq A_i^{t'}$ (Fig. 10a) or $A_{i+1}^t=A_{i+1}^{t'}=\hat{a}_i$, then we have a situation as in the case Q is strictly convex. In none of the cases under the considered hypothesis do we get $w_i(F^t)>w_i(F^{t'})$, in view of convexity of Q. Hence w_i is monotonic.

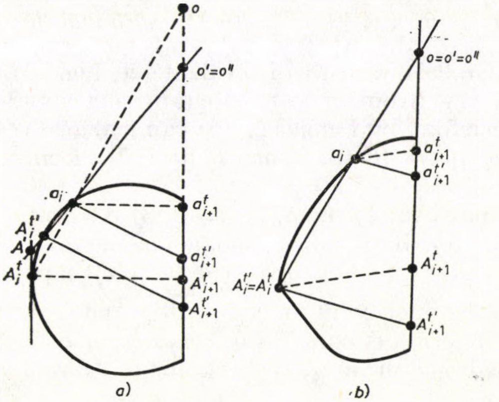

Fig. 10

Note that w_i (and w_{i+1}, too) cannot be constant on the whole of T because A_{i+1}^t for some $t\in T$ takes the place of $\hat{a}_i$ (if a_{i+1}^t tends to $\hat{a}_i$) and then for $t+\Delta t\in T$ $(\Delta t>0)$ we have $A_{i+1}^{t+\Delta t}=A_{i+1}^t$ $(A_{i+1}^d\in\overset{\frown}{\hat{a}_{i-1}\hat{a}_i}$ for each $d\in T$, in view of Remark 4) and $A_i^{t+\Delta t}\neq A_i^t$, in view of the fixing of a_i, but in this case w_i is increasing.

COROLLARY. *For any plane body Q and for the family of fixing systems $F^t(Q)=$ $=\{a_{i-1},a_i,a_{i+1}^t\}$ $(i\in\{0,1,2\})$, where $a_{i-1},a_i\in\mathrm{bd}\,Q$ are fixed, if a_{i+1}^t tends along $\overset{\frown}{\hat{a}_{i-1}\hat{a}_i}\subset\mathrm{bd}\,Q$ from $\hat{a}_{i-1}$ to $\hat{a}_i$, then $w_{i+1}(F^t)$ is decreasing at least in some η-neighbourhood of $\hat{a}_{i-1}$ and $w_i(F^t)$ is increasing at least on some ε-neighbourhood of $\hat{a}_i$. For a strictly convex body Q $w_{i+1}(F^t)$ is decreasing and $w_i(F^t)$ is increasing on the whole arc $\overset{\frown}{\hat{a}_{i-1}\hat{a}_i}$.*

Lemma 2. *For any plane body Q, different from a parallelogram, there exists a fixing system $F(Q)$ for which $w_i(F)=w_{i+1}(F)\overset{\mathrm{df}}{=}B$ $\big(i\in\{0,1,2\}$, addition is taken* mod 3$\big)$, *where $B\in(b,1)$ for some real b.*

Proof. Let $\{a_{i-1}^d, a_i^d, a_{i+1}^d\}\in\mathrm{Fix}\,(Q)$ be an arbitrary fixing system for Q. This means, by Remark 3, that a_{i+1}^d belongs to the interior of the arc $\overset{\frown}{\hat{a}_{i-1}^d\hat{a}_i^d}\subset\mathrm{bd}\,Q$, where $\hat{a}_{i-1}^d$, $\hat{a}_i^d$ are quasi-antipodal to a_{i-1}^d, a_i^d, respectively. Fix the points a_{i-1}^d, a_i^d and change the position of a_{i+1}^d on $\overset{\frown}{\hat{a}_{i-1}^d\hat{a}_i^d}$. Then, in view of Remark 5 and of Lemma 1, there exists a real number $B\in\big(\max(q_i, q_{i+1}), 1\big)$ which is the value of $w_{i+1}(F^{d'})$ and $w_i(F^{d'})$ simultaneously for some $F^{d'}(Q)=\{a_{i-1}^d, a_i^d, a_{i+1}^{d'}\}$. By the arbitrariness of the choice of $\{a_{i-1}^d, a_i^d, a_{i+1}^d\}$ we obtain a subfamily $\overline{\mathrm{Fix}}\,(Q)\subset\mathrm{Fix}\,(Q)$ for each member $F^{d'}(Q)$ of which we have $w_i(F^{d'})=w_{i+1}(F^{d'})\overset{\mathrm{df}}{=}B^{d'}$. The values $B^{d'}$ for all d' form a set $\mathscr{B}(Q)$. By the compactness of $\mathrm{bd}\,Q$ there exists $\inf\mathscr{B}(Q)=b\notin\mathscr{B}(Q)$. Hence for each d we have $B^{d'}>b$ and simultaneously, by the definition of $w_i(F)$ (cf. (1)), we have $B^{d'}<1$.

Theorem 1. *For any plane body Q there exists a fixing system $F(Q)=\{a_0, a_1, \ldots, a_{p-1}\}$ $\big(p\in\{3,4\}\big)$ for which $w_0(F)=w_1(F)=\ldots=w_{p-1}(F)\overset{\mathrm{df}}{=}D$, where D is the minimal value of w_s $\big(s\in\{0,1,\ldots,p-1\}\big)$ such that this equality holds.*

Proof. For a parallelogram this is evident (cf. Fig. 7d). Hence consider any plane body Q different from a parallelogram and its 3-point fixing system (cf. Remark FT). Therefore, by Lemma 2, one can find an $F(Q)=\{a_0, a_1, a_2\}\subset\mathrm{bd}\,Q$ such that $w_i(F)=w_{i+1}(F)=B$ for some $i\in\{0,1,2\}$. Denote by A the value of $w_{i-1}(F)$.

Three cases are possible: 1) $A=B$, 2) $A>B$, 3) $A<B$. In the first case we just have the $F(Q)$ looked for, by a simple compactness argument. In the second case we have to decrease $w_{i-1}(F)$. Denote by a_j^t $\big(j\in\{i-1, i\}\big)$ a point not quasi-antipodal to a_{i+1} belonging to the interior of the arc $\overset{\frown}{a_j\hat{a}_{i+1}}\subset\mathrm{bd}\,Q$ which does not include a_{i+1} and such that $\langle a_{i-1}^t, a_i^t\rangle$ is parallel to $\langle a_{i-1}, a_i\rangle$, and denote by a_{i+1}^t the point of $\mathrm{bd}\,Q$ which together with a_{i-1}^t, a_i^t, by Lemma 2, forms a fixing system $F^t(Q)$ in which the equality

$$w_i(F^t)=w_{i+1}(F^t)\overset{\mathrm{df}}{=}B^t \tag{7}$$

holds (Fig. 11). It is evident, by the construction of $F^t(Q)$, that $A^t<A$, where

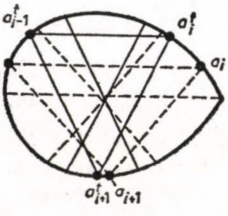

Fig. 11

$A^t=w_{i-1}(F^t)$. To prove

$$B^t \geqq B \quad (\text{i.e. } w_{i+1}(F^t) \geqq w_{i+1}(F) \quad \text{and} \quad w_i(F^t) \geqq w_i(F)), \tag{8}$$

note that by Remark 5 we have

$$w_{i+1}(F) \leqq \frac{\operatorname{dist}(a_{i+1}, a^t_{i-1})}{D(a_{i+1}, a^t_{i-1})} \overset{\text{df}}{=} w_{i+1}(\tilde{F})$$

for fixed points a_{i+1} and a_i (or a^t_i) and $w_{i+1}(\tilde{F}) \leqq w_{i+1}(F^t)$ for fixed points a^t_{i-1} and a^t_i; this implies (8). Simultaneously, by the same Remark 5,

$$w_i(F) \leqq \frac{\operatorname{dist}(a_{i+1}, a^t_i)}{D(a_{i+1}, a^t_i)} \overset{\text{df}}{=} w_i(\tilde{F})$$

for fixed points a_{i+1} and a_{i-1} (or a^t_{i-1}) and $w_i(\tilde{F}) \leqq w_i(F^t)$ for fixed points a^t_i and a^t_{i-1}. By a continuous variation of t (i.e. by a continuous variation of the positions of a^t_{i-1} and a^t_i such that $\langle a^t_{i-1}, a^t_i \rangle$ is always parallel to $\langle a_{i-1}, a_i \rangle$), in view of monotonicity of w_s $(s \in \{0, 1, 2\})$, for some t_0 one can get a position of $a^{t_0}_{i-1}$ and of $a^{t_0}_i$ such that $A^{t_0}=B^{t_0}$. Therefore $F^{t_0}(Q)$ is a fixing system for Q in which $w_0(\tilde{F}^{t_0})=$ $=w_1(F^{t_0})=w_2(F^{t_0})$.

In the third case the consideration is the same as in the second one; it differs from the previous one in the way of the choice of a^t_j. Each of them must be chosen in the interior of the arc $\overset{\frown}{a_j a_{i+1}} \subset \operatorname{bd} Q$ which does not include $\hat{a}_{i+1}$ and has to approach the respective endpoint of the greatest chord of Q parallel to $\langle a^t_{i-1}, a^t_i \rangle$.

So, we have shown that for Q there exists some $F^u(Q)=\{a^u_0, a^u_1, a^u_2\}$ for which the equality $w_0(F^u)=w_1(F^u)=w_2(F^u) \overset{\text{df}}{=} D^u$ holds; it is easy to see that D^u is the holding coefficient of $F^u(Q)$. It is possible that there exist more than one such systems as mentioned above for the fixed Q (e.g. for a triangle there exist uncountable many such systems — the points of each of them divide each side of the triangle in the same ratio, cf. Fig. 12). Hence we may consider a family $\{F^u(Q)\}_{u \in U}$ of such systems and, as a consequence, a set $\mathscr{D}(Q)=\{D^u \colon u \in U\}$. It is evident, by the compactness of $\operatorname{bd} Q$, that $\inf \mathscr{D}(Q) \overset{\text{df}}{=} D$ exists. From the next theorem it follows that D is the value of the coefficient of Q.

THEOREM 2. *$F(Q)$ is a pessimal primitive fixing system for a plane body Q iff each its passing-coefficient is equal to the coefficient of Q.*

PROOF. For a parallelogram P this is evident (Fig. 13); each point of $F_0(P)$ is the midpoint of a side of P; $w(P)=\frac{1}{2}$. Therefore consider a plane body Q different from a parallelogram; for it, by Remark 1, $w(Q)$ and $F_0(Q)$ exist; the

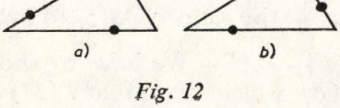

Fig. 12

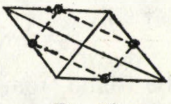

Fig. 13

latter, by Remark FT, consists of three points: a_0, a_1, a_2. According to the interpretation of $w(Q)$ for a plane body we have

$$w(Q) = \inf_{F \in \mathrm{Fix}(Q)} \{\overline{w}(F)\colon\ \overline{w}(F) = \max\big(w_0(F), w_1(F), w_2(F)\big)\}.$$

I. Let $\tilde{F}(Q)$ be a pessimal fixing system for Q. Assume that $w(Q)=w_s(\tilde{F})>w_j(\tilde{F})$ for some $s\in\{0, 1, 2\}$ $\big(s\neq j\in\{s-1, s+1\}\big)$. By moving a_{s+1} on the arc $\overset{\frown}{\hat{a}_{s-1}\hat{a}_s}$, the value of $w_s(\tilde{F})$ can be diminished, in view of Corollary after Lemma 1, and then, by Theorem 1, one can find $F'(Q)\in\mathrm{Fix}\,(Q)$ such that $w_s(F')=w_j(F')$ for each j and $w_s(F')<w_s(\tilde{F})$. This implies that $w_s(\tilde{F})$ is not $\inf\limits_{F\in\mathrm{Fix}(Q)}\{\overline{w}(F)\}$ and, as a consequence, $\tilde{F}(Q)$ is not pessimal which contradicts the hypothesis.

II. Let $w_0(F)=w_1(F)=w_2(F)=w(Q)$ for some $F(Q)=\{a_0, a_1, a_2\}\in\mathrm{Fix}\,(Q)$. Choose an arbitrary point $a_i\in F(Q)$ $\big(i\in\{0, 1, 2\}\big)$ and move it to $\hat{a}_{i-1}$ along the arc $\overset{\frown}{\hat{a}_{i+1}\hat{a}_{i-1}}\subset\mathrm{bd}\,Q$ which does not contain a_i provided a_{i-1} and a_{i+1} are fixed. Then, by Remark 5, there exists a position $a_i'\in\mathrm{int}\,\overset{\frown}{\hat{a}_{i+1}\hat{a}_{i-1}}$ such that

$$w_{i-1}(F') = \frac{\mathrm{dist}\,(a_{i-1}, a_i')}{D(a_{i-1}, a_i')} > w_{i-1}(F),$$

and by the definition of the pessimal fixing system the system $F'(Q)=\{a_{i-1}, a_i', a_{i+1}\}$ obtained is not pessimal because of $\overline{w}(F')=w_{i-1}(F')>w_{i-1}(F)=w(Q)$. This means, in view of the arbitrariness of the choice of a_i to move, that every moving of the points of $F(Q)$ leads to a fixing system the holding coefficient of which is not less than $w(Q)$. Therefore $F(Q)$ is pessimal.

The pessimal fixing systems for some bodies are given in Fig. 7. The one for an equilateral trapezoid T^p, in which the ratio of the upper base to the lower one is equal to p, is given in Fig. 14.

Proposition. *Let T^p denote an equilateral trapezoid in which C is the length of a lateral side and the ratio of the upper base to the lower one is equal to p. A fixing system for T^p is pessimal if one of its points is the midpoint of the lower base and each of the remaining two points of this system belongs to the respective lateral side of T^p and lies at a distance of $\dfrac{1-p}{2-p}C$ from the respective vertex of the upper base.*

Proof. Each of the lateral side and of the lower base of T^p contains only one point of a fixing system for T^p; the upper base of T^p does not contain any point of $F(T^p)$. Let $F(T^p)=\{a_0, a_1, a_2\}$ be a fixing system for $T^p=b_1b_2c_2c_1$ such that a_0 is the midpoint of the lower base and for each $j\in\{1, 2\}$ a_j is a point of the lateral side $\langle b_j, c_j\rangle$ such that $\mathrm{dist}\,(a_j, c_j)=\dfrac{1-p}{2-p}C$ (Fig. 14). Note that for $F(T^p)$ we have $w_0(F)=w_1(F)=w_2(F)=\overline{w}(F)$ because $w_1(F)=p+x$, where $x=\dfrac{(1-p)^2}{2-p}$, and $w_s(F)=\dfrac{1-p-x}{1-p}$ for each $s\in\{0, 2\}$; x is the ratio of a_jc_j to b_jb_0 (Fig. 14) and it may be found from the equality $w_1(F)=w_s(F)$. We have to show that for each $F'(T^p)$ different from $F(T^p)$, $\overline{w}(F')\geqq\overline{w}(F)$ which means that $F(T^p)$ is pessimal.

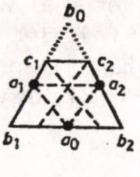

Fig. 14

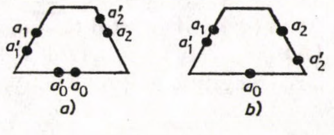

Fig. 15

Let $a_0', a_1', a_2' \in \operatorname{bd} T^p$ form a fixing system $F'(T^p)$ different from $F(T^p)$ (Fig. 15). Two cases are possible for the position of a_0', a_1', a_2' with respect to the position of a_0, a_1, a_2: 1° for some $i \in \{0, 1, 2\}$ two of the points a_0', a_1', a_2' belong to the interior of $\overset{\smile}{a_ia_{i+1}}$ (Fig. 15a); the third of these points may be not different from some point of $F(T^p)$; recall that $\overset{\smile}{ab}$ denotes always the arc $\overset{\smile}{ab} \subset \operatorname{bd} T^p$ which does not contain $\hat{a}$ and $\hat{b}$), 2° the interior of each arc $\overset{\smile}{a_ia_{i+1}}$ includes not more than one of the points a_0', a_1', a_2' (Fig. 15b). In the first case, if a_i', a_{i+1}' belong to the interior of $\overset{\smile}{a_ia_{i+1}}$, then by Thales' theorem we have $w_i(F') < w_i(F)$ because of

$$\frac{b_1a_1'}{b_1c_1} < \frac{b_1a_1}{b_1c_1} \text{ for } i=0, \quad \frac{b_0a_1'}{b_0b_1} < \frac{b_0a_1}{b_0b_1} \text{ for } i=1 \text{ and } \frac{b_2a_2'}{b_2c_2} < \frac{b_2a_2}{b_2c_2} \text{ for } i=2$$

(Fig. 16). Simultaneously, also by Thales' theorem, $w_{i-1}(F') \geqq w_{i-1}(F)$ and $w_{i+1}(F') \geqq w_{i+1}(F)$. Therefore, $w_i(F') < w_s(F')$ $(s \in \{i-1, i+1\})$ for each i which implies, in view of Theorem 2, that $F'(T^p)$ is not pessimal.

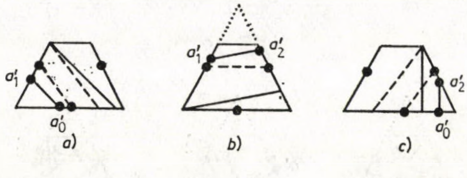

Fig. 16

In the second case three subcases are possible: (I) the interior of only one arc $\overset{\smile}{a_ia_{i+1}}$ includes one of the points a_0', a_1', a_2'; (II) the interior of only one arc $\overset{\smile}{a_ia_{i+1}}$ includes none of a_0', a_1', a_2'; (III) the interior of each arc $\overset{\smile}{a_ia_{i+1}}$ includes some of a_0', a_1', a_2'. In the first subcase, if $a_i' \in \operatorname{int} \overset{\smile}{a_ia_{i+1}}$ (Fig. 17), then by Thales' theorem we have $w_i(F') < w_i(F)$, $w_{i-1}(F') > w_{i-1}(F)$ for $i=2$, $w_i(F') = w_i(F)$, $w_{i-1}(F') >$

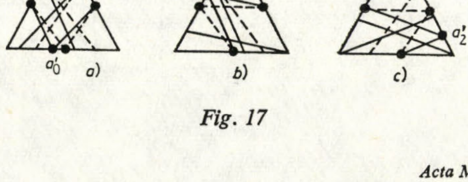

Fig. 17

$w_{i-1}(F)$ for $i=1$ and $w_i(F')=w_i(F)$, $w_{i-1}(F')\geqq w_{i-1}(F)$ for $i=0$ which implies that only for $F'(T^p)=\{a_0', a_1, a_2\}$ do we have $\overline{w}(F')\geqq\overline{w}(F)$; for $i\in\{1, 2\}$ we have $\overline{w}(F')>\overline{w}(F)$. If $a_i'\in\operatorname{int} a_{i-1}a_i$ (Fig. 18), then by Thales' theorem we have $w_i(F')\geqq w_i(F)$, $w_{i-1}(F')=w_{i-1}(F)$ for $i=0$ and $w_i(F')>w_i(F)$ for each $i\in\{1, 2\}$ which implies the same as previously.

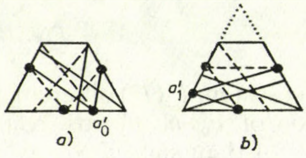

Fig. 18

In the second subcase, if $a_i'\in\operatorname{int}\overset{\smile}{a_i\hat{a}_{i-1}}$ and $a_{i+1}'\in\operatorname{int}\overset{\frown}{a_{i+1}\hat{a}_i}$ (Fig. 19), then by Thales' theorem we get $w_{i-1}(F')>w_{i-1}(F)$ for $i\in\{1, 2\}$ and $w_i(F')>w_i(F)$ for $i=0$ which means that for each $i\in\{0, 1, 2\}$ $\overline{w}(F')>\overline{w}(F)$. If $a_i'\in\operatorname{int}\hat{a}_{i+1}a_i$ and $a_{i+1}'\in\operatorname{int}\hat{a}_{i-1}a_{i+1}$ (Fig. 20), then $w_i(F')>w_i(F)$ for $i\in\{1, 2\}$ and $w_{i+1}(F')>w_{i+1}(F)$ for $i=0$ which means the same as previously. If $a_i'\in\operatorname{int}\overset{\smile}{a_i\hat{a}_{i+1}}$ and $a_{i+1}'\in\operatorname{int}\overset{\frown}{a_{i+1}\hat{a}_i}$ (Fig. 21), then $w_{i+1}(F')>w_{i+1}(F)$ for $i\in\{1, 2\}$ and $w_{i-1}(F')>w_{i-1}(F)$ for $i=0$ which means the same as previously, i.e. $\overline{w}(F')>\overline{w}(F)$.

