Acta Mathematica Hungarica

VOLUME 47, NUMBERS 1–2, 1986

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0236—5294

ACTA MATHEMATICA
HUNGARICA

Acta Mathematica publishes papers on mathematics in English, German, French and Russian

Acta Mathematica is published in two volumes of four issues a year by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Allkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to

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Acta Mathematica is indexed in Current Contents.

ACTA MATHEMATICA HUNGARICA

VOLUME 47

AKADÉMIAI KIADÓ, BUDAPEST
1986

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VOLUME 47

86-2919 — Szegedi Nyomda — F. v.: Surányi Tibor igazgató

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PRINTED IN HUNGARY
Szegedi Nyomda, Szeged

Acta Math. Hung.
47 (1—2) (1986), 3—6.

BI-IDEALS IN REGULAR SEMIGROUPS AND IN ORTHOGROUPS

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Introduction

Several authors (Lajos, Steinfeld, Kuroki, Szász, etc.) have started a deeper study of regular semigroups and of some of their sub-classes through bi-ideals, obtaining also interesting characterizations.

Recently Steinfeld [10] has proved that the complete regularity of a semigroup is sufficient for the complete regularity of each of its bi-ideals. In the present work we prove that the regularity of a semigroup is not sufficient to ensure the regularity of each of its bi-ideals.

Besides we prove that there exists an isomorphism between the semigroups $\mathfrak{B}(S)$ and $\mathfrak{B}(S/\mathfrak{H})$ of the bi-ideals of S and $S/\mathfrak{H}$, quotient of S with respect to Green's relation $\mathfrak{H}$, when S is a regular semigroup and $\mathfrak{H}$ is a congruence.

The authors we have quoted made use often of the semigroup $\mathfrak{B}(S)$ of the bi-ideals of S in order to characterize certain classes of semigroups S through properties of $\mathfrak{B}(S)$. Among other things we find the connexion between $\mathfrak{B}(S)$ and the band E of the idempotents of an orthogroup S; in particular we prove that there is an isomorphism between $\mathfrak{B}(S)$ and $\mathfrak{B}(E)$.

Recall that a subsemigroup $B(\neq\emptyset)$ of a semigroup S is said to be a *bi-ideal* of S if $BSB\subseteq B$.

It is well known that if S is regular, then $BSB=B$ for every bi-ideal B of S.

THEOREM 1. *If every principal bi-ideal of a semigroup S is regular, then S is completely regular.*

PROOF. Let $a\in S$ and let $(a)_b$ be the principal bi-ideal generated by a. Since $(a)_b$ is by assumption regular and $(a)_b=aSa$, there exist $x\in(a)_b$ and $s\in S$ such that $a=axa=aasaa=a^2sa^2$. It follows that a is a completely regular element of S. □

In [10] Steinfeld proved that the complete regularity of a semigroup is equivalent to the complete regularity of each one of its bi-ideals. It follows immediately from this and from Theorem 1 that the regularity of a semigroup does not imply the regularity of its principal bi-ideal.

One usually denotes by $\mathfrak{B}(S)$ the semigroup of the bi-ideals of the semigroup S.

THEOREM 2. *Let S be a regular semigroup and let $\mathfrak{H}$ be a congruence on S. Then the semigroups $\mathfrak{B}(S)$ and $\mathfrak{B}(S/\mathfrak{H})$ are isomorphic.*

PROOF. Let the mapping $f\colon \mathfrak{B}(S)\to\mathfrak{B}(S/\mathfrak{H})$ be defined by $f(B)=H_B$, where $H_B=\{H_b\}_{b\in B}$. Then f is an isomorphism. Notice, first of all, that $B=\bigcup_{b\in B} H_b$ for every bi-ideal B of S. Indeed, for every $x\in\bigcup_{b\in B} H_b$, there exists $b'\in B$ such that $x\mathfrak{H}b'$; therefore $x=xax=b'sx=b's'b'$ (for some a, s, s'), i.e. $x\in B$. This immediately implies that f is one-to-one.

On the other hand, if $T\subseteqq S$ is such that $H=\{H_t\}_{t\in T}$ is a bi-ideal of $S/\mathfrak{H}$, then $\bigcup_{t\in T} H_t$ is a bi-ideal of S. Indeed if $x, y\in\bigcup_{t\in T} H_t$ and $s\in S$, there exist $t_1, t_2\in T$ such that $x\in H_{t_1}$, $y\in H_{t_2}$ and, since H is a bi-ideal of $S/\mathfrak{H}$, $H_{t_1}\cdot H_s\cdot H_{t_2}=H_{t'}$ where $t'\in T$. Therefore

$$xsy\in H_{t_1}H_sH_{t_2}=H_{t'}\subseteqq\bigcup_{t\in T} H_t$$

i.e. $\bigcup_{t\in T} H_t$ is a bi-ideal of S, hence f is onto.

Finally let B, B' be two bi-ideals of S and let $b\in B$ and $b'\in B'$; then $H_bH_{b'}=H_{bb'}$ and hence $H_BH_{B'}\subseteqq H_{BB'}$. Besides let $x\in BB'$, $b\in B$, $b'\in B'$ such that $x=bb'$. Then if $h\in H_x$, $hS=bb'S$, $Sh=Sbb'$. It follows from this and the regularity of S that $h=hah=bb'sh=bb's'bb'$ where a, s, s' are suitable elements of S. Therefore $h\in H_{bb's'b}H_{b'}$. Besides $k\neq h$ is in H_x, there $k=bb's''bb'$, where s'' is a suitable elemen of S and $bb's''b\mathfrak{H}bb's'b$. Then $H_x=H_{bb's'b'b}H_{b'}$ and hence $H_{BB'}\subseteqq H_BH_{B'}$. □

Recall that a semigroup S is said to be *intra-regular* if $a=xa^2y$ where x, y are suitable elements of S, for every $a\in S$.

Pastijn [6] has proved that if S is regular, $\mathfrak{B}(S)$ is a normal band (i.e. for every $A, B, C\in\mathfrak{B}(S)$, $ABCA=ACBA$) if and only if S is intra-regular. Through this property of $\mathfrak{B}(S)$ and since S is regular and intra-regular if and only if $\mathfrak{B}(S)$ is a band (cf. [9]), we get quickly the following theorem.

THEOREM 3. *A semigroup S is regular and intra-regular if and only if $\mathfrak{B}(S)$ is a normal band.*

Since an orthogroup is a particular regular and intra-regular semigroup, if S is an orthogroup, $\mathfrak{B}(S)$ is a normal band. It is interesting to find the connexion between $\mathfrak{B}(S)$ and the band E of the idempotents of an orthogroup S.

THEOREM 4. *A semigroup S is an orthogroup if and only if every bi-ideal of S is an orthogroup.*

The proof of this theorem descends from a similar theorem for completely regular semigroups given by Steinfeld [10].

COROLLARY 5. *A semigroup S is an orthogroup with band of idempotents of type $\mathfrak{P}$, where $\mathfrak{P}$ is any of the types of band classified by M. Petrich in* [7], *if and only if every bi-ideal of S is an orthogroup with the band of idempotents of type $\mathfrak{P}$.*

If S is an orthogroup where $\mathfrak{H}$ is a congruence, the bands E and $S/\mathfrak{H}$ are isomorphic. Then from Theorem 2 we know that bands $\mathfrak{B}(S)$ and $\mathfrak{B}(E)$ are isomorphic. But, if S is an orthogroup, the isomorphism between $\mathfrak{B}(S)$ and $\mathfrak{B}(E)$ exists even if $\mathfrak{H}$ is not a congruence. In fact the following theorem holds.

THEOREM 6. *If S is an orthogroup with the band of idempotents E then the bands $\mathfrak{B}(S)$ and $\mathfrak{B}(E)$ are isomorphic.*

PROOF. Let the mapping $Z\colon \mathfrak{B}(S)\to\mathfrak{B}(E)$ be defined by $Z(B)=E(B)$, where $E(B)$ is the band of idempotents of B. We prove that Z is an isomorphism.

Let B and B' be two bi-ideals of S such that $E(B)=E(B')$ and let b be an element of B. According to Theorem 4, there is an element $x\in B$ such that $b=bxb$ and $xb=bx$, then, bx being an idempotent of B and so of B', $b=bxb=bxbxb$ is an element of B'. Analogously we can prove that, if b' is an element of B', b' is an element of B. Therefore $B=B'$, namely Z is one-to-one.

Now, let B be a bi-ideal of E and B' the bi-ideal of S generated by B. Then every element of B is an idempotent of B'; moreover, if e is an idempotent of B', since $B'=BSB$, there are $b_1, b_2\in B$ and $s\in S$ such that:

$$e = b_1 s b_2 = b_1^2 s b_2^2 = b_1 e b_2,$$

then e is an element of B. Therefore B' is a bi-ideal of S whose band of idempotents is B, so Z is onto.

Finally let B and B' be two bi-ideals of S; then, obviously, $E(B)E(B')\subseteq \subseteq E(BB')$. Moreover, if e is an element of $E(BB')$, there are $b\in B$ and $b'\in B'$ such that $e=bb'$. If we denote by $\hat{b}'$ the identity of $\mathfrak{H}$-class of B' containing b' and by $\hat{b}$ the identity of $\mathfrak{H}$-class of B containing b, we have

$$e\hat{b} = bb'\hat{b} = \hat{b}bb'\hat{b} = \hat{b}e\hat{b}\in E(B), \quad \hat{b}'e = \hat{b}'bb' = \hat{b}'bb'\hat{b}' = \hat{b}'e\hat{b}'\in E(B').$$

Then, if b^{-1} is the inverse of b in $\mathfrak{H}$-class of B containing it and if b'^{-1} is the inverse of b' in $\mathfrak{H}$-class of B' containing it,

$$e\hat{b}\hat{b}'e = bb'\hat{b}\hat{b}'bb' = bb'\hat{b}'\hat{b}\hat{b}'\hat{b}bb' = bb'(\hat{b}'\hat{b})^2bb' = bb'\hat{b}'\hat{b}bb' = bb'bb' = e^2 = e,$$

then e is an element of $E(B)\cdot E(B')$. Therefore Z is a homomorphism. □

We recall that a band E is said *left* [respectively, *right*] *regular* iff $ax=axa$ [resp. $xa=axa$] for every $a, x\in E$. S. Lajos ([3], [4]) has characterized the orthogroups with left [resp. right] regular band of idempotents E. In fact he has proved the following theorem (of which, obviously, the dual holds).

THEOREM 7. *A semigroup S is an orthogroup with left regular band of idempotents E if and only if $\mathfrak{B}(S)$ is a regular semigroup whose S is a right identity.*

The following theorem (of which the dual holds) characterizes the same class of semigroups, but pointing out the relation between E and $\mathfrak{B}(S)$.

THEOREM 8. *A semigroup S is an orthogroup with left regular band of idempotents E if and only if $\mathfrak{B}(S)$ is a left regular band.*

PROOF. Let S be an orthogroup with left regular band of idempotents E and let A, X be two bi-ideals of E. Then, if $a\in A$ and $x\in X$, $ax=axa$, i.e. $AX\subseteq AXA$. Moreover, if $a, a'\in A$ and $x\in X$, $axa'=a(xa'x)$, then $AXA\subseteq AX$. Therefore $\mathfrak{B}(E)$ is a left regular band. Hence, from Theorem 6, it follows that $\mathfrak{B}(S)$ is a left regular band.

Conversely, if $\mathfrak{B}(S)$ is a left regular band, then $\mathfrak{B}(S)$ is a regular semigroup and $BS=BSB=B$ for every bi-ideal B of S. From this and from Theorem 7 it follows that S is an orthogroup with left regular band of idempotents E. □

We recall that a band E is said *left* [resp. *right*] *normal* iff $efg=egf$ [resp. $gfe=fge$] for every $e, f, g \in E$. Obviously a left [resp. right] normal band is left [resp. right] regular.

THEOREM 9. *A semigroup S is an orthogroup with left regular band of idempotents if and only if $\mathfrak{B}(S)$ is a left normal band.*

PROOF. If S is an orthogroup with left regular band of idempotents E, by Theorem 8, $\mathfrak{B}(S)$ is a left regular band. In addition let A, X, Y be bi-ideals of S, then

$$AXY = AXYX = AXAYX \subseteqq AYX, \quad AYX = AYXY = AYAXY \subseteqq AXY.$$

It follows that $\mathfrak{B}(S)$ is a left normal band. The converse, from Theorem 8, is obvious. □

The following characterization, by Kuroki (cf. [5]), of the orthogroups in which the band of idempotents E is a semilattice, is an immediate consequence of Theorem 9 and its dual.

THEOREM 10. *A semigroup S is an orthogroup with semilattice E of idempotents if and only if $\mathfrak{B}(S)$ is a semilattice.*

Since, as we previously observed, if S is an orthogroup $\mathfrak{B}(S)$ is a normal band, from Theorems 8, 9, 10 and their duals we can finally obtain the following theorem.

THEOREM 11. *If a semigroup S is an orthogroup with the band of idempotents of type $\mathfrak{P}$, where $\mathfrak{P}$ is any of the types of band classified by M. Petrich in* [7], *then $\mathfrak{B}(S)$ is a band of type $\mathfrak{P}$. In addition if the type $\mathfrak{P}$ is that of left regular, or right regular band or semilattice, then and only then the converse holds.*

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(Received October 14, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 7—12.

ON A PROBLEM OF KAPLANSKY

I. A. AMIN (Cairo)

Dedicated to Professor R. Wiegandt on his 50th birthday

Classes of torsion-free abelian groups having inequivalent indecomposable decompositions

In this section we construct classes of torsion-free abelian groups with the aim of enriching our knowledge of the antimonies of various indecomposable decompositions of torsion-free abelian groups of countable ranks. Jonsson's suprising discovery in this respect [8] (see also [9]) has shaken, by then, our firm belief in the role of the notion of isomorphism. Since this discovery of Jonsson, various authors published results, using the basic ideas of Jonsson, demonstrating the different aspects of such study [7]. In this section we rely heavily, as others, on the original technique introduced by Jonsson. However, the groups constructed here depend on integral parameters that can be chosen in different ways to enlighten our knowledge in this respect. Also, known results (see [7]) can be even drawn alternatively from our general setting. Our main results are

THEOREM 1. *For every finite cardinal m, there exist indecomposable torsion-free abelian groups A and B each of rank 2 such that $A^{m+1}\cong B^{m+1}$ while $A^s\not\cong B^s$ for $s=1, \dots, m$.*

THEOREM 2. *For every finite cardinal m, there exist indecomposable torsion-free abelian groups A and B each of ω-rank, ω is the first infinite cardinal number, such that $A^{m+1}\cong B^{m+1}$ while $A^s\not\cong B^s$ for $s=1, \dots, m$.*

Construction

1. Let m and n be given positive integers with $n>2$, and let V be an mn-dimensional vector space over the field Q of rational numbers. Take $\{x_{ij}: i=1, \dots, n; j=1, \dots, m\}$ as a basis for V. Let furthermore P_j; r_{ts}, $j=1, \dots, n$, $t=1, \dots, n-1$ and $s=1, \dots, m$, be pairwise disjoint primes and choose positive integers a_{ts}, $t=1, \dots, n$ and $s=1, \dots, m$, such that a_{ts} is not divisible by r_{ts} for each t, s. Consider the following subgroup of $\langle V, +\rangle$

$$A(a_{11}, \dots, a_{n-1,1}, \dots, a_{im}, \dots, a_{n-1,m}; n, m) =$$

$$= \frac{1}{p_i^\infty} x_{ij}, \frac{1}{r_{tj}} \Big(\sum_{u=1}^{m} x_{tu} + a_{tj} \sum_{u=1}^{m} x_{t+1,u}\Big) : i = 1, \dots, n;\ j = 1, \dots, m;\ \ t = 1, \dots, n-1;$$

wherein $\langle B\rangle$ is understood to be the subgroup of $\langle V, +\rangle$ generated by a non-empty subset B of V. In the sequel, if no ambiguity may arise, we shall write $A(a_{ts}; n; m)$, and sometimes $A(n, m)$, to stand for the above constructed subgroup of $\langle V, +\rangle$. If moreover $n=1$, we may take a further abbreviation by writing $A(a_t; n)$, or $A(n)$, for the same group. The group $A(n, m)$ is obviously a torsion-free abelian group of rank mn. Furthermore, one can easily check that

$$A(a_{ts}; n, m) = A(a_{11}, \ldots, a_{n-1,1}; n, 1) \oplus \ldots \oplus A(a_{1m}, \ldots, a_{n-1,m}; n, 1).$$

2. In a similar fashion to that given in [2], one can prove that

$$A(a_{ij}, \ldots, a_{n-1,j}; n, 1)$$

is an indecomposable torsion-free abelian group of rank n. Furthermore if in $A(a_t, n)$ we replace the parameters $a_1, \ldots, a_{n-1}$ by parameters $b_1, \ldots, b_{n-1}$ of the same type we can prove, as in [2], that $A(a_t, n) \cong A(b_t, n)$ if and only if there exist integers $s_1, \ldots, s_n$ satisfying the following relations

$$p_{k+1}^{s_{k+1}} a_k \equiv \pm p_k^{s_k} b_k \pmod{r_k}, \quad k = 1, \ldots, n-1.$$

3. Consider now a group $A(b_{tz}; n, m)$ of the same type as that of $A(a_{tz}; n, m)$, where the concerned primes are the same in both groups. We prove that $A(a_{tz}; n, m) \cong A(b_{tz}; n, m)$ implies that

$$a_{t1} a_{t2} \ldots a_{tm} p_{k+1}^{s_{k+1}} \equiv \pm b_{t1} b_{t2} \ldots b_k^{s_k} \pmod{r_z} \quad k = 1, \ldots, n-1, \; z = 1, \ldots, m$$

$$\text{and} \quad s_k \in Z.$$

We first observe that an isomorphism $\psi\colon A(a_{tz}; n, m) \to A(b_{tz}; n, m)$ can be extended to an isomorphism between the injective hulls of these groups (in fact ψ is extendable to an endomorphism of V). This simply means that ψ admits multiplication by elements of Q. On the other hand it is a well known fact that the class of p-divisible abelian groups is homomorphically closed. So, one can infer that the image of x_{ij} under ψ should lie in the p_i-divisible subgroup of $A(b_{tz}; n, m)$. Thus, for some integers u_i; $c_1, \ldots, c_m$ we have

$$\psi(x_{ij}) = \frac{1}{p_i^{u_i}} \sum_{t=1}^{m} c_t x_{it}.$$

Using the same argument for ψ^{-1}, it is not hard to see, after effecting on both sides of the last equation by ψ^{-1}, that each c_t is either equal to zero or a rational number of the form $d_t p_i^{v_t}$, $d_t, f_t \in Z$. But we also know that both

$$\frac{1}{r_{ij}}\Big(\sum_{u=1}^{m} x_{iu} + b_{ij} \sum_{u=1}^{m} x_{i+1,u}\Big) \quad \text{and} \quad \frac{1}{r_{ij}}\Big(\sum_{u=1}^{m} \psi(x_{iu}) + a_{ij} \sum_{u=1}^{m} \psi(x_{i+1,u})\Big)$$

are elements of $\operatorname{Im}\psi = A(b_{ts}; n, m)$. So, any linear combination of such terms is again an element of $\operatorname{Im}\psi$. Thus routine calculation gives rise to congruences yielding eventually the following relations

$$a_{t1} a_{t2} \ldots a_{tm} p_{t+1}^{s_{t+1}} \equiv \pm b_{t1} b_{t2} \ldots b_t^{s_t} \pmod{r_{tz}}, \quad t = 1, \ldots, n-1, \; z = 1, \ldots, m$$

$$\text{and} \quad s_t, s_{t+1} \in Z.$$

This establishes the required assertion.

4. If ω is the first infinite cardinal number, we prove that the congruences derived in 3 above are sufficient to establish the isomorphism $A(a_{tz}; \omega, m) \cong A(b_{tz}; \omega, m)$. We first prove our assertion in the case for which $m=1$, and simple induction draws what is to be proven. To this end consider an ω-dimensional vector space over Q with a basis x_i, y_i: $i \in Z$ and let p_i, r_{2i}: $i=1, 2, \dots$ be a set of pairwise disjoint primes. Choose positive integers a_{2i} and b_{2i}; $i \in Z$ such that r_{2i} does not divide neither a_{2i} nor b_{2i} for each i. Construct as above the subgroups $A(b_{2i}; \omega, 1)$, $A(a_{2i}b_{2i}; \omega, 1)$ of the group $\langle y_i\colon i \in Z\rangle$ and the subgroups $A(a_i; \omega, 1)$, $A(1; \omega, 1)$ of $\langle x_i\colon i \in Z\rangle$, where $A(1; \omega, 1)$ stands for $A(\alpha_{2i}; \omega, 1)$ and $\alpha_{2i}=1$ for each $i \in Z$. Define a linear mapping ψ of V such that $\psi(x_{2n-1})=x_{2n-1}$, $\psi(x_{2n}) = c_{2n}+d_{2n}r_{2n}y_{2n}y_{2n}$, $\psi(y_{2n-1})=y_{2n-1}$ and $\psi(y_{2n})=r_{2n}^2+a_{2n}y_{2n}$, where the integers c_{2n}, d_{2n} are chosen so that $c_{2n}a_{2n}=1+d_{2n}r_{2n}^2$ for each $n \in Z$. Obviously ψ is non-singular and $\psi^{-1}(x_{2n})=a_{2n}x_{2n}-d_{2i}r_{2n}y_{2n}$, $\psi^{-1}(y_{2n})=c_{2n}-r_{2n}^2x_{2n}$. Now if F designates the free abelian group generated by the base elements of V, fairly direct computation shows that ψ maps the generators of $A(a_{2i}; \omega, 1) \oplus A(b_{2i}; \omega, 1)$ into those of $A(1; \omega, 1) \oplus A(a_{2i}b_{2i}; \omega, 1)$; and similarly ψ^{-1} acts on the concerned generators. Thus we have

$$A(1;\, \omega, 1) \oplus A(a_{2i}b_{2i};\, \omega, 1) \cong A(a_{2i};\, \omega, 1) \oplus A(b_{2i};\, \omega, 1).$$

Having done this, we can relabel and replace $2i$ by i, and so simple induction on m shows in this case that if the congruences cited in 3 above are satisfied with $t \in Z$ we should have

$$A^m(1;\, \omega, 1) \oplus A(a_{t1}a_{t2}\dots a_{tm};\, \omega, 1) \cong \bigoplus_{j=1}^{m} A(a_{tj};\, \omega, 1),$$

where $t \in Z$ and $A^m(1; \omega, 1)$ means the direct sum of m copies of $A(1; \omega, 1)$. But since

$$A(a_{t1}a_{t2}\dots a_{tm};\, \omega, 1) \cong A(b_{t1}b_{t2}\dots b_{tm};\, \omega, 1)$$

if and only if the congruence relations given in 3 above are satisfied for each $t \in Z$ (see [3] and [2]), we conclude that

$$\bigoplus_{j=1}^{m} A(a_{tj};\, \omega, 1) \cong \bigoplus_{j=1}^{m} A(b_{tj};\, \omega, 1)$$

whenever the above relations are satisfied for each $t \in Z$. Thus the prementioned relations are sufficient for

$$A(a_{tz};\, \omega, n) \cong A(b_{tz};\, \omega, m)$$

where ω is the first infinite cardinal number. The necessity condition for the countable cardinal case is given by 3 above.

5. It is obvious from the argument used in 4 above that $\bigoplus_{i=l}^{m} A(a_i;\ 2,1) \simeq \bigoplus_{i=l}^{m} A(b_i;2,1)$ if and only for some integers s, k,

$$a_1a_2\dots a_m p_1^s \equiv \pm\, b_1b_2\dots b_m p_2^s \pmod r$$

bearing in mind that the only primes involved here are p_1, p_2 and r. It seems most likely that the assertion is true for any finite cardinal n.

6. The above results show that any of the required isomorphisms breaks down if only one of the concerned congruence relations fails to be satisfied. Now take $n=2$; $p_1=7$, $p_2=17$, $r=11$, $a=1$ and $n=2$. It is easy to see from our conditions that $\bigoplus_{i=1}^{m} A(1;2,1)$ is not isomorphic to $\bigoplus_{i=1}^{m} A(2;2,1)$ for $s\leqq 4$, yet $\bigoplus_{i=1}^{5} A(1;2,1)\cong$ $\cong\bigoplus_{i=1}^{5} A(2;2,1)$. In fact rudiments of number theory will always insure, by proper choices of the concerned primes and parameters, that for any finite cardinal number m there exist primes p_1, p_2, r and integers a, b neither of them is divisible by r such that $A^m(a;2,1)\cong A^m(b;2,1)$; whilst $A^s(a;2,1)\not\cong A^s(b;2,1)$ for $s=1,\ldots,m-1$. Our remarks show that this result can be extended to the case in which n is equal to the first infinite cardinal number ω. Thus by choosing appropriate concerned parameters, we see that for every given finite cardinal m, there exist ω-rank torsion-free indecomposable abelian groups C and D such that $C^m\cong D^m$; whereas $C^s\cong D^s$ for $s=1,\ldots,m-1$. These results establish Theorems 1 and 2.

On a theorem of Kaplansky

This section is mainly concerned with the indecomposable decompositions of countable rank reduced torsion-free modules over a discrete valuation ring. Theorems 3 and 7 are the most important results of this section. Theorem 3 sharpens an embedding theorem of Kaplansky for such modules. We also hint that the technique used in proving Theorem 3 can be applied to draw at once known important results. Theorem 7 gives a counter example disproving a result of the author; whilst Theorem 8 gives a modified version of this false result.

In this section R designates a PID, R_p is the discrete valuation ring obtained by localizing at the prime element p of R. R_p^* is the p-adic completion of R_p, and A^* is understood to be the p-adic completion of a reduced R_p-module A. In the following theorem we give a generalization to Kaplansky's theorem concerning finite rank reduced torsion-free R_p^*-modules [10] pp. 46—53. Theorem 3 extends on the one hand Kaplansky's theorem (see Theorems 20, 22 and 23 of [10]) to cover the countable rank case, and on the other hand it gives a computational scheme for the finite rank case.

THEOREM 3. *A countable rank reduced torsion-free R_p-module of p-rank α is a dense submodule of the direct sum of α purely indecomposable R_p-modules. If A is of finite rank $n=\alpha+k$, then the rank of each direct summand does not exceed $k+1$. Furthermore, the tensor product of two finite rank R-modules A and B is a dense pure submodule of a free R_p^*-module.*

PROOF. If B is a basic submodule of A, then $r(B)=r(R_p^*\otimes B)=r_p(R_p^*\otimes A)$ and $R_p^*\otimes A=R_p^*\otimes B\oplus D$, where D is the maximal divisible submodule of $R_p^*\otimes A$. But since $R_p^*\otimes B$ is free and $R_p^*\otimes M\cong M$ for any R_p^*-module M, we see that $R_p^*\oplus A$ is the direct sum of copies of R_p^* and its quotient field, a result we obtained without appealing to Kaplansky's fundamental decomposition theorem. Now flatness of A shows that the map $\psi\colon A\to R_p^*\otimes A$, $\psi(a)=1\otimes a$ is indeed a monomorphism whose image generates the R_p^*-module $R_p^*\otimes A$. But since the equation $px=$

$=1\otimes a$, $0\neq a\in A$, has no solution in $R_p^*\otimes A$ if and only if the equation $px=a$ has no solution in A (see [1]), we infer that A is embedded isomorphically into the free R_p^*-module $R_p^*\otimes A$. In fact such a monomorphism can be effected by an injection β such that $\beta(a)=c$, where $\psi(a)=c+d$, $c\in R_p^*\oplus A$ and $d\in D$. A further application of Lemma 4.1 of [1] shows that $\alpha(A)$ is pure in $R_p^*\otimes A$. Take a basic $\{e_i\colon e\in\alpha\}$ for $R_p^*\oplus B$ and identify A with its image under β. Since a pure R_p^* is purely indecomposable, we deduce that the pure closure A_i of the i-th components of A in the free decomposition of $R_p^*\otimes B$ is purely indecomposable. Thus A can be regarded as a pure submodule of $\bigoplus_{i\in\alpha} A_i$ as asserted. Suppose now that the p-rank α of A, consequently $R_p^*\otimes B$, is finite. Consider a p-independent set $\{b_1, \ldots, b_\alpha\}$ of $R_p^*\otimes B$ and extend it to a maximal linearly independent subset $\{b_i\colon i>1\}$ of $R_p^*\otimes B$ and then express each b_i, $i>\alpha$, as an $R_p^*=$linear combination of $b_1, \ldots, b_m$. So, if $r(A)$ is finite and equals $\alpha+k$, fairly direct computation shows that $r(A_i)$ cannot exceed $k+1$ as asserted. In the general case in which α is countable, we recall that A and $R_p^*\otimes A$ have equal p-ranks. The final required result concerning $A\otimes B$ can be thus effected by using Theorem 5.13 of [1]. The proof of the theorem is complete.

Remark 4. The argument used in the first part of the proof Theorem 3 gives an implicit proof of Kaplansky's theorem on the direct decomposition of a countable rank reduced torsion-free R_p^*-module (see [10] Theorem 20, p. 48).

Remark 5. Theorem 3 shows that $\bigoplus_{i\in\alpha} A_i/A$ is divisible.

Remark 6. The argument used in the first part of Theorem 3 can be applied to get a direct proof of the freeness of a countable rank deduced torsion-free R_p^*-module (see [10], Theorem 20).

Now we give a counter-example disproving Theorem 5.3 of the author's paper [1]. In that paper the factorization given by equation 2 is dubious.

Theorem 7. *There exists a rank 3 reduced torsion-free indecomposable module over a non-complete discrete valuation ring that does not possess the exchange property.*

Proof. Consider the irreducible polynomial $f(x)=x^3-2x^2-x-3$ over Z. Thus $K=Zx/\langle f(x)\rangle$ can be regarded as a rank 3 abelian group. But since $f(x)=(x-1)(x+1)(x-2)$ in $Z/\langle 5\rangle$, $K/5K$ can be represented as the product of three fields. Let now Z_5 be the discrete valuation ring constructed by localizing at the prime 5. This means that localizing at 5 shows that K_5 is a rank 3 free Z_5-module. Moreover, the third isomorphism theorem shows that K_5 has exactly three maximal ideals. Now let R be the obtained by inverting one of these maximal ideals after localizing. $R/5R$ is a domain of order 25 that can be represented as the direct sum of two fields. Thus R is not a local ring. Also R is a rank 3 torsion-free Z_5-module. So, the abelian group $\langle R, +\rangle$ is quasi-isomorphic to a free module over the centre A of $\operatorname{End}(R, +)$ [7]. The rank of this quasi-isomorphic image should be 1 or 3. But since $\langle R, +\rangle$ cannot be quasi-isomorphic to a free Z_5-module, we conclude that, up to isomorphism, $A=\langle R, +, .\rangle$. Thus $\operatorname{End}(R, +)\cong\langle R, +, .\rangle$. Thus $\langle R, +\rangle$ is a rank 3 reduced torsion-free indecomposable Z_5-module whose endomorphism ring is not local.

However we have

THEOREM 8. *A finite rank reduced tonion-free R_p-module having a p-rank not exceeding* 3 *possesses the Krull—Schmidt property.*

PROOF. If A is such a module, the number of summands in any of its indecomposable decompositions cannot exceed 3. If the number is exactly 3, then each summand is necessarily purely indecomposable. But since any purely indecomposable module has the exchange property, we see that A has the Krull—Schmidt property. So, the only alternative to be investigated is the case in which we have two decompositions $A=B\oplus C=D\oplus E$ of A, where $r_p(A)=3$. So, $r_p(B)=r_p(D)=1$, say. Thus B and D have the exchange property and so $B\oplus C\cong B\oplus D\oplus K$ and $D\oplus E\cong$ $\cong B\oplus D\oplus L$ for some modules K and L such that $C\cong D\oplus K$ and $E\cong B\oplus L$ (see [4]). But since a purely indecomposable module is cancellable, we conclude that $K\cong L$. This completes the proof.

This means that Theorem 5.6 of [1] becomes valid if we replace "Azumaya-Fitting" by "has the Krull—Schmidt property".

The author thanks Professor B. Warfield, Jr. for a communication concerning Theorem 7.

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(Received October 14, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 13—27.

STRONG APPROXIMATIONS OF RENEWAL PROCESSES AND THEIR APPLICATIONS

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1. Introduction

Let (X, Y), $\{(X_n, Y_n),\ Y_n=(Y_n^{(1)}, \ldots, Y_n^{(d)}),\ n\geqq 1\}$ be a sequence of random vectors with values in R^{d+1}. Many authors (see, for example, the Introduction and Chapter 2 in Csörgő and Révész [8]) studied the rate of strong approximation of the partial sums $\big(U(t), S(t)\big)$,

$$U(t)=\sum_{i=1}^{[t]} X_i, \quad S(t)=\sum_{i=1}^{[t]} Y_i$$

by a $(d+1)$-dimensional Wiener process. Horváth [13] obtained that a strong invariance principle for the partial sums of independent, indentically distributed random variables (i.i.d.r.v's) with positive expectation always implies a strong approximation for the corresponding renewal process. The renewal process, being the inverse of the partial sum process is defined as

$$N(t)=\inf\{s\colon\ U(s)>t\},$$
$$=\infty, \quad \text{if} \quad \{s\colon\ U(s)>t\}=\emptyset.$$

First we show that Theorem 2.1 in [13] remains true if we drop the independence and identical distribution assumption on the summands. A joint approximation of $N(t)$ and $S\big(N(t)\big)$ will be proved in Section 2. Partial sums indexed by a renewal process appear in the mathematical theory of risk processes and queuing processes. Gut and Janson [10] gave some other interesting examples of the use of $S\big(N(t)\big)$ in the theory of chromatography, classical renewal theory, chemistry, physics, replacement policies and economics.

In the last section we consider some applications of our main theorems. We obtain that our method gives the best possible joint approximation of $U(t)$, $S(t)$, $N(t)$ and $S\big(N(t)\big)$.

We can assume without loss of generality that our probability space $(\Omega, \mathscr{A}, P)$ is so rich that every r.v. and all processes introduced later on are defined on it. Throughout this paper we use the maximum norm in R^k denoted by $\|x\|_k=\max_{1\leqq i\leqq k} |x_i|$, $x=(x_1, \ldots, x_k)$. The transpose of a row-vector x is a column-vector denoted by x^T. Let $a\wedge b=\min(a, b)$, $a\vee b=\max(a, b)$. We use the abbrevations $\xi_T \overset{\text{a.s.}}{=} o\big(a(T)\big)$ and $\xi_T \overset{\text{a.s.}}{=} O\big(b(T)\big)$, where $\{\xi_T, a(T), b(T), T\geqq 0\}$ are stochastic processes, to mean that

$$\lim_{T\to\infty} \xi_T/a(T)=0 \quad \text{a.s.}$$

and

$$P\{\limsup_{T\to\infty} |\xi_T|/b(T) = \infty\} = 0,$$

respectively. We say that $a(T)$ is not greater than $b(T)$ almost surely $\left(a(T) \overset{\text{a.s.}}{\leqq} b(T)\right)$, if for almost all $\omega \in \Omega$ there is an integer $n_0 = n_0(\omega)$ such that $a(T) \leqq b(T)$ for $T \geqq n_0$.

2. Strong approximations of the renewal process and the partial sums indexed by the renewal process

Several authors proved strong invariance principles for sums of random variables or random vectors under different conditions. We do not want to summarize these results in a single statement and hence we are not going to list the different sets of conditions (moment and dependence conditions) allowing such strong approximations. We will simply assume that the partial sums can be approximated by a Gaussian process and strong invariance principles for $N(t)$ and $S(N(t))$ will follow from this assumption of strong approximation.

CONDITION A. *We can define a* $(d+1)$*-dimensional Wiener process*

$$\{W(t) = (W^{(1)}(t), \ldots, W^{(d+1)}(t)),\ t \geqq 0\}, \quad EW(t) = 0,\ EW^T(t)W(s) = \Gamma \min(t, s)$$

such that

$$\sup_{0\leqq t\leqq T} \|(U(t)-\mu t,\ S(t)-mt)-W(t)\|_{d+1} \overset{\text{a.s.}}{=} o(r(T)), \tag{2.1}$$

where $\Gamma = \{\gamma_{i,j}\}$, $1 \leqq i, j \leqq d+1$ *is a nonsingular covariance matrix,* (μ, m) *is a constant vector,* $r(T)$ *is nondecreasing, regularly varying at infinity and*

$$r(T) = O((T \log\log T)^{1/2}). \tag{2.2}$$

For the sake of simplicity we use the notation $\sigma^2 = \gamma_{1,1}$. Condition A in the following theorem (and Condition B in Theorem 2.2 below) is meant only for the first component.

THEOREM 2.1. *If* $\mu > 0$ *then Condition* A *implies that*

$$\sup_{0\leqq t\leqq T} |\mu^{-1}t - N(t) - \mu^{-1}W^{(1)}(\mu^{-1}t)| \overset{\text{a.s.}}{=} o(r(T)),$$

if

$$(T\log\log T)^{1/4}(\log T)^{1/2} = o(r(T)) \tag{2.3}$$

and

$$\limsup_{T\to\infty} (T\log\log T)^{-1/4}(\log T)^{-1/2} \sup_{0\leqq t\leqq T} |\mu^{-1}t - N(t) - \mu^{-1}W^{(1)}(\mu^{-1}t)| =$$

$$= 2^{1/4}\sigma^{3/2}\mu^{-7/4} \quad a.s.,$$

if

$$r(T) = O((T\log\log T)^{1/4}(\log T)^{1/2}). \tag{2.4}$$

It is very important that the partial sums and the renewal process are approximated by the *same* Wiener process. It follows from this theorem that the rate of the best joint approximation of partial sums and the renewal process is the Strassen rate.

PROOF. Basically we follow the line of the proof of Theorem 2.1 in [13] but we use only our condition and the properties of the Wiener process. First we note that the regular variation of $r(T)$ implies that

$$\limsup_{T\to\infty} \frac{r((1+\varepsilon)T)}{r(T)} < \infty \tag{2.5}$$

for every $\varepsilon>0$ and by the monotonicity of $r(T)$ we get

$$\limsup_{T\to\infty} \frac{r(T+\varepsilon(T))}{r(T)} < \infty \tag{2.6}$$

for every $\varepsilon(T)\geqq 0$ such that $\varepsilon(T)=O(T)$ as $T\to\infty$. Conditions (2.1), (2.2) and the law of the iterated logarithm for the Wiener process imply that

$$\limsup_{T\to\infty} (2T\log\log T)^{-1/2} \sup_{0\leqq t\leqq T} |U(t)-\mu t| = \sigma \quad \text{a.s.} \tag{2.7}$$

Therefore,

$$U(T) \leqq (1+\varepsilon)\mu T \tag{2.8}$$

for each $\varepsilon>0$. Using the Lemma in [13], (2.5) and (2.6) we get

$$\begin{aligned} \sup_{0\leqq t\leqq T} |N(t\mu)-t| &\leqq \sup_{0\leqq t\leqq N(T\mu)} |\mu^{-1}U(t)-t| \overset{\text{a.s.}}{\leqq} \\ &\overset{\text{a.s.}}{\leqq} \sup_{0\leqq t\leqq (1+\varepsilon)T} |\mu^{-1}U(t)-t| \leqq h_\varepsilon(T), \end{aligned} \tag{2.9}$$

where

$$h_\varepsilon(T) = (1+\varepsilon)^{1/2}\mu^{-1}\sigma(2T\log\log T)^{1/2}.$$

Now consider the decomposition

$$t-N(t\mu) = \mu^{-1}\big(U(N(t\mu))-\mu N(t\mu)\big)+\mu^{-1}\big(\mu t-U(N(t\mu))\big). \tag{2.10}$$

First we approximate the first term in (2.10) by the help of (2.9), (2.6) and our condition. We get

$$\sup_{0\leqq t\leqq T} \big|U(N(t\mu))-\mu N(t\mu)-W^{(1)}(N(t\mu))\big| \overset{\text{a.s.}}{=} o(r(T)). \tag{2.11}$$

We estimate the random increment of the Wiener process by (2.9) and Theorem 1.2.1 of Csörgő and Révész [8]. We have for all $\varepsilon>0$

(2.12)

$$\begin{aligned} \sup_{0\leqq t\leqq T} \big|W^{(1)}(N(t\mu))-W^{(1)}(t)\big| &\overset{\text{a.s.}}{\leqq} \sup_{0\leqq t\leqq (1+\varepsilon)T-h_\varepsilon(T)} \ \sup_{0\leqq s\leqq h_\varepsilon(T)} |W^{(1)}(t+s)-W^{(1)}(t)| \overset{\text{a.s.}}{\leqq} \\ &\overset{\text{a.s.}}{\leqq} (1+\varepsilon)^{1/4}2^{1/4}\sigma^{3/2}\mu^{-1/2}(T\log\log T)^{1/4}(\log T)^{1/2}. \end{aligned}$$

We show that the second term in (2.10) is almost surely less than the rate of increments in (2.12) and we prove that this term is again $o(r(T))$. It follows from the definition of $N(t)$ that $\sup_{0 \leqq t \leqq T} |\mu t - U(N(t\mu))|$ equals the largest jump of $\mu t - N(t)$ on $[0, U(N(T\mu))]$. Using again (2.8), (2.6) and Theorem 1.2.1 of [8] we obtain

$$\sup_{0 \leqq t \leqq T} |\mu t - U(N(t\mu))| \overset{\text{a.s.}}{\leqq} \sup_{0 \leqq t \leqq (1+\varepsilon)T} |U(t) - \mu t - W^{(1)}(t)| +$$

$$+ \sup_{0 \leqq t \leqq (1+\varepsilon)T} \sup_{0 \leqq s \leqq 1} |W^{(1)}(t+s) - W^{(1)}(t)| \overset{\text{a.s.}}{=} o(r(T)) + O((\log T)^{1/2}).$$

Hence we proved the theorem if (2.3) is satisfied and obtained in the case of (2.4) that

$$\text{(2.13)} \quad \limsup_{T \to \infty} (T \log \log T)^{-1/4} (\log T)^{-1/2} \sup_{0 \leqq t \leqq T} \left| t - N(t\mu) - \frac{1}{\mu} W^{(1)}(t) \right| \overset{\text{a.s.}}{=}$$

$$\overset{\text{a.s.}}{=} \limsup_{T \to \infty} (T \log \log T)^{-1/4} (\log T)^{-1/2} \sup_{0 \leqq t \leqq T} \frac{1}{\mu} \left| W^{(1)}(N(t\mu)) - W^{(1)}(t) \right| \leqq$$

$$\leqq 2^{1/4} \sigma^{3/2} \mu^{-3/2} \quad \text{a.s.}$$

In order to finish the proof of the second part of the theorem it is enough to show the opposite of the inequality (2.13). The approximation in (2.11) implies that the limit points of the processes $\{(2\sigma^2 \mu^{-2} T \log \log T)^{-1/2} (N(Tt\mu) - Tt), 0 \leqq t \leqq 1\}$ and $\{(2\sigma^2 T \log \log T)^{-1/2} W^{(1)}(Tt), 0 \leqq t \leqq 1\}$ are the same as $T \to \infty$, the so-called Strassen set $\mathcal{S}$. This is the set of absolutely continuous functions f (with respect to the Lebesgue measure) such that

$$f(0) = 0 \quad \text{and} \quad \int_0^1 (f'(t))^2 \, dt \leqq 1.$$

The function

$$h_\delta(t) = \begin{cases} t, & 0 \leqq t \leqq 1-\delta \\ 1-\delta, & 1-\delta \leqq t \leqq 1, \end{cases}$$

is an element of $\mathcal{S}$ for each $0 < \delta < 1$, and therefore there is a sequence of random variables $n_k = n_k(\omega)$ such that

$$\lim_{k \to \infty} \sup_{0 \leqq t \leqq 1} \left| (2\sigma^2 \mu^{-2} n_k \log \log n_k)^{-1/2} (N(n_k t\mu) - n_k t) - h_\delta(t) \right| = 0 \quad \text{a.s.}$$

Using a modification of the proof of Theorem 1.2.1 (iii) in [8] we get

$$\lim_{T \to \infty} (T \log \log T)^{-1/4} (\log T)^{-1/2} \sup_{1-\delta \leqq t \leqq 1} \left| W^{(1)}(Tt + h_\delta(t)(2\sigma^2 \mu^{-2} T \log \log T)^{-1/2}) - \right.$$

$$\left. - W^{(1)}(Tt) \right| = (1-\delta)^{1/2} 2^{1/4} \sigma^{3/2} \mu^{-1/2} \quad \text{a.s.}$$

We obtained that

$$\limsup_{T \to \infty} (T \log \log T)^{-1/4} (\log T)^{-1/2} \sup_{0 \leqq t \leqq T} \frac{1}{\mu} \left| W^{(1)}(N(t\mu)) - W^{(1)}(t) \right| \geqq$$

$$\geqq 2^{1/4} \sigma^{3/2} \mu^{-3/2} \quad \text{a.s.,}$$

which implies the opposite of the inequality on the right side of (2.13) and the proof of Theorem 2.1 is complete.

Condition A guarantees that the Strassen-type law of the iterated logarithm holds for the partial sums. When $\mu=0$ we have to use an other type of the law of the iterated logarithm for partial sums and we have to assume a stronger condition on the rate of approximation in (2.1).

CONDITION B. *We assume that* (2.1) *is satisfied with rate*

$$r(T) = o\big(T^{1/2}(\log T)^{-2}\big) \tag{2.14}$$

and $r(T)$ *is nondecreasing and regularly varying at infinity.*

Introduce the following processes:

$$U^*(t) = \sup_{0 \leqq s \leqq t} U(s)$$

and

$$L(t) = \inf\{x\colon W^{(1)}(x) > t\}$$

$$= \inf\big\{x\colon \sup_{0 \leqq s \leqq x} W^{(1)}(s) > t\big\}, \quad 0 \leqq t < \infty.$$

The process $L(\sigma t)$ is well-known in the stochastic literature as the first passage time for the standard Wiener process (cf. Itô and McKean [16], Chapter 1.7).

THEOREM 2.2. *If* $\mu=0$ *and* $\sigma>0$ *then Condition* B *implies that there is an almost surely finite random variable* $t_0=t_0(\omega)$ *such that*

$$L\big(t-r(t^2(\log t)^3)\big) \leqq N(t) \leqq L\big(t+r(t^2(\log t)^3)\big),$$

if $t \geqq t_0$.

PROOF. By the theorem of Hirsch [12] and (2.14) we get that U^* and the supremum of $W^{(1)}$ have the same lower and upper classes of functions, for example

$$U^*(T) \overset{\text{a.s.}}{\geqq} T^{1/2}(\log T)^{-1}(\log\log T)^{-2}. \tag{2.15}$$

Using (2.15), (2.6) and Condition B we obtain

$$N(t) = \inf\{x\colon U^*(x) > t\} = \inf\{x\colon 0 \leqq x \leqq t^2(\log t)^3 \text{ and } U^*(x) > t\} \leqq$$

$$\leqq \inf\big\{x\colon 0 \leqq x \leqq t^2(\log t)^3 \text{ and } W^{(1)}(x) > t+r\big(t^2(\log t)^3\big)\big\} = L\big(t+r(t^2(\log t)^3)\big)$$

for almost all ω and all large enough t depending on ω. The second part of the inequality follows in a similar way.

Theorem 2.2 immediately implies strong laws for $N(t)$. Let $\mathfrak{B}$ be the set of all continuous, non-decreasing, real valued functions, l, defined on $[1, \infty)$ with $l(1)>0$.

COROLLARY 2.1. *We assume that* $\mu=0$, $\sigma>0$ *and Condition* B *is satisfied. Then*

$$\liminf_{t\to\infty} t^{-2}(\log\log t)N(t) = \frac{1}{2}\sigma^{-2} \quad a.s.$$

and if $l \in \mathfrak{B}$ *then*

$$P\{N(\sigma t) > t^2 l(t) \ i.o. \text{ as } t \to \infty\}$$

equals 1 *or* 0 *according as* $\int\limits_1^\infty t^{-1}(l(t))^{-2}\,dt$ *equals, or is less than* ∞

$$P\{N(\sigma t) < t^2(l(t))^{-1/2} \text{ i.o. as } t\to\infty\} =$$

$$= \begin{cases} 1, & \text{if } \int\limits_0^\infty (l(t))^{-1/2}t^{-1}\exp\left(-\frac{1}{2}\frac{1}{l(t)}\right)dt<\infty \\ 0, & \text{if } \int\limits_0^\infty (l(t))^{-1/2}t^{-1}\exp\left(-\frac{1}{2}\frac{1}{l(t)}\right)dt=\infty \end{cases}$$

and

$$1 \leqq \liminf_{t\to\infty}(N(t))^{1/\log\log t} \leqq \limsup_{t\to\infty}(N(t))^{1/\log\log t} \leqq e^2 \quad a.s.$$

PROOF. By (2.14) $\lim\limits_{t\to\infty} t^{-1}r(t^2(\log t)^3)=0$ and therefore in the light of Theorem 2.2 it is enough to determine the corresponding strong laws for $L(t)$. On the other hand, $L(\sigma t)$ is equal in distribution to the first passage time of a standard Wiener process, therefore we can use the classical theorems of Khintchine [20], Breiman [5] and Mijnheer [23] (cf. Theorems 8.2.1, 4.2.1 and 8.1.2 in [23], respectively).

Our processes $N(t)$ and $L(t)$ are continuous from the right and have finite limits from the left so it is very natural to use the Skorohod metric $\varrho_{[0,T]}$ on $\mathscr{D}[0, T]$ in our case.

COROLLARY 2.2.

$$\varrho_{[0,T]}(T^{-2}N(t), T^{-2}L(t))$$

goes to zero in probability as $T\to\infty$.

PROOF. Let $\varepsilon>0$. By Theorem 2.2 there exists a $T_0=T_0(\varepsilon)$ such that

$$P\{L(t-r(T^2(\log T)^3)) \leqq N(t) \leqq L(t+r(T^2(\log T)^3)),\ T_0\leqq t\leqq T\} > 1-\varepsilon.$$

It is easy to see that

$$T^{-2}\sup_{0\leqq t\leqq \tau} N(t)\to 0 \text{ a.s.} \quad\text{and}\quad T^{-2}\sup_{0\leqq t\leqq\tau} L(t)\to 0 \quad \text{a.s.}$$

as $T\to\infty$ for each $\tau>0$, so it is enough to show that

$$\varrho_{[0,T]}(T^{-2}L(t), T^{-2}L(t-r(T^2(\log T)^3))) \overset{\mathscr{D}}{=} \varrho_{[0,1]}(L(t), L(t-T^{-1}r(T^2(\log T)^3)))$$

and

$$\varrho_{[0,T]}(T^{-2}L(t), T^{-2}L(t+r(T^2(\log T)^3))) \overset{\mathscr{D}}{=} \varrho_{[0,1]}(L(t), L(t+T^{-1}r(T^2(\log T)^3)))$$

$(L(t)=0,\ t\leqq 0)$ go to zero in probability when $T\to\infty$. Let $\mathscr{K}$ denote the following function:

$$\mathscr{K}(t) = \begin{cases} \frac{1}{2}t & \text{if } 0\leqq t\leqq 2T^{-1}r(T^2(\log T)^3), \\ t-T^{-1}r(T^2(\log T)^3) & \text{if } 2T^{-1}r(T^2(\log T)^3)<t<1-T^{-1}, \\ (1+r(T^2(\log T)^3))\,t-r(T^2(\log T)^3) & \text{if } 1-T^{-1}\leqq t\leqq 1. \end{cases}$$

We obtain

$$\varrho_{[0,1]}\big(L(t), L(t-T^{-1}r(T^2(\log T)^3))\big) \leqq \sup_{0\leqq t\leqq 1} |\mathscr{K}(t)-t|+$$

$$+\sup_{0\leqq t\leqq 1}\big|L(\mathscr{K}(t))-L\big(t-T^{-1}r(T^2(\log T)^3)\big)\big| \leqq T^{-1}r\big(T^2(\log T)^3\big)+$$

$$+2L\big(2T^{-1}r(T^2(\log T)^3)\big)+\sup_{1-1/T\leqq t\leqq 1}\big(L((1+r(T^2(\log T)^3)\,t-r(T^2(\log T)^3))-$$

$$-L(t-T^{-1}r(T^2(\log T)^3))\big) \leqq$$

$$\leqq T^{-1}r\big(T^2(\log T)^3\big)+2L\big(2T^{-1}r(T^2(\log T)^3)\big)+L(1)-$$

$$-L\big(1-T^{-1}-T^{-1}r(T^2(\log T)^3)\big)$$

for L is a nondecreasing process. $L(t)$ has stationary increments so we get

$$P\big\{L(1)-L\big(1-T^{-1}-T^{-1}r(T^2(\log T)^3)\big) > C\sigma^2\big(T^{-1}+T^{-1}r(T^2(\log T)^3)\big)^2\big\} \leqq$$

$$\leqq \left(\frac{2}{\pi}\right)^{1/2} C^{-1/2}$$

for every $C>0$. Condition (2.14) implies that $T^{-1}r\big(T^2(\log T)^3\big)\to 0$, $T\to\infty$, so we proved that $\varrho_{[0,1]}\big(L(t), L(t-T^{-1}r(T^2(\log T)^3))\big)$ goes to zero in probability, because $L(0)=0$ and L is a.s. continuous at zero. In a similar way we can check that $\varrho_{[0,1]}\big(L(t), L(t+T^{-1}r(T^2(\log T)^3))\big)$ also goes to zero in probability.

The weak convergence of $\{T^{-2}N(Tt),\ 0\leqq t\leqq 1\}$ in the usual Skorohod space $\mathscr{D}[0,1]$ follows from Corollary 2.2. Kennedy [19] obtained estimates of the rate of convergence in limit theorem for $T^{-2}N(T)$. Siegmund and Yuh [27] proved a one-term Edgeworth expansion for certain first passage distributions for random walks.

The focus of the following theorem is the vector-valued process

$$M(t) = S\big(N(t)\big)-m\mu^{-1}t,\quad 0\leqq t<\infty,\quad \mu>0.$$

Gut and Janson [10] proved the weak convergence of $T^{-1/2}M(Tt)$ to a Wiener process in the case $d=1$. Borovkov [4] obtained lower and upper bounds for the Lévy—Prohorov distance between $M(t)$ and its limiting process for a general $d\geqq 1$. Assuming that $\{(X_i, Y_i),\ i\geqq 1\}$ are i.i.d.r. vectors, Horváth [14] studied the rate of strong approximation of $M(t)$. The following theorem states that Condition A always implies a strong approximation for the stopped sums $M(t)$ as well. Let $\{G(t),\ t\geqq 0\}$ be a d-dimensional Gaussian process defined by

$$\text{(2.16)}\qquad \begin{cases} G(t) = \big(G^{(1)}(t), \ldots, G^{(d)}(t)\big),\\ G^{(i)}(t) = W^{(i+1)}(t)-m_i\mu^{-1}W^{(1)}(t), \quad 1\leqq i\leqq d,\ m=(m_1. \ldots, m_d). \end{cases}$$

An easy computation shows that G has the covariance matrix $\Gamma^*=\{\gamma^*_{i,j}\}$, where

$$\gamma^*_{i,j} = \gamma_{i+1,j+1}-\mu^{-1}(\gamma_{i+1,1}m_j+\gamma_{j+1,1}m_i)+m_im_j\mu^{-2}\gamma_{1,1},\quad 1\leqq i,j\leqq d.$$

THEOREM 2.3. *If* $\mu>0$ *then Condition* A *implies*

$$\sup_{0\leqq t\leqq T}\|M(t)-G(\mu^{-1}t)\| \overset{\text{a.s.}}{=} o\big(r(T)\big),$$

if

$$(T\log\log T)^{1/4}(\log T)^{1/2}=o(r(T)) \tag{2.17}$$

and

$$2^{1/4}\sigma\mu^{-3/4}\max_{1\le i\le d}(|\gamma_{i+1,i+1}^{1/2}-\sigma m_i\mu^{-1}|)\le\limsup_{T\to\infty}(T\log\log T)^{-1/4}(\log T)^{-1/2}\times$$
$$\times\sup_{0\le t\le T}\|M(t)-G(\mu^{-1}t)\|_d\le 2^{1/4}\sigma\mu^{-3/4}\max_{1\le i\le d}(\gamma_{i+1,i+1}^{1/2}+\sigma m_i\mu^{-1})\quad a.s.,$$

if

$$r(T)=O((T\log\log T)^{1/4}(\log T)^{1/2}). \tag{2.18}$$

PROOF. We use the following decomposition of the i-th component of the processes:

$$M^{(i)}(t)-G^{(i)}(\mu^{-1}t)=S^{(i)}(N(t))-m_iN(t)-W^{(i+1)}(N(t))+W^{(i+1)}(N(t))-$$
$$-W^{(i+1)}(\mu^{-1}t)+m_i(\mu^{-1}W^{(1)}(\mu^{-1}t)-(\mu^{-1}t-N(t)))=A_1^{(i)}(t)+A_2^{(i)}(t)+A_3^{(i)}(t).$$

First we note that Theorem 2.1 and the law of the iterated logarithm for the Wiener process imply that

$$\limsup_{T\to\infty}(T\log\log T)^{-1/2}\sup_{0\le t\le T}|N(t)-\mu^{-1}t|=2^{1/2}\sigma\mu^{-3/2}\quad\text{a.s.} \tag{2.19}$$

Using (2.19), (2.6) and (2.1) we get

$$\sup_{0\le t\le T}|A_1^{(i)}(t)|\overset{\text{a.s.}}{=\!=}o(r(T)).$$

We estimate the increments of the Wiener process by the help of Theorem 1.2.1 in [8] and get

(2.20)

$$\limsup_{T\to\infty}(T\log\log T)^{-1/4}(\log T)^{-1/2}\sup_{0\le t\le T}|A_2^{(i)}(t)|\le 2^{1/4}\gamma_{i+1,i+1}^{1/2}\sigma^{1/2}\mu^{-3/4}\quad\text{a.s.}$$

In the same way as we proved the opposite of the inequality in (2.13) in the proof of Theorem 2.1 we can prove the opposite of (2.20) and get

$$\limsup_{T\to\infty}(T\log\log T)^{-1/4}(\log T)^{-1/2}\sup_{0\le t\le T}|A_2^{(i)}(t)|=2^{1/4}\gamma_{i+1,i+1}^{1/2}\sigma\mu^{-3/4}\quad\text{a.s.}$$

We have already estimated $A_3^{(i)}(t)$ in Theorem 2.1 and, putting together the obtained bounds, the theorem is proved.

3. Applications

EXAMPLE 1. "Collective risk theory". Collective risk theory is concerned with the random fluctuations of the total assets, the risk reserve, of an insurance company. The policyholders pay premiums regularly and at certain random times make claims to the company. We shall assume that the initial risk reserve of the company is $R_0>0$ and that the policyholders pay premium of a per unit time. Let X, $\{X_i, i\ge 1\}$ be a sequence of i.i.d. positive r.v's with $EX=\mu>0$, $0<\sigma^2=E(X-\mu)^2<\infty$. The random variable X_i will represent the time between the $(i-1)$th claim and the ith claim. The number of claims in time t is $N(t)-1$, $0 \leqq t < \infty$. When the ith claim occurs the company pays the policyholder a positive amount Y_i. We assume that Y, $\{Y_i, i \geqq 1\}$ are i.i.d. r.v's, $m=EY$, and $\gamma^2=E(Y-m)^2$, and the sequence $\{Y_i\}$ is also independent of the sequence $\{X_i\}$. The risk reserve at time t is $R(t)=R_0+ +at-S(N(t)-1)$.

If we also assume that $E|X|^p<\infty$ and $E|Y|^p<\infty$ for some $p>2$ then it follows form the Komlós—Major—Tusnády theorem (cf. Theorem 2.6.3 in [8]) that Condition A is satisfied with rate $T^{1/p}$ and $W^{(1)}$ and $W^{(2)}$ are independent. Using Theorem 2.3 we can approximate $R(t)$ with the process $D(t)=R_0+ +(a-\mu^{-1}m)t-G(\mu^{-1}t)$. If $2<p<4$ then

$$\sup_{0 \leqq t \leqq T} |R(t)-D(t)| \overset{\text{a.s.}}{=} o(T^{1/p}) \tag{3.1}$$

and if $p \geqq 4$ then

$$\sup_{0 \leqq t \leqq T} |R(t)-D(t)| \overset{\text{a.s.}}{=} O\big((T \log\log T)^{1/4} (\log T)^{1/2}\big). \tag{3.2}$$

If we calculate the covariance function of $G(\mu^{-1}t)$ we find a representation for it in distribution by means of a standard Wiener process $\{\hat{W}(t), t \geqq 0\}$. This is

$$\{G(\mu^{-1}t),\ t \geqq 0\} \overset{\mathcal{D}}{=} \{(\gamma^2\mu^{-1}+m^2\mu^{-3}\sigma^2)^{1/2}\, \hat{W}(t),\ t \geqq 0\}.$$

This representation together with (3.1) and (3.2) not only gives an improved version of Theorem 6 of Iglehart [15] but gives a rate of the approximation of the risk reserve process.

Naturally, if we assume only that (X, Y), $\{(X_i, Y_i), i \geqq 1\}$ is a sequence of i.i.d.r. vectors, $EX=\mu>0$, $E|X|^p<\infty$, $E|Y|^p<\infty$ for some $2<p\leqq 3$ then Theorem 3 of Berkes and Philipp [3] implies that Condition A holds with rate T^ϱ, $\varrho>\frac{1}{2}-\frac{p-2}{160}$. An immediate consequence of Theorem 2.3 is that

$$\sup_{0 \leqq t \leqq T} |R(t)-D(t)| \overset{\text{a.s.}}{=} o(T^\varrho),$$

$\varrho > 1/2-(p-2)/160$ and

$$\{G(\mu^{-1}t),\ t \geqq 0\} \overset{\mathcal{D}}{=} \{(\gamma^2\mu^{-1}-2\mu^{-2}m\gamma_{1,2}+m^2\mu^{-3}\sigma^2)^{1/2}\, \hat{W}(t),\ t \geqq 0\},$$

where $\gamma_{1,2}=E(X-\mu)(Y-m)$ and $\hat{W}$ is a standard Wiener process.

EXAMPLE 2. (Example D in [10].) Let X, $\{X_i, i \geqq 1\}$ be a sequence of i.i.d.r.v's, $EX=\mu>0$. How large is the sum of the squares when the sum of X_i first reaches the level t? In this example $Y=X^2$, $Y_i=X_i^2$, $i \geqq 1$, $m=EY=EX^2$. First we have to introduce some notations. Let $\mathcal{F}(x)$ denote the distribution function of X and let $\mathcal{F}_t(x)$ be the empirical distribution function of $X_1, \ldots, X_{[t]}$ defined by

$$\mathcal{F}_t(x) = \frac{1}{[t]} \#\{1 \leqq i \leqq [t]\colon X_i < x\}, \quad t \geqq 1.$$

The partial sums $U(t)$ and $S(t)$ can be written in the form

$$U(t)-\mu[t] = [t] \int_{-\infty}^{\infty} x d(\mathscr{F}_t(x)-\mathscr{F}(x))$$

and

$$S(t)-m[t] = [t] \int_{-\infty}^{\infty} x^2 d(\mathscr{F}_t(x)-\mathscr{F}(x)). \tag{3.3}$$

Komlós, Major and Tusnády proved (cf. Theorem 4.4.3 in [8]) that we can define a two-parameter Gaussian process $\{K_t(x),\ t\geqq 0,\ -\infty<x<\infty\}$ such that

$$\sup_{0\leqq t\leqq T} \sup_{-\infty<x<\infty} \left|[t](\mathscr{F}_t(x)-\mathscr{F}(x))-K_t(x)\right| \overset{\text{a.s.}}{=} O((\log T)^2) \tag{3.4}$$

and

$$EK_t(x) = 0,\quad EK_t(x)K_s(y) = (t\wedge s)(\mathscr{F}(x\wedge y)-\mathscr{F}(x)\mathscr{F}(y)).$$

First we show that if $E|X|^p<\infty$ with $p>4$, then

$$\sup_{0\leqq t\leqq T} \left|U(t)-\mu t-\int_{-\infty}^{\infty} x dK_t(x)\right| \overset{\text{a.s.}}{=} O(T^{\varrho}) \tag{3.5}$$

and

$$\sup_{0\leqq t\leqq T} \left|S(t)-mt-\int_{-\infty}^{\infty} x^2 dK_t(x)\right| \overset{\text{a.s.}}{=} O(T^{\varrho}) \tag{3.6}$$

for some $\varrho>\frac{2}{p}$. We prove only (3.6), the proof of (3.5) is the same. Let $x_T^{(1)}=-T^{1/p+\delta}$ and $X_T^{(2)}=T^{1/p+\delta}$ with some $\delta>0$. Using representation (3.3) we obtain

$$\sup_{0\leqq t\leqq T}\left|U(t)-\mu t-\int_{-\infty}^{\infty} x^2\,dK_t(x)\right| \leqq \sup_{0\leqq t\leqq T}[t]\int_{-\infty}^{x_T^{(1)}} |\mathscr{F}_t(x)-\mathscr{F}(x)|\,dx^2+$$

$$+\sup_{0\leqq t\leqq T}[t]\int_{x_T^{(2)}}^{\infty} |\mathscr{F}_t(x)-\mathscr{F}(x)|\,dx^2+2T^{2/p+\delta}\sup_{0\leqq t\leqq T}\sup_{-\infty<x<\infty}\big|[t](\mathscr{F}_t(x)-$$

$$-\mathscr{F}(x))-K_t(x)\big|+\sup_{0\leqq t\leqq T}\int_{-\infty}^{x_T^{(1)}}|K_t(x)|\,dx^2+\sup_{0\leqq t\leqq T}\int_{x_T^{(2)}}^{\infty}|K_t(x)|\,dx^2 =$$

$$= A_1(T)+\ldots+A_5(T).$$

By our moment condition

$$\lim_{x\to-\infty}|x|^p\mathscr{F}(x)=0,\quad \lim_{x\to\infty}|x|^p(1-\mathscr{F}(x))=0,$$

and therefore we get by the help of Theorem of James [16] that

$$A_1(T)\overset{\text{a.s.}}{\leqq}(T\log\log T)^{1/2}\int_{-\infty}^{x_T^{(1)}}(\mathscr{F}(x))^{1/2-\delta}\,dx^2\overset{\text{a.s.}}{\leqq}$$

$$\overset{\text{a.s.}}{\leqq} T^{(1/p+\delta)(-p/2+2+p\delta)}(T\log\log T)^{1/2}\overset{\text{a.s.}}{=}O(T^{2/p+4\delta+p\delta^2}).$$

Replacing James' law by Corollary 1.15.2 in [8], we get

$$A_4(T) \overset{\text{a.s.}}{=} O(T^{2/p+4\delta+p\delta^2}).$$

A similar argument shows that

$$A_2(T) \overset{\text{a.s.}}{=} O(T^{2/p+4\delta+p\delta^2})$$

and

$$A_5(T) \overset{\text{a.s.}}{=} O(T^{2/p+4\delta+p\delta^2}).$$

An estimation of $A_3(T)$ follows from (3.4):

$$A_3(T) \overset{\text{a.s.}}{=} O(T^{2/p+2\delta}).$$

These estimations give the proof of (3.6), since $\delta>0$ can be taken as close to zero as we wish. The process defined by (2.13) has the form

$$G(t) = \int_{-\infty}^{\infty} x^2\, dK_t(x) - \mu^{-1} m \int_{-\infty}^{\infty} x\, dK_t(x)$$

and we can easily prove that the following distributional representation holds:

$$\{G(\mu^{-1}t),\ t \geqq 0\} \overset{\mathcal{D}}{=} \{\tau^{1/2}\hat{W}(t),\ t \geqq 0\},$$

where $\hat{W}$ denotes a standard Wiener process and

$$\tau = \frac{1}{\mu}\{EX^4 - m^2 - 2\mu^{-1}m(EX^3 - m\mu) + \mu^{-2}m^2\sigma^2\},$$

$$\mu = EX, \quad m = EX^2, \quad \sigma^2 = E(X-\mu)^2 = m-\mu^2.$$

Theorem 2.3 implies that in this case

$$\sup_{0 \leqq t \leqq T} |M(t) - G(\mu^{-1}t)| \overset{\text{a.s.}}{=} o(T^{\varrho}),$$

$\varrho > \dfrac{2}{p}$, if $E|X|^p < \infty$ with $4 < p \leqq 8$ and

$$\sup_{0 \leqq t \leqq T} |M(t) - G(\mu^{-1}t)| \overset{\text{a.s.}}{=} O\big((T \log\log T)^{1/4} (\log T)^{1/2}\big),$$

if $E|X|^p < \infty$, with $p > 8$.

The method of proof in Example 2 always works if Y is a function of X, i.e. $\{(X_i, Y_i),\ i \geqq 1\} = \{(X_i, g(X_i)),\ i \geqq 1\}$ with some function g. This kind of connection between X and Y is very usual in renewal and reliability theory and in replacement policies (cf. [9], [8], Chapters 3 and 4.2 in [1]).

EXAMPLE 3. "Renewal process based on m-dependent r.v's." Let X, $\{X_i, i \geqq 1\}$ be a stationary m-dependent sequence of r.v's. The nonnegative integer m will be fixed. The renewal process $N(t)$ based on m-dependent r.v's was studied by Janson [18]. Janson proved that if $EX = \mu > 0$ then $t^{-1}N(t)$ goes to μ^{-1} a.s. and if

$EX^2<\infty$ then $t^{-1/2}(N(t)-t\mu^{-1})$ has a normal limit distribution as $t\to\infty$. We show that the result of [13] can be extended to m-dependent r.v's.

We assume that $EX^2<\infty$ and let $\mu=EX$. Heyde and Scott [11] proved that Condition A is satisfied with the rate $(T\log\log T)^{1/2}$ and the variance of $W^{(1)}$ is

$$\sigma^2 = \operatorname{var} X_1 + 2\sum_{i=1}^{m-1} \operatorname{cov}(X_1, X_{i+1}).$$

Theorem 2.1 then gives

$$\sup_{0\le t\le T} |\mu^{-1}t - N(t) - \mu^{-1}W^{(1)}(\mu^{-1}t)| \overset{\text{a.s.}}{=} o\big((T\log\log T)^{1/2}\big),$$

which immediately implies the law of iterated logarithm:

$$\limsup_{T\to\infty} (T\log\log T)^{-1/2} \sup_{0\le t\le T} |\mu^{-1}t - N(t)| = 2^{1/2}\sigma\mu^{-3/2} \text{ a.s.}$$

If we assume that $E|X|^p<\infty$, for some $p>2$, then Theorem 4.1 of Philipp and Stout [24] says that Conditions A and B hold with rate T^ϱ, $\varrho>\frac{5}{12}+\frac{1}{6p}$ and therefore we get from Theorems 2.1 and 2.2 that

$$\sup_{0\le t\le T} |\mu^{-1}t - N(t) - \mu^{-1}W^{(1)}(\mu^{-1}t)| \overset{\text{a.s.}}{=} o(T^\varrho),$$

if $EX=\mu>0$ and the strong laws in Corollary 2.1 hold for $N(t)$, we have the weak convergence of $N(t)$ in Corollary 2.2 when $\mu=0$.

EXAMPLE 4. "Processes of runs." Let $\{\xi_i\}$ be a sequence of i.i.d.r.v's with distribution function $\mathscr{F}$. We consider only two types of runs down. We say that $\xi_k, \xi_{k+1}, \dots, \xi_{k+p}$ is a run down of length p or more, if $\xi_k>\xi_{k+1}>\dots>\xi_{k+p}$. The random sequence $\xi_k, \xi_{k+1}, \dots, \xi_{k+q}$ is a run down of length q if $\xi_{k-1}\le\xi_k$, $\xi_k>>\xi_{k+1}>\dots>\xi_{k+q}$, $\xi_{k+q}\le\xi_{k+q+1}$. The number of runs down of length p or more in the sequence $\{\xi_1, \dots, \xi_{n+p}\}$ will be denoted by $U(n)$. It is easy to see that $U(n)$ is the number of $\xi_1, \dots, \xi_n$ which are initial points of a run down of length p or more. We can define $S(n)$, the number of runs down of length q in $\xi_1, \dots, \xi_n$ in a similar way. First we introduce some notations:

$$\mu = \mu(p) = \lim_{n\to\infty} \frac{1}{n} EU(n), \tag{3.7}$$

$$\sigma^2 = \sigma^2(p) = \lim_{n\to\infty} E\left(\frac{1}{n}U(n)-\mu\right)^2 \tag{3.8}$$

and

$$m = m(q) = \lim_{n\to\infty} \frac{1}{n} ES(n) \tag{3.9}$$

$$\gamma = \lim_{n\to\infty} E\left(\frac{1}{n}S(n)-m\right)^2, \tag{3.10}$$

$$\gamma_{1,2} = \lim_{n\to\infty} E\left(\frac{1}{n}S(n)-m\right)\left(\frac{1}{n}U(n)-\mu\right). \tag{3.11}$$

Whenever $\mathcal{F}$ is continuous, the limiting distributions of the normalized $U(n)$ and $S(n)$ do not depend on $\mathcal{F}$. Assuming the continuity of $\mathcal{F}$, Levene and Wolfowitz [22] determined the limiting expectation and variance of U and S. They obtained that

$$\mu(p)=\frac{1}{(p+1)!},\quad m(q)=\frac{q^3+3q+1}{(q+3)!}$$

$$\sigma^2(p)=\frac{1}{(p+1)!}-\frac{2p+1}{((p+1)!)^2}+2\sum_{k=p+2}^{2p+1}\frac{1}{k!}$$

and

$$\gamma(q)=\frac{1}{q!(q+3)!}(q^3+3q+1)(q^3+2q^2+2q-4).$$

Wolfowitz [28] proved that the random variables $n^{-1/2}(U(n)-\mu n)$ and $n^{-1/2}(S(n)-mn)$ have a two-dimensional normal limit distribution as $n\to\infty$.

In addition to the number of runs, the initial points of runs are also investigated in the stochastic literature. Let $N(n)$ denote the initial point of the nth run down of length p or more. The random variable $S(N(n))$ is the number of initial points of runs down of length q in the shortest sequence which contains exactly n initial points of runs down of length p or more. Pittel [25] proved in the case $p=1$ that the finite-dimensional distributions of the sequence $\left(\frac{2}{3}n\right)^{-1/2}(2n-N(n))$ converge to the corresponding finite-dimensional distributions of the Wiener process. Révész [26] proved a strong invariance principle for $\left(\frac{3}{2}\right)^{1/2}(N(n)-2n)$. We show that these results follow from Theorems 2.1 and 2.3.

Introduce the sequences $\{(X_i, Y_i),\ i\geqq 1\}$:

$$X_i=I\{\xi_i>\xi_{i+1}>\ldots>\xi_{i+p}\},\ i\geqq 1,\quad Y_1=I\{\xi_1>\xi_2>\ldots>\xi_{q+1},\ \xi_{q+1}\leqq\xi_{q+2}\}$$

and

$$Y_i=I\{\xi_{i-1}\leqq\xi_i,\ \xi_i>\xi_{i+1}>\ldots>\xi_{i+q},\ \xi_{i+q}\leqq\xi_{i+q+1}\},\quad i\geqq 2,$$

where $I\{A\}$ denotes the indicator of the event A. It is easy to see that $\mathcal{U}(t)=\sum_{i=1}^{[t]}X_i$, $S(t)=\sum_{i=1}^{[t]}Y_i$ and $N(t)=N([t])$ is the renewal process based on the sequence $\{X_i,\ i\geqq 1\}$. Theorem 4 of Kuelbs and Philipp [21] implies that Condition A holds with rate T^ϱ, $1/4<\varrho<1/2$, and we get the following result:

$$\sup_{0\leqq t\leqq T}|(\mu^{-1}t-N(t))-\mu^{-1}W^{(1)}(\mu^{-1}t)|\overset{\text{a.s.}}{=}o(T^\varrho),\quad 1/4<\varrho<1/2$$

and

$$\sup_{0\leqq t\leqq T}\left|S(N(t))-m\mu^{-1}t-G(\mu^{-1}t)\right|\overset{\text{a.s.}}{=}o(T^\varrho),\quad 1/4<\varrho<1/2,$$

where G is defined by (2.13). Computing the covariance of the limiting processes we obtain representations of the Gaussian processes:

$$\{\mu^{-1}W^{(1)}(\mu^{-1}t),\ t\geqq 0\}\overset{\mathcal{D}}{=}\{\sigma\mu^{-3/2}\hat{W}(t),\ t\geqq 0\}$$

and

$$\{G(\mu^{-1}t),\ t \geqq 0\} \overset{\mathcal{D}}{=} \{(\hat{\gamma})^{1/2}\hat{W}(t),\ t \geqq 0\},$$

where $\hat{W}$ is a standard Wiener process,

$$\hat{\gamma} = \mu^{-1}\{\gamma - 2\mu^{-1}\gamma_{1,2} + m^2\mu^{-2}\sigma^2\}$$

and μ, γ, $\gamma_{1,2}$, m, σ are defined by (3.7)—(3.11).

We considered only processes of runs down but with the same method we can develop strong approximations of processes of other types of runs (runs up, runs down or up, and turning points).

EXAMPLE 5. "First passage times of lacunary trigonometric series." Let $\{n_k, k \geqq 1\}$ be a lacunary sequence of positive real numbers (not necessarily integers), that is, a sequence satisfying

$$\frac{n_{k+1}}{n_k} \geqq q, \quad k \geqq 1, \tag{3.12}$$

for some $q>1$. We consider pure cosine series of the form

$$U(t) = \sum_{k=1}^{[t]} X_k, \quad X_k = 2^{1/2}\cos(2\pi n_k \xi),$$

where ξ is a r.v. uniformly distributed on $(0, 1)$. The renewal process $N(t)$ is the first passage time of the lacunary trigonometric series $U(t)$. Theorem 3.1 of Philipp and Stout [24] says that Condition B is satisfied with rate T^{ϱ}, $\varrho>5/12$ and $W^{(1)}$ is a standard Wiener process. Instead of (3.12) Berkes [2] assumed the weaker condition

$$\frac{n_{k+1}}{n_k} \geqq 1+k^{-\alpha}, \quad k \geqq 1, \tag{3.13}$$

with some $\alpha<1/2$ and he proved that Condition B holds with $r(T)=T^{\varrho}$, $\varrho=\varrho(\alpha)<$ $<1/2$ and $\sigma=1$. Thus, if (3.12) or (3.13) is satisfied, then Theorem 2.2 and Corollaries 2.1 and 2.2 hold for the first passage time of the lacunary trigonometric series.

EXAMPLE 6. "The zeros of a random walk." Let ξ, $\{\xi_k, k\geqq 1\}$ be a sequence of i.i.d.r.v's taking on integer values with $P(\xi=k)=p_k$ $(k=\pm 1, \pm 2, \ldots)$. We assume that

$$E\xi = 0, \quad E\xi^2 = \sigma^2 < \infty \quad \text{and g.c.d. } \{k\colon p_k > 0\} = 1. \tag{3.14}$$

We define the following sequence of r.v's:

$$X_k = \begin{cases} 1, & \text{if } \sum_{i=1}^{k} \xi_i = 0 \\ 0, & \text{if } \sum_{i=1}^{k} \xi_i \neq 0. \end{cases}$$

The occupation time of the recurrent random walk is defined by

$$U(t) = \#\Big\{k\colon 0 < k \leqq [t],\ \sum_{i=1}^{k} \xi_i = 0\Big\} = \sum_{i=1}^{[t]} X_i = U^*(t),$$

because X_i, $i>1$ are nonnegative r.v's. Csáki and Révész [7] proved that if $E|\zeta|^m < <\infty$ for some $3<m<4$, then there exists a Wiener process $W^{(1)}(t)$, $EW^{(1)}(t)=0$ and $EW^{(1)}(t)\,W^{(1)}(s)=\frac{1}{\sigma^2}\min(t,s)$ such that

$$\sup_{0\le t\le T}\left|U^*(t)-\sup_{0\le s\le t}W^{(1)}(s)\right| \overset{\text{a.s.}}{=} o(T^\lambda)$$

for every $\lambda>1/m$.

If $N(t)$ is the inverse of $\mathcal{U}(t)$ then $N(k)$ denotes the time when the random walk returns to zero at $k+1$-th times, so $0, N(1), N(2), \ldots$ are the zeros of the random walk. When we proved Theorem 2.2 and Corollaries 2.1 and 2.2 we used only that Condition B implies the strong approximation of the partial sums with the supremum of a Wiener process. Thus, under condition (3.14), Theorem 2.2 and Corollaries 2.1 and 2.2 hold for the zeros of the random walk. Corollary 2.1 gives a characterization of the upper and lower classes of the zeros, Corollary 2.2 says that $\{\sigma^{-2}T^2N(Tt),\ 0\le t\le 1\}$ converges weakly to the first passage time of the Wiener process. For the case of symmetric random walk $\big(P(\xi=1)=P(\xi=-1)=1/2\big)$ Chung and Hunt [6] obtained upper and lower classes for the zeros.

Acknowledgement. I thank Professor Allan Gut for sending me his re- and pre-prints. I am very grateful to Miklós Csörgő, Sándor Csörgő and Eric Willekens for their helpful comments on the first version of this paper.

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(Received October 17, 1983; revised February 8, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 29—32.

ON THE EXISTENCE OF CERTAIN SEMI-BOUNDED SELF-ADJOINT OPERATORS IN HILBERT SPACE

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Let B be a densely defined self-adjoint operator in a Hilbert space H which is bounded below by one, that is B satisfies

$$\|x\|^2 \leqq (Bx, x) \quad (x \in \mathcal{D}(B)) \tag{1}$$

where $\mathcal{D}(B)$ denotes the domain of B.

It is natural to ask: given a function b defined on a subset of H with values in H, under what condition does there exist such a semi-bounded operator B which extends b? A closely related question is treated in [2] for bounded B and yields, as a consequence, the classical Krein's extension theorem among others. The present result on extension of this type (Theorem 1) is a generalization of the known Friedrichs' extension theorem [1]. We treat also some factorization problems continuing observations of the author in [2], [3]. Our constant reference is [1].

THEOREM 1. *Let b be a function given on a dense subset H_0, in the Hilbert space H, taking values in H. There exists a semi-bounded self-adjoint operator B in H satisfying* (1) *which extends b if and only if*

$$\Big\|\sum_x c_x x\Big\|^2 \leqq \Big(\sum_x c_x b(x), \sum_x c_x x\Big) \tag{2}$$

holds for any finite sequence $\{c_x\}$ of complex numbers indexed by elements of H_0.

PROOF. The necessity of (2) is obvious, since the existence of such an extension implies

$$\Big\|\sum_x c_x x\Big\|^2 \leqq \Big(B\Big(\sum_x c_x x\Big), \sum_x c_x x\Big) = \Big(\sum_x c_x Bx, \sum_x c_x x\Big) = \Big(\sum_x c_x b(x), \sum_x c_x x\Big).$$

To prove the sufficiency assume (2) and let Y denote the set of complex valued functions of finite support on H_0. We introduce a semi-inner product on Y by

$$\Big\langle \sum_x c_x \delta_x, \sum_y d_y \delta_y \Big\rangle := \Big(\sum_x c_x b(x), \sum_y d_y y\Big), \tag{3}$$

where $\delta_x = \delta_x(x')$ denotes the function defined in case $x, x' \in H_0$, by 1 if $x' = x$ and by 0 if $x' \neq x$. After factorization of Y by the nullspace of $\langle\, ,\rangle$ and completion with respect to the norm arising from the norm inherited from the inner product on this factor space, we get a Hilbert space K. For simplicity we denote the scalar product and the image of elements of Y by the original symbols. By (3) and (2) we can get a contraction V from K into H, defining it by

$$V\Big(\sum_x c_x \delta_x\Big) := \sum_x c_x x \tag{4}$$

on Y. V satisfies for any x in H

$$\|V(V^*x)\|^2 \leqq \|V^*x\|^2 = (V(V^*x), x). \tag{5}$$

Its adjoint satisfies for any x in H_0

$$V^*(b(x)) = \delta_x \quad (\text{in } K), \tag{6}$$

since

$$\langle V^*(b(x)), \sum_y d_y \delta_y \rangle = (b(x), V(\sum_y d_y \delta_y)) = (b(x), \sum_y d_y y) = \langle \delta_x, \sum_y d_y \delta_y \rangle$$

holds for any $\sum_y d_y \delta_y$ in Y.

If $y \in H$ and $VV^* y = 0$, then for any $x \in H_0$, by (6) and (4), we have

$$0 = (VV^* y, b(x)) = \langle V_y^*, V^*(b(x)) \rangle = \langle V^* y, \delta_x \rangle = (y, V(\delta_x)) = (y, x).$$

This implies $y = 0$, because H_0 is (by assumption) a dense subset in H. The self-adjoint (contraction) VV^* on H has then a densely defined inverse B which is self-adjoint (see § 119 in [1]). $B = (VV^*)^{-1}$ has the property (1) as a consequence of (5) and extends b, since by (6) we have $VV^*(b(x)) = V(\delta_x) = x$ implying that $b(x) = (VV^*)^{-1} x = Bx$ holds for any x in H_0. The proof is complete.

The next corollary is known as "Friedrichs' extension theorem" of a semi-bounded symmetric operator to a self-adjoint one.

COROLLARY 1. *Let b be a linear operator on a dense subset H_0 of H, which is symmetric:*

$$(bx, y) = (x, by) \quad (x, y \in H_0) \tag{7}$$

and bounded below by 1, ***i.e.***

$$\|x\|^2 \leqq (bx, x). \tag{8}$$

Then there exists a self-adjoint extension B of b with property (1).

PROOF. In case b is linear, (8) is the same as (2). We remark that (7) is superfluous or exactly is a consequence of (8) (in a complex Hilbert space), since (bx, x) is non-negative, hence real, for any x in H_0.

The following two theorems are closely related to Corollary 1 in [2].

THEOREM 2. *Let A and C be densely defined operators in H such that the range of $C|_{\mathscr{D}(A)}$ is dense in H. There exists a self-adjoint operator B with property* (1) *and such that*

$$A \subset BC \tag{9}$$

if and only if $\mathscr{D}(A) \subset \mathscr{D}(C)$ and

$$\|Cx\|^2 \leqq (Ax, Cx) \quad (x \in \mathscr{D}(A)). \tag{10}$$

PROOF. The necessity of (10) follows from (9) by (1) at once. Indeed,

$$(Ax, Cx) = (B(Cx), Cx) \geqq \|Cx\|^2$$

holds for any x in $\mathscr{D}(A)$.

To prove the sufficiency of property (10), by a similar argument as in the proof of Theorem 1, define a semi-inner product on $\mathscr{D}(A)$ by

$$\langle x, y\rangle := (Ax, Cy) \quad (x, y\in\mathscr{D}(A)). \tag{11}$$

The so arising Hilbert space K is the completion of the quotient space $\mathscr{D}(A)/N$ with respect to the norm inherited from the inner product $\langle\, ,\rangle$ on this space, where N is the nullspace of $\langle\, ,\rangle$ in $\mathscr{D}(A)$. If for any x in $\mathscr{D}(A)$, Jx denotes its image in K, by (10), the map

$$V(Jx) := Cx \quad (x\in\mathscr{D}(A)) \tag{12}$$

is a densely defined linear contraction from K into H. Its unique continuous extension, as a contraction operator of K into H, is denoted also by V. Now we have

$$VV^*(Ax) = Cx \quad (x\in\mathscr{D}(A)). \tag{13}$$

Indeed, for any x, y in $\mathscr{D}(A)$

$$\langle V^*(Ax), Jy\rangle = \big(Ax, V(Jy)\big) = (Ax, Cy) = \langle Jx, Jy\rangle$$

by (11) and the definition of J hence $VV^*(Ax)=V(Jx)=Cx$ holds by (12).

Now

$$\|V(V^*x)\|^2 \leqq \|V^*x\|^2 = (VV^*x, x)$$

holds also for any x in $\mathscr{D}(A)$. If $VV_x^*=0$, then $V_x^*=0$ and thus

$$0 = \langle V^*x, Jy\rangle = \big(x, V(Jy)\big) = (x, Cy)$$

follows for any y in $\mathscr{D}(A)$ implying $x=0$, since the range of $C|_{\mathscr{D}(A)}$ is dense in H. So VV^* is a self-adjoint invertible operator, the inverse of which $B=(VV^*)^{-1}$ is the desired semi-bounded self-adjoint operator in H. It has property (1) in the same manner as before by (5) in the proof of Theorem 1 and also satisfies (9) by (13). The proof is complete.

THEOREM 3. *Let A and C be densely defined operators in the Hilbert space H. There exists a bounded operator B on H which is positive and satisfies* (9) *if and only if $\mathscr{D}(A)\subset\mathscr{D}(C)$ and*

$$\|Ax\|^2 \leqq M(Ax, Cx) \tag{14}$$

holds with some constant $M\geqq 0$ independent of x.

PROOF. The necessity of (14) is a simple consequence of the positivity of B (using a Schwarz-type inequality as follows):

$$\|Ax\|^2 = \|B(Cx)\|^2 \leqq \|B\|\big(B(Cx), Cx\big) = \|B\|\,(Ax, Cx)$$

holds for any x in $\mathscr{D}(A)$, proving (14). To prove the sufficiency, we are in the position to take a Hilbert space K arising from $\mathscr{D}(A)$ by a semi-inner product given under (11), only the map V will be defined by a suitable modification as

$$V(Jx) := Ax \quad (x\in\mathscr{D}(A)) \tag{15}$$

thus giving a continuous linear operator V from K into H. Its adjoint operator satisfies

$$V^*(Cx) = Jx \tag{16}$$

for any x in $\mathscr{D}(A)$ since by (11) for any y in $\mathscr{D}(A)$ we have

$$\langle Jy, V^*(Cx)\rangle = \big(V(Jy), Cx\big) = (Ay, Cx) = \langle Jy, Jx\rangle.$$

In consequence $B = VV^*$ is a suitable operator, since

$$B(Cx) = (VV^*)(Cx) = V\big(V^*(Cx)\big) = V(Jx) = Ax$$

holds indeed by (16) and (15) for any x in $\mathscr{D}(A)$ so proving (9). The proof is ended.

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(Received October 19, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 33—44.

APPROXIMATION BY VILENKIN—FOURIER SUMS

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Introduction. In this paper we deal with the connection (in different spaces) among the Vilenkin—Fourier sums, the modulus of continuity and the Lebesgue-constants (with respect to the Vilenkin-system). We give two sided estimates for an expression containing these quantities. The corresponding problem for the trigonometric system was considered by Lebesgue [7] and Oskolkov [8].

1.

Let $m:=(m_k,\ k\in\mathbf{N})$ $(\mathbf{N}=\{0, 1, \dots\})$ be a sequence of natural numbers, whose terms are not less than 2. Denote by Z_{m_k} $(k\in\mathbf{N})$ the discrete cyclic group of order m_k, and define G_m as the direct product of Z_{m_k}'s (endowed with the product topology and measure). G_m is a compact Abelian group with the normalized Haar measure μ. The elements of G_m are of the form $x=(x_0, x_1, \dots, x_k, \dots)$ $(0\leqq x_k<m_k,\ k, x_k\in\mathbf{N})$. Further we need the following subsets of G_m:

$$I_0 := G_m, \quad I_{n+1} := \{x\in G_m | x_0 = \dots = x_n = 0\} \quad (n\in\mathbf{N})$$

and

$$I_n(x) := \{x \dotplus y\in G_m | y\in I_n\}$$

(where $\dotplus$ is the note of the group operation, and $\dot{-}$ is its inverse).

Introducing the notations

$$M_0 := 1, \quad M_{n+1} := \prod_{i=0}^{n} m_i \quad (n\in\mathbf{N})$$

we have $\mu(I_n)=\dfrac{1}{M_n}$.

We denote by $\hat{G}_m=(\psi_n,\ n\in\mathbf{N})$ the character system of G_m ordered in the Walsh—Paley sense (see [11]). $\hat{G}_m$ is a complete orthonormal system, and the functions ψ_n $(n\in\mathbf{N})$ are defined as follows. Let

$$r_k(x) := \exp\frac{2\pi i x_k}{m_k} \quad (x\in G_m,\ k\in\mathbf{N}).$$

Each $n\in\mathbf{N}$ can be written uniquely in the form

$$n = \sum_{k=0}^{\infty} n_k M_k \quad (0 \leqq n_k < m_k,\ n_k\in\mathbf{N}).$$

Thus

$$\psi_n = \sum_{k=0}^{\infty} r_k^{n_k} \quad (n \in \mathbf{N}).$$

ψ_n $(n \in \mathbf{N})$ are called Vilenkin functions, and $\hat{G}_m$ forms a group for the multiplication. The Dirichlet kernels are

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbf{N})$$

for which it is well-known (see e.g. [10]) that

$$(1) \qquad D_{M_n}(x) = \begin{cases} M_n, & x \in I_n \\ 0, & x \notin I_n \end{cases} \quad (n \in \mathbf{N}).$$

$L_k := \int\limits_{G_m} |D_k| \, d\mu$ is the k-th Lebesgue constant with respect to $\hat{G}_m$ $(k \in \mathbf{N})$. The spaces $C(G_m)$ (the space of continuous, complex valued functions defined on G_m) and $L^p(G_m)$ (with respect to the Haar-measure μ) $(1 \leqq p < \infty)$ are defined in the usual way. If $f \in L^1(G_m)$, then

$$\hat{f}(k) := \int\limits_{G_m} f \bar{\psi}_k \, d\mu \quad (k \in \mathbf{N})$$

is the k-th (so-called) Vilenkin—Fourier coefficient, and

$$S_k f := \sum_{i=0}^{k-1} \hat{f}(i) \psi_i \quad (k \in \mathbf{N} \setminus \{0\})$$

is the k-th partial sum of the Vilenkin—Fourier series of f. Now we define a Hardy-type space by means of a martingal maximal function. Let $f \in L^1(G_m)$, and

$$f^*(x) := \sup_n M_n \Big| \int\limits_{I_n} \tau_x f \, d\mu \Big| \quad (x \in G_m)$$

where

$$\tau_x f(y) := f(x \dotplus y) \quad (y \in G_m).$$

It is said that f belongs to the Hardy-space $H(G_m)$ iff $f^* \in L^1(G_m)$, and its norm is defined by $\|f\|_H := \|f^*\|_1$. If the sequence m is bounded, then $H(G_m)$ has an atomic structure [1]. We shall use the common notation $Y(G_m)$ for the spaces $C(G_m)$, $L^p(G_m)$ $(1 \leqq p < \infty)$ and $H(G_m)$. It is known that $\lambda \colon G_m \to [0, 1]$, $\lambda(x) := \sum_{k=0}^{\infty} \frac{x_k}{M_{k+1}}$ is an almost one-to-one and measure preserving mapping. The modulus of continuity of a function $f \in Y(G_m)$ is defined as

$$\omega(f, Y, \delta) := \sup_{\lambda(h) < \delta} \|f - \tau_h f\|_Y \quad (\delta > 0).$$

It is easy to see that the analogue of the nice property of the classical modulus of continuity, i.e.

$$\omega(f, Y, \delta) \leqq \left(\left[\frac{\delta}{\delta'} \right] + 1 \right) \omega(f, Y, \delta')$$

fails to hold for all $f \in Y(G_m)$ and $\delta, \delta' > 0$. However, if we see a special case, i.e. if we assume that there exists $n \in \mathbf{N}$ such that

$$\frac{1}{M_{n+1}} < \delta' < \delta \leqq \frac{1}{M_n},$$

then it is not hard to check that

$$\omega(f, Y, \delta) \leqq 2\left(\left[\frac{\delta}{\delta'}\right]+1\right)\omega(f, Y, \delta') \quad (f \in Y(G_m)) \tag{2}$$

(where $[a]$ is the entire part of the real number a). In this paper we deal with the expression

$$\frac{\|f-S_k f\|_Y}{\omega(f, Y, k^{-1})L_k} \quad (k \in \mathbf{N},\ f \in Y(G_m)). \tag{3}$$

Trigonometric system. The analogous question for the trigonometric system was considered by Lebesgue [7]. He showed the existence of $f \in C(1)$ (the space of continuous functions with period 1) for which

$$\limsup_{k\to\infty} \frac{\|f-S_k(f)\|_C}{\bar{\omega}(f, C, 1/k)\tilde{L}_k} > 0$$

(where $S_k(f)$ is the k-th partial sum of the trigonometric Fourier series of f, $\tilde{L}_k$ is the k-th Lebesgue constant with respect to the trigonometric system, and $\bar{\omega}$ is the classical modulus of continuity).

Oskolkov [8] improved the Lebesgue's result. He proved that there exists $f \in C(1)$ such that

$$\liminf_{k\to\infty} \frac{\|f-S_k(f)\|_C}{\bar{\omega}(f, C, 1/k)\tilde{L}_k} > 0$$

Walsh-system on $[0, 1]$. Gulicev [5] studied the corresponding question for the Walsh-system on $[0, 1]$. He proved that for all $f \in C(1)$

$$\liminf_{k\to\infty} \frac{\|f-S_k f\|_C}{\bar{\omega}(f, C, 1/k)L_k} = 0 \quad (k \in \mathbf{N})$$

(where L_k is the k-th Lebesgue constant with respect to the Walsh-system and $S_k f$ is the k-th partial sum of the Walsh—Fourier series of f).

Throughout this paper $C > 0$ will denote an absolute and $C_p > 0$ an only on p depending constant (not necessarily the same at different occurrences).

2. Results

Vilenkin-system on G_m. We deal with the expression (3). First we consider the case $L^p(G_m)$ $(1 < p < \infty)$.

THEOREM 1. *For all sequences m and $f \in L^p(G_m)$ $(1 < p < \infty)$*

$$\liminf_{k\to\infty} \frac{\|f-S_k f\|_p}{\omega(f, L^p, 1/k)L_k} = 0.$$

The proof of this theorem is based on the fact that $\hat{G}_m$ is a basis in $L^p(G_m)$ $(1<p<\infty)$ [9]. This is not true for $C(G_m)$, nevertheless the following theorem is valid.

THEOREM 2. *For all sequences m and $f\in C(G_m)$*

$$\liminf_{k\to\infty}\frac{\|f-S_kf\|_c}{\omega(f,c,1/k)L_k}=0.$$

This is not the case in $L^1(G_m)$, what is showed in

THEOREM 3. *For all sequences m there exists $f\in L^1(G_m)$ such that*

$$\liminf_{k\to\infty}\frac{\|f-S_kf\|_1}{\omega(f,L^1,1/k)L_k}>0.$$

It is known [1] that $H(G_m)$ separates the sets $L^1(G_m)$ and $L^p(G_m)$ $(1<p<\infty)$. On account of this it is of interest what is the case in $H(G_m)$. The answer is given in

THEOREM 4. *For all sequences m there exists $f\in H(G_m)$ such that*

$$\liminf_{k\to\infty}\frac{\|f-S_kf\|_H}{\omega(f,H,1/k)L_k}>0.$$

For the proof of Theorem 4 we need a lemma, which is the extension to $H(G_m)$ of the following well-known Efimov's result [2]

$$\|f-S_{M_n}f\|_p\leqq\omega(f,L^p,M_n^{-1})\leqq 2\|f-S_{M_n}f\|_p\quad (f\in L^p(G_m),\ p\geqq 1,\ n\in\mathbf{N}),\tag{4}$$

$$\|f-S_{M_n}f\|_C\leqq\omega(f,C,M_n^{-1})\leqq 2\|f-S_{M_n}f\|_C\quad (f\in C(G_m),\ n\in\mathbf{N}).$$

LEMMA 1. *For all sequences m and $f\in H(G_m)$*

$$\|f-S_{M_n}f\|_H\leqq\omega(f,H,1/M_n)\leqq 2\|f-S_{M_n}f\|_H\quad (n\in\mathbf{N}).$$

We remark that it will be clear from the proof of Lemma 1 that the above inequalities remain true, if we write $E_{M_n}(f,H)$ instead of $\|f-S_{M_n}f\|_H$ (where $E_{M_n}(f,H)$ is the distance in $H(G_m)$ between f and the subspace generated by $\{\psi_i|0\leqq i<M_n\}$).

REMARKS. 1. Theorem 2 means that a result of Oskolkov's type is not valid for $\hat{G}_m$.

2. The analogue of Lebesgue's result is trivial for $\hat{G}_m$, since $\omega(f,Y,M_n^{-1})\leqq$ $\leqq 2\|f-S_{M_n}f\|_Y$ (see [2] and Lemma 1) and $L_{M_n}=1$ $(n\in\mathbf{N})$, consequently

$$\limsup_{k\to\infty}\frac{\|f-S_kf\|_Y}{\omega(f,Y,1/k)L_k}\geqq\frac{1}{2}\quad (k\in\mathbf{N},f\in Y(G_m)).$$

We get lower estimate for "lim sup" in Remark 2. In the next theorem we deal with upper estimation.

THEOREM 5. i) *If the sequence m is bounded, then for all $f\in Y(G_m)$*

$$\limsup_{k\to\infty}\frac{\|f-S_kf\|_Y}{\omega(f,Y,1/k)L_k}<\infty$$

ii) *If the sequence m is unbounded, then there exists $f\in Y(G_m)$ such that*

$$\limsup_{k\to\infty}\frac{\|f-S_kf\|_Y}{\omega(f,Y,1/k)L_k}=\infty$$

Vilenkin-system on $[0, 1]$. Since the Vilenkin functions can be regarded even as complex valued functions defined on $[0, 1]$, therefore the analogue of the statements of Theorems 1, 2, 3 are to be studied in $C(1)$, $L^p(1)$ $(1\leqq p<\infty)$ with respect to the Vilenkin-system and $\bar{\omega}$. We denote the corresponding statements as Theorem 1′, 2′, 3′, and we shall prove that they are true.

REMARKS. 1. Only the proof of Theorem 3′ is essentially different from the proof of Theorem 3.

2. Theorem 2′ was proved for the dyadic case by Gulicev in [5]. He announced Theorem 3′ without proof for the dyadic case in [4].

3. Proofs

PROOF OF THEOREM 1. Let $f\in L^p(G_m)$ $(1<p<\infty)$. It is known [10] that there exists a sequence of natural numbers $(k_n,\ n\in\mathbf{N})$ such that $M_n<k_n<2M_n$ $(n\in\mathbf{N})$ and $L_{k_n}>C\log M_n$. By (2) we have $4\omega\left(f,L^p,\frac{1}{k_n}\right)\geqq\omega\left(f,L^p,\frac{1}{M_n}\right)$. Since $\hat{G}_m$ is a basis in $L^p(G_m)$ $(1<p<\infty)$ [9], thus there exists $C_p>0$ such that

$$\|f-S_{k_n}f\|_p\leqq\|f-S_{M_n}f\|_p+\|S_{k_n}(f-S_{M_n}f)\|_p\leqq(1+C_p)\|f-S_{M_n}f\|_p.$$

From (4) we have

$$\omega\left(f,L^p,\frac{1}{M_n}\right)\geqq\|f-S_{M_n}f\|_p.$$

Thus

$$\frac{\|f-S_{k_n}f\|_p}{\omega(f,L^p,1/k_n)L_{k_n}}\leqq\frac{(1+C_p)\|f-S_{M_n}f\|_p}{1/4\,\|f-S_{M_n}f\|\,C\log M_n}<C_p\frac{1}{\log M_n}\quad(n\in\mathbf{N})$$

consequently Theorem 1 is true.

PROOF OF THEOREM 2. We follow the method of Gulicev [5] in the proof of this theorem. First we verify

LEMMA 2. *Let $s<n$ and $k<M_s$ $(s,n,k\in\mathbf{N})$. Define q_j and p_j as*

$$q_j:=M_n+(m_{n-1}-1)M_{n-1}+\ldots+(m_{n-j}-1)M_{n-j},$$

$$p_j:=q_j+k\quad(j=0,1,\ldots,n-s-1).$$

Then for all $f \in C(G_m)$

$$\frac{1}{n-s}\sum_{j=0}^{n-s-1}\|S_{p_j}f\|_C \leqq 3\|f\|_C + \left(\frac{M_s^3}{n-s}\right)^{1/2}\|f\|_c.$$

PROOF OF LEMMA 2. From the definition of the Vilenkin functions we have $D_{p_j}=D_{q_j}+\psi_{q_j}D_k$, hence

$$\frac{1}{n-s}\sum_{j=0}^{n-s-1}\|S_{p_j}f\|_C = \frac{1}{n-s}\sum_{j=0}^{n-s-1}\Big\|\int_{G_m} f(t)\,D_{p_j}(x \dot- t)\,d\mu(t)\Big\|_C \leqq$$

$$\leqq \frac{1}{n-s}\sum_{j=0}^{n-s-1}\Big\|\int_{G_m} f(t)\,D_{q_j}(x \dot- t)\,d\mu(t)\Big\|_C +$$

$$+\frac{1}{n-s}\sum_{j=0}^{n-s-1}\Big\|\int_{G_m} f(t)\,\psi_{q_j}(x \dot- t)\,D_k(x \dot- t)\,d\mu(t)\Big\|_C =: \sigma_1+\sigma_2.$$

Since $2M_n=q_j+M_{n-j}$ and by (1) $L_{M_i}=1$ $(i\in \mathbf{N})$, therefore $L_{q_j}\leqq 3$, consequently $\sigma_1 \leqq 3\|f\|_C$.

Let us consider σ_2. From $k \leqq M_s$ it follows that D_k is constant on each set $I_s(x)$ $(x\in G_m)$. The sets $I_s(x)$ $(x\in G_m)$ decompose G_m into M_s pieces of disjoint sets denoted by $I_{n,u}$ $(u=0, 1, \ldots, M_s-1)$. Denote χ_u the characteristic function of $I_{s,u}$ and define the function K_u as follows:

$$K_u(x) := D_k(x \dot- t) \quad (t\in I_{s,u},\ u=0, \ldots, M_s-1).$$

According to $|K_u(x)|\leqq M_s$ $(x\in G_m,\ u=0, \ldots, M_s-1)$ and

$$\psi_{q_j}(x \dot- t) = \psi_{q_j}(x)\overline{\psi_{q_j}(t)} \quad (x, t\in G_m)$$

we have

$$\sigma_2 = \frac{1}{n-s}\sum_{j=0}^{n-s-1}\Big\|\sum_{u=0}^{M_s-1} K_u \int_{G_m} f\chi_u\bar{\psi}_{q_j}\Big\|_C \leqq \frac{M_s}{n-s}\sum_{u=0}^{M_s-1}\sum_{j=0}^{n-s-1}\Big|\int_{G_m} f\chi_u\bar{\psi}_{q_j}\Big|.$$

By the Cauchy—Schwarz and Bessel inequalities we obtain

$$\sum_{j=0}^{n-s-1}|(f\chi_u)^\wedge(q_j)| \leqq \sqrt{n-s}\sum_{j=0}^{n-s-1}|(f\chi_u)^\wedge(q_j)|^2 \leqq$$

$$\leqq \sqrt{n-s}\Big(\int_{G_m} f^2\chi_u\Big)^{1/2} \leqq \sqrt{n-s}\,M_s^{-1/2}\|f\|_C,$$

whence

$$\sigma_2 \leqq \left(\frac{M_s^3}{n-s}\right)^{1/2}\|f\|_C.$$

The proof of Lemma 2 is complete.

Applying Lemma 2 we prove Theorem 2.

Let $f\in C(G_m)$ and

$$s_n := \max\left\{v\in\mathbf{N}\,\Big|\,\frac{M_v^3}{n-v}\leqq 1\right\} \quad (n\in\mathbf{N}).$$

Obviously $s_n<n$ for all $n\in\mathbf{N}$ and the sequence $(s_n,\ n\in\mathbf{N})$ tends monotonously to $+\infty$.

There exists [10] $k<M_{s_n}$ (depending on n) for which

$$\|D_k\|_1 > c\log M_{s_n}\quad (n\in\mathbf{N})$$

and if n is large enough, then

$$L_{p_j}\geqq L_k-L_{q_j}>C\log M_{s_n}-3>Cs_n\quad (j=0,1,\ldots,n-s_n-1).$$

It is clear that

$$\|f-S_{p_j}f\|_C\leqq\|f-S_{M_n}f\|_C+\|S_{p_j}(f-S_{M_n}f)\|_C.$$

By $\|f-S_{M_n}f\|_C\leqq\omega(f,C,M_n^{-1})$ and by the definition of s_n we get from Lemma 2

$$\frac{1}{n-s_n}\sum_{j=0}^{n-s_n-1}\|f-S_{p_j}f\|_C\leqq\omega(f,C,M_n^{-1})+4\omega(f,C,M_n^{-1})=5\omega(f,C,M_n^{-1})\qquad (n\in\mathbf{N}).$$

Since $1<\frac{p_j}{M_n}<2$, by (2) $\omega(f,C,M_n^{-1})\leqq 4\omega(f,C,p_j^{-1})$. The above estimations yield

$$\min_{0\leqq j\leqq n-s_n-1}\frac{\|f-S_{p_j}f\|_C}{\omega(f,C,p_j^{-1})L_{p_j}}\leqq C\frac{\omega(f,C,M_n^{-1})}{\omega(f,C,M_n^{-1})s_n}=C\frac{1}{s_n}.$$

Theorem 2 is proved.

Proof of Theorem 3. Let $(A_i,\ i\in\mathbf{N})$ be a sequence of positive real numbers tending monotonously to zero, and $\sum_{i=1}^{\infty}A_i<\infty$. Define the function f as follows:

$$f:=\sum_{i=0}^{\infty}A_{i+1}(D_{M_{i+1}}-D_{M_i})\oplus A_1D_{M_0}. \tag{5}$$

It is obvious that f belongs to $L^1(G_m)$. Let $M_n\leqq k<M_{n+1}$ $(k,n\in\mathbf{N})$, thus

$$\|f-S_kf\|_1\geqq\|S_{M_{n+1}}(f-S_kf)\|_1=\|S_{M_{n+1}}f-S_kf\|_1=A_{n+1}\|D_{M_{n+1}}-D_k\|_1.$$

It is easy to show that $\|D_{M_{n+1}}-D_k\|_1\geqq 1$, thus by $\|D_{M_{n+1}}\|_1=1$ and by $\|D_k\|_1\geqq 1$ we have $\|D_{M_{n+1}}-D_k\|_1\geqq\max(1,L_k-1)$, i.e. $\|D_{M_{n+1}}-D_k\|_1\geqq 1/2\,L_k$. Hence

$$\|f-S_kf\|_1\geqq 1/2\,A_{n+1}L_k.$$

Applying (1) and (4) we obtain

$$\omega(f,L^1,1/k)\leqq\omega(f,L^1,1/M_n)\leqq 2\|f-S_{M_n}f\|_1=$$

$$=2\Big\|\sum_{i=n}^{\infty}A_{i+1}(D_{M_{i+1}}-D_{M_i})\Big\|_1<4\sum_{i=n+1}^{\infty}A_i.$$

From the above estimations it follows that

$$\frac{\|f-S_kf\|_1}{\omega(f,L^1,1/k)L_k}>\frac{1/2\,A_{n+1}L_k}{4\sum_{i=n+1}^{\infty}A_iL_k}=\frac{1}{8}\frac{A_{n+1}}{\sum_{i=n+1}^{\infty}A_i}.$$

By choosing $A_l := 2^{-l}$ $(l \in \mathbf{N})$ the conditions concerning $(A_i,\ i \in \mathbf{N})$ are satisfied and

$$\frac{\|f - S_k f\|_1}{\omega(f, L^1, 1/k) L_k} > \frac{1}{8} \quad (k \in \mathbf{N}).$$

The proof of Theorem 3 is complete.

Proof of Lemma 1. Let $f \in H(G_m)$ and $n \in \mathbf{N}$). We have

$$\|f - S_{M_n} f\|_H = \Big\| \int_{G_m} (f(x) - f(x \dotplus t)) D_{M_n}(t)\, d\mu(t) \Big\|_H =$$

$$= \int_{G_m} \sup_k M_k \Big| \int_{I_k(y)} \int_{G_m} (f(x) - f(x \dotplus t)) D_{M_n}(t)\, d\mu(t)\, d\mu(x) \Big|\, d\mu(y) \leqq$$

$$\leqq \int_{G_m} \sup_k M_k \int_{I_n} M_n \Big| \int_{I_k(y)} (f(x) - f(x \dotplus t))\, d\mu(x) \Big|\, d\mu(t)\, d\mu(y) \leqq$$

$$\leqq \int_{I_n} M_n \int_{G_m} \sup_k M_k \Big| \int_{I_k(y)} (f(x) - f(x \dotplus t))\, d\mu(x) \Big|\, d\mu(y)\, d\mu(t) \leqq$$

$$\leqq \int_{I_n} M_n \omega\left(f, H, \frac{1}{M_n}\right) d\mu(t) = \omega\left(f, H, \frac{1}{M_n}\right).$$

The proof of the second inequality is very easy. Since $\tau_h(S_{M_n} f) = S_{M_n} f$ $(h \in I_n)$, therefore

$$\omega\left(f, H, \frac{1}{M_n}\right) = \sup_{\lambda(h) < 1/M_n} \|f - \tau_h f\|_H \leqq \sup_{\lambda(h) < 1/M_n} \|f - S_{M_n} f\|_H +$$

$$+ \sup_{\lambda(h) < 1/M_n} \|\tau_h(f - S_{M_n} f)\|_H = 2\|f - S_{M_n} f\|_H,$$

which was to be proved.

Proof of Theorem 4. It is easy to see that $(D_{M_{n+1}} - D_{M_n})^* = |D_{M_{n+1}} - D_{M_n}|$ consequently $\|D_{M_{n+1}} - D_{M_n}\|_H = \|D_{M_{n+1}} - D_{M_n}\|_1 < 2$ $(n \in \mathbf{N})$. Let f be the function as in (5). Thus for $M_n \leqq k < M_{n+1}$ $(k, n \in \mathbf{N})$ we have as above that

$$\|f - S_k f\|_H \geqq \|f - S_k f\|_1 \geqq \frac{1}{2} A_{n+1} L_k$$

and by Lemma 1

$$\omega(f, H, 1/k) \leqq \omega\left(f, H, \frac{1}{M_n}\right) \leqq 2\|f - S_{M_n} f\|_H =$$

$$= 2\Big\| \sum_{i=n}^{\infty} A_{i+1}(D_{M_{i+1}} - D_{M_i}) \Big\|_H < 4 \sum_{i=n+1}^{\infty} A_i.$$

Let $A_i := 2^{-i}$ $(i\in\mathbf{N})$, then the conditions concerning $(A_i,\ i\in\mathbf{N})$ are satisfied and

$$\frac{\|f-S_kf\|_H}{\omega(f,H,1/k)L_k} \geqq \frac{\frac{1}{2}A_{n+1}L_k}{4\sum_{i=n+1}^{\infty}A_iL_k} \geqq \frac{1}{8} \quad (k\in\mathbf{N}).$$

Theorem 4 is proved.

For the proof of Theorems 1′ and 2′ it is enough to observe that all the considerations used in the proof of Theorems 1 and 2 are valid in this case too. We need only to take into account the inequality $\|f-S_{M_n}f\|_c<\bar\omega\left(f,C,\frac{1}{M_n}\right)$ $(f\in C(1),\ n\in\mathbf{N})$ [3] and to make such changes like to write $\sum_{i=0}^{k-1}\psi_i(x)\psi_i(t)$ $(x,\ t\in[0,1])$ instead of $D_k(x+t)$ $(x,\ t\in G_m)$ and $\left[0,\frac{1}{M_n}\right]$ instead of I_n $(n\in\mathbf{N})$ etc.

Proof of Theorem 3′. Let f be the same function as in (5). (D_{M_i} $(i\in\mathbf{N})$ are of course the Dirichlet kernels of the Vilenkin system defined on [0, 1].) It is easy to see (by means of the mapping λ) that the estimation for $\|f-S_kf\|_1$ is valid in this case too, i.e.

$$\|f-S_kf\|_1 \geqq \frac{1}{2}A_{n+1}L_k \quad (k,\ n\in\mathbf{N},\ M_n\leqq k<M_{n+1}).$$

This is not the case for $\bar\omega(f,L^1,1/k)$ $(k\in\mathbf{N})$, because the relation $\bar\omega(f,L^1,1/M_n)\leqq \leqq 2\|f-S_{M_n}f\|_1$ does not hold for all $f\in L^1(1)$ and $n\in\mathbf{N}$.

Let us estimate $\bar\omega(f,L^1,1/k)$ in the following way. It is clear that

$$\bar\omega(f,L^1,1/k)\leqq\bar\omega\left(f,L^1,\frac{1}{M_n}\right)\leqq\bar\omega\left(f-S_{M_n}f,\frac{1}{M_n}\right)+\bar\omega\left(S_{M_n}f,L^1,\frac{1}{M_n}\right).$$

Obviously

$$\bar\omega\left(f-S_{M_n}f,L^1,\frac{1}{M_n}\right)\leqq 2\|f-S_{M_n}f\|_1<4\sum_{i=n+1}^{\infty}A_i.$$

It is to be seen from the form of $S_{M_n}f$ that $S_{M_n}f$ is constant on the intervals $\left(\frac{1}{M_{i+1}},\frac{1}{M_i}\right)$ $(i=1,\ldots,n-1)$ and $\left(0,\frac{1}{M_n}\right)$. From this it follows that

$$\bar\omega\left(S_{M_n}f,L^1,\frac{1}{M_n}\right)=\int_0^1\left|S_{M_n}f(x)-S_{M_n}f\left(x+\frac{1}{M_n}\right)\right|dx=$$

$$=\sum_{i=2}^{n}|S_{M_n}f_{(1/M_i,1/M_{i-1})}-S_{M_n}f_{(1/M_{i-1},1/M_{i-2})}|\frac{1}{M_n}+$$

$$+|S_{M_n}f_{(0,1/M_n)}-S_{M_n}f_{(1/M_n,1/M_{n-1})}|\frac{1}{M_n}+|S_{M_n}f_{(0,1/M_n)}|\frac{1}{M_n},$$

(where $S_{M_n}f_{(1/M_{i+1},1/M_i)}$ $(i=1,\ldots,n-1)$, $S_{M_n}f_{(0,1/M_n)}$ are the above mentioned constants). Since $(A_i,\ i\in\mathbf{N})$ is monoton, therefore

$$S_{M_n}f_{(0,1/M_n)}=\sum_{j=0}^{n-1}A_j(M_{j+1}-M_j)$$

and

$$S_{M_n}f_{(1/M_i,1/M_{i-1})}=\sum_{j=0}^{i-2}A_{j+1}(M_{j+1}-M_j)-A_iM_{i-1}.$$

Thus

$$\bar{\omega}\left(f,L^1,\frac{1}{M_n}\right)=\sum_{i=2}^{n}(A_{i-1}-A_i)M_{i-1}\frac{1}{M_n}+A_nM_n\frac{1}{M_n}+$$

$$+\sum_{j=0}^{n-1}A_{j+1}(M_{j+1}-M_j)\frac{1}{M_n}=:a_1+a_2+a_3.$$

From the monotonicity of $(A_i,\ i\in\mathbf{N})$

$$a_1+a_2\leqq\sum_{j=1}^{n}A_jM_j\frac{1}{M_n},$$

on the other hand

$$a_3\leqq\sum_{j=1}^{n}A_jM_j\frac{1}{M_n},$$

hence

$$\bar{\omega}\left(S_{M_n}f,L^1,\frac{1}{M_n}\right)\leqq 2\sum_{j=1}^{n}A_jM_j.$$

From the above estimation we have

$$\frac{\|f-S_kf\|_1}{\bar{\omega}(f,L^1,1/k)L_k}\geqq\frac{1/2\,A_{n+1}}{4\sum\limits_{i=n+1}^{\infty}A_i+2\sum\limits_{j=1}^{n}A_j\dfrac{M_j}{M_n}}\geqq\frac{1/2\,A_{n+1}}{4\sum\limits_{i=n+1}^{\infty}A_i+2\sum\limits_{j=1}^{n}A_j\left(\dfrac{1}{2}\right)^{n-j}}.$$

By choosing $A_i:=\left(\frac{3}{4}\right)^i\ (i\in\mathbf{N})$ we get

$$\frac{\|f-S_kf\|_1}{\bar{\omega}(f,L^1,1/k)L_k}>\frac{1}{48}\quad(k\in\mathbf{N}).$$

The proof of Theorem 3′ is complete.

PROOF OF THEOREM 5. i) Let $M_n\leqq k<M_{n+1}$ $(k,n\in\mathbf{N})$. Applying the convolution theorem concerning homogeneous Banach spaces (see [6]) we obtain from (4) and Lemma 2 for all $f\in Y(G_m)$

$$\|f-S_kf\|_Y\leqq\|f-S_{M_n}f\|_Y+\|S_k(f-S_{M_n}f)\|_Y\leqq$$

$$\leqq(1+L_k)\|f-S_{M_n}f\|_Y\leqq 2L_k\,\omega(f,Y,M_n^{-1}).$$

By (2) we have

$$\omega(f,Y,M_n^{-1})\leqq 2m_n\,\omega(f,Y,k^{-1})$$

consequently

$$\sup_k \frac{\|f-S_k f\|_Y}{\omega(f, Y, k^{-1})L_k} \leqq \sup m_n < \infty.$$

ii) Let $(A_i, i\in\mathbf{N})$ be a sequence of real numbers tending monotone decreasingly to zero, and $\sum_{i=0}^{\infty} A_i \|D_{M_i}\|_Y < \infty$. (The sequence $(A_i, i\in\mathbf{N})$ depends on Y.) Define the function f as follows:

$$f := \sum_{i=0}^{\infty} A_i r_i^{m_i-1} D_{M_i} = \sum_{i=0}^{\infty} A_i \bar{r}_i D_{M_i}.$$

($\bar{r}_i$ is the complex conjugate of r_i.) Obviously $f\in Y(G_m)$.

Let

$$d_j := (m_j-1)M_j \quad (j\in\mathbf{N}),$$

then

$$\|f-S_{d_j}f\|_Y = \|f-S_{M_j}f\|_Y = \Big\|\sum_{i=j}^{\infty} A_i\bar{r}_i D_{M_i}\Big\|_Y \geqq$$

$$\geqq A_j\|\bar{r}_j D_{M_j}\|_Y - \sum_{i=j+1}^{\infty} A_i \|\bar{r}_i D_{M_i}\|_Y.$$

It is clear that

$$\omega(f, Y, d_j^{-1}) \leqq \omega(S_{M_{j+1}}f, Y, d_j^{-1}) + \omega(f-S_{M_{j+1}}f, Y, d_j^{-1}).$$

Furthermore the assumption

$$\omega(f-S_{M_{j+1}}f, Y, d_j^{-1}) \leqq 2\|f-S_{M_{j+1}}f\|_Y \leqq 2\sum_{i=j+1}^{\infty} A_i \|\bar{r}_i D_{M_i}\|_Y$$

is trivial. From the definition of d_j $(j\in\mathbf{N})$ we have $\dfrac{1}{M_{j+1}} < d_j^{-1} \leqq \dfrac{2}{M_{j+1}}$. Since $S_{M_{j+1}}f$ is constant on every set $I_{j+1}(x)$ $(j\in\mathbf{N},\ x\in G_m)$, therefore

$$\omega(S_{M_{j+1}}f, Y, d_j^{-1}) = \|S_{M_{j+1}}f - \tau_{e_j} S_{M_{j+1}}f\|_Y =$$

$$= A_j \|(\bar{r}_j - \tau_{e_j}\bar{r}_j) D_{M_j}\|_Y, \quad (e_j := (\overset{0}{\breve{0}}, \dots, \overset{j-1}{\breve{0}}, \overset{j}{\breve{1}}, 0, \dots)\in G_m).$$

It is easy to show that

$$\|\bar{r}_i D_{M_i}\|_C = M_i \quad \text{and} \quad \omega(S_{M_{j+1}}f, C, d_j^{-1}) < cA_j \frac{M_j}{m_j},$$

$$\|\bar{r}_i D_{M_i}\|_p = M_i^{1-1/p} \quad \text{and} \quad \omega(S_{M_{j+1}}f, L^p, d_j^{-1}) < cA_j M_j^{1-1/p}\frac{1}{m_j} \quad (i, j\in\mathbf{N},\ 1\leqq p).$$

Since $(\bar{r}_i D_{M_i})^* = D_{M_i}$ and $((\bar{r}_j - \tau_{e_j}\bar{r}_j)D_{M_j})^* = |(\bar{r}_j - \tau_{e_j}\bar{r}_j)D_{M_j}|$, we have

$$\|\bar{r}_i D_{M_i}\|_H = 1 \quad \text{and} \quad \omega(S_{M_{j+1}}f, H, d_j^{-1}) < CA_j\frac{1}{m_j} \quad (i, j\in\mathbf{N}).$$

Let

$$A_i := \begin{cases} M_i^{-2} M_i^{-1}, & \text{if} \quad Y(G_m) = C(G_m) \\ M_i^{-2}(M_i^{1-1/p})^{-1}, & \text{if} \quad Y(G_m) = L^p(G_m) \ (1 \leqq p < \infty) \\ M_i^{-2}, & \text{if} \quad Y(G_m) = H(G_m) \quad (i \in \mathbf{N}), \end{cases}$$

then

$$\|f - S_{d_j} f\|_Y \geqq \frac{1}{2} M_j^{-2}$$

and

$$\omega(f, Y, d_j^{-1}) < 4M_{j+1}^{-2} + CM_{j+1}^{-1} M_j^{-1} < CM_j^{-2} m_j^{-1}.$$

Applying that $L_{(m_j-1)M_j} \leqq 2$ we have

$$\frac{\|f - S_{d_j} f\|_Y}{\omega(f, Y, d_j^{-1}) L_{d_j}} > Cm_j \quad (j \in \mathbf{N}).$$

Theorem 5 is proved.

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(Received October 25, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 45—52.

DUALITY OF BANACH FUNCTION SPACES AND THE RADON—NIKODYM PROPERTY

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1. Introduction. In this paper we shall study the duality of Banach function spaces. Throughout the paper, we let X be a Banach space and X^* its dual space. Let (Ω, Γ, μ) be a measure space with a fixed positive measure μ on the σ-field Γ of the set Ω. We consider the space $L^p(\Omega, X)$ of all (equivalence classes of) X-valued strongly measurable functions f on Ω such that

$$\|f\|_p = \Big(\int_\Omega \|f(t)\|^p \, d\mu(t)\Big)^{1/p} < +\infty \quad \text{for} \quad 1 \leqq p < \infty;$$

and the space $L^\infty(\Omega, X)$ of all X-valued essentially bounded strongly measurable functions f on Ω such that

$$\|f\|_\infty = \text{ess sup}\,\{\|f(t)\|;\ t \in \Omega\} < +\infty.$$

Let q be such that $\dfrac{1}{p}+\dfrac{1}{q}=1$ for $1 \leqq p < \infty$, and consider $L^q(\Omega, X^*)$. An exact determination of the relationship between $L^p(\Omega, X)$ and $L^q(\Omega, X^*)$ is useful for applications. For example, if $\Omega = G$ is a locally compact group with Haar measure, then the characterization of

$$L^p(G, X)^* \cong L^q(G, X^*), \quad 1 \leqq p < \infty \tag{1}$$

is useful and important in the study of multiplier problems of Banach function spaces (cf. [6], [10]).

In [3], Diestel and Uhl have shown the following

THEOREM A ([3, Theorem 4.1.1]). *For a finite measure space (Ω, Γ, μ), a necessary and sufficient condition in order that*

$$L^p(\Omega, X)^* \cong L^q(\Omega, X^*), \quad 1 \leqq p < \infty, \ \frac{1}{p}+\frac{1}{q} = 1 \tag{2}$$

holds is that X^* has the Radon—Nikodym property with respect to μ.

DEFINITION. A Banach space X has the *Radon—Nikodym Property* (RNP) with respect to a finite measure space (Ω, Γ, μ) if for each μ-continuous vector measure $\psi\colon \Gamma \to X$ of bounded variation, there exists a Bochner integrable function $g\colon \Omega \to X$ such that $\psi(E) = \int_E g \, d\mu$ for all $E \in \Gamma$.

Many authors investigated the RNP, for examples, [1, 3, 4, 5, 8, 9].

The dual space X^* has the RNP with respect to a finite measure space if and only if X is an Asplund space.

DEFINITION. A Banach space X is said to be an *Asplund space* (what Asplund [1] called a strong differentiable space) if every continuous, real-valued, convex function defined on an open subset of X is Frèchet differentiable on a dense subset G_δ of its domain.

In this note we shall establish the relationship (2) in a more general measure space.

2. Preliminaries. Let (Ω, Γ, μ) be a measure space, M be the set of all real-valued measurable functions on Ω and M^+ the set of all nonnegative measurable functions on Ω.

A mapping ϱ: $M^+ \to \bar{R} = R \cup \{+\infty\}$ is called a *functional norm* if for any $f, g \in M^+$, it satisfies the conditions:

i) $\varrho(f) \geqq 0$ and $\varrho(f)=0$ if and only if $f=0$.

ii) $\varrho(\alpha f) = \alpha\varrho(f)$ for $\alpha > 0$

iii) $\varrho(f+g) \leqq \varrho(f)+\varrho(g)$

iv) $f \leqq g$ implies $\varrho(f) \leqq \varrho(g)$.

This ϱ can be extended to M by defining $\varrho(f)=\varrho(|f|)$ for all $f \in M$. Then

$$L_\varrho = L_\varrho(\Omega, \Gamma, \mu) \rightleftharpoons \{f \in M;\ \varrho(f) < +\infty\}$$

is a Banach space with norm ϱ called a *Banach function space*. The *dual* norm ϱ' of the functional norm ϱ is defined by

$$\varrho'(g) = \sup\Big\{\int_\Omega |fg|\, d\mu\colon\ \varrho(f) \leqq 1\Big\} \quad \text{for} \quad g \in M$$

and the *dual function space* is defined by

$$L_{\varrho'} = L_{\varrho'}(\Omega, \Gamma, \mu) = \{g \in M,\ \varrho'(g) < +\infty\}$$

which is also a Banach space.

Obviously, all $L^p(\Omega, \Gamma, \mu)$, $1 \leqq p \leqq +\infty$ and Orlicz spaces are Banach function spaces.

A sequence $\{\Omega_n\}$ of measurable subsets of Ω is said to be admissible with respect to ϱ if

$$\Omega_n \uparrow \Omega, \quad \mu(\Omega_n) < +\infty \quad \text{and} \quad \varrho(\chi_{\Omega_n}) < +\infty.$$

Here and in what follows χ denotes the characteristic function.

A set $E \subset \Omega$ is called *unfriendly* if for any $B \subset E$, $B \in \Gamma$ with $\mu(B) > 0$ we have $\varrho(\chi_B) = +\infty$.

By the definition of L_ϱ, especially in $L^p(\Omega)$, $1 \leqq p < \infty$, we can assume that Γ always contains no unfriendly sets and always has an admissible sequence in (Ω, Γ, μ).

A function $f \in L_\varrho$ has *absolutely continuous* norm if

a) $\lim\limits_n \varrho(\chi_{E_n})=0$ whenever $E_n \subset E$, $E_n \in \Gamma$ such that $\varrho(\chi_E) < +\infty$ and $\lim\limits_n \mu(E_n)=0$.

b) $\lim_n \varrho(f\chi_{\Omega-\Omega_n})=0$ for any admissible sequence $\{\Omega_n\}$.

Define $L_\varrho^a=\{f\in L_\varrho | f$ has absolutely continuous norm$\}$ and $L_\varrho^\pi=\overline{\text{span}}\,\{f\in L_\varrho | f$ is bounded and has support in Ω_n for some admissible sequence $\pi=\{\Omega_n\}\}$.

It was known that (cf. Gretsky and Uhl [5] or Gretsky [4])

1) $L_\varrho^a=\bigcap_\pi L_\varrho^\pi$;

2) $(L_\varrho^a)^*\cong L_{\varrho'}$ if and only if there exists an admissible sequence π such that $L_\varrho^\pi=L_\varrho^a$.

Now let $\Gamma_0=\{E\in\Gamma;\ \varrho(\chi_E)<+\infty\}$, $\Gamma_0'=\{E\in\Gamma,\ \varrho'(\chi_E)<+\infty\}$ and $\Gamma_{00}=\Gamma_0\cap\Gamma_0'$. Then the absence of unfriendly sets (with respect to ϱ and ϱ') guarantees that the σ-finite measure space Ω can be written as a countable union of sets from Γ_{00}.

We write $L_\varrho(X)$, $L_\varrho^a(X)$ and $L_\varrho^\pi(X)$ instead of L_ϱ, L_ϱ^a and L_ϱ^π for the X-valued functions which are strongly measurable. The dual function space of $L_\varrho(X)$ is given by

$$L_{\varrho'}(X^*)=\{g\colon\ \Omega\to X^* \text{ strongly measurable and } \varrho'(\|g\|)<\infty\}$$

where

$$\varrho'(\|g\|)=\sup_{\varrho(\|f\|)\le 1}\Big\{\int_\Omega |\langle f(t), g(t)\rangle|\,d\mu;\ f\in L_\varrho(X)\Big\}.$$

Gretsky and Uhl [5; Theorem 3.2] established the following result.

THEOREM B. *Let (Ω, Γ, μ) be σ-finite and $L_\varrho^a(X)=L_\varrho^\pi(X)$. Then*

$$L_\varrho^a(X)^*\cong L_{\varrho'}(X^*) \tag{3}$$

under the correspondence $F\in L_\varrho^a(X)^*$ and $g\in L_{\varrho'}(X^*)$ given by

$$F(f)=\int_\Omega \langle f(t), g(t)\rangle\,d\mu \quad \textit{for all} \quad f\in L_\varrho^a(X)$$

if and only if each $K\in\Gamma_{00}$, X^* has the RNP with respect to μ_K which is defined on $\Gamma\cap K$ by $\mu_K(E)=\mu(E\cap K)$ for all $E\in\Gamma$.

In particular, if X is reflexive, or X^* is separable, or μ is purely atomic, then X^* has the RNP and so (3) holds.

For convenience, we say that a Banach space has the *wide RNP* if it has the property formulated in Theorem B. We give

DEFINITION. Let (Ω, Γ, μ) be a measure space. A Banach space X is said to have the *wide RNP* with respect to μ if for every $K\in\Gamma$ with $\mu(K)<\infty$, X has the RNP with respect to μ_K defined by $\mu_K(E)=\mu(K\cap E)$ for all $E\in\Gamma$.

The following problem arises: if $L_\varrho(X)=L^p(\Omega, X)$, $1\le p<\infty$, then what is the space $L_{\varrho'}(X^*)$? Our main goal is to characterize $L_{\varrho'}(X^*)$ to be $L^q(\Omega, X^*)$.

3. The dual space of $L^p(\Omega, X)$ on a measure space (Ω, Γ, μ). First we note that if $L_\varrho(X)=L^p(\Omega, X)$, $1\le p<\infty$, then

$$\$=\{E\in\Gamma;\ \mu(E)<\infty\}=\Gamma_0=\Gamma_0'=\Gamma_{00} \quad \text{(cf. Sect. 2).} \tag{4}$$

In fact $E\in\$$ if and only if $\chi_E\colon\ \Omega\to X$ satisfies $\|\chi_E\|_p=\varrho(\chi_E)<+\infty$, that is, $E\in\Gamma_0$. On the other hand if $\chi_E\colon\ \Omega\to X^*$ which can be written $\chi_E(t)=x^*\alpha_E(t)$ where

$x^* \in X^*$ and α_E is the scalar valued characteristic function, then, by Hölder inequality, we have

$$\varrho'(\chi_E) = \varrho'(\|\chi_E\|) = \sup \Big\{\int_\Omega |\langle f(t), \chi_E(t)\rangle|\, d\mu;\ \|f\|_p \leqq 1\Big\} =$$

$$= \sup \Big\{\int_\Omega \langle f(t), x^* > \alpha_E(t)\, d\mu\colon\ \|f\| \leqq 1\Big\} \leqq \begin{cases} \|x^*\|\mu(E)^{1/q} & \text{if } 1 < p < \infty \\ \|x^*\|\mu(E) & \text{if } p = 1. \end{cases}$$

Hence $E \in \Gamma_0'$ if and only if $\mu(E) < \infty$, that is, $E \in \$$.

Further, we want to characterize the spaces $L_\varrho^a(X)$ and $L_\varrho^\pi(X)$ when $L_\varrho(X) = L^p(\Omega, X)$. We give the following

LEMMA 1. *If $L_\varrho(X) = L^p(\Omega, X)$ for $1 \leqq p < \infty$, then $L_\varrho^a(X) = L^p(\Omega, X)$, and so for any admissible sequence π,*

$$L_\varrho^a(X) = L_\varrho^\pi(X) = L^p(\Omega, X). \tag{5}$$

PROOF. For any $f \in L^p(\Omega, X)$, there exists an admissible sequence $\pi = \{\Omega_n\}$ such that

$$\|f\alpha_{\Omega - \Omega_n}\|_p = \Big(\int_{\Omega - \Omega_n} \|f(t)\|^p\, d\mu\Big)^{1/p} < \frac{1}{n}.$$

Then $\lim_n \varrho(f\alpha_{\Omega - \Omega_n}) = 0$. On the other hand,

$$\|\chi_E\|_p = \varrho(\chi_E) = \Big(\int_\Omega \|x\alpha_E(t)\|^p\, d\mu\Big)^{1/p} = \|x\|\, \mu(E)^{1/p}, \quad x \in X.$$

$\varrho(\chi_E) < \infty$ if and only if $\mu(E) < \infty$. Thus for any sequence $\{E_n\}$, $E_n \subset E$, $\varrho(\chi_E) < \infty$ such that $\lim_n \mu(E_n) = 0$ we have $\lim_n \varrho(\chi_{E_n}) = 0$. This shows that $f \in L_\varrho^a(X)$ and so $L_\varrho^a(X) = L^p(\Omega, X)$.

Now for any admissible sequence π, (5) follows from $L_\varrho^a(X) \subset L_\varrho^\pi(X) \subset L_\varrho(X) = L^p(\Omega, X)$. Q.E.D.

REMARK. For $p = \infty$, $L^\infty(\Omega, X) = L_\varrho(X) \neq L_\varrho^a(X)$ since if Ω is purely nonatomic then $L_\varrho^a(X) = \{0\}$.

Our main result can be stated as follows.

THEOREM 2. *Let (Ω, Γ, μ) be a measure space and $L_\varrho(X) = L^p(\Omega, X)$, $1 < p < \infty$. Then*

$$L^p(\Omega, X)^* \cong L^q(\Omega, X^*), \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{6}$$

under the correspondence $F \in L^p(\Omega, X)^*$ and $g \in L^q(\Omega, X^*)$ defined by

$$F(f) = \int_\Omega \langle f, g\rangle(t)\, d\mu \quad \textit{for all } f \in L^p(\Omega, X) \tag{7}$$

if and only if X^* has the wide RNP.

PROOF. By Lemma 1 and Theorem B, we have only to show that

$$L_{\varrho'}(X^*) \cong L^q(\Omega, X^*).$$

The Hölder inequality guarantees that every $g \in L^q(\Omega, X^*)$ defines a continuous linear functional $F \in L^p(\Omega, X)^*$ such that $\|F\| \leqq \|g\|_q$, while the isometry can be shown by the same argument as in [3] (cf. also [5, Theorem 3.2]).

We prove that for every $F \in L^p(\Omega, X)^*$ there corresponds a $g \in L^q(\Omega, X^*)$ such that $\|F\| = \|g\|_q$. It sufficies to show that F is identically equal to zero if F is restricted to the functions in $L^p(\Omega, X)$ not belonging to $\bigcup\limits_{E \in \$} L^p(E, X)$ where $\$$ is the family of measurable subsets $E \subset \Omega$ with $\mu(E) < \infty$.

Let $E \in \Gamma$, $L^p(E, X) = \{f \in L^p(\Omega, X) \mid f = 0 \text{ outside } E\}$ and write $F(f) = F_E(f_E)$ as the continuous linear functional on $L^p(E, X)$. If $E \in \$$, then by Theorem A, there exists a unique $g_E \in L^q(E, X^*)$ such that $F(f) = F_E(f_E) = \int\limits_E \langle f, g_E \rangle(t)\, d\mu$ for all $f \in L^p(E, X)$ and $\|g_E\|_q = \|F_E\| \leqq \|F\|$ if and only if X^* has the RNP with respect to μ_E.

Let E_1 and E_2 be two disjoint subsets of finite measures. Then

$$g_{E_1 \cup E_2} = g_{E_1} + g_{E_2}, \quad \|g_{E_1 \cup E_2}\|_q^q = \|g_{E_1}\|_q^q + \|g_{E_2}\|_q^q. \tag{8}$$

Indeed, for any $f \in L^p(E_1 \cup E_2, X)$, we have $f = f_1 + f_2$ with $f_i \in L^p(E_i, X)$, $i = 1, 2$, and so

$$F(f) = F(f_1) + F(f_2) = \int\limits_\Omega \langle f_1, g_{E_1} \rangle(t)\, d\mu + \int\limits_\Omega \langle f_2, g_{E_2} \rangle(t)\, d\mu =$$

$$= \int\limits_\Omega \langle f, g_{E_1} + g_{E_2} \rangle(t)\, d\mu.$$

By the uniqueness of $g_{E_1 \cup E_2}$ corresponding to F, we see that (8) holds.

Now let $\gamma = \sup\limits_{E \in \$} \|g_E\|_q$. Since L^q, $1 < q < \infty$, permits an admissible sequence in Ω, we can consider a monotonously increasing sequence of sets $E_n \in \$$ such that $E_0 = \lim\limits_n E_n$ and $g(t) = \lim\limits_n g_{E_n}(t)$. Then g is strongly measurable and, by Lebesgue dominated convergence theorem, we obtain

$$\|g\|_q = \lim_n \|g_{E_n}\|_q = \gamma \leqq \|F\|.$$

Note that E_0 is σ-finite. If $E \in \$$ with $E \cap E_0 = \emptyset$ then $g_E = 0$. Indeed, $E_n \subset E_0$ for all n, thus it follows from (8) that for large n $\|g_{E \cup E_n}\|_q^q = \|g_E\|_q^q + \|g_{E_n}\|_q^q$, thus if $\|g_E\|_q \neq 0$, then $\|g_{E \cup E_n}\|_q > \gamma$ which is a contradiction. Hence for any $E \in \$$ and any $f \in L^p(E, X)$, there is $g_E \in L^q(E, X^*)$ corresponding to $F \in L^p(\Omega, X)^*$ such that

$$F(f) = \int\limits_\Omega \langle f, g_E \rangle(t)\, d\mu = \int\limits_\Omega \langle f, g_{E \cup E_0} \rangle(t)\, d\mu = \int\limits_\Omega \langle f, g \rangle(t)\, d\mu.$$

Finally, for any $f \in L^p(\Omega, X)$, it has absolutely continuous norm, and L^p permits an admissible sequence, there exist a monotonously increasing sequence of sets Ω_n, $\mu(\Omega_n) < +\infty$, $\lim\limits_n \Omega_n = \Omega_0$ and a sequence of functions $f_n \in L^p(\Omega_n, X)$ such that $f_n \to f$ in $L^p(\Omega, X)$. By the argument given above, we see that as $n \to \infty$, the equality

$$F(f_n) = \int\limits_\Omega \langle f_n, g_{\Omega_n} \rangle(t)\, d\mu = \int\limits_\Omega \langle f_n, g_{\Omega_0} \rangle(t)\, d\mu$$

tends to the representation of (7), and

$$\|g\|_q = \lim_n \|g_{\Omega_n}\|_q \leqq \gamma \leqq \|F\|.$$

The converse inequality $\|F\| \leqq \|g\|_q$ follows from Hölder inequality. Therefore our theorem follows from Theorem B, that is, (6) and (7) hold if and only if X^* has the wide RNP. Q.E.D.

