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ON THE EMBEDDING OF λ-NUCLEAR SPACES INTO PRODUCT OF BANACH SPACES

J. M. GARCÍA-LAFUENTE (Badajoz)

0. Introduction

If E is a nuclear space, Grothendieck [1] proved that for every 0-neighborhood U in E, there is an absolutely convex 0-neighborhood V in E, $V\subset U$ such that $\tilde{E}_V$ is norm-isomorphic to a subspace of l^p (whatever $p\in[1,\infty)$). As usual, $\tilde{E}_V$ is the completion of the linear space $E_V=E/\bigcap_n n^{-1}V$ normed with the gauge of V.

If E is the nuclear space s of rapidly decreasing sequences and F is any infinite-dimensional Banach space with Schauder basis, then for every 0-neighborhood U in E there exists an absolutely convex 0-neighborhood $V\subset U$ in E such that $\tilde{E}_V$ is norm-isomorphic to F (see [6]). This result of Saxon was improved later on by Valdivia [8] proving its validity when E is an arbitrary nuclear space and F any infinite-dimensional separable Banach space.

In the present paper we bring up these results into the context of λ-nuclearity. Namely, we will prove that for certain sequence spaces λ, a Mackey space σ can be found satisfying the following condition: If F is any infinite-dimensional Banach space with Schauder basis, for every 0-neighborhood U in σ there is an absolutely convex 0-neighborhood V in σ such that $\tilde{\sigma}_V$ is norm-isomorphic to F. As a consequence we prove an embedding theorem of λ-nuclear spaces into some product of any given infinite-dimensional Banach space with Schauder basis.

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1. Definitions and previous results

The linear sequence spaces λ we will deal to are assumed throughout to be normal, additive, decreasing rearrangement invariant and such that $\lambda\subset l^q$ for some $q>0$ (see [4] for definitions). The diametral dimension of a Hausdorff locally convex space E, denoted by $\Delta(E)$, is the collection of all sequences (δ_n) of non-negative numbers such that for every 0-neighborhood U in E there is a 0-neighborhood V in E, $V\subset U$ such that $\delta_n(V,U)\leqq M\delta_n$ for every $n\in\mathbf{N}$ and for some $M\geqq 0$. Here $\delta_n(V,U)$ is the n-th Kolmogoroff diameter of V with respect to U. This notion of diametral dimension, different of that of Terzioglu [7], fits better in our context of λ-nuclearity which is defined as follows: A Hausdorff locally convex space E is said to be λ-nuclear if for each $p>0$ the following condition holds: For every 0-neighborhood U in E there is a 0-neighborhood V in E, $V\subset U$ such that $(\delta_n(V,U))\in\lambda^p$ (a sequence (ξ_n) is in λ^p iff (ξ_n^p) is in λ).

Following [4] we denote by λ^+ the subset of all sequences (ξ_n) in λ such that

$\xi_1 \geqq \xi_2 \geqq \ldots \geqq 0$ and we define the linear space $\sigma = \bigcap_{p>0} \bar{\lambda}^p$ where $\bar{\lambda} = \{\tau = (\tau_n);$ $|\tau_n| \leqq \xi_n$ for some $\xi = (\xi_n) \in \lambda^+\}$. The space σ is always assumed to be furnished with its Mackey topology $\mu(\sigma, \sigma^\times)$, where $\sigma^\times$ is the α-dual of σ.

As usual, given a non-bounded increasing sequence $\alpha = (\alpha_n)$ of nonnegative numbers, the power series space of infinite type

$$\Lambda(\alpha) = \Big\{\beta = (\beta_n);\ p_k(\beta) = \sum_{n=1}^{\infty} e^{k\alpha_n} |\beta_n| < +\infty \ \ \forall k \in \mathbf{N}\Big\}$$

is endowed with its natural Fréchet topology defined by the seminorms p_k, $k = 1, 2, \ldots$. If V_k is the unit ball of the seminorm p_k it is known that for every $n \in \mathbf{N}$, $m \in \mathbf{N}$, $k \in \mathbf{N}$, $\delta_n(V_{k+m}, V_k) = e^{-m\alpha_n}$.

We have the following alternative description of a power series space of infinite type.

PROPOSITION 1.1. *If* $\xi_n > 0$ *for every* $n \in \mathbf{N}$ *we define the linear space* $\lambda(\xi) = \{\beta = (\beta_n);\ \sup_{n \in \mathbf{N}} \xi_n^{-1} |\beta_n| < +\infty\}$. *We then have* $\Lambda(\alpha) = \bigcap_{p>0} \lambda(\xi)^p$ *for* $\alpha_n = \ln \xi_n^{-1}$, $n \in \mathbf{N}$.

PROOF. Since $\beta \in \Lambda(\alpha)$ iff $\sum_n \xi_n^{-k} |\beta_n| < +\infty$ for every $k \in \mathbf{N}$, the inclusion $\Lambda(\alpha) \subset \bigcap_{p>0} \lambda(\xi)^p$ is trivial. Conversely, if $\beta \in \lambda(\xi)^p$ for every $p > 0$, given $k \in \mathbf{N}$ let us choose $M > 0$ such that $|\beta_n| \leqq M \xi_n^{k+q}$ for every $n \in \mathbf{N}$ (remember $\lambda \subset l^q$). Then we have $\sum_n \xi_n^{-k} |\beta_n| \leqq M \sum_n \xi_n^q < +\infty$ and $\beta \in \Lambda(\alpha)$.

Let us denote by $s = \Lambda\big((\log n)_n\big)$ the Fréchet nuclear space of rapidly decreasing sequences. We have

PROPOSITION 1.2. *For each* $(\xi_n) \in \lambda \cap \Delta(s^N)$, σ *is topologically isomorphic to* $\Lambda(\alpha)$ *where* $\alpha_n = \ln \xi_n^{-1}$.

PROOF. If we prove $\Lambda(\alpha) = \sigma$ we are done, because the Mackey topology $\mu(\sigma, \sigma^\times)$ in σ is then equal to the (Mackey) topology of $\Lambda(\alpha)$ since $\Lambda(\alpha)' = \sigma^\times$ (cf. [3] § 30). Let $\beta \in \Lambda(\alpha)$. By Proposition 1.1, for every $p > 0$, $|\beta_n|^p \leqq M_p \xi_n$ for every $n \in \mathbf{N}$ and some $M_p > 0$. Since $\xi \in \lambda^+$ we conclude $\beta \in \bigcap_p \bar{\lambda}^p = \sigma$. Conversely, let $\tau \in \lambda^+$. If we exclude the trivial case $\tau_n = 0$ for all but finitely many $n \in \mathbf{N}$, we have $\tau_n > 0$ for all $n \in \mathbf{N}$ and we define $\gamma_n = \ln \tau_n^{-1}$, $n \in \mathbf{N}$. The Fréchet power series space $E = \Lambda(\gamma)$ is even nuclear; indeed, if $q > 0$ is such that $\lambda \subset l^q$, then we have $\sum_n (e^{-q})^{\gamma^n} < +\infty$ and the Grothendieck—Pietsch criterion applies. By the Kōmura—Kōmura Theorem [2], E is topologically isomorphic to some subspace of s^N and hence $\xi \in \Delta(s^N) \subset \Delta(E)$. Taking into account the evaluation of the n-th diameter of Kolmogoroff just before Proposition 1.1, we deduce that there is $m \in \mathbf{N}$ and $M > 0$ such that $\tau_n^m \leqq M \xi_n$ for every $n \in \mathbf{N}$. Thus $\bigcap_{p>0} \bar{\lambda}^p \subset \bigcap_{p>0} \lambda(\xi)^p$ and by Proposition 1.1, $\sigma \subset \Lambda(\alpha)$.

COROLLARY 1.3. *If* $\lambda=l^q$ *for some* $q>0$ *then* $(n^{-p})_n\in\lambda$ *for every* $p>q^{-1}$. *Conversely if* $(n^{-p})_n\in\lambda$ *for some* $p>0$ *then* $\sigma=s$ *(topologically as well).*

PROOF. The first assertion is trivial. For the second one, let us note that, since s^N is nuclear, $(n^{-p})\in\lambda\cap\Delta(s^N)$ ([5], 9.4.1). By the above proposition $\sigma=\Lambda(\alpha)=s$ for $\alpha_n=\ln n^p$, $n\in\mathbf{N}$.

2. The isomorphism theorem

DEFINITION 2.1. *A sequence* (ξ_n), $\xi_n>0$ *for every* $n\in\mathbf{N}$, *is said to be pseudo-λ-nuclear if for every* $k\in\mathbf{N}$ *there is* $k'\in\mathbf{N}$ *and* $r>0$ *such that* $\sum_n \xi_n^{-k}|\beta_n|\leqq r\sup_n \xi_n^{-k'}|\beta_n|$ *for every* $\beta=(\beta_n)\in\sigma$.

It is clear that any sequence $(n^{-p})_n$ for $p>1/2$, is pseudo-λ-nuclear. Associated with any pseudo-λ-nuclear sequence ξ, we construct the power series space of infinite type $\Lambda^\xi=\Lambda(\alpha)$ where $\alpha_n=\ln\xi_n^{-1}$, $n\in\mathbf{N}$, endowed with its natural topology.

LEMMA 2.2. *Let F be any infinite-dimensional Banach space with Schauder basis, and let* ξ *be a pseudo-λ-nuclear sequence. Then for every neighborhood U of* 0 *in* Λ^ξ *there is an absolutely convex neighborhood V of* 0 *in* Λ^ξ *such that* $\widetilde{\Lambda_V^\xi}$ *is norm-isomorphic to F.*

PROOF. The guidelines of the proof are those of [6] modelled to fit our particular situation. Let $\{x_n\}$ be a Schauder basis in F with $\|x_n\|=1$ and with coefficient functionals $\{u_n\}$. Since $\{u_n\}$ is a weakly bounded sequence, the Uniform Boundedness Theorem applies to conclude that $W=\{u_n\}^0$ is a 0-neighborhood in F and its (continuous) gauge q_W satisfies $\sup_n|u_n(x)|=q_W(\sum_n u_n(x)x_n)\leqq M\|x\|$ for every $x=\sum_n u_n(x)x_n\in F$ and for some non-negative constant M. Let U be a 0-neighborhood in Λ^ξ. There is no loss of generality in assuming that $U=\{\beta\in\Lambda^\xi;\ p_k(\beta)\leqq 1\}$ holds for some seminorm p_k generating the topology of Λ^ξ (see Section 1). Let $k'\in N$ and $r>0$ be the numbers associated to k in the Definition 2.1. The (linear) map $\Phi\colon \Lambda^\xi\to F$ given by

$$\Phi(\beta)=rM\sum_{n=1}^{\infty}\xi_n^{-k'}\beta_n x_n,\quad \beta=(\beta_n)\in\Lambda^\xi$$

is well defined since the series $\sum_n \xi_n^{-k'}\beta_n$ is absolutely summable and F is complete. Φ is injective because $\{x_n\}$ is a Schauder basis and Φ has dense range as well. Therefore Λ^ξ endowed with the norm $q(\beta)=\|\Phi(\beta)\|$, $\beta\in\Lambda^\xi$, denoted then (Λ^ξ, q), is norm-isomorphic to a dense subspace of F. The norm q is continuous in Λ^ξ because $q(\beta)\leqq rM\sum_n \xi_n^{-k'}|\beta_n|=rMp_{k'}(\beta)$ for all $\beta\in\Lambda^\xi$ and, thus, $V=\{\beta\in\Lambda^\xi;\ q(\beta)\leqq 1\}$ is an absolutely convex neighborhood of 0 in Λ^ξ. Furthermore $V\subset U$ because

$$p_k(\beta)\leqq r\sup_n\xi_n^{-k'}|\beta_n|=rq_W(\sum_n\xi_n^{-k'}\beta_n x_n)\leqq\|\Phi(\beta)\|=q(\beta).$$

Since q is a norm, Λ_V^ξ is norm-isomorphic to (Λ^ξ, q) and by the preceding remark, $\widetilde{(\Lambda^\xi, q)}$ is norm-isomorphic to F.

We then get the main result. λ is a sequence space and $\sigma = \bigcap_{p>0} \bar{\lambda}^p$ is endowed with the Mackey topology $\mu(\sigma, \sigma^\times)$.

THEOREM 2.3. *If $\lambda \cap \Delta(s^N)$ contains a pseudo-λ-nuclear sequence, for each infinite-dimensional Banach space F with Schauder basis and each 0-neighborhood U in σ, there is an absolutely convex 0-neighborhood V in σ such that $\tilde{\sigma}_V$ is norm-isomorphic to F.*

PROOF. Let ξ be a pseudo-λ-nuclear sequence in $\lambda \cap \Delta(s^N)$. By Lemma 2.2, a neighborhood V can be found such that $\widetilde{\Lambda^\xi_V}$ is norm-isomorphic to F. But Λ^ξ is the space σ by Proposition 1.2.

Assuming the hypothesis of the above proposition, there exists a basis $\mathscr{V}$ of absolutely convex neighborhoods of 0 in σ such that $\tilde{\sigma}_V$ is norm-isomorphic to F for every $V \in \mathscr{V}$. Thus, Moscatelli's Embedding Theorem [4] jointly with Theorem 2.3 yields the following embedding theorem for λ-nuclear spaces

COROLLARY 2.4. *Under the hypothesis of the theorem, every λ-nuclear space E can be embedded into some I-fold topological product of any infinite-dimensional Banach space with Schauder basis.*

PROOF. By [4], Lemma 3, E is topologically isomorphic to a subspace of a product of σ's. On the other hand, σ is topologically isomorphic to a subspace of $\prod_{V \in \mathscr{V}} \tilde{\sigma}_V$. The conclusion now follows from the above remark. It is worth noting that if E is Fréchet, the index set I can be chosen countable because σ is Fréchet as well (Proposition 1.2).

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A THEOREM ON INFINITE DISTRIBUTIVITY FOR POST L-ALGEBRAS

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It is known that every Boolean algebra B satisfies the infinite distributive laws

$$(D_1) \qquad y \sum_{i\in I} x_i = \sum_{i\in I} yx_i,$$

and

$$(D_2) \qquad y + \prod_{i\in I} x_i = \prod_{i\in I} (y+x_i),$$

while an arbitrary distributive lattice may fail to satisfy either or both of these laws. In this note we show that both (D_1) and (D_2) hold for a Post L-algebra $P=(B, L)$ with a finite lattice of constants L.

Post L-algebras were introduced by T. P. Speed in [2] and further investigated by the author in [3] and [4]. It is shown in [2] that if L is a bounded distributive lattice, then every Post L-algebra P is isomorphic to the coproduct of a Boolean algebra B and L, where the coproduct is taken in the category of bounded distributive lattices and lattice homomorphisms preserving 0 and 1. P will be denoted by $P=(B, L)$.

All lattices considered in this note will be distributive lattices with 0 and 1, and all lattice homomorphisms will preserve 0 and 1. We shall use the terminology and notation of [1]. In particular, if L' is a sublattice of L and $S\subseteq L'$, then the least upper bound of S in L' and L will be denoted (whenever they exist) by $\sum\limits_{x\in S}^{L'} x$ and $\sum\limits_{x\in S}^{L} x$ respectively. Similar notation will be used for the greatest lower bounds of S in L' and L. We recall the definition of coproduct.

DEFINITION. Let L_1, L_2 and L be distributive lattices and let $i_1\colon L_1\to L$ and $i_2\colon L_2\to L$ be lattice monomorphisms. The pair $(L, \{i_1, i_2\})$ will be called the *coproduct (=free product)* of L_1 and L_2 if for every distributive lattice D and every pair of lattice homomorphisms $h_1\colon L_1\to D$ and $h_2\colon L_2\to D$, there is a unique lattice homomorphism $h\colon L\to D$ such that $hi_1=h_1$ and $hi_2=h_2$.

We shall denote the coproduct of L_1 and L_2 by $L_1 * L_2$ and to simplify the notation we shall identify L_1 and L_2 with their isomorphic images $i_1(L_1)$ and $i_2(L_2)$. With this convention, $L_1 * L_2$ can then be characterized as follows (cf. [1], Theorem VII.1):

LEMMA 1. *Let L be a distributive lattice generated by the union $L_1\cup L_2$ of two sublattices L_1 and L_2. Then L is the coproduct of L_1 and L_2 if and only if for every $a_1, b_1\in L_1$ and $a_2, b_2\in L_2$, $a_1a_2\leqq b_1+b_2$ implies $a_1\leqq b_1$ or $a_2\leqq b_2$.*

With the above convention every element x in the coproduct $L_1 * L_2$ can be expressed as

(1) $$x = \sum_{i=1}^{n} a_i b_i,$$

or

(2) $$x = \prod_{j=1}^{m} (a_i + b_i),$$

where all $a_i, a_j \in L_1$ and all $b_i, b_j \in L_2$.

We begin by showing that each L_i, $i=1, 2$, is a *regular* sublattice of $L=L_1 * L_2$; that is, for every $S \subseteqq L_i$, if $\sum_{x\in S}^{L_i} x$ exists, then $\sum_{x\in S}^{L} x$ also exists and $\sum_{x\in S}^{L_i} x = \sum_{x\in S}^{L} x$, and similarly for greatest lower bounds.

THEOREM 1. *Let $L=L_1 * L_2$ be the coproduct of two distributive lattices L_1 and L_2. Then L_1 and L_2 are regular sublattices of L.*

PROOF. Let $S=\{x_t\}_{t\in T} \subseteqq L_1$ and suppose that $\sum_{t\in T}^{L_1} x_t = a$. We shall show that $a = \sum_{t\in T}^{L} x_t$. Let $u \in L$ be an upper bound of S and express u as $u = \prod_{i=1}^{n} (a_i + b_i)$, where for all $i \in \{1, 2, \dots, n\}$, $a_i \in L_1$, $b_i \in L_2$ and $b_i \neq 1$. Then for a fixed i and for every $t \in T$, $a_t \cdot 1 = a_t \leqq u \leqq a_i + b_i$. Therefore by Lemma 1 and the fact that $b_i \neq 1$, $a_t \leqq a_i$ for all $t \in T$. Hence $a \leqq a_i \leqq a_i + b_i$ and it follows that $a \leqq \prod_{i=1}^{n} (a_i + b_i) = u$. Thus a is the least upper bound of S in L. Similarly we show that greatest lower bounds in L_1 agree with those in L.

Our aim is to show that a Post L-algebra $P=(B, L)$ with a finite lattice of constants L satisfies the distributive laws (D_1) and (D_2). But first we note that $P=(B, L)$ need not satisfy either of (D_1) or (D_2) if L is infinite. Indeed, it follows from Theorem 1 that if L does not satisfy (D_i), $i=1$ or 2, then any Post L-algebra will also fail to satisfy (D_i). It is not known whether, for infinite L, (D_i) holds in $P=(B, L)$ whenever it holds in L.

Let $P=A * L$, where A and L are distributive lattices and $L=\{l_1, l_2, \dots, l_n\}$ is finite. Then every $x \in P$ has a representation

(3) $$x = \sum_{i=1}^{n} a_i l_i,$$

where each $a_i \in A$. Of course this representation of x is not unique; however, among all these representations there is one which we shall denote by

(4) $$x = \sum_{i=1}^{n} a_i^* l_i$$

and which has the property that for any other representation (3) of x, $a_i^* \geqq a_i$ for every $i \in \{1, 2, \dots, n\}$. The representation (4) will be called the *largest representation* of x and it can be obtained as follows: Let (3) be an arbitrary representation of x. Then

$$x = 1\cdot x = (l_1+l_2+\ldots+l_n)\sum_{i=1}^{n} a_i l_i = \sum_{i=1}^{n} a_i^* l_i.$$

Note that each a_i^* is the least upper bound of $\{a_j\colon l_j \geqq l_i\}$. Moreover, for every $i\in\{1, 2, \ldots, n\}$, $a_i^* = xl_i$, hence $\sum_{i=1}^{n} a_i^* l_i$ is the largest representation of x.

For the remainder of this note we shall assume that $L=\{l_1, l_2, \ldots, l_n\}$ and that every element $x\in P=A*L$ has been expressed by its unique largest representation (4).

LEMMA 2. *Let* $P=A*L$, *where* $L=\{l_1, l_2, \ldots, l_n\}$, *and for every* $t\in T$, *let* $x_t=\sum_{i=1}^{n} a_{t,i}^* l_i$. *If* $\sum_{t\in T}^{P} x_t$ *exists, then for every* $i\in\{1, 2, \ldots, n\}$, $\sum_{t\in T}^{P} a_{t,i}^* l_i$ *exists and*

$$\sum_{t\in T}^{P} x_t = \sum_{i=1}^{n}\Big(\sum_{t\in T}^{P} a_{t,i}^* l_i\Big).$$

PROOF. Let $\sum_{t\in T}^{P} x_t = u$, where $u=\sum_{i=1}^{n} a_i^* l_i$ is the largest representation of u. We shall show that for every $i\in\{1, 2, \ldots, n\}$, $a_i^* l_i = \sum_{t\in T}^{P} a_{t,i}^* l_i$. Indeed, let $i'\in\{1, 2, \ldots, n\}$ be fixed. Then since $a_{i'}^* l_{i'} = ul_{i'}$, $a_{i'}^* l_{i'}$ is an upper bound of $S=$ $S=\{a_{t,i}^* l_i\}_{t\in T}$. Moreover, if $v=\sum_{i=1}^{n} b_i^* l_i$ is another upper bound of S, then it follows from the property of the largest representation that $b_{i'}^* l_{i'}$ is also an upper bound of S. Hence if $b_{i'}^* l_{i'} < a_{i'}^* l_{i'}$, then the element $b_{i'}^* l_{i'} + \sum_{\substack{i=1\\ i\neq i'}}^{n} a_i^* l_i$ would be an upper bound of $\{x_t\}_{t\in T}$ which is less than u. It follows from this contradiction that $a_{i'}^* l_{i'}$ is the least upper bound of S, and the conclusion of the lemma is now clear.

THEOREM 2. *Let A and L be distributive lattices where* $L=\{l_1, l_2, \ldots, l_n\}$ *is finite. Then the coproduct* $P=A*L$ *satisfies the distributive law* (D_i), $i=1, 2$, *if and only if A satisfies* (D_i).

PROOF. By Theorem 1, A is a regular sublattice of $P=A*L$. Hence if (D_i) holds in P, it will also hold in A. Conversely, suppose that (D_1) holds in A. We shall first show that if $\{a_i\}_{i\in I}\subseteqq A$ such that $\sum_{i\in I}^{A} a_i$ exists and if $l\in L$, then $\sum_{i\in I}^{P} la_i$ exists and

$$l\sum_{i\in I}^{A} a_i = \sum_{i\in I}^{P} la_i. \tag{5}$$

Indeed, let $a=\sum_{i\in I}^{A} a_i$. Then la is an upper bound of $S=\{a_i l\}_{i\in I}$ in P. Moreover, if $u=\prod_{j=1}^{k}(b_j+m_j)$ is another upper bound of S in P, then for all $i\in I$ and all $j\in K=$

$=\{1, 2, ..., k\}$, $a_i l \leqq b_j + m_j$ and hence by Lemma 1, $a_i \leqq b_j$ of $l \leqq m_j$. Let $J = \{j \in K: l \leqq m_j\}$ and $J' = \{j \in K: b_j \geqq \sum_{i=1}^{A} a_i = a\}$. Then

$$la \leqq m_j \leqq b_j + m_j, \quad \text{when} \quad j \in J,$$

and

$$la \leqq b_j \leqq b_j + m_j, \quad \text{when} \quad j \in J'.$$

Therefore $la \leqq b_j + m_j$, for all $j \in J \cup J' = K$, so that $la \leqq u$. Thus la is the least upper of S in P and (5) holds. Now, to show that (D_1) holds in P, let $y \in P$ and $\{x_t\}_{t \in T} \subseteqq P$ such that $\sum_{t \in T}^{P} x_t$ exists. Let $y = \sum_{i=1}^{n} a_i^* l_i$ and for every $t \in T$, $x_t = \sum_{i=1}^{n} a_{t,i}^* l_i$ be the largest representations of y and x_t. Then using Lemma 2, equality (5), and the fact that (D_1) holds in A,

$$y \sum_{t \in T} x_t = \Big(\sum_{j=1}^{n} a_j^* l_j\Big)\Big(\sum_{i=1}^{n} \Big(\sum_{t \in T}^{P} a_{t,i}^* l_i\Big)\Big) = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \Big(a_j^* l_j \sum_{t \in T}^{P} a_{t,i}^* l_i\Big)\Big) =$$

$$= \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \Big(a_j^* l_j l_i \sum_{t \in T}^{A} a_{t,i}^*\Big)\Big) = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \Big(l_j l_i \sum_{t \in T}^{A} a_{t,i}^* a_j^*\Big)\Big) =$$

$$= \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \Big(\sum^{P} a_{t,i}^* l_i a_j^* l_j\Big)\Big) = \sum_{i=1}^{n} \Big(\sum_{t \in T}^{P} \Big(a_{t,i}^* l_i \Big(\sum_{j=1}^{n} a_j^* l_j\Big)\Big)\Big) = \sum^{P} x_t y.$$

Thus P satisfies (D_1). To show that (D_2) holds in P whenever it holds in A, we dualize Lemma 2 and the above argument. This completes the proof of the theorem.

Since every Post L-algebra is isomorphic to the coproduct of a Boolean algebra B and L, and B satisfies both (D_1) and (D_2), Theorem 2 gives the following

COROLLARY. Every Post L-algebra $P = (B, L)$ with a finite lattice of constants L satisfies both (D_1) and (D_2).

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A DECOMPOSITION OF CONTINUITY

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1. Introduction

The α-set was first introduced by Njåstad [16]. α-continuity was studied in [15]. In this paper, we introduce the notion of $\mathscr{A}$-set and $\mathscr{A}$-continuous mapping, and prove that a mapping is continuous if and only if it is both α-continuous and $\mathscr{A}$-continuous. Our decomposition of continuity is different from Levine's [10]. We also discuss some properties of $\mathscr{A}$-continuous mappings, and point out that $\mathscr{A}$-continuity is different from twelve known continuities in weak sense.

2. Preliminaries

We recall some known definitions.

DEFINITION 2.1 ([16]). A subset S of a topological space X is said to be an α-set if $S\subset \mathrm{Int}\,(\mathrm{Cl}(\mathrm{Int}\,(S)))$.

DEFINITION 2.2 ([15]). A mapping $f\colon X\to Y$ is said to be α-continuous if for each open set $V\subset Y$, $f^{-1}(V)$ is an α-set in X.

DEFINITION 2.3 ([11]). A subset S of a topological space X is said to be semiopen if there is an open set U in X such that $U\subset S\subset \mathrm{Cl}(U)$. A mapping $f\colon X\to Y$ is said to be semicontinuous if for each open set $V\subset Y$, $f^{-1}(V)$ is a semiopen set in X.

DEFINITION 2.4 ([1, p. 92]). An open (closed) subset of a topological space X is said to be a regular open (regular closed) set if $S=\mathrm{Int}\,(\mathrm{Cl}\,(S))$ $(S=\mathrm{Cl}\,(\mathrm{Int}\,(S)))$.

DEFINITION 2.5 ([10]). A mapping $f\colon X\to Y$ is said to be weakly continuous if for each open neighborhood of $f(x)$ in Y, there is an open neighborhood U of x in X such that $f(U)\subset \mathrm{Cl}\,(V)$. A mapping $f\colon X\to Y$ is said to be weak* continuous if for each open set V in Y, $f^{-1}(\mathrm{Fr}\,(V))$ is closed in X, where $\mathrm{Fr}\,(V)=\mathrm{Cl}\,(V)\setminus \mathrm{Int}\,(V)$.

The following decomposition of continuity was given by Levine [10]:

THEOREM 2.1. *A mapping $f\colon X\to Y$ is continuous if and only if it is both weakly continuous and weak* continuous.*

Rose [18] generalized the above theorem. He weakened weak* continuity to be locally weak* continuity.

3. $\mathscr{A}$-sets

DEFINITION 3.1. A subset of a topological space X is said to be an $\mathscr{A}$-set if $S=U\setminus W$, where U is an open set and W is a regular open set.

It is easily seen that a subset S is an $\mathscr{A}$-set if and only if $S=U\cap C$, where U is an open set and C is a regular closed set. An open set U is an $\mathscr{A}$-set since $U=U\setminus\Phi$.

THEOREM 3.1. *An $\mathscr{A}$-set is semiopen.*

PROOF. Let $S=U\cap C$ be an $\mathscr{A}$-set, where U is open and $C=\mathrm{Cl}\,(\mathrm{Int}\,(C))$. Since $S=U\cap C$, we have $\mathrm{Int}\,(S)\supset U\cap\mathrm{Int}\,(C)$. It is easily seen that $\mathrm{Int}\,(S)\subset S\subset C$, hence $\mathrm{Int}\,(S)=\mathrm{Int}\,(\mathrm{Int}\,(S))\subset\mathrm{Int}\,(C)$. But $\mathrm{Int}\,(S)\subset S\subset U$, hence $\mathrm{Int}\,(S)\subset \subset U\cap\mathrm{Int}\,(C)$. Therefore $\mathrm{Int}\,(S)=U\cap\mathrm{Int}\,(C)$. Now we prove $S\subset\mathrm{Cl}\,(\mathrm{Int}\,(S))$. Let $x\in S$ and V be an arbitrary open set containing x. Then $U\cap V$ is also an open set containing x. Since $x\in C=\mathrm{Cl}\,(\mathrm{Int}\,(C))$, there is a point $z\in\mathrm{Int}\,(C)$ such that $z\neq x$ and $z\in U\cap V$. Hence $z\in U\cap\mathrm{Int}\,(C)=\mathrm{Int}\,(S)$. Therefore $x\in\mathrm{Cl}\,(\mathrm{Int}\,(S))$ and $S\subset\mathrm{Cl}\,(\mathrm{Int}\,(S))$. From $\mathrm{Int}\,(S)\subset S\subset\mathrm{Cl}\,(\mathrm{Int}\,(S))$ we know that S is semiopen.

The most significant property of $\mathscr{A}$-set is the following:

THEOREM 3.2. *Let X be a topological space. Then a subset of X is open if and only if it is both α-set and $\mathscr{A}$-set.*

PROOF. Necessity is trivial.

To prove the sufficiency, let $S=U\cap C$ be an $\mathscr{A}$-set, where U is open and $C=\mathrm{Cl}\,(\mathrm{Int}\,(C))$. Since S is an α-set, we have

$$U\cap C\subset\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(U\cap C)))=\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(U)\cap\mathrm{Int}\,(C)))=$$
$$=\mathrm{Int}\,(\mathrm{Cl}\,(U\cap\mathrm{Int}\,(C)))\subset\mathrm{Int}\,(\mathrm{Cl}\,(U)\cap\mathrm{Cl}\,(\mathrm{Int}\,(C)))=\mathrm{Int}\,(\mathrm{Cl}\,(U)\cap C)=$$
$$=\mathrm{Int}\,(\mathrm{Cl}\,(U))\cap\mathrm{Int}\,(C).$$

Since $U\subset\mathrm{Int}\,(\mathrm{Cl}\,(U))$, we have $U\cap C=(U\cap C)\cap U\subset\mathrm{Int}\,(\mathrm{Cl}\,(U))\cap\mathrm{Int}\,(C)\cap U= =U\cap\mathrm{Int}\,(C)$. Notice $U\cap C\supset U\cap\mathrm{Int}\,(C)$, we have $U\cap C=U\cap\mathrm{Int}\,(C)$, therefore $S=U\cap C$ is an open set.

The $\mathscr{A}$-set is different from semiopen set or α-set. We have the following example.

EXAMPLE 3.1. Let $X=\{a,b,c\}$ with topology $\tau=\{\emptyset,\{a\},X\}$. Then $\{a,b\}$ is semiopen because $\{a\}\subset\{a,b\}\subset X=\mathrm{Cl}\,(\{a\})$. $\{a,b\}$ is also an α-set since $\mathrm{Int}\,(\mathrm{Cl}\,(\{a,b\}))=\mathrm{Int}\,(X)=X\supset\{a,b\}$. But $\{a,b\}$ is not an $\mathscr{A}$-set since $\{a,b\}\neq X$, $\{a,b\}\neq\{a\}$, $\{a,b\}\neq\emptyset$, and $\{a,b\}\neq X\setminus\{a\}$.

4. A decomposition of continuity

DEFINITION 4.1. A mapping $f\colon X\to Y$ is said to be $\mathscr{A}$-continuous if for each open set $V\subset Y$, $f^{-1}(V)$ is an $\mathscr{A}$-set in X.

By Theorem 3.2, we have the following decomposition of continuity:

THEOREM 4.1. *A mapping $f\colon X\to Y$ is continuous if and only if it is both α-continuous and $\mathscr{A}$-continuous.*

Although it is known [15] that α-continuity implies weak continuity, our decomposition is different from Levine's decomposition because $\mathcal{A}$-continuity is independent of both weak and weak* continuities.

EXAMPLE 4.1. Weak continuity does not imply $\mathcal{A}$-continuity.

Let $X=\{a, b, c\}$ with topology $\tau=\{\emptyset, \{a\}, X\}$, $Y=\{a, b, c\}$ with topology $\sigma=\{\emptyset, \{a\}, \{a, b\}, Y\}$. Let $f: X\to Y$ be the identity mapping. Then f is weakly continuous but not $\mathcal{A}$-continuous since $f^{-1}(\{a, b\})=\{a, b\}$ is not an $\mathcal{A}$-set in X.

REMARK 4.1. The mapping f in Example 4.1 is also α-continuous.

EXAMPLE 4.2. $\mathcal{A}$-continuity does not imply weak continuity.

Let $I=[0, 1]$ with the usual topology, $X=I\times I$, $Y=I$, $E=(1/3, 2/3)$. Let $f: X\to Y$ be defined as follows:

$$f((x, y)) = \begin{cases} 0 & \text{if } (x, y)\in I\times E; \\ x & \text{if } (x, y)\in I\times(I\setminus E). \end{cases}$$

Then if $V\subset Y$ is an open set, $0\notin V$, we have

$$f^{-1}(V) = V\times(I\setminus E) = (V\times I)\setminus(V\times E) = (V\times I)\setminus(I\times E),$$

where $I\times E$ is a regular open set in X, hence $f^{-1}(V)$ is an $\mathcal{A}$-set. If $0\in V$, then $f^{-1}(V)=(V\times(I\setminus E))\cup(I\times E)=(V\times I)\cup(I\times E)$, hence $f^{-1}(V)$ is an open set. Therefore f is $\mathcal{A}$-continuous.

The mapping f is not weakly continuous since for $x=(1/2, 1/3)\in X$, $V=(1/4, 3/4)\ni 1/2=f(x)$, there is no open set $U\ni x$, such that $f(U)\subset \mathrm{Cl}\,(V)=[1/4, 3/4]$.

EXAMPLE 4.3. Weak* continuity does not imply $\mathcal{A}$-continuity.

Let $X=\{a, b, c\}$ with topology $\tau=\{\emptyset, \{a\}, X\}$, $Y=\{a, b, c\}$ with discrete topology. Let $f: X\to Y$ be the identity mapping. Then f is weak* continuous but not $\mathcal{A}$-continuous.

EXAMPLE 4.4. $\mathcal{A}$-continuity does not imply weak* continuity.

Let $X=\{a, b, c, d\}$ with topology $\tau=\{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, X\}$, $Y=\{a, b, c, d\}$ with topology $\sigma=\{\emptyset, \{c\}, \{c, d\}, \{a, b, c\}, Y\}$. Let $f: X\to Y$ be the identity mapping. Then f is $\mathcal{A}$-continuous since $f^{-1}(\{c, d\})=\{c, d\}=X\setminus\{a, b\}$ and

$$\mathrm{Int}_X\,(\mathrm{Cl}_X\,(\{a, b\})) = \mathrm{Int}_X\,(\{a, b, d\}) = \{a, b\}.$$

But f is not weak* continuous because $f^{-1}(\mathrm{Fr}\,(\{c, d\})) = f^{-1}(Y\setminus\{c, d\})= =f^{-1}(\{a, b\}) = \{a, b\}$, and $\{a, b\}$ is not closed in X.

EXAMPLE 4.5. Weak continuity and $\mathcal{A}$-continuity do not imply continuity.

Let $X=\{a, b, c, d\}$ with topology $\tau=\{\emptyset, \{b\}, \{c, d\}, \{b, c, d\}, X\}$, $Y=\{a, b, c, d\}$ with topology $\sigma=\{\emptyset, \{a\}, Y\}$. Let $f: X\to Y$ be defined by $f(a)=f(b)=a$, $f(c)=c$, $f(d)=d$. Then f is weakly continuous since $\mathrm{Cl}_Y(\{a\})=Y$; f is $\mathcal{A}$-continuous since $f^{-1}(\{a\})=\{a, b\}=X\setminus\{c, d\}$ and $\mathrm{Int}_X\,(\mathrm{Cl}_X\,(\{c, d\}))=\mathrm{Int}_X\,(\{a, c, d\})=\{c, d\}$. It is easily seen that f is not continuous.

EXAMPLE 4.6. Weak* continuity and $\mathcal{A}$-continuity do not imply continuity.

Let $X=\{a, b, c\}$ with topology $\tau=\{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $Y=\{a, b, c\}$ with discrete topology. Let f: $X \to Y$ be defined by $f(a)=a$, $f(b)=f(c)=b$. Then f is weak* continuous since $\mathrm{Fr}\,(A)=\emptyset$ for every subset A of Y; f is $\mathscr{A}$-continuous since $f^{-1}(\{b\})=\{b, c\}=X\setminus\{a\}$ and $\mathrm{Int}_X\,(\mathrm{Cl}_X\,(\{a\}))=\mathrm{Int}_X\,(\{a, b\})=\{a\}$. It is easily seen that f is not continuous.

5. Properties of $\mathscr{A}$-continuous mappings

Continuities in weak sense have been extensively investigated. We have Singal and Singal's almost continuity [9]; Husain's almost continuity [7, 8]; Stallings' almost continuity [20]; Frolík's almost continuity and weeb continuity [3]; Fomin's θ-continuity [2]; Kempsty's quasicontinuity [9]; Gentry and Hoyle's somewhat continuity [5]; and Levine's weak, weak* and semi-continuities [10,11]. In the following, we will show that $\mathscr{A}$-continuity is different from all of them.

By Theorem 3.1, we have an immediate result.

THEOREM 5.1. *Continuity* $\Rightarrow$ $\mathscr{A}$*-continuity* $\Rightarrow$ *semicontinuity.*

In the above theorem, no converse implication holds. $\mathscr{A}$-continuity does not imply continuity by Example 4.5 or 4.6. To show that semicontinuity does not imply $\mathscr{A}$-continuity, we have the following example.

EXAMPLE 5.1. Let $X=\{a, b, c\}$ with topology $\tau=\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $Y=\{a, b, c\}$ with topology $\sigma=\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Let f: $X\to Y$ be the identity mapping. Then f is semicontinuous since $f^{-1}(\{a, c\})=\{a, c\}$ and $\{a\}\subset\{a, c\}=\mathrm{Cl}_X\,(\{a\})$. But f is not $\mathscr{A}$-continuous because $f^{-1}(\{a, c\})=\{a, c\}$ and $X\setminus\{b\}$ is the only way for $\{a, c\}$ to be expressed as a difference of two open sets in X, while $\mathrm{Int}_X\,(\mathrm{Cl}_X(\{b\}))=\mathrm{Int}_X\,(\{b, c\})=\{b, c\}\neq\{b\}$.

EXAMPLE 5.2. Let $X=\{a, b, c\}$ with topology $\tau=\{\emptyset, \{a\}, X\}$, $Y=\{a, b, c\}$ with topology $\sigma=\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Let f: $X\to Y$ be the identity mapping. Then f is almost continuous in the sense of Singal and Singal, Husain, Stallings, Frolík, and is weebly, quasi-, somewhat, and θ-continuous, but f is not $\mathscr{A}$-continuous.

$\mathscr{A}$-continuous mappings possess a property which the twelve continuities in weak sense (eleven mentioned in the beginning of this section and α-continuity) do not possess.

Let f: $X\to Y$ be a surjective 1—1 mapping, where Y is a T_i space $(i=0, 1, 2)$. If f is continuous, then X is also T_i space $(i=0, 1, 2)$. If f is $\mathscr{A}$-continuous, the assertion is true for $i=0$.

THEOREM 5.2. *If* f: $X\to Y$ *is an* $\mathscr{A}$*-continuous,* 1—1 *mapping and* Y *is a* T_0 *space, then* X *is also a* T_0 *space.*

PROOF. Let $x, y\in X$ and $x\neq y$. Then $f(x), f(y)\in Y$ and $f(x)\neq f(y)$. Since Y is T_0, at least one of $f(x), f(y)$, say $f(x)$, has an open neighborhood V such that $f(x)\in V$ and $f(y)\notin V$. Let $S=f^{-1}(V)$. By $\mathscr{A}$-continuity of f, we have $S=U\cap C$, where U is open and C is a regular closed set, and $x\in S$, $y\notin S$. There are two possible cases for $y \notin S$: (i) $y \notin U$ and (ii) $y \notin C$. In case (i), $x \in U$, $y \notin U$; in case (ii), $y \in X \setminus C$ and $x \in C$, $X \setminus C$ is open and $x \notin X \setminus C$. Therefore X is a T_0 space.

REMARK 5.1. In Example 5.3, Y is T_0 but X is not T_0; in Example 4.1, Y is T_0 but X is not T_0; in Example 4.3, Y is T_2 but X is not T_0.

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THE PRINCIPLE OF LOCALIZATION FOR THE $[J, f(x, y)]$ MEANS

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The $[J, f(x, y)]$ transform is the two dimensional analogue of the $[J, f(x)]$ transform introduced by Jakimovski [2], and is defined in a fully analogous manner. Let $\{s_{mn}\}$ be a sequence and let $f(x, y)$ be differentiable infinitely often with respect to each variable on the positive real line. Let

$$f^{(mn)}(x, y) = \frac{\partial^{m+n}}{\partial x^m \partial y^n} f(x, y),$$

and

$$t(x, y) = \sum_{0,0}^{\infty,\infty} \frac{(-x)^m}{m!} \frac{(-y)^n}{n!} f^{(mn)}(x, y) s_{mn}. \tag{1}$$

If $t(x, y)$ converges for all large x and y, and if $\lim t(x, y) = t$, $x, y \to \infty$, then t is called the $[J, f(x, y)]$ limit, or simply the J-limit, of the sequence $\{s_{mn}\}$.

Ismail [8] has shown that the $[J, f(x, y)]$ transform is regular if and only if

$$f(x, y) = \int_{0,0}^{\infty,\infty} \exp\{-(ux+vy)\} dg(u, v), \quad x, y \geqq 0,$$

so that

$$f^{(mn)}(x, y) = \int_{0,0}^{\infty,\infty} (-u)^m (-v)^n \exp\{-(ux+vy)\} dg(u, v), \tag{2}$$

where $g(u, v)$ is of bounded variation in the sense of Hardy—Krause [1] with total variation equal to $1, 0 \leqq u, v < \infty$, $g(0, 0) = g(0, v) = g(u, 0) = g(0^+, v) = g(u, 0^+) = 0$, and $g(u, v) \to 1$, $u, v \to \infty$.

We use the notation of [7] and let $[c, d; a, b]$ denote a rectangle with vertices at $(a, b), (a, d), (c, b)$ and (c, d), $a < c$; $b < d$. For $0 < \delta < \pi$, let $R(\delta) = [\delta, \delta; -\delta, -\delta]$, $N(\delta) = [\pi, \delta; -\pi, -\delta] \cup [\delta, \pi; -\delta, -\pi]$, $C(\delta) = [\pi, \pi; -\pi, -\pi] \sim N[\delta]$, and $E(\delta) = N(\delta) \sim R(\delta)$. For $0 < \tau < \infty$, let $\Delta(t) = (\infty, \infty; \tau, \tau]$, and let $\theta(\tau) = (\infty, \infty; 0, 0] \sim \Delta(t)$. Then $N(\delta)$ is a cross-neighborhood of the origin, and $\theta(\tau)$ is the τ-neighborhood of the boundary of $(\infty, \infty; 0, 0]$.

Suppose that $h(s, t)$ is defined on the rectangle $R = [c, d; a, b]$, and for each fixed $t, b \leqq t \leqq d$, let $V(h; s)$ be the total variation of h as a function of s alone, $a \leqq s \leqq c$. If the set $\{V(h; s) : b \leqq t \leqq d\}$ is bounded uniformly for almost all t, then $h(s, t)$ is said to be of bounded variation in the restricted Tonelli sense with respect to s on R. It follows that the integral $\int_b^d V(h; s)\, dt$ exists. Bounded variation with respect to t

on R is defined in a similar manner [6]. Bounded variation in this sense does not imply and is not implied by boundedness. Other definitions of bounded variation for functions of two variables are given by Clarkson and Adams [1].

In this paper, we consider the $[J, f(x, y)]$ means of double Fourier series and establish the following result:

THEOREM. *Let $h(s, t)$ be periodic, of period 2π, of bounded variation with respect to s on $[\delta, \pi; -\delta, -\pi]$, and with respect to t on $[\pi, \delta; -\pi, -\delta]$, and let $h(s, t)$ be bounded on $N(\delta)$. Then the $[J, f(x, y)]$ transform of the Fourier series of $h(s, t)$ displays the principle of localization at* $(0, 0)$.

Neither the condition of boundedness nor that of bounded variation in this sense can be relaxed in the statement of the theorem. That boundedness cannot be relaxed follows from a remark of Zygmund ([9], Vol. II, p. 305) relating to the classical Abel means which are $[J, f(x, y)]$ means. That bounded variation in this sense cannot be relaxed follows a similar argument, using the result [4] that there exist $[J, f(x)]$ transforms which display the du Bois Reymond singularity.

PROOF OF THE THEOREM. Let $\{s_{mn}(s, t)\}$ be the sequence of partial sums of the Fourier series of $h(s, t)$. As in [7], it is sufficient to consider the sequence $\{s_{mn}(0, 0)\}$, denoted simply by $\{s_{m,n}\}$. Then

$$(3)\qquad s_{mn} = \frac{1}{4\pi^2} \int\limits_{-\pi,-\pi}^{\pi,\pi} h(s, t) \frac{\sin\left(m+\frac{1}{2}\right)s}{\sin(s/2)} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin(t/2)} \, ds\, dt =$$

$$= \frac{1}{4\pi^2}\Big\{ \int\limits_{R(\delta)} + \int\limits_{E(\delta)} + \int\limits_{C(\delta)} \Big\} = r_{mn} + e_{mn} + c_{mn}.$$

If a regular linear sequence to function transform $[J, f(x, y)]$ is applied to the sequence $\{s_{mn}\}$, and if $j(x, y)$ denotes the corresponding transform, then

$$(4)\qquad j(x, y) = J\{s_{mn}\} = J\{r_{mn}\} + J\{e_{mn}\} + J\{c_{mn}\} = \alpha(x, y) + \beta(x, y) + \gamma(x, y).$$

Since $h(s, t)$ is Lebesgue integrable, the sequence $\{c_{mn}\}$ is a null sequence by the Riemann—Lebesgue lemma. Since the transform is regular, it follows that the J-limit of the sequence $\{c_{mn}\}$ is zero. Thus to prove that the usual concept of the principle of localization holds for the $[J, f(x, y)]$ transform, it is necessary and sufficient to prove that the J-limit of the sequence $\{e_{mn}\}$ is zero. Then the J-limit of the sequence $\{s_{mn}\}$ equals the J-limit of the sequence $\{r_{mn}\}$ and the theorem follows.

We consider the $[J, f(x, y)]$ transform of the sequence $\{e'_{mn}\}$, where

$$e'_{mn} = \frac{1}{4\pi^2} \int\limits_{\delta,0}^{\pi,\delta} h(s, t) \frac{\sin\left(m+\frac{1}{2}s\right)}{\sin(s/2)} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin(t/2)} \, ds\, dt.$$

The results obtained for the transform of this sequence are then easily extended to results for the transform of the sequence $\{e_{mn}\}$ by using the symmetries of the set $E(\delta)$ and interchanging the roles of the variables in a suitable manner. Let $D_{mn}(s, t)$

be the Dirichlet kernel, and by (1), (2) and (3),

$$\beta'(x, y) = J\{e'_{mn}\} = \frac{1}{\pi^2}\sum_{0,0}^{\infty,\infty}\frac{(-x)^m}{m!}\frac{(-y)^n}{n!}\times$$

$$\times\int_{0,0}^{\infty,\infty}(-u)^m(-v)^n\exp\{-(ux+vy)\}dg(u, v)\int_{\delta,0}^{\pi,\delta}h(s, t)D_{mn}(s, t)\,ds\,dt =$$

$$=\frac{1}{\pi^2}\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}\exp\big(-(ux+vy)\big)h(s, t)\left\{\sum_{0,0}^{\infty,\infty}\frac{(ux)^m}{m!}\frac{(vy)^n}{n!}D_{mn}(s, t)\right\}ds\,dt\,dg(u, v) =$$

$$=\frac{1}{\pi^2}\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}h(s, t)\exp\{ux(\cos s-1)\}\exp\{vy(\cos t-1)\}\times$$

$$\times\frac{\sin(ux\sin s+s/2)}{2\sin(s/2)}\frac{\sin(vy\sin t+t/2)}{2\sin(t/2)}ds\,dt\,dg(u, v) = \frac{1}{4\pi^2}\times$$

$$\times\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}h(s, t)\exp\{ux(\cos s-1)\}\exp\{vy(\cos t-1)\}\{\sin(ux\sin s)\sin(vy\sin t)\times$$

$$\times\operatorname{ctn}(s/2)\operatorname{ctn}(t/2)+\sin(ux\sin s)\cos(vy\sin t)\operatorname{ctn}(s/2)+\cos(ux\sin s)\times$$

$$\times\sin(vy\sin t)\operatorname{ctn}(t/2)+\cos(ux\sin s)\cos(vy\sin t)\}\,ds\,dt\,dg(u, v) = I_1+I_2+I_3+I_4,$$

where we have used the dominated convergence theorem, an elementary trigonometric identity, and a result of Lorch ([3], p. 82) by which

$$\sum_0^\infty\frac{(ux)^m}{m!}\sin\left(m+\frac{1}{2}\right)s = \{\exp(ux\cos s)\}\{\sin(ux\sin s+s/2)\}.$$

Now $g(u, v)$ is of bounded variation in the sense of Hardy—Krause and also $g(0, v)=g(0^+, v)=g(u, 0)=g(u, 0^+)$, so that for any $\varepsilon>0$, and any fixed positive K, it is possible to choose τ small enough so that

$$K\int_{\theta(\tau)}|dg(u, v)| < \varepsilon.$$

Then

$$I_4 = \frac{1}{4\pi^2}\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}\{h(s, t)\exp\{u, x(\cos s-1)\}\exp\{vy(\cos t-1)\}\times$$

$$\times\cos(ux\sin s)\cos(vy\sin t)\}\,ds\,dt\,dg(u, v) =$$

$$=\frac{1}{4\pi^2}\Big\{\int_{\theta(\tau)}+\int_{\Delta(\tau)}\Big\}\int_{\delta,0}^{\pi,\delta}\{\dots\}\,ds\,dt\,dg(u, v)$$

and

$$(5)\qquad |I_4| < \Big\{\int_{\theta(\tau)}|dg(u, v)|+\exp\{\tau y(\cos\delta-1)\}\int_{\Delta(\tau)}|dg(u, v)|\Big\}\int_{\delta,0}^{\pi,\delta}|h(s, t)|\,ds\,dt =$$

$$=K\int_{\theta(\tau)}|dg(u, v)|+\exp\{\tau y(\cos\delta-1)\}\int_{\Delta(\tau)}|dg(u, v)| < \varepsilon+o(1),\quad y\to\infty,$$

since in the region of integration $\Delta(\tau)$,

$$\exp\{vy(\cos t-1)\} \leqq \exp\{\tau y(\cos\delta-1)\}, \quad \exp\{ux(\cos s-1)\} \leqq 1,$$

$h(s,t)$ is integrable, $g(u,v)$ is of bounded variation in the sense of Hardy—Krause and ε is arbitrary. Here, we have taken $K=\int_{\delta,0}^{\pi,\delta}|h(s,t)|\,ds\,dt$.

In the same manner, it is equally easy to show that

$$(6) \qquad I_3 = o(1), \quad y\to\infty$$

since

$$|I_3| < \frac{\operatorname{ctn}(\delta/2)}{4\pi^2}\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}|h(s,t)|\exp\{ux(\cos s-1)\}\exp\{vy(\cos t-1)\}ds\,dt|dg(u,v)|.$$

To estimate I_2, we first simplify the integral in the manner of Leviatan and Lorch [4]

$$(7)\quad I_2=\frac{1}{4\pi^2}\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}h(s,t)\exp\{ux(\cos s-1)\}\exp\{vy(\cos t-1)\}\sin(ux\sin s)\times$$

$$\times\cos(vy\sin t)\cos 2/s)\left\{\frac{2}{s}-\left(\frac{2}{s}-1/\sin(s/2)\right\}ds\,dt\,dg(u,v)=\right.$$

$$=\frac{1}{2\pi^2}\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}h(s,t)\exp\{ux(\cos s-1)\}\exp\{vy(\cos t-1)\}\times$$

$$\times\cos(s/2)\cos(vy\sin t)\frac{\sin(ux\sin s)}{s}ds\,dt\,dg(u,v)+I_2'.$$

Since $(2/s-1/\sin(s/2))$ is bounded, $0<s\leqq\pi$, $I_2'=o(1)$ as $y\to\infty$ by an argument similar to the one used in getting the estimate for I_4. As for the remaining integral, $\cos s-1<-s^2/3$, $0<s\leqq 1$, and so

$$\exp\{ux(\cos s-1)\} < \exp\{-uxs^2/3\}.$$

Also,

$$|\sin uxs-\sin(ux\sin s)| \leqq ux(s-\sin s) \leqq uxs^3/3$$

in the region of integration. Hence we may equate the remaining integral to $I_2''+I_2'''$, where

$$I_2''=\frac{1}{2\pi^2}\int_{0,0}^{\infty,\infty}\int_{\delta,0}^{\pi,\delta}h(s,t)\exp\{ux(\cos s-1)\}\exp\{vy(\cos t-1)\}\times$$

$$\times\cos(s/2)\cos(vy\sin t)\frac{\sin(uxs)}{s}ds\,dt\,dg(u,v)$$

and

$$I_2''' = \frac{1}{2\pi^2} \int_{0,0}^{\infty,\infty} \int_{\delta,0}^{\pi,\delta} h(s,t) \exp\{ux(\cos s-1)\} \exp\{vy(\cos t-1)\} \times$$
$$\times \{\sin(ux \sin s) - \sin uxs\} \frac{ds\,dt}{s}\, dg(v,u),$$

so that

$$|I_2'''| < \int_{0,0}^{\infty,\infty} \int_{\delta,0}^{\pi,\delta} |h(s,t)| \exp\{-uxs^2/3\} \exp\{vy(\cos t-1)\} \{uxs^3/3\}\, ds\,dt\, |dg(u,v)| <$$
$$< e^{-1}\delta \int_{0,0}^{\infty,\infty} \int_{\delta,0}^{\pi,\delta} |h(s,t)| \exp\{vy(\cos t-1)\}\, ds\,dt\, |dg(u,v)| = o(1), \quad y \to \infty,$$

by the same reasoning as used in getting the estimate for I_4. Here we have used the inequality $ze^{-z} \leqq e^{-1}$, $z \geqq 0$. Then by (7) and (8), and the estimates for I_2' and I_2'''. we have

$$(9) \qquad I_2 = \frac{1}{2\pi^2} \int_{0,0}^{\infty,\infty} \int_{\delta,0}^{\pi,\delta} h(s,t) \exp\{ux(\cos s-1)\} \exp\{vy(\cos t-1)\} \times$$
$$\times \cos(s/2) \cos(vy \sin t) \frac{\sin uxs}{s}\, ds\,dt\, dg(u,v) + o(1), \quad y \to \infty.$$

Now $h(s,t)$ is of bounded variation in the restricted Tonelli sense with respect to s on $[\delta, \pi; 0, \delta]$, and so for each fixed t, it can be expressed as the differencee of two non-negative functions, each monotonically decreasing in s. Doing this for all t, it follows that $h(s,t)$ can be expressed as the difference of two functions, each non-negative and monotonically decreasing in s. Thus the integral in (9) can be split into two integrals, where in each case, $h(s,t)$ is replaced by a function which is non-negative and monotonically decreasing in t. Thus we may assume that $h(s,t)$ is already such a function. Then so, also, is the function $h(s,t) \exp\{ux(\cos s-1)\}$ for every non-negative u and x.

If $ux=0$, then the integrand in (9) is zero. Otherwise, we can write

$$\left| \int_{\delta,0}^{\pi,\delta} h(s,t) \exp\{ux(\cos s-1)\} \exp\{vy(\cos t-1)\} \{\cos(s/2) \cos(vy \sin t) \frac{\sin(uxs)}{s}\, ds\,dt \right| =$$
$$= \left| \int_{\delta}^{\pi} \cos(vy \sin t) \exp\{vy(\cos t-1)\} \int_{0}^{\delta} h(s,t) \exp\{ux(\cos s-1)\} \cos(s/2) \times \right.$$
$$\left. \times \frac{\sin uxs}{s}\, ds\,dt \right| \leqq \int_{\delta}^{\pi} \exp\{vy(\cos t-1)\} \sum_n{}' (-1)^n \int_{(n-1)\pi/ux}^{n\pi/ux} h(s,t) \exp\{ux(\cos s-1)\} \times$$
$$\times \left| \frac{\sin uxs}{s} \right| ds\,dt < \exp\{vy(\cos\delta-1)\} \int_{\delta}^{\pi} \left\{ \int_{0}^{\pi/ux} h(s,t)\, ux\, ds \right\} dt <$$
$$< B\pi^2 \exp\{vy(\cos\delta-1)\} = o(1), \quad y \to \infty.$$

Here, B is the upper bound for $h(s, t)$, and the prime in the summation indicates summation from $n=1$ to $n=[\delta ux/\pi]$, with the additional last term of

$$(-1)^{n+1} \int_{n\pi/ux}^{\delta} h(s, t) \exp\{ux(\cos s-1)\} \left|\frac{\sin uxs}{s}\right| ds, \quad n = [\delta ux/\pi].$$

Thus, the integral in (9) is majorized by

$$\pi^2 B \int_{0,0}^{\infty,\infty} \exp\{vy(\cos\delta-1)\}\,|dg(u, v)|+o(1) = o(1)+o(1) = o(1), \quad y \to \infty$$

by applying the same argument as in (5) above. It follows that under the assumed restrictions on $h(s, t)$,

(10) $$I_2 = o(1), \quad y \to \infty.$$

That $I_1=o(1)$, $y\to\infty$, follows the same argument as in obtaining the estimate for I_2, since $(\sin(t/2))^{-1}$ is bounded in the interval $0<\delta\leqq t\leqq\pi$, and so can be absorbed in $h(s, t)$. Collecting the results, it follows that $\beta'(x, y)=o(1)$, $y\to\infty$.

As indicated, the same argument can be applied to get an estimate for the contribution to $\beta(x, y)$ as a result of integrating over $[0, \pi;\ -\delta, \delta]$, and then over $[\delta, -\delta;\ -\delta, -\pi]$. Adding these contributions, it follows that their sum is $o(1)$, $y\to\infty$. Similarly, the contribution to $\beta(x, y)$ due to integration over

$$[-\delta, \delta;\ -\pi, -\delta]\cup[\pi, \delta;\ \delta, -\delta]$$

is $o(1)$, $x\to\infty$. Combining the results, it follows that $\beta(x, y)=o(1)$ as $x, y\to\infty$, and the proof is complete.

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NOTES ON THE INAUGURAL LECTURE DELIVERED BY FREDERIC RIESZ IN 1925 AS RECTOR OF SZEGED UNIVERSITY[1]

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Several examples have been given by Frederic Riesz in his inaugural lecture in Szeged to the problem of keeping theorems of higher mathematics so that one uses only tools of elementary mathematics and certain general principles [4], [5]. One of these examples is the following.

Let a circular disc of fixed radius ϱ run along a curve lying in a bounded planar domain of area T. Suppose that all of the discs obtained in this way are confined to the interior of this domain. Then the length of the envelope of these discs cannot exceed the value $\frac{2T}{\varrho}$.

The proof of Riesz is based on the observation that if the length of the curve is nonessential then so is the fact that the centers of the discs lie on a curve. Thus he generalized the example as follows.

A family of circular discs of fixed radius ϱ lies in a bounded planar domain of area T. Then the length of the envelope of the discs cannot exceed $\frac{2T}{\varrho}$.

First Riesz establishes a fully elementary proof in a particular case, namely if the family in question is finite, and then he applies the following principle.

PRINCIPLE 1. *A sequence of curves each of length $\leqq z$ cannot have a limit curve of length $>z$.*

REMARK 1. By the envelope of a family of sets $\{..., N_\alpha, ...\}$ one clearly has to understand here the boundary of its union i.e. the set

$$\operatorname{Fr}(\bigcup_\alpha N_\alpha) = [\bigcup_\alpha N_\alpha] \setminus \langle \bigcup_\alpha N_\alpha \rangle = [\bigcup_\alpha N_\alpha] \cap [R^2 \setminus \bigcup_\alpha N_\alpha].^2$$

REMARK 2. The envelope of an infinite family of circular discs may have an infinite number of components and each of these components might possess an infinite number of ramification points. Therefore the usual notion of length cannot be

[1] This is a slightly modified version of a paper written in Hungarian in 1958 [1]. However, the main part of the lecture of Riesz has later been translated into French [5]. So it seems to be useful to translate into English the "Notes".

[2] $[M]$ is the closure and $\langle M \rangle$ is the interior of the set M. R^2 is the plane and $M \setminus N$ is the difference of M and N. $\bigcup_\alpha N_\alpha$ is the union of the sets N_α and $\operatorname{Fr} M = [M] \setminus \langle M \rangle$ is the boundary of M.

used here. Thus we need some kind of generalization of the concept of length. In this paper we shall apply the outer linear measure of Carathéodory [2].

DEFINITION 1. Let M be a set in the plane, $\varepsilon > 0$, and

$$L_\varepsilon(M) = \inf \sum_{i=1}^{\infty} \delta(M_i)$$ [3]

where $M = \bigcup_{i=1}^{\infty} M_i$ and for $i = 1, 2, \dots$ $\delta(M_i) < \varepsilon$. Let

$$L^*(M) = \lim_{\varepsilon \to 0} L_\varepsilon(M).$$

(Since $L_\varepsilon(M)$ is a monotone decreasing function of ε it follows the existence of this limit. This limit may reach the value ∞ or even $L_\varepsilon(M)$ may attain the value ∞ for all the ε-s.)

$L^*(M)$ is said to be the Carathéodory outer linear measure of M.

Let us mention some elementary properties of this concept (see [2]).

(A) Each set in the plane has a uniquely determined Carathéodory outer linear measure and this measure is either a nonnegative real number or ∞.

(B) If $B \subset A$ then $L^*(B) \leqq L^*(A)$.

(C) If $A = \bigcup_{j=1}^{\infty} A_j$ then $L^*(A) \leqq \sum_{j=1}^{\infty} L^*(A_j)$.

(D) If the distance of the sets A and B is positive then

$$L^*(A \cup B) = L^*(A) + L^*(B).$$

An immediate corollary of this property (D) is the following:

(E) If $A_1, \dots, A_n$ are mutually disjoint closed sets of the plane so that at most one of them is unbounded then

$$L^*(A_1 \cup \dots \cup A_n) = L^*(A_1) + \dots + L^*(A_n).$$

The last two elementary properties are as follows.

(F) If $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$

then

$$L^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} L^*(A_n).$$

(G) The outer linear measure of a simple arc coincides with its usual length.

REMARK 3. Principle 1 is true if each curve in question is connected. However, since the envelope of a finite family of circular discs need not be connected or even more, since the number of components of the envelope might tend to infinity if so does the number of discs, it follows that Principle 1 cannot be applied. E.g. consider the following example.

[3] $\delta(N)$ is the diameter of the set N. If $N = \emptyset$ then let $\delta(N) = 0$.

For $k=1, 2, \ldots$ let G_k be the curve consisting of the intervals

$$\frac{2i}{2^{k+1}} \leqq x \leqq \frac{2i+1}{2^{k+1}}, \quad y = \frac{1}{k} \quad (i = 0, 1, \ldots, 2^k-1).$$

The limit curve of the G_k-s is the interval $0 \leqq x \leqq 1$, $y=0$. The length of each G_k is $\frac{1}{2}$, nevertheless the length of the limit curve is 1.

However, it is still true that if the Carathéodory outer linear measure of the envelope of each finite subfamily of a family of circular discs of constant radius lying in a bounded planar domain does not exceed an upper bound z then so does the outer linear measure of the envelope of the whole family.

We can now seek for a principle generalizing this phenomenon.

It would seem plausible to omit the condition of constantness of the radii of the discs. However, accepting this modification the previous statement becomes false. This can be shown by the following example.

Set the rationals of the closed interval [0, 7] into an infinite sequence $r_1, \ldots, r_n, \ldots$. Consider the system of circular discs of the plane

$$(x-r_n)^2+\left(y-\frac{1}{2^n}\right)^2 < \frac{1}{4^n} \quad (n = 1, 2, \ldots).$$

While the outer linear measure of the envelope of an arbitrary finite subsystem of these discs does not exceed the upper bound $\sum_{n=1}^{\infty} 2\pi \frac{1}{2^n} = 2\pi$, the interval [0, 7] of the X-axis is confined to the envelope of the whole system and thus the outer linear measure of this latter envelope is at least 7 and so it is larger than 2π.

If, however, we replace the condition of constantness of the radii of the discs by the following weaker one: all the diameters of the discs are not less than a positive lower bound w, then the statement above remains true. The statement even remains true if, instead of circular discs, we take connected sets such that all of their diameters are not less than a positive lower bound w.

We can omit the condition that all the sets are lying in a bounded domain of the plane, moreover the sets in question need not even be bounded sets. Finally, we can omit the condition of connectedness of the sets. We only need to suppose that the diameters of all the components of the sets are not less than a positive lower bound w.

Thus the systems of sets in question are the following:

DEFINITION 2. Let w be a positive and z a nonnegative real number. A system $\{\ldots, N_\alpha, \ldots\}$ of sets of the plane is said to be a (w, z)-system if

(i) The components of the sets N_α are of diameter $\geqq w$.

(ii) For every finite subsystem $N_{\alpha_1}, \ldots, N_{\alpha_k}$ of the system we have

$$L^* (\mathrm{Fr}\, (\bigcup_{j=1}^{k} N_{\alpha_j})) \leqq z.$$

Our result is now the following:

Principle 2. *Let* $w>0$ *and* $z\geqq 0$. *Let* $\{\dots, N_\alpha, \dots\}$ *be a* (w, z)*-system in the plane. Then*

$$L^*(\mathrm{Fr}(\bigcup_\alpha N_\alpha)) \leqq z.$$

The proof of Principle 2 proceeds in seventeen steps.

1. *Let* F *be the boundary curve of a closed bounded nonlinear convex set of the plane. Then* $L^*(F)$ *is finite.*

In fact, the usual length $h(F)$ of F is finite. However, F is the union of two simple arcs C_1 and C_2 with the same endpoints and thus for the usual length $h(C_1)$ and $h(C_2)$ of these arcs we have obviously $h(C_1)+h(C_2)=h(F)$. On the other hand in view of (G) for $i=1, 2$ we have $L^*(C_i)=h(C_i)$ and thus by $F=C_1\cup C_2$ (C) shows that

$$L^*(F) \leqq L^*(C_1)+L^*(C_2) = h(F) < \infty.$$

2. *Let* M *be a bounded nonempty connected set of the plane. Then* $L^*(M)\geqq\delta(M)$.

To prove this statement we only need to show that for any $\varepsilon>0$ we have

$$L_\varepsilon(M) \geqq \delta(M)-\varepsilon.$$

Now let $\varepsilon>0$ and $M=M_1\cup\dots\cup M_i\cup\dots$ a decomposition of M where for $i=1, 2, \dots$ $\delta(M_i)<\varepsilon$. For $i=1, 2, \dots$ let

$$G_i = O\left(M_i, \frac{\varepsilon}{3\cdot 2^i}\right). \text{[4]}$$

Hence

$$\sum_{i=1}^{\infty}\delta(G_i) \leqq \sum_{i=1}^{\infty}\delta(M_i)+\frac{2}{3}\varepsilon. \tag{1}$$

Let p and q be points of M such that

$$r(p, q) \geqq \delta(M)-\frac{\varepsilon}{3}. \text{[5]} \tag{2}$$

Since $G_1, \dots, G_i, \dots$ is an open covering of the connected set M it follows the existence of a chain $G_{i_1}, \dots, G_{i_n}$ such that $i_1, \dots, i_n$ are pairwise distinct, $p\in G_{i_1}$, $q\in G_{i_n}$, and any two consecutive elements of the chain are intersecting. Let $p=p_0$, $q=p_n$ and for $j=1, \dots, n-1$ let $p_j\in G_{i_j}\cap G_{i_{j+1}}$. We then have

$$r(p, q) \leqq \sum_{j=0}^{n-1} r(p_j, p_{j+1}) \leqq \sum_{j=1}^{n}\delta(G_{i_j}) \leqq \sum_{i=1}^{\infty}\delta(G_i),$$

consequently in view of (1) and (2)

$$\delta(M)-\frac{\varepsilon}{3} \leqq \sum_{i=1}^{\infty}\delta(M_i)+\frac{2}{3}\varepsilon$$

[4] For any $\eta>0$, $O(N, \eta)$ is the η-neighbourhood of the subset N of the plane.

[5] $r(p, q)$ is the distance of the points p and q of the plane.

holds. Thus

$$\sum_{i=1}^{\infty} \delta(M_i) \geqq \delta(M)-\varepsilon.$$

This relation is true for every decomposition $M=\bigcup_{i=1}^{\infty} M_i$ of M into parts M_i of diameter $<\varepsilon$, consequently

$$L_\varepsilon(M) \geqq \delta(M)-\varepsilon$$

as required.

3. *Let $w>0$ and let N be a set in the plane each component of which is of diameter $\geqq w$. Let C be a bounded closed nonlinear convex set in the plane and let F be the boundary of C. Let $N'=(N\cap C)\cup F$. Then each component of N' is of diameter $\geqq \min(\delta(C), w)$.*

In order to prove the statement we only need to show that each component of N' contains either a component of N or the curve F.

Now let M' be a component of N' and let $p\in M'$.

Since F is connected it follows that in the case $p\in F$ we have $F\subset M'$.

Suppose that $p\notin F$. Then $p\in N\cap\langle C\rangle$ and thus p belongs to a component M of N. If $M\subset C$ then $M\subset(N\cap C)\cup F=N'$ and thus, taking also the connectedness of M into account, we have $M\subset M'$.

Suppose now that $M\not\subset C$. Then there exists a point of M in the exterior of C and thus $p\in\langle C\rangle$ and the connectedness of M imply $M\cap F\neq\emptyset$. However F is also connected, consequently $M\cup F$ is a connected set, too. The set $(M\cup F)\cap C= =(M\cap C)\cup F$ is connected as well. In fact if $(M\cap C)\cup F$ had a decomposition $(M\cap C)\cup F=G_1\cup G_2$ into disjoint nonempty sets G_1 and G_2 open in $(M\cap C)\cup F$, then F would be confined to one of the G_i-s, say to G_1. Hence there would hold $G_2\subset\langle C\rangle$. However, in that case $G_1\cup(M\setminus C)$ would be open in

$$(M\setminus C)\cup(M\cap C)\cup F = M\cup F$$

and thus there would exist a nontrivial decomposition $M\cup F=G_2\cup(G_1\cup(M\setminus C))$ of the set $M\cup F$ into nonempty disjoint sets open in $M\cup F$, contradicting the fact that $M\cup F$ is connected. Now by $p\in(M\cap C)\cup F$ the connected set $(M\cap C)\cup F$ is contained in the component M' of $N'=(N\cap C)\cup F$. Hence in this case M' contains again the curve F.

4. *Let G be a connected nonempty open subset of the 2-sphere S^2. Let M be a component of $S^2\setminus G$. Then the boundary* $\mathrm{Fr}(M)=M\setminus\langle M\rangle=M\cap[S^2\setminus M]$ *of M is connected.*

In fact, let M' be a component of $S^2\setminus G$ distinct from M. Joining a point p of M' with a point q of G by a simple arc $\widehat{pq}$ and going on $\widehat{pq}$ from p to q, let s be the first limit point of G. The point s clearly belongs to M' and $G\cup\{s\}$ is connected. Hence by $M'\cap(G\cup\{s\})\neq\emptyset$ the set $M'\cup(G\cup\{s\})=M'\cup G$ is connected. This is true for each component of $S^2\setminus G$ distinct from M and thus $S^2\setminus M$ is connected. Consequently $[S^2\setminus M]$ is connected as well.

Now if $\mathrm{Fr}\, M = M \cap [S^2 \setminus M]$ were nonconnected then by the connectedness of M and $[S^2 \setminus M]$ the set $M \cup [S^2 \setminus M]$ would divide the sphere S^2 in at least two parts (see [3] p. 354) but in view of $M \cup [S^2 \setminus M] = S^2$ this is impossible.

5. *Let C be a bounded closed nonlinear convex set in the plane. Let F be the boundary of C. Let N be a set in the plane containing F. Let $N' = N \cap C$. Then*

$$\mathrm{Fr}(N') = F \cup (\mathrm{Fr}(N) \cap C).$$

In fact, we have

$$[N] \cap \langle C \rangle \subset [N \cap \langle C \rangle] \subset [N \cap C].$$

On the other hand $F \subset N$ implies

$$[N] \cap F = F \subset [N \cap C].$$

Now these two relations imply

$$[N] \cap C = ([N] \cap \langle C \rangle) \cup ([N] \cap F) \subset [N \cap C],$$

but this yields $[N \cap C] = [N] \cap C$. On the other hand we have $\langle N \cap C \rangle = \langle N \rangle \cap \langle C \rangle$ and thus

$$\mathrm{Fr}(N') = \mathrm{Fr}(N \cap C) = [N \cap C] \setminus \langle C \cap N \rangle =$$

$$= (([N] \cap C) \setminus \langle C \rangle) \cup (([N] \cap C) \setminus \langle N \rangle) =$$

$$= ([N] \cap F) \cup (\mathrm{Fr}(N) \cap C) = F \cup (\mathrm{Fr}(N) \cap C)$$

as required.

6. *Let $S = \{\dots, N_\alpha, \dots\}$ be an arbitrary system of sets lying in a bounded planar domain. Let $\eta > 0$. Then there exists a finite subsystem $N_{\alpha_1}, \dots, N_{\alpha_k}$ of S such that*

$$\mathrm{Fr}(\bigcup_\alpha N_\alpha) \subset O(\mathrm{Fr}(\bigcup_{j=1}^k N_{\alpha_j}), \eta).$$

In fact, the compact set $[\bigcup_\alpha N_\alpha]$ is covered by the open sets $O(N_\alpha, \eta)$. Hence there is a finite subsystem $N_{\alpha_1}, \dots, N_{\alpha_k}$ of S such that

$$[\bigcup_\alpha N_\alpha] \subset \bigcup_{j=1}^k O(N_{\alpha_j}, \eta). \tag{3}$$

However

$$\bigcup_{j=1}^k O(N_{\alpha_j}, \eta) = O(\bigcup_{j=1}^k N_{\alpha_j}, \eta) = \langle \bigcup_{j=1}^k N_{\alpha_j} \rangle \cup O(\mathrm{Fr}(\bigcup_{i=1}^k N_{\alpha_j}), \eta). \tag{4}$$

On the other hand we have

$$\mathrm{Fr}(\bigcup_\alpha N_\alpha) = [\bigcup_\alpha N_\alpha] \setminus \langle \bigcup_\alpha N_\alpha \rangle$$

and thus $\mathrm{Fr}\,(\bigcup_\alpha N_\alpha)$ fails to meet the set $\langle \bigcup_{j=1}^{k} N_{\alpha_j}\rangle$, consequently (3) and (4) imply

$$\mathrm{Fr}\,(\bigcup_\alpha N_\alpha) = [\bigcup_\alpha N_\alpha] \setminus \langle \bigcup_\alpha N_\alpha \rangle \subset O(\mathrm{Fr}\,(\bigcup_{j=1}^{k} N_{\alpha_j}), \eta)$$

as required.

7. *Let $S=\{\dots, N_\alpha, \dots\}$ be an arbitrary system of sets lying in a bounded planar domain. Let $\eta>0$ and let $N_{\alpha_1}, \dots, N_{\alpha_k}$ be a finite subsystem of S such that*

$$\mathrm{Fr}\,(\bigcup_\alpha N_\alpha) \subset O(\mathrm{Fr}\,(\bigcup_{j=1}^{k} N_{\alpha_j}), \eta).$$

Let $N_{\alpha_{k+1}}, \dots, N_{\alpha_{k+m}}$ be an arbitrary finite subsystem of S. Then

$$\mathrm{Fr}\,(\bigcup_\alpha N_\alpha) \subset O(\mathrm{Fr}\,(\bigcup_{j=1}^{k+m} N_{\alpha_j}), \eta).$$

In fact, by assumption we have

$$[\bigcup_\alpha N_\alpha] \setminus \langle \bigcup_\alpha N_\alpha \rangle = \mathrm{Fr}\,(\bigcup_\alpha N_\alpha) \subset O(\mathrm{Fr}\,(\bigcup_{j=1}^{k} N_{\alpha_j}), \eta) \subset O([\bigcup_{j=1}^{k} N_{\alpha_j}], \eta) \subset$$

$$\subset O([\bigcup_{j=1}^{k+m} N_{\alpha_j}], \eta) = \langle \bigcup_{j=1}^{k+m} N_{\alpha_j} \rangle \cup O(\mathrm{Fr}\,(\bigcup_{j=1}^{k+m} N_{\alpha_j}), \eta).$$

However, $\langle \bigcup_{j=1}^{k+m} N_{\alpha_j} \rangle \subset \langle \bigcup_\alpha N_\alpha \rangle$, consequently $\mathrm{Fr}\,(\bigcup_\alpha N_\alpha)$ does not meet the set $\langle \bigcup_{j=1}^{k+m} N_{\alpha_j} \rangle$ and thus

$$\mathrm{Fr}\,(\bigcup_\alpha N_\alpha) \subset O(\mathrm{Fr}\,(\bigcup_{j=1}^{k+m} N_{\alpha_j}), \eta)$$

indeed.

8. *Let $w>0$. Let $S=\{\dots, N_\alpha, \dots\}$ be a system of bounded subsets of the plane such that $\bigcup_\alpha N_\alpha$ is unbounded and each component of an N_α is of diameter $\geqq w$. Then to every $z \geqq 0$ there is a finite subsystem $N_{\alpha_1}, \dots, N_{\alpha_k}$ of S such that*

$$L^*(\mathrm{Fr}\,(\bigcup_{j=1}^{k} N_{\alpha_j})) > z.$$

PROOF. Let $z \geqq 0$. Let k be an integer satisfying the relation

$$k > \frac{z}{w}. \tag{5}$$

Let us take points $p_1, \dots, p_k$ in $\bigcup_\alpha N_\alpha$ such that $j_1 \neq j_2$ implies

$$r(p_{j_1}, p_{j_2}) > z. \tag{6}$$

This can clearly be done. For each p_j, select a set N_{α_j} such that $p_j \in N_{\alpha_j}$. Then the set

$$N' = [\bigcup_{j=1}^{k} N_{\alpha_j}]$$

is clearly compact.

Let G be the unbounded component of the open complement of N'. Let us consider the components of $R^2 \setminus G$ containing the points $p_1, \ldots, p_k$. They should be in turn $M_1, \ldots, M_k$. Each M_j is a compact set. Let us extend the plane R^2 by addition of an ideal point q to a 2-sphere S^2. Let $G' = G \cup \{q\}$. G' is obviously a connected open set in S^2 and since $S^2 \setminus G' = R^2 \setminus G$ it follows that $M_1, \ldots, M_k$ are components of $S^2 \setminus G'$, too.

We are going to show that for $j=1, \ldots, k$ the relation

$$\mathrm{Fr}(M_j) \subset \mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j})$$

holds.

In fact, let $p \in \mathrm{Fr}\, M_j$. Then $p \notin G$ but each neighbourhood of p meets G. Thus $p \in N'$ and $p \notin \langle N' \rangle$. Hence $p \in \mathrm{Fr}\, N'$, consequently for $j=1, \ldots, k$ we have

$$\mathrm{Fr}(M_j) \subset \mathrm{Fr}(N') \subset \mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j}), \tag{7}$$

indeed.

For $j=1, \ldots, k$ let M'_j be the component of N_{α_j} containing p_j. Since $p_j \in M_j$, it follows $M_j \cap M'_j \neq \emptyset$ and thus $M'_j \subset R^2 \setminus G$ implies $M'_j \subset M_j$, consequently

$$\delta(M_j) \geqq w \quad (j = 1, \ldots, k). \tag{8}$$

On the other hand, since each M_j is a compact set in the plane we have

$$\delta(\mathrm{Fr}(M_j)) = \delta(M_j \setminus \langle M_j \rangle) = \delta(M_j) \quad (j = 1, \ldots, k). \tag{9}$$

Moreover Step 4 shows that for $j=1, \ldots, k$ $\mathrm{Fr}(M_j)$ is connected and thus by Step 2 we get

$$L^*(\mathrm{Fr}(M_j)) \geqq \delta(\mathrm{Fr}(M_j)). \tag{10}$$

Now we have two possibilities:

(α) $M_{j_1} \neq M_{j_2}$ provided that $j_1 \neq j_2$.

Then the boundaries of $M_1, \ldots, M_k$ are mutually disjoint compact sets. Thus by (5), (7), (8), (9), (10) and by (B) and (E) we have

$$L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j})) \geqq L^*(\bigcup_{j=1}^{k} \mathrm{Fr}(M_j)) = \sum_{j=1}^{k} L^*(\mathrm{Fr}(M_j)) \geqq kw > z.$$

(β) There exist two different indices j_1 and j_2 such that $M_{j_1} = M_{j_2}$.

Then by $\delta(M_{j_1}) \geqq r(p_{j_1}, p_{j_2})$ and by (6), (7), (9), (10) and (B) we have

$$L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j})) \geqq L^*(\mathrm{Fr}(M_{j_1})) \geqq r(p_{j_1}, p_{j_2}) > z.$$

The assertion has been proved.

Now we introduce two new concepts.

DEFINITION 3. Let H be a set in the plane. An infinite sequence $C_1, \dots, C_i, \dots$ consisting of compact convex nonlinear sets of the plane is said to be a convex covering system of H if

(α) $H \subset \bigcup_{i=1}^{\infty} \langle C_i \rangle$.

(β) The sets C_i are pairwise disjoint.

(γ) Each bounded set in the plane can meet only a finite number of sets C_i.

(δ) $\sum_{i=1}^{\infty} \delta(C_i) < \infty$.

DEFINITION 4 (sweeping procedure). Let ζ be either a positive real number or ∞. Let $Q_1, \dots, Q_t$ be a finite system consisting of bounded planar sets. The ζ-sweeping procedure is the following:

We declare that the 0-th step can be done and its result is the system

$$\Sigma_0 = \{Q_1, \dots, Q_t\}.$$

n-th step ($n=1, 2, \dots$). Suppose that the $(n-1)$-th step can be done and its result is the system of sets Σ_{n-1}. If it is possible, take two members out of Σ_{n-1} such that their diameters are less than ζ and their closed convex hulls intersect. Delete these two members from Σ_{n-1} and add their union to the reduced system. The system obtained in this way is said to be the result of the n-th step and it is denoted by Σ_n. We also declare in this case that the n-th step can be done.

If such members do not exist in Σ_{n-1} then we say that the n-th step cannot be done. In this case Σ_{n-1} is said to be the result of the ζ-sweeping procedure.

If the n-th step can be done then we try to continue the procedure. Since the number of members of the system reduces by each step it follows that after a finite number of steps we must stop. Consequently the procedure ought to have a result. Denote this result by Σ.

We add some elementary remarks to this sweeping procedure.

The union of the system Σ is the same as that of Σ_0.

If Σ_0 is an open system then so is Σ.

Since the diameter of the closed convex hull of a bounded set is the same as the diameter of the set itself it follows that the sum of the diameters of the members of Σ is not bigger than that of Σ_0.

The closed convex hulls of members of diameter $<\zeta$ of Σ are mutually disjoint.

9. *Let H be a nonbounded closed set in the plane such that $L^*(H)$ is finite. Then H can be covered by a convex covering system.*

PROOF. Let $L^*(H)=z$. Then by Definition 1 there is a decomposition $H=\bigcup_{k=1}^{\infty} H_k$ of H into bounded sets H_k such that

$$\sum_{k=1}^{\infty} \delta(H_k) \leqq z+1.$$

For $k=1, 2, \ldots$ let $G_k = O(H_k, 1/2^k)$. We then have

$$H \subset \bigcup_{k=1}^{\infty} G_k \tag{11}$$

and

$$\sum_{k=1}^{\infty} \delta(G_k) \leqq z+3. \tag{12}$$

Let $K_1, \ldots, K_n, \ldots$ be a monotone increasing sequence of concentric discs in the plane such that the difference of the radii of two consecutive discs is at least $2z+7$ and the boundary of each disc meets the set H. There exists obviously such a sequence. Let K_0 be the empty set and for $n=1, 2, \ldots$ let

$$H^n = [K_n \setminus K_{n-1}] \cap H.$$

The compact set H^n is covered by the open G_k-s. Hence there is a finite subsystem of $\{G_k; k=1, 2, \ldots\}$ which covers H^n too. Let $G_{n,1}, \ldots, G_{n,p_n}$ be such a finite subcovering. We also assume that for $j=1, \ldots, p_n$ $G_{n,j} \cap H^n \neq \emptyset$ and $G_{n,j_1} \neq G_{n,j_2}$ provided that $j_1 \neq j_2$. In such a way we get a system of nonempty sets

$$\begin{cases} G_{1,1}, \ldots, G_{1,p_1} \\ \vdots \\ G_{n,1}, \ldots, G_{n,p_n} \\ \vdots \end{cases} \tag{13}$$

and we clearly have

$$H = \bigcup_{n=1}^{\infty} H^n \subset \bigcup_{n=1}^{\infty} \Big(\bigcup_{j=1}^{p_n} G_{n,j} \Big). \tag{14}$$

Each G_k can occur in the system (13) at most twice in two consecutive rows.

In fact, if G_k appeared in two nonconsecutive rows then its diameter would be at least $2z+7$. However, (12) shows that $\delta(G_k) \leqq z+3$.

Hence taking also (12) into account we get

$$\sum_{n=1}^{\infty} \Big(\sum_{j=1}^{p_n} \delta(G_{n,j}) \Big) \leqq 2z+6. \tag{15}$$

We are going to construct the sets $G'_{n,j}$ as follows.

Let us apply the ∞-sweeping procedure to the system $G_{1,1}, \ldots, G_{1,p_1}$. The result of the procedure should be

$$G'_{1,1}, \ldots, G'_{1,q_1}.$$

Now let $n \geqq 2$ and suppose that the system

$$G'_{n-1,1}, \ldots, G'_{n-1,q_{n-1}}$$

has already been constructed. Let us regard the system

$$G_{n,1}, \ldots, G_{n,p_n}, G'_{n-1,1}, \ldots, G'_{n-1,q_{n-1}}$$

and apply the ∞-sweeping procedure to this system. The result of the procedure should be

$$G'_{n,1}, \dots, G'_{n,q_n}.$$

Now considering the system of sets

$$(16)\qquad \begin{matrix} G'_{1,1}, \dots, G'_{1,q_1} \\ \vdots \\ G'_{n,1}, \dots, G'_{n,q_n} \\ \vdots \end{matrix}$$

let us mention some properties of it.

(a) Each $G'_{n,j}$ is the union of certain $G_{n',j'}$-s.

(b) $\sum_{j=1}^{q_n} \delta(G'_{n,j}) \leqq 2z+6 \quad (n=1, 2, \dots).$

In fact by the construction we have

$$\sum_{j=1}^{q_n} \delta(G'_{1,j}) \leqq \sum_{j=1}^{p_n} \delta(G_{1,j}),$$

and for $n=2, 3, \dots$.

$$\sum_{j=1}^{q_n} \delta(G'_{n,j}) \leqq \sum_{j=1}^{p_n} \delta(G_{n,j}) + \sum_{j=1}^{q_{n-1}} \delta(G'_{n-1,j}).$$

Taking into account also (15) these two relations verify the required formula (b).

(c) For any fixed n, G'_{n,j_1} and G'_{n,j_2} have disjoint closed convex hulls provided that $j_1 \neq j_2$.

(d) Each $G_{n,j}$ is confined to precisely one of the $G'_{n,j'}$-s. (c) shows that the other members of the system $G'_{n,1}, \dots, G'_{n,q_n}$ are disjoint from $G_{n,j}$.

(e) Each $G'_{n,j}$ is confined to precisely one of the $G'_{n+1,j'}$-s. (c) shows that the other members of the system $G'_{n+1,1}, \dots, G'_{n+1,q_{n+1}}$ are disjoint from $G'_{n,j}$.

A corollary of this fact is the following:

(f) Let $G'_{n_1,j_1}, \dots, G'_{n_s,j_s}$ be a finite subsystem of mutually intersecting sets of (16). Then there is a member G'_{n_r,j_r} containing all the other members of the given subsystem.

Now two members of the system (13) are said to be equivalent if there is a member of (16) containing both of them. This relation is obviously symmetric and by (d) it is reflexive. Also, it is transitive. In fact let $G_{n_1,j_1} \cup G_{n_2,j_2} \subset G'_{n',j'}$ and $G_{n_2,j_2} \cup G_{n_3,j_3} \subset \subset G'_{n'',j''}$. Then $G'_{n',j'}$ meets $G'_{n'',j''}$ and thus by (f) one of them is confined to the other, say $G'_{n',j'} \subset G'_{n'',j''}$. However, in this case $G'_{n'',j''}$ contains both of G_{n_1,j_1} and G_{n_3,j_3}, and thus G_{n_1,j_1} and G_{n_3,j_3} are equivalent as well.

We are going to consider the equivalence classes of this relation.

First we show that to each class there belongs only a finite number of members of the system (13).

In fact let the members G_{n_1,j_1} and G_{n_2,j_2} of the system (13) belong to the same class and let $G'_{n',j'}$ be a member of the system (16) containing both of these members of (13). (b) shows that $\delta(G'_{n',j'}) \leqq 2z+6$. Hence by the assumption on the discs K_n

the members G_{n_1,j_1} and G_{n_2,j_2} occur either in the same row or in two consecutive rows of (13). However, only a finite number of members is confined to two consecutive rows of (13) and thus only a finite number of members of (13) belong to a class.

(g) For each equivalence class the union of its member belongs to the system (16).

In fact let

$$\Omega = \{G_{n_1,j_1}, \dots, G_{n_s,j_s}\}$$

be an equivalence class. Let $G'_{n'_1,j'_1}, \dots, G'_{n'_s,j'_s}$ be members of the system (16) such that for $t=1, \dots, s$

$$G_{n_1,j_1} \cup G_{n_t,j_t} \subset G'_{n'_t,j'_t}$$

holds true. By the equivalence of G_{n_1,j_1} and G_{n_t,j_t} there exist such $G'_{n'_t,j'_t}$-s. Any two of the $G'_{n'_t,j'_t}$-s are intersecting and thus by (f) one of them, say $G'_{n'_r,j'_r}$, contains all the others. Hence

$$\bigcup_{t=1}^{s} G_{n_t,j_t} \subset G'_{n'_r,j'_r}.$$

On the other hand (a) shows that $G'_{n'_r,j'_r}$ is the union of certain $G_{n,j}$-s and these $G_{n,j}$-s belong to the same equivalence class, thus they clearly belong to Ω. Consequently

$$\bigcup_{t=1}^{s} G_{n_t,j_t} = G'_{n'_r,j'_r}$$

and this is what we had to prove.

(h) Let $G'_{n,j}$ be the union of an equivalence class Ω. Let $G'_{n,j} \subset G'_{n+1,j''}$ (see (e)). Then $G'_{n,j} = G'_{n+1,j''}$.

In fact, according to (a) $G'_{n+1,j''}$ is the union of certain $G_{n',j'}$-s and these $G_{n',j'}$-s belong to the same equivalence class which clearly coincides with Ω. Consequently $G'_{n+1,j''} \subset G'_{n,j}$ and thus $G'_{n,j} = G'_{n+1,j''}$.

Now for each equivalence class let us form the union of it. We can clearly arrange these unions in an infinite sequence $G'_1, \dots, G'_i, \dots$. Each G'_i appears in the system (16) and so it is an open bounded set. Moreover we have

$$\bigcup_{i=1}^{\infty} G'_i = \bigcup_{n=1}^{\infty} \Big(\bigcup_{j=1}^{p_n} G_{n,j} \Big). \tag{17}$$

Now for $i=1, 2, \dots$ let C_i be the closed convex hull of G'_i. The system

$$C_1, \dots, C_i, \dots \tag{18}$$

consists of nonlinear compact convex sets of the plane. Hence we need only to verify that conditions (α), (β), (γ) and (δ) of Definition 3 are fulfilled.

First (14) and (17) show that

$$H \subset \bigcup_{i=1}^{\infty} G'_i \subset \bigcup_{i=1}^{\infty} \langle C_i \rangle.$$

Thus (α) is satisfied.

Next, let C_i and $C_{i'}$ be two distinct members of the system (18). C_i is the closed convex hull of G_i' and $C_{i'}$ is that of $G_{i'}'$. Let $G_i'=G_{n,j}'$ and $G_{i'}'=G_{n',j'}'$. There are two possibilities. Either $n=n'$ or $n\neq n'$.

The latter case can be reduced to the former one. In fact let $n<n'$. This can be supposed without loss of generality. Then (e) and (h) show that there is a unique $G_{n',j''}'$ for which $G_{n,j}'=G_{n',j''}'$.

Since $G_{n,j}'$ and $G_{n,j'}'$ ($G_{n',j''}'$ and $G_{n',j'}'$ respectively) are distinct sets it follows by (c) that their closed convex hulls C_i and $C_{i'}$ are disjoint. Condition (β) is fulfilled as well.

Now let M be a bounded set in the plane. Then one of the discs $K_1, \dots$, say K_n, contains M. Since in the first $n+1$ rows of the system (13) there are only a finite number of sets it follows that the number of equivalence classes containing one of them is finite as well. So the number of C_i-s containing the unions of these equivalence classes is finite.

Let us consider now a G_i' which fails to contain any $G_{n',j'}$ from the first $n+1$ rows of (13). Then by (b) and (g) we have

$$\delta(C_i) = \delta(G_i') \leqq 2z+6.$$

There is a point on the boundary or in the exterior of K_{n+1} belonging to G_i' and so to C_i. Thus by the construction of the discs $K_1, \dots$ the set C_i does not meet K_n and so it fails to intersect the set M contained in K_n.

Hence M meets only a finite number of C_i-s. Condition (γ) is satisfied, too.

Finally, let us consider an initial section $C_1, \dots, C_m$ of the sequence (18). For $i=1, \dots, m$ select the couples (n_i, j_i) such that

$$G_i' = G_{n_i, j_i}' \quad \text{(see (g))}.$$

Let $n\geqq\max\{n_i;\ i=1, \dots, m\}$ and for $i=1, \dots, m$ let

$$G_{n, j_i'}' = G_{n_i, j_i}'.$$

According to (e) and (h) there exist such $G_{n,j_i'}'$-s and for $i_1\neq i_2$ we have $G_{n,j_{i_1}'}'\neq G_{n,j_{i_2}'}'$. Consequently, since $\delta(C_i)=\delta(G_i')=\delta(G_{n,j_i'}')$ it follows by (b) that

$$\sum_{i=1}^{m} \delta(C_i) = \sum_{i=1}^{m} \delta(G_{n,j_i'}') \leqq 2z+6.$$

This implies

$$\sum_{i=1}^{\infty} \delta(C_i) \leqq 2z+6.$$

Condition (δ) is satisfied as well.

The proof of the statement is complete.

10. *Let H be a compact set in the plane. Then H can be covered by a convex covering system.*

In fact, H can be covered by a single open bounded nonempty convex set G. Let C_1 be the closure of G. C_1 can clearly be extended to a convex covering system $C_1, \dots, C_i, \dots$ of H.

11. *Let N be an unbounded set in the plane R^2 such that the Carathéodory outer linear measure of its boundary is finite and each component of it is of diameter $\geqq w$ for fixed $w>0$. Let $C_1, \ldots, C_i, \ldots$ be a convex covering system of the boundary* $\mathrm{Fr}(N)$ *of N. Then*

$$\Big(R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle\Big) \subset \langle N \rangle.$$

We first show that $R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle$ is connected.

In fact, for $i=1, 2, \ldots$ let F_i be the boundary of C_i. The F_i-s are clearly closed Jordan curves. Let p and q be points of $R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle$. The segment $[p, q]$ is a bounded set and so according to Definition 3 (γ), $[p, q]$ meets only a finite number of C_i-s in their interior. Without loss of generality we can suppose that these C_i-s are in turn $C_1, \ldots, C_m$. The sets $\langle C_1 \rangle, \ldots, \langle C_m \rangle$ intersect $[p, q]$ in disjoint open intervals. Deleting these intervals from $[p, q]$ and adding the boundaries $F_1, \ldots, F_m$ of $C_1, \ldots, C_m$ respectively, we get a connected set in $R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle$ with the points p and q. If $[p, q]$ does not meet any $\langle C_i \rangle$ then $[p, q]$ itself is such a connected set. Hence $R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle$ is connected indeed.

The connected set $R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle$ has no common point with the subset $\mathrm{Fr}(N)$ of $\bigcup_{i=1}^{\infty} \langle C_i \rangle$. Thus $R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle$ lies either in the interior $\langle N \rangle$ or in the exterior $R^2 \setminus [N]$ of N.

The latter case cannot occur. Otherwise $[N] \subset \bigcup_{i=1}^{\infty} \langle C_i \rangle$ would hold. Since N is an unbounded set, it would intersect an infinite number of C_i-s. However, condition (δ) of Definition 3 says that $\sum_{i=1}^{\infty} \delta(C_i) < \infty$ and so there would exist a C_m meeting N such that $\delta(C_m) < w$. Since the $\langle C_i \rangle$-s are disjoint open sets by $N \subset \bigcup_{i=1}^{\infty} \langle C_i \rangle$, a component M of N would lie in C_m and so $\delta(M) \leqq \delta(C_m) < w$ would hold contradicting the assumption.

Thus

$$\Big(R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle\Big) \subset \langle N \rangle$$

indeed.

12. *Let $w>0$ and $z \geqq 0$. Let $\{\ldots, N_\alpha, \ldots\}$ be a (w, z)-system in the plane R^2 and N_{α_1} an unbounded element of this system. Let $C_1, \ldots, C_i, \ldots$ be a convex covering system of* $\mathrm{Fr}(N_{\alpha_1})$. *Then*

$$\mathrm{Fr}\Big(\bigcup_{\alpha} N_\alpha\Big) \subset \bigcup_{i=1}^{\infty} \langle C_i \rangle.$$

In fact, Step 11 shows that

$$R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle \subset \langle N_{\alpha_1} \rangle,$$

and thus $\langle N_{\alpha_1} \rangle \subset \langle \bigcup_{\alpha} N_\alpha \rangle$ implies

$$R^2 \setminus \bigcup_{i=1}^{\infty} \langle C_i \rangle \subset \langle \bigcup_{\alpha} N_\alpha \rangle.$$

Consequently in view of

$$\mathrm{Fr}(\bigcup_{\alpha} N_\alpha) = [\bigcup_{\alpha} N_\alpha] \setminus \langle \bigcup_{\alpha} N_\alpha \rangle$$

we have

$$\mathrm{Fr}(\bigcup_{\alpha} N_\alpha) \subset \bigcup_{i=1}^{\infty} \langle C_i \rangle$$

indeed.

13. *Let $C_1, \ldots, C_i, \ldots$ be mutually disjoint compact sets in the plane. Let H be a closed subset of $\bigcup_{i=1}^{\infty} C_i$ and for $i=1, 2, \ldots$ let $H_i = H \cap C_i$. Then*

$$L^*(H) = \sum_{i=1}^{\infty} L^*(H_i).$$

In fact, by (F) and $H = \bigcup_{i=1}^{\infty} H_i$ we have

$$L^*(H) = \lim_{n\to\infty} L^*(H_1 \cup \ldots \cup H_n).$$

On the other hand, since $H_1, \ldots, H_n$ are closed subsets of the mutually disjoint compact sets $C_1, \ldots, C_n$ respectively, in view of (E) one has

$$L^*(H_1 \cup \ldots \cup H_n) = \sum_{i=1}^{n} L^*(H_i)$$

and thus, taking also into account the preceding relation, we get

$$L^*(H) = \lim_{n\to\infty} \sum_{i=1}^{n} L^*(H_i) = \sum_{i=1}^{\infty} L^*(H_i),$$

indeed.

14. *Let $w>0$ and $z \geqq 0$. Let $\{\ldots, N_\alpha, \ldots\}$ be a (w, z)-system and N_{α_0} a member of it. For each α let $N_\alpha^* = N_{\alpha_0} \cup N_\alpha$. Then $\{\ldots, N_\alpha^*, \ldots\}$ too is a (w, z)-system, and the envelope of this latter system is the same as that of the foregoing one.*

In fact, considering an arbitrary member $N_\alpha^* = N_\alpha \cup N_{\alpha_0}$ of the second system, each component of N_α and N_{α_0} is confined to N_α^*; hence it is confined to a component of N_α^*. Consequently each component M of N_α^* is the union of certain components of N_α and N_{α_0} and thus the diameter of M cannot be less than w.

Next, consider a finite subsystem $N^*_{\alpha_1}, \ldots, N^*_{\alpha_k}$ of $\{\ldots, N^*_\alpha, \ldots\}$. Since

$$\bigcup_{j=1}^{k} N^*_{\alpha_j} = \Big(\bigcup_{j=1}^{k} N_{\alpha_j}\Big) \cup N_{\alpha_0}$$

and by assumption

$$L^*\Big(\mathrm{Fr}\,\Big(\Big(\bigcup_{j=1}^{k} N_{\alpha_j}\Big) \cup N_{\alpha_0}\Big)\Big) \leqq z$$

it follows

$$L^*\Big(\mathrm{Fr}\,\Big(\bigcup_{j=1}^{k} N^*_{\alpha_j}\Big)\Big) \leqq z.$$

$\{\ldots, N^*_\alpha, \ldots\}$ is a (w, z)-system indeed.

Finally, the relation

$$\bigcup_\alpha N^*_\alpha = \bigcup_\alpha (N_{\alpha_0} \cup N_\alpha) = \Big(\bigcup_\alpha N_\alpha\Big) \cup N_{\alpha_0} = \bigcup_\alpha N_\alpha$$

evidently implies

$$\mathrm{Fr}\,\Big(\bigcup_\alpha N^*_\alpha\Big) = \mathrm{Fr}\,\Big(\bigcup_\alpha N_\alpha\Big)$$

as required.

15. *Let $w>0$ and $\{\ldots, N_\alpha, \ldots\}$ a system of nonempty sets lying in a bounded domain of the plane R^2 such that each component of an N_α is of diameter $\geqq w$. Let $z \geqq 0$, $\xi > 0$, $0 < \varepsilon < 6w$, and $\eta = \varepsilon\xi/(12(3z+2\xi))$. Suppose the existence of a finite subsystem $N_{\alpha_1}, \ldots, N_{\alpha_k}$ of the system $\{\ldots, N_\alpha, \ldots\}$ such that*

$$\mathrm{Fr}\,\Big(\bigcup_\alpha N_\alpha\Big) \subset O\Big(\mathrm{Fr}\,\Big(\bigcup_{j=1}^{k} N_{\alpha_j}\Big), \eta\Big) \quad \text{and} \quad L^*\Big(\mathrm{Fr}\,\Big(\bigcup_{j=1}^{k} N_{\alpha_j}\Big)\Big) \leqq z.$$

Then

$$L_\varepsilon\Big(\mathrm{Fr}\,\Big(\bigcup_\alpha N_\alpha\Big)\Big) < z + \xi.$$

PROOF. Let us introduce the notations

$$H = \mathrm{Fr}\,\Big(\bigcup_\alpha N_\alpha\Big), \;\; H' = \mathrm{Fr}\,\Big(\bigcup_{j=1}^{k} N_{\alpha_j}\Big).$$

By assumption we have $L^*(H') \leqq z$, and thus

$$L_{\varepsilon/6}(H') \leqq z.$$

Consequently, there exists a decomposition $H' = H'_1 \cup \ldots \cup H'_i \cup \ldots$ such that for $i = 1, 2, \ldots$

(19) $$\delta(H'_i) < \frac{\varepsilon}{6}, \;\; H_i \neq \emptyset$$

and

(20) $$\sum_{i=1}^{\infty} \delta(H'_i) < z + \frac{\xi}{3}.$$

For $i=1, 2, \dots$ let

$$\varrho_i = \min\left(\frac{\xi}{6\cdot 2^i}, \frac{\varepsilon}{12}\right) \quad \text{and} \quad G_i = O(H_i', \varrho_i).$$

According to (19) and (20) we clearly have

$$\delta(G_i) < \frac{\varepsilon}{3}, \quad G_i \neq \emptyset$$

and

$$\sum_{i=1}^{\infty} \delta(G_i) < z + \frac{2\xi}{3}.$$

The system $\{G_1, \dots, G_i, \dots\}$ is an open covering of the compact set H'. Hence there exists a positive integer n such that $G_1, \dots, G_n$ is a covering of H' too. Let us apply the $\varepsilon/6$-sweeping procedure to this latter finite system (see Definition 4). The result is a finite system

$$G_1', \dots, G_p', G_{p+1}', \dots, G_r'$$

of nonempty open sets such that

$$H' \subset \bigcup_{i=1}^{r} G_i',$$

$$\sum_{i=1}^{r} \delta(G_i') \leq \sum_{i=1}^{n} \delta(G_i) < z + \frac{2\xi}{3}, \tag{21}$$

$$\frac{\varepsilon}{6} \leq \delta(G_i') < \frac{\varepsilon}{3} \quad (i = 1, \dots, p), \tag{22}$$

$$\delta(G_i') < \frac{\varepsilon}{6} \quad (i = p+1, \dots, r)$$

and the closed convex hulls of the sets $G_{p+1}', \dots, G_r'$ are mutually disjoint.

Observe that by (21) and (22) we have

$$p \leq \frac{z + \frac{2\xi}{3}}{\frac{\varepsilon}{6}} = 2 \cdot \frac{3z + 2\xi}{\varepsilon}. \tag{23}$$

Now we arrange the sets $G_{p+1}', \dots, G_r'$ in two classes. The sets whose closed convex hulls are disjoint from $\bigcup_{i=1}^{p} G_i'$ belong to the first class. The others are assigned to the second. Without loss of generality we can suppose that $G_{p+1}', \dots, G_q'$ belong to the first class and $G_{q+1}', \dots, G_r'$ to the second.

Now for $i=q+1, \dots, r$ let $m(i)$ be an integer such that $m(i) \leq p$ and $G_{m(i)}'$ meets the closed convex hull of G_i'. For $i=1, \dots, p$ let

$$G_i'' = G_i' \cup \left(\cup \{G_{i'}';\ m(i') = i\}\right).$$

Thus we obtain a finite system of open sets

$$G_1'', \dots, G_p'', G_{p+1}', \dots, G_q'$$

such that

$$H' \subset (G_1'' \cup \dots \cup G_p'' \cup G_{p+1}' \cup \dots \cup G_q'),$$

$$\sum_{i=1}^{p} \delta(G_i'') + \sum_{i=p+1}^{q} \delta(G_i') \leqq \sum_{i=1}^{r} \delta(G_i') < z + \frac{2\xi}{3}, \tag{24}$$

$$\frac{\varepsilon}{6} \leqq \delta(G_i'') < \frac{2\varepsilon}{3} \quad (i = 1, \dots, p),$$

$$\delta(G_i') < \frac{\varepsilon}{6} \quad (i = p+1, \dots, q).$$

Let the closed convex hulls of $G_{p+1}', \dots, G_q'$ be in turn $C_{p+1}, \dots, C_q$. These hulls are mutually disjoint and they have no common point with any of the sets $G_1'', \dots, G_p''$.

Let us consider the boundaries $F_{p+1}, \dots, F_q$ of $C_{p+1}, \dots, C_q$, respectively. They are closed Jordan curves. By the construction the curves $F_{p+1}, \dots, F_q$ containing the sets $G_{p+1}', \dots, G_q'$ respectively in their insides have no common points with the sets $G_1', \dots, G_r'$. So they are disjoint from H'. Moreover, for $i=p+1, \dots, q$ we clearly have

$$\delta(F_i) = \delta(C_i) = \delta(G_i'). \tag{25}$$

Now we are going to show that for $i=p+1, \dots, q$ the relation

$$F_i \subset \langle \bigcup_{j=1}^{k} N_{\alpha_j} \rangle \tag{26}$$

holds.

As we have seen, the connected set F_i has no common point with

$$H' = [\bigcup_{j=1}^{k} N_{\alpha_j}] \setminus \langle \bigcup_{j=1}^{k} N_{\alpha_j} \rangle$$

and so it lies either in the interior $\langle \bigcup_{j=1}^{k} N_{\alpha_j} \rangle$ or in the exterior

$$R^2 \setminus [\bigcup_{j=1}^{k} N_{\alpha_j}] \quad \text{of} \quad \bigcup_{j=1}^{k} N_{\alpha_j}.$$

The latter case cannot occur. In fact H' meets the open set G_i' which lies in the inside of F_i and thus some of the N_{α_j}-s meet G_i' too. Hence there is a component M of an N_{α_j} intersecting G_i'. Now if F_i lay in $R^2 \setminus [\bigcup_{j=1}^{k} N_{\alpha_j}]$ then in view of $M \subset \bigcup_{j=1}^{k} N_{\alpha_j}$, M would be disjoint from F_i and so by the connectedness of M this

M would lie in the inside of F_i which would imply $M \subset C_i$ and thus

$$\delta(M) \leqq \delta(C_i) < \frac{\varepsilon}{6} < \frac{6w}{6} = w$$

contradicting the assumption $\delta(M) \geqq w$.

Relation (26) holds true indeed.

Next we are going to show that

$$H \subset (\bigcup_{i=1}^{p} O(G_i'', \eta) \cup \bigcup_{i=p+1}^{q} C_i). \tag{27}$$

In fact let $s \in H$. Let t be one of the points of H' that have a minimal distance from s. Since $H \subset O(H', \eta)$ we have $r(s, t) < \eta$. Thus we need only to show that for $i = p+1, \dots, q$ the relation $t \in C_i$ implies $s \in C_i$.

We reason by contradiction. Suppose that $t \in C_i$ and $s \notin C_i$. Let u be the cut point of the segment $[s, t]$ and F_i. Since $s \in H$ it follows $s \notin \langle \bigcup_{\alpha} N_\alpha \rangle$ and so s fails to belong to the subset $\langle \bigcup_{j=1}^{k} N_{\alpha_j} \rangle$ of $\langle \bigcup_{\alpha} N_\alpha \rangle$. However, in view of (26) we have $u \in \langle \bigcup_{j=1}^{k} N_{\alpha_j} \rangle$ and so there is a common point v of the segment $[s, u]$ and the set $H' = \mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j})$. $v \in H'$ is nearer to s than $t \in H'$ contradicting the assumption that $r(s, t)$ is the distance of s and H'.

Relation (27) holds true indeed.

Now let

$$H_i = \begin{cases} O(G_i'', \eta) \cap H & \text{for } 1 \leqq i \leqq p \\ C_i \cap H & \text{for } p+1 \leqq i \leqq q. \end{cases} \tag{28}$$

In view of (27) we have

$$H = \bigcup_{i=1}^{q} H_i.$$

Moreover for $i = 1, \dots, p$ $\delta(G_i'') < \frac{2}{3}\varepsilon$ and $\eta = \frac{\varepsilon\xi}{12(3z+2\xi)} < \frac{\varepsilon}{6}$ together show that $\delta(H_i) < \varepsilon$. Taking into account also the relation

$$\delta(C_i) = \delta(G_i') < \frac{\varepsilon}{6} \quad (i = p+1, \dots, q)$$

we get

$$\delta(H_i) < \varepsilon \quad (i = 1, \dots, q)$$

and in view of (23), (24), (25) and (28) one has

$$\sum_{i=1}^{q} \delta(H_i) \leqq \sum_{i=1}^{p} \delta(G_i'') + 2p\eta + \sum_{i=p+1}^{q} \delta(C_i) < z + \frac{2\xi}{3} + \frac{\xi}{3} = z + \xi.$$

We have got a decomposition of $H=\mathrm{Fr}(\bigcup_\alpha N_\alpha)$ into parts of diameter $<\varepsilon$ such that the sum of the diameters of these parts is less than $z+\xi$. Hence the relation

$$L_\varepsilon(\mathrm{Fr}(\bigcup_\alpha N_\alpha)) < z+\xi$$

holds.

16. Let $w>0$, $z\geqq 0$, and let $S=\{..., N_\alpha, ...\}$ be a (w, z)-system lying in a bounded domain of the plane. Let H be the envelope of this system. Let $\eta>0$ and

$$z_\eta^* = \inf L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j})) \tag{29}$$

where $N_{\alpha_1}, ..., N_{\alpha_k}$ runs over all the finite subsystems of S satisfying the relation

$$H\subset O(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j}), \eta).$$

(According to Step 6, such a subsystem exists.) Since for each finite subsystem $N_{\alpha_1}, ...$ $..., N_{\alpha_k}$ of S we have by assumption $L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j}))\leqq z<\infty$ (see Definition 2), $0\leqq z_\eta^*\leqq z$ follows. Hence z_η^* is a bounded monotone decreasing function of η. Let $z^*=\lim_{\eta\to 0} z_\eta^*$. Then $z^*\leqq z$. We now show that $L^*(H)\leqq z^*$.

In fact, we only need to prove that if S consists of nonempty sets then for each $\xi>0$ one has $L^*(H)<z^*+\xi$ and in order to verify this latter relation we only need to show that for $0<\varepsilon<6w$ we have $L_\varepsilon(H)<z^*+\xi$.

Now let $\xi>0$, $0<\varepsilon<6w$, $\xi'=\frac{\xi}{2}$ and $z'=z^*+\frac{\xi}{2}$. Let $\eta=\frac{\varepsilon\xi'}{12(3z'+2\xi')}$. Finally let $N_{\alpha_1}, ..., N_{\alpha_k}$ be a finite subsystem of S such that

$$H\subset O(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j}), \eta)$$

and

$$L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j})) \leqq z'.$$

Since $z_\eta^*\leqq z^*$, in view of (29) such a subsystem exists. Now according to Step 15 we get

$$L_\varepsilon(H) < z'+\xi' = z^*+\xi.$$

Observe that for (w, z)-systems lying in a bounded planar domain the proof of Principle 2 is complete.

17. We are going to prove Principle 2.

Let $w>0$ and $z\geqq 0$. Let $S=\{..., N_\alpha, ...\}$ be a (w, z)-system in the plane R^2. If it lies in a bounded planar domain then as we have seen in Step 16, the principle is true. Thus we can suppose that S does not lie in a bounded planar domain. Under this assumption it is impossible that each N_α should be bounded. For if S consisted only of bounded sets then according to Step 8 there would be a finite subsystem $N_{\alpha_1}, ...$

..., N_{α_k} of S such that $L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j}))>z$ and this contradicts our original assumption. Hence there is at least one unbounded set among the N_α-s, say N_{α_0}. For each α let $N_\alpha^*=N_\alpha\cup N_{\alpha_0}$. Then by Step 14, $S^*=\{..., N_\alpha^*, ...\}$ is a (w, z)-system with the same envelope as that of S. Thus we need only to verify our principle for this latter system.

Let $C_1, ..., C_i, ...$ be a convex covering system of $\mathrm{Fr}(N_{\alpha_0}^*)$. Taking into account also $L^*(\mathrm{Fr}(N_{\alpha_0}^*))\leqq z$, Steps 9 and 10 show the existence of such a covering system. Observe that we have by Step 11

$$(R^2\setminus\bigcup_{i=1}^{\infty}\langle C_i\rangle)\subset\langle N_{\alpha_0}^*\rangle=\langle N_{\alpha_0}\rangle.$$

Since for each α' we have $\langle N_{\alpha_0}^*\rangle\subset\langle N_{\alpha'}^*\rangle$ it follows

(30) $$(R^2\setminus\bigcup_{i=1}^{\infty}\langle C_i\rangle)\subset\langle N_{\alpha'}^*\rangle\subset\langle\bigcup_{\alpha} N_\alpha^*\rangle.$$

Now for $i=1, 2, ...$ let F_i be the boundary of the compact convex domain C_i and for each α and each i let $N_\alpha^i=N_\alpha^*\cap C_i$. Since for each α' we have by (30)

(31) $$F_i\subset\langle N_{\alpha'}^*\rangle\subset N_{\alpha'}^*\subset\bigcup_{\alpha} N_\alpha^*$$

it follows

(32) $$N_{\alpha'}^i=(N_{\alpha'}^*\cap C_i)\cup F_i.$$

Now we are going to show that for an arbitrary finite subsystem $N_{\alpha_1}^*, ..., N_{\alpha_k}^*$ of S^* and for $i=1, 2, ...$ we have

(33) $$L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j}^i))=L^*(F_i)+L^*(\mathrm{Fr}(\bigcup_{j=1}^{k} N_{\alpha_j}^*)\cap C_i)$$

and

(34) $$L^*(\mathrm{Fr}(\bigcup_{\alpha} N_\alpha^i))=L^*(F_i)+L^*(\mathrm{Fr}(\bigcup N_\alpha^*)\cap C_i).$$

In fact, let $\{..., N_{\alpha'}^*, ...\}$ be a nonempty (eventually non proper) subsystem of S^*. Let $N^*=\bigcup_{\alpha'} N_{\alpha'}^*$ and

$$N^i=N^*\cap C_i=\bigcup_{\alpha'}(N_{\alpha'}^*\cap C_i).$$

Then in view of (31) and Step 5 we have

$$\mathrm{Fr}(N^i)=F_i\cup(\mathrm{Fr}(N^*)\cap C_i).$$

Consequently for $i=1, 2, ...$ one has

$$\mathrm{Fr}(\bigcup_{\alpha'} N_{\alpha'}^i)=F_i\cup(\mathrm{Fr}(\bigcup_{\alpha'} N_{\alpha'}^*)\cap C_i).$$

On the other hand in view of (30) we obtain

$$\mathrm{Fr}(\bigcup_{\alpha'} N_{\alpha'}^*)\subset\bigcup_{i=1}^{\infty}\langle C_i\rangle$$

and thus for $i=1, 2, \ldots$ the compact sets F_i and $\operatorname{Fr}(\bigcup_{\alpha'} N^*_{\alpha'}) \cap C_i$ are disjoint. Accordingly by (D) we have

$$L^*(\operatorname{Fr}(\bigcup_{\alpha'} N^i_{\alpha'})) = L^*(F_i) + L^*(\operatorname{Fr}(\bigcup_{\alpha'} N^*_{\alpha'}) \cap C_i).$$

Now (33) and (34) are particular cases of this last relation.

Next, since S^* is a (w, z)-system, (B) shows that for $i=1, 2, \ldots$ and for any finite subsystem $N^*_{\alpha_1}, \ldots, N^*_{\alpha_k}$ of S^* we have

$$L^*(\operatorname{Fr}(\bigcup_{j=1}^{k} N^*_{\alpha_j}) \cap C_i) \leqq L^*(\operatorname{Fr}(\bigcup_{j=1}^{k} N^*_{\alpha_j})) \leqq z$$

and thus (33) yields

$$L^*(\operatorname{Fr}(\bigcup_{j=1}^{k} N^i_{\alpha_j})) \leqq L^*(F_i) + z$$

(see also Step 1). Hence in view of Step 3 and (32) for $i=1, 2, \ldots$ $\{\ldots, N^i_\alpha, \ldots\}$ is a $(\min(\delta(C_i), w), L^*(F_i)+z)$-system lying in a bounded planar domain (namely in C_i).

Let $\eta>0$. For $i=1, 2, \ldots$ let

$$z^i_\eta = \inf L^*(\operatorname{Fr}(\bigcup_{j=1}^{k} N^i_{\alpha_j})) \tag{35}$$

where $N^i_{\alpha_1}, \ldots, N^i_{\alpha_k}$ runs over all those finite subsystems of $\{\ldots, N^i_\alpha, \ldots\}$ for which

$$\operatorname{Fr}(\bigcup_{\alpha} N^i_\alpha) \subset O(\operatorname{Fr}(\bigcup_{j=1}^{k} N^i_{\alpha_j}), \eta) \quad \text{(see also Step 6).}$$

Let

$$z^i = \lim_{\eta \to 0} z^i_\eta. \tag{36}$$

We clearly have $z^i \leqq L^*(F_i)+z$ and thus z^i is finite. Now in view of Step 16 and (34) we have

$$L^*(\operatorname{Fr}(\bigcup_{\alpha} N^*_\alpha) \cap C_i) = L^*(\operatorname{Fr}(\bigcup_{\alpha} N^i_\alpha)) - L^*(F_i) \leqq z^i - L^*(F_i). \tag{37}$$

The following step is to show that

$$\sum_{i=1}^{\infty} (z^i - L^*(F_i)) \leqq z. \tag{38}$$

In fact we clearly need only to prove that the relation

$$\sum_{i=1}^{n} (z^i - L^*(F_i)) \leqq z + \varepsilon \tag{39}$$

holds for each $\varepsilon>0$ and for an arbitrary positive integer n.

Now let $\varepsilon>0$ and let n be a positive integer. For $i=1, \ldots, n$ let η_i be a positive real number such that

$$z^i_{\eta_i} \geqq z^i - \frac{\varepsilon}{n}. \tag{40}$$

In view of (36) such real numbers η_i exist. For $i=1, \ldots, n$ let $N^*_{\alpha_i,1}, \ldots, N^*_{\alpha_i,k_i}$ be a finite subsystem of S^* such that

$$\operatorname{Fr}\big(\bigcup_{\alpha} N^i_\alpha\big) \subset O\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k_i} N^i_{\alpha_i,j}\big), \eta_i\big). \tag{41}$$

In view of Step 6, such finite subsystems exist. Consider now all the sets $N^*_{\alpha_i,j}$ $(i=1, \ldots, n; j=1, \ldots, k_i)$ and arrange them into a sequence $N^*_{\alpha_1}, \ldots, N^*_{\alpha_k}$. Then for $i=1, \ldots, n$ we have by (41) and Step 7

$$\operatorname{Fr}\big(\bigcup_{\alpha} N^i_\alpha\big) \subset O\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^i_{\alpha_j}\big), \eta_i\big)$$

and thus (35) and (40) show that

$$L^*\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^i_{\alpha_j}\big)\big) \geqq z^i_{\eta_i} \geqq z^i - \frac{\varepsilon}{n}.$$

However by (33) this yields the relation

$$L^*\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^*_{\alpha_j}\big)\cap C_i\big) \geqq z^i - L^*(F_i) - \frac{\varepsilon}{n}$$

and since the C_i-s are mutually disjoint compact sets and thus the $\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^*_{\alpha_j}\big)\cap C_i$-s are mutually disjoint and compact as well, it follows by (E) that

$$L^*\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^*_{\alpha_j}\big)\cap (C_1\cup\ldots\cup C_n)\big) = L^*\big(\bigcup_{i=1}^{n} \big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^*_{\alpha_j}\big)\cap C_i\big)\big) = \tag{42}$$

$$= \sum_{i=1}^{n} L^*\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N_{\alpha_j}\big)\cap C_i\big) \geqq \sum_{i=1}^{n} \Big(z^i - L^*(F_i) - \frac{\varepsilon}{n}\Big) = \sum_{i=1}^{n} (z^i - L^*(F_i)) - \varepsilon.$$

Recall that S^* is a (w, z)-system and thus taking also (B) into account we find

$$L^*\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^*_{\alpha_j}\big)\cap (C_1\cup\ldots\cup C_n)\big) \leqq L^*\big(\operatorname{Fr}\big(\bigcup_{j=1}^{k} N^*_{\alpha_j}\big)\big) \leqq z. \tag{43}$$

Consequently in virtue of (42) and (43) we have

$$z \geqq \sum_{i=1}^{n} (z^i - L^*(F_i)) - \varepsilon$$

and thus

$$\sum_{i=1}^{n} (z^i - L^*(F_i)) \leqq z + \varepsilon.$$

Now in view of (37), (38), Steps 12 and 13 we get

$$L^*(\mathrm{Fr}(\bigcup_\alpha N_\alpha^*)) = \sum_{i=1}^{\infty} L^*(\mathrm{Fr}(\bigcup_\alpha N_\alpha^*)\cap C_i) \leqq \sum_{i=1}^{\infty} (z^i - L^*(F_i)) \leqq z.$$

The proof of Principle 2 is complete.

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A NOTE ON (0, 2) INTERPOLATION

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1. Introduction

In the famous problem collection of P. Turán [6], Problem 29 deals with (0, 2) interpolation on the roots of Jacobi polynomials and is as follows:

Find all Jacobi matrices $P(\alpha, \beta)$, $\alpha \neq \beta$ *for which the* (0, 2) *interpolation problem does have a unique solution.*

By a Jacobi matrix $P(\alpha, \beta)$ Turán means the triangular matrix $\{x_{k,n}\}_{k=1}^{n}{}_{n=1}^{\infty}$ whose n-th row consists of the roots of the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$.

The case $\alpha = \beta$ was analysed by Turán and his coworkers in [4] and [1].

Recently A. M. Chak, A. Sharma and J. Szabados [2] solved this general problem and gave explicit representations of the fundamental polynomials.

The weight function of the Jacobi polynomials is always greater than zero *within* $(-1, 1)$. It would be interesting to know what can be said in the case when the weight function has zeros in $(-1, 1)$. So let us consider the weight function $|x|^{2\alpha+1}(1-x^2)^\beta$ $(\alpha, \beta > -1)$. The corresponding orthogonal polynomials are called Lascenov-polynomials [3]. In the following we shall investigate the problem of (0, 2) interpolation on the zeros of the Lascenov-polynomials.

2. Preliminaries and main result

Let us denote by $R_n^{(\alpha,\beta)}(x)$ the n-th Lascenov polynomial. The weight function $|x|^{2\alpha+1}(1-x^2)^\beta$ is even, so the roots of $R_n^{(\alpha,\beta)}$ are symmetrical to zero. By a theorem of Turán and Surányi [4], for odd n's such node systems are not regular (i.e. there are no unique solutions for the (0, 2) interpolation problem). Therefore in the following we suppose that n is even $(n=2m)$.

