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A THEOREM OF PICARD TYPE

M. NIIMURA (Tokyo)

Let R be a hyperbolic Riemann surface, let $g(a)$ be the Green function for R with pole at a fixed point a_0 chosen arbitrarily in R, and put $R(\alpha)=\{a: g(a)>\alpha\}$ for any $\alpha>0$. In this note we assume the following conditions (1) and (2):

(1) The closure $\overline{R(\alpha)}$ of $R(\alpha)$ is compact for any $\alpha>0$.

(2) $\int\limits_0^\infty B(\alpha)\,d\alpha<\infty$, where $B(\alpha)$ denotes the first Betti number of $R(\alpha)$.

The classical Picard theorem on exceptional values is well-known. E. F. Collingwood and M. L. Cartwright have shown the following theorem of Picard type (see [1], p. 139): If $f(z)$ is meromorphic in $\{|z|<1\}$, then every point of $\{|z|=1\}$ belongs either to P or to F'. Here P denotes the set of points $e^{i\theta}$ for which the complement of the range of $f(z)$ contains at most two values, and F' denotes the derived set of points $e^{i\theta}$ at which $f(z)$ has the angular limit.

In this note we shall give a theorem on Green's lines. As an application we shall obtain an extension of Collingwood—Cartwright's theorem to a class of hyperbolic Riemann surfaces satisfying (1) and (2).

For the definitions concerning Green's lines we refer to [5] and henceforth consider Green's lines issuing from a_0. Let $h(a)$ be the conjugate harmonic function of $g(a)$. We map the union R' of $\{a_0\}$ and all the Green's lines onto a subregion of $\{|z|<1\}$ by $z=\varphi(a)=e^{-g(a)-ih(a)}$. The infimum of $g(a)$ over every Green's line, which does not terminate at a point of $Z=\{a_n: \text{grad } g(a_n)=0\}$, $n=1, 2, \dots, m\leqq\infty$, is equal to zero by (1). It is seen from the proof of Theorem 1 (H. Widom) of [3] that $\sum g(a_n)\leqq$ $\leqq\int\limits_0^\infty B(\alpha)\,d\alpha$ for $a_n\in Z$. The total length of the slits of $\varphi(R')$ is hence finite by (2).

Let R^* denote the Martin compactification of R, let $\xi=f(a)$ denote a function meromorphic on R, and put $\Delta=R^*-R$. At a point $e^{i\theta}$ of $\{|z|=1\}\cap\partial\varphi(R')$, where ∂ denotes boundary, let there be the tangent line to $\partial\varphi(R')$ and a sector region $A(e^{i\theta})$, included in $\varphi(R')$, having vertex at $e^{i\theta}$ and bisected by the radius λ_θ drawn to $e^{i\theta}$. If $\varphi^{-1}(\lambda_\theta)$ terminates at a point p on Δ, then we denote by $S(p)$ the set of $\varphi^{-1}(A(e^{i\theta}))$ for all $A(e^{i\theta})$ at $e^{i\theta}$. If f tends uniformly to some limit inside any $S\in S(p)$, then we say that f has angular limit at p.

$F(f)$ denotes the set of points p on Δ such that f has angular limit at p. $F'(f)$ denotes the derived set of $F(f)$. $P(f)$ denotes the set of points p on Δ such that every value of $\{|\xi|\leqq\infty\}$ is taken by f infinitely often in every neighborhood of p with two possible exceptions.

THEOREM. *If $\xi = f(a)$ is a function meromorphic on R, and if, for any r-neighborhood $U(p, r)$ of a point p on Δ, $\Delta \cap U(p, r)$ is of positive harmonic measure, then either $p \in P(f)$ or $p \in F'(f)$.*

PROOF. Suppose that $p \notin P(f)$, and choose any $r' > 0$. Then there is a $U(p, r_1)$, $0 < r_1 < r'$ such that f does not take three distinct values in $R \cap U(p, r_1)$. By Theorem 7 of [3], radial paths ending at almost all points of $\{|z|=1\} \cap \partial\varphi(R')$ are mapped by φ^{-1} to Green's lines in R' terminating at almost all points of Δ with respect to harmonic measure. Therefore there are two Green's lines L_θ and $L_{\theta'}$, with distinct coordinates θ and θ', terminating at points of $\Delta \cap U(p, r_2)$, $0 < r_2 < r_1$. It is possible to form a simply connected, rectangle-shaped region G in $\{|z|<1\}$, bounded by parts of the radial paths $\varphi(L_\theta)$ and $\varphi(L_{\theta'})$, and arcs of $\{|z|=r_0,\ 0<r_0<1\}$ and $\{|z|=1\}$ such that $\varphi^{-1}(G-M)$, where M denotes the set of the slits of $\varphi(R')$, is included in $U(p, r_1)$. Let G' be a component of $G-M$. G' is a simply connected region, and the length of $\partial G'$ is finite.

We map G' onto $\{|w|<1\}$ by a univalent holomorphic function $w = \psi(z)$. Now $f \circ \varphi^{-1} \circ \psi^{-1}(w)$ does not take three distinct values in $\{|w|<1\}$. We may assume that it does not take the values 0, 1 and ∞ in $\{|w|<1\}$. Let H be the inverse of the modular function for the half plane. We consider the function

$$H^*(\xi) = (H(\xi) - H(\xi_0))/(H(\xi) - \overline{H(\xi_0)}),$$

where ξ_0 is a number which is not real. Put $H_0(w) = H^* \circ f \circ \varphi^{-1} \circ \psi^{-1}(w)$. Then $|H_0(w)| < 1$ in $\{|w|<1\}$. By the monodromy theorem, H_0 is one-valued.

It is easy to see the following results from page 5 of [4]: At almost all points $e^{i\theta}$ of $\{|z|=1\} \cap \partial G'$, there are sector regions $A(e^{i\theta})$, included in G', having vertex at $e^{i\theta}$ and bisected by the radius drawn to $e^{i\theta}$. At the point $e^{i\theta'}$ of $\{|w|=1\}$ corresponding to $e^{i\theta}$, under the homeomorphism of the frontiers included by $\psi^{-1}(w)$, there is a sector region A_w in $\{|w|<1\}$ such that $\psi(A(e^{i\theta})) \subset A_w$, having vertex at $e^{i\theta'}$ and bisected by the radius drawn to $e^{i\theta'}$.

By applying Fatou—Lindelöf's theorem to H_0, it is seen that $H_0 \circ \psi(z)$ tends uniformly to limits inside $A(e^{i\theta^*})$ and $A(e^{i\theta^{*\prime}})$, $\theta^* \neq \theta^{*\prime}$, in G', and that both L_{θ^*} and $L_{\theta^{*\prime}}$ terminate at points of $\Delta \cap U(p, r_2)$. Thus there is a point p', $p' \neq p$, on $\Delta \cap U(p, r_2)$ such that $H^* \circ f$ tends uniformly to a limit inside any $S \in S(p')$. Thus $f = (H^*)^{-1} \circ H^* \circ f$ has the angular limit at a point, different from p, on $\Delta \cap U(p, r')$. The assertion of the Theorem is proved.

COROLLARY 1. *If f is a function meromorphic on R and for every p on Δ and for any $r>0$, $\Delta \cap U(p, r)$ is of positive harmonic measure, then the points of $P(f)$ and $F(f)$ are everywhere dense on Δ.*

COROLLARY 2. *If, under the hypothesis of Corollary 1, $P(f)$ is empty, then the points of $F(f)$ are everywhere dense on Δ.*

As an application of Corollary 2 we obtain an extension of Corollary 22.1 of [1].

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STRONG POROSITY FEATURES OF TYPICAL CONTINUOUS FUNCTIONS

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1. Introduction

In 1931 Banach [1] and Mazurkiewicz [12] proved independently that the "typical continuous function" is nowhere differentiable. (Here, and in the sequel, we use the term "typical continuous function" to mean that the set of functions which exhibit the property we are discussing is residual in the complete metric space $\mathscr{C}[0, 1]$). Shortly thereafter, Marcinkiewicz [11] and Jarnik [7] [8] [9] used the Baire Category Theorem to show that typical continuous functions exhibit a great deal of pathology with respect to differentiation properties. More recently, other authors have obtained a number of similar results: negative ones, showing that the typical continuous function is nowhere differentiable with respect to some generalized derivative; and positive ones, showing that it *is* differentiable in certain senses. (See [2] for a summary of such results.)

In the present work we show that if $\mathscr{K}$ is a σ-compact subset of $\mathscr{C}$, then the graph of the typical continuous function intersects the graph of each member of $\mathscr{K}$ in a "very thin" set. We show that this implies some of the known negative results, mentioned above, (as well as some new negative results), and we also observe that the result clarifies some of the known positive results.

2. Main results

We mentioned in the Introduction that typical continuous functions agree with function in a σ-compact subset of $\mathscr{C}$ on "very thin" sets. Our main result is Theorem 2.5 below which makes this idea precise. We begin with a definition:

DEFINITION 2.1. A set $E \subseteqq \mathbf{R}$ is called *bilaterally strongly porous* (bsp) at a point y_0 provided $p_+(E, y_0) = p_-(E, y_0) = 1$ where $p_+(E, y_0)$ is the right-porosity of E at y_0:

$p_+(E, y_0) = \overline{\lim\limits_{h \downarrow 0}} \dfrac{\lambda(E, y_0, h^+)}{h}$ where $\lambda(E, x, h^+)$ is the length of the largest open interval in $E^c \cap (x, x+h)$. We define $p_-(E, y_0)$ similarly.

E is bsp if E is bsp at all its members. (When working in [0, 1], we require strong one-sided porosity at the endpoints.)

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The first two lemmas require some notation. Let n be a positive integer, $(n\geqq 2)$, and let $0<\delta_n<\frac{1}{n}$. Define sets A_k for given x and $k\in\{1,2,3,4\}$ by:

$$A_1=\left[x+\frac{\delta_n}{2n^2},\,x+\frac{\delta_n}{2n}\right],\quad A_2=\left[x+\frac{\delta_n}{2n^2}+\frac{\delta_n}{n},\,x+\delta_n\right],$$

$$A_3=\left[x-\frac{\delta_n}{2n},\,x-\frac{\delta_n}{2n^2}\right],\quad A_4=\left[x-\delta_n,\,x-\frac{\delta_n}{2n^2}-\frac{\delta_n}{n}\right].$$

The next three lemmas are variants of results found in Goffman and Pedrick [6], which are suitable for our purposes.

LEMMA 2.2. *Let σ be a modulus of continuity, $1/2>\varepsilon>0$, n a positive integer, $(n\geqq 2)$. Then there exists $\delta_n\in\left(0,\frac{1}{n}\right)$ and $\eta>0$ such that*

(a) *there exists $f\in\mathscr{C}$, with $\|f\|\leqq\varepsilon$, so that for every $x\in\left[0,1-\frac{1}{n}\right]$ and every $g\in\mathscr{C}$ with $\|f-g\|<\eta$ there exists $k\in\{1,2\}$ such that $|g(x)-g(y)|>\sigma(|x-y|)$ for all $y\in A_k$; and*

(b) *there exists $\tilde{f}\in\mathscr{C}$, with $\|\tilde{f}\|\leqq\varepsilon$, so that for every $x\in\left[\frac{1}{n},1\right]$ and every $g\in\mathscr{C}$ with $\|\tilde{f}-g\|<\eta$ there exists $k\in\{3,4\}$ such that $|g(x)-g(y)|>\sigma(|x-y|)$ for all $y\in A_k$.*

PROOF. Choose $\delta_n\in\left(0,\frac{1}{n}\right)$ such that $\sigma(\delta_n)\leqq\frac{\varepsilon}{4n^2}$. Let $\eta=\frac{1}{2}\sigma\left(\frac{\delta_n}{2n^2}\right)$. For part (a):

Let f be the polygonal function on $[0,1]$ with slopes $\pm\varepsilon/\delta_n$ so that $0\leqq f(x)\leqq\varepsilon$, $f(1)=0$, and all but (possibly) the first segment have length $\sqrt{\varepsilon^2+\delta_n^2}$. Fix $x\in\left[0,1-\frac{1}{n}\right]$.

Case 1: $(y,f(y))$ is on the same segment as $(x,f(x))$ for every $y\in A_1$. Then, for every $y\in A_1$,

$$|f(x)-f(y)|=\frac{\varepsilon}{\delta_n}|x-y|>\frac{4n^2}{\delta_n}\sigma(\delta_n)|x-y|\geqq\frac{4n^2\sigma(\delta_n)\delta_n}{\delta_n 2n^2}=2\sigma(\delta_n)\geqq 2\sigma(|x-y|)$$

since $\frac{\delta_n}{2n^2}\leqq|x-y|\leqq\delta_n$.

Case 2: Case 1 does not hold so that $(y,f(y))$ is on an adjacent segment to $(x,f(x))$ for every $y\in A_2$. Then, for every $y\in A_2$,

$$|f(x)-f(y)|\geqq\left|f\left(x+\frac{\delta_n}{n}\right)-f(y)\right|=\frac{\varepsilon}{\delta_n}\left|x+\frac{\delta_n}{n}-y\right|\geqq\frac{\varepsilon}{\delta_n}\left(|x-y|-\frac{\delta_n}{n}\right)>$$

$$>\frac{4n^2\sigma(\delta_n)}{\delta_n}\frac{\delta_n}{2n^2}\geqq 2\sigma(|x-y|)$$

since $\frac{\delta_n}{n}+\frac{\delta_n}{2n^2}\leqq|x-y|\leqq\delta_n$. So, for $x\in\left[0, 1-\frac{1}{n}\right]$ there exists $k\in\{1, 2\}$ such that for every $y\in A_k$ $|f(x)-f(y)|>2\sigma(|x-y|)$.

Let $g\in\mathscr{C}$ with $\|f-g\|<\eta$. Then, for $x\in\left[0, 1-\frac{1}{n}\right]$ there exists $k\in\{1, 2\}$ such that for every $y\in A_k$

$$|g(x)-g(y)|>|f(x)-f(y)|-2\eta>2\sigma(|x-y|)-\sigma\left(\frac{\delta_n}{2n^2}\right)\geqq$$
$$\geqq 2\sigma(|x-y|)-\sigma(|x-y|)=\sigma(|x-y|).$$

To prove (b) take $\tilde{f}$ to be like f except that $\tilde{f}(0)=0$ and all but (possibly) the last segment have length $\sqrt{\varepsilon^2+\delta_n^2}$. The rest of the argument is analogous to that for (a).

LEMMA 2.3. *Let σ, n be as in 2.2 and let*

$$E_n^1=\left\{f\in\mathscr{C}\,|\,\exists\delta_n\in\left(0, \frac{1}{n}\right) \text{ so that } \forall x\in\left(0, 1-\frac{1}{n}\right] \ \exists k\in\{1, 2\} \text{ such that}\right.$$
$$\left.|f(x)-f(y)|>\sigma(|x-y|)\quad\forall y\in A_k\right\}$$

and

$$E_n^2=\left\{f\in\mathscr{C}\,|\,\exists\delta_n\in\left(0, \frac{1}{n}\right) \text{ so that } \forall x\in\left[\frac{1}{n}, 1\right] \ \exists k\in\{3, 4\} \text{ such that}\right.$$
$$\left.|f(x)-f(y)|>\sigma(|x-y|)\quad\forall y\in A_k\right\}.$$

Then E_n^1 and E_n^2 contain dense open subsets of C.

The proof of both results is the same as that given by Goffman and Pedrick [6].

LEMMA 2.4. *Let σ be a modulus of continuity and $\mathscr{C}(\sigma)$ denote the corresponding equicontinuous family of functions. The set of functions $\mathscr{E}$ in $\mathscr{C}$ such that if $f\in\mathscr{E}$ and $g\in\mathscr{C}(\sigma)$, then $\{x: f(x)=g(x)\}$ is bilaterally strongly porous, is residual in $\mathscr{C}$.*

PROOF. Let E_n^1 and E_n^2 be as in 2.3, $\mathscr{E}^1=\bigcap_{n=2}^{\infty}E_n^1$, and $\mathscr{E}^2=\bigcap_{n=2}^{\infty}E_n^2$. Then $\mathscr{E}^1$ and $\mathscr{E}^2$ are residual subsets of $\mathscr{C}$. We show that for $f\in\mathscr{E}^1$ and $g\in\mathscr{C}(\sigma)$, $H=\{t: f(t)=g(t)\}\cap\cap[0, 1)$ has right porosity one at all of its members.

Let $x\in H$, N a positive integer so that $x\in\left[0, 1-\frac{1}{N}\right]$, and $B=\{y: |f(x)-f(y)|>>\sigma(|x-y|)\}$. Then $B\subseteq H^c$ since $g\in\mathscr{C}(\sigma)$. For $n\geqq N$, $f\in E_n^1$ implies either:

Case 1: $A_1\subseteq B$ infinitely often, (for n), where $A_1=\left[x+\frac{\delta_n}{2n^2}, x+\frac{\delta_n}{2n}\right]$, $\delta_n\in\left(0, \frac{1}{n}\right)$. $\frac{\delta_n}{2n}\downarrow 0$ implies

$$p_+(H, x)=\overline{\lim_{h\downarrow 0}}\frac{\lambda(H, x, h^+)}{h}\geqq\overline{\lim_{n\to\infty}}\frac{\lambda\left(B^c, x, \frac{\delta_n^+}{2n}\right)}{\frac{\delta_n}{2n}}\geqq\lim_{n\to\infty}\frac{\frac{\delta_n}{2n}\left(1-\frac{1}{n}\right)}{\frac{\delta_n}{2n}}=1;$$

or

Case 2: $A_2 \subseteqq B$ infinitely often, (for n).
Then we obtain

$$p_+(H, x) \geqq \lim_{n\to\infty}\left(1-\frac{1}{n}-\frac{1}{2n^2}\right) = 1.$$

This shows that H has right porosity one at all of its members. Similarly, one can show that for $f\in\mathscr{E}^2$ and $g\in\mathscr{C}(\sigma)$, $\{f=g\}\cap(0, 1]$ has left porosity one at all of its members.

Take $\mathscr{E}=\mathscr{E}^1\cap\mathscr{E}^2$. Then $\mathscr{E}$ is a residual subset of $\mathscr{C}$, and for $f\in\mathscr{E}$ and $g\in\mathscr{C}(\sigma)$, $\{f=g\}$ is bsp.

Because of Ascoli's Theorem, the following is an immediate consequence of 2.4.

THEOREM 2.5. *Let $\mathscr{K}$ be a σ-compact subset of $\mathscr{C}$. The set of functions $\mathscr{F}$ in $\mathscr{C}$ such that if $f\in\mathscr{F}$ and $g\in\mathscr{K}$, then $\{x: f(x)=g(x)\}$ is bilaterally strongly porous, is residual in $\mathscr{C}$.*

Obviously, 2.5 is valid if we assume merely that $\mathscr{K}$ is contained in a σ-compact subset of $\mathscr{C}$.

3. Consequences of Theorem 2.5

Theorem 2.5 has a number of consequences; some are entirely new and others provide alternate proofs of known results.

THEOREM* 3.1. *Let $\mathscr{L}_\alpha$ be the class of Lipschitz functions of order α on* [0, 1]. *The set of functions $\mathscr{F}$ in $\mathscr{C}$ such that if $f\in\mathscr{F}$ and $g\in\mathscr{L}_\alpha$ then $\{x: f(x)=g(x)\}$ is bilaterally strongly porous, is residual in $\mathscr{C}$.*

We note that 3.1 extends a recent result of Thomson's [14]. There, the author obtained the corresponding result for the class of constant functions.

Observe that an immediate consequence of Theorem 3.1 is that a typical continuous function cannot agree with a function having a bounded derivative except on a bilaterally strongly porous set. What may be a bit more surprising is that we can replace the term "bounded" by the term "finite" even though the class of functions with finite derivatives is not relatively compact.

THEOREM 3.2. *Let Δ denote the class of differentiable functions on* [0, 1], *(with finite one-sided derivatives at the endpoints). The set of functions $\mathscr{F}$ in $\mathscr{C}$ such that if $f\in\mathscr{F}$ and $g\in\Delta$, then $\{x: f(x)=g(x)\}$ is bilaterally strongly porous, is residual in $\mathscr{C}$.*

PROOF. Let $\mathscr{F}$ be as in 3.1. Take $f\in\mathscr{F}$, $g\in\Delta$, and assume $\{x: f(x)=g(x)\}$ is not bsp. Then there exists x_0, $\{x_n\}_{n=1}^{\infty}$ with $x_n\in\{f=g\}$, $n=0, 1, 2, \dots$ such that $x_n\to x_0$, $\overline{\lim\limits_{n\to\infty}}\dfrac{|x_n-x_{n+1}|}{|x_n-x_0|}<1$, and $\lim\limits_{n\to\infty}\dfrac{f(x_n)-f(x_0)}{x_n-x_0}=\mu<\infty$. We can assume $\mu=0$, $x_0=0$, $f(x_0)=0$, and $x_n\downarrow 0$. There exists a positive integer k, $(k\geqq 2)$, so that $p_+(\{x_n\}, 0)<\dfrac{k-1}{k}$. Since $p_+\left(\left\{\dfrac{1}{k^j}\right\}, 0\right)=\dfrac{k-1}{k}$, there exists J such that $(*)$ for every

* Bernt Oksendal has observed that 3.1 exhibits a contrast between typical behavior in our sense and in the sense of Brownian motion. In the latter sense, the typical function *is* Lipschitz α for every $\alpha<\dfrac{1}{2}$.

$j \geqq J$ there exists x_n so that $x_n \in [1/k^{j+1}, 1/k^j]$. Since $x_n \downarrow$, we can choose a subsequence $\{x_{n_i}\} \subseteq \{x_n\}$ such that

$$x_{n_i} \in \left[\frac{1}{k^{J+2i-1}}, \frac{1}{k^{J+2i-2}}\right], \quad i = 1, 2, 3, \ldots$$

and $\left|\dfrac{f(x_{n_i})}{x_{n_i}}\right| < M$, $i=1, 2, 3, \ldots$ for some $M > 0$.

Consider the continuous function F obtained by joining sucessive points $(x_{n_i}, f(x_{n_i}))$ and $(x_{n_{i+1}}, f(x_{n_{i+1}}))$ by line segments for $i=1, 2, 3, \ldots$. Then F is a Lipschitz function. To see this, observe that the greatest possible slope of any segement of F occurs when $x_{n_i} = \dfrac{1}{k^j}$ and $x_{n_{i+1}} = \dfrac{1}{k^{j+1}}$, (where $(x_{n_i}, f(x_{n_i}))$ and $(x_{n_{i+1}}, f(x_{n_{i+1}}))$ are endpoints of that segment and $j = J+2i-1$). Then,

$$\left|\frac{f(x_{n_i}) - f(x_{n_{i+1}})}{x_{n_i} - x_{n_{i+1}}}\right| \leqq \frac{|f(x_{n_i}) + f(x_{n_{i+1}})|}{\dfrac{k-1}{k^{j+1}}} \leqq$$

$$\leqq M\left(\frac{1}{k^j} + \frac{1}{k^{j+1}}\right)\frac{k^{j+1}}{k-1} = M\frac{k+1}{k-1} \leqq 3M,$$

(since $k \geqq 2$). Thus, $F \in \mathscr{L}$. We also claim $p_+(\{x_{n_i}\}, 0) < 1$.

The largest "gaps" in $\{x_{n_i}\}$ occur when $x_{n_i} = \dfrac{1}{k^j}$ and $x_{n_{i+1}} = \dfrac{1}{k^{j+3}}$, where $j = J+2i-2$. Thus,

$$p_+(\{x_{n_i}\}, 0) = \overline{\lim_{i\to\infty}} \frac{x_{n_i} - x_{n_{i+1}}}{x_{n_i}} \leqq \lim_{j\to\infty} \frac{\dfrac{1}{k^j} - \dfrac{1}{k^{j+3}}}{\dfrac{1}{k^j}} = \frac{k^3-1}{k^3} < 1.$$

This implies $\{t : f(t) = F(t)\}$ is not bsp, giving us a contradiction and completing the proof.

Theorem 3.2 sheds some light on a number of theorems concerning nowhere generalized differentiability of typical continuous functions. To see this, suppose $x_0 \in [0, 1]$, $x_n \to x_0$, and $\lim\limits_{n\to\infty} \dfrac{f(x_n) - f(x_0)}{x_n - x_0}$ exists and is finite. It is easy to construct a differentiable function g such that $f(x_n) = g(x_n)$ for all $n = 0, 1, 2, \ldots$. By Theorem 3.2., for typical f, the sequence $\{x_k\}$ must be bilaterally strongly porous at x_0. In particular, we have the following general result.

THEOREM 3.3. *Let $E = \{E_x : x \in [0, 1]\}$ be a system of paths* [5]. *If for each $x \in [0, 1]$, E_x is not bilaterally strongly porous at x, then the class of functions which are nowhere E differentiable is a residual subset of $\mathscr{C}$. In particular, the typical continuous function is nowhere unilaterally approximately differentiable, nowhere unilaterally preponderantly differentiable, and nowhere qualitatively differentiable.*

Theorems 3.2 and 3.3 also clarify certain positive theorems. The remarkable results of Marcinkiewicz [11], Jarnik [9], and Scholz [13] may seem a bit less remarkable once one sees that the relevant paths are bilaterally strongly porous. Consider, further, the following positive result of Laczkovich [10]: Let f be measurable on the measurable set $H \subseteqq [0, 1]$ with $\lambda(H) > 0$. Then either

(i) there exists a perfect $P \subseteqq H$ such that $f|P$ is infinitely differentiable on P (with finite derivatives) such that $(f|P)^{(n)} = 0$ for n sufficiently large, or

(ii) for every $\varepsilon > 0$ there exists a perfect $P \subseteqq H$ such that $\lambda(P) > \lambda(H) - \varepsilon$ and $f|P$ is infinitely differentiable with finite derivatives (λ denotes Lebesgue measure).

Since (ii) implies approximate differentiability on a large set, option (i) must hold, (for all H), for the typical continuous f and P must be bilaterally strongly porous.

Finally we note that in [3] one finds a study of the manner in which the graph of a typical continuous f intersects straight lines. In particular, "most" such intersections are nowhere dense perfect sets. Theorem 3.2 (or 2.5 or 3.1) shows that all such intersections are bilaterally strongly porous.

4. Convex boundaries

Let f be a bounded function on [0, 1] and let H be the convex hull of the graph of f. The boundary of H can be decomposed into the graphs of two functions h_1 and h_2, of which h_1 is convex and h_2 is concave. These functions are differentiable except, perhaps, on some denumerable sets. Even if f is well-behaved, say a Lipschitz function, the infinite exceptional sets of nondifferentiability may exist. Surprisingly, for typical continuous f, the functions h_1 and h_2 have finite derivatives everywhere on (0, 1) and infinite derivatives at 0 and at 1.

THEOREM 4.1. *The set of functions in $\mathscr{C}$ for which h_1 and h_2 have the properties listed below is residual in $\mathscr{C}$.*

(i) *h_1' and h_2' exist and are continuous on* (0, 1);

(ii) $h_1'(0) = -\infty$, $h_1'(1) = \infty$, $h_2'(0) = \infty$, $h_2'(1) = -\infty$;

(iii) *h_1' and h_2' are unbounded Cantor-like functions;*

(iv) *the Cantor-like sets of support are bilaterally strongly porous in* (0. 1).

PROOF. Lemma 4.5 of [3] states: there exists a residual set of functions f in $\mathscr{C}$ such that for every open rational interval $I \subseteqq [0, 1]$, the slopes of the lines that support the graph of f in I from above at more than one point form a dense set in $\mathbf{R}$, and the same holds for the lines that support the graph of f in I from below at more than one point.

Define $\mathscr{T}$ to be the intersection of this residual set of functions and $\mathscr{F}$ from 3.2. Let $f \in \mathscr{T}$ and h_2 be the concave upper boundary of the convex hull of the graph of f. We show properties (i) through (iv) hold for h_2.

By the referred lemma, the slopes of the lines that support f from above on (0, 1) at more than one point form a dense subset of $\mathbf{R}$. A segment of each such line must be in the upper boundary of the convex hull of the graph of f. No two such segments can abut, otherwise an entire interval of slopes is missed. Thus, h_2 must be differentiable — a concave function is right and left differentiable everywhere; if the unilateral derivates disagree at a point an entire interval of slopes will be missed. Thus, since h_2 is concave and h_2' exists, h_2' is continuous on (0, 1). This proves (i).

Since the upper right derivate of f at zero is infinite and the lower left derivate of f at one is negative-infinite, $h_2'(0) = \infty$ and $h_2'(1) = -\infty$, proving (ii).

It is clear that h_2' is constant on every interval $I \subseteq (0, 1) \setminus \{x: h_2(x) = f(x)\}$. Since $\{x: h_2(x) = f(x)\}$ is nowhere dense, (iii) is shown, and since $\{x: h_2(x) = f(x)\}$ is actually bsp in $(0, 1)$, (iv) is proved.

One can similarly show h_1 satisfies properties (i) through (iv), completing the proof.

5. Concluding remarks

The results in Section 3 indicate that any path relative to which a typical continuous f has a finite derivative must be bilaterally strongly porous. (The term "finite" cannot be dropped from this statement, since the typical f does have infinite unilatral derivatives on uncountable sets.)

We view a set A which is bilaterally strongly porous at a point x_0 as being very thin at x_0 because A has huge gaps near x_0. This view is a bit misleading, however, because it is possible that both A and $\sim A$ are bilaterally strongly porous at x_0. It is therefore somewhat misleading to assert that the typical f is nowhere differentiable except with respect to "very thin" paths. Indeed, the complements of these paths may also be "very thin". For example, Scholz's result [13] shows that the typical continuous f is almost everywhere essentially differentiable to an arbitrary preassigned measurable function! And Zajíček's recent results [15] show that for *every* continuous f the extreme derivates are essential derived numbers except on some first category set. They may be infinite, of course.)

On the other hand, the sets of relative differentiability of a typical continuous f are "genuinely very thin": If $f|A$ is differentiable, then A is bilaterally strongly porous at every point of A — in particular A is a nowhere dense null set.

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REGULARITY PROPERTIES OF POLYNOMIALS ON GROUPS

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1. Introduction

Polynomials play an important role in the theory of linear functional equations. In general cases all solutions of linear functional equations are polynomials ([1], [2], [4], [6], [7], [8], [10], [11]). In this paper we deal with regularity properties of polynomials on groups. Our results can be used to characterize regular solutions of general linear functional equations on commutative topological groups and also extend some classical results concerning the wellknown Cauchy-, Pexider-, Fréchet-, and square-norm-equations ([1], [2], [5], [6], [9]). We note that similar results can be obtained for exponential polynomials but these results will be treated elsewhere.

Let G be an abelian group, X a complex linear space and $f\colon G\to X$ a function. For all $y\in G$ we define the difference operator Δ_y on f by the equation

$$\Delta_y f(x)=f(x+y)-f(x)$$

whenever $x\in G$. Further for all positive integer n and for $y_1, \dots, y_{n+1}\in G$ let

$$\Delta_{y_1,\dots,y_{n+1}}f(x)=\Delta_{y_{n+1}}(\Delta_{y_1,\dots,y_n}f)(x)$$

whenever $x\in G$. Especially, if $y_1=\dots=y_{n+1}=y$ then we use the notation

$$\Delta_y^{n+1}f(x)=\Delta_{y_1,\dots,y_{n+1}}f(x)$$

for all $x\in G$. The function f is called a polynomial of degree at most n, if

$$\Delta_{y_1,\dots,y_{n+1}}f(x)=0$$

whenever $x, y_1, \dots, y_{n+1}\in G$.

Let n be a positive integer and $A\colon G^n\to X$ a function. The function A is called n-additive if it is a homomorphism of G into X in each variable. For the sake of brevity we use the notation $G^0=G$ and we call constant functions from G to X 0-additive.

If $A\colon G^n\to X$ is n-additive, then we use the notation

$$A^{(n)}(y)=A(u_1, \dots, u_n)$$

whenever $u_1=\dots=u_n=y$. It is easy to prove that for any $A\colon G^n\to X$ n-additive, symmetric function the equality

$$\Delta_{y_1,\dots,y_k}A^{(n)}(x)=\begin{cases} n!A(y_1,\dots,y_n) & \text{for} \quad k=n\\ 0 & \text{for} \quad k>n\end{cases}$$

holds, whenever $x, y, y_1, \dots, y_n\in G$.

It is wellknown ([3]), that the function $f\colon G\to X$ is a polynomial of degree at most n if and only if there exist $A_k\colon G^k\to X$ k-additive, symmetric functions

$(k=0, \ldots, n)$ such that

$$f(x) = \sum_{k=0}^{n} A_k^{(k)}(x)$$

holds for all $x \in G$. Further this expression for f is unique in the sense that the A_k's different from zero are uniquely determined.

2. Zeros of polynomials

THEOREM 2.1. *Let G be a topological abelian group which is generated by any neighborhood of the zero and let X be a complex linear space. If a polynomial $p\colon G \to X$ vanishes on a nonvoid open set, then it vanishes everywhere.*

PROOF. Let $p(x) = \sum_{k=0}^{n} A_k^{(k)}(x)$ where $A_k\colon G^k \to X$ is a k-additive, symmetric function $(k=0, \ldots, n)$. Obviously, it is enough to prove that $A_n = 0$. As any translate of a polynomial is a polynomial again, we may suppose that p vanishes on the neighborhood U of the zero. Then we choose a neighborhood V of the zero such that for all $x, y_1, \ldots, y_n \in V$ we have $x + y_1 + \ldots + y_n \in U$. We obtain

$$A_n(y_1, \ldots, y_n) = \frac{1}{n!} \Delta_{y_1, \ldots, y_n} p(x) = \frac{1}{n!} \sum_{0 \leqq i_1 < \ldots < i_n \leqq n} (-1)^{n-k} p(x + y_{i_1} + \ldots + y_{i_k}) = 0$$

whenever $x, y_1, \ldots, y_n \in V$, that is, A_n vanishes on V^n. Using the n-additivity of A_n and the fact that V^n generates G^n, our statement follows.

THEOREM 2.2. *Let G be a locally compact abelian group which is generated by any neighborhood of the zero and let X be a complex linear space. If a polynomial $p\colon G \to X$ vanishes on a measurable set of positive measure, then it vanishes everywhere.*

PROOF. Using the same notations as in the preceding theorem we show that $A_n^{(n)} = 0$. Let $K \subset G$ be a compact set with positive Haar-measure $\lambda K > 0$ such that p vanishes on K. It is wellknown that the function $x \to \lambda(K \cap K - x \cap \ldots \cap K - nx)$ is continuous on G, and as its value is positive at the zero, we have that there is a neighborhood $U \subset G$ of the zero for which $y \in U$ implies $\lambda(K \cap K - y \cap \ldots \cap K - ny) > 0$. That is, for all $y \in U$ there exists an $x \in K$ such that $x + ky \in K$ for $k = 1, 2, \ldots, n$. Then we have

$$A_n^{(n)}(y) = \frac{1}{n!} \Delta_y^n p(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} p(x+ky) = 0,$$

whenever $y \in U$. This means that the polynomial $A_n^{(n)}$ vanishes on U, and hence by the preceding theorem it vanishes everywhere.

3. Boundedness of polynomials

THEOREM 3.1. *Let G be an abelian group and let X be a locally convex topological linear space. If a polynomial $p\colon G\to X$ is bounded on G, then it is constant.*

PROOF. Using the same notations as in Theorem 2.1, we show that $A_n^{(n)}=0$ for $n>0$. From the expression

$$A_n(y_1, \dots, y_n) = \frac{1}{n!}\Delta_{y_1, \dots, y_n} p(x) = \frac{1}{n!}\sum_{0\leqq i_1<\dots<i_k<n} (-1)^{n-k} p(x+y_{i_1}+\dots+y_{i_k})$$

which holds for all $x, y_1, \dots, y_n\in G$ we see that A_n is bounded. On the other hand, for $n>0$ the n-additivity of A_n implies

$$A_n^{(n)}(my) = m^n A_n^{(n)}(y)$$

for all $y\in G$ and for any positive integer m. Suppose, that $A_n^{(n)}(x_0)\neq 0$ for some $x_0\in G$. We choose a balanced and absorbing neighborhood $W\subset X$ of the zero such that $A_n^{(n)}(x_0)\notin W$. As $A_n^{(n)}$ is bounded, there is a real α for which

$$m^n A_n^{(n)}(x_0) = A_n^{(n)}(mx_0)\in \alpha W$$

for all positive integers m. Then $\alpha m^{-n}<1$ for some m, and we have

$$A_n^{(n)}(x_0) = m^{-n} A_n^{(n)}(mx_0)\in \alpha m^{-n} W \subset W,$$

which is a contradiction, hence the theorem is proved.

4. Continuity of polynomials

THEOREM 4.1. *Let G be a topological abelian group and let X be a topological linear space. If a polynomial of degree n $p\colon G\to X$ is continuous, then there are $A_k\colon G^k\to X$ continuous, symmetric, k-additive functions $(k=0, 1, \dots, n)$, such that $p(x)=$ $=\sum_{k=0}^{n} A_k^{(k)}(x)$ holds for $x\in G$.*

PROOF. This is a consequence of the formula

$$A_n(y_1, \dots, y_n) = \frac{1}{n!}\Delta_{y_1, \dots, y_n} p(x)$$

which is valid for all polynomials $p\colon G\to X$ of the form

$$p(x) = \sum_{k=0}^{n} A_k^{(k)}(x)\ (x, y_1, \dots, y_n\in G).$$

THEOREM 4.2. *Let G be a topological abelian group which is generated by any neighborhood of the zero, and let X be a topological linear space. If a polynomial $p\colon G\to X$ is continuous at a point, then it is continuous on G.*

PROOF. Using the notations of Theorem 2.1, it is obvious from the formula

$$A_n(y_1; \dots ; y_n) = \frac{1}{n!}\Delta_{y_1, \dots, y_n} p(x) = \frac{1}{n!} \sum_{0 \leqq i_1 < \dots < i_k \leqq n} (-1)^{n-k} p(x+y_{i_1}+\dots+y_{i_k})$$

that A_n is continuous at the point $(0, \dots, 0)$. It is enough to show, that A_n is continuous on G^n, and we may suppose $n \geqq 1$. First we prove that for all $k=1, 2, \dots, n$ and $x_{k+1}, \dots, x_n \in G$ the function $(g_1, \dots, g_k) \to A_n(g_1, \dots, g_k, x_{k+1}, \dots, x_n)$ is continuous at the zero. Let $W \subset X$ be a neighborhood of the zero and $U \subset G$ be a neighborhood of the zero for which $A_n(U, \dots, U) \subset W$. As G is generated by U, there exists a positive integer N, such that $x_{k+1}, \dots, x_n \in NU$. Further there exists a neighborhood $V \subset G$ of the zero such that $N^{n-k} V \subset U$. Let $g_1, \dots, g_k \in V$, then $N^{n-k} g_1, \dots, g_k \in U$ and

$$A_n(g_1, \dots, g_k, x_{k+1}, \dots, x_n) =$$
$$= \frac{1}{N^{n-k}} A_n(N^{n-k} g_1, g_2, \dots g_k, y_1^{(k+1)}+\dots+y_N^{(k+1)}, \dots, y_1^{(n)}+\dots+y_N^{(n)})$$

where $y_1^{(i)}, \dots, y_N^{(i)} \in U$ $(i=k+1, \dots, n)$. By the n-additivity of A_n this latter element of X can be expressed as a sum of N^{n-k} terms, each of them belonging to $N^{n-k}W$, and hence it belongs to W, and our first statement follows.

Now let $x_1, \dots, x_n \in G$ be arbitrary, then for all $g_1, \dots, g_n \in G$ the difference $A_n(x_1+g_1, \dots, x_n+g_n) - A_n(x_1, \dots, x_n)$ can be expressed as a sum each term of which belongs to an arbitrary given neighborhood of the zero of X whenever $g_1, \dots, g_n$ is chosen from an appropriate neighborhood of the zero of G by the statement proved above. This implies the continuity of A_n at $(x_1, \dots, x_n)$.

THEOREM 4.3. *Let G be a topological abelian group which is generated by any neighborhood of the zero and let X be a locally convex topological linear space. If a polynomial $p\colon G \to X$ is bounded on a nonvoid open set, then it is continuous.*

PROOF. Using the notations of Theorem 2.1, one can see that A_n is bounded on a neighborhood of the zero in G^n. Let $U \subset G$ be a neighborhood of the zero for which $A_n(U, \dots, U)$ is bounded in X. It is enough to prove that $A_n^{(n)}$ is continuous at the zero. Supposing the contrary there exists a balanced and absorbing neighborhood $W \subset X$ of the zero such that every neighborhood of zero in G contains an element z for which $A_n^{(n)}(z) \notin W$. On the other hand, there exists a positive integer N with the property $A_n(U, \dots, U) \subset NW$. Let $m > \sqrt[n]{N}$ be an integer and let $V \subset G$ be a neighborhood of the zero for which $mV \subset U$ and $z \in V$ such that $A_n^{(n)}(z) \notin W$. Then $mz \in U$, and hence $A_n^{(n)}(mz) \in NW$, but $A_n^{(n)}(mz) = m^n A_n^{(n)}(z) \notin m^n W$. On the other hand, $m^n W \supset NW$, which is a contradiction and our theorem is proved.

5. Measurability of polynomials

THEOREM 5.1. *Let G be a locally compact abelian group which is generated by any neighborhood of the zero and let X be a locally convex topological linear space. If a polynomial $p\colon G \to X$ is bounded on a measurable set of positive measure, then it is continuous.*

PROOF. By the conditions p is bounded on a compact set $K\subset G$ with $\lambda K>0$. Let $U\subset G$ be a neighborhood of the zero, for which $x\in U$ implies $K\cap K-x\cap\ldots$ $\ldots\cap K-nx\neq\emptyset$. Then there exists $y\in G$ such that $y, y+x, \ldots, y+nx\in K$. Using the notations of Theorem 2.1 we have

$$A_n^{(n)}(x)=\frac{1}{n!}\Delta_x^n p(y)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}p(y+kx),$$

and this means that $A_n^{(n)}$ is bounded on U. Then by Theorem 4.3, our statement follows.

THEOREM 5.2. *Let G be a locally compact abelian group which is generated by any neighborhood of the zero and let X be a locally convex and locally bounded topological linear space. If a polynomial $p: G\to X$ is measurable on a measurable set of positive measure, then it is continuous.*

PROOF. By the conditions p is measurable on some compact set K with $\lambda K>0$. Let $W\subset X$ be a bounded, balanced and absorbing neighborhood of the zero and let

$$A_n=\{x: p(x)\in nW\}\cap K \quad (n=1, 2, \ldots).$$

As $\bigcup_{n=1}^{\infty} nW=X$, it follows $\bigcup_{n=1}^{\infty} A_n=K$, and $A_n\subset A_{n+1}$ implies $\lim_{n\to\infty}\lambda A_n=\lambda K$. Finally we have that p is bounded on some measurable set of positive measure, and by Theorem 5.1 our statement follows.

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REGULARITY PROPERTIES OF EXPONENTIAL POLYNOMIALS ON GROUPS

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1. Introduction

In a recent paper [13] we have studied regularity properties of polynomials on groups. The results extend some classical theorems and can be applied in the theory of functional equations [1], [8], [9], [10]. Besides polynomials, another important class for the applications is the class of exponential polynomials. The present paper is devoted to the study of regularity properties of exponential polynomials on Abelian groups.

Concerning polynomials we shall use the same notations and terminology as in [13]. Concerning exponential polynomials one can find further investigations and results in [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12].

If G is an Abelian group, then every homomorphism of G into the multiplicative group of nonzero complex numbers is called an exponential. If H is a complex vectorspace and a function $f\colon G\to H$ can be expressed in the form $f=\sum_{i=1}^{n} m_i p_i$, where $p_i\colon G\to H$ is a polynomial and $m_i\colon G\to \mathbf{C}$ is an exponential, then f is called an exponential polynomial. If further $|m_i|=1$ and p_i is constant for all i, then f is called a trigonometric polynomial.

As in the case of polynomials, here we shall also use difference operators on G. Concerning the notations and definitions see [13].

We shall often use the following two lemmas which can be proved easily by induction:

LEMMA 1.1. *Let G be an Abelian group, H a complex vectorspace, $m\colon G\to\mathbf{C}$ an exponential, $p\colon G\to H$ a function and $f=mp$. Then for all $x, y\in G$ we have*

$$\Delta_y^N f(x) = m(x)\sum_{k=0}^{N}\binom{N}{k}\big(m(y)-1\big)^{N-k}\Delta_y^k p\big(x+(N-k)y\big).$$

LEMMA 1.2. *Let G be an Abelian group, H a complex vectorspace, $A\colon G\to H$ an n-additive symmetric function, $q\colon G\to H$ a polynomial of degree at most n—1, and $f=m(A^{(n)}+q)$. Then for all $y\in G$ there exists a polynomial $q_y\colon G\to H$ of degree at most $n-1$ such that for all $x\in G$ we have*

$$\Delta_y^N f(x) = m(x)\big[(m(y)-1)^N A^{(n)}(x)+q_y(x)\big]$$

(Here $A^{(n)}$ denotes the diagonalization of A, that is the homogeneous polynomial of degree n generated by the n-additive symmetric function A.)

2. Zeros of exponential polynomials

THEOREM 2.1. *Let G be an Abelian group, H a complex vectorspace, m_i: $G \to \mathbf{C}$ an exponential and p_i: $G \to H$ a polynomial $(i=1, \ldots, n)$. If $\sum_{i=1}^{n} m_i p_i = 0$, then $p_i = 0$ $(i=1, \ldots, n)$.*

PROOF. We prove the statement by induction on n. It is trivial for $n=1$, hence we suppose that it is proved for $n \leqq k-1$. Assume that $n=k \geqq 2$, and let $p_i = A_i^{(n_i)} + q_i$ where A_i: $G^{n_i} \to H$ is an n-additive symmetric function with diagonalization $A_i^{(n_i)}$, and q_i: $G \to H$ is a polynomial of degree at most $n_i - 1$ $(i=2, \ldots, n)$. Further suppose that p_1 is of degree N. Obviously, it is enough to show that $A_2^{(n_2)} = 0$. Let $y \in G$ such that $m_2(y) \neq m_1(y)$. By Lemma 1.2 we have

$$\Delta_y^{N+1} p_1(x) + \sum_{i=2}^{n} m_i(x) m_1(x)^{-1} [(m_i(y) m_1(y)^{-1} - 1)^{N+1} A_i^{(n_i)}(x) + q_{i,y}(x)] = 0$$

where $q_{i,y}$: $G \to H$ is a polynomial of degree at most $n_i - 1$. Then we obtain the equality

$$\sum_{i=2}^{n} m_i(x) [(m_i(y) m_1(y)^{-1} - 1)^{N+1} A_i^{(n_i)}(x) + q_{i,y}(x)] = 0,$$

and by our assumption it follows for $i=2, \ldots, n$ and $x \in G$

$$(m_i(y) m_1(y)^{-1} - 1)^{N+1} A_i^{(n_i)}(x) + q_{i,y}(x) = 0.$$

Here $q_{i,y}$ is of degree at most $n_i - 1$, and by $m_2(y) m_1(y)^{-1} \neq 1$ the first term for $i=2$ is of degree n_2, hence our statement follows.

We remark that by the above theorem the representation of exponential polynomials with different exponentials is unique. In the topological case the above theorem has two generalizations concerning the zeros of exponential polynomials. Here we cite only the results, each of them is proved in [11].

THEOREM 2.2. *Let G be a topological Abelian group which is generated by any neighborhood of zero and let H be a complex vectorspace. If an exponential polynomial f: $G \to H$ vanishes on some nonvoid open set, then it vanishes everywhere.*

THEOREM 2.3. *Let G be a locally compact Abelian group which is generated by any neighborhood of zero and let H be a complex vectorspace. If an exponential polynomial f: $G \to H$ vanishes on a measurable set of positive measure, then it vanishes everywhere.*

3. Boundedness of exponential polynomials

THEOREM 3.1. *Let G be an Abelian group, H a locally convex topological vectorspace, and f: $G \to H$ an exponential polynomial. If f is bounded, then it is a trigonometric polynomial.*

PROOF. Let $f = \sum_{i=1}^{n} m_i p_i$, where m_i; $G \to \mathbf{C}$ is an exponential, $m_i \neq m_j$ for $i \neq j$, p_i: $G \to H$ is a polynomial and $p_i \neq 0$ $(i, j=1, \ldots, n)$. We prove by induction on n. Let $n=1$ and $f=mp$ be bounded. If p is of degree zero, that is constant, then m is bounded. Indeed, otherwise there would exist a sequence $\{x_n\}\subset G$ for which $|m(x_n)|\geqq n$. On the other hand, if $W\subset H$ is a balanced neighborhood of zero and $p\notin W$, then there is an $\alpha>0$ for which $m(x_n)p\in\alpha W$ $(n=1,2\ldots)$. As $|nm(x_n)^{-1}|\leqq 1$, hence $np=nm(x_n)^{-1}m(x_n)p\in\alpha W$. For $n>\alpha$ we have $\alpha n^{-1}<1$, and hence $p\in\alpha n^{-1}W\subset W$, which is a contradiction. That is, m is bounded. If at a point $x_0\in G$, $|m(x_0)|\neq 1$, then $|m(x_0)|$ or $|m(-x_0)|$ is greater than 1, and hence $m(nx_0)$ or $m(-nx_0)$ has arbitrary great absolute value, contradicting the boundedness of m. Suppose now that we have proved our statement whenever p is of degree at most $N-1$, and let p be of degree $N\geqq 1$. Then there exists $y\in G$ such that $p(x+y)\neq p(x)$, that is $\Delta_y p$ is not identically zero, and it is a polynomial of degree at most $N-1$. Further, for $x\in G$

$$m(x)\Delta_y p(x) = m(y)^{-1}m(x+y)p(x+y)-m(x)p(x)$$

is valid, which implies that $m\Delta_y p$ is bounded. Hence by induction $|m|=1$, and this implies that p is also bounded. Then p is constant by the results of [13].

Now suppose that we have proved the theorem for $n-1$ and let $n\geqq 2$. Let $p_i=A_i^{(n_i)}+q_i$, where $A_i\colon G^{n_i}\to H$ is n_i-additive and symmetric with diagonalization $A_i^{(n_i)}$ and $q_i\colon G\to H$ is a polynomial of degree at most n_i-1 $(i=2,\ldots,n)$. It is enough to prove that $|m_2|=1$ and $A_2^{(n_2)}$ is constant. Let $y\in G$ such that $m_2(y)\neq$ $\neq m_1(y)$. If N denotes the degree of p_1, then by our assumptions and by Lemma 1.1 we have

$$\sum_{i=2}^{n} m_i(x)\big[(m_i(y)m_1(y)^{-1}-1)^{N+1}A_i^{(n_i)}(x)+q_{i,y}(x)\big] =$$

$$= \sum_{k=0}^{N+1}\binom{N+1}{k}(m_1(y)^{-1}-1)^{N+1-k}\Delta_y^k f(x+(N+1-k)y).$$

Here $q_{i,y}\colon G\to H$ is a polynomial of degree at most n_i-1 $(i=2,\ldots,n)$ and $f\colon G\to H$ is a bounded function. By induction it follows that $(m_i(y)m_1(y)^{-1}-1)^{N+1}$ $A_i^{(n_i)}\neq 0$ implies $|m_i|=1$ and $A_i^{(n_i)}$ is bounded. As the condition surely holds for $i=2$, our theorem is proved.

4. Continuity of exponential polynomials

LEMMA 4.1. *Let G be a topological Abelian group, H a locally convex topological vectorspace and $f\colon G\to H$ a continuous exponential polynomial. Then $f=\sum_{i=1}^{n} p_i m_i$ where $p_i\colon G\to H$ is a continuous polynomial and $m_i\colon G\to \mathbf{C}$ is a continuous exponential $(i=1,\ldots,n)$.*

PROOF. Using the notation of Theorem 3.1, we can prove similarly, with the only difference in the case $n=1$. Let $f=mp$ be continuous, where $p\colon G\to H$ is a polynomial, $p\neq 0$, and $m\colon G\to\mathbf{C}$ is an exponential. If p is of degree 0, that is constant, then we show that m is continuous at zero. Obviously $m(0)=1$, and if m were not continuous at zero, then we could find an $\varepsilon>0$ and a sequence $\{x_n\}\subset G$ with $x_n\to 0$ and $|m(x_n)-1|\geqq\varepsilon$. Let $W\subset H$ be a balanced neighborhood of zero with $p\notin W$.

As $x \to m(x)p$ is continuous, hence $m(x_n)p - p \in \varepsilon W$, whenever n is large enough. On the other hand, $\varepsilon |m(x_n)-1|^{-1} \leqq 1$ hence $\varepsilon p \in \varepsilon W$ and $p \in W$, which is impossible. That is, m is continuous at zero, and by the exponential property it follows its continuity everywhere. The remaining parts can be proved in the same way as in Theorem 3.1.

THEOREM 4.2. *Let G be a topological Abelian group, which is generated by any neighborhood of zero, H a locally convex topological vectorspace, $f\colon G \to H$ an exponential polynomial. If f is continuous on some nonvoid open set, then it is continuous on G. If f is bounded on some nonvoid open set, then it can be expressed in the form $f = \sum_{i=1}^{n} m_i \gamma_i p_i$, where $\gamma_i\colon G \to \mathbf{C}$ is an exponential with $|\gamma_i| = 1$, $m_i\colon G \to \mathbf{C}$ is a continuous exponential and $p_i\colon G \to H$ is a continuous polynomial $(i = 1, \ldots, n)$.*

PROOF. We use the notations of Theorem 3.1 supposing that f is continuous on the neighborhood $U \subset G$ of zero. We prove the statement by induction on n. First let $f = mp$, and suppose that p is of degree 0, that is constant. Then m is continuous on u, and its exponential property implies its continuity everywhere. Suppose that we have proved our statement for all p of degree at most $N-1$ and let the degree of p be $N > 1$. Let $V \subset G$ be a neighborhood of zero for which $V + V \subset U$. As G is generated by V, it is easy to see, that there exists a $y \in V$ for which $\Delta_y p$ is not identically zero on V. On the other hand, the degree of $\Delta_y p$ is at most $N-1$ and in the equation

$$m(x)\Delta_y p(x) = m(y)^{-1} f(x+y) - f(x)$$

the right hand side is continuous on V, hence by induction we have the continuity of m. Then, of course, p is continuous on u, and hence on G by [13].

Returning to the original statement of the theorem we suppose that it is valid for $n-1$. Then for all $x \in G$ we have

$$p_1(x) + \sum_{i=2}^{n} m_i(x) m_1(x)^{-1} p_i(x) = m_1(x)^{-1} f(x).$$

We show that for example m_2 and p_2 are continuous. If p_1 is of degree N, then by Lemmas 1.1 and 1.2 we obtain for all $x, y \in G$

$$\sum_{i=2}^{n} m_i(x)\left[(m_i(y) m_1(y)^{-1} - 1)^{N+1} A_i^{(n_i)}(x) + q_{i,y}(x)\right] =$$

$$= \sum_{k=0}^{N+1} \binom{N+1}{k} (m_1(y)^{-1} - 1)^{N+1-k} \Delta_y^k f(x + (N+1-k)y).$$

Here n_i is the degree of p_i, $A_i\colon G^{n_i} \to H$ is n_i-additive and symmetric, and $q_{i,y}\colon G \to H$ is a polynomial of degree at most $n_i - 1$ $(i = 2, \ldots, n)$. Let $V \subset G$ be a neighborhood of zero for which $x, y \in V$ implies $x + (N+1-k)y \in U$ $(k = 0, 1, \ldots, N+1)$. Then the right hand side of the above equation is continuous on V for all fixed $y \in V$. As $m_1 \neq m_2$, there exists $y \in V$ such that $m_2(y) m_1(y)^{-1} - 1 \neq 0$. Indeed, supposing the contrary the exponential property of m_1 and m_2 together with the fact that V generates G would imply $m_1 = m_2$. Returning to the above equation, choosing $y \in V$ with this property, the coefficient of m_2 on the left hand side is different from zero, hence by induction, m_2 and its coefficient are continuous on G. As $q_{2,y}$ is of degree at most n_2-1, hence A_2 is continuous. Now we can apply our argument for the function

$$\sum_{\substack{i=1\\ i\neq 2}}^{n} p_i m_i + (p_2 - A_2^{(n_2)}) m_2$$

and hence the first statement of the theorem is proved.

For the proof of the second part we assume that f is bounded on U, and we prove by induction again. The case $n=1$ means that $f=mp$ is bounded on U. Here we prove by induction on the degree of p. As above, it is enough to show, that if the exponential $m\colon G\to \mathbf{C}$ has the property that the function $x\to m(x)h$ is bounded on U for some nonzero $h\in H$, then $m=\mu\gamma$, where $\gamma\colon G\to\mathbf{C}$ is an exponential with $|\gamma|=1$, and $\mu\colon G\to\mathbf{C}$ is a continuous exponential. Let $W\subset H$ be a balanced and absorbing neighborhood of zero for which $h\notin W$, and let $\alpha>0$ be such that $m(x)h\in\alpha W$ for all $x\in G$. If m is unbounded on U, then there exists $x\in U$ with $|m(x)|>\alpha$ and hence $\alpha m(x)^{-1}W\subset W$, further $h\in\alpha m(x)^{-1}W\subset W$, which is a contradiction. Let $\mu=|m|$ and $\gamma=m|m|^{-1}$. Obviously $|\gamma|=1$ and $\mu>0$, further $\gamma,\mu\colon G\to\mathbf{C}$ are exponentials. If $A=\ln\mu$, then A is additive and bounded on U, hence by [13] it is continuous. Finally $m=\gamma\exp A$ which was to be proved in the case $n=1$. From now on the proof continues just like in the case of the first statement.

5. Measurability of exponential polynomials

THEOREM 5.1. *Let G be a locally compact Abelian group which is generated by any neighborhood of zero, H a locally convex and locally bounded topological vectorspace, $f\colon G\to H$ an exponential polynomial. If f is measurable on some measurable set of positive measure, then it is continuous. If f is bounded on some measurable set of positive measure, then it can be expressed in the form $f=\sum_{i=1}^{n} m_i\gamma_i p_i$, where m_i,γ_i,p_i are the same as in Theorem* 4.2.

PROOF. Using the notations of 4.2, let $K\subset G$ be a compact set with $\lambda K>0$ and f measurable on K (λ denotes a Haar-measure on G). Using the local boundedness of H we may assume that f is bounded on K. First let $n=1$ and $f=mp$, where p is constant. By the same method as above we obtain that m is bounded on K, and hence on $K+K$ too, but this latter set by the Steinhaus theorem has a nonvoid interior. By theorem 4.2. $m=\mu\gamma$, where $\mu,\gamma;\ G\to\mathbf{C}$ are exponentials, μ is continuous and $|\gamma|=1$. Thus γ is measurable on K. Let $\varepsilon>0$ be arbitrary. It is wellknown, that there exists a neighborhood $U\subset G$ of zero such that

$$\int_K |\gamma(x+y)-\gamma(y)|\,d\lambda(y) < \varepsilon\lambda K$$

for $x\in U$. On the other hand,

$$|\gamma(x)-1| = \frac{1}{\lambda K}\int_K |\gamma(x)-1|\,d\lambda(y) = \frac{1}{\lambda K}\int_K |\gamma(x+y)-\gamma(y)|\,d\lambda(y) < \varepsilon$$

whenever $x \in U$, that is, γ is continuous at zero. But this implies the continuity of γ, and of m everywhere.

Now let p be of degree $N>1$ and suppose that the first statement is proved if $n=1$ and p is of degree at most $N-1$. Let $U \subset G$ be a neighborhood of zero for which $y \in U$ implies $\lambda(K \cap (K-y))>0$. Let $y \in U$ be an element, for which $\Delta_y p$ is not constant. These choises are possible in view of Steinhaus theorem and the fact that U generates G. By the identity

$$m(x)\Delta_y p(x) = m(y)^{-1} f(x+y) - f(x)$$

we see that $m\Delta_y p$ is measurable on the set $K \cap (K-y)$ of positive measure, and hence by induction we have our statement for $n=1$. From this step the proof continues just like in Theorem 4.2.

The second statement of our theorem can be proved similarly, and even in the case $n=1$ the above proof works literally except the measurability of γ, but in this case it is not needed.

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ON α-CONTINUITY IN TOPOLOGICAL SPACES

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1. Introduction

Let S be a subset of a topological space $(X, \mathcal{T})$. We denote the closure of S and the interior of S with respect to $\mathcal{T}$ by $\mathcal{T}$ cl S and $\mathcal{T}$ int S respectively, although we may suppress the $\mathcal{T}$ when there is no possibility of confusion.

Definition 1. A subset S of $(X, \mathcal{T})$ is called
(i) an α-set if $S \subset \mathcal{T}$ int $(\mathcal{T}$ cl $(\mathcal{T}$ int $S))$,
(ii) a semi-open set if $S \subset \mathcal{T}$ cl $(\mathcal{T}$ int $S)$,
(iii) a preopen set if $S \subset \mathcal{T}$ int $(\mathcal{T}$ cl $S)$.

These three concepts were introduced by Njåstad [5], Levine [1], and Mashhour et al [3], respectively. Njåstad used the term β-set for a semi-open set. Any open set in $(X, \mathcal{T})$ is an α-set, and each α-set is semi-open and preopen, but the separate converses are false. Theorem 3 below shows that a subset of $(X, \mathcal{T})$ is an α-set if and only if it is semi-open and preopen.

Following Njåstad [5] we denote the family of all α-sets in $(X, \mathcal{T})$ by $\mathcal{T}^\alpha$, rather than by the notation $\alpha(X)$ of [2] and [4]. The families of all semi-open sets and of all preopen sets in $(X, \mathcal{T})$ are denoted by SO (X) and PO (X) respectively. Njåstad [5, Proposition 2] proved that $\mathcal{T}^\alpha$ is a topology on X. It is unusual for either SO (X) or PO (X) to be a topology on X. For example, Njåstad [5, Corollary to Proposition 7] has shown that SO (X) is a topology on X if and only if $(X, \mathcal{T})$ is extremally disconnected. The complement of an α-set in $(X, \mathcal{T})$ is called an α-closed set. Semi-closed and preclosed sets in X are similarly defined.

The concepts of α-continuity and α-open mapping have recently been introduced by Mashhour et al. [4].

Definition 2. A function $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called
(i) α-continuous if the inverse image of each open set in $(Y, \mathcal{U})$ is an α-set in $(X, \mathcal{T})$,
(ii) α-open if the image of each open set in $(X, \mathcal{T})$ is an α-set in $(Y, \mathcal{U})$.

One purpose of this paper is to show that the distinction made in [4] between the concepts of α-continuity and continuity, and between the notions of α-open mapping and open mapping, must be interpreted very strictly. Indeed, we observe (in Theorem 1 below) that if the domain space of an α-continuous mapping f is retopologized in an obvious way, then the mapping f is simply a continuous mapping. Furthermore, as Theorem 2 indicates, if the codomain space of an α-open mapping g is retopologized appropriately, then the mapping g is simply an open mapping. These observations

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put these notions of α-continuity and α-open mappings into a more natural setting, and enable us to provide immediate proofs of many of the results in [4]. In Section 4, we give factorization results for these classes of mappings, and Section 5 relates these classes to the α-irresolute mappings of [2].

2. α-continuity

Here we examine the relationship between the concepts of continuous mappings and α-continuous mappings.

THEOREM 1. *The mapping* $f\colon (X,\mathcal{T})\to(Y,\mathcal{U})$ *is α-continuous if and only if* $f\colon (X,\mathcal{T}^\alpha)\to(Y,\mathcal{U})$ *is continuous.*

PROOF. We have $f\colon (X,\mathcal{T})\to(Y,\mathcal{U})$ is α-continuous if and only if $f^{-1}(B)\in\mathcal{T}^\alpha$ for all $B\in\mathcal{U}$, that is if and only if $f\colon (X,\mathcal{T}^\alpha)\to(Y,\mathcal{U})$ is continuous. $\square$

The observation of Mashhour et al. [4] that continuity implies α-continuity is simply a reflection of the fact that $\mathcal{T}\subset\mathcal{T}^\alpha$ in the lattice of topologies on X. They show by example [4, Example 1.4] that if one takes the topology on X as fixed, then the notions of continuity and α-continuity are distinct. Theorem 1 indicates that these concepts are the same if one is willing to change the topology on X in the appropriate manner. Then [4, Example 1.4] can be regarded as the statement that the set $C\big((X,\mathcal{T}),Y\big)$ of continuous functions from $(X,\mathcal{T})$ to Y is a proper subset of $C\big((X,\mathcal{T}^\alpha),Y\big)$, and this is not surprising.

PROPOSITION 1. *If A is a subset of $(X,\mathcal{T})$ then*

$$\mathcal{T}\ \mathrm{cl}\,A\supset\mathcal{T}^\alpha\,\mathrm{cl}\,A\supset\mathcal{T}\ \mathrm{cl}\big(\mathcal{T}\ \mathrm{int}\,(\mathcal{T}\ \mathrm{cl}\,A)\big).$$

PROOF. (i) $\mathcal{T}$ cl A is $\mathcal{T}$ closed and hence $\mathcal{T}^\alpha$ closed, and $A\subset\mathcal{T}$ cl A so that $\mathcal{T}^\alpha$ cl $A\subset\mathcal{T}$ cl A.

(ii) We use the fact that a set B in $(X,\mathcal{T})$ is $\mathcal{T}^\alpha$ closed if and only if $\mathcal{T}\ \mathrm{cl}\,(\mathcal{T}\ \mathrm{int}\,(\mathcal{T}\ \mathrm{cl}\,B))\subset B$. If we take $B=\mathcal{T}^\alpha$ cl A then $A\subset B$ so that $\mathcal{T}\ \mathrm{cl}\,(\mathcal{T}\ \mathrm{int}\,(\mathcal{T}\ \mathrm{cl}\,A))\subset\mathcal{T}\ \mathrm{cl}\,(\mathcal{T}\ \mathrm{int}\,(\mathcal{T}\ \mathrm{cl}))\subset B$ since B is $\mathcal{T}^\alpha$ closed. $\square$

The following examples show that the inclusions in Proposition 1 are proper. If A is a non-empty $\mathcal{T}$ closed set with empty $\mathcal{T}$ interior, then $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,A))=\emptyset\neq\neq\mathcal{T}^\alpha$ cl A. On the other hand, if $(X,\mathcal{T})$ is the reals with the usual topology and $A=[1,2]\cup\left\{\frac{1}{n}\colon n\in\mathbf{N}\right\}$, then $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,A))=[1,2]\subset A$ so that A is $\mathcal{T}^\alpha$ closed and $A=\mathcal{T}^\alpha$ cl A, while we have $\mathcal{T}$ cl $A=A\cup\{0\}$.

From Proposition 1 we can obtain equivalences for parts (iv) and (v) of Theorem 1.1. of [4].

PROPOSITION 2. *If* $f\colon (X,\mathcal{T})\to(Y,\mathcal{U})$ *is a mapping, then*

(iv) $f(\mathcal{T}\ \mathrm{cl}\,(\mathcal{T}\mathrm{int}\,(\mathcal{T}\ \mathrm{cl}\,A)))\subset\mathcal{U}\ \mathrm{cl}\,f(A)$ *for each* $A\subset X$

if and only if

(iv)′ $f(\mathcal{T}^\alpha\,\mathrm{cl}\,A)\subset\mathcal{U}\ \mathrm{cl}\,f(A)$ *for each* $A\subset X$.

PROOF. (iv)′ implies (iv) since by Proposition 1

$$f\big(\mathcal{T}\ \mathrm{cl}\,(\mathcal{T}\ \mathrm{int}\,(\mathcal{T}\ \mathrm{cl}\,A))\big)\subset f(\mathcal{T}^\alpha\,\mathrm{cl}\,A).$$

Conversely, to show that (iv) implies (iv)′, let $A\subset X$, $C=\mathcal{U}\operatorname{cl} f(A)$ and $B=f^{-1}(C)$. Then by (iv) $f(\mathcal{T}\operatorname{cl}(\mathcal{T}\operatorname{int}(\mathcal{T}\operatorname{cl} B)))\subset\mathcal{U}\operatorname{cl} f(B)\subset\mathcal{U}\operatorname{cl} C=C$. Hence $\mathcal{T}\operatorname{cl}(\mathcal{T}\operatorname{int}(\mathcal{T}\operatorname{cl} B))\subset f^{-1}(C)=B$, so that B is $\mathcal{T}^\alpha$ closed. Now $A\subset f^{-1}(f(A))\subset \subset f^{-1}(C)=B$, so that $\mathcal{T}^\alpha\operatorname{cl} A\subset\mathcal{T}^\alpha\operatorname{cl} B=B$. Thus we have that $f(\mathcal{T}^\alpha\operatorname{cl} A)\subset \subset f(B)\subset C=\mathcal{U}\operatorname{cl} f(A)$ as required. □

The following result can be proved in a similar way.

PROPOSITION 3. *If* $f\colon (X,\mathcal{T})\to(Y,\mathcal{U})$ *is a mapping, then*

(v) $\mathcal{T}\operatorname{cl}(\mathcal{T}\operatorname{int}(\mathcal{T}\operatorname{cl} f^{-1}(M)))\subset f^{-1}(\mathcal{U}\operatorname{cl} M)$ *for each* $M\subset Y$,

if and only if

(v)′ $\mathcal{T}^\alpha\operatorname{cl} f^{-1}(M)\subset f^{-1}(\mathcal{U}\operatorname{cl} M)$ *for each* $M\subset Y$. □

Propositions 2 and 3 reveal that Theorem 1.1 of Mashhour et al. [4] is a restatement of the standard equivalent characterizations of the continuity of the mapping $f\colon (X,\mathcal{T}^\alpha)\to(Y,\mathcal{U})$.

Lemma 1.1 and Theorem 1.3 of [4] raise the question of when $\mathcal{T}^\alpha|A$ equals $(\mathcal{T}|A)^\alpha$ for a subset A of $(X,\mathcal{T})$. We observe that Lemma 1.1 is no longer true for $A\notin \mathrm{PO}(X)$. If $(X,\mathcal{T})$ is the space of Example 1.4 of [4], and $A=\{b,c\}$ then $\mathcal{T}\operatorname{int} A=\emptyset$ so $A\notin\mathrm{PO}(X)$. Let $B=\{a,b\}$. Then $B\in\alpha(X)$ but $B\cap A=\{b\}\notin\alpha(A)$. We have that $\mathcal{T}|A$ is indiscrete so that $(T|A)^\alpha$ is indiscrete, while $\mathcal{T}^\alpha|A$ is the discrete topology on A.

Lemma 1.2 of [4] follows immediately from the observation that $\mathcal{T}^\alpha$ is a topology and the definition of subspace topology. Using our Theorem 1, Theorem 1.4 of [4] can be regarded as part of the standard result of the preservation of continuity under the piecewise definition of maps. There is a similar result if the family $\{U_i\colon i\in I\}$ of [4, Theorem 1.4] is a locally-finite collection of $\mathcal{T}^\alpha$ closed sets.

3. α-open and α-closed mappings

The relationship between the notions of α-open mapping and open mapping is given by the following result whose proof is an immediate consequence of the definitions.

THEOREM 2. *The function* $f\colon (X,\mathcal{T})\to(Y,\mathcal{U})$ *is an* α*-open mapping if and only if* $f\colon (X,\mathcal{T})\to(Y,\mathcal{U}^\alpha)$ *is an open mapping.* □

The fact that $\mathcal{U}\subset\mathcal{U}^\alpha$ implies that each open mapping is an α-open mapping. If we take the topology on Y as fixed, then a distinction can be made as in [4, Example 2.3] between the notions of α-open mapping and open mapping. Theorem 2 above indicates that conceptually there is no distinction if one is willing to retopologize the codomain in a suitable fashion.

Exactly similar comments apply to the class of α-closed mappings introduced and studied in [4], since $f\colon (X,\mathcal{T})\to(Y,\mathcal{U})$ is α-closed if and only if $f\colon (X,\mathcal{T})\to \to(Y,\mathcal{U}^\alpha)$ is closed.

Definition 2.2 of [4] is simply that of the $\mathcal{T}^\alpha$ closure of a subset in $(X,\mathcal{T})$. This observation together with Theorem 2 above indicates that Theorems 2.1, 2.2 and 2.3 of [4] provide restatements of the basic characterizations and properties of open and closed mappings.

4. Comparisons

In this section we provide 'factorizations' of the notions of α-continuity and α-open (α-closed) mappings. The following classes of mappings were introduced in [1] and [3].

DEFINITION 3. A function $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ is called

(i) semi-continuous (abbreviated as s.c.) if the inverse image of each open set in Y is semi-open in X,

(ii) semi-open if the image of each open set in X is semi-open in Y,

(iii) precontinuous (abbreviated as p.c.) if the inverse image of each open set in Y is preopen in X,

(iv) preopen if the image of each open set in X is preopen in Y.

Our 'factorizations' depend on the next result that $\mathcal{T}^\alpha=\mathrm{SO}\,(X)\cap \mathrm{PO}\,(X)$.

THEOREM 3. *A subset S of a topological space $(X, \mathcal{T})$ is an α-set if and only if S is semi-open and preopen.*

PROOF. One implication, namely $\mathcal{T}^\alpha\subset \mathrm{SO}\,(X)\cap \mathrm{PO}\,(X)$, is clear since closure and interior respect inclusion.

Conversely, let S be semi-open and preopen. Then since S is semi-open we have $S\subset \mathrm{cl}\,(\mathrm{int}\, S)$, so that $\mathrm{cl}\, S\subset \mathrm{cl}\,(\mathrm{cl}\,(\mathrm{int}\, S))=\mathrm{cl}\,(\mathrm{int}\, S)$, and hence $\mathrm{int}\,(\mathrm{cl}\, S)\subset \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\, S))$. But since S is preopen, $S\subset \mathrm{int}\,(\mathrm{cl}\, S)$ so that $S\subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\, S))$, that is S is an α-set. □

Mashhour et al. [4, Theorem 3.1] have given an alternative proof of the following characterization of α-continuity.

COROLLARY 1. *The function $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ is α-continuous if and only if it is semi-continuous and precontinuous.*

PROOF. That f is α-continuous implies f is s.c. and f is p.c. follows immediately from the definitions, as observed in [4].

Conversely, let f be s.c. and p.c., and let V be an open set in Y. Then $f^{-1}(V)\in \in \mathrm{SO}\,(X)\cap \mathrm{PO}\,(X)$, so that $f^{-1}(V)\in\mathcal{T}^\alpha$ by Theorem 3, and hence f is α-continuous.

COROLLARY 2. *The function $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ is α-open if and only if it is semi-open and preopen.*

PROOF. One implication is immediate from the definitions, see [4].

Conversely, let f be semi-open and preopen, and let U be an open set in X. Then $f(U)\in \mathrm{SO}\,(Y)\cap \mathrm{PO}\,(Y)$, so by Theorem 3, $f(U)\in\mathcal{U}^\alpha$, and hence f is α-open. □

We have the analogous result for the class of α-closed mappings, $f\colon (X, \mathcal{T})\to \to(Y, \mathcal{U})$ is α-closed if and only if it is semi-closed and preclosed.

5. α-irresolute mappings

The class of α-irresolute mappings was introduced by Maheshwari and Thakur [2].

DEFINITION 4. A function $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ is α-irresolute if the inverse image of every α-set in Y is an α-set in X.

An alternative characterization of α-irresolute mappings is available immediately from the definition. We have that $f\colon (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-irresolute if and only if $f\colon (X, \mathcal{T}^\alpha) \to (Y, \mathcal{U}^\alpha)$ is continuous. It is clear that f is α-irresolute implies that f is α-continuous. Example 1 shows that the converse is false.

EXAMPLE 1. Let $X=\{a, b, c, d\}$ and $Y=\{x, y, z\}$, and define topologies $\mathcal{T}=\{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{U}=\{\emptyset, Y, \{x\}\}$. We define $f\colon X\to Y$ by $f(a)=x$, $f(b)=y$, $f(c)=f(d)=z$. Note that $\mathcal{T}^\alpha=\mathcal{T}$ and $\mathcal{U}^\alpha=\{\emptyset, X, \{x\}, \{x, y\}, \{x, z\}\}$. Then f is α-continuous, but not α-irresolute since $f^{-1}(\{x, y\})=\{a, b\}\notin \mathcal{T}^\alpha$.

We observe that Theorem 3.2 of [4] shows that if f is preopen as well as α-continuous then f is α-irresolute. Our next example shows that this implication is not an equivalence.

EXAMPLE 2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be defined as in Example 1. Then we have that $\mathrm{PO}\,(Y)=\mathcal{U}^\alpha$. We define $h\colon X\to Y$ by $h(a)=h(d)=z$, $h(b)=h(c)=x$. Then h is α-irresolute, but h is not preopen since $h(\{a\})=\{z\}$ is not preopen in Y.

Thus our next result is an improvement of Theorem 3.3 of Mashhour et al. [4].

PROPOSITION 4. *If* $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ *is α-irresolute and* $g\colon (Y, \mathcal{U})\to(Z, \mathcal{V}$, *is α-continuous, then* $g\circ f\colon (X, \mathcal{T})\to(Z, \mathcal{V})$ *is α-continuous.* □

DEFINITION 5. The function $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ is called

(i) irresolute if the inverse image of each semi-open set in Y is semi-open in X,

(ii) pre-irresolute if the inverse image of each preopen set in Y is preopen in X.

We observe immediately that if f is irresolute then it is semi-continuous, and if f is pre-irresolute then it is precontinuous. In Example 1, $\mathrm{SO}\,(X)=\mathcal{T}\cup\{\{a, d\}, \{b, c, d\}\}$ and $\mathrm{SO}\,(Y)=\mathcal{U}^\alpha$ so that the function f is semi-continuous but not irresolute since $f^{-1}(\{x, y\})\notin \mathrm{SO}\,(X)$. In Example 3 below the function j is precontinuous but not pre-irresolute.

In Theorem 3.4 of [4] it is shown that if f is α-open in addition to being precontinuous then f is pre-irresolute. Example 1 shows that the converse is false. There we have that $\mathrm{PO}\,(X)=\mathcal{T}\cup\{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, d\}, ;\{a, c, d\}\}$ and $\mathrm{PO}(Y)=\mathcal{U}^\alpha$. The function f is pre-irresolute but it is not α-open since $f(\{b, c\})=\{y, z\}\notin\mathcal{U}^\alpha$.

Corresponding to Proposition 4 we have the following results, the proofs of which are straightforward.

PROPOSITION 5. *Let* $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ *and* $g\colon (Y, \mathcal{U})\to(Z, \mathcal{V})$.

(i) *If f is irresolute and g is semi-continuous then $g\circ f$ is semi-continuous.*

(ii) *If f is pre-irresolute and g is precontinuous then $g\circ f$ is precontinuous.* □

Our next result relates these classes of 'irresolute' mappings.

PROPOSITION 6. *If* $f\colon (X, \mathcal{T})\to(Y, \mathcal{U})$ *is irresolute and pre-irresolute then f is α-irresolute.*

PROOF. Let V be an α-set in Y. By Theorem 3, $\mathcal{U}^\alpha=\mathrm{SO}\,(Y)\cap\mathrm{PO}\,(Y)$. Since f is irresolute and $V\in\mathrm{SO}\,(Y)$ we have $f^{-1}(V)\in\mathrm{SO}\,(X)$. Similarly, f is pre-irresolute and $V\in\mathrm{PO}\,(Y)$ implies $f^{-1}(V)\in\mathrm{PO}\,(X)$. Hence $f^{-1}(V)\in\mathrm{SO}\,(X)\cap\mathrm{PO}\,(X)=\mathcal{T}^\alpha$, so that f is α-irresolute.

That the converse of Proposition 6 is false is shown by the following example.

EXAMPLE 3. Let $(Y, \mathscr{U})$ and $(X, \mathscr{T})$ be as in Example 1, and define the function $j\colon (Y, \mathscr{U}) \to (X, \mathscr{T})$ by $j(x)=b$, $j(y)=c$, and $j(z)=d$. Then j is α-irresolute. But j is not irresolute since $j^{-1}(\{a, d\})=\{z\} \notin \mathrm{SO}(Y)$, nor is j pre-irresolute since $j^{-1}(\{c\})= =\{y\} \notin \mathrm{PO}(Y)$.

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ON HYPERSURFACES OF FINSLER SPACES CHARACTERIZED BY THE RELATION $M_{\alpha\beta}=\varrho h_{\alpha\beta}$

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The relation between induced and intrinsic connection parameters of a Finsler hypersurface and the heridity properties of special Finsler spaces depends, to a large extent, upon a tensor $M_{\alpha\beta}$ ([1], [14], [15]). Some properties of Finsler hypersurfaces with $M_{\alpha\beta}=0$ have been studied by Brown [1]. In this paper we shall study those Finsler hypersurfaces for which $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ where ϱ is a scalar function and $h_{\alpha\beta}$ are the components of an angular metric tensor. In a hypersurface of a c-reducible Finsler space $M_{\alpha\beta}$ is of this form. It has been shown that the hypersurfaces of some special Finsler spaces are Finsler spaces of the same type if the hypersurfaces are characterized by the relation $M_{\alpha\beta}=\varrho h_{\alpha\beta}$.

1. Fundamental formulae

Let F_n be an n-dimensional Finsler space equipped with metric function $F(x, \dot{x})$. Let F_{n-1} be a hypersurface of F_n given by the equation $x^i=x^i(u^\alpha)$*. Suppose that the functions $x^i(u^\alpha)$ are at least of class C^2 and the matrix whose elements are $B^i_\alpha\left(=\dfrac{\partial x^i}{\partial u^\alpha}\right)$ is of rank $n-1$. The element of support $\dot{x}^i$ of F_n is to be taken tangential to F_{n-1}, i.e.

$$\dot{x}^i = B^i_\alpha \dot{u}^\alpha. \tag{1.1}$$

If $g_{ij}(x, \dot{x})$ denotes the metric tensor of F_n then the induced metric tensor on F_{n-1} is given by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x}) B^i_\alpha B^j_\beta. \tag{1.2}$$

At each point of the hypersurface a unit normal vector $N^i(x, \dot{x})$ is defined by

$$\text{(a)}\quad g_{ij}(x, \dot{x}) N^i(x, \dot{x}) B^j_\alpha = 0, \qquad \text{(b)}\quad g_{ij}(x, \dot{x}) N^i(x, \dot{x}) N^j(x, \dot{x}) = 1. \tag{1.3}$$

If (N_i, B^α_i) are the vectors dual to the vectors (N^i, B^i_α) then we have

$$\begin{cases} \text{(a)}\quad B^\alpha_i(x, \dot{x}) = g^{\alpha\beta}(u, \dot{u}) g_{ij}(x, \dot{x}) B^j_\beta, \\ \text{(b)}\quad N_i(x, \dot{x}) = g_{ij}(x, \dot{x}) N^j(x, \dot{x}), \end{cases} \tag{1.4}$$

$$g^{\alpha\beta} B^i_\alpha B^j_\beta = g^{ij} - N^i N^j. \tag{1.5}$$

* Throughout this paper the Latin indices $h, i, j, k, \dots$ vary from 1 to n while Greek indices $\alpha, \beta, \eta, \delta, \dots$ vary from 1 to $n-1$.

If $\Gamma^{*i}_{jk}(x, \dot{x})$ denotes the Cartan's connection parameter of F_n then the induced connection parameter $\Gamma^{*\alpha}_{\beta\eta}(u, \dot{u})$ of F_{n-1} is given by

$$\Gamma^{*\alpha}_{\beta\eta}(u, \dot{u}) = B^{\alpha}_i(B^i_{\beta\eta} + \Gamma^{*i}_{jk} B^j_\beta B^k_\eta), \tag{1.6}$$

where $B^i_{\beta\eta} = \dfrac{\partial^2 x^i}{\partial u^\beta \partial u^\eta}$. If h_{ij} and C_{ijk} denote the angular metric tensor and $(h)hv$ the torsion tensor of F_n then from (1.1) and (1.2) it follows that the corresponding tensors of F_{n-1} will be given by

$$\text{(a)} \quad h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \qquad \text{(b)} \quad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma. \tag{1.7}$$

If $\hat{\Gamma}^\alpha_{\beta\eta}$ is the intrinsic connection parameter of F_{n-1} then [14]

$$\hat{\Gamma}^\alpha_{\beta\eta} - \Gamma^{*\alpha}_{\beta\eta} = \Lambda^\alpha_{\beta\eta}, \tag{1.8}$$

$$\begin{aligned} \Lambda_{\beta\alpha\eta} = g_{\alpha\delta}\Lambda^\delta_{\beta\eta} &= (M_{\eta\alpha}\Omega_{\beta 0} + M_{\beta\alpha}\Omega_{\eta 0} - M_{\beta\eta}\Omega_{\alpha 0}) - \\ &\quad -(M_{\delta\beta} C^\delta_{\alpha\eta} + M_{\delta\eta} C^\delta_{\alpha\beta} - M_{\delta\alpha} C^\delta_{\beta\eta})\Omega_{00}, \end{aligned} \tag{1.9}$$

where

$$\text{(a)} \quad M_{\alpha\beta} = C_{ijk} N^k B^i_\alpha B^j_\beta, \qquad \text{(b)} \quad \Omega_{\alpha\beta} = N_i(B^i_{\alpha\beta} + \Gamma^{*i}_{jk} B^j_\alpha B^k_\beta)^* \tag{1.10}$$

and 0 denote the contraction with the element of support $\dot{u}^\alpha$.

2. Finsler hypersurfaces with $M_{\alpha\beta} = \varrho h_{\alpha\beta}$

A Finsler space F_n is said to be a c-reducible [4] Finsler space if its $(h)hv$ torsion tensor C_{ijk} is of the form

$$C_{ijk} = \frac{1}{n+1}(C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) \tag{2.1}$$

where $C_i = C_{ijk} g^{jk}$ is the torsion vector. From (1.1) and (1.3) it follows that

$$h_{ij} N^i N^j = 1, \quad h_{ij} B^i_\alpha N^j = 0 \tag{2.2}$$

The relations (1.7)a, (1.10)a, (2.1) and (2.2) lead to $M_{\alpha\beta} = \varrho h_{\alpha\beta}$ where $\varrho = \dfrac{1}{n+1} C_i N^i$. Hence we have the following

PROPOSITION 1. *In a hypersurface of a c-reducible Finsler space,* $M_{\alpha\beta} = \varrho h_{\alpha\beta}$.

A semi c-reducible Finsler space is characterized by the relation [9]

$$C_{ijk} = \frac{p}{n+1}(C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) + \frac{q}{C^2} C_i C_j C_k, \tag{2.3}$$

where $C^2 = g^{ij} C_i C_j$ and $p + q = 1$ $(pq \neq 0)$. From (1.7)a, (1.10)a, (2.2) and (2.3) we get

$$M_{\alpha\beta} = \frac{p\mu}{n+1} h_{\alpha\beta} + \frac{q\mu}{C^2} L_\alpha L_\beta,$$

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* $M_{\alpha\beta}$ differ from those in the paper [1] by a factor F.

where $\mu=C_iN^i$ and $L_\alpha=C_iB^i_\alpha$. Now if $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ then

$$\left(\varrho-\frac{p\mu}{n+1}\right)h_{\alpha\beta}=\frac{q\mu}{C^2}L_\alpha L_\beta. \tag{2.4}$$

Since the rank of the matrix $h_{\alpha\beta}$ is $n-2$, from (2.4) it follows that either $n=3$ or $\varrho=\frac{p\mu}{n+1}$. If $\varrho=\frac{p\mu}{n+1}$ then (2.4) leads to either $\mu=0$ or $L_\alpha=0$. Hence we have the following

PROPOSITION 2. *In a hypersurface of a semi c-reducible Finsler space F_n if $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ then either the dimension of the enveloping space is three or C_i is tangential to the hypersurface or C_i is normal to the hypersurface.*

COROLLARY. *If C_i is tangential to the hypersurface of a semi c-reducible Finsler space then $M_{\alpha\beta}=0$.*

Now we establish the following

THEOREM 1. *If the hypersurface of a Finsler space is characterized by the relation $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ then the induced and intrinsic connections of the hypersurface are identical if and only if either $\varrho=0$ or $\Omega_{\alpha 0}=0$.*

PROOF. From (1.9) it follows that if $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ and $\Lambda^\alpha_{\beta\eta}=0$ then

$$\varrho(h_{\alpha\beta}\Omega_{\eta 0}+h_{\alpha\eta}\Omega_{\beta 0}-h_{\beta\eta}\Omega_{\alpha 0}-C_{\alpha\beta\eta}\Omega_{00})=0,$$

from which we have either $\varrho=0$ or

$$-C_{\alpha\beta\eta}\Omega_{00}+h_{\alpha\beta}\Omega_{\eta 0}+h_{\alpha\eta}\Omega_{\beta 0}-h_{\beta\eta}\Omega_{\alpha 0}=0. \tag{2.5}$$

Contracting (2.5) with $\dot u^\alpha$ we get $\Omega_{00}=0$ which in view of (2.5) leads to $\Omega_{\alpha 0}=0$.

Conversely, if $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ and either $\varrho=0$ or $\Omega_{\alpha 0}=0$ then from (1.9) we get $\Lambda^\alpha_{\beta\eta}=0$. This proves the theorem.

3. Heredity property of some special Finsler spaces

The hypersurface of a c-reducible Finsler space is a c-reducible Finsler space [15]. Similarly the hypersurface of a semi c-reducible Finsler space is a semi c-reducible Finsler space [16]. Thus the c-reducibility and semi c-reducibility properties of a Finsler space are heredity properties. In the following we shall study the heredity properties of Landsberg space. S3-like and S4-like Finsler spaces and the Finsler spaces depending upon T-tensor.

A Finsler space is said to be a Berwald's affinely connected space ([13], p. 81) if its Cartan's and Berwald 's connection parameters are equal and they are functions of coordinate only. A Landsberg space is characterized by the relation $P^i_{jk}=0$ ([11], [12]), where $P^i_{jk}=C^i_{jk|s}\dot x^s$.

Differentiating (1.6) with respect to $\dot u^\delta$ and using the relation [1] $\frac{\partial B^\alpha_i}{\partial \dot u^\delta}=2M^\alpha_\delta N_i$

we get

$$(3.1)\qquad \frac{\partial \Gamma^{*\alpha}_{\beta\eta}}{\partial \dot{u}^\delta} = 2M^\alpha_\delta \Omega_{\beta\eta} + B^\alpha_i \frac{\partial \Gamma^{*i}_{jk}}{\partial \dot{x}^h} B^j_\beta B^k_\eta B^h_\delta .$$

This gives the following

PROPOSITION 3. *In the hypersurface of a Berwald's affinely connected space the induced connection parameter is independent of the directional argument if and only if either* $M_{\alpha\beta}=0$ *or* $\Omega_{\alpha\beta}=0$.

Contracting (3.1) with $\dot{u}^\eta$ and using the relations (1.1) and $P^i_{hj} = \frac{\partial \Gamma^{*i}_{jk}}{\partial \dot{x}^h} \dot{x}^k$ ([13], p. 81) we get

$$\frac{\partial \Gamma^{*\alpha}_{\beta\eta}}{\partial \dot{u}^\delta} \dot{u}^\eta = 2M^\alpha_\delta \Omega_{\beta 0} + B^\alpha_i P^i_{hj} B^h_\delta B^j_\beta .$$

This relation gives the following

THEOREM 2. *If the hypersurface of a Landsberg space is characterized by the relation* $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ *then* $\frac{\partial \Gamma^{*\alpha}_{\beta\eta}}{\partial \dot{u}^\delta} \dot{u}^\eta$ *vanishes if and only if either* $\varrho=0$ *or* $\Omega_{\alpha 0}=0$.

Since in a hypersurface $P^\alpha_{\beta\delta} = \frac{\partial \hat{\Gamma}^\alpha_{\beta\eta}}{\partial \dot{u}^\delta} \dot{u}^\eta$, we have by virtue of Theorems 1 and 2,

THEOREM 3. *If the induced and intrinsic connection parameters of a hypersurface of a Landsberg space are identical and* $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ *then the hypersurface is a Landsberg space.*

A Finsler space $F_n (n \geqq 4)$ is called S3-like [3] if the v-curvature tensor S_{hijk} is of the form

$$(3.2)\qquad F^2 S_{hijk} = \lambda (h_{hj} h_{ik} - h_{kh} h_{ij}),$$

where the scalar λ is called the v-curvature and it is a function of coordinate only. It is well known [6] that S_{hijk} of a 3-dimensional Finsler space is of the form (3.2). Since we have to discuss the S3-likeness of a hypersurface of F_n we consider $n \geqq 5$. Contraction of (3.2) with g^{ik} gives $F^2 S_{hj} = \lambda (n-2) h_{hj}$ where $S_{hj} = S_{hijk} g^{ik}$ is a v-Ricci tensor. Again contracting this with g^{hj} we get $\lambda = F^2 S/(n-1)(n-2)$ where $S = S_{hj} g^{hj}$ is a v-scalar curvature. Thus a S3-like Finsler space is characterized by the relation

$$(3.3)\qquad S_{hijk} = \frac{S}{(n-1)(n-2)} (h_{hj} h_{ik} - h_{hk} h_{ij}).$$

The Gauss equation for the v-curvature tensor is given by [17]

$$(3.4)\qquad S_{\alpha\beta\eta\delta} = S_{hijk} B^h_\alpha B^i_\beta B^j_\eta B^k_\delta + M_{\alpha\eta} M_{\beta\delta} - M_{\alpha\delta} M_{\beta\eta} .$$

If $\bar{S}$ denotes the v-scalar curvature of F_{n-1}, i.e. $\bar{S} = S_{\alpha\beta\eta\delta} g^{\alpha\eta} g^{\beta\delta}$, then from (1.5), (2.2), (3.3) and (3.4) it follows that

$$(3.5)\qquad \bar{S} = \frac{n-3}{n-1} S + M$$

where $M=M_\alpha^\alpha M_\beta^\beta-M_\beta^\alpha M_\alpha^\beta$. The relations (1.7)a, (3.3), (3.4) and (3.5) lead to

$$S_{\alpha\beta\eta\delta}=\frac{\bar{S}}{(n-2)(n-3)}(h_{\alpha\eta}h_{\beta\delta}-h_{\beta\eta}h_{\alpha\delta})+M_{\alpha\eta}M_{\beta\delta}-M_{\alpha\delta}M_{\beta\eta}-\tag{3.6}$$

$$-\frac{M}{(n-2)(n-3)}(h_{\alpha\eta}h_{\beta\delta}-h_{\beta\eta}h_{\alpha\delta}).$$

This relation gives the following

THEOREM 4. *A hypersurface of an* S3*-like Finsler space is an* S3*-like Finsler space if and only if*

$$M_{\alpha\eta}M_{\beta\delta}-M_{\alpha\delta}M_{\beta\eta}=\frac{M}{(n-2)(n-3)}(h_{\alpha\eta}h_{\beta\delta}-h_{\beta\eta}h_{\alpha\delta}).\tag{3.7}$$

Now if $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ then $M=\varrho^2(n-2)(n-3)$ and (3.7) holds identically. Thus

THOREM 5. *A hypersurface of an* S3*-like Finsler space is an* S3*-like Finsler space provided the hypersurface is characterized by the relation* $M_{\alpha\beta}=\varrho h_{\alpha\beta}$.

The v-curvature tensor S_{hijk} of any four dimensional Finsler space is written in the form [8]

$$F^2S_{hijk}=h_{hj}M_{ik}+h_{ik}M_{hj}-h_{hk}M_{ij}-h_{ij}M_{hk}\tag{3.8}$$

where M_{ij} is a symmetric tensor and satisfies $M_{0j}=0$. Matsumoto and Shibata [9] introduced the concept of S4-likeness. A Finsler space $F_n (n\geqq 5)$ is called S4-like if S_{hijk} is of the form (3.8). Thus in order to introduce the S4-like hypersurface of F_n we take $n\geqq 6$. Matsumoto and Shimada [10] have obtained the value of M_{ij} in terms of the v-Ricci tensor and the v-scalar curvature in the form

$$M_{ij}=\frac{F^2}{n-3}\left(S_{ij}-\frac{S}{2(n-2)}h_{ij}\right).$$

Thus an S4-like Finsler space is characterized by the relation

$$S_{hijk}=\frac{S}{(n-2)(n-3)}(h_{ij}h_{hk}-h_{ik}h_{hj})+\frac{1}{n-3}(h_{hj}S_{ik}+h_{ik}S_{hj}-h_{hk}S_{ij}-h_{ij}S_{hk})\tag{3.9}$$

From (1.5), (2.2), (3.4) and (3.9) we get

$$S_{\alpha\eta}=S_{\alpha\beta\eta\delta}g^{\beta\delta}=\frac{n-4}{n-3}S_{hj}B_\alpha^hB_\eta^j+\frac{1}{n-3}\left(\frac{S}{n-2}-S_{ij}N^iN^j\right)h_{\alpha\eta}+Q_{\alpha\eta},\tag{3.10}$$

$$\bar{S}=S_{\alpha\eta}g^{\alpha\eta}=S-2S_{ij}N^iN^j+M\tag{3.11}$$

where $Q_{\alpha\eta}=M_\beta^\beta M_{\alpha\eta}-M_\alpha^\beta M_{\beta\eta}$ and $M=M_\alpha^\alpha M_\beta^\beta-M_\beta^\alpha M_\alpha^\beta$. From (1.7)a, (3.4), (3.9), (3.10) and (3.11) it follows that

$$S_{\alpha\beta\eta\delta}=\frac{\bar{S}}{(n-3)(n-4)}(h_{\beta\eta}h_{\alpha\delta}-h_{\beta\delta}h_{\alpha\eta})+\frac{1}{n-4}(h_{\alpha\eta}S_{\beta\delta}+\tag{3.12}$$

$$+h_{\beta\delta}S_{\alpha\eta}-h_{\alpha\delta}S_{\beta\eta}-h_{\beta\eta}S_{\alpha\delta})+Q_{\alpha\beta\eta\delta},$$

where

$$Q_{\alpha\beta\eta\delta} = M_{\alpha\eta} M_{\beta\delta} - M_{\alpha\delta} M_{\beta\eta} - \frac{M}{(n-3)(n-4)} (h_{\beta\eta} h_{\alpha\delta} - h_{\beta\delta} h_{\alpha\eta}) - $$
$$- \frac{1}{n-4} (h_{\beta\delta} Q_{\alpha\eta} + h_{\alpha\eta} Q_{\beta\delta} - h_{\beta\eta} Q_{\alpha\delta} - h_{\alpha\delta} Q_{\beta\eta}).$$

Hence we have the following

THEOREM 6. *A hypersurface of an* S4*-like Finsler space is an* S4*-like Finsler space if and only if* $Q_{\alpha\beta\eta\delta}=0$.

Now if $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ then $Q_{\alpha\beta}=\varrho^2(n-3)h_{\alpha\beta}$, $M=\varrho^2(n-2)(n-3)$ which give $Q_{\alpha\beta\eta\delta}=0$. Hence we have the following

THEOREM 7. *A hypersurface of an* S4*-like Finsler space is an* S4*-like Finsler space provided the hypersurface is characterized by the relation* $M_{\alpha\beta}=\varrho h_{\alpha\beta}$.

The T-tensor in a Finsler space F_n is defined as [5]

$$T_{hijk} = FC_{hij}|_k + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij},$$

where l_i are the covariant components of the unit vector in the direction of an element of support $\dot{x}^i$ and $|_k$ stands for v-covariant differentation. If T_{hijk} vanishes identically then F_n is said to satisfy the T-condition [7]. Furthermore Ikeda [2] introduced a special form of T-tensor given by

$$T_{hijk} = \lambda (h_{hi} h_{jk} + h_{hj} h_{ik} + h_{hk} h_{ij}), \tag{3.13}$$

where $\lambda(x, \dot{x})$ is a scalar function. If a T-tensor of a Finsler space F_n is of the form (3.13) then we shall say that F_n is a T-reducible Finsler space. It is well known [2] that every c-reducible Finsler space is T-reducible.

If $T_{\alpha\beta\eta\delta}$ denotes the T-tensor of F_{n-1} then we have by (1.5) and (1.10) (a)

$$T_{\alpha\beta\eta\delta} = T_{hijk} B^h_\alpha B^i_\beta B^j_\eta B^k_\delta + F(M_{\alpha\beta} M_{\eta\delta} + M_{\alpha\eta} M_{\beta\delta} + M_{\alpha\delta} M_{\beta\eta}). \tag{3.14}$$

The following theorems are direct consequences of (3.14).

THEOREM 8. *If the hypersurface of a Finsler space satisfying the T-condition is characterized by the relation* $M_{\alpha\beta}=\varrho h_{\alpha\beta}$ *then the hypersurface is T-reducible.*

THEOREM 9. *The hypersurface of a T-reducible Finsler space is T-reducible provided that the hypersurface is characterized by the relation* $M_{\alpha\beta}=\varrho h_{\alpha\beta}$.

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ON ALMOST NILPOTENT RINGS

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Our object here is to answer two questions left open in [1]. We recall from [1] that α_1, α_2, α_3 are defined respectively to be the classes of all rings R such that every non-zero subring, left ideal, ideal of R strictly contains a power of R. It is shown there that

$$\beta \subseteq \mathscr{L}\alpha_1 \subsetneqq \mathscr{L}\alpha_2 \subseteq \mathscr{L}\alpha_3 \subseteq \beta_\varphi,$$

where β denotes the lower Baer radical, $\mathscr{L}\alpha_i$ the lower radical generated by α_i and β_φ the Andrunakievic antisimple radical. We show here that $\beta = \mathscr{L}\alpha_1$ and that $\mathscr{L}\alpha_2 \neq \mathscr{L}\alpha_3$. The example given for this second result contradicts Theorem 3 of [1] and we point out a possible error in the proof there of Lemma 3. Further we prove that $\mathscr{L}\alpha_3$ is an N-radical.

THEOREM 1. *α_1 is the class of all nilpotent rings. $\mathscr{L}\alpha_1$ is equal to the lower Baer radical β.*

PROOF. Since it is clear that every nilpotent ring belongs to α_1 to prove the Theorem it is sufficient to show that each ring R in α_1 is nilpotent.

Suppose first that R contains a non-zero nilpotent element a. Let S be the subring generated by a. Then the elements of S are the polynomials in a with integer coefficients and zero constant term. Thus $a^m = 0$ implies $S^m = 0$. Since $R \in \alpha_1$ there exists n such that $R^n \subset S$. Therefore $R^{nm} = 0$ and so R is nilpotent.

Suppose next that R contains a non-zero element a with non-zero right annihilator. As above it follows that S has non-zero right annihilator in R and so R^n has this property also. From this it follows that R has non-zero right annihilator. Hence R contains a non-zero nilpotent element and so R is nilpotent.

Finally we may suppose that each non-zero element of R has zero right annihilator. Let $a \in R$, $a \neq 0$ and let T be the subring generated by a^2. Then, for some integer n, $R^n \subset T$. Let m be an odd integer, $m \geqq n$. Then $a^m \in R^n$ and so $a^m \in T$. It follows that a^m is equal to a polynomial in a^2 with integer coefficients and zero constant term. Using the fact that a has zero right annihilator we see that there exist polynomials in a of the following type

$$ka + k_2 a^{f_2} + \ldots + k_s a^{f_s} = 0,$$

where each term ka, $k_i a^{f_i}$ is non-zero, $1 < f_2 < \ldots < f_s$ and $s \geqq 2$. Amongst all such expressions assume that we have chosen one with the smallest value of s. Let $b = k_2 a^{f_2 - 1} + \ldots + k_s a^{f_s - 1}$. Then it follows that $b \neq 0$ from the choice above and the fact that a has zero right annihilator. For any $r \in R$, $a(kr + br) = 0$ and so $kr + br = 0$. In particular $b^2 = -kb$. Let U be the subring generated by b. Then $U = Zb$. Let p be a prime not dividing k. Then either $pb=0$ or $R^q \subset pU$ for some integer q. In the second $b^q=ptb$ for some integer t and so $(pt-(-k)^{q-1})b=0$. In each case b has finite order with respect to addition and so U is a finite ring without non-zero nilpotent elements. It follows from the Wedderburn—Artin Theorem that U is a direct sum of fields. This gives $U^2=U$ which is impossible with $U \subseteqq R$ and $R \in \alpha_1$. It follows that R is nilpotent.

We now give the example which shows that $\mathscr{L}\alpha_2 \neq \mathscr{L}\alpha_3$. In [1] a ring W is used which is defined to be the set of all rational numbers m/n where m is an even and n is an odd integer. It is shown that $W \in \alpha_2$ but that $W \notin \mathscr{L}\alpha_1$. Let R denote the 2×2 matrix ring with entries from W. We shall show that $R \in \alpha_3$ but that $R \notin \mathscr{L}\alpha_2$. The statements $W \in \alpha_2$, $R \notin \mathscr{L}\alpha_2$ contradict Theorem 3 of [1]. The present author cannot follow the assertion $M_n \subseteqq P$ in the last line of the proof of Lemma 3. This assertion is true if P is an ideal rather than a left ideal. So the remarks after Theorem 3 in [1] about $\mathscr{L}\alpha_1$ and $\mathscr{L}\alpha_3$ (which is misprinted as $\mathscr{L}\alpha_2$ on p.14, line 20) are true because of Theorem 1 and this fact.

Let A be a non-zero ideal of R and let a occur as an entry in a matrix of A. Let rE_{ij} denote the matrix with r in position (i, j) and all other entries zero. Then it is clear that $4aE_{ij}$ belongs to A for all $i, j \in \{1, 2\}$. Let A_{ij} denote the set of entries occurring in position (i, j) of matrices in A. Then each A_{ij} is an ideal of W and so $A_{ij} \supset W^{n_{ij}}$ for some integer n_{ij}. Let $m > n_{ij}+1$ for all i, j. Then, from the above, it follows that $R^m \subsetneqq A$. It follows that $R \in \alpha_3$. Now suppose that $R \in \mathscr{L}\alpha_2$. In [1] it is shown that $\mathscr{L}\alpha_2$ is supernilpotent; hence R contains a non-zero ideal A in α_2. Let L be the set of matrices in A whose second columns are zero. Then L is a non-zero left ideal of A. No power of A has the property that every matrix in it has second column equal to zero. Hence no power of A is contained in L. This contradicts $A \in \alpha_2$. Therefore $R \notin \mathscr{L}\alpha_2$. Thus $\mathscr{L}\alpha_2 \neq \mathscr{L}\alpha_3$ and so $\alpha_2 \neq \alpha_3$.

An example has been given in [5] to show that $\mathscr{L}\alpha_3$ is strictly contained in β_φ. This question was raised first in [3].

It is shown in [1] that $\mathscr{L}\alpha_2$ is left hereditary. The above example shows that $\mathscr{L}\alpha_2$ is not left strong. We now present further results of this type for these radicals. The notations used in the next result may be found in [2].

THEOREM 2. *$\mathscr{L}\alpha_3$ is an N-radical.*

PROOF. Let (R, V, W, S) be a Morita context such that $VsW=0 \Rightarrow s=0$ and such that R is in the semi-simple class of the radical $\mathscr{L}\alpha_3$. We need to show that S is also semi-simple. If not then S contains a non-zero ideal A in α_3. Since $\bigcap A^m$ can properly contain no power of A we have $\bigcap A^m=0$. Let $B=VAW$, then B is a non-zero ideal of R. Let K be a non-zero ideal of B. $WKV=0$ implies $VWKVW=0$, which implies $K^3=0$. Since $\beta \subset \mathscr{L}\alpha_3$, R and B are semi-prime rings and so $K \neq 0$ implies $K^3 \neq 0$. Hence WKV is a non-zero ideal of A. Since $A \in \alpha_3$ there exists an integer n such that $A^n \subset WKV$. Then

$$B^{n+2} = (VAW)^{n+2} = VAWV(AWV)^{n-1}AWVAW \subseteqq VAWVA^nWVAW \subseteqq$$

$$\subseteqq VAWVWKVWVAW \subseteqq VAWKVAW \subseteqq K.$$

If no power of B is strictly contained in K then we must have $0\neq K=B^{n+2}=B^{n+3}==\ldots=B^{m}=\ldots$. Let $k\in K$. Then $WkV\in WB^{m}V=W(VAW)^{m}V\subseteqq A^{m}$ for all m. Hence $WkV=0$ and so $VWkVW=0$. Since R and VW are semiprime rings it follows that $k=0$. This contradiction implies that K properly contains some power of B. Hence $B\in\alpha_3$. This contradicts the fact that R is semisimple. Therefore S is semisimple. It follows from Theorem 10 of [2] that $\mathcal{L}\alpha_3$ is a normal radical. Since $\beta\subset\mathcal{L}\alpha_3$ it follows that $\mathcal{L}\alpha_3$ is a supernilpotent normal radical and so an N-radical [4].

In [1] it was shown that $\mathcal{L}\alpha_3$ is hereditary. From Theorem 2 it follows that $\mathcal{L}\alpha_3$ is left and right hereditary and left and right strong.

Having seen that $\alpha_1\subsetneqq\alpha_2\subsetneqq\alpha_3$ it is natural to consider the class dual to α_2, i.e. the class α_2' defined using right ideals instead of left ideals. One has dually $\alpha_1\subseteqq\alpha_2'\subseteqq\alpha_3$ and so the question arise as to whether $\alpha_2=\alpha_2'$ or $\mathcal{L}\alpha_2=\mathcal{L}\alpha_2'$. We have not been able to settle these questions. In Lemma 1 of [1] it is shown that α_2 is left hereditary and so, dually, α_2' is right hereditary. We now show that the dual results hold also.

LEMMA. *α_2 is right hereditary and α_2' is left hereditary.*

PROOF. Clearly we need only prove one of these results. Let $R\in\alpha_2$ and let L be a non-zero left ideal of a right ideal A of R. Then $L+RL$ is a non-zero left ideal of R and so there exists n such that $R^n\subset L+RL$. Then $A^{n+1}\subseteqq AR^n\subseteqq A(L+RL)\subseteqq\subseteqq AL\subseteqq L$. Since $\bigcap R^m=0$ and this implies $\bigcap A^m=0$, some power of A is strictly contained in L. Hence $A\in\alpha_2$. Thus α_2 is right hereditary.

It follows that the radicals $\mathcal{L}\alpha_2$ and $\mathcal{L}\alpha_2'$ are each left and right hereditary. We have also seen that $\mathcal{L}\alpha_3$ is left and right hereditary. It should be noted that the class α_3 itself is neither left nor right hereditary. We saw that the 2×2 matrix ring R with entries from W belongs to α_3. However the left ideal L of R consisting of all matrices of R whose second columns are zero belongs to $\mathcal{L}\alpha_3$ but not to α_3. For if A is the subset of L whose entries in position (1,1) are zero then A is a non-zero ideal of L and $L^n\not\subset A$ for any n. Similarly by using rows one can show that α_3 is not right hereditary.

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(Γ, Γ')-FREE BORDISMS, CHARACTERISTIC NUMBERS AND STATIONARY POINT SETS

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Introduction

C. N. Lee and Wasserman [7] developed the notion of characteristic numbers for G-manifolds and proved their G-bordism invariance. In [2] we defined characteristic numbers for an unoriented singular G-bordism and proved their invariance with regard to singular G-bordism. The case of oriented singular G-bordism is considered in [3] and [4]. One of our primary aims in this paper is to develop these notions for (Γ, Γ')-free singular bordisms, $\Gamma' \subset \Gamma$ being families of subgroups in a finite group G. (For definition see [6]). In [2] we tackled this problem for some special pairs of families (for so called "almost adjacent" pairs). In an effort to consider more general pairs of families, we get an analogue of Stong's result [6, Proposition 2] for finite abelian groups in §3. In this section we prove that if (M^n, θ) is a G-manifold with stationary point free induced action of the subgroup G_2, then (M^n, θ) is a G-boundary, G a finite abelian group. Lastly in §4 this analogue has been used to show that if the fixed point set F of G_2 in M^n is nonempty and if F has an equivariant trivial normal bundle in M^n, then (M^n, θ) is a G-boundary.

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Characteristic numbers for an almost adjacent pair (Γ, Γ')

Let G be a finite group and X be a G-space. Let h^* be an equivariant cohomology theory and h_* be the associated equivariant homology theory [1] given by $h^* = H^*A$ and $h_* = H_*A$, where A is a functor from the category of G-spaces and G-maps to the category of topological spaces and continuous maps, H^* is the singular cohomology theory and H_* is the associated singular homology theory. Let

$$\langle , \rangle \colon h^*(X; G) \underset{H^*(pt.)}{\otimes} h_*(X; G) \to H_*(pt.)$$

be the Kronecker pairing.

Suppose for each compact G-manifold W there exists a class $[W, \partial W] \in h_*(W, \partial W\colon G)$ such that

a) $[W_1 \cup W_2, \partial W_1 \cup \partial W_2] = [W_1, \partial W_1] + [W_2, \partial W_2]$

and

b) $\partial_*[W, \partial W] = [\partial W]$.

Such an element $[W, \partial W] \in h_*(W, \partial W; G]$ is called a *topological class* of W.

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Following Stong [5], a family Γ in G, is a collection Γ of subgroups of G such that (i) $H\in\Gamma$ implies that all the subgroups of H also belong to Γ (ii) $H\in\Gamma$ implies that $gHg^{-1}\in\Gamma$, $\forall g\in G$. Let $\Gamma'\subset\Gamma$ be families in G such that each member of $\Gamma-\Gamma'$ is maximal in G. Such a pair (Γ, Γ') is called a pair of almost adjacent families. For any subgroup H of G, let $K=\frac{N}{H}$, N being the normalizer of H in G. Let $F_H(X)$ be the set of $x\in X$ such that $hx=x$, $\forall h\in H$. Consider the action of K on $F_H(X)$ by $(gH)x=gx$. Let EK be the total space of the universal K-bundle. For a pair (Γ, Γ') of almost adjacent families, consider the equivariant cohomology and equivariant homology

$$h^*(X;\,G) = \bigoplus_H H^*((EK\times F_H(X)/K;\,\mathbf{Z}_2)$$

$$h_*(X;\,G) = \bigoplus_H H_*((EK\times F_H(X)/K;\,\mathbf{Z}_2),$$

the summation is over the set of all representatives of the conjugacy classes of subgroups H in $\Gamma-\Gamma'$. Let X be a G-space and $[M^n, \partial M^n, \varphi, \theta, f]$ be an element of (Γ, Γ')-free bordism group $\varkappa_n(G;\,\Gamma, \Gamma')\,(X)$ [5]. Then

$$h_*(M^n;\,G) \approx \bigoplus_H \bigoplus_{k=0}^{n} H_*(F_H^k(M^n)/K;\,\mathbf{Z}_2),$$

where $F_H^k(M^n)$ is the union of k-dimensional submanifolds in $F_H(M^n)$. We define a topological class $[M, \partial M]$ of M^n in $h_*(M^n, \partial M^n; G)$ to be $\bigoplus_H \bigoplus_{k=0}^{n} \sigma_k^H$ where $\sigma_k^H\in H_k(F_H^k(M^n)/K;\,\mathbf{Z}_2)$ is the fundamental class of $F_H^k(M^n)/K$. Let $u\in h^*(X\times B(0,\,G)_n;\,G)$. Let $\tau_M n\colon\, M^n\to B(0,\,G)_n$ be the tangent map.

DEFINITION 2.1. We define the *u-characteristic number* of an (Γ, Γ')-free element $(M^n, \partial M^n, \varphi, \theta, f)$ by $\langle (f\times\tau_M n)^*(u),\ [M, \partial M]\rangle\in\mathbf{Z}_2$.

Regarding the bordism invariance, we establish

THEOREM 2.2. $[M^n, \partial M^n, \varphi, \theta, f]\in\varkappa_n(G;\,\Gamma, \Gamma')\ (X)$ *is zero if and only if all the u-characteristic numbers (corresponding to the theory h^*) of the (Γ, Γ')-free element $(M^n, \partial M^n, \varphi, \theta, f)$ are zero.*

PROOF. The G-equivariant map $f\colon\, M^n\to X$ gives K-equivariant map $f\colon\, F_H^k(M^n)\to\to F_H(X)$. Let

$$\nu_k^H\colon\, F_H^k(M^n)\to F_H\big(B(0,\,N)_{n-k}\big)$$

be the normal map. In fact the image of ν_k^H will be contained in $F_H'\big(B(0,\,N)_{n-k}\big)$ the union of path components of $p\in F_H(B(0,\,N)_{n-k})$ for which the fibre $(\gamma^{n-k})_p$ at p contains no trivial H-representation, γ^{n-k} being the universal real N-vector bundle. Let $\alpha_k^H\colon\, F_H^k(M^n)\to EK$ be the cover of the classifying map for K-bundle $F_H^k(M^n)\to\to\big(F_H^k(M^n)\big)/K$. Let f_k^H be the map obtained from $\alpha_k^H\times(f\times\nu_k^H)$ on passing to quotients. This gives the map

$$\eta\colon\,\varkappa_n(G;\,\Gamma, \Gamma')(X)\to\bigoplus_H\bigoplus_{k=0}^{n}\varkappa_k\big((EK\times\{F_H(X)\times F_H'(B(0,\,N)_{n-k})\})/K\big)$$

defined by

$$\eta([M^n, \partial M^n, \varphi, \theta, f]) = \bigoplus_H \bigoplus_{k=0}^{n} [F_H^k(M^n)/K, f_k^H].$$

We know that η is an isomorphism [5] and thus $[M^n, \partial M^n, \varphi, \theta, f]$ is zero if and only if $[F_H^k(M^n)/K, f_k^H]$ is zero, $\forall k$ and H. Next the group $h^*(X\times B(0, G)_n; G)$ is isomorphic to

$$\bigoplus_H \bigoplus_{k=0}^{n} [H^*((EK\times\{F_H(X)\times F'_H(B(0, N)_{n-k})\})/K; \mathbf{Z}_2)\otimes H^*(B0_k; \mathbf{Z}_2)].$$

Also $(f\times\tau_{M^n})^* = \bigoplus_H \bigoplus_{k=0}^{n} (f_k^H\times\tau_k^H)^*$ where $\tau_k^H\colon F_H^k(M^n)/K\to B0_k$ is the tangent map. Thus the u-characteristic number

$$\langle (f\times\tau_{M^n})*(u), [M, \partial M]\rangle = \langle \bigoplus_H \bigoplus_{k=0}^{n} (f_k^H\times\tau_k^H)*(u_k^H), \bigoplus_H \bigoplus_{k=0}^{n} \sigma_k^H\rangle$$

where u_k^H is given by $u = \bigoplus_H \bigoplus_{k=0}^{n} u_k^H$. This together with the fact that $[M^n, \partial M^n, \varphi, \theta, f]$ is zero if and only if $[F_H^k(M^n)/K, f_k^H]$ is zero gives the theorem.

An analogue of Stong's result and characteristic numbers for a more general pair of families

So far we confined ourselves to a pair of almost adjacent families. In an effort to get rid of almost adjacent families as much as possible, we come across an analogue of Stong's result [6, Proposition 2] for general groups. For this let G be a finite abelian group and Π be the family of all subgroups of G. Let $\Gamma'\subset\Gamma$ be families in G such that there exists an element a in G of order 2 such that

(1) $H\in\Gamma'\Rightarrow[H\cup\{a\}]\in\Gamma'$,

(2) $a\notin H$, $\forall H\in\Gamma-\Gamma'$,

(3) the intersection S of all the members of $\Gamma-\Gamma'$ is in $\Gamma-\Gamma'$. We call such a pair of families an *admissible pair* with respect to $a\in G$.

EXAMPLE 3.1. Let G be a finite abelian group of even order given by $G^2\times H$, where H is a finite group of odd order and $G^2 = \bigtimes_{i=1}^{r} (\mathbf{Z}_2 i)^{n_i}$. Let $\Gamma_k = \{U\times V\colon V$ is a subgroup of H and U is a subgroup of G^2 not containing $\mathbf{Z}_2^k = [t_1, \dots, t_k]$, $1\le k\le\gamma_r$ where $t_1, \dots, t_{\gamma_r}$ generate $(\mathbf{Z}_2)^{\gamma_r}$, $\gamma_r = \sum_{i=1}^{r} n_i\}$. It is simple to see that (Γ_{k+1}, Γ_k) is an admissible pair with respect to t_{k+1}. Here by Γ_{γ_r+1} we mean the family of all subgroups of G.

THEOREM 3.2. *If (Γ, Γ') is an admissible pair of families in G with respect to a, then a (Γ, Γ')-free element in $\varkappa_*(G; \Gamma, \Gamma')$ is zero as an element of $\varkappa_*(G; \Pi, \Gamma')$.*

PROOF. It is enough to show that the homomorphism $i_*\colon \varkappa_*(G; \Gamma, \Gamma')\to\varkappa_*(G; \Pi, \Gamma')$ induced from the inclusion $i\colon (\Gamma, \Gamma')\to(\Pi, \Gamma')$ is the constant homomorphism. Let $[M, \theta]$ be an element of $\varkappa_*(G; \Gamma, \Gamma')$. Let F be the closed submanifold of M consisting of all points of M fixed by S, S being the intersection of all the members of $\Gamma-\Gamma'$. Let ν be the normal bundle of the imbedding of F in the interior of M and let $D(\nu)$ be its disc bundle with the action θ^* of G on $D(\nu)$ induced by the real vector bundle maps covering the action θ on F. Since F is the fixed point set of S, $a\notin H$, $\forall H\in\Gamma-\Gamma'$ and no point of F is fixed by the subgroup $[S\cup\{a\}]$ generated by $S\cup\{a\}$, a will act freely on F and hence on $D(\nu)$. Let $F'=F/[a]$ and $D'(\nu)==D(\nu)/[a]$. Since G is abelian, the actions θ and θ^* on F and $D(\nu)$ induce actions θ' and $\theta^{*\prime}$ on F' and $D'(\nu)$, respectively. Consider the quotient maps $q_1\colon D(\nu)\to D'(\nu)$ and $q_2\colon F\to F'$ which are equivariant and double covers over $D'(\nu)$ and F', respectively. Let C_1 and C_2 be the mapping cylinders of q_1 and q_2 and ψ_1^* and ψ_2^* be the induced actions on C_1 and C_2, respectively. We have the following commutative diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\gamma} & D'(\nu) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\nu'} \\ C_2 & \xrightarrow{\beta} & F' \end{array}$$

where $\alpha\colon C_1\to C_2$ is the map induced from $\nu'\colon D'(\nu)\to F'$ by going to mapping cylinder. It is simple to see that $\partial C_2\approx F$ and the action ψ_1^* on $\alpha^{-1}(\partial C_2)$ is isomorphic to the action θ^* on $D(\nu)$. Consider $W=M\times[0,1]\cup C_1/\sim$ where $\sim$ is the equivalence relation obtained by identifying $D(\nu)\times\{1\}$ with $\alpha^{-1}(\partial C_2)$. Let the action Θ of G on W be given by $\Theta|M^n\times I=\theta\times 1$ and $\Theta|C_1=\psi_1^*$. Take V to be

$$(\partial M\times I)\cup(M\times\{1\}-(D(\nu)\times\{1\})^0)\cup(\partial C_1-(\alpha^{-1}(\partial C_2))^0)$$

where ^{0} is the interior operator. Clearly V is (Γ',Γ')-free and ∂W is isomorphic to $M\cup V$ by identifying ∂V with ∂M. This shows that $[M,\theta]$ is zero in $\varkappa_*(G;\Pi,\Gamma')$.

THEOREM 3.3. *Let Γ be a family in a finite abelian group G such that there exists an element a in G of order 2 with $[a]\notin\Gamma$, $[a]$ being the subgroup of G generated by a. Then the homomorphism $i_*\colon \varkappa_*(G;\Gamma)\to\varkappa_*(G;\Pi)$ induced grom the inclusion $i\colon k\to\Pi$ is the zero homomorphism.*

PROOF. Let $[M,\theta]\in\varkappa_*(G;\Gamma)$. Since $[a]\notin\Gamma$, a will act freely on M and therefore the quotient map $q\colon M\to M/[a]$ will be a double cover over $M/[a]$. Let C be the mapping cylinder of the double cover with the induced action ψ of G on C. Clearly the boundary ∂C is isomorphic to M with $\psi|\partial C=\theta$. Consider $W=M\times[0,1]\cup C/\sim$ where $\sim$ is the equivalence relation obtained by identifying $M\times\{1\}$ with ∂C. Let the action Θ of G on W be given by $\Theta|M\times 1=\theta\times 1$ and $\Theta|C=\psi$. Clearly $\partial(W,\Theta)==(M,\theta)$. This shows that $[M,\theta]$ is zero in $\varkappa_*(G;\Pi)$.

Let G be a finite abelian group and $\mathbf{P}$ be the family of all proper subgroups of G. Suppose Γ' is another family in G such that there exists a chain of families $\Gamma'=\Gamma_1\subset\ldots\subset\Gamma_{r+1}=\mathbf{P}$ with (Γ_{i+1},Γ_i) being an admissible pair of families with respect to an element a_i, $i=1,\ldots,r$. By repeated application of Theorem 3.2 one concludes that the homomorphism $j_*\colon\varkappa_*(G;\Pi,\Gamma')\to\varkappa_*(G;\Pi,\mathbf{P})$ induced by the inclusion $j\colon(\Pi,\Gamma')\to(\Pi,\mathbf{P})$ is a monomorphism. Therefore we can give characteristic numbers for an (Π,Γ')-free element, since $(\Pi,\mathbf{P})$ is an almost adjacent pair. In this case we define equivariant homology and cohomology as $h_*(X;G)==H_*(F_G(X);\mathbf{Z}_2)$ and $h^*(X,G)=H^*(F_G(X);\mathbf{Z}_2)$, for a G-space X, $F_G(X)$ being the fixed points set of X under G. Thus corresponding to the equivariant homology and cohomology defined as above, using Theorems 2.2 and 3.2, one gets the following

THEOREM 3.4. *An (Π, Γ')-free element in $\varkappa_*(G; \Pi, \Gamma')$ is zero if and only if all the characteristic numbers are zero.*

Special cases. We will consider two cases $G=\mathbf{Z}_2^k$ and $G=\bigtimes_{i=1}^{r} (\mathbf{Z}_2 i)^{n_i}\times H$, H being any finite abelian group of odd order and each element of H commuting with each element of $\bigtimes_{i=1}^{r} (\mathbf{Z}_2 i)^{n_i}$.

Case I: $G=\mathbf{Z}_2^k=[t_1, \ldots, t_k]$. Let $\Gamma_i=\{U: U$ is a subgroup of $\mathbf{Z}_2^k$ not containing $\mathbf{Z}_2^i=[t_1, \ldots, t_i]$, $1\leqq i\leqq k\}$. It is easy to see that (Γ_{i+1}, Γ_i) is an admissible pair with respect to t_{i+1} and $\Gamma\subset\Gamma_2\subset\ldots\subset\Gamma_k=\mathbf{P}$. Therefore by repeated applications of Theorem 3.2 one infers that the homomorphism

$$j_{1*}\colon \varkappa_*(\mathbf{Z}_2^k; \Pi, \Gamma_1) \to \varkappa_*(\mathbf{Z}_2^k; \Pi, \mathbf{P})$$

induced from the inclusion $j_1\colon (\Pi, \Gamma_1)\to(\Pi, \mathbf{P})$ is an injection. Also $[t_1]\notin\Gamma_1$, therefore by Theorem 3.3 the homomorphism

$$j_{2*}\colon \varkappa_*(\mathbf{Z}_2^k; \Pi) \to \varkappa_*(\mathbf{Z}_2^k; \Pi, \Gamma_1)$$

induced from the inclusion $j_2\colon (\Pi, \Phi)\to(\Pi, \Gamma_1)$ is a monomorphism. Thus $j_*\colon \varkappa_*(\mathbf{Z}_2^k; \Pi)\to\varkappa_*(\mathbf{Z}_2^k; \Pi, \mathbf{P})$ is a monomorphism, $j\colon (\Pi, \varphi)\to(\Pi, \mathbf{P})$. Let us define the equivariant homology and cohomology as follows: $h_*(X, \mathbf{Z}_2^k)=H_*(F_{\mathbf{Z}_2^k}(X); \mathbf{Z}_2)$ and $h^*(X; \mathbf{Z}_2^k) = H^*(F_{\mathbf{Z}_2^k}(X); \mathbf{Z}_2)$, where H_* and H^* are singular homology and cohomology, respectively. Using the monomorhism of

$$j_*\colon \varkappa_*(\mathbf{Z}_2^k; \Pi) \to \varkappa_*(\mathbf{Z}_2^k; \Pi, \mathbf{P})$$

and Theorem 2.2 for the almost adjacent pair $(\Pi, \mathbf{P})$, one immediately gets

THEOREM 3.5. *A Π-free element in $\varkappa_*(G; \Pi)$ is zero if and only if all its characteristic numbers (corresponding to the theories h^* and h_* defined as above) are zero.*

Case II: G is a finite abelian group of even order given by $G^2\times H$ where H is a finite group of odd order and G^2 is the 2-group $\bigtimes_{i=1}^{r} (\mathbf{Z}_2 i)^{n_i}$. Let $\mathbf{P}$ be the family of all subgroups of G of the type $U\times V$ where V is a subgroup of H and U is a subgroup of G^2 not containing $G_2=(\mathbf{Z}_2)^{\gamma_r}$, $\gamma_r=\sum_{i=1}^{r} n_i$. Let $(\mathbf{Z}_2)^{\gamma_r}$ be generated by $\{t_k\}$, $1\leqq k\leqq\gamma_r$.

THEOREM 3.6. *The following sequence is exact:*

$$0 \to \varkappa_*(G; \Pi) \xrightarrow{j_*} \varkappa_*(G; \Pi, \mathbf{P}) \xrightarrow{\partial_*} \varkappa_*(G; \mathbf{P}) \to 0.$$

PROOF. By Example 3.1, (Γ_{k+1}, Γ_k) is an admissible pair with respect to t_{k+1}, $1\leqq k\leqq\gamma_r$. Therefore by repeated application of Theorem 3.2, one infers that the homomorphism

$$(j_1)_*\colon \varkappa_*(G;\Pi,\Gamma_1)\to\varkappa_*(G;\Pi,\mathbf{P})$$

is a monomorphism, $j_1\colon(\Pi,\Gamma_1)\to(\Pi,\mathbf{P})$. Since $[t_1]\notin\Gamma_1$, using Theorem 3.3 we get $(j_2)_*\colon\varkappa_*(G;\Pi)\to\varkappa_*(G;\Pi,\Gamma_1)$ to be monomorphism, $j_2\colon(\Pi,\varphi)\to(\Pi,\Gamma_1)$. Thus the inclusion $j\colon(\Pi,\varphi)\to(\Pi,\mathbf{P})$ induces monomorphism j_*. This completes the poof of the Theorem.

COROLLARY 3.7. *Let (M^n,θ) be a G-manifold and the induced action of the subgroup G_2 be stationary point free. Then (M^n,θ) bounds as a G-manifold.*

The stationary point set $F_{G_2}(M^n)$

As in Case II of §3, let $G=G^2\times H$ be a finite abelian group of even order where H is an odd order group and $G^2=\mathop{\times}\limits_{i=1}^{r}(\mathbf{Z}_{2^i})^{n_i}$. Let us denote the subgroup $(\mathbf{Z}_2)^{\gamma_r}$ by by G_2, $\gamma_r=\sum\limits_{i=1}^{r}n_i$. Let $\mathbf{R}$ be the field of real numbers and $GL(\mathbf{R},j)$ be the set of all isomorphisms of the vector space $\mathbf{R}^j$ onto itself, $j=1,2$. Any irreducible real representation of G will be either one dimensional or two dimensional. Let $\varrho^j\colon G\to GL(\mathbf{R},j)$ be any nontrivial irreducible real j-dimensional representation of G, $j=1,2$.

THEOREM 4.1. Ker ϱ^j *contains a subgroup of G isomorphic to* $(\mathbf{Z}_2)^{\gamma_r-1}$.

PROOF. It is simple to see that the image (ϱ^j/G_2) is either the trivial subgroup or is the subgroup consisting of just two elements, namely the identity element and the isomorphism $\theta\colon\mathbf{R}^j\to\mathbf{R}^j$ given by $\theta(x)=(-1)x$, for every $x\in\mathbf{R}^j$. Therefore Ker (ϱ^j/G_2) is either G_2 itself or is isomorphic to $(\mathbf{Z}_2)^{\gamma_r-1}$.

Let (M^n,θ) be a closed G-manifold and $F=F_{G_2}(M^n)$ be the stationary point set of M^n under the subgroup G_2. Consider the decomposition $F=\bigcup\limits_{l=0}^{n}F^l$ where F^l is the l-dimensional component. Let $D(\nu_l)$ be the normal disc bundle of F^l in M^n with the induced action θ_l. Let Π be the family of all subgroups of G and $\mathbf{P}$ be the family of subgroups of G of the type $U\times V$ where V is a subgroup of H and U is a subgroup of G^2 not containing G_2.

DEFINITION 4.2. F is said to have an equivariant *trivial normal bundle* in M^n if G/G_2 acts trivially on F and there exists some positive dimensional real G-representations (W_l,φ_l) such that in $\varkappa_*(G,\Pi,\mathbf{P})$ $[D(\nu_l),\theta_l]=[F^l][D(W_l),\varphi_l]$ where $D(W_l)$ is the unit disc of W_l.

Given a real representation $\varrho\colon G\to GL(\mathbf{R},j)$ of G one gets a j-dimensional vector space $\mathbf{R}^j$ over $\mathbf{R}$ with the action $\psi\colon G\times\mathbf{R}^j\to\mathbf{R}^j$ given by $\psi(g,x)=\varrho(g)(x)$. We say $(\mathbf{R}^j,\psi)$ a representation space of G or by abuse of language, a representation of G. Let $\{(V_k,\psi_k)\}_{1\le k\le m}$ be the finite set of all nonisomorphic nontrivial irreducible real representations of G. Let $\mathbf{Z}^+$ be the set of nonnegative integers. Given any map $f\colon\{1,\dots,m\}\to\mathbf{Z}^+$ one has a real representation $(V(f),\psi(f))$ of G given by $V(f)=\bigoplus\limits_{k=1}^{m}(V_k,\psi_k)^{f(k)}$ where $(V_k,\psi_k)^{f(k)}$ is the direct sum of $f(k)$ copies of (V_k,ψ_k). Let us denote the unit disc and unit sphere of $V(f)$ by $D(f)$ and $S(f)$, respectively.

THEOREM 4.3. *If F has an equivariant trivial normal bundle in (M^n, θ) then it is the boundary of some manifold and (M^n, θ) itself is the boundary of some G-manifold (N, Θ).*

PROOF. Since F has an equivariant trivial normal bundle in M^n, we have $[D(v_l), \theta_l]=[F^l][D(W_l), \varphi_l]$ for some positive dimensional real representations (W_l, φ_l) of G. Also $(W_l, \varphi_l)=(V(f_l), \psi(f_l))$ for some map $f_l\colon \{1, \dots, m\}\to\mathbf{Z}^+$. Therefore $[D(v_l), \theta_l] = [F^l][D(f_l), \psi(f_l)]$. Let $j_*\colon \varkappa_*(G; \Pi)\to\varkappa_*(G; \Pi, \mathbf{P})$ be the homomorphism induced by the inclusion $j\colon (\Pi, \Phi)\to(\Pi, \mathbf{P})$. We have

$$j_*[M^n, \theta] = \sum_{l=0}^{n} [D(v_l), \theta_l] = \sum_{l=0}^{n} [F^l][D(f_l), \psi(f_l)].$$

Therefore from Theorem 3.6, one gets

$$(\partial_* j_*)[M^n, \theta] = \partial_* \sum_{l=0}^{n} [F^l][D(f_l), \psi(f_l)] = \sum_{l=0}^{n} [F^l][S(f_l), \psi(f_l)] = 0$$

in $\varkappa_*(G; \mathbf{P})$. Therefore there exists a $\mathbf{P}$-free G-manifold (D, η) such that

$$(1) \qquad (\partial D, \eta) = \bigcup_{l=0}^{n} \left(F^l\times(S(f_l), \psi(f_l))\right)$$

Since each (W_l, φ_l) is a positive dimensional real representation of G, there exists a member $k(l)$ in the set $\{1, \dots, m\}$ such that $f_l(k(l))\neq 0$. By Theorem 4.1 there exists a subgroup $H_{k(l)}$ of G isomorphic to $(\mathbf{Z}_2)^{\gamma_r-1}$ fixing $V_{k(l)}$. Let us fix some β, $0\leqq\beta\leqq n$. Let $A_{k(\beta)}$ be the largest subset of $\{1, \dots, m\}$ such that $H_{k(\beta)}$ fixes V_j, $j\in A_{k(\beta)}$. Let $A_{k(\beta)}$ be the disjoint union of $B_{k(\beta)}$ and $C_{k(\beta)}$ where $B_{k(\beta)}$ consists of all $j\in A_{k(\beta)}$ such that V_j is one dimensional and $C_{k(\beta)}$ consists of all $j\in A_{k(\beta)}$ such that V_j is two dimensional. Let

$$\sum_{j\in B_{k(\beta)}} f_l(j)+\sum_{j\in C_{k(\beta)}} 2f_l(j) = \Delta(l, \beta)\in\mathbf{Z}^+.$$

Since $f_\beta(k(\beta))\neq 0$ and $A_{k(\beta)}$ contains $k(\beta)$, $\Delta(\beta, \beta)$ cannot be zero. From (1) we get

$$(2) \qquad F_{H_{k(\beta)}}(\partial D, \eta) = F_{H_{k(\beta)}}\Big(\bigcup_{l=0}^{n} (F^l\times(S(f_l), \psi(f_l)))\Big).$$

Suppose $F_{H_{k(\beta)}}(D)=F^*$ and $\mathbf{Z}_{2,\beta}\approx\mathbf{Z}$ be the complement of $H_{k(\beta)}$ in $(\mathbf{Z}_2)^{\gamma_r}$. Then from (2) we have

$$(\partial F^*, \eta|\mathbf{Z}_{2,\beta}) = \bigcup_{l=0}^{n} \left(F^l\times(S^{\Delta(l,\beta)-1}, a)\right)$$

where a is the antipodal involution. Since D is $\mathbf{P}$-free, F^* will have stationary point free action of $\mathbf{Z}_{2,\beta}$. Therefore $[\partial F^*, \eta|\mathbf{Z}_{2,\beta}]$ is zero in $\varkappa_*(\mathbf{Z}_{2,\beta}, \Gamma_1)$ so that

$$(3) \qquad \sum_{l=0}^{n} [F^l][S^{\Delta(l,\beta)-1}, a] = 0$$

in $\varkappa_*(\mathbf{Z}_{2,\beta}, \Gamma_1)$ where Γ_1 is the family in $\mathbf{Z}_{2,\beta}$ consisting of only trivial subgroup of $\mathbf{Z}_{2,\beta}$. We know that $\varkappa_*(\mathbf{Z}_2, \Gamma_1)$ is the free $\varkappa_*$-module with generators $\{[S^n, a], n\in\mathbf{Z}^+\}$.

Therefore (3) gives $[F^\beta]=0$, since $\Delta(\beta,\beta)\neq 0$. By varying β, one concludes that $[F^\beta]=0$, $\beta=0,\dots,n$. Hence $[F]=0$ in $\varkappa_*$. Also

$$j_*[M^n,\theta]=\sum_{l=0}^{n}[F^l][D(f_l),\psi(f_l)]=0.$$

By Theorem 3.6, j_* is a monomorphism and therefore one infers that $[M^n,\theta]$ is zero in $\varkappa_*(G;\Pi)$. This completes the proof of the Theorem.

Combining Corollary 3.7 and Theorem 4.3, one infers that the fixed point data of the subgroup G_2 determines the equivariant bordism class of a G-manifold (M^n,θ).

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ON SUNS AND COSUNS IN FINITE DIMENSIONAL NORMED REAL VECTOR SPACES

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0. Introduction

We are concerned with some "dual" aspects between best approximation and best coapproximation in finite dimensional normed real vector spaces which are especially apparent in the plane.

In the first section we study the asymptotic behavior of Leibniz planes. Section 2 deals with so-called cosuns of best coapproximation, the counterparts of suns in the theory of best approximation. We show that in strictly convex spaces every existence set of best coapproximation is already a cosun. This is done by using a structural property shared by the metric projection and the metric coprojection as well. The asymptotic behavior of Leibniz planes plays an essential part in the description of cochebychev sets. We interpret the respective results of Westphal [22] on cochebychev sets in smooth l_p-spaces from a geometrical point of view and give a complete description of all cochebychev sets in a strictly convex plane.

In Section 3 we sharpen a known result of Busemann [3] on the "symmetry" of Birkhoff-orthogonality in the plane and show that the sun and cosun are just dual notions in the plane. The duality permits to give an intrinsic characterization of the important variants of suns in the plane. We show that a closed subset of the plane is a sun iff for each element of the plane its elements of best approximation are a nonempty contractible subset. More precisely, this subset is the compact part of an angle. We further study selection properties of the metric projection for a sun in the plane. In particular we show that there are always ray selections of the metric projection and any such ray selection is strongly contractive with respect to the dual *norm.

As to the notation we denote by X a finite dimensional normed real vector space (of dimension n) with norm $\|\,.\,\|$, its dual space is X'. For real numbers we use small Greek letters, for the set of strictly positive (negative) real numbers $\mathbf{R}_+$ $(\mathbf{R}_-)$. The closed ball centered at x with radius ϱ is denoted by $B(x, \varrho)$; $\dot{B}(x, \varrho)$ denotes its interior. For $K \subseteq X$ the distance function is given by $d(., K)$ and the metric projection by P_K. For the boundary of K we write ∂K, for its convex hull co K. We call K norm-convex if for any two points in K there exists a metrical midpoint in K [18]. The closed segment between $x, y \in X$ is given by $[x, y]$. We repeatedly use the semi-inner product (s.i.p.) defined by the Gateaux differential of $\|\,.\,\|^2/2$ on $X \times X$, i.e.

$$\langle y, x\rangle_s = \lim_{\lambda \downarrow 0} \frac{\|x+\lambda y\|^2 - \|x\|^2}{2\lambda}, \quad x, y \in X.$$

* The paper contains parts of the authors doctoral dissertation.

For its basic properties see [10]. We always use orthogonality in the sense of Birkhoff, e.g.

$$x \perp y \Leftrightarrow \|x+\lambda y\| \geqq \|x\|, \quad \forall \lambda \in \mathbf{R}.$$

It is easy to see that

$$x \perp y \Leftrightarrow \langle y, x\rangle_s \geqq 0 \quad \text{and} \quad \langle -y, x\rangle_s \geqq 0.$$

1. The asymptotic behavior of Leibniz planes

Following Busemann [8] and Gruber [12] we call $E(x, y)=\{z\in X\colon \|x-z\|=\|y-z\|\}$ the *Leibniz plane* determined by $x, y\in X$, $x\neq y$. The plane is symmetric about $(x+y)/2$. We call $E_x(x, y)=\{z\in X\colon \|x-z\|<\|y-z\|\}$ the open *Leibniz halfspace* containing x.