If $p=1$, $q=\infty$ and (Ω, Γ, μ) is σ-finite, that is, $\Omega = \bigcup_{n=1}^{\infty} E_n$, $\mu(E_n) < +\infty$, then for any $F \in L^1(\Omega, X)^*$, by Theorem A, there is a $g_n \in L^\infty(E_n, X^*)$ such that

$$F|_{L^1(E_n, X)}(f) = F_{E_n}(f) = \int_\Omega \langle f, g_n \rangle(t)\, d\mu, \quad \text{for all} \quad f \in L^1(E_n, X)$$

and

$$\|g_n\|_\infty = \|F_{E_n}\| \leqq \|F\|$$

if and only if X^* has the RNP with respect to μ_{E_n}, and so $\|g\|_\infty \leqq \|F\|$. Thus if $f \in L^1(\Omega, X)$, then there exists a sequence $f_n \in L^1(E_n, X)$ with $f_n \to f$ in $L^1(\Omega, X)$ such that

$$F(f_n) = \int_{E_n} \langle f_n, g_n \rangle(t)\, d\mu = \int_\Omega \langle f_n, g \rangle(t)\, d\mu.$$

Since F is continuous, both sides of the equality converge to

$$F(f) = \int_\Omega \langle f, g \rangle(t)\, d\mu \quad \text{with} \quad \|g\|_\infty \leqq \|F\|.$$

It follows that $\|g\|_\infty = \|F\|$. Hence we obtain the following

THEOREM 3. *If (Ω, Γ, μ) is a σ-finite measure space, then*

$$L^1(\Omega, X)^* \cong L^\infty(\Omega, X^*) \tag{9}$$

under the correspondence $F \in L^1(\Omega, X)^*$ and $g \in L^\infty(\Omega, X^*)$ defined by

$$F(f) = \int_\Omega \langle f, g \rangle(t)\, d\mu \quad \textit{for all} \quad f \in L^1(\Omega, X) \tag{10}$$

if and only if X^* has the wide RNP.

4. The dual space of $L^1(\Omega, X)$ on a locally compact Hausdorff space Ω. Let Ω be a locally compact Hausdorff space and $M(\Omega)$ the space of regular Radon measures, that is, the dual space $C_c(\Omega)^*$ of $C_c(\Omega)$, the set of all continuous functions on Ω with compact support. Let μ be a positive Radon measure, that is,

$$\mu(f) = \int_\Omega f(t)\, d\mu \geqq 0 \quad \text{for} \quad f(t) \geqq 0 \quad \text{in} \quad C_c(\Omega).$$

Then the space $C_c(\Omega, X)$ of continuous X-valued functions with compact support in Ω is dense in $L^p(\Omega, X)$, $1 \leqq p < \infty$. In this case Theorem 3 holds without σ-finite condition on Ω, we shall give a characterization of the dual space $L^1(\Omega, X)^*$.

Let $L^\infty_{\text{l.a.e.}}(\Omega, X)$ be the space of locally bounded almost everywhere X-valued functions on Ω, that is, $g \in L^\infty_{\text{l.a.e.}}(\Omega, X)$ is uniformly bounded outside the local null sets in Ω. A set $E\subset\Omega$ is said to be *local null* if $\mu(K\cap E)=0$ for any compact subset K in Ω. The norm of $g\in L^{\infty}_{\mathrm{l.a.e.}}(\Omega, X)$ is given by

$$\|g\|_{\infty,\mathrm{l.a.e}} = \sup_K \left[\operatorname*{ess\,sup}_{t\in K} \|g(t)\|\right] < \infty$$

where the supremum is taken over all compact subsets k in Ω (see Dieudonne [2, p. 83]). Evidently, $\|g\|_{\infty,\mathrm{l.a.e}}\leqq\|g\|_{\infty}$ and $L^{\infty}(\Omega, X)\subset L^{\infty}_{\mathrm{l.a.e}}(\Omega, X)$. Hence we have the following

THEOREM 4. *Let Ω be a locally compact Hausdorff space with a positive Radon measure μ. Then*

$$L^1(\Omega, X)^* \cong L^{\infty}_{\mathrm{l.a.e}}(\Omega, X^*) \tag{11}$$

under the correspondence $F\in L^1(\Omega, X)^*$ and $g\in L^{\infty}_{\mathrm{l.a.e}}(\Omega, X^*)$ if and only if X^* has the wide RNP with respect to the positive Radon measure μ.

PROOF. For any $f\in C_c(\Omega, X)$ and $g\in L^{\infty}_{\mathrm{l.a.e.}}(\Omega, X^*)$, we have

$$\left|\int_{\Omega} \langle f, g\rangle(t)\, d\mu\right| \leqq \|f\|_1 \|g\|_{\infty,\mathrm{l.a.e}}.$$

Since $C_c(\Omega, X)$ is dense in $L^1(\Omega, X)$, it follows that g defines a bounded linear functional $F\in L^1(\Omega, X)^*$ such that $\|F\|\leqq\|g\|_{\infty,\mathrm{l.a.e}}$.

Conversely for any $F\in L^1(\Omega, X)^*$ and any compact subset K of Ω, the restriction $F|_{L^1(K,X)}=F_K$ corresponds to a function $g_K\in L^{\infty}(K, X^*)\subset L^{\infty}_{\mathrm{l.a.e.}}(K, X^*)$ such that $F(f)=\int_{\Omega}\langle f, g_K\rangle(t)\, d\mu$ for all $f\in L^1(K, X)$, and $\|g_K\|=\|F_K\|\leqq\|F\|$ (by Theorem 3) if and only if X^* has the wide RNP. Since $\overline{\bigcup_K L^1(K, X)}=L^1(\Omega, X)$ and $L^1(\Omega, X)$ permits an admissible sequence of subsets in Ω, we have

$$\sup_K \|g\|_{\infty,\mathrm{l.a.e.}} \leqq \sup_K \|g_K\|_{\infty} = \sup_K \|F_K\| \leqq \|F\|$$

where the supremum is taken over all compact subsets of Ω. Define $g(t)=g_K(t)$ locally for almost every $t\in K$. Then we have $\|g\|_{\infty,\mathrm{l.a.e.}}\leqq\|F\|$, and so (11) holds. Q.E.D.

REMARK. It is of some importance to remark that if the locally compact Hausdorff space Ω is the union of (possibly uncountable) σ-finite subsets Ω_α such that
(i) if $\mu(E)<\infty$ then $E\cap\Omega_\alpha\neq\emptyset$ for at most a countable number of α; and
(ii) any Borel set K with $\mu(K\cap\Omega_\alpha)=0$ for all α implies $\mu(K)=0$.
Then $L^{\infty}_{\mathrm{l.a.e.}}(\Omega, X^*)=L^{\infty}(\Omega, X^*)$.

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(Received October 27, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 53—64.

ON LINEATIONS

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In 1978 on the mathematical competition which is held in memoriam Miklós Schweitzer for students in Hungary the first author raised the following problem.

Let f be a superjective mapping of the hyperbolic plane onto itself such that the image of collinear points is again collinear. Prove that f must be an isometry.

To give a proof we first need to show that the map f is injective.

This can be verified by using a lemma of Nándor Simányi. Simányi was a participant of this competition.

First we say that a map $f\colon M \to M$ of a euclidean, hyperbolic or projective space M of arbitrary dimension into itself is a lineation of M if for each straight line e of M $f(e)$ is confined to a straight line of M.

Now the lemma of Simányi sounds as follows.

LEMMA 1. *Let M be the euclidean or hyperbolic plane and let $f\colon M \to M$ be an onto lineation of M. Then for each element Q of M the set $f^{-1}(\{Q\})$ is convex.*

PROOF. We argue by contradiction.

Suppose the existence of distinct points A, B, C in M such that

$$f(A) = f(B) = Q, \quad f(C) = P \neq Q$$

and that C lies on the segment $[A, B]$.

Let e_1 and e_2 be lines in M both perpendicular to e_{PQ} and such that $P \in e_1$ and $Q \in e_2$ (see Fig. 1). Thus

(1) $$e_1 \cap e_2 = \emptyset.$$

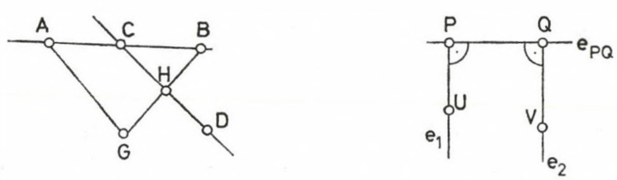

Fig. 1

Let $U \in e_1 \setminus \{P\}$ and $V \in e_2 \setminus \{Q\}$. Let $D \in f^{-1}(\{U\})$ and $G \in f^{-1}(\{V\})$. f is an onto map hence there exist such points D and G. Moreover the points A, B, C, D, G are pairwise distinct.

Since f is a lineation and the points P, Q, U, V are pairwise distinct we have obviously

(2) $$f(e_{AB}) \subset e_{PQ},$$

(3) $$f(e_{AG} \cup e_{BG}) \subset e_2,$$

(4) $$f(e_{CD}) \subset e_1.$$

Further since $f(D)=U \notin e_{PQ}$ and $f(G)=V \notin e_{PQ}$ it follows by (2) that $D \notin e_{AB}$ and $G \notin e_{AB}$. However in view of the axiom of Pasch the line e_{CD} intersects either the segment $[A, G]$ or that of $[B, G]$. Hence

$$e_{CD} \cap (e_{AG} \cup e_{BG}) \neq \emptyset.$$

Let $H \in e_{CD} \cap (e_{AG} \cup e_{BG})$. Then by (3) and (4) we have $f(H) \in e_1 \cap e_2$ and this contradicts relation (1). □

We turn now to the following

Lemma 2. *Each onto lineation of the euclidean or hyperbolic plane is an injective map.*

Proof. We argue again by contradiction.

Let f: $M \to M$ be an onto lineation of the euclidean or hyperbolic plane M and suppose the existence of two distinct points $A, B \in M$ such that $f(A)=f(B)$. Let $P=f(A)$ and let e_1 be a line in M for which

(5) $$f(e_{AB}) \subset e_1.$$

We then have obviously $P \in e_1$.

Let e' be a line in M going through P and distinct from e_1. Let Q be a point of $e' \setminus \{P\}$ and $C \in f^{-1}(\{Q\})$ (see Fig. 2). Then in view of (5) we find $C \notin e_{AB}$.

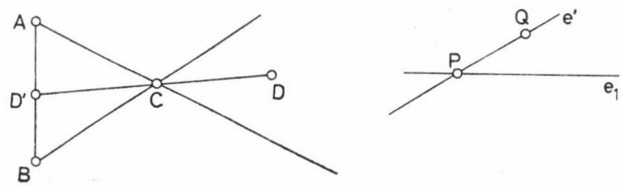

Fig. 2

Let D be an interior point of the angular region opposite to $\sphericalangle ACB$. Then the line e_{CD} meets the segment $[A, B]$ in a point D'. In view of Lemma 1 we have $f(D')=P$ and thus

$$f(e_{CD}) = f(e_{CD'}) \subset e_{PQ} = e'.$$

Consequently $f(D) \in e'$.

$f(D) \neq P$ since otherwise by $C \in [D, D']$ and because of Lemma 1 there would hold $f(C)=P$ contradicting the assumption $f(C)=Q \neq P$. Hence $f(D) \in e' \setminus \{P\}$, $D \in f^{-1}(e' \setminus \{P\})$.

Thus $f^{-1}(e' \setminus \{P\})$ contains the interior of an angular region. Consequently it contains a nonempty open set of M.

Observe that if e_1' and e_2' are distinct lines going through P then the sets $f^{-1}(e_1' \setminus \{P\})$ and $f^{-1}(e_2' \setminus \{P\})$ are disjoint. For that reason, since the set of lines of M going through P and distinct from e_1 is uncountable, there exists an uncountable family of mutually disjoint nonempty open subsets of M contradicting the fact that each family of this kind is countable.

The injectivity of f is proved. $\square$

Thereafter observe that for any bijective lineation $f\colon M \to M$ the map $f^{-1}\colon M \to M$ is obviously a lineation too. Moreover f maps each straight line e of M onto a straight line e' of M, and distinct lines should be transformed into distinct ones.

Now in order to prove the original problem observe first that the hyperbolic plane M may be considered as the interior of a conic k lying in the real projective plane P^2. (Interior of k means the set of points of P^2 that fail to lie on any tangent of k.)

Next, the bijective lineation $f\colon M \to M$ may be extended to a bijective lineation $\tilde{f}\colon P^2 \to P^2$ as follows.

For $Q \in P^2$ let e_1 and e_2 be distinct straight lines going through Q and intersecting M. For $i=1, 2$ let e_i' be the straight line containing the set $f(M \cap e_i)$ and let $\tilde{f}(Q)$ be the common point of e_1' and e_2' (see Fig. 3). According to the Desargues' theorem $\tilde{f}\colon P^2 \to P^2$ is uniquely defined. $\tilde{f}$ is clearly a bijective transformation of P^2 and $\tilde{f}|_M = f$. Moreover by Desargues' theorem $\tilde{f}$ is a lineation of P^2.

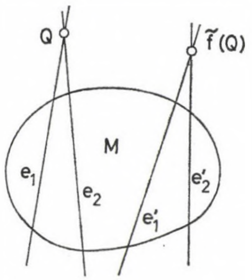

Fig. 3

Thus $\tilde{f}$ transforms k onto a conic k' and the interior M of k onto the interior M' of k'. However $\tilde{f}(M) = f(M) = M$. Consequently $k' = k$ and thus $\tilde{f}$ belongs to the group of congruences of M. The map $\tilde{f}$ and also the map $f = \tilde{f}|_M$ is a congruence of M indeed.

This is a solution of the problem in question. $\square$

Later in 1979 the first author raised the following question.

Is Lemma 2 true if we replace the euclidean or hyperbolic plane by the real projective plane?

The question was affirmatively answered by the second author. He also proved that if n is an integer bigger than 1 and M is an n-dimensional euclidean or real projective space then each onto lineation of M is injective.

First let us see the case $n=2$.

LEMMA 3. *Let M be the real projective plane and let $f\colon M\to M$ be a lineation such that $f(M)$ contains at least 4 points in general position (i.e. each triple of them is non-collinear). Then $f\colon M\to M$ is a bijective map.*

PROOF OF THE LEMMA. We only need to show that any lineation of the real projective plane keeping fixed 4 points in general position is an identity map of the plane.

Now let $f\colon M\to M$ be a lineation of M keeping fixed the points A_1, A_2, A_3 and A_4 such that each triple of them is noncollinear. Obviously we only need to prove that each point of $e_{A_1A_2}$ is a fixpoint of f. However the set of fixpoints of the map f lying on $e_{A_1A_2}$ is apparently dense in $e_{A_1A_2}$. Moreover $f(e_{A_1A_2})\subset e_{A_1A_2}$. Thus for proving the lemma we only need to show that for any three distinct fixpoints A, B, C of f lying on $e_{A_1A_2}$ and for each point P of $e_{A_1A_2}$ both of the relations $(AB, CP)>0$ and $(AB, Cf(P))<0$ cannot hold true. ((AB, CP) is the cross ratio of the points A, B, C, P).

Now let A, B, C be distinct fixpoints of f lying on $e_{A_1A_2}$. Let $D=A_3$ and let G be a fixpoint of f lying on the line e_{BD} and being distinct from D and B. There exists obviously such a fixpoint.

Introducing the symbol ∞ the relations

$$h(P)=(AB, CP)\quad (P\in e_{AB})\quad\text{and}\quad h'(Q)=(DB, GQ)\quad (Q\in e_{DB})$$

define clearly bijective maps

$$h\colon e_{AB}\to R\cup\{\infty\}\quad\text{and}\quad h'\colon e_{DB}\to R\cup\{\infty\}.$$

Now let

$$m=h\circ f|_{e_{AB}}\circ h^{-1}\colon R\cup\{\infty\}\to R\cup\{\infty\}$$

and

$$m'=h'\circ f|_{e_{DB}}\circ h'^{-1}\colon R\cup\{\infty\}\to R\cup\{\infty\}.$$

Since A, B and C are fixpoints of f it follows

(6) $$m(1)=1,\quad m(0)=0,\quad m(\infty)=\infty.$$

The remaining part of the proof proceeds in several steps.

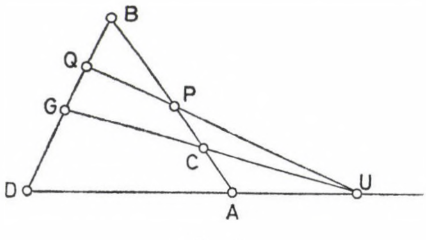

Fig. 4

1° First we show that $m=m'$.

In fact let U be the common point of e_{AD} and e_{GC} (see Fig. 4). U is clearly a fixpoint of f. Let $x\in R\cup\{\infty\}$ and $P=h^{-1}(x)$. Then $(AB, CP)=x$. Let Q be the common point of e_{UP} and e_{DB}. Then we have $(DB, GQ)=(AB, CP)$ and thus $Q=h'^{-1}(x)$. By the collinearity of P, Q, U and by $f(U)=U$ we find that $U, f(P)$ and $f(Q)$ are collinear as well and thus $(AB, Cf(P))=(DB, Gf(Q))$. Consequently $m(x)=m'(x)$ indeed.

2° Let us introduce the relations

$$0=\frac{0}{a}=\frac{0}{\infty}=\frac{a}{\infty}\quad (a\in R\setminus\{0\}),\quad \infty=\frac{\infty}{a}=\frac{\infty}{0}=\frac{a}{0}\quad (a\in R\setminus\{0\}).$$

Let V and W be points of e_{AB} where the cases $V=W=A$ and $V=W=B$ are excluded. We then have obviously

$$\frac{(AB, CW)}{(AB, CV)}=(AB, VW).$$

3° Let V and W be points of e_{AB} where the cases $V=W=A$ and $V=W=B$ are excluded. Let Z be the common point of e_{GV} and e_{AD} and let S be the common point of e_{ZW} and e_{DG}. Then $(AB, VW)=(DB, GS)$.

The statement is obviously true if $V\in e_{AB}\setminus\{A, B\}$. (See Fig. 5.)

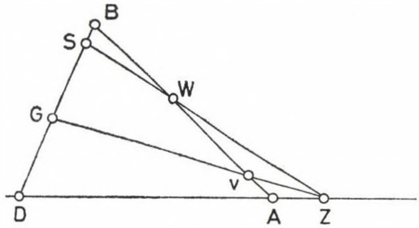

Fig. 5

If $V=A$ then $Z=A$, $S=B$ and thus $(AB, VW)=0=(DB, GS)$. On the other hand if $V=B$ then $Z=D$, $S=D$ and thus $(AB, VW)=\infty=(DB, GS)$.

4° Let x, y be elements of $R\setminus\{0\}$ where the cases $m(x)=m(y)=0$ and $m(x)=$ $=m(y)=\infty$ are excluded. Then

$$m\left(\frac{x}{y}\right)=\frac{m(x)}{m(y)}.$$

In fact let $P_1=h^{-1}(y)$ and $P_2=h^{-1}(x)$. Then by 2° we have

$$(AB, P_1P_2)=\frac{(AB, CP_2)}{(AB, CP_1)}=\frac{x}{y}. \tag{7}$$

Let T be the common point of e_{GP_1} and e_{AD} and Q the common point of e_{TP_2} and e_{DB}. Then 3° and (7) show that $(DB, GQ)=(AB, P_1P_2)=\dfrac{x}{y}$ and thus

$$Q=h'^{-1}\left(\frac{x}{y}\right). \tag{8}$$

Since the case $m(x)=m(y)=0$ is excluded it follows that the case $f(P_1)=$ $=f(P_2)=B$ cannot occur. Likewise the case $f(P_1)=f(P_2)=A$ cannot occur either. Now since f is a lineation $f(T)$ must be the common point of $e_{Gf(P_1)}$ and e_{AD}. Moreover $f(Q)$ is the common point of $e_{f(T)f(P_2)}$ and e_{DB}. Thus we have according to (8), 3°, 2° and 1°

$$m\left(\frac{x}{y}\right)=(DB, Gf(Q))=(AB, f(P_1)f(P_2))=\frac{(AB, Cf(P_2))}{(AB, Cf(P_1))}=\frac{m(x)}{m(y)}$$

as required.

5° Taking also (6) into account 4° shows that for any $y\in R\cup\{\infty\}$ we have $m\left(\frac{1}{y}\right)=\frac{1}{m(y)}$.

We are going to finish the proof of the lemma.

6° Let P be a point of e_{AB} such that $(AB, CP)>0$ and $(AB, CP)\neq\infty$. Let $x=(AB, CP)$ and $y=+\sqrt{x}$. Then

$$(AB, Cf(P))=m(y^2)=m\left(\frac{y}{\frac{1}{y}}\right).$$

However 5° shows that $m(y)=m\left(\frac{1}{y}\right)$ can only occur in the case $m(y)=\pm 1$ and thus in view of 4° and 5° we have

$$(AB, Cf(P))=\frac{m(y)}{\frac{1}{m(y)}}.$$

Consequently $(AB, Cf(P))$ cannot be a negative real number.

The proof of Lemma 3 is complete. □

And now let us formulate again the theorem of the second author.

THEOREM 1. *Let n be an integer bigger than 1 and let M be an n-dimensional euclidean or real projective space. Then each onto lineation of M is injective.*

PROOF. According to Lemmas 2 and 3 we need only consider the case $n>2$. Now suppose that the theorem is true if we decrease the integer n (at most until 2).

Let $f\colon M\to M$ be an onto lineation of M.

First we call a system $(q_1, \ldots, q_k)$ of points of M independent if $k\leq n+1$ and if there is no $(k-2)$-plane of M containing all the points $q_1, \ldots, q_k$.

Each subsystem of an independent system of points is independent as well. Moreover for any independent system of points $(q_1, \ldots, q_k)$ there is a unique $(k-1)$-plane of M containing all the points $q_1, \ldots, q_k$. We say that this plane is spanned by $q_1, \ldots, q_k$ and we denote it by $S(q_1, \ldots, q_k)$.

7° Let $(p_1, \ldots, p_k)$ be an independent system of points in M such that the system $(f(p_1), \ldots, f(p_k))$ is independent as well. Then

$$f(S(p_1, \ldots, p_k))\subset S(f(p_1), \ldots, f(p_k)).$$

To prove this statement we proceed by induction.

If $k=1$ then the statement is obviously true. Now let $1<t\leqq n+1$ and suppose that the statement is true for $k=t-1$. Let $(p_1, \ldots, p_t)$ be an independent system of points in M such that $(f(p_1), \ldots, f(p_t))$ is independent as well and let $q\in S(p_1, \ldots, p_t)$.

For $i=1, \ldots, t$ let $S_i=S(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_t)$ and

$$S_i' = S(f(p_1), \ldots, f(p_{i-1}), f(p_{i+1}), \ldots, f(p_t)).$$

Then there is obviously an index i $(1\leqq i\leqq t)$ and a point q' in S_i such that q belongs to the line $e_{p_i q'}$. (If M is a real projective space then we may select $i=1$). By the induction hypothesis $f(q')\in S_i'$ and thus $f(q')\neq f(p_i)$. Hence $f(q)$ belongs to the line $e_{f(q')f(p_i)}$ and thus to the plane $S(f(p_1), \ldots, f(p_t))$ as required.

Accordingly the statement is true for $k=t$ which completes the proof.

8° Let $(p_1, \ldots, p_k)$ be a sequence of points in M such that $(f(p_1), \ldots, f(p_k))$ is an independent system. Then $(p_1, \ldots, p_k)$ is an independent system too.

In fact let S be the intersection of all planes of M containing the set $\{p_1, \ldots, p_k\}$. S is a plane and we may select an independent subsequence $(p_{i_1}, \ldots, p_{i_t})$ of the sequence $(p_1, \ldots, p_k)$ such that $S=S(p_{i_1}, \ldots, p_{i_t})$. Consequently in view of 7°

$$f(S)\subset S(f(p_{i_1}), \ldots, f(p_{i_t}))$$

and thus for $i=1, \ldots, k$ we have

$$f(p_i)\in S(f(p_{i_1}), \ldots, f(p_{i_t})).$$

Hence $t=k$ and $i_j=j$ for $j=1, \ldots, k$. Accordingly $(p_1, \ldots, p_k)$ is an independent system of points as required.

9° For $0\leqq k\leqq n$ and for any plane S of dimension k the set $f(S)$ is contained in a k-plane of M.

In fact otherwise $f(S)$ would contain an independent system of points $(p_1', \ldots, p_{k+2}')$. Let $p_i\in S\cap f^{-1}(\{p_i'\})$. Then by 8° the sequence $(p_1, \ldots, p_{k+2})$ would be independent as well and it would lie in S which is impossible.

10° Let $1\leqq k\leqq n$ and let S be a k-plane of M. Suppose that $f(S)$ is confined to the union of a finite number of $(k-1)$-planes of M. Then there is a $(k+1)$-plane S' in M such that $f(S')$ is confined to the union of a finite number of k-planes of M.

In fact let $f(S)\subset S_1\cup\ldots\cup S_r$ where $S_1, \ldots, S_r$ are $(k-1)$-planes of M. Let $q'\in M\setminus(S_1\cup\ldots\cup S_r)$. There exists obviously such a point q'. Let $q\in f^{-1}(\{q'\})$. Then $q\notin S$. Let S' be the $(k+1)$-plane containing the set $\{q\}\cup S$ and for $i=1, \ldots, r$ let S_i' be the k-plane containing $\{q'\}\cup S_i$. If M is a euclidean space then let S'' be the k-plane going through q and parallel to S and let S_{r+1}' be a k-plane containing $f(S'')$. In view of 9° there exists such a plane S_{r+1}'. If M is a real projective space then let S_{r+1}' be an arbitrary k-plane. In both cases we obviously have

$$f(S')\subset S_1'\cup\ldots\cup S_{r+1}'.$$

11° Let $1\leqq k\leqq n$. Then for each k-plane S in M $f(S)$ can not be confined to the union of a finite number of $(k-1)$-planes.

Since the only n-plane of M is M itself and f is an onto map the assertion is obviously true for $k=n$. Hence according to 10° it is true for every k $(k \leqq n)$.

12° Let S be a 2-plane of M. Let S' be a 2-plane of M containing $f(S)$ (see 9°). Then $f(S)=S'$.

We argue again by contradiction. Suppose the existence of a point q' in $S' \setminus f(S)$. Let $q \in f^{-1}(\{q'\})$. Then $q \notin S$. Let S_1 be the 3-plane of M containing $S \cup \{q\}$. If M is a euclidean space then let S_2 be the 2-plane going through q and parallel to S and let S_2' be a 2-plane of M containing $f(S_2)$. If M is a real projective space then let S_2' be an arbitrary 2-plane of M.

Now in both cases $f(S_1)$ is clearly contained in $S' \cup S_2'$ which is impossible by 11°.

We are going to finish the proof of the theorem.

13° Let p and q be distinct points of M and let S be a 2-plane of M containing p and q. In view of 12° $f(S)$ is a 2-plane of M. Let $g\colon f(S) \to S$ be a bijective linear map (that means g takes each line into a line). There exists obviously such a map. Then

$$g \circ f|_S\colon S \to S$$

is an onto lineation and thus according to Lemmas 2 and 3 $g(f(p)) \neq g(f(q))$. Thus $f(p) \neq f(q)$.

The theorem is proved.

Now we can raise the following question.

Is Lemma 2 true if we replace the euclidean or hyperbolic plane in it by the hyperbolic space?

Since the hyperbolic space may be considered as an open ball in the euclidean 3-space where the straight lines of the hyperbolic space are the nonempty intersections of the straight lines of the euclidean space with the open ball, the following theorem of the first author gives an affirmative answer to this question.

THEOREM 2. *Let K be a nonempty open convex set in the euclidean n-space R^n where $n \geqq 2$. Let $f\colon K \to R^n$ be a mapping satisfying the following two conditions:*

(a) *for each straight line e of R^n there exists a straight line e' of R^n such that*

$$f(e \cap K) \subset e',$$

(b) *$f(K)$ is open in R^n.*

Then the map $f\colon K \to f(K)$ is bijective and $f(K)$ is a convex set.

PROOF. 14° Let $(p_1, \dots, p_k)$ be an independent system of points in K such that $(f(p_1), \dots, f(p_k))$ is independent as well. Then

$$f(S(p_1, \dots, p_k) \cap K) \subset S(f(p_1), \dots, f(p_k)).$$

To prove this statement we proceed by induction.

If $k=1$ then the statement is obviously true.

Now let $1 < t \leqq n+1$ and suppose that the statement is true for $k=t-1$. Let $(p_1, \dots, p_t)$ be an independent system of points in K such that $(f(p_1), \dots, f(p_t))$ is independent as well and let $q \in S(p_1, \dots, p_t) \cap K$. For $i=1, \dots, t$ let $S_i = S(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_t)$ and $S_i' = S(f(p_1), \dots, f(p_{i-1}), f(p_{i+1}), \dots, f(p_t))$. Then there is obviously an index i $(1 \leqq i \leqq t)$ and a point q' in $S_i \cap K$ such that q belongs to the line $e_{p_i q'}$. By the induction hypothesis $f(q')\in S'_i$ and thus $f(q')\neq f(p_i)$. Hence $f(q)$ belongs to the line $e_{f(q')f(p_i)}$ and thus to the plane $S(f(p_1), \dots, f(p_t))$ as required.

Accordingly the statement is true for $k=t$ which completes the proof.

15° Let $(p_1, \dots, p_k)$ be a sequence of points in K such that $(f(p_1), \dots, f(p_k))$ is an independent system of points. Then $(p_1, \dots, p_k)$ is an independent system of points too.

The proof is nearly the same as that of 8° only 7° must be replaced by 14° and $f(S)$ by $f(S\cap K)$.

16° For $0\leqq k\leqq n$ and for any k-plane S of R^n the set $f(S\cap K)$ is contained in a k-plane of R^n.

The proof is the same as that of 9°. Only $f(S)$ must be replaced by $f(S\cap K)$ and 8° by 15°.

17° For each $q\in R^n$ $f^{-1}(\{q\})$ is a convex set.

We argue by contradiction. Suppose that a, b, c are points of K such that c lies on the segment $[a, b]$ and $f(a)=f(b)=q, f(c)=q'\neq q$. Let S' be a hyperplane of R^n going through q' and missing q and let e' be a straight line of R^n going through q and parallel to S' (i.e. $e'\cap S'=\emptyset$). Let $q'_2, \dots, q'_n$ be points of $S'\cap f(K)$ such that the system $(q', q'_2, \dots, q'_n)$ should be independent. Since $f(K)$ is open in R^n and $q'\in f(K)$ there exists such a system. Let $p_2, \dots, p_n$ be points of K such that for $i=2, \dots, n$ the relation $f(p_i)=q'_i$ holds. In view of 15° $(c, p_2, \dots, p_n)$ is an independent system of points.

Let $S=S(c, p_2, \dots, p_n)$. According to 16° we have $f(S\cap K)\subset S'$ and since $f(a)=$ $=f(b)=q\notin S'$ it follows $\{a, b\}\cap S=\emptyset$. However $c\in S$ and thus $[a, b]\cap S\neq\emptyset$, a and b lie on different open halfspaces of R^n bounded by S.

Let q^* be a point of $e'\cap f(K)$ distinct from q. Since $f(K)$ is open in R^n and $q\in f(K)$ there exists such a q^*. Let $p^*\in f^{-1}(\{q^*\})$. Then $p^*\neq a$ and $p^*\neq b$. Moreover

$$f(e_{ap^*}\cap K)\subset e',\ f(e_{bp^*}\cap K)\subset e'. \tag{9}$$

By the convexity of K the segments $[a, p^*]$ and $[b, p^*]$ belong to K and thus in view of (9) we have

$$f([a, p^*]\cup[b, p^*])\subset e'. \tag{10}$$

However since a and b lie in different halfspaces bounded by S it follows that $([a, p^*]\cup[b, p^*])\cap S\neq\emptyset$ and thus (10) implies $e'\cap S'\neq\emptyset$ which contradicts the assumption $e'\cap S'=\emptyset$.

The assertion is proved.

18° Suppose the existence of distinct points $a, b\in K$ such that $f(a)=f(b)$. Then there exists a sequence $p_0, \dots, p_{n-2}$ of points of K satisfying the following three conditions.

(i) $p_0=b$,

(ii) $(a, p_0, \dots, p_{n-2})$ is an independent system of points in R^n,

(iii) $(f(p_0), \dots, f(p_{n-2}))$ is an independent system of points as well.

We shall construct such a sequence by a recursive way.

The systems (a, b) and $(f(b))$ are clearly independent.

Suppose that $0<k\leqq n-2$ and the points $p_0, \ldots, p_{k-1}$ of K have been taken such that $p_0=b$ and the systems $(a, p_0, \ldots, p_{k-1})$ and $(f(p_0), \ldots, f(p_{k-1}))$ should be independent. Let $S=S(a, p_0, \ldots, p_{k-1})$ and let S' be a k-plane of R^n containing the set $f(S\cap K)$. According to 16° there exists such an S'. Observe that for $i=0, \ldots, k-1$ $f(p_i)$ obviously belongs to S'. Let q_k be a point of $f(K)\setminus S'$. Since $f(K)$ is nonempty and open in R^n there exists such a point q_k. The system $(f(p_0), \ldots, f(p_{k-1}), q_k)$ is then clearly independent.

Let $p_k\in f^{-1}(\{q_k\})$. Then $p_k\notin S$ and thus the system $(a, p_0, \ldots, p_k)$ is likewise independent.

The existence of a sequence $(p_0, \ldots, p_{n-2})$ with the desired properties is proved.

19° Under the same assumption as in 18° and for the same sequence $p_0, \ldots, p_{n-2}$ let us consider the simplex s^{n-1} with vertices $a, p_0, \ldots, p_{n-2}$. By the convexity of K one obviously has $s^{n-1}\subset K$.

We are going to show that

$$f(s^{n-1}) \subset S(f(p_0), \ldots, f(p_{n-2})).$$

We proceed by induction.

For $i=1, \ldots, n-1$ let s^i be the simplex of R^n with the vertices $a, p_0, \ldots, p_{i-1}$ and let $S^{i-1}=S(f(p_0), \ldots, f(p_{i-1}))$. Since $f(a)=f(p_0)$ it follows by 17° that $f(s^1)\subset S^0$.

Suppose that $1<i\leqq n-1$ and that $f(s^{i-1})\subset S^{i-2}$ holds true.

Let $d\in s^i$. Then d lies on a segment $[d', p_{i-1}]$ where $d'\in s^{i-1}$ and thus by the induction hypothesis $f(d')\in S^{i-2}$. However since the system of points $(f(p_0), \ldots, f(p_{i-1}))$ is independent it follows that $f(p_{i-1})\notin S^{i-2}$ and thus $f(p_{i-1})\neq$ $\neq f(d')$. Consequently

$$f([d', p_{i-1}]) \subset f(e_{d' p_{i-1}} \cap K) \subset e_{f(d') f(p_{i-1})} \subset S^{i-1}$$

and thus $f(d)\in S^{i-1}$ which proves $f(s^i)\subset S^{i-1}$.

20° We are going to prove that the map f is injective

We argue again by contradiction.

Suppose the existence of distinct points $a, b\in K$ such that $f(a)=f(b)$. Let $p_0, \ldots, p_{n-2}$ be the same as in 18°. Moreover let s^{n-1} and S^{n-2} be the same as in 19°. According to 19° we have

$$f(s^{n-1}) \subset S^{n-2}. \tag{11}$$

Let $S=S(a, p_0, \ldots, p_{n-2})$ and let S_0' be an $(n-1)$-plane of R^n containing $f(S\cap K)$. In view of 16° there exists such a plane and we obviously have $S^{n-2}\subset S_0'$. Let S' be a hyperplane of R^n distinct from S_0' and containing S^{n-2}. Let $q'\in(S'\cap f(K))\setminus S^{n-2}$. Since $f(K)$ is open in R^n and by (11) $S'\cap f(K)\neq\emptyset$ there exists such a point q'.

Let $q\in f^{-1}(\{q'\})$. Since $q'=f(q)\notin f(S\cap K)$ it follows $q\notin S$ and thus the system of points $(a, p_0, \ldots, p_{n-2}, q)$ is independent. Hence we get an n-simplex s_q^n in R^n with the vertices $a, p_0, \ldots, p_{n-2}, q$ where s^{n-1} is a face of s_q^n.

For $i=0, \ldots, n-2$ let Φ_i be the open halfspace of R^n bounded by the hyperplane $S(a, p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n-2}, q)$ of R^n and missing the point p_i. Let Φ_a

be the open halfspace of R^n bounded by $S(p_0, ..., p_{n-2}, q)$ and missing the point a. Then

$$G_q = \Phi_a \cap \Phi_0 \cap \ldots \cap \Phi_{n-2} \cap K$$

is obviously a nonempty open set in R^n and $q \notin G_q$.

Let $u \in G_q$. The straight line e_{qu} meets s^{n-1} in a point u^* and q belongs to the segment $[u, u^*]$. Moreover we have $e_{qu} = e_{qu^*}$. Since $u^* \in s^{n-1}$ it follows

$$f(u^*) \in S^{n-2} \subset S'. \tag{12}$$

On the other hand

$$f(q) = q' \in S' \setminus S^{n-2} \tag{13}$$

and thus

$$f(e_{qu}) = f(e_{qu^*}) \subset e_{q'f(u^*)} \subset S'. \tag{14}$$

According to (12) and (13) the line $e_{q'f(u^*)}$ intersects S^{n-2} in the only point $f(u^*)$. Moreover by (14) we have

$$f(u) \in e_{q'f(u^*)}. \tag{15}$$

$f(u) = f(u^*)$ can not occur since otherwise in view of $q \in [u, u^*]$ and 17° $f(q) = f(u^*)$ would hold and this is impossible by (12) and (13). Thus $f(u) \neq f(u^*)$ consequently by (15) and (14) we have $f(u) \in S' \setminus S^{n-2}$. This yields the relation

$$G_q \subset f^{-1}(S' \setminus S^{n-2}). \tag{16}$$

Consequently $f^{-1}(S' \setminus S^{n-2})$ contains a nonempty open subset of R^n.

However for distinct hyperplanes S_1' and S_2' containing S^{n-2} the sets $S_1' \setminus S^{n-2}$ and $S_2' \setminus S^{n-2}$ are disjoint and thus the sets $f^{-1}(S_1' \setminus S^{n-2})$ and $f^{-1}(S_2' \setminus S^{n-2})$ are disjoint as well.

The family of hyperplanes of R^n containing S^{n-2} and distinct from S_0' is uncountable and thus there would exist an uncountable family of mutually disjoint nonempty open sets of R^n contradicting the fact that each family of mutually disjoint nonempty open sets of R^n is countable.

The injectivity of the map f is proved.

What about the convexity of $f(K)$? First observe that the space R^n can be considered as $P^n \setminus P^{n-1}$ where for $i = n-1$, n P^i is the i-dimensional real projective space. Now the bijective map f: $K \to f(K)$ may be extended to a bijective lineation $\bar{f}$: $P^n \to P^n$ in the same way what we have done at the proof of the original problem.

Now let a', b' be distinct points of $f(K)$ and let $a = f^{-1}(a')$, $b = f^{-1}(b')$. Then $f([a, b]) \subset f(K) \subset P^n \setminus P^{n-1}$ and $f([a, b])$ is a projective segment in P^n with the endpoints a', b'. However since $f([a, b]) \cap P^{n-1} = \emptyset$ $f([a, b])$ is the euclidean segment in R^n joining a' and b' and thus $[a', b'] \subset f(K)$. $f(K)$ is convex indeed. The proof of Theorem 2 is complete. □

Now we can raise the following question:

Does Lemma 3 remain true if we replace number 4 by number 3 in it?

It is easy to see that the answer is negative. Or even more there exist several lineations f: $M \to M$ of the real projective plane M such that $f(M)$ consists of three noncollinear points. Each lineation of this kind clearly gives rise to a colouring of M in three colours so that no straight line contains points of all three colours.

In fact if $f(M)=\{a, b, c\}$ then let us colour the points of $f^{-1}(\{a\})$ by red those of $f^{-1}(\{b\})$ white and those of $f^{-1}(\{c\})$ blue. On the other hand each colouring of this kind corresponds obviously to a unique lineation $f\colon M\to M$ such that $f(M)$ consists of three noncollinear points.

Now by a colouring of a projective plane we understand in the sequel a colouring of the plane in three colours say red, white and blue so that no line contains points of all three colours.

One type of colouring say radial colouring can be obtained by colouring a point p red, say and colouring each line through p with p delated either solid white or solid blue, randomly. Another type say axial colouring is obtained by colouring the points of a line e either white or blue randomly and colouring all points not on e red. We shall call colourings of either of the above type trivial. The trivial colourings are precisely the ones on which some colour is confined to a line.

Thereafter the first author posed the following question:

Does there exist a nontrivial colouring of the real projective plane?

The question has been answered affirmatively by A. W. Hales and E. G. Straus [2]. They also proved in 1980 that the projective plane $P^2(F)$ over the commutative field F has no nontrivial colouring if and only if F is an algebraic extension of a finite field.

Finally we refer the reader to the paper of David S. Carter and Andrew Vogt [1]. They solved the problem of characterization of all lineations of projective or affine Desarguesian planes. However as to the hyperbolic plane the same problem is still unsolved.

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(Received November 15, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 65—72.

PROPERTIES OF THE RELATIVE ENTROPY OF STATES OF VON NEUMANN ALGEBRAS

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In an operator algebra the measure of the Kullback—Leibler information was defined by Umegaki [26]. Nowadays it is quite common that a non-commutative algebra with a specified positive functional may serve as the basic object of an algebraic (or non-commutative or quantum) probability theory. In case when $\mathcal{A}$ is a von Neumann algebra possessing a faithful normal semifinite trace τ then any normal state of $\mathcal{A}$ can be described by a positive selfadjoint operator affiliated with $\mathcal{A}$. In the finite dimensional case this operator is simply the corresponding density matrix. To be more concrete, if $\varphi_i(a)=\tau(\varrho_i a)$ ($a\in\mathcal{A}$, $i=1, 2$), the relative entropy of φ_1 and φ_2 is then defined as follows:

$$S(\varphi_1, \varphi_2) = \varphi_2(\log \varrho_2 - \log \varrho_1) = \tau\big(\varrho_2(\log \varrho_2 - \log \varrho_1)\big).$$

We ought to note here that the relative entropies occurring in the literature differ slightly sometimes both in sign and order of the states. For example the entropy S_{B-R} in [5] is related to this one as

$$S_{B-R}(\varphi_1, \varphi_2) = -S(\varphi_2, \varphi_1).$$

Some properties of the relative entropy have important physical interpretations ([5], [18], [27]). We do not treat them in details but we look at the following phenomena. If $\mathcal{B}$ is another algebra and a stochastical mapping $\alpha\colon \mathcal{B}\to\mathcal{A}$ carries the states ω_1, ω_2 into φ_1, φ_2, respectively, then intuitively it is more difficult to make distinction between φ_1 and φ_2 than ω_1 and ω_2, consequently it is quite natural that $S(\omega_1, \omega_2)\leqq S(\varphi_1, \varphi_2)$.

If $\mathcal{A}$ does not have a trace then one can not compare two states by means of their densities. In an arbitrary von Neumann algebra Araki defined the relative entropy of two normal states using the relative modular operator ([2], [3]). By now all the important properties of the Kullback—Leibler relative entropy have been verified in general von Neumann algebras. Among them the convexity must have the largest literature. In this paper we do not touch this area and we refer to the recent paper [10] and the older ones [13], [14]. We prove the strong superadditivity of the relative entropy utilizing an idea of Lindblad ([16]).

Auxiliary results

In this section we prove some auxiliary results. Most of them may be known but have not succeeded in finding satisfactory references. First we look at theorems of interpolation type.

Let Δ be a positive selfadjoint operator on a Hilbert space $\mathcal{H}$ and let $\varepsilon>0$ be fixed for a while. The domain $\mathcal{D}(\Delta^t)$ becomes a Hilbert space $\mathcal{H}_t$ with the norm

$$\|\xi\|_t = (\varepsilon\|\xi\|^2+\|\Delta^t\xi\|^2)^{1/2}$$

for any $0\leqq t\leqq 1$. The pair $(\mathcal{H}_0, \mathcal{H}_1)$ is a compatible couple in the sense of interpolation theory and it may be the starting point of an interpolation giving a scale of spaces $(\mathcal{H}_0, \mathcal{H}_1)_t$ $(0\leqq t\leqq 1)$. Since we use only few results of interpolation theory we do not want to introduce its complete machinery. Instead, we refer to some standard books [4], [21] and [24]. In [24] one can find the following theorem: $(\mathcal{H}_0, \mathcal{H}_1)_t=\mathcal{H}_t$ (1.18.10). It will be used to prove

PROPOSITION 1. *Let Δ_j be a positive selfadjoint operator on $\mathcal{H}^j$ $(j=1,2)$. If $T\colon \mathcal{H}^1\to\mathcal{H}^2$ is a bounded operator such that*

(i) $T\mathcal{D}(\Delta_1)\subset\mathcal{D}(\Delta_2)$,

(ii) $\|\Delta_2 T\xi\| \leqq \|T\|\cdot\|\Delta_1\xi\| \quad (\xi\in\mathcal{D}(\Delta_1))$,

then we have for every $0\leqq t\leqq 1$ and $\xi\in\mathcal{D}(\Delta_1^t)$

$$\|\Delta_2^t T\xi\| \leqq \|T\|\cdot\|\Delta_1^t\xi\|.$$

PROOF. For every fixed $\varepsilon>0$ we have $(\mathcal{H}_0^j, \mathcal{H}_1^j)_t=\mathcal{H}_t^j$ $(j=1,2$ and $0\leqq t\leqq 1)$ by the above cited result. Since $\|T\xi\|_{\mathcal{H}_0^2}\leqq\|T\|\cdot\|\xi\|_{\mathcal{H}_0^1}$ and $\|T\eta\|_{\mathcal{H}_1^2}\leqq\|T\|\cdot\|\eta\|_{\mathcal{H}_1^2}$ for every $\xi\in\mathcal{H}_0^1$ and $\eta\in\mathcal{H}_1^1$ the Calderon—Lions interpolation theorem (see, for example, [4, 8.12] or [21, IX.20]) gives that

$$\|T\xi\|_{\mathcal{H}_t^2} \leqq \|T\|\,\|\xi\|_{\mathcal{H}_t^1}$$

holds for any $\xi\in\mathcal{D}(\Delta_1^t)=\mathcal{H}_t^1$. Equivalently,

$$\varepsilon\|T\xi\|^2+\|\Delta_2^t T\xi\|^2 \leqq \|T\|^2(\varepsilon\|\xi\|^2+\|\Delta_1^t\xi\|^2).$$

Letting $\varepsilon\to 0$ we obtain the Proposition.

LEMMA 1. *If Δ is a positive selfadjoint operator and $\xi\in\mathcal{D}(\Delta)$ then $\|\Delta^t\xi\|\leqq\leqq\|\xi\|^{1-t}\cdot\|\Delta\xi\|^t$ for any $0\leqq t\leqq 1$.*

PROOF. It is sufficient to apply the three lines theorem [7, p. 520] to the bounded analytical function $z\mapsto\Delta^z\xi$ $(0\leqq \operatorname{Re} z\leqq 1)$. (For a more elementary proof, see [17, p. 141].)

LEMMA 2. *Let Δ be a positive selfadjoint operator and $\xi\in\mathcal{D}(\Delta)$. Then*

$$\lim_{t\to+0} t^{-1}(\|\Delta^{t/2}\xi\|^2-\|\xi\|^2)$$

exists. It is finite or $-\infty$ *and equals* $\int_0^\infty \log \lambda d\langle E_\lambda \xi, \xi\rangle$ *where* $\int_0^\infty \lambda\, dE_\lambda$ *is the spectral resolution of* Δ.

PROOF. By the spectral theorem

$$t^{-1}(\|\Delta^{t/2}\xi\|^2-\|\xi\|^2) = \int_0^\infty t^{-1}(\lambda^t-1)\, d\,\langle E_\lambda \xi, \xi\rangle.$$

Using the monotone convergence theorem we have

$$\int_1^\infty t^{-1}(\lambda^t-1)\, d\,\langle E_\lambda \xi, \xi\rangle \xrightarrow[t\to+0]{} \int_1^\infty \log\lambda\, d\langle E_\lambda \xi, \xi\rangle$$

and the limit is finite. According to the Fatou lemma

$$\int_0^1 \log\lambda\, d\langle E_\lambda \xi, \xi\rangle \geqq \limsup_{t\to+0} \int_0^1 t^{-1}(\lambda^t-1)\, d\langle E_\lambda \xi, \xi\rangle$$

and on the other hand $\log \lambda \leqq t^{-1}(\lambda^t-1)$. Hence

$$\int_0^1 t^{-1}(\lambda^t-1)\, d\langle E_\lambda \xi, \xi\rangle \xrightarrow[t\to+0]{} \int_0^1 \log\lambda\, d\langle E_\lambda \xi, \xi\rangle.$$

The strong convergence of bounded operators admits an extension to selfadjoint ones. Let Δ_n be a selfadjoint operator ($n=1, 2, \ldots, \infty$). Then $\Delta_n \to \Delta_\infty$ strongly in the generalized sense if $(\Delta_n+i\lambda I)^{-1} \to (\Delta_\infty+i\lambda I)^{-1}$ strongly for some $\lambda \in \mathbf{R}\setminus\{0\}$ ([20] VIII. 7).

PROPOSITION 2 ([19, p. 312]). *For selfadjoint operators* Δ_n ($n=1, 2, \ldots, \infty$) *the following conditions are equivalent.*

(i) $\exp(it\Delta_n) \xrightarrow{so} \exp(it\Delta_\infty)$ *for every* $t\in\mathbf{R}$.

(ii) $f(\Delta_n) \xrightarrow{so} f(\Delta_\infty)$ *for every continuous bounded function* f,

(iii) *If* $\Delta_n = \int_{-\infty}^{\infty} \lambda\, dE_\lambda^n$ *is the spectral resolution then* $E_\lambda^n(a, b) \xrightarrow{so} E_\lambda^\infty(a, b)$ *for every* $a, b \in \mathbf{R}-\sigma_{pp}(\Delta_\infty)$.

(iv) $\Delta_n \to \Delta_\infty$ *strongly in the generalized sense.*

LEMMA 3. *Let* Δ_n *be a positive selfadjoint operator* ($n=1, 2, \ldots, \infty$) *and* $f\colon \mathbf{R}^+\to\mathbf{R}$ *a function such that* $f(0)=0$ *and* f *is continuous and strictly monotone increasing on* $(0, +\infty)$. *If* $\Delta_n\to\Delta_\infty$ *strongly in the generalized sense and* $\operatorname{Ker}\Delta_n \to \operatorname{Ker}\Delta_\infty$ *then* $f(\Delta_n)\to f(\Delta_\infty)$ *strongly in the generalized sense.*

PROOF. Let $g=f|(0, +\infty)$. If $0\notin(a, b)$ then $\chi_{(a,b)} f(\Delta_n) = \chi_{(g^{-1}(a), g^{-1}(b))}(\Delta_n)$ and in case of $0\in(a, b)$ we have $\chi_{(a,b)} f(\Delta_n) = \chi_{(g^{-1}(a), g^{-1}(b))}(\Delta_n) + \operatorname{Ker}\Delta_n$. When $a, b \notin \sigma_{pp}(\Delta_\infty)$ then $g^{-1}(a), g^{-1}(b) \notin \sigma_{pp}(\Delta_\infty)$ and condition (iii) in the previous Proposition is satisfied.

The relative entropy

Throughout this section $\mathscr{A}$ will be a von Neumann algebra with a normal faithful positive functional φ. We assume that $\mathscr{A}$ acts on a Hilbert space $\mathscr{H}$ and φ is given by a cyclic and separating vector ξ.

LEMMA 4. *Suppose that $\omega(a)=\langle a\eta, \eta\rangle$ for some $\eta\in\mathscr{H}$. Then the quadratic form q: $a\xi\mapsto\omega(aa^*)$ $(a\in\mathscr{A})$ and the conjugate linear operator S: $a\xi\mapsto a^*\xi$ $(a\in\mathscr{A})$ are closable. Moreover, $\Delta=S^*\bar{S}$ is the associated selfadjoint operator for the closure $\bar{q}$ of q. (This means that $\mathscr{D}(\Delta^{1/2})=\mathscr{D}(\bar{q})$ and $\bar{q}(\zeta)=\|\Delta^{1/2}\zeta\|$ for $\zeta\in\mathscr{D}(\bar{q})$).*

PROOF. Introducing the operator F: $a'\xi\mapsto a'^*\xi$ $(a'\in\mathscr{A}')$ we have

$$\langle Sa\xi, a'\xi\rangle = \langle Fa'\xi, a\xi\rangle$$

and so $S\subset F^*$ and $F\subset S^*$. Hence S is closable and $\Delta=S^*\bar{S}$ is selfadjoint. Now it is easy to see that q is also closable. ([9] Theorem 1.17, p. 315.)

Let $\zeta\in\mathscr{D}(\Delta)$. Then there is a sequence $(a_n\xi)$ such that $a_n\xi\to\zeta$, $a_n^*\xi\to\bar{S}\zeta\in\mathscr{D}(S^*)$. In this case we have $\zeta\in\mathscr{D}(\bar{q})$. For $\mu\in\mathscr{D}(\bar{q})$, $\bar{q}(\mu)=\langle S^*\bar{S}\mu, \mu\rangle$ and by Theorem 2.1 on p. 322 of [9] Δ is associated to $\bar{q}$.

The operator $\Delta=\Delta(\omega, \varphi)$ is called relative modular operator by Araki [2]. An interesting consequence of Lemma 4 is that $\Delta(\omega, \varphi)$ does not depend on the representing vector η. In Araki's definition η is to be chosen from the natural positive cone.

The relative entropy is defined as follows:

$$S(\omega, \varphi) = -\lim_{t\to+0} t^{-1}(\|\Delta^{t/2}\xi\|^2-\|\xi\|^2).$$

This definition is essentially due to Uhlmann [25] who formulated it by means of a quadratic interpolation machinery. According to Lemma 2 it is equivalent to Araki's form

$$S(\omega, \varphi) = -\langle\log\Delta(\omega, \varphi)\xi, \xi\rangle$$

where the right hand side is defined via the spectral resolution of $\Delta(\omega, \varphi)$.

THEOREM 1 ([3]). *Let $\mathscr{A}$, φ and ω be as above. Then for $\lambda_1, \lambda_2>0$ we have*

(i) $S(\lambda_1\omega, \lambda_2\varphi) = \lambda_2 S(\omega, \varphi)-\lambda_2\varphi(1)\log\lambda_1/\lambda_2$,

(ii) $S(\omega, \varphi) \geqq \varphi(1)[\log\varphi(1)-\log\omega(1)]$.

PROOF. (i) is straightforward from the definition. To prove (ii) one can use Lemma 1:

$$t^{-1}(\|\Delta^{t/2}\xi\|^2-\|\xi\|^2) \leqq t^{-1}(\|\xi\|^{2(1-t)}\|\Delta^{1/2}\xi\|^{2t}-\|\xi\|^2).$$

Letting $t\to+\infty$ we obtain (ii).

The next result is due to Uhlmann ([25]). In the proof we try to exclude interpolation theory as much as we can but we need Proposition 1 very heavily. We recall that α: $\mathscr{A}_1\to\mathscr{A}_2$ is a Schwarz map if $\alpha(a)^*\alpha(a)\leqq\alpha(a^*a)$ for every $a\in\mathscr{A}_1$.

THEOREM 2 ([25]). *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be von Neumann algebras with positive normal functionals ω_1, φ_1 and ω_2, φ_2, respectively. Assume that φ_1, φ_2 are faithful and*

$\varphi_1(1)=\varphi_2(1)=1$. *If* $\alpha\colon \mathcal{A}_1 \to \mathcal{A}_2$ *is a unit preserving Schwarz map such that* $\varphi_2 \circ \alpha \leqq \varphi_1$ *and* $\omega_2 \circ \alpha \leqq \omega_1$ *then*

$$S(\omega_2, \varphi_2) \geqq S(\omega_1, \varphi_1).$$

PROOF. We may suppose that φ_i is given by a cyclic separating vector $\xi_i \in \mathcal{H}_i$ $(i=1, 2)$. Since

$$\varphi_2(\alpha(a_1)^* \alpha(a_1)) \leqq \varphi_2(\alpha(a_1^* a_1)) \leqq \varphi_1(a_1^* a_1)$$

the formula $Ta_1\xi_1 = \alpha(a_1)\xi_2$ $(a_1 \in \mathcal{A}_1)$ defines a contraction of $\mathcal{H}_1$ into $\mathcal{H}_2$.

For the relative modular operators Δ_1 and Δ_2 we have $\omega_i(a_i a_i^*) = \|\Delta_i^{1/2} a_i \xi_i\|^2$ $(a_i \in \mathcal{A}_i, i=1, 2)$. Here $\mathcal{A}_i \xi_i$ is a core for $\Delta_i^{1/2}$ $(i=1, 2$, see [9, p. 331]). Let $\mu_1 \in \mathcal{D}(\Delta_1^{1/2})$. Then there is a sequence $a_i^n \xi_1 \to \mu_1$ such that $\Delta_1^{1/2} a_1^n \xi_1 \to \Delta_1^{1/2} \mu_1$. Since

$$\|\Delta_2^{1/2} T a_1^n \xi_1 - \Delta_2^{1/2} T a_1^n \xi_1\|^2 = \omega_2(\alpha(a_1^n - a_1^m)\alpha(a_1^{n*} - a_1^{m*})) \leqq \|\Delta_1^{1/2} a_1^n \xi_1 - \Delta_1^{1/2} a_1^m \xi_1\|^2$$

we obtain that $(\Delta_2^{1/2} T a_1^n \xi_1)$ is a Cauchy sequence. On the other hand $Ta_1^n \xi_1 \to T\mu_1$ and so $T\mu_1 \in \mathcal{D}(\Delta_2^{1/2})$. Now we have infered that $T\mathcal{D}(\Delta_1^{1/2}) \subset \mathcal{D}(\Delta_2^{1/2})$ and can apply Proposition 1:

$$t^{-1}(\|\Delta_2^{t/2}\xi_2\|^2 - \|\xi_2\|^2) \leqq t^{-1}(\|\Delta_1^{t/2}\xi_1\|^2 - \|\xi_1\|^2).$$

Taking the limit $t \to +0$ completes the proof.

COROLLARY 1. *If* $\omega_2 \leqq \omega_1$ *then* $S(\omega_2, \varphi) \geqq S(\omega_1, \varphi)$.

COROLLARY 2. *If* $\mathcal{B}$ *is a subalgebra of* $\mathcal{A}$ *and* $\varphi_0 = \varphi|\mathcal{B}$, $\omega_0 = \omega|\mathcal{B}$ *then* $S(\omega_0, \varphi_0) \leqq S(\omega, \varphi)$.

This monotonicity property of the relative entropy was proved by Araki [2] for special subalgebras.

If φ and ω are states then $S(\omega, \varphi) \geqq 0$ by (ii) in Theorem 1. When $S(\omega, \varphi)=0$ then $S(\omega|\mathcal{B}, \varphi|\mathcal{B})=0$ for every commutative subalgebra. Hence $\varphi(a)=\omega(a)$ for every $a \in \mathcal{A}^{sa}$ as it is known from information theory ([12]). Consequently, $\varphi=\omega$. A stronger result of this type is obtained by Hiai, Ohya and Tsukada.

THEOREM 3 ([8]). $\|\varphi-\omega\|^2 \leqq 2S(\omega, \varphi)$.

Properties of the relative entropy

Let φ, ω be faithful normal states of a von Neumann algebra $\mathcal{A}$. The following theorem shows how $S(\omega, \varphi)$ can be expressed by means of the unitary cocycle $[D\omega, D\varphi]_t$ ([6], [22, 3.1]).

THEOREM 4. *Assume that* $S(\omega, \varphi) < +\infty$. *Then*

$$S(\omega, \varphi) = i \lim_{t \to +0} t^{-1}(\varphi([D\omega, D\varphi]_t) - 1).$$

PROOF. $[D\omega, D\varphi]_t = \Delta(\omega, \varphi)^{it} \Delta(\varphi, \varphi)^{it}$ and we have

$$\varphi([D\omega, D\varphi]_t) = \langle \Delta(\omega, \varphi)^{it}\xi, \xi \rangle$$

since $\Delta(\varphi,\varphi)\xi=\xi$. Let $\int_0^\infty \lambda\, dE_\lambda$ be the spectral resolution of $\Delta(\omega,\varphi)$. So

$$t^{-1}(\varphi([D\omega, D\varphi]_t)-1) = \int_0^\infty t^{-1}(\lambda^{it}-1)\,d\mu(\lambda) =$$

$$= \int_0^\infty t^{-1}(\cos(t\log\lambda)-1)\,d\mu(\lambda)+i\int_0^\infty t^{-1}\sin(t\log\lambda)\,d\mu(\lambda)$$

where $d\mu(\lambda)=d\langle E_\lambda\xi,\xi\rangle$. We take the integrals as $\int_0^1+\int_1^\infty$ and apply the dominated convergence theorem. The condition $S(\omega,\varphi)<+\infty$ is equivalent to $\int_0^1 -\log\lambda\, d\mu(\lambda)<$ $<\infty$ so $\int_0^1 t^{-1}(\cos(t\log\lambda)-1)\,d\mu(\lambda)\to 0$ $(|t^{-1}(\cos t\log\lambda-1)|\leqq-\log\lambda)$ and $\int_0^1 t^{-1}\sin(t\log\lambda)\,d\mu(\lambda)\to\int_0^1\log\lambda\,d\mu(\lambda)$. On the other hand, $\int_0^\infty\lambda\,d\mu(\lambda)<+\infty$ and we have

$$\int_1^\infty t^{-1}(\cos(t\log\lambda)-1)\,d\mu(\lambda)\to 0,\quad \int_1^\infty t^{-1}\sin(t\log\lambda)\,d\mu(\lambda)\to\int_1^\infty\log\lambda\,d\mu(\lambda).$$

The proof is complete.

COROLLARY. *Let ω_1, φ_1, ω_2, φ_2 be faithful normal states of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. If $S(\omega_1\otimes\omega_2,\varphi_1\otimes\varphi_2)$ is finite then*

$$S(\omega_1\otimes\omega_2,\varphi_1\otimes\varphi_2) = S(\omega_1,\varphi_1)+S(\omega_2,\varphi_2).$$

PROOF. Since $[D\omega_1\otimes\omega_2, D\varphi_1\otimes\varphi_2]_t=[D\omega_1,D\varphi_1]_t\otimes[D\omega_2,D\varphi_2]_t$ ([22], 8.6) we can obtain the Corollary by derivation.

Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. If $\mathcal{B}$ is invariant under the modular automorphism group of ω then Takesaki's theorem provides an ω-preserving conditional expectation E_ω of $\mathcal{A}$ onto $\mathcal{B}$ ([23], [22, 9.1]). Denote $\varphi\circ E_\omega$ by φ'. With this notation we have

THEOREM 5. *Assume that φ, ω are faithful and $S(\omega,\varphi)$, $S(\varphi',\varphi)$ are finite. Then*

$$S(\omega,\varphi) = S(\omega|\mathcal{B},\varphi|\mathcal{B})+S(\varphi',\varphi).$$

PROOF By the chain rule we have

$$(*)\qquad [D\omega, D\varphi]_t = [D\omega, D\varphi']_t[D\varphi', D\varphi]_t.$$

Here $[D\omega, D\varphi']_t=[D\omega|\mathcal{B}, D\varphi|\mathcal{B}]_t\in\mathcal{B}$ (see [22, 10.5]) and we apply Theorem 4. Let u_t, v_t, w_t be the cocycles occurring in $(*)$. Then

$$\varphi(u_t)-1 = (\langle v_t\xi,\xi\rangle-1)+(\langle w_t\xi,\xi\rangle-1)+\langle(w_t-1)\xi,(v_t^*-1)\xi\rangle.$$

Since the last term divided by t tends to 0 as $t\to+0$ we obtain the result.

Theorem 5 generalizes Theorem 3.2 in [8] where it is assumed that $\omega|\mathscr{B}$ is tracial. However, for the sake of perfect comparison we should admit that our condition on the finiteness of the entropies is slightly stronger than that in [8].

Let $\mathscr{A}=\mathscr{A}_1\otimes\mathscr{A}_2$ and assume that $\omega=\omega_1\otimes\omega_2$ and $\varphi=\varphi_{12}$ are faithful normal states of $\mathscr{A}$. Now $\mathscr{A}_1$ and $\mathscr{A}_2$ are subalgebras of $\mathscr{A}$ under the natural identifications $\mathscr{A}_1\cong\mathscr{A}_2\otimes\mathbf{C}$ and $\mathscr{A}_2\cong\mathbf{C}\otimes\mathscr{A}_2$. There exists an ω-preserving conditional expectation E_2 of $\mathscr{A}$ onto $\mathscr{A}_2$ satisfying $E_2(a\otimes b)=\omega_1(a)1\otimes b$ (E_2 is called Fubini mapping, see [22, 9.8]). Denote $\varphi_{12}|\mathscr{A}_i$ by φ_i $(i=1,2)$. So for φ' in Theorem 5 we have

$$\varphi'(a\otimes b)=\varphi\big(E_2(a\otimes b)\big)=\omega_1(a)\varphi_2(b)$$

and $\varphi'=\omega_1\otimes\varphi_2$. Theorem 5 tells us that

$$S(\omega_1\otimes\omega_2,\varphi_{12})=S(\omega_2,\varphi_2)+S(\omega_1\otimes\varphi_2,\varphi_{12})$$

provided that each entropy is finite. Therefore,

$$S(\omega_1\otimes\omega_2,\varphi_{12})\geqq S(\omega_2,\varphi_2)+S(\omega_1,\varphi_1)$$

and we may call this inequality the superadditivity of the relative entropy. (When ω_1, ω_2 are traces then taking the negative we obtain the usual subadditivity property, see [18, 7.2.10] and [5, p. 273]). By the same method we can deduce the strong superadditivity.

THEOREM 6. *Let* $\mathscr{A}=\mathscr{A}_1\otimes\mathscr{A}_2\otimes\mathscr{A}_3$ *and let* φ_{123}, ω *be faithful normal states of* $\mathscr{A}$. *Assume that* $\omega=\omega_1\otimes\omega_2\otimes\omega_3$ *and denote* φ_{123} *restricted to* $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_1\otimes\mathscr{A}_2$, $\mathscr{A}_2\otimes\mathscr{A}_3$ *by* φ_1, φ_2, φ_3, φ_{12}, φ_{23}, *respectively. Then*

$$S(\omega,\varphi_{123})+S(\omega_2,\varphi_2)\geqq S(\omega_1\otimes\omega_2,\varphi_{12})+S(\omega_2\otimes\omega_3,\varphi_{23})$$

if all terms are finite.

PROOF. On the one hand

$$S(\omega,\varphi_{123})=S(\omega_1\otimes\omega_2,\varphi_{12})+S(\omega_1\otimes\omega_2\otimes\varphi_3,\varphi_{123})\geqq S(\omega_1\otimes\omega_2,\varphi_{12})+S(\omega_2\otimes\varphi_3,\varphi_{23})$$

and on the other hand

$$S(\omega_2\otimes\varphi_3,\varphi_{23})+S(\omega_2,\varphi_2)=S(\omega_2\otimes\omega_3,\varphi_{23}).$$

We note that the idea of this argument is due to Lindblad [16] who proved a similar result for $\mathscr{A}_i=\mathscr{B}(\mathscr{H})$.

Finally, we treat some continuity property of the relative entropy. Araki proved that the relative modular operator $\Delta(\omega,\varphi)$ is a continuous function of ω and φ: If $\omega_n\to\omega$ and $\varphi_n\to\varphi$ in norm then $\Delta(\omega_1,\varphi_n)\to\Delta(\omega,\varphi)$ strongly in the generalized sense ([3]). The continuity of the relative modular operator can be used to prove the lower semicontinuity in norm of the relative entropy ([3], 3.7).

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(Received November 15, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 73—80.

ON FUNCTIONS DEFINED BY DIGITS OF REAL NUMBERS

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1. Introduction. Let $p>1$ be a natural number. Let $\mathfrak{M}_p$ denote the set of all functions $F\colon [0,1)\to\mathbf{R}$ for which there exist

$$f_n\colon \{0, 1, \ldots, p-1\} \to \mathbf{R} \quad (n\in\mathbf{N})$$

with the property

$$\sum_{n=1}^{\infty} |f_n(k)| < +\infty \quad (k = 0, 1, \ldots, p-1)$$

such that

$$F(x) = \sum_{n=1}^{\infty} f_n(\varepsilon_n(x)) \quad \text{whenever} \quad x\in[0, 1) \tag{1.1}$$

where

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{p^n} \quad (\varepsilon_n(x)\in\{0, 1, \ldots, p-1\}) \tag{1.2}$$

is the unique p-base expansion of x. Expansion (1.2) is named unique, if numbers of the form $\frac{l}{p^N}$, so-called p-base rational numbers, have finite expansions (1.2). (See for example Galambos [1].) Let

$$\mathfrak{M}_p^0 = \{F \mid F\in\mathfrak{M}_p,\ F(0) = 0\}.$$

In this paper we shall study the properties of $\mathfrak{M}_p$. As the main result the following theorem will be proved:

THEOREM 1. *If $p>1$ and $q>1$ are relative prime numbers and $F\in\mathfrak{M}_p\cap\mathfrak{M}_q$ then there exist $A, B\in\mathbf{R}$ such that $F(x)=Ax+B$ whenever $x\in[0, 1)$.*

2. Reduction of the problem. Our investigations are made easier by the following two lemmas.

LEMMA 1. *If $F\in\mathfrak{M}_p$ then there exists an $\hat{F}\in\mathfrak{M}_p^0$ for which*

$$F(x) = \hat{F}(x)+F(0) \quad \textit{whenever} \quad x\in[0, 1). \tag{2.1}$$

PROOF. Let $F\in\mathfrak{M}_p$. Then there exist functions

$$f_n\colon \{0, 1, \ldots, p-1\} \to \mathbf{R}$$

with the property

$$\sum_{n=1}^{\infty} |f_n(k)| < \infty \quad (k = 0, 1, \ldots, p-1)$$

such that

$$F(x) = \sum_{n=1}^{\infty} \{f_n(\varepsilon_n(x)) - f_n(0)\} + \sum_{n=1}^{\infty} f_n(0)$$

where

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{p^n}$$

is the unique p-base expansion. Using $F(0) = \sum_{n=1}^{\infty} f_n(0)$ and writing

$$\hat{f}_n(k) := f_n(k) - f_n(0) \quad (k = 0, 1, \ldots, p-1)$$

and

$$\hat{F}(x) := \sum_{n=1}^{\infty} \hat{f}_n(\varepsilon_n(x))$$

we get

$$F(x) = \hat{F}(x) + F(0) \quad (x \in [0, 1))$$

and

$$\sum_{n=1}^{\infty} |\hat{f}_n(k)| < \infty \quad (k = 0, 1, \ldots, p-1).$$

This means that $\hat{F} \in \mathfrak{M}_p$ and $\hat{F}(0) = 0$, i.e. $\hat{F} \in \mathfrak{M}_p^0$.

LEMMA 2. *$F \in \mathfrak{M}_p^0$ if and only if F: $[0, 1) \to \mathbf{R}$,*

$$F(0) = 0, \quad \sum_{n=1}^{\infty} \left| F\left(\frac{k}{p^n}\right) \right| < \infty \quad (k = 0, 1, \ldots, p-1),$$

and

$$F(x) = F\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{p^n}\right) = \sum_{n=1}^{\infty} F\left(\frac{\varepsilon_n(x)}{p^n}\right) \tag{2.2}$$

for each unique expansion

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{p^n} \quad x \in [0, 1).$$

PROOF. Let $F \in \mathfrak{M}_p^0$ and $x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{p^n}$. Then by $F(0) = 0$ and $F\left(\frac{\varepsilon_n(x)}{p^n}\right) = f_n(\varepsilon_n(x))$ (1.1) implies (2.2) with $\sum_{n=1}^{\infty} \left| F\left(\frac{k}{p^n}\right) \right| < \infty$ $(k = 0, 1, \ldots, p-1)$. Conversely, if F: $[0, 1) \to \mathbf{R}$, $F(0) = 0$ and (2.2) holds then for $f_n(k) := F\left(\frac{k}{p^n}\right)$ $(k = 0, 1, \ldots, p-1)$ (1.1) is satisfied and $\sum_{n=1}^{\infty} |f_n(k)| < \infty$ $(k = 0, 1, \ldots, p-1)$. This proves that $F \in \mathfrak{M}_p^0$.

Equation (2.2) essentially says that the unique p-adic series expansion is additively transformed by F into an absolutely convergent series.

3. Elementary study of the cases $p=2$ and $q=3$. Let $F\in\mathfrak{M}_2^0\cap\mathfrak{M}_3^0$ and let $\mathbf{Q}_p$ denote the set of p-base rational numbers in $[0, 1)$. If $x\in(0, 1)$ and $x\notin\mathbf{Q}_2$, then $\varepsilon_n(1-x)=1-\varepsilon_n(x)$. By Lemma 2 we have

$$F(x)+F(1-x) = \sum_{n=1}^{\infty}\left\{F\left(\frac{\varepsilon_n(x)}{2^n}\right)+F\left(\frac{1-\varepsilon_n(x)}{2^n}\right)\right\} = \sum_{n=1}^{\infty}F\left(\frac{1}{2^n}\right) =: A \tag{3.1}$$

where $x=\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{2^n}$. There exists an $x\in(0, 1)$ such that $x\notin\mathbf{Q}_2\cup\mathbf{Q}_3$ and in the expansion $x=\sum_{n=1}^{\infty}\frac{\delta_n(x)}{3^n}$ each $\delta_n(x)$ is 0 or 2. Then $\delta_n(1-x)=2-\delta_n(x)$ and (3.1) implies

$$A = F(x)+F(1-x) = \sum_{n=1}^{\infty}\left\{F\left(\frac{\delta_n(x)}{3^n}\right)+F\left(\frac{2-\delta_n(x)}{3^n}\right)\right\} = \sum_{n=1}^{\infty}F\left(\frac{2}{3^n}\right). \tag{3.2}$$

Now let $N\in\mathbf{N}$ be fixed. Then there exists an $x\in(0, 1)$ for which $x\notin\mathbf{Q}_2\cup\mathbf{Q}_3$ and in the expansion $x=\sum_{n=1}^{\infty}\frac{\delta_n(x)}{3^n}$ each $\delta_n(x)$ is 0 or 2 if $n\neq N$, but $\delta_N(x)=1$. By $\delta_n(x)=2-\delta_n(1-x)$ we have $\delta_N(1-x)=1$. (3.1) and (3.2) imply

$$A = F(x)+F(1-x) = \sum_{\substack{n=1\\ n\neq N}}^{\infty}\left\{F\left(\frac{\delta_n(x)}{3^n}\right)+F\left(\frac{2-\delta_n(x)}{3^n}\right)\right\}+2F\left(\frac{1}{3^N}\right) =$$

$$= A-F\left(\frac{2}{3^N}\right)+2F\left(\frac{1}{3^N}\right),$$

i.e.

$$F\left(\frac{2}{3^N}\right) = 2F\left(\frac{1}{3^N}\right). \tag{3.3}$$

Because of $\mathbf{Q}_2\cap\mathbf{Q}_3=\{0\}$, (3.1) implies by Lemma 2 and by (3.3)

$$A = F\left(\frac{1}{3^N}\right)+F\left(\frac{3^N-1}{3^N}\right) = F\left(\frac{1}{3^N}\right)+F\left(\sum_{n=1}^{N}\frac{2}{3^n}\right) = F\left(\frac{1}{3^N}\right)+\sum_{n=1}^{N}2F\left(\frac{1}{3^n}\right) \tag{3.4}$$

for any $N\in\mathbf{N}$. Putting $N=1$ we obtain $F\left(\frac{1}{3}\right)+2F\left(\frac{1}{3}\right)=A$ i.e.

$$F\left(\frac{1}{3}\right) = A. \tag{3.5}$$

Using induction, we get from (3.4) by (3.5) that $F\left(\frac{1}{3^n}\right)=\frac{A}{3^n}$, i.e. by Lemma 2

$$F(x) = F\left(\sum_{n=1}^{\infty}\frac{\delta_n(x)}{3^n}\right) = \sum_{n=1}^{\infty}F\left(\frac{\delta_n(x)}{3^n}\right) = \sum_{n=1}^{\infty}\delta_n(x)\,F\left(\frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\frac{\delta_n(x)A}{3^n} = Ax.$$

This proves that there exists $A\in\mathbf{R}$ such that $F(x)=Ax$. Applying Lemma 1 we see that Theorem 1 holds for $p=2$ and $q=3$.

This proof strongly relies on equation (3.1), which in the general case $p>2$ fails to be trivially satisfied.

4. Continuity properties of $F\in\mathfrak{M}_p^0$. We shall prove the following result interesting also in itself.

THEOREM 2. *If $F\in\mathfrak{M}_p^0$ then F is continuous at each point $x\in[0, 1)\setminus\mathbf{Q}_p$ and right continuous at each point $x\in\mathbf{Q}_p$.*

PROOF. Let $x\in[0, 1)$ and $\varepsilon>0$. Since the series $\sum_{n=1}^{\infty} a_n$ defined by

$$a_n := \sum_{k=0}^{p-1}\left|F\left(\frac{k}{p^n}\right)\right|$$

is absolutely convergent, there exists an $N\in\mathbf{N}$ for which $\sum_{n=N+1}^{\infty} a_n<\frac{\varepsilon}{2}$. Let us choose a natural number $0\leqq j<p^N$ for which

$$\frac{j}{p^N}\leqq x<\frac{j+1}{p^N}.$$

If $\frac{j}{p^N}\leqq y<\frac{j+1}{p^N}$ then the digits in the p-base expansion of y are $\varepsilon_n(y)=\varepsilon_n\left(\frac{j}{p^N}\right)$ for $1\leqq n\leqq N$. This implies $\varepsilon_n(x)=\varepsilon_n(y)$ if $n=1, 2, \ldots, N$. Hence by Lemma 2

$$|F(x)-F(y)| = \left|\sum_{n=1}^{\infty} F\left(\frac{\varepsilon_n(x)}{p^n}\right)-\sum_{n=1}^{\infty} F\left(\frac{\varepsilon_n(y)}{p^n}\right)\right| \leqq$$

$$\leqq \sum_{n=N+1}^{\infty}\left|F\left(\frac{\varepsilon_n(x)}{p^n}\right)-F\left(\frac{\varepsilon_n(y)}{p^n}\right)\right| \leqq 2\sum_{n=N+1}^{\infty} a_n<\varepsilon$$

whenever $\frac{j}{p^N}\leqq y<\frac{j+1}{p^N}$. This means that F is continuous at x if $\frac{j}{p^N}<x$ (that is, $x\notin\mathbf{Q}_p$), and F is right continuous at x if $\frac{j}{p^N}=x\in\mathbf{Q}_p$.

REMARK. Theorem 2 is sharp in the following sense: There exists an $F\in\mathfrak{M}_p^0$ which is discontinuous at each point $x\in\mathbf{Q}_p$, $x>0$. For example, the function

$$F(x) := \sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{n^2} \tag{4.1}$$

defined by the unique expansion $x=\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{p^n}$ has this property. This function is an element of $\mathfrak{M}_p^0$, hence it is enough to prove that if $x\in\mathbf{Q}_p$, $x>0$, then F is not left continuous at x. Consider therefore a positive p-adic rational number

$$x=\sum_{n=1}^{N-1}\frac{\varepsilon_n(x)}{p^n}+\frac{k+1}{p^N}\quad (k\in\{0, 1, \ldots, p-2\}).$$

For $K>N$ put

$$x_K = \sum_{n=1}^{N-1} \frac{\varepsilon_n(x)}{p^n} + \frac{k}{p^n} + \sum_{n=N+1}^{K} \frac{p-1}{p^n}.$$

This monotone increasing sequence converges (from the left) to x, so if F were continuous at x we would have

$$\lim_{K\to\infty} F(x_K) = F(x). \tag{4.2}$$

(4.1) and (4.2) together imply

$$\sum_{n=1}^{N-1} \frac{\varepsilon_n(x)}{n^2} + \frac{k+1}{N^2} = \lim_{K\to\infty} \left\{ \sum_{n=1}^{N-1} \frac{\varepsilon_n(x)}{n^2} + \frac{k}{N^2} + \sum_{n=N+1}^{K} \frac{p-1}{n^2} \right\},$$

and from this we get

$$\frac{1}{N^2} = \frac{\pi^2}{6} - (p-1) \sum_{n=1}^{N} \frac{1}{n^2}$$

which is a contradiction.

5. The case of continuity. We shall prove the following result:

THEOREM 3. *If* $F\in\mathfrak{M}_p^0$ *is continuous on* $[0, 1)$, *then there exists an* $A\in\mathbf{R}$ *such that* $F(x)=Ax$ *whenever* $x\in[0, 1)$.

PROOF. Let $k\in\{0, 1, ..., p-2\}$ and let $N\in\mathbf{N}$ be fixed. Moreover, let

$$x := \sum_{n=1}^{N-1} \frac{\varepsilon_n(x)}{p^n} + \frac{k+1}{p^N}$$

be the unique p-base expansion, and for each $K>N$ let

$$x_K := \sum_{n=1}^{N-1} \frac{\varepsilon_n(x)}{p^n} + \frac{k}{p^N} + \sum_{n=N+1}^{K} \frac{p-1}{p^n}.$$

Then $\lim\limits_{K\to\infty} x_K=x$, hence (2.2) and the continuity of F imply

$$F(x) = \lim_{K\to\infty} F(x_K) = \lim_{K\to\infty} \left\{ \sum_{n=1}^{N-1} F\left(\frac{\varepsilon_n(x)}{p^n}\right) + F\left(\frac{k}{p^N}\right) + \sum_{n=N+1}^{K} F\left(\frac{p-1}{p^n}\right) \right\} =$$

$$= \sum_{n=1}^{N-1} F\left(\frac{\varepsilon_n(x)}{p^n}\right) + F\left(\frac{k}{p^N}\right) + \sum_{n=N+1}^{\infty} F\left(\frac{p-1}{p^n}\right).$$

Hence by

$$F(x) = \sum_{n=1}^{N-1} F\left(\frac{\varepsilon_n(x)}{p^n}\right) + F\left(\frac{k+1}{p^n}\right)$$

we have

$$F\left(\frac{k+1}{p^N}\right) = F\left(\frac{k}{p^N}\right) + A - \sum_{n=1}^{N} F\left(\frac{p-1}{p^n}\right) \tag{5.1}$$

where $A := \sum\limits_{n=1}^{\infty} F\left(\frac{p-1}{p^n}\right)$.

Writing $k=0$ in (5.1) we get

$$A-F\left(\frac{1}{p^N}\right) = \sum_{n=1}^{N} F\left(\frac{p-1}{p^n}\right).$$

Putting this back into (5.1) we see that

$$F\left(\frac{k+1}{p^N}\right) = F\left(\frac{k}{p^N}\right)+F\left(\frac{1}{p^N}\right). \tag{5.2}$$

This implies

$$F\left(\frac{l}{p^N}\right) = lF\left(\frac{1}{p^N}\right). \tag{5.3}$$

On the other hand, substituting $k=0$, $N=1$ in (5.1) and using (5.3) we obtain

$$F\left(\frac{1}{p}\right) = A-F\left(\frac{p-1}{p}\right) = A-(p-1)F\left(\frac{1}{p}\right),$$

i.e. $F\left(\frac{1}{p}\right) = \frac{A}{p}$.

Moreover, for $k=0$ (5.1) yields

$$F\left(\frac{1}{p^N}\right) = A-\sum_{n=1}^{N} F\left(\frac{p-1}{p^n}\right) = A-\sum_{n=1}^{N}(p-1)F\left(\frac{1}{p^n}\right). \tag{5.4}$$

Using induction on N, if $F\left(\frac{1}{p^n}\right)=\frac{A}{p^n}$ for $n<N$ (this is true for $n=1$) then we get by (5.4)

$$F\left(\frac{1}{p^N}\right) = A-\sum_{n=1}^{N-1}(p-1)\frac{A}{p^n}-(p-1)F\left(\frac{1}{p^N}\right),$$

hence

$$F\left(\frac{1}{p^N}\right) = \frac{A}{p^N}.$$

By (2.2) we now obtain

$$F(x) = F\left(\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{p^n}\right) = \sum_{n=1}^{\infty} F\left(\frac{\varepsilon_n(x)}{p^n}\right) = \sum_{n=1}^{\infty}\frac{\varepsilon_n(x)A}{p^n} = Ax.$$

6. The proof of Theorem 1. Let $p>1$ and $q>1$ be relatively prime numbers and $F\in\mathfrak{M}_p\cap\mathfrak{M}_q$. Then by Lemma 1 there exists an $\hat{F}\in\mathfrak{M}_p^0\cap\mathfrak{M}_q^0$ for which

$$F(x) = \hat{F}(x)+F(0) = \hat{F}(x)+B.$$

By Theorem 2, $\hat{F}$ is continuous if $x\in[0, 1)\setminus\mathbf{Q}_p$ or if $x\in[0, 1)\setminus\mathbf{Q}_q$ i.e. everywhere, because of $\mathbf{Q}_p\cap\mathbf{Q}_q=\{0\}$. Hence by Theorem 3 there exists an $A\in\mathbf{R}$ for which $\hat{F}(x)=Ax$, i.e.

$$F(x) = Ax+B \quad \text{whenever} \quad x\in[0, 1).$$

REMARKS. (i) There arises the question whether the condition $(p,q)=1$ is necessary? In the proof we have used the fact that if $x>0$ and $x\in\mathbf{Q}_p$ then $x\notin\mathbf{Q}_q$. We shall prove that if $(p,q)=r>1$ then there exists an $F\in\mathfrak{M}_p^0\cap\mathfrak{M}_q^0$ which is not of the form Ax. One of the simplest examples is the function

$$F(x):=\begin{cases} x & \text{if } \ 0\leqq x<\dfrac{1}{r} \\ 1+x & \text{if } \ \dfrac{1}{r}\leqq x<1. \end{cases} \tag{6.1}$$

Putting $s:=\dfrac{p}{r}$ and $t:=\dfrac{q}{r}$ we have here two cases:

If $x=\sum_{n=1}^{\infty}\dfrac{\varepsilon_n(x)}{p^n}$ and $x<\dfrac{1}{r}=\dfrac{s}{p}$, then $\dfrac{\varepsilon_n(x)}{p^n}<\dfrac{1}{r}=\dfrac{s}{p}$ for each $n\in\mathbf{N}$, i.e.

$$F(x)=x=\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{p^n}=\sum_{n=1}^{\infty}F\left(\frac{\varepsilon_n(x)}{p^n}\right).$$

If, on the other hand, $x\geqq\dfrac{1}{r}=\dfrac{s}{p}$ then by $\varepsilon_1(x)\geqq s$ we get $\dfrac{\varepsilon_1(x)}{p}\geqq\dfrac{1}{r}=\dfrac{s}{p}$. Moreover, $\dfrac{\varepsilon_n(x)}{p^n}<\dfrac{1}{r}=\dfrac{s}{p}$ if $n=2,3,\ldots$, and so

$$F(x)=1+x=1+\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{p^n}=1+\frac{\varepsilon_1(x)}{p}+\sum_{n=2}^{\infty}\frac{\varepsilon_n(x)}{p^n}=F\left(\frac{\varepsilon_1(x)}{p}\right)+\sum_{n=2}^{\infty}F\left(\frac{\varepsilon_n(x)}{p^n}\right)$$

which implies $F\in\mathfrak{M}_p^0$. The proof of $F\in\mathfrak{M}_q^0$ is quite similar: In view of $\dfrac{1}{r}=\dfrac{t}{q}$ we have only to replace s by t and p by q.

On the basis of this example we may formulate the following result:

THEOREM 4. *Let $p>1$ and $q>1$ be natural numbers. Every $F\in\mathfrak{M}_p\cap\mathfrak{M}_q$ is a linear function if and only if p and q are relatively prime numbers.*

PROOF. If $(p,q)=1$, then F is linear by Theorem 1. If $(p,q)\neq 1$ then by the previous example there exists a nonlinear $F\in\mathfrak{M}_p\cap\mathfrak{M}_q$.

(ii) The original problem can also be formulated thus: Let $p>1$, $q>1$, $F\in\mathfrak{M}_p$ and $G\in\mathfrak{M}_q$. If $F(x)=G(x)$ for $x\in[0,1)$, what can we then say about F? Now this formulation makes it natural to ask: what can we say if $F(x)=G(x)$ whenever $x\in S\subset[0,1)$ where S is a set having some given property?

THEOREM 5. *If $(p,q)=1$ and $F\in\mathfrak{M}_p$, $G\in\mathfrak{M}_q$ moreover S is a dense subset of $[0,1)$ and $F(x)=G(x)$ whenever $x\in S$, then there exist real numbers A and B for which*

$$F(x)=G(x)=Ax+B \quad \textit{whenever} \quad x\in[0,1).$$

PROOF. By Lemma 1 $F(x)=\hat{F}(x)+F(0)$ and $G(x)=\hat{F}(x)+G(0)$ where $\hat{F}\in\mathfrak{M}_p^0$ and $\hat{G}\in\mathfrak{M}_q^0$. By Theorem 2 the functions $\hat{F}$ and $\hat{G}$ are right continuous, hence F and G are right continuous, too. This proves that

$$F(x) = \lim_{\substack{s \downarrow x \\ s \in S}} F(s) = \lim_{\substack{s \downarrow x \\ s \in S}} G(s) = G(x).$$

Using Theorem 1, we now get $F(x)=G(x)=Ax+B$ whenever $x \in [0, 1)$.

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(Received November 30, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 81—87.

RANDOM GRAPHS OF BINOMIAL TYPE WITH SPARSELY-EDGED INITIAL GRAPHS

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I. Introduction. Erdős and Rényi [1] have considered a random graph $K_{n,p}$ obtained from the complete graph K_n by an independent deletion of each edge with probability $1-p$, $p=p(n)$. They have shown that for a given balanced graph H with k vertices and l edges the function $\bar{p}=n^{-k/l}$ is such that $\mathrm{Prob}\,(K_{n,p}\supset H)\to 0$ if $p/\bar{p}\to 0$ and tends to one if $p/\bar{p}\to\infty$ as $n\to\infty$.

Later Schürger [7] proved a similar result dealing with a random square lattice on n vertices $L_{n,p}$. It says that for every graph H with l edges which can be embedded into a square lattice L_n the function $n^{-1/l}$ is a threshold for the event $\{L_{n,p}\supset H\}$ in the above sense.

In this paper we show existential and distributional results about small subgraphs of a class of random graphs, which will follow from a general theorem proved in [6]. Moreover, in Section III the distributions of vertices by degrees are given and in Section IV the probability of connectedness and the orders of components of random graphs are considered. We will generalize and sharpen results of [2], [3] and [7].

Throughout this paper we will write $V(G)$, $e(G)$, $a(G)$ and $\Delta(G)$ for the set of vertices, number of edges, number of automorphisms and maximum degree of a given graph G, respectively. The notation $X_n \rightsquigarrow \mathrm{Po}\,(\lambda)$ and $X_n\rightsquigarrow N(0,1)$ means that a sequence of random variables X_n, $n=1,2,\dots$, has asymptotically (as $n\to\infty$) the Poisson and standard normal distribution, respectively. Moreover EX denotes the expectation of a random variable X whereas $\mathrm{Var}\,X$ stands for its variance. For convenience we shall write $f(n)\sim g(n)$ if $f(n)/g(n)\to 1$ as $n\to\infty$. We say that an event A_n holds almost surely (a.s.) if $\mathrm{Prob}\,(A_n)\to 1$ as $n\to\infty$.

A d-dimensional lattice $L_n(d)$, $d\geqq 2$, is the graph with the vertex set $\{(x_1,\dots,x_d)\colon 0\leqq x_i\leqq n-1,\ x_i \text{ are integers}\}$ such that there is an edge between $(x_1,\dots,x_d)$ and $(y_1,\dots,y_d)$ if and only if $\sum_{i=1}^{d}|y_i-x_i|=1$. The graph $Q_n=L_2(n)$ is called an *n-cube*. We will denote by $L_n^{(i)}$, $i=3,4,6$ the triangular, square and hexagonal *plane lattice* on n vertices, respectively.

Let $\{\mathrm{ING}\,(n)\}$ be a sequence of graphs on n vertices and suppose that each edge of $\mathrm{ING}\,(n)$ is independently deleted with probability $1-p$, $p=p(n)$. Actually, we have a sequence of probability spaces $(\mathcal{G}_n, P_n)$, where $\mathcal{G}_n$ is the family of all spanning subgraphs of $\mathrm{ING}\,(n)$ and for every $G\in\mathcal{G}_n$,

$$P_n(G)=p^{e(G)}(1-p)^{e(\mathrm{ING}(n))-e(G)}.$$

Such a random graph is denoted by $\mathrm{ING}_p\,(n)$. (More correctly $\mathrm{ING}_p\,(n)=(\mathcal{G}_n, P_n)$.) It is natural to call the graph $\mathrm{ING}\,(n)$ *the initial graph* of a random graph $\mathrm{ING}_p\,(n)$.

We will say that $f(n)$ is *a slow function* if for every $\varepsilon>0$, $f(n)=o(n^\varepsilon)$. (In particular, $f(n)=O(1)$.) The name of *sparsely-edged random graph* we restrict to random graphs for which $e(\mathrm{ING}\,(n))/|V(\mathrm{ING}\,(n))|$ is a slow function.

II. Subgraphs. First, let us recall a general theorem which was proved in [6]. For a given graph K, $V(K)=\{v_1, \ldots, v_k\}$ we call a graph F with $V(F)=\{s_1, \ldots, s_k\}$ *a copy* of K if the bijection $v_i \mapsto s_i$, $i=1, \ldots, k$, is an isomorphism between K and F. Let us denote by $b_n(K)$ the number of k-element sequences of vertices of $\mathrm{ING}\,(n)$ which induce in $\mathrm{ING}\,(n)$ a subgraph containing a copy of K.

Let H be a given connected graph with k vertices and l edges. A graph isomorphic to H is called *a H-graph*. Denote by rH a sum of r disjoint H-graphs and by $\mathscr{H}_r$ any other sum of rH-graphs. Finally, let $X_n=X_n(H)$ be the number of H-graphs contained as subgraphs in $\mathrm{ING}_p\,(n)$.

THEOREM 1. *Let $n\to\infty$. Then*

(A) *if* $b_n(H)p^l\to 0$ *then* $\mathrm{Prob}\,(X_n>0)=o(1)$,

(B) *if* $b_n(H)p^l\to c>0$ *and for every* $r=2, 3, \ldots$ $b_n(rH)\sim b_n^r(H)$ *whereas* $b_n(\mathscr{H}_r)p^{e(\mathscr{H}_r)}=o(1)$ *then* $X_n \rightsquigarrow \mathrm{Po}\,(c/a(H))$,

(C) *if* $b_n(H)p^l\to\infty$ *and* $b_n(2H)\sim b_n^2(H)$ *whereas* $b_n(\mathscr{H}_2)p^{e(\mathscr{H}_2)}=o\big(b_n^2(H)p^{2l}\big)$ *then* $\mathrm{Prob}\,(X_n=0)=o(1)$,

(D) *if* $b_n(H)p^l\to\infty$ *and for every* $r=2, 3, \ldots$ $b_n(rH)\sim b_n^r(H)$ *whereas*

$$b_n(\mathscr{H}_r)b_n^r(H)p^{e(\mathscr{H}_r)+rl} = o(1)$$

then $(X_n-EX_n)/(\mathrm{Var}\,X_n)^{1/2} \rightsquigarrow N(0, 1)$.

Now, we shall show that for random graphs with sparsely-edged initial graphs the conditions of Theorem 1 can be considerably simplified.

Let us introduce a sequence of initial graphs $\mathrm{ING}\,(n)$ with $V(\mathrm{ING}\,(n))= =\{v_1, \ldots, v_n\}$ such that $\Delta(\mathrm{ING}\,(n))$ is a slow function. For every connected graph H with k vertices and l edges denote by $\varrho_i=\varrho_i(H)$ the number of $(k-1)$-element sequences $(s_2, \ldots, s_k)$ of vertices of $\mathrm{ING}\,(n)$ which together with the vertex v_i as s_1 induce a subgraph containing a copy of H in $\mathrm{ING}\,(n)$, $i=1, \ldots, n$.

THEOREM 2. *Suppose that* $|\{i\colon \varrho_i=0, i=1, \ldots, n\}|=o(n)$ *and denote* $\mu=p^l\sum_{i=1}^n \varrho_i$. *Then, as* $n\to\infty$,

$$\mathrm{Prob}\,(X_n>0)\to\begin{cases}0 & \text{if } \mu\to 0,\\ 1 & \text{if } \mu\to\infty,\end{cases}\qquad X_n \rightsquigarrow \mathrm{Po}\,(c/a(H)) \quad \text{if} \quad \mu\to c>0$$

and

$$(X_n-EX_n)/(\mathrm{Var}\,X_n)^{1/2} \rightsquigarrow N(0, 1) \quad \text{if} \quad \mu\to\infty$$

but μ is a slow function.

PROOF. First, we shall show that

$$b_n(rH)\sim b_n^r(H), \quad r=2, 3, \ldots . \tag{1}$$

Note that for every sequence $(a_1, \ldots, a_n)$ of positive numbers the inequality

$$0 \leqq \Big(\sum_{i=1}^n a_i\Big)^r - \sum_{1\leqq i_1<\ldots<i_r\leqq n} a_{i_1}\ldots a_{i_r} \leqq \binom{r}{2}\Big(\sum_{i=1}^n a_i\Big)^{r-2}\sum_{i=1}^n a_i^2$$

holds. In our case $b_n(H) = \sum_{i=1}^{n} \varrho_i$, so

$$0 \leqq b_n^r(H) - b_n(rH) \leqq \binom{r}{2} b_n^{r-2}(H) \sum_{i=1}^{n} \varrho_i^2.$$

Since H is connected, $\varrho_i \leqq \Delta^k$, $i=1, \ldots, n$. Therefore

$$\sum_{i=1}^{n} \varrho_i^2 / b_n^2(H) \leqq n\Delta^{2k} / (n - o(n))^2 = o(1),$$

because Δ^{2k} is a slow function. The statement (1) is proved. For every graph $\mathcal{H}_r$ let us denote by v and ω the number of vertices and components of $\mathcal{H}_r$, respectively. Then

$$b_n(\mathcal{H}_r) = O(n^{\omega} \Delta^{v-\omega}) \quad \text{and} \quad e(\mathcal{H}_r) \geqq l\omega + 1.$$

So, if μ is a slow function then

$$b_n(\mathcal{H}_r) p^{e(\mathcal{H}_r)} = O\big(f(n) n^{-1/l}\big),$$

where $f(n)$ is also a slow function. Thus all assumptions of Theorem 1 are fulfilled and Theorem 2 follows.

Note that now the function $\bar{p} = (\sum_{i=1}^{n} \varrho_i)^{-1/l}$ is the threshold for $\{\mathrm{ING}_p(n) \supset H\}$ and that $\bar{p} = n^{-1/l} f^{-1}(n)$, where $f(n)$ is a slow function (compare with mentioned results of [1] and [7]).

Examples. Let $\mathrm{ING}(n) = L_n(d)$. We are interested, for instance, in the distribution of d-cubes Q_d contained as subgraphs in a random graph $L_{n,p}(d)$. Notice that in this case

$$b_n(Q_d)/a(Q_d) = (n-1)^d.$$

So, putting $p = cn^{-1/2^{d-1}}$, it follows from Theorem 2 that $X_n(Q_d) \rightsquigarrow \mathrm{Po}\,(c^{d2^{d-1}})$.

Now let $\mathrm{ING}(n) = Q_n$. Then

$$b_n(Q_d)/a(Q_d) = \binom{n}{d} 2^{n-d}$$

and putting $p = c(n^d 2^n)^{-1/d2^{d-1}}$ we obtain also that $X_n(Q_d) \rightsquigarrow \mathrm{Po}\,(c^{d2^{d-1}})$. In fact, we confirm a speculation on the threshold for small subgraphs of a random n-cube given by Erdős and Spencer in [2].

III. Vertex degrees. In such a general approach almost nothing is known about the parameters ϱ_i but in particular cases. The most obvious one is when we put $H = K_{1,l}$, the star with l edges. Then $\varrho_i(K_{1,l}) = (d_i)_l = d_i(d_i-1)\ldots(d_i-l+1)$, where d_i is the degree of the vertex v_i in $\mathrm{ING}(n)$. Note that a star $K_{1,l}$ corresponds to a vertex of degree l in a graph if there are no bigger stars in it. The next result deals with vertex degrees in a random graph $\mathrm{ING}_p(n)$ which has *an almost d-regular* initial graph, i.e. when $|\{i\colon d_i \neq d\}| = o(n)$ and $\Delta(\mathrm{ING}(n)) = d$.

Theorem 3. *Let $Y_n^{(l)}$ denote the number of vertices of degree l in a random graph $\mathrm{ING}_p(n)$ with an almost d-regular initial graph $\mathrm{ING}(n)$, $l=1, 2, \ldots$ and let $d=d(n)$*

be a slow function. Then, if $p=c((d)_l n)^{-1/l}$ *then* $Y_n^{(l)} \rightsquigarrow \mathrm{Po}(c^l/l!)$ *for* $l>1$ *and* $Y_n^{(1)}/2 \rightsquigarrow \mathrm{Po}(c)$ *for* $l=1$; *if* $p=f(n)((d)_l n^{-1/l})$, *where* $f(n)\to\infty$ *is a slow function, then* $(Y_n^{(l)}-EY_n^{(l)})/(\mathrm{Var}\, Y_n^{(l)})^{1/2} \rightsquigarrow N(0, 1)$.

PROOF. In the first case $\mu\to c^l$ whereas in the second one $\mu\to\infty$ and it is a slow function. Moreover $a(K_{1,l})=l!$ and for every $i>l$ $\mu\to 0$, so $Y_n^{(l)}=X_n(K_{1,l})$ a.s. and the statement follows from Theorem 2. (For $l=1$, $Y_n^{(1)}=2X_n(K_2)$.)

EXAMPLE. Let us return to the d-dimensional random lattice $L_{n,p}(d)$ and put $p=cn^{-d/l}$. Vertices of degree $2d$ in $L_n(d)$ are called inner vertices. We can restrict ourselves only to the random variable $\bar{Y}_n^{(l)}$ which counts inner vertices of degree l in $L_{n,p}(d)$, because $E(Y_n^{(l)}-\bar{Y}_n^{(l)})=o(1)$. So we have $\bar{Y}_n^{(l)} \rightsquigarrow \mathrm{Po}\left(c^l\binom{2d}{l}\right)$, $l=2, 3, \ldots$ $\ldots, 2d$. On the other hand, considering a random graph $L_{n,1-p}(d)$ as a complement of $L_{n,p}(d)$ in $L_n(d)$, it is obvious that if $p=1-cn^{-d/l}$ then $\bar{Y}_n^{(2d-l)} \rightsquigarrow \mathrm{Po}\left(c^l\binom{2d}{l}\right)$, $l=2, \ldots, 2d$. So, the results we have obtained are very complete, because they cover all $l=0, \ldots, 2d$. Replacing the constant c by a slow function $f(n)\to\infty$ we change the Poisson distribution of $\bar{Y}_n^{(l)}$ to a standardized normal distribution. All these above results agree with our intuition that for every $l=1, \ldots, 2d-1$ the number of vertices of degree l increases first and then decreases in the process of evolution of a random d-dimensional lattice (for the notion of evolution see [1]). For $d=2$, i.e. for the random square lattice $L_{n,p}^{(4)}$ these results were observed by Z. Palka and L. V. Quintas (a personal communication).

IV. Components and connectedness. In this section we will consider the order η_1 of the largest and the order η_2 of the second largest component of a random graph $\mathrm{ING}_p(n)$ as well as the probability of connectedness. Theorems 4—7 below cover not necessarily the same class of random graphs but all of them deal with sparsely-edged random graphs and all of them can be applied to random plane lattices.

Denote by $\Gamma^k(v)$ the set of vertices lying at the distance at most k from a given vertex v of $\mathrm{ING}(n)$ and put $B(k)=\max_{v\in V(\mathrm{ING}(n))}|\Gamma^k(v)|$.

THEOREM 4. *Suppose that* $\Delta=\Delta(\mathrm{ING}(n))$, $p=p(n)$, $k=k(n)$ *are such that* $n((\Delta-1)p)^k=o(1)$. *Then* $\eta_1(\mathrm{ING}_p(n))\leqq B(k)$ *a.s.*

PROOF. The probability that there is a path of length k in $\mathrm{ING}_p(n)$ is $O(n((\Delta-1)p)^k)=o(1)$. Consider the largest component S of $\mathrm{ING}_p(n)$ and the longest path P in it. Then $|S|\leqq B(|P|)$ but the length of P is less than k a.s. Thus the proof is complete.

Note that, in fact, the above theorem is practically useful when $\mathrm{ING}(n)$ is a sparsely-edged graph.

COROLLARY 1. (a) *If* $p=o(1/n)$ *then* $\eta_1(Q_{n,p})=o(2^n)$ *a.s.*

(b) *If* $p<1/(2d-1)$ *then* $\eta_1(L_{n,p}(d))=O((\log n)^d)$ *a.s.*

(c) *If* $p<1/(d_i-1)$ *then* $\eta_1(L_{n,p}^{(i)})=O((\log n)^2)$ *a.s., where* $i=3, 4, 6$ *and* $d_4=4$, $d_6=3$.

PROOF. Put $k=n/f$, where $f=f(n)\to\infty$ but $f=o(1/np)$. Then the assumption $d_3=6$, of Theorem 4 holds and

$$\eta_1(Q_{n,p}) \leqq B(k) = \sum_{l=0}^{k}\binom{n}{l} \leqq k(ef)^k = o(2^n).$$

Let us estimate $B(k)$ in the case of $L_n(d)$. It is easy to show that $B(k)$ is equal to the sum over l running from 0 to k of the numbers of integer solutions $(x_1, \dots, x_d)$ of the equation $\sum_{i=1}^{d}|x_i|=l$. This sum is less than $2^d\binom{d+k}{k}$. So, putting $k=c\log n$ (c is large enough) we arrive at Statement (b).

Finally, notice that for lattices $L_n^{(i)}$, $i=3, 4, 6$, if $k=k(n)\to\infty$ then $B(k)==O(k^2)$ and the proof of (c) follows on similar lines as (b).

Let us concentrate now on the order η_2 of the second largest component of $\mathrm{ING}_p(n)$ in the case when $\mathrm{ING}(n)$ is a plane lattice.

THEOREM 5. *If* $p>(i-2)/(i-1)$ *then* $\eta_2(L_{n,p}^{(i)})=O((\log n)^2)$ *a.s.*, $i=3, 4, 6$.

PROOF. Note that each minimal cutset of $L_n^{(i)}$ has either 0 or 2 common edges with a given face and therefore edges of any minimal cutset can be ordered in such a way that any two consecutive edges belong to the same face. The probability of the existence of such sequence of $c\log n$ edges in $L_{n,1-p}^{(i)}$ is $O(n((i-1)(1-p))^{c\log n})==o(1)$ for c large enough. Thus our random process of deleting edges from $L_n^{(i)}$ can only cut out a component of $L_{n,p}^{(i)}$ of order $O((\log n)^2)$ a.s., $i=3, 4, 6$.

Let us return to the parameter η_1 in the case when $\Delta(\mathrm{ING}(n))=O(1)$.

THEOREM 6. *Suppose that* $\Delta=\Delta(\mathrm{ING}(n))=O(1)$ *and p is fixed, i.e. p does not depend on n,* $0<p<1$. *Then*

$$\lim_{n\to\infty}\eta_1(\mathrm{ING}_p(n))/n < 1.$$

PROOF. Suppose that the above limit is equal to 1. It means that there is a function $f=f(n)\to\infty$ such that $\eta_1=n-n/f$. But the probability that $\mathrm{ING}_p(n)$ has a connected subgraph of order $n-n/f$ is, based on the inequality (5) from [5], at most

$$\binom{n}{n/f}(1-(1-p)^{\Delta})^{n-n/f} = o(1).$$

The previous result implies that a random graph $\mathrm{ING}_p(n)$ with small $\Delta(\mathrm{ING}(n))$ becomes connected just when $p=p(n)\to 1$. The following theorem sharpens this fact for a special class of random graphs. Denote by f_i the number of cutsets of $\mathrm{ING}(n)$ which have exactly i edges not all incident to the same vertex, $i=1, 2, \dots$.

THEOREM 7. *Let* $\mathrm{ING}(n)$ *be an almost d-regular plane graph with F edges in the largest inner face. Suppose that F is a slow function, d is fixed and* $f_i=o(n^{i/d})$, $i=1, \dots, d$. *Then for* $p=1-cn^{-1/d}$

$$\mathrm{Prob}(\mathrm{ING}_p(n) \textit{ is connected}) \to \exp(-c^d) \quad \textit{as} \quad n\to\infty.$$

PROOF. First, we will prove that besides a giant component there are only isolated vertices a.s. Similarly as in the proof of Theorem 5 it is easy to show that the probability of disconnection of ING (n) by deleting a minimal cutset with more than d edges is $o(1)$. Thus the probability of disconnection of ING (n) in another way than isolation of a vertex of degree d is $O\left(\sum_{i=1}^{d} f_i(1-p)^i\right)=o(1)$. To complete the proof it is enough to observe that the number of such vertices in $\mathrm{ING}_p(n)$ is the same as the number of vertices of degree d in $\mathrm{ING}_{1-p}(n)$. So, applying Theorem 3 we arrive at the statement.

COROLLARY 2. *Let $f=f(n)\to\infty$ as $n\to\infty$. Then*

$$\mathrm{Prob}\,(L_{n,p}^{(i)} \textit{ is connected}) \to \begin{cases} 0, & \textit{if } p=1-fn^{-1/d_i}, \\ \exp(-c^{d_i}), & \textit{if } p=1-cn^{-1/d_i}, \\ 1, & \textit{if } p=1-f^{-1}n^{-1/d_i}, \end{cases}$$

$i=3,4,6,\quad d_3=6,\ d_4=4,\ d_6=3.$

COMMENT. The results of this section confirm (compare with [1], [2]) that in the process of evolution of a random graph $\mathrm{ING}_p(n)$ first (when p is small) there are only small components. Next, the largest one grows more and more and orders of others decrease. If p is large enough ($1/2$ for $Q_{n,p}$, $1-c/\bar{n}$ for $L_{n,p}^{(4)}$) then there are only isolated vertices outside the largest component and finally (see Theorem 7) $\mathrm{ING}_p(n)$ becomes connected.

REMARKS. Theorem 4 and 5 were proved in [3] for $L_{n,p}^{(4)}$. Let us notice that Füredi's statement (a) of Theorem 2 ([3]) can not be deduced from his proof, because the information about the order of the second largest component does not imply that we have a giant component in $L_{n,p}^{(4)}$. We have to mention also that Theorem 7 for $L_{n,p}^{(4)}$ was proved independently in [3] and [4] and the methods of proof in Section IV are mainly those of [3].

Let $v\in V(\mathrm{ING}(n))$. A simple observation that $\mathrm{ING}_p(n)$ is disconnected if and only if there is a component in $\mathrm{ING}_p(n)$ not containing v leads to a generalization of Theorem 7 over such initial graphs ING (n) that ING $(n)-v$ is an almost d-regular plane graph. Thus, in particular, Theorem 7 can be applied to wheels $W_n=v\times C_n$. Finally, let us point out that some stronger results about the largest and the second largest component of $L_{n,p}^{(4)}$, with proofs involving percolation theory, can be found in [8].

Acknowledgement. I wish to thank the referee for drawing my attention to the paper of Grimmett and W. Kordecki and Z. Palka for valuable comments.

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(Received April 29, 1983; revised January 10, 1985)

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Acta Math. Hung.
47 (1—2) (1986), 89—93.

ON DISPERSION AND MARKOV CONSTANTS

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Let $\{x_n\}$ be a sequence of numbers, $0\leqq x_n\leqq 1$. H. Niederreiter in [3] introduced a measure of denseness of such a sequence as follows. For each N let

$$d_N = \sup_{0\leqq x\leqq 1}\left\{\min_{1\leqq n\leqq N} |x-x_n|\right\}$$

and put $D(\{x_n\})=\limsup_N Nd_N$. In particular he carried out investigations of sequences of the form $x_n=n\vartheta(\mathrm{mod}\ 1)$ for irrational ϑ's. For such a ϑ he defined the dispersion constant by $D(\vartheta)=D(\{n\vartheta(\mathrm{mod}\ 1)\})$. It turns out that $D(\vartheta)<\infty$ if and only if the continued fraction expansion of ϑ has bounded partial quotients. Moreover if ϑ_1 and ϑ_2 are equivalent, then $D(\vartheta_1)=D(\vartheta_2)$. (Two numbers are called equivalent if their continued fraction expansions coincide from some point on.) He also shows that if ϑ is equivalent to $\vartheta_1=(1+\sqrt{5})/2$ then $D(\vartheta)=(5+3\sqrt{5})/10=$ $=1.170...$, if ϑ is equivalent to $\vartheta_2=\sqrt{2}$ then $D(\vartheta)=(1+\sqrt{2})/2=1.207...$ and if ϑ is not equivalent to either ϑ_1 or ϑ_2 then $D(\vartheta)\geqq 3-\sqrt{3}=1.267...$. Thus the dispersion spectrum **D**, i.e. the set of all possible values of $D(\vartheta)$, contains gaps. Niederreiter also identifies another gap in **D**: $((1+\sqrt{3})/2,\ (13+7\sqrt{13})/26)=(1.366...,\ 1.470...)$. All this suggests an analogy with Lagrange (or Markov) constants and Lagrange spectrum. For each irrational ϑ the Lagrange (or Markov) constant is defined by

$$M(\vartheta)^{-1} = \liminf_n n\,\|n\vartheta\|$$

where $\|t\|$ denotes the distance from t to the nearest integer. The Lagrange spectrum **L** is then the collection of all possible values of $M(\vartheta)$. (The set **L** has lots of gaps like those above; for an extensive description of these see [1].) In particular Niederreiter asks whether $M(\vartheta_1)<M(\vartheta_2)$ implies $D(\vartheta_1)<D(\vartheta_2)$. The purpose of this paper is to show that this conjecture is not quite true, in fact if

$$\vartheta_1 = \frac{323+\sqrt{950\,629}}{930} = 1.3957...,$$

$$\vartheta_2 = \frac{96\,228\,962+\sqrt{10\,015\,935\,143\,166\,740}}{27\,494\,036} = 7.14004...$$

then $M(\vartheta_1)=9.2857...$, $M(\vartheta_2)=9.2498...$, and $D(\vartheta_1)=2.8276...$, $D(\vartheta_2)=$ $=2.8394...$. The peculiar nature of the numbers ϑ_1 and ϑ_2 will be explained later.

The constants $M(\vartheta)$ and $D(\vartheta)$ are however closely related through the following theorem.

THEOREM. *For every ϑ we have*

$$\frac{1}{4}(M(\vartheta)+2) \leqq D(\vartheta) \leqq \frac{1}{4}\left(M(\vartheta)+\frac{1}{M(\vartheta)}+2\right). \tag{1}$$

Thus, generally at least, the values of $D(\vartheta)$ tend to increase with $M(\vartheta)$. Before proceeding with the proof we introduce some notation. For any number ϑ, its continued fraction expansion

$$\vartheta = c_0(\vartheta)+\frac{1}{c_1(\vartheta)}+\frac{1}{c_2(\vartheta)}+\dots$$

will be denoted by

$$\vartheta = [c_0(\vartheta), c_1(\vartheta), c_2(\vartheta), \dots] = [c_0, c_1, c_2, \dots].$$

We set

$$\lambda_i(\vartheta) = \lambda_i = [0, c_i, c_{i-1}, c_{i-2}, \dots, c_1], \quad \Lambda_i(\vartheta) = \Lambda_i = [c_{i+1}, c_{i+2}, c_{i+3}, \dots],$$

$$\mu_i(\vartheta) = \mu_i = [0, c_{i+2}, c_{i+3}, \dots] = \Lambda_i - c_{i+1}, \quad M_i(\vartheta) = M_i = \lambda_i+\Lambda_i = \lambda_i+c_{i+1}+\mu_i.$$

As usual we put $q_{-1}=0$, $p_{-1}=1$, $q_0=1$, $p_0=c_0$ and

$$p_{k+1} = c_{k+1}p_k+p_{k-1}, \quad q_{k+1} = c_{k+1}q_k+q_{k-1}.$$

We also set $\delta_k=(-1)^k(q_k\vartheta-p_k)=|q_k\vartheta-p_k|$. The following are standard facts about continued fractions:

$$c_{k+1} = \left[\frac{\delta_{k-1}}{\delta_k}\right], \quad q_k\delta_k = \frac{1}{M_k}, \quad \frac{q_{k-1}}{q_k} = \lambda_k,$$

where [.] denotes the greatest integer function. For all this see [2] chapter 3, for instance. With this notation we know that the Markov constant $M(\vartheta)$ is given by $M(\vartheta)=\limsup_k M_k(\vartheta)$.

The following identity will be used repeatedly

$$M_{i-1} = \frac{M_i}{\lambda_i\Lambda_i} \quad \text{or} \quad \frac{1}{M_{i-1}} = \frac{\lambda_i\Lambda_i}{M_i}. \tag{2}$$

This follows immediately from

$$M_{i-1} = [0, c_{i-1}, c_{i-2}, \dots, c_1]+c_i+[0, c_{i+1}, c_{i+2}, \dots] =$$

$$= \frac{1}{[0, c_i, c_{i-1}, \dots, c_1]}+\frac{1}{c_{i+1}+[0, c_{i+2}, c_{i+3}, \dots]} = \frac{1}{\lambda_i}+\frac{1}{\Lambda_i} = \frac{\lambda_i+\Lambda_i}{\lambda_i\Lambda_i}.$$

For each ϑ and each i we also introduce the quadratic polynomial

$$\psi_i(x, \vartheta) = \psi_i(x) = \frac{1}{M_i}\{-x^2+(\Lambda_i-\lambda_i-1)x+\Lambda_i(1+\lambda_i)\}.$$

The polynomial $\psi_i(x)$ assumes its maximum at the point $x_i=\frac{1}{2}(\Lambda_i-\lambda_i-1)$ and the value at this point is given by

$$\psi_i(x_i) = \frac{1}{4}\left(M_i+\frac{1}{M_i}+2\right) \tag{3}$$

as can be easily checked. If n_i is the integer which is closest to x_i then $|x_i-n_i|\leqq\frac{1}{2}$ and

$$\psi_i(x_i)-\frac{1}{4M_i}\leqq\psi_i(n_i)\leqq\psi_i(x_i). \tag{4}$$

The proof of the theorem consists then in showing that

$$D(\vartheta)=\limsup_i \psi_i(n_i), \tag{5}$$

for it is clear that (3), (4) and (5) imply (1).

We recollect now some facts regarding the distribution of the sequence $n\vartheta \pmod 1$. These results can be found in [4]. We denote $n\vartheta \pmod 1$ by $\{n\vartheta\}$. For each fixed n let $1\leqq a_n\leqq n$ be such that $\{a_n\vartheta\}$ is the smallest among $\{\vartheta\}, \{2\vartheta\}, \dots, \{n\vartheta\}$ and let $1\leqq b_n\leqq n$ be such that $\{b_n\vartheta\}$ is the largest. Put $\alpha_n=\{a_n\vartheta\}$ and $\beta_n=1-\{b_n\vartheta\}$. The interval $[0, 1]$ is then divided by $\{\vartheta\}, \{2\vartheta\}, \dots, \{n\vartheta\}$ into $(n+1)$ subintervals as follows: $n+1-a_n$ of them are of length α_n, $a_n+b_n-(n+1)$ of them are of length $\alpha_n+\beta_n$ and $n+1-b_n$ of them are of length β_n. Notice also that the left-most subinterval has length α_n and the right-most interval has length β_n. One can actually find $a_n, b_n, \alpha_n, \beta_n$ in terms of continued fraction expansion of $\vartheta=[c_0, c_1, c_2, \dots]$. Given n to find a_n and α_n set

$$n=q_{2m}+rq_{2m+1}+s,\quad 0\leqq r<c_{2m+2},\quad 0\leqq s<q_{2m+1} \tag{6}$$

so that $q_{2m}\leqq n<q_{2m+2}$. One has then

$$a_n=q_{2m}+rq_{2m+1},\quad \alpha_n=\delta_{2m}-r\delta_{2m+1}. \tag{7}$$

To find b_n and β_n we express n as

$$n=q_{2m-1}+uq_{2m}+v,\quad 0\leqq u<d_{2m+1},\quad 0\leqq v<q_{2m} \tag{8}$$

so that $q_{2m-1}\leqq n<q_{2m+1}$. We have then

$$b_n=q_{2m-1}+uq_{2m},\quad \beta_n=\delta_{2m-1}-u\delta_{2m}. \tag{9}$$

We are now ready to prove (5) and hence the theorem. First of all it is clear that one has $D(\vartheta)=\limsup_n (n+1)d_n$ since plainly $d_n\to 0$. The equation (5) will then follow from

$$\operatorname*{Max}_{q_k\leqq n<q_{k+1}} (n+1)d_n=\operatorname*{Max}_{0\leqq b\leqq c_{k+1}} \psi_k(b) \tag{10}$$

and

$$0\leqq n_k<c_{k+1}. \tag{11}$$

We break the proof into two cases, depending whether k is even or odd and present the detailed arguments only in the even case, the case of k being odd is completely analogous. Assume then that $k=2m$ and $q_{2m}\leqq n<q_{2m+1}$. To establish (10) we show that for each n in this range one has

$$(n+1)d_n\leqq\psi_{2m}(b) \tag{12}$$

for some $0\leqq b<c_{2m+1}$, and conversely, for each such b there is a corresponding n for which the equality holds in (12). We break up the interval $[q_{2m}, q_{2m+1})$ as follows

Case I. $q_{2m} \leqq n < q_{2m-1}+q_{2m}$.

Case II. $q_{2m-1}+bq_{2m} \leqq n < q_{2m-1}+(b+1)q_{2m}, \quad b = 1, 2, \ldots, c_{2m+1}-1$.

In case I $\alpha_n=\delta_{2m-1}$, $\beta_n=\delta_{2m}$ (see (6)—(9)) and the largest value of d_n occurs when $x=0$ in which case $d_n=\delta_{2n-1}$ or when $x=1$ in which case $d_n=\delta_{2m}$ or when x is a midpoint of one of the intervals of length $\alpha_n+\beta_n$, in which case $d_n=\frac{1}{2}(\alpha_n+\beta_n)$. Since $\alpha_n<\beta_n$ we see that for n in this range $d_n=\delta_{2m-1}$ and the largest value assumed by $(n+1)d_n$ is for $n+1=q_{2m-1}+q_{2m}$, that value being

$$(q_{2m-1}+q_{2m})\delta_{2m-1} = \frac{1}{M_{2m-1}}+\frac{1}{\lambda_{2m}}\frac{1}{M_{2m-1}} =$$

$$= \frac{\lambda_{2m}\Lambda_{2m}}{M_{2m}}\left(1+\frac{1}{\lambda_{2m}}\right) = \frac{1}{M_{2m}}\Lambda_{2m}(1+\lambda_{2m}) = \psi_{2m}(0).$$

(Equation (2) was used here to deduce the second equality.) In case II, for each fixed $1\leqq b<c_{2m+1}$ we get $\alpha_n=\delta_{2m}$, $\beta_n=\delta_{2m-1}-b\delta_{2m}$. Since $b<c_{2m+1}$, $\delta_{2m-1}-b\delta_{2m}>\delta_{2m}$ so $\alpha_n<\beta_n$ hence $d_n=\beta_n$ and the largest value assumed by $(n+1)d_n$ is when $n+1=q_{2m-1}+(b+1)q_{2m}$, that value being

$$(q_{2m-1}+(b+1)q_{2m})(\delta_{2m-1}-b\delta_{2m}) =$$

$$= \frac{1}{M_{2m-1}}-b\frac{\lambda_{2m}}{M_{2m}}+(b+1)\frac{1}{\lambda_{2m}}\frac{1}{M_{2m-1}}-b(b+1)\frac{1}{M_{2m}}.$$

Replacing M_{2m-1} by the expression from (2) this quantity becomes

$$\frac{1}{M_{2m}}\{-b^2+(\Lambda_{2m}-\lambda_{2m}-1)b+\Lambda_{2m}(1+\lambda_{2m})\} = \psi_{2m}(b).$$

Thus for $q_{2m}\leqq n<q_{2m+1}$,

$$(n+1)d_n \leqq \psi_{2m}(b) \quad \text{for some} \quad 0 \leqq b < c_{2m+1}$$

and for every such b there is an n for which the equality holds. To complete the proof we must show that the maximum on the right hand side of (10) does not occur for $b=c_{2m+1}$, or what amounts to the same thing we must show (11). However $n_{2m}\leqq x_{2m}+\frac{1}{2}$ and $x_{2m}=\frac{1}{2}(d_{2m+1}+\mu_{2m}-\lambda_{2m}-1)$ so (11) is equivalent to $d_{2m+1}+\mu_{2m}-\lambda_{2m}<2d_{2m+1}$ or $\mu_{2m}-\lambda_{2m}<d_{2m+1}$ which is clear since $\mu_{2m}<1$, $\lambda_{2m}>0$ and $d_{2m+1}\geqq 1$. The left inequality of (11) follows from $n_{2m}\geqq x_{2m}-\frac{1}{2}$ in the same way. Thus the Theorem is proved.

The inequality (1) is actually the best possible in the sense that both

$$D(\vartheta) = \frac{1}{4}(M(\vartheta)+2) \tag{13}$$

and

$$D(\vartheta) = \frac{1}{4}(M(\vartheta)+M(\vartheta)^{-1}+2) \tag{14}$$

can occur. Indeed consider

$$\vartheta = [\underbrace{11\dots1}_{m_1} A\, \underbrace{11\dots1}_{m_2} A\, \underbrace{11\dots1}_{m_3} A\dots], \quad A > 3$$

where $m_1 < m_2 < m_3 \dots \to \infty$. It follows from the proof of the Theorem that for A odd, $x_i - n_i \to 0$, thus (14) holds and for A even $|x_i - n_i| \to \frac{1}{2}$, thus (13) holds.

The equation (10) can be used to calculate the values of $D(\vartheta)$ for quadratic irrationalities ϑ, i.e. those ϑ's whose continued fraction is periodic. If ϑ has a continued fraction expansion $\vartheta = [\overline{c_0, c_1, c_2, \dots, c_{p-1}}]$ where the bar indicates the period, let

$$\bar{\lambda}_i = [0, c_i, c_{i-1}, \dots], \quad \bar{\Lambda}_i = [c_i, c_{i+1}, c_{i+2}, \dots], \quad \overline{M}_i = \bar{\lambda}_i + \bar{\Lambda}_i$$

where the sequence $c_0, c_1, \dots, c_{p-1}$ is extended periodically in both directions. As before put

$$\bar{\psi}_i(x) = \frac{1}{\overline{M}_i}\{-x^2 + (\bar{\Lambda}_i - \bar{\lambda}_i - 1)x + \bar{\Lambda}_i(1+\bar{\lambda}_i)\},$$

$$\bar{x}_i = \frac{1}{2}(\bar{\Lambda}_i - \bar{\lambda}_i - 1), \quad \bar{n}_i = \text{the integer closest to } \bar{x}_i.$$

It is evident that $D(\vartheta) = \max\limits_{0 \le i \le p-1} \bar{\psi}_i(\bar{n}_i)$. The calculation of $D(\vartheta)$ can now be easily accomplished using this identity. The numbers ϑ_1 and ϑ_2 mentioned at the beginning of the paper are

$$\vartheta_1 = [\overline{1, 2, 1, 1, 8, 1, 2, 2, 1}], \quad \vartheta_2 = [\overline{7, 7, 7, 9, 9, 9, 7, 7, 7}].$$

It would be interesting to find examples with shorter period and/or smaller continued fraction digits.

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(Received July 21, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 95—107.

EMBEDDING AND COMPACTNESS THEOREMS FOR IRREGULAR AND UNBOUNDED DOMAINS IN WEIGHTED SOBOLEV SPACES

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Introduction. It is well known that embedding theorems, due to Sobolev, Gagliardo, Nirenberg, and compactness theorems, due to Kondracev and Rellich, for the classical Sobolev spaces, essentially require that the domain is bounded and verifies the cone property.

Consequently, a large amount of work has been carried out by several authors to weaken these assumptions, often in the more general context of weighted spaces.

Typical results (cf. for example Avantaggiati [2], Benci and Fortunato [3], Matarasso and Troisi [8]) ensure the compactness with rather weak assumptions, fulfilled, for instance, by cusps on the domain, when the embedding is already known and using weights infinitesimal in the singular set and to the infinity. In this way, unconditional results are in particular obtained for unbounded domains which verify the cone property. For other results in this direction, but with regard only to the unboundedness of the domain, we refer to the works of Berger and Schechter [4], Edmunds and Evans [6] and to the book of Adams [1].

In this type of results, the weights often play the role of balancing irregularities of the domain. Our purpose in this work is to further investigate the correlations between irregular domains and weights. We obtain embedding and compact embedding theorems with the same exponents of the theorems of Sobolev. As a consequence, we reobtain some results of Adams [1], Benci and Fortunato [3], Berger and Schechter [4], Edmunds and Evans [6], Matarasso and Troisi [8], Muckenhoupt and Wheeden [10]. We also obtain rather precise results for a class of domains, including cusps, having simple geometrical properties. We point out that the embedding with the Sobolev exponents without weights fails also for the simplest types of cusps; in this direction, Campanato [5] obtains embedding theorems with optimal exponents, depending on the irregularity of the domain.

We take the pleasure to thank Prof. L. Carbone for the helpful discussions on the subject.

1. Notations and statement of the results

Let Ω be an open set of R^n and let $\{\Omega_i\}_{i\in N}$ be a family of bounded open sets, contained in Ω, which covers Ω and fulfils the finite intersection property, and such that every Ω_i verifies the cone property, with cone of height h_i and aperture θ, θ independent of i (see Part 3). We call "weight function" on Ω every measurable function $\sigma\colon \Omega\to R^+$ a.e. positive in Ω. We set

$$\|u\|_{L^p(\Omega,\sigma)} = \Big(\int_\Omega \sigma(x)\,|u(x)|^p\,dx\Big)^{1/p}. \tag{1.1}$$

Let us point out that (1.1) defines a norm if and only if σ is a.e. positive in Ω. We denote by $L^p(\Omega, \sigma)$ the Banach space of the functions u defined on Ω such that $u|_{\Omega_i} \in L^p(\Omega_i)$ and $\|u\|_{L^p(\Omega,\sigma)}$ is finite, equipped with the norm (1.1), and with $W^r_{p_0,p_1}(\Omega, \alpha, \beta)$ the Banach space of the distributions u on Ω such that $u \in L^{p_0}(\Omega, \alpha)$ and $D^\nu u \in L^{p_1}(\Omega, \beta)$ for $|\nu| = r$, equipped with the norm

$$(1.2)\qquad \|u\|_{W^r_{p_0,p_1}(\Omega,\alpha,\beta)} = \|u\|_{L^{p_0}(\Omega,\alpha)} + \sum_{|\nu|=r} \|D^\nu u\|_{L^{p_1}(\Omega,\beta)}.$$

We also set

$$(1.3)\qquad |D^r u|_{L^{p_1}(\Omega,\beta)} = \sum_{|\nu|=r} \|D^\nu u\|_{L^{p_1}(\Omega,\beta)} = \Big(\int_\Omega \beta(x) |D^r u|^{\beta_1}\, dx\Big)^{1/\beta_1}.$$

Let α, β, γ be weight functions and ϱ, σ, Q measurable nonnegative functions and define

$$(1.4)\qquad N_{r,t,\tau}(\varrho, \sigma, \Omega) = |\varrho|^r_{t,\Omega} |\sigma|_{\tau,\Omega},$$

$$(1.5)\qquad N_0(M) = \sup_{i \geqq M} \Big\{ h_i^{\lambda_0\left(\frac{q\tau}{\tau-1}, \frac{p_0 t_0}{t_0+1}\right)\frac{\tau-1}{\tau}} N_{\frac{q}{p_0}, t_0, \tau}(\alpha^{-1}, \gamma Q^q, \Omega_i) \Big\},$$

$$(1.6)\qquad N_1(M) = \sup_{i \geqq M} \Big\{ h_i^{\lambda_1\left(\frac{q\tau}{\tau-1}, \frac{p_1 t_1}{t_1+1}\right)} N_{\frac{q}{p_1}, t_1, \tau}(\beta^{-1}, \gamma Q^q, \Omega_i) \Big\},$$

$$(1.7)\qquad N_0 = N_0(1), \quad N_1 = N_1(1),$$

where

$$(1.8)\qquad \lambda_0(p, p_0) = -\Big\{|k|p + n\Big(\frac{p}{p_0} - 1\Big)\Big\},$$

$$(1.9)\qquad \lambda_1(p, p_1) = p(r - |k|) - n\Big(\frac{p}{p_1} - 1\Big).$$

Our main results are the following:

Embedding Theorem. *Let $r \in N^+$, k a multiindex, $|k| < r$ and*

$$(1.10)\qquad \frac{|k|}{r} \leqq a < 1,$$

$$(1.11)\qquad \frac{\tau-1}{\tau}\frac{1}{q} \geqq a\Big(\frac{1}{p_1}\frac{t_1+1}{t_1} - \frac{r}{n}\Big) + (1-a)\frac{1}{p_0}\frac{t_0+1}{t_0} + \frac{|k|}{n},$$

with

$$(1.12)\qquad 1 \leqq p_i \leqq q < +\infty, \quad \frac{1}{p_i - 1} \leqq t_i \leqq +\infty, \quad 1 < \tau \leqq +\infty.$$

Then the following inequality holds:

$$(1.13)\qquad |QD^k u|_{L^q(\Omega,\gamma)} \leqq C\big\{N_0 \|u\|_{L^{p_0}(\Omega,\alpha)} + N_0^{1-a} N_1^a |u|^{1-a}_{L^{p_0}(\Omega,\alpha)} |D^r u|^a_{L^{p_1}(\Omega,\beta)}\big\}.$$

COMPACT EMBEDDING THEOREM. *With the notations and assumptions of the Embedding Theorem, suppose also that* $a>0$ *and*

$$N_0+N_1<+\infty, \tag{1.14}$$

$$\lim_{M\to+\infty} N_0(M)=0. \tag{1.15}$$

Then the operator

$$u\in W^r_{p_0,p_1}(\Omega,\alpha,\beta)\to QD^k u\in L^q(\Omega,\gamma)$$

is compact.

The sequel of the paper is organized as follows: Part 2 is devoted to the proof of the theorems stated above; Part 3 examines the decomposability of the domains, and the relation with the functions

$h(x,\theta)=\sup\{r\in(0,1]\,|\,\exists$ a closed cone of vertex x, aperture θ and height r, contained in $\Omega\}$,

$h_1(x,\theta)=\sup\{r\in(0,1]\,|\,\exists$ an open cone of vertex x, aperture θ and height r, contained in $\Omega\}$,

Moreover, we give in Theorem 3.14 an explicit formulation of embedding and compactness results in terms of $\alpha,\beta,\gamma,Q,h(x,\theta)$ for a class of domains not verifying the cone property; in Part 4 we shall show that this class includes cusps.

Part 4 applies the results obtained before to cusps, evaluating the order of magnitude of $h(x,\theta)$ for these domains. Finally, we also consider a case in which the weights "explode" on an unbounded domain of small measure.

2. Proofs

In the sequel we shall denote by C, C' and C_i some constants.

LEMMA 2.1. *Let* Ω_h *be an open set which verifies the cone property with cone of height* h *and let* $p,p_0,p_1\in[1,+\infty)$, $r\in N$, $k\in N^n$, $a\in\left[\frac{|k|}{r},1\right)$ *such that* $|k|<r$ *and*

$$\frac{1}{p}\geqq a\left(\frac{1}{p_1}-\frac{r}{n}\right)+\frac{1-a}{p_0}+\frac{|k|}{n}.$$

Then we have

$$\int_{\Omega_h}|D^k u(x)|^p\,dx\leqq C\Big\{h^{\lambda_0(p,p_0)}\Big(\int_{\Omega_h}|u(x)^{p_0}\,dx\Big)^{p/p_0}+$$
$$+h^{(1-a)\lambda_0(p,p_0)+a\lambda_1(p,p_1)}\Big(\int_{\Omega_h}|u(x)|^{p_0}\,dx\Big)^{(1-a)p/p_0}\Big(\int_{\Omega_h}|D^r u(x)|^{p_1}\,dx\Big)^{ap/p_1}\Big\}, \tag{2.1}$$

where the constant involved in (2.1) *is independent of* h.

PROOF. We consider the function $y=\varphi(x)=\dfrac{x-x_0}{h}$, $x_0\in\Omega$ fixed. Clearly, $\Omega=\varphi(\Omega_h)$ verifies the cone property with cone of height 1. Let $u\colon\Omega_h\to R$,

$\tilde{u}=u\circ\varphi^{-1}$. Then, by a known result (for example Adams [1] p. 106, Gagliardo [7], C. Miranda [9]), we have

(2.2)
$$\int_\Omega |D^k\tilde{u}|^p\,dy \leqq C\Big\{\Big(\int_\Omega |\tilde{u}|^{p_0}\,dy\Big)^{p/p_0}+\Big(\int_\Omega |\tilde{u}|^{p_0}\,dy\Big)^{(1-a)p/p_0}\Big(\int_\Omega |D^r\tilde{u}|^{p_1}\,dy\Big)^{ap/p_1}\Big\}.$$

With easy computations, we get for every $i\in N^n$

$$\int_\Omega |D^i\tilde{u}(y)|^p\,dy = h^{|i|p-n}\int_{\Omega_h} |D^i u(x)|^p\,dx. \tag{2.3}$$

Now, we immediately obtain our statement from (2.2), (2.3).

Proof of the Embedding Theorem. From Lemma 2.1, setting

$$\lambda_0=\lambda_0\left(\frac{q\tau}{\tau-1},\ \frac{p_0t_0}{t_0+1}\right),\quad \lambda_1=\lambda_1\left(\frac{q\tau}{\tau-1},\ \frac{p_1t_1}{t_1+1}\right),$$

we obtain

(2.4)
$$|D^k u|^{\frac{q\tau}{\tau-1}}_{\frac{q\tau}{\tau-1},\,\Omega_i} \leqq C\Big\{h_i^{\lambda_0}|u|^{\frac{q\tau}{\tau-1}}_{\frac{p_0t_0}{t_0+1},\,\Omega_i}+\big(h_i^{\lambda_0}|u|^{\frac{q\tau}{\tau-1}}_{\frac{p_0t_0}{t_0+1},\,\Omega_i}\big)^{1-a}\big(h_i^{\lambda_1}|D^r u|^{\frac{q\tau}{\tau-1}}_{\frac{p_1t_1}{t_1+1},\,\Omega_i}\big)^a.$$

Moreover, applying the Hölder inequality we have

$$\int_\Omega \gamma(x)\,|Q(x)D^k u(x)|^q\,dx \leqq \sum_{i=1}^\infty \int_{\Omega_i}\gamma(x)\,|Q(x)D^k u(x)|^q\,dx \leqq \tag{2.5}$$
$$\leqq \sum_{i=1}^\infty |\gamma Q^q|_{\tau,\Omega_i}|D^k u|^q_{\frac{q\tau}{\tau-1},\,\Omega_i};$$

$$\int_{\Omega_i}|u(x)|^{\frac{pt}{t+1}}\,dx \leqq \Big(\int_{\Omega_i}\alpha(x)^{-t}\,dx\Big)^{\frac{1}{t+1}}\Big(\int_{\Omega_i}\alpha(x)\,|u(x)|^p\,dx\Big)^{\frac{t}{t+1}}.$$

From (2.4) and (2.5) we get

$$|QD^k u|_{L^q(\Omega,\gamma)} \leqq C\sum_{i=1}^\infty\Big\{h_i^{\left(\frac{\tau-1}{\tau}\right)\lambda_0}|\gamma Q^q|_{\tau,\Omega_i}|\alpha^{-1}|^{q/p_0}_{t_0,\Omega_i}\Big\}|u|^q_{L^{p_0}(\Omega_i,\alpha)}+ \tag{2.6}$$
$$+\Big(\sum_{i=1}^\infty\{\dots\}|u|^q_{L^{p_0}(\Omega_i,\alpha)}\Big)^{1-a}\Big(\sum_{i=1}^\infty\Big\{h_i^{\left(\frac{\tau-1}{\tau}\right)\lambda_1}|\gamma Q^q|_{\tau,\Omega_i}|\beta^{-1}|^{q/p_1}_{t_1,\Omega_i}\Big\}|D^r u|^q_{L^{p_1}(\Omega_i,\beta)}\Big)^a.$$

Now, recalling the notations (1.5), (1.6), (1.7), formula (2.6) becomes

(2.7)
$$|QD^k u|^q_{L^q(\Omega,\gamma)} \leqq C\Big\{N_0\sum_{i=1}^\infty\|u\|^q_{L^{p_0}(\Omega_i,\alpha)}+N_0^{1-a}N_1^a\sum_{i=1}^\infty\|u\|^{q(1-a)}_{L^{p_0}(\Omega_i,\alpha)}|D^r u|^{qa}_{L^{p_1}(\Omega_i,\beta)}\Big\}.$$

Since $q\geqq p_0$, $q\geqq p_1$, our statement follows from (2.7) by the Hölder inequality.

REMARK. When the terms of the sums of the right member of (2.6) are infinitesimal, the upper bound (2.6) is lower than the upper bound (1.13). This loss causes conditions non optimal in general on the weights; but in this manner we are able to obtain embedding and compactness results with the Sobolev exponents.