It is well-known [3] that

$$R_n^{(\alpha,\beta)}(x) = P_m^{(\alpha,\beta)}(1-2x^2). \tag{2.1}$$

We shall need some known properties of the Jacobi polynomials [5]. They satisfy the differential equation

$$(1-x^2)y'' + [(\beta+1)(1-x)-(\alpha+1)(1+x)]y' + n(n+\alpha+\beta+1) = 0. \tag{2.2}$$

We shall use the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \tag{2.3}$$

and in this case it is known that

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}. \tag{2.4}$$

We shall prove the following

THEOREM. *The problem* (0, 2) *interpolation on the roots of* $R_n^{(\alpha,\beta)}$ $(\alpha, \beta > -1)$ *is uniquely solvable if and only if*

$$D_{n,i}(\alpha, \beta) \neq 0 \quad (i = \pm 1) \tag{2.5}$$

where $(n=2m)$

$$D_{n,i}(\alpha, \beta) = \sum_{k=0}^{m} \frac{(-1)^k}{\binom{m}{k}} \binom{m+\alpha}{m-k} \binom{m+\beta}{k} \binom{\frac{2\alpha+i}{4}}{k} \binom{\frac{\beta+1}{2}}{m-k}. \tag{2.6}$$

REMARKS. a) The proof follows the ideas of [2].

b) If $\alpha = -\frac{1}{2}$ then $R_n^{(-1/2,\beta)}(x) = P_n^{(\beta,\beta)}(x)$ i.e. then R_n is an ultraspherical polynomial. Then

$$D_{n,1}\left(-\frac{1}{2}, \beta\right) = \binom{m-\frac{1}{2}}{m} \binom{\frac{\beta+1}{2}}{m}$$

and

$$D_{n,-1}\left(-\frac{1}{2}, \beta\right) = \frac{\binom{n}{m}}{2^n} \sum_{k=0}^{m} \binom{m+\beta}{k} \binom{\frac{\beta+1}{2}}{k} = 2^{-n} \binom{n}{m} \binom{m+\frac{3\beta+1}{2}}{m}$$

so in this case the problem is regular if and only if β is not an odd integer. This was proved by J. Surányi and P. Turán [4] in a different way.

c) If β and 2α are odd integers and if $n > \alpha+\beta+\frac{3}{2}$ then the (0, 2) interpolation on the roots of $R_n^{(\alpha,\beta)}$ is not regular, because then $\binom{\frac{\beta+1}{2}}{m-k} = 0$ for $k = 0, 1, \ldots$ $\ldots, m-\frac{\beta+1}{2}-1$ and for higher k's either $\binom{\frac{2\alpha+1}{4}}{k} = 0$ or $\binom{\frac{2\alpha-1}{4}}{k} = 0$.

d) If only one of 2α and β is odd and if $n > \alpha+\beta+\frac{3}{2}$, then $D_{n,i} \neq 0$. For if β is odd and 2α is not, then the summation in (2.6) extends from $k = m-\frac{\beta+1}{2}$ to $k = m$. Also

$$\operatorname{sgn} \binom{\frac{2\alpha+i}{4}}{k} = (-1)^{k-l_i}, \; l_i = \left[\frac{2\alpha+i}{4}\right]$$

so that $\operatorname{sgn} D_{n,i}(\alpha, \beta)=(-1)^{m-l_i}$. The case when 2α is odd and β is not is similar. Thus in this case the problem is regular.

e) The explicit formulae of the fundamental polynomials can be determined the same way as in [2] with the necessary modifications evident from the proof of our theorem.

3. Proof of the Theorem

Let $\{x_j\}_{j=1}^{n}$ be the zeros of $P_m^{(\alpha,\beta)}(x)$, then by (2.1)

$$y_j = -y_{-j} = \sqrt{\frac{1-x_j}{2}} \quad (j = 1, 2, \dots, m) \tag{3.1}$$

are the roots of $R_n^{(\alpha,\beta)}(x)$ $(n=2m)$.

We shall show that the only polynomial $F \in \Pi_{4n-1}$ which satisfies the following homogeneous conditions of (0, 2) interpolation

$$F^{(k)}(y_j) = 0 \quad (k = 0, 2;\ j = \pm 1, \pm 2, \dots, \pm m), \tag{3.2}$$

is identically zero if and only if $D_{n,i}(\alpha, \beta) \neq 0$ $(i=\pm 1)$.

Let $G(x)=F(x)+F(-x)$ and $H(x)=F(x)-F(-x)$. Obviously (3.2) is true for G and H too and F is identically zero if and only if G and H are so.

From its definition, $G(x)$ is even and by (3.2) and (2.1)

$$G(x) = P_m(1-2x^2)\, g_m(1-2x^2)$$

where $\deg g_m \leq m-1$.

Then from (3.2) when $k=2$ we get

(3.3)

$$0 = G''(y_j) = 2(-4y_j P'(x_j))(-4y_j g_m'(x_j)) + (-4P'(x_j)+16y_j^2 P''(x_j))\, g_m(x_j) =$$

$$= 4P'(x_j)\left\{4(1-x_j)g_m'(x_j) - g_m(x_j) + 2(1-x_j)\frac{P''(x_j)}{P'(x_j)}\, g_m(x_j)\right\}$$

where we used (3.1), too.

From (2.2), we have

$$\frac{P''(x_j)}{P'(x_j)} = -\frac{(\beta+1)(1-x_j)-(\alpha+1)(1+x_j)}{1-x_j^2}$$

so that (3.3) yields (for $j=1, 2, \dots, n$),

(3.4)

$$0 = \frac{4P'(x_j)}{1+x_j}\{4(1-x_j^2)g_m'(x_j) - [2(\beta+1)(1-x_j)-(2\alpha+1)(1+x_j)]\, g_m(x_j)\} =$$

$$= \frac{4P'(x_j)}{1+x_j}\, \mathscr{L}_1 g_m(x_j)$$

where the differential operator $\mathcal{L}_1$ is given by

$$\mathcal{L}_1 g(x) = -4(x^2-1)g'(x)+[2(\beta+1)(x-1)+(2\alpha+1)(x+1)]g(x).$$

Then from (3.4), we have

$$\mathcal{L}_1 g_m(x) = c_1 P_m^{(\alpha,\beta)}(x). \tag{3.5}$$

$H(x)$ is odd by its definition so by (3.2) we have

$$H(x) = xP_m(1-2x^2)h_m(1-2x^2)$$

where $\deg h_m \leqq n-1$.

Similarly as for G we get with an easy computation

$$\mathcal{L}_{-1} h_m(x) = c_{-1} P_m^{(\alpha,\beta)}(x) \tag{3.6}$$

where

$$\mathcal{L}_{-1} h(x) = -4(x^2-1)h'(x)+[2(\beta+1)(x-1)+(2\alpha-1)(x+1)]h(x). \tag{3.7}$$

Let

$$q_m(x) = \sum_{k=0}^{m-1} \alpha_k (x-1)^k (x+1)^{m-1-k}.$$

Evidently, every polynomial of degree $m-1$ has such a representation which is unique. Then by an easy calculation, we have $(i=\pm 1)$

$$\mathcal{L}_i q_m(x) = (2\alpha+i)\alpha_0(x+1)^m + \sum_{k=1}^{m-1}(x-1)^k(x+1)^{m-k}\big[\alpha_k(2\alpha+i+4k)+$$

$$+\alpha_{k-1}\big(2(\beta+1)+4(m-k)\big)\big]+2(\beta+1)\alpha_{m-1}(x-1)^m.$$

Using (2.4) and comparing the coefficients of $(x-1)^k(x+1)^{n-k}$ on both sides in (3.5) and (3.6), we have the following system of equations:

$$\begin{cases} (2\alpha+i)\alpha_0 & = c_i 2^{-m}\binom{m+\alpha}{m} \\ (2\alpha+i+4k)\alpha_k+\big(2(\beta+1)+4(n-k)\big)\alpha_{k-1} & = c_i 2^{-m}\binom{m+\alpha}{m-k}\binom{m+\beta}{k} \\ & \quad (k=1,2,\dots,m-1) \\ 2(\beta+1)\alpha_{m-1} & = c_i 2^{-m}\binom{m+\beta}{m}. \end{cases} \tag{3.8}$$

Expanding the determinant of this system in terms of the elements corresponding to c_i we get (2.6). The non vanishing of these determinants is necessary and sufficient for the system of equations (3.8) to have the identically zero solution.

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AN INTERPOLATION PROBLEM FOR REAL POLYNOMIALS BY THEIR MEANS BETWEEN CONSECUTIVE ZEROS

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§1. Introduction. It is well known that besides the classical interpolation problems of Lagrange and Hermite types there are additional, well posed, interpolation problems. Among these are the interpolation by extreme values [4], [6] in the real case and [1], [7] in the complex case, and the mixed-type interpolation by extreme points and extreme values [8]. In this note we consider the interpolation problem for real polynomials by their mean value between pairs of consecutive zeros. These mean values have important properties and were treated by several authors [2], [3], [9]. The general approach is similar to that employed in [8]. The existence and uniqueness of the mean value interpolating polynomial is reduced to the nonvanishing of a certain Cauchy type determinant. Several conjectures are stated supported by numerical calculations.

§ 2. Several lemmas. Following are three lemmas required in the proof of the main theorem.

LEMMA 1. [5] *Let* $f\colon S^n \to S^n$ *be a continuous function defined on an n-dimensional sphere* S^n *such that for some* $p \in S^n$ *we have*

(1) (a) $f(p)=p$, (b) $f(S^n-\{p\}) \subset S^n-\{p\}$, *and* (c) $f|_{S^n-\{p\}}$

is a local homeomorphism. Then f is a homeomorphism of S^n *onto itself.*

LEMMA 2. *Let* $f(x)$ *be a positive real-valued function defined for positive, x bounded away from zero and integrable on every finite interval of the positive axis. For a fixed set of* x_j, $j=0, 1, \ldots, n$, $0=x_0<x_1<\ldots<x_n$, *define* $p(x)=\prod_{k=0}^{n}(x-x_k)$ *and*

$$A_i = (-1)^{n-j+1}\frac{1}{x_j-x_{j-1}}\int_{x_{j-1}}^{x_j} f(x)p(x)\,dx \quad (j=1,\ldots,n). \tag{2}$$

Then, for fixed n, as

$$\alpha=\alpha(x_1,\ldots,x_n)=\min_{1\le j\le n}(x_j-x_{j-1}, x_n^{-1})\to 0$$

also

$$\beta=\beta(x_1,\ldots,x_n)=\min_{1\le j\le n}(A_j, A_j^{-1})\to 0.$$

PROOF. It is enough to show that there is a positive function $h(d)$ defined for positive d such that (i) $h(d)\to 0$ as $d\to 0$, (ii) $\beta \geqq h$ implies $\alpha \geqq d^{2n+2}$ for all sufficiently small positive d. One verifies that $\beta>0$.

Assume $\beta \geqq h$ and $f(x) \geqq m > 0$. Then $h \leqq A_j \leqq h^{-1}$ for $j=1, 2, \ldots, n$. Choose μ such that $\Delta x_\mu = x_\mu - x_{\mu-1} \geqq x_n/n$. We then have

$$h^{-1} \geqq A_\mu = (-1)^{n-\mu+1} \frac{1}{\Delta x_\mu} \int_{x_{\mu-1}}^{x_\mu} f(x) p(x)\, dx \geqq \frac{m}{\Delta x_\mu} \int_{x_{\mu-1}}^{x_\mu} (x - x_{\mu-1})^\mu (x_\mu - x)^{n-\mu+1} dx =$$

$$= m \Delta x_\mu^{n+1} \frac{\mu!(n-\mu+1)!}{(n+2)!} \geqq \frac{m x_n^{n+1}}{n^{n+1}(n+2)!}.$$

So that

$$x_n \leqq c h^{-1/(n+1)} \tag{3}$$

where c is a positive constant depending only on n and m. On the other hand, for $j=1, 2, \ldots, n$

$$h \leqq A_j \leqq \frac{x_n^{n-1}}{\Delta x_j} \int_{x_{j-1}}^{x_j} f(x)(x - x_{j-1})(x_j - x)\, dx \leqq \Delta x_j x_n^{n-1} \int_0^{x_n} f(x)\, dx. \tag{4}$$

Define $\varphi(t) = \int_0^t f(x)\, dx$, $\psi = \varphi^{-1}$. Let $d = [\varphi(c h^{-1/(n+1)})]^{-1}$. Then $h = h(d) = c^{n+1}[\psi(d^{-1})]^{-(n+1)}$. $\psi(s) \to \infty$ as $s \to \infty$ so $h(d) \to 0$ as $d \to 0$. By (3) and (4) we have

$$\Delta x_j \geqq h x_n^{-(n-1)} \Big(\int_0^{x_n} f(x)\, dx \Big)^{-1} \geqq h c^{-(n-1)} h^{(n-1)/(n+1)} [\varphi(c h^{-1/(n+1)})]^{-1} = \tag{5}$$

$$= c^{-(n-1)} h^{2n/(n+1)} d.$$

Now $\varphi(t) \geqq mt$ implies $\psi(t) \leqq t/m$ for $t > 0$. Therefore by (3)

$$x_n^{-1} \geqq c^{-1} h^{1/(n+1)} = [\psi(d^{-1})]^{-1} \geqq md \geqq d^{2n+2} \tag{6}$$

for all sufficiently small $d > 0$. Also

$$h(d) = c^{n+1}[\psi(d^{-1})]^{-(n+1)} \geqq c^{n+1} m^{n+1} d^{n+1}.$$

Thus, by (5)

$$\Delta x_j \geqq c^{n+1} m^{2n} d^{2n+1} \geqq d^{2n+2} \tag{7}$$

for all sufficiently small $d > 0$ and all $j = 1, 2, \ldots, n$. By (6) and (7) $\alpha \geqq d^{2n+2}$ for all sufficiently small $d > 0$. This completes the proof.

Let

$$P = \{(y_1, \ldots, y_n) \in R^n,\quad 0 < y_1 < y_2 < \ldots < y_n\}$$

and let Q, $Q \supset P$ be defined by

$$Q = \{(z_1, \ldots, z_n),\ z_j > 0,\quad j = 1, 2, \ldots, n\}.$$

We have

LEMMA 3. *Let $f(x)$ be as in Lemma 2. Define a mapping F: $P \to Q$ as follows. For $x = (x_1, \ldots, x_n) \in P$ construct $p(x) = \prod_{k=0}^{n} (x - x_k)$, $x_0 = 0$, and let A_j be given by* (2).

Then the mapping F defined by $F(x)=A=(A_j)$ *is a homeomorphism of P onto Q if F is a local homeomorphism.*

PROOF. Since P and Q are homeomorphic to R^n it is possible to extend the mapping F to S^n by adjoining points $\infty_P \in P$ and $\infty_Q \in Q$ and letting $F(\infty_P)=\infty_Q$. If F is a local homeomorphism on P, then since by Lemma 2 F is continuous on S^n, F satisfies conditions (1) of Lemma 1 with $p=\infty_P$. Thus, F is a homeomorphism of S^n onto itself, hence of P onto Q.

§ 3. The main theorem and some conjectures. We are ready now to formulate the main result.

THEOREM. *In order that for any given positive numbers* $A_1, A_2, \ldots, A_n$ *there exist a unique polynomial of the form* $p(x)=\prod_{k=0}^{n}(x-x_k)$, $0=x_0<x_1<\ldots<x_n$ *such that*

$$\frac{1}{x_j-x_{j-1}}\int_{x_{j-1}}^{x_j} p(x)\,dx=(-1)^{n-j+1}A_j,\quad j=1,2,\ldots,n \tag{8}$$

it is sufficient that the determinant of the matrix $B=(b_{ij})$ *where*

$$b_{ij}=\sum_{k=j}^{n}\frac{1}{y_i-x_k}+\delta_{ij}\frac{1}{\Delta x_j},\quad i,j=1,2,\ldots,n, \tag{9}$$

be nonzero for all $0=x_0<y_1<x_1<y_2<\ldots<y_n<x_n$. δ_{ij} *is the Kronecker delta.*

PROOF. By Lemma 3 it is enough to show that the nonvanishing of the determinant of matrix (9) implies that the mapping F is a local homeomorphism. This in turn amounts to showing that $J=\|\partial(F_1, F_2, \ldots, F_n)/\partial(x_1, x_2, \ldots, x_n)\|\neq 0$ at every point of the set P. Now

$$F_j(x_1, x_2, \ldots, x_n)=\frac{(-1)^{n-j+1}}{x_j-x_{j-1}}\int_{x_{j-1}}^{x_j}\prod_{k=0}^{n}(x-x_k)\,dx,\quad j=1,2,\ldots,n.$$

Let

$$G_j=(-1)^{n-j+1}F_j,\ \Delta x_j=x_j-x_{j-1},\ p(x)=\prod_{k=0}^{n}(x-x_k).$$

We have

$$B_{jj}=\frac{\partial G_j}{\partial x_j}=-\frac{1}{\Delta x_j^2}\int_{x_{j-1}}^{x_j}p(x)\frac{x-x_{j-1}}{x-x_j}\,dx,$$

$$B_{jj-1}=\frac{\partial G_j}{\partial x_{j-1}}=\frac{1}{\Delta x_j^2}\int_{x_{j-1}}^{x_j}p(x)\frac{x-x_j}{x-x_{j-1}}\,dx,$$

and if $k\neq j-1, j$

$$B_{jk}=\frac{\partial G_j}{\partial x_k}=-\frac{1}{\Delta x_j^2}\int_{x_{j-1}}^{x_j}p(x)\frac{\Delta x_j}{x-x_k}\,dx.$$

Passing to a determinant with elements $\gamma_{jk}=-\Delta x_j^2 B_{jk}$, it is clear that $J=0$ if and only if there exists a nonzero vector $\lambda=(\lambda_k)$ such that

$$\sum_{k=1}^{n} \lambda_k \gamma_{jk} = 0, \quad j = 1, 2, \ldots, n.$$

A short calculation shows that

$$\int_{x_{j-1}}^{x_j} p(x) R_j(x)\,dx = 0, \quad j = 1, 2, \ldots, n \tag{10}$$

where

$$R_j(x) = \sum_{k=1}^{n} \frac{\lambda_k}{x-x_k} + \frac{\Delta\lambda_j}{\Delta x_j}, \quad j = 1, 2, \ldots, n, \ \lambda_0 = 0.$$

Obviously (10) implies the existence of numbers $y_j, x_{j-1}<y_j<x_j$ such that $R_j(y_j)=0$, $j=1, 2, \ldots, n$. Considering the last equations as a homogeneous, linear system for λ one deduces the vanishing of the determinant of $A=(a_{jk})$ with

$$a_{jk} = \frac{1}{y_j-x_k} + (\delta_{jk}-\delta_{j\,k+1}) \frac{1}{\Delta x_j}, \quad j, k = 1, 2, \ldots, n.$$

Finally it is easy to verify that $\|A\|=\|B\|$. As a result we have shown that $J=0$ implies $\|B\|=0$. This completes the proof.

Actually numerical calculations support the following conjectures.

CONJECTURE A. The determinant $\|B\|$ in (9) satisfies

$$\operatorname{sign} \|B\| = (-1)^n, \quad n = 1, 2, \ldots.$$

A related conjecture arises by considering the Jacobian matrix

$$\partial(F_1, F_2, \ldots, F_n)/\partial(x_0, x_1, \ldots, x_{n-1}).$$

CONJECTURE B. The determinant $\|D\|$ of the matrix with elements

$$d_{ij} = \sum_{k=0}^{j-1} \frac{1}{y_i-x_k} - \delta_{ij} \frac{1}{\Delta x_i}, \quad i, j = 1, 2, \ldots, n,$$

$x_0<y_1<x_1<y_2<\ldots<y_n<x_n$, is positive for $n=1, 2, \ldots$.

As before the last determinant is equal to the determinant of the matrix $C=(c_{ij})$, where

$$c_{ij} = \frac{1}{y_i-x_{j-1}} + (\delta_{i+1\,j}-\delta_{ij}) \frac{1}{\Delta x_i}, \quad i, j = 1, 2, \ldots, n.$$

It is also clear from the preceding that Conjecture A can be put in another form.

CONJECTURE A$_1$. Given a nontrivial rational function $R(x)$ of the form

$$R(x) = \sum_{k=1}^{n} \frac{\lambda_k}{x-x_k}$$

$0<x_1<x_2<\ldots<x_n$, there is at least one interval (x_{j-1}, x_j), $j=1, 2, \ldots, n$, where $R(x)+\Delta\lambda_j/\Delta x_j$ does not vanish. Here $\lambda_0=x_0=0$.

Finally Lemma 2 and the results in [8] also suggest strengthening the main theorem by replacing $p(x)$ in (8) by $p(x)f(x)$ where $f(x)$ is a continuous function which satisfies Lemma 2. No numerical calculations have been done, however, in this case.

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PERIODIC SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

We shall consider the following nonlinear second order differential equation:

$$x''+\lambda x+\mu g(x) = \varepsilon h(t) \tag{E}$$

in which λ, μ and ε are real parameters and g and h are real valued continuous functions. In addition to the assumption that h is a periodic function of period τ, under certain conditions, we prove the existence of τ-periodic solutions for (E). We shall first consider a periodic boundary value problem (PBVP for short) associated with (E) and obtain the existence of solutions by employing the methods involving differential inequalities [1], and the alternative method for nonlinear problems at resonance [2]. These solutions will then be related to the τ-periodic solutions of (E).

In this study, we not only allow g to be nonlinear and unbounded, but no hypotheses concerning the existence and uniqueness of solutions of the initial value problems for (E) are required. In Section 2 of this paper, we include the preliminaries on the alternative method for problems at resonance and also mention an existence result for such problems [3], which is crucial for our study. Section 3 contains the main results.

Indeed, some special cases of (E) have been considered in [4], [6] and [7]. Our results include and improve some of these results. We note that our techniques and methods differ from earlier studies.

2. Preliminaries

Let X be a real Hilbert space with inner product $(\cdot, \cdot)$, and norm $\|\cdot\|$. Let L: $D(L)\subset X\to X$ be a linear operator with finite dimensional kernel X_0 and range $X_1=X_0^\perp$ so that $X=X_0\oplus X_1$. Let P: $X\to X$ be the projection operator with the range X_0 and let H: $X_1\to D(L)\cap X_1$ be the partial inverse of L so that $H(I-P)L=$ $=LH(I-P)=I-P$. Since L has a nontrivial null space, this case is known as the "problem at resonance."

We are interested in solving the operator equation

$$Lx = Nx, \quad x\in D(L), \tag{2.1}$$

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in which $N: X \to X$ is a nonlinear operator. Thus, $x=x_0+x_1$, where $x_0 \in X_0$, $x_1 \in X_1$ is a solution of (2.1) if and only if [2],

$$x_1 = H(I-P)\,N(x_0+x_1) \tag{2.2}$$

and

$$PN(x_0+x_1) = 0. \tag{2.3}$$

The equations (2.2) and (2.3) are called the auxiliary and the bifurcation equations respectively.

The following result of [3], will be used in our subsequent discussion.

THEOREM 2.1. *Assume that H is compact and the following conditions hold:*
(i) *there exists a constant $J>0$ such that $\|Nx\| \leqq J$ for all $x \in X$;*
(ii) *there exists a constant $R>0$ such that for all $x=x_0+x_1$ in X, $x_0 \in X_0$, $x_1 \in X_1$ with $\|x_0\|=R$ and $x_1=H(I-P)N(x_0+x_1)$, we have $(Nx, x_0) \leqq 0$ (or $\geqq 0$). Then the equation* (2.1) *has at least one solution.*

3. Main results

We now consider the second order equation

$$-x'' = f(t, x) \tag{3.1}$$

in which, we assume $f(t, x)$ is continuous on $S=\{(t, x)\colon t\in \mathbf{R},\ x \in \mathbf{R}\}$. A function $\alpha(t) \in C^2[0, \tau]$ will be called a lower solution of (3.1) on $[0, \tau]$ if $-\alpha'' \leqq f(t, \alpha)$ on $[0, \tau]$. Similarly, $\beta(t) \in C^2[0, \tau]$ will be called an upper solution of (3.1) on $[0, \tau]$ if $-\beta'' \geqq f(t, \beta)$ on $[0, \tau]$.

We shall prove the following result:

THEOREM 3.1. *Assume that there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of* (3.1) *on $[0, \tau]$ with $\alpha(0)=\alpha(\tau)$, $\beta(0)=\beta(\tau)$ and such that $\alpha(t) \leqq \beta(t)$ on $[0, \tau]$. Furthermore, assume that the inequalities $\alpha'(0) \geqq \alpha'(\tau)$ and $\beta'(0) \leqq \beta'(\tau)$ hold. Then the periodic boundary value problem*

$$-x'' = f(t, x), \quad x(0) = x(\tau), \quad x'(0) = x'(\tau) \tag{3.2}$$

has at least one solution.

PROOF. Let F be the function defined as follows:

$$F(t, x) = f\big(t, p(t, x)\big) + r(t, x)$$

where $p(t, x) = \max\big\{\alpha(t), \min\{x, \beta(t)\}\big\}$ and

$$r(t, x) = \begin{cases} \dfrac{\beta(t)-x}{1+x^2} & \text{if } x > \beta(t) \\ 0 & \text{if } \alpha(t) \leqq x \leqq \beta(t) \\ \dfrac{\alpha(t)-x}{1+x^2} & \text{if } x < \alpha(t); \end{cases}$$

and consider the modified PBVP

(3.3) $$-x'' = F(t, x), \quad x(0) = x(\tau), \quad x'(0) = x'(\tau).$$

Clearly, F is continuous and bounded on $[0, \tau]\times\mathbf{R}$. Following standard arguments ([1, p. 19]), one can show that, if x is a colution of (3.3) then $\alpha(t)\leqq x(t)\leqq\beta(t)$ for all $t\in[0, \tau]$; and consequently x will be a solution of the PBVP (3.2). By employing the alternative method, we shall show that the PBVP (3.3) has at least one solution.

Let $X=L^2[0, \tau]$ and $Lx=-x''$ where $D(L)=\{x\in X\colon x$ and x' are absolutely continuous, $x''\in X$, $x(0)=x(\tau)$ and $x'(0)=x'(\tau)\}$. Let N be the Nemytskii operator generated by F. Then, the PBVP (3.3) can be translated into the operator equation (2.1). It is easy to see that X_0, the kernel of L, consists of all constant functions and the range of L, X_1 is the class of all functions whose average is zero and $X=X_0\oplus X_1$. We define the operators $P\colon X\to X$ and $H\colon X_1\to D(L)\cap X_1$ as follows: $Px=\frac{1}{\tau}\int_0^\tau x(s)ds$ and $Hx_1=y_1$ if and only if $-y_1''=x_1$, $y_1(0)=y_1(\tau)$, $y_1'(0)=y_1'(\tau)$ and $Py_1=0$.

Since F is bounded, there exist positive constants J and B such that $\|Nx\|\leqq J$ for all $x\in X$ and $|x_1(t)|\leqq B$ for all $t\in[0, \tau]$ and x_1 solution of (2.2). Notice that B is independent of $x_0\in X_0$.

Now, choose $R>0$ such that

(3.4) $$R-B > \max_{t\in[0,\tau]}\beta(t) \quad\text{and}\quad -R+B < \min_{t\in[0,\tau]}\alpha(t).$$

Choose $\bar{R}=\frac{R}{\sqrt{2}}$. If $x_0\in X_0$ and $\|x_0\|=\bar{R}$, then either $x_0=R$ or $x_0=-R$. Hence, if $x_0=R$ using the definition of F, (3.4) and the hypotheses, we see that

$$\big(N(x_0+x_1), x_0\big) = R\int_0^\tau F(t, R+x_1(t))dt = R\int_0^\tau f(t,\beta(t))dt+R\int_0^\tau \frac{\beta(t)-R-x_1(t)}{1+(R+x_1(t))^2}dt\leqq$$
$$\leqq R\int_0^\tau f(t,\beta(t))dt \leqq -R\big(\beta'(\tau)-\beta'(0)\big) \leqq 0.$$

Similarly, if $x_0=-R$ then we also see that $\big(N(x_0+x_1), x_0\big)\leqq 0$. Thus, by Theorem 2.1, it follows that the PBVP (3.3) has a solution and this completes the proof.

Remark 3.2. Notice that if, in addition to the assumptions of Theorem 3.1, $f(t, x)$ is a periodic function of period τ in t, that is $f(t+\tau, x)=f(t)$ for all $t\in\mathbf{R}$, then the solution x obtained in Theorem 3.1 is a τ-periodic solution of the differential equation (3.1).

If $f(t, x)=\lambda x+\mu g(x)-\varepsilon h(t)$, where $g\in C(\mathbf{R}, \mathbf{R})$, $h\in C[0, \tau]$ and λ, μ and ε are real parameters, we have, as an immediate consequence of Theorem 3.1, the following result.

Theorem 3.3. *Assume that there exist functions* $\alpha, \beta\in C^2[0, \tau]$ *satisfying*

(i) $\alpha(t)\leqq\beta(t)$, $t\in[0, \tau]$, $\alpha(0)=\alpha(\tau)$, $\beta(0)=\beta(\tau)$;

(ii) $\alpha'(0) \geqq \alpha'(\tau)$ and $\alpha''+\lambda\alpha+\mu g(\alpha)\geqq\varepsilon h(t)$, $t\in[0, \tau]$; *and*

(iii) $\beta'(0)\leqq\beta'(\tau)$ and $\beta''+\lambda\beta+\mu g(\beta)\leqq\varepsilon h(t)$, $t\in[0, \tau]$.

Then the PBVP

$$(3.5) \qquad -x'' = \lambda x + \mu g(x) - \varepsilon h(t), \quad x(0) = x(\tau), \quad x'(0) = x'(\tau),$$

has a solution x *such that* $\alpha(t) \leqq x(t) \leqq \beta(t)$, *for all* $t \in [0, \tau]$.

COROLLARY 3.4. *Assume that there exist constants* a *and* b *such that* $a \leqq b$ *and* $\lambda a + \mu g(a) - \varepsilon h(t) \geqq 0 \geqq \lambda b + \mu g(b) - \varepsilon h(t)$, *for all* $t \in [0, \tau]$. *Then the PBVP* (3.5) *has a solution* x *such that* $a \leqq x(t) \leqq b$ *for all* $t \in [0, \tau]$.

PROOF. The functions $\alpha, \beta\colon [0, \tau] \to \mathbf{R}$ defined by $\alpha(t) \equiv a$ and $\beta(t) \equiv b$ for all $t \in [0, \tau]$ are lower and upper solutions, respectively, satisfying the hypotheses of Theorem 3.3 and hence the conclusion.

The following example illustrates Theorem 3.3.

EXAMPLE 3.5. Consider the PBVP

$$(3.6) \qquad x'' + x - x^3 = -\sin^3 t, \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi).$$

Choose $\alpha(t) = -2 + \sin t$ and $\beta(t) \equiv 2$ for $t \in [0, 2\pi]$. It is easy to see that all the hypotheses of Theorem 3.3 are satisfied and hence the PBVP (3.6) has a solution x, satisfying the inequality $-2 + \sin t \leqq x(t) \leqq 2$ for all $t \in [0, 2\pi]$. Furthermore, since $\sin^3 t$ is 2π-periodic, in view of the Remark 3.2, it follows that $x(t)$ is a 2π-periodic solution to the differential equation $x'' + x - x^3 = -\sin^3 t$, $t \in \mathbf{R}$. Indeed, the function $x(t) = \sin t$ is such a solution.

REMARK 3.6. If the function f in (3.1) also depends on x', a result similar to Theorem 3.1 may be proved by requiring that f satisfies a Nagumo condition relative to the lower and upper solutions α and β ([1, p. 25]). For details, we refer the reader to [5].

Now we shall consider the PBVP (3.5) and prove another result on existence of solutions. This result does not require the existence of upper and lower solutions as in Theorem 3.3.

For $\alpha > 0$, define

$$(3.7) \qquad \varphi_\alpha(x) = \begin{cases} \alpha \operatorname{sgn} x & \text{if } |x| \geqq \alpha \\ x & \text{if } |x| \leqq \alpha \end{cases}$$

and consider the modified PBVP

$$(3.8) \qquad x'' + \lambda \varphi_\alpha(x) + \mu g(\varphi_\alpha(x)) = h(t), \quad x(0) = x(\tau), \quad x'(0) = x'(\tau).$$

We shall prove the following result for the problem (3.8).

LEMMA 3.7. *The PBVP* (3.8) *has at least one solution provided there exists an* $\alpha > 0$ *such that*

$$(3.9) \qquad \begin{cases} \textit{either} & \text{(i) } \lambda\alpha + \mu g(\alpha) \geqq \varepsilon Ph \geqq -\lambda\alpha + \mu g(-\alpha) \\ \textit{or} & \text{(ii) } \lambda\alpha + \mu g(\alpha) \leqq \varepsilon Ph \leqq -\lambda\alpha + \mu g(-\alpha) \end{cases}$$

where $Ph = \dfrac{1}{\tau} \displaystyle\int_0^\tau h(s)\,ds$.

PROOF. The proof is again an application of Theorem 2.1. As in the proof of Theorem 3.1, we consider the space $X=L^2[0, \tau]$ with the usual inner product and norm, and define the operators L, P and H and the subspaces X_0 and X_1 as before. Let $N_\alpha\colon X\to X$ be the nonlinear operator defined by $(N_\alpha(x))(t)=\varphi_\alpha(x(t))+ +\mu g(\varphi_\alpha(x(t)))-\varepsilon h(t)$. Clearly $D(N_\alpha)=X$ since φ_α is bounded. Thus, the PBVP (3.8) may be translated into the operator equation

(3.10) $$Lx = N_\alpha x.$$

For $x\in X$, set $x=x_0+x_1$ where $x_0=Px$ and $x_1=(I-P)x$. Thus (3.10) is equivalent to the coupled system of equations

(3.11) $$x_1 = H(I-P)N_\alpha(x_0+x_1)$$

and

(3.12) $$PN_\alpha(x_0+x_1) = 0.$$

Since φ_α is bounded, there exists a constant $J>0$ such that $\|N_\alpha x\|\leqq J$ for all $x\in X$. Also, given $x_0\in X_0$, we can find a constant $c>0$, independent of $x_0\in X_0$, such that $\max_{0\leqq t\leqq \tau} |x_1(t)|\leqq c$, for any solution x_1 of (3.11).

Now choose $R>0$ large enough so that

(3.13) $$R-c \geqq \alpha \quad \text{and} \quad -R+c \leqq -\alpha.$$

Proceeding exactly as in the proof of Theorem 3.1, we see that all the hypotheses of Theorem 3.1 are satisfied and hence the equation (3.10) has a solution. This completes the proof.

LEMMA 3.8. *Assume that* $Ph=0$ *and* $g(0)=0$. *Then, for any* $\alpha>0$ *and for any of the following choices of* λ, μ *and* g *given by*

(i) $\lambda>0$, $\mu\leqq 0$, $xg(x)\leqq 0$ *for all* x;

(ii) $\lambda>0$, $\mu\geqq 0$, $xg(x)\geqq 0$ *for all* x;

(iii) $\lambda<0$, $\mu\geqq 0$, $xg(x)\leqq 0$ *for all* x;

(iv) $\lambda<0$, $\mu\leqq 0$, $xg(x)\geqq 0$ *for all* x,

the modified PBVP (3.8) *has a solution* x. *Furthermore. for each solution* x *of* (3.8) *there is some* $\eta\in[0, \tau]$ *such that* $x(\eta)=0$.

PROOF. Clearly, for any $\alpha>0$ and for each choice of λ, μ and g the hypotheses of Lemma 3.7 are satisfied and hence the PBVP (3.8) has a solution x on $[0, \tau]$.

Integrating the differential equation in (3.8) on $[0, \tau]$, we get

(3.14) $$\int_0^\tau [\lambda\varphi_\alpha(x(t))+\mu g(\varphi_\alpha(x(t)))]\,dt = 0.$$

Suppose (i) holds. If $x(t)\neq 0$ for all $t\in[0, \tau]$, then by the continuity of x, either $x(t)>0$ or $x(t)<0$ for all $t\in[0, \tau]$. If $x(t)>0$ for all t, then $\lambda\varphi_\alpha(x(t))>0$ and $\mu g(\varphi_\alpha(x(t)))\geqq 0$ and hence (3.14) cannot be satisfied. By the same reasoning $x(t)\not< 0$ for all t and hence $x(\eta)=0$ for some $\eta\in[0, \tau]$. Similar arguments complete the proof in all cases.

We now set $\delta=\max_{0\leqq t\leqq \tau} |h(t)|$ and $g_\alpha=\max_{-\alpha\leqq x<\alpha} |g(x)|$, and prove the following result.

THEOREM 3.9. *Assume that the hypotheses of Lemma* 3.8 *hold. If there exists an* $\alpha>0$ *such that*

$$(|\lambda|\alpha+|\mu|g_\alpha+|\varepsilon|\delta)\tau^2 \leqq \alpha \tag{3.15}$$

then the PBVP (3.5) *has a solution.*

PROOF. From Lemma 3.8, we see that the modified PBVP (3.8) has a solution x on $[0,\tau]$ and $x(\eta)=0$ for some $\eta\in[0,\tau]$. The boundary condition $x(0)=x(\tau)$ implies that $x'(\xi)=0$ for some $\xi\in[0,\tau]$. Also since $|\varphi_\alpha(x)|\leqq\alpha$, for all x, it follows that

$$|x''(t)| \leqq |\lambda|\alpha+|\mu|g_\alpha+|\varepsilon|\delta \quad \text{for all} \quad t\in[0,\tau]. \tag{3.16}$$

From the identity $x'(t)=x'(\xi)+\int_\xi^t x''(s)\,ds$, we have in view of (3.16)

$$|x'(t)| \leqq (|\lambda|\alpha+|\mu|g_\alpha+|\varepsilon|\delta)\tau. \tag{3.17}$$

Since $x(\eta)=0$ for some $\eta\in[0,\tau]$, using the identity $x(t)=\int_\eta^t x'(s)\,ds$, and (3.17), we get

$$|x(t)| \leqq (|\lambda|\alpha+|\mu|g_\alpha+|\varepsilon|\delta)\tau^2.$$

It is easy to see that if the inequality (3.15) is satisfied, then $x(t)$ will be a solution to the PBVP (3.5) and the proof is complete.

REMARK 3.10. In Theorem 3.9 the choice that g is an even function of x is not permitted. However, the following examples illustrate the case when g is an even function of x.

EXAMPLE 3.11. Consider the equation

$$x''+\lambda x+\mu x^{2n} = \varepsilon h(t) \tag{3.18}$$

where n is a positive integer, $\varepsilon\neq 0$, $\lambda,\mu>0$ and h is τ-periodic and $Ph=0$.
The modified problem is given by

$$x''+\lambda\varphi_\alpha(x)+\mu(\varphi_\alpha(x))^{2n} = \varepsilon h(t). \tag{3.19}$$

Choose $\alpha>0$ so that

$$0 < \alpha < (\lambda/\mu)^{1/(2n-1)}. \tag{3.20}$$

It follows from Lemma 3.7 that the equation (3.19) has a solution x satisfying the boundary conditions

$$x(0) = x(\tau), \quad x'(0) = x'(\tau). \tag{3.21}$$

Integrating (3.19), we get

$$\int_0^\tau \left[\lambda\varphi_\alpha(x(t))+\mu(\varphi_\alpha(x(t)))^{2n}\right]dt = 0. \tag{3.22}$$

It is easy to see that $x(t)$ cannot be positive for all $t\in[0,\tau]$. Suppose that $x(t)<0$ for all $t\in[0,\tau]$. Then $\varphi_\alpha(x(t))<0$ on $[0,\tau]$ and using (3.20), we get

$$\lambda\varphi_\alpha(x(t))+\mu(\varphi_\alpha(x(t)))^{2n} = \mu\varphi_\alpha(x(t))\left[\frac{\lambda}{\mu}+(\varphi_\alpha(x(t)))^{2n-1}\right] < 0,$$

a contradiction to (3.22). Thus $x(t)$ is a solution of the PBVP (3.18)—(3.21) if

$$(\lambda\alpha+\mu\alpha^{2n}+|\varepsilon|\delta)\tau^2 \leqq \alpha. \tag{3.23}$$

Observe that (3.23) is possible only if $\lambda\tau^2<1$. Thus, if $0<\alpha<\left(\frac{1-\lambda\tau^2}{\mu\tau^2}\right)^{\frac{1}{2n-1}}$, we have $(\lambda\alpha+\mu\alpha^{2n})\tau^2<\alpha$. Therefore for ε sufficiently small the PBVP (3.18)—(3.21) has a solution provided $\lambda\tau^2<1$.

REMARK 3.12. If we choose $\lambda=n=1$ and $h(t)=\cos\omega t$, where $\omega>0$, then (3.18) reduces to

$$x''+x+\mu x^2 = \varepsilon\cos\omega t \tag{3.24}$$

with $\tau=2\pi\omega^{-1}$. In [4], it is proved that, if $\omega>2\pi$, then the equation (3.24) has at least one $2\pi\omega^{-1}$-periodic solution for all sufficiently small ε. The same conclusion may be obtained from Example 3.11. Now, we shall improve this result using Corollary 3.4, by removing the restriction $\omega>2\pi$ and also by giving an estimate for ε.

The change of variable $s=\omega t$ transforms (3.24) into

$$x''+\omega^{-2}[x+\mu x^2-\varepsilon\cos s] = 0 \tag{3.25}$$

in which $x=x(s)$ and $x''=\frac{d^2x}{ds^2}$; and the boundary conditions (3.21) into

$$x(0) = x(2\pi),\quad x'(0) = x'(2\pi). \tag{3.26}$$

Notice that the PBVP (3.25)—(3.26) is equivalent to the PBVP (3.2) with $f(s,x)=$ $=\omega^{-2}[x+\mu x^2-\varepsilon\cos s]$ and 2π replacing τ in the boundary conditions.

Let $a_1\leqq a_2$ be the roots of $\mu a^2+a-|\varepsilon|=0$. If $|\varepsilon|<\frac{1}{4\mu}$, then $\mu b^2+b+|\varepsilon|=0$ has real roots and let $b_1\leqq b_2$ be those roots. Choose constants a and b such that $b\in[b_1,b_2]$ and $a\leqq\min\{a_1,b\}$. Then, we have

$$b+\mu b^2-\varepsilon\cos s \leqq b+\mu b^2+|\varepsilon| \leqq 0 \quad \text{for all} \quad s\in[0,2\pi],$$

and

$$a+\mu a^2-\varepsilon\cos s \geqq a+\mu a^2-|\varepsilon| \geqq 0 \quad \text{for all} \quad s\in[0,2\pi].$$

From Corollary 3.4, it follows that, if $|\varepsilon|<\frac{1}{4\mu}$, then the PBVP (3.25)—(3.26) has a 2π-periodic solution. Consequently the equation (3.24) has a $2\pi\omega^{-1}$-periodic solution provided that $|\varepsilon|<\frac{1}{4\mu}$. Thus, we see that the restriction $\omega>2\pi$ in [4] is not crucial to conclude the existence of $2\pi\omega^{-1}$-periodic solutions for (3.24). Also, we notice that the inequality $|\varepsilon|<\frac{1}{4\mu}$ gives an estimate for ε.

REMARK 3.13. One may obtain a similar estimate for ε in terms of the parameters λ and μ by retaining λ in (3.18). Also, the assumption that $\mu>0$ in the equations (3.18) and (3.24) is no loss of generality since the transformation $x\to -x$ leads to the same equations except that μ and ε are replaced by $-\mu$ and $-\varepsilon$ respectively. In that case, we see that (3.24) has a $2\pi\omega^{-1}$-periodic solution provided $|\varepsilon|\leq\frac{1}{4|\mu|}$.

4. Concluding remarks

In this paper we have concentrated on the question of existence of τ-periodic solutions for the nonlinear equation (E) which contains real parameters λ, μ and ε. An important problem is to obtain periodic solutions of (E) which reduce for $\varepsilon=\lambda=0$ to those of the equation

$$(\hat{\text{E}}) \qquad x''+\mu g(x) = 0$$

which have a period in common with $h(t)$. If $g(0)=0$, then the identically zero solution of (Ê) is the simplest periodic solution, and periodic solutions near to it may be studied by employing perturbation techniques. A more interesting case is that of periodic solutions of (E) near the nonconstant periodic solutions of (Ê). Although we have not attempted to answer these questions in this paper, it seems likely that some satisfactory answers can be given to these questions.

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НЕОБХОДИМЫЕ И ДОСТАТОЧНЫЕ УСЛОВИЯ СХОДИМОСТИ РАСШИРЕННОГО ИНТЕРПОЛЯЦИОННОГО ПРОЦЕССА ЭРМИТА—ФЕЙЕРА В МЕТРИКЕ L_p

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1. Введем следующие обозначения: C — множество всех функций, непрерывных в $[-1, 1]$. Δ — подмножество из C, содержащее все функции $f(x)$, имеющие правую и левую производные соответственно в точках $x=\mp 1$. Пусть задана матрица чисел

$$\text{(M)} \qquad \{x_k^{(n)}\},\ k=1,\dots,n;\ -1<x_n^{(n)}<x_{n-1}^{(n)}<\dots<x_1^{(n)}<1, \quad n=1,2,\dots$$

и пусть $H_n(f,x)$ — полином степени $2n-1$, однозначно определяющийся из условий

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}),\ H_n'(f, x_k^{(n)}) = 0, \quad k=1,2,\dots,n.$$

Классическая теорема Л. Фейера [1] утверждает, что если n-я строчка матрицы (M) состоит из чисел

$$(1) \qquad x_k^{(n)} = \cos\frac{(2k-1)\pi}{2n}, \quad k=1,2,\dots,n,$$

то для любой $f\in C$ выполняется равномерно в $[-1, 1]$ соотношение

$$(2) \qquad H_n(f,x)\to f(x), \quad n\to\infty.$$

Хорошо известно, что процесс $\{H_n(f,x)\}$ называется интерполяционным процессом Эрмита—Фейера.

Пусть полином $H_n(f,x)$ построен для n-й строчки произвольной матрицы узлов вида (M). Наряду с $H_n(f,x)$ рассмотрим полином $F_n(f,x)$ степени $2n+3$, который однозначно определяется из условий

$$F_n(f, x_k^{(n)}) = f(x_k^{(n)});\ F_n(f,\pm 1) = f(\pm 1);\ F_n'(f, x_k^{(n)}) = F_n'(f,\pm 1) = 0,\ k=1,\dots,n.$$

Интерполяционный процесс $\{F_n(f,x)\}$ естественно называть расширенным интерполяционным процессом Эрмита—Фейера. В [2—7] автор изучал процесс $\{F_n(f,x)\}$ для узлов

(3)

$$x_0^{(n+2)} = 1, \quad x_k^{(n)} = \cos\frac{(2k-1)\pi}{2n}, \quad k=1,\dots,n, \quad x_{n+1}^{(n+2)} = -1, \quad n=1,2,\dots.$$

Было доказано, что этот процесс, построенный для $f(x)=|x|$, или для $f(x)=x^2$ расходится всюду в $(-1, 1)$. При $f(x)=x$ процесс расходится при $x\neq 0$ из $(-1, 1)$. С другой стороны, легко доказать, что процесс $\{F_n(f,x)\}$ равномерно сходится в $(-1, 1)$ при $f(x)=2x^2+x^4$. Поэтому возникла задача об определении тех функций для которых проуесс $\{F_n(f, x)\}$ сходится, и тех функций для которых он расходится. В [8] эта задача решена для класса функций C^2, имеющих непрерывные вторые производные в $[-1, 1]$. В [9] эта задача решена для более широкого класса функций Δ. В [8] также изучается вопрос о сходимости процесса $\{F_n(f, x)\}$ в метрике L_2. После этого, вполне естественно изучать сходимость процесса $\{F_n(f, x)\}$ в метрике L_p, $0<p<\infty$, когда расстояние между функциями f и g задается по формуле

$$\varrho(f, g) = \Big(\int_{-1}^{1} |f-g|^p \, dx \Big)^{1/p}. \tag{4}$$

Этому вопросу и посвящена эта заметка. Переход от метрики L_2 к метрике L_p, $0<p<\infty$, связан с некоторыми трудностями, ибо в случае L_2 можно было соответствующие интегралы непосредственно вычислить, в случае же метрики L_p этого сделать нельзя.

2. Докажем следующую теорему

Теорема. *Пусть интерполяционный процесс* $\{F_n(f, x)\}$ *построен при узлах* (3) *для* $f\in\Delta$. *Для того чтобы он сходился в метрике* L_p, $0<p<\infty$, *необходимо и достаточно, чтобы* $f'(-1)=f'(1)=0$.

Для доказательства теоремы введем функционалы

$$\alpha_n(f) = \frac{H_n'(f, 1)}{2} + \frac{\omega_n'(1)}{\omega_n(1)}[f(1)-H_n(f, 1)], \quad \omega_n(x) = \prod_{i=1}^{n} (x-x_i^{(k)}),$$

$$\beta_n(f) = \frac{H_n'(f, -1)}{2} + \frac{\omega_n'(-1)}{\omega_n(-1)}[f(-1)-H_n(f, -1)],$$

где $f\in\Delta$ и $H_n(f, x)$ интерполяционный полином Эрмита—Фейера, построенный для n строчки матрицы (M).

Можно доказать, что [11]

$$\alpha_n(f) = -\frac{1}{2}\sum_{k=1}^{n} \frac{f(x_k)}{1-x_k} h_k(1) - \frac{1}{2}\sum_{k=1}^{n} \frac{f(x_k)}{1-x_k} l_k^2(1) + \frac{\omega_n'(1)}{\omega_n(1)} f(1), \tag{5}$$

где, как обычно,

$$l_k(x) = \frac{\omega_n(x)}{(x-x_k)\omega_n'(x_k)}; \quad h_k(x) = [l_k(x)]^2 v_k^{(n)}(x);$$

$$v_k^{(n)}(x) = 1 - \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}(x-x_k^{(n)}).$$

Положим в (5) $f(x)\equiv 1$ и учтем, что при этом $\alpha_n(f)=0$. Тогда из (5) получим, что

$$\frac{\omega_n'(1)}{\omega_n(1)} = \frac{1}{2}\sum_{k=1}^{n} \frac{h_k(1)}{1-x_k} + \frac{1}{2}\sum_{k=1}^{n} \frac{l_k^2(1)}{1-x_k}. \tag{6}$$

Из (5) и (6) следует, что

(7) $$\alpha_n(f) = \frac{1}{2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{1-x_k}h_k(1)+\frac{1}{2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{1-x_k}l_k^2(1).$$

Поэтому

(8)
$$\alpha_n(f)-\frac{f'(1)}{2}\sum_{k=1}^{n}[l_k^{(n)}(1)]^2 = \frac{1}{2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{1-x_k}h_k(1)+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{f(1)-f(x_k)}{1-x_k}-f'(1)\right)l_k^2(1).$$

Л. Фейер [10] доказал, что при узлах (1) справедливо равенство

$$\lim_{n\to\infty}\sum_{k=1}^{n}[l_k^{(n)}(1)]^2 = 2.$$

Отсюда и из (8) можно вывести, что при узлах (1) имеет место равенство

(9) $$\lim_{n\to\infty}\alpha_n(f) = \frac{3}{2}f'(1).$$

В [8] доказано более обшее равенство, но относительно функции $f(x)$ предполагается, что она из класса C^2. Для случая $f\in\Delta$ упомянутое равенство доказано в [11]. Воспользуемся случаем, чтобы отметить, что требование, чтобы $f(1)=0$ из леммы статьи [11] является излишним, что непосредственно вытекает из (7).

Аналогичным образом доказывается, что при узлах (1) и $f\in\Delta$ выполняется равенство

(10) $$\lim_{n\to\infty}\beta_n(f) = \frac{3}{2}f'(-1).$$

Введем обозначение $r_n=\varrho(H_n(f), F_n(f))$, где $\varrho(f, g)$ определяется согласно (4). Ясно, что

(11) $$r_n-\varrho(f, H_n(f)) \leqq \varrho(f, F_n(f)) \leqq r_n+\varrho(f, H_n(f)).$$

В силу (2) и (11) получим, что для любой $f\in C$

$$r_n-o(1) \leqq \varrho(f, F_n(f)) \leqq r_n+o(1), \quad n\to\infty.$$

Поэтому для доказательства теоремы нужно доказать, что $r_n\to 0$, $n\to\infty$. Из определения полиномов $H_n(f, x)$ и $F_n(f, x)$ следует, что

$$d_n = F_n(f, x)-H_n(f, x) = T_n^2(x)(A_n x^3+B_n x^2+C_n x+D_n),$$

где коэффициенты A_n, B_n, C_n, D_n определяются из условий

(12) $$A_n+B_n+C_n+D_n = f(1)-H_n(f, 1);$$

(13) $$-A_n+B_n-C_n+D_n = f(-1)-H_n(f, -1);$$

(14) $$2n^2(A_n+B_n+C_n+D_n)+(3A_n+2B_n+C_n) = -H_n'(f, 1);$$

(15) $$-2n^2(-A_n+B_n-C_n+D_n)+(3A_n-2B_n+C_n) = -H_n'(f, -1).$$

Из (12) и (13), в силу (2), получим, что $A_n+C_n\to 0$, $B_n+D_n\to 0$, $n\to\infty$. Поэтому

(16) $$d_n = T_n^2(x)[(x^2-1)(A_n x+B_n)+o(1)].$$

Из системы (12—15) следует, что

$$A_n = -\frac{H_n'(f, 1)}{2} - n^2\big(f(1)-H_n(f, 1)\big) - \frac{1}{2}[f(1)-H_n(f, 1)-f(-1)+H_n(f, -1)] - B_n,$$

$$B_n = \frac{1}{4}[H_n'(f, 1)-H_n'(f, 1)-2n^2\big(f(1)-H_n(f, 1)+f(-1)-H_n(f, -1)\big)],$$

и в силу (9) и (10)

(17) $$A_n \to -\frac{3}{4}\big(f'(-1)+f'(1)\big), \quad n\to\infty,$$

(18) $$B_n \to \frac{3}{4}\big(f'(-1)-f'(1)\big), \quad n\to\infty.$$

Из (17—18) и (16) заключаем, что $r_n\to 0$, $n\to\infty$ тогда и только тогда, когда

$$U_n^{(p)} \equiv \int_{-1}^{1} T_n^{2p}(x)|f'(-1)-f'(1)-\big(f'(-1)+f'(1)\big)x|^p(1-x^2)^p\,dx \to 0, \quad n\to\infty.$$

Очевидно, что

(19) $$U_n^{(p)} = \int_0^{\pi} \cos^{2p} n\theta\,|a-b\cos\theta|^p \sin^{2p+1}\theta\,d\theta,$$

где $a=f'(-1)-f'(1)$; $b=f'(-1)+f'(1)$. Для дальнейшего нам нужна

Лемма. *Пусть $\varphi(\theta)$ непрерывна в $[0, \pi]$. Если $\varphi(\theta)\geqq 0$, $\theta\in[0, \pi]$ и при некотором p, $0\leqq p<\infty$, выполняется равенство*

(20) $$\lim_{n\to\infty}\int_0^{\pi}\varphi(\theta)\cos^{2p} n\theta = 0,$$

то $\varphi(\theta)\equiv 0$, $\theta\in[0, \pi]$.

Доказательство. Допустим, что в некоторой точке $\gamma\in[0, \pi]$, $\varphi(\gamma)=L>0$, тогда в силу непрерывности $\varphi(\theta)$ существует интервал $(\alpha, \beta)\subset[0, \pi]$, такой, что при $\theta\in(\alpha, \beta)$, $\varphi(\theta)\geqq\frac{L}{2}$. Поэтому

(21) $$I_n^{(p)} = \int_0^{\pi}\varphi(\theta)\cos^{2p} n\theta\,d\theta \geqq \frac{L}{2}\int_{\alpha}^{\beta}\cos^{2p} n\theta\,d\theta.$$

При $0<p<1$ имеем $\cos^{2p} n\theta\geqq\cos^2 n\theta$. Отсюда выводим, что

$$I_n^{(p)} \geqq \frac{L}{2}\left(\frac{\beta-\alpha}{2}+\frac{\sin 2n\beta-\sin 2n\alpha}{2n}\right).$$

Стало быть, (20) не выполняется при $0 \leqq p \leqq 1$. Рассмотрим теперь случай $1 < p < \infty$. Заметим, что $\cos^{2p} n\theta \geqq \cos^{2([p]+1)} n\theta$, где $[p]$ — целая часть числа p. Поэтому достаточно рассматривать лишь целые $p > 0$. Воспользуемся теперь хорошо известной формулой

$$\cos^{2p} n\theta = \frac{1}{2^p}\left(\sum_{k=0}^{p-1} 2\binom{2p}{k}\cos 2(p-k)n\theta + \binom{2p}{p}\right).$$

Отсюда следует, что

$$\text{(22)} \qquad \lim_{n\to\infty} \int_{\alpha}^{\beta} \cos^{2p} n\theta \, d\theta = \frac{1}{2^p}\binom{2p}{p}(\beta-\alpha).$$

Из (21) и (22) следует, что $I_n^{(p)}$ не удовлетворяет (20). Итак, лемма доказана. Применим эту лемму к интегралу (19), где $\varphi(\theta) = |a - b\cos\theta| \sin^{2p+1}\theta$. Тогда получим, что $a = b = 0$. Стало быть, $f'(-1) = f'(1) = 0$. Теорема доказана.

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A GENERALIZATION OF DISTRIBUTIVE IDEALS TO CONVEX SUBLATTICES*

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1. Introduction

The notion of standard sublattices was introduced in [3] by generalizing the notion of standard ideals to convex sublattices. In this connection G. Grätzer posed the following problem.

Problem 1.1 (Problem III. 1 of [4]). Generalize the concepts of distributive and neutral ideals to convex sublattices.

The first part of this problem is answered in Section 2 by introducing the notion of D-sublattices.

The notion of modular elements introduced in [2] has been extended to ideals in Section 3 and it is shown to preserve some properties of standard ideals. A generalization of modular ideals to convex sublattices — called M-sublattices — is also obtained using which a new characterization of standard sublattices is found.

Definition 1.1 [3]. A convex sublattice S of a lattice L is called a standard sublattice of L if

$$I\wedge\langle S, K\rangle = \langle I\wedge S,\ I\wedge K\rangle$$

and

$$I\vee\langle S, K\rangle = \langle I\vee S,\ I\vee K\rangle$$

hold for any pair $\{I, K\}$ of convex sublattices of L, whenever neither $S\cap K$ nor $I\cap\langle S, K\rangle$ is empty.

Definition 1.2 [2]. An element d of a lattice L is called modular iff for $x, y\in L$, $x\geqq y$, $d\wedge x=d\wedge y$, $d\vee x=d\vee y$ imply $x=y$.

For additional notations and basic results we refer to [4].

2. D-sublattices

Definition 2.1. A convex sublattice S of a lattice L is called a D-sublattice iff the congruence relation $\mathscr{H}[S]$ generated by S can be described as follows:

For $x, y\in L$, $x\equiv y$ $(\mathscr{H}[S])$ iff there exists a sequence $x\vee y=z_1\geqq z_2\geqq\ldots\geqq z_n= =x\wedge y$ and for each i there exist $a_i, b_i\in S$ such that the following condition (1) or

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(2) holds:

(1) $$z_{i+1} = z_i\wedge(z_{i+1}\vee b_i) \quad\text{and}\quad z_i\vee a_i = z_{i+1}\vee a_i,$$

(2) $$z_i = z_{i+1}\vee(z_i\wedge a_i) \quad\text{and}\quad z_i\wedge b_i = z_{i+1}\wedge b_i,$$

for $i=1, \dots, n-1$.

Conditions (1) and (2) of this definition are equivalent to the following conditions (1') and (2'), respectively, since by monotonicity $a_i \geqq b_i$ can be assumed:

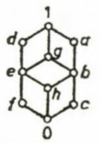

Fig. 1

(1') $$z_i/z_{i+1} \underset{w}{\nearrow} z_i\vee a_i/z_{i+1}\vee b_i \underset{w}{\searrow} a_i/b_i,$$

(2') $$z_i/z_{i+1} \underset{w}{\searrow} z_i\wedge a_i/z_{i+1}\wedge b_i \underset{w}{\nearrow} a_i/b_i.$$

We observe that in the lattice of Fig. 1 the convex sublattice $S=\{a, b, c\}$ is a D-sublattice but it is not a standard sublattice.

THEOREM 2.1. *An ideal I of a lattice L is distributive iff it is a D-sublattice.*

PROOF. Let I be a distributive ideal and $x, y\in L$, $x\equiv y$ $(\mathscr{H}[I])$. Then $x\vee y\equiv x\wedge y$ $(\mathscr{H}[I])$. Hence by a Theorem of [4] there exists $a\in I$ such that $(x\vee y)\vee a = (x\wedge y)\vee a$. Set $b=x\wedge y\wedge a$. Clearly $b\in I$. Now $x\wedge y=(x\vee y)\wedge((x\wedge y)\vee b)$ and $(x\vee y)\vee a=(x\wedge y)\vee a$. Hence (1) of Definition 2.1 holds for the chain $\{x\vee y, x\wedge y\}$, proving I is a D-sublattice.

On the other hand if I is a D-sublattice and $x, y\in L$, $x\equiv y$ $(\mathscr{H}[I])$, then either of (1) and (2) of Definition 2.1 imply $z_i\vee a_i=z_{i+1}\vee a_i$ for $i=1, \dots, n-1$. Set $a=a_1\vee\dots\vee a_{n-1}$. Then clearly $a\in I$. Also $(x\wedge y)\vee a\geqq z_n\vee a_{n-1}=z_{n-1}\vee a_{n-1}$, which implies $(x\wedge y)\vee a\geqq z_{n-1}\vee a$ and so on. Finally we obtain $(x\wedge y)\vee a\geqq z_1\vee a=(x\vee y)\vee a$. Hence $(x\wedge y)\vee a=(x\vee y)\vee a$, which yields $x\vee a=y\vee a$, proving I is distributive as desired.

LEMMA 2.1. *Every standard sublattice is a D-sublatrice.*

PROOF. Let S be a standard sublattice and $x\equiv y$ $(\mathscr{H}[S])$. Then by a Theorem of [3] there exist $a, b\in S$ such that $x\wedge y=(x\vee y)\wedge((x\wedge y)\vee b)$ and $x\vee y=(x\wedge y)\vee((x\vee y)\wedge a)$. Now $\{x\vee y, x\wedge y\}$ is a chain satisfying (1) of Definition 2.1, which proves the lemma.

THEOREM 2.2. *If S is a D-sublattice, then S is a congruence class by the congruence relation $\mathscr{H}[S]$.*

PROOF. Let $x\equiv y$ $(\mathscr{H}[S])$, $x>y$, $y\in S$. Since S is a D-sublattice there exists a chain $y=z_1<z_2<\dots<z_n=x$ and for each i there exist $a_i, b_i\in S$ satisfying one of the following:

(2.3) $$z_i = z_{i+1}\wedge(z_i\vee b_i) \quad\text{and}\quad z_i\vee a_i = z_{i+1}\vee a_i,$$

(2.4) $$z_{i+1} = z_i\vee(z_{i+1}\wedge a_i) \quad\text{and}\quad z_i\wedge b_i = z_{i+1}\wedge b_i,$$

for $i=1, \dots, n-1$.

Set $a=a_1\vee\ldots\vee a_{n-1}$. Clearly $a\in S$. Now by (2.3) or (2.4),

$$y\vee a=(z_1\vee a_1)\vee a_2\vee\ldots\vee a_{n-1}=(z_2\vee a_1)\vee a_2\vee\ldots\vee a_{n-1}=$$

$$=a_1\vee(z_2\vee a_2)\vee\ldots\vee a_{n-1}=a_1\vee(z_3\vee a_2)\vee\ldots\vee a_{n-1}=$$

$$=\ldots=(a_1\vee a_2\vee\ldots\vee a_{n-1})\vee z_n=x\vee a.$$

But $y<x\leqq x\vee a=y\vee a$, y, $y\vee a\in S$ imply $x\in S$ by convexity. The dual argument completes the proof.

THEOREM 2.3. *If S and T are D-sublattices in L such that $S\cap T\neq\emptyset$, then $\langle S, T\rangle$ is also a D-sublattice.*

PROOF. Let $x\equiv y$ $(\mathscr{H}[\langle S,T\rangle])$. Since $S\cap T\neq\emptyset$, $\mathscr{H}[\langle S,T\rangle]=\mathscr{H}[S]\vee\mathscr{H}[T]$ by a lemma of [1]. Hence there exists a finite chain $x\vee y=z_1\geqq\ldots\geqq z_n=x\wedge y$ and for each i either $z_i\equiv z_{i+1}$ $(\mathscr{H}[S])$ or $z_i\equiv z_{i+1}(\mathscr{H}[T])$ for $i=1,\ldots,n-1$. In any case, since S and T are D-sublattices, for each i, $i=1,\ldots,n-1$, there exist a finite chain $z_i=t_1^i\geqq t_2^i\geqq\ldots\geqq t_{k_i}^i=z_{i+1}$ and $a_j, b_j\in S$ or T such that either $t_j^i=t_{j+1}^i\vee(t_j^i\wedge a_j)$ and $t_j^i\wedge b_j=t_{j+1}^i\wedge b$, or $t_{j+1}^i=t_j^i\wedge(t_{j+1}^i\vee b_j)$ and $t_j^i\vee a_j=t_{j+1}^i\vee a_j$ hold for $j=1,\ldots,k_i-1$. This completes the proof.

However, under the assumptions of the above theorem $S\cap T$ need not be a D-sublattice, which is already known when S and T are ideals.

3. Modular ideals and M-sublattices

DEFINITION 3.1. An ideal I of a lattice L is called a modular ideal iff it is a modular element of the ideal lattice of L.

It follows from Lemma 1 of [5] that an ideal is standard iff it is distributive and modular. Now we give a characterization of modular ideals.

THEOREM 3.1. *An ideal I of a lattice L is modular iff for any $x, y, z\in L$ with $z>y>x$, $z/x\searrow a/b$, $a, b\in I$ imply the existence of $a_1, b_1\in I$ such that $y/x\searrow a_1/b_1$.*

PROOF. Let I be a modular ideal of L. Let $z>y>x$, $x, yz\in L$ and $z/x\searrow a/b$, $a, b\in I$. Then $I\vee(x]\supseteqq(a]\vee(x]=(a\vee x]=(z]=(a\vee y]=(a]\vee(y]$. Hence $I\vee(x]\supseteqq I\vee(a]\vee(y]=I\vee(y]$. But $I\vee(x]\subseteqq I\vee(y]$ since $(x]\subseteqq(y]$. Thus $I\vee(x]=I\vee(y]$.

Since I is modular and $(x]\subseteqq(y]$, by a lemma of [2], we get

$$(x]\vee(I\wedge(y])=((x]\vee I)\wedge(y]=(y],$$

which implies $y=x\vee a_1$ for some $a_1\in I\wedge(y]\subseteqq I$. Set $b_1=x\wedge a_1$. Then $y/x\searrow a_1/b_1$, $a_1, b_1\in I$ as desired.

Conversely, if I is not modular, then $I(L)$ has a sublattice of the type given in Fig. 2. Hence there exists $x\in J$ such that $x\notin K$. Clearly $x\in I\vee J=I\vee K$. Therefore there exist $a\in I$, $y\in K$ such that $x\leqq a\vee y$. Now $x\vee y\in J$ but $x\vee y\notin K$ since $x\notin K$. Also $a\vee y/y\searrow a/a\wedge y$, $a, a\wedge y\in I$ and $a\vee y>x\vee y>y$ imply by assumption the existence of $a_1, b_1\in I$ such that $x\vee y/y\searrow a_1/b_1$. But then $a_1\in I\wedge J\subseteqq K$ and hence $x\vee y=a_1\vee y\in K$, a contradiction, which completes the proof.

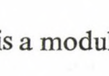

Fig. 2

Now we extend a property of standard ideals (see [5]) to modular ideals.

THEOREM 3.2. *Let I be a modular ideal and J be any ideal of a lattice L. If $I \vee J$ and $I \wedge J$ are principal ideals, then J is also a principal ideal.*

PROOF. Let $I \vee J=(a]$, $I \wedge J=(b]$, $a, b \in L$. Since $a \in I \vee J$ there exist $x \in I$, $y \in J$ such that $a \leqq x \vee y$. But $x \vee y \in (a]$. Hence $a=x \vee y$.

Now if $z \geqq y \vee b$, $z \in J$, then $(a] \supseteqq I \vee (z] \supseteqq I \vee (y \vee b] \supseteqq I \vee (y] \supseteqq (a]$, which implies $I \vee (z]=I \vee (y \vee b]$.

Further, $(b]=I \wedge J \supseteqq I \wedge (z] \supseteqq I \wedge (y \vee b] \supseteqq I \wedge (b]=(b]$, which implies $I \wedge (z]=$ $=I \wedge (y \vee b]$.

The above two equalities together with $(z] \supseteqq (y \vee b]$ imply that $(z]=(y \vee b]$ since I is modular. Hence there are no elements in J greater than $y \vee b$, proving $J=(y \vee b]$ as required.

COROLLARY 3.1. *If the meet and the join of two modular ideals are principal, then both ideals are principal.*

COROLLARY 3.2. *In a modular lattice if the meet and the join of two ideals are principal, then both ideals are principal.*

DEFINITION 3.2. A convex sublattice S of a lattice L is called an M-sublattice iff it satisfies the following conditions.

(1) For $x, y \in L$, $a, b \in S$ with $y=x \wedge (y \vee b)$ and $x \vee a=y \vee a$ imply $x=y \vee (x \wedge a_1)$ for some $a_1 \in S$.

(2) For $x, y \in L$, $a, b \in S$ with $x=y \vee (x \wedge a)$ and $x \wedge b=y \wedge b$ imply $y=x \wedge (y \vee b_1)$ for some $b_1 \in S$.

Clearly every convex sublattice of a modular lattice is an M-sublattice.

THEOREM 3.3. *An ideal I of a lattice L is an M-sublattice iff it is a modular ideal*

PROOF. It follows from Theorem 3.1 that an ideal I is modular iff I satisfies the following condition:

(3.1) For $x, y \in L$ with $x \geqq y$ and $x \vee a=y \vee a$, $a \in I$ imply $x=y \vee a_1$ for some $a_1 \in I$.

Let I be a modular ideal. Trivially (2) of Definition 3.2 holds for any ideal (take $b_1=y \wedge b$). Also if $x, y \in L$, $a, b \in I$ with $y=x \wedge (y \vee b)$ and $x \vee a=y \vee a$, then $y \leqq x$ and by (3.1) $x=y \vee a_1$ for some $a_1 \in I$, which implies (1) of Definition 3.2.

On the other hand if I is an M-sublattice and $x, y \in L$, $x \geqq y$, $x \vee a=y \vee a$, $a \in I$, then $b=y \wedge a \in I$. Now $y=x \wedge (y \vee b)$ and hence by Definition 3.2 there exists $a_1 \in I$ such that $x=y \vee (a_1 \wedge x)$, which implies (3.1) since $a_1 \wedge x \in I$. This completes the proof.

LEMMA 3.1. *Every standard sublattice is an M-sublattice.*

PROOF. Let S be a standard sublattice of a lattice L. Let $x, y \in L$, $a, b \in S$, $y=x \wedge (y \vee b)$ and $x \vee a=y \vee a$. We can take $a \geqq b$ by monotonicity. Then

$x/y \underset{w}{\nearrow} y\vee a/y\vee b \underset{w}{\searrow} a/b$, which implies $x\equiv y$ $(\mathscr{H}[S])$. Since S is standard, by a theorem of [3], there exists $a_1\in I$ such that $x=y\vee(x\wedge a_1)$, which shows (1) of Definition 3.2 holds. The dual argument completes the proof.

THEOREM 3.4. *A convex sublattice S of a lattice L is standard iff it is a D-sublattice and an M-sublattice.*

PROOF. Necessity follows from Lemmas 2.1 and 3.1.

To prove sufficiency, let $x, y\in L$, $x\equiv y$ $(\mathscr{H}[S])$. Since S is a D-sublattice there exists a finite chain $x\vee y=z_1\geqq z_2\geqq\ldots\geqq z_n=x\wedge y$ and for each i there exist $a_i, b_i\in S$ such that (1) or (2) of Definition 2.1 holds for $i=1, \ldots, n-1$.

If for an i, $1\leqq i<n$, $a_i, b_i\in S$ and (1) of Definition 2.1 holds, then since S is an M-sublattice there exists $a_i'\in S$ such that $z_i=z_{i+1}\vee(z_i\wedge a_i')$. Also $z_{i+1}\wedge b_i=z_i\wedge b_i$. Hence the pair a_i', b_i satisfies (2) of Definition 2.1. By duality we conclude that for each i there exist elements $a_i, b_i, a_i', b_i'\in S$ such that the pair a_i, b_i satisfies (1) of Definition 2.1 and the pair a_i', b_i' satisfies (2) of Definition 2.1, for $i=1, \ldots, n-1$.

Set $a=\bigvee_{i=1}^{n-1} a_i'$, $b=\bigwedge_{i=1}^{n-1} b_i$. Clearly $a, b\in S$. Also $(x\vee y)\wedge((x\wedge y)\vee b)\leqq$ $\leqq z_1\wedge(z_2\vee b_1)=z_2$. Therefore

$$(x\vee y)\wedge((x\wedge y)\vee b)\leqq z_2\wedge(z_n\vee b)\leqq z_2\wedge(z_3\vee b_2)=z_3,$$

and so on. Finally $(x\vee y)\wedge((x\wedge y)\vee b)\leqq z_n=x\wedge y$, which implies $(x\vee y)\wedge$ $\wedge((x\wedge y)\vee b)=x\wedge y$. Similarly $(x\wedge y)\vee((x\vee y)\wedge a)=x\vee y$. Hence S is standard by a theorem of [3].

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AN EVERYWHERE DIVERGENT HERMITE—FEJÉR TYPE INTERPOLATION PROCESS OF HIGHER ORDER

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1. Let $f(x)$ be a given function on $[-1, 1]$ and let $X=\{x_{k,n}\}_{k=1}^{n}$, $n=1, 2, \ldots$ be a given matrix of nodes. Let $F_n(f, X, x)$ be the polynomial of degree $\leqq 4n-1$, uniquely determined by the conditions

$$(1)\quad F_n(f, X, x_{k,n}) = f(x_{k,n}),\quad F_n^{(j)}(f, X, x_{k,n}) = 0,\quad k = 1, 2, \ldots, n;\ j = 1, 2, 3.$$

The process $\{F_n(f, X, x)\}$ is usually called Hermite—Fejér type (HFT) interpolation of higher order. If $X=T$, i.e.

$$x_{k,n} = \cos\left(k-\frac{1}{2}\right)\frac{\pi}{n},\quad k = 1, \ldots, n,$$

which are the zeros of the n^{th} Tchebycheff polynomial of first kind $T_n(x)$, then [10]

$$(2)\qquad F_n(f, T, x) = \frac{1}{n^4}\sum_{k=1}^{n} f(x_{k,n})\left[\frac{T_n(x)}{x-x_{k,n}}\right]^4\left[(1-xx_{k,n})^2+\right.$$

$$\left.+(x-x_{k,n})^2\left\{\frac{2}{3}(n^2-1)(1-xx_{k,n})-\frac{1}{2}xx_{k,n}\right\}\right].$$

It was shown by Krylov and Steuermann [6] and Laden [7] that if $f(x)\in C[-1, 1]$, then

$$(3)\qquad \lim_{n\to\infty} F_n(f, T, x) = f(x)$$

uniformly in $[-1, 1]$.

Several authors, namely Stancu [11], Florica [3], Mills [8], Prasad [9], Knoop [5] have obtained the error estimates in terms of the modulus of continuity $\omega(f, \delta)$ of $f(x)$. We mention the following recent result of Goodenough and Mills [4] which shows the interpolation character of the polynomial $F_n(f, T, x)$. It states that if $f(x)\in C[-1, 1]$, then

$$|F_n(f, T, x)-f(x)| \leqq C_1\frac{T_n^4(x)}{n}\sum_{k=1}^{n}\omega\left(f;\ \frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)+C_2\omega\left(f;\ \frac{|T_n(x)|}{n}\right), -1\leqq x\leqq 1,$$

where C_1 and C_2 are absolute constants.

Cook and Mills [2] studied the HFT interpolation on the extended Tchebycheff nodes, i.e. on the points $x_{0,n+2}=1$, $x_{k,n+2}=\cos\left(k-\frac{1}{2}\right)\frac{\pi}{n}$, $k=1, \ldots, n$, $x_{n+1,n+2}=-1$ which are the roots of $(1-x^2)T_n(x)$. We denote this polynomial by $K_n(f, T, x)$. Thus $K_n(f, T, x)$ is a polynomial of degree $\leqq 4n+7$ uniquely determined by the conditions

$$(4) \qquad K_n(f, T, x_{k,n+2}) = f(x_{k,n+2}), \quad K_n^{(j)}(f, T, x_{k,n+2}) = 0,$$

$$k = 0, 1, \ldots, n+1, \quad j = 1, 2, 3.$$

It was shown in [2] that $\{K_n(f, T, x)\}$ is a divergent process at $x=0$. In fact if $f(x)=(1-x^2)^3$, then

$$(5) \qquad \lim_{n\to\infty} K_n\big((1-x^2)^3, T, 0\big) = -\frac{95}{3}.$$

This result is unexpected in view of the aforementioned results of Krylov and Steuermann [6] and Laden [7]. The question of divergence of this process at $x\neq 0$ remains open. With the help of a method other than in [2] we give here the complete solution of this problem.

THEOREM. *The HFT interpolation process $\{K_n(f, T, x)\}$ constructed for $f(x)=(1-x^2)^3$ diverges at each point in $(-1, 1)$. Moreover, for each $x\in(-1, 1)$, there exists a sequence $\{n_k\}_{k=1}^{\infty}\uparrow\infty$ such that*

$$(6) \qquad \lim_{k\to\infty} K_{n_k}\big((1-x^2)^3, T, x\big) = -\frac{95}{3}(1-x^2)^3.$$

Our method of proof consists of constructing the polynomial $K_n(f, T, x)$ in terms of the polynomial $F_n(f, T, x)$. For this we shall need few preliminary results formulated as lemmas. We only hint at the proofs and omit the details. From now on we shall write $K_n(f, x)$ for $K_n(f, T, x)$, $F_n(f, x)$ for $F_n(f, T, x)$ and x_k for $x_{k,n}$, etc.

LEMMA 1. *If*

$$A_{k,n}(x) = A_k(x) = \left(\frac{T_n(x)}{x-x_k}\right)^4$$

and

$$B_{k,n}(x) = B_k(x) = (1-xx_k)^2+(x-x_k)^2\left\{\frac{2}{3}(n^2-1)(1-xx_k)-\frac{1}{2}xx_k\right\},$$

then

$$A_k(1) = \frac{1}{(1-x_k)^4},$$

$$A_k'(1) = -\frac{4}{(1-x_k)^5} + \frac{4n^2}{(1-x_k)^4},$$

$$A_k''(1) = \frac{20}{(1-x_k)^6} - \frac{32n^2}{(1-x_k)^5} + \frac{4n^2(10n^2-1)}{3(1-x_k)^4},$$

$$A_k'''(1) = -\frac{120}{(1-x_k)^7} + \frac{240n^2}{(1-x_k)^6} - \frac{16n^2(10n^2-1)}{(1-x_k)^5} + \frac{8n^2(68n^4-25n^2+2)}{15(1-x_k)^4};$$

$$B_k(1) = \frac{(1-x_k)^2}{2} + \frac{4n^2-1}{6}(1-x_k)^3,$$

$$B_k'(1) = \frac{4n^2-1}{6}(1-x_k)^3 + \frac{4n^2+11}{6}(1-x_k)^2 - 3(1-x_k);$$

$$B_k''(1) = \frac{8n^2+4}{3}(1-x_k)^2 - \frac{4n^2+11}{3}(1-x_k) + 1,$$

$$B_k'''(1) = (4n^2-1)[(1-x_k)-1].$$

PROOF. Just differentiate and use

$$T_n(1) = 1 = (-1)^n T_n(-1),$$

$$T_n'(1) = n^2 = (-1)^{n-1} T_n'(-1),$$

$$T_n''(1) = \frac{n^2(n^2-1)}{3} = (-1)^n T_n''(-1),$$

$$T_n'''(1) = \frac{n^2(n^2-1)(n^2-4)}{15} = (-1)^{n-1} T_n'''(-1).$$

LEMMA 2. *We have*

$$n^4 F_n(f, 1) = \sum_{k=1}^n f(x_k)\left[\frac{1}{2(1-x_k)^2} + \frac{4n^2-1}{6(1-x_k)}\right],$$

$$n^4 F_n'(f, 1) = \sum_{k=1}^n f(x_k)\left[-\frac{5}{2(1-x_k)^3} + \frac{5}{2(1-x_k)^2} + \frac{16n^4-1}{6(1-x_k)}\right],$$

$$n^4 F_n''(f, 1) = \sum_{k=1}^n f(x_k)\left[\frac{35}{(1-x_k)^4} - \frac{100n^2+65}{3(1-x_k)^3} - \frac{28n^4-50n^2-8}{3(1-x_k)^2} + \right.$$

$$\left. + \frac{10n^2(4n^2-1)(2n^2+1)}{9(1-x_k)}\right],$$

$$n^4 F_n'''(f, 1) = \sum_{k=1}^{n} f(x_k)\left[-\frac{252}{(1-x_k)^5}+\frac{392n^2+175}{(1-x_k)^4}-\frac{120n^4+228n^2+27}{(1-x_k)^3}-\right.$$

$$\left.-\frac{928n^6-1040n^4-338n^2}{15(1-x_k)^2}+\frac{2n^2(544n^6+264n^4-144n^2+11)}{45(1-x_k)}\right].$$

PROOF. Notice that

$$F_n(f, x) = \frac{1}{n^4}\sum_{k=1}^{n} f(x_k)[A_k(x)B_k(x)].$$

Differentiate and use Lemma 1.

LEMMA 3. *We have*

$$\sum_{k=1}^{n}\frac{(1-x_k^2)^3}{1-x_k} = \frac{3}{8}n, \quad \sum_{k=1}^{n}\frac{(1-x_k^2)^3}{(1-x_k)^2} = \frac{5}{8}n,$$

$$\sum_{k=1}^{n}\frac{(1-x_k^2)^3}{(1-x_k)^3} = \frac{5}{2}n, \quad \sum_{k=1}^{n}\frac{(1-x_k^2)^3}{(1-x_k)^4} = \frac{16n^2-15n}{2},$$

$$\sum_{k=1}^{n}\frac{(1-x_k^2)^3}{(1-x_k)^5} = \frac{16n^4-28n^2+15n}{3}.$$

PROOF. Use

$$\sum_{k=1}^{n} x_k = 0, \quad \sum_{k=1}^{n} x_k^2 = \frac{n}{2}, \quad \sum_{k=1}^{n} x_k^4 = \frac{3n}{8}, \quad \sum_{k=1}^{n}\frac{1}{1-x_k} = n^2, \quad \sum_{k=1}^{n}\frac{1}{(1-x_k)^2} = \frac{2n^4+n^2}{3}.$$

LEMMA 4. *If* $f(x)=(1-x^2)^3$, *then*

$$n^4 F_n((1-x^2)^3, 1) = \frac{n^3+n}{4}, \quad n^4 F_n'((1-x^2)^3, 1) = n^5-11n,$$

$$n^4 F_n''((1-x^2)^3, 1) = \frac{1}{3}(10n^7-15n^5-220n^3+840n^2-945n),$$

$$n^4 F_n'''((1-x^2)^3, 1) = \frac{1}{30}(272n^9-1028n^7-7772n^5+53\,760n^4-104\,872n^3+$$

$$+112\,560n^2-79\,200n).$$

PROOF. Put $f(x)=(1-x^2)^3$ in Lemma 2 and use Lemma 3.
We now prove the main

LEMMA 5. *Let* $f(x)\in C[-1, 1]$ *be an even function. Then*

$$K_n(f, T, x) = F_n(f, T, x)+[a_{n,2}x^6+a_{n,4}x^4+a_{n,6}x^2+a_{n,8}]T_n^4(x),$$

where

$$48a_{n,2} = -F_n'''(f,1)+(12n^2+3)F_n''(f,1)-(56n^4+28n^2+3)F_n'(f,1)+$$

$$+\frac{1}{15}(1504n^6+1120n^4+256n^2)\{F_n(f,1)-f(1)\},$$

$$16a_{n,4} = F_n'''(f,1)-(12n^2+5)F_n''(f,1)+(56n^4+44n^2+5)F_n'(f,1)-$$

$$-\frac{1}{15}(1504n^6+1680n^4+416n^2)\{F_n(f,1)-f(1)\},$$

$$16a_{n,6} = -F_n'''(f,1)+(12n^2+7)F_n''(f,1)-(56n^4+60n^2+15)F_n'(f,1)+$$

$$+\frac{1}{15}(1504n^6+2240n^4+1056n^2)\{F_n(f,1)-f(1)\}$$

and

$$48a_{n,8} = F_n'''(f,1)-(12n^2+9)F_n''(f,1)+(56n^4+76n^2+33)F_n'(f,1)-$$

$$-\frac{1}{15}(1504n^6+2800n^4+2176n^2+720)\{F_n(f,1)-f(1)\}.$$

PROOF. Let us write

$$K_n(f,x) = F_n(f,x)+\Big[\sum_{i=1}^{8} a_i x^{8-i}\Big]T_n^4(x),$$

where we write a_i for $a_{n,i}$. We see that $K_n(f,x)$ is a polynomial of degree $4n+7$, which owing to (1) satisfies all the conditions of (4) except those at ± 1. If we apply these conditions, we get 8 equations in the unknowns a_i, $1 \leqq i \leqq 8$.

On adding and subtracting these equations and using the fact that, for $f(x)$ even, $a_1=a_3=a_5=a_7=0$, and

$$F_n(f,T,x) = F_n(f,T,-x),$$

we obtain the following 4 equations:

$$a_2+a_4+a_6+a_8 = f(1)-F_n(f,1),$$

$$4(a_2+a_4+a_6+a_8)n^2+(6a_2+4a_4+2a_6) = -F_n'(f,1),$$

$$4(a_2+a_4+a_6+a_8)\frac{n^2(10n^2-1)}{3}+8(6a_2+4a_4+2a_6)n^2+$$

$$+(30a_2+12a_4+2a_6) = -F_n''(f,1),$$

and

$$8(a_2+a_4+a_6+a_8)\frac{n^2(68n^4-25n^2+2)}{15}+12(6a_2+4a_4+2a_6)\frac{n^2(10n^2-1)}{3}+$$

$$+12(30a_2+12a_4+2a_6)n^2+(120a_2+24a_4)=-F_n'''(f,1).$$

On solving this system of equations we have a_2, a_4, a_6, a_8, as given in the lemma.

Proof of the Theorem. We now complete the proof of our theorem. Let $f(x)=(1-x^2)^3$. Then owing to Lemma 4, the coefficients a_2, a_4, a_6 and a_8 on simplification become

$$a_2=\frac{98}{3}-\frac{4}{n}-\frac{182}{3n^2}+\frac{36}{n^3},\quad a_4=-98+\frac{10}{n}+\frac{147}{n^2}-\frac{70}{n^3},$$

$$a_6=98-\frac{15}{2n}-\frac{112}{n^2}+\frac{75}{2n^3},\quad a_8=\frac{98}{3}+\frac{5}{4n}+\frac{77}{3n^2}-\frac{15}{4n^3}.$$

Hence from Lemma 5, we have

$$K_n((1-x^2)^3,x)=F_n((1-x^2)^3,x)-\frac{98}{3}(1-x^2)^3T_n^4(x)+$$

$$+T_n^4(x)\left[\frac{-16x^6+40x^4-30x^2+5}{4n}+\frac{1}{n^2}\left(-\frac{182}{3}x^6+147x^4-112x^2+\frac{77}{3}\right)+\right.$$

$$\left.+\frac{1}{n^3}\left(36x^6-70x^4+\frac{75}{2}x^2-\frac{15}{4}\right)\right].$$

It is known from Berman [1] that given any $x\in(-1,1)$, a sequence $\{x_{n_k}\}$ can be found such that

$$\lim_{k\to\infty}T_{n_k}^2(x)=1.$$

According to (3),

$$\lim_{n\to\infty}(F_n(1-x^2)^3,x)=(1-x^2)^3.$$

Therefore

$$\lim_{k\to\infty}K_{n_k}((1-x^2)^3,x)=(1-x^2)^3-\frac{98}{3}(1-x^2)^3=-\frac{95}{3}(1-x^2)^3. \tag{7}$$

From this it follows that the process $\{K_n(f,T,x)\}$ diverges at all points of the interval $(-1,1)$. The result (5) of Cook and Mills [2] is obtained from (7) for $x=0$.

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ON A BOUNDARY VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

Consider the Picard boundary value problem

(E) $$x''+f(t, x, x') = 0,$$

(B. C) $$x(a) = x_a, \quad x(b) = x_b$$

where $f \in C[[a, b]\times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n]$.

There is a large, constantly growing literature on the subject of boundary value problem for ordinary differential equations. The book of Bernfeld and Lakshmikantham [1] gave a reasonably complete view of this field. On the other hand, many authors extended results from ordinary to functional differential equations. Among them, Bernfeld, Ladde and Lakshmikantham [2] studied the boundary value problem

(E_1) $$x''+f(t, x, x_t, x') = 0, \quad x(t) = h_i(t), \quad t \in J_i, \quad i = 1, 2$$

where $f \in C[[a, b]\times \mathbf{R}^n \times C([-r, 0], \mathbf{R}^n)\times \mathbf{R}^n, \mathbf{R}^n]$, $J_1 = [a-r, a]$, $J_2 = [a, b+r]$.

Also the author [5] studied the two point boundary value problem

(E_2) $$(\varrho x')+f(t, x_t, x') = 0, \quad x(t) = h(t), \quad -r \leqq t \leqq 0, \quad x(T) = A$$

where $f \in C[[0, T]\times C([-r, 0], \mathbf{R}^n)\times \mathbf{R}^n, \mathbf{R}^n]$ and ϱ is a positive continuous function defined on $[0, T]$.

Recently, Fabry and Habets [3] studied the periodic boundary value problem (E)+(B.C.) i.e. $x_a=0=x_b$ with the condition $|x(t)|\leqq\varphi(t)$, where φ is a prespecified function.

In this paper, the existence of solutions is studied for the following boundary value problem for second order functional differential equations:

(1) $$\big(\varrho(t)x'(t)\big)'+f\big(t, x_t, x'(t)\big) = 0,$$

(2) $$x(t) = h(t), \quad -r \leqq t \leqq 0, \quad h(0) = 0, \quad x(T) = 0$$

and

$$|x(t)| \leqq \varphi(t),$$

where $f \in C[[0, T]\times C([-r, 0], \mathbf{R}^n)\times \mathbf{R}^n, \mathbf{R}^n]$, $\varrho \in C\big([0, T], (0, 1]\big)$ and φ is a prespecified function.

The result of this paper generalizes a previous one due to Fabry and Habets [3] (which concern ordinary differential equations with $\varrho(t)\equiv 1$) and it is very closely related to previous results from [5]. The method we use here is similar, in its basic steps, with that of [3]. We use the a-priori bound method.

2. Main result

Let $\mathbf{R}^n$ be the real Euclidean n-space with norm $|\cdot|$ and scalar product $\langle.,.\rangle$. Let C_r denote the space of all continuous functions x: $[-r, 0]\to\mathbf{R}^n$ furnished with the sup-norm $\|x\|=\sup\{|x(t)|: t\in[-r, 0]\}$. For a continuous function x: $[-r, T]\to\mathbf{R}^n$ and $t\in[0, T]$ define $x_t\in C_r$ by $x_t(\theta)=x(t+\theta)$, $\theta\in[-r, 0]$.

Finding a solution of the boundary value problem (1)—(2) is equivalent to finding a solution of the integral equation

$$x(t) = \int_0^T G(t, s) f(s, x_s, x'(s))\, ds$$

where $G(t, s)$ is the Green's function associated with the boundary value problem

$$(\varrho(t)x'(t))' = 0, \quad x(0) = 0 = x(T)$$

i.e.

(3)
$$G(t, s) = \begin{cases} (1-m\psi(t))\psi(s), & 0 \leqq s \leqq t \leqq T \\ \psi(t)(1-m\psi(s)), & 0 \leqq t \leqq s \leqq T \end{cases}$$

with

$$\psi(t) = \int_0^t \frac{ds}{\varrho(s)}, \quad m = [\psi(T)]^{-1}.$$

Let B be the linear Banach space of all continuous functions x: $[0, T]\to\mathbf{R}^n$ having continuous first derivatives on $[0, T]$ and norm

(4)
$$\|x\|^* = \max\{\sup_{t\in[0,T]} |x(t)|, \sup_{t\in[0,T]} |x'(t)|\}.$$

Our main existence result is given by the following theorem:

THEOREM. *Assume that there exist a twice differentiable function* φ: $[0, T]\to\mathbf{R}^+-\{0\}$ *and a continuous function* F: $[0, T]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *satisfying the following conditions: for any* (t, u, v) *with* $t\in[0, T]$, $\|u\|=\varphi(t)$, $\langle u(0), v\rangle=\|u\|\varphi'(t)$ *one has:*

(i) $\langle u(0), f(t, u, v)\rangle \leqq \varphi(t) F(t, \varphi(t), \varphi'(t))+\varrho(t)|v|^2-\varrho(t)\varphi'(t)^2$,

(ii) $(\varrho(t)\varphi'(t))'+F(t, \varphi(t), \varphi'(t)) \leqq 0$.

Assume moreover that there exist numbers $a\in[0, 1)$ *and* $b\geqq 0$ *such that, for any* (t, u, v) *with* $t\in[0, T]$, $\|u\|\leqq\varphi(t)$ *and* $v\in\mathbf{R}^n$

(iii) $\langle u(0), f(t, u, v)\rangle \leqq a\varrho(t)|v|^2+b$,

(iv) $|\langle v, f(t, u, v)\rangle| \leqq H(\varrho(t)|v|)|v|$

where H: $\mathbf{R}^+\to\mathbf{R}^+-\{0\}$ *is an increasing continuous function which satisfies the condition*

$$\int_0^\infty \frac{s^2\, ds}{H(s)} = +\infty.$$

Then the boundary value problem (1)—(2) *has at least one solution x such that* $|x(t)|\leqq\leqq\varphi(t)$ *for any* $t\in[0, T]$.

The proof of our existence theorem is based on the theory of of completely continuous mappings. More precisely, we apply the following two lemmas.

LEMMA 1 [3, Theorem 1]. *Let X be a Banach space,* $A: X\to X$ *be a completely continuous mapping such that I—A is one to one, and let* Ω *be a bounded set such that* $0\in(I-A)(\Omega)$. *Then the completely continuous mapping* $S: \Omega\to X$ *has a fixed point in* $\bar{\Omega}$ *if for any* $\lambda\in(0, 1)$, *the equation*

$$x = \lambda Sx+(1-\lambda)Ax \tag{5}$$

has no solution x on the boundary $\partial\Omega$ *of* Ω.

LEMMA 2. *Let* $x: [0, T]\to\mathbf{R}^n$ *be an absolutely continuous function with an absolutely continuous derivative. Assume that for almost all* $t\in[0, T]$ *one has*

$$|\langle x'(t), \ (\varrho(t)x'(t))'\rangle| \leqq H(\varrho(t)|x'(t)|)|x'(t)|$$

where $H: \mathbf{R}^+\to\mathbf{R}^+-\{0\}$ *is continuous and satisfies*

$$\int_0^{+\infty} \frac{s^2}{H(s)}ds = +\infty.$$

Then

$$|x'(t)| \leqq \frac{1}{\hat{\varrho}}\, g\Big(\int_0^T \varrho^2(t)|x'(t)|^2dt\Big),$$

where $\hat{\varrho}=\min\limits_{0\leqq t\leqq T}\varrho(t)$ *and g is defined by*

$$\int_{\frac{1}{\hat{\varrho}}\sqrt{\frac{z}{T}}}^{g(z)} \frac{s^2}{H(s)}ds = z.$$

We omit the proof of Lemma 2, since it is similar to that of given by Mawhin [4].

To apply Lemma 1 we actually have to find a fixed point for the mapping $S: B\to B$ defined by

$$(Sy)(t) = \int_0^T G(t, s)f(s, y_s, y'(s))ds \tag{6}$$

if a suitable operator A and a set Ω can be built. Then the function

$$x(t) = \begin{cases} h(t), & -r\leqq t\leqq 0\\ y(t), & 0\leqq t\leqq T\end{cases}$$

will be a solution of the boundary value problem (1)—(2), since $h(0)=0$.

PROOF OF THE THEOREM. By (iii) and (iv), f maps bounded subsets of $[0, T]\times C_r\times\mathbf{R}^n$ into bounded subsets of $\mathbf{R}^n$, so the mapping S defined by (6) is compact.

Now define $A\colon B\to B$ and the set Ω by

$$(Ax)(t) = -\int_0^T G(t,s)\varkappa^2 \frac{1}{\varrho(s)} x(s)\,ds, \quad \varkappa \neq 0 \tag{7}$$

and

$$\Omega_\eta = \{x\in B\colon \ \forall t\in[0,T],\ |x(t)| < \varphi(t),\ |x'(t)| < \eta\} \tag{8}$$

for a certain $n>0$.

It is clear that Ω_η is open and bounded in B and A is a completely continuous operator.

We will show that $I-A$ is one to one and that if η and $\varkappa$ are chosen large enough, no solution of (5) with S defined by (6), can belong to $\partial\Omega_\eta$ for $\lambda\in(0,1)$. Then the existence of a solution $x\in\bar{\Omega}$ is implied by Lemma 1.

In order to prove that the linear operator $I-A$ is one to one, we must show that $x=Ax$ implies $x=0$. Indeed, if x satisfies the equation $x-Ax=0$ then it is a solution of the boundary value problem

$$\big(\varrho(t)x'(t)\big)' = \varkappa^2 \frac{1}{\varrho(t)} x(t), \tag{9}$$

$$x(0) = 0 = x(T). \tag{10}$$

By using the transformation $w=\int_0^t \frac{ds}{\varrho(s)}$, equation (9) becomes

$$z''(w) = \varkappa^2 z(w) \tag{11}$$

with $z(0)$ and $z\left(\int_0^T \frac{ds}{\varrho(s)}\right) = 0$. The latter problem has the unique solution $x=0$, see [3]. Therefore the operator $I-A$ is one to one.

Secondly, we prove that if $x\in\partial\Omega_\eta$ is a solution of (5) then there exists no $\xi\in(0,T)$ such that either $|x(t)|^2-\varphi^2(t)$ reaches the maximum value 0 at $t=\xi$, or $|x'(\zeta)|=\eta$.

Assume that $|x(t)|^2-\varphi^2(t)$ reaches the maximum value 0 at $t=\xi\in(0,T)$. Then we have the following relations:

$$|x(\xi)| = \varphi(\xi), \quad \langle x(\xi), x'(\xi)\rangle - \varphi(\xi)\varphi'(\xi) = 0,$$

$$\langle x_\xi(0), x'(\xi)\rangle - \varphi(\xi)\varphi'(\xi) = 0, \quad \langle u(0), \varrho(\xi)x'(\xi)\rangle - \varrho(\xi)\varphi(\xi)\varphi'(\xi) = 0, \tag{12}$$

$$J \equiv \big\langle u(0), \big(\varrho(\xi)x'(\xi)\big)'\big\rangle + \varrho(\xi)|x'(\xi)|^2 - \big(\varrho(\xi)\varphi'(\xi)\big)'\varphi(\xi) - \varrho(\xi)\varphi'^2(\xi) \leqq 0. \tag{13}$$

Assume that x is a solution of (5). That is

$$\big(\varrho(t)x'(t)\big)' + \lambda f(t, x_t, x'(t)) = (1-\lambda)\varkappa^2 \frac{1}{\varrho(t)} x(t). \tag{14}$$

For $\lambda\in(0, 1)$, by our assumptions and (12) we have:

$$J \equiv \langle u(0), -\lambda f(t, u, v)\rangle + \varrho|v|^2 - (\varrho\varphi')'\varphi - \varrho\varphi'^2 + (1-\lambda)\frac{\varkappa^2}{\varrho}\langle u(0), u(0)\rangle =$$

$$= -\lambda\langle u(0), f(t, u, v)\rangle + (1-\lambda)\frac{\varkappa^2}{\varrho}\langle u(0), u(0)\rangle + \varrho|v|^2 - (\varrho\varphi')'\varphi - \varrho\varphi'^2 =$$

$$= -\lambda\langle u(0), f(t, u, v)\rangle + (1-\lambda)\frac{\varkappa^2}{\varrho}\varphi^2 + \varrho|v|^2 - (\varrho\varphi')'\varphi - \varrho\varphi'^2 \geqq$$

$$\geqq \lambda[-\langle u(0), f(t, u, v)\rangle + \varphi F + \varrho|v|^2 - \varrho\varphi'^2] +$$

$$+(1-\lambda)\left[\varrho|v|^2 - \varrho\frac{\langle u(0), v\rangle^2}{\|u\|^2} + \varphi\left\{\frac{\varkappa^2}{\varrho}\varphi - (\varrho\varphi')'\right\}\right] = J_1 + (1-\lambda)J_2.$$

But $J_1 \geqq 0$ by (i) and $J_2 \geqq 0$ by Cauchy—Schwarz inequality. Consequently $J>0$ if $\varkappa^2$ is large enough which contradicts (13).

Next, we will prove that for any solution x of (5) satisfying the condition $|x(t)| \leqq \varphi(t)$, $|x'(t)|$ has an upper bound which does not depend on $\lambda\in[0, 1]$.

Let x be a solution of (5); multiply both sides of (14) by $x(t)$, and integrate by parts over $[0, T]$. Then by (2) we have:

$$\langle x(t), (\varrho(t)x'(t))'\rangle + \lambda\langle u(0), f(t, u, v)\rangle = (1-\lambda)\frac{\varkappa^2}{\varrho(t)}\langle u(0), u(0)\rangle,$$

$$-\int_0^T \varrho(t)|x'(t)|^2 dt + \lambda\int_0^T \langle u(0), f(t, u, v)\rangle dt = (1-\lambda)\varkappa^2\int_0^T \frac{1}{\varrho(t)}|x(t)|^2 dt$$

or

$$-\int_0^T \varrho(t)|x'(t)|^2 dt + \lambda bT + a\lambda\int_0^T \varrho(t)|x'(t)|^2 dt \geqq 0,$$

from which we have

$$\int_0^T \varrho(t)|x'(t)|^2 dt \leqq \frac{\lambda bT}{1-a\lambda} \leqq \frac{bT}{1-a} = K.$$

Since $\varrho(t) \leqq 1$ we obtain

$$\int_0^T \varrho^2(t)|x'(t)|^2 dt \leqq \int_0^T \varrho(t)|x'(t)|^2 dt \leqq K.$$

Using Lemma 2 and (iii) we have an upper bound for $|x'(t)|$:

$$|x'(t)| \leqq \frac{1}{\hat{\varrho}} g(K), \quad \forall t\in[0, T].$$

Therefore any number larger than $\frac{1}{\hat{\varrho}} g(K)$ can be taken for η in the definition of Ω_η and thus the proof of the theorem is complete.

REMARK 1. If $\varrho(t)>1$, then divide (1) by $\sup\limits_{0\leqq t\leqq T} \varrho(t)\equiv N$ and obtain an equation of the form

$$(1^*) \qquad (\omega(t)x'(t))'+\tilde{f}(t, x_t, x'(t)) = 0$$

where $\omega(t)\leqq 1$ and $\tilde{f}=\frac{1}{N}f$. Then $\tilde{f}$ also satisfies the conditions of the Theorem by a suitable choice of constants. Therefore the above method can be applied also in the case when $\varrho(t)>1$.

REMARK 2. In case when $\varphi(t)$ is a constant function — say b — our result here leads to the corresponding one recently proved by the author [5, Theorem 3] in a different way.

3. An application

Consider the differential equation

$$(15) \qquad x''(t)+q(t)L(x_t)+m(t)x'(t) = 0$$

where $q(t)$, $m(t)$ are bounded functions with bounds Q and M respectively and $L(\varphi)$ is a linear bounded operator in C_r.

We shall show that for equation (15) the conditions of the Theorem are fulfilled for a suitable choice of the function F. Let $\|L\|$ be the norm of L, set $A=Q\|L\|$ and define F by

$$F(t, x, y) = \left(A+\frac{M^2}{4}\right)\|x\|+M|y|.$$

Then we have

$$(16) \qquad \langle u(0), f(t, u, v)\rangle = q(t)\langle L(u), u(0)\rangle+m(t)\,u(0)v \leqq A\|u\|^2+Mu(0)v.$$

Now following [3] we see that

$$0 \leqq \left[\frac{M}{2}\|u\|-\frac{|\langle u(0), v|}{\|u\|}+|v|\right]^2 \leqq$$

$$\leqq \frac{M^2}{4}\|u\|^2+\frac{|\langle u(0), v\rangle|^2}{\|u\|^2}+|v|^2-M|\langle u(0), v\rangle|+M\|u\|\,|v|-2\frac{|\langle u(0), v\rangle|}{\|u\|}|v| \leqq$$

$$\leqq \frac{M^2}{4}\|u\|^2+\frac{|\langle u(0), v\rangle|}{\|u\|}\left\{\frac{|\langle u(0), v\rangle|}{\|u\|}-2|v|\right\}+|v|^2-M|\langle u(0), v\rangle|+$$

$$+M\|u\|\,.\,|v| \leqq \frac{M^2}{4}\|u\|^2-\frac{|\langle u(0), v\rangle|^2}{\|u\|^2}+|v|^2-M|\langle u(0), v\rangle|+M\|u\|\,.\,|v|.$$

Consequently from (16) we have

$$\langle u(0), f(t, u, v)\rangle \leqq A\|u\|^2+\frac{M^2}{4}\|u\|^2-\frac{|\langle u(0), v\rangle|^2}{\|u\|^2}+|v|^2+M\|u\|.|v| =$$

$$= \|u\|\left\{\left(A+\frac{M^2}{4}\right)\|u\|+M|v|\right\}+|v|^2-\frac{|\langle u(0), v\rangle|^2}{\|u\|^2} = \varphi F(t, \varphi, \varphi')+|v|^2-\varphi'^2$$

i.e. condition (i) is satisfied for $\varphi=\|u\|$.

Condition (ii) is also satisfied, see [3, p. 194]. For (iii) we notice that

(17)
$$\langle u(0), f(t, u, v)\rangle = q(t)\langle L(u), u(0)\rangle+m(t)u(0)v \leqq A\|u\|^2+Mv|u(0)| \leqq A^*+B^*|v|$$

(where $A^*=AC$, $B^*=MC$, C is the bound of φ). If $|v|<1$, then (iii) is obvious from (17). If $|v|>1$, then (iii) follows from the inequality

$$B^* \geqq B^*|v|-B_1|v|^2, \quad \text{for each} \quad B_1 \geqq 0$$

indeed, we have

$$A^*+B^*|v| = A+B_1|v|^2+B^*|v|-B_1|v|^2 \leqq A^*+B^*+B_1|v|^2.$$

Finally, it is easy to see that condition (iv) is satisfied.

Hence by the Theorem we conclude that the boundary value problem (15), (2) has a solution.

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RADICAL THEORY FOR ALGEBRAS WITH A SCHEME OF OPERATORS

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Introduction

It is fairly well known that the Kurosh—Amitsur theory of radicals works for any variety or, more generally, any "universal class", of *multioperator groups* as defined by Higgins [5]: details can be found in papers of Mlitz [10] and Ryabukhin [13]. Rings, groups and modules are examples of the structures covered by this observation. Various authors, e.g. Shul'geifer [15] and Andrunakievich and Ryabukhin [1] have devised axiom systems for *categories* in which radical theory works. Unfortunately, the examples of such categories which are presented are all categories (not necessarily varieties) of multioperator groups. Recently some radical theorists have begun to direct attention towards graded rings; see Divinsky, Suliński and Anderson [2], Suliński [16], Suliński and Watters [17]. Now (externally) graded rings are not multioperator groups, and even internally graded rings (effectively the same thing), while they are rings, do not form a universal class of rings.

In this paper we shall consider a generalization of multioperator groups, where instead of a single underlying set there are several (possibly infinitely many, but the number is fixed). The structures we shall look at are special cases of the algebras with schemes of operators introduced by Higgins [6]. For these latter there are notions of homomorphism, congruence, identity, variety, etc., which coincide with the familiar notions in the special case (one underlying set) of universal algebras. Varieties in the pertinent special cases turn out to be categories satisfying the axioms of Shul'geifer [15], and therefore suitable environments for radical theory, while not (necessarily) classes of universal algebras. They also include classes of graded rings, modules over variable rings and group representations. Thus we get some further justification and motivation for a categorical approach to radical theory, while presenting a systematic way of treating various possible settings for radical theory without the necessity of introducing *ad hoc* definitions for normal subobjects, factor objects, and so on.

After developing the theory and establishing the axioms of Shul'geifer [15], we present a number of examples of the structures we are concerned with. One conceivable way of getting a radical property for a multiset structure is by combining in some way radical properties of the "components" (which are multioperator groups). We consider this in the final section, and are led to the notion of a normal family of radicals, a generalization of normal radicals as defined by Jaegermann [7].

1. Background

Let $\langle I, \Omega\rangle$ be an algebra with a set Ω of (possibly partial) finitary operations. An $\langle I, \Omega\rangle$*-graded algebra* (which wherever possible we will subsequently call simply an *algebra*) is a collection $A=\{A_i|i\in I\}$ of non-empty sets such that for every $\omega\in\Omega$ and every $i_1, \dots, i_n, i\in I$ such that $\omega(i_1, \dots, i_n)=i$, there is an associated function $A_{i_1}\times\dots\times A_{i_n}\to A_i$. This function will then also be called ω, or sometimes ω^A. We adopt the convention that a nullary operation in Ω is a function from $\{\emptyset\}$ to I: if this function picks out i_0, the corresponding function attached to A picks out an element of A_{i_0}, i.e. is a nullary operation on A_{i_0}. (Although some operations in Ω will be partial, we do not consider empty nullary operations.)

A *homomorphism* from an algebra $A=\{A_i|i\in I\}$ to another, $B=\{B_i|i\in I\}$, is a set $\{\theta_i|i\in I\}$ where, for each i, θ_i: $A_i\to B_i$ is a function, and if $\omega(i_1, \dots, i_n)=i$, then for $a_1\in A_{i_1}, \dots, a_n\in A_{i_n}$, we have

$$\theta_i\big(\omega^A(a_1, \dots, a_n)\big) = \omega^B\big(\theta_{i_1}(a_1), \dots, \theta_{i_n}(a_n)\big).$$

(In particular, if ω is nullary and picks out i, then $\theta_i(\omega^A)=\omega^B$.)

"Subalgebra" has its expected meaning: an algebra $S=\{S_i|i\in I\}$ is a *subalgebra* of $A=\{A_i|i\in I\}$ if for each i, $S_i\subseteq A_i$ and the inclusions $S_i\to A_i$ form a homomorphism.

An $\langle I, \Omega\rangle$*-congruence* on an algebra $A=\{A_i|i\in I\}$ is a set $\sigma=\{\sigma_i|i\in I\}$ where for each i, σ_i is an equivalence relation on A_i and the expected compatibility conditions obtain:

$$\omega(i_1, \dots, i_n) = i \,\&\, a_1, a_1'\in A_{i_1}, \dots, a_n, a_n'\in A_{i_n} \,\&\, a_1\sigma_{i_1}a_1', \dots, a_n\sigma_{i_n}a_n' \Rightarrow$$
$$\Rightarrow \omega(a_1, \dots, a_n)\sigma_i\omega(a_1', \dots, a_n').$$

There is the obvious notion of factor structure and we have the predictable intimate connection between congruences and homomorphisms. We shall have more to say of this shortly: in the cases we shall discuss, things are somewhat simpler.

We shall impose the following additional conditions on Ω.

(i) For each $i\in I$, there is a nullary operation $O_i\in\Omega$ with $O_i(\emptyset)=i$.

(ii) There is a partial binary operation $+\in\Omega$ such that $i+j=+(i,j)$ is defined only for $i=j$ and then $i+i=i$.

(iii) There is a unary operation $-\in\Omega$ such that $-i=i$ for every i.

There are further restrictions also: we shall work in a (sub) variety of algebras as so far described. We therefore need to define identities.

Let $X=\{X_i|i\in I\}$ be a set of countably infinite sets. We define $\langle I, \Omega\rangle$-words of type i as follows.

If $x\in X_i$, then x is a word of type i.

If $\omega(i_1, \dots, i_n)=i$ and if $w_1, \dots, w_n$ are words of types $i_1, \dots, i_n$ respectively, then $\omega(w_1, \dots, w_n)$ is a word of type i. (Thus, in particular, nullary operations are words.) Let W_i be the set of words of type i, for each i. From the definition it is clear that $W=\{W_i|i\in I\}$ is an algebra. In fact W is a free algebra on X in the sense that every family $\{f_i\colon X_i\to A_i|i\in I\}$ (for an algebra $A=\{A_i|i\in I\}$) extends uniquely to a homomorphism from W to A.

One then defines an identity as follows. Let $w=w(x_{i_1}, ..., x_{i_n})$ and $u=u(x_{j_1}, ..., x_{j_m})$ be words of the same type. An algebra $A=\{A_i|i\in I\}$ satisfies the identity $w=u$ if

$$w(a_1, ..., a_n) = u(b_1, ..., b_m)$$

for every substitution a_1 for $x_{i_1}, ..., a_n$ for x_{i_n}, b_1 for $x_{j_1}, ..., b_m$ for x_{j_m}, where each a is of the same "type" as the corresponding x and each b is of the same "type" as the corresponding y.

Identities define varieties and the latter are characterized by closure under subalgebras, homomorphic images and (componentwise) cartesian products. All the foregoing was introduced by Higgins [6].

Our universal class will be a variety of $\langle I, \Omega\rangle$-algebras satisfying, in addition to (i), (ii) and (iii), the following identities.

(iv) $\forall i\in I, \forall x, y, z\in X_i, \quad (x+y)+z = x+(y+z)$.

(v) $\forall i\in I, \forall x\in X_i, \quad x+0_i = x = 0_i+x$.

(vi) $\forall i\in I, \forall x\in X_i, \quad x+(-x) = 0_i = (-x)+x$.

(vii) If $\omega(i_1, ..., i_n) = i$, then $\omega(0_{i_1}, ..., 0_{i_n}) = 0_i$.

In particular, if ω is nullary, then $\omega=0_i$ for some i.

These identities make our universal class, in effect, a category with a forgetful functor to graded (not necessarily abelian) groups.

2. Radical theory

Shul'geifer [15] showed that Kurosh—Amitsur radical theory can be developed in any category satisfying the following conditions.

(1) For every pair of objects A, B there is a zero morphism $A\to B$.

(2) Every morphism has a kernel.

(3) Every morphism has an image.

(4) Conormal epimorphisms take normal subalgebras (=kernels) to normal subobjects.

(5) Well-ordered ascending chains of normal subobjects have joins.

We shall now show that any variety $\mathcal{W}$ of $\langle I, \Omega\rangle$-algebras satisfying (i)—(vii) of §1 forms a category (with $\langle I, \Omega\rangle$-homomorphisms as morphisms) satisfying Shul'geifer's conditions.

PROPOSITION 2.1. *The O_i are the only nullary operations in Ω.*

PROOF. Let e be a nullary operation which picks out $i_0\in I$, and let $A=\{A_i|i\in I\}$ be an algebra. Then by (vii) e picks out O_{i_0} in A_{i_0}. □

PROPOSITION 2.2. *The algebra $\langle I, \Omega\rangle$ as $\{\{i\}|i\in I\}$, is a zero object for $\mathcal{W}$.*

PROOF. The nullary operation O_i certainly picks out an element of $\{i\}$, for each i. If, on the other hand, $\omega\in\Omega$ and $\omega(i_1, \ldots, i_n)=i$ $(n\geqq 1)$ there is obviously an induced map

$$\{i_1\}\times\ldots\times\{i_n\} = \{(i_1, \ldots, i_n)\} \to \{i\}.$$

Thus I is an $\langle I, \Omega\rangle$-algebra. Let $w(x_{i_1}, \ldots, x_{i_n})=u(x_{j_1}, \ldots, x_{j_m})$ be an identity satisfied by all algebras in $\mathscr{W}$, where $x_{i_1}\in X_{i_1}, \ldots, x_{i_n}\in X_{i_n}, y_{j_1}\in X_{j_1}, \ldots, y_{j_m}\in X_{j_m}$. Then we must have $w(i_1, \ldots, i_n)=u(j_1, \ldots, j_m)$ (as for elements to have the same value they must be in the same A_i). But this simply means that I satisfies all identities; in particular those defining $\mathscr{W}$. Thus I is in $\mathscr{W}$.

Let $A=\{A_i|i\in I\}$ be any algebra. Define $\nu^A\colon A\to I$ by

$$\nu_i^A(a) = i, \quad \forall a\in A_i, \quad \forall i.$$

In particular, $\nu_i^A(O_i)=i$, so ν^A respects nullaries. If $\omega(i_1, \ldots, i_n)=i$, then for $a_1\in A_{i_1}, \ldots, a_n\in A_{i_n}$, we have

$$\omega\big(\nu_{i_1}^A(a_1), \ldots, \nu_{i_n}^A(a_n)\big) = \omega(i_1, \ldots, i_n) = i = \nu_i^A\big(\omega(a_1, \ldots, a_n)\big).$$

Hence ν^A is a homomorphism. Now define $\mu^A\colon I\to A$ by

$$\mu_i^A(i) = O_i.$$

Utilizing (vii), we see that μ_i^A is a homomorphism. It is clear that μ^A and ν^A are the only homomorphisms between A and I. □

Thus $\mathscr{W}$ has zero morphisms, namely those that factor through I.

Now in view of the identities (iv)—(vii) we have imposed on $\mathscr{W}$, homomorphisms are, *inter alia*, collections of group homomorphisms. Our characterization of zero maps thus admits the following paraphrase: a homomorphism

$$\theta\colon A = \{A_i|i\in I\} \to B = \{B_i|i\in I\}$$

is zero if and only if each $\theta_i\colon A_i\to B_i$ is the trivial group homomorphism. To establish the existence of kernels, it is enough that we show that the group theoretic kernels of the components of a homomorphism form a subalgebra.

PROPOSITION 2.3. *Let*

$$\theta\colon A = \{A_i|i\in I\} \to B = \{B_i|i\in I\}$$

be a homomorphism. For each $i\in I$, let K_i denote the kernel of θ_i as a group homomorphism (with respect to $+$). Then $K=\{K_i|i\in I\}$ is a subalgebra of A, and a kernel for θ.

(Note that we are pragmatically adopting the view that kernels are subalgebras rather than equivalence classes of monomorphisms; in the present context they are, of course.)

PROOF. As far as the nullary operations are concerned, K behaves appropriately, being a collection of subgroups. If $\omega\in\Omega$ is n-ary, $n\geqq 1$, and if $\omega(i_1, \ldots, i_n)=i$, then for $k_1\in K_{i_1}, \ldots, k_n\in K_{i_n}$, we have

$$\theta_{i_1}(k_{i_1}) = O_{i_1}^B = \theta_{i_1}(O_{i_1}^A), \ldots, \theta_{i_n}(k_{i_n}) = O_{i_n}^B = \theta_{i_n}(O_{i_n}^A),$$

so, since equality under a homomorphism defines a congruence (see §1), we have

$$\theta_i(\omega(k_{i_1}, \dots, k_{i_n})) = \theta_i(\omega(O^A_{i_1}, \dots, O^A_{i_n})) = \theta_i(O^A_i) = O^B_i$$

(where we have invoked (vii)), i.e. $\omega(k_{i_1}, \dots, k_{i_n}) \in K_i$. Thus K is a subalgebra. Our remarks above now make it clear that K is a kernel for θ. □

We shall call a subalgebra *normal* if it is the kernel of a homomorphism.

PROPOSITION 2.4. *A subalgebra $K=\{K_i|i\in I\}$ of an algebra $A=\{A_i|i\in I\}$ is normal if and only if each K_i is a normal subgroup of A_i and K satisfies the condition*

$$(*)\quad \omega(i_1, \dots, i_n) = i \,\&\, a_1, a_1'\in A_{i_1}, \dots, a_n, a_n'\in A_{i_n} \,\&\, a_1-a_1'\in K_{i_1}, \dots, a_n-a_n'\in K_{i_n} \Rightarrow$$
$$\Rightarrow \omega(a_1, \dots, a_n)-\omega(a_1', \dots, a_n')\in K_i.$$

PROOF. Let K be the kernel of some homomorphism θ issuing from A. Under the hypotheses of (*), we have

$$\theta_{i_1}(a_1)-\theta_{i_1}(a_1') = \theta_{i_1}(a_1-a_1') = O^B_{i_1}, \dots, \theta_{i_n}(a_n)-\theta_{i_n}(a_n') = O^B_{i_n},$$

i.e.

$$\theta_{i_1}(a_1) = \theta_{i_1}(a_1'), \dots, \theta_{i_n}(a_n) = \theta_{i_n}(a_n').$$

Since equality under θ is a congruence, we then have

$$\theta_i(\omega(a_1, \dots, a_n)) = \theta_i(\omega(a_1', \dots, a_n')),$$

so

$$\theta_i(\omega(a_1, \dots, a_n)-\omega(a_1', \dots, a_n')) = O^B_i,$$

i.e.

$$\omega(a_1, \dots, a_n)-\omega(a_1', \dots, a_n')\in K_i.$$

Thus (*) is satisfied.

Conversely, if (*) is satisfied and each K_i is a normal subgroup of the corresponding A_i, we can form the collection $\{A_i/K_i|i\in I\}$. For $\omega\in\Omega$, $\omega(i_1, \dots, i_n)=i$ and $a_1\in A_{i_1}, \dots, a_n\in A_{i_n}$, we define

$$\omega(a_1+K_{i_1}, \dots, a_n+K_{i_n}) = \omega(a_1, \dots, a_n)+K_i.$$

Then (*) ensures that ω is well-defined. It follows that $A/K=\{A_i/K_i|i\in I\}$ is an algebra and the natural homomorphism from A to A/K has kernel K. Note that (*) implies that K is a subalgebra, by (vii). □

Condition (*) of Proposition 2.4 is equivalent to

$$\omega(i_1, \dots, i_n) = i \,\&\, a_1\in A_{i_1}, \dots, a_n\in A_{i_n} \,\&\, k_1\in K_{i_1}, \dots, k_n\in K_{i_n} \Rightarrow$$
$$\Rightarrow \omega(a_1, \dots, a_n)-\omega(a_1+k_1, \dots, a_n+k_n)\in K_i.$$

Since

$$\omega(a_1, \dots, a_n)-\omega(a_1+k_1, \dots, a_n+k_n) = \omega(a_1, \dots, a_n)-\omega(a_1+k_1, a_2, \dots, a_n)+$$
$$+\omega(a_1+k_1, a_2, \dots, a_n)-\omega(a_1+k_1, a_2+k_2, \dots, a_n)+\dots$$
$$\dots+\omega(a_1+k_1, a_2+k_2, \dots, a_{n-1}+k_{n-1}, a_n)-\omega(a_1+k_1, \dots, a_n+k_n),$$

and since any of $k_1, \dots, k_n$ can be zero, we get

COROLLARY 2.5. *A subalgebra K is normal in an algebra A if and only if for each i, K_i is a normal subgroup of A_i and K satisfies the condition*

$$\forall i\, \forall j \omega(i_1, \dots, i_n) = i \,\&\, a_1 \in A_{i_1}, \dots, a_n \in A_{i_n} \,\&\, k \in K_{i_j} \Rightarrow$$
$$\Rightarrow \omega(a_1, \dots, a_n) - \omega(a_1, \dots, a_{j-1}, a_j + k, a_{j+1}, \dots, a_n) \in K_i.$$

Let $\theta\colon A \to B$ be a homomorphism. Then $\{\theta(A_i) | i \in I\}$ is a subalgebra of B which is easily seen to be an image of θ.

One establishes the existence of cokernels by using group theory, just as we did with kernels. It follows that the conormal epimorphisms are the homomorphisms whose components are surjective.

Now let $\theta\colon A \to B$ be a homomorphism with surjective components and let K be a normal subalgebra of A. Then each $\theta_i(K_i)$ is a normal subgroup of the corresponding B_i. If $\omega(i_1, \dots, i_n) = i$ and if $b_1 \in B_{i_1}, \dots, b_n \in B_{i_n}$ and $l \in \theta_{i_j}(K_{i_j})$, $1 \leq j \leq n$, let $b_1 = \theta_{i_1}(a_1), \dots, b_n = \theta_{i_n}(a_n)$, $l = \theta_{i_j}(k_{i_j})$, $k_{i_j} \in K_{i_j}$. Then

$$\omega(b_1, \dots, b_n) - \omega(b_1, \dots, b_{j-1}, b_j + l, b_{j+1}, \dots, b_n) =$$
$$= \omega(\theta_{i_1}(a_1), \dots, \theta_{i_n}(a_n)) - \omega(\theta_{i_1}(a_1), \dots, \theta_{i_{j-1}}(a_{j-1}),$$
$$\theta_{i_j}(a_j) + \theta_{i_j}(l), \theta_{i_{j+1}}(a_{j+1}), \dots, \theta_{i_n}(a_n)) =$$
$$= \theta_i(\omega(a_1, \dots, a_n)) - \theta_i(\omega(a_1, \dots, a_{j-1}, a_j + l, a_{j+1}, \dots, a_n)) =$$
$$= \theta_i(\omega(a_1, \dots, a_n) - \omega(a_1, \dots, a_{j-1}, a_j + l, a_{j+1}, \dots, a_n)) \in \theta_i(K_i).$$

By Corollary 2.5, $\{\theta_i(K_i) | i \in I\}$ is normal in $\{B_i | i \in I\}$.

We observe, finally, that the componentwise set union of any infinite ascending chain of normal subalgebras is a normal subalgebra (by Proposition 2.4).

We summarize all this in

THEOREM 2.6. *The variety $\mathcal{W}$, as a category, satisfies conditions* (1)—(4). □

Thus, as pointed out by Shul'geifer [15] we can now proceed to develop radical theory exactly as in Kurosh [9]: radical classes have the same characterizations, lower and upper radicals are defined as usual, and so on.

Examples

(1) *Graded modules*. Let R be an associative ring with identity, S a monoid. For each $s \in S$, let $+, -, O_s$ be defined as in (i)—(vii) of §1. For each $r \in R$, let ω_r be a unary operation on S such that $\omega_r(s) = s$ for each s. Let

$$\Omega = \{+, -\} \cup \{O_s | s \in S\} \cup \{\omega_r | r \in R\}.$$

An $\langle S, \Omega \rangle$-algebra $A = \{A_s | s \in S\}$ is called an *S-graded R-module* if it satisfies the following identities

$$x + y = y + x; \quad \omega_1(x) = x; \quad \omega_r(x+y) = \omega_r(x) + \omega_r(y) \forall r;$$
$$\omega_{r+t}(x) = \omega_r(x) + \omega_t(x); \quad \omega_r(\omega_t(x)) = \omega_{rt}(x)\ \forall r, t.$$

The condition for normality of B in A (Corollary 2.5) here is simply that

$$\omega_r(a)-\omega_r(a+b)\in B_s$$

i.e. $\omega_r(b)\in B_s$ for every $r\in R$, $b\in B_s$, $s\in S$, i.e. that each B_s be a submodule of the corresponding A_s: all subalgebras are normal.