One should be cautious not to overvalue the term "plane". For $n\geqq 3$ each Leibniz plane is truly "flat" if and only if the space is Euclidean, Mann [17]. Furthermore, to each segment parallel to $x-y$ in $\partial B(0, 1)$ there corresponds a 2-dimensional solid angle in $E(x, y)$, thus $E(x, y)$ might have interior points.

Kalisch and Straus [14] study Leibniz planes in connection with their investigations of *determining sets* (a set K is determining if each $x\in X$ is uniquely determined by its set of distances $\{\|x-k\|: k\in K\}$). We shall be concerned with the asymptotic behavior of Leibniz planes.

Blaschke ([5], p. 157) calls

$$SG(x) = \partial B(0, 1) \cap (B(0, 1)+\mathbf{R}x), \quad x\in X\setminus\{0\},$$

the *shadow-boundary* for x. If it does not contain a segment parallel to x, we call $SG(x)$ *strict*. The *upper rim*, resp. the *lower rim* of $SG(x)$ is given by

$$SG^0(x) = \{z\in SG(x)\colon z+\mathbf{R}_+x \notin SG(x)\},$$

resp.

$$SG^u(x) = \{z\in SG(x)\colon z-\mathbf{R}_+x \notin SG(x)\}.$$

With $SG(x)$ we associate the *shadow-cone*

$$SK(x) = \bigcup_{\lambda\geqq 0} \lambda SG(x)$$

with its *upper*, resp. *lower rim*

$$SK^0(x) = \bigcup_{\lambda\geqq 0} \lambda SG^0(x), \quad \text{resp.} \quad SK^u(x) = \bigcup_{\lambda\geqq 0} \lambda SG^u(x).$$

See Figure 1 for a typical Leibniz plane and its shadow-cone in a smooth, 3-dimensional l_p-space. $SK(x)$ separates X into two parts, the *upper*, resp. *lower halsfpace* of the shadow-cone

$$SK^+(x) = SK^0(x)+\mathbf{R}_+x, \quad \text{resp.} \quad SK^-(x) = SK^u(x)-\mathbf{R}_+x.$$

Using s.i.p., we have the following description of these sets.

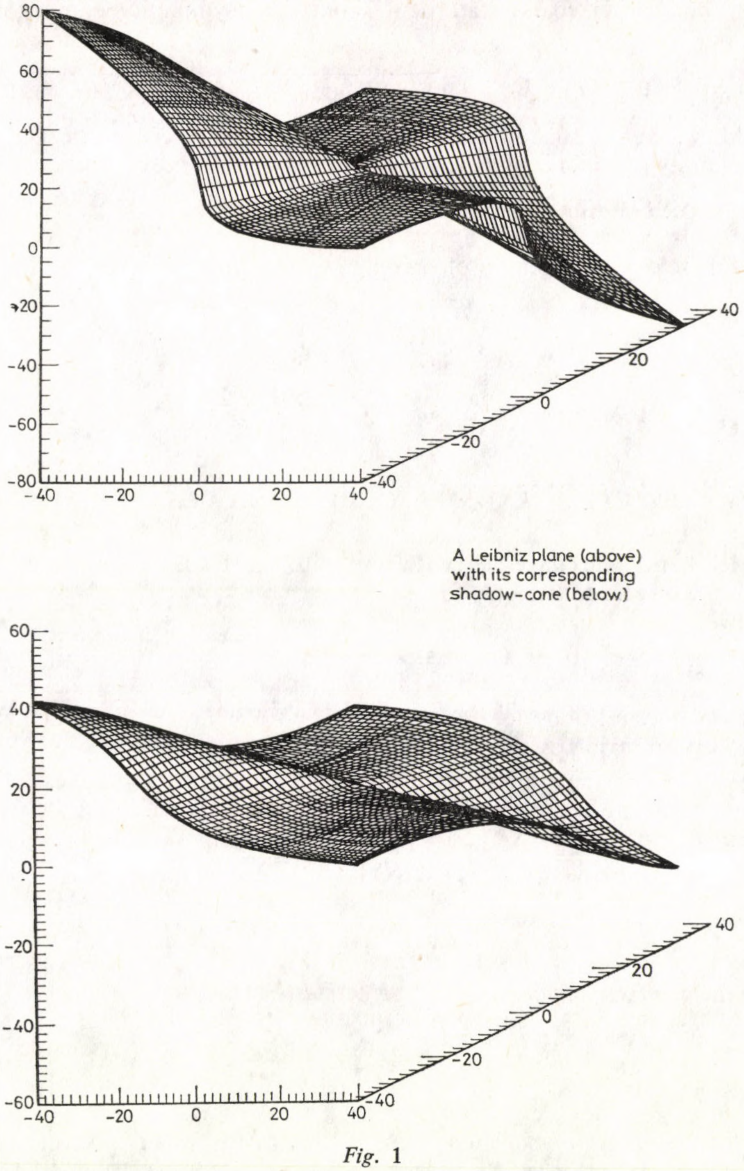

Fig. 1

LEMMA 1.1.

$$SK^+(x) = \{z \in X\colon \langle -x, z\rangle_s < 0\},$$

$$SK(x) = \{z \in X\colon \langle x, z\rangle_s \geqq 0 \quad \text{and} \quad \langle -x, z\rangle_s \geqq 0\},$$

$$SK^-(x) = \{z \in X\colon \langle x, z\rangle_s < 0\}.$$

A first, but rather coarse statement about the asymptotic behavior of Leibniz plane is

PROPOSITION 1.2. $E(x, y) \subseteq \mathbf{C}\overline{(x+SK^+(x-y))} \cap \mathbf{C}\overline{(y+SK^-(x-y))}$.

It follows that the shadow-cone is the limit (with respect to the Hausdorff distance) of a family of Leibniz planes.

PROPOSITION 1.3. $\lim_{\lambda \downarrow 0} E(\lambda x, 0) = SK(x)$.

PROOF. Let $\|x\|=1$. By a simple geometric reasoning we have

$$\max_{z \in \partial E_{\lambda x}(\lambda x, 0)} d(z, SK^0(x)+\lambda x) \leqq \lambda$$

and

$$\max_{z \in \partial E_0(\lambda x, 0)} d(z, SK^u(x)) \leqq \lambda.$$

Clearly,

$$\mathbf{C}\,\overline{SK^-(x)} \cap \mathbf{C}\overline{(\lambda x+SK^+(\lambda x))} = \mathbf{C}\overline{SK^-(x)} \cap (\lambda x+\mathbf{C}\overline{SK^+(x)}) \supseteqq SK(x),$$

where the left hand side converges (with respect to the Hausdorff distance) to $SK(x)$. Thus $E(\lambda x, 0)$ converges to $SK(x)$ for $\lambda \downarrow 0$. □

We always have

$$\partial E_x(x, y) \subseteqq C\overline{(y+SK^+(x-y))}.$$

But under which conditions do $\partial E_x(x, y)$ and $y+SK^0(x-y)$ approach each other? This is just a geometric way to look at property (S) given by Bruck and Reich in [7].

Let $y=0$. If $SG(x)$ is not strict, the upper rim of $E(x, 0)$ touches $SK^0(x)$. Thus let us suppose for the rest of this section that $SG(x)$ is strict. Then $E(x, 0)$ contains no solid angle and $SK(x)=SK^0(x)=SK^u(x)$. Furthermore $E(x, 0)$ lies strictly between $SK(x)$ and $x+SK(x)$. For $z \in SK(x) \cap \mathring{B}(0, 1)$ let

$$s_0 = \partial B(0, 1) \cap (\mathbf{R}_+ x+z), \quad \text{resp.} \quad s_u = \partial B(0, 1) \cap (\mathbf{R}_- x+z).$$

We define the positive function C_x on $SK(x) \cap \mathring{B}(0, 1)$ via

$$C_x(z) = \frac{\|s_0 - z\|}{\|s_u - z\|},$$

the so-called *chord ratio* associated with x. This ratio gives a faithful image of the behavior of $E(x, 0)$ between $SK(x)$ and $x+SK(x)$. Thus $E(x, 0)$ is strictly bounded away from 0 and $+\infty$ on $SK(x) \cap \mathring{B}(0, 1)$. Indeed, it is the behavior of $C(z)$ on $SK(x) \cap \mathring{B}(0, 1)$ for $\|z\| \to 1$, which determines the asymptotic behavior of $E(x, 0)$.

PROPOSITION 1.4. *Let the norm be twice continuously differentiable on $SG(x)$ and let*

$$x^T H(u) x > 0, \quad \forall\, u \in SG(x),$$

where $H(u)$ is the Hessian of the norm at u. Then

$$\lim_{\substack{\|z\|\to 1 \\ z\in SK(x)\cap \mathring{B}(0,1)}} C_x(z) = 1,$$

i.e., the chord ratio asymptotically equals 1.

PROOF. Let $u_n\in SK(x)\cap \mathring{B}(0,1)$, $\|u_n\|\to 1$ and $\lambda_n^+, \lambda_n^-\in \mathbf{R}$ such that $u_n+\lambda_n^+ x$ and $u_n+\lambda_n^- x$ belong to $\partial B(0,1)$. We have

$$\left\|\frac{u_n}{\|u_n\|}+\frac{\lambda_n^+}{\|u_n\|}x\right\| = \left\|\frac{u_n}{\|u_n\|}+\frac{\lambda_n^-}{\|u_n\|}x\right\|.$$

For sufficiently large n we can write the Taylor series expansion of this equation up to the second order term as

$$\left(\frac{\lambda_n^+}{\|u_n\|}\right)^2 x^T H\left(\frac{u_n}{\|u_n\|}\right)x+O\left(\frac{1}{\|u_n\|^2}\right) = \left(\frac{\lambda_n^-}{\|u_n\|}\right)^2 x^T H\left(\frac{u_n}{\|u_n\|}\right)x+O\left(\frac{1}{\|u_n\|^2}\right),$$

where the first order term vanishes since $u_n\in SK(x)$. Multiplying with $\|u_n\|^2$ and taking the limit, we have that the chord ratio asymptotically equals one. □

Thus a "reasonable" curvature behavior of the unit sphere at $SG(x)$ gives a Euclidean asymptotic behavior of $E(x,0)$, i.e. $E(x,0)$ converges "uniformly" to $(x/2)+SK(x)$. This is quite plausible because from a great distance a point with positive curvature looks like "being Euclidean".

In general it is rather difficult to relate the strict boundedness of C_x on $SK(x)\cap \cap\mathring{B}(0,1)$ to curvature relations on $SG(x)$. In the plane however we can compute C_x in this manner.

Let $SK(x)=\mathbf{R}u$ with $\|u\|=1$ and let us orient the unit circle so that it has the same orientation at u as the ray $u-\mathbf{R}_+x$. Take e as the Euclidean normal at u to $SK(x)$ which points in the direction given by the orientation of $\partial B(0,1)$ at u. Let $m_-(m_+)$ be the slope of the left (right) halftangent line $T_-(T_+)$ to $\partial B(0,1)$ at u and m the slope of $SK(x)$ in the Cartesian system whose origin is u with the "x-axis" given by e and the "y-axis" given by $-u$. For $0<\lambda<1$ let $s_-(\lambda)$ $(s_+(\lambda))$ be the cutpoints of $\lambda u+\mathbf{R}_+x$ $(\lambda u-\mathbf{R}_+x)$ with $\partial B(0,1)$. Furthermore, let $r_-(\lambda)$ $(r_+(\lambda))$ be the radius of

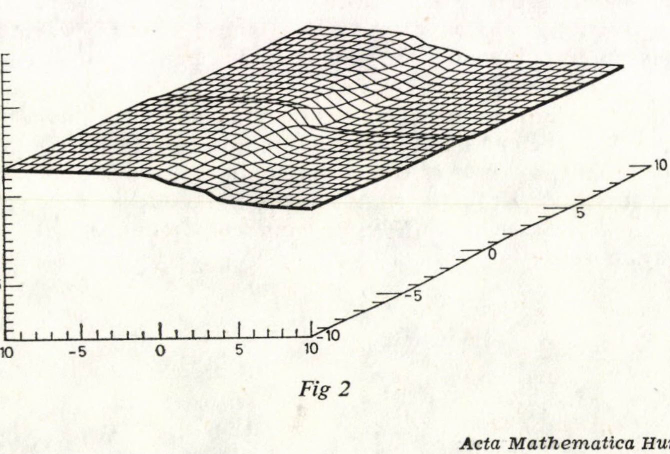

Fig 2

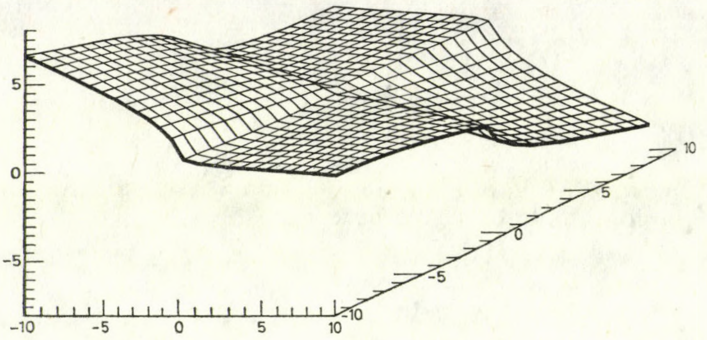

Fig 3

the Euclidean circle which passes through $s_-(\lambda)$ $(r_+(\lambda))$ and u and which is tangent to $T_-(T_+)$. Then the chord ratio is given asymptotically as follows by using a suitable Taylor series expansion ([13]).

PROPOSITION 1.5. (i) *If* $m_- \neq 0$ *or* $m_+ \neq 0$, *then*

$$\lim_{\lambda \uparrow 1} C_x(\lambda u) = \frac{m + |m_-|}{m - |m_+|} \cdot \frac{|m_+|}{|m_-|}$$

with a natural interpretation given for the singular cases.

(ii) *If* $m_- = m_+ = 0$, *then*

$$\varliminf_{\lambda \uparrow 1} C_x(\lambda u) = \varliminf_{\lambda \uparrow 1} \sqrt{\frac{r_-(\lambda)}{r_+(\lambda)}}, \quad \varlimsup_{\lambda \uparrow 1} C_x(\lambda u) = \varlimsup_{\lambda \uparrow 1} \sqrt{\frac{r_-(\lambda)}{r_+(\lambda)}}$$

From here we can obtain a complete description of the asymptotic behavior of a Leibniz line in the plane, given $E(x, y)$ for some pair (x, y). If a supporting line parallel to $x-y$ at the unit circle is not a halftangent line, $E(x, y)$ converges asymptotically to $z + SK(x-y)$ strictly between $x + SK(x-y)$ and $y + SK(x-y)$. If this supporting line is a halftangent line, but not a tangent line, the respective asymptotes of $E(x, y)$ are given by $x + SK(x-y)$ and $y + SK(x-y)$. If the supporting line is a tangent line, the asymptotic behavior of $E(s, y)$ is determined by the curvature relations in the support point. If the curvature exists and is not zero, $(x+y)/2 + SK(x-y)$ gives the asymptote of $E(x, y)$. Even if the curvature does not exist, but instead the lower and the upper curvatures from the right and from the left are different from 0 and $+\infty$, the minimal distance of $E(x, y)$ from $x + SK(x-y)$ and $y + SK(x-y)$ remains always positive.

2. Existence sets of best coapproximation

For $K \subseteq X$, $x \in X$ we define

$$B_K(x) = \{z \in X: \|z-k\| \leqq \|x-k\|, \quad \forall k \in K\}.$$

$B_K(x)$ is compact and convex. A geometric interpretation of this definition is given by

$$B_K(x) = \{z \in X \setminus \{x\}: K \subseteq \complement E_x(x, z)\} \cup \{x\}.$$

B_K is an upper semi-continuous projection. For each $x \in X$, x is an extremal point of $B_K(x)$. The *metric coprojection* from X to K (see [20]) is given by

$$R_K(x) = B_K(x) \cap K.$$

The elements of $R_K(x)$ are called the *elements of best coapproximation* of x in K. Of course, $R_K(x)$ may be void. We call K an *existence set of best coapproximation,* if $R_K(x)$ is never void. We speak of K as a *cochebyshev set,* if $|R_K(x)| = 1$ $\forall x \in X$.

PROPOSITION 2.1. *An existence set of best coapproximation is closed and norm-convex.*

PROOF. Let K be an existence set of best coapproximation. Of course, $k \in \overline{K} \setminus K$ cannot have an element of best coapproximation in K. Thus K has to be closed.

Now let $k, k' \in K$, $k \neq k'$ and let z be a metrical midpoint to k and k' in X, i.e.

$$\|k-k'\| = \|k-z\| + \|z-k'\|.$$

Then each $k_0 \in R_K(z)$ is a metrical midpoint to k and k' belonging to K. □

However for $n \geqq 3$, all closed norm-convex sets are existence sets of best coapproximation if and only if the space is Euclidean. This is an easy consequence of the theorem of Kakutani (see [13]).

For $x \in X$ we define $f_K(x, .) \in C(K)$ as

$$f_K(x, k) = \|x-k\|, \quad \forall k \in K.$$

$\{f_K(x, .): x \in X\}$ supplied with the order of $C(K)$ is a partially ordered set satisfying the descending chain condition. Thus there are minimal elements in it. Following Beauzamy and Maurey [2], we call $x \in X$ *minimal in the weak sense,* if $f_K(x, .)$ is minimal. If K is determining, X supplied with the partial order

$$y \leqq_K x \Leftrightarrow y \in B_K(x)$$

is order-isomorphic to $\{f_K(x, .): x \in X\}$. An $x \in X$ is called *minimal* (with respect to K) if $B_K(x) = \{x\}$ ([2]). Let us denote the set of minimal elements (in the weak sense) by $\min(K)$ ($\mathrm{Min}(K)$). We have $\min(K) \subseteq \mathrm{Min}(K)$. If K is determining, $\min(K)$ equals $\mathrm{Min}(K)$. We call ([2]) K *optimal (in the restricted sense),* if $\min(K) \subseteq K$ ($\mathrm{Min}(K) \subseteq K$). Since for each $x \in X$ there exists some $y \in \mathrm{Min}(K)$, such that $y \leqq_K x$, we have

PROPOSITION 2.2. *K is an existence set of best coapproximation if and only if K is optimal in the restricted sense.*

In a non strictly convex space there are optimal sets which are not existence sets of best coapproximation. In a strictly convex space, however, the optimal sets are just the existence sets of best coapproximation. Especially optimal sets are convex in strictly convex spaces.

To give another interpretation of existence sets of best coapproximation let us recall the definition of a quasi-nonexpansive mapping. T is said to be quasi-nonexpansive if

$$\|Tx-p\| \leqq \|x-p\|$$

for all $x \in X$ and each fixed point p of T. Clearly an existence set of best coapproximation is the fixed point set of a quasi-nonexpansive mapping. Conversely we have (see [13]).

PROPOSITION 2.3. *Let K be the fixed point set of a quasi-nonexpansive mapping. If K is determining, then K is an existence set of best coapproximation.*

Hence in a strictly convex space the optimal sets are just the fixed point sets of quasi-nonexpansive mappings.

The *Browder approximation region* for K and x is given by

$$A_K(x) = \{z \in X\colon \langle x-z, z-k\rangle_s \geqq 0\}$$

(see [6]) or taking the geometrical viewpoint

$$A_K(x) = \{z \in X \setminus \{x\}\colon K \subseteqq \mathbf{C}(z+SK^+(x-z))\} \cup \{x\}.$$

we have

$$A_K(x) \subseteqq B_K(x).$$

We say that K is a *cosun* if $A_K(x) \cap K \neq \emptyset$, $\forall x \notin K$. Thus K is a cosun if for each $x \notin K$ there is some $k \in K$ such that $K \cap (k+SK^+(x-k)) = \emptyset$. From this characterization we see the analogy with the notion of a sun (Vlasov [21]).

PROPOSITION 2.4. *K is a cosun if and only if $\lambda I+(1-\lambda)R_K$ is surjective for all $\lambda > 1$.*

PROOF. The fact that a cosun fulfills the surjectivity condition is easy to see. Thus let us prove the other direction. Let $x \in X \setminus K$. For $m \geqq 2$ there exists $(x_m, k_m) \in R_K$ with

$$x = mx_m+(1-m)k_m.$$

We have $x_m - k_m \to 0$. Since $k_m \in R_K(x)$, let us suppose that $\{k_m\}$ converges to some $k_0 \in R_K(x)$. Let $\delta \in (0, 1)$. We have

$$\|k_0-k\| = \lim_{m\to\infty} \|k_m-k\| \leqq \lim_{m\to\infty} \|(1-\delta)x_m+\delta x-k\| = \|(1-\delta)k_0+\delta x-k\|$$

for all $k \in K$. $\square$

For each $\lambda > 1$, $\lambda I+(1-\lambda)R_K$ has the interesting property of being *outward directed* in case K is an existence set of best coapproximation. By that we mean, given a $k_0 \in K$, for each $x \in X \setminus K$ and $y \in R_K(x)$ the vector $\lambda x+(1-\lambda)y$ has its endpoint in the complement of the tangent cone to $B(k_0, \|x-k_0\|)$ at x. If X is strictly convex, we know that $\lambda I+(1-\lambda)R_K$ is upper semi-continuous, compact- and convex-valued.

Thus we can use Theorem 3.14 in ([16], p. 47) to conclude

$$(\lambda I+(1-\lambda)R_K)(B(k_0, \|x-k_0\|)) \supseteqq B(k_0, \|x-k_0\|).$$

This means that $\lambda I+(1-\lambda)R_K$ is surjective, from which we obtain via proposition 2.4.

THEOREM 2.5. *Let X be strictly convex. Then an optimal set is a cosun.*

From Theorem 2.5 we can conclude quite easily a result of Beauzamy (see [3]).

COROLLARY 2.6. *In a strictly convex, smooth finite dimensional vector space, the optimal sets are just the contractive ray retracts of the space. The unique contractive ray retraction is given by*

$$x \to R_K(x) = A_K(x) \cap K.$$

Since $|R_K(x)|=1$ for a cochebyshev set, we can use the same argument leading to Theorem 2.5 to prove

PROPOSITION 2.7. *Every cochebyshev set is a cosun.*

For an elementary proof see Westphal [22].

Let as call $x \in \partial B(0, 1)$ *non-rigid*, if there exists some $\lambda > 0$ such that

$$[-\lambda x, x] \subseteqq B_{CSK^+(x)}(x).$$

Thus x is non-rigid if and only if $E(0, x)$, which belongs to $SK^+(x)$ is strictly bounded away from $SK^0(x)$, a question we discussed in section 1.

We call X *non-rigid*, if every $x \in \partial B(0, 1)$ is non-rigid. A non-rigid space has to be strictly convex. Also

PROPOSITION 2.8. *If X is non-rigid, every cochebyshey set is an affine subspace.*

PROOF. Let K be a non trivial cochebyshev set containing the origin. Since X is strictly convex, K has to be convex. Let x be in the linear hull of K, but $x \notin K$, and suppose $x \in \partial B(0, 1)$ and $R_K(x)=0$. There is a $\lambda > 0$ satisfying $-\lambda x \in B_K(x)$. Thus $B_K(-\lambda x) \subseteqq B_K(x)$ and, consequently, $R_K(-\lambda x)=R_K(x)$. Since K is a cosun, K belongs to $SK(x)$. But then x would have to belong to $SK(x)$, too. □

The results of Westphal [22] as well as those of Bruck and Reich [7] show that smooth l_p-spaces are non-rigid.

It is easy to see that in strictly convex and smooth spaces, the linear cochebyshev sets are exactly the subspaces admitting a linear contraction (see [2]). From this we obtain the following theorem of Westphal [22].

THEOREM 2.9. *The following statements are equivalent in a smooth l_p-space.*

(i) *K is cochebyshev*

(ii) *K is the translate of the range of a contractive linear projection*

(iii) *K is is the translate of a subspace which is an l_p-space.*

The equivalence of (ii) and (iii) is just the famous result of Ando [1] about the intrinsic characterization of the ranges of linear contractions in l_p-spaces.

In general normed spaces it seems hopeless to give a complete description of all cochebyshev sets. But we have succeeded in doing so for a strictly convex plane by using the detailed study of the asymptotic behavior of Leibniz lines in Section 1 (see [13]).

THEOREM 2.10. *In a strictly convex plane a cochebyshev set, being not a single point, is a parallel strip whose direction meets $\partial B(0, 1)$ in a point of smoothness. This parallel strip is not a line if and only if the point of smoothness is extreme and the chord ratio belonging to it is not strictly bounded away from* 0 *or* $+\infty$.

The meaning of a parallel strip and its direction is evident.

In a non strictly convex plane the situation becomes more complicated. Let L be a norm-segment with respect to the l_∞-norm joining $(-1, 0)$ and $(1, 0)$. Then

$$L\cup\{(\xi,\eta)\colon\ |\eta|\leqq|\xi+1|,\quad \xi\leqq-1\}\cup\{(\xi,\eta)\colon\ |\eta|\leqq|\xi-1|,\quad \xi\geqq 1\}$$

is a cochebyshev set in 1^2_∞.

3. Suns and cosuns in the plane

In this section X will be a two-dimensional space. Via the quadratic and skew-symmetric form

$$Q(x, y)=\eta_1\xi_2-\eta_2\xi_1,\quad x=(\xi_1,\xi_2),\quad y=(\eta_1,\eta_2)\in X$$

we define, following Karlowitz [15], the *dual *norm*

$$\|x\|^*=\sup\{Q(x, y)\colon\ y\in B(0, 1)\},\quad x\in X.$$

All entities referring to this norm are supplied with a "*". $B^*(0, 1)$ is just the dual unit ball rotated by 90°. According to Busemann [9], $\partial B^*(0, 1)$ is the homothetic to a solution of the Minkowskian isoperimetric problem in X. He pointed out that

$$x\perp y\Leftrightarrow y\perp^* x,\quad \forall x, y\in X.$$

More generally, we show

PROPOSITION 3.1.

$$\langle x, y\rangle_s\geqq 0\Leftrightarrow\langle y, x\rangle_s^*\geqq 0,\quad \forall x, y\in X.$$

PROOF. Let x and y be linearly independent. We have

$$\|x+y\|^*=\sup_{\mu\in\mathbf{R}}\frac{|Q(x+\lambda y,\ y+\mu(x+\lambda y))|}{\|y+\mu(x+\lambda y)\|}=Q(x, y)/\inf_{\mu\in\mathbf{R}}\|y+\mu(x+\lambda y)\|,\quad \lambda>0,$$

and

$$\|x\|^*=Q(x, y)/\inf_{\mu\in\mathbf{R}}\|y+\mu x\|.$$

For $\lambda>0$ the ray $y+\mathbf{R}_+(x+\lambda y)$ lines in the interior of the angle with the sides $y+\mathbf{R}_+y$ and $y+\mathbf{R}_+x$. If $\langle x, y\rangle_s=0$, then $y+\mathbf{R}x$ is tangent to $B(0, 1)$ at y, thus

$$d(0,\ y+\mathbf{R}(x+y))\leqq\|y\|=d(0,\ y+\mathbf{R}x).$$

If $\langle x, y\rangle_s>0$, then

$$d(0,\ y+\mathbf{R}x)=\|y+\mu_0 x\|$$

with $\mu_0<0$. Thus $y+\mathbf{R}(x+\lambda y)$ cuts the interior of the segment $[0, y+\mu_0 x]$, e.g.,

$$d(0, y+\mathbf{R}(x+\lambda y)) < \|y+\mu_0 x\| = d(0, y+\mathbf{R}x).$$

Thus we always get

$$\|x+\lambda y\|^*/\|x\|^* \geqq 1 \quad \text{or} \quad \langle y, x\rangle_s^* \geqq 0. \quad \square$$

The *cone of decrease* $C(x, y)$ for $x, y \in X$ is given by

$$C(x, y) = \{z \in X \colon \langle x-z, y-x\rangle_s < 0\}.$$

Clearly, $C(x, y)$ is the open convex tangent cone of $B(y, \|x-y\|)$ at x. We recall that K is a *sun* if for each $x \notin K$ there exists some $k \in K$ satisfying $C(k, x) \cap K = \emptyset$. Using Proposition 3.1 we get

PROPOSITION 3.2. $C^*(0, x) = SK^+(x)$.

PROOF.

$$\begin{aligned} C^*(0, x) &= \{z \in X \colon \langle -z, x\rangle_s^* < 0\} = \mathbf{C}\{z \in X \colon \langle -z, x\rangle_s^* \geqq 0\} = \\ &= \mathbf{C}\{z \in X \colon \langle x, -z\rangle_s \geqq 0\} = \{z \in X \colon \langle x, -z\rangle_s < 0\} = \\ &= \{-z \in X \colon \langle x, z\rangle_s < 0\} = -SK^-(x) = SK^+(x). \quad \square \end{aligned}$$

Thus the best coapproximation in the plane is closely related to the best approximation in the plane. Indeed,

PROPOSITION 3.3. *K is a cosun with respect to $\|\,.\,\|$ if and only if K is a sun with respect to $\|\,.\,\|^*$.*

This reinterpretation of the cone of decrease as the upper halfspace of the shadow-cone with respect to some "dual" norm is a strictly 2-dimensional phenomenon. For $n \geqq 3$ this can be done if and only if the space is Euclidean. Nevertheless part of the cones of decrease may allow such a reinterpretation. Take for example a 3-dimensional l_∞-space. The cones of decrease which can be interpreted as SK^+ in the l_1-norm are just the cones defined by vectors pointing strictly into the interior of the faces of the l_∞-norm.

We shall use the duality between suns and cosuns in the plane to give the following intrinsic characterization.

PROPOSITION 3.4. *The suns in the plane are exactly the closed, *norm-convex subsets.*

PROOF. "$\Rightarrow$": Since a sun is a *cosun, it is *norm-convex by Proposition 2.1.

"$\Leftarrow$": Gruber [12] has shown that the closed, norm-convex subsets of the plane are the contractive ray retracts of the plane, and, consequently, cosuns. (A proof using best approximation methods is given in [13].) $\square$

To fully appreciate this proposition one should be able to construct all closed, norm-convex sets in the plane. The structure of a compact, norm-convex set in the plane is given by (see [12])

PROPOSITION 3.5. *Let K be compact. K is norm-convex if and only if K is obtained from* co (K) *by cutting out a countable number of disjoint subsets each of them bordered* *by a closed segment S in* co (K) *and a normsegment joining the two endpoints of S in K.*

The structure of unbounded suns can be obtained by using the following *localization principle.*

Let K be norm-convex and closed and P a parallelogram whose sides are parallel to lines through the origin cutting $\partial B(0, 1)$ in extreme points. Then $\{K \cap mP\colon m \in \mathbf{N}\}$ gives an increasing sequence of compact, norm-convex sets converging to K.

Another elementary description of suns is that pointed out by Berens in [4].

PROPOSITION 3.6. *K is a sun if and only if $\lambda I+(1-\lambda)P_K$ is surjective for all $\lambda \in (0, 1)$.*

We can considerably weaken this proposition in the plane. For $K \subseteq X$, $x \in X \setminus K$ and $u \in \partial B(0, 1)$ we let $x(K, u)=x-d(x, K)u$ and $W(K, x, u)=C(x(K, u), x)$. Evidently, $W(K, x, u)$ is the interior of the convex tangent cone of $B(x, d(x, K))$ with vertex at $x(K, u)$. Therefore the boundary of this tangent cone is a line or an angle with vertex at $x(K, u)$. Let us orient $\partial B(0, 1)$ counterclockwise. Then we are able to orient $\partial W(K, x, u)$ correspondingly and we can speak of the left (right) side $\partial W_l(K, x, u)$ $(\partial W_r(K, x, u))$ of $\partial W(K, x, u)$. Let

$$S(K, x) = \{u \in \partial B(0, 1)\colon\ K \cap W(K, x, u) = \emptyset\}.$$

Of course,

$$S(K, x) = \{u \in \partial B(0, 1)\colon\ P_K(x) \subseteq \partial W(K, x, u)\}.$$

With a proof similar to that of Proposition 3.6 (see Berens [4]), we get

LEMMA 3.7. *If $\lambda I+(1-\lambda)P_K$ is surjective for some $\lambda \in (0, 1)$, then $S(K, x) \neq \emptyset$ for all $x \notin K$.*

Also we have

LEMMA 3.8. *Let $x \in X \setminus K$, $u \in S(K, x)$ and $k \in P_K(x)$. Then k and $x(k, u)$ belong to one face of $B(x, d(x, K))$.*

If $S(K, x) \neq \emptyset$, there are at most three different tangent cones of $B(x, d(x, K))$, which contain $P_K(x)$. More precisely:

(1) If $P_K(x)$ does not belong to any proper face of $B(x, d(x, K))$, then $|P_K(x)|=1$ and $S(K, x)=\{(x-P_K(x))/\|x-P_K(x)\|\}$. There is only one tangent cone of $B(x, d(x, K))$ which contains $P_K(x)$, namely the one with vertex at $P_K(x)$.

(2) If a face of $B(x, d(x, K))$ contains elements of $P_K(x)$, then $x(K, u)$ belongs to this face for all $u \in S(K, x)$. If $x(K, u)$ is interior to this face, $\partial W(K, x, u)$ is the line generated by $P_K(x)$, e.g., $P_K(x)$ belongs completely to this face. $S(K, x)$ is the corresponding antipodal face of $B(0, 1)$. The tangent cones of $B(x, d(x, K))$ containing $P_K(x)$ are the halfplane bordered by the line generated by $P_K(x)$ and the two tangent cones having their vertex at the endpoints of the corresponding face of $B(x, d(x, K))$, respectively.

(3) The linear hull of $P_K(x)$ may be two-dimensional, but only if the face of $B(x, d(x, K))$ containing elements of $P_K(x)$ is joint by another face of $B(x, d(x, K))$ containing elements of $P_K(x)$. Then there is exactly one tangent cone of $B(x, f(x, K))$

containing $P_K(x)$, namely the one given by these two faces of $B(x, d(x, K))$. The two faces touch in $x(K, u)$ and $S(K, x)=\{u\}$.

LEMMA 3.9. *If* $\lambda I+(1-\lambda)P_K$ *is surjective for some* $\lambda\in(0, 1)$, *then for all* $x\in X$ *we have that* $P_K(x)$ *is contractible.*

PROOF. Let $x\in X\setminus K$. By Lemma 3.7 we know that $S(K, x)\neq\emptyset$. For $|P_K(x)|=1$ nothing is left to show. Thus let p_a be the first and p_e the last point of a face of $B(x, d(x, K))$ containing more than one element of $P_K(x)$. Let k_1 and k_2 be elements of $P_K(x)$ in this face with no other element of $P_K(x)$ lying in between. We set $x'=(k_1+k_2)/2$. Let u equal $(p-p_e)/\|x-p_e\|$. Suppose $k_1, k_2\in\partial W_1(K, x, u)$. Let $v_1=(p_a-p_e)/\|p_a-p_e\|$ and let v_2 be the unit vector in direction of $\partial W_r(K, x, u)$. It may happen that $v_1=-v_2$. Let $u'\in\partial B(0, 1)$. If u' lies beyond u and in front of $-u$ on $\partial B(0, 1)$, but $u'\neq u$, then $k_1\in W(K, x', u')$. If u' lies beyond $-u$ and in front of u on $\partial B(0, 1)$, but $u'\neq -u$, then $k_2\in W(K, x', u')$. Thus $S(K, x')=\emptyset$ contradicting our assumption. We conclude that the intersection of $P_K(x)$ with a face of $B(x, d(x, K))$ is a segment.

If $P_K(x)$ belongs to two faces of $B(x, d(x, K))$ touching each other, then $S(K, x)=\{v\}$ and $x(K, v)$ is the point of contact of these faces. According to our assumption there is some $(y, k_y)\in P_K$ such that $x=\lambda y+(1-\lambda)k_y$. Thus $P_k(x)\subseteq$ $\subseteq P_K(y)$. Consequently there is only one tangent cone of $B(y, d(y, K))$ containing $P_K(y)$ namely the one which contains the two faces of $B(y, d(y, K))$ containing $P_K(x)$. This tangent cone has to coincide with $\overline{W(K, x, v)}$. Thus we have $y\in x+\mathbf{R}_+v$, i.e., $k_y=x(k, v)$. Thus the vertex of $\partial W(K, x, v)$ belongs to $P_K(x)$. $\square$

We conclude that for a sun $P_K(x)$ is a single point, a segment or the compact connected part of an angle.

It is quite easy to see that $\lambda I+(1-\lambda)P_K$ is *outward directed* for all $\lambda\in(0, 1)$ in the sense we have defined it in Section 2. Just as in Section 2, replacing convexity by contractibility, we see that $\lambda I+(1-\lambda)P_K$ has to be surjective for all $\lambda\in(0, 1)$, if it is surjective for some $\lambda\in(0, 1)$. Thus we get

THEOREM 3.9. *The following statements are equivalent in the plane.*

(i) *K is a sun*

(ii) *$\lambda I+(1-\lambda)P_K$ is surjective for some $\lambda\in(0, 1)$*

(iii) *$P_K(x)$ is contractible for all $x\in X$.*

From this we can derive an intrinsic description of other variants of suns in the plane. Recall that K is a *strict sun,* if for each $x\in X$ and all $k\in P_K(x)$ we have $K\cap$ $\cap C(k, x)=\emptyset$.

PROPOSITION 3.10. *The following statements are equivalent in the plane.*

(i) *K is a strict sun*

(ii) *K is *norm-convex and if there is a segment in its boundary which is parallel to a face of $B(0, 1)$, then the line generated by this segment has to support K.*

Also K is a *Chebyshev set,* if $|P_K(x)|=1$ for all $x\in K$.

PROPOSITION 3.11. *The following statements are equivalent in the plane.*

(i) *K is a chebyshev set*

(ii) *K is *norm-convex and it contains no segments in its boundary, which are parallel to a proper face of $B(0, 1)$.*

Having given a complete description of suns let us now take a close look at selection properties of P_K. We call

$$P_K^s(x) = \{k\in P_K(x)\colon\ k+\mathbf{R}_+(x-k) \subseteq P_K^{-1}(x)\}$$

the set of *sun-points* of x (with respect to K). Evidently, K is a sun if and only if $P_K^s(x)\neq\emptyset$ for all $x\in X\setminus K$.

Let K be a sun. Then $P_K^s(x)$ contains one element or equals $P_K(x)$. Since the set-valued map P_K^s is continuous, we obtain using Michael's selection theorem [19] at least one continuous selection of P_K^s. Accounting for the special properties of $P_K^s(x)$ we can even construct continuous ray selections of P_K^s. It is easy to see that any ray selection of P_K in the plane has to be continuous.

There are ray selections. Indeed, one such selection S_K of P_K^s is given by

$$S_K(x) = \text{right endpoint of } P_K^s(x), \quad x\notin K.$$

We recall that $P_K^s(x)$ is a possibly trivial segment in $\partial B(x, d(x, K))$. Thus by the orientation of $\partial B(x, d(x, K))$ it makes sense to speak of the left and the right endpoint of $P_K^s(x)$. Using the contractibility of $P_K(x)$ it is not hard to see that S_K possesses the ray property. We could have chosen

$$S_K(x)=\text{left endpoint of } P_K^s(x),\ x\notin K,$$

as well. These two selections are in some way "extreme". Indeed one can determine selections "in between". In case that $P_K^s(x)$ is a single point we always have to set $S_K(x)=P_K^s(x)$. But if $P_K^s(x)$ is a non-trivial segment, we look at the largest segment, halfline or line L containing $P_K^s(x)$ which belongs to ∂K. The line G generated by L supports K. Suppose L is a segment. Let m be the midpoint of the smallest circle containing L in its boundary lying on the same side of G as K. Set $S_K(x)=[m, x]\cap L$. If L is a halfline or a line we can use this definition as well by selecting m at infinity. We will not go into the details.

Let us recall that a mapping $S\colon X\to X$ is called *orthogonal* to K, if S is a selection of A_K (see Bruck [16]). Using the interpretation of a cone of decrease as SK^+-set we see that a ray selection of P_K is orthogonal to K with respect to the *norm.

THEOREM 3.12. *Each ray retraction S in X orthogonal to $S(X)$ is strongly contractive, i.e.*

$$\|S(x_1)-S(x_2)\|^2 \leqq \langle x_1-x_2,\ S(x_1)-S(x_2)\rangle_s, \quad \forall x_1, x_2\in X.$$

PROOF. Suppose $x_1\neq x_2$ and $S(x_1)\neq S(x_2)$. Let

$$F(x) = \{g\in X'\colon\ g(x) = \|g\|^2 = \|x\|^2\}, \quad x\in X.$$

It is known (see [10]) that

$$\langle y, x\rangle_s = \max\{g(y)\colon\ g\in F(x)\} \quad y, x\in X.$$

Let

$$c = \langle (x_1-S(x_1))-(x_2-S(x_2)),\ S(x_1)-S(x_2)\rangle_s.$$

(1) For $S(x_1)-S(x_2)\in SK^-(x_2-S(x_2))$ and $S(x_2)-S(x_1)\in SK^-(x_1-S(x_1))$ we have

$$c \geqq g(x_1-S(x))-g(x_2-S(x_2)) \quad \forall g\in F(S(x_1)-S(x_2)).$$

Thus

$$c \geqq -\langle x_1 - S(x_1), S(x_1) - S(x_2)\rangle_s - \langle x_2 - S(x_2), S(x_1) - S(x_2)\rangle_s \geqq 0.$$

(2) Let $S(x_1) - S(x_2) \in SK^-(x_2 - S(x_2))$ and $S(x_2) - S(x_1) \in SK(x_1 - S(x_1))$. There exists $g_0 \in F(S(x_1) - S(x_2))$ with $g_0(x_1 - S(x_1)) = 0$. Thus

$$c \geqq g_0(x_1 - S(x_1)) - \langle x_2 - S(x_2), S(x_1) - S(x_2)\rangle_s =$$
$$= -\langle x_2 - S(x_2), S(x_1) - S(x_2)\rangle_s \geqq 0.$$

(3) Analogously for $S(x_1) - S(x_2) \in SK(x_2 - S(x_2))$ and $S(x_2) - S(x_1) \in SK^- \times \times (x_1 - S(x_1))$ we obtain $c \geqq 0$.

(4) Let $S(x_1) - S(x_2) \in SK(x_1 - S(x_1)) \cap SK(x_2 - S(x_2))$. There exist $g_i \in \in F(S(x_1) - S(x_2))$, $i = 1, 2$, satisfying

$$g_i(x_i - S(x_i)) = 0, \quad i = 1, 2.$$

If $x_1 - S(x_1)$ is parallel to $x_2 - S(x_2)$ we can choose $g_1 = g_2$ and conclude $c \geqq 0$. Therefore let us suppose that $x_1 - S(x_1)$ is not parallel to $x_2 - S(x_2)$. Then the lines $S(x_1) + \mathbf{R}(x_1 - S(x_1))$ and $S(x_2) + \mathbf{R}(x_2 - S(x_2))$ have a unique cutpoint s given by

$$S(x_1) + \delta(x_1 - S(x_1)) = S(x_2) + \tau(x_2 - S(x_2)).$$

Suppose that $g_1(x_2 - S(x_2)) > 0$ and $g_2(x_1 - S(x_1)) < 0$. Then the uniquely determined δ and τ are given by

$$\delta = -\|S(x_2) - S(x_1)\| / g_2(x_1 - S(x_1))$$

and

$$\tau = \|S(x_1) - S(x_2)\| / g_1(x_2 - S(x_2)).$$

Since δ and τ are positive, we conclude $S(x_1) = S(s) = S(x_2)$ contradicting our assumption. Thus let $g_1(x_2 - S(x_2)) \leqq 0$. We have

$$c \geqq g_1(x_1 - S(x_1)) - g_1(x_2 - S(x_2)) = -g_1(x_2 - S(x_2)) \geqq 0.$$

In the case $g_2(x_1 - S(x_1)) \geqq 0$ we argue analogously. □

Applying this theorem to best approximation in the plane we get

THEOREM 3.13. *Let K be a sun in the normed plane. There exist ray selections of P_K and any such selection is strongly contractive with respect to the *norm.*

From the point of view of best approximation a normed plane is "almost" a Euclidean plane.

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DISTRIBUTION STATISTIQUE DE L'ORDRE D'UN ELEMENT DU GROUPE SYMETRIQUE

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I. Introduction

Soit S_n le groupe des permutations de n objets. P. Erdős et P. Turán ont démontré: (cf. [5])

$$\lim_{n\to\infty} \operatorname{Prob}\left\{\frac{\log(\text{ordre de } \sigma)-(1/2)\log^2 n}{(1/\sqrt{3})\log^{3/2} n} < x\right\} = \Phi(x) \tag{1}$$

avec

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$$

en mettant sur S_n la mesure d'équiprobabilité. P. Erdős et P. Turán annoncent qu'il est possible d'obtenir un terme d'erreur dans la formule (1).

Pour chaque permutation $\sigma\in S_n$, nous désignerons par

$$n_1 < n_2 < \ldots < n_k$$

les différentes longueurs de cycles de σ, et par $m_1, \ldots, m_k$ leur multiplicité, de telle sorte que

$$\sum_{1\leqq i\leqq k} m_i n_i = n.$$

La démonstration de (1) est basée d'abord sur le résultat (cf. [4]): excepté $o(n!)$ permutations, tous les éléments $\sigma\in S_n$ vérifient:

$$\exp\left(-3\log n\,(\log\log n)^4\right) \leqq \frac{\text{ordre de } \sigma}{n_1 n_2 \ldots n_k} \leqq 1 \tag{2}$$

et ensuite sur la distribution des valeurs de la fonction

$$f(\sigma) = \sum_{1\leqq i\leqq k} \log n_i$$

à l'aide de sa fonction caractéristique.

Par la suite, M. R. Best [1] et J. D. Bovey [2] ont redémontré la loi limite vérifiée par f, par des méthodes plus élémentaires.

Nous allons démontrer le théorème suivant, qui améliore (2):

THÉORÈME 1. *Si l'on enlève de S_n un ensemble de $O(n!/\sqrt{\log n})$ permutations, celles qui restent vérifient:*

$$\log(\text{ordre de } \sigma) = f(\sigma) - \log n \log\log n + O(\log n \log\log\log n).$$

La démonstration du théorème 1 repose sur l'idée suivante, fournie par P. Erdős: dans une permutation aléatoire, la moitié des cycles est de longueur paire, un tiers des cycles est de longueur multiple de 3, etc... et le nombre de cycles étant environ $\log n$, la contribution des nombres premiers $p \leqq \log n$ dans la différence $f(\sigma) - \log$ (ordre de σ) est approximativement:

$$\sum_{p \leqq \log n} \log p \frac{\log n}{p} = \log n \left(\log\log n + O(1)\right).$$

La contribution des nombres premiers $p > \log n$ est négligeable. La proposition 1 permet d'évaluer très précisement le nombre de $\sigma \in S_n$ qui ont exactement j cycles de longueur multiple de α, et je remercie M. Szalay, qui m'a signalé la référence [11]. La proposition 2, qui m'a été suggérée par A. Odlyzko, majore le nombre de $\sigma \in S_n$ pour lesquelles $n_1 n_2 \dots n_k$ est divisible par une puissance assez grande d'un produit de nombres premiers.

Nous montrerons ensuite:

THÉORÈME 2. *On a uniformément en* $x \in \mathbf{R}$:

$$\operatorname{Prob}\left\{\frac{f(\sigma) - (1/2)\log^2 n}{(1/\sqrt{3})\log^{3/2} n} < x\right\} = \Phi(x) + O\left(1/\sqrt{\log n}\right).$$

La démonstration du théorème 2 reprend les calculs originaux de P. Erdős et P. Turán. En fait un calcul similaire a été fait dans [10], pour étudier une fonction voisine de f, définie sur l'ensemble $\mathbf{F}_q^{(n)}[X]$ des polynômes unitaires de degré n sur un corps fini.

On déduit immédiatement des théorèmes 1 et 2:

THÉORÈME 3. *On a uniformément en* $x \in \mathbf{R}$:

$$\operatorname{Prob}\left\{\frac{\log(\text{ordre de } \sigma) - (1/2)\log^2 n + \log n \log\log n}{(1/\sqrt{3})\log^{3/2} n} < x\right\} = \Phi(x) + O\left(\frac{\log\log\log n}{\sqrt{\log n}}\right).$$

Nous conjecturons que l'on peut supprimer le $\log\log\log n$ dans le reste du théorème 1, et du théorème 3.

NOTATIONS. Nous écrirons, pour simplifier

$$l = \log n; \quad l_2 = \log\log n; \quad l_3 = \log\log\log n.$$

Pour $1 \leqq \nu \leqq n$, et pour $\sigma \in S_n$ fixé, nous poserons: $N(\nu) = \sum_{\substack{i=1 \\ \nu | n_i}}^{k} 1$, le nombre de longueurs distinctes de cycles de σ multiples de ν. La lettre p, indicée ou non désignera toujours un nombre premier. Enfin nous noterons $[x]$ la partie entière de x.

II. Démonstration du théorème 1

Enonçons d'abord quelques lemmes:

LEMME 1. *Soit* $x>0$. *On a:*
pour $u \geqq x$:

$$\sum_{m \geqq u} \frac{x^m}{m!} \leqq \left(\frac{ex}{u}\right)^u$$

et pour $0<v\leqq x$:

$$\sum_{0\leqq m \leqq v} \frac{x^m}{m!} \leqq \left(\frac{ex}{v}\right)^v.$$

DÉMONSTRATION. Elle est facile (cf. [8], p. 149).

LEMME 2. *Soit* $1\leqq \nu_1<\nu_2<\ldots<\nu_r$ *avec* $\sum_{i=1}^{r} \nu_i \leqq n$. *Le nombre de permutations de* S_n *ayant au moins un cycle de longueur* ν_1, *un cycle de longueur* $\nu_2, \ldots,$ *un cycle de longueur* ν_r *est* $\leqq n!/(\nu_1\nu_2\ldots\nu_r)$.

DÉMONSTRATION. Il *y* a

$$\frac{n!}{\nu_1!\,\nu_2!\ldots\nu_r!\,(n-\nu_1-\nu_2-\ldots-\nu_r)!}$$

façons de choisir r parties de $\{1, 2, \ldots, n\}$ de cardinal $\nu_1, \ldots, \nu_r$. Dans chacune de ces parties on doit avoir une permutation circulaire ce qui donne $(\nu_1-1)!\ldots(\nu_r-1)!$ choix possibles. Dans ce qui reste, n'importe quelle permutation marche, il *y* en a $(n-\nu_1-\ldots-\nu_r)!$. Cette démonstration est voisine de celle de la formule de Cauchy (cf [3], t.2, p. 75).

LEMME 3. *Soit* λ *réel vérifiant* $0<\lambda<1$. *On a:*

$$\sum_{p\leqq x} \frac{\log p}{p^\lambda} = O\left(\frac{x^{1-\lambda}}{1-\lambda}\right); \quad \sum_{p^m\leqq x} \frac{\log p}{p^{\lambda m}} = O\left(\frac{x^{1-\lambda}}{1-\lambda}\right);$$

$$\sum_{p\leqq x} \frac{\log p}{p} = \log x + O(1); \quad \sum_{p^m\leqq x} \frac{\log p}{p^m} = \log x + O(1).$$

Si $x\geqq 2$ *et si* $\lambda \geqq 2$, *on a:*

$$\sum_{p\geqq x} \frac{1}{p^\lambda} \leqq \frac{3}{(\log x)\, x^{\lambda-1}}.$$

DÉMONSTRATION. Soit $\theta(x) = \sum_{p\leqq x} \log p$ la fonction de Čebyšev. On a, par l'intégrale de Stieltjes

$$\sum_{p\leqq x} \frac{\log p}{p^\lambda} = \int_{2^-}^{x} \frac{d[\theta(t)]}{t^\lambda} = \frac{\theta(x)}{x^\lambda} + \int_{2}^{x} \frac{\lambda\theta(t)}{t^{\lambda+1}}\, dt$$

et comme $\theta(t)=O(t)$, cette quantité est:

$$O\Bigl(x^{1-\lambda}+\int_0^x \lambda t^{-\lambda}\,dt\Bigr)=O(x^{1-\lambda}/(1-\lambda)).$$

Pour évaluer $\sum_{p^m\le x}(\log p)p^{-\lambda m}$, on procède de même en remplaçant $\theta(x)$ par $\psi(x)=\sum_{p^m\le x}\log p$, la seconde fonction de Čebyšev. L'estimation de $\sum_{p\le x}(\log p)/p$ est classique (cf. [9], ch. 22).

De même, soit $\pi(x)=\sum_{p\le x}1$; on a:

$$\sum_{p\ge x}p^{-\lambda}=\int_{x^-}^{+\infty}\frac{d[\pi(t)]}{t^\lambda}\le-\frac{\pi(x)-1}{x^\lambda}+\int_x^{+\infty}\frac{\lambda\pi(t)}{t^{\lambda+1}}\,dt.$$

Or on sait que $\pi(t)\le(3/2)(t/\log t)$ pour tout t, donc

$$\sum_{p\ge x}p^{-\lambda}\le\frac{3/2}{\log x}\int_x^\infty \lambda t^{-\lambda}\,dt\le\frac{3}{(\log x)\,x^{\lambda-1}}.$$

LEMME 4. *Soit* $20\le\omega_1,\ \omega_2\le n$. *Si l'on enlève de* S_n *un nombre de permutations* $\le 3n!\left(\dfrac{1}{\omega_1}+\dfrac{1}{\omega_2!}\right)$, *les permulations restantes ont la propriété suivante: les cycles de longueur* $>\omega_1$ *sont uniques et les cycles de longueur* $\le\omega_1$, *ont une multiplicité* $\le\omega_2$.

DÉMONSTRATION. Ce lemme est le lemme III de [4].

PROPOSITION 1. *Soit:*

$$c_j=(1/n!)\,\mathrm{Card}\Bigl\{\sigma\in S_n;\ \sum_{\substack{1\le i\le k\\ \alpha|n_i}}m_i=j\Bigr\}$$

la probabilité qu'une permutation de S_n *ait exactement* j *cycles dont les longueurs sont multiples de* α. *Alors, si l'on pose* $r=[n/\alpha]$, *on a:*

$$c_j=\sum_{k=j}^{r}\frac{1}{r!}\frac{|s(r,k)|}{\alpha^k}\binom{k}{j}(\alpha-1)^{k-j}$$

où $s(r,k)$ *désigne le nombre de Stirling de* 1ère *espèce, et on a la majoration, pour* $r\ge2$

$$c_j\le e^\gamma r^{-1/\alpha}\frac{H^{j-1}}{j!\,\alpha^j}(j+H)$$

où γ *désigne la constante d'Euler, et* $H=1+\dfrac{1}{2}+\ldots+\dfrac{1}{r-1}$.

DÉMONSTRATION. D'après la formule 0.27, p. 183 de [11], on a:

$$\sum_{j=0}^{r}c_jx^j=\binom{(x-1)/\alpha+r}{r}$$

et d'après la définition des nombres de Stirling de première espèce, le deuxième membre ci-dessus est égal à: (cf. [3], t.2, p. 48)

$$\frac{1}{r!}\sum_{0\leqq k\leqq r}|s(r,k)|\frac{(x+\alpha-1)^k}{\alpha^k}=\frac{1}{r!}\sum_{0\leqq k\leqq r}\frac{|s(r,k)|}{\alpha^k}\sum_{j=0}^{k}\binom{k}{j}x^j(\alpha-1)^{k-j}=$$

$$=\sum_{j=0}^{r}x^j\sum_{k=j}^{r}\frac{1}{r!}\frac{|s(r,k)|}{\alpha^k}\binom{k}{j}(\alpha-1)^{k-j}.$$

Ce qui nous donne la première formule de la proposition.

On utilise ensuite la majoration:

$$|s(n,k)|\leqq\frac{(n-1)!}{(k-1)!}\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}\right)^{k-1}.$$

Cette majoration peut être démontrée par récurrence en utilisant la formule:

$$|s(n+1,k+1)|=|s(n,k)|+n|s(n,k+1)|.$$

On obtient alors:

$$c_j\leqq\sum_{k=j}^{r}\frac{1}{r!}\frac{(r-1)!}{(k-1)!}H^{k-1}\frac{k!}{j!(k-j)!}\frac{1}{(\alpha-1)^j}\left(1-\frac{1}{\alpha}\right)^k$$

et en posant $i=k-j$,

$$c_j\leqq\frac{H^{j-1}}{rj!\alpha^j}\sum_{i=0}^{r-j}\frac{H^i(1-1/\alpha)^i}{i!}(i+j).$$

La sommation est majorée par:

$$\sum_{i=0}^{\infty}\frac{(H(1-1/\alpha))^i}{i!}(i+j)=\big(H(1-1/\alpha)+j\big)\exp\big(H(1-1/\alpha)\big).$$

Compte tenu de ce que

$$H=1+\frac{1}{2}+\ldots+\frac{1}{r-1}\leqq\gamma+\log r$$

on obtient le résultat annoncé.

Corollaire. *Si l'on enlève de S_n un nombre de permutations $O\big(n!/(\log n)^2\big)$, celles restantes ont la propriété: Pour tout $\alpha\leqq l/l_2^2$, le nombre de cycles dont la longueur est multiple de α est* $\dfrac{\log n}{\alpha}+O\left(\dfrac{\log n}{\alpha}\right)^{0,8}$.

Démonstration. Fixons d'abord α. On pose, avec les notations de la proposition précédente, $x=H/\alpha$, $j_0=x-x^u$, $j_1=x+x^u+1$. On choisira $u=0{,}8$. Le nombre de permutations à enlever, c'est-à-dire celles qui ont moins de j_0 (ou plus de j_1) cycles dont la longueur est multiple de α, vaut:

$$S=\sum_{j\leqq j_0}c_j+\sum_{j\geqq j_1}c_j\leqq 2e^{\gamma}r^{-1/\alpha}\left(\sum_{j\leqq j_0}\frac{x^j}{j!}+\sum_{j\geqq j_1}\frac{x^{j-1}}{(j-1)!}\right)$$

et, par le lemme 1,

$$S \leqq 2e^{\gamma} r^{-1/\alpha} \left(\left(\frac{ex}{j_0} \right)^{j_0} + \left(\frac{ex}{j_1 - 1} \right)^{j_1 - 1} \right).$$

Or on a:

$$j_0 \log (ex/j_0) = x - \frac{1}{2} x^{2u-1} + O(x^{3u-2})$$

$$(j_1 - 1) \log \big(ex/(j_1 - 1)\big) = x - \frac{1}{2} x^{2u-1} + O(x^{3u-2}).$$

Ce qui nous donne:

$$S = O\big(r^{-1/\alpha} \exp (x - \delta x^{2u-1})\big)$$

pour un certain $\delta > 0$. On remarque ensuite que

$$H = 1 + \frac{1}{2} + \ldots + \frac{1}{r-1} = \log r + O(1)$$

et donc $e^x = O(r^{1/\alpha})$. Enfin, comme $r = [n/\alpha]$, on a: $x = (1/\alpha)\big(l + O(l_2)\big)$; comme $\alpha \leqq l/l_2^2$, on voit que $x \geqq l_2^2/2$ pour n assez grand, et donc

$$S = O\big(\exp(-\delta_1 l_1^{1,2})\big) = O(l^{-3})$$

en choisissant $u = 0{,}8$.

En faisant le même raisonnement pour tous les $\alpha \leqq l/l_2^2$, on obtient qu'excepté $O(n!/l^2)$ permutations, le nombre de cycles dont la longueur est multiple de α est compris entre $x - x^u$ et $x + x^u + 1$, ce qui achève la démonstration du corollaire.

Pour démontrer le théorème 1, nous allons construire des sous-ensembles $S_n^{(1)} \subset \subset S_n, \ldots, S_n^{(i+1)} \subset S_n^{(i)}$, tous tels que $\operatorname{Card} (S_n - S_n^{(i)}) = O\big(n!/\sqrt{l}\big)$.

Construction de $S_n^{(1)}$. On utilise le lemme 4 avec $\omega_1 = \big[\sqrt{\log n}\big]$ et $\omega_2 = l_2$. On a bien:

$$1/\omega_1 + 1/\omega_2! = O\big(1/\sqrt{l}\big)$$

et les $\sigma \in S_n^{(1)}$ ont la propriété

$$P_1\colon \big(n_i > \sqrt{l} \Rightarrow m_i = 1\big) \quad \text{et} \quad \big(n_i \leqq \sqrt{l} \Rightarrow m_i \leqq l_2\big).$$

On désignera par k_0 le nombre entier tel que $n_{k_0} \leqq \sqrt{l} < n_{k_0+1}$. La propriété P_1 s'écrit alors:

$$P_1\colon (1 \leqq i \leqq k_0 \Rightarrow m_i \leqq l_2) \quad \text{et} \quad (k_0 < i \leqq k \Rightarrow m_i = 1).$$

Construction de $S_n^{(2)}$. On utilise le corollaire de la proposition 1. Les $\sigma \in S_n^{(2)}$ auront la propriété P_1 et la propriété P_2:

$$P_2\colon \forall \alpha \leqq l/l_2^2, \qquad \sum_{\substack{1 \leqq i \leqq k \\ \alpha | n_i}} m_i = l/\alpha + O(l/\alpha)^{0{,}8}.$$

Minoration dans le théorème. 1. Nous allons montrer que pour $\sigma \in S_n^{(2)}$, on a:

$$f(\sigma) - \log (\text{ordre de } \sigma) \geqq l l_2 + O(l l_3).$$

On remarque d'abord que, à cause de P_1,

$$\sum_{\substack{1\le i\le k\\ \alpha|n_i}} m_i = \sum_{\substack{1\le i\le k\\ \alpha|n_i}} 1+\sum_{\substack{1\le i\le k_0\\ \alpha|n_i}} (m_i-1) = N(\alpha)+O\left(\frac{l_2\sqrt{l}}{\alpha}\right).$$

On a donc, par P_2, et si $\alpha\leqq l/l_2^2$:

(3) $$N(\alpha) = (l/\alpha)+O(l/\alpha)^{0,8}.$$

On a ensuite:

$$f(\sigma)-\log(\text{ordre de }\sigma) = \log(n_1\dots n_k)-\log(\text{p.p.c. }m(n_1,\dots,n_k)) \geqq$$
$$\geqq \sum_{p\le l/l_2^2} \log p(N(p)-1).$$

Cette dernière somme vaut:

$$\sum_{p\le l/l_2^2} l\frac{\log p}{p}+O\left(l^{0,8}\frac{\log p}{p^{0,8}}\right) = ll_2+O(ll_3)$$

par le lemme 3.

Construction de $S_n^{(3)}$. On remarque d'abord, en faisant $\alpha=1$ dans la formule (3), que l'on a pour tout $\sigma\in S_n^{(2)}$ la propriété P_2':

$$P_2'\colon\ k = N(1) = \log n+O(\log n)^{0,8}.$$

En fait, on aurait pu obtenir P_2' à partir des résultats de Gončarov [7], comme l'ont fait P. Erdős et P. Turán [5].

On impose ensuite les propriétés suivantes:

$$P_3\colon\ \forall\alpha\geqq(\log n)^3,\quad N(\alpha)\leqq 1,$$
$$P_3'\colon\ \forall\alpha\geqq(\log n)^{3/2},\quad N(\alpha)\leqq 4,$$
$$P_3''\colon \forall y,\quad l/l_2^2\leqq y\leqq l,\quad \forall\alpha\geqq y,\quad N(\alpha)\leqq ll_2/y.$$

Fixons $\alpha\geqq l/l_2^2$ et $j_0\geqq l/\alpha$. Avec les notations de la proposition 1, on a $r\leqq n/2$,

$$H = 1+\dots+\frac{1}{r-1}\leqq\gamma+\log r\leqq\log n$$

et le nombre de $\sigma\in S_n$ pour lesquelles $N(\alpha)\geqq j_0$ est majoré par:

$$n!\sum_{j\ge j_0} c_j \leqq 2n!\sum_{j\ge j_0}\left(\frac{(l/\alpha)^j}{j!}+\frac{(l/\alpha)^{j-1}}{\alpha(j-1)!}\right) = O\left(\frac{el}{\alpha j_0}\right)^{j_0} n!$$

par le lemme 1, à condition que $j_0\geqq 2$ et $j_0\leqq l$.

Pour P_3, on fait $j_0=2$. Le nombre de $\sigma\in S_n$ qui font exceptions à P_3 est majoré par

$$n!O\big(l^2\sum_{\alpha\ge l^3}\alpha^{-2}\big) = O(n!/l).$$

Pour P_3' on raisonne de même avec $j_0=5$.

Pour P_3'', on fixe y, et α, avec $y\leqq\alpha\leqq l^{3/2}$. (Pour $\alpha>l^{3/2}$, on a par P_3', $N(\alpha)\leqq$ $\leqq 4\leqq ll_2/y$). On choisit $j_0=ll_2/y$. On a bien $l/\alpha\leqq j_0\leqq l$ et on remarque que $j_0\geqq l_2$.

Le nombre d'exceptions est donc:

$$O\left(\frac{ey}{\alpha l_2}\right)^{j_0} n! = O\left(\frac{e}{l_2}\right)^{l_2} n! = O(n!/l^4).$$

Comme il y a au plus l valeurs de y et $l^{3/2}$ valeurs de α, le nombre d'exceptions à P_3'' est négligeable.

PROPOSITION 2. *Soit n assez grand, $m \geqq 1$, $t \geqq 2$, $tm \leqq l/2$, $y \geqq \sqrt{l}$. Le nombre de $\sigma \in S_n^{(2)}$ pour lesquelles il existe m nombres premiers $p_1 < p_2 < \ldots < p_m$, avec $y \leqq p_1$ tels que $N(p_1) \geqq t$, ..., $N(p_m) \geqq t$ est majoré par:*

$$n!\left(\left(\frac{30 \log n}{yt}\right)^t \frac{9y}{m \log y}\right)^m.$$

DÉMONSTRATION. Fixons d'abord $p_1, \ldots, p_m$. Comme $tm \leqq (1/2)\log n$, par la propriété P_2' si σ est telle que chacun de ces p_i divisent au moins la longueur de t cycles distincts, on peut trouver μ cycles de σ, de longueurs $\nu_1 < \nu_2 < \ldots < \nu_\mu$ avec $t \leqq \mu \leqq \leqq tm$ et $\nu_1 \geqq \sqrt{\log n}$ tels que $P = p_1^t \ldots p_m^t$ divise $\nu_1 \ldots \nu_\mu$. Le nombre de telles σ est donc majoré d'après le lemme 2, par:

$$Q \leqq \sum_{\mu=t}^{tm} \sum_{\substack{\sqrt{\log n} \leqq \nu_1 < \nu_2 < \ldots < \nu_\mu \leqq n \\ P \mid \nu_1 \nu_2 \ldots \nu_\mu}} \frac{n!}{\nu_1 \ldots \nu_\mu} \leqq \sum_{\mu=t}^{tm} \frac{n!}{(\mu)!} \sum_{\substack{\nu_1 = 1 \\ \vdots \\ \nu_\mu = 1 \\ P \mid \nu_1 \ldots \nu_\mu}}^{n} \frac{1}{\nu_1 \ldots \nu_\mu}$$

On peut mettre $1/P$ en facteurs dans la dernière somme, à condition de multiplier par $\tau_\mu(P) = \binom{\mu+t-1}{t}^m$. La fonction $\tau_r(n)$ est définie par:

$$\tau_2(n) = \sum_{d \mid n} 1$$

pour $r \geqq 3$,

$$\tau_r(n) = \sum_{d \mid n} \tau_{r-1}(d).$$

Il s'ensuit que:

$$Q \leqq \sum_{\mu=t}^{tm} \frac{n!}{(\mu)!\,P} \left(\sum_{\substack{\nu_1' = 1 \\ \vdots \\ \nu_\mu' = 1}}^{n} \frac{1}{\nu_1' \ldots \nu_\mu'} \right) \binom{\mu+t-1}{t}^m \leqq$$

$$\leqq \sum_{\mu=t}^{tm} \frac{n!}{(\mu)!\,P} (2 \log n)^\mu \frac{(\mu+t)^{tm}}{(t!)^m} \leqq \frac{n!}{P} \left(\frac{4e^2 \log n}{t}\right)^{tm} \frac{\sum_{\mu=t}^{tm} \left(\frac{\mu}{\log n}\right)^{tm-\mu}}{(\sqrt{2\pi t})^{m+1}} \leqq$$

$$\leqq \frac{n!}{P} \left(\frac{4e^2 \log n}{t}\right)^{tm} \frac{tm}{(\sqrt{2\pi t})^{m+1}} \leqq \frac{n!}{p_1^t \ldots p_m^t} \left(\frac{30 \log n}{t}\right)^{tm}$$

en minorant $u!$ par $\sqrt{2\pi u}\, u^u e^{-u}$, et en majorant $\mu+t$ par 2μ et $4e^2$ par 30.

Maintenant, on fait varier les nombres premiers $p_1, \dots, p_m$. On a:

$$\sum_{y \leqq p_1 < \dots < p_m} \frac{1}{p_1^t \dots p_m^t} \leqq \frac{1}{m!}\Big(\sum_{p \geqq y} \frac{1}{p^t}\Big)^m \leqq \Big(\frac{e}{m} \sum_{p \geqq y} \frac{1}{p^t}\Big)^m \leqq \Big(\frac{9}{m y^{t-1} \log y}\Big)^m$$

d'après le lemme 3, ce qui achève la démonstration.

Construction de $S_n^{(4)}$. On va imposer aux $\sigma \in S_n^{(3)}$ la condition suivante; avec $c_0 = 18 \exp(60/e)$

$$P_4\colon \forall t, \quad 2 \leqq t \leqq l_2, \quad m_t = \operatorname{Card}\{p \geqq l;\ N(p \geqq t\} \leqq c_0 2^{-t} l/l_2.$$

Fixons d'abord t; on applique la proposition 2, avec $y = \log n$, $m = [c_0 2^{-t} l/l_2] + 1$. Le nombre d'exceptions est allors majoré par:

$$n!\, 2^{-c_0 2^{-l_2 l/l_2}} = O(n!\, l^{-2})$$

et on fait ensuite varier t.

Construction de $S_n^{(5)}$. La condition supplémentaire imposée est

$$P_5\colon \forall y, \quad l/l_2^2 \leqq y \leqq l, \quad \forall s, \quad 2 \leqq s \leqq l_2,$$

$$m_s' = \operatorname{Card}\{p \geqq y;\ N(p) > 60ls/y\} \leqq 36 \cdot 2^{-s} y/l_2.$$

On fixe d'abord y et s, on applique la proposition 2 avec $t = [60\, ls/y] + 1$ et $m = [36 \cdot 2^{-s} y/l_2] + 1$, et on termine comme précédemment.

Démonstration de la majoration dans le théorème. Soit $\sigma \in S_n^{(5)}$, nous devons majorer $f(\sigma) - \log$ (ordre de σ). Cette quantité est d'abord majorée par:

$$\sum_{\substack{p, a \\ N(p^a) \geqq 2}} (\log p) N(p^a) = \sum_{i=1}^{7} T_i$$

où les sommes partielles T_i portent sur les couples (p, a), p premier, $a \geqq 1$ vérifiant $N(p^a) \geqq 2$ et:

$i = 1 \quad p^a > l^3$

$i = 2 \quad p^a \leqq l/l_2^2$

$i = 3 \quad a = 1$ et $l/l_2^2 < p \leqq l$

$i = 4 \quad a = 1$ et $l < p \leqq l^3$

$i = 5 \quad a \geqq 2$ et $l/l_2^2 < p \leqq l^{3/2}$

$i = 6 \quad a \geqq 2, \ p \leqq l$ et $l^{3/2} < p^a \leqq l^3$

$i = 7 \quad a \geqq 2, \ p > l$ et $l^{3/2} < p^a \leqq l^3$.

Par la propriété P_3, la somme T_1 est vide.

Par P_2, on a:

$$T_2 \leqq \sum_{p^a \leqq l/l_2^2} l \frac{\log p}{p} + O\Big(l^{0,8} \frac{\log p}{p^{0,8a}}\Big) \leqq l l_2 + O(l l_3),$$

par le lemme 3.

Dans T_5 le nombre de termes est $O(l^{3/4})$, et par P_3'', on a:

$$T_5 = O(l^{3/4} l_2^4).$$

Dans T_6, en utilisant P_3', on a:

$$T_6 \leqq \sum_{p \leqq l} \log p \sum_{a \leqq 3l_2/\log p} 4 \leqq 12 l_2 \pi(l) = O(l).$$

Dans T_7, on remarque que a vaut exactement 2 et comme $N(p^a) \leqq N(p)$, on constate que $T_7 \leqq T_4$. On a ensuite:

$$T_4 = \sum_{\substack{l<p \leqq l^3 \\ N(p) \geqq 2}} (\log p) N(p) \leqq 3l_2 \sum_{\substack{l<p \leqq l^3 \\ N(p) \geqq 2}} N(p).$$

D'après P_3'', $N(p) \leqq l_2$ et par la propriété P_4,

$$T_4 \leqq 3l_2 \sum_{t=2}^{l_2} t m_t = O(l).$$

Pour évaluer T_3, choisissons y, $l/l_2^2 \leqq y \leqq l/2$ et considérons

$$W_y = \sum_{y<p \leqq 2y} (\log p) N(p) \leqq l_2 \sum_{y<p \leqq 2y} N(p).$$

Nous allons montrer que pour tout y, $W_y = O(l)$. Il en résultera que $T_3 = O(ll_3)$ ce qui achèvera la démonstration du thèoreme. Par la propriété P_3'', on a $N(p) \leqq ll_2/y$. Ensuite, par P_5,

$$\sum_{y<p \leqq 2y} N(p) \leqq \frac{120l}{y} \Big(\sum_{y<p \leqq 2y} 1 \Big) + \sum_{s=2}^{[l_2]} \frac{60l(s+1)}{y} m_s' = O(l/l_2).$$

Pour supprimer le «l_3» dans le reste du théorème 1, il faudrait pouvoir montrer que:

$$T_3 = 2ll_3 + O(l).$$

III. Démonstration du théorème 2

Nous utiliserons la notation suivante pour les séries entières:

$$\sum_{n=0}^{\infty} a_n z^n \ll \sum_{n=0}^{\infty} b_n z^n$$

signifie: pour tout $n \geqq 0$, $|a_n| \leqq b_n$.

LEMME 5. *On a:*

$$\sum_{m=2}^{\infty} \frac{\log m}{m} z^m - \frac{1}{2} \log^2 \frac{1}{1-z} \ll \log \frac{1}{1-z} = \sum_{m=1}^{\infty} \frac{z^m}{m}.$$

LEMME 6. *On a:*

$$\sum_{m=2}^{\infty} \frac{\log^2 m}{m} z^m - \frac{1}{3} \log^3 \frac{1}{1-z} \ll 2 \sum_{m=2}^{\infty} \frac{\log m}{m} z^m.$$

LEMME 7. *Soit* $a>0$ *et une suite de coefficients* a_k *vérifiant* $|a_k| \leqq a/k$ *pour* $1 \leqq k \leqq n$. *On pose:*

$$\exp\Big(\sum_{k\geqq 1} \alpha_k z^k\Big) = 1 + \sum_{k=1}^{\infty} b_k z^k.$$

Alors, on a, pour $1 \leqq k \leqq n$

$$|b_k| \leqq a e^a k^{a-1}.$$

LEMME 8. *Soit* $n \geqq 3$ *et* $t \in \mathbf{R}$, *vérifiant* $|t| \leqq \sqrt{\log n}$. *On pose:*

$$h(z) = \frac{1}{1-z} \exp\left\{\frac{it}{2\log^{3/2} n} \log^2 \frac{1}{1-z} - \frac{t^2}{6\log^3 n} \log^3 \frac{1}{1-z}\right\} = \sum_{m=0}^{\infty} e_m z^m.$$

Alors on a, $e_0 = e_1 = 1$, *et*

$$e_n = \exp\left\{\frac{it\sqrt{\log n}}{2} - \frac{t^2}{6}\right\} + O\left(e^{-t^2/6} \frac{|t|}{\sqrt{\log n}}\right) + O\left(\frac{1}{n}\right)$$

et pour $2 \leqq m \leqq n$,

$$|e_m| = O\left(\exp\left\{-\frac{t^2}{6} \frac{\log^3 m}{\log^3 n}\right\}\right) + O(2^{-m})$$

où les O sous entendent des constantes explicites.

La démonstration des lemmes 5 à 8 se trouve dans [10]. Sous une forme un peu moins précise, ces lemmes figuraient dans [5].

LEMME 9. *On a pour tout x réel et m entier* $\geqq 1$,

$$\left|e^{ix} - 1 - ix - \ldots - \frac{(ix)^{m-1}}{(m-1)!}\right| \leqq \frac{|x|^m}{m!}.$$

La démonstration est facile, et se trouve par exemple dans [6], p. 512.

LEMME 10. *Soit a un nombre réel vérifiant* $0 \leqq a \leqq 1/6$. *Soit* a_k *une suite de coefficients vérifiant* $|a_k| \leqq a/k^2$ *pour* $k \geqq 1$. *On pose:*

$$\exp\Big(\sum_{k\geqq 1} a_k z^k\Big) = 1 + \sum_{k\geqq 1} b_k z^k.$$

Alors on a pour $k \geqq 1$:

$$|b_k| \leqq 2a/k^2.$$

DÉMONSTRATION. On pose:

$$y(z) = \exp\Big(\sum_{k\geqq 1} a z^k/k^2\Big) = 1 + \sum_{k\geqq 1} u_k z^k.$$

On a évidemment $|b_k| \leqq u_k$ pour tout $k \geqq 1$. D'autre part, y vérifie l'équation différentielle:

$$y' = ay\Big(\sum_{k\geqq 1} z^{k-1}/k\Big)$$

d'où l'on déduit, pour $n \geqq 0$

$$(n+1)u_{n+1} = \frac{a}{n+1} + \sum_{j=1}^{n} a u_j/(n-j+1).$$

Montrons par récurrence sur n, que l'on a $u_n \leqq 2a/n^2$. La relation est vérifiée pour $n=1$, puisque $u_1=a$. Supposons la vérifiée jusqu'à n $(n\geqq 1)$ et montrons la pour $n+1$. On a:

$$(n+1)u_{n+1} \leqq \frac{a}{n+1}+\sum_{j=1}^{n}\frac{2a^2}{j^2(n-j+1)} =$$

$$= \frac{a}{n+1}+2a^2\sum_{j=1}^{n}\left(\frac{1}{(n+1)j^2}+\frac{1}{(n+1)^2 j}+\frac{1}{(n+1)^2(n+1-j)}\right).$$

Or on a:

$$\frac{1}{n+1}\sum_{j=1}^{n}\frac{1}{j} \leqq \frac{1+\log n}{n+1} \leqq 0{,}6 \quad \text{et} \quad \sum_{j=1}^{n} 1/j^2 \leqq \pi^2/6 \leqq 5/3,$$

ce qui donne:

$$(n+1)^2 u_{n+1} \leqq a+\frac{10a^2}{3}+2{,}4a^2 \leqq 2a \quad \text{lorsque} \quad a \leqq 1/6.$$

Remarque. Cette condition $a\leqq 1/6$ n'est pas indispensable. En utilisant les méthodes de l'analyse complexe, *H.* Delange peut démontrer que pour tout a fixé, le coefficient u_k ci-dessus vérifie $u_k \sim ae^a/k^2$.

Lemme 11. *Soit* $n_1<n_2<\ldots<n_k$ *les longueurs distinctes des cycles de* $\sigma\in S_n$. *La valeur moyenne de* $f(\sigma)=\sum_{1\leqq i\leqq k}\log n_i$ *vérifie:*

$$M_n = \frac{1}{n!}\sum_{\sigma\in S_n} f(\sigma) = \frac{1}{2}\log^2 n+O(1).$$

Démonstration. Rappelons d'abord le résultat classique: soit k, $1\leqq k\leqq n$; le nombre de permutations de S_n qui n'ont aucun cycle de longueur j est:

$$n!\sum_{j=0}^{[n/k]}(-1)^j/(j!\,k^j).$$

Lorsque $k=1$, c'est le problème des chapeaux (cf. [3], p. 10). Il s'ensuit que, le nombre $d(n, k)$ de permutations qui ont au moins un cycle de longueur k vérifie:

$$\frac{n!}{k}\left(1-\frac{1}{2k}\right) \leqq d(n, k) \leqq n!/k.$$

On remarque ensuite que l'on a:

$$M_n = \frac{1}{n!}\sum_{1\leqq k\leqq n}(\log k)\,d(n, k) = \sum_{1\leqq k\leqq n}\frac{\log k}{k}+O(1) =$$

$$= \int_1^n \frac{\log x}{x}\,dx+O(1) = \frac{1}{2}\log^2 n+O(1).$$

1ère *étape*. Avec la notation du lemme 11, on pose:

$$F_n(x) = \mathrm{Prob}\,\{f(\sigma)-M_n < x\log^{3/2} n\}.$$

On associe à la distribution de probabilités F_n sa fonction caractéristique, définie par l'intégrale de Stieltjes:

$$\varphi_n(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF_n(x).$$

On a, d'après [5], p. 313:

$$\varphi_n(t) = \exp\left\{\frac{-itM_n}{\log^{3/2} n}\right\} \cdot \text{Coeff. de } z^n \text{ dans } \frac{1}{1-z} \exp D_n(z, \tau) \tag{4}$$

avec

$$\tau = t(\log n)^{-3/2}$$

et

$$D_n(z, \tau) = \sum_{j=2}^{n} \log\{1+(j^{i\tau}-1)(1-e^{-z^j/j})\}.$$

2ème *étape*. P. Erdős et P. Turán ont montré que pour t fixé,

$$\lim \varphi_n(t) = \exp(-t^2/6). \tag{5}$$

En vue d'une estimation du terme d'erreur dans (5), nous supposerons que $|t| \leqq \beta \sqrt{\log n}$, où β est une constante assez petite. Les majorations qui suivent, y compris les «O» seront valides pour $|t| \leqq \beta \sqrt{\log n}$ et $n \geqq n_0$, où n_0 est une constante absolue.

On écrit comme dans [5]:

$$D_n(z, \tau) = h_1(z)+h_2(z)+h_3(z)+h_4(z)$$

avec

$$h_1(z) = \sum_{j=2}^{n} \left\{i\tau \frac{\log j}{j} - \frac{\tau^2}{2} \frac{\log^2 j}{j}\right\} z^j,$$

$$h_2(z) = \sum_{j=2}^{n} \left\{\frac{j^{i\tau}-1}{j} - i\tau \frac{\log j}{j} + \frac{\tau^2}{2} \frac{\log^2 j}{j}\right\} z^j,$$

$$h_3(z) = \sum_{j=2}^{n} (j^{i\tau}-1)(1-e^{-z^j/j}-z^j/j),$$

$$h_4(z) = \sum_{j=2}^{n} \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{r} (j^{i\tau}-1)^r (1-e^{-z^j/j})^r.$$

On a, par le lemme 9, avec la notation $\ll$:

$$h_2(z) \ll \sum_{j=2}^{n} \frac{|\tau|^3 \log^3 j}{6j} z^j \ll \sum_{j=2}^{n} \frac{|t|^3}{6(\log n)^{3/2}} \frac{z^j}{j}.$$

Et, P. Erdős et P. Turán donnent:

$$h_4(z) \ll c_4 \frac{t^2}{\log n} \sum_{j=4}^{\infty} \frac{z^j}{j^2}$$

et

$$h_3(z) \ll c_3 \frac{|t|}{\sqrt{\log n}} \sum_{j=4}^{\infty} \frac{z^j}{j^2}.$$

On remarque que la condition: $n > \exp(10^4 t^2)$ est assurée en choisissant $\beta < 10^{-2}$. On peut donc écrire:

$$D_n(z, \tau) = h_1(z) + \sum_{j=2}^{\infty} a_j^{(1)} z^j \tag{6}$$

avec

$$|a_j^{(1)}| \leqq c^{(1)} \frac{|t|}{\sqrt{\log n}} \left(\frac{1}{j^2} + \frac{t^2}{j \log n} \right). \tag{7}$$

3ème *étape*. Compte tenu de (6), et en observant que les puissances de z d'exposant $> n$ n'interviennent pas, (4) devient:

$$\varphi_n(t) = \exp\left\{ -\frac{itM_n}{\log^{3/2} n} \right\} \cdot \text{Coeff. de } z^n \text{ dans}$$

$$\frac{1}{1-z} \exp\left\{ i\tau \sum_{j=2}^{\infty} \frac{\log j}{j} z^j - \frac{\tau^2}{2} \sum_{j=2}^{\infty} \frac{\log^2 j}{j} z^j \right\} \exp\left\{ \sum_{j=2}^{\infty} a_j^{(1)} z^j \right\}.$$

On utilise alors les lemmes 5 et 6, et avec la définition de la fonction h dans le lemme 8, on obtient:

$$\varphi_n(t) = \exp\left\{ \frac{-itM_n}{\log^{3/2} n} \right\} \cdot \text{Coeff. de } z^n \text{ dans } h(z) \exp\left\{ \sum_{j=2}^{\infty} a_j^{(2)} z^j \right\} \tag{8}$$

avec

$$|a_j^{(2)}| \leqq c^{(2)} \left(\frac{|t|}{j^2 \sqrt{\log n}} + \frac{|t^3| + |t|}{j \log^{3/2} n} \right),$$

pour $2 \leqq j \leqq n$.

4ème *étape*. On pose:

$$\begin{cases} \exp\left(\sum_{j=2}^{\infty} a_j^{(2)} z^j \right) = 1 + \sum_{j=2}^{\infty} a_j^{(3)} z^j, \\ \exp\left(\sum_{j=2}^{\infty} \frac{c^{(2)} |t|}{j^2 \sqrt{\log n}} z^j \right) = 1 + \sum_{j=2}^{\infty} b_j^{(1)} z^j, \\ a = a(t, n) = \dfrac{c^{(2)} (|t| + |t^3|)}{\log^{3/2} n}. \end{cases} \tag{9}$$

(On impose à β d'être assez petit de façon à avoir $a \leqq 1/3$)

$$\exp\left(\sum_{j=2}^{\infty} a(t, n) \frac{z^j}{j} \right) = 1 + \sum_{j=2}^{\infty} b_j^{(2)} z^j,$$

$$\left(1 + \sum_{j=2}^{\infty} b_j^{(1)} z^j\right)\left(1 + \sum_{j=2}^{\infty} b_j^{(2)} z^j\right) = 1 + \sum_{j=2}^{\infty} b_j z^j.$$

On aura alors, pour $2 \leqq j \leqq n$,

$$|a_j^{(3)}| \leqq b_j.$$

Il reste à estimer b_j. D'après le lemme 7, on aura:

$$b_j^{(2)} \leqq e^{1/3} a j^{a-1} \leqq 2a j^{a-1}$$

et d'après le lemme 10, en choisisant $\beta < 1/6c^{(2)}$,

$$b_j^{(1)} \leqq \frac{2c^{(2)}|t|}{\sqrt{\log n}\, j^2}.$$

Il s'ensuit que:

$$b_j \leqq b_j^{(1)} + b_j^{(2)} + \sum_{r=2}^{j-2} 4c^{(2)} \frac{|t|}{\sqrt{\log n}\, r^2} \frac{a}{(j-r)^{1-a}}.$$

On coupe la somme en deux, suivant que $r \leqq j/2$ ou non. On a:

$$\sum_{2 \leqq r \leqq j/2} \frac{a}{r^2 (j-r)^{1-a}} = O(aj^{a-1})$$

et

$$\sum_{j/2 < r \leqq j-2} \frac{a}{r^2 (j-r)^{1-a}} \leqq \frac{4}{j^2} \sum_{r=2}^{[j/2]} ar^{a-1} = O(j^{-5/3}).$$

On a donc, pour $2 \leqq j \leqq n$:

$$a_j^{(3)} = O\left(\frac{|t|}{\sqrt{\log n}} j^{-5/3} + aj^{-1+a}\right)$$

d'où l'on déduit:

(10) $$a_j^{(3)} = O(j^{-2/3}) \quad (2 \leqq j \leqq n)$$

et

(11) $$\sum_{j=2}^{n} |a_j^{(3)}| = O\left(\frac{|t|}{\sqrt{\log n}} + \exp\left\{c^{(2)} \frac{|t| + |t^3|}{\sqrt{\log n}}\right\} - 1\right).$$

5e *étape*. Compte tenu de (9), et avec les notations du lemme 8, (8) devient:

$$\varphi_n(t) = \exp\left\{\frac{-itM_n}{\log^{3/2} n}\right\}\left(e_n + \sum_{j=2}^{n-1} a_j^{(3)} e_{n-j} + a_n^{(3)}\right).$$

Le même calcul que dans [10] donne, en tenant compte du lemme 8, de (10) et de (11):

(12)

$$\varphi_n(t) = \exp(-t^2/6) + (\exp(-t^2/48))O\left(\frac{|t|}{\sqrt{\log n}} + \exp\left(c^{(2)} \frac{|t| + |t^3|}{\sqrt{\log n}}\right) - 1\right) + O(n^{-1/6}).$$

6e *étape*. On pose:

$$F(x) = \sqrt{\frac{3}{2\pi}} \int_{-\infty}^{x} e^{-(3/2)u^2}\, du.$$

Sa fonction caractéristique vaut:

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x) = e^{-t^2/6}$$

et l'on a la formule (cf. [Fel], p. 538):

(13) $$|F_n(x) - F(x)| \leqq \frac{1}{\pi} \int_{-T}^{T} \left|\frac{\varphi_n(t) - \varphi(t)}{t}\right| dt + \frac{24 F'(0)}{\pi T}$$

valable pour tout x réel et tout $T>0$. On choisit $T=\beta\sqrt{\log n}$ avec β assez petit. Le même calcul que dans [10] permet de déduire des formules (12) et (13) que l'on a, uniformément en x:

$$F_n(x)-F(x) = O\left(1/\sqrt{\log n}\right). \tag{14}$$

La démonstration du théorème 2 découle alors de (14), puisque l'on a:

$$\operatorname{Prob}\left\{\frac{f(\sigma)-(1/2)\log^2 n}{\left(1/\sqrt{3}\right)\log^{3/2} n} < x\right\} = F_n(y)$$

avec

$$y = x/\sqrt{3}+\left((1/2)\log^2 n - M_n\right)/\log^{3/2} n = x/\sqrt{3}+O\left(\log^{-3/2} n\right)$$

par le lemme 11.

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ON AN INTERPOLATIONAL PROCESS WITH APPLICATIONS TO FOURIER SERIES

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1. Let $\tilde{S}_n[g]$ denote the trigonometrical polynomial of degree at most n that interpolates the function g of period 2π at $m=2n+1$ equidistant nodes

$$t_i = \tau + \frac{2i\pi}{m} \quad (i = 0, \pm 1, \pm 2, \ldots). \tag{1}$$

Following Kis-Névai [7] we consider the polynomials

$$\tilde{S}_{kn}[g](t) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \tilde{S}_n[g]\left(t + \frac{k-2j}{m}\pi\right), \tag{2}$$

which for $k=1$ and $k=2$ were first introduced by S. N. Bernstein [1]. If not noted otherwise we take $m>k$ so that the arguments of $\tilde{S}_n[g]$ in (2) lie within a period of length 2π.

In [7] the expressions

$$\tilde{M}_{kn}(t) = \sup_{\substack{g\in C_{2\pi} \\ g\not\equiv \text{const.}}} \frac{|\tilde{S}_{kn}[g](t) - g(t)|}{\omega\left(g, \frac{2\pi}{m}\right)}, \quad c_{kn} = \sup_{-\infty<t<\infty} \tilde{M}_{kn}(t), \tag{3}$$

were investigated. Similarly, we define

$$\tilde{R}_{kn}(t) = \sup_{g\in \mathrm{Lip}_M 1} \frac{|\tilde{S}_{kn}[g](t) - g(t)|}{M\frac{\pi}{m}}, \quad \delta_{kn} = \sup_{-\infty<t<\infty} \tilde{R}_{kn}(t), \tag{4}$$

where $g\in\mathrm{Lip}_M 1$ means $|g(x)-g(y)|\leqq M|x-y|$ for all $x, y\in\mathbf{R}$. Without loss of generality we can choose $\tau=0$ in (1)—(4). Similarly to Theorem 1 in [7] we have

$$\tilde{R}_{kn}(-t) = \tilde{R}_{kn}(t), \quad \tilde{R}_{kn}(t) = \tilde{R}_{kn}\left(t + \frac{2\pi}{m}\right).$$

Therefore it suffices to study $\tilde{R}_{kn}$ for $0\leqq t\leqq \pi/m$ only. $\tilde{S}_{kn}[g]$ can be written in the form

$$\tilde{S}_{kn}[g](t) = \sum_{i=-n}^{n} g(t_i)s_i(t), \tag{5}$$

where

(6) $$\begin{cases} s_i(t) = \dfrac{1}{2^k} \sum\limits_{j=0}^{k} \dbinom{k}{j} d_n\left(t + \dfrac{k-2j}{m}\pi - t_i\right), \\ d_n(t) = \dfrac{1}{m} + \dfrac{2}{m} \sum\limits_{i=1}^{n} \cos it = \dfrac{1}{m} \sin \dfrac{mt}{2} \operatorname{cosec} \dfrac{t}{2}. \end{cases}$$

Let finally

(7) $$\begin{cases} \sigma_i(t) = \sum\limits_{j=-n}^{i} s_j(t) \quad (i = -n, -n+1, \dots, 0), \\ \sigma_i(t) = \sum\limits_{j=i}^{n} s_j(t) \quad (i = 1, 2, \dots, n). \end{cases}$$

We get from [3] as a special case

$$\tilde{R}_{on}(t) \leqq \sum_{i=-n}^{n} |d_n(t-t_i)|.$$

To complete the proof in [3] we first verify

THEOREM 1. $\delta_{on} = \sup\limits_{-\infty<t<\infty} \sum\limits_{i=-n}^{n} |d_n(t-t_i)|.$

In [4] it was shown that

$$\delta_{on} = \frac{2}{\pi} \ln m + C + \varepsilon_{on}, \quad 0 < \varepsilon_{on} < \frac{\pi}{72m^2}, \quad C = \frac{2}{\pi}\left(\gamma + \ln \frac{8}{\pi}\right) = 0{,}9625\dots,$$

($\gamma = 0{,}5772\dots$ denotes Euler's constant).

Now let $k=1$. As a main result of this paper we prove

THEOREM 2. *For* $0 \leqq t \leqq \pi/m$, $n > 0$, *we have*

$$\tilde{R}_{1n}(t) = 1 + \frac{m}{\pi} t \sum_{i=1}^{n} \left[d_n\left(t + \frac{2i-1}{m}\pi\right) - d_n\left(t - \frac{2i+1}{m}\pi\right)\right] + \left(\frac{m}{\pi} t - (-1)^n\right) d_n(t-\pi),$$

$$\tilde{R}_{1n}(t) = 1 + \frac{4}{\pi} t \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \sin(2j-1)\left(\frac{\pi}{m} - t\right) \operatorname{ctg} \frac{2j-1}{m}\pi + r_n,$$

$$-\frac{2}{n} < r_n < 0, \quad \left(\tilde{R}_{10}(t) = \frac{m}{\pi} t\right).$$

Now the same methods as in [5] can be used to estimate δ_{1n}. We first notice that

(8) $$\delta_{1n} = \delta + O\left(\frac{1}{n}\right), \quad \delta > 1 + \frac{1}{\pi}(\sqrt{2} - \ln(1+\sqrt{2})) = 1{,}16960\dots.$$

It is somewhat difficult to get a precise upper bound. We show

THEOREM 3. $\delta_{1n} = \delta + \varepsilon_{1n}$, $\delta = 1{,}16\,968\dots$, $\varepsilon_{1n} < 1/n^2$. (δ *coincides with the constant* c *in* [5]).

We mention that $\delta_{10}=1$, $\delta_{11}=1$, $\delta_{12}=1{,}058\ldots$, $\delta_{13}=1{,}080\ldots$, $\delta_{14}=1{,}100\ldots$. For $k>1$ we get

THEOREM 4. $\delta_{2n}>1{,}36-\frac{1-2/\pi}{m^2}$, $\delta_{kn}\geqq\frac{3}{2}$ $(k>2)$, $(\delta_{20}=1)$.

As a corollary we can conclude that $k=1$ is the best choice for k.

2. It is natural to extend some of the results obtained in Section 1 to Fourier sums and Bernstein—Rogosinski means of Fourier sums. We denote by $S_n[g]$ the n^{th} partial sum of the trigonometric Fourier series of $g\in C_{2\pi}$, which may be represented in the form

$$S_n[g](x)=\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)D_n(x-t)\,dt,\quad D_n(z)=\frac{m}{2}d_n(z). \tag{9}$$

Similarly to (2) and (3) we have

$$S_{kn}[g](x)=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}S_n[g]\left(x+\frac{k-2j}{m}\pi\right), \tag{10}$$

$$M_{kn}(x)=\sup_{\substack{g\in C_{2\pi}\\ g\not\equiv\text{const.}}}\frac{|S_{kn}[g](x)-g(x)|}{\omega\left(g,\frac{2\pi}{m}\right)}.$$

It is well known that M_{kn} does not depend on x thus $M_{kn}(x)=M_{kn}(0)=M_{kn}$. This is valid also for

$$R_{kn}(x)=\sup_{g\in \mathrm{Lip}_M 1}\frac{|S_{kn}[g](x)-g(x)|}{M\frac{\pi}{m}}. \tag{11}$$

As pointed out in [7] we have for $k=0,1,2,\ldots$

$$M_{kn}\leqq\frac{m}{\pi}\int_0^{\pi/m}\tilde{M}_{kn}(t)\,dt,\quad R_{kn}\leqq\frac{m}{\pi}\int_0^{\pi/m}\tilde{R}_{kn}(t)\,dt. \tag{12}$$

From this we get

THEOREM 5. $R_{on}\leqq\frac{1}{\pi}\int_{-\pi}^{\pi}|D_n(t)|dt$.

It is known that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|D_n(t)|\,dt=\frac{4}{\pi^2}\ln m+O(1).$$

Finally we state

THEOREM 6. $R_{1n}\leqq\gamma+\gamma_n$, $\gamma=1{,}11\,261\ldots$, $\gamma_n<1/n^2$.

We mention that $R_{10}=0{,}5$, $R_{11}\leqq0{,}893$, $R_{12}\leqq0{,}989$, $R_{13}\leqq1{,}022$, $R_{14}\leqq1{,}043$.

In the following sections we will prove the results.

3. Proof of Theorem 1. Since the proof for $n=0$ is easy we may assume $n>0$. We start with (5) and [7] (23). By Abel transformation we obtain from (7)

$$\tilde{S}_{kn}[g](t)-g(t)=\sum_{i=-n}^{-1}\{g(t_i)-g(t_{i+1})\}\sigma_i(t)+ \tag{13}$$

$$+\{g(t_0)-g(t)\}\sigma_0(t)+\{g(t_1)-g(t)\}\sigma_1(t)+\sum_{i=2}^{n}\{g(t_i)-g(t_{i-1})\}\sigma_i(t).$$

Let $k=0$. In [3] it was proved that

$$\tilde{R}_{on}(t)\leqq\sum_{i=-n}^{n}|d_n(t-t_i)|. \tag{14}$$

It is known from [2] that for $t'=(1/2)(t_0+t_1)$ we have

$$\sum_{i=-n}^{n}|d_n(t'-t_i)|=\sup_{-\infty<t<\infty}\sum_{i=-n}^{n}|d_n(t-t_i)|.$$

Thus in order to verify Theorem 1 it suffices to construct a function $g\in\mathrm{Lip}_M 1$ such that

$$\tilde{S}_{on}[g](t')-g(t')=\frac{M\pi}{m}\sum_{i=-n}^{n}|d_n(t'-t_i)|. \tag{15}$$

We choose a 2π-periodic function g, $g(t')=M\pi/m$, and for $-n\leqq i\leqq n$

$$g(t_i)=\begin{cases}2M\pi/m, & i=0,-2,-4,\dots \quad\text{and}\quad i=1,3,5,\dots\\ 0, & i=-1,-3,-5,\dots \quad\text{and}\quad i=2,4,6,\dots\end{cases}$$

(g is linear in the remaining intervals). By virtue of (13) and Lemma 2 in [7] this yields

$$\tilde{S}_{on}[g](t')-g(t')=\frac{M\pi}{m}\Big\{2\sum_{\substack{i=-n\\ i\neq 0,1}}^{n}|\sigma_i(t')|+1\Big\}=\frac{M\pi}{m}\Big\{2\sum_{i=-n}^{n}|\sigma_i(t')|-1\Big\}.$$

Now applying [7] (11) and (9) we get (15) which was to be proved.

4. Proof of Theorem 2. Let $k=1$, $n>0$. From (13) we immediately get for $g\in\mathrm{Lip}_M 1$, $0\leqq t\leqq\pi/m$,

$$|\tilde{S}_{1n}[g](t)-g(t)|\leqq\frac{2M\pi}{m}\sum_{\substack{i=-n\\ i\neq 0,1}}^{n}|\sigma_i(t)|+Mt\sigma_0(t)+M\left(\frac{2\pi}{m}-t\right)\sigma_1(t). \tag{16}$$

From [7] (37) we conclude that

$$\sum_{\substack{i=-n\\ i\neq 0,1}}^{n}|\sigma_i(t)|=\sum_{1\leqq|i|\leqq n}|\sigma_i(t)|-|\sigma_1(t)|=\frac{1}{2}-\sigma_1(t)-\frac{1}{2}(-1)^n d_n(t-\pi). \tag{17}$$

Further we infer from $\sigma_0+\sigma_1\equiv 1$

$$t\sigma_0(t)+\left(\frac{2\pi}{m}-t\right)\sigma_1(t)=\left(\frac{1}{2}-\sigma_1(t)\right)\left(2t-\frac{2\pi}{m}\right)+\frac{\pi}{m}. \tag{18}$$

Now (17) and (18) can be used to transform (16) to

$$|\tilde{S}_{1n}[g](t)-g(t)| \leqq \frac{M\pi}{m}\left\{1+\frac{2m}{\pi}t\left(\frac{1}{2}-\sigma_1(t)\right)-(-1)^n d_n(t-\pi)\right\}, \tag{19}$$

$0\leqq t\leqq \pi/m$. Let us now construct a 2π-periodic function $g\in \mathrm{Lip}_M 1$ such that equality holds in (16) and (19). We can choose a piecewise linear function g satisfying

$$g(t_i) = \begin{cases} \dfrac{2M\pi}{m} & \text{for } i = 0, -2, -4, \ldots \\ \dfrac{4M\pi}{m} & \text{for } i = -1, -3, -5, \ldots \end{cases} \qquad (-n \leqq i),$$

$$g(t) = M\left(\frac{2\pi}{m}-t\right),$$

$$g(t_i) = \begin{cases} M\left(\dfrac{4\pi}{m}-2t\right) & \text{for } i = 1, 3, 5, \ldots \\ M\left(\dfrac{2\pi}{m}-2t\right) & \text{for } i = 2, 4, 6, \ldots, \end{cases} \qquad i \leqq n.$$

Now starting with (13) and [1], Lemma 2, it is easy to verify that equality holds in (16) and (19). Using $1/2-\sigma_1=\sigma_0-1/2$ this yields

$$\tilde{R}_{1n}(t) = 1+\frac{2m}{\pi}t\left(\sigma_0(t)-\frac{1}{2}\right)-(-1)^n d_n(t-\pi). \tag{20}$$

From [7] (34) we see that

$$\sigma_0(t)-\frac{1}{2} = \frac{1}{2}\left\{\sum_{i=1}^{n}\left[d_n\left(t+\frac{2i-1}{m}\pi\right)-d_n\left(t-\frac{2i+1}{m}\pi\right)\right]+d_n(t-\pi)\right\}.$$

The first part of Theorem 2 now follows from this and (20). To prove the second part it is useful to introduce some abbreviations, $R_{1n}=1+\alpha+\beta$,

$$\alpha(t) = \frac{mt}{\pi}\sum_{i=1}^{n}\left[d_n\left(t+\frac{2i-1}{m}\pi\right)-d_n\left(t-\frac{2i+1}{m}\pi\right)\right], \tag{21}$$

$$\beta(t) = \left(\frac{mt}{\pi}-(-1)^n\right)d_n(t-\pi). \tag{22}$$

From $d_n(t)=\frac{1}{m}+\frac{2}{m}\sum_{j=1}^{n}\cos jt$ it follows that

$$d_n\left(t+\frac{2i-1}{m}\pi\right)-d_n\left(t-\frac{2i+1}{m}\pi\right) =$$

$$= \frac{2}{m}\sum_{j=1}^{n}\left[\cos j\left(t+\frac{2i-1}{m}\pi\right)-\cos j\left(t-\frac{2i+1}{m}\pi\right)\right] =$$

$$= \frac{4}{m}\sum_{j=1}^{n}\sin j\left(\frac{\pi}{m}-t\right)\sin\frac{2ij\pi}{m},$$

thus we have

(23)

$$\sum_{i=1}^{n}\left[d_n\left(t+\frac{2i-1}{m}\pi\right)-d_n\left(t-\frac{2i+1}{m}\pi\right)\right]=\frac{4}{m}\sum_{j=1}^{n}\sin j\left(\frac{\pi}{m}-t\right)\left\{\sum_{i=1}^{n}\sin\frac{2ij\pi}{m}\right\}.$$

Using

$$\sum_{i=1}^{n}\sin i\frac{2j\pi}{m}=-\frac{1}{2\sin\frac{j\pi}{m}}\left[\cos j\pi-\cos\frac{j\pi}{m}\right],$$

$$\cos j\pi-\cos\frac{j\pi}{m}=-2\sin\left(\frac{j\pi}{2}+\frac{j\pi}{2m}\right)\sin\left(\frac{j\pi}{2}-\frac{j\pi}{2m}\right)=\begin{cases}2\sin^2\frac{j\pi}{2m}, & j \text{ even}\\ -2\cos^2\frac{j\pi}{2m}, & j \text{ odd}\end{cases}$$

and $2\sin\frac{j\pi}{m}=4\sin\frac{j\pi}{2m}\cos\frac{j\pi}{2m}$, we obtain

$$\sum_{i=1}^{n}\sin\frac{2ij\pi}{m}=\begin{cases}-\frac{1}{2}\tan\frac{j\pi}{2m}, & j=2,4,6,\dots\\ \frac{1}{2}\operatorname{ctg}\frac{j\pi}{2m}, & j=1,3,5,\dots.\end{cases}\tag{24}$$

Let n be even. From

$$\sin(2\nu-1)\left(\frac{\pi}{m}-t\right)\frac{1}{2}\operatorname{ctg}\frac{2\nu-1}{2m}\pi-\sin 2\nu\left(\frac{\pi}{m}-t\right)\frac{1}{2}\tan\frac{2\nu}{2m}\pi\leqq$$

$$\leqq\sin(2\nu-1)\left(\frac{\pi}{m}-t\right)\frac{1}{2}\operatorname{ctg}\frac{2\nu-1}{2m}\pi-\sin(2\nu-1)\left(\frac{\pi}{m}-t\right)\frac{1}{2}\tan\frac{2\nu-1}{2m}\pi=$$

$$=\sin(2\nu-1)\left(\frac{\pi}{m}-t\right)\operatorname{ctg}\frac{2\nu-1}{m}\pi,\quad 0\leqq t\leqq\frac{\pi}{m},\quad 1\leqq 2\nu-1\leqq n,$$

and by virtue of (21), (23) and (24) we get

$$\alpha(t)\leqq\frac{4}{\pi}t\sum_{\nu=1}^{n/2}\sin(2\nu-1)\left(\frac{\pi}{m}-t\right)\operatorname{ctg}\frac{2\nu-1}{m}\pi\quad(n\text{ even}).\tag{25}$$

Let n be odd. The same methods now yield

(26)

$$\alpha(t)\leqq\frac{4}{\pi}t\sum_{\nu=1}^{\frac{n+1}{2}}\sin(2\nu-1)\left(\frac{\pi}{m}-t\right)\operatorname{ctg}\frac{2\nu-1}{m}\pi+\frac{2}{\pi}t\sin n\left(\frac{\pi}{m}-t\right)\tan\frac{n\pi}{2m}.$$

But for odd n we have

$$\beta(t)=-\left(\frac{mt}{\pi}+1\right)\frac{1}{m}\cos\frac{m}{2}t\sec\frac{t}{2}\leqq-\frac{2}{\pi}t\cos\left(nt+\frac{t}{2}\right).$$

It follows that

$$\frac{2}{\pi} t \sin n\left(\frac{\pi}{m}-t\right) \tan \frac{n\pi}{2m} + \beta(t) \leqq 0,$$

if we consider $\tan (n\pi/(2m)) \leqq 1$ and

$$\sin n\left(\frac{\pi}{m}-t\right) = \cos\left(nt+\frac{\pi}{2m}\right) \leqq \cos\left(nt+\frac{t}{2}\right).$$

It can easily be checked that $\beta(t) \leqq 0$, for n even. Thus we conclude from (25) and (26)

$$\text{(27)} \quad \tilde{R}_{1n}(t) \leqq 1+\frac{4}{\pi} t \sum_{\nu=1}^{\left[\frac{n+1}{2}\right]} \sin(2\nu-1)\left(\frac{\pi}{m}-t\right) \operatorname{ctg} \frac{2\nu-1}{m}\pi, \quad (n = 1, 2, 3, \ldots).$$

To get a lower bound we proceed like this. Let n be even in (23) and (24). Estimating in the following manner

$$-\sin 2\nu\left(\frac{\pi}{m}-t\right)\frac{1}{2}\tan\frac{2\nu\pi}{2m}+\sin(2\nu+1)\left(\frac{\pi}{m}-t\right)\frac{1}{2}\operatorname{ctg}\frac{2\nu+1}{2m}\pi \geqq$$

$$\geqq -\sin(2\nu+1)\left(\frac{\pi}{m}-t\right)\frac{1}{2}\tan\frac{2\nu+1}{2m}\pi+\sin(2\nu+1)\left(\frac{\pi}{m}-t\right)\frac{1}{2}\operatorname{ctg}\frac{2\nu+1}{2m}\pi =$$

$$= \sin(2\nu+1)\left(\frac{\pi}{m}-t\right)\operatorname{ctg}\frac{2\nu+1}{m}\pi, \quad 0 \leqq t \leqq \frac{\pi}{m}, \quad 1 \leqq 2\nu+1 \leqq n,$$

we get from (21), (23) and (24)

$$\alpha(t) \geqq \frac{4}{\pi} t \sum_{j=1}^{n/2} \sin(2j-1)\left(\frac{\pi}{m}-t\right)\operatorname{ctg}\frac{2j-1}{m}\pi - \frac{2}{\pi} t \sin n\left(\frac{\pi}{m}-t\right)\tan\frac{n\pi}{2m} +$$

$$+\frac{2}{\pi} t \sin\left(\frac{\pi}{m}-t\right)\tan\frac{\pi}{2m}.$$

We simply have for $n>0$,

$$-\frac{2}{\pi} t \sin n\left(\frac{\pi}{m}-t\right)\tan\frac{n\pi}{2m} > -\frac{1}{n},$$

$$\beta(t) = \left(\frac{m}{\pi}t-1\right)\frac{1}{m}\cos\frac{mt}{2}\sec\frac{t}{2} > -\frac{1}{n}.$$

By the same methods we obtain for n odd,

$$\alpha(t) \geqq \frac{4}{\pi} t \sum_{j=1}^{\frac{n+1}{2}} \sin(2j-1)\left(\frac{\pi}{m}-t\right)\operatorname{ctg}\frac{2j-1}{m}\pi + \frac{2}{\pi} t \sin\left(\frac{\pi}{m}-t\right)\tan\frac{\pi}{2m},$$

$$\beta(t) \geqq -2\frac{1}{m}\cos\frac{mt}{2}\sec\frac{t}{2} > -\frac{2}{n}.$$

Thus we have proved the second part of Theorem 2.

5. Proof of Theorem 3. By Theorem 2 and the well known series expanonsi $\sin x = \sum_{\nu=1}^{\infty} a_\nu x^\nu$ we transform $\tilde{R}_{1n}(t)$ to

(28)
$$\tilde{R}_{1n}(t) = 1+\frac{4}{\pi}\frac{mt}{2}\sum_{\nu=1}^{\infty} a_\nu\left(\frac{\pi}{2}-\frac{mt}{2}\right)^\nu \left[\frac{2^\nu}{\pi^{\nu+1}}\sum_{j=1}^{\left[\frac{n+1}{2}\right]}\frac{2\pi}{m}\left(\frac{2j-1}{m}\pi\right)^\nu \operatorname{ctg}\frac{2j-1}{m}\pi\right]+r_n.$$

The sum in brackets can be interpreted as a Riemann sum of $f_\nu(x)=x^\nu \operatorname{ctg} x$ with step $h=2\pi/m$. Thus the same methods as in [5] can be used to prove Theorem 3 and the first statement of (8). In addition we remark that the final substitution $z=mt/2$, $0\leqq t\leqq \pi/m$, leads to an expression of the form

$$\tilde{R}_{1n}(t) = f(z)+f_n(z), \quad 0\leqq z\leqq \frac{\pi}{2}, \tag{29}$$

$f_n(z)=O\left(\frac{1}{n}\right)$, $f_n(z)<1/n^2$. Unfortunately f does not reach its maximum at the mid-point $z=\pi/4$, resp. $t=\pi/(2m)$, thus complicating the evaluation of $\delta_{1n}=\sup_t \tilde{R}_{1n}(t)$. Simply substituting $t=\pi/(2m)$ in Theorem 2 yields

$$\delta_{1n} > \tilde{R}_{1n}\left(\frac{\pi}{2m}\right) = 1+\frac{2}{\pi}\sum_{j=1}^{\left[\frac{n+1}{2}\right]}\frac{\pi}{m}\sin(2j-1)\frac{\pi}{2m}\operatorname{ctg}\frac{2j-1}{m}\pi+r_n.$$

Now $\sin x \operatorname{ctg} 2x = \sin x\left(\frac{1}{2}\operatorname{ctg} x-\frac{1}{2}\tan x\right)=\cos x-\frac{1}{2}\sec x$, thus

$$\tilde{R}_{1n}\left(\frac{\pi}{2m}\right) \approx 1+\frac{2}{\pi}\int_0^{\pi/4}\cos x\,dx-\frac{1}{\pi}\int_0^{\pi/4}\sec x\,dx = 1+\frac{1}{\pi}\left(\sqrt{2}-\ln\left(1+\sqrt{2}\right)\right),$$

which proves the second statement of (8).

6. Proof of Theorems 4—6. For $k=2$ consider a piecewise linear function g, $g\left(\frac{\pi}{m}\right)=\frac{M\pi}{m}$, satisfying

$$g(t_i) = \begin{cases} 2M\pi/m, & i=1,3,5,\dots \text{ and } i=0,-2,-4,\dots \\ 4M\pi/m, & i=2,4,6,\dots \text{ and } i=-1,-3,-5,\dots \end{cases} \quad |i|\leqq n.$$

Applying (13) and [7] Lemma 2 we get

$$\tilde{S}_{2n}[g]\left(\frac{\pi}{m}\right)-g\left(\frac{\pi}{m}\right) = \frac{M\pi}{m}\left(1+2\sum_{\substack{i=-n\\ i\neq 0,1}}^{n}\left|\sigma_i\left(\frac{\pi}{m}\right)\right|\right).$$

As shown in [7] (44) we have

$$\sum_{\substack{i=-n\\ i\neq 0,1}}^{n}\left|\sigma_i\left(\frac{\pi}{m}\right)\right| = \frac{1}{2}-\frac{1}{4}d_n\left(\frac{\pi}{m}\right)-\frac{1}{4}d_n\left(-\frac{\pi}{m}\right) = \frac{1}{2}\left(1-d_n\left(\frac{\pi}{m}\right)\right).$$

Finally it suffices to mention that

$$d_n\left(\frac{\pi}{m}\right) \leqq \frac{2}{\pi}+\frac{1-\frac{2}{\pi}}{m^2},$$

which follows easily from (6) and [7] (41).

To prove the second part of Theorem 4 we let k be odd. From (2) we immediately get for $g\in \mathrm{Lip}_M 1$,

$$\tilde{S}_{kn}[g]\left(\frac{\pi}{m}\right)=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\tilde{S}_n[g]\left(\frac{k+1-2j}{m}\pi\right)=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}g\left(\frac{k+1-2j}{m}\pi\right),$$

$$\left|\tilde{S}_{kn}[g]\left(\frac{\pi}{m}\right)-g\left(\frac{\pi}{m}\right)\right|=\frac{1}{2^k}\left|\sum_{j=0}^{k}\binom{k}{j}\left[g\left(\frac{k+1-2j}{m}\pi\right)-g\left(\frac{\pi}{m}\right)\right]\right|\leqq$$

$$\leqq\frac{M\pi}{m}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}|k-2j|.$$

It is easy to construct a function g such that equality holds $(m>k)$. Evaluating the last sum as in the proof of Theorem 2 in [6] we get

$$\delta_{2\nu-1,n}\geqq\frac{2\nu}{2^{2\nu}}\binom{2\nu}{\nu}.$$

Similarly we can obtain for k even,

$$|\tilde{S}_{kn}[g](0)-g(0)|=\frac{M\pi}{m}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}|k-2j|,\quad \delta_{2\nu,n}\geqq\frac{2\nu}{2^{2\nu}}\binom{2\nu}{\nu}.$$

As it can be shown by induction, the expression

$$\eta_\nu=\frac{2\nu}{2^{2\nu}}\binom{2\nu}{\nu},\quad \nu=1,2,3,\ldots$$

as a function of ν is monotonously increasing, thus

$$\delta_{kn}\geqq\eta_2=\frac{3}{2},\quad k\geqq 3,$$

which completes the proof of Theorem 4.

By virtue of (12), (14) and (9) we easily deduce Theorem 5, integrating (28), resp. (29) we obtain Theorem 6.

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ON THE MAXIMUM RANK OF A TENSOR PRODUCT

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1. Introduction

The tensor product $V=V_1\otimes V_2\otimes\ldots\otimes V_m$ of the real vector spaces $V_1, V_2, \ldots, V_m$ of dimensions $n_1, n_2, \ldots, n_m$, respectively, contains some special elements called decomposable tensors of the form:

$$v_1\otimes v_2\otimes, \ldots \otimes, v_m \quad \text{with} \quad v_i\in V_i, \quad \text{for} \quad i=1,2,\ldots,m.$$

In general each tensor $t\in V$ has a representation as a sum of such elements:

$$t=\sum_{i=1}^{k} v_1^{(i)}\otimes v_2^{(i)}\otimes\ldots\otimes v_m^{(i)}.$$

The minimal k for which t has such a representation is called the *rank* of t and is denoted by $r(t)$. The maximum value of $r(t)$ for all $t\in V$ is called the *maximum rank of* V and is denoted by $r(n_1, n_2, \ldots, n_m)$. No general method for the computation of either $r(t)$ or $r(n_1, n_2, \ldots, n_m)$ is known. In the case of a product $V=V_1\otimes V_2$ of two spaces, tensors t can be identified with matrices and $r(t)$ coincides with the usual rank of the corresponding matrix. Also $r(n_1, n_2)=\min(n_1, n_2)$.

In the case of a product $V=V_1\otimes V_2\otimes V_3$ of three vector spaces the situation is radically different. Some lower bounds for the rank $r(t)$ of a tensor can be found, but such simple tensors as the one that performs the matrix multiplication are of unknown rank ([1], [3]). Some of the numbers $r(n_1, n_2, n_3)$ for small values of n_1, n_2, n_3 have been calculated. The computations involved are quite formidable and remain unpublished. It has been asserted e.g. that

$$r(2,2,2)=3, \quad r(3,3,3)=5, \quad r(3,3,4)=6, \quad r(3,3,5)=6 \quad \text{or} \quad 7.$$

See [2] and the references given therein. Several results bearing to special triples of integers are also known ([2], [3]), e.g.

$$r(m,n,2)=\min(2m, 2n, m+[n/2], n+[m/2]),$$

$$r(m,n,mn-k)=mn-k^2+r(k,k,k^2-k).$$

It is conjectured ([2]) that $r(m,n,nm-k)=mn-[k/2]$, for $m,n\geqq k$. There are also some inequalities ([2], [3]) such as:

$$r(n,n,3)\leqq 2n, \quad r(m,n,p)\leqq m+[p/2]n, \quad \text{if} \quad m\leqq n,$$

$$r(n,n,p)\leqq (p+1)n/2, \quad r(m,n,p)\geqq mnp/(m+n+p-2).$$

I could not find in the literature analogous results for products of more than three spaces. In this note I generalize the last inequality for an arbitrary tensor product and prove that

$$r(n_1, n_2, \dots, n_m) \geqq n_1 n_2 \dots n_m/(n_1+n_2+\dots+n_m-m+1).$$

The principal role in my proof is played by the manifold M_1 of decomposable elements of V discussed in §2, and a certain construction, which imitates the Stiefel manifold $S_k(R^n)$ of k-frames in R^n and is discussed in §3. For general definitions, elementary facts and notations I refer to Marcus [4], Brickell and Clark [5], and Hirsch [6].

2. The manifold M_1 of decomposable tensors

Without affecting the generality we can suppose that each V_i carries an inner product $\langle \dots, \dots \rangle_i$. The linear isometries of this inner product constitute the orthogonal group $O(V_i)$, isomorphic to $O(n_i)$, the group of orthogonal $n_i \times n_i$ matrices. On V we can then define an inner product:

$$\langle v_1 \otimes \dots \otimes v_m, w_1 \otimes \dots \otimes w_m \rangle = \prod_{i=1}^{m} \langle v_i, w_i \rangle,$$

for all decomposable tensors. For the remaining tensors the product is defined by linear extension. Let now M_1 denote the set of decomposable tensors of length 1 with respect to $\langle \dots, \dots \rangle$. It is easy to see that M_1 is identical with the set of tensors $v_1 \otimes \dots \otimes v_m$ with $\|v_1\| = \|v_2\| = \dots = \|v_m\| = 1$. The group product

$$G = O(V_1) \times O(V_2) \times \dots \times O(V_m)$$

operates on V in a natural way via the tensor product of linear maps

$$(A_1, A_2, \dots, A_m)(v_1 \otimes v_2 \otimes \dots \otimes v_m) = (A_1 v_1) \otimes \dots \otimes (A_m v_m)$$

for all decomposable tensors. For the remaining tensors the definition is extended linearly. Obviously this action is transitive on the set M_1. Using orthogonal coordinates in each V_i one can easily calculate the isotropy group on an element $v_1 \otimes \dots \otimes V_m \in M_1$. This group is isomorphic to the group of matrices

$$G_0 = \left\{ \bigotimes_{i=1}^{m} \begin{pmatrix} \varepsilon_i & 0 \\ 0 & Y_i \end{pmatrix} \middle|\, Y_i \in O(n_i - 1),\ \varepsilon_i = \pm 1 \text{ for } i = 1, \dots, m,\ \varepsilon_1 \varepsilon_2 \dots \varepsilon_m = 1 \right\}.$$

Then according to well known theorems ([5], Proposition 13.3.1, p. 250) M_1 is a C^∞-differentiable submanifold of V of dimension

$$\dim(M_1) = \dim(G) - \dim(G_0) = n_1(n_1-1)/2 + n_2(n_2-1)/2 + \dots + n_m(n_m-1)/2 -$$
$$-(n_1-1)(n_2-2)/2 - \dots - (n_m-1)(n_m-2)/2 = n_1+n_2+\dots+n_m-m.$$

3. The restricted Stiefel manifold $S_k(M)$

Let M be a submanifold of a real finite dimensional vector space W; for simplicity take $W \equiv R^n$. For fixed $k \leqq n$ let $S_k(M)$ denote the set of linear k-frames $(e_1, \ldots, e_k)$ of R^n with $e_i \in M$ for $i=1, 2, \ldots, k$. We call $S_k(M)$ the restricted Stiefel manifold of M. For $M \equiv R^n$, $S_k(R^n)$ is the well known Stiefel manifold of k-frames of R^n ([5], p. 92). Mapping each frame $(e_1, \ldots, e_k)$ of R^n to the $n \times k$ matrix whose columns are the coordinates of $e_1, \ldots, e_k$, we see that $S_k(R^n)$ is diffeomorphic to the open subset of $(R^n)^k \cong R^{nk}$ of $n \times k$ matrices of rank k. Similarly $S_k(M)$ can be identified with an open subset of $M \times M \times \ldots \times M$ (k times) which is a submanifold of $(R^n)^k$, namely

$$S_k(M) = (M \times M \times \ldots \times M) \cap S_k(R^n).$$

Hence $S_k(M)$ carries the structure of an open submanifold of $M \times M \times \ldots \times M$ (k times), and is of dimension mk, where $m = \dim(M)$.

4. Proof of the inequality

Let M_1 be as in §2, the submanifold of decomposable tensors of V. For a fixed $k \leqq n_1 n_2 \ldots n_m$ consider the map

$$p_k \colon S_k(M_1) \times R^k \to V$$

$$p_k\big((e_1, \ldots, e_k), \quad (\lambda_1, \ldots, \lambda_k)\big) = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_m e_m$$

for each k-frame $(e_1, \ldots, e_k)$ of elements of M_1. Obviously p_k is a C^∞-differentiable map. Would k coincide with $r(n_1, \ldots, n_m)$ then p_k should be surjective. But higher dimensional manifolds cannot be covered by lower dimensional ones via differentiable maps ([6], Proposition 1.2, p. 69). Hence we should have

$$\dim\big(S_k(M_1) \times R^k\big) \geqq \dim(V), \quad k(n_1 + n_2 + \ldots + n_m - m) + k \geqq n_1 n_2 \ldots n_m,$$

which proves the stated inequality.

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STABILITY AND CONVERGENCE OF AMARTS IN FRÉCHET SPACES

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0. Introduction

The measure amarts in Banach spaces were considered in [3]. The extensions of Bochner integrals and of Banach space-valued amarts to the Fréchet space case are natural [5, 8]. Therefore, one can study the measure amarts in Fréchet spaces. The Riesz decomposition, stability properties and convergence of amarts in Fréchet spaces are considered in Section 2. For terminologies and notations we refer to Section 1.

1. Terminologies and notations

Throughout the paper, let E be a Fréchet space, $U(E)$ a fundamental countable family of closed absolutely convex sets which forms a 0-neighborhood base for E, E' the topological dual of E and $(\Omega, \mathcal{A}, \gamma)$ a probability space. Given $U\in U(E)$, the polar U^0 and the continuous seminorm $p_U(\cdot)$ associated with U are defined as

$$U^0 = \{e\in E';\ |\langle x, e\rangle| \leqq 1;\ x\in U\}$$

and

$$p_U(x) = \inf\{\delta > 0;\ \delta^{-1}x\in U\},$$

respectively.

We shall denote by $\mathfrak{M}(E)=\mathfrak{M}(\Omega, \mathcal{A}, E)$ the class of all E-valued measures on $\mathcal{A}$. Given $\mu\in\mathfrak{M}(E)$, $U\in U(E)$, the semivariation (or the total variation, resp.) seminorm $S_U(\mu)$ (or $V_U(\mu)$, resp.) is defined by

$$S_U(\mu) = \sup\{|\langle\mu, e\rangle|(\Omega);\ e\in U^0\},$$

$$V_U(\mu) = \sup\Big\{\sum_{j=1}^{k} p_U(\mu(A_j));\ \langle A_j\rangle_{j=1}^{k}\in\pi(\Omega)\Big\},$$

where $\pi(\Omega)$ denotes the set of all measurable finite partitions of Ω.

By $S(E)=S(\Omega, \mathcal{A}, E)$ (or $V(E)=V(\Omega, \mathcal{A}, E)$, resp.) we mean the space of all S-equivalence classes (or V-equivalence classes, resp.) of S-bounded (or V-bounded. resp.) measures in $\mathfrak{M}(E)$. Thus, the proof of the following property will be omitted since the argument is analogous to that contained in [7] for $l_I^1(E)$ and $l_I^1\{E\}$.

PROPERTY 1.1. *Both $S(E)$ and $V(E)$ are Fréchet spaces.*

Let $L_1(E)=L_1(\Omega, \mathcal{A}, E)$ be the space of all Bochner integrable functions $f\colon \Omega\to E$. Thus every $f\in L_1(E)$ is Pettis integrable and if we identify f with the measure μ_f, given by

$$\mu_f\colon \mathcal{A}\to E;\quad \mu_f(A) = \int_A f\,d\gamma\quad (A\in\mathcal{A})$$

then according to [5], $L_1(E)$ can be regarded as a closed (hence by Property 1.1, Fréchet) subspace of $V(E)$ with

$$V_U(f) = V_U(\mu_f) = \int_\Omega p_U(f)\,d\gamma \quad (f\in L_1(E),\ U\in U(E)).$$

The following result can be established as in the Banach space case

PROPERTY 1.2. *Let $\mu\in S(E)$, $U\in U(E)$ and $f\in L_1(\Omega, \mathscr{B}, E)$ for some sub σ-field $\mathscr{B}\subset\mathscr{A}$. Then*

(1) $$S_U(\mu) \leqq V_U(\mu);\quad P_U(\mu) \leqq S_U(\mu) \leqq 4P_U(\mu),$$

where $P_U(\mu)=\sup\{P_U(\mu(A));\ A\in\mathscr{A}\}$.

(2) $$P_U(f) = P_U(\mu_f) \leqq S_U(f) = S_U(\mu_f) \leqq 4P_U(\mu_f).$$

2. The Riesz decomposition stability and convergence of amarts

Hereafter, we shall fix an increasing sequence $\langle\mathscr{A}_n\rangle$ of sub σ-fields with $\Sigma = \bigcup_N \mathscr{A}_n$ and $\mathscr{A}=\sigma-(\Sigma)$. Put

$$S(\langle\mathscr{A}_n\rangle, E) = \{\langle\mu_n\rangle;\ \mu_n\in S^n(E) = S(\mathscr{A}_n, E)\},$$

$$L(\langle\mathscr{A}_n\rangle, E) = \{\langle f_n\rangle;\ f_n\in L_1^n(E) = L_1(\Omega, \mathscr{A}_n, E)\}.$$

T=the set of all bounded stopping times. Given $\tau\in T$, $\langle\mu_n\rangle\in S(\langle\mathscr{A}_n\rangle, E)$ and $\langle f_n\rangle\in L(\langle\mathscr{A}_n\rangle, E)$, let

$$\mathscr{A}_\tau = \{A\in\mathscr{A};\ A\cap\{\tau = n\}\in\mathscr{A}_n,\ (n\in N)\},$$

$$\mu_\tau\colon \mathscr{A}_\tau \to E\colon\ \mu_\tau(A) = \sum_N \mu_j(A\cap\{\tau = j\}),$$

$$f_\tau\colon \Omega \to E\colon\ f_\tau(\omega) = f_j(\omega) \quad (\omega\in\{\tau = j\}),\quad j\in N.$$

It is easily checked that for each $\tau\in T$, $\mathscr{A}_\tau$ is a sub σ-field of $\mathscr{A}$; $\mu_\tau\in S^\tau(E) = S(\Omega, \mathscr{A}_\tau, E)$ and $f_\tau\in L_1^\tau(E)=L_1(\Omega, \mathscr{A}_\tau, E)$. Furthermore, if $\sigma, \tau\in T$ with $\sigma\geqq\tau$ then $\mathscr{A}_\tau\subset\mathscr{A}_\sigma$.

DEFINITION 2.1. Let $\langle\mu_n\rangle\in S(\langle\mathscr{A}_n\rangle, E)$. Call $\langle\mu_n\rangle$ a martingale, if for all $m\geqq n\in N$, one has $\mu_{m,n}=\mu_m|\mathscr{A}_n=\mu_n$.

It is easily checked that, if this occurs then for all $\tau, \sigma\in T$ if $\sigma\geqq\tau$ then $\mu_{\sigma,\tau} = \mu_\sigma|_{\mathscr{A}_\tau}=\mu_\tau$ and the element $\mu_\tau(\Omega)$ does not depend upon the choice of $\tau\in T$.

Call $\langle\mu_n\rangle$ an amart if the net $\{\mu_\tau(\Omega)\}$ converges. Clearly, every martingale is an amart.

The following result is fundamental in the theory of measure amarts. A result very closed to this was proved by Edgar and Sucheston [6]

THEOREM 2.2. *Let $\langle\mu_n\rangle\in S(\langle\mathscr{A}_n\rangle, E)$. Then the following conditions are equivalent:*

(1) *$\langle\mu_n\rangle$ is an amart.*

(2) $\lim_{\tau\in T}\sup_{\sigma\geqq\tau} S_U^\tau(\mu_{\sigma,\tau}-\mu_\tau)=0\ (U\in U(E))$.

(3) *$\langle\mu_n\rangle$ has a Riesz decomposition: $\mu_n=\alpha_n+\beta_n$ $(n\in N)$, where $\langle\alpha_n\rangle$ is a martingale and $\langle\beta_n\rangle$ a potential, i.e.*

$$\lim_{\tau\in T} S_U^\tau(\beta_\tau)=0.$$

(4) *There is a finitely additive measure $\mu_\infty\colon \Sigma\to E$ such that each $\mu_{\infty,n}=\mu_\infty|_{\mathscr{A}_n}\in \in S^n(E)$ and*

$$\lim_{\tau\in T} S_U^\tau(\mu_\tau-\mu_{\infty,\tau})=0 \quad (U\in U(E)).$$

PROOF. $(1\to 2)$ Let $\langle\mu_n\rangle$ be an amart in $S(\langle\mathscr{A}_n\rangle, E)$, then by definition the net $\{\mu_\tau(\Omega)\}$ converges, hence is Cauchy in E. Thus if $U\in U(E)$ and $\varepsilon>0$ are given, one can choose some $\tau(\varepsilon)\in T$ such that if $\tau_1,\sigma_1\in T$ with $\tau_1,\sigma_1\geqq\tau(\varepsilon)$ then

$$P_U\big(\mu_{\sigma_1}(\Omega)-\mu_{\tau_1}(\Omega)\big)\leqq\varepsilon/4. \tag{2.1}$$

Now let $\sigma,\tau\in T$ with $\sigma\geqq\tau$ and $A\in\mathscr{A}_\tau$. Define $\tau_1=\tau$; $\sigma_1=\sigma$ on A and $\tau_1=\sigma_1=m$ on $A^c=\Omega\setminus A$ for some $m>\max(\tau,\sigma)$. Then as it has been noted in [4], $\sigma_1,\tau_1\in T$ and $\sigma_1\geqq\tau_1\geqq\tau(\varepsilon)$. Hence by (2.1), one has

$$P_U\big(\mu_\sigma(A)-\mu_\tau(A)\big)=P_U\big(\mu_{\sigma_1}(\Omega)-\mu_{\tau_1}(\Omega)\big)\leqq\varepsilon/4.$$

Consequently, by Property 1.2, we get

$$\sup_{\sigma\geqq\tau\geqq\tau(\varepsilon)} S_U^\tau(\mu_{\sigma,\tau}-\mu_\tau)\leqq 4\sup_{\sigma\geqq\tau\geqq\tau(\varepsilon)} P_U^\tau(\mu_{\sigma,\tau}-\mu_\tau)\leqq\varepsilon. \tag{2.2}$$

This proves (2).

$(2\to 3)$ Fix $\eta\in T$ and let $\varepsilon>0$ be given. Choose $\tau(\varepsilon)\geqq\eta$ such that (2.2) holds. Therefore, if $\sigma\geqq\tau\geqq\tau(\varepsilon)$, one has

$$S_U^\eta(\mu_{\sigma,\eta}-\mu_{\tau,\eta})\leqq S_U^{\tau(\varepsilon)}\big(\mu_{\sigma,\tau(\varepsilon)}-\mu_{\tau,\tau(\varepsilon)}\big)\leqq$$
$$\leqq S_U^{\tau(\varepsilon)}\big(\mu_{\sigma,\tau(\varepsilon)}-\mu_{\tau(\varepsilon)}\big)+S_U^{\tau(\varepsilon)}\big(\mu_{\tau,\tau(\varepsilon)}-\mu_{\tau(\varepsilon)}\big)\leqq 2\varepsilon.$$

This means that the net $\{\mu_{\tau,\eta}\}_{\tau\geqq\eta}$ is S-Cauchy in $S(E)$. Consequently, by Property 1.1, there is an element $\langle\alpha_\eta\rangle$ in $\prod_{\eta\in T} S^\eta(E)$ such that

$$S^\eta-\lim_{\eta\leqq\tau\in T}\mu_{\tau,\eta}=\alpha_\eta \quad (\eta\in T). \tag{2.3}$$

This implies that $\langle\alpha_\eta\rangle$ is a martingale and

$$\alpha_\eta(A)=\sum_{j=1}^{\infty}\alpha_j(A\cap\{\eta=j\}) \quad (A\in\mathscr{A}_\eta).$$

Now put $\beta_n=\mu_n-\alpha_n$ $(n\in N)$. It is easy to check that by (2) and (2.3), $\langle\beta_n\rangle$ is a potential, i.e.

$$\lim_{\tau\in T} S_U^\tau(\beta_\tau)=0 \quad (U\in U(E))$$

which proves (3).

$(3\to 4)$. Define $\mu_\infty\colon\Sigma\to E$ by

$$\mu_\infty(A)=\alpha_n(A) \quad (A\in\mathscr{A}_n;\ n\in N).$$

Obviously, μ satisfies all assertions of (4). Finally, since $(4\to 1)$ is trivial, the proof of the theorem is completed.

Note that every paper concerning the vector amarts contains a version of the above theorem. But the main idea was contained first in [4].

The following result is due to Bellow [1].

THEOREM 4. *Let E be a Banach space and $\langle f_n\rangle$ an amart in $L(\langle \mathcal{A}_n\rangle, E)$. Suppose that $\sup_N E(\|f_n\|)<\infty$. Then $\langle f_n\rangle$ is uniformly bounded in the Pettis norm, i.e.*

$$\sup_{\tau\in T}\sup_{A\in\mathcal{A}_\tau}\Big\|\int_A f\,d\gamma\Big\|<\infty.$$

We note that the above result can be generalized as follows:

THEOREM 2.3. *Let E be a Fréchet space and $U\in U(E)$. Suppose that $\langle\mu_n\rangle\in \in S(\langle\mathcal{A}_n\rangle, E)$ is an amart. Then*

$$\sup_{\tau\in T} P_U^\tau(\mu_\tau)\leqq\sum_{j=1}^{n(U)}P_U^j(\mu_j)+2\,(\liminf_m P_U^m(\mu_m)+1), \tag{2.4}$$

where $n(U)=\inf\{k\in N;\ \sup_{\tau\geqq k}P_U^\tau(\mu_\tau-\mu_{\infty,\tau})\leqq 1\}<\infty$.

PROOF. Let $U\in U(E)$ and $\langle\mu_n\rangle\in S(\langle\mathcal{A}_n\rangle, E)$ an amart. We shall show first that

$$P_U^\Sigma(\mu_\infty)=\sup\{P_U(\mu_\infty(A));\ A\in\Sigma\}\leqq\liminf_m P_U^m(\mu_m). \tag{2.5}$$

Indeed, let $A\in\Sigma$ then $A\in\mathcal{A}_{n(A)}$ for some $n(A)\in N$. Thus if $m\geqq n(A)$, one has

$$P_U(\mu_\infty(A))\leqq P_U(\mu_m(A))+P_U(\mu_m(A)-\mu_\infty(A))=P_U(\mu_m(A))+P_U(\mu_m(A)-\mu_{\infty,m}(A)).$$

This with Theorem 2.2 yields

$$P_U(\mu_\infty(A))\leqq\liminf_m P_U(\mu_m(A))+\limsup_m P_U(\mu_m(A)-\mu_{\infty,m}(A))\leqq$$

$$\leqq\liminf_m P_U^m(\mu_m)+\limsup_m P_U^m(\mu_m-\mu_{\infty,m})=\liminf_m P_U^m(\mu_m)$$

which implies (2.5), hence

$$P_U^\tau(\mu_{\infty,\tau})\leqq P_U^\Sigma(\mu_\infty)\leqq\liminf_m P_U^m(\mu_m)\quad(\tau\in T). \tag{2.6}$$

Now define

$$n(U)=\inf\{k\in N;\ \sup_{\tau\geqq k}P_U^\tau(\mu_\tau-\mu_{\infty,\tau})\leqq 1\}.$$

It follows from Theorem 2.2 (4) and Property 1.2 that $n(U)<\infty$. Let $\tau\in T$ and $A\in\mathcal{A}_\tau$, then

$$\mu_\tau(A)=\sum_{j=1}^\infty\mu_j(A\cap\{\tau=j\}),$$

$$\mu_{\tau\vee n(U)}(A)=\sum_{j>n(U)}\mu_j(A\cap\{\tau=j\})+\mu_{n(U)}(A\cap\{\tau\leqq n(U)\}).$$

Consequently,

$$\mu_\tau(A)=\sum_{j\leqq n(U)}\mu_j(A\cap\{\tau=j\})+\mu_{\tau\vee n(U)}(A)-\mu_{n(U)}(A\cap\{\tau\leqq n(U)\}).$$

But by (2.6),

$$P_U^{\tau\vee n(U)}(\mu_{\infty,\tau\vee n(U)}),\quad P_U^{n(U)}(\mu_{n(U)}) \leqq \liminf_m P_U^m(\mu_m),$$

and Theorem 2.2 with $\tau\vee n(U)$, $n(U)\geqq n(U)$ implies

$$P_U(\mu_\tau(A)) \leqq \sum_{j\leqq n(U)} P_U^j(\mu_j)+\big(P_U(\mu_{\infty,\tau\vee n(U)}(A))+1\big)+\big(P_U(\mu_{\infty,n(U)}(A))+1\big) \leqq$$

$$\leqq \sum_{j\leqq n(U)} P_U^j(\mu_j)+\big(P_U^{\tau\vee n(U)}(\mu_{\infty,\tau\vee n(U)})+1\big)+\big(P_U^{n(U)}(\mu_{n(U)})+1\big) \leqq$$

$$\leqq \sum_{j\leqq n(U)} P_U^j(\mu_j)+2\big(\liminf_m P_U^m(\mu_m)+1\big).$$

This proves (2.4) and the theorem.

In what follows we shall need the following result which is interesting in itself.

LEMMA 2.4. *A Fréchet space E is nuclear if and only if $S(E)\equiv V(E)$ for every probability space $(\Omega.\ \mathcal{A},\gamma)$.*

PROOF. ($\rightarrow$) Let E be a nuclear Fréchet space. Then by ([9], 4.1.5) for each $U\in\in U(E)$ there is some $C\in U(E)$ and a finite positive Radon measure φ on C^0 such that

$$P_U(x) \leqq \int_{C^0} |\langle x, e\rangle|\, d\varphi(e) \quad (x\in E). \tag{2.7}$$

Let $\mu\in S(E)$ and $\langle A_j\rangle_{j=1}^k\in\pi(\Omega)$. Applying (2.7) to each $\mu(A_j)$ we get

$$\sum_{j=1}^k P_U(\mu(A_j)) \leqq \sum_{j=1}^k \int_{C^0} |\langle\mu(A_j), e\rangle|\, d\varphi(e) = \int_{C^0} \sum_{j=1}^k |\langle\mu(A_j), e\rangle|\, d\varphi(e) \leqq$$

$$\leqq \varphi(C^0)\sup\Big\{\sum_{j=1}^k |\langle\mu(A_j), e\rangle|;\ e\in C^0\Big\} \leqq$$

$$\leqq \varphi(C^0)\sup\{|\langle\mu, e\rangle|(\Omega);\ e\in C^0\} = \varphi(C^0)S_C(\mu).$$

Hence,

$$V_U(\mu) \leqq \varphi(C^0)S_C(\mu). \tag{2.8}$$

This shows that $S(E)=V(E)$ and the S-topology is stronger than the V-topology. Thus, in view of Property 1.2 $S(E)\equiv V(E)$. The proof of the necessity condition is completed. We shall see in the next theorem that $S(E)\equiv V(E)$ is also a sufficiency condition under which E must be nuclear.