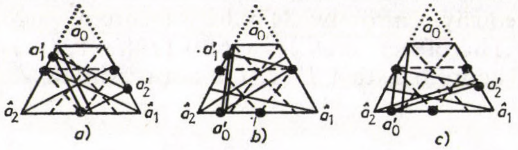

Fig. 19

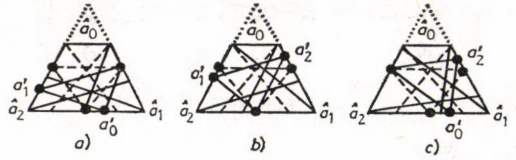

Fig. 20

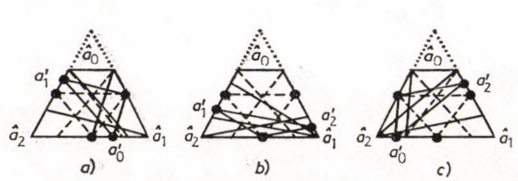

Fig. 21

In the third subcase one can observe that for each $i\in\{0, 1, 2\}$, $a_i'\in\operatorname{int}\overset{\frown}{a_i a_{i+1}}$ or $a_i'\in\operatorname{int}\overset{\frown}{a_{i-1} a_i}$ (Fig. 22). In both of these cases by Thales' theorem we have $w_i(F')>$ $>w_i(F)$ for each $i\in\{0, 1, 2\}$; this implies that $\bar{w}(F')>\bar{w}(F)$. Therefore, in each considered case we have $\bar{w}(F')\geqq\bar{w}(F)$ which means that $F(T^p)$ is pessimal.

The consideration above in the first subcase of the second case lets us to point out the following

COROLLARY. *If T^p denotes an equilateral trapezoid in which C is the length of a lateral side, B is the length of the lower base and the ratio of the upper base to the lower one is equal to p, then each fixing system for T^p in which a point belonging to a lateral side lies at a distance $\frac{1-p}{2-p}C$ from the respective vertex of the upper base and a point belonging to the lower base lies at a distance at most $\frac{p}{2(2-p)}B$ from the midpoint of the lower base, is pessimal.*

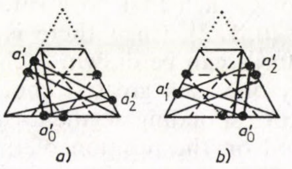

Fig. 22

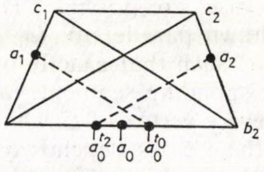

Fig. 23

Indeed, it is easy to see that if $F(T^p)$ is the pessimal fixing system for T^p mentioned in the Proposition and $F^t(T^p)=\{a_0^t, a_1, a_2\}$ differs from $F(T^p)$ only by a position of a_0 which is moved from the midpoint to an endpoint of the lower base, then the function $w_j(F^t)$ $(j\in\{0, 2\})$ is constant on the segment $\langle a_0, a_0^{t_j}\rangle$, where $a_0^{t_j}$ is a point of the lower base such that $\langle a_1, a_0^{t_0}\rangle$ is parallel to the diagonal $\langle c_1, b_2\rangle$ and $\langle a_2, a_0^{t_2}\rangle$ is parallel to $\langle c_2, b_1\rangle$ (Fig. 23). The length of $\langle a_0, a_0^{t_j}\rangle$ is equal to $\frac{p}{2(2-p)}B$.

3. Some remarks connected with the coefficient of a plane body

Let $F(Q)=\{a_0, a_1, a_2\}$ be a fixing system for any plane body Q different from a parallelogram. The greatest chord $\langle A_i^{i+2}, A_{i+1}^{i+2}\rangle$ of Q (Fig. 24), parallel to $\langle a_i, a_{i+1}\rangle$, is contained in the zone between the straight line (a_i, a_{i+1}) and the line passing through the point $a_{i+2}\in F(Q)$ and parallel to the previous one.

Let L_i be a supporting line of Q passing through $a_i\in F(Q)$. All these lines, i.e. L_0, L_1, L_2, form a triangle $C=\Delta c_0c_1c_2$ (where $c_{i+2}=L_i\cap L_{i+1}$), which will be called *the fixing triangle* for Q; it includes Q (Fig. 25). Denote by $T(a_i, a_{i+1})$ the length of the greatest chord of C parallel to the segment $\langle a_i, a_{i+1}\rangle$. Then for each

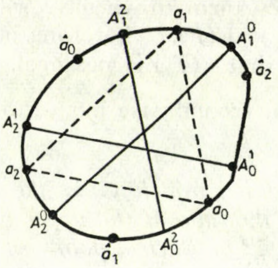

Fig. 24

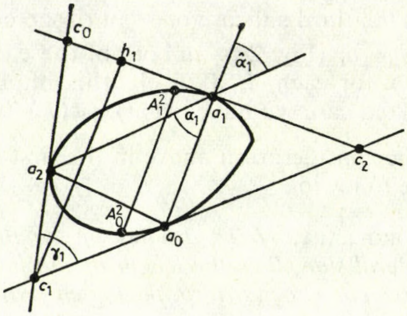

Fig. 25

$i\in\{0, 1, 2\}$ we have

$$D(a_i, a_{i+1}) \leqq T(a_i, a_{i+1}). \tag{9}$$

Two cases are possible: 1) there is a vertex of C such that no greatest chord can be drawn parallel to $\langle a_i, a_{i+1}\rangle$ for every $i\in\{0, 1, 2\}$ (then there is another vertex of C such that exactly two such greatest chords can be drawn, each parallel to a corresponding segment $\langle a_i, a_{i+1}\rangle$); 2) exactly one such greatest chord passes through every vertex of C which is parallel to a corresponding segment $\langle a_i, a_{i+1}\rangle$. Each of these cases depends on the form of Q and on the position of the points of $F(Q)$. Denote by α_i the angle of $\Delta a_0 a_1 a_2$ at the vertex a_i, by $\hat{\alpha}_i$ its opposite angle, and by γ_i the angle of C at vertex c_i. Translate γ_i by the vector $\overrightarrow{c_i a_i}$ and let $\tau(\gamma_i)$ be the image of γ_i at this translation.

REMARK 6. None of the above greatest chords of C can be drawn from the vertex c_i if $\tau(\gamma_i)\subset \operatorname{int} \hat{\alpha}_i$, and two of the above greatest chords of C can be drawn if $\tau^{-1}(\hat{\alpha}_i)\subset\gamma_i$. If $\tau(\gamma_i)$ contains only one side of $\hat{\alpha}_i$, then only one of the above greatest chords of C can be drawn from the vertex c_i.

THEOREM 3. *The coefficient of any plane body Q belongs to the segment* $\left\langle\frac{1}{2}, 1\right)$.

PROOF. For a parallelogram this is evident. Hence consider any plane body Q different from a parallelogram. Assume that

$$w(Q) < \frac{1}{2}. \tag{10}$$

This means that for every $i\in\{0, 1, 2\}$ we have

$$\operatorname{dist}(a_i, a_{i+1})/D(a_i, a_{i+1}) < \frac{1}{2}, \tag{11}$$

where $\{a_0, a_1, a_2\}=F_0(Q)$. For this $F_0(Q)$ construct the fixing triangle $C=\Delta c_0 c_1 c_2$ of Q, and denote by $T(a_i, a_{i+1})$ the length of its greatest chord parallel to the segment $\langle a_i; a_{i+1}\rangle$ (Fig. 25). It is evident that (9) holds for each i. This implies, in view of (11), that $\operatorname{dist}(a_i, a_{i+1})/T(a_i, a_{i+1}) < \frac{1}{2}$ which leads to

$$\operatorname{dist}(a_i, a_{i+1}) < \frac{1}{2} T(a_i, a_{i+1}). \tag{12}$$

By Remark 6 we have two cases: 1^0 there exists a vertex of C which is an endpoint of two of the above greatest chords of C or 2^0 each vertex of C is an endpoint of one of the above greatest chords only.

The first case, without loss of generality, is illustrated in Fig. 26. In view of (12), using similar triangles we get

$$c_1 a_0 < a_0 h_0^2 \leqq a_0 c_2 \quad \text{(taking in account the angle } a_0 c_1 a_2), \tag{13}$$

$$c_2 a_0 < a_0 h_0^1 \leqq a_0 c_1 \quad \text{(for the angle } a_1 c_2 a_0) \tag{14}$$

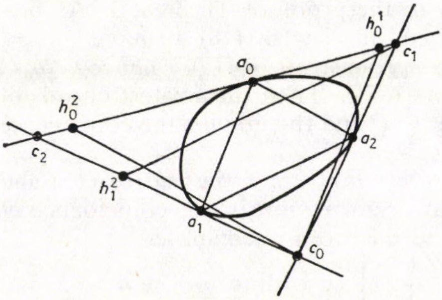

Fig. 26

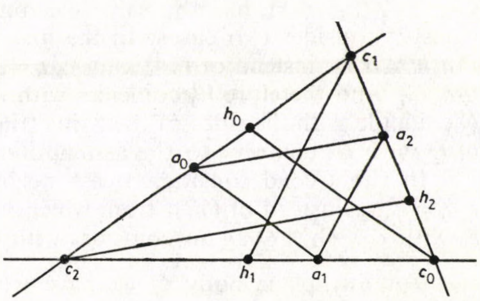

Fig. 27

which imply

$$c_1 a_0 < a_0 c_1. \tag{15}$$

Thus (10) is false, i.e. we have $w(Q) \geqq \frac{1}{2}$.

The second case if given in Fig. 27; by similar triangles, in view of (12), we get

$$c_2 a_0 < a_0 h_0 \leqq a_0 c_1 \quad \text{(for the angle } a_1 c_2 a_0) \tag{16}$$

$$c_1 a_0 < a_0 c_2 \quad \text{(for the angle } a_0 c_1 a_2) \tag{17}$$

which imply (15). This means that in this case we again have $w(Q) \geqq \frac{1}{2}$.

It is evident that if a_i is the midpoint of $\langle c_{i+1}, c_{i+2}\rangle$ for each $i \in \{0, 1, 2\}$, then $\{a_0, a_1, a_2\}$, in view of Theorem 2, is the pessimal fixing system for the triangle C. Hence we have

REMARK 7. For any triangle T we have $w(T) = \frac{1}{2}$.

Moreover, the following theorem holds.

THEOREM 4. *The class of triangles T is the unique class of plane bodies, different from a parallelogram, for which* $w(T)=\frac{1}{2}$.

PROOF. Assume the contrary, i.e. there exists a body Q different from a triangle and from a parallelogram, for which $w(Q)=\frac{1}{2}$. If $F_0(Q)$, consisting of the points a_0, a_1, a_2, is a pessimal fixing system for Q, then for every $i\in\{0, 1, 2\}$, by (9) and Theorem 3, we have

$$\operatorname{dist}(a_i, a_{i+1})/D(a_i, a_{i+1}) = \frac{1}{2}$$

and

$$\operatorname{dist}(a_i, a_{i+1})/T(a_i, a_{i+1}) = \frac{1}{2}, \tag{18}$$

where $T(a_i, a_{i+1})$ has the same meaning as in the proof of Theorem 3. As previously, consider two cases. In the first of them, in view of (18) we have $c_2 a_0 = a_0 h_0^1 \leqq a_0 c_1$ instead of (14) and $c_1 a_0 = a_0 h_0^2 \leqq a_0 c_2$ instead of (13), whence $c_2 a_0 \leqq \leqq a_0 c_2$, and therefore h_0^j coincides with c_j $(j\in\{1, 2\})$. Then the greatest chords of Q coincide with the sides of the fixing triangle for Q and this implies the coincidence of Q with C, contrary to the assumption.

In the second considered case we have $c_2 a_0 = a_0 h_0 \leqq a_0 c_1$ instead of (16) and $c_1 a_0 = a_0 c_2$ instead of (17), from which it follows, as previously, the coincidence of h_0 with c_1. Thus Q again coincides with C, and the proof is complete.

For any plane body Q we have $w(Q)\in\left\langle\frac{1}{2}, 1\right)$ according to Theorem 3, but one can ask about a body Q for which $w(Q)=s$, where s is any number belonging to $\left\langle\frac{1}{2}, 1\right)$. The answer to this question is given by

THEOREM 5. *For any* $s\in\left\langle\frac{1}{2}, 1\right)$ *there exists a non-empty class of plane bodies* Q *for which* $w(Q)=s$.

PROOF. Consider a family of trapezoids T^p each with lower base length 1 and upper one p $(p\in(0, 1))$. Each member of this family can be obtained as an image of a parallelogram P at some transformation which shortens one side of P from 1 to p. In the limiting cases for p we have the parallelogram P (for $p=1$) and a triangle (for $p=0$). In other words, the considered family, together with its limiting cases, can be obtained by a continuous variation of p in the segment $\langle 0, 1\rangle$.

For each T^p there exists $F_0(T^p)$ and $w(T^p)$ by Remark 1; the last quantity depends on p. Hence for the considered family the coefficient $w(T^p)$ is a continuous function of p. Its minimal value, by Theorem 3 and Remark 7, is equal to $\frac{1}{2}$ (for $p=0$). On the other hand it is evident that for p approaching 1 the value of $w(T^p)$ tends to 1 (because exactly two of the points of $F_0(T^p)$ must belong to the different lateral sides of T^p). By continuity $w(T^p)$ assumes all values between $\frac{1}{2}$ and 1.

It is not difficult to find the form of the mentioned function $w(T^p)$. Consider an equilateral trapezoid $T^p=b_1b_2c_2c_1$ (Fig. 14) in which $b_1b_2=1$, $c_1c_2=p$ and $b_jc_j=1-p$ $(j\in\{1,2\})$. The pessimal fixing system for T^p, in view of Proposition, consists of the midpoint of the lower base and of two points such that each of them does belong to the respective lateral side and its distance from the respective vertex of the upper base is equal to $\frac{(1-p)^2}{2-p}$. Denoting the points of the pessimal fixing system $F_0(T^p)$ as in Fig. 14 we have $w_1(F_0)=p+\frac{(1-p)^2}{2-p}=\frac{1}{2-p}$. On the other hand we have $w(T^p)=w_1(F_0)$ by Theorem 2 and the definition of the coefficient of T^p. This implies that

$$w(T^p)=\frac{1}{2-p}. \tag{19}$$

As it is seen, $w(T^p)$ is an increasing continuous function of p, with values in $\left\langle\frac{1}{2},1\right)$ for $p\in\langle 0,1)$. This means that for every $s=\frac{1}{2-p}\in\left\langle\frac{1}{2},1\right)$ there exists $p=2-\frac{1}{s}$ such that $w(T^p)=s$.

4. Homothetic covering of a plane body stabilized by its pessimal fixing system

The least number of bodies K' homothetic to a body K with the ray ratio $k\in(0,1)$ which form a covering of K will be denoted by $b(K)$. Gohberg and Markus [7] have proved

THEOREM G—M. *If a plane body Q is different from a parallelogram, then $b(Q)=3$; in the opposite case $b(Q)=4$.*

It is not so if we consider a body K together with some of its fixing systems and if we cover K with the homothetic bodies by translating them from "outside". The points of $F(K)$ impede these translations and force the homothetic bodies to have a respective size.

Let $w(Q)=q$ for some plane body Q. Then $F_0(Q)$ is an impedient system for Q' homothetic to Q if the ray ratio for Q' is less than or equal to q. Hence Q can be covered with several homothetic copies of Q by translation of the latter if the ray ratio for each of them is not greater than q. The least number of such Q' which may cover Q in this way (i.e. Q' cover Q but every point of $F_0(Q)$ is not covered) is denoted by $h(Q)$. Therefore, the natural question appears: what value does $h(Q)$ assume for a plane body Q.

Before giving an answer to this question we shall prove

LEMMA 3. *If $\{a_0,a_1,a_2\}=F(Q)$ for a plane body Q and $\Delta c_0c_1c_2$ is the fixing triangle for Q determined by $F(Q)$, then for each $i\in\{0,1,2\}$ the part $Q_i=Q\cap$*

$\cap \Delta a_i a_{i+1} c_{i+2}$ *of* Q *can be covered with a copy of* Q *which is an image of* Q *under a homothety with the ray ratio equal to* $w_i(F)$.

PROOF. Take a homothety $\chi^{k_i}_{o_{i-1}}$, where $k_i = w_i(F)$ and $o_{i-1} = (A_i^{i-1}, a_i) \cap \cap (A_{i+1}^{i-1}, a_{i+1})$; the latter belongs to the opposite angle to $\sphericalangle c_{i+1} c_{i-1} c_i$ of the fixing triangle for Q (Fig. 28). Then $\chi^{k_i}_{o_{i-1}}(Q)$ is a copy of Q in which the greatest chord parallel to $\langle a_i, a_{i+1} \rangle$ is identical with $\langle a_i, a_{i+1} \rangle$. We have to show that $\chi^{k_i}_{o_{\cdot -1}}(Q)$ covers Q_i.

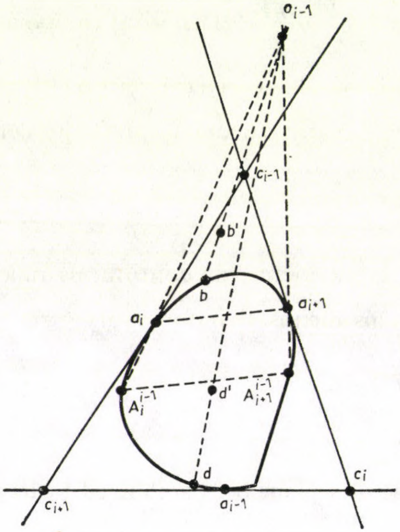

Fig. 28

Let $b \in \text{bd}\, Q \cap \Delta a_i a_{i+1} c_{i-1}$ be an arbitrary point. Its image under $\chi^{k_i}_{o_{i-1}}$ lies nearer to o_{i-1} than b which means that $\overset{\frown}{a_i a_{i+1}} \subset \text{bd}\, Q$ is covered with $\chi^{k_i}_{o_{i-1}}(Q)$. Let $\overset{\frown}{A_i^{i-1} A_{i+1}^{i-1}} \subset \text{bd}\, Q$ be the arc which includes a_{i-1} and take $d \in \text{int}\, \overset{\frown}{A_i^{i-1} A_{i+1}^{i-1}}$. The image of d under $\chi^{k_i}_{o_{i-1}}$ belongs to the straight line (d, o_{i-1}) and lies on the opposite side of $\langle a_i, a_{i+1} \rangle$ than o_{i-1} (or lies on $\langle a_i, a_{i+1} \rangle$ if $\langle A_i^{i-1}, A_{i+1}^{i-1} \rangle \subset \text{bd}\, Q$). The segment $\langle a_i, a_{i+1} \rangle$ is covered with the image of $\langle A_i^{i-1}, A_{i+1}^{i-1} \rangle$. Hence $Q_i \subset \chi^{k_i}_{o_{i-1}}(Q)$.

LEMMA 4. *If* $\{a_0, a_1, a_2\} = F_0(Q)$ *for a plane body* Q *then* $\Delta a_0 a_1 a_2 \subset Q$ *can be covered with three copies of* Q *each of which is an image of* Q *under a homothety with the ray ratio equal to* $w(Q)$.

PROOF. For each $i \in \{0, 1, 2\}$ take the homothety $\chi^{w(Q)}_{a_i}$. It is evident, by Belousov's theorem [1, p. 15] and by the considerations in the Preliminaries about the liminal number of a plane body and about the passing coefficient, that

$$\Delta a_0 a_1 a_2 \subset \bigcup_{i=0}^{2} \chi^{w(Q)}_{a_i}(\Delta a_0 a_1 a_2) \subset \bigcup_{i=0}^{2} \chi^{w(Q)}_{a_i}(Q)$$

if $w(Q)\geqq\frac{2}{3}$. In view of Theorem 3 we have to show that this is so also in the case $w(Q)\in\left\langle\frac{1}{2},\frac{2}{3}\right)$.