(2) *Graded rings.* Let S be a monoid. Let $+$, $-$ and O_s be defined as in §1. Let

$$\Omega = \{+, -, \cdot\}\cup\{O_s|s\in S\}$$

where $\cdot$ is the multiplication of S. (Note that the condition that S be a monoid can be described by means of identities.) An $\langle S, \Omega\rangle$-algebra $A=\{A_s|s\in S\}$ is called an *S-graded ring* if it satisfies the identities

$$x+y = y+x; \quad (xy)z = x(yz); \quad x(y+z) = xy+xz; \quad (x+y)z = xz+yz.$$

(Here $\cdot$ maps $A_s\times A_k$ to A_{sk}.) In this case the normal subalgebras are entirely determined by $\cdot$: B is normal if and only if

$$a_1a_2-a_1(a_2+b_2)\in B_{i_1i_2}, \quad a_1a_2-(a_1+b_1)a_2\in B_{i_1i_2},$$

i.e. $a_1b_2, b_1a_2\in B_{i_1i_2}$ whenever $a_1\in A_{i_1}$, $b_1\in B_{i_1}$, $a_2\in A_{i_2}$, $b_2\in B_{i_2}$, $i_1, i_2\in S$. Note that if e is the identity of S, we have $B_eB_e\subseteq B_e$.

(3) *Group homomorphisms.* Let $I=\{1, 2\}$, let $+$, $-$, O_1, O_2 be as in §1 and let ω be a partial unary operation such that $\omega(1)=2$ and $\omega(2)$ is not defined. Let $\Omega=\{+, -, O_1, O_2, \omega\}$ and consider the $\langle I, \Omega\rangle$-algebras defined by the identities

$$\omega(x+y) = \omega(x)+\omega(y).$$

If (A_1, A_2) and (C_1, C_2) are such algebras, a homomorphism θ: $(A_1, A_2)\to(C_1, C_2)$ is a pair of group homomorphisms θ_i: $A_i\to C_i$ such that $\theta_2(\omega(a))=\omega(\theta_1(a))$, i.e. the sides of a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\omega^A} & A_2 \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\theta_2} \\ C_1 & \xrightarrow{\omega^C} & C_2 \end{array}$$

of group homomorphisms. In this case (B_1, B_2) is normal in (A_1, A_2) if and only if

$$\omega(a)-\omega(a+b)\in B_2, \quad \text{i.e.} \quad \omega(b)\in B_2$$

for all ($a\in A_1$ and) $b\in B_1$, i.e. all subalgebras are normal.

(4) *Complexes.* Let R be an associative ring with identity. We introduce some further structure on the variety of Z-graded R-modules in the sense of (1). We introduce a partial unary operation χ on Z by setting $\chi(n)=n+1$. A complex of R-modules is a $\langle Z, \bar{\Omega}\rangle$-algebra, where $\bar{\Omega}=\Omega\cup\{\chi\}$ (with Ω as in (1)), which satisfies the identities

$$\chi(x+y) = \chi(x)+\chi(y); \quad \chi\omega_r(x) = \omega_r\chi(x)\forall r; \quad \chi\chi(x) = 0.$$

A subalgebra B of A is normal if (as in (1)) each B_n is a submodule of the corresponding A_n and

$$\chi(b) = \chi(a) - \chi(a+b) \in B_{n+1} \quad \text{whenever} \quad b \in B_n.$$

Again, all subalgebras are normal.

(5) *Modules over various rings* [3], [12]. Let $I = \{1, 2\}$. Let $+, -, O_1, O_2$ be as in §1, let ω be a partial binary operation such that $\omega(1, 2) = 2$ and $\omega(i, j)$ is undefined otherwise and let $1 \cdot 1 = 1$ with $i \cdot j$ undefined otherwise. Let $\Omega = \{+, -, O_1, O_2, \omega, \cdot\}$ and let $\mathscr{W}$ denote the variety of $\langle I, \Omega\rangle$-algebras defined by the identities

$$\omega(x, y, z) = x \cdot \omega(y, z); \quad \omega(x+y, z) = \omega(x, z) + \omega(y, z);$$

$$\omega(x, y+z) = \omega(x, y) + \omega(x, z); \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z;$$

$$x \cdot (y+z) = x \cdot y + x \cdot z; \quad (x+y) \cdot z = x \cdot z + y \cdot z.$$

A homomorphism $\theta \colon A \to B$ is a pair $\{\theta_1, \theta_2\}$ where each $\theta_i \colon A_i \to B_i$ is a group homomorphism and

$$\theta_1(x \cdot y) = \theta_1(x) \cdot \theta_1(y); \quad \theta_2(\omega(x, y)) = \omega(\theta_1(x), \theta_2(y)).$$

A subalgebra B of A is normal if

$$\omega(a_1, a_2) - \omega(a_1, b_2 + a_2) = \omega(a_1, b_2) \in B_2,$$

$$\omega(a_1, a_2) - \omega(a_1 + b_1, a_2) = \omega(b_1, a_2) \in B_2,$$

$$a \cdot a' - a \cdot (a' + b') = -a \cdot b' \in B_1 \quad \text{and} \quad a \cdot a' - (a+b) \cdot a' = -b \cdot a' \in B_1$$

for all $a_1, a, a' \in A_1$, $a_2 \in A_2$, $b_1, b, b' \in B_1$, $b_2 \in B_2$ (i.e. B_2 is an A_1-module, $B_1 A_2 \subseteq B_2$ and B_1 is an ideal of A_1).

(6) *Group representations* [11], [18]. This example is somewhat similar to the previous one. Let $I = \{1, 2\}$ and let R be an associative ring with identity, e. Let $+, -, O_1, O_2$ be as in §1 and for each $r \in R$ let ω_r be a partial unary operation such that $\omega_r(2) = 2$ and $\omega_r(1)$ is undefined. Let μ be a partial binary operation with $\mu(1, 2) = 2$. Let $\Omega = \{+, O_i | i = 1, 2\} \cup \{-\} \cup \{\omega_r | r \in R\} \cup \{\mu\}$. An $\langle I, \Omega\rangle$-algebra $\{A_1, A_2\}$ is called an R-group representation if A_2 is a unital R-module with scalar multiplication given by the ω_r and if μ defines a group homomorphism from A_1 to the automorphism group of A_2. (All these conditions can be described by identities.) A subalgebra B of A is normal if and only if $B_1 \lhd A_1$ and the following conditions are satified:

$$\omega_r(b) = \omega_r(a) - \omega_r(a+b) \in B_2 \quad \text{for all} \quad (a \in A_2 \text{ and}) \quad b \in B_2, \quad r \in R$$

(i.e. B_2 is an R-submodule of A_2),

$$-\mu(a_1, b_2) = \mu(a_1, a_2) - \mu(a_1, a_2 + b_2) \in B_2 \quad \text{for all} \quad a_1 \in A_1, \quad b_2 \in B_2$$

(i.e. B_2 is invariant under the automorphisms of A_2 defined by μ on A_1),

$$\mu(a_1, a_2 - \mu(b_1, a_2)) = \mu(a_1, a_2) - \mu(a_1, \mu(b_1, a_2)) = \mu(a_1, a_2) - \mu(a_1 + b_1, a_2) \in B_2$$

for all $a_1 \in A_1$, $b_1 \in B_1$, $a_2 \in A_2$, i.e. $\mu(-a_1, \mu(a_1, a_2 - \mu(b_1, a_2))) \in B_2$, i.e. $a_2 - \mu(b_1, a_2) \in B_2$ (i.e. B_1 acts trivially on A_2/B_2).

(7) *Morita contexts.* Let $I=\{(1,1),(1,2),(2,1),(2,2)\}$. As well as $+$, $-$ and O_{ij}, we let Ω contain partial binary operations μ_{ij}, $i,j=1,2$, acting as follows:

$$\mu_{ij}((i,k),(k,j)) = (i,j).$$

An $\langle I, \Omega\rangle$-algebra is called a Morita context if it satisfies the following identities

$$\mu_{lj}(x, \mu_{ij}(y,z)) = \mu_{lj}(\mu_{lk}(x,y),z),$$

$$\mu_{ij}(x+y,z) = \mu_{ij}(x,z)+\mu_{ij}(y,z),$$

$$\mu_{ij}(x,y+z) = \mu_{ij}(x,z)+\mu_{ij}(y,z).$$

A subalgebra B of A is normal if it satisfies the following conditions.

$$\mu_{ij}(a_{ik},a_{kj})-\mu_{ij}(a_{ik},a_{kj}+b_{kj}) = -\mu_{ij}(a_{ik},b_{kj})\in B_{ij},\ \forall a_{ik}\in A_{ik},\ b_{kj}\in B_{kj},$$

$$\mu_{ij}(a_{ik},a_{kj})-\mu_{ij}(a_{ik}+b_{ik},a_{kj}) = -\mu_{ij}(b_{ik},a_{kj})\in B_{ij},\quad \forall a_{kj}\in A_{kj},\ b_{ik}\in B_{ik}.$$

Thus a Morita context consists of a pair of rings, a bimodule over each such that the ring multiplications, the scalar multiplication and certain bilinear mappings from the cartesian products of modules to the rings are appropriately associative. Following Sands [14] we can represent a Morita context in matrix form

$$\begin{bmatrix} R & M \\ N & S \end{bmatrix}$$

where R and S are the rings, M and N the modules. This set of matrices is then an associative ring and its normal subalgebras *qua* context are among its ideals *qua* ring.

3. Decomposable radicals

Let $A=\{A_i|i\in I\}$ be an $\langle I, \Omega\rangle$-graded algebra. Let $j\in I$ be a *universal idempotent,* i.e. whenever $\omega\in\Omega$ and $\omega(j,j,\ldots,j)$ is defined, we have $\omega(j,j,\ldots,)=j$. Then each induced function on a cartesian product $A_j\times A_j\times\ldots\times A_j$ takes its values in A_j, i.e. is an operation (full) on A_j. This A_j is then a multioperator group (see (iv)—(vii)).

For any variety $\mathcal{W}$ of $\langle I, \Omega\rangle$-graded algebras, the "restriction" of the defining identities to a universal idempotent j will thus define a variety $\mathcal{W}_j$ of multioperator groups. We get a functor $\Phi_j\colon \mathcal{W}\to\mathcal{W}_j$ by setting $\Phi_j(A)=A_j$ and $\Phi_j(\theta)=\theta_j$. The characterization of normal subalgebras given in Proposition 2.4 makes it clear that the functor Φ_j is exact and preserves joins of ascending chains of normal subalgebras. Now $\mathcal{W}_j$ is also a category satisfying Shul'geifer's conditions. (This is old hat — see, e.g. [10] — but in any case varieties of multioperator groups are special cases of the varieties we are considering.) Now functors with the abovementioned preservation properties "reflect" radical classes [4]. Thus we have

PROPOSITION 3.1. *Let $\mathcal{W}$ be a variety of $\langle I, \Omega\rangle$-graded algebras, j a universal idempotent in I, $\mathcal{W}_j$ the corresponding variety of multioperator groups. If $\mathcal{R}$ is a radical class in $\mathcal{W}_j$, then $\mathcal{R}_j^*=\{A\in\mathcal{W}|A_j\in\mathcal{R}\}$ is a radical class in $\mathcal{W}$.*

Since intersections of radical classes are radical classes we have the following immediate consequence.

COROLLARY 3.2. *Let J be a set of universal idempotents in I and for each $j \in J$ let $\mathcal{R}_j$ be a radical class in $\mathcal{W}_j$. Then $\mathcal{R}_J^* = \{A \in \mathcal{W} | A_j \in \mathcal{R}_j \forall j\}$ is a radical class in $\mathcal{W}_j$.*

In particular, one can take for J the set of all universal idempotents.

DEFINITION 3.3. Let J_0 be the set of universal idempotents of I. A radical class of $\mathcal{W}$-algebras is *decomposable* if it has the form $\mathcal{R}_{J_0}^*$ in the sense of Corollary 3.2.

DEFINITION 3.4. A *$\mathcal{W}$-normal family* of radical classes is a set $\{\mathcal{R}_j | j \in J_0\}$, where $\mathcal{R}_j$ is a radical class in $\mathcal{W}_j$ for each $j \in J_0$ and such that for each $A \in \mathcal{W}$, we have

$$\mathcal{R}_{J_0}^*(A) = \mathcal{R}_{J_0}^*(\{A_i | i \in I\}) = \{B_i | i \in I\},$$

where $B_i = \mathcal{R}_j(A_j)$ for each $j \in J_0$.

It is clear that for every $\mathcal{W}$-normal family, the corresponding radical class in $\mathcal{W}$ is decomposable. As we shall see, the converse is false.

DEFINITION 3.5. Let $A = \{A_i | i \in I\}$ be a $\mathcal{W}$-algebra, J_0 the set of universal idempotents of I. For each $j \in J_0$, let N_j be a normal subobject of A_j. The set $\{N_j | j \in J_0\}$ is called a *compatible family* for A if A has a normal subalgebra $\{B_i | i \in I\}$ with $B_j = N_j$ for all $j \in J_0$.

We shall need the following result.

LEMMA 3.6. *Let $B = \{B_i | i \in I\}$ and $C = \{C_i | i \in I\}$ be normal subalgebras of $A = \{A_i | i \in I\}$. Then in the lattice of normal subalgebras of A, the join of B and C is $\{B_i + C_i | i \in I\}$.*

PROOF. We only have to show that $\{B_i + C_i | i \in I\}$ is a normal subalgebra. Let $\omega(i_1, \dots, i_n) = i$, $a_1 \in A_{i_1}, \dots, a_n \in A_{i_n}$, $b \in B_{i_k}$, $c \in C_{i_k}$. Then

$$\omega(a_1, \dots, a_k, \dots, a_n) - \omega(a_1, \dots, a_k + b + c, \dots, a_n) =$$

$$= \omega(a_1, \dots, a_k, \dots, a_n) - \omega(a_1, \dots, a_k + b, \dots, a_n) +$$

$$+ \omega(a_1, \dots, a_k + b, \dots, a_n) - \omega(a_1, \dots, (a_k + b) + c, \dots, a_n) \in B_i + C_i.$$

Since $B_i + C_i$ is a normal subgroup of A_i for each i, we have what we want. □

PROPOSITION 3.7. *Let $\mathcal{R}_{J_0}^*$ be a decomposable radical class of $\mathcal{W}$-algebras, defined by $\mathcal{R}_j$, $j \in J_0$. Then $\{\mathcal{R}_j | j \in J_0\}$ is a $\mathcal{W}$-normal family if and only if $\{\mathcal{R}_j(A_j) | j \in J_0\}$ is a compatible family in A, for every A.*

PROOF. Suppose $\{\mathcal{R}_j(A_j) | j \in J_0\}$ is a compatible family, for each A. Let $D = \{D_i | i \in I\}$ be the join of all normal subalgebras of A with j-component $\mathcal{R}_j(A_j)$ for each $j \in J_0$. It follows from Lemma 3.5 and its iterations that $D_j = \mathcal{R}_j(A_j)$ for all $j \in J_0$. Now $A/D = \{A_i/D_i | i \in I\}$ has j-component $\{O_j\}$ for every $j \in J_0$. Let $\{E_i/D_i | i \in I\}$ be a normal subalgebra of A/D such that $E_j/D_j = 0$ for each $j \in J_0$. Then (see, e.g. [15]) $\{E_i | i \in I\}$ is a normal subalgebra of A with $E_j = D_j = \mathcal{R}_j(A_j)$ for all $j \in J_0$. Hence $E_i \subseteq D_i$ for all $i \in I$. It follows that $\{O_i | i \in I\}$ is the only normal subalgebra of A/D belonging to $\mathcal{R}^*_{J_0}$. On the other hand, clearly $D\in\mathcal{R}^*_{J_0}$. Thus

$$\mathcal{R}^*_{J_0}(A) = D = \{D_i | i\in I\},$$

where $D_j=\mathcal{R}_j(A_j)$ for every $j\in J_0$, i.e. $\{\mathcal{R}_j | j\in J_0\}$ is a $\mathcal{W}$-normal family.

Conversely, if $\{\mathcal{R}_j | j\in J_0\}$ is a $\mathcal{W}$-normal family, then for every A we have

$$\mathcal{R}^*_{J_0}(A) = \{F_i | i\in I\},$$

where $F_j=\mathcal{R}_j(A_j)$ for every $j\in J_0$. Thus $\{\mathcal{R}_j(A_j) | j\in J_0\}$ is compatible for every A. □

The term "normal family" is inspired by the term "normal radical" (see, e.g., [7]). As our first example, therefore, we shall consider normal families for the variety of Morita contexts. In this case (Example (7)) the universal idempotents of I are (1,1) and (2,2). In the following proof we shall find it convenient to represent the partial binary operations μ_{ij} by juxtaposition. The standard matrix form of Morita contexts will be used. The algebras A_{11} and A_{22} are, of course, rings in this case.

PROPOSITION 3.8. *A family* $\{\mathcal{R}_{11}, \mathcal{R}_{22}\}=\{\mathcal{U}, \mathcal{S}\}$ *of ring radical classes is normal for Morita contexts if and only if* $\mathcal{U}=\mathcal{S}$ *and* $\mathcal{U}$ *is a normal radical class.*

PROOF. Suppose $\{\mathcal{U}, \mathcal{S}\}$ is a normal family. Then every Morita context $\begin{bmatrix} A & B \\ C & D\end{bmatrix}$ has a normal subalgebra, or, equivalently, a ring ideal, of the form $\begin{bmatrix} \mathcal{U}(A) & * \\ * & \mathcal{S}(D)\end{bmatrix}$. Thus we have the inclusions $B\mathcal{S}(D)C\subseteq\mathcal{U}(A)$ and $C\mathcal{U}(A)B\subseteq\mathcal{S}(D)$, so if $\mathcal{U}=\mathcal{S}$, this radical class is normal [7].

Let A be a ring with identity. Then the matrix ring $\begin{bmatrix} A & A \\ A & A\end{bmatrix}$ can be viewed as a Morita context, and by what we have just shown, we have

$$\mathcal{S}(A) = A\mathcal{S}(A)A \subseteq \mathcal{U}(A) \quad\text{and}\quad \mathcal{U}(A) = A\mathcal{U}(A)A \subseteq \mathcal{S}(A).$$

Thus $\mathcal{U}$ and $\mathcal{S}$ agree on rings with identity. We now consider two possibilities.

If $\mathcal{U}(Z)=\mathcal{S}(Z)=0$, then for every ring R, denoting by $R*Z$ the standard unital extension, we have

$$\mathcal{U}(R) = R\cap\mathcal{U}(R^*Z) = R\cap\mathcal{S}(R^*Z) = \mathcal{S}(R),$$

so $\mathcal{U}=\mathcal{S}$.

Let us then consider the case where $\mathcal{U}(Z)=\mathcal{S}(Z)\neq 0$. Consider the context $\begin{bmatrix} Z & Z \\ Z & Z^0\end{bmatrix}$ where Z^0 denotes the zeroring on the integers, Z is unital when viewed as a Z-module and has trivial scalar multiplication when viewed as a Z^0-module, and where the partial operations $Z\times Z\to Z$ and $Z\times Z\to Z^0$ are given by $(m, n)\mapsto 0$ and $(m, n)\mapsto mn$ respectively. (This context was used by Jaegermann [7].) In effect we are considering 2×2 integer matrices with the multiplication

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} e & f \\ g & h\end{bmatrix} = \begin{bmatrix} ae & af \\ ce & cf\end{bmatrix}.$$

We have

$$\begin{aligned}\mathcal{S}(Z^0) &\supseteq Z\cdot\mathcal{U}(Z)\cdot Z \text{ ("context product")}\\ &= (Z\mathcal{U}(Z)Z)^0 \text{ ("standard product")}\\ &= \mathcal{U}(Z)^0 \neq 0.\end{aligned}$$

It follows that $\mathcal{I}$ contains all nilpotent rings. In the same way, using the analogously defined context

$$\begin{bmatrix} Z^0 & Z \\ Z & Z \end{bmatrix},$$

we see that $\mathcal{U}$ contains all nilpotent rings. Now let B be any ring in $\mathcal{U}$, and consider the context $\begin{bmatrix} B & B \\ B & B \end{bmatrix}$. We have

$$B^3 = BBB = B\mathcal{U}(B)B \subseteqq \mathcal{I}(B),$$

so $B/\mathcal{I}(B)$ is nilpotent. But $\mathcal{I}$ contains all nilpotent rings, so B is in $\mathcal{I}$ and $\mathcal{U} \subseteqq \mathcal{I}$. Similarly $\mathcal{I} \subseteqq \mathcal{U}$.

By our initial remarks, $\mathcal{U} = \mathcal{I}$ is a normal radical class.

Conversely, if $\mathcal{U}$ is any normal radical class, then by Theorem 2 of Jaegermann [8], for every context $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, viewed as a ring, we have

$$\mathcal{U}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathcal{U}(A) & X \\ Y & \mathcal{U}(D) \end{bmatrix}$$

for some X, Y. The right-hand term is an ideal, and thus a normal subalgebra of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Thus $\{\mathcal{U}, \mathcal{U}\}$ is a normal family. □

Note that this implies that not every decomposable radical class is defined by a normal family.

As we have defined varieties of graded rings, only homomorphisms of degree zero have been mentioned. If (for the cases where the grading monoid is Z or Z_n) one uses homomorphisms of different degrees, then one gets a sort of quotient category or, alternatively, one can view the investigations in [2], [16] and [17] as being concerned with radical classes satisfying extra conditions — closure under isomorphisms of all degrees. With these provisos we can view Theorem 4 of Divinsky, Suliński and Anderson [2] as a special case of Proposition 3.7.

As a final example we consider normal families for modules over various rings (Example (5)). The associated multioperator groups here are rings and abelian groups. Let $\{R, M\}$ be an R-module. For any radical $\mathcal{R}$ of rings and radical $\mathcal{U}$ of abelian groups, we have $\mathcal{R}(R)\mathcal{U}(M) \subseteqq \mathcal{U}(M)$, as $\mathcal{U}(M)$ is a fully invariant subgroup. Thus $\{\mathcal{R}, \mathcal{U}\}$ is a normal family if and only if $\mathcal{R}(R)M \subseteqq \mathcal{U}(M)$ for all R and M. Thus, if all rings in $\mathcal{R}$ are additively in $\mathcal{U}$, then $\{\mathcal{R}, \mathcal{U}\}$ is normal. On the other hand, if $\{\mathcal{R}, \mathcal{U}\}$ is normal, let A be in $\mathcal{R}$ and let A^1 denote the standard unital extension of A. Then in an obvious way $\{A, A^1\}$ is an A-module, so

$$A = AA^1 = \mathcal{R}(A)A^1 \subseteqq \mathcal{U}(A^1).$$

Now as groups we have $\mathcal{U}(A^1)/A \subseteqq A^1/A \cong Z$, with $\mathcal{U}(A^1)/A \in \mathcal{U}$. If $\mathcal{U}(A^1)/A \cong Z$, then $\mathcal{U}$ contains all abelian groups, so $\mathcal{R}$-rings are certainly addively in $\mathcal{U}$, while if $\mathcal{U}(A^1)/A = 0$, we have $A = \mathcal{U}(A^1) \in \mathcal{U}$.

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ON THE ASYMPTOTIC REGULARITY OF NONEXPANSIVE MAPPINGS

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0. Introduction

Given a normed space X, a self-mapping P of X is said to be asymptotically regular if for every point x in X we have $\|P^{n+1}x-P^nx\|\to 0$ as $n\to\infty$.

The notion of asymptotic regularity plays an important role in the theory of nonexpansive mappings, especially in the study of the convergence of algorithms for computing fixed points of such mappings (see e.g. [2], [5], [7] and [8]). The first classic results on the asymptotic regularity of nonexpansive mappings can be resumed in the following theorem:

THEOREM A ([2]). *Let X be a uniformly convex Banach space and T be a nonexpansive self-mapping of X with a nonempty set F of fixed points. For a given constant t with $0<t<1$ let*

$$P=(1-t)I+tT$$

where I is the identity mapping of X. Then P is asymptotically regular.

In [4], S. Ishikawa pointed out that the theorem can be proved without the assumptions on uniform convexity and completeness, even in a much stronger variant:

THEOREM B ([4]). *Let D be a subset of a normed space X and $T\colon D\to X$ be a nonexpansive mapping. Given a sequence (x_n) in D and a sequence (t_n) of real numbers satisfying*

(i) $0\leqq t_n\leqq t<1$ *and* $\sum_{1}^{\infty} t_n=\infty$,

(ii) $x_{n+1}=(1-t_n)x_n+t_nTx_n$ *for* $n=1,2,\dots,$

if (x_n) is bounded then $\|Tx_n-x_n\|\to 0$ as $n\to\infty$.

The aim of this paper is to present some results concerning the notion of asymptotic regularity for nonexpansive mappings. First we give a symple proof to the theorem of S. Ishikawa. Then motivated by the fact that the identity mapping itself is nonexpansive and asymptotically regular, a sufficient condition is given under which the identity mapping can be replaced by such mappings. Finally we extend Theorem A to multi-valued mappings which are nonexpansive in the sense defined by S. Nadler (see [6]).

The paper consists of three sections. In the first one we prove an elementary lemma on which the proofs of all of our results are based. The following two sections present our main theorems concerning the asymptotic regularity of single- and multi-valued mappings.

1. Preliminaries

LEMMA 1. *Let X be a normed space and (a_n) and (b_n) be sequences in X. If there is a sequence (t_n) of real numbers satisfying*

(i) $\|a_n\| \to r$ *as* $n \to \infty$,

(ii) $\limsup_{n\to\infty} \|b_n\| \leqq r$ *and* $\left(\sum_{i=1}^{n} t_i b_i\right)_n$ *is bounded,*

(iii) $0 \leqq t_n \leqq t < 1$ *and* $\sum_{1}^{\infty} t_n = \infty$,

(iv) $a_{n+1} = (1-t_n) a_n + t_n b_n$,

then $r=0$.

PROOF. Suppose that $r>0$ and let

$$M = \sup\left\{\left\|\sum_{i=1}^{n} t_i b_i\right\| : n = 1, 2, 3, \ldots\right\}.$$

Choose a number $N>\max\{2M/r, 1\}$ and then choose a positive ε such that

$$1-2\varepsilon \exp\frac{N+1}{1-t} > \frac{1}{2}.$$

From (iii) it follows that there exists a natural k such that

$$N < \sum_{i=1}^{k} t_i < N+1.$$

Since $\|a_n\| \to r$ and $\limsup \|b_n\| \leqq r$, without loss of generality we may assume that for every n

$$r(1-\varepsilon) < \|a_n\| < r(1+\varepsilon) \quad \text{and} \quad \|b_n\| < r(1+\varepsilon).$$

Now taking $s_n = 1-t_n$, upon simple computation one obtains from (iv)

$$a_{k+1} = s_1 s_2 \ldots s_k a_1 + t_1 s_2 s_3 \ldots s_k b_1 + \ldots + t_{k-1} s_k b_{k-1} + t_k b_k,$$

$$a_{k+1} \in B := \operatorname{co}\{a_1, b_1, b_2, \ldots, b_k\}.$$

Let

$$x := T^{-1} \sum_{i=1}^{k} t_i b_i \quad \text{where} \quad T = \sum_{i=1}^{k} t_i \quad (\text{so } N+1 > T > N > 1)$$

and

$$y = (1-S)^{-1} a_{k+1} - S(1-S)^{-1} x \quad \text{where} \quad S = \prod_{i=1}^{k} s_i.$$

Then it is clear that $x \in B$, $a_{k+1} = Sx + (1-S)y$ and y belongs to the affine hull of a_{k+1} and x, and so to the affine hull of $\{a_1, b_1, b_2, \ldots, b_k\}$. Moreover one can write

$$y = S(1-S)^{-1}\{a_1 + t_1(s_1^{-1} - T^{-1}) b_1 + t_2(s_1^{-1} s_2^{-1} - T^{-1}) b_2 + \ldots + t_k(S^{-1} - T^{-1}) b_k\}.$$

Since $0 \leqq s_i \leqq 1$ and $T > 1$, all the coefficients of $a_1, b_1, \ldots, b_k$ are nonnegative. Thus the affine combination on the right hand side is also convex and then $y \in B$. Therefore

$$\|y\| \leqq \max \{\|a_1\|, \|b_1\|, \ldots, \|b_k\|\} < r(1+\varepsilon).$$

Hence

$$r(1-\varepsilon) < \|a_{k+1}\| \leqq S\|x\| + (1-S)\|y\| \leqq S\|x\| + (1-S)r(1+\varepsilon)$$

which implies

$$\|x\| > r(1-S^{-1}(2-S)\varepsilon) > r(1-S^{-1}2\varepsilon) = r\big(1-2\varepsilon \prod_{i=1}^{k} (1-t_i)^{-1}\big).$$

On the other hand $\|x\| = T^{-1}\big\|\sum_{i=1}^{k} t_i b_i\big\| \leqq T^{-1}M$. Now, since $\log(1+u) \leqq u$ for $-1 < u < \infty$, as in [4] we have

$$M \geqq rT\big(1-2\varepsilon \prod_{i=1}^{k} (1-t_i)^{-1}\big) = rT\big(1-2\varepsilon \exp \sum_{i=1}^{k} \log(1+t_i(1-t_i)^{-1})\big) \geqq$$

$$\geqq rT\big(1-2\varepsilon \exp \sum_{i=1}^{k} t_i(1-t_i)^{-1}\big) \geqq rT\big(1-2\varepsilon \exp T(1-t)^{-1}\big) \geqq$$

$$\geqq rN\big(1-2\varepsilon \exp(N+1)(1-t)^{-1}\big) > rN\frac{1}{2} > M,$$

a contradiction.

COROLLARY 1. *Let X be a normed vector space and (a_n) and (b_n) be sequences in X. If for a constant t with $0 < t < 1$ we have*

(i) $\|a_n\| \to r$ *as* $n \to \infty$,

(ii) $\limsup \|b_n\| \leqq r$ *and* $\sum_{1}^{\infty} b_n$ *is bounded,*

(iii) $a_{n+1} = (1-t)a_n - tb_n$,

then $r = 0$.

2. Asymptotic regularity of single-valued mappings

We begin this section by giving a simple proof to Theorem B.

PROOF OF THEOREM B. Setting $a_n = Tx_n - x_n$ and $b_n = t_n^{-1}(Tx_{n+1} - Tx_n)$ we have $a_{n+1} = (1-t_n)a_n + t_n b_n$ and $\|b_n\| = t_n^{-1}\|Tx_{n+1} - Tx_n\| \leqq t_n^{-1}\|x_{n+1} - x_n\| = \|a_n\|$. Then $\|a_{n+1}\| \leqq (1-t_n)\|a_n\| + t_n\|b_n\| \leqq \|a_n\|$ and the nonincreasing sequence $(\|a_n\|)$ will converge to some $r \geqq 0$. Clearly $\limsup_{n\to\infty} \|b_n\| \leqq \lim_{n\to\infty} \|a_n\| = r$. Finally we have

$$\big\|\sum_{i=1}^{n} t_i b_i\big\| = \big\|\sum_{i=1}^{n} (Tx_{i+1} - Tx_i)\big\| = \|Tx_{n+1} - Tx_1\| \leqq \|x_{n+1} - x_1\|$$

which is bounded. Hence Lemma 1 applies and the theorem is proved.

The main result of this section is the following

THEOREM 1. *Let D be a convex and bounded subset of a normed space X and S and T be nonexpansive self-mappings of D. For a constant t with $0<t<1$ let $P=(1-t)S+tT$ and suppose that*

(i) *S is asymptotically regular,*

(ii) *$P(Sx)=S(Px)$ for every $x\in D$.*
Then P is asymptotically regular.

PROOF. For an arbitrary $x_1\in D$ put $x_{n+1}=Px_n$ for $n\geqq 1$ and let

$$a_n = x_{n+1}-x_n = Px_n-Px_{n-1},\quad b_n = Tx_{n+1}-Tx_n,$$

$$c_n = Sx_{n+1}-Sx_n,\quad e_n = Sx_n-x_n.$$

S and T are nonexpansive, so is their convex combination P, hence the nonincreasing sequence $(\|a_n\|)$ converges to some $r\geqq 0$ as $n\to\infty$.

One can write

$$a_{n+1} = (1-t)c_n+tb_n \quad\text{and}\quad c_n = a_n+e_{n+1}-e_n.$$

Then

$$a_{n+1} = (1-t)a_n+tb_n+(1-t)(e_{n+1}-e_n) = (1-t)a_n+tb_n'$$

where

$$b_n' = b_n+t^{-1}(1-t)(e_{n+1}-e_n).$$

Let us accept for a moment the following lemma.

LEMMA 2. *Under the hypotheses of Theorem 2 we have $\|e_n\|\to 0$ as $n\to\infty$.*

Then

$$\limsup_{n\to\infty}\|b_n'\| \leqq \limsup_{n\to\infty}\left(\|b_n\|+t^{-1}(1-t)\|e_{n+1}-e_n\|\right) \leqq$$

$$\leqq \lim_{n\to\infty}\left(\|a_n\|+t^{-1}(1-t)\|e_{n+1}-e_n\|\right) = r$$

and

$$\left\|\sum_{i=1}^{n} b_i'\right\| = \left\|\sum_{i=1}^{n} b_i+t^{-1}(1-t)\sum_{i=1}^{n}(e_{i+1}-e_i)\right\| \leqq \|Tx_{n+1}-Tx_1\|+t^{-1}(1-t)\|e_{n+1}-e_1\|$$

which is bounded since D is bounded, hence Lemma 1 implies Theorem 1.

PROOF OF LEMMA 2. Let $\alpha_n^m=\|S^{m+1}x_n-S^mx_n\|$. We have to prove $\alpha_n^0\to 0$ as $n\to\infty$. First notice that

$$\alpha_{n+1}^m = \|S^{m+1}Px_n-S^mPx_n\| = \|PS^{m+1}x_n-PS^mx_n\| \leqq (1-t)\alpha_n^{m-1}+t\alpha_n^m.$$

By simple induction on n one obtains

$$\alpha_n^m \leqq \sum_{i=0}^{n}\binom{n}{i}(1-t)^i t^{n-i}\alpha_0^{m+i}.$$

In particular, for $m=0$ we have

$$\alpha_n^0 \leqq \sum_{i=0}^{n}\binom{n}{i}(1-t)^i t^{n-i}\alpha_0^i.$$

But S being asymptotically regular, $\alpha_0^i \to 0$ as $i \to \infty$. Thus it follows from the classical Silverman—Toeplitz' theorem (cf. [3], p. 75) that $\alpha_n^0 \to 0$ as $n \to \infty$.

REMARK. The identity mapping I of X is clearly an asymptotically regular and nonexpansive mapping which commutes with every self-mapping of X. Hence Theorem A follows immediately from Theorem 1.

3. Asymptotic regularity of multi-valued mappings

In this section we extend the notion of asymptotic regularity for multi-valued mappings and sequences of multi-valued mappings.

DEFINITION. Let D be a subset of a normed space X. A sequence of multi-valued mappings P_n from D into X is called asymptotically regular if for every x_1 in D there exists a sequence (x_n) in X satisfying

(i) $x_{n+1} \in P_n x_n$ for every $n \geqq 1$,

(ii) $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

A multi-valued mapping P is asymptotically regular if and only if the constant sequence $P_n = P$ is asymptotically regular. When P reduces to a single-valued mapping one obtains the classical notion of asymptotic regularity.

NOTATION. For a subset D of a normed space X we denote by $CB(D)$ the set of all closed and bounded subsets of D and provide $CB(D)$ with the Hausdorff metric

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$.

Recall that a multi-valued mapping $P\colon D \to CB(D)$ is called nonexpansive is for every x and y in D we have

$$H(Px, Py) \leqq \|x - y\|.$$

(This notion was introduced by S. B. Nadler in [6] under the name 'multi-valued contraction'.)

The aim of this section is to prove the following generalization of Theorem B for multi-valued mappings.

THEOREM 2. *Let D be a convex and bounded subset of a normed space X and $T\colon D \to CB(D)$ be a nonexpansive mapping. For a sequence (t_n) of real numbers satisfying $0 < t_n \leqq t < 1$ and $\sum_1^\infty t_n < \infty$ let $P_n = (1 - t_n) I + t_n T$. Then (P_n) is asymptotically regular.*

In order to prove the theorem we require an elementary lemma.

LEMMA 3. *Given a sequence* (ε_n) *of positive numbers with* $\sum_1^\infty \varepsilon_n<\infty$, *let* (α_n) *be a sequence of real numbers which is bounded from below and satisfies* $\alpha_{n+1}\leqq\alpha_n+\varepsilon_n$. *Then* (α_n) *converges.*

PROOF. There exists a subsequence (α_{n_k}) converging to α where $\alpha=\liminf_{n\to\infty}\alpha_n$. For any given positive ε one can find an integer N such that $\sum_{i=N}^\infty \varepsilon_i<\varepsilon$. Now since $\alpha_{n_k}\leqq\alpha$, for every $n\geqq n_{k_0}$ we have

$$\alpha_n \leqq \alpha_{n_{k_0}}+\sum_{i=n_{k_0}}^{n-1}\varepsilon_i \leqq \alpha+\varepsilon$$

which implies $\limsup_{n\to\infty}\alpha_n\leqq\alpha+\varepsilon$ for an arbitrary positive ε, that is $\limsup_{n\to\infty}\alpha_n\leqq\alpha=$ $=\liminf_{n\to\infty}\alpha_n$ or $\lim_{n\to\infty}\alpha_n=\alpha$.

PROOF OF THEOREM 2. Let (ε_n) be a sequence of positive numbers with $\sum_{n=1}^\infty \varepsilon_n<\infty$ and $t_n^{-1}\varepsilon_n\to0$ as $n\to\infty$. For x_1 in D we construct a sequence (x_n) as follows:

Choosing y_1 in Tx_1, we put $x_2=(1-t_1)x_1+t_1y_1\in P_1x_1$. Suppose that x_{n-1}, x_n and y_{n-1} have been constructed so that $y_{n-1}\in Tx_{n-1}$ and $x_n=(1-t_{n-1})x_{n-1}+$ $+t_{n-1}y_{n-1}\in P_{n-1}x_{n-1}$. Then we choose y_n in Tx_n satisfying

$$\|y_n-y_{n-1}\| \leqq d(y_{n-1}, Tx_n)+\varepsilon_{n-1}$$

and put $x_{n+1}=(1-t_n)x_n+t_ny_n\in P_nx_n$. So the construction can go on iteratively.

Now since $y_{n-1}\in Tx_{n-1}$, it follows from nonexpansiveness of T that

$$\|y_n-y_{n-1}\| \leqq d(y_{n-1}, Tx_n)+\varepsilon_{n-1} \leqq H(Tx_{n-1}, Tx_n)+\varepsilon_{n-1} \leqq \|x_n-x_{n-1}\|+\varepsilon_{n-1}.$$

Setting $a_n=y_n-x_n$ and $b_n=t_n^{-1}(y_{n+1}-y_n)$ we have

$$a_{n+1} = (1-t_n)a_n+t_nb_n$$

and

$$\|b_n\| = t_n^{-1}\|y_{n+1}-y_n\| \leqq t_n^{-1}\|x_{n+1}-x_n\|+t_n^{-1}\varepsilon_n = \|a_n\|+t_n^{-1}\varepsilon_n.$$

Then

$$\|a_{n+1}\| \leqq (1-t_n)\|a_n\|+t_n\|b_n\| \leqq \|a_n\|+\varepsilon_n.$$

Since $\sum_{n=1}^\infty \varepsilon_n<\infty$ it follows from Lemma 3 that $\|a_n\|$ converges to some r as $n\to\infty$. On the other hand since $t_n^{-1}\varepsilon_n\to0$ as $n\to\infty$ we have

$$\limsup_{n\to\infty}\|b_n\| \leqq \lim_{n\to\infty}(\|a_n\|+t_n^{-1}\varepsilon_n) = r.$$

Finally $\sum_{i=1}^n t_ib_i=y_{n+1}-y_1$ is bounded as D is and Lemma 1 applies.

COROLLARY 2. *Let* D *be a convex and bounded subset of a normed space* X *and* T: $D\to CB(D)$ *be a nonexpansive (multi-valued) mapping. For a constant* t *with* $0<t<1$ *let* $P=(1-t)I+tT$. *Then* P *is asymptotically regular.*

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APPLICATION OF THE OPERATIONAL CALCULUS IN SOLVING PARTIAL DIFFERENCE EQUATIONS

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We are given an operational calculus $CO(L^0, L^1, S, T(q), s(q), Q)$, where L^0, L^1 are real linear spaces, the linear operations S, $T(q)$ and $s(q)$ are called derivative, integral and limit condition, resp. (Axioms, properties of operational calculus etc. see [1], [2].)

§1. Scalar product. Examples

Given the assumptions

(a) $L^1 \subset L^0$,
(b) L^1, L^0 are commutative algebras over the field of real numbers,
(c) L^0 is a Mikusiński's space such that $\bigwedge_{x\in L^0} x^2 \geqq 0$,
(d) $[T(q_0)-T(q_1)]$ is a non-negative operation,
(e) $[T(q_0)-T(q_1)]x^2=0$ implies $x=0$ for $x\in L^0$,

we may introduce the product

$$\langle x, y\rangle = [T(q_0)-T(q_1)](xy) \tag{1}$$

of the elements $x, y\in L^0$. This product satisfies the following properties:

$$\langle x, y\rangle \in \operatorname{Ker} S, \quad \langle x, y\rangle = \langle y, x\rangle, \quad \langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \quad \alpha\in R,$$

$$\langle x+y, z\rangle = \langle x, z\rangle+\langle y, z\rangle, \quad \langle x, x\rangle \geqq 0 \quad \text{and} \quad \langle x, x\rangle \neq 0 \quad \text{for} \quad x\neq 0, \quad \langle 0, 0\rangle = 0.$$

So we will call it the scalar product of the elements $x, y\in L^0$.

EXAMPLES. 1. $L^0=C^0([0, a], R)$, $L^1=C^2([0, a], R)$. Introduce an operational calculus with the derivative $S=d^2/dt^2$, integral

$$T(t_0)\{f(t)\} = \Big\{\int_{t_0}^{t} (t-\tau)f(\tau)\,d\tau\Big\}, \quad f\in L^0,\ t, t_0\in[0, a]$$

and limit condition

$$s(t_0)\{u(t)\} = \{u'(t_0)t+u(t_0)\}, \quad u\in L^1,\ t, t_0\in[0, a].$$

Define the order in L^0 by the formula $\{f(t)\}=f\geqq 0 \Leftrightarrow f(t)\geqq 0$ for $t\in[0, a]$. Then

$$[T(0)-T(a)]\{f(t)\} = \Big\{\Big(\int_0^a f(\tau)\,d\tau\Big)t+\int_0^a (a-\tau)f(\tau)\,d\tau\Big\}$$

is a non-negative operation for $t\in[0, a]$. The scalar product has the form

$$\langle x, y\rangle = \Big\{\Big(\int_0^a x(\tau)y(\tau)\,d\tau\Big)t+\int_0^a (a-\tau)x(\tau)y(\tau)\,d\tau\Big\}, \quad t\in[0, a].$$

REMARK. The elements x and y are orthogonal if and only if

$$\int_0^a x(t)y(t)\,dt = 0 \quad \text{and} \quad \int_0^a (a-\tau)x(\tau)y(\tau)\,d\tau = 0$$

i.e. when x and y are orthogonal with respect to the weights $\varrho=\{1\}$, $t\in[0, a]$ and $\varrho=\{\varrho(t)\}=a-t\}$, $t\in[0, a]$.

$\{x(t)\}=\{t\}$ and $\{y(t)\}=\{29t^{27}-5t^3-4t^2+2\}$ are orthogonal with respect to the scalar product introduced in Example 1 with $a=1$.

2. Let us consider the derivative $S=\partial/\partial x$, the integral

$$T(x_0)\{f(x, y)\} = \Big\{\int_{x_0}^{x} f(\tau, y)\,d\tau\Big\}$$

and the limit condition $s(x_0)=1$. We accept here $L^0=C^0([x_0, x_1]\times[y_0, y_1], R)$, $L^1=C([x_0, x_1]\times[y_0, y_1], R)$ and the order in L^0 defined by

$$\{f(x, y)\}=f\geqq 0 \Leftrightarrow f(x, y)\geqq 0 \quad \text{for} \quad (x, y)\in[x_0, x_1]\times[y_0, y_1].$$

We may easily notice that the operation $[T(x_0)-T(x_1)]$ is non-negative and it satisfies the assumption (e). In this example the scalar product has the form

$$\langle f, g\rangle = \Big\{\int_{x_0}^{x_1} f(\tau, y)g(\tau, y)\,d\tau\Big\} = \{\varphi(y)\}\in\operatorname{Ker}\frac{\partial}{\partial x}.$$

We see from the last formula that the surfaces $z_1=\{f(x, y)\}$ and $z_2=\{g(x, y)\}$ are orthogonal if and only if the respective intersections of these surfaces with the plane $y=y_\alpha$ are orthogonal (y_α is arbitrary belonging to the interval $[y_0, y_1]$).

3. In case of the operational calculus with the derivative

$$S\{u(x_1, x_2, \dots, x_n)\} = \Big\{\sum_{i=1}^{n} b_i\,\frac{\partial u(x_1, x_2, \dots, x_n)}{\partial x_i}\Big\}, \quad b_i\in R, \quad i = 1, 2, \dots, n$$

the integral

$$T(x_n^0)\{f(x_1, x_2, \dots, x_n)\} =$$

$$= \Big\{\frac{1}{b_n}\int_{x_n^0}^{x_n} f\Big(x_1-\frac{b_1}{b_n}(x_n-\tau), x_2-\frac{b_2}{b_n}(x_n-\tau), \dots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau), \tau\Big)\,d\tau\Big\}$$

and the limit condition

$$s(x_n^0)\{u(x_1, x_2, \dots, x_n)\} =$$

$$= \Big\{u\Big(x_1-\frac{b_1}{b_n}(x_n-x_n^0), x_2-\frac{b_2}{b_n}(x_n-x_n^0), \dots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-x_n^0), x_n^0\Big)\Big\}$$

where $u\in L^1=C^2(R^{n-1}\times[x_n^1, x_n^2], R)$, $f\in L^0=C^1(R^{n-1}\times[x_n^1, x_n^2], R)$, $x_n^0\in[x_n^1, x_n^2]$, $b_n>0$ (see [4]). The scalar product defined by (1) has the form

$$\langle f, g\rangle = [T(x_n^1)-T(x_n^2)](fg) =$$

$$= \left\{\frac{1}{b_n}\int\limits_{x_n^1}^{x_n^2} f\left(x_1-\frac{b_1}{b_n}(x_n-\tau), \dots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) g\left(x_1-\frac{b_1}{b_n}(x_n-\tau), \dots, x_{n-1}-\right.\right.$$

$$\left.\left.-\frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau\right\}.$$

In the space $C^0(R^{n-1}\times[x_n^1, x_n^2], R)$ we may introduce the order

$$\{f(x_1, x_2, \dots, x_n)\} = f \geqq 0 \Leftrightarrow f(x_1, x_2, \dots, x_n) \geqq 0$$

for $(x_1, x_2, \dots, x_n)\in R^{n-1}\times[x_n^1, x_n^2]$. With this order, the operation $[T(x_n^1)-T(x_n^2)]$ is non-negative (see [7]), and it satisfies (e).

4. Let $L^0=L^1=C(N)$ be the space of sequences of real numbers $x=\{x_k\}$, $k=0, 1, 2, \dots$. The derivative S, the integral $T(k_0)$ and the limit condition $s(k_0)$ are defined by

$$S\{x_k\} \overset{\text{df}}{=} \{x_{k+1}-x_k\},$$

$$T(k_0)\{x_k\} \overset{\text{df}}{=} \begin{cases} 0 \quad \text{for} \quad k = k_0 \\ x_{k_0}+x_{k_0+1}+\dots+x_{k-1} & \text{for} \quad k_p < k \\ -x_{k_0-1}-x_{k_0-2}-\dots-x_k & \text{for} \quad k_0 > k, \end{cases}$$

$$s(k_0)\{x_k\} \overset{\text{df}}{=} \{x_{k_0}\}.$$

By the order $x\geqq 0$ if and only if $x_k\geqq 0$ for $k=0, 1, 2, \dots$ and the modulus $|x| \overset{\text{df}}{=} \{|x_k|\}$, $C(N)$ is Mikusiński's space.

For $k_0<k_1$, $[T(k_0+1)-T(k_1+1)]$ is a non-negative operation (see [5]). For condition (e) to be satisfied we must consider sequences in which $x_{k_0+1}=x_{k_1+1}=0$ and restrict ourselves to considering only a part of a sequence with coordinates $x_{k_0+2}, x_{k_0+3}, \dots, x_{k_1}$. Introduce the following relation of equality of sequences:

$$\{x_k\} \overset{\text{df}}{=} \{y_k\} \Leftrightarrow x_k = y_k \quad \text{for} \quad k = k_0+2, \dots, k_1.$$

Now we are ready to define the scalar product of the elements $\{x_k\}$, $\{y_k\}$:

$$\langle\{x_k\}, \{y_k\}\rangle = [T(k_0+1)-T(k_1+1)]\{x_k y_k\}.$$

In a similar way we may define the scalar product of the elements $\{x_k\}$, $\{y_k\}$ with weight $\varrho=\{\varrho_k\}$, $\varrho_k>0$ for $k=k_0+2, k_0+3, \dots, k_1$:

$$\langle\{x_k\}, \{y_k\}\rangle = [T(k_0+1)-T(k_0+1)]\{\varrho_k x_k y_k\}.$$

§2. Properties of the Wrońskian. Examples

Let the assumptions (a) and (b) from §1 be satisfied.

DEFINITION.

$$W(u, v) \overset{\text{df}}{=} \begin{vmatrix} u & v \\ Su & Sv \end{vmatrix} = u(Sv) - v(Su)$$

will be called the Wrońskian of a given system of elements $u, v \in L^1$.

Additionally, let us assume that L^1, L^0 are commutative algebras with unity e over the field of real numbers with the multiplication $f \cdot g$ such that

$$(2) \qquad S(fg) = (Sf)g + f(Sg), \quad f, g \in L^1.$$

The properties of such a multiplication are listed in [3].

LEMMA. *If* $u, v \in L^1$ *and* v *has the inverse* v^{-1} *then* $S(uv^{-1}) = v^{-2}[(Su)v - u(Sv)]$.

THEOREM 1. *If* $u = cv$, $c \in \operatorname{Ker} S$, $v \in L^1$ *then* $W(u, v) = 0$.

Proofs are implied by properties of the derivative S.

THEOREM 2. *If* $W(u, v) = 0$, $u, v \in L^1$ *and if* u *has an inverse* u^{-1} *(if* v *has an inverse* v^{-1}*) then* $v = cu$, $c \in \operatorname{Ker} S$ $(u = c_1 v$, $c_1 \in \operatorname{Ker} S)$.

PROOF. From the assumption it follows that

$$W(u, v) = \begin{vmatrix} u & v \\ Su & Sv \end{vmatrix} = (Sv)u - v(Su) = 0.$$

As u has an inverse u^{-1}, so multiplying the last equality by u^{-2} we get

$$u^{-2}[u(Sv) - v(Su)] = 0.$$

On the basis of the last theorem we have

$$S(vu^{-1}) = 0, \quad \text{i.e.} \quad vu^{-1} = c \in \operatorname{Ker} S,$$

so $v = cu$, $c \in \operatorname{Ker} S$. Analogously, we prove the theorem if v has an inverse.

REMARK. We should realize the fact that in the last two theorems the coefficient of proportionality c may be a real number α. It is only an apparent real number because $u = \alpha v = \alpha e v = cv$, then it is also a coefficient $c = \alpha e$ of Ker S.

EXAMPLES. 5. For the derivative $S = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$ a specific case of the operational calculus from Example 3, from the condition $W(u, v) = 0$, $u, v \in L^1$, $u(x, y) \neq 0$ for $(x, y) \in R \times [y_1, y_2]$ on the basis of Theorem 4, $\{v(x, y)\}$ may be given in the form of the product of $\varphi(x-y)$ and $\{u(x, y)\}$. φ is of the class C^1.

6. In the space $C(N)$ of real sequences $a = \{a_k\}$, let us introduce the derivative S by

$$S\{a_k\} = \{a_{k+1}\}.$$

Introducing in $C(N)$ a multiplication of the sequences $a=\{a_k\}$, $b=\{b_k\}$ according to the formula

$$a*b=\left\{\sum_{i=0}^{k}\binom{k}{i}a_i b_{k-i}\right\},$$

we may prove that the condition

$$S(a*b)=(Sa)*b+a*(Sb)$$

is satisfied (see [1], p. 19). In this case $W(\{u_k\},\{v_k\})=0$ and $u_0\neq 0$ mean that the sequence $\{v_k\}$ can be represented in the form of a product of the sequences $\{u_k\}=(u_0, u_1, u_2, \dots)$ and $\{c_k\}=(c_0, 0, 0, 0, \dots)$, i.e. $\{v_k\}=\{c_k\}*\{u_k\}$.

7. In $C(N)$ the difference derivative S defined by the formula

$$S\{a_k\}\overset{\text{df}}{=}\{a_{k+1}-a_k\}$$

does not satisfy Condition 3, given the following multiplication of sequences:

$$\{a_k\}\{b_k\}\overset{\text{df}}{=}\{a_k b_k\}.$$

S satisfies the condition

$$S(\{a_k\}\{b_k\})=\{a_{k+1}\}(S\{b_k\})+\{b_k\}(S\{a_k\})$$

(see [5]) which is similar to Condition (2).

With this derivative, Theorem 2 is also true because if

$$\begin{vmatrix}\{u_k\} & \{v_k\}\\ S\{u_k\} & S\{v_k\}\end{vmatrix}=\{u_k\}\{v_{k+1}-v_k\}-\{v_k\}\{u_{k+1}-u_k\}=0$$

and $v_k\neq 0$ for $k=0, 1, 2, \dots$ then $u_k v_{k+1}=v_k u_{k+1}$ and $v_k\neq 0$ for $k=0, 1, 2, \dots$ i.e.

$$\frac{u_k}{v_k}=\frac{u_{k+1}}{v_{k+1}} \quad \text{for} \quad k=0, 1, 2, \dots, \quad \{u_k\}=\{\alpha v_k\}=\alpha\{v_k\}.$$

COROLLARY. *The Wrońskian of the elements* $\{u_k\}$, $\{v_k\}$ *equals zero if and only if the Casorati's determinant of the elements* $\{u_k\}$, $\{v_k\}$ *equals zero.*

REMARK. The assumption $v_k\neq 0$ for $k=0, 1, 2, \dots$ is essential. Notice that for the sequences $\{u_k\}=(u_0, u_1, \theta, u_3, u_4, u_5, \dots)$ and $\{v_k\}=(\alpha u_0, \alpha u_1, \theta, \beta u_3, \beta u_4, \beta u_5, \dots)$, $\alpha\neq\beta, \alpha, \beta\in R$ proportional by parts, we have $W(\{u_k\},\{v_k\})=0$.

§3. Eigenvalues and eigenelements of the difference equation

We are given the operational calculus $CO(L^0, L^1, S, T(k_0), s(k_0), Q=N)$ where $L^0=L^1=C(N)$ is the space of sequences of real numbers $x=\{x_k\}$, $k=0, 1, 2, \dots$. S, $T(k_0)$ and $s(k_0)$ are defined by

$$S\{x_k\}\overset{\text{df}}{=}\{x_{k+1}-x_k\},$$

$$T(k_0)\{x_k\}\overset{\text{df}}{=}\begin{cases}0 & \text{for} \quad k=k_0\\ x_{k_0}+x_{k_0+1}+\dots+x_{k-1} & \text{for} \quad k_0<k\\ -x_{k_0-1}-x_{k_0-2}-\dots-x_k & \text{for} \quad k_0>k,\end{cases}$$

$$s(k_0)\{x_k\}\overset{\text{df}}{=}\{x_{k_0}\}.$$

REMARK. Instead of $\{x_k\}$ we will often write x_k for clarity.

In $C(N)$ an order like in Example 6 is introduced.

Let us consider the difference equation

$$S(r_k Sw_k) = -\lambda p_k w_{k+1} \tag{3}$$

with homogeneous conditions

$$s(k_0) w_{k+1} = 0 \quad \text{i.e.} \quad s(k_0+1) w_k = 0, \tag{4}$$

$$s(k_1) w_{k+1} = 0 \quad \text{i.e.} \quad s(k_1+1) w_k = 0, \tag{5}$$

where $p_k, r_k, w_k \in C(N)$, $\lambda \in R$, $p_k, r_k \neq 0$ for $k=0, 1, 2, \dots$. Additionally we assume that $k_0 \neq k_1$, $k_1 \neq k_0+1$, $k_1 \neq k_0-1$ (see Corollary 1 in [5]).

THEOREM 3. *If $\bar{w}_k$ is an eigenelement corresponding to the eigenvalue $\bar{\lambda}$ (i.e. $\bar{w}_k$ is a solution of the equation* (3) *with condition* (4), (5) *for $\lambda = \bar{\lambda}$), then $\alpha \bar{w}_k$, $\alpha \in R - \{0\}$ is also an eigenelement corresponding to $\bar{\lambda}$.*

THEOREM 4. *If $w_k \neq \bar{w}_k$ and w_k, $\bar{w}_k$ are the eigenelements corresponding to the different eigenvalues $\lambda, \bar{\lambda}$, then w_k and $\bar{w}_k$ satisfy the condition*

$$[T(k_0) - T(k_1)] \{w_{k+1} \bar{w}_{k+1} p_k\} = 0.$$

THEOREM 5. *If $p_k > 0$, $r_k > 0$, $k=0, 1, 2, \dots$ and $k_0 < k_1$ then the eigenvalues of the difference equation* (3) *with conditions* (4), (5) *are non-negative.*

THEOREM 6. *$\lambda = 0$ is not an eigenvalue of the difference equation* (3) *with conditions* (4), (5) *when $r_k > 0$, $k=0, 1, 2, \dots$ or $r_k < 0$, $k=0, 1, 2, \dots$.*

The proofs of Theorems 3, 4, 5 and 6 can be found in [5].

THEOREM 7. *If u_k and v_k are eigenelements of the equation* (3) *with conditions* (4) *and* (5) *corresponding to the same eigenvalue λ, then $v_k = a u_k$, $a \in R$, $a \neq 0$.*

PROOF. Because u_k, v_k are the eigenelements corresponding to the eigenvalue λ,

$$v_{k+1} S(r_k Su_k) - u_{k+1} S(r_k Sv_k) = 0.$$

Operating bilaterally on the last equality with the integral $T(k_0+1)$ and applying (1.5) from [5] and conditions (8) and (11) we get

$$T(k_0+1)\big(v_{k+1} S(r_k Su_k)\big) - T(k_0+1)\big(u_{k+1} S(r_k Sv_k)\big) = v_k r_k Su_k - u_k r_k Sv_k = 0,$$

i.e.

$$v_k Su_k - u_k Sv_k = 0, \quad \text{or} \quad W(v_k, u_k) = 0. \tag{6}$$

From conditions (4), (5) it follows that in the sequences u_k and v_k there appear zeros. From the fact that $W(v_k, u_k)=0$ and from the remark in Example 7 it follows that it should be checked whether the proportionality coefficients for the fragments of sequences with the elements other than zero and separated by zeros are equal. Let $u_{k_2+1}=0$ for $k_2 \neq k_0-1,\ k_0+1, k_1-1, k_1+1$. $u_{k_2}, u_{k_2+2} \neq 0$ on the basis of Corollary 1 from [5]. From (6), $v_{k_2+1}=0$. v_k is an eigenelement, so by Corollary 1 from [5], $v_{k_2}, v_{k_2+2} \neq 0$, i.e. $v_{k_2}=\alpha u_{k_2}$, $v_{k_2+2}=\beta u_{k_2+2}$, $\alpha, \beta \in R$, $\alpha, \beta \neq 0$ on the basis of the Wrońskian.

From (3) for $k=k_2$ we have

$$r_{k_2+1}u_{k_2+2}+r_{k_2}u_{k_2}=0, \quad \text{i.e.} \quad u_{k_2+2}=-\frac{r_{k_2}u_{k_2}}{r_{k_2+1}}.$$

Similarly, on the basis of (3) for $k=k_2$ we have

$$r_{k_2+1}v_{k_2+2}+r_{k_2}v_{k_2}=0.$$

Substituting into the last equation $v_{k_2}=\alpha u_{k_2}$ and $v_{k_2+2}=\beta u_{k_2+2}=-\beta \frac{r_{k_2}u_{k_2}}{r_{k_2+1}}$, we get

$$r_{k_2+1}\left(-\beta \frac{r_{k_2}u_{k_2}}{r_{k_2+1}}\right)+r_{k_2}(\alpha u_{k_2})=0, \quad \beta r_{k_2}u_{k_2}=\alpha r_{k_2}u_{k_2}$$

i.e. $\alpha=\beta$. Taking $\alpha=a$ we get $v_k=au_k$ which completes the proof of the theorem.

Let now

$$S_0\{a_k\} \stackrel{\text{df}}{=} \{a_{k+1}\},$$

$$T_0\{a_k\} \stackrel{\text{df}}{=} \begin{cases} 0 & \text{for} \quad k=0, \\ a_{k-1} & \text{for} \quad k \neq 0. \end{cases}$$

$$s_0\{a_k\} \stackrel{\text{df}}{=} \{a_0, 0, 0, \ldots\}.$$

COROLLARY 2. *We have*

(7) $$S_0S=SS_0$$

and

(8) $$S_0(a_kb_k)=(S_0a_k)(S_0b_k).$$

THEOREM 8. *If w_k is an eigenelement of the equation*

(9) $$S(R_kST_0w_k)=-\lambda P_kw_k$$

with conditions

(10) $$s(k_0+1)w_k=0, \quad s(k_1+1)w_k=0$$

corresponding to the eigenvalue λ, *then it is also an eigenelement of the equation*

$$S(r_k Sw_k) = -\lambda \varrho_k w_{k+1}$$

with conditions

$$s(k_0+1)w_k = 0, \quad s(k_1+1)w_k = 0$$

corresponding to the eigenvalue λ *when* $R_k=T_0r_k+s_0r_k$, $P_k=T_0p_k+s_0p_k$.

Proof is implied by properties of S_0.

COROLLARY 3 (from Theorems 8, 5 and 6). *If* $r_k>0$, $p_k>0$, $k=0, 1, 2, \dots$, $k_0<k_1$ *then the eigenvalues of the difference equation* (9) *with conditions* (10) *are positive.*

Using the property of the difference derivative we may write (3) in the form

$$r_{k+1}S^2w_k + Sr_k Sw_k = -\lambda \varrho_k w_{k+1},$$

and rearranging further we get an equation of the form

$$r_{k+1}w_{k+2} + (\lambda p_k - r_k - r_{k+1})w_{k+1} + r_k w_k = 0. \tag{11}$$

Taking $k=0, 1, 2, 3, \dots$ in the last equation and using conditions (4) and (5) we get the following system of equations

$$\begin{cases} r_1 w_2 + (\lambda p_0 - r_0 - r_1)w_1 + r_0 w_0 = 0 \\ r_2 w_3 + (\lambda p_1 - r_1 - r_2)w_2 + r_1 w_1 = 0 \\ \quad\vdots \\ (\lambda p_{k_0-1} - r_{k_0-1} - r_{k_0})w_{k_0} + r_{k_0-1}w_{k_0-1} = 0 \\ r_{k_0+1}w_{k_0+2} + r_{k_0}w_{k_0} = 0 \\ r_{k_0+2}w_{k_0+3} + (\lambda p_{k_0+1} - r_{k_0+1} - r_{k_0+2})w_{k_0+2} = 0 \\ r_{k_0+3}w_{k_0+4} + (\lambda p_{k_0+2} - r_{k_0+2} - r_{k_0+3})w_{k_0+3} + r_{k_0+2}w_{k_0+2} = 0 \\ \quad\vdots \\ (\lambda p_{k_1-1} - r_{k_1-1} - r_{k_1})w_{k_1} + r_{k_1-1}w_{k_1-1} = 0 \\ r_{k_1+1}w_{k_1+2} + r_{k_1}w_{k_1} = 0 \\ r_{k_1+2}w_{k_1+3} + (\lambda p_{k_1+1} - r_{k_1+1} - r_{k_1+2})w_{k_1+2} = 0 \\ r_{k_1+3}w_{k_1+4} + (\lambda p_{k_1+2} - r_{k_1+2} - r_{k_1+3})w_{k_1+3} + r_{k_1+2}w_{k_1+2} = 0 \\ \quad\vdots \end{cases}$$

in which the unknown quantites are $w_0, w_1, \dots, w_{k_0-1}, w_{k_0}, w_{k_0+2}, w_{k_0+3}, \dots, w_{k_1-1}, w_{k_1}, w_{k_1+2}, w_{k_1+3}, \dots$. For an arbitrary set up $k \geqq k_1-1$ it is a homogeneous system of equations with the same number of unknowns and equations. To the last system of equations, there corresponds the following matrix of coefficients:

$$
\begin{bmatrix}
r_0 & \lambda p_0 - r_0 - r_1 & r_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & r_1 & \lambda p_1 - r_1 - r_2 & r_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & r_2 & \lambda p_2 - r_2 - r_3 & r_3 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & r_3 & \lambda p_3 - r_3 - r_4 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & r_4 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \cdots & r_{k_0-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \lambda p_{k_0-2} - r_{k_0-2} - r_{k_0-1} & r_{k_0-1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & r_{k_0-1} & \lambda p_{k_0-1} - r_{k_0-1} - r_{k_0} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & r_{k_0} & r_{k_0+1} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda p_{k_0+1} - r_{k_0+1} - r_{k_0+2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & r_{k_0+2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & r_{k_1-2} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda p_{k_1-2} - r_{k_1-2} - r_{k_1-1} & r_{k_1-1} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & r_{k_1-1} & \lambda p_{k_1-1} - r_{k_1-1} - r_{k_1} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & r_{k_1} & r_{k_1+1} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda p_{k_1+1} - r_{k_1+1} - r_{k_1+2} & r_{k_1+2} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & r_{k_1+2} & \lambda p_{k_1+2} - r_{k_1+2} - r_{k_1+3} & r_{k_1+3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
$$

Let A_k stand for the determinant of the matrix of coefficients of the first $k+1$ equations, $k \geqq k_1-1$. Let A be the determinant of the matrix cut off from the matrix of the system of equations i.e.

$$A \overset{\text{df}}{=} \det \begin{bmatrix} \lambda p_{k_0+1}-r_{k_0+1}-r_{k_0+2} & r_{k_0+2} & 0 & \dots & 0 & 0 \\ r_{k_0+2} & \lambda p_{k_0+2}-r_{k_0+2}-r_{k_0+3} & r_{k_0+3} & \dots & 0 & 0 \\ 0 & r_{k_0+3} & \lambda p_{k_0+3}-r_{k_0+3}-r_{k_0+4} & \dots & 0 & 0 \\ 0 & 0 & r_{k_0+4} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & r_{k_1-2} & 0 \\ 0 & 0 & 0 & \dots & \lambda p_{k_1-2}-r_{k_1-2}-r_{k_1-1} & r_{k_1-1} \\ 0 & 0 & 0 & \dots & r_{k_1-1} & \lambda p_{k_1-1}-r_{k_1-1}-r_{k_1} \end{bmatrix}$$

We may easily notice that for every $k \geqq k_1-1$, $A_k=0$ if and only if $A=0$. This means that for an arbitrary set up $k \geqq k_1-1$, a homogeneous system of equations has a non-zero solution when $A=0$.

From the condition $A=0$ it is easy to determine the eigenvalues as the roots of a polynomial of order k_1-k_0-1. Knowing the eigenvalues, from the system of equations we may determine the coordinates $w_{k_0+2}, \dots, w_{k_1}$ of the eigenelement w_k. The rest of the coordinates may be determined from (11) by calculating

(12) $$w_k = \frac{-r_{k+1}w_{k+2}-(\lambda p_k - r_k - r_{k+1})w_{k+1}}{r_k} \quad \text{for} \quad k \leqq k_0$$

or

(13) $$w_{k+2} = \frac{-r_k w_k-(\lambda p_k - r_k - r_{k+1})w_{k+1}}{r_{k+1}} \quad \text{for} \quad k \geqq k_1.$$

If we reduce the eigenelements to vectors of finite dimension including only the coordinates $w_{k_0+2}, w_{k_0+3}, \dots, w_{k_1-1}, w_{k_1}$ i.e. to vectors of the form

(14) $$W \overset{\text{df}}{=} [w_{k_0+2}, w_{k_0+3}, \dots, w_{k_1-1}, w_{k_1}] \in R^{k_1-k_0-1}$$

then on the basis of Example 4 and Theorem 4 we obtain

THEOREM 9. *The eigenelements of* (3) *with conditions* (4) *and* (5) *corresponding to various eigenvalues are orthogonal with respect to the weight*

$$\varrho = \varrho_k = \begin{cases} 0 & \textit{for} \quad k = 0 \\ p_{k-1} & \textit{for} \quad k \neq 0. \end{cases}$$

COROLLARY 4 (from Theorems 7 and 9). *If* (3) *with conditions* (4), (5) *has* k_1-k_0-1 *various eigenvalues* λ *then the eigenelements corresponding to these eigenvalues form a basis in* $R^{k_1-k_0-1}$, *i.e. an arbitrary element* $\varphi_k \in R^{k_1-k_0-1}$ *can be represented as*

(15) $$\varphi_k = \sum_{i=1}^{k_1-k_0-1} c_i w_k^i \quad \textit{for} \quad k = k_0+2,\ k_0+3, \dots, k_1$$

where

$$(16)\quad c_i = \frac{[T(k_0+1)-T(k_1+1)](\varrho_k\varphi_k w_k^i)}{[T(k_0+1)-T(k_1+1)](\varrho_k(w_k^i)^2)} \quad \textit{for} \quad i = 1, 2, 3, \ldots, k_1-k_0-1$$

and

$$\varrho_k = \begin{cases} 0 & \text{for} \quad k = 0 \\ p_{k-1} & \text{for} \quad k \neq 0. \end{cases}$$

EXAMPLE 8. Let us find the eigenvalues and eigenelements of the difference equation

$$S(k!\,Su_k) = -\lambda(k+2)\,k!\,u_{k+1}$$

with conditions $u_1=0$, $u_5=0$. We have to find the solution of the equation

$$\begin{vmatrix} 3\lambda-3 & 2 & 0 \\ 2 & 8\lambda-8 & 6 \\ 0 & 6 & 30\lambda-30 \end{vmatrix} = 0$$

taking $r_k=k!$, $p_k=(k+2)k!$. From the last equation we have

$$(\lambda-1)(60\lambda^2-120\lambda+41) = 0$$

so

$$\lambda_1 = 1-\sqrt{\frac{19}{60}}, \quad \lambda_2 = 1+\sqrt{\frac{19}{60}}, \quad \lambda_3 = 1.$$

Knowing the eigenvalues we may find the coordinates u_2^i, u_3^i, u_4^i of the eigenvalues u_k^i, $i=1, 2, 3$.

For $\lambda_1=1-\sqrt{\frac{19}{60}}$ we get $u_2^1=\frac{10}{3}$, $u_3^1=5\sqrt{\frac{19}{60}}$, $u_4^1=1$.

For $\lambda_2=1+\sqrt{\frac{19}{60}}$ we get $u_2^2=\frac{10}{3}$, $u_3^2=-5\sqrt{\frac{19}{60}}$, $u_4^2=1$.

For $\lambda_3=1$ we get $u_2^3=-3$, $u_3^3=0$, $u_4^3=1$.

Using (12) or (13) we may find the other coordinates of the eigenelement u_k. So for example for the eigenvalues λ_1, λ_2 and λ_3 from (12) we obtaim $u_0^1=-\frac{10}{3}$, $u_0^2=-\frac{10}{3}$ and $u_0^3=3$.

For $\lambda_3=1$, from (13) we have $u_6^3=-\frac{1}{5}$, $u_7^3=0$, $u_8^3=\frac{1}{35}$ etc.

With the scalar product defined in Example 4 and on the basis of Theorem 4, the eigenelements u_k^1, u_k^2, u_k^3 are orthogonal with respect to the weight

$$\varrho_k = \begin{cases} 0 & \text{for} \quad k = 0 \\ (k+1)(k-1)! & \text{for} \quad k \neq 0. \end{cases}$$

§4. Partial difference equations

Let us consider the difference equation

(17) $$S_1(r_k S_1 w_{k,l})+p_k S_0 S_2 w_{k,l} = 0$$

with conditions

(18) $$w_{k,0} = \varphi_k \quad \text{for} \quad k = k_0+2, k_0+3, \dots, k_1,$$

(19) $$w_{k_0+1,l} = 0,$$

(20) $$w_{k_1+1,l} = 0,$$

$r_k>0$, $p_k>0$ for $k=0, 1, 2, \dots$; $w_{k,l}$ are double real sequences. Define a multiplication of double sequences by

$$\{u_{k,l}\}\{v_{k,l}\} \overset{\mathrm{df}}{=} \{u_{k,l} v_{k,l}\}.$$

(r_k, p_k are also double real sequences.) Introduce the derivatives S_1, S_2 and operation S_0 by

$$S_1\{u_{k,l}\} \overset{\mathrm{df}}{=} \{u_{k+1,l}-u_{k,l}\},\ S_2\{u_{k,l}\} \overset{\mathrm{df}}{=} \{u_{k,l+1}-u_{k,l}\},\ S_0\{u_{k,l}\} \overset{\mathrm{df}}{=} \{u_{k+1,l}\}.$$

We shall seek a solution of equation (17) with conditions (18), (19), (20) of type $w_{k,l}=u_k v_l$. Substituting $w_{k,l}=u_k v_l$ into (17) we get

$$[S_1(r_k S_1 u_k)](p_k u_{k+1})^{-1}+(S_2 v_l) v_l^{-1} = 0.$$

Since

$$[S_1(r_k S_1 u_k)](p_k u_{k+1})^{-1} \in \operatorname{Ker} S_2$$

and $(S_2 v_l) v_l^{-1} \in \operatorname{Ker} S_1$, so $[S_1(r_k S_1 u_k)](p_k u_{k+1})^{-1} = -(S_2 v_l) v_l^{-1} = -\lambda$. From the last relations we have

$$S_1(r_k S_1 u_k) = -\lambda p_k u_{k+1} \quad \text{and} \quad S_2 v_l = \lambda v_l.$$

The equation $S_1(r_k S_1 u_k)=-\lambda p_k u_{k+1}$ has the conditions $u_{k_0+1}=0$ and $u_{k_1+1}=0$ implied directly by conditions (19), (20). Let the equation

$$S_1(r_k S_1 u_k) = -\lambda p_k u_{k+1}$$

with conditions $u_{k_0+1}=0$, $u_{k_1+1}=0$ have k_1-k_0-1 various eigenvalues $\lambda_n \in \boldsymbol{R}$, with corresponding eigenelements u_k^n. For the eigenvalues λ_n we find the solution of the equation $S_2 v_l^n=\lambda_n v_l^n$. It is known that the last equation has a solution of the form

$$V_l^n = (1+\lambda_n)^l v_0^n$$

(see [5], [6]). On the basis of Theorems 5 and 6, $\lambda_n>0$, so $v_l^n \neq 0$ for $l=0, 1, 2, \dots$ when $v_0^n \neq 0$. For the eigenvalues λ_n we have the solutions

(21) $$w_{k,l}^n = u_k^n v_l^n = v_0^n (1+\lambda_n)^l u_k^n$$

of equation (17) with conditions (19), (20). The sum of the solutions of (21) satisfies (17), (19) and (20). We look for the solution of (17) with conditions (18), (19) and (20) in the form of the sum of sequences

(22) $$w_{k,l} = \sum_{n=1}^{k_1-k_0-1} v_0^n (1+\lambda_n)^l u_k^n, \quad n = 1, 2, \dots, k_1-k_0-1.$$

We now show that $w_{k,l}$ satisfies also condition (18). On the basis of Corollary 4,

$$\varphi_k = \sum_{n=1}^{k_1-k_0-1} c_n u_k^n \quad \text{for} \quad k = k_0+2,\ k_0+3,\ \dots,\ k_1$$

where c_n is defined by (16). Substituting c_n in place of v_0^n in (22) we get

$$|w_{k,l}| = \sum_{n=1}^{k_1-k_0-1} c_n(1+\lambda_n)^l u_k^n = \sum_{n=1}^{k_1-k_0-1} c_n u_k^n = \varphi_k$$

for $k=k_0+2,\ k_0+3,\ \dots,\ k_1$.

EXAMPLE 9. Find the solution of the partial difference equation

$$S_1(k!\,S_1 w_{k,l})+(k+2)\,k!\,S_0 S_2 w_{k,l} = 0$$

with conditions

$$w_{2,0} = 13 = \varphi_2, \quad w_{3,0} = -5\sqrt{\frac{19}{60}} = \varphi_3, \quad w_{4,0} = 2 = \varphi_4, \quad w_{1,l} = 0,$$

$$w_{5,l} = 0, \quad l = 0, 1, 2, \dots .$$

After rearranging we get

$$S_1(k!\,S_1 u_k) = -\lambda(k+2)\,k!\,u_{k+1}, \tag{23}$$

$$S_2\, v_l = \lambda v_l, \tag{24}$$

and

$$u_1 = u_5 = 0. \tag{25}$$

Equation (23), with conditions (25) has the eigenvalues λ_1, λ_2, λ_3 and eigenelements u_k^1, u_k^2, u_k^3 determined in Example 13. (24) has the solutions

$$v_l^1 = \left(2-\sqrt{\frac{19}{60}}\right)^l v_0^1, \quad v_l^2 = \left(2+\sqrt{\frac{19}{60}}\right)^l v_0^2, \quad v_l^3 = 2^l v_0^3$$

corresponding to the eigenvalues $\lambda_1 = 1-\sqrt{\frac{19}{60}}$, $\lambda_2 = 1+\sqrt{\frac{19}{60}}$, $\lambda_3=1$. From (22)

$$w_{k,l} = c_1\left(2-\sqrt{\frac{19}{60}}\right)^l u_k^1 + c_2\left(2+\sqrt{\frac{19}{60}}\right)^l u_k^2 + c_3 2^l u_k^3.$$

Finding the coefficients c_1, c_2, c_3 from (16) we have

$$w_{k,l} = \left(2-\sqrt{\frac{19}{60}}\right)^l u_k^1 + 2\left(2+\sqrt{\frac{19}{60}}\right)^l u_k^2 - 2^l u_k^3.$$

From the last formula and Example 8 it is easy to determine $w_{k,l}$ and so, for example

$$w_{2,1} = \frac{1}{3}\left(50+10\sqrt{\frac{19}{60}}\right), \quad w_{4,1} = 2\left(2+\sqrt{\frac{19}{60}}\right), \quad w_{0,0} = -13 \quad \text{etc.}$$

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ZU EINER AUFGABE VON J. M. WILLS

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Gewidmet seinem verehrten Lehrer Prof. Wilhelm Maier

1. In der von W. Moser verbreiteten Sammlung [2] findet man unter Nummer 21 die folgende Aufgabe von J. M. Wills:

Man betrachte die konvexen Körper

$$K_1^n = \left\{(x_1, \ldots, x_n) \colon \big(\sum_{i=1}^{n} x_i^2\big)^{1/2} \leqq \pi^{-1/2}\left(\Gamma\left(\frac{n}{2}+1\right)\right)^{1/n}\right\},$$

$$K_2^n = \left\{(x_1, \ldots, x_n) \colon \sum_{i=1}^{n} |x_i| \leqq \frac{1}{2}(n!)^{1/n}\right\},$$

$$K_3^n = \left\{(x_1, \ldots, x_n) \colon \max_{1 \leqq i \leqq n} |x_i| \leqq \frac{1}{2}\right\},$$

also eine Kugel, ein Kreuzpolytop und einen Würfel im R^n, mit dem Volumen $V(K_i^n)=1$. Man setze $K_{pq}^n := K_p^n \cap K_q^n$ für $p \neq q$ und untersuche, ob die Folgen der Volumina $V(K_{pq}^n)$ Grenzwerte besitzen. Wenn dies der Fall sein sollte, gilt

$$\lim_{n\to\infty} V(K_{pq}^n) = 0 \quad \text{oder} \quad \lim_{n\to\infty} V(K_{pq}^n) \neq 0?$$

Inhalte von Teilen einer Kugel zu bestimmen bereitet in Räumen beliebiger Dimension bekanntlich bereits in den einfachsten Fällen beträchtliche Schwierigkeiten. Aus diesem Grunde werden hier nur die Durchschnitte der Polyeder K_2^n und K_3^n betrachtet. Es zeigt sich, daß die Folge $V(K_{23}^n)$ in der Tat einen Grenzwert besitzt; und zwar gilt

(1) $$\lim_{n\to\infty} V(K_{23}^n) = 0.$$

Um dieses Ergebnis zu gewinnen, wird ein Weg verfolgt, den J. Fáry und L. Rédei [1] vor längerer Zeit bei einer ähnlichen Aufgabe eingeschlagen haben.

2. Das Kreuzpolytop K_2^n und das Maßpolytop K_3^n sind so gelagert, daß sie bei Spiegelung an den Ebenen $x_i=0$, $i=1, \ldots, n$ in sich übergehen. Aus diesem Grunde ist

(2) $$V_n := V(K_2^n \cap K_3^n) = 2^n V(\tilde{K}_2^n \cap \tilde{K}_3^n) = V(2\tilde{K}_2^n \cap 2\tilde{K}_3^n)$$

wobei mit $\tilde{K}_2^n$ und $\tilde{K}_3^n$ etwa jene Teile von K_2^n bzw. K_3^n gemeint sind, die im Kegel $x_i \geqq 0$, $i=1, \ldots, n$ liegen. Um die Bezeichnungen zu vereinfachen werde $2\tilde{K}_2^n =: S$

und $2\tilde{K}_3^n =: S^*$ gesetzt. Zu ermitteln ist also $V_n = V(S \cap S^*)$ wobei S und S^* durch

$$(3) \qquad \begin{cases} S = \{(x_1, \dots, x_n)\colon \sum_{i=1}^{n} x_i \leq \sqrt[n]{n!}; \quad x_i \geq 0, \quad i = 1, \dots, n\}, \\ S^* = \{(x_1, \dots, x_n)\colon \quad 0 \leq x_i \leq 1; \quad i = 1, \dots, n\} \end{cases}$$

gegeben sind. Offenkundig ist S ein Simplex. Ist M eine Teilmenge von S, so soll mit $\overline{M}$ die abgeschlossene Hülle von $S \setminus M$ bezeichnet werden. Bekannterweise gilt dann für jede natürliche Zahl $r \geq 1$ die Identität

$$(4) \qquad \overline{M_1 \cap M_2 \cap \dots \cap M_r} = \overline{M}_1 \cup \overline{M}_2 \cup \dots \cup \overline{M}_r$$

und sicher ist

$$(5) \qquad V_n = V(S \cap S^*) = V(S) - V(\overline{S \cap S^*}).$$

Werden nun noch durch

$$H_p^+ := \{(x_1, \dots, x_n)\colon x_p \geq 1\}, \quad H_q^- := \{(x_1, \dots, x_n)\colon x_q \leq 1\}$$

abgeschlossene Halbräume gekennzeichnet, deren Ränder zur Begrenzung von S^* gehören, so gilt

$$(6) \qquad S \cap S^* = S \cap H_1^- \cap \dots \cap H_n^- = \bigcap_{i=1}^{n} (S \cap H_i^-)$$

und weil

$$\overline{S \cap H_i^-} = S \cap H_i^+, \quad i = 1, \dots, n$$

ist, ergibt sich aus (5) mit (6) und (4) zunächst

$$(7) \qquad V_n = V(S) - V\Big(\bigcup_{i=1}^{n} (S \cap H_i^+)\Big).$$

Für den Inhalt einer endlichen Vereinigung meßbarer Mengen gilt aber — wie man durch Induktion leicht bestätigt — die Beziehung

$$V(A_1 \cup \dots \cup A_r) = \Sigma V(A_{i_1}) - \Sigma V(A_{i_1} \cap A_{i_2}) + - \dots + (-1)^{r-1} \Sigma V(A_{i_1} \cap \dots \cap A_{i_r})$$

wobei in der k-ten Summe über alle Teilmengen $\{i_1, \dots, i_k\} \subset \{1, \dots, r\}$ zu summieren ist. Wird zur Abkürzung

$$(8) \qquad (S \cap H_{i_1}^+) \cap \dots \cap (S \cap H_{i_k}^+) = S \cap H_{i_1}^+ \cap \dots \cap H_{i_k}^+ =: S_{i_1 \dots i_k}$$

gesetzt, so erhält man demnach statt (7)

$$V_n = V(S) - \Sigma V(S_{i_1}) + \Sigma V(S_{i_1 i_2}) - + \dots + (-1)^n V(S_{i_1 \dots i_n}).$$

Man sieht nun sofort, daß für jedes k die $\binom{n}{k}$ Mengen $S_{i_1 \dots i_k}$ untereinander, und damit insbesondere zu $S_{1 \dots k}$ kongruent sind. Je k Halbräume $H_{i_1}^+, \dots, H_{i_k}^+$ können durch Tausch der Koordinaten auf die Halbräume $H_1^+, \dots, H_k^+$ abgebildet werden. Dabei geht aber S in sich über und mithin $S_{i_1 \dots i_k}$ in $S_{1 \dots k}$. Man gewinnt also die Darstellung

$$(9) \qquad V_n = V(S) + \sum_{k=1}^{n} (-1)^k \binom{n}{k} V(S_{1 \dots k}).$$

Von ausschlaggebender Bedeutung ist nun, daß das Volumen von $S_{1\dots k}$ sofort bestimmt werden kann. Ist $k>\sqrt[n]{n!}$, so ist $S_{1\dots k}$ leer — also $V(S_{1\dots k})=0$, da dann die Forderungen

$$\sum_{i=1}^{n} x_i \leqq \sqrt[n]{n!} \quad \text{und} \quad x_i \geqq 1, \quad i=1, \dots, k$$

nicht mehr zu vereinbaren sind. Enthält $S_{1\dots k}$ mehr als einen Punkt, dies ist für $k<\sqrt[n]{n!}$ der Fall, so ist $S_{1\dots k}$ ein n-dimensionales Simplex, dessen Wände zu den Wänden von S parallel verlaufen. Sowohl S als auch $S_{1\dots k}\subset S$ werden von der Ebene

$$H=\left\{(x_1, \dots, x_n)\colon \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i = \frac{\sqrt[n]{n!}}{\sqrt{n}}\right\}$$

begrenzt, und liegen im gleichen Halbraum bezüglich dieser Ebene. Mithin ist $V(S_{1\dots k})=\lambda_k^n V(S)$; $0<\lambda_k<1$ und dieser Faktor λ_k ergibt sich sofort durch Vergleich der Abstände jener Ecken von S bzw. $S_{1\dots k}$ die nicht in H liegen von dieser Ebene zu

$$\lambda_k=\frac{\sqrt[n]{n!}-k}{\sqrt[n]{n!}}.$$

Zusammenfassend gewinnt man demnach mit Rücksicht auf $V(S)=1$ die Darstellung

$$V_n=\frac{1}{n!} \sum_{0\leqq k\leqq \sqrt[n]{n!}} (-1)^k \binom{n}{k} (\sqrt[n]{n!}-k)^n. \tag{10}$$

3. Die Beziehung (10) ist durchaus brauchbar, um V_n für kleine Werte von n zu berechnen. Für Grenzbetrachtungen dürfte sie allerdings kaum geeignet sein. Die bereits genannte Arbeit von Fáry und Rédei enthält aber einen Hinweis von P. Turán, wie ein derartiger Ausdruck unter Umständen geeignet umgeformt werden kann. Man beachte nämlich, daß mit einer reellen Zahl α

$$\frac{1}{2\pi i} \int_l \frac{e^z}{z^{n+1}}\, dz = \begin{cases} \dfrac{\alpha^n}{n!} & \text{für} \quad \alpha \geqq 0 \\ 0 & \text{für} \quad \alpha \leqq 0 \end{cases} \tag{11}$$

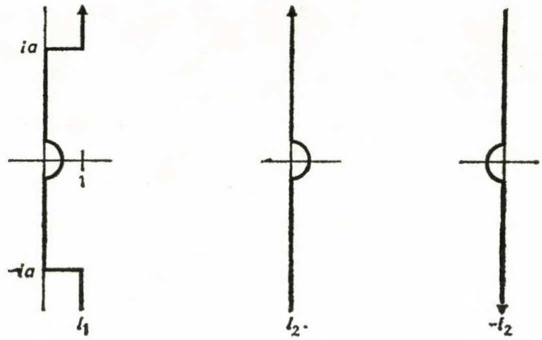
gilt, falls als Integrationsweg l die Gerade $z=1+it$ gewählt wird. Diese Beziehung nutzend kann statt (10) auch

$$V_n=\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2\pi i} \int_l \frac{e^{(\sqrt[n]{n!}-k)z}}{z^{n+1}}\, dz = \frac{1}{2\pi i} \int_l dz \frac{e^{\sqrt[n]{n!}\,z}}{z^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{-kz}$$

geschrieben werden, womit man zu der komplexen Darstellung

$$V_n=\frac{1}{2\pi i} \int_l dz \frac{e^{\sqrt[n]{n!}\,z}}{z^{n+1}} (1-e^{-z})^n \tag{12}$$

gelangt. Offenbar ist hier für jedes $n \geqq 1$ der Integrand $f_n(z)$ im Gebiet $0<|z|<\infty$ regulär. Die Integration kann darum auch über einen Weg l_1 erstreckt werden, wie er in der Abbildung angedeutet ist.



Auf den zur reellen Achse parallelen Stücken $z=\pm ia+t$, $a>0$, $0 \leqq t \leqq 1$ dieses Weges ist aber

$$|f_n(z)| \leqq \frac{|e^{\pm ia\sqrt[n]{n!}}|\, e^{\sqrt[n]{n!}\,t}(1+|e^{\pm ia}|\,e^{-t})^n}{|\pm ia+t|^{n+1}} \leqq \frac{e^{\sqrt[n]{n!}}\,2^n}{a^{n+1}}$$

Folglich wird auf diesen Wegstücken $f_n(z)$ beliebig klein, sobald a genügend groß gewählt wird, und mithin kann die Integration auch über den angegebenen Weg l_2 erfolgen. Man zerlege nun $f_n(z)$ in den geraden und den ungeraden Anteil. Es wird

$$f_n^+(z) := \frac{1}{2}\left(f_n(z)+f_n(-z)\right) = \frac{1}{2}\left(\frac{e^{z/2}-e^{-z/2}}{z}\right)^n \frac{e^{-a_n z/2}-e^{a_n z/2}}{z},$$

$$f_n^-(z) := \frac{1}{2}\left(f_n(z)-f_n(-z)\right) = \frac{1}{2}\left(\frac{e^{z/2}-e^{-z/2}}{z}\right)^n \frac{e^{-a_n z/2}+e^{a_n z/2}}{z}$$

mit

(13) $$a_n := n-2\sqrt[n]{n!}\,.$$

Man erkennt, daß $f_n^+(z)$ für $|z|<\infty$ regulär ist. Statt über den Weg l_2 kann $f_n^+(z)$ folglich auch längs der imaginären Achse integriert werden. Die Funktionen $f_n^-(z)$ besitzen bei $z=0$ einen Pol erster Ordnung mit dem Residuum $+1$. Da $f_n^-(z)$ ungerade ist wird

$$2\int_{l_2} f_n^-(z)\,dz = \int_{l_2} f_n^-(z)\,dz + \int_{-l_2} f_n^-(z)\,dz = \int_k f_n^-(z)\,dz,$$

worin k für einen positiv umlaufenen Kreis mit dem Ursprung als Mittelpunkt steht; und man erhält

$$\frac{1}{2\pi i}\int_{l_2} f_n^-(z)\,dz = \frac{1}{2\pi i}\int_k \frac{1}{2} f_n^-(z) = \frac{1}{2}.$$

Man gelangt damit zu der reellen Integraldarstellung

$$V_n = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\sin \frac{t}{2}}{\frac{t}{2}} \right)^n \frac{\sin(-a_n)\frac{t}{2}}{t}\, dt,$$

also zu

(14)
$$V_n = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \left(\frac{\sin x}{x} \right)^n \frac{\sin(n - 2\sqrt[n]{n!})x}{x}\, dx.$$

4. Um zu zeigen, daß V_n wie behauptet gegen Null strebt, muß nachgewiesen werden, daß

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_0^\infty \left(\frac{\sin x}{x} \right)^n \frac{\sin a_n x}{x}\, dx = \frac{\pi}{2}$$

gilt. Dies ist keineswegs sofort ersichtlich. Bezüglich der Begriffe und Sätze, die im folgenden benutzt werden, sei auf [3] verwiesen. Zunächst setze man $I_n = I_n' + I_n''$ mit

$$I_n' = \int_0^1 \left(\frac{\sin x}{x} \right)^n \frac{\sin a_n x}{x}\, dx; \quad I_n'' = \int_1^\infty \left(\frac{\sin x}{x} \right)^n \frac{\sin a_n x}{x}\, dx,$$

und erhält einmal

(15)
$$|I_n''| \leqq \int_1^\infty \frac{dx}{x^{n+1}} = \frac{1}{n}; \quad \text{also} \quad \lim_{n\to\infty} I_n'' = 0.$$

Andererseits wird

(16)
$$I_n' = \int_0^n \left(\frac{\sin \frac{x}{n}}{\frac{x}{n}} \right)^n \frac{\sin \frac{1}{n} a_n x}{x}\, dx = \int_0^\infty \varphi_n(x) \frac{\sin \frac{1}{n} a_n x}{x}\, dx$$

mit

(17)
$$\varphi_n(x) = \begin{cases} \dfrac{\sin \frac{x}{n}}{\frac{x}{n}} & \text{für} \quad 0 \leqq x \leqq n \\ 0 & \text{für} \quad n < x < \infty. \end{cases}$$

$\left(\text{Selbstverständlich ist } \frac{\sin x}{x} \text{ für } x=0 \text{ der Wert 1 zuzuweisen.}\right)$ Nun gilt

(18)
$$\lim_{n\to\infty} \varphi_n(x) \frac{\sin \frac{1}{n} a_n x}{x} = \frac{\sin\left(1 - \frac{2}{e}\right)x}{x}$$

und in jedem abgeschlossenen Intervall $[0, a]$ $(a>0)$, erfolgt die Konvergenz sogar gleichmäßig. Wegen $|\sin ux - \sin vx| \leqq |u-v||x|$ ist ja einerseits

$$\left| \frac{\sin \frac{1}{n} a_n x}{x} - \frac{\sin\left(1-\frac{2}{e}\right)x}{x} \right| \leqq \left| \frac{2}{e} - 2\frac{\sqrt[n]{n!}}{n} \right|$$

woraus mit $\frac{1}{n}\sqrt[n]{n!} \to \frac{1}{e}$ die gleichmäßige Konvergenz des einen Faktors abzulesen ist. Andererseits wähle man, sobald a vorgegeben ist, eine natürliche Zahl n_0 derart aus, daß $a<n_0$ gilt. Für jedes $n>n_0$ und jedes x aus $[0, a]$ gilt dann nach (17)

$$1 \geqq \varphi_n(x) \geqq \varphi_n(a) = \frac{\sin \frac{a}{n}}{\frac{a}{n}},$$

da $\left(\frac{\sin x}{x}\right)^n$ im Intervall $[0, 1]$ streng monoton fällt. Daß dann $\varphi_n(x) \to 1$ gelichmäßig in $[0, a]$ gilt, ist aus

$$\lim_{n\to\infty} \left(\frac{\sin \frac{a}{n}}{\frac{a}{n}} \right)^n = \lim_{n\to+0} \left(\frac{\sin ax}{ax}\right)^{1/x} = e^{\lim\limits_{x\to+0} \frac{1}{x} \ln \frac{\sin ax}{ax}} = e^0$$

zu folgern.

Bei gleichmäßiger Konvergenz des Integranden in jedem endlichen abgeschlossenen Teilintervall von $(0, \infty)$ gegen eine integrable Funktion ist es aber statthaft, Integration und Grenzübergang zu vertauschen, sobald das betreffende Integral gleichmäßig für alle genügend großen Werte von n existiert ([3], p. 244). Nun sind die gemäß (17) erklärten Funktionen $\varphi_n(x)$ für jedes n in $[0, \infty)$ monoton fallend und integrabel, außerdem gilt $0 \leqq \varphi_n(x) \leqq 1$. Nach dem Monotoniekriterium ([3], p. 240) existiert dann das Integral I_n' gleichmäßig für jene Werte von n, für die auch das Integral

$$I_n^* = \int_0^\infty \frac{\sin \frac{1}{n} a_n x}{x} dx$$

gleichmäßig existiert. Weil hier der Integrand wegen

$$\left| \frac{\sin \frac{1}{n} a_n x}{x} \right| \leqq \left| \frac{1}{n} a_n \right| = \left| 1 - 2\frac{\sqrt[n]{n!}}{n} \right| \to 1 - \frac{2}{e}$$

gleichmäßig beschränkt ist, genügt es zu zeigen, daß I_n^* für genügend große Werte von n gleichmäßig nach ∞ integriert werden kann. Dies läßt sich auf bekannte

Weise leicht erreichen. Da $1-\frac{2}{e}>0$ ist, gibt es sicher eine positive Zahl q und eine natürliche Zahl n_0, so daß für $n>n_0$ stets $\frac{1}{n}a_n>q$ gilt. Weiter gibt es wegen

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

zu jedem positiven ε eine Zahl $p>0$, so daß

$$\left|\int_{pq}^\infty \frac{\sin t}{t}\,dt\right| < \varepsilon$$

ausfällt, und diese Beziehung bleibt erhalten, wenn die untere Grenze vergrößert wird. Mithin ist für $n>n_0$ auch

$$\left|\int_{\frac{pa_n}{n}}^\infty \frac{\sin t}{t}\,dt\right| = \left|\int_p^\infty \frac{\sin \frac{a_n}{n}x}{x}\,dx\right| < \varepsilon$$

und dies gilt auch noch, wenn p durch eine beliebige größere Zahl ersetzt wird. Es existiert also I_n^* und folglich auch I_n' gleichmäßig für $n>n_0$ und man erhält

$$\lim_{n\to\infty} I_n' = \int_0^\infty \frac{\sin\left(1-\frac{2}{e}\right)x}{x}\,dx = \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2},$$

womit der Beweis abgeschlossen ist.

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SEMIHOMEOMORPHISMS

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Introduction

In 1972, S. G. Crossley and S. K. Hildebrand [3] have introduced and studied the concepts semihomeomorphisms, irresolute functions and presemiopen functions. A bijection is a semihomeomorphism if and only if it is irresolute and presemiopen. Every homeomorphism is a semihomeomorphism but not conversely. Every continuous, open function is irresolute as well as presemiopen. The purpose of this paper is to discuss under what additional conditions on the function and on the range and domain of the function, the following holds: (i) Every semihomeomorphism is a homeomorphism. (ii) Every irresolute, presemiopen function is continuous. (iii) Every irresolute, presemiopen function is open.

1. Preliminaries

Let X be a topological space and A be a subset of X. We will denote the complement of A in X by $X-A$, and the closure and interior of A in X by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A is *regular open* if $A=\mathrm{Int}(\mathrm{Cl}(A))$. R.O.$(X)$ is the family of all regular open sets in X. A is *semiopen* [4] if there exists an open set G such that $G\subset A\subset \mathrm{Cl}(G)$. S.O.$(X)$ is the family of all semiopen sets in X. The complement of a semiopen set is *semiclosed* [2]. The *semiclosure* of A [2], denoted by $\mathrm{sCl}(A)$, is the smallest semiclosed set that contains A. The *semiinterior of* A [2], denoted by $\mathrm{sInt}(A)$, is the largest semiopen set contained in A. Therefore, A is semiopen if and only if $A=\mathrm{sInt}(A)$ and A is semiclosed if and only if $A=\mathrm{sCl}(A)$. A is an *α-set* [6] if $A\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. The family of all α-sets in a space (X,τ), denoted by τ^α, is again a topology on X which is finer than τ and $\mathrm{S.O.}(X,\tau)=\mathrm{S.O.}(X,\tau^\alpha)$. Moreover, by Proposition 4 of [6] and Theorem 2 of [1], $\tau^\alpha=F(\tau)$, where $F(\tau)$ is the finest element in the equivalence class of all topologies on X having the same collection of semiopen sets as τ [2, 3]. Henceforth, by a space X, we will always mean the space (X,τ) and the space $(X, F(\tau))$ is denoted by X^*. Sets closed in X^* are called *α-closed sets*. A is *nowhere dense* [9] if $\mathrm{Int}(\mathrm{Cl}(A))=\emptyset$. A topology τ on X is an *α-topology* [6] if $\tau=F(\tau)$, or equivalently, all nowhere dense sets are closed [6, Corollary to Proposition 4]. Since $F(F(\tau))=F(\tau)$ by Propositions 1 and 3 of [6], $F(\tau)$ is an α-topology. We will call a space with an α-topology an *α-space*. Clearly for any space X, X^* is an α-space. A space X is *dense in itself* [9] if no point of X is open in X. X is *semiregular* [9] if for each open set G and $x\in G$, there exists a regular open set H such that $x\in H\subset G$. A function $f\colon X\to Y$ is *irresolute* [3] if $f^{-1}(A)\in\mathrm{S.O.}(X)$ for all $A\in\mathrm{S.O.}(Y)$. f is *presemiopen* [3] if $f(A)\in\mathrm{S.O.}(Y)$ for all $A\in\mathrm{S.O.}(X)$. f is *almost continuous* [8] if $f^{-1}(R)$ is open in X for all $R\in\mathrm{R.O.}(Y)$.

f is *almost open* [8] if $f(R)$ is open in Y for all $R \in$ R.O. (X). The following lemmas are useful in the sequel.

LEMMA 1.1. *If* $A \neq \emptyset$ *is semiopen, then* $\operatorname{Int}(A) \neq \emptyset$.
(Follows from the definition.)

LEMMA 1.2. *If A is semiopen and* $A \subset B \subset \operatorname{Cl}(A)$, *then B is semiopen* [4].

LEMMA 1.3. *If a singleton is semiopen, then it is open.*
(Follows from the definition.)

LEMMA 1.4. *If A is semiopen, then* $A - \operatorname{Int}(A)$ *is nowhere dense in X.*

LEMMA 1.5. *A is semiclosed if and only if* $\operatorname{Int}(\operatorname{Cl}(A)) \subset A$ [2].

LEMMA 1.6. *The intersection of a semiopen set and an open set is always semiopen* [2].

LEMMA 1.7. *The spaces X and X^* have the same collection of nowhere dense sets* [6].

LEMMA 1.8. *A is nowhere dense if and only if* $\operatorname{sInt}(\operatorname{sCl}(A)) = \emptyset$ [3].

LEMMA 1.9. *Nowhere dense sets are α-closed.*
(A is nowhere dense in X implies that A is nowhere dense in X^*, by Lemma 1.7. Since X^* is an α-space, A is α-closed).

LEMMA 1.10. *For any space X,* R.O. $(X)=$ R.O. (X^*) [6].

LEMMA 1.11. *A function* $f\colon X \to Y$ *is almost continuous if and only if* $\operatorname{Cl}(f^{-1}(A)) \subset f^{-1}(\operatorname{Cl}(A))$ *for all* $A \in$ S.O. (Y) [5].

2. Semihomeomorphisms

DEFINITION 2.1. A bijection $f\colon X \to Y$ is a *semihomeomorphism* [3] if f is irresolute and presemiopen.

REMARK 2.2. Every homeomorphism is a semihomeomorphism but not conversely [3].

THEOREM 2.3. *A function* $f\colon X \to Y$ *is a semihomeomorphism if and only if* $f\colon X^* \to Y^*$ *is a homeomorphism.*

PROOF. Necessity follows from Theorem 2.6 of [3]. Sufficiency follows from the fact that S.O. $(X)=$ S.O. (X^*) for any space X.

COROLLARY 2.4. *If X and Y are α-spaces, then every semihomeomorphism* $f\colon X \to Y$ *is a homeomorphism.*

LEMMA 2.5. *The inverse images of regular open sets under a semihomeomorphism are again regular open.*

PROOF. Theorem 2.3 and Lemma 1.10.

DEFINITION 2.6. A space X is said to be *s-semiregular* if X is semiregular at all points which are semiclosed in X.

REMARK 2.7. Every semiregular space is s-semiregular. It is not difficult to see that the converse fails.

LEMMA 2.8. *Singletons which are semiclosed in a space X are either regular open or nowhere dense.*

PROOF. Suppose $\{x\}$ is semiclosed in X. By Lemma 1.5, $\operatorname{Int}(\operatorname{Cl}(\{x\}))\subset\{x\}$. If $\operatorname{Int}(\operatorname{Cl}(\{x\}))=\emptyset$, $\{x\}$ is nowhere dense. If $\operatorname{Int}(\operatorname{Cl}(\{x\}))\neq\emptyset$, then $\operatorname{Int}\operatorname{Cl}((\{x\}))=\{x\}$ which implies that $\{x\}$ is regular open.

LEMMA 2.9. *The following are equivalent for any space X:*
(i) *X is s-semiregular.*
(ii) *X is semiregular at all points which are α-closed.*
(iii) *X is semiregular at all points which are nowhere dense.*

PROOF. (i)$\Rightarrow$(ii) is clear. (ii)$\Rightarrow$(iii) follows from Lemma 1.9. (iii)$\Rightarrow$(i) follows from Lemma 2.8.

THEOREM 2.10. *If $f\colon X\to Y$ is a semihomeomorphism where Y is s-semiregular, then f is continuous.*

PROOF. Suppose f is not continuous. Then there exists a non-empty open set G in Y such that $f^{-1}(G)$ is not open in X. Since $f^{-1}(G)$ is semiopen, by Lemma 1.1, $\operatorname{Int}(f^{-1}(G))\neq\emptyset$ and so there exists $x\in f^{-1}(G)$ such that $x\notin\operatorname{Int}(f^{-1}(G))$. By Lemma 1.4, it follows that $\{x\}$ is nowhere dense in X and so by Theorem 1.12 of [3], $\{f(x)\}$ is nowhere dense in Y. Since Y is s-semiregular, there exists a regular open set H such that $f(x)\in H\subset G$, which implies that $x\in f^{-1}(H)\subset f^{-1}(G)$. Since $f^{-1}(H)\in\in$ R.O. (X) by Lemma 2.5, $x\in\operatorname{Int}(f^{-1}(G))$, a contradiction to the assumption. Hence f is continuous.

COROLLARY 2.11. *Let $f\colon X\to Y$ be a closed or open function where Y is s-semiregular. If f is a semihomeomorphism. then f is a homeomorphism.*

COROLLARY 2.12. *If X and Y are s-semiregular spaces, then every semihomeomorphism $f\colon X\to Y$ is a homeomorphism.*

COROLLARY 2.13. *If X and Y are semiregular spaces, then every semihomeomorphism $f\colon X\to Y$ is a homeomorphism.*

REMARK 2.14. Piotrowski [7] has established the following: If $f\colon X\to Y$ is a closed function from a regular, T_1 space X onto a regular, T_1 and dense in itself space Y, then f is a semihomeomorphism if and only if f is a homeomorphism [Theorem 17]. The above Corollaries 2.11, 2.12 and 2.13 are strong generalizations of this result.

The following example shows that in Corollaries 2.4 and 2.12, the condition 'α-space' or 's-semiregular' on X or Y cannot be dropped. Moreover, it also shows that α-spaces and s-semiregular spaces are not preserved by semihomeomorphisms.

EXAMPLE 2.15. Let R be the real line with the usual topology. Then $R \neq R^*$ [6, Example, p. 963]. If i: $R \to R^*$ is the identity function, then i and i^{-1} are semihomeomorphisms but not homeomorphisms. R^* is not s-semiregular by Theorem 2.10.

The following example shows that the converses of Corollaries 2.4, 2.12 and 2.13 are not true.

EXAMPLE 2.16. Let R be the real line with the usual topology. Clearly, every semihomeomorphism of R(resp. R^*) onto itself is a homeomorphism, but R is not an α-space and R^* is not an s-semiregular space.

DEFINITION 2.17. A space X is said to be *s^*-semiregular* if every space semihomeomorphic to X is s-semiregular.

LEMMA 2.18. *The following are equivalent for any space X.*

(i) *X is s^*-semiregular.*

(ii) *X^* is s-semiregular.*

(iii) *X is an s-semiregular α-space.*

PROOF. (i)$\Rightarrow$(ii) follows from the fact that X and X^* are semihomeomorphic. (ii)$\Rightarrow$(iii) follows from Theorem 2.10. For (iii)$\Rightarrow$(i), consider a semihomeomorphism f: $X \to Y$ from X onto any space Y. By (iii) and Theorem 2.3, f: $X \to Y^*$ is a homeomorphism and so Y^* is s-semiregular. By Lemmas 1.7 and 2.10, it follows that Y is s-semiregular. Hence X is s^*-semiregular.

REMARK 2.19. Every s^*-remiregular space is s-semiregular (Lemma 2.18) but not conversely, since the real line R is s-semiregular but not s^*-semiregular.

THEOREM 2.20. *If either X or Y is s^*-semiregular, then every semihomeomorphism f: $X \to Y$ is a homeomorphism.*

PROOF. Corollary 2.12.

REMARK 2.21. Example 2.16 shows that the converse of Theorem 2.20 is not true.

LEMMA 2.22. *If every space semihomeomorphic to X is homeomorphic to X, then X is an α-space.*

PROOF. Since X and X^* are semihomeomorphic, by hypothesis, $X = X^*$. Hence X is an α-space.

THEOREM 2.23. *Let X be an s-semiregular space. A necessary and sufficient condition that every semihomeomorphism f: $X \to Y$ is a homeomorphism for all Y is that X is an s^*-semiregular space.*

PROOF. Theorem 2.20 and Lemma 2.22.

3. Irresolute and presemiopen functions

In [3], S. G. Crossley and S. K. Hildebrand have established that every continuous, open function is irresolute as well as presemiopen [Theorem 1.8]. The following example shows that a semihomeomorphism and hence an irresolute, presemiopen function need not be either continuous or open.

EXAMPLE 3.1. Let $X=\{a, b, c, d\}$ with the topology $\tau=\{X, \{a, b, c\}, \{a, b\}, \emptyset\}$ and $Y=X$ with the topology $\sigma=\{Y, \{a, b, d\}, \{a, b\}, \emptyset\}$. If i: $X\to Y$ is the identity function, then i is a semihomeomorphism but neither continuous nor open.

LEMMA 3.2. *If f: $X\to Y$ is an irresolute, presemiopen function where Y is dense in itself, then f is almost continuous.*

PROOF. Suppose f is not almost continuous. By Lemma 1.11, there exists $V\in$ S.O. (Y) such that $\mathrm{Cl}\,(f^{-1}(V))\not\subset f^{-1}(\mathrm{Cl}\,(V))$. Hence there exists $x\in \mathrm{Cl}(f^{-1}(V))$ such that $x\notin f^{-1}(\mathrm{Cl}\,(V))$. Since f is irresolute, $f^{-1}(V)\in$ S.O. (X) and so by Lemma 1.2, $f^{-1}(V)\cup\{x\}\in$ S.O. (X). Since f is presemiopen, $H=f(f^{-1}(V)\cup\{x\})\in$ S.O. (Y). Now $x\notin f^{-1}(\mathrm{Cl}\,(V))$ implies that $f(x)\notin \mathrm{Cl}\,(V)$ and so there exists an open set G containing $f(x)$ such that $G\cap V=\emptyset$. By Lemma 1.6, $G\cap H\in$ S.O. (Y). Moreover, $f(x)\in G\cap H\subset G\cap(V\cup\{f(x)\})=\{f(x)\}$ and so $\{f(x)\}=G\cap H$. By Lemma 1.3, $\{f(x)\}$ is open in Y, a contradiction to the fact that Y is dense in itself. Hence f is almost continuous.

The following example shows that the condition 'dense in itself' on the space Y in Lemma 3.2 cannot be dropped. Moreover, it also shows that an irresolute, presemiopen function from a space into an s-semiregular (semiregular) space need not be almost continuous.

EXAMPLE 3.3. Let $X=\{a, b, c, d\}$ with the topology $\tau=\{X, \{a, b, c\}, \{a, c\}, \{a, b\}, \{a\}, \{b\}, \emptyset\}$ and $Y=\{a, b\}$ with the discrete topology. Define f: $X\to Y$ by $f(a)=f(c)=a$ and $f(b)=f(d)=b$. f is an irresolute, presemiopen function but not almost continuous, since $f^{-1}(\{b\})=\{b, d\}$ is not open in X.

Piotrowski [7] has established that if f: $X\to Y$ is an irresolute, presemiopen function where Y is regular, T_1 and dense in itself, then f is continuous (Corollary 8).

The following theorem is a generalization of this result.

THEOREM 3.4. *If f: $X\to Y$ is an irresolute, presemiopen function where Y is semiregular and dense in itself, then f is continuous.*

PROOF. Lemma 3.2 and Theorem 2.4 of [8].

REMARK 3.5. Example 3.1 shows that the condition 'semiregular' on the space Y in Theorem 3.4 cannot be dropped. Example 3.3 shows that the condition 'dense in itself' on the space Y cannot be dropped.

DEFINITION 3.6. A function f: $X\to Y$ is said to be *presemiclosed* if $f(A)$ is semiclosed in Y whenever A is semiclosed in X.

REMARK 3.7. A bijection f: $X\to Y$ is presemiopen if and only if f is presemiclosed.

LEMMA 3.8. *Let f: $X\to Y$ be a presemiclosed function where Y is dense in itself. If $\{x\}$ is semiclosed in X, then $\{f(x)\}$ is nowhere dense in Y.*

PROOF. Since f is presemiclosed, $\{f(x)\}$ is semiclosed in Y. By Lemma 2.8, since Y is dense in itself, $\{f(x)\}$ is nowhere dense in Y.

THEOREM 3.9. *If f: $X\to Y$ is an irresolute, presemiopen and presemiclosed function where Y is s-semiregular and dense in itself, then f is continuous.*

PROOF. Suppose f is not continuous. Then there exists a non-empty open set G in Y such that $f^{-1}(G)$ is not open in X, and so there exists $x \in f^{-1}(G)$ such that $x \notin \operatorname{Int}(f^{-1}(G))$. Since $\{x\}$ is nowhere dense and hence semiclosed in X, by Lemma 3.8, $\{f(x)\}$ is nowhere dense in Y. Since Y is s-semiregular, there exists a regular open set H such that $f(x) \in H \subset G$ and so $x \in f^{-1}(H) \subset f^{-1}(G)$. Since $f^{-1}(H)$ is open in X by Lemma 3.2, $x \in \operatorname{Int}(f^{-1}(G))$, a contradiction to the assumption. Hence f is continuous.

Example 3.1 shows that the condition 's-semiregular' on the space Y in Theorem 3.9 cannot be dropped. Example 3.3 shows that the condition 'dense in itself' on the space Y cannot be dropped. The following example shows that the condition 'presemiclosed' on the function cannot be dropped.

EXAMPLE 3.10. Let the space X be the space X of Example 3.1 and $Y=\{x, y, u\}$ with the topology $\delta=\{Y, \{x, y\}, \emptyset\}$. The space Y is s-semiregular and dense in itself. Define $f\colon X \to Y$ by $f(a)=x$, $f(b)=f(d)=y$ and $f(c)=u$. f is both irresolute and presemiopen but neither presemiclosed nor continuous.

LEMMA 3.11. *If $f\colon X \to Y$ is an irresolute, presemiopen and presemiclosed function, then f is almost open.*

PROOF. Suppose f is not almost open. Then there exists a non-empty regular open set G in X such that $f(G)$ is not open in Y and so there exists $x \in G$ such that $f(x) \notin \operatorname{Int}(f(G))$. Since regular open sets are semiclosed and f is presemiclosed, $A=Y-f(G)$ is a non-empty semiopen set not containing $f(x)$. Now $f(x) \notin \operatorname{Int}(f(G))$ implies that $f(x) \in \operatorname{Cl}(A)$ and so by Lemma 1.2, $A \cup \{f(x)\} \in \text{S.O.}(Y)$. Since f is irresolute, $f^{-1}(A \cup \{f(x)\}) \in \text{S.O.}(X)$. By Lemma 1.6, $G \cap f^{-1}(A \cup \{f(x)\}) = B \in \text{S.O.}(X)$. Since f is presemiopen, $f(B) \in \text{S.O.}(Y)$. But $f(x) \in f(B) \subset f(G) \cap (A \cup \{f(x)\}) = \{f(x)\}$ which implies that $\{f(x)\} = f(B)$. By Lemma 1.3, $\{f(x)\}$ is open in Y, a contradiction to the assumption that $f(x) \notin \operatorname{Int}(f(G))$. Hence f is almost open.

The following example shows that the condition 'presemiclosed' on the function f in Lemma 3.11 cannot be dropped.

EXAMPLE 3.12. Let the space X be the space X of Example 3.1 and $Y=X$ with the topology $\sigma=\{Y, \{a, b\}, \emptyset\}$. Define $f\colon X \to Y$ by $f(a)=a$, $f(b)=f(c)=b$ and $f(d)=c$. f is an irresolute, presemiopen function but neither presemiclosed nor almost open.

LEMMA 3.13. *Let $f\colon X \to Y$ be an irresolute and presemiopen function. If $\{y\}$ is nowhere dense in Y, then $f^{-1}(y)$ is nowhere dense in X.*

PROOF. If $\{y\}$ is nowhere dense in Y, by Theorem 1.4 of [3], $f^{-1}(y)$ is semiclosed in X. Suppose $\operatorname{sInt}(f^{-1}(y)) \neq \emptyset$. Let $x \in \operatorname{sInt}(f^{-1}(y))$. Then there exists a semiopen set G such that $x \in G \subset f^{-1}(y)$. Since f is presemiopen, $f(G)=\{y\}$ is semiopen in Y, a contradiction to the fact that $\{y\}$ is nowhere dense in Y. Therefore $\operatorname{sInt}(f^{-1}(y))=\emptyset$. By Lemma 1.8, $f^{-1}(y)$ is nowhere dense in X.

THEOREM 3.14. *If $f\colon X \to Y$ is an irresolute, presemiopen and presemiclosed function where X is s-semiregular, then f is open.*

Proof. Suppose f is not open. Then there exists a non-empty open set G in X such that $f(G)$ is not open in Y. Since $\mathrm{Int}(f(G))\neq\emptyset$, there exists $x\in G$ such that $f(x)\notin\mathrm{Int}(f(G))$. Since $\{f(x)\}$ is nowhere dense and hence semiclosed in Y, by Lemma 3.13, $f^{-1}(\{f(x)\})$ is nowhere dense in X and so $\{x\}$ is nowhere dense in X. Since X is s-semiregular, there exists a regular open set H such that $x\in H\subset G$ and so $f(x)\in f(H)\subset f(G)$. By Lemma 3.11, $f(H)$ is open in Y and so $f(x)\in\mathrm{Int}(f(G))$, a contradiction to the assumption. Hence f is open.

Example 3.1 shows that the condition 's-semiregular' on the space X in Theorem 3.14 cannot be dropped.

Theorem 3.15. *If f: $X\to Y$ is a presemiclosed function from an s-semiregular space X into an s-semiregular and dense in itself space Y, then f is continuous and open if and only if f is irresolute and presemiopen.*

Proof. Theorems 3.9 and 3.14.

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ON DIVISORS OF SUMS OF INTEGERS. I

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§ 1. Introduction. Let N be a positive integer and let $A_1, \dots, A_k$ be non-empty subsets of $\{1, \dots, N\}$. Let $|A_i|$ denote the cardinality of A_i. For any integer n larger than one let $P(n)$ denote the greatest prime factor of n. In [1], Balog and Sárközy proved, by means of the large sieve inequality, that if $|A_1||A_2|>100N(\log N)^2$ and N is sufficiently large then there exist $a_1 \in A_1$ and $a_2 \in A_2$ such that

$$P(a_1+a_2) > \frac{1}{16}\frac{(|A_1||A_2|)^{1/2}}{\log N}.$$

In the same article they obtained a slightly weaker result by means of the Hardy—Littlewood method. We propose to employ the Hardy—Littlewood method in connection with this problem in a sequel to this article. However, the purpose of this note is to estimate $P(a_1+\dots+a_k)$ where $a_1, \dots, a_k$ are chosen from the k sets $A_1, \dots, A_k$ respectively. Put

$$T = \Big(\prod_{j=1}^{k} |A_i|\Big)^{1/k}.$$

THEOREM. *Let $A_1, \dots, A_k$ be non-empty subsets of $\{1, \dots, N\}$ with $|A_1|=\min_i |A_i|$ and $k>1$, and let ε be a positive real number. If*

$$\sum_{i=1}^{k} |A_i| > (1+\varepsilon)N,$$

then for any prime p with $N<p<\left(1+\frac{\varepsilon}{2}\right)N$, there exist $a_i \in A_i$, for $i=1, \dots, k$, such that

$$P(a_1+\dots+a_k) = p, \tag{1}$$

whenever $N>N_0(\varepsilon, k)$. If $T>8N^{1/2}\log N$, then there exist $a_i \in A_i$, for $i=1, \dots, k$, such that

$$P(a_1+\dots+a_k) > \frac{kT}{14\log T}, \tag{2}$$

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for $N > N_1(k)$. *Further, there exist* $a_i \in A_i$, *for* $i=1, \ldots, k$, *such that*

$$P(a_1+\ldots+a_k) > \frac{|A_1|}{N^{1/k+\varepsilon}}, \tag{3}$$

for $N > N_2(\varepsilon, k)$. *Here* $N_0(\varepsilon, k)$, $N_1(k)$ *and* $N_2(\varepsilon, k)$ *are numbers which are effectively computable in terms of* ε *and* k, k, *and* ε *and* k *respectively.*

To prove (1) we appeal to the Cauchy—Davenport Lemma. Note that we are able to specify the greatest prime factor of $a_1+\ldots+a_k$ in this case. For the proof of (2) we use the large sieve inequality in conjunction with the Cauchy—Davenport Lemma. If $k=2$ then (2) yields the result of Balog and Sárközy referred to above. Finally, (3) is obtained using the Cauchy—Davenport Lemma and Gallagher's larger sieve.

In the following result, which is an immediate consequence of our theorem, we require that all the summands be taken from a single set.

COROLLARY. *Let* A *be a non-empty subset of* $\{1, \ldots, N\}$, *let* ε *be a positive real number and let* k *be an integer larger than one. If* $|A| > (1+\varepsilon)N/k$ *and* p *is any prime number with* $N < p < \left(1+\frac{\varepsilon}{2}\right)N$ *then there exist* $a_1, \ldots, a_k$ *in* A *such that*

$$P(a_1+\ldots+a_k) = p,$$

for N *sufficiently large in terms of* ε *and* k. *Further, if* $|A| > 8N^{1/2}\log N$ *then there exist* $a_1, \ldots, a_k$ *in* A *such that*

$$P(a_1+\ldots+a_k) > \frac{k|A|}{14\log|A|},$$

for N *sufficiently large in terms of* k. *Furthermore, there exist* $a_1, \ldots, a_k$ *in* A *such that*

$$P(a_1+\ldots+a_k) > \frac{|A|}{N^{1/k+\varepsilon}},$$

for N *sufficiently large in terms of* ε *and* k.

It would be interesting if one could obtain results of comparable strength to the above for subsets of $\{1, \ldots, N\}$ of cardinality less than $N^{1/k}$. The only result of which we are aware in this connection is due to Erdős and Turán [4]. They showed, in 1934, by means of an elementary argument that for any finite set of positive integers A there exist integers a_1 and a_2 from A such that

$$P(a_1+a_2) > c\log|A|,$$

for a positive constant c.

§2. Preliminary lemmas. Let $\mathbf{Z}$ denote the set of integers.

LEMMA 1 (Cauchy—Davenport [2], [3]). *Let* p *be a prime number and let* A *and* B *be a subsets of* $\mathbf{Z}/p\mathbf{Z}$. *If* $|A|=m$ *and* $|B|=n$ *then* $|A+B| \geqq \min\{m+n-1, p\}$; *here* $A+B=\{a+b \mid a\in A, b\in B\}$.

LEMMA 2 (large sieve). *Let $\mathcal{N}$ be a set of integers in the interval $[M+1, M+N]$. For each prime p let $v(p)$ denote the number of residue classes modulo p that contain an element of $\mathcal{N}$. Then for any positive integer Q we have*

$$|\mathcal{N}| \leqq \frac{N+Q^2}{L}, \quad for \quad L = \sum_{q\leqq Q}' \prod_{p|q} \frac{p-v(p)}{v(p)},$$

where the summation is taken over square-free positive integers q.

PROOF. See Theorem 7.1 of [8].

LEMMA 3 (Gallagher [5]). *Let $\mathcal{N}$ be a set of integers in the interval $[M+1, M+N]$. For each prime p let $v(p)$ denote the number of residue classes modulo p that contain an element of $\mathcal{N}$. Then for any finite set of primes S we have*

$$|\mathcal{N}| \leqq \frac{\sum_{p\in S} \log p - \log N}{\sum_{p\in S} \frac{\log p}{v(p)} - \log N},$$

provided that the denominator is positive.

We shall also require the following result.

LEMMA 4. *Let p and k be integers with $k\geqq 2$ and $p-1\geqq (k-1)^k$. Let $D=\left\{(x_1, \ldots, x_k)\in \mathbf{R}^k \,|\, x_1+\ldots+x_k \leqq 1+\frac{k-2}{p} \text{ and } \frac{1}{p}\leqq x_i \leqq \frac{p-1}{p} \text{ for } i=1, \ldots, k\right\}$. Then*

$$\min_D \prod_{i=1}^{k} \left(\frac{1}{x_i}-1\right) = \left(\frac{k}{1+\frac{k-2}{p}}-1\right)^k \tag{4}$$

and

$$\min_D \sum_{i=1}^{k} \frac{1}{x_i} = \frac{k^2}{1+\frac{k-2}{p}}. \tag{5}$$

PROOF. First we shall establish (4) by induction on k. It is readily checked that (4) holds for $k=2$ and so we may assume that $k>2$. Our inductive hypothesis is that (4) holds with $k-1$ in place of k. We observe that the minimum of $\prod_{i=1}^{k}\left(\frac{1}{x_i}-1\right)$ in D, occurs in D_0 where $D_0=\left\{(x_1, \ldots, x_k)\in D \,|\, x_1+\ldots+x_k=1+\frac{k-2}{p}\right\}$. Note also that $\prod_{i=1}^{k}\left(\frac{1}{x_i}-1\right)$ and $\sum_{i=1}^{k} \log\left(\frac{1}{x_i}-1\right)$ achieve their minimum value in D_0 at the same points. Applying the method of Lagrange multipliers we conclude that if $\sum_{i=1}^{k} \log\left(\frac{1}{x_i}-1\right)$ has a local minimum at $(x_1, \ldots, x_k)$ in the interior of D_0 then for all integers i and j with $1\leqq i,j\leqq k$ either $x_i=x_j$ or $x_i=1-x_j$. If $x_i=x_j$ for all i and j then

$$\prod_{i=1}^{k}\left(\frac{1}{x_i}-1\right)=\left(\frac{k}{1+\frac{k-2}{p}}-1\right)^{k}.$$

On the other hand if $x_i=1-x_j$ for some i and j then $x_i+x_j=1$ and by the definition of D_0 we have $x_l=\frac{1}{p}$ for some integer l. Similarly if $(x_1, \ldots, x_k)$ is on the boundary of D_0 then we again have $x_l=\frac{1}{p}$ for some integer l. However, if $x_l=\frac{1}{p}$ and $(x_1, \ldots, x_k)$ is in D_0 then

$$\prod_{i=1}^{k}\left(\frac{1}{x_i}-1\right)=(p-1)\prod_{\substack{i=1\\ i\neq l}}^{k}\left(\frac{1}{x_i}-1\right)\geqq(p-1)\min_{D'}\prod_{\substack{i=1\\ i\neq l}}^{k}\left(\frac{1}{x_i}-1\right),$$

where

$$D'=\{(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_k)\in\mathbf{R}^{k-1}\,|\,x_1+\ldots+x_{l-1}+x_{l+1}+\ldots+x_k\leqq 1+\frac{k-3}{p}$$
$$\text{and}\quad \frac{1}{p}\leqq x_i\leqq\frac{p-1}{p}\quad\text{for}\quad i=1, \ldots, l-1, l+1, \ldots, k\}.$$

By our inductive hypothesis the minimum over D' of $\prod_{\substack{i=1\\ i\neq l}}^{k}\left(\frac{1}{x_i}-1\right)$ is $\left(\frac{k-1}{1+\frac{k-3}{p}}-1\right)^{k-1}$, which is at least 1 since $k>2$. Therefore if $(x_1, \ldots, x_k)$ is a point in D_0 with $x_l=\frac{1}{p}$ then

$$\prod_{i=1}^{k}\left(\frac{1}{x_i}-1\right)\geqq p-1\geqq(k-1)^k>\left(\frac{k}{1+\frac{k-2}{p}}-1\right)^{k},$$

consequently the minimum of $\prod_{i=1}^{k}\left(\frac{1}{x_i}-1\right)$ on D occurs with

$$x_1=\ldots=x_k=\frac{1+\frac{k-2}{p}}{k}.$$

Thus (4) holds and this completes the induction.