PROOF OF THE COMPACT EMBEDDING THEOREM. We obviously have

$$|QD^k u|_{L^q(\Omega,\gamma)} < C\{|QD^k u|_{L^q(\bigcup_{i\leq M}\Omega_i,\gamma)} + |QD^k u|_{L^q(\bigcup_{i>M}\Omega_i,\gamma)}\}. \tag{2.8}$$

In our hypothesis, $\bigcup_{i\leq M}\Omega_i$ verifies the cone property with cone of height $\bar{h}\geq\min_{i<M} h_i$. Since $N_0+N_1<+\infty$, we have by the Embedding Theorem

$$|QD^k u|_{L^q(\bigcup_{i\leq M}\Omega_i,\gamma)} \leq C(M)\{\varepsilon|D^r u|_{L^{\frac{p_1t_1}{t_1+1}}(\bigcup_{i\leq M}\Omega_i)} + C(\varepsilon)\|u\|_{L^{\frac{p_0t_0}{t_0+1}}(\bigcup_{i\leq M}\Omega_i)}\}, \tag{2.9}$$

$$|QD^k u|_{L^q(\bigcup_{i>M}\Omega_i,\gamma)} \leq \delta|D^r u|_{L^{p_1}(\bigcup_{i>M}\Omega_i,\beta)} + C(\delta)N_0(M)\|u\|_{L^{p_0}(\bigcup_{i>M}\Omega_i,\alpha)}. \tag{2.10}$$

From (2.8), (2.9) and (2.10), we obtain for every M, $\varepsilon,\delta>0$

$$|QD^k u|_{L^q(\Omega,\gamma)} \leq \delta|D^r u|_{L^{p_1}(\Omega,\beta)} + C(\delta)N_0(M)\|u\|_{L^{p_0}(\bigcup_{i>M}\Omega_i,\alpha)} + \tag{2.11}$$

$$+\varepsilon C(M)|D^r u|_{L^{\frac{p_1t_1}{t_1+1}}(\Omega)} + C(\varepsilon, M)\|u\|_{L^{\frac{p_0t_0}{t_0+1}}(\bigcup_{i\leq M}\Omega_i)}\}.$$

Let now $\{u_n\}_{n\in N}$ be a sequence of functions in $W^r_{p_0,p_1}(\Omega,\alpha,\beta)$ weakly convergent to zero and

$$\|u_n\|_{W^r_{p_0,p_1}(\Omega,\alpha,\beta)} \leq C. \tag{2.12}$$

Since $\bigcup_{i\leq M}\Omega_i$ is bounded and verifies the cone property, we have the compact embedding

$$W^r_{\frac{p_1t_1}{t_1+1},\frac{p_0t_0}{t_0+1}}(\bigcup_{i\leq M}\Omega_i) \to L^{\frac{p_1t_1}{t_1+1}}(\bigcup_{i\leq M}\Omega_i)$$

(cf. Lemma 6.6 and Theorem 6.2 of Adams [1]).

Hence

$$\lim_{n\to+\infty}\|u_n\|_{L^{\frac{p_1t_1}{t_1+1}}(\bigcup_{i\leq M}\Omega_i)} = 0. \tag{2.13}$$

Using (2.12), formula (2.11) gives, for every M, $\varepsilon,\delta>0$

(2.14)

$$|QD^k u_n|_{L^q(\Omega,\gamma)} < C\{\delta + C(\delta)N_0(M) + \varepsilon C(M) + C(\varepsilon, M)\|u_n\|_{L^{\frac{p_1t_1}{t_1+1}}(\bigcup_{i\leq M}\Omega_i)}\}$$

where the constant is independent of δ, ε, M, n.

Choosing $\delta<\lambda$ and M such that $C(\delta)N_0(M)<\lambda$, as we can do by (1.15), and then ε such that $\varepsilon C(M)<\lambda$, formula (2.11) gives for every $\lambda\in R^+$

$$\|QD^k u_n\|_{L^q(\Omega,\gamma)} < C\{\lambda + C(\lambda)\|u_n\|_{L^{\frac{p_1 t_1}{t_1+1}}(\bigcup_{i\le M}\Omega_i)}\}, \tag{2.15}$$

where the constant is independent of λ, n.

Finally, using (2.13), we get by (2.15) for every λ and for $n>n_0(\lambda)$

$$|QD^k u_n|_{L^q(\Omega,\gamma)} < C\lambda. \tag{2.16}$$

Then, the arbitrarity of λ implies

$$\lim_{n\to+\infty} |QD^k u_n|_{L^q(\Omega,\gamma)} = 0.$$

3. Decomposability of domains

We shall give a decomposition of the open set Ω using a function $a(x)\in C^0(\bar\Omega)$ and strictly positive on Ω; for this purpose, we introduce the following

DEFINITION. We say that Ω has the $(\mathscr{D})$-property (with respect to $a(x)$) if for every ε_1, ε_2, with $\varepsilon_2>\varepsilon_1>0$, the open set $\Omega(\varepsilon_1,\varepsilon_2)=\{x\in\Omega|\varepsilon_1<a(x)<\varepsilon_2\}$ is such that $\inf\limits_{x\in\Omega(\varepsilon_1,\varepsilon_2)} h_\Omega(x,\theta)>0$.

THEOREM 3.1. *Let $a(x)\in C^0(\bar\Omega)$, $a(x)>0$ in Ω, Ω having the $(\mathscr{D})$-property. Then there exists an admissible decomposition $\{\Omega_i\}_{i\in N}$ of Ω. Moreover, if $a(x)\in \mathrm{Lip}\,(\bar\Omega)$ and for some $\theta>0$*

$$C_1 a(x) < h(x,\theta) < C_2 a(x), \quad \forall x\in\Omega, \tag{3.1}$$

then $h_i = h(\Omega_i) \geqq C \inf\limits_{x\in\Omega_i} h(x,\theta)$.

PROOF. We can assume, without loss of generality, that Ω is bounded, because otherwise we can work on

$$\Omega_\lambda = \Omega\cap\{x\in R^n|\lambda_i < x_i < \lambda_{i+1}\}, \quad \lambda = (\lambda_1, \dots, \lambda_n)\in N^n.$$

We define the open sets

$$\Omega^+(b) = \{x\in\Omega|\, a(x) > b\}, \quad \Omega^-(b) = \{x\in\Omega|\, a(x) < b\},$$

$$D_k = \Omega^+(2^{-(k+1)})\cap\Omega^-(2^{-(k-1)}).$$

Clearly, we have

$$\Omega = \Omega^+(1/2)\cup(\bigcup_{i=2}^{\infty} D_i). \tag{3.2}$$

For $k_1<k_2$, we define

$$d(k_1,k_2) = d(\Omega^+(2^{-k_1}), \Omega^-(2^{-k_2})). \tag{3.3}$$

The continuity of $a(x)$ on $\bar\Omega$ bounded implies

$$d(k_1,k_2) > 0. \tag{3.4}$$

Finally, we define

(3.5)
$$\Omega_i = \bigcup_{x \in D_i} C_i(x),$$

where $C_i(x)$ is an open cone contained in Ω, such that $x \in C_i(x)$, of height h_i, with

(3.6)
$$0 < h_i \leqq C \min\{d(i+1, i+2), \inf_{x \in D_i} h(x, \theta)\}.$$

This cone exists by the ($\mathscr{D}$)-property; moreover, we can assume that h_i is a decreasing sequence. The constant C which appears in (3.6) is such that $0<C<1$ and it will be chosen in the sequel.

Clearly, the sets Ω_i are open, $\Omega_i \subset \Omega$, $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and every Ω_i verifies the cone property with cone of height greater than or equal to h_i. We show that the family $\{\Omega_i\}_{i \in N}$ has the finite intersection property if C is small enough. Indeed let $y \in \Omega_i \cap \Omega_{i+3}$. Then

$$y \in C_i(x_i), \quad y \in C_{i+3}(x_{i+3}), \quad \text{with} \quad x_i \in D_i, \quad x_{i+3} \in D_{i+3},$$

which implies

$$2^{-(i+1)} < a(x_i) < 2^{-(i-1)}, \quad 2^{-(i+4)} < a(x_{i+3}) < 2^{-(i+2)}.$$

Then we get

(3.7)
$$d(i+1, i+2) \leqq |x_i - x_{i+3}| \leqq |x_i - y| + |y - x_{i+3}| \leqq 2(h_i + h_{i+3}) \leqq$$
$$\leqq 4h_i \leqq 4Cd(i+1, i+2),$$

which is a contradiction if $C < \dfrac{1}{5}$.

To prove the last part of Theorem 3.1, we denote by L the Lipschitz constant of $a(x)$; clearly, we have by (3.1)

(3.8)
$$C_3 2^{-i} \leqq \inf_{x \in \Omega_i} h(x, \theta) \leqq C_4 2^{-i}$$

and so it suffices to show that we can choose in the construction above $h_i = C2^{-i}$ for a suitable constant C. Let $y \in \Omega^+(2^{-(i+1)})$, $z \in \Omega^-(2^{-(i+2)})$, such that $d(i+1, i+2) > \dfrac{1}{2} d(y, z)$. Then, by the Lipschitz property of $a(x)$

(3.9)
$$d(i+1, i+2) > \frac{1}{2}|y-z| \geqq \frac{1}{2L}|a(y) - a(z)| > \frac{1}{2L}\left(\frac{1}{2^{k+1}} - \frac{1}{2^{k+2}}\right) = \frac{1}{8L} 2^{-k}.$$

Formulas (3.7) and (3.9) give $\dfrac{1}{8L} 2^{-i} \leqq 4h_i \leqq 4C2^{-i}$ which is absurd for $C < \dfrac{1}{40L}$, proving the finite intersection property, and the Theorem.

In view of the applications, we shall consider the following choices of the functions $a(x)$:

(i) $a(x) = d(x, \partial\Omega)$

(ii) $a(x) = d(x, S)$ for $S \subset \partial\Omega$

(iii) $a(x) = h(x, \theta)$ for some θ, when $h(x, \theta) = h_1(x, \theta)$.

The first function is clearly in $C^0(\bar{\Omega})$ (in fact, it is in $\mathrm{Lip}(\bar{\Omega})$) for every domain Ω, and so it shows that an admissible decomposition always exists for every open set Ω. Unfortunately, the values of h_i are far from what one expects (consider for instance Ω having the cone property).

The choice (ii) requires the ($\mathscr{D}$)-property; it often gives good values of h_i with an appropriate choice of S, which can be considered in a sense the "singular part" of $\partial\Omega$.

For the choice (iii), we prove the following

LEMMA 3.2. *The function $h(x, \theta)$ is lower semicontinuous (in x).*

PROOF. Let $h(x_0, \theta)=1$; then, for every $\varepsilon>0$, there exists a cone $C(x_0, \theta, 1-\varepsilon) \subset\subset \Omega$, that is $\overline{C(x_0, \theta, 1-\varepsilon)} \subset \Omega$; let $d=d(C, \partial\Omega)>0$. Consider the cone $C(x, \theta, 1-\varepsilon)$, obtained translating the cone $C(x_0, \theta, 1-\varepsilon)$ with vertex in x in spite of x_0. If $|x-x_0|<d/2$, we have

$$d(y, C(x, \vartheta, 1-\varepsilon)) < d/2, \quad \forall y \in C(x, \theta, 1-\varepsilon),$$

whence $C(x, \theta, 1-\varepsilon) \subset\subset \Omega$. This means that

$$h(x, \theta) \geqq 1-\varepsilon, \quad \forall x \text{ with } |x-x_0| < d/2,$$

from which we infer

$$h(x_0, \theta) \leqq \liminf_{x \to x_0} h(x, \theta).$$

LEMMA 3.3. *The function $h_1(x, \theta)$ is upper semicontinuous (in x).*

PROOF. Let $x_n \to x$ in Ω, $\{C_n\}_{n \in N}$ a sequence of cones of vertex x_n, opening θ and height h_n, with $h_n \to h$. Passing to a subsequence, we can assume that the bisectors b_n of C_n converge to b. We show that the cone C of vertex x, height h, opening θ and bisector b is contained in Ω. Clearly, we have

$$C = \left\{ y = x + \varrho(b + y_b) \,|\, 0 < \varrho < ch,\ y_b \in (b)^\perp, \frac{\|y_b\|}{\|b\|} < c(\theta) \right\}$$

where $c(\theta)$ is a constant which depends only on the opening. Let

$$\tilde{C} = \left\{ y = x + \mu b + \varrho(b + y_b) \,|\, 0 < \varrho < ch',\ \mu > 0,\ h' + \mu < h, \frac{\|y_b\|}{\|b\|} < c(\theta) \right\}$$

be a cone of vertex $x+\mu b$ well contained in C and let $y \in \tilde{C}$, $y = x + \mu b + \varrho(b + y_b)$, $v_n = (x - x_n) + (\varrho + \mu)(b - b_n)$. Then $y = x_n + \varrho(b_n + y_b) + \{\mu b_n + v_n\}$.

We denote by P_n the projection on the linear space generated by b_n, and by Q_n the projection on $(b_n)^\perp$. We observe that $\varepsilon_n = \|x - x_n\| + \|b - b_n\| \to 0$, and get $y = x_n + (\varrho + \mu) b_n + \varrho y_b + P_n v_n + Q_n v_n$. Then

(i) $(\varrho + \mu) b_n + P_n v_n = \alpha b_n$, with $0<\alpha<\varrho+\mu+C\varepsilon_n<h'+\mu+C\varepsilon_n<h$ if $n>n_0$ because $h'+\mu<h$ and $\varepsilon_n\to 0$;

$$\text{(ii)}\quad \frac{\|\varrho y_b+Q_n v_n\|}{\|(\varrho+\mu)b_n+P_n v_n\|}<\frac{\varrho}{\varrho+\mu}\frac{\|y_b\|}{\|b_n\|}+C\varepsilon_n<c(\theta)$$

if $n>n_0$, because $\dfrac{\|y_b\|}{\|b_n\|}<c(\theta)$, $\mu>0$ and $\varepsilon_n\to 0$.

This shows that $y\in C_n$, hence $\tilde{C}\subset C_n\subset\Omega$ for every $\tilde{C}\subset\subset C$, which implies $C\subset\Omega$. We deduce

$$h_1(x_0,\theta)\geqq \max\lim_{x\to x_0} h_1(x,\theta).$$

By these Lemmas, if, for some θ, $h(x,\theta)=h_1(x,\theta)$ and this function (continuous in Ω) can be continuously extended to $\bar{\Omega}$, the conditions of Theorem 3.1 are fulfilled.

In several cases, also for simple domains having the cone property, the functions $h(x,\theta)$ and $h_1(x,\theta)$ are really discontinuous in some points, but for our purposes it is enough to have a lipschitzian function $a(x)$ of the same order of magnitude of $h(x,\theta)$. We have indeed the following explicit version of the compact embedding theorem, that we state for the sake of simplicity only in the case of weights of L^∞-type.

THEOREM 3.4. *Let Ω be bounded, $a(x)\in\mathrm{Lip}(\bar{\Omega})$ and assume that, for suitable constants C_1, C_2, we have*

$$(3.10)\qquad C_1 h_1(x,\theta)\leqq a(x)\leqq C_2 h(x,\theta),\quad \forall x\in\Omega.$$

Define $S=\{x\in\partial\Omega|a(x)=0\}$ and $A_\varepsilon=\{x\in\Omega|\varepsilon<a(x)<C_3\varepsilon\}$ and assume that

$$(3.11)\qquad \alpha^{-1},\ \beta^{-1},\ \gamma,\ Q\in L^\infty_{\mathrm{loc}}(\bar{\Omega}\setminus S),$$

$$(3.12)\qquad \sup_{x,y\in A_\varepsilon}\left|\frac{\alpha(x)}{\alpha(y)}\right|<C,\quad \forall\varepsilon>0,$$

with C_4, β, γ, Q independent of ε,

$$(3.13)\qquad \frac{|k|}{r}\leqq a<1,\quad a>0;$$

$$(3.14)\qquad 1\leqq p_i\leqq q<+\infty\quad\text{for}\quad i=0,1,$$

$$(3.15)\qquad \frac{1}{q}\geqq a\left(\frac{1}{p_1}-\frac{r}{n}\right)+(1-a)\frac{1}{p_0}+\frac{|k|}{r},$$

$$(3.16)\qquad \lim_{x\to S}\frac{h(x,\theta)^{\lambda_0(q,p_0)}\gamma(x)Q^q(x)}{\alpha^{q/p_0}(x)}=0,$$

$$(3.17)\qquad \sup_{x\in\Omega}\frac{h(x,\theta)^{\lambda_1(q,p_1)}\gamma(x)Q^q(x)}{\beta^{q/p_1}(x)}<+\infty,$$

where λ_0, λ_1 have been defined in (1.8), (1.9).

Then the operator

$$u \in W^r_{p_0, p_1}(\Omega, \alpha, \beta) \to QD^k u \in L^q(\Omega, \gamma)$$

is compact.

Proof. By (3.10), Ω has the $(\mathscr{D})$-property for $a(x)$. Then, we take the decomposition given by Theorem 3.1 and apply the compact embedding theorem choosing $t_0 = t_1 = \tau = +\infty$.

Now, our theorem follows computing $N_0(M)$ and $N_1(M)$ by (3.12), (3.16) and (3.17).

4. Examples

We show that the assumptions of Theorem 3.4 are fulfilled for the following simple bidimensional cusp

$$\Omega = \{(x_1, x_2) \in R^2 | 0 < x_1 < 1, \ -f(x_1) < x_2 < f(x_1)\} \tag{4.1}$$

with $f \in C^2([0, 1])$; $f(0) = f'(0) = 0$; $f(x_1), f'(x_1), f''(x_1) \in R^+$.

Proposition 4.1. *Let Ω be defined by* (4.1). *Then there exists θ_0 such that for every $x = (x_1, x_2) \in \Omega$ and $\theta < \theta_0$, we have*

$$C_1 f(x_1) \leqq h(x, \theta) \leqq C_2 f(x_1). \tag{4.2}$$

Proof. The straight line of angle θ passing for x intersects the curve $X_2 = f(X_1)$ for $\bar{X}_1 \operatorname{tg} \theta - f(\bar{X}_1) = x_1 \operatorname{tg} \theta - x_2$. Since in our hypothesis $\dfrac{f(x)}{x} = o(1)$, we have for x_1 small enough

$$\bar{X}_1 < C\|x\|, \quad |\bar{X}_2| = |f(\bar{X}_1)| < Cf(\|x\|). \tag{4.3}$$

Similar estimates hold for the curve $X_2 = -f(X_1)$.

Since every cone of vertex x and opening θ obviously intersects one of the graphs $X_2 = f(X_1)$, $X_2 = -f(X_1)$, we obtain by (4.3)

$$h(x, \theta) \leqq C \min(\bar{X}_1, \bar{X}_2) \leqq Cf(\|x\|). \tag{4.4}$$

Moreover, if we set

$$\partial_1 \Omega = \{(x_1, f(x_1))\}, \quad \partial_2 \Omega = \{(x_1, -f(x_1))\}$$

we have

$$d(x, \partial_1 \Omega) + d(x, \partial_2 \Omega) \geqq f(x_1) \geqq Cf(\|x\|)$$

and (4.2) is completely proved because $x_1 < \|x\| < Cx_1$.

By Proposition 4.1, we can apply Theorem 3.4 choosing $a(x) = f(x_1)$. It turns out that $S = \{0\}$, which is the expected singular point of our cusp.

Similar results also hold for general cusps (see Adams [1] p. 124) with a suitable regular function $h(x)$ of the same order of magnitude of $h(x, \theta)$.

Finally, we show in the following example that the compact embedding can hold also if the new weight is considerably bigger that the previous one, assuming the condition that this happens on a set of suitably small measure.

PROPOSITION 4.2. *Let* $\Omega=R^+\times R^+$, $q>p\geqq 1$, $\alpha(x)=1+|x_1-x_2|x_1^s$, $\beta(x)=1+x_1^{p/q}$, $\gamma(x)=1+x_1$, $t\geqq 1$ *such that*

(4.5) $$\frac{1}{q}>\frac{1}{p}>\frac{t+1}{t}-\frac{1}{2},$$

(4.6) $$s>1+\frac{p}{q},$$

(4.7) $$s>\frac{pt}{q}.$$

Then we have the compact embedding $W^1_{p,p}(\Omega,\alpha,\beta)\to L^q(\Omega,\gamma)$.

PROOF. Clearly, there exists an admissible decomposition $\{\Omega_i\}_{i\in N}$ of Ω by squares Ω_i such that $C_1\leqq h_i\leqq C_2$ and $C_1\leqq\mu(\Omega_i)\leqq C_2$, for every i. Then, we can apply Theorem 1.2 with $t_0=t_1=t$, $\tau=+\infty$ and obtain that the stated compact embedding holds if conditions (1.14), (1.15) are fulfilled. In our case this means that

(4.8) $$N_0(M)=\sup_{i>M}\{|\alpha^{-1}|^{q/p}_{t,\Omega_i}\,|\gamma|_{\tau,\Omega_i}\}\to 0\quad(M\to+\infty),$$

(4.9) $$N_1=\sup_{i>1}\{|\beta^{-1}|^{q/p}_{t,\Omega_i}\,|\gamma|_{\tau,\Omega_i}\}<+\infty.$$

Now, (4.9) is immediate and we come to the study of $N_0(M)$. For $\lambda>t$, $\lambda<\dfrac{sq}{p}$ we have, setting $\Omega(x,d)=\{y\in\Omega\,|\,\|x-y\|<d\}$,

(4.10) $$|\alpha^{-1}|_{t,\Omega(x,d)}|\gamma|^{p/q}_{\tau,\Omega(x,d)}<C\,|\alpha^{-1}\gamma^{p/q}|_{t,\Omega(x,d)}<$$
$$<|\alpha^{-1}\gamma^{p/q}|_{t,\Omega(x,d)\setminus\Omega_\varepsilon}+|\alpha^{-1}\gamma^{p/q}|_{\theta,\Omega_\varepsilon\cap\Omega(x,d)}\,\mu(\Omega_\varepsilon\cap\Omega(x,d))^{1/t-1/\theta},$$

where

$$\Omega_\varepsilon=\{x\in\Omega\,\big|\,|\alpha^{-1}(x)\gamma(x)^{p/q}|>\varepsilon\}.$$

Then, our Proposition follows at once by (4.10) if we are able to show

(i) $\sup\limits_{x\Omega}|\alpha^{-1}\gamma^{p/q}|_{\theta,\Omega(x,d)}<+\infty$,

(ii) $\lim\limits_{|x|\to+\infty}\mu(\Omega_\varepsilon\cap\Omega(x,d))=0,\quad\forall\varepsilon\in R^+$.

With regard to (i), we observe that, since $\lambda<sq/p$,

(4.11) $$\int_0^{x_1}\frac{dx_2}{\{1+(x_1-x_2)x_1^s\}^\theta}=\int_0^{x_1}\frac{dy}{(1+x_1^s y)^\theta}<C\int_0^{x_1}\frac{dy}{1+x_1^{s\theta}y^\theta}.$$

We obtain, dividing $[0,1]$ to intervals of the type $\left[\dfrac{i}{A^{1/\theta}},\dfrac{i+1}{A^{1/\theta}}\right]$,

(4.12)
$$\int_0^1\frac{dy}{1+Ay^\theta}<\sum_{0\leqq i\leqq A^{1/\theta}}\frac{1}{1+A\left(\dfrac{i}{A^{1/\theta}}\right)^\theta}\,\frac{1}{A^{1/\theta}}=\frac{1}{A^{1/\theta}}\sum\frac{1}{1+i^\theta}<CA^{-1/\theta}.$$

Moreover, if $x_1>1$

$$\int_1^{x_1} \frac{dy}{1+y^\theta x_1^{s\theta}} < Cx_1^{-s\theta} \int_1^\infty \frac{dy}{y^\theta} < Cx_1^{-s\theta}. \tag{4.13}$$

By (4.12), (4.13), we get

$$\int_0^{x_1} \frac{dx_2}{\{1+(x_1-x_2)x_1^s\}^\theta} < Cx_1^{-s}. \tag{4.14}$$

Still using (4.12), we have

$$\int_{x_1}^{+\infty} \frac{dx_2}{\{1+(x_2-x_1)x_1^s\}^\theta} = \int_0^{+\infty} \frac{dy}{\{1+x_1^s y\}^\theta} < Cx_1^{-s}. \tag{4.15}$$

By (4.14), (4.15) we obtain

$$\int_0^{+\infty} \frac{dx_1}{\alpha^\theta(x)} < C(1+x_1)^{-s}. \tag{4.16}$$

It follows for x_1 large enough (which can obviously be assumed)

$$\int_{\Omega(x_0,d)} \left(\frac{\gamma(x)^{p/q}}{\alpha(x)}\right)^\theta dx < \int_{x_{0,1}-d/2}^{x_{0,1}+d/2} \gamma(x)^{\theta p/q} \int_0^{+\infty} \frac{dx_2}{\alpha^\theta(x)} <$$

$$< C \int_{x_{0,1}-d/2}^{x_{0,1}+d/2} (1+x_1)^{\theta p/q-s}\, dx_1 < +\infty$$

if $(\theta p/q)-s<0$.

Finally, (ii) follows because $s>\frac{p}{q}+1$, observing that

$$\Omega_\varepsilon = \left\{x\in\Omega \middle| x_1 - \frac{\varepsilon^{-1}(1+x_1)^{p/q}-1}{x_1^s} < x_2 < x_1 + \frac{\varepsilon^{-1}(1+x_1)^{p/q}-1}{x_1^s}\right\}.$$

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(Received July 26, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 109—115.

К РАСШИРЕННОМУ ИНТЕРПОЛЯЦИОННОМУ ПРОЦЕССУ ЭРМИТА—ФЕЙЕРА

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1. Введем следующие обозначения: через C обозначим множество всех функций, непрерывных в $[-1, 1]$. C^2 обозначает подмножество из C, состоящее из всех функций $f(x)$, имеющих в $[-1, 1]$ непрерывные производные $f''(x)$. Δ_1 обозначает подмножество из C, состоящее из всех функций $f(x)$, имеющих левую производную $f'(1)$. Аналогичным образом определяется множество функций Δ_2. Пусть задана матрица чисел

$$\text{(м)} \qquad \{x_k^{(n)}\},\ \ k=1, 2, \ldots, n,\ \ n=1, 2, \ldots,\ \ -1 < x_n^{(n)} < x_{n-1}^{(n)} < \ldots < x_1^{(n)} < 1,$$

и пусть $H_n(f, x)$ — полином степени $2n-1$, однозначно определяющийся из условий $H_n(f, x_k^{(n)}) = f(x_k^{(n)})$, $H_n'(f, x_k^{(n)}) = 0$, $k=1, 2, \ldots, n$. Классическая теорема Л. Фейера [1] утверждает, что если n-я строчка, $\{x_k^{(n)}\}_{k=1}^n$, состоит из чисел $x_k^{(n)} = \cos\dfrac{2k-1}{2n}\pi$, $k=1, 2, \ldots, n$, то для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение $H_n(f, x) \to f(x)$, $n \to \infty$. Хорошо известно, что процесс $\{H_n(f, x)\}$ называется интерполяционным процессом Эрмита—Фейера.

Пусть полином $H_n(f, x)$ построен для n-й строчки произвольной матрицы узлов вида (м). Наряду с полиномом $H_n(f, x)$ рассмотрим полином $F_n(f, x)$ степени $2n+3$, который однозначно определяется из условий

$$F_n(f, x_k^{(n)}) = f(x_k^{(n)}); \quad F_n(f, \pm 1) = f(\pm 1), \quad F_n'(f, x_k^{(n)}) = F_n'(f, \pm 1) = 0,$$
$$k = 1, 2, \ldots, n.$$

Интерполяционный процесс $\{F_n(f, x)\}$ естественно называть расширенным интерполяционным процессом Эрмита—Фейера. В [2], [3] автор изучал процесс $\{F_n(f, x)\}$ для случая узлов

(1)
$$x_0^{(n+2)} = 1, \quad x_k^{(n+2)} = \cos\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n, \quad x_{n+1}^{(n+2)} = -1, \quad n = 1, 2, \ldots.$$

Оказалось, что этот процесс при $f(x) = |x|$ расходится при $x = 0$. Метод из [2—3] не позволил изучить поведение процесса $\{F_n(|z|, x)\}$ при $x \neq 0$. Поэтому этот метод был заменен другим методом, сущность которого изложена в работах [4—7]. С помощью этого метода было доказано, что процесс $\{F_n(|z|, x)\}$, построенный при узлах (1), расходится всюду в $(-1, 1)$. Такое утверждение имеет место для $f(x) = x^2$ и для $f(x) = x$ при $x \neq 0$. См. [6—7]. Упомянутые результаты были в [8] обобщены в двух направлениях. Во-первых, вместо матрицы узлов (1) рассматривается некоторый общий класс матриц узлов, включающий матрицу (1). Во-вторых, вместо функций x и x^2 рассматривается произвольная функция из класса C^2. В связи с [8] возник вопрос о замене класса C^2 более широким классом функций. Этот вопрос в случае сходимости в среднем, и был поставлен в [8]. В основе всех результатов из [8] лежит

Лемма. *Пусть*

$$\alpha_n(f) = \frac{1}{2} H_n'(f, 1) + \frac{\omega_n'(1)}{\omega_n(1)} [f(1) - H_n(f, 1)], \tag{2}$$

где $H_n(f, x)$ — *интерполяционный полином Эрмита—Фейера, построенный при корнях многочлена* $\omega_n(x) = \prod_{i=1}^{n} (x - x_i^{(n)})$. *Пусть матрица узлов* (м) *удовлетворяет условиям:* 1) (м) *является* ϱ*-нормальной;*[1] 2) *корни* ω_n *расположены симметрично относительно* $x=0$; 3) $\left|\omega_n'(1)(\omega_n(1))^{-1}\right| = O(n^2)$; 4) *для любого* $x \in (-1, 1)$ *существует такая последовательность* $\{n_i\}$, *что* $\lim_{i\to\infty} \omega_{n_i}^2(x)/\omega_{n_i}^2(1) > C(x) > 0$, *где* $C(x)$ *зависит только от* x. *Тогда существует такая последовательность натуральных чисел* $\{m_i\}$, *что для любой* $f \in C^2$ *выполняется равенство*

$$\lim_{i\to\infty} \alpha_{m_i}(f) = af'(1),$$

где константа a *зависит только от матрицы узлов.*

При доказательстве леммы был использован следующий известный факт: Для любой $f \in C^2$ существует такой многочлен $P_n(x)$ степени n, что всюду в $[-1, 1]$

$$|f^{(s)}(x) - P_n^{(s)}(x)| \leqq A\left(\frac{\sqrt{1-x^2}}{n}\right)^{2-s} \omega\left(f'', \frac{\sqrt{1-x^2}}{n}\right), \quad n \geqq 13, \quad s = 0, 1, 2, \tag{3}$$

где ω — модуль непрерывности $f''(x)$.

В настоящей работе доказывается эта лемма без использования неравенств (3), что позволяет отказаться от требования, что $f \in C^2$. При этом лемма значительно усиливается, а стало быть усиливаются все результаты из [8].

2. Пусть n-я строчка матрицы (м) состоит из корней полинома $\omega(x) = \omega_n(x) = \prod_{i=1}^{n} (x - x_i^{(n)})$. Согласно Л. Фейеру [9] матрица (м) называется ϱ-нормальной, если существует такое число $\varrho > 0$, что всюду в $[-1, 1]$ выполняется неравенство

$$v_k(x) = 1 - (x - x_k^{(n)})\omega_n''(x_k^{(n)})\big(\omega_n'(x_k^{(n)})\big)^{-1} > \varrho > 0, \quad k = 1, 2, \dots, n, \quad n = 1, 2, \dots,$$

где $\{x_k^{(n)}\}_{k=1}^n$ — корни $\omega_n(x)$. Л. Фейер [9] доказал, что если матрица (м) составлена из корней полиномов Якоби $J_n^{(\alpha_n, \beta_n)}(x)$, где $-1 \leqq \alpha_n, \beta_n < -\gamma < 0$,

[1] Определение ϱ-нормальной матрицы узлов дано ниже.

$n=1, 2, 3, \dots$, а γ — сколь угодно малое фиксированное число, то она ϱ-нормальная.

Среди всевозможных матриц вида (м) выделим класс матриц K. Будем говорить, что матрица (м)$\in K$, если (м) удовлетворяет следующим условиям: 1) (м) ϱ-нормальная; 2) корни полинома $\omega(x)=\prod_{i=1}^{n}(x-x_i^{(n)})$ расположены симметрично относительно точки $x=0$; 3) для любого $x\in(-1, 1)$ существует такая последовательность $\{n_i\}_{i=1}^{\infty}$ натуральных чисел, что выполняется неравенство

$$\lim_{i\to\infty}\frac{\omega_{n_i}^2(x)}{\omega_{n_i}^2(1)}\geqq C(x)>0, \tag{4}$$

где $C(x)$ зависит лишь от x.

Введем числа[2] $d_n=\sum_{i=1}^{n}[l_i^{(n)}(1)]^2$. Известно [9], что для ϱ-нормальной матрицы выполняется всюду в $[-1, 1]$ неравенство

$$\sum_{i=1}^{n}[l_i^{(n)}(x)]^2\leqq\frac{1}{\varrho}. \tag{5}$$

Поэтому $d_n\leqq\dfrac{1}{\varrho}$, $n=1, 2, \dots$. Следовательно, имеется такая последовательность натуральных чисел $\{n_i\}_{i=1}^{\infty}$, что существует конечный предел

$$\lim_{i\to\infty}d_{n_i}=d. \tag{6}$$

В нижеследующей лемме рассматривается именно такая последовательность $\{n_i\}_{i=1}^{\infty}$.

Лемма. *Пусть линейный функционал $\alpha_n(f)$ построен при матрице узлов (м)$\in K$. Пусть $\{n_i\}_{i=1}^{\infty}$ удовлетворяет условию* (6). *Тогда для любой $f\in\Delta_1$, удовлетворяющей условию $f(1)=0$ выполняется равенство*

$$\lim_{i\to\infty}\alpha_{n_i}(f)=\frac{1+d}{2}f'(1). \tag{7}$$

Доказательство. Хорошо известно [9], что

$$H_n(f, x)=\sum_{k=1}^{n}f(x_k^{(n)})[l_k^{(n)}(x)]^2v(x),$$

где

$$l_k(x)=l_k^{(n)}(x)=\frac{\omega_n(x)}{(x-x_k^{(n)})\omega_n'(x_k^{(n)})}. \tag{8}$$

Стало быть,

$$H_n'(f, 1)=\sum_{k=1}^{n}f(x_k^{(n)})[l_k^2(1)v_k'(1)+2l_k(1)l_k'(1)v_k(1)]. \tag{9}$$

[2] $l_k^{(n)}(x)$ — фундаментальный полином Лагранжа, который определяется согласно (8).

Очевидно, что[3]

$$l_k^2(1)v_k'(1) = -\frac{\omega_n^2(1)\omega_n''(x_k)}{(\omega_n'(x_k))^3(1-x_k)^2}. \tag{10}$$

После простых вычислений, получим, что

$$l_k'(1) = -\frac{\omega_n(1)}{\omega_n'(x_k)(1-x_k)^2}\left(1-\frac{\omega_n'(1)}{\omega_n(1)}(1-x_k)\right).$$

Поэтому

$$2l_k(1)l_k'(1)v_k(1) = -2\frac{\omega_n^2(1)}{(\omega_n'(x_k))^2(1-x_k)^3}\left(1-\frac{\omega_n'(1)}{\omega_n(1)}(1-x_k)\right)v_k(1). \tag{11}$$

Из (10) и (11) следует, что

$$l_k^2(1)v_k'(1)+2l_k(1)l_k'(1)v_k(1) = \left(\frac{\omega''(x_k)}{\omega'(x_k)}-\frac{2}{1-x_k}\right)l_k^2(1)+$$
$$+2\frac{\omega'(1)}{\omega(1)}h_k(1), \quad h_k(x) = [l_k^{(n)}(x)]^2 v_k(x).$$

Отсюда и из (9) получим, что

$$H_n'(f,1) = -\sum_{k=1}^{n}\frac{f(x_k)}{1-x_k}h_k(1)-\sum_{k=1}^{n}\frac{f(x_k)}{1-x_k}l_k^2(1)+2\frac{\omega'(1)}{\omega(1)}H_n(f,1). \tag{12}$$

Из (2) и (12) вытекает, что

$$\alpha_n(f) = -\sum_{k=1}^{n}\frac{f(x_k)h_k(1)}{2(1-x_k)}-\sum_{k=1}^{n}\frac{f(x_k)l_k^2(1)}{2(1-x_k)}+\frac{\omega'(1)}{\omega(1)}f(1). \tag{13}$$

По условию $f(1)=0$, поэтому

$$\alpha_n(f) = -\sum_{k=1}^{n}\frac{f(x_k)h_k(1)}{2(1-x_k)}-\sum_{k=1}^{n}\frac{f(x_k)l_k^2(1)}{2(1-x_k)} \equiv S_1+S_2. \tag{14}$$

Очевидно, что

$$S_1 = \frac{1}{2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{1-x_k}h_k(1).$$

По условию существует $f'(1)$. Поэтому по $\varepsilon>0$ можно найти такое $\delta>0$, что

$$\left|\frac{f(1)-f(x_k)}{1-x_k}-f'(1)\right| < \varepsilon, \tag{15}$$

если $1-x_k<\delta$. Так как $\sum_{k=1}^{n}h_k(1)=1$, то

$$S_1-\frac{f'(1)}{2} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{f(1)-f(x_k)}{1-x_k}-f'(1)\right)h_k(1). \tag{16}$$

[3] Ради простоты письма иногда опускается верхний индекс n у $x_k^{(n)}$ и $l_k^{(n)}(x)$.

Из (15) и (16) вытекает, что

$$(17)\qquad \left|S_1-\frac{f'(1)}{2}\right| \leqq \frac{\varepsilon}{2}\sum_{1-x_k<\delta}|h_k(1)|+\frac{1}{2}\sum_{1-x_k\geqq\delta}\left|\frac{f(1)-f(x_k)}{1-x_k}-f'(1)\right|h_k(1).$$

Согласно условию матрица узлов (м) ϱ-нормальная, поэтому $h_k(x)\geqq 0$, $x\in[-1,1]$, $k=1,\dots,n$. В частности, $h_k(1)\geqq 0$, $k=1,\dots,n$. Поэтому из (17) получаем, что

$$(18)\qquad \left|S_1-\frac{f'(1)}{2}\right| \leqq \frac{\varepsilon}{2}+\frac{1}{2}\left(\frac{2\|f\|}{\delta}+|f'(1)|\right)\sum_{1-x_k\geqq\delta}h_k(1),$$

где $\|f\| = \operatorname*{Max}_{x\in[-1,1]}|f(x)|$.

Заметим, что

$$(19)\qquad \sum_{1-x_k\geqq\delta}h_k(1)\leqq\sum_{k=1}^{n}\frac{1-x_k}{\delta}h_k(1)$$

и что для ϱ-нормальной матрицы

$$(20)\qquad \lim_{n\to\infty}\sum_{k=1}^{n}(1-x_k^{(n)})h_k^{(n)}(1)=0.$$

Из (19) и (20) выводим, что

$$(21)\qquad \lim_{n\to\infty}\sum_{1-x_k^{(n)}\geqq\delta}h_k^{(n)}(1)=0.$$

Из (18) и (21) получаем, что

$$(22)\qquad \lim_{n\to\infty}S_1^{(n)}=\frac{f'(1)}{2}.$$

Рассмотрим теперь $S_2^{(n)}$. Очевидно, что

$$S_2^{(n)}-\frac{f'(1)\,d_n}{2}=-\frac{1}{2}\sum_{k=1}^{n}\left(\frac{f(x_k)}{1-x_k}+f'(1)\right)[l_k^{(n)}(1)]^2,$$

где $d_n=\sum_{k=1}^{n}l_k^2(1)$. Ясно, что

$$(23)\qquad \left|S_2^{(n)}-\frac{f'(1)\,d_n}{2}\right|\leqq\frac{\varepsilon}{2}\sum_{k=1}^{n}l_k^2(1)+\frac{1}{2}\sum_{1-x_k\geqq\delta}\left|\frac{f(x_k)}{1-x_k}+f'(1)\right|l_k^2(1).$$

Учтем теперь неравенство (5), тогда из (23) выводим, что

$$(24)\qquad \left|S_2^{(n)}-\frac{f'(1)\,d_n}{2}\right|\leqq\frac{\varepsilon}{2\varrho}+\frac{1}{2}\left(\frac{\|f\|}{\delta}+|f'(1)|\right)\sum_{1-x_k\geqq\delta}[l_k^{(n)}(1)]^2.$$

Очевидно, что

$$(25)\qquad \sum_{1-x_k\geqq\delta}[l_k^{(n)}(1)]^2\leqq\frac{1}{\delta}\sum_{k=1}^{n}(1-x_k^{(n)})[l_k^{(n)}(1)]^2.$$

Согласно Г. Грюнвальду для ϱ-нормальных матриц [10]

$$\lim_{n\to\infty}\sum_{k=1}^{n}(1-x_k^{(n)})[l_k^{(n)}(1)]^2=0.$$

Поэтому из (24) и (25) заключаем, что

$$\lim_{n\to\infty} S_2^{(n)}=\frac{f'(1)\,d}{2}, \tag{26}$$

ибо $\lim\limits_{n\to\infty} d_n=d$. Из (14), (22), (26) выводим (7).

Теперь можно доказать следуюшую теорему:

Теорема. *Для того чтобы расширенный интерполяционный процесс* $\{F_n(f,x)\}$, *построенный для четной функции* $f\in\Delta_1$, *расходился всюду в* $(-1,1)$ *при любой матрице узлов из класса матриц K необходимо и достаточно, чтобы* $f'(1)$ *было отлично от нуля.*

Доказательство достаточности. Положим $r_n=r_n(f,x)=F_n(f,x)-$ $-H_n(f,x)$. Очевидно, что можно положить $f(1)=0$, ибо в противном случае мы бы рассматривали функцию $\varphi(x)=f(x)-f(1)$ и воспользовались бы равенством $r_n(f,x)=r_n(\varphi,x)$. По условию $f(x)$ — четная функция и корни полинома $\omega_n(x)$ расположены симметрично, поэтому $H_n(f,x)$ и $F_n(f,x)$ — четные полиномы. Следовательно, из определения H_n и F_n получаем, что

$$r_n(f,x)=\omega_n^2(x)(A_nx^2+B_n), \tag{27}$$

где A_n и B_n определяются из системы уравнений

$$\begin{cases}\omega_n^2(1)(A_n+B_n)=f(1)-H_n(f,1);\\ 2\omega_n(1)\,\omega_n'(1)(A_n+B_n)+2A_n\,\omega_n^2(1)=-H_n'(f,1).\end{cases}$$

Отсюда и из (27), после простых вычислений, получим, что

$$r_n(f,x)=\frac{\omega_n^2(x)}{\omega_n^2(1)}\,[\alpha_n(f)(1-x^2)+f(1)-H_n(f,1)], \tag{28}$$

где $\alpha_n(f)$ определяется согласно (2). Поскольку матрица узлов (м) из класса K, то выполняется неравенство (4). Кроме того, согласно теореме Л. Фейера [9] при ϱ-нормальной матрице узлов для любой $f\in C$ выполняется равномерно в $[-1,1]$ равенство $\lim\limits_{n\to\infty} H_n(f,x)=f(x)$. Поэтому из леммы и (28) вытекает, что

$$\varlimsup_{j\to\infty}|r_{n_j}(f,x)|\geqq\frac{C(x)(1+d)\,|f'(1)|}{2}(1-x^2). \tag{29}$$

По условию $f'(1)\neq 0$, поэтому из (29) следует, что $\varlimsup\limits_{j\to\infty} r_{n_j}(f,x)\neq 0$ при $x\in(-1,1)$. Итак, достаточность доказана.

Необходимость доказана в [8] (стр. 9). Надо только класс функций C^2 заменить классом функций Δ_1.

Из теоремы, в частности, вытекает такой результат: *пусть матрица узлов* (м) *состоит из корней ультрасферических полиномов* $J_n^{(\alpha_n,\alpha_n)}$, $-1<\alpha_n\leqq-\frac{1}{2}$,

$n=1, 2, \ldots$. *Пусть* $f\in\Delta_1$, *четная и* $f'(1)\neq 0$. *Тогда процесс* $\{F_n(f, x)\}_{n=1}^{\infty}$ *расходится всюду в* $(-1, 1)$.

Доказательство этого утверждения находится в [8]. При этом класс C^2 следует заменить классом Δ_1. Лемма из этой заметки позволяет усилить и остальные теоремы из [8]. Усиление состоит в том, что класс функций C^2 заменяется классом функций Δ_1, или классом функций Δ_2. Одновременно с [8] была опубликована интересная статья [11] R. Bojanic. Между [8] и [11] много общего, но между ними имеется и существенное различие. В [8] рассмотрение проводится для некоторого класса матриц узлов, включающего узлы Чебышева, но при этом предполагается, что функция из класса C^2. В [11] рассматриваются только узлы Чебышева, но зато функция из класса Δ_1, или из класса Δ_2.

В заключение отметим, что для процесса $\{F_n(f, x)\}$ характерно, что концевые точки $x=\pm 1$ являются узлами интерполяции. Этому вопросу посвящено исследование P. Вертеши [12].

Замечание. В лемме требуется, чтобы $f(x)$ удовлетворяла условию $f(1)=0$. Докажем, что это требование можно отбросить. Положим в (14) $f(x)\equiv 1$. Из определения $\alpha_n(f)$ следует, что в этом случае $\alpha_n(f)=0$. Поэтому из (14) выводим, что

$$\frac{\omega'(1)}{\omega(1)} = \sum_{k=1}^{n} \frac{h_k(1)}{2(1-x_k)} + \sum_{k=1}^{n} \frac{l_k^2(1)}{2(1-x_k)}. \tag{$*$}$$

Из $(*)$ и (14) получаем, что

$$\alpha_n(f) = \frac{1}{2}\sum_{k=1}^{n} \frac{f(1)-f(x_k)}{1-x_k}\, h_k(1) + \frac{1}{2}\sum_{k=1}^{n} \frac{f(1)-f(x_k)}{1-x_k}\, l_k^2(1).$$

Остальная часть доказательства леммы сохраняется.

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(Поступило 15. 9. 1983.)

ЛЕНИНГРАД, 192238
БАССЕЙНАЯ 68, КВ. 90
СССР

Acta Math. Hung.
47 (1—2) (1986), 117—119.

ON THE ERDŐS—STRAUS NON-AVERAGING SET PROBLEM

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A set S of integers is said to be non-averaging if the arithmetic mean of two or more members of S does not belong to S. Denote by $f(n)$ the size of a largest non-averaging subset of $\{0, 1, 2, \dots, n\}$. Straus [7] raised the problem of estimating $f(n)$ and proved ($c_1, c_2, \dots$ are positive absolute constants)

$$f(n) > \exp\left(c_1 \sqrt{\log n}\right).$$

Erdős and Straus [5] proved that

$$f(n) < c_2 n^{2/3}$$

and conjectured that for some c_3

$$f(n) < \exp\left(c_3 \sqrt{\log n}\right).$$

This conjecture was shown to be false by the author [1] who proved that

$$f(n) > c_4 n^{1/10}. \tag{1}$$

In this paper we obtain a further improvement on the lower bound for $f(n)$ by proving the following result:

THEOREM. *For some c_5 and all sufficiently large n*

$$f(n) > c_5 n^{1/5}, \tag{2}$$

and for some c_6 and infinitely many n

$$f(n) > c_6 n^{1/5} (\log\log n)^{2/5}. \tag{3}$$

PROOF. We give the proof of (3). The proof of (2) runs along similar lines, but the technical details are simpler.

Let B be the product of the first r primes. Let p be the least prime exceeding $(\log\log B)^{1/30}$ and let q be the least prime exceeding p^{10}. Then, as $r \to \infty$,

$$p \sim (\log\log B)^{1/30} \tag{4}$$

and

$$q \sim p^{10} \sim (\log\log B)^{1/3}. \tag{5}$$

It follows from the prime number theorem (or a weaker result) that, for large r, p and $2q$ do not exceed the rth prime so that B/p is an integer and $B/2q$ is an odd integer. We set

$$n = (B/p)^{10} \tag{6}$$

and our goal is to show that for the integers n given by (6), (3) holds.

Let $m=(B/2q)^2$ and consider the equation

$$a_0^2+a_1^2+a_2^2+a_3^2 = m. \tag{7}$$

It is a well known theorem of Jacobi (see, for example [6], pp. 312—314) that the number of solutions of (7) in integers a_0, a_1, a_2, a_3 is given by $8\sigma(m)$, where $\sigma(m)$ is the sum of the divisors of m. We shall be interested only in solutions in non-negative integers. The number l of such solutions is easily seen to satisfy

$$\frac{1}{2}\sigma(m) \leqq l \leqq 4\sigma(m). \tag{8}$$

If we use (i) the fact that B is the product of the first r primes, (ii) the explicit expression for $\sigma(m)$ in terms of the prime factorization of m, (iii) the theorem of Mertens (see [6] pp. 351) and (iv) the relations given by (4), (5), (6) and (8) we find after some routine calculations that

$$c_7 n^{1/5}(\log\log n)^{2/5} < l < c_8 n^{1/5}(\log\log n)^{2/5}. \tag{9}$$

We omit the details of these calculations.

We associate with a solution of (7) the lattice point (a_0, a_1, a_2, a_3) in R^4 and we note that the points corresponding to solutions lie on a sphere. We also associate with a solution of (7) the number

$$a = a_0+a_1B^3+a_2B^6+a_3B^9.$$

Consider the set of l numbers obtained in this way and let S be a subset of those whose cardinality satisfies

$$|S| = [cl] \tag{10}$$

where $c<\min(1, 2c_8)$. Note that if $a\in S$ then, since $0\leqq a_i\leqq\sqrt{m}=B/2q$,

$$a \leqq \frac{B}{2q}(1+B^3+B^6+B^9)\sim\frac{B^{10}}{2q}\sim\frac{1}{2}\left(\frac{B}{q}\right)^{10} < n$$

where we used (5) and (6). Thus $S\subset\{0, 1, 2, \ldots, n\}$.

Next we show that S is a non-averaging set. Suppose that this is not the case. Then for some k, $2\leqq k<|S|$, there are $k+1$ distinct numbers $a^{(1)}, a^{(2)}, \ldots, a^{(k+1)}$ in S such that

$$a^{(1)}+a^{(2)}+\ldots+a^{(k)} = ka^{(k+1)} \tag{11}$$

where

$$a^{(j)} = a_0^{(j)}+a_1^{(j)}B^3+a_2^{(j)}B^6+a_3^{(j)}B^9. \tag{12}$$

and $a_0^{(j)}, a_1^{(j)}, a_2^{(j)}, a_3^{(j)}$ is a solution of (7) for $j=1, 2, \ldots, k+1$. Then from (11) and (12) we get

$$A_0+A_1B^3+A_2B^6+A_3B^9 = 0 \tag{13}$$

where

$$A_i = ka_i^{(k+1)} - \sum_{j=1}^{k} a_i^{(j)}. \tag{14}$$

Now by (5), (9), (6), (4) and (10)

$$A_0 \leqq ka_0^{(k+1)} < |S| \frac{B}{2q} \sim \frac{\mathrm{cl}\, B}{2p^{10}} < cc_8 n^{1/5} (\log\log n)^{2/5} \sim$$

$$\sim cc_8 \frac{B^3 (\log\log B)^{2/5}}{2p^{12}} \sim \frac{1}{2} cc_8 B^3 < B^3.$$

Similarly, we find that $A_0 > -B^3$ so that $|A_0| < B^3$. It then follows from (13) that $A_0 = 0$. The same argument shows that $A_1 = A_2 = A_3 = 0$. We then get from (14) that

$$a_i^{(k+1)} = \frac{1}{k} \sum_{j=1}^{k} a_i^{(j)} \quad \text{for} \quad i = 0, 1, 2, 3. \tag{15}$$

However, if $P_1, P_2, \ldots, P_{k+1}$ are the lattice points in R^4 corresponding to the numbers $a^{(1)}, a^{(2)}, \ldots, a^{(k+1)}$ then (15) is just the assertion that P_{k+1} is the centroid of $P_1, P_2, \ldots, P_k$, contradicting the fact that $P_1, P_2, \ldots, P_{k+1}$ are distinct points on a sphere. Thus S is a non-averaging set. We therefore have

$$f(n) \geqq |S| = [cl] > c_2 n^{1/5} (\log\log n)^{2/5},$$

by (9) and (10), so that (3) holds.

We remark in conclusion that many of the results of [2] and the principal result in [3] may be improved simply by using (2) instead (1). We remark also that the geometric aspects of the proof of our theorem have their roots in [4].

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(Received October 31, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 121—126.

COUNTABLE ADDITIVITY OF MULTIPLICATIVE, OPERATOR-VALUED SET FUNCTIONS

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Let X be a locally convex Hausdorff space, always assumed to be quasi-complete, with dual space X'. The space of all continuous linear operators on X, equipped with the strong operator topology, that is, the topology of pointwise convergence on X, is denoted by $L(X)$. The identity operator is denoted by I.

An important problem in the spectral theory of linear operators is to determine when a given additive, multiplicative, $L(X)$-valued map P, defined on a σ-algebra $\mathcal{M}$ of subsets of some set Ω and satisfying $P(\Omega)=I$, is a spectral measure, that is, to determine when P is strongly σ-additive. Of course, the multiplicativity of P means that $P(E\cap F)=P(E)P(F)$ for every $E\in\mathcal{M}$ and $F\in\mathcal{M}$. Since the dual space of $L(X)$ is the (algebraic) tensor product $X\otimes X'$, it follows from the Orlicz—Pettis lemma that the σ-additivity of P is equivalent to the σ-additivity of each of the complex-valued set functions defined by

$$\langle Px, x'\rangle\colon E\mapsto\langle P(E)x, x'\rangle,\quad E\in\mathcal{M},$$

for each $x\in X$ and $x'\in X'$. In [3], T. A. Gillespie showed that there is a large class of Banach spaces X, including all weakly sequentially complete and all separable spaces, such that P is σ-additive whenever there exists a total set of functionals $\Gamma\subseteqq X'$ such that for each $x\in X$, each of the functions $\langle Px, x'\rangle$, $x'\in\Gamma$, is σ-additive. This is a considerable simplification in practice, since the total set Γ may be substantially smaller than all of X'. A further improvement, incorporated into this note, is to admit for the possibility of varying with respect to $x\in X$, the total set of functionals Γ such that the functions $\langle Px, x'\rangle$, $x'\in\Gamma$, are σ-additive.

Since a satisfactory spectral analysis of operators, even for those defined in Banach spaces, often requires a consideration of operators defined in locally convex spaces, it is useful to have available sufficient conditions, such as those given in [3], but valid in more general spaces X, which guarantee the strong σ-additivity of a large class of additive, multiplicative, $L(X)$-valued maps defined on σ-algebras of sets. The aim of this note is to formulate some such conditions. Most of the criteria presented are based on the special role played by the sequence spaces c_0 and l^∞ in the theory of vector measures.

1. By a spectral measure of class $(\mathcal{M};\Gamma(x), x\in X)$, where $\mathcal{M}$ is a σ-algebra of subsets of some set Ω and $\Gamma(x)$ is a total subset of X' for each $x\in X$, is meant an additive, multiplicative map $P\colon\mathcal{M}\to L(X)$ satisfying $P(\Omega)=I$, such that the range $P(\mathcal{M})=\{P(E); E\in\mathcal{M}\}$, of P, is an equicontinuous part of $L(X)$ and for each $x\in X$, the set functions $\langle Px, x'\rangle$, $x'\in\Gamma(x)$, are σ-additive. It follows from a result of A. Grothendieck (see Proposition 0.5 in [6] for example) that if the space

X is metrizable, then the requirement of equicontinuity of the range of P is redundant. If X is a Banach space and there is a total subset $\Gamma \subseteq X'$ such that $\Gamma(x)=\Gamma$ for each $x \in X$, then P is a spectral measure of class $(\mathcal{M}, \Gamma)$ in the sense of N. Dunford [2], p. 324.

A locally convex space X is said to be weakly Σ-complete ([6], p. 5) if every sequence $\{x_n\}_{n=1}^{\infty}$ of its elements such that $\{\langle x_n, x' \rangle\}_{n=1}^{\infty}$ is absolutely summable for each $x' \in X'$, is itself summable with the sum belonging to X. Weakly sequentially complete spaces, in particular reflexive spaces, are weakly Σ-complete.

PROPOSITION 1. *Each of the following conditions is sufficient to guarantee that every spectral measure of arbitrary class $(\mathcal{M}; \Gamma(x), x \in X)$, where $\mathcal{M}$ is a σ-algebra of sets and $\Gamma(x)$ is a total subset of X' for each $x \in X$, is strongly σ-additive.*

(i) *X does not contain a closed subspace isomorphic to l^{∞}.*

(ii) *X is weakly Σ-complete.*

(iii) *X is metrizable and separable.*

(iv) *X has the properties that a subspace of X' is weak-$*$ closed if its intersection with weak-$*$ closed, bounded subsets is weak-$*$ closed and any continuous linear mapping from l^{∞} to X is weakly compact.*

REMARKS. Proposition 1 (i) is an extension of part of Theorem 1 in [3]. Furthermore, since a space X is weakly Σ-complete if and only if it does not contain a copy of c_0 (see Theorem 4 in [7] for example), it is clear that (ii) follows from (i). Similarly (iii) follows from (i) also, since any subspace of a separable, second countable space is itself separable. Since the class of weakly sequentially complete spaces is a genuine subset of the class of weakly Σ-complete spaces (an example is discussed in [4], p. 73), it is clear that parts (ii) and (iii) of Proposition 1 are an extension of the Corollary in [3]. It is interesting to note that if in addition to being weakly Σ-complete the space X is metrizable, then part (ii) of Proposition 1 is a simple consequence of a well known result in the theory of vector measures (Theorem 0.4 of [6]). However, the multiplicativity of any spectral measure of class $(\mathcal{M}; \Gamma(x), x \in X)$ makes it possible to omit the metrizability condition on X. Finally, Proposition 1 (iv) follows from Proposition 0.7 of [6]. Accordingly, to prove Proposition 1 it suffices to prove part (i). However, first some preliminary lemmas are needed.

Let P be an additive, multiplicative, $L(X)$-valued map, defined on a σ-algebra $\mathcal{M}$ of subsets of some set Ω, such that its range $P(\mathcal{M})$ is an equicontinuous part of $L(X)$. Let $sim(\mathcal{M})$ denote the vector space of all $\mathcal{M}$-simple functions on Ω. If $f=\sum_{i=1}^{n} \alpha_i \chi_{E_i}$ is an element of $sim(\mathcal{M})$, where α_i, $1 \leq i \leq n$, are complex numbers and E_i, $1 \leq i \leq n$, are elements of $\mathcal{M}$, then $P(f)$ will denote the continuous operator $\sum_{i=1}^{n} \alpha_i P(E_i)$. It is well known that $P(f)$ is independent of the particular representation of f as a finite linear combination of characteristic functions of members of $\mathcal{M}$.

LEMMA 1. *The family of operators*

$$\{P(f);\ f \in sim(\mathcal{M}),\ \|f\|_{\infty} \leq 1\} \tag{1}$$

is an equicontinuous part of $L(X)$.

PROOF. See the proof of Proposition 1.1 in [5], for example.

Let $x \in X$. Then Px: $E \mapsto P(E)x$, $E \in \mathcal{M}$, is an additive, X-valued measure. Furthermore, for each continuous seminorm q on X, the equicontinuity of $P(\mathcal{M})$ implies that

$$\beta(q, x) = \sup\{q(P(E)x);\ E \in \mathcal{M}\} < \infty.$$

Accordingly, the well known inequality

$$q(P(f)x) \leqq 4\beta(q, x)\|f\|_\infty, \quad f \in sim(\mathcal{M}), \tag{2}$$

follows; see the proof of Proposition 11 in Chapter 1 of [1], for example.

If f is a bounded, $\mathcal{M}$-measurable function on Ω such that $0 \leqq f \leqq 1$, then there exists a sequence of $\mathcal{M}$-simple functions f_n, $n=1, 2, \ldots$, satisfying $0 \leqq f_n \leqq 1$, $n=1, 2, \ldots$, such that $f_n \to f$ uniformly on Ω. Since the topology of $L(X)$ is generated by the seminorms $T \mapsto q(Tx)$, $T \in L(X)$, for each $x \in X$ and each continuous seminorm q on X, it follows from (2) that $\{P(f_n)\}_{n=1}^\infty$ is a Cauchy sequence in $L(X)$. Since closed, equicontinuous subsets of $L(X)$ are complete and $\{P(f_n)\}_{n=1}^\infty$ is contained in the set (1), it follows from Lemma 1 that there exists an operator $P(f) \in L(X)$ such that $P(f_n) \to P(f)$, in $L(X)$. It follows that the domain of the integration map $f \mapsto P(f)$, $f \in sim(\mathcal{M})$, can be extended to the space of all bounded, $\mathcal{M}$-measurable functions.

The following result is a simple consequence of the multiplicativity of P.

LEMMA 2. *Let f be a bounded, $\mathcal{M}$-measurable function on Ω. Then*

$$P(f\chi_E) = P(E)P(f) = P(f)P(E),$$

for each $E \in \mathcal{M}$.

PROOF OF PROPOSITION 1 (i). We proceed as in the proof of Theorem 1 in [3]. Let P: $\mathcal{M} \to L(X)$ be a spectral measure of class $(\mathcal{M}; \Gamma(x), x \in X)$, where $\mathcal{M}$ is a σ-algebra of subsets of some set Ω and $\Gamma(x)$ is a total subset of X' for each $x \in X$.

Suppose that P is not strongly σ-additive. Then there exists an element $x \in X$ such that the X-valued set function Px is not σ-additive. Arguing as in the proof of Theorem 1, p. 42, of [3], it follows (using the totality of $\Gamma(x) \subseteq X'$) that there exists a sequence $\{\sigma_n\}_{n=1}^\infty$ of mutually disjoint elements of $\mathcal{M}$ such that the series $\sum_{n=1}^{\infty} P(\sigma_n)x$ is not convergent in X hence, not Cauchy in X. Accordingly, there exists a basic neighbourhood of zero in X of the form $U(\varepsilon, p) = \{z \in X;\ p(z) < \varepsilon\}$, where $\varepsilon > 0$ and p is a continuous seminorm on X, such that for every positive integer N there exist integers $m, n > N$ (with $m < n$, say) such that

$$\sum_{i=m+1}^{n} P(\sigma_i)x \notin U(\varepsilon, p).$$

It follows that there exists a sequence of mutually disjoint elements E_n, $n=1, 2, \ldots$, of $\mathcal{M}$, such that

$$p(P(E_n)x) \geqq \varepsilon, \quad n = 1, 2, \ldots. \tag{3}$$

Given an element $\alpha=\{\alpha_n\}_{n=1}^{\infty}$ in l^∞, let f_α denote the bounded, $\mathcal{M}$-measurable function $\sum_{n=1}^{\infty}\alpha_n\chi_{E_n}$. Then the assignment

$$\Phi\colon \alpha\mapsto P(f_\alpha)x, \quad \alpha\in l^\infty,$$

is a linear map of l^∞ into X. It follows from (2) that the inequalities

$$q(\Phi\alpha)\leqq 4\beta(q,x)\|f_\alpha\|_\infty = 4\beta(q,x)\|\alpha\|_\infty, \quad \alpha\in l^\infty, \tag{4}$$

are valid for each continuous seminorm q on X.

Fix $\alpha=\{\alpha_n\}_{n=1}^{\infty}\in l^\infty$. Then it follows from (3) and Lemma 2 that

$$\varepsilon|\alpha_n|\leqq|\alpha_n|p(P(E_n)x) = p(\alpha_n P(E_n)x) = p(P(E_n)P(f_\alpha)x) = p(P(E_n)\Phi\alpha), \tag{5}$$

for each $n=1,2,\ldots$. If U_p^0 denotes the polar of the subset, $p^{-1}([0,1])$, of X, then it follows from the equicontinuity of $P(\mathcal{M})$ that the function

$$p^*\colon z\mapsto \sup\{|\langle z, P(E_n)'x'\rangle|;\ x'\in U_p^0,\ n=1,2,\ldots\}, \quad z\in X,$$

is a continuous seminorm on X. Furthermore, (5) implies that p^* satisfies the inequalities

$$\varepsilon\|\alpha\|_\infty\leqq p^*(\Phi\alpha), \quad \alpha\in l^\infty. \tag{6}$$

It follows from (4) and (6) that the range of Φ is a closed subspace of X isomorphic to l^∞. This contradicts the hypothesis on the space X and hence, the proof of Proposition 1 is complete.

2. Proposition 1 gives sufficient conditions which ensure that *every* spectral measure of class $(\mathcal{M};\Gamma(x),x\in X)$, where $\mathcal{M}$ is a σ-algebra of sets and $\Gamma(x)$ is a total subset of X' for each $x\in X$, is strongly σ-additive. It is just as desirable, perhaps even more so, to have available criteria which can be used to determine the strong σ-additivity of *particular* spectral measures of some given class $(\mathcal{M};\Gamma(x),x\in X)$. Propositions 2 and 3 below give two such criteria.

Let P be an additive, multiplicative, $L(X)$-valued set function defined on some σ-algebra of sets $\mathcal{M}$. For each $x\in X$, let $\mathcal{M}_P[x]$ denote the cyclic subspace of X generated by x with respect to P, that is, the closed subspace of X generated by $\{P(E)x; E\in\mathcal{M}\}$.

LEMMA 3. *Let P be an additive, multiplicative, $L(X)$-valued set function with equicontinuous range, defined on some σ-algebra of sets $\mathcal{M}$. Let $x\in X$. If $x'\in X'$ is a functional such that $\langle Px, x'\rangle$ is σ-additive, then also $\langle P\xi, x'\rangle$ is σ-additive, for each $\xi\in\mathcal{M}_P[x]$.*

PROOF. It follows easily from the σ-additivity of $\langle Px,x'\rangle$, that $\langle P\xi,x'\rangle$ is σ-additive whenever ξ belongs to the linear span of $\{P(E)x; E\in\mathcal{M}\}$.

Let $\xi\in\mathcal{M}_P[x]$. Then there is a net of elements $\{\xi_\alpha\}$ such that each ξ_α belongs to the linear span of $\{P(E)x; E\in\mathcal{M}\}$ and $\xi_\alpha\to\xi$ in X.

It follows from the equicontinuity of $P(\mathcal{M})$ that the function q given by

$$q(z)=\sup\{|\langle z,y'\rangle|;\ y'\in\{P(E)'x';\ E\in\mathcal{M}\}\}, \quad z\in X,$$

is a continuous seminorm on X. Let E_n, $n=1,2,\ldots$, be a sequence of elements from $\mathcal{M}$ which decreases to the empty set $\emptyset$. Then the inequalities

$$|\langle P(E_n)\xi, x'\rangle| \leqq |\langle P(E_n)(\xi-\xi_\alpha), x'\rangle| + |\langle P(E_n)\xi_\alpha, x'\rangle| \leqq$$
$$\leqq q(\xi-\xi_\alpha) + |\langle P(E_n)\xi_\alpha, x'\rangle|,$$

are valid for each α and $n=1,2,\ldots$. If $\varepsilon>0$, then there exists an index $\alpha(\varepsilon)$ such that $q(\xi-\xi_{\alpha(\varepsilon)})<\varepsilon/2$. Since $\langle P\xi_{\alpha(\varepsilon)}, x'\rangle$ is σ-additive and $E_n\downarrow\emptyset$, there exists N such that $|\langle P(E_n)\xi_{\alpha(\varepsilon)}, x'\rangle|<\varepsilon/2$ for each $n\geqq N$. It follows that $\langle P(E_n)\xi, x'\rangle\to 0$ as $n\to\infty$. Hence, $\langle P\xi, x'\rangle$ is σ-additive.

PROPOSITION 2. *Let P be a spectral measure of class $(\mathcal{M}; \Gamma(x), x\in X)$, where $\mathcal{M}$ is a σ-algebra of sets and $\Gamma(x)$ is a total subset of X' for each $x\in X$. If the cyclic space $\mathcal{M}_P[x]$ satisfies any one of the criteria* (i)—(iv) *of Proposition* 1, *for each $x\in X$, then P is strongly σ-additive.*

PROOF. Fix $x\in X$. Then $\mathcal{M}_P[x]$ is an invariant subspace for each operator $P(E)$, $E\in\mathcal{M}$, and the restriction $\Gamma_0(x)$, of $\Gamma(x)$ to $\mathcal{M}_P[x]$ is a total subset of $\mathcal{M}_P[x]'$. Let $\Gamma_x(\xi)=\Gamma_0(x)$ for each $\xi\in\mathcal{M}_P[x]$. It follows from Lemma 3 that if P^x denotes the restriction of P to $\mathcal{M}_P[x]$, then P^x is a spectral measure of class $(\mathcal{M}; \Gamma_x(\xi), \xi\in\mathcal{M}_P[x])$ in $L(\mathcal{M}_P[x])$. Proposition 1 implies that P^x is σ-additive. In particular, $P^x(E_n)x\to 0$ in $\mathcal{M}_P[x]$ whenever E_n, $n=1,2,\ldots$, is a sequence of sets in $\mathcal{M}$ decreasing to $\emptyset$. Hence, $P(E_n)x\to 0$ in X. This shows that Px is σ-additive for each $x\in X$, that is, P is σ-additive.

REMARK. For X a Banach space, a version of Proposition 2 was proved in [3] (Theorem 2). It is worth noting that in practice, Proposition 2 has a larger range of application than Proposition 1 since the subspaces $\mathcal{M}_P[x]$, $x\in X$, may be substantially smaller than all of X.

A locally convex space X is said to be essentially separable with respect to a weaker topology τ ([6], p. 12), when any countable subset of X is contained in a linear subspace of X which is separable with respect to the given topology on X and closed with respect to τ.

PROPOSITION 3. *Let P be a spectral measure of class $(\mathcal{M}; \Gamma(x), x\in X)$, where $\mathcal{M}$ is a σ-algebra of sets and $\Gamma(x)$ is a total subset of X' for each $x\in X$. Let $\Gamma_0(x)$ be the restriction of $\Gamma(x)$ to $\mathcal{M}_P[x]$, for each $x\in X$. If $\mathcal{M}_P[x]$ is metrizable and essentially separable with respect to the topology $\sigma(\mathcal{M}_P[x], \Gamma_0(x))$, for each $x\in X$, then P is σ-additive.*

PROOF. Fix $x\in X$. In the notation of the proof of Proposition 2, P^x is a spectral measure of class $(\mathcal{M}; \Gamma_x(\xi), \xi\in\mathcal{M}_P[x])$. Accordingly, P^x is σ-additive by Theorem 0.3 of [6] and hence, it follows that P is σ-additive.

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(Received November 8, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 127—135.

STRONGLY CONVERGENT TRIGONOMETRIC SERIES AS FOURIER SERIES

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A study of strong convergence of trigonometric and Fourier series was recently introduced in [8]. Its interest is justified by the fact that it lies between absolute and ordinary convergence in regard to which trigonometric and Fourier series have been thoroughly investigated.

In this paper we are concerned with the question under which conditions is an a.e. strongly convergent trigonometric series, of index $\lambda \geqq 1$, a Fourier series. We show that, if a trigonometric series is strongly convergent of index $\lambda > 1$, on a set of positive measure or on a set of second category, then it is necessarily a Fourier series of a function f which belongs to L^p for each $p \geqq 1$, and that this result can not be improved as to conclude that $f \in L^\infty$. A similar statement is not true for the strong convergence of index $\lambda = 1$. Namely, there are trigonometric series that are strongly convergent a.e. to a function which is not in L^p for $p > 3/2$. The question whether such series are Fourier—Lebesgue or Fourier—Stieltjes series remains unsolved. We also prove several simple statements concerning the above problem and the convergence in the norm

1. Definitions and preliminaries

A real or a complex valued sequence (s_k) is strongly C_1 summable to a number t, of index $\lambda > 0$, and we write $s_k \to t\ [C_1]_\lambda$ if

$$\frac{1}{n+1} \sum_{k=0}^{n} |s_k - t|^\lambda = o(1). \tag{1.1}$$

This definition was introduced by Hardy and Littlewood in connection with the Fourier series, see [1] and [9].

A notion of strong convergence developed with the natural extensions of the above concept of strong summability, to Cesàro methods C_α, $\alpha \geqq 0$, and other summability methods, see [5] and [6] and the references cited there.

A real or a complex valued sequence (s_k) is strongly convergent to a number t, of index $\lambda > 0$, and we write $s_k \to t\ [I]_\lambda$ if

$$\frac{1}{n+1} \sum_{k=0}^{n} |(k+1)(s_k - t) - k(s_{k-1} - t)|^\lambda = o(1). \tag{1.2}$$

Here and in the other similar expressions $s_{-1} = 0$.

If A and B are two convergence (summability) methods we write $A \Rightarrow B$ if $s_k \to t\, A$ implies $s_k \to t\, B$. By I we shall denote the ordinary convergence, that is $s_k \to t\, I$ if $s_k \to t$ in the ordinary sense.

The following properties of strong convergence $[I]_\lambda$ and strong summability $[C_1]_\lambda$ can be easily verified, for more general results see [5] and [6]:

(i) $[I]_\lambda \Rightarrow [I]_\mu$ and $[C_1]_\lambda \Rightarrow [C_1]_\mu$ for $\lambda > \mu > 0$.

(ii) $[I]_\lambda \Rightarrow I \Rightarrow [C_1]_\lambda \Rightarrow C_1$ for $\lambda \geqq 1$.

(iii) If $\lambda \geqq 1$ then the following are equivalent:

1. $s_k \to t\ [I]_\lambda$,

2. $s_k \to t$ and

$$\frac{1}{n+1} \sum_{k=0}^{n} k^\lambda |s_k - s_{k-1}|^\lambda = o(1) \tag{1.3}$$

3. $s_k \to t\ [C_1]_\lambda$ and (1.3) holds,

4. $s_k \to t\ C_1$ and (1.3) holds.

Statements (i), (ii) and the equivalence of 1. and 2. in (iii) are corollaries of Theorem 1 in [6] and Theorem 5 in [5]. The equivalence of 1. and 3. follows from (ii) and the Minkowski inequality. Moreover by (ii) clearly 3. implies 4. and from (i) it follows that 4. implies 2.

Our next definition contains by now the well established concept of absolute convergence of index $\lambda > 0$, see [4] and [6].

A real or complex valued sequence (s_k) is absolutely convergent to a number t, of index $\lambda > 0$, and we write $s_k \to t\ |I|_\lambda$ if $s_k \to t$ and

$$\sum_{k=1}^{\infty} k^{\lambda-1} |s_k - s_{k-1}|^\lambda < \infty. \tag{1.4}$$

If $\lambda = 1$ then clearly (1.4) implies that $s_k \to t$ for some t and this is not true for $\lambda > 1$, see [8].

The following statement shows the relationship between these types of convergence, see [6]:

(iv) $|I|_\lambda \Rightarrow [I]_\lambda \Rightarrow I$ for $\lambda \geqq 1$.

The absolute and strong convergence of index 1 we shall call simply the absolute and strong convergence, denoted by $|I|$ and $[I]$, respectively.

Clearly, if the above definitions are applied to sequences of real or complex valued functions on the real line, in particular to trigonometric or Fourier series, then the corrosponding statements (i) through (iv) are valued for the appropriate pointwise or uniform convergence on some subset of real line.

For $p \geqq 1$, let L^p denote the set of all real or complex valued functions f such that $\|f\|_p = \left(\frac{1}{2\pi} \int |f|^p\right)^{1/p}$ is finite, with the integral being taken over any interval of length 2π. Let C denote the set of all continuous real or complex valued 2π-periodic functions. For $f\in L^p$, $p\geqq 1$, and a point x let

$$\Phi_{x,p}(t) = \int_0^t \left|\frac{1}{2}(f(x+u)+f(x-u))-f(x)\right|^p du.$$

Then x is called a Lebesgue point of $f\in L^p$ if $\Phi_{x,p}(t)=o(t)$ as $t\to 0^+$. By a theorem due to Lebesgue almost all points are Lebesgue points of f.

2. Strong convergence of trigonometric series; introductory results and remarks

Given a trigonometric series

$$a_0/2+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx) \tag{2.1}$$

let $s_n(x)$ and $\sigma_n(x)$ denote the n-th partial sum and the n-th Cesàro C_1 partial sum of (2.1) respectively. If (2.1) is a Fourier series of a function $f\in L^1$ we shall write $s_n f$ and $\sigma_n f$ for the corresponding partial sums s_n and σ_n.

The strong convergence $[I]_\lambda$ of trigonometric and Fourier series, of index $\lambda\geqq 1$ was first considered by the present author in [8]. Some results about the absolute convergence $|I|_\lambda$ of index $\lambda>1$ were recently obtained in [4] as a special case of more general results on absolute summability of trigonometric series. The absolute convergence $|I|_\lambda$ for $\lambda=1$ is just the ordinary absolute convergence in regard to which trigonometric series have been thoroughly investigated.

The results obtained in [8] already indicate that the strong convergence of trigonometric series has some of the characteristics of both absolute and ordinary convergence. Moreover, they put in a new light some of the wellknown properties of trigonometric and Fourier series, in a sense that some theorems concerning ordinary convergence can be extended to strong, but not to absolute convergence and that some statements about absolute convergence hold for strong convergence under weaker assumptions; see Theorems 5, 6 and 7, their corollaries and remarks in [8].

An obvious similarity with absolute convergence (also with absolute convergence of index $\lambda>1$, see [4]) is exhibited by the following analogies of the classical theorems due to Denjoy and Lusin, see [1] or [9]:

THEOREM A. (Theorems 1 and 2 in [8].) *If a trigonometric series* (2.1) *is* $[I]_\lambda$ *convergent of index* $\lambda\geqq 1$, *on a set of positive measure or on a set of second category then*

$$\frac{1}{n+1}\sum_{k=0}^{n}k^\lambda|a_k|^\lambda = o(1) \quad\text{and}\quad \frac{1}{n+1}\sum_{k=0}^{n}k^\lambda|b_k|^\lambda = o(1) \tag{2.2}$$

that is $ka_k\to 0$ $[C_1]_\lambda$ *and* $kb_k\to 0$ $[C_1]_\lambda$.

Applying this to Fourier series we have:

THEOREM B. (Theorem 3 in [8] is slightly improved.) *Let* $\lambda \geqq 1$.

i) *If* (2.1) *is a Fourier series of a function* $f \in L^p$ *for some* $p \geqq 1$, *then* $s_n f \to f$ $[I]_\lambda$ *at every Lebesgue point of f if and only if its coefficients satisfy* (2.2).

ii) *If* (2.1) *is a Fourier series of a function* $f \in C$ *then* $s_n f \to f$ $[I]_\lambda$ *uniformly if and only if its coefficients satisfy* (2.2).

REMARK. The above statement i) differs from Theorem 3 in [8] only in the case if $f \in L^1$ and $f \notin L^p$, $p > 1$, and that only in the sufficiency part. Namely, from the equivalence of 1. and 4. in (iii) and the Fejér—Lebesgue theorem it follows that $s_n f \to f$ $[I]_\lambda$ at every Lebesgue point of f whenever (2.2) holds. The author overlooked this fact in the proof of Theorem 3 in [8], using instead the equivalence of 1. and 3. in (iii) and the corresponding more difficult results on $[C_1]_\lambda$ summability of Fourier series, see [8], [1] and [9]. Thus, if $f \in L^1$ and $f \notin L^p$ for $p > 1$ and (2.2) holds, we were only able to conclude that $s_n f \to f$ $[I]_\lambda$ a.e.

Due to the above mentioned similarity with the absolute convergence and the fact that an a.e. absolutely convergent trigonometric series is necessarily a Fourier series of a continuous function, it is natural to ask whether an a.e. strongly $[I]_\lambda$ convergent trigonometric series, of index $\lambda \geqq 1$, is a Fourier—Lebesgue series. The answer is positive for the strong convergence of index $\lambda > 1$ and moreover such series are necessarily Fourier series of functions belonging to L^p for each $p \geqq 1$. For the strong convergence of index 1, the question is much more difficult and we give only some partial results.

3. Strong convergence $[I]_\lambda$ and Fourier series

THEOREM 1. *If the coefficients of a trigonometric series* (2.1) *satisfy* (2.2) *for some* $\lambda > 1$ *then* (2.1) *is a Fourier series of a function f which belongs to* L^p *for each* $p \geqq 1$ *and* $a_n \to f$ $[I]_\lambda$ *at every Lebesgue point of f.*

PROOF. If (2.2) holds for some $\lambda > 1$, that is if $ka_k \to 0$ $[C_1]_\lambda$ and $kb_k \to 0$ $[C_1]_\lambda$ then by (i) clearly (2.2) holds for each q, $1 < q \leqq \min(\lambda, 2)$. Therefore, by partial summation:

$$\sum_{k=1}^{n} |a_k|^q = \sum_{k=1}^{n} \frac{1}{k^q} k^q |a_k|^q = \sum_{k=1}^{n-1} \left(\Delta \frac{1}{k^q}\right) \sum_{i=1}^{k} i^q |a_i|^q + \frac{1}{n^q} \sum_{i=1}^{n} i^q |a_i|^q =$$

$$= O(1) \sum_{k=1}^{n-1} \frac{1}{k^{q+1}} \sum_{i=1}^{k} i^q |a_i|^q + o(1) = O(1) \sum_{k=1}^{n-1} \frac{1}{k^q} + o(1).$$

Here and throughout the paper, for a sequence (x_k), $\Delta x_k = x_k - x_{k+1}$ for $k = 0, 1, 2, \ldots$.

Consequently $\sum |a_k|^q < \infty$ and using the same argument it follows that $\sum |b_k|^q < \infty$. By Hausdorff—Young theorem, see [1] or [9], there is a function $f \in L^p$, where $1/p + 1/q = 1$, such that (2.1) is the Fourier series of f. Since this is true for each q, $1 < q \leqq \min(\lambda, 2)$, we have $f \in L^p$ for each

$$p \geqq \min(\lambda, 2)/(\min(\lambda, 2) - 1).$$

Thus $f \in L^p$ for each $p \geqq 1$.

From Theorem B we conclude moreover that $s_n \to f\ [I]_\lambda$ at every Lebesgue point of f.

The following result is just a simple corollary of the above Theorem 1 and Theorem A, i.e. Theorems 1 and 2 in [8]:

THEOREM 2. *If a trigonometric series* (2.1) *is* $[I]_\lambda$ *convergent for some* $\lambda > 1$, *on a set of positive measure or on a set of second category, then* (2.1) *is a Fourier series of a function f which belongs to L^p for each* $p \geqq 1$ *and* $s_n \to f\ [I]_\lambda$ *at every Lebesgue point of f.*

COROLLARY 1. *A trigonometric series* (2.1) *is* $[I]_\lambda$ *convergent a.e. to a function f, for some* $\lambda > 1$, *if and only if* (2.2) *holds and* (2.1) *is a Fourier—Lebesgue series.*

PROOF. This is an immidiate consequence of Theorem 2 and Theorem B.

REMARK 1. Theorem 2 and consequently Theorem 1, can not be improved as to conclude that $f \in L^\infty$. The following example shows that there is a trigonometric series which is strongly $[I]_\lambda$ convergent a.e. for every $\lambda \geqq 1$, so that it is a Fourier series of a function f which belongs to L^p for each $p \geqq 1$, but such that $f \notin L^\infty$. Namely it is well known that the series $\sum_{k=2}^{\infty} \frac{1}{k \log k} \cos kx$ converges a.e. to a function $f \in L^1$, see 7.3 in [2] Vol. 1. Since the coefficients of this series clearly satisfy (2.2) for each $\lambda \geqq 1$, by Theorem 1 we are able even to conclude that the series is strongly $[I]_\lambda$ convergent a.e. to a function f belonging to L^p for each $p \geqq 1$. However, by what was shown in 12.8.3, [2] Vol. 2, $f \notin L^\infty$.

We prove now that the above results can not be extended to strong convergence of index 1.

THEOREM 3. *There are trigonometric series that are strongly convergent a.e. to a function which is not in L^p for* $p > 3/2$.

PROOF. Consider the cosine series

$$a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx \tag{3.1}$$

with coefficients defined below.

Given a lacunary sequence (k_j), that is a sequence of positive integers such that $k_{j+1} \geqq \varrho k_j$ for $j = 1, 2, \ldots$ and some $\varrho > 1$, let

$$a_k = \frac{1}{j^{1/2} \ln (j+1)} \quad \text{for} \quad k = k_j, k_j + 1, \ldots, k_j + [j^{1/2}]; \quad j = 1, 2, \ldots \tag{3.2}$$

$$a_k = 0 \quad \text{otherwise.}$$

We now prove that there exists a function f such that the series (3.1) with coefficients given by (3.2), is strongly convergent to f a.e. and $f \notin L^p$ for $p > 3/2$. That is we claim:

1. $s_n \to f\ [I]$ a.e. for some function f.
2. $f \notin L^p$ for $p > 3/2$ where (s_n) is the sequence of the partial sums of (3.1).

PROOF OF 1. By statement (iii) of Section 1 it suffices to show that

$$\frac{1}{n+1}\sum_{k=0}^{n} k\,|s_k(x)-s_{k-1}(x)| = o(1) \tag{3.3}$$

uniformly in x and that for some function f

$$s_n(x)\to f(x) \tag{3.4}$$

for almost all x. For $n\geqq k_2$, let j_n be the largest integer j such that $k_j\leqq n$. Then from (3.2) clearly

$$\frac{1}{n+1}\sum_{k=0}^{n} k\,|a_k| = \frac{1}{n+1}\sum_{j=1}^{j_n-1}\sum_{k=k_j}^{k_{j+1}-1} k\,|a_k| + \frac{1}{n+1}\sum_{k=k_{j_n}}^{n} k\,|a_k| \leqq$$

$$\leqq \frac{1}{n+1}\sum_{j=1}^{j_n-1} k_{j+1}\frac{1}{\ln(j+1)} + \frac{1}{\ln(j_n+1)}.$$

By the lacunarity of the sequence (k_j), $k_{j+1}\leqq\frac{\varrho}{\varrho-1}(k_{j+1}-k_j)$ and the above inequality

$$\frac{1}{n+1}\sum_{k=0}^{n} k\,|a_k| = o(1). \tag{3.5}$$

Now (3.5) clearly implies (3.3).

Next we prove that (3.4) holds for some function f. In order to see this let us write applying partial summation

$$s_n(x) = a_0/2+\sum_{k=1}^{n} a_k\cos kx = \sum_{k=0}^{n-1}\Delta a_k D_k(x)+a_n D_n(x), \tag{3.6}$$

where D_n denotes the Dirichlet kernel.

Since $a_n\to 0$ and $D_k(x)=\dfrac{\sin(k+1/2)x}{2\sin x/2}$ for $x\neq 0 \pmod{2\pi}$ by (3.6) $s_n(x)$ converges a.e. if and only if the series

$$\sum_{k=0}^{\infty}\Delta a_k\sin(2k+1)x \tag{3.7}$$

converges almost everywhere.

From (3.2) clearly

$$\sum_{k=0}^{\infty}|\Delta a_k|^2 = 2\sum_{j=1}^{\infty}\frac{1}{j\ln^2(j+1)}<\infty$$

and consequently (3.7) is a Fourier series of a function $g\in L^2$, see 8.3.1, [2], Vol. 1. Since (3.5) clearly implies that

$$\frac{1}{n+1}\sum_{k=0}^{n} k\,|\Delta a_k| = o(1),$$

from Theorem B it follows that (3.7) is strongly convergent to g for almost all x. Therefore $s_n(x)\to f(x)$ a.e. where the function f is uniquely determined a.e. by the equation

$$f(x) = g(x/2)/2 \sin x/2 \quad \text{a.e.,} \tag{3.8}$$

which completes the proof of 1.

PROOF OF 2. Suppose, on the contrary, that $f\in L^p$ for some $p>3/2$. We may assume that $2\geqq p>3/2$. By what we have seen already, (3.7) is the Fourier series of $g\in L^2$, converging a.e. to that function. Hence by (3.8), (3.7) is the Fourier series of $2\sin x f(2x)$, where f is integrable. Consequently,

$$\Delta a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} 2\sin x f(2x)\sin(2k+1)x\,dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}[\cos 2kx-\cos(2k+2)x]f(2x)\,dx \quad \text{for} \quad k=0,1,2,\ldots.$$

But $a_n-a_0=-\sum_{k=0}^{n-1}\Delta a_k$ and therefore

$$a_n-a_0=\frac{1}{2\pi}\int_{-2\pi}^{2\pi}[\cos nx-1]f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx-\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx.$$

Since $a_n\to 0$ and f is integrable, by the Riemann—Lebesgue lemma it follows that $a_0=\frac{1}{\pi}\int_0^{2\pi}f$. Hence

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx \quad \text{for} \quad n=0,1,2,\ldots$$

that is, (3.1) is the Fourier series of f.

Now by assumption $f\in L^p$ for some p, $3/2<p\leqq 2$ and therefore by the Hausdorff—Young theorem, the sequence of its Fourier coefficients (a_k) must satisfy the inequality

$$\sum_{k=1}^{\infty}|a_k|^q \leqq \|f\|_p^q$$

where $1/p+1/q=1$. This implies, by (3.2), that

$$\sum_{j=1}^{\infty}\frac{1}{j^{q/2}\ln^q(j+1)}j^{1/2} \leqq \|f\|_p^q$$

which is impossible, since the last series diverges for $q<3$. Hence $f\notin L^p$ for $p>3/2$.

REMARK. The present author was unable to show whether the sum function of the above cosine series is integrable. The question is whether such series are Fourier—Lebesgue or Fourier—Stieltjes series at all. In view of the following results it would be sufficient to establish that $\|s_n\|_1\neq O(1)$.

CONJECTURE. There are trigonometric series that are strongly convergent a.e. and are not Fourier—Lebesgue or Fourier—Stieltjes series.

Our next task is to prove several simple statements showing the relationship between the strong convergence and the ordinary convergence in the norm.

THEOREM 4. i) *Let* (2.1) *be* $[I]$ *convergent a.e. to a function* f. *If* $f\in L^1$ *and* (2.1) *is the Fourier series of* f *then* $\|s_n-f\|_1=o(1)$ *as* $n\to\infty$.

ii) *Let* (2.1) *be* $[I]$ *convergent a.e. Then* (2.1) *is a Fourier—Lebesgue series if and only if* $\|s_m-s_n\|_1=o(1)$ *as* $m, n\to\infty$, *and* (2.1) *is a Fourier—Stieltjes series if and only if* $\|s_n\|_1=O(1)$ *as* $n\to\infty$.

iii) *Let* (2.1) *be* $[I]_\lambda$ *convergent a.e. to a function* f, *for some* $\lambda>1$. *Then* $f\in L^p$ *for each* $p\geqq 1$, (2.1) *is the Fourier series of* f *and* $\|s_n-f\|_p=o(1)$ *as* $n\to\infty$.

PROOF. We first note that by the assumption in all three statements (2.1) is $[I]$ convergent a.e. Therefore by Theorem A (2.2) holds for $\lambda=1$.

Now clearly,

$$s_n(x)-\sigma_n(x)=\frac{1}{n+1}\sum_{k=1}^{n}k(a_k\cos kx+b_k\sin kx)$$

so that by above relation

$$s_n(x)-\sigma_n(x)=o(1) \tag{3.9}$$

uniformly in x.

i) If $f\in L^1$ and (2.1) is the Fourier series of f then from (5.5) Ch. IV in [9] it follows that $\|\sigma_n-f\|_1=o(1)$. By (3.9) clearly $\|s_n-\sigma_n\|_1=o(1)$ and consequently $\|s_n-f\|_1=o(1)$.

ii) Statement ii) follows in the same way from (3.9) and the well known facts that: (2.1) is a Fourier—Lebesgue series if and only if $\|\sigma_m-\sigma_n\|_1=o(1)$ as $m, n\to\infty$; (2.1) is a Fourier—Stieltjes series if and only if $\|\sigma_n\|_1=O(1)$, see (4.3) and (5.5) Ch. IV in [9] or Theorems 4 and 5, § 60, Ch. I in [1].

iii) Although statement iii) can be regarded as a simple consequence of Theorem 2 and the well known theorem about the convergence in the norm of Fourier series of functions in L^p, $p>1$, one should prefer the trivial argument used above. By Theorem 2 clearly $f\in L^p$ for each $p\geqq 1$ and (2.1) is the Fourier series of f. Consequently from (5.12) Ch. IV in [9] or Theorem 3, § 60, Ch. I in [1] it follows that $\|\sigma_n-f\|_p=o(1)$. By (3.9) clearly $\|s_n-\sigma_n\|_p=o(1)$ and the conclusion follows.

REMARK 1. The assumption that (2.1) is $[I]$ convergent a.e., respectively $[I]_\lambda$ convergent a.e. for some $\lambda>1$, in statements i) through iii) can be replaced by the assumption that the corresponding convergence holds on a set of positive measure or on a set of second category, or simply by the assumption that the coefficients satisfy (2.2) for $\lambda=1$ respectively for some $\lambda>1$.

REMARK 2. There are trivial examples of Fourier series of functions in L^p, $p\geqq 1$, that converge to that function in the corresponding norm and whose coefficients do not satisfy (2.2).

The following cosine series with monotone coefficients illustrate this fact; $\sum_{k=1}^{\infty}\frac{1}{k}\cos kx$ and $\sum_{k=2}^{\infty}\frac{1}{\log^2 k}\cos kx$.

By 7.3 in [2] Vol. 1, the first series converges a.e. to a function $f \in L^2$, is the Fourier series of f and $\|s_n - f\|_2 = o(1)$; and the second series converges a.e. to $f \in L^1$, is the Fourier series of f and $\|s_n - f\|_1 = o(1)$. However clearly the coefficients of either series do not satisfy (2.2).

In conclusion of this paper we remark that the results presented here show that the trigonometric series which are strongly a.e. convergent of index $\lambda > 1$, are quite well behaved, being the Fourier—Lebesgue series of their sums, a subclass of $\bigcap_{p \geqq 1} L^p$, while the situation is much more intriguing with the strong convergence of index 1. It is well known that there are a.e. convergent trigonometric series that are not Fourier—Lebesgue or Fourier—Stieltjes series. We have conjectured that this statement extends to strong convergence. Whether this conjecture is true is a very interesting question in view of the fact that $|I| \Rightarrow [I] \Rightarrow I$ properly and that an a.e. absolutely convergent trigonometric series is necessarily a Fourier series of a continuous function.

The work on this paper was partially supported by the Research Council of S. R. Bosna and Hercegovina.

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(Received November 28, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 137—143.

AN EXTREMUM PROBLEM CONCERNING ALGEBRAIC POLYNOMIALS

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Let S_n be the set of all polynomials whose degree does not exceed n and whose all zeros are real but lie outside $(-1, 1)$. Similarly, we say $p_n \in Q_n$ if $p_n(x)$ is a real polynomial whose all zeros lie outside the open disk with center at the origin and radius 1. Further we will denote by H_n the set of all polynomials of degree $\leqq n$ and of the form

$$p_n(x) = \sum_{k=0}^{n} a_k q_{nk}(x), \quad \text{with} \quad a_k \geqq 0, \quad k = 0, 1, 2, \ldots, n, \tag{1.1}$$

where $q_{nk}(x) = (1+x)^k(1-x)^{n-k}$. Elements of H_n are called polynomials with positive coefficients (in $1-x$ and $1+x$) by G. G. Lorentz.

The following inequalities for derivatives of polynomials of special type are known:

THEOREM A (P. Erdős). *Let $p_n \in S_n$ then*

$$\max_{-1 \leqq x \leqq 1} |p_n'(x)| \leqq \frac{1}{2} en \max_{-1 \leqq x \leqq 1} |p_n(x)|.$$

Further, the constant $\frac{1}{2}e$ can not be replaced by a smaller one.

THEOREM B (G. G. Lorentz). *Let $p_n \in H_n$ then for each $r = 1, 2, \ldots$ there exists a constant C_r for which*

$$\max_{-1 \leqq x \leqq 1} |p_n^{(r)}(x)| \leqq C_r n^r \max_{-1 \leqq x \leqq 1} |p_n(x)|. \tag{1.2}$$

THEOREM C (J. T. Scheick). *If $p_n \in H_n$ and $n \geqq 1$ then*

$$\max_{-1 \leqq x \leqq 1} |p_n'(x)| \leqq \frac{1}{2} en \max_{-1 \leqq x \leqq 1} |p_n(x)|, \tag{1.3}$$

$$\max_{-1 \leqq x \leqq 1} |p_n''(x)| \leqq en(n-1) \max_{-1 \leqq x \leqq 1} |p_n(x)|. \tag{1.4}$$

THEOREM D (A. K. Varma). *Let $p_n \in S_n$, then we have*

$$\int_{-1}^{1} (1-x^2)(p_n'(x))^2\, dx \leqq \frac{n(n+1)(2n+3)}{4(2n+1)} \int_{-1}^{1} (1-x^2) p_n^2(x)\, dx \tag{1.5}$$

with equality for $p_n(x)=(1+x)^n$ *or* $p_n(x)=(1-x)^n$. *Moreover if* $p_n(1)=p_n(-1)=0$ *then for* $n\geqq 2$

$$\int_{-1}^{1}(p_n'(x))^2\,dx \leqq \frac{n(2n+1)(n-1)}{4(2n-3)}\int_{-1}^{1}(p_n(x))^2\,dx, \tag{1.6}$$

equality holds for only $p_n(x)=c(1+x)(1-x)^{n-1}$ *or* $p_n(x)=c(1-x)(1+x)^{n-1}$.

It is known [2] that if $p_n\in S_n$ (or $p_n\in Q_n$) then $p_n\in H_n$ or $-p_n\in H_n$. Thus Theorem B as well Theorem C can be looked as a generalization of Theorem A. Similarly Theorem D is an extension of Theorem A in L_2 norm for $p_n\in S_n$. The object of this paper is to extend Theorem B as well as Theorem D in L_2 norm for $p_n\in H_n$.

THEOREM 1. *Let* $p_n\in H_n$ *then for* $n\geqq 2$

$$\int_{-1}^{1}(p_n'(x))^2\,dx \leqq \frac{n(n-1)(2n+1)}{4(2n-3)}\int_{-1}^{1}(p_n(x))^2\,dx, \tag{1.7}$$

equality holds iff $p_n(x)=c(1+x)^{n-1}(1-x)$ *or* $p_n(x)=c(1+x)(1-x)^{n-1}$.

THEOREM 2. *Let* $p_n\in H_n$ *then*

$$\int_{-1}^{1}(1-x^2)(p_n'(x))^2\,dx \leqq \frac{n(n+1)(2n+3)}{4(2n+1)}\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx \tag{1.8}$$

with equality for $p_n(x)=(1+x)^n$ *or* $p_n(x)=(1-x)^n$.

COROLLARY. *If* $p_n\in Q_n$ *then* (1.7) *and* (1.8) *are valid.*

2. Some lemmas. For the proof of Theorem 1 and Theorem 2 we need the following lemmas.

LEMMA 2.1. *Let* $p_n\in H_n$. *Then we have*

$$\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx \geqq \frac{2(2n+1)}{(n+1)(2n+3)}\int_{-1}^{1}p_n^2(x)\,dx. \tag{2.1}$$

PROOF. From (1.1) we have

$$p_n^2(x) = \sum_{p+q=2n} a_{pq}(1+x)^p(1-x)^q,\quad a_{pq}\geqq 0. \tag{2.2}$$

Hence we may write

$$\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx = \sum_{p+q=2n} a_{pq}\int_{-1}^{1}(1+x)^{p+1}(1-x)^{q+1}\,dx.$$

But on using

$$\frac{\int_{-1}^{1}(1+x)^{p+1}(1-x)^{q+1}\,dx}{\int_{-1}^{1}(1+x)^p(1-x)^q\,dx} = \frac{4(p+1)(q+1)}{(p+q+3)(p+q+2)} \tag{2.3}$$

and simple computation the lemma follows. Note that equality in (2.1) holds for $p_n^2(x)=(1+x)^{2n}$ or $p_n^2(x)=(1-x)^{2n}$.

LEMMA 2.2. *Let* $p_n \in H_n$ *and suppose that*

$$p_n(1) = p_n(-1) = 0. \tag{2.4}$$

Then for $n \geqq 2$ *we have*

$$\frac{\int_{-1}^{1} (p_n'(x))^2\,dx}{\int_{-1}^{1} (p_n(x))^2\,dx} \leqq \frac{n(n-1)(2n+1)}{4(2n-3)}, \tag{2.5}$$

equality iff $p_n(x)=(1+x)(1-x)^{n-1}$ *or* $p_n(x)=(1-x)(1+x)^{n-1}$.

PROOF. From (1.1) and (2.4) we may write

$$p_n(x) = \sum_{k=1}^{n-1} a_{kn}(1-x)^k(1+x)^{n-k}, \quad a_{kn} \geqq 0, \quad 1 \leqq k \leqq n-1. \tag{2.6}$$

Therefore

$$\int_{-1}^{1} p_n^2(x)\,dx = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} a_{kn}a_{jn} \int_{-1}^{1} (1+x)^{2n-k-j}(1-x)^{k+j}\,dx.$$

On using the known formula

$$\int_{-1}^{1} (1-x)^p(1+x)^q\,dx = \frac{2^{p+q+1}\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}, \tag{2.7}$$

we have

$$\int_{-1}^{1} p_n^2(x)\,dx = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{a_{kn}a_{jn}2^{2n+1}\Gamma(k+j+1)\Gamma(2n-k-j+1)}{\Gamma(2n+2)}. \tag{2.8}$$

Next, we turn to prove that

$$\int_{-1}^{1} (p_n'(x))^2\,dx \leqq \frac{2^{2n-2}(n-1)}{(2n-3)} \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{a_{kn}a_{jn}\Gamma(k+j+1)\Gamma(2n-k-j)}{\Gamma(2n)}. \tag{2.9}$$

To prove (2.9) we first note that if

$$q_{kn}(x) = (1-x)^k(1+x)^{n-k}, \tag{2.10}$$

then

$$q_{kn}'(x) = -k(1-x)^{k-1}(1+x)^{n-k}+(n-k)(1-x)^k(1+x)^{n-k-1}, \tag{2.11}$$

and on using (2.7) we have

$$\begin{aligned} I_{k,j} = \int_{-1}^{1} q_{kn}'(x)q_{jn}'(x)\,dx = \frac{2^{2n-1}}{\Gamma(2n)}\big[&kj\Gamma(k+j-1)\Gamma(2n-j-k+1)+\\ &+(n-k)(n-j)\Gamma(k+j+1)\Gamma(2n-k-j-1)-\\ &-((j(n-k)+k(n-j))\Gamma(k+j)\Gamma(2n-j-k)\big]. \end{aligned} \tag{2.12}$$

After a simple computation it can be shown that

$$I_{k,j} = \frac{2^{2n-1}(n-1)\Gamma(k+j+1)\Gamma(2n-j-k+1)}{\Gamma(2n)}\mu_{k,j} \tag{2.13}$$

where

$$\mu_{k,j} = \frac{n(k+j)-2kj-n(k-j)^2}{(k+j)(k+j-1)(2n-j-k)(2n-j-k-1)}.$$

Next, we will show that for $k, j=1, 2, \ldots, n-1$

$$\mu_{k,j} \leqq \frac{1}{2(2n-3)}, \tag{2.14}$$

equality holds only for $k=1$, $j=1$, or $k=n-1$, $j=n-1$. In (2.13) let $k+j=l$ then (2.14) is equivalent to

$$l(l-1)(2n-l)(2n-l-1) \geqq (2n-3)[2nl-2n(k-j)^2-4kj]$$

or

$$l(l-1)(2n-l)(2n-l-1) \geqq (2n-3)\{2nl-l^2-(2n-1)(k-j)^2\}$$

or

$$l(2n-l)(l-2)(2n-l-2)+(2n-1)(2n-3)(k-j)^2 \geqq 0.$$

This proves (2.14). Now, one using (2.13) and (2.14) we have

$$\int_{-1}^{1} q'_{kn}(x)q'_{jn}(x)\,dx \leqq \frac{2^{2n-2}(n-1)\Gamma(k+j+1)\Gamma(2n-j-k+1)}{(2n-3)\Gamma(2n)}. \tag{2.15}$$

Now, on using (2.15), (2.10), (2.11), we obtain (2.9). Further from (2.9) and (2.8) we have (2.5). This proves Lemma 2.2.

3. Proof of Theorem 1. Let $p_n \in H_n$. Then from (1.1) we have

$$p_n(x) = a_0(1+x)^n + a_n(1-x)^n + q_n(x) \tag{3.1}$$

where

$$q_n(x) = \sum_{k=1}^{n-1} a_k(1+x)^{n-k}(1-x)^k, \quad a_k \geqq 0. \tag{3.2}$$

We note that $q_n(1)=q_n(-1)=0$, therefore on using Lemma 2.2 we have

$$\frac{\int_{-1}^{1} q'_n(x)^2\,dx}{\int_{-1}^{1} q_n(x)^2\,dx} \leqq \frac{n}{4}\,\frac{(2n+1)(n-1)}{2n-3}, \quad n \geqq 2. \tag{3.3}$$

Next, from (3.1) and (3.2) we have

$$\int_{-1}^{1} p_n'(x)^2\,dx = \frac{n^2 2^{2n-1}}{2n-1}(a_0^2+a_n^2) + \int_{-1}^{1} q_n'(x)^2\,dx + \tag{3.4}$$

$$+2n\int_{-1}^{1}\left(a_0(1+x)^{n-1}-a_n(1-x)^{n-1}\right)q_n'(x)\,dx - 2a_0a_nn^2\int_{-1}^{1}(1-x^2)^{n-1}\,dx.$$

By integrating by parts we obtain

$$\int_{-1}^{1} q_n'(x)\{a_0(1+x)^{n-1}-a_n(1-x)^{n-1}\}\,dx = \tag{3.5}$$

$$= -(n-1)\int_{-1}^{1} q_n(x)\{a_0(1+x)^{n-2}+a_n(1-x)^{n-2}\}\,dx \leqq 0.$$

From (3.4) and (3.5) we obtain

$$\int_{-1}^{1} p_n'(x)^2\,dx \leqq \int_{-1}^{1} q_n'(x)^2\,dx + \frac{2^{2n-1}n^2(a_0^2+a_n^2)}{2n-1}. \tag{3.6}$$

Also from (3.1) it follows that

$$\int_{-1}^{1} p_n^2(x)\,dx \geqq \frac{(a_0^2+a_n^2)2^{2n+1}}{2n+1} + \int_{-1}^{1} q_n^2(x)\,dx. \tag{3.7}$$

Therefore by (3.6) and (3.7) we have

$$\frac{\int_{-1}^{1} p_n'(x)^2\,dx}{\int_{-1}^{1} p_n^2(x)\,dx} \leqq \frac{\int_{-1}^{1} q_n'(x)^2\,dx + \frac{2^{2n-1}n^2(a_0^2+a_n^2)}{2n-1}}{\int_{-1}^{1} q_n^2(x)\,dx + \frac{(a_0^2+a_n^2)2^{2n+1}}{2n+1}}. \tag{3.8}$$

It is easy to verify that

$$\frac{2^{2n-1}n^2(a_0^2+a_n^2)}{2n-1} < \frac{2^{2n+1}(a_0^2+a_n^2)}{(2n+1)}\,\frac{n}{4}\,\frac{(2n+1)(n-1)}{(2n-3)}. \tag{3.9}$$

Using (3.9) and (3.3) we obtain

$$\frac{\int_{-1}^{1} p_n'(x)^2\,dx}{\int_{-1}^{1} p_n^2(x)\,dx} \leqq \frac{n}{4}\,\frac{(2n+1)(n-1)}{(2n-3)}.$$

This proves Theorem 1.

4. Proof of Theorem 2. Let $p_n \in H_n$. First we write

(4.1)
$$\frac{\int_{-1}^{1}(1-x^2)p_n'(x)^2\,dx}{\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx} = \frac{\int_{-1}^{1}(1-x^2)p_n'(x)^2\,dx}{\int_{-1}^{1}p_n^2(x)\,dx}\,\frac{\int_{-1}^{1}p_n^2(x)\,dx}{\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx}.$$

On using Lemma (2.1) we obtain

$$\frac{\int_{-1}^{1}p_n^2(x)\,dx}{\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx} \leqq \frac{(n+1)(2n+3)}{2(2n+1)}, \tag{4.2}$$

equality holds for $p_n(x)=(1+x)^n$ or $p_n(x)=(1-x)^n$. Next, we will prove that for $p_n \in H_n$

$$\frac{\int_{-1}^{1}(1-x^2)p_n'(x)^2\,dx}{\int_{-1}^{1}p_n^2(x)\,dx} \leqq \frac{n}{2}, \tag{4.3}$$

equality holds for $p_n(x)=(1+x)^k(1-x)^{n-k}$ $k=0,1,\dots,n$. Let $p_n \in H_n$. Then we may write

$$p_n(x) = \sum_{k=0}^{n} a_{kn}(1-x)^k(1+x)^{n-k} \equiv \sum_{k=0}^{n} a_{kn}q_{kn}(x). \tag{4.4}$$

Following the proof of Lemma 2.2 we first note that

$$\int_{-1}^{1} q_{kn}'(x)q_{jn}'(x)(1-x^2)\,dx = \frac{2^{2n+1}\Gamma(k+j+1)\Gamma(2n-k-j+1)}{\Gamma(2n+2)}\mu_{kj} \tag{4.5}$$

where by $k+j=l$,

$$\mu_{kj} = \frac{(2n-l)(2n-l+1)kj+(n^2-nl+kj)l(l+1)-l(2n-l)(nl-2kj)}{l(2n-l)} = \tag{4.6}$$

$$= \frac{\frac{n}{2}l(2n-l)-\frac{n}{2}(2n+1)(k-j)^2}{l(2n-l)} \leqq \frac{n}{2}$$

equality holds iff $k=j$, $k=0,1,\dots,n$. Therefore

$$\int_{-1}^{1} q_{kn}'(x)q_{jn}'(x)(1-x^2)\,dx \leqq 2^{2n+1}\frac{n}{2}\frac{\Gamma(k+j+1)\Gamma(2n-k-j+1)}{\Gamma(2n+2)}. \tag{4.7}$$

By using (4.4), (4.7) we have

$$\int_{-1}^{1} (1-x^2)p_n'(x)^2\,dx \leqq \frac{2^{2n+1}}{\Gamma(2n+2)}\frac{n}{2}\sum_{k=0}^{n}\sum_{j=0}^{n} a_{kn}a_{jn}\Gamma(k+j+1)\Gamma(2n-k-j+1) =$$

$$= \frac{n}{2}\int_{-1}^{1} p_n^2(x)\,dx.$$

This proves (4.3). Now, using (4.1)—(4.3) we have

$$\frac{\int_{-1}^{1} p_n'(x)^2(1-x^2)\,dx}{\int_{-1}^{1} p_n(x)^2(1-x^2)\,dx} \leqq \frac{n}{2}\frac{(n+1)(2n+3)}{2(2n+1)}.$$

This proves Theorem 2 as well.

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(Received December 2, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 145—151.

A REPRESENTATION THEORY FOR ORTHOMODULAR LATTICES BY MEANS OF CLOSURE SPACES

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1. Introduction

In [2] a representation theorem by means of sets and a topological representation theory for orthomodular lattices were developed, using orthogonal spaces and ordered topological spaces. A summary of these results has been reported in [4].

In the set-theoretical representation theorem the underlying set is taken as the set of all ultrafilters in the orthomodular lattice carrying an orthogonality relation.

By contrast, in the topological theorem the base set of the representation is given by the set of all proper filters in the orthomodular lattice.

Orthomodular lattices are non-distributive generalizations of Boolean algebras. The representation theorem by sets mentioned above gives, in the distributive case, the Stone representation for Boolean algebras [2, Corollary 1], as expected. But this is not the case for the topological representation. This fact naturally led to the formulation of one of the open problems stated in [2]: do there exist more "economical" underlying sets for the topological representation of an orthomodular lattice? In this paper we give an answer to this question in terms of closure spaces.

The results established here have been presented at the First Meeting on Ordered Sets and Applications, C.N.R.S., held at Villeurbanne, France, in March 1982.

2. Preliminaries

In this section we explain the terminology and notation to be adopted here. We also collect, without proof, some results which will be needed in the sequel.

An abstract system $(P; 0, 1, \leqq, ')$ is an *orthomodular ordered set* if it is an orthocomplemented ordered set satisfying for all $a, b \in P$ the following conditions:

1) if $a \leqq b'$ then the join $a \vee b$ exists in P
2) $a \leqq b$ implies $b = a \vee (a' \wedge b)$.

We will refer to an orthomodular ordered set P, for short.

Note that if $a \leqq b$ the right hand side of 2) exists because the meet $a' \wedge b$ exists and $(a' \wedge b) \leqq a'$.

In the presence of 1), the condition 2) can be replaced by 2') $a \leqq b$ and $a' \wedge b = 0$ imply $a = b$ [1].

Let P be an orthocomplemented ordered set. For $a, b \in P$ we say that *a commutes with b*, in symbols aCb if $a \wedge b$ and $a \wedge b'$ exist and a is their join, i.e. $a = (a \wedge b) \vee (a \wedge b')$. The *center* Z of P is the set of all $a \in P$ such that a commutes with p for all $p \in P$. We recall the following result

LEMMA 1 [5, p. 254]. *The center Z of an orthomodular ordered set P is a Boolean algebra.*

An *orthomodular lattice* L is an orthocomplemented lattice which satisfies the orthomodular law, i.e. for all $a, b \in L$, if $a \leqq b$ and $a' \wedge b = 0$ then $a = b$.

Orthomodular lattices are generalizations of Boolean algebras. This fact is precisely described by the following result

LEMMA 2 [5]. *For an orthomodular ordered set P the following conditions are equivalent:*

(i) *P is a Boolean algebra*

(ii) *$P = Z$*

(iii) *P is a lattice and $x \wedge y = 0$ implies $x \leqq y'$.*

A (0, 1)-lattice homomorphism h from an orthomodular lattice L_1 to another L_2 is said to be an *ortho-homomorphism* if $h(a') = (h(a))'$ for all $a \in L_1$.

A *closure space* (X, C) is a non-empty set X and a mapping $C\colon P(X) \to P(X)$ satisfying the following conditions, for all $A, B \in P(X)$: C0) $C\emptyset = \emptyset$, C1) $A \subseteq CA$, C2) $A \subseteq B$ implies $CA \subseteq CB$, C3) $CCA = CA$. A subset A is *closed* if $CA = A$ and A is *open* if $-A$ is closed. The mapping $I\colon P(X) \to P(X)$ defined by $I = -C-$ satisfies the properties dual of those of C above, and A is open if $IA = A$. Let $\mathrm{CO}(X, C)$ be the family of all subsets A of X which are both closed and open. If C also satisfies C4) $C(A \cup B) = CA \cup CB$ the closure operator C is said to be *additive* and (X, C) is a topological space.

A non-empty subset F of P is *increasing* if $a \in F$ and $a \leqq b$ imply $b \in F$.

A *Z-filter* in P is a non-empty subset F of P such that [5, p. 255]

F1) F is increasing

F2) if $a, b \in F \cap Z$ then $a \wedge b \in F$.

In particular $[a) = \{b \in L\colon a \leqq b\}$ is a Z-filter. A Z-filter F is said to be *proper* if $a \in F$ implies $a' \notin F$. If $a \neq 0$ the Z-filter $[a)$ is proper. The kernel of an ortho-homomorphism is a proper Z-filter. If F is a proper Z-filter in an orthomodular ordered set P then $F \cap Z$ is a proper filter in Z, as a consequence of the definition.

If A is a subset of an orthomodular ordered set P we note $\langle A \rangle = \{x \in P\colon$ there is $a \in A$ such that $a \leqq x\}$. $\langle A \rangle$ is the increasing set generated by A.

LEMMA 3. *Let $\{F_i\}_{i \in I}$ be a family of Z-filters in an orthomodular ordered set P. Then the Z-filter generated by $\{F_i\}_{i \in I}$ denoted $\overline{(F_i)}_{i \in I}$ is equal to $\bigcup_i F_i \cup \langle \bigvee_i (F_i \cap Z) \rangle$, where $\bigvee_i (F_i \cap Z)$ is the filter in Z generated by the family $\{F_i \cap Z\}_{i \in I}$ of filters in Z.*

PROOF. Let $G = \bigcup_i F_i \cup \langle \bigvee_i (F_i \cap Z) \rangle$. We have $F_i \subseteq G$, for all i. Suppose $x \in \langle \bigvee_i (F_i \cap Z) \rangle$, so that there exists $a \in \bigvee_i (F_i \cap Z)$ such that $a \leqq x$. As $a \in \bigvee_i (F_i \cap Z)$ then there are $f_{i_1} \in F_{i_1} \cap Z, \ldots, F_{i_n} \in F_{i_n} \cap Z$ such that $f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_n} \leqq a \leqq x$. But $f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_n} \in \overline{(F_i)}_{i \in I}$ so $x \in \overline{(F_i)}_{i \in I}$. It remains to show that G is a Z-filter. The condition F1) is clearly satisfied. Suppose $a, b \in G \cap Z = \bigvee_i (F_i \cap Z)$. Then there are $f_{i_1} \in F_{i_1} \cap Z, \ldots, f_{i_n} \in F_{i_n} \cap Z$, and $f_{j_1} \in F_{j_1} \cap Z, \ldots, f_{j_m} \in F_{j_m} \cap Z$ such that $f_{i_1} \wedge \ldots$

$\ldots\wedge f_{i_n}\leqq a$ and $f_{j_1}\wedge\ldots\wedge f_{j_m}\leqq b$, so that $f_{i_1}\wedge\ldots\wedge f_{i_n}\wedge f_{j_1}\wedge\ldots\wedge f_{j_m}\leqq a\wedge b$. This yields $a\wedge b\in\bigvee_i (F_i\cap Z)\subseteq G$. Thus the lemma is proved.

A Z-filter F is a *Z-ultrafilter* if it is proper and if it is contained in no other proper Z-filter. By Zorn's lemma every proper Z-filter in an orthomodular ordered set P is contained in a Z-ultrafilter. If M is a Z-ultrafilter in P then $M\cap Z$ is an ultrafilter in Z.

A useful characterization of Z-ultrafilters is the following

LEMMA 4 [5, p. 256]. *For a proper Z-filter F in an orthomodular ordered set P the following conditions are equivalent:*

(i) *F is a Z-ultrafilter*

(ii) *for every $a\in P$, $a\notin F$ implies $a'\in F$.*

We end the present section by recalling the following statement

LEMMA 5 [5, p. 257]. *The family of all Z-ultrafilters in an orthomodular ordered set P is a separating family.*

3. Representation theory

Let L be a non-trivial orthomodular lattice. We now outline the construction of a representation of L by sets.

THEOREM 1. *Every orthomodular lattice L can be ortho-embedded in an orthomodular lattice of sets.*

PROOF. Let X be the family of all Z-ultrafilters in the orthomodular lattice L. We define the map $u\colon L\to P(X)$ in the following way: $a\mapsto u(a)=\{M\in X\colon a\in M\}$. The map u is an order isomorphism of L onto $\mathscr{P}=\{u(a)\colon a\in L\}$ such that $u(a')=-u(a)$ [5, p. 257]. In particular $u(0)=\emptyset$ and $u(1)=X$. Hence u is an ortho-isomorphism of L onto $\mathscr{P}=\{u(a)\colon a\in L\}$. Nevertheless $\mathscr{P}$ is not, in general, a sublattice of $P(X)$.

In order to characterize the orthomodular lattice $\mathscr{P}=\{u(a)\colon a\in L\}$ we are going to supply X with a closure operator C.

Following [3] we define for any $A\subseteq X$ the set $CA=\bigcap\{u(a)\colon A\subseteq u(a)\}$. The map $C\colon P(X)\to P(X)$ satisfies the axioms C0)—C3) above and the system (X, C) is a closure space. We claim that $\mathscr{P}\subseteq \mathrm{CO}(X, C)$. In fact suppose $u(a)\in\mathscr{P}$ so $Cu(a)=u(a)$. Since $-u(a)=u(a')\in\mathscr{P}$ we infer that $C-u(a)=-Iu(a)=-u(a)$, i.e. $Iu(a)=u(a)$. Hence $u(a)\in \mathrm{CO}(X, C)$.

REMARK 1. If G is an open set of (X, C) we have $G=-CF=-\bigcap\{u(a)\colon F\subseteq u(a)\}=\bigcup\{u(a')\colon u(a')\subseteq G\}$, i.e. every open set in (X, C) is a union of subsets of $\mathscr{P}$.

REMARK 2. The family $\mathscr{P}$ in Theorem 1 is an orthomodular lattice of clopen sets. This means that for every $A, B\in\mathscr{P}$ there exist $A\wedge B$ and $A\vee B$ which are clopen, i.e. that $I(A\cap B)$ and $C(A\cup B)$ are closed and open sets respectively and $A\wedge B=I(A\cap B)$, $A\vee B=C(A\cup B)$. In addition $\mathscr{P}$ satisfies the orthomodular law, that is, if $A\subseteq B$ and $I(-A\cap B)=\emptyset$ then $A=B$.

Using Lemma 2 we state

COROLLARY. *If L is a Boolean algebra, then Theorem* 1 *gives the Stone representation.*

The following facts are noted for future use.

LEMMA 6. *If a, b belong to the center Z of an orthomodular lattice L then $u(a\vee b)=u(a)\cup u(b)$, $u(a\wedge b)=u(a)\cap u(b)$ and C is additive on $\{u(a)\colon a\in Z\}$.*

LEMMA 7. *Let L be an orthomodular lattice, and (X, C) the associated representation closure space as above. The family $C(X, C)$ of all closed sets in (X, C) satisfies the following property:*

K) *If $\{C_i\}_{i\in I}$ is a family of closed sets in (X, C) with $\bigcap_i C_i=\emptyset$ then there exist $i_1, \ldots, i_n\in I$ such that $C_{i_1}\cap\ldots\cap C_{i_n}=\emptyset$.*

PROOF. Since each closed set C_i is a meet of subsets of $\mathcal{P}$ it is enough to show that if $\bigcap_i u(a_i)=\emptyset$ then there exist $a_{i_1}, \ldots, a_{i_n}$ such that $u(a_{i_1})\cap\ldots\cap u(a_{i_n})=\emptyset$. Assume $\bigcap_i u(a_i)=\emptyset$. This means that there is no $M\in X$ such that $M\in u(a_i)$, for all $i\in I$, i.e. such that $[a_i)\subseteq M$, for all $i\in I$. Hence the Z-filter $\overline{([a_i))}$ is not proper. By Lemma 3 this means that there exists x such that

$$x, x'\in\bigcup_i [a_i)\cup\langle\bigvee_i ([a_i)\cap Z)\rangle.$$

Case 1. If $x, x'\in\bigcup_i [a_i)$ then there are a_i, a_j with $a_i\leqq x$ and $a_j\leqq x'\leqq a_i'$. We obtain $u(a_i)\cap u(a_j)\subseteq u(a_i)\cap u(a_i')=u(a_i)\cap -u(a_i)=\emptyset$ and the result follows.

Case 2. If $x, x'\in\langle\bigvee_i ([a_i)\cap Z)\rangle$ there are $b, c\in\bigvee_i ([a_i)\cap Z)$ such that $b\leqq x$ and $c\leqq x'$. Since $b, c\in\bigvee_i ([a_i)\cap Z)$ we infer that there are $b_{i_1}\in[a_{i_1})\cap Z, \ldots, b_{i_n}\in[a_{i_n})\cap Z$ and $c_{j_1}\in[a_{j_1})\cap Z, \ldots, c_{j_m}\in[a_{j_m})\cap Z$ with $b_{i_1}\wedge\ldots\wedge b_{i_n}\leqq b\leqq x$ and $c_{j_1}\wedge\ldots\wedge c_{j_m}\leqq c\leqq x'\leqq b'$. This yields $u(a_{i_1})\cap\ldots\cap u(a_{i_n})\cap u(a_{j_1})\cap\ldots\cap u(a_{j_m})\subseteq u(b_{i_1})\cap\ldots\cap u(b_{i_n})\cap u(c_{j_1})\cap\ldots\cap u(c_{j_m})=u(b_{i_1}\wedge\ldots\wedge b_{i_n}\wedge c_{j_1}\wedge\ldots\wedge c_{j_m})\subseteq u(b\wedge b')=u(0)=\emptyset$ which was to be proved.

Case 3. Assume $x\in\bigcup_i [a_i)$ and $x'\in\langle\bigvee_i ([a_i)\cap Z)\rangle$. There are $a_{i_0}\in\{a_i\}$ and $b_{i_1}\in[a_{i_1})\cap Z, \ldots, b_{i_n}\in[a_{i_n})\cap Z$ such that $a_{i_0}\leqq x$ and $b_{i_1}\wedge\ldots\wedge b_{i_n}\leqq x'$. Thus $u(a_{i_0})\subseteq u(x)$ and $u(a_{i_1})\cap\ldots\cap u(a_{i_n})\subseteq u(b_{i_1}\wedge\ldots\wedge b_{i_n})\subseteq u(x')\subseteq u(a_{i_0}')$. We obtain $u(a_{i_0})\cap u(a_{i_1})\cap\ldots\cap u(a_{i_n})\subseteq u(a_{i_0})\cap u(a_{i_0}')=\emptyset$. This completes the proof.

LEMMA 8. *Let L be an orthomodular lattice and $u\colon L\to \mathrm{CO}(X, C)$ the embedding considered in Theorem* 1. *Every clopen set A in (X, C) is a finite meet $(\wedge)$ of members of $\mathcal{P}=\{u(a)\colon a\in L\}$ i.e. A belongs to $\mathcal{P}$.*

PROOF. Suppose $A\in\mathrm{CO}(X, C)$. Because A is closed, $A=\bigcap_i A_i$ with $A_1=u(a_i)$. By assumption A is also open so $A=\bigcup_j B_j$, $B_j=u(b_j)$, by Remark 1. Hence $\emptyset=A\cap -A=\bigcap_i A_i\cap\bigcap_j -B_j$. By Lemma 7 there are $i_1, \ldots, i_n$, $j_1, \ldots, j_m$

such that $A_{i_1}\cap\ldots\cap A_{i_n}\cap-(B_{j_1}\cup\ldots\cup B_{j_m})=\emptyset$ so $A_{i_1}\cap\ldots\cap A_{i_n}\subseteqq B_{j_1}\cup\ldots\cup B_{j_m}\subseteqq A\subseteqq$ $\subseteqq A_{i_1}\cap\ldots\cap A_{i_n}$. We infer $A=A_{i_1}\cap\ldots\cap A_{i_n}$. Thus $A=IA=I(u(a_{i_1})\cap\ldots\cap u(a_{i_n}))=$ $=u(a_{i_1})\wedge\ldots\wedge u(a_{i_n})\in\mathcal{P}$ by Remark 2. This completes the proof.

LEMMA 9. *Let L be an orthomodular lattice and (X,C) the closure space as above. The family $\mathrm{CO}(X,C)$ of all clopen sets in (X,C) satisfies the following property:*

P) *If $\{A_i\}_{1\leqq i\leqq n}$ is a family of clopen sets and $A_1\cap A_2\cap\ldots\cap A_n=\emptyset$ then one of the following conditions holds:*

(i) *there are $1\leqq i,j\leqq n$ such that $A_i\cap A_j=\emptyset$,*

(ii) *there are $1\leqq i_0\leqq n$ and $B_{i_1},\ldots,B_{i_p}\in Z(\mathrm{CO}(X,C))$ such that $A_{i_1}\subseteqq B_{i_1},\ldots,$ $A_{i_p}\subseteqq B_{i_p}$ and $A_{i_0}\cap B_{i_1}\cap\ldots\cap B_{i_p}=\emptyset$.*

PROOF. Assume $u(a_1)\cap\ldots\cap u(a_n)=\emptyset$. This implies that there is no $M\in X$ such that $M\in u(a_i)$ for $1\leqq i\leqq n$, i.e. such that $[a_i)\subseteqq M$ for all $1\leqq i\leqq n$. Hence the Z-filter $\overline{([a_1),\ldots,[a_n))}$ is not proper. This means that there exists x such that $x,x'\in\overline{([a_1),\ldots,[a_n))}=\bigcup_{i=1}^{n}[a_i)\cup\langle\bigvee_{i=1}^{n}([a_i)\cap Z)\rangle$. It is straightforward to complete the proof.

The results above suggest the following definitions. An abstract closure space $S=(X,C)$ is said to be *compact* if the family $C(X,C)$ of all closed sets in (X,C) satisfies the property K) in Lemma 7. S is a *Hausdorff* closure space if for any $x,y\in X$, $x\neq y$, there exist open sets A, B such that $x\in A$, $y\in B$ and $A\cap B=\emptyset$.

Let $S=(X,C)$ be a closure space. A family $\mathcal{B}\subseteqq P(X)$ is called a *base* for C if each closed set of (X,C) is the intersection of members of $\mathcal{B}$.

If L is an orthomodular lattice, the closure space $S(L)=(X,C)$ is called the *dual closure space*. Let $L(S(L))$ be its base, ordered by inclusion.

An abstract closure space $S=(X,C)$ is called an *orthomodular closure space* if 1) S is a compact Hausdorff closure space, 2) the family $L(S)$ of all clopen subsets of S, ordered by inclusion, is an orthomodular lattice and C is additive on the center $Z(L(S))$, 3) the family $L(S)$ is a base satisfying property P) (in Lemma 9).

If S is an orthomodular closure space, the operations on $L(S)$ are given by the equalities $A\wedge B=I(A\cap B)$, $A\vee B=C(A\cup B)$ and $-A$ is the complement of A. $L(S)$ is called the *dual orthomodular lattice* of $S(L)$.

Summing up the results above and in view of these definitions we can formulate

THEOREM 2. *If L is an orthomodular lattice, $S(L)$ is an orthomodular closure space and the map $a\mapsto u(a)$ is an orthoisomorphism between L and the dual orthomodular lattice $L(S(L))$ of its dual closure space $S(L)$.*

4. Duality theory

If S_1 and S_2 are orthomodular closure spaces a map $f\colon S_1\to S_2$ is called *C-continuous* if the inverse image of each set in $L(S_2)$ is in $L(S_1)$. A bijection $f\colon S_1\to S_2$ is said to be a *closure isomorphism* if f and f^{-1} are C-continuous.

The next theorem gives a characterization of the representation closure space of an orthomodular lattice.

THEOREM 3. *If S is an orthomodular closure space then $L(S)$ is an orthomodular lattice and S is closure isomorphic to $S(L(S))$.*

PROOF. Let $S(L(S))$ be the dual space of the orthomodular lattice $L(S)$. For each $x \in X$ define $\Phi(x)=\{A \in L(S): x \in A\}$. Clearly $\Phi(x)$ is an increasing subset of $L(S)$. Let $A, B \in \Phi(x) \cap Z(L(S))$ so $x \in A \cap B = A \wedge B$ by the additivity of C on the center; hence $\Phi(x)$ is a Z-filter in $L(S)$. $\Phi(x)$ is maximal since $A \notin \Phi(x)$ implies $x \notin A$, with $A \in L(S)$, that is $x \in -A$ and $-A \in L(S)$; hence $-A \in \Phi(x)$ and $\Phi: S \to S(L(S))$. To show Φ is onto let $M_0=\{A_i\}_{i \in I}$ be a Z-ultrafilter in $L(S)$ and consider $\bigcap_{A_i \in M_0} A_i$. We have $\bigcap_i A_i \neq \emptyset$; otherwise by compactness there are $i_1, \dots, i_n \in I$ such that $A_{i_1} \cap \dots \cap A_{i_n} = \emptyset$. To avoid a cumbersome notation we write $A_1 \cap \dots \cap A_n = \emptyset$. By hypothesis $L(S)$ satisfies property P). If there are $1 \leqq p, q \leqq n$ such that $A_p \cap A_q = \emptyset$ we infer $A_p \subseteqq -A_q$. Hence $-A_q \in M_0$ and M_0 is not proper, a contradiction. Suppose there are A_{k_0} and $B_{k_1}, \dots, B_{k_r} \in Z(L(S))$ such that $A_{k_1} \subseteqq B_{k_1}, \dots, A_{k_r} \subseteqq B_{k_r}$ with $A_{k_0} \cap B_{k_1} \cap \dots \cap B_{k_r} = \emptyset$, i.e. $A_{k_0} \subseteqq -B_{k_1} \cup \dots \cup -B_{k_r} \in M_0$. But M_0 is an ultrafilter in $Z(L(S))$ so there is k_s, with $1 \leqq s \leqq r$, such that $-B_{k_s} \in M_0$ and M_0 is not proper, an impossibility. These contradictions imply that there is $x \in \bigcap_i A_i$ so $A_i \in \Phi(x)$ for all i and $M_0 \subseteqq \Phi(x)$. By maximality $M_0 = \Phi(x)$. Φ is one to one because S is a Hausdorff closure space and $L(S)$ is a base for C. Finally let $u: L(S) \to S(L(S))$ be as in Theorem 1. Suppose $A \in L(S)$. Then $x \in A$ means that $A \in \Phi(x)$. Since $\Phi(x)$ is a Z-ultrafilter this is equivalent to $\Phi(x) \in u(A)$. Thus $\Phi(A) = u(A)$. This implies that Φ and Φ^{-1} are C-continuous. This completes the proof.

The point of the present approach is exhibited by the following result.

COROLLARY. *If L is a Boolean algebra, $S(L)$ becomes the Stone space of L.*

FINAL REMARK. In Section 3 we may take the family $\mathscr{P}=\{u(a): a \in L\}$ as a subbase for a topology τ on X. But the topological space (X, τ) obtained in this way does not characterize, in general, the orthomodular lattice L, i.e. there exist non-ortho-isomorphic orthomodular lattices which provide the same topological space. For instance let $MO2=\{0, a, b, a', b', 1\}$ be the orthomodular lattice with four atoms and B the Boolean algebra with four atoms. They are not ortho-isomorphic because they do not have the same number of elements. In both cases X contains four points and the associated topology τ is the discrete one. We conclude that the topological spaces are homeomorphic while the orthomodular lattices are not ortho-isomorphic.

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(Received December 9, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 153—155.

AN INDIVIDUAL ERGODIC THEOREM FOR SUPERADDITIVE PROCESSES

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1. Introduction

Let T be an invertible positive operator on L_1 of a σ-finite measure space. Under certain norm conditions on T^n, we shall prove an individual ergodic theorem for superadditive processes with respect to T.

2. The theorem

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T a positive linear operator on $L_1(\mu)=L_1(X, \mathcal{F}, \mu)$. A *superadditive process* (with respect to T) is a sequence $\{F_n\}_{n\geqq 1}$ of functions in $L_1(\mu)$ such that $F_{n+k}\geqq F_n+T^n F_k$ for all $n, k\geqq 1$. Applying Ito's theorem [6] and Akcoglu—Sucheston's theorem [2], which is a generalization of Kingman's deep theorem [8], it may be readily seen that if T is a Markovian (i.e. $\int Tf\,d\mu=\int f\,d\mu$ for all $f\in L_1(\mu)$) and satisfies the L_1-mean ergodic theorem, then for any superadditive process $\{F_n\}_{n\geqq 1}$, with

$$\sup_n \frac{1}{n}\int F_n\,d\mu <\infty,$$

the pointwise limit $\lim\limits_n \frac{1}{n} F_n$ exists a.e. on X. It is well known that this assertion does not necessarily hold if the hypothesis that T satisfies the L_1-mean ergodic theorem is not assumed (see e.g. [3]). On the other hand, by a theorem of Akcoglu—Chacon [1], the hypothesis may be replaced by $\|T\|_p\leqq 1$ for some $1<p\leqq\infty$, where $\|T\|_p=\|T\|_{L_p(\mu)}$ denotes the operator norm of T as an operator on $L_p(\mu)$. In this note, however, we do not assume T Markovian, nor $\|T\|_p\leqq 1$ for some $1<p\leqq\infty$. Instead of these conditions we assume T invertible and power bounded as an operator on $L_1(\mu)$ and $L_p(\mu)$, respectively. The theorem which we are going to prove is as follows.

THEOREM. *Let T be an invertible positive operator on $L_1(\mu)$, where μ is a σ-finite measure, such that*

(1) $$\sup_{-\infty<n<\infty} \|T^n\|_1 = K_1 <\infty,$$

and also such that for some $1<p\leqq\infty$

(2) $$\sup_{-\infty<n<\infty} \|T^n\|_p = K_p <\infty.$$

Let $\{F_n\}_{n\geqq 1}$ *be a superadditive process in* $L_1(\mu)$ *such that*

$$\sup_{n\geqq 1}\frac{1}{n}\int F_n\,d\mu=\gamma<\infty. \tag{3}$$

Then $\lim_n \frac{1}{n}F_n$ *exists a.e. on* X.

PROOF. Let LIM denote a Banach limit (see e.g. [4]), and define

$$m(A)=\mathrm{LIM}\Big(\int T^n 1_A\,d\mu\Big)\quad\text{for}\quad A\in\mathcal{F}\quad\text{with}\quad \mu(A)<\infty,$$

where 1_A denotes the indicator function of A. When $\mu(A)=\infty$, choosing a sequence $\{A_i\}_{i\geqq 1}$ of sets in $\mathcal{F}$ so that $\mu(A_i)<\infty$, $A_i\subset A_{i+1}$ and $\lim_i A_i=A$, we define $m(A)=$ $=\lim_i m(A_i)$. It is then easily checked that m is a σ-finite measure on $\mathcal{F}$ satisfying

$$\frac{1}{K_1}\mu\leqq m\leqq K_1\mu.$$

To see that T is a Markovian operator on $L_1(m)$, let $\mu(A)<\infty$ and $\varepsilon>0$. Take a simple function $h=\sum_{i=1}^{k}a_i 1_{E_i}$ with $E_i\cap E_j=\emptyset$ for $i\neq j$, so that $T1_A\geqq h$ and $\int(T1_A-h)\,d\mu<\varepsilon$. Then we obtain

$$m(A)=\mathrm{LIM}\Big(\int T^n 1_A\,d\mu\Big)=\mathrm{LIM}\Big(\int T^{n+1}1_A\,d\mu\Big)\geqq$$

$$\geqq \mathrm{LIM}\Big(\int T^n h\,d\mu\Big)=\int h\,dm\geqq \mathrm{LIM}\Big(\int T^{n+1}1_A\,d\mu-K_1\varepsilon\Big)=m(A)-K_1\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this proves that $m(A)=\int T1_A\,dm$. By an approximation argument, T is a Markovian operator on $L_1(m)$.

Next, let us fix an r with $1<r<p$. We shall consider T as an operator on $L_r(m)$. First, by the Riesz convexity theorem, T may be regarded as an operator on $L_r(\mu)$ satisfying

$$\sup_{-\infty<n<\infty}\|T^n\|_{L_r(\mu)}=K_r<\infty.$$

Then we have

$$\frac{1}{K_1}\int|f|^r\,d\mu\leqq\int|f|^r\,dm\leqq K_1\int|f|^r\,d\mu$$

for any function f on X, and thus

$$\sup_{-\infty<n<\infty}\|T^n\|_{L_r(m)}\leqq K_1^{2/r}K_r.$$

Hence, by [11] (see also Remarks 3.1 in [7]), if we set

$$f^*(x)=\sup_{n\geqq 1}\frac{1}{n}\sum_{i=0}^{n-1}|T^i f(x)|\quad\text{for}\quad f\in L_r(m)$$

then $f^*\in L_r(m)$. This together with standard arguments (see e.g. the proof of Corollary in [5]), shows that for any $f\in L_r(m)$ $(=L_r(\mu))$ the pointwise limit $\lim_n\frac{1}{n}\sum_{i=0}^{n-1}T^i f$ exists a.e. on X.

Lastly, to complete the proof, take any $0<f\in L_1(m)\cap L_r(m)$. Since

$$\frac{1}{n}F_n=\left(\frac{1}{n}\sum_{i=0}^{n-1}T^if\right)\frac{F_n}{\sum_{i=0}^{n-1}T^if}\quad \text{on } X,$$

it is enough to show the almost everywhere convergence of $F_n/(\sum_{i=0}^{n-1}T^if)$. Here we may assume without loss of generality that $F_n\geqq 0$ for all $n\geqq 1$ (see e.g. Introduction in [2]). Then we have $\sup_{n\geqq 1}\frac{1}{n}\int F_n\,dm\leqq K_1\gamma<\infty$, and hence by Akcoglu—Sucheston's ratio ergodic theorem [2] for superadditive processes, the almost everywhere convergence of $F_n/(\sum_{i=0}^{n-1}T^if)$ follows. The proof is complete.

REMARK. The theorem holds even if the norm condition (2) is replaced by the following:

$$\text{(4)}\qquad \sup_{n\geqq 1}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^i\right\|_\infty<\infty.$$

In fact, it is known (see e.g. [9], p. 420) that if a positive operator T satisfies $\|T\|_1\leqq 1$ and (4), then for any $f\in L_p$, with $1\leqq p<\infty$, the almost everywhere convergence of the average $\frac{1}{n}\sum_{i=0}^{n-1}T^if$ follows. Using this result and the above argument, the remark may be readily checked. It should be noted here that this generalizes the corollary in [10], where T was assumed to be an operator on $L_1(\mu)$, with μ finite.

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(Received December 20, 1983; revised July 26, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 157—160.

A DIRECT DEFINITION OF DISTRIBUTIVE EXTENSIONS OF PARTIALLY ORDERED ALGEBRAS

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It is shown that distributive extensions of partially ordered algebras can be defined without any reference to the Priestley-spaces of the distributive lattices in question. Namely, given a *partially ordered algebra*

$$\mathfrak{A} = (A; F, \leqq)^{*}$$

and a bounded distributive lattice

$$\mathfrak{D} = (D; \wedge, \vee, 0, 1)$$

let $A[D]$, the underlying set of the extension $\mathfrak{A}[\mathfrak{D}]$ consist of all functions $\xi\colon A \to D$ satisfying

(1) the range of ξ is finite, i.e. $|\xi(A)|$ is finite;

(2) $\xi(a) \wedge \xi(b) = \bigvee_{c \geqq a, b} \xi(c)$ for all $a, b \in A$;

(3) $\bigvee_{a \in A} (\xi(a) \setminus \bigvee_{b > a} \xi(b)) = 1$ in the Boolean algebra generated by $\mathfrak{D}$ (here $x \setminus y$ means $x \wedge y'$ and the empty join is 0).

The operations are defined as follows: for any n-ary operational symbol μ from the type of $\mathfrak{A}$ ($n > 0$), for any $\xi_1, \ldots, \xi_n \in A[D]$ let $\mu(\xi_1, \ldots, \xi_n)$ be the map $A \to D$ for which

$$\mu(\xi_1, \ldots, \xi_n)(a) = \bigvee_{\substack{b_1, \ldots, b_n \in A \\ \mu(b_1, \ldots, b_n) \geqq a}} (\xi_1(b_1) \wedge \ldots \wedge \xi_n(b_n)) \tag{4}$$

for any $a \in A$. If μ is 0-ary, and takes the value $d \in A$ in $\mathfrak{A}$, then let μ assign in $A[D]$ the function $\xi\colon A \to D$ carrying d to 1 and the other elements of A to 0. The partial order in $\mathfrak{A}[\mathfrak{D}]$ is taken componentwise, i.e. $\xi \leqq \eta$ iff $\xi(a) \leqq \eta(a)$ in $\mathfrak{D}$ for every $a \in A$.

The definition contains that of Boolean extensions of algebras as a special case when $\mathfrak{D}$ is Boolean and the order of $\mathfrak{A}$ is trivial (i.e., the equality).