THEOREM 2.5. *Let E be a Fréchet space. Then the following conditions are equivalent:*

(1) *E is nuclear.*
(2) *Every potential in $L(\langle\mathcal{A}_n\rangle, E)$ is of class (B).*
(3) *Every potential is Bochner bounded.*

PROOF. (1)$\rightarrow$(2) Let E be a nuclear space and $\langle f_n\rangle$ a potential in $L(\langle\mathcal{A}_n\rangle, E)$. Consequently, $\langle f_n\rangle$ is Pettis bounded. Therefore, it follows from Theorem 2.3 that $\langle f_n\rangle$ is Pettis uniformly bounded. Thus, applying (2.8) to each $\mu_{f_\tau}(\tau\in T)$, the Pettis uniform boundedness of $\langle f_\tau\rangle$ implies the Bochner uniform boundedness of $\langle f_\tau\rangle$, i.e. $\langle f_n\rangle$ is of class (B). This proves (2).

$(2\to 3)$ is trivial.

$(3\to 1)$. Suppose that E fails to be nuclear. Consequently, by ([9], 4.2.4) there is a summable sequence $\langle x_n\rangle$ in E which fails to be absolutely summable, i.e. $\sum_{n=1} P_U(x_n)=\infty$ for some $U\in U(E)$. Therefore, one can suppose that $P_U(x_n)\neq 0$ $(n\in N)$ and one can choose $n_1<n_2<n_3<\ldots<n_k<\ldots$ such that

$$\text{(a)}\quad n_k+2\leqq n_{k+1}\quad (k\in N)$$

and

$$\text{(b)}\quad \sum_{j=n_k+1}^{n_{k+1}} P_U(x_j)=\alpha_k\geqq k\quad (k\in N).$$

In the proof we shall use only Property (b). Property (a) will be needed in the sequel.

For each $k\in N$, let $\langle A_{k,j}\rangle_{j=n_k+1}^{n_{k+1}}\in\pi([0,1])$ with

$$\gamma(A_{k,j})=P_U(x_j)/\alpha_k\quad (n_k+1\leqq j\leqq n_{k+1},\, k\in N),$$

where γ is the Lebesgue measure on $\mathscr{B}_{[0,1]}$.

Define

$$f_k=\sum_{j=n_k+1}^{n_{k+1}}\alpha_k x_j l_{A_{k,j}}/P_U(x_j)\quad (k\in N),\quad \sigma-(f_1,f_2,\ldots,f_k)=\mathscr{A}_k\quad (k\in N).$$

We shall show that $\langle f_k\rangle$ is a potential which fails to be Bochner bounded.

Indeed, let $\tau\in T$. Define

$$\underline{\tau}=\inf\{n;\ \gamma(\{\tau=n\})>0\},\quad \bar{\tau}=\max\{n;\ \gamma(\{\tau=n\})>0\}.$$

We have

$$\int_0^1 f_\tau\,d\gamma=\int_0^1\sum_{k=\underline{\tau}}^{\bar{\tau}}\sum_{j=n_k+1}^{n_{k+1}}\alpha_k x_j l_{B_{k,j}}/P_U(x_j)\,d\gamma=\sum_{k=\underline{\tau}}^{\bar{\tau}}\sum_{j=n_k+1}^{n_{k+1}}\alpha_k x_j\gamma(B_{k,j})/P_U(x_j),$$

where $B_{k,j}=A_{k,j}\cap\{\tau=k\}\subset A_{k,j}$ $(n_k+1\leqq j\leqq n_{k+1})$. But since $\gamma(B_{k,j})\leqq P_U(x_j)/\alpha_k$ $(n_k+1\leqq j\leqq n_{k+1})$ by ([9], 1.3.6) the net $\left\{\int_0^1 f_\tau\,d\gamma\right\}$ converges to 0. Equivalently, $\langle f_k\rangle$ is a potential.

Finally, since

$$\int_0^1 P_U(f_k)\,d\gamma=\sum_{j=n_k+1}^{n_{k+1}}\alpha_k P_U(x_j)\gamma(A_{k,j})/P_U(x_j)=\sum_{j=n_k+1}^{n_{k+1}}P_U(x_j)\geqq k$$

clearly, $\langle f_k\rangle$ cannot be Bochner bounded. This contradicts (3). The theorem is proved.

Note that the above example shows at the same time the sufficiency condition of Lemma 2.4.

LEMMA 2.6. *Every uniform potential $\langle f_n\rangle$ in $L(\langle\mathscr{A}_n\rangle, E)$,*

$$\lim_{\tau\in T}\int_\Omega P_U(f_\tau)\,d\gamma=0\quad (U\in U(E)),\tag{2.9}$$

converges strongly a.s. to 0, where E is an arbitrary Fréchet space.

PROOF By (2.9) and by definition, as a sequence of real-valued functions, $\langle P_U(f_n)\rangle$ is a uniform potential. Hence by [2] or [3], $\langle P_U(f_n)\rangle$ converges strongly to 0, a.s. $(U\in U(E))$. But we note that since $U(E)$ is countable then $\langle f_n\rangle$ itself converges strongly a.s. to 0 in E. This completes the proof.

Note that the proof of the main result ([8], Theorem 4) given by Egghe was mostly devoted to showing that a uniform potential in a nuclear Fréchet space converges strongly to 0, a.s. But as we have seen that it is easy even for uniform potentials in Fréchet spaces.

COROLLARY 2.7. *Let E be a Fréchet space. Then the following conditions are equivalent:*

(1) *E is nuclear.*

(2) *Every potential is uniform.*

(3) *Every potential converges strongly to* 0, *a.s.*

(4) *Every Bochner-convergent potential converges strongly to* 0, *a.s.*

PROOF. $(1 \leftrightarrow 2)$ It follows from definition of potentials and Lemma 2.4.

$(2\to 3)$ follows from Lemma 2.6.

$(3\to 4)$ is trivial.

$(4\to 1)$. Suppose that E is not nuclear. Take $\langle x_n\rangle$, $U\in U(E)$, $n_1<n_2<\dots$ as in the proof of $(3\to 1)$ in Theorem 2.5.

Define $f_j(\omega)=x_j I_{A_{k,j}}/P_U(x_j)$, $(n_k+1\leqq j\leqq n_{k+1},\ k\in N)$,

$$\mathcal{A}_j=\sigma-(f_1,f_2,\dots,f_j)\quad (j\in N).$$

It is easily seen that the same proof of $(3\to 1)$ in Theorem 2.5 implies

(1) $\langle f_j\rangle$ is potential,

(2) $\int\limits_\Omega P_C(f_j)\,d\gamma=\alpha_k^{-1}P_C(x_j)\to 0$ as $j\uparrow\infty$,

and

(3) It follows from property (a) of $\langle n_k\rangle$ and definition of $\langle f_j\rangle$ that $\langle f_j\rangle$ cannot converge strongly, a.s.

The proof of the corollary is completed.

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SYNTOPOGENOUS SPACES WITH PREORDER. IV (REGULARITY, NORMALITY)

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In the fourth part of the series ([8]—[10]) the notions of a regularly, completely regularly and normally preordered syntopogenous spaces will be introduced. The theory developed here has three different sources: the separation axioms (S_i) of topological spaces that originate from (T_i) without the assumption of (T_0) $(i=1, 2, 3, \pi, 4$; see e.g. [5], ch. 2.5), their extension to syntopogenous spaces ([11], [6]), finally the notion of (strongly) regularly [12], (monotonically) completely regularly [14] and normally [13] ordered topological spaces.

0. Introduction

A *preordered space* $(E, \leqq)$ is a pair consisting of a set E and a reflexive and transitive relation $\leqq$ on E called *preorder*. Its *graph* is $G(\leqq)=\{(x, y)\in E\times E\colon x\leqq y\}$. A preorder $\leqq$ is called an *order* iff $x\leqq y$, $y\leqq x$ imply $x=y$. $\leqq$ is *linear* iff either $x\leqq y$ or $y\leqq x$ holds for every $x, y\in E$. $X\subset E$ is *increasing (decreasing)* iff $x\leqq y$ $(y\leqq x)$ and $x\in X$ imply $y\in X$. For an arbitrary $X\subset E$, $i(X)=\{y\in E\colon x\leqq y, x\in X\}$ and $d(X)=\{y\in E\colon y\leqq x\in X\}$ are the smallest of all increasing and decreasing sets respectively, which contain X. A mapping f of the preordered space $(E, \leqq)$ into another one $(E', \leqq')$ is said to be *preorder preserving (inversing)* iff $x\leqq y$ implies $f(x)\leqq' f(y)\big(f(y)\leqq' f(x)\big)$. The *product* of the preordered spaces $(E_j, \leqq_j)$ $(j\in J\neq\emptyset)$ is a preordered space $(E, \leqq)$, where $E=\bigtimes_{j\in J} E_j$, and $(x_j)\leqq(y_j)$ is equivalent to $x_j\leqq_j y_j$ for each $j\in J$.

The terminology and the notations concerning *syntopogenous spaces* will be taken over from the monograph [4]. A local syntopogenous *structure* [11] is an order family on a set E such that

(L1) $<, <'\in\mathscr{S}$, implies $<\cup<'\subset<''$ for some $<''\in\mathscr{S}$;

(L2) for $<\in\mathscr{S}$, there exists $<'\in\mathscr{S}$ so that $x\in E$, $x<B$ imply $x<'C<'B$ for some $C\subset E$.

A *preordered (local) syntopogenous space* is a triplet $(E, \mathscr{S}, \leqq)$, where E is a set, $\mathscr{S}$ is a (local) syntopogenous structure and $\leqq$ is a preorder on E (cf. [1]—[3], [8]—[10]). If $\mathscr{A}$ is an order family on E, we write $G(\mathscr{A})=\bigcap\{G(<)\colon <\in\mathscr{A}\}$, where $G(<)=\{(x, y)\in E\times E\colon x<E-y$ is false$\}$. The preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ (or shortly $\mathscr{S}$) is said to be *increasing (decreasing)* iff $G(\leqq)\subset G(\mathscr{S})$ $\big(G(\leqq)^{-1}\subset G(\mathscr{S})\big)$ ([8], cf. [1]—[2]). $\mathscr{S}^u(\mathscr{S}^l)$ denotes the finest of all increasing (decreasing) syntopogenous structures coarser than $\mathscr{S}$ (see [2]). $(E, \mathscr{S}, <)$ is called ^{a}-convex, where ^{a} is an arbitrary elementary operation ([4], p. 69), iff $\mathscr{S}\sim(\mathscr{S}^u\vee\mathscr{S}^l)^a$ ([8], cf. [1]—[2] for $^a={}^i$ or ^{p}). $(E, \mathscr{S}, \leqq)$ is *symmetrizable* [8] iff there exists a symmetrical

^{i}-convex syntopogenous structure $\mathscr{S}_0$ on $(E, \leqq)$ such that $\mathscr{S}_0 < \mathscr{S} < \mathscr{S}_0^p$. $(E, \mathscr{S}, \leqq)$ is called *continuous* iff, for $<\in\mathscr{S}$, there is $<_1\in\mathscr{S}$ such that $A<B$ implies $i(A)<_1 i(B)$ and $d(A)<_1 d(B)$ (see [9]). If $\mathscr{A}$ and $\mathscr{B}$ are order families on E, we use the order family $\mathscr{A}\mathscr{B}=\{<_1<_2\colon <_1\in\mathscr{A},\ <_2\in\mathscr{B}\}$, where $<_1<_2$ is the topogenous order defined by

$$A(<_1<_2)B \text{ iff } A<_1 C<_2 B \text{ for some } C\subset E$$

(see [6]). $(E, \mathscr{S}, \leqq)$ is *T_0-preordered* iff $G(\mathscr{S}^u)\cap G(\mathscr{S}^{lc})=G(\leqq)$ [10], *T_1-preordered* iff $G(\mathscr{S}^u)=G(\mathscr{S}^{lc})=G(\leqq)$ [2], and *T_2-prerdered* iff $G(\mathscr{S}^u\mathscr{S}^{lc})=G(\leqq)$ [10].

1. Regularly preordered spaces

(1.1) LEMMA. *Let $(E, \mathscr{S}, \leqq)$ be a preordered syntopogenous space and consider the following conditions:*
(S_1) $\mathscr{S}^{ub}\sim\mathscr{S}^{lcb}$;
(S_2) $\mathscr{S}^u<(\mathscr{S}^u\mathscr{S}^{lc})^b$ *and* $\mathscr{S}^l<(\mathscr{S}^l\mathscr{S}^{uc})^b$;
(S_3) $\mathscr{S}^u<(\mathscr{S}^u\mathscr{S}^{lc})^p$ *and* $\mathscr{S}^l<(\mathscr{S}^l\mathscr{S}^{uc})^p$.
Then $(S_3)\Rightarrow(S_2)\Rightarrow(S_1)$.

PROOF. $(S_3)\Rightarrow(S_2)$ is trivial. $(S_2)\Rightarrow(S_1)$: $\mathscr{S}^u<(\mathscr{S}^u\mathscr{S}^{lc})^b<\mathscr{S}^{lcb}$, $\mathscr{S}^{lc}<(\mathscr{S}^l\mathscr{S}^{uc})^{bc}<$ $<\mathscr{S}^{ucbc}=\mathscr{S}^{uccb}=\mathscr{S}^{ub}$ (cf. [4], (5.17)), thus $\mathscr{S}^{ub}<\mathscr{S}^{lcbb}=\mathscr{S}^{lcb}$ and $\mathscr{S}^{lcb}<\mathscr{S}^{ubb}=$ $=\mathscr{S}^{ub}$. □

$(E, \mathscr{S}, \leqq)$ will be said to be *S_i-preordered* $(i=1, 2, 3)$, if it satisfies condition (S_i). An S_3-preordered space will be called *regularly preordered.*

(1.2) REMARKS. (1.2.1) In a discretely ordered syntopogenous space $(E, \mathscr{S}, =)$ we have $\mathscr{S}^u\mathscr{S}^{lc}\sim\mathscr{S}\mathscr{S}^c$, and the latter is equivalent to the order family $\mathscr{S}^\Delta$ introduced in [11]. Thus $(E, \mathscr{S}, =)$ is S_i-preordered iff $[E, \mathscr{S}]$ is an S_i-space in the sense of [11].

(1.2.2) Let r be the classical topology associated with the topology $\mathscr{T}$ on $(E, \leqq)$. Then $(E, \mathscr{T}, \leqq)$ is S_1-preordered iff, for arbitrary points $x, y\in E$, the condition
(*) *$x\in V$ and $y\notin V$ for an increasing r-open $V\subset E$* is equivalent to the following one:
(**) *$y\in W$ and $x\notin W$ for a decreasing r-open $W\subset E$.* $(E, \mathscr{T}, \leqq)$ is S_2-preordered iff either (*) or (**) implies,
(***) *$x\in V'$, $y\in W'$, $V'\cap W'=\emptyset$ for an increasing r-open $V'\subset E$ and a decreasing r-open $W'\subset E$.*

$(E, \mathscr{T}, \leqq)$ is S_3-preordered iff it is "strongly" regularly preordered in the sense of McCartan [12].

(1.2.3) In respect of $(S_1)\not\Rightarrow(S_2)\not\Rightarrow(S_3)$ one can refer to [5], p. 95, 97. □

(1.3) THEOREM (cf. [11], (1.3)). *If a space is both T_0- and S_i-preordered, then it is T_i-preordered $(i=1, 2)$. For topogenous spaces the converse is also true.*

PROOF. Suppose $(E, \mathscr{S}, \leqq)$ is T_0-and S_1-preordered. Then

$$G(\mathscr{S}^u)=G(\mathscr{S}^{ub})=G(\mathscr{S}^{lcb})=G(\mathscr{S}^{lc}),$$

thus

$$G(\leqq)=G(\mathscr{S}^u)\cap G(\mathscr{S}^{lc})=G(\mathscr{S}^u)=G(\mathscr{S}^{lc}),$$

i.e. $(E, \mathcal{S}, \leqq)$ is T_1-preordered. If $(E, \mathcal{S}, \leqq)$ is T_0- and S_2-preordered, then $\mathcal{S}^u <$ $< (\mathcal{S}^u \mathcal{S}^{lc})^b$ and

$$\mathcal{S}^{lc} < (\mathcal{S}^l \mathcal{S}^{uc})^{bc} = (\mathcal{S}^l \mathcal{S}^{uc})^{cb} = (\mathcal{S}^{ucc} \mathcal{S}^{lc})^b = (\mathcal{S}^u \mathcal{S}^{lc})^b,$$

therefore

$$G(\mathcal{S}^u \mathcal{S}^{lc}) = G\big((\mathcal{S}^u \mathcal{S}^{lc})^b\big) \subset G(\mathcal{S}^u \vee \mathcal{S}^{lc}) = G(\mathcal{S}^u) \cap G(\mathcal{S}^{lc}) = G(\leqq),$$

thus $(E, \mathcal{S}, \leqq)$ is in fact T_2-preordered (cf. [1], 3.3, [10], (1.2)).

Conversely, assume that $\mathcal{T}$ is a topogenous structure on a preordered space $(E, \leqq)$. Then $\mathcal{T}^u \sim \mathcal{T}^{ut}$ and $\mathcal{T}^l \sim \mathcal{T}^{lt}$ (see [2], 3.1, 3.2; [8], (1.7)). If $(E, \mathcal{T}, \leqq)$ is T_1- or T_2-preordered, then it is T_0-preordered, too. Let now $(E, \mathcal{T}, \leqq)$ be T_1-preordered. Then $G(\leqq) = G(\mathcal{T}^u) = G(\mathcal{T}^{lc})$, thus

$$\mathcal{T}^{ub} \sim \mathcal{T}^{utb} = \mathcal{T}^{lctb} = \mathcal{T}^{ltcb} \sim \mathcal{T}^{lcb}$$

by [10], (0.1). If $(E, \mathcal{T}, \leqq)$ is T_2-preordered, then

$$G(\mathcal{T}^u \mathcal{T}^{lc}) = G(\leqq) \subset G(\mathcal{T}^u) \cap G(\mathcal{T}^{lc}),$$

therefore

$$\mathcal{T}^u < \mathcal{T}^{utb} < (\mathcal{T}^u \mathcal{T}^{lc})^{tb} = (\mathcal{T}^{ut} \mathcal{T}^{lct})^b = (\mathcal{T}^{ut} \mathcal{T}^{ltc})^b \sim (\mathcal{T}^u \mathcal{T}^{lc})^b,$$

and similarly $\mathcal{T}^{lc} < (\mathcal{T}^u \mathcal{T}^{lc})$, hence

$$\mathcal{T}^l = \mathcal{T}^{lcc} < (\mathcal{T}^u \mathcal{T}^{lc})^{bc} = (\mathcal{T}^u \mathcal{T}^{lc})^{cb} = (\mathcal{T}^l \mathcal{T}^{uc})^b. \quad \square$$

(1.4) EXAMPLE. [11], (1.4) shows that, in general, the implication $(T_i) \Rightarrow (S_i)$ is not true. $\square$

We shall say that a (local) syntopogenous structure $\mathcal{S}_0$ *spans* the preordered syntopogenous space $(E, \mathcal{S}, \leqq)$ iff ($\mathcal{S}_0^c$ is also a local syntopogenous structure), $\mathcal{S}_0 <$ $< \mathcal{S}^u < \mathcal{S}_0^p$ and $\mathcal{S}_0^c < \mathcal{S}^l < \mathcal{S}_0^{cp}$.

If $\mathcal{S}$ is a syntopology, then $\mathcal{S}^u \sim \mathcal{S}^{up}$, $\mathcal{S}^l \sim \mathcal{S}^{lp}$ ([2], 3.1, 3.2), thus in this case $\mathcal{S}_0$ spans $(E, \mathcal{S}, \leqq)$ iff $\mathcal{S}_0^p \sim \mathcal{S}^u$ and $\mathcal{S}_0^{cp} \sim \mathcal{S}^l$.

(1.5) THEOREM. *A preordered syntopogenous space is regularly preordered iff there exists a local syntopogenous structure spanning it.*

PROOF. Let $\mathcal{S}_0$ be a local syntopogenous structure spanning the preordered syntopogenous space $(E, \mathcal{S}, \leqq)$. For $< \in \mathcal{S}^u$, there exists $<' \in \mathcal{S}_0$ with $< \subset <'^p$. Choose $<'' \in \mathcal{S}_0$ for $<'$ in accordance with $(L2)$, and suppose $<_1 \in \mathcal{S}^u$, $<_2 \in \mathcal{S}^l$ such that $<'' \subset <_1$ and $<''^c \subset <_2$. Then $A < B$ implies $x <' B$ for every $x \in A$. If $x <'' C_x <'' B$, then we have $x <_1 C_x <_2^c B$, thus $x(<_1 <_2^c) B$ for each $x \in A$, i.e. $A(<_1 <_2^c)^p B$, therefore $< \subset (<_1 <_2^c)^p$. This means $\mathcal{S}^u < (\mathcal{S}^u \mathcal{S}^{lc})^p$. The proof of $\mathcal{S}^l < (\mathcal{S}^l \mathcal{S}^{uc})^p$ is analogous, hence $(E, \mathcal{S}, \leqq)$ is regularly preordered.

Conversely, let $(E, \mathcal{S}, \leqq)$ be regularly preordered. It will be verified that $\mathcal{S}_0 = \mathcal{S}^u \mathcal{S}^{lc}$ is a local syntopogenous structure spanning $(E, \mathcal{S}, \leqq)$. It is easy to see that $\mathcal{S}_0$ satisfies $(L1)$. After this assume $< \in \mathcal{S}^u$, $<' \in \mathcal{S}^l$. Select $<_1 \in \mathcal{S}^u$, for which $< \subset <_1^2$, and suppose $<_2 \in \mathcal{S}^u$, $<_2' \in \mathcal{S}^l$ with $<_1 \subset (<_2 <_2')^p$ by (S_3). Finally put $<_1 \cup <_2 \subset <_3 \in \mathcal{S}^u$ and $<' \cup <_2' \subset <_3' \in \mathcal{S}^l$. Then $x \in E$, $x(< <'^c) B$ implies $x <_1 X <_1$ $<_1 C <'^c B$ for some $X, C \subset E$, hence $x <_2 Y <_2'^c X <_1 C <'^c B$ for a suitable $Y \subset E$, therefore $x(<_3 <_3'^c) X (<_3 <_3'^c) B$, thus $(L2)$ is also satisfied by $\mathcal{S}_0$. It can be similarly shown that $\mathcal{S}_0^c=\mathcal{S}^l\mathcal{S}^{uc}$ is also a local syntopogenous structure. From (S_3) we get $\mathcal{S}_0<\mathcal{S}^u<\mathcal{S}_0^p$ and $\mathcal{S}_0^c<\mathcal{S}^l<\mathcal{S}_0^{cp}$, that is $\mathcal{S}_0$ spans $(E, \mathcal{S}, \leqq)$. □

Considering the discrete order $=$ of E, the above theorem yields a generalization of [11], (1.10):

(1.6) Corollary. *The syntopogenous space* $[E, \mathcal{S}]$ *is regular iff there exists a symmetrical local syntopogenous structure* $\mathcal{S}_0$ *on E such that* $\mathcal{S}_0<\mathcal{S}<\mathcal{S}_0^p$.

Proof. If $[E, \mathcal{S}]$ is regular, then $(E, \mathcal{S}, =)$ is regularly ordered, thus by $\mathcal{S}^u\sim\sim\mathcal{S}^l\sim\mathcal{S}$, we have $\mathcal{S}'<\mathcal{S}<\mathcal{S}'^p$ and $\mathcal{S}'^c<\mathcal{S}<\mathcal{S}'^{cp}$, where $\mathcal{S}'$ is a local syntopogenous structure such that $\mathcal{S}'^c$ is also of this type. Then clearly $\mathcal{S}'^s$ is also a local syntopogenous structure on E, and $\mathcal{S}'^s\sim(\mathcal{S}'\cup\mathcal{S}'^c)^g<\mathcal{S}<\mathcal{S}'^p<\mathcal{S}'^{sp}$, therefore $\mathcal{S}_0=\mathcal{S}'^s$ satisfies the condition. Conversely, if $\mathcal{S}_0$ is a symmetrical local syntopogenous structure such that $\mathcal{S}_0<\mathcal{S}<\mathcal{S}_0^p$, then it spans $(E, \mathcal{S}, =)$, thus it is regularly ordered and $[E, \mathcal{S}]$ is regular. □

Another particular case of (1.5) is the following one:

(1.7) Corollary (cf. [15], Th. 3.2). *Let* $(E, \mathcal{S}, \leqq)$ *be a* ^{p}*-convex regularly preordered syntopological space. Then there exists a local syntopogenous structure* $\mathcal{S}_0$ *on E such that* $\mathcal{S}_0^c$ *is also a local syntopogenous structure,*

$$\mathcal{S}\sim\mathcal{S}_0^{sp} \tag{1.7.1}$$

and

$$G(\leqq)\subset G(\mathcal{S}_0). \tag{1.7.2}$$

If $(E, \mathcal{S}, \leqq)$ *is* T_0*-preordered, then* $\mathcal{S}_0$ *can be chosen so that*

$$G(\leqq)=G(\mathcal{S}_0). \tag{1.7.3}$$

Proof. As we showed in the proof of (1.5), $\mathcal{S}_0=\mathcal{S}^u\mathcal{S}^{lc}$ is a local syntopogenous structure spanning $(E, \mathcal{S}, \leqq)$. Because of that $\mathcal{S}$ is perfect, we have $\mathcal{S}_0^p\sim\mathcal{S}^u$ and $\mathcal{S}_0^{cp}\sim\mathcal{S}^l$. Thus

$$\mathcal{S}\sim(\mathcal{S}^u \vee \mathcal{S}^l)^p=(\mathcal{S}^u\cup\mathcal{S}^l)^{gp}\sim(\mathcal{S}_0^p\cup\mathcal{S}_0^{cp})^{gp}\sim(\mathcal{S}_0\cup\mathcal{S}_0^c)^{gp}\sim\mathcal{S}_0^{sp}$$

(see [4], (8.18), (8.48)). Since $\mathcal{S}^u$ is increasing, and $\mathcal{S}_0<\mathcal{S}^u$, we get $G(\leqq)\subset G(\mathcal{S}^u)\subset\subset G(\mathcal{S}_0)$. If $(E, \mathcal{S}, \leqq)$ is T_0-preordered, then it is also T_2-preordered by (1.1) and (1.3). Thus $G(\leqq)=G(\mathcal{S}^u\mathcal{S}^{lc})=G(\mathcal{S}_0)$. □

2. On a false lemma of Singal and Sunder Lal

I cannot prove the converse of (1.7) (i.e. that (1.7.1)—(1.7.2) imply that $(E, \mathcal{S}, \leqq)$ is regularly preordered), and my conjecture is that this implication is in fact impossible. In any case, it is sure that (1.7.1)—(1.7.2) do not imply that $\mathcal{S}_0$ spans $(E, \mathcal{S}, \leqq)$.

(2.1) Example. On the real line $\mathbf{R}$ there exists an order $\leqq$, a topology $\mathcal{T}$ and a topogenous structure $\mathcal{T}_0$ such that $\mathcal{T}_0^{sp}=\mathcal{T}$, $G(\leqq)=G(\mathcal{T}_0)$, but $\mathcal{T}_0^p\sim\mathcal{T}^u$ is not true (i.e. $\mathcal{T}_0$ does not span $(\mathbf{R}, \mathcal{T}, \leqq)$).

Let Q be the set of the rational numbers, and, for $x, y \in \mathbf{R}$, define

(2.1.1) $$x \leqq y \quad \text{iff either} \quad x \leqq y \in Q \quad \text{or} \quad x = y \notin Q.$$

Let $\mathcal{T}$ denote the topology on $\mathbf{R}$, in which a set $V \subset \mathbf{R}$ is open iff $V \cap Q$ is open in the natural topology of Q. Putting

(2.1.2) $A <_0 B$ iff $A \subset B$, and there exists $\varepsilon > 0$ such that $(x-\varepsilon, \infty) \cap Q \subset B$ for any $x \in A$,

a topogenous structure $\mathcal{T}_0 = \{<_0\}$ can be determined on $\mathbf{R}$ (for the verification of $<_0 \subset <_0^2$, in the case of $A <_0 B$, consider $C = A \cup \{y \in Q: \exists x \in A, x - \varepsilon/2 < y\}$; then $A <_0 C <_0 B$). It is easy to show that

(2.1.3) $A <_0^c B$ iff $A \subset B$, and there exists $\varepsilon > 0$ such that $(-\infty, x+\varepsilon) \subset B$ for each $x \in A \cap Q$.

$V \subset \mathbf{R}$ is $\mathcal{T}_0^{sp}$-open iff, for each $x \in V \cap Q$, there is $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \cap Q \subset V$, that is $V \cap Q$ is open in the natural topology of Q, thus $\mathcal{T}_0^{sp} = \mathcal{T}$. The equality $G(\leqq) = G(\mathcal{T}_0)$ is also obvious. ($\mathcal{T}$ is also ^{p}-convex by [8], (2.12), (2.5).)

Now if $x \in \mathbf{R} - Q$, then $[x, \infty)$ is increasing $\mathcal{T}$-open, but $x <_0 [x, \infty)$ is impossible, therefore $\mathcal{T}^u < \mathcal{T}_0^p$ is not true. $\square$

I should like to call the attention of the reader to an error in a paper of Singal—Sunder Lal ([15], Lemma 3.1). To my knowledge, in this respect correction was not yet published. The quoted lemma deals with local proximity preordered spaces $(X, \delta^*, \leqq)$ defined as follows:

(1) There exists a relation δ on 2^X (so-called PR-proximity) such that (i): $A\delta B \Rightarrow A \neq \emptyset \neq B$; (ii): $A \cup B \delta C \Leftrightarrow A\delta C$ or $B\delta C$, and $A \delta B \cup C \Leftrightarrow A\delta B$ or $A\delta C$; (iii): $A \cap B \neq \emptyset \Rightarrow A\delta B$; (iv): $x \in X$, $\{x\}\bar{\delta}B \Rightarrow \exists C \subset X$: $\{x\}\bar{\delta}C$ and $X - C\bar{\delta}B$.

(2) δ generates δ^*, i.e. $A\delta^* B$ iff from every finite covering $\underset{\sim}{A}$ and $\underset{\sim}{B}$ of A and B respectively, sets $A_0 \in \underset{\sim}{A}$, $B_0 \in \underset{\sim}{B}$ can be chosen such that $A_0 \delta B_0$ and $B_0 \delta A_0$.

(3) δ generates $\leqq$, i.e. $x \leqq y$ iff $\{x\}\delta\{y\}$. The topology $r(\delta^*)$ of this space is defined so that $V \subset X$ is open iff $\{x\}\overline{\delta^*}X - V$ for any $x \in V$. Denoting by $D(A)$ the intersection of all decreasing $r(\delta^*)$-closed sets containing $A \subset X$, Lemma 3.1 of [15] states

$$D(A) = \bigcap \{B \subset X: X - B\bar{\delta}A\}.$$

If this were true, then, for any increasing $r(\delta^*)$-open set V, we would have

$$V = X - D(X-V) = X - \bigcap \{B \subset X: X - B\bar{\delta}X - V\} =$$

$$= \bigcup \{C \subset X: C\bar{\delta}X - V\} = \{x \in X: \{x\}\bar{\delta}X - V\},$$

which means that each increasing $r(\delta^*)$-open set is open in the topology induced by δ. Example (2.1) shows that in general this is not valid (in fact, let us consider the correspondences $\mathcal{T}_0 \leftrightarrow \delta$, $\mathcal{T}_0^s \leftrightarrow \delta^*$ and $\mathcal{T} \leftrightarrow r(\delta^*)$ defined in the usual way).

Since the cited lemma is used essentially in the proof of Theorem 3.1 of [15] (which states that $r(\delta^*)$ is always regularly preordered), this latter is probably also incorrect.

3. Completely regularly preordered spaces

A preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ will be said *completely regularly preordered* iff there exists a syntopogenous structure spanning it.

(3.1) LEMMA. *Any completely regularly preordered syntopogenous space is regularly preordered.*

PROOF. See (1.5). □

(3.2) EXAMPLE. *A discretely ordered syntopogenous space $(E, \mathscr{S}, =)$ is completely regularly ordered iff it is symmetrizable.*

In this case $\mathscr{S}^u \sim \mathscr{S}^l \sim \mathscr{S}$, therefore, if $\mathscr{S}_0$ spans $(E, \mathscr{S}, =)$, then $\mathscr{S}_0 < \mathscr{S} < \mathscr{S}_0^p$ and $\mathscr{S}_0^c < \mathscr{S} < \mathscr{S}_0^{cp}$, hence $\mathscr{S}_0^s < \mathscr{S} < \mathscr{S}_0^p < \mathscr{S}_0^{sp}$. Since any syntopogenous structure on $(E, =)$ is ^{i}-convex, $\mathscr{S}$ is symmetrizable. The converse is trivial. □

There is an important class of preordered syntopogenous spaces $(E, \mathscr{S}', \leqq)$, in which the following condition is satisfied:

$$(\bullet) \qquad \mathscr{S}'^{up} \sim \mathscr{S}'^{pu} \quad \text{and} \quad \mathscr{S}'^{lp} \sim \mathscr{S}'^{pl}$$

(see [8], (5.3); [9], (3.5)). This is shown by the next theorem, too:

(3.3) THEOREM. *A ^{p}-convex preordered syntopological space $(E, \mathscr{S}, \leqq)$ is completely regularly preordered iff there exists a symmetrical ^{i}-convex syntopogenous structure $\mathscr{S}'$ satisfying $(\bullet)$ on $(E, \leqq)$ such that $\mathscr{S} \sim \mathscr{S}'^p$.*

PROOF. Suppose that there exists such a structure $\mathscr{S}'$. Then $\mathscr{S}_0 = \mathscr{S}'^u$ spans $(E, \mathscr{S}, \leqq)$. In fact, $\mathscr{S}' < \mathscr{S}$, hence $\mathscr{S}_0 < \mathscr{S}^u$ and $\mathscr{S}_0^c = \mathscr{S}'^{uc} \sim \mathscr{S}'^l < \mathscr{S}^l$ (cf. [8], (1.8)). On the other hand $\mathscr{S}^u \sim \mathscr{S}'^{pu} \sim \mathscr{S}'^{up} \sim \mathscr{S}_0^p$, and $\mathscr{S}^l \sim \mathscr{S}'^{pl} \sim \mathscr{S}'^{lp} \sim \mathscr{S}'^{ucp} = \mathscr{S}_0^{cp}$ (cf. [8], (1.8)).

Conversely, if $(E, \mathscr{S}, \leqq)$ is completely regularly preordered, then $\mathscr{S}_0 < \mathscr{S}^u < \mathscr{S}_0^p$ and $\mathscr{S}_0^c < \mathscr{S}^l < \mathscr{S}_0^{cp}$ (consequently $\mathscr{S}_0^p \sim \mathscr{S}^{up} \sim \mathscr{S}^u$ and $\mathscr{S}_0^{cp} \sim \mathscr{S}^{lp} \sim \mathscr{S}^l$) for a suitable syntopogenous structure $\mathscr{S}_0$. $\mathscr{S}_0$ is increasing by [8], (1.1.1), therefore $\mathscr{S}' = \mathscr{S}_0^s$ is ^{i}-convex on $(E, \leqq)$ (see [8], (2.12)). We have

$$\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^l)^p \sim (\mathscr{S}_0^p \vee \mathscr{S}_0^{cp})^p = (\mathscr{S}_0 \vee \mathscr{S}_0^c)^p \sim \mathscr{S}_0^{sp} = \mathscr{S}'^p$$

(cf. [4], (8.99) and (8.107)). $\mathscr{S}'$ fulfils $(\bullet)$, because $\mathscr{S}_0 < \mathscr{S}'^u$, and

$$\mathscr{S}'^{pu} \sim \mathscr{S}^u \sim \mathscr{S}_0^p < \mathscr{S}'^{up} < \mathscr{S}'^{pu}$$

(see [8], (1.7)), furthermore similarly $\mathscr{S}_0^c < \mathscr{S}'^{uc} \sim \mathscr{S}'^l$, thus

$$\mathscr{S}'^{pl} \sim \mathscr{S}^l \sim \mathscr{S}_0^{cp} < \mathscr{S}'^{lp} < \mathscr{S}'^{pl}. \quad \square$$

(3.4) COROLLARY. *Any ^{p}-convex completely regularly preordered syntopological space is symmetrizable.* □

(3.5) COROLLARY. *Any symmetrizable linearly preordered syntopological space is completely regularly preordered (and ^{p}-convex, too).*

PROOF. [9], (1.6), (3.5). □

(3.6) EXAMPLE. There exists a completely regularly preordered topological space, which fails to be symmetrizable.

Let $(\mathbf{R}, \leqq)$ be the naturally ordered real line, $Q \subset \mathbf{R}$ be the set of the rationals. Define a relation $<$ for $A, B \subset \mathbf{R}$ by $A < B$ iff, for any $x \in A$, there exists $\varepsilon > 0$ with $\{x\} \cup ((x-\varepsilon, x+\varepsilon) \cap Q) \subset B$. It is easy to show that $\mathscr{T} = \{<\}$ is a topology on $(\mathbf{R}, \leqq)$, which is not ^{p}-convex (and even not locally convex), thus it cannot be symmetrizable (cf. [3], (5.3)).

If V is an increasing $\mathscr{T}$-open set, and $x \in V$, then $W = \{x\} \cup ((x-\varepsilon, x+\varepsilon) \cap Q) \subset \subset V$ for some $\varepsilon > 0$, therefore $(x-\varepsilon, \infty) = i(W) \subset V$. This shows $\mathscr{T}^u < \mathscr{I}^{ctp}$. Since $\mathscr{I}^{ct}$ is increasing and $\mathscr{I}^{ct} < \mathscr{T}$ is trivial, we get $\mathscr{T}^u \sim \mathscr{I}^{ctp}$. The equivalence $\mathscr{T}^l \sim \mathscr{I}^{tp}$ is analogous, consequently the topogenous structure $\mathscr{T}_0 = \mathscr{I}^{ct}$ spans $(\mathbf{R}, \mathscr{T}, \leqq)$. □

(3.7) PROBLEM. *Find symmetrizable, but non completely regularly preordered syntopogenous (particularly syntopological) spaces.* (Such a space cannot be both linearly preordered and syntopological by (3.5). This problem is in a close conntecion with the study of property (●), too.) □

As the following theorem shows, the class of completely regularly preordered syntopogenous spaces is an extension of that of Priestley's monotonically completely regular spaces [14]:

(3.8) THEOREM. *A preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is completely regularly preordered iff, for each $< \in \mathscr{S}^u$ $(< \in \mathscr{S}^l)$, there exists an $(\mathscr{S}, \mathscr{I}^s)$-continuous functional family φ such that whenever $x < V$, then a preorder inversing (preserving) function $f \in \varphi$ can be found with the properties $f(E) \subset [0, 1]$, $f(x) = 0$ and $f(y) = 1$ for $y \in E - V$.*

In this case, denoting by Φ the set of all $(\mathscr{S}, \mathscr{I}^s)$-continuous ordering families consisting of preorder inversing functions, $\mathscr{S}_\Phi$ is the finest syntopogenous structure spanning $(E, \mathscr{S}, \leqq)$.

REMARK. The comparison of this theorem with [8], (4.8) makes more clear both the similarity and the difference existing between the notions of symmetrizability and completely regular preorderedness.

PROOF. We use the terminology of ch. 12 of [4]. It is easy to see that an ordering family φ is $(\mathscr{S}, \mathscr{I}^s)$-continuous and consists of preorder inversing functions if, and only if, $-\varphi = \{-f : f \in \varphi\}$ is $(\mathscr{S}, \mathscr{I}^s)$-continuous and consists of preorder preserving functions (cf. [8], (4.9)).

Let Φ denote the set of all $(\mathscr{S}, \mathscr{I}^s)$-continuous ordering families of preorder inversing functions. We show that under our conditions $\mathscr{S}_\Phi$ spans $(E, \mathscr{S}, \leqq)$. In fact, $\mathscr{S}_\Phi$ is increasing on $(E, \leqq)$ by [8], (1.5), (1.1.6), and any $\varphi \in \Phi$ is $(\mathscr{S}, \mathscr{I})$-continuous, therefore $\mathscr{S}_\Phi < \mathscr{S}$ (see [4], (12.33)), consequently $\mathscr{S}_\Phi < \mathscr{S}^u$. In view of the above said, we have

$$\mathscr{S}_\Phi^c = \Big(\bigvee_{\varphi \in \Phi} \mathscr{S}_\varphi\Big)^c = \bigvee_{\varphi \in \Phi} \mathscr{S}_\varphi^c = \bigvee_{\varphi \in \Phi} \mathscr{S}_{-\varphi} = \bigvee_{\psi \in \Psi} \mathscr{S}_\psi = \mathscr{S}_\Phi,$$

where Ψ denotes the set of all $(\mathscr{S}, \mathscr{I}^s)$-continuous ordering families consisting of preorder preserving functions. Applying again [8], (1.5), (1.1.6), [4], (12.33), we obtain $\mathscr{S}_\Psi < \mathscr{S}^l$, thus $\mathscr{S}_\Phi^c < \mathscr{S}^l$ is also valid. If $< \in \mathscr{S}^u$, and φ is the functional family determined by the condition, then, for any $\varepsilon>0$, there exists $<_{(\varepsilon)}\in\mathcal{S}$ such that

$$(+)\qquad f^{-1}(<_\varepsilon^s)\subset<_{(\varepsilon)}$$

for each $f\in\varphi$. Denote by φ' the set of all bounded, preorder inversing functions on $(E, \leqq)$, which have property $(+)$ for every $\varepsilon>0$. It can be easily shown that φ' is an ordering family, thus $\varphi'\in\Phi$. Because of the choice of φ, $<\subset<_{\varphi',1}^p$, that is $\mathcal{S}^u<\mathcal{S}_\Phi^p$. In the same manner $\mathcal{S}^l<\mathcal{S}_\Phi^p=\mathcal{S}_\Phi^{cp}$.

Conversely, suppose that $(E, \mathcal{S}, \leqq)$ is completely regularly preordered, and let $\mathcal{S}_0$ be a syntopogenous structure spanning it. Denoting by Φ_0 the set of all $(\mathcal{S}_0, \mathcal{I})$-continuous ordering families, $\mathcal{S}_0\sim\mathcal{S}_{\Phi_0}$ holds ([8], (1.11)). On the basis of $\mathcal{S}_0<\mathcal{S}^u<\mathcal{S}$ and $\mathcal{S}_0^c<\mathcal{S}^l<\mathcal{S}$, the inequality $\mathcal{S}_0^s<\mathcal{S}$ holds, thus each $\varphi\in\Phi_0$ is $(\mathcal{S}_0^s, \mathcal{I})$-, $(\mathcal{S}_0^s, \mathcal{I}^s)$-, and $(\mathcal{S}, \mathcal{I}^s)$-continuous. Because of $\mathcal{S}_0<\mathcal{S}^u$ the structure $\mathcal{S}_0$ is increasing, hence every $\varphi\in\Phi_0$ consists of preorder inversing functions (see [8], (1.6)). For $<\in\mathcal{S}^u$, there exists $<_0\in\mathcal{S}_0$ with $<\subset<_0^p$. Using theorem (12.41) of [4], an ordering family $\varphi\in\Phi_0$ can be chosen such that $<_0\subset<_{\varphi,1}$, which means $<\subset<_{\varphi,1}^p$. Thus, for $x<V$, there is $f\in\varphi$ with $f(E)\subset[0, 1]$, $f(x)=0$ and $f(y)=1$ for $y\in E-V$. Turning to $\mathcal{S}_0^c$ and $\mathcal{S}^l$ from $\mathcal{S}_0$ and $\mathcal{S}^u$ respectively, the remaining part of the condition can be verified.

We showed that if $(E, \mathcal{S}, \leqq)$ is completely regularly preordered, then $\mathcal{S}_\Phi$ spans it. From the above consideration it follows also $\Phi_0\subset\Phi$, hence $\mathcal{S}_0<\mathcal{S}_\Phi$ holds.

REMARK. With a slight modification of the proof one can see that in the above theorem $\mathcal{H}=\mathcal{I}^{sb}$, $\mathcal{I}^{st}$ or $\mathcal{H}^{tp}$ can be written instead of $\mathcal{I}^s$, provided $\mathcal{S}$ is perfect, simple or topological, respectively. □

In order to show an example for completely regularly preordered spaces, let us recall that a syntopogenous structure $\mathcal{S}$ on a set E is *locally compact*, if there is an order $<\in\mathcal{S}$ such that, for each $x\in E$, an $\mathcal{S}$-compact set $K_x\subset E$ can be found with the property $x<K_x$ (see [11]).

(3.9) THEOREM (cf. [11], (2.6)). *Any locally compact, continuous, S_2-preordered syntopological space is completely regularly preordered.*

First of all let us consider the following lemma:

(3.10) LEMMA. *Let $(E, \mathcal{S}, \leqq)$ be a locally compact, continuous, S_2-preordered syntopological space, and $<\in\mathcal{S}^u$ $(<\in\mathcal{S}^l)$. There exist $<'\in\mathcal{S}^u$ and $<''\in\mathcal{S}^l$ such that, if $K_0<V$ for an $\mathcal{S}$-compact set K_0, then there is an $\mathcal{S}$-compact set K with the property $i(K_0)<'i(K)<''^cX<'V$ $(d(K_0)<''d(K)<'^cX<''V)$.*

PROOF. Choose $<_1\in\mathcal{S}^u$ with $<\subset<_1^4$, and $<_2\in\mathcal{S}^u$, $<_2'\in\mathcal{S}^l$ with $<_1\subset$ $\subset(<_2<_2'^c)^b$ (see (S_2)). Suppose $<_3\in\mathcal{S}^u$, $<_3'\in\mathcal{S}^l$, for which $<_2\subset<_3^2$, $<_2'\subset<_3'^2$. Since $\mathcal{S}^l\sim\mathcal{S}^{lp}$, an order $<_4'\in\mathcal{S}^l$ can be selected such that $<_2'^p\subset<_4'$. In view of the local compactness there is $<_0\in\mathcal{S}$ such that $x\in E$ implies $x<_0K_x$ for a suitable compact $K_x\subset E$. Assume $<_1'\in\mathcal{S}$, $<_3\subset<_1'$ and $<_0\subset<_1'^2$. Let us choose the required orders $<'\in\mathcal{S}^u$, $<''\in\mathcal{S}^l$ so that, on the one hand $<_1\cup i(<_1')\subset<'$ (see [9], (2.4.5)), and on the other hand $<_4'\subset<''^2$.

If $K_0<V$, where K_0 is $\mathcal{S}$-compact, then there exist sets $F, J, X\subset E$ such that

$$(3.10.1)\qquad K_0<_1F<_1J<_1X<_1V.$$

We show that

(3 10.2) there exists a compact $K \subset J$ with $i(K_0) <' i(K)$.

In fact, in view of the choice of $<_0, <_1'$, for each $x \in E$, there is $H_x \subset E$ such that $x <_1' H_x <_1' K_x$. If H denotes the union of the finite number of the sets H_x covering K_0, then joining the corresponding sets K_x, we obtain a set K' such that

(3.10.3) $K_0 \subset H <_1' K'$ and K' is compact.

If $K' \subset J$, then the verification of (3.10.2) is finished, because from $i(<_1') \subset <'$ and $K_0 <_1' K'$ the inequality $i(K_0) <' i(K')$ follows. Therefore let us suppose $K' \not\subset J$, and put $J' = K' - J$, $F' = K' - F$. If $x \in K_0$ is fixed, then

(3.10.4) $x <_2 A_{xy} <_3'^c B_{xy} <_3'^c E - y$

for any $y \in F'$ by (3.10.1). Because of (3.10.1) again, there exists a closed set S such that $E - J \subset S \subset E - F$, thus $S \cap K'$ is closed in $\mathcal{S}|K'$, for which $J' \subset S \cap K' \subset F'$. (3.10.4) yields that $(E - B_{xy}) \cap K'$ is a neighbourhood of any $y \in F'$ in $\mathcal{S}|K'$, therefore J' is covered by a finite subsystem of the sets $E - B_{xy}$. Denoting by $E - B_x$ and $E - A_x$ the union of the corresponding sets $E - B_{xy}$ and $E - A_{xy}$, respectively, we obtain

$$x <_2 A_x <_3'^c B_x \subset E - J' = (E - K') \cup J.$$

Taking a set C_x for each $x \in K_0$ such that $x <_3 C_x <_3 A_x$, and denoting by C, A and B the union of the finite number of the sets C_x covering K_0, and the corresponding sets A_x and B_x, respectively, we have

(3.10.5) $K_0 \subset C <_3 A <_3'^c B \subset (E - K') \cup J$.

Further on let $\bar{A}$ be the $\mathcal{S}$-closure of A. It is easy to show that $K = K' \cap \bar{A}$ is also compact in $\mathcal{S}$. Owing to (3.10.5) $\bar{A} \subset B$, thus $K \subset J$. Since from (3.10.3) and (3.10.5) we get $K_0 <_1' K'$ and $K_0 <_1' A \subset \bar{A}$, it is obvious that $K_0 <_1' K$, hence $i(K_0) <' i(K)$.

After the verification of (3.10.2), let K be the compact set determined there. For a fixed $y \in E - X$ and for every $k \in K$, $k <_2 V_{ky} <_2'^c E - y$ issues from (3.10.1). Therefore a set V_y can be found such that $K \subset V_y <_2'^c E - y$. Applying this for any $y \in E - X$, we obtain $E - X <_2'^p E - K$, hence $E - X <_4' E - K$. Put $E - X <'' W <'' E - K$. Then by [8], (1.2) $d(W) \subset E - K$, consequently $i(K) \subset E - d(W) \subset E - W - W <''^c X$, that is

(3.10.6) $i(K) <''^c X$.

Finally, on the basis of the choice of $<'$, $X <' V$ holds, thus (3.10.2), (3.10.6) give the proof of the first part of the lemma. The other part is the dual of the first one. $\square$

The proof of the theorem will be continued with a construction of an order $[<''^c, <']$ for any $<' \in \mathcal{S}^u$, $<'' \in \mathcal{S}^l$ as follows: Define, for $A, B \subset E$, the relations $I(<''^c, <')$ and $D(<'^c, <'')$ so that

(3.11.1) $A(<''^c, <')B$ iff either $A = \emptyset$, or $B = E$, or $A \subset i(K) <''^c X <' B$,

where K is compact, and dually

(3.11.2) $AD(<'^c, <'')B$ iff either $A = \emptyset$, or $B = E$, or $A \subset d(K) <'^c X <'' B$, where K is compact. It is obvious that $I(<''^c, <')$ and $D(<'^c, <'')$ are semi-topogenous orders on E.

(3.12) LEMMA. *Under the conditions of* (3.10)—(3.11) *consider the topogenous order*

$$[<''^c, <'] = \big(I(<''^c, <')\cup D(<'^c, <'')^c\big)^q. \tag{3.12.1}$$

Then the order family

$$\mathcal{S}^+ = \{[<''^c, <']\colon <'\in\mathcal{S}^u, <''\in\mathcal{S}^l\} \tag{3.12.2}$$

is a syntopogenous structure spanning $(E, \mathcal{S}, \leqq)$.

PROOF. If $<'_1, <'_2\in\mathcal{S}^u$ and $<''_1, <''_2\in\mathcal{S}^l$, then there exist $<'\in\mathcal{S}^u$, $<''\in\mathcal{S}^l$ such that $<'_1\cup<'_2\subset<'$ and $<''_1\cup<''_2\subset<''$. It is clear that

$$I(<''^c_1, <'_1)\cup I(<''^c_2, <'_2)\subset I(<''^c, <')$$

and

$$D(<'^c_1, <''_1)\cup D(<'^c_2, <''_2)\subset D(<'^c, <''),$$

thus

$$[<''^c_1, <'_1]\cup[<''^c_2, <'_2]\subset[<''^c, <']$$

by [4], (3.25).

After this suppose $<'\in\mathcal{S}^u$, $<''\in\mathcal{S}^l$. We prove $[<''^c, <']\subset[<''^c_0, <'_0]^2$ for suitable $<'_0\in\mathcal{S}^u$, $<''_0\in\mathcal{S}^l$. In fact, for $<'\in\mathcal{S}^u$, let us choose $<'_1\in\mathcal{S}^u$, $<''_1\in\mathcal{S}^l$ in accordance with (3.10), and select $<'_2\in\mathcal{S}^u$, $<''_2\in\mathcal{S}^l$ for $<'_1$ in the same manner. If $AI(<''^c, <')B$, then $A\subset i(K)<'B$ for some compact set K, thus $K<'B$. By (3.10) $i(K)<'_1 i(K')<''^c_1 X<'_1 B$, where K' is compact. Because of $K<'_1 i(K')$ and (3.10) we have

$$A\subset i(K)<''^c_2 Y<'_2 i(K')<''^c_1 X<'_1 B,$$

therefore, if $<'_1\cup<'_2\subset<'_3\in\mathcal{S}^u$ and $<''_1\cup<''_2\subset<''_3\in\mathcal{S}^l$, then

$$I(<''^c, <')\subset I(<''^c_3, <'_3)^2. \tag{3.12.3}$$

Dually to the above reasoning, choose $<''_4\in\mathcal{S}^l$, $<'_4\in\mathcal{S}^u$ for $<''$, and similarly $<''_5\in\mathcal{S}^l$, $<'_5\in\mathcal{S}^u$ for $<''_4$ in accordance with (3.10), further consider $<''_4\cup<''_5\subset<''_6\in\mathcal{S}^l$ and $<'_4\cup<'_5\subset<'_6\in\mathcal{S}^u$. Then

$$D(<'^c, <'')\subset D(<'^c_6, <''_6)^2. \tag{3.12.4}$$

Assuming $<'_3\cup<'_6\subset<'_0\in\mathcal{S}^u$, $<''_3\cup<''_6\subset<''_0\in\mathcal{S}^l$, from (3.12.3), (3.12.4) and [4], (2.16), (3.53) the inequality $[<''^c, <']\subset[<''^c_0, <'_0]^2$ follows, thus $\mathcal{S}^+$ is in fact a syntopogenous structure.

If $<'\in\mathcal{S}^u$, $<''\in\mathcal{S}^l$, then $I(<''^c, <')\subset<'$ and $D(<'^c, <'')^c\subset<'^{cc}=<'$, besides $I(<''^c, <')^c\subset<''^{cc}=<''$ and $D(<'^c, <'')\subset<''$, thus

$$[<''^c, <']\subset<' \quad\text{and}\quad [<''^c, <']^c\subset<''. \tag{3.12.5}$$

In view of the compactness of the one point sets and (3.10), for $<'\in\mathcal{S}^u$, $<''\in\mathcal{S}^l$, there exist $<_1', <_2' \in \mathcal{S}^u$, $<_1'', <_2'' \in \mathcal{S}^l$ such that

(3.12.6)

$$<' \subset I(<_1''^c, <_1')^p \subset [<_1''^c, <_1']^p \quad \text{and} \quad <'' \subset D(<_2'^c, <_2'')^p \subset [<_2''^c, <_2']^{cp},$$

hence by (3.12.5), (3.12.6) $\mathcal{S}^+$ spans $(E, \mathcal{S}, \leqq)$. □

Thus theorem (3.9) is proved. □

4. Normally preordered spaces

A preordered syntopogenous space $(E, \mathcal{S}, \leqq)$ will be called *normally preordered* iff $\mathcal{S}^{lc}\mathcal{S}^u < \mathcal{S}^u\mathcal{S}^{lc}$.

(4.1) REMARKS. A discretely ordered syntopogenous space $(E, \mathcal{S}, =)$ is normally ordered iff $[E, \mathcal{S}]$ is normal in the sense of [11] (see also [6]). A preordered topological space is normally preordered iff its "classical" associated is normally preordered in the sense of Nachbin ([13], p. 28; see also e.g. [7], [12], [15]). □

The following theorem is a common generalization of [6], (2.20) (see also [11], (1.14)) and [15], Th. 3.4:

(4.2) THEOREM. *The preordered syntopogenous space $(E, \mathcal{S}, \leqq)$ is normally preordered iff $\mathcal{S}^{lc}\mathcal{S}^u$ is a syntopogenous structure on E. Then this is the finest of all syntopogenous structures $\mathcal{S}_0$ on E such that $\mathcal{S}_0 < \mathcal{S}^u$ and $\mathcal{S}_0^c < \mathcal{S}^l$.*

PROOF. Assume that $(E, \mathcal{S}, \leqq)$ is normally preordered. Then $\mathcal{S}^{lc}\mathcal{S}^u$ is obviously directed, moreover $\mathcal{S}^{lc}\mathcal{S}^u < \mathcal{S}^{lc2}\mathcal{S}^{u2} < \mathcal{S}^{lc}\mathcal{S}^{lc}\mathcal{S}^u\mathcal{S}^u < \mathcal{S}^{lc}\mathcal{S}^u\mathcal{S}^{lc}\mathcal{S}^u < (\mathcal{S}^{lc}\mathcal{S}^u)^2$, thus $\mathcal{S}^{lc}\mathcal{S}^u$ is a syntopogenous structure.

Conversely, if $\mathcal{S}^{lc}\mathcal{S}^u$ is a syntopogenous structure, then $\mathcal{S}^{lc}\mathcal{S}^u < (\mathcal{S}^{lc}\mathcal{S}^u)^2 < \mathcal{S}^{lc}\mathcal{S}^u\mathcal{S}^{lc}\mathcal{S}^u < \mathcal{S}^u\mathcal{S}^{lc}$, that is $(E, \mathcal{S}, \leqq)$ is normally preordered.

Finally, if $\mathcal{S}_0$ is a syntopogenous structure on E such that $\mathcal{S}_0 < \mathcal{S}^u$ and $\mathcal{S}_0^c < \mathcal{S}^l$, then $\mathcal{S}_0 < \mathcal{S}_0^2 < \mathcal{S}_0^{cc}\mathcal{S}_0 < \mathcal{S}^{lc}\mathcal{S}^u$. □

Theorem 1 of [13] can be generalized as follows:

(4.3) THEOREM. *The preordered syntopogenous space $(E, \mathcal{S}, \leqq)$ is normally preordered iff, for every $<' \in \mathcal{S}^u$, $<'' \in \mathcal{S}^l$, there exists an $(\mathcal{S}, \mathcal{I}^s)$-continuous functional family φ on E such that $A <'^c C <'' B$ implies $f(E) \subset [0, 1]$, $f(x) = 0$, $f(y) = 1$ for any $x \in A$, $y \in E - B$, where f is a suitable preorder preserving function of φ.*

PROOF. Let $(E, \mathcal{S}, \leqq)$ be normally preordered. Then $\mathcal{S}_0 = \mathcal{S}^{lc}\mathcal{S}^u$ is a syntopogenous structure on E, and $\mathcal{S}_0^c = \mathcal{S}^{uc}\mathcal{S}^l$ is decreasing, because $\mathcal{S}_0^c < \mathcal{S}^l$. By [4], (12.41), for any $< \in \mathcal{S}_0^c$, there exists an $(\mathcal{S}_0^c, \mathcal{I})$-continuous ordering family φ on E such that $A < B$ implies $f(E) \subset [0, 1]$, $f(x) = 0$, $f(y) = 1$ for each $x \in A$, $y \in E - B$, where $f \in \varphi$. Because of [8], (1.6) f is preorder preserving, and owing to

$$\mathcal{S}_0^s \sim \mathcal{S}_0 \vee \mathcal{S}_0^c < \mathcal{S}^u \vee \mathcal{S}^l < \mathcal{S}$$

φ is $(\mathcal{S}_0^s, \mathcal{I})$-, $(\mathcal{S}_0^s, \mathcal{I}^s)$-, and $(\mathcal{S}, \mathcal{I}^s)$-continuous.

Conversely, assume $<'\in\mathcal{S}^u$, $<''\in\mathcal{S}^l$, and let φ be a functional family on E satisfying the condition of the theorem. Suppose that φ_0 is the family of the preorder preserving functions of φ. Then both φ_0 and $-\varphi_0=\{-f: f\in\varphi_0\}$ are $(\mathcal{S},\mathcal{J}^s)$-, and *a fortiori* $(\mathcal{S},\mathcal{J})$-continuous, thus $\mathcal{S}_{\varphi_0}<\mathcal{S}$ and $\mathcal{S}_{-\varphi_0}<\mathcal{S}$. From this $\mathcal{S}_{\varphi_0}<\mathcal{S}^l$ and $\mathcal{S}_{-\varphi_0}<\mathcal{S}^u$ by [8], (1.5). Regarding [8], (4.9), we have $\mathcal{S}_{-\varphi_0}=\mathcal{S}^c_{\varphi_0}$. In view of the assumption concerning φ, the inequality $<'^c<''\subset<_{\varphi_0,1}$ is valid, hence

$$\{<'^c<''\}<\mathcal{S}_{\varphi_0}<\mathcal{S}^2_{\varphi_0}<\mathcal{S}_{\varphi_0}\mathcal{S}_{\varphi_0}=\mathcal{S}_{\varphi_0}\mathcal{S}^c_{-\varphi_0}<\mathcal{S}^l\mathcal{S}^{uc}.$$

With respect to the arbitrary choice of $<'$ and $<''$, we get

$$\mathcal{S}^{lc}\mathcal{S}^u=(\mathcal{S}^{uc}\mathcal{S}^l)^c<(\mathcal{S}^l\mathcal{S}^{uc})^c=\mathcal{S}^u\mathcal{S}^{lc},$$

that is $(E,\mathcal{S},\leqq)$ is normally preordered. □

It is well-known that in order that a normal topological space be completely regular the condition (S_1) is necessary and sufficient (see [5], (4.2.5)). The following theorem gives such a condition for normally preordered syntopogenous spaces (cf. [11], (1.17)).

(4.4) THEOREM. *A normally preordered syntopogenous space* $(E,\mathcal{S},\leqq)$ *is completely regularly preordered iff the following condition is satisfied:*

(P) $$\mathcal{S}^u<\mathcal{S}^{lcp} \quad \textit{and} \quad \mathcal{S}^l<\mathcal{S}^{ucp}.$$

In this case $\mathcal{S}^{lc}\mathcal{S}^u$ *is the finest of all syntopogenous structures spanning* $(E,\mathcal{S},\leqq)$.

(4.5) LEMMA (cf. [11], (1.15), (2.7)). *Condition* (P) *is weaker than* (S_3) *and stronger than* (S_1). *If* $(E,\mathcal{S},\leqq)$ *is either perfect or compact, then* (P) *and* (S_1) *are equivalent.*

PROOF. If $(E,\mathcal{S},\leqq)$ is S_3- (i.e. regularly) preordered, then $\mathcal{S}^u<(\mathcal{S}^u\mathcal{S}^{lc})^p<\mathcal{S}^{lcp}$ and $\mathcal{S}^l<(\mathcal{S}^l\mathcal{S}^{uc})^p<\mathcal{S}^{ucp}$. If $(E,\mathcal{S},\leqq)$ has the property (P), then $\mathcal{S}^{ub}<\mathcal{S}^{lcpb}=$ $=\mathcal{S}^{lcb}$ and

$$\mathcal{S}^{lcb}<\mathcal{S}^{ucpcb}=\mathcal{S}^{ucpbc}=\mathcal{S}^{ucbc}=\mathcal{S}^{ubcc}=\mathcal{S}^{ub},$$

thus $\mathcal{S}^{ub}\sim\mathcal{S}^{lcb}$ (cf. [4], (5.22), (5.33), (5.17)).

Let now $(E,\mathcal{S},\leqq)$ be perfect and S_1-preordered. Then $\mathcal{S}^u\sim\mathcal{S}^{up}$ and $\mathcal{S}^l\sim$ $\sim\mathcal{S}^{lp}$ by [8], (1.7). From here

$$\mathcal{S}^u<\mathcal{S}^{ub}\sim\mathcal{S}^{lcb}=\mathcal{S}^{lccpcp}=\mathcal{S}^{lpcp}\sim\mathcal{S}^{lcp}$$

and

$$\mathcal{S}^l<\mathcal{S}^{lb}=\mathcal{S}^{lbcc}=\mathcal{S}^{lcbc}\sim\mathcal{S}^{ubc}=\mathcal{S}^{uccpcp}=\mathcal{S}^{upcp}\sim\mathcal{S}^{ucp}$$

([4], (5.33), (5.17)).

Finally assume that $(E,\mathcal{S},\leqq)$ is a compact S_1-preordered space. For $<\in\mathcal{S}^u$ choose $<_1\in\mathcal{S}^u$ with $<\subset<_1^2$. Suppose $<_2\in\mathcal{S}^l$ such that $<_1^b\subset<_2^{cb}$, and put $<_3\in\mathcal{S}^l$, where $<_2\subset<_3^2$. Then $x<B$ implies $x<_1C<_1B$ for some $C\subset E$. If $y\in E-C$, then $y<_2E-x$, thus $y<_3V_y<_3E-x$. Because of $\mathcal{S}^u<\mathcal{S}$, $\mathcal{S}^l<\mathcal{S}$ and [4], (15.93), a finite number of the sets V_y covers $E-B$, therefore $E-B<_3E-x$, that is $x<_3^cB$. If $A<B$, then we get $x<_3^cB$ for any $x\in A$, thus $A<_3^{cp}B$, hence $<\subset<_3^{cp}$. The proof of $\mathcal{S}^l<\mathcal{S}^{ucp}$ is dual. □

PROOF OF (4.4). If $(E, \mathscr{S}, \leqq)$ is completely regularly preordered, then it has (S_3), thus satisfies (P) (see (3.1), (4.5)).

Conversely, assume that $(E, \mathscr{S}, \leqq)$ has property (P). We verify that $\mathscr{S}^{lc}\mathscr{S}^u$ spans $(E, \mathscr{S}, \leqq)$. In fact, if $<\in\mathscr{S}^u$, then there exists $<_1\in\mathscr{S}^u$ so that $<\boldsymbol{\subset}<_1^2$, further an order $<_2\in\mathscr{S}^l$ can be selected such that $<_1\boldsymbol{\subset}<_2^{cp}$. If $A<B$, then $A<_1C<_1$ $<_1B$ for a suitable $C\subset E$. Thus $A<_2^{cp}C$, which means $x<_2^cC$, i.e. $x(<_2^c<_1)B$ for any $x\in A$, hence $A(<_2^c<_1)^pB$. This shows $\mathscr{S}^u\boldsymbol{<}(\mathscr{S}^{lc}\mathscr{S}^u)^p$. The proof of $\mathscr{S}^l\boldsymbol{<}$ $\boldsymbol{<}(\mathscr{S}^{uc}\mathscr{S}^l)^p=(\mathscr{S}^{lc}\mathscr{S}^u)^{cp}$ is analogous. These give that $\mathscr{S}^{lc}\mathscr{S}^u$ spans $(E, \mathscr{S}, \leqq)$ by (4.2). If $\mathscr{S}_0$ is another syntopogenous structure spanning $(E, \mathscr{S}, \leqq)$, then from (4.2) it follows that $\mathscr{S}_0\boldsymbol{<}\mathscr{S}^{lc}\mathscr{S}^u$. □

(4.6) EXAMPLE. *Any symmetrical preordered syntopogenous space is normally preordered and has property* (P), *thus it is completely regularly preordered.*

This statement is an immediate consequence of [8], (1.8). □

Now we prove the theorem corresponding to [11], (2.8) (cf. [13], th. 4):

(4.7) THEOREM. *Any compact* S_2*-preordered syntopogenous space is normally preordered.*

PROOF. Let $(E, \mathscr{S}, \leqq)$ be compact S_2-preordered, and suppose $<\in\mathscr{S}^u$, $<'\in\mathscr{S}^l$. Assume $<_1\in\mathscr{S}^u$, $<\boldsymbol{\subset}<_1^2$, and $<_2\in\mathscr{S}^u$, $<_2'\in\mathscr{S}^l$ are such that $<_1\boldsymbol{\subset}(<_2<_2'^c)^b$, finally $<_3\in\mathscr{S}^u$, $<_3'\in\mathscr{S}^l$, for which $<_2\boldsymbol{\subset}<_3^2$ and $<_2'\boldsymbol{\subset}<_3'^2$. If $A<'^cC<B$, then there is $X\subset E$ with $A<'^cC<_1X<_1B$, therefore $x<_2C_{xy}<_2'^cE-y$ for each $x\in C$, $y\in E-X$. From this $x<_3A_{xy}<_3C_{xy}<_3'^cB_{xy}<_3'^cE-y$. Put $\mathscr{S}^{tp}=\{<^*\}$. Then $A<^{*c}C$ and $x<^*A_{xy}$ for any $x\in C$, thus from [4], (15.93) it follows that a finite number of the sets A_{xy} covers A, hence there are sets A_y, C_y, B_y such that

$$A\subset A_y<_3C_y<_3'^cB_y<_3'^cE-y.$$

Since $E-B^{*c}E-X$ and $y<^*E-B_y$ for each $y\in E-X$, the set $E-B$ is covered by a finite number of the sets $E-B_y$. Denoting by $E-B'$, $E-A'$ and $E-C'$ the union of the corresponding sets $E-B_y$, $E-A_y$ and $E-C_y$ respectively, we have

$$A\subset A'<_3C'<_3'^cB'\subset B,$$

therefore in view of the arbitrary choice of $<$, $<'$, we obtain $\mathscr{S}^{lc}\mathscr{S}^u\boldsymbol{<}\mathscr{S}^u\mathscr{S}^{lc}$, that is $(E, \mathscr{S}, \leqq)$ is normally preordered. □

(4.8) THEOREM. *A compact preordered syntopogenous space* $(E, \mathscr{S}, \leqq)$ *is completely regularly preordered iff it is* S_2*-preordered. In this case (disregarding equivalences)* $\mathscr{S}^{lc}\mathscr{S}^u$ *is the unique syntopogenous structure spanning* $(E, \mathscr{S}, \leqq)$.

PROOF. Any completely regularly preordered space is S_2-preordered. Conversely, every compact S_2-preordered space is normally preordered, and since it is S_1-preordered, it has property (P), so that it is completely regularly preordered.

Further on suppose that $\mathscr{S}_0$ is a syntopogenous structure spanning the compact S_2-preordered space $(E, \mathscr{S}, \leqq)$. If $<\in\mathscr{S}^u$ and $<'\in\mathscr{S}^l$ are arbitrary, then there exists $<_0'\in\mathscr{S}_0$ such that $<\boldsymbol{\subset}<_0'^p$, and there is $<_0\in\mathscr{S}_0$ with $<_0'\boldsymbol{\subset}<_0^2$. Putting

$A<'^c C<B$, we have $x<_0 C_x<_0 B$ for each $x\in C$ and a suitable set $C_x\subset E$. If we denote by C' the union of the finite number of the sets C_x covering A (see [4], (15.93)), then $A\subset C'<_0 B$. Thus $\mathscr{S}^{lc}\mathscr{S}^u < \mathscr{S}_0$. The inequality in the opposite direction follows from (4.4). $\square$

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BEMERKUNGEN ZU EINER ARBEIT VON INGHAM ÜBER DIE VERTEILUNG DER PRIMZAHLEN

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1. Einleitung

Riemann [9] vermutete in 1859, daß

$$(1.1) \qquad \pi(x) < \operatorname{li} x = \int_0^x \frac{dt}{\log t} \quad \text{für} \quad x > 2.$$

Im Jahre 1914 bewies Littlewood [5] jedoch, daß die Funktion $\pi(x) - \operatorname{li} x$ unendlich viele Zeichenwechsel hat. Sein Beweis war ein reiner Existenzbeweis und erlaubte nicht, ein X_0 so zu bestimmen, daß $\pi(x) - \operatorname{li} x > 0$ für mindestens ein $x < X_0$. Weiterhin war es nicht möglich, für die Anzahl $V(y)$ der Zeichenwechsel von $\pi(x) - \operatorname{li} x$ im Intervall $[2, y]$ eine Abschätzung (weder effektiv noch ineffektiv) nach unten zu geben.

Eine obere Schranke für den ersten Zeichenwechsel konnte Skewes [11] im Jahre 1955 angeben:

$$(1.2) \qquad X_0 < \exp_4 7{,}705$$

(wobei $\exp_1 x = \exp x = e^x$, $\exp_{\nu+1} x = \exp \exp_\nu x$).

Über die Anzahl der Zeichenwechsel bewies Ingham [2] im Jahre 1936:

Sei

$$(1.3) \qquad \theta = \sup_{\zeta(\varrho)=0} (\operatorname{Re} \varrho),$$

und es existiere eine Nullstelle ϱ_0 mit

$$(1.4) \qquad \operatorname{Re} \varrho_0 = \theta,$$

dann gilt:

SATZ A (Ingham). *Falls eine Nullstelle der ζ-Funktion existiert, die* (1.4) *erfüllt, dann hat* $\pi(x) - \operatorname{li} x$ *mindestens einen Zeichenwechsel in allen Intervallen der From*

$$(1.5) \qquad [y, Dy] \quad \textit{für} \quad y > Y_0,$$

wobei $D > 1$ *eine absolute Konstante ist.*

Daraus folgt unmittelbar

SATZ B (Ingham). *Unter derselben Annahme wie in Satz* A *gilt:*

$$(1.6) \qquad V(y) > \frac{\log y}{2 \log D} \quad \textit{für} \quad y > Y_1.$$

Die Methode von Ingham erlaubte es jedoch nicht, die Konstante D effektiv anzugeben: falls $\theta > 1/2$, hängt D

(i) von $\theta - 1/2$,

(ii) von der Verteilung der Nullstellen „nahe“ der Geraden $\sigma = \theta$,

(iii) von $\left(1 - \frac{\gamma'}{\gamma^*}\right)$, wobei $\gamma' = \min\{\gamma > 0 | \zeta(\theta + i\gamma) = 0\}$ und $\gamma^* = \min\{\gamma > \gamma' | \zeta(\theta + i\gamma) = 0\}$ $(\min \emptyset = \infty)$

ab;

falls $\theta = 1/2$, liegt der Grund der Ineffektivität in der Verwendung eines Satzes von Bohl aus der Theorie der diophantischen Approximation:

SATZ (Bohl). *Für beliebige reelle* $x_1, \ldots, x_N$ *und* $0 < \varepsilon < 1/2$ *existiert ein* $L = L(\varepsilon, x_1, \ldots, x_N)$, *so daß für alle* $m \in \mathbf{N}$ *ein* $n \in \mathbf{N}$, $n \leqq L$ *existiert, für welche*

$$\|(m+n)x_i\| \leqq \varepsilon \tag{1.7}$$

gilt, wobei $\|x\| = \min\{x - [x], 1 - x + [x]\}$.

Ziel unserer Arbeit ist es, für den Fall, daß die Riemannsche Vermutung wahr ist (womit insbesondere die Annahme der Sätze A and B erfüllt ist), eine effektive Form von Satz B zu beweisen. Darüber hinaus zeigen wir einen zum Satz A ähnlichen, aber viel schwächeren, jedoch effektiven Satz. Außerdem diskutieren wir noch das Problem der Effektivierbarkeit von Satz A.

Wir zeigen

SATZ 1. *Unter Annahme der Riemannschen Vermutung gilt*

$$V(y) > \frac{1}{\exp_2 7{,}707} \log y > 10^{-957} \log y \tag{1.8}$$

für $y > \exp_3 7{,}707$.

SATZ 2. *Unter Annahme der Riemannschen Vermutung hat* $\pi(x) - \mathrm{li}\, x$ *mindestens einen Zeichenwechsel im Intervall*

$$[y^{10^{-957}}, y] \tag{1.9}$$

für $y > \exp_3 7{,}707$.

Wir weisen darauf hin, daß es mit komplizierteren Methoden möglich ist, Satz 2 auch ohne Annahme zu beweisen; allerdings mit einer kleineren Konstanten als 10^{-957}.

Daß die Ineffektivität von Satz A sehr eigenartig ist, zeigt der folgende

SATZ 3. *Falls die Riemannsche Vermutung wahr ist und die Imaginärteile der ersten* 69 *Nullstellen mit positivem Imaginärteil über dem rationalen Zahlkörper unabhängig sind, dann kann man in endlich vielen Rechenschritten effektive Werte für* D *und* Y_0 *im Satz* A *angeben; jedoch kann man die Anzahl* H *der dazu nötigen Rechenschritte nicht im voraus abschätzen.*

Möglicherweise führt dieselbe Methode auch im Falle der linearen Abhängigkeit der Imaginärteile der Nullstellen in endlich vielen Rechenschritten zu effektiven Werten von D und Y_0; dies können wir aber nicht garantieren.

Zu der in Satz 3 erwähnten Anzahl H der nötigen Rechenschritte ist zu bemerken, daß sie sich mit dem in §3 beschrittenen Weg als so groß erweist, daß eine Berechnung von D und Y_0 mit heute zur Verfügung stehenden Computern nicht möglich ist: wir zeigen dort, daß $H > 10^{48}$; aber höchstwahrscheinlich gilt sogar $H > 10^{90}$.

Eine obere Abschätzung für H können wir nicht beweisen; aus heuristischen Gründen ist es jedoch wahrscheinlich, daß etwa 10^{110} Schritte genügen, um D, und etwa 10^{140} Schritte genügen, um auch Y_0 effektiv zu bestimmen. Vermutlich erhält man $D < \exp 10^{110}$ bzw. $\max \{D, Y_0\} < \exp 10^{140}$.

Im Zusammenhang mit Satz B bemerken wir, daß das erste, ohne unbewiesene Annahme gültige Ergebnis im Jahre 1962 von S. Knapowski [4] erreicht wurde. Er zeigte:

$$V(y) \geqq e^{-35} \log_4 y \quad \text{für} \quad y \geqq \exp_5 35 \tag{1.10}$$

(wobei $\log_1 y = \log y$, $\log_{\nu+1} y = \log \log_\nu y$).

Das beste effektive Ergebnis

$$V(y) > \frac{1}{\exp_3 3{,}55} \frac{\sqrt{\log y}}{\log_2 y} \quad \text{für} \quad y \geqq \exp_4 3{,}57 \tag{1.11}$$

stammt von den Autoren [1], [7].

Das beste ineffektive Ergebnis

$$V(y) > 10^{-11} \frac{\log y}{(\log_2 y)^3} \quad \text{für} \quad y > Y_1 \tag{1.12}$$

wurde von dem letztgenannten Autor [8] bewiesen.

Beim Beweis von (1.10), (1.11) und (1.12) spielt die Turànsche Methode eine entscheidende Rolle.

2. Beweis von Satz 1 und Satz 2

Im folgenden verwenden wir die folgenden Bezeichnungen:

$$\begin{cases} \Delta_0(x) = \psi(x) - x = \sum\limits_{p^m \leqq x} \log p - x, \quad \Delta_1(x) = \pi(x) - \mathrm{li}\, x = \sum\limits_{p \leqq x} 1 - \mathrm{li}\, x, \\ \Delta_2(x) = \Pi(x) - \mathrm{li}\, x = \sum\limits_{\nu \geqq 1} \frac{1}{\nu} \pi(x^{1/\nu}) - \mathrm{li}\, x, \quad \Delta(x) = \int\limits_0^x \Delta_0(t)\, dt, \\ \Delta_0^*(x) = \frac{\Delta_0(x)}{\sqrt{x}}, \quad \Delta_i^* = \frac{\Delta_i(x) \log x}{\sqrt{x}} \quad \text{für} \quad i = 1, 2. \end{cases} \tag{2.1}$$

Sei weiter $A = 500$, $B = 3600$, $\eta = \frac{3}{400}$, $C = \exp 10^3$, $\varrho = \frac{1}{2} + i\gamma$ bezeichne eine beliebige nicht triviale Nullstelle von $\zeta(s)$. Weiterhin verwenden wir die folgenden numerischen Ergebnisse:

$$\sum_{\gamma > 0} \frac{1}{\gamma^2} < 0{,}0233, \tag{2.2}$$

$$\sum_{|\gamma| < A} \frac{\sin(\gamma\omega)}{\gamma} \left(1 - \frac{|\gamma|}{A}\right) > 1{,}0262 \quad \text{bzw.} \quad < -1{,}0262, \tag{2.3}$$

falls für alle $|\gamma|<A$ gilt

(2.4) $$\|\gamma\omega-\gamma\eta\| \leqq \frac{2\pi}{B} \quad \text{bzw.} \quad \|\gamma\omega+\gamma\eta\| \leqq \frac{2\pi}{B};$$

(2.5) $$N(A) \overset{\text{def}}{=} \sum_{0<\gamma<A} 1 = 269.$$

Zu (2.2) vergleiche Rosser [10]; (2.3)—(2.5) wurde von Skewes ausgerechnet [11].

Bei den folgenden Hilfssätzen setzen wir voraus, daß die Riemannsche Vermutung wahr ist.

Hilfssatz 1. *Für* $x \geqq 1$ *gilt:*

(2.6)
$$\psi_1(x) \overset{\text{def}}{=} \int_0^x \psi(t)\,dt = \frac{x^2}{2} - \sum_{\varrho} \frac{x^{\varrho+1}}{\varrho(\varrho+1)} - x\frac{\zeta'}{\zeta}(0) + \frac{\zeta'}{\zeta}(-1) - \sum_{n=1}^{\infty} \frac{x^{1-2n}}{2n(2n-1)}.$$

Für den Beweis siehe Ingham [3], Seite 73, Theorem 28.

Hilfssatz 2. *Für* $T \geqq x^2/2$, $x \geqq C$ *gilt:*

(2.7) $$\left| \int_0^x \psi(t)\,dt - \frac{x^2}{2} + \sum_{|\gamma|<T} \frac{x^{\varrho+1}}{\varrho(\varrho+1)} \right| \leqq 5x.$$

Beweis. (2.7) folgt aus (2.6), falls wir $\frac{\zeta'}{\zeta}(0) = \log 2\pi$, $\frac{\zeta'}{\zeta}(-1) = 1{,}98\,505\ldots$ (siehe Walther [13]) und

(2.8) $$\sum_{10<K\leqq\gamma\leqq K+4} 1 < 4\log K$$

(vergleiche von Mangoldt [6]) berücksichtigen.

Aus Hilfssatz 2 und (2.2) folgt

Hilfssatz 3. *Für* $x \geqq C$ *gilt:*
$$|\Delta(x)| \leqq \frac{x^{3/2}}{20}.$$

Hilfssatz 4. *Für* $x \geqq C$ *gilt:*
$$|\Delta_0(x)| \leqq \sqrt{x}\left(\sum_{|\gamma|<x^2} \frac{\sqrt{2}}{|\gamma|} + 12\right).$$

Beweis. Aus Hilfssatz 2 folgt

(2.9)
$$\psi(x) \leqq \frac{\psi_1(x+\sqrt{x}) - \psi_1(x)}{\sqrt{x}} \leqq x + \frac{\sqrt{x}}{2} + \sum_{|\gamma|\leqq x^2} \frac{1}{|\gamma|}\frac{1}{\sqrt{x}}\left|\int_x^{x+\sqrt{x}} t^{\varrho}\,dt\right| + 11\sqrt{x}.$$

Analog erhält man die untere Abschätzung.

Aus Hilfssatz 4 folgt zusammen mit (2.8)

HILFSSATZ 5. *Für* $x \geqq 2C$ *gilt:*

$$|\Delta_0(x)| \leqq 20\sqrt{x}\log^2 x.$$

HILFSSATZ 6. *Für* $x \geqq 1$ *gilt:*

$$\Delta_2(x) - \frac{\Delta_0(x)}{\log x} = \frac{\Delta(x)}{x\log^2 x} + \int_2^x \Delta(t)\frac{\log t + 2}{t^2\log^3 t}\,dt - \tag{2.10}$$

$$-\frac{\Delta(2)}{2\log^2 2} + \frac{2}{\log 2} - \operatorname{li} 2.$$

Für den Beweis vergleiche Ingham [3], Seite 64.

Aus den Hilfssätzen 3 und 6 folgt unter Verwendung der trivialen Abschätzung $|\Delta(x)| \leqq x^2 \log x$ (für $x \leqq C$)

HILFSSATZ 7. *Für* $x \geqq C^{10}$ *gilt:*

$$\left|\Delta_2(x) - \frac{\Delta_0(x)}{\log x}\right| \leqq \frac{\sqrt{x}}{\log^2 x} \leqq 10^{-4}\frac{\sqrt{x}}{\log x}. \tag{2.11}$$

HILFSSATZ 8. *Für* $x \geqq C^{10}$ *gilt:*

$$|\Delta_2(x)| \leqq 21\sqrt{x}\log^2 x, \quad |\Delta_1(x)| \leqq 22\sqrt{x}\log^2 x. \tag{2.12}$$

BEWEIS. (2.12) folgt unmittelbar aus den Hilfssätzen 5 und 7 unter Verwendung von

$$0 \leqq \Delta_2(x) - \Delta_1(x) = \sum_{\nu \geqq 2} \frac{1}{\nu}\pi(x^{1/\nu}) \leqq \frac{\sqrt{x}}{2} + \sqrt[3]{x}\log x \leqq \sqrt{x}.$$

HILFSSATZ 9. *Für* $x \geqq C^{10}$ *gilt:*

$$\frac{x}{\log x} < \pi(x) < \frac{x}{\log x} + \frac{3x}{\log^2 x}. \tag{2.13}$$

BEWEIS. (2.13) ergibt sich aus (2.12) und

$$\frac{x}{\log x} + \frac{x}{\log^2 x} < \operatorname{li} x < \frac{x}{\log x} + \frac{2x}{\log^2 x}. \tag{2.14}$$

HILFSSATZ 10. *Für* $x \geqq C^{20}$ *gilt:*

$$\left|\Delta_2(x) - \Delta_1(x) - \frac{\sqrt{x}}{\log x}\right| \leqq 7\frac{\sqrt{x}}{\log^2 x}. \tag{2.15}$$

BEWEIS. (2.15) folgt aus (2.13) unter Verwendung von

$$\sum_{\nu \geqq 3} \frac{1}{\nu}\pi(x^{1/\nu}) < \sqrt[3]{x}\log x.$$

Aus den Hilfssätzen 7 und 10 folgt

HILFSSATZ 11. *Für* $x \geqq C^{100}$ *gilt:*

$$(2.16)\qquad \left|\Delta_1(x)+\frac{\sqrt{x}}{\log x}-\frac{\Delta_0(x)}{\log x}\right| \leqq 8\,\frac{\sqrt{x}}{\log^2 x} \leqq 8\cdot 10^{-5}\,\frac{\sqrt{x}}{\log x},$$

was gleichbedeutend ist mit:

$$(2.17)\qquad |\Delta_1^*(x)-\Delta_0^*(x)+1| \leqq 8\cdot 10^{-5}.$$

HILFSSATZ 12. *Für* $x>1$ *gilt:*

$$(2.18)\qquad \frac{\psi(x+0)+\psi(x-0)}{2} = x-\sum_{\varrho}\frac{x^{\varrho}}{\varrho}-x\frac{\zeta'}{\zeta}(0)-\frac{1}{2}\log\left(1-\frac{1}{x^2}\right).$$

Für einen Beweis vergleiche Ingham [3], Seite 77, Theorem 29.
Aus den Hilfssätzen 11 und 12 folgt

HILFSSATZ 13. *Für* $x \geqq C^{100}$ *gilt:*

$$(2.19)\qquad \left|\Delta_1^*(x)+\sum_{\varrho}\frac{x^{i\gamma}}{\varrho}+1\right| \leqq 10^{-4}.$$

Daraus ergibt sich unter Verwendung von (2.2) und $\left|\frac{1}{\varrho}-\frac{1}{i\gamma}\right| \leqq \frac{1}{2\gamma^2}$.

HILFSSATZ 14. *Für* $x \geqq C^{100}$ *gilt:*

$$(2.20)\qquad \left|\Delta_1^*(x)+\sum_{\varrho}\frac{x^{i\gamma}}{i\gamma}+1\right| < 0{,}0234.$$

Wir setzen nun

$$(2.21)\qquad G(v) = \sum_{\varrho}\frac{e^{i\gamma v}}{i\gamma}.$$

Aus Hilfssatz 14 folgt, daß $\Delta_1(x)$ in einem Intervall

$$(2.22)\qquad [e^{a_1}, e^{a_2}] \subset [\exp 10^5, y]$$

mindestens einen Zeichenwechsel hat, falls wir zeigen können, daß

$$(2.23)\qquad \max_{a_1\leqq v\leqq a_2} G(v) > 1{,}0234 \quad \text{und} \quad \min_{a_1\leqq v\leqq a_2} G(v) < -1{,}0234.$$

Dazu verwenden wir eine Idee von Ingham und betrachten

$$(2.24)\qquad \begin{cases} I_1(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-AB}^{AB}\left(\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right)^2 G\left(\omega+\frac{u}{A}\right)du, \\[2ex] I(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left(\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right)^2 G\left(\omega+\frac{u}{A}\right)du = \sum_{|\gamma|<A}\frac{e^{i\gamma\omega}}{i\gamma}\left(1-\frac{|\gamma|}{A}\right). \end{cases}$$

Die letzte Gleichung folgt unter Verwendung von

$$(2.25)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right)^2 e^{ixu}\,du = \begin{cases} 1-|x| & \text{für } |x|\leqq 1 \\ 0 & \text{sonst.}\end{cases}$$

Mit partieller Integration folgt nun leicht:

$$(2.26)\qquad \left|\int_{AB}^{\infty}\left(\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right)^2 e^{i\gamma u/A}\,du\right| \leqq$$

$$\leqq \left|\left\{\frac{A}{i\gamma}e^{i\gamma u/A}\left(\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right)^2\right\}_{AB}^{\infty}\right| + \left|\frac{A}{i\gamma}\int_{AB}^{\infty} e^{iu\gamma/A}\frac{d}{du}\left[\left(\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right)^2\right]du\right| \leqq$$

$$\leqq \frac{A}{|\gamma|}\frac{4}{(AB)^2}+\frac{A}{|\gamma|}\frac{4}{AB} \leqq \frac{5}{B|\gamma|}.$$

Obige Abschätzung gilt analog für $\int\limits_{-\infty}^{-AB}$, und somit erhalten wir unter Verwendung von (2.2)

$$(2.27)\qquad |I(\omega)-I_1(\omega)| \leqq \frac{1}{2\pi}\sum_{\varrho}\frac{1}{|\gamma|}\frac{10}{B|\gamma|} < 10^{-4}.$$

(2.23) folgt also, falls wir zeigen können, daß

$$(2.28)\qquad \max_{a_1+B\leqq\omega\leqq a_2-B} I(\omega) > 1{,}0235, \qquad \min_{a_1+B\leqq\omega\leqq a_2-B} I(\omega) < -1{,}0235.$$

Aus (2.3) folgt für ein ω, das (2.4) erfüllt:

$$(2.29)\qquad I(\omega) > 1{,}0262 \quad \text{bzw.} \quad < -1{,}0262.$$

Im folgenden suchen wir nun solche Werte n_ν, für die

$$(2.30)\qquad \left\|\frac{\gamma_i}{2\pi}n_\nu\right\| \leqq \varepsilon = \frac{1}{B} \qquad (0<\gamma_i<A, \quad \nu = 1, \dots, M)$$

erfüllt ist, denn dann ist (2.29) mit den Werten

$$(2.31)\qquad \omega = n_\nu+\eta \quad \text{bzw.} \quad n_\nu-\eta$$

gegeben.

Um (2.30) zu sichern, benötigen wir die folgende Verallgemeinerung des Dirichletschen Approximationssatzes (vergleiche Titchmarsh [12], Seite 153).

HILFSSATZ 15. *Seien $q \geqq 2$ und M natürliche Zahlen; $x_1, \ldots, x_N$ beliebige reelle Zahlen. Dann gibt es natürliche Zahlen n_ν mit:*

$$1 \leqq n_1 < n_2 < \ldots < n_M \leqq Mq^N, \tag{2.32}$$

so daß für $1 \leqq \nu \leqq M$, $1 \leqq i \leqq N$ gilt:

$$\|n_\nu x_i\| \leqq \frac{1}{q}. \tag{2.33}$$

Falls wir nun

$$q = B; \quad x_i = \frac{\gamma_i}{2\pi} \quad (i = 1, \ldots, N(A) = N), \tag{2.34}$$

$$M = \left[\frac{\log y - B - 1}{B^{N(A)}}\right] \tag{2.35}$$

wählen, so erhalten wir natürliche Zahlen

$$1 \leqq n_1 < \ldots < n_M \leqq \log y - B - 1 \quad \text{mit} \tag{2.36}$$

$$I(n_\nu - \eta) < -1{,}0262 < 1{,}0262 < I(n_\nu + \eta). \tag{2.37}$$

Also hat $\Delta_1(x) = \pi(x) - \mathrm{li}\, x$ mindestens einen Zeichenwechsel im Intervall

$$[e^{n_\nu - B - 1}, e^{n_\nu + B + 1}] \subset [\exp 10^5, y], \quad (1{,}1 \cdot 10^5 \leqq \nu \leqq M). \tag{2.38}$$

Daher folgt für $y > \exp_3 7{,}707$:

$$V(y) > \frac{1}{7202}\left(\left[\frac{\log y - 10^4}{3600^{269}}\right] - 1{,}1 \cdot 10^5\right) > \tag{2.39}$$

$$> \frac{1}{\exp_2 7{,}707} \log y > 10^{-966} \cdot \log y.$$

Falls wir den ursprünglichen Dirichletschen Approximationssatz (vergleiche Titchmarsh [12], Seite 152) verwenden, erhalten wir die Existenz eines n_1 mit

$$n_1 \in \left[\frac{\log y - 10^4}{3600^{269}}, \log y - 10^4\right], \tag{2.40}$$

so daß (2.30) mit $\nu = 1$ erfüllt ist. Daher hat $\Delta_1(x) = \pi(x) - \mathrm{li}\, x$ mindestens einen Zeichenwechsel im Intervall

$$[e^{n_1 - B - 1}, e^{n_1 + B + 1}] \subset [y^{10^{-957}}, y] \quad \text{für} \quad y > \exp_3 7{,}707. \tag{2.41}$$

3. Bemerkungen

Aus dem Beweis der Sätze 1 und 2 ist ersichtlich, wie der Bohlsche Approximationssatz zum Ergebnis von Ingham führt. Um die weiteren Bemerkungen bezüglich der Effektivierbarkeit des Satzes A von Ingham verständlich zu machen, geben wir den folgenden Beweis für den Satz von Bohl:

Wir teilen den N-dimensionalen Einheitswürfel mod 1 auf in k^N kleinere, abgeschlossene Würfel $W_1, \ldots, W_{k^N}$ mit der Seitenlänge $1/k$, wobei k die kleinste ganze Zahl $\geqq \varepsilon^{-1}$ ist. Sei $\mathscr{W}$ die Menge derjenigen Würfel, die mindestens einen der Punkte $n\underline{x}$ enthalten $(\underline{x}=(x_1, \ldots, x_N))$. Dann gibt es eine Zahl $L(\varepsilon, \underline{x})$, so daß alle $W_i \in \mathscr{W}$ mindestens einen Punkt $n\underline{x}$ mit $n \leqq L(\varepsilon, \underline{x})$ enthalten. Da aus dem Dirichletschen Approximationssatz folgt, daß für beliebige $H>0$ der Punkt $\underline{0}= =(0, \ldots, 0) \in \overline{\{n\underline{x}\}_{n=H}^{\infty}}$, ist auch $-m\underline{x} \in \{n\underline{x}\}_{n=1}^{\infty}$. Also existiert ein Würfel W_j mit $-m\underline{x} \in W_j$, der mindestens einen Punkt $n\underline{x}$ mit $n \leqq L(\varepsilon, \underline{x})$ enthält. Dann gilt:

$$\|(n+m)x_i\| = \|nx_i-(-mx_i)\| \leqq \frac{1}{k} \leqq \varepsilon, \quad (i = 1, \ldots, N). \tag{3.1}$$

Bemerkung 1. Es ist nicht möglich, $L(\varepsilon, \underline{x})$ mit einer Funktion, die von ε und N abhängt, nach oben abzuschätzen:
für alle $\varepsilon<1/2$ und $N \geqq 1$ gilt mit kleinem η und $x_1=\eta$, $x_2, \ldots, x_N$ beliebig

$$L(\varepsilon, x_1, \ldots, x_N) \geqq \left[\frac{1-2\varepsilon}{\eta}\right]. \tag{3.2}$$

Die obige Bemerkung zeigt die Schwierigkeiten, die bei einer eventuellen Effektivierung von Satz 1 auftreten können. Insbesondere muß man berücksichtigen, daß außer dem obigen trivialen Beispiel noch andere, kompliziertere Beispiele existieren, die zeigen, daß $L(\varepsilon, \underline{x})$ beliebig groß sein kann.

Andererseits ist der obige Beweis des Bohlschen Satzes doch gewissermaßen konstruktiv, und so ergibt sich die folgende

Bemerkung 2. Falls wir bei gegebenen ε, $\underline{x}$ die Menge $\mathscr{W}=\mathscr{W}(\varepsilon, \underline{x})$ kennen, so kann man in endlich vielen Schritten eine Zahl $L(\varepsilon, \underline{x})$ bestimmen. Die Anzahl der dazu notwendigen Rechenschritte kann man jedoch nicht im voraus abschätzen.

Falls die Zahlen $x_1, \ldots, x_N$ (über dem rationalen Zahlkörper) linear unabhängig sind, dann enthält $\mathscr{W}$ nach dem Kroneckerschen Satz (vergleiche Titchmarsh [12], Seite 153) alle Würfel W_i $(i=1, \ldots, k^N)$; wir haben also

Bemerkung 3. Falls die Zahlen $x_1, \ldots, x_N$ linear unabhängig sind, so kann man eine Zahl $L(\varepsilon, x_1, \ldots, x_N)$ in endlich vielen Schritten bestimmen.

Damit haben wir das in der Einleitung erwähnte interessante Phänomen, daß die Konstante im Satz A ineffektiv ist, aber unter der genannten Zusatzannahme kann man sie in endlich vielen Schritten bestimmen.

Nun gilt

$$2 \sum_{0<\gamma<\gamma_{70}} \frac{1}{|\varrho|}\left(1-\frac{\gamma}{\gamma_{70}}\right) > 1, \tag{3.3}$$

wie sich durch explizite Berechnung ergibt. Berücksichtigt man dieses Ergebnis so ergibt sich Satz 3. Dabei ist zu bemerken, daß aufgrund der Tatsache, daß wir

$\gamma_1, \dots, \gamma_{70}$ mit beliebiger Genauigkeit berechnen können, es keine Schwierigkeiten gibt, falls wir Näherungswerte für $\gamma_1, \dots, \gamma_{70}$ verwenden*.

Da nun

$$2\sum_{i=1}^{N}\frac{1}{|\varrho_i|}\left(1-\frac{\gamma_i}{\gamma_{N+1}}\right) < 1 \quad \text{für} \quad N = 1, \dots, 68 \tag{3.4}$$

gilt, muß man ein mindestens 69 dimensionales diophantisches Problem lösen. Außerdem muß man, nach (2.30) und (2.31) $\varepsilon < 1/4$ (also $k \geqq 5$) setzen (ansonsten erhält man etwas völlig Triviales). Aus diesen beiden Überlegungen ergibt sich, daß die Anzahl der Würfel und damit die Anzahl der notwendigen Schritte bestimmt größer ist als

$$5^{69} > 10^{48}. \tag{3.5}$$

Für praktische Berechnung ist diese Anzahl natürlich schon zu groß.

(3.5) gibt nur eine absolute Mindestzahl für die Anzahl der Schritte an, die sich bei der Wahl des tatsächlich notwendigen ε natürlich wesentlich erhöhen wird.

Eine Verbesserung kann eintreten, wenn man N etwas größer wählt und damit auch

$$2\sum_{i=1}^{N}\frac{1}{|\varrho_i|}\left(1-\frac{\gamma_i}{\gamma_{N+1}}\right) - 1 \tag{3.6}$$

größer its. Z. B. erhält man für $N=100$

$$2\sum_{i=1}^{N}\frac{\sin\left(\frac{\pi}{2}-2\pi\varepsilon\right)}{|\varrho_i|}\left(1-\frac{\gamma_i}{\gamma_{N+1}}\right) > 1, \tag{3.7}$$

wobei $\varepsilon < \frac{1}{10{,}6}$ sein muß. Wählt man nun $\varepsilon = \frac{1}{11}$, so erhält man für die Anzahl der in diesem Falle nötigen Schritte mindestens

$$11^{100} > 10^{104{,}13}. \tag{3.8}$$

Orientierungshalber wollen wir nun einige Überlegungen für eine Abschätzung der Anzahl der Schritte nach oben durchführen.

Falls ein Punkt P_i mit einer Wahrscheinlichkeit von $1/M$ in einen der M Würfel fällt und die Verteilung von P_i und P_j für i, j unabhängig ist, so enthalten alle Würfel

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* Nehmen wir an, daß $\gamma_1, \dots, \gamma_{70}$ linear unabhängig sind, und sei $\underline{\gamma}' = (\gamma_1', \dots, \gamma_{70}')$ ein Näherungsvektor mit $|\gamma_\nu' - \gamma_\nu| < 1/n$. Weiterhin sei ε eine feste positive Zahl, $\varepsilon = 1/9k$, $k \in \mathbf{N}$ und $W_\nu^{3^j\varepsilon}$ $(\nu = 1, \dots, (3^{-j}\varepsilon^{-1})^N)$ das früher beschriebene Würfelsystem mod 1, wobei die Seitenlänge der Würfel $3^j\varepsilon$ $(j = 0, 1, 2)$ ist. $L(\varepsilon, \underline{x})$ sei definiert wie im *Beweis* des Satzes von Bohl.

Falls $n \geqq [(L(\varepsilon, \underline{\gamma}) + 2)\varepsilon^{-1}]$, dann enthalten alle Würfel $W_\nu^{3\varepsilon}$ mindestens einen Punkt der Reihe $\{m\underline{\gamma}'\}_{m=1}^{[\varepsilon n]}$, da $\varepsilon n \geqq L(\varepsilon, \underline{\gamma})$ und $|m \cdot \gamma_\nu' - m\gamma_\nu| < \varepsilon$. So erhalten wir nach endlich vielen Schritten eine natürliche Zahl n_0 und einen Näherungsvektor γ' mit $|\gamma_\nu' - \gamma_\nu| < 1/n_0$, so daß alle Würfel $W_\nu^{3\varepsilon}$ mindestens einen Punkt der Reihe $\{m\underline{\gamma}'\}_{m=1}^{[\varepsilon n_0]}$ enthalten (dabei wissen wir nicht, ob $n_0 \geqq [(L(\varepsilon, \underline{\gamma}) + 2)\varepsilon^{-1}]$ oder nicht). Dann enthalten alle Würfel $W_\nu^{9\varepsilon}$ mindestens einen Punkt der Reihe $\{m\underline{\gamma}\}_{m=1}^{[\varepsilon n_0]}$. Auf diese Weise erhalten wir die effektive Abschätzung $L(1/k, \underline{\gamma}) < \varepsilon n_0$.

nach $cM \log M$ $(c>1)$ Schritten mit einer Wahrscheinlichkeit

$$\geqq 1-M\left(1-\frac{1}{M}\right)^{cM\log M} > 1-M^{-(c-1)} \tag{3.9}$$

mindestens einen Punkt.

Das sagt natürlich nichts für die spezielle Punktfolge $n\left(\frac{\gamma_1}{2\pi}, \ldots, \frac{\gamma_{100}}{2\pi}\right)$, wenngleich (3.9) wahrscheinlich für fast alle Folgen $n\underline{x}$, $\underline{x}\in\mathbf{R}^{100}$ bewiesen werden kann.

Das heißt aber, daß man gute Chancen hat, daß

$$L\left(\frac{1}{11}, \frac{\gamma_1}{2\pi}, \ldots, \frac{\gamma_{100}}{2\pi}\right) < 10^{109} \tag{3.10}$$

ist. Damit existierte dann zu allen Punkten $\underline{x}\in\mathbf{R}^{100}$ ein $n<10^{109}$ mit

$$\|n\gamma_i - x_i\| < \frac{1}{11} \quad (i = 1, \ldots, 100). \tag{3.11}$$

Falls man also für die Werte γ_i Näherungswerte γ_i^* hat, die mit einer Genauigkeit von 112 Dezimalen mit γ_i übereinstimmen, dann ist für $n<10^{109}$

$$|n\gamma_i^* - n\gamma_i| < 10^{-3}. \tag{3.12}$$

Damit folgt aus (3.11)

$$\|n\gamma_i^* - x_i\| < \frac{1}{10{,}6}. \tag{3.13}$$

Ebenso kann man nun die Rollen von γ_i und γ_i^* vertauschen, d.h. (3.10) annehmen und auf (3.13) mit γ_i schließen.

Falls man nun also annimmt, daß man Näherungswerte γ_i^* für γ_i mit einer Genauigkeit von 10^{-112} in weniger als 10^{109} Schritten finden kann, was möglich zu sein scheint, so haben wir Chancen, mit etwa 10^{110} Schritten den Wert der Konstanten D im Satz A von Ingham anzugeben ($D<\exp 10^{109}$).

Falls wir auch noch Y_0 effektiv angeben möchten, dann müßten wir eine Zahl explizit bestimmen, für die

$$\left\|\frac{\omega\gamma_i}{2\pi}\right\| < \varepsilon, \quad i = 1, \ldots, 100. \tag{3.14}$$

Statt (3.7) müssen wir dann

$$2\sum_{i=1}^{100} \frac{\sin\left(\frac{\pi}{2}-2\pi\varepsilon\right)}{|\varrho_i|}\left(1-\frac{\gamma_i}{\gamma_{101}}\right) > 1 \tag{3.15}$$

voraussetzen, also $\varepsilon=1/22$ wählen.

Daher sind dann wahrscheinlich etwa 10^{140} Rechenschritte nötig, und man erhält wahrscheinlich $\max\{D, Y_0\}<\exp 10^{140}$.

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ON DISTRIBUTIVE PAIRS IN LATTICES

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1. Introduction

The concept of modular pairs in lattices has been well-investigated in the literature, especially in [3] and [4]. A lattice L is modular if and only if every pair of elements of L is join-modular (or equivalently, every pair is meet-modular), while in some non-modular lattices, interesting characterizations of modular pairs are given. For instance, in an affine matroid lattice, a pair of lines is meet-modular if and only if they are not parallel ([3], §17); in the lattice of closed subspaces of a topological vector space, a pair is join-modular if and only if their linear sum is closed ([3], §31).

In this paper, we consider two new concepts, distributive pairs and semidistributive pairs in lattices. A lattice L is distributive if and only if every pair of elements of L is join-distributive (or equivalently, every pair is meet-distributive). Any join-(meet-)distributive pair is join-(meet-)semidistributive and any join-(meet-)semidistributive pair is join-(meet-)modular. Moreover, it will be shown that a lattice is distributive if every pair of elements is join-(meet-)semidistributive and that in a modular lattice or in an atomistic lattice, any join-semidistributive pair is join-distributive.

In the last section, interesting examples of distributive pairs are given. It is well-known that the congruence lattice of a lattice is distributive, while it was proved in [1] and [5] that the congruence lattice $C(S)$ of a join-semilattice S is distributive if and only if any two non-comparable elements in S have no lower bound in S. In this paper, we give a necessary and sufficient condition for a pair in $C(S)$ to be join-(meet-)distributive. Moreover, we prove a remarkable property of $C(S)$ that any meet-modular pair in $C(S)$ is meet-distributive.

2. Distributive pairs and semidistributive pairs

DEFINITION. Let L be a lattice. A pair of elements $a, b\in L$ is called *join-distributive* (resp. *meet-distributive*), denoted by $(a, b)\,D_j$ (resp. $(a, b)\,D_m$), when

$$(a\vee b)\wedge x = (a\wedge x)\vee(b\wedge x) \quad \text{for every} \quad x\in L$$

$$(\text{resp. } (a\wedge b)\vee x = (a\vee x)\wedge(b\vee x) \ \text{ for every } \ x\in L).$$

(a, b) is called *join-modular* (resp. *meet-modular*), denoted by $(a, b)\,M_j$ (resp. $(a, b)\,M_m$), when

$$c \geqq b \quad \text{implies} \quad c\wedge(a\vee b) = (c\wedge a)\vee b$$

$$(\text{resp. } c\leqq b \ \text{ implies } \ c\vee(a\wedge b)=(c\vee a)\wedge b).$$

$(a, b)\,M_j$ (resp. $(a,b)\,M_m$) coincides with $(a, b)\,M^*$ (resp. $(a, b)\,M$) in [3].

LEMMA 1. (i) $(a, b)D_j$ *is equivalent to the following condition:*

(1) *If* $x \leqq a \vee b$ *then there exist* $a_1, b_1 \in L$ *such that* $a_1 \leqq a$, $b_1 \leqq b$ *and* $a_1 \vee b_1 = x$.

(ii) $(a, b)M_j$ *is equivalent to the following condition.*

(2) *If* $b \leqq x \leqq a \vee b$ *then there exists* $a_1 \in L$ *such that* $a_1 \leqq a$ *and* $a_1 \vee b = x$.

PROOF. (i) $(a, b)D_j$ implies (1) by putting $a_1 = a \wedge x$ and $b_1 = b \wedge x$. Conversely, if we assume (1), then, putting $y = (a \vee b) \wedge x$, there exist a_1, b_1 such that $a_1 \leqq a$, $b_1 \leqq b$ and $a_1 \vee b_1 = y$. Then, since $a_1, b_1 \leqq y \leqq x$, we have $a_1 \leqq a \wedge x$ and $b_1 \leqq b \wedge x$, whence

$$(a \vee b) \wedge x = a_1 \vee b_1 \leqq (a \wedge x) \vee (b \wedge x) \leqq (a \vee b) \wedge x.$$

(ii) $(a, b)M_j$ implies (2) by putting $a_1 = x \wedge a$. Conversely, if we assume (2), then since $b \leqq c \wedge (a \vee b) \leqq a \vee b$, there exists a_1 such that $a_1 \leqq a$ and $a_1 \vee b = c \wedge (a \vee b)$. Then, $a_1 \leqq c \wedge a$, and hence

$$c \wedge (a \vee b) = a_1 \vee b \leqq (c \wedge a) \vee b \leqq c \wedge (a \vee b).$$

DEFINITION. A pair of elements $a, b \in L$ is called *join-semidistributive*, denoted by $(a, b)SD_j$, when it satisfies the following condition:

(3) If $x \leqq a \vee b$ then there exists $a_1 \in L$ such that $a_1 \leqq a$ and $a_1 \leqq x \leqq a_1 \vee b$ (hence, $x \vee b = a_1 \vee b$).

It is easy to verify that

$$a \leqq b \Rightarrow (a, b)D_j (\Leftrightarrow (b, a)D_j) \Rightarrow (a, b)SD_j \Rightarrow (a, b)M_j$$

and that (3) is equivalent to

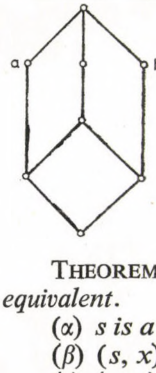
$$\{(a \vee b) \wedge x\} \vee b = (a \wedge x) \vee b \quad \text{for every} \quad x \in L.$$

Remark that conditions (1), (2) and (3) can be available for elements of a join-semilattice.

The definition of a *meet-semidistributive* pair $(a, b)SD_m$ can be given by the dual way.

LEMMA 2. *If* $(a, b)SD_j$ *and if* $(b, a_1)M_j$ *for any* $a_1 \leqq a$, *then* $(a, b)D_j$.

PROOF. If $x \leqq a \vee b$, then there exists $a_1 \in L$ such that $a_1 \leqq a$ and $a_1 \leqq x \leqq a_1 \vee b$. By $(b, a_1)M_j$ and Lemma 1, there exists $b_1 \leqq b$ such that $b_1 \vee a_1 = x$.



It follows from this lemma that $(a, b)SD_j \Leftrightarrow (a, b)D_j$ in a modular lattice. While, in the lattice given by the figure, both $(a, b)SD_j$ and $(b, a)SD_j$ hold, but $(a, b)D_j$ does not hold.

THEOREM 3. *For an element s of a lattice L, the following three statements are equivalent.*

(α) *s is a standard element, that is,* $(s, x)D_j$ *for every* $x \in L$ (see [2]).

(β) $(s, x)SD_j$ *and* $(x, s)SD_j$ *for every* $x \in L$.

(γ) $(s, x)M_j$ *and* $(x, s)SD_j$ *for every* $x \in L$.

PROOF. The implications $(\alpha)\Rightarrow(\beta)\Rightarrow(\gamma)$ are evident. It follows from Lemma 2 that $(\gamma)\Rightarrow(\alpha)$.

COROLLARY. *L is distributive if $(a, b)SD_j$ for every $a, b\in L$.*

(This result was shown in [6], p. 33.)

PROPOSITION 4. (i) *If $(a, b)D_j$ and $(a\vee b, c)D_j$ then $(a', b\vee c)D_j$ for any $a'\in\in[a, a\vee c]$.*

(ii) *If $(a, b)D_j$ then $(a', b')D_j$ for any $a'\in[a, a\vee b]$ and $b'\in[b, a\vee b]$.*

(iii) *If $(a_1, b)D_j$ and $(a_2, b)D_j$ then $(a_1\wedge a_2, b)D_j$.*

PROOF. (i) Let $x\leqq a'\vee b\vee c$. Since $a'\vee b\vee c=a\vee b\vee c$ and since $(a\vee b, c)D_j$, there exist $d\leqq a\vee b$ and $c_1\leqq c$ such that $d\vee c_1=x$. Since $(a, b)D_j$, there exist $a_1\leqq a$ and $b_1\leqq b$ such that $a_1\vee b_1=d$. Then, $a_1\leqq a'$, $b_1\vee c_1\leqq b\vee c$ and $a_1\vee(b_1\vee c_1)=x$.

(ii) Let $(a, b)D_j$ and $b'\in[b, a\vee b]$. Since $b'\leqq a\vee b$, we have $(a\vee b, b')D_j$. Hence, it follows from (i) that $(a', b')D_j$ for any $a'\in[a, a\vee b']=[a, a\vee b]$.

(iii) Let $x\leqq(a_1\wedge a_2)\vee b$. Since $x\leqq a_1\vee b$ and $(a_1, b)D_j$, there exist $c_1\leqq a_1$ and $b_1\leqq b$ such that $c_1\vee b_1=x$. Since $c_1\leqq x\leqq a_2\vee b$ and $(a_2, b)D_j$, there exist $c_2\leqq a_2$ and $b_2\leqq b$ such that $c_2\vee b_2=c_1$. Then $c_2\leqq c_1\wedge a_2\leqq a_1\wedge a_2$, $b_1\vee b_2\leqq b$ and $c_2\vee(b_1\vee b_2)=x$.

It follows from (i) and (iii) of this proposition that if s_1 and s_2 are standard elements then so are $s_1\vee s_2$ and $s_1\wedge s_2$ (cf. [2], p. 143).

PROPOSITION 5. (i) *If $(a, b)SD_j$ and $(a\vee b, c)SD_j$ then $(a', b\vee c)SD_j$ for any $a'\in[a, a\vee c]$.*

(ii) *If $(a, b)SD_j$ then $(a', b')SD_j$ for any $a'\in[a, a\vee b], b'\in[b, a\vee b]$.*

(iii) *If $(a_1, b)SD_j$ and $(a_2, b)SD_j$ then $(a_1\wedge a_2, b)SD_j$.*

PROOF. These statements can be proved similarly to Proposition 4.

LEMMA 6. *The following statement $(*)$ implies $(a, b)SD_j$.*

$(*)$ *If $x\leqq a\vee b$ and if x and a are non-comparable then $x\leqq b$.*

PROOF. Let $x\leqq a\vee b$. We put $a_1=x$ if $x\leqq a$ and put $a_1=a$ if $x\geqq a$. Then, $a_1\leqq a$ and $a_1\leqq x\leqq a_1\vee b$. If x and a are non-comparable, then we have $x\leqq b$ by $(*)$. Hence, any lower bound a_1 of $\{a, x\}$ has the desired property.

LEMMA 7. *Let p be an atom of a lattice L with 0. The following statements are equivalent.*

(α) $(p, b)SD_j$.

(β) *If $x\leqq p\vee b$ then either $x\geqq p$ or $x\leqq b$.*

PROOF. $(\alpha)\Rightarrow(\beta)$. If $x\leqq p\vee b$, then by (α) there exists $p_1\leqq p$ such that $p_1\leqq\leqq x\leqq p_1\vee b$. Since p is an atom, p_1 is either p or 0. $p_1=p$ implies $x\geqq p$, and $p_1=0$ implies $x\leqq b$.

$(\beta)\Rightarrow(\alpha)$. If $x\leqq p\vee b$ and if x and p are non-comparable, then $x\leqq b$ by (β). Hence, it follows from Lemma 6 that $(p, b)SD_j$.

REMARK. If p is an atom, then it is easy to verify that $(p, b)M_j\Leftrightarrow$"If $b\leqq x\leqq\leqq p\vee b$ then either $x\geqq p$ or $x\leqq b$".

THEOREM 8. *Let p and q be atoms of a lattice L with 0. The following three statements are equivalent.*

(α) $(p, q)D_j$.

(β) $(p, q)SD_j$ *and* $(q, p)SD_j$.

(γ) *If* $0<x<p\vee q$ *then either* $x=p$ *or* $x=q$.

PROOF. $(\alpha)\Rightarrow(\beta)$ is evident. $(\beta)\Rightarrow(\gamma)$. Let $0<x<p\vee q$. Since $(p, q)SD_j$, we have $x\geqq p$ or $x=q$ by Lemma 7. If $x>p$, then since $(q, p)SD_j$, we would have $x\geqq q$ by Lemma 7, and then $x\geqq p\vee q$, a contradiction.

$(\gamma)\Rightarrow(\alpha)$. Let $x\leqq p\vee q$. By (γ), x is one of the four elements $0, p, q, p\vee q$. Hence, it is easy to verify that there exist $p_1\leqq p$ and $q_1\leqq q$ such that $p_1\vee q_1=x$.

We remark that the above arguments from Lemma 2 can be available for join-semilattices, except (iii) of Lemma 4. By the dual way, we can obtain dual results about meet-distributive pairs and meet-semidistributive pairs, and they can be available for meet-semilattices.

3. Join-distributive pairs in atomistic lattices

Let L be an atomistic lattice (see [3]), and denote by Ω the set of all atoms of L. For every $a\in L$, we put

$$\Omega(a) = \{p\in\Omega;\ p\leqq a\}\ (\Omega(0)=\emptyset)$$

The mapping $a\to\Omega(a)$ of L into the complete lattice of subsets of Ω is order-preserving and $\Omega(\wedge_\alpha a_\alpha)=\cap_\alpha\Omega(a_\alpha)$ whenever $\wedge_\alpha a_\alpha$ exists. Moreover, this mapping is one-to-one since L is atomistic.

THEOREM 9. *Let a and b be elements of an atomistic lattice L. The following four statements are equivalent.*

(α) $(a, b)D_j$.

(β) $(a, b)SD_j$. (β') $(b, a)SD_j$.

(γ) $\Omega(a\vee b)=\Omega(a)\cup\Omega(b)$.

PROOF. Evidently, (α) implies (β) and (β'). $(\beta)\Rightarrow(\gamma)$. It is evident that $\Omega(a\vee b)\supset\supset\Omega(a)\cup\Omega(b)$. If $p\in\Omega(a\vee b)$, then by $(a, b)SD_j$, there exists $a_1\leqq a$ such that $a_1\leqq p\leqq a_1\vee b$. Since p is an atom, a_1 is either 0 or p. $a_1=0$ implies $p\leqq b$ and $a_1=p$ implies $p\leqq a$. Hence, $p\in\Omega(a)\cup\Omega(b)$. Similarly, ($\beta'$) implies ($\gamma$).

$(\gamma)\Rightarrow(\alpha)$. Let $x\leqq a\vee b$. Putting $a_1=a\wedge x$ and $b_1=b\wedge x$, we have $a_1\vee b_1\leqq x$. If p is an atom with $p\leqq x$, then since $p\in\Omega(a\vee b)=\Omega(a)\cup\Omega(b)$, we have $p\leqq a_1$ or $p\leqq b_1$, whence $p\leqq a_1\vee b_1$. Therefore, $x=a_1\vee b_1$ since L is atomistic.

4. Distributive pairs in congruence lattice of semilattices

Let S be a join-semilattice. A congruence relation θ of S is an equivalence relation having the following property:

If $x_1\equiv y_1(\theta)$ and $x_2\equiv y_2(\theta)$ then $x_1\vee x_2\equiv y_1\vee y_2(\theta)$.

The set of all congruence relations of S, denoted by $C(S)$, forms a complete lattice, where

$$x\equiv y(\wedge_\alpha\theta_\alpha)\Leftrightarrow x\equiv y(\theta_\alpha)$$

for every α. Moreover, $x\equiv y(\theta_1\vee\theta_2)$ if and only if there exist $u_0, u_1, \dots, u_n\in S$ such that $u_0=x, u_n=y$ and $u_{i-1}\equiv u_i(\theta')$ where $\theta'=\theta_1$ or $\theta'=\theta_2$ (see [5]). The greatest congruence 1 is given by $x\equiv y(1)$ for every $x, y\in S$. If J is a proper ideal of S, then there is a congruence relation $\theta(J)$ with two equivalence classes J and $S-J$. (If $J=S$, then $\theta(J)=1$.) It is evident that $\theta(J)$ is a dual-atom of $C(S)$ for any proper ideal J, and it can be verified by the following lemma that every dual-atom has such a form and that $C(S)$ is dual-atomistic (cf. [5], Theorem 1).

LEMMA 10. *Let J be an ideal of S, and let $\theta\in C(S)$. If we put*

$$\bar{J} = \{x\in S;\ x \leqq y \equiv z(\theta),\ z\in J\},$$

then $\bar{J}$ is an ideal of S and $\theta\leqq\theta(\bar{J})$. If $J=(a]\,(a\in S)$, then $\bar{J}=\{x\in S;\ x\vee a\equiv a(\theta)\}$.

PROOF. (i) It is easy to verify that $\bar{J}$ is an ideal. If $x\equiv x'(\theta)$ and $x\in\bar{J}$, then since $x\leqq y\equiv z(\theta)$ with $z\in J$, we have $x'\leqq x'\vee y\equiv x\vee y=y\equiv z(\theta)$, whence $x'\in\bar{J}$. Hence, $\theta\leqq\theta(\bar{J})$.

Let $J=(a]$. If $x\vee a\equiv a(\theta)$ then evidently $x\in\bar{J}$. Conversely, if $x\leqq y\equiv z(\theta)$ and $z\in J$, then since $z\leqq a$, we have $x\vee a=x\vee z\vee a\equiv x\vee y\vee a=y\vee a\equiv z\vee a=a(\theta)$.

LEMMA 11. *If $J_\alpha(\alpha\in A)$ are ideals of S and if $\bigcap_\alpha J_\alpha\neq\emptyset$, then $\bigwedge_{\alpha\in A}\theta(J_\alpha)\leqq\theta(\bigcap_{\alpha\in A}J_\alpha)$.*

PROOF. Assume $x\equiv y(\bigwedge_\alpha\theta(J_\alpha))$, that is, $x\equiv y(\theta(J_\alpha))$ for every α. If $x, y\in J_\alpha$ for every α, then $x\equiv y(\theta(\bigcap_\alpha J_\alpha))$ since $x, y\in\bigcap_\alpha J_\alpha$. If otherwise, $x, y\in S-J_\beta$ for some $\beta\in A$, and hence $x\equiv y\ (\theta(\bigcap_\alpha J_\alpha))$ since $x, y\in S-\bigcap_\alpha J_\alpha$.

THEOREM 12. *Let S be a join-semilattice and let θ_1, θ_2 be congruence relations of S. The following seven statements are equivalent.*

(α) *$(\theta_1, \theta_2)D_m$ in $C(S)$.*

(β) *$(\theta_1, \theta_2)SD_m$ in $C(S)$.* (β') *$(\theta_2, \theta_1)SD_m$ in $C(S)$.*

(γ) *$(\theta_1, \theta_2)M_m$ in $C(S)$.* (γ') *$(\theta_2, \theta_1)M_m$ in $C(S)$.*

(δ) *If J is a proper ideal of S and if $\theta_1\wedge\theta_2\leqq\theta(J)$, then either $\theta_1\leqq\theta(J)$ or $\theta_2\leqq\theta(J)$.*

(ε) *If J_1 and J_2 are ideals of S and if $\theta_1\leqq\theta(J_1)$, $\theta_2\leqq\theta(J_2)$ and $J_1\cap J_2\neq\emptyset$, then either $\theta_1\leqq\theta(J_1\cap J_2)$ or $\theta_2\leqq\theta(J_1\cap J_2)$.*

PROOF. Since the dual of the lattice $C(S)$ is atomistic, it follows from Theorem 9 that the four statements (α), (β), (β') and (δ) are equivalent. The implications (β)$\Rightarrow$(γ), (β')$\Rightarrow$(γ') are evident.

(γ)$\Rightarrow$(ε). Let J_1, J_2 be ideals, and let $\theta_1\leqq\theta(J_1)$, $\theta_2\leqq\theta(J_2)$ and $J_1\cap J_2\neq\emptyset$. If $\theta_1\not\leqq\theta(J_1\cap J_2)$, then there exist $u, v\in S$ such that $u\equiv v(\theta_1)$, $u\in J_1\cap J_2$ and $v\notin J_1\cap J_2$. Putting $\psi=\theta_2\wedge\theta(J_1\cap J_2)$, we have $\theta_1\wedge\theta_2\leqq\psi$, since $\theta_1\wedge\theta_2\leqq\theta(J_1)\wedge\theta(J_2)\leqq\theta(J_1\cap J_2)$ by Lemma 11. Hence, by (γ), we have $(\psi\vee\theta_1)\wedge\theta_2=\psi\vee(\theta_1\wedge\theta_2)=\psi$. We shall show $\theta_2\leqq\theta(J_1\cap J_2)$. Let $x\equiv y(\theta_2)$. We have $v\vee x\equiv v\vee y(\psi)$ since $v\vee x\equiv v\vee y(\theta_2)$ and $v\vee x$, $v\vee y\notin J_1\cap J_2$. Moreover, since $u\vee x\equiv v\vee x(\theta_1)$ and $u\vee y\equiv v\vee y(\theta_1)$, we have $u\vee x\equiv u\vee y(\psi\vee\theta_1)$. Therefore, $u\vee x\equiv u\vee y(\theta(J_1\cap J_2))$, since $u\vee x\equiv u\vee y(\theta_2)$ and $(\psi\vee\theta_1)\wedge\theta_2=\psi\leqq\theta(J_1\cap J_2)$. If $x\in J_1\cap J_2$, then since $u\vee x\in J_1\cap J_2$, we have $u\vee y\in J_1\cap J_2$, whence $y\in J_1\cap J_2$. Thus, we obtain $x\equiv y(\theta(J_1\cap J_2))$, and therefore $\theta_2\leqq\theta(J_1\cap J_2)$.

Similarly we can prove $(\gamma')\Rightarrow(\varepsilon)$.

$(\varepsilon)\Rightarrow(\delta)$. Let J be a proper ideal such that $\theta_1\wedge\theta_2\leqq\theta(J)$. We put

$$J_i=\{x\in S;\ x\leqq y\equiv z(\theta_i),\ z\in J\}\quad (i=1,2).$$

Then, it follows from Lemma 10 that J_1 and J_2 are ideals with $\theta_1\leqq\theta(J_1)$ and $\theta_2\leqq\leqq\theta(J_2)$. Since $J_i\supset J$, we have $J_1\cap J_2\supset J\neq\emptyset$, and hence, by (ε), we have either $\theta_1\leqq\theta(J_1\cap J_2)$ or $\theta_2\leqq\theta(J_1\cap J_2)$. Assuming $\theta_1\leqq\theta(J_1\cap J_2)$, we shall prove $J=J_1$. If $x\in J_1$, then $x\leqq y\equiv z(\theta_1)$ with $z\in J$. Since $y\equiv z(\theta(J_1\cap J_2))$ and $z\in J\subset J_1\cap J_2$, we have $y\in J_1\cap J_2\subset J_2$, whence $y\leqq y'\equiv z'(\theta_2)$ with $z'\in J$. We have $y\vee z\vee z'\equiv z\vee z'(\theta_1)$, and since $y'\vee z\vee z'\equiv z\vee z'(\theta_2)$, we have $y\vee z\vee z'\equiv y\vee y'\vee z\vee z'=y'\vee\vee z\vee z'\equiv z\vee z'(\theta_2)$. Hence, $y\vee z\vee z'\equiv z\vee z'(\theta(J))$ since $\theta_1\wedge\theta_2\leqq\theta(J)$. Since $z\vee z'\in J$, we obtain $x\leqq y\leqq y\vee z\vee z'\in J$. Therefore, $J_1=J$, whence $\theta_1\leqq\theta(J_1)=\theta(J)$. Similarly, if $\theta_2\leqq\theta(J_1\cap J_2)$ then $\theta_2\leqq\theta(J)$.

COROLLARY 13. *$(\theta_1,\theta_2)M_m$ implies $(\theta_2,\theta_1)M_m$, that is, $C(S)$ is M-symmetric in the sense of* [3].

COROLLARY 14. *Let J_1 and J_2 be proper ideals of S. The following statements are equivalent.*

(α) $(\theta(J_1),\ \theta(J_2))D_m$.

(β) $(\theta(J_1),\ \theta(J_2))SD_m$. $\quad(\beta')$ $(\theta(J_2),\ \theta(J_1))SD_m$.

(γ) $(\theta(J_1),\ \theta(J_2))M_m$. $\quad(\gamma')$ $(\theta(J_2),\ \theta(J_1))M_m$.

(δ) $\theta(J_1)\wedge\theta(J_2)$ *has three equivalence classes,*

(ε) $J_1\subset J_2$ *or* $J_1\supset J_2$ *or* $J_1\cap J_2=\emptyset$.

PROOF. The equivalence of (δ) and (ε) is easily verified. The rest is equivalent to (ε) by Theorem 12.

COROLLARY 15. *The following statements are equivalent.*

(α) *$C(S)$ is distributive.*

(β) *$C(S)$ is modular.*

(α') *$(\theta(J_1),\ \theta(J_2))D_m$ for any proper ideals J_1 and J_2.*

(β') *$(\theta(J_1),\ \theta(J_2))M_m$ for any proper ideals J_1 and J_2.*

(γ) *If two elements of S are non-comparable then they have no lower bound in S.*

(δ) *If two ideals J_1 and J_2 of S are not disjoint then $J_1\subset J_2$ or $J_1\supset J_2$.*

PROOF. The equivalence of (γ) and (δ) is easily verified. The implications $(\alpha)\Rightarrow\Rightarrow(\beta)\Rightarrow(\beta')$ and $(\alpha)\Rightarrow(\alpha')\Rightarrow(\beta')$ are trivial. (β') implies (δ) by Corollary 14, and (δ) implies (α) by Theorem 12.

The last corollary was partially proved in [1] and [5].

THEOREM 16. *Let S be a join-semilattice and θ_1, θ_2 be congruence relations of S. The following three statements are equivalent.*

(α) *$(\theta_1,\theta_2)D_j$ in $C(S)$.*

(β) *If $x\equiv y(\theta_1\vee\theta_2)$ then there exist $u_i\in S$ $(i=0,1,\dots,n)$ such that $x=u_0\leqq\leqq u_1\leqq\dots\leqq u_n=x\vee y$ and $u_{i-1}\equiv u_i(\theta')$ where $\theta'=\theta_1$ or $\theta'=\theta_2$.*

(γ) *If $x\equiv y(\theta_1\vee\theta_2)$ and $x\leqq y$ then there exist $u_i\in S$ $(i=0,1,\dots,n)$ such that $u_0=x$, $u_n=y$, $u_i\leqq y$ and $u_{i-1}\equiv u_i(\theta')$ where $\theta'=\theta_1$ or $\theta'=\theta_2$.*

PROOF. $(\alpha)\Rightarrow(\beta)$. Let $x\equiv y(\theta_1\vee\theta_2)$. We put $\psi=\theta((x\vee y])$. Then, we have $x\equiv x\vee y((\theta_1\vee\theta_2)\wedge\psi)$, whence $x\equiv x\vee y((\theta_1\wedge\psi)\vee(\theta_2\wedge\psi))$ by (α). Hence, there exist v_i $(i=0, 1, \ldots, n)$ such that $v_0=x$, $v_n=x\vee y$, $v_{i-1}\equiv v_i(\theta'\wedge\psi)$ where $\theta'=\theta_1$ or $\theta'=\theta_2$. Since $v_i\equiv v_n=x\vee y(\psi)$, we have $v_i\leqq x\vee y$. Putting $u_0=x$, $u_i=x\vee v_1\vee\ldots\vee v_i$, we obtain $u_0\leqq u_1\leqq\ldots\leqq u_n=x\vee y$ and $u_{i-1}\equiv u_i(\theta')$.

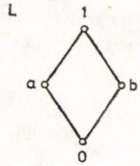
$(\beta)\Rightarrow(\gamma)$ is evident. $(\gamma)\Rightarrow(\alpha)$. It suffices to show that $(\theta_1\vee\theta_2)\wedge\psi\leqq(\theta_1\wedge\psi)\vee(\theta_2\wedge\psi)$ for every $\psi\in C(S)$. Let $x\equiv y((\theta_1\vee\theta_2)\wedge\psi)$, and we shall show $x\equiv y((\theta_1\wedge\psi)\vee(\theta_2\wedge\psi))$. We may assume $x\leqq y$. By (γ), there exist $u_i\in S$ such that $u_0=x$, $u_n=y$, $u_i\leqq y$ and $u_{i-1}\equiv u_i(\theta')$ where $\theta'=\theta_1$ or $\theta'=\theta_2$. Taking $u_i\vee x$ instead of u_i, we may assume that $x\leqq u_i\leqq y$ for every i. Since $x\equiv y(\psi)$, we have $u_i=x\vee u_i\equiv y\vee u_i=y(\psi)$. Hence, $u_{i-1}\equiv u_i(\theta'\wedge\psi)$, and therefore $x\equiv y((\theta_1\wedge\psi)\vee(\theta_2\wedge\psi))$. This completes the proof.

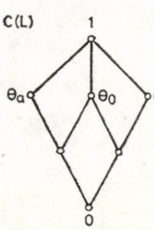
Let L be a lattice. A lattice-congruence relation θ of L is an equivalence relation having the following property: If $x_i\equiv y_i(\theta)$ $(i=1, 2)$ then $x_1\vee x_2\equiv y_1\vee y_2(\theta)$ and $x_1\wedge x_2\equiv y_1\wedge y_2(\theta)$. We denote by $C_0(L)$ the set of all lattice-congruence relations of L. $C_0(L)$ is a sublattice of $C(L)$.

COROLLARY 17. *If* $\theta_1, \theta_2\in C_0(L)$ *then* $(\theta_1, \theta_2)D_j$ *in* $C(L)$. *Especially,* $C_0(L)$ *is a distributive lattice.*

PROOF. If $\theta_1, \theta_2\in C_0(L)$, then the statement (γ) of Theorem 16 is satisfied by taking $u_i\wedge y$ instead of u_i.

Remark that for $\theta_1, \theta_2\in C_0(L)$, $(\theta_1, \theta_2)D_m$ does not necessarily hold. (See the example below.)





EXAMPLE. Let L be the Boolean lattice with four elements $\{1, a, b, 0\}$. $C(L)$ has three dual-atoms $\theta_a=\theta((a])$, $\theta_b=\theta((b])$, $\theta_0=\theta((0])$, and since $\theta_a\wedge\theta_b=0$, we have
$C(L)=\{1, \theta_a, \theta_b, \theta_0, \theta_a\wedge\theta_0, \theta_b\wedge\theta_0, 0\}$
(and $C_0(L)=\{1, \theta_a, \theta_b, 0\}$).
In $C(L)$, there are six non-comparable pairs:
(θ_a, θ_b), (θ_a, θ_0), (θ_b, θ_0), $(\theta_a, \theta_b\wedge\theta_0)$, $(\theta_b, \theta_a\wedge\theta_0)$, $(\theta_a\wedge\theta_0, \theta_b\wedge\theta_0)$. It is easy to verify the following facts.

(1) All the pairs except (θ_a, θ_b) are meet-distributive.

(2) (θ_a, θ_b) and $(\theta_a\wedge\theta_0, \theta_b\wedge\theta_0)$ are join-distributive, but the other non-comparable pairs are not.

(3) (θ_0, θ_a), (θ_0, θ_b), $(\theta_a\wedge\theta_0, \theta_b)$ and $(\theta_b\wedge\theta_0, \theta_a)$ are join-semidistributive.

(4) (θ_a, θ_0) and (θ_b, θ_0) are join-modular but not join-semidistributive.

(5) $(\theta_a, \theta_b\wedge\theta_0)$ and $(\theta_b, \theta_a\wedge\theta_0)$ are not join-modular.

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A STUDY ON THE RELATIONS OF TWO n-DIMENSIONAL UNIFIED FIELD THEORIES*

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1. Introduction

In Appendix II to his last book, "The meaning of relativity", Einstein proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. Characterizing Einstein's unified field theory as a set of geometrical postulates in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time), Hlavatý [7] gave the mathematical foundation of the 4-dimensional unified field theory $(4-g-\mathrm{UFT})$ for the first time. Since then the geometrical consequences of these postulates are developed very far.

4-dimensional $^*g$-unified field theory $(4-{}^*g-\mathrm{UFT})$, which is more useful for physical applications than the usual $4-g-\mathrm{UFT}$, was introduced in Chung's paper [1], and many consequences of this theory have been obtained so far by him. Recently, he found relations between two Einstein's 4-dimensional unified field theories and obtained n-dimensional representations of the unified field tensor $^*g^{\lambda\nu}$ (Chung—Han [4]).

The purpose of the present paper is to find the relations of two n-dimensional unified field theories, $n-g-\mathrm{UFT}$ and $n-{}^*g-\mathrm{UFT}$. This paper contains five chapters. Chapter 2 introduces some preliminaries, and Chapter 3 is devoted to the derivation of n-dimensional recurrence relations. In the fourth chapter we deal with the representations of the tensor $^*g_{\lambda\mu}$, defined by (2.7), and finally in the last chapter, we prove that two n-dimensional unified field theories, $n-g-\mathrm{UFT}$ and $n-{}^*g-$ $-\mathrm{UFT}$, are identical so far as their classification of classes is concerned.

2. Preliminary results

In this chapter we introduce the basic algebraic postulates of two n-dimensional unified field theories at first, and then some preliminaries without proofs.

A. Two n-dimensional unified field theories. In the usual Einstein's n-dimensional unified field theory $(n-g-\mathrm{UFT})$, the generalized n-dimensional Riemannian space X_n, referred to a real coordinate transformation $x^\lambda \to \bar{x}^\lambda$ for which

$$\mathrm{Det}\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0, \tag{2.1}$$

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is endowed with a real non-symmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$*:

$$g_{\lambda\mu} = h_{\lambda\mu}+k_{\lambda\mu}. \tag{2.2}$$

Here the matrices $(g_{\lambda\mu})$ and $(h_{\lambda\mu})$ are assumed to be of rank n. Therefore we may define a unique tensor $h^{\lambda\nu}=h^{\nu\lambda}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu^\nu, \tag{2.3}$$

and the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and lowering indices of tensors in X_n in the usual manner in $n-g-$UFT.

On the other hand, n-dimensional $^*g$-unified field theory ($n-{}^*g-$UFT) in the same space X_n is based upon the basic real tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu^\nu. \tag{2.4}$$

It may also be decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu}+{}^*k^{\lambda\nu}. \tag{2.5}$$

Since Det $(^*h^{\lambda\nu})\neq 0$ (Hlavatý [1], p. 41), we may define a unique tensor $^*h_{\lambda\mu}$ by

$$^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta_\mu^\nu. \tag{2.6}$$

In $n-{}^*g-$UFT we use both $^*h_{\lambda\mu}$ and $^*h^{\lambda\nu}$ as tensors for raising and lowering indices of all tensor quantities in X_n in the usual manner. Then we may define new tensors $^*g_{\lambda\mu}$, $^*k_{\lambda\mu}$, and $^*k_\lambda{}^\nu$ by

$$^*g_{\lambda\mu} = {}^*g^{\alpha\beta}{}^*h_{\lambda\alpha}{}^*h_{\mu\beta}, \quad {}^*k_{\lambda\mu} = {}^*k^{\alpha\beta}{}^*h_{\lambda\alpha}{}^*h_{\mu\beta}, \quad {}^*k_\lambda^\nu = {}^*k^{\alpha\nu}{}^*h_{\alpha\lambda}, \tag{2.7}$$

respectively, so that

$$^*g_{\lambda\mu} = {}^*h_{\lambda\mu}+{}^*k_{\lambda\mu}. \tag{2.8}$$

B. Some preliminaries. This section is a collection of notations and basic results which are needed in our subsequent considerations.

In what follows, the following densities, scalars, and tensors are frequently used:

(2.9)a $$\mathfrak{g} \overset{\text{df}}{=} \text{Det}\,(g_{\lambda\mu}), \quad \mathfrak{h} \overset{\text{df}}{=} \text{Det}\,(h_{\lambda\mu}), \quad \mathfrak{k} \overset{\text{df}}{=} \text{Det}\,(k_{\lambda\mu});$$

(2.9)b $$g \overset{\text{df}}{=} \mathfrak{g}/\mathfrak{h}, \quad k \overset{\text{df}}{=} \mathfrak{k}/\mathfrak{h};$$

(2.9)c $$^{(o)}k_\lambda{}^\nu \overset{\text{df}}{=} \delta_\lambda{}^\nu, \quad {}^{(p)}k_\lambda{}^\nu \overset{\text{df}}{=} {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu \quad (p = 1, 2, \ldots);$$

(2.9)d
$$M_p \overset{\text{df}}{=} E^{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_n} E^{\beta_1\ldots\beta_p\beta_{p+1}\ldots\beta_n} k^{\alpha_1\beta_1}\cdots k^{\alpha_p\beta_p} h^{\alpha_{p+1}\beta_{p+1}}\ldots h^{\alpha_n\beta_n} \quad (p = 0, 1, 2, \ldots);$$

(2.9)e $$K_p \overset{\text{df}}{=} k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2}\ldots k_{\alpha_p]}{}^{\alpha_p}, \quad \overline{K}_p \overset{\text{df}}{=} \sum_{x=0}^{p} K_x \quad (p = 0, 1, 2, \ldots).$$

* Throughout the present paper, Greek and Roman indices are used for the holonomic and nonholonomic components of tensor, resp. Both indices take the values $1, 2, \ldots, n$ unless otherwise stated.