To establish (5) requires only a routine application of the method of Lagrange multipliers. Alternatively (5) can be deduced from the arithmetic-harmonic mean inequality.

§3. Proof of the main theorem. We shall first prove (1). Let p be a prime with $N<p<\left(1+\frac{\varepsilon}{2}\right)N$. Assume that $N>\max\left\{\frac{2(k-1)}{\varepsilon}, k\right\}$ and put $A_i(p)=\{a+p\mathbf{Z}|a\in A_i\}$ for $i=1, \ldots, k$. By repeated application of Lemma 1 we find that

$$(6)\qquad |A_1(p)+\ldots+A_k(p)| \geqq \min\left\{\sum_{i=1}^{k}|A_i(p)|-(k-1), p\right\}.$$

Since $A_i\subseteqq\{1, \ldots, N\}$ and $p>N$, $|A_i(p)|=|A_i|$. Therefore

$$\sum_{i=1}^{k}|A_i(p)| > (1+\varepsilon)N$$

and, since $\frac{\varepsilon}{2}N>k-1$, the minimum on the right hand side of (6) is p. Accordingly, $|A_1(p)+\ldots+A_k(p)|=p$, hence $A_1(p)+\ldots+A_k(p)=\mathbf{Z}/p\mathbf{Z}$. Therefore there exist $a_i\in A_i$, for $i=1, \ldots, k$, with $p|a_1+\ldots+a_k$. Since $a_1+\ldots+a_k\leqq kN$, $k<N$ and $p>N$, $P(a_1+\ldots+a_k)=p$ as required.

To prove (2) we assume that $T\geqq 8N^{1/2}\log N$ and we put $Q=\frac{kT}{7\log T}$. Further, we shall suppose that N is chosen sufficiently large for the subsequent argument; in particular, large enough that $\frac{Q}{2}>(k-1)^k$, $T^{1/7}>k$ and $N^{1/2}>8\log N$. We shall now show that the assumption that $P(a_1+\ldots+a_k)<\frac{Q}{2}$ whenever $a_i\in A_i$, $i=1, \ldots, k$, leads to a contradiction and this will establish (2).

Applying Lemma 2 with $M=0$, we find that

$$|A_i| < \frac{N+Q^2}{\sum\limits_{Q/2<p<Q}\frac{p-\nu_i(p)}{\nu_i(p)}},$$

where the summation in the denominator is taken over primes p and where $\nu_i(p)$ is the number of residue classes modulo p that contain an element of A_i. Thus

$$(7)\qquad T<\frac{N+Q^2}{H},$$

where

$$H=\left(\prod_{i=1}^{k}\sum_{Q/2<p<Q}\frac{p-\nu_i(p)}{\nu_i(p)}\right)^{1/k}.$$

By a generalization of the Cauchy—Schwarz inequality (see 81.3, page 68 of [7]),

$$(8)\qquad H\geqq\sum_{Q/2<p<Q}\left(\prod_{i=1}^{k}\frac{p-\nu_i(p)}{\nu_i(p)}\right)^{1/k}.$$

Define $A_i(p)$ as above and notice that, by Lemma 1, we again obtain (6). However, for each prime p with $\frac{Q}{2}<p<Q$, $A_1(p)+\ldots+A_k(p)$ does not contain the zero

residue class hence $|A_1(p)+\ldots+A_k(p)|\leqq p-1$. Further, $\nu_i(p)=|A_i(p)|$ and therefore

(9) $$\nu_1(p)+\ldots+\nu_k(p)\leqq p+k-2.$$

Certainly $1\leqq \nu_i(p)\leqq p-1$ and thus putting $\frac{\nu_i(p)}{p}=x_i$ and applying (4) of Lemma 4 we find, since $p>\frac{Q}{2}>(k-1)^k$, that

(10) $$\left(\prod_{i=1}^{k}\left(\frac{p}{\nu_i(p)}-1\right)\right)^{1/k} \geqq \frac{k}{1+\frac{k-2}{p}}-1 \geqq \frac{k}{2}.$$

By the prime number theorem,

(11) $$\sum_{Q/2<p<Q}\frac{k}{2} > \frac{kQ}{5\log Q},$$

for N sufficiently large. Combining (7), (8) (10) and (11) we obtain

$$T<\frac{N+Q^2}{\frac{kQ}{5\log Q}}.$$

By assumption $N^{1/2}>8\log N$ and so $N<\frac{1}{5}Q^2$. Thus

$$\frac{kT}{6}<Q\log Q\leqq \frac{kT}{7\log T}\log kT,$$

and, since $T^{1/7}>k$,

$$\frac{kT}{7\log T}\log kT<\frac{kT}{6}.$$

This gives the required contradiction.

Finally, we shall prove (3). We may assume without loss of generality that ε is less than one. Put

$$Q=\frac{|A_1|}{N^{1/k+\varepsilon/2}} \quad\text{and}\quad Q_1=|A_1|.$$

We shall assume that N is sufficiently large in terms of ε and k for the validity of the argument to follow. Further we shall assume that $Q>N^{\varepsilon/2}$ and that $P(a_1+\ldots+a_k)\leqq Q$, whenever $a_i\in A_i$, $i=1,\ldots,k$. Let $\nu_i(p)$ denote the number of residue classes modulo p that contain an element of A_i. By Lemma 3,

(12) $$|A_i|\leqq \frac{\sum_{Q<p<Q_1}\log p-\log N}{\sum_{Q<p<Q_1}\frac{\log p}{\nu_i(p)}-\log N},$$

for $i=1, \ldots, k$, whenever the denominator is positive; here the summations are taken over all primes p between Q and Q_1. We shall show that for at least one integer i the denominator in (12) is at least $\frac{\varepsilon}{2}\log N$. As before we find that (9) holds for each prime p with $Q<p<Q_1$. Since $1\leqq v_i(p)\leqq p-1$, on putting $\frac{v_i(p)}{p}=x_i$ and applying (5) of Lemma 4 we find that

$$\frac{1}{k}\sum_{i=1}^{k}\frac{\log p}{v_i(p)} \geqq \frac{\log p}{p}\,\frac{k}{1+\frac{k-2}{p}}.$$

Since $p>Q\geqq N^{\varepsilon/2}$,

$$\frac{k}{1+\frac{k-2}{p}} > k-\frac{\varepsilon}{8}$$

for N sufficiently large. Thus

$$\frac{1}{k}\sum_{i=1}^{k}\left(\sum_{Q<p<Q_1}\frac{\log p}{v_i(p)}-\log N\right) = \sum_{Q<p<Q_1}\left(\frac{1}{k}\sum_{i=1}^{k}\frac{\log p}{v_i(p)}\right)-\log N \geqq$$

$$\geqq \sum_{Q<p<Q_1}\left(k-\frac{\varepsilon}{8}\right)\frac{\log p}{p}-\log N.$$

By Theorem 425 of [6] there is a constant C such that the right hand side of the above inequality is

$$\geqq \left(k-\frac{\varepsilon}{8}\right)(\log Q_1-\log Q-C)-\log N \geqq \left(k-\frac{\varepsilon}{4}\right)\log(Q_1/Q)-\log N.$$

This in turn is $>\frac{\varepsilon}{2}\log N$, since $k\geqq 2$, $\varepsilon<1$ and $Q_1/Q=N^{1/k+\varepsilon/2}$. Since the average of the denominators in (12) is at least $\varepsilon/2\log N$, for at least one integer i,

$$|A_i| \leqq \frac{\sum_{Q<p<Q_1}\log p-\log N}{\frac{\varepsilon}{2}\log N}.$$

Hence, by the prime number theorem,

$$|A_i| \leqq \frac{4Q_1}{\varepsilon\log N} = \frac{4|A_1|}{\varepsilon\log N},$$

for N sufficiently large. But $|A_1|\leqq|A_i|$ and so we have a contradiction for N sufficiently large. Therefore either $P(a_1+\ldots+a_k)>Q$ for some $a_1\in A_i$, $i=1, \ldots, k$ or $Q<N^{\varepsilon/2}$. Consequently, for some $a_i\in A_i$, $i=1, \ldots, k$,

$$P(a_1+\ldots+a_k) > \frac{Q}{N^{\varepsilon/2}} = \frac{|A_i|}{N^{1/k+\varepsilon}},$$

as required.

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SOME MONOTONICITY PROPERTIES OF THE ZEROS OF ULTRASPHERICAL POLYNOMIALS*

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1. Introduction

A well known result of Stieltjes [4, p. 121] asserts that the k^{th} positive zero $x_{nk}^{(\lambda)}$, $k=1, 2, \ldots, \left[\frac{n}{2}\right]$ of the ultraspherical Jacobi polynomial $P_n^{(\lambda)}(x)$ decreases as λ increases, for $\lambda>0$. Recently one of the present authors [3] proved that the function $\lambda x_{nk}^{(\lambda)}$ increases as λ increases provided $0<\lambda<1$ and $k=1, 2, \ldots, \left[\frac{n}{2}\right]$. Ahmed [1] extended this result to the larger interval $0<\lambda\leqq 3/2$.

In this paper we prove further monotonicity properties of $x_{nk}^{(\lambda)}$ and $\theta_{nk}^{(\lambda)}= =\arccos x_{nk}^{(\lambda)}$. Our principal tool is the Sturmian comparison theorem which will be enunciated in the following lemma.

LEMMA (Sturm comparison theorem). *Let the functions $y(x)$ and $Y(x)$ be nontrivial solutions of the differential equations*

$$y''(x)+f(x)y(x) = 0, \quad Y''(x)+F(x)Y(x) = 0,$$

and let them have consecutive zeros at $x_1, x_2, \ldots, x_m$ and $X_1, X_2, \ldots, X_m$ respectively on the intervals (a, b) and (a, B).

Suppose that the coefficients $f(x)$, $F(x)$ are continuous, that $f(x)<F(x)$ on the interval (a, X_m) and that

$$\lim_{x\to a^+} [y'(x)Y(x)-y(x)Y'(x)] = 0. \tag{1.1}$$

Then $X_k<x_k$ for $k=1, 2, \ldots, m$.

This version of the Sturmian theorem, taken from [2, p. 247] differs from the usual formulation in the fact that here we require the inequality $f(x)<F(x)$ only on the interval (a, X_m), and not on the larger interval (a, x_m) or on (a, b).

The connection between the two differential equations under the conditions of the Lemma is simply characterised by saying that the second differential equation is a Sturmian majorant of the first one.

The main results of this paper are the following theorems.

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THEOREM 1. *Let* i, j, n, m *be natural numbers such that* $m \geqq n \geqq 2$, $1 \leqq i \leqq \left[\frac{n}{2}\right]$, $1 \leqq j \leqq \left[\frac{n}{2}\right]$. *Then for* $-1/2 < \lambda \leqq 3/2$ *the determinant*

$$\text{sign}\, l \cdot \begin{vmatrix} x_{ni}^{(\lambda)} & x_{n,i+l}^{(\lambda)} \\ x_{mj}^{(\lambda)} & x_{m,j+l}^{(\lambda)} \end{vmatrix} \tag{1.2}$$

is positive provided $x_{mj}^{(\lambda)} > x_{ni}^{(\lambda)}$ *and the integer* $l \neq 0$ *satisfies the relations* $-\min\{i-1, j-1\} < l < \min\{n-i, m-j\}$.

THEOREM 2. *Suppose that* $-1/2 < \lambda < \lambda'$, $0 \leqq \lambda' \leqq 1/2$ *and* $n \geqq 4$. *Then*

$$\text{sign}(i-k) \begin{vmatrix} \theta_{ni}^{(\lambda)} & \theta_{ni}^{(\lambda')} \\ \theta_{nk}^{(\lambda)} & \theta_{nk}^{(\lambda')} \end{vmatrix} > 0$$

provided $1 \leqq k \leqq \left[\frac{n}{2}\right]$, $i \neq k$.

2. The proofs

In this section we shall give the proofs of the theorems announced above and also two corollaries of Theorem 1.

PROOF OF THEOREM 1. The function

$$u_{n\lambda}(x) = (1-x^2)^{\lambda/2+1/4} P_n^{(\lambda)}(x)$$

satisfies the differential equation

$$y''(x) + p_{n,\lambda}(x) y(x) = 0, \quad -1 < x < 1, \tag{2.1}$$

where

$$p_{n,\lambda}(x) = -\frac{(n+\lambda)^2}{1-x^2} + \frac{2+4\lambda-4\lambda^2+x^2}{4(1-x^2)^2}. \tag{2.2}$$

Substituting x by $x_{ni}^{(\lambda)} t$ we have that the function $u_{n\lambda}(x_{ni}^{(\lambda)} t)$ is a solution of the differential equation

$$z''(t) + q_{n,\lambda,i}(t) z(t) = 0, \tag{2.3}$$

where

$$q_{n,\lambda,i}(t) = (x_{ni}^{(\lambda)})^2 p_{n,\lambda}(x_{ni}^{(\lambda)} t). \tag{2.4}$$

Besides (2.3) we consider the differential equation

$$w''(t) + q_{m,\lambda,j}(t) w(t) = 0 \tag{2.5}$$

satisfied by $u_{m\lambda}(x_{mj}^{(\lambda)}t)$. The functions $u_{n\lambda}(x_{ni}^{(\lambda)}t)$ and $u_{m\lambda}(x_{mj}^{(\lambda)}t)$ have consecutive zeros at

$$t_{nk}=\frac{x_{nk}^{(\lambda)}}{x_{ni}^{(\lambda)}},\quad k=1,2,\dots,n,$$

$$t_{mk}=\frac{x_{mk}^{(\lambda)}}{x_{mj}^{(\lambda)}},\quad k=1,2,\dots,m,$$

respectively.

For the sake of short notation we introduce ξ and η by $\xi=x_{ni}^{(\lambda)}$, $\eta=x_{mj}^{(\lambda)}$. Then by the conditions of Theorem 1, $0<\xi<\eta$. We claim that the differential equation (2.5) is a Sturmian majorant for (2.3) if $-1/\eta<t<1/\eta$. To prove this it is sufficient to show that the function Δ defined by

$$\Delta=4(1-\xi^2t^2)^2(1-\eta^2t^2)^2[q_{m,\lambda,j}(t)-q_{n,\lambda,i}(t)]$$

is positive. Using (2.2) and (2.4) we get

$$\Delta=4(m-n)(m+n+2\lambda)(1-\xi^2t^2)^2(1-\eta^2t^2)\eta^2+$$
$$+(\eta^2-\xi^2)[4N+A-(4N-1)(\xi^2+\eta^2)t^2+(4N-A-2)\xi^2\eta^2t^4]=\Delta_1+\Delta_2,$$

where $N=(n+\lambda)^2$, $A=2+4\lambda-4\lambda^2$. It is clear that $\Delta_1\geqq 0$. Let α,β be defined by $\alpha=\xi^2t^2$, $\beta=\eta^2t^2$, then $0<\alpha<\beta<1$; thus the term Δ_2 can be written as $\Delta_2=(\eta^2-\xi^2)d_2$, where

$$d_2=d_2(\alpha,\beta)=4N+A-(4N-1)(\alpha+\beta)+(4N-A-2)\alpha\beta.$$

The values of d_2 depend linearly on α and β, hence

$$d_2\geqq\min\{d_2(0,0),\ d_2(0,1),\ d_2(1,0),\ d_2(1,1)\}=\min\{4N+A,\ A+1,\ 0\}.$$

By our restrictions on λ, n we have

$$A+1=-4\lambda^2+4\lambda+3=(-2\lambda+3)(2\lambda+1)\geqq 0$$

and

$$d_2(0,0)=4N+A\geqq 4N-1=(2n+2\lambda+1)(2n+2\lambda-1)>0.$$

Hence Δ is positive and the differential equation (2.5) is a Sturmian majorant for (2.3). Now the functions $u_{n\lambda}(x_{ni}^{(\lambda)}t)$ and $u_{m\lambda}(x_{mj}^{(\lambda)}t)$ have a commo zero at $t=1$ and the limit condition (1.1) is trivially satisfied with $a=1$ in the present case. Applying our Lemma to the differential equations (2.3) and (2.5) we obtain the inequalities

$$t_{n,i+l}<t_{m,j+l}\quad\text{for}\quad l=1,2,\dots,\min\{n-i,m-j\},$$

$$t_{n,i-l}>t_{m,j-l}\quad\text{for}\quad l=1,2,\dots,\min\{i-1,j-1\},$$

which is equivalent to the positive sign of the determinant (1.2). Thus the proof of Theorem 1 is complete.

COROLLARY 1. *Let* $1\leqq i\leqq\left[\frac{n}{2}\right]$ *and* $-1/2<\lambda\leqq 3/2$. *Then the quotient* $x_{nk}^{(\lambda)}/x_{ni}^{(\lambda)}$ *increases with respect to* n *for* $k>i$ *and decreases for* $1\leqq k<i$.

PROOF. The conclusions of the Corollary follow from Theorem 1 when we consider the case $i=j$ and $m>n$. In fact under these hypotheses we know that $x_{ni}^{(\lambda)}<x_{mi}^{(\lambda)}$ and then Theorem 1 yields

$$\operatorname{sign}(k-i)\cdot\begin{vmatrix} x_{ni}^{(\lambda)} & x_{nk}^{(\lambda)} \\ x_{mi}^{(\lambda)} & x_{mk}^{(\lambda)} \end{vmatrix} > 0,$$

from which the desired result follows.

COROLLARY 2. *Let* $1\leqq j<i\leqq\left[\frac{n}{2}\right]$ *and* $-\frac{1}{2}<\lambda\leqq\frac{3}{2}$. *Then the quotient* $x_{n,i+l}^{(\lambda)}/x_{n,j+l}^{(\lambda)}$ *increases when the integer* l *increases from* 0 *to* $n+1-i$.

PROOF. Since $x_{ni}^{(\lambda)}<x_{nj}^{(\lambda)}$ we can apply Theorem 1 with $m=n$ obtaining the desired result.

PROOF OF THEOREM 2. Let us consider the function $v_{n\lambda}(\theta)$ defined by

$$v_{n\lambda}(\theta)=(\sin\theta)^{\lambda}P_n^{(\lambda)}(\cos\theta)$$

which satisfies the differential equation

$$\frac{d^2v}{d\theta^2}+r_{n\lambda}(\theta)v=0, \tag{2.6}$$

where

$$r_{n\lambda}(\theta)=(n+\lambda)^2+\frac{\lambda(1-\lambda)}{\sin^2\theta}. \tag{2.7}$$

For the sake of simplicity we introduce the notations c and d by $c=\theta_{nk}^{(\lambda)}$ and $d=\theta_{nk}^{(\lambda')}$. Due to the restrictions on λ,λ' we have $c<d$. The functions $v_1(\theta)=v_{n\lambda}(c\theta)$ and $v_2(\theta)=v_{n\lambda}(d\theta)$ satisfy the differential equations

$$\frac{d^2v_1}{d\theta^2}+s_{n\lambda}(\theta)v_1=0 \tag{2.8}$$

and

$$\frac{d^2v_2}{d\theta^2}+s_{n\lambda'}(\theta)v_2=0, \tag{2.9}$$

respectively, where $s_{n\lambda}(\theta)=c^2r_{n\lambda}(c\theta)$ and $s_{n\lambda'}(\theta)=d^2r_{n\lambda'}(d\theta)$. However the zeros of $v_1(\theta)$ and $v_2(\theta)$ are at $\theta_{nj}^{(\lambda)}/c$, $j=1,\dots,n$ and $\theta_{nj}^{(\lambda')}/d$, $j=1,\dots,n$, respectively, satisfying the relations

$$0<\frac{\theta_{n1}^{(\lambda)}}{c}<\frac{\theta_{n2}^{(\lambda)}}{c}<\dots<\frac{\theta_{n,k-1}^{(\lambda)}}{c}<1<\frac{\theta_{n,k+1}^{(\lambda)}}{c}<\dots<\frac{\theta_{nn}^{(\lambda)}}{c}<\frac{\pi}{c}$$

and

$$0<\frac{\theta_{n1}^{(\lambda')}}{d}<\frac{\theta_{n2}^{(\lambda')}}{d}<\dots<\frac{\theta_{n,k-1}^{(\lambda')}}{d}<1<\frac{\theta_{n,k+1}^{(\lambda')}}{d}<\dots<\frac{\theta_{nn}^{(\lambda')}}{d}<\frac{\pi}{d}.$$

We claim again that the differential equation (2.9) is a Sturmian majorant for (2.8) if $0<\theta<\pi/d$. In order to prove this statement we must show that the inequality

$$d^2\left[(n+\lambda')+\frac{\lambda'(1-\lambda')}{\sin^2(d\theta)}\right]>c^2\left[(n+\lambda)^2+\frac{\lambda(1-\lambda)}{\sin^2(c\theta)}\right]$$

holds. Since $n+\lambda'>n+\lambda>0$ and $\lambda'(1-\lambda')\geqq\max\{0,\lambda(1-\lambda)\}$, it is sufficient to show the validity of the inequality

$$\frac{d^2}{\sin^2(d\theta)}>\frac{c^2}{\sin^2(c\theta)}\quad \text{if}\quad 0<\theta<\frac{\pi}{d}.$$

But this holds because the function $\theta/\sin\theta$ increases for $0<\theta<\pi$ and we have $0<c<d$ and $0<d\theta<\pi$. Moreover, in this case the common zero of $v_1(\theta)$ and $v_2(\theta)$ is at $\theta=1$ and the limit condition (1.1) in our Lemma is satisfied with $a=1$. So an application of the Sturmian comparison theorem gives

$$\frac{\theta_{nj}^{(\lambda')}}{d}>\frac{\theta_{nj}^{(\lambda)}}{c}\quad \text{for}\quad j=1,2,\dots,k-1,$$

$$\frac{\theta_{nj}^{(\lambda')}}{d}<\frac{\theta_{nj}^{(\lambda)}}{c}\quad \text{for}\quad j=k+1,\dots,n.$$

This completes the proof of Theorem 2.

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A STRONG APPROXIMATION THEOREM FOR SUMS OF RANDOM VECTORS IN THE DOMAIN OF ATTRACTION TO A STABLE LAW

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1. Introduction. Let $x_1, x_2, \ldots$ be independent, identically distributed symmetric random variables in the domain of normal attraction to a stable law G_α of exponent $0<\alpha<2$. As is well-known this is the case if and only if the distribution function F of x_1 satisfies

$$1-F(t) = ct^{-\alpha}+\beta(t)t^{-\alpha}, \quad t>0 \tag{1.1}$$

where $c>0$ and $\beta(t)\to 0$ as $t\to\infty$. Further, if this condition is satisfied, then $\{x_i\}$ are outside of the domain of partial attraction of the normal distribution and thus, by a theorem of Heyde [7], for any positive numerical sequence $\{a_n\}$, we have

$$\lim_{n\to\infty} \frac{x_1+\ldots+x_n}{a_n} = 0 \quad \text{a.s.} \quad \text{or} \quad \overline{\lim_{n\to\infty}} \frac{x_1+\ldots+x_n}{a_n} = +\infty \quad \text{a.s.} \tag{1.2}$$

according as $\sum_{n=1}^{\infty} P\{|x_1|>a_n\}$ converges or diverges. We say that $\{a_n\}$ belongs to the upper (lower) class if the first (second) relation of (1.2) holds.

In [12] Stout proved an almost sure invariance principle, the first of its kind, which can be summarized as follows: Let $x_1, x_2, \ldots$ be independent symmetric random variables with common distribution function F satisfying (1.1) with $c>0$, $0<\alpha<2$, $\beta(t)\to 0$. Let y be a symmetric stable random variable belonging to the same parameters c, α, i.e.

$$P\{y>t\} \sim ct^{-\alpha} \quad \text{as} \quad t\to\infty. \tag{1.3}$$

Let $\{a_n\}$ be an upper class sequence. Then under some regularity conditions on β there exists a sequence of independent random variables $\{y_i,\ i=1,2,\ldots\}$ each having the same distribution as y such that

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = O\big(a_n\beta(a_n)\big) \quad \text{a.s.} \tag{1.4}$$

Obviously, $a_n=(n\log n)^{1/\alpha}(\log n)^\gamma$ belongs to the upper class iff $\gamma>0$. Thus, if $\beta(t)=O(t^{-\lambda})$ for some $\lambda>0$, then the remainder term in (1.4) is $O(n^{1/\alpha-\tau})$ for some $\tau>0$. Similarly, if $\beta(t)=O\big((\log t)^{-\lambda}\big)$ for some $\lambda>1/\alpha$, then the remainder term in (1.4) is $O\big(n^{1/\alpha}(\log n)^{-\tau}\big)$ for some $\tau>0$. However, $a_n=(n\log n)^{1/\alpha}$ belongs to the lower class and thus if $\beta(t)\sim(\log t)^{-\lambda}$ for $\lambda<1/\alpha$, then no matter how we choose the upper class sequence $\{a_n\}$, the remainder term in (1.4) cannot be $O(n^{1/\alpha})$.

Thus, if $\beta(t)$ tends to zero very slowly, Stout's theorem does not even imply the weak limit theorem

$$(1.5) \qquad n^{-1/\alpha}(x_1+\ldots+x_n) \xrightarrow{D} G_\alpha.$$

One purpose of our paper is to extend Stout's theorem to cover functions β tending to zero arbitrarily slowly.

THEOREM 1. *Let $x_1, x_2, \ldots$ be independent identically distributed symmetric random variables with distribution function $F(x)$ satisfying*

$$(1.6) \qquad 1-F(t) = ct^{-\alpha}+\beta(t)t^{-\alpha}, \quad t \geqq t_0$$

where $c>0$, $0<\alpha<2$, $\beta(t)\to 0$ as $t\to+\infty$ and $\beta(t)t^{-\alpha}$ is decreasing for $t\geqq t_0$. Then, if the probability space is rich enough, there exist sequences $\{y_i, i\geqq 1\}$ and $\{z_i, i\geqq 1\}$ of independent random variables such that

$$(1.7) \qquad \sum_{i=1}^{n}(x_i-y_i-z_i) \ll n^{1/\alpha-\lambda}$$

where $\lambda>0$ is a constant, the common characteristic function of the y_i's is $\exp(-c_1|t|^\alpha)$ for some $c_1>0$ and the z_i's have a common symmetric distribution function $F_1(x)$ satisfying

$$(1.8) \qquad 1-F_1(t) = \beta(t)t^{-\alpha}, \quad t\geqq t_0.$$

Finally, the sequences $\{y_i, i\geqq 1\}$ and $\{z_i, i\geqq 1\}$ are independent of each other.

Note that in Theorem 1 we approximate the partial sums $\sum_{i=1}^{n} x_i$ not by $\sum_{i=1}^{n} y_i$ of i.i.d. stable random variables but by $\sum_{i=1}^{n} y_i+\sum_{i=1}^{n} z_i$, where y_i are i.i.d. stable random variables and z_i are small "correcting" random variables which are i.i.d. and also independent of the y_i's. As Theorem 1 shows, such an approximation is always possible (no matter how slowly β tends to zero) with a remainder term $O(n^{1/\alpha-\lambda})$, $\lambda>0$. Rewriting (1.7) in the form

$$(1.9) \qquad \sum_{i=1}^{n} x_i-\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i+O(n^{1/\alpha-\lambda}) \quad \text{a.s.}$$

we see that Theorem 1 enables one to decompose the remainder term $O(a_n\beta(a_n))$ in Stout's theorem (1.4) in the form $\sum_{i=1}^{n} z_i+O(n^{1/\alpha-\lambda})$ where the z_i's are independent random variables with common distribution function F_1 defined by (1.8). Consequences of this decomposition will be discussed in Section 4. As we shall see, (1.9) is more informative than Stout's theorem for $\beta(t)$ tending to zero more slowly than any negative power of t; for example, it enables one to derive limit theorems for $\{x_n\}$ even when the remaider term $O(a_n\beta(a_n))$ in Stout's theorem (1.4) is too weak to do that. Moreover, (1.9) leads to an improvement of Stout's theorem for $\alpha<1$, see (4.7).

Stout's proof is based on the quantile transform method and is thus restricted to the treatment of real-valued random variables. Since our proof of Theorem 1 is based on a multidimensional approximation theorem of Berkes and Philipp [3] we can also treat vector-valued random variables. The second purpose of this note is to prove such an invariance principle.

Let $\{x_j, j \geqq 1\}$ be a sequence of independent $\mathbf{R}^d$-valued random vectors with common distribution function F, belonging to the domain of normal attraction to a stable law G_α of exponent $0<\alpha<2$. Let

$$\nu(m) = \int_{\mathbf{R}^d} |x|^m |F-G_\alpha|(dx), \quad m = 1, 2, \ldots \tag{1.10}$$

be the pseudomoments of $F-G_\alpha$. Then we have

THEOREM 2. *Let* $\{x_j, j \geqq 1\}$ *be as above and suppose that*

$$\nu(1+[\alpha]) < \infty. \tag{1.11}$$

Then, without changing its probability law, we can redefine the sequence $\{x_j, j \geqq 1\}$ *on a new probability space together with a sequence* $\{y_j, j \geqq 1\}$ *of independent random variables with common distribution* G_α *such that*

$$\sum_{j \leqq n} (x_j - y_j) \ll n^{1/\alpha - \lambda} \quad a.s. \tag{1.12}$$

with probability 1 *where* $\lambda = \dfrac{1-\{\alpha\}}{32(d+1)\alpha}$.

Here, $[\alpha]$ *and* $\{\alpha\}$ *denote the integral and fractional part of* α, *respectively.*

Theorem 2 gives new information only for $d>1$ since for $d=1$ (1.11) implies that the distribution functions F and G_α of x_1 and y_1 satisfy $|F(t)-G_\alpha(t)| = O(t^{-[\alpha]})$, $t \to \pm\infty$ and thus

$$1-F(t) = ct^{-\alpha} + O(t^{-\alpha-\lambda}) \quad (\lambda > 0).$$

But then, Stout's theorem also yields (in the symmetric case) a remainder term $O(n^{1/\alpha-\lambda_1})$, $\lambda_1>0$. Note also that for $\alpha=2$, Theorem 2 is well-known (see [11], Theorem 2). At the cost of some further calculations the theorem could be generalized to B-valued random variables under the assumption that their partial sums can be well approximated by their finite-dimensional projections with a certain degree of accuracy.

We shall first give the proof of Theorem 2, which needs less preparations than the proof of Theorem 1 and thus makes the method more transparent. Section 3 then contains the proof of Theorem 1 and in the final section we make some supplementary remarks in connection with Theorem 1.

2. Proof of Theorem 2. We first give some definitions and lemmas. For nonnegative $m, s_1, \ldots, s_d$ with $s_1+\ldots+s_d=m$ we put

$$\mu(s_1, \ldots, s_d) = \int_{\mathbf{R}^d} u_1^{s_1} \ldots u_d^{s_d} (F-G_\alpha)(du).$$

Define $S_n = n^{-1/\alpha} \sum_{j \leqq n} x_j$ and write

$$f_n(t) = E \exp i\langle t, S_n\rangle, \quad \varphi_\alpha(t) = \int_{\mathbf{R}^d} \exp(i\langle t, u\rangle) G_\alpha(du)$$

and

$$\psi(t) = n^{-(r-\alpha)/\alpha} \sum_{s_1+\ldots+s_d=r} \frac{\mu(s_1, \ldots, s_d)}{s_1! \ldots s_d!} (it_1)^{s_1} \ldots (it_d)^{s_d}.$$

Here and throughout the paper $r=1+[\alpha]$. It is well-known that for some constants $c_1>0$ and c_2 we have

$$\varphi_\alpha(t) = \exp\left(-|t|^\alpha(c_1+ic_2)\right).$$

LEMMA 2.1 (Banys [2]). *Suppose that* (1.11) *holds. Then there exists a positive constant A such that for all* $|t| \leqq A^{-1}n^{(r-\alpha)/\alpha}$

$$\left|f_n(t)-(1+\psi(t))\varphi_\alpha(t)\right| \leqq An^{-(r-\alpha)/\alpha}(|t|^r+|t|^{r+\alpha}+|t|^{2r})\exp\left(-\frac{1}{2}c_1|t|^\alpha\right).$$

COROLLARY. *Suppose that* (1.11) *holds. Then there exists a positive constant A such that for all* $|t| \leqq A^{-1}n^{(r-\alpha)/\alpha}$

$$|f_n(t)-\varphi_\alpha(t)| \leqq An^{-(r-\alpha)/\alpha}. \tag{2.1}$$

PROOF. This follows at once from

$$|f_n(t)-\varphi_\alpha(t)| \leqq \left|f_n(t)-(1+\psi(t))\varphi_\alpha(t)\right| + |\psi(t)\varphi_\alpha(t)| \leqq$$

$$\leqq An^{-(r-\alpha)/\alpha}(|t|^r+|t|^{r+\alpha}+|t|^{2r})\exp\left(-\frac{1}{2}c_1|t|^\alpha\right)+An^{-(r-\alpha)/\alpha}|t|^r\exp\left(-\frac{1}{2}c_1|t|^\alpha\right).$$

LEMMA 2.2. *Let* $\{\xi_j\colon 1\leqq j\leqq n\}$ *be a finite sequence of independent identically distributed* $\mathbf{R}^d$*-valued random variables with sum* $S=\sum_{j\leqq n}\xi_j$. *Assume that the distribution function of* $|\xi_1|$ *is continuous. Then L, defined by* $|\xi_L|=\max_{1\leqq j\leqq n}|\xi_j|$, *is with probability one, a well defined random variable that is independent of S and has uniform distribution on* $\{1, 2, \ldots, n\}$.

The proof of this simple lemma is given in [5].

Turning to the proof of Theorem 2, we first observe that there is no loss of generality in assuming that F is continuous, since if it is not, we may convolve F with a Gaussian distribution of sufficiently small second moment so that (1.11) remains valid. The conclusion of the theorem remains uneffected since for a sequence of independent identically distributed Gaussian random vectors the law of the iterated logarithm holds and since for all $0<\alpha<2$ we get by elementary calculations $(1/\alpha)-\lambda>1/2$.

The proof of Theorem 2 is now as follows. Put

$$\varrho = 8(d+1)/(r-\alpha), \quad \gamma = 4/\alpha, \tag{2.2}$$

$$t_k = [k^{\varrho+1}],\ n_k = t_{k+1}-t_k,\ T_k = k^\gamma,\ H_k = (t_k, t_{k+1}], \tag{2.3}$$

$$X_k = n_k^{-1/\alpha} \sum_{j\in H_k} x_j. \tag{2.4}$$

We define L_k by $|x_{t_k+L_k}|=\max_{j\in H_k}|x_j|$. By Lemma 2.2 L_k is defined with probability 1, has uniform distribution on $[0, n_k]\cap \mathbf{Z}$ and is independent of X_k.

Let f_k be the characteristic function of X_k. Then by Lemma 2.1, (2.2) and (2.3) we have for all sufficiently large k and all $|t|\leqq T_k$

(2.5) $$|f_n(t)-\varphi_\alpha(t)| \leqq cn_k^{-(r-\alpha)/\alpha}.$$

Here c is a constant. Moreover,

(2.6) $$G_\alpha\left\{u\colon |u| \geqq \frac{1}{4}T_k\right\} \ll T_k^{-\alpha} \ll k^{-4}.$$

After possibly enlarging the basic probability space $(\Omega, \mathcal{F}, P)$ we may assume that on this space there exists a sequence $\{u_i, i\geqq 1\}$ of independent random variables, uniformly distributed over [0, 1] and independent of $\{x_i, i\geqq 1\}$. From (2.5) and (2.6) we get, using Theorem 1 of Berkes and Philipp [3], that there exists a sequence $\{Y_k, k\geqq 1\}$ of independent random variables with common distribution G_α such that

(2.7) $$P\{|X_k-Y_k| \geqq \alpha_k\} \leqq \alpha_k$$

where

(2.8) $$\alpha_k \ll T_k^{-1}\log T_k+T_k^d n_k^{-(r-\alpha)/2\alpha}+k^{-4} \ll k^{-3/2}.$$

Moreover, as the proof of Theorem 1 of Berkes and Philipp [3] shows, $\{Y_k, k\geqq 1\}$ can be chosen in such a way that Y_k is measurable with respect to the σ-field generated by $X_1, \ldots, X_k, u_1, \ldots, u_k$.

Now, $\{L_k, k\geqq 1\}$ is measurable with respect to the σ-field generated by $\{x_i, i\geqq 1\}$, hence is independent of $\{u_i, i\geqq 1\}$. Since by Lemma 2.2 $\{L_k, k\geqq 1\}$ is also independent of $\{X_k, k\geqq 1\}$, we finally get that $\{L_k, k\geqq 1\}$ is independent of $\{Y_k, k\geqq 1\}$.

Let $\{y_i, i\geqq 1\}$ be a sequence of independent random variables, defined on some probability space and with common distribution G_α. Denote by L_k^* the random variable defined by $|y_{t_k+L_k^*}|=\max_{j\in H_k}|y_j|$. By Lemma 2.2 L_k^* is well-defined, has uniform distribution on $[0, n_k]\cap\mathbf{Z}$ and is independent of $Y_k^*=n_k^{-1/\alpha}\sum_{j\in H_k} y_j$. Hence the sequence $\{(Y_k^*, L_k^*), k\geqq 1\}$ has the same distribution as $\{(Y_k, L_k), k\geqq 1\}$. We apply Lemma A1 of Berkes and Philipp [3] to the joint law F of the sequences $\{x_i, i\geqq 1, X_k, k\geqq 1\}$ and $\{(Y_k, L_k), k\geqq 1\}$ and the joint law G of the sequences $\{(Y_k^*, L_k^*), k\geqq 1\}$ and $\{y_i, i\geqq 1\}$ and the spaces $S_1=(\mathbf{R}^d)^\infty\times(\mathbf{R}^d)^\infty$, $S_2=(\mathbf{R}^d\times\mathbf{N})$, $S_3=(\mathbf{R}^d)^\infty$. We obtain a joint law Q with marginals F and G, which we realize on some probability space Ω'. Hence, keeping the same notation we can set $Y_k=Y_k^*$ and $L_k=L_k^*$. (For the details see [5], pp. 487—488.)

In summary, we have redefined the sequences $\{x_i, i\geqq 1\}$, $\{Y_k, k\geqq 1\}$ and $\{L_k, k\geqq 1\}$ without changing their joint law on a (possibly) new probability space, which for convenience we denote again by $(\Omega, \mathcal{F}, P)$, together with a sequence $\{y_i, i\geqq 1\}$ of i.i.d. random variables with common distribution G_α with the following properties:

(2.9) $$Y_k = n_k^{-1/\alpha}\sum_{i\in H_k} y_i; \quad |Y_{t_k+L_k}| = \max_{i\in H_k}|y_i|.$$

i.e. the location t_k+L_k of $\max_{i\in H_k}|x_i|$ and $\max_{i\in H_k}|y_i|$ is the same.

This together with (2.4) yields:

$$\sum_{j\in H_k}(x_j-y_j) = n_k^{1/\alpha}(X_k-Y_k). \tag{2.10}$$

From (2.7) and (2.8) we get, using the Borel—Cantelli lemma,

$$X_k-Y_k \ll \alpha_k \ll k^{-3/2} \quad \text{a.s.} \quad \text{as} \quad k\to\infty$$

and hence using (2.2) and (2.3) we find

$$\sum_{i\le t_k}(x_i-y_i) \ll \sum_{j\le k} n_j^{1/\alpha_j-3/2} \ll k^{(\varrho/\alpha)-1/2} \ll t_k^{(1/\alpha)-\lambda} \quad \text{a.s.} \tag{2.11}$$

This proves the desired estimate for all n of the form $n=t_k$. For general n we need the following lemmas.

LEMMA 2.3. *As* $k\to\infty$ *we have*

$$\max_{n\in H_k}\min\Big\{\Big|\sum_{t_k<j\le n}x_j\Big|,\ \Big|\sum_{n<j\le t_{k+1}}x_j\Big|\Big\} \ll t_k^{(1/\alpha)-\lambda}$$

with probability 1.

PROOF. By Theorem 6.1 of de Acosta and Giné [1] we have for $p<\alpha$:

$$\sup_{n\ge 1} E\Big|n^{-1/\alpha}\sum_{j\le n}x_j\Big|^p<\infty.$$

Thus the hypotheses of Theorem 12.1 of Billingsley [4] are satisfied with $u_j=1$, $\gamma_B=p$ and $\alpha_B=p/\alpha$. Hence the probability of the left side exceeding the right side in the statement of the lemma is bounded by $n_k^{2p/\alpha}t_k^{-((1/\alpha)-\lambda)2p}\ll k^{-5/4}$ by (2.2) and (2.3). Applying the Borel—Cantelli lemma we obtain the result.

LEMMA 2.4. *With probability* 1 *there exists a* k_0 *such that for all* $k\ge k_0$ *there is at most one index* $j\in H_k$ *with* $|x_j|>t_k^{(1/\alpha)-\lambda}$.

PROOF. Since $\sup_{t\ge 0} t^\alpha P(|x_1|>t)<\infty$ we have

$$P\{\min(|x_i|,|x_j|)>t_k^{1/\alpha-\lambda} \text{ for some } i\ne j\in H_k\} \le n_k^2P^2\{|x_1|>t_k^{(1/\alpha)-\lambda}\} \le$$
$$\le n_k^2 t_k^{2(\alpha\lambda-1)} \ll k^{-3/2}$$

by (2.2) and (2.3). The result follows now from the Borel—Cantelli Lemma.

Now let $n\ge 1$ be given and choose k such that $n\in H_k$. Assume first that $\omega\in E_k:=\{|X_k|\le 6t_k^{(1/\alpha)-\lambda}\}$. Then we have by Lemma 2.3 and (2.11)

$$\Big|\sum_{j\le n}(x_j-y_j)\Big| \le \Big|\sum_{j\le t_k}(x_j-y_j)\Big| + \Big|\sum_{j\le t_{k+1}}(x_j-y_j)\Big| + \min\Big\{\Big|\sum_{t_k<j\le n}x_j\Big|,\Big|\sum_{n<j\le t_{k+1}}x_j\Big|\Big\}+$$
$$+\min\Big\{\Big|\sum_{t_k<j\le n}y_j\Big|,\Big|\sum_{n<j\le t_{k+1}}y_j\Big|\Big\}+\Big|\sum_{j\in H_k}x_j\Big| \ll t_k^{1/\alpha-\lambda}.$$

If on the other hand $\omega\in E_k^c$ then by Lemma 2.4 there is with probability 1 at most one $j\in H_k$ with $|x_j|>t_k^{(1/\alpha)-\lambda}$. Since $\omega\in E_k^c$ Lemma 2.3 implies that there is with probability 1 a k_0 such that for $k\ge k_0$:

$$\Big|\sum_{j\le n}x_j-\sum_{j\le t_k}x_j\Big| \le t_k^{(1/\alpha)-\lambda}, \quad t_k<n<L_k$$

and

$$\Big|\sum_{j\le n} x_j - \sum_{j\le t_{k+1}} x_j\Big| \le t_k^{(1/\alpha)-\lambda}, \quad L_k \le n \le t_{k+1}.$$

Indeed, as n runs through H_k the minimum in Lemma 2.3 must switch at least at one location from the first term to the second term. Since $\omega\in E_k^c$ each such switch must occur at a location j where $|x_j|\ge 4_k^{1/\alpha-\lambda}$. Hence, in view of Lemma 2.4 there is exactly one such switch and this takes place at the location t_k+L_k of the largest $|x_j|$. These inequalities remain valid if x_j is replaced by y_j since the locations of the maxima in the blocks are the same for the x- and the y-process. Thus if $t_k<n<L_k$ by (2.11)

$$\Big|\sum_{j\le n}(x_j-y_j)\Big| \le \Big|\sum_{j<t_k}(x_j-y_j)\Big|+\Big|\sum_{t_k<j\le n} x_j\Big|+\Big|\sum_{t_k<j\le n} y_j\Big| \ll t_k^{(1/\alpha)-\lambda}.$$

We obtain the same estimate for $L_k\le n\le t_{k+1}$ in a similar fashion and thus the proof of Theorem 2 is finished.

3. Proof of Theorem 1. We first need two lemmas.

LEMMA 3.1. *Let F be a symmetric distribution function such that $1-F(t)=$ $=ct^{-\alpha}+\beta(t)t^{-\alpha}$ for $t\ge t_0$ where $0<\alpha<2$, $c\ge 0$, $\beta(t)\to 0$ as $t\to+\infty$ and $\beta(t)t^{-\alpha}$ is decreasing for $t\ge t_0$. Extend $\beta(t)$ for all $t\ge 0$ by defining $\beta(t)=0$ for $0\le t<t_0$. Then for the characteristic function f of F we have*

$$f(u) = 1-c_1u^\alpha-\beta_1(u)u^\alpha+O(u^2) \quad as \quad u\downarrow 0 \tag{3.1}$$

where

$$c_1 = 2c\int_0^\infty \frac{\sin y}{y^\alpha}\,dy, \quad \beta_1(u) = 2\int_0^\infty \frac{\beta\left(\frac{y}{u}\right)\sin y}{y^\alpha}\,dy. \tag{3.2}$$

(Here both integrals are convergent as Riemann integrals.) Moreover, $\beta_1(t)\to 0$ as $t\downarrow 0$.

PROOF. The assumptions made on $\beta(t)$ imply that $A=\sup_{t\ge 0}|\beta(t)|<+\infty$. Fix $u>0$ and let $k_0=\left[\frac{t_0u}{\pi}\right]+1$. The monotonicity of $\beta(t)t^{-\alpha}$ for $t\ge t_0$ implies that the terms of the sum

$$\sum_{k=k_0}^\infty \int_{k\pi}^{(k+1)\pi} \frac{\beta\left(\frac{y}{u}\right)\sin y}{y^\alpha}\,dy$$

are alternately positive and negative and their absolute values tend monotonically to zero. Hence the sum is convergent and thus the integral defining $\beta_1(u)$ is convergent as a Riemann integral. It follows also that for $k\ge k_0$ (and consequently for $k\ge 1$ if $0<u\le\pi/t_0$)

$$\left|\int_{k\pi}^\infty \frac{\beta\left(\frac{y}{u}\right)\sin y}{y^\alpha}\,dy\right| \le \left|\int_{k\pi}^{(k+1)\pi} \frac{\beta\left(\frac{y}{u}\right)\sin y}{y^\alpha}\,dy\right| \le A\int_{k\pi}^{(k+1)\pi}\frac{1}{y^\alpha}\,dy \le A\pi^{1-\alpha}\frac{1}{k^\alpha}. \tag{3.3}$$

On the other hand, as $\beta(t)\to 0$ for $t\to+\infty$ we have for every fixed $k\geqq 1$,

$$\int_0^{k\pi}\frac{\beta\left(\frac{y}{u}\right)\sin y}{y^\alpha}\,dy\to 0\quad\text{as}\quad u\downarrow 0$$

since the integrand tends to 0 for every fixed y and $\left|\beta\left(\frac{y}{u}\right)\sin y\cdot y^{-\alpha}\right|\leqq A|\sin y/y^\alpha|$ which is integrable over $[0, k\pi]$. As (3.3) holds for $k\geqq 1$ provided $0<u\leqq\pi/t_0$, it follows that $\beta_1(u)\to 0$ as $u\to+0$. To show (3.1) we write, using the symmetry of F and integration by parts,

$$f(u)=1-2u\int_0^\infty(1-F(x))\sin ux\,dx=1-2u\int_{t_0}^\infty(1-F(x))\sin ux\,dx+O(u^2)=$$

$$=1-2u\int_{t_0}^\infty cx^{-\alpha}\sin ux\,dx-2u\int_{t_0}^\infty\frac{\beta(x)\sin ux}{x^\alpha}\,dx+O(u^2)=$$

$$=1-2u^\alpha\int_{t_0u}^\infty\frac{c\sin y}{y^\alpha}\,dy-2u^\alpha\int_{t_0u}^\infty\frac{\beta\left(\frac{y}{u}\right)\sin y}{y^\alpha}\,dy+O(u^2)=$$

$$=1-2u^\alpha\int_0^\infty\frac{c\sin y}{y^\alpha}\,dy-2u^\alpha\int_0^\infty\frac{\beta\left(\frac{y}{u}\right)\sin y}{y^\alpha}\,dy+O(u^2)=$$

$$=1-c_1u^\alpha-\beta_1(u)u^\alpha+O(u^2).\quad\square$$

LEMMA 3.2. *Let F and F_1 be symmetric distribution functions such that*

$$1-F(t)=ct^{-\alpha}+\beta(t)t^{-\alpha},\quad t\geqq t_0,$$

$$1-F_1(t)=\beta(t)t^{-\alpha},\quad t\geqq t_0,$$

where $c>0$, $0<\alpha<2$, $\beta(t)\to 0$ as $t\to+\infty$ and $\beta(t)t^{-\alpha}$ is decreasing for $t\geqq t_0$. Let f and f_1 be the characteristic functions of F and F_1, respectively. Then

$$f\left(\frac{u}{n^{1/\alpha}}\right)^n=\exp(-c_1|u|^\alpha)f_1\left(\frac{u}{n^{1/\alpha}}\right)^n+O(n^{-\delta})\tag{3.4}$$

for $|u|\leqq n^\delta$; here c_1 is the number defined by (3.2) and δ is a positive constant.

PROOF. Since f and f_1 are even functions, it suffices to consider the case $u\geqq 0$. Let $\beta_1(u)$ be defined as in Lemma 1 then $\beta_1(u)\to 0$ as $u\downarrow 0$ and thus

$$\left(1-\frac{u^\alpha}{n}\beta_1\left(\frac{u}{n^{1/\alpha}}\right)\right)^{-1}=1+O(n^{-1/\alpha})$$

provided $0\leqq u\leqq n^{1/2\alpha}$. Thus we have by Lemma 1, setting $\varrho=\min\left(2/\alpha-1,\frac{1}{2}\right)$

and assuming $u^\alpha \leqq n^{1/4}$, $u^2 \leqq n^{\varrho/2}$,

$$f\left(\frac{u}{n^{1/\alpha}}\right) = 1 - \frac{u^\alpha}{n}\beta_1\left(\frac{u}{n^{1/\alpha}}\right) - c_1\frac{u^\alpha}{n} + O\left(\frac{u^2}{n^{1+\varrho}}\right) =$$

$$= \left(1 - \frac{u^\alpha}{n}\beta_1\left(\frac{u}{n^{1/\alpha}}\right)\right)\left[1 + \left(-c_1\frac{u^\alpha}{n} + O\left(\frac{u^2}{n^{1+\varrho}}\right)\right)(1 + O(n^{-1/2}))\right] =$$

$$= \left(1 - \frac{u^\alpha}{n}\beta_1\left(\frac{u}{n^{1/\alpha}}\right)\right)\left(1 - c_1\frac{u^\alpha}{n} + O(n^{-(1+\varrho/2)})\right) =$$

$$= \left(1 - \frac{u^\alpha}{n}\beta_1\left(\frac{u}{n^{1/\alpha}}\right)\right)\exp\left\{-c_1\frac{u^\alpha}{n} + O(n^{-(1+\varrho/2)})\right\}.$$

Applying Lemma 1 with $c=0$ we get

$$1 - u^\alpha\beta_1(u) = f_1(u) + O(u^2) = f_1(u)\bigl(1 + O(u^2)\bigr)$$

provided $|f_1(u)| \geqq \frac{1}{2}$ which is satisfied if $|u| \leqq u_0$ with a small u_0. Using this, we get for $|u| \leqq n^\delta$, and δ sufficiently small,

$$\left(1 - \frac{u^\alpha}{n}\beta_1\left(\frac{u}{n^{1/\alpha}}\right)\right)^n = f_1\left(\frac{u}{n^{1/\alpha}}\right)^n\left(1 + O\left(\frac{u^2}{n^{1+\varrho}}\right)\right)^n = f_1\left(\frac{u}{n^{1/\alpha}}\right)^n(1 + O(n^{-\varrho/2})).$$

Thus finally

$$f\left(\frac{u}{n^{1/\alpha}}\right)^n = f_1\left(\frac{u}{n^{1/\alpha}}\right)^n(1 + O(n^{-\varrho/2}))\cdot\exp\{-c_1u^\alpha + O(n^{-\varrho/2})\} =$$

$$= f_1\left(\frac{u}{n^{1/\alpha}}\right)^n\exp(-c_1u^\alpha)(1 + O(n^{-\varrho/2})) = f_1\left(\frac{u}{n^{1/\alpha}}\right)^n\exp(-c_1u^\alpha) + O(n^{-\varrho/2})$$

for $|u| \leqq n^\delta$, if δ is sufficiently small. □

We can now complete the proof of Theorem 1. Obviously, Theorem 1 can be formulated equivalently as follows:

THEOREM 1*. *Let* $x_1, x_2, \ldots$ *be independent identically distributed symmetric random variables with distribution function* F *satisfying* (1.6) *where* $c>0$, $0<\alpha<2$, $\beta(t)\to 0$ *as* $t\to+\infty$ *and* $\beta(t)t^{-\alpha}$ *is decreasing for* $t \geqq t_0$. *Let* y *and* z *be independent symmetric random variables such that the characteristic function of* y *is* $\exp(-c_1|u|^\alpha)$ *with* c_1 *defined by* (3.2) *and the distribution function* F_1 *of* z *satisfies* (1.8); *put* $w=y+z$. *Then there exist independent identically distributed random variables* $w_1, w_2, \ldots$ *each distributed as* w *such that*

$$(x_1 + \ldots + x_n) - (w_1 + \ldots + w_n) = O(n^{1/\alpha - \lambda}) \quad a.s.$$

with a positive constant λ.

Theorem 1* is analogous to Theorem 2 and can be proved in the same way. Set

$$f_n(u) = E\exp\{iun^{-1/\alpha}(\sum_{k=1}^n x_k)\}, \quad g_n(u) = E\exp\{iun^{-1/\alpha}(\sum_{k=n}^n w_k)\}$$

where $w_1, w_2, \ldots, w_n$ are independent random variables distributed as w. Then by Lemma 2 we have

(3.5) $$|f_n(u) - g_n(u)| \leqq Cn^{-\delta} \quad \text{for} \quad |u| \leqq n^\delta$$

with positive constants C and δ. Also,

$$P\{w \geqq t\} \sim P\{y \geqq t\} \sim ct^{-\alpha} \quad \text{as} \quad t \to +\infty$$

i.e. w also belongs to the domain of normal attraction of the stable law G_α. Consequently, by the theorem of de Acosta and Giné above we have

$$\sup_{n \geqq 1} E\left|n^{-1/\alpha} \sum_{k=1}^n w_k\right|^{\alpha/2} < +\infty$$

which implies

(3.6) $$P\{n^{-1/\alpha}\left|\sum_{k=1}^n w_k\right| \geqq t\} \ll t^{-\alpha/2} \quad \text{for} \quad t \geqq 1, n \geqq 1.$$

By (3.5) and (3.6) the Prohorov distance of the distributions belonging to $f_n(u)$ and $g_n(u)$ is $\ll n^{-\varepsilon}$ for some $\varepsilon > 0$. From this point on, we can follow the proof of Theorem 2 without any change (the w_i playing the role of the y_i) and we get Theorem 1*.

4. Remarks. In this section we discuss applications of Theorem 1 and its relation to Stout's theorem. First we shall deal with the almost sure approximation. If $\beta(x) \leqq x^{-\gamma}$ for some $\gamma > 0$ then $\sum_{i=1}^n z_i = O(n^{1/\alpha - \lambda_1})$ a.s. for some $\lambda_1 > 0$ by Theorems 1 and 2 of Feller [6] and thus, as $\sum_{i=1}^n z_i$ can be absorbed into the error term of (1.9), we obtain in this case

(4.1) $$\sum_{i \leqq n} x_i - \sum_{i \leqq n} y_i = O(n^{1/\alpha - \lambda}) \quad \text{a.s.}$$

for some $\lambda > 0$. This is equivalent to the conclusion of Stout's theorem applied to this case.

On the other hand if β tends to zero slower than any negative power of x, the situation is different. For the discussion it is convenient to formulate the following fact explicitly.

LEMMA 4.1. *Suppose that β is non-increasing and that*

(4.2) $$t^\delta \beta(t) \to \infty \quad \textit{for all} \quad \delta > 0$$

and

(4.3) $$K := \sup_{t \geqq 1} \beta(t)/\beta(t^2) < \infty.$$

Then

(4.4) $$P\{\sum_{i \leqq n} z_i \geqq a_n(\beta(a_n))^{1/\alpha} \text{ i.o.}\} = 0 \quad \textit{or} \quad 1$$

according as $\{a_n\}$ belongs to the upper or lower class of $\{x_n\}$ i.e.

$$\sum_{n\geqq 1} a_n^{-\alpha} < \infty \quad or \quad = \infty. \tag{4.5}$$

REMARK. If for instance $\beta(t)=(\log t)^{-\lambda}$ or $(\log\log t)^{-\lambda}$ etc. for some $\lambda>0$ then the lemma applies.

PROOF. By (1.8) $E|z_i|^\gamma=\infty$ for any $\alpha<\gamma$ and so z_i is outside the domain of partial attraction to the normal law (see [8], p. 117). Thus by [7, Theorem 1] a sequence $\{c_n\}$ with $c_n>0$ belongs to the upper class for $\{z_n,\ n\geqq 1\}$ if and only if

$$\sum_{n\geqq 1} P(|z_1| \geqq c_n\} < \infty. \tag{4.6}$$

Let $c_n:=a_n(\beta(a_n))^{1/\alpha}$ with $a_n\to\infty$ then $a_n^{1/2}\leqq c_n\leqq a_n$ for $n\geqq n_0$ and thus

$$P\{|z_1| \geqq c_n\} = \beta(c_n)c_n^{-\alpha} \geqq a_n^{-\alpha}, \quad n \geqq n_1$$

and

$$P\{|z_1| \geqq c_n\} \leqq \beta(c_n)a_n^{-\alpha}/\beta(a_n) \leqq Ka_n^{-\alpha}, \quad n \geqq n_1.$$

Consequently, the series in (4.5) and (4.6) converge and diverge simultaneously. □

As an immediate consequence of Lemma 4.1 we see that if $\{a_n\}$ is an upper class sequence for $\{x_n,\ n\geqq 1\}$ then

$$\sum_{i\leqq n} x_i - \sum_{i\leqq n} y_i = \sum_{i\leqq n} z_i + O(n^{1/\alpha-\lambda}) = O\big(a_n(\beta(a_n))^{1/\alpha}\big) \quad \text{a.s.} \tag{4.7}$$

This is better than Stout's result (1.4) for $\alpha<1$, but worse for $\alpha>1$.

Passing from the almost sure approximation to approximation in probability we note first that if β satisfies the hypotheses of Theorem 1 then

$$n^{-1/\alpha} \sum_{i\leqq n} z_i \to 0 \quad \text{in probability.} \tag{4.8}$$

This follows immediately from the classical degenerate convergence theorem or from Lemma 3.1 with $c=0$. Hence by Lévy's maximal inequality we have

$$n^{-1/\alpha} \max_{1\leqq m\leqq n} \Big|\sum_{i\leqq m} z_i\Big| \to 0 \quad \text{in probability.}$$

and hence by Theorem 1

$$n^{-1/\alpha} \max_{1\leqq m\leqq n} \Big|\sum_{i\leqq m} (x_i-y_i)\Big| \to 0 \quad \text{in probability.} \tag{4.9}$$

Of course, (4.9) also follows immediately from [10, Theorem 1].

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OSCILLATORY PROPERTIES OF ARITHMETICAL FUNCTIONS. I

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1. Introduction

The first general theorem concerning sign changes of partial sums of arithmetical functions has been proved by E. Landau [8] in 1905 and sounds slightly reformulated as follows (we shall use the notation $s=\sigma+it$ throughout this paper):

THEOREM (Landau). *Let $f(x)$ be real for $x\geqq x_0$; suppose $F(s)=\int_{x_0}^{\infty} f(x)x^{-s-1}\,dx$ is regular for $\sigma>\theta$ but not regular in any half-plane $\sigma>\theta-\varepsilon$ with $\varepsilon>0$. If $F(s)$ is regular at $s=\theta$ then $f(x)$ changes sign infinitely often as $x\to\infty$.*

Unfortunately, this very beautiful and general theorem does not yield any information about the frequency of sign changes. For any real function $f(x)$ defined for $x>0$ we may define the number $V(f,Y)$ of sign changes in the interval $(0, Y]$ as follows:

$$V(f,Y)=\sup\big\} N;\ \exists\{x_i\}_{i=1}^{N},\quad 0<x_1<\ldots<x_N\leqq Y,\tag{1.1}$$
$$f(x_i)\neq 0,\quad \operatorname{sgn} f(x_i)\neq \operatorname{sgn} f(x_{i+1}),\quad 1\leqq i<N\big\}.$$

We shall say, that $V(f,Y)>h(Y)$ with combined oscillation of size $g(x)$ if there exists a series $\{x_i\}_{i=1}^{h(Y)}$ with $\operatorname{sgn} f(x_i)\neq\operatorname{sgn} f(x_{i+1})$ and $|f(x_i)|>g(x_i)$.

Imposing more conditions on the function f, Pólya [11] was able to deduce another general theorem concerning the behaviour of the function $V(f,Y)$.

THEOREM (Pólya). *Let $f(x)$ and $F(s)$ satisfy the conditions of Landau's theorem, further let $F(s)$ be meromorphic in some half-plane $\sigma\geqq\theta-c_0$, $c_0>0$. Let $\gamma=\inf\{|t|;$ $F(s)$ is not regular at $s=\theta+it\}$ and let $\gamma=\infty$ if $F(s)$ is regular on $\sigma=\theta$. Then*

$$\varlimsup_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}.\tag{1.2}$$

Finally, Grosswald [3] succeeded in generalizing the theorem of Pólya, for the case when logarithmic singularities with principal part $P_n(s-s_n)\log(s-s_n)$ are allowed in the strip $\theta-c_0\leqq\sigma\leqq\theta$ too, where $P_n(u)$ are polynomials with $\sup_n \deg P_n(u)<\infty$.

The aim of this work is to show that the ideas of the first named author [5] which led to the proof of

$$\varliminf_{Y\to\infty}\frac{V(\psi(x)-x,Y)}{\log Y}>0,\quad \varliminf_{Y\to\infty}\frac{V(\Pi(x)-\operatorname{li}x,Y)}{\log Y}>0\tag{1.3}$$

can be extended as to show Grosswald's theorem with $\underline{\lim}$ instead of $\overline{\lim}$. This will be our Theorem 1. The corresponding sharpening of Pólya's theorem will be formulated as Corollary 1. Corollary 1 enables us to prove at least $c \log Y$ sign changes for the partial sums of many number theoretic functions, including

$$\psi(x, q, l_1)-\psi(x, q, l_2), \quad (l_1, q) = (l_2, q) = 1, \quad l_1 \not\equiv l_2(q), \tag{1.4}$$

$$M(x) = \sum_{n \leqq x} \mu(n), \tag{1.5}$$

$$R_k(x) = Q_k(x)-\frac{x}{\zeta(k)} = \sum_{n \leqq x} \sum_{d^k|n} \mu(d)-\frac{x}{\zeta(k)}, \tag{1.6}$$

in case of (1.4) using the hypothesis that there are no real positive zeros of L-functions mod q. This condition is necessary in some sense, due to the explicit formula

$$\psi(x, q, l_1)-\psi(x, q, l_2) = \sum_{\chi(q)} \left(\bar\chi(l_2)-\bar\chi(l_1)\right) \sum_{\varrho=\varrho_\chi} \frac{x^\varrho}{\varrho}+O(\log x). \tag{1.7}$$

In the case of (1.4) Knapowski and Turán proved [7] a very weak lower bound for the number of sign changes under the same condition as our Corollary 2 (in part V). Further they showed (in part VI) $V(f, Y)>c \log\log Y$ under the additional condition that the domain $\sigma>\frac{1}{2}$, $|t| \leqq cq^{10}$ is free of zeros of $L(s, \chi, q)$ functions. In case of (1.5) and (1.6) the best known lower bounds for $V(f, Y)$ were $c \log Y/\log_2^{3/2} Y$ (Pintz [10]) and $c \log\log Y$ (Kátai [6]), resp.

The only general theorem, existing in the literature, which yields concrete lower estimate of $V(f, Y)$ for every value of Y, seems to be Theorem 2 of Kátai [6] which ensures the inequality $V(f, Y) \gg \log\log Y$ for a wide class of functions. It is too long to quote exactly his theorem; however, we may remark that this class includes the functions (1.5) and (1.6) (but not (1.4), for general q). His theorem, although it refers to a smaller class of functions and it gives a weaker lower bound for $V(f, Y)$ has two advantages over our Theorem:

(i) it usually yields effective lower bounds for $V(f, Y)$;

(ii) it ensures a larger (in some cases, apart from a constant factor, optimal) size of oscillation.

Due to some theoretic reasons the method presented does not allow to obtain optimal oscillation. However, it is possible to prove a restricted version of it, Theorem 2, which leads to $V(f, Y)>c \log Y$, as an effective estimate for a rather wide class of functions. The conditions imposed for $F(s)$ are similar to that of Kátai's Theorem 2 [6].

Unfortunately the type of singularities, as required in (3) of our Theorem 1 are not general enough to cover the most important applications with logarithmic singularities as $\pi(x)-\mathrm{li}\, x$, e.g. (which was dealt with in Kaczorowski [5] using more complicated arguments). Thus, we have to remark that it is stated erroneously in Grosswald [3] that his Theorem 6 follows from Theorem 2. (Similarly Theorem D of his paper [4] does not imply Theorems 3, 5a, 18, 20, 22, 24.) Another extension of Pólya's theorem for the case of functions having logarithmic singularities is due to

Levinson [9], although his theorem needs also some modifications to yield the needed applications.

In the 2nd part of this work, we shall extend Theorem 1 for a larger class of functions (including $\pi(x)-\text{li}\, x$, $\pi(x,q,l_1)-\pi(x,q,l_2)$ and some other important arithmetical functions).

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2. Statement of results

We shall prove first a general theorem which, however, yields in most cases ineffective results.

THEOREM 1. *Let $f(x)$ be real for $x>0$ and suppose that the integral $\int\limits_0^\infty f(x)x^{-s-1}dx$ converges absolutely for $\sigma\geqq\sigma_1$ and represents in that half plane a function $F(s)$ having the following properties:*

(1) *$F(s)$ is regular for $\sigma>\theta$ but not in any half plane $\sigma>\theta-\varepsilon$ with $\varepsilon>0$;*

(2) *there exists a denumerable (finite or infinite) set $S=\{\varrho_\nu=\beta_\nu\pm i\gamma_\nu\}$ $(\gamma_\nu\geqq 0)$ without finite limit point satisfying $\theta-c_0\leqq\beta_\nu\leqq\theta$ for some $c_0>0$ and such that $F(s)$ can be continued as a meromorphic function in the open set D obtained by making the cuts $s=\sigma\pm i\gamma_n$ $(\sigma\leqq\beta_n)$ in the half-plane $\sigma>\theta-c_0$;*

(3) *for $s\to\varrho_\nu$ $(s\in D)$ $F(s)=P_\nu(s-s_\nu)\log(s-s_\nu)+F_\nu(s)$ where $F_\nu(s)$ is meromorphic at $s=\varrho_\nu$, and P_ν is a polynomial ($P_\nu\equiv 0$ is possible too). Let $\gamma=\min\limits_{\beta_\nu=\theta}\gamma_\nu$ and $\gamma=\infty$ if $\beta_\nu<\theta$ for all $\nu=1,2,\ldots$.*

Under these conditions we have

$$\varliminf_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}, \tag{2.1}$$

and every interval of the form

$$[Y^{1-\varepsilon},Y],\quad Y>Y_0(\varepsilon) \tag{2.2}$$

contains at least one sign change of $f(s)$. The sign changes in (2.1) *and* (2.2) *are combined with an oscillation of size (cf. the definition following* (1.1)*)*

$$x^{\theta-\varepsilon}, \tag{2.3}$$

for arbitrary $\varepsilon>0$.

Theorem 1 yields the following sharpening of Pólya's theorem.

COROLLARY 1. *If $f(x)$ is real for $x>0$, $F(s)=\int\limits_0^\infty f(x)x^{-s-1}dx$ converges absolutely for $\sigma\geqq\sigma_1$ and*

(1′) *$F(s)$ is regular for $\sigma>\theta$ but not in any half-plane $\sigma>\theta-\varepsilon$ with $\varepsilon>0$;*

(2′) *$F(s)$ is meromorphic for $\sigma\geqq\theta-c_0$ with some $c_0>0$.*

Then relations (2.1) to (2.3) holds.

We remark that Grosswald [3] needs additionally the condition that $\sup\limits_{1\leqq\nu<\infty} \deg P_\nu<\infty$. His result is with the additional condition

$$\varliminf_{Y\to\infty} \frac{V(f,Y)}{\log Y} \geqq \frac{\gamma}{\pi}. \tag{2.4}$$

The example $f(x)=0$ for $x<1$, $f(x)=x^{\beta+i\gamma}+x^{\beta-i\gamma}(x\geqq1)$ (with the corresponding $F(s)=(s-\beta-i\gamma)^{-1}+(s-\beta+i\gamma)^{-1}$) shows that inequality (2.1) (unlike (2.2)) is best possible, since in this case $V(f,Y)\sim\frac{\gamma}{\pi}\log Y$.

Since in the proof of Theorem 1 many singularities of $F(s)$ may occur and in concrete applications we do not have enough information about the distribution of them (this being the case in the most important number theoretic problems when singularities of $F(s)$ are zeros of the Riemann zeta or Dirichlet's L-functions) we shall prove a second theorem which yields effective results as well. Here only one singularity of $F(s)$ occurs and therefore the conditions might be checked in concrete cases (although they are stronger in some sense than in Theorem 1).

For the aim of concrete applications we give the formulation of Theorem 2 only for meromorphic functions but this can be extended in the same way for the case of logarithmic singularity as Theorem 1.

THEOREM 2. *If $f(x)$ is real for $x>0$, $F(s)=\int\limits_0^\infty f(x)x^{-s-1}dx$ is absolutely convergent for $\sigma>\sigma_1$ and*

(1) *$F(s)$ has a pole at $\varrho_0=\beta_0+i\gamma_0$, $\gamma_0>0$, $\beta_0>0$ with principal part $\sum\limits_{j=1}^{k+1} h_j(s-\varrho_0)^{-j}$;*

(2) *apart from the poles ϱ_0 and $\overline{\varrho_0}$, $F(s)$ is regular on and to the right of the broken line L defined by*

$$L=\begin{cases} |t|\geqq\Gamma, & \sigma=\sigma_1+a_0\\ \beta_0+a_1\leqq\sigma\leqq\sigma_1+a_0, & |t|=\Gamma\\ H\leqq|t|\leqq\Gamma, & \sigma=\beta_0+a_1\\ \beta_0-a_2\leqq\sigma\leqq\beta_0+a_1, & |t|=H\\ |t|\leqq H, & \sigma=\beta_0-a_2,\end{cases} \tag{2.5}$$

where $a_0>0$, $0<a_2<\beta_0$, $-a_2\leqq a_1\leqq\sigma_1+a_0-\beta_0$, $\Gamma\geqq H>\gamma_0$ further

$$d_1=\max\left(\frac{\sigma_1+a_0-\beta_0}{\log\frac{|\sigma_1+a_0+i\Gamma|}{|\varrho_0|}}, \frac{a_1}{\log\frac{|\beta_0+a_1+iH|}{|\varrho_0|}}\right)<\frac{a_2}{\log\frac{|\varrho_0|}{\beta_0-a_2}}=d_2; \tag{2.6}$$

(3) *$|F(s)|\leqq M$ for $s\in L$.*

Then for every $\varepsilon>0$ *and* $Y>Y_0=Y_0(\varepsilon, a_0, a_1, a_2, \sigma_1, \beta_0, \gamma_0, H, M, \Gamma, h_1, \dots$ $\dots, h_{k+1})$, *effective constant, we have*

$$V(f, Y) > \left(1-\frac{d_1}{d_2}-\varepsilon\right)\frac{\gamma_0}{\pi}\log Y. \tag{2.7}$$

We remark that if the singularities of $F(s)$ are the zeros of $\zeta(s)$ then by the calculations of Brent, van de Lune, te Riele and Winter [1] (see also the remark in Zbl. 486 10027), the first 300 million zeros are on the critical line and therefore we may choose

$$\sigma_1 = 1, \beta_0 = \frac{1}{2}, \quad \gamma_0 = 14.13\dots, \quad H = 21, \quad \Gamma = 10^8, \tag{2.8}$$

and arbitrary values a_i with $a_0>0$, $0<a_1<\frac{1}{2}$, $0<a_2<\frac{1}{2}$.

The most important applications of Theorem 1 (more precisely of Corollary 1), apart from those discussed in [5] (cf. (1.3)) are the following.

COROLLARY 2. *If* $(l_1, q)=(l_2, q)=1$, $l_1\not\equiv l_2(q)$ *and the L-functions* mod q *have no real zeros in* $\left[\frac{1}{2}, 1\right]$ *then for*

$$\Delta_3(x, q, l_1, l_2) = \Delta_3(x) = \sum_{\substack{n\equiv l_1(q)\\ n\le x}} \Lambda(n) - \sum_{\substack{n\equiv l_2(q)\\ n\le x}} \Lambda(n) \tag{2.9}$$

(where $\Lambda(n)$ *is von Mangoldt's function) we have*

$$\varliminf_{Y\to\infty} \frac{V(\Delta_3, Y)}{\log Y} > 0, \tag{2.10}$$

with combined oscillation of size $x^{1/2-\varepsilon}$.

COROLLARY 3. *If* $(l_1, q)=(l_2, q)=1$, $l_1\not\equiv l_2(q)$, *both* l_1 *and* l_2 *are quadratic non-residues or both are quadratic residues and the L-functions* mod q *have no real zeros in* $\left[\frac{1}{2}, 1\right]$ *then for*

$$\Delta_4(x, q, l_1, l_2) = \Delta_4(x) = \sum_{p\equiv l_1(q)} \log p - \sum_{p\equiv l_2(q)} \log p \tag{2.11}$$

we have

$$\varliminf_{Y\to\infty} \frac{V(\Delta_4, Y)}{\log Y} > 0, \tag{2.12}$$

with combined oscillation of size $x^{1/2-\varepsilon}$.

We remark that the condition concerning the absence of real zeros of L-functions was verified by Spira [12] for all $q<25$. So for these moduli (2.9)—(2.12) hold without any unproved hypotheses.

COROLLARY 4. *Let* $\mu(n)$ *denote the Möbius-function,* $\theta = \sup_{\zeta(\varrho)=0} \operatorname{Re} \varrho$, $\gamma_1 = \min_{\operatorname{Im} \varrho > 0} \operatorname{Im} \varrho = 14.13\ldots$, *a arbitrary real number and*

$$M_a(x) = \begin{cases} 0 & \text{for } x < 1 \\ \sum_{n \leqq x} \mu(n) n^{-a} - \dfrac{1}{\zeta(a)} & \text{for } x \geqq 1. \end{cases} \tag{2.13}$$

Then we have

$$\varlimsup_{Y \to \infty} \frac{V(M_a, Y)}{\log Y} \geqq \frac{\gamma_1}{\pi} \tag{2.14}$$

with combined oscillation of size

$$x^{\theta - a - \varepsilon}. \tag{2.15}$$

Hence in case of $a < \theta$ the constant $1/\zeta(a)$ can be deleted from the definition of $M_a(x)$ or can be substituted by an arbitrary other constant.

By the aid of Theorem 2 we are able to prove a similar but effective theorem for a restricted range of a, however. It is possible to prove e.g., the following version:

COROLLARY 5. *If* $-10^{-8} < a < 1/4$ *and*

$$\overline{M_a}(x) = \sum_{n \leqq x} \mu(n) n^{-a} \tag{2.16}$$

then for $Y > c_1(a)$, *effective constant, we have*

$$V(M_a, Y) > \frac{1}{3} \log Y. \tag{2.17}$$

Further, Theorem 2 yields

COROLLARY 6. *Let* $Q_k(x)$ *denote the number of k-free numbers not exceeding x, where* $k \geqq 2$ *is a natural number and let* $R_k(x) = 0$ *for* $x < 1$ *and*

$$R_k(x) = Q_k(x) - \frac{x}{\zeta(k)} = \sum_{\substack{n \leqq x \\ p|n \to p^k \nmid n}} 1 - \frac{x}{\zeta(k)} \quad \text{for} \quad x \geqq 1. \tag{2.18}$$

Then we have for $Y > c_2(k)$, *effective constant,*

$$V(R_k, Y) > \frac{5}{2k} \log Y. \tag{2.19}$$

3. Proof of Theorem 1

We are entitled to assume $\theta > 0$ since otherwise we can work with $f(x) \cdot x^c$. Let us define the operation δ by

$$\delta f(x) = \int_0^x \frac{f(\xi)}{\xi} d\xi, \tag{3.1}$$

and let δ_k be the k times iterated operation, i.e. $\delta_1=\delta$, $\delta_n=\delta\delta_{n-1}$. It is easy to see that

$$\delta_2(f(x)) = \int_0^x \frac{1}{t}\int_0^t \frac{f(u)}{u}\,du\,dt = \int_0^x \frac{f(u)}{u}\log\frac{x}{u}\,du = \tag{3.2}$$

$$= \int_0^\infty \frac{f(u)}{u}\cdot\frac{1}{2\pi i}\int_{(\sigma_1)}\frac{(x/u)^s}{s^2}\,ds = \frac{1}{2\pi i}\int_{(\sigma_1)} F(s)\frac{x^s}{s^2}\,ds$$

and we obtain by induction according to n that

$$\delta_n f(x) = \frac{1}{2\pi i}\int_{(\sigma_1)} F(s)\frac{x^s}{s^n}\,ds. \tag{3.3}$$

Let us consider first the case when $F(s)$ is meromorphic for $\sigma\geqq\theta-c_0$ (or at least for $\sigma\geqq\theta-\eta$, $|t|\leqq\Gamma$ with some $\eta>0$ and with a sufficiently large Γ).

We may suppose $\gamma>0$ otherwise we have nothing to prove. Let us choose $\eta>0$ in such a way that $\eta<c_0$, $\eta<\frac{\theta}{2}$ and that the following region and line,

$$\sigma>\theta-\eta,\quad |t|\leqq\gamma \quad\text{and}\quad \sigma=\theta-\eta,\quad -\infty<|t|<\infty, \tag{3.4}$$

resp., should contain no singularity of $F(s)$ except $\theta\pm i\gamma$, if $\gamma<\infty$. If $\gamma=\infty$, let η be defined so that $\eta<c_0$, $\eta<\frac{\theta}{2}$ and that the segment $[\theta-\eta,\theta]$ should be free of singularities of $F(s)$.

Later on we shall choose a sufficiently large constant Γ so that there should be no singularity ϱ of $F(s)$ on the broken line L defined by

$$L=\begin{cases} |t|\geqq\Gamma, & \sigma=\sigma_1\\ \theta-\eta\leqq\sigma\leqq\sigma_1, & |t|=\Gamma\\ |t|\leqq\Gamma, & \sigma=\theta-\eta\end{cases} \tag{3.5}$$

but there should be at least one singularity to the right of L. Then we have

$$\delta_n f(x) = 2\sum_{\nu=1}^{m}\operatorname{Re}\left\{\operatorname{Res}\left(F(s)\frac{x^s}{s^n}\right)_{s=\varrho_\nu}\right\}+O\left(\frac{x^{\theta-\eta}}{(\theta/2)^n}\right)+O\left(\frac{x^{\sigma_1}}{n\Gamma^{n-1}}\right) = \tag{3.6}$$

$$=: 2\sum_{\nu=1}^{m}\operatorname{Re}\left\{A_\nu(x)\frac{x^{\varrho_\nu}}{\varrho_\nu^n}\right\}+R_1+R_2,$$

where $\varrho_1,\dots,\varrho_m$ $(m\geqq1)$ are the singularities of $F(s)$ above the real axis and right of L, numerated according to $0<\gamma_1\leqq\dots\leqq\gamma_m$.

Let

$$n=[b\log Y],\quad x=Y^\alpha,\quad \sqrt{b}\leqq\alpha\leqq1. \tag{3.7}$$

If we fix b satisfying

$$b<b_0=b_0(\eta)<\min\left(\frac{1}{100},\left(\frac{\theta}{4}\right)^2\right) \tag{3.8}$$

where b_0 is chosen sufficiently small (but independently of Γ) then with a positive d_1

$$R_1 \ll \frac{x^{\beta_1}}{|\varrho_1|^n} Y^{-d_1}. \tag{3.9}$$

Now we fix Γ satisfying

$$\Gamma > \Gamma_0(b, \eta) \tag{3.10}$$

where Γ_0 is chosen sufficiently large. Then with a positive d_2 we have

$$R_2 \ll \frac{x^{\beta_1}}{|\varrho_1|^n} Y^{-d_2}. \tag{3.11}$$

Later on we shall choose Δ sufficiently small with

$$\Delta < b \min_{\substack{1 \leqq \nu < \mu \leqq n \\ \beta_\nu = \beta_\mu}} \big|\log|\varrho_\nu| - \log|\varrho_\mu|\big| = \Delta_0. \tag{3.12}$$

Then it is easy to see that the inequality

$$\big|(\beta_\nu \alpha - b \log|\varrho_\nu|) - (\beta_\mu \alpha - b \log|\varrho_\mu|)\big| > \Delta \tag{3.13}$$

holds for all ν, μ with $1 \leqq \nu \leqq \mu \leqq m$ and for all $\alpha \in [0, 1]$ apart from finitely many intervals of total length at most

$$\Delta D, \quad D = D(\Gamma, \eta) = 2 \sum_{\substack{1 \leqq \nu < \mu \leqq m \\ \beta_\nu \neq \beta_\mu}} \frac{1}{|\beta_\nu - \beta_\mu|}. \tag{3.14}$$

Let us choose

$$\Delta = \min\left(\frac{\sqrt{b}}{D}, \Delta_0\right). \tag{3.15}$$

In such a way we obtain disjoint intervals of the form $(e_\nu, e_\nu') \subset [\sqrt{b}, 1]$ ($e_\nu = e_\nu'$ is possible) of total length at least

$$1 - \Delta D - \sqrt{b} \geqq 1 - 2\sqrt{b} \tag{3.16}$$

such that for $1 \leqq \nu \leqq m$

$$\max_{\substack{1 \leqq \mu \leqq m \\ \mu \neq \nu}} \frac{x^{\beta_\mu}}{|\varrho_\mu|^n} < \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-\Delta} \quad \text{if} \quad x \in (Y^{e_\nu}, Y^{e_\nu'}). \tag{3.17}$$

Taking into account that with the notation in (3.16)

(3.18)
$$\left(\frac{d}{ds^j} \frac{x^s}{s^n}\right)_{s=\varrho_\mu} = \frac{x^{\varrho_\mu}}{\varrho_\mu^n} \sum_{l=0}^{j} \binom{j}{l} \log^{j-l} x \cdot (-n) \dots (-n-l+1) \varrho_\mu^{-l} \ll_j \frac{x^{\beta_\mu}}{|\varrho_\mu|^n} \left(\frac{\log Y}{\theta/2}\right)^j$$

we obtain by (3.9), (3.11) and (3.17) with a positive d_3

$$\delta_n f(x) = 2 \operatorname{Re}\left\{A_\nu(x) \frac{x^{\varrho_\nu}}{\varrho_\nu^n}\right\} + O\left(\frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-d_3}\right) \quad \text{if} \quad x \in (Y^{e_\nu}, Y^{e_\nu'}). \tag{3.19}$$

If the principal part of $F(s)$ at $s=\varrho_\nu$ has the form $(k\geqq 0,\ h_{k+1}\neq 0)\ \sum_{j=1}^{k+1} h_j(s-\varrho_\nu)^j$ then we have by (3.18)

$$A_\nu(x)=\frac{h_{k+1}}{k!}\sum_{l=0}^{k}\binom{k}{l}\log^{k-l}x\cdot(-n)\dots(-n-l+1)\varrho_\nu^{-l}+O(\log^{k-1}Y)=$$

$$=\frac{h_{k+1}}{k!}\sum_{l=0}^{k}\binom{k}{l}\log^{k-l}x(-n)^l\varrho_\nu^{-l}+O(\log^{k-1}Y)= \tag{3.20}$$

$$=\frac{h_{k+1}}{k!}\left(\log x-\frac{n}{\varrho_\nu}\right)^k+O(\log^{k-1}Y)=\frac{h_{k+1}}{k!}\left(\log x-\frac{n}{\varrho_\nu}\right)^k\left(1+O\left(\frac{1}{\log Y}\right)\right),$$

owing to (3.7)—(3.8). If $v=\log x$ runs over an interval

$$v\in I=\left(v_0,\ v_0+\frac{2\pi}{\gamma_\nu}(1+b)\right)\subset(Y^{e_\nu},Y^{e'_\nu}) \tag{3.21}$$

then by (3.7)—(3.8)

$$\left(v-\frac{n}{\varrho_\nu}\right)^k=\left(v_0-\frac{n}{\varrho_\nu}\right)^k\left(1+O\left(\frac{1}{\log Y}\right)\right). \tag{3.22}$$

Therefore we have for $J=\left(X_0,\ X_0\exp\left(\frac{2\pi}{\gamma_\nu}(1+b)\right)\right)\subset(Y^{e_\nu},Y^{e'_\nu})$

$$x_1,x_2,x_3\in J,\quad x_1<x_2<x_3 \tag{3.23}$$

such that for $j=1,2$ and $j=1,2,3$, resp.

$$\operatorname{sgn}\delta_n f(x_j)\neq\operatorname{sgn}\delta_n f(x_{j+1}),\quad |\delta_n f(x_j)|\gg\frac{x_j^{\beta_\nu}}{|\varrho_\nu|^n}\geqq\frac{x_j^{\beta_1}}{|\varrho_1|^n}. \tag{3.24}$$

This implies that the number of sign changes of $\delta_n f(x)$ in the interval $(Y^{e_\nu},Y^{e'_\nu})$ is at least

$$2\left[\frac{(e'_\nu-e_\nu)\log Y}{(2\pi/\gamma_\nu)(1+b)}\right]\geqq 2\left[\frac{(e'_\nu-e_\nu)\log Y}{(2\pi/\gamma_1)(1+b)}\right]. \tag{3.25}$$

Taking into account (3.16) we obtain for Y sufficiently large

$$V(\delta_n f,Y)>\left(1-3\sqrt{b}\right)\frac{\gamma_1}{\pi}\log Y. \tag{3.26}$$

Now we have only to note that if $\gamma<\infty$ then by (3.4) we have $\gamma_1=\gamma$. If $\gamma=\infty$ then for every constant C we have $\gamma_1>C$ if we choose η so small that the domain $\sigma\geqq\theta-\eta$, $|t|\leqq C$ is free of singularities of $F(s)$. Remarking further that for an arbitrary function g

$$V(g,Y)\geqq V(\delta g,Y) \tag{3.27}$$

we see that

$$\varliminf_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}. \tag{3.28}$$

We remark further that if for an arbitrary g the function δg has at least $k+1$ sign changes in an interval $[A, B]$ then g has at least k sign changes in $[A, B]$. Since owing to (3.16) and (3.23)—(3.24) the interval

$$\left[Y^{1-3\sqrt{b}}\exp\left(-n\frac{2\pi}{\gamma_1}\right), Y\right] \subset [Y^{1-3\sqrt{b}-(2\pi b/\gamma_1)}, Y] \tag{3.29}$$

contains at least $n+1$ sign changes of $\delta_n f$, the function $f(x)$ has at least one sign change in the interval

$$[Y^{1-4\sqrt{b}}, Y], \quad Y > Y_0 \tag{3.30}$$

if b was chosen sufficiently small.

What concerns the order of magnitude of the oscillation of $f(x)$ we obtain the assertion of our Theorem if we can show the same assertion with $f(x)$ replaced by

$$\bar{f}(x) = \begin{cases} f(x), & 0 \leqq x < 1 \\ f(x) \pm x^{\theta-\varepsilon}, & x \geqq 1. \end{cases} \tag{3.31}$$

But we have obviously for the corresponding function

$$\bar{F}(s) = F(s) \pm \frac{1}{s-\theta+\varepsilon} \tag{3.32}$$

and since we do not use in the proof any properties of $F(s)$ in the halfplane $\sigma < \theta - \eta$, everything remains actually unchanged if at the beginning we choose

$$\eta \leqq \varepsilon/2. \tag{3.33}$$

If $F(s)$ has logarithmic singularities too, then we proceed similarly with the choice of the parameters and broken line L. But, using the idea of Grosswald [3] we consider now the functions

$$F^{(K)}(s) = \int_0^\infty (-1)^K \log^K x \cdot f(x) x^{-s}\, dx, \quad K = \max_{1 \leqq \nu \leqq m} (\deg P_\nu + 1) \tag{3.34}$$

instead of $f(s)$ and $F(s)$. Thus, similarly to [3, p. 215] we obtain that $F^{(K)}(s)$ is meromorphic on L and to the right of L and therefore the argumentation (3.1)—(3.33) can be applied to $F^{(K)}(s)$. So we have the same conclusions for the function $\tilde{f}(x) = (-1)^K \log^K x \cdot f(x)$ in place of f. Since we have obviously

$$|V(f, Y) - V(\tilde{f}, Y)| \leqq 1 \tag{3.35}$$

and

$$f(x) \gg \log^{-K} x \cdot \tilde{f}(x), \tag{3.36}$$

all assertions of Theorem 1 hold for $f(x)$ in this case too.

4. Proof of Theorem 2

Since the proof of Theorem 2 is very similar to that of Theorem 1, we shall be brief. We obtain similarly to (3.1)—(3.6)

$$\delta_n f(x) = 2\,\mathrm{Re}\left\{A_0(x)\frac{x^{\varrho_0}}{\varrho_0^n}\right\} + O\left(M|\sigma_1+a_0+i\Gamma|\left(\frac{x^{\beta_0-a_2}}{(\beta_0-a_2)^n}+\right.\right. \tag{4.1}$$

$$\left.\left.+\frac{x^{\beta_0+a_1}}{|\beta_0+a_1+iH|^n}+\frac{x^{\sigma_1+a_0}}{|\sigma_1+a_0+i\Gamma|^n}\right)\right)$$

with absolute constants in the O symbols.

If we choose now

$$n = [d_1 \log Y + \sqrt{\log Y}], \quad x \in [Y^{d_1/d_2}\exp(\log^{3/4} Y), Y] \tag{4.2}$$

then easy calculation shows that the three error terms in (4.1) are all

$$\ll \exp(-\log^{1/5} Y)\frac{x^{\beta_0}}{|\varrho_0|^n} \quad \text{if} \quad Y > Y(\beta_0, \sigma_1, a_i, d_j, M, \Gamma). \tag{4.3}$$

Further we obtain, similarly to (3.18)—(3.24), at least two sign changes of $\delta_n f$ in every interval of the form ($\varepsilon > 0$ is arbitrary)

$$J = \left(X_0, X_0 \exp\left(\frac{2\pi}{\gamma_0}(1+\varepsilon)\right)\right) \subset [Y^{d_1/d_2}\exp(\log^{3/4} Y), Y], \tag{4.4}$$

if $Y > Y_0$. This gives the desired inequality (2.7) similarly to (3.25)—(3.28).

5. Proofs of Corollaries 2 to 6

In case of Corollary 2 the corresponding function $F(s)$ is, as well known,

$$F(s) = \frac{1}{s}\sum_{\substack{n\equiv l_1(q)\\ n\le x}}\frac{\Lambda(n)}{n^s} - \sum_{\substack{n\equiv l_2(q)\\ n\le x}}\frac{\Lambda(n)}{n^s} = \frac{1}{\varphi(q)s}\sum_{\chi(q)}(\bar\chi(l_2)-\bar\chi(l_1))\frac{L'}{L}(s,\chi), \tag{5.1}$$

which is meromorphic in the whole plane and has singularities in the half-plane $\sigma \geqq \frac{1}{2}$ (see Grosswald [2]). This proves Corollary 2. If l_1 and l_2 are both quadratic non-residues then we have

$$\Delta_4(x) = \Delta_3(x) + O(x^{1/3}) \tag{5.2}$$

whilst the oscillation of $\Delta_3(x)$, ensured by Corollary 2 is at least $x^{1/2-\varepsilon}$ and therefore Corollary 3 is true in this case. If l_1 and l_2 are both quadratic residues then let $\alpha_1', \ldots, \alpha_N'$ and $\alpha_1'', \ldots, \alpha_N''$ denote the solutions of the congruences $x^2 \equiv l_1 \pmod q$ and $x^2 \equiv l_2 \pmod q$. (The number of solutions of the two congruences is equal.) If we define

$$\bar\Delta_4(x) = \sum_{\substack{n\equiv l_1(q)\\ n\le x}}\Lambda(n) - \sum_{\substack{n\equiv l_2(q)\\ n\le x}}\Lambda(n) - \sum_{j=1}^{N}\Big\{\sum_{\substack{n\equiv\alpha_j'(q)\\ n^2\le x}}\Lambda(n^2) - \sum_{\substack{n\equiv\alpha_j''(q)\\ n^2\le x}}\Lambda(n^2)\Big\} \tag{5.3}$$

then we have clearly

$$\bar{\Delta}_4(x) = \Delta_4(x) + O(x^{1/3}). \tag{5.4}$$

On the other hand, the function

$$\int_0^\infty \frac{\bar{\Delta}_4(x)}{x^{s+1}}\,dx = \frac{1}{\varphi(q)s}\sum_{\chi(q)}\left(\bar{\chi}(l_2)-\bar{\chi}(l_1)\right)\frac{L'}{L}(s,\chi) - \tag{5.5}$$

$$-\frac{1}{\varphi(q)s}\sum_{j=1}^{N}\sum_{\chi(q)}\left(\bar{\chi}(\alpha_j'')-\bar{\chi}(\alpha_j')\right)\frac{L'}{L}(2s,\chi)$$

is also meromorphic in the whole plane and has also singularities in the half plane $\sigma \geqq \frac{1}{2}$, since the second summand is regular for $\sigma \geqq \frac{1}{2}$. Therefore Corollary 1, applied to $\bar{\Delta}_4(x)$ and (5.4) prove Corollary 3.

To prove Corollary 4 we have only to note that the function

$$\int_0^\infty \frac{M_a(x)}{x^{s+1}}\,dx = \frac{1}{s}\sum_{n=1}^{\infty}\frac{\mu(n)n^{-a}}{n^s} - \frac{1}{s\zeta(a)} = \frac{1}{s\zeta(s+a)} - \frac{1}{s\zeta(a)} \tag{5.6}$$

is regular for real $s > -a-2$, meromorphic in the whole plane and has its "lowest" non-real singularity at $s = \frac{1}{2} - a + i\gamma_1$, $\gamma_1 = 14.13\ldots$ so Corollary 4 follows from Corollary 1.

In the proof of Corollary 5 we have the identity

$$\int_0^\infty \frac{\overline{M}_a(x)}{x^{s+1}}\,dx = \frac{1}{s\zeta(s+a)}. \tag{5.7}$$

Thus, in view of (2.8) we may choose

$$\sigma_1 = 1-a, \quad \beta_0 = \frac{1}{2}-a, \quad \gamma_0 = 14.13\ldots, \quad H = 21, \quad \Gamma = 10^8, \tag{5.8}$$

$$a_0 = 10^{-3}, \quad a_1 = 10^{-20}, \quad a_2 = \min\left(\frac{4}{5}\left(\frac{1}{2}-a_0\right), 2\right)$$

and with some calculations this leads to $d_1/d_2 < 0.9$. Hence we obtain (2.17) by Theorem 2.

In case of Corollary 6 we have

$$\int_0^\infty \frac{R_k(x)}{x^{s+1}}\,dx = \frac{\zeta(s)}{s\zeta(ks)} - \frac{1}{(s-1)\zeta(k)}, \tag{5.9}$$

which is regular for real $s > 0$, meromorphic in the whole plane and has simple poles at $\frac{1}{k}\left(\frac{1}{2}\pm i\gamma_0\right)$, $\gamma_0=14.13\ldots$. Further, in view of (2.8) we can choose

$$\sigma_1=\frac{1}{k},\quad H=\frac{21}{k},\quad \Gamma=\frac{10^8}{k},\quad a_0=a_1=\frac{10^{-3}}{k},\quad a_2=\frac{2}{5k} \tag{5.10}$$

which leads to $d_1/d_2<2/5$. Thus Theorem 2 implies Corollary 6.

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MULTIPLICATIVE FUNCTIONS OVER THE GAUSSIAN INTEGERS WITH REGULARITY PROPERTIES

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1. The letters **C**, **Z**, **N** denote the set of complex numbers, rational integers and natural numbers, respectively.

Let G^* denote the multiplicative semigroup of the nonzero Gaussian integers. We shall say that a function $F\colon G^* \to \mathbf{C}$ is completely multiplicative if

$$F(\alpha\beta) = F(\alpha)\,F(\beta) \tag{1.1}$$

holds for each $\alpha, \beta \in G^*$. From (1.1) we get immediately that $F(i)^4 = F(-i)^4 = F(-1)^2 = F(1) = 1$, except the case when F is identically zero.

Let $\gamma \in G^*$. We are interested in characterizing all multiplicative functions satisfying the condition

$$F(\alpha+\gamma) - F(\alpha) \to 0 \quad \text{as} \quad |\alpha| \to \infty, \tag{1.2}$$

or something like that.

Presently we are unable to determine all solutions.

Let $\mathscr{F}$ denote the set of all functions ψ satisfying the following conditions:

a) $\psi(x)$ is defined on $(0, \infty)$;

b) $\psi(x)$ is positive, tending to zero monotonically,

c) $\sum_{l=1}^{\infty} \psi(2^l) < \infty$.

THEOREM. *Let $F\colon G^* \to \mathbf{C}$ be a completely multiplicative function such that $|F(\alpha)| = 1$ for each $\alpha \in G^*$. Assume that for some $\psi \in \mathscr{F}$,*

$$|F(\alpha+\gamma) - F(\alpha)| \leqq \psi(|\alpha|) \tag{1.3}$$

holds for each $\alpha \in G^*$, $\alpha \neq -\gamma$, where $\gamma \in G^*$ is an arbitrary fixed Gaussian integer. Then

$$F(\alpha) = e^{i\tau \log|\alpha|} \cdot e^{ik \arg \alpha} \tag{1.4}$$

where τ is a real number, k is an integer.

REMARK. It is obvious that F defined by (1.4) is completely multiplicative and satisfies (1.3).

2. Lemmata. Assume that the conditions stated in Theorem hold.

LEMMA 1. *For every* $\delta\in G^*$ *there exists* $\psi_1\in\mathcal{F}$ *such that, for* $\alpha\neq-\delta$,

$$|F(\alpha+\delta)-F(\alpha)| \leqq \psi_1(|\alpha|). \tag{2.1}$$

ψ_1 *may depend on* δ.

PROOF. Let $\mathcal{B}$ denote the set of those δ for which (2.1) holds. It is obvious that $\delta_1, \delta_2\in\mathcal{B}$ implies $\delta_1+\delta_2\in\mathcal{B}$. Furthermore by substituting $\alpha\delta$ into α in (2.1) and observing that $|F(\delta)|=1$, we deduce that $1\in\mathcal{B}$. Let now $\varepsilon\in\{i, -i, -1, 1\}$, i.e. a unit in G^*. Since $F(\alpha+\bar\varepsilon\gamma)=F(\varepsilon\alpha+\gamma)F(\bar\varepsilon)$, $F(\alpha)=F(\varepsilon\alpha)F(\bar\varepsilon)$, we get $\bar\varepsilon\gamma\in\mathcal{B}$ if $\gamma\in\mathcal{B}$. So $\{i, -i, -1, 1\}\in\mathcal{B}$, consequently the assertion is true. □

LEMMA 2. *If* f: $\mathbf{N}\to\mathbf{C}$ *is a completely multiplicative function such that* $|f(n)|=1$ *and*

$$|f(n+1)-f(n)| \leqq \psi_2(n) \tag{2.2}$$

with a suitable $\psi_2\in\mathcal{F}$, *then* $f(n)=n^{i\tau}$ *with a real* τ.

This is Theorem 4.1 in [1].

For a real z let $\|z\|$ denote the distance of z to the nearest integer.

LEMMA 3. *Let* ξ *be an irrational,* η *be a real number with the following property: if* $m_1<m_2<\dots$ *is an infinite sequence of natural numbers such that* $\|m_\nu\xi\|\to0$, *then* $\|m_\nu\eta\|\to0$. *Then* $\eta=k\xi+l$, k *and* l *are integers.*

For the proof see [2].

Let $\alpha\in G^*$. It is known that the relation

$$\nu \arg\alpha \equiv 0 \pmod{2\pi} \tag{2.3}$$

can be solved with nonzero integer ν, if and only if

$$\arg\alpha\equiv 0\left(\mathrm{mod}\,\frac{\pi}{4}\right).$$

We shall say that $\alpha\in G^*$ is a primitive element, if $\arg\alpha\not\equiv0\left(\mathrm{mod}\,\frac{\pi}{4}\right)$, and D/α, $D\in\mathbf{N}$ implies $D=1$.

We shall say that $\alpha\in G^*$ is a square-free number if in its prime-decomposition $\alpha=\pi_1^{a_1}\dots\pi_r^{a_r}$ each exponent $a_j=1$ $(j=1, \dots, r)$.

LEMMA 4. *Let* α, β *be square-free primitive, coprime elements, i.e.* $(\alpha, \beta)=1$, *furthermore* $\beta\neq\bar\alpha$. *Then the relation*

$$\frac{A}{B}=\alpha^\nu\beta^\mu$$

with rational integers A, B, ν, μ *cannot hold, except when* $\nu=0$, $\mu=0$, *or when* $\nu=\mu$ *and* $\beta=\bar\alpha\eta\delta^{-1}$ *with* $\delta=1+i$ *and a suitable* $\eta|2$.

PROOF. Let us assume on the contrary that there exists another solution. We may assume that $(A, B)=1$, and $\nu>0$. From $\mu=0$ it would follow $2\nu\arg\alpha\equiv0$

mod 2π, which is excluded by the assumption that α is primitive. Since $(\alpha,\beta)=1$, therefore $\alpha|A$ and so $\bar{\alpha}|A$. Since α is primitive, therefore $(\alpha,\bar{\alpha})=1$ or δ, and so $\alpha\bar{\alpha}|A$. Let us assume that $\gamma=(\bar{\alpha}, B)$, and γ is not a unit in G^*. Then as above we deduce that $\gamma\bar{\gamma}|B$, which by $N(\gamma)|N(\alpha)|A$, contradicts the assumption $(A, B)=1$.

It has remained the case when $\mu>0$, $\bar{\alpha}|\beta^{\mu}\delta^{\nu}$. But then $\bar{\alpha}|\beta\delta$ since $\bar{\alpha}$ is a square-free integer. Since $\bar{\alpha}^{\nu}|A$, therefore $\mu\geqq\nu$. Since μ is positive, the roles of α and β can be exchanged. Hence it follows that $\nu=\mu$ and $\bar{\beta}|\alpha\delta$. Thus $\delta\beta=\bar{\alpha}\Delta$, $\delta\alpha=\bar{\beta}H$, $\Delta, H\in G^*$, whence $\Delta|2$.

The lemma is proved.

LEMMA 5. *Let* $k\colon G^*\to \mathbf{Z}$ *be a function such that*

$$H(\alpha):=e^{ik(\alpha)\arg\alpha}$$

is completely multiplicative. Then $k(\alpha)=k$ *is constant for all* $\alpha\in G^*$, $\arg\alpha\not\equiv 0\left(\mathrm{mod}\,\frac{\pi}{4}\right)$.

PROOF. Since $H(\alpha\beta)=H(\alpha)H(\beta)$, therefore

$$1=H(\alpha\beta)H^{-1}(\alpha)H^{-1}(\beta)=e^{i\arg[(\alpha\beta)^{k(\alpha\beta)}\alpha^{-k(\alpha)}\beta^{-k(\beta)}]},$$

whence we have that

$$\Gamma:=(\alpha\beta)^{k(\alpha\beta)}\alpha^{-k(\alpha)}\beta^{-k(\beta)} \tag{2.4}$$

is a positive real number. Since Γ is an element of the field of Gaussian numbers, therefore $\Gamma=\frac{A}{B}$, A, B are rational integers. From (2.4) we have

$$\frac{A}{B}=\alpha^{k(\alpha\beta)-k(\alpha)}\beta^{k(\alpha\beta)-k(\beta)}.$$

Let now α and β be such integers for which the condition of Lemma 4 holds. Then, from Lemma 4 we get $k(\alpha)=k(\beta)$. This implies $k(\alpha)=k=$constant if α runs over the set of primitive, square-free elements.

Let now $\pi\in G^*$ be a primitive prime. Then

$$1=H(\pi^a)H(\pi)^{-a}=e^{ik(\pi^a)\arg(\pi^a)-ik(\pi)a\arg\pi}$$

implies

$$\pi^{ak(\pi^a)}\pi^{-ak(\pi)}=\frac{A}{B},$$

i.e. $ak(\pi^a)=ak(\pi)$, $k(\pi^a)=k(\pi)=k$.

Let now $\alpha=\pi_1^{a_1}\dots\pi_r^{a_r}$ be composed from primitive primes. Starting from

$$1=H(\alpha)H^{-a_1}(\pi_1)\dots H^{-a_r}(\pi_r),$$

we deduce as above that

$$\pi_1^{a_1(k(\alpha)-k)}\dots\pi_r^{a_r(k(\alpha)-k)}=\frac{A}{B}, \tag{2.5}$$

A, B, are positive rational integers. The left hand side can be represented as $\alpha^{k(\alpha)-k}$.

Let us assume that $k(\alpha)\neq k$. Then α satisfies (2.3) with $v=k(\alpha)-k$, and α is not a primitive element. Similarly, $k((1+i)\alpha)=k(\alpha)=k$.

So we have proved that $k(\alpha)=k$ for each primitive element $\alpha\in G^*$. Finally, $k((1+i)n\alpha)=k(n\alpha)=k(\alpha)=k$.

Lemma 6. *The set of functions F satisfying the conditions stated in the theorem is a group under the multiplication $F_1\cdot F_2(\alpha)=F_1(\alpha)F_2(\alpha)$, furthermore $F^{-1}(\alpha)=\overline{F(\alpha)}$.*

This is an easy consequence of the triangle inequality so the proof is omitted.

Lemma 7. *Let $\mathcal{D}:=\{0,1,i,1+i\}$, $A\in G^*$ such that $|A|<2^M$. Then A can be represented as*

$$A=\delta_0+\delta_1\cdot 2+\ldots+\delta_M\cdot 2^M+\beta\cdot 2^{M+1},$$

where $\delta_j\in\mathcal{D}$ $(j=0,\ldots,M)$, $\beta\in G$, $|\beta|\leqq 2$.

Proof. $\mathcal{D}$ is a full-residue system mod 2 in G. Therefore we can define the algorithm $\delta_0\equiv A \pmod 2$, $\delta_0\in\mathcal{D}$, $A=\delta_0+2A_1$, $A_1=\delta_1+2A_2,\ldots$. Hence

$$|A_{j+1}|\leqq\frac{|A_j|}{2}+\frac{\sqrt{2}}{2},$$

and so

$$|A_M|\leqq\frac{|A_{M-1}|}{2}+\frac{\sqrt{2}}{2}\leqq\frac{|A_{M-2}|}{2^2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2^2}\leqq\ldots\leqq\frac{|A|}{2^M}+\sqrt{2},$$

$$|A_{M+1}|<\frac{1}{2}+\sqrt{2}.$$

This gives the desired result.

3. Proof of the Theorem. From Lemma 1 we deduce that

$$|F(n+1)-F(n)|\leqq\psi_1(n)$$

as $n\to\infty$ over $n\in\mathbf{N}$, which by Lemma 2 gives $F(n)=e^{i\tau\log n}$ with a suitable real τ.

Let now

$$H(\alpha):=e^{-i\tau\log|\alpha|}, \tag{3.1}$$

and

$$F_1(\alpha):=F(\alpha)H(\alpha). \tag{3.2}$$

Since H satisfies the conditions stated for F in the theorems, by Lemma 6 we get that they are valid for F_1 as well.

First we observe that F_1 depends only on the argument of α. Indeed, if $\arg\alpha=\arg\beta$, $\alpha,\beta\in G^*$, then $\beta=r\alpha$, $r>0$ real, therefore $r=\frac{A}{B}=$rational. We may assume that $A,B>0$, and so we have

$$F_1(B\beta)=F_1(\beta)=F_1(A\alpha)=F_1(\alpha)$$

since $F_1(n)=1$ if $n\in\mathbf{N}$.

Let now $\alpha=|\alpha|e^{2\pi i\xi}$, ξ irrational. Let furthermore m_l be any sequence of positive integers, $m_l\to\infty$, such that $\|m_l\xi\|\to 0$. Let r_l be a monotonic sequence of positive integers, t_l and s_l be such positive integers for which the inequality

$$\left|\frac{t_l|\alpha|^{m_l}}{s_l2^{r_l}}-1\right| < 1/l^2$$

holds. Then

$$\left|\frac{t_l\alpha^{m_l}}{s_l2^{r_l}}-1\right| < \frac{1}{2^{h_l}} \tag{3.3}$$

for a suitable monotonic sequence $h_1\leqq h_2\leqq \dots$ of integers, where $h_l\to\infty$ $(l\to\infty)$. Then, for $E:=t_l\alpha^{m_l}-s_l2^{r_l}$ we have

$$|E| < 2^{r_l-h_l}s_l. \tag{3.4}$$

Let K be the integer for which $2^{K-1}\leqq s_l<2^K$. Then by Lemma 7 E can be written in the form

$$E=\delta_0+\delta_1\cdot 2+\dots+\delta_{r_l-h_l+K}\cdot 2^{r_l-h_l+K}+\beta\cdot 2^{r_l-h_l+K+1},$$

where $\delta_j\in\mathcal{D}$, $|\beta|\leqq 2$.

Let now a be the nonnegative residue of s_l mod (2^{K-h_l+2}), i.e. $s_l=a+b\cdot 2^{K-h_l+2}$, $0\leqq a<2^{K-h_l+2}$. Let

$$a=e_0+e_1\cdot 2+\dots+e_{K-h_l+1}\cdot 2^{K-h_l+1}$$

be the binary expansion of a, i.e. $e_j\in\{0,1\}$. Hence we get immediately that $M:=E+s_l2^{r_l}$ can be written in the form

$$M=\theta_0+\theta_1\cdot 2+\dots+\theta_H\cdot 2^H+b\cdot 2^{H+1},$$

where $H=r_l+K-h_l+1$, $\theta_j\in G$, $|\theta_j|\leqq 3$. Since $2^{K-1}\leqq s_l<2^K$, we have

$$2^{K-1}-2^{K-h_l+2}<b\cdot 2^{K-h_l+2}<2^K, \quad \text{and so} \quad 2^{h_l-3}-1<b<2^{h_l-2}.$$

Let now $M_1, M_2, \dots$ be defined as follows:

$$(M_0:=)M=\theta_0+2M_1,\quad M_1=\theta_1+2M_2,\ \dots,\quad M_H=\theta_H+2M_{H+1},\quad M_{H+1}=b.$$

From Lemma 1 we get, with a suitably chosen $\psi_2\in\mathcal{F}$,

$$|F_1(M_j)-F_1(2M_{j+1})|\leqq\psi_2(2|M_{j+1}|)$$

$(j=0,\dots,H)$, if $b=M_{H+1}\geqq 3$. Since $F_1(2)=1$, $F_1(b)=1$, we get

$$|F_1(M)-1|\leqq\sum_{j=0}^{H}\psi_2(2|M_{j+1}|).$$

Since $|M_j|\geqq 2|M_{j+1}|-3$, and $M_{H+1}>2^{h_l-3}-1$, by Property c) of ψ_2 we get immediately that $|F_1(M)-1|\leqq\varrho_l$, where $\varrho_l\to 0$ as $l\to\infty$. Now we observe that $M=t_l\alpha^{m_l}$, and so from $F_1(t_l)=1$ we get $|F_1(\alpha^{m_l})-1|\leqq\varrho_l$, and so by $F_1(\alpha):=e^{2\pi i\eta}$, $\|\eta m_l\|\to 0$. Thus from Lemma 3 $\eta=k\xi+l$, $k, l\in\mathbf{Z}$. Let now ξ be a rational number. Then ξ is a multiple of 1/8, i.e. $\xi\equiv 0 \pmod{1/8}$. Let $\alpha=1+i$, giving the value $\xi=1/8$. Then

α^8=positive integer, consequently $F_1(\alpha^8)=F_1(\alpha)^8=e^{2\pi i 8\eta}=1$, and so 8η=integer, $\eta=\frac{n}{8}$, $n\in\mathbf{Z}$. All the rational ξ's are determined by α^j ($j=1, \dots, 8$), the corresponding values of ξ and η are $j/8$, $nj/8$ (mod 1). So we have proved that $\eta=k(\alpha)\xi+l(\alpha)$, $k(\alpha), l(\alpha)\in\mathbf{Z}$ holds for $\alpha\in G^*$. Since $2\pi\xi=\arg\alpha$, the conditions of Lemma 5 are satisfied for $H=F_1$. Consequently $F_1(\alpha)=e^{ik\arg\alpha}$ with a suitable integer k, assuming that $\arg\alpha\not\equiv 0\left(\operatorname{mod}\frac{\pi}{4}\right)$.

Let now $\alpha_n=(1+i)n$. Then $F_1(\alpha_n+1)-F_1(\alpha_n)\to 0$ as $n\to\infty$, $F_1(\alpha_n)=F_1(1+i)$. Since

$$\arg(\alpha_n+1) = \arg\left(1+i-\frac{i}{n+1}\right) \to \arg(1+i),$$

from the continuity of the functions arg and exp we get that $F_1(1+i)=e^{ik\arg(1+i)}$ as well. So

$$F_1((1+i)^j) = e^{ikj\arg(1+i)} = e^{ik\arg(1+i)^j} \quad \text{for} \quad j=1, \dots, 8.$$

By using the multiplicativity of F_1 we get immediately that $F_1(\alpha)=e^{ik\arg\alpha}$ for $\alpha\in G^*$. The proof of the Theorem is finished.

4. Remarks. 1. Perhaps the same assertion can be deduced from (1.2) instead of (1.3). Our method is not strong enough to do this. Recently E. Wirsing proved that if f is completely multiplicative, defined on $\mathbf{N}$, $f(n+1)-f(n)\to 0$ ($n\to\infty$), $|f(n)|=1$ ($n=1, 2, \dots$) then $f(n)=e^{i\tau\log n}$. By using this theorem we can deduce immediately the following assertion.

If $F\colon G^*\to\mathbf{C}$ is completely multiplicative, $|F(n)|=1$ satisfying (1.2) then

$$F(\alpha) = e^{i\tau\log|\alpha|}H(\alpha),$$

where H satisfies (1.1) and it depends only on the argument of α.

2. A similar theorem can be proved for multiplicative functions defined on the set of integers in other imaginary quadratic fields.

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ON CONSECUTIVE SUMS IN SEQUENCES

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Introduction

Let $A \subset \mathbf{N}$. We define B as the following set of integers, called the set of the consecutive sums in the sequence A:

$$B = \Big\{ \sum_{u \leqq i \leqq v} a_i \,|\, A = \{a_1, a_2, \ldots, a_n\},\ 1 \leqq u \leqq v \leqq n \Big\}.$$

We will call it shortly the set of c-sums.

In [1] Erdős and Harzheim asked that if $1 \leqq a_1, a_2, \ldots, a_k \leqq n$, can we find cn a_i's so that all c-sums are different? (They conjectured that this is not true if $a_1 < a_2 < a \ldots < a_k$ is also assumed.)

We now answer the first question and investigate several related problems.

Some authors examined the case of c-sums with just two terms (see Segal, Odlyzko in [1] and see also [2] and [3]).

1. In this section we answer the question of Erdős and Harzheim. We prove the following

THEOREM 1. *Let $k = f(n)$ be the maximum number of integers so that $1 \leqq a_1, a_2, \ldots$ $\ldots, a_k \leqq n$ and all c-sums are different. Then*

$$\left(\frac{1}{3} + o(1)\right) n \leqq f(n) \leqq \left(\frac{2}{3} + o(1)\right) n.$$

PROOF. First we prove the lower bound. Let

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \ldots, \quad s_k = a_1 + a_2 + \ldots + a_k.$$

The condition that all c-sums are different implies, that

$$(1.1) \qquad s_v - s_{u-1} = \sum_{u \leqq i \leqq v} a_i \neq \sum_{r \leqq j \leqq t} a_j = s_t - s_{r-1}.$$

However, this means that the sequence $\{s_i\}$ is a Sidon sequence and obviously

$$s_{i+1} - s_i \leqq n, \quad i = 1, 2, \ldots, k-1.$$

Let now p be a prime satisfying

$$(1.2) \qquad (1 - \varepsilon) \frac{n}{3} \leqq p \leqq \frac{n}{3}$$

and let

$$(1.3) \qquad a_{i+1} = 2p + [(i+1)^2]_p - [i^2]_p, \quad i = 0, 1, \ldots, p-1$$

where

$$[i^2]_p \equiv i^2 \bmod p \quad \text{and} \quad 0 \leqq [i^2]_p \leqq p-1.$$

It follows from (1.2) and (1.3) that

$$a_{i+1} = 2p+[(i+1)^2]_p-[i^2]_p < 3p \leqq n$$

and

$$s_i = 2pi+[i^2]_p.$$

It is well-known that the sequence $\{s_i\}$ is a Sidon sequence (see [4] p. 90) and this proves the lower bound.

Upper bound. Let t be sufficiently large but fixed. Consider all c-sums where the number of terms is at most t:

$$b_{u,r} = \sum_{u+1}^{u+r} a_i, \quad 1 \leqq r \leqq t, \quad 1 \leqq u+r \leqq k. \tag{1.4}$$

Summing all such $b_{u,r}$-s, we observe that an a_i appears in the sum at most $\binom{t+1}{2}$ times (namely if $1 \leqq r \leqq t$, then a_i occurs at most in $b_{i-1,r}, b_{i-2,r}, \ldots, b_{i-r,r}$ that is at most r times). So

(1.5)

$$\sum_{\substack{1 \leqq r \leqq t \\ 1 \leqq u+r \leqq k}} b_{u,r} \leqq \binom{t+1}{2} \sum_{i=1}^{k} a_i \leqq \frac{(t+1)^2}{2}[n-k+1+\ldots+n] = \frac{(t+1)^2}{2} \cdot \frac{k(2n-k+1)}{2}.$$

On the other hand $b_{u,r} \neq b_{v,t}$ so

$$\sum_{\substack{1 \leqq r \leqq t \\ 1 \leqq u+r \leqq k}} b_{u,r} = (b_{1,1}+b_{2,1}+\ldots+b_{k-1,1})+(b_{1,2}+b_{2,2}+\ldots+b_{k-2,2})+\ldots \tag{1.6}$$

$$\ldots+(b_{1,t}+\ldots+b_{k-t,t}) = c_1+c_2+\ldots+c_{kt-\binom{t+1}{2}} \geqq 1+2+\ldots$$

$$\ldots+\left(kt-\binom{t+1}{2}\right) > \frac{\left(kt-\binom{t+1}{2}-1\right)^2}{2}$$

where $c_1 < c_2 < \ldots$ are the $b_{u,r}$ monotonously ordered. But t is fixed so we have

$$\sum_{\substack{1 \leqq r \leqq t \\ 1 \leqq u+r \leqq k}} b_{u,r} > \frac{k^2 t^2 (1-\varepsilon)}{2} \tag{1.7}$$

if n is large enough.

From (1.5) and (1.7) we have

$$(1-\varepsilon)\frac{k^2 t^2}{2} < \frac{(t+1)^2}{2} \frac{k(2n-k+1)}{2}$$

so $k < (1+\varepsilon')\dfrac{2}{3} n$. Q.e.d.

REMARK. P. Erdős observed that the upper bound follows also from the Erdős—Turán argument on finite Sidon sequences (see [4], p. 86).

In the next sections we will investigate monotone sequences i.e. where $a_1 < a_2 < \dots < a_k$.

2. Translating problem. Let $A = \{a_1 < a_2 < \dots < a_n\}$ be a finite sequence and as usual

$$A+t = \{a_1+t < a_2+t < \dots < a_k+t\}, \quad t \in \mathbf{N}.$$

It is easy to see that for every finite sequence there exists an integer t such that every c-sum is different.

In fact

$$\text{(i)} \qquad \sum_{i=u+1}^{u+r} (a_i+t) = \sum_{j=v+1}^{v+s} (a_j+t)$$

implies $r \neq s$ (the sequence is monotone) and

$$\text{(ii)} \qquad \sum_{u+1}^{u+r} a_i - \sum_{v+1}^{v+s} a_j = (s-r)t.$$

So if

$$t-1 = \sum_{i=2}^{k} a_i = \max_{u,r,s,v} \Big\{\sum_{u+1}^{u+r} a_i - \sum_{v+1}^{v+s} a_j\Big\}$$

then (i) and (ii) do not hold.

But t may be smaller than $\sum_{2}^{k} a_i$.

DEFINITION 1. Denote by $t = t_A(k)$ the minimal value of t for which $A+t = \{a_1+t < a_2+t < \dots < a_k+t\}$ is such a sequence where all c-sums are different.

We will show that for $A = \{1, 2, \dots, k\}$ $t_A(k)$ is not much smaller than $\sum_{i=2}^{k} i \sim k^2/2$.

In fact we prove the following

THEOREM 2. *We have*

$$\big(1+o(1)\big)\frac{k^2}{25} < t_A(k) < \frac{k^2}{4}.$$

PROOF. First we prove that $t_A(k)$ does not exceed $k^2/4$. Let $t = \left[\frac{k^2}{4}\right]$, so

$$A+t = \left\{\left[\frac{k^2}{4}\right]+1, \dots, \left[\frac{k^2}{4}\right]+k\right\}.$$

Assume

$$(2.1) \qquad \sum_{i=u+1}^{u+r} (i+t) = \sum_{j=v+1}^{v+s} (j+t)$$

or

$$(2.2) \qquad \sum_{i=u+1}^{u+r} i - \sum_{j=v+1}^{v+s} j = (s-r)t, \quad r < s.$$

But

$$\sum_{i=u+1}^{u+r} i - \sum_{j=v+1}^{v+s} j = dr-m \tag{2.3}$$

where

$$m = \sum_{j=v+1}^{v+s-r} j \quad \text{and} \quad d = (u+r)-(v+s).$$

Since $r<s$, therefore $d+r<k$ so from the right hand side of **(2.3)** we have

$$(s-r)t = dr-m < \left(\frac{d+r}{2}\right)^2 - m < \frac{k^2}{4} - m < t,$$

a contradiction.

Now we deduce that if $t \leqq \frac{k^2}{25}(1+o(1))$ then there exists at least one pair of equal c-sums.

Denote by $B(x, y)$ a block of consecutive integers with y terms — where y is an odd number — and with centre x. I.e. let

$$B(x, y) = \left[x-\frac{y-1}{2};\ \dots;\ x;\ \dots;\ x+\frac{y-1}{2}\right].$$

Put

$$|B(x, y)| = \sum_{i=x-(y-1)/2}^{x+(y-1)/2} i = xy.$$

Let $m=\left[\frac{k}{10}\right]$. We shall consider certain B-blocks of the form $B\big((2i-1)(2i+1; u\big)$ and $\big(B(2i-1)(2i+3); v\big)$ with suitably chosen u and v:

$$B_{i,1} = B\big((2i-1)(2i+1);\ 2i+3\big) = [c_{i,1};\ d_{i,1}],$$

$$B_{i,2} = B\big((2i-1)(2i+3);\ 2i+1\big) = [c_{i,2};\ d_{i,2}],$$

$$B_{i,3} = B\big((2i+1)(2i+3);\ 2i-1\big) = [c_{i,3};\ d_{i,3}],$$

$i=1, 2, \dots, m-1$. Clearly $|B_{i,1}|=|B_{i,2}|=|B_{i,3}|$.

Case I: $c_{i,1} \leqq t \leqq c_{i,2}$. Since

$$d_{i,3}-t \leqq d_{i,3}-c_{i,1} = 10i+4 < k$$

therefore both $B_{i,2}$ and $B_{i,3}$ are completely in the interval $[t+1;\ t+k]$.

Case II: $c_{i,2} \leqq t < c_{i+1,1}$. Since

$$d_{i+1,2}-t \leqq d_{i+1,2}-c_{i,2} = 10i+9 < k$$

therefore both $B_{i+1,1}$ and $B_{i+1,2}$ are completely in the interval $[t+1, t+k]$.

We show that for any

$$t \leqq c_{m-1,2} = 4m^2-5m-2 = \frac{k^2}{25}(1+o(1))$$

the interval $[t+1; t+k]$ necessarily contains two equal c-sums, namely either both $B_{i,2}$ and $B_{i,3}$ or both $B_{i+1,1}$ and $B_{i+1,2}$, for some i. Q.e.d.

3. We have seen in the first section that there exists a sequence with cn terms till n, so that all c-sums are different. Prof. Erdős asked the following question in connection with this (personal communication). Is it true that if $\{a_i\}$ is an increasing sequence and

$$a_{i+1}-a_i \leqq K, \quad K\in \mathbf{N}, \tag{3.1}$$

then there exist at least two c-sums which are equal if a_i is large enough?

The answer is yes and we establish this statement in a quantitative form.

Let $f(a, K)$ be the largest integer with the following property: There exists an increasing sequence

$$a = a_1 < a_2 < \ldots < a_s = f(a, K)$$

such that

$$a_{i+1}-a_i \leqq K, \quad i = 1, 2, \ldots, s-1$$

and all c-sums are different.

It is easy to see that

$$f(1, 1) = 2,\ f(2, 1) = 4,\ f(1, 2) = 7,\ f(2, 2) = 10$$

and we have seen in the preceding section that

$$a+(1+o(1))2\sqrt{a} < f(a, 1) < a+(1+o(1))5\sqrt{a} \quad \text{for} \quad a > a_0.$$

We give an upper bound for $f(a, k)$, wich proves that $f(a, k)$ exists for all a and k.