REMARK. The idea of working out such a definition is due to A. P. Huhn. The result was announced at the Czechoslovakian Summer School, Kroměříž, 8 September, 1980.

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* The operations in F are assumed to be isotone with respect to $\leqq$.

MOTIVATIONS. The notion of the extension of a (universal) algebra by a Boolean algebra was introduced in the fifties by A. L. Foster [2] (see also Grätzer [3], Burris [1]). Foster's definition is the following: let

$$\mathfrak{A} = (A; F)$$

be an algebra, $\mathfrak{B}=(B, \wedge, \vee, ', 0, 1)$ a Boolean algebra. For the extension $\mathfrak{A}[\mathfrak{B}]= =(A[B]; F)$ the underlying set $A[D]$ is exactly the set of all mappings $\xi\colon A \to B$ with the following properties:

(1_0) $|\xi(A)|$ is finite;

(2_0) $\xi(a) \wedge \xi(b) = 0$ if $a, b \in A$, $a \neq b$;

(3_0) $\bigvee\limits_{a \in A} \xi(a) = 1$ in $\mathfrak{B}$.

For n-ary ($n>0$) operations μ from the type of $\mathfrak{A}$,

(4_0) $$\mu(\xi_1, \ldots, \xi_n)(a) = \bigvee_{\substack{b_1, \ldots, b_n \in A \\ \mu(b_1, \ldots, b_n)=a}} (\xi_1(b_1) \wedge \ldots \wedge \xi_n(b_n))$$

($a \in A$, $\xi_1, \ldots, \xi_n \in A[B]$); the 0-ary operations are defined just as after (4). (Note that Burris denotes this construction by $\mathfrak{A}[\mathfrak{B}]^*$ and he means by $\mathfrak{A}[\mathfrak{B}]$ a little bit more general notion: if $\mathfrak{B}$ is complete, then (1_0) is omitted and infinite joins are also allowed in (3_0) and (4_0).)

There is another possibility for introducing Boolean extensions. Namely, it is well known that every Boolean algebra can be represented as the Boolean algebra of all clopen subsets of its Stone space. The Stone space of a Boolean algebra $\mathfrak{B}$ consists of all dual prime ideals of $\mathfrak{B}$, and all sets of form $\mathscr{P}_d = \{P \mid P$ is a dual prime of $\mathfrak{B}$ and $d \in P\}$ (where $d \in \mathfrak{B}$ is arbitrarily fixed) and the complements of these sets give a subbase (in fact, a base) for the topology (M. H. Stone [7]). Let us denote this space by $\mathscr{S}_{\mathfrak{B}}$: then we can consider all continuous functions $f\colon \mathscr{S}_{\mathfrak{B}} \to \mathfrak{A}$ (with respect to the discrete topology on $\mathfrak{A}$), they form a subalgebra of the power $\mathfrak{A}^{S_{\mathfrak{B}}}$, and it is isomorphic to $\mathfrak{A}[\mathfrak{B}]$. An isomorphism is given by the correspondence $f \to \xi_f$, where $\xi_f \in \mathfrak{A}[\mathfrak{B}]$ having the property

(5) $$\xi_f(a) = f^{-1}(a) \quad \text{for every} \quad a \in A$$

(here for any $d \in B$, d is identified with $\mathscr{P}_d$, this identification gives the canonical representation isomorphism of $\mathfrak{B}$ with the lattice of all clopen subsets of $\mathscr{S}_{\mathfrak{B}}$ (the latter is called the dual lattice of $\mathscr{S}_{\mathfrak{B}}$)).

But bounded distributive lattices have also representation spaces, the so-called Priestley-spaces; these are defined in the same way as in the case of Boolean algebras with the further specification that this space is partially ordered under set inclusion. The canonical representation isomorphism is the same as mentioned at the end of the previous paragraph. (For the details consult H. A. Priestley [4], [5].)

Now, it is clear that the second definition of $\mathfrak{A}[\mathfrak{B}]$ above can be generalized to partially ordered algebras and distributive lattices by considering all continuous monotone mappings from the Priestley-space of $\mathfrak{D}$ into $\mathfrak{A}$ (the latter is endowed with discrete topology). (This idea seems to go back to E. T. Schmidt [6].) Then the question naturally arises, whether one can construct a definition for distributive extension like that of Boolean extensions listed in (1_0)—(4_0) (i.e., a *direct* definition, not using representation spaces), so that the correspondence under (5) gives an isomorphism between the two constructions. We shall see that the definition in the abstract fits, even it generalizes the direct definition of Boolean extensions.

THE JUSTIFICATION OF THE DEFINITION. Let $\mathfrak{D}$ be a bounded distributive (in what follows, 0, 1-distributive) lattice, $(X; \mathcal{T}, \leqq)$ its Priestley-space as defined above (with X being the set of all dual prime ideals of $\mathfrak{D}$, $\mathcal{T}$ being the topology mentioned and $\leqq$ the set inclusion between elements of X), $\mathfrak{A}$ an ordered algebra. Let us identify $\mathfrak{D}$ with the lattice of all clopen increasing subsets of $(X; \mathcal{T}, \leqq)$. Given an arbitrary continuous monotone map $f\colon (X, \mathcal{T}, \leqq)\to\mathfrak{A}$, we want to construct a function $\xi_f\colon A\to D$. Of course, its values cannot be determined as the f-inverse images of the one-element sets, because these images are not increasing in general (although they are clopen), and so do not belong to $\mathfrak{D}$. But we get elements of $\mathfrak{D}$ by taking the f-inverse images of increasing subsets of $\mathfrak{A}$; and these sets completely determine f, because for any $a\in A, f^{-1}(a)=f^{-1}([a))\setminus f^{-1}([a)\setminus\{a\})$ (here $[x)=\{y|y\geqq x\}$). Furthermore, it is sufficient to know the f-inverse images of all sets of form $[t)$ since for any increasing subset A_0 of A we have $A_0=\bigcup\{[b)|b\in A_0\}$. Now consider the sets $f^{-1}([a))$, there are only finitely many of them (even if A is infinite!), because the range of f, $\mathcal{R}(f)$ is finite, being f a continuous map of a compact space into a discrete, and so $f^{-1}([a))=f^{-1}([a)\cap\mathcal{R}(f))$. It is easy to see that

$$(6)\qquad f^{-1}([a))\cap f^{-1}([b)) = \bigcup_{c\geqq a,b} f^{-1}([c)) \quad \text{for every } a, b\in A \text{ and}$$

$$\bigcup_{a\in A}\Big(f^{-1}([a))\setminus\bigcup_{b>a} f^{-1}([b))\Big) = X.$$

Note that the Boolean lattice of all clopen subsets of X can be viewed as Boolean algebra generated by $\mathfrak{D}$, introducing the operations of taking complements and assigning $\emptyset$ and X as 0 and 1, respectively; see Priestley [5]. Define the function ξ_f associated with f as the map $A\to D$ satisfying

$$(7)\qquad \xi_f(a) = f^{-1}([a)),\quad a\in A.$$

(7) trivially turns into (5), if $\mathfrak{A}$ is trivially ordered and $\mathfrak{D}$ is Boolean (lattice or algebra).

With respect to (6), the remark preceding it and (7), we introduce a

DEFINITION. The extension of $\mathfrak{A}$ by $\mathfrak{D}$ is the partially ordered algebra $\mathfrak{A}[\mathfrak{D}]= =(A[D]; F, \leqq)$ described in the introduction.

THEOREM. *The definition is correct, and the correspondence $f\mapsto\xi_f$ is an isomorphism between the subalgebra of $\mathfrak{A}$ consisting of all continuous monotone maps $f\colon (X, \mathcal{T}, \leqq)\to\mathfrak{A}$ and the $\mathfrak{A}[\mathfrak{D}]$ in our previous direct definition.*

PROOF. The joins in (2) and (3) as well as in (4) are always finite (because of (1)), so the definition is correct.

Let φ denote the correspondence $f\mapsto\xi_f$. First we shall see that φ is onto. For this, let ξ be any function satisfying (1)—(3), and consider the map $f\colon (X, \mathcal{T}, \leqq)\to\mathfrak{A}$ for which

$$(8)\qquad f^{-1}(a) = \xi(a)\setminus\bigcup_{b>a}\xi(b),\quad a\in A.$$

f is well-defined: If $a<b$, then $f^{-1}(a)\subseteq\xi(a)\setminus\xi(b)$, $f^{-1}(b)\subseteq\xi(b)$, so $f^{-1}(a)\cap\cap f^{-1}(b)=\emptyset$. If $a\|b$ and still $x\in f^{-1}(a)\cap f^{-1}(b)$, then we would also have $x\in\xi(a)\cap\xi(b)$, so by (2), for suitable element c ($c\geqq a, b$), $x\in\xi(c)$. But $c\neq a$ or $c\neq b$, for example, $c>a$, and then $f^{-1}(a)\subseteq\xi(a)\setminus\xi(c)$, a contradiction, because $x\in f^{-1}(a)$. Therefore, in the case of $a\neq b$, $f^{-1}(a)\cap f^{-1}(b)=\emptyset$ always holds.

f is defined everywhere: By (8) and (3), the sets $f^{-1}(a)$ cover X.

f is continuous: Trivial, since the $\xi(a)$'s are clopen and the union in (8) is finite.

f is monotone: From (8) we see that $\xi(a)=f^{-1}([a))$, and from this the assertion follows easily; we also see that $\xi=\xi_f$, proving that φ is onto.

Now let $f_1, \dots, f_n$ be $(X, \mathcal{T}, \leqq)\to\mathfrak{A}$ continuous monotone functions $(n>0)$, μ an n-ary operation, $f=\mu(f_1, \dots, f_n)$. Then by the monotonicity of μ

$$f^{-1}([a)) = (\mu(f_1, \dots, f_n))^{-1}([a)) = \bigcup_{\mu(b_1, \dots, b_n)\geqq a} (f_1^{-1}(b_1)\cap\dots\cap f_n^{-1}(b_n)) =$$

$$= \bigcup_{\mu(b_1, \dots, b_n)\geqq a} (f_1^{-1}([b_1))\cap\dots\cap f_n^{-1}([b_n))).$$

This shows that φ preserves operations (the case of 0-ary operations is left to the reader).

Finally, $f\leqq g$ is equivalent to $f^{-1}([a))\subseteq g^{-1}([a))$ for every $a\in A$, i.e. $\xi_f(a)\leqq\leqq\xi_g(a)$, which is $\xi_f\leqq\xi_g$, and this means that φ is an order-isomorphism, too. The proof is complete.

The remark preceding the definition establishes that we indeed generalize Foster's definition concerning Boolean extensions.

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(Received December 27, 1983)

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Acta Math. Hung.
47 (1—2) (1986), 161—163.

AN INEQUALITY FOR THE DIRICHLET DISTRIBUTION

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1. Introduction and notations

The classical Chebyshev inequality and its generalizations (Bernstein, Kolmogorov, Rényi—Hajek inequalities etc.) are useful tools for proving weak and strong laws of large numbers. There is a considerable literature on this topic. Using different techniques, Berge [1], Lal [3], Marshal [4], Olkin [4], [5] and Pratt [5] investigated the multivariate generalization possibilities of the Chebyshev inequality. All of these inequalities are valid for wide classes of the probability measures, for example, for the class of all probability measures having second moments. However, the bounds for some important multivariate distributions seem to be of interest for the applications, especially, for the optimization problems. In this paper one such bound is given for the Dirichlet distribution. The Dirichlet distribution is defined by the following density:

$$f(x_1, \ldots, x_n) = \frac{\Gamma(v_1+\ldots+v_{n+1})}{\Gamma(v_1)\Gamma(v_2)\ldots\Gamma(v_{n+1})} x_1^{v_1-1}\ldots x_n^{v_n-1}(1-x_1-\ldots-x_n)^{v_{n+1}-1}$$

if

$$x\in[S_n] = \left\{x\colon\ x = (x_1, \ldots, x_n),\ x_i \geqq 0,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i \leqq 1\right\}$$

and $f(x_1, \ldots, x_n)=0$ otherwise. $v_1, \ldots, v_{n+1}$ are positive parameters. Following Wilks [7] we use the notation $D(x_1, \ldots, x_n;\ v_1, \ldots, v_n;\ v_{n+1})$ for the Dirichlet distribution function, i.e.,

(1)
$$D(x_1, \ldots, x_n;\ v_1, \ldots, v_n;\ v_{n+1}) =$$

$$= \frac{\Gamma(v_1+\ldots+v_{n+1})}{\Gamma(v_1)\Gamma(v_2)\ldots\Gamma(v_{n+1})} \underset{\substack{0\leqq y_i\leqq x_i,\, i=1,\ldots,n\\ \sum_{i=1}^{n} y_i\leqq 1}}{\int\ldots\int} y_1^{v_1-1}\ldots y_n^{v_n-1}(1-y_1-\ldots-y_n)^{v_{n+1}-1}\, dy_1\ldots dy_n.$$

2. Some auxiliary lemmas

We make the following change of variables in (1):

(2)
$$y_1 = z_1,\ y_2 = z_2(1-z_1),\ \ldots,\ y_n = z_n(1-z_1)\ldots(1-z_{n-1}).$$

LEMMA 1. *For* (2) *the following relation holds:*

$$1-y_1-\ldots-y_i = (1-z_1)\ldots(1-z_i),\quad i = 1, \ldots, n.$$

PROOF. When $i=1$ the statement is trivial since $1-y_1=1-z_1$. Suppose that the lemma is true for $i\leqq k$, and we prove it for $k+1$.

$$1-y_1-\ldots-y_{k+1}=(1-z_1)\ldots(1-z_k)-y_{k+1}=$$
$$=(1-z_1)\ldots(1-z_k)-z_{k+1}(1-z_1)\ldots(1-z_k)=(1-z_1)\ldots(1-z_k)(1-z_{k+1}),$$

hence the lemma is proved.

LEMMA 2. *The change of variables* (2) *of the integration maps*

$$S_n=\left\{y\colon y=(y_1,\ldots,y_n),\ y_i>0,\ i=1,\ldots,n,\ \sum_{i=1}^n y_i<1\right\}$$

onto

$$I_n=\{z\colon z=(z_1,\ldots,z_n),\ 0<z_i<1,\ i=1,\ldots,n\}.$$

PROOF. If $(y_1,\ldots,y_n)\in S_n$, then $y_i>0, i=1,\ldots,n, \sum_{i=1}^n y_i<1$. This implies that $0<y_i<1$, $i=1,\ldots,n$. Thus, we have $0<z_1<1$ and it is easy to see that $z_i>0$, $i=2,\ldots,n$ by induction. We need to prove that $z_i<1, i=2,\ldots,n$. Indeed, otherwise there exists $k>1$ such that $z_i<1$, for $i<k$, and $z_k>1$. This fact implies the existence of such a point $y=(y_1,\ldots,y_n)$ for which we have $y_k>1$. This is a contradiction to $(y_1,\ldots,y_n)\in S_n$. Hence, $z_i<1$, $i=1,\ldots,n$. In a similar way it is easy to see that each $z\in I_n$ is an image. Hence the lemma is proved.

LEMMA 3. *The change of variables* (2) *of integration maps*

$$\left\{y\colon y=(y_1,\ldots,y_n),\ 0<y_i<x_i,\ i=1,\ldots,n,\ \sum_{i=1}^n y_i<1\right\}$$

onto

$$\left\{z\colon z=(z_1,\ldots,z_n),\ 0<z_1<x_1,\ 0<z_i<\right.$$
$$\left.<\min\left\{\frac{x_i}{(1-z_1)\ldots(1-z_{i-1})},1\right\},\ i=2,\ldots,n\right\}.$$

This is an immediate consequence of (2).

3. An inequality

THEOREM.

$$D(x_1,\ldots,x_n;\nu_1,\ldots,\nu_n;\nu_{n+1})= \tag{3}$$
$$=\frac{\Gamma(\nu_1+\ldots+\nu_{n+1})}{\Gamma(\nu_1)\ldots\Gamma(\nu_{n+1})}\int_0^{x_1}dz_1\int^{\min\left(\frac{x_2}{1-z_1},1\right)}dz_2\ldots\int^{\min\left(\frac{x_n}{(1-z_1)\ldots(1-z_{n-1})},1\right)}z_1^{\nu_1-1}\times$$
$$\times(1-z_1)^{\nu_2+\ldots+\nu_{n+1}-1}z_2^{\nu_2-1}(1-z_2)^{\nu_3+\ldots+\nu_{n+1}-1}\ldots z_n^{\nu_n-1}(1-z_n)^{\nu_{n+1}-1}\,dz_n,$$

$$D(x_1,\ldots,x_n;\nu_1,\ldots,\nu_n;\nu_{n+1})\geqq\frac{\Gamma(\nu_1+\ldots+\nu_{n+1})}{\Gamma(\nu_1)\ldots\Gamma(\nu_{n+1})}\times \tag{4}$$
$$\times B(x_1;\nu_1;\nu_2+\ldots+\nu_{n+1})B(x_2;\nu_2;\nu_3+\ldots+\nu_{n+1})\ldots B(x_n;\nu_n;\nu_{n+1})$$

where $B(x;\alpha;\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt.$

PROOF. $\frac{D(y_1, \ldots, y_n)}{D(z_1, \ldots, z_n)} = (1-z_1)^{n-1}(1-z_2)^{n-2}\ldots(1-z_{n-1})$. Using the previous lemmas the integrand is transformed into the following expression:

$$y_1^{\nu_1-1}\ldots y_n^{\nu_n-1}(1-y_1-\ldots-y_n)^{\nu_{n+1}-1}\,dy_1\ldots dy_n =$$
$$= z_1^{\nu_1-1}[z_2(1-z_1)]^{\nu_2-1}\ldots[z_n(1-z_1)\ldots(1-z_{n-1})]^{\nu_n-1}\times$$
$$\times[(1-z_1)\ldots(1-z_n)]^{\nu_{n+1}-1}(1-z_1)^{n-1}\ldots(1-z_{n-1})\,dz_1\ldots dz_n =$$
$$= z_1^{\nu_1-1}(1-z_1)^{\nu_2+\ldots+\nu_{n+1}-1}z_2^{\nu_2-1}(1-z_2)^{\nu_3+\ldots+\nu_{n+1}-1}\ldots z_n^{\nu_n-1}(1-z_n)^{\nu_{n+1}-1}\,dz_1\ldots dz_n.$$

This implies (3). (4) is a consequence of (3) and of the fact that $x_i \leqq \frac{x_i}{(1-z_1)\ldots(1-z_{i-1})}$, for $i=1, \ldots, n$, and $(z_1, \ldots, z_n)\in I_n$. The theorem is proved.

5. Concluding remarks

The Dirichlet distribution is one of the important multivariate distributions that appear in the applications, especially, in order statistics, probabilistic constrained programming models, delivery problems etc. [6]. The bound is useful, since the gamma and the beta functions are widely tabulated.

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(Received January 9, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 165—178.

ON THE OPTIMAL LEBESGUE CONSTANTS FOR POLYNOMIAL INTERPOLATION

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1. Introduction

Let $X=\{x_{kn}\}$, $k=1,2,\ldots,n$; $n=1,2,\ldots$, be any triangular matrix with

(1.1) $$-1\equiv x_{n+1,n}\leqq x_{nn}<x_{n-1,n}<\ldots<x_{2n}<x_{1n}\leqq x_{0n}\equiv 1,\quad n=1,2,\ldots.$$

As it is well-known, in the study of the Lagrange interpolation the behaviour of the Lebesgue functions

(1.2) $$\lambda_n(X,x)=\sum_{k=1}^{n}|l_{kn}(X,x)|$$

and the Lebesgue constants

(1.3) $$\lambda_n(X)=\max_{-1\leqq x\leqq 1}\lambda_n(X,x)$$

is of fundamental importance. In (1.2) and (1.3), sometimes omitting the superfluous notations,

$$l_{kn}(x)=\omega_n(x)[\omega_n'(x_k)(x-x_k)]^{-1},\quad \omega_n(x)=c_n\prod_{k=1}^{n}(x-x_k).$$

In this paper, roughly speaking, we are going to prove the following relation: If $\gamma=0.577215\ldots$ is the Euler constant and $\chi:=\frac{2}{\pi}\left(\gamma+\ln\frac{4}{\pi}\right)=0.521251\ldots$, then

$$\lambda_n^*:=\min_X\lambda_n(X)=\frac{2}{\pi}\ln n+\chi+o(1).$$

(For the precise formula see (3.5).)

2. Preliminary results

From our point of view, the most important results are as follows (for further references see [12]—[15]).

2.1. In 1914 G. Faber [1], in 1916 S. Bernstein [2] proved that for arbitrary X

$$\lambda_n(X)>\frac{\ln n}{8\sqrt{\pi}},\quad n=1,2,\ldots.^{1}$$

[1] The minimum is attained (see Theorem 2.1).

P. Erdős [3] in 1961 obtained that

$$\frac{2}{\pi}\ln n - c_1 \leqq \lambda_n^* \leqq \frac{2}{\pi}\ln n + c_2. \tag{2.1}$$

(Here and later $c, c_1, c_2, \ldots$, const., denote absolute, positive, not necessarily different real numbers.)

In 1981 P. Erdős and P. Vértesi [4] established the Erdős conjecture on the Lebesgue function and proved as follows.

Let $\varepsilon>0$ be any given number. Then for arbitrary matrix X there exist sets H_n with $|H_n|\leqq\varepsilon$ and $\eta(\varepsilon)>0$ such that

$$\lambda_n(X,x) > \eta(\varepsilon)\ln n \quad \textit{if} \quad x\in[-1,1]\setminus H_n \quad \textit{and} \quad n\geqq n_0(\varepsilon).$$

This result was sharpened by P. Vértesi [5] and [6].

2.2. Another conjecture of P. Erdős and S. Bernstein for λ_n^* was proved by T. A. Kilgore [7] and C. De Boor and A. Pinkus [8] in 1978. To formulate the result, first let us see some observations.

A simple argument shows that for $n\geqq 2$, $\lambda_n(X,x)$ is a piecewise polynomial with $\lambda_n(X,x)\geqq 1$ and $\lambda_n(X,x)=1$ iff $x=x_{kn}$, $1\leqq k\leqq n$. Between consecutive nodes, $\lambda_n(X,x)$ has a single maximum, in $(-1,x_{nn})$ and $(x_{1n},1)$ it is convex and monotone (see e.g. F. W. Luttmann and T. J. Rivlin [9]).

Let us denote the local maxima by

$$\mu_{kn}(X) := \max_{x_{kn}\leqq x\leqq x_{k-1,n}} \lambda_n(X,x), \quad k=1,2,\ldots,n+1;\ n\geqq 3. \tag{2.2}$$

Another simple observation is that to obtain λ_n^*, "without loss of generality we (can) restrict our attention to those nodal configurations where $-1\equiv x_{nn}$ and $1\equiv x_{1n}$". (See Kilgore [7, p. 274].) We call these X *canonical* matrices.

Now the statement is:

THEOREM 2.1 ([7], [8]). *Let the matrix X be canonical. Then $\lambda_n(X,x)$ equioscillates, i.e.,*

$$\mu_{2n}(X) = \mu_{3n}(X) = \ldots = \mu_{nn}(X), \tag{2.3}$$

iff $\lambda_n(X)=\lambda_n^*$. Moreover, for arbitrary canonical X

$$\min_{2\leqq k\leqq n} \mu_{kn}(X) \leqq \lambda_n^* \leqq \max_{2\leqq k\leqq n} \mu_{kn}(X), \quad n\geqq 3. \tag{2.4}$$

Here, the above (so called) optimal matrix X^* (which has (2.3)*) is unique.*

2.3. Using the analogous result of [8] it turns out that *trigonometric interpolation on $[0,2\pi)$ at equidistant nodes is optimal.*

For the corresponding Lebesgue constants $\tilde\lambda_n^*$, the values

$$\tilde\lambda_{2n}^* = \frac{1}{n}\sum_{k=1}^{n}\cot\frac{2k-1}{4n}\pi, \quad n=1,2,\ldots, \tag{2.5}$$

$$\tilde\lambda_{2n+1}^* = \tilde\lambda_{2(2n+1)}^*, \quad n=1,2,\ldots, \tag{2.6}$$

were obtained by H. Ehlich and K. Zeller [12, (2.4)].

The complex case, when the nodes are on the unit circle line Γ, was treated by L. Brutman [10] and L. Brutman, A. Pinkus [11]. They proved that again *the case of the equidistant nodes (on Γ) is optimal* and the corresponding Lebesgue constants $\bar{\lambda}_n^*$ are:

(2.7) $$\bar{\lambda}_n^* = \tilde{\lambda}_{2n}^*, \quad n = 1, 2, \ldots$$

(see [10, (23)]).

Very recently P. N. Shivakumar and R. Wong [13] further V. K. Dzjadik and V. V. Ivanov [14] obtained *asymptomic expansions* for the right side of (2.5). Especially, in [13] the expansion

(2.8) $$\tilde{\lambda}_{2n}^* \overset{A}{\sim} \frac{2}{\pi}\ln n+\chi+\frac{2}{\pi}\ln 2+\sum_{k=1}^{\infty}\frac{a_k}{n^{2k}}, \quad a_k = \frac{(-1)^{k+1}(2^{2k-1}-1)^2\pi^{2k-1}B_{2k}^2}{4^{k-1}k(2k)!}$$

was established as $n\to\infty$ (B_k are the Bernoulli numbers). Further, the error has the same sign as, and is in absolute value less than, the first term neglected (compare R. Güntner [18, Theorem 1] and (3.2)). I.e., by (2.5)—(2.8) we see that the problem of the optimal nodes and the corresponding Lebesgue constants is settled considering the trigonometric or the complex interpolation.

3. Asymptotic for λ_n^*

3.1. If $X\subset[-1, 1]$, neither the optimal system, nor λ_n^* has been known. But there are some estimates for λ_n^*. The mentioned Erdős theorem (see (2.1)) gives a fairly sharp evaluation, especially if we take into account that he could not use Theorem 2.1 and its very useful relation (2.4) (see further (3.1)).

Let us remark that for arbitrary (not only for canonical) X we have (2.4). Indeed, if $1-x_{1n}>0$, say, consider the "intermediate" matrix $z_{kn}=x_{kn}-(x_{1n}+x_{nn})/2$ finally the matrix $X_c:=Y=\{y_{kn}\}$ where $y_{kn}=z_{kn}/z_{1n}$, $1\leqq k\leqq n$; $n=3, 4, \ldots$. It is easy to see that Y is canonical, from where (2.4) holds true for Y. Moreover, by construction $\mu_{kn}(X)=\mu_{kn}(Y)$ if $2\leqq k\leqq n$, which gives the statement.

By the above remark *we can write for arbitrary X the relation*

(3.1) $$\min_{1\leqq k\leqq n+1}\mu_{kn}(X) \leqq \min_{2\leqq k\leqq n}\mu_{kn}(X) \leqq \lambda_n^* \leqq \max_{2\leqq k\leqq n}\mu_{kn}(X) \leqq \max_{1\leqq k\leqq n+1}\mu_{kn}(X)$$

which can be used to obtain estimates for λ_n^* applying special matrices X and evaluating the differences

$$\delta_n(X) := \max_{2\leqq k\leqq n}\mu_{kn}(X)-\min_{2\leqq k\leqq n}\mu_{kn}(X),$$

$$\Delta_n(X) := \max_{1\leqq k\leqq n+1}\mu_{kn}(X)-\min_{1\leqq k\leqq n+1}\mu_{kn}(X).$$

3.2. Two very natural choices for the special X are the Chebyshev matrix $T=\left\{\cos\frac{2k-1}{2n}\pi\right\}$, $k=1, 2, \ldots, n$, $n=1, 2, \ldots$ and the matrix of the Chebyshev extremum nodes $V=\left\{\cos\frac{k\pi}{n-1}\right\}$, $k = 0, 1, \ldots, n-1$, $n = 2, 3, \ldots$.

Ehlich and Zeller [12] proved that

$$\lambda_n(T)=\lambda_{n+1}(V)=\tilde{\lambda}^*_{2n},\quad n=1,3,5,\ldots,$$

$$\lambda_n(T)=\lambda_{n+1}(V)+\varrho_n=\tilde{\lambda}^*_{2n},\quad n=2,4,6,\ldots,\quad 0<\varrho_n<\frac{1}{n^2},$$

from where by (2.8)

$$\lambda_n(T)=\frac{2}{\pi}\ln n+\chi+\frac{2}{\pi}\ln 2+\frac{\pi}{72n^2}-\varepsilon_n,\quad n\geqq 1 \tag{3.2}$$

where $0\leqq\varepsilon_n<0.0088n^{-4}$. Using analogous estimations and (3.1), Dzjadik and Ivanov [14] got the value of λ_n^* within the error 0.45.

They had no knowledge of the paper of L. Brutman [15] written in 1978, where using a quite serious analysis of $\lambda_n(T,x)$, he proved that

$$\delta_n(T)<0.201\quad\text{if}\quad n\geqq 3 \tag{3.3}$$

from where by (3.1) we can obtain λ_n^* within the error 0.201.

By further analysis R. Günttner [18] obtained that

$$\delta_n(T)=\frac{2}{\pi}\ln 2-\frac{4}{3\pi}+o(1), \tag{3.4}$$

(for a different proof see 4.4) i.e., the error can be lessened to 0.01686... . But we can not obtain a better estimation for λ_n^* using T.

Further calculations show that for other special matrices X, $\delta_n(X)>\delta_n(T)$ (see e.g. [9] and [12]).

3.3. The main goal of this paper is to prove

THEOREM 3.1. *We have the relations*

$$\frac{\text{const}}{(\ln n)^{1/3}}>\lambda_n^*-\frac{2}{\pi}\ln n-\chi>\begin{cases}\dfrac{\pi}{18n^2}+O\left(\dfrac{1}{n^4}\right) & \textit{if}\quad n=2m,\\[2mm] -\dfrac{2}{\pi n}+O\left(\dfrac{1}{n^2}\right) & \textit{if}\quad n=2m+1.\end{cases} \tag{3.5}$$

4. Proof. Properties of the matrix T

4.1. First we quote and verify some important properties of the matrix T. As it was proved by N. A. Pogodiceva [16] (see further [17, 8.2.5])

$$\lambda_n(T,x)=\frac{2}{\pi}|T_n(x)|\ln n+O(1),\quad -1\leqq x\leqq 1,\quad n=1,2,\ldots \tag{4.1}$$

uniformly in x and n. Here $T_n(x)=\cos n\vartheta$, $x=\cos\vartheta$, is the n-th Chebyshev polynomial.

Let $I_{kn}=[\vartheta_{k-1,n},\vartheta_{kn}]$, $k=1,2,\ldots,n+1$, where $\vartheta_k=\dfrac{2k-1}{2n}\pi$, $k=1,2,\ldots,n$, $\vartheta_0=0$, $\vartheta_{n+1}=\pi$. Let $t_k[\vartheta]\equiv t_{kn}[T,\vartheta]$ be the trigonometric polynomial coinciding with $\lambda_n[\vartheta]-\lambda_n[T,\vartheta]:=\lambda_n(T,\cos\vartheta)$ on I_k. By virtue of the symmetry of the nodes we will examine $t_{kn}[\vartheta]$ only for $1\leqq k\leqq m+1$ where $m=\left[\frac{n}{2}\right]$. Let $v_{kn}=\cos\xi_{kn}$ where ξ_{kn} are the local maximum places of t_k on I_k. Then we have

(i) $\tau_k:=\dfrac{k-1}{n}\pi\leqq\xi_k$, t_k is a concave function in $[\tau_k,\xi_k]$, $1\leqq k\leqq m+1$, $\tau_1=$ $=\xi_1=0$, further $\tau_{m+1}=\xi_{m+1}=\dfrac{\pi}{2}$ if n is even.

$$\text{(ii)}\ \begin{cases}\lambda_n(T)=t_1[\xi_1]>t_2[\xi_2]>\ldots>t_m[\xi_m]>t_{m+1}[\xi_{m+1}],\\ \lambda_n(T)=t_1[\tau_1]>t_2[\tau_2]>\ldots>t_m[\tau_m]>t_{m+1}[\tau_{m+1}].\end{cases}$$

Further

$$t_k[\tau_k]-t_{k+1}[\tau_{k+1}]=\lambda_n[\tau_k]-\lambda_n[\tau_{k+1}]= \tag{4.2}$$

$$=\frac{\cos\vartheta_k(1-\cos\vartheta_1)}{n\sin\vartheta_k\sin\frac{\vartheta_{2k}}{2}\sin\frac{\vartheta_{2k-1}}{2}}=\frac{4}{\pi}\frac{1+O\left(\frac{k^2}{n^2}\right)}{(4k-1)(4k-3)(2k-1)},\quad 1\leqq k\leqq m,$$

if n is big enough which henceforward will be supposed.

(iii) $t_1'[\tau_1]=0$, further

$$\frac{8n}{9\pi^2}\left[1+O\left(\frac{1}{n^2}\right)\right]=t_2'[\tau_2]>t_3'[\tau_3]>\ldots>t_{m+1}'[\tau_{m+1}]\geqq 0 \tag{4.3}$$

(see Brutman [15, Theorem 1 and Lemma 1, further (13), (15) and (17)]); in (4.2) and (4.3) we used

$$\begin{cases}\sin x=x-\eta_1x^3=x(1-\eta_1x^2),\quad 0<\eta_1<\dfrac{1}{6}\\[2mm] \cos x=1-\eta_2x^2,\quad 0<\eta_2<\dfrac{1}{2}.\end{cases} \tag{4.4}$$

They will be applied later, too.

4.2. First we prove the right hand side of (3.5). If $n=2m$, by (ii) and (3.1) $\lambda_n^*\geqq\lambda_n\left[\frac{\pi}{2}\right]$. But by [15 (24)], $\lambda_n\left[\frac{\pi}{2}\right]=\lambda_m(T)$, from where using (3.2) and $m=n/2$

$$\lambda_n^*\geqq\lambda_n\left[\frac{\pi}{2}\right]=\lambda_m(T)=\frac{2}{\pi}\ln\frac{n}{2}+\chi+\frac{2}{\pi}\ln 2+\frac{\pi}{72\left(\frac{n}{2}\right)^2}+O\left(\frac{1}{n^4}\right)$$

from where we get the corresponding part of (3.5).

Let now $n=2m+1$. Then again $\lambda_n^* \geqq \lambda_n[\xi_{m+1}]$. Here by [15 (26), (16) and Lemma 1]

$$0<\lambda_n[\xi_{m+1}]-\lambda_n[\tau_{m+1}]<\frac{1}{n^2},$$

where by [15 (27)]

$$\lambda_n[\tau_{m+1}]=\frac{2}{2m+1}\sum_{k=1}^{m}\cot\frac{2k-1}{2(2m+1)}\pi+\frac{1}{n}\tan\frac{\pi}{4n}:=P_n+R_n.$$

Here $0<R_n<n^{-2}$.

Estimating P_n, we shall use the expansion

$$\cot x=\frac{1}{x}-\sum_{r=1}^{\infty}2^{2r}|B_{2r}|\frac{x^{2r-1}}{(2r)!},\quad x\in(-\pi,\pi),$$

holding uniformly in any closed subinterval, from where we get

$$\lambda_n(T)=\frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1}-\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{|B_{2r}|}{(2r)!}\frac{\pi^{2r}}{(2n)^{2r}}\sum_{k=1}^{n}(2k-1)^{2r-1}$$

(cf. (2.5), 3.2 or [13, (3.2) and (3.3)]). Now, using that the function $\cot x$ is monotone decreasing in $(0,\pi/2)$, we have, using the previous two relations

$$P_n=\frac{4}{\pi}\sum_{k=1}^{m}\frac{1}{2k-1}-\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{|B_{2r}|}{(2r)!}\frac{\pi^{2r}}{(2m+1)^{2r}}\sum_{k=1}^{m}(2k-1)^{2r-1}>\lambda_m(T).$$

Here by (3.2) and $m=(n/2)(1-1/n)$ further $\ln(1+x)=x-\frac{x^2}{2}+\dots$ $(|x|<1)$ we get

$$\lambda_m(T)=\frac{2}{\pi}\ln\left(\frac{n}{2}\left(1-\frac{1}{n}\right)\right)+\chi+\frac{2}{\pi}\ln 2+O\left(\frac{1}{n^2}\right)=\frac{2}{\pi}\ln n+\chi-\frac{2}{\pi n}+O\left(\frac{1}{n^2}\right),$$

from where we get the right side of (3.5) when $n=2m+1$. Later on we use that

$$P_n=\frac{2}{2m+1}\sum_{k=1}^{m+1}\cot\frac{2k-1}{2(2m+1)}\pi=\frac{4}{\pi}\sum_{k=1}^{m+1}\frac{1}{2k-1}-$$

$$-\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{|B_{2r}|}{(2r)!}\frac{\pi^{2r}}{(2m+1)^{2r}}\sum_{k=1}^{m+1}(2k-1)^{2r-1}<\lambda_{m+1}(T)$$

(since the $(m+1)$th term equals 0 in P_n).

Further, we get, as above, that

$$\lambda_{m+1}(T)=\frac{2}{\pi}\ln n+\chi+\frac{2}{\pi n}+O\left(\frac{1}{n^2}\right),$$

i.e. we can state that

$$\lambda_n[\tau_{m+1}]=\frac{2}{\pi}\ln n+\chi+O\left(\frac{1}{n}\right)\quad\text{if}\quad n\geqq n_0, \tag{4.5}$$

whatsoever is n.

4.3. To prove the left hand side is more tedious. First we prove that the local maximum points of $\lambda_n(T, x)$ are "close" to τ_k. More preciously

$$\xi_k = \tau_k + \delta_k \quad \text{with} \quad 0 \leqq \delta_k = O\left(\frac{1}{n\sqrt{\ln n}}\right), \quad 1 \leqq k \leqq m+1, \tag{4.6}$$

uniformly in n and k.

Indeed, obviously $\lambda_n[\xi_k] \geqq \lambda_n[\tau_k] = \frac{2}{\pi}\ln n + O(1)$ (see (4.1)). Again by (4.1), we can suppose $n\delta_k \leqq \frac{\pi}{4} < 1$. Now by (4.1)

$$\frac{2}{\pi}\ln n + O(1) = \lambda_n[\xi_k] = \frac{2}{\pi}|\cos n\xi_k| \ln n + O(1) =$$

$$= \frac{2}{\pi}(\cos n\delta_k)\ln n + O(1) = \frac{2}{\pi}\left(1 - \frac{n^2\delta_k^2}{2} + \eta\frac{n^4\delta_k^4}{24}\right)\ln n + O(1) =$$

$$= \frac{2}{\pi}\ln n - \frac{n^2\delta_k^2}{\pi}\left(1 - \eta\frac{n^2\delta_k^2}{12}\right)\ln n + O(1) \leqq \frac{2}{\pi}\ln n - \frac{11}{12\pi}n^2\delta_k^2 \ln n + O(1)$$

$$(0 < \eta < 1).$$

By these, $n^2\delta_k^2 \ln n = O(1)$, which gives (4.6). An important consequence of (4.3) and (4.6) is the estimation

$$\lambda_n[\tau_k] \leqq \lambda_n[\xi_k] \leqq \lambda_n[\tau_k] + O\left(\frac{1}{\sqrt{\ln n}}\right), \quad 1 \leqq k \leqq m+1, \tag{4.7}$$

uniformly in n and k. This can be obtained using (i), (4.3) and (4.6) for t_k at τ_k.

4.4. By the above results we can verify (3.4). Indeed, according to (4.7), (ii), (4.2), (4.5) and (3.2)

$$\delta_n(T) = \lambda_n[\xi_2] - \lambda_n[\xi_{m+1}] = \lambda_n[\tau_2] - \lambda_n[\tau_{m+1}] + O\left(\frac{1}{\sqrt{\ln n}}\right) = \lambda_n[\tau_1] -$$

$$-(\lambda_n[\tau_1] - \lambda_n[\tau_2]) - \lambda_n[\tau_{m+1}] + O\left(\frac{1}{\sqrt{\ln n}}\right) = \frac{2}{\pi}\ln 2 - \frac{4}{3\pi} + O\left(\frac{1}{\sqrt{\ln n}}\right),$$

as it was stated.

The matrix D

4.5. The main idea of the proof is to construct another matrix D, which is "close" to T, moreover, for which $\lambda_n(D_c) \approx \lambda_n[T, \tau_{m+1}] \approx \frac{2}{\pi}\ln n + \chi$. For this aim let

$$D = \{y_1, x_2, \ldots, x_{n-1}, y_n\}, \quad n = 1, 2, \ldots, \tag{4.8}$$

where $x_k = x_{kn}$ are the Chebyshev roots, $1 \leqq k \leqq n$,

$$y_1 = -y_n = \cos(\vartheta_1 + \psi_n) \quad \text{with} \quad 0 < \psi_n = \frac{c}{n \ln n}, \tag{4.9}$$

where c will be determined later. Obviously

$$(4.10)\qquad D_n(x) := \omega_n(D, x) = \frac{T_n(x)(y_1^2-x^2)}{x_1^2-x^2} = \left(1-\frac{x_1^2-y_1^2}{x_1^2-x^2}\right) T_n(x)$$

from where

$$D_n'(x) = \frac{[T_n'(x)(y_1^2-x^2)-2xT_n(x)](x_1^2-x^2)+2xT_n(x)(y_1^2-x^2)}{(x_1^2-x^2)^2}$$

which means

$$(4.11)\qquad \begin{cases} D_n'(x_k) = \dfrac{T_n'(x_k)(y_1^2-x_k^2)}{x_1^2-x_k^2} = \dfrac{T_n'(x_k)}{1+\dfrac{x_1^2-y_1^2}{y_1^2-x_k^2}}, & k = 2, 3, \dots, n-1, \\[2ex] D_n'(y_1) = -\dfrac{2y_1T_n(y_1)}{x_1^2-y_1^2}, \quad D_n'(y_n) = -\dfrac{2y_nT_n(y_n)}{x_1^2-y_n^2}. \end{cases}$$

4.6. First we verify that

$$(4.12)\qquad \lambda_n(D, x) = \lambda_n(T, x)+O(1) = \frac{2}{\pi} |\cos n\vartheta| \ln n+O(1).$$

By

$$(4.13)\qquad \cos^2\alpha-\cos^2\beta = \sin(\alpha+\beta)\sin(\beta-\alpha) =$$

$$= (\alpha+\beta)(\beta-\alpha)[1-\eta_1(\alpha+\beta)^2][1-\eta_2(\alpha-\beta)^2], \quad 0 < \eta_1, \eta_2 < \frac{1}{6},$$

we have

$$(4.14)\qquad x_1^2-y_1^2 = \left(\frac{\pi}{n}+\psi_n\right)\psi_n\left[1+O\left(\frac{1}{n^2}\right)\right][1+O(\psi_n^2)] = \frac{\pi\psi_n}{n}[1+O(n\psi_n)].$$

Now, by (4.10), (4.11), (4.14) and $T_n(y_1) = -\sin n\psi_n = -n\psi_n[1+O(n^2\psi_n^2)]$,

(4.15)

$$l_1(D, x) = \frac{T_n(x)(y_1+x)}{(x_1-x)(x_1+x)}\,\frac{x_1^2-y_1^2}{2y_1T_n(y_1)} = -\frac{T_n(x)}{(x_1-x)(x_1+x)}\,\frac{y_1+x}{2y_1}\,\frac{\pi}{n^2}[1+O(n\psi_n)],$$

from where we obtain $|l_1(D, x)| = O(1)$. Similarly, $|l_n(D, x)| = O(1)$, too.

Consider now $\lambda_n(D, x)$. By (4.10), (4.11) and using the boundedness of $|l_1(D)|$, $|l_n(D)|$, $|l_1(T)|$ and $|l_n(T)|$,

$$(4.16)\qquad \pm\lambda_n(D, x) = \left(1+\frac{y_1^2-x_1^2}{x_1^2-x^2}\right)\sum_{k=2}^{n-1}\left(1+\frac{x_1^2-y_1^2}{y_1^2-x_k^2}\right)|l_k(T, x)| \pm$$

$$\pm(|l_1(D, x)|+|l_n(D, x)|) = \lambda_n(T, x)+\frac{y_1^2-x_1^2}{x_1^2-x^2}\sum_{k=2}^{n-1}|l_k(T, x)|+$$

$$+\sum_{k=2}^{n-1}\frac{x_1^2-y_1^2}{y_1^2-x_k^2}|l_k(T, x)|+\frac{y_1^2-x_1^2}{x_1^2-x^2}\sum_{k=2}^{n-1}\frac{x_1^2-y_1^2}{y_1^2-x_k^2}|l_k(T, x)|+O(1) :=$$

$$:= \lambda_n(T, x)+\Sigma_1+\Sigma_2+\Sigma_3+O(1).$$

Here, if $\min_{1\leqq k\leqq n} |\vartheta-\vartheta_k| = |\vartheta-\vartheta_j|$ and $x\geqq 0$, say,

$$\Sigma_1 = O(1)\frac{\psi_n}{n}\sum_{\substack{k=2\\ k\neq j}}^{n-1}\left|\frac{T_n(x)\sin\vartheta_k}{n(x_1-x)(x_1+x)(x-x_k)}\right|+O(1) =$$

$$= O(1)\frac{\psi_n}{n}\sum_{k=2}^{n-1}\frac{1}{n}\frac{n^2}{j^2}\frac{k}{n}\frac{n^2}{(k+j)(|k-j|+1)}+O(1) =$$

$$= O(n\psi_n)\frac{1}{j^2}\sum_{k=2}^{n-1}\frac{k}{(k+j)(|k-j|+1)} = O\left(\frac{n\psi_n}{j^2}\right)\Big(\sum_{k<j/2}+\sum_{j/2\leqq k<2j}+\sum_{k>2j}\Big)+$$

$$+O(1) = O\left(\frac{n\psi_n \ln n}{j^2}\right)+O(1),$$

using $|x-x_k| = \left|2\sin\frac{\vartheta-\vartheta_k}{2}\sin\frac{\vartheta+\vartheta_k}{2}\right| \sim \frac{k+j}{n}\frac{|k-j|+1}{n}$ $(k\neq j)$ and $|T_n(x)|\,|x-x_j|^{-1}\sim$ $\sim |T_n'(x_j)|\sim n^2 j^{-1}$. By similar arguments

$$\Sigma_2 = O\left(\frac{n\psi_n\ln(j+1)}{j^2}\right) \quad\text{and}\quad \Sigma_3 = O\left(\frac{n^2\psi_n^2\ln(j+1)}{j^4}\right),$$

i.e. $|\Sigma_1|+|\Sigma_2|+|\Sigma_3|+O(1)\leqq K$ for a certain $K>0$. Now, if $\lambda_n(T,x)\leqq 2K$, by (4.16), $\lambda_n(D,x)\leqq 3K$. If $\lambda_n(T,x)>2K$, by (4.16), $\lambda_n(T,x)-K\leqq\lambda_n(D,x)\leqq\lambda_n(T,x)+$ $+K$, which was to be proven.

4.7. According to (4.12), $\lambda_n[D,\tau_k]=\frac{2}{\pi}\ln n+O(1)$, $1\leqq k\leqq n+1$, from where for the local maximum place $z_k=\cos\xi_k$ of $\lambda_n(D,x)$ we get

$$\xi_k = \tau_k+\varrho_k \quad\text{with}\quad |\varrho_k| = O\left(\frac{1}{n\sqrt{\ln n}}\right),\quad 1\leqq k\leqq n+1, \tag{4.17}$$

uniformly in n and k (see the corresponding argument in 4.3; $\lambda_n[D,\vartheta]:=\lambda_n(D,\cos\vartheta)$).

4.8. To estimate the local maximum values of $\lambda_n(D,x)$, we prove the following. If $c>0$ is fixed,

$$\lambda_n(D,x) = \left[1-\frac{\pi\psi_n}{n(x_1^2-x^2)}\right]\lambda_n(T,x)+O(n\psi_n) \quad\text{if}\quad n|\vartheta-\vartheta_1|\geqq c, \tag{4.18}$$

from where we immediately get, uniformly in n and k,

$$\lambda_n[D,\tau_{k+1}] = \left(1-\frac{4}{\pi}\frac{n\psi_n}{4k^2-1}\right)\lambda_n[T,\tau_{k+1}]+O(n\psi_n),\quad k=0,1,\ldots,m+1. \tag{4.19}$$

Indeed, if we verified (4.18), then by (4.13)

$$\frac{\pi\psi_n}{n(x_1^2-\cos^2\tau_{k+1})}\lambda_n[T,\tau_{k+1}] = \frac{4}{\pi}\frac{n\psi_n}{4k^2-1}\lambda_n[T,\tau_{k+1}]\left[1+O\left(\frac{(k+1)^2}{n^2}\right)\right] =$$

$$= \frac{4}{\pi}\frac{n\psi_n}{4k^2-1}\lambda_n[T,\tau_{k+1}]+O\left(\frac{\psi_n\ln n}{n}\right), \quad k=0,1,\dots,m+1,$$

from where we obtain (4.19).

To prove (4.18) first we compare $l_1(T,x)$ and $l_1(D,x)$. By definition

$$l_1(T,x) = \frac{T_n(x)\sin\vartheta_1}{n(x-x_1)} = \frac{\pi}{2n^2}\frac{T_n(x)}{x-x_1}+O\left(\frac{1}{n^2}\right). \tag{4.20}$$

To estimate $l_1(D,x)$ we write

$$l_1(D,x) = \left(1-\frac{x_1^2-y_1^2}{x_1^2-x^2}\right)\frac{T_n(x)}{x-y_1}\frac{y_1^2-x_1^2}{2y_1T_n(y_1)}.$$

Here, if $n|\vartheta-\vartheta_1|\geqq c$,

$$\frac{y_1^2-x_1^2}{(x-y_1)2y_1T_n(y_1)} = \frac{\pi\psi_n[1+O(n\psi_n)]}{n(x-x_1+x_1-y_1)2(1+y_1-1)\sin n\psi_n} =$$

$$= \frac{\pi\psi_n[1+O(n\psi_n)]}{n(x-x_1)[1+O(n\psi_n)]2[1+O(n^{-2})]n\psi_n[1+O(n^2\psi)]} = \frac{\pi[1+O(n\psi_n)]}{2n^2(x-x_1)}$$

(see (4.14) and (4.15)), which means

$$l_1(D,x) = \left(1-\frac{x_1^2-y_1^2}{x_1^2-x^2}\right)l_1(T,x)+O(n\psi_n) \quad \text{if} \quad n|\vartheta-\vartheta_1|\geqq c. \tag{4.21}$$

Using similar arguments for $l_n(T,x)$ and $l_n(D,x)$, we can write as follows

$$\lambda_n(D,x) = \left(1-\frac{x_1^2-y_1^2}{x_1^2-x^2}\right)\sum_{k=2}^{n-1}|l_k(T,x)|+|l_1(D,x)|+|l_n(D,x)|+\Sigma_2+\Sigma_3 =$$

$$= \left(1-\frac{x_1^2-y_1^2}{x_1^2-x^2}\right)\lambda_n(T,x)+O(n\psi_n), \quad n|\vartheta-\vartheta_1|\geqq c$$

(see (4.16) and (4.21)), from where by (4.14) we get (4.18).

4.9. Now we are ready to prove that

$$\lambda_n[D,\tau_k] \leqq \lambda_n[D,\xi_k] \leqq \lambda_n[D,\tau_k]+O\left(\frac{1}{\sqrt{\ln n}}\right), \quad k=1,2,\dots,m+1, \tag{4.22}$$

uniformly in k and n.

Here the first inequality is trivial. To obtain the second one, we write by (4.18) and (4.19)

$$\lambda_n(D, z_k) = \left[1 - \frac{\pi\psi_n}{n(x_1^2 - z_k^2)}\right] \lambda_n(T, z_k) + O(n\psi_n) = \tag{4.23}$$

$$= \left[1 - \frac{\pi\psi_n}{n(x_1^2 - \cos^2\tau_k)}\right] \lambda_n[T, \tau_k] + \left[1 - \frac{\pi\psi_n}{n(x_1^2 - \cos^2\tau_k)}\right] \cdot$$

$$\cdot (\lambda_n[T, \xi_k] - \lambda_n[T, \tau_k]) + \left[\frac{\pi\psi_n}{n(x_1^2 - \cos^2\tau_k)} - \frac{\pi\psi_n}{n(x_1^2 - z_k^2)}\right] \lambda_n[T, \xi_k] + O(n\psi_n) :=$$

$$:= \lambda_n[D, \tau_k] + J_2 + J_3 + O(n\psi_n).$$

First we remark that by (4.17) and (4.13)

$$|J_3| = O(1) \frac{n\psi_n}{\sqrt{\ln n}\,(k+1)^3} \lambda_n[T, \xi_k] \leqq \frac{c_1}{\sqrt{\ln n}}, \quad k = 1, 2, \ldots, m+1.$$

Now if $\lambda_n[T, \xi_k] \geqq \lambda_n[T, \tau_k]$, then by (4.17) and (4.3) the difference is $O\left(\frac{1}{\sqrt{\ln n}}\right)$, from where $J_2 = O\left(\frac{1}{\sqrt{\ln n}}\right)$ which gives (4.22). We can use the same argument whenever $0 \leqq \lambda_n[T, \tau_k] - \lambda_n[T, \xi_k] \leqq \frac{c}{\sqrt{\ln n}}$, $k=1, 2, \ldots, m+1$, $n=3, 4, \ldots$. On the other hand if for any fixed N there would exist an $n \geqq N$ and a k such that

$$\lambda_n[T, \tau_k] - \lambda_n[T, \xi_k] > \frac{3c_1}{\sqrt{\ln n}},$$

then

$$\lambda_n[D, \xi_k] < \lambda_n[D, \tau_k] - \frac{2c_1}{\sqrt{\ln n}} + \frac{c_1}{\sqrt{\ln n}} + O(n\psi_n) < \lambda_n[D, \tau_k]$$

if $N \geqq n_0$, a contradiction. I.e., (4.22) is proved.

4.10. Next we verify that for $k=1, 2, \ldots, m$

$$\lambda_n[D, \tau_{k+1}] - \lambda_n[D, \tau_k] = \tag{4.24}$$

$$= \frac{4}{\pi}\left[\frac{8n\psi_n \ln n}{\pi} \frac{1}{(2k-3)(2k-1)(2k+1)} - \frac{1}{(4k-3)(4k-1)(2k-1)}\right] + O(n\psi_n)$$

uniformly in k and n.

Indeed, by the equation

$$[4(k-1)^2 - 1]^{-1} - (4k^2 - 1)^{-1} = 4[(2k-3)(2k-1)(2k+1)]^{-1},$$

further by (4.19), (4.1) and (4.2) we can write

$$\lambda_n[D, \tau_{k+1}]-\lambda_n[D, \tau_k] = \frac{4n\psi_n}{\pi}\left[\frac{1}{4(k-1)^2-1}-\frac{1}{4k^2-1}\right]\lambda_n[T, \tau_{k+1}]-$$

$$-\left[1-\frac{4}{\pi}\frac{n\psi_n}{4(k-1)^2-1}\right](\lambda_n[T, \tau_k]-\lambda_n[T, \tau_{k+1}])+O(n\psi_n) =$$

$$= \frac{4n\psi_n}{\pi}\frac{4}{(2k-3)(2k-1)(2k+1)}\left[\frac{2}{\pi}\ln n+O(1)\right]-\frac{4}{\pi}\frac{1+O\left(\frac{k^2}{n^2}\right)}{(4k-3)(4k-1)(2k-1)}+$$

$$+O\left(\frac{n\psi_n}{k^5}\right)+O(n\psi_n) = \frac{4}{\pi}\left[\frac{8n\psi_n\ln n}{\pi}\frac{1}{(2k-3)(2k-1)(2k+1)}-\right.$$

$$\left.-\frac{1}{(4k-3)(4k-1)(2k-1)}\right]+O(1)(n\psi_n k^{-3}+k^{-1}n^{-2}+n\psi_n k^{-5}+n\psi_n),$$

which gives (4.24).

4.11. Now let $\psi_n=\pi/(8n\ln n)$. Then by (4.22) and (4.24) we have for $k=1, 2, \dots, m$

(4.25)
$$\lambda_n(D, z_{k+1})-\lambda_n(D, z_k) = \frac{4}{\pi}\left[\frac{1}{(2k-3)(2k-1)(2k+1)}-\frac{1}{(4k-3)(4k-1)(2k-1)}\right]+$$

$$+O\left(\frac{1}{\sqrt{\ln n}}\right).$$

By (4.25) we investigate $d_{kn}(D):=\lambda_n(D, z_{k+1})-\lambda_n(D, z_k)$. Obviously $d_1(D)\approx$ $\approx -8/(3\pi)<0$, but for any fixed M by a simple computation we can verify $d_k(D)>0$, $k=2, 3, \dots, M$. Now let M be big enough. Then by (4.25)

$$d_k(D) = \frac{4}{\pi}\left[\frac{1}{8k^3\left[1+O\left(\frac{1}{k}\right)\right]}-\frac{1}{32k^3\left[1+O\left(\frac{1}{k}\right)\right]}\right]+O\left(\frac{1}{\sqrt{\ln n}}\right) =$$

$$= \frac{3}{8\pi k^3}+O\left(\frac{1}{k^4}\right)+O\left(\frac{1}{\sqrt{\ln n}}\right)$$

from where $d_k(D)>0$ if $M\leqq k\leqq c_1(\ln n)^{1/6}$. Thus *we obtained the relations*

(4.26) $$d_k(D) > 0 \quad \text{if} \quad 2\leqq k\leqq c_1(\ln n)^{1/6}$$

with a certain $c_1>0$ (of course, $n\geqq n_0$).

4.12. *To complete the proof the only thing we have to prove is the relation*

(4.27) $$\lambda_n(D, z_{k+1}) = \frac{2}{\pi}\ln n+\chi+O\left(\frac{1}{(\ln n)^{1/3}}\right) \quad \textit{if} \quad c_1(\ln n)^{1/6}\leqq k\leqq m,$$

considering (4.26), (4.27), finally (3.1). To this end first we remark that

$$\lambda_n[D, \xi_{k+1}] = (\lambda_n[D, \xi_{k+1}] - \lambda_n[D, \tau_{k+1}]) + (\lambda_n[D, \tau_{k+1}] - \lambda_n[T, \tau_{k+1}]) + \lambda_n[T, \tau_{k+1}].$$

Here by (4.22)

$$\lambda_n[D, \xi_{k+1}] - \lambda_n[D, \tau_{k+1}] = O\left(\frac{1}{\sqrt{\ln n}}\right), \tag{4.28}$$

moreover, by (4.19)

$$\lambda_n[D, \tau_{k+1}] - \lambda_n[T, \tau_{k+1}] = O\left(\frac{1}{(\ln n)^{1/3}}\right) \quad \text{if} \quad c_1(\ln n)^{1/6} \leqq k \leqq m. \tag{4.29}$$

Finally, by (4.5) and (4.2)

$$\lambda_n[T, \tau_{k+1}] = \lambda_n[T, \tau_{m+1}] + \sum_{j=k+1}^{m} (\lambda_n[T, \tau_j] - \lambda_n[T, \tau_{j+1}]) = \tag{4.30}$$

$$= \frac{2}{\pi} \ln n + \chi + O\left(\frac{1}{n}\right) + \frac{4}{\pi} \sum_{j=k+1}^{m} \frac{1 + O\left(\frac{j^2}{n^2}\right)}{(4j-3)(4j-1)(2j-1)} =$$

$$= \frac{2}{\pi} \ln n + \chi + O\left(\frac{1}{n}\right) + O\left(\frac{1}{k^2}\right) = \frac{2}{\pi} \ln n + \chi + O\left(\frac{1}{(\ln n)^{1/3}}\right)$$

if $c_1(\ln n)^{1/6} \leqq k \leqq m$, i.e. by (4.28)—(4.30) we obtain (4.27), as it was to be proven.

4.13. Let us remark that for our matrix D we have

$$\lambda_n(D_c) = \lambda_n^* + O\left(\frac{1}{(\ln n)^{1/3}}\right)$$

i.e. D_c has the smallest possible Lebesgue constants at least asymptotically.

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(Received January 10, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 179—180.

ON A CERTAIN CLASS OF COMPLETE REGULARITY

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Given two topological spaces X and E, X is said to be E-completely regular if X is homeomorphic to a subspace of E^m for some cardinal m. $\mathfrak{C}(E)$ represents the class of all E-completely regular spaces, and it is said to be a class of complete regularity. $\mathfrak{F}^*$ denotes the space consisting of three points 0, 1, 2 in which the only proper open set is $\{0\}$. $C(X, E)$ is the set of all continuous functions from X to E. For other references and notations see [1].

Let T be the space consisting of the points 0, 1, 2 where $\{0\}$ and $\{1, 2\}$ are the only proper open subsets.

THEOREM. *$\mathfrak{C}(T)$ is the class of all zero-dimensional spaces.*

PROOF. Assume that X is zero-dimensional. For two distinct points x, y in X, the function f with values in $\{1, 2\}$ and $f(x)\neq f(y)$ belongs to $C(X, T)$. Also if x is not in the closed subset F of X and G_x is a clopen neighbourhood of x disjoint with F, then the function g such that $g(y)=1$ if $y\in G_x$ and $g(y)=0$ elsewhere, belongs to $C(X, T)$ and $g(x)\notin \overline{g(F)}$. From [1, Theorem 3.8], X is T-completely regular.

Since the property of being zero-dimensional is productive and hereditary, the conclusion follows.

LEMMA. *If X is a zero-dimensional non-indiscrete space, then each point in X has a proper neighbourhood.*

PROOF. Suppose that X is the only neighbourhood of some point p. Let G be a clopen subset. If $p\in G$ then $G=X$ and if $p\notin G$ then $p\in X\sim G$ which is open, thus $G=\emptyset$.

PROPOSITION 1. *$\mathfrak{C}(E)$ is the class of all zero-dimensional spaces if and only if E is a zero-dimensional, non-T_0, non-indiscrete space.*

PROOF. Obviously, E is not indiscrete. Since $E\in\mathfrak{C}(E)$, E is zero-dimensional. It follows from [1, 3.11] that E is not T_0. Conversely, it is trivial from [1, 3.5] that $\mathfrak{C}(E)\subset\mathfrak{C}(T)$. On the other hand, it suffices to show that $T\in\mathfrak{C}(E)$. Since E is not T_0, there are two distinct points a, b in E with the same neighbourhoods. From the previous lemma, there is a proper clopen subset G such that $\{a, b\}\subset G$. The subspace $E_1=\{a, b, c\}$, where $c\in E\sim G$, is homeomorphic to T.

In [1, page 171], S. Mrówka says: "$\mathfrak{C}(E)$ is the class of all topological spaces if and only if E contains a non-trivial (i.e., containing more than one point) T_0-subspace and a non-trivial indiscrete subspace."

From our Theorem, it is evident that the space T verifies all the above conditions, nevertheless $\mathfrak{C}(T)\neq\mathfrak{C}(\mathfrak{F}^*)$; therefore, Mrówka's statement is not correct. In the following, we give a correct version of Mrówka's result.

PROPOSITION 2. *$\mathfrak{C}(E)$ is the class of all topological spaces if and only if E is a non-T_0 space with a proper closed subset F such that there are two points $a\in F$, $b\notin F$, with the property that every neighbourhood of a is also a neighbourhood of b.*

PROOF. Assume that $\mathfrak{C}(E)$ is the class of all topological spaces. Since $\mathfrak{F}^*$ is E-completely regular, from [1, Theorem 3.8], there is a finite number n and $f\in C(\mathfrak{F}^*, E^n)$ which verifies $f(0)\notin\overline{f(\{1, 2\})}$ and thus every neighbourhood of $f(1)$ is a neighbourhood of $f(0)$; if f_i is a projection of f on E such that a suitable neighbourhood of $f_i(0)$ does not contain $f_i(1)$, then $a=f_i(1)$ and $b=f_i(0)$ satisfy the conclusion. Moreover, there is $g\in C(\mathfrak{F}^*, E)$ which verifies $g(1)\neq g(2)$ and thus, $g(1)$ and $g(2)$ have the same neighbourhoods.

Conversely, it is sufficient to see that $\mathfrak{F}^*\in\mathfrak{C}(E)$. For this purpose we make use of [1, Theorem 3.8]. The functions $g_i\in C(\mathfrak{F}^*, E)$, $i=1, 2$, such that $g_i(1)$ and $g_i(2)$ are two distinct points with the same neighbourhoods and $g_i(0)=g_i(i)$, verify condition (a). The function $f\in C(\mathfrak{F}^*, E)$ with $f(0)=b$ and $f(1)=f(2)=a$, satisfies condition (b) of the above mentioned theorem.

Acknowledgement. We must sincerely thank Professor M. Lòpez Pellicer for his valuable help and constant stimulation.

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(Received January 11, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 181—185.

NEW PROOFS OF A THEOREM OF KOMLÓS

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The theorem of Komlós [3] states that from an L^1-bounded sequence of random variables, a subsequence can be extracted so that every further subsequence converges Cesàro a.s. to the same limit. In the following, this theorem is proved in two new ways. The first proof is classical in nature, using a Kolmogorov-type maximal inequality, while the second parallels the argument given by Etemadi [2] in showing the strong law of large numbers for pairwise independent, identically distributed random variables. These are in contrast to the proof given by Komlós which uses martingale difference sequences.

We begin with some lemmas similar to those in [3]. Throughout, we are considering a probability space $(\Omega, \mathfrak{F}, P)$. For $a>0$, denote $F_a(x)=xI_{[-a,a]}(x)$.

LEMMA 1. *Suppose $\{X_n\}$ is a sequence of random variables satisfying $\sup_n E|X_n|<\infty$. Then there exists a subsequence $\{X_n^0\}$ such that for each further subsequence $\{X_n'\}$,*

(1) $$\{F_n(X_n')\} \text{ is uniformly integrable,}$$

(2) $$\sum_{n=1}^{\infty} P(|X_n'|>n)<\infty,$$

and

(3) $$\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} E|F_n(X_n')|^{1+\varepsilon}<\infty, \quad \text{for all} \quad \varepsilon>0.$$

PROOF. For each $j\geqq 1$, there exists a subsequence $\{X_{j,n}\}$ of $\{X_{j-1,n}\}$ (taking $\{X_{0,n}\}=\{X_n\}$) and a scalar $M_j\geqq 0$ such that

$$\lim_{n\to\infty} \int_{j-1<|X_{j,n}|\leqq j} |X_{j,n}|\,dP = M_j.$$

By a diagonal argument, we can choose $\{X_n^0\}$ such that for any further subsequence $\{X_n'\}$,

(4) $$\frac{M_j}{2} < \int_{j-1<|X_n'|\leqq j} |X_n'|\,dP < M_j+\frac{1}{j^2}, \quad 1\leqq n, \quad 1\leqq j\leqq n^2.$$

Since $\{X_n\}$ is L^1-bounded, we have $\sum_{j=1}^{\infty} M_j<\infty$. From this and (4), (1)—(3) can be readily shown.

From (1) and (2), we see the known result that $\{X_n\}$ has a subsequence equivalent to a uniformly integrable sequence. Denote $X_n \to X\ \sigma(L^1, L^\infty)$, if $\{X_n\}$ converges weakly in L^1 to X.

LEMMA 2. *Suppose* $X_n \to X\sigma(L^1, L^\infty)$ *and* $F_k(X_n) \to \beta_k\ \sigma(L^1, L^\infty)$, *for random variables* X, $\beta_k \in L^1$, $k \geqq 1$. *Then* $\beta_k \to X$ *a.s. and in* L^1.

PROOF. We can suppose that (4) holds for all subsequences of $\{X_n\}$. Then,

$$\sum_{k=1}^{\infty} E|\beta_k - \beta_{k-1}| \leqq \sum_{k=1}^{\infty} \lim_n E|F_k(X_n) - F_{k-1}(X_n)| \leqq \sum_{k=1}^{\infty} \left(M_k + \frac{1}{k^2}\right) < \infty.$$

Hence, $\beta_k \to \beta$ a.s. and in L^1 for some $\beta \in L^1$. From this, it can be shown that $\beta = X$ a.s.

The next lemma provides some groundwork for the maximal inequality following, and an estimate to be used later. (Cf. Révész [4].)

LEMMA 3. *Let* $\{X_n\}$ *be any sequence of random variables. Then there exists a subsequence* $\{X_n^0\}$ *of* $\{X_n\}$, *and a sequence* $\{\beta_n\}$ *of bounded random variables such that for any further subsequence* $\{X_n'\}$ *of* $\{X_n^0\}$,

$$F_k(X_n') \to \beta_k\ \sigma(L^1, L^\infty), \quad k \geqq 1, \tag{5}$$

and

$$\left|\int (F_n(X_n')\beta_n - \beta_n^2)\, dP\right| \leqq 1, \quad \text{for all} \quad n \geqq 1. \tag{6}$$

PROOF. For each $j \geqq 1$, there exists a subsequence $\{X_{j,n}\}$ of $\{X_{j-1,n}\}$, and a random variable β_j, $|\beta_j| \leqq j$ a.s., such that $F_j(X_{j,n}) \to \beta_j\ \sigma(L^1, L^\infty)$. In particular, we choose $\{X_{j,n}\}$ to satisfy $\left|\int (F_j(X_{j,n})\beta_j - \beta_j^2)\, dP\right| \leqq 1$, for all $n \geqq 1$. Letting $X_n^0 = X_{n,n}$, (5) and (6) follow.

LEMMA 4 (maximal inequality). *Let* $\{X_n\}$ *be any sequence of random variables and let* $\{\alpha_n\}$ *be a sequence of random variables in* L^2. *Then there exists a subsequence* $\{X_n^0\}$ *of* $\{X_n\}$, *and a sequence* $\{\beta_n\}$ *of bounded random variables such that for any subsequence* $\{X_n'\}$ *of* $\{X_n^0\}$, *and for* $0 < \varepsilon < \frac{\sqrt{2}}{2}$, $1 \leqq m < n$,

$$P\left(\max_{m \leqq j \leqq n} \left|\sum_{i=m}^{j} \alpha_i Z_i\right| > \varepsilon\right) \leqq \frac{1}{\varepsilon^2}\left(2 \sum_{i=m}^{n} E(\alpha_i Z_i)^2 + \frac{1}{2^m}\right),$$

where we write $Z_i = F_i(X_i') - \beta_i$, $i \geqq 1$.

PROOF. By Lemma 3, we obtain $\{\beta_n\}$ and can suppose (5) holds for any subsequence of $\{X_n\}$. In fact, we have $F_k(X_n) \to \beta_k\ \sigma(L^\infty, L^1)$, $k \geqq 1$.

Let $X_1^0 = X_1$, and suppose X_i^0, $i \leqq j-1$ have been chosen. By the weak convergence, we can choose X_j^0 satisfying the finitely many conditions below:

$$\left|E[\alpha_r(F_r(X_i^0) - \beta_r)\alpha_s(F_s(X_j^0) - \beta_s)]\right| < 1/(48 \cdot 2^{j-1}), \tag{7}$$

for $1 \leqq i \leqq j-1$, $r \leqq i$, $s \leqq j$;

$$\left|\int_{A_i(a_1, \ldots, a_i;\, b_1, \ldots, b_i)} \alpha_r(F_r(X_i^0) - \beta_r)\alpha_s(F_s(X_j^0) - \beta_s)\, dP\right| < 1/(48 \cdot 2^{j-1}), \tag{8}$$

for $1\leqq i\leqq j-1$, $r\leqq i$, $s\leqq j$, $1\leqq l\leqq j-1$, all finite sequences $\{a_1, \dots, a_t\}$, $\{b_1, \dots, b_t\}$ satisfying $1\leqq a_1<\dots<a_t\leqq j-1$, $1\leqq b_1<\dots<b_t$, and $b_k\leqq a_k$ for $1\leqq k\leqq t$, where

$$A_l(a_1, \dots, a_t;\ b_1, \dots, b_t) = \Big\{\max_{1\leqq u\leqq t-1}\Big|\sum_{i=1}^{u} \alpha_{b_i}\big(F_{b_i}(X^0_{a_i})-\beta_{b_i}\big)\Big| \leqq 2^{-l/2},$$

$$\Big|\sum_{i=1}^{t} \alpha_{b_i}\big(F_{b_i}(X^0_{a_i})-\beta_{b_i}\big)\Big| > 2^{-l/2}\Big\},$$

$$A_l(a_1;\ b_1) = \Big\{\big|\alpha_{b_1}\big(F_{b_1}(X^0_{a_1})-\beta_{b_1}\big)\big| > 2^{-l/2}\Big\}.$$

Let $\{X'_n\}=\{X^0_{c_n}\}$ be a subsequence of $\{X^0_n\}$; denote $Z_n=F_n(X'_n)-\beta_n$, $n\geqq 1$. For $1\leqq N\leqq m<n$, let $\varepsilon=2^{-N/2}$ and denote

$$A_\varepsilon = \Big\{\max_{m\leqq j\leqq n}\Big|\sum_{i=m}^{j} \alpha_i Z_i\Big| > \varepsilon\Big\}.$$

We have $A_\varepsilon=\bigcup_{k=m}^{n} B_k$, where we write $B_k=A_N(c_m, \dots, c_k;\ m, \dots, k)$. Then,

$$E\Big(\sum_{i=m}^{n} \alpha_i Z_i\Big)^2 \geqq \int_{A_\varepsilon}\Big(\sum_{i=m}^{n} \alpha_i Z_i\Big)^2 dP = \sum_{k=m}^{n} \int_{B_k}\Big(\sum_{i=m}^{n} \alpha_i Z_i\Big)^2 dP \geqq$$

$$\geqq \sum_{k=m}^{n} \int_{B_k}\Big(\sum_{i=m}^{k} \alpha_i Z_i\Big)^2 dP+2\sum_{k=m}^{n}\ \sum_{j=k+1}^{n}\ \sum_{i=m}^{k} \int_{B_k} \alpha_i Z_i \alpha_j Z_j\, dP \geqq \varepsilon^2 P(A_\varepsilon)-1/3\cdot 2^m,$$

by definition of B_k and (8). On the other hand,

$$E\Big(\sum_{i=m}^{n} \alpha_i Z_i\Big)^2 = \sum_{i=m}^{n} E(\alpha_i Z_i)^2+2\sum_{j=m+1}^{n}\ \sum_{i=m}^{j-1} E(\alpha_i Z_i \alpha_j Z_j) \leqq$$

$$\leqq \sum_{i=m}^{n} E(\alpha_i Z_i)^2+1/6\cdot 2^m,$$

by (7). Hence,

$$P(A_\varepsilon) \leqq \frac{1}{\varepsilon^2}\Big(\sum_{i=m}^{n} E(\alpha_i Z_i)^2 +1/2\cdot 2^m\Big).$$

If $2^{-(N+1)/2}<\varepsilon<2^{-N/2}$ for some $1\leqq N\leqq m-1$, then we get the stated result, while if $0<\varepsilon<2^{-m/2}$, the inequality holds trivially.

The Komlós theorem follows from Lemma 1 and the next result.

LEMMA 5. *Suppose $X_n\to X$ $\sigma(L^1, L^\infty)$ for some random variable $X\in L^1$. Then there exists a subsequence $\{X^0_n\}$ of $\{X_n\}$ such that for each further subsequence $\{X'_n\}$,*

$$\frac{1}{n}\sum_{i=1}^{n} X'_i \to X \quad \textit{a.s. (and in } L^1).$$

PROOF. We can assume (2), (3), (5) and (6) hold for subsequences of $\{X_n\}$. By Lemma 4, there exists a subsequence $\{X^0_n\}$ of $\{X_n\}$ such that for $\varepsilon>0$, $1\leqq m<n$

and any further subsequence $\{X_n'\}$,

$$P\left(\max_{m\leq j\leq n}\left|\sum_{i=m}^{j}\frac{Z_i}{i}\right|>\varepsilon\right)\leq\frac{1}{\varepsilon^2}\left(2\sum_{i=m}^{n}\frac{EZ_i^2}{i^2}+\frac{1}{2^m}\right),$$

where $Z_i=F_i(X_i')-\beta_i$, $i\geq 1$. By (6) and (3),

$$\sum_{i=1}^{\infty}\frac{EZ_i^2}{i^2}\leq\sum_{i=1}^{\infty}\frac{EF_i(X_i')^2}{i^2}+2\sum_{i=1}^{\infty}\frac{1}{i^2}<\infty.$$

Consequently, $\sum_{i=1}^{\infty}\frac{Z_i}{i}$ converges a.s. By the Kronecker Lemma, Lemma 2 and (2), $\frac{1}{n}\sum_{i=1}^{n}X_i'\to X$ a.s.

REMARK. Since $\{X_n\}$ is uniformly integrable, we have convergence in L^1 as well. In addition, X is the only possible limit under the given hypothesis.

THEOREM 6 (Komlós). *Suppose $\{X_n\}$ is a sequence of random variables satisfying $\sup_n E|X_n|<\infty$. Then there exists a subsequence $\{X_n^0\}$ and a random variable $\beta\in L^1$ such that for each further subsequence $\{X_n'\}$,*

$$\frac{1}{n}\sum_{i=1}^{n}X_i'\to\beta\quad a.s. \tag{9}$$

PROOF. By Lemma 1, we can split a subsequence $\{X_{k_n}\}$ into $X_{k_n}=Y_n+Z_n$, where $Y_n\to Y$ $\sigma(L^1,L^\infty)$ for some $Y\in L^1$, and $Z_n\to 0$ a.s. Applying Lemma 5 to $\{Y_n\}$, the result follows.

The Komlós theorem can be proved using an argument similar to that given by Etemadi [2] in proving the strong law of large numbers for pairwise independent, identically distributed random variables.

THEOREM 7 (Komlós). *If $\sup_n E|X_n|<\infty$, then (9) holds.*

PROOF. Without loss of generality, we can assume $X_n\geq 0$, $n\geq 1$. By Lemmas 1, 3 and (7), we can suppose for any subsequence $\{X_n'\}$

$$\sum_{n=1}^{\infty}P(X_n'>n)<\infty, \tag{10}$$

$$\sum_{n=1}^{\infty}\frac{EZ_n^2}{n^2}<\infty; \tag{11}$$

and

$$\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}|E(Z_iZ_j)|<\infty,\quad\text{where}\quad Z_n=F_n(X_n')-\beta_n,\quad n\geq 1. \tag{12}$$

Furthermore, by Lemma 2,

$$\beta_k\to\beta\quad\text{a.s. for some}\quad\beta\in L^1. \tag{13}$$

Let $\{X_n'\}$ be a subsequence of $\{X_n\}$, and define

$$S_n = \sum_{i=1}^{n} X_n', \quad S_n^* = \sum_{i=1}^{n} F_i(X_i'), \quad T_n = \sum_{i=1}^{n} \beta_i.$$

Let $\varepsilon>0$, $\alpha>1$, and write $k_n=[\alpha^n]$, $n\geqq 1$. Now, by (11) and (12),

$$\sum_{n=1}^{\infty} P(|S_{k_n}^* - T_{k_n}| > k_n\varepsilon) \leqq c \sum_{n=1}^{\infty} \frac{1}{k_n^2} E(S_{k_n}^* - T_{k_n})^2 \leqq$$

$$\leqq c \sum_{n=1}^{\infty} \frac{1}{k_n^2} \sum_{i=1}^{k_n} EZ_i^2 + c \sum_{n=1}^{\infty} \frac{1}{k_n^2} \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} |E(Z_i Z_j)| \leqq c \sum_{i=1}^{\infty} \frac{EZ_i^2}{i^2} + c \sum_{n=1}^{\infty} \frac{1}{k_n^2} < \infty$$

where c denotes a constant, possibly different at each appearance. By the Borel—Cantelli lemma,

$$\frac{S_{k_n}^* - T_{k_n}}{k_n} \to 0 \quad \text{a.s.}$$

By (10) and (13), $\frac{S_{k_n}}{k_n} \to \beta$ a.s. By monotonicity of $\{S_n\}$, we get

$$\frac{1}{\alpha}\beta \leqq \varliminf_n \frac{S_n}{n} \leqq \varlimsup_n \frac{S_n}{n} \leqq \alpha\beta \quad \text{a.s.}$$

Since this holds for all $\alpha>1$, we conclude $\frac{S_n}{n} \to \beta$ a.s.

REMARK. With slight modifications, both proofs will work under the weaker hypothesis that $\{X_n\}$ contains a subsequence $\{Y_n\}$ such that $Y_n \to \mu$, $\int |x|\, d\mu(x) < \infty$. (Cf. Aldous [1].)

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(Received January 19, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 187—190.

A RULE OF SIGNS FOR REAL EXPONENTIAL POLYNOMIALS

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1. Introduction. A real exponential polynomial is given by $f(x)=\sum_{i=1}^{N} P_i(x)e^{\lambda_i x}$, where λ_i are distinct real numbers and P_i are real polynomials. Some classical results provide upper bounds on $Z(f)$, the number of real zeros of f (each counted according to its multiplicity). In the particular case that the P_i are non-zero constants, say $P_i=a_i$ $(i=1, \dots, N)$, Laguerre's rule of signs [1] bounds $Z(f)$ by the number of changes of sign in the sequence $a_1, \dots, a_N$. This number, which we denote by $W(a_1, \dots, a_N)$, is defined as the number of pairs a_{m-k}, a_m $(k \geqq 1)$ such that $a_{m-k}a_m<0$ and $a_{m-\nu}=0$ for $\nu=1, \dots, k-1$. With this notation, Laguerre's rule can be formulated as follows [2, V.77].

THEOREM A (Laguerre). *Let* $f(x)=\sum_{i=1}^{N} a_i e^{\lambda_i x}$, *where the* a_i *and* λ_i *are real,* $a_i \neq 0$ $(i=1, \dots, N)$ *and* $\lambda_1<\dots<\lambda_N$. *Then* $Z(f) \leqq W(a_1, \dots, a_N)$, *and* $Z(f)$ *is of the same parity as* $W(a_1, \dots, a_N)$.

In the general case we have [2, V.75]

THEOREM B. *Let* $f(x)=\sum_{i=1}^{N} P_i(x)e^{\lambda_i x}$, *where* P_i *is a real polynomial of degree* n_i, $P_i(x) \not\equiv 0$ $(i=1, \dots, N)$ *and the* λ_i *are distinct real numbers. Then*

$$Z(f) \leqq \sum_{i=1}^{N} n_i+N-1. \tag{1}$$

Theorem B does not contain Theorem A, since $W(a_1, \dots, a_N) \leqq N-1$. We will replace (1) by an estimate that implies it, and reduces to Laguerre's when each P_i is a constant. The proof uses Theorem A. Then we will discuss the sharpness, or possible lack of sharpness, of this bound for $Z(f)$.

2. A rule of signs. In (1), $N-1$ can be replaced by an expression that is equal to $W(a_1, \dots, a_N)$ if $n_i=0$ and $P_i(x)=a_i$ for $i=1, \dots, N$.

THEOREM 1. *Let* $f(x)=\sum_{i=1}^{N} P_i(x)e^{\lambda_i x}$, *where* P_i *are real polynomials and* $\lambda_1<\dots$ $\dots<\lambda_N$. *Suppose no* P_i *is identically zero; let* n_i *be the degree and* a_i *the leading coefficient of* P_i. *Then,*

$$Z(f) \leqq \sum_{i=1}^{N} n_i+W_f, \tag{2}$$

where

(3) $$W_f = W\big((-1)^{m_1}a_1, (-1)^{m_2}a_2, \dots, (-1)^{m_N}a_N\big)$$

with $m_k = n_1 + \dots + n_k$ $(1 \leqq k \leqq N)$. *Moreover, both sides of* (2) *are of the same parity.*

PROOF. Let

$$g_1(x) = e^{\lambda_1 x}\big(e^{-\lambda_1 x} f(x)\big)^{(n_1)}$$

and for $2 \leqq k \leqq N$, let

$$g_k(x) = e^{\lambda_k x}\big(e^{-\lambda_k x} g_{k-1}(x)\big)^{(n_k)}.$$

Then

$$g_N(x) = b_1 e^{\lambda_1 x} + \dots + b_N e^{\lambda_N x},$$

where $b_1, \dots, b_N$ are non-zero constants, given by

(4) $$b_i = (n_i)!\, a_i \prod_{\substack{j=1 \\ j \neq i}}^{N} (\lambda_i - \lambda_j)^{n_j}.$$

By Rolle's theorem, $Z(f) \leqq \sum_{i=1}^{N} n_i + Z(g_N)$, and by Laguerre's rule, $Z(g_N) \leqq W(b_1, \dots, b_N)$, whence

$$Z(f) \leqq \sum_{i=1}^{N} n_i + W(b_1, \dots, b_N).$$

Now from (4) we have

$$\operatorname{sgn} b_j = (-1)^{n_{j+1} + \dots + n_N} \operatorname{sgn} a_j \quad (1 \leqq j \leqq N-1)$$

and $\operatorname{sgn} b_N = \operatorname{sgn} a_N$, so that on multiplying the sequence $b_1, \dots, b_N$ by $(-1)^{m_N}$ we get

$$W(b_1, \dots, b_N) = W\big((-1)^{m_1}a_1, \dots, (-1)^{m_N}a_N\big),$$

and (3) is proved. The statement concerning the parity of $Z(f)$ follows easily from [2, V.8], since $f(x) \sim a_N x^N e^{\lambda_N x}$ $(x \to +\infty)$ and $f(x) \sim a_1 x^{n_1} e^{\lambda_1 x}$ $(x \to -\infty)$.

3. Sharpness. If $f(x) = \sum_{i=1}^{N} P_i(x) e^{\lambda_i x}$, we define m_k $(1 \leqq k \leqq N)$ and W_f as in Theorem 1, and set $d = m_N = \sum_{i=1}^{N} n_i$. The following result shows that equality can hold in (2).

THEOREM 2. *Make any choice of integers* $N \geqq 1$, W $(0 \leqq W \leqq N-1)$ *and* $n_i \geqq 0$ $(i = 1, \dots, N)$ *and of real numbers* λ_i *with* $\lambda_1 < \lambda_2 < \dots < \lambda_N$. *Prescribe* $d+W$ *real numbers* x_ν, *with* $x_1 \leqq x_2 \leqq \dots \leqq x_{d+W}$. *There exist* N *real polynomials* $P_i(X) \not\equiv 0$ *with* $\deg P_i = n_i$, *such that if* $f(x) = \sum_{i=1}^{N} P_i(x) e^{\lambda_i x}$, *then* $W_f = W$, $Z(f) = d + W_f$ *and the real zeros of* f *are the* x_ν, *each with a multiplicity equal to the number of its occurrences in the sequence* $\{x_\nu\}_{\nu=1}^{d+W}$.

PROOF. Consider the $d+N$ coefficients of $P_1, \dots, P_N$ as unknowns; let a_i denote the coefficient of x^{n_i} in P_i. If we add to the conditions

(5) $$f(x_\nu) = 0, \quad \nu = 1, \dots, d+W$$

(modified appropriately if multiple zeros are prescribed) the $(N-1)-W$ equations

$$a_i-(-1)^{n_{i+1}}a_{i+1} = 0, \quad i = 1, \dots, N-1-W, \tag{6}$$

we have a system of $d+N-1$ homogeneous linear equations for $d+N$ unknowns. We shall show that any non-trivial solution of this system is such that

$$a_i \neq 0 \quad \text{for} \quad i = 1, \dots, N. \tag{7}$$

Then, the corresponding real exponential polynomial f has all the required properties. Indeed, each P_i has the prescribed degree n_i. Also, (7) implies that $W_f = W((-1)^{m_1}a_1, (-1)^{m_2}a_2, \dots, (-1)^{m_N}a_N)$. Then because of (6),

$$W_f = W((-1)^{m_{N-W}}a_{N-W}, \dots, (-1)^{m_N}a_N) \leqq W. \tag{8}$$

Finally, $Z(f) \leqq d+W_f \leqq d+W$ by (2) and (8), so that $Z(f)=d+W$ and $W=W_f$ because of (5).

To establish (7) we first observe that if $a_i=0$ for some i then either $P_i(X)\equiv 0$ or $\deg P_i \leqq n_i-1$. Consider now the a_i with $i \leqq N-W$; by (6), they are all zero if one of them is. But $a_i=0$ for each $i\leqq N-W$ would entail, by the preceding observation and (2), $Z(f) \leqq \sum_{i=1}^{N} n_i-(N-W)+W_f \leqq d+W-1$, in contradiction with (5). Hence (7) holds for $i\leqq N-W$; the same type of argument as in (8) then yields $W_f \leqq W$. Consequently, if at least one a_i with $i>N-W$ were zero, (2) would give $Z(f) \leqq \sum_{i=1}^{N} n_i-1+W$, again contradicting (5). This completes the proof of (7).

In contrast to Theorem 2, we shall now show that equality can fail to hold in (2), by a margin as wide as we please. (Remember that $d+W_f-Z(f)$ must be even.) The notation is the same as in Theorem 2. (Theorem 2 is the case $M=0$ of Theorem 3; we treat that case separately since its proof is somewhat simpler.)

THEOREM 3. *Choose integers* $N\geqq 1$, W $(0\leqq W \leqq N-1)$, $n_i \geqq 0$ $(i=1, \dots, N)$ *and* M *(*M *even,* $0\leqq M \leqq d+W$*), and reals* λ_i $(\lambda_1<\lambda_2<\dots<\lambda_N)$. *Prescribe* $d+W-M$ *real numbers* x_ν, *with* $x_1 \leqq x_2 \leqq \dots \leqq x_{d+W-M}$. *There exist N real polynomials* $P_i(x) \not\equiv 0$ *with* $\deg P_i = n_i$ *such that if* $f(x)=\sum_{i=1}^{N} P_i(x)e^{\lambda_i x}$, *then* $W_f=W$, $Z(f)=d+W-M$ *and each* x_ν *is a zero of f with a multiplicity equal to the number of its occurrences in the sequence* $\{x_\nu\}_{\nu=1}^{d+W-M}$.

PROOF. We may assume $\lambda_1>0$. We also assume $x_\nu<x_{\nu+1}$ for all ν (the proof goes through *mutatis mutandis* in case of multiple zeros). Choose some $x^*>x_{d+W-M}$, and consider the following system of $d+N-1$ homogeneous linear equations for the $d+N$ coefficients of the P_i:

$$f(x_\nu) = 0, \quad \nu = 1, \dots, d+W-M, \tag{9}$$

$$a_i-(-1)^{n_{i+1}}a_{i+1} = 0, \quad i = 1, \dots, N-1-W, \tag{10}$$

$$f^{(k)}(x^*) = 0, \quad k = 1, \dots, M. \tag{11}$$

Any non-trivial solution of this system is such that $a_i \neq 0$ for $1 \leqq i \leqq N$. To see this, we consider f'. Since $\lim\limits_{x \to -\infty} f(x)=0$, and because of (9), f' has at least $d+W-M$ zeros on $(-\infty, x_{d+W-M})$ [2, V.16]. With (11) it follows that $Z(f') \geqq d+W$.

Now $f'(x)=\sum\limits_{i=1}^{N} Q_i(x)e^{\lambda_i x}$ with $Q_i(x)=\lambda_i P_i(x)+P_i'(x)$; since $\lambda_i \neq 0$, either $Q_i(x) \equiv P_i(x) \equiv 0$, or $\deg Q_i = \deg P_i$. On combining this with (2) we see that $a_i=0$ for all $i \leqq N-W$ would imply $Z(f') \leqq d-(N-W)+W_{f'} \leqq d+W-1$, which is not true. Hence $a_i \neq 0$ for $i \leqq N-W$, by (10), and $W_{f'} \leqq W$ (the sequence involved has at most N terms, and the first $N-W$ have the same sign). This in turn implies $a_i \neq 0$ for $N-W<i \leqq N$ (else $Z(f') \leqq d-1+W$). So $a_i \neq 0$, and $\deg P_i = n_i$, for each i. Also, $W_f = W_{f'}$ ($\lambda_i a_i \neq 0$, so $\lambda_i a_i$ is the leading coefficient of Q_i; and $\lambda_i>0$). Then, $d+W \leqq Z(f') \leqq d+W_{f'} \leqq d+W$ gives $W_f = W$.

It remains to show that $Z(f)=d+W-M$. If not, then f would have at least 2 more zeros, because of the parity statement in Theorem 1 ($W=W_f$ and M is even). But then f' would have a zero distinct from those already enumerated, which is impossible. This concludes the proof.

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(Received January 24, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 191—199.

ON THE CONVERGENCE OF EIGENFUNCTION EXPANSION IN THE NORM OF SOBOLEFF SPACES

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1. Let $S_k \subset R^n$ $(n \geqq 3;\ k=1, \dots, l)$ be manifolds of dimension $\dim S_k = m_k \leqq \leqq n-3$ having smooth projection to R^{m_k}, i.e. there exist coordinates $(\xi, y) = =(\xi_1, \dots, \xi_{m_k};\ y_1, \dots, y_{n-m_k})$ and functions $\varphi_j^k \in C^1(R^{m_k} \to R^{n-m_k})$ such that

$$S_k = \{(\xi, y) \in R^n\colon\ y_j = \varphi_j^k(\xi),\ |\nabla \varphi_j^k(\xi)| \leqq C_j^k\}, \quad S = \bigcup_{k=1}^{l} S_k.$$

Let $q \in C^\infty(R^n \setminus S)$ be a real-valued function, for which

(1) $$|D^\alpha q(x)| \leqq C[\operatorname{dist}(x, S)]^{-\tau-|\alpha|}, \quad (x \in R^n,\ 0 \leqq |\alpha| \leqq 1),$$

holds, for some $\tau \geqq 0$.

Consider the Schrödinger operator $L_0 = -\Delta + q(x)\,\cdot$, $D(L_0) = C_0^\infty(R^n)$. Such operators occur as the Hamiltonian of molecules [6—12]. E.g., in the case of Li (or H_2) molecule we have $n=6$, $m=3$, $x \in R^3$, $y \in R^3$, $q(x, y) = c_1|x|^{-1} + c_2|y|^{-1} + +c_3|x-y|^{-1}$, $H = -\Delta + q(x, y)\,\cdot$. In the case of homogeneous and isotropic space the manifolds S_k are subspaces in R^n.

It is easy to see that for $\dim S \leqq n-3$ we have $q \in L^2_{\mathrm{loc}}(R^n)$ if $\tau < 3/2$. Indeed, taking into account

$$l^{-1} \sum_{k=1}^{l} [\operatorname{dist}(x, S_k)]^{-1} \leqq [\operatorname{dist}(x, S)]^{-1} \leqq \sum_{k=1}^{l} [\operatorname{dist}(x, S_k)]^{-1},$$

it is enough to prove this for $S = S_k$, $\dim S = m \leqq n-3$,

$$S = \{(\xi, y) \in R^n\colon\ y_j = \varphi_j(\xi),\ |\nabla \varphi_j(\xi)| \leqq C_j;\ j = 1, \dots, n-m\}.$$

Using the coordinate transformation $(\xi, y) \to (\xi, z)$, $z_j = y_j - \varphi_j(\xi)$ we have for the Jacobian $D(\xi, z)/D(\xi, y) = 1$ and for any $0 \leqq \eta \in C_0^\infty(R^n)$

(2) $$\int_{R^n} |q(x)|^2 \eta(x)\, dx = \int_{R^m} d\xi \int_{R^{n-m}} |q(\xi, z+\varphi(\xi))|^2 \eta(\xi, z+\varphi(\xi))\, dz,$$

where $\varphi = (\varphi_1, \dots, \varphi_{n-m}) \in C^1(R^m \to R^{n-m})$. On the other hand for any $x = (\xi, y) \in R^n$ and $u = (\tilde{\xi}, \varphi(\tilde{\xi})) \in S$ we have

$$|y-\varphi(\xi)| \leqq |y-\varphi(\tilde{\xi})| + |\varphi(\tilde{\xi}) - \varphi(\xi)| \leqq |y-\varphi(\tilde{\xi})| + |\nabla\varphi(\xi^*)| \cdot |\tilde{\xi} - \xi| \leqq$$
$$\leqq C(|y-\varphi(\tilde{\xi})| + |\tilde{\xi} - \xi|),$$

hence

$$|y-\varphi(\xi)|^2 \leqq 2C^2(|y-\varphi(\tilde{\xi})|^2 + |\tilde{\xi} - \xi|^2) = 2C^2|x-u|^2,$$

i.e. $|y-\varphi(\xi)| \leqq C \operatorname{dist}(x, S)$, consequently

$$|q(\xi, z+\varphi(\xi))| \leqq C[\operatorname{dist}((\xi, z+\varphi(\xi)), S)]^{-\tau} \leqq C|z|^{-\tau}.$$

According to (2) we have

$$\int_{R^n} |q(x)|^2 \eta(x)\, dx \leqq C \int_{R^m} d\xi \int_{R^{n-m}} |z|^{-2\tau} \eta(\xi, z+\varphi(\xi))\, dz < \infty \tag{3}$$

if $2\tau < n-m$. But we assume in this work that $m \leqq n-3$, i.e. $n-m \geqq 3$ and hence for $\tau < 3/2$ we get $2\tau < 3 \leqq n-m$. It follows from Lemma 3 of the present work that the operator L_0 is bounded below, i.e. $(L_0 f, f) = (-\Delta f, f) + (gf, f) = (\nabla f, \nabla f) + (qf, f) \geqq -c(f, f)$ for every $f \in C_0^\infty(R^n)$ and hence, by a theorem of K. O. Friedrichs [3] the operator L_0 has a selfadjoint extension L with $L \geqq -cI$. Denote $L = \int_{-c}^{\infty} \lambda\, dE_\lambda$ the spectral expansion of L and consider for any $f \in L_2(R^n)$ the expansion $E_\lambda f$.

It is proved in [5]: if $\tau = 1$ and $0 \leqq s \leqq 1$, then $\|E_\lambda f - f\|_{H^s(R^n)} \to 0$ as $\lambda \to \infty$. $H^s(R^n)$ denotes the space of functions from $L_2(R^n)$, with the norm [6, 2.3.3]

$$\|f\|_{H^s(R^n)} := \|(I-\Delta)^{s/2} f\|_{L_2(R^n)} = \|(1+|\xi|^2)^{s/2} \hat{f}(\xi)\|_{L_2(R^n)}.$$

Later on this theorem was extended in [4] for $\tau = 1$ and $0 \leqq s \leqq 2$. The localization of E_λ was investigated in [8]. Our aim is to prove the following

THEOREM. *Suppose $\tau \in [0, 3/2)$ and $0 \leqq s \leqq 2$ or $\tau \in [0, 1/2)$ and $0 \leqq s < \frac{7}{2} - \tau$. Then, for any $f \in H^s(R^n)$ we have*

$$\|E_\lambda f - f\|_{H^s(R^n)} \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{4}$$

It follows from Lemma 3 below — among others — taking into account the Kato—Rellich theorem [11, X.2] that the operator L_0 is essentially selfadjoint, further $D(\bar{L}_0) = D(L) = H^2(R^n)$. Our theorem seems to be true for arbitrary $\tau \in [0, 3/2)$ and $0 \leqq s < \frac{7}{2} - \tau$ but our Lemma 9 is not enough to prove this. According to the ideas of L. L. Stachó [15] this last result does not seem to be refinable, namely we can not replace $\tau = 3/2$ or $s = \frac{7}{2} - \tau$.

The author is indebted to professor Š. A. Alimov and to V. S. Serov for their valuable suggestions.

2. For the proof we need some lemmas.

LEMMA 1. *Let $k \geqq 3$, $1 \leqq p < k$, $0 \leqq s < k/p$. Then for any $f \in L_p^s(R^k)$*

$$\||x|^{-s} f(x)\|_{L_p(R^k)} \leqq C \|f\|_{L_p^s(R^k)} \tag{5}$$

holds.

Here and below in this work C is a constant independent of f and not necessarily the same in each occurrences.

PROOF. Using the notation $I:=\| |x|^{-s}f(x)\|^p_{L_p(R^k)}$ we get by Hölder's inequality

$$I=\int_{R^k}|x|^{-sp}|f(x)|^p\,dx\leq p\int_{\theta}d\theta\int_0^\infty r^{k-1-sp}\int_r^\infty |f|^{p-1}\left|\frac{\partial f}{\partial t}\right|dt\,dr=$$

$$=\frac{p}{k-sp}\int_{R^k}|x|^{-sp+1}|f|^{p-1}\left|\left(\nabla_x f,\frac{x}{|x|}\right)\right|dx\leq$$

$$\leq C\Big(\int_{R^k}(|x|^{-s}|f(x)|)^p\,dx\Big)^{(p-1)/p}\Big(\int_{R^k}|\nabla f|^p|x|^{(-s+1)p}\,dx\Big)^{1/p}=$$

$$=CI^{(p-1)/p}\Big(\int_{R^k}|\nabla f|^p|x|^{(-s+1)p}\,dx\Big)^{1/p},$$

hence

(6) $$\| |x|^{-s}f(x)\|_{L_p(R^k)}\leq C\| |x|^{(-s+1)}\nabla f(x)\|_{L_p(R^k)}.$$

If s is an integer, then iterating (6) s times we get (5).

Now define

$$s_0:=\begin{cases}\dfrac{k}{p}-1, & \text{when } \dfrac{k}{p} \text{ is an integer}\\[2mm] \left[\dfrac{k}{p}\right] & \text{otherwise.}\end{cases}$$

Taking into account Theorem 4.3.2/2 of Triebel [6]:

$$L^s_p(R^k)=\big(L_p(R^k),W^{s_0}_p(R^k)\big),\quad s=\theta s_0,\quad 0<\theta<1;$$

we obtain

(7) $$\| |x|^{-s}f(x)\|_{L_p(R^k)}\leq C\|f\|_{L^s_p(R^k)}\quad (0\leq s\leq s_0,\ p<k/s).$$

Now let $s\in(s_0,k/p)$. It follows from (7) that for $1\leq p_0<k/s_0$

(8) $$\| |x|^{-s_0}f(x)\|_{L_{p_0}(R^k)}\leq C\|f\|_{L^{s_0}_{p_0}(R^k)}$$

holds. On the other hand, for any $1\leq p_1<k/(s_0+1)$ we get from (7)

(9)

$$\| |x|^{-s_0}f(x)\|_{L^1_{p_1}(R^k)}\leq C[\| |x|^{-s_0}f(x)\|_{L_{p_1}(R^k)}+\| |x|^{-s_0}\nabla f(x)\|_{L_{p_1}(R^k)}+\| |x|^{-s_0-1}f(x)\|_{L_{p_1}(R^k)}]\leq$$

$$\leq C[\|f\|_{L^{s_0}_{p_1}(R^k)}+\|f\|_{L^{s_0+1}_{p_1}(R^k)}]\leq C\|f\|_{L^{s_0+1}_{p_1}(R^k)}.$$

Taking into account $(L_{p_0},L^1_{p_1})_\delta=L^\delta_p$ $(0<\delta<1,\ p^{-1}=(1-\delta)p_0^{-1}+\delta p_1^{-1})$ (cf. Triebel [6], 2.4.2/1) we obtain from (8) and (9) the estimate

(10) $$\| |x|^{-s_0}f(x)\|_{L^\delta_p(R^k)}\leq C\|f\|_{L^{s_0+\delta}_p(R^k)}\quad (\forall\, 0<\delta<1).$$

Now, using (10) we prove (5) for $s_0<s<k/p$. Define $\delta=s-s_0$. It is easy to see that $\delta\in(0,1)$. Indeed, if k/p is an integer, then $\delta=s-s_0=\left(\frac{k}{p}-\varepsilon\right)-\left(\frac{k}{p}-1\right)=1-\varepsilon$

$\left(s=\frac{k}{p}-\varepsilon,\ 0<\varepsilon<1\right)$. If k/p is not an integer, then $\delta=\left(\frac{k}{p}-\varepsilon\right)-\left[\frac{k}{p}\right]<1-\varepsilon$. Consequently, from (10) we get

$$\||x|^{-s}f(x)\|_{L_p(R^k)}=\left\||x|^{-\delta}(|x|^{-s_0}f(x))\right\|_{L_p(R^k)}\leqq$$
$$\leqq C\||x|^{-s_0}f(x)\|_{L_p^{\delta}(R^k)}\leqq C\|f\|_{L_p^s(R^k)}.$$

Lemma 1 is proved.

LEMMA 2. *For any natural number* $k\geqq 3$, $0\leqq s<3/2$ *and* $f\in C_0^\infty(R^k)$

$$\||x|^{-s}f(x)\|^2_{L_2(R^k)}\leqq C\|f\|_{H^1(R^k)}\|f\|_{H^2(R^k)}. \tag{11}$$

PROOF. First we prove (11) for $s\geqq 1$. Using (6) at $p=2$ and taking into account the inequality $|x|^{-2s+2}\leqq |x|^{-1}+1$ $(0\leqq 2s-2\leqq 1)$ we get

$$\||x|^{-s}f(x)\|^2_{L_2(R^k)}\leqq C\||x|^{-s+1}\nabla f(x)\|^2_{L_2(R^k)}\leqq$$
$$\leqq C[\||x|^{-1/2}\nabla f(x)\|^2_{L_2(R^k)}+\|\nabla f(x)\|^2_{L_2(R^k)}].$$

Hence, taking into account the following estimate (cf. [4, Lemma 1])

$$\||x|^{-1/2}f(x)\|^2_{L_2(R^k)}\leqq C\|f\|_{H^1(R^k)}\|f\|_{L_2(R^k)}\quad (k\geqq 3, f\in C_0^\infty(R^k)) \tag{12}$$

we obtain (11) for the case $1\leqq s<3/2$. If $0\leqq s\leqq 1$, then (11) follows from (5) immediately. Lemma 2 is proved.

LEMMA 3. *For any* $\tau\in[0,3/2)$ *and* $\varepsilon>0$ *there exists* $C(\varepsilon)>0$ *such that for every* $f\in C_0^\infty(R^n)$ $(n\geqq 3)$ *the following estimate holds:*

$$\|qf\|^2_{L_2(R^n)}\leqq \varepsilon\|f\|^2_{H^2(R^n)}+C(\varepsilon)\|f\|^2_{L_2(R^n)}. \tag{13}$$

PROOF. Using (3) for $\eta=|f|^2$, applying (11) for $k=n-m$ and taking into account the inequality

$$ab\leqq \varepsilon a^2+\frac{1}{4\varepsilon}b^2\quad (a,b,\varepsilon>0), \tag{14}$$

we get

$$\|qf\|^2_{L_2(R^n)}\leqq C\|f\|_{H^1(R^n)}\|f\|_{H^2(R^n)}\leqq \frac{\varepsilon}{2}\|f\|^2_{H^2(R^n)}+C(\varepsilon)\|f\|^2_{H^1(R^n)}.$$

Hence, taking into consideration the estimate

$$\|f\|^2_{H^1(R^n)}\leqq \varepsilon_1\|f\|^2_{H^2(R^n)}+C(\varepsilon_1)\|f\|^2_{L_2(R^n)},$$

we obtain

$$\|qf\|^2_{L_2(R^n)}\leqq \frac{\varepsilon}{2}\|f\|^2_{H^2(R^n)}+\varepsilon_1 C(\varepsilon)\|f\|^2_{H^2(R^n)}+C(\varepsilon_1)C(\varepsilon)\|f\|^2_{L_2(R^n)}.$$

If we choose ε_1 so that $\varepsilon_1 C(\varepsilon)<1/2$, then (13) follows. Lemma 3 is proved.

COROLLARY. *For any* $\tau\in[0,3/2)$ *the operator* L_0 *is essentially selfadjoint and* $D(\bar{L}_0)=D(L)=H^2(R^n)$.

PROOF. From (13) we obtain for any $\varepsilon>0$ the estimate

$$\|qf\|_{L_2(R^n)} \leqq \varepsilon\|(I-\Delta)f\|_{L_2(R^n)}+C(\varepsilon)\|f\|_{L_2(R^n)}. \tag{15}$$

Since $I-\Delta$ is essentially selfadjoint and $D(\overline{I-\Delta})=H^2(R^n)$, the Corollary follows by Kato—Rellich's theorem [11, X.2].

REMARK. For the essential selfadjointness of L_0 it is enough to prove the estimate

$$\|qf\|_{L_2(R^n)} \leqq C\|f\|_{H^2(R^n)}\|f\|_{H^{2-\delta}(R^n)},$$

for some $\delta>0$, because

$$\|qf\|_{L_2(R^n)} \leqq \varepsilon\|f\|_{H^2(R^n)}+C(\varepsilon)\|f\|_{H^{2-\delta}(R^n)} \leqq$$
$$\leqq \varepsilon\|f\|_{H^2(R^n)}+\varepsilon_1 C(\varepsilon)\|f\|_{H^2(R^n)}+C(\varepsilon_1)C(\varepsilon)\|f\|_{L_2(R^n)}.$$

LEMMA 4. *For any* $f\in H^2(R^n)$

$$\|Lf\|_{L_2(R^n)} \leqq C\|f\|_{H^2(R^n)}. \tag{16}$$

PROOF. Using (13) we obtain for any $f\in H^2(R^n)$

$$\|Lf\|_{L_2(R^n)} = \|-\Delta f+qf\|_{L_2(R^n)} \leqq \|\Delta f\|_{L_2(R^n)}+\|qf\|_{L_2(R^n)} \leqq$$
$$\leqq C[\|f\|_{H^2(R^n)}+\|f\|_{L_2(R^n)}] \leqq C\|f\|_{H^2(R^n)}.$$

Lemma 4 is proved.

LEMMA 5. *There exist constants* $C_1>0$ *and* $C_2>0$ *such that for every* $f\in H^2(R^n)$

$$\|Lf\|^2_{L_2(R^n)} \geqq C_1\|f\|^2_{H^2(R^n)}-C_2\|f\|^2_{L_2(R^n)}. \tag{17}$$

PROOF. Using (14), applying the Cauchy—Bunyakovsky inequality, further taking into account the identity

$$\|Lf\|^2_{L_2(R^n)} = \|\Delta f\|^2_{L_2(R^n)}-2(qf,\Delta f)+\|qf\|^2_{L_2(R^n)},$$

we obtain

$$|(qf,\Delta f)| \leqq \|qf\|_{L_2(R^n)}\|\Delta f\|_{L_2(R^n)} \leqq \varepsilon\|\Delta f\|^2_{L_2(R^n)}+C(\varepsilon)\|qf\|^2_{L_2(R^n)}$$

and

$$\|Lf\|^2_{L_2(R^n)} \geqq \|\Delta f\|^2_{L_2(R^n)}-2|(qf,\Delta f)|+\|qf\|^2_{L_2(R^n)} \geqq$$
$$\geqq \|\Delta f\|^2_{L_2(R^n)}-\varepsilon\|\Delta f\|^2_{L_2(R^n)}-C(\varepsilon)\|qf\|^2_{L_2(R^n)} \geqq$$
$$\geqq (1-\varepsilon)\|\Delta f\|^2_{L_2(R^n)}-C(\varepsilon)\|qf\|^2_{L_2(R^n)}.$$

Now applying (13) for some $\varepsilon_1>0$, it follows

$$\|Lf\|^2_{L_2(R^n)} \geqq (1-\varepsilon)\|\Delta f\|^2_{L_2(R^n)}-\varepsilon_1 C(\varepsilon)\|f\|^2_{H^2(R^n)}-C(\varepsilon,\varepsilon_1)\|f\|^2_{L_2(R^n)}.$$

On the other hand

$$\|\Delta f\|_{L_2(R^n)} = \|\Delta f-f+f\|_{L_2(R^n)} \geqq \|(\Delta-I)f\|_{L_2(R^n)}-\|f\|_{L_2(R^n)}$$

consequently

$$\|Lf\|^2_{L_2(R^n)} \geqq \big(1-\varepsilon-\varepsilon_1 C(\varepsilon)\big)\|f\|^2_{H^2(R^n)}-C(\varepsilon,\varepsilon_1)\|f\|^2_{L_2(R^n)}$$

and hence (17) follows if we set $\varepsilon=1/2$ and ε_1 is small enough. Lemma 5 is proved.

LEMMA 6. *There exists* $\mu_0>0$ *such that for any* $\mu\geqq\mu_0$ *and* $f\in C_0^\infty(R^n)$ *we have*

$$\|L_\mu f\|_{L_2(R^n)} \geqq C_\mu \|f\|_{H^2(R^n)} \quad (L_\mu := L+\mu I). \tag{18}$$

The constant C_μ *does not depend on* f.

PROOF. It follows from (17) using the spectral theorem that

$$\|f\|^2_{H^2(R^n)} \leqq C_1\|Lf\|^2_{L_2(R^n)}+C_2\|f\|^2_{L_2(R^n)} \leqq$$

$$\leqq C \int_{-C_0}^{\infty} (\lambda^2+1)\, d(E_\lambda f, f) \leqq C \int_{-C_0}^{\infty} (\lambda+\mu)^2\, d(E_\lambda f, f) = C\|L_\mu f\|^2_{L_2(R^n)},$$

if $\mu\geqq\mu_0$ and μ_0 is large enough, because in this case we have $\lambda^2+1\leqq(\lambda+\mu)^2$ $(\lambda\geqq -C_0,\ \mu\geqq\mu_0)$. Lemma 6 is proved.

LEMMA 7 [4, Lemma 6]. *Let* A *and* B *be strongly positive selfadjoint operators in the Hilbert space* H. *Suppose that the conditions*

$$D(B)\subset D(A), \tag{19}$$

$$\|Af\|_H \leqq C\|Bf\|_H \quad (f\in D(B)), \tag{20}$$

are fulfilled. Then for any $\theta\in[0, 1]$ *we have*

$$\|A^\theta f\|_H \leqq C_\theta \|B^\theta f\|_H \quad (f\in D(B)). \tag{21}$$

LEMMA 8. *For any* $\mu\geqq\mu_0$, $s\in\left[0, \frac{7}{2}-\tau\right)$ *and* $f\in H^s(R^n)$

$$\|L_\mu^{s/2} f\|_{L_2(R^n)} \leqq C_s\|f\|_{H^s(R^n)}. \tag{22}$$

PROOF. First we prove (22) for $0\leqq s\leqq 2$. It is trivial for $s=0$ and it was proved in Lemma 4 for $s=2$. Now apply Lemma 7 for $A=L_\mu$, $B=I-\Delta$, $D(B)=H^2(R^n)$. We obtain:

$$\|L_\mu^\theta f\|_{L_2(R^n)} \leqq C\|f\|_{H^{2\theta}(R^n)} \quad (0\leqq\theta\leqq 1). \tag{23}$$

Now let $2<s<\frac{7}{2}-\tau$. Using Lemma 1 we obtain for any $p_0<3/\tau$ the estimate

$$\|L_\mu f\|_{L_{p_0}(R^n)} \leqq C[\|f\|_{L_{p_0}(R^n)} + \|f\|_{L^2_{p_0}(R^n)} + \|qf\|_{L_{p_0}(R^n)}] \leqq \tag{24}$$

$$\leqq C[\|f\|_{L^2_{p_0}(R^n)} + \|f\|_{L^\tau_{p_0}(R^n)}] \leqq C\|f\|_{L^2_{p_0}(R^n)}.$$

On the other hand, using Lemma 1 once again, we obtain for any $p_1<3/(\tau+1)$ and $f\in L^3_{p_1}(R^n)$ the estimate

$$\|\nabla L_\mu f\|_{L_{p_1}(R^n)} \leqq C[\|f\|_{L^3_{p_1}(R^n)} + \|(\nabla q)f\|_{L_{p_1}(R^n)} + \tag{25}$$

$$+\|q\nabla f\|_{L_{p_1}(R^n)}] \leqq C[\|f\|_{L^3_{p_1}(R^n)} + \|f\|_{L^{1+\tau}_{p_1}(R^n)}] \leqq C\|f\|_{L^3_{p_1}(R^n)}.$$

Using (24), (25), the equality $(L_{p_0}, L^1_{p_1})_\delta = L^\delta_p$ $(0<\delta<1,\ p^{-1}=(1-\delta)p_0^{-1}+\delta p_1^{-1})$ of Triebel [6, 2.4.2/1] and taking into account that in our case $p<3/(\tau+\delta)$, we obtain for any $\delta\in(0,1)$ and $f\in L^{2+\delta}_p(R^n)$ the estimate

(26) $$\|L_\mu f\|_{L^\delta_p(R^n)} \leqq C\|f\|_{L^{2+\delta}_p(R^n)}.$$

Now we are in the position to prove (22) for $2<s<\dfrac{7}{2}-\tau$. Set $\delta:=s-2$. Then $\delta<\dfrac{7}{2}-\tau-2<\dfrac{3}{2}$, further we obtain from (26) that for any $f\in H^s(R^n)$ we have $L_\mu f\in H^\delta(R^n)$. Using (23) and then (26) we obtain

$$\|L^{s/2}_\mu f\|_{L_2(R^n)} = \|L^{\delta/2}_\mu (L_\mu f)\|_{L_2(R^n)} \leqq C\|L_\mu f\|_{L^\delta_2(R^n)} \leqq C\|f\|_{L^{2+\delta}_2(R^n)} = C\|f\|_{H^s(R^n)}.$$

Lemma 8 is proved.

LEMMA 9. *Suppose* $0\leqq s\leqq 2$, $0\leqq\tau<3/2$ *or* $0\leqq\tau<1/2$ *and* $0\leqq s<\dfrac{7}{2}-\tau$. *Then for any* $\mu\geqq\mu_0$ *and* $g\in H^s(R^n)$ *we have*

(27) $$\|g\|_{H^s(R^n)} \leqq C\|L^{s/2}_\mu g\|_{L_2(R^n)}.$$

PROOF. (27) is trivial for $s=0$ and it was proved in Lemma 6 for $s=2$. Hence, using Lemma 7 for $B=L_\mu$, $A=I-\Delta$, $D(A)=H^2(R^n)$, we obtain

(28) $$\|g\|_{H^s(R^n)} \leqq C\|L^{s/2}_\mu g\|_{L_2(R^n)} \quad (0\leqq s\leqq 2,\ 0\leqq\tau<7/2-\tau).$$

Now suppose $0\leqq\tau<1/2$ and $2<s<\dfrac{7}{2}-\tau$. Let $\delta:=s-2$. For any $g\in C^\infty_0(R^n)$ we have obviously by (28)

(29) $$\|g\|_{H^s(R^n)} = \|(I-\Delta)g\|_{H^\delta(R^n)} \leqq C\|L^{\delta/2}_\mu (I-\Delta)g\|_{L_2(R^n)} \leqq$$
$$\leqq C[\|L^{\delta/2}_\mu g\|_{L_2(R^n)} + \|L^{\delta/2}_\mu (L_\mu - q)g\|_{L_2(R^n)}] \leqq C[\|L^{-1}_\mu (L^{s/2}_\mu g)\|_{L_2(R^n)} + \|L^{\delta/2}_\mu (qg)\|_{L_2(R^n)} +$$
$$+\|L^{s/2}_\mu g\|_{L_2(R^n)}] \leqq C[\|L^{s/2}_\mu g\|_{L_2(R^n)} + \|L^{\delta/2}_\mu (qg)\|_{L_2(R^n)}].$$

Now we estimate $\|L^{\delta/2}_\mu (qg)\|_{L_2}$. We obtain from (3) and (5)

(30) $$\|qg\|_{L_2(R^n)} \leqq C\|g\|_{H^\tau(R^n)} \quad (0\leqq\tau<3/2)$$

and

(31) $$\|gq\|_{H^1(R^n)} \leqq \|q\nabla g\|_{L_2} + \|\nabla q g\|_{L_2} + \|gq\|_{L_2} \leqq C\|g\|_{H^{\tau+1}(R^n)} \quad (0\leqq\tau<1/2).$$

We apply the interpolation theorem of Stein [13]. To this suppose δ is such that $\tau+\delta<3/2$ and choose $\varepsilon>0$ so that $\tau(\delta)=\tau$, where $\tau(z):=z(0{,}5-\varepsilon)+(1{,}5-\varepsilon)(1-z)$. Define the operators A_z and T_z as follows:

$$A_z g := |q(x)|^{\tau(z)/\tau}(\operatorname{sgn} q(x))g(x), \quad T_z g := (I-\Delta)^{z/2}A_z g.$$

From (30) and (31) we obtain for any $g\in C^\infty_0(R^n)$:

$$\|T_z g\|_{L_2(R^n)} = \|A_z g\|_{L_2(R^n)} \leqq C\|g\|_{H^{3/2-\varepsilon}(R^n)} \quad (\mathrm{Re}\, z=0),$$

and

$$\|T_z g\|_{L_2(R^n)} \leqq \|A_z g\|_{H^1(R^n)} \leqq C\|g\|_{H^{3/2-\varepsilon}(R^n)} \quad (\mathrm{Re}\, z=1),$$

hence by Stein's interpolation theorem [13] we get for $z=\delta$:

$$\|T_\delta g\|_{L_2} \leqq C\|g\|_{H^{3/2-\varepsilon}}, \quad \|A_\delta g\|_{L_2} \leqq C\|g\|_{H^{3/2-\varepsilon}},$$

i.e. using also (14) we obtain

$$\|qg\|_{H^\delta(R^n)} \leqq \varepsilon\|g\|_{H^2(R^n)}+C(\varepsilon)\|g\|_{L_2(R^n)}.$$

Hence and from (29) the desired estimate (27) follows. Lemma 9 is proved.

PROOF OF THE THEOREM. Using (22) and (27) we obtain for $f\in H^s(R^n)$:

$$\|f-E_\lambda f\|_{H^s} = \|L_\mu^{-s/2}L_\mu^{s/2}(I-E_\lambda)f\|_{H^s} \leqq$$
$$\leqq C\|L_\mu^{s/2}(I-E_\lambda)f\|_{L_2} = C\|(I-E_\lambda)(L_\mu^{s/2}f)\|_{L_2} \to 0 \quad (\lambda\to\infty).$$

The Theorem is proved.

REMARK. If the S_k's are subspaces, then we can state the Theorem for any $\tau\in[0, 3/2)$ and $s\in\left[0, \frac{7}{2}-\tau\right)$ because in this case we can prove Lemma 9 in a more general form. This follows from the following fact: if $g(z)\in C_0^\infty(R^k\setminus\{0\})$ is a function for which $|D^\alpha g(z)|\leqq C|z|^{-\tau-|\alpha|}$ $(z\neq 0)$ holds, then $g\in H^s(R^n)$ for any $s<\frac{k}{2}-$ $-\tau=:\delta$. For the proof of this fact it is enough to show that $g\in H_1^{k-\tau}(R^k)$ (here H denotes the Nikol'skiĭ's class of functions), because taking into account the well known imbeddings $H_1^{k-\tau}\subset H_2^\delta\subset B_{2,2}^{\delta-\varepsilon}\subset L_2^{\delta-\varepsilon}$ our statement follows. We use here the notations of [14]. For the proof we must show the estimate

$$I := \omega_2^{(2)}(D^\alpha g, t) := \sup_{|h|\leqq t}\int_\Omega |\Delta_h^2 D^\alpha g|\,dz = O(t^{s-|\alpha|}) \quad (\operatorname{supp} g\subset\Omega).$$

The desired estimate follows immediately for $|z|<2h$ and $|z|\geqq 2h$, resp. from the following estimates:

$$\sup_{|h|\leqq t}\int_{\Omega'}|\Delta_h^2 D^\alpha g(z)|\,dz \leqq 4\sup_{|h|\leqq t}\int_{\Omega'}|D^\alpha g(z)|\,dz \leqq$$
$$\leqq C\sup_{|h|\leqq t}\int_0^{2|h|}|z|^{-\tau-|\alpha|+k-1}\,dz = \sup_{|h|\leqq t} O(|h|^{k-\tau-|\alpha|}) =$$
$$= O(t^{s-|\alpha|}), \quad \Omega' := \{z\in\Omega:\ |z|<2|h|\};$$

$$|\Delta_h^2 D^\alpha g| = \left|\sum_{i,j=1}^k \frac{\partial^2}{\partial z_i\partial z_j}(D^\alpha g)(z^*)h_i h_j\right| \leqq$$
$$\leqq C\sum_{|\beta|=2}(D^{\alpha+\beta}g)(z^*)|h|^2, \quad z^*\in[z-h, z+h],$$

hence

$$I \leqq \sup_{|h|\leqq t}|h|^2\sum_{|\beta|=2}\int_{\Omega''}|D^{\alpha+\beta}g|\,dz+O(t^{s-|\alpha|}) \leqq$$
$$\leqq \sup_{|h|\leqq t}\int_{2|h|}^{A}|z|^{-\tau-|\alpha|-2+k-1}\,dz+O(t^{s-|\alpha|}) = O(t^{s-|\alpha|}), \quad \Omega'' := \{z\in\Omega;\ |z|\geqq 2|h|\}.$$

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(Received February 2, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 201—221.

RE-PROXIMITIES

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1. Introduction

D. Harris [8] has introduced *R*-proximities in order to investigate regular-closed extensions of regular topological spaces. With a slight modification of his definition, we say that a binary relation δ on the power set $\mathfrak{P}(X)$ of X is an *R-proximity* iff

R1. $A\delta B$ implies $B\delta A$,
R2. $\emptyset\bar{\delta}X$ ($\bar{\delta}$ means non-δ),
R3. $A\neq\emptyset$ implies $A\delta A$,
R4. $A\delta(B\cup C)$ iff $A\delta B$ or $A\delta C$,
R5. $\{x\}\bar{\delta}X-V$ implies the existence of W such that $\{x\}\bar{\delta}X-W$, $W\bar{\delta}X-V$.

We omit the condition assumed in [8] $\{x\}\delta\{y\}$ implies $x=y$.

Clearly the concept of an *R*-proximity is a generalization of that of an Efremovič proximity (see e.g. [3], p. 63). Similarly to the case of Efremovič proximities, an *R*-proximity δ induces a topology on X if we put

$$x\in \mathrm{cl}\, A \quad \text{iff} \quad \{x\}\delta A. \tag{1.1}$$

This topology is always regular ([8], Lemma 1; in the present paper regularity does not include T_1).

Conversely, if X is a regular topological space, there are *R*-proximities *compatible* with X (i.e. such that they induce the topology of X). One of them is defined by

$$A\delta B \quad \text{iff} \quad \mathrm{cl}\, A\cap \mathrm{cl}\, B\neq\emptyset; \tag{1.2}$$

more generally, if Y is a regular extension of X (i.e. a regular space containing X as a dense subspace), we obtain an *R*-proximity on X by putting for $A, B\subset X$

$$A\delta B \quad \text{iff} \quad \mathrm{cl}_Y A\cap \mathrm{cl}_Y B\neq\emptyset \tag{1.3}$$

([8], Lemma 2).

The purpose of the present paper is to investigate those *R*-proximities that are defined by (1.3); we shall call them *RE-proximities*. A special kind of *RE*-proximities is the concept of an *RC*-proximity; this is an *R*-proximity obtained by (1.3) from a regular-closed extension Y of X.

For technical reasons, we shall use some concepts and results from the theory of syntopogenous spaces (see [3]). We also need some results of a recent paper of K. Matolcsy [9]; the author is very thankful to him for some essential contributions to the content of this paper.

2. Local syntopogenous structures

Let δ be a relation on $\mathfrak{P}(X)$ satisfying R1—R4 (such a relation is called *proximity* in [1], *basic proximity* in [2]), and define, for $A, B \subset X$,

$$(2.1) \qquad A < B \quad \text{iff} \quad A\bar{\delta}X-B.$$

Then $<$ is a symmetrical topogenous order on X, and conversely, if $<$ is a symmetrical topogenous order on X, and δ is defined for $A, B \subset X$ by

$$(2.2) \qquad A\delta B \quad \text{iff} \quad A < X-B \quad \text{does not hold,}$$

then δ satisfies R1—R4 (which can be easily seen using the argument in [3], pp. 62—63).

The relations δ and $<$ obtained from each other with the help of (2.1) and (2.2) will be said to be *associated* with each other.

In particular, δ is an R-proximity iff the symmetrical topogenous order $<$ associated with δ satisfies

$$(2.3) \qquad \{x\} < V \quad \text{implies the existence of } W \text{ such that} \quad \{x\} < W < V$$

(this is another formulation of R5). Let us agree in calling *R-order* a symmetrical topogenous order fulfilling (2.3).

According to the terminology of [10], a *local syntopogenous structure* on X is a system $\mathscr{L}$ of topogenous orders on X satisfying

L1. For $<_1, <_2 \in \mathscr{L}$, there is $< \in \mathscr{L}$ such that $<_1 \cup <_2 \subset <$.

L2. For $< \in \mathscr{L}$, there is $<_0 \in \mathscr{L}$ such that $\{x\} < V$ implies the existence of W with $\{x\} <_0 W <_0 V$.

(2.4) LEMMA. *$<$ is an R-order iff it is a symmetrical topogenous order such that $\{<\}$ is a local syntopogenous structure.* □

In the following we collect some simple facts concerning local syntopogenous structures. As it has been observed in [5], pp. 2—3, the majority of concepts defined in [3] for syntopogenous spaces can be generalized for *order structures* (i.e. systems $\mathscr{L}$ of topogenous orders satisfying L1), so in particular for local syntopogenous structures.

(2.5) LEMMA. *If $\mathscr{L}$ is a local syntopogenous structure then so is $\mathscr{L}^t$.*

PROOF. Let $\mathscr{L}^t = \{<'\}$. If $x \in X$, $V \subset X$, and $x <' V$, then $x < V$ for some $< \in \mathscr{L}$. Let $<_0 \in \mathscr{L}$ correspond to $<$ according to L2. Then $x <_0 W <_0 V$ for some $W \subset X$, hence $x <' W <' V$. □

(2.6) LEMMA ([10], (1.9)). *If $\mathscr{L}$ is a local syntopogenous structure then $\mathscr{L}^p$ is a perfect syntopogenous structure.*

PROOF. For a given $< \in \mathscr{L}$, let $<_0 \in \mathscr{L}$ be chosen according to L2. Then $A <^p B$ implies $x < B$ for $x \in A$, hence $x <_0 C_x <_0 B$ for some $C_x \subset X$. Therefore

$$A <_0^p C = \bigcup_{x \in X} C_x <_0^p B. \quad \square$$

(2.7) COROLLARY. *If $\mathscr{L}$ is a local syntopogenous structure then $\mathscr{L}^{tp}$ is a topology.* □

(2.8) LEMMA. *If $\mathscr{L}_i$ $(i\in I\neq\emptyset)$ is a local syntopogenous structure on X, then so is*

$$\mathscr{L} = \bigvee_{i\in I} \mathscr{L}_i = (\bigcup_{i\in I} \mathscr{L}_i)^g.$$

PROOF. An arbitrary order $<\in\mathscr{L}$ can be written in the form

$$< = (\bigcup_1^n <_k)^q$$

where $<_k\in\mathscr{L}_{i_k}$, $i_k\in I$. Choose $<'_k\in\mathscr{L}_{i_k}$ for $<_k$ according to L2. Then $x<V$ implies by [3], (3.7)

$$V = \bigcap_1^n V_k, \quad x <_k V_k \quad (k = 1, ..., n),$$

consequently $x<'_k W_k <'_k V_k$ for suitable sets W_k, and finally $x<'W<'V$ for

$$W = \bigcap_1^n W_k, \quad <' = (\bigcup_1^n <'_k)^q \in \mathscr{L}. \quad \square$$

(2.9) LEMMA. *If f: $X\to Y$ and $\mathscr{L}$ is a local syntopogenous structure on Y then so is $f^{-1}(\mathscr{L})$ on X.*

PROOF. For $<, <_0\in\mathscr{L}$ satisfying L2, we find that $xf^{-1}(<)V$ implies $f(x)< <Y-f(X-V)$, hence

$$f(x) <_0 U <_0 Y-f(X-V)$$

for a suitable set $U\subset Y$. Setting $W=f^{-1}(U)$, we have

$$f(x) <_0 U \subset Y-f(X-W), \quad f(W) \subset U <_0 Y-f(X-V),$$

i.e. $xf^{-1}(<_0)Wf^{-1}(<_0)V$. □

(2.10) COROLLARY. *If $\mathscr{L}_i$ $(i\in I\neq\emptyset)$ is a local syntopogenous structure on X_i then so is $\mathscr{L}=\bigtimes_{i\in I}\mathscr{L}_i$ on $X=\bigtimes_{i\in I}X_i$.* □

By generalizing the respective definitions formulated for syntopogenous structures in [3], p. 224 and [4], p. 240, we say that a filter base $\mathfrak{r}$ in X is *compressed* with respect to an order structure $\mathscr{L}$ on X iff $<\in\mathscr{L}$, $A<B$ implies the existence of $R\in\mathfrak{r}$ satisfying either $R\subset B$ or $R\cap A=\emptyset$, and that $\mathfrak{r}$ is *round* with respect to $\mathscr{L}$ iff $R\in\mathfrak{r}$ implies the existence of $<\in\mathscr{L}$ and $R_1\in\mathfrak{r}$ such that $R_1<R$. It is easy to check that the statements [3] (15.47), (15.48), (15.50), (15.51), (15.52), (15.54), (15.55) and [4] (16.41) to (16.43) remain valid if we replace syntopogenous structures by arbitrary order structures.

(2.11) LEMMA. *If $\mathscr{L}$ is an order structure on X and $\mathfrak{s}$ is a round, compressed filter then $\mathfrak{s}$ is a maximal round filter.*

PROOF. Let $\mathfrak{s}'\supset\mathfrak{s}$ be a round filter and $S'\in\mathfrak{s}'$. Then there are $S_1'\in\mathfrak{s}'$ and $<\in\mathscr{L}$ such that $S_1'<S'$. Since $S_1'\cap S\neq\emptyset$ for $S\in\mathfrak{s}$, there is $S\in\mathfrak{s}$ such that $S\subset S'$, i.e. $S'\in\mathfrak{s}$. □

(2.12) LEMMA. *If $\mathscr{L}$ is an order structure on X and $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are round filter bases, then*

$$\mathfrak{r}=\mathfrak{r}_1(\cap)\mathfrak{r}_2=\{R_1\cap R_2\colon R_1\in\mathfrak{r}_1,\ R_2\in\mathfrak{r}_2\}$$

is a round filter base provided its elements are non-empty.

PROOF. $R_i'<_i R_i$, $R_i, R_i'\in\mathfrak{r}_i$, $<_i\in\mathscr{L}$ $(i=1,2)$ imply $<_1\cup<_2\subset<$ for a suitable $<\in\mathscr{L}$, hence $R_1'\cap R_2'<R_1\cap R_2$. □

(2.13) COROLLARY. *If $\mathscr{L}$ is an order structure, $\mathfrak{r}$ is a round filter base and $\mathfrak{s}$ is a maximal round filter then $\emptyset\notin\mathfrak{r}(\cap)\mathfrak{s}$ implies $\mathfrak{r}\subset\mathfrak{s}$. Consequently, if $\mathfrak{s}\neq\mathfrak{s}'$ are maximal round filters then there are $S\in\mathfrak{s}$, $S'\in\mathfrak{s}'$ satisfying $S\cap S'=\emptyset$.* □

(2.14) LEMMA. *If $\mathscr{L}$ is an order structure then every round filter base is contained in a maximal round filter.*

PROOF. Apply the Kuratowski—Zorn lemma. □

(2.15) LEMMA. *Let $\mathscr{L}$ be a local syntopogenous structure on X. If $\mathfrak{r}$ is a compressed filter base, and $x\in X$ is a cluster point of $\mathfrak{r}$ with respect to the topology $\mathscr{L}^{tp}$, then $\mathfrak{r}\to x$.*

PROOF. For an $\mathscr{L}^{tp}$-neighbourhood V of x, we have $x<V$ for some $<\in\mathscr{L}$, hence $x<_0W<_0V$ for some $<_0\in\mathscr{L}$ and $W\subset X$. Then $R\cap W\neq\emptyset$ for $R\in\mathfrak{r}$ so that $R_0\subset V$ for some $R_0\in\mathfrak{r}$. □

3. *R*-orders

Let δ be an R-proximity on X and $<$ the R-order associated with δ, $\mathscr{L}=\{<\}$. We shall refer to concepts connected with the local syntopogenous structure $\mathscr{L}$ as to concepts connected with $<$ or δ; e.g. we shall speak of $<$-round filters or δ-compressed filters, etc.

By (2.7) $\mathscr{T}=\{<^p\}$ is a topology; it coincides with the (classical) topology induced by δ in the sense of (1.1). In fact, for the latter $A\subset\operatorname{int}B$ iff $\{x\}\bar{\delta}X-B$ for $x\in A$, i.e. iff $x<B$ for $x\in A$, or equivalently iff $A<^pB$.

(3.1) LEMMA ([8] Lemma 1, 3.1, 3.2). *Let $<$ be an R-order on X, $\mathscr{T}=\{<^p\}$. Then:*

(a) *The topology $\mathscr{T}$ is regular.*

(b) *Every $<$-round filter is regular.*

(c) *Every $\mathscr{T}$-neighbourhood filter is maximal $<$-round.*

PROOF. (c): For a $\mathscr{T}$-neighbourhood V of $x\in X$, we have $x<V$, hence $x<W<V$ for some W. Thus W is a $\mathscr{T}$-neighbourhood of x, and the neighbourhood filter $\mathfrak{v}$ of x is $<$-round. If $\mathfrak{s}\supset\mathfrak{v}$ is a $<$-round filter, and $S\in\mathfrak{s}$, choose $S_1, S_2\in\mathfrak{s}$

such that $S_1 < S$, $S_2 < S_1$. Now $x \notin S_1$ would imply $x < X - S_2$, i.e. $X - S_2 \in \mathfrak{v} \subset \mathfrak{s}$. Hence $x \in S_1$ and $x < S$, $S \in \mathfrak{v}$.

(b): If $\mathfrak{s}$ is a $<$-round filter and $S \in \mathfrak{s}$, then there is $S_1 \in \mathfrak{s}$ such that $S_1 < S$. Hence $S_1 <^p S$, $S_1 \subset \operatorname{int} S$, $\operatorname{int} S \in \mathfrak{s}$. On the other hand, $X - S < X - S_1$, $X - S \subset \subset \operatorname{int}(X - S_1)$, $\operatorname{cl} S_1 \subset S$, $\operatorname{cl} S_1 \in \mathfrak{s}$.

(a): By (b) and (c) the $\mathcal{T}$-neighbourhood filters are regular, i.e. $\mathcal{T}$ is regular. □

(3.2) LEMMA. *Let $\{<_0\}$ be a regular topology on X. Then*

$$< \; = \; <_0 \cap <_0^c \tag{3.3}$$

is the finest R-order compatible with $<_0$. The R-proximity δ associated with $<$ is given by

$$A \bar{\delta} B \quad \textit{iff} \quad A \cap \operatorname{cl} B = \operatorname{cl} A \cap B = \emptyset. \tag{3.4}$$

PROOF. $<$ is a symmetrical topogenous order. $x < V$ implies $x <_0 V$, hence there are open sets G_i satisfying $x \in G_2 \subset \operatorname{cl} G_2 \subset G_1 \subset \operatorname{cl} G_1 \subset V$. Thus $x <_0 G_1 <_0 V$, $x <_0^c G_1 <_0^c V$, i.e. $x < G_1 < V$, and $<$ is an R-order.

By the above argument $x <_0 V$ implies $x < V$ while the converse is obvious. Hence $<$ is compatible with $<_0$.

If $<'$ is an R-order (or, more generally, a symmetrical topogenous order) such that $<'^p = <_0$ then $<' \subset <_0$, $<' \subset <_0^c$, hence $<' \subset <$.

Finally $A \bar{\delta} B \Leftrightarrow A < X - B \Leftrightarrow A \subset \operatorname{int}(X - B)$ and $B \subset \operatorname{int}(X - A) \Leftrightarrow A \cap \operatorname{cl} B = B \cap \cap \operatorname{cl} A = \emptyset$. □

(3.5) LEMMA. *Let X be a regular topological space with the topology $\{<_0\}$. Then*

$$<' \; = \; <_0^c <_0 \tag{3.6}$$

is an R-order compatible with $<_0$, and

$$A <' B \quad \textit{iff} \quad \operatorname{cl} A \subset \operatorname{int} B, \tag{3.7}$$

hence the R-proximity δ' associated with $<'$ is given by

$$A \delta' B \quad \textit{iff} \quad \operatorname{cl} A \cap \operatorname{cl} B \neq \emptyset. \tag{3.8}$$

PROOF. $<'$ is a symmetrical topogenous order. $x <_0^c <_0 V$ implies $x <_0 V$. Choose again open sets G_i such that

$$x \in G_3 \subset \operatorname{cl} G_3 \subset G_2 \subset \operatorname{cl} G_2 \subset G_1 \subset V.$$

Then

$$x <_0^c \operatorname{cl} G_3 <_0 G_2 <_0^c \operatorname{cl} G_2 <_0 G_1 \subset V,$$

i.e.

$$x <' G_2 <' V,$$

and $<'$ is an R-order. Also $x <' V \Leftrightarrow x <_0 V$, thus $<'$ is compatible with $<_0$. Finally

$$A <' B \Leftrightarrow A <_0^c C <_0 B \quad \text{for some } C$$

$$\Leftrightarrow \operatorname{cl} A \subset C \subset \operatorname{int} B \quad \text{for some} \quad C \Leftrightarrow \operatorname{cl} A \subset \operatorname{int} B,$$

hence

$$A \bar{\delta}' B \Leftrightarrow A <' X - B \Leftrightarrow \operatorname{cl} A \subset \operatorname{int}(X - B) \Leftrightarrow \operatorname{cl} A \cap \operatorname{cl} B = \emptyset. \quad \square$$

Let us introduce the notation $<_X$ and δ_X for the relations $<'$ and δ', respectively.

(3.9) LEMMA. *In a regular space X, the $<_X$-round filters coincide with the regular filters.*

PROOF. (3.1) and (3.7). □

(3.10) LEMMA. *Let X be a regular space and $<$ a topogenous order on X. We have $<\subset<_X$ iff every neighbourhood filter is $<$-compressed.*

PROOF. The neighbourhood filter $\mathfrak{v}$ of x is $<$-compressed iff $A<B$, $x\in\operatorname{cl} A$ implies $x\in\operatorname{int} B$. Hence all neighbourhood filters are $<$-compressed iff $A<B$ implies $\operatorname{cl} A\subset\operatorname{int} B$ i.e. $A<_X B$ by (3.7). □

In general, there are compatible R-orders $<$ in a regular space X that do not satisfy the condition $<\subset<_X$. E.g., on the real line $\mathbf{R}$ with the usual topology,

$$\operatorname{cl}(0,1)\cap(1,2)=(0,1)\cap\operatorname{cl}(1,2)=\emptyset$$

but

$$\operatorname{cl}(0,1)\cap\operatorname{cl}(1,2)\neq\emptyset$$

so that $(0,1)\bar{\delta}(1,2)$ for the R-proximity (3.4) but $(0,1)\delta_{\mathbf{R}}(1,2)$.

On the other hand, a class of compatible R-orders coarser than $<_X$ is obtained from the following:

(3.11) LEMMA. *Let X be a regular space, $n\in\mathbf{N}$, and define $A<_n B$ iff there are open sets $G_1, \ldots, G_n$ such that*

$$\operatorname{cl} A\subset G_1\subset\operatorname{cl} G_1\subset G_2\subset\ldots\subset G_n\subset\operatorname{cl} G_n\subset\operatorname{int} B.$$

Then $<_n$ is a compatible R-order on X, and

$$<_{n+1}\subset<_n\subset<_X \quad \textit{for} \quad n\in\mathbf{N}. \tag{3.12}$$

The $<_n$-round filters coincide with the regular filters.