Here $E^{\alpha_1 \dots \alpha_n}$ denotes the n-dimensional contravariant indicator. It should be remarked that the corresponding starred quantities can be similarly defined as in (2.9) just by putting "*" to all quantities in (2.9).

It has been shown that the following relations hold in X_n (Chung—Lee [2]):

$$M_p = p!(n-p)!\mathfrak{h}K_p \quad (p = 0, 1, 2, \dots). \tag{2.10}$$

As direct consequences of (2.10), we have

$$M_0 = n!\mathfrak{h}, \quad M_n = n!\mathfrak{k}; \tag{2.11a}$$

$$K_0 = \overline{K}_0 = 1, \quad K_n = k, \quad \text{if} \quad n \quad \text{is even}; \tag{2.11b}$$

$$M_p = K_p = 0, \quad \text{if} \quad p \quad \text{is odd}; \tag{2.11c}$$

$$\mathfrak{g} = \mathfrak{h} \sum_{p=0}^{n} K_p = \sum_{p=0}^{n} \frac{1}{p!(n-p)!} M_p; \tag{2.11d}$$

$$g = \sum_{p=0}^{n} K_p = \overline{K}_n. \tag{2.11e}$$

It has also been shown (Chung—Han [4]) that the n-dimensional representations of the tensors $^*h^{\lambda\nu}$ and $^*k^{\lambda\nu}$ are

$$^*h^{\lambda\nu} = \frac{1}{g} \sum_{p=0}^{n-1} (K_0{}^{(p)}k^{\lambda\nu} + K_2{}^{(p-2)}k^{\lambda\nu} + \dots + K_{p-2}{}^{(2)}k^{\lambda\nu} + K_p h^{\lambda\nu}), \tag{2.12a}$$

$$^*k^{\lambda\nu} = \frac{1}{g} \sum_{p=2}^{n} (K_0{}^{(p-1)}k^{\lambda\nu} + K_2{}^{(p-3)}k^{\lambda\nu} + \dots + K_{p-4}{}^{(3)}k^{\lambda\nu} + K_{p-2}k^{\lambda\nu}). \tag{2.12b}$$

An eigenvector a^μ of $k_{\lambda\mu}$ which satisfies

$$(Mh_{\lambda\mu} + k_{\lambda\mu})a^\mu = 0 \quad (M \text{ is an arbitrary scalar}) \tag{2.13}$$

is called a basic vector of X_n, and the corresponding eigenvalue of $k_{\lambda\mu}$ a basic scalar of X_n. It has also been shown in Chung—Lee [2] that the basic scalars M satisfy

$$M^n + K_2 M^{n-2} + \dots + K_{n-2} M^2 + K_n = 0, \quad \text{if } n \text{ is even}; \tag{2.14a}$$

$$M(M^{n-1} + K_2 M^{n-3} + \dots + K_{n-3} M^2 + K_{n-1}) = 0, \quad \text{if } n \text{ is odd}. \tag{2.14b}$$

Hlavatý [1] also proved that the nonholonomic components of the tensors $^{(p)}k_\lambda{}^\nu$, $^{(p)}k_{\lambda\nu}$, and $^{(p)}k^{\lambda\nu}$ are given by

$$^{(p)}k_x{}^i = \underset{x}{M}{}^p \delta_x^i, \tag{2.15a}$$

$$^{(p)}k_{xi} = \underset{x}{M}{}^p h_{xi}, \quad (p = 0, 1, 2, \dots) \tag{2.15b}$$

$$^{(p)}k^{xi} = \underset{x}{M}{}^p h^{xi}. \tag{2.15c}$$

3. n-Dimensional recurrence relations in X_n

In this chapter we derive several powerful recurrence relations in X_n using the concept of basic scalars.

AGREEMENT (3.1). In this and in what follows, we use the following notation:

$$\sigma \overset{\text{df}}{=} \begin{cases} 0, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases} \tag{3.1}$$

Furthermore, the index f is assumed to take the values $0, 2, 4, \ldots, n-\sigma$ in our subsequent considerations.

Now, we are ready to derive an important n-dimensional recurrence relation which holds in X_n.

THEOREM (3.2). (Main recurrence relation). *We have*

$$\text{(3.2)a} \qquad {}^{(n+p)}k_\lambda{}^\nu + K_2{}^{(n+p-2)}k_\lambda{}^\nu + \ldots + K_{n-\sigma-2}{}^{(\sigma+p+2)}k_\lambda{}^\nu + K_{n-\sigma}{}^{(\sigma+p)}k_\lambda{}^\nu = 0,$$

which may be condensed to

$$\text{(3.2)b} \qquad \sum_{f=0}^{n-\sigma} K_f{}^{(n+p-f)}k_\lambda{}^\nu = 0,$$

where $p=0, 1, 2, \ldots$.

PROOF. Let $\underset{x}{M}$ be a basic scalar. Then, in virtue of (2.14), we have

$$\sum_{f=0}^{n-\sigma} K_f \underset{x}{M}^{n-f} = 0. \tag{3.3}$$

Multiplying δ_x^i to both sides of (3.7) and making use of (2.15)a, we have

$$\text{(3.4)a} \qquad \sum_{f=0}^{n-\sigma} K_f{}^{(n-f)}k_x{}^i = 0,$$

whose holonomic form may be given by

$$\text{(3.4)b} \qquad \sum_{f=0}^{n-\sigma} K_f{}^{(n-f)}k_\lambda{}^\alpha = 0.$$

Our recurrence relation (3.2) follows from (3.4)b by multiplying ${}^{(p)}k_\alpha{}^\nu$ to both sides of (3.4)b.

As a variation of (3.2), we may easily have the following useful recurrence relation:

$$\sum_{f=0}^{n-\sigma} K_f{}^{(n+p-\sigma-f)}k_\lambda{}^\nu = 0 \quad (p = 1, 2, 3, \ldots). \tag{3.5}$$

For an arbitrary symmetric tensor $X_{\omega\mu}$, introduce the following abbreviations:

$$\overset{(p)}{X}_{\omega\mu} \overset{\text{df}}{=} {}^{(p)}k_\omega{}^\alpha X_{\alpha\mu} \quad (p = 0, 1, 2, \ldots). \tag{3.6}$$

We then have

$$(3.7)\qquad \overset{(o)}{X}_{\omega\mu} \overset{\mathrm{df}}{=} X_{\omega\mu},\quad {}^{(p)}k_{\omega}{}^{\alpha}\overset{(q)}{X}_{\alpha\mu} = \overset{(p+q)}{X}_{\omega\mu}\quad (p, q = 0, 1, 2, \ldots).$$

The following variations of our main recurrence relation (3.2) are needed in the proof of Theorem (4.2).

THEOREM (3.3). (Variations of recurrence relation). *The tensor* $\overset{(p)}{X}_{\omega\mu}$ *satisfies*

$$(3.8)\text{a}\qquad \sum_{f=0}^{n-\sigma} K_f \overset{(n+p-f)}{X_{\omega\mu}} = 0 \quad (p = 1, 2, \ldots),$$

which is equivalent to

$$(3.8)\text{b}\qquad \sum_{f=0}^{n-\sigma} K_f \overset{(n+\sigma-f)}{X_{\omega\mu}} = 0.$$

PROOF. The relation (3.8)a follows from (3.2) in virtue of (3.6). The relation (3.8)b is a useful variation of (3.8)a. It may be obtained from (3.8)a, putting $p=\sigma$ and noting that $K_{n+1}=0$.

4. The tensor $^*g_{\lambda\mu}$ in X_n

In this chapter we derive useful representations of the unified field tensor $^*g_{\lambda\mu}$ in X_n, using the recurrence relations obtained in the previous chapter.

THEOREM (4.1). *Another n-dimensional representations of the unified field tensor* $^*g^{\lambda\nu}$ *may be given by*

$$(4.1)\text{a}\qquad {}^*h^{\lambda\nu} = \frac{1}{g}\sum_{f=0}^{n+\sigma-2} \overline{K}_f{}^{(n+\sigma-2-f)}k^{\lambda\nu},$$

$$(4.1)\text{b}\qquad {}^*k^{\lambda\nu} = \frac{1}{g}\sum_{f=0}^{n-\sigma-2} \overline{K}_f{}^{(n-\sigma-1-f)}k^{\lambda\nu}.$$

PROOF. Using (2.9)e and (2.11)b, c, (2.12) may be written as

(4.2)a

$$ {}^*h^{\lambda\nu} = \begin{cases} \dfrac{1}{g}\left({}^{(n-2)}k^{\lambda\nu}+\overline{K}_2{}^{(n-4)}k^{\lambda\nu}+\ldots+\overline{K}_{n-4}{}^{(2)}k^{\lambda\nu}+\overline{K}_{n-2}h^{\lambda\nu}\right), & \text{if } n \text{ is even,}\\ \dfrac{1}{g}\left({}^{(n-1)}k^{\lambda\nu}+\overline{K}_2{}^{(n-3)}k^{\lambda\nu}+\ldots+\overline{K}_{n-3}{}^{(2)}k^{\lambda\nu}+\overline{K}_{n-1}h^{\lambda\nu}\right), & \text{if } n \text{ is odd;}\end{cases}$$

(4.2)b

$$ {}^*k^{\lambda\nu} = \begin{cases} \dfrac{1}{g}\left({}^{(n-1)}k^{\lambda\nu}+\overline{K}_2{}^{(n-3)}k^{\lambda\nu}+\ldots+\overline{K}_{n-4}{}^{(3)}k^{\lambda\nu}+\overline{K}_{n-2}k^{\lambda\nu}\right), & \text{if } n \text{ is even,}\\ \dfrac{1}{g}\left({}^{(n-2)}k^{\lambda\nu}+\overline{K}_2{}^{(n-4)}k^{\lambda\nu}+\ldots+K_{n-5}{}^{(3)}k^{\lambda\nu}+\overline{K}_{n-3}k^{\lambda\nu}\right), & \text{if } n \text{ is odd.}\end{cases}$$

The expressions (4.1) are condensed forms of (4.2).

THEOREM (4.2). *The n-dimensional representations of the tensor $^*h_{\lambda\mu}$ in X_n may be given by*

(4.3) $$^*h_{\lambda\nu} = h_{\lambda\nu} - {}^{(2)}k_{\lambda\mu}.$$

PROOF. In order to prove (4.3), consider a symmetric tensor $X_{\lambda\mu}$ uniquely defined by

(4.4) $$^*h^{\lambda\nu} X_{\lambda\mu} = \delta_\mu^\nu.$$

Substituting (4.1)a into (4.4), we have

(4.5)a $$\sum_{f=0}^{n+\sigma-2} \overline{K}_f {}^{(n+\sigma-2-f)}k^{\lambda\nu} X_{\lambda\mu} = g\delta_\mu^\nu.$$

Multiplying $h_{\omega\nu}$ to both sides of (4.5)a, we have in virtue of (3.6) and of the tensor ${}^{(n+\sigma-2-f)}k^{\lambda\nu}$ being symmetric

(4.5)b $$\sum_{f=0}^{n+\sigma-2} \overline{K}_f \overset{(n+\sigma-2-f)}{X_{\omega\mu}} = gh_{\omega\mu},$$

which is equivalent to

(4.5)c $$\sum_{f=0}^{n+\sigma-4} \overline{K}_f \overset{(n+\sigma-2-f)}{X_{\omega\mu}} + \overline{K}_{n+\sigma-2} X_{\omega\mu} = gh_{\omega\mu}.$$

On the other hand, multiplying ${}^{(2)}k_\lambda{}^\omega$ to both sides of (4.5)b and making use of (3.7), we have

(4.6)a $$\sum_{f=0}^{n+\sigma-2} \overline{K}_f \overset{(n+\sigma-f)}{X_{\omega\mu}} = g^{(2)}k_{\omega\mu}.$$

Substitution for $\overset{(n+\sigma)}{X_{\omega\mu}}$ into left side of (4.6)a from the recurrence relation (3.8)b gives

(4.6)b $$\sum_{f=0}^{n+\sigma-4} \overline{K}_f \overset{(n+\sigma-2-f)}{X_{\omega\mu}} - K_{n+\sigma} X_{\omega\mu} = g^{(2)}k_{\omega\mu}$$

in virtue of (2.11)b, c. Now, subtracting (4.6)b from (4.5)c, we finally have (4.3) in virtue of $\overline{K}_{n+\sigma} = g$ in X_n and (2.6).

THEOREM (4.3). *The n-dimensional representation of the tensor $^*k_{\lambda\mu}$ in X_n may be given by*

(4.7) $$^*k_{\lambda\mu} = k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu}.$$

PROOF. In order to prove (4.7), start with the obvious equations

(4.8) $$g^*k_{\lambda\mu} = g^*k^{\alpha\beta} {}^*h_{\lambda\alpha} {}^*h_{\mu\beta}.$$

Substituting from (4.1)b and (4.3) into (4.8) and rearranging the range of summations, we have

$$(4.9)\qquad g^*k_{\lambda\mu} = \sum_{f=0}^{n-\sigma-2} \bar{K}_f{}^{(n-\sigma-1-f)}k^{\alpha\beta}(h_{\lambda\alpha}-{}^{(2)}k_{\lambda\alpha})(h_{\mu\beta}-{}^{(2)}k_{\mu\beta}) =$$

$$= \sum_{f=0}^{n-\sigma-2} \bar{K}_f({}^{(n-\sigma-1-f)}k_{\lambda\mu}-2{}^{(n-\sigma+1-f)}k_{\lambda\mu}+{}^{(n-\sigma+3-f)}k_{\lambda\mu}) =$$

$$= \sum_{f=0}^{n-\sigma-2} \bar{K}_f{}^{(n-\sigma-1-f)}k_{\lambda\mu}-2\sum_{f=0}^{n-\sigma-4} \bar{K}_{f+2}{}^{(n-\sigma-1-f)}k_{\lambda\mu}+$$

$$+\sum_{f=0}^{n-\sigma-6} \bar{K}_{f+4}{}^{(n-\sigma-1-f)}k_{\lambda\mu}+\{(\bar{K}_2-2){}^{(n-\sigma+1)}k_{\lambda\mu}+{}^{(n-\sigma+3)}k_{\lambda\mu}\} =$$

$$= \sum_{f=0}^{n-\sigma-6} (\bar{K}_f-2\bar{K}_{f+2}+\bar{K}_{f+4}){}^{(n-\sigma-1-f)}k_{\lambda\mu}+$$

$$+(\bar{K}_{n-\sigma-4}-2\bar{K}_{n-\sigma-2}){}^{(3)}k_{\lambda\mu}+\bar{K}_{n-\sigma-2}k_{\lambda\mu}+$$

$$+\{(\bar{K}_2-2){}^{(n-\sigma+1)}k_{\lambda\mu}+{}^{(n-\sigma+3)}k_{\lambda\mu}\} =$$

$$= \sum_{f=0}^{n-\sigma-6} (K_{f+4}-K_{f+2}){}^{(n-\sigma-1-f)}k_{\lambda\mu}-(\bar{K}_{n-\sigma-2}+K_{n-\sigma-2}){}^{(3)}k_{\lambda\mu}+$$

$$+\bar{K}_{n-\sigma-2}k_{\lambda\mu}+\{(K_2-2){}^{(n-\sigma+1)}k_{\lambda\mu}+{}^{(n-\sigma+3)}k_{\lambda\mu}\}.$$

On the other hand, in virtue of the recurrence relation (3.5), the last term of (4.9) may be reduced to

$$(4.10)\qquad (\bar{K}_2-2){}^{(n-\sigma+1)}k_{\lambda\mu}+{}^{(n-\sigma+3)}k_{\lambda\mu} =$$

$$= (K_2-1){}^{(n-\sigma+1)}k_{\lambda\mu}-(K_2{}^{(n-\sigma+1)}k_{\lambda\mu}+K_4{}^{(n-\sigma+1)}k_{\lambda\mu}+\ldots+K_{n-\sigma-2}{}^{(5)}k_{\lambda\mu}+K_{n-\sigma}{}^{(3)}k_{\lambda\mu}) =$$

$$= -{}^{(n-\sigma+1)}k_{\lambda\mu}-(K_4{}^{(n-\sigma-1)}k_{\lambda\mu}+\ldots+K_{n-\sigma-2}{}^{(5)}k_{\lambda\mu}+K_{n-\sigma}{}^{(3)}k_{\lambda\mu}) =$$

$$= -(K_2{}^{(n-\sigma-1)}k_{\lambda\mu}+K_4{}^{(n-\sigma-3)}k_{\lambda\mu}+\ldots+K_{n-\sigma-4}{}^{(5)}k_{\lambda\mu}+K_{n-\sigma-2}{}^{(3)}k_{\lambda\mu}+K_{n-\sigma}k_{\lambda\mu})-$$

$$-(K_4{}^{(n-\sigma-1)}k_{\lambda\mu}+\ldots+K_{n-\sigma-2}{}^{(5)}k_{\lambda\mu}+K_{n-\sigma}{}^{(3)}k_{\lambda\mu}) =$$

$$= \sum_{f=0}^{n-\sigma-6} (K_{f+2}-K_{f+4}){}^{(n-\sigma-1-f)}k_{\lambda\mu}+(K_{n-\sigma-2}-K_{n-\sigma}){}^{(3)}k_{\lambda\mu}+K_{n-\sigma}k_{\lambda\mu}.$$

Substitution of (4.10) into (4.9) finally proves our assertion (4.7) since $\bar{K}_{n-\sigma}=g$.

REMARK (4.4). The representation (4.3) for $^*h_{\lambda\mu}$ is coincident with Chung's 4-dimensional result ((3.1a), Chung [1], p. 1304). However, the representation (4.7) for $^*k_{\lambda\mu}$ obtained in the present paper is more refined and useful than Chung's previous lengthy result for 4-dimensional case ((3.1)b, Chung [1], p. 1304):

$$(4.11)\text{a}\qquad {}^*k_{\lambda\mu} = \frac{1}{g}\Big\{(1-k)k_{\lambda\mu}-2\left(1+\frac{1}{2}K_2\right){}^{(3)}k_{\lambda\mu}+$$

$$+\frac{\varkappa}{2}\sqrt{\mathfrak{k}}\,(e_{\lambda\mu\alpha\beta}k^{\alpha\beta}+2e_{\alpha\beta\gamma[\lambda}{}^{(2)}k_{\mu]}{}^{\alpha}k^{\beta\gamma}+e_{\alpha\beta\gamma\delta}k^{\alpha\beta(2)}k_\lambda{}^{\gamma(2)}k_\mu{}^{\delta})\Big\},$$

where

(4.11)b $$\varkappa \overset{\text{df}}{=} \operatorname{sgn}(E^{\omega\mu\lambda\nu} k_{\omega\mu} k_{\lambda\nu}).$$

The coincidence of (4.7) and (4.11) follows from

(4.12)a $$\sqrt{\mathfrak{k}}\, E^{\omega\mu\lambda\nu} = \varkappa\mathfrak{h}(k^{\omega\mu}k^{\lambda\nu}+k^{\omega\lambda}k^{\nu\mu}+k^{\omega\nu}k^{\mu\lambda}),$$

(4.12)b $$\sqrt{\mathfrak{k}}\, e_{\omega\mu\lambda\nu} = \varkappa(k_{\omega\mu}k_{\lambda\nu}+k_{\omega\lambda}k_{\nu\mu}+k_{\omega\nu}k_{\mu\lambda}),$$

which may be verified by using (2.10) (Hlavatý, [7], p. 6) and the skew-symmetry of $k^{\lambda\nu}$. Here $e_{\omega\mu\lambda\nu}$ denotes 4-dimensional covariant indicator.

5. Relations between $n-g-$UFT and $n-{}^*g-$UFT

In this chapter, we investigate the relations between two n-dimensional unified field theories as an application of the results obtained in the previous chapter.

DEFINITION (5.1). *The tensor* $\begin{cases} k_{\lambda\mu} \text{ of } n-g-\text{UFT} \\ {}^*k_{\lambda\mu} \text{ of } n-{}^*g-\text{UFT}\end{cases}$ *is said to belong to*

(1) *the first class if* $\begin{cases} K_{n-\sigma}\neq 0, \\ {}^*K_{n-\sigma}\neq 0,\end{cases}$

(2) *the second class with* j^{th} *category if*

$$\begin{cases} K_{2j}\neq 0, & K_{2j+2}=K_{2j+4}=\ldots=K_{n-\sigma}=0, \\ {}^*K_{2j}\neq 0, & {}^*K_{2j+2}={}^*K_{2j+4}=\ldots={}^*K_{n-\sigma}=0,\end{cases}$$

(3) *the third class if* $\begin{cases} K_2=K_4=\ldots=K_{n-\sigma}=0 \\ {}^*K_2={}^*K_4=\ldots={}^*K_{n-\sigma}=0.\end{cases}$

THEOREM (5.2). *We have*

(5.1) $${}^*k_\lambda{}^\nu = k_\lambda{}^\nu.$$

PROOF. Using (4.1)b and (4.3) and rearranging the range of summation, we have

(5.2) $$\begin{aligned} g^*k_\lambda{}^\nu &= g^*k^{\alpha\nu *}h_{\alpha\lambda} = \sum_{f=0}^{n-\sigma-2} \overline{K}_f{}^{(n-\sigma-1-f)}k^{\alpha\nu}(h_{\alpha\lambda}-{}^{(2)}k_{\alpha\lambda}) = \\ &= \sum_{f=0}^{n-\sigma-2} \overline{K}_f({}^{(n-\sigma-1-f)}k_\lambda{}^\nu - {}^{(n-\sigma+1-f)}k_\lambda{}^\nu) = \\ &= \Big(\sum_{f=0}^{n-\sigma-4} \overline{K}_f{}^{(n-\sigma-1-f)}k_\lambda{}^\nu + \overline{K}_{n-\sigma-2}k_\lambda{}^\nu\Big) - \\ &\quad -\Big(\sum_{f=0}^{n-\sigma-4} K_{f+2}{}^{(n-\sigma-1-f)}k_\lambda{}^\nu + {}^{(n-\sigma+1)}k_\lambda{}^\nu\Big) = \\ &= -\Big(\sum_{f=0}^{n-\sigma-4} K_{f+2}{}^{(n-\sigma-1-f)}k_\lambda{}^\nu + {}^{(n-\sigma+1)}k_\lambda{}^\nu\Big) + \overline{K}_{n-\sigma-2}k_\lambda{}^\nu = \\ &= -\sum_{f=0}^{n-\sigma-2} K_f{}^{(n-\sigma+1-f)}k_\lambda{}^\nu + \overline{K}_{n-\sigma-2}k_\lambda{}^\nu. \end{aligned}$$

On the other hand, in virtue of the recurrence relation (3.5) for the case $p=1$, the first term of the right-hand side of (5.2) may be reduced to

$$-\sum_{f=0}^{n-\sigma-2} K_f{}^{(n-\sigma+1-f)}k_\lambda{}^\nu = K_{n-\sigma}k_\lambda{}^\nu. \tag{5.3}$$

Substituting (5.3) into (5.2) and noting that $\overline{K}_{n-\sigma}=g$, we have (5.1).

We finally have the following two theorems, which are direct results of (2.9)e, (5.1), and Definition (5.1).

THEOREM (5.3). *We have*

$$^*K_p = K_p \quad (p = 0, 1, 2, \ldots). \tag{5.4}$$

THEOREM (5.4). *The classification of the tensor field* $k_{\lambda\mu}$ *in* $n-g-$UFT *is identical to that of the tensor field* $^*k_{\lambda\mu}$ *in* $n-{}^*g-$UFT.

In addition to the relation stated in Theorem (5.4), the following theorem (Chung [1], p. 1307) gives the complete relationships between two n-dimensional unified field theories.

THEOREM (5.5). *The signature of the tensor* $h_{\lambda\mu}$ *in* $n-g-$UFT *is identical to that of the tensor* $^*h_{\lambda\mu}$ *in* $n-{}^*g-$UFT.

REMARK (5.6). The relation in Theorem (5.4) is coincident with Chung's previous result for 4-dimensional case (Theorem (3.6), Chung—Yang [3], p. 48).

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CONDITIONS FOR INCLUSION BETWEEN NÖRLUND SUMMABILITY METHODS

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1. Introduction

Let $p=\{p_n\}_{n\geqq 0}$ denote a sequence of complex numbers, let $P_n=\sum_{k=0}^{n} p_k$ and let $p(z)=\sum_{n=0}^{\infty} p_n z^n$. A sequence $\{s_n\}_{n\geqq 0}$ is Nörlund summable (N, p) to l if $P_n\neq 0$ for $n\geqq 0$ and $\lim_{n\to\infty} \sum_{\nu=0}^{n} p_{n-\nu}s_\nu/P_n=l$. We use the same notation with other letters in place of p, P. It is well known that necessary and sufficient conditions for (N, p) to be regular (i.e., finite limit preserving) are

$$\text{(a)}\quad \sum_{\nu=0}^{n} |p_\nu| = O(|P_n|) \quad\text{and}\quad \text{(b)}\quad p_n = o(P_n),$$

cf. Theorem 16 of [2] where Hardy considers the special case $p_n\geqq 0$ so that (a) is automatically satisfied. In this paper we make a contribution to the solution of an open problem raised by Theorem 19 of [2] and mentioned explicitly on page 91 of [2]. In particular, we consider the question whether the condition $\sum_{\nu=0}^{n} |k_\nu|=O(|Q_n|)$ alone is necessary and sufficient for (N, p) to imply (N, q) when $P_n=O(1)$, $|Q_n|\to\infty$, both (N, p) and (N, q) are regular, the sequence $\{k_n\}_{n\geqq 0}$ being obtained from the generating function $k(z)=q(z)/p(z)$. We can solve the problem completely for $p(z)$ a polynomial, and for a wide class of functions $p(z)$ with algebraic and logarithmic singularities on $|z|=1$, but the general case leads to delicate questions that escape our analysis.

2. The main problem

In Theorem 19 of [2], under the hypotheses that (N, p) and (N, q) are both regular, Hardy shows that the two conditions

$$\text{(A)}\qquad \sum_{\nu=0}^{n} |k_{n-\nu}P_\nu| = O(|Q_n|),$$

$$\text{(B)}\qquad k_n = o(Q_n),$$

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are necessary and sufficient for (N, p) to imply (N, q)[2]. Following his argument (for the case $p_n \geqq 0$, $q_n \geqq 0$) it is not difficult to verify that (B) may be omitted in the cases (i) $|P_n| \to \infty$, (ii) $P_n = O(1)$ and $Q_n = O(1)$. In the remaining case, $P_n = O(1)$ and $|Q_n| \to \infty$, it is natural to conjecture that (A) alone is necessary and sufficient for (N, p) to imply (N, q). To deal with this problem we consider regular Nörlund methods (N, p) with $P_n = O(1)$. It is easy to see from the regularity conditions that this is equivalant to considering sequences $\{p_n\}$ with $\sum_{n=0}^{\infty} |p_n| < \infty$, $p(1) \neq 0$ and $P_n \neq 0$ for $n \geqq 0$.

Given $\sum_{n=0}^{\infty} |p_n| < \infty$, $p_0 \neq 0$ and $p(1) \neq 0$, the little Nörlund method (Z, p) is defined as follows:

$$s_n \to l(Z, p) \quad \text{if} \quad \lim_{n \to \infty} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu = l p(1).$$

This method is regular, and equivalent to (N, p) when (N, p) is regular and $P_n = O(1)$. In this case (A) is equivalent to

$$\text{(C)} \qquad \sum_{\nu=0}^{n} |k_\nu| = O(|Q_n|)$$

provided (N, q) is regular. A simple direct argument shows that, provided (Z, p) is defined and (N, q) is regular, (B) and (C) are necessary and sufficient for (Z, p) to imply (N, q).

In Section 3 we prove that the conjecture is true when $p(z)$ has no zeros on $|z| = 1$, and in Sections 4 and 5 we investigate what happens when $p(z)$ has zeros on $|z| = 1$ and when (N, q) is the Cesàro method (C, α) respectively.

3. The case $p(z) \neq 0$ for $|z| = 1$

Before considering this case we show that (C) does imply that (B) holds in the (C, δ) sense for every $\delta > 0$. In fact we prove slightly more.

THEOREM 1. *Suppose that (Z, p) is defined, (N, q) is regular and*

$$\text{(1)} \qquad k_n = O(|Q_n|).$$

Then

$$\frac{k_n}{Q_n} \to 0 \quad (Z, p).$$

PROOF. Consider the identity

$$\sum_{\nu=0}^{n} p_{n-\nu} \frac{k_\nu}{Q_\nu} = \sum_{\nu=0}^{n} p_\nu \frac{k_{n-\nu}}{Q_{n-\nu}} = \frac{q_n}{Q_n} + \sum_{\nu=0}^{n} p_\nu \frac{k_{n-\nu}}{Q_{n-\nu}} \left(1 - \frac{Q_{n-\nu}}{Q_n}\right).$$

[2] Since Hardy only considers Nörlund methods with $p_n \geqq 0$, $q_n \geqq 0$ his conditions have to be modified in the obvious way.

The first term on the right-hand side tends to 0 by the regularity of (N, q). By the Weierstrass M-test, the series on the right-hand side is absolutely and uniformly convergent with respect to n since

$$\left|p_\nu \frac{k_{n-\nu}}{Q_{n-\nu}}\left(1-\frac{Q_{n-\nu}}{Q_n}\right)\right| \leqq M|p_\nu|\ ^3$$

by (1) and the regularity of (N, q), and so the second term on the right-hand side tends to 0 (by taking the limit as $n\to\infty$ inside the sum). This completes the proof.

COROLLARY. *Under the hypotheses of Theorem* 1,

$$\frac{k_n}{Q_n} \to 0 \quad (C, \delta)$$

for every $\delta > 0$.

PROOF. Let $t_n = \sum_{\nu=0}^{n} p_{n-\nu} s_\nu$ where $s_\nu = k_\nu/Q_\nu$. Then, by (1), $s(z) = \sum_{n=0}^{\infty} s_n z^n$ is analytic in $|z|<1$, and $(1-z)s(z)=(1-z)t(z)/p(z)\to 0$ as $z\to 1$ through real values in $|z|<1$, since $t_n\to 0$ and $p(1)\neq 0$. It follows that $s_n\to 0$ (Abel) and the result is now a consequence of Théorème VI' (sequence version) of [5] or Theorems 70 and 92 of [2].

We give an example to show that we cannot replace $\delta>0$ by $\delta=0$ in the corollary. Let $\{p_n\}$, $\{q_n\}$ be defined from the generating functions $p(z)=1+z$, $q(z)=(1-z^2)^{-1}$ so that $k(z)=[(1+z)(1-z^2)]^{-1}$. Then $Q(z)=(1-z)^{-1}q(z)$ and so $Q(-z)=k(z)$, i.e., $Q_n=(-1)^n k_n$. It is clear that the hypotheses of Theorem 1 hold, but that in this case $k_n/Q_n=(-1)^n\to 0$ (C, δ) for all $\delta>0$ whereas $k_n/Q_n \not\to 0$ as $n\to\infty$. We remark that this example does not satisfy (C) and so is not a counterexample to the conjecture.

If $p(z)$ has no zeros on $|z|=1$, we can use Theorem 1 together with the following tauberian result to establish the conjecture in this case.

THEOREM 2. *Let (Z, p) be defined. Then (Z, p) sums no bounded divergent sequence if and only if $p(z)\neq 0$ for $|z|=1$.*

PROOF. For the sufficiency of the condition we first observe that $p(z)$ has only a finite number of zeros in $|z|<1$ (otherwise they would accumulate on the boundary). Let these be at the points $z=z_i$ with multiplicity λ_i $(i=1, 2, \dots, l)$. Then, by Theorem 1 of [7], we have that $s_n\to 0$ (Z, p) if and only if $s_n = t_n + \sum_{i=1}^{l} f_i(n) z_i^{-n}$ where $\{t_n\}$ converges to 0 and $f_i(n)$ is a polynomial in n of degree $(\lambda_i - 1)$. By Lemma 2 of [8], $\{\sum_{i=1}^{l} f_i(n) z_i^{-n}\}_{n\geqq 0}$ is unbounded unless $f_i(n)\equiv 0$ $(i=1, 2, \dots, l)$. Hence the only sequences summable (Z, p) are convergent or unbounded.

To prove the necessity of the condition suppose $p(\beta)=0$, $|\beta|=1$, $\beta\neq 1$. Since we are assuming $\sum_{n=0}^{\infty}|p_n|<\infty$, $p(z)=\sum_{n=0}^{\infty} p_n z^n$ converges for $|z|\leqq 1$ and so $p(\beta)=$

[3] We use M to denote a positive constant, independent of the variables, that may be different at each occurrence.

$=\sum_{n=0}^{\infty} p_n\beta^n=0$. It is now easy to see that the bounded divergent sequence $\{\beta^{-n}\}$ is summable to 0 (Z, p), and the result follows.

COROLLARY. *Suppose that (Z, p) is defined, $p(z)\neq 0$ for $|z|=1$, (N, q) is regular and* (C) *holds. Then (Z, p) implies (N, q).*

PROOF. By the remarks at the end of Section 2 it is sufficient to show that (B) holds. Since (C) implies that (1) holds, Theorem 1 gives that the bounded sequence $\{k_n/Q_n\}$ is summable (Z, p) to 0, and Theorem 2 shows that it must converge to 0, i.e. (B) must hold.

4. The case where $p(z)$ may have zeros on $|z|=1$

A summability method based on a regular, normal (i.e., lower triangular with non-zero diagonal) sequence to sequence matrix $A=(a_{nk})$ is said to be perfect if $\sum_{n=\nu}^{\infty} \alpha_n a_{n\nu}=0$ $(\nu=0, 1, \ldots)$ together with $\sum_{n=0}^{\infty} |\alpha_n|<\infty$ implies $\alpha_n=0$ $(n=0, 1, \ldots)$. See [4] and [9] for some basic properties. For the methods (N, p) and (Z, p) we have $a_{n\nu}$ equal to $p_{n-\nu}/P_n$ and $p_{n-\nu}$ respectively. It is clear that neither (N, p) nor (Z, p) is perfect if $p(z)$ has a zero in $|z|<1$ (since, if $p(w)=0$ with $0<|w|<1$, then $\alpha_n= =P_n w^n$ is a non-zero term of an absolutely convergent series that satisfies the conditions for perfectness of (N, p), and likewise with $\alpha_n=w^n$ for (Z, p)). This observation also settles an undecided question mentioned on page 707 of [4]. Hill asks whether the Nörlund method (N, p) with generating function $p(z)=(1+az)(1-z)^{-2}$ is perfect for $a>1$. Since $p(z)$ has a zero at $z=-1/a$ which is in $|z|<1$, (N, p) cannot be perfect.

THEOREM 3. *Suppose that (Z, p) is perfect, (N, q) is regular and* (C) *holds. Then (Z, p) implies (N, q).*

PROOF. This follows directly from Theorem II. 8 of [9] with $(Z, p)=A, (N, q)=B$, and the observation that (C) is necessary and sufficient for every sequence summable to 0 (Z, p) to be bounded (N, q).

The remainder of this section is devoted to finding examples of perfect (Z, p) methods. We introduce the notation $\{c_n\}$ for the coefficients of the generating function $c(z)=1/p(z)$. It follows from Theorem 8 of [4] that when (Z, p) is defined then $c_n=O(1)$ is a sufficient condition for it to be perfect.

LEMMA 1. *If $p(z)=\left(1-\frac{z}{\beta}\right)^{\lambda}$ where $\beta\neq 1$, $|\beta|=1$, $\lambda>0$, then (Z, p) is perfect.*

PROOF. We have $p_n=A_n^{-\lambda-1}\beta^{-n}$ where $A_n^{-\lambda-1}=\binom{n-\lambda-1}{n}$ is defined from the relation

$$(1-z)^{\lambda} = \sum_{n=0}^{\infty} A_n^{-\lambda-1} z^n, \tag{2}$$

so that $\sum_{n=0}^{\infty} |p_n|<\infty$, $p_0=1$ and $p(1)\neq 0$. Suppose that $\sum_{n=0}^{\infty} |\alpha_n|<\infty$ and $\sum_{n=\nu}^{\infty} \alpha_n p_{n-\nu}=0$

($\nu=0, 1, \ldots$). This can be written as

$$\sum_{n=\nu}^{\infty} \alpha_n A_{n-\nu}^{-\lambda-1} \beta^{\nu-n} = \beta^{\nu} \sum_{n=\nu}^{\infty} A_{n-\nu}^{-\lambda-1}(\alpha_n \beta^{-n}) = 0,$$

and using the notation for fractional differences (see [1]) this is equivalent to

$$\Delta^{\lambda}(\alpha_\nu \beta^{-\nu}) = 0 \quad (\nu = 0, 1, \ldots).$$

If $\lambda \in \mathbf{N}$, then an inductive argument (as on page 706 of [4]) shows that $\alpha_\nu=0$ ($\nu=0, 1, \ldots$). If $\lambda \in (N, N+1)$ for $N \in \mathbf{N}$, then

$$\Delta^{N+1-\lambda}\big(\Delta^{\lambda}(\alpha_\nu \beta^{-\nu})\big) = \Delta^{N+1}(\alpha_\nu \beta^{-\nu}) = 0$$

by the absolute convergence of the double series involved, and so the result follows from the integer case. Thus (Z, p) is perfect.

The following lemma is a special case of Theorem 5 of [4].

LEMMA 2. *If (Z, m), (Z, l) are perfect and $p(z)=m(z)l(z)$, then (Z, p) is perfect.*

LEMMA 3. *If $\sum_{n=0}^{\infty} |r_n| < \infty$ and $r(z) \neq 0$ for $|z| \leqq 1$, then (Z, r) is perfect.*

PROOF. By the Wiener—Levy theorem (page 246 of [12]), $1/r(z) = \sum_{n=0}^{\infty} t_n z^n$ where $\sum_{n=0}^{\infty} |t_n| < \infty$. Suppose $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $\sum_{n=s}^{\infty} \alpha_n r_{n-s} = 0$ ($s=0, 1, \ldots$). Then, for $\nu \geqq 0$,

$$0 = \sum_{s=\nu}^{\infty} t_{s-\nu} \sum_{n=s}^{\infty} \alpha_n r_{n-s} = \sum_{n=\nu}^{\infty} \alpha_n \sum_{s=\nu}^{n} r_{n-s} t_{s-\nu} = \alpha_\nu,$$

the interchange of order of summation being legitimate because the double series involved is absolutely convergent. Hence (Z, r) is perfect.

As an immediate consequence of Lemmas 1 and 2 we see that, if (Z, r) is perfect and $p(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{\beta_i}\right)^{\lambda_i} r(z)$ where $\beta_i \neq 1$, $|\beta_i| = 1$, $\lambda_i > 0$ ($i=0, 1, \ldots, n$), then (Z, p) is perfect. Thus Theorem 3 holds for such a (Z, p) method.

LEMMA 4. *If $p(z) = \left(1 - \frac{z}{\beta}\right)^{\lambda} \left(-\frac{\beta}{z} \log\left(1 - \frac{z}{\beta}\right)\right)^{\mu}$ where $\beta \neq 1$, $|\beta| = 1$, $0 < \lambda < 1$ and $\mu \in \mathbf{R}$, then (Z, p) is perfect.*

PROOF. If $\mu=0$, this is a case of Lemma 1. Suppose $\mu \neq 0$. Then we have

$$p_n \sim M n^{-\lambda-1} (\log n)^{\mu} \beta^{-n}$$

by page 93 of [6]. (Although Littlewood gives this formula only for $\lambda < 0$ we can establish the result in our case by using backward induction and the differential equation on page 93 of [6].) Hence $\sum_{n=0}^{\infty} |p_n| < \infty$, $p_0 = 1$ and $p(1) \neq 0$. Moreover, $c(z)=$

$=1/p(z)=\left(1-\frac{z}{\beta}\right)^{-\lambda}\left(-\frac{\beta}{z}\log\left(1-\frac{z}{\beta}\right)\right)^{-\mu}$, so that again by Littlewood's result

$$c_n \sim Mn^{\lambda-1}(\log n)^{-\mu}\beta^{-n}.$$

Hence $c_n=O(1)$, and so (Z, p) is perfect by Theorem 8 of [4].

By using Lemma 2, we see that if $p(z)$ is any finite product of functions of the form of those in Lemmas 1 and 4, then (Z, p) is perfect and Theorem 3 holds for such a (Z, p) method. In view of the results above, it would be of interest to know whether every (Z, p) method with $p(z)$ having no zeros inside the unit circle is perfect. A likely candidate for a counterexample can be obtained by considering generalized Laguerre polynomials. Let

$$p(z)=\left(1-\frac{z}{\lambda}\right)^{-\alpha-1}\exp\left(\frac{-z}{\lambda-z}\right) \quad \text{for} \quad \lambda\neq 1, \quad |\lambda|=1, \quad \alpha\in\mathbf{R},$$

so that

$$p_n\lambda^n = L_n^{\alpha}(1)\sim Mn^{(\alpha/2)-(1/4)}\cos\left(2\sqrt{n}+\theta\right)$$

by (8.22.1) of [10]; where θ is a constant depending only on α. Thus, if $\alpha<-3/2$, then $\sum_{n=0}^{\infty}|p_n|<\infty$, $p_0=1$ and $p(1)\neq 0$. However, in this case (8.22.3) of [10] gives

$$c_n\lambda^n = L_n^{-\alpha-2}(-1)\sim Mn^{-(\alpha/2)-(5/4)}\exp\left(2\sqrt{n}\right),$$

and this leads us to suspect that (Z, p) need not be perfect but we are unable to prove it.

THEOREM 4. *Suppose that (Z, r) is perfect and that*

$$p(z)=\prod_{j=1}^{m}\left(1-\frac{z}{\alpha_j}\right)^{\nu_j}\prod_{i=1}^{n}\left(1-\frac{z}{\beta_i}\right)^{\lambda_i}\left(-\frac{\beta_i}{z}\log\left(1-\frac{z}{\beta_i}\right)\right)^{\mu_i} r(z) \quad \textit{where} \quad |\alpha_j|<1, \quad \nu_j\in\mathbf{N}$$

$(j=1, 2, \dots, m)$, $\beta_i\neq 1$, $|\beta_i|=1$, $\lambda_i>0$, $\mu_i\in\mathbf{R}$ $(i=1, 2, \dots, n)$. *Suppose that* (N, q) *is regular and that* (C) *holds. Then* (Z, p) *implies* (N, q).

Note that, by Lemma 3, sufficient conditions for (Z, r) to be perfect are that $\sum_{n=0}^{\infty}|r_n|<\infty$ and that $r(z)\neq 0$ for $|z|\leqq 1$.

PROOF OF THEOREM 4. Let $s(z)=\prod_{j=1}^{m}\left(1-\frac{z}{\alpha_j}\right)^{\nu_j}$ and $t(z)=p(z)/s(z)$. Then $k(z)s(z)t(z)=q(z)$. Define $l(z)=k(z)s(z)$ so that $l(z)t(z)=q(z)$. By Lemmas 1, 2 and 4, (Z, t) is perfect and

$$\sum_{\nu=0}^{n}|l_\nu| = \sum_{\nu=0}^{n}\left|\sum_{\mu=0}^{\nu}k_{\nu-\mu}s_\mu\right| \leqq \sum_{\mu=0}^{n}|s_\mu|\sum_{\nu=\mu}^{n}|k_{\nu-\mu}| = O(|Q_n|)$$

by (C). Thus, by Theorem 3, (Z, t) implies (N, q). Similarly, using the corollary to Theorem 2 in place of Theorem 3, we get that (Z, s) implies (N, q). Since $p(z)=$ $=s(z)t(z)$, by Corollary 3 of [7], we see that $w_n\to 0$ (Z, p) if and only if $w_n=a_n+b_n$ where $a_n\to 0$ (Z, s) and $b_n\to 0$ (Z, t). Hence, by the above, it is easy to see that (Z, p) implies (N, q).

5. The case $(N, q)=(C, \alpha)$

Although we cannot settle the general case with an arbitrary regular (N, q) method, consideration of the special case when (N, q) is the Cesàro method (C, α) leads to some interesting questions on the summability of the power series $\sum_{n=0}^{\infty} c_n z^n$ on its circle of convergence. The Cesàro method (C, α) for $\alpha > -1$ is the Nörlund method (N, q) with $q_n = A_n^{\alpha-1}$ where this is defined by (2). For (N, q) to be regular and $Q_n \to \infty$ we have to consider $\alpha > 0$. In this case $k(z)=(1-z)^{-\alpha}/p(z)=(1-z)^{-\alpha}c(z)$ so that $k_n = C_n^{\alpha-1}$ where we use the notation for Cesàro sums (see for example, page 96 of [2] with c_n replacing a_n). For the question under consideration, Hardy's Theorem 19 becomes: if (N, p) is regular, $P_n = O(1)$ and $\alpha > 0$, then the conditions

$$\sum_{\nu=0}^{n} |C_\nu^{\alpha-1}| = O(n^\alpha), \tag{3}$$

$$C_\nu^{\alpha-1} = o(n^\alpha), \tag{4}$$

are necessary and sufficient for (N, p) to imply (C, α) (where $p(z)c(z)=1$). The problem is to show that (4) follows from (3) and the other hypotheses.

THEOREM 5. *If (N, p) is regular, $P_n = O(1)$, $\alpha > 0$, then* (3) *is sufficient for (N, p) to imply $(C, \alpha+\delta)$ for every $\delta > 0$.*

PROOF. By the corollary to Theorem 1, $C_n^{\alpha-1}/A_n^{\alpha} \to 0$ (C, δ), i.e., $c_n \to 0$ $(C, \delta) \times \times (C, \alpha)$, the iterated Cesàro method, and by page 23 of [5] or Ch. 11 of [2] this is equivalent to $c_n \to 0$ $(C, \alpha+\delta)$, i.e., (4) with α replaced by $(\alpha+\delta)$. Also, (3) implies that (3) holds with α replaced by $(\alpha+\delta)$, since (3) is exactly the condition for the series $\sum_{n=0}^{\infty} c_n$ to be strongly bounded $[C, \alpha]_1$ (see page 488 of [11]). Hence, by Hardy's result, (N, p) implies $(C, \alpha+\delta)$.

We are unable to decide whether we can take $\delta = 0$ in Theorem 5. It is clear that (3) alone does not imply (4) (consider $C_n^{\alpha-1} = n^\alpha$ if $n = 2^s$ $(s=0, 1, \ldots)$ and 0 otherwise) but we have been unable to construct an example with the c_n's satisfying the further hypotheses that $c(z)p(z)=1$, (N, p) regular and $P_n = O(1)$. We can, however, make the following simplification.

THEOREM 6. *If (N, p) is regular, $P_n = O(1)$, $\alpha > 0$, then* (3) *and*

$$c_n = o(n^\alpha) \tag{5}$$

are necessary and sufficient for (N, p) to imply (C, α).

PROOF. By the remarks before Theorem 5 it is enough to show that, under the other hypotheses of the theorem, (4) is equivalent so (5). Now (4) says that $c_n \to 0$ (C, α), and so by the limitation theorem for (C, α) (Theorem 46 of [2]) (5) must hold. Conversely, by the convergence of $\sum_{n=0}^{\infty} |p_n|$ and the regularity of (N, p), we see that $p(z)$ is continuous at $z=1$ and $p(z) \to p(1) \neq 0$ as $z \to 1$ in any manner from within the unit circle. Also, (3) implies that $\sum_{n=0}^{\infty} c_n z^n$ is convergent for $|z|<1$ and, by the continuity of $\sum_{n=0}^{\infty} c_n z^n = 1/p(z)$ at $z=1$, we have that $\sum_{n=0}^{\infty} c_n z^n \to 1/p(1)$ as $z \to 1$ in any manner from within the unit circle. Hence, by a result of Dienes (cf. Théorème XXVI of [5] or Theorem 9.23 of [12]), (5) implies that $\sum_{n=0}^{\infty} c_n$ is summable (C, α). By the remarks at the bottom of page 102 of [2], $c_n \to 0$ (C, α), i.e., (4) holds, and this proves the result.

If we only require an implication from (N, p) to Cesàro summability of some positive order then we have a more complete result, cf. [3].

THEOREM 7. *Suppose that (N, p) is regular and $P_n = O(1)$. In order that (N, p) should imply Cesàro summability of some positive order it is necessary and sufficient that $c_n = O(n^\gamma)$ for some $\gamma > 0$.*

PROOF. To show that the condition is necessary, suppose (N, p) implies (C, α) for $\alpha > 0$. Then $\sum_{n=0}^{\infty} c_n = 1/p(1)$ (N, p) and so $\sum_{n=0}^{\infty} c_n = 1/p(1)$ (C, α). Hence, by the limitation theorem for (C, α), $c_n = o(n^\alpha)$ and so the condition holds.

For the sufficiency part, $c_n = O(n^\gamma)$ implies that $\sum_{n=0}^{\infty} c_n z^n$ is convergent for $|z|<1$ and that $c_n = o(n^\delta)$ for $\delta > \gamma$. Hence, by Dienes' theorem, as in the proof of Theorem 6, $\sum_{n=0}^{\infty} c_n = 1/p(1)$ (C, δ). Thus, $c_n = o(n^\delta)$ and, by II of [11], $\sum_{n=0}^{\infty} c_n = = 1/p(1)$ $[C, \delta+1]_1$, and so (3) and (5) hold with α replaced by $\delta+1$. Therefore, by Theorem 6, (N, p) implies $(C, \delta+1)$.

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A NOTE ON AN ARTICLE BY ARTIKIS

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1. Introduction

Let $p>0$ and assume that $\Phi(t)$ is the characteristic function (ch. f.) of a random variable (r.v.) $Z=XU^{1/p}$ where X is some r.v. and U is a uniform r.v. on (0, 1) independent of X. Obviously, Φ is of the form

$$\Phi(t) = p\int_0^1 \psi(ut)u^{p-1}\,du, \quad -\infty<t<\infty,$$

where ψ is some ch. f. When $p=1$, Z satisfies Khintchine's characterization of unimodality and hence it is unimodal. Generally, r.v.'s of the form $Z=XU^{1/p}$ are referred to as p-unimodal or p-star unimodal (see, e.g., Olshen and Savage, 1970).

It may be noted that $\Phi(t)$ also takes the form

$$\Phi(t) = \frac{p}{t|t|^{p-1}}\int_0^t \psi(u)|u|^{p-1}\,du, \quad -\infty<t<\infty. \tag{1}$$

Observe here that the right hand side of (1) has been expressed by Lukacs [3], p. 321 as $(p/t^p)\int_0^t \psi(u)u^{p-1}$ du; $-\infty<t<\infty$. Clearly, the expression in this form requires some clarifications for $t<0$ because in this case t^p and u^{p-1} are not uniquely defined in general. Further, his assumption that $p\geqq 1$ is not necessary and $\Phi(t)$ is a ch.f. for every positive real p.

In this note we shall characterize $\Phi(t)$ for which $\psi(t)$ is a certain power of $\Phi(t)$ itself. Further, we shall prove that $\Phi(t)$ and hence $\psi(t)$ in this case are self-decomposable and therefore unimodal. The first part of the problem was solved for $p=1$ by Shanbhag [5] and, using essentially Shanbhag's techniques, for $p\geqq 1$ by Artikis [2]. We also use Shanbhag's technique but for minor simplifications.

2. The characterization

Let $\Phi(t)$ be a ch.f. given by (1) and let $d=\sup\{b\in R^+:\ \Phi(t)\neq 0$ for all $t\in(-b, b)\}$ and $d=\infty$ if $\Phi(t)\neq 0$ for all $t\in R$.

THEOREM 1. *Let $\Phi(t)$ in* (1) *be non-degenerate. Then* $\psi(t)=\left(\Phi(t)\right)^{(p+r-1)/p}$ *for all* $t\in(-d, d)$ *if and only if*

$$\Phi(t) = \left\{1+i\delta(\gamma)\mu t+c|t|^\gamma\left[1+i\theta\frac{t\omega(\gamma)}{|t|}\right]\right\}^{-p/\gamma}, \quad -\infty<t<\infty, \tag{2}$$

where $\gamma=r-1,\ 1<r\leqq 3,\ c\geqq 0,\ |\theta|\leqq 1,\ \mu$ *real,*

$$\delta(\gamma)=\begin{cases}1 & \text{if } \gamma=1\\ 0 & \text{if } \gamma\neq 1\end{cases}\quad \text{and}\quad \omega(\gamma)=\begin{cases}0 & \text{if } \gamma=1\\ \tan(\pi\gamma/2) & \text{if } \gamma\neq 1.\end{cases}$$

PROOF. Let $\psi(t)=(\Phi(t))^{(p+r-1)/p}$ for all $t\in(-d,d)$. Then,

$$\Phi(t)=\frac{p}{t|t|^{p-1}}\int_0^t (\Phi(u))^{(p+r-1)/p}|u|^{p-1}\,du$$

and hence for $t\in(0,d)$ we have

$$t^p\Phi'(t)+pt^{p-1}\Phi(t)=p\big(\Phi(t)\big)^{(p+r-1)/p}t^{p-1}. \tag{3}$$

The case $r=1$ implies that $t^p\Phi'(t)=0$ and since $t\neq 0$ we have $\Phi(t)=1$. But this is not possible because $\Phi(t)$ is non-degenerate and hence $r\neq 1$. Rewriting (3) as

$$\{t^p\Phi(t)\}^{-(p+r-1)/p}\frac{d}{dt}\{t^p\Phi(t)\}=pt^{-r}$$

and integrating on both sides, it follows that

$$\Phi(t)=(1+k_1t^{r-1})^{-p/(r-1)}$$

where k_1 is a constant independent of t with $\operatorname{Re}(k_1)\geqq 0$. Similarly, we can see that

$$\Phi(t)=(1+k_2|t|^{r-1})^{-p/(r-1)}\quad \text{for}\quad t\in(-d,0)$$

where K_2 is a constant with $\operatorname{Re}(K_2)\geqq 0$.

Obviously, the form of $\Phi(t)$ implies that $\Phi(t)\neq 0$ for all real t and thus $d=\infty$. Since $\lim_{t\to 0}\Phi(t)=1$ we have that $r>1$. Further, because a non-degenerate ch.f. can not be of the form $1+o(t^2)$ as $t\to 0$, from the form of $\Phi(t)$ we have that $r\leqq 3$. Hence, $1<r\leqq 3$ and we can write

$$\Phi(t)=\begin{cases}(1+k_1t^{r-1})^{-p/(r-1)} & t\geqq 0\\ (1+k_2t^{r-1})^{-p/(r-1)} & t<0.\end{cases} \tag{4}$$

Now, define $\Phi_m(t)=[\Phi(t/m^{1/(r-1)})]^m$. Evidently, $\lim_{m\to\infty}\Phi_m(t)=\Phi^*(t)$ is given by

$$\Phi^*(t)=\begin{cases}\exp\left\{-\dfrac{pk_1}{r-1}t^{r-1}\right\} & t\geqq 0\\[2mm] \exp\left\{-\dfrac{pk_2}{r-1}t^{r-1}\right\} & t<0.\end{cases}$$

From the well-known continuity theorem (cf. [3], p. 48), it follows that $\Phi^*(t)$ is a ch. f. Clearly, this is the ch. f. of a stable distribution with characteristic exponent $r-1$. Using a characterization of stable distributions (cf. [3], p. 136), we can see that $\Phi(t)$ has the form given in (2).

The converse is immediate by the fact that if

$$h(t) = \left\{i\delta(\gamma)\mu t + c\,|t|^{\gamma}\left(1 + i\theta\,\frac{t\omega(\gamma)}{|t|}\right)\right\}$$

we have $h'(t) = \frac{\gamma}{t}h(t)$, and then

$$\psi(t) = p^{-1}\{t\Phi'(t) + p\Phi(t)\} = p^{-1}\left\{-\frac{tp}{\gamma}\,h'(t)\big(1+h(t)\big)^{-p/\gamma-1} + p\big(1+h(t)\big)^{-p/\gamma}\right\} =$$

$$= \big(1+h(t)\big)^{-(p+\gamma)/\gamma} = \big(\Phi(t)\big)^{(p+r-1)/p}.$$

As a simple modification of this theorem $\left(r = \frac{2p-1}{p-1}\right)$ we have

COROLLARY 1. *Let $f(t)$ be some ch. f. and Φ in* (1) *be nondegenerate. Then, $\Phi(t) = (f(t))^{p-1}$ and $\psi(t) = (f(t))^{p}$ for all $t \in (-d, d)$ if and only if $p \geqq 2$ and*

$$f(t) = \left\{1 + c\,|t|^{\gamma}\left[1 + i\theta\,\frac{t}{|t|}\tan(\pi\gamma/2)\right]\right\}^{-1},$$

where $\gamma = p/(p-1)$, $c \geqq 0$, $|\theta| \leqq 1$.

The technique we have used here, except for minor simplifications, is basically that of Shanbhag [5]. In his paper, Shanbhag proves the theorem for the special case when $p=1$. Artikis [2] following Shanbhag's exact line of proof generalizes the result for $p \geqq 1$ assuming implicitly that $d = \infty$. His paper contains two theorems. The second of these is as stated above a direct corollary for $p \geqq 1$ and $d = \infty$ of our Theorem 1. (Incidentally we also refine his argument.) Furthermore, his first theorem is our Corollary 1 with the implicit assumption that $d = \infty$.

Obviously, $\Phi(t)$ (and hence $\psi(t)$) given by (2) is the ch.f. of an infinitely divisible r.v. Then, as a direct consequence of the following theorem, it follows that in fact $\Phi(t)$ is self-decomposable (i.e., $\Phi(t)/\Phi(\lambda t)$ is a ch.f. for all $\lambda \in (0, 1)$) for all $p > 0$ and hence by Yamazato ([6]) unimodal for all $p > 0$.

THEOREM 2. *Let $\psi_1(t)$ be the ch.f. of stable distributions of the form given in Theorem* 1, *i.e.,*

$$\psi_1(t) = \exp\left\{-i\delta(\gamma)\mu t - c\,|t|^{\gamma}\left(1 + i\theta\,\frac{t\omega(\gamma)}{|t|}\right)\right\}.$$

Then

$$\Phi_1(t) = \int_0^{\infty} (\psi_1(t))^x\,dF(x) \tag{5}$$

with F as some self-decomposable distribution function on $[0, \infty)$ *is the c.h.f. of a self-decomposable distribution.*

PROOF. Φ_1 is a ch.f. because ψ_1 is infinitely divisible. Let $\xi(s)$, $\mathrm{Re}(s) \geqq 0$, denote the Laplace—Stieltjes transform (L. S. T.) of F and put $s = i\,\delta(t)\mu t + c|t|^{\gamma}\left(1 + i\theta\,\frac{t\omega(\gamma)}{|t|}\right)$. Then we have $\Phi_1(t) = \xi(s)$ and $\Phi_1(\lambda t) = \xi(\lambda^{\gamma}s)$ for all $\lambda \in (0, 1)$.

Hence,

$$\Phi_1(t)/\Phi_1(\lambda t) = \xi(s)/\xi(\lambda^\gamma s) \tag{6}$$

for every $\lambda \in (0, 1)$.

Self-decomposability of F implies that the right hand side of (6) is the L. S. T. of some distribution F_λ. Thus the left hand side of (6) is of the form of (5) with F replaced by F_λ and hence is a ch.f. Therefore, $\Phi_1(t)$ is self-decomposable.

If we take F in Theorem 2 to be a gamma distribution with L. S. T. $\xi(s) = (1+s)^{-p/\gamma}$, then the ch.f. $\Phi_1(t)$ in Theorem 2 coincides with $\Phi(t)$ given by (2). Therefore, $\Phi(t)$ (and hence $\psi(t)$) is self-decomposable and so unimodal. It may be worth mentioning that in this case when $\mu=\theta=0$ and $p/\gamma=a$,

$$\Phi_1(t) = (1+c|t|^\gamma)^{-a}, \quad a>0, \quad \gamma \in (0, 2].$$

Hence, the ch.f. $(1+|t|^\gamma)^{-1}$, $\gamma \in (0,2]$ proved by Linnik and Laha (cf. [3], p. 96—7) to be unimodal is a special case of $\Phi_1(t)$ which more specifically has been shown to be self-decomposable above.

Generally, mixtures of self-decomposable distributions are not necessarily self-decomposable even if the mixing distribution is also self-decomposable (see [1]). It follows, however, from Theorem 2 that

COROLLARY 2. *Power mixturs of all strictly stable ch.f.'s are self-decomposable when the mixing distribution is self-decomposable with support on* $[0, \infty)$.

(E.g., variance mixtures of normal $(0, \sigma^2)$ distributions are self-decomposable when the mixing r.v. σ^2 is a self-decomposable r.v.)

From what we have seen we have that besides 1-unimodal distributions the class of p-unimodal distributions (in Olshen—Savage sense) given in Theorem 1 is also unimodal (in the usual sense).

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ON THE CHARACTERIZATION OF COMPLEX-VALUED MULTIPLICATIVE FUNCTIONS. II

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1. If f is a monotonic, real-valued multiplicative function then $f(n)=n^k$, if $f(m)\neq 0$ for any $m\in\mathbf{N}$ (see [1], [4]). Similarly to the theorems proved for additive functions in [2] it is easy to show, that to any function $g(n)$ there exists a "rare" set $A=\{a_n\}_1^\infty$, such that $a_n>g(n)$ and if f is monotonic on A, then $f(n)=n^k$, if $f(a)\neq 0$ for any $a\in A$.

Let us consider the case when f is a complex-valued multiplicative function. If $|f|$ is monotonic on the same suitable "rare" set A, we get $|f(n)|=n^k$, since $|f|$ is a real-valued multiplicative function. Making stronger assumptions we can prove the following.

THEOREM. *To any real-valued function $g(n)$ there exists a set $A=\{a_n\}_1^\infty$, such that $a_n>g(n)$, and if at least two of the functions* $\operatorname{Re} f$, $\operatorname{Im} f$ *and* $|f|$ *are monotonic on A, then $f(n)=n^k$, except the cases* a) $\operatorname{Im} f=0$ *and* b) $f(a)=0$ *for all $a\in A$ $a\geqq a_0$.*

2. To prove the theorem first we introduce the notations $f_1:=\operatorname{Re} f$ and $f_2:=\operatorname{Im} f$; further $f_i\geqq 0$ denotes that $f_i(n)\geqq 0$ for all $n\in\mathbf{N}$.

For every triplet (m,s,j) of natural numbers choose a prime $w_{msj}>\max(m,s)$ (different primes for different indices). Let A^* be the set of all numbers ot the form mw_{msj} or sw_{msj}. Let A_s be the set of numbers that can be written as a product of different primes of the form w_{2sj}. Finally we put $A=\bigcup_{s=1}^{\infty} A_s\cup A^*$. It is easy to see that if the set $\{w_{msj}\}$ is sufficiently rare, this set A will satisfy our requirement $a_n>g(n)$.

PROPOSITION 1. *Let B be a set and suppose $\{b_j\}_{j=1}^\infty\subset B$, where for any $i\neq j$ $(b_i,b_j)=1$ and $\prod_{j\in M} b_j\in B$ for any finite non-empty set $M\subset\mathbf{N}$. Then the following assertions hold:*

(i) *If $f_1(a)\geqq 0$ $(f_1(a)\leqq 0)$ for any $a\in B_j$ $a\geqq a_0$, then* $\lim_{i\to\infty}\operatorname{arc} f(b_i)=0$.

(ii) *If $f_2(a)\geqq 0$ $(f_2(a)\leqq 0)$ for any $a\in B$, $a\geqq a_0$, then* $(\{\operatorname{arc} f(b_i)\}_{i=1}^\infty)'\subset\{0,\pi\}$, *where $(M)'$ denotes the set of cluster points of M.*

For the proof of Proposition 1 see [3, Proposition 2].

LEMMA 1. *Let $i=1$ or 2. If $f_i(a)\geqq 0$ $(f_i(a)\leqq 0)$ for all $a\in A$, $a\geqq a_0$, then $f_i\geqq 0$ $(f_i\leqq 0)$.*

PROOF. Let us assume that there exists an s_0 such that $f_i(s_0)<0$ $(f_i(s_0)>0)$.

a) In case $i=1$ using Proposition 1 $(B:=A_{s_0})$

$$\lim_{j\to\infty} \operatorname{arc} f(w_{1s_0j}) = 0, \quad \text{so} \quad \lim_{j\to\infty} \operatorname{arc} f(s_0 w_{1s_0j}) = \operatorname{arc} f(s_0),$$

which is a contradiction, namely $s_0 w_{1s_0j}\in A$, too.

b) In case $i=2$, Proposition 1 gives for the sequence $(b_j)=(w_{1s_0j})$ the property $(\{\operatorname{arc} f(b_j)\}_{j=1}^{\infty})'\subset\{0, \pi\}$. If $0\in(\{\operatorname{arc} f(b_j)\}_{j=1}^{\infty})'$, then we have a subsequence (b_{j_k}) with $\lim_{k\to\infty} \operatorname{arc} f(b_{j_k})=0$ and then we work with the sequence (b_{j_k}) instead of the original (b_j). If $\lim_{j\to\infty} \operatorname{arc} f(b_j)=\pi$, we work with the sequence $(b_{2j-1}b_{2j})$ instead of the original (b_j). Thus we get $\lim_{j\to\infty} \operatorname{arc} f(s_0 b_j)=\operatorname{arc} f(s_0)$, which is a contradiction.

Lemma 1 implies

COROLLARY 1. *Let* $i=1$ *or* 2. *If* f_i *is monotonic on* A, *then* $f_i\geqq 0$ *or* $f_i\leqq 0$.

Further we can show.

COROLLARY 2. *Let* $i=1$ *or* 2. *If* $f_i\geqq 0$ $(f_i\leqq 0)$ (*for example* f_1 *or* f_2 *is monotonic on* A), *then in case* $i=1$, $\lim_{j\to\infty} \operatorname{arc} f(c_j)=0$ *and in case of* $i=2$, $\lim_{j\to\infty} \operatorname{arc} f(c_j)=0$ *or* $f_2=0$, *whenever* (c_j) *is a sequence of pairwise coprime numbers.*

PROOF. This follows for $i=1$ from Lemma 1 and Proposition 1 $(B:=\mathbf{N})$. For $i=2$, Lemma 1 and Proposition 1 imply $(\{\operatorname{arc} f(c_j)\}_{j=1}^{\infty})'\subset\{0, \pi\}$ and $f_2\geqq 0$ or $f_2\leqq 0$. Now let us assume the existence of an x_0 with $f_2(x_0)\neq 0$ and a subsequence $\{c_{j_\nu}\}_{\nu=1}^{\infty}$ with $\lim_{\nu\to\infty} \operatorname{arc} f(cj_\nu)=\pi$. Then as $(c_l, c_j)=1$ for $l\neq j$, we have $(x_0, c_j)=1$ for $j\geqq j_0$. Thus

$$\lim_{\nu\to\infty} \operatorname{arc} f(x_0 c_{j_\nu}) = \pi+\operatorname{arc} f(x_0),$$

which implies

$$\operatorname{sgn} f_2(x_0 c_{j_\nu}) = -\operatorname{sgn} f_2(x_0)$$

for $\nu>\nu_0$, a contradiction. □

LEMMA 2. *If* f_1 *is monotonic on* A, *then* $f_1\geqq 0$.

PROOF. If there exists an $a\in A$, such that $f_1(a)>0$, then using Corollary 1, we have $f_1\geqq 0$. Thus we shall prove the existence of an $a\in A$ with $f(a)>0$.

There exist $a_j\in A$ $(j=1, 2, \ldots, 9)$ pairwise relatively prime numbers, such that $\prod_{j\in M} a_j\in A$ for any $M\subset\{1, 2, \ldots, 9\}$ (see A_s). If

$$\frac{\pi}{2} \leqq \operatorname{arc} f(a_j) \leqq \frac{3\pi}{2} \quad \operatorname{mod} 2\pi \quad (j = 1, 2, \ldots, 9),$$

then we can easily verify that

$$\frac{\pi}{2} \leqq \operatorname{arc} f(a_j a_k) \leqq \frac{3\pi}{2} \mod 2\pi \quad (1 \leqq j < k \leqq 9),$$

$$\frac{\pi}{2} \leqq \operatorname{arc} f(a_j a_k a_s) \leqq \frac{3\pi}{2} \mod 2\pi \quad (1 \leqq j < k < s \leqq 9),$$

$$\frac{\pi}{2} \leqq \operatorname{arc} f(a_j a_k a_s a_t) \leqq \frac{3\pi}{2} \mod 2\pi \quad (1 \leqq j < k < s < t \leqq 9)$$

is not possible at the same time.
(We have either
a) $\geqq 4$ elements with $\operatorname{arc} f(a) = \pi/2$
b) $\geqq 4$ elements with $\operatorname{arc} f(a) = 3\pi/2$
or
c) $\geqq 3$ elements with $\pi/2 < \operatorname{arc} f(a) < 3\pi/2$.
Cases a) and b) can be settled trivially. In case c) we can distinguish further
c/(i) there are a', a'' with $\pi/2 < \operatorname{arc} f(a') \leqq \pi \leqq \operatorname{arc} f(a'') < 3\pi/2$,
c/(ii) there are a', a'', a''' with

$$\frac{\pi}{2} < \operatorname{arc} f(a') \leqq \operatorname{arc} f(a'') \leqq \operatorname{arc} f(a''') \leqq \pi$$

c/(iii) there are a', a'', a''' with

$$\pi \leqq \operatorname{arc} f(a') \leqq \operatorname{arc} f(a'') \leqq \operatorname{arc} f(a''') < \frac{3\pi}{2},$$

and simple considerations yield the assertion.) □

For f_2, Corollary 1 gives

LEMMA 3. *If f_2 is monotonic on A, then $f_2 \geqq 0$ or $f_2 \leqq 0$.*

LEMMA 4. *If $|f|$ is monotonic on A, then with the exception of the functions described in* b) *of the theorem, we have $|f(n)| = n^k$.*

PROOF. a) If $|f|$ is monotonically increasing on A, let $s = m+1$. So

$$|f(m)|\,|(fw_{m,m+1,j})| = |f(mw_{m,m+1,j})| \leqq \left|f\big((m+1)w_{m,m+1,j}\big)\right| = \\ = |f(m+1)|\,|f(w_{m,m+1,j})|.$$

Using that $|f(w_{m,m+1,j})| \neq 0$ for $j \geqq j_0$, we get $|f(m)| \leqq |f(m+1)|$ for all $m \in \mathbf{N}$, i.e. $|f(m)| = m^k$ by Erdős' theorem [1].
b) If $|f|$ is monotonically decreasing, the proof is similar. □

3. PROOF OF THE THEOREM. We shall distinguish the following cases; I. Both $|f|$ and f_1 are monotonic on A. I/a) Both $|f|$ and f_1 are monotonically increasing on A.
Using Lemma 4 we have $|f(n)| = n^k$, further Corollary 2 implies $\lim_{j\to\infty} \operatorname{arc} f(w_{msj}) = 0$ for any $(m, s) \in \mathbf{N}^2$. Let $s = m+1$. So

$$f_1(mw_{m,m+1,j}) \leqq f_1\big((m+1)w_{m,m+1,j}\big),$$

i.e.

$$m^k w_{m,m+1,j}^k \cos [\operatorname{arc} f(m)+\operatorname{arc} f(w_{m,m+1,j})] \leqq$$
$$\leqq (m+1)^k w_{m,m+1,j}^k \cos [\operatorname{arc} f(m+1)+\operatorname{arc} f(w_{m,m+1,j})].$$

Dividing by $w_{m,m+1,j}^k$ for $j \mapsto \infty$ we obtain

$$f_1(m) = m^k \cos \operatorname{arc} f(m) \leqq (m+1)^k \cos \operatorname{arc} f(m+1) = f_1(m+1),$$

consequently f_1 is monotonically increasing. So both f_1 and $|f|$ are monotonic. Now we can use the theorem proved in [3], according to which if f denotes a complex-valued multiplicative function and at least two of the functions $\operatorname{Re} f$, $\operatorname{Im} f$ and $|f|$ are monotonic on a set A having upper density one, then $f(n)=n^k$, except the cases a) $f(a)=0$ for all $a \in A$, $a \geqq a_0$ and b) $\operatorname{Im} f=0$.

The other cases I/b), c), d) and II/a), b), c), d), when $|f|$ and f_1 are monotonic or $|f|$ and f_2 are monotonic can be treated similarly.

III/a) Both of f_1 and f_2 are monotonically increasing on A.

Using Lemmas 2 and 3, $f_1 \geqq 0$, further $f_2 \geqq 0$ or $f_2 \leqq 0$.

(i) If $f_1 \geqq 0$ and $f_2 \geqq 0$, then $|f|$ is monotonically increasing on A too. So we can use I/a).

(ii) If $f_1 \geqq 0$ and $f_2 \leqq 0$, then $-\pi/2 \leqq \operatorname{arc} f(n) \leqq 0 \pmod{2\pi}$, so $\operatorname{arc} f|_A$ is monotonically increasing. Thus for every $x_0 \in \mathbf{N}$ there exists $w_{1x_0j} > x_0$ and so $\operatorname{arc} f(x_0 w_{1x_0j}) = \operatorname{arc} f(x_0) + \operatorname{arc} f(w_{1x_0j})$, consequently $\operatorname{arc} f(x_0)=0$. So f is real-valued, contradicting the theorem.

III/b) If both f_1 and f_2 are monotonically decreasing, then similarly to III/a) we have two possibilities.

(i) If $f_1 \geqq 0$ and $f_2 \geqq 0$, then $|f|$ is monotonically decreasing too, so we have the case I/b).

(ii) If $f_1 \geqq 0$ and $f_2 \leqq 0$, then $-\pi/2 \leqq \operatorname{arc} f(n) \leqq 0 \pmod{2\pi}$ and $\operatorname{arc} f|_A$ is monotonically decreasing. By Corollary 2 $\lim_{j \to \infty} \operatorname{arc} f(c_j)=0$ or $f_2 \equiv 0$ whenever (c_j) is a sequence of pairwise coprime numbers. This implies $f_2 \equiv 0$, which was excluded by the formulation of the theorem. The other two cases III/c), d) lead to case I and II respectively, by considering $1/f$. □

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ON THE CONVERGENCE OF HERMITE—FEJÉR INTERPOLATION

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1. Introduction

We give a convergence estimation for the Hermite—Fejér operator sequence based on the roots of the orthogonal polynomials generated by the weight $(1-x^2)^\beta |x|^{2\alpha+1} (-1<\alpha, \beta)$.

2. Notations and preliminary results

2.1. For an arbitrary $f \in C[-1, 1]$ we construct the uniquely defined Hermite—Fejér interpolatory polynomials $H_n(f, X, x)$ of degree $2n-1$ satisfying

(2.1)
$$H_n(f, X, x_{k,n}) = f(x_{k,n}), \quad H_n'(f, X, x_{k,n}) = 0 \quad (k = 1, 2, \dots, n, \quad n = 1, 2, \dots)$$

where X stands for the matrix

$$-1 < x_{n,n} < x_{n-1,n} < \dots < x_{2,n} < x_{1,n} < 1 \quad (n = 1, 2, \dots). \tag{2.2}$$

As it is well known we have the representation

$$H_n(f, X, x) = \sum_{k=1}^{n} f(x_{k,n}) v_k(x) l_k^2(x) \tag{2.3}$$

where $l_k(x) = l_{k,n}(X, x)$ are the fundamental polynomials of Lagrange interpolation and $v_k(x) = v_{k,n}(X, x)$ are appropriate linear functions $(k = 1, 2, \dots, n)$.

2.2. If for any $x \in [-1, 1]$ $v_{k,n}(X, x) \geqq \varrho > 0$ $(k = 1, 2, \dots, n;\ n = 1, 2, \dots)$ then the matrix X is called *ϱ-normal*. This definition was introduced by L. Fejér [7]. Concerning ϱ-normal matrices G. Grünwald proved the following important theorem [8]:

THEOREM A. *If X is ϱ-normal then $\{H_n(f, X, x)\}$ uniformly tends to $f(x) \in C[-1, 1]$ in $[-1, 1]$.*

2.3. The case $X^{(\alpha,\beta)} = \{x_{k,n}^{(\alpha,\beta)}\}$ $(\alpha, \beta > -1)$ i.e. when the nodes are the corresponding roots of the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ was investigated by several authors (see e.g. [1], [2], [3]). In [1] G. Szegő proved

THEOREM B. *The necessary and sufficient condition that for any* $f\in C[-1,1]$

$$\lim_{n\to\infty}\|H_n(f, X^{(\alpha,\beta)}, x)-f(x)\| = 0$$

is that $-1\leqq\alpha,\ \beta<0$.

In [2] P. Vértesi proved

THEOREM C. *For every continuous function* $f(x)$ *we have*

$$|H_n(f, X^{(\alpha,\beta)}, x)-f(x)| = \tag{2.4}$$

$$= O\left(\sum_{i=1}^{n}\left[\omega\left(f;\ \frac{i\sqrt{1-x^2}}{n}\right)+\omega\left(f;\ \frac{i^2|x|}{n^2}\right)\right]i^{2\gamma-1}\right)\quad (x\in[-1,1]).$$

Here the O sign depends on α and β; $\gamma=\max(\alpha,\beta,-1/2)$.

2.4. Let us consider the weight function

$$p(\alpha,\beta;\ x) = |x|^{2\alpha+1}(1-x^2)^{\beta}\quad (\alpha,\beta>-1,\ |x|\leqq 1)$$

and denote the corresponding orthogonal system by $\{R_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$. One can prove (see Lascenov [4])

$$\begin{cases} R_{2n}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(1-2x^2) \\ R_{2n+1}^{(\alpha,\beta)}(x) = xP_n^{(\alpha+1,\beta)}(1-2x^2)\end{cases}\quad (n=0,1,\ldots) \tag{2.5}$$

from where the roots $\{y_{k,n}^{(\alpha,\beta)}\}=Y$ of $\{R_n^{(\alpha,\beta)}(x)\}$ are

$$y_{k,n}^{(\alpha,\beta)} = -y_{-k,n}^{(\alpha,\beta)} = \begin{cases}\sin\frac{1}{2}\,\theta_{k,m}^{(\alpha,\beta)} & (n=2m;\ k=1,2,\ldots,m),\\ \sin\frac{1}{2}\,\theta_{k,m}^{(\alpha+1,\beta)} & (n=2m+1,\ k=0,1,\ldots,m),\end{cases} \tag{2.6}$$

$$\theta_{k,n}^{(\alpha,\beta)} = -\theta_{-k,n}^{(\alpha,\beta)}\quad (k=1,2,\ldots,m)$$

where $x_{k,n}^{(\alpha,\beta)}=\cos\theta_{k,n}^{(\alpha,\beta)}$ $(k=1,2,\ldots,n)$ and $\theta_{0,n}^{(\alpha,\beta)}=0$.

2.5. If X is ϱ normal then the root $s_{k,n}$ of $v_{k,n}(x)$ is outside of $[-1,1]$. The points $\{s_{k,n}\}$ are the *conjugate points*. By this definition we can formulate two problems raised by P. Turán [9].

1. Construct a matrix X for which the set $\{s_{k,n}\}$ is dense in $[-1,1]$ and $\lim_{n\to\infty}\|H_n(f,X)-f\|=0$ if $f\in C[-1,1]$.

2. Construct a weight-function $p(x)$ vanishing and continuous for a certain $x_0\in(-1,1)$ such that for the corresponding matrix P $\lim_{n\to\infty}\|H_n(f,P)-f\|=0$ if $f\in C[-1,1]$.

These questions were solved by J. Balázs [10]. He proved that the matrix Y satisfies both 1., and 2. A more general result is due to P. Vértesi [5]:

THEOREM D. *The necessary and sufficient condition that for any* $f \in C[-1, 1]$

$$\lim_{n\to\infty} \|H_n(f, Y, x)-f(x)\| = 0$$

is that the relations

$$-1 < \alpha, \beta < 0 \quad \textit{and} \quad \beta-\alpha \leqq 0.5 \tag{2.7}$$

hold.

3. The main result

3.1. The aim of this paper is to give a convergence estimation for the system of nodes Y (see Theorem C).

Let $x=\sin \theta/2$ $(-\pi \leqq \theta \leqq \pi)$ and let $0<\varepsilon$ be an arbitrary small fixed number. With these notations we want to prove the following

THEOREM. *Let* $f \in C[-1, 1]$. *Then*

(i) *if* $0 \leqq \theta \leqq \varepsilon$ *then*

$$|H_n(f, Y, x)-f(x)| = O\left(\omega(f;\ |x-y_j|)+[n(\theta-\theta_j)]^2 \sum_{i=1}^{n} \omega\left(f;\ \frac{i}{n}\right) i^{2\gamma-1}\right),$$

(ii) *if* $\varepsilon \leqq \theta \leqq \pi-\varepsilon$ *then*

$$|H_n(f, Y, x)-f(x)| = O\left(\omega(f;\ |x-y_j|)+[n(\theta-\theta_j)]^2 \sum_{i=1}^{n} \omega\left(f;\ \frac{i}{n}\right) \frac{1}{i^2}\right),$$

(iii) *if* $\pi-\varepsilon \leqq \theta \leqq \pi$ *then*

$$|H_n(f, Y, x)-f(x)| = O\big(\omega(f;\ |x-y_j|)+[n(\theta-\theta_j)]^2 \times$$

$$\times\left\{\sum_{i=1}^{n} \omega\left(f;\ \frac{i^2}{n^2}\right) i^{2\beta-1}+\sum_{i=1}^{n} \omega\left(f;\ \frac{i}{n}\right) \frac{1}{i^2}+\frac{|2\alpha+1|}{n^{2\alpha+1-2\beta}} \sum_{i=1}^{n} \omega\left(f;\ \frac{i}{n}\right) i^{2\alpha}\right\}\Bigg)$$

where $\gamma=\max(\alpha, -1/2)$, y_j *denotes the nearest root to* x *and the* O *sign depends on* α, β *and* $\|f\|$.

REMARK 1. Evidently similar statements are true for $-\pi \leqq \theta \leqq 0$ as well.

2. The estimation reflects the interpolatory property of $H_n(f, x)$.

3. If $f \in \mathrm{Lip}\, \eta$ $(0<\eta<1)$ then e.g. (ii) may be reduced to the following

(ii)′ if $\varepsilon \leqq \theta \leqq \pi-\varepsilon$ then

$$|H_n(f, Y, x)-f(x)| = O(n^{-\eta}).$$

I.e. the estimation is, in a certain sense, the best possible.

4. If $2\alpha+1=0$ i.e. when Y corresponds to the roots of the ultraspherical polynomials we get the estimation of Theorem C when $\alpha=\beta$.

4. Proof

4.1. First we verify the following useful relation

LEMMA 4.1. *Let* ω *be a not identically* 0 *modulus of continuity. Then*

$$\max_{1\leqq i\leqq n} \omega\left(\frac{i}{n}\right)\left(\frac{i}{n}\right)^{\gamma} = O\left(\sum_{i=1}^{n} \omega\left(\frac{i}{n}\right) \frac{i^{\gamma-1}}{n^{\gamma}}\right). \tag{4.1}$$

PROOF. Evidently

$$(4.2)\qquad \sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\frac{i^{\gamma-1}}{n^{\gamma}} \geqq \frac{1}{n}\sum_{\frac{1}{2}\leqq\frac{i}{n}\leqq 1}\omega\left(\frac{i}{n}\right)\left(\frac{i}{n}\right)^{\gamma-1} \sim \int_{\frac{1}{2}}^{1}\omega(t)t^{\gamma-1}\,dt \sim 1.$$

Let $s=s_n$ be defined by

$$s = \max\left\{l\colon \omega\left(\frac{l}{n}\right)\left(\frac{l}{n}\right)^{\gamma} = \max_{1\leqq i\leqq n}\omega\left(\frac{i}{n}\right)\left(\frac{i}{n}\right)^{\gamma}\right\}.$$

If $\gamma\geqq 0$ then obviously $s=n$ and the statement of the Lemma is trivial. If $s\geqq n/2$ then by (4.2) we again get our statement. Hence we suppose that $\gamma<0$ and $1\leqq s\leqq n/2$. If $s<c_1$ where c_1 is a constant satisfying

$$(4.3)\qquad \frac{-\gamma}{1-2^{\gamma}} < c_1$$

then

$$\omega\left(\frac{s}{n}\right)\left(\frac{s}{n}\right)^{\gamma} \leqq c_1\omega\left(\frac{s}{n}\right)\frac{s^{\gamma-1}}{n^{\gamma}} \leqq c_1\sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\frac{i^{\gamma-1}}{n^{\gamma}}.$$

Now we deal with the case $c_1\leqq s\leqq n/2$. Then using (4.3) and the fact that both $\omega(t)$ and t^{γ} are monotonic functions we get

$$\frac{1}{n}\sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\left(\frac{i}{n}\right)^{\gamma-1} \geqq \omega\left(\frac{s}{n}\right)\frac{1}{n}\sum_{i=s}^{n}\left(\frac{i}{n}\right)^{\gamma-1} \geqq$$

$$\geqq \omega\left(\frac{s}{n}\right)\left\{\int_{\frac{s}{n}}^{1}t^{\gamma-1}\,dt - \frac{1}{n}\left(\frac{s}{n}\right)^{\gamma-1}\right\} \geqq c_2\left(\frac{s}{n}\right)^{\gamma}\omega\left(\frac{s}{n}\right)$$

where $c_2=\dfrac{1-2^{\gamma}}{-\gamma}-\dfrac{1}{c_1}>0.$

4.2. We shall use the following lemma of P. Vértesi [6, Lemma 3.2].

LEMMA 4.2. *We have*

$$(4.4)\qquad |P_n^{(\alpha,\beta)}(\cos\theta)| \sim |\cos\theta-\cos\theta_j|\,\theta_j^{-\alpha-3/2}n^{1/2} \sim |\theta-\theta_j|\,\theta_j^{-\alpha-1/2}n^{1/2} \quad (\alpha,\beta>-1)$$

uniformly in $\theta\in[0,\pi-\varepsilon]$ *where* $\cos\theta_j$ *is the nearest root to* $\cos\theta$.

4.3. Using [1] we have

$$(4.5)\qquad \frac{c_3}{n} \leqq \theta_{k+1,n}^{(\alpha,\beta)}-\theta_{k,n}^{(\alpha,\beta)} \leqq \frac{c_4}{n} \quad (k=0,1,\ldots,n)$$

with $0<c_3=c_3(\alpha,\beta)$, $c_4=c_4(\alpha,\beta)$, $\theta_0^{(\alpha,\beta)}=0$, $\theta_{n+1}^{(\alpha,\beta)}=\pi$. Further

$$P_n^{(\alpha,\beta)}(x)=(-1)^n P_n^{(\beta,\alpha)}(-x),\quad P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}\sim n^\alpha, \tag{4.6}$$

$$|P_n^{(\alpha,\beta)}(\cos\theta)|=\begin{cases} O(\theta^{-\alpha-1/2}n^{-1/2}) & \text{if } \dfrac{c_5}{n}\leqq\theta\leqq\pi-\varepsilon,\\ O(n^\alpha) & \text{if } 0\leqq\theta\leqq\dfrac{c_5}{n},\end{cases} \tag{4.7}$$

$$|P_n^{(\alpha,\beta)}(x_k)'|\sim\begin{cases} k^{-\alpha-3/2}n^{\alpha+2} & \text{if } 0<\theta_k\leqq\pi-\varepsilon,\\ (n-k+1)^{-\beta-3/2}n^{\beta+2} & \text{if } \varepsilon\leqq\theta_k<\pi,\end{cases} \tag{4.8}$$

$$\frac{P_n^{(\alpha,\beta)}(x_k)''}{P_n^{(\alpha,\beta)}(x_k)'}=\frac{\alpha-\beta+(\alpha+\beta+2)x_k}{1-x_k^2} \tag{4.9}$$

(see [1], (4.1.3), (4.1.1), (7.32.5), (8.9.2), (8.9.9), (4.21.7) and (4.5.1)).

By (2.5) and a simple computation we have

(4.10)
$$l_{k,n}(Y,x)=\begin{cases} -\dfrac{P_m(1-2x^2)}{4y_kP_m'(x_{|k|})(x-y_k)} & (n=2m;\ k=\pm1,\pm2,\dots,\pm m),\\ -\dfrac{xP_m(1-2x^2)}{4y_k^2P_m'(x_{|k|})(x-y_k)} & (n=2m+1;\ k=\pm1,\pm2,\dots,\pm m),\\ \dfrac{P_m(1-2x^2)}{P_m(1)} & (n \text{ is odd},\quad k=0).\end{cases}$$

(Here and later on if n is even then the corresponding parameters of P_m, P_m' and $x_{k,n}$ are (α,β); for odd n they are $(\alpha+1,\beta)$.) Further

(4.11)
$$v_{k,n}(Y,x)=$$
$$=\begin{cases} 1+\dfrac{x-y_k}{y_k}\left[2\alpha+1-2(\beta+1)\dfrac{y_k^2}{1-y_k^2}\right] & (n=1,2,\dots;\ k=\pm1,\pm2,\dots,\pm m),\\ 1 & (n \text{ is odd},\ k=0).\end{cases}$$

In the following sometimes we shall use the above relations without reference.

4.4. Now we turn to the proof of our Theorem. First let n be even, $n=2m$.

4.4.1. Let $0<\eta\leqq\theta\leqq\pi$ and $-\pi/2<\theta_k\leqq\mu<0$ where η and μ are constants. Then $|v_k(x)|=O(1)$ and by (2.6), (4.8) and (4.10)

$$l_k^2(x)=O\left(\frac{P_m^2(1-2x^2)}{P_m'(x_{|k|})^2}\right)=O\left(\frac{P_m^2(1-2x^2)}{m}\right) \tag{4.12}$$

so

$$\sum_{-\frac{\pi}{2}<\theta_k\leqq\mu}|v_k(x)\,l_k^2(x)|=O\big(P_m^2(1-2x^2)\big). \tag{4.13}$$

If $0<\eta\leqq\theta\leqq\pi$ and $\pi/2\leqq-\theta_k<\pi$ then it follows from (2.6) and (4.11) that $|v_k(x)|= =O\left(\left(\frac{m}{m+1-k}\right)^2\right)$ so by (4.8)

(4.14)
$$\sum_{\frac{\pi}{2}\leqq-\theta_k<\pi} |v_k(x)\, l_k^2(x)| =$$
$$= O\left(P_m^2(1-2x^2) \sum_{\frac{\pi}{2}\leqq-\theta_k\leqq\pi} \left(\frac{m}{m+1-k}\right)^2 \frac{(m+1-k)^{2\beta+3}}{m^{2\beta+4}}\right) = O\big(P_m^2(1-2x^2)\big).$$

Hence by (4.13) and (4.14) we have

(4.15)
$$\sum_{-\pi<\theta_k\leqq\mu} |v_k(x)\, l_k^2(x)| = O\big(P_m^2(1-2x^2)\big)$$

where $0<\eta\leqq\theta\leqq\pi$.

4.4.2. Now let $\varepsilon\leqq\theta\leqq\pi$ and $|\theta_k|<\varepsilon/2$. Then we may write
$$\sum_{|\theta_k|<\frac{\varepsilon}{2}} |\Delta_k(x)| =$$
$$= P_m^2(1-2x^2) \sum_{0<\theta_k<\frac{\varepsilon}{2}} [4y_k P_m'(x_{|k|})]^{-2} \left\{\frac{f(x)-f(y_k)}{(x-y_k)^2}\, v_k(x) + \frac{f(x)-f(-y_k)}{(x+y_k)^2}\, v_{-k}(x)\right\}$$

where

(4.16)
$$\Delta_k(x) = \big(f(x)-f(y_k)\big) v_k(x)\, l_k^2(x).$$

But by (4.11)
$$\{\dots\} = \left[\frac{f(x)-f(y_k)}{(x-y_k)^2} + \frac{f(x)-f(-y_k)}{(x+y_k)^2}\right] -$$
$$-\left(2(\beta+1)\frac{y_k^2}{1-y_k^2} - (2\alpha+1)\right)\frac{1}{y_k}\left[\frac{f(x)-f(y_k)}{x-y_k} - \frac{f(x)-f(-y_k)}{x+y_k}\right] = T_1 - T_2.$$

Evidently $|T_1|=O(1)$ and
$$|T_2| = O\left((y_k^2+|2\alpha+1|)\left(\frac{|f(x)|}{|x^2-y_k^2|} + \frac{|f(y_k)|}{y_k|x-y_k|} + \frac{|f(-y_k)|}{y_k|x+y_k|}\right)\right) =$$
$$= O\left((y_k^2+|2\alpha+1|)\left(1+\frac{\omega(y_k)}{y_k}\right)\right)$$

so using again (2.6), (4.5) and (4.8) we get

(4.17)
$$\sum_{|\theta_k|<\frac{\varepsilon}{2}} |\Delta_k(x)| = O\left(P_m^2(1-2x^2) \sum_{|\theta_k|<\frac{\varepsilon}{2}} \frac{|k|^{2\alpha+1}}{m^{2\alpha+2}} \left(\frac{k^2}{m^2}+|2\alpha+1|\right)\left(1+\frac{\omega\left(\frac{|k|}{m}\right)m}{|k|}\right)\right) =$$
$$= O\left(P_m^2(1-2x^2)\left\{1+\frac{|2\alpha+1|}{n^{2\alpha+1}} \sum_{i=1}^{m} \omega\left(\frac{i}{m}\right) i^{2\alpha}\right\}\right).$$

4.4.3. Now we turn to the proof of (i) so let $0\leqq\theta\leqq\varepsilon$.

(A) First we deal with the case $|\theta_k|<2\varepsilon$ and $4j\leqq|k|$. Then $|v_k(x)|=O(1)$ so by (4.4) and (4.10)

(4.18)

$$\Big|\sum_{\substack{4j\leqq|k|\\|\theta_k|<2\varepsilon}}\Delta_k(x)\Big| = O\big(\Sigma\omega(|y_k-x|)\,l_k^2(x)\big) = O\left(P_m^2(1-2x^2)\Sigma\omega(|y_k|)\frac{1}{(y_kP_m'(x_{|k|}))^2}\frac{1}{y_k^2}\right) =$$

$$= O\left(\frac{(\theta-\theta_j)^{2n}}{\theta_j^{2\alpha+1}}\sum_{4j\leqq k}\omega\left(\frac{k}{n}\right)\frac{k^{2\alpha-1}}{n^{2\alpha}}\right) = O\left([m(\theta-\theta_j)]^2\sum_{i=1}^{m}\omega\left(\frac{i}{m}\right)i^{2\alpha-1}\right).$$

(B) If $j/2\leqq|k|\leqq4j$ then $|v_k(x)|=O(1)$ and so similarly as above

(4.19)
$$\Big|\sum_{\frac{1}{2}j\leqq|k|\leqq4j}\Delta_k(x)\Big| = O\big(\omega(|y_k-x|)\,l_k^2(x)\big) =$$

$$= O\left(\omega(|x-y_j|)+P_m^2(1-2x^2)\sum_{\frac{1}{2}j\leqq|k|\leqq4j}\omega(|x-y_k|)\frac{|k|^{2\alpha+1}}{n^{2\alpha+2}}\frac{1}{(x-y_k)^2}\right) =$$

$$= O\left(\omega(|x-y_j|)+\frac{(\theta-\theta_j)^2m}{\theta_j^{2\alpha+1}}\frac{j^{2\alpha+1}}{m^{2\alpha+2}}\sum_{k\neq j}\omega(|x-y_k|)\frac{1}{(x-y_k)^2}\right) =$$

$$= O\left(\omega(|x-y_j|)+[n(\theta-\theta_j)]^2\sum_{i=1}^{n}\omega\left(\frac{i}{m}\right)\frac{1}{i^2}\right).$$

(C) If $|k|\leqq j/2$ and $k\neq0$ then $|x|>c_6/n$. Then

$$\Big|\sum_{|k|\leqq\frac{j}{2}}\Delta_k(x)\Big| =$$

$$= O\Bigg(P_m^2(1-2x^2)\sum_{|k|\leqq j/2}\omega(x-y_k)\left\{1+\frac{x-y_k}{|y_k|}\left[|2\alpha+1|+2(\beta+1)\frac{y_k^2}{1-y_k^2}\right]\right\}\times$$

$$\times\big(y_kP_m'(x_{|k|})(x-y_k)\big)^{-2}\Bigg) = O\Bigg(P_m^2(1-2x^2)\omega(x)\times$$

$$\times\sum_{0<k\leqq j/2}\left\{\frac{1}{x^2}\frac{k^{2\alpha+1}}{m^{2\alpha+2}}+|2\alpha+1|\frac{k^{2\alpha}}{m^{2\alpha+1}}\frac{1}{x}+\frac{k^{2\alpha+3}}{m^{2\alpha+4}}\frac{1}{x^2}\right\}\Bigg) =$$

$$= O\left(P_m^2(1-2x^2)\omega(x)\left\{\frac{1}{x^2}\left(\frac{j}{m}\right)^{2\alpha+2}+\frac{|2\alpha+1|}{x}\frac{\sum_{k=1}^{j}k^{2\alpha}}{m^{2\alpha+1}}+\frac{1}{x^2}\left(\frac{j}{m}\right)^{2\alpha+4}\right\}\right) =$$

$$= O\left(P_m^2(1-2x^2)\omega(x)\left\{x^{2\alpha}+\frac{|2\alpha+1|}{x}\frac{\sum_{k=1}^{j}k^{2\alpha}}{m^{2\alpha+1}}+x^{2(\alpha+1)}\right\}\right) = O(I_1+I_2+I_3).$$

Applying Lemmas 4.1 and 4.2 we get

(4.20)
$$I_1 = O\left(\frac{P_m^2(1-2x^2)}{m^{2\alpha}} \sum_{i=1}^{m} \omega\left(\frac{i}{m}\right) i^{2\alpha-1}\right) = O\left([m(\theta-\theta_j)]^2 \sum_{i=1}^{m} \omega\left(\frac{i}{m}\right) i^{2\alpha-1}\right).$$

If $\alpha > -1/2$ then

$$I_2 = O\left(P_m^2(1-2x^2)\frac{\omega(x)}{x}\left(\frac{j}{n}\right)^{2\alpha+1}\right) = O\big(P_m^2(1-2x^2)\omega(x)x^{2\alpha}\big) = O(I_1).$$

If $\alpha = -1/2$ then $I_2 = 0$ so we have to deal only with the case of $\alpha < -1/2$. Then applying again Lemmas 4.1 and 4.2

$$I_2 = O\left(\frac{P_m^2(1-2x^2)\omega(x)}{m^{2\alpha+1}x}\right) = O\left([m(\theta-\theta_j)]^2 \sum_{i=1}^{m} \omega\left(\frac{i}{m}\right)\frac{1}{i^2}\right).$$

As $I_3 = O(I_1)$ we get

(4.21)
$$\Big|\sum_{|k|\leqq \frac{j}{2}} \Delta_k(x)\Big| = O\left([m(\theta-\theta_j)]^2 \sum_{i=1}^{m} \omega\left(\frac{i}{m}\right) i^{2\gamma-1}\right).$$

(D) To prove (i) we have to give an estimation in the case of $2\varepsilon < |\theta_k| < \pi$ as well. Now, similarly to 4.4.1

$$\sum_{2\varepsilon < |\theta_k| < \pi} |v_k(x)\, l_k^2(x)| = O\left(P_m^2(1-2x^2)\sum\left(\frac{m}{m+1-k}\right)^2 \frac{(m+1-k)^{2\beta+3}}{n^{2\beta+4}}\right) =$$
$$= O\big(P_m^2(1-2x^2)\big).$$

By Lemma 4.2

$$P_m^2(1-2x^2) = O\big([m(\theta-\theta_j)]^2\big) = \begin{cases} O(n^{2\alpha}) & \text{if} \quad \alpha \geqq -\dfrac{1}{2} \\ O\left(\dfrac{1}{n}\right) & \text{if} \quad \alpha \leqq -\dfrac{1}{2} \end{cases}$$

and with this we proved (i).

4.4.4. Now we prove (ii). Let $\varepsilon \leqq \theta \leqq \pi - \varepsilon$. We may write

$$\sum_{|k|=1}^{m} \Delta_k(x) = \sum_{\theta_k \leqq \frac{\varepsilon}{2}} \Delta_k + \sum_{\frac{\varepsilon}{2} < \theta_k < \pi - \frac{\varepsilon}{2}} \Delta_k + \sum_{\pi - \frac{\varepsilon}{2} \leqq \theta_k < \pi} \Delta_k = S_1 + S_2 + S_3.$$

If we apply 4.4.1 and 4.4.2 with $\eta = \varepsilon$ and $\mu = \varepsilon/2$ then we get

$$|S_1| = O\big(P_m^2(1-2x^2)\big)$$

i.e. by Lemma 4.2

(4.22)
$$|S_1| = O\left([m(\theta-\theta_j)]^2 \frac{1}{m}\right).$$

If $\varepsilon/2<\theta_k<\pi-\varepsilon/2$ then $|v_k(x)|=O(1)$ so

(4.23)

$$|S_2| = O\Big(\sum_{\frac{\varepsilon}{2}<\theta_k<\pi-\frac{\varepsilon}{2}} \omega(|x-y_k|)\, l_k^2(x)\Big) = O\left(P_m^2(1-2x^2)\sum \frac{\omega(|x-y_k|)}{n(x-y_k)^2}\right) =$$

$$= O\left([n(\theta-\theta_j)]^2 \sum \frac{\omega(|x-y_k|)}{[n(x-y_k)]^2} = O\big(\omega(|x-y_j|)+[n(\theta-\theta_j)]^2\big) \sum_{i=1}^{n} \omega\left(\frac{i}{n}\right)\frac{1}{i^2}\right).$$

If $\pi-\varepsilon/2\leqq\theta_k<\pi$ then $v_k(x)=O\left(\left(\frac{m}{m+1-k}\right)^2\right)$ so

$$|S_3| = O\left(P_m^2(1-2x^2) \sum_{\pi-\frac{\varepsilon}{2}\leqq\theta_k<\pi} \left(\frac{m}{m+1-k}\right)^2 \frac{(m+1-k)^{2\beta+3}}{m^{2\beta+4}}\right) = \tag{4.24}$$

$$= O\big(P_m^2(1-2x^2)\big) = O\left(\frac{[m(\theta-\theta_j)]^2}{m}\right).$$

From (4.22)—(4.24) we get (ii).

4.4.5. Finally we prove (iii). Let $\pi-\varepsilon\leqq\theta\leqq\pi$.
If $\theta_k<\varepsilon/2$ then by (4.15) and by (4.17)

$$\Big|\sum_{\theta_k<\frac{\varepsilon}{2}} \Delta_k(x)\Big| = O\left(P_m^2(1-2x^2)\left\{1+\frac{|2\alpha+1|}{m^{2\alpha+1}} \sum_{i=1}^{m} \omega\left(\frac{i}{m}\right) i^{2\alpha}\right\}\right). \tag{4.25}$$

If $\varepsilon/2\leqq\theta_k\leqq\pi-2\varepsilon$ then $|v_k(x)|=O(1)$, $l_k^2(x)=O\left(\frac{P_m^2(1-2x^2)}{m}\right)$ so

$$\sum_{\frac{\varepsilon}{2}<\theta_k<\pi-2\varepsilon} |v_k(x)\, l_k^2(x)| = O\big(P_m^2(1-2x^2)\big). \tag{4.26}$$

Let us use the notations $K=m+1-k$ and $J=m+1-j$. If $\theta<\theta_m$ and $2K\leqq J$ then $|v_k(x)|=O\left(\frac{J^2-K^2}{K^2}\right)$ so

$$\sum_{2K\leqq J} (|x-y_j|)\, |v_k(x)\, l_k^2(x)| =$$

$$= O\left(P_m^2(1-2x^2) \sum_{2K\leqq J} \omega\left(\frac{J^2-K^2}{m^2}\right) \frac{J^2-K^2}{K^2}\, \frac{K^{2\beta+3}}{m^{2\beta+4}}\, \frac{m^4}{(J^2-K^2)^2}\right) =$$

$$= O\left(P_m^2(1-2x^2)\frac{1}{J^2}\,\frac{1}{n^{2\beta}}\,\omega\left(\frac{J^2}{m^2}\right) \sum_{K=1}^{\left[\frac{J}{2}\right]} K^{2\beta+1}\right) =$$

$$= O\left(P_m^2(1-2x^2)\left(\frac{J}{m}\right)^{2\beta} \omega\left(\frac{J^2}{m^2}\right)\right).$$

Applying Lemma 4.1 for $\omega(\sqrt{t})$ we have

$$(4.27)\qquad \Big|\sum_{2K\leq J}\Delta_k(x)\Big| = O\left(\left(\frac{P_m^2(1-2x^2)}{m^{2\beta}}\right)\sum_{i=1}^{m}\omega\left(\frac{i^2}{m^2}\right)i^{2\beta-1}\right).$$

If $J/2\leq K\leq 2J$ then $|v_k(x)|=O(1)$ so

$$\Big|\sum_{\frac{1}{2}J\leq K\leq 2J}\Delta_k(x)\Big| = O\left(\omega(|x-y_j|)+P_m^2(1-2x^2)\sum_{\substack{\frac{1}{2}J\leq K\leq 2J\\ K\neq J}}\omega(|x-y_k|)\frac{K^{2\beta+3}}{n^{2\beta+4}}\frac{1}{(x-y_k)^2}\right)=$$

$$= O\left(\omega(|x-y_j|)+\frac{P_m^2(1-2x^2)}{m^{2\beta}}J^{2\beta+3}\sum_{\frac{1}{2}J\leq K\leq 2J}\omega\left(\frac{|K^2-J^2|}{m^2}\right)\frac{1}{(K^2-J^2)^2}\right).$$

But

$$\sum_{\frac{J}{2}\leq K\leq 2J}\omega\left(\frac{|K^2-J^2|}{m^2}\right)\frac{1}{(K^2-J^2)^2} = O\left(\sum_{i=1}^{m}\omega\left(\frac{iJ}{m^2}\right)\frac{1}{i^2J^2}\right) = O\left(\frac{1}{J^2}\sum_{i=1}^{m}\omega\left(\frac{i}{n}\right)\frac{1}{i^2}\right)$$

so by Lemma 4.2

$$(4.28)\qquad \Big|\sum_{\frac{J}{2}\leq K\leq 2J}\Delta_k(x)\Big| = O\left(\omega(|x-y_j|)+[m(\theta-\theta_j)]^2\sum_{i=1}^{m}\omega\left(\frac{i}{m}\right)\frac{1}{i^2}\right).$$

If $2J<K$ then $|v_k(x)|=O(1)$ hence

$$(4.29)\qquad \Big|\sum_{2J<K}\Delta_k(x)\Big| = O\left(P_m^2(1-2x^2)\,\Sigma\omega\left(\frac{K^2-J^2}{m^2}\right)\frac{K^{2\beta+3}}{m^{2\beta+4}}\left(\frac{m^2}{K^2-J^2}\right)^2\right)=$$

$$= O\left(\frac{P_m^2(1-2x^2)}{m^{2\beta}}\sum_{i=1}^{m}\omega\left(\frac{i^2}{m^2}\right)i^{2\beta-1}\right) = O\left([n(\theta-\theta_j)]^2\sum_{i=1}^{m}\omega\left(\frac{i^2}{m^2}\right)i^{2\beta-1}\right).$$

From (4.25)—(4.29) (iii) already follows.

4.5. For odd n, following P. Vértesi [5], using (4.10) and the corresponding formulae for the Jacobi polynomials and roots, it is easy to see that

$$|l_{k,2n+1}(Y;\,x)| = O(1)\,|l_{k,2n}(Y;\,x)|\left|\frac{\theta-\theta_{j,2n+1}}{\theta-\theta_{j,2n}}\right|\qquad (k\neq 0)$$

and $|l_{0,2n+1}(Y;\,x)|=O(1)$. So this case can be considered as the above one. This proves our Theorem for each n.

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ON THE DISTRIBUTION OF p^θ mod 1

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1. Introduction

This paper is devoted to the following results.

THEOREM 1. *Let* $1/3<\theta<1$, $0<\varepsilon\leqq\varepsilon(\theta)$, $0\leqq w<1$ *be fixed real numbers and* $1\leqq a\leqq q$, $(a, q)=1$ *integers. If* $x\geqq 1$ *and*

$$x^{-1/4+31\varepsilon}\leqq \Delta\leqq 1/2 \tag{1.1}$$

then

$$\#\{p\leqq x,\ p\equiv a\,(q),\ \|p^\theta-w\|<\Delta\}=2\Delta\pi(x, q, a)+O(\Delta x^{1-\varepsilon}+x^{(1+\theta)/2+25\varepsilon}). \tag{1.2}$$

The implied constant depends at most on ε *and* θ.

THEOREM 2. *Let* $1/3<\theta<1$, $0<\varepsilon$, $0\leqq w<1$ *be fixed real numbers. If* $x\geqq 1$ *and*

$$x^{-3/10}\leqq \Delta\leqq 1/2 \tag{1.3}$$

then

$$\#\{p\leqq x,\ \|p^\theta-w\|<\Delta\}\geqq\frac{2}{3}\Delta\pi(x)+O(x^{(1+\theta)/2+\varepsilon}). \tag{1.4}$$

The implied constant depends at most on ε *and* θ.

An easy consequence of Theorem 2 is

COROLLARY 1. *Let* $2/5\leqq\theta<1$, $0<\varepsilon$, $0\leqq w<1$ *be fixed real numbers. There are infinitely many primes p with*

$$\|p^\theta-w\|<p^{-(1-\theta)/2+\varepsilon}. \tag{1.5}$$

Theorem 2 has also a q-analogue which can be proved in the same manner. But the result is a little bit more complicated as we have to change the right hand side of (1.4) into

$$2\Delta\pi(x, q, a)-\frac{4}{3}\Delta\frac{x}{\varphi(q)\log\dfrac{x}{q}}+O(x^{(1+\theta)/2+\varepsilon}), \tag{1.6}$$

(where the implied constant depends at most on ε and θ). This implies that Corollary 1 remains true even if we confine ourselves to primes in a given arithmetic progression.

We prove these theorems in two steps, first we transform a sum over primes into bilinear forms, and then we estimate the resulting forms by analytic methods. The most convenient way of doing this is to use Vaughan's method. Theorem 1 can, in fact, be proved in this way (see the previous paper of the author [1] where the history of the problem and its connections to other problems can also be found), but the proof of Theorem 2 requires some new ideas. The key argument is the sieve identity method due to Harman [4] and Heath—Brown [5]. The main advantage of this method is that we can control the signs of coefficients in the occurring bilinear forms. Thus we can omit some "bad" subsums at the expense of getting only a lower bound instead of an asymptotic formula.

The essence of the sieve identity method is the repeated application of the Bushtab identity

$$S(\mathscr{C}, z) = S(\mathscr{C}, z') - \sum_{z' \leqq p < z} S(\mathscr{C}_{p,p}). \tag{1.7}$$

(We use the standard sieve notations, see e.g. [3].) However we do not utilize (1.7) explicitly as the resulting combinatorial decomposition can be obtained directly from Richert's fundamental identity. The arguments go through with any function $F(\mathscr{C}, z)$ in place of $S(\mathscr{C}, z)$ if it satisfies a recursion similar to (1.7). For example let $\mathscr{C}$ be a finite set of integers and let $F(\mathscr{C}, z)$ denote the number of elements of $\mathscr{C}$ having only prime factors less than z. We have

$$F(\mathscr{C}, z) = F(\mathscr{C}, z') - \sum_{z \leqq p < z'} F(\mathscr{C}_{p,p}).$$

This means, one can prove the following theorem ($P(n)$ denotes the largest prime factor of n).

THEOREM 3. *Let* $1/3 < \theta < 1$, $0 < \varepsilon$, $0 \leqq w < 1$ *be fixed real numbers. If* $x \geqq 1$, $z > 2$ *and*

$$x^{-1/4+35\varepsilon} \leqq \Delta \leqq 1/2$$

then

$$\# \{n \leqq x, P(n) \leqq z, \|n^\theta - w\| < \Delta\} =$$

$$= 2\Delta \# \{n \leqq x, P(n) \leqq z\} + O(\Delta x^{1-\varepsilon} + x^{(1+\theta)/2+30\varepsilon}).$$

The implied constant depends at most on ε *and* θ.

This implies

COROLLARY 2. *Let* $1/2 \leqq \theta < 1$, $0 < \varepsilon$, $0 \leqq w < 1$ *be fixed real numbers. There is a* $K(\varepsilon) > 0$ *with the property: For any* $K \geqq K(\varepsilon)$ *fixed real number there are infinitely many numbers n with* $P(n) \leqq \log^K n$ *and* $\|n^\theta - w\| < n^{-(1-\theta)/2+\varepsilon}$.

The main tool in estimating bilinear forms is the Approximate Functional Equation of the zeta function. We can use it for a reflection argument which allows us to change a special Dirichlet series into an other one with shorter length.

In our theorems $x^{-1/4}$ is a critical limit for getting asymptotic results but for getting lower bounds $x^{-3/10}$ is only a practical limit. Below $x^{-3/10}$ the calculations of Section 8 become very clumsy.

Of course, we can get non-trivial results for $\theta \leqq 1/3$ as well, but we cannot get an error term as good as $x^{(1+\theta)/2}$. We do not prove these weaker results because we wish to maintain the error term $x^{(1+\theta)/2}$, since this is the same one that the Riemann Hypothesis would imply. The aim of this paper is to show a new field in which the Riemann Hypothesis can be repleaced by elementary methods.

There is a quite interesting feature of Theorem 1. If $q=x^\delta$, no asymptotic formula is known for $\pi(x, q, a)$, because it depends strongly on the existence of Siegel zeros. (1.2) shows that the irregularities of $\#\{p\leqq x,\ p\equiv a(q),\ \|p^\theta-w\|<\Delta\}$ are parallel to the irregularities of $\pi(x, q, a)$.

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2. Notations

First we list the conditions assumed throughout the paper.

(2.1) $$\frac{1}{3}<\theta<1, \quad 0\leqq w<1, \quad 0<\Delta\leqq\frac{1}{2}, \quad 0<\varepsilon\leqq\varepsilon(\theta)$$

(θ, w, Δ and ε are real numbers) and

(2.2) $$1\leqq a\leqq q, \quad (a, q)=1$$

(a and q are integers).

We define

(2.3) $$\Phi(n)=\Phi(n;\ \theta, w, \Delta)=\begin{cases}1 & \text{if } \|n^\theta-w\|<\Delta,\\ 0 & \text{otherwise.}\end{cases}$$

In other words $\Phi(n)$ is the characteristic function of integers n to which a number of type integer$+w$ corresponds between $n^\theta-\Delta$ and $n^\theta+\Delta$.

We will use a_m and b_n as general coefficients of linear or bilinear forms. They may be arbitrary complex numbers satisfying

(2.4) $$a_m\ll m^\eta, \quad b_n\ll n^\eta \quad \text{for all} \quad \eta>0.$$

$m\sim M$ indicates the inequality $M\leqq m<2M$.

We will use the standard sieve notations. For a given finite set of integers $\mathscr{C}$ we define $\mathscr{C}_d$ as the set of integers from $\mathscr{C}$ divisible by d, $|\mathscr{C}_d|$ as the number of elements of $\mathscr{C}_d$ and $S(\mathscr{C}, z)$ as the number of integers from $\mathscr{C}$ not divisible by primes less than z. We will use the product

(2.5) $$P(z)=\prod_{p<z} p$$

(note that in the introduction $P(n)$ had different meaning). $p(n)$ denotes the smallest prime factor of n. We have always

(2.6) $$z_0=2.$$

Clearly $S(\mathscr{C}, z_0)=|\mathscr{C}|$.

For our purpose we take

$$\mathscr{A} = \{n \leqq x, \quad n \equiv a(q), \quad \|n^{\theta}-w\| < \Delta\}, \tag{2.7}$$

$$\mathscr{B} = \{n \leqq x, \quad n \equiv a(q)\}. \tag{2.8}$$

From Section 7 on, we will use $\mathscr{A}$ and $\mathscr{B}$ only in the special case $a=q=1$.

Concerning the analytic arguments used in Section 3 we refer the reader to [6], [7] and for the sieve results see [3].

3. Estimation of bilinear forms

LEMMA 1. *Under* (2.1)—(2.4) *we have*

$$\sum_{\substack{mn\leqq x\\ m\sim M\\ mn\equiv a(q)}} a_m b_n \Phi(mn) = 2\Delta \sum_{\substack{mn\leqq x\\ m\sim M\\ mn\equiv a(q)}} a_m b_n + O(\Delta x^{1-\varepsilon}+x^{(1+\theta)/2+9\varepsilon}) \tag{3.1}$$

whenever

$$\Delta^{-1}x^{14\varepsilon} \ll M \ll \Delta^2 x^{1-20\varepsilon} \quad or \quad \Delta^{-2}x^{20\varepsilon} \ll M \ll \Delta x^{1-14\varepsilon} \tag{3.2}$$

or

$$b_n = 1 \quad for\ all\ n\ and \quad M \ll \Delta x^{1-14\varepsilon}. \tag{3.3}$$

PROOF. As the proof contains quite a few standard technical steps, first we give an outline, and will present the proof only briefly because the same techniques were given in details in [1].

For the sake of simplicity we can assume that $a=q=1$. We estimate the sum

$$S = \sum_{m\sim M} \sum_{n\sim \frac{x}{M}} a_m b_n \Phi_U(mn)$$

where

(3.4)

$$\Phi_U(k) = \begin{cases} 1 & \text{if there is an integer } m \text{ with } \quad k^{\theta}\left(1-\dfrac{1}{U}\right) < m+w \leqq k^{\theta}\left(1+\dfrac{1}{U}\right), \\ 0 & \text{otherwise} \end{cases}$$

and $U=x^{\theta}/\Delta$. Taking

$$L(S, w) = \sum_{k\sim x^{\theta}} (k+w)^{-s}, \quad A(s) = \sum_{m\sim M} a_m m^{-s},$$

$$B(s) = \sum_{n\sim \frac{x}{M}} b_n n^{-s}, \quad H(s) = \frac{\left(1+\dfrac{1}{U}\right)^{s} - \left(1-\dfrac{1}{U}\right)^{s}}{s}, \quad T = \frac{x^{\theta+5\varepsilon}}{\Delta}$$

from the Perron formula we get

$$S = \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} L(s, w)A(-\theta s)B(-\theta s)H(s)\,ds + O(\Delta x^{1-\varepsilon}).$$

The correct definition of U, T, L, A, B and H will be given later on. The main part of the integral is between $1/2-ix^\theta$ and $1/2+ix^\theta$ — this constitutes the main term of S. This follows from the special shape of $L(s, w)$; we know its asymptotic behaviour when $|t|\leqq x^\theta$. We have the approximate functional equation

$$L\left(\frac{1}{2}+it, w\right) = \chi\left(\frac{1}{2}+it\right) \sum_{k\sim\frac{|t|}{x^\theta}} e^{2\pi ikw}k^{-s}+O(1)$$

provided $x^{3\theta}\geqq|t|>x^\theta$. Here $\chi(s)$ is the weight function occuring in the functional equation of zeta. This has the property $|\chi(1/2+it)|=1$. Taking

$$K(s, w) = \sum_{k\sim\frac{T}{x^\theta}} e^{2\pi ikw}k^{-s} \quad \text{we get} \quad \left|L\left(\frac{1}{2}+it, w\right)\right| \approx \left|K\left(\frac{1}{2}+it, w\right)\right|.$$

Finally we use classical mean value theorems to estimate the integral

$$I = \int_{-T}^{T}\left|K\left(\frac{1}{2}+it, w\right)A\left(-\frac{\theta}{2}-i\theta t\right)B\left(-\frac{\theta}{2}-i\theta t\right)\right|dt.$$

By the Cauchy—Schwarz inequality

$$I \leqq \Big(\int_{-T}^{T}|KA|^2\Big)^{1/2}\Big(\int_{-T}^{T}|B|^2\Big)^{1/2} \ll \left(T+\frac{T}{x^\theta}M\right)^{1/2}(M^{1+\theta})^{1/2}\left(T+\frac{x}{M}\right)^{1/2}\left(\frac{x}{M}\right)^{(1+\theta)/2}x^\varepsilon$$

and, of course, we can change the roles of A and B. Collecting these estimates we get

$$|S-\text{main term}| \ll \Delta x^{1-\varepsilon}+$$
$$+Ux^{(1+\theta)/2+\varepsilon}\min\left(\left(T+\frac{TM}{x^\theta}\right)\left(T+\frac{x}{M}\right), \left(T+\frac{Tx^{1-\theta}}{M}\right)(T+M)\right)^{1/2}.$$

(This has the required size exactly when M satisfies the given conditions.) When $b_n=1$ for all n, we have the fourth power moment estimation for $B(s)$. By the Cauchy—Schwarz inequality

$$I \leqq \Big(\int_{-T}^{T}|K|^4\Big)^{1/4}\Big(\int_{-T}^{T}|B|^4\Big)^{1/4}\Big(\int_{-T}^{T}|A|^2\Big)^{1/2} \ll T^{1/2}(T+M)^{1/2}x^{(1+\theta)/2+\varepsilon}$$

and we get the same conclusion.

The above argument becomes more complicated in details for two reasons. First S is not exactly the bilinear form we want to investigate; second the reflection of $L(s, w)$ into $K(s, w)$ requires an additional average argument.

Now we give the detailed proof. First we consider the case of (3.2). Observe that the condition $mn\equiv a(q)$ can be dropped, because the orthogonality of characters gives

$$(3.5)\qquad \sum_{\substack{m\sim M\\ mn\leqq x\\ mn\equiv a(q)}} a_m b_n \Phi(mn) = \frac{1}{\varphi(q)}\sum_{\chi(q)}\bar{\chi}(a)\sum_{\substack{m\sim M\\ mn\leqq x}} a_m\chi(m)b_n\chi(n)\Phi(mn).$$

If (3.1) is proved for $a=q=1$ then (3.5) proves the general result. Writing

$$a_m = \max(0, \operatorname{Re} a_m) - \max(0, -\operatorname{Re} a_m) + i \max(0, \operatorname{Im} a_m) - i \max(0, -\operatorname{Im} a_m)$$

(and similarly with b_n) we can assume that $a_m \geqq 0$ and $b_n \geqq 0$. An easy splitting up argument shows that it is enough to prove

$$\sum_{\substack{m\sim M \\ x_1<mn<x_2}} a_m b_n \Phi(mn) = 2\Delta \sum_{\substack{m\sim M \\ x_1<mn<x_2}} a_m b_n + O(\Delta x^{1-4\varepsilon} + x^{(1+\theta)/2+6\varepsilon}) \tag{3.6}$$

where

$$\max(M, x^{(1+\theta)/2}) \leqq x_1 < x_2 \leqq x, \quad x_2 - x_1 \leqq \frac{x_1}{x^{2\varepsilon}}, \tag{3.7}$$

x_1 and x_2 are halves of integers.

We define $\Phi_U(n)$ by (3.4) and taking $U_j = x_j{}^\theta/\Delta$, $j=1, 2$ we have

$$\Phi_{U_2}(n) \leqq \Phi(n) \leqq \Phi_{U_1}(n) \quad \text{for} \quad x_1 < n < x_2. \tag{3.8}$$

Since

$$\left(\frac{mn}{x_j}\right)^\theta = 1 + O(x^{-2\varepsilon}), \quad j = 1, 2$$

whenever $x_1 < mn < x_2$, it is enough to prove

$$\sum_{\substack{m\sim M \\ x_1<mn<x_2}} a_m b_n \Phi_U(mn) = \frac{2}{U} \sum_{\substack{m\sim M \\ x_1<mn<x_2}} a_m b_n (mn)^\theta + O(\Delta x^{1-4\varepsilon} + x^{(1+\theta)/2+6\varepsilon}) \tag{3.9}$$

where

$$\frac{x_1{}^\theta}{\Delta} \ll U \ll \frac{x_1{}^\theta}{\Delta}. \tag{3.10}$$

We can assume that

$$x_1^{-\frac{1-\theta}{2}} \leqq \Delta \leqq 1/2 \tag{3.11}$$

— the case of $\Delta < x_1^{-(1-\theta)/2}$ follows from the monotonicity in Δ.

Take

$$\begin{cases} L_y(s, w) = \displaystyle\sum_{\frac{1}{3}y<k\leqq 3y} (k+w)^{-s}, \quad x_1{}^\theta \leqq y \leqq 2x_1{}^\theta, \\ C(s) = \displaystyle\sum_{\substack{m\sim M \\ x_1<mn<x_2}} a_m b_n (mn)^{-s} = \sum_{x_1<l<x_2} c_l l^{-s}, \\ H(s) = \dfrac{\left(1+\frac{1}{U}\right)^s - \left(1-\frac{1}{U}\right)^s}{s}, \quad T = \dfrac{x_1{}^\theta x^{5\varepsilon}}{\Delta} (< x_1). \end{cases} \tag{3.12}$$

We start with the Perron integral

$$\Phi_U(n) = \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} L_y(s, w) n^{\theta s} H(s)\, ds + O(R_n) \tag{3.13}$$

where $x_1<n<x_2$, $x_1{}^\theta \leqq y \leqq 2x_1{}^\theta$ and

$$(3.14)\quad R_n = \sum_{\frac{1}{3}x_1{}^\theta<k\leqq 6x_1{}^\theta} \min\left(1, \frac{1}{T\left|\log\frac{n^\theta\left(1+\frac{1}{U}\right)}{k+w}\right|}+\frac{1}{T\left|\log\frac{n^\theta\left(1-\frac{1}{U}\right)}{k+w}\right|}\right).$$

It is easy to show that

$$(3.15)\quad \sum_{x_1<n<x_2} c_n R_n \ll \frac{x_1{}^{1+\theta+\varepsilon}}{T} \leqq \Delta x^{1-4\varepsilon}$$

and for every $x_1{}^\theta \leqq y \leqq 2x_1{}^\theta$ we get

$$(3.16)\quad \sum_{\substack{m\sim M\\ x_1<mn<x_2}} a_m b_n \Phi_U(mn) = \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} L_y(s, w)C(-\theta s)H(s)\,ds + O(\Delta x^{1-4\varepsilon}).$$

Applying the Perron formula, the mean value theorem of Dirichlet polynomials and

$$(3.17)\quad \begin{cases} H(s) = \dfrac{2}{U}+O(|s-1|U^{-2}), \quad H(s) \ll \dfrac{1}{U}, \\[2ex] L_y(s, w) = \dfrac{(3y)^{1-s}-\left(\frac{1}{3}y\right)^{1-s}}{1-s}+O(x_1{}^{-\theta/2}) \end{cases}$$

for $s=\frac{1}{2}+it, \quad |t| \leqq T_0 = x_1{}^\theta, \quad x_1{}^\theta \leqq y \leqq 2x_1{}^\theta$

we can get

$$(3.18)\quad \frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} L_y(s, w)C(-\theta s)H(s)\,ds =$$

$$= \frac{2}{U}\sum_{\substack{m\sim M\\ x_1<mn<x_2}} a_m b_n (mn)^\theta + O(\Delta x_1{}^{1-(\theta/2)}),$$

and

$$(3.19)\quad \sum_{\substack{m\sim M\\ x_1<mn<x_2}} a_m b_n \Phi_U(mn) = \frac{2}{U}\sum_{\substack{m\sim M\\ x_1<mn<x_2}} a_m b_n (mn)^\theta +$$

$$+O\left(\Delta x^{1-4\varepsilon}+\frac{\Delta}{x_1{}^\theta}\int_{T_0}^{T}\left|L_y\left(\frac{1}{2}+it, w\right)C\left(-\frac{\theta}{2}-i\theta t\right)\right|dt\right)$$

for all $x_1{}^\theta \leqq y \leqq 2x_1{}^\theta$.

Now it remains to prove

$$(3.20)\quad \frac{\Delta}{x_1{}^{2\theta}}\int_{x_1{}^\theta}^{2x_1{}^\theta}\int_{T_0}^{T}\left|L_y\left(\frac{1}{2}+it, w\right)C\left(-\frac{\theta}{2}-i\theta t\right)\right|dt\,dy \ll \Delta x^{1-4\varepsilon}+x^{(1+\theta)/2+6\varepsilon}.$$

We can shorten $L_y(s, w)$ by using the approximate functional equation of $\zeta(s, w)$ (see Chudakov [2]). We have

(3.21)
$$L_y\left(\frac{1}{2}+it, w\right) = \chi\left(\frac{1}{2}+it\right) \sum_{\frac{1}{3}v<k\leqq 3v} e^{2\pi ikw} k^{-1/2+it} + O(y^{-1/2}+y^{1/2}t^{1/2}+y^{-3/2}t^{1/2})$$

where $x_1^{\theta} \leqq y \leqq 2x_1^{\theta}$, $x_1^{\theta} \leqq t$, $v = t/2\pi y$ and $|\chi(1/2+it)| = 1$. (χ is the weight function occuring in the functional equation of the Riemann zeta function.) We get uniformly for $x_1^{\theta} \leqq t \leqq x_1$

$$\frac{1}{x_1^{\theta}} \int_{x_1^{\theta}}^{2x_1^{\theta}} \left|Ly\left(\frac{1}{2}+it, w\right)\right| d_y \ll \int_{\frac{t}{4\pi x_1^{\theta}}}^{\frac{t}{2\pi x_1^{\theta}}} \left|K_v\left(\frac{1}{2}+it, w\right)\right| \frac{dv}{v} + O(1) \tag{3.22}$$

where

$$K_v(s, w) = \sum_{\frac{1}{3}v<k\leqq 3v} e^{2\pi ikw} k^{-s}.$$

Taking

$$A(s) = \sum_{m\sim M} a_m m^{-s}, \quad B(s) = \sum_{\frac{x_1}{2M}\leqq n<\frac{x_2}{M}} b_n n^{-s} \tag{3.23}$$

and using the Perron formula again we get

$$C\left(-\frac{\theta}{2}-i\theta t\right) = \tag{3.24}$$

$$= \frac{1}{2\pi i} \int_{\frac{1+\theta}{2}-iT}^{\frac{1+\theta}{2}+iT} A\left(-\frac{\theta}{2}-i\theta t+z\right) B\left(-\frac{\theta}{2}-i\theta t+z\right) \frac{x_2^2-x_1^2}{2}\, dz + O\left(\frac{x_1^{1+(\theta/2)+\varepsilon}}{T}\right) \ll$$

$$\ll x_1^{(1+\theta)/2} \int_{-T}^{T} \left|A\left(\frac{1}{2}-i\theta t+i\tau\right) B\left(\frac{1}{2}-i\theta t+i\tau\right)\right| \frac{d\tau}{1+|\tau|} + \frac{x_1^{1+(\theta/2)+\varepsilon}}{T}$$

for $T_0 \leqq t \leqq T$. Substituting (3.22) and (3.24) into the left hand side of (3.20) and using the mean-value theorem of Dirichlet polynomials we get

$$\frac{\Delta}{x_1^{2\theta}} \int_{x_1^{\theta}}^{2x_1^{\theta}} \int_{T_0}^{T} \left|L_y\left(\frac{1}{2}+it, w\right) C\left(-\frac{\theta}{2}-i\theta t\right)\right| dt\, dy \ll \tag{3.25}$$

$$\ll \Delta_1^{1-(\theta/2)+\varepsilon} + \Delta x_1^{(1-\theta)/2} \int_{-T}^{T} \int_{T_0}^{T} \left|AB\left(\frac{1}{2}-i\theta t+i\tau\right)\right| \frac{dt\, d\tau}{1+|\tau|} +$$

$$+ x_1^{(1-\theta)/2} \int_{-T}^{T} \int_{\frac{1}{4}\pi}^{\frac{T}{2\pi x_1^{\theta}}} \int_{2\pi x_1^{\theta} v}^{4\pi x_1^{\theta} v} \left|K_v\left(\frac{1}{2}+it\right) AB\left(\frac{1}{2}-i\theta t+i\tau\right)\right| \frac{dt\, dv\, d\tau}{v(1+|\tau|)} \ll$$

$$\ll \Delta x_1^{1-4\varepsilon} + x_1^{(1+\theta)/2+6\varepsilon} + \Delta^{1/2} x_1^{1-(\theta/2)+3\varepsilon} + F^{1/2} x_1^{(1/2)+6\varepsilon} + F^{-1/2}\Delta^{1/2} x_1^{1+1\varepsilon}$$

where $F=M$ or $F=x_1/M$. (3.20) follows from (3.25) if M satisfies (3.2). The first part of the proof is finished.

The case of (3.3) is quite similar, we indicate only the changes. In this case we retain the condition $mn\equiv a\,(q)$, and so the new definition of $C(s)$ is

$$C(s) = \sum_{\substack{m\sim M\\ x_1<mn<x_2\\ mn\equiv a(q)}} a_m (mn)^{-s}. \tag{3.26}$$

Instead of (3.23) we have

$$A(s,\chi) = \sum_{m\sim M} a_m \chi(m) m^{-s}, \quad \mathscr{L}(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \tag{3.27}$$

and instead of (3.24) we have

$$C\left(-\frac{\theta}{2}-i\theta t\right) \ll \tag{3.28}$$

$$\ll \frac{x_1^{(1+\theta)/2}}{\varphi(q)} \sum_{\chi(q)} \int_{-T}^{T} \left| A\mathscr{L}\left(\frac{1}{2}-i\theta t+i\tau,\chi\right)\right| \frac{d\tau}{1+|\tau|} + \frac{x_1^{1+(\theta/2)+\varepsilon}}{1+|t|}.$$

After that, the argument runs parallel to the one given above, but at the end we have to use

$$\sum_{\chi(q)} \int_{-T}^{T} \left| \mathscr{L}\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \ll qT \log^4 qT \tag{3.29}$$

which is a simple consequence of the fourth power moment estimate for $\mathscr{L}(1/2+it,\chi)$ with primitive characters, see Montgomery [6]. The lemma is proved.

4. Auxiliary results

In order to prove our theorems we will compare $S(\mathscr{A}, x^{1/2})$ and $2\Delta S(\mathscr{B}, x^{1/2})$. This comparison will be based on sieve identities. This is the reason why we are interested in proving formulae of type

$$\sum_{m\sim M} a_m S(\mathscr{A}_m, z_m) = 2\Delta \sum_{m\sim M} a_m S(\mathscr{B}_m, z_m) + O(\Delta x^{1-\varepsilon} + x^{(1+\theta)/2+25\varepsilon}). \tag{4.1}$$

The wider the range of M and z_m are the stronger theorems we get. Since

$$\sum_{m\sim M} a_m S(\mathscr{A}_m, z_m) = \sum_{\substack{m\sim M\\ mn\le x\\ mn\equiv a(q)\\ (mn, P(z_m))=1}} a_m \Phi(mn) \tag{4.2}$$

and

$$\sum_{m\sim M} a_m S(\mathscr{B}_m, z_m) = \sum_{\substack{m\sim M\\ mn\le x\\ mn\equiv a(q)\\ (mn, P(z_m))=1}} a_m, \tag{4.3}$$

Lemma 1 implicitly contains results of type (4.1). It seems to be a problem that m and n are not entirely independent of each other in (4.2) and (4.3) but this does not cause any confusion.

In this section we write Lemma 1 into the (4.1) form.

LEMMA 2. *Under* (2.1), (2.2) *and* (2.4) *we have*

$$\sum_{m\sim M} a_m S(\mathcal{A}_m, z_0) = 2\Delta \sum_{m\sim M} a_m S(\mathcal{B}_m, z_0)+O(\Delta x^{1-\varepsilon}+x^{(1+\theta)/2+9\varepsilon}) \tag{4.4}$$

for $M\ll \Delta x^{1-14\varepsilon}$.

PROOF. This is a trivial consequence of Lemma 1 in case of (3.3) coupled with (4.2) and (4.3). (We recall that $z_0=2$.)

LEMMA 3. *Under* (2.1), (2.2) *and* (2.4) *for*

$$\Delta^{-1}x^{35\varepsilon} \ll M \ll \Delta^2 x^{1-50\varepsilon} \quad or \quad \Delta^{-2}x^{50\varepsilon} \ll M \ll \Delta x^{1-35\varepsilon} \tag{4.5}$$

and for any $N\geqq 1$, $z_m\geqq 2$ *we have*

$$\sum_{\substack{m\sim M\\ n\sim N}} a_m b_n S(\mathcal{A}_{mn}, z_m) = \tag{4.6}$$

$$= 2\Delta \sum_{\substack{m\sim M\\ n\sim N}} a_m b_n S(\mathcal{B}_{mn}, z_m)+O(\Delta x^{1-\varepsilon}+x^{(1+\theta)/2+24\varepsilon}).$$

PROOF. We can assume (as we did in the proof of Lemma 1) that $a_m\geqq 0$, $b_n\geqq 0$ and $z_m\leqq x$ whenever $a_m\neq 0$. Denoting the least prime greater than z_m by p_m we have

$$S(\mathcal{C}, z_m) = S(\mathcal{C}, p_m). \tag{4.7}$$

We divide the m-s into disjoint classes $\mathcal{M}_1, \dots, \mathcal{M}_r$ having the following properties: If $m\in\mathcal{M}_j$ for some $j\leqq r$ then

$$\zeta_j \leqq p_m < \zeta_{j+1} \leqq \zeta_j(1+x^{-(5/4)\varepsilon}). \tag{4.8}$$

The ζ_j-s can be chosen in such a way that

$$r \ll x^{(3/2)\varepsilon}. \tag{4.9}$$

Of course, some of the $\mathcal{M}_j$-s may be empty.

If the interval $[\zeta_j, \zeta_{j+1})$ contains only one prime then

$$\sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{A}_{mn}, z_m) = \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{A}_{mn}, \zeta_j) =$$

$$= \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ (mn, P(\zeta_j))=1}} \sum_{\substack{k\leqq \frac{x}{mn}\\ kmn\equiv a(q)\\ (k, P(\zeta_j))=1}} a_m b_n \Phi(kmn) = \sum_{\substack{m\in\mathcal{M}_j\\ ml\leqq x\\ ml\equiv a(q)\\ (m, P(\xi_j))=1}} a_m c_l \Phi(ml)$$

where

$$c_l = \sum_{\substack{l=nk\\ n\sim N\\ (l, P(\zeta_j))=1}} b_n \tag{4.10}$$

and similarly

$$\sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{B}_{mn}, z_m) = \sum_{\substack{m\in\mathcal{M}_j\\ ml\leqq x\\ ml\equiv a(q)\\ (m, P(\zeta_j))=1}} a_m c_l$$

and Lemma 1 with $(5/2)\varepsilon$ in place of ε gives

$$\sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{A}_{mn}, z_m) = \tag{4.11}$$

$$= 2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{B}_{mn}; z_m) + O(\Delta x^{1-(5/2)\varepsilon} + x^{(1+\theta)/2+(45/2)\varepsilon}).$$

If the interval $[\zeta_j, \zeta_{j+1})$ contains more than one prime then

$$\zeta_j \geqq x^{(5/4)\varepsilon}. \tag{4.12}$$

For such an $\mathcal{M}_j$ we have

$$\sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{A}_{mn}, z_m) = \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ (mn, P(2m))=1}} \sum_{\substack{mnk\leqq x\\ mnk\equiv a(q)\\ (k, P(2m))=1}} a_m b_n \Phi(mnk) \leqq$$

$$\leqq \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ (m, P(2m))=1\\ (n, P(\zeta_j))=1}} \sum_{\substack{mnk\leqq x\\ mnk\equiv a(q)\\ (k, P(\zeta_j))=1}} a_m b_n \Phi(mnk) = 2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ (m, P(2m))=1\\ (n, P(\zeta_j))=1}} \sum_{\substack{mnk\leqq x\\ mnk\equiv a(q)\\ (k, P(\zeta_j))=1}} a_m b_n +$$

$$+O(\Delta x^{1-(5/2)\varepsilon} + x^{(1+\theta)/2+(45/2)\varepsilon})$$

by using Lemma 1 with $(5/2)\varepsilon$ in place of ε. Similarly we have

$$\sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{A}_{mn}, z_n) \geqq$$

$$\geqq 2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ (m, P(2m))=1\\ (n, P(\zeta_{j+1}))=1}} \sum_{\substack{mnk\leqq x\\ mnk\equiv a(q)\\ (k, P(\zeta_{j+1}))=1}} a_m b_n + O(\Delta x^{1-(5/2)\varepsilon} + x^{(1+\theta)/2+(45/2)\varepsilon}).$$

Trivially

$$2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ mnk\leqq x\\ mnk\equiv a(q)\\ (m, P(2m))=1\\ (nk, P(\zeta_{j+1}))=1}} a_m b_n \leqq 2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N}} a_m b_n S(\mathcal{B}_{mn}, z_m) \leqq 2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ mnk\leqq x\\ mnk\equiv a(q)\\ (m, P(\zeta_m))=1\\ (nk, P(\zeta_j))=1}} a_m b_n.$$

The difference between the left and the right hand side is less than

$$2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ pn\sim N\\ \zeta_j\leqq p<\zeta_{j+1}\\ mpnk\leqq x}} a_m b_n + 2\Delta \sum_{\substack{m\in\mathcal{M}_j\\ n\sim N\\ \zeta_j\leqq p<\zeta_{j+1}\\ mnpk\leqq x}} a_m n_n \ll \tag{4.13}$$

$$\ll \Delta x^{1+\eta} \sum_{m\in\mathcal{M}_j} \frac{a_m}{m} \sum_{\zeta_j\leqq p<\zeta_{j+1}} \frac{1}{p} \ll \Delta x^{1-(9/8)\varepsilon} \sum_{m\in\mathcal{M}_j} \frac{a_m}{m}.$$

This shows that (4.11) with the added error term (4.13) is valid for these kind of j-s. Summing over all the j-s we get (in view of (4.9))

$$\sum_{\substack{m\sim M\\ n\sim N}} a_m b_n S(\mathcal{A}_{mn}, z_m) = 2\Delta \sum_{\substack{m\sim M\\ n\sim N}} a_m b_n S(\mathcal{B}_{mn}, z_m) + $$
$$+O\left(\Delta x^{1-\varepsilon}+x^{(1+\theta)/2+24\varepsilon}+\Delta x^{1-(9/8)\varepsilon}\sum_{m\sim M}\frac{a_m}{m}\right)$$

which proves the lemma.

5. The sieve identities

We start with the Richert's Fundamental Identity. As its proof is very short we present it here for the sake of completeness. We note that from now on $\chi(n)$ and $\varphi(n)$ differ from their standard arithmetic meanings.