Let $w(Q)=\frac{1}{2}+s=\frac{1+2s}{2}$, where $s\in\left\langle 0,\frac{1}{6}\right)$; then $\bigcup_{i=0}^{2}\chi_{a_i}^{w(Q)}(\Delta a_0a_1a_2)$ does not cover $\Delta a_0a_1a_2$ (Fig. 29) (cf. [1], p. 12). Assume that $\bigcup_{i=0}^{2}\chi_{a_i}^{w(Q)}(Q)$ does not cover $\Delta a_0a_1a_2$ either. (In the sequel, for the generality of the considerations and of a notation, we shall write $\Delta a_{i-1}a_ia_{i+1}$ (the addition is taken mod 3) instead of $\Delta a_0a_1a_2$.) Let $b\in\operatorname{int}\Delta a_{i-1}a_ia_{i+1}$ be a point which belongs to none of $\chi_{a_i}^{w(Q)}(Q)$ and for each i take into account the image b_i of b under the homothety $\chi_{a_i}^{k}$, where $k=w(Q)=\frac{1+2s}{2}$. Since $b\notin\bigcup_{i=0}^{2}\chi_{a_i}^{k}(Q)$ thus $b_i\notin Q$ for each i but only one of these points may be a vertex of the fixing triangle for Q. Moreover, it is possible that in some neighbourhood of b_i there exists a point belonging to Q. Note that $\langle b_{i+1},b_i\rangle$ is parallel to $\langle a_i,a_{i+1}\rangle$, $b_ib_{i+1}<a_ia_{i+1}$ and b is the common point of the diagonals of the trapezoid $a_ia_{i+1}b_ib_{i+1}\overset{\text{df}}{=}T_{i-1}$. Therefore, Q is included in a polygon $d_id'_{i-1}d_{i+1}d'_id_{i-1}d'_{i+1}$ (Fig. 30) and for each $i-1$ the greatest chord

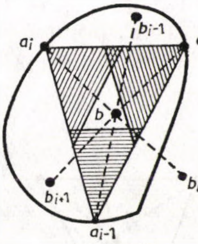

Fig. 29

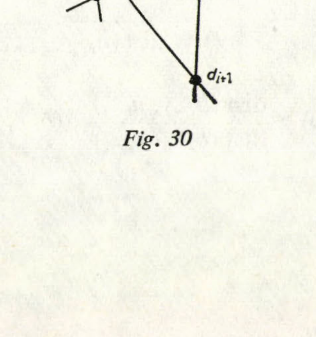

Fig. 30

$\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle$ of Q must be intersecting T_{i-1} and its image by $\chi^k_{a_{i-1}}$ must have a length equal to the length of $\langle a_i, a_{i+1}\rangle$.

It is evident that if T_{i-1} has the empty common part with $\chi^k_{a_{i-1}}(\Delta a_{i-1}a_ia_{i+1})$ (as a consequence of this we have $\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle \cap \chi^k_{a_{i-1}}(\Delta a_{i-1}a_ia_{i+1})=\emptyset$) then $\chi^k_{a_{i-1}}(\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle)$ must be lying between the prolongated lateral sides of T_{i-1} nearer to a_{i-1} than $\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle$. This implies that the length of $\chi^k_{a_{i-1}}(\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle)$ is less than the length of $\langle a_i, a_{i+1}\rangle$ which is impossible. Note that the same occurs if $T_{i-1}\cap\chi^k_{a_{i-1}}(\Delta a_{i-1}a_ia_{i+1})\neq\emptyset$ but $T_{i-1}\cap\chi^k_{a_{i-1}}(\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle)=\emptyset$. We must look at the case when $T_{i-1}\cap\chi^k_{a_{i-1}}(\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle)\neq\emptyset$.

Take into account the segment $\langle a_{i-1}, b_{i-1}\rangle$ (Fig. 31); it is passing through b and does intersect $\langle a_i, a_{i+1}\rangle$ in a point $\tilde a_{i-1}$, $\langle b_i, b_{i+1}\rangle$ in a point $\tilde b_{i-1}$, $\chi^k_{a_{i-1}}(\langle a_i, a_{i+1}\rangle)$ in a point c_{i-1}, $\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle$ in a point e_{i-1}; denote by D the length of $\langle a_{i-1}, \tilde a_{i-1}\rangle$ and assume that $\operatorname{dist}(b, \tilde a_{i-1})=p_{i-1}\cdot D$, where $p_{i-1}\in\left(2s, \dfrac{1-2s}{2}\right)$. Then

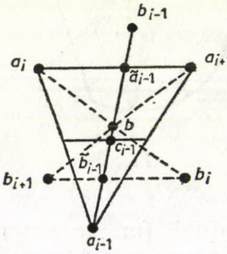

Fig. 31

$$(20)\quad\begin{cases}\operatorname{dist}(a_{i-1}, b_{i-1}) = \dfrac{2}{1+2s}(1-p_{i-1})D,\\[2ex] \operatorname{dist}(b_{i-1}, \tilde a_{i-1}) = \dfrac{1-2s-2p_{i-1}}{1+2s}D \quad\text{as}\quad a_{i-1}b_{i-1}-a_{i-1}\tilde a_{i-1},\\[2ex] \operatorname{dist}(b_{i-1}, c_{i-1}) = \dfrac{3-4s-4s^2-4p_{i-1}}{2(1+2s)}D \quad\text{as}\quad a_{i-1}b_{i-1}-a_{i-1}c_{i-1},\\[2ex] \operatorname{dist}(b_{i-1}, \tilde b_{i-1}) = \dfrac{1-2s}{1+2s}D \quad\text{as}\quad b_{i-1}\tilde a_{i-1}+\tilde a_{i-1}\tilde b_{i-1} \quad \left(\tilde a_{i-1}\tilde b_{i-1} = \dfrac{2p_{i-1}}{1+2s}D\right).\end{cases}$$

Assume that $\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle\cap\chi^k_{a_{i-1}}(\Delta a_{i-1}a_ia_{i+1})\neq\emptyset$; then

$$\operatorname{dist}(b_{i-1}, c_{i-1}) \leqq \operatorname{dist}(b_{i-1}, e_{i-1}) \leqq \operatorname{dist}(b_{i-1}, \tilde b_{i-1}),$$

i.e.

$$\frac{3-4s-4s^2-4p_{i-1}}{2(1+2s)}D \leqq \operatorname{dist}(b_{i-1}, e_{i-1}) \leqq \frac{1-2s}{1+2s}D$$

in view of $(20)_{3,4}$. Since $w(Q)\geqq\dfrac{\operatorname{dist}(b_{i-1}, a_{i-1})}{\operatorname{dist}(b_{i-1}, e_{i-1})}$ (b_{i-1} belongs to the fixing triangle for Q) and

$$\frac{1-2s-2p_{i-1}}{1-2s} \leqq \frac{\operatorname{dist}(b_{i-1}, a_{i-1})}{\operatorname{dist}(b_{i-1}, e_{i-1})} \leqq \frac{2(1-2s-2p_{i-1})}{3-4s-4s^2-4p_{i-1}}$$

hence we must have $\frac{3}{2}-2s-2s^2-2p_{i-1} \leqq 1-2s$. This implies $p_{i-1} \geqq \frac{1}{2}\left(\frac{3}{2}-2s^2\right)$ but $p_{i-1} \in \left(2s, \frac{1-2s}{2}\right)$ thus $\frac{1}{2}\left(\frac{3}{2}-2s^2\right) < \frac{1-2s}{2} \Leftrightarrow 0 < 2s^2-2s-\frac{1}{2}$ which does not hold for $s \in \left\langle 0, \frac{1}{6}\right)$. This means that if there exists $b \in Q$ which belongs to none of the homothetic copies of Q then for every i the greatest chord of Q parallel to $\langle a_i, a_{i+1}\rangle$ does not intersect $\chi^k_{a_{i-1}}(\Delta a_{i-1} a_i a_{i-1})$. Therefore it is possible that $\chi^k_{a_{i-1}}(\langle A_i^{i-1}, A_{i+1}^{i-1}\rangle)$ intersects T_{i-1} and its length is equal to the length of $\langle a_i, a_{i+1}\rangle$. However in this case Q is included in the quadrangle $a_{i-1} d_{i-1} b_{i-1} d'_{i-1}$ (Fig. 30) which implies that the greatest chord $\langle A^i_{i+1}, A^i_{i-1}\rangle$ (and $\langle A^{i+1}_{i-1}, A^{i+1}_i\rangle$, too) cannot be longer than $\langle a_{i-1}, b_{i-1}\rangle$ and, in view of $(20)_1$ and of the property of a triangle concerning its sides, the following inequalities hold:

$$w_{i+1}(F_0) = \frac{\operatorname{dist}(a_{i+1}, a_{i-1})}{\operatorname{dist}(A^i_{i+1}, A^i_{i-1})} \geqq \frac{\operatorname{dist}(a_{i+1}, a_{i-1})}{\operatorname{dist}(a_{i-1}, b_{i-1})} > \frac{D}{\frac{2(1-p_{i-1})}{1+2s}D} =$$

$$= \frac{1+2s}{2(1-p_{i-1})} > \frac{1+2s}{2} = w(Q).$$

The inequality $w_{i+1}(F_0) > w(Q)$ implies, in view of Theorem 2, that the considered set $\{a_{i-1}, a_i, a_{i+1}\}$ is not a pessimal fixing system for Q which contradicts the hypothesis.

THEOREM 6. *For any plane body Q the inequalities*

$$3 \leqq h(Q) \leqq 6$$

hold.

PROOF. The first inequality follows from Theorem G—M. The second one is a simple consequence of Lemmas 3 and 4.

THEOREM 7. *For each integer $q \in \{3, 4, 5, 6\}$ there exists a class of plane bodies Q stabilized by a three-point system for which $h(Q)=q$.*

PROOF. It is easy to see that we have $h(C)=3$ if C is a circle. The points of $F_0(C)$ are the vertices of an equilateral triangle inscribed in C (cf. Fig. 1a or Fig. 7c), hence $w(C)=\frac{\sqrt{3}}{2}$. Simultaneously $k^3(C)=\frac{\sqrt{3}}{2}$, thus $w(C)=k^3(C)<k^2(C)$ which means that C can be covered with three homothetic circles (but not with two ones) in the presence of any $F_0(C)$. For a triangle T we have $w(T)=\frac{1}{2}$ because the points of $F_0(T)$ are the midpoints of sides of T (cf. Proposition for $p=0$) and

$k^6(T)=\frac{1}{2}$, $k^5(T)=\frac{8}{15}$ (cf. [1], p. 15), thus $w(T)=k^6(T)<k^5(T)$; this implies $h(T)=6$.

Now consider a one-parameter family of trapezoids T^p, the same as in the Proposition, and recall that $w(T^p)$ is a continuous function in $\langle 0, 1)$ which assumes all values between $\frac{1}{2}$ and 1. On the other hand for each fixed $j\in\{4, 5\}$ $k^j(T^p)$ is also a continuous function of the parameter p and, in view of [2, p. 353] $k^j(T^p)\in\left\langle\frac{1}{2}, \frac{2}{3}\right)$ for any $p\in\langle 0, 1)$; moreover, for each $p\in\langle 0, 1)$ there holds $k^3(T^p)>k^4(T^p)>k^5(T^p)$. Both of these functions, $w(T^p)$ and $k^j(T^p)$, are continuous, hence for some $p=p_j\in\langle 0, 1)$ we have $w(T^p)=k^j(T^p)$. Therefore $h(T^{p_j})=j$ for each j.

One can obtain the same values of $h(T^p)$, i.e. 4 and 5, by (19) in view of [5, Theorem 2] and the Theorem in [2]. For $p=\frac{1}{4}$ we get $w(T^{1/4})=\frac{4}{7}=k^4(T^{1/4})<$ $<k^3(T^{1/4})$ which implies $h(T^{1/4})=4$; the setting $p=\frac{1}{8}$ in (19) leads to $h(T^{1/8})=5$.

Remark also that from (19) we have $w(T^0)=\frac{1}{2}=k^6(T^0)<k^5(T^0)$, which implies $h(T^0)=6$ pointed out in the proof of Theorem 7, because T^0 is a triangle; also $w(T^{1/2})=\frac{2}{3}=k^3(T^{1/2})$ by [5, Theorem 1] which implies $h(T^{1/2})=3$. This means that the trapezoids $T^{1/2}$ form another class of plane bodies Q, different from the class of circles, for which $h(Q)=3$.

Further, the following observation can be made.

REMARK 8. For a parallelogram P we have $h(P)=4$.

Indeed, a parallelogram P is stabilized by a four-point system. Such a system is pessimal if each of its points is the midpoint of a side of the parallelogram. Then $w(P)=\frac{1}{2}$, but the same number is the value of $k^4(P)$ [5, formula (1.4)], which implies $h(P)=4$.

5. Some remarks and open questions

Depending on the form of a body $K\subset E^n$, a pessimal fixing system for K is determined uniquely (e.g. for a simplex and for a parallelotope) or not (e.g. for a ball).

(i) For what bodies $K\subset E^n$ $(n\geqq 2)$ is the system $F_0(K)$ determined uniquely? For which ones are there a finite number of $F_0(K)$?

The number of the points belonging to $F_0(K)$ for $K\subset E^n$ will be called *the fixation index of* K and will be denoted by $FI(K)$.

(ii) To what interval does the coefficient of $K\subset E^n$ belong if $FI(K)=n+r$ $(r\in\{1, 2, \dots\})$? Is $\left\langle\frac{n-1}{n}, 1\right)$ this interval for each n if $FI(K)=n+1$?

The last seems to be true because $w(S^n)=\frac{n-1}{n}$ for an n-dimensional simplex S^n, but

(iii) Is an n-dimensional simplex the body in E^n $(n\geqq 3)$ for which the coefficient assumes the minimal value among all possible ones? Is it the unique such body?

For plane bodies Q we have $h(Q)\in\{3, 4, 5, 6\}$ (Theorem 6).

(iv) Characterize the plane bodies Q satisfying $h(Q)=q$ for each $q\in\{3, 4, 5, 6\}$.

(v) What value does $h(G)$ assume for $G\subset E^n$ if $n\geqq 3$? The minimal value of $h(G)$ must be equal to $n+1$ (attained for G which is a ball), but what is the maximal one?

For a 3-dimensional simplex S^3 we have $\frac{9}{13}=k^5(S^3)>k^6(S^3)=\frac{2}{3}$ [5, Theorem 3], [6, Theorem 2] and $w(S^3)=\frac{2}{3}$ because each point of $F_0(S^3)$ is the barycentre of one of its faces. Hence $w(S^3)=k^6(S^3)$, which implies $h(S^3)=6$.

(vi) Is the number 6 the maximal value of $h(G)$ for $G\subset E^3$ if $F_0(G)$ is a four-point system?

References

[1] Ю. Ф. Белоусов, Теоремы о покрытии плоских фигур, *Укр. Геом. Сборник*, **20** (1977), 11—17.

[2] H. Brzozowska, S. Fudali, Homothetic coverings of trapezoids, *Demonstratio Math.*, **16** (1983), 329—356.

[3] L. W. Danzer, Review of the paper L. Fejes Tóth: On primitive polyhedra, *Math. Reviews*, **26** (1963), 569—570.

[4] L. Fejes Tóth, On primitive polyhedra, *Acta Math. Acad. Sci. Hung.*, **13** (1962), 379—382.

[5] S. Fudali, Homothetic covering of frustums of simplices, *Demonstratio Math.*, **16** (1983), 133—150.

[6] S. Fudali, A. Wasilewska, Homothetic covering of the tetrahedron, *Matematyka*, **35** (1982), 337—349 (in Polish).

[7] И. Ц. Гохберг, А. С. Маркус, Одна задача о покрытии выпуклых фигур подобными, *Изв. Молд. филиала АН СССР*, **10** (1960), 87—90.

(Received October 26, 1983; revised August 23, 1984 and November 11, 1986)

INSTITUTE OF MATHEMATICS
UNIVERSITY
SZCZECIN
POLAND

Acta Math. Hung.
50 (3—4) (1987), 227—240.

RECURRENCE IN TOPOLOGICAL DYNAMICS AND THE RIEMANN HYPOTHESIS

B. BAGCHI (Calcutta)

1. Introduction

The object of this paper is to point out some surprising connections between two apparently unrelated branches of mathematics. On the one hand we have topological dynamics with its historical origin in the study of particles in motion; on the other hand is the theory of Riemann zeta function and other allied Dirichlet series which arose in the abstract number-theoretic deliberations of Euler, Dirichlet and Riemann.

In its abstract formulation, topological dynamics studies continuous actions of topological groups on topological spaces. In the present work, we allow the additive group of reals (or the additive group of integers) to act on a certain function space H. The Riemann zeta function ζ (as also the other Dirichlet L-functions) may be regarded as a point in the resulting "flow" H. Implicit in [1] was the result that the celebrated Riemann hypothesis on the location of the zeros of zeta is equivalent to strong recurrence (in the sense of Gottschalk and Hedlund in [3]) of zeta regarded as a point in this flow. Analogous results hold for the other Dirichlet L-functions. Here we further elaborate and develop this relationship to show that the generalised Riemann hypothesis holds if and only if all the functions in a naturally defined class of Dirichlet series (viz., those with periodic coefficients) are strongly recurrent points of the flow H. Further, we show that the generalised Riemann hypothesis holds if certain finite subsets of H, each point of which is shown to be strongly recurrent, are jointly strongly recurrent. This makes it interesting to study flows in which separate strong recurrence of points in a finite subset implies joint strong recurrence. By a coherent flow we understand a flow in which this implication is valid. In terms of this definition, the generalized Riemann hypothesis is equivalent to coherence of certain "subspace restrictions" of the flow H. So it is pertinent to ask for sufficient conditions for coherence. To make a beginning in this direction, we show that locally equi-continuous flows are coherent.

2. Flows and other relevant notions from topological dynamics

Throughout this paper G will stand for either the additive group $\mathbf{R}$ of reals or the additive group $\mathbf{Z}$ of integers — as the case may be.

A *flow* is a triplet (X, G, π) where X is a topological space, $G=\mathbf{R}$ or $\mathbf{Z}$, and $\pi\colon X\times G\to X$ is a continuous map satisfying

$$\text{(a)}\quad \pi(x,0)=x,\qquad \text{(b)}\quad \pi(x,t_1+t_2)=\pi\big(\pi(x,t_1),t_2\big)\quad \text{for}\quad x\in X,\quad t_1,t_2\in G.$$

In this paper we shall consider flows on metric spaces only. Thus each space X

will be considered equipped with a metric ϱ. If $X_1, X_2, \ldots, X_n$ are metric spaces with metrics $\varrho_1, \ldots, \varrho_n$ respectively, then the product space $X = X_1 \times \ldots \times X_n$ will be assigned the metric ϱ given by $\varrho(x, y) = \max_{1 \leqq j \leqq n} \varrho_j(x_j, y_j)$. A flow (X, G, π) is said to be a *continuous flow* or a *discrete flow* according as $G = \mathbf{R}$ or $G = \mathbf{Z}$.

If (X, G, π) is a flow and $t \in G$, the map $\pi^t \colon X \to X$ is defined by $\pi^t(x) = \pi(x, t)$, $x \in X$. Clearly each π^t is a homeomorphism of X onto itself. In case of a discrete flow, π^n is clearly the n^{th} iterate of π^1, so that the flow is in this case determined by the surjective homeomorphism π^1. Conversely, each homeomorphism of a space X onto itself induces a discrete flow on X. Thus discrete flow might alternatively be described as a space together with a homeomorphism of the space onto itself.

If (X, G, π) is a flow and Y is a subspace of X, then Y is said to be *invariant* in case whenever $x \in Y$ and $t \in G$, we have $\pi(x, t) \in Y$. In symbols, we have $\pi(Y, G) = Y$. In this case the restriction of π to $Y \times G$, which again we denote by π, determines a flow (Y, G, π). This is called a *subspace restriction* of (X, G, π).

If (X, G, π) is a flow and $A \subseteqq X$, let us define the *span of A*, to be denoted by $\mathrm{Sp}\,(A)$, as the minimal invariant set containing A. Notice that we *do not* require the span to be closed. Thus $\mathrm{Sp}\,(A) = \{\pi(x, t) \colon x \in A,\ t \in G\} = \pi(A, G)$. In particular, for any point x in X, $\mathrm{Sp}\,(\{x\}) = \pi(x, G)$ is called the *orbit* of x, and is denoted by $O(x)$.