PROOF. Clearly $<_n$ is a symmetrical topogenous order. $x<_n V$ implies $x\in\operatorname{int} V$, hence there are open sets $G_n, \ldots, G_1, H_{n+1}, \ldots, H_1$ satisfying

$$\operatorname{int} V\supset\operatorname{cl} G_n\supset G_n\supset\operatorname{cl} G_{n-1}\supset\ldots\supset G_1\supset$$

$$\supset\operatorname{cl} H_{n+1}\supset H_{n+1}\supset\ldots\supset\operatorname{cl} H_1\supset H_1\supset\operatorname{cl}\{x\},$$

so that $x<_n H_{n+1}<_n V$. Thus $<_n$ is an R-order. The same argument shows $x<_n V$ for every neighbourhood V of x so that $<_n$ is compatible with X.

(3.12) is obvious. Hence $<_n$-round filters are $<_X$-round and regular by (3.9). Conversely if $\mathfrak{s}$ is a regular filter then $S\in\mathfrak{s}$ implies the existence of open sets $G_i\in\mathfrak{s}$ satisfying

$$\operatorname{int} S\supset\operatorname{cl} G_n\supset G_n\supset\ldots\supset\operatorname{cl} G_1\supset G_1\supset\operatorname{cl} G_0,$$

so that $G_0<_n S$. □

Observe that the R-proximity δ_1 associated with $<_1$ satisfies

(3.13) $A\bar{\delta}_1 B$ iff there are open sets G and H such that $\operatorname{cl} A\subset G$, $\operatorname{cl} B\subset H$, $G\cap H=\emptyset$.

(3.14) COROLLARY. *$\delta_1=\delta_X$ iff X is normal.* □

4. *RE*-orders

Let X be a regular space, and Y a regular extension of X. Then $<_Y$ is an R-order on Y, compatible with Y, and by (2.4) and (2.9) $<_Y|X$ is an R-order on X compatible with X. We call *RE-order* on X an R-order obtained in this way from a regular extension of X; more generally, an RE-order on a set X is an R-order $<$ on X such that $< = <_Y|X$ for a suitable regular topological space Y containing X, equipped with the topology $\{<^p\}$, as a dense subspace. An RE-proximity is an R-proximity associated with an RE-order.

(4.1) Lemma. *Let Y be a regular extension of X, and δ be the RE-proximity associated with $<_Y|X$. Then, for $A, B\subset X$,*

(4.2) $$A\delta B \quad \textit{iff} \quad \mathrm{cl}_Y A\cap \mathrm{cl}_Y B \neq \emptyset.$$

Consequently

(4.3) $$A\bar{\delta}B \quad \textit{implies} \quad \mathrm{cl}_X A\bar{\delta}\,\mathrm{cl}_X B;$$

in particular

(4.4) $$<_Y|X \subset <_X.$$

Proof. $A\bar{\delta}B$ iff $A(<_Y|X)X-B$, i.e. iff $A<_Y(X-B)\cup(Y-X)=Y-B$, or equivalently iff $\mathrm{cl}_Y A\cap\mathrm{cl}_Y B=\emptyset$ by (3.8). □

We shall see (cf. (8.3)) that (4.3) is not sufficient for δ in order to be an RE-proximity.

Our next purpose is to show that the description of RE-orders is closely related with their round, compressed filters.

(4.5) Lemma. *Let Y be a regular space, f: $X\to Y$, $f(X)$ dense, $\mathfrak{s}$ a regular filter in Y. Then*

(4.6) $$f^{-1}(\mathfrak{s}) = \{f^{-1}(S)\colon\ S\in\mathfrak{s}\}$$

generates in X an $f^{-1}(<_Y)$-round filter $\mathfrak{r}$, and

(4.7) $$\{\mathrm{cl}\, f(R)\colon\ R\in\mathfrak{r}\}$$

is a filter base that generates $\mathfrak{s}$.

Proof. Clearly $f(X)\cap S\neq\emptyset$ for $S\in\mathfrak{s}$ so that (4.6) is a filter base in X. By (3.9) $S\in\mathfrak{s}$ implies $S_1<_Y S$ for some $S_1\in\mathfrak{s}$, whence

$$f^{-1}(S_1)f^{-1}(<_Y)f^{-1}(S),$$

and (4.6) generates an $f^{-1}(<_Y)$-round filter $\mathfrak{r}$.

For $S\in\mathfrak{s}$ choose an open $G\in\mathfrak{s}$ such that $G\subset\mathrm{cl}\, G\subset S$. Then $f^{-1}(S)\subset R\subset X$ implies $\mathrm{cl}\, f(R)\supset\mathrm{cl}\, f\big(f^{-1}(G)\big)=\mathrm{cl}\,\big(G\cap f(X)\big)\supset G\in\mathfrak{s}$, hence $\mathrm{cl}\, f(R)\in\mathfrak{s}$. On the other hand, $S\supset\mathrm{cl}\, G=\mathrm{cl}\,\big(G\cap f(X)\big)=\mathrm{cl}\, f\big(f^{-1}(G)\big)$, $f^{-1}(G)\in\mathfrak{r}$. □

(4.8) Lemma. *Let f: $X\to Y$, δ be an R-proximity on Y satisfying the condition*

(4.9) $$A\bar{\delta}B \quad \textit{implies} \quad \mathrm{cl}\, A\bar{\delta}\,\mathrm{cl}\, B,$$

$<$ the R-order associated with δ, and $f(X)$ dense in Y (always with respect to the topology induced by δ). Then, for a $<$-round filter $\mathfrak{s}$ in Y, the filter base $f^{-1}(\mathfrak{s})$ generates an $f^{-1}(<)$-round filter $\mathfrak{r}$ in X, and

$$(4.10) \qquad \{\mathrm{cl}\, f(R):\ R\in\mathfrak{r}\}$$

generates the filter $\mathfrak{s}$. Conversely if $\mathfrak{r}$ is an $f^{-1}(<)$-round filter in X, then (4.10) *generates a $<$-round filter $\mathfrak{s}$ in Y, and $f^{-1}(\mathfrak{s})$ generates $\mathfrak{r}$.*

Proof. By (3.1) a $<$-round filter $\mathfrak{s}$ is regular so that $f^{-1}(\mathfrak{s})$ generates a filter $\mathfrak{r}$ in X, and (4.10) generates $\mathfrak{s}$ by (4.5). $\mathfrak{r}$ is $f^{-1}(<)$-round by [4], (16.43).

For an arbitrary $f^{-1}(<)$-round filter $\mathfrak{r}$, the filter base (4.10) generates in Y a $<$-round filter $\mathfrak{s}$. In fact, suppose $\mathrm{cl}\, f(R)\subset S\subset Y$, $R\in\mathfrak{r}$, and choose $R_1\in\mathfrak{r}$ such that $R_1 f^{-1}(<)R$. Then $f(R_1)<Y-f(X-R)$, hence by (4.9)

$$\mathrm{cl}\, f(R_1) < Y-\mathrm{cl}\, f(X-R) \subset \mathrm{cl}\, f(R) \subset S,$$

$\mathrm{cl}\, f(R_1)\in\mathfrak{s}$. On the other hand, for the same sets S, R, R_1, $f^{-1}(S)\supset R\supset f^{-1}(\mathrm{cl}\, f(R_1))$ because $\mathrm{cl}\, f(R_1)\cap f(X-R)=\emptyset$, therefore $f^{-1}(\mathfrak{s})$ generates $\mathfrak{r}$. $\square$

(4.11) Corollary (cf. [8], Lemma 2). *Let Y be a regular extension of a topological space X, and $<=<_Y|X$. Then the $<$-round filters $\mathfrak{r}$ coincide with the traces in X of the regular filters $\mathfrak{s}$ in Y, and the formulas*

$$(4.12) \qquad \mathfrak{r} = \mathfrak{s}|X = \{S\cap X:\ S\in\mathfrak{s}\},$$

$$(4.13) \qquad \mathfrak{s} = \{S\subset Y:\ S\supset \mathrm{cl}_Y R \text{ for some } R\in\mathfrak{r}\}$$

establish a bijection between these two classes of filters.

Proof. By (3.9) the regular filters in Y are precisely the $<_Y$-round filters, and $\delta=\delta_Y$ clearly satisfies (4.9). $\square$

(4.14) Lemma. *Let Y be a regular extension of X, $y\in Y$, $\mathfrak{v}$ the neighbourhood filter of y, $\mathfrak{r}=\mathfrak{v}|X$, and $<=<_Y|X$. Then $\mathfrak{r}$ is a $<$-round, $<$-compressed filter.*

Proof. By (3.10) $\mathfrak{v}$ is $<_Y$-compressed, hence $\mathfrak{r}$ is $<$-compressed ([3], (15.48) and (15.51)). Since $\mathfrak{v}$ is regular in Y, $\mathfrak{r}$ is $<$-round by (4.11). $\square$

5. Preregular systems of filters

The above results permit us to characterize *RE*-orders (and *RE*-proximities) with the help of suitable systems of filters. The method is similar to that followed in [9] for characterizing *RC*-proximities.

(5.1) Theorem. *Let Y be a regular extension of the topological space X, $<=<_Y|X$. Denote by $\mathfrak{R}$ the system of all traces in X of the neighbourhood filters of the points $y\in Y$. Then*

(5.2) *$x\in X$ implies that there is an $\mathfrak{r}\in\mathfrak{R}$ such that $x\in\bigcap\mathfrak{r}$,*

(5.3) *$R_0\in\mathfrak{r}\in\mathfrak{R}$ implies that there is $R_1\in\mathfrak{r}$ such that $R_0\in\mathfrak{r}'$ whenever $\mathfrak{r}'\in\mathfrak{R}$ and $\emptyset\notin\mathfrak{r}'|R_1$.*

The system $\mathfrak{R}$ *determines* $<$ *in the following manner:*

(5.4) $$A < B \quad \textit{iff} \quad \mathfrak{r}\in\mathfrak{R},\ \ \emptyset\notin\mathfrak{r}|A \quad \textit{implies} \quad B\in\mathfrak{r},$$

or equivalently, for the R-proximity δ *associated with* $<$,

(5.5) $$A\bar{\delta}B \quad \textit{iff} \quad \mathfrak{r}\in\mathfrak{R},\ \ \emptyset\notin\mathfrak{r}|A \quad \textit{implies} \quad X-B\in\mathfrak{r}.$$

Proof. (5.2) holds for $\mathfrak{r}=\mathfrak{v}|X$ if $\mathfrak{v}$ is the neighbourhood filter of x in Y. By (4.14) every $\mathfrak{r}\in\mathfrak{R}$ is $<$-round and $<$-compressed, thus for $R_0\in\mathfrak{r}\in\mathfrak{R}$ there is $R_1\in\mathfrak{r}$ such that $R_1<R_0$; then (5.3) holds since $\mathfrak{r}'\in\mathfrak{R}$ is $<$-compressed. (5.5) is true because, by (4.2), $A\bar{\delta}B$ iff $\mathrm{cl}_Y A\cap \mathrm{cl}_Y B=\emptyset$, and (5.4) means the same as (5.5). □

We shall prove a certain converse of (5.1). For this purpose, let us say that $\mathfrak{R}$ is a *preregular system of filters* on X iff $\mathfrak{R}$ is a set of filters in X and satisfies (5.2) and (5.3).

(5.6) Lemma. *If* $\mathfrak{R}$ *is a system of filters satisfying* (5.3), *and* $\mathfrak{r},\mathfrak{r}'\in\mathfrak{R}$, $\mathfrak{r}\neq\mathfrak{r}'$, *then there are* $R\in\mathfrak{r}$, $R'\in\mathfrak{r}'$ *such that* $R\cap R'=\emptyset$.

Proof. Suppose $R_0\in\mathfrak{r}-\mathfrak{r}'$ and choose $R_1\in\mathfrak{r}$ according to (5.3). Then $\emptyset\in\mathfrak{r}'|R_1$. □

(5.7) Corollary. *If* $\mathfrak{R}$ *is a preregular system of filters on* X *then, for* $x\in X$, *there is a unique* $\mathfrak{r}\in\mathfrak{R}$ *such that* $x\in\bigcap\mathfrak{r}$. □

(5.8) Theorem. *Let* $\mathfrak{R}$ *be a preregular system of filters on* X, *and define, for* $A,B\subset X$, *a relation* $<$ *by* (5.4). *Then* $<$ *is an R-order on* X.

Proof. Clearly $\emptyset<\emptyset$ and $X<X$, further $A'\subset A<B\subset B'$ implies $A'<B'$. If $A<B$ then $A\subset B$; in fact, $x\in A-B$ would imply $x\in\bigcap\mathfrak{r}$ for some $\mathfrak{r}\in\mathfrak{R}$, hence $B\notin\mathfrak{r}$, which contradicts (5.4). Thus $<$ is a semi-topogenous order on X.

If $A<B$, $\mathfrak{r}\in\mathfrak{R}$, and $\emptyset\notin\mathfrak{r}|X-B$, then $B\notin\mathfrak{r}$, hence $\emptyset\in\mathfrak{r}|A$, $X-A\in\mathfrak{r}$. Hence $<$ is symmetrical. Further $A=A_1\cap A_2$, $B=B_1\cap B_2$, $A_i<B_i$ $(i=1,2)$ implies that $B_i\in\mathfrak{r}$ whenever $\mathfrak{r}\in\mathfrak{R}$ and $\emptyset\notin\mathfrak{r}|A$, consequently in this case $B\in\mathfrak{r}$. Hence $<$ is a topogenous order.

Let $x<R_0$. If $\mathfrak{r}$ is the unique filter in $\mathfrak{R}$ that satisfies $x\in\bigcap\mathfrak{r}$ (cf. (5.7)) then $R_0\in\mathfrak{r}$ by (5.4). Choose $R_1\in\mathfrak{r}$ according to (5.3). Then $R_1<R_0$ by (5.4), and $x<R_1$ because $\emptyset\notin\mathfrak{r}'|\{x\}$, $\mathfrak{r}'\in\mathfrak{R}$ implies $\mathfrak{r}'=\mathfrak{r}$ by (5.7). □

If $\mathfrak{R}$ is a preregular system of filters on X, and a relation $<$ satisfies (5.4), then we shall say that $\mathfrak{R}$ *induces* the R-order $<$ as well as the R-proximity associated with $<$ (by (5.8) $<$ is an R-order in fact). Now (5.1) can be interpreted by saying that an *RE*-order is always induced by a preregular system of filters.

(5.9) Lemma. *If* $\mathfrak{R}$ *is a preregular system of filters and* $<$ *is the R-order induced by* $\mathfrak{R}$, *then every* $\mathfrak{r}\in\mathfrak{R}$ *is* $<$*-round and* $<$*-compressed.*

Proof. The first statement follows from (5.3), the second one from (5.4). □

(5.10) Lemma. *Under the hypotheses of* (5.9), *if* $x\in X$, $\mathfrak{r}\in\mathfrak{R}$, *and* $x\in\bigcap\mathfrak{r}$, *then* $\mathfrak{r}$ *coincides with the neighbourhood filter of* x *with respect to the topology* $\{<^p\}$.

PROOF. $x <^p V \Leftrightarrow x < V \Rightarrow V \in \mathfrak{r}$ by (5.4). Conversely $V \in \mathfrak{r}$ implies $x < V$ because $\emptyset \notin \mathfrak{r}' | \{x\}$, $\mathfrak{r}' \in \mathfrak{R}$ cannot hold unless $\mathfrak{r}' = \mathfrak{r}$ (cf. (5.7)). □

Consider now an arbitrary preregular system $\mathfrak{R}$ of filters on a set X. By (5.8) $\mathfrak{R}$ induces an R-order $<$. Let us equip X with the topology $\{<^p\}$ (regular by (3.1)), and denote by $\mathfrak{R}'$ the subset of $\mathfrak{R}$ composed of the free filters $\mathfrak{r} \in \mathfrak{R}$. Since every $\mathfrak{r} \in \mathfrak{R}'$ is regular ((5.9) and (3.1)), hence open, we can construct a strict extension ([6], (6.1.8)) Y of X such that the elements of $Y - X$ are in a one-to-one correspondence with the elements of $\mathfrak{R}'$, and $\mathfrak{r} \in \mathfrak{R}'$ is the trace in X of the neighbourhood filter of the corresponding point of $Y - X$. By (5.10) the elements of $\mathfrak{R}$ coincide with the traces of the neighbourhood filters of the points $x \in Y$, hence (5.3) implies by a theorem of [7] that Y is a regular space, and it is a *reduced* extension of X, i.e. $x \in Y - X$, $y \in Y$, $x \neq y$ implies that the neighbourhood filters of x and y are distinct. It follows from (5.4) and (5.1) that $< = <_Y | X$.

Thus we have proved the following converse of (5.1):

(5.11) THEOREM (K. Matolcsy). *Let $\mathfrak{R}$ be a preregular system of filters on X, and $<$ the R-order induced by $\mathfrak{R}$. Then $<$ is an RE-order; more precisely, there exists a regular, reduced extension Y of X (equipped with $\{<^p\}$) such that the filters $\mathfrak{r} \in \mathfrak{R}$ coincide with the traces in X of the neighbourhood filters of the points of Y, and $< = <_Y | X$.* □

(5.12) COROLLARY. *A topogenous order is an RE-order iff it can be induced by a preregular system of filters.*

PROOF. (5.1) and (5.11). □

Let us call, for a given preregular system $\mathfrak{R}$ of filters on X, an extension Y of the space X equipped with $\{<^p\}$, where $<$ is induced by $\mathfrak{R}$, *associated* with $\mathfrak{R}$ iff it has the properties described in (5.11) (i.e. iff Y is a reduced, regular extension of X and the trace filters of the neighbourhood filters of the points of Y coincide with the filters $\mathfrak{r} \in \mathfrak{R}$).

The following proposition motivates a certain partial ordering in the class of all preregular systems of filters on X:

(5.13) LEMMA. *Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two preregular systems of filters on X, $<_1$ and $<_2$ the RE-orders induced by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, and Y_1, Y_2 two extensions associated with $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively. Then the following statements are equivalent:*

(a) *For $\mathfrak{r}_2 \in \mathfrak{R}_2$ there is $\mathfrak{r}_1 \in \mathfrak{R}_1$ such that $\mathfrak{r}_1 \subset \mathfrak{r}_2$.*

(b) *There is a continuous extension $f\colon Y_2 \to Y_1$ of* id_X.

PROOF. (a)⇒(b): By (5.7) and (5.10) the topology $\{<_1^p\}$ is coarser than $\{<_2^p\}$, and the trace filter in X of the neighbourhood filter of every $y \in Y_2$ converges in Y_1. Since Y_1 is regular, the existence of a continuous $f\colon Y_2 \to Y_1$, $f | X = \mathrm{id}_X$ follows by [6], (6.2.2).

(b)⇒(a): If $\mathfrak{r}_2$ is the trace of the neighbourhood filter of $y_2 \in Y_2$ then $f(\mathfrak{r}_2) \to y_1 \in Y_1$, i.e. $\mathfrak{r}_1 \subset \mathfrak{r}_2$ for the trace $\mathfrak{r}_1$ of the neighbourhood filter of y_1. □

Let us say, for two preregular systems $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of filters on X that $\mathfrak{R}_1$ is *coarser* than $\mathfrak{R}_2$, $\mathfrak{R}_2$ is *finer* than $\mathfrak{R}_1$ iff (5.13) (a) holds.

(5.14) LEMMA. *If, under the hypotheses of* (5.13), *both* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *are finer than the other one then* $\mathfrak{R}_1=\mathfrak{R}_2$. *This is true iff* Y_1 *and* Y_2 *are equivalent extensions of* X *(i.e. iff there is a homeomorphism* $f\colon Y_2\to Y_1$ *such that* $f|X=\mathrm{id}_X$*).*

PROOF. If $\mathfrak{r}_2'\subset\mathfrak{r}_1\subset\mathfrak{r}_2$, $\mathfrak{r}_1\in\mathfrak{R}_1$, $\mathfrak{r}_2, \mathfrak{r}_2'\in\mathfrak{R}_2$, then $\mathfrak{r}_2'=\mathfrak{r}_2$ by (5.6), hence $\mathfrak{r}_1=\mathfrak{r}_2$ and $\mathfrak{R}_2\subset\mathfrak{R}_1$. Similarly $\mathfrak{R}_1\subset\mathfrak{R}_2$. If Y_1 and Y_2 are equivalent extensions then $\mathfrak{R}_1=\mathfrak{R}_2$ by (5.13) and the statement established already. Conversely if $\mathfrak{R}_1=\mathfrak{R}_2$ then there are continuous mappings $f\colon Y_2\to Y_1$ and $g\colon Y_1\to Y_2$ such that $f|X=g|X=\mathrm{id}_X$. Now $g\circ f\colon Y_2\to Y_2$ is a continuous extension of id_X, whence it coincides with id_{Y_2} because a point $y_2\in Y_2-X$ and another point $y_2'\in Y_2$ have distinct neighbourhood filters, consequently disjoint neighbourhoods by the regularity of Y_2 ([6], (2.5.24)). Similarly $f\circ g=\mathrm{id}_{Y_1}$. □

(5.15) LEMMA. *If* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *are preregular systems of filters on* X *and* $\mathfrak{R}_2$ *is finer than* $\mathfrak{R}_1$ *then* $<_1\subset<_2$ *for the respective induced RE-orders.*

PROOF. If $A<_1B$, $\mathfrak{r}_2\in\mathfrak{R}_2$, and $\emptyset\notin\mathfrak{r}_2|A$, then choose $\mathfrak{r}_1\in\mathfrak{R}_1$ such that $\mathfrak{r}_1\subset\mathfrak{r}_2$. Clearly $\emptyset\notin\mathfrak{r}_1|A$, hence $B\in\mathfrak{r}_1\subset\mathfrak{r}_2$. □

The following theorem constructs compatible *RE*-orders from arbitrary sufficiently coarse compatible *R*-orders of a regular space X:

(5.16) THEOREM. *Let* X *be a regular space,* $<$ *an R-order compatible with* X *and satisfying* $<\subset<_X$. *Then the set* $\mathfrak{R}$ *of all* $<$*-round,* $<$*-compressed filters is a preregular system of filters, and the RE-order* $<'$ *induced by* $\mathfrak{R}$ *satisfies* $<\subset<'$ *and is compatible with* X. $<$ *is an RE-order iff* $<=<'$.

PROOF. By (3.1) and (3.10) the neighbourhood filters belong to $\mathfrak{R}$, so (5.2) is fulfilled. If $R_0\in\mathfrak{r}\in\mathfrak{R}$ and $R_1<R_0$, $R_1\in\mathfrak{r}$, then (5.3) holds because $\mathfrak{r}'\in\mathfrak{R}$ is $<$-compressed. Thus $\mathfrak{R}$ is preregular.

By (5.4) $A<B$ implies $A<'B$. By (5.10) the $\{<'^p\}$-neighbourhood filter of $x\in X$ is the same as its neighbourhood filter in X (it is namely the unique $\mathfrak{r}\in\mathfrak{R}$ satisfying $x\in\bigcap\mathfrak{r}$, cf. (5.7)). Hence $<'$ is compatible with X.

$<=<'$ implies that $<$ is an *RE*-order. Conversely if $<$ is an *RE*-order, then there is by (5.1) a preregular system $\mathfrak{R}_0$ of filters that induces $<$. By (5.9) $\mathfrak{R}_0\subset\mathfrak{R}$, hence $\mathfrak{R}_0$ is finer than $\mathfrak{R}$, so $<'\subset<$ by (5.15). □

(5.17) COROLLARY. *An arbitrary RE-order* $<$ *is induced by the set* $\mathfrak{R}$ *of all* $<$*-round,* $<$*-compressed filters;* $\mathfrak{R}$ *is the largest preregular system of filters inducing* $<$.

PROOF. (5.16) and (5.9). □

(5.18) COROLLARY. *Let* X *be a regular space,* $<$ *a compatible RE-order,* $\mathfrak{R}$ *the system of all* $<$*-round,* $<$*-compressed filters, and* Y *an extension associated with* $\mathfrak{R}$. *Then every reduced, regular extension* Z *of* X *such that* $<=<_Z|X$ *is equivalent to a subspace* Z' *of* Y *satisfying* $X\subset Z'\subset Y$*; these subspaces* Z' *constitute a non-empty ascending system in* Y.

PROOF. $<=<_Y|X$ by (5.17) and (5.11). An extension Z with the above properties is associated by (5.1) with a preregular system $\mathfrak{R}_0$ of filters that induces $<$; hence $\mathfrak{R}_0\subset\mathfrak{R}$ by (5.17). Denote by Z' the subspace of Y composed of those points

$y \in Y$ for which the traces of their neighbourhood filters belong to $\mathfrak{R}_0$. By (5.14) the extensions Z and Z' are equivalent. If $Z' \subset Z'' \subset Y$ then clearly

$$< \, = \, <_Y|X \subset \, <_{Z''}|X \subset \, <_{Z'}|X = \, <,$$

hence $<_{Z''}|X = \, <$. $\square$

It is convenient to say the extensions Y described in (5.18) to be *associated* with the *RE*-order $<$. It can happen (see (8.8)) that some proper subspaces $Z' \subset Y$ fulfil the condition $< \, = \, <_{Z'}|X$.

6. *RC*-orders

According to [9], the concept of a regular-closed space in the sense of [8] can be generalized in the following way: a topological space X is said to be *T_3-closed* iff every maximal regular filter is fixed in X; for T_3-spaces this condition furnishes the regular-closed spaces of [8].

Let us call *RC-orders* the *RE*-orders on X that have the form $< \, = \, <_Y|X$ where Y is a T_3-closed, regular extension of X. An *RC-proximity* is an *RE*-proximity associated with an *RC*-order; if Y is supposed to be a T_3-space, we obtain the *RC*-proximities in the sense of [8].

Now we can prove the following characterization of *RC*-orders:

(6.1) THEOREM. *Let X be a regular space and $<$ an RE-order compatible with X. $<$ is an RC-order iff every maximal $<$-round filter is $<$-compressed. If this condition is fulfilled then all maximal $<$-round filters constitute a preregular system $\mathfrak{R}$ of filters such that the extensions Y associated with $\mathfrak{R}$ are reduced, T_3-closed extensions, associated with $<$.*

PROOF. Let Z be a regular extension of X such that $< \, = \, <_Z|X$. By (4.11) the $<$-round filters coincide with the traces of the regular filters in Z, and larger $<$-round filters correspond to larger regular filters ((4.12) and (4.13)). Hence maximal $<$-round filters are traces of maximal regular filters. If Z is T_3-closed, the latters are fixed in Z, i.e. neighbourhood filters of points $z \in Z$. Such a filter is $<_Z$-compressed by (3.10), and its trace is $<$-compressed by (4.14).

Suppose now that $<$ is an *RE*-order, compatible with X, with the property that every maximal $<$-round filter is $<$-compressed. Since conversely $<$-compressed, $<$-round filters are maximal $<$-round by (2.11), therefore the system $\mathfrak{R}$ of all maximal $<$-round filters is the same as the system of all $<$-round, $<$-compressed filters, consequently it is a preregular system of filters inducing $<$ (see (5.17)), and an extension Y associated with $\mathfrak{R}$ is associated with $<$ as well. If $\mathfrak{s}$ is a maximal regular filter in Y then $\mathfrak{r} = \mathfrak{s}|X$ is maximal $<$-round by the first part of the proof, so $\mathfrak{r} \in \mathfrak{R}$, and $\mathfrak{r} = \mathfrak{v}|X$ for the neighbourhood filter $\mathfrak{v}$ of some $y \in Y$. By (4.11) $\mathfrak{s} = \mathfrak{v}$, and $\mathfrak{s}$ is fixed, Y is T_3-closed. $\square$

An *RE*-order need not be an *RC*-order (see (8.4)). However, the extension associated with an arbitrary *RE*-order has a property similar to but weaker than T_3-closedness.

For this purpose, let us say that an extension Y of a topological space X is *disjunctive* iff $\mathrm{cl}_Y A \cap \mathrm{cl}_Y B = \emptyset$ whenever A, B are disjoint, closed subsets of X.

(6.2) LEMMA. *A regular extension Y of X is disjunctive iff* $<_X = <_Y|X$.

PROOF. The equality $<_X = <_Y|X$ is equivalent, by (4.4), to $<_X \subset <_Y|X$, i.e. to the condition that $\mathrm{cl}_X A \cap \mathrm{cl}_X B = \emptyset$ implies $\mathrm{cl}_Y A \cap \mathrm{cl}_Y B = \emptyset$ ((3.8) and (4.2)). □

A regular space X will be said to be *D-closed* iff there is no proper reduced, regular, disjunctive extension of X.

(6.3) THEOREM. *A regular space X is D-closed iff every $<_X$-compressed, regular filter is fixed.*

PROOF. If Y is a reduced, regular, disjunctive extension, and $y \in Y - X$, then the trace $\mathfrak{r} = \mathfrak{v}|X$ of the neighbourhood filter $\mathfrak{v}$ of y is $<_X$-compressed, $<_X$-round by (6.2) and (4.14), hence $<_X$-compressed and regular by (3.9); but $\mathfrak{r}$ is free because Y is a reduced, regular extension ([6], (2.5.24)).

Conversely, suppose that there is in X a free $<_X$-compressed, regular (i.e. $<_X$-round) filter $\mathfrak{r}$. Let $\mathfrak{R}$ be the system of all $<_X$-round, $<_X$-compressed filters; $\mathfrak{R}$ is preregular by (5.16). Let Y be an extension associated with $\mathfrak{R}$. In other words, Y is an extension associated with $<_X$, hence it is a proper (because $\mathfrak{r} \in \mathfrak{R}$), reduced, regular extension of X, and $<_X = <_Y|X$ by (5.18). By (6.2) Y is a disjunctive extension. □

Now we can prove:

(6.4) THEOREM. *Let $<$ be a compatible RE-order in a regular space X and Y an extension associated with $<$. Then Y is a D-closed space.*

PROOF. If a filter $\mathfrak{s}$ is $<_Y$-compressed and regular, i.e. $<_Y$-round by (3.9), then $\mathfrak{r} = \mathfrak{s}|X$ is $<$-round by (4.11) and $<$-compressed by [3], (15.48) and (15.51). Therefore $\mathfrak{r} = \mathfrak{v}|X$ for the neighbourhood filter $\mathfrak{v}$ of some $y \in Y$. By (4.11) $\mathfrak{s} = \mathfrak{v}$ so that $\mathfrak{s}$ is fixed and (6.3) can be applied. □

Unfortunately D-closedness does not characterize the extension Y associated with $<$ because there may exist proper D-closed subspaces $Z \subset Y$ satisfying $< = <_Z|X$ (see (8.8)).

It is easy to obtain all reduced, regular, disjunctive extensions of a space X:

(6.5) THEOREM. *Let X be a regular space and Y an extension associated with $<_X$. Then the reduced, regular, disjunctive extensions of X are, up to equivalence, the subspaces Z of Y such that $X \subset Z \subset Y$.*

PROOF. (6.2) and (5.18) show that every extension in question is equivalent to a subspace Z of Y such that $X \subset Z \subset Y$. Conversely (6.2) implies that Y itself is disjunctive and then the same is obviously true for every Z lying between X and Y. □

7. $(\mathfrak{R}, \mathfrak{R}')$-continuous mappings

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be preregular systems of filters on X and X', respectively, and $f\colon X \to X'$. The mapping f will be said to be $(\mathfrak{R}, \mathfrak{R}')$*-continuous* iff $\mathfrak{r} \in \mathfrak{R}$ implies that the filter base

$$f(\mathfrak{r}) = \{f(R)\colon R \in \mathfrak{r}\}$$

is finer than some filter $\mathfrak{r}' \in \mathfrak{R}'$. Thus, in the case $X = X'$, $\mathfrak{R}$ is finer than $\mathfrak{R}'$ iff id_X is $(\mathfrak{R}, \mathfrak{R}')$-continuous.

Some of our preceding results can be easily generalized for $(\mathfrak{R}, \mathfrak{R}')$-continuous mappings. In the following propositions $\mathfrak{R}$ and $\mathfrak{R}'$ always are preregular systems on X and X', $<$ and $<'$ the *RE*-orders induced by $\mathfrak{R}$ and $\mathfrak{R}'$, X and X' are equipped with the topologies $\{<^p\}$ and $\{<'^p\}$, respectively. $(<, <')$-continuity of $f\colon X \to X'$ means $(\{<\}, \{<'\})$-continuity.

(7.1) LEMMA. *If $f\colon X \to X'$ is $(\mathfrak{R}, \mathfrak{R}')$-continuous then it is $(<, <')$-continuous.*

PROOF. $A <' B$ implies $f^{-1}(A) < f^{-1}(B)$ since, if $\emptyset \notin \mathfrak{r} | f^{-1}(A)$ for some $\mathfrak{r} \in \mathfrak{R}$, then $\emptyset \notin \mathfrak{r}' | A$ for an $\mathfrak{r}' \in \mathfrak{R}'$ coarser than $f(\mathfrak{r})$, hence $B \in \mathfrak{r}'$, $f(R) \subset B$ for some $R \in \mathfrak{r}$, and $f^{-1}(B) \in \mathfrak{r}$. □

There are some partial converses of (7.1).

(7.2) LEMMA. *Let X be a D-closed regular space, $\mathfrak{R}$ the system of all $<_X$-compressed, regular filters, and $f\colon X \to X'$ continuous. Then f is $(\mathfrak{R}, \mathfrak{R}')$-continuous.*

PROOF. By (3.9) and (5.17) $\mathfrak{R}$ is preregular and induces $<_X$. By (6.3) every $\mathfrak{r} \in \mathfrak{R}$ is fixed, hence it is the neighbourhood filter of some $x \in X$ by (5.10). Hence $f(\mathfrak{r}) \to f(x)$ with respect to $\{<'^p\}$, and $f(\mathfrak{r})$ is finer than the neighbourhood filter of $f(x)$ that belongs to $\mathfrak{R}'$ by (5.10) again. □

(7.3) LEMMA. *If $f\colon X \to X'$ is continuous then it is $(<_X, <_{X'})$-continuous.*

PROOF. $\mathrm{cl}_X A \cap \mathrm{cl}_X B \neq \emptyset$ implies $\mathrm{cl}_{X'} f(A) \cap \mathrm{cl}_{X'} f(B) \neq \emptyset$. □

Observe that $(<, <')$-continuity of f does not imply its $(\mathfrak{R}, \mathfrak{R}')$-continuity in general (see (8.5)).

(7.4) LEMMA. *Let Y and Y' be extensions of X and X' associated with $\mathfrak{R}$ and $\mathfrak{R}'$, respectively. If $g\colon Y \to Y'$ is continuous and $g(X) \subset X'$ then $f = g|X$ is $(\mathfrak{R}, \mathfrak{R}')$-continuous.*

PROOF. Every $\mathfrak{r} \in \mathfrak{R}$ is the trace in X of the neighbourhood filter $\mathfrak{v}$ of some $y \in Y$. Hence $f(\mathfrak{r}) \to g(y)$ in Y' so that $f(\mathfrak{r})$ is finer than the trace $\mathfrak{r}'$ of the neighbourhood filter of $g(y)$; clearly $\mathfrak{r}' \in \mathfrak{R}'$. □

(7.5) LEMMA. *Let Y and Y' denote the same as in* (7.4). *If $f\colon X \to X'$ is $(\mathfrak{R}, \mathfrak{R}')$-continuous then there is a continuous extension $g\colon Y \to Y'$ of f.*

PROOF. By (7.1) f is $(\{<^p\}, \{<'^p\})$-continuous, and if $\mathfrak{r}$ is the trace in X of the neighbourhood filter of some $y \in Y$, i.e. if $\mathfrak{r} \in \mathfrak{R}$, then $f(\mathfrak{r})$ is finer than some $\mathfrak{r}' \in \mathfrak{R}'$ so that $f(\mathfrak{r}) \to y'$ for a point $y' \in Y'$ whose neighbourhood filter $\mathfrak{v}'$ satisfies $\mathfrak{r}' = \mathfrak{v}' | X'$. Hence [6], (6.2.2) applies. □

Now let $\mathfrak{R}$ and $\mathfrak{R}'$ denote the systems of all $<$-round, $<$-compressed and $<'$-round, $<'$-compressed filters, where $<$ and $<'$ are *RE*-orders on X and X', respectively (cf. (5.17)). We say that $f\colon X \to X'$ is *strongly* $(<, <')$*-continuous* iff it is $(\mathfrak{R}, \mathfrak{R}')$-continuous. In particular, if $X = X'$, $<$ is said to be *strongly finer* than $<'$, $<'$ *strongly coarser* than $<$ iff id_X is strongly $(<, <')$-continuous, i.e. iff $\mathfrak{R}$ is finer than $\mathfrak{R}'$. By (7.1) strong $(<, <')$-continuity implies $(<, <')$-continuity, and if $<$ is strongly finer than $<'$ then $<$ is finer than $<'$; the converses are not true (see (8.5), (8.9), (8.11)).

(7.6) LEMMA. *Let $\mathfrak{R}$, $\mathfrak{R}'$, $\mathfrak{R}''$ be preregular systems of filters on X, X', X'' respectively, $f\colon X\to X'$, $g\colon X'\to X''$. If f is $(\mathfrak{R}, \mathfrak{R}')$-continuous and g is $(\mathfrak{R}', \mathfrak{R}'')$-continuous, then $g\circ f$ is $(\mathfrak{R}, \mathfrak{R}'')$-continuous.*

PROOF. For $\mathfrak{r}\in\mathfrak{R}$, there are $\mathfrak{r}'\in\mathfrak{R}'$, $\mathfrak{r}''\in\mathfrak{R}''$ such that $f(\mathfrak{r})$ is finer than $\mathfrak{r}'$, $g(\mathfrak{r}')$ is finer than $\mathfrak{r}''$. Then $g(f(\mathfrak{r}))$ is finer than $g(\mathfrak{r}')$, hence than $\mathfrak{r}''$. □

(7.7) COROLLARY. *If $<$, $<'$, $<''$ are RE-orders on X, X', X'' respectively, $f\colon X\to X'$, $g\colon X'\to X''$, and f is strongly $(<, <')$-continuous, g is strongly $(<', <'')$-continuous, then $g\circ f$ is strongly $(<, <'')$-continuous.* □

(7.8) COROLLARY. *The relation "strongly finer" is transitive.* □

On a regular space X, $<_X$ is the finest compatible *RE*-order by (4.4). However, it need not be the strongly finest one (see (8.12)).

(7.9) LEMMA. *Let $<_i$ $(i\in I\neq\emptyset)$ be R-orders on a set X, and*

$$< = \Big(\bigcup_{i\in I} <_i\Big)^q.$$

Then $<$ is the coarsest R-order finer than every $<_i$.

PROOF. Clearly

$$\{<\} = \Big(\bigvee_{i\in I} \{<_i\}\Big)^t,$$

hence the statement follows from (2.4), (2.8) and (2.5). □

The analogous question for *RE*-orders is more delicate.

(7.10) LEMMA (K. Matolcsy). *Let $\mathfrak{R}_i$ $(i\in I\neq\emptyset)$ be preregular systems of filters on X. Then there is a preregular system $\mathfrak{R}$ of filters that is the coarsest one of all preregular systems finer than every $\mathfrak{R}_i$.*

PROOF. Consider all centred systems of the form $\bigcup_{i\in I}\mathfrak{r}_i$ where $\mathfrak{r}_i\in\mathfrak{R}_i$, and let $\mathfrak{R}$ denote the system of filters generated by these centred systems.

For $x\in X$, there is $\mathfrak{r}_i\in\mathfrak{R}_i$ $(i\in I)$ such that $x\in\bigcap\mathfrak{r}_i$. Clearly $\bigcup_{i\in I}\mathfrak{r}_i$ is centred and generates a filter $\mathfrak{r}\in\mathfrak{R}$ with $x\in\bigcap\mathfrak{r}$.

Now let $\mathfrak{r}\in\mathfrak{R}$ be generated by $\bigcup_{i\in I}\mathfrak{r}_i$, $\mathfrak{r}_i\in\mathfrak{R}_i$, and $R\in\mathfrak{r}$. Then

$$R\supset\bigcap_{k=1}^{n} R_k,\quad R_k\in\mathfrak{r}_{i_k},\quad i_k\in I.$$

Choose $R_k'\in\mathfrak{r}_{i_k}$ such that $\mathfrak{r}^*\in\mathfrak{R}_{i_k}$, $\emptyset\notin\mathfrak{r}^*|R_k'$ implies $R_k\in\mathfrak{r}^*$. Then

$$R' = \bigcap_{k=1}^{n} R_k'\in\mathfrak{r}.$$

If $\mathfrak{r}''\in\mathfrak{R}$ is generated by $\bigcup_{i\in I}\mathfrak{r}_i''$, $\mathfrak{r}_i''\in\mathfrak{R}_i$, $\emptyset\notin\mathfrak{r}''|R'$, and $R''\in\mathfrak{r}_{i_k}''$ is arbitrary, then $R''\in\mathfrak{r}''$ so that $R''\cap R_k'\supset R''\cap R'\neq\emptyset$. Hence $\emptyset\notin\mathfrak{r}_{i_k}''|R_k'$, thus $R_k\in\mathfrak{r}_{i_k}''$, $\bigcap_{k=1}^{n}R_k\in\mathfrak{r}''$, $R\in\mathfrak{r}''$. Therefore $\mathfrak{R}$ is a preregular system of filters.

If $\mathfrak{r}\in\mathfrak{R}$ is generated by $\bigcup_{i\in I}\mathfrak{r}_i$, $\mathfrak{r}_i\in\mathfrak{R}_i$, then $\mathfrak{r}_i\subset\mathfrak{r}$; hence $\mathfrak{R}$ is finer than every $\mathfrak{R}_i$.

Finally if $\mathfrak{R}'$ is a preregular system finer than each $\mathfrak{R}_i$, and $\mathfrak{r}'\in\mathfrak{R}'$, then there are $\mathfrak{r}_i\in\mathfrak{R}_i$ such that $\mathfrak{r}_i\subset\mathfrak{r}'$; clearly $\bigcup_{i\in I}\mathfrak{r}_i$ is centred and generates a filter $\mathfrak{r}\in\mathfrak{R}$, $\mathfrak{r}\subset\mathfrak{r}'$. □

(7.11) THEOREM (K. Matolcsy). *Let $<_i$ $(i\in I\neq\emptyset)$ be RE-orders on X. Then there is an RE-order $<$ on X finer than each $<_i$ and such that $<$ is strongly coarser than every RE-order $<'$ strongly finer than each $<_i$.*

PROOF. Denote by $\mathfrak{R}_i$ the system of all $<_i$-round, $<_i$-compressed filters, and consider the coarsest of all preregular systems finer than each $\mathfrak{R}_i$ (cf. (7.10)). If $<$ is the RE-order induced by this $\mathfrak{R}$ then it is finer than each $<_i$ by (5.15). If $<'$ is an RE-order strongly finer than each $<_i$, and $\mathfrak{R}'$ denotes the system of all $<'$-round, $<'$-compressed filters, then $\mathfrak{R}'$ is finer than each $\mathfrak{R}_i$, thus it is finer than $\mathfrak{R}$ and also finer than the system of all $<$-round, $<$-compressed filters (larger than $\mathfrak{R}$) so that $<'$ is strongly finer than $<$. □

We cannot assert that $<$ is strongly finer than each $<_i$; in fact it can happen that, for a family $\{<_i\colon i\in I\}$ of RE-orders on X, there is no RE-order strongly finer than each $<_i$ (see (8.10)).

(7.12) LEMMA (K. Matolcsy). *Under the hypotheses of* (7.10), *let $<_i$ and $<$ be the RE-orders induced by $\mathfrak{R}_i$ and $\mathfrak{R}$, respectively, further $<^*=(\bigcup_{i\in I}<_i)^q$. Then every $\mathfrak{r}\in\mathfrak{R}$ is $<^*$-round and $<^*$-compressed.*

PROOF. If $\mathfrak{r}$ is generated by $\bigcup_{i\in I}\mathfrak{r}_i$, $\mathfrak{r}_i\in\mathfrak{R}_i$, and $R\in\mathfrak{r}$, then $R\supset\bigcap_{k=1}^{n}R_k$, $R_k\in\mathfrak{r}_{i_k}$, $i_k\in I$. By (5.9) $\mathfrak{r}_{i_k}$ is $<_{i_k}$-round, hence there are $R_k'\in\mathfrak{r}_{i_k}$ such that $R_k'<_{i_k}R_k$. Now $R'=\bigcap_{k=1}^{n}R_k'\in\mathfrak{r}$, and $R'<^*R$, i.e. $\mathfrak{r}$ is $<^*$-round.

For the same $\mathfrak{r}$, suppose $\emptyset\notin\mathfrak{r}|A$, $A<^*B$. Then there are sets A_{jk}, B_{jk} such that

$$A\subset\bigcup_{j=1}^{m}\bigcap_{k=1}^{n_j}A_{jk},\quad \bigcup_{j=1}^{m}\bigcap_{k=1}^{n_j}B_{jk}\subset B,$$

$$A_{jk}<_{i(j,k)}B_{jk},\quad i(j,k)\in I.$$

Clearly there is a j such that $\emptyset\notin\mathfrak{r}|A_{jk}$ for each $k=1,\dots,n_j$. Since $\mathfrak{r}_{i(j,k)}\subset\mathfrak{r}$, we have $\emptyset\notin\mathfrak{r}_{i(j,k)}|A_{jk}$, hence $B_{jk}\in\mathfrak{r}_{i(j,k)}\subset\mathfrak{r}$ because $\mathfrak{r}_{i(j,k)}$ is $<_{i(j,k)}$-compressed by (5.9). Thus $\bigcap_{k=1}^{n_j}B_{jk}\in\mathfrak{r}$ and $B\in\mathfrak{r}$ so that $\mathfrak{r}$ is $<^*$-compressed. □

(7.13) COROLLARY (K. Matolcsy). *Under the hypotheses of* (7.12), *$<$ and $<^*$ induce the same topology, namely the supremum of the topologies $\{<_i^p\}$.*

PROOF. $<^*$ induces this supremum since

$$<^{*p} = (\bigcup_{i\in I} <_i)^{qp} = (\bigcup_{i\in I} <_i^p)^{qp}.$$

Now denote by $\mathfrak{R}^*$ the system of all $<^*$-round, $<^*$-compressed filters; by (7.12) $\mathfrak{R}\subset\mathfrak{R}^*$. Consequently $\mathfrak{R}^*$ is a preregular system of filters. In fact, (5.3) is valid for $\mathfrak{R}^*$ because its elements are $<^*$-round and $<^*$-compressed, and (5.2) is guaranteed by $\mathfrak{R}\subset\mathfrak{R}^*$. Now the $\{<^p\}$-neighbourhood filter of $x\in X$ is by (5.10) the unique filter $\mathfrak{r}\in\mathfrak{R}$ such that $x\in\bigcap\mathfrak{r}$; the same $\mathfrak{r}$ is by (5.7) the unique filter belonging to $\mathfrak{R}^*$ with the same property. But $\mathfrak{r}$ is $<^*$-round, $<^*$-compressed, hence maximal $<^*$-round by (2.11). By (3.1) the $\{<^{*p}\}$-neighbourhood filter of x is maximal $<^*$-round as well, hence it is identical with $\mathfrak{r}$ by (2.13). □

8. Counter-examples

We pointed out several times that some more or less plausible statements are not true. We give now the counter-examples necessary for this purpose.

Let X be the set of the ordinals $\leqq\omega$ and Y that of the ordinals $\leqq\omega_1$, both equipped with the order topology. Denote $N=X-\{\omega\}$, $N_1=Y-\{\omega_1\}$. Consider the product space $Z=X\times Y$, and denote by T the subspace $T=Z-\{(\omega,\omega_1)\}$. Thus T is the famous Tychonoff plank.

(8.1) LEMMA. *In T there is a single free, regular filter, namely the trace $\mathfrak{r}_0$ in T of the neighbourhood filter $\mathfrak{s}_0$ of $(\omega,\omega_1)\in Z$.*

PROOF. Z is a regular space, hence $\mathfrak{s}_0$ is regular, and the same holds for $\mathfrak{r}_0$ in T; $\mathfrak{r}_0$ is obviously free.

Now let $\mathfrak{s}$ be a free, regular filter in T. Z is compact, hence $\mathfrak{s}$ has a cluster point in Z which cannot be distinct from (ω,ω_1) since $\mathfrak{s}$ is free. Z is compact T_2, hence $\mathfrak{s}\to(\omega,\omega_1)$. Hence $\mathfrak{r}_0\subset\mathfrak{s}$.

Consider $S\in\mathfrak{s}$, and let G_i be open and F a closed set in T such that (for the closures in T)

$$S\supset \mathrm{cl}\, G_1\supset G_1\supset \mathrm{cl}\, G_2\supset G_2\supset \mathrm{cl}\, G_3\supset G_3\supset F\in\mathfrak{s}.$$

Suppose first that $F\cap(N\times\{\omega_1\})$ is infinite; say $(k_n,\omega_1)\in F$, $k_n\in N$ for $n\in N$. Then there are ordinals $\alpha_n<\omega_1$ such that $(k_n,\alpha)\in G_3$ for $\alpha_n<\alpha\leqq\omega_1$. If $\alpha_n\leqq\beta<\omega_1$ for each n, then $(\omega,\alpha)\in\mathrm{cl}\, G_3$ whenever $\beta<\alpha<\omega_1$. For every such α there is $n_\alpha\in N$ such that $(n,\alpha)\in G_2$ for $n_\alpha<n\leqq\omega$. There exists a $k\in N$ such that $n_\alpha=k$ for uncountably many α, whence

$$(8.2)\qquad (n,\omega_1)\in\mathrm{cl}\, G_2\quad\text{for}\quad k<n<\omega.$$

We can find, for every $k<n<\omega$, a $\gamma_n<\omega_1$ such that $\gamma_n<\alpha\leqq\omega_1$ implies $(n,\alpha)\in G_1$, so that if $\gamma_n\leqq\gamma<\omega_1$ then $(n,\alpha)\in\mathrm{cl}\, G_1$ whenever $k<n\leqq\omega$, $\gamma<\alpha\leqq\omega_1$ (except the case $n=\omega$, $\alpha=\omega_1$). Therefore $\mathrm{cl}\, G_1\in\mathfrak{r}_0$, $S\in\mathfrak{r}_0$ in this case.

Suppose now that $F\cap(N\times\{\omega_1\})$ is finite. Then there is an $n_0\in N$ such that, for $n_0<n<\omega$, there exists $\varepsilon_n<\omega_1$ satisfying $(n,\alpha)\notin F$ for $\varepsilon_n<\alpha\leqq\omega_1$. For an ε such that $\varepsilon_n\leqq\varepsilon<\omega_1$ $(n_0<n<\omega)$, we have $(n,\alpha)\notin F$ whenever $n_0<n<\omega$, $\varepsilon<\alpha\leqq\omega_1$. But (ω,ω_1) is a limit point of F in Z, which is possible only if $F\cap(\{\omega\}\times N_1)$ is uncountable. Hence there are uncountably many $\alpha<\omega_1$ and, to each of these α, an $n_\alpha\in N$ such that $n_\alpha<n\leqq\omega$ implies $(n,\alpha)\in G_3\subset G_2$. There is a $k\in N$ such that $n_\alpha=k$ for uncountably many α, and then (8.2) is valid. By the above reasoning this implies $S\in\mathfrak{r}_0$ so that $\mathfrak{s}\subset\mathfrak{r}_0$. □

(8.3) COROLLARY. *Consider in the space T the R-oder $<_1$ defined in* (3.11) *and the R-proximity δ_1, associated with $<_1$, given by* (3.13). *Then $<_1\subset<_T$ and δ_1 satisfies* (4.3), *but $<_1$ is not an RE-order.*

PROOF. We know from (3.12) that $<_1$ is coarser than $<_T$ and (3.13) furnishes (4.3) for $\delta=\delta_1$. By (3.11) the $<_1$-round filters coincide with the regular filters of T. Among them there is a single which is free, namely the filter $\mathfrak{r}_0$ in (8.1).

Now $\mathfrak{r}_0$ is not $<_1$-compressed. In fact, let $A\subset N$ be the set of the even numbers and $B\subset N$ that of the odd numbers. Then $A\times\{\omega_1\}\bar{\delta}_1\ B\times\{\omega_1\}$ since $A\times Y$ and $B\times Y$ are disjoint open subsets of T. However, every element of $\mathfrak{r}_0$ meets both $A\times\{\omega_1\}$ and $B\times\{\omega_1\}$.

Apply (5.16) for $X=T$, $<=<_1$. Then $\mathfrak{R}$ consists of the neighbourhood filters only so that $<'=<_T$. By (3.14) $<_1\neq<_T$, hence $<_1$ is not an *RE*-order. □

(8.4) COROLLARY. *The filter $\mathfrak{r}_0$ is not $<_T$-compressed, consequently the space T is D-closed without being T_3-closed. The extension associated with $<_T$ is T itself; hence $<_T$ is not an RC-order.*

PROOF. The unique free, regular filter $\mathfrak{r}_0$ in T is not $<_T$-compressed; in fact, $A=N\times\{\omega_1\}$ and $B=\{\omega\}\times N_1$ satisfy

$$A\bar{\delta}_T B,\quad \emptyset\notin\mathfrak{r}_0|A,\quad \emptyset\notin\mathfrak{r}_0|B.$$

Hence every $<_T$-compressed, regular filter is fixed. On the other hand, $\mathfrak{r}_0$ is a free maximal regular filter.

A $<_T$-round, $<_T$-compressed filter is regular in T by (3.9), therefore none of these filters is free, and the extension associated with $<_T$ is T itself. By (5.18) T is the unique extension T' of itself such that $<_T=<_{T'}|T$; consequently there is no T_3-closed extension T' with this property, and $<_T$ is not an *RC*-order. □

(8.5) LEMMA. *Consider the subspace $T_1=X\times N_1$ of T, and let $f\colon T_1\to T$ denote the canonical injection. Then f is $(<_{T_1},<_T)$-continuous without being strongly $(<_{T_1},<_T)$-continuous.*

PROOF. f is continuous, hence its $(<_{T_1},<_T)$-continuity follows from (7.3). Let $\mathfrak{R}$ and $\mathfrak{R}_1$ denote the systems of all $<_T$-round, $<_T$-compressed and $<_{T_1}$-round, $<_{T_1}$-compressed filters, respectively. We know from (8.1) and (8.4) that $\mathfrak{R}$ consists of the neighbourhood filters in T ((3.9), (3.10), (5.17), (5.10)).

On the other hand, the trace filter $\mathfrak{r}_1=\mathfrak{r}_0|T_1$ belongs to $\mathfrak{R}_1$. In fact $\mathfrak{r}_1$ is regular in T_1, hence $<_{T_1}$-round by (3.9). It is also $<_{T_1}$-compressed because if A and B are disjoint, closed subsets of T_1 then $A\cap(\{\omega\}\times N_1)$ and $B\cap(\{\omega\}\times N_1)$ are disjoint, closed subsets of the space $\{\omega\}\times N_1$. The latter is homeomorphic to $N_1\subset Y$, thus, by a well-known property of N_1, there is a $\beta<\omega_1$ such that, say, $(\omega,\alpha)\notin A$ for $\beta\leqq\alpha<\omega_1$. For every α of this kind, there are $n_\alpha<\omega$ and an open neighbourhood U_α of α in N_1 such that

$$A\cap(V_\alpha\times U_\alpha)=\emptyset,$$

where

$$V_\alpha=\{x\in X:\ n_\alpha<x\leqq\omega\}.$$

Let G_n be the union of those sets U_α for which $n_\alpha=n$. The sets G_n constitute a countable cover of the subspace

$$C=\{\alpha\in N_1:\ \beta\leqq\alpha<\omega_1\}\subset N_1.$$

Since N_1 and consequently C are countably compact, $C\subset\bigcup_{k=1}^{p}G_{n_k}$. For $m=\max\{n_k:\ k=1,\dots,p\}$, we have $A\cap(D\times C)=\emptyset$, where

$$D=\{x\in X:\ m<x\leqq\omega\};$$

in fact, $x\in D$, $y\in C$ implies $y\in G_{n_k}$ for some k, hence $y\in U_\alpha$ for an α such that $n_\alpha=n_k\leqq m<x\leqq\omega$, so $x\in V_\alpha$. Now $D\times C$ clearly belongs to $\mathfrak{r}_1$.

If $f(\mathfrak{r}_1)=\mathfrak{r}_1$ were finer than some $\mathfrak{r}\in\mathfrak{R}$ then we would have $\mathfrak{r}_1\to z$ for some $z\in T$, which is impossible. □

(8.6) Corollary. *For the space T_1 in* (8.5), *we have*

$$(8.7)\qquad <_{T_1}=<_T|T_1=<_Z|T_1,$$

and Z is an extension of T_1 associated with $<_{T_1}$.

Proof. Let A and B be disjoint, closed subsets of T_1. We have seen in the proof of (8.5) that $(\omega,\omega_1)\notin \mathrm{cl}_Z A\cap \mathrm{cl}_Z B$. Consider (n,ω_1) for some $n\in N$, and $V=\{n\}\times N_1$. $A\cap V$ and $B\cap V$ are disjoint, closed subsets of the subspace $V\subset T_1$, homeomorphic to N_1, thus there is a $\beta<\omega_1$ such that, say, $(n,\alpha)\notin A$ for $\beta<\alpha\leqq\omega_1$. Now $\{(n,\alpha):\ \beta<\alpha\leqq\omega_1\}$ is a neighbourhood of (n,ω_1) in Z so that $(n,\omega_1)\notin\mathrm{cl}_Z A$.

Therefore $<_{T_1}=<_Z|T_1$. Clearly

$$<_Z|T_1\subset<_T|T_1\subset<_{T_1}$$

so that (8.7) is established.

We show that the $<_{T_1}$-round, $<_{T_1}$-compressed filters in T_1 coincide with the traces of the neighbourhood filters in Z. From (8.7) we obtain by (4.14) that these traces are $<_{T_1}$-round and $<_{T_1}$-compressed. Conversely, a $<_{T_1}$-round, $<_{T_1}$-compressed filter in T_1 is, by (2.11), maximal $<_{T_1}$-round, and by (4.11) it is the trace of a maximal regular filter in Z. Since Z is compact T_2, such a filter is a neighbourhood filter. □

(8.8) Corollary. *For the above space T_1, the extension Z associated with $<_{T_1}$ contains a proper D-closed subspace T such that $<_{T_1}=<_T|T_1$.*

Proof. (8.6) and (8.4). □

(8.9) Lemma. *Let $<$ denote the discrete topogenous order on T. Then $<_T\subset<$ but $<$ is not strongly finer than $<_T$.*

PROOF. $\{<\}$ is a symmetrical topogenous structure, hence $<$ is an *RE*-ordet (the extension associated with $<$ is the Čech—Stone compactification of the discrete space T). Let $\mathfrak{r}$ be an ultrafilter in T, finer than the filter $\mathfrak{r}_0$. Clearly $\mathfrak{r}$ is $<$-round, $<$-compressed, but it is not finer than any $<_T$-round, $<_T$-compressed filter because the latters are, by (3.9), (8.1), (2.11) and (3.1), neighbourhood filters in T. □

(8.10). COROLLARY. *On the set T, there is no RE-order that would be strongly finer than any RE-order on T.*

PROOF. Such an *RE*-order would be finer than any *RE*-order on T, thus it would coincide with the discrete order $<$. However, the latter is not strongly finer than $<_T$ by (8.9). □

(8.11) LEMMA. *Consider the subspace $T_2=N\times Y$ of T. Then $<_T|T_2\subset <_{T_2}$ but $<_{T_2}$ is not strongly finer than $<_T|T_2$.*

PROOF. The first assertion is obvious by (4.4). Let $\mathfrak{s}$ be an ultrafilter in N finer than the filter base $\{R_n: n\in N\}$, $R_n=\{x\in N: x\geqq n\}$. Denote by $\mathfrak{r}$ the filter in T_2 generated by the filter base

$$\{S\times V_\alpha:\ S\in\mathfrak{s},\ \alpha<\omega_1\},$$

where $V_\alpha=\{y\in Y: \alpha<y\leqq\omega_1\}$. Clearly $\mathfrak{r}$ is regular in T_2 because S is clopen in N, V_α is open in Y, and $\alpha<\beta<\omega_1$ implies $V_\alpha\supset \mathrm{cl}_Y V_\beta$. Hence $\mathfrak{r}$ is $<_{T_2}$-round by (3.9).

We show that $\mathfrak{r}$ is $<_{T_2}$-compressed. In fact, let $A, B\subset T_2$ be disjoint, closed subsets of T_2. For $n\in N$, either $A\cap(\{n\}\times V_{\alpha_n})$ or $B\cap(\{n\}\times V_{\alpha_n})$ is empty for some $\alpha_n<\omega_1$. Let C be the set of those $n\in N$ for which this holds for $A\cap(\{n\}\times Y)$. Either C or $N-C$ belongs to $\mathfrak{s}$.

In the first case there is an $\alpha<\omega_1$ such that $\alpha_n\leqq\alpha$ for $n\in C$; then $C\times V_\alpha\in\mathfrak{r}$ does not intersect A. In the second one there is $\alpha<\omega_1$ such that $\alpha_n\leqq\alpha$ for $n\in N-C$, and then $(N-C)\times V_\alpha$ does not meet B.

By (2.11) and (4.11) the $<_T|T_2$-round, $<_T|T_2$-compressed filters are traces in T_2 of maximal regular filters in T, and the latters are by (8.1) either neighbourhood filters in T or equal to $\mathfrak{r}_0$. None of the formers is coarser than $\mathfrak{r}$ because $\mathfrak{r}\to(\omega,\omega_1)$ in Z, and $\mathfrak{r}_0|T_2$ is not $<_T|T_2$-compressed. In fact, if P and Q denote the sets of the even and the odd numbers in N, respectively, then $P\times\{\omega_1\}$ and $Q\times\{\omega_1\}$ are disjoint, closed subsets of T, and each element of $\mathfrak{r}_0|T_2$ intersects both of them. Hence $<_{T_2}$ is not strongly finer than $<_T|T_2$. □

(8.12) COROLLARY. *In the space T_2, there is no compatible RE-order that would be strongly finer than all compatible RE-orders.*

PROOF. Such an *RE*-order would be finer than all compatible *RE*-orders, i.e. it would coincide with $<_{T_2}$ by (4.4). However, $<_{T_2}$ is not strongly finer than the compatible *RE*-order $<_T|T_2$. □

Similar questions, answered negatively in the above examples, have been raised in [8] for *RC*-orders instead of *RE*-orders. Our results may be considered as first tentatives towards an answer, although they do not solve the problems because the order $<_T$ involved in them is not an *RC*-order.

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(Received February 9, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 223—225.

A REMARK ON A THEOREM OF H. DABOUSSI

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1. Let $e(\alpha)=e^{2\pi i\alpha}$. Let $\mathcal{M}$ denote the class of complex valued multiplicative functions, and $\mathcal{A}$ denote the class of real valued additive functions.

Let $\mathcal{F}\subseteq\mathcal{M}$ be the set of those multiplicative functions f for which $|f(n)|\leqq 1$ holds for every natural number n. Some years ago H. Daboussi has shown that for every irrational α

$$\frac{1}{N}\sum_{n\leqq N} f(n)e(n\alpha)\to 0 \quad (N\to\infty) \tag{1.1}$$

uniformly in $f\in\mathcal{F}$. Later this result has been extended and improved in [1] and [2].

An immediate consequence of Daboussi's theorem is the following result.

If α is an irrational number and $F\in\mathcal{A}$, then the sequence

$$\xi_n = F(n)+\alpha n \tag{1.2}$$

is uniformly distributed mod 1. Moreover, there exists a sequence $\varrho_n=\varrho_n(\alpha)>0$ monotonically tending to zero such that

$$\sup_{F\in\mathcal{A}}\ \sup_{0\leqq\gamma<\delta<1}\frac{1}{N}\Big(\#\big\{n\leqq N,\ \{\xi_n\}\in[\gamma,\delta)\big\}-N(\delta-\gamma)\Big)\leqq\varrho_N \quad (N\geqq 1).$$

2. We shall say that a sequence of real numbers $t(n)$ $(n=1,2,\dots)$ belongs to $\mathcal{T}$ if $F(n)+t(n)$ $(n=1,2,\dots)$ is uniformly distributed mod 1 for every $F\in\mathcal{A}$. It would be interesting to characterize $\mathcal{T}$. We are unable to do this, but we can prove that

$$\frac{1}{N}\sum_{n\leqq N} f(n)e\big(t(n)\big)\to 0 \quad (N\to\infty) \tag{2.1}$$

holds uniformly for $f\in\mathcal{F}$ for a quite large set of $t(n)$.

THEOREM 1. *Let us assume that for every positive K there exists a finite set $\mathcal{P}_K$ of primes $p_1<p_2<\dots<p_R$ such that*

$$A_{\mathcal{P}_K}:=\sum_{i=1}^{R}1/p_i>K, \tag{1}$$

(2) *for the sequences $\eta_{i,j}(m)=t(p_im)-t(p_jm)$ the relation*

$$\frac{1}{x}\sum_{m=1}^{x} e\big(\eta_{i,j}(m)\big)\to 0 \quad (x\to\infty) \tag{2.2}$$

holds, whenever $i\neq j$, $i,j\in\{1,\dots,R\}$.

Then there exists a sequence $\varrho_n > 0$ *monotonically tending to zero such that*

$$\sup_{f\in\mathcal{F}}\left|\frac{1}{N}\sum_{n<N} f(n)e(t(n))\right| \leqq \varrho_n. \tag{2.3}$$

THEOREM 2. *Let us assume that for every positive* K *there exists a finite set* $\mathcal{P}_K$ *of primes* $p_1 < \ldots < p_R$ *such that*

(1) $$A_{\mathcal{P}_K} := \sum_{i=1}^{R} 1/p_i > K,$$

(2) *the sequences* $\eta_{i,j}(m) = t(p_i m) - t(p_j m)$ $(m = 1, 2, \ldots)$ *are uniformly distributed* mod 1 *for every* $i \neq j$, $i, j \in \{1, \ldots, R\}$.

Then $t \in \mathcal{T}$. *Furthermore, for the discrepancy sequence*

$$D_N(F) := \sup_{0\leqq\gamma<\delta<1}\left|\frac{1}{N}\#\{\{F(n)+t(n)\}\in[\gamma,\delta)\}-(\delta-\gamma)\right|$$

we have

$$\limsup_{N\to\infty}\sup_{F\in\mathcal{A}} D_N(F) = 0. \tag{2.4}$$

3. PROOF OF THEOREM 1. Let $c, c_1, c_2, \ldots$ be absolute positive constants, $B, B_1, B_2, \ldots$ be real numbers majorized by absolute constants.

After fixing a K we put $\mathcal{P}_K = \mathcal{P}$, and $\omega_{\mathcal{P}}(n) = \sum_{\substack{p|n\\ p\in\mathcal{P}}} 1$. From the Turán—Kubilius inequality we get immediately

$$\sum_{n\leqq x} |\omega_{\mathcal{P}}(n) - A_{\mathcal{P}}| \leqq c_1 x (A_{\mathcal{P}})^{1/2}. \tag{3.1}$$

Let

$$S(x) = S(x, f) = \sum_{n\leqq x} f(n)e(t(n)), \tag{3.2}$$

$$H(x) = H(x, f) = \sum_{n\leqq x} f(n)e(t(n))\omega_{\mathcal{P}}(n). \tag{3.3}$$

From (3.1) we deduce

$$|H(x) - A_{\mathcal{P}} S(x)| \leqq c_1 x' \sqrt{A_{\mathcal{P}}}. \tag{3.4}$$

Furthermore,

$$H(x) = \sum_{\substack{pm\leqq x\\ p\in\mathcal{P}}}\sum f(pm)e(t(pm)).$$

For $(p, m) = 1$ we may write $f(pm) = f(p)f(m)$. The contribution of the pairs p,m satisfying $(p, m) > 1$ can be majorized by $x \sum 1/p_i^2$, consequently

$$H(x) = \sum_{m\leqq x/p_1} f(m) \sum_{p_i\leqq x/m} f(p_i)e(t(p_i m)) + B_1 x = \sum_{m\leqq x/p_1} f(m)\Sigma_m + B_1 x. \tag{3.5}$$

Since $(a+b)^2 \leqq 2(a^2+b^2)$ for real a, b, using the Cauchy-inequality, we get

$$|H(x)|^2 \leqq 2\Big(\sum_{m\leqq x/p_1} |f(m)|^2\Big)\Big(\sum_{m\leqq x/p_1} |\Sigma_m|^2\Big) + 2B_1^2 x^2 = 2UV + 2B_1^2 x^2. \tag{3.6}$$

We have $U \leqq x$. Furthermore,

$$V=\sum_{m\leqq x/p_1}\ \sum_{p_i,p_j\leqq x/m} f(p_i)\overline{f(p_j)}e\big(t(p_im)-t(p_jm)\big).$$

The contribution of the terms $p_i=p_j$ is $\sum\left[\frac{x}{p_i}\right]<xA_{\mathscr{P}}$. Consequently

$$V \leqq xA_{\mathscr{P}}+\sum_{\substack{p_i,p_j\in\mathscr{P}\\ i\neq j}}\Big|\sum_{m\leqq\min\left(\frac{x}{p_i},\frac{x}{p_j}\right)} e\big(t(p_im)-t(p_jm)\big)\Big|. \tag{3.7}$$

Collecting our inequalities (3.4), (3.6), (3.7), we get

$$\frac{|S(x)|^2A^2}{x^2} \leqq c_2A_{\mathscr{P}}\sum_{\substack{p_i,p_j\in\mathscr{P}\\ i\neq j}}\frac{1}{x}\Big|\sum_{m\leqq\min\left(\frac{x}{p_i},\frac{x}{p_j}\right)} e\big(t(p_im)-t(p_im)\big)\Big|. \tag{3.8}$$

Let $B(x)=\sup_{f\in\mathscr{F}}|S(x,f)|$. Since the right hand side of (3.8) does not depend on f, it holds for $B(x)$ instead of $|S(x,f)|$ as well. Consequently

$$\limsup\left(\frac{B(x)^2}{x}\right) \leqq c_2/A_{\mathscr{P}}. \tag{3.9}$$

Since $\mathscr{P}=\mathscr{P}_K$ can be chosen such that $A_{\mathscr{P}}>K$ for arbitrary large K, (3.9) implies that $B(x)=o(x)$ $(x\to\infty)$.

4. Proof of Theorem 2. Let k be an arbitrary nonzero integer. By putting $f(n)=e\big(kF(n)\big)$, $t(n)\to kt(n)$ the conditions of Theorem 1 are satisfied. By using a quantitative form of the Weyl-criterion, for example Erdős—Turán inequality, we get our theorem immediately.

5. Some remarks. 1. Theorems 1 and 2 remain valid assuming only that p_i, p_j are coprime integers.

2. If $t\in\mathscr{T}$, then $t(n)$ is uniformly distributed mod 1. This is obvious, since the zero-function is additive.

3. There exists a $t(n)$ uniformly distributed mod 1 which does not belong to $\mathscr{T}$. Indeed, let $\omega(n)$ be the number of prime divisors of n. Then $\alpha\omega(n)$ is uniformly distributed mod 1 if α is an irrational number, this can be proved in several ways. By putting $t(n)=\alpha\omega(n)$, $F(n)=-\alpha\omega(n)\in\mathscr{A}$, we get that $0=F(n)+t(n)$, which cannot be uniformly distributed.

4. Let $t(n)=\alpha_kn^k+\ldots+\alpha_1n$ be a polynomial of n such that at least one of the coefficients $\alpha_1,\ldots,\alpha_k$ is irrational. Then the conditions of Theorems 1 and 2 hold.

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(Received February 14, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 227—231.

ON SOME FUNCTIONS DEFINED BY THE CANONICAL EXPANSION OF COMPLEX NUMBERS

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1. We shall say that a Gaussian integer $\vartheta=a+bi$ is a canonical number base if every Gaussian integer α can be represented uniquely in the form

$$\alpha = a_0+a_1\vartheta+\ldots+a_r\vartheta^r \tag{1.1}$$

where $a_j\in A=\{0, 1, \ldots, N(\vartheta)-1\}$. I. Kátai and J. Szabó proved in [1], that ϑ is a canonical number base if and only if $\operatorname{Re}\vartheta<0$ and $\operatorname{Im}\vartheta=\pm 1$. In the same paper they proved that if ϑ is a canonical number base then every complex number z can be written in the form

$$z = \sum_{l=-\infty}^{k} a_l\vartheta^l. \tag{1.2}$$

These investigations have been extended for arbitrary quadratic fields by I. Kátai and B. Kovács [2], [3], and for some other algebraic fields by B. Kovács [4].

The geometrical properties of sets of complex numbers that have an expansion (1.2) with a given integer part have been considered by W. J. Gilbert [5], [6].

In their paper [7] Z. Daróczy, A. Járai and I. Kátai determined those functions for which

$$F\left(\sum_{k=1}^{\infty}\frac{\delta_k}{q^k}\right) = \sum_{k=1}^{\infty} F\left(\frac{\delta_k}{q^k}\right) \tag{1.3}$$

holds for every $\delta_k=0, 1, \ldots, q-1$, $k=1, 2, \ldots$, where $q\geqq 2$ is an integer. Namely they proved that if F satisfies (1.3) then $F(z)=az+b$ holds with suitable constants a, b.

Let $\vartheta=-A\pm i$ be a Gaussian number base. Let H denote the set of all complex numbers that have at least one representation in the form

$$z = \sum_{l=1}^{\infty}\frac{a_l}{\vartheta^l},\quad a_l\in A = \{0, 1, \ldots, N(\vartheta)-1\}. \tag{1.4}$$

THEOREM. *Let F be a complex valued function defined on H, $F(0)=0$. Assume that*

$$F(z) = \sum_{l=1}^{\infty} F\left(\frac{a_l}{\vartheta^l}\right) \tag{1.5}$$

holds for every $z\in H$. Then

$$F(z) = cz+d\bar{z}, \tag{1.6}$$

with suitable constants c, d.

2. Proof of the theorem

LEMMA 1. *Under the conditions stated in Theorem we have*

$$\sum_{l=1}^{\infty}\sum_{k=1}^{N(\vartheta)-1}\left|F\left(\frac{k}{\vartheta^l}\right)\right| < \infty. \tag{2.1}$$

PROOF. It is obvious that $\operatorname{Re} F(z)$ and $\operatorname{Im} F(z)$ satisfy the relation (1.5), consequently we may assume that F is a real valued function. For $k \in A$ let

$$E_k = \left\{l \middle| F\left(\frac{k}{\vartheta^l}\right) \geqq 0\right\}, \quad F_k = \left\{l \middle| F\left(\frac{k}{\vartheta^l}\right) < 0\right\}.$$

Let z_k and w_k be defined by

$$z_k = \sum_{l \in E_k} \frac{k}{\vartheta^l}, \quad w_k = \sum_{l \in F_k} \frac{k}{\vartheta^l}.$$

Since the series represent complex numbers $z_k, w_k \in H$, therefore

$$\sum_{l \in E_k} F\left(\frac{k}{\vartheta^l}\right) = F(z_k), \quad \sum_{l \in F_k} -F\left(\frac{k}{\vartheta^l}\right) = -F(w_k)$$

are absolutely convergent. Making this for each k, we get (2.1) immediately.

LEMMA 2. *Let B and C be positive coprime integers, $D=B+C$, and $X_0=0$, $X_1, \dots, X_D$ be arbitrary complex numbers satisfying the following relations:*

$$X_{C+n} = X_C + X_n \quad (n = 0, \dots, B) \tag{2.2}$$

$$X_{B+m} = X_B + X_m \quad (m = 0, \dots, C). \tag{2.3}$$

For every integer $n \in \{1, 2, \dots, D\}$, if $n = qC - sB$ with nonnegative integers q, s, then $X_n = qX_C - sX_B$. Consequently $X_n = nX_1$.

PROOF. First we shall prove that

$$X_{qC-sB} = qX_C - sX_B \tag{2.4}$$

whenever $1 \leqq qC - sB \leqq D$. We shall prove it by induction with respect to $t(q, s) = q + s$.

(2.4) is obviously true if $t(q, s) = 1$. In this case $q=1$, $s=0$. Let us assume that (2.2) has been proved for every q, s satisfying $t(q, s) \leqq r-1$. Let $t(q, s) = r$, $n = qC - sB$. If $n < C$, then $s \geqq 1$, $n + b = n_1 \leqq D$, $n_1 = qC - (s-1)B$, from (2.3) $X_{n_1} = X_n + X_B$.

Furthermore $t(q, (s-1)) = r-1$, and so $X_{n_1} = qX_C - (s-1)X_B$. Consequently $X_n = X_{n_1} - X_B = qX_C - sX_B$, (2.4) holds.

Let $n > C$. Then $n_1 = n - C = (q-1)C - sB$, from (2.2) and by the induction argument

$$X_n = X_C + X_{n-C} = X_C + q - 1, \quad X_C - sX_B = qX_C - sX_B.$$

The case $n = C$ is obvious.

To prove the second assertion we observe that $1=\xi C-\eta B$ with suitable positive integers ξ and η. Consequently $X_1=\xi X_C-\eta X_B$. Furthermore $n=(n\xi)C-(n\eta)B$, and so $X_n=n\xi X_C-n\eta X_B=n(\xi X_C-\eta X_B)=nX_1$.

To prove our theorem we are looking for such complex numbers that have two different expansions of the form (1.4), and we shall derive some relations among the values $F\left(\frac{a}{\vartheta^l}\right)$.

The minimal polynomial of ϑ has the form $\varphi(z)=z^2+2Az+A^2+1$, whence we deduce immediately that

$$\vartheta^3-1=-B\vartheta^2-C\vartheta+A^2, \tag{2.5}$$

with

$$B=2A-1, \quad C=(A-1)^2. \tag{2.6}$$

From (2.5) we get

$$\left(-\frac{B}{\vartheta}-\frac{C}{\vartheta^2}+\frac{A^2}{\vartheta^3}\right)\frac{1}{1-1/\vartheta^3}=1. \tag{2.7}$$

Since

$$\frac{1}{1-1/\vartheta^3}=1+\frac{1}{\vartheta^3}+\frac{1}{\vartheta^6}+\dots,$$

from (2.7) we get

$$1+\sum_{j=0}^{\infty}\frac{B}{\vartheta^{1+3j}}+\sum_{j=0}^{\infty}\frac{C}{\vartheta^{2+3j}}=\sum_{j=0}^{\infty}\frac{A^2}{\vartheta^{3+3j}}. \tag{2.8}$$

Let us divide both sides of (2.8) by ϑ^l, $l\geqq 1$. Then

$$\frac{1}{\vartheta^l}+\sum_{j=0}^{\infty}\frac{B}{\vartheta^{l+3j+1}}+\sum_{j=0}^{\infty}\frac{C}{\vartheta^{2+3j+l}}=\sum_{j=0}^{\infty}\frac{A^2}{\vartheta^{3+3j+l}}. \tag{2.9}$$

Let $B=x-y$, $x, y\in A$. Then

$$\frac{1}{\vartheta^l}+\frac{x}{\vartheta^{1+l}}+\sum_{j=1}^{\infty}\frac{B}{\vartheta^{1+3j+l}}+\sum_{j=0}^{\infty}\frac{C}{\vartheta^{2+3j+l}}=\frac{y}{\vartheta^{1+l}}+\sum_{j=0}^{\infty}\frac{A^2}{\vartheta^{3+3j+l}}. \tag{2.10}$$

Since $l, x, y, B, C, A^2\in A$, from (1.5) we get that

$$F\left(\frac{x}{\vartheta^{1+l}}\right)-F\left(\frac{y}{\vartheta^{1+l}}\right)=\text{constant} \tag{2.11}$$

for $x-y=B$, $x, y\in A$.

Similarly, if we put $C=u-v$ in C/ϑ^{2+l}, we get

$$F\left(\frac{u}{\vartheta^{2+1}}\right)-F\left(\frac{v}{\vartheta^{2+1}}\right)=\text{constant} \tag{2.12}$$

for $u-v=C$, $u, v\in A$.

That is

$$F\left(\frac{B+y}{\vartheta^\nu}\right)-F\left(\frac{y}{\vartheta^\nu}\right)=F\left(\frac{B}{\vartheta^\nu}\right) \tag{2.13}$$

for $v \geqq 2$, and

$$F\left(\frac{C+v}{\vartheta^v}\right)-F\left(\frac{v}{\vartheta^v}\right)=F\left(\frac{C}{\vartheta^v}\right) \tag{2.14}$$

for $v \geqq 3$.

Let us assume that $v \geqq 3$. We put $X_n=F\left(\frac{n}{\vartheta^v}\right)$. Then from (2.13) and (2.14) we get

$$X_{B+y}=X_B+X_y, \quad X_{C+v}=X_C+X_v.$$

The condition $(B, C)=1$ obviously holds. Consequently from Lemma 2 we get $X_n=nX_1$, that is

$$F\left(\frac{n}{\vartheta^v}\right)=nF\left(\frac{1}{\vartheta^v}\right) \quad (v \geqq 3, n \in A). \tag{2.15}$$

Let

$$\Lambda_r=F(1/\vartheta^r), \quad S_r=\sum_{j=r}^{\infty} \Lambda_j.$$

From (2.9) we get immediately that

$$\Lambda_l+\sum_{j=0}^{\infty} B\Lambda_{1+3j+l}+\sum_{j=0}^{\infty} C\Lambda_{2+3j+l}=A^2 \sum_{j=0}^{\infty} \Lambda_{3+3j+l}, \tag{2.16}$$

whenever $l \geqq 2$. Let us consider this equation for $l=R, R+1, R+2$, and take the sum of both sides of these equations. We get immediately that $\Lambda_R+\Lambda_{R+1}+\Lambda_{R+2}+BS_{R+1}+CS_{R+2}=A^2S_{R+3}$ if $R \geqq 2$.

Since $B+C=A^2$, we have

$$\Lambda_R+\Lambda_{R+1}+\Lambda_{R+2}+B(\Lambda_{R+1}+\Lambda_{R+2})+C\Lambda_{R+2}=0.$$

Consequently

$$\Lambda_R+(1+B)\Lambda_{R+1}+(1+B+C)\Lambda_{R+2}=0. \tag{2.7}$$

Observing that $1+B=2A$, $1+B+C=A^2+1$, we get that the characteristic polynomial of this recursion is $K(w)=(A^2+1)w^2+2Aw+1$.

The roots of $K(w)$ are $1/\vartheta$ and $1/\bar{\vartheta}$. Consequently $\Lambda_R=c(1/\vartheta)^R+d(1/\bar{\vartheta})^R$ $(R \geqq 2)$ with suitable constants c, d. Hence we get $F(z)=cz+d\bar{z}$, if z can be expressed in the form

$$z=\sum_{j=2}^{\infty} \frac{a_j}{\vartheta^j}. \tag{2.18}$$

Let us consider now (2.9) for $l=1$. Let

$$\xi=\sum_{j=0}^{\infty} \frac{B}{\vartheta^{2+3j}}+\sum_{j=0}^{\infty} \frac{C}{\vartheta^{3+3j}}, \quad \eta=\sum_{j=0}^{\infty} \frac{A^2}{\vartheta^{4+3j}}.$$

From (2.9) we get $\frac{1}{\vartheta}+\xi=\eta$, consequently $\frac{a+1}{\vartheta}+\xi=\frac{a}{\vartheta}+\eta$, $a \in \{0, \ldots, A^2-1\}$. Since ξ and η can be expressed in the form (2.18), therefore $F(\xi)=c\xi+d\bar{\xi}$, $F(\eta)=$

$=c\eta+d\bar{\eta}$, furthermore

$$F\left(\frac{a+1}{\vartheta}\right)+F(\xi) = F\left(\frac{a}{\vartheta}\right)+F(\eta),$$

we get

$$F\left(\frac{a+1}{\vartheta}\right)-F\left(\frac{a}{\vartheta}\right) = c(\eta-\xi)+d\overline{(\eta-\xi)} = c\frac{1}{\vartheta}+d\left(\frac{1}{\bar{\vartheta}}\right).$$

By using $F(0)=0$, this gives $F\left(\frac{a}{\vartheta}\right)=c\frac{a}{\vartheta}+d\left(\frac{\bar{a}}{\bar{\vartheta}}\right)$.

So we proved that $F\left(\frac{a}{\vartheta^\nu}\right)=c\left(\frac{a}{\vartheta^\nu}\right)+d\left(\frac{\bar{a}}{\bar{\vartheta}^\nu}\right)$, for every $a\in A$, $\nu\geqq 1$. Thus our theorem follows immediately.

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(Received March 8, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 233—238.

ON THE EXPECTED TIME OF THE FIRST OCCURRENCE OF EVERY k BIT LONG PATTERNS IN THE SYMMETRIC BERNOULLI PROCESS

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Let $T_k(\omega)$ denote the first occurrence time of every k bit long sequence in the realization $\omega=\omega_1\omega_2\dots$ ($\omega_i=0$ or 1, $i=1, 2, \dots$) of the symmetric Bernoulli process. Using Theorem 3.1 from Shou-Yen Robert Li [1] we give an elementary proof of

$$2^k(k\ln 2-\ln k) \leqq E(T_k(\omega)) \leqq 2^{k+2}(k\ln 2+c+O(2^{-k})).$$

Let U_k denote the set of different zero-one sequences of length k. We say the sequence $B=b_1b_2\dots b_l$ is an *i-continuation* of the sequence $A=a_1a_2\dots a_n$, if

$$a_n = b_{l-i},\ a_{n-1} = b_{l-i-1},\ \dots,\ a_{n-(l-i)+1} = b_1.$$

In the case when i is the smallest number for which B is i-continuation of A, we say B is *strict i-continuation* of A.

Let $H\subseteqq U_k$ be a subset of U_k. We define for the sequences $A\in H$ the continuation mean of A in H by

$$l^{(H)}(A) = \sum_{i=1}^{k-1} 2^{-i} l_i^{(H)}(A), \tag{1}$$

where $l_i^{(H)}(A)$ is the number of sequences in H that are strict i-continuations of A. It is easy to see that $l^{(H)}(A)$ is the expected number of the different elements of H which occur in a random, symmetric and independent continuation of A with $k-1$ bit.

In [1] Shou-Yen Robert Li gives the continuation measure $A*B$ of two patterns of series of independent, identically distributed discrete random variables. Applying it for the symmetric Bernoulli process, we get for $A, B\in U_k$:

$$A*B = \sum_{i=1}^{k-1} c_i 2^{k-i}, \tag{2}$$

where

$$c_i = \begin{cases} 1, & \text{if } B \text{ is } i\text{-continuation of } A \\ 0, & \text{else.} \end{cases}$$

LEMMA. *Let $N^{(H)}(\omega)$ be the time of the first occurrence of an element from H, where $\omega=\omega_1\omega_2\dots$ is a realization of the symmetric Bernoulli porocess, i.e.:*

$$N^{(H)}(\omega) = \min_N \{\omega_{N-k+1}\omega_{N-k+2}\dots\omega_N\in H\}.$$

Let

$$P_A = P\{\omega_{N^{(H)}_{(\omega)}-k+1}\dots\omega_{N^{(H)}_{(\omega)}} = A\},$$

and $|H|$ *be the cardinality of* H. *Then*

$$2^k \sum_{A\in H} P_A(l^{(H)}(A)+1) \leqq |H|E(N^{(H)}(\omega)) \leqq 2^{k+1} \sum_{A\in H} P_A(l(A)+1). \tag{3}$$

REMARK. The expression $\sum_{A\in H} P_A l^{(H)}(A)$ has a special meaning. It is the expected number of elements from H that occur in the subsequence of ω: $\omega_{N^{(H)}-k+1}\dots\omega_{N^{(H)}}\dots\omega_{N^{(H)}+k-1}$. Denoting it by $n(H)$, we get from (3):

$$2^k(n(H)+1) \leqq |H|E(N^{(H)}(\omega)) \leqq 2^{k+1}(n(H)+1). \tag{3'}$$

PROOF. Theorem 3.1 of [1] states that for every $A\in H$

$$\sum_{B\in H} P_B B*A = E(N^{(H)}(\omega)). \tag{4}$$

Summing it with respect to A and exchanging the order of summation we obtain:

$$\sum_{B\in H} P_B \sum_{A\in H} B*A = |H|E(N^{(H)}(\omega)). \tag{5}$$

We can estimate $B*A$ by

$$2^{k-i} \leqq B*A \leqq 2\cdot 2^{k-i} \tag{6}$$

when A is strict i-continuation of B.

Let $H(B,i)$ denote the subset of H that consists of the strict i-continuations of B, obviously $|H(B,i)|=l_i^{(H)}(B)$. Using this notation we get

$$\sum_{A\in H} B*A = B*B+\sum_{i=1}^{k-1} \sum_{A\in H(B,i)} B*A. \tag{7}$$

The term $\sum_{A\in H(B,i)} B*A$ can easily be estimated from (6):

$$\sum_{A\in H(B,i)} B*A \leqq l_i^{(H)}(B)-2^{k-i+1} = \frac{l_i^{(H)}(B)}{2^i}2^{k+1} \tag{8}$$

and

$$\sum_{A\in H(B,i)} B*A \geqq \frac{l_i^{(H)}(B)}{2^i}2^k.$$

Using the trivial inequalities $2^k \leqq B*B < 2^{k+1}$ we get from (1), (7) and (8)

$$2^k(1+l^{(H)}(B)) \leqq \sum_{A\in H} B*A \leqq 2^{k+1}(1+l^{(H)}(B)). \tag{9}$$

Estimating the left hand side of (5) by (9) we obtain the inequality (3) of our lemma.

Let $(\Omega, \mathcal{A}, P)$ be the probability space describing the symmetric Bernoulli process. The elements ω of Ω are the infinite binary sequences $\omega=\omega_1\omega_2\dots$, where $\omega_i=0$ or 1, $i=1,2,\dots$. The first occurrence time of every element of U_k is defined by

$$T_k(\omega) = \min\{T \mid \text{for every } A\in U_k$$
$$\text{there exists } i\leqq T-k+1 \text{ such that } \omega_i\omega_{i+1}\dots\omega_{i+k-1} = A\}.$$

For the subsequence $\omega_1\omega_2\ldots\omega_{T_k}$ we define a set of different, disjoint subsequences

$$\begin{aligned}\beta_1 &= \omega_1\ldots\omega_k,\\ \beta_2 &= \omega_{i_2}\ldots\omega_{i_2+k-1},\\ &\vdots\\ \beta_s &= \omega_{i_s}\ldots\omega_{i_s+k-1},\\ &\vdots\\ \beta_{N(\omega)} &= \omega_{i_{N(\omega)}}\ldots\omega_{i_{N(\omega)}+k-1}\end{aligned}$$

in the following way. Let us suppose that we have already defined β_s. Then we choose i_{s+1} so that

$$i_{s+1} = \min_{j\geqq i_s+k} \{\omega_j\omega_{j+1}\ldots\omega_{j+k-1} \neq \omega_l\omega_{l+1}\ldots\omega_{l+k-1} \text{ for every } l < i_s\}.$$

It can be easily seen that the sequence $\beta_1(\omega), \beta_2(\omega), \ldots, \beta_{N(\omega)}(\omega)$ is uniquely defined for every $\omega\in\Omega$.

Let $H_j(\omega)$ be the set of k-long sequences that do not occur in ω before β_j, and let $M_j(\omega)=|H_j(\omega)|$.

It follows from the construction of the sequence $\beta_1, \beta_2, \ldots, \beta_N$, that $M_j(\omega)-M_{j+1}(\omega)=1+X_j(\omega)$, where $0\leqq X_j(\omega)\leqq k-1$ and $X_j(\omega)$ is explicitly the number of different elements of $H_j(\omega)$ that occur in the subsequence $\omega_{i_j+1}\omega_{i_j+2}\ldots\omega_{i_{j+1}-1}$. The last notation we need is the distance $Y_j(\omega)$ between β_j and β_{j+1}:

$$Y_j(\omega) = i_{j+1}(\omega)-i_j(\omega), \quad \text{for} \quad j = 1, \ldots, N(\omega), \tag{10}$$

where $i_1=1$ and $i_{N(\omega)+1}=T_k(\omega)$.

Using our notations, we can construct $T_k(\omega)$ as a sum of random numbers of weakly dependent non-negative variables:

$$T_k(\omega) = \sum_{j=0}^{N(\omega)} Y_j(\omega). \tag{11}$$

We shall use this form to obtain a lower and upper estimation of $E(T_k(\omega))$. In order to simplify our formulae in the sequel we omit the argument ω of the random variables.

THEOREM 1. *We have*

$$E(T_k) \geqq 2^k(k\ln 2-\ln k-c),$$

where c is the Euler-constant.

PROOF. Taking the expectation of (11) and neglecting Y_N from the right side we get

$$E(T_k) \geqq E\Big(\sum_{j=1}^{N-1} Y_j\Big). \tag{12}$$

We can calculate the expectation on the right side as the expectation of the conditional expectation with respect to $H_1, H_2, \dots$ and H_l:

$$(13)\qquad E(\sum_{j=1}^{N-1} Y_j) = E[E(\sum_{j=1}^{N-1} Y_j|H_1, H_2, \dots, H_l)] =$$

$$= E[E(\sum_{j=1}^{l-1} Y_j|H_1, H_2, \dots, H_l)] + E[E(Y_l|H_1 H_2 \dots H_l)] + E[E(\sum_{j=l+1}^{N-1} Y_j|H_1 \dots H_l)].$$

As $E(Y_l|H_1 H_2 \dots H_l) = E(Y_l|H_l)$, and

$$E[E(\sum_{j=1}^{l-1} Y_j|H_1, H_2, \dots, H_l)] = E[E(\sum_{j=1}^{l-1} Y_j|H_1, H_2, \dots, H_{l-1})],$$

it follows that

$$(14)\qquad E(\sum_{j=1}^{N-1} Y_j) = E(\sum_{j=1}^{N-1} E(Y_l|H_l)).$$

Let us observe that if H_j is given, Y_j is equal to the first occurrence time $N^{(H_j)}$ of H_j, then

$$(15)\qquad E(Y_j|H_j = E(N^{(H_j)}).$$

Using inequality (3′) of the lemma, we get

$$(16)\qquad E(Y_j|H_j) \geqq 2^k \frac{n(H_j)+1}{M_j^{-1}}.$$

From (14) and (16) it follows that

$$(17)\qquad E(\sum_{j=1}^{N-1} Y_j) \geqq 2^k E\left(\sum_{j=1}^{N-1} \frac{n(H_j)+1}{M_j}\right).$$

Using simple transformations we get

$$(18)\qquad E\left(\sum_{j=1}^{N-1} \frac{n(H_j)+1}{M_j}\right) = E\left(\sum_{j=1}^{N-1} \frac{X_j+1+n(H_j)-X_j}{M_j}\right) =$$

$$= \sum_{i=1}^{2^k} \frac{1}{i} + E\left(\sum_{j=1}^{N-1} \left(\frac{X_j+1}{M_j} - \sum_{i=M_{j+1}}^{M_j} \frac{1}{i}\right)\right) + E\left(\sum_{j=1}^{N-1} \frac{n(H_j)-X_j}{M_j}\right).$$

As $M_j - M_{j+1} = X_j + 1$, and $0 \leqq X_j \leqq k-1$ the second term on the left side of (18) has its minimal value $\ln k + 0(k^{-1})$ when $X_j = k-1$, for $j = 1, 2, \dots, N-1$.

Using the same arguments as in the steps between (13) and (14) we get

$$(19)\qquad E\left(\sum_{j=1}^{N-1} \frac{n(H_j)-X_j}{M_j}\right) = E\left(\sum_{j=1}^{N-1} E\left(\frac{n(H_j)-X_j}{M_j}|H_j\right)\right).$$

As $n(H_j)$ and M_j are measurable with respect to H_j,

$$E\left(\frac{n(H_j)-X_j}{M_j}\Big|M_j\right)=\frac{1}{M_j}(n(H_j)-E(X_j|H_j)). \tag{20}$$

It is not true that $E(X_j|H_j)=n(H_j)$, but $n(H_j)\geqq E(X_j|H_j)$, since counting X_j we neglect the elements of H_j that have occurred after β_j and before β_{j+1}. Combining this with (19) and (20), we get that the third term in (18) satisfies the inequality

$$E\left(\sum_{j=1}^{N-1}\frac{n(H_j)-X_j}{M_j}\right)\geqq 0. \tag{21}$$

As the first term in (18) is equal to $\ln 2^k+c+O(2^{-k})$, from (12), (17) and (18) we get $E(T_k)\geqq 2^k(\ln 2^k-\ln k+c)$, and this proves Theorem 1.

THEOREM 2. $E(T_k)\leqq 2^{k+2}(\ln 2^k+c+O(2^{-k}))$.

PROOF. Starting from (11) and using the method of getting (14) and (16) we can show that

$$E(T_k)\leqq E\Big(\sum_{j=1}^{N}E(Y_l|H_l)\Big)\leqq 2^{k+1}E\left(\sum_{j=1}^{N}\frac{n(H_j)+1}{M_j}\right). \tag{22}$$

As the second term on the right side of (18) is less than or equal to zero, we can neglect it, and

$$E\left(\sum_{j=1}^{N}\frac{n(H_j)+1}{M_j}\right)\leqq\sum_{i=1}^{2^k}\frac{1}{i}+E\left(\sum_{j=1}^{N}\frac{n(H_j)-X_j}{M_j}\right). \tag{23}$$

According to (20)

$$E\left(\sum_{j=1}^{N}\frac{n(H_j)-X_j}{M_j}\right)=E\left(\sum_{j=1}^{N}\frac{n(H_j)-E(X_j|H_j)}{M_j}\right). \tag{24}$$

Let $H_j'=H_j-H_{j+1}-\beta_{j+1}$ be the set of subsequences of the sequence $\omega_{i_j-k+1}\dots\dots\omega_{i_j}\dots\omega_{i_j+k-1}$ belonging to H_j. By a simple extension of the notions $l^{(H)}(A)$ and $l_i^{(H)}(A)$ for the case when $A\notin H$, we can investigate the difference $l_i^{(H_j)}(A)-l_i^{(H_j-H_j')}(A)$.

It follows from the structure of the set H_j' that two elements of H_j' cannot be the strict i-continuations of A with the same i.

So $l_i^{(H_j)}(A)-l_i^{(H_j-H_j')}(A)\leqq 1$, and

$$l^{(H_j)}(A)-l^{(H_j-H_j')}(A)\leqq 1. \tag{25}$$

As $n(H_j)=\sum_{A\in H_j}P_A l^{(H_j)}(A)$ and $E(X_j|H_j)=\sum_{A\in H_j}P_A l^{(H_j-H_j')}(A)$, we get

$$n(H_j)-E(X_j|H_j)=\sum_{A\in H_j}P_A(l^{(H_j)}(A)-l^{(H_j-H_j')}(A))\leqq\sum_{A\in M_j}P_A=1. \tag{26}$$

If we use the trivial estimation

$$E\left(\sum_{j=1}^{N}\frac{1}{M_j}\right)\leqq\sum_{i=1}^{2^k}\frac{1}{i},$$

from (22), (23), (24) and (26) it follows that $E(T_k) \leqq 2^{k+2}(\ln 2^k + c + O(2^{-k}))$, which proves Theorem 2.

ACKNOWLEDGEMENT. The author is grateful to I. Kátai and T. Móri for their helpful discussions.

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(Received March 19, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 239—260.

ON SOME DE LA VALLÉE POUSSIN TYPE DISCRETE LINEAR OPERATORS

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0. Introduction. We shall define a de la Vallée—Poussin type trigonometric kernel (see (3) below), and the associated trigonometric as well as algebraic approximating operator. This kernel depends on several parameters (j, k, l, and m), and depending on the relation among these parameters, the corresponding operator shows fairly good properties. For some combination of the parameters we shall determine the exact norm of the operator, for other values we give reasonable estimates for the norms (see Sections 7 and 10). We shall prove the Jackson, Timan, and Telyakowski—Gopengauz theorems with explicit constants (see Section 15), and will see that for a certain choice of the parameters our operator converges in the order of best approximation. (Compare e.g. [5], where the constant given for the Telyakowski—Gopengauz theorem is of order 10^3, while our constant is ~ 5.7; cf. Section 16). At the same time our operator interpolates at certain nodes, and reproduces polynomials of certain degree. For some combination of the parameters all of these good properties listed so far can be achieved with the same operator. At the end of the paper we give a constructive solution to a problem raised by G. Freud and A. Sharma [2].

1. Notations. t is a real variable, g is a real or complex valued 2π-periodic continuous function of t. The positive integers j, m and the nonnegative integers k, l are such that the number

$$n=\frac{1}{2}(jm+km-k-1+l) \tag{1}$$

is a nonnegative integer. Further let

$$t_\nu=\frac{2\pi\nu}{jm}\quad(\nu=0,\pm1,\pm2,\ldots), \tag{2}$$

$$s_{jklm}(t)=\frac{\sin\frac{jmt}{2}\sin^k\frac{mt}{2}\cos^l\frac{t}{2}}{jm^{k+1}\sin^{k+1}\frac{t}{2}},\quad\text{if}\quad \sin\frac{t}{2}\neq0, \tag{3}$$

$$s_{jklm}(t)=\lim_{\tau\to t}s_{jklm}(\tau)=1,\quad\text{if}\quad\sin\frac{t}{2}=0, \tag{4}$$

$$S_{jklm}(g,t)=\sum_{\nu=0}^{jm-1}g(t_\nu)s_{jklm}(t-t_\nu). \tag{5}$$

2. REMARKS. $S_{100m}(g, t)$ (m odd) and $S_{101m}(g, t)$ (m even) are ordinary interpolatory polynomials. $S_{110m}(g, t)$ and $S_{310m}(g, t)$ were investigated by D. Jackson [4] and J. Szabados [6], respectively.

$s_{100m}(t)$ (m odd) is the Dirichlet kernel, $s_{110m}(t)$ is the Fejér kernel, $s_{310m}(t)$ is the de la Vallée—Poussin kernel, and $s_{130m}(t)$ is the Jackson kernel.

The estimates given in this paper should be viewed as $m\to\infty$ (or $n\to\infty$), while the other parameters (j, k, l) remain fixed. We separate the indices by commas if it can be misunderstood (e.g. we write $s_{j,k+1,l,m}(t)$, but $s_{jklm}(t)$).

3. LEMMA. *$S_{jklm}(g, t)$ is a trigonometric polynomial of order at most n, and*

(6) $$S_{jklm}(g, t_\nu) = g(t_\nu) \quad (\nu = 0, \pm 1, \pm 2, \ldots).$$

4. REMARK. Proofs will start in Section 20, where the coefficients of $S_{jklm}(g, t)$ will also be calculated.

5. NOTATIONS. $E_\nu^T(g)$ $(\nu=0, 1, 2, \ldots)$ denotes the best approximation of g by trigonometric polynomials of order at most ν in uniform metric,

(7) $$L_{jklm}(t) = \sum_{\nu=0}^{jm-1} |s_{jklm}(t-t_\nu)|;$$

(8) $$q = \frac{1}{2}(jm-km+k-1-l).$$

6. REMARK. Since j, m and n are integers, so is $q=jm-n-1$.

7. THEOREM. *If $q\geqq 0$, then*

(9) $$|S_{jklm}(g, t)-g(t)| \leqq \big(1+L_{jklm}(t)\big)E_q^T(g);$$

(10) $$\|L_{jklm}\| = \max_t L_{jklm}(t)$$

is a decreasing function of k and l;

(11)—(12) $$\|L_{j10m}\| = \begin{cases} \dfrac{1}{j}\left(1+2\displaystyle\sum_{\nu=1}^{(j-1)/2} 1/\cos\frac{\nu\pi}{j}\right) & \textit{if } j \textit{ is odd}, \\ \dfrac{2}{j}\displaystyle\sum_{\nu=1}^{j/2} 1/\cos\frac{2\nu-1}{2j}\pi & \textit{if } j \textit{ and } m \textit{ are even}; \end{cases}$$

(13) $$\|L_{j11m}\| \leqq \frac{2}{\pi}\log j+2.283 \quad \textit{if } j \textit{ is even, } m \textit{ is odd};$$

(14) $$\|L_{221m}\| \leqq \frac{2}{\sqrt{3}} < 1.155;$$

(15) $$\|L_{332m}\| = \frac{11}{9}-\frac{2}{9m^2} < \frac{11}{9} < 1.223.$$

8. REMARKS. Let λ_j denote the Lebesgue constant of the Lagrange interpolation with respect to the Chebyshev nodes $\cos\dfrac{2\nu-1}{2j}$ $(\nu=1, \ldots, j)$. It is known

[1] that

$$\lambda_j = \frac{1}{j} \sum_{\nu=1}^{j} \cot \frac{2\nu-1}{4j} \pi \quad (j = 1, 2, \ldots).$$

This together with (11)—(12) easily yield

$$L_{j10m} = \begin{cases} \frac{1}{j} \left(1 + 2 \sum_{\nu=1}^{(j-1)/2} 1/\sin \frac{2\nu-1}{2j} \pi\right) = \lambda_j & (j = 1, 3, \ldots) \\ \frac{2}{j} \sum_{\nu=1}^{j/2} 1/\sin \frac{2\nu-1}{2j} \pi = \lambda_j & (j = 2, 4, \ldots). \end{cases}$$

By [3]

$$\lambda_j = \frac{2}{\pi} \log j + c + \alpha_j \quad (j = 1, 2, \ldots),$$

where

$$c = \frac{2}{\pi} \left(\gamma + \log \frac{8}{\pi}\right) = 0.9625\ldots,$$

and $\gamma = 0.5772\ldots$ is the Euler constant, $0 < \alpha_j < \frac{\pi}{72j^2}$ $(j = 1, 2, \ldots)$.

For certain values of the indices, Lemma 29 will give the explicit form of the Lebesgue function $L_{jklm}(t)$.

If j and m are odd, $k = l = 0$, then in (9) $n = q = \frac{jm-1}{2}$, but $\|L_{j00m}\| \sim \sim \frac{2}{\pi} \log (jm)$ is an unbounded function of n. If $k \geqq 1$ and $l \geqq 0$ are fixed and j is large, then n/q is close to 1, but $\|L_{jklm}\| \sim \frac{2}{\pi} \log j$ is relatively large.

9. Notations. The modulus of continuity of g is denoted by $\omega(g, \delta)$;

$$M_{jklm}(t) = \sum_{\nu=0}^{jm-1} \left| \frac{\sin m \frac{t-t_\nu}{2}}{m \cdot \sin \frac{t-t_\nu}{2}} \right|^k \left| \cos \frac{t-t_\nu}{2} \right|^l. \tag{16}$$

(This is a decreasing function of k and l.)

10. Theorem. *If $k \geqq 1$, $l \geqq 1$ and $q \geqq 0$, then*

$$|S_{jklm}(g, t) - g(t)| \leqq \omega\left(g, \frac{\pi}{jm}\right) \left\{L_{jklm}(t) + \frac{2}{\pi} M_{j,k,l-1,m}(t)\right\}; \tag{17}$$

$$M_{j20m}(t) = j \quad \textit{if} \quad j \geqq 1; \tag{18}$$

$$M_{j31m}(t) < \frac{\sqrt{2}}{\sqrt{3}} j \quad \textit{if} \quad j \geqq 2; \tag{19}$$

$$M_{j42m}(t) = \frac{2}{3} j - \frac{j}{6m^2} < \frac{2}{3} j \quad \textit{if} \quad j \geqq 2. \tag{20}$$

11. COROLLARY. (17), (14), (18) *and* (1) *imply*

$$\|S_{221m}(g,t)-g(t)\| \leqq \left(\frac{2}{\sqrt{3}}+\frac{4}{\pi}\right)\omega\left(g,\frac{\pi}{n+1}\right). \tag{21}$$

12. REMARKS. The coefficient on the right hand side of (21) is less than 2.428. Lemma 39 will give the explicit form of $M_{jklm}(t)$ for certain values of the indices.

(9) and the Korneičuk inequality

$$E_q^T(g) \leqq \omega\left(g,\frac{\pi}{q+1}\right) \tag{22}$$

yield

$$\|S_{jklm}(g,t)-g(t)\| \leqq (1+\|L_{jklm}(t)\|)\omega\left(g,\frac{\pi}{q+1}\right).$$

Nevertheless, the estimate (17) can be more exact than this.

13. NOTATIONS. x is a variable in $[-1,1]$, $t=\arccos x$, $f(x)$ is a real valued continuous function of x; $E_\nu(f)$ $(\nu=0,1,2,\ldots)$ is its best approximation by algebraic polynomials of degree at most ν;

$$P_{jklm}(f,x) = S_{jklm}(f\circ\cos,t); \tag{23}$$

$$x_\nu = \cos\frac{2\pi\nu}{jm} \quad (\nu=0,\pm1,\pm2,\ldots). \tag{24}$$

14. REMARK. $P_{100m}(f,x)$ (m odd) and $P_{101m}(f,x)$ (m even) is the Lagrange interpolating polynomial associated with the nodes $\cos\frac{2\pi\nu}{m}$.

15. THEOREM. $P_{jklm}(f,x)$ *is an algebraic polynomial of degree at most n, and*

$$P_{jklm}(f,x_\nu) = f(x_\nu) \quad (\nu=0,\pm1,\pm2,\ldots). \tag{25}$$

If $q\geqq 0$, *then*

$$|P_{jklm}(f,x)-f(x)| \leqq (1+L_{jklm}(t))E_q(f) \quad (|x|\leqq 1). \tag{26}$$

If $k\geqq 1$ *and* $q\geqq 0$, *then*

$$|P_{jklm}(f,x)-f(x)| \leqq \omega\left(f,\frac{\pi}{jm}\right)\left[L_{jklm}(t)+\frac{2}{\pi}M_{jklm}(t)\right] \quad (|x|\leqq 1). \tag{27}$$

If $k\geqq 2$ *and* $q\geqq 0$, *then*

$$|P_{jklm}(f,x)-f(x)| \leqq \omega\left(f,\frac{\pi\sqrt{1-x^2}}{jm}\right)\left[L_{jklm}(t)+\frac{2}{\pi}M_{j,k,l+1,m}(t)\right]+ \tag{28}$$

$$+\omega\left(f,\frac{\pi|x|}{j^2m^2}\right)\left[L_{jklm}(t)+\frac{2j}{\pi^2}M_{j,k-1,l,m}(t)\right] \quad (|x|\leqq 1).$$

If j is odd, $k \geqq 2$ is even, and $q \geqq 0$, then

$$(29)\qquad |P_{jklm}(f, x)-f(x)| \leqq \omega\left(f, \frac{2\pi\sqrt{1-x^2}}{m}\right) L_{jklm}(t)+\omega\left(f, \frac{2\pi^2|x|}{m^2}\right)\cdot$$

$$\cdot\left[2L_{jklm}(t)+\frac{2}{\pi j} M_{j,k,l+1,m}(t)\right] \quad (|x| \leqq 1).$$

If j or m is even, $k \geqq 2$ and $q \geqq 0$, then

$$(30)\qquad |P_{jklm}(f, x)-f(x)| \leqq \omega\left(f, \frac{\pi\sqrt{1-x^2}}{jm}\right)\cdot$$

$$\cdot\left[L_{jklm}(t)+\frac{2}{\pi} M_{j,k,l+1,m}(t)+\frac{j|x|}{\pi} M_{j,k-1,l,m}(t)\right] \quad (|x| \leqq 1).$$

If j is odd, $k \geqq 2$ and m are even, and $q \geqq 0$, then

(31)

$$|P_{jklm}(f, x)-f(x)| \leqq \omega\left(f, \frac{2\pi\sqrt{1-x^2}}{m}\right)\left[2L_{jklm}(t)+\left(1+\frac{|x|}{2}\right) M_{jklm}(t)\right] \quad (|x| \leqq 1).$$

16. Corollaries. (27), (14), (18) and (1) yield

$$\|P_{221m}(f,x)-f(x)\| \leqq \left(\frac{2}{\sqrt{3}}+\frac{4}{\pi}\right) \omega\left(f, \frac{\pi}{n+1}\right).$$

(28), (15), (19), (18) and (1) give (since $M_{jklm}(t)$ is a decreasing function of l)

$$|P_{332m}(f, x)-f(x)| \leqq \left(\frac{11}{9}+\frac{2\sqrt{6}}{\pi}\right)\omega\left(f, \frac{\pi\sqrt{1-x^2}}{n+1}\right)+\left(\frac{11}{9}+\frac{18}{\pi^2}\right)\omega\left(f, \frac{\pi^2|x|}{(n+1)^2}\right) \quad (|x| \leqq 1).$$

(The coefficients on the right hand side are less than 2.782 and 3.047, respectively.)

(30), (15), (19), (10) and (1) yield (since $M_{jklm}(t)$ is a decreasing function of l)

$$|P_{332m}(f, x)-f(x)| \leqq \left(\frac{11}{9}+\frac{2\sqrt{6}+9}{\pi}\right) \omega\left(f, \frac{\pi\sqrt{1-x^2}}{n+1}\right) \quad (|x| \leqq 1).$$

(Here the coefficient is less than 5.647.)

17. Remarks. (26) and the Korneičuk inequality

$$E_q(f) \leqq \omega\left(f, \frac{\pi}{q+1}\right)$$

yield

$$|P_{jklm}(f, x)-f(x)| \leqq \left(1+L_{jklm}(t)\right) \omega\left(f, \frac{\pi}{q+1}\right) \quad (|x| \leqq 1),$$

but (27) may be more exact than this.

In general, for $k \geqq 2$, (28) is more precise than (27). For $k=2$ in (28) and (30), $\|M_{j1lm}\| \sim \frac{2}{\pi} \log (jm)$ is an unbounded function of m, while in this case in (29) and (31) the coefficients are bounded (provided j is fixed).

If $-1<x<1$ and m is large enough, then (28) is more exact than (30).

18. NOTATIONS. Let

$$a_{\nu j0m} = \begin{cases} 1, & \text{if } 0 \leqq \nu < jm, \\ 0, & \text{if } \nu < 0 \text{ or } \nu \geqq jm; \end{cases} \tag{32}$$

$$a_{\nu jkm} = \begin{cases} \sum_{h=0}^{m-1} a_{\nu-h,\,j,\,k-1,\,m}, & \text{if } 0 \leqq \nu < jm+km-k, \quad k \geqq 1, \\ 0, & \text{if } \nu < 0 \text{ or } \nu \geqq jm+km-k, \quad k \geqq 1. \end{cases} \tag{33}$$

19. LEMMA. *We have*

$$a_{\nu jkm} = \sum_{h=0}^{[\nu/m]} (-1)^h \binom{k}{h} \binom{\nu+k-hm}{k}, \quad \textit{if} \quad 0 \leqq \nu < jm; \tag{34}$$

(35)

$$a_{\nu jkm} = \sum_{h=0}^{[\nu/m]} (-1)^h \binom{k}{h} \binom{\nu+k-hm}{k} - \sum_{h=0}^{[\nu/m]-j} (-1)^h \binom{k}{h} \binom{\nu+k-hm-jm}{k}, \quad \textit{if} \quad \nu \geqq jm.$$

Especially

$$a_{\nu 1km} = \sum_{h=0}^{[\nu/m]} (-1)^h \binom{k+1}{h} \binom{\nu+k-hm}{k}. \tag{36}$$

20. PROOF. We prove by mathematical induction that if $z \neq 1$ is an arbitrary complex number, then

$$\frac{(1-z^{jm})(1-z^m)^k}{(1-z)^{k+1}} = \sum_{\nu=0}^{jm+km-k-1} a_{\nu jkm} z^\nu. \tag{37}$$

For $k=0$ the statement follows from (32):

$$\frac{1-z^{jm}}{1-z} = \sum_{\nu=0}^{jm-1} z^\nu = \sum_{\nu=0}^{jm-1} a_{\nu j0m} z^\nu.$$

If (37) holds for $k-1$ in place of k, then by (33) it holds for k, too:

$$\frac{(1-z^{jm})(1-z^m)^k}{(1-z)^{k+1}} = \frac{(1-z^{jm})(1-z^m)^{k-1}}{(1-z)^k} \frac{1-z^m}{1-z} = \sum_{\nu=0}^{jm+km-m-k} a_{\nu,j,k-1,m} z^\nu \sum_{h=0}^{m-1} z^h =$$

$$= \sum_{\nu=0}^{km+jm-k-1} a_{\nu jkm} z^\nu.$$

If $|z|<1$, then

$$\frac{(1-z^{jm})(1-z^m)^k}{(1-z)^{k+1}} = (1-z^{jm})\sum_{h=0}^{k}(-1)^h\binom{k}{h}z^{hm}\sum_{\nu=0}^{\infty}\binom{\nu+k}{k}z^\nu =$$

$$= (1-z^{jm})\sum_{\nu=0}^{\infty}z^\nu\sum_{h=0}^{[\nu/m]}(-1)^h\binom{k}{h}\binom{\nu+k-hm}{k} = \sum_{\nu=0}^{jm-1}z^\nu\sum_{h=0}^{[\nu/m]}(-1)^h\binom{k}{h}\binom{\nu+k-hm}{k}+$$

$$+\sum_{\nu=jm}^{\infty}z^\nu\left\{\sum_{h=0}^{[\nu/m]}(-1)^h\binom{k}{h}\binom{\nu+k-hm}{k}-\sum_{h=0}^{[\nu/m]-j}(-1)^h\binom{k}{h}\binom{\nu+k-hm-jm}{k}\right\}.$$

This together with (37) implies (34)—(35).

Now if $|z|<1$ again, then

$$\left(\frac{1-z^m}{1-z}\right)^{k+1} = \sum_{h=0}^{k+1}(-1)^h\binom{k+1}{h}z^{hm}\sum_{\nu=0}^{\infty}\binom{\nu+k}{k}z^\nu =$$

$$= \sum_{\nu=0}^{\infty}z^\nu\sum_{h=0}^{[\nu/m]}(-1)^h\binom{k+1}{h}\binom{\nu+k-hm}{k}.$$

This and (37) yield (36).

21. NOTATION. Let

$$b_{\nu jklm} = \frac{1}{2^l jm^{k+1}}\sum_{h=0}^{l}\binom{l}{h}a_{\nu+n-h,j,k,m} \quad (\nu = 0, \pm 1, \pm 2, \dots). \tag{38}$$

22. LEMMA. *We have*

$$b_{-\nu,j,k,l,m} = b_{\nu jklm}, \quad \textit{if} \quad \nu > 0; \tag{39}$$

$$b_{\nu jklm} = \frac{1}{jm}, \quad \textit{if} \quad q \geqq 0 \quad \textit{and} \quad -q \leqq \nu \leqq q; \tag{40}$$

$$s_{jklm}(t) = b_{0jklm}+2\sum_{\nu=1}^{n}b_{\nu jklm}\cos \nu t. \tag{41}$$

23. PROOF. Let $z=e^{it}$. We get from (3)

$$s_{jklm}(t) = \frac{\dfrac{z^{jm/2}-z^{-jm/2}}{2i}\left(\dfrac{z^{m/2}-z^{-m/2}}{2i}\right)^k\left(\dfrac{z^{1/2}+z^{-1/2}}{2}\right)^l}{jm^{k+1}\left(\dfrac{z^{1/2}-z^{-1/2}}{2i}\right)^{k+1}}. \tag{42}$$

Hence and from (1) and (37)

$$s_{jklm}(t) = \frac{(1-z^{jm})(1-z^m)^k(1+z)^l}{2^l jm^{k+1}(1-z)^{k+1}z^n} = \frac{\sum\limits_{h=0}^{jm+km-k-1}a_{hjkm}z^h\sum\limits_{r=0}^{l}\binom{l}{r}z^r}{2^l jm^{k+1}z^n}.$$

Thus we get from (38)

$$s_{jklm}(t)=\sum_{h=-n}^{n} b_{hjklm} z^h. \tag{43}$$

(42) does not change if we replace t by $-t$. Hence (43) yields (39).

(43), (39) and (4) imply (41).

We obtain from (32)—(33) by mathematical induction:

$$a_{\nu jkm}=m^k, \quad \text{if} \quad km-k \leqq \nu < jm. \tag{44}$$

If $q\geqq 0$, $-q\leqq \nu \leqq q$ and $0\leqq h\leqq l$, then by (1) and (8) $km-k\leqq \nu+n-h<jm$, and therefore (38) and (44) yield (40).

24. PROOF OF LEMMA 3. By (5) and (41)

$$S_{jklm}(g,t)=\sum_{\nu=0}^{jm-1} g(t_\nu)\Big(b_{0jklm}+2\sum_{h=1}^{n} b_{hjklm}\cos h(t-t_\nu)\Big)= \tag{45}$$

$$=b_{0jklm}\sum_{\nu=0}^{jm-1} g(t_\nu)+\sum_{h=1}^{n}\cos ht\Big(2b_{hjklm}\sum_{\nu=0}^{jm-1} g(t_\nu)\cos ht_\nu\Big)+$$

$$+\sum_{h=1}^{n}\sin ht\Big(2b_{hjklm}\sum_{\nu=0}^{jm-1} g(t_\nu)\sin ht_\nu\Big).$$

Thus $S_{jklm}(g,t)$ is a trigonometric polynomial of order at most n.

(2)—(4) imply

$$s_{jklm}(t_\nu)=\begin{cases}1, & \text{if} \quad (jm)|\nu,\\ 0, & \text{if} \quad (jm)\nmid \nu.\end{cases}$$

This together with (5) yields (6).

25. LEMMA. *If* $q\geqq 0$ *and* $y(t)$ *is a trigonometric polynomial of order at most* q, *then*

$$S_{jklm}(y,t)=y(t). \tag{46}$$

26. PROOF. If $q\geqq 0$ and

$$y_r(t)=z^r \quad (z=e^{it},\ r=0,\pm 1,\pm 2,\dots,\pm q), \tag{47}$$

then with the notation $z_\nu=e^{it_\nu}$ $(\nu=0,1,\dots,jm-1)$ we get by (5) and (43)

$$S_{jklm}(y_r,t)=\sum_{\nu=0}^{jm-1} z_\nu^r \sum_{h=-n}^{n} b_{hjklm}\left(\frac{z}{z_\nu}\right)^h=\sum_{h=-n}^{n} b_{hjklm}z^h\sum_{\nu=0}^{jm-1} z_\nu^{r-h}. \tag{48}$$

Now (2) yields

$$\sum_{\nu=0}^{jm-1} z_\nu^v=\sum_{\nu=0}^{jm-1} z_v^\nu=\begin{cases} jm, & \text{if} \quad (jm)|v,\\ 0 & \text{otherwise.}\end{cases} \tag{49}$$

By (1) and (8), we have $jm-1=n+q\geqq r-h\geqq -n-q=1-jm$ in (48). Hence and by (48), (49), (40) and (47) we obtain

$$S_{jklm}(y_r,t)=jmb_{rjklm}z^r=z^r=y_r(t). \tag{50}$$

But any trigonometric polynomial of order at most q can be written in the form

$$\sum_{r=-q}^{q} c_r y_r(t),$$

where c_r are complex numbers. Thus the statement follows from (5) and (50).

27. Proof of (9). If y denotes the best approximating trigonometric polynomial of g of order at most q, then by (46), (5) and (7) we have

$$|S_{jklm}(g,t)-g(t)| = |S_{jklm}(g,t)-S_{jklm}(y,t)+y(t)-g(t)| =$$

$$= \Big|\sum_{\nu=0}^{jm-1}(g(t_\nu)-y(t_\nu))s_{jklm}(t-t_\nu)+y(t)-g(t)\Big| \le \sum_{\nu=0}^{jm-1}|g(t_\nu)-y(t_\nu)|\cdot$$

$$\cdot |s_{jklm}(t-t_\nu)|+|y(t)-g(t)| \le (L_{jklm}(t)+1)E_q^T(g).$$

28. Proof of the second statement of Theorem 7. Since $\left|\cos\frac{t}{2}\right|\le 1$ and $\left|\sin\frac{mt}{2}\right|\le m\left|\sin\frac{t}{2}\right|$, thus by (3), (7) and (10), $|s_{jklm}(t)|$, $L_{jklm}(t)$ and $\|L_{jklm}\|$ are decreasing functions of k and l.

29. Lemma. *$L_{jklm}(t)$ is a $\frac{2\pi}{jm}$-periodic function. If $-\frac{2\pi}{jm}\le t\le 0$, jk is odd and l is even, then*

$$L_{jklm}(t) = m\left(b_{0jklm}+2\sum_{\nu=1}^{[n/m]} b_{\nu m,j,k,l,m}\cos\nu\left(mt+\frac{\pi}{j}\right)\Big/\cos\frac{\nu\pi}{j}\right). \tag{51}$$

If j, l and m are even, k is odd then

(52)

$$L_{jklm}(t) = 2m\sum_{\nu=1}^{[n/m+1/2]} b_{m(\nu-1/2),j,k,l,m}\cos\left(\nu-\frac{1}{2}\right)\left(mt+\frac{\pi}{j}\right)\Big/\cos\left(\nu-\frac{1}{2}\right)\frac{\pi}{j}.$$

30. Proof. By (7) and (2)

$$L_{jklm}\left(t+\frac{2\pi}{jm}\right) = \sum_{\nu=0}^{jm-1}\left|s_{jklm}\left(t+\frac{2\pi}{jm}-t_\nu\right)\right| = \sum_{\nu=0}^{jm-1}|s_{jklm}(t-t_{\nu-1})| = \sum_{\nu=-1}^{jm-2}|s_{jklm}(t-t_\nu)|.$$

By (41), $s_{jklm}(t)$ is 2π-periodic, hence

$$s_{jklm}(t-t_{-1}) = s_{jklm}\left(t+\frac{2\pi}{jm}\right) = s_{jklm}\left(t-2\pi+\frac{2\pi}{jm}\right) = s_{jklm}(t-t_{jm-1}),$$

i.e. $L_{jklm}(t)$ is, indeed, $\frac{2\pi}{jm}$-periodic:

$$L_{jklm}\left(t+\frac{2\pi}{jm}\right) = \sum_{\nu=0}^{jm-1}|s_{jklm}(t-t_\nu)| = L_{jklm}(t).$$

Because of (41), $s_{jklm}(t)$ is an even function. If jk is odd and l is even, then by (2)—(3)

$$\operatorname{sgn} s_{jklm}(t-t_{hj+r}) = \operatorname{sgn} s_{jklm}(t_{hj+r}-t) = (-1)^r \quad (0 \leqq h < m,\ 0 \leqq r < j)$$

when $-\dfrac{2\pi}{jm} < t < 0$. Hence and from (7), (43) and (2) we get

$$L_{jklm}(t) = \sum_{r=0}^{j-1}\sum_{h=0}^{m-1}(-1)^r s_{jklm}(t_{hj+r}-t) = \sum_{r=0}^{j-1}(-1)^r \sum_{h=0}^{m-1}\sum_{\nu=-n}^{n} b_{\nu jklm}\left(\frac{z_{hj+r}}{z}\right)^{\nu} =$$

$$= \sum_{r=0}^{j-1}(-1)^r \sum_{\nu=-n}^{n} b_{\nu jklm}\left(\frac{z_r}{z}\right)^{\nu} \sum_{h=0}^{m-1} z_{\nu}^{hj}.$$

Here

$$\sum_{h=0}^{m-1}(z_\nu^j)^h = \begin{cases} m, & \text{if } \ m|\nu, \\ 0 & \text{otherwise.}\end{cases} \tag{53}$$

Thus by (2)

$$L_{jklm}(t) = m \sum_{\nu=-[n/m]}^{[n/m]} b_{m\nu,j,k,l,m} z^{-m\nu} \sum_{r=0}^{j-1}(-1)^r z_\nu^{mr}.$$

Here

$$\sum_{r=0}^{j-1}(-z_\nu^m)^r = \frac{2}{1+e^{2\pi i\nu/j}} = \frac{e^{-\pi i\nu/j}}{\cos\dfrac{\pi\nu}{j}}, \qquad e^{i\nu(\pi/j+mt)}+e^{-i\nu(\pi/j+mt)} = 2\cos\nu\left(mt+\frac{\pi}{j}\right),$$

i.e. (51) holds true, indeed.

If j, l and m are even, k is odd, then by (2)—(3)

$$\operatorname{sgn} s_{jklm}(t_{hj+r}-t) = (-1)^{h+r} \quad (0 \leqq h < m,\ 0 \leqq r < j)$$

for $-\dfrac{2}{jm} < t < 0$. Therefore

$$L_{jklm}(t) = \sum_{r=0}^{j-1}\sum_{h=0}^{m-1}(-1)^{h+r} s_{jklm}(t_{hj+r}-t) = \sum_{r=0}^{j-1}(-1)^r \sum_{\nu=-n}^{n}(-1)^h \sum_{h=0}^{m-1} b_{\nu jklm}\left(\frac{z_{hj+r}}{z}\right)^{\nu} =$$

$$= \sum_{r=0}^{j-1}(-1)^r \sum_{\nu=-n}^{n} b_{\nu jklm}\left(\frac{z_r}{z}\right)^{\nu} \sum_{h=0}^{m-1}(-z_\nu^j)^h.$$

Here

$$\sum_{h=0}^{m-1}(-z_\nu^j)^h = \begin{cases} m, & \text{if } \ 2\nu/m \text{ is an odd integer,} \\ 0, & \text{otherwise.}\end{cases}$$

Therefore

$$L_{jklm}(t) = m \sum_{\nu=-[n/m-1/2]}^{[n/m+1/2]} b_{m(\nu-1/2),j,k,l,m} z^{-m(\nu-1/2)} \sum_{r=0}^{j-1}(-z_{2\nu-1}^{m/2})^r.$$

Here

$$\sum_{r=0}^{j-1}(-z_{2\nu-1}^{m/2})^r=\frac{2}{1+e^{(2\nu-1)\pi i/j}}=\frac{e^{-(2\nu-1)i\pi/(2j)}}{\cos\frac{2\nu-1}{2j}\pi},$$

$$e^{-i(\nu-1/2)(mt+\pi/j)}+e^{i(\nu-1/2)(mt+\pi/j)}=2\cos\left(\nu-\frac{1}{2}\right)\left(mt+\frac{\pi}{j}\right),$$

which proves (52).

31. Proof of (11). If j is odd, $k=1$, and $l=0$, then by (1) $n=\frac{j+1}{2}m-1$ and $\left[\frac{n}{m}\right]=\frac{j-1}{2}$. (8) shows that $q=\frac{j-1}{2}m$. Thus by (40), (51) can be written in the form

(54) $$L_{j10m}(t)=\frac{1}{j}\left(1+2\sum_{\nu=1}^{(j-1)/2}\cos\nu\left(mt+\frac{\pi}{j}\right)\Big/\cos\frac{\nu\pi}{j}\right).$$

Since $L_{jklm}(t)$ is $\frac{2\pi}{jm}$-periodic, this yields (11).

32. Proof of (12). If j and m are even, $k=1$, and $l=0$, then $n=\frac{j+1}{2}m-1$, $\left[\frac{n}{m}+\frac{1}{2}\right]=\frac{j}{2}$, and $q=\frac{j-1}{2}m$. Hence and by (40), (52) can be written in the form

(55) $$L_{j10m}(t)=\frac{2}{j}\sum_{\nu=1}^{j/2}\cos\left(\nu-\frac{1}{2}\right)\left(mt+\frac{\pi}{j}\right)\Big/\cos\left(\nu-\frac{1}{2}\right)\frac{\pi}{j}.$$

Since $L_{jklm}(t)$ is $\frac{2\pi}{jm}$-periodic, this implies (12).

33. Proof of (13). Let j be even, m odd and $0\le t\le\frac{2\pi}{jm}$. Since $L_{jklm}(t)$ is $\frac{2\pi}{jm}$-periodic, (7) can be written in the form

(56) $$L_{j11m}(t)=\sum_{\nu=1-jm/2}^{jm/2}|s_{j11m}(t-t_\nu)|.$$

In Section 28 we already mentioned that

(57) $$|s_{j11m}(t)|\le|s_{j01m}(t)|.$$

By (3),

$$\operatorname{sgn}s_{j01m}(t-t_\nu)=\operatorname{sgn}s_{j01m}(t+t_{\nu-1})\quad(\nu=1,2,\ldots,jm/2).$$

Hence

$$|s_{j01m}(t-t_\nu)|+|s_{j01m}(t+t_{\nu-1})|=|s_{j01m}(t-t_\nu)+s_{j01m}(t+t_{\nu-1})|.$$

Here by (2)

$$s_{j01m}(t-t_\nu)+s_{j01m}(t+t_{\nu-1}) = s_{j01m}\left(t-\frac{\pi}{jm}-\frac{2\nu-1}{jm}\pi\right)+s_{j01m}\left(t-\frac{\pi}{jm}+\frac{2\nu-1}{jm}\pi\right)$$

is the fundamental polynomial of Lagrange interpolation at $\cos\left(t-\frac{\pi}{jm}\right)$ based on the Chebyshev nodes. Its absolute value in $\left[0, \frac{2\pi}{jm}\right]$ attains its maximum at $t=\frac{\pi}{jm}$. Hence, by (2), (3), (57) and the inequality $\cot x<1/x$ $(0<x<\pi/2)$ we obtain

$$|s_{j11m}(t-t_\nu)|+|s_{j11m}(t+t_{\nu-1})| \leqq 2\left|s_{j01m}\left(\frac{2\nu-1}{jm}\pi\right)\right| = \tag{58}$$

$$= \frac{2}{jm}\cot\frac{2\nu-1}{2jm}\pi < \frac{4}{\pi}\frac{1}{2\nu-1} \quad (1 \leqq \nu \leqq jm/2).$$

(3) and the inequality $\frac{\cos x}{\sin^2 x}<\frac{1}{x^2}$ $(0<x<\pi/2)$ yields

$$|s_{j11m}(t-t_\nu)|+|s_{j11m}(t+t_{\nu-1})| \leqq \frac{1}{jm^2}\left\{\frac{\cos\frac{t_\nu-t}{2}}{\sin^2\frac{t_\nu-t}{2}}+\frac{\cos\frac{t_{\nu-1}+t}{2}}{\sin^2\frac{t_{\nu-1}+t}{2}}\right\} \leqq$$

$$\leqq \frac{4}{jm^2}\left\{\frac{1}{(t_\nu-t)^2}+\frac{1}{(t_{\nu-1}+t)^2}\right\}.$$

For $\nu\geqq 2$, here the right hand side attains its maximum for $t=0$ or $t=t_1$. The value of this maximum is $\frac{j}{\pi^2}\left(\frac{1}{\nu^2}+\frac{1}{(\nu-1)^2}\right)$. This together with (56) and (58) implies

$$L_{j11m}(t) \leqq \frac{4}{\pi}\sum_{\nu=1}^{[j/\pi]+1}\frac{1}{2\nu-1}+\frac{j}{\pi^2}\frac{1}{([j/\pi]+1)^2}+\frac{2j}{\pi^2}\sum_{\nu=[j/\pi]+2}^{\infty}\frac{1}{\nu^2} <$$

$$< \frac{2}{\pi}(2+\log([j/\pi]+1))+\frac{j}{\pi^2([j/\pi]+1)^2}+\frac{2j}{\pi^2([j/\pi]+1)} < \frac{2}{\pi}\log j+$$

$$+\frac{2}{\pi}\left(2+\log\left(\frac{1}{\pi}+\frac{1}{2}\right)\right)+\frac{1}{2}+\frac{2}{\pi} < \frac{2}{\pi}\log j+2.283.$$

Since $L_{jklm}(t)$ is $\frac{2\pi}{jm}$-periodic, this gives (13).

34. Proof of (14). By (7), (3), (16), (11), and (20), using Cauchy—Schwarz inequality,

$$L^2_{221m}(t) = \left\{\sum_{v=0}^{2m-1}\left|\frac{\sin m(t-t_v)}{2m\cdot\sin\frac{t-t_v}{2}}\right|\left(\frac{\sin m\frac{t-t_v}{2}}{m\cdot\sin\frac{t-t_v}{2}}\right)^2\left|\cos\frac{t-t_v}{2}\right|\right\}^2 \leqq$$

$$\leqq \sum_{v=0}^{2m-1}\left(\frac{\sin m(t-t_v)}{2m\cdot\sin\frac{t-t_v}{2}}\right)^2\sum_{v=0}^{2m-1}\left(\frac{\sin m\frac{t-t_v}{2}}{m\cdot\sin\frac{t-t_v}{2}}\right)^4\left(\cos\frac{t-t_v}{2}\right)^2 =$$

$$= L_{1,1,0,2m}(t)M_{242m}(t) < \frac{4}{3},$$

which yields the statement.

35. Proof of (15). If $j=k=3$ and $l=2$, then by (1) $n=3m-1$, $\left[\frac{n}{m}\right]=2$, and by (8) $q=0$, by (40) $b_{0332m}=\frac{1}{3m}$. Also by (38) and (39)

$$b_{m332m} = b_{-m,3,3,2,m} = \frac{1}{12m^4}(a_{2m-1,3,3,m}+2a_{2m-2,3,3,m}+a_{2m-3,3,3,m}),$$

$$b_{2m,3,3,2,m} = b_{-2m,3,3,2,m} = \frac{1}{12m^4}(a_{m-1,3,3,m}+2a_{m-2,3,3,m}+a_{m-3,3,3,m}).$$

(34) gives

$$a_{m-v,3,3,m} = \binom{m+3-v}{3},\quad a_{2m-v,3,3,m} = \binom{2m+3-v}{3}-3a_{m-v,3,3,m}\quad (v=1,2,3).$$

These relations imply

$$b_{m332m} = \frac{5}{18m}-\frac{1}{36m^3},\quad b_{2m,3,3,2,m} = \frac{1}{18m}+\frac{1}{36m^3}.$$

Thus we obtain from (51)

$$L_{332m}(t) = m\left(b_{0332m}+4b_{m332m}\cos\left(mt+\frac{\pi}{3}\right)-4b_{2m,3,3,2,m}\cos 2\left(mt+\frac{\pi}{3}\right)\right) =$$

$$= \frac{1}{3}+\left(\frac{10}{9}-\frac{1}{9m^2}\right)\cos\left(mt+\frac{\pi}{3}\right)-\left(\frac{2}{9}+\frac{1}{9m^2}\right)\cos 2\left(mt+\frac{\pi}{3}\right).$$

Since

$$L'_{332m}(t) = \left(\frac{4m}{9}+\frac{2}{9m}\right)\sin 2\left(mt+\frac{\pi}{3}\right)-\left(\frac{10m}{9}-\frac{1}{9m}\right)\sin\left(mt+\frac{\pi}{3}\right)$$

and $|\sin 2t|\leqq 2|\sin t|$, we have $L'_{332m}(t)>0$ if $-\frac{2\pi}{3m}<t<-\frac{\pi}{3m}$, and $L'_{332m}(t)<0$ if $-\frac{\pi}{3m}<t<0$. Using the $\frac{2\pi}{3m}$-periodicity of $L_{332m}(t)$, we get the statement:

$$\|L_{332m}\| = L_{332m}\left(-\frac{\pi}{3m}\right) = \frac{11}{9}-\frac{2}{9m^2} < \frac{11}{9}.$$

36. PROOF OF (17). If $q \geqq 0$, then by (5) and (46)

$$\sum_{\nu=0}^{jm-1} s_{jklm}(t-t_\nu) = S_{jklm}(1, t) = 1,$$

whence

(59) $$S_{jklm}(g, t)-g(t) = \sum_{\nu=0}^{jm-1} (g(t_\nu)-g(t))s_{jklm}(t-t_\nu).$$

If $t_h \leqq t \leqq t_{h+1}$, then by (2), e.g. for jm even, this can be written in the form

(60) $$S_{jklm}(g, t)-g(t) = \sum_{\nu=h+1-jm/2}^{h+jm/2} (g(t_\nu)-g(t))s_{jklm}(t-t_\nu),$$

being g and s_{jklm} 2π-periodic functions. (The case of odd jm can be treated similarly.) Therefore

(61) $$|S_{jklm}(g, t)-g(t)| \leqq \sum_{\nu=h+1-jm/2}^{h+jm/2} |g(t_\nu)-g(t)| \cdot |s_{jklm}(t-t_\nu)|.$$

Here

(62) $$|g(t_\nu)-g(t)| \leqq \omega(|t_\nu-t|) \leqq \omega\left(g, \frac{\pi}{jm}\right)\left(1+\frac{jm}{\pi}|t_\nu-t|\right).$$

If t is a node, then by (6), (17) obviously holds. Otherwise by (3) we get for $k \geqq 1$, $l \geqq 1$

(63) $$\frac{jm}{\pi}|t_\nu-t| \cdot |s_{jklm}(t-t_\nu)| = \frac{1}{\pi}\left|\sin jm\frac{t-t_\nu}{2}\right| \cdot \left|\frac{\sin m\dfrac{t-t_\nu}{2}}{m \cdot \sin\dfrac{t-t_\nu}{2}}\right|^k \cdot$$
$$\cdot \left|\cos\frac{t-t_\nu}{2}\right|^{l-1} \cdot \left|\cot\frac{t-t_\nu}{2}\right| \cdot |t-t_\nu|.$$

Here

(64) $$\left|\sin jm\frac{t-t_\nu}{2}\right| \leqq 1,$$

(65) $$\left|\cot\frac{t-t_\nu}{2}\right| \cdot |t-t_\nu| \leqq 2.$$

It is easy to see that the summation in (7) and (16) can be taken from any integer u to $u+jm-1$. Thus (61)—(65), (7) and (16) imply (17).

37. PROOF OF (18). By (7), (3), (2) and (54)

(66) $$L_{110m}(t) = \sum_{\nu=0}^{m-1}\left(\frac{\sin m\left(\dfrac{t}{2}-\dfrac{\pi\nu}{m}\right)}{m \cdot \sin\left(\dfrac{t}{2}-\dfrac{\pi\nu}{m}\right)}\right)^2 = 1.$$

(16), (2) and (66) yield

$$M_{j20m}(t)=\sum_{\nu=0}^{jm-1}\left(\frac{\sin m\left(\frac{t}{2}-\frac{\pi\nu}{jm}\right)}{m\cdot\sin\left(\frac{t}{2}-\frac{\pi\nu}{jm}\right)}\right)^2= \tag{67}$$

$$=\sum_{h=0}^{j-1}\sum_{\nu=0}^{m-1}\left(\frac{\sin m\left(\frac{t}{2}-\frac{\pi\nu}{m}-\frac{\pi h}{jm}\right)}{m\cdot\sin\left(\frac{t}{2}-\frac{\pi\nu}{m}-\frac{\pi h}{jm}\right)}\right)^2=\sum_{h=0}^{j-1}L_{110m}\left(t-\frac{2\pi h}{jm}\right)=j.$$

38. Proof of (19). By using Cauchy—Schwarz inequality, (16), (18) and (20) give

$$M_{j31m}^2(t)=\left\{\sum_{\nu=0}^{jm-1}\left|\frac{\sin m\frac{t-t_\nu}{2}}{m\cdot\sin\frac{t-t_\nu}{2}}\right|^3\left|\cos\frac{t-t_\nu}{2}\right|\right\}^2\leqq$$

$$\leqq\sum_{\nu=0}^{jm-1}\left(\frac{\sin m\frac{t-t_\nu}{2}}{m\cdot\sin\frac{t-t_\nu}{2}}\right)^2\sum_{\nu=0}^{jm-1}\left(\frac{\sin m\frac{t-t_\nu}{2}}{m\cdot\sin\frac{t-t_\nu}{2}}\right)^4\cos^2\frac{t-t_\nu}{2}\leqq M_{j20m}(t)M_{j42m}(t)<\frac{2}{3}j^2,$$

which yields the statement. (We used (20) which will be proved in the next two sections.)