THEOREM 3. *We have*

$$f(a, K) < (a+K/2)e^{K+1}+Ke^{2K+2}.$$

PROOF. Let D be arbitrary but fixed. We estimate how many blocks have c-sums less than D. (3.1) implies

$$a_{i+1} \leqq a+iK \tag{3.2}$$

so

$$a_{i+1}+\ldots+a_{i+j} \leqq ja+K\left[\binom{i+j+1}{2}-\binom{i+1}{2}\right] = j(a+K/2)+\frac{K}{2}(2ij+j^2). \tag{3.3}$$

Thus if

$$j(a+K/2)+\frac{K}{2}(2ij+j^2) \leqq D \tag{3.4}$$

then the block $a_{i+1}+\ldots+a_{i+j}$ clearly has a c-sum less than D. Rearranging (3.4) we obtain

$$i \leqq \frac{D}{K\cdot j}-\frac{a+K/2}{K}-\frac{j}{2}.$$

This means that there are at least

$$\left[\frac{D}{K\cdot j}-\frac{a+K/2}{K}-\frac{j}{2}\right]$$

blocks of length j and c-sum less than D. Summing for all j from 1 to A we have at least

$$S=\sum_{j=1}^{A}\left[\frac{D}{K\cdot j}-\frac{a+K/2}{K}-\frac{j}{2}\right] \tag{3.5}$$

blocks with c-sums less than D. Now

$$S>S'=\frac{D}{K}\log A-\frac{A(a+K/2)}{K}-\frac{(A+2)^2}{4}. \tag{3.6}$$

If

$$S'\geqq D \tag{3.7}$$

then obviously we must have two blocks with equal c-sums (since all c-sums are less than D). (3.7) means

$$D(\log A-K)>A(a+K/2)+\frac{K(A+2)^2}{4}.$$

If we choose $A=[e^{K+1}]$ then we obtain

$$D>L=e^{K+1}(a+K/2)+Ke^{2K+2}.$$

This means that we necessarily have equal c-sums among those blocks whose c-sum is in the range $[1, L]$.

Thus we must have also $f(a, K)\leqq L$. Q.e.d.

4. It is well-known that if $A=\{a_1<a_2<\ldots<a_n\}$ then the difference set $D(A)= =\{a_i-a_j|a_i, a_j\in A\ \ i>j\}$ contains at least $n-1$ elements and $A=\{d, 2d, \ldots, nd\}$ shows that this is sharp. But what happens if we assume that the set of the consecutive gaps is an increasing sequence?

Is it true that if

$$g(n)=\min_A\left\{|D(A)|\,\middle|\, b_1<b_2<\ldots<b_n,\ \ b_i=a_{i+1}-a_i\right\}$$

then $\lim g(n)/n=\infty$? I conjecture that to every $\varepsilon>0$ there is an n_0 such that for every $n>n_0$, $g(n)>n^{2-\varepsilon}$. We are very far from being able to prove this, but we prove the following weaker

THEOREM 4. *We have*

$$g(n)>cn\frac{\log n}{\log\log n}.$$

REMARK. Erdős and Harzheim noted in [1] that for $a_i=i^2$, $|D(A)|=o(n^2)$ which shows $g(n)=o(n^2)$.

It is not too difficult to see that for $a_i=i^2$

$$|D(A)|=O\big(n^2/(\log n)^\alpha\big)\quad\text{for some}\quad \alpha>0.$$

PROOF. It is clear that $D(A)$ consists of the c-sums of the set $\{a_{i+1}-a_i\}$. We distinguish two cases.

Case A: For

$$k=\left[\frac{n}{3}\right],\quad \frac{b_{2k}}{\log n}\leqq b_k.$$

Then consider the sums

(4.1) $$b_k+b_{k+1},\ b_{k+1}+b_{k+2},\ \dots,\ b_{2k-1}+b_{2k}.$$

It is clear that these sums are different and do not exceed $2b_{2k}$.

There exists an integer i_1 such that

(4.2) $$\frac{2b_{2k}}{i_1}\leqq\frac{b_{2k}}{\log n};$$

for example $i_1=2\,([\log n]+1)$. But then

(4.3) $$2b_{2k}\leqq 2\log n\cdot b_k<i_1b_k<b_k+b_{k+1}+\dots+b_{k+i_1-1}<\dots$$
$$\dots<b_{2k-i_1+1}+\dots+b_{2k}<i_1b_{2k}$$

are different and are not equal to any of the sums in (4.1). And generally, if i_j has already been defined then define i_{j+1} so that

(4.4) $$\frac{i_jb_{2k}}{i_{j+1}}\leqq\frac{b_{2k}}{\log n}.$$

For example let $i_{j+1}=2([\log n]+1)^{j+1}$. But then

(4.5) $$i_jb_{2k}<b_k+b_{k+1}+\dots+b_{k+i_{j+1}-1}<\dots<b_{2k-i_{j+1}+1}+\dots+b_{2k}<i_{j+1}b_{2k}.$$

The process runs until

(4.6) $$2([\log n]+1)^{j+1}<n/3,$$

i.e. $j<c\dfrac{\log n}{\log\log n}$.

We now count how many sums we have obtained. In the first step we got $n/3$ sums, in the i_j-th step we got $\dfrac{n}{3}-i_j+2$ sums. So we have

(4.7) $$\frac{n}{3}\,\frac{\log n}{\log\log n}-\sum_{j=1}^{c\frac{\log n}{\log\log n}}i_j>c'n\frac{\log n}{\log\log n}$$

(since $\Sigma i_j=\Sigma 2([\log n]+1)^j\ll n$) and this proves the theorem in Case A.

Case B: $\dfrac{b_{2k}}{\log n}>b_k$ or

(4.8) $$b_{2k}\cdot\frac{n}{3\log n}>\frac{n}{3}\,b_k.$$

Let

(4.9) $$T=\left[\frac{n}{3\log n}\right]+1$$

and consider the sums

(4.10) $$b_{2k+iT+1}+\ldots+b_{2k+jT}, \quad 1\leqq i<j\leqq \log n$$

and

(4.11) $$b_u+\ldots+b_v, \quad 1\leqq u<v\leqq n/3.$$

The sums in (4.10) are larger than $b_{2k}\cdot T\geqq b_{2k}\cdot n/3\log n$, and the sums in (4.11) do not exceed $b_k\cdot n/3$.

So (4.8) implies that the sums in (4.10) and (4.11) are different.

Now consider the sums

(4.12) $$b_i+\ldots+b_{2k+jT}, \quad 1\leqq i\leqq n/3, \quad 1\leqq j\leqq \log n.$$

We assert that the sums in (4.12) are different. Indirectly if we find two equal sums

(4.13) $$b_i+\ldots+b_{2k+jT}=b_u+\ldots+b_{2k+vT},$$

$i<u$, $j<v$, then we have

(4.14) $$b_i+\ldots+b_{u-1}=b_{2k+jT+1}+\ldots+b_{2k+vT}$$

which is impossibile.

So from (4.12) we obtain $\frac{n}{3}\log n$ different sums, wich proves the theorem in Case B and thus the proof of Theorem 4 is complete.

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PROBLEMS AND RESULTS ON ADDITIVE PROPERTIES OF GENERAL SEQUENCES. II

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1. Let $\mathcal{A}=\{a_1, a_2, \dots\}$ $(a_1<a_2<\dots)$ be an infinite sequence of positive integers. Denote by $R(n)$ the number of solutions of $a_i+a_j=n$. Starting from a problem of Sidon, P. Erdős [1] proved the following theorem (by using probabilistic methods): There is a sequence $\mathcal{A}$ so that there are two constants c_1 and c_2 for which for every n

$$c_1 \log n < R(n) < c_2 \log n. \tag{1}$$

On the other hand, an old conjecture of Erdős states that for no sequence $\mathcal{A}$ can we have

$$\frac{R(n)}{\log n} \to c \quad (0 < c < +\infty). \tag{2}$$

(See [2] and [4] for further related results and problems.)

These problems led us to study the question: how regular can be the behaviour of the function $R(n)$? In part I [3] of this paper, we proved the following results:

THEOREM 1. *If $F(n)$ is an arithmetic function such that*

$$F(n) \to +\infty, \tag{3}$$

$$F(n+1) \geqq F(n) \quad \textit{for} \quad n \geqq n_0 \tag{4}$$

and

$$F(n) = o\left(\frac{n}{(\log n)^2}\right), \tag{5}$$

and we write

$$\Delta(N) = \sum_{n=1}^{N} (R(n)-F(n))^2,$$

then $\Delta(N)=o(NF(N))$ cannot hold.

COROLLARY 1. *If $F(n)$ is an arithmetic function satisfying* (3), (4) *and* (5), *then*

$$\max_{n\leqq N} |R(n)-F(n)| = o((F(n))^{1/2}) \tag{6}$$

cannot hold.

(In fact, Theorem 1 says that (6) is impossible in square mean.)

The aim of this paper is to show that the above results are nearly best possible. We will prove the following theorem:

THEOREM 2. *If $F(n)$ is an arithmetic function satisfying*

$$F(n) > 36 \log n \quad \text{for} \quad n > n_0, \tag{7}$$

and there exist a real function $g(x)$, defined for $0<x<+\infty$, and real numbers x_0, n_1 such that

(i) *$g'(x)$ exists and it is continuous for $0<x<+\infty$,*
(ii) *$g'(x)\leqq 0$ for $x\geqq x_0$,*
(iii) *$0<g(x)<1$ for $x\geqq x_0$,*
(iv) $\left|F(n)-2\int_0^{n/2} g(x)g(n-x)\,dx\right|<(F(n)\log n)^{1/2}$ *for* $n>n_1$,

then there exists a sequence $\mathcal{A}$ such that

$$|R(n)-F(n)| < 8\big(F(n)\log n\big)^{1/2} \quad \text{for} \quad n > n_2.$$

By choosing $F(n)$ and $g(x)$ in Theorem 2 in an appropriate way, the following corollaries can be derived from Theorem 2:

COROLLARY 2. *If β is an arbitrary real number such that $\beta>8/\pi^{1/2}$, then there exists an infinite sequence $\mathcal{A}$ such that* (1) *holds with* $(0<)c_1=\beta^2\pi-8\beta\pi^{1/2}$, $c_2=\beta^2\pi+8\beta\pi^{1/2}$.

(So that, e.g., choosing $\beta=5$, we obtain that (1) holds with $c_1=6<\beta^2\pi-8\beta\pi^{1/2}$ and $c_2=151>\beta^2\pi+8\beta\pi^{1/2}$.)

COROLLARY 3. *If $G(x)$ is a real function defined in $(0, +\infty)$ and such that*

(i) $\displaystyle\lim_{x\to+\infty}\frac{G(x)}{\log x}=+\infty,$
(ii) $G(x)=o(x)$,
(iii) *$G'(x)$ exists and it is continuous for $0<x<+\infty$,*
(iv) *$G'(x)>0$ for $x>x_0$*

and

(v) $G'(x) = o\left(\dfrac{G(x)}{x}\right)$,

then there exists a sequence $\mathcal{A}$ such that

$$\lim_{n\to+\infty}\frac{R(n)}{G(n)}=1.$$

(So that, e.g., there exists a sequence $\mathcal{A}$ with $R(n)\sim\log n \log\log n$.)

COROLLARY 4. *If $0<\alpha<1$, then there exists a sequence $\mathcal{A}$ such that*

$$|R(n)-n^{\alpha}| < 8n^{\alpha/2}(\log n)^{1/2} \quad \text{for} \quad n > n_0.$$

In fact, in order to derive Corollaries 2, 3 and 4 from Theorem 2, we have to use Theorem 2 with $\beta\left(\dfrac{\log x}{x}\right)^{1/2}$, $\left(\dfrac{G(x)}{\pi x}\right)^{1/2}$, $cx^{(\alpha-1)/2}$ (where $c=c(\alpha)$) and $\beta^2\pi\log n$, $2\int_1^{n/2} g(x)g(n-x)\,dx$, n^{α} in place of $g(x)$ and $F(n)$, respectively.

2. Sections 2, 3 and 4 will be devoted to the proof of Theorem 2. The proof is based on the probabilistic method of Erdős and Rényi [1], [2]. The Halberstam—Roth book [4] contains an excellent exposition of this method thus we use the terminology and notation of this book. In this section, we give a survey of those notations, facts and results connected with this probabilistic method which will be needed in the proof of Theorem 2.

Let Ω denote the set of the strictly increasing sequences of positive integers.

LEMMA 1. *Let*

$$\alpha_1, \alpha_2, \alpha_3, \dots \tag{8}$$

be real numbers satisfying

$$0 < \alpha_n < 1 \quad (n = 1, 2, \dots). \tag{9}$$

Then there exists a probability space (Ω, S, μ) *with the following two properties:*

(i) *For every natural number n, the event* $B^{(n)} = \{\omega\colon\ \omega \in \Omega,\ n \in \omega\}$ *is measurable, and* $\mu(B^{(n)}) = \alpha_n$.

(ii) *The events* $B^{(1)}, B^{(2)}, \dots$ *are independent.*

This lemma is identical with Theorem 13 in [4], p. 142.

We denote by $\varrho_n(\omega)$ the characteristic function of the event $B^{(n)}$:

$$\varrho_n(\omega) = \begin{cases} 1 & \text{if} \quad n \in \omega, \\ 0 & \text{if} \quad n \notin \omega. \end{cases}$$

For some $\omega = \{a_1, a_2, \dots\} \in \Omega$, we denote by $r_n = r_n(\omega)$ the number of solutions of

$$a_x + a_y = n, \quad a_x \in \omega, \quad a_y \in \omega, \quad a_x < a_y \tag{10}$$

so that

$$|R(n) - 2r_n(\omega)| \leqq 1 \tag{11}$$

(where $R(n)$ is the number of solutions of (10) without the restriction $a_x < a_y$) and

$$r_n(\omega) = \sum_{1 \leqq j < \frac{1}{2}n} \varrho_j(\omega)\, \varrho_{n-j}(\omega).$$

Furthermore, we put

$$\delta_n(j) = \mu(\{\omega\colon\ j \in \omega,\ n-j \in \omega\}) = \alpha_j \alpha_{n-j} \quad \text{for} \quad j < n/2,$$

$$\lambda_n = M(r_n(\omega)) = \sum_{1 \leqq j < \frac{n}{2}} \delta_n(j)$$

(where $M(\xi)$ denotes the expectation of the random variable ξ),

$$P_n(d) = \mu(\{\omega\colon\ r_n(\omega) = d\}) = \tag{12}$$

$$= \sum_{1 \leqq j_1 < \dots < j_d < n/2} \delta_n(j_1)(1-\delta_n(j_1))^{-1} \dots \delta_n(j_d)(1-\delta_n(j_d))^{-1} \prod_{1 \leqq j < n/2} (1-\delta_n(j))$$

$$\text{for} \quad 0 \leqq d \leqq n$$

and

(13) $$f(z) = \sum_{d=0}^{n} P_n(d) z^d =$$

$$= \sum_{d=0}^{n} \Big(\sum_{1 \leqq j_1 < \ldots < j_d < n/2} \delta_n(j_1)(1-\delta_n(j_1))^{-1} \ldots \delta_n(j_d)(1-\delta_n(j_d))^{-1} \prod_{1 \leqq j < n/2} (1-\delta_n(j)) \Big) z^d =$$

$$= \prod_{1 \leqq j < n/2} ((1-\delta_n(j)) + \delta_n(j) z)$$

(for any complex number z).

We shall also need the Borel—Cantelli lemma:

LEMMA 2. *Let (X, S, μ) be a probability space and let $E_1, E_2, \ldots$ be a sequence of measurable events. If*

$$\sum_{j=1}^{+\infty} \mu(E_j) < +\infty,$$

then, with probability 1, *at most a finite number of the events E_j can occur.*

(See [4], p. 135.)

3. The proof of Theorem 2 will be based on Lemma 3 and Theorem 3 below.

LEMMA 3. *If the sequence* (8) *satisfies* (9), $n \geqq 3$, *and Δ is a real number satisfying*

(14) $$0 < \Delta < \lambda_n,$$

then we have

(15) $$\mu(\{\omega:\ |r_n(\omega) - \lambda_n| \geqq \Delta) < 2 \exp(-\Delta^2/4\lambda_n).$$

(Note that (9) implies $\lambda_n > 0$ for $n \geqq 3$.)

PROOF OF LEMMA 3. First we estimate $\mu(\{\omega:\ r_n(\omega) \geqq \lambda_n + \Delta\})$. In view of (13) and (14), for $1 < x < 2$ we have

(16) $$\mu(\{\omega:\ r_n(\omega) \geqq \lambda_n + \Delta\}) = \sum_{d \geqq \lambda_n + \Delta} P_n(d) \leqq \sum_{d \geqq \lambda_n + \Delta} P_n(d) x^{d-(\lambda_n+\Delta)} =$$

$$= x^{-(\lambda_n+\Delta)} \sum_{d \geqq \lambda_n + \Delta} P_n(d) x^d \leqq x^{-(\lambda_n+\Delta)} \sum_{d=0}^{n} P_n(d) x^d = x^{-(\lambda_n+\Delta)} f(x) =$$

$$= (1+(x-1))^{-(\lambda_n+\Delta)} \prod_{1 \leqq j < n/2} (1+(x-1)\delta_n(j)) <$$

$$< \exp\left(-(\lambda_n+\Delta)\left((x-1) - \frac{(x-1)^2}{2}\right)\right) \prod_{1 \leqq j < n/2} \exp((x-1)\delta_n(j)) =$$

$$= \exp\left(-(\lambda_n+\Delta)\left((x-1) - \frac{(x-1)^2}{2}\right) + (x-1) \sum_{1 \leqq j < n/2} \delta_n(j)\right) =$$

$$= \exp\left(-(\lambda_n+\Delta)\left((x-1) - \frac{(x-1)^2}{2}\right) + (x-1)\lambda_n\right) =$$

$$= \exp\left(-\Delta(x-1) + (\lambda_n+\Delta)\frac{(x-1)^2}{2}\right) < \exp(-\Delta(x-1) + \lambda_n(x-1)^2)$$

since we have $1+u<e^u$ for $u>0$ and

$$1+u=\exp(\log(1+u))=\exp\left(u-\frac{u^2}{2}+\frac{u^3}{3}-\dots\right)>\exp\left(u-\frac{u^2}{2}\right) \quad \text{for} \quad 0\leqq u<1.$$

Writing $x=1+\Delta/2\lambda_n$ in (16) (then $1<x<2$ holds by (14)), we obtain that

(17) $$\mu(\{\omega:\ r_n(\omega)\geqq\lambda_n+\Delta\})<\exp(-\Delta^2/2\lambda_n+\Delta^2/4\lambda_n)=\exp(-\Delta^2/4\lambda_n).$$

Similarly, for $0<x<1$ we have

(18) $$\mu(\{\omega:\ r_n(\omega)\leqq\lambda_n-\Delta\})=\sum_{d\leqq\lambda_n-\Delta}P_n(d)\leqq\sum_{d\leqq\lambda_n-\Delta}P_n(d)x^{d-(\lambda_n-\Delta)}=$$

$$=x^{-(\lambda_n-\Delta)}\sum_{d\leqq\lambda_n-\Delta}P_n(d)x^d\leqq x^{-(\lambda_n-\Delta)}\sum_{d=0}^{n}P_n(d)x^d=x^{-(\lambda_n-\Delta)}f(x)=$$

$$=(1-(1-x))^{-(\lambda_n-\Delta)}\prod_{1\leqq j<n/2}(1-(1-x)\delta_n(j))<$$

$$<\exp((1-x)(\lambda_n-\Delta))\prod_{1\leqq j<n/2}\exp\left(-(1-x)\delta_n(j)+\frac{(1-x)^2(\delta_n(j))^2}{2}\right)=$$

$$=\exp\left((1-x)(\lambda_n-\Delta)-(1-x)\sum_{1\leqq j<n/2}\delta_n(j)+\frac{(1-x)^2}{2}\sum_{1\leqq j<n/2}(\delta_n(j))^2\right)\leqq$$

$$\leqq\exp\left((1-x)(\lambda_n-\Delta)-(1-x)\sum_{1\leqq j<n/2}\delta_n(j)+\frac{(1-x)^2}{2}\sum_{1\leqq j<n/2}\delta_n(j)\right)=$$

$$=\exp\left((1-x)(\lambda_n-\Delta)-(1-x)\lambda_n+\frac{(1-x)^2}{2}\lambda_n\right)=\exp\left(-\Delta(1-x)+\frac{(1-x)^2}{2}\lambda_n\right)$$

since for $0<x<1$ we have

$$\exp(-u)<1-u=\exp(\log(1-u))=\exp\left(-u+\frac{u^2}{2}-\frac{u^3}{3}+\dots\right)<\exp\left(-u+\frac{u^2}{2}\right).$$

Writing $x=1-\Delta/\lambda_n$ in (18) (then $0<x$ holds by (14)), we obtain

(19) $$\mu(\{\omega:\ r_n(\omega)\leqq\lambda_n-\Delta\})<\exp(-\Delta^2/\lambda_n+\Delta^2/2\lambda_n)=\exp(-\Delta^2/2\lambda_n).$$

(17) and (19) yield (15).

THEOREM 3. *If the sequence* (8) *satisfies* (9), *and there exists a positive integer* n_0 *such that*

(20) $$\lambda_n=\sum_{1\leqq j<n/2}\alpha_j\alpha_{n-j}>9\log n \quad \textit{for} \quad n\geqq n_0,$$

then, with probability 1, *there exists a number* $n_1=n_1(\omega)$ *such that*

$$|R(n)-2\lambda_n|<7(\lambda_n\log n)^{1/2} \quad \text{for} \quad n>n_1.$$

PROOF. By using Lemma 3 with $\Delta=3(\lambda_n \log n)^{1/2}$ (then (14) holds by (20)), we obtain

$$\sum_{n=1}^{+\infty} \mu(\{\omega:\ |r_n(\omega)-\lambda_n| \geqq 3(\lambda_n \log n)^{1/2}\}) =$$

$$= O(1)+\sum_{n=n_0}^{+\infty} \mu(\{\omega:\ |r_n(\omega)-\lambda_n| \geqq 3(\lambda_n \log n)^{1/2}\}) <$$

$$< O(1)+2\sum_{n=n_0}^{+\infty} \exp\left(-(3(\lambda_n \log n)^{1/2})^2/4\lambda_n\right) = O(1)+2\sum_{n=n_0}^{+\infty} n^{-9/4} < +\infty.$$

Thus by the Borel—Cantelli lemma (Lemma 2), with probability 1, at most a finite number of the events

$$|r_n(\omega)-\lambda_n| \geqq 3(\lambda_n \log n)^{1/2}$$

can occur, i.e., with probability 1, there exists a number $n_2=n_2(\omega)$ such that

$$|r_n(\omega)-\lambda_n| < 3(\lambda_n \log n)^{1/2} \quad \text{for} \quad n > n_2.$$

By (11) and (20), for such a sequence ω, for large n we have

$$|R(n)-2\lambda_n| \leqq |R(n)-2r_n(\omega)|+2|r_n(\omega)-\lambda_n| < 1+6(\lambda_n \log n)^{1/2} < 7(\lambda_n \log n)^{1/2}$$

which completes the proof of Theorem 3.

4. In this section, we complete the proof of Theorem 2. We put

$$\alpha_n = \begin{cases} 1/2 & \text{for} \quad 1 \leqq n \leqq x_0, \\ g(n) & \text{for} \quad x_0 < n < +\infty. \end{cases}$$

Defining the sequence (8) in this way, (9) holds trivially. Furthermore, in view of (iii) in Theorem 2, we have

$$\lambda_n = \sum_{1\leqq j<n/2} \alpha_j \alpha_{n-j} = \sum_{x_0<j\leqq n/2} g(j)g(n-j)+O(1). \tag{21}$$

By (i) in Theorem 2, we may use the Euler—Maclaurin summation formula in order to estimate the last sum. In view of (i), (ii) and (iii), we obtain

(22)

$$\sum_{x_0<j\leqq n/2} g(j)g(n-j) = \int_{x_0}^{n/2} g(x)g(n-x)\,dx - \left[g(x)g(n-x)\left(x-[x]-\frac{1}{2}\right)\right]_{x_0}^{n/2} +$$

$$+\int_{x_0}^{n/2} \left(g'(x)g(n-x)-g(x)g'(n-x)\right)\left(x-[x]-\frac{1}{2}\right)dx =$$

$$= \left(\int_1^{n/2} g(x)g(n-x)\,dx+O(1)\right)+O\left((g(n/2))^2+g(x_0)\,g(n-x_0)\right)+$$

$$+O\left(\int_{x_0}^{n/2} \left(|g'(x)|+|g'(n-x)|\right)dx\right) =$$

$$= \int_1^{n/2} g(x)g(n-x)\,dx+O(1)+O\left(\int_{x_0}^{n/2} \left(-g'(x)-g'(n-x)\right)dx\right) =$$

$$= \int_1^{n/2} g(x)g(n-x)\,dx+O(1)+O\left([-g(x)+g(n-x)]_{x_0}^{n/2}\right) = \int_1^{n/2} g(x)g(n-x)\,dx+O(1).$$

(21) and (22) yield

$$\lambda_n = \int_1^{n/2} g(x)g(n-x)\,dx + O(1).$$

Thus by (7) and (iv) in Theorem 2,

$$(23)\quad \left|F(n)-2\lambda_n\right| \leqq \left|F(n)-2\int_1^{n/2} g(x)g(n-x)\,dx\right| + 2\left|\int_1^{n/2} g(x)g(n-x)\,dx-\lambda_n\right| <$$

$$< \big(F(n)\log n\big)^{1/2}+O(1) < 2\big(F(n)-\log n\big)^{1/2}$$

for large n, hence in view of (7),

$$\lambda_n > \frac{1}{2}F(n)-\big(F(n)\log n\big)^{1/2} > \frac{1}{2}F(n)-\left(F(n)\cdot\frac{F(n)}{36}\right)^{1/2} = \frac{1}{3}F(n) > 12\log n$$

so that also (20) holds.

Thus all the conditions in Theorem 3 hold. By using Theorem 3, we obtain that, with probability 1, for large n we have

$$(24)\quad |R(n)-2\lambda_n| < 7(\lambda_n\log n)^{1/2}.$$

In view of (7), (23) and (24) yield for large n

$$|R(n)-F(n)| \leqq |R(n)-2\lambda_n|+|2\lambda_n-F(n)| < 7(\lambda_n\log n)^{1/2}+|2\lambda_n-F(n)| \leqq$$

$$\leqq 7\left(\left(\frac{1}{2}F(n)+\frac{1}{2}|2\lambda_n-F(n)|\right)\log n\right)^{1/2}+|2\lambda_n-F(n)| <$$

$$< 7\left(\left(\frac{1}{2}F(n)+\big(F(n)\log n\big)^{1/2}\right)\log n\right)^{1/2}+2\big(F(n)\log n\big)^{1/2} <$$

$$< 7\left(\left(\frac{1}{2}F(n)+\left(F(n)\cdot\frac{F(n)}{36}\right)^{1/2}\right)\log n\right)^{1/2}+2\big(F(n)\log n\big)^{1/2} =$$

$$= 7\left(\frac{2}{3}F(n)\log n\right)^{1/2}+2\big(F(n)\log n\big)^{1/2} < 8\big(F(n)\log n\big)^{1/2}$$

which completes the proof of Theorem 3.

5. So far we have estimated the probabilities $P_n(d)$ for d "far" from the expectation $\lambda_n=M\big(r_n(\omega)\big)$. In [2], Erdős and Rényi gave lower and upper bounds for $P_n(d)$ for *all* d. These estimates give the right order of magnitude of $P_n(d)$ for d "near" λ_n, provided $\alpha_j=O(j^{-1/4})$. Furthermore, they determined the limit distribution of $r_n(\omega)$. Sharpening and generalizing these estimates, we are going to complete this paper by giving an asymptotics for $P_n(d)$ for d "near" λ_n.

THEOREM 4. *If the sequence* (8) *satisfies* (9),

$$(25)\quad \lim_{n\to+\infty}\alpha_n = 0$$

and

(26)
$$\lambda_n = \sum_{1 \leq j < n/2} \alpha_j \alpha_{n-j} > 3 \quad \textit{for} \quad n \geqq n_0,$$

and we put

$$\lambda_n' = \sum_{1 \leq j < n/2} \alpha_j \alpha_{n-j}(1-\alpha_j\alpha_{n-j}) = \sum_{1 \leq j < n/2} \delta_n(j)(1-\delta_n(j)),$$

then for $n > n_1$ *and all* d *we have*

$$\left| P_n(d) - \frac{1}{(2\pi\lambda_n')^{1/2}} e^{-(\lambda_n - d)^2/2\lambda_n'} \right| < 13 \frac{(\log \lambda_n)^2}{\lambda_n}$$

where $P_n(d)$ *is defined by* (12).

(Thus the limit distribution of the random variable $\frac{r_n(\omega)-\lambda_n}{(\lambda_n')^{1/2}}$ is the normal distribution.)

Proof. Throughout the proof, θ will denote a complex number with absolute value $\leqq 1$. (In other words, $u = \theta v$ means that $|u| \leqq |v|$.)

We denote the characteristic function of the random variable $r_n(\omega)$ by $\varphi(t)$, so that in view of (13)

$$\varphi(t) = M(e^{ir_n(\omega)t}) = f(e^{it}) = \sum_{d=0}^{n} P_n(d) e^{idt} = \prod_{1 \leq j < n/2} \big((1-\delta_n(j)) + \delta_n(j) e^{it}\big).$$

Furthermore, we put

$$\eta = 2\left(\frac{\log \lambda_n}{\lambda_n}\right)^{1/2}.$$

Then we have

(27)
$$P_n(d) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varphi(t) e^{-idt} dt =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(1/2)\lambda_n' t^2} e^{i(\lambda_n - d)t} dt - \frac{1}{2\pi} \int_{\eta \leq |t|} e^{-(1/2)\lambda_n' t^2} e^{i(\lambda_n - d)t} dt +$$

$$+ \frac{1}{2\pi} \int_{|t| \leq \eta} (e^{-i\lambda_n t} \varphi(t) - e^{-(1/2)\lambda_n' t^2}) e^{i(\lambda_n - d)t} dt +$$

$$+ \frac{1}{2\pi} \int_{\eta \leq |t| \leq \pi} \varphi(t) e^{-idt} dt = J - J_1 + J_2 + J_3.$$

First we estimate J. Substituting $t = (\lambda_n')^{-1/2} x$, we obtain

(28)
$$J = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(1/2)\lambda_n' t^2} e^{i(\lambda_n - d)t} dt =$$

$$= \frac{1}{2\pi(\lambda_n')^{1/2}} \int_{-\infty}^{+\infty} e^{-(1/2)x^2} e^{i(\lambda_n - d)(\lambda_n')^{-1/2} x} dx = \frac{1}{(2\pi\lambda_n')^{1/2}} e^{-(\lambda_n - d)^2/2\lambda_n'}$$

since it is well-known that

$$\int_{-\infty}^{+\infty} e^{iux-x^2/2}\,dx = (2\pi)^{1/2} e^{-u^2/2}$$

(see, e.g., [5], p. 261).

In order to estimate J_2 and J_3, we need an estimate for $\varphi(t)$. For $|z|<1/2$, we have

$$\left|e^z-\left(1+z+\frac{z^2}{2}\right)\right| = \left|\sum_{k=3}^{+\infty}\frac{z^k}{k!}\right| \leqq \sum_{k=3}^{+\infty}\frac{|z|^k}{3} = \frac{|z|^3}{6}\frac{1}{1-|z|} < \frac{|z|^2}{3}$$

and

$$1-z = \exp(\log(1-z)) = \exp\left(-z+\frac{z^2}{2}-\frac{z^3}{3}+\dots\right) =$$

$$= \exp\left(-z+\theta\left(\frac{|z|^2}{2}+\frac{|z|^3}{3}+\dots\right)\right) = \exp\left(-z+\theta\left(\frac{|z|^2}{2}+\frac{|z|^3}{2}+\dots\right)\right) =$$

$$= \exp\left(-z+\theta\frac{|z|^2}{2}\frac{1}{1-|z|}\right) = \exp(-z+\theta|z|^2).$$

Thus in view of (25), for large n, $1\leqq j<n/2$ and $|t|\leqq 1/2$ we have

$$e^{-i\delta_n(j)t}\big((1-\delta_n(j))+\delta_n(j)e^{it}\big) = e^{-i\delta_n(j)}\big(1-\delta_n(j)(1-e^{it})\big) =$$

$$= \left(1-i\delta_n(j)\,t-\frac{1}{2}(\delta_n(j))^2t^2+\frac{\theta}{3}(\delta_n(j))^3t^3\right)\left(1-\delta_n(j)\left(-it+\frac{t^2}{2}+\frac{\theta}{3}t^3\right)\right) =$$

$$= 1-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3 =$$

$$= \exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3+\theta\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3\right)^2\right) =$$

$$= \exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3+\frac{2\theta}{3}(\delta_n(j))^2t^4\right) =$$

$$= \exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\theta\delta_n(j)t^3\right)$$

hence

(29) $$e^{-i\lambda_n t}\varphi(t) = \prod_{1\leqq j<n/2} e^{-i\delta_n(j)t}\big((1-\delta_n(j))+\delta_n(j)e^{it}\big) =$$

$$= \prod_{1\leqq j<n/2}\exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\theta\delta_n(j)t^3\right) = e^{-(1/2)\lambda_n' t^2+\theta\lambda_n t^3}$$

(for large n and $|t|\leqq 1/2$).

Furthermore, in view of (25), for large n and $|t|\leqq\pi$ we have

(30)
$$|\varphi(t)| = \prod_{1\leqq j<n/2} |1-\delta_n(j)+\delta_n(j)e^{it}| =$$
$$= \prod_{1\leqq j<n/2} ((1-\delta_n(j)+\delta_n(j)e^{it})(1-\delta_n(j)+\delta_n(j)e^{-it}))^{1/2} =$$
$$= \prod_{1\leqq j<n/2} ((1-\delta_n(j))^2+(\delta_n(j))^2+2\delta_n(j)(1-\delta_n(j))\cos t)^{1/2} =$$
$$= \prod_{1\leqq j<n/2} (1+2\delta_n(j)(1-\delta_n(j))(\cos t-1))^{1/2} =$$
$$= \prod_{1\leqq j<n/2} (1-4\delta_n(j)(1-\delta_n(j))(\sin t/2)^2)^{1/2} \leqq$$
$$\leqq \prod_{1\leqq j<n/2} \left(1-3\delta_n(j)\left(\frac{2}{\pi}\cdot\frac{t}{2}\right)^2\right)^{1/2} = \prod_{1\leqq j<n/2} \left(1-\frac{3}{\pi^2}\delta_n(j)t^2\right)^{1/2} \leqq$$
$$\leqq \prod_{1\leqq j<n/2} \left(1-\frac{1}{4}\delta_n(j)t^2\right)^{1/2} < \prod_{1\leqq j<n/2} \left(1-\frac{1}{8}\delta_n(j)t^2\right) <$$
$$< \prod_{1\leqq j<n/2} e^{-(1/8)\delta_n(j)t^2} = e^{-(1/8)\lambda_n t^2}$$

(for large n and $|t|\leqq\pi$), since

$$|\sin x| \geqq \frac{2}{\pi}|x| \quad \text{for} \quad |x| \leqq \pi/2,$$

$$(1-u)^{1/2} < 1-\frac{u}{2} \quad \text{for} \quad 0\leqq u<1$$

and

$$(0<)\ 1-x<e^{-x} \quad \text{for} \quad 0\leqq x<1.$$

By (25), (29) and (30), for large n we have

(31)
$$|J_1|+|J_3| \leqq \frac{1}{2\pi}\int_{\eta\leqq|t|} e^{-(1/2)\lambda_n' t^2}dt+\frac{1}{2\pi}\int_{\eta\leqq|t|\leqq\pi} |\varphi(t)|dt \leqq$$
$$\leqq \frac{1}{2\pi}\Big(\int_{\eta\leqq|t|} e^{-(1/2)\lambda_n' t^2}dt+\int_{\eta\leqq|t|\leqq\pi} e^{-(1/8)\lambda_n t^2}dt\Big) \leqq$$
$$\leqq \frac{1}{2\pi}\Big(\int_{\eta\leqq|t|} e^{-(1/8)\lambda_n t^2}dt+\int_{\eta\leqq|t|} e^{-(1/8)\lambda_n t^2}dt\Big) = \frac{1}{\pi}\int_{\eta\leqq|t|} e^{-(1/8)\lambda_n t^2}dt =$$
$$= \frac{2}{\pi}\int_{\eta}^{+\infty} e^{-(1/8)\lambda_n t^2}dt \leqq \frac{2}{\pi}\int_{\eta}^{+\infty} \frac{t}{\eta}e^{-(1/8)\lambda_n t^2}dt =$$
$$= -\frac{8}{\pi\eta\lambda_n}[e^{-(1/8)\lambda_n t^2}]_{\eta}^{+\infty} = \frac{8}{\pi\eta\lambda_n}e^{-(1/8)\lambda_n \eta^2} =$$
$$= \frac{4}{\pi(\lambda_n\log\lambda_n)^{1/2}}e^{-(1/2)\log\lambda_n} < \frac{2}{\lambda_n(\log\lambda_n)^{1/2}}$$

and

(32)
$$|J_2| \leqq \frac{1}{2\pi} \int\limits_{|t| \leqq \eta} |e^{-i\lambda_n t}\varphi(t) - e^{-(1/2)\lambda'_n t^2}|\,dt = \frac{1}{2\pi} \int\limits_{|t| \leqq \eta} e^{-(1/2)\lambda'_n t^2} |e^{\theta(t)\lambda_n t^3} - 1|\,dt \leqq$$
$$\leqq \frac{1}{2\pi} \int\limits_{|t| \leqq \eta} 2\lambda_n |t|^3 dt \leqq \frac{1}{\pi}\lambda_n \int\limits_{|t| \leqq \eta} \eta^3 dt = \frac{2}{\pi}\lambda_n \eta^4 < 11 \frac{(\log \lambda_n)^2}{\lambda_n}$$

since
$$|e^z - 1| = \left| z + \frac{z^2}{2!} + \frac{z_3}{3!} + \dots \right| \leqq |z| + |z|^2 + |z|^3 + \dots = \frac{|z|}{1-|z|} \leqq 2|z|.$$

In view of (26), (27), (28), (31) and (32) yield for large n that
$$|P_n(d) - J| = \left| P_n(d) - \frac{1}{(2\pi\lambda'_n)^{1/2}} e^{-(\lambda_n - d)^2/2\lambda'_n} \right| \leqq$$
$$\leqq |J_1| + |J_2| + |J_3| < \frac{2}{\lambda_n (\log n)^{1/2}} + 11 \frac{(\log \lambda_n)^2}{\lambda_n} < 13 \frac{(\log \lambda_n)^2}{\lambda_n}$$

which completes the proof of Theorem 4.

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u-ISOMORPHIC SEMIGROUPS OF CONTINUOUS FUNCTIONS

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0. Introduction. Let X be a topological space, and denote by $C(X)$ the set of all real-valued continuous functions defined on X, by $C^*(X)$ the subset of $C(X)$ composed of bounded functions. Both $C(X)$ and $C^*(X)$ can be considered as a ring under pointwise addition and multiplication of functions, or as a semigroup under pointwise multiplication. For a completely regular Hausdorff space X, let βX and vX denote the Čech—Stone compactification and the Hewitt realcompactification of X, respectively (see e. g. [2]).

The following propositions are well-known for completely regular Hausdorff spaces X and Y:

A. *If $C(X)$ and $C(Y)$ are ring isomorphic, then vX and vY are homeomorphic.*

B. *If $C^*(X)$ and $C^*(Y)$ are ring isomorphic, then βX and βY are homeomorphic.*

It is also known that, both in Propositions A and B, ring isomorphy can be replaced by semigroup isomorphy. Moreover, for A, an essentially stronger result is valid. In order to formulate it, let us recall (cf. [1]) that, in a semigroup S, we write $a \rhd b$ for $a, b \in S$, iff there exists an element $c \in S$ such that $a = cb$; for two semigroups S_1 and S_2 and the respective relations $\rhd_1$ and $\rhd_2$, a bijection $\varphi\colon S_1 \to S_2$ is said to be a *d-isomorphism* iff

$$a \rhd_1 b \Leftrightarrow \varphi(a) \rhd_2 \varphi(b).$$

C. (cf. [1], Theorem 3). *For two completely regular Hausdorff spaces X and Y, if $C(X)$ and $C(Y)$ are d-isomorphic, then vX and vY are homeomorphic.*[1]

In fact, [1], Theorem 3 says still more: we can replace $C(X)$ and $C(Y)$ by $S_1(X)$ and $S_2(Y)$, respectively, where $S_1(X)$ and $S_2(Y)$ denote the semigroups composed of all continuous functions from X to $\mathbf{S}_1$ and from Y to $\mathbf{S}_2$, respectively, $\mathbf{S}_1$ and $\mathbf{S}_2$ are quasi-real semigroups, and we consider pointwise multiplication of functions.

As to the concept of a quasi-real semigroup ([1], p. 133), $\mathbf{S}$ is said to be a quasi-real semigroup iff it is a topological semigroup containing the subsemigroup $[0, +\infty)$ of the real line $\mathbf{R}$ equipped with the usual topology and the multiplication of real numbers as a topological subsemigroup, the numbers 0 and 1 are a zero element and a unity element in $\mathbf{S}$, respectively, for $a \in \mathbf{S}$, $a \neq 0$ there is $b \in \mathbf{S}$ such that $a \cdot b$ti-$= b \cdot a = 1$, this element b is a continuous function of $a \in \mathbf{S} - \{0\}$, there is a con=

[1] In [1], owing to a misprint, X and Y stand instead of vX and vY.

nuous homomorphism $a \mapsto |a|$ from $\mathbf{S}$ into $[0, +\infty)$ satisfying $|a|=a$ for $a \in [0, +\infty)$, finally the sets $V_\varepsilon = \{x \in \mathbf{S}: |x| < \varepsilon\}$ $(\varepsilon > 0)$ constitute a neighbourhood base of 0 in $\mathbf{S}$.

It is quite natural to ask whether we can replace, in Proposition B, ring isomorphy by some weakened form of semigroup isomorphy (like d-isomorphy or something similar), and $C^*(X)$, $C^*(Y)$ by semigroups composed of continuous functions from X and Y to suitable topological semigroups $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. The purpose of this paper is to answer this question (a somewhat similar but, at least partly, weaker result is contained in [4], Theorem 5.11).

1. u-isomorphic semigroups. Let S be a semigroup. According to [4], we introduce a relation $\succ$ in S (called in [4] *canonical order* in S) by defining $a \succ b$ iff $a = ab$ $(a, b \in S)$, i. e. iff b is a relative right unity element with respect to a.

If S_1 and S_2 are two semigroups, a bijection $\varphi: S_1 \to S_2$ is said to be a *u-isomorphism* iff

$$a \succ_1 b \Leftrightarrow \varphi(a) \succ_2 \varphi(b)$$

for $a, b \in S_1$ and the respective relations $\succ_1$ and $\succ_2$. S_1 and S_2 are said to be *u-isomorphic* iff there exists a u-isomorphism $\varphi: S_1 \to S_2$.

A semigroup isomorphism is obviously a u-isomorphism. Conversely, two u-isomorphic semigroups need not be semigroup isomorphic. In fact, in a group G (considered as a semigroup), $a \succ b$ iff b is the unity element of the group. Hence two groups of the same cardinality are always u-isomorphic without being necessarily (semi)group isomorphic.

In general, u-isomorphy introduced here and d-isomorphy considered in [1] are independent of each other.

EXAMPLE 1.1. Let S be the interval $(0, 1]$ equipped with the multiplication of real numbers. In this semigroup, clearly

$$a \rhd b \Leftrightarrow a \leqq b, \quad a \succ b \Leftrightarrow b = 1.$$

Hence S is u-isomorphic to any group of cardinality 2^ω (e. g. to $(0, +\infty)$ with the multiplication of real numbers) without being d-isomorphic to a group (in which $a \rhd b$ for any two elements). □

EXAMPLE 1.2. Let $S_1 = [0, 1)$, $S_2 = (-1, 1)$, the semigroup operation coinciding in both cases with the multiplication of real numbers. Now clearly

$$a \rhd_1 b \quad \Leftrightarrow \quad \text{either} \quad a = b = 0 \quad \text{or} \quad a < b,$$

$$a \rhd_2 b \quad \Leftrightarrow \quad \text{either} \quad a = b = 0 \quad \text{or} \quad |a| < |b|,$$

$$a \succ_1 b \Leftrightarrow a = 0, \quad a \succ_2 b \Leftrightarrow a = 0$$

for $a, b \in S_1$ or $a, b \in S_2$, respectively. Hence a bijection $\varphi: S_1 \to S_2$ such that $\varphi(0) = 0$ is a u-isomorphism while S_1 and S_2 fail to be d-isomorphic because, for $a, b \in S_1$, $a \neq b$, either $a \rhd_1 b$ or $b \rhd_1 a$ is always valid, but neither $c \rhd_2 -c$ nor $-c \rhd_2 c$ holds for $0 < c < 1$. □

EXAMPLE 1.3. Let $S_1=S_2=(0, 1]$, the semigroup operation being defined in S_1 as the multiplication of real numbers, in S_2 by $\min(a, b)$ $(a, b\in S_2)$. Now

$$a \rhd_1 b \Leftrightarrow a \rhd_2 b \Leftrightarrow a \leqq b,$$

$$a \succ_1 b \Leftrightarrow b = 1, \quad a \succ_2 b \Leftrightarrow a \leqq b.$$

Hence S_1 and S_2 are *d*-isomorphic without being *u*-isomorphic. □

If, instead of considering semigroups in general, we restrict ourselves to semigroups of the form $C(X)$, the situation changes essentially. In fact, by [1], Theorem 4, if X and Y are arbitrary topological spaces, the *d*-isomorphy of $C(X)$ and $C(Y)$ implies their ring isomorphy and, *a fortiori*, their semigroup and *u*-isomorphy. On the other hand, we have

EXAMPLE 1.4. Let X be a non-compact, realcompact space, $Y=\beta X$. Both X and Y are realcompact, hence $C(X)$ and $C(Y)$ are not *d*-isomorphic by statement C. We show that $C(X)$ and $C(Y)$ are *u*-isomorphic. Since $C(Y)$ and $C^*(X)$ are ring isomorphic, hence semigroup isomorphic and *u*-isomorphic, it suffices to establish the *u*-isomorphy of $C(X)$ and $C^*(X)$:

THEOREM 1.5. *For an arbitrary topological space X, $C(X)$ and $C^*(X)$ are u-isomorphic.*

PROOF. For $A, B\subset X, A\cap B=\emptyset$, let C_{AB} and C^*_{AB} denote the set of those $f\in C(X)$ and $f\in C^*(X)$, respectively, for which

$$Z(f) = A, \quad Z(1-f) = B,$$

where, for $f\colon X\to\mathbf{R}$,

$$Z(f) = \{x\in X\colon f(x) = 0\}.$$

For distinct pairs (A, B) and (A', B'), the sets C_{AB} and $C_{A'B'}$ are disjoint, and the same holds for C^*_{AB} and $C^*_{A'B'}$. Clearly

$$C(X) = \bigcup\{C_{AB}\colon A, B\subset X, \ A\cap B = \emptyset\},$$

$$C^*(X) = \bigcup\{C^*_{AB}\colon A, B\subset X, \ A\cap B = \emptyset\}.$$

Now $|C_{AB}|=|C^*_{AB}|$ for every pair $A, B\subset X$, $A\cap B=\emptyset$. In fact, $C^*_{AB}\subset C_{AB}$, hence $|C^*_{AB}|\leqq|C_{AB}|$. On the other hand, let $\psi\colon \mathbf{R}\to(-1, 2)$ be a homeomorphism such that $\psi(0)=0$, $\psi(1)=1$. Then $\chi(f)=\psi\circ f$ defines an injection from C_{AB} into C^*_{AB} so that $|C_{AB}|\leqq|C^*_{AB}|$.

Therefore there exists a bijection $\varphi_{AB}\colon C_{AB}\to C^*_{AB}$ for every pair (A, B) in question, and from these bijections we obtain a bijection $\varphi\colon C(X)\to C^*(X)$ satisfying

$$\varphi|C_{AB} = \varphi_{AB} \quad (A, B\subset X, A\cap B = \emptyset).$$

Now, for $f, g\in C(X)$, clearly

$$f=fg \Leftrightarrow Z(f)\cup Z(1-g) = X,$$

$$\varphi(f) = \varphi(f)\varphi(g) \Leftrightarrow Z(\varphi(f))\cup Z(1-\varphi(g)) = X.$$

However, $f \in C_{AB}$ implies $\varphi(f) \in C^*_{AB}$, hence

$$Z(f) = Z(\varphi(f)) = A, \quad Z(1-f) = Z(1-\varphi(f)) = B,$$

so that $Z(f) = Z(\varphi(f))$, $Z(1-g) = Z(1-\varphi(g))$ for $f, g \in C(X)$, and φ is a u-isomorphism. □

Consequently we can say that, for semigroups of the form $C(X)$, d-isomorphy is stronger than u-isomorphy. In particular, Example 1.4 shows that, in Proposition C, d-isomorphy cannot be replaced by u-isomorphy.

The same argument furnishes, if we denote by ψ a homeomorphism

$$\psi\colon [0, +\infty) \to [0, 2)$$

such that $\psi(0)=0$, $\psi(1)=1$, the following

THEOREM 1.6. *Let X be a topological space, $S(X)$ and $S^*(X)$ the sets of all continuous functions $f\colon X \to [0, +\infty)$ and of all bounded continuous functions $f\colon X \to [0, +\infty)$, respectively, equipped with a semigroup operation defined by pointwise multiplication. Then $S(X)$ and $S^*(X)$ are u-isomorphic.*

2. Segment-like semigroups. On generalizing Proposition A, the role of the multiplicative semigroup $\mathbf{R}$ of the real numbers was given to quasi-real semigroups. Now we introduce a larger class of topological semigroups in order to play a similar role in the generalization of B.

A topological semigroup $\mathbf{S}$ will be said to be a *segment-like semigroup* iff

(2.1) The topological subsemigroup $[0, 1]$ of $\mathbf{R}$ is a topological subsemigroup of $\mathbf{S}$,

(2.2) $0 \in [0, 1]$ is a zero element in $\mathbf{S}$,

(2.3) $1 \in [0, 1]$ is a unity element in $\mathbf{S}$,

(2.4) $a, b \in \mathbf{S}$, $ab=a$ implies either $a=0$ or $b=1$,

(2.5) There exists a continuous homomorphism $a \mapsto |a|$ from $\mathbf{S}$ into $\mathbf{R}$,

(2.6) $|a|=a$ for $a \in [0, 1]$,

(2.7) $|a|=0$, $a \in \mathbf{S}$ implies $a=0$.

The definition in [1], p. 133 shows that every quasi-real semigroup is segment-like; similarly, in a quasi-real semigroup, all elements x satisfying $|x| \leqq 1$ constitute a segment-like subsemigroup.

Further examples of segment-like semigroups are e.g. the subsemigroups of $\mathbf{R}$ of the form

$$[c, 1] \quad \text{for} \quad -1 \leqq c \leqq 0,$$

or

$$(c, 1] \quad \text{for} \quad -1 \leqq c < 0.$$

More generally, let $\mathbf{S}_0$ be a segment-like (in particular a quasi-real) semigroup, and consider the topological subsemigroups for $0 < c \leqq 1$:

$$\mathbf{S}_1 = [0, 1] \cup \{x \in \mathbf{S}_0\colon\ |x| \leqq c\},$$

$$\mathbf{S}_2 = [0, 1] \cup \{x \in \mathbf{S}_0\colon\ |x| < c\}.$$

Then $\mathbf{S}_1$ and $\mathbf{S}_2$ are segment-like semigroups.

Let us note that condition (2.7) is independent of the others:

EXAMPLE 2.8. Let $\mathbf{S}=H\cup\{0\}$ where

$$H=\{(x, n)\colon x\in(0, +\infty), \quad n=0, 1, 2, \ldots\},$$

0 is distinct from the elements of H, and the latter is an open subset of $\mathbf{S}$ equipped with the topology inherited from $\mathbf{R}^2$, while a neighbourhood base of 0 is composed of the sets

$$V_\varepsilon=\{(x, n)\in H\colon x<\varepsilon\}\cup\{0\} \quad (\varepsilon>0).$$

We define

$$(x, n)\cdot(y, m)=(xy, n+m),$$

$$0\cdot(x, n)=(x, n)\cdot 0=0\cdot 0=0$$

for $(x, n), (y, m)\in H$. Then $\mathbf{S}$ is a (commutative) topological semigroup and, after having identified $(x, 0)\in H$ with $x\in R$, conditions (2.1) to (2.4) are fulfilled.

Now let

$$|0|=0, \quad |(x, 0)|=x, \quad |(x, n)|=0 \quad \text{for} \quad n\geqq 1.$$

Then (2.5) and (2.6) hold; however, (2.7) fails to be valid. □

The same holds for condition (2.4):

EXAMPLE 2.9. Let S be the set $(0, 1]\times(0, 1]$ equipped with the topology inherited from $\mathbf{R}^2$ and with the semigroup operation

$$(x, y)\cdot(x', y')=\big(xx', \min(y, y')\big).$$

Then define $\mathbf{S}=S\cup\{0\}$ where $0\notin S$ and

$$0\cdot(x, y)=(x, y)\cdot 0=0\cdot 0=0 \quad \text{for} \quad (x, y)\in S.$$

Let S be open in $\mathbf{S}$ and the neighbourhood base of 0 composed of the sets

$$V_\varepsilon=\{(x, y)\in S\colon x<\varepsilon\}\cup\{0\} \quad (\varepsilon>0).$$

Clearly $\mathbf{S}$ is a topological semigroup satisfying (2.1) to (2.3), provided $(x, 1)\in S$ is identified with $x\in\mathbf{R}$ and $0\in\mathbf{S}$ with $0\in\mathbf{R}$.

Define $|(x, y)|=x$ for $(x, y)\in S$, $|0|=0$. Then (2.5) to (2.7) are true.

However, (2.4) fails to hold because e. g.

$$\left(1, \frac{1}{3}\right)\cdot\left(1, \frac{2}{3}\right)=\left(1, \frac{1}{3}\right). \quad \square$$

3. *u*-ideals. Let S be a semigroup. By a slight modification of the definition of an O-ideal in [4], we say that a subset $U\subset S$ is a *u-ideal* iff

(3.1) $\emptyset\neq U\neq S$,

(3.2) $a\in S$, $b\in U$, $a\succ b$ imply $a\in U$,

(3.3) $a, b\in U$ implies the existence of $c\in U$ with $a\succ c$, $b\succ c$.

We get from (3.3) by putting $a=b$:

LEMMA 3.4. *If U is a u-ideal, then, for $a\in U$, there is $c\in U$ such that $a\succ c$.* □

Now let X be a completely regular Hausdorff space, $\mathbf{S}$ a segment-like semigroup, and denote by $S(X)$ the set of all continuous functions from X into $\mathbf{S}$. $S(X)$ is a semigroup under pointwise multiplication of functions.

For $f\in S(X)$, we define the function $|f|\colon X\to\mathbf{R}$ by

$$|f|(x) = |f(x)|.$$

By (2.5) $|f|\in C(X)$ for $f\in S(X)$. We denote by $S^*(X)$ the set of all $f\in S(X)$ such that $|f|\in C^*(X)$; clearly $S^*(X)$ is a subsemigroup of $S(X)$. Similarly, let $S_0(X)$ denote the subsemigroup of $S(X)$ composed of those f for which $f(X)\subset[0, 1]$. Clearly $S_0(X)\subset S^*(X)\subset S(X)$.

Now let S denote any subsemigroup of $S(X)$ such that $S_0(X)\subset S\subset S(X)$. For $f\in S$, we introduce the notations

$$Z(f) = \{x\in X\colon f(x) = 0\},\ E(f) = \{x\in X\colon f(x) = 1\}.$$

For $f\in S$ again, we denote by f^* the continuous extension of $|f|\in C(X)$ to βX with range contained in the one-point compactification $\mathbf{R}^*=\mathbf{R}\cup\{\infty\}$ of $\mathbf{R}$. Similarly to the above notations, we put

$$Z(f^*) = \{x\in\beta X\colon f^*(x) = 0\},\ E(f^*) = \{x\in\beta X\colon f^*(x) = 1\};$$

both $Z(f^*)$ and $E(f^*)$ are closed subsets of βx.

LEMMA 3.5. *For $f, g\in S$, $f\succ g$ holds iff*

$$Z(f)\cup E(g) = X,$$

and the latter equality implies

$$Z(f^*)\cup E(g^*) = \beta X.$$

PROOF. The first statement results from (2.4). If $Z(f)\cup E(g)=X$, then $\overline{Z(f)}\cup \overline{E(g)}=\beta X$, the closures being understood with respect to βX. By (2.6)

$$\overline{Z(f)}\subset Z(f^*),\quad \overline{E(g)}\subset E(g^*),$$

hence

$$Z(f^*)\cup E(g^*) = \beta X. \quad \square$$

The following two theorems are contained in [5] for a compact X and $\mathbf{S}=\mathbf{R}$:

THEOREM 3.6. *If $\emptyset\neq F\subset\beta X$ is a compact subset and U is composed of those $f\in S$ for which $F\subset\operatorname{int} Z(f^*)$, then U is a u-ideal.*

PROOF. By (2.6) $Z(f^*)=\beta X$ if $f(x)=0$ for $x\in X$, hence $U\neq\emptyset$. Similarly $Z(f^*)=\emptyset$ if $f(x)=1$ for $x\in X$, hence $U\neq S$.

Suppose $f\in S$, $g\in U$, $f\succ g$. By 3.5

$$Z(f^*)\cup E(g^*) = \beta X,\ Z(g^*)\subset\beta X-E(g^*)\subset Z(f^*),$$

and

$$F\subset\operatorname{int} Z(g^*)\subset\operatorname{int} Z(f^*)$$

shows $f\in U$.

Now suppose $f, g \in U$. Then

$$F \subset \operatorname{int} Z(f^*) \cap \operatorname{int} Z(g^*)$$

so that, by the normality of βX, there are open subsets G, G_1 and a closed subset F_1 of βX such that

$$F \subset G_1 \subset F_1 \subset G \subset Z(f^*) \cap Z(g^*).$$

Let h_0: $\beta X \to [0, 1]$ be a continuous function such that

$$h_0(x) = 0 \quad \text{for} \quad x \in F_1,$$

$$h_0(x) = 1 \quad \text{for} \quad x \in \beta X - G.$$

Then by (2.1) $h = h_0 | X \in S$, by (2.6) $h^* = h_0$, and

$$F \subset G_1 \subset F_1 \subset Z(h^*)$$

shows $h \in U$. Finally $f \succ h$, $g \succ h$ because, say, $x \in X$, $f(x) \neq 0$ implies by (2.7) $x \notin Z(f^*)$, $x \notin G$, hence $h_0(x) = h(x) = 1$. □

THEOREM 3.7. *For a u-ideal U in S,*

$$F = \bigcap \{Z(f^*) \colon f \in U\} \neq \emptyset$$

is a closed subset of βX such that U is obtained from F by the construction described in 3.6.

PROOF. For $f \in U$, by 3.4 there is a $g \in U$ such that $f \succ g$; thus $Z(f^*) \neq \emptyset$, otherwise (2.7) and 3.5 would imply $Z(f) = \emptyset$, $E(g) = X$, hence $h \succ g$ for every $h \in S$, consequently $U = S$ by (3.2), in contradiction with (3.1).

We show that the non-empty sets $Z(f^*)$ $(f \in U)$ constitute a filter base. In fact, if $f, g \in U$, let $h \in U$ satisfy $f \succ h$, $g \succ h$. Then, by 3.5,

$$Z(f^*) \cup E(h^*) = Z(g^*) \cup E(h^*) = \beta X,$$

$$Z(h^*) \subset \beta X - E(h^*) \subset Z(f^*) \cap Z(g^*).$$

By the compactness of βX,

$$F = \bigcap \{Z(f^*) \colon f \in U\} \neq \emptyset.$$

We show, for $f \in S$,

$$f \in U \Leftrightarrow F \subset \operatorname{int} Z(f^*).$$

In fact, for $f \in U$, by 3.4 we can choose $g \in U$ such that $f \succ g$, and then by 3.5

$$Z(f^*) \cup E(g^*) = \beta X,$$

$$Z(g^*) \subset \beta X - E(g^*) \subset Z(f^*),$$

so that

$$F \subset Z(g^*) \subset \operatorname{int} Z(f^*)$$

because $E(g^*)$ is closed.

Conversely, suppose $f \in S$, $F \subset \operatorname{int} Z(f^*)$. Then, again by the compactness of βX, there is a $g \in U$ such that

$$Z(g^*) \subset \operatorname{int} Z(f^*) \subset Z(f^*).$$

By 3.4 there is an $h \in U$ fulfilling $g \succ h$. Then, by 3.5,

$$Z(g) \cup E(h) = X.$$

Now by (2.6) and (2.7) $Z(g) \subset Z(f)$ so that

$$Z(f) \cup E(h) = X,$$

$f \succ h$ by 3.5, and $f \in U$ by (3.2). □

COROLLARY 3.8. *The formulas*

$$f \in U \Leftrightarrow F \subset \operatorname{int} Z(f^*) \tag{3.9}$$

and

$$F = \bigcap \{Z(f^*): f \in U\} \tag{3.10}$$

establish a bijection between the set of all u-ideals in S and that of all non-empty closed subsets of βX. If U_1, U_2 are u-ideals and F_1, F_2 denote the corresponding closed subsets, then

$$U_1 \subset U_2 \quad \textit{iff} \quad F_1 \supset F_2; \tag{3.11}$$

consequently the singletons in βX (i.e. the points of βX) correspond to the maximal u-ideals in S.

PROOF. The first statement results from (3.6) and (3.7) if we observe that, if $F \subset \beta X$ is closed and $x \in \beta X$, $x \notin F$, then there is a $g \in C(\beta X)$ such that $0 \leqq g \leqq 1$, $g=0$ on a neighbourhood of F, $g(x)=1$; then $f=g|X$ satisfies $f \in S$, $f^*=g$, $F \subset \operatorname{int} Z(f^*)$, $x \notin Z(f^*)$. (3.11) is clear from (3.9) and (3.10). □

4. Main results. The following theorem yields an answer to the question raised in the Introduction:

THEOREM 4.1. *Let X_1 and X_2 be two completely regular Hausdorff spaces, $\mathbf{S}_1$ and $\mathbf{S}_2$ two segment-like semigroups, and denote by $S_i(X_i)$ the semigroup composed of all continuous functions from X_i to $\mathbf{S}_i$, by S_i a subsemigroup of $S_i(X_i)$ such that*

$$S_0(X_i) \subset S_i \subset S_i(X_i)$$

where

$$S_0(X_i) = \{f \in S_i(X_i): f(X_i) \subset [0, 1]\}.$$

If S_1 and S_2 are u-isomorphic, then βX_1 and βX_2 are homeomorphic.

PROOF. A u-isomorphism from S_1 onto S_2 carries the u-ideals in S_1 to the u-ideals in S_2 by keeping their inclusion relations invariant. By 3.8 this furnishes a homeomorphism from βX_1 onto βX_2. □

If X is an arbitrary topological space, then there exist a compact Hausdorff space Y and a mapping $h: X \to Y$ such that

$$C^*(X) = \{f \circ h: f \in C(Y)\}$$

(see e.g. [3]). From this, it is easy to deduce:

THEOREM 4.2. *Let X_1 and X_2 be arbitrary topological spaces, $\mathbf{S} \supset [0, 1]$ a topological subsemigroup of* $\mathbf{R}$ *(equipped with the multiplication of real numbers and the* *usual topology), S_i the semigroup under pointwise multiplication of all bounded continuous functions $f\colon X_i \to \mathbf{S}$. If S_1 and S_2 are u-isomorphic then they are semigroup isomorphic.*

PROOF. Let Y_i be a compact Hausdorff space and $h_i\colon X_i \to Y_i$ a mapping satisfying

$$C^*(X_i) = \{f \circ h_i\colon f \in C(Y_i)\}.$$

Then S_i is semigroup isomorphic to S_i', the semigroup of all continuous functions from Y_i into $\mathbf{S}$. Since $\mathbf{S}$ is easily seen to be segment-like, 4.1 yields that Y_1 and Y_2 are homeomorphic, hence S_1' and S_2' are semigroup isomorphic, and the same holds for S_1 and S_2. □

It is easy to see that the subsemigroups of $\mathbf{R}$ containing $[0, 1]$ are precisely

$$\mathbf{R},\ [0, +\infty),\ [c, 1]\ (-1 \leqq c \leqq 0),\ (c, 1] (-1 \leqq c < 0).$$

In the case $\mathbf{S} = \mathbf{R}$, the above argument furnishes:

THEOREM 4.3. *Let X_1 and X_2 be arbitrary topological spaces. If $C^*(X_1)$ and $C^*(X_2)$ are u-isomorphic, then they are ring isomorphic.* □

Let us note that a similar argument to that one applied in the proof of 4.2 yields the following analogon of [1], Theorem 4:

THEOREM 4.4. *Let X_1 and X_2 be arbitrary topological spaces, and S_i the semigroup of all continuous functions from X_i to $[0, +\infty)$ under pointwise multiplication. If S_1 and S_2 are d-isomorphic, then they are semigroup isomorphic.*

PROOF. Notice that $[0, +\infty)$ is quasi-real so that [1], Theorem 3 can be applied for two realcompact Hausdorff spaces Y_1 and Y_2 and mappings $h_i\colon X_i \to Y_i$ such that

$$C(X_i) = \{f \circ h_i\colon f \in C(Y_i)\}. \quad \square$$

It is natural to ask whether *u*-isomorphy can be replaced by *d*-isomorphy in 4.2:

PROBLEM 4.5. *Let X_1, X_2, $\mathbf{S}$, S_1, S_2 be the same as in Theorem 4.2, and suppose that S_1 and S_2 are d-isomorphic. Is it true that they are necessarily semigroup isomorphic?*

The answer is positive in the cases $\mathbf{S} = \mathbf{R}$ and $\mathbf{S} = [0, +\infty)$. In fact, the argument in the proof of 4.2 reduces the question to the case when X_1 and X_2 are compact Hausdorff spaces and then [1], Theorem 4 or 4.4 of the present paper applies. However, the question remains open if

$$\mathbf{S} = [c, 1] \quad (-1 \leqq c \leqq 0) \quad \text{or} \quad \mathbf{S} = (c, 1] \quad (-1 \leqq c < 0).$$

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EXPONENTIAL SUMS OVER PRIMES IN SHORT INTERVALS

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Introduction

As it is well known, non-trivial estimates for the exponential sum

$$S(\alpha) = \sum_{n \leqq x} \Lambda(n)\, e(n\alpha),$$

where Λ is the von Mangoldt function and $e(x)=e^{2\pi i x}$, may be obtained by the method of Vinogradov [8], Ch. IX, by the zero-density method (see Montgomery [3], ch. 16) and by Vaughan's method [6] and [7]. The estimates of $S(\alpha)$ depend on q, the denominator of a "good" rational approximation to α, and are non-trivial in the range $\log^c x \leqq q \leqq x \log^{-c} x$, with a suitable constant $c>0$.

The method of Vinogradov and the zero-density method were used by Pan Chen-Dong [4] and subsequently by Chen Jing-Run [1] in order to give estimates for

$$S(x, y, \alpha) = \sum_{x-y<n\leqq x} \Lambda(n) e(n\alpha).$$

The results of Chen are non-trivial only when $y \gg x^{2/3+\varepsilon}$ and q is in a certain range.

In the present paper we use Heath—Brown's identity [2], together with analytic techniques, in order to obtain estimates for $S(x, y, \alpha)$ in the wider range $y \gg x^{3/5+\varepsilon}$. Precisely, our result is the following

THEOREM. *Let* $1 \leqq a < q$, $(a, q)=1$ *and* $1 \leqq y \leqq x$. *Then*

$$S\left(x, y, \frac{a}{q}\right) \ll (x^{1/2} q^{1/2} + y q^{-1/2} + x^{3/10} y^{1/2}) \log^{100} x.$$

The exponent of the log-factor is far from being optimal.

Pan [4] and Chen [1] used their results to prove a localized form of the Vinogradov three-primes theorem, i.e. they proved that every large positive odd integer N can be written as

$$(1) \qquad N = p_1+p_2+p_3, \; p_i = N/3+O(N^\theta), \; i = 1, 2, 3,$$

and $\theta \geqq 2/3+\varepsilon$ (Chen [1]). The estimates for $S(x, y, \alpha)$ are used when α lies on certain "minor arcs". A disadvantage of our method is that it works only for rational α, and it seems that our estimate does not lead to an improvement on Chen's value of θ in (1).

PROOF OF THE THEOREM. We may clearly assume $2 \leqq q \leqq x$, otherwise the result is trivial. Using the well known bound for the Gaussian sums (see [3], ch. 16) we have

$$S\left(x, y, \frac{a}{q}\right) = \sum_{b=1}^{q}{}^{*} e\left(\frac{ab}{q}\right) \sum_{\substack{x-y<n\leqq x\\ n\equiv b\,(\mathrm{mod}\, q)}} \Lambda(n) + O\Big(\sum_{\substack{x-y<n\leqq x\\ (n,q)\neq 1}} \Lambda(n)\Big) = \tag{2}$$

$$= \frac{1}{\Phi(q)} \sum_{\chi} \left(\sum_{b=1}^{q}{}^{*} \bar\chi(b) e\left(\frac{ab}{q}\right)\right) \sum_{x-y<n\leqq x} \Lambda(n)\chi(n) + O(\log^2 x) \ll$$

$$\ll \frac{q^{1/2}}{\Phi(q)} \sum_{\chi} \Big|\sum_{x-y<n\leqq x} \Lambda(n)\chi(n)\Big| + \log^2 x,$$

where the * means that the sum is restricted to $(b, q)=1$.

Let us recall Heath—Brown's identity [2]. Let $k\geqq 1$ and $n\leqq 2x$; then

$$\Lambda(n) = \sum_{j=1}^{k} (-1)^j \binom{k}{j} \sum_{\substack{n_1\dots n_j m_1 \dots m_j = n\\ m_1,\dots,m_j\leqq x^{1/k}}} \!\!\cdots\!\! \sum \mu(m_1)\dots\mu(m_j)\log n_1. \tag{3}$$

We choose $k=5$ in (3), and use (3) in order to treat the character sum in (2). Removing the log-factor by partial summation and using a splitting-up argument we are led to consider multilinear forms of the type

$$L = L(N_1, \dots, N_j, M_1, \dots, M_j) = \tag{4}$$

$$= \sum_{\chi} \Big| \sum_{\substack{x-y<m_1\dots m_j n_1 \dots n_j\leqq x\\ M_i<m_i\leqq 2M_i\\ N_i<n_i\leqq 2N_i}} \!\!\cdots\!\! \sum \mu(m_1)\dots\mu(m_j)\chi(m_1\dots m_j n_1\dots n_j)\Big|.$$

We have

$$S\left(x, y, \frac{a}{q}\right) \ll \max_{L} \frac{L}{q^{1/2}} \log^{12} x + \log^2 x, \tag{5}$$

where the max is taken over all the multilinear forms L of the form (4) satisfying the following conditions

$$\begin{cases} j\leqq 5, \quad 1/2\leqq M_i\leqq x^{1/5}, \quad i=1,\dots,j,\\ 1/2\leqq N_i\leqq x, \quad i=1,\dots,j, \quad M_1\dots M_j N_1\dots N_j\leqq x. \end{cases} \tag{6}$$

We write

$$M_i(s,\chi) = \sum_{M_i<m\leqq 2M_i} \mu(m)\chi(m)m^{-s}, \quad N_i(s,\chi) = \sum_{N_i<n\leqq 2N_i} \chi(n)n^{-s},$$

and evaluate L by Perron's formula (see Prachar [5], p. 376), thus obtaining

$$L = \frac{1}{2\pi} \sum_{\chi} \left| \int_{-T}^{T} \prod_{i=1}^{j} M_i\left(\frac12+it,\chi\right) N_i\left(\frac12+it,\chi\right) w\left(\frac12+it,x,y\right) dt\right| + O\left(\frac{qx^{3/2}}{T}\right), \tag{7}$$

where

$$w(s, x, y) = \frac{x^s-(x-y)^s}{s}.$$

Since

$$w\left(\frac{1}{2}+it, x, y\right) \ll \min\left(\frac{y}{x^{1/2}}, \frac{x^{1/2}}{|t|+1}\right),$$

putting $T=x$ and $T_0=\frac{x}{y}$ and using a splitting-up argument we get from (7)

$$(8)\qquad L \ll \frac{y}{x^{1/2}} \sum_{\chi} \int_{-T_0}^{T_0} \left| \prod_{i=1}^{j} M_i\left(\frac{1}{2}+it, \chi\right) N_i\left(\frac{1}{2}+it, \chi\right)\right| dt +$$

$$+ \max_{T_0 \leqq T' \leqq T} \frac{x^{1/2}}{T'} \log x \sum_{\chi} \int_{T'/2}^{T'} \left| \prod_{i=1}^{j} M_i\left(\frac{1}{2}+it, \chi\right) N_i\left(\frac{1}{2}+it, \chi\right)\right| dt + qx^{1/2} \ll$$

$$\ll \max_{T_0 \leqq T' \leqq T} \frac{x^{1/2}}{T'} \log x \sum_{\chi} \int_{-T'}^{T'} \left| \prod_{i=1}^{j} M_i\left(\frac{1}{2}+it, \chi\right) N_i\left(\frac{1}{2}+it, \chi\right)\right| dt + qx^{1/2}.$$

From (5) and (8) we finally get

$$(9)\qquad S\left(x, y, \frac{a}{q}\right) \ll$$

$$\ll \left(\max_{(L)} \max_{T_0 \leqq T' \leqq T} \frac{x^{1/2}}{T' q^{1/2}} \sum_{\chi} \int_{-T'}^{T'} \left| \prod_{i=1}^{j} M_i\left(\frac{1}{2}+it, \chi\right) N_i\left(\frac{1}{2}+it, \chi\right)\right| dt + q^{1/2} x^{1/2}\right) \log^{13} x,$$

where $\max_{(L)}$ means that the maximum is taken over j, M_i and N_i satisfying the conditions (6).

We now show that we can group the factors $M_i(s, \chi)$, $N_i(s, \chi)$, $i=1, \dots, j$, in such a way that the resulting product has either the form

$$(10)\qquad \prod_{i=1}^{j} M_i(s, \chi) N_i(s, \chi) = M(s, \chi) N(s, \chi),$$

with

$$M(s, \chi) = \sum_{M < m \leqq CM} a(m)\chi(m) m^{-s}, \quad N(s, \chi) = \sum_{N < n \leqq CN} b(n)\chi(n) n^{-s},$$

$$C \leqq 2^9, \quad |a(n)|, \quad |b(n)| \leqq d_9(n) \quad \text{and} \quad M, N \leqq x^{3/5},$$

or

$$(11)\qquad \prod_{i=1}^{j} M_i(s, \chi) N_i(s, \chi) = M(s, \chi) N_1(s, \chi) N_2(s, \chi),$$

where $M(s, \chi)$ is as before, with $M \leqq x^{3/5}$, and we allow the possibility of $N_2(s, \chi) \equiv 1$ (i.e. $N_2=1/2$). It is understood that $M(s, \chi)$ and $N(s, \chi)$ are products of some $M_i(s, \chi)$ and $N_i(s, \chi)$. Moreover, such grouping will be independent of the characters χ.

First we subdivide the set $\{M_1(s, \chi), \dots, M_j(s, \chi), N_1(s, \chi), \dots, N_j(s, \chi)\}$ into two subsets $\mathcal{M}$ and $\mathcal{N}$ where $\mathcal{M}$ contains the factors of length $\leqq x^{1/5}$ and $\mathcal{N}$ those of length $> x^{1/5}$, where the length of a Dirichlet polynomial $M_i(s, \chi)$ (or $N_i(s, \chi)$)

is just M_i (or N_i). It is clear that $\mathcal{N}$ contains only factors of type $N_i(s,\chi)$, and $|\mathcal{N}|\leqq 4$.

We have to consider two cases separately, according to the size of $\Pi\mathcal{N}=$ $=\prod_{N_i(s,\chi)\in\mathcal{N}} N_i$ (an analogous definition holds for $\Pi\mathcal{M}$).

Case I: $\Pi\mathcal{N}\leqq x^{2/5}$. If $\Pi\mathcal{N}\Pi\mathcal{M}\leqq x^{2/5}$ then (10) is true with $M(s,\chi)=$ $=\prod_{i=1}^{j} M_i(s,\chi)N_i(s,\chi)$, $N(s,\chi)\equiv 1$. If instead we have $\Pi\mathcal{N}\Pi\mathcal{M}>x^{2/5}$ then there is a (not necessarily unique) subset $\mathcal{M}^*\subset\mathcal{M}$ such that

$$x^{2/5}<\Pi\mathcal{M}^*\Pi\mathcal{N}\leqq x^{3/5},$$

since the length of the polynomials contained in $\mathcal{M}^*$ is $\leqq x^{1/5}$. Hence, taking

$$M(s,\chi)=\prod_{M_i(s,\chi)\in\mathcal{M}^*} M_i(s,\chi)\prod_{N_i(s,\chi)\in\mathcal{M}^*\cup\mathcal{N}} N_i(s,\chi),$$

$$N(s,\chi)=\prod_{M_i(s,\chi)\in\mathcal{M}\setminus\mathcal{M}^*} M_i(s,\chi)\prod_{N_i(s,\chi)\in\mathcal{M}\setminus\mathcal{M}^*} N_i(s,\chi),$$

we get a grouping of the form (10).

Case II: $\Pi\mathcal{N}>x^{2/5}$. There is always a subset $\mathcal{N}^*\subset\mathcal{N}$ (not necessarily unique) such that $|\mathcal{N}^*|\leqq 2$ and $\Pi\mathcal{N}^*>x^{2/5}$, since the length of the Dirichlet polynomials contained in $\mathcal{N}$ is $>x^{1/5}$. Taking

$$M(s,\chi)=\prod_{M_i(s,\chi)\in\mathcal{M}} M_i(s,\chi)\prod_{N_i(s,\chi)\in\mathcal{M}\cup(\mathcal{N}\setminus\mathcal{N}^*)} N_i(s,\chi),$$

$$\mathcal{N}^*=\{N_1(s,\chi),\ N_2(s,\chi)\}$$

(in the case $|\mathcal{N}^*|=1$ we choose $N_2(s,\chi)\equiv 1$), we get a grouping of the form (11). As we have already remarked, the grouping depends on the length of the polynomials $M_i(s,\chi)$ and $N_i(s,\chi)$ but is independent of χ.

We now show that either of the grouping (10) or (11) lead to the proof of the Theorem.

We need the following two lemmas.

LEMMA 1. *For any* $T\geqq 2$, $M\geqq 1/2$ *and complex numbers* $a(m)$ *we have*

$$\sum_{\chi\,(\mathrm{mod}\,q)}\int_{-T}^{T}\Big|\sum_{m\leqq M} a(m)\chi(m)m^{-1/2-it}\Big|^2 dt\ll(qT+M)\sum_{m\leqq M}\frac{|a(m)|^2}{m}.$$

This is Theorem 6.4 of Montgomery [3].

LEMMA 2. *For any* $T\geqq 2$ *and* $N\geqq 1/2$ *we have*

$$\sum_{\chi\,(\mathrm{mod}\,q)}\int_{-T}^{T}\Big|\sum_{N<n\leqq 2N}\chi(n)n^{-1/2-it}\Big|^4 dt\ll qT\log^8 qNT.$$

PROOF. We give only the sketch of the quite standard argument. Using Perron's formula and shifting the line of integration to $\sigma=0$ we have

$$\sum_{N<n\leqq 2N}\chi(n)\,n^{-1/2-it}=\frac{1}{2\pi i}\int_{-iU}^{iU}\mathscr{L}(1/2+it+s,\chi)\frac{(2N)^s-N^s}{s}\,ds+O(1)\ll$$

$$\ll\int_{-U}^{U}|\mathscr{L}(1/2+it+iu,\chi)|\frac{1}{1+|u|}\,du+O(1).$$

provided that U is large enough. Rising both sides to the fourth power, using the Hölder's inequality, integrating over t and summing over χ we arrive at

$$\sum_{\chi}\int_{-T}^{T}\Big|\sum_{N<n\leqq 2N}\chi(n)\,n^{-1/2-it}\Big|^4dt\ll\log^3 U\sum_{\chi}\int_{-T}^{T}\int_{-U}^{U}|\mathscr{L}(1/2+it+iu,\chi)|^4\frac{1}{1+|u|}\,du\,dt+$$

$$+O(qT)\ll T\log^4 U\max_{T\leqq V\leqq U}\frac{1}{V}\sum_{\chi}\int_{-V}^{V}\mathscr{L}|(1/2+iv,\chi)|^4dv+O(qT).$$

Lemma 2 follows from Theorem 10.1 of Montgomery [3]. (Note that bounding the contribution of the main character requires a different argument.)

We have to estimate the quantity

$$I=I(M_1,\dots,M_j,N_1,\dots,N_j,T')=\sum_{\chi}\int_{-T'}^{T'}\left|\prod_{i=1}^{j}M_i\left(\frac{1}{2}+it,\chi\right)N_i\left(\frac{1}{2}+it,\chi\right)\right|dt.$$

If $\prod_{i=1}^{j}M_i(s,\chi)N_i(s,\chi)$ has the form (10) then by the Cauchy—Schwarz inequality and Lemma 1 we have

$$(12)\quad I\ll(qT'+M)^{1/2}(qT'+N)^{1/2}\left(\sum_{M<m\leqq CM}\frac{d_9(m)^2}{m}\right)^{1/2}\left(\sum_{N<n\leqq CN}\frac{d_9(n)^2}{n}\right)^{1/2}\ll$$

$$\ll\big(qT'+q^{1/2}(T')^{1/2}x^{3/10}+x^{1/2}\big)\log^{80}x.$$

If $\prod_{i=1}^{j}M_i(s,\chi)N_i(s,\chi)$ has the form (11) then by the Cauchy—Schwarz inequality, Lemmas 1 and 2 we get

(13)

$$I\ll(qT'+M)^{1/2}\left(\sum_{M<m\leqq CM}\frac{d_9(m)^2}{m}\right)^{1/2}(qT'\log^8 qT'N_1)^{1/4}(qT'\log^8 qT'N_2)^{1/4}\ll$$

$$\ll\big(qT'+q^{1/2}(T')^{1/2}x^{3/10}\big)\log^{50}x.$$

Finally, from (9), (12) and (13) we obtain in any case

$$S\left(x,y,\frac{a}{q}\right)\ll\max_{(L)}\max_{T_0\leqq T'\leqq T}\left(\frac{x}{q}\right)^{1/2}\left(q+\frac{q^{1/2}x^{3/10}}{(T')^{1/2}}+\left(\frac{x}{T'}\right)^{1/2}\right)\log^{95}x\ll$$

$$\ll(q^{1/2}x^{1/2}+y^{1/2}x^{3/10}+yq^{-1/2})\log^{100}x,$$

and the theorem is proved.

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A CORRECTION TO MY PAPER "MULTIPLICATIVE FUNCTIONS WITH REGULARITY PROPERTIES. I"

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As E. Wirsing called my attention, the assertion of the Lemma in [1], p. 306 is erroneous. Here I give the correct version of it.

LEMMA. *Let α be an irrational, β be a real number, with the following property: if $m_1<m_2<\dots$ is an infinite sequence of natural numbers such that $\|m_\nu\alpha\|\to 0$, then $\|m_\nu\beta\|\to 0$.*

Then $\beta=k\alpha+l$, k and l are integers.

PROOF. If $1, \alpha, \beta$ are rationally independent, then the sequence $(m\alpha, m\beta)$ $(m=1, 2, \dots)$ is uniformly distributed mod 1 and the property stated in the lemma cannot hold. Consequently, we may assume that $1, \alpha, \beta$ are rationally dependent, i.e. $A\alpha+\beta B=C$ holds with suitable integers A, B, C. Then $B\neq 0$, since α is irrational. If $A=0$, then $\beta=\frac{C}{B}=\frac{c}{b}$, $(c, b)=1$. If β is not an integer, then $|b|>1$. Let l be such a residue mod $|b|$ for which $cl\equiv 1 \pmod{|b|}$. Then for $m_\nu=l+\nu|b|$ we have $m_\nu\beta\equiv\frac{1}{b} \pmod 1$ while $m_\nu\alpha \pmod 1$ is everywhere dense, so it contains a suitable subsequence $m_{\nu_t}\alpha$, $\|m_{\nu_t}\alpha\|\to 0$. This contradicts the assumption stated in the lemma. Assume now that $A\neq 0$. Then $\beta=\frac{C}{A}-A\frac{\alpha}{B}$. First we show that $\frac{C}{A}=$integer. Let $\frac{p_\nu}{q_\nu}$ be the sequence of partial quotients to $\frac{\alpha}{B}$. We have

$$\frac{\alpha}{B}=\frac{p_\nu}{q_\nu}+\frac{\theta_\nu}{q_\nu^2}, \quad |\theta_\nu|\leqq 1.$$

Then

$$q_\nu\alpha = Bp_\nu+\frac{\theta_\nu B}{q_\nu}, \tag{1}$$

$$q_\nu\beta = \frac{C}{A}q_\nu-Ap_\nu-\frac{A\theta_\nu}{q_\nu}. \tag{2}$$

From (1) $\|q_\nu\alpha\|\to 0$ follows. Therefore by the assumption $\|q_\nu\beta\|\to 0$, so from (2) we get $\left\|\frac{C}{A}q_\nu\right\|\to 0$. Let us assume that $\frac{C}{A}=\frac{c}{a}$, $(c, a)=1$. Then $\left\|\frac{c}{a}q_\nu\right\|\to 0$, whence we have $a|q_\nu$ for every large ν. This contradicts the well-known relation

$$|p_\nu q_{\nu-1}-p_{\nu-1}q_\nu| = 1.$$

It has remained to consider the case when $\beta=E-\frac{A}{B}\alpha$, A, B, E are integers. We have to prove that $\frac{A}{B}$=integer. Assume in the contrary that $\frac{A}{B}=\frac{P}{Q}$, $(P, Q)=1$, $Q>1$. Let $\frac{P_\nu}{Q_\nu}$ be the sequence of partial quotients to α. We have

$$\alpha=\frac{P_\nu}{Q_\nu}+\frac{|\theta_\nu|}{q_\nu^2}, \quad |\theta_\nu| \leqq 1, \quad \beta=E-\frac{P}{Q}\cdot\frac{P_\nu}{Q_\nu}-\frac{P}{Q}\cdot\frac{\theta_\nu}{Q_\nu^2}.$$

By choosing now $m_\nu=Q_\nu$, we get

$$Q_\nu\beta=EQ_\nu-\frac{P}{Q}P_\nu+o_\nu(1),$$

and so $\left\|\frac{P}{Q}P_\nu\right\|\to 0$. Hence $Q|P_\nu$ holds for every large ν, a contradiction to $|P_\nu Q_{\nu-1}-P_{\nu-1}Q_\nu|=1$. The lemma is proved.

All the theorems the proof of which was based on the incorrect form of Lemma, can be deduced from this by minor changes in the argumentation.

Reference

[1] I. Kátai, Multiplicative functions with regularity properties. I, *Acta Math. Hung.*, **42** (1983), 295—308.

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[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

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NONEQUIVALENT SPLITTINGS OF ABELIAN GROUPS

S. SZABÓ (Budapest)

1. Introduction. Let G be a finite abelian group written additively; S a subset of G and let M be a set of integers. We say that the multiplier set M splits G with splitting set S if every nonzero element of G has a unique representation as a product ms, where $m \in M$ and $s \in S$.

Splittings arise in connection with the problem of tiling the Euclidean space by translates of certain symmetric star polytopes composed of unit cubes called crosses and semicrosses. The existence of cross and semicross tilings were investigated in a number of papers (see the references).

The purpose of this paper to study the geometrically nonequivalent cross and semicross tilings.

2. Definitions. If τ is a point set in the n-dimensional Euclidean space and $\mathbf{L}$ is a vector set, then $(\tau, \mathbf{L})$ will denote the family of translates of τ by elements of $\mathbf{L}$.

The system $(\tau, \mathbf{L})$ is called an integer lattice tiling if the interiors of its elements are disjoint, the system covers the whole n-space, $\mathbf{L}$ is a lattice and every vector of $\mathbf{L}$ has integer coordinates.

Denote $\mathbf{X}$ the lattice spanned by the coordinate unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

Let τ be the union of finitely many closed n-dimensional unit cubes whose edges are parallel to the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and whose centers are $P_1, \ldots, P_r$. Assume the following: interiors of the cubes are disjoint, the neighbouring cubes meet along entire $(n-1)$-dimensional faces, P_1 is a fixpoint of every motion which takes τ into itself, and the set $\mathbf{T}=\{\overrightarrow{P_1P_1}, \overrightarrow{P_1P_2}, \ldots, \overrightarrow{P_1P_r}\}$ has n linearly independent vectors.

We say that the integer lattice systems $(\tau, \mathbf{L})$ and $(\tau, \mathbf{L}')$ are congruent if there exists an orthogonal mapping $\alpha\colon \mathbf{X}\to\mathbf{X}$ such that $\alpha(\mathbf{T})=\mathbf{T}$ and $\alpha(\mathbf{L})=\mathbf{L}'$.

3. Noncongruent tilings. Our first purpose is to characterise the orthogonal mappings $\alpha\colon \mathbf{X}\to\mathbf{X}$ for which $\alpha(\mathbf{T})=\mathbf{T}$. Since α is an orthogonal mapping and $\mathbf{T}$ has n linearly independent vectors, $\alpha(\mathbf{X})=\mathbf{X}$ and α maps the set of pairs $\{-\mathbf{e}_1, \mathbf{e}_1\}, \ldots, \{-\mathbf{e}_n, \mathbf{e}_n\}$ onto itself. Divide the pairs into two parts. If there exists an orthogonal α for which $\alpha(\mathbf{T})=\mathbf{T}$ and $\alpha(-\mathbf{e}_i)=\mathbf{e}_i$, then $\{-\mathbf{e}_i, \mathbf{e}_i\}$ belongs to the second class. We may assume that $\{-\mathbf{e}_1, \mathbf{e}_1\}, \ldots, \{-\mathbf{e}_k, \mathbf{e}_k\}$ form the first part. Thus α maps the set of elements $\mathbf{e}_1, \ldots, \mathbf{e}_k$ onto itself and the set of pairs $\{-\mathbf{e}_{k+1}, \mathbf{e}_{k+1}\}, \ldots, \{-\mathbf{e}_n, \mathbf{e}_n\}$ onto itself as well. We can divide the previous elements into transitivity classes. If there exists an orthogonal α such that $\alpha(\mathbf{T})=\mathbf{T}$ and $\alpha(\mathbf{e}_i)=\mathbf{e}_j$, then $\mathbf{e}_i$ and $\mathbf{e}_j$ belong to the same class. Similarly, if there exists α such that $\alpha\{-\mathbf{e}_i, \mathbf{e}_i\}=\{-\mathbf{e}_j, \mathbf{e}_j\}$, then these pairs belong to the same class. Denote $\{\mathbf{e}_i\colon i\in I_1\}, \ldots, \{\mathbf{e}_i\colon i\in I_s\}$ and $\{\{-\mathbf{e}_i, \mathbf{e}_i\}\colon i\in \bar{I}_1\}, \ldots, \{\{-\mathbf{e}_i, \mathbf{e}_i\}\colon i\in \bar{I}_{\bar{s}}\}$ the transitivity classes.

We can summarize our results in the following theorem.

THEOREM 3.1. *If* α: $\mathbf{X}\to\mathbf{X}$ *is a linear mapping, then* α *is an orthogonal mapping from* $\mathbf{X}$ *onto itself and* $\alpha(\mathbf{T})=\mathbf{T}$ *if and only if*

$$\alpha\{\mathbf{e}_i\colon i\in I_j\} = \{\mathbf{e}_i\colon i\in I_j\} \quad \textit{for} \quad 1 \leqq j \leqq r, \tag{3.1}$$

$$\alpha\{\{-\mathbf{e}_i, \mathbf{e}_i\}\colon i\in \bar{I}_j\} = \{\{-\mathbf{e}_i, \mathbf{e}_i\}\colon i\in \bar{I}_j\} \quad \textit{for} \quad 1 \leqq j \leqq \bar{r}. \tag{3.2}$$

Now we shall formulate the congruence of two integer lattice tilings by means of finite abelian groups. Denote T the set of coordinates of vectors of $\mathbf{T}$, i.e. let

$$T = \{(t_1, \ldots, t_n)\colon t_1\mathbf{e}_1+\ldots+t_n\mathbf{e}_n\in\mathbf{T}\}.$$

If G is an abelian group written additively and there are $g_1, \ldots, g_n\in G$ and each element g of G is uniquely represented in the form $g=t_1g_1+\ldots+t_ng_n$, $(t_1, \ldots, t_n)\in T$, then we shall use the notation $G=T(g_1, \ldots, g_n)$. If $G=T(g_1, \ldots, g_n)$ and $G=T(g_1', \ldots, g_n')$ and there is an automorphism $\gamma\colon G\to G$ such that

$$\gamma\{g_i\colon i\in I_j\} = \{g_i'\colon i\in I_j\}, \quad 1 \leqq j \leqq r, \tag{3.3}$$

$$\gamma\{\{-g_i, g_i\}\colon i\in \bar{I}_j\} = \{\{-g_i', g_i'\}\colon i\in \bar{I}_j\}, \quad 1 \leqq j \leqq \bar{r}, \tag{3.4}$$

then we shall say that $(g_1, \ldots, g_n)$ are equivalent to $(g_1', \ldots, g_n')$. Obviously, this relation is an equivalence relation. Denote $h(G, T)$ the number of equivalence classes. If there is only one empty equivalence class, then let $h(G, T)=0$.

The main result of this section is the following theorem.

THEOREM 3.2. *The number of pair-wise noncongruent integer lattice tilings* $(\tau, \mathbf{L})$ *is* $\Sigma(G, T)$, *where the summation is extended over all pairwise non-isomorphic finite abelian groups* G.

PROOF. Let $(\tau, \mathbf{L})$ and $(\tau, \mathbf{L}')$ be integer lattice tilings. We shall show that there are finite abelian groups G, G' and $g_1, \ldots, g_n\in G$, $g_1', \ldots, g_n'\in G'$ such that $G=T(g_1, \ldots, g_n)$ and $G'=T(g_1', \ldots, g_n')$. The system $(\tau, \mathbf{L})$ is an integer lattice tiling if and only if each element $\mathbf{x}$ in $\mathbf{X}$ is uniquely represented in the form

$$\mathbf{x} = \mathbf{l}+t_1\mathbf{e}_1+\ldots+t_n\mathbf{e}_n, \quad \mathbf{l}\in\mathbf{L}, \quad (t_1, \ldots, t_n)\in T. \tag{3.5}$$

Let G be the factor group $\mathbf{X}/\mathbf{L}$ and g_i the coset $\mathbf{e}_i+\mathbf{L}$ so $G=T(g_1, \ldots, g_n)$. The construction of G' can be done in an analogous way.

Now assume that $(\tau, \mathbf{L})$ and $(\tau, \mathbf{L}')$ are congruent integer lattice tilings. So there exists an orthogonal mapping $\alpha\colon \mathbf{X}\to\mathbf{X}$ such that $\alpha(\mathbf{T})=\mathbf{T}$ and $\alpha(\mathbf{L})=\mathbf{L}'$. Using the facts $\alpha(\mathbf{X})=\mathbf{X}$, $\alpha(\mathbf{L})=\mathbf{L}'$, $G=\mathbf{X}/\mathbf{L}$ and $G'=\mathbf{X}/\mathbf{L}'=\alpha(\mathbf{X})/\alpha(\mathbf{L})$ we conclude that the mapping $\gamma\colon G\to G'$ given by $\gamma(\mathbf{L}+\mathbf{x})=\alpha(\mathbf{L}+\mathbf{x})=\mathbf{L}'+\alpha(\mathbf{x})$, $\mathbf{x}\in\mathbf{X}$ is a well defined mapping and it is an isomorphism. Now we prove that γ satisfies the conditions (3.3) and (3.4). Indeed, according to (3.1) and (3.2) we have

$$\begin{aligned}\gamma\{g_i\colon i\in I_j\} &= \gamma\{\mathbf{L}+\mathbf{e}_i\colon i\in I_j\} = \{\gamma(\mathbf{L}+\mathbf{e}_i)\colon i\in I_j\} = \{\alpha(\mathbf{L}+\mathbf{e}_i)\colon i\in I_j\} = \\ &= \{\mathbf{L}'+\alpha(\mathbf{e}_i)\colon i\in I_j\} = \mathbf{L}'+\{\alpha(\mathbf{e}_i)\colon i\in I_j\} = \mathbf{L}'+\alpha\{\mathbf{e}_i\colon i\in I_j\} = \\ &= \mathbf{L}'+\{\mathbf{e}_i\colon i\in I_j\} = \{\mathbf{L}'+\mathbf{e}_i\colon i\in I_j\} = \{g_i'\colon i\in I_j\}.\end{aligned}$$

3.4) can be verified in a similar manner.

Assume that there are abelian groups G and G' and $g_1, \dots, g_n \in G$, $g'_1, \dots, g'_n \in G'$ such that $G=T(g_1, \dots, g_n)$ and $G'=T(g'_1, \dots, g'_n)$. We shall prove that there are integer lattice tilings $(\tau, \mathbf{L})$ and $(\tau, \mathbf{L}')$. Let φ be a homomorphism from $\mathbf{X}$ onto G defined by $\varphi(z_1\mathbf{e}_1+\dots+z_n\mathbf{e}_n)=z_1g_1+\dots+z_ng_n$, $z_1, \dots, z_n$ are integers. According to the homomorphism theorem $\mathbf{L}=\operatorname{Ker}\varphi$ is a subgroup of $\mathbf{X}$; in other words $\mathbf{L}$ is an integer lattice. Since each g in G is uniquely represented in the form $g=t_1g_1+\dots$ $\dots+t_ng_n$, $(t_1, \dots, t_n)\in T$, each $\mathbf{x}$ in $\mathbf{X}$ is uniquely represented in the form (3.5). Thus the system $(\tau, \mathbf{L})$ is an integer lattice tiling. $(\tau, \mathbf{L}')$ can be constructed in a similar way.

Now assume that $(g_1, \dots, g_n)$ and $(g'_1, \dots, g'_n)$ are equivalent, i.e. there exists an isomorphism $\gamma\colon G\to G'$ satisfying (3.3) and (3.4). We shall show that the tilings $(\tau, \mathbf{L})$ and $(\tau, \mathbf{L}')$ are congruent, i.e. there exists an orthogonal mapping $\alpha\colon \mathbf{X}\to\mathbf{X}$ such that $\alpha(\mathbf{T})=\mathbf{T}$ and $\alpha(\mathbf{L})=\mathbf{L}'$. Let ψ' be the restriction of φ' to the set $\{-\mathbf{e}_1, \dots, -\mathbf{e}_n, \mathbf{e}_1, \dots, \mathbf{e}_n\}$. We can easily see that ψ' is a bijection between the sets $\{-\mathbf{e}_1, \dots, -\mathbf{e}_n, \mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{-g'_1, \dots, -g'_n, g'_1, \dots, g'_n\}$. Let α be the linear mapping from $\mathbf{X}$ onto itself given by $\alpha(\mathbf{e}_i)=\psi'^{-1}\gamma\varphi(\mathbf{e}_i)$, $1\leqq i\leqq n$. Now we prove that α satisfies (3.1) and (3.2). From this we would conclude that α is an orthogonal mapping and $\alpha(\mathbf{T})=\mathbf{T}$. Indeed, using (3.3) we have $\alpha\{\mathbf{e}_i\colon i\in I_j\}=\{\alpha(\mathbf{e}_i)\colon i\in I_j\}=$ $=\{\psi'^{-1}\gamma\varphi(\mathbf{e}_i)\colon i\in I_j\}=\{\psi'^{-1}\gamma(g_i)\colon i\in I_j\}=\psi'^{-1}\gamma\{g_i\colon i\in I_j\}=\psi'^{-1}\{g'_i\colon i\in I_j\}=$ $\{\psi'^{-1}(g'_i)\colon i\in I_j\}=\{\mathbf{e}_i\colon i\in I_j\}$, $1\leqq j\leqq r$. (3.2) can be verified in an analogous way.

It remains to prove that $\alpha(\mathbf{L})=\mathbf{L}'$. Since $\mathbf{L}=\operatorname{Ker}\varphi$ and $\mathbf{L}'=\operatorname{Ker}\varphi'$ so we want to prove that from $\mathbf{x}\in\mathbf{X}$ and $\varphi(\mathbf{x})=0$ it follows $\varphi'\alpha(\mathbf{x})=0$. Let $\mathbf{x}=z_1\mathbf{e}_1+\dots$ $\dots+z_n\mathbf{e}_n$, where $z_1, \dots, z_n$ are integers. Then

$$\varphi'\alpha(\mathbf{x}) = z_1\varphi'\alpha(\mathbf{e}_1)+\dots+z_n\varphi'\alpha(\mathbf{e}_n) = z_1\varphi'\psi'^{-1}\gamma\varphi(\mathbf{e}_1)+\dots+z_n\varphi'\psi'^{-1}\gamma\varphi(\mathbf{e}_n) =$$

$$= z_1\gamma\varphi(\mathbf{e}_1)+\dots+z_n\gamma\varphi(\mathbf{e}_n) = \gamma\varphi(z_1\mathbf{e}_1+\dots+z_n\mathbf{e}_n) = \gamma\varphi(\mathbf{x}) = \gamma(0) = 0.$$

Thus $\alpha(\mathbf{L})\subseteqq\mathbf{L}'$. The converse embedding $\alpha(\mathbf{L})\supseteqq\mathbf{L}'$ can be proved using the fact $\mathbf{L}\supseteqq\alpha^{-1}(\mathbf{L}')$.

4. Nonequivalent splittings. In this section we specialise the set τ to cross and semicross. Let M be a set of integers, $\mathbf{T}=\{\mathbf{0}, m\mathbf{e}_i\colon m\in M,\ 1\leqq i\leqq n\}$, and

$$T=\{(0, \dots, 0), (m, 0, \dots, 0), \dots, (0, \dots, 0, m)\colon m\in M\}.$$

In case $M=\{1, \dots, k\}$ the set τ belonging to the above set $\mathbf{T}$ is the union of $kn+1$ cubes formed of n arms of length k meeting a corner cube; it is called a semicross. In case $M=\{-k, \dots, -1, 1, \dots, k\}$ the set τ is called a cross which is a union of $2kn+1$ cubes formed of $2n$ arms of length k attached to a central cube.

According to Theorem 3.2 there exists an integer lattice tiling $(\tau, \mathbf{L})$ if and only if there exists an abelian group and elements $g_1, \dots, g_n$ in it such that $G=T(g_1, \dots, g_n)$. Note that $G=T(g_1, \dots, g_n)$ if and only if M splits G with splitting set $S=\{g_1, \dots, g_n\}$.

If each element of M is relatively prime to the order of G, then the splitting is called nonsingular, othrewise singular.

We want to study the noncongruent integer lattice tiling $(\tau, \mathbf{L})$, hence we need to give the orhogonal mappings which carry the set $\mathbf{T}$ onto itself. The linear mapping $\alpha\colon\mathbf{X}\to\mathbf{X}$ is an orthogonal mapping and $\alpha(\mathbf{T})=\mathbf{T}$ if and only if

$$\alpha\{\{-\mathbf{e}_1, \mathbf{e}_1\}, \dots, \{-\mathbf{e}_n, \mathbf{e}_n\}\} = \{\{-\mathbf{e}_1, \mathbf{e}_1\}, \dots, \{-\mathbf{e}_n, \mathbf{e}_n\}\},$$

$$\alpha\{\mathbf{e}_1, \dots, \mathbf{e}_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$$

respectively, depending on M is partitioned in the form $M=(-\bar{M})\cup\bar{M}$ or not. So the splitting sets S and S' of G which belong to the same multiplier set M are equivalent if there exists an automorphism $\gamma\colon G\to G$ such that $\gamma(S\cup(-S))=S'\cup(-S')$, $\gamma(S)=S'$, respectively. Note that if M splits G with the splitting set S, then M is partitioned in the form $M=(-\bar{M})\cup\bar{M}$ if and only if $\bar{M}$ splits G with the splitting set $(-S)\cup S$. This enables us to restrict ourselves to the case when M is not partitioned in the form $N=(-\bar{M})\cup\bar{M}$, but we have to distinguish cases when S is partitioned in the form $S=(-\bar{S})\cup\bar{S}$ or not. Denote $\Omega(G, M)$ and $\omega(G, M)$ the number of nonequivalent splittings of G by M, respectively.

We shall use the following theorem due to W. Hamaker and S. K. Stein [3].

THEOREM 4.1. *Let G_1, G_2 and G_3 be finite abelian groups and*

$$\{0\}\to G_1\xrightarrow{\alpha} G_2\xrightarrow{\beta} G_3\to\{0\}$$

an exact sequence. Assume that M splits the groups G_1 and G_3 with splitting sets S_1 and S_3 such that the splitting of G_3 is nonsingular. Then M splits G_2 with the splitting set $S_2=\alpha(S_1)\cup\beta^{-1}(S_3)$.

Evidently, if S_1 and S_3 are partitioned in the forms $S_1=(-\bar{S}_1)\cup\bar{S}_1$ and $S_3=(-\bar{S}_3)\cup\bar{S}_3$, then S_2 is partitioned in the form $S_2=(-\bar{S}_2)\cup\bar{S}_2$.

THEOREM 4.2. *If M splits the finite abelian group G nonsingularly and H is a characteristic subgroup of G, then*

$$\omega(G, M)\geqq\omega(H, M)\omega(G/H, M) \quad \textit{and} \quad \Omega(G, M)\geqq\Omega(H, M)\Omega(G/H, M).$$

PROOF. For the sake of brevity we shall prove only the first statement. Assume that M splits the groups H and G/H. Obviously, these splittings are nonsingular. If α is the inclusion mapping and β is the natural homomorphism between G and G/H, then

$$\{0\}\to H\xrightarrow{\alpha} G\xrightarrow{\beta} G/H\to\{0\}$$

is an exact sequence. According to Theorem 4.1, M splits the group G with the splitting sets $S\cup\beta^{-1}(V)$ and $S'\cup\beta^{-1}(V')$, where S, S' and V, V' are splitting sets of H and G/H, respectively. Assume that $S\cup\beta^{-1}(V)$ and $S'\cup\beta^{-1}(V')$ are equivalent, i.e. there exists an automorphism $\gamma\colon G\to G$ such that $\gamma(S\cup\beta^{-1}(V))=$ $=S'\cup\beta^{-1}(V')$. From this we conclude $\beta\gamma(S\cup\beta^{-1}(V))=\beta(S'\cup\beta^{-1}(V'))$ and so $\beta\gamma(S)\cup\beta\gamma\beta^{-1}(V)=\beta(S')\cup V'$. Since H is a characteristic subgroup so $\gamma(H)=H$ and since $S, S'\subseteq H$, hence $\beta\gamma(S)=\beta(S')=0$. Thus $\beta\gamma\beta^{-1}(V)=V'$. Using the facts $\gamma(G)=G$ and $\gamma(H)=H$ we can verify that the mapping $\bar{\gamma}\colon G/H\to G/H$ given by $\bar{\gamma}(g+H)=\gamma(g)+H$, $g\in G$ is well defined and it is an automorphism. Since $\bar{\gamma}\beta(g)=\bar{\gamma}(g+H)=\gamma(g)+H=\beta\gamma(g)$ so $V'=\beta\gamma\beta^{-1}(V)=\bar{\gamma}\beta\beta^{-1}(V)=\bar{\gamma}(V)$ hence V' and V are equivalent splitting sets of G/H.

Obviously, $S=H\cap(S\cup\beta^{-1}(V))$ and $S'=H\cap(S'\cup\beta^{-1}(V'))$. So $\gamma(S)=$ $=\gamma(H\cap(S\cup\beta^{-1}(V)))=\gamma(H)\cap\gamma(S\cup\beta^{-1}(V))=H\cap(S'\cup\beta^{-1}(V'))=S'$, i.e. S and S' are equivalent as well. This completes the proof.

The following theorem is sharper than the previous one for cyclic groups whose orders are prime powers. Denote $C(r)$ the cyclic group of order r.

THEOREM 4.3. *Let p be an odd prime and t a positive integer. Assume that each element in M is relatively prime to p. Then*

$$\omega(C(p^t), M) \geqq [\omega(C(p), M)]^t |M|^{t-1},$$
$$\Omega(C(p^t), M) \geqq [\Omega(C(p), M)]^t (|M|/2)^{t-1}.$$

PROOF. We shall prove only the first statement. Let c be a generator element of $C(p^t)$ and let $p^i C(p^t)$ be a cyclic group of order p^{t-i} with generator $p^i c$.

Assume that M splits $p^{t-1}C(p^t)$. Each splitting set of $p^{t-1}C(p^t)$ is represented in the form $p^{t-1}cV$, where V is a set of integers and we can choose these sets such that $p^{t-1}cV$ and $p^{t-1}cV'$ are equivalent if and only if $V=V'$. For $m\in M$ the mapping $\mu\colon C(p^t)\to C(p^t)$ given by $\mu(c)=mc$ is an automorphism, because m is relatively prime to p. Obviously, $p^{t-1}cV$ is a splitting set of $p^{t-1}C(p^t)$ if and only if $p^{t-1}cmV= =\mu(p^{t-1}cV)$ is a splitting set of $p^{t-1}C(p^t)$.

Consider the exact sequences

$$\{0\} \to p^{t-1}C(p^t) \to p^{t-i-1}C(p^t) \to p^{t-i}C(p^t) \to \{0\}$$

for $0\leqq i\leqq t-1$. If M splits $p^{t-1}C(p^t)$ with the splitting set $p^{t-1}m_t cV_t$, then according to Theorem 4.1, M splits $p^{t-1}C(p^t)$ with splitting set $p^{t-1}m_t cV_t \cup p^{t-2}m_{t-1}cV_{t-1}+ +p^{t-1}C(p^t)$. Again according to Theorem 4.1, M splits $p^{t-3}C(p^t)$ with splitting set $p^{t-1}m_t cV_t \cup p^{t-2}m_{t-1}cV_{t-1}+p^{t-1}C(p^t)\cup p^{t-3}m_{t-2}cV_{t-2}+p^{t-2}C(p^t)$ and finally M splits $C(p^t)$ with splitting set

$$S = \bigcup_{i=0}^{t-1} p^{t-i-1}m_{t-i}cV_{t-i}+p^{t-i}C(p^t).$$

Assume that

$$S' = \bigcup_{i=0}^{t-1} p^{t-i-1}m'_{t-i}cV'_{t-i}+p^{t-i}C(p^t)$$

is a splitting set as well and $m_1=m_1'$. If S and S' are equivalent then there exists an automorphism $\gamma\colon C(p^t)\to C(p^t)$ such that $\gamma(S)=S'$ and so

$$\gamma(p^{t-i-1}m_{t-i}cV_{t-i}+p^{t-i}C(p^t)) = p^{t-i-1}m'_{t-i}cV'_{t-i}+p^{t-i}C(p^t) \quad \text{for} \quad 0\leqq i\leqq t-1$$

because γ preserves the orders of the elements. Multiplying by p^i we conclude that $\gamma(p^{t-1}m_{t-i}cV_{t-i})=p^{t-1}m'_{t-i}cV'_{t-i}$ for $0\leqq i\leqq t-1$. Thus $\mu'^{-1}_{t-i}\mu_{t-i}\gamma(p^{t-1}cV_{t-i})= =p^{t-1}cV'_{t-i}$, i.e. $p^{t-1}cV_{t-i}$ and $p^{t-1}cV'_{t-i}$ are equivalent splitting sets, therefore $V_{t-i}=V'_{t-i}$. If $\gamma\colon C(p^t)\to C(p^t)$ is defined by $\gamma(c)=dc$, where d is an integer, then $p^{t-1}dm_{t-i}\equiv p^{t-1}m'_{t-1} \pmod{p^t}$ and so $dm_{t-i}\equiv m'_{t-i} \pmod p$. Now using the fact $m_1=m_1'$ we have $d\equiv 1 \pmod p$ and $m_{t-i}\equiv m'_{t-i} \pmod p$. Since M splits $p^{t-1}C(p^t)$ whose order is p, so elements of M are pair-wise incongruent modulo p and therefore from $m_{t-i}\equiv m'_{t-i} \pmod p$ and $m_1, \dots, m_t, m_1', \dots, m_t'\in M$ it follows that $m_{t-i}=m'_{t-i}$ for $0\leqq i\leqq t-1$. Thus there exist $[\omega(C(p), M)]^t|M|^{t-1}$ nonequivalent splitting sets of $C(p^t)$.

Now we specialise our result to the case $M=\{1, 2\}$.

THEOREM 4.4. *If 2 is a primitive root modulo p^t for each t, then $\omega(C(p^t), \{1, 2\})= =2^{t-1}$ and if $4|(p-1)$, then $\Omega(C(p^t), \{1, 2\})=2^{t-1}$. (For example p may be* 3, 5, 11, 13, 19.)

PROOF. According to the previous theorem $\omega(C(p^t), \{1, 2\}) \geqq \omega(C(p), \{1, 2\})2^{t-1}$. At first we prove $\omega(C(p), \{1, 2\})=1$ so it will be sufficient to prove $2^{t-1} \geqq \omega(C(p^t), \{1, 2\})$.

In our case 2 is a primitive root modulo p so we can construct only two splitting sets $\{c, 4c, 16c, \ldots\}$ and $\{2c, 8c, 32c, \ldots\}$, where c is a generator of $C(p)$. But these are equivalent.

Now we shall prove that $2^{t-1} \geqq \omega(C(p^t), \{1, 2\})$. Consider the autmorphism defined by $c \to 2c$. This is a permutation of elements of $C(p^t)$, and it consists of t cyclics $(p^i c, 2p^i c, \ldots, -p^i c, -2p^i c, \ldots)$. Hence we can construct only 2^t splitting sets. But among these there are only 2^{t-1} nonequivalent. Indeed, a splitting set containing the elements $c, 4c, \ldots$ from the first cycle is equivalent to an other splitting set containing the elements $2c, 8c, \ldots$.

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SOME PROPERTIES OF GENERALIZED MEASURES ON THE DYADIC GROUP

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1. Introduction. Let

$$\mu = \sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x)$$

be a Walsh series and let m_μ be the *dyadic measure* defined by the equation

$$m_\mu(I) = \lim_{n\to\infty} \int_I \Big(\sum_{k=0}^{n} \hat{\mu}(k) w_k(x)\Big)\, dx$$

for each dyadic interval I. I_n^p denotes the set of all 0—1 sequences $(\varepsilon_1 \varepsilon_2 \dots)$ such that

$$\sum_{k=1}^{n} \varepsilon_k/2^k = p/2^n, \quad n = 0, 1, \dots, \quad p = 0, 1, \dots, 2^n - 1. \tag{1}$$

In this paper, we shall identify $(\varepsilon_1 \varepsilon_2 \dots)$ and $p/2^n$ if (1) holds. I_n^p is called a *dyadic interval of rank n* and $I_n(x)$ is the dyadic interval of rank n which contains x.

When a dyadic measure m_μ satisfies $\hat{\mu}(k) = O(1)$ as $k \to \infty$, it is called a *pseudo measure*. It is easy to see that if a dyadic measure m_μ satisfies

$$\int_0^1 |S_{2^n}(\mu, x)|\, dx = \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)| = O(1) \quad \text{as} \quad n \to \infty, \tag{2}$$

where $S_k(\mu, x)$ is k-th partial sum of μ, then there exists a unique Radon measure m^* on the dyadic group such that $m^*(I) = m_\mu(I)$ for all dyadic intervals I. We shall identify m^* and m_μ.

2. Pseudo measures and Radon measures. It is easy to construct a dyadic measure m_μ such that

$$m_\mu(I_n(x)) \neq O(1) \quad \text{everywhere as} \quad n \to \infty.$$

But we have the following:

THEOREM 1. *If m_μ is a pseudo measure, then*
(i) $m_\mu(I_n(x)) = o\big(n(\log n)^{1+\varepsilon}/\sqrt{2^n}\big)$ *a.e. as* $n \to \infty$ *for each* $\varepsilon > 0$;
(ii) *the above convergence is quasi-uniform.*

PROOF. Since

$$S_{2^n}(\mu, x) \equiv S_{2^n}(x) = \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 2^n m_\mu(I_n(x)),$$

we have

$$\int_0^1 \{\sum_{n=2}^{\infty} 2^{1/2(n-2\log n-2(1+\varepsilon)\log\log n)}|S_{2^n}(x)|/2^n\}^2\,dx =$$

$$= \int_0^1 \{\sum_{n=2}^{\infty} 1/\big(n(\log n)^{1+\varepsilon}\big)\big(\sum_{n=2}^{\infty} n(\log n)^{1+\varepsilon}\times$$

$$\times 2^{(n-2\log n-2(1+\varepsilon)\log\log n)}|S_{2^n}(x)|^2/2^{2n}\big)\,dx =$$

$$= \big(\sum_{n=2}^{\infty} 1/n(\log n)^{1+\varepsilon}\big)\big(\sum_{n=2}^{\infty}(\log n)^{1+\varepsilon}/(2^n n(\log n)^{2(1+\varepsilon)})\big)\int_0^1 |S_{2^n}(x)|^2\,dx \leqq$$

$$\leqq \big(\sum_{n=2}^{\infty} 1/\big(n(\log n)^{1+\varepsilon}\big)\big)\big(\sum_{n=2}^{\infty} 1/\big(n(\log n)^{1+\varepsilon}\big)\sum_{k=0}^{2^n-1}|\hat{\mu}(k)|^2/2^n\big) \leqq$$

$$\leqq \big(\sum_{n=2}^{\infty} 1/\big(n(\log n)^{1+\varepsilon}\big)^2 < \infty,$$

for each $\varepsilon>0$, from which (i) follows.

Since

$$(\sup_n |m_\mu(I_n(x))|)^2 \leqq \sup_n \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^2 =$$

$$= \sup_n 1/2^n \int_0^1 |S_{2^n}(x)|^2\,dx = \sup_n \big(1/2^n \sum_{k=0}^{2^n-1}|\hat{\mu}(k)|^2\big)$$

and the last term is finite, the majorant function of $m_\mu(I_n(x))$ is bounded. From Theorem 3 of [5], we have (ii).

REMARK. When m_μ is a Radon measure,

$$m_\mu(I_n(x)) = O(1/2^n) \quad \text{a.e. as} \quad n\to\infty.$$

THEOREM 2. *For a countable dense set E, there exists a positive Radon measure m_μ such that*

$$m_\mu(I_n(x)) \neq o(1) \quad as \quad n\to\infty \quad on\ E.$$

PROOF. Let $\{\theta_n\}_n$ satisfy $\sum_{n=1}^{\infty}\theta_n = 1$ and $\theta_n>0$ for all n, and set $E=\{x_1, x_2, \ldots\}$. Define m_μ by the equation

$$m_\mu(I) = \sum_{x_k\in I}\theta_k$$

for all dyadic intervals I. Obviously, m_μ is a positive Radon measure and it satisfies

$$\lim_{n\to\infty} m_\mu(I_n(x_k)) = \theta_k \neq 0$$

for all k. The proof is complete.

THEOREM 3. *If m_μ is a Radon measure, then the following conditions are equivalent:*

(i) $(1/n)\sum_{k=0}^{n-1}|\hat{\mu}(k)|^2=o(1)$ *as* $n\to\infty$,

(ii) $m_\mu(I_n(x))=o(1)$ *everywhere as* $n\to\infty$,

(iii) $m_\mu(I_n(x))=o(1)$ *uniformly as* $n\to\infty$,

(iv) *when* $\varphi(t)$ *is continuous and* $\varphi(t)\downarrow 0$ *as* $t\to 0+$,

$$(1/n)\sum_{k=0}^{n-1}\varphi(|\hat{\mu}(k)|)=o(1) \quad as \quad n\to\infty.$$

PROOF. We shall first prove (ii) from (i). By (i), we have

$$(1/2^n)\sum_{k=0}^{2^n-1}|\hat{\mu}(k)|^2=\sum_{p=0}^{2^n-1}|m_\mu(I_n^p)|^2=o(1) \quad \text{as} \quad n\to\infty,$$

then (ii) follows.

Set

$$F_n(t)=\begin{cases}\sum_{k=0}^{p}|m_\mu(I_n^k)|, & \text{for} \quad p/2^n<t\leqq(p+1)/2^n, \quad p=0,1,\dots,2^n-1,\\ 0, & \text{for} \quad t=0.\end{cases}$$

By assumption, there exists a limit function $F(t)=\lim_{n\to\infty}F_n(t)$ which is increasing. By (ii), $F(t)$ is continuous. Then it is uniformly continuous. By

$$F((p+1)/2^n)-F(p/2^n)\geqq|m_\mu(I_n^p)|,$$

we have (iii).

By (iii), we have

$$\sum_{p=0}^{2^n-1}|m_\mu(I_n^p)|^2\leqq\max_p|m_\mu(I_n^p)|\Big(\sum_{k=0}^{2^n-1}|m_\mu(I_n^k)|\Big)=o(1), \quad \text{as} \quad n\to\infty.$$

Then (i) follows.

By assumption, we can set $|\hat{\mu}(k)|\leqq K$ for all k. For each $\varepsilon>0$, let t_ε be a positive number such that $\varphi(t_\varepsilon)<\varepsilon$. Hence we have

$$(1/n)\sum_{k=0}^{n-1}\varphi(|\hat{\mu}(k)|)=(1/n)\Big(\sum_{|\hat{\mu}(k)|\leqq t_\varepsilon}+\sum_{|\hat{\mu}(k)|>t_\varepsilon}\Big)\leqq$$

$$\leqq(1/n)\Big(\sum_{|\hat{\mu}(k)|\leqq t_\varepsilon}\varepsilon+\sum_{|\hat{\mu}(k)|>t_\varepsilon}K\Big)\leqq\varepsilon+K(1/n)\#\{k;\ t_\varepsilon<|\hat{\mu}(k)|,\ 0\leqq k<n\}\leqq$$

$$\leqq\varepsilon+K(1/t_\varepsilon^2)(1/n)\sum_{k=0}^{n-1}|\hat{\mu}(k)|^2<\varepsilon+\varepsilon=2\varepsilon,$$

for sufficiently large n. Then (iv) follows from (i).

The converse is obvious.

REMARK. When m_μ is a Radon measure, (iii) holds if and only if

(3) $$(1/n)\sum_{k=0}^{n-1}|\hat{\mu}(k)|=o(1) \quad \text{as} \quad n\to\infty.$$

THEOREM 4. *If m_μ is a Radon measure, then*

$$\int_0^1 |S_{2^n}(\mu, x)|^{1+\varepsilon_n}\, dx = O(1) \quad as \quad n \to \infty,$$

where $\varepsilon_n = O(1/n)$ *as* $n \to \infty$.

PROOF. By the relations

$$\int_0^1 |S_{2^n}(\mu, x)|^{1+\varepsilon_n}\, dx = 2^{n\varepsilon_n} \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^{1+\varepsilon_n} = O\Big(\big(\sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|\big)^{1+\varepsilon_n}\Big) =$$

$$= O\Big(\Big(\int_0^1 |S_{2^n}(\mu, x)|\, dx\Big)^{1+\varepsilon_n}\Big) = O(1) \quad \text{as} \quad n \to \infty,$$

the conclusion follows.

THEOREM 5. *If a Radon measure m_μ satisfies*

$$\int_0^1 |S_{2^n}(\mu, x)|^{1+\varepsilon_n}\, dx = O(1) \quad as \quad n \to \infty, \tag{4}$$

where $\varepsilon_n = o(1)$ *and* $n\varepsilon_n \uparrow \infty$ *as* $n \to \infty$, *then* m_μ *satisfies* (3).

PROOF. Let us assume that

$$\sum_{p=0}^{2^n-1} |m_\mu(I_n^p)| < K$$

and

$$\int_0^1 |S_{2^n}(\mu, x)|^{1+\varepsilon_n}\, dx < K \quad \text{for all } n,$$

where $K \geqq 1$. Therefore we have

$$(1/2^n) \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 = \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^2 = \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^{1+\varepsilon_n+1-\varepsilon_n} \leqq$$

$$\leqq \max_p |m_\mu(I_n^p)|^{1-\varepsilon_n} \Big\{\sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^{1+\varepsilon_n}\Big\} \leqq$$

$$\leqq \Big(\sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|\Big)^{1-\varepsilon_n} 2^{n\varepsilon_n} \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^{1+\varepsilon_n} (1/2^{n\varepsilon_n}) \leqq$$

$$\leqq K^{1-\varepsilon_n} \int_0^1 |S_{2^n}(\mu, x)|^{1+\varepsilon_n}\, dx\, (1/2^{n\varepsilon_n}) \leqq K^2 (1/2^{n\varepsilon_n}) = o(1) \quad \text{as} \quad n \to \infty.$$

From Theorem 3 and Remark, the proof is complete.

REMARK. Let m_μ be a positive Radon measure such that

$$m_\mu(I_0^0)=1,\quad m_\mu(I_{n+1}^{2p})=(1/2+\theta_{n+1})m_\mu(I_n^p),\quad m_\mu(I_{n+1}^{2p+1})=(1/2-\theta_{n+1})m_\mu(I_n^p),$$

for $0<\theta_n<1/2$ and $n=0,1,\ldots,\ p=0,1,\ldots,2^n-1$.

If $\liminf_{n\to\infty}\theta_n<1/2$, then (iii) of Theorem 3 is valid. Moreover for $\varepsilon_n\downarrow 0$ we have

$$\int_0^1 |S_{2^n}(\mu,x)|^{1+\varepsilon_n}\,dx = 2^{n\varepsilon_n}\sum_{p=0}^{2^n-1}|m_\mu(I_n^p)|^{1+\varepsilon_n}=$$

$$=2^{n\varepsilon_n}\prod_{k=1}^{n}\{(1/2-\theta_k)^{1+\varepsilon_n}+(1/2+\theta_k)^{1+\varepsilon_n}\}.$$

The above product is bounded if and only if

$$A_n\equiv n\varepsilon_n+\sum_{k=1}^{n}\log_2\big((1/2-\theta_k)^{1+\varepsilon_n}+(1/2+\theta_k)^{1+\varepsilon_n}\big)$$

is bounded from above. Since

$$(1/2\pm\theta)^{1+\varepsilon}=(1/2)^{1+\varepsilon}+(1+\varepsilon)(1/2)^{\varepsilon}\theta+(1+\varepsilon)\varepsilon(1\pm\theta')^{\varepsilon-1}\theta^2$$

for some θ' such that $0<\theta'<\theta$, we have

$$(1/2-\theta)^{1+\varepsilon}+(1/2+\theta)^{1+\varepsilon}=$$

$$=(1/2)^{1+\varepsilon}+\varepsilon(1+\varepsilon)\{(1/2+\theta'')^{\varepsilon-1}+(1/2-\theta''')^{\varepsilon-1}\}\theta^2$$

for some $0<\theta'',\theta'''<\theta$. By

$$\log_2\big((1/2)^{\varepsilon}+t\big)\leqq-\varepsilon+(2^{\varepsilon}/\log_e 2)\,t,$$

we have

$$A_n\leqq n\varepsilon_n-n\varepsilon_n+\sum_{k=1}^{n}(2^{\varepsilon_n}/\log_e 2)\,\varepsilon_n(1+\varepsilon_n)\times$$

$$\times(1/2+\theta_n'')^{\varepsilon_n-1}+(1/2-\theta_k''')^{\varepsilon_n-1}\theta_k^2\leqq C\varepsilon_n\sum_{k=1}^{n}\theta_k^2,$$

where C is a positive constant.