LEMMA 4. *Let $\chi(d)$ and $\varphi(d)$ be arbitrary functions with*

$$\chi(1) = 1. \tag{5.1}$$

We define $\bar{\chi}(d)$ as

$$\bar{\chi}(1) = 0, \quad \bar{\chi}(d) = \chi\left(\frac{d}{p(d)}\right) - \chi(d), \quad d > 1. \tag{5.2}$$

We have for any $z \geqq 2$

$$\sum_{d|P(z)} \mu(d)\varphi(d) = \sum_{d|P(z)} \mu(d)\chi(d)\varphi(d) + \sum_{d|P(z)} \mu(d)\bar{\chi}(d) \sum_{t|P(p(d))} \mu(t)\varphi(dt). \tag{5.3}$$

PROOF. $d|P(z)$ and $t|P(p(d))$ imply that $(d, t)=1$ and $dt|P(z)$. Collecting the terms $\delta=dt$ we get for the second sum on the right hand side of (5.3)

$$\sum_{d|P(z)} \mu(d)\bar{\chi}(d) \sum_{t|P(p(d))} \mu(t)\varphi(dt) = \sum_{\delta|P(z)} \mu(\delta)\varphi(\delta) \sum_{\substack{\delta=dt\\ t|P(p(d))}} \bar{\chi}(d). \tag{5.4}$$

Take $\delta=p_1 \ldots p_r$ where $p_1 > \ldots > p_r$. $\delta=dt$ and $t|P(p(d))$ are simultaneously satisfied iff $d=p_1 \ldots p_j$ for some $j \leqq r$. Using (5.1) and (5.2) we arrive at

$$\sum_{\substack{\delta=dt\\ t|P(p(d))}} \bar{\chi}(d) = \sum_{j=1}^{r} \bar{\chi}(p_1 \ldots p_j) = 1-\chi(\delta). \tag{5.5}$$

It is also true for $\delta=1$. (5.4) and (5.5) give (5.3), and this completes the proof.

LEMMA 5. *Under* (2.1), (2.2) *and* (2.4) *we have*

$$\sum_{m\sim M} a_m S(\mathcal{A}_m, z) = 2\Delta \sum_{m\sim M} a_m S(\mathcal{B}_m, z) + O(\Delta x^{1-\varepsilon} + x^{(1+\theta)/2+25\varepsilon}) \tag{5.6}$$

for

$$M \ll \Delta x^{1-36\varepsilon}, \quad 2 < z \leqq \Delta^3 x^{1-87\varepsilon}, \quad x^{-1/3+30\varepsilon} \leqq \Delta \leqq \frac{1}{2}. \tag{5.7}$$

PROOF. We can assume that $a_m=0$ whenever m has a prime factor less than z. Let $\mathscr{C}$ be a finite set and $y>1$ be a real number. We apply Lemma 4 with

$$\varphi(d)=\sum_{m\sim M} a_m|\mathscr{C}_{md}|,\quad \chi(d)=\begin{cases}1 & \text{if } d<y,\\ 0 & \text{otherwise.}\end{cases}\tag{5.8}$$

In this case we have

$$\bar{\chi}(d)=\begin{cases}1 & \text{if } d\geqq y \text{ and } \dfrac{d}{p(d)}<y,\\ 0 & \text{otherwise.}\end{cases}\tag{5.9}$$

The well-known identity (the sieve of Erathostenes)

$$S(\mathscr{C},z)=\sum_{d|P(z)}\mu(d)|\mathscr{C}_d|$$

coupled with (5.3) gives

$$\sum_{m\sim M} a_m S(\mathscr{C}_m,z)=\sum_{m\sim M}\sum_{\substack{d|P(z)\\ d<y}} a_m\mu(d)S(\mathscr{C}_{md},z_0)+\tag{5.10}$$
$$+\sum_{m\sim M}\sum_{\substack{d|P(z)\\ d\geqq y,\ d<yp(d)}} a_m\mu(d)S(\mathscr{C}_{md},p(d)).$$

(We remember that $z_0=2$.) As $a_m\neq 0$ implies $p(m)\geqq z>p(d)$, after collecting the $md=n$ terms (5.10) will have the shape

$$\sum_{m\sim M} a_m S(\mathscr{C}_m,z)=\sum_{n<2My} b_n S(\mathscr{C}_n,z_0)+\sum_{My\leqq n<2My2} c_n S(\mathscr{C}_n,p(n)).\tag{5.11}$$

We turn to the proof of (5.6). If $M\geqq\Delta^{-2}x^{51\varepsilon}$ then (5.6) immediately follows from Lemma 3. Next we can assume $M<\Delta^{-2}x^{51\varepsilon}$ and so $y=(\Delta^{-2}x^{51\varepsilon})/M>1$.

First we use (5.11) for $\mathscr{C}=\mathscr{A}$. The condition on z and the definition of y ensure that $2Myz\ll\Delta x^{1-36\varepsilon}$ and thus after an easy splitting up argument Lemmas 2 and 3 are applicable with 1.01ε in place of ε. This leads to

$$\sum_{m\sim M} a_m S(\mathscr{A}_m,z)=2\Delta\sum_{n<2My} b_n S(\mathscr{B}_n,z_0)+$$
$$+2\Delta\sum_{My\leqq n<2My2} c_n S(\mathscr{B}_n,p(n))+O(\Delta x^{1-\varepsilon}+x^{(1+\theta)/2+25\varepsilon}).$$

Finally using (5.11) again (for $\mathscr{C}=\mathscr{B}$) we get (5.6) which proves the lemma.

6. Proof of Theorem 1

Now we are in a position to prove Theorem 1. Clearly we have

$$\#\{p\leqq x,\ p\equiv a(q),\ \|p^\theta-w\|<\Delta\}=S(\mathscr{A},x^{1/2})+O(x^{1/2}).\tag{6.1}$$

The Bushtab identity (1.7) with $z'=\Delta^3x^{1-87\varepsilon}$ gives

$$S(\mathscr{A},x^{1/2})=S(\mathscr{A},z')-\sum_{z'\leqq p<x^{1/2}} S(\mathscr{A}_{p,p}).\tag{6.2}$$

Lemma 5 with $M=1$ and $a_1=1$ settles the first term. Since $\Delta \geqq x^{-1/4+31\varepsilon}$ implies that the interval $[z', x^{1/2})$ can be covered by intervals of type $[M, 2M)$ where M satisfies the condition (4.5) given in Lemma 3 with 1.01ε in place of ε, an easy splitting up argument gives

$$(6.3)\qquad S(\mathcal{A}, x^{1/2}) = 2\Delta S(\mathcal{B}, z') - 2\Delta \sum_{z' \leqq p < x^{1/2}} S(\mathcal{B}_{p,p}) + O(\Delta x^{1-\varepsilon} + x^{(1+\theta)/2+25\varepsilon}).$$

Using the Bushtab identity again we get

$$(6.4)\qquad S(\mathcal{A}, x^{1/2}) = 2\Delta S(\mathcal{B}, x^{1/2}) + O(\Delta x^{1-\varepsilon} + x^{(1+\theta)/2+25\varepsilon}) =$$
$$= 2\Delta\pi(x, q, a) + O(\Delta x^{1-\varepsilon} + x^{(1+\theta)/2+25\varepsilon}).$$

(6.1) and (6.4) prove Theorem 1.

7. Combinatorial decomposition

The considerations of the previous section were based on the fact that the intervals

$$(7.1)\qquad (1, \Delta^3 x),\quad (\Delta^{-1}, \Delta^2 x),\quad (\Delta^{-2}, \Delta x)$$

were overlapping (omitting some unimportant factors). However this does not happen when Δ is small. In the case $\Delta < x^{-1/4}$ there are gaps between the intervals of (7.1).

We will give such a combinatorial decomposition of $S(\mathcal{A}, x^{1/2})$ that most of the constituents can be estimated by our lemmas. It will be satisfactory provided all the "bad" subsums belonging to the gaps have only non-negative coefficients.

For the sake of simplicity we confine ourselves to the case $a=q=1$. As we are satisfied with an error term like $o(\Delta x/\log x)$, we will take ε as a sufficiently small fixed positive number and we will use our lemmas with 0.01ε in place of ε.

As Theorem 1 has already been proved we can assume that

$$(7.2)\qquad x^{-1/3+\varepsilon} \leqq \Delta \leqq x^{-1/4+\varepsilon}.$$

Our goal with the following curious definitions is to ensure that some sums of the form $\sum_d S(\mathcal{C}_d, p(d))$ can be separated into the form $\sum_m \sum_n S(\mathcal{C}_{mn}, p(m))$ where m has the acceptable size (7.1).

DEFINITION OF $\mathcal{P}$. First we define the sequence Q_j by

$$(7.3)\qquad Q_s < \ldots < Q_0 = x^{1/2},\quad Q_{j-1} = 2Q_j$$

and we choose s by

$$(7.4)\qquad \frac{1}{2}\Delta^3 x^{1-\varepsilon} \leqq Q_s < \Delta^3 x^{1-\varepsilon}.$$

$\mathcal{P}$ is the set of sequences $(P_1, \ldots, P_r)$ including the empty sequence where $P_1 \geqq \ldots \geqq P_r$ and all the $P-s$ are $Q-s$.

Note that

$$(7.5)\qquad s \leqq \log x.$$

Definition of "Bad" Numbers. We say that a real number P is *bad* if

$$(7.6)\qquad \Delta^3 x^{1-\varepsilon} < P \leqq \Delta^{-1} x^{\varepsilon} \quad \text{or} \quad \Delta^2 x^{1-\varepsilon} < P \leqq \Delta^{-2} x^{\varepsilon} \quad \text{or} \quad \Delta x^{1-\varepsilon} < P.$$

We say that P is *good* if it is not bad.

Definition of $\mathscr{P}_0$. Let R be a fixed integer $\geqq 1$. $\mathscr{P}_0$ is the set of sequences $(P_1, \dots, P_r) \in \mathscr{P}$ satisfying the following conditions

$$(7.7)\qquad \text{all subproducts of } P_1 \dots P_r \text{ are bad (except 1)},$$

$$(7.8)\qquad P_1 \dots P_{2j-1} P_{2j}^{\,2} < \Delta x^{1-\varepsilon} \quad \text{for all} \quad j \leqq [r/2],$$

$$(7.9)\qquad r \leqq 2R-1,$$

$$(7.10)\qquad P_1, \dots, P_r \text{ differ from each other whenever } r \text{ is even.}$$

(Note that the empty sequence is in $\mathscr{P}_0$.)

Definition of "Belonging". We say that a square free number d *belongs to the sequence* $(P_1, \dots, P_r)$ if $d = p_1 \dots p_r$ and

$$(7.11)\qquad P_j \leqq p_j < 2P_j \quad (j = 1, \dots, r).$$

$d=1$ belongs to the empty sequence. We say that d *belongs to the set* $\mathscr{P}_k$ iff d belongs to a sequence contained by $\mathscr{P}_k$. We indicate these facts by $d \in (P_1, \dots, P_r)$ or $d \in \mathscr{P}_k$, respectively.

Note that d belongs to $\mathscr{P}$ iff d is square free and all the prime factors of d are between Q_s and Q_0 (or $d=1$). If d belongs to $\mathscr{P}$ then there is exactly one sequence in $\mathscr{P}$ such that d belongs to it.

For a given finite set of integers $\mathscr{C}$ we apply Lemma 4 with $\varphi(d) = |\mathscr{C}_d|$, $z = Q_0 = x^{1/2}$ and

$$(7.12)\qquad \chi(d) = \begin{cases} 1 & \text{if} \quad d = d_0 d_1, \quad d_0 \in \mathscr{P}_0, \quad d_1 | P(Q_s), \\ 0 & \text{otherwise.} \end{cases}$$

We compute $\bar{\chi}(d)$. As in the case $p(d) < Q_s$ we have clearly $\chi(d/p(d)) = \chi(d)$. It is enough to investigate the case $d \in \mathscr{P}$. $\bar{\chi}(d)$ differs from zero iff

$$(7.13)\qquad \frac{d}{p(d)} \in \mathscr{P}_0 \quad \text{but} \quad d \notin \mathscr{P}_0,$$

$$(7.14)\qquad d \in \mathscr{P}_0 \quad \text{but} \quad \frac{d}{p(d)} \notin \mathscr{P}_0.$$

For $d \in (P_1, \dots, P_r)$ (7.13) is valid whenever $(P_1, \dots, P_{r-1})$ satisfies the conditions (7.7)—(7.10) and $(P_1, \dots, P_r)$ does not satisfy at least one of them. (7.14) is valid whenever $(P_1, \dots, P_r)$ satisfies the conditions (7.7)—(7.10) and $(P_1, \dots, P_{r-1})$ fails (7.10). We need the following definitions.

Definition of $\mathscr{P}_1$. $\mathscr{P}_1$ is the set of sequences $(P_1, \dots, P_r) \in \mathscr{P}$ not satisfying (7.7), satisfying (7.10) and the subsequence $(P_1, \dots, P_{r-1})$ satisfies (7.7)—(7.10).

DEFINITION OF $\mathcal{P}_2$. $\mathcal{P}_2$ is the set of sequences $(P_1, \dots, P_{2r})\in\mathcal{P}$ satisfying (7.7), not satisfying (7.8) and the subsequence $(P_1, \dots, P_{2r-1})$ satisfies (7.7)—(7.10).

DEFINITION OF $\mathcal{P}_3$. $\mathcal{P}_3$ is the set of sequences $(P_1, \dots, P_{2R})\in\mathcal{P}$ satisfying (7.7) and the subsequence $(P_1, \dots, P_{2R-1})$ satisfies (7.7)—(7.10).

DEFINITION OF $\mathcal{P}_4$. $\mathcal{P}_4$ is the set of sequences $(P_1, \dots, P_{2r})\in\mathcal{P}$ not satisfying (7.10) and the subsequence $(P_1, \dots, P_{2r-1})$ satisfies (7.7)—(7.10).

DEFINITION OF $\mathcal{P}_5$. $\mathcal{P}_5$ is the set of sequences $(P_1, \dots, P_{2r+1})\in\mathcal{P}$ satisfying (7.7)—(7.10) and the subsequence $(P_1, \dots, P_{2r})$ does not satisfy (7.10).

Note that these sets are not all disjoint but $\mathcal{P}_1$ and $\mathcal{P}_5$ are disjoint from the others. It is easy to check that (7.13) is valid for some d iff $d\in\mathcal{P}_1\cup\dots\cup\mathcal{P}_4$ and (7.14) is valid for some d iff $d\in\mathcal{P}_5$. Thus we have

$$\bar{\chi}(d)=\begin{cases} 1 & \text{if } d\in\mathcal{P}_1\cup\mathcal{P}_2\cup\mathcal{P}_3\cup\mathcal{P}_4,\\ -1 & \text{if } d\in\mathcal{P}_5,\\ 0 & \text{otherwise.}\end{cases} \tag{7.15}$$

By applying Lemma 4, (5.3) gives our main decomposition:

$$S(\mathcal{C}, Q_0) = \sum_{d\in\mathcal{P}_0}\mu(d)S(\mathcal{C}_d, Q_s)+\sum_{d\in\mathcal{P}_1}\mu(d)S(\mathcal{C}_d, p(d))+ \\ +\sum_{d\in\mathcal{P}_2\cup\dots\cup\mathcal{P}_5} S(\mathcal{C}_d, p(d)). \tag{7.16}$$

Using this for $\mathcal{C}=\mathcal{A}$, and omitting some non-negative terms, we get

$$S(\mathcal{A}, x^{1/2}) \geqq \sum_{d\in\mathcal{P}_0}\mu(d)S(\mathcal{A}_d, Q_s)+\sum_{d\in\mathcal{P}_1}\mu(d)S(\mathcal{A}_d, p(d)). \tag{7.17}$$

We are going to show that Lemmas 5 and 3 are applicable for the first and the second sum respectively. This is clear for the first sum since $Q_s<\Delta^3x^{1-\varepsilon}$ by (7.4) and $d<\Delta x^{1-\varepsilon}$ by (7.8). This is why we need the condition (7.8). When $(P_1, \dots, P_r)\in\mathcal{P}_1$, $P_1\dots P_r$ has at least one good subproduct. As $(P_1, \dots, P_{r-1})$ satisfies (7.7) i.e. $P_1\dots P_{r-1}$ has only bad subproducts, P_r is always a factor of any good subproduct of $P_1\dots P_r$. $P_1, \dots, P_{r-1}$ are all different because of (7.10) is true for $(P_1, \dots, P_{r-1})$ in case r is odd, and for $(P_1, \dots, P_r)$ in case r is even. If $P_r=P_{r-1}$ then P_{r-1} is always a factor of any good subproduct of $P_1\dots P_r$ since otherwise $P_1\dots P_{r-1}$ would posses a good subproduct contradicting $(P_1, \dots, P_{r-1})\in\mathcal{P}_0$. Thus any sequence $(P_1, \dots, P_r)\in\mathcal{P}_1$ can be divided into two disjoint subsequences say $\mathcal{M}$ and $\mathcal{N}$ where $\mathcal{M}\in\mathcal{P}$, $\mathcal{N}\in\mathcal{P}$; P_r is in $\mathcal{M}$, the product of the constituents of $\mathcal{M}$ is good and all the constituents of $\mathcal{M}$ differ from the constituents of $\mathcal{N}$. Thus we have

$$\sum_{d\in P_1,\dots,P_r)}\mu(d)S(\mathcal{C}_d, p(d)) = (-1)^r\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} S(\mathcal{C}_{mn}, p(m)), \tag{7.18}$$

and Lemma 3 is applicable. As $|\mathscr{P}_1|<\log^{2R} x$ by (7.5) from (7.17) we get

(7.19) $$S(\mathscr{A}, x^{1/2}) \geqq 2\Delta \sum_{d\in\mathscr{P}_0} \mu(d) S(\mathscr{B}_d, Q_s)+$$

$$+2\Delta \sum_{d\in\mathscr{P}_1} \mu(d) S(\mathscr{B}_d, p(d))+O\left(\frac{\Delta x}{\log^2 x}+x^{(1+\theta)/2+\varepsilon}\right)=$$

$$=2\Delta S(\mathscr{B}, x^{1/2})-2\Delta \sum_{d\in\mathscr{P}_2\cup\ldots\cup\mathscr{P}_5} S(\mathscr{B}_d, p(d))+O\left(\frac{\Delta x}{\log^2 x}+x^{(1+\theta)/2+\varepsilon}\right).$$

At the end of this section we show that some terms of the last sum are negligable. From now on we have to assume that

(7.20) $$Q_s > x^{0.001}$$

(which is obviously satisfied by $\Delta\geqq x^{-3/10}$). If $d\in\mathscr{P}_2\cup\ldots\cup\mathscr{P}_5$, $d=p_1\ldots p_r$ and $p_1\ldots p_{r-1}p_r^2>x\geqq p_1\ldots p_r$ then $S(\mathscr{B}_d, p(d))=1$. By (7.8) either $d=p_1p_2$ or $r\geqq 3$ and $p_1\ldots p_{r-1}<\Delta x^{1-\varepsilon}$. The contributions of $d<x^{1-\varepsilon}$ (and also $d>x$) are trivial. Thus for $r\geqq 3$ and $p_1\ldots p_r>x^{1-\varepsilon}$ we have $p_r>\Delta^{-1}$. But from (7.7) we also have $p_r\leqq \Delta^{-1}x^{\varepsilon}$ as $p_1>p_2>p_3>\Delta^{-1}x^{\varepsilon}$ implies $p_1>p_2>p_3>\Delta^2x^{1-\varepsilon}$ and so $d\geqq p_1p_2p_3> >\Delta^6x^{3-3\varepsilon}>x$ by (7.2). The contribution of the terms $d\in\mathscr{P}_2\cup\ldots\cup\mathscr{P}_5$, $dp(d)>x$ is less than

(7.21) $$\ll x^{1-\varepsilon}+\sum_{p_2<p_1<x^{1/2}} 1+\sum_{\substack{p_r<\ldots<p_1\\ p_1\ldots p_r\leqq x\\ \Delta^{-1}<p_r\leqq\Delta^{-1}x^{\varepsilon}}} 1\ll$$

$$\ll\frac{x}{\log^2 x}+\sum_{\Delta^{-1}<p\leqq\Delta^{-1}x^{\varepsilon}} S(\mathscr{B}_{p,p})\ll\frac{\varepsilon x}{\log x}.$$

In the last step we were using the following consequence of Selberg's upper bound sieve:

(7.22) $$S(\mathscr{B}_d, p(d))\ll\frac{x}{d\log x}$$

whenever $dp(d)\leqq x$ and $p(d)>x^{0.001}$. Of course, the implied constant in (7.21) is independent of ε.

As $d\in\mathscr{P}_4\cup\mathscr{P}_5$ implies that there is at least one pair of prime divisors p_j, p_{j+1} of d with $P_j\leqq p_{j+1}<p_j<2P_j$ by (7.20), (7.22) and (7.5) we have

(7.23) $$\sum_{\substack{d\in\mathscr{P}_4\cup\mathscr{P}_5\\ dp(d)\leqq x}} S(\mathscr{B}_d, p(d))\ll\frac{x}{\log x}\sum_{\substack{d\in\mathscr{P}_4\cup\mathscr{P}_5\\ dp(d)\leqq x}}\frac{1}{d}\ll$$

$$\ll\frac{x}{\log x}\sum_{j=1}^{s}\left(\sum_{Q_j\leqq p<2Q_j}\frac{1}{p}\right)^2\left(1+\sum_{Q_s\leqq p<x^{1/2}}\frac{1}{p}\right)^{2R-2}\ll\frac{x}{\log^2 x}.$$

Finally we consider the case $dp(d)\leqq x$, $p(d)>x^{0.001}$ and there exists a divisor t of d with $x^c\leqq t\leqq x^{c+2\varepsilon}$ for a fixed $c\geqq 0.001$. We can write $d=puv$ where $x^c\leqq$

$\leqq pu \leqq x^{c+2\varepsilon}$, $p(u) > p$ or $u=1$. The contribution of this part is less than

$$(7.24)\qquad \ll \frac{x}{\log x}\sum \frac{1}{d} \ll \frac{x}{\log x}\sum \frac{1}{v}\sum \frac{1}{u}\sum \frac{1}{p} \ll$$

$$\ll \frac{\varepsilon x}{\log x}\sum \frac{1}{v}\sum \frac{1}{u} \ll \frac{\varepsilon x}{\log x}\left(1+\sum_{Q_s \leqq p < x^{1/2}} \frac{1}{p}\right)^{2R-1} \ll \frac{\varepsilon x}{\log x}.$$

Writing (7.21), (7.23) and (7.24) into (7.19) we get our most important inequality

$$(7.25)\qquad S(\mathscr{A}, x^{1/2}) \geqq 1.99\Delta\pi(x) - 2\Delta \sum_{d \in \mathscr{D}} S(\mathscr{B}_{d, p(d)}) + O(x^{(1+\theta)/2+\varepsilon})$$

whenever $1/2 \geqq \Delta \geqq x^{-0.33}$ and where $\mathscr{D}$ is the set of d-s satisfying $d = p_1 \dots p_{2r}$ with the following conditions:

$$(7.26)\qquad r \leqq R, \quad \Delta^3 x \leqq p_{2r} < \dots < p_1 < x^{1/2},$$

$$(7.27)\qquad \Delta^3 x < t \leqq \Delta^{-1} \quad \text{or} \quad \Delta^2 x < t \leqq \Delta^{-2} \quad \text{or} \quad \Delta x < t$$

for all $t \mid p_1 \dots p_{2r}$, $t \neq 1$,

$$(7.28)\qquad p_1 \cdots p_{2j-1} p_{2j}^2 \leqq \Delta x \quad \text{for} \quad j < r,$$

$$(7.29)\qquad p_1 \cdots p_{2r-1} p_{2r}^2 > \Delta x \quad \text{for} \quad r \neq R,$$

$$(7.30)\qquad p_1 \cdots p_{2r-1} p_{2r}^2 \leqq x \quad \text{for} \quad r \leqq R.$$

8. Proof of Theorem 2

Our starting point is (7.25). We will use the following sharper form of (7.22) (see Theorem 3.6 of [3]):

$$(8.1)\qquad S(\mathscr{B}_d, p(d)) \leqq \frac{x}{d \log p(d)} + O\left(\frac{x}{d \log^2 x}\right)$$

whenever $dp(d) \leqq x$, $p(d) > x^{0.001}$.

We will show that for $\Delta \geqq x^{-3/10}$ $\mathscr{D}$ is a relatively small set — combining this with (8.1) gives that the sum in (7.25) is small enough. It would be, of course, more desirable to find a direct treatment for investigating the sum itself without determining the possible choices of d.

We choose $R=3$. As $\mathscr{D}$ has a monotonic property with respect to Δ, for $\Delta \geqq x^{-3/10}$ we have $\mathscr{D} \subset \mathscr{D}^*$ where $\mathscr{D}^*$ is the set of p_1p_2, $p_1p_2p_3p_4$ and $p_1p_2p_3p_4p_5p_6$ satisfying the conditions (7.26)—(7.30) with $\Delta = x^{-3/10}$ and $R=3$. We have

$$(8.2)\qquad S(\mathscr{A}, x^{1/2}) \geqq 1.98\Delta\pi(x) - \sum_{d \in \mathscr{D}^*} \frac{2\Delta x}{d \log p(d)} + O(x^{(1+\theta)/2+\varepsilon})$$

whenever $1/2 \geqq \Delta \geqq x^{-3/10}$.

First we consider the d-s having six prime factors. From $p_1 > \dots > p_6 \geqq x^{0.1}$ we have $p_1p_2p_3 > x^{0.3}$ but as $p_1p_2p_3$ is bad we have $p_1p_2p_3 > x^{0.4}$. From this $p_1p_2p_3p_4p_5 > x^{0.6}$ but as $p_1p_2p_3p_4p_5$ is also bad we have $p_1p_2p_3p_4p_5 > x^{0.7}$. This contradicts $p_1p_2p_3p_4^2 \leqq x^{0.7}$ showing that $\mathscr{D}^*$ has no elements having six prime factors.

Next we consider the d-s having four prime factors. A treatment similar to the one above shows that $\mathcal{D}^*$ has no elements having four prime factors with $p_1 > x^{0.3}$. For all $p_1p_2p_3p_4 \in \mathcal{D}^*$ we have

$$x^{0.1} \leqq p_4 < \ldots < p_1 \leqq x^{0.3}. \tag{8.3}$$

As $p_1p_2p_3p_4$ is bad we distinguish two cases

$$p_1p_2p_3p_4 > x^{0.7} \tag{8.4}$$

or

$$p_1p_2p_3p_4 \leqq x^{0.6}. \tag{8.5}$$

(8.4) implies that $p_1p_2 > x^{0.4}$ thus $p_1 > x^{0.2} \cdot p_1p_2^2 \leqq x^{0.7}$ implies $p_1p_2p_3 \leqq x^{0.6}$ and thus $p_2p_3 \leqq x^{0.4}$ which implies

$$p_2p_3 \leqq x^{0.3}. \tag{8.6}$$

It is easy to check that (8.3), (8.4) and (8.6) imply (7.26)—(7.30). Similarly we can get that (8.3),

$$p_1p_2 \leqq x^{0.3} \tag{8.7}$$

and

$$p_1p_2p_3p_4^2 > x^{0.7} \tag{8.8}$$

are necessary and sufficient for the satisfaction of (7.26)—(7.30) and (8.5).

We can consider the case $d = p_1p_2$ in a similar manner, and we can conclude that (7.26)—(7.30) are satisfied iff one of the following three cases holds:

$$x^{0.1} \leqq p_2 < p_1 \leqq x^{0.3}, \tag{8.9}$$

$$p_1p_2^2 > x^{0.7} \tag{8.10}$$

or

$$x^{0.4} \leqq p_1 < x^{0.5} \quad \text{and} \quad x^{0.1} \leqq p_2 \leqq x^{0.3}, \tag{8.11}$$

$$p_1p_2 > x^{0.7}, \tag{8.12}$$

$$p_1p_2^2 \leqq x \tag{8.13}$$

or

(8.11), (8.10) and

$$p_1p_2 \leqq x^{0.6}. \tag{8.14}$$

Using the elementary facts that

$$\sum_{x^a < p \leqq x^b} \frac{1}{p} \sim \int_a^b \frac{d\alpha}{\alpha}, \quad \sum_{x^a < p \leqq x^b} \frac{1}{p \log p} \sim \frac{1}{\log x} \int_a^b \frac{d\alpha}{\alpha^2}$$

we get

$$S(\mathcal{A}, x^{1/2}) \geqq \tag{8.15}$$

$$\geqq \left(1.97 - 2\iint \frac{d\alpha\, d\beta}{\alpha\beta^2} - 2\iiiint \frac{d\alpha\, d\beta\, d\gamma\, d\delta}{\alpha\beta\gamma\delta^2}\right) \Delta\pi(x) + O(x^{(1+\theta)/2+\varepsilon})$$

where the two dimensional integral is taken over the regions

$$\begin{aligned} &T_1: \quad \frac{7}{30} \leqq \alpha \leqq \frac{3}{10}, \quad \frac{7}{20} - \frac{\alpha}{2} \leqq \beta \leqq \alpha, \\ &T_2: \quad \frac{4}{10} \leqq \alpha \leqq \frac{5}{10}, \quad \frac{7}{10} - \alpha \leqq \beta \leqq \frac{1}{2} - \frac{\alpha}{2}, \\ &T_3: \quad \frac{4}{10} \leqq \alpha \leqq \frac{5}{10}, \quad \frac{7}{20} - \frac{\alpha}{2} \leqq \beta \leqq \frac{6}{10} - \alpha, \end{aligned} \tag{8.16}$$

and the four dimensional integral is taken over the regions

$$T_4: \begin{cases} \dfrac{7}{50} \leqq \alpha \leqq \dfrac{3}{70}, \quad \dfrac{7}{40} - \dfrac{\alpha}{4} \leqq \beta \leqq \alpha, \\ \dfrac{7}{30} - \dfrac{\alpha+\beta}{3} \leqq \gamma \leqq \beta, \quad \dfrac{7}{20} - \dfrac{\alpha+\beta+\gamma}{2} \leqq \delta \leqq \gamma, \end{cases} \tag{8.17}$$

$$T_5: \begin{cases} \dfrac{3}{20} \leqq \alpha \leqq \dfrac{1}{6}, \quad \dfrac{7}{40} - \dfrac{\alpha}{4} \leqq \beta \leqq \dfrac{3}{10} - \alpha, \\ \dfrac{7}{30} - \dfrac{\alpha+\beta}{3} \leqq \gamma \leqq \beta, \quad \dfrac{7}{20} - \dfrac{\alpha+\beta+\gamma}{2} \leqq \delta \leqq \gamma, \end{cases}$$

$$T_6: \begin{cases} \dfrac{1}{4} \leqq \alpha \leqq \dfrac{3}{10}, \quad \dfrac{7}{30} - \dfrac{\alpha}{3} \leqq \beta \leqq \dfrac{3}{20}, \\ \dfrac{7}{20} - \dfrac{\alpha+\beta}{2} \leqq \gamma \leqq \beta, \quad \dfrac{7}{10} - \alpha - \beta - \gamma \leqq \delta \leqq \gamma, \end{cases}$$

$$T_7: \begin{cases} \dfrac{1}{4} \leqq \alpha \leqq \dfrac{3}{10}, \quad \dfrac{3}{20} \leqq \beta \leqq \alpha - \dfrac{1}{10}, \\ \dfrac{7}{20} - \dfrac{\alpha+\beta}{2} \leqq \gamma \leqq \dfrac{3}{10} - \beta, \quad \dfrac{7}{10} - \alpha - \beta - \gamma \leqq \delta \leqq \gamma. \end{cases}$$

The integrals over T_1, T_2 and T_3 can be calculated with elementary integrals. The integrals over T_4, T_5, T_6 and T_7 can be estimated by increasing the ranges into parallelepiped. Easy numerical calculations give Theorem 2.

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CONSTRUCTION OF MINIMAL SUFFICIENT OR PAIRWISE SUFFICIENT σ-FIELD

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1. Introduction

The notion of sufficiency is one of the fundamental concepts in the mathematical statisfics. In the dominated case the class of sufficient subfields has very nice properties, in particular there exists a minimal sufficient σ-field, whereas this does not hold for arbitrary statistical spaces. In order to preserve this property, generalizations of domination have been introduced. These are the notion of compactness introduced by T. S. Pitcher [16], the notion of coherent statistical space due to M. Hasegawa and M. D. Perlmann [10], the notion of weakly dominated class of measures defined by D. Mussmann [14]. These properties are practically the same.

In this paper we consider a condition of topological nature which is also equivalent to these notions and which enables us to construct the minimal sufficient σ-field. Furthermore, it turns out that this condition is equivalent to a pair of conditions one of which is quite intuitive and implies the existence of minimal sufficient σ-field, while the other is less suggestive but implies that — roughly speaking — a σ-field containing another sufficient σ-field is also sufficient.

2. The uniform structure

Let Ω be a set, $\mathscr{A}$ a field consisting of some subsets of Ω. Let P be a nonnegative, finitely additive set function defined on $\mathscr{A}$, such that $P(\Omega)=1$. The members of $\mathscr{A}$ can be divided into P-equivalence classes. Denote the equivalence class of an event $A\in\mathscr{A}$ by the symbol $\tilde{A}$, and the set of equivalence classes by the symbol $\mathfrak{A}$. As the field $\mathscr{A}$ is a Boolean algebra of subsets, the space $\mathfrak{A}$ can be regarded as a Boolean algebra, too. Moreover, by means of the set function P defined on $\mathscr{A}$ we can define a function on the set $\mathfrak{A}$ in a natural way; this will be also denoted by P.

On this space one can define a metric as follows: $\varrho(\tilde{A}, \tilde{B})=P(A\cdot B)$, where $A\cdot B$ denotes the symmetric difference of the events A and B, i.e. $A\cdot B=(A\setminus B)+(B\setminus A)$.

The following proposition is true:

PROPOSITION 1. *The space $\mathfrak{A}$ is metrically complete if and only if the space $\mathfrak{A}$ as a Boolean algebra is σ-complete and the function P is σ-additive on the space $\mathfrak{A}$.*

PROOF. Suppose that $\mathfrak{A}$ is metrically complete. Let $(\tilde{A}_n)_{n\in N}$ be an increasing sequence of equivalence classes. We will show that there exists an equivalence class $\tilde{A}$ such that $\tilde{A}=\bigvee_{n\in N} A_n$, and $P(\tilde{A})=\lim_{n\to\infty} P(A_n)$. The sequence $\tilde{A}_n$ is a Cauchy sequence. For, if $d=\sup P(A_n)$ and $\varepsilon>0$ is an arbitrary positive number, then there exists a number n_0 such that $P(A_{n_0})>d-\varepsilon$. For every pair of indices $n>m>n_0$ we have

$$\varrho(\tilde{A}_n, \tilde{A}_m) = P(A_n \circ A_m) = P(A_n \setminus A_m) \leqq d - P(A_{n_0}) < \varepsilon.$$

Since $\mathfrak{A}$ is metrically complete, there exists an equivalence class $\tilde{A}$ such that $\varrho(\tilde{A}, \tilde{A}_n)\to\to 0$, i.e. $P(A\circ A_n)=P(A\setminus A_n)+P(A_n\setminus A)\to 0$. The sequence $\tilde{A}_n$ increases, so $P(A_n\setminus A)=0$ for every $n\in N$. This means that $\tilde{A}_n<\tilde{A}$ in the Boole-algebra $\mathfrak{A}$, and $P(A_n)\leqq P(A)$ for every $n\in N$. On the other hand $P(A)\leqq P(A\setminus A_n)+P(A_n)$; letting n tend to infinity we get $P(A)\leqq d$. Summing up, $P(A)=d$ and so $\tilde{A}=\bigvee_{n\in N}\tilde{A}_n$.

The converse is a well-known measure theoretic fact. See Halmos [9]. □

REMARK. As the Boolean algebra $\mathfrak{A}$ satisfies the countable chain condition, since $P(\Omega)=1$, the Boolean algebra $\mathfrak{A}$ is complete if it is σ-complete.

Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a statistical space, i.e. Ω is a set, $\mathscr{A}$ is a σ-field (for simplicity) consisting of subsets of Ω, $\mathscr{P}$ is a family of σ-additive (for simplicity) probability measures defined on $\mathscr{A}$. Furthermore, since it is irrevelant from the point of view of the notions to be introduced whether the class of measures $\mathscr{P}$ is closed under linear convex combinations or not, we suppose to simplify the description that if $(P_i)_{i=1,\dots,n}\subset\subset\mathscr{P}$, $(c_i)_{i=1,\dots,n}\subset\mathbf{R}$, $c_i\geqq 0$ $i=1,\dots,n$, $\Sigma c_i=1$ then $\Sigma c_iP_i\in\mathscr{P}$. First we will define the equivalence classes, then a uniform structure on these equivalence classes, as in the case of a single measure P.

Let $\mathscr{N}(\mathscr{P})=\{A\in\mathscr{A}\colon P(A)=0$ for every $P\in\mathscr{P}\}$. We say that the events A and B belong to the same equivalence class if $A\circ B\in\mathscr{N}(\mathscr{P})$. Denote by $\mathfrak{A}$ the set of equivalence classes; this can be made into a Boole-algebra. $\tilde{A}<\tilde{B}$ means $A\setminus B\in\in\mathscr{N}(\mathscr{P})$. Now every $P\in\mathscr{P}$ defines a semi-metric on the space $\mathfrak{A}$ — in the same manner as above —, denote this by ϱ_P. The uniform structure induced by these semi-metrics will be denoted by $\mathscr{U}$. (cf. Császár [5].) I.e., for every $P\in\mathscr{P}$ and $\varepsilon>0$ let $U(P,\varepsilon)=\{(\tilde{A},\tilde{B})\colon \varrho_p(\tilde{A},\tilde{B})<\varepsilon\}$; these generate the uniform neighborhood-base of the uniform structure $\mathscr{U}$.

The *fundamental assumption* of this paper is the following:

The space $\mathfrak{A}$ with the uniform structure $\mathscr{U}$ is complete. (Shortly, $\mathfrak{A}$ is $\mathscr{U}$-complete.)

Precisely, this means that if $(\tilde{A}_i)_{i\in I}\subset\mathfrak{A}$ is a generalized Cauchy sequence (Cauchy net) for every semi-metric ϱ_P, $P\in\mathscr{P}$, then there exists an event $A\in\mathscr{A}$, such that $\tilde{A}_i$ converges to $\tilde{A}$ for every semi-metric ϱ_P. This condition implies that the space $\mathfrak{A}$ has very beautiful and useful properties.

First we prove that this property is a generalization of the domination. This is a simple consequence of the following assertion.

PROPOSITION 2. *Denote by $\mathscr{P}'$ the closure of $\mathscr{P}$ in the total variation norm-topology. Then $\mathscr{P}$ and $\mathscr{P}'$ induce the same uniform structure on the set $\mathfrak{A}$.*

PROOF. First observe that obviously $\mathscr{N}(\mathscr{P})=\mathscr{N}(\mathscr{P}')$. Consider a measure $P\in\mathscr{P}'$ and a number $\varepsilon>0$. There exists a measure $Q\in\mathscr{P}$ for which $\operatorname{Var}|P-Q|<\varepsilon/2$. In this case

$$\left\{(\tilde{A}, \tilde{B})\colon \tilde{A}, \tilde{B}\in\mathfrak{A}, Q(\tilde{A}, \tilde{B}) < \frac{\varepsilon}{2}\right\} \subset \{(\tilde{A}, \tilde{B})\colon \tilde{A}, \tilde{B}\in\mathfrak{A}, P(\tilde{A}, \tilde{B}) < \varepsilon\}.$$

This yields the required assertion. □

Consider now the dominated case. It is well-known (cf. Halmos—Savage [8]) that in this case there exists a countable subfamily $(P_i)_{i\in\mathbf{N}}\subset\mathcal{P}$ and a sequence of nonnegative real numbers $(c_i)_{i\in\mathbf{N}}$ for which $\Sigma c_i=1$, such that the measure $P^*=\mathcal{A}c_iP_i$ dominates the family $\mathcal{P}$. P^* belongs to $\mathcal{P}'$, and every element of P' is absolutely continuous with respect to P^*. Obviously ϱ_p^* and $\mathcal{P}'$ induce the same uniform structure. Using Proposition 2 we conclude that the space $\mathfrak{A}$ is metrizable. Perhaps it is worth mentioning that the condition $(\Omega, \mathcal{A}, \mathcal{P})$ is dominated is the same as $\mathfrak{A}$ is metrizable.

Examine the properties of $\mathfrak{A}$ under our fundamental assumption.

PROPOSITION 3. *If $\mathfrak{A}$ is $\mathcal{U}$-complete then $\mathfrak{A}$ is a complete Boole-algebra.*

PROOF. Let $(\tilde{A}_i)_{i\in I}\subset\mathfrak{A}$ be an increasing generalized sequence of equivalence classes. Then — as we have seen in the proof of Proposition 1 — this is a Cauchy sequence for every semi-metric ϱ_P, so it is a $\mathcal{U}$-Cauchy sequence. From the assumption it follows that there exists an equivalence class $\tilde{A}$ such that $\tilde{A}_i$ converges to $\tilde{A}$ in every semi-metric ϱ_P. So $P(A_i\setminus A)=0$ for every $i\in I$ and $P\in\mathcal{P}$, and $P(A\setminus A_i)\to 0$. This means $\tilde{A}=\bigvee_{i\in I}\tilde{A}_i$. □

From the proof of this proposition easily follows that if $\mathfrak{A}$ is $\mathcal{U}$-complete then every measure belonging to $\mathcal{P}$ is continuous for the lattice operations sup, inf, taken for arbitrary — not necessarily finite or countable — many members of $\mathfrak{A}$. Namely the following proposition can be easily proved.

PROPOSITION 4. *Suppose that $\mathfrak{A}$ is $\mathcal{U}$-complete. If $\mathfrak{B}\subset\mathfrak{A}$ is a subset closed for finite union, then*

$$P(\bigvee_{\tilde{A}\in\mathfrak{B}}\tilde{A})=\sup_{\tilde{A}\in\mathfrak{B}}P(\tilde{A}).$$

Outline of the proof. Since for every pair of equivalence classes $\tilde{A}$, $\tilde{B}$ belonging to $\mathfrak{B}$ the union $\tilde{A}\vee\tilde{B}$ belongs to $\mathfrak{B}$, the set $\mathfrak{B}$ is a generalized sequence for the lattice relation $<$. Similarly, as we have seen before it is a Cauchy sequence for every semi-metric ϱ_p. So, using the assumption, this generalized sequence is convergent to an equivalence class, which is exactly $\bigvee_{\tilde{A}\in\mathfrak{B}}\tilde{A}$.

Similarly, as we have seen before, the measure of the latter equals $\sup_{\tilde{A}\in\mathfrak{B}}P(\tilde{A})$. □

Let $P\in\mathcal{P}$. Obviously $\mathcal{N}(\mathcal{P})\subset\mathcal{N}(P)$, so one can construct the factor Boolean-algebra $\mathcal{N}(P|\mathcal{P})=\mathcal{N}(P)/\mathcal{N}(\mathcal{P})$.

COROLLARY 1. *If $\mathfrak{A}$ is $\mathcal{U}$-complete, then for every probability measure $P\in\mathcal{P}$ $\mathcal{N}(P|\mathcal{P})$ is a principal ideal in $\mathfrak{A}$.*

PROOF. $\mathcal{N}(P|\mathcal{P})$ is a subset of $\mathfrak{A}$ closed under finite union, so the preceding proposition assures

$$P(\bigvee_{\tilde{A}\in\mathcal{N}(P|\mathcal{P})}\tilde{A})=\sup_{\tilde{A}\in\mathcal{N}(P|\mathcal{P})}P(\tilde{A})=0.$$

Thus $\mathcal{N}(P|\mathcal{P})$ contains a maximal element, it is a principal ideal.

We can reformulate this corollary using the σ-field $\mathcal{A}$ instead of the Boolean-algebra $\mathfrak{A}$.

COROLLARY 2. *If $\mathfrak{A}$ is $\mathscr{U}$-complete, then for every $P\in\mathscr{P}$ there exists a set A_P such that $P(A_P)=1$ and if*

(*) *$B\subset A_p$ is any measurable set with $\mathscr{P}$-measure zero then $B\in\mathscr{N}(\mathscr{P})$, i.e. $Q(B)=0$ for every probability measure Q belonging to the family $\mathscr{P}$.*

A statistical space having property (*) will be called "parcellable". The set A_p or the equivalence class $\tilde{A}_p$ is called the parcel of the measure P.

We already have seen two consequences of $\mathscr{U}$-completeness, namely, the Boolean algebra $\mathfrak{A}$ is complete and the measures $P\in\mathscr{P}$ are continuous with respect to the sup, inf operations. Moreover, these properties imply the uniform completeness. Precisely, the following proposition holds.

PROPOSITION 5. *$\mathfrak{A}$ is $\mathscr{U}$-complete if and onty if*
(i) *$\mathfrak{A}$ is complete as a Boolean algebra and*
(ii) *$\mathfrak{A}$ is a parcellable.*

PROOF. The $\mathscr{U}$-completeness implies (i) and (ii) by Propositions 3 and 4. Now we prove the converse assertion.

Let $(\tilde{A}_i)_{i\in I}$ be a generalised $\mathscr{U}$-Cauchy sequence from $\mathfrak{A}$. Then for every $P\in\mathscr{P}$, $(\tilde{A}_i\wedge\tilde{A}_p)_{i\in I}$ is a ϱ_p-Cauchy sequence, consequently it has a ϱ_p limit; denote this by $\tilde{B}_p$.

Since $P(\Omega\setminus A_p)=0$, we may suppose that $B_p\subset A_p$. This B_p is "mod $\mathscr{N}(\mathscr{P})$" uniquely determined and $\tilde{B}_p\wedge\tilde{A}_Q=\tilde{B}_Q\wedge\tilde{A}_p$, for every pair $P,Q\in\mathscr{P}$. Let $\tilde{B}=\sup_{P\in\mathscr{P}}\tilde{B}_p$. This exists, since the Boole-algebra $\mathfrak{A}$ is complete. In this case

$$\tilde{B}\wedge\tilde{A}_Q=\sup_{P\in\mathscr{P}}(\tilde{B}_p\wedge\tilde{A}_Q)=\sup_{P\in\mathscr{P}}(\tilde{B}_p\wedge\tilde{A}_Q)=\tilde{B}_Q$$

for every $Q\in\mathscr{P}$. This means that $Q(B\setminus B_Q)=0$, so the generalised sequence $(\tilde{A}_i)_{i\in I}$ converges to the equivalence class $\tilde{B}$ in the uniform structure $\mathscr{U}$. Thus $\mathfrak{A}$ is $\mathscr{U}$-complete.

The properties examined above are closely related to other ones having been introduced to assure the existence of minimal sufficient or pairwise sufficient σ-field in a statistical space.

Namely E. Hasegawa and M. D. Perlmann [10] have introduced the notion of coherent statistical space. If $(g_P)_{P\in\mathscr{P}}$ is a class of random variables such that $0\leqq g_P\leqq 1$ then this class is called to be

(i) *finitely coherent* if for every finite subsystem $(P_i)_{i=1,\dots,n}\subset\mathscr{P}$ there exists a function $g_{P_1,\dots,P_n}$ such that $g_{P_1,\dots,P_n}=g_{P_i}$ P_i-a.e. $(i=1,\dots,n)$;

(ii) *coherent* if there exists a function g such that $g=g_P$ P-a.e. for every $P\in\mathscr{P}$.

The statistical space $(\Omega,\mathscr{A},\mathscr{P})$ is said to be *coherent* if every finitely coherent class of random variables is coherent. They have shown that this is a generalization of the domination. On the other hand, the statistical space $(\Omega,\mathscr{A},\mathscr{P})$ is coherent if the elements of $\mathscr{P}$ are discrete measures and the σ-field $\mathscr{A}$ contains every subset of Ω.

D. Mussmann [14] examined the notion of weakly dominated statistical space. The statistical space $(\Omega,\mathscr{A},\mathscr{P})$ is weakly dominated if there exists a localizable measure λ for which $\mathscr{N}(\lambda)\subset\mathscr{N}(\mathscr{P})$. (A measure λ is said to be *localizable* if every $A\in\mathscr{A}$ — for which $\lambda(A)\neq 0$ — has a subset $B\subset A$ with positive finite measure, and the lattice structure of the space $L^\infty(\Omega,\mathscr{A},\lambda)$ is complete.)

E. Siebert [17] proved that a statistical space is weakly dominated if and only if it is coherent. He has introduced a notion similar to weak domination, the notion of majorized statistical space. A statistical space is said to be *majorized* if there exists a measure λ on $(\Omega, \mathscr{A})$ for which $\mathscr{N}(\lambda)\subset\mathscr{N}(\mathscr{P})$, and the Radon—Nikodym derivate dP exists for every $P\in\mathscr{P}$.

These notions are in a close connection with the notion of uniform completeness and the existence of parcels.

PROPOSITION 6. *If the statistical space $(\Omega, \mathscr{A}, \mathscr{P})$ is majorized then it is parcellable.*

PROOF. Let λ be a measure which majorizes the measure family $\mathscr{P}$. Denote by f_p the Radon—Nikodym derivative $dP/d\lambda$. Let $A_p=(f_p>0)$. If $B\subset A_p$, $B\in\mathscr{A}$ and $P(B)=0$ then

$$\int_B f_p\, d\lambda = 0,$$

consequently $\lambda(B)=0$, so $B\in\mathscr{N}(\mathscr{P})$. Thus $\tilde{A}_p$ is the parcel belonging to the measure P. □

PROPOSITION 7. *The statistical space $(\Omega, \mathscr{A}, \mathscr{P})$ is weakly dominated if and only if $\mathfrak{A}$ is $\mathscr{U}$-complete.*

PROOF. We shall show that the $\mathscr{U}$-completeness is the same as to be coherent.

Suppose first that $\mathfrak{A}$ is $\mathscr{U}$-complete. Let $(g_p)_{P\in\mathscr{P}}$ be a class of random variables, $0\leqq g_p\leqq 1$, having the property that for every finite subsystem $(P_i)_{i=1,\dots,n}\subset\mathscr{P}$ the functions $(g_{p_i})_{i=1,\dots,n}$ have "common version". Since $\mathfrak{A}$ is parcellable we can consider the parcels $\tilde{A}_p$, $P\in\mathscr{P}$. On the event A_p the measure P dominates the class $\mathscr{P}$ so for every $Q\in\mathscr{P}$ $g_Q\chi_{A_p}=g_p\chi_{A_p}$ Q-a.e.

Consider the function g_p: $A_p\to\mathbf{R}$. Its inverse mapping determines a σ-homomorphism h_p from $\mathfrak{B}$ — the Borel subsets of the real line — into the principal ideal generated by $\tilde{A}_p$ of the Boolean algebra $\mathfrak{A}$. Since

$$g_Q\chi_{A_p}\chi_{A_Q} = g_p\chi_{A_p}\chi_{A_Q} \mod(\mathscr{N})$$

we have $h_p(B)\wedge\tilde{A}_Q=h_Q(B)\wedge\tilde{A}_p$ for every $B\in\mathscr{B}$. Define now a σ-homomorphism h: $\mathscr{B}\to\mathfrak{A}$ as follows:

$$h(B) = \bigvee_{P\in\mathscr{P}} h_p(B) \quad B\in\mathscr{B}.$$

(This exists since $\mathfrak{A}$ is complete as a Boolean algebra.) The homomorphism h can be induced by means of a function g: $\Omega\to\mathbf{R}$. (cf. R. Sikorski [18].)

Then, since $h(B)\wedge\tilde{A}_p=h_p(B)$ for every $B\in\mathscr{B}$, $P\in\mathscr{P}$ we have $g=g_p$ P-a.e. for every $P\in\mathscr{P}$. Thus $(\Omega, \mathscr{A}, \mathscr{P})$ is coherent.

Conversely, suppose that $(\Omega, \mathscr{A}, \mathscr{P})$ is coherent. Let $\mathfrak{B}\subset\mathfrak{A}$ be a subclass closed under finite union. In the factor Boolean algebra $\mathfrak{A}/\mathscr{N}(P|\mathscr{P})(=\mathscr{A}/\mathscr{N}(P))$ it has supremum i.e. there exists an event $B_p\in\mathscr{A}$ for which $P(\tilde{B}\setminus B_p)=0$ for every $\tilde{B}\in\mathfrak{B}$ and $P(B_p)=\sup_{\tilde{B}\in\mathfrak{B}} P(B)$. Consider the family of random variables $(\chi_{B_p})_{P\in\mathscr{P}}$. Let $(P_i)_{i=1,\dots,n}\subset\mathscr{P}$. In this case $Q=\frac{1}{n}\sum_{i=1}^{n} P_i$ belongs to $\mathscr{P}$, and $\chi_{B_Q}=\chi_{B_{P_i}}$ P_i-a.e. $(i=1, \dots, n)$. Thus, using our assumption, there exists a function g for which $g=\chi_{B_p}$

P-a.e. for every $P\in\mathcal{P}$. Obviously $P(B_p\circ(g=1))=0$, consequently

$$\widetilde{(g=1)} = \bigvee_{\tilde{B}\in\mathfrak{B}} \tilde{B}$$

which means that $\mathfrak{A}$ is complete as a Boolean algebra, and $P(g=1)=\sup\limits_{\tilde{B}\in\mathfrak{B}} P(B)$. Applying this for $\mathfrak{B}=\mathcal{N}(P|\mathcal{P})$ we get that $\mathfrak{A}$ is parcellable. □

LEMMA 1. *Let $\mathcal{F}\subset\mathcal{A}$ be a σ-field for which $\mathcal{N}(\mathcal{P})\subset\mathcal{F}$. In this case the factor Boole-algebra $\mathcal{F}/\mathcal{N}(\mathcal{P})$ is a subset of the uniform space $\mathfrak{A}$. The closure of this set (in the $\mathcal{U}$-uniform structure) can be obtained in the following way: $\mathcal{U}$-closure $[\mathcal{F}/\mathcal{N}(\mathcal{P})]=\overline{\overline{\mathcal{F}}}/\mathcal{N}(\mathcal{P})$ where $\overline{\overline{\mathcal{F}}}=\bigcup\limits_{P\in\mathcal{P}}\sigma(\mathcal{F},\mathcal{N}(P))$.*

PROOF. Let $(\tilde{A}_i)_{i\in I}\subset\mathcal{F}/\mathcal{N}(\mathcal{P})$ be a net, which converges to an equivalence class $\tilde{A}$. We have to prove $\tilde{A}\in\overline{\overline{\mathcal{F}}}/\mathcal{N}(\mathcal{P})$. For every $P\in\mathcal{P}$ the net $(\tilde{A}_i)_{i\in I}$ is a Cauchy net in the semi-metric ϱ_P, consequently there exists a $B_p\in\mathcal{F}$ for which $P(B_p\circ A_i)\to 0$. At the same time we know that $P(A\circ A_i)\to 0$. Thus $P(A\circ B_p)=0$, i.e. $A\in\sigma(\mathcal{F},\mathcal{N}(P))$ for every $P\in\mathcal{P}$.

Conversely, let $\tilde{A}\in\overline{\overline{\mathcal{F}}}/\mathcal{N}(\mathcal{P})$. For every $P\in\mathcal{P}$ there exists a $B_p\in\mathcal{F}$ for which $P(A\circ B_p)=0$. This means that an arbitrary $\mathcal{U}$-neighborhood of $\tilde{A}$ intersects the set $\mathcal{F}/\mathcal{N}(\mathcal{P})$, thus $\tilde{A}$ belongs to the closure of this set. □

LEMMA 2. *Suppose that $\mathfrak{A}$ is parcellable. Let $\mathcal{F}\subset\mathcal{A}$ be a σ-field such that $\mathcal{N}(\mathcal{P})\subset\mathcal{F}$ and for every measure $P\in\mathcal{P}$ the parcel A_p belongs to $\mathcal{F}$. In this case $\mathcal{U}$-closure $[\mathcal{F}/\mathcal{N}(\mathcal{P})]=\{\tilde{A}\in\mathfrak{A}|\tilde{A}\wedge\tilde{A}_p\in\mathcal{F}/\mathcal{N}(\mathcal{P})$ for every $P\in\mathcal{P}\}$.*

PROOF. Since $\tilde{A}_p$ is the parcel of the measure P we have

$$\{\tilde{A}\in\mathfrak{A}|\tilde{A}\wedge\tilde{A}_p\in\mathcal{F}/\mathcal{N}(\mathcal{P})\} = \sigma(\mathcal{F},\mathcal{N}(\mathcal{P}))/\mathcal{N}(\mathcal{P}).$$

Thus

$$\{\tilde{A}\in\mathfrak{A}|\tilde{A}\wedge\tilde{A}_p\in\mathcal{F}/\mathcal{N}(\mathcal{P}) \quad \text{for every } P\in\mathcal{P}\} =$$
$$= \bigcap_{P\in\mathcal{P}}\{\tilde{A}\in\mathfrak{A}|\tilde{A}\wedge\tilde{A}_p\in\mathcal{F}/\mathcal{N}(\mathcal{P})\} =$$
$$= \bigcap_{P\in\mathcal{P}}[\sigma(\mathcal{F},\mathcal{N}(\mathcal{P}))/\mathcal{N}(\mathcal{P})] = \mathcal{U}\text{-closure } [\mathcal{F}/\mathcal{N}(P)]. \ \square$$

It can occur that the space $\mathfrak{A}$ is not $\mathcal{U}$-complete. But if we extend simultaneously the space Ω and the σ-field $\mathcal{A}$ we can achieve the uniform completeness. Namely, the following lemma holds.

LEMMA 3. *Let $(\Omega,\mathcal{A},\mathcal{P})$ be a statistical space. There exists two other statistical spaces $(\Omega_1,\mathcal{A}_1,\mathcal{P}_1)$, $(\Omega_0,\mathcal{A}_0,\mathcal{P}_0)$ such that — roughly speaking — $(\Omega_0,\mathcal{A}_0,\mathcal{P}_0)$ is isomorphic to $(\Omega,\mathcal{A},\mathcal{P})$, and $(\Omega_1,\mathcal{A}_1,\mathcal{P}_1)$ is a uniformly complete extension of $(\Omega_0,\mathcal{A}_0,\mathcal{P}_0)$. Precisely, denote by $\mathfrak{A}_1$, $\mathfrak{A}_0$ the corresponding $\mathcal{N}(\mathcal{P}_1)$, $\mathcal{N}(\mathcal{P}_0)$ equivalence classes of the σ-fields $\mathcal{A}_1$, $\mathcal{A}_0$. Then there exists a measurable mapping $f\colon\Omega_1\to\Omega_0$ and a Boolean algebra isomorphism $g\colon\mathfrak{A}\to\mathfrak{A}_0$ such that*

(i) *$\mathcal{P}_0=\mathcal{P}\circ g^{-1}$,*

(ii) *f is onto, measurable, $\mathcal{P}_0=\mathcal{P}_1\circ f^{-1}$, and if $P_1,Q_1\in\mathcal{P}_1$, $P_1|_{f^{-1}(\mathcal{A}_0)}=Q_1|_{f^{-1}(\mathcal{A}_0)}$ then $P_1=Q_1$ on the whole σ-field $\mathcal{A}_1$,*

(iii) *the set* $\mathfrak{A}_1$ *is complete under the uniform structure* $\mathscr{U}_1$ *determined by the measures belonging to* $\mathscr{P}_1$.

PROOF. Consider the space $\mathfrak{A}$ with the uniform structure $\mathscr{U}$.

If $\mathfrak{A}$ is not $\mathscr{U}$-complete then we can take its $\mathscr{U}$-completion, that is a uniform space — denote this by $\mathfrak{A}_1$ — which is complete, and in which $\mathfrak{A}$ is everywhere dense. Since the Boolean algebra operations of $\mathfrak{A}$ are uniformly continuous, these can be extended to $\mathfrak{A}_1$. Furthermore, it can be easily checked that $\mathfrak{A}_1$ will be a Boolean algebra with these extended operations. Similarly, each function $P\in\mathscr{P}$ is uniformly continuous, so each has a continuous and hence uniformly continuous extension, which is a nonnegative, σ-additive (since finitely additive and continuous) function defined on the Boolean algebra $\mathfrak{A}_1$. Thus these extensions are measures on the Boole algebra $\mathfrak{A}_1$. Denote their family by $\mathscr{P}_1$. We choose the spaces Ω_0, Ω_1 as the spaces of the Stone representations of the Boolean algebras $\mathfrak{A}$, $\mathfrak{A}_1$, i.e. $\mathfrak{A}$, $\mathfrak{A}_1$ will be isomorphic to the σ-field generated by the clopen subsets of Ω_0, Ω_1 modulo the sets of the first category. Denote $\mathscr{A}_0$, $\mathscr{A}_1$ the σ-fields generated by the clopen subsets of the space Ω_0, Ω_1, and let $\mathscr{N}_0$, $\mathscr{N}_1$ the ideal of the sets of the first category. In this case $\mathfrak{A}_0=\mathscr{A}_0/\mathscr{N}_0$, $\mathfrak{A}_1=\mathscr{A}_1/\mathscr{N}_1$. The Boolean algebras $\mathfrak{A}$, $\mathfrak{A}_0$ are isomorphic. Denote this isomorphism by $g\colon \mathfrak{A}\to\mathfrak{A}_0$. Since $g^{-1}(\mathfrak{A}_0)=\mathfrak{A}\subset\mathfrak{A}_1$, there exists a measurable function $f\colon \Omega_1\to\Omega_0$ inducing the imbedding. (See Sikorski [18].) Define the measure family $\mathscr{P}_0$ on $\mathfrak{A}_0$ by means of the isomorphism g, and $\mathscr{P}_0$ and $\mathscr{P}_1$ on the σ-fields $\mathscr{A}_0$ and $\mathscr{A}_1$ in such a way that the measure of an event from $\mathscr{A}_0$ or $\mathscr{A}_1$ be equal to the measure of its equivalence class (mod $\mathscr{N}_0$ and mod $\mathscr{N}_1$, respectively). In this case $\mathscr{N}_0=\mathscr{N}(\mathscr{P}_0)$, $\mathscr{N}_1=\mathscr{N}(\mathscr{P}_1)$. The second part of (ii) is an immediate consequence of the fact that $\mathfrak{A}$ is dense in $\mathfrak{A}_1$. $\square$

3. Sufficiency

We begin this part with some well-known considerations. Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a statistical space. A σ-field is called sufficient if for every $A\in\mathscr{A}$ the conditional expectations $E_p(\chi_A|\mathscr{F})$ have a common version (cf. P. R. Halmos—L. J. Savage [8]). A subfield $\mathscr{F}\subset\mathscr{A}$ is said to be pairwise sufficient if it is sufficient for every pair of measures belonging to the class $\mathscr{P}$.

The sufficiency of a σ-field $\mathscr{F}$ is equivalent to the sufficiency of the σ-field generated by $\mathscr{F}$ and $\mathscr{N}(\mathscr{P})$. It is easily seen that a σ-field $\mathscr{F}$ is pairwise sufficient if and only if the σ-field $\bigcap_{P\in\mathscr{P}} \sigma(\mathscr{F}, \mathscr{N}(\mathscr{P}))=\overline{\mathscr{F}}$ is pairwise sufficient.

On the other hand, if $\mathscr{F}_1$ and $\mathscr{F}_2$ are two sufficient σ-fields but they do not necessarily contain $\mathscr{N}(\mathscr{P})$ then it can occur that the σ-field $\mathscr{F}_1\cap\mathscr{F}_2$ is not sufficient. For these reasons we will suppose without explicitly mentioning that if we say "$\mathscr{F}$ is a sufficient (or pairwise sufficient) σ-field", then $\mathscr{N}(\mathscr{P})\subset\mathscr{F}$. (I.e., we are speaking only about $\mathscr{P}$-complete sufficient σ-fields.) In this case we may form the factor Boole algebra $\mathscr{F}/\mathscr{N}(\mathscr{P})$ which is a subset of the Boole algebra $\mathfrak{A}$. Denote by S (or $S(\Omega, \mathscr{A}, \mathscr{P})$) the set of sufficient σ-fields and by V (or $V(\Omega, \mathscr{A}, \mathscr{P})$) the set of pairwise sufficient σ-fields. The following lemma will connect the notion of the sufficiency and that of the uniform structure.

LEMMA 4. *If* $\mathscr{F}$ *is a sufficient* σ*-field, then* $\mathscr{F}/\mathscr{N}(\mathscr{P})$ *is a closed subset of* $\mathfrak{A}$ *in the topology induced by the uniform structure* $\mathscr{U}$.

PROOF. Let $(\tilde{A}_i)_{i\in I}\subset\mathscr{F}/\mathscr{N}(\mathscr{P})$ be a generalized sequence which converges to an equivalence class $\tilde{A}\in\mathfrak{A}$. Since $\mathscr{F}$ is a sufficient σ-field, there exists a random variable which is a common version of the conditional probabilities $E_p(\chi_A|\mathscr{F})$ for every $P\in\mathscr{P}$. Denote this random variable by $E(\chi_A|\mathscr{F})$. We will prove that the event $\{E(\chi_A|\mathscr{F})=1\}\in\mathscr{F}$ determines the same equivalence class as A, implying $\tilde{A}\in\mathscr{F}/\mathscr{N}(\mathscr{P})$.

Since the generalized sequence $(A_i)_{i\in I}$ converges, it is a Cauchy sequence for every semi-metric ϱ_p. Since $A_i\in\mathscr{F}$, and $\mathscr{F}$ is a σ-field, by Proposition 1 there exists an event $B_p\in\mathscr{F}$ such that $P(A_i\circ B_p)\to 0$ for every $P\in\mathscr{P}$. On the other hand, $P(A_i\circ A)\to 0$, thus $P(A\circ B_p)=0$. This implies that $P(\{E(\chi_A|\mathscr{F})=1\}\circ B_p)=0$. Thus $P(A_i\circ\{E(\chi_A|\mathscr{F})=1\})\to 0$. This was to be proved. □

We have seen that the uniform completeness is equivalent to the properties that $\mathfrak{A}$ is parcellable and it is complete as a Boolean algebra. We examine further the consequences of these properties.

PROPOSITION 8. *Let $(\Omega,\mathscr{A},\mathscr{P})$ be a statistical space, $\mathscr{F}\subset\mathscr{A}$ be a sufficient σ-field. Suppose that $\mathscr{F}/\mathscr{N}(\mathscr{P})$ is parcellable. In this case $\mathfrak{A}$ is parcellable, too.*

PROOF. According to our assumption, for every $P\in\mathscr{P}$ there exists an equivalence class $\tilde{A}_p^{\mathscr{F}}\in\mathscr{F}/\mathscr{N}(\mathscr{P})$ for which $P(\tilde{A}_p^{\mathscr{F}})=1$ and

$$\tilde{A}_p^{\mathscr{F}}=\wedge\{\tilde{A}\in\mathscr{F}/\mathscr{N}(\mathscr{P}):\ P(\tilde{A})=1\}$$

where the $\wedge$ is taken in the Boolean algebra $\mathscr{F}/\mathscr{N}(\mathscr{P})$. Let $A\in\mathscr{A}$ be an event for which $P(A)=1$. Suppose that $\tilde{A}<\tilde{A}_p^{\mathscr{F}}$. Since $\mathscr{F}$ is sufficient the random variable $E(\chi_A|\mathscr{F})$ exists. Let $B=\{E(\chi_A|\mathscr{F})=1\}$. In this case $B\in\mathscr{F}$, $\tilde{B}<\tilde{A}_p^{\mathscr{F}}$ and $P(B)=1$. Using the definition of $\tilde{A}_p^{\mathscr{F}}$ we have $\tilde{B}=\tilde{A}_p^{\mathscr{F}}$ and consequently $\tilde{A}=\tilde{A}_p^{\mathscr{F}}$. Thus

$$\tilde{A}_p^{\mathscr{F}}=\wedge\{\tilde{A}\in\mathfrak{A}:\ P(\tilde{A})=1\}.\ \square$$

PROPOSITION 9. *Let $(\Omega,\mathscr{A},\mathscr{P})$ be a statistical space and let $\mathscr{F}\subset\mathscr{A}$ be a sufficient σ-field. Then if $(\tilde{A}_i)_{i\in I}\subset\mathscr{F}/\mathscr{N}(\mathscr{P})$ is a family of equivalence classes for which $\tilde{A}=\bigvee_{i\in I}\tilde{A}_i$ exists in the Boolean algebra $\mathfrak{A}$, we have $\tilde{A}\in\mathscr{F}/\mathscr{N}(\mathscr{P})$. In particular, if $\mathfrak{A}$ is complete as a Boolean algebra, then $\mathscr{F}/\mathscr{N}(\mathscr{P})$ is a complete subalgebra of this space.*

PROOF. Since $\mathscr{F}$ is sufficient the random variable $E(\chi_A|\mathscr{F})$ exists. Let $B=\{E(\chi_A|\mathscr{F})=1\}$. Then $\tilde{B}<\tilde{A}$. Since $A_i\in\mathscr{F}$ and $P(A_i\setminus A)=0$ for every $P\in\mathscr{P}$, consequently $P(A_i\setminus B)=0$. On the other hand, if $C\in\mathscr{F}$ and $P(A_i\setminus C)=0$ for every $P\in\mathscr{P}$ then $P(A\setminus C)=0$, consequently $P(B\setminus C)=0$. Thus $\tilde{B}=\bigvee_{i\in I}\tilde{A}_i$ where the supremum is taken in the Boolean algebra $\mathscr{F}/\mathscr{N}(\mathscr{P})$. Recalling $\tilde{B}<\tilde{A}$ this proves that $\tilde{B}=\tilde{A}$. □

In the sequel we shall deal with the existence of minimal pairwise sufficient or minimal sufficient σ-field. A σ-field $\mathscr{F}_0\in V$ is said to be *minimal* if for every $\mathscr{G}\in V$ we have $\mathscr{F}_0\subset\overline{\mathscr{G}}$ where the bar denotes the same as in Lemma 1. A σ-field $\mathscr{F}\in S$ is said to be *minimal* if for every $\mathscr{G}\in S$ we have $\mathscr{F}\subset\mathscr{G}$. (Notice that if $\mathscr{G}\in S$ then $\overline{\mathscr{G}}=\mathscr{G}$ by Lemmas 1 and 4.)

Let $P,Q\in\mathscr{P}$. Consider the generalized Radon—Nikodym decomposition (i.e. the Lebesgue-decomposition and the Radon—Nikodym derivative of the absolutely continuous part) of the measure P with respect to Q. According to this there exists a set N with Q-measure zero and a random variable X such that for every $A\in\mathscr{A}$ we have

$$P(A)=\int_{A\setminus N} X\,dQ+P(A\cap N).$$

PROPOSITION 10. *If $\mathscr{F}\in\mathscr{V}$ then the above random variable X can be chosen to be $\mathscr{F}$ measurable and the set N to belong to $\mathscr{F}$.*

PROOF. Since $\mathscr{F}$ is sufficient with respect to the class of measures $(1/2(P+Q), Q)$ the Radon—Nikodym derivative $Y=\dfrac{dP}{d(1/2(P+Q))}$ can be chosen to be $\mathscr{F}$-measurable (cf. P. R. Halmos—L. J. Savage [8]). Let $N=(Y=2)$ and

$$X=\begin{cases}\dfrac{Y}{2-Y} & \text{on the set } \Omega\setminus N,\\ 0 & \text{on the set } N.\ \square\end{cases}$$

Suppose now that $\mathfrak{A}$ is parcellable. For every $P,\ Q\in\mathscr{P}$ take such a version of the generalized Radon—Nikodym derivative dP/dQ which vanishes off the set A_Q. This can be done since $Q(\Omega\setminus A_Q)=0$. Denote by $\mathscr{F}_{p,Q}$ the σ-field generated by this Radon—Nikodym derivate and the ideal $\mathscr{N}(\mathscr{P})$. Let

$$(*)\qquad \mathscr{F}_0=\sigma(A_Q,\mathscr{F}_{p,Q},\ P,Q\in\mathscr{P}).$$

THEOROM 1. *If $\mathfrak{A}$ is parcellable, then $\mathscr{F}_0$ is a minimal pairwise sufficient σ-algebra.*

PROOF. First we show that $\mathscr{F}_0$ is pairwise sufficient. Let $P_1,P_2\in\mathscr{P}$. Denote $Q=1/2(P_1+P_2)$. Since $A_Q\in\mathscr{F}_0$ and the Radon—Nikodym derivatives have $\mathscr{F}_0$ measurable versions we have that for every $A\in\mathscr{A}$

$$E_Q(\chi_A|\mathscr{F}_0)=E_{P_1}(\chi_A|\mathscr{F}_0|)\quad P_1\text{-a.e.}$$

$$E_Q(\chi_A|\mathscr{F}_0)=E_{P_2}(\chi_A|\mathscr{F}_0)\quad P_2\text{-a.e.}$$

Now let $\mathscr{G}\in V$. We shall show that $A_p\in\overline{\mathscr{G}}$. Since $\mathscr{G}$ is pairwise sufficient, in view of Proposition 10 for every $Q\in\mathscr{P}$ there exists an event $B_1\in\mathscr{G}$ for which $Q(B_1\circ(A_p\setminus A_Q))=0$ and another $B_2\in\mathscr{G}$ for which $Q(B_2\circ(A_p\cap A_Q))=0$. Consequently

$$A_p=(A_pA_Q)\cup(A_p\cap A_Q)\in\sigma(\mathscr{G},\mathscr{N}(Q)).$$

Thus $A_p\in\mathscr{G}$. On the other hand, on the set A_Q the measure Q dominates the class of measures $\mathscr{P}$, so on this event the ideal $\mathscr{N}(Q)$ is the same as the ideal $\mathscr{N}(\mathscr{P})$. Consequently — since $A_Q\in\overline{\mathscr{G}}$ — we have $\mathscr{F}_{P,Q}\subset\overline{\mathscr{G}}$. Summing up, $\mathscr{F}_0\subset\overline{\mathscr{G}}$. $\square$

Observe that $\mathscr{F}_0=\overline{\mathscr{F}}_0$ does not necessarily hold. Denote $\mathscr{F}_1=\overline{\overline{\mathscr{F}}}_0$. According to Lemma 2, we have

$$\mathscr{F}_1=\{A\in\mathscr{A}\,|\,\tilde A\wedge\tilde A_p\in\mathscr{F}_0/\mathscr{N}(\mathscr{P})\quad\text{for every } P\in\mathscr{P}\}.$$

THEOREM 2. *If $\mathfrak{A}$ is $\mathscr{U}$-complete, then $\mathscr{F}_1$ is the minimal sufficient σ-field.*

PROOF. It is enough to show that $\mathscr{F}$ is a sufficient σ-field since it is minimal pairwise sufficient (by Theorem 1) and obviously every sufficient σ-field is pairwise sufficient. Let $A\in\mathscr{A}$. Denote

$$\mathscr{F}|_{\tilde{A}_p} = \{B\in\mathscr{F}_1/\mathscr{N}(\mathscr{P})\,|\,\tilde{B} < \tilde{A}_p\}.$$

Since the conditional expectation $E_p(\chi_A|\mathscr{F}_1)$ is unique "mod $\mathscr{N}(\mathscr{P})$" on the event A_p consequently the inverse mapping of $E_p(\chi_A|\mathscr{F}_1)$: $A_p\to\mathbf{R}$ determines a homomorphism h_p: $\mathscr{B}\to\mathscr{F}|_{\tilde{A}_p}$ (where $\mathscr{B}$ denotes the Borel subsets of the real line). In this case $h_p(B)\wedge\tilde{A}_Q=h_Q(B)\wedge\tilde{A}_p$ for every $B\in\mathscr{B}$, $P,Q\in\mathscr{P}$. Let $h(B)=\bigvee_{P\in\mathscr{P}} h_p(B)$.

Since $\mathscr{F}_1/\mathscr{N}(\mathscr{P})$ is $\mathscr{U}$-closed and $(h_p(B))_{P\in\mathscr{P}}$ converges to $h(B)$ in the $\mathscr{U}$-uniform structure we have $h(B)\in\mathscr{F}_1/\mathscr{N}(\mathscr{P})$. Consequently there exists an $\mathscr{F}_1$ measurable random variable X: $\Omega\to\mathbf{R}$ which induces just this homomorphism. On the other hand $h(B)\wedge\tilde{A}_p=h_p(B)$ for every $B\in\mathscr{B}$, $P\in\mathscr{P}$ so X is the common version of the conditional expectations $E_p(\chi_A|\mathscr{F}_1)$. Thus $\mathscr{F}_1$ is a sufficient σ-field. □

COROLLARY 3. *Just as we have proved that $\mathscr{F}_1$ is a sufficient σ-field, it can be shown — under the assumption that $\mathfrak{A}$ is $\mathscr{U}$-complete — that if $\mathscr{G}$ is a σ-field which contains the σ-field $\mathscr{F}_1$ and for which $\mathscr{G}/\mathscr{N}(\mathscr{P})$ is a $\mathscr{U}$-closed subset of $\mathfrak{A}$ then $\mathscr{G}$ is a sufficient σ-field.*

Thus a σ-field $\mathscr{G}$ is sufficient if and only if it contains the σ-field $\mathscr{F}_1$ and its factor algebra — mod $\mathscr{N}(\mathscr{P})$ — is $\mathscr{U}$-closed. Observe that the assumption of this corollary can be weakened, since we used the $\mathscr{U}$-closedness to imply the lattice completeness when we defined the homomorphism h.

Thus a σ-field $\mathscr{G}$ is sufficient if and only if it contains the σ-field $\mathscr{F}_1$, and its factor lattice $\mathscr{G}/\mathscr{N}(\mathscr{P})$ is complete as a Boolean algebra. In the dominated case this means that a σ-field is sufficient if and only if it contains the minimal sufficient σ-field. This is well-known.

COROLLARY 4. *If $(\mathscr{G}_i)_{i\in I}$ is an arbitrary — not necessarily countable — sequence of sufficient σ-fields then the σ-field $\mathscr{G}=\bigcap_{i\in I}\mathscr{G}_i$ is also a sufficient σ-field.*

PROOF. Since $\mathscr{F}_1\subset\mathscr{G}_i$ for every $i\in I$, $\mathscr{F}_1\subset\mathscr{G}$. On the other hand, $\mathscr{G}/\mathscr{N}(\mathscr{P})=$ $=\bigcap_{i\in I}\mathscr{G}_i/\mathscr{N}(\mathscr{P})$, the sets $\mathscr{G}_i/\mathscr{N}(\mathscr{P})$ are $\mathscr{U}$-closed, thus their intersection is also $\mathscr{U}$-closed. This was to be proved. □

REMARK. We will use the notation of Lemma 3. Suppose that $\mathfrak{A}$ is not $\mathscr{U}$-complete. According to Lemma 3 there exists a uniformly complete extension $\mathfrak{A}_1$ of the space $\mathfrak{A}$ which can be regarded as the equivalence classes of a suitable statistical space $(\Omega_1,\mathscr{A}_1,\mathscr{P}_1)$. Here the measures $P\in\mathscr{P}$ are in a one-to-one correspondence with measures $\bar{P}\in\mathscr{P}_1$ established by the mappings g and f.

Given a sufficient σ-field $\mathscr{F}\subset\mathscr{A}$ denote

$\mathscr{F}_g=\{A\in\mathscr{A}_0\colon\ \tilde{A}\in g(\mathscr{F}/\mathscr{N}(\mathscr{P}))\}$,

$\bar{\mathscr{F}}=\{A\in\mathscr{A}_1\colon$ there exists $B\in\mathscr{F}_g\colon f^{-1}(B)=A\}$

$\mathfrak{G}=\mathscr{U}_1$-closure $(\bar{\mathscr{F}}/\mathscr{N}(\mathscr{P}_1))$, $\mathscr{G}=\{A\in\mathscr{A}_1\colon\ \tilde{A}\in\mathfrak{G}\}$.

$\mathscr{F}_g$ and $\bar{\mathscr{F}}$ are automatically sufficient σ-fields for the statistical space $(\Omega_0,\mathscr{A}_0,\mathscr{P}_0)$

and $(\Omega_1, f^{-1}(\mathscr{A}_0), \mathscr{P}_1|f^{-1}(\mathscr{A}_0))$, respectively. We will show that $\mathscr{G}$ is *also* a *sufficient* σ-field for $(\Omega_1, \mathscr{A}_1, \mathscr{P}_1)$.

Let $A\in\mathscr{A}_1$ be an arbitrary event, and consider another event $B\in\mathscr{G}$. There exist two generalized sequences $(\tilde{A}_i)_{i\in I}\subset\mathfrak{A}_0$, $(\tilde{B}_j)_{j\in J}\subset\mathfrak{A}_0$, for which $(f^{-1}(A_i))_{i\in I}$ converges to the equivalence class $\tilde{A}$, $(f^{-1}(B_j))_{j\in J}$ converges to $\tilde{B}$. Being $\overline{\mathscr{F}}$ sufficient, the following equation holds

$$\int_{f^{-1}B_j} E(\chi_{f^{-1}(A_i)}|\overline{\mathscr{F}})\,d\overline{P} = \int_{f^{-1}(B_j)} \chi_{f^{-1}(A_i)}\,d\overline{P}$$

where $E(\chi_{f^{-1}(A_i)}|\overline{\mathscr{F}})$ is the common version of the conditional expectations $E_{\overline{P}}(\chi_{f^{-1}(A_i)}|\overline{\mathscr{F}})$. First take the limit with respect to the index $j\in J$. Since $\overline{P}(B\circ f^{-1}(B_j))$ converges to zero for every $\overline{P}\in\mathscr{P}_1$, and $E(f^{-1}(A_i)|\overline{\mathscr{F}})$ are bounded ($i\in I$), we get

$$\int_B E(\chi_{f^{-1}(A_i)}|\overline{\mathscr{F}})\,d\overline{P} = \int_B \chi_{f^{-1}(A_i)}\,d\overline{P}.$$

The random variables $E(\chi_{f^{-1}(A_i)}|\overline{\mathscr{F}})$ form a stochastically Cauchy convergent sequence for every $\overline{P}\in\mathscr{P}_1$. For, if $\varepsilon>0$ then

$$\overline{P}(|E(\chi_{f^{-1}(A_i)}|\overline{\mathscr{F}})-E(\chi_{f^{-1}(A_{i'})}|\overline{\mathscr{F}})|>\varepsilon) \leqq$$
$$\leqq \overline{P}(E(|\chi_{f^{-1}(A_i)}-\chi_{f^{-1}(A_{i'})}||\overline{\mathscr{F}})>\varepsilon) \leqq \overline{P}(f^{-1}(A_i)\circ f^{-1}(A_{i'}))/\varepsilon.$$

Consequently for every $\overline{P}\in\mathscr{P}_1$ there is an $\overline{\mathscr{F}}$-measurable random variable $X_{\overline{P}}$-which is the stochastic limit of the sequence $E(\chi_{f^{-1}(A_i)}|\overline{\mathscr{F}})$ under the measure $\overline{P}$. $\mathfrak{G}$ is a $\mathscr{U}_1$-closed subset of the $\mathscr{U}_1$-complete space $\mathfrak{A}_1$, consequently it is also $\mathscr{U}_1$-complete. Thus $\mathfrak{G}\cap\mathscr{N}(\overline{P})$ is a principal ideal — see Corollary 1 — for every $\overline{P}\in\mathscr{P}_1$, and so there is a $\mathscr{G}$-measurable event $B_{\overline{P}}$ such that $\overline{P}(B_{\overline{P}})=1$, and if $B\in\mathscr{G}$ and $B\subset B_{\overline{P}}$, $\overline{P}(B)=0$ then $B\in\mathscr{N}(\mathscr{P}_1)$. Consequently, each $X_{\overline{P}}$ is unique mod $\mathscr{N}(\mathscr{P}_1)$ on this event. Similarly, if $\overline{P}, \overline{Q}\in\mathscr{P}_1$ then the restriction of these measures to the σ-field $\mathscr{G}$ are absolutely continuous on the event $B_{\overline{P}}\cap B_{\overline{Q}}$ with respect to the other one, thus $X_{\overline{P}}\neq X_{\overline{Q}})\cap B_{\overline{P}}\cap B_{\overline{Q}}\in\mathscr{N}(\mathscr{P}_1)$.

We may define homomorphisms $h_{\overline{P}}$ from the Borel subsets of the real line into $\overline{\mathscr{F}}/\mathscr{N}(\mathscr{P}_1)$ using the restriction of the random variables $X_{\overline{P}}$ to the set $A_{\overline{P}}$ as in the proof of Theorem 1. Let $h(B)=\sup\limits_{\overline{P}\in\mathscr{P}_1} h_{\overline{P}}(B)$ for every $B\in\mathscr{B}$. (The sup is taken in the lattice $\mathfrak{G}$.) Observe that $h(B)\in\mathfrak{G}$. Thus there exists a $\mathscr{G}$-measurable random variable X for which $P(X\neq X_p)=0$ for every $\overline{P}\in\mathscr{P}_1$. This is the required conditional expectation. □

The converse of this assertion is not true. Namely, if $\mathscr{G}$ is a sufficient σ-field of $(\Omega_1, \mathscr{A}_1, \mathscr{P}_1)$ then the σ-field

$$\mathscr{F} = \{A\in\mathscr{A}\,|\,\text{there exists a } B\in\mathscr{G} \text{ for which } g(\tilde{A}) = \widetilde{f(B)}\}$$

is not necessarily sufficient in the statistical space $(\Omega, \mathscr{A}, \mathscr{P})$.

Denoting by $\mathscr{F}_1$ the minimal sufficient σ-field of $(\Omega_1, \mathscr{A}_1, \mathscr{P}_1)$ — this always exists —, the following assertion is true.

PROPOSITION 11. *There exists a minimal pairwise sufficient σ-field in the statistical space $(\Omega, \mathscr{A}, \mathscr{P})$ if and only if the σ-field $\mathscr{F}=\{A\in\mathscr{A}|$ there exists a $B\in\mathscr{F}_1$ for which*

$g(\tilde{A})=f(\widetilde{B})\}$ *is pairwise sufficient. In this case this* $\mathscr{F}$ *is the minimal pairwise sufficient* σ*-field.*

We omit the proof of this proposition. Observe that the σ-field $\mathscr{F}$ is — roughly speaking — the intersection of the σ-field $\mathscr{F}_1$ and $\mathscr{A}$.

We have seen in Proposition 9 that if the Boole-algebra $\mathfrak{A}$ is complete and $\mathscr{F}$ is a sufficient σ-field then the Boole-algebra $\mathscr{F}/\mathscr{N}(\mathscr{P})$ is also complete. In addition to this, the Boole-algebra completeness of $\mathfrak{A}$ has also other consequences, namely in this case an assertion similar to Corollary 3 of Theorem 2 holds.

More precisely, let $\mathscr{F}$ and $\mathscr{G}$ be two σ-fields, such that $\mathscr{F}\subset\mathscr{G}$, $\mathscr{N}(\mathscr{P})\subset\mathscr{F}$ and the factor algebras $\mathscr{F}/\mathscr{N}(\mathscr{P})$ and $\mathscr{G}/\mathscr{N}(\mathscr{P})$ are complete. In this case if $\mathscr{F}$ is a sufficient σ-field, then $\mathscr{G}$ is also sufficient. The proof of this assertion is tedious and will be the subject of another paper.

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LARGE DEVIATION RESULTS FOR WAITING TIMES IN REPEATED EXPERIMENTS

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1. Introduction

Let $Z_1, Z_2, \dots$ be a sequence of independent, identically distributed random variables with a finite set of possible values Σ. Let A_i $(i=1, 2, \dots, r)$ be finite sequences of length k over Σ and denote by τ_i the waiting time until A_i occurs as a run in the process $Z_1, Z_2, \dots$.

We shall prove a large deviation type result on the asymptotic independence and exponentiality of the stopping times τ_i. The method of proof is a refinement of the reasoning applied in Móri and Székely [6], where a limit theorem is proved for the waiting time for pure runs (homogenous patterns) of arbitrary events by reducing the joint distribution problem to the asymptotic exponentiality of the minimum waiting time.

Our results verify the observation that the limit behaviour of extremals of waiting times for runs can be computed in the same way as extremals of independent exponentially distributed random variables. For example, if n urns are given and balls are placed at random in these urns one after another till there is at least one ball in every urn, then the number of balls needed has a double exponential limit distribution as n tends to infinity (Erdős and Rényi [1]). The same limiting behaviour is found for the maximum of n independent, exponentially distributed random variables (with expectation n). A great number of papers is devoted to the systematic study of similar problems in more general situations; here we refer only to Ivanov and Novikov [3] and the handbook of Kolchin, Sevast'yanov and Chistyakov [4].

2. Results

In order to formulate our results we need first some definitions and notations.

Our calculations are facilitated by the leading number algorithm described in the paper of Li [5]. Let $A=(a_1, a_2, \dots, a_k)$ and $B=(b_1, b_2, \dots, b_l)$ be two sequences over Σ. For every pair of indices (i, j) write

$$\delta_{ij} = \begin{cases} P(Z=b_j)^{-1} & \text{if } 1 \leqq i \leqq k, \quad 1 \leqq j \leqq l \quad \text{and} \quad a_i = b_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then define

$$A * B = \delta_{11}\delta_{22}\dots\delta_{kk} + \delta_{21}\delta_{32}\dots\delta_{k,k-1} + \dots + \delta_{k1}.$$

This quantity describes the overlapping between A and B, and for any finite collection of sequences of possible outcomes it provides a method to compute either the expected waiting time till one of them is observed in a run of experiments, or the probability for each sequence to the first to appear. E.g. the expected waiting time for a single sequence A is exactly $A*A$.

For $A=(a_1, \dots, a_k)$ let $P(A)$ denote the probability of obtaining A in k successive independent trials, i.e. $P(A)=\sum_{i=1}^{k} P(Z_i=a_i)$.

Consider the sequences $A_1, A_2, \dots, A_r \in \Sigma^k$ and the related stopping times $\tau_1, \tau_2, \dots, \tau_r$ defined in the Introduction. Denote

$$b = k \sum_{i=1}^{r} P(A_i), \quad c = \max_{1\le j\le r} \sum_{i\ne j} \frac{A_j * A_i}{A_i * A_i}.$$

THEOREM 1. *Let* $x_1, x_2, \dots, x_r$ *be arbitrary positive numbers. Then*

$$\left(\frac{1-7b}{1-6b}\right)^{r-1} \exp\left(-\frac{b}{1-5b}\sum_{i=1}^{r} x_i\right) \le$$

$$\le P\left(\frac{\tau_1}{A_1*A_1} > x_1, \dots, \frac{\tau_r}{A_r*A_r} > x_r\right) \exp\left(\sum_{i=1}^{r} x_i\right) \le \left(\frac{1-4b}{1-5b}\right)^{r-1} \exp\left(c \sum_{i=1}^{r} x_i\right)$$

if $b \le 1/8$.

Let us specialize our result to obtain some wellknown theorems. For applications we must be sure that c is small enough. This trivially holds if the sequences A_i are pure runs of different outcomes, so the urn model described in the Introduction is an important field of immediate application. As an example we prove

COROLLARY 1. (i) *Let* Σ *consist of* n *equally probable outcomes and denote by* $\nu_m(n, k)$ *the waiting time until pure runs of length not less than* k $(k>1)$ *are observed for all but* m *elements of* Σ. *Let* m *be fixed,* k *may vary with* n. *Then for every real number* y

$$\lim_{n\to\infty} P(n^{-k}\nu_m - \log n < y) = e^{-e^{-y}} \sum_{j=0}^{m} \frac{1}{j!} e^{-jy}.$$

(ii) *Trying* N *experiments we denote by* $\xi_N(n, k)$ *the number of those elements of* Σ *for which we have not observed a run of length not less than* k. *If* n *and* N *tend to infinity such that* $n \exp(-n^{-k}N) \to \lambda > 0$, *then the limit distribution of* $\xi_N(n, k)$ *is Poissonian with expectation* λ.

The two parts of Corollary 1 are equivalent and they are usually proved by means of generating functions (cf. Ivanov and Novikov [3]), or by using general theorems on Poisson approximation such as Theorem 1 of Sevast'yanov [7].

The requirement $c\to 0$ is met also in the case when r is fixed and $A_1, A_2, \dots, A_r$ are starting sequences of length k of given infinite sequences over Σ not containing each other. Thus let $A_i=(a_{i1}, a_{i2}, \dots)$, $i=1, 2, \dots, r$ be infinite sequences over Σ such that for every pair of indices (i, j), $i\ne j$ and nonnegative integer m, $A_j \ne (a_{i,m+1}, a_{i,m+2}, \dots)$. Let $A_i(k)$ stand for the starting subsequence of length k of A_i and denote by $\tau_i(k)$ the corresponding waiting time. Let π_i be defined as $P(A_i(m))$ if A_i is a pure periodic sequence with period m (i.e. m is the smallest integer for which $a_{i,j}=a_{i,j+m}$, $j=1, 2, \dots$), otherwise let $\pi_i=0$.

COROLLARY 2. *We have*

$$\lim_{k\to\infty} P(P(A_i(k))\tau_i(k) > y_i, \quad i=1,2,\dots,r) = \exp\Big(-\sum_{i=1}^{r}(1-\pi_i)y_i\Big)$$

for arbitrary positive numbers $y_1, y_2, \dots, y_r$.

3. Remarks

1. Theorem 1 asserts that the waiting times τ_i divided by their expectation behave approximately like independent, exponentially distributed random variables, at least in the domain

$$\sum_{i=1}^{r} x_i = o(b^{-1}), \tag{1}$$

$$\sum_{i=1}^{r} x_i = o(c^{-1}). \tag{2}$$

The first relation here is for the exponentiality while the second one implies the independence. Our theorem is sharp in the sense that neither (1) nor (2) can be weakened. We give three simple examples to show this.

(i) Let τ denote the waiting time until a given outcome of probability p occurs in repeated experiments. Then $E(\tau)=1/p$ and by Theorem 1 $P(\tau>n)e^{np}\to 1$ if $p\to 0$ and $np^2\to 0$. Let $np^2\to\lambda>0$, then

$$P(\tau>n)e^{np} = \exp\big(n\log(1-p)+np\big) = \exp\big(-\lambda+o(1)\big).$$

(ii) In the following example we consider the waiting time for a k-run of a given outcome with (fixed) probability p. A detailed discussion of this case is given in Feller [2], XIII. 7. It is shown there that

$$\left|P(\tau>n)-\frac{1-pz}{(k+1-kz)\,qz^{n+1}}\right| \le \frac{2p^2}{q}\,p^n$$

where $q=1-p$ and $z=1+qp^k+(k+1)q^2p^{2k}+O(k^3p^{3k})$ as $k\to\infty$.

Since

$$E(\tau) = p^{-1}+p^{-2}+\dots+p^{-k} = \frac{1-p^k}{qp^k},$$

Theorem 1 asserts that

$$P(\tau>n)\exp\left(\frac{qp^k}{1-p^k}\,n\right)\to 1$$

if $k\to\infty$ and $nkp^{2k}\to 0$. Suppose conversely $nkp^{2k}\to\lambda>0$, then by a short calculation we obtain

$$P(\tau>n)\exp\left(\frac{qp^k}{1-p^k}\,n\right)\sim z^{-n}\exp(qp^k n)\sim$$

$$\sim\exp\left(-\left(k+\frac{1}{2}\right)q^2p^{2k}n\right)\sim\exp(-q^2\lambda).$$

(iii) The necessity of condition (2) is verified below. Tossing a coin we consider two competing sequence: a pure head run of length $2k$ and a run of k heads followed by k tails. Denote the corresponding waiting times by τ_1 and τ_2, resp. Then $E(\tau_1)=2^{2k+1}-2$, $E(\tau_2)=2^{2k}$, $b=k2^{-2k+2}$, $c=2^{-k}$ and from Theorem 3.1 of Li [5] $E(\alpha)=(2^{4k+1}-2^{2k+1})/(3\cdot 2^{2k}-2^k-2)$ where $\alpha=\min(\tau_1, \tau_2)$. Our Theorem 1 asserts that

$$P(2^{-2k}\alpha > x) = P\left(\frac{\tau_1}{E(\tau_1)} > \frac{2^{2k}}{2^{2k+1}-2}x, \frac{\tau_2}{E(\tau_2)} > x\right) \sim \exp\left(-\frac{3}{2}x\right)$$

if $x=o(2^k)$. Now let $x=\lambda 2^k$ $(\lambda>0)$, then Lemma 3 in the proof of our Theorem 1 implies

$$P(2^{-2k}\alpha > x)\exp\left(\frac{3}{2}x\right) \sim \exp\left(\left(\frac{3}{2}-\frac{2^{2k}}{E(\alpha)}\right)x\right) = \exp\left(\frac{x}{2^{k+1}+2}\right) \sim \exp\left(\frac{\lambda}{2}\right).$$

2. Our results can be generalized by allowing the sequences $A_1, A_2, \ldots, A_r$ to have different lengths $k_1, k_2, \ldots, k_r$. In this case b should be defined as $\left(\max_{1\le i\le r} k_i\right) \sum_{i=1}^{r} P(A_i)$. If r is fixed and we are not interested in large deviations, then the same proof as in Móri—Székely [6] shows that conditions $c\to 0$ and $\sum_{i=1}^{r} k_i P(A_i)\to 0$ are sufficient for the asymptotic independence and exponentiality of the waiting times τ_i. When $c\not\to 0$, the joint distribution is approximately of Marshall—Olkin type.

3. *An open problem.* The following question is beyond our reach. Tossing a coin let us denote by τ the number of experiments tried until every possible head-and-tail pattern of length k appears as a connected subsequence of the outcomes. Describe the limit behaviour of τ or at least of $E(\tau)$. Theorem 1 does not apply to this case, because

$$b = k\sum_{i=1}^{2^k} P(A_i) = k, \qquad \sum_{i=1}^{2^k} A_j * A_i = k2^k \quad \text{hence} \quad c \sim \frac{k}{2}-1.$$

Theorem 3.1 of Li [5] offers a method for computing the exact value of $E(\tau)$, since the maximal waiting time can be expressed in terms of the partial minimums. This sieve formula, however, seems to be hardly evaluable even for small values of k.

4. Proofs

The proof of Theorem 1 will be based on a sequence of elementary lemmas. The first one is perhaps not new. It asserts that, roughly speaking, a nearly memoryless distribution is nearly exponential.

LEMMA 1. *Let α be a nonnegative random variable. Suppose*

$$\varkappa_1 P(\alpha > x+y) \le P(\alpha > x)P(\alpha > y) \le \varkappa_2 P(\alpha > x+y)$$

for arbitrary positive x, y, where $0<\varkappa_1\le 1\le \varkappa_2<\infty$. Then there exists a positive number λ such that

(3) $$\varkappa_1 \le \lambda E(\alpha) \le \varkappa_2$$

and

(4) $$\varkappa_1 e^{-\lambda x} \leqq P(\alpha > x) \leqq \varkappa_2 e^{-\lambda x}$$

hold for every positive x.

PROOF. Denote $G(x)=P(\alpha>x)$. It is easy to see that

$$\varkappa_1^m G(mx) \leqq G(x)^m \leqq \varkappa_2^m G(mx),$$

hence

$$\varkappa_1^{m/n}\varkappa_2^{-1} G\left(\frac{m}{n}x\right) \leqq G(x)^{m/n} \leqq \varkappa_2^{m/n}\varkappa_1^{-1} G\left(\frac{m}{n}x\right)$$

for every positive integers m, n. Thus

$$\frac{\varkappa_1^{1/x}}{\varkappa_2^{1/y}} \leqq \frac{G(x)^{1/x}}{G(y)^{1/y}} \leqq \frac{\varkappa_2^{1/x}}{\varkappa_1^{1/y}}$$

where x, y can be arbitrary positive numbers. From this it follows that $G(x)^{1/x}$ converges, as $x\to\infty$, to a positive limit $e^{-\lambda}(\lambda\geqq 0)$. Then (4) obviously holds (hence $\lambda>0$), and integrating this inequality from 0 to $+\infty$ with respect to x one obtains (3).

In the sequel we often refer to the following basic inequality. Let $0<x_0\leqq \leqq x_1 \leqq \ldots \leqq x_r$ and denote $\alpha=\min\limits_{1\leqq i\leqq r}\tau_i$. Then

(5) $$\begin{aligned} P(\tau_1 > x_1, \ldots, \tau_r > x_r) &\leqq P(\alpha > x_0) P(\tau_1 > x_1 - x_0, \ldots, \tau_r > x_r - x_0) \leqq \\ &\leqq P(\tau_1 > x_1, \ldots, \tau_r > x_r) + bP(\alpha > x_0 - k + 1)\times \\ &\times P(\tau_1 > x_1 - x_0 - k + 1, \ldots, \tau_r > x_r - x_0 - k + 1). \end{aligned}$$

In words, the probability of the event that the sequence A_i is not observed as a run up to time x_i $(i=1, 2, \ldots, r)$ can be majorized by interrupting and then recommencing the observation at the time x_0 (which is the earliest of all x_i's). The error of this estimate does not exceed the probability of the event that there is a run A_i beginning before x_0 and ending after x_0. From this explanation inequality (5) appears immediately.

LEMMA 2. *We have*

$$\frac{P(\alpha > x - k + 1)}{P(\alpha > x)} \leqq \frac{1}{1-2b} \quad \textit{if} \quad b \leqq 1/4.$$

PROOF. Let us abbreviate the fraction on the left side by $t(x)$. Clearly $t(x)=1$ for $x<k$. Substituting $x_1=x_2=\ldots=x_r=x$ and $x_0=k-1$ into (5) we obtain

$$t(x) \leqq 1 + bt(x)t(x-k+1).$$

By induction it follows that $t(x)$ is less than the smaller root of the equation $t=1+bt^2$, thus

$$t(x) < \frac{2}{1+\sqrt{1-4b}} \leqq \frac{1}{1-2b}$$

if $b\leqq 1/4$, but this holds by supposition.

LEMMA 3. *We have*

$$\exp\left(-\frac{bx}{1-5b}\right) \leqq P\left(\frac{\alpha}{E(\alpha)} > x\right) e^x \leqq \frac{1-4b}{1-5b} \tag{6}$$

and

$$\exp\left(-\frac{bx}{1-5b}\right) \leqq P\left(\alpha \sum_{i=1}^{r} \frac{1}{A_i * A_i} > x\right) e^x \leqq \frac{1-4b}{1-5b} \exp\left(\frac{cx}{1+c}\right). \tag{7}$$

PROOF. The substitution $x_0=x$, $x_1=x_2=\ldots=x_r=y$ turns inequality (5) into

$$P(\alpha > x+y) \leqq P(\alpha > x)P(\alpha > y) \leqq$$

$$\leqq P(\alpha > x+y)+bP(\alpha > x-k+1)P(\alpha > y-k+1).$$

Applying Lemma 2 we find α meeting the conditions of Lemma 1 with

$$\varkappa_1 = 1, \quad \varkappa_2 = \left(1-\frac{b}{(1-2b)^2}\right)^{-1} < \frac{1-4b}{1-5b}.$$

Thus Lemma 1 gives

$$\varkappa_1 \exp\left(-(\varkappa_2-1)x\right) \leqq P\left(\frac{\alpha}{E(\alpha)} > x\right) e^x \leqq \varkappa_2 \exp\left((1-\varkappa_1)x\right),$$

which is just the assertion of (6).

On the other hand, Theorem 3.1 of Li [5] says that

$$\sum_{j=1}^{r} p_j(A_j * A_i) = E(\alpha) \quad (i = 1, 2, \ldots, r),$$

where $p_j=P(\alpha=\tau_j)$. Dividing by $A_i * A_i$ and summing it from $i=1$ to r we obtain

$$1 \leqq E(\alpha) \sum_{i=1}^{r} \frac{1}{A_i * A_i} \leqq 1 + \sum_{j=1}^{r} p_j \sum_{i \neq j} \frac{A_j * A_i}{A_i * A_i} \leqq 1+c.$$

Now (7) follows from (6) by easy computation.

LEMMA 4. *We have*

$$\frac{P(\tau_1 < x_1-k+1, \ldots, \tau_r < x_r-k+1)}{P(\tau_1 < x_1, \ldots, \tau_r < x_r)} \leqq \frac{1}{1-4b}.$$

PROOF. Denote by t_r the supremum in x_i's of the left side. By Lemma 2, $t_1 \leqq \leqq \frac{1}{1-2b}$. From (5) we have

$$P(\tau_1 > x_1-k+1, \ldots, \tau_r > x_r-k+1) \leqq$$

$$\leqq P(\alpha > x_1-k+1)P(\tau_2 > x_2-x_1, \ldots, \tau_r > x_r-x_1),$$

further

$$P(\tau_1>x_1,\dots,\tau_r>x_r)\geqq P(\alpha>x_1)P(\tau_2>x_2-x_1,\dots,\tau_r>x_r-x_1)-$$
$$-bP(\alpha>x_1-k+1)P(\tau_2>x_2-x_1-k+1,\dots,\tau_r>x_r-x_1-k+1)\geqq$$
$$\geqq(1-2b)P(\alpha>x_1-k+1)P(\tau_2>x_2-x_1,\dots,\tau_r>x_r-x_1)-$$
$$-bt_{r-1}P(\alpha>x_1-k+1)P(\tau_2>x_2-x_1,\dots,\tau_r>x_r-x_1).$$

Hence $t_r\leqq(1-2b-bt_{r-1})^{-1}$, thus t_r is less than the smaller solution of the equation $t(1-2b-bt)=1$, i.e.

$$t_r<\frac{1}{2b}\left(1-2b-((1-2b)^2-4b)^{1/2}\right)\leqq\frac{1}{1-4b},\quad \text{if}\quad b\leqq 1/8.$$

Now we are in position to verify our main result.

Proof of Theorem 1. Denote $y_i=x_iE(\tau_i)=x_i(A_i*A_i)$. Without loss of generality we may assume $y_1\leqq y_2\leqq\dots\leqq y_r$. First we show that

$$\left(\frac{1-7b}{1-6b}\right)^{r-1}\leqq\frac{P(\tau_1>y_1,\dots,\tau_r>y_r)}{P(\alpha_1>y_1)P(\alpha_2>y_2-y_1)\dots P(\alpha_r>y_r-y_{r-1})}\leqq 1 \tag{8}$$

where $\alpha_i=\min(\tau_i,\tau_{i+1},\dots,\tau_r)$.

Starting from the basic inequality (5) we can write

$$P(\tau_1>y_1,\dots,\tau_r>y_r)\geqq P(\alpha_1>y_1)P(\tau_2>y_2-y_1,\dots,\tau_r>y_r-y_1)-$$
$$-bP(\alpha_1>y_1-k+1)P(\tau_2>y_2-y_1-k+1,\dots,\tau_r>y_r-y_1-k+1)\geqq$$
$$\geqq(1-bt_1t_{r-1})P(\alpha_1>y_1)P(\tau_2>y_2-y_1,\dots,\tau_r>y_r-y_1).$$

Applying Lemma 2 we obtain

$$1-bt_1t_{r-1}\geqq 1-\frac{b}{(1-2b)(1-4b)}>\frac{1-7b}{1-6b},$$

and (8) follows by induction.

The terms of the denominator in (8) can be estimated by Lemma 3 in the following way.

$$P(\alpha_i>y_i-y_{i-1})=P\left(\alpha_i\sum_{j=i}^{r}\frac{1}{A_j*A_j}>z_i\right)$$

where

$$z_i=\sum_{j=i}^{r}\frac{1}{A_j*A_j}\left((A_i*A_i)x_i-(A_{i-1}*A_{i-1})x_{i-1}\right),$$

hence

$$\exp\left(-\frac{b}{1-5b}z_i\right)\leqq P(\alpha_i>y_i-y_{i-1})\exp(z_i)\leqq\frac{1-4b}{1-5b}\exp(cz_i). \tag{9}$$

This calculation remains valid also for $i=1$ by defining $x_0=y_0=0$. Since z_1+ $+z_2+\dots+z_r=x_1+x_2+\dots+x_r$, we can complete the proof by combining (8) and (9).

PROOF OF COROLLARY 1. The event $\{v_m(n,k)>z\}$ occurs if and only if more than m of the events $A_i=\{\tau_i>z\}$ $(i=1,2,\dots,r)$ occur; so we can apply the well-known identity (cf. Feller [2], IV. 5)

$$P(v_m(n,k)>z)=\sum_{j=0}^{n-m-1}(-1)^j\binom{j+m}{m}S_{j+m+1} \tag{10}$$

where

$$S_l=\sum_{1\le i_1<\dots<i_l\le r}P(A_{i_1}A_{i_2}\dots A_{i_l}),$$

i.e. S_l is the sum of probabilities of simultaneous occurrence for every l-tuple of events. Moreover, if summation on the right side of (10) is taken from 0 to $2s$, resp. $2s+1$, we obtain an upper, resp. lower estimate for the probability $P(v_m(n,k)>z)$. Hence the limit of the left side of (10) can be evaluated by taking limit by terms on the right side.

In our case $z=n^k(\log n+y)$ and by Theorem 1

$$\binom{j+m}{m}S_{j+m+1}\sim\binom{j+m}{m}\binom{n}{j+m+1}\exp\big(-(j+m+1)(\log n+y)\big)\sim$$

$$\sim\frac{1}{j+m+1}\,\frac{1}{j!\,m!}\exp\big(-(j+m+1)y\big).$$

Thus

$$\lim_{n\to\infty}P(n^{-k}v_m-\log n<y)=1-\sum_{j=0}^{\infty}\frac{1}{(j+m+1)j!\,m!}\exp\big(-(j+m+1)y\big),$$

from which Part (i) follows by easy calculation.

To prove Part (ii) we observe that $P(\xi_N(n,k)\le m)=P(v_m(n,k)\le N)$.

PROOF OF COROLLARY 2. This assertion will follow from Theorem 1 if we show that $b\to0$, $c\to0$ and

$$P(A_i(k))(A_i(k)*A_i(k))\to(1-\pi_i)^{-1}$$

as $k\to\infty$. Introducing the notations

$$p_{\max}=\max\{P(Z=\sigma):\ \sigma\in\Sigma\},\quad p_{\min}=\min\{P(Z=\sigma):\ \sigma\in\Sigma\}$$

we can write $b\le krp^k_{\max}$, thus $\lim\limits_{k\to\infty}b=0$. On the other hand, if the length of maximal overlapping between $A_j(k)$ and $A_i(k)$ does not exceed $k-m$ (i.e. for $t=1,2,\dots,m$ one can find u, $1\le u\le k-j$ such that $a_{j,t+u}\ne a_{i,u}$), then

$$A_j(k)*A_i(k)\le P(A_i(k))^{-1}(p^m_{\max}+p^{m+1}_{\max}+\dots+p^{k-1}_{\max})<$$

$$<(A_i(k)*A_i(k))\,p^m_{\max}(1-p_{\max})^{-1}.$$

Since the infinite sequences A_j and A_i do not contain each other as a tail, $k\to\infty$ implies $m\to\infty$, thus $c\to0$. In a similar way one can see that for large k, $A_i(k)$ overlaps itself at length $k-m$ ($m>0$ fixed) if and only if m is a multiple of the period of A_i. Hence

$$A_i(k)*A_i(k)\sim P(A_i(k))^{-1}(1+\pi_i+\pi_i^2+\dots),$$

completing the proof.

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ON THE DIRICHLET KERNELS AND A HARDY SPACE WITH RESPECT TO THE VILENKIN SYSTEM

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1. Introduction. Given a so-called Vilenkin system, first we prove some inequalities for the Direchlet kernels with respect to this system. Furthermore we shall show a Hardy type estimation for the Vilenkin—Fourier coefficients. Our goal is to investigate the role of the boundedness of the sequence which generates the Vilenkin system.

2. In this section we introduce some definitions and notations. Let $m=(m_0, m_1, \ldots, m_k, \ldots)$ be a sequence of natural numbers, $m_k \geqq 2$ $(k\in \mathbf{N}:=\{0, 1, \ldots\})$ and denote Z_{m_k} $(k\in\mathbf{N})$ the $m_k$$^{\text{th}}$ discrete cyclic group where Z_{m_k} is represented by $\{0, 1, \ldots, m_k-1\}$. If we define G_m as the complete direct product of Z_{m_k}'s then G_m is a compact Abelian group with Haar measure 1. The elements of G_m are of the form $(x_0, x_1, \ldots, x_k, \ldots)$ $(x_k\in Z_{m_k}, k\in\mathbf{N})$ and the topology of G_m is completely determined by the sets

$$I_n := \{(x_0, x_1, \ldots, x_k, \ldots)\in G_m\colon\ x_j = 0 \quad (j = 0, \ldots, n-1)\}$$

$(n\in\mathbf{N}^* := \mathbf{N}\setminus\{0\}, I_0 := G_m)$. Let us denote the cosets of I_n's by $I_n(x) := x+I_n$ $(x\in G_m, n\in\mathbf{N})$. Furthermore let

$$I_n(x, k) := \{(y_0, y_1, \ldots)\in I_n(x)\colon\ y_n = k\}$$

$(x\in G_m, k=0, 1, \ldots, m_n-1,\ n\in\mathbf{N})$. It is well-known [8] that the characters of G_m form a complete orthonormal system $\hat{G}_m$ in $L^1(G_m)$. The elements of $\hat{G}_m$ can be obtained as follows. Define the sequence $(M_k, k\in\mathbf{N})$ as $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k\in\mathbf{N})$, then all $n\in\mathbf{N}$ have a unique representation of the following form

$$n = \sum_{k=0}^{\infty} n_k M_k \quad (n_k = 0, 1, \ldots, m_k-1, k\in\mathbf{N}).$$

If

$$r_n(x) := \exp\frac{2\pi i x_n}{m_n} \quad (n\in\mathbf{N}; x = (x_0, x_1, \ldots)\in G_m, i := \sqrt{-1})$$

then the elements of $\hat{G}_m$ are nothing but the functions [8]

$$\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k} \quad (n\in\mathbf{N}).$$

$(\Psi_n, n\in\mathbf{N})$ is the so-called Vilenkin system.

The Fourier coefficients of a function $f\in L^1(G_m)$ with respect to $\hat{G}_m$ are denoted by $\hat{f}(k)$ $(k\in\mathbf{N})$ and let $S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\Psi_k$ $(n\in\mathbf{N})$. The kernels of Dirichlet type are of the form $D_n := \sum_{k=0}^{n-1} \Psi_k \, (n\in\mathbf{N})$. It is known [8] that

$$(1) \qquad D_{M_n}(x) = \begin{cases} M_n & (x\in I_n) \\ 0 & (x\in G_m \setminus I_n) \end{cases} \quad (n\in\mathbf{N}).$$

For an $x=(x_0, x_1, \dots)\in G_m$ we shall write $|x| := \sum_{k=0}^{\infty} \frac{x_k}{M_{k+1}}$, furthermore let $\tilde{x} := := (\tilde{x}_0, \tilde{x}_1, \dots)\in G_m$, where $\tilde{x}_k := \min\{x_k, m_k - x_k\}$ $(k\in\mathbf{N})$.

The concept of the Hardy space [4] can be defined in various manners, e.g. by a maximal function $f^* := \sup_n |S_{M_n} f|$ $(f\in L^1(G_m))$, saying that f belongs to the Hardy space if $f^*\in L^1(G_m)$. This definition is suitable if the sequence m is bounded. In this case a good property of the space $\{f\in L^1(G_m) : f^*\in L^1(G_m)\}$ is the atomic structure [4]. To the definition of our space of Hardy type for an arbitrary m, first we give the concept of the atoms [7]. A set $I\subset G_m$ is called an interval if for some $x\in G_m$ and $n\in\mathbf{N}$, I is of the form $I=\bigcup_{k\in U} I_n(x, k)$ where U is obtained from Z_{m_n} by dyadic partition. (The sets $U_1, U_2, \dots \subset Z_{m_n}$ are obtained by means of such a partition if

$$U_1 = \left\{0, \dots, \left[\frac{m_n}{2}\right]-1\right\}, \quad U_2 = \left\{\left[\frac{m_n}{2}\right], \dots, m_n - 1\right\},$$

$$U_3 = \left\{0, \dots, \left[\frac{[m_n/2]-1}{2}\right]-1\right\}, \quad U_4 = \left\{\left[\frac{[m_n/2]-1}{2}\right], \dots, \left[\frac{m_n}{2}\right]-1\right\}, \dots$$

etc.; [ ] denotes the entire part.) We define the atoms as follows: the function $a\in L^\infty(G_m)$ is called an atom if either $a\equiv 1$ or there exists an interval I for which $\operatorname{supp} a\subset I$, $|a|\leqq |I|^{-1}$ and $\int_I a = 0$ hold. ($|I|$ denotes the Haar measure of I.) Now we can define the space $H(G_m)$ as the set of all functions $f=\sum_{i=0}^{\infty} \lambda_i a_i$ where a_i's are atoms and for the coefficients λ_i we have $\sum_{i=0}^{\infty} |\lambda_i| < \infty$. $H(G_m)$ is a Banach space with respect to the norm $\|f\| := \inf \sum_{k=0}^{\infty} |\lambda_k|$.

(The infimum is taken over all decompositions $f=\sum_{i=0}^{\infty} \lambda_i a_i$.) It is known [7] that $\|f\|$ is equivalent to $\|f^{**}\|_1$ $(f\in L^1(G_m))$, where $f^{**}(x) := \sup_I |I|^{-1} \left|\int_I f\right|$ $(x\in G_m, x\in I$ and I is interval). Since by (1)

$$f^*(x) = \sup_n |I_n(x)|^{-1} \left| \int_{I_n(x)} f \right| \quad (x\in G_m),$$

therefore $f^* \leqq f^{**}$ and thus $H(G_m)\subset \{f\in L^1(G_m) : f^*\in L^1(G_m)\}$. Moreover these spaces coincide if the sequence m is bounded.

$C>0$ will denote an absolute, although not always the same, constant.

3. First we are concerned with the functions D_n $(n\in\mathbf{N})$.

THEOREM 1. *If* $n=\sum_{k=0}^{\infty} n_k M_k \in \mathbf{N}$ *and* $x \in G_m \setminus \{0\}$ *then*

(i) $|D_n(x)| \leqq \min\left\{n, \dfrac{n_j+1}{|x|}\right\}$

where $j \in \mathbf{N}$ *is determined by the condition* $x \in I_j \setminus I_{j+1}$.

(ii) $|D_n(x)| \leqq \min\left\{n, \dfrac{2}{|\tilde{x}|}\right\}$.

We remark that these two estimations can be essentially different from each other if m is not bounded. However, for a bounded m, $\dfrac{1}{|x|}$ is equivalent to $\dfrac{1}{|\tilde{x}|}$. It is evident that in the bounded case (i.e. for bounded m's) we have $\dfrac{n_j+1}{|x|} \leqq \dfrac{\sup_k m_k}{|x|}$ from which follows

COROLLARY 1. *If* m *is bounded then for* $n \in \mathbf{N}$ *and* $x \in G_m \setminus \{0\}$

$$|D_n(x)| \leqq C_m \min\left\{n, \frac{1}{|x|}\right\}$$

is true, where $C_m > 0$ *depends only on* m.

This is a known result [8]. Next we show that the boundedness of m in the preceding estimation is necessary.

THEOREM 2. *If the sequence* m *is not bounded then* $\sup |D_n(x)| \cdot |x| = \infty$, *where the supremum is taken over all* $x \in G_m$ *and* $n \in \mathbf{N}$.

Let us denote by $L_n := \|D_n\|_1$ $(n \in \mathbf{N})$ the n-th Lebesgue constant with respect to the system $\hat{G}_m$. It is well-known [8] that $L_n = O(\log n)$ $(n \to \infty)$. In the next theorem we shall show a stronger estimation, namely

THEOREM 3. *For all* m *we have*

$$\int_{G_m} \sup_{0 \leqq k \leqq n} |D_k| = O(\log n) \quad (n \to \infty).$$

From Theorem 3 there follow some corollaries for multiple Vilenkin series by the application of a paper of F. Móricz [6]. (The d-multiple $(d=1, 2, \ldots)$ characters are defined as $\Psi_N(X) := \prod_{i=1}^{d} \Psi_{N_i}(x^{(i)})$ $(N=(N_1, \ldots, N_d) \in \mathbf{N}^d,\ X=(x^{(1)}, \ldots, x^{(d)}) \in G_m^d)$. Furthermore let $S_N^{(d)} f$ the N-th rectangular partial sum of $f \in L^1(G_m^d)$ with respect to the system $(\Psi_N,\ N \in \mathbf{N}^d)$.)

COROLLARY 2. *For* $f \in L^2(G_m^d)$ *set*

$$\tilde{f} := \sup_{N \in \mathbf{N}^d} \frac{|S_N^{(d)} f|}{\sqrt{\operatorname{Log}(N+2)}},$$

where $\operatorname{Log}(N+2) := \log(N_1+2) \cdot \ldots \cdot \log(N_d+2)$. *Then* $\|\tilde{f}\|_2 \leqq C \|f\|_2$.

COROLLARY 3. *If $f\in L^2(G_m^d)$ and a_N is the N-th $(N\in\mathbf{N}^d)$ Fourier coefficient of f with respect to the d-multiple Vilenkin system then the rectangular partial sums $S_N^{(d)}f$ converge almost everywhere, provided that*

$$\sum_{N\in\mathbf{N}^d} |a_N|^2 \operatorname{Log}(N+2) < \infty.$$

The statements of the above corollaries for the bounded case can be found in a work of J. D. Chen [3]. (His proof for Corollary 2 is correct only for bounded m's.) For $d=1$ Corollaries 2 and 3 are well-known [8], [9] (Corollary 2 only in the bounded case).

Further we investigate the space $H(G_m)$. First we shall prove a Hardy type inequality for the Vilenkin—Fourier coefficients. This claims that for a function $f\in H(G_m)$ the quantity $\sum_{k=1}^{\infty} k^{-1}|\hat{f}(k)|$ is finite, more precisely the following theorem is true.

THEOREM 4. *There exists an absolute constant $C>0$ such that*

$$\sum_{k=1}^{\infty} k^{-1}|\hat{f}(k)| \leqq C\|f\| \quad (f\in H(G_m)).$$

For the bounded case Theorem 4 was proved by J. A. Chao [2]. The proof is based only on the atomic structure of $H(G_m)$ and the next theorem shows that, in general, this property cannot be replaced by the integrability of f^*.

THEOREM 5. *If $\limsup m=\infty$ then there exists an $f\in L^1(G_m)$ such that $f^*\in L^1(G_m)$ and $\sum_{k=1}^{\infty} k^{-1}|\hat{f}(k)|=\infty$.*

This statement gives a negative answer to a question of J. A. Chao, i.e. that an atomic structure for the space $\{f\in L^1(G_m)\colon f^*\in L^1(G_m)\}$ cannot be available, if the sequence m is not bounded. Thus $H(G_m)=\{f\in L^1(G_m)\colon f^*\in L^1(G_m)\}$ is true if and only if m is bounded.

In [7] we showed on some operators that these mappings are bounded from $H(G_m)$ into $L^1(G_m)$, for example in the bounded case (and only in this case) the maximal operator of the $(C, 1)$-means of the partial sums forms such a mapping. It is of interest that, on the contrary, the maximal operator of the so-called strong $(C, 1)$-means is not (H, L^1)-bounded, namely if

$$\sigma f := \sup_n \frac{1}{n} \sum_{k=1}^{n} |S_k f| \quad (f\in L^1(G_m))$$

then the following theorem is true.

THEOREM 6. *For all sequences m*

$$\sup\{\|\sigma f\|_1\colon\ f\in H(G_m),\quad \|f\| \leqq 1\} = \infty.$$

If a_n $(n\in\mathbf{N})$ denotes the function $a_n:=r_n\cdot D_{M_n}$, then the a_n's are atoms and

$$\|\sigma a_n\|_1 \geqq \frac{1}{2M_n}\sum_{k=1}^{2M_n}\|S_k a_n\|_1 = \frac{1}{2M_n}\sum_{k=M_n+1}^{2M_n}\|S_k a_n\|_1 = \frac{1}{2M_n}\sum_{k=1}^{M_n}\|D_k\|_1.$$

Thus Theorem 6 is a simple consequence of the next lemma.

LEMMA. *There exist absolute constants $C_1>0$ and $C_2>0$ such that*

$$C_1\log n \leqq \frac{1}{n}\sum_{k=1}^{n} L_k \leqq C_2\log n \quad (n=2,3,\ldots). \tag{2}$$

The right side of the lemma follows immediately from the above mentioned relation $L_n=O(\log n)$ $(n\to\infty)$, the left side is a special case of a well-known result of S. V. Bockariev [1]. Since the proof of the general Bockariev's theorem is rather complicated we shall show the lemma in a simple way.

3. Proofs. PROOF OF THEOREM 1. The statement of Theorem 1 is a simple consequence of the following identity [8]:

$$D_n = \Psi_n \sum_{k=0}^{\infty}\sum_{j=m_k-n_k}^{m_k-1} r_k^j D_{M_k} \quad \Big(n=\sum_{k=0}^{\infty} n_k M_k\in\mathbf{N}\Big). \tag{3}$$

Indeed, if $x\in I_j\setminus I_{j+1}$ for some $j\in\mathbf{N}$ then by (1) we get $D_{M_k}(x)=0$ $(k>j)$ therefore

$$|D_n(x)| = \Big|\sum_{k=0}^{j} M_k \sum_{s=m_k-n_k}^{m_k-1}(r_k(x))^s\Big| \leqq \sum_{k=0}^{j-1}(m_k-1)M_k+n_jM_j<(n_j+1)M_j.$$

On the other hand, it follows from $x\in I_j\setminus I_{j+1}$ that $\dfrac{1}{M_{j+1}}\leqq|x|\leqq\dfrac{1}{M_j}$ and thus $|D_n(x)|\leqq\dfrac{n_j+1}{|x|}$. Since $\|D_n\|_\infty\leqq n$ $(n\in\mathbf{N})$, this completes the proof of (i).

To the proof of (ii) we use again (3), i.e.,

$$|D_n(x)| < M_j + M_j\Big|\sum_{s=1}^{n_j}(r_j(x))^s\Big|,$$

if — as above — $x\in I_j\setminus I_{j+1}$. Furthermore we have

$$\Big|\sum_{s=1}^{n_j}(r_j(x))^s\Big| = \frac{\left|\exp\dfrac{2\pi i n_j x_j}{m_j}-1\right|}{\left|\exp\dfrac{2\pi i x_j}{m_j}-1\right|} = \frac{\left|\sin\dfrac{\pi n_j x_j}{m_j}\right|}{\sin\dfrac{x_j\pi}{m_j}} \leqq$$

$$\leqq \frac{1}{\sin\dfrac{\pi x_j}{m_j}} = \frac{1}{\sin\dfrac{\pi\tilde{x}_j}{m_j}} \leqq \frac{m_j}{2\tilde{x}_j},$$

from which $|D_n(x)|\leqq M_j+\dfrac{M_{j+1}}{2\tilde{x}_j}\leqq\dfrac{M_{j+1}}{\tilde{x}_j}\leqq\dfrac{2}{|\tilde{x}|}$ follows.

This completes the proof of Theorem 1.

PROOF OF THEOREM 2. Let $x\in G_m\setminus\{0\}$ be given and denote by $j\in\mathbf{N}$ the same index as in Theorem 1. Then by (1) and (3) we have for all $n\in\mathbf{N}$

$$|D_n(x)| = \Big|\sum_{k=0}^{j} M_k \sum_{s=m_k-n_k}^{m_k-1} (r_k(x))^s\Big| \geqq M_j \Big|\sum_{s=m_j-n_j}^{m_j-1} (r_j(x))^s\Big| -$$

$$-\sum_{k=0}^{j-1} (m_k-1)M_k > M_j \Big|\sum_{s=1}^{n_j} (r_j(x))^s\Big| - M_j,$$

i.e.

$$M_j \Big|\sum_{s=1}^{n_j} (r_j(x))^s\Big| < |D_n(x)| + M_j. \tag{4}$$

Since $\Big|\sum_{s=1}^{n_j} (r_j(x))^s\Big| = \dfrac{\left|\sin \dfrac{\pi n_j x_j}{m_j}\right|}{\sin \dfrac{\pi x_j}{m_j}}$ thus for $n_j := \left[\dfrac{m_j}{2}\right]$ and $x_j := m_j - 1$ it follows that

$$\Big|\sum_{s=1}^{n_j} (r_j(x))^s\Big| = \frac{\left|\sin \dfrac{\pi [m_j/2]}{m_j}\right|}{\sin \dfrac{\pi}{m_j}} \geqq \frac{m_j}{\pi\sqrt{2}}.$$

Furthermore if x is so selected then $|x| \geqq \dfrac{x_j}{M_{j+1}} = \dfrac{m_j-1}{M_{j+1}} \geqq \dfrac{1}{2M_j}$, i.e. $M_j \geqq \dfrac{1}{2|x|}$. From this we get by (4)

$$|D_n(x)| > \frac{1}{2|x|}\left(\frac{m_j}{\pi\sqrt{2}} - 1\right),$$

which gives our statement.

We remark that on the basis of the above proof we obtain a lower bound for C_m in Corollary 1.

PROOF OF THEOREM 3. Let $n = \sum_{k=0}^{\infty} n_k M_k \in \mathbf{N}$ be given and choose $s\in\mathbf{N}$ such that $M_s \leqq n < M_{s+1}$. Then

$$\int_{G_m} \sup_{0\leqq k\leqq n} |D_k| = \int_{I_{s+1}} \sup_{0\leqq k\leqq n} |D_k| + \int_{G_m\setminus I_{s+1}} \sup_{0\leqq k\leqq n} |D_k| =: J_1 + J_2.$$

For J_1 it follows evidently that $J_1 \leqq \dfrac{n}{M_{s+1}} < 1$. Furthermore let us write J_2 in the following form:

$$J_2 = \sum_{j=0}^{s} \int_{I_j\setminus I_{j+1}} \sup_{0\leqq k\leqq n} |D_k| =: \sum_{j=0}^{s} J_{2j}.$$

By Theorem 1 the following estimation is true for J_{2j} $(j=0, \ldots, s-1)$:

$$J_{2j} \leqq 2 \int\limits_{I_j \setminus I_{j+1}} \frac{dx}{|\tilde{x}|} = 2 \sum_{k=1}^{m_j-1} \int\limits_{I_j(0,k)} \frac{dx}{|\tilde{x}|} \leqq 4 \sum_{k=1}^{\left[\frac{m_j}{2}\right]} \frac{1}{k} \leqq C \log m_j.$$

For $j=s$ it is necessary to give a finer estimation, namely by (1) and (3) for $x \in I_s \setminus I_{s+1}$ we get

$$\sup_{0 \leqq k \leqq n} |D_k(x)| = \sup_{0 \leqq k \leqq n} \Big| \sum_{t=0}^{\infty} \sum_{v=m_t-k_t}^{m_t-1} (r_t(x))^v D_{M_t}(x) \Big| =$$

$$= \sup_{0 \leqq k \leqq n} \Big| \sum_{t=0}^{s} M_t \sum_{v=m_t-k_t}^{m_t-1} (r_t(x))^v \Big|,$$

where $k = \sum_{t=0}^{s} k_t M_t \in \{0, \ldots, n\}$. Therefore

$$\sup_{0 \leqq k \leqq n} |D_k(x)| \leqq \sum_{t=0}^{s-1} (m_t-1) M_t + M_s \sup_{0 \leqq k \leqq n} \Big| \sum_{v=1}^{k_s} (r_s(x))^v \Big| <$$

$$< M_s + M_s \sup_{0 \leqq t \leqq n_s} \frac{\left| \sin \dfrac{\pi x_s t}{m_s} \right|}{\sin \dfrac{\pi x_s}{m_s}}$$

and thus

$$J_{2s} \leqq 1 + M_s \int\limits_{I_s \setminus I_{s+1}} \sup_{0 \leqq t \leqq n_s} \frac{\left| \sin \dfrac{\pi x_s t}{m_s} \right|}{\sin \dfrac{\pi x_s}{m_s}} \, dx.$$

Consequently

$$J_{2s} < 1 + \frac{1}{m_s} \sum_{x=1}^{m_s-1} \sup_{0 \leqq t \leqq n_s} \frac{\left| \sin \dfrac{\pi x t}{m_s} \right|}{\sin \dfrac{\pi x}{m_s}}$$

and we need only to prove that

$$\frac{1}{m_s} \sum_{x=1}^{m_s-1} \sup_{0 \leqq t \leqq n_s} \frac{\left| \sin \dfrac{\pi x t}{m_s} \right|}{\sin \dfrac{\pi x}{m_s}} \leqq C \log (n_s+1).$$

To this end let $q:=\left[\frac{m_s}{2n_s}\right]$, then we have

$$\frac{1}{m_s}\sum_{x=1}^{m_s-1}\sup_{0\le t\le n_s}\frac{\left|\sin\frac{\pi xt}{m_s}\right|}{\sin\frac{\pi x}{m_s}}\leqq 2\frac{1}{m_s}\sum_{x=1}^{[m_s/2]}\sup_{0\le t\le n_s}\frac{\left|\sin\frac{\pi xt}{m_s}\right|}{\sin\frac{\pi x}{m_s}}=$$

$$=2\left(\frac{1}{m_s}\sum_{x=1}^{q}\sup_{0\le t\le n_s}\frac{\left|\sin\frac{\pi xt}{m_s}\right|}{\sin\frac{\pi x}{m_s}}+\frac{1}{m_s}\sum_{x=q+1}^{[m_s/2]}\sup_{0\le t\le n_s}\frac{\left|\sin\frac{\pi xt}{m_s}\right|}{\sin\frac{\pi x}{m_s}}\right)\leqq$$

$$\leqq 2\left(\frac{1}{m_s}\sum_{x=1}^{m_s-1}\frac{\left|\sin\frac{\pi xn_s}{m_s}\right|}{\sin\frac{\pi x}{m_s}}+\frac{1}{m_s}\sum_{x=q+1}^{[m_s/2[}\frac{1}{\sin\frac{\pi x}{m_s}}\right).$$

It is known [5] that

$$\frac{1}{m_s}\sum_{x=1}^{m_s-1}\frac{\left|\sin\frac{\pi xn_s}{m_s}\right|}{\sin\frac{\pi x}{m_s}}\leqq C\log(n_s+1).$$

Furthermore for $x\in\{q+1,\ldots,[m_s/2]\}$ the elementary estimation already used above

$$\frac{1}{\sin\frac{\pi x}{m_s}}\leqq\frac{m_s}{2x}$$

is true, accordingly

$$\frac{1}{m_s}\sum_{x=q+1}^{[m_s/2]}\frac{1}{\sin\frac{\pi x}{m_s}}\leqq\frac{1}{2}\sum_{x=q+1}^{[m_s/2]}\frac{1}{x}\leqq C\log\frac{[m_s/2]}{q}\leqq C\log(n_s+1).$$

Summarizing the above facts we get

$$J_2\leqq C\Big(\sum_{j=0}^{s-1}\log m_j+\log(n_s+1)\Big)\leqq C\log n_sM_s\leqq C\log n,$$

which completes the proof of Theorem 3.

Proof of Theorem 4. Let $f\in H(G_m)$ be given, then by the definition of our Hardy space the function f can be represented as $f=\sum_{i=0}^{\infty}\lambda_ia_i$, where a_i's are atoms and

$\sum_{i=0}^{\infty} |\lambda_i| < \infty$. Therefore

$$\sum_{k=1}^{\infty} k^{-1}|\hat{f}(k)| = \sum_{k=1}^{\infty} k^{-1} \Big| \sum_{i=0}^{\infty} \lambda_i \hat{a}_i(k) \Big| \leqq \sum_{i=0}^{\infty} |\lambda_i| \sum_{k=1}^{\infty} k^{-1}|\hat{a}_i(k)|,$$

thus it is enough to prove that $\sup \sum_{k=1}^{\infty} k^{-1}|\hat{a}(k)| < \infty$, where the supremum is taken over all atoms $a \in H(G_m)$. To this end let $a \in H(G_m)$ be an atom and I denote such an interval for which $|a| < |I|^{-1}$, $\operatorname{supp} a \subset I$ and $\int_I a = 0$. (We may assume evidently that $a \neq 1$.) Define $n \in \mathbf{N}$ and $y \in G_m$ such that $I \subset I_n(y)$ holds and n is maximal. Since Ψ_k $(k=0, \ldots, M_n-1)$ is constant on $I_n(y)$ thus $\hat{a}(k)=0$ $(k=0, \ldots, M_n-1)$, therefore

$$\sum_{k=1}^{\infty} k^{-1}|\hat{a}(k)| = \sum_{k=M_n}^{\infty} k^{-1}|\hat{a}(k)| = \sum_{k=M_n}^{M_{n+1}-1} k^{-1}|\hat{a}(k)| + \sum_{k=M_{n+1}}^{\infty} k^{-1}|\hat{a}(k)| =: \Sigma^1 + \Sigma^2.$$

Applying the Cauchy—Buniakowski inequality we get for Σ^2

$$\Sigma^2 \leqq \Big(\sum_{k=M_{n+1}}^{\infty} |\hat{a}(k)|^2 \Big)^{1/2} \Big(\sum_{k=M_{n+1}}^{\infty} k^{-2} \Big)^{1/2} \leqq C \frac{\|a\|_2}{\sqrt{M_{n+1}}} \leqq C.$$

Let us write Σ^1 in the form

$$\Sigma^1 = \sum_{j=1}^{m_n-1} \sum_{k=0}^{M_n-1} \frac{|\hat{a}(jM_n+k)|}{jM_n+k}$$

and observe that $|\hat{a}(jM_n+k)| = |\hat{a}(jM_n)|$ $(j=1, \ldots, m_n-1, k=0, \ldots, M_n-1)$. From this it follows that

$$\Sigma^1 = \sum_{j=1}^{m_n-1} |\hat{a}(jM_n)| \sum_{k=0}^{M_n-1} \frac{1}{jM_n+k} \leqq \sum_{j=1}^{m_n-1} \frac{|\hat{a}(jM_n)|}{j}.$$

If the function A is defined on Z_{m_n} as

$$A(k) := m_n \int_{I_n(y,k)} a \quad (k \in Z_{m_n}),$$

then $\hat{a}(jM_n)$ is the j-th $(j=1, \ldots, m_n-1)$ Fourier coefficient $\hat{A}(j)$ of A with respect to the (discrete trigonometric) system

$$e_k(t) := \exp \frac{2\pi ikt}{m_n} \quad (t \in Z_{m_n}, k = 0, \ldots, m_n-1).$$

Let us define $a, b \in \{0, \ldots, m_n-1\}$ so that $I = \bigcup_{k=a}^{b} I_n(y, k)$ and let $c := a + \left[\frac{b-a}{2}\right]$.

Then supp $A \subset \{a, \dots, b\}$, $\sum_{k=a}^{b} A(k)=0$ and $|A(k)| \leqq \frac{m_n}{b-a}$ $(k \in \{a, \dots, b\})$. Moreover,

$$|\hat{a}(jM_n)| = \left| \frac{1}{m_n} \sum_{k=a}^{b} A(k) \left(\exp \frac{2\pi ikj}{m_n} - \exp \frac{2\pi icj}{m_n} \right) \right| \leqq$$

$$\leqq C \sum_{k=a}^{b} \frac{1}{m_n} |A(k)| \frac{|k-c|j}{m_n} \leqq C \frac{(b-a)j}{m_n^2} \sum_{k=a}^{b} \frac{m_n}{b-a} = C \frac{(b-a)j}{m_n},$$

from which we can establish the desired estimation in the following way (for the idea see [4]):

$$\sum_{j=1}^{m_n-1} \frac{|\hat{a}(jM_n)|}{j} = \sum_{j=1}^{m_n-1} \frac{|\hat{A}(j)|}{j} = \sum_{j \leqq m_n/b-a} + \sum_{j > m_n/b-a} \leqq$$

$$\leqq C + \sqrt{\sum_{j > m_n/b-a} j^{-2}} \sqrt{\frac{1}{m_n} \sum_{k=a}^{b} |A(k)|^2} \leqq C + C \sqrt{\frac{b-a}{m_n}} \sqrt{\sum_{k=a}^{b} \frac{m_n}{(b-a)^2}} \leqq C.$$

The proof of Theorem 4 is thus complete.

Proof of Theorem 5. Without loss of generality we may assume that $m_n \geqq 6$ $(n \in \mathbf{N})$. Let $\Delta_k := [m_k/2]+1$ $(k \in \mathbf{N})$ and define the functions f_k as follows:

$$f_k(x) := \begin{cases} M_{k+1} & x \in I_k(0, 1) \\ -M_{k+1} & x \in I_k(0, \Delta_k) \quad (x \in G_m,\ k \in \mathbf{N}). \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the supports of f_k's, resp. f_k^*'s are disjoint and $\|f_k\|_1 = \|f_k^*\|_1 = 2$ $(k \in \mathbf{N})$. Therefore for the function $f := \sum_{k=0}^{\infty} \lambda_k f_k$ $(\lambda_k > 0\ (k \in \mathbf{N}),\ \sum_{k=0}^{\infty} \lambda_k < \infty)$ the relation $\|f\|_1 = \|f^*\|_1 = 2 \sum_{k=0}^{\infty} \lambda_k < \infty$ is true. On the other hand

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(k)|}{k} = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k-1} \sum_{s=0}^{M_k-1} \frac{|\hat{f}(jM_k+s)|}{jM_k+s},$$

where for fixed j, k and s we have

$$|\hat{f}(jM_k+s)| = \left| \sum_{t=0}^{\infty} \lambda_t \hat{f}_t(jM_k+s) \right| = \lambda_k |\hat{f}_k(jM_k+s)| = \lambda_k \left| \int_{I_k} f_k \bar{r}_k^j \right| =$$

$$= \lambda_k \left| \exp \frac{2\pi ij}{m_k} - \exp \frac{2\pi ij\Delta_k}{m_k} \right| = \lambda_k \left| 1 - \exp \frac{2\pi ij[m_k/2]}{m_k} \right| = 2\lambda_k \left| \sin \frac{\pi j[m_k/2]}{m_k} \right| \geqq C\lambda_k,$$

if $\Delta_k \geqq j \equiv 1 \pmod 2$. Thus

$$\sum_{k=1}^{\infty} k^{-1} |\hat{f}(k)| \geqq \sum_{k=0}^{\infty} \sum_{\substack{j=1 \\ j \equiv 1(2)}}^{\Delta_k} \sum_{s=0}^{M_k-1} \frac{C\lambda_k}{jM_k+s} \geqq C \sum_{k=0}^{\infty} \lambda_k \sum_{\substack{j=1 \\ j \equiv 1(2)}}^{\Delta_k} j^{-1} \geqq C \sum_{k=0}^{\infty} \lambda_k \log m_k.$$

Since $\limsup m = \infty$ we can choose a sequence of indices $(n_k, k \in \mathbf{N})$ such that $\sum_{k=0}^{\infty} \frac{1}{\log m_{n_k}} < \infty$. Define

$$\lambda_j := \begin{cases} \dfrac{1}{\log m_{n_k}} & \text{if } j = n_k \text{ for some } k \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases} \quad (j \in \mathbf{N}).$$

Then $\sum_{k=1}^{\infty} k^{-1} |\hat{f}(k)| = \infty$, as was stated.