39. Lemma. *If k and l are even, $j\geqq k/2\geqq 2$, and $m\geqq\frac{k+l}{2}>l$, then*

$$M_{jklm}(t)=\frac{j}{2^l m^{k-1}}\sum_{\nu=0}^{l}\binom{l}{\nu}\sum_{h=0}^{k/2-1}(-1)^h\binom{k}{h}\binom{\frac{km+k+l}{2}-1-\nu-hm}{k-1}. \tag{68}$$

40. Proof. Just like in case of (67) we get

$$M_{jklm}(t)=\sum_{\nu=0}^{j-1}L_{1,k-1,l,m}\left(t-\frac{2\pi\nu}{jm}\right). \tag{69}$$

By (51)

$$L_{1,k-1,l,m}(t)=\left(b_{0,1,k-1,l,m}+2\sum_{h=1}^{[n/m]}b_{hm,1,k-1,l,m}\cos hmt\right). \tag{70}$$

(1) shows that now $n=\frac{1}{2}(km-k+l)$ and hence

$$\left[\frac{n}{m}\right]=\frac{k}{2}-1. \tag{71}$$

Since

$$\sum_{\nu=0}^{j-1} \cos hm\left(t-\frac{2\pi\nu}{jm}\right) = \sum_{\nu=0}^{j-1} \cos\left(hmt-\frac{2\pi h\nu}{j}\right) = 0, \tag{72}$$

(69)—(72) imply $M_{jklm}(t)=jmb_{0,1,k-1,l,m}$. Hence and by (38), (36) and (71) we obtain (68).

41. Proof of (20). (68) yields

(73)

$$M_{j42m}(t) = \frac{j}{4m^3}\sum_{\nu=0}^{2}\binom{2}{\nu}\sum_{h=0}^{1}(-1)^h\binom{4}{h}\binom{2m+2-\nu-hm}{3} = \frac{2j}{3}-\frac{j}{6m^2} < \frac{2j}{3}.$$

42. Notations.

$$p_0(x) = s_{jklm}(t), \tag{74}$$

$$p_\nu(x) = s_{jklm}(t-t_\nu)+s_{jklm}(t+t_\nu), \quad 0<\nu<jm/2, \tag{75}$$

$$p_{jm/2}(x) = s_{jklm}(t-t_{jm/2}), \quad \text{if } jm \text{ is even}, \tag{76}$$

$$T_h(x) = \cos ht \quad (h=0, 1, 2, \ldots), \tag{77}$$

$$g = f\circ\cos. \tag{78}$$

43. Proof of the first statement of Theorem 15. If jm is even, then by (23), (5), (2), (24), and (74)—(76)

(79)

$$P_{jklm}(f, x) = \sum_{\nu=0}^{jm-1} f(x_\nu)s_{jklm}(t-t_\nu) = f(1)p_0(x)+\sum_{\nu=1}^{jm/2-1} f(x_\nu)p_\nu(x)+f(-1)p_{jm/2}(x).$$

If jm is odd, then

$$P_{jklm}(f, x) = f(1)p_0(x)+\sum_{\nu=1}^{(jm-1)/2} f(x_\nu)p_\nu(x). \tag{80}$$

We get from (41), (79) and (80) that

$$p_0(x) = b_{0jklm}+2\sum_{h=1}^{n} b_{hjklm}T_h(x),$$

$$p_\nu(x) = 2b_{0jklm}+4\sum_{h=1}^{n} b_{hjklm}\cos ht_\nu T_h(x), \quad 0<\nu<jm/2,$$

$$p_{jm/2}(x) = b_{0jklm}+2\sum_{h=1}^{n}(-1)^h b_{hjklm}T_h(x),$$

as well as $P_{jklm}(f, x)$ are algebraic polynomials of degree at most n, since the Chebyshev polynomial $T_h(x)$ is of degree h.

44. Proof of (25). This follows from (23), (24), (2) and (6).

45. Proof of (26). If $q\geqq 0$ and $w(x)$ is an algebraic polynomial of degree at most q, then $y=w\circ\cos$ is a trigonometric polynomial of order at most q, and hence by (23), (78), and (46)

$$P_{jklm}(w, x) = S_{jklm}(w \circ \cos, t) = w(\cos t) = w(x).$$

Thus, just like in Section 27, by (79)—(80) we obtain

$$|P_{jklm}(f, x) - f(x)| \leqq \Big(1 + \sum_{\nu=0}^{[jm/2]} |p_\nu(x)|\Big) E_q(f). \tag{81}$$

By (75)

$$|p_\nu(x)| \leqq |s_{jklm}(t-t_\nu)| + |s_{jklm}(t+t_\nu)| \quad (0 < \nu < jm/2). \tag{82}$$

(81), (82), (74), (76) and (7) imply (26).

46. Remark. In general, (81) is more exact than (26).

47. Proof of (27). Let e.g. jm even (the case of odd jm can be treated similarly), and $x_{h+1} \leqq x \leqq x_h$. Now (61) can be written in the form

$$|P_{jklm}(f, x) - f(x)| \leqq \sum_{\nu=h+1-jm/2}^{h+jm/2} |f(x_\nu) - f(x)| \cdot |s_{jklm}(t-t_\nu)|. \tag{83}$$

Here $|x - x_\nu| = |\cos t - \cos t_\nu| \leqq 2 \left|\sin \frac{t-t_\nu}{2}\right|$, and hence

$$|f(x) - f(x_\nu)| \leqq \omega\left(f, \frac{\pi}{jm}\right)\left(1 + \frac{2jm}{\pi}\left|\sin \frac{t_\nu - t}{2}\right|\right). \tag{84}$$

Since

$$\frac{2jm}{\pi}\left|\sin \frac{t_\nu - t}{2}\, s_{jklm}(t-t_\nu)\right| = \frac{2}{\pi}\left|\sin jm \frac{t-t_\nu}{2}\right| \left|\frac{\sin m \frac{t-t_\nu}{2}}{m \cdot \sin \frac{t-t_\nu}{2}}\right|^k \cdot \left|\cos \frac{t-t_\nu}{2}\right|^l \tag{85}$$

and here $\left|\sin jm \frac{t-t_\nu}{2}\right| \leqq 1$, (83)—(85), (7) and (16) yield (27).

48. Proof of (28). Since

$$x - x_\nu = \cos t - \cos t_\nu = \cos t - \cos(t_\nu - t + t) = \cos t\big(1 - \cos(t_\nu - t)\big) + \tag{86}$$

$$+ \sin t \cdot \sin(t_\nu - t) = 2\sin^2 \frac{t_\nu - t}{2} \cos t + 2 \sin t \cdot \sin \frac{t_\nu - t}{2} \cos \frac{t_\nu - t}{2},$$

we have

$$|f(x) - f(x_\nu)| \leqq \omega\left(f, \frac{\pi \sin t}{jm}\right)\left(1 + \frac{2jm}{\pi}\left|\sin \frac{t_\nu - t}{2} \cdot \cos \frac{t_\nu - t}{2}\right|\right) + \tag{87}$$

$$+ \omega\left(f, \frac{\pi^2 |\cos t|}{j^2 m^2}\right)\left(1 + \frac{2j^2 m^2}{\pi^2} \sin^2 \frac{t_\nu - t}{2}\right).$$

Since

$$(88)\qquad \frac{2jm}{\pi}\left|\sin\frac{t_\nu-t}{2}\cdot\cos\frac{t_\nu-t}{2}\cdot s_{jklm}(t-t_\nu)\right| = \frac{2}{\pi}\left|\frac{\sin m\dfrac{t_\nu-t}{2}}{m\cdot\sin\dfrac{t_\nu-t}{2}}\right|^k\cdot$$

$$\cdot\left|\cos\frac{t_\nu-t}{2}\right|^{l+1}\left|\sin jm\frac{t_\nu-t}{2}\right|,$$

and here the last factor is $\leqq 1$, further

$$(89)\qquad \frac{2j^2m^2}{\pi^2}\sin^2\frac{t_\nu-t}{2}\,|s_{jklm}(t-t_\nu)| = \frac{2j}{\pi^2}\left|\frac{\sin m\dfrac{t-t_\nu}{2}}{m\cdot\sin\dfrac{t-t_\nu}{2}}\right|^{k-1}\cdot$$

$$\cdot\left|\cos\frac{t-t_\nu}{2}\right|^l\left|\sin jm\frac{t-t_\nu}{2}\cdot\sin m\frac{t-t_\nu}{2}\right|,$$

and here the last two factors are $\leqq 1$, we obtain from (83), (87)—(89), (7) and (16) that (28) holds.

49. Proof of (29). Let us write (60) in the form

$$P_{jklm}(f,x)-f(x) = \sum_{\nu=1}^{j}\sum_{r=-m/2}^{m/2-1}\big(f(x_{\nu+h+jr})-f(x)\big)s_{jklm}(t_{\nu+h+jr}-t),$$

provided m is even (the case of odd m can be treated similarly). Applying Abel transformation we get

$$(90)\qquad \sum_{r=-m/2}^{m/2-1}\big(f(x_{\nu+h+jr})-f(x)\big)s_{jklm}(t_{\nu+h+jr}-t) =$$

$$=\big(f(x_{\nu+h})-f(x)\big)\sum_{u=0}^{m/2-1}s_{jklm}(t_{\nu+h+ju}-t)+\sum_{r=1}^{m/2-1}\big(f(x_{\nu+h+jr})-f(x_{\nu+h+jr-j})\big)\cdot$$

$$\cdot\sum_{u=r}^{m/2-1}s_{jklm}(t_{\nu+h+ju}-t)+\big(f(x_{\nu+h-j})-f(x)\big)\sum_{u=-m/2}^{-1}s_{jklm}(t_{\nu+h+ju}-t)+$$

$$+\sum_{r=-m/2}^{-2}\big(f(x_{\nu+h+rj})-f(x_{\nu+h+j+jr})\big)\sum_{u=-m/2}^{r}s_{jklm}(t_{\nu+h+ju}-t).$$

Since j is odd and k is even, it follows from the definition of $s_{jklm}(t)$ that for $0\leqq u< <m/2$

$$\operatorname{sgn} s_{jklm}(t_{\nu+h+ju}-t) = \operatorname{sgn} s_{jklm}(t_{\nu+h-j-ju}-t) = (-1)^{u+\nu+1},$$

and the sequences $|s_{jklm}(t_{v+h+ju}-t)|$, $|s_{jklm}(t_{v+h-j-ju}-t)|$ $(u=0, 1, \ldots, m/2-1)$ are decreasing. Therefore

$$\left|\sum_{u=r}^{m/2-1} s_{jklm}(t_{v+h+ju}-t)\right| \leqq |s_{jklm}(t_{v+h+jr}-t)| \quad (1 \leqq r \leqq m/2-1), \tag{91}$$

$$\left|\sum_{u=-m/2}^{r} s_{jklm}(t_{v+h-j-ju}-t)\right| \leqq |s_{jklm}(t_{v+h-j-jr}-t)| \quad (-m/2 \leqq r \leqq -2). \tag{92}$$

Evidently

$$x_{v-j}-x_v = \cos t_{v-j}-\cos t_v = 2\sin\frac{\pi}{m}\sin\left(t_v-\frac{\pi}{m}\right) = \tag{93}$$

$$= 2\sin\frac{\pi}{m}\left\{\sin t\cdot\cos\left(t_v+\frac{\pi}{m}-t\right)+\cos t\cdot\sin\left(t_v+\frac{\pi}{m}-t\right)\right\} =$$

$$= 2\sin\frac{\pi}{m}\left\{\sin t\cdot\cos\left(t_v+\frac{\pi}{m}-t\right)+\cos t\left[\sin\frac{\pi}{m}\cos(t_v-t)+\right.\right.$$

$$\left.\left.+2\cos\frac{\pi}{m}\sin\frac{t_v-t}{2}\cos\frac{t_v-t}{2}\right]\right\}.$$

Therefore

$$|x_{v-j}-x_v| \leqq \frac{2\pi}{m}\sin t+|\cos t|\left(\frac{2\pi^2}{m^2}+\frac{4\pi}{m}\left|\sin\frac{t_v-t}{2}\cdot\cos\frac{t_v-t}{2}\right|\right),$$

and

(94)

$$|f(x_{v-j})-f(x_v)| \leqq \omega\left(f, \frac{2\pi\sin t}{m}\right)+\omega\left(f, \frac{2\pi^2|\cos t|}{m^2}\right)\left(2+\frac{2m}{\pi}\left|\sin\frac{t_v-t}{2}\cos\frac{t_v-t}{2}\right|\right).$$

The same holds for $|f(x_{v+j})-f(x_v)|$, too.

(86) yields

$$|x-x_{v+h}| \leqq \frac{2\pi}{m}\sin t+\frac{2\pi^2}{m^2}|\cos t| \quad (t_h \leqq t \leqq t_{h+1}),$$

i.e.

$$|f(x)-f(x_{v+h})| \leqq \omega\left(f, \frac{2\pi}{m}\sin t\right)+\omega\left(f, \frac{2\pi^2}{m^2}|\cos t|\right), \tag{95}$$

and the same holds for $|f(x)-f(x_{v+h-j})|$, too. (90)—(95), (88), (7) and (16) give (29).

50. Proof of (30). We may assume that $-1<x<1$, since for $x=\pm 1$ we have zeros on both sides of (30). From (86)

(96)

$$|f(x)-f(x_v)| \leqq \omega\left(f, \frac{\pi\sin t}{jm}\right)\left(1+\frac{2jm}{\pi}\left|\sin\frac{t_v-t}{2}\cos\frac{t_v-t}{2}\right|+\frac{2jm}{\pi}|\cot t|\sin^2\frac{t_v-t}{2}\right).$$

(3) and (2) imply

$$\frac{2jm}{\pi}|\cot t|\sin^2\frac{t_\nu-t}{2}|s_{jklm}(t_\nu-t)| = \tag{97}$$

$$=\frac{2}{\pi}\left|\frac{\sin jmt/2}{m\cdot\sin t}\right|\left|\frac{\sin m\dfrac{t-t_\nu}{2}}{m\cdot\sin\dfrac{t-t_\nu}{2}}\right|^{k-1}\cdot\left|\cos\frac{t-t_\nu}{2}\right|^{l}|\cos t|\cdot\left|\sin m\frac{t-t_\nu}{2}\right|.$$

Here the second factor is $\leqq j/2$ (being jm even), and the last factor is $\leqq 1$. (83), (88), (96) and (97) give (30).

51. Proof of (31). We obtain from (93)

$$|x_{v-j}-x_v|\leqq\frac{2\pi}{m}\sin t\left\{1+|\cot t|\left(\sin\frac{\pi}{m}+2\left|\sin\frac{t_v-t}{2}\cos\frac{t_v-t}{2}\right|\right)\right\}.$$

Hence

$$|f(x_{v-j})-f(x_v)|\leqq\omega\left(f,\frac{2\pi\sin t}{m}\right)\left\{2+|\cot t|\left(\sin\frac{\pi}{m}+2\left|\sin\frac{t_v-t}{2}\cos\frac{t_v-t}{2}\right|\right)\right\}. \tag{98}$$

The same holds for $|f(x_{v+j})-f(x_v)|$. (86) gives

$$|x-x_{\nu+h}|\leqq\frac{2\pi}{m}\sin t\left(1+\left|\cot t\cdot\sin\frac{t_{\nu+h}-t}{2}\right|\right).$$

Therefore

$$|f(x)-f(x_{\nu+h})|\leqq\omega\left(f,\frac{2\pi\sin t}{m}\right)\left(2+\left|\cot t\cdot\sin\frac{t_{\nu+h}-t}{2}\right|\right). \tag{99}$$

A similar estimate holds for $|f(x)-f(x_{\nu+h-j})|$. In (98) we have

$$\sin\frac{\pi}{m}\leqq\sin\frac{t_v-t}{2}\quad\left(t+\frac{2\pi}{m}\leqq t_v\leqq\pi+t\right). \tag{100}$$

Thus

$$\left|\cot t\cdot\sin\frac{t_v-t}{2}s_{jklm}(t_v-t)\right|=\left|\frac{\sin jmt/2}{jm\cdot\sin t}\cdot\cos t\right|\cdot \tag{101}$$

$$\cdot\left|\frac{\sin m\dfrac{t_v-t}{2}}{m\cdot\sin\dfrac{t_v-t}{2}}\right|^{k-1}\cdot\left|\cos\frac{t_v-t}{2}\right|^{l},$$

where

$$\left|\sin\frac{jmt}{2}\right|\leqq\frac{jm}{2}\sin t. \tag{102}$$

(90)—(92), (98)—(102) and (7) imply (31).

52. THEOREM. *Let* $0<\varepsilon\leqq 1$, $N\geqq 20/\varepsilon^2$ *an arbitrary integer*, $f(x)$ *an arbitrary continuous function in* $[-1, 1]$. *Then there exists a polynomial* $J_N(f, x)$ *of degree at most* $N(1+\varepsilon)$ *which depends on* $f(x)$ *linearly, interpolates* $f(x)$ *in at least* N *nodes, and*

$$|f(x)-J_N(f, x)| \leqq \frac{13}{\varepsilon^2}\,\omega\left(f, \frac{\pi\sqrt{1-x^2}}{2N}\right) \quad (-1\leqq x\leqq 1). \tag{103}$$

53. REMARK. Theorem 52 gives a constructive answer for a problem raised by G. Freud and A. Sharma [2]. Later the problem was solved by them [2, addendum] and by P. Vértesi [7, Theorem 2.6] without giving an estimate for the constant figuring on the right hand side of (103).

54. PROOF OF THEOREM 52. Let $j=[5/\varepsilon]$, $k=3$, $l=0$, $m=2\left[\frac{N-1}{j}\right]+2$. Then by (1) the degree of the polynomial $P_{j30m}(f, x)$ is

$$n=\frac{jm}{2}+\frac{3m}{2}-2\leqq N-1+j+3\,\frac{N-1}{j}+3-2 < N+j+\frac{3N}{j}=N\left(1+\frac{j}{N}+\frac{3}{j}\right)\leqq$$

$$\leqq N\left(1+\frac{5}{\varepsilon}\cdot\frac{\varepsilon^2}{20}+3\,\frac{\varepsilon}{4}\right)=N(1+\varepsilon)$$

(n is an integer, since m is even). (79) implies that $P_{j30m}(f, x)$ depends linearly on $f(x)$. By (25), $P_{j30m}(f, x)$ interpolates $f(x)$ at the nodes $\cos\frac{2\pi v}{jm}$ $(v=0, 1, \dots, jm/2)$, i.e. at least in

$$\frac{jm}{2}+1=j\left[\frac{N-1}{j}\right]+j+1\geqq N+1 \tag{104}$$

points. (30) yields with $J_N(f, x)=P_{j30m}(f, x)$

$$|J_N(f, x)-f(x)|\leqq\omega\left(f, \frac{\pi\sqrt{1-x^2}}{jm}\right)\left(L_{j30m}(t)+\frac{2}{\pi}\,M_{j31m}(t)+\frac{j}{\pi}\,M_{j20m}(t)\right).$$

(The conditions are fulfilled, since m is even, $j\geqq 5$, and hence and by (8), $q>0$.) Now (10)—(12), the first statement in Section 8, (18)—(19) and (103) give

$$|J_N(f, x)-f(x)|\leqq\omega\left(f, \frac{\pi\sqrt{1-x^2}}{2N}\right)\left(1+\frac{2}{\pi}\log j+\frac{2}{\pi}\sqrt{\frac{2}{3}}\,j+\frac{j^2}{\pi}\right)\leqq \tag{105}$$

$$\leqq\omega\left(f, \frac{\pi\sqrt{1-x^2}}{2N}\right)\left(1+\frac{2}{\pi}\log\frac{5}{\varepsilon}+\frac{10}{\pi}\sqrt{\frac{2}{3}}\cdot\frac{1}{\varepsilon}+\frac{25}{\pi\varepsilon^2}\right).$$

The function $\varepsilon^2+\frac{2}{\pi}\,\varepsilon^2\log\frac{5}{\varepsilon}+\frac{10}{\pi}\sqrt{\frac{2}{3}}\,\varepsilon$ has a maximum of $1+\frac{2}{\pi}\log 5+\frac{10}{\pi}\sqrt{\frac{2}{3}}$ in $(0, 1]$. Therefore

$$1+\frac{2}{\pi}\log\frac{5}{\varepsilon}+\frac{10}{\pi}\sqrt{\frac{2}{3}}\cdot\frac{1}{\varepsilon}+\frac{25}{\pi\varepsilon^2}\leqq\left(1+\frac{2}{\pi}\log 5+\frac{10}{\pi}\sqrt{\frac{2}{3}}+\frac{25}{\pi}\right)\cdot\frac{1}{\varepsilon^2}<\frac{13}{\varepsilon^2},$$

which, together with (105), implies (103).

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(Received April 6, 1984)

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Acta Math. Hung.
47 (1—2) (1986), 261—280.

ON THE EQUICONVERGENCE OF EIGENFUNCTION EXPANSIONS ASSOCIATED WITH ORDINARY LINEAR DIFFERENTIAL OPERATORS

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The equiconvergence theorems are very useful in the spectral investigation of differential operators because many results known for the most special operators may be transferred by their application to more general ones. For the case of orthonormal bases consisting of eigenfunctions of second-order operators several results have been obtained since the beginning of this century, see e.g. [1], [2], [3], [5], [7], [8], [14], [19], [22], [23]. All these results are contained in a result of I. Joó and the author [10]. It concerns also the non-selfadjoint case i.e. when eigenfunctions of higher order are also used and when the system is not orthonormal but only a Riesz basis. (On the existence of such Riesz bases see [4], [11], [20], [21].) The proof was based on an efficient method due to V. A. Il'in (see e.g. [2]) of the constant application of some mean value formulas.

The aim of the present paper is to extend this result for differential operators of higher order. In some special cases this was already done in [15]. Our main tool will be a generalized Titchmarsh type formula derived in [12]. We note that it is not a mean value formula if the order of the differential operator is odd. In some cases a simpler expression for its coefficients was found by I. Joó [9]; these expressions are important because they make possible to obtain sharp estimates for the coefficients. We shall need several results proved in [10], [17] and [18], too.

By and large the following result will be proved: all Riesz bases (and in particular all orthonormal bases) consisting of eigenfunctions (maybe also of higher order) of some n-order linear differential operator, locally behave in the same way. Here we stress two circumstances:

— there are no assumptions on the distribution of the eigenvalues: they can be arbitrary complex numbers;
— there are no boundary conditions.

As an immediate consequence of this result we note that (for example) Carleson's theorem remains valid for "all" eigenfunction expansions.

All the preliminary results used in this paper are contained in [10], [12], [17] and [18]. All the results of the papers [12] and [17] are needed. From [18] we need the case $i=0$ of Theorem 3; for its proof it is not necessary to apply the results of the paper [16]. From [10] we use only a result of technical character (Lemma 6). For the reader's convenience we collect in Section 2 all preliminary results used in this paper.

1. Formulation of the main result

Let G be an open interval on the real line, n a natural number, $q_s \in H^{n-s}_{\text{loc}}(G)$ a complex function $(s=2, \dots, n)$ and consider the differential operator

$$Lu = u^{(n)} + q_2 u^{(n-2)} + q_3 u^{(n-3)} + \dots + q_n u$$

defined on $H^n_{\text{loc}}(G)$. (Recall that, by definition, $H^k_{\text{loc}}(G)$ is the set of all complex functions $v \in L^2_{\text{loc}}(G)$ having distributional derivatives in $L^2_{\text{loc}}(G)$ of order up to k.)

As usual, a function $u \not\equiv 0$ is called an eigenfunction of order 0 (of the operator L) with some eigenvalue $\lambda \in \mathbf{C}$ if $Lu = \lambda u$. Furthermore, a function u is called an eigenfunction of order k (of the operator L) with some eigenvalue $\lambda \in \mathbf{C}$ $(k=1, 2, \dots)$ if the function $u^* := Lu - \lambda u$ is an eigenfunction of order $k-1$ with the same eigenvalue λ.

Let us now given a system $(u_r)_{r=1}^{\infty}$ of eigenfunctions and denote o_r (resp. λ_r) the order (resp. the eigenvalue) of u_r. Assume that the following three conditions are satisfied:

(C 1) (u_r) is a Riesz basis i.e. (u_r) is the image of an orthonormal basis under a linear topological isomorphism of $L^2(G)$;

(C 2) $\sup o_r < \infty$;

(C 3) if $o_r \geqq 1$ then $u_r^* = u_s$ for some index s.

By (C 1) there exists a unique system (v_r) in $L^2(G)$ such that $\langle u_r, v_s \rangle = \delta_{rs}$ (the Kronecker symbol). Introduce the following notations:

$$|\nu_r| = \max \{|\operatorname{Im} \mu| \colon \mu \in \mathbf{C} \text{ and } \mu^n = \lambda_r\}, \tag{1}$$

$$\sigma_\nu(f, x) = \sum_{|\nu_r| < \nu} \langle f, v_r \rangle u_r(x) \quad (\nu > 0,\ f \in L^2(G),\ x \in G),$$

$$S_\nu(f, x, R) = \int_{x-R}^{x+R} \frac{\sin \nu(y-x)}{\pi(y-x)} f(y)\, dy \quad (\nu > 0,\ f \in L^2(G),\ x \pm R \in G).$$

The aim of this paper is to prove the following result:

THEOREM. *To any compact subinterval K of G there exists a number $R_0 > 0$ such that*

$$\lim_{\nu \to \infty} \sup_{x \in K} |S_\nu(f, x, R) - \sigma_\nu(f, x)| = 0$$

whenever $f \in L^2(G)$ and $0 < R < R_0$. □

REMARKS. (i) We note that $S_\nu(f, x, R)$ does not depend on the system (u_r).

(ii) The conditions of the theorem are very weak. The assumptions (C 1)—(C 3) are practically satisfied for almost all known Riesz bases of eigenfunctions. We emphasize that there are no assumptions on the distribution of the eigenvalues λ_r. Several sufficient conditions are known for the existence of orthonormal bases of eigenfunctions (see e.g. [21]), and more generally, for the existence of Riesz bases of eigenfunctions (see [4], [11], [20]).

(iii) For second-order operators several equiconvergence theorems were proved from the beginning of this century, see e.g. [1], [2], [3], [5], [7], [8], [14], [19], [22]. These results are contained in a theorem of Joó and Komornik, proved in [10] by developing an important method of V. A. Il'in [2]. This result is also slightly stronger than the case $n=2$ of the above theorem: instead of $q_2 \in L^2_{\mathrm{loc}}(G)$ it was sufficient to assume that $q_2 \in L^1_{\mathrm{loc}}(G)$.

(iv) The proof of the just mentioned result of Joó and Komornik is not applicable for the general case. However, by integrating by parts we obtain the desired estimates also in this case. On the other hand, in [12] a new method (based on a suitable generalization of the well-known Titchmarsh formula) for the spectral investigation of n-order differential operators was developed. Using this method, several results were proved, see e.g. [12]—[18]. The present paper represents a new evidence for the efficiency of this method. Some special cases of the theorem of the present paper were proved in [15].

2. Preliminary results

A) We shall need the following estimate, which is a consequence of some results of [17], [18]: putting

(2) $$|\varrho_r| = \min\{|\mathrm{Re}\,\mu| \colon \mu \in \mathbf{C} \text{ and } \mu^n = \lambda_r\},$$

to any compact intervals $K_1 \subset G$, $K_2 \subset \mathrm{int}\, K_1$ there exists a positive constant ε_0 such that

(3) $$\|u_r\|_{L^\infty(K_2)} e^{|\varrho_r|\varepsilon_0} \leqq \frac{1}{\varepsilon_0} \|u_r\|_{L^2(K_1)} \quad (r = 1, 2, \ldots).$$

B) In [12] we derived a generalization of the well-known Titchmarsh formula for n-order operators; the results cited in A) were proved by the use of this formula. Now we need another generalization of the Titchmarsh formula.

Denote $\omega_1, \ldots, \omega_n$ the n-th roots of unity and set $m = \left[\frac{n+1}{2}\right]$. For $0 \neq \mu \in \mathbf{C}$, $t > 0$ and $-mt < y < (n-m)t$ we denote by $f_k(\mu, t)$ the elementary symmetric polynomial of degree $m-k$ of the variables $e^{\mu\omega_1 t}, \ldots, e^{\mu\omega_n t}$ with the main coefficient $(-1)^k$ $(k = m-n, \ldots, m)$ and we put

$$F(\mu, t, y) = \sum_{k=1+[-y/t]}^{m} f_k(\mu, t) \sum_{p=1}^{n} \frac{\omega_p}{n\mu^{n-1}} e^{\mu\omega_p(y+kt)}.$$

One can easily see that f_k and F can be continuously extended for all $\mu \in \mathbf{C}$, $t \geqq 0$ and $-mt \leqq y \leqq (n-m)t$. Furthermore, the extended function F has the following properties for any fixed $\mu \in \mathbf{C}$ and $t > 0$:

(4) $F(\mu, t, \cdot)$ is $n-2$ times continuously differentiable in $(-mt, (n-m)t)$ and
$$D_3^i F(\mu, t, -mt+0) = D_3^i F(\mu, t, (n-m)t-0) = 0 \quad (i = 0, \ldots, n-2).$$

(5) $F(\mu, t, \cdot)$ is n times continuously differentiable in $(kt, (k+1)t)$ and $D_3^n F = \mu^n F$ $(-m \leqq k \leqq n-m-1)$.

(6) $$-D_3^{n-1} F(\mu, t, (n-m)t-0) = f_{m-n}(\mu, t),$$
$$D_3^{n-1} F(\mu, t, -kt+0) - D_3^{n-1} F(\mu, t, -kt-0) = f_k(\mu, t) \quad (m-n < k < m),$$
$$D_3^{n-1} F(\mu, t, -mt+0) = f_m(\mu, t).$$

Using these properties and integrating by parts we obtain for any $u \in H^n_{\mathrm{loc}}(G)$ the formula

$$(7) \qquad \sum_{k=m-n}^{m} f_k(\mu, t) u(x+kt) + \int_{x+(m-n)t}^{x+mt} F(\mu, t, x-\tau)[\mu^n u(\tau) - u^{(n)}(\tau)]\, d\tau = 0$$

whenever $t>0$, $x+(m-n)t \in G$ and $x+mt \in G$.

C) Apply the formula (7) to the eigenfunction u_r. Denoting by μ_r an arbitrary n-th root of λ_r, we obtain

(8)
$$\sum_{k=m-n}^{m} f_k(u_r, t) u_r(x+kt) + \int_{x+(m-n)t}^{x+mt} F(\mu_r, t, x-\tau) \Big[\sum_{s=2}^{n} q_s(\tau) u_r^{(n-s)}(\tau) - u_r^*(\tau)\Big]\, d\tau = 0$$

whenever $t>0$, $x+(m-n)t \in G$ and $x+mt \in G$.

For $n=2$ and $o_r=0$ this reduces to the Titchmarsh formula. For $n=2$ and $o_r \neq 0$ it was found by Joó [6]. For $o_r=0$, n arbitrary (then $u_r^* \equiv 0$) the formula (8) is a special case of a more general formula derived in [12]. We note that the above simple form of the coefficients f_k (which has great importance to obtain some estimates in the sequel) was proved by Joó [9].

We shall frequently use two equivalent forms of the formula (8). Index the n-th roots of λ_r so that

$$\operatorname{Re} \mu_{r,1} \geqq \ldots \geqq \operatorname{Re} \mu_{r,n}$$

and put $\mu_r = \mu_{r,m}$, $\varrho_r = \operatorname{Re} \mu_r$, $\nu_r = \operatorname{Im} \mu_r$. These notations are consistent with the former ones used in (1), (2), (3) and (8). Denote $g_k(\mu_r, t)$ and $G(\mu_r, t, y)$ (resp. $h_k(\mu_r, t)$ and $H(\mu_r, t, y)$) the functions obtained from $f_k(\mu_r, t)$ and $F(\mu_r, t, y)$ by dividing by $e^{(\mu_{r,1}+\dots+\mu_{r,m-1})t}$ (resp. $e^{(\mu_{r,1}+\dots+\mu_{r,m})t}$). Then from (8) we obtain the following two formulas:

(9)
$$\sum_{k=m-n}^{m} g_k(\mu_r, t) u_r(x+kt) + \int_{x+(m-n)t}^{x+mt} G(\mu_r, t, x-\tau) \Big[\sum_{s=2}^{n} q_s(\tau) u_r^{(n-s)}(\tau) - u_r^*(\tau)\Big]\, d\tau = 0,$$

(10)
$$\sum_{k=m-n}^{m} h_k(\mu_r, t) u_r(x+kt) + \int_{x+(m-n)t}^{x+mt} H(\mu_r, t, x-\tau) \Big[\sum_{s=2}^{n} q_s(\tau) u_r^{(n-s)}(\tau) - u_r^*(\tau)\Big]\, d\tau = 0.$$

D) It follows obviously from (4) that

$$(11) \qquad D_3^i G(\mu_r, t, -mt+0) = D_3^i G(\mu_r, t, (n-m)t-0) = 0,$$

$$D_3^i H(\mu_r, t, -mt+0) = D_3^i H(\mu_r, t, (n-m)t-0) = 0 \quad (0 \leqq i \leqq n-2,\ r = 1, 2, \ldots).$$

The following estimates follow directly from the definition of the coefficients in the formulas (8), (9), (10); we refer to [15] and [17] for some details. In all these estimates we assume that $\varrho_r \geqq 0$ and $t>0$.

(12) $g_k(\mu_r, t)$, $h_k(\mu_r, t)$, $D_2 g_k(\mu_r, t)$ and $D_2 h_k(\mu_r, t)$ tend to 0 if $|k| \geqq 2$ and $|\mu_r, t| \to \infty$.

(13) $g_1(\mu_r, t)$ and $g_{-1}(\mu_r, t)$ remain bounded, $D_2 g_1(\mu_r, t)$, $D_2 g_{-1}(\mu_r, t)$ and $g_0(\mu_r, t) - e^{\mu_r t}$ tend to 0 if $\varrho_r t \to \infty$ and $\dfrac{|v_r|}{\varrho_r} \to \infty$.

(14) $h_1(\mu_r, t)$, $h_{-1}(\mu_r, t)$, $D_2 h_1(\mu_r, t)$, $D_2 h_{-1}(\mu_r, t)$ and $h_0(\mu_r, t) - 1$ tend to 0 if $\varrho_r t \to \infty$ and $\dfrac{|v_r|}{\varrho_r}$ remains bounded.

(15) $g_{-1}(\mu_r, t)$, $g_1(\mu_r, t)$ and $h_{-1}(\mu_r, t)$ remain bounded, $D_2 g_{-1}(\mu_r, t)$, $D_2 g_1(\mu_r, t)$, $g_1(\mu_r, t)+1$, for n odd $D_2 h_{-1}(\mu_r, t)$ and $h_0(\mu_r, t)-1$, for n even $g_{-1}(\mu_r, t)+1$ tend to 0 if $|v_r t| \to \infty$ and $\varrho_r t$ remains bounded.

(16) For any real number v the fractions

$$\frac{g_0(\mu_r, t) - g_0(iv, t)}{t} \quad \text{and} \quad \frac{h_1(\mu_r, t) - h_1(iv, t)}{t}$$

remain bounded (uniformly in v) if $\varrho_r t$ and $|\mu_r - iv|$ remain bounded.

$$\frac{D_3^i G(\mu_r, t, y)}{|\mu_r|^{i+1-n} e^{\varrho_r (t-|y|)}} \quad \text{and} \quad \frac{D_3^i H(\mu_r, t, y)}{|\mu_r|^{i+1-n} e^{-\varrho_r |y|}} \tag{17}$$

are uniformly bounded $(i=0, \ldots, n-1,\ r=1, 2, \ldots)$.

E) It follows from Theorem 2 in [12] that for any compact subinterval K of G there exists a constant $C>0$ such that

$$\|u_r'\|_{L^\infty(K)} \leqq C(1+|\mu_r|)\|u_r\|_{L^\infty(K)} \quad (r = 1, 2, \ldots).$$

F) Finally we recall two important properties of the Riesz bases: the generalized Bessel inequality and the generalized Parseval identity. First, there exists a constant C such that

$$\sum_{r=1}^{\infty} |\langle u_r, w\rangle|^2 \leqq C\|w\|^2_{L^2(G)} \quad \forall w \in L^2(G). \tag{18}$$

secondly,

$$\langle f, w\rangle = \sum_{r=1}^{\infty} \langle f, v_r\rangle\langle u_r, w\rangle \quad \forall f, w \in L^2(G). \tag{19}$$

3. Estimation of the sum of squares of the eigenfunctions

In this section, under the conditions of the Theorem we shall prove the following strong estimate:

PROPOSITION. *To any compact subinterval K of G there exists a positive number ε such that*

$$\sup_{\nu\geqq 1} \sum_{|\nu-|\nu_r||\leqq 1} \|u_r\|^2_{L^\infty(K)} e^{|\varrho_r|^\varepsilon} < \infty.$$

This result will follow from several lemmas.

LEMMA 1.

$$\sup_{\nu\geqq 1} \sum_{\substack{|\nu-\nu_r|\leqq 1\\ \nu_r\geqq A\varrho_r\\ \varrho_r\geqq A}} \left(\frac{\|u_r\|_{L^2(K)}}{\varrho_r}\right)^2 = O(1) \quad (A\to\infty).$$

PROOF. Fix $R>0$ such that $K_{2nR}\subset G$ where $K_\delta:=\{x:\ \mathrm{dist}\,(x,K)\leqq\delta\}$. For any $\nu\geqq 1$, $x\in K$, $R\leqq t\leqq 2R$ and $r\in I_\nu(A):=\{r:\ |\nu-\nu_r|\leqq 1,\ \nu_r\geqq A\varrho_r,\ \nu_r\geqq A\}$, by the application of (9) we obtain

$$-\int_R^{2R} e^{-i\nu t} g_0(\mu_r,t)\,dt\,u_r(x) = \sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0}} \int_R^{2R} e^{-i\nu t} g_k(\mu_r,t) u_r(x+kt)\,dt -$$

$$-\int_R^{2R} e^{-i\nu t} \int_{x+(m-n)t}^{x+mt} G(\mu_r,t,x-\tau)\Big[u_r^*(\tau)-\sum_{s=2}^{n} q_s(\tau)u_r^{(n-s)}(\tau)\Big]\,d\tau\,dt.$$

Integrating by parts and using (11) hence we obtain

$$-\int_R^{2R} e^{-i\nu t} g_0(\mu_r,t)\,dt\,u_r(x) = \sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0}} g_k(\mu_r,2R) \int_R^{2R} e^{-i\nu t} u_r(x+kt)\,dt -$$

$$-\sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0}} \int_R^{2R} D_2 g_k(\mu_r,t) \int_R^{t} e^{-i\nu t} u_r(x+k\xi)\,d\xi\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_R^{2R} e^{-i\nu t}\int_{x+(m-n)t}^{x+mt} D_3^{n-s-i+1}G(\mu_r,t,x-\tau)\cdot$$

$$\cdot\int_{x+(m-n)t}^{\tau} q_s^{(i)}(\xi)u_r(\xi)\,d\xi\,d\tau\,dt -$$

$$-\int_R^{2R} e^{-i\nu t}\int_{x+(m-n)t}^{x+mt} D_3 G(\mu_r,t,x-\tau)\int_{x+(m-n)t}^{\tau} u_r^*(\xi)\,d\xi\,d\tau\,dt.$$

The following estimates will be uniform in ν, x, r, t, τ when $A\to\infty$. Using the estimates (12), (13), (17), with suitably defined functions $w_1, w_2, w_3, w_4\in L^2(G)$ (which depend also on v, x, t, τ, k, s, i) we obtain

$$(1-o(1))\left|\frac{u_r(x)}{\varrho_r}\right| \leq \sum_{\substack{m-n\leq k\leq m\\ k\neq 0}} O(1)|\langle w_1, u_r\rangle| + \sum_{\substack{m-n\leq k\leq m\\ k\neq 0}} \int_R^{2R} O(1)|\langle w_2, u_r\rangle|\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}\int_R^{2R}\int_{x+(m-n)t}^{x+mt} O(1)|\langle w_3, u_r\rangle|\,d\tau\,dt + \int_R^{2R}\int_{x+(m-n)t}^{x+mt} O(1)\,|\langle w_4, u_r^*\rangle|\,d\tau\,dt.$$

Taking the square of this inequality, summarizing for $r\in I_v(A)$, using (18) and (C 3) we obtain

$$\sum_{r\in I_v(A)}\left|\frac{u_r(x)}{\varrho_r}\right|^2 \leq \sum_{\substack{m-n\leq k\leq m\\ k\neq 0}} O(1)\,\|w_1\|^2_{L^2(G)} + \sum_{\substack{m-n\leq k\leq m\\ k\neq 0}} \int_R^{2R} O(1)\,\|w_2\|^2_{L^2(G)}\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}\int_R^{2R}\int_{x+(m-n)t}^{x+mt} O(1)\,\|w_3\|^2_{L^2(G)}\,d\tau\,dt + \int_R^{2R}\int_{x+(m-n)t}^{x+mt} O(1)\,\|w_4\|^2_{L^2(G)}\,d\tau\,dt.$$

Furthermore, one can easily see that $\|w_i\|^2_{L^2(G)} = O(1)$, $i=1, 2, 3, 4$, therefore

$$\sum_{r\in I_v(A)}\left|\frac{u_r(x)}{\varrho_r}\right|^2 = O(1).$$

Integrating on K, we obtain the required estimate. □

LEMMA 2.

$$\sup_{v\geq 1}\sum_{\substack{|v+v_r|\leq 1\\ v_r\leq -A\varrho_r\\ \varrho_r\geq A}}\left(\frac{\|u_r\|_{L^2(K)}}{\varrho_r}\right)^2 = O(1) \quad (A\to\infty).$$

PROOF. Quite similar to that of Lemma 1, replacing e^{-ivt} by e^{ivt}. □

LEMMA 3. *For any fixed* $A>0$ *we have*

$$\sum_{\substack{|v_r|<A\varrho_r\\ \varrho_r\geq B}}\|u_r\|^2_{L^2(K)} = O(1) \quad (B\to\infty).$$

PROOF. Fix $R>0$ such that $K_{2nR}\subset G$. For any $x\in K$, $R\leq t\leq 2R$, $r\in I(B):=\{r:\ |v_r|<A\varrho_r,\ \varrho_r\geq B\}$, applying now (10), integrating by parts and using (11), we obtain

$$-\int_R^{2R} h_0(\mu_r, t)\,dt\,u_r(x) = \sum_{\substack{m-n\leq k\leq m\\ k\neq 0}} h_k(\mu_r, 2R)\int_R^{2R} u_r(x+kt)\,dt -$$

$$-\sum_{\substack{m-n\leq k\leq m\\ k\neq 0}}\int_R^{2R} D_2 h_k(\mu_r, t)\int_R^{t} u_r(x+k\xi)\,d\xi\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_R^{2R}\int_{x+(m-n)t}^{x+mt} D_3^{n-s-i+1}H(\mu_r, t, x-\tau)\int_{x+(m-n)t}^{\tau} q_s^{(i)}(\xi)u_r(\xi)\,d\xi\,d\tau\,dt -$$

$$-\int_R^{2R}\int_{x+(m-n)t}^{x+mt} D_3 H(\mu_r, t, x-\tau)\int_{x+(m-n)t}^{\tau} u_r^*(\xi)\,d\xi\,d\tau\,dt.$$

The following estimates will be uniform in x, r, t, τ when $B\to\infty$. Using the estimates (12), (14), (17) with suitably defined functions $w_5, w_6, w_7, w_8\in L^2(G)$ (having also the parameters x, t, τ, k, s, i) we obtain

$$(1-o(1))|u_r(x)| \leqq \sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0}} O(1)|\langle w_5, u_r\rangle| + \sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0}} \int_R^{2R} O(1)|\langle w_6, u_r\rangle|\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}\int_R^{2R}\int_{x+(m-n)t}^{x+mt} O(1)|\langle w_7, u_r\rangle|\,d\tau\,dt + \int_R^{2R}\int_{x+(m-n)t}^{x+mt} O(1)|\langle w_8, u_r^*\rangle|\,d\tau\,dt.$$

Furthermore we have $\|w_i\|^2_{L^2(G)}=O(1)$, $i=5, 6, 7, 8$ and the proof can be finished by the same way as in Lemma 1. □

LEMMA 4. *For any fixed* $B>0$ *we have*

$$\sup_{v\geqq 1} \sum_{\substack{|v-v_r|\leqq 1\\ 0\leqq \varrho_r< B\\ v_r\geqq D}} \|u_r\|^2_{L^2(K)} = O(1) \quad (D\to\infty).$$

PROOF. Fix $0<R_0<\dfrac{|K|}{4}$ such that $K_{4nR_0}\subset G$ ($|K|$ denotes the length of K). We will show that for any fixed $0<R<R_0$ we have the estimate

$$\sum_{r\in I}|u_r(y)|^2 \leqq C\big(1+o(1)\big)R\sum_{r\in I}\|u_r\|^2_{L^2(K)}+O(1) \tag{20}$$

($D\to\infty$) uniformly in $v\geqq 1$, $y\in K$ and uniformly for any finite subset I of

$$J_v(D):=\{r\colon\ |v-v_r|\leqq 1,\ 0\leqq \varrho_r<B,\ v_r\geqq D\}$$

(C is an absolute constant). Hence the lemma will follow easily. Indeed, integrating on K we obtain

$$\sum_{r\in I}\|u_r\|^2_{L^2(K)} \leqq C|K|\big(1+o(1)\big)R\sum_{r\in I}\|u_r\|^2_{L^2(K)}+O(1).$$

Choose at the beginning of the proof R so small that $C|K|R<\dfrac{1}{2}$. Then, being all the terms finite by the choice of I,

$$\sum_{r\in I}\|u_r\|^2_{L^2(K)} = O(1),$$

and, being $I\subset J_v(D)$ arbitrary,

$$\sum_{r\in J_v(D)}\|u_r\|^2_{L^2(K)} = O(1)$$

as stated in the lemma.

Denote c the centre of K. We prove (20) differently in the following three cases: a) $y\geqq c$, b) $y\leqq c$ and n is even, c) $y\leqq c$ and n is odd.

a) Applying (10) with $x=y-t$ we obtain

$$-\int_R^{2R} g_1(\mu_r, t)\,dt\,u_r(y) = \int_R^{2R} (g_0(u_r, t)-g_0(iv, t))u_r(y-t)\,dt +$$

$$+\int_R^{2R} g_0(iv, t)u_r(y-t)\,dt + \sum_{\substack{m-n\leq k\leq m\\ k\neq 0,1}} g_k(\mu_r, 2R)\int_R^{2R} u_r(y-t+kt)\,dt -$$

$$-\sum_{\substack{m-n\leq k\leq m\\ k\neq 0,1}} \int_R^{2R} D_2 g_k(\mu_r, t)\int_R^t u_r(y-\xi+k\xi)\,d\xi\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_R^{2R}\int_{y-t+(m-n)t}^{y-t+mt} D_3^{n-s-i+1}G(\mu_r, t, y-t-\tau)\cdot$$

$$\cdot\int_{y-t+(m-n)t}^{\tau} q_s^{(i)}(\xi)u_r(\xi)\,d\xi\,d\tau\,dt -$$

$$-\int_R^{2R}\int_{y-t+(m-n)t}^{y-t+mt} D_3G(\mu_r, t, y-t-\tau)\int_{y-t+(m-n)t}^{\tau} u_r^*(\xi)\,d\xi\,d\tau\,dt.$$

The following estimates are uniform in v, y, r, t, τ when $D\to\infty$. Introducing the functions $w_9, \dots, w_{13}\in L^2(G)$ (depending also on the parameters v, y, τ, t, k, s, i) in a suitable way, by (12), (15), (16) and (17) we have

$$(R-o(1))|u_r(y)| \leq CR^{3/2}\|u_r\|_{L^2(K)} + |\langle w_9, u_r\rangle| +$$

$$+\sum_{\substack{m-n\leq k\leq m\\ k\neq 0,1}} O(1)|\langle w_{10}, u_r\rangle| + \sum_{\substack{m-n\leq k\leq m\\ k\neq 0,1}}\int_R^{2R} O(1)|\langle w_{11}, u_r\rangle|\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}\int_R^{2R}\int_{y-t+(m-n)t}^{y-t+mt} O(1)|\langle w_{12}, u_r\rangle|\,d\tau\,dt + \int_R^{2R}\int_{y-t+(m-n)t}^{y-t+mt} O(1)|\langle w_{13}, u_r^*\rangle|\,d\tau\,dt.$$

Taking the square of both sides, summarizing for $r\in I$, taking into account that $\|w_i\|_{L^2(G)}=O(1)$, $i=9, \dots, 13$ and using (18) we obtain (20).

b) Applying (10) with $x=y+t$ we obtain almost the same formula as in the preceding case, with the following changes:

— $y-t$ is replaced by $y+t$ everywhere,

— instead of $g_1(\mu_r, t)$, $g_{-1}(\mu_r, t)$ is placed on the left hand side.

(20) may be derived exactly by the same manner as before because $g_{-1}(\mu_r, t)= = -1+o(1)$ by (15) (this does not remain true if n is odd).

c) We apply now (11):

$$-\int_R^{2R} h_0(\mu_r, t)\,dt\,u_r(y) = \int_R^{2R} (h_1(\mu_r, t) - h_1(iv, t))u_r(y+t)\,dt +$$

$$+\int_R^{2R} h_1(iv, t)u_r(y+t)\,dt + \sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0,1}} h_k(\mu_r, 2R)\int_R^{2R} u_r(y+kt)\,dt -$$

$$-\sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0,1}} \int_R^{2R} D_2 h_k(\mu_r, t)\int_R^{t} u_r(y+k\xi)\,d\xi\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_R^{2R}\int_{y+(m-n)t}^{y+mt} D_3^{n-s-i+1}H(\mu_r, t, y-\tau)\cdot$$

$$\cdot\int_{y+(m-n)t}^{\tau} q_s^{(i)}(\xi)u_r(\xi)\,d\xi\,d\tau\,dt -$$

$$-\int_R^{2R}\int_{y+(m-n)t}^{y+mt} D_3H(\mu_r, t, y-\tau)\int_{y+(m-n)t}^{\tau} u_r^*(\xi)\,d\xi\,d\tau\,dt.$$

Using (12), (15), (16) and (17), with obvious notation we obtain

$$(R-o(1))|u_r(y)| \leqq CR^{3/2}\|u_r\|_{L^2(K)} + |\langle w_{14}, u_r\rangle| +$$

$$+\sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0,1}} O(1)|\langle w_{15}, u_r\rangle| + \sum_{\substack{m-n\leqq k\leqq m\\ k\neq 0,1}}\int_R^{2R} O(1)|\langle w_{16}, u_r\rangle|\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}\int_R^{2R}\int_{y+(m-n)t}^{y+mt} O(1)|\langle w_{17}, u_r\rangle|\,d\tau\,dt + \int_R^{2R}\int_{y+(m-n)t}^{y+mt} O(1)|\langle w_{18}, u_r^*\rangle|\,d\tau\,dt.$$

Furthermore $\|w_i\|_{L^2(G)} = O(1)$, $i=14, \dots, 18$ and (20) can be obtained as in Part a). □

LEMMA 5. *For any fixed* $B>0$ *we have*

$$\sup_{v\geqq 1}\sum_{\substack{|v+v_r|\leqq 1\\ 0\leqq \varrho_r<B\\ v_r\leqq -D}} \|u_r\|^2_{L^2(K)} = O(1) \quad (D\to\infty).$$

PROOF. It is quite similar to that of Lemma 4, replacing in the formulas the term $g_0(iv, t)$ (resp. $h_1(iv, t)$) by $g_0(-iv, t)$ (resp. $h_1(-iv, t)$). □

LEMMA 6. *For any fixed* $B, D>0$ *we have*

$$\sum_{\substack{|\varrho_r|<B\\ |v_r|<D}} \|u_r\|^2_{L^2(K)} < \infty.$$

PROOF. We will show the existence of a constant C such that

$$R^2 \sum_{r\in I} |u_r(y)|^2 \leqq CR^3 \sum_{r\in I} \|u_r\|^2_{L^2(K)} + C \tag{21}$$

for any $y\in K$, $0<R<\dfrac{|K|}{2n}$ and for any finite subset I of $J:=\{r\colon |\varrho_r|<B,\ |\nu_r|<D\}$.
Indeed, then choosing R such that $CR|K|\leqq\dfrac{1}{2}$, integrating on K and taking into account that I (and therefore $\sum_{r\in I} \|u_r\|^2_{L^2(K)}$) is finite, we obtain

$$\sum_{r\in I} \|u_r\|^2_{L^2(K)} \leqq \frac{2C|K|}{R^2}$$

uniformly in I; hence the lemma follows.

Denote again c the centre of K. To prove (21), we distinguish three cases: a) $y\geqq c$, b) $y\leqq c$ and $n\geqq 2$, c) $y\leqq c$ and $n=1$.

a) Apply (8) with $x=y-mt$, $0<t<R$, then we obtain

$$-\int_0^R f_m(\mu_r, t)\,dt\, u_r(y) = \sum_{k=m-n}^{m-1} \int_0^R (f_k(\mu_r, t)-f_k(0,0))u_r(y-mt+kt)\,dt +$$

$$+\sum_{k=m-n}^{m-1} \int_0^R f_k(0,0)u_r(y-mt+kt)\,dt +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_0^R \int_{y-nt}^{y} D_3^{n-s-i+1}F(\mu_r, t, y-\tau)\int_{y-nt}^{\tau} q_s^{(i)}(\xi)u_r(\xi)\,d\xi\,d\tau\,dt -$$

$$-\int_0^R \int_{y-nt}^{y} D_3F(\mu_r, t, y-\tau)\int_{y-nt}^{\tau} u_r^*(\xi)\,d\xi\,d\tau\,dt.$$

Being f_k, F smooth the functions

$$\frac{f_k(\mu_r, t)-f_k(0,0)}{t} \quad\text{and}\quad D_3^{n-s-i+1}F(\mu_r, t, y-\tau)$$

are bounded for $|\varrho_r|<B$, $|\nu_r|<D$, $0<t<R$ and $y-nt<\tau<y$. Furthermore, $|f_m(\mu_r, t)|\equiv 1$. Therefore, introducing the functions (depending on the parameters y, τ, k, s, i) $w_{19}, w_{20}, w_{21}\in L^2(G)$ in a suitable way, we obtain the estimates

$$R|u_r(y)| \leqq C_1R^{3/2}\|u_r\|_{L^2(K)} + \sum_{k=m-n}^{m-1} |\langle w_{19}, u_r\rangle| +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}\int_0^R \int_{y-nt}^{y} C_1|\langle w_{20}, u_r\rangle|\,d\tau\,dt + \int_0^R \int_{y-nt}^{y} C_1|\langle w_{21}, u_r^*\rangle|\,d\tau\,dt$$

and $\|w_i\|_{L^2(G)}\leqq C_1$, $i=19, 20, 21$ for some constant C_1. Hence (21) follows by the usual way.

b) Apply now (8) with $x=y+(n-m)t,\ 0<t<R$, then

$$-\int_0^R f_{m-n}(\mu_r, t)\,dt\,u_r(y) = \sum_{k=m-n+1}^{m} \int_0^R (f_k(\mu_r, t)-f_k(0,0))u_r(y+(n-m)t+kt))\,dt +$$

$$+\sum_{k=m-n+1}^{m} \int_0^R f_k(0,0)\,u_r(y+(n-m)t+kt)\,dt +$$

$$+\sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_0^R \int_y^{y+nt} D_3^{n-s-i+1} F(\mu_r, t, y+nt-\tau) \int_y^{\tau} q_s^{(i)}(\xi)\,u_r(\xi)\,d\xi\,d\tau\,dt -$$

$$-\int_0^R \int_y^{y+nt} D_3 F(\mu_r, t, y+nt-\tau) \int_y^{\tau} u_r^*(\xi)\,d\xi\,d\tau\,dt.$$

Hence one can proceed as in the preceding case because $n \geqq 2$ implies also $|f_{m-n}(\mu_r, t)| \equiv 1$.

c) This case can be treated quite similarly to the case b), with a sole change: before the estimations we divide the formula written just above by $e^{\mu_r t}$. Then

$$\left| \frac{f_{m-n}(\mu_r, t)}{e^{\mu_r t}} \right| \equiv 1$$

and the usual procedure works. □

Proof of the Proposition. It follows from Lemmas 1—6 that

$$\sup_{\nu \geqq 1} \sum_{\substack{|\nu-|\nu_r|| \leqq 1 \\ \varrho_r \geqq 0}} \left(\frac{\|u_r\|_{L^2(K)}}{1+\varrho_r} \right)^2 < \infty.$$

If n is even then the condition $\varrho_r \geqq 0$ is always satisfied. If n is odd then we have also

$$\sup_{\nu \geqq 1} \sum_{\substack{|\nu-|\nu_r|| \leqq 1 \\ \varrho_r \leqq 0}} \left(\frac{\|u_r\|_{L^2(K)}}{1-\varrho_r} \right)^2 < \infty$$

by a reflection principle described in the introduction of [17]. Therefore we have in both cases

$$\sup_{\nu \geqq 1} \sum_{|\nu-|\nu_r|| \leqq 1} \left(\frac{\|u_r\|_{L^2(K_1)}}{1+|\varrho_r|} \right)^2 < \infty$$

for any compact subinterval K_1 of G. Applying (2) (choose $K_2=K$) hence we obtain the proposition $\left(\text{with } \varepsilon=\frac{\varepsilon_0}{2} \text{ for example}\right)$. □

4. Proof of the Theorem

The idea of the proof is the following. Putting

$$\delta(\nu, |\nu_r|) = \begin{cases} 1 & \text{if } \nu > |\nu_r|, \\ \frac{1}{2} & \text{if } \nu = |\nu_r|, \\ 0 & \text{if } \nu < |\nu_r| \end{cases}$$

and

$$w(x+t) = \begin{cases} \dfrac{\sin \nu t}{\pi t} & \text{if } |t| < R, \\ 0 & \text{otherwise,} \end{cases}$$

by the application of the proposition proved in the preceding section we will show that for any compact subset K of G

$$\sup_{\nu>0} \sup_{x\in K} \sum_{r=1}^{\infty} \left|\langle u_r, w\rangle - \delta(\nu, |\nu_r|)\, u_r(x)\right|^2 < \infty \tag{22}$$

whenever R is sufficiently small $\big(w\in L^2(G)$ depends on the parameters ν and $R\big)$. Taking into account that

$$S_\nu(f, x, R) = \langle f, w\rangle = \sum_{r=1}^{\infty} \langle f, v_r\rangle \langle u_r, w\rangle,$$

$$\sigma_\nu(f, x) = \sum_{r=1}^{\infty} \langle f, v_r\rangle \delta(\nu, |\nu_r|)\, u_r(x) - \frac{1}{2} \sum_{|\nu_r|=\nu} \langle f, v_r\rangle u_r(x),$$

applying the Cauchy—Schwarz inequality, (22) and the proposition again, we obtain

$$\sup_{\nu>0} \sup_{x\in K} |S_\nu(f, x, R) - \sigma_\nu(f, x)| \le C \|f\|_{L^2(G)} \quad \big(\forall f\in L^2(G)\big)$$

with some constant C independent of f. Now it suffices to show that

$$\lim_{\nu\to\infty} \sup_{x\in K} |S_\nu(f, x, R) - \sigma_\nu(f, x)| = 0$$

for any f from a dense subset of $L^2(G)$. But this last property is satisfied for any finite linear combination f of the eigenfunctions u_r because then f is continuously differentiable and $\sigma_\nu(f, x) \equiv f(x)$ for ν sufficiently large, therefore one can apply a classical result of the theory of Fourier series (see [24], Volume 1, p. 55).

The rest of this section is devoted to the proof of the estimate (22). In the sequel we shall consider only the case $n>1$ because the case $n=1$ (then $Lu=u'$) can be easily led to the case $n=2$ $(Lu=u'')$.

LEMMA 7. *We have* $\left|\dfrac{1}{R} \displaystyle\int_R^{2R} e^{\mu t}\, dt\right| \to 0$ *if* $\mu\in\mathbf{C}$, $\operatorname{Re}\mu \le 0$, $R>0$ *and* $|\mu R| \to \infty$.

PROOF. It suffices to show that

$$\left|\frac{1}{R}\int_R^{2R} e^{\mu t}\,dt\right| \leqq e^{\operatorname{Re}\mu R}\min\left\{1, \frac{4}{|\operatorname{Im}\mu R|}\right\}.$$

For this, first we note that obviously

$$\left|\frac{1}{R}\int_R^{2R} e^{\mu t}\,dt\right| \leqq e^{\operatorname{Re}\mu R}.$$

On the other hand, applying the theorem of Bonnet, there exist $R \leqq R_1$, $R_2 \leqq 2R$ such that

$$\left|\frac{1}{R}\int_R^{2R} e^{\mu t}\,dt\right| \leqq \left|\frac{1}{R}\int_R^{2R} e^{\operatorname{Re}\mu t}\cos\operatorname{Im}\mu t\,dt\right| +$$

$$+\left|\frac{1}{R}\int_R^{2R} e^{\operatorname{Re}\mu t}\sin\operatorname{Im}\mu t\,dt\right| = \left|\frac{1}{R}e^{\operatorname{Re}\mu R}\int_R^{R_1}\cos\operatorname{Im}\mu t\,dt\right| +$$

$$+\left|\frac{1}{R}e^{\operatorname{Re}\mu R}\int_R^{R_1}\sin\operatorname{Im}\mu t\,dt\right| \leqq \frac{4}{|\operatorname{Im}\mu R|}e^{\operatorname{Re}\mu R}. \quad \square$$

LEMMA 8. *For any compact intervals* $K_1 \subset G$, $K \subset \operatorname{int} K_1$ *there exists* $R_0 > 0$ *such that for any fixed* $0 < R < R_0$

$$\sup_{x\in K}\int_0^R \left|\frac{u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t}{t}\right| dt \leqq$$

$$\leqq C\frac{\ln|\mu_r|}{|\mu_r|}\left(\|u_r\|_{L^\infty(K_1)} + \|u_r^*\|_{L^\infty(K_1)}\right)$$

whenever $|\mu_r|$ *is sufficiently large.*

PROOF. We shall use the notations of Section 2. We shall assume that $\varrho_r \geqq 0$. The case $\varrho_r < 0$ hence can be obtained by the reflection principle mentioned at the end of Section 3.

Putting

$$v_r(y) = u_r(y) + \int_x^y \sum_{p=1}^n \frac{\omega_p}{n\mu_r^{n-1}} e^{\mu_r\omega_p(y-\tau)}\Big(\sum_{s=2}^n q_s(\tau)u_r^{(n-s)}(\tau) - u_r^*(\tau)\Big)\,d\tau$$

one can readily verify that $v_r \in H^n_{\mathrm{loc}}(G)$ and $v_r^{(n)} = \mu_r^n v_r$. Consequently v_r is a linear combination of the functions $e^{\mu_r\omega_1(y-x)}, \ldots, e^{\mu_r\omega_n(y-x)}$. By (3) we can fix $R_0 > 0$ such that $K_{4nR_0} \subset K_1$ and

$$\|u_r\|_{L^\infty(K_{4nR_0})} e^{2\varrho_r R_0} \leqq C\|u_r\|_{L^\infty(K_1)} \quad (r = 1, 2, \ldots). \tag{23}$$

We shall distinguish two cases: a) n is odd and $n > 1$ i.e. $n = 2m-1$, $m \geqq 2$. b) n is even i.e. $n = 2m$, $m \geqq 1$.

a) For any $x \in K$, $0<S<2R_0$ and $0<t<S$ the determinant

$$\begin{vmatrix} v_r(x-mS) & \dots & e^{-m\mu_{r,p}S} & \dots \\ \vdots & & \vdots & \\ v_r(x-2S) & \dots & e^{-2\mu_{r,p}S} & \dots \\ (v_r(x-t)+v_r(x+t)-2v_r(x)\operatorname{ch}\mu_r t) & \dots & (2\operatorname{ch}\mu_{r,p}t-2\operatorname{ch}\mu_r t) & \dots \\ v_r(x+2S) & \dots & e^{2\mu_{r,p}S} & \dots \\ \vdots & & \vdots & \\ v_r(x+(m+1)S) & \dots & e^{(m+1)\mu_{r,p}S} & \dots \end{vmatrix}$$

$(p=1, \dots, n)$ vanishes. Expanding it according to the first column, with obvious notation we obtain

$$d(\mu_r, S)\big(u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t\big) = \sum_{\substack{-m \leqq k \leqq m+1 \\ |k| \geqq 2}} d_k(\mu_r, S, t)\, u_r(x+kS) +$$

$$+\int_{x-mS}^{x+(m+1)S} D(\mu_r, S, t, x-\tau)\Big(\sum_{s=2}^{n} q_s(\tau)u_r^{(n-s)}(\tau)-u_r^*(\tau)\Big)\,d\tau.$$

One can easily see (cf. (4)) that

$$D_4^j D\big(\mu_r, S, t, -(m+1)S\big) = D_4^j D(\mu_r, S, t, mS) = 0, \quad j=0, \dots, n-2.$$

Therefore the above formula implies

$$d(\mu_r, S)\big(u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t\big) = \sum_{\substack{-m \leqq k \leqq m+1 \\ |k| \geqq 2}} d_k(\mu_r, S, t)\, u_r(x+kS) +$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_{x-mS}^{x+(m+1)S} D_4^{n-s-i}D(\mu_r, S, t, x-\tau)\,q_s^{(i)}(\tau)u_r(\tau)\,d\tau -$$

$$-\int_{x-mS}^{x+(m+1)S} D(\mu_r, S, t, x-\tau)u_r^*(\tau)\,d\tau.$$

Putting

$$Q(\mu_r, S) = e^{((m+1)\mu_{r,1}+\dots+2\mu_{r,m}-2\mu_{r,m+1}-\dots-m\mu_{r,n})S},$$

by the method of [17] we obtain the estimates

$$|d_k(\mu_r, S, t)| \leqq C_1 |Q(\mu_r, S)| \cdot |\mu_r t| \big(|e^{-\mu_{r,m-1}S}| + |e^{(\mu_{r,m}+\mu_{r,m+1})S}|\big),$$

$$|D_4^j D(\mu_r, S, t, x-\tau)| \leqq C_1 |Q(\mu_r, S)| \cdot |\mu_r|^{j+1-n} \min\{1, |\mu_r t|\} e^{\varrho_r S};$$

furthermore, being n odd there exists a constant $\alpha>0$ with

$$\operatorname{Re}\mu_{r,m-1} \geqq \alpha|\mu_r| \quad \text{and} \quad \operatorname{Re}\mu_{r,m+1} \leqq -\alpha|\mu_r|, \quad \forall r.$$

(In the above estimates C_1 denotes an absolute constant.) Using these estimates from the formula we obtain

$$\left|\frac{d(\mu_r, S)}{Q(\mu_r, S)}\right| \cdot \left|\frac{u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t}{t}\right| \leqq C_2|\mu_r|e^{-\alpha|\mu_r|}\cdot\|u_r\|_{L^\infty(K_{2nS})}e^{\varrho_r S}+$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s} C_2|\mu_r|^{2-s-i}\min\{|\mu_r t|^{-1}, 1\}\|u_r\|_{L^\infty(K_{2nS})}e^{\varrho_r S}+$$

$$+C_2|\mu_r|^{2-n}\min\{|\mu_r t|^{-1}, 1\}\|u_r^*\|_{L^\infty(K_{2nS})}e^{\varrho_r S}$$

with another absolute constant C_2.

Let us now fix $0<R<R_0$ arbitrarily. If $|\mu_r|>\dfrac{1}{R}$ then

$$\int_0^R \min\{|\mu_r t|^{-1}, 1\}\,dt \leqq \int_0^{|\mu_r|^{-1}} 1\,dt+\int_{|\mu_r|^{-1}}^R |\mu_r t|^{-1}\,dt = |\mu_r|^{-1}(1+\ln R+\ln|\mu_r|),$$

therefore if $R<S<2R$ and $|\mu_r|>\max\left\{1, \dfrac{1}{R}\right\}$ then

$$\left|\frac{d(\mu_r, S)}{Q(\mu_r, S)}\right| \int_0^R \left|\frac{u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t}{t}\right| dt \leqq$$

$$\leqq C_3\frac{1+\ln|\mu_r R|}{|\mu_r|}\left(\|u_r\|_{L^\infty(K_{4nR})}+\|u_r^*\|_{L^\infty(K_{4nR})}\right)e^{2\varrho_r R}.$$

If $|\mu_r|$ is sufficiently large then by Lemma 7 we have

$$\int_R^{2R}\left|\frac{d(\mu_r, S)}{Q(\mu_r, S)}\right| dS > \frac{R}{2}$$

whence

$$\int_0^R \left|\frac{u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t}{t}\right| dt \leqq$$

$$\leqq C_4\frac{\ln|\mu_r|}{|\mu_r|}\left(\|u_r\|_{L^\infty(K_{4nR})}+\|u_r^*\|_{L^\infty(K_{4nR})}\right)e^{2\varrho_r R}$$

with some constant C_4 depending only on R. Taking into account (23) and the condition (C 3) the Lemma follows.

b) For any $x\in K$, $0<S<2R_0$ and $0<t<S$ the determinant

$$\begin{vmatrix} v_r(x) & \dots 1 & \dots \\ (v_r(x-t)+v_r(x+t)-2v_r(x)\operatorname{ch}\mu_r t) & \dots (2\operatorname{ch}\mu_{r,p}t-2\operatorname{ch}\mu_r t) & \dots \\ (v_r(x-2S)+v_r(x+2S)) & \dots (2\operatorname{ch}2\mu_{r,p}S) & \dots \\ \vdots & \vdots & \\ (v_r(x-mS)+v_r(x+mS)) & \dots (2\operatorname{ch}m\mu_{r,p}S) & \dots \end{vmatrix}$$

$(p=1, \dots, m)$ vanishes. Expanding it according to the first column, with obvious notation we obtain the formula

$$d(\mu_r, S)\big(u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t\big)=$$
$$=\sum_{\substack{0\le k\le m\\ k\ne 1}} d_k(\mu_r, S, t)\big(u_r(x+kS)+u_r(x-kS)\big)+$$
$$+\int_{x-mS}^{x+mS} D(\mu_r, S, t, x-\tau)\Big(\sum_{s=2}^{n} q_s(\tau)u_r^{(n-s)}(\tau)-u_r^*(\tau)\Big)\,d\tau.$$

One can easily see that $D_4^j D(\mu_r, S, t, \pm mS)=0$, $j=0, \dots, n-2$, therefore

$$d(\mu_r, S)\big(u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t\big)=$$
$$=\sum_{\substack{0\le k\le m\\ k\ne 1}} d_k(\mu_r, S, t)\big(u_r(x+kS)+u_r(x-kS)\big)+$$
$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_{x-mS}^{x+mS} D_4^{n-s-i}D(\mu_r, S, t, x-\tau)q_s^{(i)}(\tau)u_r(\tau)\,d\tau-$$
$$-\int_{x-mS}^{x+mS} D(\mu_r, S, t, x-\tau)u_r^*(\tau)\,d\tau.$$

Putting $Q(\mu_r, S)=e^{(m\mu_{r,1}+\dots+2\mu_{r,m-1})S}$ we have the estimates

$$|d_k(\mu_r, S, t)|\le C_1|Q(\mu_r, S)|\cdot|\mu_r t|\cdot\big|e^{(2\mu_{r,m}-\mu_{r,m-1})S}\big| \quad \text{if} \quad m\ge 2,$$
$$|D_4^j D(\mu_r, S, t, x-\tau)|\le C_1|Q(\mu_r, S)|\cdot|\mu_r|^{j+1-n}\min\{1, |\mu_r t|\}e^{\varrho_r S}.$$

Furthermore, $d_0(\mu_r, S, t)\equiv 0$ if $m=1$ and there exists a constant $\alpha>0$ such that $\operatorname{Re}\mu_{r,m-1}\ge\alpha|\mu_r|$ $(r=1, 2, \dots)$ if $m\ge 2$. Therefore, fixing $0<R<R_0$ arbitrarily, we obtain

$$\left|\frac{d(\mu_r, S)}{Q(\mu_r, S)}\right|\int_0^R\left|\frac{u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t}{t}\right|dt\le$$
$$\le C_2\frac{1+\ln|\mu_r R|}{|\mu_r|}\big(\|u_r\|_{L^\infty(K_{4nR})}+\|u_r^*\|_{L^\infty(K_{4nR})}\big)e^{2\varrho_r R}$$

if $R<S<2R$ and $|\mu_r|>\max\left\{1, \frac{1}{R}\right\}$ and the proof can be finished as in Part a). □

LEMMA 9. *For any $R>0$ there exists a constant $C>0$ such that*

$$\left|\frac{2}{\pi}\int_0^R\frac{\sin\nu t\operatorname{ch}\mu_r t}{t}\,dt-\delta(\nu, |\nu_r|)\right|\le\frac{Ce^{|\varrho_r|R}}{2+\big|\nu-|\nu_r|\big|}$$

for all $\nu>0$ and $r=1, 2, \dots$.

PROOF. See [10]. □

Let us now prove (22). Given a compact interval $K\subset G$ arbitrarily, fix another compact interval $K_1\subset G$ such that $K\subset \operatorname{int} K_1$, and then a number $R_1>0$ such that $0<R_1<R_0$ (R_0 is defined as in Lemma 8), $K_{4nR_1}\subset K_1$,

$$\sup_{\nu\geqq 1}\sum_{|\nu-|\nu_r||\leqq 1}(\|u_r\|_{L^\infty(K_1)}e^{|\varrho_r|R_1})^2<\infty. \tag{24}$$

This choice is possible by the proposition of Section 3.

Fix $0<R<R_1$ arbitrarily and fix a constant $A=A(R)>2$ such that the assertion of Lemma 8 holds true whenever $|\mu_r|>A$. In the sequel C denotes diverse constants independent of $\nu\geqq 1$, $x\in K$ and $r=1,2,\ldots$.

Consider first the case when $|\mu_r|>A$. Applying Lemmas 8 and 9 we have

$$|\langle u_r, w\rangle-\delta(\nu, |\nu_r|)u_r(x)|=\left|\int_0^R\frac{\sin\nu t}{\pi t}(u_r(x-t)+u_r(x+t))\,dt-\delta(\nu, |\nu_r|)u_r(x)\right|\leqq$$

$$\leqq\left|\int_0^R\frac{\sin\nu t}{\pi t}(u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t)\,dt\right|+$$

$$+\left|\frac{2}{\pi}\int_0^R\frac{\sin\nu t\operatorname{ch}\mu_r t}{t}\,dt-\delta(\nu, |\nu_r|)\right|\cdot|u_r(x)|\leqq$$

$$\leqq C\left(\frac{\ln|\mu_r|}{|\mu_r|}+(2+|\nu-|\nu_r||)^{-1}\right)(\|u_r\|_{L^\infty(K_1)}+\|u_r^*\|_{L^\infty(K_1)})e^{|\varrho_r|R_1},$$

for all $\nu\geqq 1$ and $x\in K$. Using (24) and (C 3) with any fixed $0<\varepsilon<1$ we obtain

$$\sum_{|\mu_r|>A}|\langle u_r, w\rangle-\delta(\nu, |\nu_r|)u_r(x)|^2\leqq$$

$$\leqq C\sum_{|\mu_r|>A}((1+|\nu_r|)^{-2+\varepsilon}+(2+|\nu-|\nu_r||)^{-2})(\|u_r\|_{L^\infty(K_1)}+\|u_r^*\|_{L^\infty(K_1)})^2e^{2|\varrho_r|R_1}\leqq$$

$$\leqq C\sum_{i=1}^\infty(i^{-2+\varepsilon}+(1+|\nu-i|)^{-2})\sum_{i-1\leqq|\nu_r|<i}(\|u_r\|_{L^\infty(K_1)}e^{|\varrho_r|R_1})^2\leqq$$

$$\leqq C\sum_{i=1}^\infty(i^{-2+\varepsilon}+(1+|\nu-i|)^{-2})\leqq C$$

i.e.

$$\sum_{|\mu_r|>A}|\langle u_r, w\rangle-\delta(\nu, |\nu\,|)u_r(x)|^2\leqq C. \tag{25}$$

Consider now the case when $|\mu_r|\leqq A$. For any $\nu\geqq 1$ and $x\in K$, integrating by parts and taking into account that the improper integral $\int_0^\infty\frac{\sin x}{x}\,dx$ is convergent, we obtain

$$|\langle u_r, w\rangle|=\left|\int_0^R\frac{\sin\nu t}{\pi t}\,dt(u_r(x-R)+u_r(x+R))+\right.$$

$$\left.+\int_0^R\int_0^t\frac{\sin\nu\xi}{\pi\xi}\,d\xi(u_r'(x-t)-u_r'(x+t))\,dt\right|\leqq C(\|u_r\|_{L^\infty(K_1)}+\|u_r'\|_{L^\infty(K_1)}).$$

But $|\mu_r|$ is bounded, therefore by the result mentioned in Section 2, Part E) we can conclude that

$$|\langle u_r, w\rangle| \leqq C\|u_r\|_{L^\infty(K_1)}$$

and

$$\left|\langle u_r, w\rangle - \delta(v, |v_r|)u_r(x)\right| \leqq C\|u_r\|_{L^\infty(K_1)}.$$

Using again (24) we obtain

$$\sum_{|\mu_r|\leqq A} \left|\langle u_r, w\rangle - \delta(v, |v_r|)u_r(x)\right|^2 \leqq C. \tag{26}$$

(25) and (26) imply (22) and the proof of the Theorem is finished. □

REMARK. We note that in the proof of the Proposition in Section 3 we did not use the full assumption (C 1) but only its consequence (18). Thus our result remains valid for all (not necessarily complete) orthonormal systems consisting of eigenfunctions of order 0 (for example). □

OPEN PROBLEMS. 1. It would be interesting to know whether the assumption (C 3) is necessary for the validity of the Proposition.

2. From the viewpoint of applications the Theorem proved in this paper seems to be very general and satisfactory. However, from a pure mathematical viewpoint it would be useful to enlighten whether the result remains true for the more general differential operator

$$Lu = u^{(n)} + q_1 u^{(n-1)} + \ldots + q_n u, \quad q_s \in L^1_{\text{loc}}(G), \quad s = 1, \ldots, n.$$

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(Received April 9, 1984)

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Index: 26.014

Acta Mathematica Hungarica

VOLUME 47, NUMBERS 3–4, 1986

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0236—5294

ACTA MATHEMATICA
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Acta Mathematica publishes papers on mathematics in English, German, French and Russian

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Acta Math. Hung.
47 (3—4) (1986), 281—285.

EXTENDING FAMILIES OF DISCRETE ZERO SETS

C. E. AULL (Blacksburg)

1. Introduction. In [2], the extending of families of disjoint zero sets of a set S to a family of disjoint zero sets of the space X were studied. Here we restrict the disjoint zero sets of S to discrete families or at least to families discrete in the unions.

DEFINITION 1. A set S is TC^*-embedded in X (DC^*-embedded) [$\bar{D}C^*$-embedded] if any family of disjoint (discrete) [discrete in their union] zero sets may be extended to a disjoint family of zero sets of X. A zero set, Z, of S is extended to a zero set $E(Z)$ of X, if $E(Z)\cap S=Z$. It is clear that for a set $S\subset X$, TC^*-embedding $\Rightarrow$ $\Rightarrow \bar{D}C^*$-embedding $\Rightarrow DC^*$-embedding. Since TC^*-embedding was studied in [2], our emphasis will be on DC^*-embedding and $\bar{D}C^*$-embedding including with cardinality restrictions on the discrete families and relations to analogous type embeddings of cozero sets and open sets.

The following lemma proved in [1] will be useful in proving results in this paper.

LEMMA A. *Suppose each set of type A in S can be extended to a set of type A, and if $\{F_b\}$ is a disjoint family of sets of type A in S and $\{H_b\}$ is a disjoint family of sets of type A in X such that $F_b\subset H_b$, and the intersection of two sets of type A is of type A, then $\{F_b\}$ may be extended to a disjoint family of sets of type A.*

2. Some cardinality restrictions. In [2] it was shown that a denumerable family (family of cardinal ω_1) of disjoint zero sets of a subset S may be extended to a family of disjoint zero sets of a space X if S is C^*-embedded (C-embedded) in X.

DEFINITION 2 (Zenor [10]). A space X is a Z-space if given 2 disjoint closed sets of X one a zero set, they are completely separated in X.

THEOREM 1. *Let S be a Z-space and let $\{Z_\alpha\colon \alpha\in\Omega\}$ be a discrete family of zero sets of cardinality ω_1. Suppose S is C^*-embedded in a Tychonoff space X. Then the family $\{Z_\alpha\}$ may be extended disjointly to X.*

PROOF. For every Z_α, there exist disjoint zero sets H_α and H^*_α, of X such that $Z_\alpha\subset H_\alpha$, and $\bigcup\{Z_\beta\colon \beta\neq\alpha\}\subset H^*_\alpha$. Let $M_\alpha=H_\alpha\cap\bigcap_\beta\{H^*_\beta\colon \beta<\alpha\}$, then $\{M_\alpha\}$ is a disjoint family of zero sets of X such that $Z_\alpha\subset M_\alpha$. An application of Lemma A completes the proof.

With the continuum hypothesis, we may replace ω_1 by c in the above proof. In the case of normal spaces, we may make the replacement without CH.

THEOREM 2. *Let S be normal and let $\{Z_a\colon a\in A, |A|\leqq c\}$ be a discrete family of zero sets. Then $\{Z_\alpha\}$ may be extended disjointly to any Tychonoff space X that S is C^*-embedded in.*

Proof. Let $\{Z_a\}$ be indexed by R the real numbers. Construct a continuous function f on $B=\bigcup\{Z_a\colon a\in R\}$ such that $f^{-1}(a)=Z_a$. Then since B is closed f may be extended to S and hence to X so that there exist disjoint zero sets of $\{H_a\}$ of X such that $Z_a\subset H_a$. An application of Lemma A completes the proof.

If all the zero sets are regular closed, we may replace normal by metanormal (every regular closed set is C^*-embedded).

Corollary 2. *If D is discrete and $|D|\leqq c$, then D is TC^*-embedded in any space D is C^*-embedded in.*

Theorem 3. *Let S be normal and hereditarily extremally disconnected and C^*-embedded in X. Let $\{Z_a\colon a\in A\colon |A|\leqq c\}$ be a family of zero sets discrete in their union. Then $\{Z_a\}$ may be extended disjointly to X.*

Proof. Similar to Theorem 2.

3. DC^*-embeddings. Definition 4 (Junnila [7]). A topological space X is collectionwise δ-normal ($C_\delta-N$) if for every discrete family of closed sets $\{F_a\}$ of X, there is a disjoint family of G_δ-sets, $\{G_a\}$ such that $F_a\subset G_a$.

It is clear that we may replace G_δ-sets by zero sets in the above definition in a normal space.

Theorem 4. *The following are equivalent for a normal space X:*

(a) *X is $C_\delta-N$,*

(b) *every closed set is DC^*-embedded.*

Proof. (a)→(b). If $\{Z_a\}$ is a discrete family of zero sets of a closed set F, $\{Z_a\}$ is a discrete family of closed sets of X and can be expanded to a disjoint family of zero sets of X. An application of Lemma A completes the proof.

(b)→(a). Let F_a be a discrete family of closed sets. Since each F_a is a zero set in $\bigcup F_a$, a closed set, the result follows.

Definition 5. A set S is T_z-embedded (TG-embedded) in a space X if any disjoint family of cozero (open) sets $\{G_a\}$ of S may be extended to a disjoint family of cozero (open) sets of X. If $\{G_a\}$ is restricted to discrete families, we say S is D_z-embedded (DG-embedded) in X.

In [1], it was proved that a space is collectionwise normal iff every closed set is D_z-embedded (DG-embedded).

Corollary 4. *If every closed set is D_z-embedded (DG-embedded), then every closed set is DC^*-embedded.*

4. DC^*-embeddings. Definition 6. A topological space X is collectionwise $\bar{\delta}$-normal ($C_{\bar{\delta}}-N$) if for F_a closed in X, $\{F_a\}$ discrete in $\bigcup F_a$, there exists a disjoint family of G_δ-sets $\{G_a\}$ such that $F_a\subset G_a$.

Again, we may replace G_δ-set by a zero set in a normal space in the above definition.

Theorem 5. *The following are equivalent for a normal space X:*

(a) *X is $C_{\bar{\delta}}$-normal.*

(b) *Every closed set of X is $\bar{D}C^*$-embedded in X.*

PROOF. The proof of (a)→(b) is similar to that of Theorem 4. (b)→(a). Let $\{F_a\}$ be a family of closed sets of X discrete in $\cup F_a$. Each F_a is open in $\cup F_a$. So there is a family of disjoint open sets, $\{G_a\}$, of $F=\overline{\cup F_a}$ by Theorem 2 of [1] such that $F_a \subset G_a$. By the normality of X, there is a family of zero sets of F, $\{H_a\}$, such that $F_a \subset H_a \subset G_a$. By (b), $\{H_a\}$ can be extended to a disjoint family of zero sets $\{Z_a\}$ such that $F_a \subset Z_a$.

In [1], it was shown that every closed set is T_z-embedded iff for any family of set $\{F_a\}$ discrete in their union such that each F_a is an F_σ-set of X. Then there exists a family of disjoint open sets $\{G_a\}$ such that $F_a \subset G_a$. Also from [1] if every closed set is TG-embedded, then every closed set is T_z-embedded.

COROLLARY 5. *If every closed set of a normal space X is TG- or T_z-embedded, then every closed set of X is $\bar{D}C^*$-embedded.*

The following lemma might be compared with Corollary 7B of [1].

LEMMA 6. *The following are equivalent for a normal hereditary extremally disconnected space X and* (a)↔(b) *in normal spaces:*

(a) *X is $HC_\delta - N$ (every subset is $C_\delta - N$).*

(b) *Given a collection of subsets $\{F_a\}$ of X discrete in $\cup F_a$, there exists a disjoint family of zero sets $\{Z_a\}$ such that $F_a \subset Z_a$.*

(c) *X is $C_{\bar\delta} - N$.*

(One might Compare (b) with McAuley's [8] equivalence of hereditary collectionwise normality.)

PROOF. (b)→(a) and (b)→(c) are immediate. (c)→(b). Let $\{F_a\}$ be discrete in $\cup F_a$. Each F_a can be extended to an open set G_a in $F=\overline{\cup F_a}$ and $\{G_a\}$ is disjoint. Then $\bar{G}_a$ in F is closed in X and $\{\bar{G}_a\}$ is discrete in $\cup \bar{G}_a$ since $\bar{G}_a$ is open in F. So (b) is satisfied.

(a)→(c). Let $\{F_a\}$ be discrete in $\cup F_a$ and let F_a be closed in X. There exists a family $\{G_a\}$ of disjoint open sets of $F=\overline{\cup F_a}$ such that $F_a \subset G_a$ again by Theorem 2 of [1].

There exists an open set G of X such that $G \cap \overline{\cup F_a} = \cup G_a$. There exists a family of disjoint G_δ-sets of G, $\{H_a\}$, such that $F_a \subset H_a$ since G is $C_\delta - N$ by (a) and since $\{F_a\}$ is closed and discrete in G. Each H_a is also a G_δ of X.

THEOREM 6. *The following are equivalent for a hereditarily extremally disconnected space X.*

(a) *X is normal and $HC_\delta - N$.*

(b) *Every subset of X is DC^*-embedded.*

(c) *Every subset of X is $\bar{D}C^*$-embedded.*

(d) *Every closed subset of X is $\bar{D}C^*$-embedded.*

(e) *X is normal and $C_{\bar\delta} - N$.*

PROOF. (a)→(c). Let $\{Z_a\}$ be a family of zero sets of a set S, discrete in $\cup Z_a$. There is a family of disjoint zero sets of X, $\{H_a\}$, such that $Z_a \subset H_a$ by (a) and Lemma 6. An application of Lemma A completes the proof.

(c)→(b) and (c)→(d) are immediate. (b)→(c). Let $\{Z_a\}$ be a family of zero sets of

S discrete in $M=\bigcup Z_a$. Since M is DC^*-embedded in X, there exists zero sets $\{H_a\}$ of X such that $Z_a \subset H_a$. An application of Lemma A completes the proof.

(d)$\to$(e) follows from Theorem 5.

(e)$\to$(a) follows from Lemma 6.

COROLLARY 6. *Let X be hereditarily extremally disconnected. Then any of the following imply conditions* (a) *to* (e) *of Theorem* 6.

(a) *Every closed subset of X is TG-embedded.*

(b) *Every closed subset of X is T_z-embedded.*

(c) *Every subset of X is DG-embedded.*

PROOF. Theorem 6 and Theorems 4 and 5 of [1].

We note that since (a) and (c) are equivalent to HCN we only need extremally disconnected in the corollary statement in these cases.

5. Some examples and questions. EXAMPLE 1. R. Fox has shown that R is not TC^*-embedded in βR. Since R is Lindelöf (hereditary Lindelöf) R is DC^*-embedded ($\bar{D}C^*$-embedded) in βR.

EXAMPLE 2. The one point compactification X of a discrete space of cardinality greater than c, is DC^*-embedded in any product of closed intervals it is embedded in since X is compact, but is not $\bar{D}C^*$-embedded in such a product since Engelking [5] has shown the product has at most c disjoint zero sets.

We have thus shown that TC^*-embedding, $\bar{D}C^*$-embedding and DC^*-embedding are distinct properties. However Theorem 6 shows that if every set is DC^*-embedded then every set is $\bar{D}C^*$-embedded. So we have the following set of relations (a)$\to$(b)$\to$$\to$(c)$\leftrightarrow$(d)$\to$(e) for a space X:

(a) X is perfectly normal and extremally disconnected.

(b) Every subset of X is TC^*-embedded.

(c) Every subset of X is $\bar{D}C^*$-embedded.

(d) Every subset of X is DC^*-embedded.

(e) X is normal and hereditary extremally disconnected.

In [2], it was shown that under the existence of measurable cardinals or the assumption of club (b)$\to$(a) based on work of Blair [5] and Wage [10]. The question then arises, does (c)$\to$(b) and does (e)$\to$(d), particularly if we use MA$+\sim$CH?

6. A mapping theorem. The following theorem can be proved by the same methods used in the proof of Theorem 2 of [1].

THEOREM 7. *Let $S\subset X$ and let f be continuous on X such that $f^{-1}(f(S))=S$. If f is cozero set preserving and S is TC^*-embedded (DC^*-embedded) [$\bar{D}C^*$-embedded] in X then $f(S)$ is TC^*-embedded (DC^*-embedded) [$\bar{D}C^*$-embedded] in $f(x)$.*

COROLLARY 7. *Under cozero set preserving maps, the combination normality and $C_\delta-N$ (Theorem 5), and the combination hereditarily extremally disconnected, normality and $HC_\delta-N$ (Theorem 6) are preserved.*

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(Received September 17, 1982)

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Acta Math. Hung.
47 (3—4) (1986), 287—297.

LOCAL EXPANSIONS ON GRAPHS AND ORDER OF A POINT

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Let X and Y be metric spaces with metrics d_X and d_Y, respectively. A mapping $f\colon X\to Y$ of X onto Y is said to be a local expansion if it is continuous and if for every point x of X there exist an open neighborhood U of x and a constant $M>1$ such that for every two points y and z of U the inequality

$$d_Y\big(f(y), f(z)\big) \geqq M d_X(y, z)$$

holds.

Discussing some properties of local expansions on metric continua ([1], [6]), especially on linear graphs ([3]), we have observed that the order of a point does not decrease under a local expansion ([1], Proposition 4.2; [3], Proposition 5). Investigating this fact more carefully for linear graphs, we have discovered some reasons of it. The results are presented in this paper.

We use the concept of order of a point in a space in the sense of Menger—Urysohn (see [4], p. 274 or [7], p. 48). A point of order 1 in the space X is called an end point of X; the set of all end points of X is denoted by $E(X)$. A point of order 3 or more in the space X is called a ramification point; the set of all ramification points of X is denoted by $R(X)$. By an n-od we mean a set homeomorphic to the one-point union of n closed intervals.

We recall several properties of local expansions. All of them are easy to verify and some of these properties have already been established for local expansions $f\colon X\to X$ of a metric space X onto itself ([3], Properties 1—4) and they are stated here for a more general case of local expansions from one metric space onto another.

PROPOSITION 1. *Let two metric spaces (X, d_X) and (Y, d_Y) be given, and let $f\colon X\to Y$ be a local expansion. Then*

(i) *f is locally one-to-one: for every point $x\in X$ and for the open neighborhood U of x as in the definition of the local expansion, the partial mapping $f|U\colon U\to f(U)\subset Y$ is one-to-one;*

(ii) *for every point $x\in X$ and for the open neighborhood U of x as in the definition of the local expansion, every arc $ab\subset U$ is mapped onto an arc $f(a)f(b)$ homeomorphically under f;*

(iii) *for each simple closed curve $S\subset X$, its image $f(S)$ does not contain end points of itself;*

(iv) *for each arc $ab\subset X$ no point of $ab\setminus\{a, b\}$ is mapped on an end point of $f(ab)$;*

(v) *if X is compact then the inverse image $f^{-1}(y)$ of every point $y\in Y$ is finite.*

Definitions of notions undefined here can be found in [3] and [5].

The following example shows that a continuous image of a linear graph (even of an arc) under a local expansion need not be a linear graph.

EXAMPLE 1. There is a local expansion f: $[0, 1]\to K$ of $[0, 1]$ onto K such that:

1° K is a plane hereditarily locally connected curve metrized by a convex metric;

2° K has exactly one end point and countably many points of order 4; other points of K are of order 2;

3° f is a local expansion with $M=2$ for every point $x\in[0, 1]$;

4° $f^{-1}(f(x))=\{x\}$ for all $x\in[0, 1]$ save a countable set

$$\left\{\frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \dots, 0, 1\right\}$$

for elements x of which we have $f^{-1}(f(x))=\{x, 1-x\}$;

5° $\mathrm{ord}_y K=2\cdot \mathrm{card} f^{-1}(y)$ for all $y\in K\setminus\{e\}$, where e is the only end point of K.

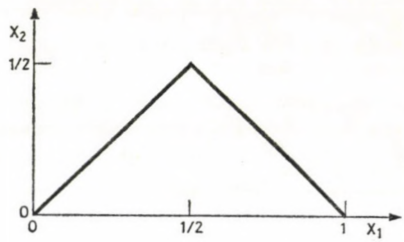

Fig. 1

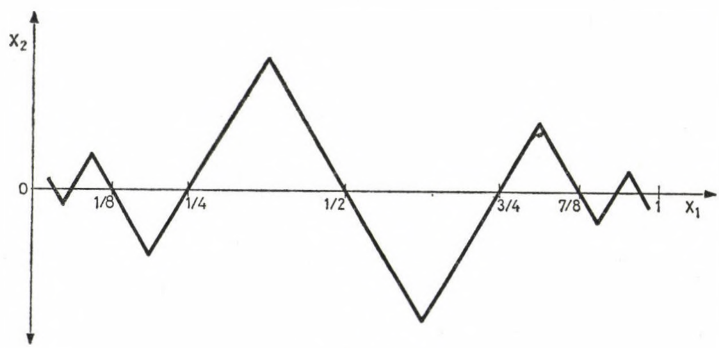

Fig. 2

To construct the example consider two auxiliary functions f_1, f_2: $[0, 1]\to R$, the graphs of which are pictured in Fig. 1 and Fig. 2, define a mapping f from $[0, 1]$ into the plane R^2 by

$$f(x) = (f_1(x), f_2(x)) \quad \text{for} \quad x\in[0, 1], \tag{1}$$

put $K=f([0, 1])$, and take the length of the shortest arc in K joining its two points as a metric for K. This metric is obviously convex. The continuum K is depicted in Fig. 3.

The continuum K can also be described in some other way. Namely, let G be the graph of the function f_2 (see Fig. 2) and let a mapping g from G into the plane R^2 be defined by

$$g\big((x_1, x_2)\big) = \begin{cases} (x_1, x_2), & \text{if } x_1 \in \left[0, \frac{1}{2}\right], \\ (1-x_1, x_2), & \text{if } x_1 \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{2}$$

Then $K=g(G)$. In other words, K is obtained from G under the mapping g which is the identity on the left half of G and the symmetry with respect to the straight line $x_1=\frac{1}{2}$ on the right half of G.

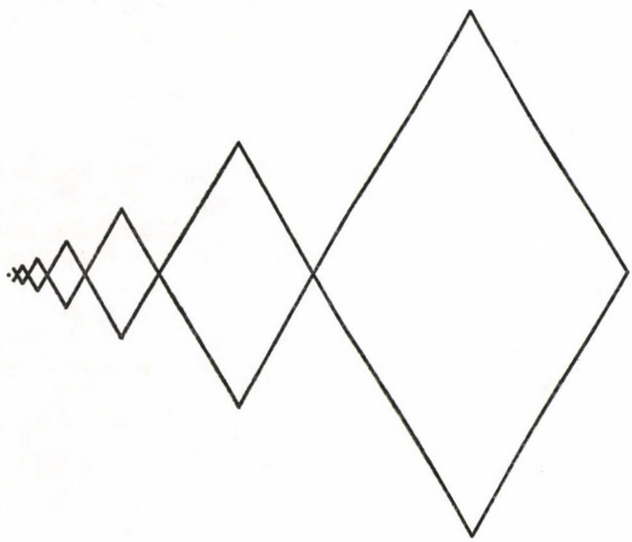

Fig. 3

Now properties 1° and 2° of K are evident from the construction; in particular $e=(0, 0)$ is the only end point of K. By the definition of the metric on K we see that any subsegment of $[0, 1]$ contained in $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$ or in $\left[\frac{1}{4}, \frac{3}{4}\right] \setminus \left\{\frac{1}{4}, \frac{3}{4}\right\}$ is expanded twice under f, whence 3° follows. Properties 4° and 5° easily follow from the definition of f, and therefore the argumentation is complete.

Now we introduce some auxiliary notions and notations. Let X be a linear graph and let x be a point of X. We denote by $K(X, x)$ the closure of an arbitrary component of $X \setminus \{x\}$. The set $K(X, x)$ is said to be of the first kind provided that it contains a simple closed curve. Otherwise it is said to be of the second kind.

Consider an arc $xp \subset X$ and let $K(X, x; p)$ denote the closure of the component of $X \setminus \{x\}$ that contains the point p. Then obviously $xp \subset K(X, x; p)$. The arc xp is said to be of the first kind with respect to x provided that $K(X, x; p)$ is of the first kind, i.e., if it contains a simple closed curve. Otherwise xp is said to be of the second kind with respect to x.

Given a point x of a linear graph X, let U_x be an open connected neighborhood of x containing neither end points nor ramification points except, perhaps, x itself. Thus

$$(U_x \setminus \{x\}) \cap (E(X) \cup R(X)) = \emptyset, \tag{3}$$

and $\bar{U}_x$ is an n-od, where $n = \operatorname{ord}_x X$:

$$\bar{U}_x = \bigcup \{xp_i \colon i = 1, 2, \ldots, \operatorname{ord}_x X\}. \tag{4}$$

The number of the arcs xp_i which are of the first kind with respect to x will be called the cyclic order of X at x, and it will be denoted by $c(X, x)$; the number of the arcs xp_i which are of the second kind with respect to x will be called the tree order of X at x, and it will be denoted by $t(X, x)$. Therefore we have

$$c(X, x) + t(X, x) = \operatorname{ord}_x X \tag{5}$$

for every point $x \in X$.

LEMMA 1. *Let X and Y be linear graphs and let $f\colon X \to Y$ be a local expansion of X onto Y. For each point $x \in X$ let U_x be an open connected neighborhood of x satisfying* (3) *and such that $\bar{U}_x \subset U$, where U is an open neighborhood of x as in the definition of the local expansion f. If an arc xp_i of* (4) *is of the first kind with respect to x, then its image $f(xp_i)$ is an arc of the first kind with respect to its end point $f(x)$.*

PROOF. Since $xp_i \subset U$, hence f maps xp_i homeomorphically onto an arc $f(x)f(p_i)$ (see Proposition 1 (ii)). Suppose on the contrary that $f(x)f(p_i)$ is of the second kind with respect to $f(x)$. Put $T = K(Y, f(x); f(p_i))$. Thus T is a tree in Y containing the arc $f(x)f(p_i)$. Consider two cases. If xp_i lies on a simple closed curve S, then $f(x)f(p_i) \subset f(S)$, whence $f(S)$ has a nondegenerate intersection with T. We see that $T \cap f(S)$ is a tree as a subcontinuum of T, and therefore $f(S)$ contains an end point of itself, contrary to (iii) of Proposition 1. If xp_i is contained in no simple closed curve, then, since it is of the first kind with respect to x, there exists an arc xq and a simple closed curve S_1 such that $xp_i \subset xq \subset xq \cup S_1 \subset K(X, x; p_i)$ and $xq \cap S_1 = \{q\}$. Since no point of $xq \setminus \{x, q\}$ is mapped to an end point of $f(xq)$ by (iv) of Proposition 1 and since the arc $f(x)f(p_i)$ is contained in $T \cap f(xq)$, we conclude that $f(xq) \subset T$ and, moreover, $f(xq)$ is an arc with $f(x) \neq f(q)$. Thus $f(S_1)$ has a nondegenerate intersection with T, which implies, as in the previous case, a contradiction with (iii) of Proposition 1. So the lemma is proved.

Since $\bar{U}_x \subset U$ and $f|U$ is one-to-one by (i) of Proposition 1, hence $f|\bar{U}_x$ is a homeomorphism, and therefore we conclude from Lemma 1 the following

COROLLARY 1. *Let X and Y be linear graphs and let $f\colon X \to Y$ be a local expansion of X onto Y. Then the cyclic order of a point is never decreased, i.e., for every point $x \in X$ we have*

$$c(X, x) \leqq c(Y, f(x)). \tag{6}$$

LEMMA 2. *Let two linear graphs T_1 and T_2 be given such that T_2 contains no simple closed curve, and let $f\colon T_1 \to f(T_1) \subset T_2$ be a local expansion of T_1 into T_2. Then T_1*

also contains no simple closed curve, f is an embedding, and for every arc $A\subset T_1$ we have

$$\lambda(A) < \lambda(f(A)), \tag{7}$$

where $\lambda(A)$ and $\lambda(f(A))$ are defined by (3) *of* [5], *p.* 80.

PROOF. Suppose S is a simple closed curve in T_1. Then $f(S)$ is a nondegenerate subcontinuum of the tree T_2, so it is a tree. Hence it contains an end point of itself, contrary to (iii) of Proposition 1. Further, it is known that for every arc $ab\subset T_1$ the partial mapping $f|ab$: $ab\to f(ab)$ is a homeomorphism (see [1], Corollary 4.1), whence f is one-to-one, and thus it is an embedding because T_1 is compact. Inequality (7) has been proved in Proposition 7 of [3] for local expansions of linear graphs onto itself (i.e., under an additional assumption $T_1=T_2$ in our notation), but the whole argumentation remains true without this assumption. So the lemma is proved.

Let (X, d_X) and (Y, d_Y) be metric spaces. A continuous mapping f: $X\to Y$ of X into Y is said to be an expansive embedding provided that for every two points $y, z\in X$ we have $d_X(y, z)<d_Y(f(y), f(z))$. It is easy to verify that if f is an expansive embedding, then f: $X\to f(X)\subset Y$ is a homeomorphism.

LEMMA 3. *Let X and Y be linear graphs and let f: $X\to Y$ be a local expansion of X onto Y. Further, let $K(X, x)$ and $K(Y, f(x))$ be of the second kind, and let there exist an arc $xp\subset K(X, x)$ whose image $f(xp)$ is contained in $K(Y, f(x))$. Then $f(K(X, x))\subset \subset K(Y, f(x))$, and $f|K(X, x)$: $K(X, x)\to K(Y, f(x))$ is an expansive embedding.*

PROOF. Suppose on the contrary that there exists a point $q\in K(X, x)$ such that $f(q)\in Y\setminus K(Y, f(x))$. Note that $q\neq x$. Since $K(X, x)$ is of the second kind, it is a tree for which the point x is an end point, and we see that $xp\cap xq\subset K(X, x)$ is a nondegenerate arc, the image of which under f is contained in $K(Y, f(x))$. Consider the component of the set $xq\cap f^{-1}(K(Y, f(x)))$, i.e., the maximal subarc xr of the arc xq such that $f(xr)\subset K(Y, f(x))$. Since $f(q)$ is out of $K(Y, f(x))$ we see that $r\neq q$, i.e., xr is a proper subarc of the arc xq. The set $K(Y, f(x))$ being of the second kind, the continuum $f(xr)$ is a tree for which $f(x)$ is the only boundary point. Thus we conclude from continuity of f that $f(r)=f(x)$. Since every tree has at least two end points, there is an end point w of $f(xr)$ which is distinct from $f(x)$. Let $s\in xr\cap f^{-1}(w)$. So $s\in xr\setminus\{x, r\}$ and $f(s)=w$ is an end point of $f(xr)$ contrary to (iv) of Proposition 1. The further part of the conclusion of the lemma is a straight consequence of Lemma 2. Thus the proof is complete.

PROPOSITION 2. *Let X and Y be linear graphs and let f: $X\to Y$ be a local expansion of X onto Y. If, for a point $x\in X$, we have*

$$\operatorname{ord}_x X > c(Y, f(x)), \tag{8}$$

then, among $t(X, x)$ sets $K(X, x)$ of the second kind there exist at least $m(x)=\operatorname{ord}_x X- -c(Y, f(x))=t(X, x)-(c(Y, f(x))-c(X, x))$ of them, say $K_i(X, x)$ for $i=1, 2, \dots \dots, m(x)$, such that the partial mapping $f|\bigcup\{K_i(X, x)$: $i=1, 2, \dots, m(x)\}$ is an expansive embedding of $\bigcup\{K_i(X, x)$: $i=1, 2, \dots, m(x)\}$ into $\bigcup\{K_j(Y, f(x))$: $j=1, 2, \dots, t(Y, f(x))\}$, where all $K_j(Y, f(x))$ are of the second kind.

PROOF. Let U be an open neighborhood of x as in the definition of the local expansion f. To simplify notation, put $n=\operatorname{ord}_x X$, $c=c(X, x)$ and $t=t(X, x)$. So

$n=c+t$ by (5). Consider an n-od at x contained in U, which is the union of n arcs L_k $(k=1, 2, \ldots, n)$ emanating from x and disjoint out of this point. Label these arcs in such a manner that the indices $k \leqq c$ correspond to arcs of the first kind with respect to x: $L_1, L_2, \ldots, L_c$, while further indices correspond to arcs of the second kind with respect to x: $L_{c+1}, L_{c+2}, \ldots, L_{c+t}$.

Since $L_k \subset U$ for $k=1, 2, \ldots, n$, it follows from (ii) of Proposition 1 that $f|L_k$: $L_k \to f(L_k)$ is a homeomorphism. So the image under f of every L_k is an arc $f(L_k)$ having $f(x)$ as its end point. We know by Lemma 1 that for $k \leqq c$ the arcs $f(L_k)$ are of the first kind with respect to $f(x)$. Hence every arc L_k for $c+1 \leqq k \leqq c+t=n$ is mapped under f onto an arc which can be either of first or of second kind with respect to $f(x)$. Since in an $\operatorname{ord}_{f(x)} Y$-od at $f(x)$ we have $c(Y, f(x))$ arcs of the first kind with respect to $f(x)$ and since $c=c(X, x)$ of them are already occupied (in the sense that they have nondegenerate intersection with the arcs $f(L_k)$ for $k=1, 2, \ldots$ $\ldots, c)$, hence at most $c(Y, f(x))-c$ arcs L_k with $k=c+1, c+2, \ldots, c+t$ can be mapped into arcs of the first kind with respect to $f(x)$ (note that $c(Y, f(x))-c \geqq 0$ by (6) of Corollary 1). Therefore it remains at least $m(x)=t-(c(Y, f(x))-c)$ arcs L_{k_i} (where $k_i \in \{c+1, c+2, \ldots, c+t\}$ and $i=1, 2, \ldots, m(x)$) of the second kind with respect to x, and every of them is homeomorphically mapped onto an arc of the second kind with respect to $f(x)$. Observe that $m(x)>0$ by the hypothesis. Since every such arc L_{k_i} is contained in a set $K(X, x)$ of the second kind (being the closure of a component of $X \setminus \{x\}$), we have at least $m(x)$ sets $K_i(X, x)$ of the second kind with $L_{k_i} \subset K_i(X, x)$ for $i=1, 2, \ldots, m(x)$. Since the arc $f(L_{k_i})$ is of the second kind with respect to $f(x)$, it is contained in some set $K_j(Y, f(x))$ of the second kind, where $j \in \{1, 2, \ldots, t(Y, f(x))\}$. Therefore $K_j(Y, f(x))$ is a tree and we conclude from Lemma 3 that $f|K_i(X, x)$: $K_i(X, x) \to K_j(Y, f(x))$ is an expansive embedding. Thus the proof is complete.

As a consequence of Corollary 1 and of Proposition 2 we get the following

THEOREM 1. *Let X and Y be linear graphs and let f: $X \to Y$ be a local expansion of X onto Y. Then for every point $x \in X$ we have*

$$c(X, x) \leqq c(Y, f(x)) \tag{6}$$

and either $\operatorname{ord}_x X \leqq c(Y, f(x))$, *or — if* $\operatorname{ord}_x X > c(Y, f(x))$ *— there are* $\operatorname{ord}_x X - c(Y, f(x))$ *trees being the closures of components of $X \setminus \{x\}$ which are expansively embedded under f into the corresponding trees being the closures of some components of $Y \setminus \{f(x)\}$. Furthermore, this expansive embedding is one-to-one with respect to the trees in the sense that no two different trees $K_{i_1}(X, x)$ and $K_{i_2}(X, x)$ are embedded into the same tree $K_j(Y, f(x))$.*

COROLLARY 2. *Let X and Y be linear graphs and let f: $X \to Y$ be a local expansion of X onto Y. Then for every point $x \in X$ we have*

$$\operatorname{ord}_x X \leqq \operatorname{ord}_{f(x)} Y. \tag{9}$$

Indeed, if $\operatorname{ord}_x X \leqq c(Y, f(x))$, then (9) trivially holds by (5) applied to Y at $f(x)$. Otherwise we have (8), and by Proposition 3 there exist $m(x)$ arcs in X of the second kind with respect to x which are mapped onto corresponding arcs of the second kind with respect to $f(x)$. Thus we have the inequality $m(x) \leqq t(Y, f(x))$, i.e., $t(X, x)-(c(Y, f(x))-c(X, x)) \leqq t(Y, f(x))$, and using (5) we have (9).

COROLLARY 3. *Let $f\colon X\to Y$ be a local expansion of a linear graph X onto a linear graph Y. Then*

$$\text{card}\, E(Y) \leqq \text{card}\, E(X). \tag{10}$$

Indeed, for every end point y of Y only end points of X can be in $f^{-1}(y)$, i.e., $f^{-1}(y)\subset E(X)$ for all $y\in E(Y)$ by (9), whence (10) follows.

EXAMPLE 2. There exists a local expansion $f\colon [0,1]\to S$ of the unit interval $[0,1]$ onto the unit circumference $S=\{(x_1,x_2)\colon x_1^2+x_2^2=1\}$ such that none of inequalities (6), (9) and (10) can be replaced by the corresponding equality.

Indeed, put $f(x)=\exp(2\pi i x)$ for $x\in[0,1]$. Then f is a local expansion with the coefficient $M=2\pi$ for all x, and none of the three inequalities mentioned above turns into equality.

EXAMPLE 3. There exists a local expansion $f\colon X\to Y$ of a simple triod X onto a circle with a tail Y, and there are two points x and x' of X such that for x inequality (8) is satisfied, while for x' it is not.

To describe the example, let x_1, x_2 be the cartesian rectangular coordinates of a point in the euclidean plane. Put $A=\{(x_1,0)\colon 0\leqq x_1\leqq 1\}$, $B=\{(0,x_2)\colon 0\leqq x_2\leqq 1\}$ and $C=\{(0,x_2)\colon -1\leqq x_2\leqq 0\}$ and define $X=A\cup B\cup C$ with the natural convex metric. Further, put $D=\{(x_1,0);\ 0\leqq x_1\leqq 2\}$, $S=\{(x_1,x_2)\colon (x_1+1)^2+x_2^2=1\}$ and define $Y=D\cup S$ with the natural convex metric. Finally define $f\colon X\to Y$ as follows.

$$f\big((x_1,x_2)\big)=$$
$$=\begin{cases}(2x_1,0) & \text{if}\quad x_1\in[0,1] \quad\text{and}\quad x_2=0, \quad \text{i.e., if}\quad (x_1,x_2)\in A,\\ (1+\cos\pi x_2, \sin\pi x_2) & \text{if}\quad x_1=0 \quad\text{and}\quad x_2\in[-1,1], \quad \text{i.e., if}\, (x_1,x_2)\subset B\cup C.\end{cases}$$

Thus $f(A)=D$ and $f(B\cup C)=S$. We see that for all points $x\in A\setminus\{(1,0)\}$ we have (8). In particular, for $x=(0,0)$ we have $\text{ord}_x X=3$ and $c\big(Y,f(x)\big)=2$, and for $x\in A\setminus\{(0,0),(1,0)\}$ we have $\text{ord}_x X=2$ and $c\big(Y,f(x)\big)=1$. For $x=(1,0)$ we have $\text{ord}_x X=c\big(Y,f(x)\big)=1$. If $x=(0,1)$ or $x=(0,-1)$, then $\text{ord}_x X=1$ and $c\big(Y,f(x)\big)=2$. If $x\in B\cup C\setminus\{(0,0),(0,1),(0,-1)\}$, then $\text{ord}_x X=c\big(Y,f(x)\big)=2$.

Let us consider now a particular case of $Y=X$. Under this additional assumption one can prove some further properties of expansive embeddings of the trees mentioned in the conclusion of the theorem. To show these properties we start with the mapping of the set of end points.

PROPOSITION 3. *Let $f\colon X\to X$ be a local expansion of a linear graph X onto itself. Then $f|E(X)\colon E(X)\to E(X)$ is a one-to-one and onto mapping (i.e., f permutes the end points of X).*

In fact, by Proposition 6 of [3] we have $f\big(E(X)\big)\subset E(X)$. The inverse inclusion is a consequence of inequality (9), whence we see that f maps the set $E(X)$ onto itself. Since $E(X)$ is finite, the conclusion holds.

Under the assumption $Y=X$ we get from Propositions 2 and 3 the following corollary, in which notation of Proposition 2 is used.

COROLLARY 4. *If $f\colon X\to X$ is a local expansion of a linear graph X onto itself, and if for a point $x\in X$ we have $\text{ord}_x X>c\big(X,f(x)\big)$, then there are at least $m(x)=\text{ord}_x X-c\big(X,f(x)\big)$ sets $K_i(X,x)$ $\big(i=1,2,\ldots,m(x)\big)$ of the second kind such that* *the partial mapping* $f|\bigcup\{K_i(X, x)\colon\ i=1, 2, \dots, m(x)\}$ *is an expansive embedding of* $\bigcup\{K_i(X, x)\colon i=1, 2, \dots, m(x)\}$ *into* $\bigcup\{K_j(X, f(x))\colon j=1, 2, \dots, t(X, f(x))\}$, *where all* $K_j(X, f(x))$ *are of the second kind. Furthermore, for every* $i=1, 2, \dots, m(x)$ *and for some properly chosen* $j\in\{1, 2, \dots, t(X, f(x))\}$ *we have* $f(E(K_i(X, x)))\subset \subset E(K_j(X, f(x)))$.

As a consequence of the above corollary we get Theorem 2 of [3], which we reformulate in the form of

COROLLARY 5. *If* $f\colon X\to X$ *is a local expansion of a linear graph onto itself, then there exists a point* $p\in X$ *at which* X *is of the maximal order, and such that for every component of* $X\setminus\{p\}$ *its closure* $K(X, p)$ *is of the first kind.*

PROOF. Consider the set M of all points x of X at which $\operatorname{ord}_x X$ is maximal, and let P be a subset of M composed of these points p of M for which $c(X, p)$ is maximal. By inequality (9) we have $f(M)\subset M$, whence by (6) it follows $f(P)\subset P$. Suppose on the contrary that there is a point $p\in P$ having a set $K(X, p)$ of the second kind, i.e. such that $t(X, p)>0$. Note that the number $t(X, p)$ is constant for all points $p\in P$ by the definition of P. Now consider an arc p_0e_0 such that (a) $p_0\in P$, (b) $e_0\in E(X)$, (c) p_0e_0 is contained in some tree $K(X, p_0)$, and (d) $\lambda(p_0e_0)$ is the greatest possible for all arcs satisfying (a), (b) and (c). Let U_0 be an open connected neighborhood of p_0 such that $\overline{U}_0\subset U$, where U is an open neighborhood of p_0 as in the definition of the local expansion f. Then $f|\overline{U}_0$ is a homeomorphism by (i) of Proposition 1, whence we conclude, using Lemma 1, that $f(K(X, p_0))$ is contained in some tree of the form $K(X, f(p_0))$. Applying Lemma 2 we see that $f|p_0e_0$ is an expansive embedding with $\lambda(p_0e_0)<\lambda(f(p_0)f(e_0))$ which contradicts to condition (d) of the definition of p_0e_0. This completes the proof.

Recall that the condition formulated in Corollary 5 is not only necessary but also sufficient for the existence of a local expansion of a linear graph onto itself (see [3], Theorem 1).

Frequently local expansions are considered together with openness of the mapping. We shall see now that not only the order ([5], Proposition 1) but also the cyclic order and the tree order of a point are invariants of open local expansions of linear graphs. We begin with the following

LEMMA 4. *Let* X *and* Y *be linear graphs and let* $f\colon X\to Y$ *be an open mapping of* X *onto* Y. *Then for every two points* x *and* p *of* X *we have*

$$K(Y, f(x);\ f(p))\subset f(K(X, x;\ p)).$$

PROOF. Suppose on the contrary that there exists a point $y\in K(Y, f(x);\ f(p))\setminus \setminus f(K(X, x;\ p))$. Thus $y\neq f(x)$ and since $K(Y, f(x);\ f(p))\setminus\{f(x)\}$ is just the component of $Y\setminus\{f(x)\}$ containing the point $f(p)$, hence there exists an arc $f(p)y$ in this component. Thus it does not contain $f(x)$. Let us order this arc from $f(p)$ to y. Since $f(p)$ is in $f(K(X, x;\ p))$ while y is not, there exists a last point y_0 of the arc $f(p)y$ which belongs to $f(K(X, x;\ p))$. Then obviously $y_0\neq y$. Let $q\in K(X, x;\ p)$ be such that $f(q)=y_0$. Thus $q\neq x$. Indeed, if $q=x$, then $f(x)=y_0\in f(p)y$, contrary to the choice of the arc $f(p)y$. Hence $U=K(X, x;\ p)\setminus\{x\}$ is an open neighborhood of q. Since f is open, $f(U)$ is an open set around y_0, and therefore the intersection of

$f(U)$ with the subarc y_0y of the arc $f(p)y$ is nondegenerate, contrary to the definition of y_0.

Taking $X=[-1, 1]$, $Y=[0, 1]$ and $f\colon X\to Y$ defined by $f(x)=|x|$ we see that for $x=1/2$ and $p=1/3$ the inclusion in Lemma 4 cannot be replaced by equality.

THEOREM 2. *Let $f\colon X\to Y$ be an open local expansion of a linear graph X onto a metric space Y. Then:*

1° Y is a linear graph;

2° f is a local homeomorphism;

3° for every point $x\in X$ we have

$$\operatorname{ord}_x X = \operatorname{ord}_{f(x)} Y, \tag{11}$$

$$c(X, x) = c\big(Y, f(x)\big), \tag{12}$$

$$t(X, x) = t\big(Y, f(x)\big); \tag{13}$$

4° for every continuum $Q\subset Y$ the inverse image $f^{-1}(Q)$ has finitely many components, and every one of them is mapped onto the whole Q under f;

5° if a point $y\in Y$ lies in a simple closed curve Q (contained in Y), then every point of the inverse image $f^{-1}(y)$ also lies in a simple closed curve S (contained in X), and $f|S\colon S\to Q$ maps openly S onto Q;

6° for every point $x\in X$ and for a (properly chosen) point $p\in X$ the kind of an arc xp is an invariant of the mapping f in the following sense. Given a point $x\in X$, let U_x be an open connected neighborhood of x satisfying condition (4) *and such that $\overline{U}_x\subset U$, where U is an open neighborhood of x as in the definition of the local expansion f. If an arc xp_i of* (4) *is of the first (resp. second) kind with respect to x, then its image $f(xp_i)$ is an arc of the first (resp. second) kind with respect to $f(x)$.*

PROOF. Properties 1° and 2° are known: namely the property of being a linear graph is preserved under open mappings ([7], Chapter X, Theorem 1.1, p. 182), and every open local expansion is a local homeomorphism ([1], Proposition 3.7). Let x be a point of X. To prove (11) of 3° observe that one inequality is proved in Corollary 2 as (9), and the opposite inequality is a consequence of openness of the mapping and it is proved as Corollary 7.31 of [7], p. 147. Properties (12) and (13) of 3° will be shown in the final part of the proof.

To verify 4° note that if Q is a one-point set, then the conclusion follows from (v) of Proposition 1, because f is a local expansion. So, let Q be nondegenerate. Since Y is a linear graph by 1°, it follows that Q has the non-empty interior (cf. [2], p. 54). It is known that every open mapping of a locally connected continuum is quasi-monotone ([7], Chapter VIII, Corollary 8.11, p. 152) which means that for every sub-continuum Q of Y having the non-empty interior the inverse image $f^{-1}(Q)$ has finitely many components and every one of them is mapped onto Q under f. Since the linear graph X is a locally connected continuum indeed, 4° is established.

To show 5° let us take a point $y\in Q\subset Y$, where Q is a simple closed curve. Since $f^{-1}(Q)$ is an inverse set ([7], p. 137), it follows by (7.2) of [7], p. 147 that the partial mapping $f|f^{-1}(Q)$ is open. Since $f^{-1}(Q)$ has finitely many components by 4°, say $C_1, C_2, \ldots, C_m$, we conclude that every one of them is an open subset of $f^{-1}(Q)$, whence it follows that $f|C_k\colon C_k\to Q$ is open for every $k=1, 2, \ldots, m$. Moreover, we have $f(C_k)=Q$ for every k by 4°, and we see that every component C_k is a linear graph as a subcontinuum of X. Therefore (11) can be applied to the open local expansion $f|C_k$ of C_k onto Q $(k=1, 2, ..., m)$, whence we have $\operatorname{ord}_p C_k=2$ for every point $p\in C_k$. Thus every C_k is a simple closed curve ([4], §51, V, Theorem 6, p. 294). Now let $x\in f^{-1}(y)\subset f^{-1}(Q)=\bigcup\{C_k\colon k=1, 2, ..., m\}$. Then $x\in C_k$ for some k, and 5° follows.

One part of 6°, namely for arcs of the first kind, has been proved previously as Lemma 1 (even without openness of f). To prove the other part, for arcs of the second kind, let us take an arc xp_i of the second kind with respect to x and observe that the set $K(X, x;\ p_i)$ is a tree. Denote it by T. Obviously $xp_i\subset T$. Since $xp_i\subset U$, hence f maps xp_i homeomorphically onto an arc $f(x)f(p_i)$ (see Proposition 1 (ii)). Suppose on the contrary that $f(x)f(p_i)$ is of the first kind with respect to $f(x)$, i.e. that it is contained in the closure $K=K(Y, f(x);\ f(p_i))$ of the component of $Y\setminus\{f(x)\}$ containing the point $f(p_i)$ and such that a simple closed curve Q is contained in K. By Lemma 4 we have $K\subset f(T)$. Thus $f(T)$ contains Q. Let $y\neq f(x)$ be a point of Q, and take a point x_1 of T such that $f(x_1)=y$. Thus $x_1\neq x$. Since the closure T of the corresponding component of $X\setminus\{x\}$ is a tree, and since $x_1\in T$, there is no simple closed curve in X containing x_1; but this contradicts 5°. Therefore 6° is established.

Now equalities (12) and (13) of 3° are immediate consequences of 6° by (11) and (5). Thus the proof of the theorem is complete.

Remark that the role of X and Y are not reversible in 5° of Theorem 2 in the sense that if $f\colon X\to Y$ is an open local expansion from a linear graph X onto a linear graph Y and if a point $x\in X$ lies in a simple closed curve contained in X, then $f(x)$ need not lie in any simple closed curve in Y. To see this take four different points a_0, a_1, a_2, a_3 and join them by arcs (named edges) as follows: every a_i is joined by exactly one edge with a_{3-i} and by exactly two edges with a_{1-i}, where indices are considered modulo 4, and where the edges are assumed to be disjoint out of their end points. No other edges are considered, in particular there is no edge joining a_i with a_{i+2}. Let X be the union of all six edges. Metrize X by a convex metric assuming that the length of every edge is equal to 1. So X is a linear graph. Note that every point of X lies in a simple closed curve. Further, let A and B be two disjoint circumferences of length 2 each and let C be a straight line segment also of length 2 joining a point p of A with a point q of B. Metrize the union $Y=A\cup B\cup C$ by the convex metric generated by lengths of arcs in A, B and C respectively. Let a mapping $f\colon X\to Y$ of X onto Y be defined as follows. $f(a_0)=f(a_1)=p$; $f(a_2)=f(a_3)=q$; f expands each of the two edges joining a_0 and a_1 twice, mapping it onto A, each of the two edges joining a_2 with a_3 is expanded onto B, and finally the edges a_1a_2 and a_0a_3 are expanded onto C. It can be easily seen that f is an open local expansion (with the coefficient of expansion $M=2$ at each point $x\in X$), and that the interior points of a_1a_2 and a_0a_3 are mapped on some interior points of C, so their images do not belong to any simple closed curve in Y.

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(Received May 23, 1983)

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Acta Math. Hung
47 (3—4) (1986), 299—312.

MONOIDS WITH DISJUNCTIVE IDENTITY AND THEIR CODES*

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1. Introduction

In this paper we prove several properties of monoids with disjunctive identity. As is well-known a disjunctive identity of a finitely generated monoid corresponds to a biprefix code, in fact a strong code in the sense of [16, 17]. Whereas finite strong codes have been characterized completely [16, 17], little is known about infinite strong codes in general.

In this paper, several properties of strong codes in general, as well as of a special class of strong codes are proved. To a great extent, the Dyck languages are characteristic for the structure of these special strong codes. Some insight into the structure of these codes as well as that of monoids with disjunctive identity will also be gained from considering special monoid presentations, a subject which has recently received renewed interest for applications in other areas of computer science, too [3].

2. Notation and definitions

Let X be an alphabet, that is a non-empty set. X^* denotes the free monoid generated by X. Let $X^+=X^*=X^*\setminus\{1\}$.

If S is a semigroup then $S^1=S$ if S is a monoid, and $S^1=S\cup\{1\}$ otherwise with 1 acting as the identity of S^1. For $L\subseteq S$, $x\in S$, let

$$L..x = \{(u, v)|u, v\in S^1, uxv\in L\}.$$

The *principal congruence* σ_L of L is defined by

$$x\sigma_L y \leftrightarrow L..x = L..y.$$

The *residue* of L is the set

$$W(L) = \{x|x\in S, L..x = \emptyset\}.$$

L is said to be *disjunctive* in S if σ_L is the equality; it is *quasidisjunctive in* S if

$$x\sigma_L y \to x=y \quad \text{or} \quad x, y\in W(L).$$

For $x\in S$ and any equivalence ϱ on S let $[x]_\varrho$ be the ϱ-class of x and for $M\subseteq S$ let M/ϱ be the set of ϱ-classes of elements in M; thus, in particular, $\{x\}/\varrho=[x]_\varrho$. L/σ_L

* This research was supported by grant A7877 of the Natural Sciences and Engineering Council of Canada and also by Deutsche Forschungsgemeinschaft.

is disjunctive in S/σ_L. As $W(L)$ is either empty or an ideal of S the set $W(L)/\sigma_L$ is either empty or consists of a single element, the zero of S/σ_L, and $W(L)/\sigma_L = W(L/\sigma_L)$.

We shall also have to consider the congruence σ_L^+ of L for $L \subseteqq S$ defined by

$$x\sigma_L^+ y \leftrightarrow (\forall u, v \in S: uxv \in L \leftrightarrow uyv \in L).$$

Clearly σ_L^+ and σ_L coincide on monoids. σ_L^+ is the principal congruence in the sense of Dubreil (see [5]). Whereas L is always a union of σ_L-classes, it need not be a union of σ_L^+-classes at all.

A semigroup is said to be *totally disjunctive* if each singleton subset of it is disjunctive. If there is no risk of confusion we do not distinguish between a singleton set and its element notationally.

3. Monoids with disjunctive identity — basic properties

The examples one would have in mind when starting to consider monoids with disjunctive identity would be:

— groups,

— the bicyclic monoid B,

— the polycyclic monoids.

We shall prove some propositions which seem to indicate that the monoids listed are somehow characteristic for the class of monoids with disjunctive identity, and we shall later give further classes of examples.

LEMMA 3.1. *Let S be a semigroup with a non-zero disjunctive element. Then S has a unique* [0-]*minimal ideal which contains all disjunctive elements of S.*

PROOF. Let $\emptyset \neq L \neq \{0\}$, $L \subseteqq S$ be disjunctive. If I is a non-zero ideal of S then $\emptyset \neq L \cap I \neq \{0\}$. Hence, if $L = \{x\}$ then x is in the intersection K of all non-zero ideals of S. Therefore $K = S^1 \times S^1$ and K is the unique [0-]minimal ideal of S. □

As an immediate consequence of 3.1 one observes that for x disjunctive in S either $W(x) = \emptyset$ or $W(x) = \{0\}$ if S has a zero. For $|W(x)| \leqq 1$ by disjunctivity. If $W(x) \neq \emptyset$ then $W(x) = \{0\}$ as $W(x)$ is an ideal of S.

PROPOSITION 3.2. *Let M be a monoid with disjunctive identity. Then M is a simple or* 0*-simple monoid.*

PROOF. By Lemma 3.1, the identity is in the unique [0-]minimal ideal of M. But $M1M = M$, so M is simple or 0-simple. □

Evidently, Proposition 3.2 generalises to monoids with quasidisjunctive identity. In that case $M/W(1)$ is a [0-]simple monoid. That not every simple monoid has its identity element disjunctive will be shown later in Example 6.2.

For a while we conjectured that Proposition 3.2 could be strengthened to say that a monoid with disjunctive identity is an inverse [0-]bisimple monoid. This would combined well with the following result due to B. M. Schein:

PROPOSITION 3.3 [15]. *Each non-zero element of a* [0-]*bisimple inverse semigroup is disjunctive.*[1]

Semigroups with each element disjunctive, called totally disjunctive in [10], were studied to a certain extent in [9, 10].

The above conjecture is "settled" by the following counterexamples.

EXAMPLE 3.4. Consider the Bruck-extension ([5, §8.5] and [10]) $BR(M, \theta)$ of the monoid $M=\{1, 0\}$ with $10=01=00=0$, $11=1$ and $\theta(M)=1$. M is totally disjunctive, and therefore $BR(M, \theta)$ is totally disjunctive [10]. However, $BR(M, \theta)$ is not [0-]bisimple as M is not bisimple [5].

The following example shows that even a simple monoid with disjunctive identity need not be inverse.

EXAMPLE 3.5. Let $X=\{x, y\}$ and consider the monoid M with the presentation $\langle X|x^2y=1\rangle$. Each element of M can be represented by a word w of the form

$$y^n xy^{m_1} xy^{m_2} \dots xy^{m_k} x^l$$

with $n, k, l \geqq 0, m_1, m_2, \dots, m_k \geqq 1$. Multiplying w by

$$x^l x^{2m_k-1} \dots x^{2m_2-1} x^{2m_1-1} x^{2n}$$

from the left and by y^l from the right yields 1; hence M is simple.

The $\mathcal{R}$- and $\mathcal{L}$-classes of w are

$$R_w = \{y^n xy^{m_1} \dots xy^{m_k} x^{\bar{l}} | \bar{l} \geqq 0\}$$

and

$$L_w = \{y^{\bar{n}} xy^{m_1} \dots xy^{m_k} x^l | \bar{n} \geqq 0\};$$

hence M is not bisimple; in fact the $\mathcal{D}$-class of w is

$$D_w = \{y^{\bar{n}} xy^{m_1} \dots xy^{m_k} x^{\bar{l}} | \bar{n}, \bar{l} \geqq 0\}.$$

M is regular, each $\mathcal{D}$-class containing idempotents of the form

$$y^{\bar{n}} xy^{m_1} \dots xy^{m_k} x^{\bar{l}} \quad \text{with} \quad \bar{l} = 2(\bar{n}+m_1+\dots+m_k)-k.$$

We now show that M is not an inverse monoid and that this also obtains for the monoid X^*/σ_L where L is the set of words in X^* which represent the identity of M. There are surjective homomorphisms $\varphi, \varphi_L, \varphi_M$,

$$\begin{tikzcd} X^* \arrow[rr, "\varphi_L"] \arrow[dr, "\varphi_M"'] & & X^*/\sigma_L \arrow[dl, "\varphi"] \\ & M & \end{tikzcd}$$

such that φ_L is given by $w \mapsto [w]_{\sigma_L}$, φ_M is defined by the presentation of M, and $\varphi(m)=\varphi_L \varphi_M^{-1}(m)$. In particular, $\varphi_M^{-1}(1)=L$. Clearly, yxx and xyx denote non-commuting idempotents in M. So M is not inverse.

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[1] The definition of disjunctivity is slightly different in [15]. However, it does not matter for this case.

On the other hand yxx and xyx are both not in L, and from $(x)(yxx)(xy) \underset{M}{=} xyx \notin L$, $(x)(xyx)(xy) \underset{M}{=} 1 \in L$ we see that $\varphi(yxx) \neq \varphi(xyx)$. Similarly, $\varphi((xyx)(yxx)) \neq \neq \varphi((yxx)(xyx)) = \varphi(yxx)$. Hence X^*/σ_L is not inverse; however, 1 is disjunctive in X^*/σ_L by the construction.

Even though the disjunctivity of its identity has quite severe implications for the structure of a monoid M it turns out that this property is still weak enough to allow for the following embeddability theorem.

PROPOSITION 3.6. *Each monoid can be embedded into a bisimple monoid with a disjunctive identity.*

PROOF. Consider the monoid $\mathcal{M}(A)$ of [5, § 8.6], that is, A is a set with $|A|$ an infinite regular cardinal number and $\mathcal{M}(A)$ is the set of mappings ξ of A into itself such that $|\xi(A)| = |A|$ and $|\xi^{-1}(a)| < |A|$ for all $a \in \xi(A)$. Each monoid can be embedded into $\mathcal{M}(A)$ for some A, and $\mathcal{M}(A)$ is a bisimple monoid [5] with identity ι, the identity mapping. It is sufficient to prove that ι is disjunctive in $\mathcal{M}(A)$. Consider $\delta, \xi \in \mathcal{M}(A)$, $\delta \neq \xi$, $\delta(a) \neq \xi(a)$ say. Let C be a cross-section of $\xi^{-1}\xi$ containing a. Let η be any bijection of A onto C and $\bar{a} = \eta^{-1}(a)$. Clearly $\eta \in \mathcal{M}(A)$. We now choose $\theta \in \mathcal{M}(A)$ such that $\theta\xi\eta = \iota$ and $\theta\xi\eta \neq \iota$. For $b \in \xi(A)$ let $\theta(b) = \eta^{-1}\xi^{-1}(b)$; for $b \notin \xi(A)$ let $\theta(b) = b'$ for some $b' \in A$ such that $b' \neq \bar{a}$.

Clearly θ is a mapping as $|\eta^{-1}\xi^{-1}(b)| = 1$ for all $b \in \xi(A)$, and $\theta \in \mathcal{M}(A)$. For $c \in A$ obviously $\theta\xi\eta(c) = c$ and $\theta\xi\eta = \iota$. Now consider $\theta\delta\eta(\bar{a}) = \theta\delta(a)$. If $\delta(a) \notin \xi(A)$ then $\theta\delta(a) = b' \neq a$ and $\theta\delta\eta \neq \iota$. Otherwise $\delta(a) \neq \xi(a)$ implies that $\xi^{-1}\delta(a) \cap \cap \xi^{-1}\xi(a) = \emptyset$ and thus $\bar{a} = \theta\xi\eta(a) = \eta^{-1}\xi^{-1}\xi(a) \notin \eta^{-1}\xi^{-1}\delta(a)$; again $\theta\delta\eta \neq \iota$. □

It is interesting to note that $\mathcal{M}(A)$ is in fact totally disjunctive; thus each monoid can be embedded into a totally disjunctive bisimple monoid.

4. The pre-image of 1

Let M be a monoid with disjunctive identity. We decsribe the properties of languages $L \subseteq X^*$ for an appropriate alphabet X such that $X^*/\sigma_L \cong M$ and $L/\sigma_L = 1$.

If $\bar{M}$ is a submonoid of a monoid M, then $\bar{M}$ is said to be *expansion* and *contraction closed* if it satisfies the following two conditions:

$(\mathscr{E}_0)$ $$\forall x, y, m \in \boldsymbol{M}(xy \in \bar{M},\ m \in \bar{M} \rightarrow xmy \in \bar{M}),$$

$(\mathscr{C}_0)$ $$\forall x, y, m \in \boldsymbol{M}(xmy \in \bar{M},\ m \in \bar{M} \rightarrow xy \in \bar{M}),$$

PROPOSITION 4.1. *Let M_1, M_2 be monoids, φ a homomorphism of M_1 onto M_2. If M_2 has a disjunctive identity then M_1 has an expansion and contraction closed submonoid $\bar{M}$, such that $\varphi^{-1}(1) = \bar{M}$. In this case then $M_2 \cong M_1/\sigma_{\bar{M}}$ with $\bar{M}/\sigma_{\bar{M}} = 1$.*

PROOF. Let 1 be disjunctive in M_2 and define $\bar{M} = \varphi^{-1}(1)$. Clearly $\bar{M}$ is a submonoid which satisfies $(\mathscr{E}_0)$ and $(\mathscr{C}_0)$. As 1 is disjunctive in M_2, the congruence defined by φ is the coarsest one saturating $\bar{M}$. Therefore it coincides with $\sigma_{\bar{M}}$. Hence the rest of the claim. □

The statement of Proposition 4.1 has a converse as follows:

PROPOSITION 4.2. *Let M be a monoid and $\bar{M}$ a submonoid of M which is expansion and contraction closed. Then $M/\sigma_{\bar{M}}$ is a monoid with disjunctive identity.*

PROOF. It is sufficient to prove that $\bar{M}$ is a $\sigma_{\bar{M}}$-class. If $(x, y)\in\bar{M}..m$ for $x, y\in M$, $m\in\bar{M}$ then $xmy\in\bar{M}$ and therefore $xy\in\bar{M}$ by $(\mathscr{C}_0)$. This implies $(x, y)\in\bar{M}..1$.

On the other hand, if $(x, y)\in\bar{M}..1$ for $x, y\in M$ then $xy\in\bar{M}$ and therefore $xmy\in\bar{M}$ for $m\in\bar{M}$ by $(\mathscr{E}_0)$. Thus $(x, y)\in\bar{M}..m$. This proves $1\sigma_{\bar{M}}m$ for all m in $\bar{M}$. □

The family $\mathscr{F}_0M$ of expansion and contraction closed submonoids of a monoid M is a complete lower semilattice with smallest element $\{1\}$ with respect to intersection. If $\{M_i|i\in I\}$ is any family of $M_i\in\mathscr{F}_0M$, then $\bar{M}=\bigcap_{i\in I} M_i$ is non-empty, as $1\in M_i$ for all M_i. If $xy\in\bar{M}$ and $m\in\bar{M}$ then $xy\in M_i$ and $m\in M_i$ for all i; by $(\mathscr{E}_0)$ $xmy\in M_i$ and therefore $xmy\in\bar{M}$. The proof for $(\mathscr{C}_0)$ is similar.

A submonoid $\bar{M}$ of a monoid M is called *left unitary* if $\bar{m}\in\bar{M}$, $\bar{m}m\in M$ implies $m\in\bar{M}$. By duality one defines *"right unitary"*. $\bar{M}$ is *unitary* if it is both left and right unitary. A code is called *biprefix* if it is both a prefix and suffix code.

PROPOSITION 4.3. *Each $\bar{M}\in\mathscr{F}_0M$ is a unitary submonoid of M.*

PROOF. Consider $m\in M$, $\bar{m}\in\bar{M}$ with $\bar{m}m=1\bar{m}m\in\bar{M}$. By $(\mathscr{C}_0)$ also $1m\in\bar{M}$. Thus $\bar{M}$ is left unitary. Dually, $m\bar{m}\in\bar{M}$ again implies $m\in\bar{M}$, and $\bar{M}$ is right unitary. □

COROLLARY 4.4 (see also [6]). *Each $M\in\mathscr{F}_0X^*$ is generated by the biprefix code $M^-\setminus(M^-)^2$ where $M^-=M\setminus\{1\}$.*

For codes the properties $(\mathscr{E}_0)$ and $(\mathscr{C}_0)$ are expressed as follows: Let X be an alphabet, $C\subseteq X^+$ a code. C is said to be *strong* [16, 17] if C satisfies the following two conditions:

$$(\mathscr{E}_1) \qquad \forall x, y, m\in X^+(xy\in C, m\in C \rightarrow xmy\in C^+),$$

$$(\mathscr{C}_1) \qquad \forall x, y, m\in X^*(xmy, m\in C^+ \rightarrow xy\in C^*).$$

Every strong code is a biprefix code. However, not every biprefix code is strong; for instance $C=\{ab\}$ over $X=\{a, b\}$ is not strong but of course biprefix.

As one might intuit, strong codes are closely related to expansion and contraction closed monoids.

PROPOSITION 4.4. *Each $M\in\mathscr{F}_0X^*$ is generated by a strong code; conversely, if $C\subseteq X^*$ is a strong code then $C^*\in\mathscr{F}_0X^*$.*

Finite strong codes were completely characterised in [16, 17]. Let X and $C\subseteq X^+$ be finite, and let $\operatorname{alph} C=\{u\in X|X^*uX^*\cap C\neq\emptyset\}$. Then, if C is a strong code, $C=(\operatorname{alph} C)^n$ for some n. The converse is obvious. Clearly $X^*/\sigma_{C^*}\cong Z_n$ or Z_n^0, where Z_n is the cyclic group of order n and Z_n^0 is Z_n with a zero element added, depending on whether $\operatorname{alph} C=X$ or $\operatorname{alph} C\neq X$.

As will be seen in the sequel, the situation is far more complicated if C is infinite.