If

$$(1/n)\sum_{k=1}^{n}\theta_k^2=O\big(1/(n\varepsilon_n)\big)\quad\text{as}\quad n\to\infty,$$

then m_μ satisfies (4). Let $\Delta m_\mu(I_n^p)=m_\mu(I_{n+1}^{2p})-m_\mu(I_{n+1}^{2p+1})$. Then by easy computation we have

$$\sum_{p=0}^{2^n-1}|\Delta m_\mu(I_n^p)|=2\theta_{n+1}.$$

REMARK. Let m_μ be a non-negative Radon measure which is defined by the following equations:

$$m_\mu(I_0^0)=1,\quad m_\mu(I_1^0)=m_\mu(I_1^1)=1/2,$$

$$m_\mu(I_2^0)=m_\mu(I_2^2)=1/2,\quad m_\mu(I_2^1)=m_\mu(I_2^3)=0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$m_\mu(I_{l_1}^0)=m_\mu(I_{l_1}^{2^{l_1-1}})=1/2,\quad m_\mu(I_{l_1}^p)=0\quad\text{for}\quad p\neq 0, 2^{l_1-1},$$

and

$$\begin{cases} m_\mu(I_{l_1+1}^0)=m_\mu(I_{l_1+1}^1)=m_\mu(I_{l_1+1}^{2^{l_1}})=m_\mu(I_{l_1+1}^{2^{l_1}+1})=1/4,\\ m_\mu(I_{l_1+1}^p)=0\quad\text{for}\quad p\neq 0,1,2^{l_1},2^{l_1}+1.\end{cases}$$

Continuing in this way m_μ is defined for $l_n\uparrow\infty$ as $n\to\infty$. m_μ has the following properties:

$$m_\mu(I_n(x))=o(1)\quad\text{everywhere as } n\to\infty,$$

$$\sum_{p=0}^{2^n-1}|\Delta m_\mu(I_n^p)|=\begin{cases}1, & \text{for}\quad n=l_k,\\ & \qquad\qquad k\neq 1,2,\dots,\\ 0, & \text{for}\quad n\neq l_k,\end{cases}$$

and

$$2^n m_\mu(I_n(x))=o(1)\quad\text{a.e. as } n\to\infty.$$

Let us assume that $l_s=sN$ for $s=1,2,\dots$ where N is a natural number. When $l_s\leqq n<l_{s+1}$, we have

$$\int_0^1 |S_{2^n}(\mu,x)^{1+\varepsilon_n}\,dx=2^{n\varepsilon_n}\sum_{p=0}^{2^n-1}|m_\mu(I_n^p)^{1+\varepsilon_n})|=$$

$$=2^{n\varepsilon_n}(1/2^s)^{1+\varepsilon_n}=2^{(n-s)\varepsilon_n}\geqq$$

$$\geqq 2^{(l_s-s)\varepsilon_{l_{s+1}}}\geqq 2^{(N-1)\varepsilon_{l_{s+1}}}=2^{((N-1)/(N+1))(N+1)s\varepsilon_{N(s+1)}}.$$

The last term tends to infinity if $n\varepsilon_n\uparrow\infty$ as $n\to\infty$. Then we have

$$\int_0^1 |S_{2^n}(\mu,x)|^{1+\varepsilon_n}\,dx\to\infty\quad\text{as}\quad n\to\infty.$$

THEOREM 6. *If f is an integrable function, then*

$$\sum_{p=0}^{2^n-1}|\Delta m_f(I_n^p)|=o(1)\quad \textit{as}\quad n\to\infty.$$

PROOF. It is well known that

$$\int_0^1 |S_{2^n}(f,x)-f(x)|\,dx=o(1)\quad\text{as}\quad n\to\infty.$$

Hence we have

$$\sum_{p=0}^{2^n-1} |\Delta m_f(I_n^p)| = \int_0^1 |S_{2^{n+1}}(f, x) - S_{2^n}(f, x)|\, dx =$$

$$\leqq \int_0^1 |S_{2^{n+1}}(f) - f|\, dx + \int_0^1 |f - S_{2^n}(f)|\, dx = o(1)$$

as $n \to \infty$. The proof is complete.

REMARK. Let $\eta_0 = 1$ and $\eta_n \equiv 2\theta_n \downarrow 0$ as $n \to \infty$. Set

$$g(x) = \begin{cases} 2^{k+1}(\eta_k - \eta_{k+1}) & \text{on } I_{k+1}^1, \quad \text{for} \quad k = 0, 1, \ldots, \\ \infty, \quad \text{for} \quad x = 0. \end{cases}$$

Obviously, $g(x)$ satisfies $g(x) \neq 0$ and $\int_0^1 |g(x)|\, dx = 1$. On the other hand, we have

$$\int_0^1 |S_{2^n}(g, x)|^{1+\varepsilon_n}\, dx = 2^{n\varepsilon_n} \sum_{p=0}^{2^n-1} |m_g(I_n^p)|^{1+\varepsilon_n} \geqq 2^{n\varepsilon_n} \eta_n^{1+\varepsilon_n}$$

and

$$\sum_{p=0}^{2^n-1} |\Delta m_g(I_n^p)| \geqq |\eta_n - 2\eta_{n+1}|.$$

Let us assume $\theta_{n+1} \geqq (3/4)\theta_n$ for all n. Then we have

$$2\eta_{n+1} - \eta_n = 4\theta_{n+1} - 2\theta_n = 4\theta_{n+1} - 3\theta_n + \theta_n \geqq \theta_n.$$

When $n\varepsilon_n \uparrow \infty$ as $n \to \infty$, let $\{\eta_n\}_n$ satisfy

$$2^{n\varepsilon_n} \eta_n^{1+\varepsilon} \uparrow \infty \quad \text{as} \quad n \to \infty.$$

Then

$$\int_0^1 |S_{2^n}(g, x)|^{1+\varepsilon_n}\, dx \uparrow \infty \quad \text{as} \quad n \to \infty$$

and

$$\sum_{p=0}^{2^n-1} |\Delta m_g(I_n^p)| \geqq \theta_n \quad \text{for all } n.$$

THEOREM 7. *For each* $0 < \alpha < 1$, *there exists a pseudo measure* m_μ *such that*

$$\sum_{p=0}^{2^n-1} |\Delta m_\mu(I_n^p)|^{1+\alpha} \to \infty \quad as \quad n \to \infty,$$

but $\hat{\mu}(k) = o(1)$ *as* $k \to \infty$.

PROOF. Let θ be a positive number such that

$$(1/2)^{2/(1+\theta)} < \theta < 1/2.$$

When $0\leqq p<4^n$, set $p=4^{n-1}p_{n-1}+\ldots+4p_1+p_0$, where $p_i=0, 1, 2, 3$ and let $N(p)$ be the number of $p_i=3, i=1, 2, \ldots, n-1$. Set

$$\Delta m_\mu(I_{2n}^p)=\theta^n(-1)^{N(p)}$$

and

$$\Delta m_\mu(I_{2n+1}^p)=\begin{cases}\theta(-1)^{N(p)}, & \text{for even } p,\\ 0, & \text{otherwise.}\end{cases}$$

When $4^n\leqq k<2\cdot 4^n-1$, we have

$$\hat{\mu}(k)=\hat{\mu}(4^n+k')=\sum_{p=0}^{4^n-1}\Delta m_\mu(I_{2n}^p)w_{k'}(p/4^n)=$$

$$=\theta^n\Big\{\sum_{p_0=0}^{3}\ldots\sum_{p_{n-1}=0}^{3}(-1)^{N(p)}w_{k'}(p_{n-1}/4)\ldots w_{k'}(p_0/4)\Big\}=$$

$$=\theta^n\prod_{i=1}^{n-1}\Big(\sum_{p_i=0}^{3}(-1)^{N(p_i)}w_{k'}(p_i/4^{n-i})\Big)$$

where $k'=4^{n-1}k_{n-1}+\ldots+4k_1+k_0$. On the other hand, we have

$$\sum_{p_i=0}^{3}(-1)^{N(p_i)}w_{k'}(p_i/4^{n-i})=$$

$$=w_{k'}(0/4^{n-i})+w_{k'}(1/4^{n-i})+w_{k'}(2/4^{n-i})-w_{k'}(3/4^{n-i})=$$

$$=\begin{cases}2, & \text{for } k_{n-i-1}=0, 1, 2,\\ -2, & \text{for } k_{n-i-1}=3.\end{cases}$$

Hence we get

$$\hat{\mu}(k)=\theta^n\prod_{i=0}^{n-1}\big(2(-1)^{N(k_{n-i-1})}\big)=\theta^n 2^n(-1)^{N(k')}.$$

Similarly we can prove that

$$\hat{\mu}(k)=\theta^n\sum_{p=0}^{4^n-1}(-1)^{N(p)}w_{k'}(p/4^n)$$

for $2\cdot 4^n\leqq k\equiv 2\cdot 4^n+k'<4^{n+1}$. When $0\leqq k'<2\cdot 3^n$, quite similarly to the first case, we have

$$\hat{\mu}(k)=\theta^n 2^n(-1)^{N(k')}.$$

When $4^n\leqq k'\equiv 4^n+k''<2\cdot 4^n$, we have

$$\hat{\mu}(k)=\theta^n\sum_{p=0}^{4^n-1}(-1)^{N(p)}w_{4^n}(p/4^n)w_{k''}(p/4^n)=$$

$$=\theta^n\sum_{p=0}^{4^n-1}(-1)^{N(p)}w_{k''}(p/4^n)=\theta^n 2^n(-1)^{N(k'')}.$$

By assumption it is obvious that $\hat{\mu}(k)=o(1)$ as $k\to\infty$. On the other hand, we can prove that

$$\begin{cases} \sum_{p=0}^{4^n-1} |\Delta m_\mu(I_n^p)|^{1+\alpha} = \theta^{(1+\alpha)n}4^n = (4\theta^{1+\alpha})^n \to \infty, \\ \sum_{p=0}^{2\cdot 4^n-1} |\Delta m_\mu(I_n^p)|^{1+\alpha} = 4^n\theta^{(1+\alpha)\alpha} \to \infty, \quad \text{as} \quad n\to\infty. \end{cases}$$

The proof is complete.

Helson [2] solved the problem of Steinhaus (see [1], p. 244—252) and gave a positive answer, that is, *if*

$$\int_0^1 \Big|\sum_{k=-n}^{n} |\hat{\mu}(k)\exp(ikx)|\,dx = O(1) \quad \text{as} \quad n\to\infty,$$

then $\hat{\mu}(k)=o(1)$ *as* $k\to\infty$. Sidon [4] showed that this problem has a negative answer in case of Walsh series. However we can prove the following:

THEOREM 8. *If a Walsh series* μ *satisfies*

$$\int_0^1 \Big|\sum_{k=0}^{n} \hat{\mu}(k)w_k(x)\Big|\,dx = O(1) \quad \text{as} \quad n\to\infty,$$

then m_μ *is a Radon measure and*

$$(1/n)\sum_{k=0}^{n-1}|\hat{\mu}(k)|^2 = O(1/\log n) \quad \text{as} \quad n\to\infty.$$

PROOF. By Olevskii's inequality (see [3] p. 6), we have

$$\Big(\sup_{0\le k\le n}|\hat{\mu}(k)|\Big)\Big(\max_{0\le k\le n}\int_0^1\Big|\sum_{j=0}^{k}\hat{\mu}(j)w_j(x)\Big|\,dx\Big) \ge K\Big(\sum_{j=0}^{n}|\hat{\mu}(j)|^2/n\Big)\log n,$$

where K is a positive constant, from which the conclusion follows.

3. Application to a certain stochastic process. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X(\omega)=\{X_n(\omega)\}_{n=0}^{\infty}$ a sequence of random variables which take the values 0 or 1. Let m_μ be a dyadic measure defined by

$$m_\mu(I) = P\big(\omega\in\Omega;\ X(\omega)\in I\big)$$

for each dyadic interval I.

THEOREM 9. (i) *X is a sequence of independent random variables if and only if*

$$\hat{\mu}(n) = \prod_{j=1}^{s}\hat{\mu}(2^{k_j})$$

for $n = 2^{k_1}+2^{k_2}+\ldots+2^{k_s}, \quad k_1<k_2<\ldots<k_s,$

(ii) *X is a Markov-chain if and only if*

$$\det\begin{pmatrix}\hat\mu(0) & \hat\mu(k)\\ \hat\mu(2^n) & \hat\mu(2^n+k)\end{pmatrix}+\det\begin{pmatrix}\hat\mu(2^{n-1}) & \hat\mu(2^{n-1}+k)\\ \hat\mu(3\cdot 2^{n-1}) & \hat\mu(3\cdot 2^{n-1}+k)\end{pmatrix}=0 \tag{5}$$

for $n=1,2,\dots;\ k=0,1,\dots,2^{n-1}-1$.

PROOF. From the definition of m_μ it follows that

$$\hat\mu(0)=m_\mu(I_0^0)=P(X\in I_0^0)=1.$$

For $\varepsilon=0$ or 1, we have

$$P(X_k=\varepsilon)=\sum_{s=0}^{2^{k-1}-1}P\big(X\in I_k(s/2^{k-1}+\varepsilon/2)\big)=$$

$$=\sum_{s=0}^{2^{k-1}-1}m_\mu\big(I_k(s/2^{k-1}+\varepsilon/2^k)\big)=\sum_{s=0}^{2^{k-1}-1}1/2^k\sum_{j=0}^{2^k-1}\hat\mu(j)w_j(s/2^{k-1}+\varepsilon/2^k)=$$

$$=1/2^k\sum_{j=0}^{2^k-1}\hat\mu(j)w_j(\varepsilon/2^k)\sum_{s=0}^{2^{k-1}-1}w_j(s/2^{k-1})=1/2^k\big(\hat\mu(0)+\hat\mu(2^{k-1})w_{2^{k-1}}(\varepsilon/2^k)\big)2^{k-1}=$$

$$=1/2\big(1+\hat\mu(2^{k-1})w_{2^{k-1}}(\varepsilon/2^k)\big).$$

On the other hand, for each $x=(\varepsilon_1\varepsilon_2\dots\varepsilon_n\dots)$ we have

$$\sum_{k=0}^{2^n-1}\hat\mu(k)w_k(x)=2^n m_\mu\big(I_n(x)\big)=2^nP\big(X\in I_n(x)\big)=$$

$$=2^n\prod_{k=1}^{n}P(X_k=\varepsilon_k)=\prod_{k=1}^{n}\big(1+\hat\mu(2^{k-1})w_{2^{k-1}}(\varepsilon_k/2^k)\big)=$$

$$=\prod_{k=1}^{n}\big(1+\hat\mu(2^{k-1})w_{2^{k-1}}(x)\big),$$

from which (i) follows.

(ii) can be proved in a similar way.

REMARK. A Rademacher series satisfies (5).

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A CLASSIFICATION OF KAEHLERIAN MANIFOLDS SATISFYING SOME SPECIAL CONDITIONS

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1. Introduction

Let us consider an n $(=2m)$ real dimensional Kaehlerian manifold K^n with (g, F) as its Kaehlerian structure. That is, g is a Riemannian metric and F is a complex structure in K^n such that

$$F_t^h F_i^t = -\delta_i^h, \quad F_{ij} = -F_{ji}, \quad F^i{}_{j,k} = 0, \quad F_j^t F_i^s g_{ts} = g_{ji},$$

where g_{ji} and F_i^h are local components of g and F respectively. It is well known that $F_{ji} = F_j^t g_{ti}$ is skew-symmetric.

The holomorphically projective (or H-projective for brevity) curvature tensor $P_{kji}{}^h$ of a Kaehlerian manifold is given by

$$P_{kji}{}^h = R_{kji}{}^h + \frac{1}{n+2}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + S_{ki}F_j{}^h - S_{ji}F_k{}^h + 2S_{kj}F_i^h), \tag{1.1}$$

where R_{kjih}, R_{ji} and R denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively, and $S_{ji} = F_j{}^a R_{ai}$. This tensor corresponds to the Weyl's projective curvature tensor of a Riemannian space and is invariant under any holomorphically projective correspondence.

On the other hand, as a complex analogue to the Weyl's conformal curvature tensor, Bochner [10] (see also Yano and Bochner [6] introduced in a Kaehlerian manifold the curvature tensor

$$\begin{aligned} B_{kji}{}^h = R_{kji}{}^h + \frac{1}{n+4}[R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + g_{ki}R_j{}^h - g_{ji}R_k{}^h + \\ + S_{ki}F_j{}^h - S_{ji}F_k{}^h + F_{ki}S_j{}^h - F_{ji}S_k{}^h + 2S_{kj}F_i^h + 2S_i^h F_{kj}] + \\ + \frac{R}{(n+2)(n+4)}[g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h + F_{ki}F_j{}^h - F_{ji}F_k{}^h + 2F_{kj}F_i^h]. \end{aligned} \tag{1.2}$$

Bochner introduced the curvature tensor using a complex local coordinate system. We call this the Bochner curvature tensor. The form (1.2) of this Bochner curvature tensor with respect to the real coordinate system has been given by Tachibana [9] (see also Yano [5]).

In order to shorten complicated calculations, we put

$$L_{ji} = -\frac{1}{n+4}R_{ji} + \frac{R}{2(n+2)(n+4)}g_{ji}, \tag{1.3}$$

where

$$L_k{}^h = L_{kt}g^{th}, \quad M_{ij} = -L_{jt}F_i^t, \quad M_k^h = M_{kt}g^{th}$$

and

(1.4) $$D_{kji}{}^{h} = \delta_j{}^h L_{ki} - \delta_k{}^h L_{ji} + L_j{}^h g_{ki} - L_k{}^h g_{ji} + F_j{}^h M_{ki} - - F_k{}^h M_{ji} + M_j{}^h F_{ki} - M_k{}^h F_{ji} + 2(F_{kj} M_i^h + M_{kj} F_i^h).$$

Then from (1.2) we have

(1.5) $$B_{kji}{}^{h} = R_{kji}{}^{h} - D_{kji}{}^{h}$$

and

(1.6) $$B_{kji}{}^{h}{}_{,l} = R_{kji}{}^{h}{}_{,l} - D_{kji}{}^{h}{}_{,l}.$$

Moreover, L_{ji} and $D_{kji}{}^h$ satisfy the following relations:

(1.7) $$L = L_{ji} g^{ji} = -\frac{R}{2(n+2)},$$

$$D_{kjih} = D_{kji}{}^{a} g_{ah} = -D_{jkih} = -D_{kjhi} = D_{ihkj}.$$

§2 deals with some properties of Kaehlerian manifolds satisfying $R_{ij,k} = R_{ik,j}$. §3 is concerned with Kaehlerian manifolds with parallel Bochner curvature tensor which are Ricci-recurrent. In §4 a classification of such manifolds whose Ricci tensor is Codazzi is given.

2. Kaehlerian manifolds satisfying $R_{ij,k} = R_{ik,j}$

DEFINITION 2.1. A symmetric tensor T is called a Codazzi tensor if

(2.1) $$T_{ij,k} = T_{ik,j}.$$

REMARK 2.1. In a Kaehlerian manifold whose Ricci-tensor is a Codazzi tensor the scalar curvature is covariant constant.

PROOF. Let R_j be a Codazzi tensor, then we have

(2.2) $$R_{ij,k} = R_{ik,j}.$$

Transvecting with g^{ji} we get

(2.3) $$R_{,k} = 0,$$

i.e. R is a covariant constant.

REMARK 2.2. In a Kaehlerian manifold whose Ricci tensor is a Codazzi tensor the relation

(2.4) $$P_{kji}{}^{a}{}_{,a} = 0$$

is satisfied.

PROOF. From (1.1) it follows that

$$P_{kji}{}^{a}{}_{,a} = \frac{n-2}{n+2}(R_{ji,k} - R_{ki,j}). \tag{2.5}$$

Equations (2.2) and (2.5) yield (2.4).

THEOREM 2.1. *In a Kaehlerian manifold K^n the H-projective curvature tensor satisfies the identity*

$$\begin{aligned} P_{kjih,l} + P_{jlih,k} + P_{lkih,j} - \frac{1}{n+2}[g_{jh}P_{lki}{}^{a}{}_{,a} - g_{kh}P_{lji}{}^{a}{}_{,a} + \\ + g_{lh}P_{kji}{}^{a}{}_{,a} + F_{jh}F_i^a P_{kal}{}^{b}{}_{,b} - F_{kh}F_i^a P_{jal}{}^{b}{}_{,b} + \\ + F_{lh}F_i^a P_{jak}{}^{b}{}_{,b} + F_{ih}(F_k^a P_{laj}{}^{b}{}_{,b} + F_j^a P_{kal}{}^{b}{}_{,b} + F_l^a P_{jak}{}^{b}{}_{,b})] = 0 \end{aligned} \tag{2.6}$$

PROOF. Multiplying (1.1) by g_{ht} and summing for t we have

$$P_{kjih} = R_{kjih} + \frac{1}{n+2}(g_{jh}R_{ki} - g_{kh}R_{ji} + F_{jh}S_{ki} - F_{kh}S_{ji} + 2F_{ih}S_{kj}). \tag{2.7}$$

From (2.7) we obtain

$$\begin{aligned} P_{kjih,l} + P_{jlih,k} + P_{lkih,j} = \frac{1}{n+2}[g_{jh}(R_{ki,l} - R_{li,k}) - \\ - g_{kh}(R_{ji,l} - R_{li,j}) + g_{lh}(R_{ji,k} - R_{ki,j}) + F_{jh}(S_{ki,l} - S_{li,k}) - \\ - F_{kh}(S_{ji,l} - S_{li,j}) + F_{lh}(S_{ji,k} - S_{ki,j}) + 2F_{ih}(S_{kj,l} + S_{jl,k} + S_{lk,j})] = 0. \end{aligned} \tag{2.8}$$

In virtue of (2.5), (2.8) reduces to (2.6).

REMARK 2.3. We shall call (2.6) the Bianchi identities for the H-projective curvature tensor.

In view of Remark 2.2 and Theorem 2.1 we have the following

COROLLARY. *In a Kaehlerian manifold K^n, whose Ricci tensor is a Codazzi tensor we have*

$$P_{kjih,l} + P_{jlih,k} + P_{lkih,j} = 0. \tag{2.9}$$

REMARK 2.4. In a Kaehlerian manifold K^n, whose Ricci tensor is a Codazzi tensor, the tensor B_{kji} vanishes identically (S. Tachibana [9]).

PROOF. The tensor

$$B_{kji} \stackrel{\text{def}}{=} R_{ji,k} - R_{ki,j} + \frac{1}{2(n+2)}[g_{ki}\delta_j^a - g_{ji}\delta_k^a + F_{ki}F_j^a - F_{ji}F_k^a + 2F_{kj}F_i^a]R_{,a}$$

satisfies the identity

$$B_{kji}{}^{a}{}_{,a} = \frac{n}{n+4}B_{kji}. \tag{2.10}$$

Hence the statement.

A Kaehlerian manifold is called symmetric in the sense of Cartan (resp. H-projective symmetric) if the Riemannian curvature tensor (respectively H-projective curvature tensor) is parallel.

A Kaehlerian manifold for which the Bochner curvature tensor $B_{kji}{}^h$ satisfies the relation

$$B_{kji}{}^h{}_{,l} = 0, \tag{2.11}$$

is called a Kaehlerian manifold with parallel Bochner curvature tensor. For the sake of brevity such a manifold shall be called a BS_n-manifold.

The following characterizations of a Kaehlerian manifold satisfying $R_{ij,k} = R_{ik,j}$ are direct consequences of some of the results in [3, 7, 8].

REMARK 2.5. A Kaehlerian manifold which is symmetric in the sense of Cartan is a Kaehlerian manifold whose Ricci tensor is a Codazzi tensor.

REMARK 2.6. A Kaehlerian H-projective symmetric manifold is a Kaehlerian manifold whose Ricci tensor is a Codazzi tensor.

REMARK 2.7. A Kaehlerian manifold with parallel and nonvanishing Bochner curvature tensor is a Kaehlerian manifold whose Ricci tensor is Codazzi tensor.

Moreover, we have

REMARK 2.8. In a Kaehlerian manifold whose Ricci tensor is a Codazzi tensor the relation

$$B_{kjih,l} = R_{kjih,l} \tag{2.12}$$

holds.

3. Ricci-recurrent BS_n-manifolds

A Kaehlerian manifold is called Ricci-recurrent if the Ricci tensor is non-zero and is recurrent. That is, $R_{ij,l} = A_l R_{ij}$ for a non-zero vector A_l.

We shall prove the following

LEMMA 3.1. *A necessary and sufficient condition for R_{ji} to be a recurrent tensor is that L_{ji} be a recurrent tensor.*

PROOF. If we put

$$L_{ji,l} = A_l L_{ji}, \tag{3.1}$$

we get

$$L_{,l} = A_l L \tag{3.2}$$

or equivalently $R_{,l} = A_l R$. Consequently, from (1.3) we have

$$R_{ij,l} = A_l R_{ij}. \tag{3.3}$$

Conversely, if we have (3.3), then (3.1) holds by virtue of (1.3).

Lemmas 3.2 and 3.3 below were proved by A. G. Walker [1], [2].

LEMMA 3.2 (A. G. Walker). *The curvature tensor of a Riemannian space satisfies the identity*

$$R_{kjih,lm} - R_{kjih,ml} + R_{ihlm,kj} - R_{ihlm,jk} + R_{lmkj,ih} - R_{lmkj,hi} = 0, \tag{3.4}$$

where $R_{kjih,lm} = R_{kjih,l,m}$.

LEMMA 3.3 (A. G. Walker). *If $a_{\alpha\beta}$, b_α are numbers satisfying*

$$a_{\alpha\beta} = a_{\beta\alpha}, \quad a_{\alpha\beta} b_\nu + a_{\beta\nu} b_\alpha + a_{\nu\alpha} b_\beta = 0 \tag{3.5}$$

for $\alpha, \beta, \nu = 1, 2, \ldots, N$, then either all the $a_{\alpha\beta}$ are zero, or all the b_α are zero.

Using these lemmas, we can prove the following

THEOREM 3.1. *In a BS_n-manifold, if $R_{ji,l} = A_l R_{ji}$ for a non-zero vector A_l and a non-zero tensor R_{ji} then the recurrence vector A_l is gradient.*

PROOF. If we assume that $R_{ji,l} = A_l R_{ji}$ or equivalently $L_{ji,l} = A_l L_{ji}$, then we have

$$D_{kji}{}^h{}_{,l} = A_l D_{kji}{}^h. \tag{3.6}$$

Consequently, from (1.6) we get

$$R_{kji}{}^h{}_{,l} = A_l D_{kji}{}^h. \tag{3.7}$$

Differentiating (3.7) covariantly and using (3.6) we have

$$R_{kji}{}^h{}_{,lm} = A_{l,m} D_{kji}{}^h + A_l A_m D_{kji}{}^h.$$

Hence

$$R_{kjih,lm} - R_{kjih,ml} = D_{lm} D_{kjih},$$

where $A_{lm} = A_{l,m} - A_{m,l}$. The identity (3.4) now gives

$$A_{lm} D_{kjih} + A_{kj} D_{ihlm} + A_{ih} D_{lmkj} = 0, \tag{3.8}$$

which is of the form (3.5) because $D_{kjih} = D_{ihkj}$, the indices α, β, ν being replaced by kj, ih, lm. Since R_{ij} is a non-zero tensor, D_{kjih} is not all zero. Accordingly, from Lemma 3.3 it follows that $A_{lm} = 0$. Hence $A_{l,m} = A_{m,l}$, which is the condition for A_l to be gradient.

The following theorem can easily be proved.

THEOREM 3.2. *A BS_n-manifold satisfying $R_{ji} = 0$ is symmetric in the sense of Cartan.*

Now we shall prove the following

LEMMA 3.4. *In a Ricci-recurrent BS_n-manifold the relation*

$$(n+4) L_{ma} B_{kji}{}^a + L B_{kjim} = 0 \tag{3.9}$$

holds.

PROOF. Let us assume that a BS_n-manifold satisfies

$$L_{ji,l} = A_l L_{ji} \tag{3.10}$$

for a non-zero vector A_l. If $L_{ji} = 0$, then $R_{ji} = 0$ and therefore, the manifold is symmetric in the sense of Cartan by virtue of Theorem 3.2. So we assume that L_{ji} is a non-zero tensor. Then, from Lemma 3.1 and Theorem 3.1 it follows that the recurrence vector A_l is gradient. Consequently we have

$$L_{ji,m,l} - L_{ji,l,m} = 0. \tag{3.11}$$

When the Ricci identity and (1.5) are applied to (3.11), this expression becomes

(3.12) $$L_{ai}B_{lmj}{}^{a}+L_{ja}B_{lmi}{}^{a}+L_{ai}D_{lmj}{}^{a}+L_{ja}D_{lmi}{}^{a}=0.$$

Differentiating (3.12) covariantly and using (2.11), (3.6) and (3.10), we have

$$A_s(L_{ai}B_{lmj}{}^{a}+L_{ja}B_{lmi}{}^{a})+2A_s(L_{ai}D_{lmj}{}^{a}+L_{ja}D_{lmi}{}^{a})=0$$

from which follows

(3.13) $$L_{ai}B_{lmj}{}^{a}+L_{ja}B_{lmi}{}^{a}+2(L_{ai}D_{lmj}{}^{a}+L_{ja}D_{lmi}{}^{a})=0,$$

because of the A_1 being non-zero. Therefore, from (3.12) and (3.13) we find

(3.14) $$L_{ai}B_{lmj}{}^{a}+L_{ja}B_{lmi}{}^{a}=0,$$

which implies

(3.14)′ $$M_{ai}B_{lmj}{}^{a}-M_{aj}B_{lmi}{}^{a}=0.$$

Transvecting (3.15) with g^{im}, we get

(3.15) $$L^{am}B_{jalm}=0.$$

On the other hand, it follows from (2.11) that

$$B_{kji}{}^{h}{}_{,l,m}-B_{kji}{}^{h}{}_{,m,l}=0.$$

Applying the Ricci identity and (1.5) to this equation we have

(3.16) $$B_{lma}{}^{h}B_{kji}{}^{a}-B_{lmk}{}^{a}B_{aji}{}^{h}-B_{lmj}{}^{a}B_{kai}{}^{h}-B_{lmi}{}^{a}B_{kja}{}^{h}+D_{lma}{}^{h}B_{kji}{}^{a}-$$
$$-D_{lmk}{}^{a}B_{aji}{}^{h}-D_{lmj}{}^{a}B_{kai}{}^{h}-D_{lmi}{}^{a}B_{kja}{}^{h}=0.$$

Differentiating (3.16) covariantly and using (2.11) and (3.6), we have

$$A_s(D_{lma}{}^{h}B_{kji}{}^{a}-D_{lmk}{}^{a}B_{aji}{}^{h}-D_{lmj}{}^{a}B_{kai}{}^{h}-D_{lmi}{}^{h}B_{kja}{}^{h})=0,$$

from which follows

(3.17) $$D_{lma}{}^{h}B_{kji}{}^{a}-D_{lmk}{}^{a}B_{aji}{}^{h}-D_{lmj}{}^{a}B_{kai}{}^{h}-D_{lmi}{}^{a}B_{kja}{}^{h}=0.$$

Contraction with respect to h and l gives

$$D_{lma}{}^{l}B_{kji}{}^{a}-D_{lmk}{}^{a}B_{aji}{}^{l}-D_{lmj}{}^{a}B_{kai}{}^{l}-D_{lmi}{}^{a}B_{kja}{}^{l}=0.$$

After some calculations we have

$$(n+4)L_{ma}B_{kji}{}^{a}-LB_{kjim}+F_m{}^{a}[M_{lk}B_{aji}{}^{l}+M_{lj}B_{kai}{}^{l}+M_{li}B_{kja}{}^{l}]+$$
$$+M_m{}^{a}[F_{lk}B_{aji}{}^{l}+F_{lj}B_{kai}{}^{l}+F_{li}B_{kja}{}^{l}]+2F_{lm}[M_k{}^{a}B_{aji}{}^{l}+M_j{}^{a}B_{kai}{}^{l}+M_i^{a}B_{kja}{}^{l}]+$$
$$+2M_{lm}[F_k{}^{a}B_{aji}{}^{l}+F_j{}^{a}B_{kai}{}^{l}+F_i^{a}B_{kja}{}^{l}]=0.$$

Making some lengthy but straightforward subsequent computations with the help of (3.14), (3.15) and $B_{kjih}+B_{jikh}+B_{ikjh}=0$, we obtain the required relation (3.9).

4. Classification of Kaehlerian manifolds satisfying some special conditions

We shall prove the following main result:

THEOREM 4.1. *In a BS_n-manifold whose Ricci tensor is Codazzi and recurrent, the following cases occur:*

i) *the manifold has vanishing Bochner curvature tensor and is recurrent,*
ii) *the manifold is symmetric in the sense of Cartan and $R_{ij}=0$,*
iii) *the scalar curvature vanishes and the recurrence vector A_l is null.*

PROOF. Making use of

$$R_{ij,k} = R_{ik,j}, \tag{4.1}$$

we have

$$L_{ij} A_k = L_{ik} A_j. \tag{4.2}$$

From (4.2) by transvection with g^{ik} follows

$$A^a L_{aj} = LA_j. \tag{4.3}$$

Transvecting (3.9) with A^m and using (4.3), we get

$$LA^a B_{kjia} = 0. \tag{4.4}$$

Hence, either

$$A^a B_{kjia} = 0, \tag{4.5}$$

or

$$L = 0 \quad \text{or equivalently} \quad R = 0. \tag{4.6}$$

We first consider the case when (4.5) holds. Transvecting (4.2) with $B_{hlm}{}^j$ and using (4.5) we have $A_k L_{ia} B_{hlm}{}^a=0$, and consequently

$$L_{ia} B_{hlm}{}^a = 0, \tag{4.7}$$

substituting (4.7) in (3.9), we obtain $LB_{kjim}=0$. Hence, we find either $L=0$ or $B_{jkim}=0$.

We consider the case when $B_{kji}{}^h=0$. In general, when a BS_n-manifold satisfies (3.10) we have

$$R_{kji}{}^h{}_{,l} + A_l(B_{kji}{}^h - R_{kji}{}^h) = 0. \tag{4.8}$$

Substituting $B_{kji}{}^h=0$, we find $R_{kji}{}^h{}_{,1}=A_1 R_{kji}{}^h$, that is, the manifold is recurrent.

Next, we consider the case when (4.6) holds. In this case from (4.3) we have

$$A^a R_{ja} = 0. \tag{4.9}$$

From (4.1) we obtain

$$R_{ij} A_k = R_{ik} A_j,$$

which on transvection by A^k reduces to

$$A^k A_k R_{ji} - A_j A^k R_{ki} = 0.$$

Using (4.9) we get $A^k A_k R_{ji}=0$, which implies

$$A^k A_k L_{ji} = 0. \tag{4.10}$$

Since L_{ji} is a non-zero tensor by our assumption, from (4.10) we find that A_k is a null vector. This completes the proof of the theorem.

COROLLARY 4.1. *A BS_n-manifold with non-vanishing scalar curvature whose Ricci tensor is Codazzi and recurrent is a Kaehlerian manifold with vanishing Bochner curvature tensor and recurrent.*

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MATRIX TRANSFORMATIONS OF CESÀRO SUMMABLE SERIES

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1. In a recent paper [1] Alpár gave necessary and sufficient conditions for an infinite matrix A to transform every (C, α) summable series into a (C, β) summable series where α and β are any non-negative real numbers. Now Cesàro summability (C, α) is usually considered for the range $\alpha \geqq -1$ (see [4]) and it is the object of this note to extend Alpár's results to the cases $\alpha, \beta \geqq -1$. Since summability $(C, -1)$ requires a separate definition, we have to consider the cases $\alpha = -1$, $-1 < \alpha \leqq 0$ separately and our proof uses the functional analytic properties of these spaces of Cesàro summable series.

2. We shall basically use the notation of [1] and in this section we give the extra notation and definitions needed.

We say $\sum_{n=0}^{\infty} a_n = s$ $(C, -1)$ if the series to sequence transformation

$$t_n = \sum_{k=0}^{n-1} a_k + (n+1) a_n \tag{1}$$

tends to s as n tends to infinity. Thus the infinite matrix associated with the $(C, -1)$ transformation is a normal matrix i.e. a lower triangular matrix with all terms on the main diagonal being non-zero.

The convergence domain of an infinite matrix A is defined to be

$$c_A = \{\mathbf{x} = (x_n)_{n \geqq 0} \,|\, t_n = \sum_{k=0}^{\infty} a_{nk} x_k \to \text{limit as } n \to \infty\}.$$

It has a unique topology under which it is an FK space i.e. a complete, metrisable, locally convex space of sequences with continuous coordinate projections (see [5], [6]). In the special case $A = (C, -1)$ we shall use the notation c_{-1} and note that the topology on c_{-1} is given by the norm

$$\|\mathbf{a}\| = \sup_{n \geqq 0} \Big| \sum_{k=0}^{n-1} a_k + (n+1) a_n \Big| \tag{2}$$

for $\mathbf{a} = (a_n)_{n \geqq 0} \in c_{-1}$. For $A = (C, \alpha)$ where $\alpha > -1$ we use c_α instead of c_A and its norm is given by

$$\|\mathbf{a}\| = \sup_{n \geqq 0} \left| \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha} a_k}{A_n^{\alpha}} \right| \tag{3}$$

for $\mathbf{a} = (a_n)_{n \geqq 0} \in c_\alpha$.

If δ^n denotes the sequence with 1 in the n^{th} coordinate and 0's elsewhere then we say that the FK space X has AK if $\{\delta^n\}_{n\geqq 0}$ forms a Schauder basis for X i.e. for every $\mathbf{x}=(x_n)_{n\geqq 0}\in X$ we have $\mathbf{x}=\sum_{n=0}^{\infty} x_n\delta^n$ where the convergence is in the topology of X (see [7]). We make the remark (though we shall not need it) that for $\alpha>0$, c_α does not have AK whereas we shall prove in the next section that, for $-1\leqq\alpha\leqq 0$ c_α does have AK and this turns out to be crucial to our proof.

3. From (1) it is not difficult to show that $\mathbf{a}\in c_{-1}$ if and only if $\sum_{n=0}^{\infty} a_n$ is convergent and $na_n=o(1)$. Using this and (2) it is easy to verify that c_{-1} has AK. In the case $-1<\alpha\leqq 0$ if we use (3) then the main result in [2] proves directly that c_α has AK (though without using this language.) We shall need a representation of the space of continuous linear functionals on c_α $(-1\leqq\alpha\leqq 0)$ and we give this in the next two theorems.

THEOREM 1. *Let* $-1<\alpha\leqq 0$ *and suppose that f is a continuous linear functional defined on* c_α. *Then there exists a constant* $\Lambda\in\mathbf{C}$ *and a sequence* $\boldsymbol{\lambda}=(\lambda_n)_{n\geqq 0}\in l$ *(the space of absolutely convergent series) such that*

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} x_n \left(\Lambda+\sum_{k=n}^{\infty}\frac{A_{k-n}^{\alpha}\lambda_k}{A_k^{\alpha}}\right) \quad (\mathbf{x}\in c_\alpha) \tag{4}$$

and

$$\|f\| = |\Lambda|+\sum_{n=0}^{\infty}|\lambda_n|. \tag{5}$$

Conversely, given $\Lambda\in\mathbf{C}$ *and* $\boldsymbol{\lambda}\in l$ *there exists a continuous linear functional f defined on* c_α *such that* (4) *and* (5) *hold.*

PROOF. We first observe that (3) defines an isometry between c_α and c, the space of convergent sequences taken with the uniform norm; we can use this to obtain an isometry between their dual spaces and then use the known representation of the continuous dual of c (as in [5]) to obtain (4) and (5).

Hence, if $\mathbf{x}\in c_\alpha$ then $\mathbf{t}=(C,\alpha)\mathbf{x}$ belongs to c where for $n\geqq 0$ $t_n=\sum_{k=0}^{n}\frac{A_{n-k}^{\alpha}x_k}{A_n^{\alpha}}$ and $\|\mathbf{t}\|_c=\|\mathbf{x}\|_{c_\alpha}$ from (3).

Conversely, given $\mathbf{t}\in c$ there is an $\mathbf{x}\in c_\alpha$ such that these relations still hold (by just defining $\mathbf{x}$ to be the inverse (C,α) transform of $\mathbf{t}$ i.e. $x_n=\sum_{k=0}^{n}A_{n-k}^{-\alpha-2}A_k^{\alpha}t_k$ for $n\geqq 0$). Thus $f\in c_\alpha^*$ (the continuous dual of c_α) if and only if $F\in c^*$ where for all $\mathbf{x}\in c_\alpha$

$$f(\mathbf{x}) = F\big((C,\alpha)\mathbf{x}\big) = F(\mathbf{t})$$

and moreover

$$\|f\| = \sup_{\|\mathbf{x}\|_{c_\alpha}=1}|f(\mathbf{x})| = \sup_{\|\mathbf{t}\|_c=1}|F(t)| = \|F\|.$$

We know from [5] that $F\in c^*$ if and only if there exists $\Lambda\in\mathbf{C}$, $\boldsymbol{\lambda}\in l$ such that for all $\mathbf{t}\in c$

$$F(\mathbf{t}) = \Lambda \lim_{n\to\infty} t_n + \sum_{n=0}^{\infty} \lambda_n t_n$$

and $\|F\|=|\Lambda|+\sum_{n=0}^{\infty}|\lambda_n|$. Thus, if $f\in c_\alpha^*$ then for all $\mathbf{x}\in c_\alpha$ we have

$$f(\mathbf{x}) = \Lambda \lim_{n\to\infty} t_n + \sum_{n=0}^{\infty} \lambda_n \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha} x_k}{A_n^{\alpha}}$$

and (5) holds. Moreover, since $\alpha\leqq 0$, every (C, α) summable series is convergent so that $\lim_{n\to\infty} t_n=\sum_{k=0}^{\infty} x_k$, and we get

$$f(\mathbf{x}) = \Lambda \sum_{k=0}^{\infty} x_k + \sum_{n=0}^{\infty} \lambda_n \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha} x_k}{A_n^{\alpha}}. \tag{6}$$

To obtain (4) we have to prove that we can interchange the order of summation in the second sum in (6). We write this sum as

$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{k=0}^{n} (\dots) = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{n=k}^{N} (\dots) = \lim_{N\to\infty} \sum_{k=0}^{N} \Big\{\sum_{n=k}^{\infty} - \sum_{n=N+1}^{\infty}\Big\}(\dots)$$

since we know $\sum_{n=k}^{\infty} \frac{A_{n-k}^{\alpha}\lambda_n}{A_n^{\alpha}}$ is convergent as $\boldsymbol{\lambda}\in l$. Thus it remains to show that

$$\lim_{N\to\infty} \sum_{k=0}^{N} x_k \sum_{n=N+1}^{\infty} \frac{A_{n-k}^{\alpha}\lambda_n}{A_n^{\alpha}} = 0. \tag{7}$$

But

$$\sum_{k=0}^{N} x_k \sum_{n=N+1}^{\infty} \frac{A_{n-k}^{\alpha}\lambda_n}{A_n^{\alpha}} = \sum_{n=N+1}^{\infty} \lambda_n \sum_{k=0}^{N} \frac{A_{n-k}^{\alpha} x_k}{A_n^{\alpha}}$$

and so by an inequality known to Anderson and Bosanquet (see page 35 of [3]) since $-1<\alpha\leqq 0$ and $\mathbf{x}\in c_\alpha$

$$\left|\sum_{k=0}^{N} \frac{A_{n-k}^{\alpha} x_k}{A_n^{\alpha}}\right| \leqq \max_{0\leqq k\leqq N} |t_k| = O(1) \quad \text{as} \quad N\to\infty \tag{8}$$

and hence (7) holds since $\boldsymbol{\lambda}\in l$.

The converse follows similarly (again using the inequality in (8)).

COROLLARY 1. *Suppose that* $(\varepsilon_n)_{n\geqq 0}$ *is a sequence with the property that for every* $\mathbf{x}\in c_\alpha$ $(-1<\alpha\leqq 0)$, $\sum_{n=0}^{\infty} x_n\varepsilon_n$ *is convergent. Then its sum* $g(\mathbf{x})=\sum_{n=0}^{\infty} x_n\varepsilon_n$ *defines a continuous linear functional* g *on* c_α *with*

$$\|g\| = \Big|\varepsilon_0 - \sum_{n=0}^{\infty} A_n^{\alpha} \Delta^{\alpha+1}\varepsilon_n\Big| + \sum_{n=0}^{\infty} A_n^{\alpha} |\Delta^{\alpha+1}\varepsilon_n| \tag{9}$$

where $\Delta^{\alpha+1}\varepsilon_n$ *denotes the fractional difference* $\sum_{k=n}^{\infty} A_{k-n}^{-\alpha-2}\varepsilon_k$ *and* (9) *is taken as including the assertion that this series converges.*

PROOF. Clearly $g\in c_\alpha^*$ by the Banach—Steinhaus theorem since the projection mappings $\mathbf{x}\mapsto x_n$ (n fixed) are continuous and c_α is a Banach space. Suppose $\Lambda\in\mathbf{C}$ and $\boldsymbol{\lambda}\in l$ are such that

$$g(\mathbf{x}) = \sum_{n=0}^{\infty} x_n\varepsilon_n = \sum_{n=0}^{\infty} x_n\left(\Lambda+\sum_{k=n}^{\infty}\frac{A_{k-n}^{\alpha}\lambda_k}{A_k^{\alpha}}\right) \tag{10}$$

so that

$$\varepsilon_n = \Lambda+\sum_{k=n}^{\infty}\frac{A_{k-n}^{\alpha}\lambda_k}{A_k^{\alpha}}.$$

In terms of fractional differences this can be written

$$\varepsilon_n = \Lambda+\Delta^{-\alpha-1}\left(\frac{\lambda_n}{A_n^{\alpha}}\right).$$

By a result of Andersen (see page 36 of [3]), since $-1<\alpha\leqq 0$ we can solve this difference equation for $\boldsymbol{\lambda}$ to get

$$\Delta^{\alpha+1}\varepsilon_n = \Delta^{\alpha+1}\left(\Lambda+\Delta^{-\alpha-1}\left(\frac{\lambda_n}{A_n^{\alpha}}\right)\right) = \frac{\lambda_n}{A_n^{\alpha}}$$

so that $\lambda_n = A_n^{\alpha}\Delta^{\alpha+1}\varepsilon_n$. Also, from (10),

$$g(\boldsymbol{\delta}^0) = \varepsilon_0 = \Lambda+\sum_{k=0}^{\infty}\lambda_k$$

and hence (9) follows from (5).

THEOREM 2. *Suppose that f is a continuous linear functional defined on* c_{-1}. *Then there exists a constant* $\Lambda\in\mathbf{C}$ *and a sequence* $\boldsymbol{\lambda}=(\lambda_n)_{n\geqq 0}\in l$ *such that*

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} x_n\Big(\Lambda+n\lambda_n+\sum_{k=n}^{\infty}\lambda_k\Big)\quad(\mathbf{x}\in c_{-1}) \tag{11}$$

and (5) *holds.*

Conversely, given $\Lambda\in\mathbf{C}$ *and* $\boldsymbol{\lambda}\in l$ *there exists a continuous linear functional defined on* c_{-1} *such that* (11) *and* (5) *hold.*

PROOF. This follows the lines of proof of Theorem 1 (although it is simpler since no inequality like (8) is necessary).

COROLLARY 2. *Suppose that* $(\varepsilon_n)_{n\geqq 0}$ *is a sequence with the property that for every* $\mathbf{x}\in c_{-1}$, $\sum_{n=0}^{\infty} x_n\varepsilon_n$ *is convergent. Then its sum* $g(\mathbf{x})$ *defines a continuous linear functional* g *on* c_{-1} *with*

$$\|g\| = \left|\varepsilon_0-\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{\Delta\varepsilon_k}{k+1}\right|+\sum_{n=0}^{\infty}\left|\sum_{k=n}^{\infty}\frac{\Delta\varepsilon_k}{k+1}\right| \tag{12}$$

and (12) *is taken as including the assertion that* $\sum_{k=n}^{\infty}\frac{\Delta\varepsilon_k}{k+1}$ *is convergent.*

PROOF. Similar to the proof of Corollary 1. We make the remark that the converse to both corollaries was proved in [3] i.e. the finiteness of the sums in (9) and (12) ensure that $(\varepsilon_n)_{n\geqq 0}$ is a convergence factor for c_α $(-1<\alpha\leqq 0)$ and c_{-1} respectively.

4. In order to classify those matrices A that transform c_α into c_β where $\alpha,\beta\geqq -1$ we prove a more general result that gives necessary and sufficient conditions on the matrix A to transform c_α into c_B where $\alpha\geqq -1$ and B is *any* normal matrix. The special case $B=(C,\beta)$ where $\beta\geqq -1$ then gives the required extension of Alpár's results. We have to consider the cases $\alpha>0$, $-1<\alpha\leqq 0$, $\alpha=-1$ separately.

THEOREM 3. *Let* $\alpha>0$ *and* B *be a normal matrix. In order that*

$$y_n=\sum_{k=0}^{\infty} a_{nk}x_k \tag{13}$$

should exist for each $n\geqq 0$ *and that* $\mathbf{y}\in c_B$ *for every* $\mathbf{x}\in c_\alpha$ *it is necessary and sufficient that the following conditions hold:*

(I) *for each fixed* $n=0,1,\ldots,$ $a_{nk}=O(k^{-\alpha})$ *as* $k\to\infty$,

(II) *for each fixed* $k=0,1,\ldots$ *there exists* $a_k\in\mathbf{C}$ *such that*

$$\sum_{n=0}^{\infty} a_{nk}=a_k\ (B),$$

(III) $\sup_{n\geqq 0}\sum_{k=0}^{\infty} A_k^\alpha\Big|\sum_{r=0}^{\infty} b_{nr}\underset{k}{\Delta}^{\alpha+1}a_{rk}\Big|<\infty$ where $\underset{k}{\Delta}^{\alpha+1}a_{rk}=\sum_{s=k}^{\infty}A_{s-k}^{-\alpha-2}a_{rs}$ *(which converges by* (I)*).*

Moreover, if (I), (II) *and* (III) *hold and* $\mathbf{t}=(C,\alpha)\mathbf{x}$ *then*

$$\lim_{n\to\infty}\sum_{k=0}^{n} b_{nk}y_k=\Big(a_0-\sum_{k=0}^{\infty}A_k^\alpha\Delta^{\alpha+1}a_k\Big)\lim_{n\to\infty}t_n+\sum_{k=0}^{\infty}t_kA_k^\alpha\Delta^{\alpha+1}a_k \tag{14}$$

PROOF. This is modelled closely on Alpár's proof of Theorem 1 in [1]. We first prove the sufficiency of the conditions. Suppose that (I), (II) and (III) hold. From the convergence of the sum in (III) for each fixed n, we show that

$$\sum_{k=0}^{\infty}A_k^\alpha\,|\underset{k}{\Delta}^{\alpha+1}a_{nk}|<\infty \tag{15}$$

for each fixed n. In order to do this we write[1]

$$u_{nk}=\sum_{r=0}^{n} b_{nr}\underset{k}{\Delta}^{\alpha+1}a_{rk}$$

and let (b_{nr}^{-1}) denote the reciprocal matrix of B so that

$$\sum_{k=0}^{\infty}A_k^\alpha\,|\underset{k}{\Delta}^{\alpha+1}a_{nk}|\leqq\sum_{r=0}^{n}|b_{nr}^{-1}|\sum_{k=0}^{\infty}A_k^\alpha|u_{rk}|<\infty$$

by (III).

[1] This was suggested to me by Professor B. Kuttner.

Now, using (15) and (I) we see that the hypotheses of Theorem A—B on page 236 of [1] hold and so (13) exists for each $\mathbf{x}\in c_\alpha$ and since $\alpha>0$,

$$y_n=\sum_{k=0}^{\infty} a_{nk}x_k=\sum_{k=0}^{\infty} t_k A_k^\alpha \underset{k}{\Delta}^{\alpha+1} a_{nk}$$

where $\mathbf{t}=(C,\alpha)\mathbf{x}$.

Hence

$$v_n=\sum_{r=0}^{n} b_{nr}y_r=\sum_{k=0}^{\infty} t_k A_k^\alpha \sum_{r=0}^{n} b_{nr}\underset{k}{\Delta}^{\alpha+1} a_{rk} \tag{16}$$

and to complete the proof of the sufficiency we have to show that the sequence to sequence transformation from $(t_k)_{k\geqq 0}$ to $(v_n)_{n\geqq 0}$ is conservative. Since (III) is assumed to hold it remains to prove that

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} A_k^\alpha \sum_{r=0}^{n} b_{nr}\underset{k}{\Delta}^{\alpha+1} a_{rk} \quad \text{exists} \tag{17}$$

and that for $k=0,1,\ldots$

$$\lim_{n\to\infty}\sum_{r=0}^{n} b_{nr}\underset{k}{\Delta}^{\alpha+1} a_{rk} \quad \text{exists.} \tag{18}$$

To prove (17) we observe that

$$\sum_{k=0}^{\infty} A_k^\alpha \sum_{r=0}^{n} b_{nr}\underset{k}{\Delta}^{\alpha+1} a_{rk}=\sum_{r=0}^{n} b_{nr}\sum_{k=0}^{\infty} A_k^\alpha \underset{k}{\Delta}^{\alpha+1} a_{rk}$$

and using Lemma A on page 236 of [1] (since (I) and (15) hold) this simplifies to $\sum_{r=0}^{n} b_{nr}a_{r0}$ which tends to a_0 as n tends to ∞ by (II).

In order to prove (18) we first observe that for each fixed $n\geqq 0$,

$$\sum_{r=0}^{n} b_{nr}a_{rk}=O(k^{-\alpha}) \quad \text{as} \quad k\to\infty,$$

and since (III) can also be written as

$$\sup_{n\geqq 0}\sum_{k=0}^{\infty} A_k^\alpha \Big|\underset{k}{\Delta}^{\alpha+1}\Big(\sum_{r=0}^{n} b_{nr}a_{rk}\Big)\Big|<\infty, \tag{19}$$

Lemma A in [1] implies that

$$\sum_{r=0}^{n} b_{nr}a_{rk}=\sum_{s=k}^{\infty} A_{s-k}^\alpha \underset{s}{\Delta}^{\alpha+1}\Big(\sum_{r=0}^{n} b_{nr}a_{rs}\Big).$$

Hence, by (19) we get that

$$\sup_{n,k}\Big|\sum_{r=0}^{n} b_{nr}a_{rk}\Big|<\infty. \tag{20}$$

(since $A_{s-k}^{\alpha} \leqq A_s^{\alpha}$ for $s \geqq k$). Now the sum in (18) can be re-written as

$$\sum_{s=k}^{\infty} A_{s-k}^{-\alpha-2} \sum_{r=0}^{n} b_{nr} a_{rs} \tag{21}$$

and so (18) will be proved if we can show that for each fixed $k \geqq 0$, the series in (21) is uniformly convergent with respect to n, since (18) will then equal

$$\sum_{s=k}^{\infty} A_{s-k}^{-\alpha-2} \big(\lim_{n\to\infty} \sum_{r=0}^{n} b_{nr} a_{rs}\big) = \Delta^{\alpha+1} a_k$$

by (II). But the uniform convergence of (21) follows since

$$\sup_{n \geqq 0} \Big| \sum_{s=K}^{\infty} A_{s-k}^{-\alpha-2} \sum_{r=0}^{n} b_{nr} a_{rs} \Big| \leqq \sup_{n,s} \Big| \sum_{r=0}^{n} b_{nr} a_{rs} \Big| \sum_{r=K}^{\infty} |A_{s-k}^{-\alpha-2}| \to 0 \quad \text{as} \quad K \to \infty$$

by (20).

Hence the sufficiency part is proved and (14) follows from the standard results on conservative transformations (see [4]).

For the necessity of the conditions we first observe that for each $k \geqq 0$ $\boldsymbol{\delta}^k \in c_\alpha$ so that $A\boldsymbol{\delta}^k \in c_B$ and so (II) must hold. Also since (13) is assumed to hold we can apply Theorem (A—B) of [1] to deduce that (I) and (16) must hold. The necessity of (III) follows from the fact that (16) is necessarily a conservative sequence to sequence transformation.

COROLLARY 3. *In order that* $A\colon c_\alpha \to c_B$ *(where* $\alpha > 0$*) with the limits agreeing, it is necessary and sufficient that, further to* (I), (II) *and* (III) *we require that* $a_k = 1$ *for all* $k \geqq 0$.

PROOF. Clearly, if $a_k = 1$ for all $k \geqq 0$ then $\Delta^{\alpha+1} a_k = 0$ and (14) shows that in this case A preserves limits.

Conversely, if A preserves limits, then by taking $\mathbf{x} = \boldsymbol{\delta}^k$ in (13) we see that $a_k = 1$ from (II) and this proves the result.

THEOREM 4. *Let* $-1 < \alpha \leqq 0$ *and B be a normal matrix. In order that* (13) *should exist for each* $n \geqq 0$ *and that* $\mathbf{y} \in c_B$ *for every* $\mathbf{x} \in c_\alpha$ *it is necessary and sufficient that* (II) *and* (III) *hold, where we assume that* (III) *includes the assertion that the series defining* $\underset{k}{\Delta}^{\alpha+1} a_{rk}$ *is convergent. Moreover if* (II) *and* (III) *hold then so does* (14).

PROOF. Assume that (II) and (III) hold. Then (15) holds by the same argument as in the proof of Theorem 3. By Theorems 4 and 5 of [3] we see that (15) implies that (13) exists for each $n \geqq 0$ and each $\mathbf{x} \in c_\alpha$. Hence, by the corollary to Theorem 1, if we define

$$g_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$$

for $\mathbf{x} \in c_\alpha$ and $n \geqq 0$, then $g_n \in c_\alpha^*$ and so

$$B_n(\mathbf{x}) = \sum_{r=0}^{n} b_{nr} g_r(\mathbf{x}) = \sum_{k=0}^{\infty} x_k \sum_{r=0}^{n} b_{nr} a_{rk} \tag{22}$$

is also a continuous linear functional on c_α with

$$(23)\qquad \|B_n\| = \Big|\sum_{r=0}^{n} b_{nr}a_{r0} - \sum_{k=0}^{\infty} A_k^\alpha \underset{k}{\Delta}^{\alpha+1}\Big(\sum_{r=0}^{n} b_{nr}a_{rk}\Big)\Big| + \sum_{k=0}^{\infty} A_k^\alpha \Big|\underset{k}{\Delta}^{\alpha+1}\Big(\sum_{r=0}^{n} b_{nr}a_{rk}\Big)\Big|$$

from (9). Now (II) and (III) imply that $\{B_n\}_{n\ge 0}$ is a uniformly bounded sequence of continuous linear functionals on c_α; (note that we only need (II) for the case $k=0$ and that in (III)

$$\sum_{r=0}^{n} b_{nr}\underset{k}{\Delta}^{\alpha+1}a_{rk} = \underset{k}{\Delta}^{\alpha+1}\Big(\sum_{r=0}^{n} b_{nr}a_{rk}\Big)$$

since (III) includes the assertion that $\underset{k}{\Delta}^{\alpha+1}a_{rk}$ exists.) Also (II) and (22) imply that for each fixed $k\ge 0$

$$\lim_{n\to\infty} B_n(\boldsymbol{\delta}^k) = \lim_{n\to\infty}\sum_{r=0}^{n} b_{nr}a_{rk} = a_k.$$

But by the remarks at the beginning of Section 3, c_α has AK so that in particular the closed linear span of $\{\boldsymbol{\delta}^k\colon k=0,1,2,\ldots\}$ is c_α, and since $\{B_n\}_{n\ge 0}$ is a uniformly bounded sequence of continuous linear functionals on c_α with $\lim_{n\to\infty} B_n(\mathbf{x})$ existing on a dense subspace of c_α, it must exist for all $\mathbf{x}\in c_\alpha$. Hence (II) and (III) are sufficient for $A\colon c_\alpha\to c_B$. In order to show that (14) also holds we observe that by using the proof of Corollary 1, we have, from (22), for each fixed $n\ge 0$ a constant $\Lambda_n\in\mathbf{C}$ and a sequence $(\lambda_{nk})_{k\ge 0}\in l$ such that for every $\mathbf{x}\in c_\alpha$,

$$(24)\qquad B_n(\mathbf{x}) = \sum_{k=0}^{\infty} x_k\left(\Lambda_n + \sum_{s=k}^{\infty}\frac{A_{s-k}^\alpha\lambda_{ns}}{A_s^\alpha}\right)$$

where

$$\Lambda_n = \sum_{r=0}^{n} b_{nr}a_{r0} - \sum_{k=0}^{\infty}\lambda_{nk}$$

and

$$\lambda_{nk} = A_k^\alpha\underset{k}{\Delta}^{\alpha+1}\Big(\sum_{r=0}^{n} b_{nr}a_{rk}\Big).$$

Moreover, if $\mathbf{t}=(C,\alpha)\mathbf{x}$, then we can use (8) to show (24) is equal to

$$B_n(\mathbf{x}) = \Lambda_n\lim_{k\to\infty} t_k + \sum_{s=0}^{\infty}\lambda_{ns}t_s$$

which can be written as (since $(\lambda_{ns})_{s\ge 0}\in l$)

$$B_n(\mathbf{x}) = \sum_{r=0}^{\infty} b_{nr}a_{r0}\lim_{k\to\infty} t_k + \sum_{s=0}^{\infty}\lambda_{ns}\big(t_s - \lim_{k\to\infty} t_k\big).$$

Since $\lim_{n\to\infty} B_n(\mathbf{x})$ exists for every $\mathbf{x}\in c_\alpha$, this means that (λ_{ns}) transforms every convergent to zero sequence into a convergent sequence and that

$$(25)\qquad \lim_{n\to\infty} B_n(\mathbf{x}) = a_0\lim_{k\to\infty} t_k + \sum_{s=0}^{\infty}\lambda_s\big(t_s - \lim_{k\to\infty} t_k\big)$$

where $\lambda_s=\lim_{n\to\infty}\lambda_{ns}$. Also $\boldsymbol{\lambda}\in l$ (since $\sup_{n\geqq 0}\sum_{s=0}^{\infty}|\lambda_{ns}|<\infty$), so that (25) can be re-arranged to

$$\lim_{n\to\infty} B_n(\mathbf{x}) = \Big(a_0-\sum_{s=0}^{\infty}\lambda_s\Big)\lim_{k\to\infty} t_k+\sum_{s=0}^{\infty}\lambda_s t_s.$$

Hence (14) will be proved provided we show that, for each fixed $k\geqq 0$,

$$\lim_{n\to\infty} \underset{k}{\Delta}^{\alpha+1}\Big(\sum_{r=0}^{n} b_{nr}a_{rk}\Big) = \underset{k}{\Delta}^{\alpha+1}a_k. \tag{26}$$

Using (II), a sufficient condition for this to be true is that for each fixed $k\geqq 0$, the series in (21) is uniformly convergent with respect to n.

We can prove uniform convergence by using the fact that c_α has AK (since $-1<\alpha\leqq 0$) and that $(A_{s-k}^{-\alpha-2})_{s\geqq 0}\in c_\alpha$ for each fixed $k\geqq 0$. For, using (22),

$$\sup_{n\geqq 0}\Big|\sum_{s=K}^{\infty}A_{s-k}^{-\alpha-2}\sum_{r=0}^{n}b_{nr}a_{rs}\Big| = \sup_{n\geqq 0}\Big|B_n\Big(\sum_{s=K}^{\infty}A_{s-k}^{-\alpha-2}\boldsymbol{\delta}^s\Big|\leqq$$

$$\leqq\big(\sup_{n\geqq 0}\|B_n\|\big)\cdot\Big\|\sum_{s=K}^{\infty}A_{s-k}^{-\alpha-2}\boldsymbol{\delta}^s\Big\|_{c_\alpha}\to 0 \quad\text{as}\quad K\to\infty$$

since $(B_n)_{n\geqq 0}$ is uniformly bounded and c_α has AK. Thus (14) follows from (II) and (III).

In order to prove the conditions necessary, (II) must hold as in the proof of Theorem 3. Also, for every $\mathbf{x}\in c_\alpha$, (22) holds and since $(B_n(\mathbf{x}))_{n\geqq 0}$ is convergent, $(B_n)_{n\geqq 0}$ is a pointwise bounded system of continuous linear functionals on the Banach space c_α and hence is uniformly bounded and thus (23) implies that (III) holds.

COROLLARY 4. *In order that* $A\colon c_\alpha\to c_B$ *(where* $-1<\alpha\leqq 0$*) with the limits agreeing, it is necessary and sufficient that, further to* (II) *and* (III) *we require that* $a_k=1$ *for all* $k\geqq 0$.

PROOF. The same as Corollary 3.

THEOREM 5. *Let B be a normal matrix. In order that* (13) *should exist for each* $n\geqq 0$ *and that* $\mathbf{y}\in c_B$ *for every* $\mathbf{x}\in c_{-1}$ *it is necessary and sufficient that* (II) *and* (IV) *hold where*

$$\sup_{n\geqq 0}\sum_{k=0}^{\infty}\Big|\sum_{r=0}^{n}b_{nr}\sum_{s=k}^{\infty}\frac{a_{rs}-a_{r,s+1}}{s+1}\Big|<\infty \tag{IV}$$

and (IV) *is taken as including the assertion that* $\sum_{s=k}^{\infty}\frac{a_{rs}-a_{r,s+1}}{s+1}$ *is convergent. Moreover, if* (II) *and* (IV) *hold and* $\mathbf{t}=(C,-1)\mathbf{x}$ *then*

$$\lim_{n\to\infty}\sum_{k=0}^{n}b_{nk}y_k = \Big(a_0-\sum_{k=0}^{\infty}\sum_{s=k}^{\infty}\frac{\Delta a_s}{s+1}\Big)\lim_{n\to\infty}t_n+\sum_{k=0}^{\infty}t_k\sum_{s=k}^{\infty}\frac{\Delta a_s}{s+1}. \tag{27}$$

PROOF. This follows the lines of the proof of Theorem 4 and only the significant changes in detail will be mentioned. Assume that (II) and (IV) hold. From (IV) we deduce that for each $n\geqq 0$,

$$\sum_{k=0}^{\infty}\left|\sum_{r=k}^{\infty}\frac{a_{nr}-a_{n,r+1}}{r+1}\right|<\infty \tag{28}$$

(just as (15) was deduced from (III)). By Lemmas 7 and 7(a) of [3], (28) implies that (13) exists for each $n\geqq 0$ and each $\mathbf{x}\in c_{-1}$ and hence that (22) defines a continuous linear functional on c_{-1} with

$$\|B_n\| = \left|\sum_{r=0}^{n} b_{nr}a_{r0}-\sum_{k=0}^{\infty}\sum_{r=0}^{n} b_{nr}\sum_{s=k}^{\infty}\frac{a_{rs}-a_{r,s+1}}{s+1}\right|+\sum_{k=0}^{\infty}\left|\sum_{r=0}^{n} b_{nr}\sum_{s=k}^{\infty}\frac{a_{rs}-a_{r,s+1}}{s+1}\right| \tag{29}$$

from (12). The sufficiency now follows as before since c_{-1} also has AK.

To see that (27) also holds, we repeat the argument used in the previous proof, using Corollary 2 instead of Corollary 1, to get that (27) will hold provided we show that, for each fixed $k\geqq 0$,

$$\lim_{n\to\infty}\sum_{r=0}^{n} b_{nr}\sum_{s=k}^{\infty}\frac{a_{rs}-a_{r,s+1}}{s+1}=\sum_{s=k}^{\infty}\frac{\Delta a_s}{s+1}. \tag{30}$$

Using summation by parts and the fact that for each fixed $r\geqq 0$, $a_{rs}=o(s)$, (see p. 38 of [3]) the inner sum of (30) gives

$$\sum_{s=k}^{\infty}\frac{a_{rs}-a_{r,s+1}}{s+1}=\frac{a_{rk}}{k+1}+\sum_{s=k+1}^{\infty}a_{rs}\left(\frac{1}{s+1}-\frac{1}{s}\right), \tag{31}$$

and (30) will follow from

$$\lim_{n\to\infty}\sum_{r=0}^{n} b_{nr}\sum_{s=k+1}^{\infty}a_{rs}\left(\frac{1}{s+1}-\frac{1}{s}\right)=\sum_{s=k+1}^{\infty}a_s\left(\frac{1}{s+1}-\frac{1}{s}\right) \tag{32}$$

on summing the right hand side by parts as in (31) (we shall prove that $a_s=o(s)$). To obtain (32), we prove that for each fixed $k\geqq 0$, the series

$$\sum_{s=k+1}^{\infty}\left(\frac{1}{s+1}-\frac{1}{s}\right)\sum_{r=0}^{n} b_{nr}a_{rs} \tag{33}$$

is uniformly convergent with respect to n, and so (32) then follows from (II). To prove the uniform convergence of (33) we use the AK property of c_{-1} and the fact that

$$\sum_{s=k+1}^{\infty}\delta^s\left(\frac{1}{s+1}-\frac{1}{s}\right)\in c_{-1}$$

from (1) with

$$\left\|\sum_{s=k+1}^{\infty}\delta^s\left(\frac{1}{s+1}-\frac{1}{s}\right)\right\|_{c_{-1}}=\frac{1}{k+1}$$

from (2). Thus, using (22),

$$\sup_{n\geqq 0}\left|\sum_{s=K}^{\infty}\left(\frac{1}{s+1}-\frac{1}{s}\right)\sum_{r=0}^{n} b_{nr}a_{rs}\right| = \sup_{n\geqq 0}\left|B_n\left(\sum_{s=K}^{\infty}\delta^s\left(\frac{1}{s+1}-\frac{1}{s}\right)\right)\right| \leqq \sup_{n\geqq 0}\|B_n\|\cdot\frac{1}{K}\to$$

$$\to 0 \quad \text{as} \quad K\to\infty$$

since $(B_n)_{n\geqq 0}$ is uniformly bounded and hence (33) is uniformly convergent with respect to n. We get, using (II), (31) and (33) that

$$\lim_{n\to\infty}\sum_{r=0}^{n} b_{nr}\sum_{s=k}^{\infty}\frac{a_{rs}-a_{r,s+1}}{s+1} = \frac{a_k}{k+1}+\sum_{s=k+1}^{\infty} a_s\left(\frac{1}{s+1}-\frac{1}{s}\right). \tag{34}$$

From (25) we know that the right hand side of (34) is the k^{th} term of an absolutely convergent series (it corresponds to λ in (25)), so we deduce that $a_k=o(k)$, and we use (31) to obtain (30). Hence (27) holds if (II) and (IV) hold.

The proof of the necessity of (II) and (IV) is essentially the same as that given in the proof of Theorem 4 using (29) instead of (23).

COROLLARY 5. *In order that A: $c_{-1}\to c_B$ with the limits agreeing, it in necessary and sufficient that, further to* (II) *and* (IV) *we require that $a_k=1$ for all $k\geqq 0$.*

PROOF. The same as Corollary 3 but replacing (14) by (27).

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ON THE DETERMINATION OF THE JUMP OF A FUNCTION BY ITS FOURIER SERIES

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1. Joint work with C. Goffman on everywhere convergence of Fourier series [7] has led D. Waterman to the following generalization of the concept of bounded variation in [9].

DEFINITION. *Let f be a real function on an interval I, $\{I_n\}$ a sequence of nonoverlapping intervals $I_n=[a_n, b_n]\subset I$ and $f(I_n)=f(b_n)-f(a_n)$. Let Λ denote a nondecreasing sequence $\{\lambda_n\}$ of positive real numbers such that $\Sigma\, 1/\lambda_n=\infty$. A function f is said to be of Λ-bounded variation (ΛBV) if, for every $\{I_n\}$, we have*

$$\sum |f(I_n)|/\lambda_n < \infty.$$

The supremum of these sums, $V_\Lambda(f;\, I)$, is called the Λ-variation of f on I.

We shall suppose that $I=[0, 2\pi]$ and that f is 2π-periodic.

The relationship of this notion of bounded variation to another interesting generalization given by Z. A. Čanturija's *modulus of variation* [4], was discussed by the author in [1] and [2]. The existing inclusion relations yield that certain theorems on the absolute convergence of Fourier series of functions of Waterman classes $\{n^\alpha\}BV$, $0<\alpha<1$, are equivalent to the corresponding statements for Čanturija's $V[n^\gamma]$, $0<\gamma<1$, and further to the older ones for Wiener classes of bounded p-variation, [2]. The case $\alpha=1$, i.e. $\lambda_n=n$, when we also speak of *harmonic bounded variation (HBV)* is, however, a distinguished one. It appears that some important classical theorems, valid for the Jordan class BV, find in HBV their natural setting. D. Waterman has proved in [9] that the Fourier series of functions of this class converge everywhere and converge uniformly on closed intervals of continuity. (See also Čanturija [5]. Note that his class $V[v(n)]$ with $\Sigma v(k)/k^2<\infty$ is contained in HBV.) Further, the classes L and HBV are complementary in the sense that for a product of $f\in L$ and $g\in HBV$ the Parseval formula holds (Waterman, [11]). Both of these results are best possible in that they are not valid for any strictly larger class ΛBV.

In this note we shall show this is also the case with the equation

$$(*) \qquad \lim_{n\to\infty} \frac{S_n'(x_0, f)}{n} = \frac{1}{\pi}[f(x_0+0)-f(x_0-0)],$$

[1] This work was done while the author was a **DAAD** fellow at the **Mathematical Institute A** of the **University of Stuttgart.**

where $S_n(x_0, f)$ denotes the n-th partial sum of the Fourier series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$$

of a function f at a point x_0.

More precisely, our result is as follows.

THEOREM 1. *The equation* $(*)$ *holds for any function* $f\in HBV$ *at any point* x_0.

REMARKS. 1. $(*)$ was proved first by L. Fejér [6] for f satisfying the so-called Dirichlet-condition and then by P. Csillag [3] for functions of the class BV (see also Zygmund, [12] p. 107). In [8] B. I. Golubov has generalized it for the larger classes V_p, $1<p<\infty$, of Wiener's bounded p-variation.

2. Let Φ be a convex function defined on $[0, \infty)$ such that $\Phi(0)=0$, $\Phi(x)/x\to 0$ $(x\to 0+)$ and $\Phi(x)/x\to\infty$ $(x\to\infty)$. Its complementary function Ψ is given by $\Psi(x)=\sup\{xy-\Phi(y)\colon y\geqq 0\}$. If now $\Sigma\Psi(1/n)<\infty$, it follows readily *(Young's inequality)* that HBV contains the class V_Φ of all functions of *Φ-bounded variation*, i.e. of those for which there exists $c>0$ (depending on f) such that

$$\sum\Phi(c|f(I_k)|)<\infty$$

for every sequence of nonoverlapping intervals $I_k\subset[0, 2\pi]$. In particular, all Wiener classes $(\Phi(u)=u^p,\ 1<p<\infty)$ are contained in HBV. Hence Theorem 1 generalizes the corresponding statement of Golubov.

3. Actually $(*)$ says that the sequence $\{c_k(x_0)\}$, $c_k(x_0)=kb_k\cos kx_0-ka_k\sin kx_0$, is Cesàro summable. Given another regular matrix summability method $A=(A_{nk})$ with $A_{nk}=0$ for $k>n$, it is well known that Cesàro summability implies A-summability (to the same limit) if and only if

$$(.)\qquad \sum_{k=1}^{n}k|\Delta A_{nk}|=O(1)\quad(n\to\infty),$$

where $\Delta A_{nk}=A_{nk}-A_{nk+1}$. The matrices $A^{(r)}$, $r=1, 2, \ldots$, with the entries $A_{nk}^{(r)}=rk^{r-1}/n^r$ for $1\leqq k\leqq n$, $A_{nk}^{(r)}=0$ for $k>n$, are regular and satisfy (.). Further

$$\frac{S_n^{(2r+1)}(x_0, f)}{n^{2r+1}}=\frac{(-1)^r}{2r+1}\sum_{k=1}^{n}A_{nk}^{(2r+1)}c_k(x_0),$$

$$\frac{\tilde{S}_n^{(2r)}(x_0, f)}{n^{2r}}=\frac{(-1)^{r+1}}{2r}\sum_{k=1}^{n}A_{nk}^{(2r)}c_k(x_0),$$

where $\tilde{S}_n(x_0, f)$ denotes, as usual, the n-th partial sum at x_0 of the series conjugate to the Fourier series of f.

Therefrom we conclude: once Theorem 1 is proved, the following apparently more general form of it is true.

THEOREM 1′. *Let $f \in HBV$ and $r=1, 2, \dots$. Then for any point x_0 we have*

(1) $$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(x_0, f)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}[f(x_0+0)-f(x_0-0)],$$

(2) $$\lim_{n\to\infty} \frac{\tilde{S}_n^{(2r)}(x_0, f)}{n^{2r}} = \frac{(-1)^{r+1}}{2r\pi}[f(x_0+0)-f(x_0-0)].$$

4. We can prove that if $\Lambda BV \supsetneqq HBV$ then there exists a continuous function of Λ-bounded variation such that $\sum_{k=1}^{n} kb_k \neq O(n)$. Thus HBV is the limiting case in the scale of ΛBV classes for applicability of $(C, 1)$ method in matters of determination of the jump of a function from its Fourier series. For larger ΛBV classes any of the (C, α), $\alpha > 1$, methods can be used (see [12a]), p. 62).

2. PROOF OF THEOREM 1. Let $f \in HBV$. Translation of the argument reduces the problem to the case $x_0 = 0$. The function

$$f_0(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \quad \text{for} \quad 0 < x < 2\pi, \quad f_0(x+2\pi) = f_0(x),$$

has a jump at the point $x_0 = 0$, equal to π, f_0 belongs to $BV \subset HBV$ and satisfies $(*)$. Subtracting a multiple of it from f, we may suppose that f is continuous at $x_0 = 0$. Since $(*)$ is invariant with respect to subtraction of a constant from the function, we can assume $f(0) = 0$.

Now it is convenient to look at the Dirichlet kernel in the form

$$D_n(t) = \frac{\sin(n+1/2)t}{2\sin t/2} = \frac{\sin nt}{t} + g(t)\sin nt + \frac{\cos nt}{2},$$

where

$$g(t) = \frac{1}{2\operatorname{tg} t/2} - \frac{1}{t}.$$

Then

$$\frac{D_n'(t)}{n} = \frac{\cos nt}{t} - \frac{\sin nt}{nt^2} + g'(t)\frac{\sin nt}{n} + g(t)\cos nt - \frac{1}{2}\sin nt.$$

Note that both g and g' are continuous on $[-\pi, \pi]$. Hence, for an arbitrary $0 < \delta < \pi$, we have

$$-\pi \frac{S_n'(0, f)}{n} = \int_{-\pi}^{\pi} f(t)\frac{D_n'(t)}{n}\,dt = \int_{-\delta}^{\delta} f(t)\left[\frac{\cos nt}{t} - \frac{\sin nt}{nt^2}\right]dt + o(1) =$$

$$= \int_0^{\delta} [f(t) - f(-t)]\left[\frac{\cos nt}{t} - \frac{\sin nt}{nt^2}\right]dt + o(1).$$

The function

$$h_n(t) = \frac{\cos nt}{t} - \frac{\sin nt}{nt^2} \quad \text{for} \quad t \neq 0 \quad \text{and} \quad h_n(0) = 0$$

is continuous, $h_n(\pi)=(-1)^n/\pi$, and $h_n(t_0)=-\dfrac{n\sin nt_0}{2}$ at the point t_0 where $h_n'(t_0)=0$. Therefore $|h_n(t)|\leqq n$ for every t and every n, and

$$\Big|\int_0^{\pi/n}[f(t)-f(-t)]\,h_n(t)\,dt\Big|\leqq \pi \sup_{0<t<\pi/n}|f(t)-f(-t)|=o(1).$$

Now let $N+1=[n\delta/\pi]$ and $\varphi(t)=f(t)-f(-t)$. Clearly

$$\int_{(N+1)\pi/n}^{\delta}\varphi(t)\frac{\cos nt}{t}\,dt=o(1)\quad\text{and}\quad \int_{(N+1)\pi/n}^{\delta}\varphi(t)\frac{\sin nt}{nt^2}\,dt=o(1).$$

Since φ is bounded (being in HBV) and continuous at 0, we have also

$$\int_{\pi/n}^{(N+1)\pi/n}\varphi(t)\frac{\sin nt}{nt^2}\,dt=o(1).$$

In fact

$$\int_{\pi/n}^{(N+1)\pi/n}\varphi(t)\frac{\sin nt}{nt^2}\,dt=\sum_{k=1}^{N}\int_{k\pi/n}^{(k+1)\pi/n}\dots=\sum_{k=1}^{N}\int_0^{\pi}\varphi\left(\frac{t+k\pi}{n}\right)\frac{(-1)^k\sin t}{(t+k\pi)^2}\,dt.$$

The integrand here is dominated by

$$\sum_{k=1}^{N}\left|\varphi\left(\frac{t+k\pi}{n}\right)\right|\Big/k^2.$$

Given $\varepsilon>0$, choose N_0 such that $\sum_{N_0+1}^{\infty}1/k^2<\varepsilon$. Then

$$\sum_{k=1}^{N}\left|\varphi\left(\frac{t+k\pi}{n}\right)\right|\Big/k^2\leqq \sup_{0<x\leqq(N_0+1)\pi/n}|\varphi(x)|\cdot\sum_{k=1}^{N_0}1/k^2+\varepsilon\sup|\varphi(x)|$$

and the first term on the right is $o(1)$ as $n\to\infty$.

So it remains to prove

$$\int_{\pi/n}^{(N+1)\pi/n}\varphi(t)\frac{\cos nt}{t}\,dt=\int_{\pi/n}^{(N+1)\pi/n}f(t)\frac{\cos nt}{t}\,dt-\int_{\pi/n}^{(N+1)\pi/n}f(-t)\frac{\cos nt}{t}\,dt=o(1).$$

The first integral may be treated in the manner indicated by D. Waterman to get $o(1)$. We demonstrate it here for the sake of completeness. (Letting $f_1(t)=f(-t)$ we reduce the second one to this case.) We may suppose N even. Now

$$\int_{\pi/n}^{(N+1)\pi/n}f(t)\frac{\cos nt}{t}\,dt=\sum_{k=1}^{N}\int_{k\pi/n}^{(k+1)\pi/n}f(t)\frac{\cos nt}{t}\,dt=\int_0^{\pi}\sum_{k=1}^{N}f\left(\frac{t+k\pi}{n}\right)\frac{(-1)^k\cos t}{t+k\pi}\,dt.$$

The absolute value of the integrand is dominated by

$$\left|\sum_{k=1}^{N-1}{}'\left[f\left(\frac{t+k\pi}{n}\right)\frac{1}{t+k\pi}-f\left(\frac{t+(k+1)\pi}{n}\right)\frac{1}{t+(k+1)\pi}\right]\right|,$$

where $'$ indicates summation over odd indices. The general term of the sum equals

$$\left[f\left(\frac{t+k\pi}{n}\right)-f\left(\frac{t+(k+1)\pi}{n}\right)\right]\frac{1}{t+k\pi}+f\left(\frac{t+(k+1)\pi}{n}\right)\left[\frac{1}{t+k\pi}-\frac{1}{t+(k+1)\pi}\right].$$

Since $\frac{1}{t+k\pi}-\frac{1}{t+(k+1)\pi}<\frac{1}{k^2}$, the sum of the absolute values of the second terms may be treated in the same way as φ above, to conclude that it is $o(1)$ as $n\to\infty$. Finally

$$\left|\sum_{k=1}^{N-1}{}' \left[f\left(\frac{t+k\pi}{n}\right)-f\left(\frac{t+(k+1)\pi}{n}\right)\frac{1}{t+k\pi}\right| \leqq$$

$$\leqq \sum_{k=1}^{N-1}{}' \left|f\left(\frac{t+k\pi}{n}\right)-f\left(\frac{t+(k+1)\pi}{n}\right)\right| \Big/ k \leqq V_H(f;\ [0,\delta]).$$

But $V_H(f;\ [0,\delta])<\varepsilon$ if δ is sufficiently small, since f is continuous at 0 (Waterman, [10]). The theorem is proved.

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ALMOST SURE LINEARITY FOR SIGNED RANK STATISTICS IN THE NON-I.I.D. CASE

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1. Introduction. Let $X_{N1}, \ldots, X_{NN}$, $N \geqq 1$ be independent random variables with continuous distribution functions $F_{N1}, \ldots, F_{NN}$, $N \geqq 1$ respectively. For every $N \geqq 1$ and $\theta \in \mathbf{R}$, consider the signed rank order statistic

$$S_N^+(\theta) = \frac{1}{N} \sum_{i=1}^{N} \operatorname{sgn}(X_{Ni} - \theta)\, a_N\big(R_{Ni}^+(\theta)\big) \tag{1.1}$$

where $R_{Ni}^+(\theta)$ is the rank of $|X_{Ni} - \theta|$ among $|X_{N1} - \theta|, \ldots, |X_{NN} - \theta|$, $i = 1, 2, \ldots, N$, $\operatorname{sgn} x = 1$ or -1 according as $x \geqq 0$ or < 0 and $a_N(i)$, $i = 1, 2, \ldots, N$ are the scores generated by a known function $\psi(t)$, $0 < t < 1$ by

$$a_N(i) = \psi\left(\frac{i}{N+1}\right), \quad i = 1, 2, \ldots, N. \tag{1.2}$$

For any $K > 0$, $k > 0$ define

$$\omega_N(K, k) = \sup_{\theta \in I_N(K,k)} N^{1/2} |S_N^+(\theta) - S_N^+(0) + \theta C_N| \tag{1.3}$$

where

$$I_N(K, k) = [-KN^{-1/2}(\operatorname{Log} N)^k, KN^{-1/2}(\operatorname{Log} N)^k] \tag{1.4}$$

and C_N is a constant depending on ψ and $F_{N1}, \ldots, F_{NN}$ to be specified later.

For the case when $F_{N1} = \ldots = F_{NN} \equiv F$, Sen and Ghosh [5] showed under rather stringent regularity conditions on the score generating function that when F is symmetric about 0 and $k \geqq 1$,

$$\omega_N(K, k) = O\big(N^{-1/4}(\operatorname{Log} N)^{2k}\big) \quad \text{a.s. as } N \to \infty. \tag{1.5}$$

When F has a symmetric density, by adapting Jurečková's [3] method dealing with linear rank statistics, van Eeden [10] showed that under certain regularity conditions

$$\omega_N(K, 0) = o_p(1) \quad \text{for every fixed } K. \tag{1.6}$$

However, for many statistical applications it suffices merely to have an intermediate result, namely that

$$\omega_N(K, k) \to 0 \quad \text{a.s. as} \quad N \to \infty. \tag{1.7}$$

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Recently, Sen [6] has obtained this conclusion requiring certain regularity conditions but allowing F not to be symmetric about 0.

In all the above it is assumed that the observations are independent and identically distributed. The purpose here is to establish almost sure linearity of the kind (1.7) when $X_{N1}, \ldots, X_{NN}, N \geqq 1$ are independent but non-identically distributed. The results presented here enable us to get more precise information in sequential analysis based on rank statistics and in this connection we refer to Lai [4] and Steyn and Geertsema [7].

2. Main result. Concerning the score function ψ we shall make the following assumptions:

$$\psi(0)=0 \text{ and } \psi(t) \text{ is nondecreasing in } t \in (0, 1), \tag{2.1}$$

ψ is twice differentiable on (0, 1) and for all $0<t<1$

$$|\psi(t)| \leqq C(1-t)^{-1/2+\lambda}, \quad |\psi^{(i)}(t)| \leqq C(1-t)^{-i-1/2+\delta} \tag{2.2}$$

for $i=1, 2$, and some $1/4<\delta<\lambda<1/2$, $2\lambda-\delta>1/2$, where C is a positive constant.

It may be noted that (2.1) and (2.2) hold true for example if $\psi(t)=G^{-1}\left(\frac{1+t}{2}\right)$, $0 \leqq t<1$, where G is a symmetric d.f. having a smooth density and whose tail has an increasing failure rate (e.g. $G=\Phi$, i.e. standard normal d.f.).

For the ease of convenience we shall suppress, from now on, the subscript N from $X_{Ni}, F_{Ni}, R^{+}_{Ni}(\cdot)$ etc.

Define for $\theta \in \mathbf{R}$

$$\begin{cases} \hat{F}^{\theta}_N(x) = \dfrac{1}{N}\sum\limits_{i=1}^{N} u(x-X_i^{\theta}), & \bar{F}^{\theta}_N(x) = \dfrac{1}{N}\sum\limits_{i=1}^{N} F_i^{\theta}(x), \\ \hat{H}^{\theta}_N(x) = \dfrac{1}{N}\sum\limits_{i=1}^{N} u(x-|X_i^{\theta}|), & \bar{H}^{\theta}_N(x) = \dfrac{1}{N}\sum\limits_{i=1}^{N} F_i^{*,\theta}(x) \end{cases} \tag{2.3}$$

where $X_i^{\theta}=X_i-\theta$, $F_i^{\theta}(x)=P(X_i^{\theta} \leqq x)$, $F_i^{*,\theta}(x)=P(|X_i^{\theta}| \leqq x)$, $i=1, \ldots, N$ and $u(t)=1$ or 0 according as $t \geqq 0$ or <0. (In the sequel we shall suppress θ when $\theta=0$.)

Furthermore we assume that each F_i, $1 \leqq i \leqq N$, $N \geqq 1$ has a continuous density f_i such that $f_i'(x)$ exists and is continuous for almost all $x \in \mathbf{R}$ and the following condition is satisfied:

$$\sup_N \sup_x \bar{f}_N(x)\{\bar{F}_N(x)(1-\bar{F}_N(x))\}^{-1/2+\delta_0} < \infty \tag{2.4}$$

for some $0<\delta_0<\delta$, where $\bar{f}_N(x)=\dfrac{1}{N}\sum\limits_{i=1}^{N} f_i(x)$.

Set

$$\mu_N^{+}(\theta) = \int_0^{\infty} \psi\big(\bar{F}_N(x+\theta)-\bar{F}_N(-x+\theta)\big)\, d\bar{F}_N(x+\theta), \quad \theta \in \mathbf{R} \tag{2.5}$$

and note that under the assumption (2.2), $\mu_N^{+}(\theta)$ is finite for all $\theta \in \mathbf{R}$.

Now, using (2.2) and (2.4) it is easily shown that

(2.6)
$$\nu_N^+(\theta) = -(\mu_N^+)'(\theta) = 2\int_0^\infty \psi'(\bar F_N(x+\theta)-\bar F_N(-x+\theta))\bar f_N(x+\theta)\bar f_N(-x+\theta)\,dx$$

exists (finite) for all $\theta\in\mathbf{R}$.

Define

(2.7)
$$\omega_N^*(K,k) = \sup_{\theta\in I_N(K,k)} N^{1/2}|S_N^+(\theta)-S_N^+(0)-2\mu_N^+(\theta)+2\mu_N^+(0)|.$$

The main result of this paper is the following theorem:

THEOREM 2.1. *Consider the statistic* $S_N^+(\theta)$ *defined by* (1.1) *with scores given by* (1.2). *Then if* (2.1), (2.2) *and* (2.4) *are satisfied, for every* $K>0$, $k\geqq 1$ *there exist positive numbers* C_1, C_2, α *and* β *such that*

(2.8)
$$\mathbf{P}\{\omega_N^*(K,k)\geqq C_1N^{-\alpha}\}\leqq C_2N^{-1-\beta},\quad \forall N\geqq N_0.$$

If in addition we assume

(2.9)
$$\limsup_{N\to\infty}\sup_x|\bar f_N'(x)|<\infty$$

then (2.8) *also holds true for* $\omega_N^*(K,k)$ *being replaced by* $\omega_N(K,k)$, *where* $C_N=2\nu_N^+(0)$ *and we have*

(2.10)
$$\sup_{\theta\in I_N(K,k)} N^{1/2}|S_N^+(\theta)-S_N^+(0)-2\nu_N^+(0)\theta|\to 0\quad a.s.\ as\ N\to\infty.$$

3. Proof of Theorem 2.1. The proof of Theorem 2.1 rests on the following lemmas:

LEMMA 3.1. *For every* $\varepsilon>0$, *there exist* $M>0$ *and* N_0 *(both depending on* ε *only) such that for* $N\geqq N_0$ *we have*

(3.1)
$$\mathbf{P}\left\{\sup_x\frac{|\hat F_N(x)-\bar F_N(x)|}{\{\bar F_N(x)(1-\bar F_N(x))\}^{1/2-\varepsilon_0}}\leqq MN^{-1/2}\log N\right\}\geqq 1-N^{-1-\varepsilon}$$

where

$$\varepsilon_0=\frac{1+\varepsilon}{2(2+\varepsilon)}.$$

LEMMA 3.2. *For every* $\varepsilon>0$, $C>0$ *and* $k\geqq 1$ *there exist* $M>0$ *and* N_0 *(both depending on* ε, C *and* k *only) such that for* $N\geqq N_0$ *we have*

(3.2)
$$\mathbf{P}\Big\{\sup_{|F_N(x)-F_N(y)|\leqq CN^{-1/2}(\mathrm{Log}\,N)^k} N^{-1/2}|\hat F_N(x)-\hat F_N(y)-\bar F_N(x)+\bar F_N(y)|\leqq$$
$$\leqq MN^{-1/4}(\mathrm{Log}\,N)^k\Big\}\geqq 1-N^{-1-\varepsilon}.$$

The proofs of Lemmas 3.1 and 3.2 are given in the Appendix.

Now set

$$\bar{S}_N^+(\theta) = \frac{1}{N} \sum_{i=1}^{N} u(X_i^\theta)\psi\left(\frac{R_{Ni}^+(\theta)}{N+1}\right), \quad \theta \in \mathbf{R} \tag{3.3}$$

and note that

$$S_N^+(\theta) = 2\bar{S}_N^+(\theta) - \bar{\psi}_N, \quad \bar{\psi}_N = \frac{1}{N} \sum_{i=1}^{N} \psi\left(\frac{i}{N+1}\right). \tag{3.4}$$

Then, writing

$$\bar{S}_N^+(\theta) = \int_0^\infty \psi\left(\frac{N}{N+1} \hat{H}_N^\theta(x)\right) d\hat{F}_N^\theta(x) \tag{3.5}$$

we have with probability one the following Chernoff—Savage [1] type decomposition:

$$\bar{S}_N^+(\theta) - \mu_N^+(\theta) = \frac{1}{N} \sum_{i=1}^{N} Y_{Ni}(\theta) + Q_N(\theta) \tag{3.6}$$

where

$$Y_{Ni}(\theta) = \left[u(X_i^\theta)\psi\left(\bar{H}_N^\theta(|X_i^\theta|)\right) - E\left(u(X_i^\theta)\psi(\bar{H}_N^\theta(|X_i|))\right)\right] + \tag{3.7}$$

$$+ \int_0^\infty [u(x - X_i^\theta) - F_i^{*,\theta}(x)]\psi'\left(\bar{H}_N^\theta(x)\right) d\bar{F}_N^\theta(x)$$

and

$$Q_N(\theta) = \int_0^\infty \left[\psi\left(\frac{N}{N+1}\hat{H}_N^\theta\right) - \psi(\bar{H}_N^\theta)\right] d\hat{F}_N^\theta - \int_0^\infty [\hat{H}_N^\theta - \bar{H}_N^\theta]\psi'(\bar{H}_N^\theta)\, d\bar{F}_N^\theta. \tag{3.8}$$

Now it is easily seen that the first term in the right hand side of (3.7) equals

$$\int_0^\infty [F_i^\theta(x) - u(x + \theta - X_i)]\psi'\left(\bar{H}_N^\theta(x)\right) d\bar{H}_N^\theta(x),$$

and we obtain

$$A_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} Y_{Ni}(\theta) = A_{N1}(\theta) + A_{N2}(\theta) \tag{3.9}$$

where

(3.10)

$$A_{N1}(\theta) = \int_0^\infty [\hat{F}_N(x+\theta) - \bar{F}_N(x+\theta)]\psi'\left(\bar{F}_N(x+\theta) - \bar{F}_N(-x+\theta)\right) d\bar{F}_N(-x+\theta)$$

and

(3.11)

$$A_{N2}(\theta) = -\int_0^\infty [\hat{F}_N(-x+\theta) - \bar{F}_N(-x+\theta)]\psi'\left(\bar{F}_N(x+\theta) - \bar{F}_N(-x+\theta)\right) d\bar{F}_N(x+\theta).$$

We now show that under (2.4)

$$v_0^+ = \limsup_{N\to\infty} \sup_\theta |v_N^+(\theta)| < \infty. \tag{3.12}$$

Indeed, by (2.2), (2.4) and (2.6),

$$0 \leqq -\frac{(v_N^+)'(\theta)}{2} \leqq (\sup_N \sup_x \bar{f}_N(x)[\bar{F}_N(x)]^{-1/2+\delta_0})\times$$

$$\times \int_0^\infty (1-\bar{F}_N(x+\theta))^{-1+\delta-\delta_0}\bar{f}_N(x+\theta)\,dx \leqq \text{const}\,[1-\bar{F}_N(\theta)]^{\delta-\delta_0} \leqq \text{const}, \quad \forall\theta\in\mathbf{R}$$

and (3.12) follows.

Now, with $\alpha>0$ to be specified later, for every $K>0$, $k\geqq 1$ set

(3.13) $m_N=[K(\operatorname{Log} N)^k N^\alpha]+1$, $\theta_{jN}=jN^{-\alpha}$ for $j=0, \pm 1, \ldots, \pm(m_N-1)$

and $\theta_{m_N,N}=-\theta_{-m_N,N}=K(\log N)^k$ ($[\cdot]$=integer part).

Then, by using the Mean Value Theorem, (2.1) and (3.12) we get

$$0 \leqq \mu_N^+(N^{-1/2}\theta_{j,N})-\mu_N^+(N^{-1/2}\theta_{j+1,N}) \leqq v_0^+ N^{-1/2-\alpha}, \tag{3.14}$$

$$j=-m_N, -m_N+1, \ldots, m_N-1.$$

Thus (2.7), (3.4), (3.14) and the fact that $\bar{S}_N^+(\theta)$ is nonincreasing in $\theta\in\mathbf{R}$ yield that

(3.15)

$$\omega_N^*(K,k) \leqq 2\max_{|j|\leqq m_N} N^{1/2}|\bar{S}_N^+(N^{-1/2}\theta_{j,N})-\bar{S}_N^+(0)-\mu_N^+(N^{-1/2}\theta_{j,N})+\mu_N^+(0)|+2v_0^+N^{-\alpha}.$$

Now from the decomposition (3.6), using (3.9)—(3.11) we have for $|j|\leqq m_N$

$$N^{1/2}\big(\bar{S}_N^+(N^{-1/2}\theta_{j,N})-\mu_N^+(N^{-1/2}\theta_{j,N})\big) = \tag{3.16}$$

$$= N^{1/2}A_{N1}(N^{-1/2}\theta_{j,N})+N^{1/2}A_{N2}(N^{-1/2}\theta_{j,N})+N^{1/2}Q_N(N^{-1/2}\theta_{j,N}).$$

Choose $\varepsilon>0$ such that

$$1/4 < \varepsilon_0 = \frac{1+\varepsilon}{2(2+\varepsilon)} < \delta \tag{3.17}$$

and let Ω_N be the set on which both bounds provided by (3.1) and (3.2) are simultaneously satisfied. On this set the terms $N^{1/2}[A_{N1}(N^{-1/2}\theta_{j,N})-A_{N1}(0)]$ and $N^{1/2}[A_{N2}(N^{-1/2}\theta_{j,N})-A_{N2}(0)]$ are now dealt with.

Let $0<a<1/3$ and select $x_N>0$ such that

$$\bar{H}_N(x_N) = 1-N^{-a}, \quad N \geqq 1. \tag{3.18}$$

Now, with $|j|\leqq m_N$ fixed, set $N^{-1/2}\theta_{j,N}=\eta_{j,N}$. For convenience we omit in the sequel the subscripts j and N from $\eta_{j,N}$. Assume for the moment that $\eta>0$. Then, noting that

(3.19)

$$N^{1/2}A_{N1}(\eta) = \int_{-\eta}^{\infty} N^{1/2}[\hat{F}_N(x+2\eta)-\bar{F}_N(x+2\eta)]\psi'(\bar{F}_N(x+2\eta)-\bar{F}_N(-x))\,d\bar{F}_N(-x)$$

we write

$$N^{1/2}A_{N1}(\eta) = \int_{-\eta}^{0}(\ldots)+\int_0^{x_N}(\ldots)+\int_{x_N}^{\infty}(\ldots) = A_{N1}^{(1)}(\eta)+A_{N1}^{(2)}(\eta)+A_{N1}^{(3)}(\eta). \tag{3.20}$$

With the help of (2.2), (3.1) and

$$\sup_N \sup_x \bar{f}_N(x) < \infty \tag{3.21}$$

(this fact follows from (2.4)) we obtain

$$|A_{N1}^{(1)}(\eta)| \leqq \text{const.} \int_0^\eta \psi'(\bar{F}_N(-x+2\eta)-\bar{F}_N(x))\, d\bar{F}_N(x) \leqq \tag{3.22}$$

$$\leqq \text{const.} \int_0^\eta \psi'(\bar{F}_N(-x+2\eta)-\bar{F}_N(x))[\bar{f}_N(x)+\bar{f}_N(-x+2\eta)]\, dx =$$

$$= \text{const.}\, \psi(\bar{F}_N(\eta)-\bar{F}_N(0)) \leqq \text{const.}\, N^{-1/2}(\log N)^k$$

where we have used the fact that $\eta = O(N^{-1/2}(\log N)^k)$ uniformly in $|j| \leqq m_N$.
Further by (2.2), (3.1), (3.17) and $\bar{F}_N(-x_N) \leqq N^{-a}$ we get on Ω_N

$$|A_{N1}^{(3)}(\eta)| \leqq \text{const.}\, (\log N) \int_{x_N}^\infty \{\bar{F}_N(x+2\eta)(1-\bar{F}_N(x+2\eta))\}^{1/2-\varepsilon_0} \times \tag{3.23}$$

$$\times (1-\bar{F}_N(x+2\eta)+\bar{F}_N(-x))^{-3/2+\delta}\, d(-\bar{F}_N(-x)) \leqq$$

$$\leqq \text{const.}\, (\log N) \int_{x_N}^\infty [\bar{F}_N(-x)]^{-1+\delta-\varepsilon_0}\, d(-\bar{F}_N(-x)) \leqq$$

$$\leqq \text{const.}\, (\log N)[\bar{F}_N(-x_N)]^{\delta-\varepsilon_0} \leqq \text{const.}\, (\log N)\, N^{-(\delta-\varepsilon_0)a}.$$

Now

$$A_{N1}^{(1)}(\eta) - A_{N1}^{(1)}(0) = B_{N1}^{(1)}(\eta) + B_{N1}^{(2)}(\eta)$$

where

$$B_{N1}^{(1)}(\eta) = \tag{3.24}$$

$$= \int_0^{x_N} N^{1/2}[\hat{F}_N(x+2\eta)-\hat{F}_N(x)-\bar{F}_N(x+2\eta)+\bar{F}_N(x)]\psi'(\bar{F}_N(x+2\eta)-\bar{F}_N(-x))\, d\bar{F}_N(-x),$$

$$B_{N1}^{(2)}(\eta) =$$

$$= \int_0^{x_N} N^{1/2}[\hat{F}_N(x)-\bar{F}_N(x)][\psi'(\bar{F}_N(x+2\eta)-\bar{F}_N(-x))-\psi'(\bar{F}_N(x)-\bar{F}_N(-x))]\, d\bar{F}_N(-x).$$

Then, since $\bar{F}_N(x_N+2\eta)-\bar{F}_N(-x_N) = \bar{F}_N(x_N)-\bar{F}_N(-x_N)+O(N^{-1/2}(\log N)^k)$, on using (2.2), (3.2), (3.18) and (3.21), we have for sufficiently large N

$$|B_{N1}^{(1)}(\eta)| \leqq \tag{3.25}$$

$$\leqq \text{const.}\, N^{-1/4}(\log N)^k \int_{-x_N}^0 \psi'(\bar{F}_N(-x+2\eta)-\bar{F}_N(x))[\bar{f}_N(x)+\bar{f}_N(-x+2\eta)]\, dx \leqq$$

$$\leqq \text{const.}\, N^{-1/4}(\log N)^k \psi(\bar{F}_N(x_N+2\eta)-\bar{F}_N(-x_N)) \leqq \text{const.}\, N^{-1/4+a(1/2-\lambda)}(\log N)^k.$$

Finally, by the Mean Value Theorem, (2.2), (3.1), (3.21) and the fact that $\bar{F}_N(x+2\eta)-\bar{F}_N(x)=O(N^{-1/2}(\log N)^k)$ uniformly in $x\in\mathbf{R}$ and $|j|\leqq m_N$ we obtain

(3.26)

$$|B_{N1}^{(2)}(\eta)| \leqq \text{const. } N^{-1/2}(\log N)^{k+1} \int_{-x_N}^{0} (1-\bar{F}_N(-x+2\eta)+\bar{F}_N(x))^{-5/2+\delta}\, d\bar{F}_N(x) \leqq$$

$$\leqq \text{const. } N^{-1/2}(\log N)^{k+1}(1-\bar{F}_N(x_N+2\eta)+\bar{F}_N(-x_N))^{-3/2+\delta} \leqq$$

$$\leqq \text{const. } N^{-1/2+a(3/2-\delta)}(\log N)^{k+1}.$$

A similar analysis holds true for $\eta<0$. Therefore from (3.22)—(3.26) we see that there exists a $\gamma_1>0$ such that on Ω_N

$$|N^{1/2}[A_{N1}(\eta)-A_{N1}(0)]| = O(N^{-\gamma_1}). \tag{3.27}$$

Similarly one can show that there exists $\gamma_2>0$ such that on Ω_N

$$|N^{1/2}[A_{N2}(\eta)-A_{N2}(0)]| = O(N^{-\gamma_2}). \tag{3.28}$$

Now, it follows from Ralescu [8] that there exist $0<\gamma_3<1/2$ and $\tau>0$ such that for sufficiently large N and every $\theta\in\mathbf{R}$

$$\mathbf{P}\{|Q_N(\theta)| \geqq C_1' N^{-1/2-\gamma_3}\} \leqq C_2' N^{-1-\tau} \tag{3.29}$$

foı some $C_1', C_1'>0$.

Therefore using (3.27)—(3.29) we infer the existence of positive constants $\gamma, \varrho, C_1'', C_2''$ and N_0 (all possibly dependent on C, δ, λ in (2.2), K, k, a) such that

(3.30)

$$\mathbf{P}\{N^{1/2}|\bar{S}_N^+(N^{-1/2}\theta_{j,N})-\bar{S}_N^+(0)-\mu_N^+(N^{-1/2}\theta_{j,N})+\mu_N^+(0)| \geqq C_1'' N^{-\gamma}\} \leqq C_2'' N^{-1-\varrho}$$

for all $|j|\leqq m_N$, $N\geqq N_0$.

Finally, if we choose $0<\alpha<\min(\gamma,\varrho)$ in (3.13), the satisfaction of (2.8) with $\beta=\varrho-\alpha$ follows from (3.15) and (3.30).

The proof of the Theorem 2.1 will be completed if we show that by requiring also the condition (2.9) that

$$\sup_{\theta\in I_N(K,k)} |v_N^+(\theta)-v_N^+(0)| = O(N^{-b}) \quad \text{for some } b>0. \tag{3.31}$$

To this end, with x_N defined as in (3.18) we decompose

$$v_N^+(\theta) = v_{N1}^+(\theta)+v_{N2}^+(\theta) \tag{3.32}$$

where

$$v_{N1}^+(\theta) = 2\int_{\theta}^{x_N} \psi'(\bar{F}_N(x)-\bar{F}_N(-x+2\theta))\bar{f}_N(x)\bar{f}_N(-x+2\theta)\,dx$$

and

$$v_{N2}^+(\theta) = 2\int_{x_N}^{\infty} \psi'(\bar{F}_N(x)-\bar{F}_N(-x+2\theta))\bar{f}_N(x)\bar{f}_N(-x+2\theta)\,dx. \tag{3.33}$$

Assume that $\theta \in I_N(K, k)$, $\theta > 0$ (similar bounds will hold true for $\theta < 0$). Then using (2.2), (2.4) and the fact that $1-\bar{F}_N(x_N) \leqq N^{-a}$, we get

(3.34)

$$|v_{N2}^+(\theta)| \leqq \text{const.} \int_{x_N}^{\infty} (1-\bar{F}_N(x)+\bar{F}_N(-x+2\theta))^{-3/2+\delta} \bar{f}_N(x)\bar{f}_N(-x+2\theta)\,dx \leqq$$

$$\leqq \text{const.} \{\sup_N \sup_x \bar{f}_N(x)[\bar{F}_N(x)]^{-1/2+\delta_0}\} \int_{x_N}^{\infty} [1-\bar{F}_N(x)]^{-1+\delta-\delta_0} \bar{f}_N(x)\,dx \leqq$$

$$\leqq \text{const.} [1-\bar{F}_N(x)]^{\delta-\delta_0} \leqq \text{const.}\, N^{-a(\delta-\delta_0)}.$$

Furthermore, from (2.1), (2.2) and (2.9), we have for sufficiently large N

$$2 \int_0^{\theta} \psi'(\bar{F}_N(x)-\bar{F}_N(-x))\bar{f}_N(-x)\bar{f}_N(x)\,dx \leqq \tag{3.35}$$

$$\leqq \text{const.} \int_0^{\theta} \psi'(\bar{F}_N(x)-\bar{F}_N(-x))[\bar{f}_N(x)+\bar{f}_N(-x)]\,dx \leqq$$

$$\leqq \text{const.}\, \psi(\bar{F}_N(\theta)-\bar{F}_N(-\theta)) \leqq \text{const.}\, N^{-1/2}(\log N)^k.$$

Then, (3.32)—(3.25) entail

$$v_N^+(\theta)-v_N^+(0) = \tag{3.36}$$

$$= \int_{\theta}^{x_N} [\psi'(\bar{F}_N(x)-\bar{F}_N(-x+2\theta))-\psi'(\bar{F}_N(x)-\bar{F}_N(-x))]\bar{f}_N(-x+2\theta)\bar{f}_N(x)\,dx +$$

$$+ \int_{\theta}^{x_N} \psi'(\bar{F}_N(x)-\bar{F}_N(-x))[\bar{f}_N(-x+2\theta)-\bar{f}_N(-x)]\bar{f}_N(x)\,dx + O(N^{-a(\delta-\delta_0)}) +$$

$$+ O(N^{-1/2}(\log N)^k).$$

Now, using the Mean Value Theorem, (2.2), (2.4) and (3.18) we have

$$\left| \int_{\theta}^{x_N} [\psi'(\ldots)-\psi'(\ldots)]\bar{f}_N(-x+2\theta)\bar{f}_N(x)\,dx \right| \leqq \tag{3.37}$$

$$\leqq \text{const.}\, N^{-1/2}(\log N)^k \int_{\theta}^{x_N} (1-\bar{F}_N(x)+\bar{F}_N(-x))^{-5/2+\delta}(\bar{f}_N(x)+\bar{f}_N(-x))\,dx \leqq$$

$$\leqq \text{const.}\, N^{-1/2}(\log N)^k(1-\bar{F}_N(x_N)+\bar{F}_N(-x_N))^{3/2+\delta} = \text{const.}\, N^{-1/2+a(3/2-\delta)}(\log N)^k.$$

Finally, using (2.2), (2.4), (2.9) and (3.18), we get

$$\left| \int_{\theta}^{x_N} \psi'(\bar{F}_N(x)-\bar{F}_N(-x))[\bar{f}_N(-x+2\theta)-\bar{f}_N(-x)]\bar{f}_N(x)\,dx \right| \leqq \tag{3.38}$$

$$\leqq \text{const.}\, N^{-1/2}(\log N)^k \int_{\theta}^{x_N} (1-\bar{F}_N(x)+\bar{F}_N(-x))^{-3/2+\delta}(\bar{f}_N(x)+\bar{f}_N(-x))\,dx \leqq$$

$$\leqq \text{const.}\, N^{-1/2}(\log N)^k(1-\bar{F}_N(x_N)+\bar{F}_N(-x_N))^{-1/2+\delta} = \text{const.}\, N^{-1/2+a(1/2-\delta)}(\log N)^k.$$

Therefore, since $0<a<1/3$, relations (3.36)—(3.38) entail the satisfaction of (3.31). Thus (2.10) follows and the proof is complete.

4. Appendix. PROOF OF LEMMA 3.1. We first establish that for every $\varepsilon>0$ there exist $M>0$ and N_0 such that for $N\geqq N_0$

$$\text{(4.1)}\quad \mathbf{P}\left\{\sup_{N^{-1}\leqq \bar{F}_N(x)\leqq 1-N^{-1}} \frac{|\hat{F}_N(x)-\bar{F}_N(x)|}{\{\bar{F}_N(x)(1-\bar{F}_N(x))\}^{1/2}} \geqq MN^{-1/2}\operatorname{Log} N\right\} \leqq 2N^{-1-\varepsilon}.$$

Note that with probability 1

$$\text{(4.2)}\quad \sup_{N^{-1}\leqq \bar{F}_N(x)\leqq 1-N^{-1}} \frac{|\hat{F}_N(x)-\bar{F}_N(x)|}{\{\bar{F}_N(x)(1-\bar{F}_N(x))\}^{1/2}} = \sup_{N^{-1}\leqq t\leqq 1-N^{-1}} \frac{|\hat{F}_N(\bar{F}^{-1}(t))-t|}{\{t(1-t)\}^{1/2}}.$$

Set $G_N(t)=\{t(1-t)\}^{-1/2}[\hat{F}_N(\bar{F}_N^{-1}(t))-t]$, $0<t<1$ and $\alpha_{j,N}=j/N$, $i\leqq j\leqq N-1$. Using the monotonicity of $\hat{F}_N$ and $\bar{F}_N$, it is easily seen that

$$\text{(4.3)}\quad \sup_{N^{-1}\leqq t\leqq 1-N^{-1}} |G_N(t)| \leqq 2^{1/2} \max_{1\leqq j\leqq N-1} |G_N(\alpha_{j,N})|+2^{1/2}N^{-1/2}.$$

Now, with $1\leqq j\leqq N-1$ fixed, note that

$$N\hat{F}_N(\bar{F}_N^{-1}(\alpha_{j,N})) = \sum_{i=1}^{N} u(\bar{F}_N^{-1}(\alpha_{j,N})-X_i) = \sum_{i=1}^{N} Z_i$$

where Z_i, $1\leqq i\leqq N$ are independent Bernoulli r.v.'s with

$$\mathbf{P}\{Z_i=1\} = F_i(\bar{F}_N^{-1}(\alpha_{j,N})) = p_i, \quad 1\leqq i\leqq N.$$

Consequently, with $M_1>0$ to be specified later, by Bernstein's inequality (see Uspensky [9], pp. 204—205) we obtain for $1\leqq j\leqq N-1$

$$\text{(4.4)}\quad \mathbf{P}\{|G_N(\alpha_{j,N})| \geqq M_1N^{-1/2}\operatorname{Log} N\} \leqq 2\exp(-h_{j,N})$$

where

$$\text{(4.5)}\quad h_{j,N} \geqq \frac{M_1^2}{2}\,\frac{\operatorname{Log}^2 N}{1+M_1[N\alpha_{j,N}(1-\alpha_{j,N})]^{-1/2}\operatorname{Log} N}.$$

Now using the relations $[\alpha_{j,N}(1-\alpha_{j,N})]^{-1/2}\leqq(2N)^{1/2}$, for all $1\leqq j\leqq N-1$, it follows from (4.5) that if $M_1\geqq 2^{5/2}(2+\varepsilon)$, then

$$\text{(4.6)}\quad \sum_{j=1}^{N-1} \exp(-h_{j,N}) \leqq N^{-1-\varepsilon}, \quad N\geqq N_1.$$

Finally, if we let $M=2^{1/2}(1+M_1)$, the assertion (4.1) follows from (4.2)—(4.4) and (4.6).

Now, using Bernoulli's inequality we have:

$$\mathbf{P}\{\sup_{0<t<N^{-2-\varepsilon}} \hat{F}_N(\bar{F}_N^{-1}(t))=0\} \geqq 1-N^{-1-\varepsilon}$$

which implies that with probability $\geqq 1-N^{-1-\varepsilon}$

$$\text{(4.7)}\quad \sup_{0<t<N^{-2-\varepsilon}} |G_N(t)| = O(N^{-1-\varepsilon/2}).$$

Similarily one proves that with probability $\geqq 1-N^{-1-\varepsilon}$

$$\sup_{1-N^{-2-\varepsilon}<t<1} |G_N(t)| = O(N^{-1-\varepsilon/2}). \tag{4.8}$$

We now show that for any $\varepsilon>0$, there exist $M>0$ and N_0 such that

$$\mathbf{P}\Big\{\sup_{N^{-2-\varepsilon}\leqq t<N^{-1}} |K_N(t)| \geqq MN^{-1/2}\operatorname{Log} N\Big\} \leqq 2N^{-1-\varepsilon},\ N\geqq N_0 \tag{4.9}$$

where $K_N(t)=\{t(1-t)\}^{\varepsilon_0}G_N(t),\ 0<t<1$.

Observe that

$$\sup_{N^{-2-\varepsilon}\leqq t<N^{-1}} |K_N(t)| \leqq 2^{1/2} \max_{1\leqq j\leqq [N^{1+\varepsilon}]} |K_N(\beta_{j,N})| + O(N^{-3/2-\varepsilon}) \tag{4.10}$$

where $\beta_{j,N}=j/N^{2+\varepsilon}$, $1\leqq j\leqq [N^{1+\varepsilon}]$ ($[\cdot]$=integer part).

Using the Bernstein inequality again

$$\mathbf{P}\{|K_N(\beta_{j,N})| \geqq MN^{-1/2}\operatorname{Log} N\} \leqq 2\exp(-l_{j,N}) \tag{4.11}$$

where

$$l_{j,N} \geqq M2^{-1/(2(2+\varepsilon))-2}\operatorname{Log} N^{1/4}.$$

Hence, by choosing M sufficiently large we get

$$2\sum_{j=1}^{[N^{1+\varepsilon}]} \exp(-l_{j,N}) \leqq 2N^{-1-\varepsilon},$$

which together with (4.10) entails (4.9).

Similarly one proves that for every $\varepsilon>0$, there exist $M>0$ and N_0 such that

$$\mathbf{P}\Big\{\sup_{1-N^{-1}\leqq t\leqq 1-N^{-2-\varepsilon}} |K_N(t)| \geqq MN^{-1/2}\operatorname{Log} N\Big\} \leqq 2N^{-1-\varepsilon},\quad N\geqq N_0. \tag{4.12}$$

Finally, (4.1), (4.7), (4.8), (4.9) and (4.12) entail the satisfaction of (3.1). The proof of Lemma 3.1 is complete.

PROOF OF LEMMA 3.2. As in the proof of Lemma 3.1 it suffices to show that there exist $M>0$ and N_0 such that for $N\geqq N_0$

(4.13)

$$\mathbf{P}\Big\{\sup_{|s-t|\leqq CN^{-1/2}(\operatorname{Log} N)^k} N^{1/2}\big|\hat{F}_N(\bar{F}_N^{-1}(s))-\hat{F}_N(\bar{F}_N^{-1}(t))-s+t\big| \geqq MN^{-1/4}(\operatorname{Log} N)^k\Big\} \leqq N^{-1-\varepsilon}.$$

Define

$$D_N(t)=\sup_{|s-t|\leqq CN^{-1/2}(\operatorname{Log} N)^k} \big|\hat{F}_N(\bar{F}_N^{-1}(s))-\hat{F}_N(\bar{F}_N^{-1}(t))-s+t\big|,\quad 0<t<1$$

and

$$\gamma_{N,j}=j/[N^{1/2}],\quad j=0,1,\dots,[N^{1/2}],\quad ([\cdot]=\text{integer part}).$$

Observe that

$$\sup_{|s-t|\leqq CN^{-1/2}(\operatorname{Log} N)^k} \big|\hat{F}_N(\bar{F}_N^{-1}(s))-\hat{F}_N(\bar{F}_N^{-1}(t))-s+t\big| < 3\max_{1\leqq j\leqq [N^{1/2}]} D_N(\gamma_{N,j}). \tag{4.14}$$

Further set

$$T_{Nj}(s) = \hat{F}_N(\bar{F}_N^{-1}(s)) - \hat{F}_N(\bar{F}_N^{-1}(\gamma_{N,j})) - s + \gamma_{N,j}, \quad 1 \leqq j \leqq [N^{1/2}]$$

and

$$\xi_{r,N} = \gamma_{N,j} + \frac{CN^{-1/2}(\operatorname{Log} N)^k}{[N^{1/4}]}\, r,$$

where r is an integer. Then

$$\sup_{\xi_{r,N} \leqq s \leqq \xi_{r+1,N}} |T_{Nj}(s)| \leqq \max_{l=r,r+1} |T_{Nj}(\xi_{l,N})| + (\xi_{r+1,N} - \xi_{r,N})$$

which entails for N sufficiently large

$$D_N(\gamma_{N,j}) \leqq \max_{-[N^{1/4}] \leqq r \leqq [N^{1/4}]-1} |T_{Nj}(\xi_{r,N})| + CN^{-3/4}(\operatorname{Log} N)^k. \tag{4.15}$$

Now using Bernstein's inequality we obtain

$$\mathbf{P}\Big\{\max_{-[N^{1/4}] \leqq r \leqq [N^{1/4}]-1} |T_{Nj}(\xi_{r,N})| > MN^{-3/4}(\operatorname{Log} N)^k\Big\} \leqq 4N^{1/4}\exp(-\lambda_N) \tag{4.16}$$

where

$$\lambda_N = \frac{M^2}{2}\frac{(\operatorname{Log} N)^k}{C + MN^{-1/4}}.$$

From (4.14)—(4.16), we get for $M > C$,

(4.17)

$$\mathbf{P}\Big\{\sup_{|s-t| \leqq CN^{-1/2}(\operatorname{Log} N)^k} \big|\hat{F}_N(\bar{F}_N^{-1}(s)) - \hat{F}_N(\bar{F}_N^{-1}(t)) - s + t\big| \geqq 3MN^{-3/4}(\operatorname{Log} N)^k\Big\} \leqq \varrho_N$$

where

$$\varrho_N = 4N^{3/4}\exp\Big\{-\frac{(M-C)^2}{2}\frac{(\operatorname{Log} N)^k}{C + (M-C)N^{-1/4}}\Big\}.$$

Now, since $k \geqq 1$, it follows that

$$\lim_{N\to\infty} \frac{\operatorname{Log} \varrho_N}{\operatorname{Log} N} = \begin{cases} -\infty & \text{if } k > 1 \\ \dfrac{3}{4} - \dfrac{(M-C)^2}{2C} & \text{if } k = 1. \end{cases} \tag{4.18}$$

Thus, if M is chosen sufficiently large, (4.17) and (4.18) imply (3.2) and the proof of Lemma 3.2 is complete.

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SIMULTANEOUS TRIANGULARIZATION OF PROJECTOR MATRICES

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It is known that every matrix can be transformed into upper triangular form with unitary transformation. Now consider the problem of simultaneous triangularization of two matrices. Considering the results mentioned by H. Shapiro [1] it seems to be useful to find simple necessary and sufficient conditions which can be easily verified for special matrices to be simultaneously triangularisable with unitary transformation.

THEOREM. *Assume that C and D are projector matrices with complex elements. Then they can be simultaneously transformed into upper triangular form with unitary transformation if and only if $CD-DC$ is nilpotent.*

PROOF. I. It is obvious that the nilpotency of $CD-DC$ is necessary for arbitrary matrix pairs.

II. 1. First we prove that if $CD-DC$ is nilpotent then C and D have at least one common eigenvector. Let us transform these matrices with a similarity transformation which transforms D into the special form

$$P^{-1}DP = B = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \quad \text{and} \quad P^{-1}CP = A.$$

C and D have a common eigenvector if and only if A and B have.

Since

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

is a projector, the following equalities are true:

$$A_{11}A_{11}+A_{12}A_{21} = A_{11}, \tag{1}$$

$$A_{11}A_{12}+A_{12}A_{22} = A_{12}, \tag{2}$$

$$A_{21}A_{11}+A_{22}A_{21} = A_{21}, \tag{3}$$

$$A_{21}A_{12}+A_{22}A_{22} = A_{22}. \tag{4}$$

By premultiplying (2) by A_{21} and by postmultiplying (3) by A_{12} and subtracting the resulting equations we get

$$A_{22}(A_{21}A_{12}) = (A_{21}A_{12})A_{22} \tag{5}$$

and similarly

$$A_{11}(A_{12}A_{21}) = (A_{12}A_{21})A_{11}. \tag{6}$$

Since

$$AB-BA = \left[\begin{array}{c:c} 0 & -A_{12} \\ \hdashline A_{21} & 0 \end{array}\right]$$

we have

$$(AB-BA)^k = \begin{cases} \left[\begin{array}{c:c} \pm\overset{1}{\breve{A}}_{12}\overset{2}{\breve{A}}_{21}\dots\overset{k}{\breve{A}}_{21} & 0 \\ \hdashline 0 & \pm\overset{1}{\breve{A}}_{21}\overset{2}{\breve{A}}_{12}\dots\overset{k}{\breve{A}}_{12} \end{array}\right] & \text{for even } k \\ \left[\begin{array}{c:c} 0 & \pm\overset{1}{\breve{A}}_{12}\overset{2}{\breve{A}}_{21}\dots\overset{k}{\breve{A}}_{12} \\ \hdashline \pm\overset{1}{\breve{A}}_{21}\overset{2}{\breve{A}}_{12}\dots\overset{k}{\breve{A}}_{21} & 0 \end{array}\right] & \text{for odd } k. \end{cases}$$

a) If $A_{12}=0$ and $A_{21}=0$ then $AB-BA=0$, consequently A and B have a common eigenvector.

b) Let us assume that there is an integer k such that

$$\overset{1}{\breve{A}}_{12}\overset{2}{\breve{A}}_{21}\dots\dots\overset{k-1}{\breve{\ }} \neq 0 \quad \text{but} \quad \overset{1}{\breve{A}}_{12}\overset{2}{\breve{A}}_{21}\dots\dots\overset{k}{\breve{\ }} = 0 \quad \text{and} \quad \overset{1}{\breve{A}}_{21}\overset{2}{\breve{A}}_{12}\dots\dots\overset{k}{\breve{\ }} = 0. \tag{7}$$

The cases when k is even or odd will be studied in different ways.