If (X, G, π) is a flow and $h \neq 0, h \in G$, let us define $\pi_h \colon X \times Z \to X$ by $\pi_h(x, n) = \pi(x, nh)$. Clearly (X, Z, π_h) is a discrete flow. We call it the *discrete subflow modulo h*. This is clearly a particular instance of the more general notion of "subgroup restriction" in [4, p. 6].

If for $1 \leqq j \leqq n$ (X_j, G, π_j) is a flow (with a common "phase group" G) then their *product flow* (Y, G, π) is given by $Y = X_1 \times \ldots \times X_n$ and $\pi = \pi_1 \times \ldots \times \pi_n \colon Y \times G \to Y$ where $\pi(x, t) = (\pi_j(x_j, t) \colon 1 \leqq j \leqq n)$. In particular, when $X_1 = \ldots = X_n = X$, we shall denote the product flow by (X^n, G, π).

If (X_1, G, π_1) and (X_2, G, π_2) are two flows with a common phase group G, and $\varphi \colon X_1 \to X_2$ is a continuous map satisfying $\varphi(\pi_1(x, t)) = \pi_2(\varphi(x), t)$, $x \in X_1$, $t \in G$, then we say that φ is a *flow homomorphism*. If, in addition, φ is a homeomorphism between X_1 and $\varphi(X_1)$ then we say that φ is an *embedding* of (X_1, G, π_1) in (X_2, G, π_2).

If (X, G, π) is a flow and $x_0 \in X$, then we say that x_0 is *periodic* in case there exists $t_0 \in G$, $t_0 \neq 0$, such that $\pi^{t_0}(x_0) = x_0$. Strong recurrence and almost periodicity are two of the recursion notions which generalise periodicity. We must begin with a few technicalities in order to define them.

If A if a Borel subset of $\mathbf{R}$, the *upper* and *lower asymptotic density* of A, denoted by $\bar{\delta}(A)$ and $\underline{\delta}(A)$, are defined by

$$\bar{\delta}(A) = \limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} I_A(t)\, dt, \quad \underline{\delta}(A) = \liminf_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} I_A(t)\, dt.$$

Here I_A is the indicator function of A: $I_A(t) = 1$ if $t \in A$ and $= 0$ otherwise. In case $\bar{\delta}(A) = \underline{\delta}(A)$, we call the common value the *asymptotic density* of A, and denote it by $\delta(A)$.

If A is a subset of $\mathbf{Z}$, the *upper* and *lower asymptotic density* of A (discrete version), denoted respectively by $\bar{d}(A)$ and $\underline{d}(A)$, are defined by

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{\#(A\cap[-N,N])}{2N}, \quad \underline{d}(A) = \liminf_{N\to\infty} \frac{\#(A\cap[-N,N])}{2N}.$$

Here $\#(\cdot)$ stands for the number of elements of $(\cdot)$.

In case $\bar{d}(A)=\underline{d}(A)$, we call the common value the *asymptotic density* (discrete version) of A, and denote it by $d(A)$.

If $A\subseteqq\mathbf{R}$ (respectively $A\subseteqq Z$) then we say that A is *syndetic* or *relatively dense* in case there exists a compact set $K\subseteqq\mathbf{R}$ (respectively a finite set $K\subseteqq Z$) such that $A+K=\mathbf{R}$ (respectively $A+K=Z$).

Notice that equivalently a set is syndetic if there is an $l>0$ such that every interval of length l contains an element of the set.

2.1. DEFINITIONS. Let (X, G, π) be a continuous (respectively discrete) flow. Let $x_0\in X$. Then x_0 is said to be *strongly recurrent* in case for every $\varepsilon>0$ there exists a Borel set $A\subseteqq G$ such that $\bar{\delta}(A)>0$ (respectively $\bar{d}(A)>0$) and $\varrho(\pi^t(x_0), x_0)<\varepsilon$ for all $t\in A$. Here ϱ is the given metric on X. If $h\in G$, $h\neq 0$, then x_0 is said to be *strongly recurrent modulo h* in case x_0, regarded as a point in the discrete subflow (X, Z, π_h), is strongly recurrent.

More generally, a subset Y of X is said to be *jointly strongly recurrent* in case for every $\varepsilon>0$, there exists a Borel set $A\subseteqq G$ with $\bar{\delta}(A)>0$ (respectively $\bar{d}(A)>0$) such that $\varrho(\pi^t(y), y)<\varepsilon$ for all $t\in A$ and all $y\in Y$. Further, Y is said to be jointly strongly recurrent modulo h if it is jointly strongly recurrent when regarded as a set in (X, Z, π_h). Notice that strong recurrence is an *orbital property*. That is, a point x_0 in the flow (X, G, π) is strongly recurrent if and only if it is strongly recurrent when regarded as a point in the subspace restriction $(O(x_0), G, \pi)$. More generally, a subset Y of X is jointly strongly recurrent if and only if it is so when regarded as a set in $(\mathrm{Sp}(Y), G, \pi)$.

The following proposition is a special case of the "inheritance theorem" first proved by Gottschalk and Hedlund in [3] and later given in an improved form in [4].

2.2. PROPOSITION (inheritance theorem for strong recurrence). *Let (X, G, π) be a flow, $x_0\in X$, $h\in G$, $h\neq 0$. Then x_0 is strongly recurrent if and only if x_0 is strongly recurrent modulo h.*

The next two propositions are important for application but their proofs are trivial. So we omit the proof.

2.3. PROPOSITION. *Let (X_1, G, π_1) and (X_2, G, π_2) be flows with a common phase group and let φ be a flow-homomorphism between them. If a point $x_0\in X_1$ is strongly recurrent then so is $\varphi(x_0)\in X_2$.*

2.4. PROPOSITION. *Let (X, G, π) be a flow and let $Y=\{y_1, \ldots, y_n\}$ be a finite subset of X. Then Y is jointly strongly recurrent if and only if the n-tuple $(y_1, \ldots, y_n)$, regarded as a point in the product-flow (X^n, G, π), is strongly recurrent.*

Propositions 2.2 and 2.4 together yield:

2.5. **Proposition.** *If* (X, G, π) *is a flow and* Y *is a finite subset of* X, *then for each fixed* $h \in G$, $h \neq 0$, Y *is jointly strongly recurrent if and only if it is jointly strongly recurrent modulo* h.

Proof. In view of 2.4, this follows on applying 2.2 to the product-flow (X^n, G, π) with $n = \#(Y)$.

Similarly, 2.3 and 2.4 imply:

2.6. **Proposition.** *Let* $Y = \{y_1, \ldots, y_n\}$ *be a finite subset of the flow* (X, G, π). *Let* φ *be a flow-homomorphism of the product flow* (X^n, G, π) *into another flow. If* Y *is jointly strongly recurrent then* $\varphi(y_1, \ldots, y_n)$ *is strongly recurrent.*

The chief utility of Propositions 2.2 and 2.5 lie in the fact that they allow us to move back and forth between continuous and discrete flows. Because of them it often suffices to prove results regarding strong recurrence and joint strong recurrence for discrete flows only, This technique will be exploited in the concluding section. Here we use Proposition 2.5 to prove a result which will be used in Section 4:

2.7. **Proposition.** *Let* (X, G, π) *be a flow. Let* $A = \{x_1, \ldots, x_m\}$ *be a jointly strongly recurrent subset of* X. *Let* $B = \{y_1, \ldots, y_n\}$ *be a finite subset of* X *such that all points in* B *are periodic (not necessarily with the same periods). Then* $A \cup B$ *is jointly strongly recurrent.*

Proof. It clearly suffices to prove only the case $n=1$, since the result then follows by induction on n. Accordingly, let $n=1$. Let $h \in G$, $h \neq 0$ be a period of y_1. By Proposition 2.5, A is jointly strongly recurrent modulo h. But the orbit of y_1 modulo h is a singleton. Hence trivially $A \cup \{y_1\}$ is jointly strongly recurrent modulo h. Another appeal to Proposition 2.5 completes the proof.

A parallel notion of recursiveness is that of almost periodicity. A point x_0 in a flow (X, G, π) is said to be *almost periodic* in case for every $\varepsilon > 0$ there exists a syndetic set $A \subseteq G$ such that for all $t \in A$, $\varrho(\pi^t(x_0), x_0) < \varepsilon$. Here, as before, ϱ is the given metric on X. As with strong recurrence, we could proceed to define joint almost periodicity and almost periodicity modulo h. Analogues of Propositions 2.2—2.7 hold for almost periodicity also.

Clearly a syndetic subset of G has positive upper asymptotic density. Therefore almost periodicity implies strong recurrence. We shall have a brief occasion to return to almost periodicity in the concluding section.

Most of the notions and results discussed in this section are standard. The author failed to locate Proposition 2.7 in the literature. Clearly analogues of this proposition hold for all recursion notions for which the Gottschelk—Hedlund inheritance theorem is valid. So it must be familiar to experts in the field.

3. Strong recurrence in the flow *H* and the Riemann hypotheses

3.1. *The space H.* Throughout this paper Ω will stand for the open strip consisting of complex numbers z such that $\frac{1}{2}<\operatorname{Re}(z)<1$. Here $\operatorname{Re}(z)$ is the real part of the complex number z. $H=H(\Omega)$ will stand for the space of all analytic functions on Ω, with the topology of uniform convergence on compacta (u.c.c.).

Let K_n, $n\geqq 1$ be a sequence of compact subsets of Ω which increase to Ω. We also assume that for each $n\geqq 1$, K_n is contained in the interior of K_{n+1}. For $f, g\in H$, we define

$$\varrho(f, g) = \sup_{n\geqq 1} \min\left(\frac{1}{n},\quad \sup_{z\in K_n} |f(z)-g(z)|\right).$$

As is easy to check, ϱ is a metric on H which induces the u.c.c. topology. Of course, ϱ depends on the choice of the sequence K_n. We shall think of H as equipped with this metric.

3.2. *The flow* $(H, \mathbf{R}, \sigma)$. We now allow the real line to act on H by vertical shift. More precisely, we define $\sigma\colon H\times\mathbf{R}\to H$ by $\sigma(f, t)=g$ where $g(z)=f(z+it)$, $z\in\Omega$. Clearly this makes $(H, \mathbf{R}, \sigma)$ a continuous flow.

3.3. *Dirichlet series, Dirichlet L-functions and the Riemann hypotheses.* A *Dirichlet series* is a series of the type $\sum_{n=1}^{\infty} a_n n^{-z}$, where a_n is a sequence of complex numbers (called the sequence of coefficients of the series) and z is a complex variable. Of particular importance in the theory is the series $\zeta(z)=\sum_{n=1}^{\infty} n^{-z}$ which converges for $\operatorname{Re}(z)>1$, and defines an analytic function there. As is well known, the *zeta function* $\zeta(z)$ may be analytically continued to the entire complex plane except for a singularity at $z=1$. Thus zeta is analytic on Ω, and we may (and do) regard ζ as a point in H.

An *arithmetic function* is a function f from the natural numbers into the complex plane. We often write f_n for $f(n)$ and thus display f as a sequence $\{f_n\}$. An arithmetic function f is said to be *completely multiplicative* in case $f(mn)=f(m)f(n)$ for $m, n\geqq 1$ and $f(1)=1$. An arithmetic function f is said to be *periodic* in case there exists $k\geqq 1$ such that $f(n+k)=f(n)$ for all $n\geqq 1$. In this case k is said to be a period of f, and we say that f is periodic modulo k.

A *Dirichlet character modulo k* is by definition an arithmetic function nonvanishing only for integers co-prime to k which is both completely multiplicative and periodic. It is well known that given any $k\geqq 1$ there are only finitely many Dirichlet characters modulo k. Indeed, the total number of Dirichlet characters modulo k equals the number of integers in the interval $[1, k]$ which are relatively prime to k.

Corresponding to each Dirichlet character χ, one defines the *Dirichlet L-function* $L(., \chi)$ by the series $L(z, \chi)=\sum_{n=1}^{\infty}\chi(n)n^{-z}$. These are natural generalisations of the zeta function. Indeed $\zeta=L(., \chi_0)$ where χ_0 is the unique Dirichlet character modulo 1. Like zeta, all Dirichlet L-functions admit analytic continuation to the entire plane except possibly at $z=1$. Accordingly we regard the functions $L(.,\chi)$ as points in H.

The Riemann zeta function vanishes at the points $-2, -4, \dots$ the so called trivial zeros. The classical *Riemann hypothesis* conjectures that all the nontrivial zeros of zeta lie on the line $\mathrm{Re}\,(z)=\frac{1}{2}$. Likewise, the *generalised Riemann hypothesis* (GRH) for Dirichlet L-functions conjectures that for each character χ, the nontrivial zeros of $L(.,\chi)$ lie on the critical line $\mathrm{Re}\,(z)=\frac{1}{2}$. In view of the functional equation and the Euler product for $L(.,\chi)$, an equivalent formulation is: $L(z,\chi)\neq 0$ for $z\in\Omega$. That is, $\frac{1}{L(.,\chi)}\in H$. This is the form in which we shall consider these celebrated hypotheses.

3.4. *The relationship between the Riemann hypotheses and strong recurrence.* We begin with the following theorem from [1].

3.5. Proposition. *Let $k\geqq 1$, and let χ_j, $1\leqq j\leqq n$, be distinct Dirichlet characters modulo k. Let f_j, $1\leqq j\leqq n$, be members of H such that $\frac{1}{f_j}$ also belong to H. Then for every $\varepsilon>0$, the set A of all $t\in\mathbf{R}$ such that*

$$\text{(i)}\quad \sup_{1\leqq j\leqq n} \varrho\big(\sigma^t(L(.,\chi_j)), f_j\big) < \varepsilon$$

has positive upper asymptotic density $\bar{\delta}(A)>0$.

Conversely, if f_j, $1\leqq j\leqq n$, are such that for all $\varepsilon>0$ the set A of all $t\in\mathbf{R}$ satisfying (i) *satisfies $\bar{\delta}(A)>0$ then $\frac{1}{f_j}\in H$ for $1\leqq j\leqq n$.*

Proof. The direct part is a reformulation of Theorem 3.1 in [1]. The converse may be proved following the arguments in the proof of the 'if' part of Theorem 3.7 in [1].

The particular case $n=1$ of Proposition 3.5 is so important that we state it separately as:

3.6. Corollary. *Let χ be a Dirichlet character and let $f\in H$. For $\varepsilon>0$, let A_ε denote the set of all real numbers t such that $\varrho\big(\sigma^t(L(.,\chi)),f\big)<\varepsilon$. Then $\frac{1}{f}\in H$ if and only if $\bar{\delta}(A_\varepsilon)>0$ for every $\varepsilon>0$.*

Recall that in the above ϱ is the given metric on H which induces the u.c.c. topology.

Substituting $L(.,\chi)$ for f in Corollary 3.6, we immediately deduce:

3.7. Theorem. *Let χ be a Dirichlet character. Then the Riemann hypothesis holds for $L(.,\chi)$ if and only if $L(.,\chi)$ is a strongly recurrent point of the flow $(H,\mathbf{R},\sigma)$.*

Notice that in view of Proposition 2.2, Theorem 3.7 might equivalently be stated in terms of strong recurrence in discrete subflows (H, Z, σ_h), $h \neq 0$.

3.8. REMARKS. For each $k \geqq 1$, there is a unique Dirichlet character χ_0 modulo k defined by $\chi_0(n)=1$ if $(n, k)=1$, $\chi_0(n)=0$ otherwise. χ_0 is called the principal character modulo k; the other characters are called non-principal. The non-trivial zeros of $L(., \chi_0)$ coincide with the nontrivial zeros of zeta. Accordingly, the Riemann hypothesis for $L(., \chi_0)$ is equivalent to the classical Riemann hypothesis. In [2] H. Bohr essentially proved Theorem 3.7 for non-principal characters. His proof depends on the fact that for non-principal characters χ, the Riemann hypothesis is equivalent to convergence of the series $\sum \chi(p)p^{-z}$, $z \in \Omega$, as p runs over the primes. Therefore Bohr's proof does not extend to the principal character. After minor modifications, the main theorem in [2] may be stated as follows. Let $f \in H$ be given by a *convergent* Dirichlet series $\sum_p a_p p^{-z}$, $z \in \Omega$, where p runs over the primes. We also assume that

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(a+it)|^2\,dt < \infty \quad \text{for} \quad \frac{1}{2} < a < 1.$$

Then f is a strongly recurrent point of the flow $(H, \mathbf{R}, \sigma)$. Bohr deduces the non-principal case of Theorem 3.7 above from this result.

4. Joint strong recurrence and the Riemann hypotheses

Substituting $L(., \chi_j)$ for f_j in Proposition 3.5, we obtain:

4.1. THEOREM. *Let* $\chi_1, \ldots, \chi_n$ *be Dirichlet characters to a common modulus. Then the following are equivalent:*

(a) *Riemann hypothesis holds for* $L(., \chi_j)$, $1 \leqq j \leqq n$.

(b) *The set* $\{L(., \chi_j)\colon 1 \leqq j \leqq n\}$ *is a jointly strongly recurrent subset of* $(H, \mathbf{R}, \sigma)$.

For integers h, k, $1 \leqq h \leqq k$, let $\zeta_{h,k} \in H$ be defined by $\zeta_{h,k}(z) = \sum_{n=0}^{\infty} (nk+h)^{-z}$, $\operatorname{Re}(z) > 1$, and hence by analytic continuation. For $k \geqq 1$, let $h_1, \ldots, h_n$ be the integers in $[1, k]$ which are relatively prime to k, and let $\chi_1, \ldots, \chi_n$ be the Dirichlet characters modulo k. The map $\varphi\colon H^n \to H^n$ which sends $(f_1, \ldots, f_n)$ to $(g_1, \ldots, g_n)$ given by $g_j = \sum_{i=1}^{n} \chi_j(h_i) f_i$, $1 \leqq j \leqq n$, is clearly a flow homomorphism of the product flow $(H^n, \mathbf{R}, \sigma)$ onto itself. Its inverse φ^{-1} is given by $f_j = \frac{1}{n} \sum_{i=1}^{n} \overline{\chi_i(h_j)} g_i$ and is again a flow homomorphism. Therefore, by Proposition 2.3, $(f_1, \ldots, f_n)$ is a strongly recurrent point of $(H^n, \mathbf{R}, \sigma)$ if and only if $\varphi(f_1, \ldots, f_n)$ is. In particular, since

$$(L(., \chi_1), \ldots, L(., \chi_n)) = \varphi(\zeta_{h_1,k}, \ldots, \zeta_{h_n,k}),$$

it follows from Proposition 2.4 that $\{L(., \chi_j)\colon 1 \leqq j \leqq n\}$ is jointly strongly recurrent if and only if $\{\zeta_{h,k}\colon 1 \leqq h \leqq k, (h, k)=1\}$ is jointly strongly recurrent. Therefore Theorem 4.1 may be rewritten as:

4.2. THEOREM. *Let* $\chi_1, \ldots, \chi_n$ *be an enumeration of all the Dirichlet characters modulo a fixed integer* $k \geqq 1$. *Then the following are equivalent:*

(a) *Riemann hypothesis holds for* $L(., \chi_j)$, $1 \leqq j \leqq n$.

(b) *The set* $\{\zeta_{h,k}: 1 \leqq h \leqq k, (h, k) = 1\}$ *is a jointly strongly recurrent subset of* $(H, \mathbf{R}, \sigma)$.

4.3. LEMMA. *Let* $\chi_1, \ldots, \chi_n$ *be distinct Dirichlet characters to a fixed modulus. Let* $g = \sum_{j=1}^{n} a_j L(., \chi_j)$ *where at least two of the complex numbers* a_j *are non-zero. Then for any arbitrary* $f \in H$, *and for any* $\varepsilon > 0$, *the set* A_ε *of all real numbers t such that* $\varrho(\sigma^t(g), f) < \varepsilon$ *satisfies* $\delta(A_\varepsilon) > 0$.