EXAMPLE 4.5. Consider the context-free grammar $G=(V, X, P, \sigma)$, where $X=\{a, b\}$, $V=\{a, b, \sigma, \tau\}$ $P=\{\sigma\rightarrow a\tau b,\ \tau\rightarrow\tau\tau,\ \tau\rightarrow a\tau b,\ \tau\rightarrow 1\}$. Clearly, $L(G)$ satisfies $(\mathscr{E}_1)$ and $(\mathscr{C}_1)$. So $L(G)$ is a strong code, and $L(G)^*$ is the *Dyck language* [7]. $X^*/\sigma_{L(G)^*}$ is the bicyclic monoid with $L(G)^*/\sigma_{L(G)^*}=1$. We shall call $L(G)$ the *Dyck code over* $\{a, b\}$.

EXAMPLE 4.6. Consider $L=\{w|w\in X^*, |w|_a=|w|_b\}$ where $X=\{a, b\}$ and $|w|_a$, $|w_b|$ denote the number of a's and b's in w, respectively. L is a deterministic context-free language. X^*/σ_L is isomorphic with Z, the infinite cyclic group, and $L/\sigma_L=0$, the additive identity of Z. Thus L is generated by a strong code C, which by the above results of [16, 17] is not finite.

That expansion and contraction closed monoids or strong codes may be arbitrarily complex seems obvious; still we postpone a more precise statement to this extent to the next chapter. However, already at this stage, we should like to mention the example of a language $L\subseteqq\{a, b\}^*$ provided by [8] which is not context-free, but is the σ_L-class of 1:

EXAMPLE 4.7. Let M be the monoid generated by $X=\{a, b\}$ subject to the relation $ab^2a^2b=1$. Let L be the set of words equivalent to 1. Then $L/\sigma_L=1$ and M is an infinite group [8, 20].

5. Getting closer to the Dyck languages

Example 4.5, that is the Dyck language over $X=\{a, b\}$, suggests to strengthen the expansion and contraction conditions as follows:

$(\mathscr{E}_2)$ $$\forall x, y, m\in X^+(xy\in C, m\in C \to xmy\in C),$$

$(\mathscr{C}_2)$ $$\forall x, y, m\in X^*(xmy\in C, m\in C, xy\neq 1 \to xy\in C).$$

In addition we introduce the condition of *infix closure:*

$(\mathscr{I})$ $$\forall x, y, m\in X^*(xmy\in C, xy\in C, m\in X^+ \to m\in C).$$

Clearly, if C satisfies $(\mathscr{E}_2)$, $(\mathscr{C}_2)$ then it also satisfies $(\mathscr{E}_1)$, $(\mathscr{C}_1)$. The converse is not true in general. To see this take L as in Example 4.6. Then $ab\in C$, $ba\in C$ for C the code generating L, but $a(ba)b=(ab)^2\notin C$.

On the other hand, evidently the Dyck code over $\{a, b\}$ satisfies the conditions $(\mathscr{E}_2)$, $(\mathscr{C}_2)$. It is slightly more difficult to see that the code of Example 3.5 satisfies $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$, too; if L is the language of that example then the code C with $C^*=L$ is given by $C=xLxLy$ where L can also be characterised as the set of all words w over $X=\{x, y\}$ such that $|w|_x=2|w|_y$ and $|u|_x\geqq 2|u|_y$ for all prefixes u of w. This allows one to show that $(\mathscr{C}_2)$ and $(\mathscr{E}_2)$ obtain.

With $(\mathscr{E}_2)$ we associate a partial ordering $\underset{E}{\leqq}$ of X^* with respect to $C\subseteqq X^*$ as follows: Let $x, y\in X^*$. The relation $\underset{E}{\leqq}$ is the transitive closure of $\underset{E}{\leqq}'$ where $x\underset{E}{\leqq}'y$ if and only if

(a) $x, y\notin C$ and $x=y$

or

(b) $x=x_1x_2$, $x_1, x_2\in X^+$, $y=x_1mx_2$ for some $m\in C$.

Clearly,

$$\mathscr{E}_2 C = \{y | y \in X^*,\ x \underset{E}{\leqq} y \text{ for some } x \in C\}$$

is the smallest subset of X^* which contains C and satisfies $(\mathscr{E}_2)$. Let

$$\operatorname{Min}_E C = \{x | x \in C,\ y \underset{E}{\leqq} x \rightarrow y = x \text{ for } y \in C\}$$

be the set of minimal elements of C. Obviously, $\operatorname{Min}_E C$ is well-defined and non-empty for any $C \neq \emptyset, \{1\}$.

For instance, $\operatorname{Min}_E C = \{ab\}$ for the Dyck code C over $X = \{a, b\}$.

LEMMA 5.1. *If $C \subseteqq X^*$ satisfies $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$, then $\mathscr{E}_2 \operatorname{Min}_E C = C$.*

PROOF. $\mathscr{E}_2 \operatorname{Min}_E C \subseteqq C$ is obvious. Let $w \in C$. If $w \in \operatorname{Min}_E C$ then $w \in \mathscr{E}_2 \operatorname{Min}_E C$. Otherwise, $w = xmy$ with $xy \in C$, $x, y \in X^+$, $m \in C$. But $|xy| < |w|$, $|m| < |w|$. By induction $xy \in \mathscr{E}_2 \operatorname{Min}_E C$, $m \in \mathscr{E}_2 \operatorname{Min}_E C$. Thus $xmy = w \in \mathscr{E}_2 \operatorname{Min}_E C$. □

In terms of grammars, Lemma 5.1 can be expressed as follows: Let $C \subseteqq X^*$ satisfy $(\mathscr{E}_2)$. Then $C = L(G)$ for the following (generalized) context-free grammar $G_C = (V, \Sigma, P, \sigma)$:

$$V = \{\sigma, \tau\} \cup \Sigma,$$

$$P = \{\sigma \rightarrow w | w \in \operatorname{Min}_E C\} \cup \{\tau \rightarrow w | w \in \operatorname{Min}_E C\} \cup$$

$$\cup \{\sigma \rightarrow w_1 \tau w_2 | w_1, w_2 \in X^+, w_1 w_2 \in \operatorname{Min}_E C\} \cup$$

$$\cup \{\tau \rightarrow w_1 \tau w_2 | w_1, w_2 \in X^+, w_1 w_2 \in \operatorname{Min}_E C\} \cup \{\tau \rightarrow \tau\tau\}.$$

G_C is "generalized" in that P may be infinite if $\operatorname{Min}_E C$ is. Clearly, if $\operatorname{Min}_E C$ itself is context-free, then G_C can be made a context-free grammar with finitely many productions, and $L(G_C)$ is context-free.

The construction of G_C shows that if C happens not to be context-free then this fact is caused by $\operatorname{Min}_E C$ not being context-free.

PROPOSITION 5.2. *Let $C \subseteqq X^*$ satisfy $(\mathscr{E}_2)$. If $\operatorname{Min}_E C$ is generated by a type i grammar, $i = 0, 1, 2$, then also C is generated by a type i grammar.*

That $\operatorname{Min}_E C$ for a code C satisfying $(\mathscr{E}_2)$ need not be context-free can be seen in the following example:

EXAMPLE 5.3. Consider a subset C' of the set $C = \{a^n b a b^n | n \geqq 2\}$. As C is a subset of the Dyck code, which satisfies $(\mathscr{E}_2)$, also $\mathscr{E}_2 C'$ is a code. Clearly $\operatorname{Min}_E \mathscr{E}_2 C' = C'$ and C' can be chosen non-contextfree. Observe that the smallest subset of $\{a, b\}^*$ which contains C and satisfies both $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$ is the Dyck code itself. To be slightly more specific: Let $\hat{C}'$ be the smallest subset of $\{a, b\}^*$ which contains $C' \subseteqq C$ and satisfies $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$, and let

$$k = \gcd\{n - m | n > m,\ a^n bab^n \in C',\ a^m bab^m \in C'\}$$

if $|C'| > 1$. In this case

$$\operatorname{Min}_E \hat{C}' \subseteqq \{a^k b^k\} \cup \{a^n bab^n | a^n bab^n \in \hat{C}',\ n \leqq N\}$$

where

$$N = \min\{\bar{n} | \gcd\{n-m | \bar{n} \geqq n > m, a^n bab^n \in C', a^m bab^m \in C'\} = k\}$$

Thus $\mathrm{Min}_E \hat{C}'$ is finite and $\hat{C}'$ is context-free. This is trivially true for $|\hat{C}'|=1$ as well.

As property $(\mathscr{E}_2)$ implies $(\mathscr{E}_1)$, by Proposition 4.4 a code satisfying $(\mathscr{E}_2)$ is strong and hence biprefix. However, the following proposition shows that $(\mathscr{E}_2)$ is still far more restrictive.

PROPOSITION 5.4. *Let C be a code over X satisfying $(\mathscr{E}_2)$. Then for all $u \in X^+$ at least one of the sets $X^+u \cap C$ and $uX^+ \cap C$ is empty.*

PROOF. Suppose $uv \in C$, $wu \in C$ for some $v, w \in X^+$. Then, by $(\mathscr{E}_2)$, $w(uv)u = (wu)(vu) \in C$, contradicting the fact that C is a prefix code. □

This implies that all words of a code C satisfying $(\mathscr{E}_2)$ are primitive, that is, if $u^r \in C$ for some $u \in X^+$, $r \geqq 1$ then $r=1$. Such a code is antireflective [21], that is if $uv \in C$ then $vu \notin C$ for $u, v \in X^+$.

Observe that it follows from Proposition 5.4 as special cases that:

(1) the only code C over $X=\{a\}$ which satisfies $(\mathscr{E}_2)$ is $C=\{a\}$, and that,

(2) in a code C over $X=\{a, b\}$ satisfying $(\mathscr{E}_2)$ each word begins with a and ends on b or vice versa, or $C \subseteqq \{a, b\}$.

The second statement indicates that codes over $X=\{a, b\}$ and satisfying $(\mathscr{E}_2)$ have a bracket structure with right and left brackets distinguished similar to the case of the Dyck code.

PROPOSITION 5.5. *Let $C \subseteqq X^*$ be a code satisfying $(\mathscr{E}_2)$, $C \nsubseteq X$. Then C is not regular.*

PROOF. $C \nsubseteq X$ implies that C is infinite by $(\mathscr{E}_2)$. Suppose C is regular. Consider $w=xy \in C$, $|x|=1$, $|y|>0$. By $(\mathscr{E}_2)$, also $x^n y^n \in C$ for all $n \geqq 1$. By the pumping lemma for regular languages [7] $x^{mk+l} y^n \in C$ for some m, l with $m \geqq 1$, $l+m=n$, all k and n large enough. But then C is not a suffix code, a contradiction. □

PROPOSITION 5.6. *Let $C \subseteqq X^*$ be a strong code satisfying $(\mathscr{E}_2)$. Then the group of units of X^*/σ_{C^*} is trivial.*

PROOF. Let $M=X^*/\sigma_{C^*}$ and suppose that M has a non-trivial group of units. Equivalently, there are $u, v \in X^*$ satisfying $u, v \notin C^*$, $uv, vu \in C^*$. Let $|uv|$ be minimal with respect to this property. Hence u, v, uv, vu do not have any proper subword in C. Therefore $uv, vu \in C$, contradicting Proposition 5.4. □

As a simple interesting consequence of Proposition 5.6 we note that for a strong code C satisfying $(\mathscr{E}_2)$ and such that X^*/σ_{C^*} is bisimple or 0-bisimple, all groups of X^*/σ_{C^*} are trivial, that is, X^*/σ_{C^*} is *aperiodic*. One should also observe that the code generating the language L of Example 4.7 does not satisfy $(\mathscr{E}_2)$ as the corresponding monoid X^*/σ_L is in fact a non-trivial group by [8, 20]. Similarly, for the code generating the language $\{w \mid |w|_a = |w|_b\}$ over the alphabet $\{a, b\}$ the condition $(\mathscr{E}_2)$ does not obtain (see Example 4.6).

The following examples show that conditions $(\mathscr{E}_2)$, $(\mathscr{C}_2)$, and $(\mathscr{I})$ do not imply each other. We shall then prove that $(\mathscr{E}_2)$ and $(\mathscr{I})$ are contradictory for every non-trivial code C.

EXAMPLE 5.7. Consider the alphabet $X=\{a, b\}$. The code $\{a^2b^2, a^3b^3\}$ does not satisfy $(\mathscr{C}_2)$ not $(\mathscr{I})$ but satisfies $(\mathscr{E}_2)$. The code $\{a^nb^n|n\geqq 1\}\cup\{a^2bab^2\}$ is $(\mathscr{C}_2)$ but not $(\mathscr{I})$ nor $(\mathscr{E}_2)$. The code $\{a^nb^n|n\geqq 1\}\cup\{b^2aba^2\}$ is $(\mathscr{I})$ but neither $(\mathscr{C}_2)$ nor $(\mathscr{E}_2)$. Finally $\{a^nb^n|n\geqq 1\}$ is both $(\mathscr{C}_2)$ and $(\mathscr{I})$, but not $(\mathscr{E}_2)$.

PROPOSITION 5.8. *Let $C\subseteqq X^*$ be a code such that $C\nsubseteqq X$. Then C does not satisfy both $(\mathscr{E}_2)$ and $(\mathscr{I})$.*

PROOF. By Proposition 5.4, $a^n\notin C$ for $a\in X$, $n\geqq 2$. Hence by $C\nsubseteqq X$ there is $w\in C$ such that w contains at least two different symbols from X. Suppose

$$w = a_1^{n_1}a_2^{n_2}\dots a_k^{n_k},\quad a_i\neq a_{i+1},\ k\geqq 2.$$

By $(\mathscr{E}_2)$ we may assume that $k\geqq 4$ and $n_1, n_k\geqq 2$. Let $m,\overline{m}>0$ with $m+\overline{m}=n_1$. Then the word

$$a_1^m(a_1^{n_1}a_2^{n_2}\dots a_k^{n_k})a_1^{\overline{m}}a_2^{n_2}\dots a_k^{n_k}$$

is in C and is equal to

$$a_1^{n_1}(a_1^m a_2^{n_2}\dots a_k^{n_k}a_1^{\overline{m}})a_2^{n_2}\dots a_k^{n_k}.$$

By $(\mathscr{I})$ this implies that

$$a_1^m a_2^{n_2}\dots a_k^{n_k}a_1^{\overline{m}}\in C,$$

contradicting Proposition 5.4. □

The essence of Proposition 5.8 is, that a code satisfying $(\mathscr{I})$ cannot have simple "pumping" properties. A special case is treated in the following statement; recall that $L\subseteqq X^*$ is *regular-free* if no infinite subset of L is regular [18].

PROPOSITION 5.9. *A code C satisfying $(\mathscr{I})$ is regular-free.*

PROOF. Suppose C' is an infinite regular subset of C. By the pumping lemma for regular languages there is $u=xvy\in C'$ with $v\neq 1$ such that $xv^ny\in C'$ for all $n\geqq 0$. Hence $xy\in C'$ and by $(\mathscr{I})$ it follows that $v^n\in C$ for all $n\geqq 1$. Therefore C is not a code, a contradiction. □

The combination of $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$ is also quite restrictive as can be seen from the following example: Let $C\subseteqq X^*$ be a code satisfying $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$ where $\{a, b\}\subseteqq X$. If $a^mb^r, a^nb^s\in C$ with $m, r, n, s\geqq 1$, then $\frac{m}{r}=\frac{n}{s}$. For, by $(\mathscr{E}_2)$ one has $a^{mn}b^{rn}, a^{mn}b^{ms}\in C$. If $\frac{m}{r}\neq\frac{n}{s}$ then also $ms\neq nr$, $ms>rn$ without loss in generality, and C is not a prefix code, a contradiction. Thus, if C contains a^mb^r with m chosen to be minimal then

$$C\cap a^*b^* = \{a^{km}b^{kr}|k\in\mathbf{N}\}.$$

A finite strong maximal code C over the alphabet X is always equal to X^n for some n. No infinite code satisfying $(\mathscr{E}_2)$ can be maximal neither as a code nor as a prefix code nor as a suffix code. This implies, for instance, that the Dyck code D over $X=\{a, b\}$ is not maximal; this is, of course, obvious anyway, as $D\cup\{a\}$ is a suffix code. Similarly the code

$$C = \{b\}\cup\{ab^na|n\geqq 0\}$$

over $X=\{a, b\}$ which is given as an example of a maximal biprefix code in [14] cannot satisfy $(\mathscr{E}_2)$ as it is maximal even as a code; that $(\mathscr{E}_2)$ does not hold for this code can also be derived as a consequence of Proposition 5.6 as $X^*/\sigma_{C^*}\cong Z_2$, the cyclic group of order 2.

There is no obvious relationship between X^*/σ_C and X^*/σ_{C^*} even if C is a strong code. For instance, for C as in the last example, X^*/σ_C is $\mathscr{J}$-trivial and thus is aperiodic whereas X^*/σ_{C^*} is a non-trivial group. However, if C satisfies $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$ then there is a closer connection.

PROPOSITION 5.10. *Let C be a code satisfying both $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$. Then C is a σ_C-class and C^+ is contained in a σ_C^+-class in X^+/σ_C^+, that is the class of C itself.*

PROOF. Suppose that $u, v\in C$ and $(x, y)\in C..u$. If $x\neq 1\neq y$ then $xuy\in C$ implies xvy by $(\mathscr{C}_2)$ and $(\mathscr{E}_2)$. If $x=1$ or $y=1$ then $x=1=y$ as C is a prefix and suffix code and again $xuy\in C$ implies $xvy\in C$. Thus C is contained in a single σ_C-class which proves the first statement. For the second statement consider $c\in C$, $u\in C^+$ and $x, y\in X^+$. If $xcy\in C$ then $xuy\in C$ using $(\mathscr{C}_2)$ once and $(\mathscr{E}_2)$ possibly several times. Conversely if $xuy\in C$ then $xcy\in C$. Hence $u\sigma_C^+ c$ for all $u\in C^+$, $c\in C$. □

From the first part of Proposition 5.10 one may expect that the properties $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$ have a formal counterpart in X^*/σ_C. This is the contents of the next proposition:

PROPOSITION 5.11. *For a finitely generated monoid M the following properties are equivalent:*

(1) *There is a finite alphabet X and a code $C\subseteq X^+$ satisfying $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$ such that $X^*/\sigma_C\cong M$.*

(2) *There is an element $m\in M$ with the following three properties:*

(a) $mm_2=m=m_1m\to m_1=m_2=1$,

(b) $m=m_1m_2$ *with* $m_1\neq 1\neq m_2\to m_1mm_2=m$,

(c) $m_1mm_2=m$ *with* $m_1\neq 1\neq m_2\to m_1m_2=m$.

PROOF. Let C be given according to property (1), and let $m=C/\sigma_C$ by Proposition 5.10. C being a biprefix code yields (2a), whereas $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$ imply (2b) and (2c), respectively. Conversely, given M and $m\in M$ according to (2), let X be a set of generators of M and $\varphi\colon X^*\to M$ the homomorphism induced by the inclusion $X\subseteq M$. Consider $C=\varphi^{-1}(m)$. By (2a) C is a biprefix code, hence a code. Properties (2b) and (2c) imply $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$, respectively. □

Finally we observe, that if C is a strong code which satisfies $(\mathscr{E}_2)$ then C^* is very pure and C is a circular code [2]. A fortiori, this is true if C satisfies both $(\mathscr{E}_2)$ and $(\mathscr{C}_2)$.

6. Special monoid presentations

Let X be an alphabet (finite or infinite) and R a set of relations over X^*, that is, of equations of the form $u=v$ with $u, v\in X^*$. The pair $\langle X|R\rangle$ is the presentation of the monoid $M\cong X^*/\varrho_R$ where ϱ_R is the finest congruence on X^* such that $u\varrho_R v$ for all $u, v\in X^*$ with $u=v\in R$. Presentations $\langle X|R\rangle$ such that $u=v\in R$ implies that

$v=1$, the empty word, were called special in [1], trivial in [6], unitary in [3]. They will be referred to as special in the sequel.

The bicyclic and polycyclic monoids, the groups, the monoid $\langle x, y|x^2y=1\rangle$ of Example 3.5 and the monoid $\langle a, b|ab^2a^2b=1\rangle$ of [8] can serve as examples of monoids with special presentations. In general little is known about monoids that have a special presentation. From a slight generalization of Example 3.4 we shall see that there are even simple monoids which cannot be presented in this way.

LEMMA 6.1. *Let M be a monoid. M has a special presentation $\langle X|R\rangle$ and a disjunctive identity if and only if $\varrho_R=\sigma_L$ with L the ϱ_R-class of* 1.

PROOF. By definition $\varrho_R\subseteq\tau\subseteq\sigma_L$ for all congruences τ on X^* such that L is a τ-class. If 1 is disjunctive in M then M has no proper congruence $\hat{\tau}$ with 1 as a $\hat{\tau}$-class and thus $\varrho_R=\sigma_L$. On the other hand, if $\varrho_R=\sigma_L$, then 1, being the σ_L-class of L is disjunctive. □

Using Lemma 6.1 we can find a monoid with disjunctive identity which does not have any special presentation.

EXAMPLE 6.2. Consider the Bruck-extension $BR(M,\theta)$ of the monoid $M=\{1, z_1, z_2\}$, $z_iz_j=1z_i=z_i1=z_i$ for $i,j=1,2$, $11=1$ with $\theta(M)=1$. $BR(M,\theta)$ is a simple monoid with identity (0, 1, 0). However

$$(n, z_1, m)\,\sigma_{(0,1,0)}\,(n, z_2, m)$$

for all n, m. So the identity is not disjunctive. On the other hand, as the $\sigma_{(0,1,0)}$-class of the identity is just $\{(0,1,0)\}$ it follows that for any presentation $\langle X|R\rangle$ of M the ϱ_R-class L of 1 coincides with the σ_L-class. Thus, $M/\sigma_{(0,1,0)}$ does not have a special presentation. In fact, $M/\sigma_{(0,1,0)}$ is the monoid considered in Example 3.4. It can be presented by

$$\langle X|\{ab=1,\ zb=b,\ az=a,\ z^2=z\}\rangle$$

where $X=\{a, b, z\}$. Let ϱ be the congruence corresponding to this presentation. The ϱ-class of 1 is the set C^* where C is the smallest strong code containing the set

$$L=\{a^nz^ib^n|n\geqq 1,\ i\geqq 0\}\cup\{a^nb^n|n\geqq 1\}.$$

Each ϱ-class has a unique representative of the form b^na^m or b^nza^m for $n,m\geqq 0$.

Now consider the presentation $\langle X|\{u=1$ for $u\in L\}\rangle$ with its corresponding congruence ϱ'. By construction $\varrho'\subseteq\sigma_{C^*}$, and the ϱ'-class of 1 is again C^*. One then computes that the set $z^*(bz^*)^*(az^*)^*$ forms a cross-section through the set of ϱ'-classes. This may help in visualizing the gap between ϱ' and σ_{C^*}.

LEMMA 6.3. *Let M be a monoid with disjunctive identity. Then there is a special presentation $\langle X|R\rangle$ of a monoid M' and a surjective homomorphism $\varphi\colon M'\to M$ such that $\varphi^{-1}\varphi(1)=1$.*

PROOF. Let X be a set of generators of M and let $\psi\colon X^*\to M$ be the extension of the inclusion $X\subseteq M$. Then consider the presentation $\langle X|R\rangle$ with $R=\{w|\psi(w)=1\}$. Clearly $\varrho_R\subseteq\sigma_R$ which proves the lemma. □

A monoid homomorphism with the property that $\varphi^{-1}\varphi(1)=1$ will be called *identity separating*. Thus the Lemmas can be combined to yield:

PROPOSITION 6.4. *Let M be a monoid with disjunctive identity. Then M is the least identity separating homomorphic image of a monoid having a special presentation.*

We now proceed to characterize special presentations of simple monoids.

Let X be an alphabet (finite or infinite), and let $u, v \in X^+$. We say that *u meets v* if

(1) $$v \in X^* u X^*$$

or

(2) $$u = u_1\bar{u},\ v = \bar{u}v_1 \quad \text{for some} \quad \bar{u} \in X^+,\ u_1, v_1 \in X^*$$

or

(3) $$u = \bar{u}u_2,\ v = v_2\bar{u} \quad \text{for some} \quad \bar{u} \in X^+,\ u_2, v_2 \in X^*.$$

Let $L \subseteq X^*$. The word u is said to *meet L* if u meets some $v \in L$. Finally, if $\langle X|R\rangle$ is a special presentation, then u is said to *meet R* if u meets the set $L_R=\{v|v=1\in R\}$.

PROPOSITION 6.5. *A monoid M is simple if and only if it is the homomorphic image of a monoid M' having a special presentation $\langle X|R\rangle$ such that every $w \in X^+$ meets R.*

PROOF. Let X be any set of generators of M, φ_M the surjective homomorphism of X^* onto M which extends the inclusion $X \subseteq M$, and let $L=\varphi_M^{-1}(1)$, $R=\{w=1|w \in L\}$. Clearly, M is a homomorphic image of M' where M' is the monoid presented by $\langle X|R\rangle$. Consider $w \in X^*$. As M is simple there are $a, b \in M$ such that $a\varphi(w)b=1$ but then $\varphi^{-1}(a)w\varphi^{-1}(b) \subseteq R$ and w meets R.

For the converse observe that each homomorphic image of a simple monoid is also a simple monoid. It will therefore suffice to prove that a monoid M having a special presentation $\langle X|R\rangle$ such that every w meet R is simple. Let φ_R be the homomorphism of X^* onto M given by R. Suppose M is not simple. Then there exists $m \in M$ such that MmM does not contain the identity element. Choose $w \in X^*$ in such a way that w has minimal length and satisfies $1 \notin M\varphi_R(w)M$. Clearly, X^*wX^* does not contain a word v such that $(v=1) \in R$. As w meets R, however, there exists a relation $(v=1) \in R$ such that $w=\bar{w}u$, $v=u\bar{v}$ or $w=u\bar{w}$, $v=\bar{v}u$ for a non-empty word $u \in X^+$ and some $\bar{w}, \bar{w} \in X^*$. Suppose the former (the other case being the dual). Then $\varphi_R(w\bar{v})=\varphi_R(\bar{w})$ and $|\bar{w}|<|w|$. From the minimality of w it follows that there exist $a, b \in M$ such that $1=a\varphi_R(\bar{w})b=a\varphi_R(w)\varphi_R(\bar{v})b$, a contradiction! $\square$

A special presentation $\langle X|R\rangle$ is called *complete* if every word in X^* meets R. Let $L=\{w|w=1\in R\}$ if $\langle X|R\rangle$ is a special representation. A subset of X^* is *dense*, if it intersects every ideal of X^*. By Proposition 6.5 the gap between ϱ_R and σ_L can be narrowed down a bit.

PROPOSITION 6.6. *Let $C \subseteq X^*$ be a strong code with C^* dense. Then X^*/σ_{C^*}, the syntactic monoid of C^*, is the least identity separating homomorphic image of a monoid M which has a complete special presentation.*

We conclude this section with a theorem relating the groups of units of X^*/ϱ_R and X^*/ϱ_L to each other.

PROPOSITION 6.7. *Let $\langle X|R\rangle$ be a special monoid presentation, $L=[1]_{\varrho_R}$, and let G_R, G_L be the groups of units of X^*/ϱ_R and X^*/σ_L, respectively. Then $G_R\cong G_L$. To be more precise: $[u]_{\sigma_L}\in G_L$ implies $[u]_{\sigma_L}=[u]_{\varrho_R}$, and $[u]_{\varrho_R}\in G_R$ if and only if $[u]_{\sigma_L}\in G_L$.*

PROOF. Clearly, when passing from ϱ_R to σ_L, G_R is mapped into G_L. Now consider $u, v\in X^*$ such that $[u]_{\sigma_L}, [v]_{\sigma_L}$, are inverses of each other in G_L, that is,

$$\forall x, y\in X^*\colon\ xuvy\in L \leftrightarrow xvuy\in L \leftrightarrow xy\in L.$$

Letting $xy=1$, this implies $uv, vu\in L$, hence $uv\varrho_R vu\varrho_R 1$. Thus $[u]_{\varrho_R}, [v]_{\varrho_R}$ are units in G_R and inverses of each other. This proves that G_R is mapped onto G_L and that no element outside G_R is mapped into G_L. Finally consider $u, v, w\in X^*$ such that

$$[u]_{\sigma_L}[v]_{\sigma_L}=[v]_{\sigma_L}[u]_{\sigma_L}=[1]_{\sigma_L}$$

and

$$[u]_{\sigma_L}[w]_{\sigma_L}=[w]_{\sigma_L}[u]_{\sigma_L}=[1]_{\sigma_L}.$$

As above it follows that $uv, vu, uw, wu\in L$. That is $uv\varrho_R vu\varrho_R uw\varrho_R wu\varrho_R$ and $[u]_{\varrho_R}$, $[v]_{\varrho_R}, [w]_{\varrho_R}$ are units in G_R with $[u]_{\varrho_R}, [w]_{\varrho_R}$ inverses of $[u]_{\varrho_R}$. Therefore $[v]_{\varrho_R}=[w]_{\varrho_R}$. Thus $[v]_{\sigma_L}=[w]_{\sigma_L}$ and $[v]_{\sigma_L}\in G_L$ implies $[v]_{\varrho_R}=[w]_{\varrho_R}\in G_R$. □

As an immediate consequence one has the following:

COROLLARY 6.8. *Let $\langle X|R\rangle$ be a special monoid presentation of a group and let $L=[1]_{\sigma_R}$. Then $\sigma_L=\varrho_R$.*

This corollary can be applied to Example 4.7, that is the language L corresponding to $\langle\{a, b\}|ab^2a^2b=1\rangle$ in order to show that the systactic monoid X^*/σ_L of L in fact coincides with the monoid having this presentation.

Finally one should note that by a result of [4] for special monoid presentations with a single relation $w=1$, there is a linear time algorithm that given input w will determine whether the group of units of the monoid presented is trivial or non-trivially finite or infinite.

The main application of Proposition 6.7 in the context of this paper should be seen in combining it with Proposition 5.6.

COROLLARY 6.9. *Let $C\subseteqq X^*$ be a strong code satisfying $(\mathscr{E}_2)$. Then the group of units of the monoid presented by $\langle X|\{w=1, w\in C\}\rangle$ is trivial.*

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(Received July 12, 1983; revised March 20, 1985)

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Acta Math. Hung.
47 (3—4) (1986), 313—319.

ON COMPACTLY DOMINATED SPACES

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1. Introduction

A topological space X is said to be *compactly dominated* (or Morita k-space) [2] if there exists a family $\{K_\alpha\}_{\alpha\in\Delta}$ of compact subsets of X which dominates the space X ([9], p. 14), i.e.,

(i) $X=\bigcup\limits_{\alpha\in\Delta} K_\alpha$, and for each subfamily $\{K_\beta|\beta\in\Delta'\}$ of $\{K_\alpha\}_{\alpha\in\Delta}$, $\bigcup\limits_{\beta\in\Delta'} K_\beta$ is closed in X,

(ii) the topology of the subspace $Y=\bigcup\limits_{\beta\in\Delta'} K_\beta$ is the weak topology defined by the family of subspaces $\{K_\beta\}_{\beta\in\Delta'}$.

The need for studying such spaces stems from the following considerations: Let X be a topological space for which some dimension function (for example covering dimension dim) is defined. Can we determine $\dim X$ if we know $\dim K$ for each compact subset K of X? Particularly, we want to know under what conditions on X, if any, does there exist a compact subset K of X such that $\dim X=\dim K$? The answer to this question, of course, depends on the space X as well as the dimension function under consideration. For instance, in the case of covering dimension dim, the answer, in general, is no. To see this let us consider the Tychonoff plank $\Omega=[0, w_1]\times\times[0, w_0]-\{(w_1, w_0)\}$. Any compact subset K of Ω is zero-dimensional whereas $\dim\Omega=1$ ([9], p. 162). This shows that even in a locally compact space X, $\dim X$ may not be determined by dimensions of its compact subsets. On the otherhand let us consider the sheaf theoretic cohomological dimension $\dim_L$ ([1], [4], [5]) over a given ring L. For this dimension function, however, if X is any locally compact space, then $\dim_L X$ is simply $\sup\limits_K \{\dim_L K\}$ where K runs over all compact subsets of X: this of course follows from the local property of $\dim_L$ [1]. A locally compact space is clearly compactly generated, but if we take X to be compactly generated and consider the same sheaf theoretic cohomological dimension $\dim_L$, which is known to be better-behaved than covering dimension dim, then $\dim_L X$ need not be determined by $\dim_L K$ where K varies over all compact subsets of X. For example, consider the Knaster—Kuratowski space X ([8], p. 54). Then X is a one dimensional metric space and hence $\dim_Z X=\dim X=1$. X is compactly generated, but there cannot exists a compact subset K of X such that $\dim_Z K=1$. This is because X is totally disconnected and hence any compact subset K of X is also totally disconnected. Consequently, $\dim_Z K==\dim K=0$. This shows, by the way, that in general the weak topology sum theorem is valid neither for the covering dimension dim nor for the cohomological dimension $\dim_Z$. However, if the topology of X is the weak topology defined by a family of closed subsets of X in the sense of Morita ([8], p. 215), then it has been proved that the sum theorem is valid for the cohomological (for all locally paracompact spaces) as well as the covering dimension (for normal spaces) [3]. This leads us to ask the following question: *What are those spaces X whose topology is the weak topology defined by a family of compact subsets of X in the sense of Morita?*

Using the terminology of Pears [9] we will call such spaces *compactly dominated spaces*, and the objective of this paper is to investigate various interesting topological properties of such spaces. Unless mentioned otherwise all of our spaces under consideration are assumed to be Hausdorff and so by a compact space we mean a compact Hausdorff space.

2. Examples and characterizations

A topological space X is said to be dominated by a closed covering $\{A_\alpha\}_{\alpha\in\Delta}$ ([9], p. 14) if for each subset Δ' of Δ, $\bigcup\limits_{\alpha\in\Delta'} A_\alpha$ is closed in X and the subspace $\bigcup\limits_{\alpha\in\Delta'} A_\alpha$ of X has the weak topology with respect to the covering $\{A_\alpha\}_{\alpha\in\Delta'}$. Thus we have the following:

Definition 2.1. A space X is said to be compactly dominated, if it is dominated by a covering consisting of compact subspaces of X.

It is clear that any space X having a locally finite covering by a family of compact subsets is a compactly dominated space. Every compact space is trivially compactly dominated. Also note that a discrete space X is dominated by the family $\{\{x\}|x\in X\}$ consisting of singletons and so is compactly dominated. Conversely, any space X dominated by a family consisting of finite sets must be obviously discrete. For the sake of completeness we include the proofs of the following two examples which show that the concept of compactly dominated spaces is a generalization of compact spaces as well as CW-complexes.

Example 2.2. *A CW-complex is compactly dominated.*

Proof. We shall prove that for any CW-complex K its geometric realization $|K|$ (which is also a CW-complex) is compactly dominated. For, let $S=\{S_\alpha\}$ be all cells of K. Then by definition, $|K|=\bigcup\limits_\alpha |S_\alpha|$ and $|K|$ has the weak topology defined by $\{|S_\alpha|\}$. Let $Y=\bigcup\limits_{\beta\in\Delta} |S_\beta|$, $\{S_\beta\}_{\beta\in\Delta}$ is a subfamily of $\{S_\alpha\}$. Then Y is closed in $|K|$ since for any $|S_\alpha|$, $Y\cap|S_\alpha|$ is a finite union of faces of S_α, which is again closed in $|S_\alpha|$. Hence Y is closed in $|K|$. Next, let $F\subset Y$ such that $F\cap|S_\beta|$ is closed in $|S_\beta|$. For any α either $|S_\alpha|$ is a face of $|S_\beta|$ for some β or else $|S_\alpha|$ is disjoint from $|S_\beta|$. In any case $F\cap S_\beta$ is closed in $|S_\beta|$ implies that $F\cap|S_\alpha|$ is closed in $|S_\alpha|$. Hence F is closed in $|K|$. Thus $|K|$ is a compactly dominated space. Q.E.D.

Example 2.3. *A locally compact paracompact space X is compactly dominated.*

Proof. Since X is locally compact there exists ([6], p. 238) an open covering $\{V(x)\}_{x\in X}$ of X, where each $V(x)$ is relatively compact. Now, X being paracompact, there exists a locally finite closed covering $\{F_\alpha\}$ of X which refines $\{V(x)\}_{x\in X}$ and each F_α is compact. Therefore X is compactly dominated. Q.E.D.

Theorem 2.4. *A separable topologically complete space X which is not locally compact at any of its points cannot be a compactly dominated space.*

PROOF. Since X is topologically complete, it is of second category by Baire's category theorem ([6], p. 299). Also, X is not locally compact at any of its points implies that any compact subset of X is nowhere dense. Suppose X is compactly dominated. Then there exists a covering $\{A_\alpha\}_{\alpha\in\Delta}$ which dominates X and where each A_α is compact. Since X is separable, there exists a countable dense subset, say $A=\{x_1, x_2, \dots, x_n, \dots\}$, of X. Then clearly for every $x_i\in A$ there exists an $\alpha_i\in\Delta$ such that $x_i\in A_{\alpha_i}$. Thus $A\subset \bigcup_{\alpha\in\Delta} A_\alpha$. Since A is a dense subset of X we have $X=\bar{A}\subset \overline{\bigcup_{\alpha\in\Delta} A_\alpha} = \bigcup_{\alpha\in\Delta} A_\alpha$ since $\bigcup_{\alpha\in\Delta} A_\alpha$ is closed in X. Therefore $X=\bigcup_{\alpha\in\Delta} A_\alpha$, i.e., X is the countable union of nowhere dense sets of X. This is a contradiction since X is of second category. Hence X cannot be compactly dominated. Q.E.D.

The above theorem gives us the following corollary giving various counter-examples for compactly dominated spaces.

COROLLARY 2.5. *None of the following spaces is compactly dominated;*

(i) *the set P of all irrationals with the inherited subspaces topology from* $\mathbf{R}$,

(ii) *the countable product $\mathbf{R}^\omega$ of real lines,*

(iii) *the space $C[0, 1]$ of all real valued continuous functions on the unit interval I with the metric topology generated by the sup norm,*

(iv) *any infinite dimensional separable Hilbert space H.*

The following result turns out to be quite interesting:

PROPOSITION 2.6. *The set $\mathbf{Q}$ of rationals with the inherited subspace topology from $\mathbf{R}$ is not compactly dominated.*

PROOF. Suppose $\mathbf{Q}$ is compactly dominated. Therefore there is a covering $\{A_\alpha\}_{\alpha\in\Delta}$ of compact subsets of $\mathbf{Q}$ which dominates $\mathbf{Q}$.

Since $\mathbf{Q}$ is countable, there is a subfamily, say $\{A_{\alpha_i}\}_{i=1}^\infty$, of the above family which covers $\mathbf{Q}$ and dominates $\mathbf{Q}$. We shall show that this is not possible. For, consider any point x of Q. Consider a neighbourhood base at x of the form $B_1^x\supset B_2^x\supset\dots$ $\dots\supset B_n^x\supset\dots$. Select a point $x_1\in B_1^x - A_{\alpha_1}$ (since A_{α_1} is nowhere dense). Select for each $n\geqq 1$, $x_n\in B_n^x-(A_{\alpha_1}\cup\dots\cup A_{\alpha_n})$ (since $A_{\alpha_1}\cup\dots\cup A_{\alpha_n}$ is nowhere dense). So that we have a sequence $\langle x_n\rangle$ which converges to x. Therefore the infinite set $\{x_n\}_{n=1}^\infty$ is not closed. But as the set $\{x_n\}_{n=1}^\infty$ is finite in each A_{α_n}, it should be closed. This contradiction establishes the claimed result. Q.E.D.

Theorems 2.4 and 2.6 are essentially due to A. K. Desai (private communication).

The following proposition gives some necessary conditions for a space to be compactly dominated. However, none of them is sufficient as shown by Corollary 2.5.

PROPOSITION 2.7. *Suppose X is a compactly dominated space. Then*

(i) *X is paracompact,*

(ii) *X is a k-space.*

PROOF. (i) Since X is compactly dominated, there exists a covering $\{F_\alpha\}_{\alpha\in\Delta}$ which dominates X and where each F_α is compact. Since each F_α is compact, it is paracompact. Thus it follows from ([9], p. 65) that X is paracompact.

(ii) To show that X is k-space, it is enough to show ([6], p. 248) that X is the quotient of a locally compact space. Since X is compactly dominated there exists a covering $\{F_\alpha\}_{\alpha\in\Delta}$ dominating X and where each F_α is compact. Let $\sum_{\alpha\in\Delta} F_\alpha$ denote the free union of the family $\{F_\alpha\}_{\alpha\in\Delta}$, where each F_α is compact. $\sum_{\alpha\in\Delta} F_\alpha$ is evidently locally compact. By ([6], p. 132) we know that X is the quotient space of the free union of its compact subspaces, i.e., $X=\sum_{\alpha\in\Delta} F_\alpha/\sim$. Hence X is a k-space. Q.E.D.

The above proposition shows that a σ-compact space need not be compactly dominated:

EXAMPLE 2.8. *The following spaces are σ-compact, but not paracompact, and hence cannot be compactly dominated:*

(i) *The Arens Square* ([10], p. 98).
(ii) *The Double origin topology* ([10], p. 92).
(iii) *The nested interval topology* ([10], p. 76).
(iv) *The countable particular point topology* ([10], p. 44).
(v) *Roy's lattice space* ([10], p. 143).

We note that local compactness is not necessary for compactly dominated spaces. In fact, a compactly dominated space need not be locally compact at any of its points as the following example shows.

EXAMPLE 2.9. Let $S^\infty=\bigcup_{n=1}^{\infty} S^n$, where $S^1\subset S^2\subset\ldots\subset S^n\subset\ldots$ with inductive limit topology. Then S^∞ is a CW-complex and hence compactly dominated, but S^∞ is not locally compact at any of its points.

Now we come to a characterization of spaces dominated by closed covering.

THEOREM 2.10. *A space X is dominated by a closed covering $\{A_\alpha\}_{\alpha\in\Delta}$ if and only if X is a continuous closed image of a free union $\sum_{\alpha\in\Delta} A'_\alpha$, where $A'_\alpha\cong A_\alpha$, for every $\alpha\in\Delta$.*

PROOF. Suppose X is dominated by a closed covering $\{A_\alpha\}_{\alpha\in\Delta}$. For each $\alpha\in\Delta$ we pick a space A'_α so that $A'_\alpha\cong A_\alpha$. Let $h_\alpha\colon A'_\alpha\to A_\alpha$ be a homeomorphism. Consider the space $\sum_{\alpha\in\Delta} A'_\alpha$ and define a map $h\colon \sum_{\alpha\in\Delta} A'_\alpha\to X$ so that $h|A'_\alpha=h_\alpha$. Clearly h is continuous and onto. We claim that h is closed. Let F be the closed subset of the free union $\sum_{\alpha\in\Delta} A'_\alpha$. Let $\Delta'=\{\alpha\in\Delta|F\cap A'_\alpha\neq\emptyset\}$. Then $F=\bigcup_{\alpha\in\Delta'}(F\cap A'_\alpha)$ implies that $h(F)=\bigcup_{\alpha\in\Delta'} h(F\cap A'_\alpha)=\bigcup_{\alpha\in\Delta'} h(F)\cap A_\alpha$. Now $h(F)\cap A_\alpha=h_\alpha(F\cap A'_\alpha)$ is closed in A_α, since $F\cap A'_\alpha$ is closed in A'_α and $h|A'_\alpha$ is a homeomorphism. Thus we find that $h(F)\cap A_\alpha$ is closed in A_α for each $\alpha\in\Delta'$. Since $\bigcup_{\alpha\in\Delta'} A_\alpha$ is closed in X and has the weak topology defined by the covering $\{A_\alpha\}_{\alpha\in\Delta'}$, it follows that $h(F)$ is closed in $\bigcup_{\alpha\in\Delta'} A_\alpha$. But $\bigcup_{\alpha\in\Delta'} A_\alpha$ being closed in X implies that $h(F)$ is closed in X. Conversely, suppose $v\colon \sum_{\alpha\in\Delta} A'_\alpha\to X$ is a continuous closed surjection, where $v|A_\alpha\colon A'_\alpha\to A_\alpha$ is a homeomorphism. Let Δ' be a subset of Δ. Since $\bigcup_{\alpha\in\Delta'} A'_\alpha$ is closed in $\sum_{\alpha\in\Delta} A'_\alpha$, we find that $\bigcup_{\alpha\in\Delta'} A_\alpha=v\big(\bigcup_{\alpha\in\Delta'} A'_\alpha\big)$ is closed in X. Next we shall prove that $\bigcup_{\alpha\in\Delta'} A_\alpha$ has the weak topology defined by the family $\{A_\alpha|\alpha\in\Delta'\}$. Let G be a subset of $\bigcup_{\alpha\in\Delta'} A_\alpha$

such that $G\cap A_\alpha$ is closed in A_α, for every $\alpha\in\Delta'$. Then $G=\bigcup_{\alpha\in\Delta'}(G\cap A_\alpha)$. Let $G'_\alpha\subset A'_\alpha$ which is homeomorphic to $G\cap A_\alpha$. Then G'_α is closed in A'_α, which means $\bigcup_{\alpha\in\Delta'} G'_\alpha$ is closed in $\sum_{\alpha\in\Delta'} A'_\alpha$. Now ν being closed we find that $\nu(\bigcup_{\alpha\in\Delta} G'_\alpha)= =\bigcup_{\alpha\in\Delta'}(G\cap A_\alpha)=G$ must be closed in $\bigcup_{\alpha\in\Delta'} A_\alpha$. Thus $\bigcup_{\alpha\in\Delta'} A_\alpha$ has the weak topology defined by the family $\{A_\alpha|\alpha\in\Delta'\}$. Hence X is dominated by the closed covering $\{A_\alpha\}_{\alpha\in\Delta}$. Q.E.D.

The results of the following corollary are well known ([9], p. 65, 27, 34). The above theorem yields their simpler alternative proofs.

COROLLARY 2.11. *If a space X is dominated by a closed covering $\{A_\alpha|\alpha\in\Delta\}$, where each A_α is paracompact (normal or perfectly normal, respectively), then X is paracompact (normal or perfectly normal, respectively).*

The following corollary gives a characterization of compactly dominated spaces.

COROLLARY 2.12. *A space X is compactly dominated if and only if X is a continuous closed image of a free union of compact spaces.*

3. Natural questions

In this section we determine the behaviour of compactly dominated spaces with respect to subspaces, products and continuous images.

THEOREM 3.1. (a) *A closed subspace of a compactly dominated space is compactly dominated.*

(b) *An arbitrary subspace of a compactly dominated space need not be compactly dominated.*

PROOF. (a) Suppose A is a closed subspace of a compactly dominated space X. Since X is compactly dominated, there exists a covering $\{F_\alpha\}_{\alpha\in\Delta}$ of compact subsets, which dominate X. For every $\alpha\in\Delta$, $A\cap F_\alpha$ is clearly compact. We claim that A is dominated by the family $\{A\cap F_\alpha\}_{\alpha\in\Delta}$. Obviously $A=\bigcup_{\alpha\in\Delta} A\cap F_\alpha$ and for any $\Delta'\subset\Delta$, $\bigcup_{\alpha\in\Delta'}(A\cap F_\alpha)=A\cap(\bigcup_{\alpha\in\Delta'} F_\alpha)$ is closed in $\bigcup_{\alpha\in\Delta}(A\cap F_\alpha)$. Further let $B\subset\bigcup_{\alpha\in\Delta'} A\cap F_\alpha$ such that $B\cap(A\cap F_\alpha)$ is closed in $A\cap F_\alpha$ for every $\alpha\in\Delta'$, then we shall show that B is closed in $\bigcup_{\alpha\in\Delta'} A\cap F_\alpha$. Since $B\cap(A\cap F_\alpha)=B\cap F_\alpha$ for every $\alpha\in\Delta'$, it follows that $B=\bigcup_{\alpha\in\Delta'}(B\cap F_\alpha)$ is closed in $\bigcup_{\alpha\in\Delta'}(A\cap F_\alpha)$. Hence A is compactly dominated. Q.E.D.

(b) See the following example:

EXAMPLE 3.2. Consider the set **P** (respectively **Q**) of all irrationals (respectively rationals) with the inherited subspace topology from **R**. **R** being locally compact and paracompact is compactly dominated by Example 2.3 whereas **P** (respectively **Q**) is not compactly dominated by Corollary 2.5 (respectively Proposition 2.6).

THEOREM 3.3. (a) *A continuous closed image of a compactly dominated space X is compactly dominated.*

(b) *A quotient space of a compactly dominated space, however, need not be compactly dominated.*

PROOF. (a) Suppose X is compactly dominated and $f\colon X \to Y$ is continuous closed surjection. By Corollary 2.12, X is a continuous closed image of a free union say ΣZ_α, of compact spaces. Let $v\colon \Sigma Z_\alpha \to X$ be such a map. Then clearly v is continuous closed surjection. Hence by Corollary 2.12 again Y must be compactly dominated. Q.E.D.

(b) See the following example:

EXAMPLE 3.4. P, the set of all irrationals being metrizable is a k-space. Let $\{C_\alpha\}$ be the set of all compact subsets of p. Then by ([6], p. 132) $\sum_\alpha C_\alpha/\sim$ is homeomorphic to P. Here $\sum_\alpha C_\alpha$ is a compactly dominated, but by Corollary 2.5, P is not comcactly dominated. Hence the quotient of a compactly dominated space need not be pompactly dominated.

REMARKS 3.5. We note that the quotient space of a k-space is always a k-space ([6], p. 248), but in contrast with this the quotient of a compactly dominated space need not be compactly dominated, by the above example.

Using the above theorem we now obtain another useful characterization of compactly dominated spaces.

COROLLARY 3.6. *A space X is compactly dominated if and only if X is a continuous closed image of a locally compact paracompact space.*

PROOF. If X is compactly dominated, then by Corollary 2.12 it is immediate that X is a continuous closed image of a locally compact paracompact space. Conversely, a locally compact paracompact space is compactly dominated by Example 2.3, and by Theorem 3.3 a continuous closed image of such a space must be compactly dominated. Q.E.D.

THEOREM 3.7. (a) *Let X be a compactly dominated space. If Y is locally compact paracompact, then $X \times Y$ is compactly dominated.*

(b) *The product of two compactly dominated spaces need not be compactly dominated.*

PROOF. (a) Since X is compactly dominated, there exists a covering $\{A_\alpha\}_{\alpha \in \Delta}$ consisting of compact subspaces of X which dominates X. Y being locally compact implies there exists a covering $\{U_y\}$ of Y consisting of relatively compact open sets. Thus we get a covering $\{V_y\}$ of Y consisting of compact subspaces of Y, where $V_y = \overline{U}_y$ for every $y \in Y$. Again Y being paracompact implies there exists a locally finite closed refinement $\{B_\beta\}$ of $\{V_y\}$. Thus by ([7] Thm. 1) it follows that $X \times Y$ is dominated by the covering $\{A_\alpha \times B_\beta\}$ consisting of compact subspaces of $X \times Y$. Hence $X \times Y$ is compactly dominated. Q.E.D.

(b) See the following example:

EXAMPLE 3.8. Let $\mathscr{M}$ be the set of all maps of $\mathbf{Z}^+$ into itself. Consider the family

$\{u_\varphi|\varphi\in\mathcal{M}\}$ in one-to-one correspondence with the set $\mathcal{M}$ and the family $\{v_n|n\in\mathbf{Z}^+\}$. Let $V=\bigoplus_{\varphi\in\mathcal{M}}\mathbf{R}_{u_\varphi}$ and $W=\bigoplus_{n\in\mathbf{Z}^+}\mathbf{R}_{v_n}$. We take the finite topology on both V and W. Thus V and W having finite topology are CW-complexes and hence are compactly dominated. Consider $P=\left\{\left(\frac{1}{\varphi(n)}u_\varphi, \frac{1}{\varphi(n)}v_n\right)\middle|\varphi\in\mathcal{M},\ n\in\mathbf{Z}^+\right\}\subset V\times W$. We claim that $V\times W$ is not a k-space. Suppose the contrary. Then P must be closed in $V\times W$. We shall show that P is not closed in the product $V\times W$. Otherwise CP (the complement of P) would be open, and since the origin $o\in CP$, there would be a basic neighbourhood $U_1\times U_2$ with $o\in U_1\times U_2\subset CP$. Since U_1, U_2 are open in V and W, respectively, for each φ and each n, there would be an a_φ and a_n such that $\{\lambda u_\varphi|o\leqq\lambda<a_\varphi\}\subset U_1$, $\{\mu v_n|o\leqq\mu<a_n\}\subset U_2$. Let $\bar\varphi\in\mathcal{M}$ be the map $\bar\varphi(n)=\max\left[n, \frac{1}{a_n}\right]+1$ and find $\bar n$ with $\bar\varphi(\bar n)>\frac{1}{a_{\bar\varphi}}$. Then $\left(\frac{1}{\bar\varphi(\bar n)}u_{\bar\varphi}, \frac{1}{\bar\varphi(\bar n)}v_{\bar n}\right)\in U_1\times U_2$, but it is not in CP, a contradiction. Thus the product $V\times W$ is not even a k-space and hence it is not compactly dominated by Proposition 2.7.

We conlcude with the following.

REMARK 3.9. *The product of two continuous closed maps need not be closed.* Let us consider the spaces V and W of Example 3.8. Then V and W are compactly dominated. We apply Corollary 2.12 to find $v_1\colon \sum_{\alpha\in\Delta}A_\alpha\to V$ and $v_2\colon \sum_{\beta\in\Delta'}B_\beta\to W$ as two continuous closed surjections, where $\sum_{\alpha\in\Delta}A_\alpha$ and $\sum_{\beta\in\Delta'}B_\beta$ are free unions of compact spaces. Then, clearly, $\sum_{(\alpha,\beta)\in\Delta\times\Delta'}(A_\alpha\times B_\beta)$ is a free union of compact spaces. But $v_1\times v_2\colon \sum_{\alpha\in\Delta}A_\alpha\times\sum_{\beta\in\Delta'}B_\beta\to V\times W$ is not closed, for, otherwise $V\times W$ would be a compactly dominated space, in contrast with Example 3.8.

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(Received July 12, 1983; revised July 3, 1984)

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Acta Math. Hung.
47 (3—4) (1986), 321—323.

COMPARING ALMOST-DISJOINT FAMILIES

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0. Introduction

In this paper families of almost-disjoint, countably infinite sets will be considered, i.e. families of type $\mathcal{A}=\langle A_\alpha\colon \alpha\in\varkappa\rangle$ where $|A_\alpha|=\omega$ and $|A_\alpha\cap A_\beta|$ is finite whenever $\alpha<\beta<\varkappa$. One of the early results in the theory of almost-disjoint sets is the following theorem of E. W. Miller [3]: if $\mathcal{A}=\langle A_\alpha\colon \alpha<\varkappa\rangle$ is a family of infinite sets and instead of almost-disjointness the stronger $|A_\alpha\cap A_\beta|\leqq n$ holds with a fixed $n<\omega$, then $\mathcal{A}$ has the property B, i.e. there is a coloring of the union set with two colors and without a monochromatic A. This is no longer true for almost-disjoint families.

The role of property B in connection with this result is a bit misleading, because, as it was later discovered, much stronger properties hold under the Miller condition, e.g. $\mathcal{A}$ has a transversal – a one-to-one choice function. That the transversal property really implies property B, at least for families of infinite sets, is proved in [1]. The strongest property to date gained from the Miller condition is the following: $\mathcal{A}$ is essentially disjoint, i.e. there are finite sets $\langle B_\alpha\colon \alpha<\varkappa\rangle$ such that the sets $\{A_\alpha-B_\alpha\colon \alpha<\varkappa\}$ are disjoint (see [2]). This property is obviously stronger than any of property B and the transversal property.

As these properties are proved under a relatively mild restriction on the size of pairwise intersections, one may wonder whether one of them or any other nontrivial property at all can be recognised from the matrix $\{|A_\alpha\cap A_\beta|\colon \alpha<\beta<\varkappa\}$. Call two enumerated almost-disjoint families $\mathcal{A}=\{A_\alpha\colon \alpha<\varkappa\}$ and $\mathcal{B}=\{B_\alpha\colon \alpha<\varkappa\}$ similar if $|A_\alpha\cap A_\beta|=|B_\alpha\cap B_\beta|$ holds whenever $\alpha<\beta<\varkappa$. The question therefore is: which properties are similarity-invariant? It is easy to see that property B and the transversal property are not, as we can simultaneously enlarge each set with two (or infinitely many) new elements without changing any $A_\alpha\cap A_\beta$. However, essential-disjointness is similarity-invariant. Even more is true, namely a certain monotonicity holds: if $\mathcal{A}=\langle A_\alpha\colon \alpha<\varkappa\rangle$ and $\mathcal{B}=\langle B_\alpha\colon \alpha<\varkappa\rangle$ are both almost-disjoint families, $|A_\alpha\cap A_\beta|\leqq|B_\alpha\cap B_\beta|$ holds for $\alpha<\beta<\varkappa$ and B is essentially disjoint then so is $\mathcal{A}$. The proof of this statement is the main content of the present note. We will derive from it, as an immediate corollary, Miller's theorem.

In this paper the standard set theory notation is used.

1. The main argument

DEFINITION 1. A system $\mathcal{A}=\langle A_\alpha\colon \alpha<\varkappa\rangle$ is almost-disjoint, if $|A_\alpha\cap A_\beta|<\omega=|A_\alpha|, |A_\beta|$ holds for $\alpha<\beta<\varkappa$.

DEFINITION 2. A system $\mathcal{A}=\langle A_\alpha\colon \alpha<\varkappa\rangle$ is essentially disjoint if and only if

$|A_\alpha|=\omega$ for $\alpha<\varkappa$ and there exist finite sets $\{B_\alpha: \alpha<\varkappa\}$ such that the sets $\{A_\alpha-B_\alpha: \alpha<\varkappa\}$ are pairwise disjoint.

LEMMA 1. *Assume* $\varkappa>\omega$ *regular*, $\mathscr{A}=\langle A_\alpha: \alpha<\varkappa\rangle$ *is an almost-disjoint system, and for every* $\mathscr{A}'\subset\mathscr{A}$ *with* $|\mathscr{A}'|<|\mathscr{A}|$, $\mathscr{A}'$ *is essentially disjoint. Then* $\mathscr{A}$ *is essentially disjoint if and only if*

$$(1.1)\qquad L=\{\alpha<\varkappa: \text{ there is a } \beta(\alpha)\geqq\alpha \text{ with } |A_{\beta(\alpha)}\cap\bigcup_{\xi<\alpha}A_\xi|=\omega\}$$

is non-stationary.

PROOF. Assume first that L is stationary. If $\{B_\alpha: \alpha<\varkappa\}$ witnesses essential disjointness, for $\alpha\in L$ $(A_{\beta(\alpha)}-B_{\beta(\alpha)})\cup\bigcup_{\xi<\alpha}A_\xi$ is non-empty, choose an element x_α from it. It is easy to see that $\beta(\alpha)$ is one-to-one on a stationary set $L'\subset L$, for these α's the points $\{x_\alpha: \alpha\in L'\}$ are different. If $\alpha\in L'$, $x_\alpha\in A_{\gamma(\alpha)}$ for a suitable $\gamma(\alpha)<\alpha$, by the pressing-down lemma $\gamma(\alpha)=\gamma$ for a stationary $L''\subset L'$. But then $\{x_\alpha: \gamma(\alpha)=\gamma\}$ would be $\varkappa$ different elements of A_γ!

Assume now that L is non-stationary. As $0\in L$, we can choose a closed unbounded $C\subseteqq\varkappa-L$ with $0\in C$. Let $\{c_\alpha: \alpha<\varkappa\}$ be the monotone enumeration of C. By hypothesis, each of the systems $\{A_\xi: c_\alpha\leqq\xi<c_{\alpha+1}\}$ is essentially disjoint, let $\{B(\alpha,\xi): \xi\in[c_\alpha, c_{\alpha+1})\}$ witness this. For $\xi\in[c_\alpha, c_{\alpha+1})$, $D_\xi=A_\xi\cap\{\cup A_\gamma: \gamma<c_\alpha\}$ is finite, as $c_\alpha\in L$. Put, for $\xi\in[c_\alpha, c_{\alpha+1})$, $B_\xi=B(\alpha,\xi)\cup D_\xi$. Then the sets $\{A_\xi-B_\xi: \xi<\varkappa\}$ are pairwise disjoint.

LEMMA 2. *Assume that* $\varkappa>\omega$ *is regular*, $\mathscr{A}=\langle A_\alpha: \alpha<\varkappa\rangle$ *is an almost-disjoint system and for every* $\mathscr{A}'\subset\mathscr{A}$ *with* $|\mathscr{A}'|<|\mathscr{A}|$, $\mathscr{A}'$ *is essentially disjoint. Then* $\mathscr{A}$ *is essentially disjoint if and only if*

$$(1.2)\qquad N=\{\alpha<\varkappa: \text{ there is a } \beta(\alpha)\geqq\alpha \text{ with } \sup_{\gamma<\alpha}|A_{\beta(\alpha)}\cap A_\gamma|=\omega\}$$

is nonstationary.

PROOF. As $N\subseteqq L$, the only if part follows by Lemma 1.

To prove the other direction, assume that N is non-stationary, but L is stationary. Choose a closed, unbounded $C\subseteqq\varkappa-N$, increasingly enumerated as $\{c_\alpha: \alpha<\varkappa\}$, $c_0=0$. For every $\xi\in[c_\alpha, c_{\alpha+1})$, the number

$$(1.3)\qquad n(\xi)=\sup_{\gamma<C_\alpha}|A_\xi\cap A_\gamma|$$

is finite, by the definition of N. If $c_\alpha\in L$ choose an element $\beta(\alpha)\geqq c_\alpha$ with $|A_{\beta(\alpha)}\cap(\bigcup_{\gamma<c_\alpha}A_\gamma)|\geqq n(\beta(\alpha))+1$. With a successive application of the pressing-down lemma we can find a stationary subset of $L-N$, say L' such that the ordinals $\{\beta(\alpha): \alpha\in L'\}$ are different, $n(\beta(\alpha))=n$, and there is a $\gamma<\varkappa$ such that $|A_{\beta(\alpha)}\cap(\bigcup_{\xi<\gamma}A_\xi)|\geqq n+1$. As the set $[\bigcup_{\xi<\gamma}A_\xi]^{n+1}$ has cardinality less than $\varkappa$, there is an unbounded set $L''\subset L'$ such that $A_{\beta(\alpha)}\cap(\bigcup_{\xi<\gamma}A_\xi)$ contains the same $(n+1)$-element set for $\alpha\in L''$. Choose $\tau,\tau'\in L''$ with $\tau<c_\alpha<\tau'$ where c_α is in C. Then, by (1.3) $|A_\tau\cap A_{\tau'}|\leqq n$ but, as $\tau,\tau'\in L''$, they contain the same $(n+1)$-element subset, a contradiction.

2. Applications

LEMMA 3. *If* $\varkappa > \mathrm{cf}(\varkappa)$, $\mathcal{A} = \langle A_\alpha : \alpha < \varkappa \rangle$ *is a family such that every* $\mathcal{A}' \subset \mathcal{A}$ *with* $|\mathcal{A}'| < |\mathcal{A}|$ *is essentially disjoint, then* $\mathcal{A}$ *is essentially disjoint, as well.*

PROOF. See [2], Proposition 5.

THEOREM. *Assume that* $\mathcal{A} = \langle A_\alpha : \alpha < \varkappa \rangle$ *and* $\mathcal{B} = \langle B_\alpha : \alpha < \varkappa \rangle$ *are almost-disjoint systems. If* $|A_\alpha \cap A_\beta| \leqq |B_\alpha \cap B_\beta|$ *for* $\alpha < \beta < \varkappa$ *and* $\mathcal{B}$ *is essentially disjoint then* $\mathcal{A}$ *is essentially disjoint, too.*

PROOF. By induction on $\lambda \leqq \varkappa$ we prove that every subsystem of $\mathcal{A}$ with cardinality $\leqq \lambda$ is essentially disjoint. For $\lambda \leqq \omega$ this follows from almost-disjointness, see Proposition 1.b in [2]. If the statement is proved for every $\lambda' < \lambda$ and $\{A_\alpha : \alpha \in X\}$ is a subfamily of size λ, we use Lemma 3 or Lemma 2 according to whether λ is singular or regular. If λ is regular, take $\{B_\alpha : \alpha \in X\}$. By re-ordering X in order-type λ, we get two families $\mathcal{A}' = \{A'_\alpha : \alpha < \lambda\}$ and $\mathcal{B}' = \{B'_\alpha : \alpha < \lambda\}$ with the properties that $\mathcal{B}'$ is essentially disjoint, every subsystem of $\mathcal{A}'$ with cardinality less than λ is essentially disjoint and $|A'_\alpha \cap A'_\beta| \leqq |B'_\alpha \cap B'_\beta|$ for $\alpha < \beta < \lambda$. By this, if $\alpha \in N(\mathcal{A}')$ then $\alpha \in N(\mathcal{B}')$ applying (1.2). So $N(\mathcal{A}') \subseteq N(\mathcal{B}')$ and Lemma 2 gives that $\mathcal{A}'$ is essentially disjoint. If λ is singular, $\{A_\alpha : \alpha \in X\}$ is essentially disjoint by Lemma 3.

COROLLARY (Miller [3], Komjáth [2]). *If* $n < \omega$, $\mathcal{A} = \langle A_\alpha : \alpha < \varkappa \rangle$ *is an almost-disjoint system with* $|A_\alpha \cap A_\beta| \leqq n$ *for* $\alpha < \beta < \varkappa$, *then* $\mathcal{A}$ *is essentially disjoint.*

PROOF. By our Theorem, it is enough to find an essentially disjoint $\{B_\alpha : \alpha < \varkappa\}$ with $|B_\alpha \cap B_\beta| = n$ $(\alpha < \beta < \varkappa)$. For this, take the system $\{X \cup Y_\alpha : \alpha < \varkappa\}$ where $|X| = n$, $|Y_\alpha| = \omega$, and the sets $\{X, Y_\alpha : \alpha < \varkappa\}$ are all disjoint.

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(Received July 20, 1983)

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Acta Math. Hung.
47 (3—4) (1986), 325—332.

THE STRUCTURE OF INEFFABILITY PROPERTIES OF $P_\varkappa\lambda$

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Notation and basic facts

Unless we specify otherwise, $\varkappa$ denotes an uncountable regular cardinal, and λ a cardinal $\geqq\varkappa$. For any such pair, $P_\varkappa\lambda$ denotes the set $\{x\subset\lambda\colon |x|<\varkappa\}$, and $\lambda^{<\varkappa}$ is the cardinality of this set.

The basic combinatorial notions are defined here for $P_\varkappa\lambda$ as in Jech [11]. For any $x\in P_\varkappa\lambda$, $\hat{x}$ denotes the set $\{y\in P_\varkappa\lambda\colon x\subset y\}$. $X\subset P_\varkappa\lambda$ is said to be *unbounded* iff $(\forall x\in P_\varkappa\lambda)(X\cap\hat{x}\neq 0)$, and $I_{\varkappa\lambda}$ denotes the *ideal of not unbounded subsets* of $P_\varkappa\lambda$. In the sequel, an "ideal on $P_\varkappa\lambda$" is always "a proper, non-principal, $\varkappa$-complete ideal on $P_\varkappa\lambda$ extending $I_{\varkappa\lambda}$" unless we specify otherwise. Further, for any ideal I on $P_\varkappa\lambda$, I^+ denotes the set $\{X\subset P_\varkappa\lambda\colon X\notin I\}$, and I^* the filter dual to I.

$C\subset P_\varkappa\lambda$ is said to be *closed* in $P_\varkappa\lambda$ iff $(\forall X\subset C)(X$ is a chain of length$<\varkappa\Rightarrow\Rightarrow\bigcup X\in C)$, and is called a *cub* iff it is both closed and unbounded. Further, $S\subset P_\varkappa\lambda$ is said to be *stationary* in $P_\varkappa\lambda$ iff $S\cap C\neq 0$ for every cub $C\subset P_\varkappa\lambda$. Finally, $NS_{\varkappa\lambda}$ denotes the non-stationary ideal on $P_\varkappa\lambda$, and $CF_{\varkappa\lambda}$ its dual.

The *diagonal union* $\nabla(X_\alpha\colon \alpha<\lambda)$ of a λ-sequence $(X_\alpha\colon \alpha<\lambda)$ of subsets of $P_\varkappa\lambda$ is defined by $\nabla(X_\alpha\colon \alpha<\lambda)=\{x\in P_\varkappa\lambda\colon (\exists\alpha\in x)(x\in X_\alpha)\}$, and for any ideal I on $P_\varkappa\lambda$, ∇I is the set defined by $\nabla I=\{X\subset P_\varkappa\lambda\colon (\exists\,(X_\alpha\colon \alpha<\lambda)\in{}^\lambda I)\,(X=\nabla(X_\alpha\colon \alpha<\lambda))\}$. It is easy to see that ∇I is a (not necessarily proper) ideal on $P_\varkappa\lambda$ extending I.

An ideal I on $P_\varkappa\lambda$ is said to be *normal* iff $\nabla I=I$, equivalently, iff every $f\colon P_\varkappa\lambda\to\lambda$ which is regressive on a set in I^+ (i.e. has the property $\{x\in P_\varkappa\lambda\colon f(x)\in x\}\in I^+$) is constant on a set in I^+. Jech proved in [11] that $NS_{\varkappa\lambda}$ is normal, and we proved in [5] that $\nabla I_{\varkappa\lambda}\subset\nabla\nabla I_{\varkappa\lambda}=NS_{\varkappa\lambda}$ and hence that $NS_{\varkappa\lambda}$ is the *smallest* normal ideal on $P_\varkappa\lambda$.

0. Introduction

In [11] Jech provided the following $P_\varkappa\lambda$ generalizations of Jensen's notions (see [12]) of ineffability and almost ineffability.

0.1. Definition. For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda\geqq\varkappa$, $\varkappa$ is said to be

(1) *λ-ineffable* iff for every $(A_x\colon x\in P_\varkappa\lambda)$ such that $(\forall x\in P_\varkappa\lambda)(A_x\subset x)$,

$$(\exists A\subset\lambda)(H=\{x\in P_\varkappa\lambda\colon A_x=A\cap x\}\in NS^+_{\varkappa\lambda}),$$

[1] The results in this paper were first announced in [6], and are included in the author's Ph. D. dissertation (McMaster University, 1981) written under the direction of Donald H. Pelletier to whom the author is grateful.

and

(2) *almost λ-ineffable* iff for every $(A_x: x\in P_\varkappa\lambda)$ such that

$$(\forall x\in P_\varkappa\lambda)(A_x\subset x),\ (\exists A\subset\lambda)\big(H=\{x\in P_\varkappa\lambda:\ A_x=A\cap x\}\in I^+_{\varkappa\lambda}\big).$$

Magidor [13] subsequently proved that $\varkappa$ is supercompact iff $\varkappa$ is λ-ineffable for every $\lambda\geqq\varkappa$. In 3.3 below we use our results to improve this result of Magidor by showing that $\varkappa$ is supercompact iff $\varkappa$ is *almost* λ-ineffable for every $\lambda\geqq\varkappa$.

DiPrisco and Zwicker [9] defined mild λ-ineffability, and showed that it characterizes strong compactness in the same way; i.e. that $\varkappa$ is strongly compact iff $\varkappa$ is mildly λ-ineffable for every $\lambda\geqq\varkappa$. The definition we give in 0.2 below is easily seen to be equivalent to the one given in [9].

0.2. Definition. For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda\geqq\varkappa$, $\varkappa$ is said to be *mildly λ-ineffable* iff for any $(A_x: x\in P_\varkappa\lambda)$ such that $(\forall x\in P_\varkappa\lambda)$ $(A_x\subset x)$, $(\exists A\subset\lambda)(\forall x\in P_\varkappa\lambda)\big(H_x=\{y\in\hat{x}:\ A_y\cap x=A\cap x\}\neq 0\big)$.

We study this notion of DiPrisco and Zwicker in [7] and [8] where we find that it is a $P_\varkappa\lambda$ generalization of weak compactness in the sense of some of the latter's familiar characterizations.

Baumgartner [1], [2] showed that many small large cardinal properties can be viewed as properties of normal ideals on $\varkappa$. For instance, he showed in [1] that $\varkappa$ is almost ineffable iff there is a normal ideal $NAIn_\varkappa$ on $\varkappa$ such that for every $X\subset\varkappa$, $X\in NAI\imath^+_\varkappa$ iff for any $(A_\alpha: \alpha\in X)$ such that $(\forall\alpha\in X)(A_\alpha\subset\alpha)$, $(\exists A\subset\varkappa)\big(H=\{\alpha\in X:$ $A_\alpha=A\cap\alpha\}\in I^+_\varkappa\big)$ where $I_\varkappa$ is the ideal of size $<\varkappa$ subsets of $\varkappa$.

In Section 1 of this paper we give natural ideal-theoretic characterizations of λ-ineffability, almost λ-ineffability and mild λ-ineffability. Whereas we find that the ideals characterizting λ-ineffability and almost λ-ineffability are normal, the one characterizing mild λ-ineffability is not normal. Thus we came to regard mild λ-ineffability as an "ideal-theoretically weak" $P_\varkappa\lambda$ generalization of weak compactness, and sought an "ideal-theoretically stronger" $P_\varkappa\lambda$ generalization of weak compactness between mild λ-ineffability and almost λ-ineffability. We define such a notion in Section 3 below, and study it further in Section 3 and in [8].

Our definition of this new ineffability property of $P_\varkappa\lambda$ was motivated by some work of Shelah [15] together with a result of our own (1.3 below). A perusal of Shelah's paper suggests the following definition together with the formulation of his result that succeeds it.

0.3. Definition. For any uncountable regular cardinal $\varkappa$, $X\subset\varkappa$ is said to have the *Shelah property* iff for any $(f_\alpha: \alpha\in X)$ such that $(\forall\alpha\in X)(f_\alpha: \alpha\to\alpha)$, $(\exists f: \varkappa\to\varkappa)$ $(\forall\alpha<\varkappa)\big(\{\beta\in[\alpha,\varkappa)\cap X:\ f_\beta{\restriction}\alpha=f{\restriction}\alpha\}\neq 0\big)$ where $[\alpha,\varkappa)=\{\beta<\varkappa:\ \alpha\leqq\varkappa\}$. Further, define the set $NSh_\varkappa$ by $NSh_\varkappa=\{X\subset\varkappa:\ X$ does *not* have the Shelah property$\}$.

Shelah proved that $\varkappa$ is weakly compact iff $\varkappa$ has the Shelah property, and that if $\varkappa$ has the Shelah property, then $NSh_\varkappa$ is a normal ideal on $\varkappa$.

We conclude this section by noting that $\varkappa$ is easily seen to have the Shelah property iff $\varkappa$ satisfies "Baumgartner's principle" where the latter is defined in 0.4 below. This definition is easily seen to be equivalent to the version appearing in Erdős et al. [10].

0.4. Definition. An uncountable regular cardinal $\varkappa$ is said to satisfy *Baumgartner's principle* iff for any $\varkappa$-sequence $(g_\alpha:\ \alpha<\varkappa)$ of regressive functions on $\varkappa$, $(\exists f: \varkappa\to\varkappa)(\forall\alpha<\varkappa)\big(\{\beta\in[\alpha,\varkappa):\ (\forall\xi<\alpha)\big(g_\xi(\beta)=f(\xi)\big)\}\neq 0\big)$.

1. The ideals $NIn_{\varkappa\lambda}$, $NAIn_{\varkappa\lambda}$ and $NMIn_{\varkappa\lambda}$

In this section we give ideal-theoretic characterizations of λ-ineffability, almost λ-ineffability and mild λ-ineffability.

1.1. Definition. For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda\geqq\varkappa$, say that $X\subset P_\varkappa\lambda$ is

(1) *λ-ineffable* iff for any $(A_x:\ x\in X)$ such that $(\forall x\in X)(A_x\subset x)$,

$$(\exists A\subset\lambda)(H=\{x\in X:\ A_x=A\cap x\}\in NS^+_{\varkappa\lambda}),$$

(2) *almost λ-ineffable* iff for any $(A_x:\ x\in X)$ such that $(\forall x\in X)(A_x\subset x)$,

$$(\exists A\subset\lambda)(H=\{x\in X:\ A_x=A\cap x\}\in I^+_{\varkappa\lambda}),$$

(3) *mildly λ-ineffable* iff for any $(A_x:\ x\in X)$ such that $(\forall x\in X)(A_x\subset x)$,

$$(\exists A\subset\lambda)(\forall x\in P_\varkappa\lambda)(H_x=\{y\in X\cap\hat{x}:\ A_y\cap x=A\cap x\}\neq 0).$$

Notice that the condition given in the conclusion of (1) can be replaced by $(\exists H\in\mathcal{P}(X)\cap NS^+_{\varkappa\lambda})(\forall x,y\in H)(x\subset y\Rightarrow A_x=A_y\cap x)$. Similarly, the condition given in the conclusion of (2) can be replaced by

$$(\exists H\in\mathcal{P}(X)\cap I^+_{\varkappa\lambda})(\forall x,y\in H)(x\subset y\Rightarrow A_x=A_y\cap x).$$

Now define the sets $NIn_{\varkappa\lambda}$, $NAIn_{\varkappa\lambda}$, $NMIn_{\varkappa\lambda}$ by

$$NIn_{\varkappa\lambda}=\{X\subset P_\varkappa\lambda:\ X \text{ is } \textit{not}\ \lambda\text{-ineffable}\},$$

$$NAIn_{\varkappa\lambda}=\{X\subset P_\varkappa\lambda:\ X \text{ is } \textit{not} \text{ almost } \lambda\text{-ineffable}\},$$

$$NMIn_{\varkappa\lambda}=\{X\subset P_\varkappa\lambda:\ X \text{ is } \textit{not} \text{ mildly } \lambda\text{-ineffable}\}.$$

It is clear that $I_{\varkappa\lambda}\subset NMIn_{\varkappa\lambda}\subset NAIn_{\varkappa\lambda}\subset NIn_{\varkappa\lambda}$. Moreover, it is clear that $\varkappa$ is λ-ineffable iff $P_\varkappa\lambda\notin NIn_{\varkappa\lambda}$, $\varkappa$ is almost λ-ineffable iff $P_\varkappa\lambda\notin NAIn_{\varkappa\lambda}$, and $\varkappa$ is mildly λ-ineffable iff $P_\varkappa\lambda\notin NMIn_{\varkappa\lambda}$.

Note that easy arguments (e.g. see [7], [8]) show that if $\varkappa$ has any one of these ineffability properties at level λ for some $\lambda\geqq\varkappa$, then $\varkappa$ is weakly compact and has that ineffability property at every level $\gamma\in[\varkappa,\lambda]$. Moreover, it is easy to see that $\varkappa$ is ineffable iff $\varkappa$ is $\varkappa$-ineffable, that $\varkappa$ is almost ineffable iff $\varkappa$ is almost $\varkappa$-ineffable, and that $\varkappa$ is weakly compact iff $\varkappa$ is mildly $\varkappa$-ineffable.

1.2. Theorem. *For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda\geqq\varkappa$,*
(1) $\varkappa$ is λ-ineffable iff $NIn_{\varkappa\lambda}$ is a normal ideal on $P_\varkappa\lambda$, and
(2) $\varkappa$ is almost λ-ineffable iff $NAIn_{\varkappa\lambda}$ is a normal ideal on $P_\varkappa\lambda$.

Proof. We will just prove (2); (1) follows by a similar but even simpler argument. The reverse implication of (2) is clear, so it remains to prove the forward one.

The assumption that $\varkappa$ is almost λ-ineffable guarantees that $NAIn_{\varkappa\lambda}$ is proper.

Also, it is easy to see that $I_{\varkappa\lambda} \subset NAIn_{\varkappa\lambda}$ and that $(\forall X, Y \subset P_\varkappa\lambda)(X \subset Y \& Y \in NAIn_{\varkappa\lambda} \Rightarrow \Rightarrow X \in NAIn_{\varkappa\lambda})$.

Now pick $\delta > \varkappa$ and $\{X_\nu : \nu < \delta\} \subset NAIn_{\varkappa\lambda}$. For each $\nu < \delta$, let $(A_x^\nu : x \in X_\nu)$ witness $X_\nu \in NAIn_{\varkappa\lambda}$. Set $X = \bigcup \{X_\nu : \nu < \delta\}$ and suppose by way of contradiction that $X \notin NAIn_{\varkappa\lambda}$. For each $x \in X$, let $\nu(x)$ be the least $\nu < \delta$ such that $x \in X_\nu$, and set $A_x = A_x^{\nu(x)}$. Now let $A \subset \lambda$ be such that $H = \{x \in X : A_x = A \cap x\} \in I_{\varkappa\lambda}^+$. The $\varkappa$-completeness of $I_{\varkappa\lambda}$ guarantees that $(\exists \nu < \delta)(H \cap X_\nu \in I_{\varkappa\lambda}^+)$. Let γ be the least ordinal $< \delta$ such that $H \cap X_\gamma \in I_{\varkappa\lambda}^+$. Note that $(\forall x \in H \cap X_\gamma)(\nu(x) \leqq \gamma)$. Then the minimality of γ together with the minimality of the $\nu(x)$'s imply that $\{x \in X_\gamma : \nu(x) = \gamma\} \in I_{\varkappa\lambda}^+$ thus contradicting $X_\gamma \in NAIn_{\varkappa\lambda}$.

We next show that $NS_{\varkappa\lambda} \subset NAIn_{\varkappa\lambda}$ and then use this fact to prove that $NAIn_{\varkappa\lambda}$ is normal. Pick $X \in NS_{\varkappa\lambda}$, and (by [14]) let $f: X \to \lambda \times \lambda$ be such that $(\forall x \in X)$ $(f(x) \in x \times x)$ and $(\forall \alpha, \beta < \lambda)(f^{-1}(\{(\alpha, \beta)\}) \in I_{\varkappa\lambda})$. For each $x \in X$ set $f(x) = = (\alpha_x, \beta_x) \in x \times x$, and define $A_x \subset x$ by $A_x = \bigcup f(x) = \{\alpha_x, \beta_x\}$. Now notice that $(\forall x \in X)(\forall y \in X \cap \hat{x})(A_x = A_y \cap x \Rightarrow y \in f^{-1}(\{(\alpha_x, \beta_x)\}) \cup f^{-1}(\{(\beta_x, \alpha_x)\}))$. Thus we have that $(\forall x \in X)(\{y \in X \cap \hat{x} : A_x = A_y \cap x\} \in I_{\varkappa\lambda})$, so $X \in NAIn_{\varkappa\lambda}$.

Now pick $X \in NAIn_{\varkappa\lambda}^+$, let $p: \lambda \times \lambda \rightarrowtail \lambda$, and set $C_p = \{x \in P_\varkappa\lambda : p''(x \times x) \subset x\}$. Since C_p is cub in $P_\varkappa\lambda$, it follows by the previous paragraph that $X \cap C_p \in NAIn_{\varkappa\lambda}^+$. We will show that $X \cap C_p \notin \nabla NAIn_{\varkappa\lambda}$; it will then follow that $X \notin \nabla NAIn_{\varkappa\lambda}$. Suppose by way of contradiction that $X \cap C_p \in \nabla NAIn_{\varkappa\lambda}$, and let $f: X \cap C_p \to \lambda$ be $NAIn_{\varkappa\lambda}$-small and regressive on $X \cap C_p$. For each $\alpha < \lambda$, let $(A_x^\alpha : x \in f^{-1}(\{\alpha\}))$ witness $f^{-1}(\{\alpha\}) \in NAIn_{\varkappa\lambda}$. Then define $(A_x : x \in C_p \cap X)$ by $A_x = \{p(\xi, f(x)) : \xi \in A_x^{f(x)}\}$, and let $H \subset X \cap C_p$ be such that $H \in I_{\varkappa\lambda}^+$ and $(\forall x, y \in H)(x \subset y \Rightarrow A_x = A_y \cap x)$. Notice that $A_x = A_y \cap x \Rightarrow f(x) = f(y)$. This shows that $(\exists \alpha < \lambda)(\forall x \in H)(f(x) = \alpha)$, and hence that $(\exists \alpha < \lambda)(H \subset f^{-1}(\{\alpha\}))$ thus contradicting $f^{-1}(\{\alpha\}) \in NAIn_{\varkappa\lambda}$. □

As an easy consequence of the preceding theorem together with the fact that $NS_{\varkappa\lambda}$ is the smallest normal ideal on $P_\varkappa\lambda$ [5], we obtain the following useful characterizations of λ-ineffable and almost λ-ineffable subsets of $P_\varkappa\lambda$.

1.3. Corollary. *$X \subset P_\varkappa\lambda$ is λ-ineffable (almost λ-ineffable) iff for any $(f_x : x \in X)$ such that $(\forall x \in X)(f_x : x \to x)$, $(\exists f : \lambda \to \lambda)[H = \{x \in X : f_x = f \restriction x\} \in NS_{\varkappa\lambda}^+ (I_{\varkappa\lambda}^+)]$.*

Proof. The reverse implications are clear. Pick $X \in I^+$ where $I \in \{NIn_{\varkappa\lambda}, NAIn_{\varkappa\lambda}\}$ and let $(f_x : x \in X)$ be such that $(\forall x \in X)(f_x : x \to x)$. Further, let $p: \lambda \times \lambda \rightarrowtail \lambda$ and set $C_p = \{x \in P_\varkappa\lambda : p''(x \times x) \subset x\}$. The normality of I together with the minimality of $NS_{\varkappa\lambda}$ guarantees that $X \cap C_p \in I^+$.

For each $x \in X \cap C_p$, define $A_x \subset x$ by $A_x = \{p(\xi, f_x(\xi)) : \xi \in x\}$, and then let $H \subset X \cap C_p$ be such that $H \in I_{\varkappa\lambda}^+$ and $(\forall x, y \in H)(x \subset y \Rightarrow A_x = A_y \cap x)$. Notice that $A_x = A_y \cap x$ means that $\{p(\xi, f_x(\xi)) : \xi \in x\} = \{p(\xi, f_y(\xi)) : \xi \in y\} \cap x$ and hence that $f_x = f_y \restriction x$. Thus define $f: \lambda \to \lambda$ by $f(\alpha) = f_x(\alpha)$ where x is any element of $H \cap \widehat{\{\alpha\}}$. □

The ideal-theoretic characterization of mild λ-ineffability is much weaker than that given in 1.2 above for λ-ineffability and almost λ-ineffability as we now show:

1.4. Proposition. *For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda \geqq \varkappa$, $\varkappa$ is mildly λ-ineffable iff $NMIn_{\varkappa\lambda} = I_{\varkappa\lambda}$.*

Proof. It is clear that if $NMIn_{\varkappa\lambda} = I_{\varkappa\lambda}$, then $\varkappa$ is mildly λ-ineffable, and if $\varkappa$ is mildly λ-ineffable then $I_{\varkappa\lambda}\subset NMIn_{\varkappa\lambda}$. We show that if $\varkappa$ is mildly λ-ineffable, then $I^+_{\varkappa\lambda}\subset NMIn^+_{\varkappa\lambda}$. Pick $X\in I^+_{\varkappa\lambda}$ and let $(A_x\colon x\in X)$ be such that $(\forall x\in X)(A_x\subset x)$. For each $z\in P_\varkappa\lambda-X$ pick $x_z\in X\cap\hat{z}$, and for each $z\in X$ set $x_z=z$. Define $A'_z=A_{x_z}\cap z$ and let $A\subset\lambda$ be such that $(\forall z\in P_\varkappa\lambda)(\exists y\in\hat{z})(A'_y\cap z=A\cap z)$. Then $(\forall z\in P_\varkappa\lambda)(\exists y\in\hat{z})$ $(A_{x_y}\cap y\cap z=A\cap z)$, so $(\forall z\in P_\varkappa\lambda)(\exists x\in X\cap\hat{z})(A_x\cap z=A\cap z)$. $\square$

We conclude this section with some remarks on the "projections" of the ideals studied above from $P_\varkappa\varkappa$ to $\varkappa$.

It is easy to see that for any *normal* ideal I on $P_\varkappa\varkappa$, $I\restriction\varkappa=\{Y\cap\varkappa\colon Y\in I\}$ is a normal ideal on $\varkappa$. Moreover, $(I\restriction\varkappa)^+=\{Y\cap\varkappa\colon Y\in I^+\}$ and $I=\{Y\subset P_\varkappa\varkappa\colon Y\cap\varkappa\in I\restriction\varkappa\}$. As an easy consequence of these facts together with 1.2 above we obtain

1.5. Proposition. *For any uncountable regular cardinal* $\varkappa$, $NIn_\varkappa=NIn_{\varkappa\varkappa}\restriction\varkappa$ *and* $NAIn_\varkappa=NAIn_{\varkappa\varkappa}\restriction\varkappa$. $\square$

It is also easy to see that for any uncountable regular cardinal $\varkappa$, $I_\varkappa=I_{\varkappa\varkappa}\restriction\varkappa$ and hence that $NMIn_{\varkappa\varkappa}\restriction\varkappa$ is *not* the weakly compact ideal on $\varkappa$. This is not surprising for as we show in [8], mild λ-ineffability is a $P_\varkappa\lambda$ generalization of weak compactness in the sense of some of its familiar characterizations. And as Baumgartner observed in [2], p. 87, these characterizations of weak compactness do not yield the weakly compact ideal; they just yield $I_\varkappa$.

In Section 2 below we define a new ineffability property of $P_\varkappa\lambda$ "between" mild λ-ineffability and almost λ-ineffability whose associated ideal is normal, and show that the projection of this ideal from $P_\varkappa\varkappa$ to $\varkappa$ is the weakly compact ideal. We study this notion further in Section 3 below and in [8].

2. A new ineffability property of $P_\varkappa\lambda$

Motivated by Shelah's work in [15] together with out 1.3 above, we define a new ineffability property of $P_\varkappa\lambda$ as follows:

2.1. Definition. For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda\geqq\varkappa$, say that $X\subset P_\varkappa\lambda$ has the *λ-Shelah property* iff for any $(f_x\colon x\in X)$ such that $(\forall x\in X)$ $(f_x\colon x\to x)$, $(\exists f\colon\lambda\to\lambda)(\forall x\in P_\varkappa\lambda)(H_x=\{y\in X\cap\hat{x}\colon f_y\restriction x=f\restriction x\}\neq 0)$.

Further, define the set $NSh_{\varkappa\lambda}$ by $NSh_{\varkappa\lambda}=\{X\subset P_\varkappa\lambda\colon X$ does *not* have the λ-Shelah property$\}$, and say that *$\varkappa$ is λ-Shelah* iff $P_\varkappa\lambda\notin NSh_{\varkappa\lambda}$.

It is clear that $I_{\varkappa\lambda}\subset NMIn_{\varkappa\lambda}\subset NSh_{\varkappa\lambda}$, and in view of 1.3 above, it is also clear that $NSh_{\varkappa\lambda}\subset NAIn_{\varkappa\lambda}\subset NIn_{\varkappa\lambda}$.

The main result of this section is that $\varkappa$ is λ-Shelah iff $NSh_{\varkappa\lambda}$ is a normal ideal on $P_\varkappa\lambda$ (Theorem 2.3 below). We start with the following simple preliminary which was inspired by a result of Baumgartner and Laver [4].

2.2. Lemma. *Suppose that* $I\subsetneqq\mathcal{P}(P_\varkappa\lambda)$ *is such that*

$$(\forall X,Y\subset P_\varkappa\lambda)(X\subset Y\,\&\,Y\in I\Rightarrow X\in I),\tag{1}$$

$$(\forall X\in I)(\forall Y\in I_{\varkappa\lambda})(X\cup Y\in I),\quad and\tag{2}$$

$$\nabla I\subset I.\tag{3}$$

Then I is a $\varkappa$-complete normal ideal on $P_\varkappa\lambda$.

PROOF. It suffices to prove that I is $\varkappa$-complete. In fact, we will show that $I^*=\{X\in P_\varkappa\lambda\colon P_\varkappa\lambda-X\in I\}$ is $\varkappa$-complete.

Pick $\delta<\varkappa$ and let $(X_\alpha\colon \alpha<\delta)\in{}^\delta(I^*)$. For each $\beta<\lambda$ write $\beta=\delta\gamma+\alpha$ where $\gamma\geqq 0$ and $\alpha<\delta$, and set $Y_\beta=X_\alpha$. By (3), $Y=\Delta(Y_\beta\colon \beta<\lambda)=\{y\in P_\varkappa\lambda\colon (\forall\beta\in y)(y\in Y_\beta)\}\in I^*$, so it follows by (2) that $Y\cap\hat{\delta}\in I^*$. Now note that $(\forall y\in Y\cap\hat{\delta})$ $(\delta\subset y\ \&\ (\forall\beta\in y)(y\in Y_\beta))$. Thus $Y\cap\hat{\delta}\subset\bigcap\{X_\alpha\colon \alpha<\delta\}$. It now follows by (1) that $\bigcap\{X_\alpha\colon \alpha<\delta\}\in I^*$. $\square$

2.3. THEOREM. *For any uncountable regular cardinal $\varkappa$ and* any *cardinal $\lambda\geqq\varkappa$, $\varkappa$ is λ-Shelah iff $NSh_{\varkappa\lambda}$ is a normal ideal on $P_\varkappa\lambda$.*

PROOF. The reverse implication is clear. Moreover, it is clear that if $\varkappa$ is λ-Shelah, then $NSh_{\varkappa\lambda}$ is proper, and that if $X\subset P_\varkappa\lambda$ is λ-Shelah, then every $Y\supset X$ is λ-Shelah and $X\in I^+_{\varkappa\lambda}$. Thus $NSh_{\varkappa\lambda}$ satisfies (1) and (2) of the preceding lemma, so we complete the proof by showing that it also satisfies (3).

Let $(X_\nu\colon \nu<\lambda)\in{}^\lambda NSh_{\varkappa\lambda}$, and for each $\nu<\lambda$ let $(f^\nu_x\colon x\in X_\nu)$ witness $X_\nu\in NSh_{\varkappa\lambda}$. Set $X=\nabla(X_\nu\colon \nu<\lambda)$. Suppose that $X\notin NSh_{\varkappa\lambda}$; we derive the required contradiction as follows.

For each $x\in X$ let $(\alpha^x_0, \ldots, \alpha^x_\nu, \ldots, (\nu<\mathrm{ot}(x)))$ enumerate x in increasing order. Notice that in view of the fact that $I_{\varkappa\lambda}\subset NSh_{\varkappa\lambda}$, we may assume w.l.o.g. that $X\subset\widehat{\{0\}}$ and hence that $(\forall x\in X)(\alpha^x_0=0)$.

For each $x\in X$ pick $\gamma(x)\in x$ so that $x\in X_{\gamma(x)}$ and define $g_x\colon x\to x$ by

$$g_x(\xi)=\begin{cases}\gamma(x) & \text{if}\quad \xi=\alpha^x_\nu \quad\text{where}\quad \nu=0 \quad\text{or}\quad \lim(\nu)\\ f^{\gamma(x)}_x(\alpha^x_\mu) & \text{if}\quad \xi=\alpha^x_\nu \quad\text{where}\quad \nu=\mu+1.\end{cases}$$

Now let $g\colon \lambda\to\lambda$ be such that $(\forall x\in P_\varkappa\lambda)(\exists y\in X\cap\hat{x})(g_y\restriction x=g\restriction x)$, and set $\gamma=g(0)$. Finally, define $f\colon \lambda\to\lambda$ by $f(\xi)=g(\xi+1)$. We show that $(\forall x\in P_\varkappa\lambda)(\exists y\in X_\gamma\cap\hat{x})$ $(f^\gamma_y\restriction x=f\restriction x)$ thus contradicting $X_\gamma\in NSh_{\varkappa\lambda}$.

Pick $x\in P_\varkappa\lambda$ and set $x'=x\cup\{0\}\cup\{\xi+1\colon \xi\in x\}$. Now pick $y\in X\cap\hat{x}'$ such that $g_y\restriction x'=g\restriction x'$. Notice that since $0\in x'\subset y$, $g_y(0)=\gamma$, so $\gamma(y)=\gamma$. Thus observe that for each $\xi\in x$ we have $f(\xi)=g(\xi+1)=g_y(\xi+1)=f_y(\xi)$ since $\{\xi,\xi+1\}\subset x'\subset y$. $\square$

It is easy to see that $\varkappa$ has the Shelah property and hence (by [15]) is weakly compact iff $\varkappa$ is $\varkappa$-Shelah. Moreover, an easy argument using Theorem 2.3 above and the remark preceding 1.5 yields

2.4. PROPOSITION. *For any uncountable regular cardinal $\varkappa$, $NSh_\varkappa=NSh_{\varkappa\varkappa}\restriction\varkappa$.* $\square$

In view of the facts that $\varkappa$ is weakly compact iff $\varkappa$ is mildly $\varkappa$-ineffable (see [7]) and $\varkappa$ is weakly compact iff $\varkappa$ is $\varkappa$-Shelah, it is clear that $\varkappa$ is $\varkappa$-Shelah iff $\varkappa$ is mildly $\varkappa$-ineffable. In Section 3 we will see that the λ-Shelah property and mild λ-ineffability are not equivalent for arbitrary $\lambda>\varkappa$, however.

3. λ-Shelah cardinals and supercompactness

The main result of this section is that if $\varkappa$ is $2^{\lambda^{<\varkappa}}$-Shelah, then $\varkappa$ is λ-supercompact (Theorem 3.2 below). Our proof of this requires the following easy preliminary which shows that the λ-Shelah property is a $P_\varkappa\lambda$ generalization of "Baumgartner'-s principle" (see 0.4 above).

3.1. PROPOSITION. *For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda\geqq\varkappa$, $X\subset P_\varkappa\lambda$ has the λ-Shelah property iff for any λ-sequence $(g_\alpha\colon \alpha<\lambda)$ of regressive functions on $P_\varkappa\lambda$, $(\exists f\colon \lambda\to\lambda)(\forall x\in P_\varkappa\lambda)(E_x=\{y\in X\cap\hat{x}\colon (\forall\alpha\in x)(g_\alpha(y)=f(\alpha))\}\neq 0)$.*

3.2. THEOREM. *For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda\geqq\varkappa$, if $\varkappa$ is $2^{\lambda^{<\varkappa}}$-Shelah, then $\varkappa$ is λ-supercompact.*

PROOF. Set $\gamma=2^{\lambda^{<\varkappa}}$ and let $(f_\alpha\colon \alpha<\gamma)$ enumerate the set of all regressive functions on $P_\varkappa\lambda$. For each $y\in P_\varkappa\lambda$ fix an element τ_y of y, and then for each $\alpha<\gamma$ define $g_\alpha\colon P_\varkappa\lambda\to\gamma$ by

$$g_\alpha(y)=\begin{cases} f_\alpha(y\cap\lambda) & \text{if } y\cap\lambda\neq 0\\ \tau_y & \text{otherwise.}\end{cases}$$

It is clear that for each $\alpha<\gamma$, g_α is regressive and that $f_\alpha=g_\alpha{\restriction}P_\varkappa\lambda$.

Now let $g\colon \gamma\to\gamma$ be such that $(\forall y\in P_\varkappa\gamma)(E_y=\{z\in P_\varkappa\gamma\colon y\subset z\ \&\ (\forall\alpha\in y)(g_\alpha(z)= =g(\alpha))\}\in I^+_{\varkappa\lambda})$. Notice that $g\colon \gamma\to\lambda$.

For each $y\in P_\varkappa\gamma$, set $E'_y=\{z\cap\lambda\colon z\in E_y\}$. It is easy to see that:

(i) $$(\forall y\in P_\varkappa\gamma)(E'_y\in I^+_{\varkappa\lambda}),$$

(ii) $$(\forall y\in P_\varkappa\gamma)(E'_y\subset\bigcap\{E'_{\{\alpha\}}\colon \alpha\in y\}),$$

and

(iii) $$(\forall\alpha<\lambda)(E'_{\{\alpha\}}\subset\widehat{\{\alpha\}}\cap P_\varkappa\lambda).$$

An immediate consequence of (i)—(iii) above is that $\{E'_{\{\alpha\}}\colon \alpha<\gamma\}$ generates a proper $\varkappa$-complete filter F in $\mathcal{P}(P_\varkappa\lambda)$ extending $I^*_{\varkappa\lambda}$. We claim that F is a *normal ultrafilter*.

Let $f\colon P_\varkappa\lambda\to\lambda$ be such that $X=\{x\in P_\varkappa\lambda\colon f(x)\in x\}\in F$ and let β be any ordinal $<\gamma$ such that $f_\beta{\restriction}X=f{\restriction}X$. Then since $E'_{\{\beta\}}\in F$ and since $E'_{\{\beta\}}-\{0\}=\{x\in P_\varkappa\lambda\colon f_\beta(x)= =g(\beta)\}$, it follows that $f^{-1}(\{g(\beta)\})\in F$.

Pick $X\subset P_\varkappa\lambda$ and let $\chi_X\colon P_\varkappa\lambda\to\{0,1\}$ be its characteristic function. Notice that χ_X is regressive on $\widehat{\{0,1\}}\cap P_\varkappa\lambda\in F$. An argument similar to the one used in the preceding paragraph shows that $(\exists\beta<\gamma)(g(\beta)\in\{0,1\}\ \&\ \chi_X^{-1}(\{g(\beta)\})\in F)$. $\square$

An easy consequence of 1.3 and 3.2 together with fact that λ-supercompactness implies λ-ineffability (Magidor [13]) is the following improvement of Magidor's characterization [13] of supercompactness.

3.3. COROLLARY. *$\varkappa$ is supercompact iff $\varkappa$ is λ-Shelah for every $\lambda\geqq\varkappa$ iff $\varkappa$ is almost λ-ineffable for every $\lambda\geqq\varkappa$.*

PROOF. By 1.3 and 2.1 above, $NSh_{\varkappa\lambda}\subset NAIn_{\varkappa\lambda}\subset NIn_{\varkappa\lambda}$ for every $\lambda\geqq\varkappa$. The rest now follows by Magidor's result and 3.2. $\square$

We conclude this section by showing that the λ-Shelah property and mild λ-ineffability are not provably equivalent for arbitrary $\lambda>\varkappa$. Baumgartner [3] obtained a result which amounts to the same thing.

3.4. COROLLARY. *The λ-Shelah property and mild λ-ineffability are not provably equivalent for arbitrary $\lambda>\varkappa$.*

PROOF. By Theorem 3.2 together with the DiPrisco—Zwicker characterization [9] of strong compactness, and Menas' result [14] that the least measurable cardinal which is a limit of strongly compact cardinals is itself strongly compact, but is not $2^\varkappa$-supercompact. □

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(Received September 14, 1983)

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Acta Math. Hung.
47 (3—4) (1986), 333—340.

SATURATION OF AN INTERPOLATORY POLYNOMIAL OPERATOR

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1. Introduction. We obtain the saturation order and the saturation class for the interpolatory polynomial operator introduced by O. Kis and P. Vértesi [1].

2. Preliminary results. 2.1. As far as we know, D. L. Berman [1] and G. Freud [2] were the first ones who, answering a question of P. Butzer, proved the Jackson theorem via *interpolatory polynomials of degree* $\leq cn$ (see (i)—(vi)). From that time at least a dozen papers appeared proving the Jackson, Timan or the Gopengauz—Teliakovskii theorem. The authors are, among others, M. Sallay, R. B. Saxena, P. Vértesi, O. Kis, A. K. Varma, T. M. Mills, A. Sharma, J. Szabados, R. Bojanic, R. DeVore, K. K. Mathur, N. Misra and H. G. Lehnhoff. (For detailed references, see [4] and [5].)

2.2. Perhaps of the simplest form is the operator found by O. Kis and P. Vértesi [1]. The aim of this paper to find the corresponding saturation order and saturation class. To be more precise, let for $n=1, 2, \ldots$

$$t_{kn} = \frac{2k\pi}{2n+1}, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

and consider the trigonometric polynomial

$$p_n(f, t) = \sum_{k=-n}^{n} f(t_{kn}) u_{kn}(t), \quad n = 1, 2, \ldots, \tag{2.2}$$

for the continuous 2π-periodic f (shortly $f \in \tilde{C}$). Here

$$u_{kn}(t) = 4l_{kn}^3(t) - 3l_{kn}^4(t), \quad k = 0, \pm 1, \ldots, \pm n, \tag{2.3}$$

where

$$l_{kn}(t) = \frac{\sin \frac{2n+1}{2}(t-t_{kn})}{(2n+1)\sin \frac{t-t_{kn}}{2}}, \quad k = 0, \pm 1, \ldots, \pm n, \tag{2.4}$$

are the fundamental functions of trigonometric interpolation based on (2.1). Obviously

* The work was completed during the second author's visit in Gainesville, in 1983 and 1986.

(i) $\deg p_n \leqq 4n$,

(ii) $p_n(f, t_{kn}) = f(t_{kn}), \quad k = 0, \pm 1, \ldots, \pm n,$

moreover, by [1], (1.2),

(iii) $|p_n(f, t) - f(t)| \leqq \text{const.}\, \omega\left(f, \frac{1}{n}\right), \quad n = 1, 2, \ldots$

where ω is the usual modulus of continuity. ("const.", c, c_1, ... mean absolute positive numbers.)

2.3. If $g(x)$ is continuous on $[-1, 1]$ (shortly $g \in C$), let us consider

$$q_n(g, x) = \sum_{k=-n}^{n} g(\cos t_{kn})\, u_{kn}(\arccos x), \quad n = 1, 2, \ldots. \tag{2.5}$$

It can be seen that

(iv) q_n is an algebraic polynomial of degree $\leqq 4n$,

(v) $q_n(g, \cos t_{kn}) = g(\cos t_{kn}), \quad k = 0, 1, \ldots, n,$

(vi) $|q_n(g, x) - g(x)| \leqq \text{const.}\left[\omega\left(g, \frac{\sqrt{1-x^2}}{n}\right) + \omega\left(g, \frac{|x|}{n^2}\right)\right], \quad n = 1, 2, \ldots$

(see [1]).

2.4. A very natural question arises: How good is the polynomial p_n, i.e. can the order of approximation be better than $1/n$ (excluding the trivial functions, of course)?

3. Results. Let $\|f\|$ and $\|g\|$ be the usual maximum norm for $f \in \tilde{C}$ or $g \in C$, respectively. The saturation order and saturation class for p_n is contained in the following

THEOREM 3.1. *For the operator* p_n

$$\|p_n(f, t) - f(t)\| = \begin{cases} o\left(\frac{1}{n}\right) & \textit{iff } f = \textit{constant}, \\ O\left(\frac{1}{n}\right) & \textit{iff } f \in \text{Lip}\, 1. \end{cases} \tag{3.1}$$

For q_n we state

THEOREM 3.2. *If* $\varphi(t) = g(\cos t)$, *then*

$$\|q_n(g, x) - g(x)\| = \begin{cases} o\left(\frac{1}{n}\right) & \textit{iff } g = \textit{constant}, \\ O\left(\frac{1}{n}\right) & \textit{iff } \varphi \in \text{Lip}\, 1. \end{cases} \tag{3.2}$$

REMARKS. a) To prove these theorems we cannot use the usual saturation arguments. The method of our "ad hoc" proof was initiated by G. Somorjai [8].

b). By (3.2) we get

$$\|q_n(h, x)-h(x)\| = O\left(\frac{1}{n}\right) \tag{3.3}$$

if $h(x)=\sqrt{1-x^2}$ (because now $\varphi(t)=\sin t$). Considering that $h\in \mathrm{Lip}\, 1/2$ but $h\notin \mathrm{Lip}\,(1/2+\varepsilon)$ $(\varepsilon>0)$, this estimation is better than (vi).

c). Similar saturation theorems can be proved if the corresponding interpolatory operators are based on the roots of $P_n^{(\alpha,\beta)}(x)$, $(1-x^2)P_n^{(\alpha,\beta)}(x)$, etc. The main tool is in P. Erdős, P. Vértesi [9] which actually gives a correct proof for Theorem 2.4 in [6].

4. Proofs. 4.1. PROOF OF THEOREM 3.1. If, omitting the superfluous notations, t_j are (one of) the nearest nodes to t, i.e.

$$t = t_j+\frac{\alpha}{2n+1} \quad \text{where} \quad -\pi \leqq \alpha = \alpha_n \leqq \pi, \tag{4.1}$$

we shall see that the operator norm has the same order as the absolute value of the j-th term. Namely,

$$\frac{8}{\pi^3} \leqq |u_j(t)| \leqq \sum_{k=-n}^{n} |u_k(t)| \leqq \text{const.} \tag{4.2}$$

Indeed, by (2.4) and (4.1)

$$|l_j(t)| \geqq \frac{\frac{2}{\pi}\frac{2n+1}{2}|t-t_j|}{(2n+1)\frac{|t-t_j|}{2}} = \frac{2}{\pi},$$

moreover, by $|l_k|\leqq 1$, $|u_j|=|l_j|^3|4-3l_j|\geqq |l_j|^3 \geqq 8/\pi^3$. To obtain the upper estimation, by (2.3) and (4.1)

$$|u_k(t)| \leqq 7\frac{\sin^3\frac{2n+1}{2}|t-t_k|}{(2n+1)^3\sin^3\frac{|t-t_k|}{2}} \leqq \text{const}\frac{|\alpha|^3}{n^3|t-t_k|^3} \leqq \tag{4.3}$$

$$\leqq \text{const}\frac{|\alpha|^3}{|j-k|^3} \quad \text{if} \quad k=j\pm 1, j\pm 2, \ldots, j\pm n.$$

By (4.3) and $|u_j|\leqq 7$ we get (4.2), considering that

$$\sum_{k=-n}^{n} |u_k(t)| = \sum_{k=j-n}^{j+n} |u_k(t)|.$$

4.2. Let

$$M_f(t) := \varlimsup_{y\to t}\left|\frac{f(t)-f(y)}{t-y}\right|, \quad 0 \leqq M_f(t) \leqq \infty, \quad f\in \tilde{C}.$$

Then, by [9; 2.1] we have

LEMMA 4.1. *If* $E\subset[-\pi,\pi)$ *is a set such that* $[-\pi,\pi)\setminus E$ *is countable. Then* $M_f:=\sup_{t\in[-\pi,\pi)} M_f(t)=M_f(E):=\sup_{t\in E} M_f(t)$, *i.e. we have*

(a) $f=$*constant iff* $M_f(E)=0$,

(b) $f\in \mathrm{Lip}\,1$ *iff* $M_f(E)<\infty$.

4.3. Now we state

LEMMA 4.2. *If* $f\in\tilde{C}$ *then*

$$\overline{\lim_{n\to\infty}}\,\{n\,\|p_n(f,t)-f(t)\|\}\geqq \mathrm{const}\cdot M_f. \tag{4.4}$$

PROOF OF LEMMA 4.2 (Parts 4.3—4.8). First we assume that $M_f>0$; otherwise the lemma is trivial. Let

$$E:=\{x;\ x\in[-\pi,\pi),\ x/\pi \text{ is irrational}\}.$$

Then by [9] we can state

LEMMA 4.3. *Let* $x_0\in E$ *and* $\varrho>0$ *be fixed. If* $\{y_l\}$ *is an infinite sequence with* $y_l\to x_0$, $y_l\neq x_0$, *then one can find infinitely many (different) nodes* t_{j_k,n_k} *and numbers* $\{x_k\}\subset\{y_l\}$ *with*

$$|x_0-t_{j_k,n_k}|\leqq\frac{\varrho^2}{n_k},\qquad k=1,2,\dots, \tag{4.5}$$

$$\frac{\varrho}{3n_k}<|x_k-t_{j_k,n_k}|\leqq\frac{2\varrho}{n_k},\qquad k=1,2,\dots,\ x_k\in\{y_l\} \tag{4.6}$$

if $\varrho>0$ *is small enough.*

By Lemma 4.1 one can choose a $T_0\in E$ such that $M_f(T_0)\geqq\frac{M_f}{2}>0$. Let $0<T_0\leqq\frac{\pi}{2}$, say. Let $0<\varepsilon\leqq\frac{1}{2}$ be fixed. Then by definition there exists a d_0, $0<d_0<\frac{\pi}{2}$, such that

$$\left|\frac{f(t)-f(T_0)}{t-T_0}\right|\leqq(1+\varepsilon)\,M_f(T_0)\quad\text{if}\quad 0\leqq|t-T_0|\leqq d_0. \tag{4.7}$$

Let us define the sequence $\{\mathcal{T}_r\}$ such that

$$\mathcal{T}_r\neq T_0,\quad r=1,2,\dots;\quad \lim_{r=\infty}\mathcal{T}_r=T_0$$

and

$$\left|\frac{f(\mathcal{T}_r)-f(T_0)}{\mathcal{T}_r-T_0}\right|\geqq\begin{cases}(1-\varepsilon)\,M_f(T_0) & \text{if}\quad M_f<\infty,\\[1ex] \max\limits_{\substack{-\pi\leqq t<\pi\\ |t-T_0|\geqq|\mathcal{T}_r-T_0|}}\left|\dfrac{f(t)-f(T_0)}{t-T_0}\right| & \text{if}\quad M_f=\infty.\end{cases} \tag{4.8}$$

That means, in both cases

$$(4.9)\qquad \left|\frac{f(t)-f(T_0)}{t-T_0}\right| \leqq \frac{1+\varepsilon}{1-\varepsilon}\left|\frac{f(\mathcal{T}_r)-f(T_0)}{\mathcal{T}_r-T_0}\right| \quad \text{if} \quad |\mathcal{T}_r-T_0| \leqq |t-T_0| \leqq d_0,$$
$$r = 1, 2, \ldots.$$

4.4. Now, using the properties of $\{\mathcal{T}_r\}$, and Lemma 4.3 we shall prove that

$$\max\{|p_n(f, T_0)-f(T_0)|,\ |p_n(f, \mathcal{T}_r)-f(\mathcal{T}_r)|\} \geqq \frac{1}{\pi^3}|f(T_0)-f(\mathcal{T}_r)|$$

at least for proper subsequences of $\{\mathcal{T}_r\}$ and $\{n\}$.

4.5. Indeed, let us apply Lemma 4.3 with the cast $x_0=T_0$, $\{y_l\}=\{\mathcal{T}_r\}$ for the set E. Then there exist $1<m_1<m_2<\ldots$, $j_1, j_2, \ldots \mathcal{T}_1<\mathcal{T}_2<\ldots$ (we denote the elements of the subsequence by $\mathcal{T}_k$ again) such that

$$(4.10)\qquad |T_0-t_{j_k,m_k}| \leqq \frac{\varrho^2}{m_k}, \quad k = 1, 2, \ldots,$$

$$(4.11)\qquad \frac{\varrho}{3m_k} < |\mathcal{T}_k-t_{j_k,m_k}| \leqq \frac{2\varrho}{m_k}, \quad k = 1, 2, \ldots.$$

By (4.10) and (4.11) it is easy to get

$$(4.12)\qquad \frac{\varrho}{4m_k} < |\mathcal{T}_k-T_0| < \frac{3\varrho}{m_k}$$

if ϱ is small enough.

Now let $i\neq j_k$. Then

$$(4.13)\qquad \left|\frac{f(\mathcal{T}_k)-f(t_i)}{\mathcal{T}_k-t_i}\right| \leqq \left|\frac{f(\mathcal{T}_k)-f(T_0)}{\mathcal{T}_k-T_0}\right|\left|\frac{\mathcal{T}_k-T_0}{\mathcal{T}_k-t_i}\right| + \left|\frac{f(T_0)-f(t_i)}{T_0-t_i}\right|\left|\frac{T_0-t_i}{\mathcal{T}_k-t_i}\right|.$$

Here, if ϱ is small enough,

$$\left|\frac{\mathcal{T}_k-T_0}{\mathcal{T}_k-t_i}\right| \leqq \frac{\dfrac{3\varrho}{m_k}}{\dfrac{2\pi}{2m_k+1}-\dfrac{\varrho}{m_k}} \leqq \varrho$$

and, as above,

$$\left|\frac{T_0-t_i}{\mathcal{T}_k-t_i}\right| = \left|\frac{T_0-\mathcal{T}_k+\mathcal{T}_k-t_i}{\mathcal{T}_k-t_i}\right| = \left|1+\frac{T_0-\mathcal{T}_k}{\mathcal{T}_k-t_i}\right| \leqq 1+\varrho.$$

Let $|T_0-t_i|\leqq d_0$; then using (4.9) we get from (4.13)

$$\left|\frac{f(\mathcal{T}_k)-f(t_i)}{\mathcal{T}_k-\mathrm{t}_i}\right| \leqq \left[\varrho+(1+\varrho)\frac{1+\varepsilon}{1-\varepsilon}\right]\left|\frac{f(\mathcal{T}_k)-f(T_0)}{\mathcal{T}_k-t_0}\right| \leqq (1+3\varepsilon)\left|\frac{f(\mathcal{T}_k)-f(T_0)}{\mathcal{T}_k-T_0}\right|$$

whenever ε and ϱ are small enough.

On the other hand, if $M_f=\infty$, by (4.8),

$$\left|\frac{f(\mathscr{T}_k)-f(t_i)}{\mathscr{T}_k-t_i}\right| \leqq [\varrho+(1+\varrho)]\left|\frac{f(\mathscr{T}_k)-f(T_0)}{\mathscr{T}_k-T_0}\right|.$$

Summarizing, we obtain

$$\tag{4.14} \left|\frac{f(\mathscr{T}_k)-f(t_i)}{\mathscr{T}_k-t_i}\right| \leqq (1+3\varepsilon)\left|\frac{f(\mathscr{T}_k)-f(T_0)}{\mathscr{T}_k-T_0}\right| \begin{cases}\text{whenever } |T_0-t_i| \leqq d_0\\ \textit{or} \text{ for any } i \text{ if } M_f=\infty.\end{cases}$$

(4.14) will be used to estimate p_n-f. Indeed, with $j=j_k$ we can write

$$\tag{4.15} |p_{m_k}(f,\mathscr{T}_k)-f(\mathscr{T}_k)| = \Big|\sum_{r=-m_k}^{m_k}[f(t_r)-f(\mathscr{T}_k)]\,u_r(\mathscr{T}_k)\Big| \geqq$$

$$\geqq |f(\mathscr{T}_k)-f(t_j)|\cdot|u_j(\mathscr{T}_k)| - \sum_{\substack{0<|t_i-T_0|\leqq d_0\\ i\neq j}} |f(t_i)-f(\mathscr{T}_k)|\cdot|u_i(\mathscr{T}_k)| -$$

$$-\sum_{|t_i-T_0|>d_0} |f(t_i)-f(\mathscr{T}_k)|\cdot|u_i(\mathscr{T}_k)| := A_1-A_2-A_3.$$

4.6. First we suppose that for a certain k

$$\tag{4.16} |f(\mathscr{T}_k)-f(t_j)| \geqq \frac{3}{4}|f(\mathscr{T}_k)-f(T_0)|.$$

Then by (4.2) and (4.16)

$$\tag{4.17} A_1 \geqq \frac{6}{\pi^3}|f(\mathscr{T}_k)-f(T_0)|.$$

Moreover, using (4.3), (4.14), (4.11), (4.12) and (2.1),

$$\tag{4.18} A_2 \leqq \text{const.} \sum_{\substack{0<|t_i-T_0|\leqq d_0\\ i\neq j}} \left|\frac{f(\mathscr{T}_k)-f(t_i)}{\mathscr{T}_k-t_i}\right| \frac{\varrho^3}{m_k^3|\mathscr{T}_k-t_i|^2} \leqq$$

$$\leqq \text{const.}\,(1+3\varepsilon) \sum_{\substack{0<|t_i-T_0|\leqq d_0\\ i\neq j}} |f(\mathscr{T}_k)-f(T_0)| \frac{\varrho}{m_k|\mathscr{T}_k-T_0|}\,\frac{\varrho^2}{m_k^2|\mathscr{T}_k-t_i|^2} \leqq$$

$$\leqq \text{const.}\,(1+3\varepsilon)\,\varrho^2\,|f(\mathscr{T}_k)-f(T_0)|.$$

Let $0<M_f<\infty$. Then, by (4.3), we estimate A_3 as follows:

$$\tag{4.19} A_3 \leqq \text{const.}\,\|f\|\,\frac{\varrho^3}{m_k^3}\sum_{i=-m_k}^{m_k} d_0^{-3} = \frac{\text{const.}\,\|f\|}{m_k^2 d_0^3}\,\varrho^3.$$

Using (4.15)—(4.19), (4.8) and (4.12)

$$|p_{m_k}(f,\mathcal{T}_k)-f(\mathcal{T}_k)| \geqq |f(\mathcal{T}_k)-f(T_0)| \left(\frac{6}{\pi^3}-\text{const.}\,(1+3\varepsilon)\,\varrho^2-\frac{\text{const.}\,\|f\|\,\varrho^3}{|f(\mathcal{T}_k)-f(T_0)|\,d_0^3 m_k^2}\right) \geqq$$

$$\geqq |f(\mathcal{T}_k)-f(T_0)| \left(\frac{6}{\pi^3}-\text{const.}\,(1+3\varepsilon)\,\varrho^2-\frac{\text{const.}\,\|f\|\,\varrho^2}{M_f(T_0)\,d_0^3 m_k}\right) \geqq$$

$$\geqq \frac{1}{\pi^3}|f(\mathcal{T}_k)-f(T_0)|, \quad \text{if } \varrho \text{ is small enough.}$$

On the other hand, if $M_f=\infty$, we estimate as follows (see (4.3), (4.8) and (4.12)):

$$A_3 \leqq \text{const.} \sum_{|t_i-T_0|>d_0} \frac{|f(\mathcal{T}_k)-f(t_i)|}{|\mathcal{T}_k-t_i|}\,\frac{\varrho^3}{m_k^3|\mathcal{T}_k-t_i|^2} \leqq \tag{4.20}$$

$$\leqq \text{const.}\,(1+3\varepsilon)\left|\frac{f(\mathcal{T}_k)-f(T_0)}{\mathcal{T}_k-T_0}\right| \frac{\varrho^3}{m_k^3} \sum_{|t_i-T_0|>d_0} d_0^{-2} \leqq$$

$$\leqq \text{const.}\,(1+3\varepsilon)\,\frac{\varrho^2}{m_k d_0^2}\,|f(\mathcal{T}_k)-f(T_0)|.$$

Using these we get again

$$|p_{m_k}(f,\mathcal{T}_k)-f(\mathcal{T}_k)| \geqq \frac{1}{\pi^3}\,|f(T_0)-f(\mathcal{T}_k)|, \tag{4.21}$$

whenever (4.16) holds true.

4.7. Now let us suppose that (4.16) does not hold. Then obviously

$$|f(T_0)-f(t_j)| \geqq |f(T_0)-f(\mathcal{T}_k)|-|f(\mathcal{T}_k)-f(t_j)| \geqq \tag{4.22}$$

$$\geqq |f(T_0)-f(\mathcal{T}_k)|-\frac{3}{4}|f(T_0)-f(\mathcal{T}_k)| = \frac{1}{4}|f(T_0)-f(\mathcal{T}_k)|.$$

As above

$$|p_{m_k}(f,T_0)-f(T_0)| \geqq B_1-B_2-B_3, \tag{4.23}$$

where by (4.2) and (4.22)

$$B_1 := |f(T_0)-f(t_j)|\,|u_j(T_0)| \geqq \frac{2}{\pi^3}\,|f(\mathcal{T}_k)-f(T_0)|.$$

Further, by (4.3), (4.5), (4.8)

$$B_2 := \sum_{\substack{|t_i-t_0|\leqq d_0\\ i\neq j}} |f(t_i)-f(T_0)|\,|u_i(T_0)| \leqq \text{const.} \sum_{\dots} \frac{|f(t_i)-f(T_0)|}{|t_i-T_0|}\cdot$$

$$\cdot\frac{\varrho^6}{m_k^3|t_i-T_0|^2} \leqq \text{const.}\,\frac{|f(\mathcal{T}_k)-f(T_0)|}{|\mathcal{T}_k-T_0|}\,\frac{\varrho}{m_k}\,\varrho^5 \sum_{\dots} \frac{1}{m_k^2|t_i-T_0|^2} \leqq$$

$$\leqq \text{const.}\,\varrho^5\,|f(\mathcal{T}_k)-f(T_0)|.$$

Finally, if $M_f<\infty$, as in (4.19) (see (4.3))

$$B_3 := \sum_{|t_i-T_0|>d_0} |f(t_i)-f(T_0)|\,|u_i(T_0)| \leqq \frac{\text{const.}\,\|f\|\,\varrho^5}{m_k^2 d_0^3}.$$

If, on the other hand, $M_f=\infty$, we get by (4.3), (4.8), and (4.12)

$$B_3 \leqq \text{const.} \sum_{\dots} \frac{|f(t_i)-f(T_0)|}{|t_i-T_0|} \frac{\varrho^6}{m_k^3|t_i-T_0|^2} \leqq \text{const.} \frac{|f(\mathcal{T}_k)-f(T_0)|}{|\mathcal{T}_k-T_0|} \frac{\varrho}{m_k} \cdot$$

$$\cdot \varrho^5 \sum_{\dots} \frac{1}{m_k^2 d_0^2} \leqq \text{const.}\, |f(\mathcal{T}_k)-f(T_0)| \frac{\varrho^5}{m_k d_0^2}.$$

Using these estimations we get from (4.23)

$$|p_{m_k}(f, T_0)-f(T_0)| \geqq \frac{1}{\pi^3} |f(T_0)-f(\mathcal{T}_k)| \tag{4.24}$$

for a proper ϱ, whenever (4.16) does not hold. (4.21) and (4.24) complete the proof of 4.4.

4.8. Now using 4.4 and (4.12), we have

$$m_k \|p_{m_k}(f, t)-f(t)\| > \frac{\varrho}{4\pi^3} \frac{|f(T_0)-f(\mathcal{T}_k)|}{|T_0-\mathcal{T}_k|}, \quad k=1, 2, \dots,$$

from where we get (4.4) by (4.8).

4.9. Now we prove Theorem 3.1. The first part of (3.1) comes from (iii), (a) and (4.4). Now let $f \in \text{Lip}\, 1$; then by (iii), $\|p_n(f, t)-f(t)\| \leqq \text{const.}\, n^{-1}$, on the other hand if $\|p_n(f, t)-f(t)\| = O(n^{-1})$, then by (4.4) $M_f < \infty$, i.e. $f \in \text{Lip}\, 1$.

4.10. Proof of Theorem 3.2. The proof can be done as before considering the relations $\varphi(t) \equiv g(\cos t)$ and $p_n(\varphi, t) \equiv q_n(g, \cos t)$.

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(Received November 8, 1983; revised May 15, 1986)

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Acta Math. Hung.
47 (3—4) (1986), 341—359.

DISTRIBUTION OF DIGITS OF PRIMES IN q-ARY CANONICAL FORM

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1. Let $q\geqq 2$ be an integer, $\mathcal{A}_q=\{0, 1, \dots, q-1\}$. Every nonnegative integer can be uniquely written in the form

$$n = \sum_{j=0}^{\infty} a_j(n)\, q^j, \quad a_j(n)\in\mathcal{A}_q.$$

For an arbitrary subset $\mathcal{B}$ of the nonnegative integers, let $A_{\mathcal{B}}(x)$ be the counting function of the elements of $\mathcal{B}$ not exceeding x, and

$$A_{\mathcal{B}}\left(x\middle|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right) = \#\,\{n\in\mathcal{B} | n \leqq x,\ a_{j_l}(n) = b_l \ \ (l = 1, \dots, r)\}.$$

For the set $\mathcal{P}$ of all primes we shall use the notation

$$A_{\mathcal{P}}(x) = \pi(x), \quad A_{\mathcal{P}}\left(x\middle|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right) = \Pi\left(x\middle|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right).$$

If $\mathcal{B}=\mathcal{N}_0$, the whole set of the nonnegative integers, then we shall write

$$A_{\mathcal{N}_0}(x) = A(x), \quad A_{\mathcal{N}_0}\left(x\middle|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right) = A\left(x\middle|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right).$$

If $x=q^N-1$, N a positive integer, $0\leqq j_1<j_2<\dots<j_r\leqq N-1$, then

$$A\left(q^N-1\middle|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right) = \frac{q^N}{q^r} = \frac{1}{q^r}A(q^N-1)$$

for every choice of $b_1, \dots, b_r\in\mathcal{A}_q$. Let now $q^N\leqq x<q^{N+1}$, $j_1<j_2<\dots<j_r\leqq N-1$, $b_1, b_2, \dots, b_r\in\mathcal{A}_q$. Every $n\leqq x$ can be written in the form

$$\text{(1.1)} \qquad \begin{cases} n = mq^{j_r+1}+v, & 0 \leqq v < q^{j_r+1} \\ m = 0, 1, \dots, \left[\dfrac{x}{q^{j_r+1}}\right]. \end{cases}$$

If $m=\left[\dfrac{x}{q^{j_r+1}}\right]$, then the couple (m, v) gives an integer n if and only if $v\leqq\left\{\dfrac{x}{q^{j_r+1}}\right\}\times$

$\times q^{j_r+1}$. Since $a_l(n)=a_l(v)$ for $l\leqq j_r$, therefore

$$(1.2)\qquad A\left(x\left|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right.\right) = \left(\left[\frac{x}{q^{j_r+1}}\right]+1\right) A\left(q^{j_r+1}-1\left|\begin{matrix} j_1, \dots, j_r \\ b_1, \dots, b_r\end{matrix}\right.\right) + O(q^{j_r-r}) =$$

$$= \frac{x}{q^r}+O(q^{j_r-r}).$$

Here and in the sequel $[z]$ and $\{z\}$ denote the integer part and the fractional part of z, resp.

We should like to prove a similar theorem for the primes, namely that

$$(1.3)\qquad \Pi\left(x\left|\begin{matrix} 0, & j_2, \dots, j_r \\ b, & b_2, \dots, b_r\end{matrix}\right.\right) = (1+o(1))\frac{\lambda(b)}{q^r}\pi(x)$$

uniformly as $j_2<\dots<j_r<N-c\log N$, $q^N<x$, $r<c_1\log N$, where $\lambda(b)=0$ if $(b,q)>1$ and $\lambda(b)=q/\varphi(q)$ if $(b,q)=1$.

By using the results on the exponential sums with prime variables due to I. M. Vinogradov, and some theorems on the distribution of primes in arithmetical progressions we shall get such a theorem (Theorem 1) in Section 3.

If j_r is a value close to N then we may hope to have a relation like

$$(1.4)\qquad \Pi\left(x\left|\begin{matrix} 0, & j_2, \dots, j_r \\ b, & b_2, \dots, b_r\end{matrix}\right.\right) = (1+o(1))\frac{\lambda(b)}{\log x} A\left(x\left|\begin{matrix} 0, & j_2, \dots, j_r \\ b, & b_2, \dots, b_r\end{matrix}\right.\right)$$

(see Theorem 2).

In the further sections we shall apply Theorems 1 and 2 to prove the existence of the distribution of q-additive functions on the set of primes with and without normalizing factors.

2. Lemmas. $\pi(x,k,l)$ denotes the number of primes p, $\equiv l \pmod k$, $p\leqq x$. The letters $c, c_1, c_2, \dots$, denote suitable positive constants, not the same at every occurrence.

LEMMA 1. *Let* $0<\xi<1$ *and*

$$S = [1-\xi, 1)\cup\bigcup_{b=1}^{q-1}\left(\left[\frac{b}{q}-\xi, \frac{b}{q}+\xi\right]\right)\cup[0,\xi].$$

Then

$$(2.1)\qquad \#\left\{p\leqq x\left|\left\{\frac{p}{q^{j+1}}\right\}\in S\right.\right\} \leqq c\xi\pi(x)$$

uniformly for $1\leqq j\leqq N$, *where* $q^N\leqq x$. *If* $\xi q^{j+1}<1$, *then the left hand side of* (2.1) *is zero or one.*

PROOF. Let us fix a small $c_1>0$. We know that

$$(2.2)\qquad \pi(x+y)-\pi(x) \ll y/\log x,$$

if $y \gg x^{c_1}$. If

$$\left\{\frac{p}{q^{j+1}}\right\} \in (\alpha, \alpha+\xi), \quad p \leqq x, \tag{2.3}$$

then

$$mq^{j+1}+\alpha q^{j+1} < p < mq^{j+1}+(\alpha+\xi)q^{j+1}$$

with a suitable integer $m=0, 1, \ldots, [x/q^{j+1}]$.

Let us assume that $\xi q^{j+1} > x^{c_1}$. Then, by (2.2), the number of primes satisfying (2.3) is less than

$$\sum_m \pi(mq^{j+1}+\alpha q^{j+1}+\xi q^{j+1}) - \pi(mq^{j+1}+\alpha q^{j+1}) \ll \frac{\xi q^j}{\log x}\cdot\frac{x}{q^j} \ll \xi\pi(x).$$

Let $\xi q^{j+1} < x^{c_1}$. For a fixed integer $b \in \mathcal{A}_q$ the condition

$$\left\{\frac{p}{q^{j+1}}\right\} \in \left[\frac{b}{q}-\xi, \frac{b}{q}+\xi\right] \tag{2.4}$$

holds if and only if

$$\frac{p}{q^{j+1}} = \frac{b}{q}+\theta+m, \quad |\theta| \leqq \xi, \quad m = \text{integer},$$

i.e. if $p=bq^j+\theta q^{j+1}+mq^{j+1}$. Then $\theta q^{j+1}=l=\text{integer}$, $|l|<\xi q^{j+1}$. If $\xi q^{j+1}<1$, then $l=0$, so $q^j|p$. This may occur only if $q=p, j=1$. Let us assume that $\xi q^{j+1}>1$. The number of primes satisfying (2.4) can be estimated by the so called Brun—Titchmarsh inequality

$$\sum_{|l|<\xi q^{j+1}} \pi(x, q^{j+1}, l) \ll \xi\pi(x).$$

The number of primes p under the condition

$$\left\{\frac{p}{q^{j+1}}\right\} \in [0, \xi] \cup (1-\xi, 1]$$

can be estimated similarly, so we omit the details.

Let $\varphi_b(x)$ be a periodic function mod 1,

$$\varphi_b(x) = \begin{cases} 1 & \text{if } x \in \left(\frac{b}{q}, \frac{b+1}{q}\right) \\ \frac{1}{2} & \text{if } x = \frac{b}{q} \text{ or } \frac{b+1}{q} \\ 0 & \text{if } x \in [0, 1] \setminus \left[\frac{b}{q}, \frac{b+1}{q}\right]. \end{cases}$$

For the sake of brevity let $e(y)=e^{2\pi iy}$. The Fourier-expansion of $\varphi_b(x)$ has the following explicit form:

$$\varphi_b(x) = \sum_{m=-\infty}^{\infty} c_m(b)\, e(mx),$$

$$c_0(b) = \frac{1}{q}, \quad c_m(b) = -\frac{e\left(-\frac{mb}{q}\right)}{2\pi i m}\left[e\left(-\frac{m}{q}\right)-1\right].$$

Let $0<\Delta<1/2q$, and

$$f_b(x) := \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \varphi_b(x+z)\, dz.$$

Then the Fourier coefficients $d_m(b)$ in the expansion

$$f_b(x) = \sum_{m=-\infty}^{\infty} d_m(b)\, e(mx)$$

satisfy the relations

(1) $$d_0(b) = \frac{1}{q}, \quad d_m(b) = e_m(b)\frac{e\left(\frac{m\Delta}{2}\right)-e\left(-\frac{m\Delta}{2}\right)}{2\pi i m},$$

(2) $$d_m(b) = 0 \quad \text{if} \quad m \equiv 0 \pmod q, \quad m \neq 0$$

(3) $$|d_m(b)| \leqq \min\left(\frac{\Delta}{\pi m}, \frac{1}{\pi m^2}\right).$$

Furthermore, we have

a) $$0 \leqq f_b(x) \leqq 1 \quad \text{for every } x,$$

b) $$f_b(x) = 1 \quad \text{if} \quad x \in \left(\frac{b}{q}+\Delta, \frac{b+1}{q}-\Delta\right), \quad b \in \mathcal{A}_q,$$

c) $$f_b(x) = 0 \quad \text{if} \quad x \in [0, 1] \setminus \left(\frac{b}{q}-\Delta, \frac{b+1}{q}+\Delta\right), \quad b \in \mathcal{A}_q.$$

Let now $b_1, b_2, \dots, b_r \in \mathcal{A}_q$,

$$F(x_1, \dots, x_r) = f_{b_1}(x_1) \dots f_{b_r}(x_r).$$

Let $1 \leqq l_1 < l_2 < \dots < l_r \leqq N$ be integers,

$$Q = \left[\frac{1}{q^{l_1+1}}, \dots, \frac{1}{q^{l_r+1}}\right].$$

Let $\mathcal{M}$ denote the set of the vectorials $M=[m_1, \dots, m_r]$ with integer entries m_j. We shall define $t(y)$ as

$$t(y) = F\left(\frac{y}{q^{l_1+1}}, \dots, \frac{y}{q^{l_r+1}}\right).$$

Since the Fourier series of $f_{b_j}(x_j)$ are absolutely convergent, therefore

$$t(y) = \sum_M E_M e(MQy), \tag{2.5}$$

where

$$E_M = d_{m_1}(b_1) \dots d_{m_r}(b_r),$$

MQ denotes the dot product.

Let $(b_0, q) = 1$,

$$\mathscr{H}(x, b_0) = \sum_{\substack{p \le x \\ p \equiv b_0 \pmod q}} t(p), \tag{2.6}$$

$$S(M, b_0) = \sum_{\substack{p \le x \\ p \equiv b_0 \pmod q}} e(MQp). \tag{2.7}$$

From (2.5) we get

$$\mathscr{H}(x, b_0) = \frac{1}{q^r} \pi(x, q, b_0) + \sum_{M \ne 0} E_M S(M; b_0). \tag{2.8}$$

Furthermore, by Lemma 1, applying it with $\xi = \Delta$, we have

$$\Pi\left(x \left| \begin{matrix} 0, & l_1, \dots, l_r \\ b_0, & b_1, \dots, b_r \end{matrix} \right.\right) = \mathscr{H}(x, b_0) + O\big((r+1)\Delta\pi(x)\big). \tag{2.9}$$

Now we shall estimate the exponential sums $S(M, b_0)$.

Let $MQ = \dfrac{A}{H}$, $(A, H) = 1$. First we observe that $E_M = 0$ in (2.4) if $H|q$. Indeed, from

$$\frac{A}{H} q^{l_r+1} = m_{l_r} + m_{l_{r+1}} q^{l_r - l_{r-1}} + \dots,$$

we get that q is a divisor of the left hand side, so $q|m_{l_r}$, $d_{m_{l_r}}(b_r) = 0$, $E_M = 0$.

To estimate $S(M, b_0)$ for relatively small H, we start from the relation

$$S(M, b_0) = \sum_{\substack{l \bmod [H, q] \\ l \equiv b_0 \pmod q}} e\left(\frac{A}{H} l\right) \pi(x, [H, q], l) + O(1). \tag{2.10}$$

Since

$$\pi(x, k, l) - \frac{\operatorname{li} x}{\varphi(k)} \ll x e^{-d_1\sqrt{\log x}}$$

holds uniformly in $1 \le k \le (\log x)^{c_1}$ with an ineffective but positive constant d_1 ([1], Ch. IV), we have

$$S(M, b_0) = \frac{\operatorname{li} x}{\varphi([H, q])} E(b_0; H, q) + O\left(Hxe^{-d_1\sqrt{\log x}}\right), \tag{2.11}$$

where

$$E(b_0; H, q) = \sum_{\substack{l \bmod [H, q] \\ l \equiv b_0 \pmod q}} e\left(\frac{A}{H} l\right). \tag{2.12}$$

Now we observe that the sum (2.12) is zero. Indeed, we put $l=b_0+tq$, $t=0, 1, \ldots$ $\ldots, \frac{[H, q]}{q}-1$, $k=\frac{[H, q]}{q}$, and get

$$E(b_0; H, q) = \sum_{t=0}^{R-1} e\left(\frac{A}{H}(b_0+t_q)\right) = e\left(\frac{A}{H}b_0\right)\frac{e\left(\frac{A}{H}kq\right)-1}{e\left(\frac{A}{H}q\right)-1} = 0.$$

So we have proved

LEMMA 2. *If c_1 is a positive constant,* $1 \leqq H \leqq (\log x)^{c_1}$, $H \nmid q$, *then*

$$S(M, b_0) = O\left(xe^{-d_2\sqrt{\log x}}\right) \tag{2.13}$$

holds with a suitable positive constant d_2.

REMARK. A somewhat weaker estimation could be obtained by using the Barban—Tshudakov—Linnik theorem [3] and its generalization in the wider range $1 \leqq H \leqq x^c$, since H runs over the integers all prime factors of which divides q.

We shall use the following theorem of I. M. Vinogradov ([2], Theorem 2) which we state as

LEMMA 3. *If* $1 \leqq H < x$, $H \nmid q$, *then*

$$S(M, b_0) \ll \log^3 x \left[e^{-0.5\sqrt{\log x}} + \sqrt{\frac{1}{H}+\frac{H}{x}}\right]. \tag{2.14}$$

LEMMA 4. *Let ε_0 be a sufficiently small positive constant,* $y \geqq xe^{-\varepsilon_0\sqrt{\log x}}$, $\log H < \varepsilon\sqrt{\log x}$. *Then*

$$\sum_{\substack{x-y \leqq p \leqq x \\ p \equiv b_0 \pmod q}} e\left(\frac{A}{H}p\right) \ll \frac{y \log\log H}{\sqrt{H}\log x}. \tag{2.15}$$

Lemma 4 is stated and proved in [2] (Ch. X, § 3, Lemma 4) without assuming the condition $p \equiv b_0 \pmod q$. Since this little modification does not imply any important change in the proof, we omit it.

3. Now we are in a position to formulate and prove our theorems.

THEOREM 1. *Let* $q^N < x < q^{N+1}$, $0 \leqq r < \sqrt{N}$, $1 \leqq l_1 < l_2 < \ldots < l_r \leqq N$, $b_0, b_1, \ldots$ $\ldots, b_r \in \mathcal{A}_q$, $(b_0, q)=1$. *Then*

$$\Pi\left(x \,\middle|\, \begin{matrix} 0, & l_1, \ldots, l_r \\ b_0, & b_1, \ldots, b_r \end{matrix}\right) = \frac{\operatorname{li} x}{q^r\varphi(q)} + O\left(\frac{\operatorname{li} x}{q^r}e^{-d(\log x)^{1/2}}\right) + \tag{3.1}$$

$$+O\left(\frac{x}{q^r}(\log x)^3\left(\frac{q^{l_r}}{x}\right)^{1/2}\right)$$

with a suitable positive constant d, uniformly in $r, l_1, \ldots, l_r, b_1, \ldots, b_r$.

PROOF. Since $H|q^{l_r+1}$, therefore $1 \leqq H < q^{l_r+1}$, and so by Lemmas 2, 3, 4 we get

$$S(M, b_0) \ll \pi(x)\left\{e^{-c_4(\log x)^{1/2}} + (\log x)^4 \left(\frac{q^{l_r}}{x}\right)^{1/2}\right\}$$

with a suitable positive constant c_4, uniformly for every M.