α) Assume first that k is even. Then by premultiplying (2) by $\overset{1}{\breve{A}}_{21}\overset{2}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{21}$ we get

$$\overset{1}{\breve{A}}_{21}\dots\overset{k-1}{\breve{A}}_{21}A_{11}A_{12} + \underbrace{\overset{1}{\breve{A}}_{21}\dots\overset{k-1}{\breve{A}}_{21}A_{12}A_{22}}_{0} = \underbrace{\overset{1}{\breve{A}}_{21}\dots\overset{k-1}{\breve{A}}_{21}A_{12}}_{0}$$

which implies

$$\overset{1}{\breve{A}}_{21}\overset{2}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{21}A_{11}A_{12} = 0. \tag{8}$$

By postmultiplying (1) by $\overset{1}{\breve{A}}_{12}\overset{2}{\breve{A}}_{21}\dots\overset{k-1}{\breve{A}}_{12} \neq 0$ we get

$$A_{11}(A_{11}\overset{1}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{12}) = A_{11}\overset{1}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{12}. \tag{9}$$

If $A_{11}\overset{1}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{12} \neq 0$ then an arbitrary nonzero column of

$$G = \left[\begin{array}{c} A_{11}\overset{1}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{12} \\ \hdashline 0 \end{array}\right]$$

is a common eigenvector of A and B, since

$$AG = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c} A_{11}\overset{1}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{12} \\ \hdashline 0 \end{array}\right] = \left[\begin{array}{c} A_{11}A_{11}\overset{1}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{12} \\ \hdashline A_{21}A_{11}\overset{1}{\breve{A}}_{12}\dots\overset{k-1}{\breve{A}}_{12} \end{array}\right].$$

(9) implies that the upper part of the right hand side is equal to the upper part of G, and (6) and (8) imply that the lower part is the zero matrix. If

$$A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{12}} = 0 \tag{10}$$

then an arbitrary nonzero column of

$$G = \left[\begin{array}{c} \overset{1}{A_{12}}\cdots\overset{k-1}{A_{12}} \\ \hdashline 0 \end{array}\right]$$

is a common eigenvector of A and B since (10) and (7) imply $AG=0$.

β) Assume next that k is odd. Then similarly to (9) we get

$$A_{11}(A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}}) = A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}}. \tag{11}$$

If $A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}}\neq 0$, then every nonzero column of

$$G = \left[\begin{array}{c} A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}} \\ \hdashline 0 \end{array}\right] \tag{12}$$

is a common eigenvector of A and B. Observe that

$$AG = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c} A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}} \\ \hdashline 0 \end{array}\right] = \left[\begin{array}{c} A_{11}A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}} \\ \hdashline A_{21}A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}} \end{array}\right].$$

(11) implies that the upper part of the right hand side equals the upper part of G, and (6) and (7) imply that the lower part equals the zero matrix. If $A_{11}\overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}}=0$, then every nonzero column of $G=\left[\begin{array}{c} \overset{1}{A_{12}}\cdots\overset{k-1}{A_{21}} \\ 0 \end{array}\right]$ is a common eigenvector of A and B.

c) Let us now assume that there is an integer k such that $\overset{1}{A_{21}}\overset{2}{A_{12}}\cdots^{k-1}\neq 0$ but $\overset{1}{A_{21}}\overset{2}{A_{12}}\cdots^{k}=0$ and $\overset{1}{A_{12}}\overset{2}{A_{21}}\cdots^{k}=0$.

We can show similarly to the provious cases that there is at least one common eigenvector.

2. To construct the unitary matrix for transforming simultaneously these two matrices into triangular form apply the initial step of the wellknown method for transforming one matrix into triangular form with unitary matrix. Since C and D have a common eigenvector, the first step can be performed, that is there exists a unitary matrix U_1 such that

$$C = U_1^* C_1 U_1 = U_1^* \left[\begin{array}{c:c} \mu_1 & \cdots \\ \hdashline 0 & C_1^{(1)} \end{array}\right] U_1$$

and

$$D = U_1^* D_1 U_1 = U_1^* \left[\begin{array}{c:c} \lambda_1 & \cdots \\ \hdashline 0 & D_1^{(1)} \end{array}\right] U_1.$$

Then we have to prove that $C_1^{(1)}$ and $D_1^{(1)}$ are projectors, and $C_1^{(1)}D_1^{(1)}-D_1^{(1)}C_1^{(1)}$ is nilpotent, consequently the triangularisation process can be continued. Observe that

$$U_1^*\left[\begin{array}{c|c}\mu_1 & \dots \\ \hline 0 & C_1^{(1)}\end{array}\right]U_1 = C = C^2 = U_1^*\left[\begin{array}{c|c}\mu_1^2 & \dots \\ \hline 0 & (C_1^{(1)})^2\end{array}\right]U_1,$$

consequently $C_1^{(1)}=(C_1^{(1)})^2$. Then

$$0=(CD-DC)^k=[U_1^*(C_1D_1-D_1C_1)U_1]^k = U_1^*(C_1D_1-D_1C_1)^kU_1 =$$

$$= U_1^*\left[\begin{array}{c|c}0 & \dots\dots \\ \hline 0 & (C_1^{(1)}D_1^{(1)}-D_1^{(1)}C_1^{(1)})^k\end{array}\right]U_1$$

which completes the proof.

REMARK. The conditions of the theorem are not very restrictive, since there exist nonzero, noncommutative projector matrices having these properties. If one selects

$$A=\begin{bmatrix}1&0&0&0&1&-1\\0&1&0&0&0&0\\0&0&1&0&-1&1\\0&0&0&1&0&0\\1&2&1&2&0&0\\1&2&1&2&0&0\end{bmatrix},\quad B=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

then

$$AB-BA\neq 0,\quad \text{but}\quad (AB-BA)^2=0.$$

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A NOTE ON ALMOST NILPOTENT RINGS

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All rings considered in this note are associative. Let α, α_l, α_r, α_s be the classes of all rings R such that every non-zero ideal, left ideal, right ideal, subring of R, respectively, strictly contains a power of R. Obviously $\alpha \supseteq \alpha_l, \alpha_r \supseteq \alpha_s$. Rings of the class α are called almost nilpotent.

Let l_M denote the lower radical determined by a class M of rings. In [3] Heyman, Jenkins and Le Roux proved that $l\alpha_s \neq l\alpha_l$ and asked whether

a) $l\alpha_s$ is equal to the prime radical β;

b) $l\alpha_l \neq l\alpha$.

They proved that if R is the ring of Example 13 of [1] then $T = Rz \in \alpha$ and conjectured that

c) $T \notin \alpha_l$.

In [7] Sands answered the questions a) and b) and asked whether

d) $\alpha_l = \alpha_r$ or $l\alpha_l = l\alpha_r$.

Here we will make some simple observations on the subject. In particular we answer the quoted questions (arguments applied to a) and b) differ a little from those used by Sands) and give a new proof of the fact that $l\alpha$ is an N-radical.

That I is an ideal (left ideal, right ideal) of a ring R will be denoted by $I \lhd R$ ($I <_l R$, $I <_r R$).

It is known [3, 4] and easy to check that the class α consists of nilpotent and prime non-idempotent rings, the classes α, α_l, α_s are hereditary with respect to ideals, left ideals and subrings respectively and that the classes α, α_l, α_s are homomorphically closed. Let us observe also that $R \in \alpha$ if and only if $\bigcap_{n=1}^{\infty} R^n = 0$ and $0 \neq I \lhd R$ implies that R/I is nilpotent.

LEMMA 1. *Let M be a left hereditary class of rings any element of which is β-radical or β-semisimple.*

i) *if $R \in M$ and R contains a non-zero nilpotent element then $R \in \beta$.*

ii) *if M is homomorphically closed then $\alpha \subsetneqq l_M$.*

PROOF. i) Let S be the subring of R generated by a nonzero element $a \in R$ such that $a^2 = 0$ and let $L = R^1 a$, where R^1 is the ring obtained by an adjoining unity to the ring R. Since $R^1 a <_l R$, $R^1 a \in M$. Now $S <_l R^1 a$ and $S^2 = 0$. Thus $\beta(R^1 a) \neq 0$, so $R^1 a \in \beta$. In consequence $\beta(R) \neq 0$. Hence $R \in \beta$.

ii) In [4] Korolczuk proved that if R is a finitely generated prime nil ring (examples of such rings were constructed by Golod (cf [2]) then R contains a maximal ideal I with respect to the property $R^n \subsetneqq I$ and that the ring $\bar{R} = R/I$ is prime and belongs to α. We claim that $\bar{R}\notin l_M$. Namely, if not, then, since M is homorphically closed, $\bar{R}$ contains a non-zero accessible subring $A\in M$. But A is a nil ring, so i) implies that $A\in\beta$. Thus $\beta(R)\neq 0$, a contradiction.

COROLLARY 1 ([7], Theorem 1). *α_s is the class of all nilpotent rings and $l\alpha_s=\beta$.*

PROOF. Hilbert Nullstellensatz implies that if a nonzero element x of a ring R is not nilpotent then the subring of R generated by x can be homomorphically mapped onto a finite field. Thus every homomorphically closed class of rings which is hereditary with respect to subrings consists of nil rings or contains a finite field. This, i) and the fact that α_s contains no field imply that $\alpha_s\subseteq\beta$. Now the fact that all β-rings of α are nilpotent ends the proof.

The class α_1 satisfies all the assumptions of Lemma 1 ii). Thus we have

COROLLARY 2 ([7]). $\alpha\subsetneqq l\alpha_l$. *In particular* $l\alpha\neq l\alpha_l$.

This Corollary is also a consequence of the following

PROPOSITION 1. *If R is the ring of Example* 13 *of* [1] *then* $T=Rz$ *is $l\alpha_l$-semisimple.*

PROOF. If $l\alpha_l(T)\neq 0$ then T contains a non-zero accessible α_1-subring A. Since $T\lhd R$, A is also an accessible subring of R. Let A^* be the ideal of R generated by A. By Andrunakievich lemma $(A^*)^n\subseteq A$ for some n. Since $\beta(R)=0$, $(A^*)^n\neq 0$. The hereditariness of α_1 implies that $(A^*)^n\in\alpha_l$. Since all the ideals of R have the form Rz^m, $m=0, 1, \ldots$, we obtain that $(A^*)^n=Rz^m\in\alpha_l$ for some m. Now $0\neq R(1-z)z^m< {}_lRz^m$. Obviously no power of Rz^m is contained in $R(1-z)z^m$. Thus $Rz^m\notin\alpha_l$, a contradiction.

Obviously Proposition 1 confirms c).

A modification of Proposition 1 answers d). Let K be the field of rational functions in countably many variables $x_1, x_2, \ldots$ with rational coefficients and let $P=K\{z,\sigma\}=\left\{\sum_{i=0} a_iz^i\,|\,a_i\in K\right\}$ be the skew power series ring with the homomorphism $\sigma\colon K\to K$ defined by $\sigma(x_i)=x_{i+1}$.

PROPOSITION 2. *The ring Pz belongs to α_l and is $l\alpha_r$-semisimple.*

PROOF. One can easily check that if $L<_l P$ then $L\lhd P$ and $L=Pz^n$ for some integer $n\geqq 0$. Thus $T\in\alpha_l$. Now if $l\alpha_r(T)\neq 0$ then T contains a non-zero accessible subring $A\in\alpha_r$. Andrunakievich lemma and hereditariness of α_r imply that we can assume $A\lhd P$. Thus $A=Pz^n$ for some $n\geqq 1$. Now $zPz^n=\sigma(K)\{z,\sigma\}z^{n+1}$ is a non-zero right ideal of A. Since for $m=1, 2, \ldots$, $A^m=(Pz^n)^m=Pz^{n+m}\subsetneqq zPz^n$, we obtain a contradiction.

The class α is neither left nor right hereditary ([7]). However we have

PROPOSITION 3. *If L is a left (right) ideal of $R\in\alpha$ then $L/\beta(L)\in\alpha$.*

PROOF. Let $L<_l R$ and $R\in\alpha$. If R is nilpotent then $L=\beta(L)$, so $L/\beta(L)\in\alpha$. Thus let the ring R be prime. Then, since $L\beta(L)<_l R$ and $L\beta(L)\in\beta$, $L\beta(L)=0$. This proves that $\beta(L)=\{l\in L\,|\,Ll=0\}$. Hence if $0\neq I|\beta(L)\lhd L/\beta(L)$ then $LI\neq 0$ and, since R is prime, $0\neq LIR\lhd R$. Thus R/LIR is nilpotent and, consequently, $L/L\cap LIR\approx L+LIR/LIR$ is nilpotent. This and the fact that $(L\cap LIR)^2\subseteq LIRL\subseteq$

$\subseteqq LIL \subseteqq I$, prove that L/I is nilpotent. Now since $\bigcap_{n=1}^{\infty} R^n = 0$, $\bigcap_{n=1}^{\infty} (L/\beta(L))^n = 0$. Therefore $L/\beta(L) \in \alpha$.

Similarly if $L <_r R$ then $L/\beta(L) \in \alpha$.

COROLLARY 3. *The radical $l\alpha$ is left and right hereditary.*

PROOF. Let $\bar{\alpha} = \{R | R/\beta(R) \in \alpha\}$. Obviously $l\alpha = l\bar{\alpha}$. By Proposition 3, $\bar{\alpha}$ is left and right hereditary, so $l\alpha$ is left and right hereditary.

PROPOSITION 4. *Let L be a left (right) ideal of R and let L^* be the ideal of R generated by L.*

a) *If $I \lhd L^*$ and $IL \in \beta$ ($LI \in \beta$) then $I \subseteqq \beta(L^*)$.*

b) *If $L/\beta(L) \in \alpha$ then $L^*/\beta(L^*) \in \alpha$.*

PROOF. Let us assume that L is a left ideal of R (if $L <_r R$, symmetric arguments can be applied).

a) Since $IL <_l L^*$, $IL \subseteqq \beta(L^*)$. Now $IL^* = ILR^1 \subseteqq \beta(L^*)R^1 \subseteqq \beta(L^*)$. In particular $I^2 \subseteqq \beta(L^*)$, so $I \subseteqq \beta(L^*)$.

b) Let $\beta(L^*) \subsetneqq I \lhd L^*$ and let I^* be the ideal of R generated by I. Since $IL \subseteqq I^* \cap L$ and, by a) $IL \notin \beta$, we have $L \cap I^* \notin \beta$. Thus, since $L/\beta(L) \in \alpha$, $L/\beta(L) + L \cap I^*$ is nilpotent. Now $L\beta(L) <_l L^*$ and $L\beta(L) \in \beta$, so $L\beta(L) \subseteqq \beta(L^*) \subseteqq I^*$. Hence $(\beta(L))^2 \subseteqq L \cap I^*$ and, consequently, $(\beta(L) + L \cap I^*)^2 \subseteqq L \cap I^*$. This proves that $L/L \cap I^*$ is nilpotent. But L^*/I^* is equal to the ideal of R/I^* generated by $L + I^*/I^* \approx L/L \cap I^*$ and $(I^*)^3 \subseteqq I$, so L^*/I is nilpotent.

Now $\left(\bigcap_{n=1}^{\infty} (L^*)^n\right) L = \left(\bigcap_{n=1}^{\infty} (LR^1)^n\right) L \subseteqq \bigcap_{n=1}^{\infty} (LR^1)^n L = \bigcap_{n=1}^{\infty} L^{n+1} \subseteqq \beta(L)$. Hence, by a), $\bigcap_{n=1}^{\infty} (L^*)^n \subseteqq \beta(L^*)$. This ends the proof.

Proposition 4 and Proposition 3 of [5] give the following

COROLLARY 4. *The radical $l\alpha$ is left and right strong.*

Left strong and left hereditary radicals containing β are called ([6]) N-radicals. Thus $l\alpha$ is an N-radical.

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К РАСШИРЕННОМУ ИНТЕРПОЛЯЦИОННОМУ ПРОЦЕССУ КРЫЛОВА—ШТАЕРМАНА

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1. Пусть

$$x_k^{(n)} = \cos \frac{2k-1}{2n}\pi, \quad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots \tag{1}$$

и C — множество всех функций $f(x)$, непрерывных в $[-1, 1]$. Обозначим через $H_n(f, x)$ многочлен степени $2n-1$, однозначно определяемый из условий

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad H_n'(f, x_k^{(n)}) = 0, \quad k = 1, 2, \ldots, n, \tag{2}$$

где $f \in C$. Как известно, процесс $\{H_n(f, x)\}$ называется интерполяционным процессом Эрмита—Фейера. Л. Фейер [1] доказал, что для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение

$$H_n(f, x) \to f(x), \quad n \to \infty.$$

Н. М. Крылов и И. Я. Штаерман [2] удлинили процесс Эрмита—Фейера в том смысле, что они заменили полином $H_n(f, x)$ степени $2n-1$, однозначно определяемый из условий (2) на полином $P_n(f, x)$ степени $4n-1$, однозаначно определяемый из $4n$ условий:

$$P_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad P_n^{(i)}(f, x_k^{(n)}) = 0, \quad i = 1, 2, 3, \quad k = 1, 2, \ldots, n,$$

где $P_n^{(i)}(f, x)$ — производная порядка i от $P_n(f, x)$. Н. М. Крылов и И. Я. Штаерман [2] доказали, что для любой $f \in C$ выполняется в $[-1, 1]$ соотношение $P_n(f, x) \to f(x), n \to \infty$.

В 1965 г. появилась статья автора [3], в которой изучался процесс $\{H_n(f, x)\}$ для узлов

$$x_0^{(n+2)} = 1, \quad x_k^{(n+2)} = \cos \frac{2k-1}{2n}\pi, \quad k = 1, \ldots, n, \quad x_{n+1}^{(n+2)} = -1, \quad n = 1, 2, \ldots, \tag{3}$$

полученных расширением узлов (1) добавлением в качестве узлов точек ± 1. Оказалось, что этот процесс, построенный для $f(x) = |x|$, расходится при $x = 0$. Позже [4], [5], [6] было доказано, что процесс $\{H_n(f, x)\}$, построенный при узлах (3) для $f(x) = x^2$ и $f(x) = |x|$ расходится всюду в $(-1, 1)$. При $f(x) = x$ этот процесс расходится во всех точках $x \neq 0$ из $(-1, 1)$. Л. Кук и Т. М. Миллс [7] доказали, что порцесс $\{P_n(f, x)\}$ Крылова—Штаермана, построенный при узлах (3) для $f(x) = (1-x^2)^3$ расходится при $x = 0$. После этого естественно встал вопрос и построении всюду расходящегося в $[-1, 1]$ процесса Крылова—Штаермана. Этот вопрос решен в [8]. Полином $P_n(f, x)$ имеет довольно сложный вид. Для упрощения вычислений в [8] рассматривался случай четного n, $n=2m$. Тогда среди узлов (1) нет точки $x=0$. Именно эта точка и добавляется в качества узла. Таким образом, матрица узлов состоит из чисел [1]

$$x_k^{(2m)} = \cos\frac{2k-1}{4m}\pi, \quad k = 1, 2, \dots, 2m, \quad x = 0, \quad m = 1, 2, \dots . \tag{4}$$

Расширенный интерполяционный полином $\Pi_n(f, x)$ Крылова—Штаермана степени $8m+3$ однозначно определяется из условий

$$\Pi_n(f, x_k^{(2m)}) = f(x_k^{(2m)}), \quad \Pi_n^{(i)}(f, x_k^{(2m)}) = 0, \quad i = 1, 2, 3, \quad k = 1, 2, \dots, 2m,$$
$$\Pi_n(f, 0) = f(0), \quad \Pi_n^{(i)}(f, 0) = 0, \quad i = 1, 2, 3.$$

В [8] доказана следующая теорема:

Теорема 1. *Расширенный интерполяционный процесс* $\{\Pi_n(f, x)\}$ *Крылова—Штаермана, построенный при узлах* (4) *для* $f(x)=x^2$ *расходится во всех точках* $x\neq 0$ *из* $[-1, 1]$.

2. В связи с этим интересна:

Теорема 2. *Расширенный интерполяционный процесс* $\{\Pi_n(f, x)\}$ *Крылова—Штаермана, построенный при узлах* (4) *для* $f(x)=x^{2m}$, $m\geqq 2$, *сходится равномерно в* $[-1, 1]$.

Доказательство. Рассмотрим разность $r_n(f, x)=\Pi_n(f, x)-P_n(f, x)$. Из определения полиномов $\Pi_n(f, x)$ и $P_n(f, x)$ следует, что точки $\{x_k^{(2m)}\}_{k=1}^{2m}$ являются корнями четвертой кратности $r_n(f, x)$. Учтем еще, что узлы $\{x_k^{(n)}\}_{k=1}^{n}$ расположены симметрично относительно точки $x=0$ и что $f(x)=x^{2m}$ — четная функция. Поэтому

$$r_n(f, x) = T_n^4(x)(a+bx^2), \quad T_n(x) = \cos n \arccos x, \tag{5}$$

где a и b определяются из равенств

$$a = f(0)-P_n(f, 0), \quad [T_n^4(x)(a+bx^2)]''_{x=0} = -P_n''(f, 0). \tag{6}$$

С помощью дифференциального уравнения для $T_n(x)$ и формулы Лейбница из (6) получим

$$b = 2n^2\big(f(0)-P_n(f, 0)\big)-\frac{1}{2}P_n''(f, 0). \tag{7}$$

Найдем теперь $P_n(f, 0)$ из $P_n''(f, 0)$. Х. Н. Ладен [9] доказал, что при любой системе узлов $\{x_k\}_{k=1}^{n}$ интерполяционный полином Крылова—Штаермана $P_n(f, x)$ может быть вычислен по формуле

$$P_n(f, x) = \sum_{k=1}^{n} f(x_k)U_k(x)[l_k(x)]^4, \tag{8}$$

[1] Интерполяционные процессы при матрице узлов (4) также рассматривались в [10] и [11].

где

$$(9)\qquad U_k(x) = 1-2(x-x_k)\frac{\omega''(x_k)}{\omega'(x_k)}+\frac{(x-x_k)^2}{2}\left[5\left(\frac{\omega''(x_k)}{\omega'(x_k)}\right)^2-\frac{4}{3}\frac{\omega^{(3)}(x_k)}{\omega'(x_k)}\right]+$$

$$+\frac{(x-x_k)^3}{6}\left[-15\left(\frac{\omega''(x_k)}{\omega'(x_k)}\right)^3+10\frac{\omega''(x_k)\omega^{(3)}(x_k)}{(\omega'(x_k))^2}-\frac{\omega^{(4)}(x_k)}{\omega'(x_k)}\right],$$

$$(10)\qquad l_k(x) = \frac{\omega(x)}{(x-x_k)\omega'(x_k)},$$

$$\omega_n(x) = \omega(x) = \prod_{k=1}^{n}(x-x_k).$$

Из (9) с помощью дифференциального уравнения для $T_n(x)$ после элементарных вычислений получим, что

$$(11)\qquad U_k(0) = 1+\frac{2(n^2+2)x_k^2}{3(1-x_k^2)}+\frac{(2n^2+1)x_k^4}{3(1-x_k^2)^2}.$$

При узлах (1)

$$(12)\qquad l_k(0) = (-1)^{k+m}\sqrt{1-x_k^2}/nx_k.$$

Из (8), (11), (12) следует, что при $f(x)=x^{2m}$

$$(13)\qquad P_n(z^{2m},0) = \sum_{k=1}^{n} x_k^{2m}\left(1+\frac{2(n^2+2)x_k^2}{3(1-x_k^2)}+\frac{(2n^2+1)x_k^4}{3(1-x_k^2)^2}\right)\frac{(1-x_k^2)^2}{n^4x_k^4}.$$

Введем обозначения

$$\sum_{k=1}^{n} x_k^{2m-2} = \theta_1 n,\quad \sum_{k=1}^{n} x_k^{2m} = \theta_2 n$$

и напомним тождество

$$(14)\qquad \sum_{k=1}^{n}\frac{1}{x_k^2} = n^2,$$

которое нужно будет в дальнейшем. Из (13) получим

$$(15)\quad P_n(z^{2m},0) = \frac{2(n^2+2)}{3n^4}n(\theta_1-\theta_2)+\frac{(2n^2+1)\theta_2 n}{3n^4}+O\left(\frac{1}{n^3}\right) = \frac{2\theta_1}{3n}+O\left(\frac{1}{n^3}\right).$$

Найдем теперь b. С помощью формулы Лейбница из (8) вытекает

$$(16)\qquad P_n''(z^{2m},0) = \sum_{k=1}^{n} x_k^{2m}U_k''(0)\big(l_k(0)\big)^4+8\sum_{k=1}^{n} x_k^{2m}U_k'(0)\,l_k^3(0)\,l_k'(0)+$$

$$+12\sum_{k=1}^{n} x_k^{2m}U_k(0)\,l_k^2(0)\big(l_k'(0)\big)^2+4\sum_{k=1}^{n} x_k^{2m}U_k(0)\,l_k^3(0)\,l_k''(0) \equiv \sum_{i=1}^{4} S_i.$$

Из (9) с помощью дифференциального уравнения для $T_n(x)$, после некоторых вычислений, получим, что

(17) $$U_k'(0) = -\frac{2x_k(2n^2+1)}{3(1-x_k^2)} - \frac{(4n^2+1)x_k^3}{2(1-x_k^2)^2};$$

(18) $$U_k''(0) = \frac{4(n^2-1)}{3(1-x_k^2)} + \frac{4n^2x_k^2}{(1-x_k^2)^2}.$$

Из (16) и (12), (18) следует

(19) $$S_1 = \frac{1}{n^4}\sum_{k=1}^{n} x_k^{2m-4}\left[\frac{4(n^2-1)(1-x_k^2)}{3} + 4n^2x_k^2\right].$$

Если учесть $m \geqq 2$, то из (19) вытекает

(20) $$S_1 = O\left(\frac{1}{n}\right).$$

В силу (10) получим

(21) $$l_k'(0) = (-1)^{m+k}\frac{\sqrt{1-x_k^2}}{nx_k^2}.$$

Из (16), (17), (12), (21) выводим

$$S_2 = \frac{8}{n^4}\sum_{k=1}^{n} x_k^{2m-5}\left(-\frac{2x_k(2n^2+1)}{3(1-x_k^2)} - \frac{x_k^3(4n^2+1)}{2(1-x_k^2)^2}\right)(1-x_k^2)^2.$$

Поэтому при $m \geqq 2$

(22) $$S_2 = O\left(\frac{1}{n}\right).$$

Вычислим теперь S_3. В силу (16), (11), (12), (21) имеем

$$S_3 = \frac{12}{n^4}\sum_{k=1}^{n} x_k^{2m-6}\left(1 + \frac{2(n^2+2)x_k^2}{3(1-x_k^2)} + \frac{(2n^2+1)x_k^4}{3(1-x_k^2)^2}\right)(1-x_k^2)^2.$$

Отсюда при $m > 2$ непосредственно видно, что

(23) $$S_3 = O\left(\frac{1}{n}\right).$$

Если же $m=2$, то для установления (23) нужно использовать тождество (14). Вычислим S_4. Из (10) получается, что

(24) $$l_k''(0) = (-1)^{m+k+1}(n^2x_k^2-2)\sqrt{1-x_k^2}/nx_k^3.$$

Поэтому из (16), (11) (24) выводим

$$S_4 = -\frac{4}{n^4}\sum_{k=1}^{n} x_k^{2m-6}\left(1 + \frac{2(n^2+2)x_k^2}{3(1-x_k^2)} + \frac{(2n^2+1)x_k^4}{3(1-x_k^2)^2}\right)(n^2x_k^2-2)(1-x_k^2)^2.$$

Стало быть,

(24) $$S_4 = -\frac{8(n^2+2)(\theta_1-\theta_2)}{3n} - \frac{4(2n^2+1)\theta_2}{3n} + O\left(\frac{1}{n}\right) = -\frac{8n\theta_1}{3} + O\left(\frac{1}{n}\right).$$

Наконец, из (16), (20), (22), (23), (24) выводим

$$P_n''(z^{2m}, 0) = -\frac{8n\theta_1}{3} + O\left(\frac{1}{n}\right). \tag{25}$$

Из (7), (15), (25) следует, что

$$b = O\left(\frac{1}{n}\right). \tag{26}$$

В силу (5), (26) выводим, что

$$\|\Pi_n(f, x) - P_n(f, x)\| \leqq \frac{C}{n}, \quad f(x) = x^{2m}, \quad \|f\| = \operatorname*{Max}_{-1 \leqq x \leqq 1} |f(x)|,$$

где $C > 0$ — константа. Поэтому

$$\|\Pi_n(f, x) - f(x)\| \leqq \|P_n(f, x) - f(x)\| + \frac{C}{n}.$$

Согласно теореме Крылова—Штаермана $\lim\limits_{n\to\infty} \|P_n(f, x) - f(x)\| = 0$. Поэтому

$$\lim_{n\to\infty} \|\Pi_n(f, x) - f(x)\| = 0.$$

В данной статье рассматривается процесс Крылова—Штаермана при корнях полинома Чебышева. Хорошо известно, что полиномы Чебышева частный случай полиномов Якоби. Вероятно, было бы интересно рассматривать расширенный интерполяционный процесс Крылова—Штаермана при корнях полиномов Якоби.

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О ПРОЦЕССЕ УСРЕДНЕНИЯ ФУНКЦИЙ ОГРАНИЧЕННОЙ ВАРИАЦИИ

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Пусть Φ некоторый класс вещественных ограниченных функций определённых на множестве X. Следуя [1] введём:

Определение 1. Пусть $P=(S, A, \mu)$ вероятностное пространство. Скажем, что функция $F(x, s)$: $X\times S\to R$ допустимая, если удовлетворяет следующим условиям:

(i) $F(x, s)$ ограничена,
(ii) $F(\cdot, s)\in\Phi$ при любом $s\in S$,
(iii) $F(x, \cdot)$ измерима в (S, A) при любом $x\in X$.

Определение 2. Через Φ^a обозначим класс тех функций f: $X\to R$, для которых существуют вероятностные пространства $P=(S, A, \mu)$ и допустимые функции $F(x, s)$, такие что

$$f(x)=\int\limits_S F(x, s)\,d\mu(s)\quad (x\in X).$$

Через U обозначим класс ступенчатых функций определённых на [0, 1], т.е. $f\in U$ тогда и только тогда, когда существуют точки $0=x_0<x_1<\ldots<x_n=1$, такие что f постоянна на интервалах (x_{i-1}, x_i). Через V обозначим класс функций ограниченной вариации определённых на [0, 1], и через W обозначим класс функций $f(x)$ определённых на [0, 1], для которых $V_n(f)=o(n)$ где

$$V_n(f)=\sup_{\{x_i\}_{i=1}^n}\sum_{i=0}^{n-1}|f(x_i)-f(x_{i+1})|.$$

Легко видеть, что W совпадает с классом функций, имеющих только точки разрыва первого рода, а последний совпадает с классом функций, для которых существуют пределы

$$f(x+)=\lim_{z\to x+}f(z)\quad\text{и}\quad f(y-)=\lim_{z\to y-}f(z)$$

где $x\in[0, 1)$ и $y\in(0, 1]$. Легко убедиться также, что каждая функция $f\in W$ есть равномерный предел ступенчатых функций.

В работе [1] была поставлена следующая задача (см. [1]): найти класс Φ^a в том случае, когда $\Phi=V$, т. е. определить структурные свойства функций класса V^a.

В настоящей работе доказывается следующая

Теорема. $U^a=V^a=W^a=W$. *Если* $f\in W$, *то для любого* $\varepsilon>0$ *существует, определённая на* $[0, 1]\times[0, 1]$, *функция* $F(x, s)$, *такая что*

1) $\|F\|<\|f\|+\varepsilon$ ($\|\cdot\|$ *означает точная верхная грань значений функций*),
2) $F(\cdot, s)\in U$ *для любого* $s\in[0, 1]$,
3) $F(x, \cdot)$ *измеримая по Лебегу для любого* $s\in[0, 1]$,
4) $f(x)=\int\limits_0^1 F(x, s)\,ds$.

Эта теорема является решением вышеуказанной задачи.

Доказательство. Сперва докажем, что $W^a\subset W$. Пусть $P=(S, A, \mu)$ вероятностное пространство, а G: $[0, 1]\times S\to R$ допустимая функция относительно класса W. Пусть $f(x)=\int\limits_S F(x, s)\,d\mu(s)$. Если $x_n<x$ и $\lim\limits_{n\to\infty} x_n=x$, то для любого $s\in S$ $\lim\limits_{n\to\infty} G(x_n, s)=G(x+, s)$ так как $G(\cdot, s)\in W$. Тогда, используя ограниченность $G(x, s)$ и известную теорему Лебега, имеем

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\int\limits_0^1 G(x_n, s)\,d\mu(s) = \int\limits_0^1 G(x+, s)\,d\mu(s).$$

и следовательно, в силу произвольности x_n, существует $f(x+)$. То же самое верно и для $f(x-)$. Следовательно $f\in W$.

Теперь пусть заданы $f\in W$ и $\varepsilon>0$. Можно предположить, что $\|f\|=1$. Выберём $0<\eta<1$, такое что $\eta/(1-\eta)<\varepsilon$ и последовательность функций $g_k\in U$, такая что $|f(x)-g_k(x)|<\eta^{k+1}/2$ $(k=0, 1, 2, \ldots, x\in[0, 1])$. Обозначим $f_0=g_0$ и $f_k=g_k-g_{k-1}$ $(k=1, 2, \ldots)$. Имеем $f_k\in U$, $f=\sum\limits_{k=0}^{\infty} f_k$, $\|f_0\|<1+\eta/2$ и $\|f_k\|<\eta^k (k\geqq 1)$. Для любого $x\in[0, 1]$, определим $F(x, 0)=0$ и $F(x, s)=(\eta^k\eta^{k+1})^{-1}f_k(x)$, если $\eta^{k+1}<s\leqq\eta^k$ $(k=0, 1, \ldots)$. Легко убедиться, что $F(x, s)$ удовлетворяет условиям 1)—4) теоремы, и следовательно $W\subset U^a$. Соотношения $U^a\subset V^a\subset W^a$ следуют из соотношений $U\subset V\subset W$. Итак, имеем $U^a\subset V^a\subset W^a\subset W\subset U^a$. Теорема доказана.

В заключение автор выражает благодарность рецензенту, за предложенное упращение доказательства теоремы.

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ALGEBRAS WITH BOOLEAN AND STONEAN CONGRUENCE LATTICES

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1. Introduction

It is known tha the congruence lattice Con (A) of an algebra A is complete and compactly generated. Assuming congruence distributive algebras A (lattices, l-groups, Boolean algebras etc.), the lattice Con (A) enjoys an additional property: Con (A) is relatively pseudocomplemented and, in particular, pseudocomplemented. Thus a natural problem arises to determine those algebras whose congruence lattices have certain specified properties formulated in terms of (relative) pseudocomplements. For example, one can ask after those algebras whose congruence lattices are atomic, Boolean, Stonean or completely (relatively) Stonean.

There are some results in the literature along these lines. For example, T. Tanaka [23], G. Grätzer and E. T. Schmidt [11], and P. Crawley [6] have characterized those lattices whose congruence lattices are Boolean, answering a question posed by G. Birkhoff (see [5; p. 157, Problem 39]). R. Beazer [2] have described pseudocomplemented semilattices (see also H. P. Sankappanavar [21]), p-algebras (see also J. Berman [4]) and double p-algebras with Boolean congruence lattices. For complemented lattices this was done by M. F. Janowitz [15]. Tanaka's result has been generalized for universal algebras by J. Hashimoto [12]. On the other hand, O. Frink [9] described Boolean algebras with Stonean congruence lattices. T. Katriňák [17] offered a description of those lattices whose congruence lattices are Stonean. Later, for special classes of lattices, the same was done by Iqbalunissa [13] and M. F. Janowitz [14], [15]. The last known results in this area are those of R. Beazer [3] for regular double p-algebras and of H. P. Sankappanavar [21] for pseudocomplemented semilattices.

In this paper we use a technique of Tanaka [23] adapted for universal algebras and that of Hashimoto [12] to describe algebras with atomic, Boolean, Stonean and completely Stonean congruence lattices, respectively (Theorems 1, 2, 4, 5, 7, 9 and 10). We have introduced a concept of an algebra with a strong centre. The class of these algebras comprises almost all the mentioned special algebras, as p-algebras, complemented lattices and so on. Theorems 6 and 11 give information about those algebras with a strong centre whose congruence lattices are Boolean and (completely) Stonean, respectively.

2. Preliminaries

Let Con (A) denote the set of all congruence relations on a universal algebra A. Then Con (A) forms a complete and compactly generated lattice with Δ and ∇, the smallest and the largest congruence relations, respectively. Throughout this paper we consider congruence distributive algebras only, i.e. we assume that Con (A)

is a distributive lattice. It is a well known fact that distributivity of Con (A) implies infinite distributivity

$$\theta \wedge \bigvee (\alpha_i\colon i\in I) = \bigvee (\theta \wedge \alpha_i\colon i\in I)$$

for any $\theta, \alpha_i \in \mathrm{Con}(A)$. It follows that for every $\alpha, \beta\in \mathrm{Con}(A)$ there exists a largest $\tau\in \mathrm{Con}(A)$ such that $\alpha\wedge\tau\leqq\beta$. Clearly, $\tau=\bigvee(\delta\colon \alpha\wedge\delta\leqq\beta)$. (Notation: $\tau=\alpha_*\beta$.) In particular, for every $\alpha\in\mathrm{Con}(A)$ there exists a (uniquely determined) pseudocomplement $\alpha^*=\alpha_*\Delta$. Thus, Con (A) is a complete Heyting algebra.

A *pseudocomplemented semilattice* (=PCS) is an algebra $(S; \wedge, ^*, 0, 1)$ in which $(S; \wedge, 0, 1)$ is a bounded meet-semilattice and for every element $a\in S$ the element $a^*\in S$ is the *pseudocomplement* of a; that is $x\leqq a^*$ if and only if $x\wedge a=0$. If, for any PCS S, we write $B(S)$ for $\{x\in S\colon x=x^{**}\}$ (the set of *closed* elements of S) and $D(S)$ for $\{x\in S\colon x^{**}=1\}$ (the set of *dense* elements of S), then $(B(S); \sqcup, \wedge, ^*, 0, 1)$ is a Boolean algebra when $a\sqcup b$ is defined to be $(a^*\wedge b^*)^*$, for any $a, b\in B(S)$ and $D(S)$ is a filter (dual ideal) in S.

An algebra $(L; \vee, \wedge, ^*, 0, 1)$ is called a *p-algebra* or a *pseudocomplemented lattice* (=PCL) if $(L; \wedge, ^*, 0, 1)$ is a PCS and $(L; \vee, \wedge)$ is a lattice. A double *p-algebra* is an algebra $(L; \vee, \wedge, ^*, ^+, 0, 1)$ in which the deletion of $^+$ gives a p-algebra and the deletion of * gives a *dual* p-algebra, that is $a\vee x=1$ if and only if $x\geqq a^+$.

A special class of PCS's is formed by the *implicative* semilattices $(S; \wedge, _*, 0, 1)$, where $(S; \wedge, 0, 1)$ is a bounded semilattice and $x\wedge y\leqq z$ if and only if $y\leqq x_* z$. Then $x^*=x_*0$ plays the role of a pseudocomplement of x. $(L; \vee, \wedge, _*, 0, 1)$ is called a *Heyting* algebra if the deletion of $\vee$ gives an implicative semilattice. Similarly, $(L; \vee, \wedge, *, +, 0, 1)$ is called a *double Heyting* algebra, if $(L; \vee, \wedge, *, 0, 1)$ is a Heyting algebra and the dual of $(L; \vee, \wedge, +, 0, 1)$ forms also a Heyting algebra.

Frequently we shall use the following rules of computation with $^*, \vee, \wedge$.

(1) $$x\leqq y \quad \text{implies} \quad x^*\geqq y^*,$$

(2) $$x\leqq x^{**},$$

(3) $$x^* = x^{***},$$

(4) $$(x\vee y)^* = x^*\wedge y^*,$$

(5) $$(x\wedge y)^{**} = x^{**}\wedge y^{**},$$

(6) $$(x\vee y)^{**} = (x^{**}\vee y^{**})^{**} = (x^*\wedge y^*)^*.$$

For complete PCL's it is true that

(7) $$\bigwedge(x_i^{**}\colon i\in I) = \big(\bigwedge(x_i^{**}\colon i\in I)\big)^{**}.$$

A distributive p-algebra L is said to be a *Stone algebra*, if it satisfies the identity

$$x^*\vee x^{**} = 1.$$

This is equivalent with the identity $(x\wedge y)^*=x^*\vee y^*$.

The relation γ on a PCS or a PCL L defined by

$$a\equiv b(\gamma) \quad \text{if and only if} \quad a^*=b^*$$

is a congruence relation on L and is called the *Glivenko* congruence. The relation φ on a double p-algebra L defined by $a \equiv b(\varphi)$ if and only if $a^*=b^*$ and $a^+=b^+$ is a congruence relation on L and is called the *determination* congruence. A double p-algebra L is said to be regular, if $\varphi=\Delta$.

Every Boolean algebra $(B;\ \vee, \wedge, ', 0, 1)$ is a distributive p-algebra in which $a^*=a'$ for every $a\in B$. It is easily verified that a p-algebra L is a Boolean algebra if and only if $x=x^{**}$ for every $x\in L$. Distributive p-algebras satisfy the identity

$$x = x^{**} \wedge (x \vee x^*),$$

which has the following consequences: (i) A distributive p-algebra L is a Boolean algebra if and only if $D(L)$ contains only 1.

(ii) Let A be a congruence distributive algebra. If $\theta\in \mathrm{Con}(A)$ is (completely) meet-irreducible, i.e. A/θ is finitely (subdirectly) irreducible, then $\theta\in D(\mathrm{Con}(A))$ or θ is a dual atom of $B(\mathrm{Con}(A))$.

The centre $C(L)$ of a bounded lattice L is the set of all complemented, neutral elements of L and is, of course, a Boolean sublattice of L.

An *ortholattice* (=OL) is an algebra $(L;\ \vee, \wedge, ', 0, 1)$ satisfying the following identities:

(i) $(L;\ \vee, \wedge, 0, 1)$ is a bounded lattice,
(ii) $x \vee x'=1$ and $x\wedge x'=0$,
(iii) $(x \wedge y)'=x' \vee y'$ and $(x \vee y)'=x' \wedge y'$,
(iv) $x=(x')'$.

An ortholattice L is called an *orthomodular* lattice (=OML) if

$$x \leqq y \quad \text{implies} \quad x \vee (x' \wedge y) = y$$

for any $x, y\in L$.

We recall that an algebra A is called *(finitely) subdirectly irreducible* if

$$\bigwedge(\alpha_i\colon\ i\in I) = \Delta \quad \text{(for finite } I)$$

in $\mathrm{Con}(A)$ implies $\alpha_i=\Delta$ for some $i\in I$.

We refer to G. Birkhoff [5] and G. Grätzer [10] not only for the standard results about PCS's and PCL's but also for general lattice-theoretic and universal algebraic notations, terminology and result (see also P. Crawley and R. P. Dilworth [7] and E. T. Schmidt [22]). The standard results and rules of computation in double p-algebras may be found in R. Beazer [1] or in T. Katriňák [18], while those for implicative semilattices and ortholattices may be found in W. C. Nemitz [20] and G. Kalmbach [16], respectively.

3. Atomic congruence lattices

We shall start with a Tanaka's result stated for lattices. It can be generalized for congruence distributive algebras without changing the original proof. First we recall three concepts: Let $\{\alpha_i\colon\ i\in I\}\subseteq B(\mathrm{Con}(A))$ be such that $\alpha_i^*=\wedge(\alpha_j\colon\ j\in I, j\neq i)$ for every $i\in I$. Then $(A/\alpha_i\colon\ i\in I)$ is called a *canonical* subdirect factorization of A. A subdirect factorization $(A/\alpha_i\colon\ i\in I)$ of A is said to be *irredundant* if $\wedge(\alpha_j\colon\ j\in I, j\neq i)\neq\Delta$ for every $i\in I$. Eventually, a lattice with 0 is called *atomic*, if for every $a\neq 0$ there exists an atom $p\leqq a$.

THEOREM A ([23]; Theorem 1, Remark 1]). *Let A be a congruence distributive universal algebra. The following conditions on A are equivalent:*

(i) Con (A) *is an atomic lattice;*

(ii) $D(\text{Con}(A))$ *is a principal filter;*

(iii) *A has a canonical subdirect factorization with subdirectly irreducible factors;*

(iv) *A has an irredundant subdirect factorization with subdirectly irreducible factors.*

Now we can add two new conditions.

THEOREM 1. *Let A be a congruence distributive algebra. Then the conditions* (i)—(iv) *from Theorem* A *are equivalent with the following two conditions:*

(v) $B(\text{Con}(A))$ *is atomic and every dual atom of* $B(\text{Con}(A))$ *is completely meet-irreducible in* Con (A);

(vi) Con (A) *satisfies the (infinite) identity*

$$\wedge(x_i^{**}\colon i\in I) = (\wedge(x_i\colon i\in I))^{**}. \tag{8}$$

PROOF. (i)⇒(v). Let $\{\alpha_i\colon i\in I\}$ denote the set of all atoms of Con (A). Then $\{\alpha_i^*\colon i\in I\}$ comprises all dual atoms of $B(\text{Con}(A))$. Suppose that $\alpha_i^*<\delta$ for some $i\in I$. Therefore, $\delta\in D(\text{Con}(A))$, because α_i^* is a dual atom of $B(\text{Con}(A))$. Hence $\alpha_i\leqq\delta$, as α_i is an atom. Thus $\alpha_i^*<\alpha_i\vee\alpha_i^*\leqq\delta$.

(v)⇒(iii) is trivial.

(ii)⇒(vi). Assume $[\beta)=D(\text{Con}(A))$. Since $\alpha_i=\alpha_i^{**}\wedge\delta_i$ for every $i\in I$, where $\delta_i\in[\beta)$, we have

$$(\wedge(\alpha_i\colon i\in I))^{**} = (\wedge(\alpha_i^{**}\wedge\delta_i\colon i\in I))^{**} = ((\wedge(\alpha_i^{**}\colon i\in I))\wedge(\wedge(\delta_i\colon i\in I)))^{**} =$$
$$= (\wedge(\alpha_i^{**}\colon i\in I))^{**}\wedge(\wedge(\delta_i\colon i\in I))^{**} = (\wedge(\alpha_i^{**}\colon i\in I))^{**} = \wedge(\alpha_i^{**}\colon i\in I)$$

using (7) and the fact that $\wedge(\delta_i\colon i\in I)\geqq\beta$.

(vi)⇒(ii). Evidently, $\beta^{**}=(\wedge(\alpha\colon \alpha\in D(\text{Con}(A))))^{**}=\wedge(\alpha^{**}\colon \alpha\in D(\text{Con}(A)))=$ $=\nabla$. Thus $[\beta)=D(\text{Con}(A))$.

THEOREM 2. *Let A be a congruence distributive algebra. Then the following conditions are equivalent:*

(i) $B(\text{Con}(A))$ *is atomic;*

(ii) Con (A) *satisfies the (infinite) identity*

$$\wedge((\vee(x_{ts}\colon s\in S))^{**}\colon t\in T) = (\vee(\wedge(x_{t\varphi(t)}\colon t\in T))\colon \varphi\in S^T)^{**}; \tag{9}$$

(iii) *A has a canonical subdirect factorization with finitely subdirectly irreducible factors;*

(iv) *A has an irredundant subdirect factorization with finitely subdirectly irreducible factors;*

(v) *A has only one irredundant subdirect factorization with finitely subdirectly irreducible factors.*

PROOF. (i)⇒(ii). It is well known that an atomic and complete Boolean algebra satisfies the completely distributive identity. Hence $B(\text{Con}(A))$ is completely distributive. Moreover, Con (A) satisfies also the identity

$$(\vee(x_i\colon i\in I))^{**} = (\vee(x_i^{**}\colon i\in I))^{**}. \tag{10}$$

The last two facts imply readily (9).

(ii)⇒(i). Identities (9) and (10) imply complete distributivity of $B(\mathrm{Con}\,(A))$. From this (i) follows by a known theorem of Tarski (see [5; Theorem V.17]).

(i)⇒(iii). Let $\{\alpha_i\colon i\in I\}$ denote the set of all dual atoms of $B(\mathrm{Con}\,(A))$. Then $(A/\alpha_i\colon i\in I)$ is a desired factorization of A.

(iii)⇒(iv). Evidently, the factorization from (iii) is irredundant.

(iv)⇒(i). Let $(A/\alpha_i\colon i\in I)$ denote an irredundant subdirect factorization with finitely subdirectly irreducible factors. Therefore the α_i's are meet-irreducible congruence relations on A, and

$$\wedge(\alpha_i\colon i\in I)=\Delta$$

holds. We know that for every $i\in I$, α_i is a dual atom of $B(\mathrm{Con}\,(A))$ or $\alpha_i\in D(\mathrm{Con}\,(A))$. We claim that $\alpha_i\in B(\mathrm{Con}\,(A))$ for every $i\in I$. Assume to the contrary that $\alpha_i\in D(\mathrm{Con}\,(A))$ for some $i\in I$. Then $\beta=\wedge(\alpha_j\colon j\in I, j\neq i)\neq\Delta$. Hence $\Delta=\beta\wedge\alpha_i$, and consequently, $\Delta^{**}=\Delta=\beta^{**}\wedge\alpha_i^{**}=\beta^{**}$, a contradiction, as $\beta\leqq\beta^{**}$. Thus the α_i's are dual atoms of $B(\mathrm{Con}\,(A))$. Now it is easy to show that $B(\mathrm{Con}\,(A))$ is atomic.

(iv)⇔(v) is now trivial.

REMARK 1. Since for a congruence distributive algebra A is $B(\mathrm{Con}(A))$ a complete Boolean algebra, Con (A) satisfies also the following (infinite) identity:

$$(x\vee\wedge(x_i^{**}\colon i\in I))^{**}=\wedge((x\vee x_i)^{**}\colon i\in I).$$

It is easily verified that an atomic Con (A) satisfies the identity $(\wedge(\vee(x_{ts}\colon s\in S)\colon t\in T))^{**}=(\vee(\wedge(x_{t\varphi(t)}\colon t\in T))\colon \varphi\in S^T)^{**}$. It is an open question whether this identity holds for a larger class of algebras, even for all congruence distributive algebras.

EXAMPLES. 1. Evidently, for a Boolean algebra A, Con (A) is atomic if and only if A is atomic. Similarly for implicative semilattices and Heyting (brouwerian) algebras, Con (A) is atomic if and only if A is dual atomic.

2. Making use of weak projectivity between quotients of a lattice L one can easily describ ethe congruence atomic lattices. $a/b\to c/d$ is our notation for $c/d\approx_w\approx_w a/b$ (see [10; p. 130, 131]). First we need a concept: A proper quotient a/b of a lattice L (i.e. $b<a$) is said to be primitive, if $a/b\to c/d$ in L for a proper quotient c/d implies the existence of a finite $b=t_0\leqq\ldots\leqq t_k\leqq a$ such that $c/d\to t_i/t_{i-1}$ for every $i=1,\ldots,k$ (see [6] or [7]).

Now it is routine to prove: (i) $\theta\in\mathrm{Con}\,(L)$ *is an atom if and only if* $\theta=\theta(a,b)$ *for some primitive quotient* a/b;

(ii) Con (L) *of a lattice L is atomic if and only if for every proper quotient* a/b *of L there is a primitive quotient* p/q *of L such that* $a/b\to p/q$.

For weakly modular lattices (i.e. $a/b\to u/v$ for proper quotients implies $u/v\to\to a_1/b_1$ for some $b\leqq b_1<a_1\leqq a$) we get a simplification.

(iii) Con (L) *of a weakly modular lattice L is atomic if and only if every proper quotient of L contains a primitive subquotient;*

(iv) (See [6; Lemma 3.1] or [7; Theorem 10.5].) *If L is a distributive lattice, then* Con (L) *is atomic if and only if every proper quotient* a/b *contains a prime subquotient* a_1/b_1 (i.e. a_1 *covers* b_1).

We close this section with the following

LEMMA 1. *Let A be a congruence distributive algebra with atomic $B(\mathrm{Con}\,(A))$. Assume that $\{\alpha_i\colon i\in I\}$ is the set of all dual atoms of $B(\mathrm{Con}\,(A))$. Then the map*

$$M \to \theta_M = \wedge(\alpha_i\colon i\in I-M)$$

is an isomorphism between Boolean algebras 2^I and $B(\mathrm{Con}\,(A))$. Moreover, if $(A_i\colon i\in I)$ is a subdirect factorization of A with factors $A_i=A/\alpha_i$, $i\in I$, and $x=(x_i)_{i\in I}$, $y=(y_i)_{i\in I}\in A$, then $x\equiv y(\theta_M)$ if and only if $\{i\in I\colon x_i\neq y_i\}\subseteq M$.

PROOF. (Similar to [12; Theorem 3.1].) Since $B(\mathrm{Con}\,(A))$ is an atomic and complete Boolean algebra, every element of $B(\mathrm{Con}\,(A))$ has the form θ_M. Moreover, the map $M\to\theta_M$ is one-to-one and isotone. Hence $2^I\cong B(\mathrm{Con}\,(A))$. The last statement follows from the fact that $x\equiv y(\alpha_i)$ if and only if $x_i=y_i$.

4. Boolean congruence lattices

Again we begin with a Tanaka's result generalized for algebras by J. Hashimoto.

THEOREM B ([23; Theorem 2], [12; Theorem 5.1]). *Let A be a congruence distributive algebra. Then* Con (A) *is a Boolean lattice if and only if A has a discrete subdirect factorization with simple factors (*i.e. *the components of arbitrary two elements of A are identical except a finite number of components).*

Crawley's result [6; Theorem 4.1] (see also [7; Theorem 10.6]) proved for lattices can be also generalized for algebras as follows.

THEOREM 3. *Let A be a congruence distributive algebra. Then* Con (A) *is a Boolean algebra if and only if*

(i) Con (A) *is atomic and*

(ii) *for every $a, b\in A$, $a\neq b$ there exists a finite set of atoms $\theta_1, \dots, \theta_k$ of* Con (A) *such that*

$$\theta(a, b) = \theta_1\vee\dots\vee\theta_k.$$

PROOF. Let Con (A) be a Boolean lattice. Then $D(\mathrm{Con}\,(A))=[\nabla)$ and Con (A) is atomic (Theorem A). Since Con (A) is atomic and complete, every $\theta\in\mathrm{Con}\,(A)$ is a join of atoms. From this (ii) follows, as $\theta(a, b)$ is compact.

Conversely, let Con (A) satisfy (i) and (ii). Again by Theorem A, $D(\mathrm{Con}\,(A))=[\beta)$. Assume θ is an atom. Evidently $\theta=\theta^{**}\wedge\delta$, where $\delta\geqq\beta$. Since θ is an atom, we also have $\theta=\theta^{**}\wedge\beta$. Hence $\theta\leqq\beta$. Therefore,

$$\nabla = \vee(\theta_i\colon \theta_i \text{ is an atom of } \mathrm{Con}\,(A)) \leqq \beta,$$

using (ii) and the fact that $\nabla=\vee(\theta(a, b)\colon a, b\in A)$. Thus $\beta=\nabla$ and Con (A) is a Boolean lattice.

Now we shall examine algebras A with Con (A) enjoying the (infinite) identity

$$\vee(x_i^{**}\colon i\in I) = (\vee(x_i\colon i\in I))^{**}. \tag{11}$$

(Note that in a general case $\vee(x_i^{**}\colon i\in I)\leqq(\vee(x_i\colon i\in I))^{**}$.)

LEMMA 2. *Let* Con (A) *satisfy the identity* (11). *Then*
(i) Con (A) *is a Stone lattice,*
(ii) $B(\mathrm{Con}\,(A))$ *is a closed sublattice of* Con (A),
(iii) $B(\mathrm{Con}\,(A))$ *is compactly generated and*
(iv) $B(\mathrm{Con}\,(A))$ *is atomic.*

PROOF. (i) Clearly, $(\alpha\vee\alpha^*)^{**}=\nabla=\alpha^{**}\vee\alpha^*$, by (11). (ii) follows directly from (7) and (11). (iii) If θ is a compact element of Con (A), then it is easy to see that θ^{**} is a compact element of $B(\mathrm{Con}\,(A))$. The rest is a consequence of (11). (iv) follows from (iii) and Theorem A (ii) (see also [12; Theorem 4.2] or [7; Theorem 4.3]).
Lemma 2 gives rise to the following definition.

DEFINITION 1. A complete distributive PCL is called *completely Stonean* if it satisfies the identity (11).

LEMMA 3. *Let* Con (A) *be completely Stonean. Then for every* $a, b\in A$ *there exist atoms* $\theta_1, \ldots, \theta_k$ *(k is finite) of* $B(\mathrm{Con}\,(A))$ *such that*

$$\theta(a, b)\leqq\theta_1\vee\ldots\vee\theta_k.$$

PROOF. According to Lemma 2, $B(\mathrm{Con}\,(A))$ is an atomic complete Boolean algebra. Therefore, for $a\neq b$,

$$\theta(a, b)\leqq(\theta(a, b))^{**}=\bigvee(\theta_i\colon i\in I)=(\bigvee(\theta_i\colon i\in I))^{**},$$

where $\{\theta_i\colon i\in I\}$ is a set of atoms of $B(\mathrm{Con}\,(A))$. The compactness of $\theta(a, b)$ implies $\theta(a, b)\leqq\bigvee(\theta_i\colon i\in I_1)$ for some finite $I_1\subseteq I$.

THEOREM 4. *Let A be a congruence distributive algebra. Then* Con (A) *is a Boolean lattice if and only if*
(i) *A has a subdirect factorization with simple factors and*
(ii) Con (A) *is an atomic completely Stonean lattice.*

PROOF. Let Con (A) be a Boolean lattice. Then (i) follows from Theorem B. (ii) is trivially true, because $\alpha=\alpha^{**}$ for every $\alpha\in\mathrm{Con}\,(A)$. Conversely, let A satisfy (i) and (ii). Suppose that A is subdirectly decomposed into $(A/\alpha_i\colon i\in I)$, where A/α_i are simple algebras. Since $D(\mathrm{Con}\,(A))$ is a principal filter (Theorem 1), we can assume without loss of generality that $\alpha_i\in B(\mathrm{Con}\,(A))$ for every $i\in I$. Clearly, $\{\alpha_i\colon i\in I\}$ is the set of all dual atoms of $B(\mathrm{Con}\,(A))$ and the factorization is irredundant. Take $a, b\in A$. Let $a=(a_i)_{i\in I}$ and $b=(b_i)_{i\in I}$ be representations of a, b in the considered subdirect factorization of A. Clearly, $a_i=b_i$ if and only if $a\equiv b(\alpha_i)$. According to Lemma 3, $\theta(a, b)\leqq\bigvee(\alpha_i^*\colon i\in I_1)$ and I_1 is a finite subset of I. Evidently,

$$\bigvee(\alpha_i^*\colon i\in I_1)=\bigwedge(\alpha_i\colon i\in I-I_1),$$

as $\bigvee(\alpha_i^*\colon i\in I_1)\in B(\mathrm{Con}\,(A))$. Hence $\theta(a, b)\leqq\bigwedge(\alpha_i\colon i\in I-I_1)$, which is equivalent to $a_i=b_i$ for every $i\in I-I_1$. Thus $(A/\alpha_i\colon i\in I)$ is a discrete factorization of A and Theorem B concludes the proof.
As a consequence of Theorem B or Theorem 4 we obtain

COROLLARY. *Let A be a finite congruence distributive algebra. Then* Con (A) *is a Boolean lattice if and only if A has a subdirect factorization with simple factors.*

PROOF. According to Theorem 4 we have only to show that Con (A) is a Stone lattice, i.e. $\alpha^{**} \vee \alpha^* = \nabla$ for every $\alpha \in \text{Con}(A)$. This is true for every dual atom $\alpha \in \text{Con}(A)$. Assume that for $\eta \in \text{Con}(A)$ we have $\eta^{**} = \alpha^{**} \wedge \tau^{**}$ such that $\alpha^{**} \vee \alpha^* = \tau^{**} \vee \tau^* = \nabla$. Therefore,

$$\eta^{**} \vee \eta^* \geqq (\alpha^{**} \wedge \tau^{**}) \vee \alpha^* \vee \tau^* = (\alpha^{**} \vee \alpha^* \vee \tau^*) \wedge (\tau^{**} \vee \alpha^* \vee \tau^*) = \nabla.$$

Since Con (A) is finite, we see that Con (A) is a Stone lattice.

LEMMA 4. *Let an algebra A be a direct product of algebras $(A_i \colon i \in I)$ with $|A_i| \geqq 2$ for every $i \in I$. Let* Con (A) *be completely Stonean. Then I is finite.*

PROOF. Every element $x \in A$ can be written in the form $x = (x_i)_{i \in I}$, where $x_i \in A_i$ for every $i \in I$. There are congruence relations $\{\alpha_i \colon i \in I\}$ defined as follows:

$$x \equiv y(\alpha_i) \quad \text{if and only if} \quad x_i = y_i.$$

Consider the congruence relations

$$\theta_M = \wedge (\alpha_i \colon i \in I - M)$$

for every $M \subseteq I$ (see also Lemma 1). We shall use the notation θ_i for $\theta_{\{i\}}$, $i \in I$. Clearly, $\theta_M \wedge \theta_{I-M} = \Delta$ and $\theta_M \vee \theta_{I-M} = \nabla$. Hence $\theta_M \in B(\text{Con}(A))$. Now, by (11)

$$\vee (\theta_i \colon i \in I) = (\vee (\theta_i \colon i \in I))^{**} = \nabla,$$

as $(\vee (\theta_i \colon i \in I))^* = \wedge (\theta_{I-\{i\}} \colon i \in I) = \Delta$, But $x \equiv y(\vee (\theta_i \colon i \in I))$ if and only if $\{i \in I \colon x_i \neq y_i\}$ is finite. Thus I is finite.

THEOREM 5. *Let an algebra A be a direct product of algebras $(A_i \colon i \in I)$ with $|A_i| \geqq 2$ for every $i \in I$. Let A be congruence distributive. Then* Con (A) *is a Boolean lattice if and only if*

(i) *I is finite and*

(ii) Con (A_i) *is a Boolean lattice for every $i \in I$.*

PROOF. The necessity of (i) and (ii) follows from Lemma 4 and the fact that $A_i \cong A/\theta_{I-\{i\}}$, $i \in I$. For sufficiency we invoke [8] to see that every $\theta \in \text{Con}(A)$ can be written in the form $\theta_1 \times \ldots \times \theta_n$, where $\theta_i \in \text{Con}(A_i)$, $i \in I = \{1, \ldots, n\}$. Therefore, $\text{Con}(A) = \text{Con}(A_1) \times \ldots \times \text{Con}(A_n)$. Hence Con (A) is a Boolean lattice.

REMARK 2. A simplification of identities from Remark 1 can be achieved for congruence lattices satisfying identities (8) and (11):

$$x^{**} \vee \wedge (x_i^{**} \colon i \in I) = \wedge (x^{**} \vee x_i^{**} \colon i \in I);$$

$$\wedge (\vee (x_{ts}^{**} \colon s \in S) \colon t \in T) = \vee (\wedge (x_{t\varphi(t)}^{**} \colon t \in T) \colon \varphi \in S^T).$$

In the next theorem we want to apply Theorem 4 to classes of algebras as Boolean algebras, PCS's, PCL's, OL's, OML's, double p-algebras, Heyting algebras, double Heyting algebras, implicative semilattices, bouneed lattices and so on. First we need some concepts.

Let $(S; \wedge, 0, 1)$ denote a bounded meet-semilattice and $F(S)$ the lattice of all filters of S. An element $a \in S$ is said to be *central* if the filter $[a)$ is a central element of $F(S)$. $C(S)$ denotes the set of all central elements of S. It is routine to verify that

$C(S) \cong C(F(S))$. It is also evident that for lattices the introduced concept coincides with those known in the literature (see [5] or [10]).

To every element $a \in S$ one can assign a congruence relation θ_a defined as follows

$$x \equiv y(\theta_a) \quad \text{if and only if} \quad a \wedge x = a \wedge y.$$

θ_a is the smallest congruence relation with the kernel $[a)$, i.e. $\theta_a = \theta[[a)]$. Now, we shall say that an n-ary operation f on S is *centre-preserving* if for every $a \in C(S)$ $x_i \equiv y_i(\theta_a)$, $i = 1, \ldots, n$, implies

$$f(x_1, \ldots, x_n) \equiv f(y_1, \ldots, y_n)(\theta_a).$$

DEFINITION 2. A universal algebra $(S; \wedge, 0, 1, f_1, \ldots)$ is said to have a *strong centre* if $(S; \wedge, 0, 1)$ is a bounded meet-semilattice and every operation f_i is centre-preserving.

It can be readily verified that Boolean algebras and bounded lattices have a strong centre. Assume S to be a PCS, $a \in C(S)$. Therefore,

$$a \wedge x = a \wedge y \quad \text{implies} \quad a^{**} \wedge x^{**} = a^{**} \wedge y^{**}.$$

Thus $x \equiv y(\theta_a)$ implies $x^{**} \equiv y^{**}(\theta_{a^{**}})$ in the Boolean algebra $B(S)$. Hence $x^* \equiv$ $\equiv y^*(\theta_{a^{**}})$ in $B(S)$, that means $a^{**} \wedge x^* = a^{**} \wedge y^*$. It follows that $a \wedge x^* = a \wedge y^*$, as $a \leqq a^{**}$. So, PCS's have a strong centre. Similarly it can be shown that PCL's and double p-algebras have a strong centre as well. Clearly, $a \wedge x = a \wedge y$ implies $a' \vee x' = a' \vee y'$ in an ortholattice L. Hence $a \wedge x' = a \wedge y'$, using the property that $a \in C(L)$. Thus OL's, and OML's in particular, have a strong centre. Eventually, suppose that $(S; \wedge, {}_*, 0, 1)$ is an implicative semilattice. Take $a, x, y \in S$ arbitrary. It is known (see [20]) that there is a one-to-one correspondence between filters and congruence relations on S. More precisely, for $J \in F(S)$

$$x \equiv y(\theta[J]) \quad \text{if and only if} \quad x * y \wedge y * x \in J.$$

Having $J = [a)$, $x_* y \wedge y_* x \in [a)$ implies $x \wedge (x_* y \wedge y_* x) = y \wedge (x_* y \wedge y_* x) = x \wedge y$, whence $a \wedge x = a \wedge y$. Conversely, $a \wedge x = a \wedge y$ implies $a \wedge x \leqq x \wedge y$. Therefore,

$$a \leqq x * (a \wedge x) = x * a \leqq x * (x \wedge y) = x * y.$$

Similarly, $a \leqq y_* x$, whence $a \leqq x_* y \wedge y_* x$. Thus $\theta_a = \theta[[a)]$ and implicative semilattices have a strong centre. The proof for Heyting and double Heyting algebras runs in the same way.

LEMMA 5. *A nontrivial PCS (PCL, Boolean algebra, Heyting algebra, implicative semilattice, distributive lattice) S is simple if and only if $|S| = 2$. A simple double p-algebra is regular.*

PROOF. Let S be a Boolean algebra (Heyting algebra, implicative semilattice). It is known that every congruence relation on this algebra is uniquely determined by its kernel, which can be an arbitrary filter. Hence, S nontrivial and simple implies $|S| = 2$. A similar argument can be used for distributive lattices. Let L be a nontrivial simple PCS or PCL. Then $\gamma = \Delta$ for the Glivenko congruence γ. Hence, $L = B(L)$ is a Boolean algebra. Thus $|L| = 2$. Similarly, if L is a simple double p-algebra, then $\varphi = \Delta$ for the determination congruence φ. Thus L is regular.

Suppose that $(A; \wedge, 0, 1, f_1, \ldots)$ have a strong centre. If there exists a polynomial function g on A such that $g(1)=0$ and $g(0)=1$, then A is said to be *quasi-complemented* algebra and g a *quasi-complementation*. (Of course, quasi-complementation will also stand for a unary polynomial g correlated to a polynomial function having the described property.)

It is easy to verify that PCS's, PCL's, PL's, PML's, Boolean algebras, double p-algebras, implicative semilattices, Heyting algebras, double Heyting algebras and complemented lattices are examples of quasi-complemented algebras (a^*, a_*0 and a' are quasicomplements, respectively). On the other hand, bounded meet-semilattices or bounded lattices are not quasi-complemented.

We shall also have occasion to invoke the following property:

(PCC) *Any two complemented congruence relations permute.*

LEMMA 6. *Let a quasi-complemented algebra A have a subdirect factorization with two-element algebras. Let g denote a polynomial of quasi-complementation. Moreover, let h be a binary polynomial defined by*

$$h(x, y) = g(g(x) \wedge g(y)).$$

Denote by $A'=(A; \wedge, h, g, 0, 1, f_1, \ldots)$ an algebra in which the deletion of h and g gives the algebra A, and by $A''=(A; \wedge, h, g, 0, 1)$. Then A'' is a Boolean algebra, A' is quasi-complemented and $\mathrm{Con}(A)=\mathrm{Con}(A')=\mathrm{Con}(A'')$.

PROOF. Since h and g are polynomial functions on A, we see that A' is also quasi-complemented. By assumption, A is a subdirect product of algebras $(B_i; i\in I)$ with $B_i=\{0, 1\}$ for every $i\in I$. It is easy to check that $(B_i; \wedge, h, g, 0, 1)$, $i\in I$, are two-element Boolean algebras. Defining $B_i'=(B_i; \wedge, h, g, 0, 1, f_1, \ldots)$, $i\in I$, we obtain two-element Boolean algebras with additional operations $f_1, \ldots$. Since g and h are polynomials, A' is a subdirect product of the system $(B_i'; i\in I)$. Hence $A''(A')$ is a Boolean algebra (with additional operations). Clearly $\mathrm{Con}(A)=\mathrm{Con}(A')\subseteq\mathrm{Con}(A'')$. Assume $\theta\in\mathrm{Con}(A'')$. There exists a filter F on A'' such that $\theta=\theta[F]$. Clearly, $\theta[F]=\vee(\theta[[a)]: a\in F)$ in $\mathrm{Con}(A'')$. Since $C(A)=A$, we see that $\theta[[a)]\in\mathrm{Con}(A)$ for every $a\in A$. But $\mathrm{Con}(A)$ is a closed sublattice of $\mathrm{Con}(A'')$. Hence, $\theta\in\mathrm{Con}(A)$. Thus $\mathrm{Con}(A)=\mathrm{Con}(A'')$.

THEOREM 6. *Let A be an algebra with a strong centre.*

(i) *$C(A)$ is finite whenever* Con (A) *is completely Stonean and satisfies* (PCC).

(ii) *Let A be quasi-complemented and have a subdirect factorization with two-element algebras. Then* Con (A) *is a Boolean lattice if and only if A is a finite Boolean lattice.*

(iii) *Let A be congruence distributive and let A enjoy the property* (PCC). *Then* Con (A) *is a Boolean lattice if and only if A is a finite direct product of simple algebras.*

PROOF. (i) Con (A) is a Stone lattice (Lemma 2) from which it follows by (PCC) that $C(A)\cong B(\mathrm{Con}(A))$. Take a filter J of $C(A)$. Then

$$\vee(\theta_a: a\in J) = \theta[J]\in B(\mathrm{Con}(A))$$

by (11). From (PCC) we deduce $\theta[J]=\theta_b$ for some $b\in C(A)$. First we show that $\mathrm{Ker}\,\theta(J)=J$. Take $c\equiv 1(\theta[J])$. Then there exists a finite chain $1=z_0\geqq z_1\geqq \ldots\geqq z_n=c$

such that $z_{i-1} \wedge a_i = z_i \wedge a_i$ for some $a_i \in J, i=1, ..., n$. Therefore $z_i \in J$ for every $i=1, ..., n$. Thus $c \in J$, whence $\operatorname{Ker} \theta[J]=J$. Since $\operatorname{Ker} \theta_b=[b)$, we get $J=[b)$. Every filter of the Boolean algebra $C(A)$ is principal. Hence $C(A)$ is finite.

(ii) Using the notation from Lemma 6, $A''=(A; \wedge, h, g, 0, 1)$ is a Boolean algebra and $\operatorname{Con}(A)=\operatorname{Con}(A'')$. It is well known that $\operatorname{Con}(A'')$ is a Boolean lattice if and only if A is finite (see [9]). Since h and g are polynomial functions on A, A is a finite Boolean lattice. Conversely, invoking Lemma 6 we have $\operatorname{Con}(A)=\operatorname{Con}(A'')$. Since A is finite, we see that $\operatorname{Con}(A'')$ is a Boolean lattice.

(iii) Assume that $\operatorname{Con}(A)$ is a Boolean algebra. It follows by (PCC) that $C(A) \cong \operatorname{Con}(A)$. Since $\operatorname{Con}(A)$ satisfies the identity (11), we see by (i) that $C(A)$ is finite. Let $\alpha_1, ..., \alpha_n$ denote the dual atoms of $\operatorname{Con}(A)$. Again by (PCC)

$$\nabla \geqq (\alpha_1 \wedge ... \wedge \alpha_{i-1}) \circ \alpha_i > \alpha_i' \circ \alpha_i = \alpha_i \vee \alpha_i' = \nabla.$$

Therefore, A is a direct product of simple algebras $(A/\alpha_i\colon i=1, ..., n)$. Conversely, if A is a finite direct product of simple algebras $(A_i\colon i=1, ..., n)$, then by [8] every congruence relation $\theta \in \operatorname{Con}(A)$ can be represented in the form $\theta_1 \times ... \times \theta_n$, where $\theta_i \in \operatorname{Con}(A_i)$, $i=1, ..., n$. Hence $\operatorname{Con}(A)$ is a finite Boolean lattice.

COROLLARY 1. *Let L be a PCS (PCL, Boolean algebra, Heyting algebra, implicative semilattice). Then* Con (L) *is a Boolean lattice if and only if L is a finite Boolean lattice.*

Proof follows from Lemma 5 and Theorem 7 (ii).

COROLLARY 2. *Let L be a double p-algebra (double Heyting algebra, OL, OML; complemented lattice). Then* Con (L) *is a Boolean lattice if and only if L is a finite direct product of simple algebras.*

PROOF. Clearly, all considered classes of algebras are congruence distributive. First we check that the algebras in question satisfy (PCC). Let L be a double p-algebra with a Boolean $\operatorname{Con}(L)$. Since simple double p-algebras are regular (Lemma 5), we see that also L is regular (Theorem 4). Now we can start with a regular double p-algebra L. Beazer [1] has shown that any two congruence relations on a regular double p-algebra are permuting. The same is also true for double Heyting algebras, because every congruence relation has the form $\theta=\theta[J]$, where J is a special filter. Note that the lattice of all filters on a double Heyting algebra is distributive. Finally, Janowitz [15; Theorem 2] established (PCC) for complemented lattices. It follows that the same is also true for OL's and OML's. Now we can apply Theorem 7 (iii) and the proof is complete.

REMARK 3. Corollary 1 was established for PCS's by Beazer [1], [2], Sankappanavar [21] and for PCL's by Beazer [1], [2] and Berman [4] using different methods. For Boolean algebras this result is well known. Double p-algebras were investigated by Beazer [1], [2]. He obtained an intrinsic characterization of those double p-algebras L whose $\operatorname{Con}(L)$ is a Boolean lattice. For complemented lattices Corollary 2 was established by Janowitz [15].

5. Stonean congruence lattices

Recall that for atomic $B(\mathrm{Con}\,(A))$ every closed congruence relation can be written in the form θ_M, $M\subseteq I$ (Lemma 1).

THEOREM 7. *Let A be a congruence distributive algebra with atomic $B(\mathrm{Con}\,(A))$. Let θ_M, $M\subseteq I$ be congruence relations from Lemma* 1. *Then* Con (A) *is a Stone lattice if and only if*

$$\theta_{M\cup N}=\theta_M\vee\theta_N \quad \textit{for any} \quad M, N\subseteq I.$$

PROOF. Assume that Con (A) is a Stone lattice. $\theta_M\vee\theta_N\leqq\theta_{M\cup N}$ is trivially true. For the converse inclusion we need to know that

$$\theta^*_{M\cup N}=(\theta_M\vee\theta_N)^*=\theta^*_M\wedge\theta^*_N.$$

Since $\theta^*_R=\theta_{I-R}$ (Lemma 1), we see that $\theta^*_{M\cup N}=\theta_{I-(M\cup N)}$, $\theta^*_M=\theta_{I-M}$ and $\theta^*_N==\theta_{I-N}$. But $I-(M\cup N)=(I-M)\cap(I-N)$. Hence $\theta^*_{M\cup N}=\theta^*_M\wedge\theta^*_N$. Moreover,

$$(\theta^*_M\wedge\theta^*_N)^*=\theta^{**}_M\vee\theta^{**}_N=\theta_M\vee\theta_N=\theta^{**}_{M\cup N}=\theta_{M\cup N},$$

as Con (A) is a Stone lattice.

Conversely, assume $\theta_{M\cup N}=\theta_M\vee\theta_N$ for any $M, N\subseteq I$. By Lemma 1, every $\theta\in B(\mathrm{Con}\,(A))$ can be written in the form $\theta=\theta_M$. Take $\alpha\in\mathrm{Con}\,(A)$. Then $\alpha^*=\theta_M$ for some $M\subseteq I$. Evidently, $\alpha^{**}=\theta_{I-M}$. By assumption,

$$\alpha^*\vee\alpha^{**}=\theta_M\vee\theta_{I-M}=\theta_I=\nabla,$$

and the proof is complete.

COROLLARY 1. *Let A be a congruence distributive algebra. Then* Con (A) *is a Stone lattice and obeys the identity* (8) *if and only if any irredundant subdirect factorization of A with subdirectly irreducible factors $(A_i\colon i\in I)$ satisfies the following condition:*

(S) *Let $x=(x_i)_{i\in I}$, $y=(y_i)_{i\in I}$ be elements of A with $x_i, y_i\in A_i$, $i\in I$. Then for any $M, N\subseteq I$ there exists a finite sequence $x=t^0, \dots, t^n=y$ in A such that $\{i\in L\colon x_i\neq y_i\}\subseteq M\cup N$ if and only if $\{i\in I\colon t^k_i\neq t^{k+1}_i\}\subseteq M$ or $\{i\in I\colon t^k_i\neq t^{k+1}_i\}\subseteq N$ for every $k=0, \dots, n-1$.*

The proof follows from Theorems 1, 7 and Lemma 1.

COROLLARY 2. *Let A be a congruence distributive algebra. Let A be a direct product of finitely subdirectly irreducible algebras. Then* Con (A) *is a Stone lattice.*

PROOF. Assume that A is directly factorized with $(A_i\colon i\in I)$, $|A_i|\geqq 2$ and let $\{\alpha_i\colon i\in I\}$ be the corresponding set of congruence relations on A. Since $A/\alpha_i\cong A_i$, $i\in I$, we see that α_i is meet-irreducible. The factorization is irredundant. Therefore, $\{\alpha_i\colon i\in I\}$ is the set of all dual atoms of $B(\mathrm{Con}\,(A))$ (Theorem 2). Take $x=(x_i)_{i\in I}$, $y=(y_i)_{i\in I}\in A$. Assume $x\equiv y(\theta_{M\cup N})$. Take $z=(z_i)_{i\in I}$ from A defined as follows: $z_i=x_i$ for all $i\in I-N$ and $z_i=y_i$ for all $i\in N$. Then $x\equiv z(\theta_M)$ and $z\equiv y(\theta_N)$ (see Lemma 1). Hence $x\equiv y(\theta_M\vee\theta_N)$ and Theorem 7 concludes the proof.

Only a partial result has been obtained without the assumption of atomicity of $B(\mathrm{Con}\,(A))$ (see Theorem 7).

THEOREM 8. *Let A be congruence distributive and let A be a finite direct product of algebras* $(A_i\colon i=1, \dots, n)$. *Then* Con (A) *is a Stone lattice if and only if* Con (A_i) *are so for every* $i=1, \dots, n$.

PROOF. Let $\alpha_1, \dots, \alpha_n \in \mathrm{Con}(A)$ denote the congruence relations corresponding to the product $(A_i\colon 1, \dots, n)$, that means $A/\alpha_i \cong A_i$ for every $i=1, \dots, n$. Therefore,

$$\alpha_i^* = \alpha_1 \vee \dots \vee \alpha_{i-1} \vee \alpha_{i+1} \vee \dots \vee \alpha_n$$

is the complement of α_i in Con (A) for every $i=1, \dots, n$. Hence $\alpha_i=\alpha_i^{**}$, $i=1, \dots, n$. Now it is easy to verify, for every $i=1, \dots, n$, that $[\alpha_i, \nabla] \cong \mathrm{Con}(A_i)$ is a Stone lattice whenever Con (A) is so. Conversely, if Con (A_i) are Stonean for every $i=1, \dots, \dots, n$, then also Con (A) is Stonean, because $\mathrm{Con}(A)=\mathrm{Con}(A_1)\times \dots \times \mathrm{Con}(A_n)$ by [8].

THEOREM 9. *Let A be a congruence distributive algebra. Then* Con (A) *is completely Stonean if and only if*

(i) $B(\mathrm{Con}(A))$ *is atomic and*

(ii) *for every* $a, b \in A$ *there exists a finite set of atoms* $\theta_1, \dots, \theta_n$ *of* $B(\mathrm{Con}(A))$ *such that*

$$\theta(a, b) \leqq \theta_1 \vee \dots \vee \theta_n.$$

PROOF. The necessity follows from Lemmas 2 and 3. Conversely, assume that A satisfies (i) and (ii). Take an arbitrary subset $\{\alpha_i\colon i\in I\}$ of Con (A). Let $\theta(a, b) \leqq \leqq (\bigvee(\alpha_i\colon i\in I))^{**}$ for some $a, b \in A$. By (i) and (ii),

$$\theta(a, b) \leqq \theta_1 \vee \dots \vee \theta_n$$

for some atoms $\theta_1, \dots, \theta_n$ of $B(\mathrm{Con}(A))$. Without loss of generality we can assume that $\theta_1 \vee \dots \vee \theta_n \leqq (\bigvee(\alpha_i\colon i\in I))^{**}$. We claim that

$$\theta_1 \vee \dots \vee \theta_n \leqq \bigvee(\alpha_i^{**}\colon i\in I).$$

Indeed, suppose to the contrary that $\theta_j \not\leqq \bigvee(\alpha_i^{**}\colon i\in I)$ for some $1\leqq j\leqq n$. Therefore,

$$\theta_j \wedge \bigvee(\alpha_i^{**}\colon i\in I) = \bigvee(\theta_j \wedge \alpha_i^{**}\colon i\in I) = \Delta,$$

by infinite distributivity and the fact that θ_j is an atom of $B(\mathrm{Con}(A))$. Hence

$$\Delta = \theta_j \wedge \bigvee(\alpha_i^{**}\colon i\in I) = \theta_j \wedge (\bigvee(\alpha_i\colon i\in I))^{**}$$

by (5) and (10), which is a contradiction. Thus, $\theta_1 \vee \dots \vee \theta_n \leqq \bigvee(\alpha_i^{**}\colon i\in I)$, as claimed. It follows that $\theta(a, b) \leqq \bigvee(\alpha_i^{**}\colon i\in I)$ and Con (A) satisfies the identity (11), because Con (A) is compactly generated.

COROLLARY. *Let A be a finite congruence distributive algebra. Then* Con (A) *is a Stone lattice if and only if for every* $a, b \in A$ *there exists a finite set of atoms* $\theta_1, \dots, \theta_n$ *of* $B(\mathrm{Con}(A))$ *such that*

$$\theta(a, b) \leqq \theta_1 \vee \dots \vee \theta_n.$$

There is another description of completely Stonean congruence lattices.

THEOREM 10. *Let A be a congruence distributive algebra. Then* Con (A) *is (atomic and) completely Stonean if and only if*

(i) Con (A) *is a Stone lattice and*

(ii) *A has a discrete subdirect factorization with finitely subdirectly irreducible (subdirectly irreducible) factors.*

PROOF. Assume that Con (A) satisfies (11). (i) follows from Lemma 2. Lemmas 2, 3 and Theorem 2 imply (ii). Conversely, assume (i) and (ii). We shall prove that both conditions from Theorem 9 are satisfied. By the hypothesis A is a discrete subdirect product of finitely subdirectly irreducible (subdirectly irreducible) algebras $A_i=A/\alpha_i$, $i\in I$. Evidently, $\{\alpha_i\colon i\in I\}$ is a set of (completely) meet-irreducible congruence relations of A. Therefore, $\alpha_i\in D(\mathrm{Con}\,(A))$ or α_i is a dual atom of $B(\mathrm{Con}\,(A))$, for every $i\in I$. Consider $\theta(a,b)\in\mathrm{Con}\,(A)$ for some $a\neq b$ in A. By (ii),

$$\theta(a,b)\leqq\wedge(\alpha_i\colon i\in I-\{1,\ldots,n\}).$$

Since $\wedge(\alpha_i\colon i\in I)=\Delta$, we have

$$\theta(a,b)\wedge\alpha_1\wedge\ldots\wedge\alpha_n=\Delta.$$

Without loss of generality we can assume that $\alpha_1,\ldots,\alpha_k\in B(\mathrm{Con}\,(A))$ and $\alpha_{k+1},\ldots$ $\ldots,\alpha_n\in D(\mathrm{Con}\,(A))$, where $1\leqq k\leqq n$. Since $\alpha_{k+1}\ldots\alpha_n\in D(\mathrm{Con}\,(A))$, we have

$$\theta(a,b)\wedge\alpha_1\wedge\ldots\wedge\alpha_k=\Delta.$$

Therefore, $\theta(a,b)\leqq(\alpha_1\wedge\ldots\wedge\alpha_k)^*=\alpha_1^*\vee\ldots\vee\alpha_k^*$, by (i). Evidently, $\alpha_1^*,\ldots,\alpha_k^*$ are atoms of $B(\mathrm{Con}\,(A))$, because $\alpha_1,\ldots,\alpha_k$ are dual atoms of $B(\mathrm{Con}\,(A))$. Now, we have established condition (ii) of Theorem 9. It remains to show that $B(\mathrm{Con}\,(A))$ is atomic.

Assume that $\Delta\neq\theta\in B(\mathrm{Con}\,(A))$. Then there exist $a\neq b$ such that $\theta(a,b)\leqq\theta$. We have just proven that there exists a finite set of atoms $\theta_1,\ldots,\theta_n$ of $B(\mathrm{Con}\,(A))$ such that

$$\theta(a,b)\leqq\theta_1\vee\ldots\vee\theta_n.$$

We claim that $\theta_j\leqq\theta$ for some $1\leqq j\leqq n$. If $\theta_j\nleqq\theta$ for every $1\leqq j\leqq n$ then $\theta\wedge(\theta_1\vee\ldots\vee\theta_n)=(\theta\wedge\theta_1)\vee\ldots\vee(\theta\wedge\theta_n)=\Delta$, because $\theta_1,\ldots,\theta_n$ are atoms of $B(\mathrm{Con}\,(A))$. It follows that $\Delta=\theta(a,b)$, which is a contradiction with $a\neq b$. Thus $\theta_j\leqq\theta$ for some $1\leqq j\leqq n$, as claimed, and $B(\mathrm{Con}\,(A))$ is atomic. Now Theorem 9 concludes the proof. (For the atomicity of Con (A) see also Theorem 1.)

COROLLARY. *Let A be congruence distributive and let A be a direct product of algebras $(A_i\colon i\in I)$ with $|A_i|\geqq 2$ for every $i\in I$. Then* Con (A) *is (atomic and) completely Stonean if and only if*

(i) *I is finite and*

(ii) Con (A_i) *is (atomic and) completely Stonean for every $i\in I$.*

The (alternative) proof follows from Lemma 4, Theorem 8 and the fact that every $\theta\in\mathrm{Con}\,(A)$ can be written in the form $\theta=\theta_1\times\ldots\times\theta_n$ (see [8]).

REMARK 4. Comparing Theorems 3 and 4 with Theorem 9 one can ask whether an atomic completely Stonean Con (A) is Boolean. The answer is: *NO*. Namely, it is enough to consider a finite Stonean and non Boolean lattice D. It is well known (see [10; Theorem II. 17] or [22; Theorem 3.3.1]) that there exists a (finite) lattice L such that $\mathrm{Con}(L)\cong D$. (Note that Theorem 3 is valid also in the case if there "$\theta(a,b)=\theta_1\vee\dots\vee\theta_k$" is replaced by a weaker condition "$\theta(a,b)\leqq\theta_1\vee\dots\vee\theta_k$".) Finally, we can also say that the conditions of Theorem 4 are independent. In order to prove this assertion it is enough to consider the above example and an infinite non complete Boolean algebra (see [9]).

The final theorem is devoted to the algebras with a strong centre. First we need a new concept: An algebra $A=(A;\wedge,0,1,\vee,f_2,\dots)$ with a strong centre is said to be an *l-algebra* if $(A;\wedge,\vee,0,1)$ is a bounded lattice.

THEOREM 11. *Let A be an algebra with a strong centre.*

(i) *Let A obey* (PCC). *If* Con (A) *is a Stone lattice then* $C(A)$ *is a complete Boolean algebra which is* $\bigvee$*-closed in A. If in addition A is an l-algebra, then* $C(A)$ *is a closed sublattice of* $(A;\wedge,\vee,0,1)$.

(ii) *Let A be quasi-complemented and have a subdirect factorization with two-element algebras. Then* Con (A) *is a Stone lattice if and only if* $(A;\wedge,h,g,0,1)$ (see Lemma 6) *is a complete Boolean algebra. In addition,* Con (A) *is completely Stonean if and only if* Con (A) *is a Boolean lattice.*

(iii) *Let A be a congruence distributive algebra and let A obey* (PCC). *Then* Con (A) *is (atomic and) completely Stonean if and only if A is a finite direct product of finitely subdirectly irreducible (subdirectly irreducible) algebras.*

PROOF. (i) By assumption, $C(A)\cong B(\mathrm{Con}(A))$. But $B(\mathrm{Con}(A))$ is a complete Boolean lattice. Hence $C(A)$ is also complete. Take $J\subseteq C(A)$. For every $a\in J$ there is $\theta_a\in B(\mathrm{Con}(A))$. Since $\bigwedge(\theta_a\colon a\in J)\in B(\mathrm{Con}(A))$, we see by (PCC) that there exists $b\in C(A)$ with $\theta_b=\bigwedge(\theta_a\colon a\in J)$. Moreover,

$$\mathrm{Ker}\,\theta_b=[b)=\mathrm{Ker}\left(\bigwedge(\theta_a\colon a\in J)\right)=\bigwedge([a)\colon a\in J).$$

Thus $b=\bigvee(a\colon a\in J)$ and $C(A)$ is $\bigvee$-closed in A. By dual reasoning we can establish the last part of (i).

(ii) First we need to know that Con (B) of a Boolean algebra is a Stone lattice if and only if B is complete (see [9] or [17]). In one direction the statement (ii) follows from (i), the converse is easily verified. For the last part of (ii) we need to know that for a Boolean algebra B, Con (A) is completely Stonean if and only if B is finite. This follows from Theorems 1 and 4. The rest is a consequence of Lemma 6 and Theorem 6.

(iii) The proof is similar to that of Theorem 6 (iii).

COROLLARY. *Let L be a regular double p-algebra (Heyting algebra, double Heyting algebra, OL, OML, complemented lattice). If* Con (L) *is a Stone lattice then* $C(L)$ *is a complete Boolean lattice which is a closed sublattice of L. Moreover,* Con (L) *is (atomic and) completely Stonean if and only if L is a finite direct product of finitely subdirectly irreducible (subdirectly irreducible) algebras.*

PROOF. All algebras in question are congruence distributive l-algebras enjoying (PCC) (see Corollary 2 to Theorem 6). The rest follows from Theorem 11.

REMARK 5. In general, the converse implication to the first statement of Corollary is not true (see Beazer [3] for regular double p-algebras and Janowitz [15] for complemented lattices).

REMARK 6. It seems possible to generalize some of the results of this paper for commutator varieties. P. Zlatoš [24] has succeeded in establishing a general form of Theorem 8.

REMARK 7. In [19] we have characterized those quasi-modular p-algebras whose congruence lattices are Stonean, completely Stonean or relatively Stonean.

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О НЕПРЕРЫВНОСТИ ФУНКЦИИ МНОГИХ ПЕРЕМЕННЫХ ИЗ КЛАССА ОБОБЩЕННОЙ ОГРАНИЧЕННОЙ ВАРИАЦИИ

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§1. Введение

В 1924 году Н. Винер [26] получил следующий критерий для непрерывности функции ограниченной вариации: пусть 2π-периодическая функция f имеет ограниченную вариацию на периоде и в каждой точке $x\in[0, 2\pi]$

$$\min\{f(x-0), f(x+0)\} \leqq f(x) \leqq \max\{f(x-0), f(x+0)\},$$

тогда для непрерывности функции f необходимо и достаточно каждое из следующих условий:

$$\frac{1}{n}\sum_{k=1}^{n} k\varrho_k = o(1) \text{ [1]} \tag{1}$$

и

$$\frac{1}{n}\sum_{k=1}^{n} k^2\varrho_k^2 = o(1) \tag{2}$$

при $n\to\infty$, где $\varrho_k=\sqrt{a_k^2+b_k^2}$, а a_k и b_k — коэффициенты Фурье функции f.

С. М. Лозинский [16, 17] показал, что для непрерывности функции из класса ограниченной вариации необходимо и достаточно каждое из следующих условий:

$$\frac{1}{n}\sum_{k=n}^{\infty} \varrho_k^2 = o(1) \quad \text{и} \quad \sum_{k=1}^{n} \varrho_k = o(\ln n) \tag{3}$$

при $n\to\infty$.

Б. И. Голубов (см. [5], [6], [8], [10]) установил, что условия (1)—(3) необходимы и достаточны для непрерывности функций класса V_p при $1<p<2$ и достаточны при $p\geqq 2$. Он же показал, что при $p\geqq 2$ для включения $f\in C(0, 2\pi)\cap V_p$ не существует необходимых и достаточных условий, выраженных в терминах модулей коэффициентов Фурье.

Эти исследования были продолжены Е. Коэным [4] и З. А. Чантурия [24, 25], соответственно, для классов V_Φ и $V[\nu(n)]$. С другой стороны, Б. И. Голубов [7, 11] впервые исследовал вопрос об аналогах упомянутых выше теорем Н. Винера и С. М. Лозинского для функций многих переменных, имеющих ограниченную p-вариацию в смысле Харди.

[1] Как отмечал Б. И. Голубов в своей работе [7], достаточность условия (1) для непрерывности функции ограниченной вариации можно вывести из одной теоремы Фейера (см., напр., [12], стр. 177, теорема (9.3)), а необходимость этого условия доказал также Сидон [18].

В настоящей работе (для любого натурального n) вводятся классы функций $H_{\Phi(n)}^{(m_1,\dots,m_n)}$, $\overline{H}_{\Phi(n)}^{(m_1,\dots,m_n)}$, изучаются некоторые свойства функций из этих классов и установлены в основном неусиляемые (в определенном смысле) теоремы типа Винера—Лозинского.

§2. Определения и формулировки основных результатов

Пусть функция f является вещественной, конечной, измеримой функцией, определенной на n-мерном евклидовом пространстве R^n и $Q_n=[-\pi,\pi]^n$. Предположим, что она 2π-периодическая относительно каждой из переменных и $C_{r_1\dots r_n}(f)$ — коэффициенты Фурье функции f по тригонометрической системе, т. е.

$$C_{r_1\dots r_n}(f)=\frac{1}{(2\pi)^n}\int\limits_{Q_n} f(x_1,\dots,x_n)e^{-i(r_1x_1+\dots+r_nx_n)}\,dx_1\dots dx_n.$$

Обозначим через B любое непустое подмножество из $M=\{1,2,\dots,n\}$, а через x_B — множество тех координат $x_1,x_2,\dots,x_n$, индексы которых составляют множество B. Пусть $B=\{i_1,\dots,i_p\}$ $(i_1<i_2<\dots<i_p)$ и $m_1,m_2,\dots,m_n$ — натуральные числа. Положим

(4)

$$\Delta^{(m_{i_1})}_{x_{i_1}^{(k_{i_1})}}(f;\,h_{i_1}^{(k_{i_1})})=\sum_{\nu_{i_1}=0}^{m_{i_1}}(-1)^{\nu_{i_1}}C_{m_{i_1}}^{\nu_{i_1}}f(x_1,\dots,x_{i_1-1},x_{i_1}^{(k_{i_1})}+\nu_{i_1}h_{i_1}^{(k_{i_1})},x_{i_1+1},\dots,x_n),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\Delta^{(m_{i_1},\dots,m_{i_p})}_{x_{i_1}^{(k_{i_1})},\dots,x_{i_p}^{(k_{i_p})}}(f;\,h_{i_1}^{(k_{i_1})},\dots,h_{i_p}^{(k_{i_p})})=$$

$$=\Delta^{(m_{i_p})}_{x_{i_p}^{(k_{i_p})}}\big(\Delta^{(m_{i_1},\dots,m_{i_{p-1}})}_{x_{i_1}^{(k_{i_1})},\dots,x_{i_{p-1}}^{(k_{i_{p-1}})}}(f;\,h_{i_1}^{(k_{i_1})},\dots,h_{i_{p-1}}^{(k_{i_{p-1}})});\;h_{i_p}^{(k_{i_p})}\big),$$

где

$$h_{i_1}^{(k_{i_1})}=\frac{1}{m_{i_1}}(x_{i_1}^{(k_{i_1}+1)}-x_{i_1}^{(k_{i_1})}),\;\dots,\;h_{i_p}^{(k_{i_p})}=\frac{1}{m_{i_p}}(x_{i_p}^{(k_{i_p}+1)}-x_{i_p}^{(k_{i_p})}).$$

Ниже будем использовать и такие обозначения

(5) $$\tilde{\Delta}^{(m_{i_1},\dots,m_{i_p})}_{x_{i_1}^{(k_{i_1})},\dots,x_{i_p}^{(k_{i_p})}}(f;\,h_{i_1}^{(k_{i_1})},\dots,h_{i_p}^{(k_{i_p})};\;x_1,\dots,x_{i_1-1},x_{i_1+1},\dots,x_{i_p-1},x_{i_p+1},\dots,x_n)=$$

$$=\Delta^{(m_{i_1},\dots,m_{i_p})}_{x_{i_1}^{(k_{i_1})},\dots,x_{i_p}^{(k_{i_p})}}(f;\,h_{i_1}^{(k_{i_1})},\dots,h_{i_p}^{(k_{i_p})}).$$

Если $\{a_s\}_{s=1}^n$ — некоторая система вещественных чисел, то будем считать, что

$$\Pi_{a_s}=\{a_s=x_s^{(0)}<x_s^{(1)}<\dots<x_s^{(N_s)}=a_s+2\pi\}$$

— произвольное разбиение периода, а

$$\Pi_{a_B} = \Pi_{a_{i_1} a_{i_2} \dots a_{i_p}} = \Pi_{a_{i_1}} \times \Pi_{a_{i_2}} \times \dots \times \Pi_{a_{i_p}},$$

где индексы $i_1, i_2, \dots, i_p$ составляют множество B.

Допустим еще, что $\Phi(n) = \{\Phi_B\}_{B \subset M}$ — система строго возрастающих (непрерывных в точке 0) функций, определенных на $[0, +\infty[$, для которой $\Phi_B(0)=0$ $(B \subset M)$ и $\lim\limits_{x \to \infty} \Phi_B(x) = \infty$.

Введем обозначение

$$V_{\Phi_B}^{(B)}(f;\, m_{i_1}, \dots, m_{i_p}) =$$

$$= \sup_{x_{C_M B}} \sup_{a_B} \sup_{\Pi_{a_B}} \sum_{k_{i_1}=0}^{N_{i_1}-1} \dots \sum_{k_{i_p}=0}^{N_{i_p}-1} \Phi_B\left(2^{p-(m_{i_1}+\dots+m_{i_p})} \left|\Delta_{x_{i_1}^{(k_{i_1})}, \dots, x_{i_p}^{(k_{i_p})}}^{(m_{i_1}, \dots, m_{i_p})}(f;\, h_{i_1}^{(k_{i_1})}, \dots, h_{i_p}^{(k_{i_p})})\right|\right).$$

Если существует такое взаимно однозначное соответствие между системами $\{B\}_{B \subset M}$ и $\{\Phi_B\}_{B \subset M}$, что $V_{\Phi_B}^{(B)}(f;\, m_{i_1}, \dots, m_{i_p}) < \infty$ для каждого $B \subset M$, то будем говорить, что функция f является функцией ограниченной Φ-вариации $(f \in H_{\Phi(n)}^{(m_1, m_2, \dots, m_n)} \equiv H_{\Phi(n)}^{(m)})$. Если $\Phi_B(u) \equiv \Phi(u)$ для любого $B \subset M$, то класс $H_{\Phi(n)}^{(m)}$ обозначим через $H_{\Phi}^{(m)}(n)$.

Положим

$$\tilde{\Delta}_{x_1}(f;\, \delta) = f(x_1+\delta, x_2, \dots, x_n) - f(x_1-\delta, x_2, \dots, x_n),$$

$$\dots\dots\dots\dots\dots\dots$$

$$\tilde{\Delta}_{x_1, x_2, \dots, x_k}(f;\, \delta) = \tilde{\Delta}_{x_k}(\tilde{\Delta}_{x_1, \dots, x_{k-1}}(f;\, \delta);\, \delta), \quad k = 2, 3, \dots, n.$$

Множество функций f из класса $H_{\Phi(n)}^{(m)}$ для которых

(6)
$$\lim_{\delta \to 0} \tilde{\Delta}_{x_1, x_2, \dots, x_n}(f;\, \delta) = 0$$

обозначим через $\overline{H}_{\Phi(n)}^{(m)} \equiv \overline{H}_{\Phi(n)}^{(m_1, \dots, m_n)}$.

Пусть $\tilde{\varepsilon}_B$ такая точка из R^n, что все координаты точка $\tilde{\varepsilon}_B$, индексы которых составляют множество B отличны от нуля, а остальные координаты — нули. Далее, запись $\tilde{\varepsilon}_B \to 0$ будет означать, что все координаты этого вектора, индексы которых составляют множество B стремятся с определенной стороны (либо слевой, либо справой) к нулю.

При $m_1=1$, $\Phi_{\{1\}}(u) \equiv u$ — $H_{\Phi(1)}^{(m)}$ совпадает с классом V класс функций ограниченной вариации; при $m_1=1$, $\Phi_{\{1\}}(u)=u^p$ $(p>1)$ — с классом V_p Винера [26], а при $m_1=1$ и для общей Φ этот класс превращается в класс V_Φ Л. Юнгой [27]. Далее, при $n=1$, $m_1=2$ и $\Phi_{\{1\}}(u)=u$ этот класс впервые рассмотрел Ф. И. Харшиладзе [21] и изучил (см. и работы [22], [23]) некоторые свойства функций из этого класса. В случае $n=1$ и $\Phi_{\{1\}}(u) \equiv u$ класс $H_{\Phi(n)}^{(m)}$ ввел С. К. Хавпачев [19], а при $n=1$ и $\Phi_{\{1\}}(u)=u^p$ $(p>1)$ — А. А. Кельзон [14]. Класс $H_{\Phi}^{(m)}(2)$ при $m_1=1$, $m_2=1$ и $\Phi(u) \equiv u$ превращается в класс Харди [13]. Наконец, отметим, что в многомерном случае для общей Φ класс $H_{\Phi}^{(m)}(n)$ (при $m_1=m_2=\dots=m_n=1$) ввел Б. И. Голубов [9], а $H_{\Phi(n)}^{(m)}$ (при $m_1=m_2=\dots=m_n=2$) рассмотрен в нашей работе [2].

Справедливы следующие утверждения:

Теорема 1. *Любая функция из класса* $H^{(m)}_{\Phi(n)}$ *ограничена.*

Теорема 2. *Если* $f \in H^{(m)}_{\Phi(n)}$ *и* $x^{(0)} \in R^n$, *то для любого* B $(B \subset M)$ *существуют конечные пределы*

$$\lim_{\tilde{\varepsilon}_B \to 0} f(x^{(0)} + \tilde{\varepsilon}_B).$$

Теорема 3. *Пусть* $f \in H^{(m)}_{\Phi(n)}$.

(а) *Если* $\lim\limits_{u \to +0} (u^2/\Phi_M(u)) = 0$, *то каждое из следующих условий*

1) $$\prod_{k=1}^{n} p_k^{1-2m_k} \sum_{i_1=-p_1}^{p_1} \dots \sum_{i_n=-p_n}^{p_n} |C_{i_1 \dots i_n}(f)|^2 \prod_{s=1}^{n} i_s^{2m_s} = o(1),$$

2) $$\prod_{k=1}^{n} p_k^{-m_k} \sum_{i_1=-p_1}^{p_1} \dots \sum_{i_n=-p_n}^{p_n} |C_{i_1 \dots i_n}(f)| \prod_{s=1}^{n} i_s^{m_s} = o(1),$$

3) $$\sum_{i_1=-p_1}^{p_1} \dots \sum_{i_n=-p_n}^{p_n} |C_{i_1 \dots i_n}(f)| = o\Big(\prod_{s=1}^{n} \ln p_s\Big),$$

4) $$\sum_{|i_1| \geqq p_1} \dots \sum_{|i_n| \geqq p_n} |C_{i_1 \dots i_n}(f)| = o\Big\{\Big(\prod_{s=1}^{n} p_s\Big)^{-1}\Big\} \text{ при } p_1, \dots, p_n \to \infty,$$

5) $$\varliminf_{p_1, \dots, p_n \to \infty} \Big(\prod_{s=1}^{n} \ln p_s\Big)^{-1} \sum_{i_1=-p_1}^{p_1} \dots \sum_{i_n=-p_n}^{p_n} |C_{i_1 \dots i_n}(f)| = 0,$$

необходимо и достаточно чтобы $f \in \overline{H}^{(m)}_{\Phi(n)}$.

(б) *Каждое из перечисленных условий достаточно для выполнения* (6).

(в) *Если для любого* $B \subset M$ $\varliminf\limits_{u \to +0} (u^2/\Phi_B(u)) \neq 0$, *то для включения* $f \in \overline{H}^{(m)}_{\Phi(n)}$ *не существует необходимых и достаточных условий, выраженных через модули коэффициентов Фурье функции* f.

Отметим, что $H^{(m_1, \dots, m_n)}_{\Phi(n)} \subset H^{(r_1, \dots, r_n)}_{\Phi(n)}$ при $r_i \geqq m_i$, $i = 1, \dots, n$. Действительно, для этой цели достаточно показать, что $H^{(m_1, \dots, m_{i_s}, \dots, m_n)}_{\Phi(n)} \subset H^{(m_1, \dots, m_{i_s}+1, \dots, m_n)}_{\Phi(n)}$, где i_s $(1 \leqq i_s \leqq n)$ — некоторое число. Возьмем некоротое подмножество $B \subset M$, которое содержит i_s. Пусть $B = \{i_1, \dots, i_s, \dots, i_p\}$. Тогда по определению

$$V^{(B)}_{\Phi_B}(f; m_{i_1}, \dots, m_{i_s}+1, \dots, m_{i_p}) = \sup_{x_{C_M B}} \sup_{a_B} \sup_{\Pi_{a_B}} \sum_{k_{i_1}=0}^{\mathcal{N}_{i_1}-1} \dots \sum_{k_{i_s}=0}^{\mathcal{N}_{i_s}-1} \dots \sum_{k_{i_p}=0}^{\mathcal{N}_{i_p}-1}$$

$$\Phi_B\Big(2^{p - \sum\limits_{r=1}^{p} m_{i_r} - 1} \Big|\Delta^{(m_{i_1}, \dots, m_{i_s}+1, \dots, m_{i_p})}_{x_{i_1}^{(k_{i_1})}, \dots, x_{i_s}^{(k_{i_s})}, \dots, x_{i_p}^{(k_{i_p})}}(f; h_{i_1}^{(k_{i_1})}, \dots, h_{i_s}^{(k_{i_s})}, \dots, h_{i_p}^{(k_{i_p})})\Big|\Big).$$

Но, как легко видеть,

$$|\Delta^{(m_{i_1},\dots,m_{i_s}+1,\dots,m_{i_p})}_{x^{(k_{i_1})}_{i_1},\dots,x^{(k_{i_s})}_{i_s},\dots,x^{(k_{i_p})}_{i_p}}(f;\ h^{(k_{i_1})}_{i_1},\dots,h^{(k_{i_s})}_{i_s},\dots,h^{(k_{i_p})}_{i_p})| =$$

$$= |\Delta^{(m_{i_1},\dots,m_{i_s},\dots,m_{i_p})_s}_{x^{(k_{i_1})}_{i_1},\dots,x^{(k_{i_s})}_{i_s},\dots,x^{(k_{i_p})}_{i_p}}(f;\ h^{(k_{i_1})}_{i_1},\dots,h^{(k_{i_s})}_{i_s},\dots,h^{(k_{i_p})}_{i_p})-$$

$$-\Delta^{(m_{i_1},\dots,m_{i_s},\dots,m_{i_p})_s}_{x^{(k_{i_1})}_{i_1},\dots,x^{(k_{i_s})}_{i_s}+h^{(k_{i_s})}_{i_s},\dots,x^{(k_{i_p})}_{i_p}}(f;\ h^{(k_{i_1})}_{i_1},\dots,h^{(k_{i_s})}_{i_s},\dots,h^{(k_{i_p})}_{i_p})| \leqq$$

$$\leqq 2\max\{|\Delta^{(m_{i_1},\dots,m_{i_s},\dots,m_{i_p})_s}_{x^{(k_{i_1})}_{i_1},\dots,x^{(k_{i_s})}_{i_s},\dots,x^{(k_{i_p})}_{i_p}}(f;\ h^{(k_{i_1})}_{i_1},\dots,h^{(k_{i_s})}_{i_s},\dots,h^{(k_{i_p})}_{i_p})|,$$

$$|\Delta^{(m_{i_1},\dots,m_{i_s},\dots,m_{i_p})_s}_{x^{(k_{i_1})}_{i_1},\dots,x^{(k_{i_s})}_{i_s}+h^{(k_{i_s})}_{i_s},\dots,x^{(k_{i_p})}_{i_p}}(f;\ h^{(k_{i_1})}_{i_1},\dots,h^{(k_{i_s})}_{i_s},\dots,h^{(k_{i_p})}_{i_p})|\},$$

где операция $\Delta^{(m_{i_1},\dots,m_{i_s},\dots,m_{i_p})_s}_{\dots}$ совпадает с $\Delta^{(m_{i_1},\dots,m_{i_s},\dots,m_{i_p})}_{\dots}$, только с той разницей, что в $\Delta^{(m_{i_1},\dots,m_{i_s},\dots,m_{i_p})_s}_{\dots}$ $h^{(k_{i_s})}_{i_s}=\dfrac{1}{m_{i_s}+1}(x^{(k_{i_s}+1)}_{i_s}-x^{(k_{i_s})}_{i_s})$. Поэтому, получим

$$V^{(B)}_{\Phi_B}(f;\ m_{i_1},\dots,m_{i_s}+1,\dots,m_{i_p}) \leqq V^{(B)}_{\Phi_B}(f;\ m_{i_1},\dots,m_{i_s},\dots,m_{i_p}).$$

Замечание 1. Так как $H^{(m_1,\dots,m_n)}_{\Phi(n)}\subset H^{(r_1,\dots,r_n)}_{\Phi(n)}$ при $r_k\geqq m_k$, $k=1,\dots,n$, то в условиях 1) и 2) пункта (а) теоремы 3 m_k, соответственно, можно заменить числами r_k.

Теорема 4. *Предположим, что* $f\in H^{(m)}_{\Phi(n)}$.

(а) *Если* $\lim\limits_{u\to+0}\left(u^2/\Phi_{\{i\}}(u)\right)=0$ $(i=1,\dots,n)$, *то для непрерывности*[2] *функции* f *необходимо и достаточно выполнение следующих условий*

$$\sum_{|i_j|\geqq p_j}\ \sum_{-\infty}^{\infty}{}'_j\,|C_{i_1\dots i_n}(f)|^2=o(1/p_j),\quad p_j\to\infty\quad (j=1,\dots,n), \tag{7}$$

где символ $\sum\limits_{-\infty}^{\infty}{}'_j$ *означает, что суммирование производится относительно всех индексов (кроме j-го индекса) от* $-\infty$ *до* $+\infty$.

(б) *Совокупность перечисленных условий* (7) *достаточна для непрерывности функции* f.

(в) *Если для некоторого* i $(1\leqq i\leqq n)$ $\varliminf\limits_{u\to+0}\left(u^2/\Phi_{\{i\}}(u)\right)>0$, *то для непрерывности функции* f *не существует необходимых и достаточных условий, выраженных через модули коэффициентов Фурье функции* f.

Теорема 5. *Пусть* $f\in H^{(m)}_{\Phi(n)}$.

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[2] Ниже любую функцию эквивалентную (в смысле меры Лебега) непрерывной функции будем называть непрерывной.

(а) *Если* $\lim\limits_{u\to+0}(u^2/\Phi_{\{i\}}(u))=0$ $(i=1,\dots,n)$, *то для непрерывности функции* f *необходимо и достаточно выполнение следующих условий*

$$(8)\qquad p_j^{1-2m_j}\sum_{i_j=-p_j}^{p_j}\sum_{-\infty}^{\infty}{}'_j|C_{i_1\dots i_n}(f)|^2 i_j^{2m_j}=o(1),\quad p_j\to\infty\quad (j=1,\dots,n).$$

(б) *Совокупность перечисленных условий* (8) *достаточна для непрерывности функции* f.

Так как приведенные утверждения в общем случае (для любой конечной размерности) доказываются аналогично случая $n=2$ и, кроме того, в многомерном случае доказательства этих теорем имеют довольно развернутую форму, то мы собираемся доказать сформулируемые теоремы только для случая $n=2$.

Наконец, заметим, что некоторые результаты этой статьи анонсированы в [3].

§3. Доказательства теорем 1 и 2

Доказательство теоремы 1. Требуется доказать, что если $f\in H_{\Phi(2)}^{(m_1,m_2)}$, то f ограничена. В силу условия теоремы существует такая константа $C(f,\Phi_{\{1\}})>0$,[3] что для любых точек $x_1^{(k)}<x_1^{(k+1)}$ $(x_1^{(k)}\in[0,2\pi],\ k=0,1,2,\dots)$ будем иметь

$$\sup_{x_2}\Phi_{\{1\}}\Big\{2^{1-m_1}\Big|\sum_{\nu=0}^{m_1}(-1)^\nu C_{m_1}^\nu f(x_1^{(k)}+\nu h_1^{(k)},x_2)\Big|\Big\}\leqq C(f,\Phi_{\{1\}}).$$

Так как $\lim\limits_{x\to+\infty}\Phi_{\{1\}}(x)=+\infty$, то отсюда

$$\Big|\sum_{\nu=0}^{m_1}(-1)^\nu C_{m_1}^\nu f(x_1^{(k)}+\nu h_1^{(k)},x_2)\Big|\leqq C(f,\Phi_{\{1\}}).$$

Тогда из последнего неравенства и результата С. К. Хавпачева [20] (см., также, [14]) об ограниченности функции одной переменной m-изменением, заключаем, что f как функция от x_1 ограничена для каждого фиксированного x_2. Для полноты изложения дадим доказательство последнего факта. Пусть $\tilde E$ — некоторое множество положительной меры на оси x_1 и $x_1^{(*)}$ — точка плотности множества $\tilde E$. Тогда для любого достаточно малого t $(t>0)$ найдется число τ такое, что $0<\tau\leqq t/m_1$ и $x_1^{(*)}+t-k\tau\in\tilde E$, $k=1,2,\dots,m_1$. Для доказательства последнего предложения рассмотрим последовательность положительных чисел t_n для которых $\lim\limits_{n\to\infty}t_n=0$. Положим $A_n=\tilde E\cap\Big[x_1^{(*)}+\dfrac{m_1-1}{m_1}t_n,\ x_1^{(*)}+t_n\Big]$ и $\operatorname{mes}A_n=h_n$. Точки множества A_n могут быть представлены в виде $x_1^{(*)}+t_n-\tau$, где τ — расстояние точки из A_n до точки $x_1^{(*)}+t_n$. Подвергнем точки множества A_n линейному преобразованию по формуле $v=x_1^{(*)}+t_n-k\tau$ $(k=2,3,\dots,m_1)$. Этим образуются множества E_k $(k=2,3,\dots,m_1)$.

[3] В дальнейшем мы обозначаем через $C(f)$, $C(f,\Phi_{\{1\}},\tilde E)$, $C_1(m_1,m_2,f)$, ... положительные константы, зависящие лишь от указанных параметров.

Так как $x_1^{(*)}$ — точка плотности множества $\tilde{E}$, то для произвольного $\varepsilon>0$ существует $N_\varepsilon>0$, что при $n>N_\varepsilon$

$$\operatorname{mes}(\tilde{E}\cap[x_1^{(*)}, x_1^{(*)}+t_n]) > \left(1-\frac{\varepsilon}{m_1}\right) t_n.$$

В результате линейного преобразования $v=x_1^{(*)}+t_n-k\tau$ $(k=2, 3, \ldots, m_1)$, где $0<\tau\leqq t_n/m_1$, точки множества $\tilde{E}\cap\left[x_1^{(*)}+\frac{m_1-1}{m_1}t_n,\ x_1^{(*)}+\frac{km_1-1}{km_1}t_n\right]$ попадают в сегмент $\left[x_1^{(*)}+\frac{m_1-k}{m_1}t_n,\ x_1^{(*)}+\frac{m_1-1}{m_1}t_n\right]$. Пусть мера той части множества точек $\tilde{E}\cap\left[x_1^{(*)}+\frac{m_1-1}{m_1}t_n,\ x_1^{(*)}+\frac{km_1-1}{km_1}t_n\right]$, для которых точки вида $x_1^{(*)}+t_n-k\tau$ из E_k не принадлежат множеству $\tilde{E}$, равна $\alpha t_n/km_1$. Покажем, что $\alpha<\varepsilon$. Действительно, если $\alpha\geqq\varepsilon$, то

$$\operatorname{mes}\{\tilde{E}\cap[x_1^{(*)}, x_1^{(*)}+t_n]\} \leqq t_n-\left(\frac{t_n}{m_1}-h_n\right)-k\frac{\alpha t_n}{km_1} \leqq t_n-\frac{\varepsilon t_n}{m_1} = \left(1-\frac{\varepsilon}{m_1}\right) t_n.$$

В целом, множество точек $x_1^{(*)}+t_n-\tau$ из $\tilde{E}\cap\left[x_1^{(*)}+\frac{m_1-1}{m_1}t_n,\ x_1^{(*)}+\frac{2m_1-1}{2m_1}t_n\right]$, для которых точки вида $x_1^{(*)}+t_n-k\tau$ $(k=2, 3, \ldots, m_1)$, лежащие вне сегмента $\left[x_1^{(*)}+\frac{m_1-1}{m_1}t_n,\ x_1^{(*)}+t_n\right]$, не принадлежат множеству $\tilde{E}$, имеет меру, не большую, чем

$$\sum_{k=2}^{m_1}\frac{\varepsilon t_n}{km_1} = \frac{\varepsilon t_n}{m_1}\sum_{k=2}^{m_1}\frac{1}{k}.$$

Множество $\tilde{E}\cap\left[x_1^{(*)}+\frac{m_1-1}{m_1}t_n,\ x_1^{(*)}+\frac{2m_1-1}{2m_1}t_n\right]$ имеет меру не меньшую, чем $\frac{t_n}{2m_1}-\frac{\varepsilon t_n}{m_1}$. Тогда множество точек $x_1^{(*)}+t_n-\tau$ из множества $\tilde{E}\cap\left[x_1^{(*)}+\frac{m_1-1}{m_1}t_n,\ x_1^{(*)}+\frac{2m_1-1}{2m_1}t_n\right]$, для которых точки вида $x_1^{(*)}+t_n-k\tau$ $(k=2, 3, \ldots, m_1)$ принадлежат множеству $\tilde{E}$, имеет меру, не меньшую чем

$$\frac{t_n}{2m_1}-\frac{\varepsilon t_n}{m_1}-\frac{\varepsilon t_n}{m_1}\sum_{k=2}^{m_1}\frac{1}{k}.$$

Соответстующим выбором ε $(\varepsilon>0)$ последнее выражение может буть сделано сколь угодно близким $t_n/2m_1$.

Зафиксируем точку x_2 и пусть E множество положительной меры тех x_1, где отображение $x_1\to f(x_1, x_2)$ ограничена. Как известно, почти все точки множества E являются точками плотности множества E. Рассмотрим одну из них и обозначим ее через $x_1^{(*)}$. Тогда из выше сказанного следует, что для любого достаточно малого t $(t>0)$ найдется число τ, такое, что $0<\tau\leqq t/m_1$ и $x_1^{(*)}+t-k\tau\in E$, $k=1, 2, \ldots, m_1$. Имеем

$$\left|\sum_{\nu=0}^{m_1}(-1)^\nu C_{m_1}^\nu f(x_1^{(*)}+t-\nu\tau, x_2)\right|\leqq C(f, \Phi_{\{1\}}).$$

Следовательно,

$$|f(x_1^{(*)}+t, x_2)|\leqq C(f, \Phi_{\{1\}}, E, x_2)$$

для любого достаточно малого t, т. е. существует такое число δ $(\delta>0)$, что, если $0<t\leqq\delta$, то

$$|f(x_1^{(*)}+t, x_2)|\leqq C(f, \Phi_{\{1\}}, E, x_2).$$

Если учесть неравенство

$$\left|\sum_{\nu=0}^{m_1}(-1)^\nu C_{m_1}^\nu f\left(x_1^{(*)}+\nu\frac{t}{m_1-1}, x_2\right)\right|\leqq C(f, \Phi_{\{1\}}),$$

то получим

$$\left|f\left(x_1^{(*)}+\frac{m_1}{m_1-1}\,t, x_2\right)\right|\leqq C(f, \Phi_{\{1\}}, E, x_2)\quad (0<t\leqq\delta),$$

т. е.

$$|f(x_1^*+t, x_2)|\leqq C(f, \Phi_{\{1\}}, E, x_2),$$

если $0<t\leqq\dfrac{m_1\delta}{m_1-1}$. Продолжая этот процесс, через k-тое число шагов будем иметь

$$|f(x_1^{(*)}+t, x_2)|\leqq C(f, \Phi_{\{1\}}, E, x_2)\qquad \left(0<t\leqq\left(\frac{m_1}{m_1-1}\right)^k\delta\right).$$

Теперь для доказательства ограниченности отображения $x_1\to f(x_1, x_2)$ при фиксированной точки x_2 остается учесть периодичность функции f относительно x_1. Так что $\varphi(x_2)=\sup\limits_{x_1}|f(x_2, x_1)|$ — конечная функция.

Пусть F — множество положительной меры тех x_2, где φ ограничена. Положим

$$|f(x_1, x_2)|\leqq\varphi(x_2)\leqq C(f, F, \Phi_{\{1\}}),\quad x_2\in F.$$

Возьмем $x_2^{(*)}$ — некоторую точку плотности множества F. Тогда, как и выше, для любого достаточно малого s $(s>0)$ существует такое число l $(0<l<s/m_2)$, что точки $x_2^{(*)}+s-\mu l\in F$, $\mu=1, 2, \ldots, m_2$. Так как $f\in H_{\Phi(2)}^{(m_1, m_2)}$, то

$$\left|\sum_{\mu=0}^{m_2}(-1)^\mu C_{m_2}^\mu f(x_1, x_2^{(*)}+s-\mu l)\right|\leqq C(f, \Phi_{\{2\}}).$$

Следовательно,

$$|f(x_1, x_2^{(*)}+s)|\leqq C(f, \Phi_{\{1\}}, \Phi_{\{2\}}, F, x_2^{(*)})$$

для любого x_1 и достаточно малого s, т. е. существует η $(\eta>0)$, что если $0<s\leqq\eta$, то

$$|f(x_1, x_2^{(*)}+s)| \leqq C(f, \Phi_{\{1\}}, \Phi_{\{2\}}, F, x_2^{(*)})$$

равномерно относительно x_1. Продолжая это рассуждение таким же образом как и выше, через k-тое число шагов будем иметь

$$|f(x_1, x_2^{(*)}+s)| \leqq C(f, \Phi_{\{1\}}, \Phi_{\{2\}}, F, x_2^{(*)}) \quad (0 < s \leqq (m_2/(m_2-1))^k\eta).$$

Теперь ограниченность функции f следует из периодичности этой функции относительно x_2.

Доказательство теоремы 2. Покажем, что если $f\in H_{\Phi(2)}^{(m_1, m_2)}$, то в каждой фиксированной точке $(x_1^{(0)}, x_2^{(0)})$ существуют конечные пределы

$$f(x_1^{(0)}\pm 0, x_2^{(0)}) \equiv \lim_{\varepsilon\to+0} f(x_1^{(0)}\pm\varepsilon, x_2^{(0)}), \quad f(x_1^{(0)}, x_2^{(0)}\pm 0) \equiv \lim_{\varepsilon\to+0} f(x_1^{(0)}, x_2^{(0)}\pm\varepsilon),$$

$$f(x_1^{(0)}\pm 0, x_2^{(0)}\pm 0) \equiv \lim_{\substack{\varepsilon\to+0\\ \delta\to+0}} f(x_1^{(0)}\pm\varepsilon, x_2^{(0)}\pm\delta).$$

Можно ограничиться функциями $\Phi_B(u)\equiv u$ $(B\subset\{1, 2\})$, так как в общем случае доказательство проводится аналогично.

При каждом фуксированном $x_2^{(0)}$ функция f как функция переменной x_1 является из класса $H_{\Phi(1)}^{(m_1)}$; поэтому, в сущности, из результата А. А. Кельзона [14] вытекает существование конечных пределов $f(x_1^{(0)}\pm 0, x_2^{(0)})$. По аналогичной причине существуют и конечны $f(x_1^{(0)}, x_2^{(0)}\pm 0)$.

Теперь докажем существование конечных пределов $f(x_1^{(0)}\pm 0, x_2^{(0)}\pm 0)$. Покажем существование конечного предела $f(x_1^{(0)}+0, x_2^{(0)}+0)$. Существование остальных пределов доказывается аналогично.

Положим $a=x_1^{(0)}-\pi$ и $b=x_2^{(0)}-\pi$, т. е. центром квадрата $[a, a+2\pi]\times\times[b, b+2\pi]$ служит точка $(x_1^{(0)}, x_2^{(0)})$. Пусть $x_1^{(k)}$ и $\tilde{x}_1^{(k)}$ $(k=1, \dots, \infty)$ — последовательности на $[a, a+2\pi]$, а $x_2^{(i)}$ и $\tilde{x}_2^{(i)}$ $(i=1, \dots, \infty)$ — на $[b, b+2\pi]$, причем $x_1^{(k)}\to\to+x_1^{(0)}$ и $\tilde{x}_1^{(k)}\to+x_1^{(0)}$ при $k\to\infty$. Покажем (см. (4)), что

$$(9) \qquad \lim_{k, i\to\infty} \Delta_{x_1^{(k)}, x_2^{(i)}}^{(m_1, m_2)}\big(f;\ (\tilde{x}_1^{(k)}-x_1^{(k)})/m_1,\ (\tilde{x}_2^{(i)}-x_2^{(i)})/m_2\big) = 0.$$

Допустим противное, пусть

$$\varlimsup_{k, i\to\infty} |\Delta_{x_1^{(k)}, x_2^{(i)}}^{(m_1, m_2)}\big(f;\ (\tilde{x}_1^{(k)}-x_1^{(k)})/m_1,\ (\tilde{x}_2^{(i)}-x_2^{(i)})/m_2\big) = C > 0.$$

Тогда сущестуют такие подпоследовательности $x_1^{(k_r)}, \tilde{x}_2^{(k_r)}, x_2^{(i_r)}, \tilde{x}_2^{(i_r)}$, что

$$(10) \qquad \lim_{r\to\infty} \big|\Delta_{x_1^{(k_r)}, x_2^{(i_r)}}^{(m_1, m_2)}\big(f;\ (\tilde{x}_1^{(k_r)}-x_1^{(k_r)})/m_1,\ (\tilde{x}_2^{(i_r)}-x_2^{(i_r)})/m_2\big)\big| = C > 0.$$

Ясно, что из последовательностей $x_1^{(k_r)}$ и $\tilde{x}_1^{(k_r)}$ можно выделить такие убывающие подпоследовательности $x_1^{(k_{r_j})}$ и $\tilde{x}_1^{(k_{r_j})}$, что сегменты $[x_1^{(k_{r_j})}, \tilde{x}_1^{(k_{r_j})}]$ не пересекались.[4] Поэтому, учитывая (10) заключаем, что суммы вида

[4] Предполагается, что $x_1^{(k_{r_j})} \leqq \tilde{x}_1^{(k_{r_j})}$. В противном случае рассмотрим сегмент $[\tilde{x}_1^{(k_{r_j})}, x_1^{(k_{r_j})}]$.

$$\sum_{j=1}^{p}\left|\Delta^{(m_1,m_2)}_{x_1^{(k_{r_j})},x_2^{(k_{r_j})}}\left(f;\ (\tilde{x}_1^{(k_{r_j})}-x_1^{(k_{r_j})})/m_1,\ (\tilde{x}_2^{(k_{r_j})}-x_2^{(k_{r_j})})/m_2\right)\right|$$

можно сделать сколь угодно большими вместе с p. Последнее противоречит тому, что $f\in H_{\Phi(2)}^{(m_1,m_2)}$, т. е. имеет место (9).