PROOF. In view of Proposition 3.5, it suffices to show that the set $\left\{\sum_{j=1}^{n} a_j f_j: f_j \in H \text{ and } \frac{1}{f_j} \in H\right\}$ is dense in H. Let S be the set of all $f \in H$ such that $\frac{1}{f} \in H$. Since S is closed under multiplication by non-zero complex numbers and since the closure of S contains the constant function 0, it is enough to show that $S+S = \{f_1+f_2: f_1, f_2 \in S\}$ is dense in H. But if $g \in H$ is bounded then there is a complex $\alpha \neq 0$ such that $g(z) \neq \alpha$ for $z \in \Omega$. Hence, putting $f_1(z) = g(z) - \alpha, f_2(z) \equiv \alpha$, we get $g = f_1 + f_2$, $f_1, f_2 \in S$. Thus $S+S$ contains the set of all bounded members of H. Since the latter set is dense in H, we are done.

We have the representation $\zeta_{h,k} = \frac{1}{n} \sum_{i=1}^{n} \overline{\chi_j(h)} L(., \chi_j)$, for $(h, k) = 1$. Also, if $k \geqq 3$ then $n \geqq 2$ so that at least two of the coefficients in this formula are non-zero. Hence from the above lemma we deduce:

4.4. PROPOSITION. *Let* $1 \leqq h \leqq k$, $k \geqq 3$, $(h, k) = 1$. *Then for any* $f \in H$, *for any* $\varepsilon > 0$, *the set* A_ε *of all real numbers t such that* $\varrho(\sigma^t(\zeta_{h,k}), f) < \varepsilon$ *satisfies* $\delta(A_\varepsilon) > 0$.

In particular, substituting $\zeta_{h,k}$ for f in the above, we deduce.

4.5. THEOREM. *Let* $1 \leqq h \leqq k$, $k \geqq 3$, $(h, k) = 1$. *Then* $\zeta_{h,k}$ *is a strongly recurrent point of* $(H, \mathbf{R}, \sigma)$.

4.6. DEFINITION. For $k \geqq 1$, let D_k be the span of the set $\{\zeta_{h,k}: 1 \leqq h \leqq k, (h, k) = 1\}$. Thus D_k is the set of all points $\sigma^t(\zeta_{h,k})$ with $1 \leqq h \leqq k$, $(h, k) = 1$, $t \in \mathbf{R}$. $(D_k, \mathbf{R}, \sigma)$ is a subspace restriction of $(H, \mathbf{R}, \sigma)$.

Since for each $t \in \mathbf{R}$, σ^t is a flow homomorphism of $(H, \mathbf{R}, \sigma)$ into itself, Theorem 4.5 implies that each point in $(D_k, \mathbf{R}, \sigma)$ is strongly recurrent for $k \geqq 3$. If the "generating set" $\{\zeta_{h,k}: 1 \leqq h \leqq k, (h, k) = 1\}$ were jointly strongly recurrent, it would likewise follow that any finite subset of $(D_k, \mathbf{R}, \sigma)$ is jointly strongly recurrent. So it is natural to enquire when strong recurrence of the points in a finite set implies joint strong recurrence of the set. This leads us to the following:

4.7. DEFINITION. A flow (X, G, π) will be said to be *coherent* in case whenever A is a finite subset of X such that each point of A is strongly recurrent, it follows that A is jointly strongly recurrent.

In view of the discussion preceding this definition, we have:

4.8. THEOREM. *Let* $k \geqq 3$ *be an integer. Then the following are equivalent:*
(a) *Riemann hypothesis holds for* $L(.,\chi)$ *for all Dirichlet characters* χ *modulo* k.
(b) *The flow* $(D_k, \mathbf{R}, \sigma)$ *is coherent.*

In the fifth and concluding section, we shall consider the notion of coherence in some details. Here we proceed to obtain yet another equivalent formulation of the Riemann hypotheses.

A *Dirichlet polynomial* is a function of the form $z \to \sum_{n=1}^{N} a_n n^{-z}$. These are entire functions and therefore may be regarded as points in $(H, \mathbf{R}, \sigma)$.

4.9. LEMMA. *Let A be a finite subset of $(H, \mathbf{R}, \sigma)$ which is jointly strongly recurrent. Let B be a finite set of Dirichlet polynomials. Then $A \cup B$ is jointly strongly recurrent.*

PROOF. Let $A=\{f_1, \dots, f_n\}$, $B=\{g_1, \dots, g_m\}$. We may write $g_j(z)=\sum_{r=1}^{N} a_{rj} r^{-z}$. That is, $g_j=\sum_{r=1}^{N} a_{rj} h_r$, $1 \leqq j \leqq m$ where h_r is given by $h_r(z)=r^{-z}$, $z \in \Omega$. Each h_r is clearly a periodic point of $(H, \mathbf{R}, \sigma)$. Therefore by Proposition 2.7, $\{f_1, \dots, f_n, h_1, h_2, \dots, h_N\}$ is jointly strongly recurrent. That is, by Proposition 2.4, $(f_1, \dots, f_n, h_1, \dots, h_N)$ is a strongly recurrent point of the product flow $(H^{n+N}, \mathbf{R}, \sigma)$. Also, the map from H^{n+N} into H^{n+m} which sends $(\theta_1, \dots, \theta_{n+N})$ into $(\theta_1^*, \dots, \theta_{n+m}^*)$, with $\theta_j^*=\theta_j$ if $1 \leqq j \leqq n$ and $\theta_{n+j}^*=\sum_{r=1}^{N} a_{rj} \theta_{n+r}$ for $1 \leqq j \leqq m$, is a flow homomorphism of $(H^{n+N}, \mathbf{R}, \sigma)$ into $(H^{n+m}, \mathbf{R}, \sigma)$ which sends the strongly recurrent point $(f_1, \dots, f_n, h_1, \dots, h_N)$ into the point $(f_1, \dots, f_n, g_1, \dots, g_m)$. Therefore by Proposition 2.3,

$$(f_1, \dots, f_n, g_1, \dots, g_m)$$

is strongly recurrent. Hence, by Proposition 2.4, $A \cup B=\{f_1, \dots, f_n, g_1, \dots, g_m\}$ is jointly strongly recurrent.

4.10. PROPOSITION. *Let A be a finite subset of $(H, \mathbf{R}, \sigma)$ which is jointly strongly recurrent. Let B be a finite subset of $(H, \mathbf{R}, \sigma)$ such that each point in B is given by a Dirichlet series which converges uniformly over Ω. Then $A \cup B$ is jointly strongly recurrent.*

PROOF. Let $A=\{f_1, \dots, f_n\}$, $B=\{g_1, \dots, g_m\}$. Let $\varepsilon>0$ be arbitrary and let us choose $\eta>0$ and a compact set $K \subset \Omega$ such that whenever $h_1, h_2 \in H$ satisfies $\sup_{z \in K} |h_1(z)-h_2(z)|<\eta$, it follows that $\varrho(h_1, h_2)<\frac{\varepsilon}{3}$.

By assumption, there exists Dirichlet polynomials $g_1^*, \dots, g_m^*$ such that $\sup_{z \in \Omega} |g_j(z)-g_j^*(z)|<\eta$. Hence, by choice of η, we have $\varrho(g_j, g_j^*)<\frac{\varepsilon}{3}$ and $\varrho(\sigma^t(g_j), \sigma^t(g_j^*))<\frac{\varepsilon}{3}$ for all real t.

Now, by the preceding lemma, there exists a set U of reals with $\bar{\delta}(U)>0$ such that whenever $t \in U$, we have $\varrho(\sigma^t(f_j), f_j)<\varepsilon$ for $1 \leqq j \leqq n$ and $\varrho(\sigma^t(g_j^*), g_j^*)<\frac{\varepsilon}{3}$

for $1\leqq j\leqq m$. Hence the triangular inequality for ϱ implies that for $t\in U$ we also have $\varrho(\sigma^t(g_j), g_j)<\varepsilon$ for $1\leqq j\leqq m$. Since $\varepsilon>0$ was arbitrary, this implies $A\cup B=\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$ is jointly strongly recurrent.

For $k\geqq 1$ let P_k denote the set of all $f\in H$ which are obtained by analytic continuation of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$, $\mathrm{Re}\,(z)>1$, where $\{a_n\}$ is a sequence of complex numbers which is periodic modulo k. We also let $P=\bigcup_{k=1}^{\infty} P_k$. Then we have:

4.11. Theorem. *For any integer* $k\geqq 1$, *the following are equivalent:*

(a) *The Riemann hypothesis for* $L(.,\chi)$ *for all Dirichlet characters* χ *modulo* k.

(b) *Each member of* P_k *is a strongly recurrent point of* $(H, \mathbf{R}, \sigma)$.

In consequence, the generalized Riemann hypothesis holds for all Dirichlet L-functions if and only if all points of P are strongly recurrent.

Proof. Since for characters χ modulo k, $L(.,\chi)$ is in P_k, (b)⇒(a) is a consequence of Theorem 3.7. To prove (a)⇒(b), let $f\in P_k$. Then we have $f(z)=\sum_{m=1}^{\infty} a_m m^{-z}$ for $\mathrm{Re}\,(z)>1$, where $\{a_m\}$ is periodic modulo k. Let E_k be the set of all integers $m\geqq 1$ such that whenever a prime p divides m, p also divides k. Let $\chi_1, \ldots, \chi_n$ be the Dirichlet characters modulo k. For $1\leqq j\leqq n$ and $m\in E_k$, let us define $\alpha_{m,j}$ by

$$\alpha_{m,j}=\frac{1}{n}\sum_{h=1}^{k}\overline{\chi_j(h)\,a}_{mh}, \quad 1\leqq j\leqq n.$$

Let us now define g_j in H by

$$g_j(z)=\sum_{m\in E_k}\alpha_{m,j}m^{-z}, \quad z\in\Omega, \quad 1\leqq j\leqq n.$$

Clearly the above series converges absolutely for $\mathrm{Re}\,(z)>0$ and hence uniformly for z in Ω. Also, we have the formula

$$(*) \qquad f=\sum_{j=1}^{n} g_j L(.,\chi_j)$$

which may be verified by direct manipulation for $\mathrm{Re}\,(z)>1$ and hence by analytic continuation.

If (a) holds, then by Theorem 4.1, $\{L(.,\chi_j)\colon 1\leqq j\leqq n\}$ is jointly strongly recurrent. Hence, by Proposition 4.10, $\{L(.,\chi_j)\colon 1\leqq j\leqq n\}\cup\{g_j\colon 1\leqq j\leqq n\}$ is jointly strongly recurrent. Hence by Proposition 2.6, f given by the formula $(*)$ is strongly recurrent. So we are done.

4.12. Example. In view of Theorem 4.11 it might be natural to conjecture that whenever $f\in H$ is given by analytical continuation of a formula $f(z)=\sum_{n=1}^{\infty} a_n n^{-z}$, $\mathrm{Re}\,(z)>1$, with $\{a_n\}$ bounded, f is a strongly recurrent point of $(H, \mathbf{R}, \sigma)$. The following example shows that this is false. Take $a_n=(-1)^{n-1}n^{-1/2}$, $n\geqq 1$. Then

$\{a_n\}$ is bounded, $f(z)=\sum_{n=1}^{\infty} a_n n^{-z}=(1-2^{3/4-z})\zeta\left(z+\frac{1}{4}\right)$. Thus f is entire and therefore may be regarded as a point in H. However this function f is not a strongly recurrent point of $(H, \mathbf{R}, \sigma)$. To prove this, let K be the compact subset of Ω given by

$$K=\left\{x+iy\colon \frac{5}{8}\leqq x\leqq\frac{7}{8},\quad -\frac{\pi}{\log 2}\leqq y\leqq\frac{\pi}{\log 2}\right\}.$$

It is easy to see that f is nonzero on K. Therefore we may choose ε such that $0<\varepsilon<$ $<\operatorname*{Inf}_{z\in K}|f(z)|$. If f were strongly recurrent, then the set A of all real t such that $\sup_{z\in K}|f(z+it)-f(z)|<\varepsilon$ would satisfy $\bar{\delta}(A)>0$, and in particular A would be unbounded. For $t\in A$, $K-it$ contains no zero of f. Since each point $\frac{3}{4}+\frac{2\pi mi}{\log 2}$ with $m\in Z$, $m\neq 0$, is a zero of f, this can not happen if $|t|$ is sufficiently large. This contradiction shows that f is not strongly recurrent.

5. On coherent flows

We begin with an inheritance theorem for coherence:

5.1. Proposition. *Let (X, G, π) be a flow, $h\in G$, $h\neq 0$. Then (X, G, π) is coherent if and only if the discrete subflow (X, Z, π_h) is coherent.*

Proof. This is an immediate consequence of Propositions 2.2 and 2.5.

Next we consider some negative examples. Let K be the space of all bisequences of signs. That is, K consists of sequences $\varepsilon_n\colon n\in Z$ with $\varepsilon_n=\pm 1$. We give it the topology of pointwise convergence. Clearly, there is a metric on K which induces the topology of pointwise convergence. Let $\pi^1\colon K\to K$ be the homeomorphism defined by $\pi^1(\varepsilon)=\varepsilon'$ when $\varepsilon'_n=\varepsilon_{n+1}$, $n\in Z$. π^1 induces a discrete flow (K, Z, π) on K. K is clearly a compact topological group under the operation of pointwise multiplication. Let μ be the unique Haar probability measure on K.

5.2. Proposition. *The flow (K, Z, π) is not coherent. Indeed, whenever A is a Borel subset of K with $\mu(A)>0$, there exist ε, ε' in A such that both ε and ε' are strongly recurrent but $\{\varepsilon, \varepsilon'\}$ is not jointly strongly recurrent.*

Proof. Relative to the probability measure μ, π^1 is a measure preserving and ergodic transformation (see [7, pp. 40 and 53]). Hence by the individual ergodic theorem ([7, p. 31]), we have, for every nonempty open set $U\subseteqq K$,

$$\bar{d}(\{n\in Z\colon \pi^n(\varepsilon)\in U\})\geqq\mu(U)>0$$

holds for almost all (μ) ε in K.

Since the topology of K admits a countable basis, it follows that almost all (μ) points of K are strongly recurrent. That is, if L denotes the set of all strongly recurrent points of K, then $\mu(L)=1$. Therefore $\mu(A\cap L)=\mu(A)>0$. Hence, replacing A by $A\cap L$ if necessary, we may assume that $A\subseteqq L$.

Suppose, now, that each pair ε, δ in A is jointly strongly recurrent. The map from K^2 to K sending (ε, δ) to $\varepsilon\delta^{-1}=\varepsilon\delta$ is a flow homomorphism. Hence by Proposition 2.6, the image $AA^{-1}=\{\varepsilon\delta^{-1}\colon \varepsilon\in A,\ \delta\in A\}$ of A under this map consists of strongly recurrent points. That is, we have $AA^{-1}\subseteq L$. But K is a compact group and A is a Borel subset with $\mu(A)>0$. Therefore Theorem B in [5, p. 68], suitably generalised, implies that AA^{-1} contains a neighbourhood of the identity 1 of K. Hence this neighbourhood is contained in L. That is, there exists an integer $N\geqq 1$ such that whenever ε in K satisfies $\varepsilon_n=1$ for $-N\leqq n\leqq N$, it follows that ε is strongly recurrent. This absurd conclusion establishes the proposition.

5.3. PROPOSITION. *The flow* $(H, \mathbf{R}, \sigma)$ *is not coherent.*

PROOF. Let $(K, \mathbf{Z}, \pi)$ be the flow introduced above. Let $\varphi\colon K\to H$ be defined by: for ε in K, $\varphi(\varepsilon)=f_\varepsilon$ where

$$f_\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n(z-in)^{-2}, \quad z\in\Omega.$$

Clearly φ is a flow homomorphism of $(K, \mathbf{Z}, \pi)$ into the discrete subflow modulo 1 $(H, \mathbf{Z}, \sigma_1)$ of $(H, \mathbf{R}, \sigma)$. Since φ a is 1—1 continuous map from the compact space K, it is a homeomorphism of K onto $\varphi(K)$. Thus φ embeds (K, Z, π) as a subspace restriction of (H, Z, σ_1). Hence Proposition 5.2 implies that (H, Z, σ_1) is not coherent. Hence by Proposition 5.1, $(H, \mathbf{R}, \sigma)$ is not coherent.

Next we proceed to obtain a sufficient condition for coherence. We begin with:

5.4. DEFINITIONS. We say that a flow (X, G, π) is *equicontinuous* in case for every x_0 in X and for every $\varepsilon>0$, there exists an $\eta>0$ (possibly depending on both x_0 and ε) such that whenever x in X satisfies $\varrho(x, x_0)<\eta$, we have for all t in G $\varrho(\pi^t(x), \pi^t(x_0))<\varepsilon$. That is, the requirement is that the family $\{\pi^t\colon t\in G\}$ should be equicontinuous at each point of X. Notice that equicontinuity of a flow depends only on the topology of the phase space X (and of course on the behaviour of the phase projection π) and not on the particular metric ϱ.

We say that a flow (X, G, π) is *locally equicontinuous* in case for every finite subset A of X, the subspace restriction $(\mathrm{Sp}\,(A), G, \pi)$ is equicontinuous.

Discrete flows induced by isometries of the phase space are clearly equicontinuous. The next proposition shows that these examples are typical.

5.5. PROPOSITION. *Let* (X, G, π) *be a flow on a metric space* (X, ϱ). *Then* (X, G, π) *is equicontinuous if and only if there exists a metric* ϱ^* *on* X *which is equivalent to* ϱ *and relative to which each* π^t, $t\in G$, *is an isometry.*

PROOF. The 'if' part is trivial. To prove the 'only if' part, we may assume that ϱ is bounded (replace, if necessary, ϱ by the equivalent metric $\min(1, \varrho)$). Define $\varrho^*\colon X\times X\to\mathbf{R}$ by $\varrho^*(x, y)=\sup\{\varrho(\pi^t(x), \pi^t(y))\colon t\in G\}$. A routine check shows that ϱ^* works.

5.6. LEMMA. *Every equicontinuous flow is coherent.*

PROOF. Since a discrete subflow of an equicontinuous flow is clearly equicontinuous, it is sufficient to consider (due to Proposition 5.1) discrete flows. Accordingly, let $(X, \mathbf{Z}, \pi)$ be a discrete flow which is equicontinuous. In view of Proposition 5.5 we may assume that, relative to the given metric ϱ on X, π is an isometry of X.

Now let $\{x_1, \ldots, x_n\}$ be a finite subset of X such that each x_j, $1 \leqq j \leqq n$, is strongly recurrent. Let $\varepsilon > 0$. Then there exists $A_j \subseteqq Z$ with $\bar{d}(A_j) > 0$ such that for $m \in A_j$, $\varrho(\pi^m(x_j), x_j) < \frac{\varepsilon}{2}$, $1 \leqq j \leqq n$. Hence if $r, s \in A_j$, then

$$\varrho(\pi^{r-s}(x_j), x_j) = \varrho(\pi^r(x_j), \pi^s(x_j)) \leqq \varrho(\pi^r(x_j), x_j) + \varrho(\pi^s(x_j), x_j) < \varepsilon.$$

Hence, if we put $B = \bigcap_{j=1}^{n} (A_j - A_j)$, where $A_j - A_j = \{r - s\colon\ r \in A_j,\ s \in A_j\}$, then for $m \in B$, we have $\varrho(\pi^m(x_j), x_j) < \varepsilon$ for $1 \leqq j \leqq n$. Since each A_j satisfies $\bar{d}(A_j) > 0$, Theorem 1 of [6] implies that $\bar{d}(B) > 0$. Hence $\{x_1, \ldots, x_n\}$ is jointly strongly recurrent, and we are done.

5.7 THEOREM. *Every locally equicontinuous flow is coherent.*

PROOF. Let (X, G, π) be a locally equicontinuous flow. Let A be a finite subset of X such that each point in A is strongly recurrent. Since by definition of local equicontinuity the subspace restriction $(\mathrm{Sp}(A), G, \pi)$ is equicontinuous, Lemma 5.6 implies that A is jointly strongly recurrent in $(\mathrm{Sp}(A), G, \pi)$. Hence A is jointly strongly recurrent in (X, G, π) and we are done.

Theorem 5.7 is clearly not the last word on coherence. The next proposition shows that locally equicontinuous flows exhibit a property which can not be expected of general coherent flows.