Proof of the Lemma. As it was already mentioned, it remains to show only the left side of (2). First of all we observe that it is enough to prove

$$\frac{1}{jM_n} \sum_{k=1}^{jM_n} L_k \geqq C \log jM_n \quad (n \in \mathbf{N}, j = 1, \dots, m_n - 1).$$

For such a j and n we get

$$\frac{1}{jM_n} \sum_{k=1}^{jM_n} L_k = \frac{1}{jM_n} \int_{G_m} \sum_{k=1}^{jM_n} |D_k| \geqq \frac{1}{jM_n} \int_{I_n \setminus I_{n+1}} \sum_{k=1}^{jM_n} |D_k| +$$

$$+ \frac{1}{jM_n} \sum_{s=0}^{n-1} \int_{I_s \setminus I_{s+1}} \sum_{k=1}^{jM_n} |D_k| =: A_1 + A_2.$$

1° If $k \in \{1, \dots, jM_n\}$ then $k = \sum_{i=0}^{n} k_i M_i$ and $k_n \leqq j$, therefore by (1) and (3) for $x \in I_s \setminus I_{s+1}$ $(s = 0, \dots, n-1)$

$$|D_k(x)| = \Big| \sum_{t=0}^{s} \sum_{v=m_t-k_t}^{m_t-1} (r_t(x))^v M_t \Big| = M_s \left| \sum_{t=0}^{s-1} \frac{k_t M_t}{M_s} + \sum_{v=m_s-k_s}^{m_s-1} (r_s(x))^v \right| \geqq$$

$$\geqq M_s \Big(\Big| \sum_{v=1}^{k_s} (r_s(x))^v \Big| - 1 \Big).$$

From this it follows that

$$\int_{I_s \setminus I_{s+1}} \sum_{k=1}^{jM_n} |D_k| \geqq M_s \sum_{k=1}^{jM_n} \int_{I_s \setminus I_{s+1}} \Big(\Big| \sum_{v=1}^{k_s} (r_s)^v \Big| - 1 \Big) \geqq$$

$$\geqq \sum_{k=M_s}^{jM_n} m_s^{-1} \sum_{x=1}^{m_s-1} \frac{\left| \sin \dfrac{\pi x k_s}{m_s} \right|}{\sin \dfrac{\pi x}{m_s}} - jM_n = \frac{jM_n}{m_s^2} \sum_{k=1}^{m_s-1} \sum_{x=1}^{m_s-1} \frac{\left| \sin \dfrac{\pi x k}{m_s} \right|}{\sin \dfrac{\pi x}{m_s}} - jM_n,$$

whereas

$$m_s^{-2} \sum_{x=1}^{m_s-1} \sum_{k=1}^{m_s-1} \frac{\left| \sin \dfrac{\pi x k}{m_s} \right|}{\sin \dfrac{\pi x}{m_s}} \geqq C m_s^{-2} \sum_{x=1}^{[m_s/2]} \frac{m_s x}{x} \sum_{k=1}^{[m_s/2x]} \frac{xk}{m_s} = C m_s^{-2} \sum_{x=1}^{[m_s/2]} x \frac{m_s^2}{x^2} \geqq C \log m_s.$$

This implies

$$\frac{1}{jM_n}\int\limits_{I_s\setminus I_{s+1}}\sum_{k=1}^{jM_n}|D_k|\geqq C\log m_s-1,$$

i.e.

$$\frac{1}{jM_n}\int\limits_{I_s\setminus I_{s+1}}\sum_{k=1}^{jM_n}|D_k|\geqq C\log m_s\quad (s=0,\dots,n-1),$$

if the m_s's are large enough. It is not hard to see that this estimation holds for all m_s's, therefore $A_2\geqq C\log M_n$.

2° In the same manner as above we get

$$A_1=\frac{1}{jM_n}\int\limits_{I_s\setminus I_{s+1}}\sum_{k=1}^{jM_n}|D_k|\geqq\frac{1}{jM_n}\sum_{k=M_n}^{jM_n}m_n^{-1}\sum_{x=1}^{m_n-1}\frac{\left|\sin\frac{\pi x k_n}{m_n}\right|}{\sin\frac{\pi x}{m_n}}-1=$$

$$=m_n^{-1}\sum_{x=1}^{m_n-1}j^{-1}\sum_{k=1}^{j}\frac{\left|\sin\frac{\pi x k}{m_n}\right|}{\sin\frac{\pi x}{m_n}}-1\geqq C\sum_{x=1}^{[m_n/2]}\frac{1}{jx}\sum_{k=1}^{j}\left|\sin\frac{\pi x k}{m_n}\right|-1\geqq$$

$$\geqq C\sum_{x=1}^{[m_n/2]}\frac{1}{jx}\left[\frac{2xj}{m_n}\right]\sum_{k=1}^{[m_n/2x]}\frac{xk}{m_n}-1\geqq C\sum_{x=[m_n/2j]}^{[m_n/2]}m_n^{-2}xm_n^2x^2-1\geqq C\log j-1.$$

These inequalities prove our statement.

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CORRECTIONS TO "COMPLETE CONVERGENCE AND CONVERGENCE RATES FOR RANDOMLY INDEXED PARTIAL SUMS WITH AN APPLICATION TO SOME FIRST PASSAGE TIMES"

A. GUT (Uppsala)

In Section 5 of my paper [1] it is claimed that "by using obvious modifications" of previous proofs and the corresponding results in the classical case "one can obtain results", which are then given as Theorems 5.1 and 5.2. However, whereas the classical results corresponding to the results in the earlier sections are valid without restrictions on ε, this is no longer the case in Section 5 and the latter results therefore have to be modified slightly. The corrected results are presented below.

THEOREM 5.1. a) *Let* $r>2$. *Suppose that* $EX_1=0$, $EX_1^2=\sigma^2$ *and* $E|X_1|^r\cdot(\log^+|X_1|)^{-r/2}<\infty$. *If*

$$\sum_{n=1}^{\infty} n^{(r/2)-2}P(|N_n-Nn|>n\delta)<\infty \quad \textit{for all } \delta>0, \tag{5.1}$$

where N is a positive random variable, such that $P(N\geqq A)=1$ for some $A>0$, then

$$\sum_{n=1}^{\infty} n^{(r/2)-2}P\left(|S_{N_n}|>\varepsilon\sqrt{N_n\log^+ N_n}\right)<\infty, \quad \varepsilon>\sigma\sqrt{r-2}. \tag{5.2}$$

b) *Let* $r=2$. *Suppose that* $EX_1=0$ *and* $EX_1^2<\infty$. *If* (5.1) *holds with N as above, then* (5.2) *holds for all* $\varepsilon>0$.

c) *Let* $r\geqq 2$. *Suppose that* $EX_1^2=\sigma^2<\infty$ *and* $EX_1=0$. *If* (5.1) *holds with* $P(N\leqq B)=1$ *for some* $B>0$, *then*

$$\sum_{n=1}^{\infty} n^{(r/2)-2}P\left(|S_{N_n}|>\varepsilon\sqrt{n\log n}\right)<\infty, \quad \varepsilon>\sigma\sqrt{(r-2)B}. \tag{5.3}$$

THEOREM 5.2. a) *Suppose that* $EX_1=0$, $EX_1^2=\sigma^2$ *and that*

$$EX_1^2\frac{\log^+|X_1|}{\log^+\log^+|X_1|}<\infty.$$

If

$$\sum_{n=1}^{\infty}\frac{1}{n}P(|N_n-Nn|>n\delta)<\infty \quad \textit{for all } \delta>0, \tag{5.4}$$

where N is a positive random variable, such that $P(N\geqq A)=1$ for some $A>0$, then

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(|S_{N_n}|>\varepsilon\sqrt{N_n\log^+\log^+ N_n}\right)<\infty, \quad \varepsilon>2\sigma. \tag{5.5}$$

However, if (5.4) *holds with* $P(A\leqq N\leqq B)=1$ *for* $0<A\leqq B<\infty$, *then* $EX_1^2=$ $=\sigma^2<\infty$ *and* $EX_1=0$ *imply that* (5.5) *holds for* $\varepsilon>\sigma\sqrt{2\frac{B}{A}}$.

b) *Suppose that* $EX_1^2=\sigma^2<\infty$ *and* $EX_1=0$. *If* (5.4) *holds with* $P(N\leqq B)=1$ *for some* $B>0$, *then*

$$\sum_{n=3}^{\infty}\frac{1}{n}P\big(|S_{N_n}|>\varepsilon\sqrt{n\log\log n}\big)<\infty,\quad \varepsilon>\sigma\sqrt{2B}. \tag{5.6}$$

REMARRK. A minor change should also be made in Theorem 2.1, namely that the assumption should be "If, for some $\delta>0$, (2.1) holds (with ε replaced by δ)" and the conclusion should be that "(2.2) holds for all $\varepsilon>0$". As a consequence some ε have to be replaced with δ in an obvious way.

Finally, as a consequence of the above facts, the formulas in part C on page 231 (which furthermore should have been numbered as (5.7)—(5.9)) are valid for all $\varepsilon>\sigma\sqrt{2\mu^{-3}}$, $\sigma\sqrt{2}$ and $\sigma\sqrt{2\mu^{-1}}$, respectively, and formulas (5.8) and (5.9) in D should have been called (5.10) and (5.11).

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[1] A. Gut, Complete convergence and convergence rates for randomly indexed partial sums with an application to some first passage times, *Acta Math. Hung.*, **42** (1983), 225—232.

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[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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CONTENTS

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T_3-CLOSED EXTENSIONS, SYSTEMS OF FILTERS, PROXIMITIES

K. MATOLCSY (Debrecen)

0. Preliminaries

A space X having some topological property **P** is said to be **P**-closed if it is closed in any topological space possessing **P** that contains X as a subspace (or equivalently, if it has no proper extension with the property **P**). If $\mathbf{P}=$"to be T_2" then we get the well-known H-closed spaces. Another important particular case of this notion is $\mathbf{P}=$"to be T_3" (where $T_3=\text{regular}+T_0$). T_3-closed spaces were studied firstly by B. Banaschewski [2] and generalized by C. T. Scarborough—A. H. Stone [13].

From the point of view of the study of these spaces it is important to know all regular and T_3 extensions of a space. D. Doičinov [8] characterized them by means of systems $\mathfrak{R}$ of filters such that

(1) any neighbourhood filter is in $\mathfrak{R}$, and

(2) if $R\in\mathfrak{r}\in\mathfrak{R}$ then there exists $R_1\in\mathfrak{r}$, for which $\mathfrak{r}'\in\mathfrak{R}$, $\emptyset\notin\{R_1\}(\cap)\mathfrak{r}'$ imply $R\in\mathfrak{r}'$.

Y is a regular extension of X iff it is a strict extension ([4], p. 215) and the system $\mathfrak{R}=\{\mathfrak{v}_Y(y)(\cap)\{X\}: y\in Y\}$ satisfies (1)—(2), where $\mathfrak{v}_Y(y)$ is the neighbourhood filter of $y\in Y$.

Starting out from a counterexample of H. Herrlich ([10], S. 20), D. Harris [9] described the class of all T_3-spaces which have a T_3-closed extension. This method is based on R- and RC-proximities. Following essentially the terminology of [9], by an *R-proximity* on a set X we shall mean a symmetrical relation δ on $\mathfrak{P}(X)$ such that

(a) $\emptyset\bar{\delta}A$ for every $A\subset X$,

(b) $A\delta A$ for every $A\neq\emptyset$,

(c) $A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$,

(d) $x\in X$, $x<A$ implies $x<B<A$ for some $B\subset X$, where $E<F$ means $E\bar{\delta}X-F$.

A filter $\mathfrak{r}$ in X is said to be *round* iff $R\in\mathfrak{r}$ implies $R_1<R$ for some $R_1\in\mathfrak{r}$. An R-proximity δ is called an *RC-proximity* iff

(e) $A<B\Leftrightarrow\emptyset\notin\{A\}(\cap)\mathfrak{r}$ implies $B\in\mathfrak{r}$ for any maximal round filter $\mathfrak{r}$.

We say that an R-proximity is *separated,* if

(f) $x, y\in X$, $x\neq y$ imply $x\bar{\delta}y$.

Any regular (T_3) topology can be induced by a (separated) R-proximity δ as follows:

$F\subset X$ is closed iff $x\delta F$ implies $x\in F$,

or equivalently

$V\subset X$ is open iff $x\in V$ implies $x<V$.

The main result of Harris is that there exists a one-to-one correspondence between the T_3-closed extensions and compatible separated RC-proximities of a T_3-space.

1. T_3-reduced extensions

Let Y be a topological space and $\mathfrak{v}_Y(x)$ be the neighbourhood filter of $x \in Y$. Y is said to be an *S_2-space* iff $x, y \in Y$, $\mathfrak{v}_Y(x) \neq \mathfrak{v}_Y(y)$ implies $\emptyset \in \mathfrak{v}_Y(x)(\cap)\mathfrak{v}_Y(y)$ (see e.g. [4]). Thus Y is T_2 iff it is T_0 and S_2. Y is an S_3- (or a *regular*) space if whenever $x \in X$, $F \subset Y$ is closed and $x \notin F$, then x and F can be separated by disjoint open sets. Analogously Y is T_3 if and only if it is T_0 and regular.

If X is a subspace of Y then Y is called *reduced* with respect to X iff $x \in Y$, $y \in Y - X$, $x \neq y$ imply $\mathfrak{v}_Y(x) \neq \mathfrak{v}_Y(y)$ (see [4]). Y is said to be *S_2-reduced* with respect to X iff $x \in Y$, $y \in Y - X$, $\mathfrak{v}_Y(x) \neq \mathfrak{v}_Y(y)$ imply $\emptyset \in \mathfrak{v}_Y(x)(\cap)\mathfrak{v}_Y(y)$ (see [5]). Y is *T_2-reduced* with respect to X iff it is reduced and S_2-reduced, that is $x \in Y$, $y \in Y - X$, $x \neq y$ imply $\emptyset \in \mathfrak{v}_Y(x)(\cap)\mathfrak{v}_Y(y)$ (see [7] and [11]).

The above "reduced separation axioms" express that Y satisfies the corresponding separation axiom as far as it is possible in spite of the unfavourable separation properties of X.

We shall say that Y is *S_3-reduced* (or *regular-reduced*) with respect to X iff, for any $x \in Y - X$ ($x \in X$) and any closed set $F \subset Y$ such that $x \notin F$, x and F (x and $F - X$) can be separated by disjoint open sets in Y. Y is *T_3-reduced* with respect to X iff it is reduced and S_3-reduced. It is obvious that a regular space (T_3-space) is regular-reduced (T_3-reduced) with respect to each of its subspaces.

(1.1) LEMMA. *Any S_3-reduced (T_3-reduced) space is also S_2-reduced (T_2-reduced) with respect to the same subspace.*

PROOF. If $y \in Y$ and $x \in Y - X$ such that $\mathfrak{v}_Y(x) \neq \mathfrak{v}_Y(y)$ then $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. In the first case x and $F = \overline{\{y\}}$ can be separated by disjoint open sets. In the second one assume $F = \overline{\{x\}}$. Then $x \in F - X$, thus y and x have disjoint neighbourhoods even if $y \in X$. If Y is T_3-reduced with respect to X then $y \in Y$, $x \in Y - X$, $x \neq y$ imply $\mathfrak{v}_Y(x) \neq \mathfrak{v}_Y(y)$, therefore the above train of thought can be applied. $\square$

If X is a dense subspace of a topological space Y then Y is called an *extension* of X. Y is a *regular-reduced (T_3-reduced) extension* of X iff it is regular-reduced (T_3-reduced) with respect to X.

(1.2) THEOREM. *A regular-reduced (T_3-reduced) extension Y of a regular (T_3-) space X is always regular (a T_3-space).*

PROOF. Suppose that X is regular. If F is closed in Y and $x \notin F$, then in the case $x \in Y - X$, x and F can be separated by disjoint open sets. If $x \in X$ then there are disjoint open sets $V, W \subset Y$ such that $x \in V$ and $F - X \subset W$. The set $F \cap X$ is closed in X, therefore there exist open sets V_0, W_0 of X, for which $x \in V_0$, $F \cap X \subset W_0$, $V_0 \cap W_0 = \emptyset$. If V_1, W_1 are open sets in Y such that $V_0 = V_1 \cap X$ and $W_0 = W_1 \cap X$, then $V_1 \cap W_1 = \emptyset$ because of the density of X in Y. Thus $V \cap V_1$ and $W \cup W_1$ are disjoint open sets containing x and F, respectively. If, in addition, X is T_0, and Y is reduced, then Y is also T_0 ([4], (6.1.16)). $\square$

In order to characterize regular-reduced extensions let us introduce the following notion:

Denoting always by $\mathfrak{N}$ the family of all neighbourhood filters of a given topological space X, a system $\mathfrak{S}$ of filters in X will be said *regular* iff

(R1) *any filter in* $\mathfrak{S}$ *is regular,*

and

(R2) *if* $\mathfrak{r}\in\mathfrak{N}\cup\mathfrak{S}$ *then, for every* $V\in\mathfrak{r}$, *there exists* $W\in\mathfrak{r}$ *such that* $\mathfrak{s}\in\mathfrak{S}$ *implies either* $V\in\mathfrak{s}$ *or* $X-W\in\mathfrak{s}$.

$\mathfrak{S}=\emptyset$ always satisfies these axioms, thus it will be called the *trivial* regular system in X.

Let Y be an extension of the topological space X. Then $\mathfrak{s}(x)=\mathfrak{v}_Y(x)(\cap)\{X\}$ $(x\in Y)$ is an open filter in X, and it is called the *trace filter* of this extension corresponding to x. We denote the system $\{\mathfrak{s}(p)\colon p\in Y-X\}$ by $\mathfrak{S}_Y$. If $x\in X$ then $\mathfrak{s}(x)=\mathfrak{v}_X(x)$ is obvious. Y is said to be a *strict extension* of X, if the sets $s(G)=\{x\in Y\colon G\in\mathfrak{s}(x)\}$, where G is open in X, form an open base in Y (cf. [4], ch. 6.1.b).

(1.3) THEOREM. *The extension* Y *of the topological space* X *is regular-reduced iff the following conditions are satisfied:*

(1.3.1) Y *is a strict extension of* X.

(1.3.2) *The system* $\mathfrak{S}_Y$ *is regular in* X.

PROOF. If $Y=X$ then $\mathfrak{S}_Y=\emptyset$, and the theorem is trivial. Let Y be a proper regular-reduced extension of X. Suppose that $V\subset Y$ is an open neighbourhood of $x\in Y$. We show $x\in s(G)\subset V$ for some open set G of X. If $x\in V\cap X$ then there exist open sets $V_1, W_1\subset Y$ such that $x\in V_1$, $(Y-V)-X\subset W_1$ and $V_1\cap W_1=\emptyset$. Assume $G=V\cap V_1\cap X$. Then G is open in X and $x\in s(G)$. Observe $s(G)\cap X=G\subset V$. On the other hand, if $p\in s(G)-X$ and $p\notin V$, then $W_1\cap X\in\mathfrak{s}(p)$. Therefore $G\in\mathfrak{s}(p)$ implies $\emptyset=V_1\cap W_1\supset G\cap W_1=G\cap W_1\cap X\in\mathfrak{s}(p)$, which is impossible, hence $s(G)\subset V$. In the other case, when $x\in V-X$, we have open sets $V_1, W_1\subset Y$ such that $x\in V_1$, $Y-V\subset W_1$ and $V_1\cap W_1=\emptyset$. Supposing $G=V_1\cap X$, we get $x\in s(G)$ and $s(G)\subset Y-s(X-G)=\overline{G}^Y\subset\overline{V}_1^Y\subset Y-W_1\subset V$ (cf. [4], (6.1.9)(a)).

We show that $\mathfrak{S}_Y$ has the properties (R1)—(R2). $\mathfrak{s}(p)$ is an open filter in X for any $p\in Y-X$. If $G\in\mathfrak{s}(p)$ is open then $F=Y-s(G)$ is closed in Y, and $p\notin F$, thus $p\in V$, $F\subset W$, $V\cap W=\emptyset$, where V and W are open in Y. From here $p\in V\subset\overline{V}^Y\subset Y-W\subset s(G)$, consequently $F_0=\overline{V}^Y\cap X$ is a closed set in X, for which $F_0\in\mathfrak{s}(p)$ and $F_0\subset s(G)\cap X=G$, hence $\mathfrak{s}(x)$ is a regular filter in X. In order to see (R2) put $\mathfrak{r}\in\mathfrak{N}\cup\mathfrak{S}_Y$. Then $\mathfrak{r}=\mathfrak{s}(x)$ for some $x\in Y$. If $V\in\mathfrak{s}(x)$ and $G\subset V$ is an open set of $\mathfrak{s}(x)$ in X, then $Y-s(G)$ is closed in Y, and does not contain x. In this case there are open sets U, U' in Y such that $x\in U$, $(Y-s(G))-X\subset U'$ and $U\cap U'=\emptyset$. Then $W=U\cap X\in\mathfrak{s}(x)$ and $Y-s(X-W)=\overline{W}^Y\subset\overline{U}^Y\subset Y-U'\subset s(G)\cup X\subset s(V)\cup X$ so that $p\in Y-X$, $X-W\notin\mathfrak{s}(p)$ imply $V\in\mathfrak{s}(p)$ (cf. [4], (6.1.9)(a)).

Conversely, let us assume (1.3.1)—(1.3.2), and prove that Y is regular-reduced. Firstly suppose $p\in Y-X$ and let F be a closed set in Y, $p\notin F$. Then there exists an open $V\in\mathfrak{s}(p)$ such that $s(V)\subset Y-F$, and there is $W\in\mathfrak{s}(p)$, for which either $V\in\mathfrak{s}(q)$ or $X-W\in\mathfrak{s}(q)$ whenever $q\in Y-X$. In view of the regularity of $\mathfrak{s}(p)$, an open set U of X can be chosen such that $U\in\mathfrak{s}(p)$, $\overline{U}^X\subset V\cap W$. We show $\overline{s(U)}^Y\subset Y-F$. In fact, $\overline{s(U)}^Y\cap X=\overline{s(U)\cap X}^X=\overline{U}^X\subset V\subset s(V)$. At the same time $q\in\overline{s(U)}^Y-X$ implies $\emptyset\neq s(U)\cap s(S)=s(U\cap S)$ for each $S\in\mathfrak{s}(q)$, hence $U\cap S\neq\emptyset$. From this $X-U\notin\mathfrak{s}(q)$, and *a fortiori* $X-W\notin\mathfrak{s}(q)$. We get $V\in\mathfrak{s}(q)$, that is $q\in s(V)$. Thus $\overline{s(U)}^Y\subset s(V)\subset Y-F$, so that $s(U)$ and $Y-\overline{s(U)}^Y$ are disjoint open sets separating p and F. Let now x be a point of X, F be a closed set of Y, $x \notin F$. Then choose $V, W \in \mathfrak{v}_X(x) = \mathfrak{s}(x)$ in the same way as in the above train of thought, and let $U \in \mathfrak{v}_X(x)$ be open in X, $U \subset V \cap W$. A similar consideration gives that $s(U)$ and $Y - \overline{s(U)}^Y$ separate x and $F - X$. $\square$

(1.4) Corollary (Doičinov [8]). *The extension Y of the topological space X is regular iff Y is a strict extension of X and the system $\mathfrak{R} = \{\mathfrak{s}(y) \colon y \in Y\}$ satisfies condition* (2).

Proof. If the condition is satisfied then X is regular and $\mathfrak{S}_Y$ has (R1)—(R2), thus Y is also regular by (1.2) and (1.3). Conversely, if Y is regular then it is regular-reduced with respect to X, therefore it is a strict extension and $\mathfrak{S}_Y$ is regular. Since in this case the subspace X is also regular, any filter $\mathfrak{r} \in \mathfrak{R}$ is regular. If $R \in \mathfrak{r}$ then one can choose $R_1 \in \mathfrak{r}$ such that $R \in \mathfrak{s}$ or $X - R_1 \in \mathfrak{s}$ whenever $\mathfrak{s} \in \mathfrak{S}_Y$. Put $R_2 \in \mathfrak{r}$, $\overline{R}_2^X \subset \mathrm{int}_X R$ and $R_3 = R_1 \cap R_2$. It is easy to check that $\mathfrak{r}' \in \mathfrak{R}$, $\emptyset \notin \{R_3\}(\cap)\mathfrak{r}'$ implies $R \in \mathfrak{r}'$. $\square$

We say that a regular system $\mathfrak{S}$ of filters in X is *free regular*, if any filter $\mathfrak{s} \in \mathfrak{S}$ is a free filter, i.e. $\bigcap \mathfrak{s} = \emptyset$. Now (1.3) can be completed as follows:

(1.5) Corollary. *The extension Y of the topological space X is T_3-reduced iff it is strict, the system $\mathfrak{S}_Y$ is free regular, and $\mathfrak{s}(p) \neq \mathfrak{s}(q)$ whenever $p, q \in Y - X$, $p \neq q$.*

Proof. If Y is T_3-reduced, $p \in Y - X$, and $x \in X$ is such that $x \in \bigcap \mathfrak{s}(p)$, then $\mathfrak{s}(p) \subset \mathfrak{v}_X(x)$, because $\mathfrak{s}(p)$ is an open filter. For any $W \in \mathfrak{v}_X(x)$, $X - W$ cannot be in $\mathfrak{s}(p)$, thus $\mathfrak{v}_X(x) \subset \mathfrak{s}(p)$ by (R2), i.e. $\mathfrak{s}(x) = \mathfrak{v}_X(x) = \mathfrak{s}(p)$. Then the system $\{s(G) \colon G \in \mathfrak{s}(p) = \mathfrak{s}(x)\}$ is a neighbourhood base for p as well as for x, thus $p = x$ because Y is reduced, but this is a contradiction. Similarly, for $p, q \in Y - X$, $\mathfrak{s}(p) = \mathfrak{s}(q)$ implies $p = q$. Conversely, if the conditions are satisfied, then $p \in Y - X$, $y \in Y$, $p \neq y$ implies $\mathfrak{s}(p) \neq \mathfrak{s}(y)$, hence Y is reduced by [4], (6.1.17). The remaining part of the proof is contained in (1.3). $\square$

(1.6) Examples. Let X be the natural topological space of the rational numbers and Y be the real line with the topology in which $V \subset Y$ is open iff, for any $x \in V$, there exists $\varepsilon > 0$ such that $\{x\} \cup ((x - \varepsilon, x + \varepsilon) \cap X) \subset V$. Then X is T_3, but Y is not a T_3-reduced extension (in fact, it is not strict).

Let X be the real line with the deleted sequence topology of Yu. Smirnov (for $x \neq 0$, $\mathfrak{v}_X(x)$ is the natural one, and the sets $(-\varepsilon, \varepsilon) - \{1/n \colon n = 1, 2, \ldots\}$ form a base for $\mathfrak{v}_X(0)$. Suppose $Y = X \cup \{p\}$, $p \notin X$, $\mathfrak{s}(p) = \{S \subset X \colon (x, \infty) \subset S$ for some $x \in X\}$, and let Y be topologized by the strict extension of the topology of X corresponding to $\mathfrak{s}(p)$. Then Y is T_3-reduced, but not T_3 (in fact, X is not regular (see (1.2)). $\square$

2. Comparison of T_3-reduced extensions

Let Y and Z be two extensions of the topological space X. As it is usual, Y is said to be a *finer extension of X than Z* ($Z \leqq Y$) iff there exists a continuous surjection $f \colon Y \to Z$ fixing X (i.e. $f(x) = x$ for any $x \in X$). They are called *equivalent* ($Z = Y$) iff, in addition, f is a homeomorphism. The sign $=$ reflects that we do not distinguish equivalent extensions.

In order to show that in the set of all T_3-reduced extensions of a given space X the above order can be characterized by means of the systems of the trace filters corresponding to the imaginary points, let us consider two systems $\mathfrak{S}_1$, $\mathfrak{S}_2$ of filters in X. We shall say that $\mathfrak{S}_2$ is *finer* than $\mathfrak{S}_1$ ($\mathfrak{S}_1 \leqq \mathfrak{S}_2$) iff, for any $\mathfrak{s}_2 \in \mathfrak{S}_2$, there exists $\mathfrak{s}_1 \in \mathfrak{S}_1$ such that $\mathfrak{s}_1 \subset \mathfrak{s}_2$, and conversely, for any $\mathfrak{s}_1 \in \mathfrak{S}_1$, there is $\mathfrak{s}_2 \in \mathfrak{S}_2$ such that $\mathfrak{s}_1 \subset \mathfrak{s}_2$.

(2.1) THEOREM. *Let X be an arbitrary topological space.*

(2.1.1) *There exists a one-to-one correspondence between the T_3-reduced extensions Y of X and the free regular systems $\mathfrak{S}$ of filters in X. Y and $\mathfrak{S}$ correspond to each other iff $\mathfrak{S} = \mathfrak{S}_Y$.*

(2.1.2) *If Y and Z are two T_3-reduced extensions of X, then $Z \leqq Y$ iff $\mathfrak{S}_Z \leqq \mathfrak{S}_Y$ holds.*

(2.1.3) *Y and Z are equivalent iff each of them is finer than the other.*

For the verification of the theorem we need three lemmas.

(2.2) LEMMA. *Let $\mathfrak{S}$ be a free regular system of filters in X and $\mathfrak{s} \in \mathfrak{S}$. Then $\mathfrak{s}$ has no cluster point. If $\mathfrak{s}' \in \mathfrak{S}$, $\mathfrak{s} \neq \mathfrak{s}'$, then $\emptyset \in \mathfrak{s}(\cap)\mathfrak{s}'$.*

PROOF. If $x \in X$ then $x \notin S$ for some $S \in \mathfrak{s}$, and $F \subset S$ for a suitable closed $F \in \mathfrak{s}$, thus $X-F$ is a neighbourhood of x. If $\mathfrak{s}' \in \mathfrak{S}$, $\mathfrak{s} \neq \mathfrak{s}'$ and e.g. $\mathfrak{s} \not\subset \mathfrak{s}'$, then, for $V \in \mathfrak{s} - \mathfrak{s}'$, there exists $W \in \mathfrak{s}$ with $X - W \in \mathfrak{s}'$ by (R2). □

(2.3) LEMMA. *Let Y be arbitrary, and Z be a T_3-reduced extension of X such that $\mathfrak{S}_Z \leqq \mathfrak{S}_Y$. Then $Z \leqq Y$.*

PROOF. The definition of $\mathfrak{S}_Z \leqq \mathfrak{S}_Y$ shows that $\mathfrak{S}_Z = \emptyset$ is equivalent to $\mathfrak{S}_Y = \emptyset$, that is $Z = X \Leftrightarrow Y = X$, thus in this case one can consider the identity f of X. Therefore we can assume $\mathfrak{S}_Z \neq \emptyset$ and $\mathfrak{S}_Y \neq \emptyset$. Put $f(x) = x$ for each $x \in X$. If $p \in Y - X$, then define $f(p) = q$, where $q \in Z - X$ is such that $\mathfrak{s}_Z(q) \subset \mathfrak{s}_Y(p)$. From (2.2) it issues that there exists a unique point $q \in Z - X$ with this property, hence by this definition a mapping $f: Y \to Z$ can be obtained such that $f|X = id_X$. If $q \in Z - X$ then there is a point $p \in Y - X$ with $\mathfrak{s}_Z(q) \subset \mathfrak{s}_Y(p)$, thus $f(p) = q$, consequently f is a surjection. f is continuous. Indeed, if $y \in Y$ then, for each $U \in \mathfrak{v}_Z(f(y))$, there exists $P \in \mathfrak{v}_Y(y)$ such that $f(P) \subset U$. In order to see this, put $V \in \mathfrak{s}_Z(f(y))$, $s_Z(V) \subset U$. Since $\mathfrak{s}_Z(f(y)) \in \mathfrak{N} \cup \mathfrak{S}_Z$, there is $W \in \mathfrak{s}_Z(f(y))$ for which either $V \in \mathfrak{s}$ or $X - W \in \mathfrak{s}$ whenever $\mathfrak{s} \in \mathfrak{S}_Z$, finally an open set G in X can be selected such that $G \in \mathfrak{s}_Z(f(y))$ and $G \subset V \cap W$. Owing to the choice of $f(y)$, $G \in \mathfrak{s}_Y(y)$ is always true, thus the set $s_Y(G)$ is an open neighbourhood of y in Y (see [4], (6.1.9)(b)). Choose $P = s(G)$. If $x \in P \cap X$ then $x \in G \subset V$, thus $V \in \mathfrak{v}_X(x) = \mathfrak{s}_Z(x)$, i.e. $f(x) = x \in s_Z(V) \subset U$. If $x \in P - X$ then $\mathfrak{s}_Z(f(x)) \in \mathfrak{S}_Z$ by $f(x) \in Z - X$. $G \in \mathfrak{s}_Y(x)$ implies $W \in \mathfrak{s}_Y(x)$. In this case $X - W \in \mathfrak{s}_Y(f(x)) \subset \mathfrak{s}_Z(x)$ cannot hold, hence $V \in \mathfrak{s}_Z(f(x))$, that is $f(x) \in s_Z(V) \subset \subset U$. □

(2.4) LEMMA. *If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two systems of filters in X satisfying* (R2) *then $\mathfrak{S}_1 \leqq \mathfrak{S}_2$ and $\mathfrak{S}_2 \leqq \mathfrak{S}_1$ imply $\mathfrak{S}_1 = \mathfrak{S}_2$.*

PROOF. In fact, if $\mathfrak{s}_1 \in \mathfrak{S}_1$ then there exists $\mathfrak{s}_2 \in \mathfrak{S}_2$ such that $\mathfrak{s}_1 \subset \mathfrak{s}_2$, and there exists $\mathfrak{s}_1' \in \mathfrak{S}_1$ such that $\mathfrak{s}_2 \subset \mathfrak{s}_1'$. Thus $\mathfrak{s}_1 \subset \mathfrak{s}_1'$, therefore $X - W \notin \mathfrak{s}_1$ whenever

$W \in \mathfrak{s}'_1$. By (R2) this means $\mathfrak{s}'_1 \subset \mathfrak{s}_1$, i.e. $\mathfrak{s}'_1 = \mathfrak{s}_1$, hence $\mathfrak{s}_1 = \mathfrak{s}_2$, and $\mathfrak{S}_1 \subset \mathfrak{S}_2$. $\mathfrak{S}_2 \subset \mathfrak{S}_1$ is similar. □

PROOF OF (2.1). (2.1.1) is an immediate consequence of (1.5). (2.1.2): Suppose $Z \leqq Y$, and let f be a continuous surjection of Y onto Z fixing X. Then $f(Y-X) = Z-X$. In fact, if $p \in Y-X$, $f(p) \in X$, then $f(\mathfrak{v}_Y(p)) \to f(p)$ in Z, and *a fortiori* $\mathfrak{s}_Y(p) = f(\mathfrak{s}_Y(p)) \to f(p)$, which cannot be true by (2.2), thus $f(Y-X) \subset Z-X$. Conversely, if $q \in Z-X$ then there exists $y \in Y$ with $f(y) = q$, hence $y \notin X$ by $f(X) = X$, therefore $Z-X \subset f(Y-X)$. Finally it is easy to verify that if $f(p) = q$ for some $p \in Y-X$, $q \in Z-X$, then $\mathfrak{s}_Z(q) \subset \mathfrak{s}_Y(p)$ issues from $f|X = id_X$. If $\mathfrak{S}_Z \leqq \mathfrak{S}_Y$ then (2.3) gives $Z \leqq Y$. (2.1.3) follows from (1.5) and (2.4), because two reduced strict extensions of X with the same family of trace filters are always equivalent. □

3. General T_3-closed spaces

As K. Császár [7] did in connection with H-closed spaces, we generalize the notion of T_3-closedness for arbitrary topological spaces.

An arbitrary space X will be said to be *T_3-closed* iff it is closed in every topological space Y that contains X as a subspace and is T_3-reduced with respect to X.

(3.1) THEOREM. *For a topological space X, the following statements are equivalent:*

(3.1.1) *X is T_3-closed.*
(3.1.2) *X has no proper T_3-reduced extension.*
(3.1.3) *There does not exist any non-trivial free regular system of filters in X.*
(3.1.4) (cf. [10], S. 2). *In X every regular filter has a cluster point (that is Y is an $R(i)$-space).*
(3.1.5) *In X every maximal regular filter has a cluster point.*
(3.1.6) (cf. [10], S. 2a). *If $\mathfrak{V}$ ($\mathfrak{W}$) is an open (closed) covering of X such that each $V \in \mathfrak{V}$ ($W \in \mathfrak{W}$) lies in the union of a finite subsystem of $\mathfrak{W}$ ($\mathfrak{V}$), then a finite number of the elements of $\mathfrak{V}$ covers X.*

First of all we prove a lemma:

(3.2) LEMMA. *Let X be a topological space.*

(3.2.1) *Any regular filter can be included in a maximal regular filter.*
(3.2.2) *If $\mathfrak{s}$ is a regular filter and $\mathfrak{s}^*$ is a maximal regular filter then either $\emptyset \in \mathfrak{s}(\cap)\mathfrak{s}^*$ or $\mathfrak{s} \subset \mathfrak{s}^*$.*

PROOF. (3.2.1): The set of all regular filters (ordered by the set theoretical inclusion) satisfies the condition of the Kuratowski—Zorn lemma.

(3.2.2) If $\emptyset \notin \mathfrak{s}(\cap)\mathfrak{s}^*$ then $\mathfrak{s}(\cap)\mathfrak{s}^*$ is also a regular filter finer than both $\mathfrak{s}$ and $\mathfrak{s}^*$, thus $\mathfrak{s} \subset \mathfrak{s}(\cap)\mathfrak{s}^* = \mathfrak{s}^*$. □

PROOF OF (3.1). (3.1.1)⇔(3.1.2): If Y is a proper T_3-reduced extension of X then $X \neq Y = \overline{X}^Y$, therefore X is not closed in Y. Conversely, if Y is a space T_3-reduced with respect to X and X is not closed in Y, then $\overline{X}^Y$ is a proper T_3-reduced extension of X.

(3.1.2)$\Leftrightarrow$(3.1.3) is clear by (1.5).

(3.1.3)$\Leftrightarrow$(3.1.4): If $\mathfrak{s}$ is a regular filter without cluster points then it is free, and it is easy to show that the system $\{\mathfrak{s}\}$ is free regular. The converse follows from (2.2).

(3.1.4)$\Leftrightarrow$(3.1.5) in view of (3.2.1).

(3.1.4)$\Leftrightarrow$(3.1.6): If $\mathfrak{s}$ is a regular filter without cluster points then let $\mathfrak{g}$ and $\mathfrak{f}$ be an open and a closed base for $\mathfrak{s}$, and suppose $\mathfrak{V}=\{X-F\colon F\in\mathfrak{f}\}$, $\mathfrak{W}=\{X-G\colon G\in\in\mathfrak{g}\}$. Since $\bigcap\mathfrak{g}=\bigcap\mathfrak{f}=\emptyset$, $\mathfrak{V}$ is an open and $\mathfrak{W}$ is a closed covering of X respectively. Any element of $\mathfrak{V}(\mathfrak{W})$ is contained in an element of $\mathfrak{V}(\mathfrak{W})$, but X cannot be covered by a finite subsystem of $\mathfrak{V}$. Conversely, if $\mathfrak{V}(\mathfrak{W})$ is an open (closed) covering of X satisfying the condition written in (3.1.6), but X cannot be covered by a finite number of elements of $\mathfrak{V}$, then the complements of the finite unions of the elements of $\mathfrak{V}$ form a closed filter base $\mathfrak{f}$. In the same manner we get an open filter base from $\mathfrak{W}$. They are equivalent, i.e. generate the same filter in X, which is a free regular filter. This cannot have a cluster point. □

(3.3) COROLLARY. *Let us denote by $\mathfrak{R}^*$ the system of all maximal regular filters in a regular space X. Then we have $\mathfrak{N}\subset\mathfrak{R}^*$. X is T_3-closed iff $\mathfrak{N}=\mathfrak{R}^*$.*

PROOF. Any neighbourhood filter in X is regular. Every $\mathfrak{v}_X(x)$ can be included in a maximal regular filter $\mathfrak{r}(x)$. $x\in R$ for any $R\in\mathfrak{r}(x)$, otherwise there is a closed $F\in\mathfrak{r}(x)$ with $X-F\in\mathfrak{v}_X(x)$, which contradicts $\mathfrak{v}_X(x)\subset\mathfrak{r}(x)$. Thus $\mathfrak{r}(x)\subset\mathfrak{v}_X(x)$ is also true because $\mathfrak{r}(x)$ is open. These show $\mathfrak{N}\subset\mathfrak{R}^*$. If X is T_3-closed then every $\mathfrak{r}\in\mathfrak{R}^*$ has a cluster point x, and in this case $\mathfrak{r}=\mathfrak{v}_X(x)$ by (3.2.2). Conversely, if $\mathfrak{N}=\mathfrak{R}^*$ then every $\mathfrak{r}\in\mathfrak{R}^*$ converges in X, therefore X is T_3-closed. □

4. T_3-closed extensions

By a *T_3-closure* of an arbitrary topological space X we shall mean a T_3-closed, T_3-reduced extension of X. A space will be said to be *T_3-closable,* if it has at least one T_3-closure. This definition is similar to that of a *generalized absolute closure* (in other words *ordinary H-closed extension*) given by C. T. Liu [11] (and also K. Császár [7]) in connection with the property **P**="to be T_2". A T_3-space is T_3-closable iff it is *RC-regular* in the sense of Harris [9].

A free regular system $\mathfrak{S}$ of filters on the topological space X will be called *maximal* iff $\mathfrak{S}\subset\mathfrak{S}'$ implies $\mathfrak{S}=\mathfrak{S}'$ for any free regular system $\mathfrak{S}'$ of filters in X.

(4.1) THEOREM. *A T_3-reduced extension Y of a topological space X is T_3--closed iff the corresponding free regular system $\mathfrak{S}_Y$ of filters is maximal in X.*

PROOF. Suppose that Y is T_3-closed. We show that, if $\mathfrak{S}$ is a free regular system of filters in X, and $\mathfrak{S}_Y\subset\mathfrak{S}$, then $\mathfrak{S}_Y=\mathfrak{S}$. In fact, assume $\mathfrak{s}'\in\mathfrak{S}-\mathfrak{S}_Y$ and put $s(\mathfrak{s}')=\{V\subset Y\colon s(S)\subset V$ for some $S\in\mathfrak{s}'\}$. $s(S)\neq\emptyset$ for any $S\in\mathfrak{s}'$ and $s(S_1)\cap s(S_2)==s(S_1\cap S_2)\in s(\mathfrak{s}')$ for every $S_1, S_2\in\mathfrak{s}'$, thus $s(\mathfrak{s}')$ is obviously an open filter in Y. If $S\in\mathfrak{s}'$ is open in X, then there exists $S_1\in\mathfrak{s}'$ such that either $S\in\mathfrak{s}$ or $X-S_1\in\mathfrak{s}$ for any $\mathfrak{s}\in\mathfrak{S}$, in particular for every $\mathfrak{s}(p)$ $(p\in Y-X)$. Suppose $S_2\in\mathfrak{s}'$ is open and $\bar{S}_2^X\subset S\cap S_1$. It is easy to show that $\overline{s(S_2)}^Y\subset s(S)$, which means that $s(\mathfrak{s}')$ is regular in Y. $s(\mathfrak{s}')$ has a cluster point $x\in Y$. Because of $s(\mathfrak{s}')(\cap)\{X\}=\mathfrak{s}'$, we have $\emptyset\notin\mathfrak{s}'(\cap)\mathfrak{r}$

for some $\mathfrak{r}\in\mathfrak{R}\cup\mathfrak{S}_Y$. Then $\mathfrak{s}'\neq\mathfrak{r}\in\mathfrak{R}\cup\mathfrak{S}$, but this contradicts (2.2), therefore $\mathfrak{S}-\mathfrak{S}_Y=\emptyset$.

Conversely, let $\mathfrak{S}_Y$ be maximal free regular. If Y is not T_3-closed then there exists a regular filter $\mathfrak{r}'$ in Y without a cluster point. Put $\mathfrak{s}'=\mathfrak{r}'(\cap)\{X\}$. Then $\mathfrak{s}'$ is a regular filter without a cluster point in X, and what is more: $\emptyset\in\mathfrak{s}'(\cap)\mathfrak{s}$ for any $\mathfrak{s}\in\mathfrak{S}_Y$. Thus $\mathfrak{S}=\mathfrak{S}_Y\cup\{\mathfrak{s}'\}$ consists of free regular filters. Assume $\mathfrak{r}\in\mathfrak{R}\cup\mathfrak{S}_Y$, $V\in\mathfrak{r}$. There exists $V_1\in\mathfrak{r}$ such that $V\in\mathfrak{s}$ or $X-V_1\in\mathfrak{s}$ for each $\mathfrak{s}\in\mathfrak{S}_Y$, and a set $V_2\in\mathfrak{r}$ can be chosen with $X-V_2\in\mathfrak{s}'$. Putting $W=V_1\cap V_2$, we have either $V\in\mathfrak{s}$ or $X-W\in\mathfrak{s}$ for any $\mathfrak{s}\in\mathfrak{S}$. If $\mathfrak{r}=\mathfrak{s}'$ then, for every $V\in\mathfrak{s}'$, there is an open $R\in\mathfrak{r}'$ in Y with $R\cap X\subset V$. Assume $R_1\in\mathfrak{r}'$ is open in Y, $\overline{R}_1^Y\subset R$, finally put $W=R_1\cap X\in\mathfrak{s}'$. If $\mathfrak{s}\in\mathfrak{S}_Y$, say $\mathfrak{s}=\mathfrak{s}(p)$ for some $p\in Y-X$, then $X-W\notin\mathfrak{s}(p)$ implies $p\in\overline{R}_1^Y\subset R$, thus $V\in\mathfrak{s}(p)$. We proved that $\mathfrak{S}$ is a free regular system of filters in X, which contradicts the condition of the maximality of $\mathfrak{S}_Y$. This means that Y is T_3-closed. □

(4.2) THEOREM. *A topological space X is T_3-closable iff there is at least one maximal free regular system of filters in X. In this case there exists an order isomorphism between the ordered set of all T_3-closures and all maximal free regular systems of filters of X. Two T_3-closures are equivalent iff each one of them is finer than the other.*

PROOF. (2.1) and (4.1). □

Remark. A T_3-closed space is always T_3-closable. Then the unique maximal free regular system of filters is the trivial one $\mathfrak{S}=\emptyset$ (see. (3.1.3)). □

C. T. Scarborough—A. H. Stone observed that, if the topology of a space X can be obtained as the supremum of an $H(i)$ and an $R(i)$ topology then it can be embedded into an $R(i)$-space ([13], th. 3.20). We can extend this theorem for arbitrary topological spaces X as follows:

Let us denote by $\mathfrak{S}^*$ the system of all maximal regular filters without cluster points in X, and consider the strict extension αX of X in which $\mathfrak{S}_{\alpha X}=\mathfrak{S}^*$, and the correspondence existing between the points $p\in\alpha X-X$ and the filters $\mathfrak{s}(p)\in\mathfrak{S}^*$ is one-to-one. (For T_3-spaces this construction is due to P. Alexandroff [1]).

(4.3) THEOREM. *The space αX is always T_3-closed.*

PROOF. Let $\mathfrak{r}$ be a regular filter in αX. Then $\mathfrak{r}(\cap)\{X\}=\mathfrak{s}$ is also a regular filter in X. If $\mathfrak{s}$ has a cluster point x in X, then x is a cluster point of $\mathfrak{r}$ in αX. If $\mathfrak{s}$ has no cluster point in X then $\mathfrak{s}\subset\mathfrak{s}(p)$ for some $p\in\alpha X-X$ (see (3.2.1)), and p is a cluster point of $\mathfrak{r}$ in αX. □

We mention some open questions in connection with αX (see also problem III of Harris [9] and cf. Á. Császár [6]).

(4.4) PROBLEM. Look for necessary and sufficient conditions under which αX is a T_3-closure of X. (Such a trivial condition is that αX be T_3-reduced with respect to X, i.e. $\mathfrak{S}^*$ satisfy (R2).)

(4.5) PROBLEM. Does there exist a T_3-closure Y of a space X such that $Y\not\equiv\alpha X$? (This question is motivated by the evident property of αX that $\alpha X\leqq Y$ implies $\alpha X=Y$ for any T_3-reduced extension Y of X.)

(4.6) PROBLEM. Does there exist a space X with a finest T_3-closure different from αX?

It is well-known that in the partially ordered set of the T_2-compactifications of a Tychonoff space any non-empty subset has a least upper bound. As the following theorem shows, in order that the partially ordered set of all T_3-closures of an arbitrary space have this property, it is necessary and sufficient that the space have a finest T_3-closure.

(4.7) THEOREM. *If a non-empty subset of all T_3-closures of an arbitrary topological space has an upper bound, then it has a least upper bound, too.*

(4.8) LEMMA. *Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two free regular systems of filters on a topological space X. If $\mathfrak{S}_2$ is maximal and, for any $\mathfrak{s}_2\in\mathfrak{S}_2$, there exists $\mathfrak{s}_1\in\mathfrak{S}_1$ such that $\mathfrak{s}_1\subset\mathfrak{s}_2$, then $\mathfrak{S}_1\leqq\mathfrak{S}_2$.*

PROOF. It is sufficient to show that, for every $\mathfrak{s}_1\in\mathfrak{S}_1$, there exists $\mathfrak{s}_2\in\mathfrak{S}_2$ such that $\mathfrak{s}_1\subset\mathfrak{s}_2$. Suppose that $\mathfrak{s}_1\in\mathfrak{S}_1$ is such that $\mathfrak{s}_1\not\subset\mathfrak{s}_2$ for any $\mathfrak{s}_2\in\mathfrak{S}_2$. Then $\mathfrak{s}_1\notin\mathfrak{S}_2$ is evident. It will be shown that the system $\mathfrak{S}_2\cup\{\mathfrak{s}_1\}$ is free regular, which contradicts the maximality of $\mathfrak{S}_2$. In fact, this system consists of free regular filters. Suppose $S\in\mathfrak{s}_1$. There exists $S_1\in\mathfrak{s}_1$ such that either $S\in\mathfrak{s}$ or $X-S_1\in\mathfrak{s}$ for each $\mathfrak{s}\in\mathfrak{S}_1$. If $S\notin\mathfrak{s}_2\in\mathfrak{S}_2$ and $\mathfrak{s}\in\mathfrak{S}_1$, $\mathfrak{s}\subset\mathfrak{s}_2$, then $S\notin\mathfrak{s}$ implies $X-S_1\in\mathfrak{s}$, hence $X-S_1\in\mathfrak{s}_2$. On the other hand, put $S\in\mathfrak{s}_2\in\mathfrak{S}_2$, and assume either $S\in\mathfrak{s}$ or $X-S_1\in\mathfrak{s}$ whenever $\mathfrak{s}\in\mathfrak{S}_2$, where $S_1\in\mathfrak{s}_2$. One can choose $S_0\in\mathfrak{s}_1$ with $S_0\notin\mathfrak{s}_2$. Applying again (R2) for $\mathfrak{S}_1$, we can find a set $S_0'\in\mathfrak{s}_1$ such that $X-S_0'\in\mathfrak{s}_2$. Let us consider $S_2=S_1\cap\cap(X-S_0')\in\mathfrak{s}_2$. Then, for any $\mathfrak{s}^*\in\mathfrak{S}_2\cup\{\mathfrak{s}_1\}$, either $S\in\mathfrak{s}^*$ or $X-S_2\in\mathfrak{s}^*$. Finally, if $\mathfrak{v}$ is a neighbourhood filter in X then, for any $V\in\mathfrak{v}$, there exist sets $W_1, W_2\in\mathfrak{v}$ such that $X-W_1\in\mathfrak{s}_1$, and either $V\in\mathfrak{s}_2$ or $X-W_2\in\mathfrak{s}_2$ for any $\mathfrak{s}_2\in\mathfrak{S}_2$. Put $W=W_1\cap W_2$ in $\mathfrak{v}$. Then $V\notin\mathfrak{s}^*$ implies $X-W\in\mathfrak{s}^*$ for every $\mathfrak{s}^*\in\mathfrak{S}_2\cup\{\mathfrak{s}_1\}$. □

PROOF OF (4.7). Let Z_i $(i\in I\neq\emptyset)$ and Y be T_3-closures of X such that $Z_i\leqq Y$ for any $i\in I$. If $\mathfrak{S}_i$ and $\mathfrak{S}$ are the corresponding free regular systems of filters in X, we have $\mathfrak{S}_i\leqq\mathfrak{S}$ for each $i\in I$. Denote by $\mathfrak{S}^+$ the system of all filters in X which can be generated by a centred system of the form $\bigcup_{i\in I}\mathfrak{s}_i$, where $\mathfrak{s}_i\in\mathfrak{S}_i$ for any $i\in I$. If $\mathfrak{s}\in\mathfrak{S}$ then there exists $\mathfrak{s}_i\in\mathfrak{S}_i$ $(i\in I)$ such that $\mathfrak{s}_i\subset\mathfrak{s}$, and $\mathfrak{s}_i$ is unique in $\mathfrak{S}_i$ with this property by (2.2). Let $r(\mathfrak{s})$ denote the filter generated by $\bigcup_{i\in I}\mathfrak{s}_i$, then $r(\mathfrak{s})\subset\mathfrak{s}$.

The elements of $\mathfrak{S}^+$ are free regular filters. In order to show that $\mathfrak{S}^+$ satisfies (R2), suppose $\mathfrak{r}\in\mathfrak{N}\cup\mathfrak{S}^+$. If $V\in\mathfrak{r}=\mathfrak{v}_X(x)$ for some $x\in X$, then there exists $W\in\mathfrak{r}$ such that (for a fixed index $i\in I$) either $V\in\mathfrak{s}$ or $X-W\in\mathfrak{s}$ whenever $\mathfrak{s}\in S_i$. If $\mathfrak{s}^+\in\mathfrak{S}^+$ and $\mathfrak{s}^+$ is generated by the centred system $\bigcup_{i\in I}\mathfrak{s}_i$ $(\mathfrak{s}_i\in\mathfrak{S}_i,\ i\in I)$, then $V\notin\mathfrak{s}^+$ implies $V\notin\mathfrak{s}_i$, thus $X-W\in\mathfrak{s}_i\subset\mathfrak{s}^+$. After this suppose that $\mathfrak{r}\in\mathfrak{S}^+$ and $\mathfrak{r}$ is generated by $\bigcup_{i\in I}\mathfrak{s}_i\,(\mathfrak{s}_i\in\mathfrak{S}_i,\ i\in I)$. If $V\in\mathfrak{r}$ then there are indices $i_1, i_2, \dots, i_n\in I$, filters $\mathfrak{s}_j\in\mathfrak{S}_{i_j}$ and sets $V_j\in\mathfrak{s}_j$ $(j=1, 2, \dots, n)$ such that $\bigcap_{j=1}^{n} V_j\subset V$. Assume $W_j\in\mathfrak{s}_j$ $(j=1, 2, \dots, n)$, and for any $\mathfrak{s}\in\mathfrak{S}_{i_j}$, either $V_j\in\mathfrak{s}$ or $X-W_j\in\mathfrak{s}$. Then $W=\bigcap_{j=1}^{n} W_j\in\mathfrak{r}$.

If $\mathfrak{s}^+\in\mathfrak{S}^+$ and $\mathfrak{s}^+$ is generated by $\bigcup_{i\in I}\mathfrak{s}'_i$ ($\mathfrak{s}'_i\in\mathfrak{S}_i$, $i\in I$), then $V\notin\mathfrak{s}^+$ implies $V_j\notin\mathfrak{s}'_{i_j}$ for at least one index i_j, thus $X-W\supset X-W_j\in\mathfrak{s}'_{i_j}\subset\mathfrak{s}^+$, i.e. $X-W\in\mathfrak{s}^+$.

We showed that $\mathfrak{S}^+$ is free regular, hence there exists a T_3-reduced extension Z^+ of X such that $\mathfrak{S}_{Z^+}=\mathfrak{S}^+$. Using the filters $r(\mathfrak{s})$ ($\mathfrak{s}\in\mathfrak{S}$), it is easy to verify by (4.8) that $\mathfrak{S}_i\leqq\mathfrak{S}^+\leqq\mathfrak{S}$, thus $\mathfrak{Z}_i\leqq\mathfrak{Z}^+\leqq Y$ for any $i\in I$. Since Z^+ is the continuous image of the T_3-closed space Y, it is a T_3-closure of X (see (3.1.4) and [13], th. 3.6).

Finally let Y' be a T_3-closure of X such that $Z_i\leqq Y'$ for each $i\in I$. Put $\mathfrak{S}'=\mathfrak{S}_{Y'}$. If $\mathfrak{s}'\in\mathfrak{S}'$ then there exists $\mathfrak{s}_i\in\mathfrak{S}_i$ with $\mathfrak{s}_i\subset\mathfrak{s}'$ ($i\in I$), and the filter generated by $\bigcup_{i\in I}\mathfrak{s}_i$ is contained in $\mathfrak{s}'$. Thus $\mathfrak{S}^+\leqq\mathfrak{S}'$ by (4,8), that is $Z^+\leqq Y'$. □

5. On RC-proximities

The method followed in the previous chapter and the close connection existing between the T_3-closures and the compatible separated RC-proximities of a T_3-space [9] motivates to characterize RC-proximities by systems of filters. Our basic theorem is the following one (cf. [12], (6.16)):

(5.1) THEOREM. *A system $\mathfrak{R}$ of filters in a set X is the system of all maximal round filters for a suitable RC-proximity on X if and only if it satisfies the following conditions:*

(A) *For any $x\in X$, there exists $\mathfrak{r}\in\mathfrak{R}$ such that $x\in\bigcap\mathfrak{r}$.*

(B) *If $R\in\mathfrak{r}\in\mathfrak{R}$ then there is $R_1\in\mathfrak{r}$ such that, for each $\mathfrak{r}'\in\mathfrak{R}$, either $R\in\mathfrak{r}'$ or $X-R_1\in\mathfrak{r}'$.*

(C) $\mathfrak{R}$ *is maximal with respect to the property* (B).

PROOF. Let δ be an RC-proximity on X and $\mathfrak{R}$ be the family of all maximal round filters for δ. Then $\mathfrak{R}$ satisfies (A) by [9], 3.2. If $R\in\mathfrak{r}\in\mathfrak{R}$ and $R_1\in\mathfrak{r}$ is such that $R_1<R$, then either $R\in\mathfrak{r}'$ or $X-R_1\in\mathfrak{r}'$ whenever $\mathfrak{r}'\in\mathfrak{R}$ (see axiom (e)). Finally suppose that $\mathfrak{R}\subset\mathfrak{R}'$, where $\mathfrak{R}'$ is a system of filters satisfying (B). Then any filter in $\mathfrak{R}'$ is round by (e). Suppose $\mathfrak{r}'\in\mathfrak{R}'$. $\mathfrak{r}'$ is contained in a maximal round filter $\mathfrak{r}\in\mathfrak{R}$ ([9], 3.3). We show $\mathfrak{r}'=\mathfrak{r}$. In fact, if $R\in\mathfrak{r}-\mathfrak{r}'$ and $R_1\in\mathfrak{r}$, then neither $R\in\mathfrak{r}'$ nor $X-R_1\in\mathfrak{r}'$, which contradicts $\mathfrak{r},\mathfrak{r}'\in\mathfrak{R}'$ and (B).

Conversely, assume that $\mathfrak{R}$ is a system of filters fulfilling (A)—(C). Define a relation δ on $\mathfrak{P}(X)$ by letting

$$A\delta B\Leftrightarrow \text{there exists } \mathfrak{r}\in\mathfrak{R} \text{ such that } \emptyset\notin\mathfrak{r}(\cap)\{A\} \text{ and } \emptyset\notin\mathfrak{r}(\cap)\{B\}.$$

If we denote $A\bar{\delta}X-B$ by $A<B$, then

(5.1.1) $A<B\Leftrightarrow\emptyset\notin\mathfrak{r}(\cap)\{A\}$ implies $B\in\mathfrak{r}$ whenever $\mathfrak{r}\in\mathfrak{R}$.

It is easy to see that δ is symmetrical, and it satisfies (a)—(c). From (B) it follows that

(5.1.2) if $R\in\mathfrak{r}\in\mathfrak{R}$ then there exists $R_1\in\mathfrak{r}$ with $R_1<R$.

If $x\in X$, $x<A$, then $A\in\mathfrak{r}$, where $x\in\bigcap\mathfrak{r}$, $\mathfrak{r}\in\mathfrak{R}$ (see (A)). Suppose $B\in\mathfrak{r}$, $B<A$ and $C\in\mathfrak{r}$, $C<B$. Then $x\in C$ implies $x<B<A$, thus (d) is also satisfied, i.e. δ

is an R-proximity. Any filter $\mathfrak{r}\in\mathfrak{R}$ is round in δ by (5.1.2), thus it is contained in a maximal round filter $\mathfrak{r}'$. For any $R\in\mathfrak{r}'$, there exists $R_1\in\mathfrak{r}'$ such that $R_1<R$. In view of $\emptyset\notin\mathfrak{r}(\cap)\{R_1\}$ we have $R\in\mathfrak{r}$ by (5.1.1), i.e. $\mathfrak{r}=\mathfrak{r}'$, thus any filter in $\mathfrak{R}$ is maximal round. Finally let $\mathfrak{r}'$ be a maximal round filter in δ. If $\mathfrak{r}'\notin\mathfrak{R}$ then $\emptyset\in\mathfrak{r}(\cap)\mathfrak{r}'$ for any $\mathfrak{r}\in\mathfrak{R}$ (see [9], 3.4), therefore from the roundness of $\mathfrak{r}'$ and from the property (B) of $\mathfrak{R}$ it issues that the system $\mathfrak{R}\cup\{\mathfrak{r}'\}$ also satisfies (B), which contradicts (C). We verified that $\mathfrak{R}$ is exactly the family of all maximal round filters in δ, thus δ fulfils axiom (e), too (see (5.1.1)), i.e. it is an RC-proximity. □

From (5.1) and axiom (e) one can deduce:

(5.2) Corollary. *There exists a one-to-one correspondence between the systems of filters satisfying* (A)—(C) *and the RC-proximities on a set X.* □

Let δ_1 and δ_2 be two RC-proximities on a set X. Usually, δ_2 is said to be *finer than* δ_1 iff $A\delta_2B$ implies $A\delta_1B$. We shall say that δ_2 is *strongly finer* than δ_1, if in δ_2 every maximal round filter contains a filter which is maximal round in δ_1. For Efremovich proximities ([12], Definition (1.7)) these notions are equivalent.

(5.3) Lemma. *If δ_2 is strongly finer than δ_1 then δ_2 is finer than δ_1, but the converse fails to be true in general.*

Proof. If $A\delta_1B$ then there exists a maximal round filter $\mathfrak{r}$ in δ_2 such that $\emptyset\notin\mathfrak{r}(\cap)\{A\}$ and $\emptyset\notin\mathfrak{r}(\cap)\{B\}$ by axiom (e). If $\mathfrak{r}_1\subset\mathfrak{r}$ is a maximal round filter in δ_1 then $\emptyset\notin\mathfrak{r}_1(\cap)\{A\}$ and $\emptyset\notin\mathfrak{r}_1(\cap)\{B\}$ implies $A\delta_1B$ by (e). Let us now consider a T_3-space X that is T_3-closed but not compact (see [3]). Then there exists a unique separated RC-proximity δ_1 compatible with the topology of X, for which any maximal round filter agrees with some neighbourhood filter ([9], Theorem F and 3.2, 3.4). Denote by δ_2 the discrete proximity of X ($A\delta_2B\Leftrightarrow A\cap B\neq\emptyset$). Then δ_2 is obviously an RC-proximity finer than δ_1, however δ_2 is not strongly finer than δ_1, because there exists a non-convergent ultrafilter in X, and it is maximal round in δ_2. □

(5.4) Lemma. *Let X be a regular space. A system $\mathfrak{S}$ of filters in X is maximal free regular iff it is the system of all maximal round filters without cluster points for a compatible RC-proximity on X. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two maximal free regular systems on X then $\mathfrak{S}_1\leqq\mathfrak{S}_2$ iff, for the corresponding compatible RC-proximities, δ_2 is strongly finer than δ_1.*

Proof. If δ is a compatible RC-proximity on X then denoting by $\mathfrak{N}$ the family of all maximal round filters, $\mathfrak{R}=\mathfrak{N}\cup\mathfrak{S}$, where $\mathfrak{S}$ consists of the free elements of $\mathfrak{R}$ (see [9], 3.2). $\mathfrak{R}$ satisfies (A)—(C), thus $\mathfrak{S}$ is obviously free regular (cf. [9], 3.1). If $\mathfrak{S}'$ is a free regular system on X such that $\mathfrak{S}\subset\mathfrak{S}'$, then $\mathfrak{R}\subset\mathfrak{N}\cup\mathfrak{S}'=\mathfrak{R}'$, It is easy to show that $\mathfrak{R}'$ satisfies (B), hence $\mathfrak{R}=\mathfrak{R}'$ by (C). Since $\mathfrak{S}'\cap\mathfrak{N}=\emptyset$, we have $\mathfrak{S}=\mathfrak{S}'$, i.e. $\mathfrak{S}$ is maximal free regular. Conversely, if $\mathfrak{S}$ is a maximal free regular system on X then one can show that $\mathfrak{R}=\mathfrak{N}\cup\mathfrak{S}$ fulfils (A) and (B). Let $\mathfrak{R}'$ be a system of filters in X satisfying (B) and $\mathfrak{R}\subset\mathfrak{R}'$. If we denote by $\mathfrak{S}'$ the set of all free elements of $\mathfrak{R}'$, then $\mathfrak{S}'$ is free regular, and from $\mathfrak{S}'\cap\mathfrak{N}=\emptyset$ the inclusion $\mathfrak{S}\subset\mathfrak{S}'$ follows, thus $\mathfrak{S}=\mathfrak{S}'$ and $\mathfrak{R}=\mathfrak{R}'$, i.e. $\mathfrak{R}$ satisfies (C), and it is the set of all maximal round filters of an RC-proximity δ on X. δ is compatible, because $x<V$ iff $\mathfrak{r}\in\mathfrak{R}$, $x\in\cap\mathfrak{r}$ implies $V\in\mathfrak{r}$. But such an $\mathfrak{r}\in\mathfrak{R}$ is in $\mathfrak{N}$, thus it is the neighbourhood filter of x (see [9], 3.4).

Let $\mathfrak{S}_1, \mathfrak{S}_2$ be maximal free regular systems in X, and δ_1, δ_2 be the corresponding compatible RC-proximities on X. Denoting by $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the systems of all maximal round filters in δ_1 and δ_2 respectively, we have $\mathfrak{R}_i=\mathfrak{N}\cup\mathfrak{S}_i$ $(i=1,2)$. If $\mathfrak{S}_1\leqq\mathfrak{S}_2$ then δ_2 is evidently strongly finer than δ_1. Conversely, if δ_2 is strongly finer than δ_1 then, for $\mathfrak{s}_2\in\mathfrak{S}_2$, there exists $\mathfrak{r}_1\in\mathfrak{R}_1$ such that $\mathfrak{r}_1\subset\mathfrak{s}_2$. $\mathfrak{r}_1$ cannot be a neighbourhood filter in X because $\mathfrak{s}_2$ has no cluster point, therefore $\mathfrak{r}_1\in\mathfrak{S}_1$. From this $\mathfrak{S}_1\leqq\mathfrak{S}_2$ follows by (4.8). □

Harris theory of RC-proximities and T_3-closures can be completed as follows:

(5.5) THEOREM. *Let X be a regular T_3-closable space. There exists an order isomorphism between the set of all T_3-closures of X and the family of all compatible RC-proximities partially ordered by the relation of the strong fineness.*

PROOF. (4.2) and (5.4). □

In view of this theorem, Problem I of Harris [9] can be formulated as follows:

(5.6) PROBLEM. Is the converse of (5.3) true for compatible RC-proximities?

It would be easy to state the theorem corresponding to (4.7) for compatible RC-proximities on a regular space, however the statement in question is true under more general conditions, too.

(5.7) THEOREM. *If a non-empty family of RC-proximities on a set X has an upper bound, then it has a least upper bound (with respect to the relation of the strong fineness), whose topology is the supremum of the corresponding topologies.*

PROOF. Let δ_i, δ be RC-proximities on X such that δ is strongly finer than δ_i for any $i\in I\neq\emptyset$. Denote by $\mathfrak{R}_i$ and $\mathfrak{R}$ the set of all maximal round filters in δ_1 and δ respectively. We define a system $\mathfrak{R}^+$ of filters as follows: put $\mathfrak{r}^+\in\mathfrak{R}^+$ if and only if $\mathfrak{r}^+$ can be generated by a centred system of the form $\bigcup_{i\in I}\mathfrak{r}_i$, where $\mathfrak{r}_i\in\mathfrak{R}_i$ for any $i\in I$. Similarly to the train of thought followed in the proof of (4.7), one can show that $\mathfrak{R}^+$ satisfies (A)—(B), and

(5.7.1) for any $\mathfrak{r}\in\mathfrak{R}$, there exists $\mathfrak{r}^+\in\mathfrak{R}^+$ such that $\mathfrak{r}^+\subset\mathfrak{r}$.

In order to see (C) suppose that $\mathfrak{R}'$ is a system fulfilling (B) and $\mathfrak{R}^+\subset\mathfrak{R}'$. If $\mathfrak{r}_0\in\in\mathfrak{R}'-\mathfrak{R}^+$ then $\emptyset\in\mathfrak{r}_0(\cap)\mathfrak{r}$ for every $\mathfrak{r}\in\mathfrak{R}$. In fact, one can find a filter $\mathfrak{r}^+\in\mathfrak{R}^+$ such that $\mathfrak{r}^+\subset\mathfrak{r}$. From $\mathfrak{r}_0$, $\mathfrak{r}^+\in\mathfrak{R}'$, $\mathfrak{r}_0\neq\mathfrak{r}^+$ and (B) we get $\emptyset\in\mathfrak{R}_0(\cap)\mathfrak{r}^+\subset\mathfrak{r}_0(\cap)\mathfrak{r}$. Let us consider the system $\mathfrak{Z}=\mathfrak{R}\cup\{\mathfrak{r}_0\}$, which is properly larger than $\mathfrak{R}$. If $R\in\mathfrak{r}\in\mathfrak{R}$ then there exist sets $R_1, R_2\in\mathfrak{r}$ such that either $R\in\mathfrak{r}'$ or $X-R_1\in\mathfrak{r}'$ for each $\mathfrak{r}'\in\mathfrak{R}$ and $X-R_2\in\mathfrak{r}_0$. Putting $R'=R_1\cap R_2$, we have $R'\in\mathfrak{r}$, and either $R\in\mathfrak{z}\,X-R'\in\mathfrak{z}$ for every $\mathfrak{z}\in\mathfrak{Z}$. If $R_0\in\mathfrak{r}_0$ then a set $R'\in\mathfrak{r}_0$ can be chosen for which either $R_0\in\mathfrak{r}'$ or $X-R'\in\mathfrak{r}'$ whenever $\mathfrak{r}'\in\mathfrak{R}'$. $R_0\notin\mathfrak{z}\in\mathfrak{Z}$ implies $\mathfrak{z}\in\mathfrak{R}$, hence $\mathfrak{r}^+\subset\mathfrak{z}$ for some $\mathfrak{r}^+\in\mathfrak{R}^+\subset\mathfrak{R}'$. Since $R_0\notin\mathfrak{r}^+$, $X-R'\in\mathfrak{r}^+$, consequently $X-R'\in\mathfrak{z}$. Thus we showed that $\mathfrak{Z}$ fulfils (B), but this contradicts the maximality of $\mathfrak{R}$. Hence $\mathfrak{R}'-\mathfrak{R}^+=\emptyset$, that is $\mathfrak{R}^+$ is also maximal with respect to the property (B). From (5.1) it follows that there exists an RC-proximity δ^+ on X such that the maximal round filters in δ^+ are exactly the elements of $\mathfrak{R}^+$. It is evident that δ^+ is strongly finer than δ_i for each $i\in I$. Suppose that δ' is another RC-proximity strongly finer than every δ_i $(i\in I)$. If $\mathfrak{r}'$ is a maximal round filter in δ' then there is $\mathfrak{r}_i\in\mathfrak{R}_i$ for which $\mathfrak{r}_i\subset\mathfrak{r}'$, and the centred system

$\bigcup_{i\in I} \mathfrak{r}_i$ generates a filter $\mathfrak{r}^+\subset\mathfrak{r}'$. Since $\mathfrak{r}^+\in\mathfrak{R}^+$, $\mathfrak{r}^+$ is maximal round in δ^+, i.e. δ' is strongly finer than δ^+.

In the topology of δ^+ the neighbourhood filter of $x\in X$ is the element $\mathfrak{r}^+$ of $\mathfrak{R}^+$ for which $x\in\bigcap\mathfrak{r}^+$. This filter is generated by the centred system $\bigcup_{i\in I}\mathfrak{r}_i$, where $\mathfrak{r}_i\in\mathfrak{R}_i$ and $x\in\bigcap\mathfrak{r}_i$ for any $i\in I$. But this $\mathfrak{r}_i\in\mathfrak{R}_i$ is identical with the neighbourhood filter of x in the topology of δ_i $(i\in I)$, thus the last statement is also proved. □

A mapping f of an RC-proximity space (X, δ) into an RC-proximity space (Y, δ') will be called *strongly equicontinuous* iff, for any maximal round filter $\mathfrak{r}$ in (X, δ), the filter generated by the filter base $f(\mathfrak{r})=\{f(R)\colon R\in\mathfrak{r}\}$ contains a filter $\mathfrak{r}'$, which is maximal round in (Y, δ'). If $X=Y$ then id_X is strongly equicontinuous if and only if δ is strongly finer than δ'. The strong equicontinuity of f implies its equicontinuity (i.e. $A\delta B\Rightarrow f(A)\,\delta'\,f(B)$), as it can be shown analogously to (5.3).

Since in absolutely closed RC-proximity spaces the maximal round filters agree with the neighbourhood filters, the following lemma can be verified easily:

(5.8) Lemma. *A mapping between two absolutely closed RC-proximity spaces is continuous iff it is strongly equicontinuous.* □

Let now X^* be a T_3-closure of a regular space X and f be a mapping of X into a T_3-closed T_3-space Y. Denote by δ the compatible RC-proximity on X corresponding to X^* and let δ' be the unique compatible RC-proximity on Y. The following theorem answers problem II of Harris [9].

(5.9) Theorem. *In order that f have a unique continuous extension onto X^* it is necessary and sufficient that it be strongly equicontinuous with respect to δ and δ'.*

Proof. If f is strongly equicontinuous then, for any point $x\in X^*$, $f(\mathfrak{s}(x))$ is convergent in Y, where $\mathfrak{s}(x)$ is the trace of the neighbourhood filter of x in X^* (see (5.4) and (5.5)), thus f is continuous and it has a unique continuous extension onto X^* by [4], (6.2.2) and (6.2.3). Conversely, if g is a continuous extension of f onto X^* then g is strongly equicontinuous with respect to δ^* and δ', where δ^* is the unique compatible RC-proximity of X^* (see (5.8)), and from (5.4) and (5.5) it follows that f is also strongly equicontinuous with respect to δ and δ'. □

Using (5.4)—(5.5), one can show that the canonical injection of an RC-proximity space into its absolutely closed ideal space is strongly equicontinuous, thus from (5.8) and (5.9) we get our final theorem:

(5.10) Theorem. *In the category of all separated RC-proximity spaces and their strongly equicontinuous mappings the subcategory of the absolutely closed spaces is epireflective.* □

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ON BIRECURRENT AFFINE MOTIONS IN A FINSLER MANIFOLD

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1. Introduction

K. Takano and T. Imai studied affine motions generated by recurrent and birecurrent vector fields in a non-Riemannian manifold of recurrent curvature [10, 11]. Most of the results obtained by K. Takano [10] were extended to Finsler manifolds of recurrent curvature (recurrent Finsler manifolds) by R. B. Misra [3], R. B. Misra and F. M. Meher [4], F. M. Meher [2] and A. Kumar [1]. The results obtained by these authors were some necessary consequences of the above types of affine motions in special type of Finsler manifolds and could not throw any light on the behaviour of these results in a general Finsler manifold as well as on the sufficient conditions for the above vector fields to generate an affine motion. The present author [8] obtained a necessary and sufficient condition for a recurrent vector field to generate an affine motion in a general Finsler manifold. However, the problem to find the necessary and sufficient condition for a birecurrent vector field to generate an affine motion in a general Finsler manifold remained untouched. In the present paper, the author solves this problem and from the necessary sufficient condition for a birecurrent vector field to generate an affine motion the author deduces the same necessary and sufficient condition for a recurrent vector field to generate an affine motion as obtained earlier. The author also finds out the integrability condition. The rest of the paper is devoted to the study of affine motions generated by the above vector fields in some special Finsler manifolds and almost all of the results obtained by the above authors [1, 2, 3, 4] are generalized. The notation of the present paper differs from that of H. Rund [9].

2. Preliminaries

Let F_n be an n-dimensional Finsler manifold of class at least C^7 equipped with a metric function F[1] satisfying the requisite conditions [9], corresponding symmetric metric tensor **g** and the Berwald's connection **G**. The coefficients of Berwald's connection **G**, denoted by G^i_{jk}, satisfy

$$\text{(2.1)} \qquad \text{a)}\quad G^i_{jk} = G^i_{kj}, \qquad \text{b)}\quad G^i_{jk}\dot{x}^k = G^i_j, \qquad \text{c)}\quad \dot{\partial}_k G^i_j = G^i_{jk},$$

where $\dot{\partial}_k$ stands for partial differentiation with respect to $\dot{x}^k$. The partial derivatives $\dot{\partial}_h G^i_{jk}$ of the connection coefficients G^i_{jk} constitute a tensor, say G^i_{jkh}, symmetric

[1] All the geometric objects are supposed to be functions of the line elements $(x^i, \dot{x}^i)$ unless stated otherwise. The indices $i, j, k, \ldots$ take positive integral values from 1 to n.

in its lower indices and satisfy

$$G^i_{jkh}\dot{x}^h = 0. \tag{2.2}$$

The covariant operator $\mathscr{B}_k$ for Berwald's connection **G** commutes with the differential operator $\dot{\partial}_j$ according to

$$(\dot{\partial}_j \mathscr{B}_k - \mathscr{B}_k \dot{\partial}_j) T^i_h = G^i_{jkr} T^r_h - G^r_{jkh} T^i_r, \tag{2.3}$$

$$(\mathscr{B}_j \mathscr{B}_k - \mathscr{B}_k \mathscr{B}_j) T^i_h = H^i_{jkr} T^r_h - H^r_{jkh} T^i_r - (\dot{\partial}_r T^i_h) H^r_{jk}, \tag{2.4}$$

where H^i_{jkh} constitute Berwald's curvature tensor. This tensor is skew-symmetric in its first two lower indices and positively homogeneous of degree zero in $\dot{x}^i$'s. The tensor H^i_{jk} appearing above satisfies

$$\text{a) } H^i_{jkh}\dot{x}^h = H^i_{jk}, \quad \text{b) } \dot{\partial}_h H^i_{jk} = H^i_{jkh}, \quad \text{c) } y_i H^i_{jk} = 0 \text{ [7]}, \tag{2.5}$$

where $y_i = g_{ij}\dot{x}^j$ and g_{ij} are components of the metric tensor **g**. The tensor H^i_{jk} and the deviation tensor H^i_j are related by

$$\text{a) } H^i_{jk}\dot{x}^k = H^i_j, \quad \text{b) } \tfrac{1}{3}(\dot{\partial}_k H^i_j - \dot{\partial}_j H^i_k) = H^i_{jk}. \tag{2.6}$$

Let us consider an infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon v^i(x^j) \tag{2.7}$$

generated by a contravariant vector field $v^i(x^j)$, ε being an infinitesimal constant. The Lie derivatives of an arbitrary tensor field T^i_j and connection coefficients G^i_{jk} with respect to the infinitesimal transformation (2.7) are given by [12], Ch. VII. § 5, as

$$\mathscr{L}T^i_j = v^r \mathscr{B}_r T^i_j - T^r_j \mathscr{B}_r v^i + T^i_r \mathscr{B}_j v^r + (\dot{\partial}_r T^i_j) \mathscr{B}_s v^r \dot{x}^s, \tag{2.8}$$

$$\mathscr{L}G^i_{jk} = \mathscr{B}_j \mathscr{B}_k v^i + H^i_{mjk} v^m + G^i_{jkr} \mathscr{B}_s v^r \dot{x}^s. \tag{2.9}$$

An infinitesimal transformation is an affine motion if and only if [9, 12]

$$\mathscr{L}G^i_{jk} = 0. \tag{2.10}$$

A vector field v^i is said to be recurrent or birecurrent according as it satisfies [10, 11]

$$\mathscr{B}_k v^i = \mu_k v^i \tag{2.11}$$

or

$$\mathscr{B}_j \mathscr{B}_k v^i = \mu_{jk} v^i, \tag{2.12}$$

where μ_k and μ_{jk} are components of a non-zero covariant vector field and a non-zero covariant tensor field of order 2, respectively.

A Finsler manifold is said to be a Berwald manifold, Landsberg manifold, recurrent manifold and birecurrent manifold if it admits

$$G^i_{jkh} = 0, \tag{2.13}$$

$$y_i G^i_{jkh} = 0, \tag{2.14}$$

$$\mathscr{B}_m H^i_{jkh} = \lambda_m H^i_{jkh}, \quad H^i_{jkh} \neq 0, \tag{2.15}$$

and

$$\mathscr{B}_l \mathscr{B}_m H^i_{jkh} = a_{lm} H^i_{jkh}, \quad H^i_{jkh} \neq 0, \tag{2.16}$$

respectively. λ_m and a_{lm} are components of a non-zero covariant vector and covariant tensor field of order 2; and are called recurrence vector and recurrence tensor respectively.

3. Birecurrent affine motion

Let us consider an infinitesimal transformation generated by a birecurrent vector field $v^i(x^j)$ characterized by (2.12). In view of (2.9), the Lie derivative of G^i_{jk} is given by

$$\mathscr{L} G^i_{jk} = \mu_{jk} v^i + H^i_{mjk} v^m + G^i_{jkr} \mathscr{B}_s v^r \dot{x}^s. \tag{3.1}$$

If the vector field v^i generates an affine motion, then the Lie derivative of G^i_{jk} vanishes and hence

$$\mu_{jk} v^i + H^i_{mjk} v^m + G^i_{jkr} \mathscr{B}_s v^r \dot{x}^s = 0. \tag{3.2}$$

Transvecting (3.2) by $\dot{x}^k$ and using the equations (2.2) and (2.5a), we have

$$\mu_{jk} \dot{x}^k v^i + H^i_{mj} v^m = 0. \tag{3.3}$$

Transvecting (3.3) by y_i and using (2.5c), we find at least one of the following conditions

$$\text{a)} \quad \mu_{jk} \dot{x}^k = 0, \qquad \text{b)} \quad y_i v^i = 0. \tag{3.4}$$

Equation (3.4b) can not be true, for partial differentiation of (3.4b) with respect to $\dot{x}^j$ gives $(\dot{\partial}_j y_i) v^i = g_{ij} \ v^i = 0$[2], which means $v^i = 0$. Therefore condition (3.4a) necessarily holds. Thus, if the birecurrent vector field v^i characterized by (2.12) generates an affine motion, then condition (3.4a) is necessary. Conversely, suppose that the recurrence tensor μ_{jk} of a birecurrent vector field v^i characterized by (2.12) satisfies (3.4a) and this vector field generates an infinitesimal transformation. The Lie derivative of G^i_{jk} with respect to this transformation is given by (3.1). Transvecting (3.1) by $\dot{x}^k$ and using (2.1b), (2.2), (2.5a) and (3.4a), we get

$$\mathscr{L} G^i_j = H^i_{mj} v^m. \tag{3.5}$$

Differentiating (3.4a) partially with respect to $\dot{x}^m$, we get

$$\dot{x}^k \dot{\partial}_m \mu_{jk} + \mu_{jm} = 0. \tag{3.6}$$

Differentiating (2.12) partially with respect to $\dot{x}^m$ and using the commutation formula exhibited by (2.3), we have

$$\mathscr{B}_j (G^i_{mkr} v^r) + G^i_{mjr} \mathscr{B}_k v^r - G^r_{mjk} \mathscr{B}_r v^i = \dot{\partial}_m \mu_{jk} v^i; \tag{3.7}$$

which, after transvection with $\dot{x}^k$, gives

$$G^i_{mjr} \dot{x}^k \mathscr{B}_k v^r = \dot{x}^k \dot{\partial}_m \mu_{jk} v^i. \tag{3.8}$$

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[2] $\dot{\partial}_j y_i = g_{ij}$ [9].

From (3.6) and (3.8) we have

(3.9) $$\mu_{jm}v^i+G^i_{mjr}\dot{x}^k\mathscr{B}_k v^r = 0.$$

Since the tensor G^i_{mjr} is symmetric in its lower indices, the equation (3.9) shows that the recurrence tensor μ_{jm} is symmetric. Taking skew-symmetric part of (2.12), using the commutation formula exhibited by (2.4) and the symmetric property of the tensor μ_{jk}, we have

(3.10) $$H^i_{jkh}v^h = 0.$$

Transvecting the Bianchi identity [9] $H^i_{jkh}+H^i_{khj}+H^i_{hjk}=0$ by v^h and using (3.10), we have

(3.11) $$(H^i_{khj}+H^i_{hjk})v^h = 0.$$

In view of the skew-symmetric property of the curvature tensor H^i_{jkh} in its first two lower indices, (3.11) becomes

(3.12) $$H^i_{hkj}v^h = H^i_{hjk}v^h.$$

Transvecting (3.12) by $\dot{x}^j$ and using (2.5a), (2.5b) and (2.6a), we have

(3.13) $$2H^i_{hk}v^h = \dot{\partial}_k H^i_h v^h.$$

Transvecting (3.13) by y_i and using (2.5c), we have

(3.14) $$y_i\dot{\partial}_k H^i_h v^h = 0.$$

Transvecting (2.5c) by $\dot{x}^k$ and using (2.6a), we get $y_iH^i_j=0$. Differentiating this partially with respect to $\dot{x}^k$ and using $\dot{\partial}_k y_i = g_{ik}$ [9], we get

(3.15) $$y_i\dot{\partial}_k H^i_j + g_{ik}H^i_j = 0.$$

Transvecting (3.15) by v^j and using (3.14), we get $g_{ik}H^i_jv^j=0$; which, after transvection with g^{km} (the associate of g_{ij}), in view of $g^{km}g_{ik}=\delta^m_i$, gives $H^i_hv^h=0$. Consequently, the equation (3.13) reduces to

(3.16) $$H^i_{hk}v^h = 0.$$

In view of (3.16), equation (3.5) becomes

(3.17) $$\mathscr{L}G^i_j = 0.$$

Differentiating (3.17) partially with respect to $\dot{x}^k$ and using the commutative property of the operators $\mathscr{L}$ and $\dot{\partial}_k$, we find (2.10). Thus, the infinitesimal transformation considered is an affine motion. This leads to:

THEOREM 3.1. *The necessary and sufficient condition for a birecurrent vector field v^i characterized by* (2.12) *to generate an affine motion in a Finsler manifold is given by* (3.4a).

If the recurrence tensor μ_{jk} of a birecurrent vector field v^i generating an affine motion is independent of the $\dot{x}^i$'s, then partial differentiation of (3.4a) with respect to $\dot{x}^m$ gives $\mu_{jm}=0$; a contradiction. Thus, we have:

COROLLARY 3.1. *The recurrence tensor of a birecurrent vector field generating an affine motion can not be independent of the* $\dot{x}^i$'s.

We have already seen in the above discussion that whenever the recurrence tensor μ_{jk} satisfies (3.4a), we get (3.9), which shows that the recurrence tensor is necessarily symmetric. Thus, we conclude:

THEOREM 3.2. *If a birecurrent vector field generates an affine motion, then its recurrence tensor is symmetric. In other words, a birecurrent vector field with non--symmetric recurrence tensor can not generate an affine motion in a Finsler manifold.*

Now we shall find out some special Finsler manifolds which do not admit any birecurrent affine motion. In this context we propose the following:

THEOREM 3.3. *A Landsberg manifold does not admit any affine motion generated by a birecurrent vector field.*

PROOF. Let us consider a Landsberg manifold characterized by (2.14). If it admits an affine motion generated by a birecurrent vector field v^i characterized by (2.12), we have (3.4a), from which we may deduce (3.9). Transvecting (3.9) by y_i and using (2.14), we get $y_i v^i \mu_{jk}=0$. This implies $\mu_{jk}=0$ for $y_i v^i \neq 0$, a contradiction.

Since every Berwald manifold is a Landsberg manifold, in view of Theorem 3.3, we have:

COROLLARY 3.2. *A Berwald manifold does not admit any affine motion generated by a birecurrent vector field.*

THEOREM 3.4. *A non-flat Finsler manifold of scalar curvature does not admit any affine motion generated by a birecurrent vector field.*

PROOF. Let us consider a non-flat Finsler manifold of scalar curvature characterized by [9]

$$H^i_j = F^2 R(\delta^i_j - l^i l_j), \tag{3.18}$$

where R is the Riemannian curvature, $l^i=\dfrac{\dot{x}^i}{F}$ and $l_j=g_{ij}l^i$. If this manifold admits an affine motion generated by a birecurrent vector field v^i characterized by (2.12), then we have (3.4a) and (3.3); which imply $H^i_{mj}v^m=0$. Transvecting $H^i_{mj}v^m=0$ by $\dot{x}^j$ and using (2.6a), we get $H^i_m v^m=0$. Transvecting (3.18) by v^j and using $H^i_j v^j=0$, we have

$$v^i - l_j v^j l^i = 0, \tag{3.19}$$

since $R\neq 0$ for a non-flat manifold. In view of $l^i=\dfrac{\dot{x}^i}{F}$, the equation (3.19) may be rewritten as

$$v^i - (l_j v^j/F)\dot{x}^i = 0. \tag{3.20}$$

Differentiating (3.20) partially with respect to $\dot{x}^m$ we get

$$\dot{x}^i \dot{\partial}_m (l_j v^j/F) + (l_j v^j/F)\delta^i_m = 0. \tag{3.21}$$

Since F is positively homogeneous of degree one in $\dot{x}^i$'s, l^i is positively homogeneous of degree zero in the $\dot{x}^i$'s and so is the vector l_j. Hence $l_j v^j/F$ is positively homogeneous of degree -1 in the $\dot{x}^i$'s. Therefore $\dot{x}^m \dot{\partial}_m(l_j v^j/F) = -(l_j v^j)/F$. In view of this result, the contraction of indices in (3.21) yields $(n-1)l_j v^j/F=0$; which implies $l_j v^j/F=0$. In view of this, (3.20) reduces to $v^i=0$, but the vector field v^i is non-zero. This completes the proof.

Let us consider a recurrent Finsler manifold characterized by (2.15). If the curvature scalar $H \equiv \frac{1}{n-1} H^i_i$ is non-vanishing, then we have the identities [6]

$$\lambda_m H^i_{jkh} + \lambda_k H^i_{mjh} + \lambda_j H^i_{kmh} = 0, \tag{3.22}$$

and

$$H^r_j G^i_{kmr} = H^r_k G^i_{jmr}. \tag{3.23}$$

If this manifold admits an affine motion generated by a birecurrent vector field v^i characterized by (2.12), then we have (3.9) and (3.16). Transvecting (3.22) by $\dot{x}^h$ and using (2.5a), we get

$$\lambda_m H^i_{jk} + \lambda_k H^i_{mj} + \lambda_j H^i_{km} = 0. \tag{3.24}$$

The transvection of (3.24) by v^m, in view of (3.16) and the skew-symmetric property of H^i_{km} in the indices k and m, gives $\lambda_m v^m=0$, since $H^i_{jk} \neq 0$. Thus the birecurrent vector field v^i is orthogonal to the recurrence vector λ_m. Again, the transvection of (3.16) by $\dot{x}^k$ gives $H^i_h v^h=0$. Differentiating $H^i_h v^h=0$ covariantly with respect to x^l and using the fact that the deviation tensor H^i_h is recurrent in a recurrent Finsler manifold, we have

$$H^i_h \mathscr{B}_l v^h = 0. \tag{3.25}$$

Transvecting (3.23) with $\mathscr{B}_l v^k$ and using (3.25), we get

$$H^r_j G^i_{kmr} \mathscr{B}_l v^k = 0. \tag{3.26}$$

The transvection of (3.9) by H^j_p, in view of (3.26), gives $H^j_p \mu_{jm}=0$. If $\operatorname{Det} \mu_{jm} \neq 0$, then $H^j_p \mu_{jm}=0$ implies $H^j_p=0$; which, in view of (2.5b) and (2.6b), gives $H^i_{jkh}=0$, a contradiction. Hence $\operatorname{Det} \mu_{jm}=0$. This leads to:

THEOREM 3.5. *If a birecurrent vector field v^i characterized by* (2.12) *generates an affine motion in a recurrent Finsler manifold, then the recurrence vector is orthogonal to the vector field v^i and* $\operatorname{Det} \mu_{jm}=0$.

If the Finsler manifold under consideration is birecurrent characterized by (2.16), then we have the identity [5]

$$a_{lm} H^i_{jk} + a_{lk} H^i_{mj} + a_{lj} H^i_{km} = 0. \tag{3.27}$$

If it admits an affine motion generated by a birecurrent vector field v^i characterized by (2.12), then we have (3.16). Transvecting (3.27) by v^m and using (3.16), we get $a_{lm} v^m=0$ since $H^i_{jk} \neq 0$. This leads to:

THEOREM 3.6. *If a birecurrent Finsler manifold admits an affine motion generated by a birecurrent vector field v^i characterized by* (2.12), *then* $a_{lm} v^m=0$.

4. Recurrent affine motion

Let us consider a recurrent vector field characterized by (2.11). The covariant differentiation of (2.11) shows that the vector field v^i is birecurrent with the recurrence tensor $\mathscr{B}_j\mu_k+\mu_j\mu_k$, i.e.

$$\mathscr{B}_j\mathscr{B}_k v^i = (\mathscr{B}_j\mu_k+\mu_j\mu_k)v^i. \tag{4.1}$$

In view of Theorem 3.1, the necessary and sufficient condition for the above vector field to generate an affine motion is given by

$$(\mathscr{B}_j\mu_k+\mu_j\mu_k)\dot{x}^k = 0. \tag{4.2}$$

Differentiating (4.2) partially with respect to $\dot{x}^m$ and using the commutation formula exhibited by (2.3), we get

$$\mathscr{B}_j\mu_m+\mu_j\mu_m+\dot{x}^k(\mathscr{B}_j\dot{\partial}_m\mu_k+\mu_k\dot{\partial}_m\mu_j+\mu_j\dot{\partial}_m\mu_k) = 0. \tag{4.3}$$

Differentiating (2.11) partially with respect to $\dot{x}^m$ and using the commutation formula exhibited by (2.3), we get

$$G^i_{kmr}v^r = (\dot{\partial}_m\mu_k)v^i. \tag{4.4}$$

Transvecting (4.4) by $\dot{x}^k$ and using (2.2), we get

$$\dot{x}^k\dot{\partial}_m\mu_k = 0. \tag{4.5}$$

In view of (4.5), (4.3) may be written as

$$\mathscr{B}_j\mu_m+\mu_j\mu_m+\mu\dot{\partial}_m\mu_j = 0, \tag{4.6}$$

where $\mu=\mu_k\dot{x}^k$. Thus, we see that (4.2) implies (4.6). Transvecting (4.4) by $\dot{x}^m$ we get

$$\dot{x}^m\dot{\partial}_m\mu_k = 0. \tag{4.7}$$

The transvection of (4.6) by $\dot{x}^m$, in view of (4.7), gives (4.2). Therefore the condition (4.6) implies (4.2). Thus the conditions (4.2) and (4.6) are equivalent. This leads to:

THEOREM 4.1. *The necessary and sufficient condition for a recurrent vector field v^i characterized by* (2.11) *to generate an affine motion in a Finsler manifold is given by* (4.6).

From (4.4) it is clear that the tensor $\dot{\partial}_m\mu_k$ is symmetric, i.e.

$$\dot{\partial}_m\mu_k = \dot{\partial}_k\mu_m. \tag{4.8}$$

In view of this result, (4.6) shows that the tensor $\mathscr{B}_j\mu_k$ is symmetric, i.e.

$$\mathscr{B}_j\mu_k = \mathscr{B}_k\mu_j. \tag{4.9}$$

Differentiating (4.6) covariantly with respect ot x^k and then taking skew-symmetric part with respect to the indices j and k, we have

$$\mu_r H^r_{jkm}+(\dot{\partial}_r\mu_m)H^r_{jk} = 0; \tag{4.10}$$

to find this result we have utilized the commutation formulae (2.3) and (2.4), equation

(4.6) and the symmetric property of the tensor $\dot{\partial}_m \mu_k$ and G^i_{jkh}. This is an integrability condition for (4.6). In view of (4.8), this integrability condition may be rewritten as

$$\mu_r H^r_{jkm} + (\dot{\partial}_m \mu_r) H^r_{jk} = 0. \tag{4.11}$$

The transvection of (4.11) by $\dot{x}^m$, in view of (2.5a), gives

$$\mu_r H^r_{jk} = 0; \tag{4.12}$$

which after partial differentiation with respect to $\dot{x}^m$, in view of (2.5b), gives (4.11). Thus, the conditions (4.11) and (4.12) are equivalent. Therefore we may conclude that *the integrability condition for* (4.6) *is* (4.12).

If the recurrent vector field v^i characterized by (2.11) generates an affine motion, then

$$\mathcal{L}G^i_{jk} \equiv (\mathcal{B}_j \mu_k + \mu_j \mu_k) v^i + H^i_{mjk} v^m + G^i_{jkr} \mathcal{B}_s v^r \dot{x}^s = 0;$$

which after transvection with $\dot{x}^k$, in view of (2.2), (2.5a) and (4.2), gives

$$H^i_{mj} v^m = 0. \tag{4.13}$$

Differentiating (4.13) partially with respect to $\dot{x}^k$ and using (2.5b), we get

$$H^i_{mjk} v^m = 0. \tag{4.14}$$

Also, in view of (2.4) and (4.9), the skew-symmetric part of (4.1) is given by

$$H^i_{jkm} v^m = 0. \tag{4.15}$$

We know that the tensor H^i_{jk} satisfies the identity [9]

$$\mathcal{B}_m H^i_{jk} + \mathcal{B}_k H^i_{mj} + \mathcal{B}_j H^i_{km} = 0. \tag{4.16}$$

Transvecting (4.16) by v^m and using (2.11) and (4.13), we have

$$v^m \mathcal{B}_m H^i_{jk} = 0. \tag{4.17}$$

Thus, we conclude:

THEOREM 4.2. *Conditions* (4.13), (4.14), (4.15) *and* (4.17) *are the necessary consequences of an affine motion generated by a recurrent vector field* v^i *in a Finsler manifold.*

From this theorem it is clear that if a recurrent vector field v^i generates an affine motion in a general Finsler manifold, we have (4.13); which, after contraction of the indices i and j, yields $H^i_{mi} v^m = 0$. Thus, we have:

COROLLARY 4.1. *If a Finsler manifold admits an affine motion generated by a recurrent vector field* v^i, *then necessarily* $H^i_{mi} v^m = 0$.

Let the Finsler manifold considered be recurrent which is characterized by (2.15). Transvecting (2.15) by $\dot{x}^h$ and using (2.5a), we get

$$\mathcal{B}_m H^i_{jk} = \lambda_m H^i_{jk}. \tag{4.18}$$

Transvecting (4.18) by v^m and using (4.17), we have $\lambda_m v^m = 0$; for $H^i_{jk} = 0$ will imply $H^i_{jkh} = 0$, a contradiction. Thus, we have:

COROLLARY 4.2. *If a recurrent vector field v^i generates an affine motion in a recurrent Finsler manifold, then the recurrence vector is orthogonal to the vector field v^i.*

If the Finsler manifold considered is birecurrent, the results obtained after transvection of (2.16) by $\dot{x}^h$ and covariant differentiation of (4.17) with respect to x^l imply $a_{lm}v^m=0$. Thus, we have:

COROLLARY 4.3. *If a recurrent vector field v^i generates an affine motion in a bireccurent Finsler manifold with recurrence tensor a_{lm}, then $a_{lm}v^m=0$.*

Corollary 4.1 generalizes Theorem 2.1 of R. B. Misra and F. M. Meher [4] and Theorem 5.1 of R. B. Misra [3]. From (4.12) and (4.13), it is clear that the condition $\mu_r H^r_{jk} v^i = H^i_{jkr} v^r \mu_k \dot{x}^k$ is trivially true. This represents a generalization of Theorem 3.1 of R. B. Misra and F. M. Meher [4]. These authors also established that the processes of covariant differentiation commute for $\dot{x}^h \mathscr{B}_h v^i$ in a recurrent Finsler manifold admitting an affine motion generated by a recurrent vector field v^i characterized by (Theorem 3.2, [4])

$$\mathscr{B}_k v^i = -\lambda_k v^i. \tag{4.19}$$

Writing the commutation formula (2.4) for $\mathscr{B}_h v^i$ and then using (2.11), (4.10) and (4.15), we have $(\mathscr{B}_j \mathscr{B}_k - \mathscr{B}_k \mathscr{B}_j)\mathscr{B}_h v^i = 0$; which shows that the processes of covariant differentiation commute for $\mathscr{B}_h v^i$ in a general Finsler manifold admitting an affine motion generated by any recurrent vector field v^i. This generalizes the above theorem of R. B. Misra and F. M. Meher.

They also proved that in a recurrent Finsler manifold admitting an affine motion generated by a recurrent vector field v^i characterized by (4.19), there exists a scalar point function σ satisfying

$$\mathscr{B}_k(\sigma \dot{x}^h \mathscr{B}_h v^i) = 0, \tag{4.20}$$

and it is connected with the recurrence vector λ_m by

$$\mathscr{B}_k \lambda + \lambda(\sigma_k - \lambda_k) = 0, \tag{4.21}$$

where $\lambda = \lambda_h \dot{x}^h$, $\sigma_k = (\mathscr{B}_k \sigma)/\sigma = \mathscr{B}_k \log \sigma$. In view of Theorem 4.1 and the fact that the recurrence vector is at most a point function [3, 6], the condition

$$\mathscr{B}_j \lambda_m - \lambda_j \lambda_m = 0 \tag{4.22}$$

holds good. Transvecting (4.22) by $\dot{x}^m$ we get $\mathscr{B}_j \lambda - \lambda \lambda_j = 0$, and hence (4.21) may be written as $\lambda \sigma_k = 0$. This implies $\sigma_k = 0$, for $\lambda = 0$, after partial differentiation with respect to $\dot{x}^m$, gives $\lambda_m = 0$, a contradiction. Thus we see that the vector field σ_k vanishes identically. Due to unawareness of this fact F. M. Meher devoted his paper [2] to the study of the vector field σ_k.

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NOTE ON A PROBLEM OF KÁTAI

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Suppose the set $E \subset \mathbf{N} = \{1, 2, 3, \ldots\}$ has the following property: all $f \colon \mathbf{N} \to \mathbf{R}$ satisfying $f(ab) = f(a) + f(b)$ for which $f(E) \subset \mathbf{Z}$ must actually map all $\mathbf{N}$ into $\mathbf{Z}$. Kátai has conjectured that any $x \in \mathbf{N}$ must be expressible in the form $\Pi x_i^{l_i}$ for some $x_i \in E$ and $l_i \in \mathbf{Z}$. This follows immediately from the proposition below by writing $\theta(x)$ in the form $l_1\theta(x_1) + l_2\theta(x_2) + \ldots$.

The countably generated free abelian group

$$\oplus\mathbf{Z} = \{(x_1, x_2, \ldots) | x_i \in \mathbf{Z}; \quad \exists I \quad \text{with} \quad x_i = 0 \quad \text{for all} \quad i > I\}$$

under coordinatewise addition admits a unique homomorphism $\theta \colon \mathbf{N} \to \oplus\mathbf{Z}$ of monoids [(i.e. $\theta(ab) = \theta(a) + \theta(b)$] for which $\theta(p_k) = (0, \ldots, 0, 1, 0, \ldots)$, zero except in the k^{th} coordinate, where p_k is the k^{th} prime.

PROPOSITION. *For E as above, $\theta(E)$ generates $\oplus\mathbf{Z}$ as an abelian group.*

PROOF. Assuming not, we can construct a commutative diagram as follows

$$\begin{tikzcd}
\mathbf{N} \arrow[d, "\theta"'] \arrow[dr, "f"] & \\
\oplus\mathbf{Z} \arrow[r, "\beta"] \arrow[d, "\text{project}"'] & \mathbf{R} \arrow[d, "\text{project}"] \\
\oplus\mathbf{Z}/G \arrow[r, "\alpha"] & \mathbf{R}/\mathbf{Z}
\end{tikzcd}$$

where G is the subgroup generated by $\theta(E)$. First let α be any non-zero homomorphism, which exists since G is proper and $\mathbf{R}/\mathbf{Z}$ is an injective abelian group containing elements of all orders. Next let β be any homomorphism which makes the square commute, using the fact that $\oplus\mathbf{Z}$ is free. But now $f := \beta \circ \theta$ clearly contradicts the assumption on E.

NOTES. i) The converse of Kátai's conjecture is obviously true. ii) The analogous but easier question where we assume $f(E) = 0 \Rightarrow f(\mathbf{N}) = 0$ and deduce the same result except that the exponents l_i may be rational was proved by Wolke [1]. His proof may be reformulated as above.

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IDEAL EXTENSIONS OF RINGS

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1. Introduction and summary

A ring R is an extension of a ring A by a ring B if R has an ideal I for which: $I \cong A$, $R/I \cong B$. With the usual identification of A with I and B with R/I, the extension problem is as follows: given rings A and B, construct all rings R having A as an ideal and such that $R/A=B$. A solution to this problem was given by Everett [1]; this is an analogue of the Schreier theorem for group extensions and is referred to as "Everett's theorem". As in the group case, one chooses a system of representatives of the cosets of A in R, and makes them act on A multiplicatively, hence every representative induces a bitranslation of A. Everett's theorem is, however, quite involved in view of the long list of ring postulates the extension ring has to satisfy; in addition, because of having chosen representatives in different cosets, two "factor systems", one for addition and one for multiplication, have to be introduced. The additive group of the extension ring R is an abelian group extension of the additive group of A by the additive group of B, and hence follows the Schreier group extension theory.

Two extensions R and R' of a ring A are equivalent if there exists an A-isomorphism of R onto R' (i.e., leaves A elementwise fixed) which maps the cosets of A in R onto the cosets of A in R'. Given rings A and B, a function θ of B onto a set of permutable bitranslations of $A(\theta\colon b\to\theta^b\in\Omega(A))$ and two functions $[\,,\,]$, $\langle\,,\,\rangle$: $B\times B\to A$, on $R=A\times B$ define an addition and multiplication by

$$(\alpha, a)+(\beta, b) = (\alpha+\beta+[a, b], a+b),$$
$$(\alpha, a)(\beta, b) = (\alpha\beta+\alpha\theta^b+\theta^a\beta+\langle a, b\rangle, ab).$$

If the three functions satisfy certain conditions, R is an extension of A by B where A is identified with $\{(0,\alpha)|\alpha\in A\}$ and B with the quotient R/A; the ring R is an Everett sum of A and B. Conversely, every extension of A by B is equivalent to an extension of this form.

For a full discussion of ring extensions and of Everett's theorem consult Rédei ([5], §§ 52—54).

Section 2 contains a discussion of the extension problem for rings including the relevant definitions. Material concerning the translational hull of rings is exposed in Section 3. Everett sums are constructed in Section 3 and a new proof of Everett's theorem is given including an equivalence criterion for Everett sums. Strict, pure and essential extensions as well as the character of an arbitrary extension are discussed in Section 5. Extensions of rings A for which the annihilator $\mathfrak{A}(A)$ is trivial are treated in Section 6; this case admits simpler constructions and stronger statements including several ramifications. Extensions of semiprime atomic rings are treated in

Section 7 from the point of view of finding conditions on such rings which insure that they admit only direct sum extensions with certain other rings. Several problems on the subject make up the concluding Section 8.

Some of the results here were announced in [3].

2. The problem

For a rigorous treatment of ring extensions, we must consider an extension of a ring A by a ring B as a triple (φ, R, ψ) rather than as a single ring. Formally, we proceed as follows. In the entire paper, *A and B stand for arbitrary rings unless specified otherwise.*

DEFINITION 2.1. A triple (φ, R, ψ) is an *(ideal) extension* of A by B if

(i) R is a ring,
(ii) φ is an isomorphism of A onto an ideal I of R,
(iii) ψ is a homomorphism of R onto B with kernel I.

In other words, an extension of A by B is a short exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} R \xrightarrow{\psi} B \longrightarrow 0.$$

It is essential to have a criterion for distinguishing extensions of A by B, or for considering two such extensions as "equal". For this we need the following concept.

DEFINITION 2.2. Two extensions (φ, R, ψ) and (φ', R', ψ') of A by B are *equivalent* if there exists an isomorphism χ of R onto R' making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varphi'"'] & R \arrow[d, "\psi"] \arrow[dl, "\chi"'] \\ R' \arrow[r, "\psi'"] & B \end{tikzcd}$$

In such a case, we call χ an *equivalence isomorphism.*

In order to obtain an overview of all ideal extensions of A by B, up to equivalence, we may use the following strategy.

(i) We construct a special type of extension of A by B by means of an Everett sum (the analogue of a Schreier product for groups).

(ii) Next we show that every extension of A by B is equivalent to an Everett sum.

(iii) We establish a criterion for equivalence of Everett sums.

The first part of this program is the direct part of the Everett theorem; the second part is the converse of the Everett theorem. The direct part is quite involved; the amorphous mass of various conditions can be put into relative order by the construction of the translational hull of A.

3. The translational hull

We introduce here the relevant concepts and establish a few of their properties.

DEFINITION 3.1. Let R be any ring. A transformation λ (respectively ϱ) on R is a *left* (respectively *right*) *translation* of R if λ is written as a left (respectively right) operator and satisfies

$$\lambda(xy) = (\lambda x)y, \quad \lambda(x+y) = \lambda x + \lambda y,$$

(respectively $(xy)\varrho = x(y\varrho)$, $(x+y)\varrho = x\varrho + y\varrho$) for all $x, y \in R$. Moreover, the pair (λ, ϱ) is *linked* if

$$x(\lambda y) = (x\varrho)y \quad (x, y \in R),$$

and is then called a *bitranslation* of R. The set $\Omega(R)$ of all bitranslations of R with the operations of addition and multiplication defined by

$$(\lambda, \varrho) + (\lambda', \varrho') = (\lambda+\lambda', \varrho+\varrho'), \quad (\lambda, \varrho)(\lambda', \varrho') = (\lambda\lambda', \varrho\varrho')$$

is the *translational hull* of R. We will often denote $(\lambda, \varrho) \in \Omega(R)$ by a single letter θ and consider it as a *double operator*, that is

$$\theta x = \lambda x, \quad x\theta = x\varrho \quad (x \in R).$$

Note that $\lambda+\lambda'$ and $\varrho+\varrho'$ is the usual addition in an abelian group, that is,

$$(\lambda+\lambda')x = \lambda x + \lambda' x, \quad x(\varrho+\varrho') = x\varrho + x\varrho' \quad (x \in R),$$

and the compositions $\lambda\lambda'$ and $\varrho\varrho'$ are defined by

$$(\lambda\lambda')x = \lambda(\lambda' x), \quad x(\varrho\varrho') = (x\varrho)\varrho' \quad (x \in R).$$

Easy verification shows that $\Omega(R)$ is closed under its operations and that it is actually a ring. There is an important part of $\Omega(R)$ which we now define.

DEFINITION 3.2. For any $r \in R$, the functions λ_r and ϱ_r given by

$$\lambda_r(x) = rx, \quad x\varrho_r = xr \quad (x \in R),$$

are the *inner left*, respectively *right, translations of* R *induced by* r; the pair $\pi_r = (\lambda_r, \varrho_r)$ is the *inner bitranslation of* R *induced by* r. The set $\Pi(R)$ of all inner bitranslations of R is the *inner part* of $\Omega(R)$.

Note that π_0 is the zero of the ring $\Omega(R)$. Simple verification shows that for any $r, s \in R$, $\theta \in \Omega(R)$, we have

$$\theta\pi_r = \pi_{\theta r}, \quad \pi_r\theta = \pi_{r\theta}, \quad \pi_r + \pi_s = \pi_{r+s},$$

which implies that $\Pi(R)$ is an ideal of $\Omega(R)$. One sees just as easily that the mapping $\pi: r \to \pi_r$ $(r \in R)$ is a homomorphism of R with kernel

$$\mathfrak{A}(R) = \{r \in R \mid rx = xr = 0 \text{ for all } x \in R\},$$

called the *annihilator* of R.

DEFINITION 3.3. The mapping π above is the *canonical homomorphism* of R onto $\Pi(R)$. The annihilator $\mathfrak{A}(R)$ is *trivial* if $\mathfrak{A}(R)=0$.

Thus the canonical homomorphism π is an isomorphism if and only if R has trivial annihilator. This is a relatively mild restriction on the ring R and we will see that many statements concerning extensions of R simplify considerably in the case $\mathfrak{A}(R)=0$. There is one more concept we need in this context.

DEFINITION 3.4. Two bitranslations θ and θ' of R are *permutable* if

$$(\theta x)\theta' = \theta(x\theta'), \quad (\theta' x)\theta = \theta'(x\theta) \quad (x\in R).$$

A nonempty set S of bitranslations is *permutable* if any two bitranslations in S are permutable.

If we write $\theta=(\lambda, \varrho)$, $\theta'=(\lambda', \varrho')$, then the above condition becomes

$$(\lambda x)\varrho' = \lambda(x\varrho'), \quad (\lambda' x)\varrho = \lambda'(x\varrho) \quad (x\in R).$$

Note that any two inner bitranslations are permutable. We will need the following simple results.

LEMMA 3.5. *Let ψ be an isomorphism of a ring R onto a ring S. Define a mapping*

$$\bar{\psi}\colon (\lambda, \varrho) \to (\bar{\lambda}, \bar{\varrho}) \quad ((\lambda, \varrho)\in\Omega(R))$$

where

$$\bar{\lambda}s = [\lambda(s\psi^{-1})]\psi, \quad s\bar{\varrho} = [(s\psi^{-1})\varrho]\psi \quad (s\in S).$$

Then $\bar{\psi}$ is an isomorphism of $\Omega(R)$ onto $\Omega(S)$, said to be induced by ψ. Moreover, for any $r\in R$, $\pi_r\bar{\psi}=\pi_{r\psi}$.

PROOF. The straightforward verification is omitted.

LEMMA 3.6. *Let R be a ring and (λ, ϱ), $(\lambda', \varrho')\in\Omega(R)$. Then $(\lambda r)\varrho'-\lambda(r\varrho')\in \mathfrak{A}(R)$ for all $r\in R$.*

PROOF. The simple verification is omitted.

4. Everett sums

We again fix two rings A and B. Elements of A (respectively B) will be denoted by lower case Greek (respectively Latin) letters; 0 is the zero of any ring.

CONSTRUCTION 4.1. Let $(\theta; [\,,], \langle\,,\rangle)$ be a triple of functions

$$\theta\colon B \to \Omega(A), \text{ with } \theta\colon a \to \theta^a, \quad [\,,]\colon B\times B \to A, \ \langle\,,\rangle\colon B\times B \to A$$

satisfying the following conditions for all $\alpha, \beta\in A$, $a, b, c\in B$:

(i) $\theta^0=\pi_0$; $[0, a]=\langle a, 0\rangle=\langle 0, a\rangle=0$;
(ii) θ^a is permutable with θ^b;
(iii) $\theta^a+\theta^b-\theta^{a+b}=\pi_{[a,b]}$;
(iv) $\theta^a\theta^b-\theta^{ab}=\pi_{\langle a,b\rangle}$;
(v) $[a, b]+[a+b, c]=[a, b+c]+[b, c]$;
(vi) $[a, b]=[b, a]$;
(vii) $\langle ab, c\rangle-\langle a, bc\rangle=\theta^a\langle b, c\rangle-\langle a, b\rangle\theta^c$;
(viii) $\langle a, c\rangle+\langle b, c\rangle-\langle a+b, c\rangle=[a, b]\theta^c-[ac, bc]$;
(ix) $\langle a, b\rangle+\langle a, c\rangle-\langle a, b+c\rangle=\theta^a[b, c]-[ab, ac]$.

Let $R=A\times B$ be the Cartesian product of A and B with operations:

$$\text{(A)}\qquad (\alpha, a)+(\beta, b) = (\alpha+\beta+[a, b],\ a+b),$$

$$\text{(M)}\qquad (\alpha, a)(\beta, b) = (\alpha\beta+\alpha\theta^b+\theta^a\beta+\langle a, b\rangle,\ ab).$$

Define two mappings as follows:

$$\varphi\colon\ \alpha \to (\alpha, 0)\ (\alpha\in A),\quad \psi\colon\ (\alpha, a) \to a\ \big((\alpha, a)\in R\big).$$

Denote the triple (φ, R, ψ) by $E(\theta; [\,,], \langle\,,\rangle)$ and call it an *Everett sum of A and B.*

THEOREM 4.2. *The Everett sum* $(\varphi, R, \psi)=E(\theta; [\,,], \langle\,,\rangle)$ *is an extension of A by B.*

PROOF. Associativity of addition is equivalent to item (v). The identity for addition is $(0, 0)$, and $(-\alpha-[a, -a], -a)$ is the additive inverse of (α, a). Commutativity of addition is equivalent to item (vi). Hence R is an abelian group under addition.

Associativity of multiplication is verified as follows:

$$[(\alpha, a)(\beta, b)](\gamma, c) = (\alpha\beta+\alpha\theta^b+\theta^a\beta+\langle a, b\rangle, ab)(\gamma, c) =$$

$$= \big(\alpha\beta\gamma+(\alpha\theta^b)\gamma+(\theta^a\beta)\gamma+\langle a, b\rangle\gamma+(\alpha\beta)\theta^c+(\alpha\theta^b)\theta^c+(\theta^a\beta)\alpha^c+\langle a, b\rangle\theta^c+$$

$$+\theta^{ab}\gamma+\langle ab, c\rangle, abc\big),$$

$$(\alpha, a)[(\beta, b)(\gamma, c)] = (\alpha, a)(\beta\gamma+\beta\theta^c+\theta^b\gamma+\langle b, c\rangle, bc) =$$

$$= \big(\alpha\beta\gamma+\alpha(\beta\theta^c)+\alpha(\theta^b\gamma)+\alpha\langle b, c\rangle+\alpha\theta^{bc}+\theta^a(\beta\gamma)+\theta^a(\beta\theta^c)+\theta^a(\theta^b\gamma)+\theta^a\langle b, c\rangle+$$

$$+\langle a, bc\rangle, abc\big).$$

Using the properties of bitranslations, including item (ii), the equality of the above expressions is equivalent to the equality

$$\langle a, b\rangle\gamma+\langle a, b\rangle\theta^c+\langle ab, c\rangle+\alpha\theta^b\theta^c+\theta^{ab}\gamma =$$

$$= \alpha\langle b, c\rangle+\theta^a\langle b, c\rangle+\langle a, bc\rangle+\alpha\theta^{bc}+\theta^a\theta^b\gamma;$$

equivalently

$$(\langle ab, c\rangle-\langle a, bc\rangle)+\alpha(\theta^b\theta^c-\theta^{bc})-\alpha\langle b, c\rangle =$$

$$= (\theta^a\langle b, c\rangle-\langle a, b\rangle\theta^c)+(\theta^a\theta^b-\theta^{ab})\gamma-\langle a, b\rangle\gamma;$$

and using items (iv) and (vii) this is equivalent to

$$\alpha\pi_{\langle b, c\rangle}-\alpha\langle b, c\rangle = \pi_{\langle a, b\rangle}\gamma-\langle a, b\rangle\gamma$$

which holds in view of the definition of an inner bitranslation π_x. Therefore the multiplication is associative.

For the right distributive law, we write

$$[(\alpha, a)+(\beta, b)](\gamma, c) = (\alpha+\beta+[a, b], a+b)(\gamma, c) =$$
$$= ((\alpha+\beta+[a, b])\gamma+(\alpha+\beta+[a, b])\theta^c+\theta^{a+b}\gamma+\langle a+b, c\rangle, (a+b)c) =$$
$$= (\alpha\gamma+\beta\gamma+[a, b]\gamma+\alpha\theta^c+\beta\theta^c+[a, b]\theta^c+\theta^{a+b}\gamma+\langle a+b, c\rangle, \quad ac+bc),$$
$$(\alpha, a)(\gamma, c)+(\beta, b)(\gamma, c) = (\alpha\gamma+\alpha\theta^c+\theta^a\gamma+\langle a, c\rangle, ac)+$$
$$+(\beta\gamma+\beta\theta^c+\theta^b\gamma+\langle b, c\rangle, bc) = (\alpha\gamma+\alpha\theta^c+\theta^a\gamma+\langle a, c\rangle+$$
$$+\beta\gamma+\beta\theta^c+\theta^b\gamma+\langle b, c\rangle+\langle ac, bc\rangle, ac+bc).$$

The equality of these two expressions is thus equivalent to

$$\langle a, b\rangle\gamma+[a, b]\theta^c+\theta^{a+b}\gamma+\langle a+b, c\rangle = \theta^a\gamma+\theta^b\gamma+\langle a, c\rangle+\langle b, c\rangle+\langle ac, bc\rangle$$

which can be written as

$$[a, b]\gamma+[a, b]\theta^c = (\theta^a+\theta^b-\theta^{a+b})\gamma+(\langle a, c\rangle+\langle b, c\rangle-\langle a+b, c\rangle+\langle ac, bc\rangle.$$

Using items (iii) and (viii), this is equivalent to

$$[a, b]\gamma+[a, b]\theta^c = \pi_{[a,b]}\gamma+[a, b]\theta^c$$

which evidently holds. The arguments for the left distributive law are analogous.

This makes R a ring. Using parts of item (i), we see that φ is an isomorphism of A onto the ring $I=\{(\alpha, 0)|\alpha\in A\}$. It follows at once that I is an ideal of R. From (A) and (M) above it is clear that ψ is a homomorphism of R onto B with kernel I. Therefore (φ, R, ψ) is an extension of A by B.

The above theorem completes the first part of our program. For the second part, we first introduce the following construction.

Construction 4.3. Let (ξ, R, η) be an extension of A by B. Let $I=A\xi$, and $\sigma\colon B\to R$ be any function satisfying: $\sigma\eta$ is the identity mapping on B, $0\sigma=0$. Using the notation in 3.5, we define the functions θ, $[\ ,\ ]$ and $\langle\ ,\ \rangle$ by:

$$\theta\colon a \to \theta^a = (\pi_{a\sigma}|_I)\overline{\xi^{-1}} \quad (a\in B),$$
$$[a, b] = \big(a\sigma+b\sigma-(a+b)\sigma\big)\xi^{-1} \quad (a, b\in B),$$
$$\langle a, b\rangle = \big((a\sigma)(b\sigma)-(ab)\sigma\big)\xi^{-1} \quad (a, b\in B).$$

The function σ "chooses" one element in each coset of I, and in I it "chooses" the zero. It is called a *choice function.*

Theorem 4.4. *The functions* $\theta, [\ ,\ ], \langle\ ,\ \rangle$ *defined in Construction* 4.3 *satisfy conditions in Construction* 4.1, *and the extension* (ξ, R, η) *is equivalent to* $E(\theta, [\ ,\ ], \langle\ ,\ \rangle)$ *by the equivalence isomorphism*

$$\chi\colon r \to (\alpha, a) \quad (r\in R),$$

where $a=(r+I)\eta$ *and* $\alpha=(r-a\sigma)\xi^{-1}$.

PROOF. For any $a\in B$, we have $\pi_{a\sigma}|_I\in\Omega(I)$, so that 3.5 gives $\theta^a\in\Omega(A)$. Hence θ maps B into $\Omega(A)$. Further, for any $a, b\in B$, we get

$$(a\sigma+b\sigma-(a+b)\sigma)\eta = a+b-(a+b) = 0$$

since $\sigma\eta$ is the identity mapping on B. Thus $a\sigma+b\sigma-(a+b)\sigma\in I$ which implies that $[a, b]\in A$. A similar argument shows that also $\langle a, b\rangle\in A$.

The condition $0\sigma=0$ implies that condition 4.1 (i) is fulfilled. Since inner bitranslations of R are permutable, condition 4.1 (ii) also holds. For any $a, b\in R$, we get by Lemma 3.5,

$$\theta^a+\theta^b-\theta^{a+b} = (\pi_{a\sigma+b\sigma-(a+b)\sigma}|_I)\overline{\xi^{-1}} = \pi_{(a\sigma+b\sigma-(a+b)\sigma)\xi^{-1}} = \pi_{[a,p]}$$

since $a\sigma+b\sigma-(a+b)\sigma\in I$. This verifies condition 4.1 (iii); 4.1 (iv) follows similarly.

Instead of verifying the remaining conditions in Construction 4.1, we let $E= =A\times B$ with operations (A) and (M) in 4.1 and show that χ given above is an isomorphism of R onto E. This will imply that E itself is a ring. The remaining conditions in 4.1 will then follow from ring axioms without much effort.

Let $r, s\in R$ and $r\chi=(\alpha, a)$, $s\chi=(\beta, b)$. Since $r-a\sigma\in I$, we have that χ maps R into $A\times B$. Further,

$$(r+s)+I)\eta = ((r+I)+(s+I))\eta = (r+I)\eta+(s+I)\eta = a+b,$$

$$(r+s-(a+b)\sigma)\xi^{-1} = (r+s-a\sigma-b\sigma+[a, b]\xi)\xi^{-1} =$$

$$= (r-a\sigma)\xi^{-1}+(s-b\sigma)\xi^{-1}+[a, b] = \alpha+\beta+[a, b]$$

and χ is additive; one shows similarly that $(rs+I)\eta=ab$. Note that by Lemma 3.5, we have

$$\alpha\theta^b = (r-a\sigma)\xi^{-1}(\pi_{b\sigma}|_I)\overline{\xi^{-1}} = ((r-a\sigma)(b\sigma))\xi^{-1}$$

and analogously

$$\theta^a\beta = ((a\sigma)(s-b\sigma))\xi^{-1}.$$

Using this, we obtain

$$(rs-(ab)\sigma)\xi^{-1} = (rs-(a\sigma)(b\sigma)+\langle a, b\rangle\xi)\xi^{-1} = (rs-(a\sigma)(b\sigma))\xi^{-1}+\langle a, b\rangle =$$

$$= ((r-a\sigma)(s-b\sigma)+r(b\sigma)+(a\sigma)s-2(a\sigma)(b\sigma))\xi^{-1}+\langle a, b\rangle =$$

$$= (r-a\sigma)\xi^{-1}(s-b\sigma)\xi^{-1}+(r(b\sigma)+(a\sigma)s-2(a\sigma)(b\sigma))\xi^{-1}+\langle a\rangle =$$

$$= \alpha\beta+\alpha\theta^b+\theta^a\beta+\langle a, b\rangle$$

and χ is also multiplicative.

If $r\chi=(0, 0)$, then $(r+I)\eta=0$ so $r\in I$ and thus $\alpha=r\xi^{-1}=0$. Hence $r=0$, and the kernel of χ is trivial. Let $(\alpha, a)\in A\times B$. Then for $r=a\sigma+\sigma\xi$, we have

$$(r+I)\eta = (a\sigma+\alpha\xi+I)\eta = a, \quad (r-a\sigma)\xi^{-1} = \alpha\xi\xi^{-1} = \alpha,$$

so that $(\alpha, a)=r\chi$. Therefore χ is an isomorphism of R onto E. So E is a ring. Now a simple inspection of the various parts of the proof of Theorem 4.2 easily gives that the conditions 4.1 (v)—(ix) all hold.

For any $\alpha\in A$, we have

$$\alpha\xi\chi = ((\alpha\xi-0)\xi^{-1}, 0) = (\alpha, 0) = \alpha\varphi,$$

and for $r \in R$, $r\chi = (\alpha, a)$, we obtain

$$r\chi\psi = (\alpha, a)\psi = a = r\eta.$$

This proves that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\xi"'] & E \arrow[d, "\psi"] \\ R \arrow[r, "\eta"'] \arrow[ur, "\chi"] & B \end{tikzcd}$$

is commutative. As a consequence, we have that the extensions (ξ, R, η) and $E(\theta; [\,,\,], \langle\,,\,\rangle)$ are equivalent.

The third part of our program provides the form of all equivalence isomorphisms between two Everett sums thereby giving necessary and sufficient conditions for their equivalence.

THEOREM 4.5. *Let* $(\xi, R, \eta) = E(\theta; [\,,\,], \langle\,,\,\rangle$ *and* $(\xi', R', \eta') = E(\theta'; [\,,\,]', \langle\,,\,\rangle')$ *be Everett sums of* A *and* B. *Let* $\zeta\colon B \to A$ *be any function satisfying* $0\zeta = 0$, *and for all* $a, b \in B$,

(i) $\theta'^a - \theta^a = \pi_{a\zeta}$,

(ii) $[a, b]' - [a, b] = a\zeta + \zeta b\zeta - (a+b)\zeta$,

(iii) $\langle a, b\rangle' - \langle a, b\rangle = \theta^a(b\zeta) + (a\zeta)\theta^b + (a\zeta)(b\zeta) - (ab)\zeta$.

Then χ *defined by:*

$$\chi\colon (\alpha, a) \to (\alpha - a\zeta, a) \quad ((\alpha, a) \in A \times B)$$

is an equivalence isomorphism of R *onto* R'. *Conversely, every equivalence isomorphism of* R *onto* R' *is of this form for some function* ζ *satisfying the above conditions.*

PROOF. *Necessity*. Let ζ and χ be as in the statement of the theorem. It is clear that χ is a permutation of the set $A \times B$.

Additivity of χ follows by straightforward verification using condition (ii). Using conditions (i) and (iii), we obtain

$$\begin{aligned}(\alpha, a)\chi(\beta, b)\chi &= (\alpha - a\zeta, a)(\beta - b\zeta, b) = \big(\alpha\beta - \alpha(b\zeta) - (a\zeta)\beta + (a\zeta)(b\zeta) + \\ &+ \theta'^a(\beta - b\zeta) + (\alpha - a\zeta)\theta'^b + \langle a, b\rangle, ab\big) = \big(\alpha\beta - \alpha(b\zeta) - (a\zeta)\beta + (a\zeta)(b\zeta) + \\ &+ (\theta^a + \pi_{a\zeta})(\beta - b\zeta) + (\alpha - a\zeta)(\theta^b + \pi_{b\zeta}) + \theta^a(b\zeta) + (a\zeta)\theta^b + (a\zeta)(b\zeta) - (ab)\zeta + \\ &+ \langle a, b\rangle, ab\big) = (\alpha\beta - \alpha(b\zeta) - (a\zeta)\beta + (a\zeta)(b\zeta) + \theta^a\beta - \theta^a(b\zeta) + (a\zeta)\beta - (a\zeta)(b\zeta) + \\ &+ \alpha\theta^b - (a\zeta)\theta^b + \alpha(b\zeta) - (a\zeta)(b\zeta) + \theta^a(b\zeta) + (a\zeta)\theta^b + (a\zeta)(b\zeta) - (ab)\zeta + \langle a, b\rangle, ab\big) = \\ &= \big(\alpha\beta + \theta^a\beta + \alpha\theta^b + \langle a, b\rangle - (ab)\zeta, ab\big) = (\alpha\beta + \theta^a\beta + \alpha\theta^b + \langle a, b\rangle, ab\big)\chi = \\ &= \big((\alpha, a)(\beta, b)\big)\chi\end{aligned}$$

and χ is an isomorphism. It is immediate that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi'"] \arrow[d, "\varphi"'] & R' \arrow[d, "\psi'"] \\ R \arrow[r, "\psi"'] \arrow[ur, "\chi"] & B \end{tikzcd}$$

is commutative. Thus χ is an equivalence isomorphism.

Sufficiency. Let $\chi: R \to R'$ be an equivalence isomorphism. Commutativity of the above diagram immediately implies that

$$(\alpha, a)\chi = \big((\alpha, a)\gamma, a\big) \quad \big((\alpha, a) \in E\big)$$

for some function $\gamma: A \times B \to A$ satisfying $(\alpha, 0)\gamma = \alpha$ for all $\alpha \in A$. Further, letting $a\zeta = -(0, a)\gamma$, we get

$$(\alpha, a)\chi = \big((\alpha, 0)+(0, a)\big)\chi = (\alpha, 0)\chi+(0, a)\chi = (\alpha, 0)+\big((0, a)\gamma, a\big) = (\alpha - a\zeta, a),$$

and $0\zeta = 0$.

It remains to verify conditions (i)—(iii). Indeed,

$$(\theta^a\alpha, 0) = (\theta^a\alpha, 0)\chi = \big((0, a)(\alpha, 0)\big)\chi = (0, a)\chi(\alpha, 0)\chi =$$
$$= (-a\zeta, a)(\alpha, 0) = \big(-(a\zeta)\alpha + \theta'^a\alpha, 0\big)$$

which gives $(\theta'^a - \theta^a)\alpha = \lambda_{a\zeta}\alpha$. An analogous argument shows that $\alpha(\theta'^a - \theta^a) = \alpha\varrho_{a\zeta}$. Hence $\theta'^a - \theta^a = \pi_{a\zeta}$ which verifies condition (i). Straightforward computation shows that additivity of χ implies condition (ii). Finally, using condition (i) and reversing the verification in the proof of necessity above that χ is a homomorphism, we see that condition (iii) holds as well.

We have thus completed the program announced in Section II. In summary, we have the following result.

THEOREM 4.6 [1]. *Every Everett sum of rings A and B is an ideal extension of A by B. Conversely, every ideal extension of A by B is equivalent to some Everett sum. Theorem 4.5 gives a criterion for the equivalence of Everett sums.*

The usual embedding of a ring R into a ring E with an identity is an extension of R by the ring $\mathbf{Z}$ of integers with

$$[m, n] = \langle m, n\rangle = 0, \quad \theta^n r = r\theta^n = nr \quad (m, n \in \mathbf{Z},\ r \in R).$$

5. Some invariants of an extension

Let $\mathcal{R} = (\zeta, R, \eta)$ be an extension of A by B. We have seen in Construction 4.3 that a function $\theta: B \to \Omega(A)$ can be defined if a choice function $\sigma: B \to R$ is given. Now let

$$\nu = \nu_A: \Omega(A) \to \Omega(A)/\Pi(A)$$

be the natural homomorphism. Then conditions 4.1 (iii) and (iv) imply that the composition $\theta\nu: B \to \Omega(A)/\pi(A)$ is a homomorphism. Let $\sigma': B \to R$ be another choice function and $\theta' \colon B \to \Omega(A)$ be the corresponding mapping. Then letting $I = A\xi$, we obtain for any $a \in B$,

$$\theta^a - \theta'^a = (\pi_{a\sigma}|_I)\overline{\xi^{-1}} - (\pi_{a\sigma'}|_I)\overline{\xi^{-1}} = (\pi_{a\sigma - a\sigma'}|_I)\overline{\xi^{-1}} \in \Pi(A)$$

since $a\sigma - a\sigma' \in I$. It follows that $\theta\nu = \theta'\nu$, and we may introduce the following concept.

DEFINITION 5.1. With the above notation, the homomorphism

$$\chi = \chi(\mathscr{R}) = \theta\nu_A \colon B \to \Omega(A)/\Pi(A)$$

is the *character* of the extension $\mathscr{R}$ of A by B.

By Theorem 4.5, we see that two equivalent extensions have the same character. We may thus speak of the *character of the equivalence class*.

Let $\mathscr{R} = (\xi, R, \eta)$ be an extension of A by B. Let $I = A\xi$ so that I is an ideal of R. Define a function $\tau = \tau(R\colon I)$ by $\tau\colon r \to \tau^r = \pi_r|_I$ $(r \in R)$. It follows easily that $\tau\colon R \to \Omega(I)$ is a homomorphism. Note that $\tau|_I = \pi$, the canonical homomorphism $\pi\colon I \to \Omega(I)$. We have seen that $\bar{\xi}$ is an isomorphism of $\Omega(A)$ onto $\Omega(I)$. Let $\bar{\xi}_\pi$ be the isomorphism of $\Omega(A)/\Pi(A)$ onto $\Omega(I)/\Pi(I)$ induced by $\bar{\xi}$. Then we have the following simple result.

LEMMA 5.2. *With the above notation, the following diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\xi"] \arrow[d, "\xi\pi"'] & R \arrow[rr, "\eta"] \arrow[dl, "\tau"'] \arrow[dd, "\tau\nu_I"] & & B \arrow[r] \arrow[dd, "\chi"] \arrow[dr, "\theta"] & 0 \\
& \Omega(I) \arrow[dr, "\nu_I"'] & & & & \Omega(A) \arrow[dl, "\nu_A"] \\
& & \Omega(I)/\Pi(I) & & \Omega(A)/\Pi(A) \arrow[ll, "\bar{\xi}_\pi"'] &
\end{tikzcd}$$

is commutative.

PROOF. Let $r \in R$. Then $r\eta\sigma - r \in I$ and

$$\theta^{r\eta}\bar{\xi} - \tau^r = (\pi_{r\eta\sigma} - \pi_r)|_I = \pi_{r\eta\sigma - r} \in \Pi(I)$$

and the diagram commutes.

It is convenient to introduce the following concepts.

DEFINITION 5.3. With the notation as above, the image $T(R\colon I)$ of R under the homomorphism $\tau(R\colon I)$ is the *type* of the extension $\mathscr{R}$. The extension $\mathscr{R}$ is *strict* if $\tau^r \in \Pi(I)$ for all $r \in R$; it is *pure* if $\tau^r \in \Pi(I)$ implies $r \in I$.

The type $T(R\colon I)$ is thus a ring of permutable bitranslations of I containing $\Pi(I)$. In fact, $T(R\colon I) = \Pi(I)$ if and only if the extension is strict.

PROPOSITION 5.4. *With the notation as above, the following is true.*

(i) *$\mathscr{R}$ is a strict extension if and only if $\chi(\mathscr{R})$ is the zero homomorphism, or equivalently, $\theta\colon B \to \Pi(A)$.*

(ii) *$\mathscr{R}$ is a pure extension if and only if $\chi(\mathscr{R})$ is a monomorphism, or equivalently $\theta^a \in \Pi(A)$ implies $a = 0$.*

PROOF. (i) Using Lemma 5.2, we obtain
$\mathscr{R}$ is a strict extension $\Leftrightarrow\tau\colon R\to\Omega(I)\Leftrightarrow\tau\nu_I\colon R\to 0=\Pi(I)\Leftrightarrow\chi\colon B\to 0=\Pi(A)\Leftrightarrow$ $\Leftrightarrow\theta\colon B\to\Pi(A)$.

(ii) Assume that $\mathscr{R}$ is a pure extension and let $b\chi=\Pi(A)$. Then $b=r\eta$ for some $r\in R$ and thus $r\eta\chi=\Pi(A)$. By Lemma 5.2, we get $r\tau\nu_I=\Pi(I)$ so that $\tau^r\in\Pi(I)$. Since the extension is pure, we must have $r\in I$ and thus $b=r\eta=0$. Consequently χ is a monomorphism.

Next assume that χ is a monomorphism and let $\theta^a\in\Pi(A)$. It follows that $a\chi=0$ which yields $a=0$ since χ is one-to-one.

Suppose next that $\theta^a\in\Pi(A)$ implies that $a=0$ and let $\tau^r\in\Omega(I)$. Then $r(\tau\nu_I)=0$ which by Lemma 5.2 gives $(r\eta)(\theta\nu_A)=0$. This implies that $\theta^{r\eta}\in\Pi(A)$ which by hypothesis yields $r\eta=0$. But then $r\in I$ and the extension is pure.

DEFINITION 5.5. Continuing with the same notation, the set $\mathfrak{A}_R(I)=\{r\in R|ri=$ $=ir=0$ for all $i\in I\}$ is the *annihilator* of I in R. Also let

$$S(R\colon I)=\{r\in R|\pi_r|_I\in\Pi(I)\}.$$

If we call R an *extension* of any of its ideals, we have the following result.

PROPOSITION 5.6. *Both $\mathfrak{A}_R(I)$ and $S=S(R\colon I)$ are ideals of R and $S=I+$ $+\mathfrak{A}_R(I)$. Moreover, S is the greatest strict extension of I contained in R, and R is a pure extension of S.*

PROOF. Clearly $\mathfrak{A}_R(I)$ is an ideal of R; S is the complete inverse image of $\Pi(I)$ under the homomorphism $\tau(R\colon I)$, and is thus an ideal of R. Obviously $I+$ $+\mathfrak{A}_R(I)\subseteqq S$. If $s\in S$, then $\pi_s|_I\in\Pi(I)$ so $\pi_s=\pi_i$ for some $i\in I$; but then $s=i+$ $+(s-i)\in I+\mathfrak{A}_R(I)$. Hence $S\subseteqq I+\mathfrak{A}_R(I)$, and the equality prevails. It follows from the definition of S that it is the greatest strict extension of I contained in R. Let $r\in R$ be such that $\tau^r(R\colon S)\in\Pi(S)$. Then $\tau^r(R\colon S)=\pi_s$ for some inner bitranslation π_s of S. But then $\pi_s|_I=\pi_i$ for some inner bitranslation π_i of I. Hence $\tau^r(R\colon S)|_I=\pi_i$ which implies that $r\in S$. Consequently R is a pure extension of S.

We can represent the situation in the preceding proposition by the following diagram:

$$\begin{tikzcd}[arrows=-]
& R \arrow[d] & \\
& S=S(R\colon I) \arrow[dl] \arrow[dr] & \\
I \arrow[dr] & & \mathfrak{A}_R(I)=\ker\tau(R\colon I) \arrow[dl] \\
& \mathfrak{A}(I) \arrow[d] & \\
& (0) &
\end{tikzcd}$$

Letting $(\varphi, R, \psi)=E(\theta; [\,,\,], \langle\,,\,\rangle)$ be an Everett sum of the rings A and B and $I=A\varphi$, we obtain

$$S=S(R\colon I)=\{(\alpha, a)\in R|\theta^a\in\Pi(A)\},\quad \mathfrak{A}_R(I)=\{(\alpha, a)\in R|\theta^a=\pi_{-\alpha}\},$$

since $\tau^{(\alpha,a)}=\pi_{(\beta,0)}$ if and only if $\theta^a=\pi_{\beta-\alpha}$. Setting $P=\{a\in B|\theta^a\in\Pi(A)\}$, we obtain an ideal of B for which $S=P\eta^{-1}$, and R is a strict extension if and only if $P=B$, R is a pure extension if and only if $P=(0)$.

In particular, every extension of A by a simple ring B is either strict or pure.

6. The case $\mathfrak{A}(A)=0$

We will now see that in the case $\mathfrak{A}(A)=0$ most results in the theory of extensions of A by B simplify considerably. The first target is the Everret theorem. To this end, we first consider the situation in the general setting where $\mathfrak{A}(A)=0$ need not hold.

LEMMA 6.1. *Let $\chi\colon B\to\Omega(A)/\Pi(A)$ be a homomorphism. Let $\theta\colon B\to\Omega(A)$, with $\theta\colon a\to\theta^a$, be any function for which $\theta^0=\pi_0$ and $\theta\nu=\chi$. Also suppose given two functions $[\,,]$, $\langle\,,\rangle\colon B\times B\to A$ satisfying conditions* 4.1 (iii) *and* (iv), *respectively. Then all other conditions in Construction* 4.1 *hold modulo the annihilator* $\mathfrak{A}(A)$.