Если $x_1^{(k)}\to +x_1^{(0)}$, $\tilde{x}_1^{(k)}\to +x_1^{(0)}$, а $x_2^{(i)}$ — любая последовательность из $[x_2^{(0)}-\pi/(m_2+1),\ x_2^{(0)}+\pi/(m_1+1)]$, то (см. (5))

$$\lim_{i,k\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k)}}\left(f;\ (\tilde{x}_1^{(k)}-x_1^{(k)})/m_1;\ x_2^{(i)}\right)=0. \tag{11}$$

В самом деле, сначала докажем, что предел существует. Пусть

$$\varlimsup_{i,k\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k)}}\left(f;\ (\tilde{x}_1^{(k)}-x_1^{(k)})/m_1;\ x_2^{(i)}\right)=B,$$

$$\varliminf_{i,k\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k)}}\left(f;\ (\tilde{x}_1^{(k)}-x_1^{(k)})/m_1;\ x_2^{(i)}\right)=A. \tag{12}$$

Тогда существуют такие подпоследовательности $x_1^{(k_p)}$, $\tilde{x}_1^{(k_p)}$ и $x_2^{(i_p)}$, что

$$\lim_{p\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k_p)}}\left(f;\ (\tilde{x}_1^{(k_p)}-x_1^{(k_p)})/m_1;\ x_2^{(i_p)}\right)=B. \tag{13}$$

Отметим, что значение предела в (13) зависит от выбора последовательности $x_2^{(i_p)}$. Предположим, что $\tilde{x}_2^{(p)}$ — некоторая последовательность из $[x_2^{(0)}-\pi/(m_2+1),\ x_2^{(0)}+\pi/(m_2+1)]$ и

$$\varlimsup_{p\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k_p)}}\left(f;\ (\tilde{x}_1^{(k_p)}-x_1^{(k_p)})/m_1;\ \tilde{x}_2^{(p)}\right)=B(\{\tilde{x}_2^{(p)}\}),$$

$$\varliminf_{p\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k_p)}}\left(f;\ (\tilde{x}_1^{(k_p)}-x_1^{(k_p)})/m_1;\ \tilde{x}_2^{(p)}\right)=A(\{\tilde{x}_2^{(p)}\}).$$

Положим $\sup B(\{\tilde{x}_2^{(p)}\})=B^*$ и $\inf A(\{\tilde{x}_2^{(p)}\})=A^*$. Тогда $A^*\leqq B\leqq B^*$. Ясно, что найдутся такие подпоследовательности $x_1^{(k_{p_j})}$, $\tilde{x}_1^{(k_{p_j})}$ и последовательность $x_2^{*(j)}$, из $[x_2^{(0)}-\pi/(m_2+1),\ x_2^{(0)}+\pi/(m_2+1)]$, что

$$\lim_{j\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k_{p_j})}}\left(f;\ (\tilde{x}_1^{(k_{p_j})}-x_1^{(k_{p_j})})/m_1;\ x_2^{*(j)}\right)=B^*. \tag{14}$$

Далее, имеем см. (13)

$$\lim_{j\to\infty}\tilde{\Delta}^{(m_1)}_{x_1^{(k_{p_j})}}\left(f;\ (\tilde{x}_1^{(k_{p_j})}-x_1^{(k_{p_j})})/m_1;\ x_2^{(i_{p_j})}\right)=B. \tag{15}$$

Рассмотрим любое натуральное число l $(l\geqq m_2)$. Тогда в силу (9) для любого $\tau=0, 1, 2, \dots, l$ имеем

$$\lim_{j\to\infty}\sum_{\nu=\tau}^{m_2+\tau}(-1)^{\nu}C_{m_2}^{\nu-\tau}\tilde{\Delta}^{(m_1)}_{x_1^{(k_{p_j})}}\left(f;\ (\tilde{x}_1^{(k_{p_j})}-x_1^{(k_{p_j})})/m_1;\ x_2^{*(j)}+\nu(x_2^{(i_{p_j})}-x_2^{*(j)})\right)=0. \tag{16}$$

Следуя А. А. Кельзону [15], введем обозначение

$$\Delta_{j,\nu}^{(m_1)} = \tilde{\Delta}_{x_1^{(k_{p_j})}}^{(m_1)}(f;\ (\tilde{x}_1^{(k_{p_j})} - x_1^{(k_{p_j})})/m_1;\ x_2^{*(j)} + \nu(x_2^{(i_{p_j})} - x_2^{*(j)})),$$

$$\nu = 0, \ldots, m_2+l; \quad j = 1, \ldots, \infty.$$

Из последовательностей $\{\Delta_{j,0}^{(m_1)}\}$, $\{\Delta_{j,1}^{(m_1)}\}, \ldots, \{\Delta_{j,m_2+l}^{(m_1)}\}$ выделим сходящиеся последовательности. Предположим, что

$$\lim_{s\to\infty} \Delta_{j_s,\nu}^{(m_1)} = \Delta_\nu^{(m_1)}; \quad \nu = 0, \ldots, m_2+l.$$

Согласно (14) и (15) имеем

$$\Delta_0^{(m_1)} = B^*, \quad \Delta_1^{(m_1)} = B \quad \text{и} \quad C_1(m_1, m_2; f) \leqq \Delta_\nu^{(m_1)} \leqq C_2(m_1, m_2; f),$$

$$\nu = 2, \ldots, m_2+l$$

Пусть

$$A_r = \sum_{\nu=0}^{r} (-1)^\nu C_r^\nu \Delta_\nu^{(m_1)} \quad (r = 1, \ldots, \infty).$$

Учитывая (16), находим

$$\sum_{\nu=\tau}^{m_2+\tau} (-1)^\nu C_{m_2}^{\nu-\tau} \Delta_\nu^{(m_1)} = 0, \quad \tau = 0, \ldots, l. \tag{17}$$

Отсюда, в силу равенства

$$\sum_{\nu=0}^{s+1} (-1)^\nu C_{s+1}^\nu \Delta_\nu^{(m_1)} = \sum_{\nu=0}^{s} (-1)^\nu C_s^\nu \Delta_\nu^{(m_1)} + \sum_{\nu=0}^{s} (-1)^{\nu+1} C_s^\nu \Delta_{\nu+1}^{(m_1)}, \quad s = 0, \ldots, \infty,$$

получим

$$A_{m_2} = A_{m_2+1} = \ldots = A_{m_2+l} = 0.$$

Отметим, что из условия $A_{m_2}=0$ вытекает $A_{m_2-1}=0$. Последнее можно доказать так же, как это делается в работе А. А. Кельзона [15]. Мы для полноты доказательства приведем соответствующее рассуждение.

Вновь учитывая соотношение (17), находим

$$A_{m_2-1} = \sum_{\nu=\tau}^{\tau+m_2-1} (-1)^{\nu-\tau} C_{m_2-1}^{\nu-\tau} \Delta_\nu^{(m_1)} = \sum_{\nu=0}^{m_2-1} (-1)^\nu C_{m_2-1}^\nu \Delta_{\nu+\tau}^{(m_1)}. \tag{18}$$

Последнее доказывается с помощью индукции. Ясно, что 1) при $\tau=0$ имеет место (18). Пусть (18) справедливо для $\tau-1$, т. е.

$$A_{m_2-1} = \sum_{\nu=\tau-1}^{\tau+m_2-2} (-1)^{\nu-\tau+1} C_{m_2-1}^{\nu-\tau+1} \Delta_\nu^{(m_1)}$$

и докажем (18) для τ. Имеем

$$\sum_{\nu=\tau}^{\tau+m_2-1}(-1)^{\nu-\tau}C_{m_2-1}^{\nu-\tau}\Delta_\nu^{(m_1)}-\sum_{\nu=\tau-1}^{\tau+m_2-2}(-1)^{\nu-\tau+1}C_{m_2-1}^{\nu-\tau+1}\Delta_\nu^{(m_1)}=(-1)C_{m_2-1}^{0}\Delta_{\tau-1}^{(m_1)}+$$

$$+\sum_{\nu=\tau}^{\tau+m_2-2}(-1)^{\nu-\tau}[C_{m_2-1}^{\nu-\tau}+C_{m_2-1}^{\nu-\tau+1}]\Delta_\nu^{(m_1)}+(-1)^{m_2-1}C_{m_2-1}^{m_2-1}\Delta_{\tau+m_2-1}^{(m_1)}=$$

$$=\sum_{\nu=\tau}^{\tau+m_2-2}(-1)^{\nu-\tau}C_{m_2}^{\nu-\tau+1}\Delta_\nu^{(m_1)}+(-1)C_{m_2}^{0}\Delta_{\tau-1}^{(m_1)}+(-1)^{m_2-1}C_{m_2}^{m_2}\Delta_{\tau-m_2-1}^{(m_1)}=$$

$$=-\sum_{\nu=\tau-1}^{\tau-m_2-1}(-1)^{\nu-\tau+1}C_{m_2}^{\nu-\tau+1}\Delta_\nu^{(m_1)}.$$

Последнее выражение в силу (17) равняется нулю.

Соотношение (18) дает

$$(l+1)A_{m_2-1}=\sum_{\tau=0}^{l}\sum_{\nu=0}^{m_2-1}(-1)^\nu C_{m_2-1}^{\nu}\Delta_{\nu+\tau}^{(m_1)}=$$

$$=\sum_{\nu=0}^{m_2-1}(-1)^\nu C_{m_2-1}^{\nu}\sum_{\tau=0}^{l}\Delta_{\nu+\tau}^{(m_1)}=\sum_{\nu=0}^{m_2-1}\sum_{\tau=\nu}^{l+\nu}g_{\nu,\tau},$$

где $g_{\nu,\tau}=(-1)^\nu C_{m_2-1}^{\nu}\Delta_\tau^{(m_1)}$. Имеем

$$\begin{aligned}(l+1)A_{m_2-1}=\ & g_{0,0}+g_{0,1}+\ldots+g_{0,m_2-1}+g_{0,m_2}+\ldots+g_{0,l}+\\ & g_{1,1}+g_{1,2}+\ldots+g_{1,m_2-1}+g_{1,m_2}+\ldots+g_{1,l}+g_{1,l+1}+\\ & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ & g_{m_2-1,m_2-1}+g_{m_2-1,m_2}+\ldots+g_{m_2-1,l}+g_{m_2-1,l+1}+\ldots+g_{m_2-1,m_2-1+l}=\end{aligned}$$

$$=\sum_{k=0}^{m_2-2}\sum_{\nu=0}^{k}g_{\nu,k}+\sum_{k=m_2-1}^{l}\sum_{\nu=0}^{m_2-1}g_{\nu,k}+\sum_{k=l+1}^{l+m_2-1}\sum_{\nu=k-l}^{m_2-1}g_{\nu,k}=$$

$$=\sum_{k=0}^{m_2-1}\Delta_k^{(m_1)}\sum_{\nu=0}^{k}(-1)^\nu C_{m_2-1}^{\nu}+\sum_{k=m_2-1}^{l}\Delta_k^{(m_1)}\sum_{\nu=0}^{m_2-1}(-1)^\nu C_{m_2-1}^{\nu}+$$

$$+\sum_{k=l+1}^{l+m_2-1}\Delta_k^{(m_2)}\sum_{\nu=k-l}^{m_2-1}(-1)^\nu C_{m_2-1}^{\nu}.$$

Ясно, что $\sum_{\nu=0}^{m_2-1}(-1)^\nu C_{m_2-1}^{\nu}=0$. Поэтому,

$$(l+1)A_{m_2-1}=\sum_{k=0}^{m_2-1}\Delta_k^{(m_1)}\sum_{\nu=0}^{k}(-1)^\nu C_{m_2-1}^{\nu}+\sum_{k=l+1}^{l+m_2-1}\Delta_k^{(m_1)}\sum_{\nu=k-l}^{m_2-1}(-1)^\nu C_{m_2-1}^{\nu}.$$

Так как функция f ограничена, то последняя сумма оценивается константой, не зависящей от l. В силу произвольности l $A_{m_2-1}=0$. Последовательно повторяя это рассуждение, получим $A_1=0$, т. е.

$$C_1^0\Delta_0^{(m_1)}=C_1^1\Delta_1^{(m_1)}.$$

Стало быть, $B=B^*$. Аналогично можно показать, что $A^*=B=B^*$. Таким образом, предел в (13) не зависит от последовательности $x_2^{(i_p)}$ $(p=1, \ldots, \infty)$ $\left(x_2^{(i_p)} \in [x_2^{(0)}-\pi/(m_2+1), x_2^{(0)}+\pi/(m_2+1)]\right)$. Если теперь в (13) вместо $x_2^{(i_p)}$ подставить постоянную последовательность $x_2^{(i_p)} \equiv C$ $(p=1, \ldots, \infty)$, то из существования предела $f(x_1^{(0)}+0, C)$ следует, что $B=0$. Точно таким же образом можно показать (см. (12)), что $A=0$. Следовательно, имеет место (11).

Теперь покажем, что для любых последовательностей $h_1^{(k)}(h_1^{(k)} \to +0)$, $t_1^{(k)}(t_1^{(k)} \to +0)$, $x_2^{(i)}\left(x_2^{(0)}-\pi/(m_2+1) \leqq x_2^{(i)} \leqq x_2^{(0)}+\pi/(m_2+1)\right)$

(19)
$$\lim_{i,k\to\infty} [f(h_1^{(k)}, x_2^{(i)})-f(t_1^{(k)}, x_2^{(i)})] = 0.$$

Пусть

$$\overline{\lim_{i,k\to\infty}} [f(h_1^{(k)}, x_2^{(i)})-f(t_1^{(k)}, x_2^{(i)})] = P,$$

(20)
$$\underline{\lim_{i,k\to\infty}} [f(h_1^{(k)}, x_2^{(i)})-f(t_1^{(k)}, x_2^{(i)})] = Q.$$

Тогда найдутся подпоследовательности $h_1^{(k_j)}, t_1^{(k_j)}, x_2^{(i_j)}$ для которых

(21)
$$\lim_{j\to\infty} [f(h_1^{(k_j)}, x_2^{(i_j)})-f(t_1^{(k_j)}, x_2^{(i_j)})] = P.$$

Рассмотрим любое натуральное число $r \geqq m$. Тогда из соотношения (11) для любого $s=0, 1, 2, \ldots, r$ получим

(22)
$$\lim_{j\to\infty} \sum_{\nu=s}^{m_1+s} (-1)^\nu C_{m_1}^{\nu-s} f\left(h_1^{(k_j)}+\nu(t_1^{(k_j)}-h_1^{(k_j)}), x_2^{(i_j)}\right) = 0, \quad \nu = 0, \ldots, m+r, \quad j = 1, \ldots, \infty.$$

Будем предполагать, что $h_1^{(k_j)} \leqq t_1^{(k_j)}$. Если для некоторых j последнее не выполняется, то в (22) вместо $h_1^{(k_j)}$ будем предполагать $t_1^{(k_j)}$. Пусть

$$\varphi_{j,\nu} = f\left(h_1^{(k_j)}+\nu(t_1^{(k_j)}-h_1^{(k_j)}), x_2^{(i_j)}\right).$$

Как и выше, из последовательностей $\{\varphi_{j,0}\}, \{\varphi_{j,1}\}, \ldots, \{\varphi_{j,m+r}\}$ выделим сходящиеся подпоследовательности

$$\lim_{p\to\infty} \varphi_{j_p,\nu} = \varphi_\nu, \quad \nu = 0, \ldots, m+r.$$

Из равенства (21) следует, что

(23)
$$\varphi_0-\varphi_1 = P.$$

Рассмотрим

$$K_q = \sum_{\nu=0}^{q} (-1)^\nu C_q^\nu \varphi_\nu.$$

Соотношение (22) дает

$$\sum_{\nu=s}^{m+s} (-1)^\nu C_{m_1}^{\nu-s} \varphi_\nu = 0, \quad s = 0, \ldots, r.$$

В силу последнего равенства имеем

$$K_m = K_{m+1} = \ldots = K_{m+r} = 0.$$

Используя предыдущее рассуждение можно заключить, что из соотношения $K_m=0$ следует $K_{m-1}=0$, и т. д. — следует $K_1=0$. Последнее означает, что (см. (23)) $P=0$.

Аналогично можно показать, что (см. (20)) $Q=0$. Следовательно, имеет место (19). Точно таким же образом можно доказать, что

$$\lim_{i,k\to\infty} [f(x_1^{(k)}, s_2^{(i)}) - f(x_1^{(k)}, z_2^{(i)})] = 0. \tag{24}$$

где $s_2^{(i)}, z_2^{(i)} \to +x_2^{(0)}$, а $x_1^{(k)} \in [x_1^{(0)} - \pi/(m_1+1), x_1^{(0)} + \pi/(m_1+1)]$.

Наконец, докажем, что $f(x_1^{(0)}+0, x_2^{(0)}+0)$ существует. Пусть

$$\varlimsup_{\substack{x_1\to+x_1^{(0)}\\ x_2\to+x_2^{(0)}}} f(x_1, x_2) = M \quad \text{и} \quad \varliminf_{\substack{x_1\to+x_1^{(0)}\\ x_2\to+x_2^{(0)}}} f(x_1, x_2) = N.$$

Тогда найдутся такие последовательности $x_1^{(k)}, x_2^{(k)}, \tilde{x}_1^{(k)}$ и $\tilde{x}_2^{(k)}$ $(x_1^{(k)}, \tilde{x}_1^{(k)} \to +x_1^{(0)}$, а $x_2^{(k)}, \tilde{x}_2^{(k)} \to +x_2^{(0)}$ при $k\to\infty)$, что

$$\lim_{k\to\infty} f(x_1^{(k)}, x_2^{(k)}) = M, \quad \lim_{k\to\infty} f(\tilde{x}_1^{(k)}, \tilde{x}_2^{(k)}) = N.$$

В силу (19) следует

$$\lim_{k\to\infty} f(\tilde{x}_1^{(k)}, \tilde{x}_2^{(k)}) = M,$$

а из последнего (см. (24)), находим

$$\lim_{k\to\infty} f(\tilde{x}_1^{(k)}, \tilde{x}_2^{(k)}) = M.$$

Таким образом, $M=N$, т. е. существует $f(x_1^{(0)}+0, x_2^{(0)}+0)$.

Теорема 2 доказана.

§4. Вспомогательные утверждения

Докажем некоторых утверждений, которые понадобятся в дальнейшем.

Лемма 1.[5] *Пусть*

$$\omega_{\Phi_M}^{(m_1,m_2)}(f;\,\delta,\eta) = \sup_{\substack{0\le t\le\delta\\ 0\le\theta\le\eta}} \int_0^{2\pi}\int_0^{2\pi} \Phi_M\big(2^{2-m_1-m_2}|\Delta_{x_1-t/2,\,x_2-\theta/2}^{(m_1,m_2)}(f;\,t,\theta)|\big)\,dx_1\,dx_2.$$

Тогда

$$\omega_{\Phi_M}^{(m_1,m_2)}(f;\,\delta,\eta) \le 4m_1m_2\delta\eta V_{\Phi_M}^{(M)}(f;\,m_1,m_2).$$

[5] Относительно лемм 1 и 2 см. работу Б. И. Голубова [7].

Доказательство. Положим $x_1^{(i)}=im_1\delta$, $x_2^{(j)}=jm_2\eta$, где $i=0, ..., p+1$, $j=0, ..., q+1$, $0<\delta, \eta\leqq 2\pi$; p и q подобраны так, что $x_1^{(p)}\leqq 2\pi$, $x_2^{(q)}\leqq 2\pi$, $x_1^{(p+1)}>2\pi$, $x_2^{(q+1)}>2\pi$. Имеем

$$\int_0^{2\pi}\int_0^{2\pi}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1-\delta/2,\,x_2-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2=$$

$$=\sum_{i=1}^{p}\sum_{j=1}^{q}\int_{x_1^{(i-1)}}^{x_1^{(i)}}\int_{x_2^{(j-1)}}^{x_2^{(j)}}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1-\delta/2,\,x_2-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2+$$

$$+\int_{x_1^{(p)}}^{2\pi}\int_{x_2^{(q)}}^{2\pi}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1-\delta/2,\,x_2-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2+$$

$$+\int_{x_1^{(p)}}^{2\pi}\int_{0}^{x_2^{(q)}}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1-\delta/2,\,x_2-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2+$$

$$+\int_{0}^{x_1^{(p)}}\int_{x_2^{(q)}}^{2\pi}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1-\delta/2,\,x_2-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2=$$

$$=\int_0^{m_1\delta}\int_0^{m_2\eta}\sum_{i=1}^{p}\sum_{j=1}^{q}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1+x_1^{(i-1)}-\delta/2,\,x_2+x_2^{(j-1)}-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2+$$

$$+\int_0^{2\pi-x_1^{(p)}}\int_0^{2\pi-x_2^{(q)}}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1+x_1^{(p)}-\delta/2,\,x_2+x_2^{(q)}-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2+$$

$$+\int_0^{2\pi-x_1^{(p)}}\int_0^{m_2\eta}\sum_{j=1}^{q}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1+x_1^{(p)}-\delta/2,\,x_2+x_2^{(j-1)}-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2+$$

$$+\int_0^{m_1\delta}\int_0^{2\pi-x_2^{(q)}}\sum_{i=1}^{p}\Phi_M\left(2^{2-m_1-m_2}|\Delta^{(m_1,m_2)}_{x_1+x_1^{(i-1)}-\delta/2,\,x_2+x_2^{(q)}-\eta/2}(f;\,\delta,\eta)|\right)dx_1\,dx_2\leqq$$

$$\leqq 4m_1m_2\delta\eta V^{(M)}_{\Phi_M}(f;\,m_1,\,m_2).$$

Лемма 1 доказана.

Лемма 2. *Пусть* $f\in H^{(m_1,m_2)}_{\Phi(2)}$. *Для того чтобы* f *принадлежала классу* $\overline{H}^{(m_1,m_2)}_{\Phi(2)}$ *необходимо и достаточно, чтобы*

$$\lim_{\delta,\eta\to 0}\sup_{x_1,x_2}\omega^{(m_1,m_2)}_{x_1,x_2}(f;\,\delta,\eta)=0, \tag{25}$$

где

$$\omega^{(m_1,m_2)}_{x_1,x_2}(f;\,\delta,\eta)=\sup_{\substack{0\leqq t\leqq\delta\\0\leqq\theta\leqq\eta}}|\Delta^{(m_1,m_2)}_{x_1-t/2,\,x_2-\theta/2}(f;\,t,\theta)|.$$

Доказателство. Из соотношения (25) легко следует (6) при $n=2$. Пусть $f\in\overline{H}_{\Phi(2)}^{(m_1,m_2)}$, но

$$\varlimsup_{\delta,\eta\to 0}\sup_{x_1,x_2}\omega_{x_1,x_2}^{(m_1,m_2)}(f;\ \delta,\eta)>0.$$

Тогда найдутся такие последовательности $\delta_1^{(p)}$ и $\eta_2^{(p)}$ $(\delta_1^{(p)},\eta_2^{(p)}\to+0)$, что

$$\sup_{x_1,x_2}\omega_{x_1,x_2}^{(m_1,m_2)}(f;\ \delta_1^{(p)},\eta_2^{(p)})>\delta_0,$$

где δ_0 — некоторое положительное число. Отсюда следует существование таких подпоследовательностей $\delta_1^{(p_i)}$, $\eta_2^{(p_i)}$ и последовательности $(x_1^{(i)},\ x_2^{(i)})$ точек, что

(26) $$\omega_{x_1^{(i)},x_2^{(i)}}^{(m_1,m_2)}(f;\ \delta_1^{(p_i)},\ \eta_2^{(p_i)})>\delta_0.$$

Учитывая периодичность функции f, можно считать, что точки $(x_1^{(i)},\ x_2^{(i)})$ лежат в Q_2. Либо из последовательности $(x_1^{(i)},\ x_2^{(i)})$ можно выделить бесконечную стационарную последовательность $\big((x_1^{(i_k)},x_2^{(i_k)})=(x_1^{(0)},x_2^{(0)}),\ \ k=k_0,k_0+1,\ldots\big)$, либо в ней найдется бесконечное множество различных точек $\big((\tilde{x}_1^{(i_k)},\tilde{x}_2^{(i_k)})\neq\neq(\tilde{x}_1^{(i_j)},\tilde{x}_2^{(i_j)})$ при $i\neq j\big)$. Легко видеть, что в первом случае условие (26) противоречит соотношению (6) при $n=2$, а во втором — предположению $f\in H_{\Phi(2)}^{(m_1,m_2)}$.

Лемма 3. *Пусть (см.* (5)*)*

$$\omega_{\Phi_{\{1\}}}(f,\delta)=\sup_{0\le t\le\delta,x_2}\int_0^{2\pi}\Phi_{\{1\}}\big(2^{1-m_1}|\tilde{\Delta}_{x_1-t/2}^{(m_1)}(f;\ t;\ x_2)|\big)\,dx_1,$$

$$\omega_{\Phi_{\{2\}}}(f,\delta)=\sup_{x_1,0\le s\le\delta}\int_0^{2\pi}\Phi_{\{2\}}\big(2^{1-m_2}|\tilde{\Delta}_{x_2-s/2}^{(m_2)}(f;\ s;\ x_1)|\big)\,dx_2.$$

Тогда

$$\omega_{\Phi_{\{i\}}}^{(m_i)}(f,\delta)\le 2m_i\delta V_{\Phi_{\{i\}}}^{\{i\}}(f;\ m_i),\quad i=1,2.$$

Последнее утверждеие доказывается так же, как лемма 1.

Лемма 4. *Предположим, что* $f\in H_{\Phi(2)}^{(m_1,m_2)}$ *и для любой точки* (x_1,x_2)

$$f(x_1+0,x_2)=f(x_1-0,x_2),\quad f(x_1,x_2+0)=f(x_1,x_2-0).$$

Тогда

(27) $$\lim_{\delta\to 0}\sup_{x_i}\omega_{x_1,x_2}^{(m_i)}(f,\delta)=0;\quad i=1\ \ \text{либо}\ \ i=2,$$

где (см. (5)*)*

(28) $$\omega_{x_1,x_2}^{(m_1)}(f,\delta)=\sup_{0\le t\le\delta}|\tilde{\Delta}_{x_1-t/2}^{(m_1)}(f;\ t;\ x_2)|,$$

(29) $$\omega_{x_1,x_2}^{(m_2)}(f,\delta)=\sup_{0\le\theta\le\delta}|\tilde{\Delta}_{x_2-\theta/2}^{(m_2)}(f;\ \theta;\ x_1)|.$$

Доказательство. Допустим противное, пусть

$$\varlimsup_{\delta\to 0}\sup_{x_1}\omega_{x_1,x_2}^{(m_1)}(f,\delta)>0.$$

Тогда найдется последовательность $\delta_1^{(p)}$ ($\delta_1^{(p)}\to 0$ при $p\to\infty$), что

$$\sup_{x_1} \omega_{x_1, x_2}^{(m_1)}(f, \delta_1^{(p)}) \geqq \delta_0 > 0.$$

Стало быть существуют такая подпоследовательность $\delta_1^{(p_i)}$ и последовательность точек $x_1^{(i)}$, что

$$\omega_{x_1^{(i)}, x_2}^{(m_1)}(f;\ \delta_1^{(p_i)}) \geqq \delta_0.$$

Теперь рассуждая так же, как при доказательстве леммы 2, получим, что верно соотношение (27) при $i=1$. Аналогично доказывается (27) в случае $i=2$.

§5. Доказательства теорем 3,4 и 5

Доказательство теоремы 3. Сначала докажем пункт (a). Обозначим через $C_{n_1 n_2}^1$ коэффициент Фурье функции $2^{2-m_1-m_2}\Delta_{x_1-\delta/2, x_2-\eta/2}^{(m_1, m_2)}(f;\ \delta, \eta)$ (как функции относительно x_1 и x_2). Тогда простые вычисления показывают, что

$$C_{n_1 n_2}^1 = 4(-i)^{m_1+m_2} e^{i[(m_1-1)n_1\delta/2+(m_2-1)n_2\eta/2]} \sin^{m_1}\frac{n_1\delta}{2}\sin^{m_2}\frac{n_2\eta}{2}\, C_{n_1 n_2}(f). \tag{30}$$

Таким образом, в силу равенства Парсеваля имеем

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[2^{2-m_1-m_2}\Delta_{x_1-\delta/2, x_2-\eta/2}^{(m_1, m_2)}(f;\ \delta, \eta)]^2\, dx_1\, dx_2 = \tag{31}$$

$$= 16\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}|C_{n_1 n_2}(f)|^2\sin^{2m_1}\frac{n_1\delta}{2}\sin^{2m_2}\frac{n_2\eta}{2}.$$

Пусть $f\in \overline{H}_{\Phi(2)}^{(m_1, m_2)}$. Так как $\lim\limits_{u\to+0}(u^2/\Phi_M(u))=0$, то $u^2=\Phi_M(u)\alpha(u)$, где $\alpha(u)=o(1)$ при $u\to+0$. Поэтому, в силу лемм 1 и 2 находим

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[2^{2-m_1-m_2}\Delta_{x_1-\pi/(2p), x_2-\pi/(2q)}^{(m_1, m_2)}\left(f;\ \frac{\pi}{p}, \frac{\pi}{q}\right)\right]^2 dx_1\, dx_2 = \tag{32}$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Phi_M\left(2^{2-m_1-m_2}\left|\Delta_{x_1-\pi/(2p), x_2-\pi/(2q)}^{(m_1, m_2)}\left(f;\ \frac{\pi}{p}, \frac{\pi}{q}\right)\right|\right)\times$$

$$\times\alpha\left(2^{2-m_1-m_2}\left|\Delta_{x_1-\pi/(2p), x_2-\pi/(2q)}^{(m_1, m_2)}\left(f;\ \frac{\pi}{p}, \frac{\pi}{q}\right)\right|\right) dx_1\, dx_2 \leqq$$

$$\leqq \frac{1}{4\pi^2}\sup_{x_1, x_2}\alpha\left(2^{2-m_1-m_2}\left|\Delta_{x_1-\pi/(2p), x_2-\pi/(2q)}^{(m_1, m_2)}\left(f;\ \frac{\pi}{p}, \frac{\pi}{q}\right)\right|\right)\times$$

$$\times\int_0^{2\pi}\int_0^{2\pi}\Phi_M\left(2^{2-m_1-m_2}\left|\Delta_{x_1-\pi/(2p), x_2-\pi/(2q)}^{(m_1, m_2)}\left(f;\ \frac{\pi}{p}, \frac{\pi}{q}\right)\right|\right) dx_1\, dx_2 =$$

$$= o(1)\cdot O(\pi^2/pq) = o\big(1/(pq)\big) \quad \text{при} \quad p, q\to\infty.$$

Следовательно, учитывая неравенство $\sin x \geqq \frac{2}{\pi} x \ \left(0 \leqq x \leqq \frac{\pi}{2}\right)$ и (31), получим

(33) $$p^{1-2m_1} q^{1-2m_2} \sum_{i=-p}^{p} \sum_{j=-q}^{q} |C_{ij}(f)|^2 i^{2m_1} j^{2m_2} = o(1) \quad \text{при} \quad p, q \to \infty.$$

Используя неравенство Коши—Шварца, имеем

$$(1/(p^{m_1} q^{m_2})) \sum_{i=-p}^{p} \sum_{j=-q}^{q} |C_{ij}(f)|\, i^{m_1} j^{m_2} \leqq$$

$$\leqq \left\{p^{1-2m_1} q^{1-2m_2} \sum_{i=-p}^{p} \sum_{j=-q}^{q} |C_{ij}(f)|^2 i^{2m_1} j^{2m_2}\right\}^{1/2} \left\{\sum_{i=-p}^{p} \sum_{j=-q}^{q} p^{-1} q^{-1}\right\}^{1/2}.$$

Отсюда

(34) $$(1/(p^{m_1} q^{m_2})) \sum_{i=-p}^{p} \sum_{j=-q}^{q} |C_{ij}(f)|\, i^{m_1} j^{m_2} = o(1).$$

Из последнего соотношения, в свою очередь, следует, что

(35) $$\sum_{i=-p}^{p} \sum_{j=-q}^{q} |C_{ij}(f)| = o(\ln p \ln q) \quad \text{при} \quad p, q \to \infty.$$

Действительно, пусть

$$A_{M,N} = \sum_{i=1}^{M} \sum_{j=1}^{N} i^{m_1} j^{m_2} |C_{ij}(f)| = o(M^{m_1} N^{m_2}), \quad M, N \to \infty,$$

$$a_{i,N} = \sum_{j=1}^{N} j^{m_2} |C_{ij}(f)|.$$

Тогда

$$A_{M,N} = \sum_{i=1}^{M} i^{m_1} a^{i,N}.$$

Применяя преобразование Абеля, получим

$$\sum_{i=1}^{M} \sum_{j=1}^{N} |C_{ij}(f)| = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_{i,j} - a_{i,j-1}) j^{-m_2} = \sum_{i=1}^{M} \sum_{j=1}^{N-1} \left(j^{-m_2} - (j+1)^{-m_2}\right) a_{i,j} +$$

$$+ \sum_{i=1}^{N} N^{-m_2} a_{i,N} = \sum_{j=1}^{N-1} \left(j^{-m_2} - (j+1)^{-m_2}\right) \sum_{i=1}^{M} a_{i,j} + \sum_{i=1}^{N} N^{-m_2} a_{i,N}.$$

Вновь пользуясь теоремой Абеля, находим

$$\sum_{i=1}^{M} a_{i,j} = \sum_{i=1}^{M} i^{-m_1} (A_{i,j} - A_{i-1,j}) = \sum_{i=1}^{M-1} \left(i^{-m_1} - (i+1)^{-m_1}\right) A_{i,j} +$$

$$+ M^{-m_1} A_{M,j} = \sum_{i=1}^{M-1} \left(i^{-m_1} - (i+1)^{-m_1}\right) o(i^{m_1} \cdot j^{m_2}) + o(j^{m_2}) =$$

$$= o(j^{m_2} \ln M) + o(j^{m_2}) = o(j^{m_2} \ln M).$$

Следовательно,

$$\sum_{i=1}^{M}\sum_{j=1}^{N}|C_{ij}(f)| = \sum_{j=1}^{N-1}\left(j^{-m_2}-(j+1)^{-m_2}\right)o(j^{m_2}\ln M)+o(\ln M) =$$

$$= \sum_{j=1}^{N-1} o(j^{-1}\ln M)+o(\ln M) = o(\ln M\ln N).$$

Таким образом, выполняются соотношения (33)—(35) и, кроме того, (33)⇒(34)⇒(35). Из (35), в свою очередь, вытекает равенство

$$\varliminf_{p,q\to\infty}(\ln p\ln q)^{-1}\sum_{i=-p}^{p}\sum_{j=-q}^{q}|C_{ij}(f)| = 0.$$

Отсюда, применяя во внимание двумерный аналог теоремы Лукача (см. лемму 3 из работы Б. И. Голубова [7]), заключаем, что $f\in\overline{H}_{\Phi(2)}^{(m_1, m_2)}$.

Теперь покажем, что из условия

$$(36)\qquad \sum_{|i|\geqq p}\sum_{|j|\geqq q}|C_{ij}(f)|^2 = o\left(1/(pq)\right)\quad (p, q\to\infty)$$

следует (33). Действительно, легко обощить для двукратного случая одну формулу Абеля (см., напр., [1], стр. 78)

$$(37)\qquad \sum_{i=1}^{p}\lambda_i u_i = \sum_{i=2}^{p}(\lambda_i-\lambda_{i-1})\sum_{\nu=i}^{\infty}u_\nu+\lambda_1\sum_{\nu=1}^{\infty}u_\nu-\lambda_p\sum_{\nu=p+1}^{\infty}u_\nu.$$

Имеем

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\lambda_{i,j}u_{ij} = \sum_{i=2}^{p}\sum_{j=2}^{q}(\lambda_{i,j}-\lambda_{i,j-1}-\lambda_{i-1,j}+\lambda_{i-1,j-1})\sum_{r=i}^{\infty}\sum_{\nu=j}^{\infty}u_{r\nu}+$$

$$+\sum_{j=2}^{q}(\lambda_{1,j}-\lambda_{1,j-1})\sum_{r=1}^{\infty}\sum_{\nu=j}^{\infty}u_{r\nu}-\sum_{j=2}^{q}(\lambda_{p,j}-\lambda_{p,j-1})\sum_{r=p+1}^{\infty}\sum_{\nu=j}^{\infty}u_{r\nu}+$$

$$+\sum_{i=2}^{p}(\lambda_{i,1}-\lambda_{i-1,1})\sum_{r=i}^{\infty}\sum_{\nu=1}^{\infty}u_{r\nu}+\lambda_{1,1}\sum_{r=1}^{\infty}\sum_{\nu=1}^{\infty}u_{r\nu}-\lambda_{p,1}\sum_{r=p+1}^{\infty}\sum_{\nu=1}^{\infty}u_{r\nu}-$$

$$-\sum_{i=2}^{p}(\lambda_{i,q}-\lambda_{i-1,q})\sum_{r=i}^{\infty}\sum_{\nu=q+1}^{\infty}u_{r\nu}-\lambda_{1,q}\sum_{r=1}^{\infty}\sum_{\nu=q+1}^{\infty}u_{r\nu}+\lambda_{p,q}\sum_{r=p+1}^{\infty}\sum_{\nu=q+1}^{\infty}u_{r\nu}.$$

Положим

$$\lambda_{i,j} = i^{2m_1}j^{2m_2},\quad u_{ij} = |C_{ij}(f)|^2.$$

Тогда при условии (36), получим

$$\sum_{i=1}^{p}\sum_{j=1}^{q} i^{2m_1}j^{2m_2}|C_{ij}(f)|^2=\sum_{i=2}^{p}\sum_{j=2}^{q}[i^{2m_1}-(i-1)^{2m_1}][j^{2m_2}-(j-1)^{2m_2}]o\big(1/(ij)\big)+$$

$$+\sum_{j=2}^{q}[j^{2m_2}-(j-1)^{2m_2}]\,o(1/j)+\sum_{j=2}^{q}p^{2m_1}[j^{2m_2}-(j-1)^{2m_2}]\,o(1/p_j)+$$

$$+\sum_{i=2}^{p}[i^{2m_1}-(i-1)^{2m_1}]\,o(1/i)+O(1)+p^{2m_1}o(1/p)+$$

$$+\sum_{i=2}^{p}q^{2m_2}[i^{2m_1}-(i-1)^{2m_1}]\,o\big(1/(iq)\big)+q^{2m_2}o(1/q)+$$

$$+p^{2m_1}q^{2m_2}o\big(1/(pq)\big)=o(p^{2m_1-1}q^{2m_2-1})\quad\text{при}\quad p,q\to\infty.$$

Стало быть, из условия (36) следует $f\in\overline{H}_{\Phi(2)}^{(m_1,m_2)}$. Теперь обратно — докажем, что при условии теоремы 3 из соотношения (6) вытекает (36). Так как $f\in H_{\Phi(2)}^{(m_1,m_2)}$, $\lim\limits_{u\to+0}\big(u^2/\Phi_M(u)\big)=0$ и имеет место (6) и (31), то как и выше (см. (32)) можно показать, что

$$I(f)\equiv\lim_{h,\eta\to 0}\big(1/(h\eta)\big)\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|C_{ij}(f)|^2\sin^{2m_1}i\pi h\sin^{2m_2}j\pi\eta=0.$$

Далее, легко видеть, что

$$I(f)\geqq\varlimsup_{\substack{k\to\infty\\ p\to\infty}}\Big(\int_0^\infty\int_0^\infty\frac{e^{-k/x_1}e^{-p/x_2}}{x_1^3x_2^3}\,dx_1\,dx_2\Big)^{-1}\times$$

$$\times\int_0^\infty\int_0^\infty x_1x_2\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|C_{ij}(f)|^2\sin^{2m_1}\frac{i\pi}{x_1}\sin^{2m}\frac{j\pi}{x_2}\,\frac{e^{-k/x_1}e^{-p/x_2}}{x_1^3x_2^3}\,dx_1\,dx_2=$$

$$=\varlimsup_{\substack{k\to\infty\\ p\to\infty}}k^2p^2\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|C_{ij}(f)|^2\int_0^\infty\sin^{2m_1}i\pi x_1e^{-kx_1}\,dx_1\int_0^\infty\sin^{2m_2}j\pi x_2e^{-px_2}\,dx_2.$$

Но

$$\int_0^\infty\sin^{2m_1}i\pi x_1e^{-kx_1}\,dx_1=(i\pi)^{2m_1}k(2m_1)!\prod_{\tau=0}^{m_1}[k^2+(2i\tau\pi)^2]^{-1}\geqq$$

$$\geqq\pi^{2m_1}(2m_1)!\,i^{2m_1}k^{-1}[k^2+(2m_1i\pi)^2]^{-m_1}.$$

Следовательно,

$$I(f) \geqq \varlimsup_{\substack{k\to\infty \\ p\to\infty}} \pi^{2(m_1+m_2)}(2m_1)!(2m_2)! \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |C_{ij}(f)|^2 i^{2m_1} j^{2m_2} kp[k^2+(2\pi m_1 i)^2]^{-m_1} \times$$

$$\times [p^2+(2\pi m_2 j)^2]^{-m_2} \geqq \frac{(2m_1)!(2m_2)!}{2^{2(m_1+m_2)} m_1^{2m_1} m_2^{2m_2}} \varlimsup_{\substack{k\to\infty \\ p\to\infty}} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |C_{ij}(f)|^2 kp i^{2m_1} j^{2m_2} \times$$

$$\times (k^2+i^2)^{-m_1}(p^2+j^2)^{-m_2} \geqq \frac{(2m_1)!(2m_2)!}{2^{2(m_1+m_2)} m_1^{2m_1} m_2^{2m_2}} \varlimsup_{\substack{k\to\infty \\ p\to\infty}} \sum_{|i|\geqq k} \sum_{|j|\geqq p} |C_{ij}(f)|^2 kp i^{2m_1} j^{2m_2} \times$$

$$\times (2i^2)^{-m_1}(2j^2)^{-m_2} = \frac{(2m_1)!(2m_2)!}{2^{3(m_1+m_2)} m_1^{2m_1} m_2^{2m_2}} \varlimsup_{\substack{k\to\infty \\ p\to\infty}} kp \sum_{|i|\geqq k} \sum_{|j|\geqq p} |C_{ij}(f)|^2.$$

Стало быть, имеет место (36).

Относительно доказательства пункта (б) теоремы 3 при $n=2$ отметим следующее. При доказательстве достаточности условий 1)—5) пункта (а) теоремы 3 мы не пользовались усповием $\lim_{u\to+0} \left(u^2/\Phi_M(u)\right)=0$. С другой стороны, отметим, что доказательство достаточности условий 1)—4) теоремы 3 вытекает из работы Б. И. Голубова (см. [7], теорему 2).

Для доказательства пункта (в) можно использовать пример, построенный Б. И. Голубовым [7]. Действительно, положим

$$\varphi_1(x_1) = \sum_{p=1}^{\infty} \frac{\sin px_1}{p}, \quad \varphi_2(x_1) = \sum_{p=1}^{\infty} \frac{\sin p(x_1+\log p)}{p}.$$

Функция φ_1 — ограниченной вариации, а функция φ_2 — из класса Lip 1/2 (см., напр., [12], стр. 317—320). Пусть

$$f_1(x_1, x_2) = \varphi_1(x_1)\,\varphi_1(x_2) \quad \text{и} \quad f_2(x_1, x_2) = \varphi_2(x_1)\,\varphi_2(x_2).$$

В силу условия пункта (в) теоремы 3 найдутся $x_1^{(0)}$ $(x_1^{(0)}>0)$ и такая положительная константа A, что если $0\leqq x_1 \leqq x_1^{(0)}$, то

$$\Phi_B(x_1) \leqq Ax_1^2, \quad B\subset M=\{1,2\}. \tag{38}$$

Ясно, что φ_1 имеет неустранимый разрыв в точке 0. Так что $\varphi_1 \in H^{(1)}_{\Phi(1)}$ и $\varphi_1 \overline{\in} \overline{H}^{(1)}_{\Phi(1)}$, где $\Phi(u)\equiv u$. Следовательно, $f_1 \in H^{(m_1,m_2)}_{\Phi(2)}$, но $f_1 \overline{\in} \overline{H}^{(m_1,m_2)}_{\Phi(2)}$. С другой стороны, учитывая (38) находим

$$\varphi_2 \in \mathrm{Lip}\,1/2 \subset \overline{H}^{(1)}_{\Phi^*(1)},$$

где $\Phi^*(u)=u^2$. Стало быть, $f_2 \in \overline{H}^{(m_1,m_2)}_{\Phi(2)}$. Остается заметить, что модули коэффициентов Фурье функций f_1 и f_2 с одиноковой парой индексов совпадают.

Теорема 3 доказана.

Доказательство теоремы 4. Пусть $C^2_{n_1n_2}$ — коэффициенты Фурье функции $2^{1-m_1}\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\delta;x_2)$. Тогда, как и выше (см. (30)),

$$C^2_{n_1n_2} = 2(-i)^{m_1} e^{i(m_1-1)n_1\delta/2} \sin^{m_1}\frac{n_1\delta}{2}\, C_{n_1n_2}(f).$$

В силу равенства Парсеваля

(39)

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[2^{1-m_1}\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\ \delta;\ x_2)]^2\,dx_1\,dx_2 = 4\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}|C_{n_1n_2}(f)|^2\sin^{2m_1}\frac{n_1\delta}{2}.$$

Пусть $\delta=\dfrac{\pi}{p}$, тогда

(40)
$$\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[2^{1-m_1}\tilde{\Delta}^{(m_1)}_{x_1-\pi/(2p)}\left(f;\ \frac{\pi}{p};\ x_2\right)\right]^2 dx_1\,dx_2 =$$

$$= \sum_{n_1=-p}^{p}\sum_{n_2=-\infty}^{\infty}|C_{n_1n_2}(f)|^2\sin^{2m_1}\frac{n_1\pi}{2p}+\sum_{|n_1|>p}\sum_{n_2=-\infty}^{\infty}|C_{n_1n_2}(f)|^2\sin^{2m_1}\frac{n_1\pi}{2p}\leqq$$

$$\leqq (\pi/(2p))^{2m_1}\sum_{n_1=-p}^{p}\sum_{n_2=-\infty}^{\infty}|C_{n_1n_2}(f)|^2 n_1^{2m_1}+\sum_{|n_1|>p}\sum_{n_2=-\infty}^{\infty}|C_{n_1n_2}(f)|^2.$$

Используя (7) и преобразование Абеля (см. (37)) для первого слагаемого последнего соотношения, получим

$$p^{1-2m_1}\sum_{n_1=1}^{p}\sum_{n_2=-\infty}^{\infty}|C_{n_1n_2}(f)|\,n_1^{2m_1} =$$

$$= p^{1-2m_1}\sum_{n_2=-\infty}^{\infty}\sum_{n_1=2}^{p}[n_1^{2m_1}-(n_1-1)^{2m_1}]\sum_{\nu=n_1}^{\infty}|C_{\nu n_2}(f)|^2+$$

$$+p^{1-2m_1}\sum_{n_2=-\infty}^{\infty}\sum_{\nu=1}^{\infty}|C_{\nu n_2}(f)|^2-p^{1-2m_1}\sum_{n_2=-\infty}^{\infty}p^{2m_1}\sum_{\nu=p+1}^{\infty}|C_{\nu n_2}(f)|^2 =$$

$$= p^{1-2m_1}\sum_{n_1=2}^{p}[n_1^{2m_1}-(n_1-1)^{2m_1}]\,o(n_1^{-1})+O(p^{1-2m_1})+po(p^{-1}) = o(1)\quad\text{при}\quad p\to\infty.$$

Стало быть, первое слагаемое в (40) имеет порядок $o(p^{-1})$ при $p\to\infty$. Поэтому, в силу (7) имеем

(41)
$$\int_0^{2\pi}\int_0^{2\pi}\left[\tilde{\Delta}^{(m_1)}_{x_1-\pi/(2p)}\left(f;\ \frac{\pi}{p};\ x_2\right)\right]^2 dx_1\,dx_2 = o(1/p)\quad\text{при}\quad p\to\infty.$$

Так как функция f ограничена (см. теорему 1) и существуют односторонние пределы во всех точках (см. теорему 2), то

(42)
$$\int_0^{2\pi}\int_0^{2\pi}[f(x_1+0,\ x_2)-f(x_1-0,\ x_2)]^2\,dx_1\,dx_2 = 0.$$

В силу периодичности функции f будем иметь (см. (41))

$$\int_0^{2\pi}\int_0^{2\pi}\left|\tilde{\Delta}^{(m_1)}_{x_1-\pi/(2p)}\left(f;\ \frac{\pi}{p};\ x_2\pm\frac{\pi}{q}\right)\right|^2 dx_1\,dx_2 = o(1/p)\quad\text{при}\quad p,\,q\to\infty.$$

Отсюда

$$(43)\qquad \int_0^{2\pi}\int_0^{2\pi}[f(x_1+0, x_2\pm 0)-f(x_1-0, x_2\pm 0)]^2\,dx_1\,dx_2 = 0.$$

Аналогично получим

$$(44)\qquad \int_0^{2\pi}\int_0^{2\pi}[f(x_1, x_2+0)-f(x_1, x_2-0)]^2\,dx_1\,dx_2 = 0,$$

$$(45)\qquad \int_0^{2\pi}\int_0^{2\pi}[f(x_1\pm 0, x_2+0)-f(x_1\pm 0, x_2-0)]^2\,dx_1\,dx_2 = 0.$$

Таким же образом можно показать, что

$$(46)\qquad \int_0^{2\pi}\int_0^{2\pi}[f(x_1, x_2)-f(x_1+0, x_2)]^2\,dx_1\,dx_2 = 0,$$

$$(47)\qquad \int_0^{2\pi}\int_0^{2\pi}[f(x_1, x_2+0)-f(x_1+0, x_2+0)]^2\,dx_1\,dx_2 = 0.$$

Теперь объедняя равенства (42)—(47), легко заключить, что почти всюду

$$f(x_1, x_2) = f(x_1\pm 0, x_2\pm 0) = f(x_1\pm 0, x_2\mp 0) = f(x_1\pm 0, x_2) = f(x_1, x_2\pm 0).$$

Таким образом, достаточность условий (7) дозакана. Покажем их необходимость. В силу (39) и того факта, что $\lim\limits_{u\to+0}\big(u^2/\Phi_{\{1\}}(u)\big)=0$ имеем $\big(u^2=\Phi_{\{1\}}(u)\alpha_1(u)$, где $\alpha_1(u)=o(1)$ при $u\to+0\big)$

$$(48)\qquad \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[2^{1-m_1}\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\delta;x_2)]^2\,dx_1\,dx_2 =$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Phi_{\{1\}}\big(2^{1-m_1}|\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\ \delta;\ x_2)|\big)\alpha_1\big(2^{1-m_1}|\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\ \delta;\ x_2)|\big)\,dx_1\,dx_2 =$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\Big\{\sup_{x_1}\alpha_1\big(2^{1-m_1}|\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\ \delta;\ x_2)|\big)\times$$

$$\times\int_0^{2\pi}\Phi_{\{1\}}\big(2^{1-m_1}|\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\ \delta;\ x_2)|\big)\,dx_1\Big\}\,dx_2.$$

Отсюда, используя леммы 3 и 4, находим

$$(49)\qquad \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[2^{1-m_1}\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\ \delta;\ x_2)]^2\,dx_1\,dx_2 \leqq$$

$$\leqq \frac{\delta m_1}{2\pi^2}V^{\{1\}}_{\Phi_{\{1\}}}(f;\ m_1)\int_0^{2\pi}\Big\{\sup_{x_1}\alpha_1\big(2^{1-m_1}|\tilde{\Delta}^{(m_1)}_{x_1-\delta/2}(f;\ \delta;\ x_2)|\big)\,dx_2 = o(\delta)\Big\},\quad \delta\to+0.$$

Поэтому (см. (39)),

$$I_1(f) \equiv \lim_{h\to 0} (1/h) \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |C_{ij}(f)|^2 \sin^{2m_1} i\pi h = 0.$$

Стало быть,

$$I_1(f) \geqq \varlimsup_{k\to\infty} \left(\int_0^\infty \frac{e^{-k/x_1}}{x_1^3}\,dx_1\right)^{-1} \int_0^\infty x_1 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |C_{ij}(f)|^2 \sin^{2m_1} \frac{i\pi}{x_1} \frac{e^{-k/x_1}}{x_1^3}\,dx_1 =$$

$$= \varlimsup_{k\to\infty} k^2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |C_{ij}(f)|^2 \int_0^\infty \sin^{2m_1} i\pi x_1 e^{-kx_1}\,dx_1.$$

Отсюда

$$I_1(f) \geqq \varlimsup_{k\to\infty} \pi^{2m_1}(2m_1)! \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} [k^2+(2\pi m_1 i)^2]^{-m_1} |C_{ij}(f)|^2 i^{2m_1} k \geqq$$

$$\geqq \frac{(2m_1)!}{2^{2m_1} m_1^{2m_1}} \varlimsup_{k\to\infty} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} k|C_{ij}(f)|^2 i^{2m_1}(k^2+i^2)^{-m_1} \geqq$$

$$\geqq \frac{(2m_1)!}{(2m_1)^{2m_1}} \varlimsup_{k\to\infty} \sum_{|i|\geqq k} \sum_{j=-\infty}^{\infty} k|C_{ij}(f)|^2 i^{2m_1}(2i^2)^{-m_1} =$$

$$= \frac{(2m_1)!}{2^{3m_1} m_1^{2m_1}} \varlimsup_{k\to\infty} k \sum_{|i|\geqq k} \sum_{j=-\infty}^{\infty} |C_{ij}(f)|^2.$$

Следовательно,

$$\sum_{|i|\geqq k} \sum_{j=-\infty}^{\infty} |C_{ij}(f)|^2 = o(1/k), \quad k\to\infty.$$

Аналогично можно показать, что

$$\sum_{|j|\geqq k} \sum_{i=-\infty}^{\infty} |C_{ij}(f)|^2 = o(1/k), \quad k\to\infty.$$

Анализируя приведенное доказательство пункта (а) теоремы 4, легко заключить справедливость пункта (б) той же теоремы.

Пункт (в) теоремы 4 (в случае $n=2$) легко доказывается, если будем построить две функции двух переменних, постоянных относительно одной переменной, а относительно другой — построим так, как это сделано в работе Е. Коэна (см. [4], стр. 231).

Доказательство теоремы 5. Необходимая часть пункта (а) теоремы 5 следует из выше приведенного рассуждения (см. (48), (49) и (39)). Для доказательства достаточности заметим, что из условий (8) следует (см. (40)) соотношения (42), (43), (46) и (47). Аналогично можно получить (44) и (45), а из этих равенств, как и выше, следует непрерывность функции f и доказательство пункта (б) теоремы 5.

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ON A BOHR TYPE INEQUALITY

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1. Introduction. In 1952 H. Bohr proved that if

$$f(x) = \sum_{k=1}^{N} (a_k \cos \lambda_k x + b_k \sin \lambda_\lambda x), \quad \lambda_k \geqq n > 0,$$

then its integral function

$$F(x) = \int^{x} f(t)\, dt = \sum_{k=1}^{N} \frac{1}{\lambda_k} (a_k \sin \lambda_k x - b_k \cos \lambda_k x)$$

satisfies the inequality

(1) $$\max_{-\infty < x < \infty} |F(x)| \leqq \frac{\pi}{2n} \max_{-\infty < x < \infty} |f(x)|.$$

Here the constant $\pi/2$ is best possible (see H. Bohr [1]). Later many authors discussed the analogues of inequality (1) (see e.g. [2], [7]).

In 1970, G. Freud and J. Szabados [6] considered the analogue of (1) for the case of algebraic polynomials. They proved the following statement. Let $\alpha, \alpha+\beta > -1$ be real numbers, and let $\{f_n\}_{n=1}^{\infty} \subset L_1[-1, 1]$ be a sequence of functions satisfying

a) $$|f(x)| \leqq c_n (1-x^2)^{\beta} \quad (c_n > 0, x \in [-1, 1]),$$

b) $$\int_{-1}^{1} f_n(x) p_n(x) (1-x^2)^{\alpha}\, dx = 0$$

for any $p_n \in P_n$, where P_n denotes the set of all algebraic polynomials of degree at most n. Then we have

(2) $$\limsup_{n\to\infty} \frac{n \left| \int_{-1}^{x} (1-t^2)^{\alpha} f_n(t)\, dt \right|}{c_n} \leqq \pi (1-x^2)^{1/2+\alpha+\beta}, \quad x \in (-1, 1).$$

An equivalent formulation is the following: let

$$\varrho(x) = (1-x^2)^{\gamma} \quad (\gamma > -1, \quad x \in (-1, 1)).$$

If $g(x)$ satisfies the conditions

a′) $$\|g\|_\infty := \|g\|_{L^\infty(-1,1)} < \infty,$$

b′) $$\int_{-1}^{1} g(x)p(x)\varrho(x)\,dx = 0, \quad \forall p \in P_n$$

then

(3) $$\left\| \frac{1}{\sqrt{1-x^2}\,\varrho(x)} \int_{-1}^{x} g(t)\,\varrho(t)\,dt \right\|_\infty \leqq \frac{\pi}{n}\|g\|_\infty.$$

A natural question to consider is the following: what is the situation if we replace the weight ϱ by another weight function?

In 1974 G. Freud [4] considered this question for the weight $\varrho(x)=e^{-x^2/2}$ $(x\in(-\infty,\infty))$. He gave interesting applications for weighted polynomial approximation.

In this paper we consider the analogue of (3) for the weight functions

$$v(x) = v_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha, \beta \geqq -1, \quad x\in(-1,1),$$

$$u(x) = u_\lambda(x) = x^\lambda e^{-x}, \quad \lambda \geqq 0, \quad x > 0.$$

In forthcoming papers we shall give applications of these results for weighted polynomial approximation.

2. A Bohr type inequality with the weight v. For $n\geqq 1$, let $P_n^\perp(v)$ be the set of all integrable functions g satisfying

1. $\|g\|_\infty < \infty$
2. $\int_{-1}^{1} g(t)p_n(t)v(t)\,dt = 0, \quad \forall p_n\in P_n.$

We have the following

THEOREM 1. *Let* $\alpha, \beta > -1$, $n=1, 2, \ldots$. *For every* $g\in P_n^\perp(v)$ *we have*

(4) $$\left\| \frac{1}{\sqrt{1-x^2}\,v(x)} \int_{-1}^{x} g(t)v(t)\,dt \right\|_\infty \leqq \frac{c_1}{n}\|g\|_\infty,$$

where c_1 *(and later* c_k $(x, y, \ldots)$*) denotes a constant depending only on* $\alpha, \beta, (x, y, \ldots)$.

PROOF. Let

$$\Gamma_x(t) = \begin{cases} 1 & \text{for} \quad t\leqq x \\ 0 & \text{for} \quad x<t \end{cases} \quad (x, t\in\mathbf{R}),$$

and

$$E_n(\Gamma_x) \leqq \inf_{p\in P_n} \int_{-1}^{1} |\Gamma_x(t)-p(t)|\,v(t)\,dt \quad (x\in[-1,1]).$$

Fisrt we prove that

(5) $$E_n(\Gamma_x) = \frac{c_2}{n}\sqrt{1-x^2}\,v(x) \quad (x\in[-1,1]).$$

For this purpose we apply estimates of the Christoffel function. Let us denote by $J_k^{(\alpha,\beta)}(x)$ the k-th orthonormal Jacobi polynomial $(k=0, 1, 2, \dots)$, and let $\lambda_n^{(\alpha,\beta)}(x)$ be the Christoffel function of the system $\{J_k^{(\alpha,\beta)}\}_{k=0}^{\infty}$, i.e.

$$\lambda_n^{(\alpha,\beta)}(x) = \Big\{\sum_{k=0}^{n-1} [J_k^{(\alpha,\beta)}(x)]^2\Big\}^{-1}.$$

In [8], Lemma 2.1, P. G. Nevai proved the following estimate:

$$\lambda_n^{(\alpha,\beta)}(x) \leqq c_3 \begin{cases} n^{-2}v(x) \quad \text{if} \quad 1-c_4/n^2 \leqq x \leqq 1 \\ n^{-1}\sqrt{1-x^2}\,v(x) \quad \text{if} \quad -1+c_5/n^2 \leqq x \leqq 1-c_4/n^2 \\ n^{-2} \quad \text{if} \quad -1 \leqq x \leqq -1+c_5/n^2 \quad (n=1, 2, \dots). \end{cases} \tag{6}$$

Now we prove (5) in three steps:

a) If $1-c_4/n^2 \leqq x \leqq 1$ then

$$\int_x^1 v(t)\,dt \leqq c_6(1-x)v(x)$$

from which we obtain by approximation with the polynomial $g_1(x)\equiv 1$

$$E_n(\Gamma_x) \leqq \int_{-1}^{1} |\Gamma_x(t)-g_1(t)|\,v(t)\,dt = \int_x^1 v(t)\,dt \leqq c_7(1-x)v(x) \leqq$$

$$\leqq \frac{c_6 c_7}{n}\sqrt{1-x^2}\,v(x) = \frac{c_8}{n}\sqrt{1-x^2}\,v(x).$$

b) If $-1\leqq x\leqq -1+c_4/n^2$, then a similar consideration with the polynomial $g(x)\equiv 0$ gives (5).

c) Finally, the case $-1+c_4/n^2\leqq x\leqq 1-c_4/n^2$ follows from the estimate (6) by application of the following inequality (see G. Freud [3], p. 72):

$$E_n(\Gamma_x) \leqq \lambda_n^{(\alpha,\beta)}(x) \quad (x\in[-1, 1]).$$

Now we return to the proof of Theorem 1. Let $g\in P_n^{\perp}(v)$. For any $p_n\in P_n$ we have by the properties of g:

$$\Big|\int_{-1}^{x} g(t)v(t)\,dt\Big| \leqq \Big|\int_{-1}^{1} g(t)\Gamma_x(t)v(t)\,dt\Big| =$$

$$= \Big|\int_{-1}^{1} g(t)[\Gamma_x(t)-p_n(t)]v(t)\,dt\Big| \leqq \|g\|_\infty \int_{-1}^{1} |\Gamma_x(t)-p_n(t)|\,v(t)\,dt.$$

So, by (5) we have

$$\Big|\int_{-1}^{x} g(t)v(t)\,dt\Big| \leqq \|g\|_\infty E_n(\Gamma_x) \leqq \frac{c_2}{n}\sqrt{1-x^2}\,v(x)\|g\|_\infty, \quad x\in[-1, 1],$$

which is equivalent to (4). This completes the proof of our theorem.

3. A Bohr type inequality with the weight $u(x)$. Let us denote by $P_n^{\perp}(u)$, $n=1, 2, \ldots$ the set of all functions g which are integrable on $(0, \infty)$ and satisfy the following properties:

1) $\|g\|_{\infty}^{*} := \|g\|_{L^{\infty}(0,\infty)} < \infty$,

2) $\int\limits_0^{\infty} g(t)p(t)u(t)\,dt = 0, \quad \forall p \in P_n$.

The following theorem is true:

THEOREM 2. *Let* $\lambda \geqq 0$, $n \geqq 1$. *For every* $g \in P_n^{\perp}(u)$ *we have*

$$\left\| \frac{1}{\sqrt{x}\,u(x)} \int\limits_0^x g(t)u(t)\,dt \right\|_{\infty}^{*} \leqq \frac{c_1(\lambda)}{\sqrt{n}} \|g\|_{\infty}^{*}. \tag{7}$$

PROOF. The proof of (7) is analogous to that of (4). We apply the following estimate, which follows from Theorem 4.2 of G. Freud [5]:

$$\varrho_n^{\lambda}(x) := \Big\{\sum_{k=0}^{n-1} [l_k^{\lambda}(x)^2]\Big\}^{-1} \leqq \frac{c_2(\lambda)}{\sqrt{n}} \sqrt{x}\,u(x) \quad \left(\frac{1}{n} \leqq x \leqq n,\ n = 1, 2, \ldots\right),$$

where $l_k^{\lambda}(x)$, $k=0, 1, \ldots$ is the k-th Laguerre's orthonormal polynomial with respect to the parameter λ.

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ON A GROUP-MATRIX TYPE AUTOMATON WITH OUTPUT

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1. Introduction

In this paper some results of [6] are extended for a strongly connected group-matrix type automaton with output. A new type of representation of a strongly connected group-matrix type automaton of order n on G with output or an (n, G)-automaton with output is introduced. Using this representation we prove some important results on automorphism groups of a strongly connected (n, G)-automaton with output and also on a strongly connected abelian (n, G)-automaton with output.

2. Introductory concepts

DEFINITION 2.1. An automaton A is a 5-tuple $A=\langle S, \Sigma, \Delta, M, Z\rangle$ where S is a non-empty set called the set of internal states, Σ is the input alphabet, Δ is the output alphabet, M is a function from $S\times\Sigma$ to S called the transition function, Z is a function from $S\times\Sigma$ to Δ called the output function.

Σ^* denotes the free monoid generated by the elements of Σ and ε is its identity.

For all $s\in S$ and all $x, y\in\Sigma^*$ we have $M(s, \varepsilon)=s$ and $M(s, xy)=$ $=M(M(s, x), y)$.

Also we have for all $s\in S$ and $x, y\in\Sigma^*$ $Z(s, \varepsilon)=\delta$, $\delta\in\Delta^*$, where Δ^* is the output dictionary,

$$Z(s, xy) = Z(s, x)\, Z(M(s, x), y).$$

DEFINITION 2.2. Let $A=\langle S, \Sigma, \Delta, M, Z\rangle$ be an automaton. A one-to-one mapping of A onto itself $\varrho\colon A\to A$ is called an automorphism if it is operation-preserving, that is,

$$\varrho(M(s, x)) = M(\varrho(s), x)\wedge Z(s, x) = Z(\varrho(s), x)$$

for all $s\in S$ and $x\in\Sigma^*$.

The set of all operation-preserving permutations of A forms a group, the automorphism group $G(A)$ of the automaton A.

DEFINITION 2.3. An automaton $A=\langle S, \Sigma, \Delta, M, Z\rangle$ is said to be strongly connected if for any pair of states $s, t\in S$ there exist $\sigma_0, \sigma_1\in\Sigma$ such that $M(s, \sigma_0)=t$ and $M(t, \sigma)=s$.

Lemma 1 and Theorem 2 proved in [4] can be easily extended as follows:

THEOREM 2.1. *If $A=\langle S, \Sigma, \Delta, M, Z\rangle$ is a strongly connected automaton and g, h are elements in $G(A)$ such that $g(s_0)=h(s_0)$ for some $s_0\in S$, then $g(s)=h(s)$ for all $s\in S$.*

THEOREM 2.2. *If* $A=\langle S, \Sigma, \Delta, M, Z\rangle$ *is a strongly connected automaton, then* $|G(A)|$ *divides* $|S|$, *where* $|K|$ *denotes the cardinality of the set K.*

DEFINITION 2.4. The automata $A=\langle S, \Sigma, \Delta, M, Z\rangle$ and $B=\langle T, \Gamma, \beta, N, Y\rangle$ are said to be isomorphic to each other, denoted by $A\approx B$, if there exist two one-to-one and onto mappings $\varrho\colon S\to T$ and $\xi\colon \Sigma\to\Gamma$ such that

$$\varrho(M(s,\sigma)) = N(\varrho(s), \xi(\sigma))\wedge Z(s,\sigma) = Y(\varrho(s), \xi(\sigma))$$

for all $s\in S$ and $\sigma\in\Sigma$.

THEOREM 2.3. *If A and B are isomorphic automata then* $G(A)$ *is isomorphic to* $G(B)$ $(G(A)\approx G(B))$.

Now in order to construct a group-matrix type automaton of order n on G or an (n, G)-automaton we recall some important definitions introduced in [6].

DEFINITION 2.5. Let G be a finite group. Let G^0 be the set $G\cup\{0\}$ endowed with two operations $(\cdot)$ and $(+)$ satisfying the following properties:

(a) For all $g, h\in G$, $g\cdot h$ is defined as the group operation in G.

(b) For all $g\in G$, $g\cdot 0=0\cdot g=0$ and $0\cdot 0=0$.

(c) For all $g\in G$, $g+0=0+g=g$ and $0+0=0$.

(d) For any $g, h\in G$, $g+h$ is not defined.

DEFINITION 2.6. Let G be a finite group and let n be a positive integer. An $n\times n$ matrix (f_{pq}) $(1\leqq p\leqq n,\ 1\leqq q\leqq n, f_{pq}\in G^0)$ is called a group-matrix of order n on G if for each $p'(1\leqq p'\leqq n)$, there exists a unique number $q'(1\leqq q'\leqq n)$ such that $f_{p'q'}\neq 0$.

The set of all group-matrices of order n on G is denoted by $\tilde{G}_n$. Then $\tilde{G}_n$ forms a semigroup under the operation

$$(f_{pq})(g_{pq}) = \Big(\sum_{k=1}^{n} f_{pk}\, g_{kq}\Big).$$

DEFINITION 2.7. Let G be a finite group and let n be a positive integer. A vector $(f_p)(1\leqq p\leqq n, f_p\in G^0)$ is called a group-vector of order n on G, if there exists a unique number $p'(1\leqq p'\leqq n)$ such that $f_{p'}\neq 0$.

The set of all group-vectors of order n on G is denoted by $\hat{G}_n$. For all $(f_p)\in\hat{G}_n$ and $(g_{pq})\in\tilde{G}_n$, the multiplication is defined by

$$(f_p)(g_{pq}) = \Big(\sum_{k=1}^{n} f_k\, g_{kp}\Big).$$

Under this operation, $(f_p)(g_{pq})\in\hat{G}_n$.

3. (n, G)-automaton with output

DEFINITION 3.1. Let G be a finite group and let n be a positive integer. An automaton $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ is called a group-matrix type automaton of order n on G with output or an (n, G)-automaton with output, if the following conditions are satisfied:

(1) $\hat{G}_n$ is the set of states.
(2) Σ is a set of inputs.
(3) Δ is a set of outputs.
(4) M_Ψ is a state transition function defined by

$$M_\Psi(\hat{g}, x) = \hat{g}\Psi(x) \quad (\hat{g}\in\hat{G}_n,\ x\in\Sigma^*)$$

where Ψ is a mapping of Σ^* into $\tilde{G}_n$ and $\Psi(xy)=\Psi(x)\Psi(y)$ for all $\varkappa, y\in\Sigma^*$
(5) Z is a function from $\hat{G}_n\times\Sigma^*$ to Δ^* called the output function.
(Cf. Definition 2.4 in [6].)

THEOREM 3.1. *An (n, G)-automaton with output $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ is strongly connected if and only if the following condition is satisfied:*
For all $p', q'(1\leqq p'\leqq n,\ 1\leqq q'\leqq n)$ there exist an element $x=\sigma_0\sigma_1\in\Sigma^*$ such that $\psi_{p'q'}(x)=e$ (e is the identity of G) where $\sigma_0, \sigma_1\in\Sigma$ and we put $\Psi(x)=(\psi_{pq}(x))$.

PROOF. *Necessity*. Assume at first that A is strongly connected. Then there exist $\sigma_0, \sigma_1\in\Sigma$ such that

$$M_\Psi(\hat{g}_p, \sigma_0) = \hat{g}_q, \tag{3.1}$$

and

$$M_\Psi(\hat{g}_q, \sigma) = \hat{g}_{p'}. \tag{3.2}$$

Using (3.1) and (3.2) we have

$$M_\Psi(\hat{g}_p, \sigma_0\sigma_1) = \hat{g}_{p'}. \tag{3.3}$$

Now set $\sigma_0\sigma_1=x\in\Sigma^*$. Then by (3.3), $M_\Psi(\hat{g}_{p'}, x)=\hat{g}_p$, thus $\hat{g}_{p'}\Psi(x)=\hat{g}_p$. and $\psi_{p'q'}(x)=e$.

Sufficiency. Assume now that the condition of the theorem is satisfied. Then, $M_\Psi(\hat{g}_{p'}, x)=\hat{g}_{p'}, \Psi(x)$, hence $M_\Psi(\hat{g}_{p'}, \sigma_0\sigma_1)=\hat{g}_{p'}$ and $M_\Psi(M_\Psi(\hat{g}_{p'}, \sigma_0), \sigma_1)=\hat{g}_{p'}$. Set $M_\Psi(\hat{g}_{p'}, \sigma_0)=\hat{g}_{q'}$. Then we have $M_\Psi(\hat{g}_{q'}, \sigma_1)=\hat{g}_{p'}$. Thus A is strongly connected.

LEMMA 3.1. *Let $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ be a strongly connected (n, G)-automaton with output and let $h_1, h_2\colon A\to A$ be operation-preserving mappings satisfying $h_1(\hat{g}_p)=h_2(\hat{g}_p)$ for any $\hat{g}_p\in\hat{G}_n$. Then $h_1\equiv h_2$ (that is, $h_1(\hat{g})=h_2(\hat{g})$ for all $\hat{g}\in\hat{G}_n$).*

PROOF. Suppose that h_1 and h_2 satisfy the hypothesis of the lemma and let $\hat{g}\in\hat{G}_n$ be an arbitrary state. Since A is strongly connected there exists $\sigma\in\Sigma$ such that $M_\Psi(\hat{g}_p, \sigma)=\hat{g}$. Then

$$h_1(\hat{g}) = h_1[M_\Psi(\hat{g}_p, \sigma)] = M_\Psi(h_1(\hat{g}_p), \sigma) = M_\Psi(h_2(\hat{g}_p), \sigma) = h_2[M_\Psi(\hat{g}_p, \sigma)] =$$

$$= h_2(\hat{g}) \quad \text{for all} \quad \hat{g}\in\hat{G}_n.$$

Hence $h_1\equiv h_2$.

THEOREM 3.2. *Let $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ be a strongly connected (n, G)-automaton with output. If a function $\alpha\colon \hat{G}_n\to\hat{G}_n$ is in $G(A)$ then for some $x\in\Sigma^*$ $\alpha(\hat{g})\equiv$ $\equiv\hat{g}\Psi(x)=\hat{g}$.*

PROOF. Let $\alpha\in G(A)$ and $\alpha(\hat{g}_p)=\hat{g}_p$. Since A is strongly connected there exists $x\in\Sigma^*$ such that $\Psi(x)=e$. Let $\alpha'(\hat{g})\equiv\hat{g}\Psi(x)=\hat{g}$. Then

$$\alpha'[M_\Psi(\hat{g}, y)] = \alpha'[\hat{g}\,\Psi(y)] = \alpha'(\hat{g})\Psi(y) = M_\Psi(\alpha'(\hat{g}), y), \quad y\in\Sigma^*$$

Furthermore

$$Z(\alpha'(\hat{g}), y) = Z(\hat{g}\,\Psi(x), y) = Z(\hat{g}, y).$$

Thus α' is operation-preserving.

But $\alpha'(\hat{g}_p)=\alpha(\hat{g}_p)$ and so by Lemma 3.1, $\alpha(\hat{g})\equiv\alpha'(\hat{g})\equiv\hat{g}\Psi(x)=\hat{g}$. Hence the theorem is proved.

REMARK 3.1. In the above theorem α is an automorphism of A and the mapping $g\to\alpha$ is an isomorphism of G onto a subgroup of $G(A)$. (Cf. Theorem 2.1 in [6].)

DEFINITION 3.2. An (n, G)-automaton with output $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ is said to be abelian if $\Psi(xy)=\Psi(yx)$, $x, y\in\Sigma^*$.

THEOREM 3.3 *Let* $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ *be a strongly connected (n, G)-automaton with output. Then if $G(A)$ is abelian it follows that A is a strongly connected abelian (n, G)-automaton with output.*

PROOF. Let

$$\alpha(\hat{g}) \equiv \hat{g}\Psi(x) \tag{3.4}$$

and

$$\beta(\hat{g}) \equiv \hat{g}\Psi(y), \tag{3.5}$$

where $x, y\in\Sigma^*$. Now since $G(A)$ is abelian we have

$$\alpha\beta(\hat{g}) = \beta\alpha(\hat{g}). \tag{3.6}$$

Hence from (3.4) and (3.5) we obtain by (3.6),

$$\Psi(x)\,\Psi(y) = \Psi(y)\,\Psi(x) \quad \text{or} \quad \Psi(xy) = \Psi(yx).$$

Hence the theorem is verified.

REMARK 3.2. The above theorem can be proved for an (n, G)-automaton with output only.

In a similar way we can also verify the converse of the above theorem:

THEOREM 3.4. *If* $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ *is a strongly connected abelian (n, G)-automaton with output then $G(A)$ is also abelian.*

DEFINITION 3.3. An (n, G)-automaton A with output is called regular if A is strongly connected and $G(A)\approx G$.

THEOREM 3.5. *Let* $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ *be an (n, G)-automaton with output. Then if A is strongly connected, A is regular.*

PROOF. We have to prove that $G(A)\approx G$.

In order to show this, it is enough to verify that $\varrho=\alpha$ for all $\varrho\in G(A)$, where α is the mapping of $\hat{G}_n$ onto itself [see Theorem 3.2].

Now assume that $\varrho\in G(A)$. Then, we have for all $\hat{g}\in\hat{G}_n$ and $x\in\Sigma^*$

$$\varrho(M_\Psi(\hat{g}, x)) = M_\Psi(\varrho(\hat{g}), x)\wedge z(\hat{g}, x) = z(\varrho(\hat{g}), x)$$

This means that $\hat{g}\equiv\varrho(\hat{g})$ [cf. [5]]. Now Theorem 3.2 implies that $\alpha(\hat{g})=$ $=\hat{g}\Psi(x)=\hat{g}=\varrho(\hat{g})$. Hence $\alpha=\varrho$.

REMARK 3.3. The above theorem was proved in [6] only for a $(1, G)$-automaton without output.

LEMMA 3.2. *Let G and G' be two isomorphic groups. If $A=\langle\hat{G}_n, \Sigma, \Delta, M_\Psi, Z\rangle$ is an (n, G)-automaton with output then there exists an (n, G')-automaton with output $A'=\langle\hat{G}'_n, \Sigma', \Delta', M_{\Psi'}, Z'\rangle$ such that $A\approx A'$. Moreover if A is regular, then A' is regular, too.*

PROOF. Set $\Sigma'=\{\sigma';\ \sigma\in\Sigma\}$. Let Φ be an isomorphism from G onto G'.

We define $\Psi'(\sigma')=(\tilde{\Phi}(\psi_{pq}(\sigma)))$, where $\tilde{\Phi}$ is the extension of Φ to G^0 such that $\tilde{\Phi}(0)=0$ and $\Psi(\sigma)=(\psi_{pq}(\sigma))$.

Set $\Delta'=\{\delta';\ \delta\in\Delta\}$. With the above Σ', Ψ' and Δ' we define

$$A' = \langle\hat{G}'_n, \Sigma', \Delta', M_{\Psi'}, Z'\rangle.$$

In order to prove that $A\approx A'$ we set $\xi(\sigma)=\sigma$ for all $\sigma\in\Sigma$ and $\varrho(\hat{g})=(\tilde{\Phi}(g_p))$ for all $\hat{g}=(g_p)\in\hat{G}_n$.

Then, ξ and ϱ are one-to-one mappings of Σ onto Σ' and of $\hat{G}_n$ onto $\hat{G}'_n$, respectively. By definition of Ψ' and using that Φ is an isomorphism from G onto G', we can easily prove that

$$\varrho(M_\Psi(\hat{g}, \sigma)) = M_{\Psi'}(\varrho(\hat{g}), \xi(\sigma))\wedge Z(\hat{g}, \sigma) = Z'(\varrho(\hat{g}), \xi(\sigma)).$$

Hence $A\approx A'$. Furthermore, assume that A is regular. Then, $G(A)\approx G$. By Theorem 2.3, we have $G(A)\approx G(A')$. Moreover $G\approx G'$ holds by assumption. Hence we have $G(A')\approx G'$. This means that A' is regular.

(Cf. Lemma 3.1 in [6].)

THEOREM 3.6. *Let $A=\langle S, \Sigma, M, \Delta, Z\rangle$ be a strongly connected automaton with output such that $|S|=n|G(A)|$ where n is a positive integer. Moreover, assume that G is a finite group such that $G\approx G(A)$. Then there exists a regular (n, G)-automaton with output isomorphic to A.*

PROOF. Similarly to the proof of Theorem 3.1 in [6] we have

$$M_\Psi(\varrho(s), \xi(\sigma)) = \varrho(M(s, \sigma)).$$

Moreover since S is isomprhic to $\widehat{G(A)}_n$ it follows that $s\equiv\varrho(s)$ [cf. [5]]. This implies that $Z(s, \sigma)=Z'(\varrho(s), \sigma)$ where Z' is the output-function of the $(n, G(A))$-automaton. Thus

$$Z(s, \sigma) = z'(\varrho(s), \xi(\sigma)).$$

Therefore, $A \approx A'$ and A' is strongly connected. Moreover, by Theorem 2.3, $G(A) \approx G(A')$. Hence A' is regular.

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ON COMPLETELY ADDITIVE FUNCTIONS

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1. A sequence of positive real numbers (λ_n) is an *interval filling sequence* if $\lambda_{n+1}<\lambda_n$ for all $n\in\mathbf{N}$ (the set of the positive integers), $\sum_{n=1}^{\infty}\lambda_n=L\in\mathbf{R}$ (the set of the real numbers) and for any $x\in[0, L]$ there exists a sequence (ε_n) such that $\varepsilon_n\in\{0, 1\}$ $(n\in\mathbf{N})$ and $x=\sum_{n=1}^{\infty}\varepsilon_n\lambda_n$. If (λ_n) is an interval filling sequence, $\sum_{n=1}^{\infty}\lambda_n=L$ and a function $F\colon [0, L]\to\mathbf{R}$ satisfies

$$F\Big(\sum_{n=1}^{\infty}\varepsilon_n\lambda_n\Big)=\sum_{n=1}^{\infty}\varepsilon_n F(\lambda_n) \tag{1}$$

for all sequences (ε_n), $\varepsilon_n\in\{0, 1\}$ $(n\in\mathbf{N})$ then F is called a *completely additive function* (with respect to (λ_n)).

These notions have been introduced and discussed in [1]. In [1] and [2], under various further assumptions on the interval filling seqence, the completely additive functions have been determined by showing that they are linear functions.

In this paper we suppose nothing on the interval filling sequence and prove that those completely additive functions, which are nonnegative or differentiable at a point, are linear functions.

Throughout this paper (λ_n) denotes a fixed interval filling sequence with $L=\sum_{n=1}^{\infty}\lambda_n$.

2. If $F\colon [0, L]\to\mathbf{R}$ is a completely additive function then, as a simple consequence of (1), we get that $F(0)=0$ and

$$F(x)+F(L-x)=F(L) \tag{2}$$

for all $x\in[0, L]$. In our investigations we use the following two results proved in [1].

LEMMA 1. (a) *If* $x\in[0, L[$ *then there exists a sequence* (ε_n) *such that* $\varepsilon_n\in\{0, 1\}$ $(n\in\mathbf{N})$, $\varepsilon_n=0$ *for infinitely many* $n\in\mathbf{N}$ *and* $x=\sum_{n=1}^{\infty}\varepsilon_n\lambda_n$.

(b) *If* $F\colon [0, L]\to\mathbf{R}$ *is a completely additive function then* F *is continuous.*

The following observation is due to Z. Daróczy.

LEMMA 2. *If* $F\colon [0, L]\to\mathbf{R}$ *is a completely additive function and* $F(x)>0$ *for all* $x\in]0, L]$ *then* F *is strictly increasing, consequently* F *is differentiable at almost all points of* $[0, L]$.

PROOF. Let $0 \leqq t < s \leqq L$. Since F is continuous, there exists $u \in [t, s]$ such that $F(u) = \sup F([t, s])$. Suppose that $u < s$. Then, by the statement (a) of Lemma 1, there exists $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$ such that $\varepsilon_n = 0$ for infinitely many $n \in \mathbf{N}$ and $u = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n$. Therefore there is $i \in \mathbf{N}$ so that $u + \lambda_i < s$ and $\varepsilon_i = 0$. Hence, by (1), $F(u + \lambda_i) = F(u) + F(\lambda_i) > F(u)$, which is impossible. Thus $u = s$ and $F(t) \leqq F(u) = F(s)$. Similarly, there is $k \in \mathbf{N}$ so that $t + \lambda_k < s$ and $F(t + \lambda_k) = F(t) + F(\lambda_k)$. Thus $F(s) \geqq F(t + \lambda_k) > F(t)$. □

To prove our main results another lemma will also be needed.

LEMMA 3. *Let* $F\colon [0, L] \to \mathbf{R}$ *be a completely additive function.*
(a) *For all* $x \in]0, L[$ *there exists* $n(x) \in \mathbf{N}$ *such that*

$$F(\lambda_n) \in \{F(x + \lambda_n) - F(x), \ -F(x - \lambda_n) + F(x)\} \tag{3}$$

if $n(x) < n \in \mathbf{N}$.

(b) *If F is differentiable at a point* $x \in [0, L]$ *then the sequence* $\left(\frac{F(\lambda_n)}{\lambda_n}\right)$ *is convergent and*

$$\lim_{n \to \infty} \frac{F(\lambda_n)}{\lambda_n} = F'(x). \tag{4}$$

PROOF. (a) If $x \in]0, L[$ then there exists $n(x) \in \mathbf{N}$ such that $\lambda_n < \min\{x, L - x\}$ if $n(x) < n \in \mathbf{N}$. Let $x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n$ $(\varepsilon_n \in \{0, 1\},\ n \in \mathbf{N})$. If $n > n(x)$ and $\varepsilon_n = 0$ then, by (1), $F(x + \lambda_n) = F(x) + F(\lambda_n)$. If $n > n(x)$ and $\varepsilon_n = 1$ then, again by (1),

$$F(L - x + \lambda_n) = F(L - x) + F(\lambda_n)$$

whence, according to (2), we obtain

$$-F(x - \lambda_n) = -F(x) + F(\lambda_n).$$

(b) Let F be differentiable at a point $x \in [0, L]$. If $x = 0$ then (4) is trivial. If $x = L$ then, by (2)

$$F'(L) = \lim_{n \to \infty} \frac{F(L - \lambda_n) - F(L)}{-\lambda_n} = \lim_{n \to \infty} \frac{F(\lambda_n)}{\lambda_n}.$$

If $x \in]0, L[$ then it follows from part (a) that there is $n(x) \in \mathbf{N}$ so that for $n > n(x)$

$$\frac{F(\lambda_n)}{\lambda_n} \in \left\{ \frac{F(x + \lambda_n) - F(x)}{\lambda_n},\ \frac{F(x - \lambda_n) - F(x)}{-\lambda_n} \right\}$$

which implies (4).

3. In this section we prove our main results.

THEOREM 1. *If* $F\colon [0, L] \to \mathbf{R}$ *is a completely additive function and* $F(x) \geqq 0$ *for all* $x \in [0, L]$ *then there exists* $c \in \mathbf{R}$ *such that*

$$F(x) = cx \tag{5}$$

for all $x \in [0, L]$.

PROOF. (i) First we suppose that $F(x)>0$ if $x\in]0, L]$. Then Lemmas 2 and 3 (b) imply that F is differentiable almost everywhere and for some $c\in\mathbf{R}$, $F'(x)=c$ for almost all $x\in[0, L]$. Furthermore, there exists $K\in\mathbf{R}$ such that $F(\lambda_n)\leqq K\lambda_n$ for all $n\in\mathbf{N}$. Because of (1), this implies $F(x)\leqq Kx$ for all $x\in[0, L]$. Define the function F_1 on $[0, L]$ by $F_1(x)=(K+1)x-F(x)$. Then $F_1\colon [0, L]\to\mathbf{R}$ is completely additive too, and $F_1(x)>0$ if $x\in]0, L]$. Therefore, by Lemma 2, F_1 is increasing thus for all $0\leqq x\leqq y\leqq L$ we have

$$F(y)-F(x) \leqq (K+1)(y-x).$$

This implies that F satisfies the Lipschitz condition, consequently, F is absolutely continuous. Thus for all $x\in[0, L]$

$$F(x) = F(x)-F(0) = \int_0^x F' = \int_0^x c = cx.$$

(ii) If $F(x)\geqq 0$ for all $x\in[0, L]$ then define the function F_2 on $[0, L]$ by $F_2(x)=$ $=F(x)+x$ and apply the result proved in (i) to F_2.

THEOREM 2. *If $F\colon [0, L]\to\mathbf{R}$ is a completely additive function and F is differentiable at a point of* $[0, L]$ *then we have* (5) *with some constant* $c\in\mathbf{R}$.

PROOF. It follows from part (b) of Lemma 3 that there is $K\in\mathbf{R}$ so that $F(\lambda_n)\leqq$ $\leqq K\lambda_n$ for all $n\in\mathbf{N}$. This implies that the function f defined by $f(x)=Kx-F(x)$, $x\in[0, L]$ is nonnegative and completely additive. Thus Theorem 2 is a consequence of Theorem 1.

This theorem implies that if a completely additive (consequently continuous) function is not linear then it must be nowhere differentiable. The existence of such a function is an open problem.

The proof of Theorem 2 shows that if F is a competely additive function and the sequence $\left(\frac{F(\lambda_n)}{\lambda_n}\right)$ is bounded from one side then F is linear. It follows from (3) that this condition is satisfied if

$$\sup F([t, x]) = \inf F([x, s]) \quad \text{or} \quad \inf F([t, x]) = \sup F([x, s])$$

for some $0\leqq t<x<s\leqq L$.

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BEMERKUNG ÜBER DIE PARWEISE UNABHÄNGIGEN ZUFALLIGEN GRÖßEN

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1. Der folgende Satz ist wohlbekannt. (S.z.B. [3], S. 87.)

SATZ A. *Es sei* $\{\xi_n\}_1^\infty$ *eine Folge von orthonormierten zufälligen Größen. Ist*

$$\sum_{n=1}^{\infty} \frac{D^2(\xi_n)}{n^2} \log^2 n < \infty, \tag{1}$$

dann gilt

$$\frac{\xi_1 + \ldots + \xi_n}{n} \to 0 \quad (n \to \infty)$$

mit Wahrscheinlichkeit 1.

Wir haben bewiesen, daß die Bedingung (1) genau ist. Es gilt nämlich der folgende Satz [4].

SATZ B. *Es sei* $\{D_n\}_1^\infty$ *eine Folge von positiven Zahlen mit*

$$\frac{D_n}{n} \geqq \frac{D_{n+1}}{n+1} \quad (n = 1, 2, \ldots)$$

und

$$\sum_{n=1}^{\infty} \frac{D_n^2}{n^2} \log^2 n = \infty.$$

Dann gibt es eine Folge $\{\xi_n\}_1^\infty$ *von orthonormierten zufälligen Größen derart, daß* $D(\xi_n) = D_n$ $(n=1, 2, \ldots)$ *und*

$$\overline{\lim_{n\to\infty}} \frac{|\xi_1 + \ldots + \xi_n|}{n} = \infty$$

mit Wahrscheinlichkeit 1 *bestehen.*

BEMERKUNG. Mit in der Arbeit [4] angewandter Methode kann man zeigen, daß im Satz B die Folge $\{\xi_n\}_1^\infty$ derart angewählt werden kann, daß $|\xi_n| \leqq D_n K$ $(n=1, 2, \ldots)$ mit einer endlichen Konstante K mit Wahrscheinlichkeit 1 besteht, und die ξ_n im Wahrscheinlichkeitsfeld $((0, 1), \mathscr{A}, \mu)$ definiert sind, wobei $\mathscr{A}$ die σ-Algebra der in Lebesgueschen Sinne meßbaren Untermengen vom Intervall (0, 1) und μ das gewöhnliche Lebesguesche Maß sind.

In der Arbeit [1] ist das Problem aufgeworfen, ob die Bedingung (1) auch für paarweise unabhängige zufällige Größen genau ist.

In dieser Note werden wir den folgenden Satz beweisen.

SATZ. *Unter den Bedingungen des Satzes* B *gibt es eine Folge* $\{\xi_n\}_1^\infty$ *von paarweise unabhängigen zufälligen Größen derart, daß die Folge*

$$\left\{\frac{\xi_1+\ldots+\xi_n}{n}\right\}_{n=1}^{\infty}$$

mit Wahrscheinlichkeit 1 *divergiert.*

2. Zum Beweis des Satzes werden wir den folgenden Hilfssatz anwenden.

HILFSSATZ. *Es sei* $\{\varphi_n(x)\}_1^\infty$ *eine Folge von in* (0, 1) *orthonormierten Funktionen, für die*

$$|\varphi_n(x)| = 1 \quad (x\in(0,1),\ n = 1, 2, \ldots) \tag{2}$$

und

$$\int_0^1 \varphi_n(x)\,dx = 0 \quad (n = 1, 2, \ldots) \tag{3}$$

bestehen. Dann sind die Funktionen $\varphi_n(x)$ $(n=1, 2, \ldots)$ *im Intervall* (0, 1) *paarweise unabhängig.*

Diese Bemerkung stammt von S. V. Botschkarjev. (Wörtliche Mitteilung.)

BEWEIS DES HILFSSATZES. Es seien n, m $(n\neq m)$ positive ganze Zahlen und l_1, l_2 nichtnegative ganze Zahlen. Dann gilt

$$\int_0^1 (\varphi_n(x))^{l_1}(\varphi_m(x))^{l_2}\,dx = \int_0^1 (\varphi_n(x))^{l_1}\,dx \int_0^1 (\varphi_n(x))^{l_2}\,dx. \tag{4}$$

Sind l_1, l_2 gerade Zahlen, dann gilt (4) wegen (2) und (3). Ist l_1 gerade und l_2 ungerade, dann gilt (4) wegen

$$\int_0^1 (\varphi_n(x))^{l_1}(\varphi_m(x))^{l_2}\,dx = \int_0^1 \varphi_m(x)\,dx = 0 = \int_0^1 \varphi_n(x)\,dx \int_0^1 \varphi_n(x)\,dx =$$

$$= \int_0^1 (\varphi_n(x))^{l_1}\,dx \int_0^1 (\varphi_m(x))^{l_2}\,dx.$$

Sind l_1, l_2 ungerade Zahlen, dann folgt (4) wegen (2), (3) und wegen der Orthogonalität der Funktionen $\varphi_n(x)$. Aus (4) erhalten wir durch Anwendung eines bekannten Satzes [2], daß die Funktionen $\varphi_n(x)$ paarweise unabhängig sind.

BEWEIS DES SATZES. Auf Grund der Bemerkung gibt es eine Folge $\{\varphi_n(x)\}_1^\infty$ der in (0, 1) orthonormierten Funktionen, für die

$$|\varphi_n(x)| \leqq K(<\infty) \quad (x\in(0,1);\ n = 1, 2, \ldots)$$

gilt, und die Folge

$$\left\{\frac{D_1\varphi_1(x)+\ldots+D_n\varphi_n(x)}{n}\right\}_{n=1}^{\infty}$$

in (0, 1) fast überall divergiert.

Durch Anwendung eines bekannten Satzes [5] folgt daraus, daß ein orthonormiertes System $\{\psi_n(x)\}_1^\infty$ im Intervall (0, 1) derart existiert, daß

$$|\psi_n(x)| = 1 \quad (x\in(0,1);\ n = 1, 2, \ldots)$$

besteht, und die Folge

$$\left\{\frac{D_1\psi_1(x)+\ldots+D_n\psi_n(x)}{n}\right\}_{n=1}^{\infty}$$

in (0, 1) fast überall divergiert.

Es sei

$$\chi_n(x) = \begin{cases} \psi_n(2x), & x\in(0, 1/2), \\ -\psi_n(2(x-1/2)), & x\in(1/2, 1) \end{cases}$$

$(n=1, 2, \ldots)$. Das System $\{\chi_n(x)\}_1^\infty$ ist offensichtlich orthonormiert in (0, 1); die Folge

$$\left\{\frac{D_1\chi_1(x)+\ldots+D_n\chi_n(x)}{n}\right\}_{n=1}^{\infty}$$

divergiert in (0, 1) fast überall, weiterhin gelten

$$|\chi_n(x)| = 1 \quad (x\in(0,1);\ n = 1, 2, \ldots)$$

und

$$\int_0^1 \chi_n(x)\,dx = 0 \quad (n = 1, 2, \ldots).$$

Durch Anwendung des Hilfssatzes ergibt sich, daß die Funktionen $\chi_n(x)$ paarweise unabhängig sind.

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MULTIPLICATIVE FUNCTIONS OVER THE GAUSSIAN INTEGERS. II

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1. We shall use the following standard notations: $\mathbf{N}$=natural numbers, Z= rational integers, Q=rational numbers, $\mathbf{R}$=real numbers, $\mathbf{C}$=complex numbers, $\mathbf{F}=Q(i)$=the simple extension of Q by i, G=the Gaussian integers, i.e. the ring of the integers in $\mathbf{F}$. Let furthermore $G^*=G\setminus\{0\}$ be the multiplicative group of G; $Q_x, \mathbf{R}_x, \mathbf{F}_x, \mathbf{C}_x$ be the multiplicative group of the positive rationals, positive reals, of $\mathbf{F}$ and of the complex numbers, respectively.

The letters $c, c_1, c_2, \ldots, K$ shall denote suitable positive constants not the same at every occurrence.

We shall say that a function $F\colon G^*\to\mathbf{C}$ is completely multiplicative if

$$F(\alpha\beta) = F(\alpha)\,F(\beta) \tag{1.1}$$

holds for every $\alpha, \beta\in G^*$. We shall denote by $\mathcal{M}^*$ the class of the completely multiplicative functions.

Let $N(\alpha)=\alpha\bar{\alpha}$ denote the norm of α.

The domain of $F\in\mathcal{M}^*$ will be extended by $F(0)=0$ onto G.

If $F\in\mathcal{M}^*$ is nowhere zero in G^*, then by

$$F\left(\frac{\alpha}{\beta}\right) := F(\alpha)\cdot F^{-1}(\beta), \tag{1.2}$$

we can extend the domain of F onto $\mathbf{F}$, furthermore the relation

$$F(\gamma\delta) = F(\gamma)\,F(\delta) \tag{1.3}$$

will be true for each $\gamma, \delta\in\mathbf{F}$.

It is obvious that if $F\in\mathcal{M}^*$ is not identically zero, then $F(1)=1$, $F(i)\in\{1, -1, i, -i\}$.

We are interested now in those $F\in\mathcal{M}^*$ for which some regularity condition holds.

In the first paper of this series [1] we considered those $F\in\mathcal{M}^*$ for which $|F(\alpha+\gamma)-F(\alpha)|\leqq\varepsilon(|\alpha|)$ hold for each $\alpha\in G^*$ with a suitable fixed $\gamma\in G^*$ and with a suitable monotonically decreasing $\varepsilon(x)$ satisfying $\sum_{a=1}^{\infty}\varepsilon(2^a)<\infty$, and concluded that F has to be of the form $F(\alpha)=|\alpha|^{i\tau}e^{ik\arg\alpha}$ with $\tau\in\mathbf{R}$, $k\in\mathbf{N}$. Here we stated our conjecture that the same assertion can be deduced from the weaker assumption $F(\alpha+\gamma)-F(\alpha)\to 0$ $(|\alpha|\to\infty)$ as well. This was proved by E. Wirsing recently. Namely he proved the following result.[1] If $F\in\mathcal{M}^*$,

$$F(\alpha+\gamma)-F(\alpha)\to 0 \quad (|\alpha|\to\infty), \quad F(\gamma)\neq 0, \tag{1.4}$$

then $F(\alpha)\to 0$ $(|\alpha|\to\infty)$ or $F(\alpha)=|\alpha|^{\sigma+i\tau}e^{ik\arg\alpha}$.

We shall say that $F\in A_\gamma$, or $F\in\mathcal{L}$ if

$$\sum_{\alpha\in G^*}\frac{|F(\alpha+\gamma)-F(\alpha)|}{|\alpha|^2}<\infty \tag{A_γ}$$

or

$$\sum_{\alpha\in G^*}\frac{|F(\alpha)|}{|\alpha|^2}<\infty \tag{$\mathcal{L}$}$$

holds.

The following observations are almost obvious.

LEMMA 1. *Let $F\in A_\gamma$, $\gamma\in G^*$. Let $\mathcal{B}$ denote the set of those $\beta\in G^*$ for which $F\in A_\beta$. Assume that $F(\gamma)\neq 0$. Then $\beta=G^*$.*

LEMMA 2. *Let $F\in A_\gamma$, $\gamma\in G^*$. Assume that $F(\delta)=0$ for a suitable $\delta\in G^*$, $(\delta,\gamma)=1$. Then $F\in\mathcal{L}$.*

PROOF OF LEMMA 1. Summing over $\alpha\gamma$ in (A_γ), by $F(\gamma)\neq 0$ we immediately get $1\in\mathcal{B}$. Let us put now $-\alpha$, $i\alpha$, $-i\alpha$ into the sum (A_1) instead of α, we get $-1,-i$, $i\in\mathcal{B}$. Here we use that $F(-1), F(-i), F(i)\neq 0$, which is obvious, since $F(1)=1$. Furthermore, by the triangle inequality we get that $\beta_1,\beta_2\in\mathcal{B}$ implies $\beta_1+\beta_2\in\mathcal{B}$, and so $k\cdot 1+l\cdot i\in\mathcal{B}$ for every $k,l\in Z$. □

PROOF OF LEMMA 2. We may assume that F is not identically zero, and so $F(1)=1$, $F(i)\neq 0$, $F(-i)\neq 0$, $F(-1)\neq 0$. Let as earlier $\mathcal{B}$ denote the set of those $\beta\in G^*$ for which A_β holds. By changing α into $i\alpha$, $-i\alpha$, $-\alpha$ in (A_γ), we get $i\gamma$, $-i\gamma$, $-\gamma\in\mathcal{B}$, consequently from the property $\beta_1,\beta_2\in\mathcal{B}\Rightarrow\beta_1+\beta_2\in\mathcal{B}$, we get $\varepsilon\gamma\in\mathcal{B}$ for each $\varepsilon\in G^*$.

Let us consider now $A_{\varepsilon\gamma}$ and put $\delta\alpha$ instead of α. Then by $F(\delta)=0$ we get

$$\sum_{\alpha}\frac{|F(\delta\alpha+\varepsilon\gamma)|}{|\delta\alpha|^2}<\infty. \tag{1.5}$$

Let now $\varepsilon_0=0,\varepsilon_1,\ldots,\varepsilon_{t-1}$ be a complete residue set mod δ. Since $(\delta,\gamma)=1$ therefore $\varepsilon_j\gamma$ $(j=0,\ldots,t-1)$ is a complete residue set mod δ as well. (1.5) is obviously true for $\varepsilon=\varepsilon_0=0$, since $F(\delta)=0$. Summing the sums (1.5) choosing $\varepsilon=\varepsilon_j$ $(j=0,\ldots,t-1)$ we run over the whole set of $\alpha\in G^*$ which gives $F\in\mathcal{L}$. □

2. Some lemmata. Let us assume that $F\in\mathcal{M}^*$ is nowhere zero in G^*. Then we can consider F as a homomorphism from $\mathbf{F}_x\to\mathbf{C}_x$. We shall say that F is continuous at the point 1, if $r_\nu\in\mathbf{F}_x$, $r_\nu\to 1$ $(\nu\to\infty)$ implies $F(r_\nu)\to 1$ $(\nu\to\infty)$.

[1] Oral communication (September 7, 1984).

LEMMA 3. *Let* $F\colon \mathbf{F}_x \to \mathbf{C}_x$ *be a homomorphism continuous at the point* 1. *Then its domain can be extended onto* $\mathbf{C}_x$ *by the relation*

$$F(w) := \lim_{\substack{r_\nu \to w \\ r_\nu \in \mathbf{F}_x}} F(r_\nu)$$

uniquely. The mapping $F\colon \mathbf{C}_x \to \mathbf{C}_x$ *so obtained is a continuous homomorphism, and so*

$$F(w_1 w_2) = F(w_1) F(w_2), \quad \forall\, w_1, w_2 \in \mathbf{C}_x. \tag{2.1}$$

Consequently $F(w) = |w|^s e^{ik \arg w}$, $s \in \mathbf{C}$, $k \in Z$.

REMARK. A similar lemma was used and proved in [2].

PROOF. First we observe that F is bounded in every bounded domain. Let us assume the contrary. Then there would exist a convergent sequence $r_\nu \in \mathbf{F}_x$, $r_\nu \to w$, $w \neq 0$ such that $|F(r_\nu)| \to \infty$. Then for every ν there would exist an l_ν such that $|F(r_{l_\nu})| > \nu |F(r_\nu)|$. But $r_{l_\nu} r_\nu^{-1} \to 1$ $(v \to \infty)$, and this contradicts the assumption that F is continuous at the point 1. Since $F^{-1}(t) = F\left(\frac{1}{t}\right)$, we get that $F^{-1}(t)$ is bounded when so is $1/t$.

Let $w \in \mathbf{C}_x$, r_ν, $s_\mu \in \mathbf{F}_x$, $r_\nu \to w$, $s_\mu \to W$, $t_{\nu,\mu} = r_\nu s_\mu^{-1}$. Since $t_{\nu,\mu} \to 1$, therefore

$$F(t_{\nu,\mu}) = F(r_\nu) F^{-1}(t_\mu) \to 1. \tag{2.2}$$

Since the sequences $F(r_\nu)$, $F^{-1}(t_\mu)$ are bounded, from (2.2) we get

$$F(r_\nu) - F(t\mu) \to 0 \quad (\nu, \mu \to \infty). \tag{2.3}$$

This implies immediately the first assertion stated in the lemma. The continuity of F in $\mathbf{C}_x$ and the validity of (2.1) are straightforward, so we omit the details.

Let us consider now the explicit form of F satisfying (2.1). It is obvious that $F(w)$ is nowhere zero, since $F(z) = 0$, $z \in \mathbf{C}_x$ would imply that F is identically zero. We observe that F is a continuous function that satisfies a Cauchy-like equation in a multiplicative form. Let $G(w) = |F(w)|$. Since from (2.1), $\log G(w_1 w_2) = \log G(w_1) + \log G(w_2)$, we get $G(w) = |w|^\sigma$, $\sigma \in \mathbf{R}$. Let now $H(w) = \frac{F(w)}{G(w)}$. Then $H(w_1 w_2) = H(w_1) H(w_2)$, $|H(w)| = 1$, consequently H is a continuous character, and so it has the form $H(w) = |w|^{i\tau} e^{ik \arg \alpha}$, $\tau \in \mathbf{R}$, $k \in \mathbf{N}$. □

LEMMA 4. *Let us assume that* $F \in A_y$, *F is nowhere zero. If* $|F(p)| < 1$ *for at least one* $p \in G^*$, *then* $F \in \mathscr{L}$.

PROOF. Let $r = |F(p)| < 1$, $\mathscr{B}_p = \{a_0 = 0, a_1, \dots, a_{t-1}\}$ be a fixed complete residue system mod p. Let K be a large constant, $K = K(p)$, and

$$l(\alpha) := \max_{|\alpha - \beta| \leqq K} |F(\alpha) - F(\beta)|. \tag{2.4}$$

From Lemma 1 we get

$$\sum_{\alpha \in G^*} \frac{l(\alpha)}{|\alpha|^2} < \infty. \tag{2.5}$$

Let $P=|p|$. We have $t=N(p)=p\bar{p}=P^2$. Let us define α_1, b_0 by the following relation:

$$\alpha = p\alpha_1+b_0, \quad b_0\in\mathscr{B}_p, \quad \alpha_1\in G.$$

Hence we get

$$\frac{|\alpha|}{p}-c_1<|\alpha_1|<\frac{|\alpha|}{p}+c_1$$

with suitable $c>0$, and so for every $\varepsilon>0$

$$(1-\varepsilon)\frac{1}{p^2|\alpha_1|^2} \leqq \frac{1}{|\alpha|^2} \leqq (1-\varepsilon)\frac{1}{p^2|\alpha_1|},$$

whenever $|\alpha|$ is large enough. Since α_1 occurs exactly for $t=P^2$ of the α's, and

$$-l(p\alpha_1) \leqq |F(\alpha)|-|F(p)|\cdot|F(\alpha_1)| \leqq l(p\alpha_1),$$

we get by (2.5)

$$(2.6)\qquad (1-\varepsilon)|F(p)|B\left(\frac{x}{p}-c_1\right)-c_2 \leqq B(x) \leqq (1+\varepsilon)|F(p)|B\left(\frac{x}{p}+c_1\right)+c_2.$$

Let ε be so small that $s:=|F(p)|(1+\varepsilon)<1$. Considering the right hand side of (2.6) we get easily that $B(x)$ is bounded, i.e. $F\in\mathscr{L}$. Indeed, by choosing $x_0=1$, $x_{\nu+1}=Px_\nu-c_1$, we get

$$B(x_{\nu+1}) \leqq sB(x_\nu)+c_2 \quad (\nu = 0, 1, 2, \dots)$$

and so $B(x_\nu)$ is bounded. □

LEMMA 5. *Let $F\in A_\gamma$, F is nowhere zero, $F\notin\mathscr{L}$. Then $|F(\alpha)|=|\alpha|^\Delta$ with a fixed Δ, $0\leqq\Delta<1$.*

PROOF. If $F\in A_\gamma$, then $|F|\in A_\gamma$, so we may assume that $F(\alpha)>0$ $\forall\alpha\in G^*$. Let $p\in G^*$, $P=|p|>1$.

Let $\mathscr{B}_p, \alpha_1, b_0, B(x)$ be defined as in the proof of Lemma 4. Let $F(p)=P^r$, $r=r_p$. Consider now (2.6). By Lemma 4 we may assume that $r_p\geqq 0$. Let $Q=(1+\varepsilon)P^r$, $P^k\leqq x\leqq P^{k+1}$, $Y_0=x$, $Y_{l+1}=\frac{Y_l}{P}+c_1$ $(l=0, 1, \dots, k)$. Since

$$Y_k \leqq \frac{Y_{k-1}}{P}+c_1 \leqq \frac{Y_{k-2}}{p^2}+c_1+\frac{c_1}{P} \leqq \dots \leqq \frac{Y_0}{P^k}+c_1\left(1+\frac{1}{p}+\dots\right) \leqq P+\frac{c_1}{1-1/P}=:L,$$

from the right hand side of (2.6) we get

$$B(x_0) \leqq QB(x_1)+c_2 \leqq Q^2B(x_2)+Qc_2+c_2 \leqq \dots$$

$$\dots \leqq Q^kB(x_k)+c_2Q^k(1+Q^{-1}+\dots) < c_3Q^k.$$

This gives

$$(2.7)\qquad \overline{\lim_{x\to\infty}}\,\frac{\log B(x)}{\log x} \leqq r_p.$$

Now we prove that

$$\varliminf_{x\to\infty} \frac{\log B(x)}{\log x} \geqq r_p. \tag{2.8}$$

Since $B(y)\to\infty$, the left hand side of (2.6) gives

$$RB\left(\frac{x}{P}-c_1\right) \leqq B(x), \quad R = (1-2\varepsilon)F(p), \tag{2.9}$$

whenever $x>x_0(\varepsilon)$. By choosing $Z_0=x$, $Z_{\nu+1}=\frac{Z_\nu}{P}-c_1$, we get

$$B(x) \geqq R^\nu B(Z_\nu) \quad (Z_\nu > x_0(\varepsilon)),$$

which gives (2.8) immediately.

(2.7), (2.8) imply that $r_p=\Delta$ is a constant. Then $F(\alpha)=|\alpha|^\Delta$. Since $F\in A_\gamma$,

$$|F(\alpha+1)-F(\alpha)| = |\alpha|^\Delta\left|\left|1+\frac{1}{\alpha}\right|^\Delta-1\right|, \quad \left|1+\frac{1}{\alpha}\right|^2 = 1+\frac{2\operatorname{Re}\alpha+1}{|\alpha|^2},$$

we get $|F(\alpha+1)-F(\alpha)|\asymp|\alpha|^{\Delta-2}|2\operatorname{Re}\alpha+1|$. The condition $F\in A_1$ implies $\Delta<1$. □

3. Formulation of the theorem and some further lemmata. THEOREM. *Let* $F\in A_\gamma$ *for a suitable* $\gamma\in G^*$, *and let* $F(\gamma)\neq 0$. *Then* $f\in\mathcal{L}$ *or*

$$F(\alpha) = |\alpha|^s e^{ik\arg\alpha}. \quad (\forall\,\alpha\in G^*), \tag{3.1}$$

where $s\in\mathbf{C}$, $k\in Z$; $0\leqq\operatorname{Re} s<1$.

For an arbitrary pair $p, q\in G^*$ let $\mathcal{H}_{p,q}$ be the set of the accumulation points of the sequence $p^k q^{-l}(k, l=0, 1, 2, \ldots)$. We should like to find a pair p, q such that $\mathcal{H}_{p,q}=\mathbf{C}$.

LEMMA 6. *Let* $q=2$, $p_1=(2+i)(3+2i)(4+i)$, $p_2=(2+i)(3-2i)(4-i)$, $p_3=(2-i)(3+2i)(4-i)$. *Then at least one of the sets* $\mathcal{H}_{p_1,2}, \mathcal{H}_{p_2,2}, \mathcal{H}_{p_3,2}$ *is equal to* $\mathbf{C}$.

PROOF. Let $p=Pe^{2\pi i\varphi}$, $|p|=P$, $w=e^r\cdot e^{2\pi is}$. $w\in\mathcal{H}_{p,2}$ if there exists a sequence $(k_t, l_t)\in\mathbf{N}^2$ such that $p^{k_t}q^{-l_t}\to w$, i.e.

$$|k_t\log P-l_t\log 2-r| < \varepsilon_t, \tag{3.2}$$

$$\|k_t\varphi-s\| < \varepsilon_t \tag{3.3}$$

for a suitable $\varepsilon_t\to 0$. (3.2) can be rewritten in the form

$$\left\|k_t\frac{\log P}{\log 2}-\frac{r}{\log 2}\right\| < \frac{\varepsilon_t}{\log 2}.$$

So $\mathcal{H}_{p,2}=\mathbf{C}$ if for every $r\in\mathbf{R}$, $s\in[0, 1)$ the inequalities

$$\left\|k\frac{\log P}{\log 2}-r\right\| < \varepsilon, \quad \|k\varphi-s\| < \varepsilon$$

are satisfied for infinitely many k, for every $\varepsilon>0$. Kronecker's theorem asserts that these inequalities are satisfied for infinitely many k if $\frac{\log P}{\log 2}$, φ, 1 are linearly independent over the field of the rational numbers.

Let now φ_j $(j=1, 2, 3)$ and P be defined by

$$p_j = Pe^{2\pi i\varphi_j} \quad (j = 1, 2, 3).$$

Then $P=|p_j|=\sqrt{5\cdot 13\cdot 17}$. It has remained to prove that $\frac{\log P}{\log 2}$, φ_j, 1 are linearly independent at least for one j. Assume the contrary. Then we get immediately that $\varphi_1, \varphi_2, \varphi_3$ are linearly dependent, i.e. $r_1\varphi_1+r_2\varphi_2+r_3\varphi_3=0$, $r_i\in Z$, $\Sigma r_i^2\neq 0$. Consequently

$$p_1^{2r_1}p_2^{2r_2}\cdot p_3^{2r_3} = p^{2(r_1+r_2+r_3)}e^{2\pi i\cdot 2\cdot(r_1\varphi_1+r_2\varphi_2+r_3\varphi_3)} = 5\cdot 13\cdot 17.$$

Using the unicity of the prime decomposition over the Gaussian integers, we deduce easily that $r_1=r_2=r_3=0$, and this is a contradiction. □

LEMMA 7. *Let $F\in A_\gamma$. Assume that $|F(\alpha)|=1$ $\forall\alpha\in G^*$. Let $p\in G^*$ be a number such that $\mathscr{H}_{p,2}=\mathbf{C}$. Let $A, B\in G^*$. Then*

$$p^{k_t}\cdot 2^{-l_t}\to\frac{B}{A} \tag{3.4}$$

implies

$$F(p^{k_t}\cdot 2^{-l_t})\to F\left(\frac{B}{A}\right). \tag{3.5}$$

PROOF. Let $P=|p|$, $\mathscr{B}_p$, $\mathscr{B}_2$ be fixed complete residue sets mod p, mod 2, respectively. Let

$$C(u, v) := \{\alpha|\alpha\in G, |\alpha-u|<v\}. \tag{3.6}$$

Let ε, δ, r be small positive numbers, and let $k, l, H_1, H_2\in\mathbf{N}$ be such that

$$\left|\frac{p^kA}{2^lB}-1\right|<\varepsilon, \quad \left|\frac{p^{H_1}}{2^{H_2}}-1\right|<\delta. \tag{3.7}$$

Let

$$D_1 := C\left(Ap^{k+H_1}, \frac{1}{4}P^{k+H_1}\right), \quad D_2 = C(B\cdot 2^{l+H_2}, r\cdot 2^{l+H_2}).$$

First we observe that $D_2\subseteq D_1$ if r is sufficiently small. r may not depend on k, l, H_1, H_2, but may depend on A, B, p. Inded, let $\alpha\in D_2$. Then $|\alpha-B\cdot 2^{l+H_2}|<$ $<r\cdot 2^{l+H_2}$. But then

$$|\alpha-Ap^{k+H_1}|\leqq|\alpha-B\cdot 2^{l+H_2}|+|B\cdot 2^{l+H_2}-Ap^{k+H_1}|\leqq$$
$$\leqq r\cdot 2^{l+H_2}+|B|\cdot 2^{l+H_2}\left|\frac{Ap^{k+H_1}}{B\cdot 2^{l+H_2}}-1\right|.$$

Taking into account (3.7) we get

$$|\alpha - Ap^{k+H_1}| \leqq 2^{l+H_2}(r+|B|(2\varepsilon+2\delta))$$

and the right hand side is smaller than $\frac{1}{4}P^{k+H_1}$ if ε, δ, r are sufficiently small.

Let ε, δ, r be so small that $D_2 \subseteq D_1$. Let now $\alpha \in D_2$ and consider the following two algorithms:

$$\alpha_0 = \alpha, \quad \alpha_m = \alpha_{m+1}p + b_m; \quad b_m \in \mathscr{B}_p, \quad \alpha_{m+1} \in G \quad (m = 0, 1, 2, \ldots),$$

$$\beta_0 = \alpha, \quad \beta_m = \beta_{m+1} \cdot 2 + d_m, \quad d_m \in \mathscr{B}_2, \quad \beta_{m+1} \in G \quad (m = 0, 1, 2, \ldots).$$

By elementary computation we get

$$\left|\alpha_\nu - \frac{\alpha}{p^\nu}\right| \leqq L_1, \quad \left|\beta_\mu - \frac{\alpha}{2^\mu}\right| \leqq L_1,$$

$$(3.8) \qquad |B\alpha_k A\beta_l| \leqq L_2;$$

furthermore

$$\alpha_\nu \in C\left(Ap^{k+H_1-\nu}, \frac{1}{4}P^{k+H_1-\nu} + L_3\right), \quad \beta_\mu \in C(B \cdot 2^{l+H_2-\mu}, r \cdot 2^{l+H_2-\mu} + L_3)$$

with suitable constants L_1, L_2, L_3; $L_j = L_j(p, A, B)$.

Let $K = K(p)$ be a large constant, and $l(\alpha)$ be defined by (2.4). We have

$$|F(\alpha_m) - F(\alpha_{m+1}p)| \leqq l(\alpha_m), \quad |F(\beta_m) - F(2\beta_{m+1})| \leqq l(\beta_m),$$

and by $|F(p)| = |F(2)| = 1$ we get

$$|F(\alpha) - F(p^k\alpha_k)| \leqq l(\alpha_0) + l(\alpha_1) + \ldots + l(\alpha_{k-1}),$$

$$|F(\alpha) - F(2^l\beta_l)| \leqq l(\beta_0) + l(\beta_1) + \ldots + l(\beta_{l-1}).$$

Since α_t occurs exactly for P^{2t} distinct α's, with the notation

$$t(y) := \sum_{|\alpha| \geqq y} \frac{l(\alpha)}{|\alpha|^2}$$

we get

$$\sum_{\alpha \in D_2} |F(\alpha) - F(p^k)F(\alpha_k)| \frac{1}{|\alpha|^2} \leqq 2t\left(\frac{1}{4}P^{H_1}\right),$$

$$\sum_{\alpha \in D_2} |F(\alpha) - F(2^l)F(\beta_l)| \frac{1}{|\alpha|^2} \leqq 2t\left(\frac{1}{4} \cdot 2^{H_2}\right)$$

whenever H_1, H_2 are sufficiently large. Then

$$\sum_{\alpha \in D_2} \left|F\left(\frac{p^k}{B}\right) F(B\alpha_k) - F(2^l/A) F(\beta_l A)\right| \frac{1}{|\alpha|^2} \leqq 2t\left(\frac{1}{4}P^{H_1}\right) + 2t\left(\frac{1}{4}P^{H_2}\right),$$

and so

$$\sum_{\alpha\in D_2}\left|F\left(\frac{p^k}{B}\right)-F\left(\frac{2^l}{A}\right)\right|\cdot\frac{|F(\alpha_k)|}{|\alpha|^2}\leqq\sum_{\alpha\in D_2}\frac{|F(\alpha_k B)-F(\beta_l A)|}{|\alpha|^2}+2t\left(\frac{1}{4}P^{H_1}\right)+2t\left(\frac{1}{4}2^{H_2}\right).$$

Let us estimate now the sum on the right hand side. From (3.8) we get

$$|F(\alpha_k B)-F(\beta_l A)|\leqq l(\alpha_k B).$$

Furthermore $|\alpha_k|>\frac{1}{2}\frac{|\alpha|}{p^k}$, say, α_k occurs exactly for P^{2k} of distinct α's, so this sum is less than

$$ct\left(\frac{1}{2}P^{H_1}\right).$$

Let us consider the left hand side. This can be written in the form

$$\left|F\left(\frac{p^k}{2}\right)-F\left(\frac{B}{A}\right)\right|\sum_{\alpha\in D_2}\frac{1}{|\alpha|^2}.$$

Since $\#(D_2)=r^2\cdot 2^{2(l+H_2)}\pi(1+o(1))$, and

$$\frac{1}{2\cdot|B|^2\cdot 2^{2(l+H_2)}}<\frac{1}{|\alpha|^2}<\frac{2}{|B|^2\cdot 2^{2(l+H_2)}},$$

we get

$$\sum\frac{1}{|\alpha|^2}>c=c(r,A,B)(>0).$$

Collecting our results we get that

$$\left|F\left(\frac{p^k}{2^l}\right)-F\left(\frac{B}{A}\right)\right|<c_1\left(t\left(\frac{1}{4}P^{H_1}\right)+t\left(\frac{1}{4}2^{H_2}\right)\right)$$

with a positive constant c_1, independent of k, l, H_1, H_2.

Let now $(k,l):=(k_t,l_t)$ be such a sequence for which $p^{k_t}\cdot 2^{-l_t}\to\frac{B}{A}$. Then $H_1=H_1(t)$, $H_2=H_2(t)$ can be chosen so that $H_1(t)\to\infty$, $H_2(t)\to\infty$. This completes the proof of our Lemma. □

4. Proof of the theorem. I. *The case* $|F(\alpha)|=1$ $\forall\alpha\in G^*$. We shall prove that $F\colon \mathbf{F}_x\to\mathbf{C}_x$ is continuous at the point 1, and take into consideration Lemma 3.

Let p denote such a p_j in Lemma 6 for which $\mathscr{H}_{p_j,2}=\mathbf{C}$. Let $r_\nu=\frac{B_\nu}{A_\nu}\to 1$, $A_\nu, B_\nu\in G^*$. For each fixed r_ν there exists a sequence

$$p^{k_j(\nu)}\cdot 2^{-l_j(\nu)}\to\frac{B_\nu}{A_\nu},\quad F(p^{k_j(\nu)}2^{-l_j(\nu)})\to F\left(\frac{B_\nu}{A_\nu}\right).$$

Let ε_ν be a sequence of positive numbers such that $\varepsilon_\nu \to 0$. We select an element (a_ν, b_ν) from the sequence $(k_j(\nu), l_j(\nu))$ such that $a_\nu \geqq \nu$, $b_\nu \geqq \nu$,

$$\left|F(p^{a_\nu}\cdot 2^{-b_\nu}) - F\left(\frac{B_\nu}{A_\nu}\right)\right| < \varepsilon, \quad \left|p^{a_\nu}\cdot 2^{-b_\nu} - \frac{B_\nu}{A_\nu}\right| < \varepsilon_\nu$$

hold. Then $p^{a_\nu}\cdot 2^{-b_\nu} \to 1$, and by Lemma 7 $(A=B=1)$ we get $F(p^{a_\nu}\cdot 2^{-b_\nu}) \to 1$. But this implies $F\left(\frac{B_\nu}{A_\nu}\right) \to 1$.

The proof is complete under the assumption $|F(\alpha)|=1$ $\forall \alpha \in G^*$.

II. *Completion of the proof.* Let us assume now that $|F(\alpha)| \neq 1$ for at least one $\alpha \in G^*$. Taking into account Lemma 4 we may assume that $|F(\alpha)|>1$. Let us assume that $F \notin \mathscr{L}$. Since $F(\gamma) \neq 0$, by Lemma 2 we get that F is nowhere zero, and so by Lemma 5 that $|F(\alpha)| = |\alpha|^{\Delta}$, $0<\Delta<1$. Let $T(\alpha)$ be defined by $F(\alpha) = |\alpha|^{\Delta}\cdot T(\alpha)$. Since $F \in A_\gamma$, therefore $F \in A_1$. To complete the proof it is enough to show that $T \in A_1$. But this is an immediate consequence of the inequality

$$|T(\alpha+1)-T(\alpha)| = \left|\frac{F(\alpha+1)}{|\alpha+1|^{\Delta}} - \frac{F(\alpha)}{|\alpha|^{\Delta}}\right| \leqq \frac{1}{|\alpha|^{\Delta}}|F(\alpha+1)-F(\alpha)| + \frac{1}{|\alpha|^{\Delta}}\left|\left|\frac{\alpha}{\alpha+1}\right|^{\Delta} - 1\right| |F(\alpha+1)|.$$

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[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

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