5.8. PROPOSITION. *A point in a locally equicontinuous flow is strongly recurrent if and only if it is almost periodic.*

PROOF. As before, it suffices to consider discrete flows. Since almost periodicity always implies strong recurrence, we only need to prove the 'only if' part. Accordingly, let $(X, \mathbf{Z}, \pi)$ be a discrete locally equicontinuous flow and let x_0 be a strongly recurrent point in this flow. Arguing, with the equicontinuous flows $(O(x_0), Z, \pi)$, as in the proof of Lemma 5.6, we see that for every $\varepsilon > 0$ there exists a set $A \subseteqq Z$ with $\bar{d}(A) > 0$ such that whenever m is in $B = A - A$, we have $\varrho(\pi^m(x_0), x_0) < \varepsilon$. But by Theorem 2 of [6], B is syndetic. Hence x_0 is almost periodic.

Although the evidence is admittedly very scanty, the results of the previous section tempts one to make the following conjecture. Future work on coherence should be directed towards settling it.

5.9. CONJECTURE. Let D be the space of all $f \in H$ such that f is obtained by analytic continuation of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$, $\mathrm{Re}(z) > 1$, where a_n is a *bounded* sequence of complex numbers. Clearly D is an invariant subset of $(H, \mathbf{R}, \sigma)$. Therefore the subspace restriction $(D, \mathbf{R}, \sigma)$ is a flow in its own right. We *conjecture* that $(D, \mathbf{R}, \sigma)$ is coherent.

Since the flows $(D_k, \mathbf{R}, \sigma)$, $k \geqq 3$, of 4.6 are subspace restrictions of $(D, \mathbf{R}, \sigma)$, in view of Theorem 4.8 we have:

5.10. THEOREM. *The coherence conjecture in* 5.9 *implies the generalised Riemann hypothesis for all Dirichlet L-functions.*

Notice that Example 4.11 shows that all points of $(D, \mathbf{R}, \sigma)$ are not strongly recurrent.

Acknowledgements. This paper is a continuation of work begun in the author's doctoral thesis. The author wishes to take this opportunity to thank Professor G. J. Babu under whose able supervision the thesis was written. His heart-felt thanks are also due to Professor R. Tijdeman for kindly bringing the results of [6] to his attention.

References

[1] B. Bagchi, A joint universality theorem for Dirichlet *L*-functions, *Math. Z.*, **181** (1982), 319—334.
[2] H. Bohr, Über eine quasi-periodische Eigenschaft Dirichletschen Reihen mit Anwendung auf die Dirichletschen *L*-Funktionen, *Math. Annalen,* **85** (1922), 115—122.
[3] W. H. Gottschalk and G. A. Hedlund, Recursive properties of topological transformation groups, *Bull. Amer. Math. Soc.,* **52** (1946), 488—489.
[4] W. H. Gottschalk and G. A. Hedlund, *Topological dynamics,* Amer. Math. Soc. Colloquium publications **36** (1955).
[5] P. R. Halmos, *Measure theory,* Van Nostrand (New York, 1950).
[6] C. L. Stewart and R. Tijdeman, On infinite difference sets, *Can. J. Math.*, **31** (1979), 897—910.
[7] P. Walters, *Ergodic theory — introductory lectures,* Springer Verlag (Berlin, 1975).

(Received October 20, 1984)

THEORETICAL STATISTICS AND MATHEMATICS DIVISION
INDIAN STATISTICAL INSTITUTE
CALCUTTA, INDIA

Acta Math. Hung.
50 (3—4) (1987), 241—247.

SEMI-SIMPLICITY OF ALTERNATIVE LOOP RINGS

E. G. GOODAIRE and M. M. PARMENTER[1] (St. John's)

0. Introduction

A *loop* is a set L together with a binary operation $(g, h) \to gh$ for which there is a two-sided identity element and with the property that the left and right multiplication maps determined by any element of L are one-one and onto. Given any associative and commutative ring R, one can mimic the construction of the group ring to form the *loop ring* RL. We have for some time been intrigued by the fact that a loop ring can be an alternative (but not associative) ring; that is, there exist loop rings which are not associative but in which the *associator* $(x, y, z)=(xy)z-x(yz)$ is an alternating (skew-symmetric) function of its arguments. The variety of alternative rings (that is, rings in which the associator is an alternating function) not only includes the variety of associative rings but resembles it in many ways: many of the ideas and theorems of the associative theory have natural extensions to the alternative case. In this paper, it is our aim to establish conditions which guarantee the semi-simplicity of alternative loop rings with respect to any nil radical and with respect to the Jacobson radical.

Suppose G is a non-abelian group with involution $g \to g^*$ and g_0 is an element in the centre of G which is fixed by *. Let L be the set

$$L = G \cup Gu, \quad u \text{ an indeterminate} \tag{1}$$

together with multiplication defined by

$$\begin{cases} g(hu) = (hg)u \\ (gu)h = (gh^*)u \\ (gu)(hu) = g_0 h^* g \end{cases} \tag{2}$$

for $g, h \in G$.

In [3] it was observed that provided R has no 2-torsion, RL is an alternative ring if and only if $g+g^*$ is in the centre of the group ring RG for all $g \in G$. Thus we were able quickly to produce two alternative loop rings by taking G to be the group of quaternions, the involution to be the inverse map and g_0 either element of the centre of G.

Recent results concerning the structure of *RA loops* (by which we mean those loops whose loop rings are alternative but not associative) can be used to show that

---

[1] This research was supported in part by the Natural Sciences and Engineering Research Council, Grants No. A9087 and A8775. The first author wishes to thank for its hospitality the Department of Mathematics of the University of British Columbia and for his continual encouragement, Dr. C. T. Anderson.

the construction indicated above, at first thought to be very special, in fact describes precisely how all alternative loop rings arise. In order to see this, we first record some features of *RA* loops established in [1].

THEOREM 1. *Let L be an RA loop. Then*

(i) *The centre, $Z(L)$, and nucleus, $N(L)$, coincide.*

(ii) *$g^2 \in Z(L)$ for any $g \in L$.*

(iii) *There exists an element $e \in Z(L)$ of order 2 such that for every g and h in L, either $hg=gh$ or $hg=egh$, and for every g, h and k in L either $g(hk)=(gh)k$ or $g(hk)=e(gh)k$.*

(iv) *There exists a subgroup G of L of index 2, with $Z(G)=Z(L)$, and an element $u \in L-G$ such that every element of L can be written uniquely in the form gu^ε, $g \in G$, $\varepsilon=0$ or 1. Multiplication in L is given by*

$$(3) \qquad \begin{cases} g(hu) = (g\gamma \cdot h\gamma \cdot (gh)\gamma \cdot gh)u \\ (gu)h = (h\gamma \cdot gh)u \\ (gu)(hu) = g\gamma \cdot (gh)\gamma \cdot g_0 gh \end{cases}$$

where $g_0=u^2$ and $\gamma\colon G \to Z(G)$ is defined by

$$g\gamma = \begin{cases} 1 & \textit{if} \quad g \in Z(G) \\ e & \textit{if} \quad g \notin Z(G). \end{cases}$$

It was noted in [1] that the map γ satisfies $(gh)\gamma = g\gamma \cdot h\gamma \cdot hgh^{-1}g^{-1}$. Therefore, defining $g^*=(g\gamma)g$ we obtain an involution on G and the multiplication rules (3) are precisely those of (2). In what follows, we will say that a group G *determines* an *RA* loop L if G determines L in the sense of part (iv) of Theorem 1. We emphasize the significance of Theorem 1: every *RA* loop can be constructed from a group by the method outlined in (1) and (2).

1. The nucleus and centre

In the rest of this paper, R is always a commutative associative ring with identity in which $2x=0 \Rightarrow x=0$ and L is an *RA* loop determined by a group G. We have noted that the map $^*\colon G \to G$ defined by

$$g^* = \begin{cases} g & g \in Z(G) \\ eg & g \notin Z(G) \end{cases}$$

is an involution on G. This extends easily to an involution on the group ring RG: $\left(\sum_{g\in G} \alpha_g g\right)^* = \sum_{g\in G} \alpha_g g^*$. Now the conjugacy class of an element $g \in G$ is $\{g\}$ or $\{g, eg\}$ according as g is central or not (because of Theorem 1 (iii)) and since the class sums (i.e. complete sums of the elements in a conjugacy class) span the centre of RG, it follows that both $g+g^*$ and $g+eg$ are in $Z(RG)$ for any $g \in G$; hence

$$(4) \qquad \alpha+\alpha^* \quad \text{and} \quad (1+e)\alpha \in Z(RG) \quad \text{for any} \quad \alpha \in RG.$$

It can further be readily verified that

$$\alpha \in Z(RG) \Leftrightarrow \alpha^* = \alpha \tag{5}$$

and

$$\alpha \in RG \quad \text{and} \quad e\alpha = \alpha \Rightarrow \alpha \in Z(RG). \tag{6}$$

It is clear from Theorem 1 that any element of *RL* can be written uniquely in the form $x+yu$, where $x, y \in RG$. We refer to x and y as the *RG* and *RGu components* of $x+yu$ respectively. Multiplication in *RL* is very reminiscent of multiplication in a Cayley—Dickson algebra (see for example [4, Sec. 2.2]):

$$(x+yu)(z+wu) = (xz+g_0w^*y)+(wx+yz^*)u, \quad \text{where} \quad x, y, z, w \in RG.$$

PROPOSITION 2. *The centre, $Z(RL)$, and nucleus, $N(RL)$, of the ring RL coincide and are equal to* $\{x+yu | x, y \in Z(RG), ey=y\} = \{x+yu | x \in Z(RG)$ *and* $ey=y\}$.

PROOF. The equality of the two indicated sets follows immediately from (6). The Proposition is therefore established by proving that the nucleus and centre of *RL* are the first of these two sets. Let $r=x+yu$, $s=z+wu$, $t=p+qu$, where $x, y, z, w, p, q \in RG$. Direct calculation gives

$$(r, s, t) = g_0[(w^*y, p)+(q^*w, x)+(q^*y, z^*)]+[q(x, z)+w(x, p^*)+y(z^*, p^*)+ \\ +g_0(q, w^*)y+g_0(w^*q, y)]\,u. \tag{8}$$

Now suppose that $x, y \in Z(RG)$ and $ey=y$. Since for any $\gamma = \sum_{g\in G} \gamma_g g \in RG$, $\gamma-\gamma^* = (1-e) \sum_{g \notin Z(G)} \gamma_g g$, the condition $ey=y$ forces $y(\gamma-\gamma^*)=0$. Consequently

$$(\gamma y)^* = y^*\gamma^* = y\gamma^* = y(\gamma^*-\gamma+\gamma) = y\gamma = \gamma y.$$

In other words, γy is central for all $\gamma \in RG$. It now follows easily that $(r, s, t)=0$; i.e. that $r \in N(L)$. Conversely, suppose $r \in N(L)$ so that $(r, s, t)=0$ for all s and t. For any α and β in *RG*, the centrality of $\alpha+\alpha^*$ and $\beta+\beta^*$ implies that $(\alpha^*, \beta) = (\alpha, \beta^*) = -(\alpha, \beta)$ and $(\alpha^*, \beta^*) = (\alpha, \beta)$. Referring to (8), it follows that

$$g_0[(w^*y, p)+(q^*w, x)+(y^*q, z)]+[q(x, z)-w(x, p)+y(z, p)- \\ -g_0(q, w)y+g_0(w^*q, y)]\,u = 0.$$

Suppose $x \notin Z(RG)$. Then choosing w such that $(w, x) \neq 0$ and setting $q=1, p=z=0$, the *RG* component of (r, s, t) is $g_0(w, x) \neq 0$. Therefore x must be central and

$$g_0[(w^*y, p)+(y^*q, z)]+[y(z, p)-g_0(q, w)y+g_0(w^*q, y)]\,u = 0.$$

If $y \notin Z(RG)$, choose y so that $(y, p) \neq 0$ and set $w=1$, $z=0$. Then the *RG*-component of (r, s, t) is $g_0(y, p) \neq 0$. Thus both x and y are central elements of *RG* and

$$g_0[(w^*y, p)+(y^*q, z)]+[y(z, p)-g_0(q, w)y]u = 0.$$

Easily $y(z, p)=0$ for any $z, p \in RG$, in particular with $z=g$, $p=h$, g and h non-commuting elements of G. Then $gh-hg = gh-egh = (1-e)gh$ and $y(1-e)gh=0 \Rightarrow$

$\Rightarrow y(1-e)=0$ because gh is inverible. Thus the nucleus of RL is the required set. To see that this coincides with the centre of RL, let $r=x+yu$, $s=z+wu$ and compute

$$(r, s) = [(x, z)+g_0(w^*y-y^*w)]+[w(x-x^*)+y(z^*-z)]u.$$

Certainly if x and y are in $Z(RG)$ and $ey=y$ (implying γy central and $y(\gamma-\gamma^*)=0$ for all $\gamma \in RG$), then $(r, s)=0$. On the other hand, $(r, s)=0$ for all s implies $(x, z)=0$, $y(z^*-z)=0$ and $w^*y=y^*w$ for all z and w. With $z=g$, a non-central element in G, we get $ey=y$, and with $w=1$, $y=y^*$. So x and y have the desired properties.

Now for $r=x+yu \in RL$, define $r^*=x^*+eyu$. Then $*$ is an involution on RL. Moreover, the Proposition, together with (5), give immediately

COROLLARY 3. *$r\in Z(RL) \Leftrightarrow r^*=r$ and so, in particular, for any $r\in RL$, $r+r^*$ and rr^* are central elements of RL.*

PROPOSITION 4. *Every non-zero ideal of RL contains a non-zero central element.*

PROOF. Any element r in RL is either in the centre of RL or can be written in the form

$$(*) \qquad r = n+\sum_{g\in S} \alpha_g g$$

where $n \in Z(RL)$, S is some finite non-empty subset of $L-Z(L)$, and no α_g is 0. Suppose that r is a non-zero element of an ideal J. Multiplying by the inverse of a loop element if necessary, we can assume that the coefficient of the identity in the representation of r as a linear combination of loop elements is not zero. Now if $r\in Z(RL)$ then $r\in J\cap Z(RL)$ and we are done; otherwise, write r in the form $(*)$ and choose any $h\in L$ which fails to commute with some $g\in S$. Write $S=S_1\cup T_1$ where $S_1=\{g\in S \mid gh=hg\}$ and $T_1=S-S_1$. By assumption, $T_1\neq\emptyset$ and so $|S_1|<|S|$. Observe that by (4) and Theorem 1 (iii), $g+h^{-1}gh=g+eg\in Z(RL)$ for any $g\in T_1$. Therefore, $r_1=r+h^{-1}rh$ (an element of J)$=2n+2\sum_{g\in S_1}\alpha_g g+\sum_{g\in T_1}\alpha_g(g+eg)=n_1+\sum_{g\in S_1}\beta_g g$ where $n_1\in Z(RL)$. Since the coefficient of the identity in the representation of $h^{-1}rh$ as a linear combination of loop elements is the same as that for r, and since we are assuming that the ring R has no 2-torsion, we know that $r_1\neq 0$. So if $S_1=\emptyset$, $r_1\in J\cap Z(RL)$ and we are done; otherwise since r_1 has been expressed in the form $(*)$, we can repeat for r_1 what we did for r and obtain a non-zero element $r_2\in J\cap Z(RL)$ which can be written in the form $r_2=n_2+\sum_{g\in S_2} r_g g$, where $|S_2|<|S_1|$. Continuing in this fashion we obtain eventually a non-zero element of $J\cap Z(RL)$ because S is finite.

2. Nil ideals

In any *power-associative* ring (one in which, for each natural number n, all nth powers of any element are equal), an element x is *nilpotent* if some positive power $x^n=0$. A ring A is *nil* if each of its elements is nilpotent and *nilpotent* if for some natural number n, all products of n elements of A, with any arrangement of parentheses, are zero; in the latter case, as with associative rings, we write $A^n=0$. A non-associative ring is said to be *semi-prime* if it contains no non-zero *trivial* ideals (that is, ideals I with $I^2=0$), a condition which, if the ring is alternative, is equivalent to the absence of non-zero nilpotent ideals. In general, a semi-prime ring may still contain nil ideals; the next result is therefore of interest.

PROPOSITION 5. *Suppose RL is an alternative loop ring. Then RL is semi-prime if and only if RL contains no non-zero nil ideals. Therefore RL is semi-simple with respect to a particular nil radical if and only if RL is semi-simple with respect to any other nil radical.*

PROOF. A *nil radical* is one which lies between the Baer lower radical and the upper nil radical (see [4; Chapter 8]) and so the last statement is clear. One direction of the first statement is obvious. On the other hand, if RL is semi-prime and J is a non-zero nil ideal in RL, then, by Proposition 4, J contains a non-zero element in $Z(RL)=N(RL)$. Thus J contains a non-zero central element with square 0. Such an element clearly generates a non-zero principal trivial ideal. This contradiction establishes the result.

COROLLARY 6. *Let RL be an alternative loop ring and G any group which determines L. Then if RG is semiprime, so is RL.*

PROOF. If RL contains a non-zero trivial ideal, then $Z(RL)$ contains a non-zero element $x+yu$ with square 0. But $(x+yu)^2=(x^2+g_0y^*y)+(yx+yx^*)u=(x^2+g_0y^2)+ +2yxu$ since $x^*=x$ and $y^*=y$. Therefore, $x^2+g_0y^2=yx=0$ (in the absence of 2-torsion). Hence, $0=x^3+g_0y^2x=x^3$ (because $yx=0$). Thus x is a nilpotent element in the centre of RG. If it is 0, $y^2=0$ and so we can always find a non-zero nilpotent element in the centre of RG. Such an element clearly generates a nilpotent principal ideal of RG, contradicting the assumption that RG is semi-prime.

THEOREM 7. *An alternative loop ring RL is semiprime if and only if the group ring of the centre of L is semiprime.*

PROOF. It is a result of Connell [2] that a group ring RG is semiprime if and only if R is semiprime and G contains no finite normal subgroups whose order is a zero divisor in R. Suppose then that RZ is semiprime $(Z=Z(L))$ and G is any group which determines the RA loop L. In order to show that RL is semiprime, it suffices by Corollary 6 to show that RG is semiprime. If it is not, then by Connell's result, G must contain a finite normal subgroup H whose order is a zero divisor in R. Write $|H|=2^ab$ with b odd. Then certainly b is also a zero divisor in R. Now $HZ(G)/Z(G)\cong H/(H\cap Z(G))$, and since $[G\colon Z(G)]=4$, we conclude that $H\cap Z(G)$ has order divisible by b which is impossible since $H\cap Z(G)$ is a finite normal subgroup of Z. Conversely, if RL is semiprime, then certainly RZ can contain no non-zero nilpotent elements since any nilpotent element in RZ generates a nilpotent ideal in RL. Thus RZ is semiprime and the Theorem follows.

This theorem, together with Proposition 5 (which holds for commutative associative rings as well as for alternative loop rings) reveals

COROLLARY 8. *With respect to any nil radical, RL is semi-simple if and only if the group ring of the centre of L is semi-simple.*

3. The Jacobson radical

We now turn our attention to the Jacobson radical of RL, the largest quasi-regular ideal of RL, also known in the theory of alternative rings as the Smiley or Zhevlakov radical. Our aim is to show that Corollary 8 is also valid for this radical. To do this, we will require the concept of a *normalizing extension.*

DEFINITION. A ring S (with identity) is a *finite normalizing extension* of a subring R with *normalizing basis* $s_1, s_2, \ldots, s_k$ provided R has the same identity as S, $S=\sum_{i=1}^{k} Rs_i$ and $Rs_i=s_iR$ for all s_i.