PROOF. By hypothesis $\theta^0=\pi_0$; also for any $a\in B$,

$$\pi_{[a,0]}=\theta^a+\theta^0-\theta^a=\theta^0=\pi_0$$

and similarly $\pi_{\langle a,0\rangle}=\pi_{\langle 0,a\rangle}=\pi_0$. This verifies the assertion for condition 4.1 (i). Condition 4.1 (ii) follows from Lemma 3.6. Verification for conditions (v)—(ix) is straightforward; as a sample, we check (viii). For any $a,b,c,\in B$ we have

$$\pi_{\langle a,c\rangle}+\pi_{\langle b,c\rangle}-\pi_{\langle a+b,c\rangle}=\theta^a\theta^c-\theta^{ac}+\theta^b\theta^c-\theta^{bc}-\theta^{a+b}\theta^c-\theta^{(a+b)c},$$

$$\pi_{[a,b]}\theta^c-\pi_{[ac,bc]}=\pi_{[a,b]}\theta^c-\pi_{[ac,bc]}=(\theta^a+\theta^b-\theta^{a+b})\theta^c-(\theta^{ac}+\theta^{bc}-\theta^{ac+bc})$$

where in the second equality we have used Lemma 3.5. A simple inspection shows that the two expressions are equal. It now suffices to point out that $\mathfrak{A}(A)$ is the kernel of $\pi\colon A\to\Omega(A)$.

This lemma says that if the two functions satisfy conditions 4.1 (iii) and (iv), then the remaining conditions are "very close" to being satisfied. For the case when $\mathfrak{A}(A)=0$, the remaining conditions will be satisfied, and we may define $[\,,]$ and $\langle\,,\rangle$ by 4.1 (iii) and (iv) because in that case π is one-to-one. We do this in the next result.

THEOREM 6.2. *For the rings A and B with $\mathfrak{A}(A)=0$, let $\chi\colon B\to\Omega(A)/\Pi(A)$ be a homomorphism and $\theta\colon B\to\Omega(A)$, with $\theta\colon a\to\theta^a$, be any function for which $\theta^0=\pi_0$ and $\theta\nu=\chi$. Define the functions $[\,,]$ and $\langle\,,\rangle$ by the requirements:*

$$\pi_{[a,b]}=\theta^a+\theta^b-\theta^{a+b},\quad \pi_{\langle a,b\rangle}=\theta^a\theta^b-\theta^{ab}\quad (a,\ b\in B).$$

Then $E(\theta;[\,,],\langle\,,\rangle)$ is an Everett sum of A and B, which we denote by $E(\theta;\chi)$. Conversely, every extension of A by B is equivalent to some $E(\theta;\chi)$. Moreover, $E(\theta;\chi)$ and $E(\theta',\chi')$ are equivalent if and only if $\chi=\chi'$.

PROOF. Since the annihilator of A is the kernel of π, the latter is one-to-one, so $[\,,]$ and $\langle\,,\rangle$ are unambiguously defined. Conditions 4.1 (iii) and (iv) follow by the definition of $[\,,]$ and $\langle\,,\rangle$ and $\theta^0=\pi_0$ by the requirement on θ; the rest of the conditions in Construction 4.1 follow directly from Lemma 6.1. Now Theorem 4.2 gives the direct part. This together with Theorem 4.2 gives the converse.

Clearly $\chi(E(\theta;\chi))=\chi$, and we have noted above that equivalent extensions have the same character. Conversely, consider the extensions $E(\theta;\chi)$ and $E(\theta',\chi)$.

Let ζ be defined by the requirement: $\pi_{a\zeta}=\theta'^a-\theta^a$ $(a\in B)$.

Note that $\theta'^a-\theta^a\in\Pi(A)$ since $\theta v=\theta' v$ and that $\pi_{a\zeta}$ uniquely gives $a\zeta$. Hence $\xi: B\to A$ and clearly $0\zeta=0$. Let $a, b\in B$. Then

$$\pi_{[a,b]'}-\pi_{[a,b]}=\theta'^a+\theta'^b-\theta'^{a+b}-(\theta^a+\theta^b-\theta^{a+b})=$$
$$=(\theta'^a-\theta^a)+(\theta'^b-\theta^b)-(\theta'^{a+b}-\theta^{a+b})=\pi_{a\zeta}+\pi_{b\zeta}-\pi_{(a+b)\zeta}$$

and thus Condition 4.5 (ii) holds. Further, using Lemma 3.5, we get

$$\pi_{\theta^a(b\zeta)}+\pi_{(a\zeta)\theta^b}+\pi_{(a\zeta)(b\zeta)}-\pi_{(ab)\zeta}=$$
$$=\theta^a(\theta'^b-\theta^b)+(\theta'^a-\theta^a)\theta^b+(\theta'^a-\theta^a)(\theta'^b-\theta^b)-(\theta'^{ab}-\theta^{ab})=$$
$$=(\theta'^a\theta'^b-\theta'^{ab})-(\theta^a\theta^b-\theta^{ab})=\pi_{\langle a,b\rangle}-\pi_{\langle a,b\rangle'}$$

and condition 4.5 (iii) is satisfied as well. Thus by Theorem 4.5, the two extensions $E(\theta;\chi)$ and $E(\theta';\chi)$ are equivalent.

Another kind of extension is provided by the following concept.

DEFINITION 6.3. An ideal I of a ring R is *large* if I has a nonzero intersection with every nonzero ideal of R. An extension $\mathscr{R}=(\xi, R, \eta)$ of A by B is an *essential extension* if $A\xi$ is a large ideal of R.

PROPOSITION 6.4. *Let* $\mathscr{R}=(\xi, R, \eta)$ *be an extension of* A *by* B, *and consider the following conditions.*

(i) $\mathscr{R}$ *is an essential extension.*
(ii) $\tau(R: I)$ *is a monomorphism, where* $I=A\xi$.
(iii) $\mathscr{R}$ *is a pure extension.*

Then (i) *implies* (ii) *if* $\mathfrak{A}(A)=0$. *The implications* (ii)$\Rightarrow$(iii) *and* (iii)$\Rightarrow$(i) *always hold.*

PROOF. (i) *implies* (ii). Let K be the kernel of $\tau=\tau(R: I)$, and let $r\in K\cap I$, $i\in I$. Then $ri=\tau^r i=\pi_0 i=0$ and similarly $ir=0$. Hence $r\in\mathfrak{A}(I)=0$ and $K\cap I=0$. Since the extension is essential, we get $K=0$. Therefore τ is a monomorphism.

(ii) *implies* (iii). Let $r\in R$ be such that $\tau^r\in\Pi(I)$. Then $\tau^r=\pi_i=\tau^i$ for some $i\in I$, and since τ is one-to-one, we get $r=i\in I$. Thus the extension is pure.

(iii) *implies* (i). Let K be an ideal of R for which $I\cap K=0$ and let $k\in K$, $i\in I$. Then $\tau^k i=ki\in I\cap K=0$ and similarly $i\tau^k=0$. Thus $\tau^k=\pi_0\in\Pi(I)$ and the hypothesis implies that $k\in I$. But then $k\in I\cap K=0$. Hence $K=0$ and the extension is essential.

There is a kind of essential extension of particular interest.

DEFINITION 6.5. Let $\mathscr{R}=(\xi, R, \eta)$ be an essential extension of A by B. Then $\mathscr{R}$ is a *maximal essential extension* if for any essential extension $\mathscr{R}'=(\xi, R', \eta')$ of A by B such that $R\subseteq R'$, we have $R=R'$.

For these extensions, we have the following statement.

THEOREM 6.6. *Let* $\mathscr{R}=(\xi, R, \eta)=E(\theta; [\ ,\ ], \langle\ ,\ \rangle)$ *be an essential extension of* A *by* B *and assume that* $\mathfrak{A}(A)=0$. *Then the following conditions are equivalent.*

(i) $\mathscr{R}$ *is a maximal essential extension.*

(ii) $\tau(R\colon I)$ *maps* R *onto* $\Omega(I)$, *where* $I=A\xi$.

(iii) θ *maps* B *onto* $\Omega(A)$.

PROOF. (i) *implies* (ii). By Proposition 6.4, $\tau=\tau(R\colon I)$ is an isomorphism of R onto $T=T(R\colon I)$. The diagram

$$\begin{tikzcd}
A \arrow[rr, "\xi"] \arrow[d, "\xi"'] & & R \arrow[ddll, "\tau"'] \arrow[dd, "\eta"] \\
R \arrow[d, "\tau"'] & & \\
T \arrow[r, "\tau^{-1}"] & R \arrow[r, "\eta"] & B
\end{tikzcd}$$

obviously commutes which gives that τ is an equivalence isomorphism for the extensions $\mathscr{R}$ and $(\xi\tau, T, \tau^{-1}\eta)$. We can build an overring $R'=R\cup(\Omega(I)\setminus T)$ of R in which I is an ideal in the usual way. In order to prove that τ maps R onto $\Omega(I)$, it thus suffices to show that $R'=R$. The hypothesis implies that I is a large ideal of R. Hence it suffices to show that I is also large in R'. By the construction of R', this is equivalent to showing that $\Pi(I)$ is large in $\Omega(I)$ which we now proceed to do.

Let J be an ideal of $\Omega(I)$ and let $0\neq(\lambda, \varrho)\in J$. We assume that $\lambda\neq\lambda_0$ since the case $\varrho\neq\varrho_0$ can be treated symmetrically. There exists $a\in I$ such that $\lambda a\neq 0$. Since $\mathfrak{A}(I)=0$, either $(\lambda a)b\neq 0$ or $b(\lambda a)\neq 0$ for some $b\in I$. In the first case, $(\lambda\lambda_a)b\neq 0$ and in the second case $b(\varrho\varrho_a)=(b\varrho)a=b(\lambda a)\neq 0$ and thus in either case $(\lambda, \varrho)\pi_a\neq\pi_0$. Hence $(\lambda, \varrho)\pi_a\in J\cap\Pi(I)$ which shows that $\Pi(I)$ is large in $\Omega(I)$.

(ii) *implies* (i). Let $\mathscr{R}'=(\xi, R', \eta')$ be an essential extension of A by B such that $R\subseteq R'$. Letting $I=A\xi$, $\tau=\tau(R\colon I)$, $\tau'=\tau(R'\colon I)$, we obtain the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\tau"] \arrow[dr, "\iota"'] & \Omega(I) \\
& R' \arrow[u, "\tau'"']
\end{tikzcd}$$

where ι is the inclusion mapping and by Proposition 6.4, τ is an isomorphism and τ' a monomorphism. For any $r'\in R'$, we get $r'\tau'\tau^{-1}=r\in R$ whence $r'\tau'=r\tau$. Since $\tau'|_R=\tau$, it follows that $r'=r$. Thus $R=R'$ and $\mathscr{R}$ is a maximal essential extension.

The equivalence of items (ii) and (iii) follows easily from Lemma 5.2.

The (external) *direct sum* of the rings A and B is usually denoted by $A\oplus B$. Strictly speaking, *a direct sum* of A and B is a triple of the form (φ, R, ψ) where R is a ring, $\varphi\colon A\to R$ and $\psi\colon B\to R$ are monomorphisms, and the triple satisfies a universality condition. The Cartesian product $A\oplus B$ with coordinatewise operations, together with monomorphisms

$$\varphi\colon \alpha\to(\alpha, 0),\quad \gamma\colon a\to(0, a)\quad (\alpha\in A,\ a\in B)$$

satisfies the conditions for a direct sum. We will denote by $A\times B$ the triple (φ, R, ψ), where R and φ are defined as above on $A\times B$ and

$$\varphi\colon (\alpha, a)\to a \quad ((\alpha, a)\in A\times B).$$

Then $A\oplus B$ is an extension of A by B, which we will refer to as the *direct sum* of A and B. To simplify the notation, we will denote by $A\oplus B$ also the ring alone. In the latter context, both A and B are said to be *direct summands* of $A\oplus B$.

We now derive some consequences of the above results.

COROLLARY 6.7. *Let $\mathscr{R}=(\xi, R, \eta)$ be an extension of A by B, let $I=A\xi$ and $\tau=\tau(R\colon I)$, and assume that $\mathfrak{A}(A)=0$. Then the following statements hold.*

(i) *$\mathscr{R}$ is a strict extension if and only if $\mathscr{R}$ is equivalent to the direct sum $A\oplus B$.*

(ii) *If $\mathscr{R}$ is a pure extension, then $\mathscr{R}$ is equivalent to the extension $(\xi\tau, T(R\colon I), \tau^{-1}\eta)$.*

(iii) *$\mathscr{R}$ is a maximal essential extension if and only if τ is an isomorphism in which case $\mathscr{R}$ is equivalent to $(\xi\tau, \Omega(I), \tau^{-1}\eta)$.*

PROOF. (i) Let $\mathscr{R}$ be a strict extension. By Proposition 5.4, $\chi(\mathscr{R})$ is the zero homomorphism. Evidently $\chi(A\oplus B)$ is also the zero homomorphism, which by Theorem 6.2 gives that $\mathscr{R}$ and $A\oplus B$ are equivalent. The converse is trivial.

(ii) We have seen this in the proof of Theorem 6.6.

(iii) Let $\mathscr{R}$ be a maximal essential extension. Then τ is a monomorphism by Proposition 6.4 and an epimorphism by Theorem 6.6. It follows from part (ii) that $\mathscr{R}$ is equivalent to $(\xi\tau, \Omega(I), \tau^{-1}\eta)$. The converse also follows from Proposition 6.4 and Theorem 6.6.

COROLLARY 6.8. *Let $\mathfrak{A}(A)=0$. The set of equivalence classes of extensions of A by B is in one-to-one correspondence with homomorphisms from B into $\Omega(A)/\Pi(A)$. In this correspondence, all strict extensions correspond to the zero homomorphism, the classes of pure extensions to monomorphism and maximal essential extensions to isomorphisms.*

PROOF. The first statement follows directly from Theorem 6.2. The remaining assertions follow easily from Proposition 5.4 and Corollary 6.7.

The first statement of Corollary 6.8 was noted by Mac Lane [2]. A further interesting property of the extensions under consideration here is the following.

PROPOSITION 6.9. *Let $\mathfrak{A}(A)=0$ and $\mathscr{R}=(\xi, R, \eta)$ be an extension of A by B. Then $\mathscr{R}$ is equivalent to an extension $\mathscr{R}'=(\xi', R', \eta')$ of A by B where R' is a subdirect product of the type of $\mathscr{R}$ and B.*

PROOF. Let $I=A\xi$, $\tau=\tau(R\colon I)$ and $T=T(R\colon I)$. Define $\chi\colon r\to(\tau^r, r\eta)$ $(r\in R)$. Then χ is a homomorphism of R into the direct sum $T\oplus B$. If $r\chi=(\pi_0, 0)$, then $\pi_r=\pi_0$ and $r\in I$. Since $\mathfrak{A}(A)=0$, it follows that $r=0$. Thus the kernel of χ is trivial and χ is a monomorphism. Let $R'=R\chi$. Then R' is a subdirect product of T and B. Let

$$\xi'\colon \alpha\to(\pi_\alpha, 0) \quad (\alpha\in R), \qquad \eta'\colon (\tau^r, r\eta)\to r\eta \quad (r\in R).$$

Easy inspection shows that ξ' is a monomorphism, η' is a homomorphism of R'

onto B, and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\xi} & R \\ {\scriptstyle \xi'}\downarrow & \swarrow{\scriptstyle \chi} & \downarrow{\scriptstyle \eta} \\ R' & \xrightarrow{\eta'} & B \end{array}$$

commutes. Hence the extensions (ξ, R, η) and (ξ', R', η') are equivalent.

If we disregard the equivalence of extensions of a ring A, with $\mathfrak{A}(A)=0$, and a ring B, we may say that all extensions of A by B can be embedded into $\Omega(A)\oplus B$. We now give a new proof of a well known result.

THEOREM 6.10 [6]. *A ring A has the property that for every ring B, every extension of A by B is equivalent to $A\oplus B$ if and only if A has an identity.*

PROOF. *Necessity.* Form an extension of A by $\Omega(A)$, with $\theta\colon \Omega(A)\to\Omega(A)$ the identity mapping, and both functions $[\ ,\ ]$ and $\langle\ ,\ \rangle$ identically equal to zero. In this extension $\mathscr{R}=E(\theta; [\ ,\ ], \langle\ ,\ \rangle)$, denoting by ι the identity of $\Omega(A)$, we have that $(0, \iota)$ is the identity of the ring. By hypothesis, $\mathscr{R}$ is equivalent to the direct sum $A\oplus\Omega(A)$. Hence the latter ring has an identity. Since A is a homomorphic image of the ring $A\oplus\Omega(A)$, it must itself have an identity.

Sufficiency. Let $\mathscr{R}=(\xi, R, \eta)$ be an extension of A by B. The presence of identity in A implies that $\mathfrak{A}(A)=0$ and also that $\Omega(A)=\Pi(A)$. Hence Proposition 6.9 applies so $\mathscr{R}$ is equivalent to an extension $\mathscr{R}'=(\xi', R', \eta')$ where R' is a subdirect product of $\Pi(A)$ and B since $\Pi(A)$ must be the type of $\mathscr{R}$. In view of the isomorphism of A and $\Pi(A)$, we may assume that $R'\subseteq A\oplus B$ and is a subdirect product. For any $(\alpha, a)\in A\oplus B$, we have $(\beta, a)\in R'$ for some $\beta\in A$, and thus

$$(\alpha, a) = (\alpha-\beta, 0)+(\beta, a)$$

shows that $R'=A\oplus B$. Therefore $\mathscr{R}$ is equivalent to the direct sum $A\oplus B$, as required.

A variant of the preceding result for strict extensions follows.

THEOREM 6.11. *A ring A has the property that for every ring B, every strict extension of A by B is equivalent to $A\oplus B$ if and only if $\mathfrak{A}(A)=0$.*

PROOF. *Necessity.* Form an extension of A by $\Pi(A)$, with $\theta\colon \Pi(A)\to\Omega(A)$ the inclusion mapping, and both functions $[\ ,\ ]$, $\langle\ ,\ \rangle$ identically equal to zero. By hypothesis, this extension $E(\theta; [\ ,\ ], \langle\ ,\ \rangle)$ is equivalent to the direct sum $A\oplus\Pi(A)$. Then $A\oplus\Pi(A)$ can be regarded as the Everett sum with all these functions identically equal to zero. Theorem 4.5 provides a function $\zeta\colon \Pi(A)\to A$ satisfying $\pi_0=0$ and conditions (i)—(iii). Condition (i) becomes $\pi_\alpha=\pi_{\pi_\alpha\zeta}$, and conditions (ii) and (iii) imply that ζ is a homomorphism. The equation $\pi_\alpha=\pi_{\pi_\alpha\zeta}$ for all $\alpha\in A$ means that the mapping $\zeta\pi$ is the identity transformation on $\Pi(A)$. It then follows that ζ is a monomorphism of $\Pi(A)$ into A.

Let $C=\Pi(A)\zeta$. For any $\alpha, \beta\in A$, we obtain

$$(\pi_\alpha\zeta)\beta = (\pi_\alpha\zeta)\pi_\beta = (\pi_\alpha\zeta)\pi_{\pi_\beta\zeta} = (\pi_\alpha\zeta)(\pi_\beta\zeta) = (\pi_\alpha\pi_\beta)\zeta = \pi_{\alpha\beta}\zeta$$

and analogously $\beta(\pi_\alpha\zeta)=\pi_{\beta\alpha}\zeta$. Hence C is an ideal of A. Let $\alpha\in C\cap\mathfrak{A}(A)$. Then

$\varkappa=\pi_\beta\zeta$ for some $\beta\in A$ and $\pi_\alpha=\pi_0$. It follows that

$$\pi_0=\pi_\alpha=\pi_{\pi_\beta\zeta}=\pi_\beta$$

so that $\alpha=\pi_\beta\zeta=\pi_0\zeta=0$. Thus $C\cap\mathfrak{A}(A)=0$. Also, for any $\alpha\in A$, we have $\pi_\alpha=\pi_{\pi_\alpha\zeta}$ and hence

$$\alpha=\pi_\alpha\zeta+(\alpha-\pi_{\alpha\zeta})\in C+\mathfrak{A}(A),$$

proving that $A=C+\mathfrak{A}(A)$. But then $A=C\oplus\mathfrak{A}(A)$.

Let $d\in\mathfrak{A}(C)$. Then for any $c\in C$ and $a\in\mathfrak{A}(A)$, we have $d(c+a)=dc+da=0$ and similarly $(c+a)d=0$. It follows that $d\in\mathfrak{A}(A)$, which shows that $\mathfrak{A}(C)=0$.

Now set $D=\mathfrak{A}(A)$ so that $D^2=0$. Let B be the ring whose additive group is $(Z,+)$ and for which $B^2=0$. Fix any $d\in D$ and let

$$\theta^m=0,\ [m,n]=0,\ \langle m,n\rangle=mnd\quad(m,n\in B).$$

Conditions (i)—(ix) in Construction 4.1 are verified easily. We thus obtain a strict extension $\mathcal{R}=E(\theta;[\ ,\ ],\langle\ ,\ \rangle)$ of $C\oplus D$ by B. By hypothesis, this extension is equivalent to the direct sum of $C\oplus D$ and B. Thus Construction 4.3 provides a function $\zeta\colon B\to C\oplus D$ such that, among other conditions, $(m\cdot n)\zeta=\langle m,n\rangle$ and $0\zeta=0$. Hence

$$0=0\zeta=(1.1)\zeta=\langle 1,1\rangle=d.$$

Since $d\in D$ is arbitrary, it follows that $D=0$. But then $A=C$ and therefore $\mathfrak{A}(A)=0$, as required.

Sufficiency. This is the content of part (i) in Corollary 6.7.

7. Extensions of semiprime atomic rings

We have seen in Corollary 6.8 that for given rings A and B with $\mathfrak{A}(A)=0$, all extensions of A by B are determined, up to equivalence, by homomorphisms $\chi\colon B\to\Omega(A)/\Pi(A)$. For example, the zero homomorphism corresponds to the class of strict extensions, and any such is equivalent to the direct sum $A\oplus B$. There exist rings A and B with the property that there exists only the zero homomorphism from B into $\Omega(A)/\Pi(A)$, which means that any extension of A by B is equivalent to their direct sum. This situation occurs when, for instance, the additive group of B is torsion and the additive group of $\Omega(A)/\Pi(A)$ is torsion free. We consider below an example of such a situation and of a related one.

In order to economize with space, we refer to [4] for details concerning the concepts and statements needed here and present below only the bare minimum.

Let A be a *semiprime* (no nonzero nilpotent ideals) *atomic* (generated by its minimal right ideals) *ring*. According to ([4], II.6.1), A is a direct sum $A=\bigoplus_{\lambda\in\Lambda}A_\lambda$ of simple atomic rings. By ([4], II.1.9), for each $\lambda\in\Lambda$, $A_\lambda\cong\mathcal{F}_{U_\lambda}(V_\lambda)$, where (U_λ,V_λ) is a pair of dual vector spaces over a division ring Δ_λ and $\mathcal{F}_{U_\lambda}(V_\lambda)$ is the ring of all linear transformations on V_λ of finite rank having an adjoint in U_λ. Each $\mathcal{F}_{U_\lambda}(V_\lambda)$ is a regular ring by ([4], I.3.6) which implies that A is a regular ring and in particular $\mathfrak{A}(A)=0$. In addition

$$\Pi(A_\lambda)\cong\mathcal{F}_{U_\lambda}(V_\lambda),\quad \Pi(A)\cong\bigoplus_{\lambda\in\Lambda}\mathcal{F}_{U_\lambda}(V_\lambda),$$

and in view of ([4], II.5.10), $\Omega(A)\cong \prod_{\Lambda\in\lambda} \Omega(A_\lambda)$. Further, ([4], I.7.14) yields $\Omega(\mathscr{F}_{U_\lambda}(V_\lambda))\cong\mathscr{L}_{U_\lambda}(V_\lambda)$ where the latter is the ring of all linear transformations on V_λ having an adjoint in U_λ. Combining the last two statements, we obtain $\Omega(A)\cong$ $\cong \prod_{\lambda\in\Lambda} \mathscr{L}_{U_\lambda}(V_\lambda)$ which finally yields

$$\Omega(A)/\Pi(A) \cong \prod_{\lambda\in\Lambda} \mathscr{L}_{U_\lambda}(V_\lambda)/\bigoplus_{\lambda\in\Lambda} \mathscr{F}_{U_\lambda}(V_\lambda). \tag{1}$$

We will analyse now the ring figuring on the right hand side of (1). The first part of our discussion will provide necessary and sufficient conditions for the additive group of this ring to be torsion free, the second part deals with a related type of situation.

We denote by ch R the *characteristic* of a ring R.

LEMMA 7.1. *Let (U, V) be a pair of dual infinite dimensional vector spaces over a divising ring Δ and let $R=\mathscr{L}_U(V)/\mathscr{F}_U(V)$.*

(i) $\operatorname{ch}\Delta=0$ *if and only if the additive group of R is torsion free.*

(ii) *if* $\operatorname{ch}\Delta=p$, *a prime, then* $\operatorname{ch} R=p$.

PROOF. Assume first that the additive group of R is not torsion free. Then there exists $a\in\mathscr{L}_U(V)$ with the properties: $a\notin\mathscr{F}_U(V)$ and $na\in\mathscr{F}_U(V)$ for some natural number n. Then dim Va is infinite, so there exists an infinite linearly ordered set $x_1, x_2, x_3, \ldots$ of vectors in V such that the set $\{x_1a, x_2a, \ldots\}$ is linearly independent. On the other hand, the set $\{x_1(na), x_2(na), \ldots\}$ is linearly dependent since $\dim V(na)<\infty$. Hence there exist scalars $\delta_1, \delta_2, \ldots, \delta_k$ in Δ, not all equal to zero, for which

$$\delta_1(x_1(na))+\delta_2(x_2(na))+\ldots+\delta_k(x_k(na)) = 0.$$

It follows that

$$(n\delta_1)(x_1a)+(n\delta_2)(x_2a)+\ldots+(n\delta_k)(x_ka) = 0.$$

Since the set $\{x_1a, x_2a, \ldots, x_ka\}$ is linearly independent, we must have $n\delta_i=0$, for $i=1, 2, \ldots, k$. Thus there exists $\delta_i\in\Delta$ such that $\delta_i\neq0$, $n\delta_i=0$, $n>0$. Since then $(nl)\delta_i=0$, we must have $nl=0$ and thus $\operatorname{ch}\Delta\neq0$. This gives the direct implication in part (i).

Suppose next that $\operatorname{ch}\Delta=p$. For $a\in\mathscr{L}_U(V)$ and $v\in V$, we have

$$v(pa) = (pv)a = (pl)va = 0va = 0$$

so that $pa=0$. Since p is prime, $\operatorname{ch}\mathscr{L}_U(V)$ is either equal to p or to l; the latter would contradict the hypothesis on V. Hence $\operatorname{ch}\mathscr{L}_U(V)=p$ and thus ch R is equal to either p or l, the latter is again impossible in view of infinite dimensionality of V. Therefore $\operatorname{ch} R=p$ which establishes part (ii).

If $\operatorname{ch}\Delta\neq0$, then $\operatorname{ch}\Delta=p$ for some prime p and thus, by the above, $\operatorname{ch} R=p$, which evidently shows that the additive group of R is torsion. This proves the reverse implication in part (i).

LEMMA 7.2. *Let (U_λ, V_λ) be a pair of dual vector spaces over a division ring Δ_λ, $\lambda\in\Lambda$. Then the additive group of $R=\prod_{\lambda\in\Lambda}\mathscr{L}_{U_\lambda}(V_\lambda)/\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ is torsion free if and only if* $\operatorname{ch}\Delta_\lambda=0$ *whenever* $\dim V_\lambda$ *is infinite.*

PROOF. Assume that there exists $\mu\in\Lambda$ such that $\operatorname{ch}\Delta_\mu=p\neq 0$ and $\dim V_\mu$ is infinite. Then $pl_\mu=0$ since $\operatorname{ch}\mathscr{L}_{U_\mu}(V_\mu)=0$ as in the proof of Lemma 7.1. Letting $a_\mu=l_\mu$ and $a_\lambda=0_\lambda$ for $\lambda\neq\mu$, we obtain an element (a_λ) in $\prod_{\lambda\in\Lambda}\mathscr{L}_{U_\lambda}(V_\lambda)$ for which

$$(a_\lambda)+\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)\neq\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda),\quad p\Big((a_\lambda)+\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)\Big)=\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$$

and the additive group of R is not torsion free.

Conversely, assume that $\operatorname{ch}\Delta_\lambda=0$ whenever $\dim V_\lambda$ is infinite. Let $(a_\lambda)\in\in\prod_{\lambda\in\Lambda}\mathscr{L}_{U_\lambda}(V_\lambda)$ be such that $n(a_\lambda)\in\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ for some natural number n. Hence $na_\lambda=0_\lambda$ for all $\lambda\in\Lambda$ except for a finite number, say $\lambda_1,\lambda_2,\dots,\lambda_k$ for which $na_{\lambda_i}\in\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$, $i=1,2,\dots,k$. Hence $a_{\lambda_i}\in\mathscr{L}_{U_{\lambda_i}}(V_{\lambda_i})$ and $na_{\lambda_i}\in\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$. If $\dim V_{\lambda_i}<\infty$, then we have $a_{\lambda_i}\in\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$ automatically, otherwise the hypothesis insures that $\operatorname{ch}\Delta_{\lambda_i}=0$ and Lemma 7.1 (i) yields $a_{\lambda_i}\in\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$, $i=1,2,\dots,n$. Consequently $(a_\lambda)\in\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ which shows that the additive group of R is torsion free.

A nonzero idempotent of a ring R is *primitive* if it is minimal relative to the partial order of nonzero idempotents of R defined by: $e\leqq f\Leftrightarrow e=ef=fe$. We can now prove the first principal result of this section.

THEOREM 7.3. *Let* $A=\bigoplus_{\lambda\in\Lambda}A_\lambda$, *where* A_λ *are simple atomic rings. The additive group of* $\Omega(A)/\Pi(A)$ *is torsion free if and only if* $\operatorname{ch}eA_\lambda e=0$ *whenever* e *is a primitive idempotent of* A_λ *and the ring* A_λ *has no identity element.*

PROOF. In view of (1), we may consider the ring $\prod_{\lambda\in\Lambda}\mathscr{L}_{U_\lambda}(V_\lambda)/\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ instead of $\Omega(A)/\Pi(A)$. By ([4], I.3.18), for any primitive idempotent e of $\mathscr{F}_{U_\lambda}(V_\lambda)$, we have $\Delta_\lambda\cong e\mathscr{F}_{U_\lambda}(V_\lambda)e$ and thus $\Delta_\lambda\cong eA_\lambda e$. We have $A_\lambda\cong\mathscr{F}_{U_\lambda}(V_\lambda)$ and the latter ring has an identity element if and only if $\dim V_\lambda<\infty$. These considerations reduce the hypotheses in the statement of the theorem to those of Lemma 7.2 whence follows the desired conclusion.

Note that in the above theorem, if e is a primitive idempotent in A_λ, then $eA_\lambda e==eAe$ and $A_\lambda=AeA$ so the hypothesis of the theorem can be stated in these terms.

COROLLARY 7.4. *If* A *is a ring satisfying the conditions in Theorem* 7.3 *and* B *is a ring whose additive group is torsion, then any extension of* A *by* B *is equivalent to their direct sum.*

PROOF. We pointed out at the outset of this section that $\mathfrak{A}(A)=0$. The assertion now follows by Theorem 7.3 and Corollary 6.8.

The *socle* of a ring R is the subring of R generated by its minimal right ideals. If R has no minimal right ideals, the socle is set to be equal to 0. Hence atomic rings are precisely those which coincide with their socle.

COROLLARY 7.5. *Let* R *be a ring with the following properties:*

(i) *the socle* S *of* R *is semiprime and is an ideal of* R,

(ii) *the additive group of* R/S *is torsion,*

(iii) S *satisfies the conditions of Theorem* 7.3.

Then $R\cong S\oplus R/S$.

We now consider another type of condition on the additive group of $\Omega(A)/\Pi(A)$. Denote by $|$ the division of integers.

LEMMA 7.6. *Let* (U_λ, V_λ) *be a pair of dual vector spaces over a division ring* $\Delta_\lambda, \lambda\in\Lambda$, *and let* n *be a natural number. The ring* $R=\prod_{\lambda\in\Lambda}\mathscr{L}_{U_\lambda}(V_\lambda)/\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ *has no nonzero elements whose additive order divides* n *if and only if* $\operatorname{ch}\Delta_\lambda\nmid n$ *whenever* $\operatorname{ch}\Delta_\lambda\neq 0$ *and* $\dim V_\lambda$ *is infinite.*

PROOF. Assume that there exists $\mu\in\Lambda$ such that $\operatorname{ch}\Delta_\mu\neq 0$, $\dim V_\mu$ is infinite and $\operatorname{ch}\Delta_\mu|n$. Let $a_\mu=l_\mu$ and $a_\lambda=0_\lambda$ if $\lambda\neq\mu$. Then $(a_\lambda)+\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ is a nonzero element of R since $\dim V_\mu$ is infinite. Further, $n(a_\lambda)\in\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ since $\operatorname{ch}\Delta_\mu|n$. Hence $(a_\lambda)+\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$ is a nonzero element of R whose additive order divides n.

Conversely, assume that $\operatorname{ch}\Delta_\lambda\nmid n$ whenever $\operatorname{ch}\Delta_\lambda\neq 0$ and $\dim V$ is infinite. Let $(a_\lambda)\in\prod_{\lambda\in\Lambda}\mathscr{L}_{U_\lambda}(V_\lambda)$ and $n(a_\lambda)\in\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$. There exists $\lambda_1,\lambda_2,\dots,\lambda_k$ such that $na_{\lambda_i}\in\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$ for $i=1,2,\dots,n$ and $na_\lambda=0_\lambda$ if $\lambda\neq\lambda_i$. If $\dim V_{\lambda_i}<\infty$, we have $a_{\lambda_i}\in\mathscr{L}_{U_{\lambda_i}}(V_{\lambda_i})=\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$. Suppose that $\dim V_{\lambda_t}$ is infinite. If $\operatorname{ch}\Delta_{\lambda_i}=0$, then Lemma 7.1 (i) gives that the additive group of $\mathscr{L}_{U_{\lambda_i}}(V_{\lambda_i})/\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$ is torsion free, so we must have $a_{\lambda_i}\in\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i})$. If $\operatorname{ch}\Delta_{\lambda_i}\neq 0$, then the hypothesis in conjunction with Lemma 7.1 (ii) gives that $\operatorname{ch}\mathscr{L}_{U_{\lambda_i}}(V_{\lambda_i})/\mathscr{L}_{U_{\lambda_i}}(V_{\lambda_i})=\operatorname{ch}\Delta_{\lambda_i}$. We also have that $\operatorname{ch}\Delta_{\lambda_i}=p$ for some prime p, and the hypothesis yields that $p\nmid n$. Since p is prime, $(n,p)=1$ and there exist integers s and t such that $ns+pt=1$. In the division ring Δ_{λ_i}, we thus have $nsl_{\lambda_i}=l_{\lambda_i}$ since $\operatorname{ch}\Delta_{\lambda_i}=p$. Consequently

$$a_{\lambda_i}=s(na_{\lambda_i})\in\mathscr{F}_{U_{\lambda_i}}(V_{\lambda_i}).$$

This shows that $(a_\lambda)\in\bigoplus_{\lambda\in\Lambda}\mathscr{F}_{U_\lambda}(V_\lambda)$, which proves the assertion.

We can now easily derive the desired result.

THEOREM 7.7. *Let* $A=\bigoplus_{\lambda\in\Lambda}A_\lambda$, *where* A_λ *are simple atomic rings, and let* n *be a natural number. The ring* $\Omega(A)/\Pi(A)$ *has no nonzero elements whose additive order divides* n *if and only if* $\operatorname{ch} eA_\lambda e\nmid n$ *whenever* e *is a primitive idempotent of* A_λ. $\operatorname{ch} eA_\lambda e\neq 0$ *and the ring* A_λ *does not have an identity element.*

PROOF. The argument here goes along the same lines as in the proof of Theorem 7.3 now using Lemma 7.6 instead of Lemma 7.2. The details are omitted.

COROLLARY 7.8. *If* A *is a ring satisfying the conditions in Theorem* 7.7 *and* B *is a ring whose characteristic divides* n, *then any extension of* A *by* B *is equivalent to their direct sum.*

PROOF. See the argument in the proof of Corollary 7.4.

COROLLARY 7.9. *Let* R *be a ring with the following properties:*

(i) *the socle* S *of* R *is semiprime and is an ideal of* R,
(ii) $\operatorname{ch} R/S \mid n$,
(iii) S *satisfies the conditions of Theorem* 7.7.

Then $R \cong S \oplus R/S$.

8. Problems

Many questions may be asked concerning Everett sums of two rings, strict, pure and essential extensions of rings. The following is a modest sample of such queries.

PROBLEM 1. When is an Everett sum $E(\theta; [\,,\,], \langle\,,\,\rangle)$ an essential extension? By Proposition 6.4, every pure extension is essential, so the sought condition is at most as strong as the one in Proposition 5.4 (ii).

PROBLEM 2. What are necessary and sufficient conditions on a ring A in order that every essential extension of A be pure? In view of Proposition 6.4, a sufficient condition is $\mathfrak{A}(A)=0$. Is this condition also necessary? Theorem 6.11 can be interpreted as a dual to the sought result.

PROBLEM 3. How far can the results of Section 6 be carried without the hypothesis $\mathfrak{A}(A)=0$? In the present formulation, probably not much, but appropriate modifications may produce some interesting results.

PROBLEM 4. We may say that the rings A and B are *incompatible* if every extension of A by B is equivalent to their direct sum. Theorems 6.10, 6.11, and Corollaries 7.4 and 7.8 provide some pairs of incompatible rings. Find other criteria insuring that two rings A and B be incompatible.

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A NOTE ON ARCS IN HYPERSPACES

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Introduction

Given a metric space X, a metric can be defined on the set $\mathcal{CL}(X)$ of all non-empty closed subsets of X in a fairly natural way ([4], § 0.1). This metric was first defined by Hausdorff, and the resulting space called a hyperspace of X. Later, the same ideas were extended to the non-metric setting by defining suitable topologies on the set $\mathcal{CL}(X)$ for any topological space X. Among the various topologies, the Vietoris topology (§ 1 below) is in a sense the most reasonable extension of the original metric hyperspace topology. Broadly speaking, the term 'hyperspace of X' is used, at least in this note, for any subfamily of $\mathcal{CL}(X)$ to which all singleton subsets of X belong and that has the relativized Vietoris topology.

One early and basic result in the hyperspace theory is the arcwise connectedness of the hyperspacees $\mathcal{CL}(X)$ and $\mathcal{C}(X)$ of a metric continuum X (see [4], § 1.10 and § 1.14). Here $\mathcal{C}(X)$ is the family of all subcontinua of X. This result was proved for general continua by McWaters [2] in 1967, by using Koch's arc theorem for partially ordered topological spaces. In this note, we prove a general arc theorem for hyperspaces by straight set-topological approach and from it deduce arcwise connectedness of $\mathcal{CL}(X)$ and $\mathcal{C}(X)$, in case X is a continuum.

1. Preliminaries

All topological spaces in this note are assumed to be Hausdorff spaces. For any space X, $\mathcal{CL}(X)$ stands for the collection of all non-empty closed subsets of X. The subcollection of all compact and connected subsets is denoted by $\mathcal{C}(X)$. For any finite collection $\{U_1, U_2, \dots, U_n\}$ of non-empty open subsets of X, the subset $\{F\in\mathcal{CL}(X)\colon F\subseteqq U_1\cup U_2\cup\dots\cup U_n$ and $F\cap U_i\neq\emptyset$ for $i=1, 2, \dots, n\}$ of $\mathcal{CL}(X)$ is denoted by $\langle U_1, U_2, \dots, U_n\rangle$.

The family of all subsets of $\mathcal{CL}(X)$ of the above form is a base for a topology on $\mathcal{CL}(X)$. This topology is called the Vietoris topology of the finite topology and the resulting space and its subspace $\mathcal{C}(X)$ are called *the* hyperspaces of X. More generally any subspace $\mathcal{H}(X)$ of the hyperspace $\mathcal{CL}(X)$ that includes the collection of all singletons may be referred as a hyperspace of X.

While most of the terminology and notations are standard, it may be mentioned that in what follows an *arc* is a compact, connected, linearly ordered (not necessarily metric) topological space and we carefully use the symbols $\subseteqq$ and $\subset$ for subsets and proper subsets respectively.

Suppose A and B, $A\subset B$, are two members of a subfamily $\mathcal{S}$ of $\mathcal{CL}(X)$, then a subcollection of $\mathcal{S}$ that is linearly ordered by inclusion relation $\subseteqq$, has A and B

as members and is such that each member contains A and is contained in B is called a *chain from A to B in $\mathcal{S}$.* Any such chain is contained in a maximal chain of the same type. In case such a chain is an arc under its order topology, it is called an *order arc from A to B in $\mathcal{S}$.*

2. Results

First we state and prove two lemmas.

LEMMA 2.1. *For any topological space X and any maximal chain $\mathcal{C}$ from A to B in $\mathcal{S}$, $\mathcal{S}\subseteq\mathcal{C}\mathcal{L}(X)$, the subspace topology on $\mathcal{C}$ is larger than the order topology.*

PROOF. We need only to show that for any F in $\mathcal{C}$, $A\subset F\subset B$, the half-open intervals $[A, F)=\{E\in\mathcal{C}\colon A\subseteq E\subset F\}$ and $(F, B]=\{E\in\mathcal{C}\colon A\subseteq E\subset F\}$ are open in the subspace topology of $\mathcal{C}$. First, let E be any member of $[A, F)$. Then there is an x in $F-E$ and as for any member S of $\langle X-\{x\}\rangle\cap\mathcal{C}$, $F\subseteq S$ cannot be true, we have: $E\in\langle X-\{x\}\rangle\cap\mathcal{C}\subseteq[A, F)$. Similarly, for any member G of $(F, B]$, $G-F$ is non-empty and we have: $G\in\langle X-F, X\rangle\cap\mathcal{C}\subseteq(F, B]$.

LEMMA 2.2. *Let X be a topological space and $\mathcal{S}$ a subfamily of $\mathcal{C}\mathcal{L}(X)$. Then for any A, B in $\mathcal{S}$, $A\subset B$, each maximal chain from A to B in $\mathcal{S}$ is a closed subset of $\mathcal{S}$.*

PROOF. Let $\mathcal{C}$ be a maximal chain from A to B in $\mathcal{S}$ and suppose F is in $\mathcal{S}-\mathcal{C}$. Then, there exists an E in $\mathcal{C}$ such that (a) $A\nsubseteq F$ or (b) $F\nsubseteq B$ or (c) $A\subset F\subset B$ and $E\nsubseteq F\nsubseteq E$. In case (a) there is an x in $A-F$ and $\langle X-\{x\}\rangle\cap\mathcal{S}$ is a neighborhood of F disjoint from $\mathcal{C}$. In case (b), $\langle X-B, X\rangle\cap\mathcal{S}$ is such a neighborhood of F. Finally, in case (c) if $z\in E-F$ then $\langle X-\{z\}, X-E\rangle\cap\mathcal{S}$ is a neighborhood of F disjoint from $\mathcal{C}$.

Now we present the main theorems and corollaries.

THEOREM 2.3. *Let X be a topological space and $\mathcal{S}$ a compact, Hausdorff subspace of $\mathcal{C}\mathcal{S}(X)$. Then for any A, B in $\mathcal{S}$, $A\subset B$, any maximal chain $\mathcal{C}$ from A to B in $\mathcal{S}$ is a compact ordered subspace. Consequently, if in addition $\mathcal{C}$ is order-dense (that is for any R, T in $\mathcal{C}$, $R\subset T$, there is an $\mathcal{S}$ in $\mathcal{C}$ such that $R\subset S\subset\subset T$) then $\mathcal{C}$ is an order-arc from A to B in $\mathcal{S}$.*

PROOF. By Lemma 2.2, $\mathcal{C}$ is a compact, Hausdorff subspace of $\mathcal{S}$. By Lemma 2.1, the subspace topology on $\mathcal{C}$ is larger than the order topology and therefore in view of the fact that the order topology is always Hausdorff and compact Hausdorff topologies are minimal Hausdorff, the two topologies on $\mathcal{C}$ must be the same. Thus $\mathcal{C}$ is a compact, ordered subspace of $\mathcal{S}$. For the last part it need be only noted that the stated condition on $\mathcal{C}$ makes it connected also.

THEOREM 2.4. *Let X be a topological space and A, B members of $\mathcal{C}\mathcal{L}(X)$ such that $A\subset B$. Then there is an order-arc from A to B in $\mathcal{C}\mathcal{L}(X)$ if and only if A, B belong to a compact Hausdorff order-dense subspace $\mathcal{S}$ of $\mathcal{C}\mathcal{L}(X)$.*

PROOF. If $\mathcal{C}$ is an order-arc from A to B in $\mathcal{C}\mathcal{L}(X)$ then $\mathcal{C}$ itself may be taken as $\mathcal{S}$. Conversely, if there is an $\mathcal{S}$ having the stated properties then by Theorem 23., there is an order-arc from A to B in $\mathcal{S}$ and hence in $\mathcal{C}\mathcal{L}(X)$.

COROLLARY 2.5. *Let* X *be a topological space,* $\mathscr{P}$ *a subcollection of* $\mathscr{CL}(X)$ *and* A, B *two members of* $\mathscr{P}$. *If* A, B *belong to a compact Hausdorff order-dense subspace* $\mathscr{S}$ *of* $\mathscr{P}$ *such that* $\mathscr{S}$ *has a member* C *containing both* A *and* B, *then there is an arc from* A *to* B *in* $\mathscr{P}$.

PROOF. By Theorem 2.4, there is an order-arc $\mathscr{A}$ from A to C in $\mathscr{S}$ and an order-arc $\mathscr{B}$ from B to C in $\mathscr{S}$. The collection $\mathscr{A}\cap\mathscr{B}$ is a non-empty, closed subset of both $\mathscr{A}$ and $\mathscr{B}$, consequently has a common infimum, say D, in both $\mathscr{A}$ and $\mathscr{B}$. Now the subarcs $[A, D]$ and $[B, D]$ of $\mathscr{A}$ and $\mathscr{B}$ respectively, together provide an arc (under a modified linear order) from A to B in $\mathscr{S}$ and hence in $\mathscr{P}$.

COROLLARY 2.6. ([2], § 3). *If* X *is a compact and connected space, then* $\mathscr{C}(X)$ *is arcwise connected.*

PROOF. The space $\mathscr{C}(X)$ is compact ([3], Proposition 4.13.5) and X itself is a member of $\mathscr{C}(X)$. Consequently, in view of the fact that $\mathscr{C}(X)$ is order-dense ([1], p. 173), we get, by Corollary 2.5, that $\mathscr{C}(X)$ is arcwise connected.

COROLLARY 2.7. ([2], § 3). *If* X *is a compact and connected space, then* $\mathscr{CL}(X)$ *is arcwise connected.*

PROOF. The space $\mathscr{CL}(X)$ is compact ([3], Theorem 4.2) but is not order-dense if X has more than one point. Therefore, we need to work with suitable subcollections of $\mathscr{CL}(X)$. For this, given A in $\mathscr{CL}(X)$, consider the collection $\mathscr{S}$ of all members F of $\mathscr{CL}(X)$ such that $A\subseteq F\subseteq X$ and each component of F has points of A. Clearly, X belongs to $\mathscr{S}$ and $\mathscr{S}\subseteq\mathscr{C}(X)$ if and only if A is in $\mathscr{C}(X)$. We show that $\mathscr{S}$ satisfies the conditions of Theorem 2.4, in order to first conclude that there is an order-arc from A to X in $\mathscr{CL}(X)$. From this, in view of the proof of Corollary 2.5, it will follow that $\mathscr{CL}(X)$ is arcwise connected.

$\mathscr{S}$ is a closed, hence compact, subset of $\mathscr{CL}(X)$: If $E\in\mathscr{CL}(X)-\mathscr{S}$ then either $A\nsubseteq E$ or $A\subset E$ and there is a component K of E disjoint from A. In the first case, there is an x in $A-E$ and $\langle X-\{x\}\rangle$ is a neighborhood of E disjoint from $\mathscr{S}$. In the second case, E is not connected between A and K ([1], p. 170), hence there exists a closed subset $D\supseteq K$ of E such that $E-D\supseteq A$ is also closed. Now, X being normal, there are disjoint open subsets U and V of X such that $D\subseteq U$ and $E-D\subseteq V$, and then the neighborhood $\langle U, V\rangle$ of E is disjoint from $\mathscr{S}$.

$\mathscr{S}$ is order-dense: Given R, T in $\mathscr{S}$, $R\subset T$, let $x\in T-R$ and U be an open set containing R such that $x\notin \mathrm{Cl}(U)$. Suppose K is the component of T to which x belongs, then as $A\cap K$ is non-empty, $R\cap K$ is also non-empty. Let $y\in R\cap K$ and V stand for the set $U\cap K$. Then, V is a non-empty proper open subset of the continuum K and hence for the component C of $\mathrm{Cl}(V)$ to which y belongs, we have $C\cap(K-V)\neq\emptyset$ ([4], Theorem 20.1). Now, if $S=R\cup C$ then $S\in\mathscr{S}$ and $R\subset S\subset T$. This completes the proof.

For any A in $\mathscr{CL}(X)$, let $\mathscr{L}(A)$ and $\mathscr{M}(A)$ be respectively the collection of all those members of $\mathscr{CL}(X)$ that are contained in A and the collection of all those that contain A. Then, obviously $\mathscr{L}(A)=\mathscr{CL}(A)$, $\mathscr{M}(B)\subseteq\mathscr{M}(A)$ whenever B belongs to $\mathscr{M}(A)$, and $\mathscr{M}(A)$ is contained in each neighborhood $\langle V_1, V_2, \ldots, V_n\rangle$ of X to which A belongs. These observations coupled with the fact that the order-arc from A to X in $\mathscr{CL}(X)$ obtained above is contained in $\mathscr{M}(A)$ (in $\mathscr{M}(A)\cap\mathscr{C}(X)$ if A is in $\mathscr{C}(X)$) lead us to the following corollaries.

COROLLARY 2.8. *If X is a compact and connected space and A is a closed non-empty subset then the subspaces $\mathscr{M}(A)$ and $\mathscr{M}(A)\cap\mathscr{C}(X)$ of $\mathscr{CL}(X)$ are arcwise connected. In case A is in $\mathscr{C}(X)$ then $\mathscr{L}(A)$ is also arcwise connected.*

COROLLARY 2.9. ([2], § 4). *If X is a compact and connected space then the hyperspaces $\mathscr{CL}(X)$ and $\mathscr{C}(X)$ are locally arcwise connected at X.*

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THE CESÀRO—DENJOY—PETTIS SCALE OF INTEGRATION

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1. Introduction

In an earlier paper [5] we have introduced a scale of Cesàro—Denjoy—Bochner integration. In the present paper we introduce a scale of Cesàro—Denjoy—Pettis integrals, the $C_n D_* P$ integrals, such that the strength of the integral is increased with n, the $C_0 D_* P$ integral being the special Denjoy—Pettis integral introduced in [6].

2. Definition and terminologies

Throughout the paper, $\boldsymbol{R}$ is the real line, X is a real Banach space, $\|\cdot\|$ its norm X^* its conjugate space. The definition of Peano derivative and of $AC_n G_*$ for rea function are as in [2]. We shall frequently refer to the $C_n D$ integral of [7] and to the $C_n P$ integral of [3] for real valued functions. These integrals, viz. the $C_n D$ integral and the $C_n P$ integral are equivalent (see [8]). The Lebesgue—Bochner integral and the Lebesgue—Pettis integral will be denoted by LB and LP respectively. Unless otherwise stated, function will mean an X-valued function defined on an interval $[a, b]$.

DEFINITION 2.1. Let $F: [a, b] \to X$ and let $\xi \in [a, b]$. Let n be a positive integer. If there are constants $\alpha_1, \alpha_2, \ldots, \alpha_n \in X$ depending on ξ such that

$$x^* \left[F(t) - F(\xi) - (t-\xi)\alpha_1 - \ldots - \frac{(t-\xi)^n}{n!}\alpha_n\right] = o\big((t-\xi)^n\big) \quad (t \to \xi)$$

for all $x^* \in X^*$, then α_n is called *the weak Peano derivative of F at ξ of order n* and is denoted by $F^{\omega}_{(n)}(\xi)$. It is easily seen that if $F^{\omega}_{(n)}(\xi)$ exists then $F^{\omega}_{(k)}(\xi)$ $(1 \le k \le n)$ exists. In particular $F^{\omega}_{(1)}(\xi)$ is the weak derivative of F at ξ. For convenience we shall write $F^{\omega}_{(0)}$ to mean F. It is clear that if the strong Peano derivative $F_{(n)}$ (cf. [5]) exists at a point ξ then $F^{\omega}_{(n)}$ also exists at ξ and $F_{(n)}(\xi) = F^{\omega}_{(n)}(\xi)$.

DEFINITION 2.2. Let $F: [a, b] \to X$ and let n be a positive integer. If there are functions $F_i: [a, b] \to X$; $i = 1, 2, \ldots, n$ such that

$$x^* \left[F(t) - F(\xi) - (t-\xi)F_1(\xi) - \frac{(t-\xi)^2}{2!}F_2(\xi) - \ldots - \frac{(t-\xi)^n}{n!}F_n(\xi)\right] =$$
$$= o\big((t-\xi)^n\big) \quad (t \to \xi)$$

for almost all $\xi \in [a, b]$ for each $x^* \in X^*$ (the exceptional set of measure zero may vary with x^*), then F_n is said to be *the pseudo derivative of F on $[a, b]$ of order*

n and is denoted by $D_P^n F$. It is easily seen that $D_P^k F$ $(1 \leqq k \leqq n)$ exists if $D_P^n F$ exists and that if $F_{(k)}^{\omega}$ exists a.e. in $[a, b]$ then $D_P^k F$ exists in $[a, b]$ and $D_P^k F = F_{(k)}^{\omega}$.

Definition 2.3. Let $n \geqq 0$. A function F: $[a, b] \to X$ is called *weakly* $AC_n G_*$ *on* $[a, b]$ if $F_{(n)}^{\omega}$ exists in $[a, b]$ and if the real function $x^* F$ is $AC_n G_*$ [2] for each $x^* \in X^*$.

Since $|x^* F| \leqq \|x^*\| \, \|F\|$ it is easy to verify that strong $AC_n G_*$ defined in [5] implies weak $AC_n G_*$.

3. Preliminary results

Theorem 3.1. *Let F be weakly $AC_n G_*$ in $[a, b]$ and let $D_P^{n+1} F$ exist in $[a, b]$. If $D_P^{n+1} F = 0$ in $[a, b]$ then $F_{(n)}^{\omega}$ is constant.*

Proof. Let $x^* \in X^*$ be arbitrary. Then $x^* F$ is a real valued $AC_n G_*$ function. Since $x^* D_P^{n+1} F = 0$, $(x^* F)_{(n+1)} = 0$ a.e. So by [2; Theorem 16 coupled with Lemma 2] $(x^* F)_{(n)}$ is constant. But since $F_{(n)}^{\omega}$ exists in $[a, b]$, $(x^* F)_{(n)} = x^* F_{(n)}^{\omega}$. Hence $x^* F_{(n)}^{\omega}$ is constant. Let $\xi \in [a, b]$. Then $x^* (F_{(n)}^{\omega}(\xi) - F_{(n)}^{\omega}(a)) = 0$. Since x^* is arbitrary, the theorem is proved.

Theorem 3.2. *If $F_{(n)}^{\omega}$, $n \geqq 1$, exists in $[a, b]$, then $F_{(k)}^{\omega}$ are strongly measurable for $k = 1, 2, \dots, n$.*

Proof. Since F is weakly continuous, it is strongly measurable (cf. [6] or [4' p. 73]). Since $F_{(1)}^{\omega}$ exists in $[a, b]$, for each $t \in [a, b]$ and each $x^* \in X^*$

$$\lim_{h \to 0} x^* \frac{1}{h} [F(t+h) - F(t)] = x^* F_{(1)}^{\omega}(t).$$

Taking any sequence $\{h_r\}$ which converges to 0 we get a sequence of strongly measurable functions $\left\{\frac{1}{h_r} [F(t+h_r) - F(t)]\right\}$ which converges to $F_{(1)}^{\omega}(t)$ weakly everywhere. So, by ([4; p. 74, Theorem 3.5.4], $F_{(1)}^{\omega}$ is strongly measurable. Thus the theorem is true for $n = 1$. Suppose that it is true for $n = m-1$. Then since $F_{(k)}^{\omega}$ is strongly measurable for $k = 0, 1, \dots, m-1$ and since by the existence of $F_{(m)}^{\omega}$ we have

$$\lim_{h \to 0} x^* \left[\frac{m!}{h^m} \left\{F(t+h) - F(t) - h F_{(1)}^{\omega}(t) - \dots - \frac{h^{m-1}}{(m-1)!} F_{(m-1)}^{\omega}(t)\right\}\right] = x^* F_{(m)}^{\omega}(t)$$

for each $t \in [a, b]$ and each x^*, applying similar argument as above, $F_{(m)}^{\omega}$ is strongly measurable. The proof is thus complete by induction.

Definition 3.3. A function f: $[a, b] \to X$ is said to be $C_n D_* P$ *(Cesàro—Denjoy—Pettis) integrable* if there is a weakly $AC_n G_*$ function F: $[a, b] \to X$ such that $D_P^{n+1} F$ exists in $[a, b]$ and $D_P^{n+1} F = f$ on $[a, b]$. Then $F_{(n)}^{\omega}(t)$ is called an indefinite $C_n D_* P$ integral of f and $F_{(n)}^{\omega}(b) - F_{(n)}^{\omega}(a)$ is its definite $C_n D_* P$ integral in $[a, b]$ and is denoted by

$$(C_n D_* P) \int_a^b f(t)\, dt.$$

The definite integral of an integrable function is unique by Theorem 3.1. Clearly by Theorem 3.2 an indefinite C_nD_*P integral is strongly measurable. It can be verified that the class of all C_nD_*P integrable functions in $[a, b]$ is a linear space and the C_nD_*P integral is a linear operator from this linear space to X, and this operator is additive on abutting intervals. In [6] we have defined the D^*P integral (cf [9]) of a function $f\colon [a, b]\to X$ to be $F(b)-F(a)$ if there exists a weakly ACG_* function $F\colon [a, b]\to X$ such that $D_P^1F=f$ on $[a, b]$. Clearly if f is D^*P integrable in $[a, b]$ then f is C_0D_*P integrable and the integrals are equal.

THEOREM 3.4. *The function f is C_nD_*P integrable over $[a, b]$ if and only if there is a function $F\colon [a, b]\to X$ such that $F^{\omega}_{(n)}$ exists in $[a, b]$ and $x^*F^{\omega}_{(n)}$ is an indefinite C_nD integral of x^*f for each $x^*\in X^*$. Further, we have*

$$x^*\left(F^{\omega}_{(n)}(b)-F^{\omega}_{(n)}(a)\right)=(c_nD)\int_a^b x^*f(t)\,dt.$$

PROOF. Let f be C_nD_*P integrable. Then there is a weakly AC_nG_* function $F\colon [a, b]\to X$ such that $D_P^{n+1}F=f$ in $[a, b]$. So, for each $x^*\in X^*$, x^*F is AC_nG_* and $(x^*F)_{(n+1)}=x^*f$ a.e. and since $(x^*F)_{(n)}=x^*F^{\omega}_{(n)}$ so by the definition of C_nD integral we see that $x^*F^{\omega}_{(n)}$ is an indefinite C_nD integral of x^*f and

$$x^*F^{\omega}_{(n)}(b)-x^*F^{\omega}_{(n)}(a)=(C_nD)\int_a^b x^*f(t)\,dt.$$

Conversely, if $F^{\omega}_{(n)}$ exists and $x^*F^{\omega}_{(n)}$ is an indefinite C_nD integral of x^*f for each $x^*\in X^*$ then since $(x^*F)_{(n)}=x^*F^{\omega}_{(n)}$, x^*F is AC_nG_*. Also since $(x^*F)_{(n)}$ is an indefinite C_nD integral of x^*f we have $(x^*F)_{(n+1)}=x^*f$ a.e. and hence $D_P^{n+1}F=f$. This completes the proof.

THEOREM 3.5. *If f is C_nD_*P intagrable then f is weakly measurable.*

PROOF. By Theorem 3.4, for arbitrary $x^*\in X^*$, the real valued function x^*f is C_nD integrable and hence is C_nP integrable and so x^*f is measurable [1]. Hence, x^* being arbitrary, f is weakly measurable.

THEOREM 3.6. *If f is LP integrable then f is C_0D_*P integrable and the integrals are equal.*

PROOF. Let $x^*\in X^*$ be arbitrary and $F(t)=\int_a^t f(\xi)\,d\xi$. Then x^*f is Lebesgue integrable with indefinite integral x^*F. So x^*f is C_0P integrable with C_0P integral x^*F. Since $F^{\omega}_{(0)}=F$ the result follows by Theorem 3.4.

THEOREM 3.7. *A $C_{n-1}D_*P$ integrable function f is C_nD_*P integrable and the integrals are equal.*

PROOF. Let f be $C_{n-1}D_*P$ integrable in $[a, b]$. Let F be weakly $AC_{n-1}G_*$ and $D_P^nF=f$ in $[a, b]$. Since F is weakly continuous, it is LB integrable [6] and hence LP-integrable. Let $G(t)=\int_a^t F$. Since D_P^nF exists, for each $x^*\in X^*$ there is $E_{x^*}\subset$

$\subset[a, b]$ of measure zero such that for $\xi \in E_{x^*}$

$$x^*\left[F(t)-F(\xi)-\sum_{i=1}^{n}\frac{(t-\xi)^i}{i!}D_P^i F(\xi)\right]=o((t-\xi)^n) \quad (t\to\xi).$$

Hence since F is LP-integrable,

$$x^*\left[G(t)-G(\xi)-(t-\xi)F(\xi)-\sum_{i=1}^{n}\frac{(t-\xi)^{i+1}}{(i+1)!}D_P^i F(\xi)\right]=o((t-\xi)^{n+1}) \quad (t\to\xi)$$

Hence $D_P^{n+1}G=D_P^n F$. It can be shown that, since $F_{(n-1)}^{\omega}$ exists, $(x^*G)_{(n)}=(x^*F)_{(n-1)}=x^*F_{(n-1)}^{\omega}$ and hence $G_{(n)}^{\omega}=F_{(n-1)}^{\omega}$ and that, since F is weakly $AC_{n-1}G_*$, G is weakly AC_nG_*. Thus f is C_nD_*P integrable. Since $G_{(n)}^{\omega}=F_{(n-1)}^{\omega}$ the result follows.

THEOREM 3.8. *A C_nD_*B integrable function is C_nD_*P integrable.*

This is obvious since strongly AC_nG_* implies weakly AC_nG_* and existence of strong Peano derivative implies the existence of weak Peano derivative.

THEOREM 3.9. *If f is C_nD_*P integrable and $F(t)=(C_nD_*P)\int_a^t f$ then F is $C_{n-1}D_*P$ integrable in $[a, b]$.*

PROOF. If f is C_nD_*P integrable then by Theorem 3.4 there is $\Phi:[a, b]\to X$ such that $\Phi_{(n)}^{\omega}$ exists in $[a, b]$ and $(x^*\Phi)_{(n)}$ is an indefinite C_nD integral of x^*f for each $x^*\in X^*$. Hence $\Phi_{(n-1)}^{\omega}$ exists in $[a, b]$ and $(x^*\Phi)_{(n-1)}$ is an indefinite $C_{n-1}D$ integral of $x^*F(t)=\int_a^t x^*f$ for each $x^*\in X^*$. Hence F is $C_{n-1}D_*P$ integrable by Theorem 3.4.

4. Integration by parts

THEOREM 4.1. *Let f be C_nD_*P integrable and let $F=\int_a^t f$. If G: $[a, b]\to R$ is such that $G^{(n)}$ is absolutely continuous, then fG is C_nD_*P integrable in $[a, b]$ and*

$$(C_nD_*P)\int_a^b fG=[FG]_a^b-(C_{n-1}D_*P)\int_a^b FG'.$$

PROOF. We shall first prove the theorem for $n=1$. Let Φ: $[a, b]\to X$ be such that $\Phi_{(1)}^{\omega}=F$ and $D_P^2\Phi=f$ and Φ is weakly AC_1G_*. Since Φ is weakly continuous and $G^{(1)}$ is continuous so $\Phi G^{(1)}$ is weakly continuous and so by Lemma 4.1 of [6] it is LB integrable and so LP integrable. Now let

$$\Psi(\xi)=\Phi(\xi)G(\xi)-\int_a^{\xi}\Phi(t)G^{(1)}(t)\,dt$$

and $x^*\in X^*$ be arbitrary. Then

$$(x^*\Psi)_{(1)}=(x^*\Phi)_{(1)}G+x^*\Phi G_{(1)}-x^*\Phi G_{(1)}=(x^*\Phi)_{(1)}G=x^*(\Phi^{\omega}_{(1)}G)=x^*FG.$$

Therefore $\Psi^{\omega}_{(1)}=FG$ and also (cf. [1])

$$(x^*\Psi)_{(2)}=x^*FG'+Gx^*f \quad \text{(a.e.)}=x^*(FG'+fG) \quad \text{(a.e.)}$$

So, $D^2_P\Psi=FG'+fG$. Now since Φ is weakly AC_1G_* and since

$$x^*\Psi(\xi)=x^*\Phi(\xi)G(\xi)-\int_a^{\xi}x^*\Phi(t)G^{(1)}(t)\,dt$$

by Theorem 12 of [5], Ψ is weakly AC_1G_*. Hence $FG'+fG$ is C_1D_*P integrable and

$$[\Psi^{\omega}_{(1)}]^b_a=(C_1D_*P)\int_a^b(FG'+fG).$$

Now by Theorem 3.9, F is C_0D_*P integrable and so FG' is C_0D_*P integrable by [6] and this completes the proof for $n=1$.

Now we assume the theorem for $n=m-1$ and prove it for $n=m$. The theorem will then follow by induction. Let Φ: $[a,b]\to X$ be such that $\Phi^{\omega}_{(m)}=F$ and $D^{m+1}_P\Phi=f$ and Φ is weakly AC_mG_*. Since Φ is weakly continuous and $G^{(m)}$ is continuous so $\Phi G^{(r)}$ is LB integrable [6] and so LP integrable for $r=1,2,\ldots,m$. Setting

$$\Psi(\xi)=\Phi(\xi)G(\xi)+\sum_{r=1}^{m}(-1)^r\binom{m}{r}\frac{1}{(r-1)!}\int_a^{\xi}(\xi-t)^{r-1}\Phi(t)G^{(r)}(t)\,dt$$

we get for arbitrary $x^*\in X^*$

$$x^*\Psi(\xi)=x^*\Phi(\xi)G(\xi)+\sum_{r=1}^{m}(-1)^r\binom{m}{r}\frac{1}{(r-1)!}\int_a^{\xi}(\xi-t)^{r-1}x^*\Phi(t)G^{(r)}(t)\,dt.$$

By Theorem 12 of [5] $(x^*\Psi)_{(m)}=x^*(FG)$ for all x^* i.e. $\Psi^{\omega}_{(m)}=FG$ and $(x^*\Psi)_{(m+1)}=$ $=x^*(FG^{(1)}+fG)$ a.e. for all x^* i.e. $D^{m+1}_P\Psi=FG^{(1)}+fG$ and Ψ is weakly AC_mG_*. Hence $FG^{(1)}+fG$ is C_mD_*P integrable and

$$(FG)(t)=\int_a^t(FG^{(1)}+fG).$$

Now by Theorem 3.9, F is $C_{m-1}D_*P$ integrable and since $(G^1)^{(m-1)}=(G)^{(m)}$ is absolutely continuous, $FG^{(1)}$ is $C_{m-1}D_*P$ integrable and hence by Theorem 3.7, it is C_mD_*P integrable. So we have

$$[FG]^b_a=(C_{m-1}D_*P)\int_a^bF(t)G^{(1)}(t)\,dt+(C_mD_*P)\int_a^bf(t)G(t)\,dt,$$

completing the proof.

5. Examples

EXAMPLE 5.1. There exists a C_0D_*P integrable function which is not LP integrable.

Let f be an everywhere finite real valued function on [0, 1] which is D^* integrable but not L-integrable and let $F(t)$ be its indefinite integral with $F(0)=0$. Let $\{c_n\}\in l_2$ be fixed. Define g: $[0, 1]\to l_2$ by $g(t)=\{c_n f(t)\}$, $t\in[0, 1]$ and G: $[0, 1]\to l_2$ by $G(t)=\{c_n F(t)\}$, $t\in[0, 1]$. Now, if $x^*\in l_2^*$, then there exists a sequence $\{d_n\}\in l_2$ such that

$$x^*g(t)=f(t)\Sigma c_n d_n = Af(t)$$

where $\Sigma c_n d_n = A$. Since $f(t)$ is not L-integrable so $x^*g(t)$ is not L-integrable. So g is not LP integrable. On the other hand $f(t)$ being D^* integrable $x^*g(t)$ is D^* integrable and

$$\int_0^\xi x^*g(t)\,dt = \int_0^\xi Af(t)\,dt = A\int_0^\xi f(t)\,dt = AF(\xi) = F(\xi)\Sigma c_n d_n = x^*G(\xi).$$

Since $G_{(0)}^{\omega}=G$, by Theorem 3.4, $g(t)$ is C_0D_*P integrable on [0,1].

EXAMPLE 5.2. For each $n>0$ there exists a C_nD_*P integrable function which is not $C_{n-1}D_*P$ integrable.

Let f be a real valued finite function in [0, 1] which is C_nP integrable but not $C_{n-1}P$ integrable. Then there is a real valued function Φ in [0, 1] such that Φ is AC_nG_* and $\Phi_{(n)}(t)$ is the indefinite integral of $f(t)$. We may suppose $\Phi_{(n)}(0)=0$. Let $\{c_r\}\in l_2$. Define the function g and Ψ on [0, 1] with values in l_2 such that

$$g(t)=\{c_r f(t)\},\quad t\in[0, 1],\quad \Psi(t)=\{c_r\Phi(t)\},\quad t\in[0, 1].$$

Then the strong Peano derivative $\Psi_{(i)}$ exists at each point where $\Phi_{(i)}$ exists and $\Psi_{(i)}=\{c_r\Phi_{(i)}\}$ for $i=1, 2, \dots n+1$. Let $x^*\in l_2^*$. Then there is a sequence $\{d_r\}\in l_2$ such that

$$x^*g(t)=f(t)\Sigma c_r d_r = Af(t).$$

Since $f(t)$ is not $C_{n-1}P$ integrable so $x^*g(t)$ is not $C_{n-1}P$ integrable. Hence by Theorem 3.4, $g(t)$ is not $C_{n-1}D_*P$ integrable.

Again since $f(t)$ is C_nP integrable, we see $x^*g(t)$ is C_nP integrable on [0, 1] and

$$\int_0^\xi x^*g(t)\,dt = \int_0^\xi Af(t)\,dt = A\int_0^\xi f(t)\,dt = A\Phi_{(n)}(\xi) = \Sigma c_r\Phi_{(n)}(\xi)\,d_r = x^*\Psi_{(n)}(\xi).$$

So, by Theorem 3.4 $g(t)$ is C_nD_*P integrable on [0, 1].

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ON 3-TORSION FREE RINGS IN WHICH EVERY CUBE COMMUTES WITH EACH OTHER

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In this note rings are associative and have an identity element 1. The set of all positive integers is denoted by **N**. For two elements x and y of a ring the commutator $xy-yx$ is denoted by $[x, y]$.

Let n be in **N**. Awtar [3] proved that an $n!$-torsion free ring satisfying the identity $[x^n, y^n]=0$ is commutative, and asked if an n-torsion free ring with $[x^n, y^n]=0$ is commutative. The answer is yes when $n=2$. Abu-Khuzam and Yaqub [1], [2] showed that such a ring is commutative if certain conditions are added. Bell [4] showed that the answer is negative for $n=3$. In fact, we have

EXAMPLE. Let $S=\left\{\begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} \middle| a, b\in GF(4)\right\}$. Then S is a 3-torsion free non-commutative ring satisfying $[x^3, y^3]=0$.

A ring R is called a *P_n-ring* if R satisfies the identity $[x^n, y^n]=0$ and every commutator in R is n-torsion free, that is, $n[x, y]=0$ implies $[x, y]=0$ for any $x, y\in R$. The purpose of this note is to show that the ring S in the example is, essentially, the only non-commutative P_3-ring.

In what follows, R is a P_3-ring. Let $x, y\in R$. The equations $[x^3, y^3]=$ $=[x^3, (1+y)^3]=0$ yield $3[x^3, y+y^2]=0$. Since every commutator in R is 3-torsion free, we get

$$[x^3, y+y^2] = 0. \tag{1}$$

Substituting $1+x$ for x in (1) we obtain

$$[x+x^2, y+y^2] = 0. \tag{2}$$

Again substituting $1+y$ for y and next $1+x$ for x in (2), we have the identity

$$4[x, y] = 0. \tag{3}$$

Conversely, as is easily seen, a ring satisfying the identities $[x^3, y^3]=4[x, y]=0$ is a P_3-rings. Therefore we see that a homomorphic image of a P_3-ring is again a P_3-ring, and that a ring is a P_3-ring if and only if each of its subdirectly irreducible factors is a P_3-ring. Now the following theorem, the main result of this note, characterizes P_3-rings.

THEOREM. *Let R be a ring. R is a P_3-ring if and only if R is either a commutative ring or a subdirect product of a commutative ring and copies of the ring S given in Example.*

The proof of the theorem will be carried out stage by stage. It suffices to show that a subdirectly irreducible P_3-ring is either commutative or isomorphic to S. The following easy, known result will be used frequently. Let x and y be elements of a ring.

(I) *If* $[x,[x,y]]=0$, *then* $[x^n, y] = nx^{n-1}[x,y]$ *for all* $n\in\mathbf{N}$.

Let x and y be in R. Assume $xy=0$. Since $[x^3, y^3]=0$, we have $y^i x^j=0$ for all integers $i\geqq 3$ and $j\geqq 3$. Then, using (1) and (2) we can get $yx=0$. Thus we have

(II) *For* $x, y\in R$, $xy=0$ *implies* $yx=0$.

Let N denote the set of all nilpotent elements of R, D the commutator ideal of R and Z the center of R. Let $a, b\in N$. By double induction on nilpotency indices of a and b using (2), it is easily proved that $[a, b]=0$. It is known that for a ring with $[x^3, y^3]=0$ the nilpotent elements form an ideal and the commutator ideal is nil (see Kezlan [6] or Lihtman [7]). Thus we obtain

(III) N *is a commutative ideal and* $D\subset N$. *Therefore* N^2 *and* D^2 *are in* Z.

Let $x\in R$ and $a\in N$. Note that $a^i\in Z$ for any $i\geqq 2$ by (III). Hence by (1) we have $[x^3, a]=0$. Moreover, we find from (2) that

$$[x+x^2, a] = 0. \tag{4}$$

Therefore we have $[x, a]=-[x^2, a]=[x^4, a]=x^3[x, a]$. Thus we get

(IV) $(1-x^3)[x, a]=0$ *for any* $x\in R$ *and* $a\in N$.

By (II) there is no distinction between left and right zero-divisors in R. Let A denote the set of all zero divisors of R. For a subset T of R the left (=right) annihilator of T in R forms a two-sided ideal of R and is denoted by $\mathrm{Ann}(T)$. Let us assume that R is subdirectly irreducible. H denotes the heart (the smallest nonzero ideal) of R. Clearly $\mathrm{Ann}(H)\subset A$. Conversely, let $a\in A$. Then $\mathrm{Ann}(a)$ is a nonzero ideal of R and contains H. This implies $a\in\mathrm{Ann}(H)$. Thus we find that $A=\mathrm{Ann}(H)$ is a two-sided ideal of R. Let $a\in A$ and $b\in N$. Then $1-a^3\notin A$. Hence by (IV) we have $[a, b]=0$. Let $x\in R$. Since $xa\in A$, we have $xba=xab=bxa$. These are summarized as

(V) *If* R *is subdirectly irreducible, then* (i) A *is an ideal of* R, (ii) $[a, b]=0$ *for any* $a\in A$ *and* $b\in N$, *and* (iii) $[x, b]$ *is in* $\mathrm{Ann}(A)$ *for any* $x\in R$ *and* $b\in N$.

Next we shall prove

(VI) *Let* R *be subdirectly irreducible. If* $\mathrm{Ann}(A)\subset Z$, *then* R *is commutative.*

Assume that $\mathrm{Ann}(A)\subset Z$. Let $x\in R$ and $a\in N$. Since $[x, a]\in\mathrm{Ann}(A)$ by (iii) in (V), we have $[x,[x, a]]=0$. Hence by (I) and (3) we obtain $[x^4, a]=4x^3[x, a]=0$. Thus by (4) we get $[x, a]=0$. Let y be in R. Since $[x, y]\in N$ by (iii), we have $[x,[x, y]]=0$. Hence, again by (I) and (3) we find $[x^4, y]=4x^3[x, y]=0$. Now using (2) repeatedly, we can show $[x, y]=0$.

Now we claim

(VII) *Let R be subdirectly irreducible. If R is not commutative, then* $R/A \cong$ $\cong GF(4)$.

Indeed, assume that R is not commutative. Then by (III), $A \neq 0$ and so $\mathrm{Ann}(A) \subset$ $\subset A$. Since $\mathrm{Ann}(A)$ annihilates A, we find $\mathrm{Ann}(A) \subset N$. Moreover, (III) implies that R/A is a commutative domain. There are $x \in R$ and $a \in \mathrm{Ann}(A)$, such that $b=[x, a] \neq 0$ by (VI), so $1-x^3 \in A$ due to (IV). If $1-x \in A$, then $[x, a]=0$, a contradiction. So we find $1+x+x^2 \in A$. We have $2a=0$, because $2 \in A$ by (3). Hence by (4) we find $[x, b]=[x,[x, a]]=[x^2, a]=[x, a]=b$. Let y be an arbitrary element of $R \setminus A$. If $[y, b] \neq 0$, then $1+y+y^2 \in A$ for the same reason as above. Furthermore we have

$$0 \equiv 1+x+x^2-(1+y+y^2) \equiv (x-y)(1+x+y) \equiv (x-y)(y-x^2) \pmod{A}.$$

It follows that $y \equiv x \pmod{A}$ or $y \equiv x^2 \pmod{A}$. Thus we get

$$[y, b]=[x, b]=b. \tag{5}$$

If $[y, b]=0$, then $[xy, b]=[x, b]y=by$. Since by is nonzero, it must be equal to b by (5). It follows that $1-y \in A$. This proves our claim.

By (5), $Rb=bR$ is a nonzero two-sided ideal of R, so it contains H. Therefore there is an element z of R such that zb is a nonzero element of H. Since $z \notin A$, we have $1-z^3 \in A$ by (VII). Hence, $b=z^3 b$ is contained in H. Thus we obtain

(VIII) *If R is subdirectly irreducible, then $[x, a]$ is in H for any $x \in R$ and* $a \in \mathrm{Ann}(A)$.

Next we strengthen (VI) as follows.

(IX) *Let R be subdirectly irreducible. If $H \subset Z$, then R is commutative.*

Assume that $H \subset Z$. It suffices to show $\mathrm{Ann}(A) \subset Z$. Let $x \in R$ and $a \in \mathrm{Ann}(A)$. Since $[x, a] \in H$ by (VIII), we see $[x, a] \in Z$ and $[x,[x, a]]=0$. By (I) and (3) we find $[x^4, a]=4x^3[x, a]=0$. Thus we conclude $[x, a]=0$ by (4).

Now we shall finish the proof of the theorem. Assume that R is subdirectly irreducible and non-commutative. If $AN \neq 0$, then AN is a nonzero ideal contained in Z by (ii) in (V). So, H is also contained in Z and R must be commutative by (IX), a contradiction. Therefore we have $AN=NA=0$. Let $x \in R$ and $a, b \in A$. Since $[x, a]$ and $[x, b]$ are in N by (III), we have $[x, a]b=a[x, b]=0$. It follows that $[x, ab]=$ $=0$. This means that $ab \in Z$ and the ideal A^2 is contained in Z. Thus, for the same reason as above we find $A^2=0$, that is, $A=\mathrm{Ann}(A)$. Now, choose an element x in R so that the residue class of x modulo A is a primitive element of $R/A\ (\cong GF(4))$. Let a be a nonzero element of A. If $[x, a]=0$, then $Ra=\{0, a, xa, x^2a\}$ forms a two-sided nonzero ideal of R contained in Z. Again this is impossible. Therefore, $[x, a] \neq 0$ and we have $[x,[x, a]]=[x, a]$ by (5), that is, $[x,[x, a]-a]=0$. But, what we have just proved shows

$$[x, a]=a. \tag{6}$$

This means that for any nonzero element a of A, $Ra=aR$ forms a two-sided ideal of R of order 4. Since R is subdirectly irreducible, we conclude $A=H=Ra=aR=$ $=\{0, a, xa, x^2a\}$. We have $2=0$, otherwise 2 is a central nonzero element of A

and this is impossible. We can choose x so that $x^3=1$ holds. Indeed if $x^3=1+b$ with $b\,(\neq 0)\in A$, then $x^6=(1+b)^2=1$; take x^2 for x. Then $\{0, 1, x, x^2\}$ is a subfield of R isomorphic to $GF(4)$. Moreover, by (6) we have $ax=(x+1)a=x^2a$. It is now clear that $R\cong S$, where S is the ring S in the example.

The proof of the Theorem is complete.

REMARK. Let $n\in N$. If n has a divisor of the form $1+p^r+p^{2r}+\ldots+p^{sr}$, where r and s are positive integers and p is a prime not dividing n, then there exists a non-commutative P_n-ring (see [5, Remark 3]). We believe that similar results to our theorem in this note should hold for general P_n-rings.

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A LOSYNSKI–KHARSHILADZE THEOREM FOR MÜNTZ POLYNOMIALS

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In this note $C(\Delta)$ always denotes the space of continuous functions on the unit circle and $C_{[0,1]}$ denotes the space of continuous real valued functions on the interval $[0, 1]$; z denotes the variable in Δ and t denotes the variable in $[0, 1]$.

Let $\{\lambda_j\}_{j=0}^{\infty}$ be an increasing sequence of integers; $\lambda_0=0$. The positive solution of the Littlewood conjecture implies that for any choice of linear bounded projections q_n: $C(\Delta)\to C(\Delta)$ with range $q_n=\text{span }\{z^{\lambda_j}\}_{j=0}^{n}$,

$$\|q_n\| \geqq O(1)\log n.$$

This result prompted the second author to conjecture (cf. [4]) that for any sequence of linear bounded projections p_n: $C_{[0,1]}\to C_{[0,1]}$ with range $p_n=\text{span }\{t^{\lambda_j}\}_{i=0}^{n}$ we have $\|p_n\|\to\infty$.

This conjecture turned out to be false in general. In this paper we indicate some sequences $\{\lambda_j\}$ for which the conjecture is true and some for which it is false.

A sequence $\{\lambda_j\}$ is called lacunary if $\dfrac{\lambda_{j+1}}{\lambda_j}\geqq\varrho>1$ for all j.

THEOREM 1. *Let $\{\lambda_j\}$ be a lacunary sequence, then there exists a sequence of projections p_n: $C_{[0,1]}\to C_{[0,1]}$ with range $p_n=\text{span }\{t^{\lambda_j}\}_{j=0}^{n}$ such that $\|p_n\|=O(1)$.*

PROOF. Denote by $V=\overline{\text{span }\{t^{\lambda_j}\}_{j=0}^{\infty}}$. Then V is the space of analytic functions on $|z|\leqq 1$ whose power series include only powers of z^{λ_i} (cf. [1]). Moreover $\{t^{\lambda_j}\}$ forms a Schauder basis for V. Hence there exists a sequence of projections Q_n: $V\to V$ with range $Q_n=\text{span }\{t^{\lambda_j}\}_{j=0}^{n}$ such that $\|Q_n\|=O(1)$. Because of the minimality of the system $\{t^{\lambda_j}\}$, the space V is isomorphic to c_0 (cf. [1]) and thus (cf. [3]) it is complemented in $C_{[0,1]}$. Let P be a projection from $C_{[0,1]}$ onto V. Then the projections $p_n=Q_nP$ are the desired projections.

REMARK. Similar results are true for $L_p[0, 1]$ $(p>1)$ spaces, and for arbitrary sequences (λ_j) with $\lambda_j\in R$ and $\lambda_j\to\infty$. Namely if $\{\lambda_j\}$ is a lacunary sequence then there exists a sequence of projections $p_n: L_p\to L_p$ with range $p_n=\text{span }\{t^{\lambda_j}\}_{j=0}^{n}$ such that $\|p_n\|=O(1)$. Proof follows from [1] and is exactly the same as in Theorem 1.

THEOREM 2. *Let $\lambda_n\leqq n+o(\log n)$. Then for every sequence of projections p_n from $C_{[0,1]}$ into $C_{[0,1]}$ with range $p_n=\text{span }\{t^{\lambda_j}\}_{j=0}^{n}$, $\|p_n\|\to\infty$.*

PROOF. Let $C_{[-\pi,\pi]}$ be the space of even periodic functions with period 2π. It is sufficient to prove that for any projections p_n with range $p_n=$

$=\operatorname{span}\{\cos^{\lambda_j}\theta\}_{j=0}^n$ we have $\|p_n\|\to\infty$. Every such projection has a representation

$$(p_n x)(\theta)=\sum_{j=0}^{n}{}_i f_j^{(n)}(x)\cos^{\lambda_j}\theta$$

where $f_j^{(n)}\in(C_{[-\pi,\pi]})^*$ and $f_j^{(n)}(\cos^{\lambda_i}\theta)=\delta_{ij}$. On the other hand, the same projection looks like

$$(p_n x)(\theta)=\sum_{j=0}^{\lambda_n}{}_i l_j^{(n)}(x)\cos j\theta$$

where the linear functionals $l_j^{(n)}$ are linear combinations of the functionals $f_i^{(n)}$.

Using an inequality of Sidon [2] and usual arguments of functional analysis we get

(1) $$\|p_n\|\geqq C\sum_{j=0}^{\lambda_n}{}_1 \frac{\|l_j^{(n)}\|}{\lambda_n-j+1}.$$

The problem is reduced to estimating the norms $\|l_j^{(n)}\|$. Define $z_1(\theta)=\cos^{\lambda_1}\theta=$ $=\sum_{j=0}^{\lambda_1} a^1_{(j)}\cos j\theta$. Then $\|z_1\|=1$. On the other hand there are at most λ_1 elements in this sum and so for at least one index ν_1 we get $|a^{(1)}_{\nu_1}|\geqq\frac{1}{\lambda_1}$.

It follows from the definition of $l_j^{(n)}$ that $l_{\nu_1}^{(n)}(z_1)=a^{(1)}_{\nu_1}$. Therefore $\|l_{\nu_1}^{(n)}\|\geqq\frac{1}{\lambda_1}$.

We can now choose numbers α_1^1,α_2^1 such that the polynomial

$$z_2(\theta)=\alpha_1^1\cos^{\lambda_1}\theta+\alpha_2^1\cos^{\lambda_2}\theta=\sum_{j=0}^{\lambda_2} a_j^{(2)}\cos j\theta$$

has norm 1 and the coefficient of $\cos(\nu_1\theta)$ is zero. Then the polynomial $z_2(\theta)$ has at most λ_2-1 monomials and so for at least one index ν_2,

$$\|l_{\nu_2}^{(n)}\|\geqq|a_{\nu_2}^{(2)}|\geqq\frac{1}{\lambda_2-1}.$$

Continuing in the same way, we define inductively a polynomial

$$z_{m+1}(\theta)=\sum_{j=1}^{m+1}\alpha_j^{(m+1)}\cos^{\lambda_j}\theta=\sum_{j=1}^{\lambda_{m+1}} a_j^{m+1}\cos j\theta$$

such that $\|z_{m+1}\|=1$ and all the coefficients $a_{\nu_k}^{(m+1)}$ $(k=1,\dots,m)$ are zero. Then the polynomial z_{m+1} has at most $\lambda_{m+1}-m$ monomials and so there exists an index ν_{m+1} such that

$$\|l_{\nu_{m+1}}^{(n)}\|\geqq|a_{\nu_{m+1}}^{(m+1)}|\geqq\frac{1}{\lambda_{m+1}-m}.$$

Using inequality (1) we obtain

$$\|p_n\| \geqq C\Sigma \frac{\|l_{\nu_j}^{(n)}\|}{\lambda_n-\nu_j+1} \geqq C \min_j \|l_{\nu_j}^{(n)}\| \sum_k \frac{1}{\lambda_n-k+1} \geqq$$

$$\geqq \frac{\ln n-\ln(\lambda_n-n)}{o(\ln n)} \geqq C_1 \frac{\ln n-\ln\ln n}{o(\ln n)} \to \infty.$$

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ON q-INCREASING ELEMENTS IN SEMIGROUPS

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Introduction

There are several works on increasing elements in semigroups; here we generalize the notion of increasing element, by introducing q-increasing (or quasi-increasing) elements.

In § 1. a we determine several general properties of q-increasing elements. Some semigroups (as periodic, and completely regular semigroups) contain no q-increasing elements. In § 1.b some sufficient conditions are given in order that q-increasing elements exist in a semigroup, in particular in a regular semigroup, by means of Green's relations.

In § 2. minimal subsets relative to q-increasing elements are defined, their existence is proven, and a "reduction" theorem is given for those subsets.

In § 3. at first we consider unsolved problems about increasing elements: some of them are solved by means of Szép's decompositions, $D_L(S)$ and $D_R(S)$ or by Γ-decomposition of S. At last (§ 3.c) we prove both a necessary condition so that a semigroup, which is a semilattice of subsemigroups, have increasing elements and a result on left separative semigroups.

We use the standard notation of the algebraic theory of semigroups.

§ 1.

a) Let S be a semigroup.

DEFINITION 1.1. An element a of a subset $T\subseteqq S$ such that $aT=T[Ta=T]$ is said *a left* [*right*] *quasi-increasing (or q-increasing) element,* relative to T, if there exists a subset $T'\subset T$ with $aT'=T[T'a=T]$.

A left [right] increasing element is left [right] q-increasing too. It is easy to give examples of semigroups which contain q-increasing elements but no increasing elements.

The bicyclic semigroup, $\mathscr{C}(p,q)$, has both increasing and q-increasing elements.

THEOREM 1.2. *No element of S can be both left and right q-increasing.*

PROOF. If a is both a left and a right q-increasing element of S, there exist subsets $T'\subset T$, $V'\subset V$ such that $a\in T\cap V$, $aT=aT'=T$, $Va=V'a=V$. Since $V'a=V$, there exist $v_1, v_2\in V'$ such that $a=v_1a$, $v_1=v_2a$. Let $y\in T$; since $aT'=T$,

* This work was performed in the sphere of activity of G.N.S.A.G.A. (C.N.R. — ITALY).

there exists $y' \in T'$ such that $y = ay'$. Therefore $v_1 y = v_1 ay' = ay' = y$, hence $v_2 T = v_2 aT' = v_1 T' = T'$, and also $T' = v_2 T = v_2 aT = v_1 T = T$, a contradiction.

COROLLARY 1.3. *If S is commutative, no element of S is left [right] q-increasing.*

The proofs of the following Propositions 1.4, 1.5, 1.6 are not difficult and we omit them.

PROPOSITION 1.4. *A left [right] cancellable element of S is not a left [right] q-increasing element.*

Therefore a group has no q-increasing elements.

PROPOSITION 1.5. *An idempotent element of S is not a left [right] q-increasing element.*

PROPOSITION 1.6. *If T is a subset of S, the set $T^{(1)}$ $[T^{(r)}]$ of the left [right] q-increasing elements relative to T (if $T^{(1)}[T^{(r)}] \neq \emptyset$) is a subsemigroup of S.*

PROPOSITION 1.7. *Let a be a left [right] q-increasing element of S. Then:*
i) *a^n is a left [right] q-increasing element of S ($n \in \mathbf{N}$).*
ii) *a generates an infinite cyclic semigroup.*

PROOF. i) It follows from Proposition 1.6. ii) A finite cyclic semigroup contains an idempotent, which can not be a q-increasing element, by Proposition 1.5.

COROLLARY 1.8. *Periodic semigroups contain no left [right] q-increasing elements.*

THEOREM 1.9. *A completely regular (c. r.) element of S is not a left [right] q-increasing element of S.*

PROOF. Let $a \in S$ be completely regular and suppose that there exist subsets $T' \subset T \subseteqq S$ such that $a \in T$, $aT' = aT = T$. If $\hat{a}$ is the identity of the $\mathscr{H}$-class H_a and a^{-1} is the inverse element of a in H_a, then we have:

$$\hat{a}T' = a^{-1}aT' = a^{-1}T = a^{-1}aT = aa^{-1}T = a\hat{a}T' = aT' = T;$$

$$\hat{a}T = \hat{a}aT' = aT' = T; \quad \hat{a} = a^{-1}a \in a^{-1}T = \hat{a}T' = T.$$

Therefore $\hat{a}$ is a left q-increasing element of S, in contradiction with Proposition 1.5.

By Theorem 1.9, a completely regular semigroup contains no left [right] q-increasing elements, and this generalizes a result about increasing elements in a c.r. semigroup ([2]; Theorem 1(a)).

The following results are of "Ljapin type" (for increasing elements). If $S^{[r]}$ $[S^{[1]}]$ is the set of the right [left] q-increasing elements in S and $S^{[n]}$ is the set of the elements which are not q-increasing then S is the disjoint union of $S^{[r]}$, $S^{[1]}$ and $S^{[n]}$. One can give examples of semigroups S in which either $S^{[r]}$ or $S^{[1]}$ or both $S^{[r]}$ and $S^{[1]}$ are not subsemigroups.

THEOREM 1.10. *If $S^{[r]}$ is a subsemigroup of S, then $S^{[r]}$ has no left q-increasing elements.*

PROOF. Let $a \in S^{[r]}$ be a left q-increasing element for $S^{[r]}$; then there exist $z, z' \in S^{[r]}$ such that $a = az$, $z = az'$. Because $a \in S^{[r]}$, there exist $T, T' \subseteqq S$, $T' \subset T$

such that $a\in T$, $T=T'a=Ta$, hence, for every $t\in T$, $t=t'a(t'\in T')$, it holds $tz=(t'a)z=t'a=t$. Therefore $T'=T'z=T'az'=Tz'=Taz'=Tz=T$, a contradiction. The dual result of Theorem 1.10 holds for $S^{[1]}$.

b) Sufficient conditions in order that a semigroup S contain left [right] q-increasing elements are determined by Theorems II.3.2 and II.3.3 in [6].

The following theorems give other sufficient conditions by means of the Green's relations in a regular semigroup. Let $\mathcal{S}$ be the minimum inverse semigroup congruence on a regular semigroup S, while i is the identity mapping on S.

THEOREM 1.11. *Let S be a regular, not orthodox semigroup, such that $\mathcal{S}\subseteqq\mathcal{D}$, $\mathcal{S}\cap\mathcal{H}=i$. Then S contains left and right q-increasing elements.*

PROOF. From [1], Theorem B, S contains a copy of the bicyclic semigroup $\mathcal{C}(p,q)$.

REMARK. Let S be the semigroup consisting of all nondecreasing unbounded sequences $f\colon n\to fn$ of natural numbers $\mathbf{N}$ under the composition of functions. S has an identity $(i\colon n\to n)$, but no other units; therefore S contains no increasing elements. It is proved in [1] that S is regular, but not orthodox; moreover $\mathcal{S}\subseteqq\mathcal{D}$ and $\mathcal{S}\cap\mathcal{H}=i$. Hence, by Theorem 1.11, S contains left and right q-increasing elements.

THEOREM 1.12. *Let S be a regular semigroup. If there exist two distinct $\mathcal{D}$-related idempotents, then S contains left and right q-increasing elements.*

PROOF. By [1], Proposition, S contains a copy of $\mathcal{C}(p,q)$.

§ 2.

In a semigroup S, let a be a left increasing element relative to the subset $T\subseteqq S$, that is $a\in T$ and there esists a subset $T'\subset T$ such that $aT'=aT=T$. (1)

DEFINITION 2.1. A subset T' in (1) is said *minimal* for a with respect to T if $a(T'-x)\neq T$, for every $x\in T'$.

PROPOSITION 2.2. *Let a be a left q-increasing element relative to $T\subseteqq S$. Then there exists a subset M minimal for a with respect to T.*

PROOF. Let $aT'=T$, $aT=T$, with $T'\subset T$. Let us consider the relation λ in T defined by $x\lambda y$ if $ax=ay$ $(x,y\in T)$.

It is evident that λ is an equivalence in T; let T/λ be the quotient set. In each equivalence class $[x]\in T/\lambda$ there is some element of T'. Let us choose a representative $t_i'\in T'$ in every class $[x_i]_{i\in I}$ and let $M=\{t_i'\}_{i\in I}$. It is clear that $M\subseteqq T'\subset T$, $aM=T$, and M is minimal for a with respect to T.

Of course the duals of Definition 2.1 and Proposition 2.2 hold. However minimal subsets M above considered are not absolutely minimal.

THEOREM 2.3 (Reduction theorem). *Let $\{a_i\}_{i\in\mathbf{N}}$ be a sequence of left q-increasing elements of S such that $a_i\in T\subseteqq S$, $a_iT=a_iM=T$ $(M\subset T)$, with M mini-*

mal for a_i *with respect to* T, *for every* $i \in \mathbf{N}$. *Then, if*

$$M^{(1)} = M, \quad M^{(i)} = \{x \in M^{(i-1)} / a_{i-1} a_{i-2} \dots a_1 x \in M\} \quad (i \geqq 2),$$

we have:

1) $M^{(i+1)} \subset M^{(i)}$.
2) $T = a_{i+1} a_i \dots a_1 M^{(i+1)}$.
3) $M^{(i+1)}$ *is minimal for* $a_{i+1} a_i \dots a_1$ *with respect to* T.

PROOF. By induction on $i \in \mathbf{N}$. For $i=1$, $M^{(2)} \subseteqq M^{(1)}$. But $M \neq T$, hence, if $t \in T - M$, since $T = a_1 M^{(1)}$, there exists an element $m \in M^{(1)}$ such that $t = a_1 m$; so $m \notin M^{(2)}$ and $M^{(2)} \neq M^{(1)}$. Because $a_1 M = T$, we have $a_1 M^{(2)} = M$, hence $a_2 a_1 M^{(2)} = a_2 M = T$.

At last, $M^{(2)}$ is minimal for $a_2 a_1$. In fact, if $\overline{m} \in M^{(2)}$, $a_1(M^{(2)} - \overline{m})$ does not contain $a_1 \overline{m} (\in M)$, since M is minimal for a_1, hence $a_1(M^{(2)} - \overline{m}) \subset M$. Therefore $a_2 a_1 (M^{(2)} - \overline{m}) = a_2(M - a_1 \overline{m}) \neq S$, because M is minimal for a_2. Suppose the thesis is true for $(i-1)$, then it holds for i. In fact, $M^{(i+1)} \subseteqq M^{(i)}$. Since $M \neq T$, there exists an element $t \in T - M$; because, by induction, $T = a_i a_{i-1} \dots a_1 M^{(i)}$, there exists $m_0 \in M^{(i)}$ such that $a_i a_{i-1} \dots a_1 m_0 = t \notin M$, hence $m_0 \notin M^{(i+1)}$. Therefore $M^{(i+1)} \subset M^{(i)}$. By definition of $M^{(i+1)}$ and because $T = a_i a_{i-1} \dots a_1 M^{(i)}$, we have $a_i a_{i-1} \dots a_1 M^{(i+1)} = M$, hence $a_{i+1} a_i \dots a_1 M^{(i+1)} = a_{i+1} M = T$. At last, $M^{(i+1)}$ is minimal for $a_{i+1} a_i \dots a_1$. In fact, if $\overline{m} \in M^{(i+1)}$, $a_i a_{i-1} \dots a_1 \overline{m} \notin a_i a_{i-1} \dots a_1 \cdot (M^{(i+1)} - \overline{m})$ (because $M^{(i)}$ is minimal for $a_i a_{i-1} \dots a_1$), hence $a_i a_{i-1} \dots a_1 (M^{(i+1)} - \overline{m}) \neq M$ and $a_{i+1} a_i \dots a_1 (M^{(i+1)} - \overline{m}) \neq T$, because M is minimal for a_{i+1}. This completes the proof.

We remark that Theorem 2.3 generalizes an analogous theorem for increasing elements (Theorem 1.2 in [7]).

§ 3.

a) Let S be a semigroup with increasing elements. A problem in [6] was if among the minimal subsets relative to a left increasing element a of S there must be a subsemigroup.

The answer is negative.

In fact, let $S = aM$, with M minimal for a, M subsemigroup of S, then S contains a copy of the bicyclic semigroup $\mathscr{C}(p, q)$, by Theorem I.1 in [6], hence S contains idempotents; but there are semigroups with left increasing elements which have no idempotents (as Teissier semigroup, [3], vi, 3.14).

Therefore we can divide the set C of semigroups with left increasing elements into two classes, C_1 and C_2: if $S \in C_1$, there exists some left increasing element $a \in S$ such that $S = aM$, where $M (\neq S)$ is minimal for a and M is a subsemigroup of S; while $C_2 = C - C_1$. For the class C_1 the "Ljapin type" results hold which were proved in [6]. For the class C_2 we know only general results, as those proved in [2].

b) Let $D_L(S) = \{S_i\}$, $D_R(S) = \{D_i\}$, $i = 0, 1, \dots, 5$, be the left and the right, respectively, Szép's decompositions of a semigroup S (see [4]).

An open problem, from [4], was the following: if S is regular, $S_4 = D_1 \cup D_3$ and $D_4 = S_1 \cup S_3$? The problem is solved by means of the Γ-decomposition of S.

THEOREM 3.1. *Let S be a regular semigroup. Then:*

i) $S_4=D_1\cup D_3(\neq\emptyset)$ *and* $D_4=S_1\cup S_3(\neq\emptyset)$ *if and only if S contains an identity 1 and 1 is not $\mathcal{D}$-primitive.*

ii) *If* $1\notin S$, *then* $D_1\cup D_3\subseteq S_2$, $D_4\subseteq S_0\cup S_2$, $S_1\cup S_3\subseteq D_2$, $S_4\subseteq D_0\cup D_2$.

PROOF. From properties of Γ-decomposition of a regular semigroup (see [5]). As regards the relation between increasing elements and invertible elements in a semigroup S, the following theorem holds.

THEOREM 3.2. *If S contains no left [right] identity, then left [right] increasing elements are exactly right [left] invertible elements of S.*

PROOF. It is easy to prove that, if $S_5\neq\emptyset$, S contains a left identity. On the other hand right invertible elements of S are exactly the elements in $S_1\cup S_3\cup S_5$. Now $S_5=\emptyset$, hence the theorem holds.

c) Let S be a semilattice $\mathcal{Y}$ of subsemigroups S_α, i.e.

$$S=\bigcup_{\alpha\in\mathcal{Y}} S_\alpha,\quad S_\alpha\cap S_\beta=\emptyset\quad(\alpha\neq\beta),\quad S_\alpha S_\beta\subseteq S_{\alpha\beta},\quad \forall\alpha,\ \beta\in\mathcal{Y}. \tag{1}$$

Let a be a left increasing element of S and let $\bar\alpha\in\mathcal{Y}$ such that $a\in S_{\bar\alpha}$. Because $aS=S$, one proves that $\bar\alpha\alpha=\alpha$ (2) for every $\alpha\in\mathcal{Y}$. Hence $\mathcal{Y}$ has a last element, $\bar\alpha=\max\mathcal{Y}$.

If $aS'=S$, with $S'\subset S$, let us put $S'_{\bar\alpha}=S_{\bar\alpha}\cap S'$. We have $aS'_{\bar\alpha}=S_{\bar\alpha}$ (3), because of (1) and (2). If $S_{\bar\alpha}$ contains no left increasing elements (for itself), then $S'_{\bar\alpha}=S_{\bar\alpha}$ follows from (3), therefore a is a completely regular element (because a must be in the last component of $D_L(S_{\bar\alpha})$). But this is a contradiction, since a is a left increasing element of S.

At last, we have proven the following theorem.

THEOREM 3.3. *Let S be a semilattice $\mathcal{Y}$ of subsemigroups, according to* (1). *Then a necessary condition, in order that S contain left [right] increasing elements, is the following:*

1) *$\mathcal{Y}$ has a last element, $\bar\alpha=\max\mathcal{Y}$, and the left [right] increasing elements lie in $S_{\bar\alpha}$.*
2) *$S_{\bar\alpha}$ contains left [right] increasing elements for itself.*

Several corollaries follow from Theorem 3.3, as, for example, the result that separative semigroups contain no left [right] increasing elements, in consequence of Theorem II.6.4 in [8]. But now we generalize this last result by a direct proof. We recall the definition: a semigroup S is said left [right] separative if, for every $x, y\in S$,

$$(x^2=xy \text{ and } y^2=yx) \text{ implies } x=y$$

$$[(x^2=yx \text{ and } y^2=xy) \text{ implies } x=y].$$

THEOREM 3.4. *If S is a left [right] separative semigroup, then S contains no left [right] q-increasing elements.*

PROOF. Let $a\in S$ be a left q-increasing element, and let $T', T\subseteq S$, $T'\subset T$ such that $a\in T$ and $aT'=aT=T$. Then $a^2T=T$. Let $x\in T$ such that $a^2x=a$. We have:

$$a^2=a^2xa=a(axa),\quad (axa)^2=ax(a^2x)a=(axa)a,$$

hence $a=axa$, because S is left separative. Therefore $(xa^2)^2=x(a^2x)a^2=(xa^2)a$, $a^2=(axa)a=a(xa^2)$; hence $a=xa^2$. Then $a\in a^2S\cap Sa^2$, therefore a is completely regular (from Theorem IV, 1.2 of [8]), a contradiction for Theorem 1.9.

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THEOREMS OF THE ALTERNATIVE AND THEIR APPLICATIONS IN MULTIOBJECTIVE OPTIMIZATION

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1. Theorems of the alternative

An important technique for proving the existence of Lagrange or Kuhn—Tucker multipliers in optimization problems is to apply an appropriate theorem of the alternative, sometime called transposition theorem (see e.g., Avriel [1], Rockafellar [2]). In recent years these types of theorems have been developed and used in vector optimization (e.g., Borwein [3], Lehmann and Oettli [4]). In this section we give some general theorems of the alternative in finite dimensional spaces.

Let A and B be linear mappings from R^n into R^k and R^m, respectively. Let M be a convex cone in R^k, N be a convex cone in R^m. We say that a vector $\lambda \in R^k$ is nontrivial on M if $\langle \lambda, x\rangle \neq 0$ for some $x \in M$. The following seperation lemmas are needed:

LEMMA 1.1. *Suppose that H is a subspace in R^k. The following conditions are equivalent:*

(i) $H \cap \mathrm{ri}\, M = \emptyset$.

(ii) *There exists a vector $\lambda \in M^*$ which is nontrivial on M such that $\langle \lambda, h\rangle = 0$, for all $h \in H$.*

Here $\mathrm{ri}\, M$ *is the relative interior of M and $M^* = \{y \in R^k : \langle y, x\rangle \geqq 0$ for each $x \in M\}$.*

PROOF. It is obvious that (ii) implies (i). If (i) holds then in virtue of the separation theorem (Theorem 11.2 in [2]), there exists a hyperplane containing H and generating an open half-space which contains ri M. Let λ be a vector which is orthogonal to the hyperplane and belongs to the open half-space containing ri M, then $\langle \lambda, x\rangle > 0$ for all $x \in \mathrm{ri}\, M$ and $\langle \lambda, h\rangle = 0$ for every $h \in H$. Thus λ satisfies (ii) and the proof is complete.

LEMMA 1.2. *Assume that M_1 and M_2 are convex cones in R^k. If $M_1 \cap \mathrm{ri}\, M_2$ is empty, then there exists $\lambda \in (-M_1^*) \cap M_2^*$ which is nontrivial on $M_1 \cup M_2$. If in addition M_1 is closed then λ is nontrivial on M_2.*

PROOF. The proof of this lemma follows immediately from the separation theorem (Theorem 11.3 in [2]).

REMARK 1.1. The inverse of this lemma can be stated as follows: If there exists a vector λ from $(-M_1^*) \cap M_2^*$ such that it is nontrivial on M_2, then the intersection of M_1 with ri M_2 is empty.

COROLLARY 1.1 (Lehmann and Oettli's theorem [4].) *Let M_1, M_2 be non-*

empty convex cones with $\operatorname{int} M_2$ *nonempty. Then exactly one of the following two systems* (1) *and* (2) *has a solution:*

(1) $x \in M_1$, $x \in \operatorname{int} M_2$.
(2) $y \in -M_1^*$, $y \in M_2^*$, $y \neq 0$.

The proof of this corollary is immediate from Lemma 1.2 and Remark 1.1.

THEOREM 1.1. *For* B *and* N *as mentioned above and for an arbitrary vector* b *in* R^n, *the following statements are equivalent:*

(i) $\langle b, z \rangle \geqq 0$ *for every* $z \in R^n$ *with* $Bz' \in N$,
(ii) b *belongs to the closure* $\operatorname{cl} B'(N^*)$ *of* $B'(N^*)$, *where* $B'(N^*) = \{x \in R^n\colon x = B'y',\ y \in N^*\}$. *(The sign " ' " stands for transposition.)*

PROOF. (ii)$\Rightarrow$(i). Let $b = \lim B'y_i'$, $y_i \in N^*$. If $z \in R^n$ with $Bz' \in N$ then

$$\langle y_i, Bz' \rangle = \langle B'y_i', z \rangle \geqq 0.$$

Consequently $\langle b, z \rangle \geqq 0$.

Conversely, if b does not belong to $\operatorname{cl} B'(N^*)$, we can separate the compact set $\{b\}$ and the closed convex cone $\operatorname{cl} B'(N^*)$ by some vector $z \in R^n$, i.e.,

$$\langle b, z \rangle < 0, \quad \langle x, z \rangle \geqq 0 \tag{1}$$

for each $x \in \operatorname{cl} B'(N^*)$. The last relation shows that $\langle B'y', z \rangle \geqq 0$ for each $y \in N^*$. Hence $Bz' \in N^{**} = \operatorname{cl} N$. From this and (1) it follows that (i) does not hold. The theorem is proved.

REMARK 1.2. If $B'(N^*)$ is closed then (i) is equivalent to the following:

(ii') there exists a vector $\mu \in N^*$ such that $b = B'\mu'$. Moreover, if the inequality in (i) is strict for some $z \in R^n$ with $Bz' \in N$, then μ is nontrivial on N.

COROLLARY 1.2 (Farkas' theorem [1]). *Let* B *and* b *be as in Theorem* 1.1. *Then* $\langle b, z \rangle \geqq 0$ *holds for all vectors* z *satisfying* $Bz' \geqq 0$ *if and only if there exists a vector* $x \in R_+^m$ *such that* $b = B'x'$.

PROOF. Set $N = R_+^m$ to derive at once Farkas' theorem from Remark 1.2.

THEOREM 1.2. *For* A, B, M *and* N *as above, either*
(i) *there exists a vector* $z \in R^n$ *such that*

$$Az' \in \operatorname{ri} M \quad \text{and} \quad Bz' \in \operatorname{ri} N, \tag{2}$$

or

(ii) *there exist* $\lambda \in M^*$ *and* $\mu \in N^*$ *such that*

$$A'\lambda' + B'\mu' = 0 \tag{3}$$

and (λ, μ) *is nontrivial on* $M \times N$, *but never both.*

PROOF. The proof of this theorem presented in [7] uses Lemmas 1.1 and 1.2 and admits to know when λ is nontrivial on M and μ is nontrivial on N. Here we give another proof which is very simple by exploiting only Lemma 1.1. First note that if (ii) holds then obviously (i) does not hold. Suppose now (ii) does not hold, i.e., $\operatorname{ri}(M\times N)\cap H=\emptyset$ where $H=\{(Az', Bz')\in R^k\times R^m: z\in R^n\}$ is a linear subspace. By Lemma 1.1 there exists a vector $(\lambda, \mu)\in(M\times N)^*=M^*\times N^*$ such that it is nontrivial on $M\times N$ and $\langle(\lambda, \mu), h\rangle=0$ for all $h\in H$. Thus $\lambda Az'+\mu Bz'=0$ for all $z\in R^n$, therefore $A'\lambda'+B'\mu'=0$. The proof is complete.

THEOREM 1.3. *For A, B, M and N defined as above, if $A(B^{-1}(N))$ is closed then either*

(i) *there exists a vector $z\in R^n$ such that*

$$Az'\in\operatorname{ri} M \quad and \quad Bz'\in N, \tag{6}$$

or

(ii) *there exist $\lambda\in M^*$ which is nontrivial on M such that*

$$-A'\lambda'\in\operatorname{cl} B'(N^*), \tag{7}$$

but never both.

PROOF. If there exists a vector $z\in R^n$ which satisfies (6) then (7) is obviously impossible. Now, suppose that there is no $z\in R^n$ satisfying (6). It means that the cone $A(B^{-1}(N))=\{y\in R^k: y=Az'$ where $z\in R^n$ with $Bz'\in N\}$ has an empty intersection with $\operatorname{ri} M$. In virtue of Lemma 1.2, there exists a vector $\lambda\in M^*$ which is nontrivial on M and satisfies $\langle\lambda, y\rangle\leqq 0$ for all $y\in A(B^{-1}(N))$. That is $\lambda Az'\leqq 0$ for all $z\in R^n$ with $Bz'\in N$. Theorem 1.1 yields $-A'\lambda'\in\operatorname{cl} B'(N^*)$ completing the proof.

REMARK 1.3. If int M is nonempty then the assumption on the closedness of $A(B^{-1}(N))$ can be omitted. Indeed, $A(B^{-1}(N))\cap\operatorname{int} M=\emptyset$ implies $\operatorname{cl} A(B^{-1}(N))\cap$ $\cap\operatorname{int} M=\emptyset$ and the proof of the previous theorem is still fair.

REMARK 1.4. Many classical theorems of the alternative can be derived from Theorems 1.1, 1.2 and 1.3 by setting A, B, M and N in special forms. We refer the interested readers to [7].

2. Optimality conditions in multiobjective optimization

In this section we are dealing with the general multiobjective programming problem introduced and studied in [6]. Using the theorems of the alternative obtained in the previous section we shall derive optimality conditions for our problem. For, let f and g be vector functions defined and continuously differentiable on some open set $D\subset R^n$, with values in R^k and respectively R^m. Let $X\subset D$ denote the set of all $x\in D$ satisfying the constraint $g(x)\in -N$. Remember that program (P) is as follows: (P) Find $x^*\in X$ such that $f(x^*)\in\operatorname{Min}\{f(x): x\in X|M\}$, i.e., there is no $x\in X$ such that $f(x^*)-f(x)\in M\setminus 0$.

A point $x^*\in X$ is said to be a local solution to problem (P) if there exists a neighbourhood U of x^* in R^n so that $f(x^*)\in\operatorname{Min}[f(x): x\in X\cap U|M]$.

For the sake of simplicity in what follows it is assumed in that the interior of M and N is nonempty.

We say that a C^1 curve $\varphi\colon(-\varepsilon, \varepsilon)\to D$ is admissible relative to the given constraint g provided

(a) $\frac{d}{dt} f(\varphi(t)) \in -\operatorname{int} M$ for all $t \in (-\varepsilon, \varepsilon)$,

(b) there is a number δ, $0<\delta<\varepsilon$ such that $g(\varphi(t)) \in -N$ for all $t \in [0, \delta)$.

REMARK 2.1. If x^* is a local solution to problem (P) then there is no admissible curve passing through x^*, i.e., $\varphi(0)=x^*$ but in general the inverse fails.

THEOREM 2.1 (First-order necessary condition). *Let x^* be a local solution to problem* (P). *Then there exist vectors $\lambda^* \in M^*$ and $\mu^* \in N^*$ such that*

$$\lambda^* Df(x^*) + \mu^* Dg(x^*) = 0, \tag{8}$$

$$\langle \mu^*, g(x^*) \rangle = 0, \tag{9}$$

$$(\lambda^*, \mu^*) \neq 0. \tag{10}$$

PROOF. Define $N^*(x^*) = \{\mu \in N^* : \langle \mu, g(x^*) \rangle = 0\}$. If $N^*(x^*) = \{0\}$, take $\mu^* = 0$; we shall prove that there is a vector $\lambda^* \in M^*$ which satisfies $\lambda^* Df(x^*) = 0$ and $\lambda^* \neq 0$. Indeed, the assumption $N^*(x^*) = \{0\}$ assures that $\langle \mu, g(x^*) \rangle < 0$ for every $\mu \in N^*$, $\mu \neq 0$. The set $K = \{\mu \in N^* : \|\mu\| = 1\}$ is compact, hence there exists a positive ε such that $\langle \mu, g(x^*) \rangle < \varepsilon$ for all $\mu \in K$. By the continuity of g at x^* one can find a neighbourhood $U \subset D$ of x^* such that $\langle \mu, g(x) \rangle < -\varepsilon/2$ for every $x \in U$ and $\mu \in K$. In this way if $x \in U$ then $\langle \mu, g(x) \rangle < 0$ for every $\mu \in N^* | 0$. Hence $g(x) \in -\operatorname{ri} N^{**}$. Notice that N^{**} coincides with the closure of N, therefore $-\operatorname{ri} N^{**} \subset -N$ and so $g(x) \in -N$ for all $x \in U$. Thus we may consider $f(x^*) \in \operatorname{Min} [f(x) \colon x \in U | M]$. It is known that $Df(x^*)$ is a linear mapping from R^n to R^k. We claim that the image of $Df(x^*)$ has an empty intersection with $\operatorname{int} M$. For suppose the opposite; then there exists a vector $z \in R^n$ such that $Df(x^*) z' \in \operatorname{int} M$. Consider the curve $\varphi(t) \colon (-\varepsilon, \varepsilon) \to D$ which is given by relation $\varphi(t) = x^* - tz$. For sufficiently small ε, $\varphi(t) \in U$,

$$\frac{d}{dt} f(\varphi(t)) = Df(\varphi(t)) \frac{d}{dt} \varphi(t) = Df(\varphi(t))(-z) \in -\operatorname{int} M$$

for all $t \in (-\varepsilon, \varepsilon)$. This shows that $\varphi(t)$ is an admissible curve passing through x^* and x^* can not be a local solution to problem (P). Thus we have $\operatorname{Im}(Df(x^*)) \cap \cap \operatorname{int} M = \emptyset$. Lemma 2.1 is applied for the subspace $\operatorname{Im}(Df(x^*))$ and the cone M in R^k to obtain a vector $\lambda^* \in M^*$, $\lambda^* \neq 0$ so that $\langle y, \lambda^* \rangle = 0$ for each $y \in \operatorname{Im}(Df(x^*))$. Denoting $Df(x^*) = A$, we have $\langle \lambda^*, Az' \rangle = 0$ for all $z \in R^n$, therefore $\lambda^* A = 0$. By this it remains to prove the theorem by supposing $N^*(x^*) \neq \{0\}$. Observe first that if $N^{**}(x^*) = \{0\}$ then $N = \{0\}$. So it may be assumed that $N^{**}(x^*) \neq \{0\}$. Denoting $Dg(x^*) = B$, we assert that for every vector $z \in R^n$ satisfying $Bz' \in \in -\operatorname{ri} N^{**}(x^*)$, we can not have $Az' \in -\operatorname{int} M$. If this is true, or equivalently, there is no $z \in R^n$ such that $Bz' \in -\operatorname{ri} N^{**}(x^*)$ and $Az' \in -\operatorname{int} M$, then by Theorem 1.2 there exist vectors $\lambda^* \in M^*$ and $\mu^* \in (N^{**}(x^*))^* = N^*(x^*)$ which satisfy:

$$A' \lambda^{*\prime} + B' \mu^{*\prime} = 0 \quad \text{and} \quad (\lambda^*, \mu^*) \neq 0. \tag{11}$$

Since $\mu^* \in N^*(x^*) \subset N^*$, hence (11) gives (8) and (10), and besides, $\langle g(x^*), \mu^* \rangle = 0$ that is (9). In this way, to finish the proof we need only to verify that if $z \in R^n$ with $Bz' \in -\operatorname{ri} N^{**}(x^*)$, then $Az' \notin -\operatorname{int} M$. Assume this is wrong, i.e., for some vector $z \in R^n$ one has $Bz' \in -\operatorname{ri} N^{**}(x^*)$ and $Az' \in -\operatorname{int} M$. Construct a curve $\varphi(t)$

defined by the relation $\varphi(t)=x^*+tz$. We state that for sufficiently small positive δ,

(12) $$g(\varphi(t))\in -N \quad \text{for all} \quad t\in[0, \delta).$$

It is not difficult to see that the above relation holds if the following assertion holds: for every $\mu_0\in K$, there exists a neighbourhood V of μ_0 and a positive δ such that

(13) $$\langle g(\varphi(t)), \mu\rangle < 0 \quad \text{for all} \quad \mu\in V\cap K, \quad t\in[0, \delta).$$

For this, consider the expansion of $g(\varphi(t))$ at $t=0$,

(14) $$g(\varphi(t)) = g(x^*)+tBz'+o(tz),$$

where $o(tz)/t$ tends to 0 while t tends to 0 due to the differentiability of g. Suppose on the contrary that (13) is not true, i.e., there is a sequence $\{\mu_i\}$ converging to μ in K and a sequence $\{t_i\}$ $t_i>0$, converging to 0 such that

(15) $$\langle g(\varphi(t_i)), \mu_i\rangle \geqq 0.$$

It is clear that $\mu\in N^*(x^*)$, otherwise in view of $\langle g(x^*), \mu\rangle<0$ we would get the contradiction between (14) and (15) when t_i tends to 0. Further, as $\operatorname{int} N$ is non-empty, so is $\operatorname{int} N^{**}(x^*)$. $(N^*(x^*)\subset N^*$, therefore $N^{**}(x^*)\supset N^{**}=\operatorname{cl} N)$. From $Bz'\in -\operatorname{int} N^{**}(x)$ and $\mu\in N^*(x^*)$, $\mu\neq 0$, it follows that there exists a neighbourhood V of μ and a positive ε such that $\langle Bz', y\rangle < -\varepsilon$ for all $y\in V\cap K$. Taking in count (14) and (15) we have

$$t_i(-\varepsilon+\langle o(t_iz)/t_i, \mu_i\rangle) \geqq \langle g(x^*), \mu_i\rangle+t_i\langle Bz', \mu_i\rangle+\langle o(t_iz), \mu_i\rangle \geqq 0,$$

for i large enough. Thus $-\varepsilon+\langle o(t_iz)/t_i, \mu_i\rangle\geqq 0$ for i large enough. This contradicts the convergence of $o(t_iz)/t_i$ to 0 when i runs to ∞, and relation (12) is proved. Furthermore, from the continuous differentiability of $f(x)$ at x^* and from the assumption $Df(x^*)z\in -\operatorname{int} M$, it follows that

(16) $$\frac{d}{dt} f(\varphi(t)) = Df(\varphi(t))\frac{d}{dt}\varphi(t) = Df(\varphi(t))z\in -\operatorname{int} M,$$

for t small enough. Combine (16) with (12) to get the admissibility of the curve $\varphi(t)$ passing through x^*. This contradicts the assumption of the theorem and the proof is complete.

COROLLARY 2.1 (Smale's theorem [5]). *Suppose that x^* is a local solution to problem* $\operatorname{Min}[f(x)\colon x\in D,\ g(x)\leqq 0|R^k_+]$. *Then there exist a nonnegative vector λ^* and nonnegative numbers $\mu_1^*, \dots, \mu_m^*$ such that*

$$\lambda^*Df(x^*)+\Sigma\mu_i^*Dg_i(x^*) = 0, \quad \mu_i^* g_i(x^*) = 0, \quad i = 1, \dots, m.$$

PROOF. This theorem is immediate from Theorem 2.1 by setting $M=R^k_+$, $N=R^m_+$.

It should be remarked here that in the above necessary condition there is no guarantee that $\lambda^*\neq 0$.

The remainder of this section is concerning a regularity condition, called constraint qualification which is able to ensure that λ^* is nonzero.

Define $C(x^*)=\{z\in R^n\colon Dg(x^*)z\in -(N^*(x^*))^*\}$.

DEFINITION 2.1. A vector $z \neq 0$ is called a feasible direction vector from x^* if there exists a positive number δ such that $g(x^*+\alpha z) \in -N$ for all α, $0 \leqq \alpha < \delta$.

REMARK 2.2. If z is a feasible direction vector from x^* then z belongs to $C(x^*)$.

Recall that the closed cone of tangents to a nonempty set $A \subset R^n$ at the point $x \in A$, denoted by $S(A, x)$, is defined as $S(A, x) = \{z \in R^n$: there exists a sequence of vectors $\{x_i\} \subset A$ converging to x and a sequence of nonnegative numbers $\{t_i\}$ such that the sequence $\{t_i(x_i - x)\}$ converges to $z\}$. (See Lemma 3.5 in [1]).

LEMMA 2.1. *Suppose that x^* is a local solution to problem* (P), *then*

$$Df(x)^*(S(X, x^*)) \cap (-\operatorname{int} M) = \emptyset.$$

PROOF. Suppose the above intersection is nonempty, that is, there exists a sequence $\{x_i\}$ converging to x and a sequence of nonnegative numbers $\{t_i\}$ so that

$$z \in S(X, x^*), \quad z = \lim t_i(x_i - x), \quad Df(x^*)z \in -\operatorname{int} M.$$

By the differentiability assumption we have

$$t_i(f(x_i) - f(x^*)) = Df(x^*)t_i(x_i - x^*) + r(t_i(x_i - x^*))$$

where $r(t_i(x_i - x^*))$ tends to 0 as i tends to ∞. Hence for i large enough

$$t_i(f(x_i) - f(x^*)) \in \operatorname{int} M.$$

This contradicts the assumption that x^* is a local solution to problem (P) and the proof is complete.

THEOREM 2.2. *Let x^* be a local solution to problem* (P) *and suppose that* $\{[Dg(x^*)]'y' : y \in N^*(x^*)\}$ *is closed and*

$$Df(x^*)(C(x^*)) = Df(x^*)(S(X, x^*)).$$

Then there exist vectors $\lambda^* \in M$, $\lambda^* \neq 0$ and $\mu^* \in N^*$ such that (8) *and* (9) *are satisfied.*

PROOF. Suppose that x^* is a local solution to problem (P). In virtue of Lemma 2.1, the set

$$Df(x^*)(C(x^*)) \cap (-\operatorname{int} M) = Df(x^*)(S(X, x^*)) \cap (-\operatorname{int} M)$$

is empty. It means that if some vector $z \in R^n$ satisfies $Dg(x^*)z \in -(N^*(x^*))^*$, then $Df(x^*)z \notin -\operatorname{int} M$. The cone $[Dg(x^*)]'(N^*(x^*))$ is closed, therefore we can apply Theorem 1.3 to obtain vectors $\lambda^* \in M^*$ and $\mu \in (N^*(x^*))^{**} = N^*(x^*)$ such that $\lambda^* Df(x^*) + \mu^* Dg(x^*) = 0$, λ^* is nontrivial on M. Since $\mu^* \in N^*(x^*)$, relation (9) holds immediately and the proof is complete.

Now we are going to discuss a sufficient optimality condition for a convex multiobjective programming problem. In addition to the assumptions made on f and g and D we require also that D is a convex set, f is an M-convex function and g is an N-concave function on D. We need some preliminary results about convex functions.

LEMMA 2.2 (Theorem 25.1 in [2]). *Let h be a convex function from D into R^1. Suppose that h is differentiable at $x \in D$, then*

$$h(x)+Dh(x)(y-x)' \leqq h(x)$$

for all $y \in D$.

LEMMA 2.3. *Every local optimal solution to program* (P) *is its global optimal solution.*

PROOF. In virtue of the assumptions made above, it is abvious that X is a convex set. Suppose that x is a local solution to problem (P) and that it is not a global solution. This means that there exists another point y in X such that

$$f(x)-f(y) \in M \setminus 0. \tag{17}$$

Consider the point $z=tx+(1-t)y$, with t being in [0, 1]. By the convexity of X the point z is in X and by the M-convexity of f we have

$$tf(x)+(1-t)f(y)-f(z) \in M. \tag{18}$$

Consider the difference between $f(x)$ and $f(z)$:

$$f(x)-f(z) = (1-t)f(x)-(1-t)f(y)+tf(x)+(1-t)f(y)-f(z).$$

From (17) and (18) it follows that $f(x)-f(z) \in M \setminus 0$ if $t \neq 0$.

This contradicts the local optimality of x when t tends to 1. The proof is finished.

THEOREM 2.3 (Sufficient condition). *Assume f, g and D are as above, and there exist vectors $x^* \in D$, $\lambda^* \in M^*$, $\mu^* \in N^*$ satisfying:*

$$g(x^*) \in -N, \tag{19}$$

$$\lambda^* Df(x^*)+\mu^* Dg(x^*) = 0, \tag{20}$$

$$\mu^* g(x^*) = 0, \tag{21}$$

$\langle \lambda^*, z \rangle$ is strictly positive for each $z \in M$, $z \neq 0$.

Then x^* is a global solution to problem (P).

PROOF. First we prove that x^* is a solution to the following scalar programming problem

$$\min \lambda^* f(x) \quad \text{s.t.} \quad g(x) \in -N, \quad x \in D.$$

Indeed, as $\mu^* \in N^*$ we have $\mu^* g(x) \leqq 0$ for every $x \in D$ with $g(x) \in -N$. Hence

$$\lambda^* f(x) \geqq \lambda^* f(x)+\mu^* g(x) \tag{22}$$

for all $x \in D$ with $g(x) \in -N$.

Applying Lemma 2.2 to the scalar functions $\lambda^* f$ and $\mu^* g$ and using (22) we obtain

$$\lambda^* f(x) \geqq \lambda^* f(x^*)+\lambda^* Df(x^*)(x-x^*)'+\mu^* g(x^*)+\mu^* Dg(x^*)(x-x^*)'. \tag{23}$$

Rearranging yields

$$\lambda^* f(x) \geqq \lambda^* f(x^*)+\mu^* g(x^*)+[\lambda^* Df(x^*)+\mu^* Dg(x^*)](x-x)'$$

and by (20), (21) we have $\lambda^* f(x) \geqq \lambda^* f(x^*)$. From the assumption on λ^* it follows that λ^* belongs to the interior of M^*. Now our theorem is immediate from Lemma 6.1 in [6].

REMARK 2.3. The previous theorem can be proved directly by using Lemma 2.3 without using the results from scalar programming.

REMARK 2.4. The assumption $\langle\lambda^*, z\rangle > 0$ for each $z \in M$, $z \neq 0$ is important. If we require only $\lambda^* \in M^*$, $\lambda^* \neq 0$ then Theorem 2.3 may be false. For this, consider the following.

EXAMPLE 2.1. Let $D = \{t \in R^1 : 0 < t < 2\}$, $M = R^2_+$. Set $f(t) = (t, 0) : D \to R^2$ and delete g. Clearly

$$Df(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Taking $\lambda^* = (0, 1) \in M^*$, $\lambda^* \neq 0$ we have $\lambda^* Df(1) = 0$; however, the point 1 is not solution to the problem $\mathrm{Min}\,[f(t) : t \in D | R^2_+]$.

3. Duality under differentiability assumptions

In this section we study duality theory for multiobjective programming under differentiability assumptions. For the sake of simplicity it is assumed that $M = R^k_+$ and $N = R^m_+$. Suppose also f and g are differentiable functions from an open set $U \subset R^n$ into R^k and R^m, respectively. Consider the following two programs called primal and dual:

(P) $\mathrm{Min}\,[f(x) : x \in U,\ g(x) \in -R^m_+]$,
(D) $\mathrm{Max}\,[\Phi(x, y) : (x, y) \in V]$,

where $\Phi(x, y) = f(x) + e\langle y, g(x)\rangle$ and $V \subset U \times R^m_+$, $e = (1, \ldots, 1) \in R^k$.

THEOREM 3.1. *If x^* solves problem* (P) *and if $(\bar{x}, \bar{y}) \in V$ implies the minimum condition*

$$\Phi(\bar{x}, \bar{y}) \in \mathrm{Min}\,[\Phi(x, \bar{y}) : x \in U], \tag{1}$$

then $(x^*, y^*) \in V$ with y^* satisfying $\langle y^*, g(x^*)\rangle = 0$ solves problem (D).

PROOF. As $y \in R^m_+$ and $g(x^*) \in -R^m_+$ we have $\langle y, g(x^*)\rangle \leqq 0$, for every $y \in R^m_+$. Therefore

$$f(x^*) - f(x^*) - e\langle y, g(x^*)\rangle = f(x^*) - \Phi(x^*, y) \in M \setminus 0$$

for all $y \in R^m_+$. Since $x^* \in U$, $f(x^*) - a \in M \setminus 0$ for all $a \in \mathrm{Inf}\,[\Phi(x, y) : x \in U]$. Consider the following programming problem:

$$\mathrm{Sup}\,\big[\mathrm{Inf}\,[\Phi(x,y) : x \in U] : y \in R^m_+\big]. \tag{2}$$

We prove that (x^*, y^*) solves this problem. Indeed, $(x^*, y^*) \in V$ therefore

$$\Phi(x^*, y^*) \in \mathrm{Min}\,[\Phi(x, y^*) : x \in U].$$

Moreover, $f(x^*) = \Phi(x^*, y^*) - \Phi(x^*, y) \in M \setminus 0$, for each $y \in R^n_+$. Hence (x^*, y^*)

solves (2). Now if $(\bar{x}, \bar{y}) \in V$ then (1) holds and by (2) the value $\Phi(\bar{x}, \bar{y})$ can not be greater than $f(x^*)$. Thus $f(x^*) = \Phi(x^*, y^*)$ is an optimal value of problem (D).

REMARK 3.1. In virtue of Theorem 2.1 we see that if x^* solves problem (P) then there exists $\lambda^* \in R^k_+$ and $y^* \in R^m_+$ such that

(3) $$\lambda^* Df(x^*) + y^* Dg(x^*) = 0, \quad \langle y^*, g(x^*) \rangle = 0.$$

If $\lambda^* \neq 0$ and the minimum condition is satisfied then the pair (x^*, y^*) solves the problem

(4) $$\mathrm{Max}\,[\Phi(x, y)\colon (x, y) \in U \times R^m_+,\ \lambda^* D_x \Phi(x, y) = 0],$$

where D_x is the derivative with respect to the variable x. Indeed, without loss of generality we may assume that $\langle \lambda^*, e \rangle = 1$. Now consider

$$V = \{(x, y) \in U \times R^m_+ \colon \lambda^* D_x \Phi(x, y) = 0\}.$$

We show that $(x^*, y^*) \in V$. Indeed

$$\lambda^* D_x \Phi(x^*, y^*) = \lambda^* [Df(x^*) + e D_x \langle y^*, g(x^*) \rangle] =$$

$$= \lambda^* Df(x^*) + \langle \lambda^*, e \rangle D_x \langle y^*, g(x^*) \rangle = \lambda^* Df(x^*) + y^* Dg(x^*) = 0$$

by (3). Now the remark is immediate from Theorem 3.1.

THEOREM 3.2. *If in addition to the assumption made at the beginning of this section we require that f is R^k_+-convex, g is R^m_+-concave, U is convex and (x^*, y^*) solves the following program:*

$$\mathrm{Max}\,[\Phi(x, y)\colon (x, y) \in U \times R^m_+],$$

and satisfies $\lambda D_x \Phi(x, y) = 0$ for some $\lambda \in \operatorname{int} R^k_+$, then x^* solves problem (P).

PROOF. We begin by showing that

(5) $$g(x^*) \in -R^m_+.$$

For, suppose that (5) is not true. Then we could choose $y \in R^m_+$ such that

$$e[\langle y, g(x^*) \rangle - \langle y^*, g(x^*) \rangle] \in M \setminus 0.$$

This contradicts the optimality of (x^*, y^*). Moreover, $\langle y^*, g(x^*) \rangle = 0$ because

(6) $$\langle y^*, g(x^*) \rangle < 0$$

would imply $\Phi(x^*, y^*/2) - \Phi(x^*, y^*) \in M \setminus 0$. Furthermore, $\lambda D_x \Phi(x^*, y^*) = 0$ implies

(7) $$\lambda^* Df(x^*) + \mu^* Dg(x^*) = 0,$$

with $\lambda^* = \lambda$, $\mu^* = \langle \lambda, e \rangle y^*$. Combine (5), (6) and (7) and then apply Theorem 2.3 to complete the proof.

THEOREM 3.3. *If the assumptions in the previous theorem hold and* (x^*, y^*) *solves problem* (4) *with* $\lambda^* \in \operatorname{int} R^k_+$ *and* $g(x^*) \in -R^m_+$, *then* x^* *solves problem* (P).

PROOF. In view of Theorem 2.3 and by (7), in order to prove Theorem 3.3 it suffices only to remark that $\langle y^*, g(x^*)\rangle = 0$.

The proof is complete.

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A NOTE ON THE APPROXIMATION IN $C_{2\pi}$ BY LINEAR POLYNOMIAL OPERATORS

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1. Let H_n be the set of all trigonometric polynomials of degree at most n and H_n^* the class of all linear operators $T\colon C_{2\pi}\to H_n$. We define for $g\in C_{2\pi}$

$$E_n^L(g) = \inf_{T\in H_n^*} \max_x |T[g](x)-g(x)|, \quad (n = 1, 2, \ldots).$$

Jackson [2] proved his classical theorem

$$E_n^L(g) \leqq C_n \omega\left(g, \frac{2\pi}{m}\right), \quad C_n \text{ bounded,} \tag{1}$$

$m=2n+1$. From Korneicuk [6] and [7] it is known that C_n is asymptotically not less than 1.

To get an upper bound for C_n it is natural to examine some suitable linear operators. For example Kis and Nevai [5] investigated

$$\tilde{S}_{kn}[g](t) = \frac{1}{2^k} \sum_{j=0}^{k} \tilde{S}_n[g]\left(t+\frac{k-2j}{m}\pi\right), \tag{2}$$

where $\tilde{S}_n[g]$ denotes the trigonometrical interpolation polynomial of $g\in C_{2\pi}$ at $m=2n+1$ equidistant nodes, and

$$S_{kn}[g](t) = \frac{1}{2^k} \sum_{j=0}^{k} S_n[g]\left(t+\frac{k-2j}{m}\pi\right), \tag{3}$$

where $S_n[g]$ denotes the n^{th} partial sum of the trigonometric Fourier series. In [3] and [4] Kis has investigated the analogue of $\tilde{S}_{kn}$ if $m=2n$. Kis and Nevai [5] conclude that among these operators the Bernstein—Rogosinski means S_{1n} seem to be the best ones. If we write

$$|S_{1n}[g](t)-g(t)| \leqq C_{1n}\omega\left(g, \frac{2\pi}{m}\right), \tag{4}$$

then we can assume $C_n \leqq C_{1n}$. In [5] (49) it is shown that it is possible to choose C_{1n} such that

$$C_{1n} \leqq \frac{m}{\pi} \int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt, \quad \tilde{M}_{1n}(t) = \sup_{\substack{g\in C_{2\pi}\\ g\not\equiv \text{const.}}} \frac{|\tilde{S}_{1n}[g](t)-g(t)|}{\omega\left(g, \dfrac{2\pi}{m}\right)}. \tag{5}$$

Furthermore it is cited (Theorem 6(51), Theorem 7) that

$$\frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt \leqq 1+\frac{1}{\pi}\int_{\pi/2}^{\pi}\frac{\sin t}{t}\,dt \quad (<1.15), \tag{6}$$

but this is not correct. Apart from the upper bound being 1. 1531... the basic estimations [5](33) and [5](40) used in proof of (6) are false.

We point out this and then state

THEOREM 1.

$$\frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt = 1+\frac{1}{\pi}\left\{\sum_{k=1}^{n}\frac{1}{k}\sin\frac{k\pi}{m}-\sum_{k=1}^{n/2}\frac{1}{k}\tan\frac{k\pi}{m}\right\}, \quad n \text{ even}, \tag{7}$$

$$\frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt =$$

$$= 1+\frac{1}{\pi}\left\{\sum_{k=1}^{n}\frac{1}{k}\sin\frac{k\pi}{m}-\sum_{k=1}^{(n-1)/2}\frac{1}{k}\tan\frac{k\pi}{m}+\frac{\pi}{m}+2\sum_{k=1}^{n}\frac{(-1)^k}{k}\sin\frac{k\pi}{m}\right\}, \quad n \text{ odd}.$$

Using Riemann sums it is obvious that for $n\to\infty$ we have

$$\frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt \to c = 1+\frac{1}{\pi}\left\{\int_0^{\pi/2}\frac{\sin t}{t}\,dt-\int_0^{\pi/4}\frac{\tan t}{t}\,dt\right\} = 1.1660\ldots$$

Moreover we prove

THEOREM 2. $\frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt<c$, $n=0, 1, 2, \ldots$, *more precisely*

$$\frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt < c-\frac{1}{m}\frac{1}{\pi}+\frac{1}{m^2}\left(\frac{3}{4}-\frac{1}{\pi}\right), \quad n \text{ even}, \tag{8}$$

$$\frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt < c-\frac{1}{m}\frac{1}{\pi}+\frac{1}{m^2}\left(1-\frac{3}{2\pi}\right), \quad n \text{ odd}.$$

REMARKS. (i) It is shown in the proof that in (8) equality holds by adding a term of order $O\left(\frac{1}{m^2}\right)$.

(ii) To get an upper bound for C_n in (1), n small, we evaluate

$$C_{1n}^* = \frac{m}{\pi}\int_0^{\pi/m} \tilde{M}_{1n}(t)\,dt$$

by Theorem 1. Apart from $C_{10}^*=1$ we obtain (rounded):

n	1	2	3	4	5	6	7	8	9	10	11	12
C_{1n}^*	1.058	1.107	1.121	1.132	1.138	1.142	1.145	1.148	1.150	1.151	1.152	1.154

2. Proof of Theorem 1

We start with (34) in [5], $n>0$. Throughout this paper we can assume $\tau=0$. (34) can be written as

$$2\sigma_0(t) = 1+d_n\left(\frac{\pi}{m}+t\right)+\sum_{i=2}^{n}\left[d_n\left(\frac{2i-1}{m}\pi+t\right)-d_n\left(\frac{2i-1}{m}\pi-t\right)\right].$$

But for even i, $0<t<\frac{\pi}{m}$, it can be shown that

$$d_n\left(\frac{2i-1}{m}\pi+t\right)-d_n\left(\frac{2i-1}{m}\pi-t\right)=$$

$$=\frac{1}{m}\cos\frac{mt}{2}\left\{\operatorname{cosec}\left(\frac{2i-1}{2m}\pi-\frac{t}{2}\right)-\operatorname{cosec}\left(\frac{2i-1}{2m}\pi+\frac{t}{2}\right)\right\}>0,$$

in contrast to the proof of (36) in [5].