From (2.7), (2.8) we get

$$\Pi\left(x \middle| \begin{matrix} 0, & l_1, \dots, l_r \\ b_0, & b_1, \dots, b_r \end{matrix}\right) = \frac{\operatorname{li} x}{q^r \varphi(q)} + O((r+1)\Delta\pi(x)) + \tag{3.2}$$

$$+ O\left(K\pi(x)\, e^{-c_4(\log x)^{1/4}}\right) + O\left(K\left(\frac{q^{l_r}}{x}\right)^{1/2} \pi(x)(\log x)^4\right),$$

where $K = \Sigma |E_M|$.

From inequality (3) in Section 2 we deduce

$$K \leqq \left(\frac{1}{q} + 2\sum_{m=1}^{\infty} \min\left(\frac{\Delta}{\pi m}, \frac{1}{\pi m^2}\right)\right)^r,$$

whence

$$K \leqq \left(\frac{1}{q} + 2\Delta \log e/\Delta\right)^r \tag{3.3}$$

immediately follows. Let $\Delta = e^{-2\log q \cdot \sqrt{N}}$. Then the right hand side of (3.3) is $O(q^{-r})$, furthermore

$$(r+1)\Delta\pi(x) \ll \frac{\operatorname{li} x}{q} e^{-\sqrt{\log x}}.$$

This completes the proof of Theorem 1.

Let now $l_0 = 0$, $(1 \leqq) l_1 < l_2 < \dots < l_r \leqq N$, $q^N \leqq x < q^{N+1}$, $2^r < N^{1/5}$, $(b_0, q) = 1$, $b_0, b_1, \dots, b_r \in \mathcal{A}_q$. Let us assume that $N - l_r < N^{1/4}$ and let ν be the largest integer for which $l_\nu < 2l_{\nu+1} - N - 2N^{1/4}$ is satisfied, i.e.

$$l_s \geqq 2l_{s+1} - N - 2N^{1/4} \quad (s = r, \dots, \nu+1), \quad l_{r+1} := N$$

$$l_\nu < 2l_{\nu+1} - N - 2N^{1/4}.$$

Let

$$t = l_{\nu+1} - \left[\frac{1}{2} N^{1/4}\right].$$

We have

$$N - l_s \leqq 2N^{1/4} + 2(N - l_{s+1}) \quad (s \geqq \nu+1),$$

whence

$$N - l_{\nu+1} \leqq (2 + 2^2 + \dots + 2^{r-\nu}) N^{1/4} + 2^{r-\nu}(N - l_r) \ll N^{1/5+1/4} = N^{9/20},$$

and so

$$t \geqq N - cN^{9/20}. \tag{3.4}$$

Let the function $\delta(u)$ be defined by

$$\delta(u) = \begin{cases} 1 & \text{if } a_{l_s - t}(u) = b_s \quad (s = \nu+1, \dots, r) \\ 0 & \text{otherwise.} \end{cases}$$

The primes $p\in[1, x]$ can be written in the form

$$\left\{\begin{array}{l}p = uq^t+v \\ 0 \leqq v < q^t, \quad u = 0, 1, \dots, \left[\frac{x}{q^t}\right].\end{array}\right. \tag{3.5}$$

The fulfilment of the conditions $a_{l_h}(p)=b_h$ $(h=0, \dots, v)$ depends only on the value v, consequently the condition $a_{l_h}(p)=b_n$ $(h=0, \dots, r)$ is equivalent with the following conditions:

$$a_{l_h}(v) = b_h \quad (h = 0, \dots, v) \quad \text{and} \quad \delta(u) = 1.$$

The number of primes p with $u=0$ or $u=\left[\frac{x}{q^t}\right]$ is less than $2q^t$, therefore

$$\Pi\left(x\left|\begin{matrix}0, & l_1, \dots, l_r \\ b_0, & b_1, \dots, b_r\end{matrix}\right.\right) = \sum_{u=1}^{[x/q^t]} \delta(u)\left[\Pi\left((u+1)q^t\left|\begin{matrix}0, & l_1, \dots, l_r \\ b_0, & b_1, \dots, b_r\end{matrix}\right.\right)-\right. \tag{3.6}$$

$$\left.-\Pi\left(uq^t\left|\begin{matrix}0, & l_1, \dots, l_r \\ b_0, & b_1, \dots, b_r\end{matrix}\right.\right)\right]+O(q^t).$$

To estimate the right hand side of (3.6), we shall use Theorem 1:

$$(\Pi_r :=)\,\Pi\left(x\left|\begin{matrix}0, & l_1, \dots, l_r \\ b_0, & b_1, \dots, b_r\end{matrix}\right.\right) = \frac{1}{\varphi(q)\,q^v} \sum_{n\leqq x/q^t} \delta(u)\left[\mathrm{li}((u+1)q^t)-\mathrm{li}(uq^t)\right]+ \tag{3.7}$$

$$+O\big(q^{-v}\,\mathrm{li}\, x e^{-d(\log x)^{1/2}} \sum_{u\leqq x/q^t} \delta(u)\big)+O\left((\log x)^3 \sum_{n\leqq x/q^t} \frac{\delta(u)(uq^t)^{1/2} q^{(1/2)l_v}}{q^v}\right).$$

From (1.2) and the definition of δ we get

$$\sum_{n\leqq x/q^t} \delta(u) \ll xq^{v-r-t},$$

so the first error term on the right hand side of (3.7) is less than

$$\ll x\,\mathrm{li}\, xq^{-r-t} e^{-d(\log x)^{1/2}},$$

and by taking into account (3.4),

$$\ll \frac{\mathrm{li}\, x}{q^r} e^{-d(\log x)^{1/2}}. \tag{3.8}$$

The second error term is less than

$$\ll (\log x)^3 x^{1/2} q^{(1/2)l_v-v} \sum_{n\leqq x/q^t} \delta(u) \ll \frac{x(\log x)^3}{q^r}\,\frac{x^{1/2} q^{(1/2)l_v}}{q^t} \ll \frac{x(\log x)^3}{q^r}\, q^{(1/2)(N+l_v)-t},$$

and by observing that

(3.9)

$$\frac{1}{2}(N+l_v)-t = \frac{1}{2}(N+l_v-2l_{v+1})+\left[\frac{1}{2}N^{1/4}\right] \leqq -\frac{1}{2}N^{1/4} \ll \frac{x}{q^r} e^{-(1/4)(\log x)^{1/4}}.$$

Since for $u \geqq 1$

$$\mathrm{li}\,((u+1)\,q^t) - \mathrm{li}\,(uq^t) = \frac{q^t}{\log uq^t} + O\left(\frac{q^t}{(\log x)^2}\right) =$$

$$= \frac{q^t}{\log x} + O\left(q^t \frac{\left|\log \frac{x}{q^t u}\right| + 1}{\log^2 x}\right) = \frac{q^t}{\log x} + O\left(q^t (\log x)^{9/20-2}\right),$$

we get

$$\Pi_r = \frac{q^t}{\varphi(q)\,q^v} \frac{1}{\log x} \sum_{u \leqq x/q^t} \delta(u) + O\left(\frac{x}{q^r} e^{-(1/2)(\log x)^{1/4}}\right) + O(q^t) + \tag{3.10}$$

$$+ O\left(q^{t-v} (\log x)^{9/20-2} \sum_{u \leqq x/q^t} \delta(u)\right).$$

Arguing as in the proof of (3.5) we get

$$A\left(x\middle|\begin{matrix}0, & l_1, \dots, l_r\\ b_0, & b_1, \dots, b_r\end{matrix}\right) =$$

$$= \sum_{u=1}^{[x/q^t]} \delta(u) \left[A\left((u+1)\,q^t\middle|\begin{matrix}0, & l_1, \dots, l_v\\ b_0, & b_1, \dots, b_v\end{matrix}\right) - A\left(uq^t\middle|\begin{matrix}0, & l_1, \dots, l_v\\ b_0, & b_1, \dots, b_v\end{matrix}\right)\right] + O(q^t).$$

Since the difference just after $\delta(u)$ is exactly q^{t-v-1}, we have

$$A\left(x\middle|\begin{matrix}0, & l_1, \dots, l_r\\ b_0, & b_1, \dots, b_r\end{matrix}\right) = q^{t-v-1} \sum_{u \leqq x/q^t} \delta(u) + O(q^t).$$

Substituting this in the right hand side of (3.10), and taking into account that

$$q^t \ll q^N/q^{(1/2)N^{1/4}} \quad \text{and} \quad q^r < 2^{cr} \ll N^{c/5},$$

we get

$$\Pi\left(x\middle|\begin{matrix}0, & l_1, \dots, l_r\\ b_0, & b_1, \dots, b_r\end{matrix}\right) = \frac{q}{\varphi(q)\log x} A\left(x\middle|\begin{matrix}0, & l_1, \dots, l_r\\ b_0, & b_1, \dots, b_r\end{matrix}\right) + \tag{3.11}$$

$$+ O\left(\frac{x}{q^r} (\log x)^{9/20-2}\right).$$

So we have proved the following

THEOREM 2. *Let* $q^N \leqq x < q^{N+1}$, *r be a nonnegative integer*, $2^r < N^{1/5}$, $l_0 = 0$, $(1 \leqq) l_1 < l_2 < \dots < l_r \leqq N$; $b_0, b_1, \dots, b_r \in \mathcal{A}_q$, $(b_0, q) = 1$. *Then* (3.11) *holds uniformly in* $l_1, \dots, l_r, b_1, \dots, b_r, r$.

Let

$$(\delta_r :=)\, \delta\left(x\middle|\begin{matrix}l_0, & l_1, \dots, l_r\\ b_0, & b_1, \dots, b_r\end{matrix}\right) := \frac{1}{\mathrm{li}\, x} \Pi\left(x\middle|\begin{matrix}l_0, & l_1, \dots, l_r\\ b_0, & b_1, \dots, b_r\end{matrix}\right) - \tag{3.12}$$

$$- \frac{\varepsilon(l_0)}{x} A\left(x\middle|\begin{matrix}l_0, & l_1, \dots, l_r\\ b_0, & b_1, \dots, b_r\end{matrix}\right),$$

where $\varepsilon(0) = q/\varphi(q)$ and $\varepsilon(l) = 1$ if $l \geqq 1$.

COROLLARY. *Let* $0 \leqq l_0 < l_1 < \ldots < l_r \leqq N$. *Then under the conditions of Theorem* 1

$$\delta_r \ll \frac{1}{q^r}\left(e^{-d(\log x)^{1/2}} + (\log x)^4 \left(\frac{q^{l_r}}{x}\right)^{1/2}\right), \tag{3.13}$$

while under the conditions of Theorem 2

$$\delta_r \ll \frac{1}{q^r}(\log x)^{(9/20)-1}. \tag{3.14}$$

4. Let f be a real valued q-additive function, i.e. such that

$$f(n) = \sum_{j=0}^{\infty} f(a_j(n)q^j), \quad f(0) = 0.$$

Let

$$m_k = \frac{1}{q}\sum_{b=0}^{q-1} f(bq^k), \quad \sigma_k^2 = \frac{1}{q}\sum_{b=0}^{q-1} f^2(bq^k) - m_k^2, \tag{4.1}$$

and

$$M(x) = \sum_{k \leqq N} m_k, \quad D^2(x) = \sum_{k \leqq N} \sigma_k^2, \tag{4.2}$$

where $q^N \leqq x < q^{N+1}$.

Since for $x = q^{N+1}-1$, $f(n)$ can be interpreted as the sum of random variables, we can prove that

$$\frac{1}{q^{N+1}} \sum_{n=0}^{q^{N+1}-1} \left(f(n) - M(q^{N+1}-1)\right)^2 = D^2(q^{N+1}-1),$$

whence for every $x \geqq 1$

$$\sum_{n \leqq x} \left(f(n) - M(x)\right)^2 \leqq cxD^2(x) \tag{4.3}$$

follows.

THEOREM 3. *We have*

$$\sum_{p \leqq x} \left(f(p) - M(x)\right)^2 \leqq c\pi(x) D^2(x) \tag{4.4}$$

with a suitable positive constant $c = c(q)$.

PROOF. Let

$$H(bq^j) = f(bq^j) - m_j, \quad t_0(n) = H\left(a_0(n)\right) = f\left(a_0(n)\right) - m_0,$$

$$t_1(n) = \sum_{j=1}^{L} H\left(a_j(n)\right), \quad t_2(n) = \sum_{j=L+1}^{N} H\left(a_j(n)\right).$$

First we observe that

$$t^2(n) \leqq 3\left(t_0^2(n) + t_1^2(n) + t_2^2(n)\right).$$

We have

$$\sum_{p \leqq x} t_0^2(p) = \sum_{(b,q)=1} H^2(b)\,\pi(x,q,b) + O(\sigma_0^2) \ll \pi(x)\,\sigma_0^2.$$

Let us now consider

$$(E_i =)\ \frac{1}{\operatorname{li} x}\sum_{p\le x} t_i^2(p) - \frac{1}{x}\sum_{n\le x} t_i^2(n).$$

We have

$$E_i = \sum_j \sum_{b\in\mathcal{A}_q} H^2(bq^j)\,\delta\left(x\middle|\begin{matrix} j\\ b\end{matrix}\right) + 2\sum_{j_1<j_2}\ \sum_{b_1,b_2\in\mathcal{A}_q} H(b_1q^{j_1})H(b_2q^{j_2})\,\delta\left(x\middle|\begin{matrix} j_1, & j_2\\ b_1, & b_2\end{matrix}\right),$$

where j, j_1 and j_2 in E_1 run over the integers in $[1, L]$, while in $E_2, j, j_1, j_2\in[L+1, N]$. Let $L=N-c_1\log N$ with a constant so large that

$$\delta\left(x\middle|\begin{matrix} j_1, & j_2\\ b_1, & b_2\end{matrix}\right) \ll (\log x)^{-2},\quad \delta\left(x\middle|\begin{matrix} j\\ b\end{matrix}\right) \ll (\log x)^{-2}$$

hold uniformly for $j\le L$, $j_1<j_2\le L$. Then

$$E_1 \ll (\log x)^{-2}\Big(\sum_{j\le L}\sum_b |H(bq^j)|\Big)^2,$$

and by Cauchy inequality,

$$E_1 \ll (\log x)^{-2} L\Big(\sum_{\nu=1}^{L}\sigma_\nu^2\Big).$$

To estimate E_2 we use (3.14) and get

$$E_2 \ll (\log x)^{9/20-1}\Big(\sum_{L+1<j\le N}\sum_b |H(bq^j)|\Big)^2,$$

whence by Cauchy inequality,

$$E_2 \ll (\log x)^{9/20-1}\log N\Big(\sum_{\nu=L+1}^{\nu}\sigma_\nu^2\Big).$$

Furthermore, applying (4.3) for $t_i(n)$ instead of $f(n)-M(x)$, we deduce immediately that

$$\sum t_1^2(n) \ll x\sum_{\nu=1}^{L}\sigma_\nu^2,\quad \sum t_2^2(n) \ll x\sum_{\nu=L+1}^{N}\sigma_\nu^2.$$

Collecting our previous estimations we get (4.4).

Let $s_j(b)$ be arbitrary complex numbers defined for $b\in\mathcal{A}_q$, $j=1, 2, \ldots$. Let $N=N(x)$ be the integer satisfying the condition $q^N\le x<q^{N+1}$, and let t_N be a sequence of positive integers for which

$$(N-t_N)/\log N \to \infty \quad (N\to\infty) \tag{4.5}$$

holds. Let

$$S(n) = \sum_{j=1}^{t_N} s_j\big(a_j(n)\big), \tag{4.6}$$

$$E_k(x) := \frac{1}{\operatorname{li} x}\sum_{p\le x} S^k(p) - \frac{1}{x}\sum_{n\le x} S^k(n), \tag{4.7}$$

$k\ge 1$, integer.

Let the coefficients $C(k_j l_1, \ldots, l_r;\ \nu_1, \ldots, \nu_r)$ be defined by the identity

(4.8) $(x_1+\ldots+x_{t_N})^k = \sum_{i=1}^{k} \sum_{\substack{1\le l_1<l_2<\ldots<l_r(\le t_N)\\ \nu_1+\ldots+\nu_r=k}} C(k;\ l_1, \ldots, l_r;\ \nu_1, \ldots, \nu_r)\, x_{l_1}^{\nu_1}\ldots x_{l_r}^{\nu_r}.$

Let k be fixed. From the Corollary we get

$$\delta\left(x\left|\begin{matrix} l_1, \ldots, l_r\\ b_1, \ldots, b_r\end{matrix}\right.\right) \ll (\log x)^{-H}, \quad \text{if} \quad 1 \le l_1 < \ldots < l_r \le t_N, \quad r \le k,$$

with an arbitrary large H for $x > x(H)$. Using twice the polynomial identity (4.8), we get

$$E_k(x) = \sum_{r=1}^{k} \sum_{l_i, \nu_i} C(k;\ l_1, \ldots, l_r;\ \nu_1, \ldots, \nu_r) \sum_{b_1, \ldots, b_r} \Pi s_{l_i}^{\nu_i}(b_i)\,\delta\left(x\left|\begin{matrix} l_1, \ldots, l_r\\ b_1, \ldots, b_r\end{matrix}\right.\right) \ll$$

$$\ll (\log x)^{-H} \sum_{r} \sum_{l_i; \nu_i} C(k;\ l_1, \ldots, l_r;\ \nu_1, \ldots, \nu_r) \sum_{b_1, \ldots, b_r} \Pi |s_{l_i}^{\nu_i}(b_i)|.$$

Since

$$\sum_{b_1, \ldots, b_r} \Pi |s_{l_i}(b_i)|^{\nu_i} \le \prod_{i=1}^{r} \Big(\sum_{b_i} |s_{l_i}(b_i)|\Big)^{\nu_i},$$

by using (4.8) we get

(4.9) $$E_k(x) \ll (\log x)^{-H} \Big(\sum_{l=1}^{t_N} \sum_{b} |s_j(b)|\Big)^k.$$

So we have proved

LEMMA 5. *Under the condition* (4.5) *the inequality* (4.9) *holds for every* $k \ge 1$, *where the constant implied by* $\ll$ *in* (4.9) *may depend on* k.

5. Now assume that $f(bq^j)$ is bounded as $j \to \infty$, $b \in \mathcal{A}_q$. We are interested in the limit of the distribution functions

(5.1) $$G_x(y) = \frac{1}{x}\{n;\ 0 \le n < x \,|\, f(n) < M(x) + yD(x)\}$$

for $x \to \infty$.

Let the mutually independent random variables $\xi_0, \xi_1, \ldots$ be defined by

$$P\big(\xi_j = f(bq^j)\big) = \frac{1}{q} \quad (b \in \mathcal{A}_q,\ j = 0, 1, 2, \ldots),$$

and let $\eta_N = \xi_0 + \ldots + \xi_{N-1}$. It is obvious that

$$G_{q^N}(y) = H_N(y) = P\big(\eta_N < M(q^{N-1}) + yD(q^{N-1})\big).$$

The condition $f(bq^j) = O(1)$ implies

$$\frac{1}{\sigma_j^2} E(|\xi_j - m_j|^3) = O(1) \quad (j \to \infty)$$

and so by Berry's theorem (see e.g. [4], Ch. XVI) we have

$$|G_{q^N}(y)-\varphi(y)| \leqq \frac{c}{D(q^{N-1})},$$

where

$$\varphi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\,dt.$$

Hence we deduce that

$$\lim_{x\to\infty} G_x(y) = \varphi(y) \tag{5.2}$$

assuming that $D(x)\to\infty$ $(x\to\infty)$.

Let $\varepsilon_N\to 0$, $t_N\to\infty$, such that

$$N-t_N\to\infty,\quad N-t_N \leqq \varepsilon_N D(q^N),\ D(q^N)-D(q^{t_N}) \leqq \varepsilon_N D(q^N). \tag{5.3}$$

Since $m_k=O(1)$, we have

$$M(q^N)-M(q^{t_N}) \ll N-t_N. \tag{5.4}$$

Let $q^N<x<q^{N+1}$, and write the integers $n\leqq x$ in the form

$$n = uq^{t_N}+v,\quad 0 \leqq v < q^{t_N},\quad u = 0, 1, \ldots, q^{N-t_N}. \tag{5.5}$$

From (5.5), (5.3) we get

$$\left|\frac{f(n)-M(x)}{D(x)} - \frac{f(v)-M(q^{t_N-1})}{D(q^{t_N-1})}\right| \leqq \varrho_N,$$

where ϱ_N is a suitable sequence tending to zero. The number of integers $n\leqq x$ having a fixed residue v (mod q^{t_N}) is $x/q^{t_N}+O(1)$. Therefore we have

$$O(q^{t_N-M})+H_{t_N}(y-\varrho_N) \leqq G_x(y) \leqq H_{t_N}(y+\varrho_N)+O(q^{t_N-N}),$$

which implies (5.2).

So we proved

LEMMA 6. *If $f(bq^j)=O(1)$, $D(x)\to\infty$, then* (5.2) *holds.*

THEOREM 4. *Let $f(bq^j)$ be bounded, $D(x)\to\infty$,*

$$F_x(y) := \frac{1}{\operatorname{li} x}\{p \leqq x \mid f(p) < M(x)+yD(x)\}. \tag{5.6}$$

Then there exists a sequence x_ν $(\to\infty)$ such that

$$F_{x_\nu}(y) \to \varphi(y) \quad (\nu\to\infty) \tag{5.7}$$

for all real numbers y. If in addition

$$D(x/\log x)/D(x) \to 1 \quad (x\to\infty), \tag{5.8}$$

then

$$\lim_{x\to\infty} F_x(y) = \varphi(y) \tag{5.9}$$

holds.

PROOF. First we observe that there exists a suitable sequence $x_\nu(\to\infty)$ and a sequence $r_\nu(\to\infty)$ such that

(5.10) $$D(z(x_\nu))/D(x_\nu) \to 1 \quad (\nu\to\infty),$$

where

(5.11) $$z(x_\nu) = x_\nu e^{-r_\nu \log\log x_\nu}.$$

Assuming the contrary, we would have

$$D^2(x)-D^2(xe^{-d(\log\log x)}) > \alpha D^2(x)$$

with suitable positive constants α, d for every large x, which gives that

$$D^2(x) \geqq \beta \frac{\log x}{\log\log x} D^2(x/2),$$

with a constant $\beta>0$, but this contradicts the fact that

$$D^2(x) = \sum \sigma_\nu^2 \ll \log x.$$

Under the additional condition (5.8) we shall choose $x_\nu=\nu$ $(\nu=1, 2, \ldots)$ with an appropriate r_ν, $r_\nu\to\infty$.

Let us assume that x_ν and r_ν are chosen properly, let s_x be the integer such that $q^{s_x}\leqq z(x)<q^{s_x+1}$. Let furthermore

$$f_x(n) = \sum_{j=1}^{s_x} f(a_j(n)q^j), \quad g_x(n) = \sum_{j=s_x+1}^{N} f(a_j(n)q^j)+f(a_0(n)q^j),$$

$$\Delta(x) = M(x)-M(z(x))+m_0, \quad q^N \leqq x < q^{N+1}.$$

Let the variable x run over the values x_ν. We shall define

$$R^2(x) = \sum_{s_x<l\leqq N} \sigma_l^2.$$

From (5.10) we get

(5.11) $$R(x)/D(x) \to 0.$$

Let

$$\alpha_n = \frac{g_x(n)-\Delta(x)}{D(x)}, \quad \beta_n = \frac{f_x(n)-M(z(x))}{D(x)}.$$

From (4.3) and Theorem 3 we get

(5.12) $$\sum |\alpha_n|^2 \ll x\left(\frac{R(x)}{D(x)}\right)^2, \quad \sum |\alpha_p|^2 \ll \mathrm{li}\, x\left(\frac{R(x)}{D(x)}\right)^2.$$

From (5.12) we get

$$\frac{1}{x}\#\{n\leqq x \,\big|\, |\alpha_n|>\varepsilon\}\to 0, \quad \frac{1}{\mathrm{li}(x)}\#\{p\leqq x \,\big|\, |\alpha_p|>\varepsilon\}\to 0$$

and so the quantities $\alpha_n+\beta_n=D^{-1}(x)(f(n)-M(x))$ $(n=0, \ldots, x)$ are distributed as β_n $(n=0, \ldots, x)$ in the limit, and the same is true for $\alpha_p+\beta_p$ and β_p. Since $\alpha_n+\beta_n$ $(n=0, \ldots, x)$ are distributed in limit according to the Gaussian law, the same is true for β_n $(n=0, \ldots, x)$, consequently

$$\frac{1}{x}\sum_{n\leqq x}\beta_n^k \to \mu_k \quad (x\to\infty)$$

for every integer $k\geqq 1$. It remains to prove only that

$$\frac{1}{\operatorname{li} x}\sum_{p\leqq x}\beta_p^k \to \mu_k \quad (x\to\infty).$$

But this is an immediate consequence of Lemma 5. By taking

$$s_j(b)=\frac{f(dq^j)-m_j}{D(x)},$$

from (4.9) we get

$$\frac{1}{\operatorname{li} x}\sum_{p\leqq x}\beta_p^k-\frac{1}{x}\sum_{n\leqq x}\alpha_n^k \ll (\log x)^{-H}\Big(\sum_l\sum |s_j(b)|\Big)^k \ll$$

$$\ll (\log x)^{-H} N^{2k}\Big(\sum_l\sum_b |s_j(b)|^2\Big)^{k/2} \ll (\log x)^{-H+2k}=o(1),$$

if $H>2k$.

6. In paper [6] written jointly with J. Mogyoródi we have considered the function

$$\alpha(n)=\sum_{j=0}^{\infty} a_j(n)$$

for prime n's, and proved that under the unproved density hypothesis for the Riemann zeta function,

$$F_x(y):=\frac{1}{\operatorname{li} x}\#\{p\leqq x \mid \alpha(p)<M_x+yD_x\}=\varphi(y)+O\big((\log\log x)^{-1/3}\big)$$

holds, where

$$M_x=\frac{q-1}{2}\frac{\log x}{\log q},\quad D_x^2=\frac{q^2-1}{12}\frac{\log x}{\log q}.$$

Theorem 4 gives immediately

$$\lim_{x\to\infty}F_x(y)=\varphi(y)$$

without the unproved hypothesis since the condition (5.8) holds.

7. H. Delange [7] investigated the existence of the limit distribution of values of real q-additive functions. Let $N_x(\alpha)$ denote the number of those integers $n(\leqq x)$ for which $f(n)<\alpha$. We say that $f(n)$ has a limit distribution with the distribution function $F(\alpha)$, if

$$\frac{N_x(\alpha)}{x}\to F(\alpha)$$

for all continuity points of $F(\alpha)$. H. Delange proved that the fulfilment of the following conditions are necessary and sufficient for f to have a limit distribution:

(a) $\sum_{j=0}^{\infty}\sum_{a=1}^{q-1} f(aq^j)$ converges,

(b) $\sum_{j=0}^{\infty}\sum_{a=1}^{\infty} f^2(aq^j)$ converges.

Let $M_x(\alpha)$ denote the number of those primes $p(\leqq x)$ for which $f(p)<\alpha$. We say that $f(p)$ has a limit distribution with the distribution function $F(\alpha)$, if

$$\frac{M_x(\alpha)}{\pi(x)} \to F(\alpha) \tag{7.1}$$

at every continuity point of $F(\alpha)$. In [8] we have proved that the fulfilment of (a), (b) implies (7.1), if q is a power of an odd prime. The proof was based on the Barban—Linnik—Tshudakov theorem [3] stating that

$$\pi(x, q^r, l) = \big(1+O(\log x)^{-c}\big)\frac{\operatorname{li} x}{\varphi(q^r)}$$

whenever $x \geqq (q^r)^3$.

From Theorem 3 we can deduce another proof of our theorem. Let us assume that (b) holds. This implies that the sequences

$$m_j = \frac{1}{q}\sum_{a=1}^{q-1} f(aq^j), \quad \sigma_j^2 = \frac{1}{q}\sum_{a=0}^{q-1} \big(f(aq^j)-m_j\big)^2$$

tend to zero, furthermore that

$$\sum_{j=0}^{\infty} \sigma_j^2 < \infty.$$

Let L_N be a sequence of positive integers tending monotonically to infinity and assume that $L_N = O(\log\log N)$ holds. Let $q^N \leqq x \leqq q^{N+1}$, and

$$t_N(n) = \sum_{j=L_N+1}^{N} \big(f(a_j(n)q^j)-m_j\big).$$

Since

$$\sup_{b\in\mathcal{A}_q}\ \sup_{j\geqq L_N+1} |f(bq^j)-m_j| \leqq \varrho_N, \quad \varrho_N \to 0$$

and

$$\sum_{l=L_N}^{\infty} \sigma_l^2 \leqq \varrho_N', \quad \varrho_N' \to 0,$$

therefore from Theorem 3 we deduce immediately that

$$\sum_{p<x} t_N^2(p) \leqq \varepsilon_N \pi(x), \quad \varepsilon_N \to 0. \tag{7.2}$$

Let

$$h_N(n) = \sum_{j=0}^{L_N} \big(f(a_j(n)q^j)-m_j\big), \quad K_r = \sum_{j=0}^{r} m_j.$$

Let us consider the characteristic function

$$\varphi_x(\tau) = \frac{1}{\pi(x)} \sum_{p \leqq x} e^{i\tau(f(p)-K_N)}. \tag{7.3}$$

The convergence of $\varphi_x(\tau)$ is necessary and sufficient for the existence of the limit distribution for $f(p)-K_N$. Since

$$f(p)-M_N = h_N(p)-M_{L_N}+t_N(p),$$

from (7.2) we deduce that

$$\varphi_x(\tau) = \frac{1}{\pi(x)} \sum_{p \leqq x} e^{i\tau(h_N(p)-M_{L_N})}+O(1)$$

uniformly in $|\tau|<c$. Since $h_N(p)$ depends only on $p \pmod{q^{L_N}}$, therefore we have

$$\varphi_x(\tau) = \frac{1}{\pi(x)} \sum_{l<q^{L_N}} e^{i\tau(h_N(l)-M_{L_N})}\pi(x, q^{L_N}, l)+o(1),$$

and so by the prime number theorem for the arithmetical progressions we get (7.4)

$$\varphi_x(\tau) = \frac{1}{\varphi(q)} e^{-\tau m_0}\Big(\sum_{\substack{l=1 \\ (l,q)=1}}^{q-1} e^{i\tau f(l)}\Big) \prod_{j=1}^{L_N} \frac{1}{q}\Big(\sum_{a=0}^{q-1} e^{i\tau f(aq^j)}\Big)e^{-i\tau m_j}+o(1) = \Psi_{L_N}(\tau)+o(1).$$

Condition (b) implies that $\Psi_{L_N}(\tau)$ converges as $L_N\to\infty$. Then from (7.4) $\varphi_x(\tau)$ has a limit as well, consequently

$$\lim \frac{1}{\pi(x)} \{p \leqq x | f(p)-M_N < \alpha\} = F(\alpha) \tag{7.5}$$

exists for every continuity point of $F(\alpha)$.

Let now assume that (a) holds, i.e. that $\sum m_j$ converges. Then from (7.4) we get

$$\tilde{\varphi}_x(\tau) := \frac{1}{\pi(x)} \sum_{p \leqq x} e^{i\tau f(p)} = \tilde{\Psi}_{L_N}(\tau)+o(1), \tag{7.6}$$

where

$$\tilde{\Psi}_k(\tau) = \frac{1}{\varphi(q)}\Big(\sum_{\substack{l=1 \\ (l,q)=1}}^{q-1} e^{i\tau f(l)}\Big) \prod_{j=1}^{k} \frac{1}{q}\Big(\sum_{a=0}^{q-1} e^{i\tau f(aq^j)}\Big). \tag{7.7}$$

The convergence of $\tilde{\Psi}_k(\tau)$ $(k\to\infty)$ is guaranteed by (a), (b), so they imply the convergence of $\tilde{\varphi}_x(\tau)$ $(x\to\infty)$.

So we have proved the following

THEOREM 5. *Let f be a realvalued q-additive function such that* (b) *holds. Then* (7.5) *holds with a distribution function $F(\alpha)$ the characteristic function of which can be written as*

$$\varphi_F(\tau) = \frac{1}{\varphi(q)}\Big(\sum_{\substack{l=1 \\ (l,q)=1}}^{q-1} e^{i\tau f(l)}\Big)e^{-i\tau m_0} \prod_{j=1}^{\infty} \frac{1}{q}\Big(\sum_{a=0}^{q-1} e^{i\tau f(aq^j)}\Big)e^{-i\tau m_j}.$$

Assume in addition that (a) *holds. Then $f(p)$ has a limit distribution with the distribution function $G(\alpha)$, the characteristic function of which can be represented as*

$$\varphi_G(\tau) = \frac{1}{\varphi(q)}\Big(\sum_{l=1}^{q-1} e^{i\tau f(l)}\Big)\prod_{j=1}^{\infty}\frac{1}{q}\Big(\sum_{a=0}^{q-1} e^{i\tau f(aq^j)}\Big). \tag{7.9}$$

It seems probable that the fulfilment of (a) and (b) is necessary for the existence of the limit distribution of $f(p)$. Presently we can prove it under the condition $f(bq^j)=O(1)$.

THEOREM 6. *Let f be a realvalued q-additive function such that the limit distribution of $f(p)$ exists. Assume that $f(bq^j)$ is bounded as $j\to\infty$, $b\in\mathcal{A}_q$. Then the conditions* (a), (b) *hold.*

PROOF. Let

$$G(\alpha) = \lim \frac{1}{\pi(x)} \#\{p \leqq x \mid f(p) < \alpha\},$$

$$m_j = \frac{1}{q}\sum_{a=0}^{q-1} f(aq^j),\quad \sigma_j^2 = \frac{1}{q}\sum_{a=0}^{q-1}\big(f(aq^j)-m_j\big)^2,\quad D^2(x) = \sum_{j=0}^{N}\sigma_j^2,$$

where $N=N(x)$ is defined by $q^N \leqq x < q^{N+1}$. First we prove that $D(x)$ is bounded. Let us assume that $D(x)\to\infty$. The conditions of Theorem 4 are satisfied. Let x_ν be such a sequence for which (5.8) holds. Since for all $\gamma<\alpha$,

$$G(\alpha)-G(\gamma) = \lim \frac{1}{\pi(x_\nu)} \#\{p \leqq x_\nu \mid \gamma \leqq f(p) \leqq \alpha\} =$$

$$= \lim \frac{1}{\pi(x_\nu)} \#\left\{p \leqq x_\nu \,\middle|\, \frac{\gamma - M_N}{D^2(x_\nu)} \leqq \frac{f(p)-M_N}{D^2(x_\nu)} < \frac{\alpha-M_N}{D^2(x_\nu)}\right\} =$$

$$= \lim_\nu \left(\varphi\left(\frac{\alpha-M_N}{D^2(x_\nu)}\right) - \varphi\left(\frac{\gamma-M_N}{D^2(x_\nu)}\right)\right),$$

and the last difference is less than $O\left(\frac{\gamma-\alpha}{D^2(x_\nu)}\right)$, this implies $G(\alpha)=G(\gamma)$. This cannot be satisfied, so $D(x)$ is bounded. Since $f(0)=0$, therefore $\sigma_j^2 > \frac{1}{q} m_j^2$, and so

$$\frac{1}{q}\sum_j \sum_a f^2(aq^j) \leqq \sum (\sigma_j^2 + m_j^2) < \infty.$$

Consequently condition (b) is satisfied. Then from the first assertion stated in Theorem 5 we have that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \leqq x \mid f(p) - M_x < \alpha\} = F(\alpha)$$

holds for all continuity points of $F(\alpha)$. Now we shall prove that M_x is bounded. Assume in the contrary that $M_{x_\nu}\to\infty$ for a suitable sequence $x_\nu\to\infty$. If α is a continuity point of F, then

$$F(\alpha) = \lim \frac{1}{\pi(x_\nu)} \# \{p \leqq x_\nu | f(p) < M_{x_\nu}+\alpha\} \geqq \underline{\lim}\, G(t_\nu+\alpha),$$

where $t_\nu \leqq x_\nu$ is a sequence such that $t_\nu \to \infty$. Since $\lim_{\beta\to\infty} G(\beta)=1$, we have $F(\alpha)=1$, This is impossible. Similarly, we can get a contradiction assuming that $M_{x_\nu} \to -\infty$. So we have proved that M_x is bounded. Let now

$$\beta = \varliminf_{x\to\infty} M_x < \varlimsup_{x\to\infty} M_x = \gamma.$$

Let $M_{x_\nu} \to \beta$, $M_{y_\nu} \to \gamma$. Then

$$G(\alpha+\beta-\varepsilon) \leqq F(\alpha) \leqq G(\alpha+\beta+\varepsilon), \quad G(\alpha+\gamma-\varepsilon) \leqq F(\alpha) \leqq G(\alpha+\gamma+\varepsilon)$$

whenever α is a continuity point of F, and ε is an arbitrary positive number. Let $\varepsilon < \frac{\gamma-\beta}{2}$. Then $\alpha+\gamma-\varepsilon > \alpha+\beta+\varepsilon$, $G(\alpha+\gamma-\varepsilon) \leqq F(\alpha) \leqq G(\alpha+\beta+\varepsilon)$. From the monotony of G, $G(y)$ is constant in the interval $y \in [\alpha+\beta+\varepsilon, \alpha+\gamma-\varepsilon]$. Since the set of continuity points α of F is everywhere dense on the real line, we conclude that G is constant on the whole line. This is impossible. So $\beta=\gamma$, M_x is convergent, consequently the condition (a) is satisfied. This completes the proof of Theorem 6.

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(Received April 5, 1984)

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Acta Math. Hung.
47 (3—4) (1986), 361—370.

ON THE RATE OF CONVERGENCE OF A LACUNARY TRIGONOMETRIC INTERPOLATION PROCESS

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1. Introduction. Let $n, m \in \mathbf{N}^+$,

$$x_k = x_{kn} = \frac{2\pi k}{n} \quad (k \in \mathbf{N}). \tag{1}$$

The problem of (0, m) interpolation for the nodes (1) consists of finding trigonometric polynomials $R_n(f, x)$ of order at most $n-1$ to a given $f(x) \in C_{2\pi}$ such that

$$R_n(f, x_k) = f(x_k), \quad R_n^{(m)}(f, x_k) = 0 \quad (k = 0, 1, \ldots, n-1). \tag{2}$$

It is well-known (cf. A. Sharma—A. K. Varma [1]) that this problem has a unique solution when m is odd and n is arbitrary, or m is even and n is odd. We shall investigate the rate of convergence of this process in these cases. The mentioned relation of m and n will be assumed in the sequel.

2. The rate of convergence in terms of best approximation. Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of order at most n, let

$$E_n(f) = \inf_{T \in \mathcal{T}_n} \max_x |f(x) - T(x)| \quad (n \in \mathbf{N}_0^+)$$

be the best approximation of an $f(x) \in C_{2\pi}$, and let

$$\omega_s(f, h) = \sup_{x, |t| \le h} \left| \sum_{k=0}^{s} (-1)^k \binom{s}{k} f(x+kt) \right| \quad (s \in \mathbf{N}^+)$$

be the modulus of smoothness of $f(x)$ of order s.

THEOREM 1. *For any* $f(x) \in C_{2\pi}$ *we have*

$$\|f(x) - R_n(f, x)\| = O\Big(n^{\frac{1+(-1)^m}{2}} E_{[n/4]}(f) + n^{-m} \sum_{k=0}^{n} (k+1)^{m-1} E_k(f)\Big) \quad (m \in \mathbf{N}^+). \tag{3}$$

Here and in what follows the O-signs mean a constant depending only on m.

REMARKS. 1. For m odd, the estimate shows that the process is uniformly convergent for all $f(x) \in C_{2\pi}$. This fact was established in [1]. As for error estimates, the result of P. Vértesi [2, Theorem 2.5] cannot even give $O(n^{-m} \log n)$, while (3) can reach $O(n^{-m})$ for smooth functions (see Section 3).

2. For m even, the estimate shows that a sufficient condition for the uniform convergence is $E_n(f) = o(n^{-1})$. The latter is equivalent to $\omega_2(f, h) = o(h)$ which, as

a sufficient condition was found in [1]. In [2, Theorem 2.3] it is shown (as a special case of a more general statement) that there exists an $f_1(x)\in C_{2\pi}$ such that $\omega_2(f_1, h)=$ $=O(h)$, but $R_{2n+1}(f_1, \pi)$ does not converge to $f_1(\pi)$. It would be interesting to strengthen this counterexample to the following

CONJECTURE 1. *There exists an* $f_2(x)\in \mathrm{Lip}\, 1$ *such that* $R_{2n+1}(f_2, x)$ *does not converge uniformly to* $f_2(x)$ *(*m *even).*

As for the rate of convergence, Theorem 2.3 in [2] cannot give even $O\left(n\omega_m\left(f, \frac{1}{n}\right)\right)$, i.e. $O(n^{1-m})$, while (3) can reach $O(n^{-m})$ again (m odd, n even).

The proof of Theorem 1 is based on two lemmas. Denote by $\tilde{f}(x)$ the trigonometric conjugate of $f(x)\in C_{2\pi}$.

LEMMA 1. *If* $T(x)\in \mathscr{T}_{[n/4]}$, *then*

$$\|T(x)-R_n(T, x)\| = O(n^{-m})\{\|T^{(m)}\|+\|\tilde{T}^{(m)}\|\}.$$

PROOF. It follows from formulae (5) and (6) of [1] and the unique existence of R_n that

$$T(x)-R_n(T, x) = \sum_{k=1}^{n} T^{(m)}(x_k)\left[-\frac{i^m}{n}\sum_{j=1}^{n-1}\frac{e^{ij(x-x_k)}+(-1)^m e^{ij(x_k-x)}}{(n-j)^m-(-j)^m}+\frac{A_n(x)}{n^{m+1}}\right]$$

for any $T(x)\in \mathscr{T}_n$, with

$$A_n(x) = \begin{cases} 1-\cos nx & (m \text{ even}) \\ \sin nx & (m \text{ odd}). \end{cases}$$

Thus

(4)

$$e^{ilx}-R_n(e^{ilt}, x) = -\frac{(-l)^m}{n}\sum_{j=1}^{n-1}\frac{1}{(n-j)^m-(-j)^m}\left[e^{ijx}\sum_{k=1}^{n} e^{i(l-j)x_k}+(-1)^m e^{-ijx}\cdot\right.$$

$$\left.\cdot\sum_{k=1}^{n} e^{i(l+j)x_k}\right] = -\frac{(-l)^m\cdot e^{ilx}(1-e^{-inx})}{(n-l)^m-(-l)^m} \quad (l=0, 1, \dots, n).$$

Now let

$$\frac{1}{(n-z)^m-(-z)^m} = \sum_{k=0}^{\infty} a_{kn} z^k \quad \left(|z|<\frac{n}{2}\right), \tag{5}$$

then

$$|a_{kn}| \leqq \frac{1}{2\pi}\left|\oint_{|z|=3n/8}\frac{z^{-k-1}\,dz}{(n-z)^m-(-z)^m}\right| \leqq \frac{\left(\frac{3n}{8}\right)^{-k}}{n^m\left\{\left(\frac{5}{8}\right)^m-\left(\frac{3}{8}\right)^m\right\}} \leqq \tag{6}$$

$$\leqq \frac{8^m\left(\frac{8}{3}\right)^k\cdot n^{-k-m}}{2m3^{m-1}} = \frac{3}{2m}\cdot\left(\frac{8}{3n}\right)^{k+m} \quad (k\in \mathbf{N}_0^+).$$

We obtain from (4) and (5)

$$(7)\qquad e^{ilx}-R_n(e,^{ilt},x) = (-1)^{m+1}(1-e^{-inx})\sum_{k=0}^{\infty} a_{kn}\, i^{-k-m}(e^{ilx})^{(k+m)} \quad (l=0,1,\ldots,n).$$

Now if $T(x)\in\mathcal{T}_{[n/4]}$ is of the form

$$T(x) = \frac{a_0}{2}+\sum_{l=1}^{[n/4]}(a_l\cos lx+b_l\sin lx)$$

then evidently

$$(8)\qquad W(x) = T(x)+i\tilde{T}(x) = \frac{a_0}{2}+\sum_{l=1}^{[n/4]}(a_l-ib_l)\,e^{ilx}.$$

Thus by the linearity of the operator R_n we get from (7)

$$(9)\qquad W(x)-R_n(W,x) = (-1)^{m+1}(1-e^{-inx})\sum_{k=0}^{\infty} a_{kn}\, i^{-k-m}W^{(k+m)}(x).$$

Taking real parts on both sides of this relation, using the Bernstein—Szegő inequality and (6) we get

$$\|T(x)-R_n(T,x)\| \leqq \frac{3}{2m}\sum_{k=0}^{\infty}\left(\frac{8}{3n}\right)^{k+m}\left(\frac{n}{4}\right)^{k}\{\|T^{(m)}\|+\|\tilde{T}^{(m)}\|\} =$$

$$= \frac{3}{2m}\left(\frac{8}{3n}\right)^{m}\{\|T^{(m)}\|+\|\tilde{T}^{(m)}\|\}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k} \leqq \frac{3}{m}\left(\frac{4}{n}\right)^{m}\{\|T^{(m)}\|+\|\tilde{T}^{(m)}\|\},$$

which was to be proved.

LEMMA 2. *If $T_N(x)$ is the best approximating trigonometric polynomial of order N of $f(x)\in C_{2\pi}$ then*

$$(10)\qquad \max\{\|T_N^{(m)}\|,\|\tilde{T}_N^{(m)}\|\} \leqq 2^{2m+1}\sum_{k=0}^{N}(k+1)^{m-1}E_k(f) \quad (N\in\mathbf{N}_0^+;\ m\in\mathbf{N}^+).$$

PROOF. Since both sides of (10) remain unchanged if we replace $f(x)$ by $f(x)+$ const., we may assume that $T_0(x)=0$. Let $2^{s-1}\leqq N<2^s$ (for $N=0$ the statement is obvious), and

$$V_0(x) = T_1(x),\quad V_k(x) = T_{2^k}(x)-T_{2^{k-1}}(x)\quad (k=1,\ldots,s-1),$$
$$V_s(x) = T_N(x)-T_{2^{s-1}}(x).$$

Then obviously

$$\|V_0\| \leqq 2E_0(f),\quad \|V_k\| \leqq 2E_{2^{k-1}}(f)\quad (k=1,\ldots,s),$$

and hence by the Bernstein—Szegő inequality

$$\max\{\|V_0'\|,\|\tilde{V}_0'\|\} \leqq 2E_0(f),\quad \max\{\|V_k^{(m)}\|,\|\tilde{V}_k^{(m)}\|\} \leqq 2^{km+1}E_{2^{k-1}}(f)\quad (k=1,\ldots,s).$$

Since $T_N(x)=\sum_{k=0}^{s}V_k(x)$, we get

$$\max\{\|T_N^{(m)}\|,\|\tilde{T}_N^{(m)}\|\} \leqq 2E_0(f)+\sum_{k=1}^{s}2^{km+1}E_{2^{k-1}}(f) \leqq 2^{2m+1}\sum_{k=0}^{N}(k+1)^{m-1}E_k(f).$$

We now want to show that the order of magnitude (when $N\to\infty$) in (10) cannot be improved.

EXAMPLE 1. If

$$f_3(x)=\sum_{k=0}^{\infty}3^{-k}\cos 9^k x$$

then for the best approximating polynomial $T_N(x)$ we have

$$\min\{\|T_N^{(m)}\|,\|\tilde{T}_N^{(m)}\|\}\geqq\frac{1}{4\cdot 3^{2m-1}}\sum_{k=0}^{N}(k+1)^{m-1}E_k(f_3)\quad(N\in\mathbf{N}^+). \tag{11}$$

Namely, let $9^s\leqq N<9^{s+1}$. It is readily seen that

$$T_k(x)=\sum_{j=0}^{k}3^{-j}\cos 9^j x\in\mathcal{T}_{9^k}$$

is the best approximating polynomial of $f_3(x)$ of order at most N, and

$$E_{9^k}(f_3)=\sum_{j=k+1}^{\infty}3^{-j}=\frac{1}{2\cdot 3^k}\quad(k\in\mathbf{N}_0^+).$$

Hence

$$\sum_{k=0}^{N}(k+1)^{m-1}E_k(f_3)\leqq E_0(f_3)+\sum_{k=0}^{s}8\cdot 9^{km+m-1}E_{9^k}(f_3)\leqq \tag{12}$$

$$\leqq\frac{3}{2}+4\cdot 9^{m-1}\sum_{k=1}^{s}(3^{2m-1})^k\leqq 2\cdot 3^{(2m-1)(s+1)},$$

while

$$\|T_N^{(m)}\|\geqq\max\left\{|T_N^{(m)}(0)|,\left|T_N^{(m)}\left(\frac{\pi}{2}\right)\right|\right\}\geqq\left|\sum_{k=0}^{s}(-3)^{(2m-1)k}\right|= \tag{13}$$

$$=\left|\frac{(-3)^{(2m-1)(s+1)}-1}{(-3)^{2m-1}-1}\right|\geqq\frac{3^{(2m-1)(s+1)}-1}{3^{2m-1}+1}\geqq\frac{1}{2}\cdot 3^{(2m-1)s}$$

and similarly

$$\|\tilde{T}_N^{(m)}\|\geqq\frac{1}{2}\cdot 3^{(2m-1)s}. \tag{14}$$

(12)—(14) prove (11).

PROOF OF THEOREM 1. We shall make use of the representation

$$R_n(f,x)=\sum_{k=0}^{n-1}f(x_k)F_n(x-x_k) \tag{15}$$

where

$$F_n(x)=\frac{1}{n}\left\{1+2\sum_{j=1}^{n-1}\frac{(n-j)^m\cos jx}{(n-j)^m-(-j)^m}\right\}$$

and

$$\|R_n\|=\left\|\sum_{k=0}^{n-1}|F_n(x-x_k)|\right\|=O\left(n^{\frac{1+(-1)^m}{2}}\right) \tag{16}$$

(cf. [1], formulae (4) and (21)). Let $T(x)\in\mathcal{T}_{[n/4]}$ be the best approximating polynomial of $f(x)$ of order at most $\left[\frac{n}{4}\right]$. Then by (15), Lemmas 1 and 2 we get

$$\|f(x)-R_n(f,x)\| \leqq \|f(x)-T(x)\|+\|T(x)-R_n(T,x)\|+\|R_n(T-f,x)\| \leqq$$

$$\leqq (1+\|R_n\|)E_{[n/4]}(f)+O(n^{-m})\sum_{k=0}^{n}(k+1)^{m-1}E_k(f).$$

But evidently

$$E_{[n/4]}(f)=O(n^{-m})\sum_{k=0}^{n}(k+1)^{m-1}E_k(f),$$

and (16) yields the statement.

3. A better estimate for odd m's. While Theorem 1 gives an error estimate valid for any $f(x)\in C_{2\pi}$, the best order $O(n^{-m})$ for odd m's can be attained only under the strong condition $\sum_{k=0}^{\infty}(k+1)^{m-1}E_k(f)<\infty$. G. Sunouchi [6] proved that if $f(x)$ and $\tilde{f}(x)\in C_{2\pi}$ then

$$\|f(x)-R_n(f,x)\| = O\left(\omega_m\left(f,\frac{1}{n}\right)+\omega_m\left(\tilde{f},\frac{1}{n}\right)\right) \quad (m \text{ odd},\ n\in\mathbf{N}^+). \tag{17}$$

(In case $m=1$, this was proved in [4].) This result is not so general as Theorem 1, but it gives the optimal order under weaker conditions. In order to see that $f^{(m-1)}(x)$ and $\tilde{f}^{(m-1)}(x)\in\mathrm{Lip}\,1$ is indeed a weaker condition than $\sum_{k=0}^{\infty}(k+1)^{m-1}E_k(f)<\infty$, we shall analyze the following

EXAMPLE 2. There exists an $f_4(x)$ such that $f_4^{(m-1)}(x)$ and $\tilde{f}_4^{(m-1)}(x)\in\mathrm{Lip}\,1$, but $\liminf_{n\to\infty} n^m E_n(f_4)>0$ (for odd m's). Thus in this case Theorem 2 gives $O(n^{-m})$ for the error of approximation, while Theorem 1 gives only $O(n^{-m}\log n)$.

Let

$$F(z)=\begin{cases}(z-1)^{2m}e^{\frac{1}{z-1}} & \text{if } |z|\leqq 1,\quad z\neq 1\\ 0 & \text{if } z=1.\end{cases}$$

$F(z)$ is analytic in $|z|<1$, and

$$F^{(m)}(z)=e^{\frac{1}{z-1}}\left(-1+\sum_{k=1}^{m}c_k(z-1)^k\right)$$

where $c_1,\dots,c_m$ depend only on m. Thus

$$\sup_{\substack{|z|=1\\ z\neq 1}}|F^{(m)}(z)|<\infty,\quad \text{i.e.}\quad F^{(m-1)}(z)\in\mathrm{Lip}\,1.$$

The same is true for the real and imaginary part of $F(e^{ix})$, i.e. for

$$f_4(x)=-e^{-1/2}\left(2\sin\frac{x}{2}\right)^{2m}\cos\left(mx-\frac{1}{2}\cot\frac{x}{2}\right)$$

and

$$\tilde{f}_4(x) = -e^{-1/2}\left(2\sin\frac{x}{2}\right)^{2m}\sin\left(mx-\frac{1}{2}\cot\frac{x}{2}\right).$$

Evidently

(18)

$$f_4^{(m+1)}(x) = -e^{-1/2}(-1)^{(m+1)/2}\left(2\sin\frac{x}{2}\right)^{-2}\cos\left(mx-\frac{1}{2}\cot\frac{x}{2}\right)+O(x^{-1}) \quad (x\to 0).$$

Let $0<x_0<\frac{\pi}{2}$ be such that

$$\frac{1}{2}\cot\frac{x_0}{2}-mx_0 = \left(2\left[\sqrt{\frac{n}{16\pi(m+1)}}\right]+\frac{1+(-1)^{(m-1)/2}}{2}\right)\pi;$$

then elementary calculations yield

$$\left|x_0-2\sqrt{\frac{m+1}{n\pi}}\right| = O(n^{-1})$$

for sufficiently large n's. Now if $|x_0-\xi|\leqq\frac{m+1}{n}$, then

$$\left|2\sqrt{\frac{m+1}{n\pi}}-\xi\right| = O(n^{-1})$$

and hence

$$\left|\frac{1}{2}\cot\frac{x_0}{2}-mx_0-\left(\frac{1}{2}\cot\frac{\xi}{2}-m\xi\right)\right| = \frac{\sin\frac{|x_0-\xi|}{2}}{2\sin\frac{x_0}{2}\sin\frac{\xi}{2}}+O(n^{-1})\leqq\frac{\pi}{4}+O(n^{-1/2})$$

for sufficiently large n's, i.e. by (18)

$$f_4^{(m+1)}(\xi) = -e^{-1/2}\left(2\sin\frac{\xi}{2}\right)^{-2}\cos\left\{\frac{1}{2}\cot\frac{x_0}{2}-mx_0-\left(\frac{1}{2}\cot\frac{\xi}{2}-m\xi\right)\right\}+O(\xi^{-1})\geqq$$

$$\geqq\frac{n\pi}{4(m+1)}\cos\frac{\pi}{4}+O(\sqrt{n})\geqq\frac{n}{2m}\quad\left(|x_0-\xi|\leqq\frac{m+1}{n}\right)$$

for sufficiently large n's. By the definition of the modulus of smoothness

$$\text{(19)}\qquad \omega_{m+1}\left(f_4,\frac{1}{n}\right)\geqq\sum_{k=0}^{m+1}(-1)^k\binom{s}{k}f\left(x_0+\frac{k}{n}\right) = n^{-m-1}f^{(m+1)}(\xi)\geqq\frac{1}{2mn^m}$$

$$\left(|x_0-\xi|\leqq\frac{m+1}{n}\right).$$

On the other hand, by the generalized Bernstein inequality (see e.g. [3], p. 344), using $E_k(f_4) \leqq c_1 k^{-m}$, we get

$$\omega_{m+1}\left(f_4, \frac{1}{n}\right) \leqq c_2 n^{-m-1} \sum_{k=1}^{n} (k+1)^m E_k(f_4) \leqq c_2 n^{-m-1} \Big(\sum_{k=1}^{[\lambda n]} c_1 + E_{[\lambda n]}(f_4) \sum_{k=1}^{n} (k+1)^m \Big) \leqq$$

$$\leqq \lambda c_2 n^{-m} + c_3 E_{[\lambda n]}(f_4).$$

Thus (19) yields

$$c_3 E_{[\lambda n]}(f_4) \geqq \frac{1}{2mn^m} - \frac{\lambda c_2}{n^m} > \frac{1}{4mn^m}$$

provided $\lambda = \dfrac{1}{4c_2 m}$. This proves $\liminf_{n\to\infty} n^m E_n(f_4) > 0$.

4. The saturation problem. In case of odd m, G. Sunouchi [6] solved the saturation problem of the operator R_n. He proved that

$$\|f(x) - R_n(f, x)\| = \begin{cases} O(n^{-m}) & \text{iff } f^{(m-1)}(x) \text{ and } \tilde{f}^{(m-1)}(x) \in \operatorname{Lip} 1, \\ o(n^{-m}) & \text{iff } f(x) = \text{const.} \end{cases}$$

We note that in case $m=1$, this was proved by V. F. Vlasov [5] and J. Szabados [4] independently.

I am unable to solve the saturation problem for m even. In this connection I propose the following

CONJECTURE 2. *Let m be even, $f(x) \in C_{2\pi}$. Then*

(a) $\|f(x) - R_{2n+1}(f, x)\| = O(n^{-m})$ *iff* $\omega_{m+1}(f, h) = O(h^{m+1})$;

(b) $\|f(x) - R_{2n+1}(f, x)\| = o(n^{-m})$ *iff* $f(x) = \text{const.}$

The "if" part of (a) follows from Theorem 1, but the converse is an open problem.

6. A Voronovskaya type estimate. If we assume slightly more on $f(x)$ than it is done in the saturation theorem, then we can establish a result on the behaviour of the sequence

$$\Delta_n(f, x) = n^m\{f(x) - R_n(f, x)\} \quad (n = 1, 2, \dots).$$

THEOREM 2. *Assume that*

$$\int_0^1 \frac{\omega(f^{(m)}, h)}{h}\, dh < \infty \quad (m \text{ odd}) \text{ or } f^{(m+1)}(x) \in C_{2\pi} \ (m \text{ even}). \tag{20}$$

Then for the sequence $\Delta = \{\Delta_n(f, x)\}_{n=1}^{\infty}$ and $\bar{\Delta} = \{\Delta_{2n+1}(f, x)\}_{n=0}^{\infty}$ the following possibilities arise:

(a) *If $\dfrac{x}{\pi} = \dfrac{p}{q}$ $((p, q) = 1)$ is a rational number, then Δ and $\bar{\Delta}$ have finitely many cluster points, namely*

(21)

$$(-1)^{(m-1)/2}\left[\sin\frac{r\pi}{q} f^{(m)}\left(\frac{p\pi}{q}\right) + \left(1 - \cos\frac{r\pi}{q}\right)\tilde{f}^{(m)}\left(\frac{p\pi}{q}\right)\right]\left(r = \begin{cases} 0, 2, \dots, 2q-2 & (p \text{ even}) \\ 0, 1, \dots, 2q-1 & (p \text{ odd}) \end{cases}\right)$$

when m is odd, and

$$(22)\quad (-1)^{m/2-1}\left[\left(1-\cos\frac{r\pi}{q}\right)f^{(m)}\left(\frac{p\pi}{q}\right)-\sin\frac{r\pi}{q}\tilde{f}^{(m)}\left(\frac{p\pi}{q}\right)\right]\quad (r=0,2,\dots,2q-2)$$

when m is even, respectively.

(b) *If $\frac{x}{\pi}$ is irrational then any ϱ satisfying*

$$(23)\qquad \tilde{f}^{(m)}(x)-\sqrt{f^{(m)}(x)^2+\tilde{f}^{(m)}(x)^2}\leqq\varrho\leqq\tilde{f}^{(m)}(x)+\sqrt{f^{(m)}(x)^2+\tilde{f}^{(m)}(x)^2}$$

will be a cluster point of Δ or $\bar{\Delta}$, and there are no other cluster points.

Proof. Let $M=m+\frac{1+(-1)^m}{2}$. By (20), there exist $T_n(x)\in\mathcal{T}_n$ so that

$$(24)\qquad \|f^{(j)}(x)-T_n^{(j)}(x)\|=O(n^{j-M})\,\omega\left(f^{(M)},\frac{1}{n}\right)\quad (j=0,1,\dots,M).$$

Let first m be odd. Using the notation

$$(25)\qquad V_0(x)=T_1(x),\quad V_k(x)=T_{2^k}(x)-T_{2^{k-1}}(x)\quad (k=1,2,\dots),$$

we have $f(x)=\sum_{k=1}^{\infty}V_k(x)$. Here

$$(26)\qquad \|V_k\|=O(2^{-km})\,\omega(f^{(m)},2^{-k})\quad (k=1,2,\dots),$$

whence

$$\|\tilde{V}_k\|=O(k2^{-km})\,\omega(f^{(m)},2^{-k})\quad (k=1,2,\dots).$$

Thus $\sum_{k=1}^{\infty}\tilde{V}_k(x)$ converges uniformly, i.e. $\tilde{f}(x)=\sum_{k=1}^{\infty}\tilde{V}_k(x)$ exists. Moreover, by (26) and the Bernstein—Szegő inequality

$$\|\tilde{V}_k^{(j)}\|=O(2^{k(j-m)})\,\omega(f^{(m)},2^{-k})\quad (j=1,2,\dots,m).$$

Therefore by (20)

$$\sum_{k=1}^{s}\|\tilde{V}_k^{(j)}\|=O\Big(\sum_{k=1}^{s}\omega(f^{(m)},2^{-k})\Big)=O\left(\int_{2^{-s}}^{1}\frac{\omega(f^{(m)},h)}{h}\,dh\right)=O(1)$$

$$(j=1,2,\dots,m;\ s\to\infty),$$

i.e. $\tilde{f}^{(j)}(x)=\sum_{k=1}^{\infty}\tilde{V}^{(j)}(x)$ $(j=1,2,\dots,m)$ converge uniformly. Especially by (25)

$$(27)\qquad \lim_{s\to\infty}\|\tilde{f}^{(m)}(x)-\tilde{T}_{2^s}^{(m)}(x)\|=0.$$

Hence and by (24), using again a well-known inequality of Stečkin (cf. [3], formula 4.8 (18))

$$(28)\quad \|T_{2^s}^{(m+1)}\|=O(2^s)\,\omega(f^{(m)},2^{-s}),\ \|\tilde{T}_{2^s}^{(m+1)}\|=O(s2^s)\,\omega(f^{(m)},2^{-s})=o(2^s),$$

since (20) implies $\omega\left(f^{(m)},\frac{1}{n}\right)\log n\to 0\quad (n\to\infty)$.

Now if m is even, then (27)—(28) trivially hold by (24).

Now let $2^{s+2} \leqq n < 2^{s+3}$. Then by (24), (9), (16), (6), (27) and (28) we get

$$\Delta_n(f, x) = n^m\{f(x) - T_{2^s}(x)\} + n^m\{T_{2^s}(x) - R_n(T_{2^s}, x)\} + n^m R_n(T_{2^s} - f, x) =$$

$$= n^m(1 + \|R_n\|) O(n^{-M}) \omega\left(f^{(M)}, \frac{1}{n}\right) +$$

$$+ n^m(-1)^{m+1} \sum_{k=0}^{\infty} a_{kn} \operatorname{Re}\{(1 - e^{-inx}) i^{-k-m} [T_{2^s}^{(k+m)}(x) + i\tilde{T}_{2^s}^{(k+m)}(x)]\} =$$

$$= O\left(\omega\left(f^{(M)}, \frac{1}{n}\right)\right) + (-1)^{m+1} \operatorname{Re}\{(1 - e^{-inx}) i^{-m} [T_{2^s}^{(m)}(x) + iT_{2^s}^{(m)}(x)]\} +$$

$$+ O\left(\sum_{k=1}^{n} \left(\frac{8}{3n}\right)^k 2^{s(k-1)}\right) [\|T_{2^s}^{(m+1)}\| + \|\tilde{T}_{2^s}^{(m+1)}\|] = O\left(\omega\left(f^{(M)}, \frac{1}{n}\right)\right) \left(1 + \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k\right) +$$

$$+ o(1) \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k + (-1)^{m+1} \operatorname{Re}\{(1 - e^{-inx}) i^{-m} [f^{(m)}(x) + i\tilde{f}^{(m)}(x)]\} =$$

$$= (-1)^{m+1} \operatorname{Re}\{(1 - e^{-inx}) i^{-m} [f^{(m)}(x) + i\tilde{f}^{(m)}(x)]\} + o(1).$$

Here

$$\text{(29)} \qquad \operatorname{Re}\{\ldots\} = \begin{cases} (-1)^{(m-1)/2} [\sin nx f^{(m)}(x) + (1 - \cos nx) \tilde{f}^{(m)}(x)] & (m \text{ odd}) \\ (-1)^{m/2} [(1 - \cos nx) f^{(m)}(x) - \sin nx \tilde{f}^{(m)}(x)] & (m \text{ even}). \end{cases}$$

Now if $\frac{x}{\pi} = \frac{p}{q}$, $(p, q) = 1$ then, for a given r described in (21) or (22), choose a sequence $n_1 < n_2 < \ldots$ such that $n_k p \equiv r \pmod{2q}$. Thus $\sin n_k x = \sin \frac{r\pi}{q}$, $\cos n_k x = \cos \frac{r\pi}{q}$, and (21) and (22) follow from (29).

If $\frac{x}{\pi}$ is irrational, then to any given $0 \leqq \alpha < 2\pi$, there exists a sequence $n_1 < n_2 < \ldots \ldots < n_k < \ldots$ such that $n_k x \to \alpha$ $(k \to \infty)$ mod 2π. Being the range of the functions

$$\varphi_1(\alpha) = \sin \alpha f^{(m)}(x) + (1 - \cos \alpha) \tilde{f}^{(m)}(x) \quad (m \text{ odd})$$

$$\varphi_2(\alpha) = (1 - \cos \alpha) f^{(m)}(x) - \sin \alpha \tilde{f}^{(m)}(x) \quad (m \text{ even})$$

exactly the interval (23), the statement follows.

Remarks. 3. In the case when x/π is rational, Theorem 2 gives only an upper bound for the number of cluster points, since there may be equal numbers among them.

4. The only case when Δ has a limit is when $f^{(m)}(x) = \tilde{f}^{(m)}(x) = 0$, and in this case the limit is 0.

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(Received May 7, 1984; revised January 3, 1985)

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Acta Math. Hung.
47 (3—4) (1986), 371—382.

ON THE NUMBER OF OCCURRENCES OF SEQUENCE PATTERNS

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1. Let $B=\{0, 1\}$ and let B^n be the set of words over B having exactly n letters, i.e. $B^n=\{b_1...b_n|b_j\in B\ (j=1, ..., n)\}$. B^0 denotes the empty word, $B^*=\bigcup_{n=0}^{\infty} B^n$ is the set of all words over B having finite length. In this section Greek letters denote words over B, except λ: $\lambda(\alpha)$ denotes the length of α.

For a word $\beta=b_1...b_n$ let $(\beta)_r=b_{r+1}...b_n$, and ${}_s[\beta]=b_1...b_s$. Consequently ${}_s[(\beta)_r]=b_{r+1}...b_{r+s}$.

Let $\beta=b_1...b_n$, $\alpha=a_1...a_t$, $t\leqq n$. We shall say that α occurs in β if there exists an index $s\leqq n-t$ such that ${}_t[(\beta)_s]=\alpha$. We say furthermore that s is an index of occurrence of α in β. Let $s_1<s_2<...<s_l$ be the whole set of indices of occurrences of α in β.

The combinatorial structure of the occurrence of patterns in words has been considered earlier by several authors. We mention only the paper of L. J. Guibas and A. M. Odlyzko [2] that contains further references concerning the previous investigations.

Let α be fixed and let β run over the set of all words in B^n randomly. There are a lot of interesting questions concerning the statistical behaviour of $s_1, ..., s_l$. The most straightforward one is the distribution of l. Our main purpose in this paper is to give an estimation with a good remainder term for l (see Theorem 1).

DEFINITION 1. We shall say that h is a shifting index for $\alpha=a_1...a_t$, $1\leqq h<t$, if $a_{h+1}...a_t=a_1...a_{t-h}$. Let $\mathscr{B}(\alpha)$ be the whole set of shifting indices of α: $h_1<h_2<...<h_s$.

It may occur that $\mathscr{B}(\alpha)$ is empty. It is obvious furthermore that h_i, $h_j\in\mathscr{B}(\alpha)$, $h_i+h_j<t$ imply $h_i+h_j\in\mathscr{B}(\alpha)$. So, if $u=[t/h_1]$, then $jh_1\in\mathscr{B}(\alpha)$ for $j=1, ..., u$.

Let now $h\in\mathscr{B}(\alpha)$ such that $h_1\nmid h$. We shall prove that in this case $h+h_1>t$. Assume in the contrary that $h+h_1\leqq t$. Since $h_1\in\mathscr{B}(\alpha)$, therefore $a_{j+h_1}=a_j$ for $j=1, ..., t-h_1$, i.e. the sequence a_j is periodic with period h_1. Furthermore, since $h\in\mathscr{B}(\alpha)$, therefore $a_{j+h}=a_j$ $(j=1, ..., t-h)$. Hence we shall deduce that $h-h_1\in\mathscr{B}(\alpha)$, i.e. $a_{l+(h-h_1)}=a_l$ for $l=1, ..., t-h+h_1$.

The last relation is obviously true for $l=h_1+j$, $j=1, 2, ..., t-h$, since $a_{l+h-h_1}=a_{h+j}=a_j=a_{j+h_1}=a_l$.

For $l=1, 2, ..., h_1$ we can make use of the assumption $h_1+h<t$ in the following way: $a_{h+l}=a_l$ for $l=1, 2, ..., h_1$, so $a_{h-h_1+l}=a_{h+l}=a_l$, which proves that the relation $a_{h+l}=a_l$ holds for every $l=1, 2, ..., t-h+h_1$. Repeating this idea with $h-h_1$ instead of h several times, we conclude that there is a shifting index which is smaller than h_1. So we have proved the following

LEMMA 1. *For the set of $\mathcal{B}(\alpha)$ there are three possibilities:*

(1) $\mathcal{B}(\alpha)$ *is empty,*

(2) $\mathcal{B}(\alpha)$ *contains only elements greater than or equal to $t/2$. Then $h_1 \geqq t/2$.*

(3) $\mathcal{B}(\alpha)$ *contains at least one element less than $t/2$. Then $h_1 < t/2$, $jh_1 \in \mathcal{B}(\alpha)$ for $j=1, \ldots, [t/h_1]$. If $h \in \mathcal{B}(\alpha)$, $h_1 \nmid h$, then $h+h_1 > t$.*

2. Let $\omega := \omega_1 \omega_2 \ldots$ be a random infinite Bernoulli-trial, and let $\alpha = a_1 \ldots a_t$ be fixed. Let A_i denote the event

$$A_i\colon\ \omega_i \ldots \omega_{i+t-1} = \alpha.$$

Let furthermore $V_r(N)$ denote the probability of the event that among $A_1, \ldots$ $\ldots, A_{N-t}$ exactly r occur, and let

$$v_r(z) = \sum_N V_r(N) z^N \tag{2.1}$$

be the generator function.

Let

$$Q(z) = \sum_h \left(\frac{z}{2}\right)^h + \left(\frac{z}{2}\right)^t \frac{1}{1-z}, \tag{2.2}$$

$$H(z) = (1-z) Q(z), \tag{2.3}$$

where here and in the sequel $\sum_h$ denotes a summation over $h \in \mathcal{B}(\alpha)$.

There are several ways to get an explicit expression for $v_r(z)$. Since in [2] an outline of the deduction is given (Section 2), we state it without proof.

LEMMA 2. *We have*

$$V_r(z) = \left(\frac{z}{2}\right)^t \frac{H(z)^{r-1}}{(1-z+H(z))^{r+1}} \quad (r \geqq 1), \tag{2.4}$$

$$V_0(z) = \frac{z^t \left\{1 + \sum_h \left(\frac{z}{2}\right)^h + 2^{-t} \left(\sum_{l=0}^{t-1} z^l\right)\right\}}{1-z+H(z)}. \tag{2.5}$$

3. Let ζ_N be the random variable that counts the number of occurrences of α in a random $\omega_1 \ldots \omega_N$. Since

$$P(\zeta_N = r) = \begin{cases} V_r(N) & \text{if} \quad N \geqq t,\ r \geqq 0 \\ 1 & \text{if} \quad N < t,\ r = 0 \\ 0 & \text{if} \quad N < t,\ r \geqq 1, \end{cases}$$

by using the formulae (2.4), (2.5), and differentiating them twice we get immediately

$$M(\zeta_N) = \begin{cases} \dfrac{N-t+1}{2^t} & \text{for} \quad N \geqq t \\ 0 & \text{for} \quad N < t, \end{cases} \tag{3.1}$$

$$D^2(\zeta_N) = a\zeta_N + b \quad \text{for} \quad N \geqq 2t, \tag{3.2}$$

where

$$a = \frac{1}{2^t} + \frac{1-2t}{2^{2t}} + \frac{2}{2^t}\sum_h \frac{1}{2^h}, \tag{3.3}$$

$$b = -ta + \left(\frac{1}{2^t} + \frac{1}{2^{2t}} - \frac{2}{2^t}\sum_h \frac{h-1}{2^h}\right). \tag{3.4}$$

4. In this section we shall give an asymptotic formula for $V_r(N)$ with an optimal remainder term. For the sake of simplicity we shall assume that the length t of α is quite large, $t \geqq 10$. We should mention that a more general problem has been considered by D. A. Moskvin and A. G. Postnikov [1]. Namely they proved the following assertion. Let $[a, b] \subseteq [0, 1]$ any interval, and let χ denote its indicator function. Let

$$N_n(\zeta, [a, b]) = \sum_{k=0}^{n-1} \chi(\{2^k \zeta\}),$$

i.e. the number of those $k=0, 1, \ldots, n-1$, for which the fractional parts $\{2^k \zeta\}$ belong to $[a, b]$. Then for each nonnegative integer l,

$$\operatorname{meas}(\xi\colon 0 \leqq \xi \leqq 1,\, N_n(\xi, [a, b]) = l) =$$
$$= \frac{1}{\sigma\sqrt{2\pi n}} \exp\left(-\frac{(l-n(b-a))^2}{2n\sigma^2}\right) + O\left(\frac{\sqrt{\ln n}}{n}\right),$$

uniformly in l. Here σ is defined by

$$\sigma^2 = \lim_n \frac{1}{n} \int_0^1 (N_n(\xi, [a, b]) - (b-a)n)^2 \, d\xi.$$

Our purpose now is to improve the remainder term in this special case. To carry out the proof without a cumbersome discussion we shall assume that

$$t \geqq 10, \quad 2^t \leqq \log N. \tag{4.1}$$

We shall use the Parseval formula

$$V_r(N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v_r(e^{i\theta}) e^{-iN\theta} \, d\theta. \tag{4.2}$$

In order to estimate the right hand side we should approximate v_r near $\theta = 0$ quite well, and to give an upper estimation for v_r out of a small interval.

Let

$$\frac{N}{2^{t+1}} \leqq r \leqq \frac{N}{2^{t-1}}.$$

Let $w = e^{i\theta} - 1 = z - 1$, and let the coefficients $c_0, c_1, \ldots, c_t$ be defined by

$$H(1+w) = -w \sum \frac{1}{2^h}(1+w)^h + \frac{1}{2^t}(1+w)^t = c_0 + c_1 w + \ldots + c_t w^t. \tag{4.3}$$

So

$$c_0 = 1/2t, \quad c_1 = -\sum_h \frac{1}{2^h} + \frac{t}{2^t} \tag{4.4}$$

$$c_2 = -\sum \frac{h}{2^h} + \frac{t(t-1)}{2^{t+1}}. \tag{4.5}$$

LEMMA 3. *Let the least shifting index h_1 defined in Section 1 be not less than 2. Then*

$$\left|\frac{H(1+w)}{w}\right| \leqq 0{,}47 \tag{4.6}$$

for $|w| \geqq 8c_0$, *furthermore*

$$|\mathcal{I}_3| = \left|\frac{1}{2\pi} \int\limits_{|w| \geqq 8c_0} v_r(z) z^{-N} d\theta\right| \leqq B_1 \cdot 0{,}9^{(1/2)Nc_0}, \tag{4.7}$$

with a suitable absolute constant B_1.

The inequality (4.7) *holds in the case $h_1=1$ as well.*

PROOF. Since

$$\frac{H(1+w)}{w} = -\sum \left(\frac{z}{2}\right)^h + \frac{1}{w}\left(\frac{z}{2}\right)^t,$$

therefore

$$\left|\frac{H(1+w)}{w}\right| \leqq \sum \frac{1}{2^h} + \frac{c_0}{|w|}. \tag{4.8}$$

Let us assume that $h_1 \geqq 2$. From Lemma 1, all elements of $\mathcal{B}(\alpha)$ up to $t-h_1$ are multiples of h_1, so for $h_1 < t/2$

$$\sum_h \frac{1}{2^h} < \frac{1}{2^{h_1}-1} + \frac{1}{2^{t-h_1+1}} + \dots < \frac{1}{2^{h_1}-1} + \frac{1}{2^{t-h_1}}.$$

Here the right hand side is the largest if $h_1=2$, $t=10$. If $h_1 > t/2$, then

$$\sum_h 1/2^h < \frac{2}{2^{t/2}} \leqq \frac{1}{16}.$$

Then the right hand side of (4.8) is less than $1/3+1/2^8+1/8 < 0{,}47$.

Hence we have

$$\left|\frac{H(z)}{H(z)-w}\right| \leqq \left|\frac{H(z)}{w}\right| \frac{1}{1-\left|\dfrac{H(z)}{W}\right|} \leqq \frac{0{,}47}{1-0{,}47} < 0{,}9,$$

and

$$|H(z)-w| \geqq |w|\left(1-\left|\frac{H(z)}{w}\right|\right) \geqq 0{,}53\,|w|.$$

So, by (2.4) we get

$$|\mathscr{I}_3| \leqq \frac{c_0}{2\pi} \int\limits_{|W| \geqq 8c_0} \frac{1}{|H(z)-w|^2} \left| \frac{H(z)}{H(z)-w} \right|^{r-1} d\theta \leqq$$

$$\leqq 0{,}9^{r-1} c_0 \int\limits_{|W|>8c_0} \frac{1}{|w|^2} d\theta \leqq B_1 \cdot 0{,}9^{(1/2)Nc_0},$$

i.e. (4.7) holds.

Let us consider now the case $h_1 = 1$. Observing that

$$\left[\left(\frac{z}{2}\right)-1\right] H(z) = \left(\frac{z}{2}-1\right)(1-z)\left(\sum_{l=1}^{t-1}\left(\frac{z}{2}\right)^l\right) + \left(\frac{z}{2}-1\right)\left(\frac{z}{2}\right)^t =$$

$$= -\left(\frac{z}{2}\right)^{t+1} + 2\left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right),$$

$$\left(\frac{z}{2}-1\right)\left\{(1-z)+H(z)\right\} = -\left(\frac{z}{2}\right)^{t+1} + 2\left(\frac{z}{2}\right) - 1,$$

we get easily the formula

$$v_r(z) = \frac{\left(\frac{z}{2}\right)^t (1-z/2)^2}{\left[1-z+\left(\frac{z}{2}\right)^{t+1}\right]^2} \left[\frac{\frac{z}{2}\left(1-z+\left(\frac{z}{2}\right)^t\right)}{1-z+\left(\frac{z}{2}\right)^{t+1}}\right]^{r-1}.$$

Consequently

$$|v_r(z)| \leqq -\frac{\frac{1}{2^t}\left(1+\frac{1}{2}\right)^2}{\left(|1-z|-\left(\frac{1}{2}\right)^{t+1}\right)^2} \left[\frac{\frac{1}{2}\left(|1-z|+\frac{1}{2^t}\right)}{|1-z|-\left(\frac{1}{2}\right)^{t+1}}\right]^{r-1}.$$

The minimum of the right hand side is attained for $|1-z| = 8c_0$, so

$$|v_r(z)| \leqq \frac{\frac{9}{4} c_0}{7{,}5^2 c_0^2} \left[\frac{\frac{1}{2} \cdot 9c_0}{7{,}5c_0}\right]^{r-1} = \frac{2^t}{5^2}\left(\frac{3}{5}\right)^{r-1},$$

hence (4.7) immediately follows.

This completes the proof of Lemma 3.

LEMMA 4. *For any positive integer l and any complex number w the inequalities*

$$|(1+w)^l - 1 - lw| \leqq l(l-1)|w|^2(1+|w|)^{l-2},$$

$$|(1+w)^l - 1| \leqq l|w|(1+|w|)^{l-1}$$

hold.

PROOF. Observing that the coefficients w^k of the function $(1+w)^k$ are positive, the left hand sides of the inequalities are not greater than $(1+|w|)^l - 1 - l|w|$,

$(1+|w|)^l-1$, respectively. By using the wellknown mean-value theorems, the inequalities follow immediately.

LEMMA 5. *The inequality*

$$(4.9)\qquad |-w+H(1+w)|^2 \geqq |H(1+w)|^2+\left[1+2\sum_h \frac{\cos h\theta}{2^h}-\frac{(2t-1)}{2^t}\right]|w|^2$$

holds for each real θ, $w=e^{i\theta}-1=z-1$.

PROOF. We have

$$|-w+H(1+w)|^2 = |H(1+w)|^2+|w|^2-2\,\mathrm{Re}\,\bar{w}H(1+w).$$

Furthermore

$$\mathrm{Re}\,\bar{w}z^t = \mathrm{Re}(z^{t-1}-z^t) = \cos(t-1)\,\theta-\cos t\theta = 2\sin\frac{\theta}{2}\sin\left(t-\frac{1}{2}\right)\theta,$$

and $|w|^2=4\sin^2\theta/2$. Since $\left|\frac{\sin nx}{\sin x}\right|\leqq n$ for integer n, we have

$$|2\,\mathrm{Re}\,\bar{w}z^t| \leqq 4\sin^2\frac{\theta}{2}\,\frac{\sin(2t-1)\,\theta/2}{\sin\theta/2} \leqq (2t-1)\,|w|^2.$$

Observing that

$$-2\,\mathrm{Re}\,\bar{w}H(1+w) = |w|^2\,2\,\mathrm{Re}\sum\left(\frac{z}{2}\right)^h-2c_0\,\mathrm{Re}\,\bar{w}z^t \geqq$$

$$\geqq \left[2\sum_h \frac{\cos h\theta}{2^h}-(2t-1)\,c_0\right]|w|^2,$$

(4.9) immediately follows.

Let

$$F(z) := \frac{H(z)}{1-z+H(z)}.$$

Since $|H(z)|\leqq H(1)=c_0$, from Lemma 5 we deduce

$$|F(z)|^2 \leqq \frac{c_0^2}{c_0^2+\frac{1}{2}|w|^2},$$

in the interval $|w|\leqq 8c_0$. Let $|w|=c_0\eta$. Then

$$|F(z)|^2 \leqq \frac{1}{1+\frac{\eta^2}{2}}.$$

Let M be a small positive number to be chosen later. Assume that $M\leqq|w|\leqq 8c_0$.

Then

$$\frac{1}{|H(1+w)-w|^2} \leqq \frac{1}{M^2}\frac{|w|^2}{|H(1+w)-w|^2} \leqq \frac{2}{M^2}\left\{\frac{|H(1+w)-w|^2+|H(1+w)|^2}{|H(1+w)-w|^2}\right\} \leqq$$

$$\leqq \frac{2}{M^2}(1+|F(z)|^2),$$

and so for $v_r(z)$ we have

$$|v_r(z)| \leqq \frac{c_0 \cdot 2}{M^2}(|F(z)|^{r-1}+|F(z)|^{r+1} \leqq \frac{4c_0}{M^2}|F(z)|^{r-1} < \tag{4.10}$$

$$< \frac{4c_0}{M^2}\left(\frac{1}{1+\frac{\eta^2}{2}}\right)^{(r-1)/2} < \frac{4c_0}{M^2}e^{-(\eta^2/4)((r-1)/2)}.$$

Let

$$M = \frac{(\log N)^2}{(Nc_0)^{1/2}}. \tag{4.11}$$

From (4.11) we get immediately that

$$|\mathscr{I}_2| = \left|\frac{1}{2\pi}\int\limits_{M \leqq |w| \leqq 8c_0} v_r(z)z^{-N}\,dz\right| \leqq \frac{B_2c_0^3N^2}{\log^2 N}\exp(-B_3\log^2 N), \tag{4.12}$$

with suitable positive absolute constants B_2, B_3.

Let us assume now that $|w| \leqq M$. Let

$$r(w) = -w\sum\frac{(1+w)^h-1}{2^h}+\frac{(1+w)^t-1-tw}{2^t},$$

i.e.

$$H(1+w) = c_0+c_1w+r(w).$$

Since $(1+|w|)^t<2$, therefore by Lemma 4 we have

$$|r(w)| \leqq 3|w|^2. \tag{4.13}$$

Consequently

$$\frac{W}{H(1+w)} = \frac{W}{c_0}-c_1\left(\frac{W}{c_0}\right)^2+O\left(\left(\frac{w}{c_0}\right)^3\right), \tag{4.14}$$

$$\frac{W^2}{H^2(1+w)} = \left(\frac{W}{c_0}\right)^2+O\left(\left(\frac{W}{c_0}\right)^3\right). \tag{4.15}$$

Here and in the sequel the constants implicitly stated in the order terms are absolute, they do not depend on t.

Furthermore,

$$F(z)^{-1} = 1-\frac{w}{H(z)} = \exp\left(\ln\left(1-\frac{w}{H(z)}\right)\right) =$$

$$= \exp\left(-\frac{w}{H(z)}-\frac{1}{2}\frac{w^2}{H^2(z)}+O\left(\frac{w^3}{c_0^3}\right)\right),$$

and so by (4.14), (4.15) we get

$$F(z)^{r-1} = \exp\left((r-1)\frac{w}{c_0}+\left(\frac{1}{2}-c_1\right)(r-1)+\left(\frac{w}{c_0}\right)^2+O\left(r\left(\frac{w}{c_0}\right)^3\right)\right).$$

Furthermore,

$$z^{-N+t} = \exp\left((-N+t)\log(1+w)\right) = \exp\left([-N+t]w+\frac{N-t}{2}w^2+O(Nw^3)\right),$$

$$\frac{1}{(1-z+H(z))^2} = \frac{1}{c_0^2}\frac{1}{\left[1+\left(\frac{H(1+w)-w}{c_0}-1\right)\right]^2} = \frac{1}{c_0^2}\exp\left(-2\log(1+\Lambda)\right),$$

with

$$\Lambda = \frac{H(1+w)-w}{w}-1.$$

Observing that

$$\Lambda = (c_1-1)\frac{w}{c_0}+\frac{c_2}{c_0}w^2+O\left(\frac{|w|^3}{c_0}\right),$$

$$\Lambda^2 = (c_1-1)^2\frac{w^2}{c_0^2}+O\left(\frac{|w|^3}{c_0^2}\right), \quad \Lambda^3 = O\left(\left(\frac{w}{c_0}\right)^3\right),$$

and that

$$-2\log(1+\Lambda) = -2\Lambda+\Lambda^2+O(\Lambda^3),$$

we get

$$\frac{1}{(1-z+H(z))^2} = \frac{1}{c_0^2}\exp\left[2(1-c_1)\frac{w}{c_0}-\left[\frac{2c_2}{c_0}-\frac{(c_1-1)^2}{c_0^2}\right]w^2+O\left(\left(\frac{w}{c_0}\right)^3\right)\right].$$

Consequently

(4.16) $$v_r(z) = \frac{1}{c_0}\exp\left(A_1w+A_2w^2+O(Nw^3)\right),$$

where

(4.17) $$A_1 = \frac{r+1-2c_1}{c_0}-(N-t),$$

(4.18) $$A_2 = \frac{N-t}{2}+\left(\frac{1}{2}-c_1\right)\frac{r-1}{c_0^2}-\left[\frac{2c_2}{c_0}-\frac{(c_1-1)^2}{c_0^2}\right].$$

Let

(4.19) $$\mathscr{I}_1 := \frac{1}{2\pi} \int_{|w| \leqq M} v_r(z)\, z^{-N}\, d\theta.$$

Since

$$w = i\theta - \frac{\theta^2}{2} + O(\theta^3), \quad w^2 = -\theta^2 + O(\theta^3), \quad w^3 = O(\theta^3),$$

we have

$$\mathscr{I}_1 = \frac{1}{2\pi c_0} \int_{|\theta| \leqq M_1} \exp\left(iA_1\theta - \left(A_2 + \frac{A_1}{2}\right)\theta^2 + O(D\theta^3)\right),$$

where

$$D = |A_1| + |A_2| + N + (1/c_0)^3,$$

and M_1 is defined by $|e^{iM_1} - 1| = M$. In the range $\frac{N}{2c_0} \leqq r \leqq \frac{2N}{c_0}$ we have $D = O(N/c_0)$. Since

$$e^{O(D\theta^3)} = 1 + O(D\theta^3),$$

we get

(4.20) $$\mathscr{I}_1 = R_1 + O(R_2),$$

where

(4.21) $$R_1 = \frac{1}{2\pi c_0} \int_{-M_1}^{M} \exp\left(iA_1\theta - \left(A_2 + \frac{A_1}{2}\right)\theta^2\right) d\theta,$$

(4.22) $$R_2 = \frac{N}{c_0^2} \int_{-M_1}^{M_1} \theta^3 \exp\left(-\left(A_2 + \frac{A_1}{2}\right)\theta^2\right) d\theta.$$

Since

$$\int_{-\infty}^{\infty} \theta^3 \exp(-E\theta^2)\, d\theta = \frac{1}{E^2} \int_{-\infty}^{\infty} u^3 \exp(-u^2)\, du,$$

and $A_2 + \frac{A_1}{2} \gg N/c_0$, we get

(4.23) $$R_2 \ll 1/N.$$

Let $R_1 = R_3 - R_4$, where R_3 is defined by the right hand side of (4.21) extending the integration to the whole real line. Then

$$|R_4| \leqq \frac{1}{2\pi c_0} \int_{|\theta| \geqq M_1} \exp\left(-\left(A_2 + \frac{A_1}{2}\right)\theta^2\right) d\theta.$$

Since for positive X and Y,

$$\int_X^{\infty} \exp(-Y\theta^2)\, d\theta = \frac{1}{\sqrt{Y}} \int_{X\sqrt{Y}}^{\infty} e^{-v^2}\, dv \leqq$$

$$\leqq \frac{1}{\sqrt{Y}} \int_{X\sqrt{Y}} e^{-v^2} \frac{2v}{2X\sqrt{Y}}\, dv = \frac{1}{2XY} \int_{v^2 = X^2 Y}^{\infty} e^{-v^2}\, dv^2 = \frac{1}{2XY} e^{-X^2 Y},$$

by choosing $X=M_1$, $Y=A_2+\frac{A_1}{2}$, we get

$$R_4 \ll \frac{1}{c_0 XY} e^{-X^2Y},$$

and so

(4.24) $$R_4 \ll 1/N.$$

LEMMA 6. *Let a and b be real numbers, $b>0$,*

$$L(a, b) = \int_{-\infty}^{\infty} \exp(ia\theta - b\theta^2)\, d\theta.$$

Then

$$L(a, b) = \frac{\sqrt{\pi}}{\sqrt{b}} \exp\left(-\frac{a^2}{4b}\right).$$

PROOF. The assertion is known. For the sake of completeness we shall prove it. Since

$$b\theta^2 - ia\theta = b\left(\theta - i\frac{a}{2b}\right)^2 + \frac{a^2}{4b},$$

we have

$$L(a, b) = \exp\left(-\frac{a^2}{4b}\right) \int_{-\infty}^{\infty} \exp\left(-b\left(\theta - i\frac{a}{2b}\right)^2\right) d\theta.$$

The integral on the right hand side can be considered as the integral of $e^{-bw^2}\,dw$ taken on the line $\operatorname{Im} w = \frac{a}{2b}$. Moving this line into $\operatorname{Im} w=0$, we get

$$L(a, b) = \exp\left(-\frac{a^2}{4b}\right) \int_{-\infty}^{\infty} \exp(-bx^2)\, dx =$$

$$= \frac{1}{\sqrt{b}} \exp - \left(\frac{a^2}{4b}\right) \int_{-\infty}^{\infty} \exp(-x^2)\, dx = \frac{\sqrt{\pi}}{\sqrt{b}} \exp\left(-\frac{a^2}{4b}\right).$$

By using Lemma 6, we have

$$R_3 = \frac{1}{2c_0\sqrt{\pi b}} \exp\left(-\frac{a^2}{4b}\right),$$

where

$$a = A_1 = \frac{r}{c_0} - N + k, \quad k = \frac{1-2c_1}{c_0} + t, \quad b = A_2 + \frac{A_1}{2} = er + f,$$

$$e = \frac{1}{2c_0} + \frac{1/2 - c_1}{c_0^2}, \quad f = \frac{1-2c_1-4c_2}{2c_0} + \frac{c_1^2 - c_1 + 1/2}{c_0^2}.$$

Let $s = \frac{r}{c_0} - N$. Then

$$a^2 = s^2 + 2ks + k^2, \quad 4b = 4ec_0N + 4ec_0s + 4f,$$

and so

$$\frac{a^2}{4b} = \frac{s^2}{4ec_0 N} + O\left(\frac{|s|^3 c_0}{N^2}\right) + O\left(\frac{|s|}{N}\right) + O\left(\frac{1}{c_0 N}\right)$$

in the interval

$$|s| \leqq N^{1/2} (\log N)^2,$$

say. Let $\eta = sN^{-1/2}$. Then

$$\exp\left(-\frac{a^2}{4b}\right) = \exp\left(-\frac{\eta^2}{4ec_0}\right)\left(1 + O\left(\frac{\eta^3 c_0}{N^{1/2}}\right) + O\left(\frac{\eta}{N^{1/2}}\right) + O\left(\frac{1}{c_0 N}\right)\right),$$

Furthermore

$$\frac{1}{\sqrt{4b}} = \frac{1}{4ec_0 N^{1/2}} (1 + O(\eta/N^{1/2})),$$

and so

$$\frac{1}{\sqrt{4b}} \exp\left(-\frac{a^2}{4b}\right) = \frac{1}{(4ec_0 N)^{1/2}} \exp\left(-\frac{\eta^2}{4ec_0}\right)\left(1 + O\left(\frac{|\eta| + |\eta|^4}{c_0 N^{1/2}}\right)\right),$$

(4.26)
$$\frac{1}{c_0\sqrt{4b}} \exp\left(-\frac{a^2}{4b}\right) = \frac{1}{c_0(4ec_0 N)^{1/2}} \exp\left(-\frac{\eta^2}{4ec_0}\right)[1 + O(K)],$$

where

$$K = \frac{|\eta|}{N^{1/2}} + \frac{|\eta|^3 c_0}{N^{1/2}} + \frac{1}{c_0 N}.$$

Let $K = \frac{\eta}{\sqrt{4ec_0}}$. Since $ec_0 = O(1/c_0)$, therefore $|\eta| \ll \frac{K}{c_0^{1/2}}$ and so the order term in the right hand side of (4.26) is less than

$$\ll \frac{1}{c_0^{1/2} N^{1/2}} \exp(-K^2)\left(\frac{|K|}{c_0^{1/2} N^{1/2}} + \frac{|K|^3}{c_0^{1/2} N^{1/2}}\right) \ll O\left(\frac{1}{Nc_0}\right).$$

Consequently

$$R_3 = \frac{1}{\sqrt{\pi}\, c_0 (4ec_0 N)^{1/2}} \exp\left(-\frac{\eta^2}{4ec_0}\right) + O\left(\frac{1}{Nc_0}\right),$$

and by introducing

(4.27)
$$\sigma^2 = 2ec_0^3,$$

we get

(4.28)
$$R_3 = \frac{1}{\sigma\sqrt{2\pi n}} = \exp\left(-\frac{(r - Nc_0)^2}{2N\sigma^2}\right) + O\left(\frac{1}{Nc_0}\right).$$

Taking into account the relations (4.27), (4.24), (4.22), (4.21), (4.20), (4.19), (4.7), (4.2), we get

(4.29)
$$V_r(N) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(r - Nc_0)^2}{2N\sigma^2}\right) + O\left(\frac{1}{Nc_0}\right)$$

for every $r \in \left[\frac{c_0}{2} N, 2c_0 N\right] = S$. (4.29) is valid for the integers r out of S, moreover the sum $\sum_{r\notin S} V_r(n)$, extended for all of these r's is less than $O(1/Nc_0)$. To prove this, it is enough to use Čebishev-inequality,

$$P(|\zeta_N - M\zeta_N| \geqq \lambda D\zeta_N) \ll 1/\lambda^2,$$

and take into account (3.1), (3.2), (3.3), (3.4). Hence we get that

$$\sum_{r\notin s} V_r(n) \ll 1/\lambda^2,$$

with $\lambda \gg \frac{c_0\sqrt{N}}{\sigma}$. Since $\sigma^2 \asymp c_0$, we have $1/\lambda^2 \ll 1/Nc_0$.

We have proved the following

THEOREM 1. *Let us assume that* $t \geqq 10$, $2^t \leqq \log N$. *Then*

$$V_r(N) = \frac{1}{\sigma\sqrt{2\pi N}} \exp\left(-\frac{(r-Nc_0)^2}{2N\sigma^2}\right) + O\left(\frac{2^t}{N}\right)$$

uniformly in r, where

$$\sigma^2 = c_0^2 + \left(\frac{1}{2} - c_1\right)c_0, \quad c_0 = \frac{1}{2^t}, \quad c_1 = -\sum_{h\in\mathcal{B}(\alpha)} 2^{-h} + tc_0.$$

REMARKS. 1. From (4.19), (4.12), (4.7) we get

$$V_r(N) = I_1 = O(\exp(-B_4 \log^2 N))$$

for bounded t. Approximating $F(z)$ and z^{-N} by a function of the form exp (polynomial of θ) in the interval $|\omega| \leqq M$ we would deduce for $V_r(N)$ an asymptotic expansion as well.

2. The order of the second maximal term in the asymptotic expansion of $V_r(N)$ is $1/N$, which shows that the order of the remainder term in N is best possible.

3. We hope to generalize our theorem such that it implies the Moskvin—Postnikov result with the improved remainder term $O(1/n)$. The main difficulty is to construct the generator function and to estimate it on the unit circle.

Acknowledgement. We are indebted to the referee for calling our attention to the paper [2] and some other previous results.

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(Received May 23, 1984; revised October 5, 1984)

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Acta Math. Hung.
47 (3—4) (1986), 383—385.

A NORMAL CONNECTED LEFT-SEPARATED SPACE

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Answering the question of A. V. Arhangelskii, M. G. Tkačenko [1] constructed a regular, connected, left-separated space and posed the following problem: does there exist a normal, connected, left-separated space?

Using (CH), we construct here a regular, hereditarily Lindelöf (and hence normal), connected, left-separated space.

Recall that the space X is called left-separated, if there exists a well-ordering of $X=\{x_\alpha:\ \alpha<\tau\}$ such that for any $\alpha<\tau$ the left ray $\{x_\beta:\ \beta<\alpha\}$ is closed in X. Every ordinal is considered as the set of all preceding ordinals and cardinals are identified with the corresponding initial ordinals. We shall denote by I the unit interval [0, 1] in the natural topology.

If α is an ordinal, then $I^\alpha=\times\{I_\beta:\ \beta<\alpha\}$, where $I_\beta=I$ for each $\beta<\alpha$. Let B be a fixed countable base of I, let $I\in[\alpha]^{<\omega}$, i.e. I is a finite subset of α, and let $\varepsilon\colon I\to B$ be a function. Then we set $[\varepsilon]=\{f\in I^\alpha:\ \forall\beta\in I\ (f(\beta)\in\varepsilon(\beta))\}$. Clearly, $[\varepsilon]$ is an elementary open set in I^α, and the set $G=\{[\varepsilon]\}$ of all $[\varepsilon]$ is a base of the product topology on I^α. The domain of a function ε will be denoted by $\Gamma(\varepsilon)$.

DEFINITION. (a) Let $D=\bigcup\{[\varepsilon_n]:\ n\in\omega\}$, then D is called an E-set in I^α;

(b) Let $D=\bigcup\{[\varepsilon_n]:\ n\in\omega\}$ and $\Gamma(\varepsilon_i)\cap\Gamma(\varepsilon_j)=\emptyset$ if $i\neq j$, then D is called a D-set in I^α.

If D is an E-set (or D-set) then we put $\Gamma(D)=\bigcup\{\Gamma(\varepsilon_n):\ n\in\omega\}$ where $D=\bigcup\{[\varepsilon_n]:\ n\in\omega\}$.

The next proposition is obvious.

PROPOSITION 1. *Every D-set is open and dense in I^α.*

Let $x_\beta\in I_\beta$ and $x=(x_\beta)_{\beta\in\alpha}\in I^\alpha$, then the subspace $\sigma_x=\{y\in I^\alpha:\ |\{\beta:\ y_\beta\neq x_\beta\}|<\omega\}$ is called the σ-product of I^α defined by the point x.

The following is well-known.

PROPOSITION 2. *For each $x\in I^\alpha$, σ_x is a dense and connected subspace of I^α.*

THEOREM 1. *Let for every $i\in\omega$ D_i be a D-set in I^α and $P=\bigcap\{D_i:\ i\in\omega\}$. Then there exists a point $x\in I^\alpha$ so that $P\supseteq\sigma_x$.*

PROOF. We can restrict our attention to the coordinates in $\Gamma=\bigcup\{\Gamma(D_i):\ i\in\omega\}$ and since Γ is countable, it is sufficient to prove this theorem for $\alpha=\omega$.

First we shall prove the following lemma.

LEMMA. *For every $i\in\omega$ let γ_i be an infinite family of pairwise disjoint finite sub-*

sets of ω. *Then there exists an infinite and disjoint subfamily* $\mu\subset\bigcup\{\gamma_i\colon i\in\omega\}$ *such that* $\mu\cap\gamma_i$ *is infinite for all* $i\in\omega$.

PROOF OF THE LEMMA. Let us enumerate $\{\gamma_i\colon i\in\omega\}$ in a sequence $\lambda_0, \lambda_1, \dots$ such that every γ_i occurs in this sequence infinitely many times.

Let N_0 be an arbitrary element of λ_0. Suppose, that for each $i<k$ we have already chosen N_i so that

( ) $N_i\in\lambda_i$

(b) if $i<k$, $j<k$ and $i\neq j$, then $N_i\cap N_j=\emptyset$.

Let N_k be an arbitrary element of λ_k such that $N_k\cap N_i=\emptyset$ for every $i<k$. Such an N_k exists in λ_k, since $\bigcup\{N_i\colon i<k\}$ is finite and the elements of λ_k are disjoint. Obviously, $\mu=\{N_k\colon k\in\omega\}$ is as required.

Now let D_i be a D-set, $D_i=\bigcup\{[\varepsilon_n^i]\colon n\in\omega\}$. Let $\Gamma_n^i=\Gamma(\varepsilon_n^i)$ and $\gamma_i=\{\Gamma_n^i\colon n\in\omega\}$. According to the lemma there is a disjoint and infinite subfamily $\mu\subset\bigcup\{\gamma_i\colon i\in\omega\}$ such that $\mu\cap\gamma_i$ is infinite for all $i\in\omega$. Then there is a point $x\in I^\omega$ such that $x(j)\in\varepsilon_n^i(j)$ holds whenever $\Gamma_n^i\in\mu$ and $j\in\Gamma_n^i$.

Since $x\in[\varepsilon_n^i]$ for each $\Gamma_n^i\in\mu$, $x\in D_i$ for every $i\in\omega$ and hence $x\in\bigcap\{D_i\colon i\in\omega\}=P$.

Now, if $y\in\sigma_x$, then $y(j)$ is distinct from x only for finitely many coordinates j. Therefore, for every $i\in\omega$ there is $\Gamma_n^i\in\mu\cap\gamma_i$ such that $x/\Gamma_n^i=y/\Gamma_n^i$. Consequently, $x\in[\varepsilon_n^i]$ implies $y\in[\varepsilon_n^i]\subset D_i$. Thus $y\in\bigcap\{D_i\colon i\in\omega\}=P$, and we have proved that $P\supseteqq\sigma_x$.

The following is now obvious:

COROLLARY. *The intersection of countably many D-sets is a dense and connected subspace of* I^α.

THEOREM 2 (CH). *There exists a hereditarily Lindelöf left-separated and connected space T.*

PROOF. We shall construct this space as a subspace of (the Σ-product of) I^{ω_1}.

Let us consider the space I^{ω_1}. If $\mathscr{D}$ is the family of all the D-sets in I^{ω_1}, then it is easy to see that $|\mathscr{D}|=\mathfrak{c}$. Since we assume (CH), we may write $\mathscr{D}=\{D_\alpha\colon \omega\leqq\alpha<\omega_1\}$ and it can be assumed in a standard manner that $\Gamma(D_\alpha)\subseteqq\alpha$ for every $\alpha<\omega_1$.

Let $\mathscr{E}$ be the family of all nonempty E-sets in I^{ω_1}. Then again $|\mathscr{E}|=\mathfrak{c}$, and hence $|\mathscr{E}^2|=\mathfrak{C}$. Assuming (CH) we can enumerate $\mathscr{E}^2$ as $\mathscr{E}^2=\{(U_\alpha, V_\alpha)\colon \omega\leqq\alpha<\omega_1\}$ in such a way that for every $\alpha<\omega_1$ $\Gamma(U_\alpha)\cup\Gamma(V_\alpha)\subseteqq\alpha$.

Let $\omega\leqq\beta<\omega_1$ and let $P_\beta=\bigcap\{D_\alpha\colon \alpha<\beta\}$. Since for every $\alpha<\beta$ $\Gamma(D_\alpha)\subseteqq\alpha\subset\beta$, then $Q_\beta=pr_\beta P_\beta$ is a dense and connected subspace of I^β.

For every β with $\omega\leqq\beta<\omega_1$ we choose a point $f_\beta\in I^{\omega_1}$ such that

1. if $U_\beta\cap V_\beta=\emptyset$, then $f_\beta\in P_\beta\setminus(U_\beta\cup V_\beta)$, and if $U_\beta\cap V_\beta\neq\emptyset$, then $f_\beta\in P_\beta\cap\cap U_\beta\cap V_\beta$;
2. $f_\beta(\beta)=1$;
3. $f_\beta(\gamma)=0$ for every $\gamma>\beta$.

This can be done, because if $U_\beta\cap V_\beta=\emptyset$, then $(pr_\beta U_\beta)\cup(pr_\beta V_\beta)$ does not cover Q_β (otherwise Q_β would not be connected).

If, on the other hand, $U_\beta \cap V_\beta \neq \emptyset$, then $pr_\beta(U_\beta \cap V_\beta)$ is a nonempty open set in I^β, and therefore $pr_\beta(U_\beta \cap V_\beta) \cap Q_\beta \neq \emptyset$, because Q_β is dense in I^β.

Let $T = \{f_\alpha : \omega \leqq \alpha < \omega_1\}$.

I. *The space T is left-separated.* Indeed, if $0(f_\alpha) = (pr_\alpha^{-1}(1/2, 1]) \cap T$, then for every $\beta < \alpha$ $0 = f_\beta(\alpha) \notin (1/2, 1]$, hence $f_\beta \notin 0(f_\alpha)$.

II. *The space T is hereditarily Lindelöf.* Suppose the contrary, then, as is well-known (cf. [2]), there is a right separated sequence $\{y_\beta : \beta < \omega_1\} \subseteq T$ such that if $y_\beta = f_{\alpha(\beta)}$ then $\beta < \beta'$ implies $\alpha(\beta) < \alpha(\beta')$.

Let $[\varepsilon_\beta]$ be an elementary neighbourhood of y_β from G such that $[\varepsilon_\beta] \cap \{y_{\beta'} : \beta' > \beta\} = \emptyset$.

Let $\Gamma_\beta = \Gamma(\varepsilon_\beta)$. Since $\{\Gamma_\beta : \beta < \omega_1\}$ is an uncountable family of finite subsets of ω_1, this family contains an uncountable Δ-system [2]. Without loss of generality we may assume $\{\Gamma_\beta : \beta < \omega_1]$ to be such a family. Then for every $\beta < \omega_1$ $\Gamma_\beta = \Gamma \cup \tilde{\Gamma}_\beta$ and $\tilde{\Gamma}_\beta \cap \tilde{\Gamma}_{\beta'} = \emptyset$, if $\beta \neq \beta'$. We may also assume that for every $\beta', \beta < \omega_1$ $\varepsilon_\beta / \Gamma = \varepsilon_{\beta'} / \Gamma$.

Let for every $i \in \omega$ $\tilde{\varepsilon}_i : \tilde{\Gamma}_i \to B$ be defined as $\tilde{\varepsilon}_i = \varepsilon_i / \Gamma_i$. Let $D = \bigcup \{[\tilde{\varepsilon}_i] : i \in \omega\}$. Then D is a D-set in I^{ω_1}, hence there is an $\alpha_0 < \omega_1$, so that $D = D_{\alpha_0}$. Let $\beta \geqq \omega$ be an ordinal such that $\alpha(\beta) > \alpha_0$. Then $y_\beta \notin [\tilde{\varepsilon}_n]$ for every $n \in \omega$, and therefore $y_\beta \notin D = D_{\alpha_0}$. But according to 1, $f_{\alpha(\beta)} \in P_{\alpha(\beta)} = \bigcap \{D_\alpha : \alpha < \alpha(\beta)\} \subseteq D_{\alpha_0}$, since $\alpha_0 < \alpha(\beta)$, a contradiction. Thus T is hereditarily Lindelöf.

III. *T is connected.* Suppose it is not. Then, there exist open sets U and V so that $U \cap T \neq \emptyset$, $V \cap T \neq \emptyset$, $U \cup V \supseteq T$ and $U \cap V \cap T = \emptyset$.

Since T is hereditarily Lindelöf and G is a base in I^{ω_1}, we can assume that both U and V are E-sets and hence there is an $\alpha_0 < \omega_1$, so that $(U, V) = (U_{\alpha_0}, V_{\alpha_0})$.

Let us consider the point $f_{\alpha_0} \in T$. If $U_{\alpha_0} \cap V_{\alpha_0} = \emptyset$, then $f_{\alpha_0} \in P_{\alpha_0} \setminus (U_{\alpha_0} \cup V_{\alpha_0}) \subseteq P_{\alpha_0} \setminus T$ and this is impossible, hence $U_{\alpha_0} \cap V_{\alpha_0} \neq \emptyset$. But then $f_{\alpha_0} \in P_{\alpha_0} \cap U_{\alpha_0} \cap V_{\alpha_0}$ and thus $f_{\alpha_0} \in U \cap V \cap T = \emptyset$, again a contradiction. Consequently, T is connected.

We conclude this note by drawing attention to the fact that the space T is hereditarily Lindelöf actually because it has a property which is a straightforward generalization of the so-called HFC property, defined for subspaces of 2^{ω_1} in [3].

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(Received June 14, 1984)

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Acta Math. Hung.
47 (3—4) (1986), 387—390.

NECESSARY CONDITIONS FOR CERTAIN SOBOL'EV SPACES

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Dedicated to K. Tandori on his 60[th] birthday

Introduction

Let $]a, b[\subset \mathbf{R}$ be an open (not necessarily bounded) interval, $p\in]1, \infty[$ be a real number. The following well-known result was proved by F. Riesz:

An absolutely continuous function $f\colon]a, b[\to \mathbf{R}$ (or $\mathbf{C}$) has its derivative $f'\in L_p(a, b)$ if and only if, there exists a real number $K\geqq 0$ such that for any system $\{]a_i, b_i[\subset]a, b[\colon i=1, 2, \ldots\}$ of nonoverlapping bounded subintervals the inequality

$$\sum_i \frac{|f(b_i)-f(a_i)|^p}{|b_i-a_i|^{p-1}} \leqq K$$

holds. That is, the functions belonging to the space $W_p^1[a, b]$ can be characterized by this property.

Our main objective is to generalize this result for the multidimensional case. Our main results are the following theorem and its corollary (Theorem 2).

THEOREM 1. *Let $n\in\mathbf{N}, p, s\in\mathbf{R}$. Suppose that $p\in]1, \infty[$ and $s>1+\frac{n-1}{p}$. Let $]a_i, b_i[\subset\mathbf{R}$ $(i=1, 2, \ldots, n)$ be open (not necessarily bounded) intervals,*

$$\Omega := \mathop{\times}_{j=1}^{n}]a_j, b_j[, \quad \Omega_i = \mathop{\times}_{\substack{j=1\\ j\neq i}}^{n}]a_j, b_j[.$$

If $f\in W_p^s(\Omega)$, then there exist real numbers $K_i\geqq 0$ $(i=1, 2, \ldots, n)$, such that for any systems

$$\{]a_{ij}, b_{ij}[\subset]a_i, b_i[\colon j=1, 2, \ldots, I_i\} \quad (i=1, 2, \ldots, n, \ I_i\in\mathbf{N})$$

of nonoverlapping bounded subintervals the inequalities

$$\sum_{j=1}^{I_i} \frac{\|f_{i,b_{ij}}-f_{i,a_{ij}}\|^p_{W_p^{s-1}(\Omega_i)}}{|b_{ij}-a_{ij}|^{p-1}} \leqq K_i \quad (i=1, 2, \ldots, n)$$

hold.

For any $i=1, 2, \ldots, n$, $a\in]a_i, b_i[$, $f_{i,a}\colon \Omega_i\to\mathbf{R}$ (or $\mathbf{C}$) denotes the function $\xi\mapsto f(\xi_1, \ldots, \xi_{i-1}, a, \xi_i, \ldots, \xi_{n-1})$.

The following theorem will be mentioned as the most important consequence of Theorem 1.

THEOREM 2. *Let $n\in\mathbf{N}$, $p, s\in\mathbf{R}$. Suppose that $p\in]1, \infty[$ and $s>1+\frac{n-1}{p}$.*

Let $]a_i, b_i[\subset \mathbf{R}$ $(i=1, 2, \ldots, n)$ *be open (not necessarily bounded) intervals,* $\Omega := \mathop{\times}\limits_{j=1}^{n}]a_j, b_j[$, $\Omega_i := \mathop{\times}\limits_{\substack{j=1\\ j\neq i}}^{n}]a_j, b_j[$. *If* $f\in W_p^s(\Omega)$, *then there exist real numbers* $L_i \geqq 0$ $(i=1, 2, \ldots, n)$ *such that for any systems*

$$\{]a_{ij}, b_{ij}[\subset]a_i, b_i[: j = 1, 2, \ldots, I_i\} \quad (i = 1, 2, \ldots, n,\ I_i\in\mathbf{N})$$

of nonoverlapping bounded subintervals and sets $\{\xi_{ij}\in\Omega_i,\ j=1, 2, \ldots, I_i\}$ *the inequalities*

$$\sum_{j=1}^{I_i} \frac{|f_{i,b_{ij}}(\xi_{ij})-f_{i,a_{ij}}(\xi_{ij})|^p}{|b_{ij}-a_{ij}|^{p-1}} \leqq L_i \quad (i = 1, 2, \ldots, n)$$

hold.

1. In this section we survey some facts on Sobol'ev spaces.

Let $k, n\in\mathbf{N}$, $p\in]1, \infty[$, $\Omega\subset\mathbf{R}^n$ be an open subset. The Sobol'ev space $W_p^k(\Omega)$ consists of the functions $f\colon \Omega\to\mathbf{R}$ (or $\mathbf{C}$) satisfying the condition

$$\|f\|_{W_p^k(\Omega)} := \Big(\sum_{|\alpha|\leqq k} \|D^\alpha f\|_{L_p(\Omega)}^p\Big)^{1/p} < \infty.$$

Let $W_p^0(\Omega) := L_p(\Omega)$.

Let the integer part of a real number s be denoted by $[s]$. If $s\in\mathbf{R}_+$, then the Sobol'ev space $W_p^s(\Omega)$ consists of the functions $f\colon \Omega\to\mathbf{R}$ (or $\mathbf{C}$) satisfying the condition

$$\|f\|_{W_p^s(\Omega)} := \left(\|f\|_{W_p^{[s]}(\Omega)}^p + \sum_{|\alpha|=[s]} \int_{\Omega\times\Omega} \frac{|D^\alpha f(x)-D^\alpha f(y)|^p}{|x-y|_{(s-[s])p}^{n+(s-[s]p)}}\, dx\, dy\right)^{1/p} < \infty.$$

a) The Sobol'ev embedding theorem (see [1], [2]) is well-known: if $s > \dfrac{n}{p}$, then $W_p^s(\Omega)\subset C(\Omega)$, and the inclusion operator is continuous and linear.

b) We need a one dimensional version of the trace theorem: let $n\in\mathbf{N}$, $p, s\in\mathbf{R}$, $\Omega\subset\mathbf{R}^n$ be an open subset. Suppose that $p\in]1, \infty[$, $s > \dfrac{n-1}{p}$. If $f\in W_p^s(\Omega)$, then for each $\xi\in\mathbf{R}^{n-1}$, $i=1, 2, \ldots, n$, the function $f^{i,\xi}\colon \Omega^{i,\xi}\to\mathbf{R}$ (or $\mathbf{C}$), where

$$\Omega^{i,\xi} = \{t\in\mathbf{R}\colon (\xi_1, \ldots, \xi_{i-1}, t, \xi_i, \ldots, \xi_{n-1})\in\Omega\}$$

and $f^{i,\xi}(t) := f(\xi_1, \ldots, \xi_{i-1}, t, \xi_i, \ldots, \xi_{n-1})$, belongs to the space $W_p^\sigma(\Omega^{j,\xi})$ for each $0\leqq\sigma< s-\dfrac{n-1}{p}$ (see [2]).

2. In this section the main results (Theorems 1 and 2) will be proved.

Proof of Theorem 1. Consider an arbitrary element $\xi\in\Omega_i\subset\mathbf{R}^{n-1}$. By the trace theorem b) the functions $f^{i,\xi}$ belong to the space $W_p^1(a_i, b_i)$ for each $i=1, 2, \ldots, n$, $\xi\in\Omega_i$, because $s-\dfrac{n-1}{p} > 1$. Thus the functions $f^{i,\xi}$ are absolulety continuous, and

$$f^{i,\xi}(b) - f^{i,\xi}(a) = \int_a^b \partial_i f^{i,\xi}.$$

Estimate the norm $\|f_{i,b}-f_{i,a}\|_{W_p^{s-1}(\Omega_i)}$:

$$\|f_{i,b}-f_{i,a}\|^p_{W_p^{s-1}(\Omega_i)} = \|f^{i,\cdot}(b)-f^{i,\cdot}(a)\|^p_{W_p^{s-1}(\Omega_i)} =$$

$$= \Big\|\int_a^b \partial_i f^{i,\cdot}(t)\,dt\Big\|^p_{W_p^{s-1}(\Omega_i)} \leqq \Big(\int_a^b \|\partial_i f^{i,\cdot}(t)\|_{W_p^{s-1}(\Omega_i)}dt\Big)^p \leqq$$

$$\leqq \int_a^b \|\partial_i f^{i,\cdot}(t)\|^p_{W_p^{s-1}(\Omega_i)}dt\,(b-a)^{p-1}.$$

Let $\{]a_{ij}, b_{i,j}[\subset]a_i, b_i[: j=1, 2, \dots, I_i\}$ be a system of nonoverlapping bounded subintervals. Applying the inequality

$$\|f_{i,b}-f_{i,a}\|^p_{W_p^{s-1}(\Omega_i)} \leqq (b-a)^{p-1}\int_a^b \|\partial_i f^{i,\cdot}(t)\|^p_{W_p^{s-1}(\Omega_i)}\,dt$$

we get

$$\sum_{j=1}^{I_i}\frac{\|f_{i,b_{ij}}-f_{i,a_{ij}}\|^p_{W_p^{s-1}(\Omega_i)}}{|b_{ij}-a_{ij}|^{p-1}} \leqq \sum_{j=1}^{I_i}\int_{a_{ij}}^{b_{ij}} \|\partial_i f^{i,\cdot}(t)\|^p_{W_p^{s-1}(\Omega_i)}\,dt \leqq$$

$$\leqq \int_{a_i}^{b_i} \|\partial_i f^{i,\cdot}(t)\|^p_{W_p^{s-1}(\Omega_i)} \leqq \|\partial_i f\|^p_{W_p^{s-1}(\Omega)}.$$

If $K_i \geqq \|\partial_i f\|^p_{W_p^{s-1}(\Omega)}$, then the statement of Theorem 1 is true for K_i.

Corollary. If $\{]a_{ij}, b_{ij}[\subset]a_i, b_i[\, j=1, 2, \dots, I_i\}$ $(i=1, 2, \dots, n)$ are systems of nonoverlapping bounded intervals, then the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{I_i}\frac{\|f_{i,b_{ij}}-f_{i,a_{ij}}\|^p_{W_p^{s-1}(\Omega_i)}}{|b_{ij}-a_{ij}|^{p-1}} \leqq n\,\|f\|^p_{W_p^{s}(\Omega)}$$

holds for every $f\in W_p^s(\Omega)$.

Proof of Theorem 2. Now we combine the result of Theorem 1 with the Sobol'ev embedding theorem. Consider the spaces $W_p^{s-1}(\Omega_i)$ $(i=1, 2, \dots, n)$. The dimension of Ω_i is $n-1$, thus the inequality $s-1>\dfrac{n-1}{p}$ shows that the condition of the Sobol'ev embedding theorem is satisfied, that is there exist real numbers $M_i\geqq 0$ $(i=1, 2, \dots, n)$ such that for each $g_i\in W_p^{s-1}(\Omega_i)$ the inequalities

$$\|g_i\|_{C(\Omega_i)} \leqq M_i\,\|g_i\|_{W_p^{s-1}(\Omega_i)} \quad (i = 1, 2, \dots, n)$$

hold, thus for each $\xi_i\in\Omega_i$

$$|g_i(\xi_i)| \leqq M_i\,\|g_i\|_{W_p^{s-1}(\Omega_i)} \quad (i = 1, 2, \dots, n).$$

Let $\{]a_{ij}, b_{ij}[\subset]a_i, b_i[: j=1, 2, ..., I_i\}$ be a system of nonoverlapping bounded subintervals and $\{\xi_{ij}\in\Omega_i: j=1, 2, ..., I_i\}$ be a set. Now

$$\sum_{j=1}^{I_i} \frac{|f_{i,b_{ij}}(\xi_{ij})-f_{i,a_{ij}}(\xi_{ij})|^p}{|b_{ij}-a_{ij}|^{p-1}} \leqq$$

$$\leqq M_i^p \sum_{j=1}^{I_i} \frac{\|f_{i,b_{ij}}-f_{i,a_{ij}}\|_{W_p^{s-1}(\Omega_i)}}{|b_{ij}-a_{ij}|^{p-1}} \leqq M_i^p \|\partial_i f\|^p_{W_p^{s-1}(\Omega)}.$$

If $L_i \geqq M_i^p \|\partial_i f\|^p_{W_p^{s-1}(\Omega)}$, then the statement of Theorem 2 is true for L_i.

REMARK. Theorems 1 and 2 give a necessary condition for a function to belong to the Sobol'ev space $W_p^s(\Omega)$ $\left(s>1+\frac{n-1}{p}\right)$. It is very probable that Theorem 1 is also a sufficient condition.

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Received June 22, 1984)

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Acta Math. Hung.
47 (3—4) (1986), 391—393.

A MOMENT THEOREM FOR CONTRACTIONS ON HILBERT SPACES

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Given a subset X of a Hilbert space H which spans the space H, and a function $f\colon \mathbf{Z}\times X\to H$, where $\mathbf{Z}$, as usual, stands for the set of integers, one can ask whether there exists a contraction T on the Hilbert space H such that

$$T_n x = f(n, x) \quad (n\in\mathbf{Z},\ x\in X) \tag{1}$$

holds, where T_n is defined for $n\in\mathbf{Z}$ as follows:

$$T_n = \begin{cases} T^n & \text{if } n \geqq 0, \\ T^{*|n|} & \text{if } n < 0. \end{cases} \tag{1'}$$

For a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on the Hilbert space H, with an extension $T_{-t}=T_t^*$ $(t\geqq 0)$ to $\mathbf{R}$, the corresponding problem is that given a function $f\colon \mathbf{R}\times X\to H$, under what condition does there exist a continuous semigroup $\{T_t\}$ of contractions such that

$$T_t x = f(t, x) \quad (t\in\mathbf{R},\ x\in X). \tag{2}$$

The present note gives an answer to these problems. It is in connection with the preceding papers [1], [2] on this subject. [1] deals with equation (1) required for $X=\{x_0\}$ and $n\geqq 0$ only, i.e., the equation $T^n x_0 = x_n$ $(n=1, 2, \ldots)$, with given $\{x_n\}_0^\infty\subset H$; and analogously for the continuous one parameter case T_t $(t\geqq 0)$. On the other hand, [2] considers the case when the operators T_n we are seeking for are not derived from some contraction T as in (1′) but rather from some unitary operator U on a Hilbert space H' and from an operator $V\colon H'\to H$ of the form $T_n = VU^nV^*$ $(n\in\mathbf{Z})$; and analogously, $T_t = VU_tV^*$ $(t\in\mathbf{R})$ where U_t is a continuous one parameter group of unitaries. (In fact, [2] treats even more general groups and $*$-semigroups.)

For unitary dilation theory we refer to Sz.-Nagy [3].

THEOREM 1. *There exists a contraction T satisfying* (1) *if and only if the function f satisfies*

$$f(0, x) = x \quad (x\in X), \tag{3}$$

$$\big(f(n, x), f(m, y)\big) = \big(f(n-m, x), y\big) \quad (m, n\in\mathbf{Z}\colon mn \leqq 0;\ x, y\in X), \tag{4}$$

$$\Big\|\sum_{n,x} c_{n,x} f(n, x)\Big\|^2 \leqq \sum_{m,y}\sum_{n,x} \bar c_{m,y} c_{n,x}\big(f(n-m, x), y\big) \tag{5}$$

where $\{c_{n,x}\}$ $(n\in\mathbf{Z},\ x\in X)$ is an arbitrary finite double sequence of complex numbers.

THEOREM 2. *There exists a continuous family* $\{T_t\}$ *of contractions satisfying* (2) *if and only if f is continuous in its first variable and satisfies the identities* (3), (4), (5) *with* m, n *taking their values in* $\mathbf{R}$.

PROOF OF NECESSITY. 1. Let T be a contraction on the Hilbert space H and let U be a unitary dilation on a Hilbert space K containing H as a subspace. Then for the orthogonal projection P of K onto H we have

$$T_n x = PU^n x \quad (n \in \mathbf{Z}, x \in H) \tag{6}$$

where T_n is defined by (1′). Let further $\{C_{n,x}\}$ be a finite double sequence of complex numbers indexed by elements of the set $\mathbf{Z}\times X$. Then we have by (1) and (6), for any x, y in X,

$$f(0, x) = T_0 x = x,$$

$$\big(f(n, x), f(m, y)\big) = (T^n x, T^{*|m|} y) = (T^{n+|m|} x, y) = (T^{n-m} x, y) =$$
$$= \big(f(n-m, x), y\big) \quad \text{if} \quad m < 0,\ n \geqq 0,$$

$$\big(f(n, x), f(m, y)\big) = (T^{*|n|} x, T^m y) = (T^{*(|n|+m)} x, y) =$$
$$= (T^{*|n-m|} x, y) = \big(f(n-m, x), y\big) \quad \text{if} \quad m \geqq 0,\ n < 0;$$

and

$$\Big\|\sum_{n,x} c_{n,x} f(n, x)\Big\|^2 = \Big\|\sum_{n,x} c_{n,x} T_n x\Big\|^2 = \Big\|P \sum_{n,x} c_{n,x} U^n x\Big\|^2 \leqq \Big\|\sum_{n,x} c_{n,x} U^n x\Big\|^2 =$$
$$= \sum_{m,y}\sum_{n,x} \bar{c}_{m,y} c_{n,x} (U^n x, U^m y) = \sum_{m,y}\sum_{n,x} \bar{c}_{m,y} c_{n,x} (U^{n-m} x, y) =$$
$$= \sum_{m,y}\sum_{n,x} \bar{c}_{m,y} c_{n,x} (U^{n-m} x, Py) = \sum_{m,y}\sum_{n,x} \bar{c}_{m,y} c_{n,x} (PU^{n-m} x, y) =$$
$$= \sum_{m,y}\sum_{n,x} \bar{c}_{m,y} c_{n,x} (T_{n-m} x, y) = \sum_{m,y}\sum_{n,x} \bar{c}_{m,y} c_{n,x} \big(f(n-m, x), y\big).$$

2. The case of a continuous semigroup $\{T_t\}$ of contractions can be dealt with in a similar way by using the corresponding minimal dilation $\{U_t\}$.

PROOF OF SUFFICIENCY. 1. Let F_0 be the (complex) linear space of all finite double sequences $\{c_{n,x}\}$ $(n \in \mathbf{Z}, x \in X)$ of complex numbers with the shift operation

$$U_0\{c_{n,x}\} := \{c'_{n,x}\}, \quad \text{where} \quad c'_{n,x} = c_{n-1,x} \quad (n \in \mathbf{Z},\ x \in X).$$

Introduce a semi-inner product $\langle .\,, .\rangle$ in F_0 by

$$\langle \{c_{n,x}\}, \{d_{m,y}\}\rangle := \sum_{m,y}\sum_{n,x} \bar{d}_{m,y} c_{n,x} \big(f(n-m, x), y\big). \tag{7}$$

Positive semi-definiteness follows from (5). It also follows from (5) that the linear map V_0 defined by

$$V_0\{c_{n,x}\} := \sum_{n,x} c_{n,x} f(n, x) \quad (\{c_{n,x}\} \in F_0)$$

is a contraction from F_0 into the Hilbert space H.

Denote F the Hilbert space resulting from F_0 by factoring with respect to the null space of $\langle .\,, .\rangle$ and then by completing with respect to the norm inherited. At the same time U_0 induces a unitary operator U on the Hilbert space F and V_0 induces a contraction V from F into H. In what follows the equivalence class represented by $\{c_{n,x}\}$ is also denoted by $\{c_{n,x}\}$.

We first show that for any $x \in X$

$$V^*x = \{d_{n,x}\}, \quad \text{where} \quad d_{n,x} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

For this let $\{c_{m,y}\} \in F$ so that (7) gives, in view of (3) and (4),

$$\langle \{c_{m,y}\}, V^*x \rangle = (V\{c_{m,y}\}, x) = \sum_{m,y} c_{m,y}(f(m, y), x) =$$

$$= \sum_{m,y} \sum_{n,x} c_{m,y} \bar{d}_{n,x}(f(m-n, y), x) = \langle \{c_{m,y}\}, \{d_{n,x}\} \rangle,$$

as desired. Now we get by (8) for any x in X,

$$UV^*x = \{h_{n,x}\}, \quad \text{where} \quad h_{n,x} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$U^{-1}V^*x = \{k_{n,x}\}, \quad \text{where} \quad k_{n,x} = \begin{cases} 1, & \text{if } n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Defining $T = VUV^*$ we have a contraction on H satisfying (1). Indeed,

$$Tx = VUV^*x = V\{h_{n,x}\} = \sum_{n,x} h_{n,x} f(n, x) = f(1, x),$$

$$T^*x = VU^{-1}V^*x = V\{k_{n,x}\} = \sum_{n,x} k_{n,x} f(n, x) = f(-1, x)$$

by the definition of $\{h_{n,x}\}$ and $\{k_{n,x}\}$. We use induction on n (first for natural numbers): assuming (1) for an $n>0$ we observe that for any $y \in X$

$$(y, T^{n+1}x) = (T^*y, T^n x) = (T^*y, f(n, x)) =$$

$$= (VU^{-1}V^*y, f(n, x)) = (VU^{-1}\{d_{n,y}\}, f(n, x)) = (f(-1, y), f(n, x))$$

$$(\text{by } (4)) = (y, f(n+1, x));$$

because this shows that (1) also holds for $n+1$.

For a negative integer n the same method applies and the proof of Theorem 1 is complete.

2. The proof of Theorem 2 is similar. We have only to add the observation that continuity in the parameter is immediate from the construction.

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(Received June 25, 1984; revised January 28, 1985)

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Acta Math. Hung.
47 (3—4) (1986), 395—399.

HÖLDER-TYPE INEQUALITIES FOR QUASIARITHMETIC MEANS

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1. Introduction

Let $\varphi\colon \mathbf{R}_+ \to \mathbf{R}$ be a continuous strictly monotonic function. Define the quasi-arithmetic mean M_φ by the help of φ as follows:

$$M_\varphi(x_1, \ldots, x_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right).$$

In [2], L. Losonczi considered the Hölder-type inequality

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqq M_\varphi(x_1, \ldots, x_n) M_\psi(y_1, \ldots, y_n). \tag{1}$$

Assuming $\varphi, \psi\colon \mathbf{R}_+ \to \mathbf{R}$ to be twice differentiable with nonvanishing first derivative, he proved that (1) is satisfied for any $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}_+$ if and only if there exist $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ so that

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^q\right)^{1/q} \leqq M_\varphi(x_1, \ldots, x_n) M_\psi(y_1, \ldots, y_n) \tag{2}$$

holds for all values $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}_+$.

The aim of the present note is to prove this theorem without any differentiability assumption on φ and ψ. Our method will be completely different from Losonczi's one. An important cornerstone of the discussion is the Lemma (see below) which gives some information about nonconcave functions. I am very grateful to Prof. Gy. Szabó, since he gave me the simple but nice idea of the proof of this Lemma.

2. Results

LEMMA. *Let $T \subseteqq \mathbf{R}$ be an interval and let $f\colon I \to \mathbf{R}_+$ be a continuous strictly monotonic nonconcave function. Define the set H_f by*

$$H_f := \left\{\frac{f(x)}{f(y)} \,\middle|\, x, y \in I, \frac{f(x)+f(y)}{2} > f\left(\frac{x+y}{2}\right)\right\}.$$

Then there exists $1 < \mu$ such that the interval $]1, \mu[$ is contained in H_f.

PROOF. Since f is nonconcave and continuous, hence there exist $x, y \in I$ such that

$$\frac{f(x)+f(y)}{2} > f\left(\frac{x+y}{2}\right). \tag{3}$$

We may assume that $x<y$. Now consider the function

$$g(t) := f(t) - \frac{(t-x)f(y)+(y-t)f(x)}{y-x}, \quad x \leqq t \leqq y.$$

It is obvious that $g(x)=g(y)=0$. Further, because of (3), we have $g\left(\frac{x+y}{2}\right)<0$. Hence there exists a uniquely determined value $t_0 \in]x, y[$ such that

$$g(t_0) \leqq g(u), \quad \text{if} \quad x < u < t_0, \tag{4}$$

$$g(t_0) < g(v), \quad \text{if} \quad t_0 < v < y. \tag{5}$$

Choose $\varepsilon_0>0$ so that $[t_0-\varepsilon_0, t_0+\varepsilon_0] \subset [x, y]$. Then, applying (4) and (5), we get

$$g(t_0) < \frac{g(t_0-\varepsilon)+g(t_0+\varepsilon)}{2}$$

that is

$$f(t_0) < \frac{f(t_0-\varepsilon)+f(t_0+\varepsilon)}{2} \tag{6}$$

for $0<\varepsilon<\varepsilon_0$, Now, (6) and the definition of H_f imply

$$\frac{f(t_0-\varepsilon)}{f(t_0+\varepsilon)} \in H_f \quad \text{and} \quad \frac{f(t_0+\varepsilon)}{f(t_0-\varepsilon)} \in H_f \tag{7}$$

for $0<\varepsilon<\varepsilon_0$. Let

$$\mu := \begin{cases} \dfrac{f(t_0+\varepsilon_0)}{f(t_0-\varepsilon_0)}, & \text{if } f \text{ is increasing,} \\[2ex] \dfrac{f(t_0-\varepsilon_0)}{f(t_0+\varepsilon_0)}, & \text{if } f \text{ is decreasing.} \end{cases}$$

Since f is strictly monotonic, hence $\mu>1$. Further, the continuity of f and (7) yield $]1, \mu[\subset H_f$. $\square$

THEOREM. *Let $\varphi, \psi \colon \mathbf{R}_+ \to \mathbf{R}$ be continuous strictly monotonic functions. Inequality* (1) *is satisfied for all $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbf{R}_+$, $n \in \mathbf{N}$ if and only if there exist $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ such that*

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i^p\right)^{1/p} \leqq M_\varphi(x_1, \dots, x_n) \tag{8}$$

and

$$\left(\frac{1}{n}\sum_{i=1}^{n} y_i^q\right)^{1/q} \leqq M_\psi(y_1, \ldots, y_n) \tag{9}$$

for any $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_n$ *in* $\mathbf{R}_+$.

PROOF. Let P be the set of all values $p>0$ for which (8) is satisfied for any $x_1, \ldots, x_n \in \mathbf{R}_+$, $n \in \mathbf{N}$. Substituting $y_1 = \ldots = y_n = 1$ into (1) we easily obtain that $1 \in P$. Since, for $0<p'<p$, $x_1, \ldots, x_n \in \mathbf{R}_+$, $n \in \mathbf{N}$ we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{p'}\right)^{1/p'} \leqq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{p}\right)^{1/p}$$

(see [1, Theorem 5, p. 15]), hence $p \in P$, $0<p'<p$ implies $p' \in P$. Therefore P is an interval. If P were unbounded then (8) would hold for all positive p. Then, taking the limit $p \to \infty$ we would obtain

$$\max_{1 \leqq i \leqq n} x_i \leqq M_\varphi(x_1, \ldots, x_n)$$

(see [1, Theorem 4, p. 15]). This contradiction proves that that P is bounded. Denote by p_0 the least upper bound of P. Then, since the right hand side of (8) is a continuous function of p, we have $p_0 \in P$. In other words, we have proved that $P =]0, p_0]$ for some $1 \leqq p_0 < \infty$.

Similarly, denote by Q the set of all positive values q for which (9) is satisfied. Then we obtain that $Q =]0, q_0]$ for some $1 \leqq q_0 < \infty$.

Now we shall show that

$$\frac{1}{p_0} + \frac{1}{q_0} \leqq 1. \tag{10}$$

To get a contradiction, assume that (10) is not valid. Then there exist $p_0 < p_1$, $q_0 < q_1$ such that

$$\frac{1}{p_1} + \frac{1}{q_1} = 1.$$

But then $p_1 \notin P$ and $q_1 \notin Q$. That is, there exist $x_1, \ldots, x_n \in \mathbf{R}_+$ such that

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{p_1}\right)^{1/p_1} > \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \varphi(x_i)\right).$$

Let $s_i := \varphi(x_i)$ for $i = 1, \ldots, n$. Then we have

$$\frac{1}{n}\sum_{i=1}^{n} [\varphi^{-1}(s_i)]^{p_1} > \left[\varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} s_i\right)\right]^{p_1}$$

i.e. the function $f := (\varphi^{-1})^{p_1}$ is nonconcave. Similarly, it follows from $q_1 \notin Q$ that the function $g := (\psi^{-1})^{q_1}$ is also nonconcave.

Now, applying the Lemma, we obtain that there exist $1 < \mu, \nu$ such that $]1, \mu[\subset H_f$ and $]1, \nu[\subset H_g$ is satisfied.

But then $H_f \cap H_g$ is nonvoid. Hence there exist $s_1, s_2 \in \varphi(\mathbf{R}_+)$ and $t_1, t_2 \in \psi(\mathbf{R}_+)$ such that

$$\frac{f(s_1)}{f(s_2)} = \frac{g(t_1)}{g(t_2)} \tag{11}$$

and

$$\frac{f(s_1)+f(s_2)}{2} > f\left(\frac{s_1+s_2}{2}\right), \quad \frac{g(t_1)+g(t_2)}{2} > g\left(\frac{t_1+t_2}{2}\right). \tag{12}$$

Let $x_i = \varphi^{-1}(s_i)$ and $y_i = \psi^{-1}(t_i)$ for $i=1, 2$. Then, by (11), we have

$$\frac{x_1^{p_1}}{x_2^{p_1}} = \frac{y_1^{q_1}}{y_2^{q_2}}.$$

Therefore

$$\frac{x_1 y_1 + x_2 y_2}{2} = \left(\frac{x_1^{p_1}+x_2^{p_1}}{2}\right)^{1/p_1} \left(\frac{y_1^{q_1}+y_2^{q_1}}{2}\right)^{1/q_1}. \tag{13}$$

Applying (12), we obtain

$$\frac{x_1^{p_1}+x_2^{p_1}}{2} > \left[\varphi^{-1}\left(\frac{\varphi(x_1)+\varphi(x_2)}{2}\right)\right]^{1/p_1}, \quad \frac{y_1^{q_1}+y_2^{q_1}}{2} > \left[\psi^{-1}\left(\frac{\psi(y_1)+\psi(y_2)}{2}\right)\right]^{1/q_1}$$

i.e.

$$\left(\frac{x_1^{p_1}+x_2^{p_1}}{2}\right)^{1/p_1} > M_\varphi(x_1, x_2), \quad \left(\frac{y_1^{q_1}+y_2^{q_1}}{2}\right)^{1/q_1} > M_\psi(y_1, y_2). \tag{14}$$

Now, inequalities (13) and (14) give the desired contradiction, since then (1) is not satisfied for $x_1, x_2, y_1, y_2 \in \mathbf{R}_+$.

This contradiction validates inequality (10).

If (10) is satisfied then we can find $1 < p \leqq p_0$, $1 < q \leqq q_0$ so that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

As we have proved, $p \in P$, $q \in Q$ therefore (8) and (9) hold.

Thus the proof of the "only if" part of the Theorem is complete. We omit the the proof of the "if" part since it is based upon the classical Hölder inequality, and is very simple.

At last we mention a somewhat generalized form of the Theorem. The proof of this result can easily be made by the same way as above.

GENERAL THEOREM. *Let* $k \geqq 2$, $\varphi_1, \ldots, \varphi_k \colon \mathbf{R}_+ \to \mathbf{R}$ *be continuous monotonic functions. Then the inequality*

$$\frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{k} x_{ij} \leqq \prod_{j=1}^{k} M_{\varphi_j}(x_{1j}, \ldots, x_{nj})$$

is satisfied for all $x_{ij} \in \mathbf{R}_+$; $i=1, \ldots, n$; $j=1, \ldots, k$; $n \in \mathbf{N}$ *if and only if there exist*

$p_1, \ldots, p_k > 1$ *with* $\sum_{j=1}^{k} \frac{1}{p_j} = 1$ *such that*

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i^{p_j}\right)^{1/p_j} \leqq M_{\varphi_j}(x_1, \ldots, x_n)$$

for any $x_1, \ldots, x_n \in \mathbf{R}_+$, $n \in \mathbf{N}$ *and for each* $j = 1, \ldots, k$.

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[1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*. Cambridge Univ. Press (Cambridge, 1952).

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(Received July 17, 1984)

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[1] G. Szegő, *Orthogonal polynomials*, AMS Coll, Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

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