It will be useful to introduce the *trace* t and *norm* n, functions from RL to $Z(RL)$ defined by $t(r)=r+r^*$ and $n(r)=rr^*$ for $r\in RL$. One can verify that the trace is linear over $Z(RL)$ and *associative* $(t(rs\cdot u)=t(r\cdot su)$ for all r, s and u in RL), that the norm is quadratic over $Z(RL)$ and that RL is *quadratic* as an algebra over its centre:

$$r^2-t(r)r+n(r)=0 \quad \text{for all} \quad r\in RL.$$

PROPOSITION 9. *$Z(RL)$ is a finite normalizing extension of $RZ(L)$.*

PROOF. Let G be any group determining L. Then $Z(G)=Z(L)(=Z)$ and G/Z is the Klein 4-group [1]. Thus G is the union of cosets Z, Za, Zb, Zab for some elements a and b not in Z. Clearly, for no $z\in Z$ is za central, and so the conjugacy class of za is $\{za, eza\}$. Since $Z(RG)$ is spanned by class sums of G, any element in $Z(RG)$ is an R-linear combination of z_i, $z_ja(1+e)$, $z_kb(1+e)$, $z_lab(1+e)$, where $z_i, z_j, z_k, z_l\in Z$; i.e. $x\in Z(RG)\Rightarrow x=t_1+t_2a(1+e)+t_3b(1+e)+t_4ab(1+e)$ where the $t_i\in RZ$. Now if $x+yu\in Z(RL)$, then $x\in Z(RG)$, $y\in Z(RG)$, and $ey=y$. Writing $y=s_1+s_2a(1+e)+s_3b(1+e)+s_4ab(1+e)$ with $s_i\in RZ$, the condition $ey=y$ implies $(1-e)s_1=0$. This means that the coefficients of g and eg in the representation of s_1 as a linear combination of elements of Z are equal; i.e. $s_1=(1+e)s'$ for some $s'\in RG$. It is now apparent that $x+yu$ is an RZ-linear combination of 1, $a(1+e)$, $b(1+e)$, $ab(1+e)$, $(1+e)u$, $a(1+e)u$, $b(1+e)u$, $ab(1+e)u$, these elements all lying in $Z(RL)$ by Proposition 2. Thus these eight elements form a normalizing basis for $Z(RL)$ over RZ.

COROLLARY 10. $J(RZ)=0\Leftrightarrow J(Z(RL))=0$.

PROOF. If S is a finite normalizing extension of R, then for some integer n, $(J(S))^n\subseteqq J(R)S\subseteqq J(S)$. With $S=Z(RL)$ and $R=RZ$, it is clear that $J(Z(RL))=0\Rightarrow J(RZ)=0$. On the other hand, if $J(RZ)=0$, then RZ is semi-prime and $J(Z(RL))$ is nilpotent by the Proposition. But if $Z(RL)$ contains any non-zero nilpotent element, we can clearly find a non-zero nilpotent ideal in RL, contradicting the semi-primeness of RZ assured by Theorem 7.

THEOREM 11. $J(RL)=0\Leftrightarrow J(Z(RL))=0$.

PROOF. Since $J(RL)\cap Z(RL)\subseteqq J(Z(RL))$, if $J(Z(RL))=0$, then $J(RL)=0$ by Proposition 4. On the other hand, suppose $J(Z(RL))\neq 0$. Let a be any non-zero element in this radical. Following an argument of [4, p. 208], we observe that $t(r)a-n(r)a^2$ is in $J(Z(RL))$ and so is a quasi-regular element of RL for any

$r \in RL$. Thus $1-t(r)a+n(r)a^2$ is invertible in $Z(RL)$; call its inverse b. Let $d=ba^2n(r)$ (so that $d+b-bt(r)a=1$) and $f=-bar+d$. Remembering that $r^2=$ $=t(r)r-n(r)$ and that a, b and d are in the centre of RL, $(ar)f=-ba^2r^2+ard=$ $=ard-ba^2t(r)r+d=ar(d-bat(r))+d=ar(1-b)+d=ar+f$, from whence ar is quasi-regular. Hence aRL is a non-zero quasi-regular ideal of RL and $J(RL)\neq 0$.

Combining the result of this Theorem with Corollary 10 we obtain at once

COROLLARY 12. $J(RZ)=0 \Leftrightarrow J(RL)=0$.

When R is a commutative ring, necessary and sufficient conditions for $J(RZ)=0$ are known [2]. So it is known precisely when an alternative loop ring over a commutative ring is Jacobson semi-simple. Interestingly enough, the associative case is still unsettled, even over fields of characteristic zero.

References

[1] Orin Chein and Edgar G. Goodaire, Loops whose loop rings are alternative, to appear.

[2] Ian G. Connell, On the group ring, *Can. J. Math.*, **15** (1963), 650—667.

[3] Edgar G. Goodaire, Alternative loop rings, *Publ. Math. Debrecen,* **30** (1983), 31—38.

[4] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, A. I. Shirshov, *Rings that are Nearly Associative* (translated by Harry F. Smith), Academic Press (New York, 1982).

(Received October 30, 1984)

DEPARTMENT OF MATHEMATICS AND STATISTICS
MEMORIAL UNIVERSITY OF NEWFOUNDLAND
ST. JOHN'S, NEWFOUNDLAND
CANADA A1C 5S7

Acta Math. Hung
50 (3—4) (1987), 249—252.

PROPERTIES OF RING ELEMENTS THAT DETERMINE SUPERNILPOTENT AND SPECIAL RADICALS

S. VELDSMAN (Port Elizabeth)

§ 1. Introduction. In [5], Wiegandt considered certain conditions on properties that elements of a ring may have which determine radical classes. This approach has simplified considerably the process of determining whether a property of ring elements determines a radical class. We extend these ideas to determine which properties yield supernilpotent and special radical classes. In Section 2 we recall the essentials from [5] and in Section 3 we characterize supernilpotent and special radical classes in terms of certain properties. Section 4 is devoted to examples where we apply our theory to some well known properties. We also introduce a new property which generalizes nilpotency and takes the characteristic of the ring into consideration to obtain a special radical class. All rings considered are associative (not necessarily with identity). For the general theory of radicals, see [4] and for special and supernilpotent radicals we refer to [1], [2] and [3].

§ 2. Essentials. Let $\mathscr{P}$ be an abstract property that an element of a ring may possess. $\sim\mathscr{P}$ will denote "not $\mathscr{P}$", the logical negation of $\mathscr{P}$. We assume, for convenience, that the zero of the ring has both $\mathscr{P}$ and $\sim\mathscr{P}$. An element x of a ring A is a *$\mathscr{P}$-element* in A if it has property $\mathscr{P}$. A ring A (subring of A, ideal of A) is a *$\mathscr{P}$-ring* (*$\mathscr{P}$-subring* of A, *$\mathscr{P}$-ideal* of A) if each of their elements is a $\mathscr{P}$-element in A.

The following conditions will be frequently used:

Let I be an ideal in A.

(α) If x is a $\mathscr{P}$-element of A, then the coset $x+I$ is a $\mathscr{P}$-element of A/I.

(β) If $x\in I$ is a $\mathscr{P}$-element of A, then x is also a $\mathscr{P}$-element of I.

(δ) If $x\in I$ is a $\mathscr{P}$-element in I, then x is also a $\mathscr{P}$-element of A.

(ε) If I is a $\mathscr{P}$-ideal of A and the coset $a+I$ is a $\mathscr{P}$-element in A/I, then a is a $\mathscr{P}$-element of A.

For a property $\mathscr{P}$, the class of all $\mathscr{P}$-rings will be denoted by $\mathbf{R}(\mathscr{P})$. If $\mathscr{B}$ is a class of rings, then $\mathscr{S}\mathscr{B}$ is, as usual, defined by

$$\mathscr{S}\mathscr{B} = \{A|A \text{ has no non-zero ideals which are in } \mathscr{B}\}.$$

If $\mathscr{B}$ is a radical class, then $\mathscr{S}\mathscr{B}$ is the semisimple class of $\mathscr{B}$.

2.1. PROPOSITION. *Let $\mathscr{P}$ be a property which satisfies conditions* (α), (δ) *and* (ε). *Then* $\mathbf{R}(\mathscr{P})=\{A|A$ *is a $\mathscr{P}$-ring*$\}$ *is a radical class. If $\mathscr{P}$ also satisfies condition* (β), *then* $\mathbf{R}(\mathscr{P})$ *is a hereditary radical class. Conversely, if $\mathscr{R}$ is a radical class, then there is a property $\mathscr{P}$ which satisfies conditions* (α), (δ) *and* (ε) *such that* $\mathscr{R}=\mathbf{R}(\mathscr{P})$. *If $\mathscr{R}$ is hereditary, then $\mathscr{P}$ also satisfies condition* (β).

Proof. The first statement is just Theorem 1 of [5]. Let $\mathcal{R}$ be a radical class. If $\mathcal{R}(A)$ denotes the $\mathcal{R}$-radical of the ring A, we define a property $\mathcal{P}$ as follows: x is a $\mathcal{P}$-element of A iff $x \in \mathcal{R}(A)$. Clearly, $\mathcal{R}=\mathbf{R}(\mathcal{P})$. Because $\theta(\mathcal{R}(A)) \subseteq \mathcal{R}(\theta(A))=\mathcal{R}(B)$ for any surjective homomorphism $\theta\colon A \to B$, $\mathcal{P}$ satisfies condition (α). (δ) follows from $\mathcal{R}(I) \subseteq \mathcal{R}(A)$ for any ideal I in A. To show that $\mathcal{P}$ also satisfies condition (ε), let I be a $\mathcal{P}$-ideal in A (i.e. $I \subseteq \mathcal{R}(A)$) and let $a+I$ be a $\mathcal{P}$-element in A/I. By (α), the coset $(a+I)+\mathcal{R}(A)/I$ is a $\mathcal{P}$-element in $(A/I)/(\mathcal{R}(A)/I) \cong A/\mathcal{R}(A)$. Because $(A/\mathcal{R}(A))=0$, $a+I \in \mathcal{R}(A)/I$ and hence $a \in \mathcal{R}(A)$ follows. If $\mathcal{R}$ is hereditary, then $\mathcal{R}(I)=I \cap \mathcal{R}(A)$ for any ideal I in A, hence $\mathcal{P}$ also has (β).

Starting with a property $\mathcal{P}$ which satisfies conditions (α), (δ) and (ε), $\mathcal{R}=\mathbf{R}(\mathcal{P})$ is a radical class. Using the above proposition again, there is a property $\mathcal{P}'$ (viz. $a \in \mathcal{R}(A)$) which satisfies conditions (α), (δ) and (ε) such that $\mathbf{R}(\mathcal{P})=\mathcal{R}=\mathbf{R}(\mathcal{P}')$. What is the relationship between $\mathcal{P}$ and $\mathcal{P}'$? Obviously, if x is a $\mathcal{P}'$-element of A, then it is also a $\mathcal{P}$-element. The converse is not true. For example, let $\mathcal{P}$ be the nilpotency property. The $\mathcal{R}=\mathbf{R}(\mathcal{P})$ is a radical class and then $\mathcal{P}'$ is the property $x \in \mathcal{R}(A)$, i.e. iff (x), the ideal in A generated by x, is a nilideal. If A is the complete matrix ring of order two over the integers, then $x=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ is nilpotent (hence a $\mathcal{P}$-element), but (x) is not a nilideal of A (hence x is not a $\mathcal{P}'$-element).

In general, x is a $\mathcal{P}'$-element iff (x) is a $\mathcal{P}$-ideal of A and it is easy to see that $\mathcal{P}$ and $\mathcal{P}'$ will coincide if and only if $\mathcal{S}\mathcal{R}=\{A \mid A$ has no non-zero $\mathcal{P}$-elements$\}$.

§ 3. Special and supernilpotent radicals. We now introduce two more conditions a property $\mathcal{P}$ may possess:

(σ) If x is a $\sim\mathcal{P}$-element in A, then there is a prime ideal I in A, $x \notin I$, such that A/I has no non-zero $\mathcal{P}$-ideals.

(τ) If x is a $\sim\mathcal{P}$-element in A, then there is a semi-prime ideal I in A, $x \notin I$, such that A/I has no non-zero $\mathcal{P}$-ideals.

3.1. Proposition. *Let $\mathcal{P}$ be a property which satisfies conditions (α), (β), (δ) and (σ). Then $\mathbf{R}(\mathcal{P})$ is a special radical class. Conversely, if $\mathcal{R}$ is a special radical class, then there is a property $\mathcal{P}$ which satisfies (α), (β), (δ) and (σ) such that $\mathcal{R}=\mathbf{R}(\mathcal{P})$.*

Proof. $\mathbf{R}(\mathcal{P})$ homomorphically closed and hereditary follows from (α) and (β) respectively. If $A \notin \mathbf{R}(\mathcal{P})$, there is a non-zero $\sim\mathcal{P}$-element x in A. By (σ), there is a prime ideal I in A, $x \notin I$, such that A/I has no non-zero $\mathcal{P}$-ideals. Hence A/I is a non-zero primefactor ring of A which has no non-zero $\mathbf{R}(\mathcal{P})$-ideals by (δ). Corollary 1 in [3] then yields the desired result. Conversely, if $\mathcal{R}$ is a special radical class, there is a property $\mathcal{P}$ (viz. $a \in \mathcal{R}(A)$) which satisfies conditions (α), (β) and (δ) with $\mathcal{R}=\mathbf{R}(\mathcal{P})$ by Proposition 2.1. Using the fact that $\mathcal{R}(A)=\cap\{I \mid I$ an ideal in A such that A/I is a prime ring which has no non-zero ideals in $\mathcal{R}\}$, also (σ) follows.

Likewise, for supernilpotent radical classes, we have:

3.2. Proposition. *If property $\mathcal{P}$ satisfies conditions (α), (β), (δ) and (τ), then $\mathbf{R}(\mathcal{P})$ is a supernilpotent radical class. Conversely, if $\mathcal{R}$ is a supernilpotent radical class, then there is a property $\mathcal{P}$ which satisfies (α), (β), (δ) and (τ) such that $\mathcal{R}=\mathbf{R}(\mathcal{P})$.*

§ 4. Examples. Our first three examples are well known and use standard techniques. We merely illustrate the use of Proposition 3.1 which simplifies the calculations of verifying whether a class of rings, determined by some property of the elements of the rings, is a special radical class.

4.1. Let $\mathscr{P}$ be the nilpotency property. Then $\mathscr{P}$ satisfies conditions (α), (β) and (δ). We show (σ) is also satisfied. Suppose x is a $\sim\mathscr{P}$-element in A. Then $M=\{x, x^2, x^3, \dots\}$ is an m-system that does not contain 0. By Zorn's Lemma, choose an ideal I in A maximal with respect to $I\cap M=\emptyset$. Then I is a prime ideal and $x\notin I$. If B/I is a non-zero $\mathscr{P}$-ideal of A/I, then, for each $b\in B$, there is a positive integer n such that $b^n\in I$. Because $I\subset B$, $B\cap M\neq\emptyset$. Hence $x^m\in I$ for some m which contradicts $I\cap M=\emptyset$. Thus A/I has no non-zero $\mathscr{P}$-ideals. Hence $\mathbf{R}(\mathscr{P})$, the class of all nilrings, is a special radical class.

4.2. Let $\mathscr{P}$ be the left quasi-regular property of an element x in a ring A. Then $\mathscr{P}$ has (α), (β) and (δ). We show $\mathscr{P}$ also satisfies condition (σ). Suppose x is not left quasi-regular in A. Then $I'=\{yx-y\mid y\in A\}$ is a left ideal in A which does not contain x. By Zorn's Lemma, choose a left ideal M in A maximal with respect to $I'\subseteq M$ and $x\notin M$. Let $I=\{a\in A\mid aA\subseteq M\}$. Then I is a (two-sided) ideal in A, $I\subseteq M$ and $x\notin I$. Furthermore, M is a maximal left ideal in A. Using this, I a prime ideal in A follows. Lastly we show that A/I has no non-zero $\mathscr{P}$-ideals. Suppose J/I is a non-zero $\mathscr{P}$-ideal of A/I. Then $JA\nsubseteq M$ and $A^2\nsubseteq M$. From this $JA+{}$ $+M=A$ and $AJA\nsubseteq M$ follows. Let $a\in J$ and $b\in A$ such that $Aab\nsubseteq M$. Because $Aab+M=A$, there is a $y\in A$ such that $b-yab\in M$. From $a\in J$, $ya+I$ a $\mathscr{P}$-element in A/I follows. Hence there is a $z\in A$ such that $z\circ ya\in I$. Then $b=b-{}$ $-(z\circ ya)=(b-yab)-z(b-yab)$ is in M. This means $Aab\subseteq M$ which is a contradiction. Hence $\mathbf{R}(\mathscr{P})$, the Jacobson radical class, is a special radical class.

4.3. Let x be a $\mathscr{P}$-element in the ring A iff x is G-regular, i.e. iff

$$x\in G(x)=\{xa+a+\Sigma\,(b_ixa_i+b_ia_i)\mid a, a_i, b_i\in A\}.$$

Then $\mathscr{P}$ satisfy conditions (α) and (δ). We show the validity of (σ). Suppose x is not a G-regular element in the ring A. Then $x\notin G(x)$. By Zorn's Lemma, choose an ideal I in A maximal with respect to $G(x)\subseteq I$ and $x\notin I$. Then I is a maximal ideal in A and $x+I(\neq 0)$ is the identity element of A/I. Hence A/I is a simple ring with an identity element from which I a prime ideal follows. The additive inverse of the identity (if not zero) in any ring is never G-regular. From this, and the fact that A/I is simple, it follows that A/I has no non-zero $\mathscr{P}$-ideals. Because $\mathbf{R}(\mathscr{P})$, the class of all Brown—McCoy radical rings is hereditary, $\mathbf{R}(\mathscr{P})$ a special radical class follows.

4.4. Let $\mathscr{P}$ be the following property: x is a $\mathscr{P}$-element in A if there exists positive integers k and n such that $(nx)^k=0$. Clearly $\mathscr{P}$ satisfies (α), (β) and (δ). We show $\mathscr{P}$ also has (σ). Let x be a $\sim\mathscr{P}$-element of A. Hence $(nx)^k\neq 0$ for all n and k. Let $S=\{(nx)^k\mid n, k$ positive integers$\}$. Then S is an m-system. For $(nx)^k$ and $(mx)^l$, let $r=n^{l+1}m^{k+1}x$. Then $(nx)^k r(mx)^l=((nm)x)^{k+l+1}\in S$. Because $0\notin S$, by Zorn's Lemma, there is an ideal I in A maximal with respect to $I\cap S=\emptyset$. Then I is a prime ideal, $x\notin I$ and A/I has no non-zero $\mathscr{P}$-ideals. Indeed, let B/I be a non-zero ideal of A/I. Because $I\subset B$, there are positive integers n and k such that $(nx)^k \in B$. If $(nx)^k + I$ is a $\mathscr{P}$-element of B/I, then $\big(m(nx)^k\big)^l \in I$ for some m and l. This contradicts $I \cap S = \emptyset$. Hence $(nx)^k + I$ is a non-zero $\sim\mathscr{P}$-element of B/I. Hence $\mathbf{R}(\mathscr{P})$ is a special radical class and contains all the nilrings as well as the rings (and also fields) with non-zero characteristic.

References

[1] V. A. Andrunakievič, Radicals of associative rings, I, *Amer. Math. Soc. Transl.*, **52**, 95—128. Russian original: *Mat. Sb.*, **44** (1958), 179—212.

[2] V. A. Andrunakievič, Radicals of associative rings, II, *Amer. Math. Soc. Transl.*, **52**, 129—150. Russian original: *Mat. Sb.*, **55** (1961), 329—346.

[3] B. J. Gardner and R. Wiegandt, Characterizing and constructing special radicals. *Acta Math. Acad. Sci. Hungar.*, **40** (1982), 73—83.

[4] R. Wiegandt, Radicals and semisimple classes of rings, *Queen's Papers in Pure and Applied Mathematics* no. 37 (Kingston, Ontario, 1974).

[5] R. Wiegandt, Radicals of rings defined by means of elements, *Österreich. Akad. Wiss. Math. Naturwiss, SB II*, **184** (1975), 117—125.

(Received November 8, 1984)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PORT ELIZABETH
P.O. BOX 1600
6000 PORT ELIZABETH
REPUBLIC OF SOUTH AFRICA

Acta Math. Hung.
50 (3—4) (1987), 253—256.

TYCHONOV'S THEOREM FOR *G*-SPACES

S. A. ANTONYAN (Yerevan) and J. DE VRIES (Amsterdam)

The aim of this note is to present a version of most of the result of the paper [1] in the English language. Briefly, it concerns a version for G-spaces of the well-known result that every Tychonov space of weight τ can be topologically embedded in I^τ, the product of τ copies of the unit interval I. We shall provide full proofs for our results (in [1], only a special case is proven without indication of proofs of the more general cases). Also, we generalize the results of [1] to arbitrary locally compact, sigma-compact groups (in [1], results are stated only for compact and for locally compact second countable groups). Finally, we point out some connections with related results.

The letter G shall always denote a topological group. For terminology and notation concerning G-spaces, we refer to [7].

THEOREM 1. *Let G be locally compact and sigma-compact. Then for every G-space* $\langle X, \pi\rangle$ *with X a compact Hausdorff space of weight* $w(X)=:\tau$ *there exists an action* $\tilde{\pi}$ *of G on* $\mathbf{R}^\tau$ *such that*

(i) *the cube* I^τ *is an invariant subset of* $\mathbf{R}^\tau$ *under this action;*

(ii) *X can equivariantly be embedded in* I^τ.

Moreover, there exists a linear structure on $\mathbf{R}^\tau$ *making* $\mathbf{R}^\tau$ *(with its ordinary product topology) a locally convex topological vector space such that*

(iii) I^τ *is a convex subset of* $\mathbf{R}^\tau$;

(iv) *the action* $\tilde{\pi}$ *is linear (i.e.* $\tilde{\pi}^t\colon \mathbf{R}^\tau\to\mathbf{R}^\tau$ *is linear for every* $t\in G$*).*

PROOF. We assume that X is not finite, so that $\tau\geqq\aleph_0$ (if necessary, replace X by $X\cup I$ (disjoint union) and extend the action of G to this larger space such that all points of I remain invariant). The proof consists of several steps.

Step 1. Let $C_c(G)$ be the space of all real-valued continuous functions on G, endowed with the compact-open topology, and define an action ϱ of G on $C_c(G)$ by $\varrho^t f(s):=f(st)$ for $f\in C_c(G)$ and $s, t\in G$ (for ϱ to be continuous it is essential that G be locally compact; cf. [7; 2.1.4]). Then $C_c(G)^\tau$ is also a G-space, the action of G on $C_c(G)^\tau$ being defined coordinate-wise by ϱ. Since X can be embedded in $\mathbf{R}^\tau$, it follows from [7; 7.1.4] that, as a G-space, X has an equivariant embedding φ in the G-space $C_c(G)^\tau$.

Since G is locally compact, $C_c(G)$ and hence $C_c(G)^\tau$ are *complete* locally convex topological vector spaces. Since $\varphi[X]$ is a compact subset of this space, also the closed convex hull K of $\varphi[X]$ in $C_c(G)^\tau$ is compact [3; Ch. I, § 4, no. 1]. Moreover, the action of G on $C_c(G)^\tau$ is linear and continuous, and this implies that K is invariant in $C_c(G)^\tau$ under the action of G.

Resuming, we have a complete locally convex topological vector space $C_c(G)^\tau$, a linear action of G on it, and we have a compact convex invariant subset K in which X can equivariantly be embedded.

Step 2. This step consists in proving the following statement, which comprises essentially the main idea of [1]:

Let K_0 be an infinite-dimensional compact convex subset of a separable Fréchet space E. Then there exists a homeomorphism $\psi\colon E\to\mathbf{R}^{\aleph_0}$ such that $\psi[K]=I^{\aleph_0}$.

The proof consists of a straightforward application of three results from infinite dimensional topology. First, by the Anderson—Kadec theorem, there exists a homeomorphism $\psi_1\colon E\to\mathbf{R}^{\aleph_0}$, and, second, by Keller's theorem [2; III, Theorem 3.1], there exists a homeomorphism $\psi_2\colon K_0\to I^{\aleph_0}$. Now we have the homeomorphism $\psi_2\circ\psi_1^{\leftarrow}|\psi_1[K_0]\colon \psi_1[K_0]\to I^{\aleph_0}$ between the compact subsets $\psi_1[K_0]$ and $I^{\aleph_0}$ of the infinite dimensional separable Fréchet space $\mathbf{R}^{\aleph_0}$. According to a theorem of Klee [5], this homeomorphism has an extension to a homeomorphism $\eta\colon \mathbf{R}^{\aleph_0}\to\mathbf{R}^{\aleph_0}$. Now let $\psi:=\eta\circ\psi_1$.

Step 3. Our topological group G is assumed to be sigma-compact, so $C_c(G)$ is a Fréchet space. Now observe that we can write $\tau=\tau\cdot\aleph_0$, so the index-set for the product $C_c(G)^\tau$ may assumed to be a disjoint union of τ copies of a given countable set. This fixes a homeomorphism

$$\Phi\colon C_c(G)^\tau\to\prod_{\lambda\in\Lambda}E_\lambda$$

where Λ is a set of cardinality τ and $E_\lambda=C_c(G)^{\aleph_0}$ for every $\lambda\in\Lambda$. From this description it also follows, that Φ is linear and that Φ is equivariant. For every $\lambda\in\Lambda$, let $\Phi_\lambda\colon C_c(G)^\tau\to E_\lambda$ be the composition of Φ with the canonical projection onto E_λ. If we put $K_\lambda:=\Phi_\lambda[K]$, then K_λ is a compact convex invariant subset of E_λ.

Note that E_λ is an infinite-dimensional Fréchet space (a product of countably many Fréchet spaces) and we may assume that K_λ is also infinite dimensional. (If it is not, then proceed as follows: let $J\subseteq C_c(G)$ be the (invariant!) set of all constant functions on G with values in the interval I. Note that J is homeomorphic with I so that, in particular, J is compact. Then $J^{\aleph_0}$ is a compact subset of $C_c(G)^{\aleph_0}=$ $=E_\lambda$, hence $K_\lambda\cup J^{\aleph_0}$ is compact. If we replace K_λ by the closed convex hull of $K_\lambda\cup J^{\aleph_0}$, then we obtain an infinite-dimensional compact convex subset of E_λ, which is still invariant under the action of G.)

It follows that the closed linear subspace F_λ of E_λ generated by K_λ is an infinite dimensional Fréchet space, invariant under the action of G. Moreover, since K_λ is separable (being compact and metrizable) F_λ is separable as well.

Resuming, we have for every $\lambda\in\Lambda$ an infinite-dimensional compact convex subset K_λ of a separable Fréchet space F_λ. Moreover, G acts linearly on F_λ such that K_λ is an invariant subset of F_λ (the action of G on F_λ is, of course, the action which is inherited from the action of G on E_λ in which F_λ is an invariant subspace). Finally, note that the composition of φ (from Step 1 of the proof) and Φ is an equivariant embedding of X into the invariant compact convex subset $\prod_{\lambda\in\Lambda}K_\lambda$ of the linear G-space $\prod_{\lambda\in\Lambda}F_\lambda$.

Step 4. By Step 2, for every $\lambda\in\Lambda$ there exists a homeomorphism $\psi_\lambda\colon F_\lambda\to\mathbf{R}^{\aleph_0}$, with $\psi_\lambda[K_\lambda]=I^{\aleph_0}$. The maps ψ_λ define in the obvious way a homeomorphism $\Psi\colon \prod_{\lambda\in\Lambda} F_\lambda \to (\mathbf{R}^{\aleph_0})^\tau \cong \mathbf{R}^\tau$ such that $\Psi\left[\prod_{\lambda\in\Lambda} K_\lambda\right]=(I^{\aleph_0})^\tau\cong I^\tau$. If the linear structure and the action of G are carried over from $\prod_{\lambda\in\Lambda} F_\lambda$ to $\mathbf{R}^\tau$ via this homeomorphism Ψ, then it is clear that the properties (i) through (iv) of the theorem are valid. □

REMARKS 1. In [1], the theorem is only proved for the case that G is compact and X is a compact metric space. In that case, one needs only Steps 1 and 2 of the above proof (the case that X is finite is dealt with in a different way). Notice, that in [1] the embedding of X into a compact convex invariant set of a linear Fréchet G-space (i.e. Step 1 of the proof) is obtained in a different way, as follows: since G is compact (!), there exists an invariant metric d on X. Let $C_u(X)$ be the space of all continuous functions on X endowed with the topology of uniform convergence, and define an action σ of G on $C_u(X)$ by $\sigma^t f(x)=f(\pi^{t^{-1}}x)$ for $f\in C_u(X)$, $t\in G$, $x\in X$. Then $C_u(X)$ is a separable Fréchet space (for separability, use the Stone—Weierstrass theorem), σ is a linear action of G on $C_u(X)$ and, finally, X can equivariantly be embedded in $C_u(X)$ by means of the mapping $x\to d(x,\cdot)\colon X\to C_u(X)$ (that this mapping is equivariant follows from invariance of the metric d).

In [1], the above theorem (or rather, the stronger Theorem 2 below) is stated without proof for the case that G is locally compact and second countable.

2. For the case $\tau=\aleph_0$ the above theorem, as far as properties (i) and (ii) are concerned (so without the statements about the linear structure) follow easily from [9]. In that case, no assumptions about G need to be made. For a related result, see [2; VI. Corollary 7.1]. Compare also with [6; 3.6] (actually, Theorem 1 above is stronger than this result in [6] in that G is allowed to be only sigma-compact instead of second countable).

In Theorem 1, the linear structure and the action of G in $\mathbf{R}^\tau$ depend on the given G-space $\langle X,\pi\rangle$. The following "universal" result generalises Theorems 3, 4 and 5 in [1] where only second countable locally compact groups or compact groups are considered.

THEOREM 2. *Let G be locally compact and sigma-compact. Then for every infinite cardinal number $\tau\geqq w(G)$, the weight of G, there exists an action $\tilde{\pi}$ of G on $\mathbf{R}^\tau$ such that*

(i) *the cube I^τ is an invariant subset of $\mathbf{R}^\tau$ under this action;*

(ii) *every G-space $\langle X,\pi\rangle$ with X a Tychonov space of weight $w(X)\leqq\tau$ can equivariantly be embedded in I^τ.*

Moreover, there exists a linear structure on $\mathbf{R}^\tau$ such that $\mathbf{R}^\tau$ is a locally convex topological vector space and properties (iii) *and* (iv) *of Theorem* 1 *are valid.*

PROOF. Every G-space $\langle X,\pi\rangle$ with X a Tychonov space of weight $w(X)\leqq\tau$ can equivariantly be embedded in $\langle C_c(G)^\tau,\varrho\rangle$ (compare with Step 1 of the proof of Theorem 1; for the embedding, compactness of X need not be assumed). By [8], the G-space $\langle C_c(G)^\tau,\varrho\rangle$ can equivariantly be embedded in a G-space $\langle X^*,\pi^*\rangle$, where X^* is a compact Hausdorff space of weight

$$w(X^*)\leqq \max\{\mathcal{L}(G),\ w(C_c(G)^\tau)\}.$$

($\mathcal{L}(G)$ is the Lindelöf degree of G.) Since $\mathcal{L}(G)=\aleph_0$ and $w(C_c(G)^\tau)=\tau w(G)=\tau$, it follows that $w(X^*)=\tau$. Now apply Theorem 1 to $\langle X^*,\pi^*\rangle$. □

REMARK. Certain restriction on the group in Theorem 2 seems inevitable. The following example arose in a discussion with Jan van Mill.

Let G be the full homeomorphism group of $\mathbf{Q}$, endowed with the discrete topology, and let G act on $\mathbf{Q}$ in the obvious way. Suppose that $\mathbf{Q}$ could be equivariantly embedded in a compact subset of $\mathbf{R}^{\tau}$ with $\tau=\aleph_0=w(\mathbf{Q})$ and that the action of G on $\mathbf{Q}$ could be extended to an action of G on $\mathbf{R}^{\aleph_0}$. Then the closure X of $\mathbf{Q}$ in $\mathbf{R}^{\aleph_0}$ would be a compactification of $\mathbf{Q}$ such that every homeomorphism of $\mathbf{Q}$ extends to a homeomorphism of X. By [4], this would imply that $X\cong\beta\mathbf{Q}$, a contradiction ($\beta\mathbf{Q}$ cannot be homeomorphic with a subset of the metrizable space $\mathbf{R}^{\aleph_0}$).

References

[1] S. A. Antonyan, Tychonov's theorem in the category of topological transformation groups, *Dokl. Akad. Nauk Arm. SSR,* **71** (1980), 212—216 (Russian).

[2] C. Bessaga and A. Pełczyński, *Selected topics in infinite dimensional topology,* PWN (Wa szawa, 1975).

[3] N. Bourbaki, *Espaces Vectoriels Topologiques,* Hermann (Paris, 1953).

[4] E. K. van Douwen, Characterizations of $\beta\mathbf{Q}$ and $\beta\mathbf{R}$, *Arch. Math.,* **32** (1979), 391—393.

[5] V. Klee, Jr., Some topological properties of convex sets, *Trans. Amer. Math. Soc.,* **78** (1955), 30—45.

[6] J. de Vries, Linearizations of actions of locally compact groups, Preprint June 1979, *Trudy Mat. Inst. Steklov,* **154** (1983), 53—70 (Russian; English translation in: *Proc. Steklov Inst. Math.,* 1984, Issue 4, 57—74).

[7] J. de Vries, *Topological transformation groups: a categorical approach,* Mathematisch Centrum (Amsterdam, 1975).

[8] J. de Vries, *Equivariant embeddings of G-spaces, in: General Topology and its Relations to Modern Analysis and Algebra* IV, Part B (Proc. 4th Prague Topological Symposium, (1976), Prague, 1977, pp. 485—493.

[9] J. E. West, Extending certain transformation group actions in separable infinite dimensiona Fréchet spaces and the Hilbert cube, *Bull. Amer. Math. Soc.,* **74** (1968), 1015—1019

(Received November 15, 1984)

USSR
375049 YEREVAN—49
YEREVAN STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS

CWI
KRUISLAAN 413
1098 SJ AMSTERDAM
THE NETHERLANDS

PRINTED IN HUNGARY

Szegedi Nyomda, Szeged

Instructions for authors. Manuscripts should be typed on standard size paper (25 rows; 50 characters in each row). Mathematical symbols (except sin, log, etc.) will be set in italics, so *use handwritten letters, or indicate them by simple underlining*. The text of theorems, lemmas, corollaries are to be printed also in italics. The words THEOREM, LEMMA. COROLLARY, PROOF, REMARK, PROBLEM, etc. (or their German and French equivalents) should be typed in capital letters. Sub-headings will be set in boldface lower case; indicate them by red underlining. Other special. types of symbols (such as German, Greek, script, etc.) should also be clearly identified Mark footnotes by consecutive superscript numbers. When listing references, please follow the following pattern:

[1] G. Szegő, *Orthogonal polynomials,* AMS Coll, Publ. Vol. XXXIII (Providence 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

For abbreviation of names of journals follow the Mathematical Reviews. After the references give the author's affiliation.

Do not send abstract and second copy of the manuscript. Authors will receive only galley-proofs (one copy). Manuscripts will not be sent back to authors (neither for the purpose of proof-reading nor when rejecting a paper).

Authors obtain 100 reprints free of charge. Additional copies may be ordered from the publisher

Periodicals of the Hungarian Academy of Sciences are sold at the following addresses:

AUSTRALIA
C.B.D. LIBRARY AND SUBSCRIPTION SERVICE
Box 4886, G.P.O., *Sydney N.S.W. 2001*
COSMOS BOOKSHOP, 145 Ackland Street
St. Kilda Melbourne), Victoria 3182

AUSTRIA
GLOBUS, Höchstädtplatz 3, *1206 Wien XX*

BELGIUM
OFFICE INTERNATIONAL DE LIBRAIRIF
30 Avenue Marnix, *1050 Bruxelles*
LIBRAIRIE DE MONDE ENTIER
162 rue du Midi, *1000 Bruxelles*

BULGARIA
HEMUS, Bulvar Ruszki 6, *Sofia*

CANADA
PANNONIA BOOKS, P.O. Box 1017
Postal Station „B", *Toronto, Ontario M5T 2T8*

CHINA
CNPICOR, Periodical Department, P.O. Box 50 *Pekin*

CZECHOSLOVAKIA
MAD'ARSKÁ KULTURA, Národní třida 22
115 66 Praha
PNS DOVOZ TISKU, Vinohradská 46, *Prha 2*
PNS DOVOZ TLAČE, *Bratislava 2*

DENMARK
EJNAR MUNKSGAARD, Norregade 6
1165 Copenhagen K

FEDERAL REPUBLIC OF GERMANY
KUNST UND WISSEN ERICH BIEBER
Postfach 46, 7000 Stuttgart 1

FINLAND
AKATEEMINEN KIRJAKAUPPA, P.O. Box 128
SF-00101 Helsinki 10

FRANCE
DAWSON-FRANCE S. A., B. P. 40, *91121 Palaiseau*
EUROPÉRIODIQUES S. A., 31 Avenue de Versailles,
78170 La Celle St. Cloud
OFFICE INTERNATIONAL DE DOCUMENTATION ET LIBRAIRIE, 48 rue Gay-Lussac *75240 Paris Cedex 05*

GERMAN DEMOCRATIC REPUBLIC
HAUS DER UNGARISCHEN KULTUR
Karl Liebknecht-Straße 9, *DDR-102 Berlin*
DEUTSCHE POST ZEITUNGSVERTRIEBSAMT
Straße der Pariser Kommune 3-4, *DDR-104 Berlin*

GREAT BRITAIN
BLACKWELLS'S PERIODICALS DIVISION
Hythe Bridge Street, *Oxford OXI 2ET*
BUMPUS, HALDANE AND MAXWELL LTD.
Cowper Works, *Olney, Bucks MK46 4BN*
COLLET'S HOLDINGS LTH. Denington Estate *Wellingborough, Northants NN8 2QT*
WM. DAWSON AND SONS LTD., Cannon House *Folkstone, Kent CT19 5EE*
H. K. LEWIS AND CO., 136 Gower Street
London WC1E 6BS

GREECE
KOSTARAKIS BROTHERS INTERNATIONAL
BROOKSELLERS, 2 Hippokratous Street, *Athens-143*

HOLLAND
MEULENHOFF-BRUNA B.V, Beulingstraat 2, Amsterdam
MARTINUS NIJHOFF B.V.
Lange Voorhout 9-11, *Den Haag*
SWETS SUBSCRIPTION SERVICE
347b Heereweg, *Lisse*

INDIA
ALLIED PUBLISHING PRIVATE LTD., 13/14
Asaf Ali Road, *New Delhi 110001*
150 B-6 Mount Road, *Madras 600002*
INTERNATIONAL BOOK HOUSE PVT. LTD.
Madame Cama Road, *Bombay 400039*
THE STATE TRADING CORPORATION OF INDIA LTD.,
Books Import Division, Chandralok
36 Janpath, *New Delhi 110001*

ITALY
INTERSCIENTIA, Via Mazzé 28, *10149 Torino*
LIBRERIA COMMISSIONARIA SANSONI, Via Lamarmora 45, *50121 Firenze*
SANTO VANASIA, Via M. Macchi 58 *20124 Milano*
D. E. A., Via Lima 28, *00198 Roma*

JAPAN
KINOKUNIYA BOOK-STORE CO. LTD.
17-7 Shinjuku 3 chome, Shinjuku-ku, *Tokyo 160-91*
MARUZEN COMPANY LTD., Book Department,
P.O. Boy 5050 Tokyo International, *Tokyo 100-31*
NAUKA LTD. IMPORT DEPARTMENT
2-30-19 Minami Ikebukuro, Toshima-ku, *Tokyo 171*

KOREA
CHULPNMUL, *Phenjan*

NORWAY
TANUM-TIDSKRIFT-SENTRALEN A.S.,
Karl Johansgatan 41-43, *1000 Oslo*

POLAND
WEGIERSKI INSTYTUT KULTURY,
Marszalkowska 80, *00-517 Warszawa*
CKP W, ul. Towarowa 28, *00-958 Warszawa*

ROUMANIA
D. E. P., *Bucuresti*
ILEXIM, Calea Grivitei 64-66, *Buçuresti*

SOVIET UNION
SOJUZPECHAT—IMPORT, *Moscow*
and the post offices in each town
MEZHDUNARODNAYA KNIGA, *Moscow G-200*

SPAIN
DIAZ DE SANTOS, Lagasca 95, *Madrid 6*

SWEDEN
GUMPERTS UNIVERSITETSBOKHANDEL AB
Box 346, *401 25 Göteborg 1*

SWITZERLAND
KARGER LIBRI AG., Petersgraben 31, *4011 Basel*

USA
EBSCO SUBSCRIPTION SERVICES
P.O. Box 1943, *Birmingham, Alabama 35201*
F.W. FAXON COMPANY, INC.
15 Southwest Park, *Westwood Mass. 02090*
READ-MORE PUBLICATIONS, INC.
140 Cedar Street, *New York, N.Y. 10006*

YUGOSLAVIA
JUGOSLAVENSKA KNJIGA, Tarezije 27, *Beograd*
FORUM, Vojvode Mišića 1, *21000 Novi Sad*

Index: 26.014