Now to prove Theorem 1 we rewrite (34) in [5], $n>0$,

$$\sigma_0(t)=\frac{1}{2}+\frac{1}{2}\left\{\sum_{i=1}^{n}\left[d_n\left(t+\frac{2i-1}{m}\pi\right)-d_n\left(t-\frac{2i+1}{m}\pi\right)\right]+d_n(t-\pi)\right\}.$$

Using [5](37) and (11) we conclude

$$\tilde{M}_{1n}(t) = 1+\alpha(t)+\beta(t),$$

$$\alpha(t)=\frac{1}{2}\sum_{i=1}^{n}\left[d_n\left(t+\frac{2i-1}{m}\pi\right)-d_n\left(t-\frac{2i+1}{m}\pi\right)\right], \tag{9}$$

$$\beta(t)=\frac{1}{2}[1-(-1)^n]\,d_n(t-\pi).$$

Let n be even. Then $\beta\equiv 0$ and by the aid of [1](23) and (24) we get

$$\alpha(t)=\frac{1}{m}\left\{\sum_{j=1}^{n/2}\sin\left[(2j-1)\left(\frac{\pi}{m}-t\right)\right]\operatorname{ctg}\left[(2j-1)\frac{\pi}{2m}\right]-\right.$$

$$\left.-\sum_{j=1}^{n/2}\sin\left[2j\left(\frac{\pi}{m}-t\right)\right]\tan\left[2j\frac{\pi}{2m}\right]\right\}.$$

The first part of Theorem 1 now follows from

$$\int_0^{\pi/m}\sin\left[k\left(\frac{\pi}{m}-t\right)\right]\operatorname{ctg}\frac{k\pi}{2m}\,dt=\frac{1}{k}\sin\frac{k\pi}{m}, \tag{10}$$

$$\int_0^{\pi/m}\sin\left[k\left(\frac{\pi}{m}-t\right)\right]\tan\frac{k\pi}{2m}\,dt=\frac{2}{k}\tan\frac{k\pi}{2m}-\frac{1}{k}\sin\frac{k\pi}{m}.$$

In the same way we get, for n odd,

$$\alpha(t)=\frac{1}{m}\left\{\sum_{j=1}^{(n+1)/2}\sin\left[(2j-1)\left(\frac{\pi}{m}-t\right)\right]\operatorname{ctg}\left[(2j-1)\frac{\pi}{2m}\right]-\right.$$
$$\left.-\sum_{j=1}^{(n-1)/2}\sin\left[2j\left(\frac{\pi}{m}-t\right)\right]\tan\left[2j\frac{\pi}{2m}\right]\right\}, \tag{11}$$

which implies by (10) the first two sums in the second part of Theorem 1. But now we have to take into account in (9), for n odd,

$$\beta(t)=d_n(t-\pi)=\frac{1}{m}+\frac{2}{m}\sum_{i=1}^{n}\cos i(t-\pi)=\frac{1}{m}+\frac{2}{m}\sum_{i=1}^{n}(-1)^i\cos it.$$

By integrating we immediately obtain the remaining expression in Theorem 1, for n odd.

3. Proof of Theorem 2

Using the abbreviations

$$f(x)=\frac{\sin x}{x},\quad g(x)=\frac{\tan x}{x},\quad h=\frac{\pi}{m},$$

$\left(f'(x)<0,\ f''(x)<0,\ 0<x<\pi/2\right)$, we obtain, for any n, $n>2$,

$$\sum_{k=1}^{n}\frac{1}{k}\sin\frac{k\pi}{m}=h\left\{\frac{1}{2}f\left(\frac{\pi}{m}\right)+\sum_{k=2}^{n-1}f\left(\frac{k\pi}{m}\right)+\frac{1}{2}f\left(\frac{n\pi}{m}\right)\right\}+\frac{h}{2}\left\{f\left(\frac{\pi}{m}\right)+f\left(\frac{n\pi}{m}\right)\right\}\leqq$$
$$\leqq\int_0^{\pi/2}f(x)\,dx+\left\{\frac{h}{2}f\left(\frac{\pi}{m}\right)-\int_0^{\pi/m}f(x)\,dx\right\}+\left\{\frac{h}{2}f\left(\frac{n\pi}{m}\right)-\int_{n\pi/m}^{\pi/2}f(x)\,dx\right\}\leqq$$
$$\leqq\int_0^{\pi/2}f(x)\,dx-\frac{\pi}{2m}+\frac{1}{2}\left(\frac{h}{2}\right)^2\left|f'\left(\frac{\pi}{2}\right)\right|\leqq\int_0^{\pi/2}f(x)\,dx-\frac{\pi}{2m}+\frac{h^2}{2\pi^2},$$

since $f'(\pi/2)=-4/\pi^2$.

Similarly we get, noticing that $g'(x)>0$, $g''(x)>0$, $0<x<\pi/4$ (n even)

$$\sum_{k=1}^{n/2}\frac{1}{k}\tan\frac{k\pi}{m}=h\left\{\frac{1}{2}g\left(\frac{\pi}{m}\right)+\sum_{k=2}^{(n/2)-1}g\left(\frac{k\pi}{m}\right)+\frac{1}{2}g\left(\frac{n\pi}{2m}\right)\right\}+\frac{h}{2}\left\{g\left(\frac{\pi}{m}\right)+g\left(\frac{n\pi}{2m}\right)\right\}\geqq$$
$$\geqq\int_0^{\pi/4}g(x)\,dx+\left\{\frac{h}{2}g\left(\frac{\pi}{m}\right)-\int_0^{\pi/m}g(x)\,dx\right\}+\left\{\frac{h}{2}g\left(\frac{n\pi}{2m}\right)-\int_{n\pi/2m}^{\pi/4}g(x)\,dx\right\}\geqq$$
$$\geqq\int_0^{\pi/4}g(x)\,dx-\frac{\pi}{2m}+\left\{\frac{h}{4}g\left(\frac{n\pi}{2m}\right)-\int_{n\pi/2m}^{\pi/4}g(x)\,dx\right\}+\frac{h}{4}g\left(\frac{n\pi}{2m}\right)\geqq$$
$$\geqq\int_0^{\pi/4}g(x)\,dx-\frac{\pi}{2m}-\frac{1}{2}\left(\frac{h}{4}\right)^2\frac{16}{\pi^2}\left(\frac{\pi}{2}-1\right)+\frac{h}{4}\left(g\left(\frac{\pi}{4}\right)-\frac{h}{4}\frac{16}{\pi^2}\left(\frac{\pi}{2}-1\right)\right),$$

since $g'\left(\frac{\pi}{4}\right)=\frac{16}{\pi^2}\left(\frac{\pi}{2}-1\right)$. Thus using Theorem 1 this yields Theorem 2 (n even). Note that in both cases by the trapezoid-rule we could reach equality by taking into account a term of order $O(h^2)$. Now let n be odd. We proceed in the following manner:

$$\sum_{k=1}^{(n-1)/2}\frac{1}{k}\tan\frac{k\pi}{m}=$$

$$=h\left\{\frac{1}{2}g\left(\frac{\pi}{m}\right)+\sum_{k=1}^{(n-3)/2}g\left(\frac{k\pi}{m}\right)+\frac{1}{2}g\left(\frac{n-1}{2}\frac{\pi}{m}\right)\right\}+\frac{h}{2}\left\{g\left(\frac{\pi}{m}\right)+g\left(\frac{n-1}{2}\frac{\pi}{m}\right)\right\}\geqq$$

$$\geqq\int_0^{\pi/4}g(x)\,dx+\left\{\frac{h}{2}g\left(\frac{\pi}{m}\right)-\int_0^{\pi/m}g(x)\,dx\right\}+\left\{\frac{h}{2}g\left(\frac{n-1}{2}\frac{\pi}{m}\right)-\int_{(n-1)/2\,(\pi/m)}^{\pi/4}g(x)\,dx\right\}\geqq$$

$$\geqq\int_0^{\pi/4}g(x)\,dx-\frac{\pi}{2m}+\left\{\frac{h}{2}g\left(\frac{n-1}{2}\frac{\pi}{m}\right)-\int_{(n-1)/2\,(\pi/m)}^{(\pi/4)-(h/4)}g(x)\,dx\right\}-\int_{(\pi/4)-(h/4)}^{\pi/4}g(x)\,dx\geqq$$

$$\geqq\int_0^{\pi/4}g(x)\,dx-\frac{\pi}{2m}-\frac{1}{2}\left(\frac{h}{2}\right)^2\frac{16}{\pi^2}\left(\frac{\pi}{2}-1\right)-\frac{h}{4}g\left(\frac{\pi}{4}\right).$$

Thus we conclude by (11), (10) and (12)

$$\frac{m}{\pi}\int_0^{\pi/m}\alpha(t)\,dt\leqq\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin t}{t}\,dt-\frac{1}{\pi}\int_0^{\pi/4}\frac{\tan t}{t}\,dt+\frac{1}{m\pi}+\frac{1}{m^2}\left(1-\frac{3}{2\pi}\right).$$

But since $\beta(t)=d_n(t-\pi)\leqq-\frac{1}{m}\cos\frac{mt}{2}$, $0\leqq t\leqq\frac{\pi}{m}$, it follows that

$$\frac{m}{\pi}\int_0^{\pi/m}\beta(t)\,dt\leqq-\frac{1}{\pi}\frac{2}{m}.$$

Now from (9) we obtain Theorem 2 (n odd).

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ALMOST r-PARACONTACT CONNECTIONS

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In this paper we determine the general family of almost r-paracontact connections on a manifold M and give the conditions for an almost r-paracontact connection to be symmetric. Moreover, we deal with curvature tensors of such connections.

1. An almost r-paracontact connection on a manifold M

DEFINITION 1.1 [1]. If, on a manifold M, there exist a tensor field φ of the type (1, 1), r vector fields $\xi_1, \dots, \xi_r$ and r 1-forms $\eta^1, \dots, \eta^r$ such that

(1.1) $$\eta^i(\xi_j) = \delta^i_j, \quad i, j = 1, \dots, r$$

(1.2) $$\varphi(\xi_i) = 0, \quad i = 1, \dots, r$$

(1.3) $$\eta^i \circ \varphi = 0, \quad i = 1, \dots, r$$

(1.4) $$\varphi^2 = Id - \eta^i \otimes \xi_i$$

where the summation convention is employed on repeated indices, then the structure $\Sigma = (\varphi, \xi_{(i)}, \eta^{(i)})$ $i=1, \dots, r$ is said to be an almost r-paracontact structure on M.

If, moreover, there exists a positive definite Riemannian metric g on M, such that:

(1.5) $\eta^i(X) = g(X, \xi_i)$, $i=1, \dots, r$, for any vector field X,

(1.6) $g(\varphi X, \varphi Y) = g(X, Y) - \sum_i \eta^i(X)\eta^i(Y)$ for any vector fields X, Y

then $\Sigma = (\varphi, \xi_{(i)}, \eta^{(i)}, g)$ $i=1, \dots, r$ is called an *almost r-paracontact metric structure* on M. The metric g is called compatible Riemannian metric.

DEFINITION 1.2. Suppose that on a manifold M there is given an almost r-paracontact structure $\Sigma = (\varphi, \xi_{(i)}, \eta^{(i)})$ $i=1, \dots, r$. A linear connection Γ on a manifold M given by its covariant derivative Σ is said to be an almost r-paracontact or simply Σ-connection if and only if

(1.7) $$\nabla_X \varphi = 0$$

(1.8) $$\nabla_X \eta^i = 0, \quad i = 1, \dots, r$$

for every vector field X.

REMARK 1. From (1.7) and (1.8) it follows that

(1.9) $$\nabla_X \xi_i = 0, \quad i = 1, \dots, r.$$

If, moreover, for an almost r-paracontact metric structure $\Sigma=(\varphi, \xi_{(i)}, \eta^{(i)}, g)$ $i=1, \dots, r$, a Σ-connection Γ satisfies

(1.10) $$\nabla_X g = 0$$

then this connection is called a *metric Σ-connection.*

Now, suppose that on a manifold M with an almost r-paracontact structure Σ there is given a linear connection Γ by its covariant derivative ∇. We are going to find an almost r-paracontact connection $\hat{\Gamma}$ given by $\hat{\nabla}$ in the form

(1.11) $$\hat{\nabla}_X = \nabla_X + S_X$$

where S is a tensor field of the type (1, 2) and $S_X(Y)=S(X, Y)$. From (1.11) we have

(1.12) $$\hat{\nabla}_X Y = \nabla_X Y + S_X(Y)$$

and for φ we have

$$(\hat{\nabla}_X \varphi)Y = \hat{\nabla}_X(\varphi Y) - \varphi \hat{\nabla}_X Y = \nabla_X(\varphi Y) + (S_X \varphi)Y - \varphi(\nabla_X Y) -$$
$$-(\varphi S_X)Y = (\nabla_X \varphi)Y + (S_X \circ \varphi - \varphi \circ S_X)Y$$

or

(1.13) $$\hat{\nabla}_X \varphi = \nabla_X \varphi + S_X \circ \varphi - \varphi \circ S_X.$$

For η^i we have

$$(\hat{\nabla}_X \eta^i)Y = X(\eta^i(Y)) - \eta^i(\hat{\nabla}_X Y) = X(\eta^i(Y)) - \eta^i(\nabla_X Y) - (\eta^i \circ S_X)Y = (\nabla_X \eta^i - \eta^i \circ S_X)Y$$

or

(1.14) $$\hat{\nabla}_X \eta^i = \nabla_X \eta^i - \eta^i \circ S_X, \quad i = 1, \dots, r.$$

Since $\hat{\Gamma}$ is a Σ-connection then because of (1.7), (1.8) and (1.9) the formulas (1.12), (1.13) and (1.14) become

(1.15) $$\nabla_X \xi_i = -S_X(\xi_i), \quad i = 1, \dots, r,$$

(1.16) $$\nabla_X \varphi = \varphi \circ S_X - S_X \circ \varphi,$$

(1.17) $$\nabla_X \eta^i = \eta^i \circ S_X, \quad i = 1, \dots, r.$$

From (1.16) and because of (1.4) we have

$$(\nabla_X \varphi) \circ \varphi = \varphi \circ S_X \circ \varphi - S_X + \eta^i \otimes S_X(\xi_i)$$

and on account of (1.15) we have

$$S_X - \varphi \circ S_X \circ \varphi = -(\nabla_X \varphi) \circ \varphi - \eta^i \otimes \nabla_X \xi_i.$$

Because of

(1.18) $$\varphi \circ \nabla_X \varphi + \nabla_X \varphi \circ \varphi = -\nabla_X \eta^i \otimes \xi_i - \eta^i \otimes \nabla_X \xi_i$$

the above identity is of the form

(1.19) $$S_X - \varphi \circ S_X \circ \varphi = \varphi \circ \nabla_X \varphi + \nabla_X \eta^i \otimes \xi_i.$$

Now we introduce the following tensor fields of the type (2, 2)

$$A = \tfrac{1}{2}(\mathrm{Id}\otimes\mathrm{Id}-\varphi\otimes\varphi), \tag{1.20}$$

$$B = \tfrac{1}{2}(\mathrm{Id}\otimes\mathrm{Id}+\varphi\otimes\varphi), \tag{1.21}$$

$$C = \tfrac{1}{2}(\eta^i\otimes\xi_i\otimes\mathrm{Id}+\mathrm{Id}\otimes\eta^i\otimes\xi_i-\eta^i\otimes\xi_i\otimes\eta^j\otimes\xi_j). \tag{1.22}$$

The operations AB, AS, AS_X are expressed locally as follows: $A^{ij}_{kl}B^{ml}_{in}$, $A^{ij}_{kl}S^l_{mi}$, $A^{ij}_{kl}S^l_{xi}$, respectively.

We have the following relations

$$A+B = \mathrm{Id}\otimes\mathrm{Id},\quad AA = A-\tfrac{1}{2}C,\quad BB = B-\tfrac{1}{2}C, \tag{1.23}$$

$$AB = BA = AC = CA = BC = CB = CC = \tfrac{1}{2}C.$$

Let us put

$$F = A+C, \tag{1.24}$$

$$H = B-C. \tag{1.25}$$

From (1.23) we have

$$F+H = \mathrm{Id}\otimes\mathrm{Id},\quad HH = H,\quad FF = F,\quad FH = HF = 0. \tag{1.26}$$

Now the equation (1.19) may be written in the form

$$AS_X = \tfrac{1}{2}\varphi\circ\nabla_X\varphi+\tfrac{1}{2}\nabla_X\eta^i\otimes\xi_i. \tag{1.27}$$

Operating with C on (1.27) and because of (1.23) we have

$$CS_X = \tfrac{1}{2}(\mathrm{Id}\otimes\eta^i\otimes\xi_i+\eta^i\otimes\xi_i\otimes\mathrm{Id}-\eta^i\otimes\xi_i\otimes\eta^j\otimes\xi_j)(\varphi\nabla_X\varphi+\nabla_X\eta^i\otimes\xi_i) =$$

$$= \tfrac{1}{2}(\eta^i\circ\varphi\circ\nabla_X\varphi\otimes\xi_i+\eta^i\otimes(\varphi\circ\nabla_X\varphi)(\xi_i)-\eta^i\otimes(\eta^j\circ\varphi\circ\nabla_X\varphi)(\xi_i)\xi_j+$$

$$+\eta^i(\xi_j)\nabla_X\eta^j\otimes\xi_i+\eta^i\otimes\xi_j(\nabla_X\eta^j)(\xi_i)-(\nabla_X\eta^i)(\xi_j)\eta^k(\xi_i)\eta^j\otimes\xi_k) =$$

$$=\tfrac{1}{2}(\eta^i\otimes(\varphi\circ\nabla_X\varphi)(\xi_i)+\nabla_X\eta^i\otimes\xi_i)$$

and because of

$$(\nabla_X\varphi)(\xi_i)+\varphi(\nabla_X\xi_i) = 0 \tag{1.28}$$

we obtain

$$CS_X = \tfrac{1}{2}(-\eta^i\otimes\nabla_X\xi_i+\eta^j(\nabla_X\xi_i)\eta^i\otimes\xi_j+\nabla_X\eta^i\otimes\xi_i). \tag{1.29}$$

The equation (1.27) is equivalent to the following

$$(A+C)S_X = FS_X = \tfrac{1}{2}\varphi\circ\nabla_X\varphi+\nabla_X\eta^i\otimes\xi_i-\tfrac{1}{2}\eta^i\otimes\nabla_X\xi_i+\tfrac{1}{2}\eta^j(\nabla_X\xi_i)\eta^i\otimes\xi_j. \tag{1.30}$$

Now we need the following

LEMMA 1.1 [2]. *If F is a projection operator i.e. $FF=F$ and $H=\mathrm{Id}-F$ is such that $HH=H$ and $FH=HF=0$, then all solutions of the equation $Fx=y$ are of the form $x=y+Hw$ where w is arbitrary.*

Hence, and from (1.26) the solution of (1.30) is

$$S_X = \tfrac{1}{2}\varphi\circ\nabla_X\varphi+\nabla_X\eta^i\otimes\xi_i-\tfrac{1}{2}\eta^i\otimes\nabla_X\xi_i+\tfrac{1}{2}\eta^j(\nabla_X\xi_i)\eta^i\otimes\xi_j+HP_X \tag{1.31}$$

where P_X is a tensor field of the type (1, 1) such that $P_X(Y)=P(X, Y)$ for an arbitrary tensor field P of the type (1, 2) on a manifold M. Now we can state the following

THEOREM 1.1. *The general family of the almost r-paracontact connections on a manifold M with an almost r-paracontact structure $\Sigma=(\varphi, \xi_{(i)}, \eta^{(i)})$ $i=1, \dots, r$ is given by*

$$\hat{\nabla}_X = \nabla_X+\tfrac{1}{2}\varphi\circ\nabla_X\varphi+\nabla_X\eta^i\otimes\xi_i-\tfrac{1}{2}\eta^i\otimes\nabla_X\xi_i+\tfrac{1}{2}\eta^j(\nabla_X\xi_i)\eta^i\otimes\xi_j+HP_X \tag{1.32}$$

where ∇ is the covariant derivative with respect to an arbitrary initial connection Γ on M.

COROLLARY 1.1. *If the initial connection Γ is a Σ-connection, then the general family of an almost r-paracontact connections is given by*

$$\tilde{\nabla}_X = \nabla_X+HP_X \tag{1.33}$$

P_X being arbitrary tensor field of the type (1, 1).

Now, suppose that on M there is given an almost r-paracontact metric structure $\Sigma=(\varphi, \xi_{(i)}, \eta^{(i)}, g)$ $i=1, \dots, r$ and a Riemannian connection Γ given by ∇, associated with g. Then, from (1.5) and (1.6) we have

$$g(\xi_i, \xi_j) = \delta_{ij}, \quad i, j = 1, \dots, r \tag{1.34}$$

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{1.35}$$

for any vector fields X, Y. From (1.34) we obtain

$$g(\nabla_Z\xi_i, \xi_j)+g(\xi_i, \nabla_Z\xi_j) = 0, \tag{1.36}$$

or

$$\eta^j(\nabla_Z\xi_i)+\eta^i(\nabla_Z\xi_j) = 0. \tag{1.37}$$

From (1.5) we have

$$\eta^i(\nabla_Y X)+(\nabla_Y\eta^i)X = g(\nabla_Y X, \xi_i)+g(X, \nabla_Y\xi_i),$$

or

$$(\nabla_Y\eta^i)(X) = g(X, \nabla_Y\xi_i). \tag{1.38}$$

From (1.6) we have

$$g(\nabla_Z(\varphi X), \varphi Y)+g(\varphi X, \nabla_Z(\varphi Y)) = g(\nabla_Z X, Y)+g(X, \nabla_Z Y)-$$

$$-\sum_i (\nabla_Z\eta^i)(X)\eta^i(Y)-\sum_i \eta^i(X)(\nabla_Z\eta^i)(Y)-\sum_i \eta^i(\nabla_Z X)\eta^i(Y)-\sum_i \eta^i(X)\eta^i(\nabla_Z Y)$$

and making use of (1.5) we obtain

$$g(\nabla_Z(\varphi X), \varphi Y)+g(\varphi X, \nabla_Z(\varphi Y))-g(\varphi^2(\nabla_Z X), Y)-\\ -g(X, \varphi^2(\nabla_Z Y))+\sum_i (\nabla_Z\eta^i)(X)\eta^i(Y)+\sum_i (\nabla_Z\eta^i)(Y)\eta^i(X) = 0. \tag{1.39}$$

For the tensor field g and a connection $\hat{\nabla}$ given by (1.11) we have

$$(\hat{\nabla}_Z g)(X, Y) = -g(S_Z(X), Y)-g(X, S_Z(Y)). \tag{1.40}$$

Now, consider a Σ-connection $\hat{\nabla}$ given by (1.11), where S_X is given by (1.31) with $P_X=0$. Then, using (1.35), (1.37) and (1.39) we obtain

$$\begin{aligned}(\hat{\nabla}_Z g)(X, Y) &= -\tfrac{1}{2} g(\varphi(\nabla_Z\varphi)X, Y)+\tfrac{1}{2}\eta^i(X)g(\nabla_Z\xi_i, Y)-\nabla_Z\eta^i(X)g(\xi_i, Y)-\\ &-\tfrac{1}{2} g(X, (\varphi\nabla_Z\varphi)Y)+\tfrac{1}{2}\eta^i(Y)g(X, \nabla_Z\xi_i)-\nabla_Z\eta^i(Y)g(X, \xi_i)-\\ &-\tfrac{1}{2}\eta^i(\nabla_Z\xi_j)\eta^j(X)g(\xi_i, Y)-\tfrac{1}{2}\eta^i(\nabla_Z\xi_j)\eta^j(Y)g(X, \xi_i) =\\ &= -\tfrac{1}{2} g(\varphi\nabla_Z(\varphi X)-\varphi^2\nabla_Z X, Y)-\tfrac{1}{2} g(X, \varphi\nabla_Z(\varphi Y)-\varphi^2\nabla_Z Y)-\\ &-\tfrac{1}{2}\sum_i (\nabla_Z\eta^i)(X)\eta^i(Y)-\tfrac{1}{2}\sum_i (\nabla_Z\eta^i)(Y)\eta^i(X)-\\ &-\tfrac{1}{2}\sum_i \eta^i(\nabla_Z\xi_j)\eta^j(X)\eta^i(Y)-\tfrac{1}{2}\sum_i \eta^i(\nabla_Z\xi_j)\eta^j(Y)\eta^i(X) =\\ &= -\tfrac{1}{2} g(\nabla_Z(\varphi X), \varphi Y)+\tfrac{1}{2} g(\varphi^2\nabla_Z X, Y)-\tfrac{1}{2} g(\varphi X, \nabla_Z(\varphi Y))+\\ &+\tfrac{1}{2} g(X, \varphi^2\nabla_Z Y)-\tfrac{1}{2}\sum_i (\nabla_Z\eta^i)(X)\eta^i(Y)-\tfrac{1}{2}\sum_i (\nabla_Z\eta^i)(Y)\eta^i(X)-\\ &-\tfrac{1}{2}\sum_{i,j} \eta^i(X)\eta^j(Y)(\eta^i(\nabla_Z\xi_j)+\eta^j(\nabla_Z\xi_i)),\end{aligned}$$

or

$$\hat{\nabla}_Z g = 0. \tag{1.41}$$

Hence we get

THEOREM 1.2. *If on a manifold M there is an almost r-paracontact metric structure $\Sigma=(\varphi, \xi_{(i)}, \eta^{(i)}, g)$ $i=1, \dots, r$ and a Riemannian connection Γ given by ∇ induced by the metric g, then the connection $\hat{\Gamma}$ given by $\hat{\nabla}$ of the form:*

$$\hat{\nabla}_X = \nabla_X+\tfrac{1}{2}\varphi\circ\nabla_X\varphi+\nabla_X\eta^i\otimes\xi_i-\tfrac{1}{2}\eta^i\otimes\nabla_X\xi_i+\tfrac{1}{2}\eta^j(\nabla_X\xi_i)\eta^i\otimes\xi_j \tag{1.42}$$

is metric Σ-connection on M, i.e. $\hat{\nabla}\varphi=0$, $\hat{\nabla}\eta^i=0$, $\hat{\nabla}\xi_i=0$, $\hat{\nabla}g=0$.

2. The torsion tensor of a Σ-connection and normality of an almost r-paracontact structure Σ on M

For an almost r-paracontact structure $\Sigma=(\varphi, \xi_{(i)}, \eta^{(i)})$ $i=1, \dots, r$ on M we have defined in [1] the following tensor fields

$$\overset{1}{N}(X, Y) = N_\varphi(X, Y) - 2\,d\eta^i(X, Y)\xi_i, \tag{2.1}$$

$$\overset{2}{N}{}^i(X, Y) = (\alpha_{\varphi X}\eta^i)Y - (\alpha_{\varphi Y}\eta^i)X, \tag{2.2}$$

$$\overset{3}{N}_i(X) = -(\alpha_{\xi_i}\varphi)X, \tag{2.3}$$

$$\overset{4}{N}{}_i^j(X) = -(\alpha_{\xi_i}\eta^j)X \tag{2.4}$$

where N_φ is the Nijenhuis tensor field for φ and α_X denotes the Lie derivative with respect to a vector field X. If on M there is given a linear connection Γ by its covariant derivative ∇ with the torsion tensor field T, then the above identities may be written as follows

$$\overset{1}{N}(X, Y) = \varphi T(X, \varphi Y) + \varphi T(\varphi X, Y) - T(X, Y) - T(\varphi X, \varphi Y) + \tag{2.5}$$
$$+(\nabla_X\varphi)\varphi Y - (\nabla_Y\varphi)\varphi X + (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X - \eta^i(X)\nabla_Y\xi_i + \eta^i(Y)\nabla_X\xi_i,$$

(2.6)
$$\overset{2}{N}{}^i(X, Y) = (\nabla_X\eta^i)\varphi Y - (\nabla_{\varphi Y}\eta^i)X - (\nabla_Y\eta^i)\varphi X + (\nabla_{\varphi X}\eta^i)Y + \eta^i T(\varphi X, Y) + \eta^i T(X, \varphi Y),$$

$$\overset{3}{N}_i(X) = (\nabla_{\xi_i}\varphi)X - (\nabla_X\varphi)\xi_i - \nabla_{\varphi X}\xi_i + \varphi T(X, \xi_i) - T(\varphi X, \xi_i), \tag{2.7}$$

$$\overset{4}{N}{}_i^j(X) = (\nabla_X\eta^j)\xi_i - (\nabla_{\xi_i}\eta^j)X + \eta^j T(\xi_i, X). \tag{2.8}$$

Now, suppose that a given connection Γ is almost r-paracontact. Then we have

$$\overset{1}{N}(X, Y) = \varphi T(X, \varphi Y) + \varphi T(\varphi X, Y) - T(X, Y) - T(\varphi X, \varphi Y), \tag{2.9}$$

$$\overset{2}{N}{}^i(X, Y) = \eta^i(T(\varphi X, Y) + T(X, \varphi Y)), \tag{2.10}$$

$$\overset{3}{N}_i(X) = \varphi T(X, \xi_i) - T(\varphi X, \xi_i), \tag{2.11}$$

$$\overset{4}{N}{}_i^j(X) = \eta^j T(\xi_i, X). \tag{2.12}$$

Moreover, since $\nabla_X(\eta^i(Y)) = \eta^i(\nabla_X Y)$ we have

$$2\,d\eta^i(X, Y) = X(\eta^i(Y)) - Y(\eta^i(X)) - \eta^i[X, Y] = \eta^i(\nabla_X Y - \nabla_Y X - [X, Y]),$$

or

$$2\,d\eta^i = \eta^i \circ T, \quad i = 1, \dots, r. \tag{2.13}$$

We have the following projection operators

$$a = \eta^i \otimes \xi_i, \quad b = \mathrm{Id} - \eta^i \otimes \xi_i \tag{2.14}$$

where a projects on the distribution determined by the vector fields $\xi_1, \dots, \xi_r$ and

b projects on the distribution defined by $\eta^1=\ldots=\eta^r=0$. They have the following properties

$$a+b=\mathrm{Id},\quad aa=a,\quad bb=b,\quad ab=ba=0 \tag{2.15}$$

and moreover

$$\varphi\varphi=b,\quad a\varphi=\varphi a=0,\quad b\varphi=\varphi b=\varphi. \tag{2.16}$$

From (1.21), (1.22) and (2.14) we have $H=\frac{1}{2}(b\otimes b+\varphi\otimes\varphi)$, or

$$\varphi\otimes\varphi=2H-b\otimes b. \tag{2.17}$$

Now we express the torsion tensor field T of the connection Γ by means of the tensor field $\overset{1}{N}$ and the operator H. Since $T_X(T_X(Y)=T(X,Y))$ is a tensor field of the type (1, 1), then from (2.17) we have

$$\varphi T_Y\varphi=2HT_Y-bT_Yb \tag{2.18}$$

and for any vector field X we have

$$\varphi T(Y,\varphi X)=2H_XT_Y-bT(Y,bX). \tag{2.19}$$

From (2.18) we have

$$\varphi T_{\varphi Y}\varphi^2=2HT_{\varphi Y}\varphi-bT_{\varphi Y}\varphi b$$

or on account of (2.16) we obtain

$$\varphi T_{\varphi Y}b=2HT_{\varphi Y}\varphi-bT_{\varphi Y}\varphi. \tag{2.20}$$

Making use of (2.20) we have

$$H_YT_{\varphi X}\varphi-H_XT_{\varphi Y}\varphi-\tfrac{1}{2}(bT_{\varphi X}\varphi)Y+\tfrac{1}{2}(bT_{\varphi Y}\varphi)X=\tfrac{1}{2}(\varphi T_{\varphi X}b)Y-\tfrac{1}{2}(\varphi T_{\varphi Y}b)X.$$

Hence

$$bT_{\varphi X}\varphi)Y=\tfrac{1}{2}(\varphi T_{bY}\varphi)X-\tfrac{1}{2}(\varphi T_{bX}\varphi)Y+H_YT_{\varphi X}\varphi-H_XT_{\varphi Y}\varphi \tag{2.21}$$

and using (2.19) we have

$$(bT_{\varphi X}\varphi)Y=\tfrac{1}{2}(bT_{bX}b)Y-\tfrac{1}{2}(bT_{bY}b)X-H_Y(T_{bX}-T_{\varphi X}\varphi)+H_X(T_{bY}-T_{\varphi Y}\varphi). \tag{2.22}$$

We may write (2.9) in the following way

$$\overset{1}{N}(X,Y)=-T_XY-(aT_{\varphi X}\varphi)Y-(bT_{\varphi X}\varphi)Y+(\varphi T_X\varphi)Y-(\varphi T_Y\varphi)X.$$

Using (2.19) and (2.22) we obtain

$$\begin{aligned}\overset{1}{N}(X,Y)&=-T_XY-(aT_{\varphi X}\varphi)Y-\tfrac{1}{2}(bT_{bX}b)Y+\tfrac{1}{2}(bT_{bY}b)X-\\
&-(bT_Xb)Y+(bT_Yb)X+H_Y(T_{bX}-T_{\varphi X}\varphi)-H_X(T_{bY}-T_{\varphi Y}\varphi)+2H_YT_X-2H_XT_Y=\\
&=-T_XY-(aT_{\varphi X}\varphi)Y-(bT_{bX}b)Y-(bT_Xb)Y-(bT_{bX})Y+\\
&+H_Y(T_{2X+bX}-T_{\varphi X}\varphi)-H_X(T_{2Y+bY}-T_{\varphi Y}\varphi)\end{aligned}$$

or

$$\overset{1}{N}(X, Y) = -(aT_X)Y - (aT_{\varphi X}\varphi)Y - (bT_{X+bX})(Y+bY) + \\ + H_Y(T_{2X+bX} - T_{\varphi X}\varphi) - H_X(T_{2Y+bY} - T_{\varphi Y}\varphi). \tag{2.23}$$

From (2.9) we have

$$a\overset{1}{N}(X, Y) = -(aT_X)Y - (aT_{\varphi X}\varphi)Y. \tag{2.24}$$

Since $a+b=\mathrm{Id}$, then from (2.23) and (2.24) we obtain

$$(bT_{X+bX})(Y+bY) = -b\overset{1}{N}(X, Y) + H_Y(T_{2X+bX} - T_{\varphi X}\varphi) - H_X(T_{2Y+bY} - T_{\varphi Y}\varphi),$$

or

$$T_{X+bX}(Y+bY) = -b\overset{1}{N}(X, Y) + aT_{X+bX}(Y+bY) + \\ + H_Y(T_{2X+bX} - T_{\varphi X}\varphi) - H_X(T_{2Y+bY} - T_{\varphi Y}\varphi). \tag{2.25}$$

From (2.13) and (2.14) we have

$$a \circ T = 2d\eta^i \otimes \xi_i. \tag{2.26}$$

Now, inserting $\frac{1}{2}X + \frac{1}{2}aX$ instead of X and $\frac{1}{2}Y + \frac{1}{2}aY$ instead of Y into (2.25) and making use of the following identity $2H_{\frac{1}{2}X+\frac{1}{2}aX}H_Y = T_X T_Y$ and (2.26) we obtain

$$T(X, Y) = -\tfrac{1}{4}\, b\overset{1}{N}(X+aX, Y+aY) + 2d\eta^i(X, Y)\,\xi_i + \\ + H_Y(T_{X-\frac{1}{4}bX} - T_{\frac{1}{4}\varphi X}\varphi) - H_X(T_{Y-\frac{1}{4}bY} - T_{\frac{1}{4}\varphi Y}\varphi). \tag{2.27}$$

Now consider the connection $\tilde{\Gamma}$ given by

$$\tilde{\nabla}_X = \nabla_X - H(T_{X-\frac{1}{4}bX} - T_{\frac{1}{4}\varphi X}\varphi). \tag{2.28}$$

In virtue of Corollary 1.1 this connection is also an almost r-paracontact connection. The torsion tensor field $\tilde{T}$ of this connection is

$$\tilde{T}(X, Y) = 2\, d\eta^i(X, Y)\xi_i - \tfrac{1}{4}\, b\overset{1}{N}(X+aX, Y+aY). \tag{2.29}$$

Hence we obtained

THEOREM 2.1. *On a manifold M with an almost r-paracontact structure* $\Sigma = (\varphi, \xi_{(i)}, \eta^{(i)})$ $i=1, \dots, r$ *there exists a Σ-connection with the torsion tensor field given by* (2.29).

We also have

THEOREM 2.2. *A tensor field T of the type* $(1, 2)$ *with* $T(X, Y) = -T(Y, X)$ *is the torsion tensor field of an almost r-paracontact connection if and only if it satisfies the relations* (2.9) *and* (2.13).

PROOF. If the tensor field T satisfies the relations (2.9) and (2.13) then it satisfies the relation (2.27), since this relation was obtained using only (2.9) and (2.13).

There exists an almost r-paracontact connection $\tilde{\nabla}$ with its torsion tensor field $\tilde{T}$ given by (2.29).

Consider the connection

$$\nabla_X = \tilde{\nabla}_X + H(T_{X-\frac{1}{4}bX} - T_{\frac{1}{2}\varphi X}\varphi) \tag{2.30}$$

which is an almost r-paracontact connection whose torsion tensor field is exactly T.

We also have

THEOREM 2.3. *On a manifold M with an almost r-paracontact structure $\Sigma = (\varphi, \xi_{(i)}, \eta^{(i)})$ $i=1, \dots, r$ there exists a symmetric Σ-connection if and only if the following conditions are satisfied:*

(i) *all 1-forms η^i are closed.*

(ii) *Σ is normal, i.e. $\overset{1}{N}=0$.*

PROOF. Suppose that there exists a symmetric Σ-connection on M. Then, on account of (2.9) and (2.13) the conditions (i) and (ii) are satisfied. Conversely, if (i) and (ii) are satisfied, then according to Theorem 2.1 there exists a symmetric Σ-connection on M.

Now, in virtue of Theorem 7 from [1] we have

THEOREM 2.4. *On a manifold M with an almost r-paracontact structure $\Sigma = (\varphi, \xi_{(i)}, \eta^{(i)})$ $i=1, \dots, r$ there exists a symmetric Σ-connection Γ if and only if $[\xi_i, \xi_j]=0$, $(\alpha_X \eta^i)Y=0$ for $X \in D^+$ (or D^-), $Y \in D^-$ (or D^+) $i, j=1, \dots, r$ and the distributions $D^+, D^-, D^+ \oplus D^0, D^- \oplus D^0$ are integrable, where $D^+=\{X; \varphi X=X\}$, $D^-=\{X; \varphi X=-X\}$, $D^0=\{X; \varphi X=0\}$.*

3. The curvature tensor field of an almost r-paracontact connection

Now we give some properties of the curvature tensor field of an almost r-paracontact connection on a manifold M. The curvature tensor field of a linear connection Γ is defined by the formula

$$R_{XY} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{3.1}$$

If the connection Γ is an almost r-paracontact connection, then we obtain the following properties of the curvature tensor field R_{XY}

$$R_{XY}\xi_i = 0, \quad i = 1, \dots, r. \tag{3.2}$$

For this connection we have: $\eta^i(\nabla_X Y) = X\eta^i(Y)$ and

$$\eta^i R_{XY} Z = \eta^i(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) = XY\eta^i(Z) - YX\eta^i(Z) - [XY]\eta^i(Z).$$

Hence

$$\eta^i \circ R_{XY} = 0. \tag{3.3}$$

Because of $\nabla_X(\varphi Y) = \varphi \nabla_X Y$ we have

$$R_{XY}(\varphi Z) = \nabla_X(\varphi \nabla_Y Z) - \nabla_Y(\varphi \nabla_X Z) - \varphi \nabla_{[X,Y]} Z = \varphi(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z.$$

Hence we get

$$R_{XY}\circ\varphi = \varphi\circ R_{XY}. \tag{3.4}$$

On account of (3.3), (3.4) and (1.4) we have

$$\varphi\circ R_{XY}\circ\varphi = R_{XY}. \tag{3.5}$$

From (1.20), (1.21), (1.22), (1.24) and (1.25) we have

$$F\varphi = \tfrac{1}{2}(\varphi-\varphi^3), \quad H\varphi = \tfrac{1}{2}(\varphi+\varphi^3). \tag{3.6}$$

Hence, and because of (3.5) we have

$$FR_{XY}Z = F\varphi R_{XY}\varphi Z = \tfrac{1}{2}(\mathrm{Id}-\varphi^2)\varphi R_{XY}\varphi Z = \tfrac{1}{2}(\mathrm{Id}-\varphi^2)R_{XY}Z = \tfrac{1}{2}\eta^i(R_{XY}Z)\xi_i$$

and because of (3.3) we get

$$FR_{XY} = 0. \tag{3.7}$$

Now

$$HR_{XY}Z = H\varphi R_{XY}\varphi Z = \tfrac{1}{2}(\varphi+\varphi^3)R_{XY}\varphi Z =$$

$$= \tfrac{1}{2}(\mathrm{Id}+\varphi^2)\varphi R_{XY}\varphi Z = \tfrac{1}{2}(\mathrm{Id}+\varphi^2)R_{XY}Z = R_{XY}Z+\tfrac{1}{2}\eta^i(R_{XY}Z)\xi_i.$$

Because of (3.3) we have

$$HR_{XY} = R_{XY}. \tag{3.8}$$

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ÜBER EINE KENNZEICHNUNG DER ALTERNIERENDEN GRUPPE VOM GRADE 5

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In [2], S. 160—165 hat Lüneburg eine Kennzeichnung der alternierenden Gruppe $\mathfrak{A}_5$ bewiesen, die nach seinen Angaben auf die Diplomarbeit von Assion zurückgeht. Dieser Kennzeichnungssatz kann folgendermaßen ausgesprochen werden.

SATZ (Assion). *G sei eine endliche Gruppe mit einer Konjugiertenklasse $\mathfrak{K}$ von Untergruppen der Ordnung* 3, *die folgende Eigenschaften hat:*

(E1) *Das Erzeugnis von je zwei Untergruppen aus $\mathfrak{K}$ ist isomorph zu einer Untergruppe von $\mathfrak{A}_5$.*

(E2) *$\mathfrak{K}$ erzeugt G.*

(E3) *Es gibt $A, B\in\mathfrak{K}$ mit $\langle A, B\rangle\cong\mathfrak{A}_5$.*

Dann ist G isomorph zu $\mathfrak{A}_5$.

Ich will zeigen, daß sich der in [2] angegebene Beweis für diesen Satz ganz erheblich vereinfachen läßt. Die Schlußweise in Teil b) des Lüneburgschen Beweises ([2], S. 163, Mitte) liefert nämlich in Verbindung mit simplen Zusatzüberlegungen ein Resultat, das den Satz von Assion als Spezialfall enthält. Überdies zeigt der unten formulierte Satz, daß die Voraussetzung der Endlichkeit der Gruppe G im Satz von Assion überflüssig ist.

SATZ. *Jede Gruppe G, die eine Konjugiertenklasse $\mathfrak{K}$ von Untergruppen der Ordnung* 3 *mit den Eigenschaften* (E1) *und* (E2) *besitzt, ist isomorph zu $\mathfrak{A}_5$ oder zu Z_3 oder zum Erzeugnis der Streckungen eines affinen Raumes passender Dimension über $GF(4)$.*

BEWEIS. Ehe wir in die Detailüberlegungen eintreten, wollen wir darauf hinweisen, daß (E1) gleichwertig mit der folgenden Aussage ist: Das Erzeugnis von je zwei verschiedenen Untergruppen aus $\mathfrak{K}$ ist isomorph zu $\mathfrak{A}_4$ oder zu $\mathfrak{A}_5$.

a) *Bei beliebig vorgegebenen $A, B\in\mathfrak{K}$ gilt:*

1) *Jede Untergruppe der Ordnung* 3 *von $\langle A, B\rangle$ gehört zu $\mathfrak{K}$.*

2) *Jedes $C\in\mathfrak{K}$, das $\langle A, B\rangle$ normalisiert, gehört zu $\langle A, B\rangle$.*

BEWEIS. 1) Wegen (E1) ist jede Untergruppe der Ordnung 3 von $\langle A, B\rangle$ in $\langle A, B\rangle$ zu A konjugiert und darum Mitglied von $\mathfrak{K}$. 2) Weil die Anzahl der Untergruppen der Ordnung 3 von $\langle A, B\rangle$ kein Vielfaches von 3 ist, gibt es eine Untergruppe $D\leqq\langle A, B\rangle$ der Ordnung 3, die von C normalisiert wird. Nach 1) gilt $D\in\mathfrak{K}$. (E1) liefert jetzt $C=D$.

b) *Vorgelegt seien* $A, B, C \in \mathfrak{K}$ *mit* $\langle A, C\rangle, \langle B, C\rangle \cong \mathfrak{A}_4$. *Man wähle Erzeugende* a, b, c *für* A, B, C *mit* $o(ac)=o(bc)=2$. *Dann gilt:*

1) $\langle A, B\rangle \cong \mathfrak{A}_5 \Rightarrow C \leqq \langle A, B\rangle$.

2) $\langle A, B\rangle \cong \mathfrak{A}_4 \Rightarrow C \leqq N(\langle A, B\rangle'$ *oder* $o(ab)=2$.

Beweis. Aus $\langle b, c\rangle \cong \mathfrak{A}_4$ und $o(bc)=2$ folgt $o(b^{-1}c)=3$. Nach a1) ist $\langle b^{-1}c\rangle \in \mathfrak{K}$ und folglich $\langle a, b^{-1}c\rangle$ gemäß (E1) isomorph zu einer Untergruppe von $\mathfrak{A}_5$. Daher gilt $o:=o(ab^{-1}c)\in\{1, 2, 3, 5\}$. Wir gehen jetzt diese vier denkbaren Fälle der Reihe nach durch und überzeugen uns jeweils von der Gültigkeit der Behauptung. Dabei dürfen wir natürlich $A \neq B$ annehmen. Außerdem machen wir uns die in jedem Falle gültige Inklusion $\langle A, B\rangle \leqq N(\langle A, B\rangle')$ zunutze.

$o=1$ ist unmöglich, denn sonst wäre $ac=a(ba^{-1})$, im Widerspruch zu $o(ac)=2$. Die Beziehung $(ab^{-1}c)^2=ab^{-1}cab^{-1}c=ab^{-1}a^{-1}c^{-1}b^{-1}c=ab^{-1}a^{-1}bc^2=[a^{-1}, b]c^{-1}$ zeigt erstens, daß die Behauptung im Falle $o=2$ zutreffend ist. Dann ist nämlich $c=[a^{-1}, b]\in\langle a, b\rangle$. Zweitens erhält man bei $o=3$ $c^{-1}ba^{-1}=(ab^{-1}c)^{-1}=(ab^{-1}c)^2=$ $=[a^{-1}, b]c^{-1}$ und daraus $(ba^{-1})^c=[a^{-1}, b]$. Da außerdem $(a^{-1}b)^c=ab^{-1}$ gilt, normalisiert c die Untergruppe $K:=\langle a^{-1}b, ab^{-1}\rangle$. Bei $o(ab)\neq 3$ folgt aus $(ab)^3=$ $=(aba)(bab)=(ab^{-1})(a^{-1}b)^{-1}(ab^{-1})^{-1}(a^{-1}b)\in K$ zunächst $ab\in K$ und dann $a=(a^{-1}b)$ $(ab)^{-1}\in K$, $b=a(a^{-1}b)\in K$, Also gilt $K=\langle a, b\rangle$. a2) liefert jetzt $c\in\langle a, b\rangle$. Im Falle $o(ab)=3$ ist $\langle a, b\rangle \cong \mathfrak{A}_4$ und $K=\langle a, b\rangle'$, unsere Behauptung also ebenfalls zutreffend. Im letzten Falle $o=5$ gilt $\langle a, b^{-1}c\rangle \cong \mathfrak{A}_5$. Deutet man $a, b^{-1}c$ als Dreierzyklen von $\mathfrak{A}_5$, so sieht man, daß $a^{b^{-1}c}a$ involutorisch ist. Deshalb gilt $1=a^{b^{-1}c}aab^{-1}ca=c^{-1}bab^{-1}cac^{-1}bab^{-1}ca=c^{-1}(bab^{-1}a^{-1}b^{-1}c^{-1}ab^{-1}a^{-1}c)c=$ $=([b^{-1}, a^{-1}]b^{-1}(b^{-1})^{a^{-1}c})^c$ und dann $(b^{a^{-1}})^c\in\langle a, b\rangle$. Da außerdem $(a^{-1}b)^c=ab^{-1}$ ist, bleibt die Untergruppe $K:=\langle b^{a^{-1}}, a^{-1}b\rangle$ beim Transformieren mit c innerhalb von $\langle a, b\rangle$. Im Falle $o(ab)\neq 2$ folgt aus $(ab)^2=aba^{-1}a^{-1}b=b^{a^{-1}}a^{-1}b\in K$ zunächst $ab\in K$ und dann — ähnlich wie oben — $K=\langle a, b\rangle$. Da $\langle a, b\rangle$ endlich ist, wird $\langle a, b\rangle$ von c normalisiert. Mit a2) folgt $c\in\langle a, b\rangle$. Bei $o(ab)=2$ ist $\langle a, b\rangle \cong \mathfrak{A}_4$, unsere Behauptung also wiederum richtig.

c) *Sind* $A, B \in \mathfrak{K}$ *verschieden, so gibt es höchstens eine zu* $\mathfrak{A}_5$ *isomorphe Untergruppe von* G, *die* A *und* B *enthält.*

Beweis. $H, K \leqq G$ seien zu $\mathfrak{A}_5$ isomorphe Untergruppen mit $A, B \leqq H, K$. Weil A und B verschieden sind, gibt es eine Untergruppe $C \leqq H$ der Ordnung 3 mit $\langle A, C\rangle$, $\langle B, C\rangle \cong \mathfrak{A}_4$ und $\langle A, B, C\rangle = H$. Da C in H zu A konjugiert ist, gilt $C\in\mathfrak{K}$. Ebenso findet man ein $D\in\mathfrak{K}$ mit $\langle A, D\rangle, \langle B, D\rangle \cong \mathfrak{A}_4$ und $\langle A, B, D\rangle = K$. Wir führen jetzt die Annahme $H \neq K$ zum Widerspruch. Aus $A < \langle A, B\rangle \leqq H \cap K <$ $< H \cong \mathfrak{A}_5$ folgt mit (E1) $\langle A, B\rangle \cong \mathfrak{A}_4$. Da außerdem $\langle A, C\rangle, \langle A, D\rangle \cong \mathfrak{A}_4$ gilt, gibt es Erzeugende a, b, c, d für A, B, C, D mit $o(ab)=o(ac)=o(ad)=2$. Wegen $\langle A, B, C\rangle \cong \mathfrak{A}_5$ und $\langle B, C\rangle \cong \mathfrak{A}_4$ normalisiert A die Gruppe $\langle B, C\rangle'$ nicht. Daraufhin ist nach b2) $o(bc)=2$. Ebenso folgt $o(bd)=2$. Da C und D wegen $H \neq K$ verschieden sind, ist $\langle C, D\rangle$ gemäß (E1) isomorph zu $\mathfrak{A}_4$ oder zu $\mathfrak{A}_5$.

1. Fall: $\langle C, D\rangle \cong \mathfrak{A}_4$. Weil die Produkte ac, ad, bc, bd involutorisch sind, gilt $(ab^{-1})(c^{-1}d)=acbd=c^{-1}a^{-1}d^{-1}b^{-1}=(c^{-1}d)(ab^{-1})$, d.h. ab^{-1} und $c^{-1}d$ sind vertauschbar. Da außerdem $\langle ab^{-1}\rangle$, $\langle c^{-1}d\rangle$ verschieden sind und $\langle ab^{-1}\rangle$ zu $\mathfrak{K}$ gehört, folgt mit (E1) $\langle c^{-1}d\rangle \notin \mathfrak{K}$ und daraus mit a1) $o(c^{-1}d)\leqq 2$. Wegen $C \neq D$ gilt $o(c^{-1}d)=2$ und folglich $o(cd)=3$. b2) liefert jetzt $A, B \leqq N(\langle C, D\rangle')$. Demnach

normalisiert $\langle A, B, D\rangle = H$ die Vierergruppe $\langle C, D\rangle'$. Weil sich H durch Elemente der Ordnung 5 erzeugen läßt und diese $\langle C, D\rangle'$ zentralisieren, operiert H auf $\langle C, D\rangle'$ trivial. Dies steht im Widerspruch zu $(c^{-1}d)^a = cd^{-1} \neq c^{-1}d$.

2. Fall: $\langle C, D\rangle \cong \mathfrak{A}_5$. Mit b1) folgt $A, B \leqq \langle C, D\rangle$. Dies führt zu $H, K \leqq \langle C, D\rangle$ und damit zum Widerspruch $H = K$.

d) *Gibt es Untergruppen* $A, B \in \mathfrak{K}$ *mit* $\langle A, B\rangle \cong \mathfrak{A}_5$, *so ist* $G = \langle A, B\rangle$.

Beweis. Wegen (E2) braucht man nur zu zeigen, daß $\langle A, B\rangle$ ein beliebig vorgegebenes $C \in \mathfrak{K}$ enthält. Man darf sich $C \neq A, B$ und wegen b1) etwa $\langle A, C\rangle \cong \mathfrak{A}_5$ vorstellen. Sodann fixiere man ein $D \in \mathfrak{K}$ mit $D \leqq \langle A, C\rangle$ und $\langle A, D\rangle, \langle C, D\rangle \cong \mathfrak{A}_4$. Wegen c) genügt es, $D \in \langle A, B\rangle$ zu beweisen. Dabei darf man natürlich $B \neq D$ und wegen b1) $\langle B, D\rangle \cong \mathfrak{A}_5$ unterstellen. Nun wähle man ein $E \in \mathfrak{K}$ mit $E \leqq \langle A, B\rangle$ und $\langle A, E\rangle, \langle B, E\rangle \cong \mathfrak{A}_4$. Wiederum darf $D \neq E$ angenommen werden. Im Falle $\langle D, E\rangle \cong \mathfrak{A}_4$ folgt mit b1) zunächst $E \leqq \langle B, D\rangle$ und daraus mit c) $D \leqq \langle B, D\rangle = \langle A, B\rangle$. Analog folgt im Falle $\langle D, E\rangle \cong \mathfrak{A}_5$ zunächst $A \leqq \langle D, E\rangle$ und dann $D \leqq \langle D, E\rangle = \langle A, B\rangle$.

e) *Enthält* $\mathfrak{K}$ *wenigstens zwei Untergruppen und gilt* $\langle A, B\rangle \cong \mathfrak{A}_4$ *für je zwei verschiedene* $A, B \in \mathfrak{K}$, *so ist* G *isomorph zum Erzeugnis der Streckungen eines affinen Raumes passender Dimension über* $GF(4)$.

Beweis. Zunächst begründen wir

$(*)$ $\langle A, B\rangle'$ *ist normal in* G *für alle* $A, B \in \mathfrak{K}$.

Andernfalls gibt es Untergruppen $A, B \in \mathfrak{K}$ so, daß $\langle A, B\rangle'$ nicht normal in G ist. A und B sind dann jedenfalls verschieden. Außerdem gibt es wegen (E2) ein $C \in \mathfrak{K}$, das $\langle A, B\rangle'$ nicht normalisiert. Da nach Voraussetzung $\langle A, B\rangle \cong \mathfrak{A}_4$ gilt, liegt C außerhalb von $\langle A, B\rangle$. Wegen $\langle A, C\rangle, \langle B, C\rangle \cong \mathfrak{A}_4$ können wir Erzeugende a, b, c für A, B, C mit $o(ac) = o(bc) = 2$ fixieren. Nach b2) ist $o(ab) = 2$. Zur Gewinnung eines Widerspruchs betrachten wir jetzt $\langle a, b, c\rangle$. Wegen $(abc)^2 = abcabc = b^{-1}a^{-1}a^{-1}c^{-1}c^{-1}b^{-1} = b^{-1}acb^{-1} = b^{-1}c^{-1}a^{-1}b^{-1} = cbba = cb^{-1}a$ gehört cb^{-1} und dann auch c zu $\langle abc, bc\rangle$. Also erzeugen die beiden Elemente abc, bc zusammen $\langle a, b, c\rangle$. Außerdem ist $(abc)^4 = (cb^{-1}a)^2 = cb^{-1}acb^{-1}a = cb^{-1}c^{-1}a^{-1}b^{-1}a = ccbbaa = c^{-1}b^{-1}a^{-1} = (abc)^{-1}$, damit $(abc)^5 = 1$ und dann $o(abc) = 5$, weil $abc \neq 1$ wegen $C \nleqq \langle A, B\rangle$ gilt. Daneben hat man $o(bc) = 2$, $o((abc)(bc)) = 3$. Da eine Gruppe H mit den Erzeugenden x_1, x_2 und den Relationen $x_1^5 = x_2^2 = (x_1x_2)^3 = 1$ z. B. nach [1], S. 140, 19.9 isomorph zu $\mathfrak{A}_5$ ist, ist $\langle a, b, c\rangle$ ein nichttriviales homomorphes Bild der einfachen Gruppe $\mathfrak{A}_5$, also isomorph zu $\mathfrak{A}_5$. ab^{-1}, $b^{-1}c$ haben beide die Ordnung 3, ihr Produkt $(ab^{-1})(b^{-1}c) = abc$ hat, wie oben überlegt, die Ordnung 5. $\langle ab^{-1}\rangle$, $\langle b^{-1}c\rangle$ sind also zwei Untergruppen aus $\mathfrak{K}$ mit einem zu $\mathfrak{A}_5$ isomorphen Erzeugnis, entgegen der Voraussetzung.

$A \in \mathfrak{K}$ sei beliebig gewählt. Das Erzeugnis V des Untergruppensystems $\{\langle A, B\rangle' | B \in \mathfrak{K} - \{A\}\}$ ist nach (*) normal in G. Da AV sämtliche Mitglieder von $\mathfrak{K}$ aufnimmt, können wir mit (E2) $AV = G$ schließen. Als Erzeugnis paarweise fremder, normaler Vierergruppen ist V eine Gruppe vom Exponenten 2, außerdem $\neq \{1\}$ wegen $|\mathfrak{K}| > 1$. Wir fixieren ein Erzeugendes $a \in A$ von A und betrachten den von a in V bewirkten Automorphismus $\alpha\colon V \to V$ mit $x^\alpha := x^a$. Bei beliebig fixiertem $B \in \mathfrak{K} - \{A\}$ gilt $xx^\alpha x^{\alpha^2} = 1$ auf $\langle A, B\rangle'$. Demnach ist $1 + \alpha + \alpha^2$ der Nullendomorphismus von V. Nun sieht man, daß $K := \{\alpha, \alpha^2, \alpha^3, 0\}$ mit den vom Endomorphismenring *End* V übernommenen Rechenoperationen ein zu $GF(4)$ isomorpher Körper ist. V ist in natürlicher Weise ein K-Vektorraum. τ_x bezeichne die zum Vektor $x \in V$ gehörige Translation. Die Zuordnung $a^\varepsilon x \to \alpha^\varepsilon \tau_x$ $(\varepsilon \in \{\pm 1, 0\})$ liefert offenkundig einen Isomorphismus von G auf das Erzeugnis der Streckungen des zum K-Vektorraum V gehörigen affinen Raumes.

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T_1-CLOSED SPACES

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1. Introduction

As it is well-known, a Hausdorff space X is said to be H-closed iff it is a closed subspace in every Hausdorff space $Y \supset X$, or equivalently, iff X has no proper Hausdorff extension. This concept can be generalized (by keeping invariant its characteristic properties) for arbitrary topological spaces X (see e.g. [2]).

For this purpose, let us recall the following definitions. If X is a subspace of the topological space Y, we say that Y is *reduced* with respect to X iff $x \in Y$, $y \in Y-X$, $x \neq y$ implies that at least one of the points x and y has a neighbourhood not containing the other one. Y is said to be *strongly reduced* iff $\{y\}$ is closed for $y \in Y-X$. Y is *T_1-reduced* (*T_2-reduced*) iff $x \in Y$, $y \in Y-X$, $x \neq y$ implies that both x and y have neighbourhoods not containing the other one (x and y have disjoint neighbourhoods). Y is a T_i-space $(i=0, 1)$ iff X is T_i and Y is reduced (if $i=0$) or T_1-reduced (if $i=1$) with respect to X. The same is true for $i=2$ provided Y is an *extension* of X (i.e. X is dense in Y).

Now an arbitrary topological space X is said to be *H-closed* iff it has no proper T_2-reduced extension. For a T_2-space X, this coincides with the classical definition.

Similarly, let us agree in calling a space X *T_1-closed* iff X has no proper T_1-reduced extension. If X is a T_1-space itself, this condition means that X has no proper extension that is a T_1-space, or that X is closed in every T_1-space containing it. However, it is easy to see (2.3) that T_1-spaces with this property are finite (and discrete); on the other hand, if we drop the condition to be T_1, there are many examples of T_1-closed spaces (2.5, 2.6, 2.7).

Our purpose is to study the properties of T_1-closed spaces and of T_1-closed extensions of arbitrary spaces.

We shall need some fundamental facts of extension theory. Let Y be an extension of the space X, and $p \in Y$. Then the neighbourhood filter $\mathfrak{v}(p)$ of p has a trace

$$\mathfrak{v}(p)|X = \{V \cap X \colon V \in \mathfrak{v}(p)\}$$

in X that is an open filter in X (i.e. it is generated by a filter base consisting of open sets); in particular, if $p \in X$, then $\mathfrak{v}(p)|X$ coincides with the neighbourhood filter in X of the point p. Conversely, if X is an arbitrary topological space, $Y \supset X$ is a set, and we assign to each point $p \in Y-X$ an open filter $\mathfrak{s}(p)$ in X, then there are (in general several) topologies on Y such that X is a dense subspace of Y and $\mathfrak{s}(p)$ coincides with the trace in X of the neighbourhood filter in Y of any $p \in Y-X$. Among these topologies, there are a coarsest one and a finest one. The latter, called the *loose extension* of X with respect to the filter system $\{\mathfrak{s}(p) \colon p \in Y-X\}$, is obtained by taking, for $x \in X$, the neighbourhoods in X of x for the elements of a neighbourhood base of x in Y, and for $p \in Y-X$, the sets $S \cup \{p\}$ $(S \in \mathfrak{s}(p))$ for the elements of a neighbourhood base of p. In order to obtain the coarsest topology on Y furnishing the given trace filters $\mathfrak{s}(p)$, let us denote by $\mathfrak{s}(x)$ the neighbourhood filter in X of the point $x\in X$, and put

$$s(G) = \{y\in Y\colon\ G\in\mathfrak{s}(y)\}$$

for any open subset $G\subset X$. Now the sets $s(G)$ constitute a base for the topology we are looking for; it is called the *strict extension* of X with respect to the system $\{\mathfrak{s}(p)\colon p\in Y-X\}$.

In a space X, an *ultraopen* filter is a maximal open filter (i.e. an open filter $\mathfrak{s}$ such that if $\mathfrak{s}'$ is an open filter containing $\mathfrak{s}$ then $\mathfrak{s}'=\mathfrak{s}$). By the Kuratowski—Zorn lemma, every open filter $\mathfrak{s}$ is contained in an ultraopen filter. An open filter $\mathfrak{s}$ is ultraopen iff, for every open subset $G\subset X$, either G or $X-G$ is contained in $\mathfrak{s}$. Two distinct ultraopen filters contain disjoint elements.

A filter $\mathfrak{s}$ in X is said to be *fixed* or *free* according to whether $\bigcap\mathfrak{s}\neq\emptyset$ or $\bigcap\mathfrak{s}=\emptyset$, respectively.

2. Characterization of T_1-closed spaces

The following theorem gives simple characterizations of T_1-closed spaces:

THEOREM 2.1. *For a topological space X, the following statements are equivalent:*

(a) *X is T_1-closed,*
(b) *In X every open filter is fixed,*
(c) *In X every ultraopen filter is fixed,*
(d) *In X there is a finite dense subset.*

PROOF. (a)⇒(b): Suppose $\mathfrak{s}$ is a free, open filter in X. Set $Y=X\cup\{p\}$, $p\notin X$, and consider the loose extension of X corresponding to the trace filter $\mathfrak{s}(p)=\mathfrak{s}$. This is a proper T_1-reduced extension of X.

(b)⇒(c): obvious.

(c)⇒(d): Suppose there is no finite, dense subset in X. Then the complements of the closures of the finite subsets generate a free open filter that is contained in a free ultraopen filter.

(d)⇒(a): Let Y be an extension of X, $p\in Y-X$. The trace in X of the neighbourhood filter of p is an open filter in X. If $F\subset X$ is a finite, dense subset, then $\mathfrak{s}|F$ is a filter in F, and clearly $\bigcap(\mathfrak{s}|F)\neq\emptyset$. Hence every neighbourhood of p contains any point $x\in\bigcap(\mathfrak{s}|F)$ so that Y cannot be a T_1-reduced extension of X. □

COROLLARY 2.2. *Every finite space is T_1-closed.* □

COROLLARY 2.3. *A T_1-space is T_1-closed iff it is finite (and discrete).* □

COROLLARY 2.4. *Every extension of a T_1-closed space is T_1-closed.*

PROOF. 2.1 (d). □

THEOREM 2.5. *A space is T_1-closed iff it is an extension of a finite space.* □

By 2.1, a space is T_1-closed iff its *density* (i.e. the smallest cardinality of a dense subset) is finite. Let us call *elementary T_1-closed space* a T_1-closed space with density

1. The space $T=\{0, 1\}$, where the open sets in T are $\emptyset$, $\{0\}$, T is an elementary T_1-closed space.

THEOREM 2.6. *If X_i is an elementary T_1-closed space for $i\in I$, then $X=\bigtimes_{i\in I} X_i$ is an elementary T_1-closed space as well.*

PROOF. If $x_i\in X_i$ is chosen such that $\overline{\{x_i\}}=X_i$, then $(x_i)=x\in X$ satisfies $\overline{\{x\}}=X$. □

In particular, any power of the space T is an elementary T_1-closed T_0-space. Further examples of elementary T_1-closed spaces are obtained from

THEOREM 2.7. *Let X be an arbitrary topological space, $Y=X\cup\{p\}$, $p\notin X$, and define the open subsets of Y to be $\emptyset$ and the sets $G\cup\{p\}$ where G is open in X. Then Y is an elementary T_1-closed space containing X as a closed subspace and reduced with respect to X.* □

3. Operations on T_1-closed spaces

We look for invariance of T_1-closedness for usual operations on spaces.

THEOREM 3.1. *A continuous image of a T_1-closed space is T_1-closed.*

PROOF. If $F\subset X$ is finite and dense, $f\colon X\to Y$ is continuous and surjective, then $f(F)$ is dense in Y (2.1 (d)). □

LEMMA 3.2. *If Y is T_1-closed, and $f\colon X\to Y$ is surjective, then the inverse image topology on X is T_1-closed.*

PROOF. If $F'\subset Y$ is finite and dense, and $F\subset X$ is a finite set such that $f(F)=F'$, then F is dense with respect to the inverse image topology. □

We recall that the T_0-reflection of a space X is the quotient space Y obtained from the equivalence relation for which x and y are equivalent iff their neighbourhood filters coincide; then X has the inverse image topology with respect to Y and the canonical surjection.

THEOREM 3.3. *A space is T_1-closed iff its T_0-reflection is T_1-closed.*

PROOF. 3.1, 3.2. □

The following statements are easily obtained from 2.1 (d):

THEOREM 3.4. *If $X=\bigcup_{1}^{n} X_i$ and each of the subspaces X_i is T_1-closed then X is T_1-closed as well.* □

THEOREM 3.5. *If X_i is T_1-closed for $i=1, \ldots, n$, then $X=\bigtimes_{1}^{n} X_i$ is T_1-closed.*

PROOF. If $F_i\subset X_i$ is finite and dense then so is $F=\bigtimes_{1}^{n} F_i$ *in* X. □

THEOREM 3.6. *In a T_1-closed space every open subspace and every regularly closed subspace is T_1-closed.*

PROOF. Let F be finite and dense in X, $G\subset X$ open. Then $F\cap G$ is dense in G so that G is T_1-closed, and $\bar{G}$ is T_1-closed by 2.4. □

4. Structure of T_1-closed spaces

We show that T_1-closed spaces have a rather simple structure.

Let X be a T_1-closed space and $\mathfrak{s}$ an ultraopen filter in X. By 2.1 (c) the set $K=\bigcap\mathfrak{s}$ is nonempty. A set K of this type will be called a *kernel set* in X.

LEMMA 4.1. *Let $\mathfrak{s}$ be an ultraopen filter in a topological space X, and $x\in K=\bigcap\mathfrak{s}$. Then $\mathfrak{s}$ coincides with the neighbourhood filter of x, hence $\bar{K}=\overline{\{x\}}$. If $y\in X-K$ then $x\notin\overline{\{y\}}$, hence $K\cap\overline{\{y\}}=\emptyset$.*

PROOF. Every $S\in\mathfrak{s}$ is a neighbourhood of x. Conversely, if G is an open neighbourhood of x, then $x\in G\cap S$ for every $S\in\mathfrak{s}$, hence $G\in\mathfrak{s}$. For $z\in K$, the neighbourhood filters of x and z both coincide with $\mathfrak{s}$ so that $z\in\overline{\{x\}}$, hence $K\subset\overline{\{x\}}$ and

$$\overline{\{x\}}\subset\bar{K}\subset\overline{\{x\}}.$$

If $y\in X-K$ then there is $S\in\mathfrak{s}$ such that $y\notin S$, hence $x\notin\overline{\{y\}}$, and $K\cap\overline{\{y\}}=\emptyset$. □

LEMMA 4.2. *Let $\mathfrak{s}\neq\mathfrak{s}'$ be ultraopen filters in a topological space X, $K=\bigcap\mathfrak{s}$, $K'=\bigcap\mathfrak{s}'$. Then $K\cap\bar{K}'=\emptyset$.*

PROOF. There are $S\in\mathfrak{s}$ and $S'\in\mathfrak{s}'$ such that $S\cap S'=\emptyset$. Hence $K\cap K'=\emptyset$ and, by 4.1, $y\in K'$ implies

$$K\cap\bar{K}'=K\cap\overline{\{y\}}=\emptyset. \quad \square$$

LEMMA 4.3. *In a T_1-closed space, a finite set is dense iff it meets every kernel set.*

PROOF. Suppose F is a finite, dense set in X. By 4.1, $F\cap K\neq\emptyset$ for every kernel set K. Conversely, if F is finite and $F\cap K\neq\emptyset$ for every kernel set K, then F is dense because every open set $G\neq\emptyset$ is contained in an ultraopen filter $\mathfrak{s}$, and for $K=\bigcap\mathfrak{s}$ we have $\emptyset\neq F\cap K\subset G$. □

THEOREM 4.4. *In a T_1-closed space of density n, there are n kernel sets K_i $(i=1, \ldots, n)$. For $1\leqq i\leqq n$, $1\leqq j\leqq n$, $i\neq j$, $\bar{K}_i\cap K_j=\emptyset$, and $X=\bigcup_1^n\bar{K}_i$.*

PROOF. By 4.2 two distinct kernel sets are disjoint. Hence if $F=\{x_1, \ldots, x_n\}$ is dense in X, then it can meet n kernel sets at most, and the number of all kernel sets is finite and $\leqq n$ by 4.3. On the other hand, if $K_1, \ldots, K_m$ are the kernel sets and $y_i\in K_i$, then $\{y_1, \ldots, y_m\}$ is dense by 4.3 so that $m\geqq n$.

For two distinct kernel sets K_i and K_j, we have $\bar{K}_i\cap K_j=\emptyset$ by 4.2. Finally $y_i\in K_i$ $(i=1, \ldots, n)$ implies by 4.3

$$X=\bigcup_1^n\overline{\{y_i\}}=\bigcup_1^n\bar{K}_i. \quad \square$$

The kernel sets can be characterized by the property formulated in 4.3:

THEOREM 4.5. *In a T_1-closed space X, let $A_1, \ldots, A_n$ be pairwise disjoint subsets with the property that a finite subset $F \subset X$ is dense iff $F \cap A_i \neq \emptyset$ for $i = 1, \ldots, n$. Then n is the density of X and the sets A_i coincide with the kernel sets of X.*

PROOF. Clearly n is the smallest cardinality of a dense subset of X so that n is the density of X. Let $K_1, \ldots, K_n$ be the distinct kernel sets of X (4.4).

Suppose a set A_i does not meet any set K_j $(j=1, \ldots, n)$. Then choosing $y_j \in K_j$, the set $\{y_1, \ldots, y_n\}$ is dense and does not meet A_i, contrarily to the hypothesis. Hence every A_i meets at least one K_j. Similarly every K_j meets at least one A_i.

If $A_i \cap K_j \neq \emptyset \neq A_i \cap K_k$, $j \neq k$, then choose points $y_s \in K_s$ for $s=1, \ldots, n$ in such a manner that

$$y_j \in A_i \cap K_j, \quad y_k \in A_i \cap K_k.$$

Since two of the points y_s belong to A_i, there must exist an A_h that contains no y_s. This contradicts the hypothesis because $\{y_1, \ldots, y_n\}$ is dense. Hence every A_i meets precisely one K_j. Similarly every K_j meets precisely one A_i, and the numeration can be chosen in such a way that $A_i \cap K_j \neq \emptyset$ iff $i=j$.

If $K_i - A_i \neq \emptyset$, we can again choose points $y_s \in K_s$ $(s=1, \ldots, n)$ in such a manner that no one of them belong to A_i; this is impossible so that $K_i \subset A_i$. Similarly $A_i \subset K_i$. Hence $A_i = K_i$ for $i=1, \ldots, n$. $\square$

The following statements show that every T_1-closed space can be obtained from elementary T_1-closed spaces with the help of some simple operations.

LEMMA 4.6. *If $\mathfrak{s}$ is an ultraopen filter in a topological space X, $Y \subset X$, and every element of $\mathfrak{s}$ meets Y, then $\mathfrak{s}|Y$ is an ultraopen filter in the subspace Y.*

PROOF. $\mathfrak{s}|Y$ is an open filter in Y. If $G \subset Y$ is open in Y and $S \cap G \neq \emptyset$ for every $S \in \mathfrak{s}$, then $G = H \cap Y$, H open in X, and $S \cap H \neq \emptyset$ for $S \in \mathfrak{s}$. Hence $H \in \mathfrak{s}$, and $G \in \mathfrak{s}|Y$. Consequently $\mathfrak{s}|Y$ is ultraopen in Y. $\square$

LEMMA 4.7. *Let X be a T_1-closed space, $K_1, \ldots, K_n$ its kernel sets, and $1 \leqq m < n$,*

$$Y = \bigcup_1^m \overline{K}_i, \quad Z = \bigcup_{m+1}^n \overline{K}_i.$$

Then Y and Z are T_1-closed subspaces, their kernel sets are $K_1, \ldots, K_m$ and $K_{m+1}, \ldots, K_n$, respectively, and

$$Y \cap K_i = \emptyset \quad (m+1 \leqq i \leqq n), \quad Z \cap K_i = \emptyset \quad (1 \leqq i \leqq m).$$

PROOF. The last statements follow from 4.2. If $x_i \in K_i$ $(i=1, \ldots, m)$ then by 4.1

$$Y = \bigcup_1^m \overline{K}_i = \bigcup_1^m \overline{\{x_i\}}$$

so that $\{x_1, \ldots, x_m\}$ is dense in Y and Y is T_1-closed by 2.1. We show that $K_1, \ldots, K_m$ are the kernel sets of Y.

In fact, fix $i \leqq m$, and let $\mathfrak{s}$ be an ultraopen filter in X such that $K_i = \bigcap \mathfrak{s}$. Then by 4.6 $\mathfrak{s}|Y$ is ultraopen in Y and $K_i = \bigcap (\mathfrak{s}|Y)$. Hence $K_1, \ldots, K_m$ are all kernel sets in Y and there is no further kernel set because the above set $\{x_1, \ldots, x_m\}$ is dense in Y (4.3, 4.4). By symmetry, Z is T_1-closed with the kernel sets $K_{m+1}, \ldots, K_n$. □

LEMMA 4.8. *Suppose X is a topological space, $X = Y \cup Z$, Y and Z are T_1-closed, closed subspaces with kernel sets $K_1, \ldots, K_m$ and $K_{m+1}, \ldots, K_n$, respectively. Assume*

$$K_i \cap Y = \emptyset \quad (m+1 \leqq i \leqq n), \quad K_i \cap Z = \emptyset \quad (1 \leqq i \leqq m).$$

Then X is T_1-closed and its kernel sets are $K_1, \ldots, K_n$.

PROOF. Let $\mathfrak{s}$ be an ultraopen filter in X. Either each element of $\mathfrak{s}$ meets Y or each $S \in \mathfrak{s}$ meets Z, say $S \cap Y \neq \emptyset$ for $S \in \mathfrak{s}$. Then by 4.6 $\mathfrak{s}|Y$ is ultraopen in Y, and $\bigcap (\mathfrak{s}|Y) = K_i$ for some $1 \leqq i \leqq m$. Hence the open set $X - Z \supset K_i$ meets each element $S \in \mathfrak{s}$ so that $X - Z \in \mathfrak{s}$ and

$$Z \cap \bigcap \mathfrak{s} = \emptyset, \quad \bigcap \mathfrak{s} = \bigcap (\mathfrak{s}|Y) = K_i.$$

Similarly we obtain that $\bigcap \mathfrak{s} = K_i$ for some $m+1 \leqq i \leqq n$ provided every $S \in \mathfrak{s}$ meets Z.

By 3.4, X is T_1-closed and its kernel sets coincide with some sets K_i. Moreover, every K_i is a kernel set in X. In fact, if $i \leqq m$, $x \in K_i$, then by 4.1 $x \notin \overline{\{y\}}$ whenever $y \in K_j$, $j \leqq m$, $j \neq i$, and the same is true if $y \in K_j$, $j \geqq m+1$ because then $\overline{\{y\}} \subset Z$, $x \notin Z$. Therefore, by 4.3, K_i must occur among the kernel sets of X. A similar argument applies for $i \geqq m+1$. □

THEOREM 4.9. *If X is a T_1-closed space with the kernel sets $K_1, \ldots, K_n$, and $Y_i = \overline{K}_i$, then Y_i is an elementary T_1-closed, closed subspace with kernel set K_i, and $K_i \cap Y_j = \emptyset$ for $i \neq j$.*

Conversely, if X is a topological space, $X = \bigcup_{1}^{n} Y_i$, Y_i is an elementary T_1-closed, closed subspace with kernel set K_i, and $K_i \cap Y_j = \emptyset$ for $i \neq j$, then X is T_1-closed and its kernel sets are $K_1, \ldots, K_n$.

PROOF. The first part is contained in 4.7, and the second one is obtained from 4.8 by an easy induction. □

In a T_1-closed T_0-space, the situation is very simple:

THEOREM 4.10. *If X is a T_1-closed T_0-space then the kernel sets are singletons $\{x_i\}$ $(i = 1, \ldots, n)$, and $F = \{x_1, \ldots, x_n\}$ is the unique discrete, dense subset of X.*

PROOF. By 4.1, the kernel sets are singletons, and the subspace F is discrete. By 4.3, F is dense in X.

Let D be a discrete, dense subset of X. If D is finite, then $D \supset F$ by 4.3. But $x \in D - F$ would imply $x \in \overline{\{x_i\}}$ for some $x_i \in F$ which is impossible because D is discrete. Hence $D = F$.

If D is infinite then there are three distinct points $x, y \in D$ and $x_i \in F$ such that $x, y \in \overline{\{x_i\}}$. Select open sets U and V such that

$$U \cap D = \{x\}, \quad V \cap D = \{y\}.$$

Then $x_i \in U \cap V$ and $U \cap V \cap D = \emptyset$ which contradicts the hypothesis that D is dense. □

In order to formulate the following theorem, let us recall [1] that a space X is said to be an *S_1-space* (*S_2-space*) iff, for $x, y \in X$, whenever x has a neighbourhood not containing y, y has a neighbourhood not containing x (x and y have disjoint neighbourhoods).

THEOREM 4.11. *An S_1-space X is T_1-closed iff it is the topological sum of a finite number of indiscrete spaces.*

PROOF. Let $K_1, \ldots, K_n$ be the kernel sets of X. Each K_i is closed. In fact, if $x \in K_i$, $y \in X - K_i$, then by 4.1 $x \notin \overline{\{y\}}$ so that $y \notin \overline{\{x\}} = \overline{K}_i$. By 4.4 $X = \bigcup_1^n K_i$, the sets K_i are pairwise disjoint and closed, and each subspace K_i is indiscrete.
The converse is obvious. □

4.11 can be deduced also from 3.3. and 2.3.

5. T_1-closed extensions

By 2.7 every topological space can be embedded in a T_1-closed one. However, the question of the existence of T_1-closed *extensions* is more delicate.
In view of the fact [2] that every space has a T_2-reduced H-closed extension, one would expect that every space has a T_1-reduced T_1-closed extension. This is far from being true:

LEMMA 5.1. *If Y is a T_1-closed space, strongly reduced with respect to a subspace $X \subset Y$, then X is T_1-closed as well.*

PROOF. Let $F \subset Y$ be finite and dense. Since $X \cap \overline{F - X} = \emptyset$, necessarily $X \subset \subset \overline{F \cap X}$. □

For a space that is not T_1-closed, we cannot expect therefore more than the existence of (reduced) T_1-closed extensions. But such an extension does not always exist.

LEMMA 5.2. *Let X be a topological space, $x, y \in X$. Then $x \in \overline{\{y\}}$ iff the neighbourhood filter of y converges to x.* □

THEOREM 5.3. *Let X be a topological space. The following statements are equivalent:*

(a) *X has a T_1-closed extension.*
(b) *There is in X a finite number of open filters whose limit points cover X.*

(c) *There is in* X *a finite number of filter bases whose limit points cover* X.
(d) *There is in* X *a finite number of ultraopen filters whose limit point cover* X.
(e) *In* X *there are finitely many ultraopen filters only.*

PROOF. (a)$\Rightarrow$(b): Let Y be a T_1-closed extension of X, $F\subset Y$ finite and dense. By 5.2 every $x\in X$ is the limit point of the neighbourhood filter $\mathfrak{v}(y)$ of some $y\in F$. Then $\mathfrak{v}(y)|X$ is an open filter in X that converges to x.

(b)$\Rightarrow$(c): obvious.

(c)$\Rightarrow$(d): Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ be filter bases in X such that, for $x\in X$, there is an $\mathfrak{r}_i$ such that $\mathfrak{r}_i\to x$. Define

$$\mathfrak{s}_i = \{S \subset X\colon \operatorname{int} S \supset R\in\mathfrak{r}_i\}.$$

Then $\mathfrak{s}_i$ is an open filter such that $\mathfrak{r}_i\to x$ implies $\mathfrak{s}_i\to x$. Let $\mathfrak{s}_i'$ be an ultraopen filter containing $\mathfrak{s}_i$. Then $\mathfrak{s}_i\to x$ implies $\mathfrak{s}_i'\to x$.

(d)$\Rightarrow$(e): Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ be ultraopen filters whose limit points cover X. If $\mathfrak{s}$ were an ultraopen filter distinct from each $\mathfrak{s}_i$, then there would exist an open set $G\in\mathfrak{s}$ such that $G\notin\mathfrak{s}_i$ for $i=1, \ldots, n$. Now a point $x\in G$ cannot be limit point of any $\mathfrak{s}_i$. Hence there is no further ultraopen filter in X.

(e)$\Rightarrow$(a): Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ be the ultraopen filters in X, $1\leqq m\leqq n$, and suppose that $\mathfrak{s}_1, \ldots, \mathfrak{s}_m$ are free, $\mathfrak{s}_{m+1}, \ldots, \mathfrak{s}_n$ are fixed (if every ultraopen filter is fixed then X itself is a T_1-closed improper extension by 4.1). Define

$$Y = X\cup\{p_1, \ldots, p_m\}$$

where the points $p_i\notin X$ are pairwise distinct, and equip Y with the strict extension of X with respect to the trace filters $\mathfrak{s}(p_i)=\mathfrak{s}_i$. Then Y is a reduced extension of X because the filters $\mathfrak{s}_i$ are free and any two of them contain disjoint open elements $G_i\in\mathfrak{s}_i$, $G_j\in\mathfrak{s}_j$ so that $s(G_i)$ and $s(G_j)$ are disjoint neighbourhoods of p_i and p_j, respectively.

If $x_i\in\bigcap\mathfrak{s}_i$ for $m+1\leqq i\leqq n$ then

$$F = \{p_1, \ldots, p_m, x_{m+1}, \ldots, x_n\}$$

is dense in Y. In fact, the sets $s(G)$, where $\emptyset\neq G\subset X$ is open, constitute a base in Y and each G belongs to some $\mathfrak{s}_i$; if $i\geqq m+1$ then $x_i\in G\subset s(G)$, if $i\leqq m$ then $p_i\in s(G)$. $\square$

Let us say that a space is T_1-*closable* iff it has a T_1-closed extension. The condition 5.3 (e) shows that this is a rather peculiar class of spaces. An example of a T_1-closable space that is not T_1-closed is an infinite set X in which the proper closed subsets are all finite subsets. Then the family of all non-empty open subsets constitutes a free open filter that is clearly the only ultraopen filter in X.

In the class of S_2-spaces there are no proper T_1-closable spaces:

LEMMA 5.4. *A* T_1-*closable* S_2-*space is* T_1-*closed.*

PROOF. Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ be the ultraopen filters in the S_2-space X. Then by 5.3, $X=\bigcup\limits_1^n A_i$ where A_i denotes the set of the limit points of $\mathfrak{s}_i$. Select $x_i\in A_i$ from the non-empty sets A_i and define F to be the set of these points x_i. Since X is an S_2-

space, any $x \in A_i$ has the same neighbourhood filter as x_i so that $A_i \subset \overline{\{x_i\}}$. Hence F is dense in X. □

In the part (e)⇒(a) of the proof of 5.3, we have constructed, for a given T_1-closable space X, a reduced T_1-closed extension Y. Let us call a space Y obtained in this way a *standard T_1-closed extension* of X; in other words, a standard T_1-closed extension of X is a strict extension Y such that the trace filters in X of the neighbourhood filters of the points of $Y-X$ are the distinct free ultraopen filters in X. Hence we can say:

COROLLARY 5.5. *For an arbitrary T_1-closable space X, there exist standard T_1-closed extensions. If Y is an extension of this kind, then $Y-X$ is finite, $p, q \in Y-X$, $p \neq q$ implies that p and q have disjoint neighbourhoods in Y, finally $x \in X$, $p \in Y-X$ implies that p has a neighbourhood not containing. x* □

If Y is a standard T_1-closed extension of X and Z is another extension of X, then, obviously, Z is a standard T_1-closed extension iff Y and Z are equivalent in the usual sense (i. e. there is a homeomorphism from Y onto Z that keeps fixed every point of X).

Our next purpose is to show that all possible T_1-closed extensions of a T_1-closable space can be obtained with the help of its standard T_1-closed extensions.

LEMMA 5.6. *Let Y be an extension of a topological space X and $\mathfrak{s}$ an ultraopen filter in X. Then there is a unique ultraopen filter $\mathfrak{s}'$ in Y such that $\mathfrak{s}=\mathfrak{s}'|X$; $\mathfrak{s}'$ is generated by the collection of all open sets $G' \subset Y$ such that $G' \cap X \in \mathfrak{s}$.*

PROOF. The sets G' described in the statement clearly constitute a filter base that generates an open filter $\mathfrak{s}'$ in Y. If $H \subset Y$ is open and $S' \cap H \neq \emptyset$ for every $S' \in \mathfrak{s}'$ then $H \cap X \cap S \neq \emptyset$ for every $S \in \mathfrak{s}$, hence $H \cap X \in \mathfrak{s}$ and $H \in \mathfrak{s}'$. Therefore $\mathfrak{s}'$ is ultraopen in Y. Clearly $\mathfrak{s}'|X=\mathfrak{s}$.

If $\mathfrak{s}'' \neq \mathfrak{s}'$ is an ultraopen filter in Y then there are disjoint sets $S' \in \mathfrak{s}'$, $S'' \in \mathfrak{s}''$ so that $S'' \cap X \notin \mathfrak{s}$ and $\mathfrak{s}''|X \neq \mathfrak{s}$. □

LEMMA 5.7. *Let Y be a standard T_1-closed extension of the T_1-closable space X and let us denote by $\mathcal{T}$ and $\mathcal{T}'$ the topologies of X and Y, respectively. Let $\mathcal{T}''$ be another topology on Y such that $\mathcal{T}''$ too is an extension of $\mathcal{T}$ and the trace in X of the $\mathcal{T}''$-neighbourhood filter of any $p \in Y-X$ coincides with the trace of the $\mathcal{T}'$-neighbourhood filter of p. If $\mathcal{T}''$ is T_1-closed then $\mathcal{T}''=\mathcal{T}'$.*

PROOF. For $y \in Y$, let us denote by $\mathfrak{s}(y)$ the trace in X of the $\mathcal{T}'$-neighbourhood filter of y. By hypothesis $\mathfrak{s}(y)$ is also the trace of the $\mathcal{T}''$-neighbourhood filter of y. For an open subset $G \subset X$, we denote again

$$s(G)=\{y \in Y:\ G \in \mathfrak{s}(y)\}.$$

We know that $\mathcal{T}'$ is coarser then $\mathcal{T}''$ because $\mathcal{T}'$ is a strict extension. Now let G'' be $\mathcal{T}''$-open. It is enough to show that G'' is $\mathcal{T}'$-open as well.

If $p \in G''-X$ then the $\mathcal{T}$-open set $G=G'' \cap X$ belongs to $\mathfrak{s}(p)$. Since $Y-X$ is (finite and) $\mathcal{T}'$-discrete, there is a $\mathcal{T}$-open set H such that $s(H)-X=\{p\}$. Then $s(G \cap H)=(G \cap H) \cup \{p\} \subset G''$ so that p is a $\mathcal{T}'$-interior point of G''.

Let $x \in G = G'' \cap X$ and suppose $q \in s(G) - G''$. Then $q \in Y - X$ and $s(G)$ belongs by 5.6 to the $\mathcal{T}'$-ultraopen filter $\mathfrak{s}'$ such that $\mathfrak{s}'|X = \mathfrak{s}(q)$. There is again a $\mathcal{T}$-open set H such that

$$s(H) - X = \{q\}, \quad H \in \mathfrak{s}(q), \quad s(H) \in \mathfrak{s}',$$

hence $\bigcap \mathfrak{s}' \subset \{q\}$. On the other hand $\bigcap \mathfrak{s}' \neq \emptyset$ because $\mathcal{T}'$ is T_1-closed, hence $\bigcap \mathfrak{s}' = \{q\}$. The filter $\mathfrak{s}'$ is $\mathcal{T}''$-open so that there exists a $\mathcal{T}''$-ultraopen filter $\mathfrak{s}'' \supset \mathfrak{s}'$. Clearly

$$\mathfrak{s}''|X \supset \mathfrak{s}'|X = \mathfrak{s}(q),$$

further

$$\bigcap \mathfrak{s}'' \subset \bigcap \mathfrak{s}' = \{q\},$$

consequently $\bigcap \mathfrak{s}'' = \{q\}$, because $\bigcap \mathfrak{s}'' \neq \emptyset$ owing to the T_1-closedness of $\mathcal{T}''$. $G \in \mathfrak{s}(q)$ implies that every element of $\mathfrak{s}''$ meets G and, *a fortiori*, G'', so that $G'' \in \mathfrak{s}''$, in contradiction to the hypothesis $q \notin G''$. Thus we have shown $s(G) \subset G''$ so that x is a $\mathcal{T}'$-interior point of G''. □

LEMMA 5.8. *Let Y be a standard T_1-closed extension of the T_1-closable space X and Z an arbitrary T_1-closed extension of X. Then there is a topological embedding $f\colon Y \to Z$ such that $f|X = \mathrm{id}_X$ so that Z contains a standard T_1-closed extension of X as a subspace.*

PROOF. Define $f(x) = x$ for $x \in X$. If $p \in Y - X$, then (with the notations of 5.7) $\mathfrak{s}(p)$ is ultraopen in X, hence by 5.5 there is a unique ultraopen filter $\mathfrak{s}'(p)$ in Z such that $\mathfrak{s}'(p)|X = \mathfrak{s}(p)$; since Z is T_1-closed, $\bigcap \mathfrak{s}'(p) \neq \emptyset$, so that we can select a point from this intersection. Define $f(p)$ to be this point.

Then $f(p) \in Z - X$ for $p \in Y - X$ because $\mathfrak{s}(p)$ is free. If $p, q \in Z - X$, $p \neq q$, then $\mathfrak{s}(p) \neq \mathfrak{s}(q)$, consequently $\mathfrak{s}'(p) \neq \mathfrak{s}'(q)$ so that $\mathfrak{s}'(p)$ and $\mathfrak{s}'(q)$ contain disjoint elements, and $f(p) \neq f(q)$. Therefore $f\colon Y \to Z$ is injective.

Consider $f(Y)$ as a (dense) subspace of Z. An ultraopen filter $\mathfrak{s}$ in $f(Y)$ is the trace in $f(Y)$ of a unique ultraopen filter $\mathfrak{s}'$ in Z (5.6). Clearly $\mathfrak{s}|X = \mathfrak{s}'|X$ and this is an ultraopen filter in X (4.6). Either $\mathfrak{s}|X$ is fixed and then so is $\mathfrak{s}$, or $\mathfrak{s}|X = \mathfrak{s}(p)$ for some $p \in Y - X$ and then $\mathfrak{s}'$ is the unique ultraopen filter in Z such that $\mathfrak{s}'|X = \mathfrak{s}(p)$, i. e. $\mathfrak{s}' = \mathfrak{s}'(p)$. In this case

$$f(p) \in \bigcap (\mathfrak{s}'|f(Y)) = \bigcap \mathfrak{s}.$$

Hence $f(Y)$ is T_1-closed by 2.1 (c).

Thus $f(Y)$ is an extension of X. By 4.1 $\mathfrak{s}'(p)$ is the neighbourhood filter of $f(p)$ in Z whenever $p \in Y - X$, its trace in $f(Y)$ is the neighbourhood filter of $f(p)$ in $f(Y)$, and the latter filter has for trace in X the filter $\mathfrak{s}(p)$. Hence if we consider on Y the inverse image topology obtained from $f(Y)$ and f, we get an extension of X such that the trace of the neighbourhood filter of $p \in Y - X$ coincides with $\mathfrak{s}(p)$. By 3.2 this extension is T_1-closed. Therefore this is, by 5.7, the topology of Y as a standard T_1-closed extension, and $f\colon Y \to f(Y)$ is a homeomorphism, $f(Y)$ is a standard T_1-closed extension as well. □

Hence we obtain:

THEOREM 5.9. *The T_1-closed extensions of a T_1-closable space X coincide with the extensions of its standard T_1-closed extensions.*

PROOF. By 5.8 every T_1-closed extension is an extension of a standard T_1-closed extension. Conversely an extension of a T_1-closed extension is a T_1-closed extension by 2.4. □

We can now show that the properties listed in 5.5 are characteristic for standard T_1-closed extensions:

THEOREM 5.10. *Let Z be a T_1-closed extension of a space X such that $p, q \in Z-X$, $p \neq q$ implies that p and q have disjoint neighbourhoods, and $x \in X$, $p \in Z-X$ implies that p has a neighbourhood not containing x. Then Z is a standard T_1-closed extension of X.*

PROOF. By 5.9 there is a subspace Y such that $X \subset Y \subset Z$ and Y is a standard T_1-closed extension of X. If $p \in Z-Y$ then the trace filter in X of the neighbourhood filter of p, denoted by $\mathfrak{s}(p)$, is a free open filter in X, and is contained in a free ultraopen filter that coincides with $\mathfrak{s}(q)$ for some $q \in Y-X$. This contradicts the hypothesis that $\mathfrak{s}(p)$ and $\mathfrak{s}(q)$ contain disjoint elements. Hence $Z=Y$. □

6. T_1-closable spaces

We conclude with some properties of T_1-closable spaces.

THEOREM 6.1. *A T_1-closable space is H-closed.*

PROOF. It suffices to show [2] that every ultraopen filter is convergent in a T_1-closable space X.

Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ be the ultraopen filters in X (5.3 (e)). There is, for a given i, an open set $G \in \mathfrak{s}_i$ such that $G \notin \mathfrak{s}_j$ for $j \neq i$. If $x \in G$, then x is limit point of some $\mathfrak{s}_j$ (5.3 (d)) which can happen for $j=i$ only. Hence $\mathfrak{s}_i \to x$. □

T_1-closable spaces have invariance properties similar to those of T_1-closed spaces.

THEOREM 6.2. *Every extension of a T_1-closable space is T_1-closable.*

PROOF. By 5.3 the number of ultraopen filters is finite in the given space. By 4.6 and 5.6 the number of ultraopen filters in an extension is the same. □

THEOREM 6.3. *A continuous image of a T_1-closable space is T_1-closable.*

PROOF. Let $f\colon X \to Y$ be continuous and surjective, and $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ filter bases in X such that their limit points cover X (5.3. (c)). If $y \in Y$, $y=f(x)$, $\mathfrak{r}_i \to x$ then $f(\mathfrak{r}_i) \to f(x)$ so that $f(\mathfrak{r}_1), \ldots, f(\mathfrak{r}_n)$ have a similar property in Y. □

LEMMA 6.4. *If $f\colon X \to Y$ is surjective and Y is a T_1-closable space then the inverse image topology on X is T_1-closable.*

PROOF. Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ be filter bases in Y whose limit points cover Y. If $x \in X$ and $\mathfrak{r}_i \to f(x)$ then $f^{-1}(\mathfrak{r}_i) \to x$. □

THEOREM 6.5. *A space is T_1-closable iff its T_0-reflection is T_1-closable.* □

THEOREM 6.6. *If* $X=\bigcup_1^n X_i$ *and each subspace* X_i *is* T_1*-closable then* X *is* T_1*-closable.*

PROOF. Select a finite number of filter bases in each X_i whose limit points cover X_i. □

THEOREM 6.7. *If* $X=\mathop{\times}_1^n X_i$ *and each* X_i *is* T_1*-closable then* X *is* T_1*-closable.*

PROOF. Choose a T_1-closed extension Y_i of X_i and consider $Y=\mathop{\times}_1^n Y_i$. □

LEMMA 6.8. *Let* Y *be an open subspace of a topological space* X. *Every ultraopen filter* $\mathfrak{s}$ *in* Y *generates an ultraopen filter* $\mathfrak{s}'$ *in* X *such that* $\mathfrak{s}'|Y=\mathfrak{s}$.

PROOF. $\mathfrak{s}'$ is an open filter in X because Y is open. If $G\subset X$ is open and $S'\cap G\neq\emptyset$ for every $S'\in\mathfrak{s}'$ then, in particular,

$$S\cap G=S\cap G\cap X\neq\emptyset$$

for $S\in\mathfrak{s}$ so that $G\cap X\in\mathfrak{s}$, and $G\in\mathfrak{s}'$. Hence $\mathfrak{s}'$ is ultraopen in X. The trace $\mathfrak{s}'|Y\supset\mathfrak{s}$ is an ultraopen filter in Y, hence $\mathfrak{s}'|Y=\mathfrak{s}$. □

THEOREM 6.9. *Every open and every regularly closed subspace of a* T_1*-closable space is* T_1*-closable.*

PROOF. According to 6.8, if $Y\subset X$ is open, then the number of ultraopen filters in Y is not larger than the number of ultraopen filters in X. Hence if X is T_1-closable then so is Y, and the same holds for $\overline{Y}$ by 6.2. □

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ALMOST SUBIDEMPOTENT RADICALS AND A GENERALIZATION OF A THEOREM OF JACOBSON

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The purpose of this paper is twofold: to study almost subidempotent radical properties and from this study to give a new proof and also a generalization of the famous theory of Jacobson [19] stating that the Jacobson radical of a Noetherian ring is transfinitely nilpotent, moreover, as a corollary another approach to the famous and unsolved problem of Köthe [24] is obtained.

By the term almost subidempotent radical we mean a radical property with idempotent radical rings. Following V. A. Andrunakievič [2] a hereditary almost subidempotent radical property is simply called subidempotent. The notion of transfinitely nilpotent ring was first discussed by R. Baer [7] and is a generalization of the nilpotent ring. The ring R is said to be transfinitely nilpotent if there exists an ordinal number r such that $R^r=0$. We recall that the power R^r is defined as follows: If r is a nonlimit ordinal number then $R^r=R^{r-1}R$, while if r is a limit ordinal number then $R^r=\bigcap\limits_{b<r} R^b$.

The following second symbol for powers of rings with ordinal numbers as indices is known: If r is a nonlimit ordinal number then $R_r=\bigcap\limits_{n} R^n_{r-1}$, while if r is a limit ordinal number then $R_r=\bigcap\limits_{c<r} R_c$ (G. Krause and T. H. Lenegan [25]).

Throughout this paper all rings considered will be associative. The terminology and basic results of radical theory can be found in [10], [1], [2].

First we examine for some almost subidempotent radical property N, the class $T(N)$ of all radical properties Y with $Y\geqq N$ and $Y(R)_\pi=N(R)\ \forall R$, for some ordinal number π depending on Y and R.

In what follows we apply the results of Section 1 to trivial almost subidempotent radical property $N=\{\{0\}\}$ and we prove that in the corresponding class $T(\{0\})$ there exists the radical property $T_{\{0\}}$ which coincides with the radical property of B. J. Gardner [13]. We continue applying, in general, to almost subidempotent radical properties N with $N\cap B=\{0\}$ where B is the R. Baer's [7] radical property. Further, we apply to almost subidempotent radical properties N with $N\cap B\neq 0$.

In the last part of this paper the above mentioned theorem of Jacobson is proved with a different proof. Finally, we give an equivalent statement for the nonexistence of simple prime nil-rings and hence another approach to the famous and unsolved problem of Köthe [24].

1. Transfinitely nilpotent closure of an almost subidempotent radical property

Let N be an almost subidempotent radical property. Evidently, if for the ring R it holds $R_\pi=0$, for some ordinal number π, then R is an N-semisimple ring.

If we denote $E=\{R|R^2=R,\ R\ N\text{-semisimple}\}$ and $Z=\{R|R^2=0\}$ then it is easily proved that N coincides with the upper radical property determined by the class $E\cup Z$.

Also, let $S(N)$ be the class of all radical properties S with

$$S \geqq N \tag{1}$$

and

$$I_S = N \tag{2}$$

where I_S is the class of all idempotent S-radical rings. It is clear that $S(N)\neq\emptyset$ since $N\in S(N)$. Even if S is the upper radical property determined by the class E then $S\in S(N)$ holds since on the right hand we have $S\geqq N$ and on the other $I_S=N$. Indeed, if the idempotent and S-radical ring R is not N-radical, then the ring $R/N(R)\neq 0$ will be an idempotent and N-semisimple ring, that is, $R/N(R)\in E$, a contradiction.

Now let $T(N)$ be the subclass of $S(N)$ as follows: $T(N)=\{T|T\in S(N),\ R_\pi\in T\ \forall R\in T\}$, where π is an ordinal number depending on R such that R_π is an idempotent ring.

Evidently, from (2) we have $N(R_\pi)=R_\pi$ and since in general $N(R)\subseteqq R_\pi$ holds we have $N(R)=R_\pi\ \ \forall R\in T$. Finally, from (1) it is implied

$$T(R)_\pi = N(R) \quad \forall R. \tag{3}$$

At this point we can observe that (1) and (3) imply (2). Obviously, it is $T(N)\neq\emptyset$ since $N\in T(N)$. Also, the class $T(N)$ contains all the hereditary radical properties of $S(N)$. With the following example we will show that there exist almost subidempotent radical properties N for which the corresponding classes $T(N)$ contain non-hereditary radicals also.

EXAMPLE 1.1. Let N be the upper radical property determined by the class $E\cup Z$ where $E\neq\emptyset$ and $E=\{A|A$ simple ring, $1\in A$, $A\not\cong Z(p)\ \forall p\}$. It is obvious that N is an almost subidempotent radical. Now, let S be the upper radical property determined by the class $E\cup\{2\mathbf{Z}\}$. We have $S\geqq N$ since if R is an N-radical ring then R cannot be mapped homomorphically on a ring of the class E and, as idempotent, cannot be mapped homomorphically on a nonzero ideal of the ring $2\mathbf{Z}$ (Y. L. Lee [26]). Now we suppose that there exists an S-radical ring R for which the idempotent ideal R_π for some ordinal number π, is not S-radical. Then R_π, as an idempotent ring, can be mapped homomorphically on a simple ring with identity of the class E, that is, there exists an ideal I of R_π such that $R_\pi/I\cong A\in E$. The ideal I is also an ideal of R. Indeed, if I^* is the ideal of R generated by I then if $I\subset I^*$ is supposed then $I^*=R_\pi$ must hold and consequently by the Andrunakievič lemma [1] $R_\pi^3=I^{*3}\subseteqq I\subset R_\pi$, a contradiction. The ring R/I contains the ideal $R_\pi/I\cong A$ which must be a direct summand of R/I (N. Divinsky [10] p. 145).

Thus, if we suppose the ring isomorphism $R/I \cong R_\pi/I \oplus K/I$ then $R/K \cong \dfrac{R/I}{K/I} \cong$ $\cong R_\pi/I \cong A \in E$, a contradiction, since we have supposed R to be an S-radical ring. Consequently $S \in T(N)$. However, the radical property S is not hereditary since the ring $\mathbf{Z}$ of rational integers is S-radical and contains the S-semisimple ideals $2m \cdot \mathbf{Z}$, $m \geqq 1$.

It is now natural to ask: For which almost subidempotent radical properties N do the corresponding classes $T(N)$ contain a radical which contains every other radical of this class?

LEMMA 1.2. *If for the almost subidempotent radical property N the corresponding class $T(N)$ contains a maximal radical property T then T coincides with the upper radical property determined by the class $\tilde{E} = \{A/T(A) | A^2 = A,\ A$ N-semisimple$\} \cup \{R/T(R) | R$ S-radical, $N(R) \subset R_\pi\ \forall \pi\}$ where S is the upper radical property determined by the class $E = \{A/T(A) | A^2 = A,\ A$ N-semisimple$\}$.*

PROOF. Let $\tilde{S}$ be the upper radical property determined by the class $\tilde{E}$. Evidently, $\tilde{S} \geqq T$ holds since each admissible subring of every ring of the class $\tilde{E}$ is a T-semisimple ring (Y. L. Lee [26]). Now, if we suppose R to be an $\tilde{S}$-radical ring with $N(R) \subset R_\pi\ \forall \pi$, then, since obviously $\tilde{S} \leqq S$ holds, R will be an S-radical ring which can be mapped homomorphically on the nonzero ring $R/T(R) \in \tilde{E}$, a contradiction. Indeed, we have $R/T(R) \neq 0$ because we have supposed $N(R) \subset R_\pi$, $\forall \pi$. Consequently, $\tilde{S} \in T(N)$ and since T is a maximal radical property of $T(N)$, $T = \tilde{S}$ must hold.

As usual, let L denote the lower radical operator.

LEMMA 1.3. *Let N be an almost subidempotent radical property and R_0 a ring such that $L(N \cup \{R_0\}) \cap K(N) = \emptyset$ holds, where $K(N) = \{R | R_\pi \supset N(R)\ \forall \pi\}$. If R is an N-semisimple and $L(N \cup \{R_0\})$-radical ring then there exists an ordinal number π, depending on R, such that $R_\pi = 0$ holds.*

PROOF. Indeed, if $R_\pi \neq 0$, where R_π is an idempotent ring for some ordinal number π depending on R, then we would have $N(R) = 0 \subset R_\pi \Rightarrow R \in K(N) \Rightarrow R \in$ $\in L(N \cup \{R_0\}) \cap K(N) = \emptyset$, a contradiction.

PROPOSITION 1.4. *If N is an almost subidempotent radical property then the following statements are equivalent:*

(a) *In the class $T(N)$ there exists a radical property T_N which contains every other radical property of this class.*

(b) *There exists a class of rings $\tilde{E} = \{A/B | A^2 = A,\ A$ N-semisimple, $0 \neq A/B$ N-semisimple$\} \cup \{R/C | R$ S-radical, $R \in K(N)$, $0 \neq R/C$ N-semisimple$\}$, where S is the upper radical property determined by $E = \{A/B | A^2 = A,\ A$ N-semisimple, $0 \neq A/B$ N-semisimple$\}$ such that $L(N \cup \{R_0\}) \cap K(N) \neq \emptyset$ holds for every ideal R_0 of a ring of the class $\tilde{E}$ with $(R_0)_\pi = 0$.*

(c) *$L(N \cup \{R_i | i \in I\}) \cap K(N) = \emptyset$ holds, where $\{R_i | i \in I\}$ is the class of all rings with $(R_i)_{\pi_i} = 0$ and $L(N \cup \{R_i\}) \cap K(N) = \emptyset$.*

PROOF. (a)$\Rightarrow$(b). By Lemma 1.2, T_N must coincide with the upper radical property determined by the class $\tilde{E} = \{A/T_N(A) | A^2 = A,\ A$ N-semisimple$\} \cup \{R/T_N(R) | R$ S-radical, $N(R) \subset R_\pi\ \forall \pi\}$ where S is the upper radical property determined by the class $E=\{A/T_N(A)|A^2=A,\ A\ N\text{-semisimple}\}$. Now if $R_0\neq 0$ is an ideal of a ring of the class $\tilde{E}$ such that $(R_0)_\pi=0$, for some ordinal number π depending on R_0 and $L_0\cap K(N)=\emptyset$, where $L_0=L(N\cup\{R_0\})$, then evidently $L_0(R)\neq R\ \forall R\in K(N)$ must hold. Consequently, if we denote $\tilde{S}_0$ the upper radical property determined by the class $\tilde{E}_0=\{A/L_0(A)|A^2=A,\ A\ N\text{-semisimple}\}\cup\{R/L_0(R)|R\in K(N),\ R\ S_0\text{-radical}\}$ where S_0 is the upper radical property determined by the class $E_0=\{A/L_0(A)|A^2=A,\ A\ N\text{-semisimple}\}$, then $\tilde{S}_0\in T(N)$ must hold. Indeed, on the one hand we have $\tilde{S}_0\geqq N$, since each admissible subring of a ring of the class $\tilde{E}_0$ is an L_0-semisimple ring and consequently N-semisimple, on the other hand if R is an $\tilde{S}_0$-radical ring and $R\in K(N)$ then R will also be an S_0-radical ring and it can be mapped homomorphically on the nonzero ring $R/L_0(R)\in\tilde{E}_0$, a contradiction. However, $\tilde{S}_0\not\leqq T_N$ holds since the ring R_0 is an L_0-radical ring and consequently an $\tilde{S}_0$-radical ring, and also is a T_N-semisimple, a result which contradicts the fact that T_N contains every other radical property of the class $T(N)$.

(b)⇒(c). If we suppose that $L(N\cup\{R_i|i\in I\})\cap K(N)\neq\emptyset$ holds, where $\{R_i|i\in I\}$ is the class of all rings with $(R_i)_{\pi_i}=0$ and $L(N\cup\{R_i\})\cap K(N)=\emptyset$ then $0\neq R\in K(N)$ must exist, which is an L_0-radical, where $L_0=L(N\cup\{R_i|i\in I\})$. In this case it follows that an admissible subring $R_0\neq 0$ of a ring of the class $\tilde{E}$ must exist such that R_0 is an L_0-radical ring. Indeed, if R is not an S-radical ring then it can be mapped homomorphically onto an admissible subring $R_0\neq 0$ of a ring of the class E, which must be an L_0-radical ring, while if R is an S-radical ring then an ideal C of R must exist such that $0\neq R/C=R_0\in\tilde{E}$ and L_0-radical ring. Consequently, a ring of the class $\tilde{E}$ must have an admissible subring $R_0'\neq 0$ of first degree over $N\cup\{R_i|i\in I\}$. Evidently, R_0' must be an homomorphic image of a ring of the class $\{R_i|i\in I\}$, suppose R_{i_0}, since R_0' is N-semisimple. By assumption on the class $\{R_i|i\in I\}$ we have $L(N\cup\{R_{i_0}\})\cap K(N)=\emptyset$.

However, since R_0' is a nonzero $L(N\cup\{R_{i_0}\})$-radical ring and also an admissible subring of some ring $M\in\tilde{E}$ we have $L(N\cup\{R_{i_0}\})(M)=X\neq 0$ (N. Divinsky and A. Sulinski [11]) and by Lemma 1.3 $X_\pi=0$ must hold and thus by (b) $L(N\cup\{R_{i_0}\})\cap\cap K(N)\supset L(N\cup\{X\})\cap K(N)\neq\emptyset$ a contradiction.

(c)⇒(a). Evidently, in this case $L_0(R)\neq R\ \forall R\in K(N)$ must hold, where $L_0=L(N\cup\{R_i|i\in I\})$. Consequently, if we construct the class $\tilde{E}=\{A/L_0(A)|A^2=A,\ A\ N\text{-semisimple}\}\cup\{R/L_0(R)|R\ S\text{-radical},\ R\in K(N)\}$ where S is the upper radical property determined by the class $E=\{A/L_0(A)|A^2=A,\ A\ N\text{-semisimple}\}$ then for every ideal R_0 of a ring of the class $\tilde{E}$, with $(R_0)_\pi=0$, $L(N\cup\{R_0\})\cap K(N)\neq\emptyset$ must hold. Indeed, if $L(N\cup\{R_0\})\cap K(N)=\emptyset$ holds then R_0 must be an L_0-radical ring, a contradiction. Also, for the upper radical property $\tilde{S}$ determined by the class $\tilde{E}$, $\tilde{S}\in T(N)$ holds, since evidently we have $N\leqq\tilde{S}$ and if R is an $\tilde{S}$-radical ring and simultaneously $R\in K(N)$ then R can be mapped onto the nonzero ring $R/L_0(R)\in\tilde{E}$, a contradiction.

Finally, we will show that $\tilde{S}$ contains every other radical property of the class $T(N)$. In fact, if there exists a radical $Y\in T(N)$ such that $Y\not\leqq\tilde{S}$ then a ring $M\in\tilde{E}$ must exist with $Y(M)=R_0\neq 0$. Consequently, an ordinal number π must exist such that $(R_0)_\pi=(Y(R_0))_\pi=N(R_0)=0$ and thus $L(N\cup\{R_0\})\cap K(N)\neq\emptyset$ must hold. Now, if R is an $L(N\cup\{R_0\})$-radical and $R\in K(N)$, then since $N\leqq Y$ and R_0 is a Y-radical ring, R will be a Y-radical ring which contradicts the fact that $Y\cap K(N)=\emptyset$.

PROPOSITION 1.5. *If for the almost subidempotent radical property N one of* *the statements of Proposition* 1.4 *holds then the radical property* T_N *coincides with the upper radical property determined by a class* $\tilde{E}$ *and with the lower radical property determined by the class* $N\cup\{R_i|i\in I\}$ *as it is defined in Proposition* 1.4.

PROOF. The first part has already been established. Thus, we need only show that $T_N=L(N\cup\{R_i|i\in I\})$. Evidently, since $L(N\cup\{R_i|i\in I\})\cap K(N)=\emptyset$ it is implied that $L(N\cup\{R_i|i\in I\})\leqq T_N$. On the other hand we have $R_\pi=N(R)$, for some ordinal number π depending on R, $\forall R\in T_N$. Consequently, for the ring $R/N(R)$ the conditions $(R/N(R))_\pi=0$ and $R/N(R)\in T_N$ hold, from which $L(N\cup\{R/N(R)\})\cap\cap K(N)\subseteq T_N\cap K(N)=\emptyset$ follows. Therefore, $R/N(R)\cong R_i\in L(N\cup\{R_i|i\in I\})$ must hold for some $i\in I$, which implies that $R\in L(N\cup\{R_i|i\in I\})$, $\forall R\in T_N$, that is $T_N\leqq\leqq L(N\cup\{R_i|i\in I\})$.

2. Almost subidempotent radical properties N with $N\cap B=\{0\}$

Let N be the trivial almost subidempotent radical property i.e. $N(R)=0$, $\forall R$. If $R_0\neq 0$ is a ring with $(R_0)_\pi=0$, for some ordinal number π depending on R_0, and $L(N\cup\{R_0\})\cap K(N)=L(\{R_0\})\cap K(\{0\})=\emptyset$ then $R_0^2\subset R_0$ must obviously hold. Likewise, if we have $R_0^2\neq 0$ then $R_0^n\subset R_0^2$, $\forall n>2$ must hold. In what follows we shall show that the additive group $(R_0/R_0^n)^+$, $\forall n>2$ is torsion and divisible. Indeed, if at first R_0/R_0^n, $n>2$ were a nondivisible p-ring then the radical property $L(\{R_0\})$ would contain all the B-radical p-rings (B. J. Gardner [15], Lemma 3.1) which contradicts the fact that the class of all B-radical p-rings contains nonzero idempotent rings, that is, rings from the class $K(\{0\})$, as the following example shows:

EXAMPLE 2.1 (G. Köthe [24]). K is a ring generated by a set $\{x_1, x_2, x_3, \dots$ $\dots, x_n, \dots\}$ having the following properties:

$$px_n=0,\quad \forall n,\quad x_ix_j=x_jx_i,\quad \forall (i,j),\quad x_1^2=0,\quad x_2^2=x_1, \dots, x_{n+1}^2=x_n, \dots.$$

Consequently, if R_0/R_0^n is a torsion ring then it must be divisible since otherwise as the ring-direct sum of p-components it could be mapped onto a nondivisible nilpotent p-ring, a contradiction.

Finally, if R_0/R_0^n is not a torsion-ring then $T(R_0/R_0^n)\neq R_0/R_0^n$ and

$$R=\frac{R_0/R_0^n}{T(R_0/R_0^n)}\neq 0$$

is a nilpotent torsion-free and $L(\{R_0\})$-radical ring. Consequently, the radical class $L(\{R_0\})$ will contain the nontorsion ring $R/R^2\neq 0$. Indeed, since $R^k\neq R^{k+1}=0$ for some positive integer k, if R/R^2 were a torsion ring, then for any elements $x_1, x_2, \dots, x_k\in R$, denoting the orders of their cosets with respect to R^2 by $n_1, n_2, \dots$, $\dots, n_k$, respectively, we would have $(n_1n_2\dots n_k)(x_1x_2\dots x_k)\in R^{2k}=0$, whence, R would be a torsion ring, a contradiction. Thus $L(\{R_0\})$ contains the nonzero torsion-free zeroring

$$A=\frac{R/R^2}{T(R/R^2)}\neq 0$$

which must be divisible, since otherwise it would have the zeroring on $A/pA\neq 0$, for some prime p among its homomorphic images, a contradiction. Hence the zeroring on the group of rational numbers $\mathbf{Q}$ belongs to $L(\{R_0\})$, so $L(\{R_0\})$ contains all rings in B which have divisible torsion-free additive groups (B. J. Gardner [14], Theorem), a contradiction since some of them are idempotents, as the following example shows:

EXAMPLE 2.2 (N. Divinsky [10], Example 3, p. 19). Let A be the Zassenhaus algebra over $\mathbf{Q}$ with basis set $\{x_a|0<a<1\}$, where a are rational numbers, and the multiplication of the basic elements is defined as follows:

$$x_a x_b = x_{a+b} \quad \text{if} \quad a+b<1 \quad \text{and} \quad x_a x_b = 0 \quad \text{if} \quad a+b \geqq 1.$$

Now, since R_0/R_0^n, $\forall n>2$, is a torsion and divisible ring, it must be a zeroring (L. Fuchs [12], Theorem 120.3, p. 288), that is, $R_0^2 \leqq R_0^n \Rightarrow R_0^2 = R_0^n \Rightarrow R_0^2 = 0$. Thus, we can formulate the following:

LEMMA 2.3. *$R_0\neq 0$ is a ring with $(R_0)_\pi=0$, for some ordinal number π depending on R_0, and $L(\{R_0\})\cap K(\{0\})=\emptyset$, if and only if R_0 is a zeroring with*

$$R_0^+ \cong \bigoplus_p \big(\bigoplus_{n_p} \mathbf{Z}(p^\infty)\big), \quad n_p \geqq 0.$$

PROOF. The converse is obvious.

PROPOSITION 2.4. *In the corresponding class $T(\{0\})$ of the almost subidempotent radical property $N=\{\{0\}\}$, there exists the radical property $T_{\{0\}}$ and it coincides with Gardner's radical*

$$D_P = \big\{R|R^+ \cong \bigoplus_p \big(\bigoplus_{n_p} \mathbf{Z}(p^\infty)\big), \quad n_p \geqq 0\big\}.$$

PROOF. Indeed,

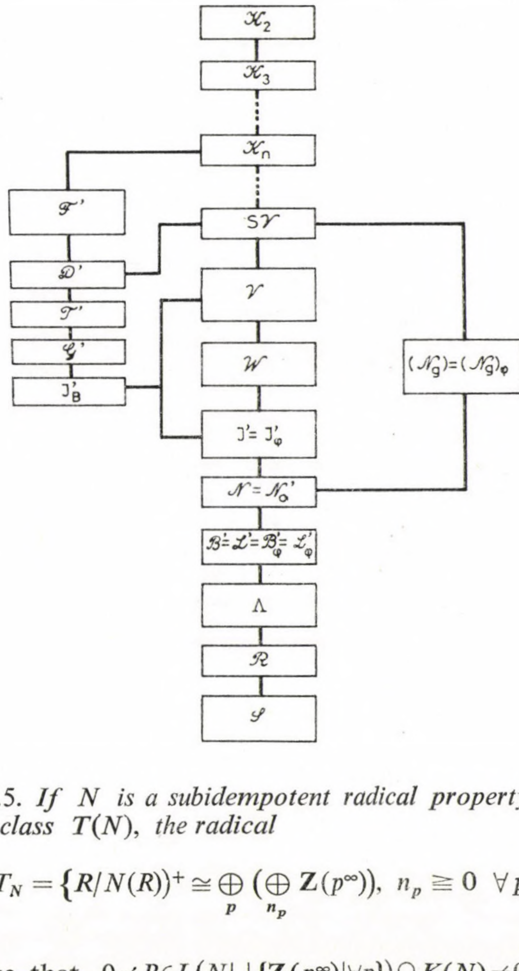
$$L\Big(\big\{R|R^+ \cong \bigoplus_p \big(\bigoplus_{n_p} \mathbf{Z}(p^\infty)\big), n_p \geqq 0\big\}\Big) = L(D_P) = D_P$$

(B. J. Gardner, [13]), and since $D_P\cap K(\{0\})=\emptyset$ by Propositions 1.4 and 1.5 $T_{\{0\}}=D_P$ is implied.

Now, let, in general, N be an almost subidempotent radical property with $N\cap B=\{0\}$. In the same way as above, it is proved that if $R_0\neq 0$ is a ring with $(R_0)_\pi=0$ for some ordinal number π depending on R_0, and $L(N\cup\{R_0\})\cap K(N)=\emptyset$, then R_0 is a zeroring with $R_0^+\cong \bigoplus_p \big(\bigoplus_{n_p} \mathbf{Z}(p^\infty)\big)$, $n_p\geqq 0$.

The converse is not true in general. Indeed, as we will show later, there exists an N with $L\big(N\cup\{\mathbf{Z}(p^\infty)\}\big)\cap K(N)\neq\emptyset$ for some or for all p. Also, for each $R\in L\big(N\cup\{\mathbf{Z}(p^\infty)|p\in P^*\}\big)$, where $P^*=\big\{p|L\big(N\cup\{\mathbf{Z}(p^\infty)\}\big)\cap K(N)=\emptyset\big\}$, it is easy to prove that R^2 is an idempotent ideal, $(R/R^2)^+\cong \bigoplus_{p\in P^*} \big(\bigoplus_{n_p} \mathbf{Z}(p^\infty)\big)$, $n_p\geqq 0$, and that $N(R)=R^2$ if $R\notin K(N)$ while $N(R)\subset R^2$ if $R\in K(N)$. There are many almost subidempotent radical properties $N\neq\{0\}$ with $N\cap B=\{0\}$. Instances of such are the radical-semisimple classes K_n, $n\geqq 2$ (R. Wiegandt [40], E. P. Armendariz [6], B. J. Gardner and P. N. Stewart [16], P. N. Stewart [35], F. Szász [38]) the von Neumann regular class V (J. von Neumann [30], B. Brown and N. H. McCoy [9]), the strongly regular class SV (R. F. Arens and L. Kaplansky [5]), the weakly regular class W (B. Brown and N. H. McCoy [8]), the complementary radicals of supernilpotent radical properties, B', N', J', J'_B, G', N'_g, $\mathcal{T}'$, D', F' (V. A. Andrunakievič [2]), the radical Λ of de la Rosa [33], and the two radicals R and S of F. A. Szász [36], [37].

The relationship among them is as follows:



PROPOSITION 2.5. *If N is a subidempotent radical property with $N \leqq W$, then in its corresponding class $T(N)$, the radical*

$$T_N = \{R/N(R))^+ \cong \bigoplus_p (\bigoplus_{n_p} \mathbf{Z}(p^\infty)),\ n_p \geqq 0\ \forall p\}$$

exists.

PROOF. Suppose that $0 \neq R \in L(N \cup \{\mathbf{Z}(p^\infty)|\forall p\}) \cap K(N) \neq \emptyset$. Without loss of generality we can assume that R is an N-semisimple ring. Then $D_P(R) \neq 0$, but $D_P(R) \subset R$ since $0 \neq R_\pi \nsubseteq D_P(R)$. Consequently, since $R/D_P(R) \neq 0$, $D_P(R) \subset \subset U \leqq R$ must exist such that $N(R/D_P(R)) = U/D_P(R) \neq 0$. Now if $D_P(R)^*$ is the set of all two-sided annihilators of $D_P(R)$ in U, it is well known that $(D_P(R)^*)^2 \cap \cap D_P(R) = 0$ (V. A. Andrunakievič [4], Lemma 12), since we have supposed that $N \leqq W$. Thus, $(D_P(R)^*)^2$ is isomorphic to an ideal of $U/D_P(R)$ and consequently an N-radical ring. Simultaneously, $(D_P(R)^*)^2$ as an ideal of the N-semisimple ring R, must be N-semisimple and this leads to $(D_P(R)^*)^2=0$, that is, $D_P(R)^*=D_P(R)$.

On the other hand, we have $U^+=D_P(R)^+\oplus S$ since $D_P(R)^+$ is a divisible group. So, if $T(S)\neq 0$, where $T(S)$ is the torsion part of S, then $T(S)D_P(R)= =D_P(R)T(S)=0\Rightarrow T(S)\subseteq D_P(R)^*\Rightarrow D_P(R)\subset D_P(R)^*$ holds, a contradiction. Finally, if $T(S)=0$ then $U/D_P(R)$ is torsion-free and, as W-radical, is a hereditarily idempotent ring. Thus, it is easy to show that $S\cong(U/D_P(R))^+$ is divisible, whence $D_P(R)S=SD_P(R)=0\Rightarrow S\subseteq D_P(R)^*\Rightarrow D_P(R)\subset D_P(R)^*$ holds, a contradiction. Consequently, $L(N\cup\{\mathbf{Z}(p^\infty)|\forall p\})\cap K(N)=\emptyset$, whence by Propositions 1.4 and 1.5 we have

$$T_N=\{R|(R/N(R))^+\cong\bigoplus_p(\bigoplus_{n_p}\mathbf{Z}(p^\infty)),\ n_p\geqq 0,\ \forall p\}.$$

COROLLARY 2.6. *For the almost subidempotent radical properties* K_n, $n\geqq 2$, SV, V, W *in the corresponding classes* $T(K_n)$, $n\geqq 2$, $T(SV)$, $T(V)$, $T(W)$ *there exist the following radical properties:*

$$T_{K_n}=\{R|(R/K_n(R))^+\cong\bigoplus_p(\bigoplus_{n_p}\mathbf{Z}(p^\infty)),\ n_p\geqq 0,\ \forall p\},\quad n\geqq 2,$$

$$T_{SV}=\{R|(R/SV(R))^+\cong\bigoplus_p(\bigoplus_{n_p}\mathbf{Z}(p^\infty)),\ n_p\geqq 0,\ \forall p\},$$

$$T_V=\{R|(R/V(R))^+\cong\bigoplus_p(\bigoplus_{n_p}\mathbf{Z}(p^\infty)),\ n_p\geqq 0,\ \forall p\},$$

$$T_W=\{R|(R/W(R))^+\cong\bigoplus_p(\bigoplus_{n_p}\mathbf{Z}(p^\infty)),\ n_p\geqq 0,\ \forall p\},$$

respectively.

COROLLARY 2.7. *If* $Y'\leqq W$ *holds for a complementary radical* Y' *of a supernilpotent radical property* Y, *then in the corresponding class* $T(Y)$ *there exists the radical property*

$$T_Y=\{R|(R/Y'(R))^+\cong\bigoplus_p(\bigoplus_{n_p}\mathbf{Z}(p^\infty)),\ n_p\geqq 0,\ \forall p\}.$$

For the radical properties $F'\leqq D'\leqq\mathcal{T}'\leqq G'$ we do not know if they are contained in W. For the other radical properties $J'\leqq N'\leqq B'\leqq A\leqq R\leqq S$ and N'_g Proposition 2.5 does not work, since we have $W<J'$ (N. Jacobson [20], Example, p. 237) and $N'_g\not\leqq W$ (J. C. Robson [32]). However, for the radical R, with the following example due to C. Hopkins [18], and in which we have changed the coefficients of y, it is shown that $T(R)=\{R\}$.

EXAMPLE 2.8. Let A be the set of all $ax+by$, where $a\in\mathbf{Q}_p=\left\{\frac{n}{m}|(p,m)=1\right\}$ and $b\in\mathbf{Z}(p^\infty)$. The addition is defined in the usual way, but multiplication is as follows: $(ax+by)(cx+dy)=acx+(c*b)y$, where $m\left(\frac{n}{m}*b\right)=nb$ if $c=\frac{n}{m}\in\mathbf{Q}_p$. It is easy to show that the coefficient $\frac{n}{m}*b$ is uniquely determined. So, the ring A contains nonzero right annihilators, since $(ax+by)y=0$, and a right unity, since $(ax+by)x=ax+by$. Also, $\mathbf{Z}(p^\infty)\cong I=\{by|b\in\mathbf{Z}(p^\infty)\}\leqq A$ and $A/I\cong\mathbf{Q}_p$ hold.

Evidently, $A \in L(R \cup \{\mathbf{Z}(p^\infty)\})$, since $\mathbf{Q}_p$ is an R-radical ring, and simultaneously $A \in K(R)$. Indeed, A is not an R-radical ring, since it has nonzero right annihilators, but is an idempotent ring. Consequently, we have $A \in L(R \cup \{\mathbf{Z}(p^\infty)\}) \cap K(R) \neq \emptyset$, $\forall p$.

3. Almost subidempotent radical properties N with $N \cap B \neq \{0\}$

At first, it is easy to prove the following:

PROPOSITION 3.1. *If Y is a radical property then the class I_Y of all idempotent Y-radical rings is an almost subidempotent radical property.*

PROOF. The class I of all idempotent rings is an almost subidempotent radical class and, evidently, the class $I \cap Y = I_Y$ is the same.

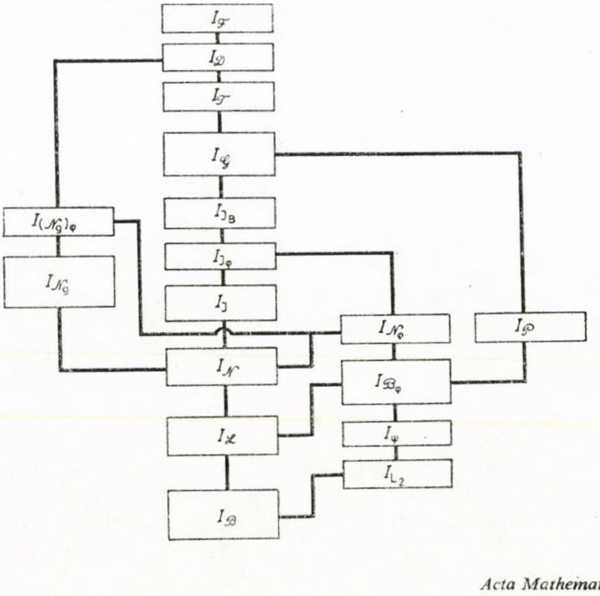
COROLLARY 3.2. *For every weakly supernilpotent radical property Y, $I_Y \cap B \neq \{0\}$ holds.*

PROOF. Indeed, every idempotent B-radical ring is an I_Y-radical ring. The rings in Examples 2.1 and 2.2 are idempotent B-radicals.

COROLLARY 3.3. *The almost subidempotent radical I_Y, where Y is a weakly supernilpotent radical property, is not subidempotent.*

PROOF. Evidently, every idempotent B-radical ring contains nonzero nilpotent ideals.

Consequently, from the known weakly supernilpotent radical properties B, N, L, Ψ (G. Tzintzis [39]), L_2 (L. C. A. van Leeuwen and G. A. P. Heyman [28]), B_φ, P, N_g, J, J_φ, N_φ, $(N_g)_\varphi$, J_B, G, $\mathcal{T}$, D, F, there arise the non subidempotent almost subidempotent radical properties I_B, I_N, I_L, I_Ψ, I_{L_2}, I_{B_φ}, I_P, I_{N_g}, I_J, I_{J_φ}, I_{N_φ}, $I_{(N_g)_\varphi}$, I_{J_B}, I_G, $I_{\mathcal{T}}$, I_D, I_F, respectively, some of which may coincide. The relationship among them is as follows:



Moreover, as for the difference among them we can say the following: $I_N < I_J$ ([10], Example 12), $I_{J_\varphi} < I_{J_B}$ ([27], Corollary 9), $I_{J_B} < I_G$ ([10], Example 11), $I_G < I_{\mathcal{T}}$ ([10], Example 8), $I_{\mathcal{T}} < I_D$ (Matrix ring $n \times n$, $n \geqq 2$, over a division ring), $I_D < I_F$ (a nonfield division ring), $I_{N_\varphi} < I_{J_\varphi}$ ([10], Example 8), $I_N < I_{N_g}$ (matrix ring $n \times n$, $n \geqq 2$ over a division ring), $I_{N_\varphi} < I_{(N_g)_\varphi}$ (matrix ring $n \times n$, $n \geqq 2$ over a division ring), $I_{(N_g)_\varphi} < I_D$ ([10], Example 8), $I_P < I_G$ ([10], Example 11), $I_J \nleqq I_P$ ([10], Example 12).

PROPOSITION 3.4. *For the almost subidempotent radical property I_{Y_φ}, where Y is a supernilpotent radical property, in the corresponding class $T(I_{Y_\varphi})$ there exists the radical $T_{I_{Y_\varphi}}$ and it coincides with Y_φ.*

PROOF. Evidently, $Y_\varphi \in T(I_{Y_\varphi})$ holds, since Y_φ is a hereditary radical property. Now, if there exists a radical class Ξ such that $\Xi \in T(I_{Y_\varphi})$ and $\Xi \nleqq Y_\varphi$, then there must exist a ring $0 \neq R \in \Xi$ and $R \notin Y_\varphi$ which may be subdirectly irreducible with Y-semisimple heart H. However, $R_\pi \supset H \neq 0 = I_{Y_\varphi}(R)$ holds for every ordinal number π, which contradicts that $R \in \Xi$.

COROLLARY 3.5. *For the almost subidempotent radical properties I_{B_φ}, I_{N_φ}, I_{J_φ}, $I_{(N_g)_\varphi}$, I_{J_B}, I_G, $I_{\mathcal{T}}$, I_D, I_F, in the corresponding classes $T(I_Y)$, where $Y = B_\varphi, N_\varphi, J_\varphi, (N_g)_\varphi, J_B, G, \mathcal{T}, D, F$, respectively, there exist the radicals T_{I_Y} and they coincide with Y.*

OBSERVATION 3.6. For the other almost subidempotent radical properties, except I_P, if some of their classes $T(I_Y)$, $Y = B, L, N, J, L_2, \Psi, N_g$, have the radical property T_{I_Y} then it is evident that $Y \leqq T_{I_Y}$ must hold since Y is hereditary. However, for the nonhereditary Jenkin's radical property P, if in the class $T(I_P)$ there exists the radical T_{I_P} then $T_{I_P} < P$ holds.

Indeed, we have $T_{I_P} \leqq P$, since otherwise T_{I_P} must contain a prime simple ring, which contradicts $T_{I_P} \cap K(P) = \emptyset$. On the other hand, there exists a P-radical ring R which is subdirectly irreducible with idempotent heart H such that $R_\omega = H$, where ω is the first limit ordinal number (L. C. A van Leeuwen and G. A. P. Heyman [27] p. 445).

4. On a theorem of N. Jacobson and a problem of G. Köthe

In 1945, N. Jacobson ([19], Theorem 10, p. 306) proved the famous theorem which we have mentioned at the beginning of this paper. The proof is based on the quasi-regularity notion and on the finitely generated left (right) ideals, as modules, of a Noetherian ring. The first natural question is whether there exists an ordinal number τ such that $J(R)^\tau = 0$, for every Noetherian ring R. There exist examples of commutative local Noetherian domains where such τ cannot be finite ([10], Example 10). However, it was known that for every commutative Noetherian ring R, $J(R)^\omega = 0$ ([41], p. 215) holds. Thus, the following assertion arises, which is usually referred to as Jacobson's conjecture: "if R is a Noetherian ring then $J(R)^\omega = 0$". But, in 1965 I. N. Herstein [17] with an example and A. V. Jategaonkar [21] later with another showed that the conjecture is not true. Later A. V. Jategaonkar [22] showed that for every ordinal number τ there exists a local p.l.i-domain R with

$J(R)^{\tau}\neq 0$. With the following, changing $J(R)^{a}$ with $J(R)_{a}$, we will show the Jacobson's theorem with a different proof.

PROPOSITION 4.1. *If R is a Noetherian ring then $J(R)_a=0$, for some ordinal number a depending on R.*

PROOF. Suppose R is a Noetherian ring. Then $I_J(R)=0$ holds. Indeed, if $I_J(R)\neq 0$ then for every ascending chain $A_1\subset A_2\subset\ldots\subset A_k\subset\ldots\subset I_J(R)$ of right (left) ideals of $I_J(R)$ if $A_1^*\subseteqq A_2^*\subseteqq\ldots\subseteqq A_k^*\subseteqq\ldots\subseteqq I_J(R)$ is the corresponding ascending chain of right (left) ideals of R, where $A_k^*=|A_k)_R=A_kR+A_k$; $\forall k=1, 2, \ldots$, there exists an integer k such that $A_k^*=A_{k+1}^*=\ldots\subseteqq I_J(R)$. If we have $A_k^*=I_J(R)$ then $I_J(R)^2=(A_k^*)^2\subseteqq A_k\subset I_J(R)$ which means that $I_J(R)$ is not an idempotent ideal, a contradiction. Thus, we must have $A_k^*\subset I_J(R)$. If now, A_k^* is not a maximal right ideal of $I_J(R)$ then there exist distinguished right ideals of $I_J(R)$ such that $A_k^*\subset B_1\subset B_2\subset\ldots\subset B_n\subset\ldots\subset I_J(R)$ and if $A_k^*\subseteqq B_1^*\subseteqq\ldots\subseteqq B_n^*\subseteqq\ldots\subseteqq I_J(R)$ is the corresponding ascending chain of right ideals of R then there exists an integer n such that $B_n^*=B_{n+1}^*=\ldots=\subset I_J(R)$. If again B_n^* is not a maximal right ideal of $I_J(R)$ then there exists a new ascending chain $B_n^*\subset C_1\subset C_2\subset\ldots\subset C_v\subset\ldots\subset I_J(R)$ of right ideals of $I_J(R)$ and so on. Consequently, after a finite number of such steps, there must exist a right ideal M of $I_J(R)$ such that $M^*\subset I_J(R)$ be a maximal right ideal. But this result contradicts that $I_J(R)$ is a J-radical ring. Now, since $J\in T(I_J)$, there exists an ordinal number π depending on R, such that $J(R)_\pi=I_J(R)=0$.

The proof of Proposition 4.1 leads to the following generalization.

PROPOSITION 4.2. *If the ring R has the A. C. C. on two-sided ideals, then for every almost subidempotent radical property $Y\leqq P$, where P is Jenkin's radical property, and for every $\tilde{Y}\in T(Y)$ there exists an ordinal number π depending both on R and $\tilde{Y}$ such that $\tilde{Y}(R)_\pi=0$.*

PROOF. Firstly, we have $Y(R)=0$. Indeed, if we use the proof of Proposition 4.1, changing only right ideals with two-sided ideals, then it is implied that there exists a maximal ideal M^* of $Y(R)$, a contradiction, since we have supposed $Y\leqq P$. Now, if $\tilde{Y}\in T(Y)$ then evidently there exists an ordinal number π depending both on R and $\tilde{Y}$ such that $\tilde{Y}(R)_\pi=Y(R)=0$.

COROLLARY 4.3. *If the ring R has the A. C. C. on two-sided ideals then there exists an ordinal number π depending on R such that $B_\varphi(R)_\pi=0$.*

PROOF. Indeed, we have $I_{B_\varphi}\leqq P$ and $T_{B_\varphi}=B_\varphi$ (Corollary 3.5).

OBSERVATION 4.4. In the class of all rings with A. C. C. on two-sided ideals, Corollary 4.3 does not work if we replace the V. A. Andrunakievič's [3] antisimple radical property B_φ with the Jacobson's [19] radical property J. Indeed, for the Sasiada's [34] prime simple ring A we have $A_\pi=A\neq 0$ for every ordinal number π.

Finally, the following corollary approaches the famous and unsolved problem of Köthe, for the nonexistence of a simple prime nil-ring.

COROLLARY 4.5. *In the class of all rings R with A. C. C. on two-sided ideals the following statements are equivalent:*

(1) $N(R)_\pi=0$, *for some ordinal number* π *depending on* R.
(2) $N(A)=0$, *for every prime simple ring* A.

PROOF. (1)$\Rightarrow$(2) is obvious. Conversely, if (2) holds then we have $I_N \leqq N \leqq P$, whence, since $N \in T(I_N)$, by Proposition 4.2 (1) follows.

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ENTIRE FUNCTIONS BOUNDED OUTSIDE A FINITE AREA

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Dedicated To G. Pólya and G. Szegő with respect and affection

0. Introduction

Let $f(z)$ be an entire function. Consider the (open) set of the z-plane defined by

(1) $$\{z\colon\ |f(z)|>B\}\quad (B>0),$$

and let

(2) $$\mu\big(|f(z)|>B\big)$$

denote its area (that is its 2-dimensional Lebesgue measure).

QUESTION. *When is it possible that*

(3) $$\mu\big(|f(z)|>B\big)<+\infty,$$

for some suitable B $(0<B<+\infty)$?

Our answer is contained in

THEOREM 1. *Let* $f(z)$ *be entire, transcendental and such that*

(4) $$\limsup_{r\to+\infty}\frac{\log\log\log M(r)}{\log r}<2\quad \big(M(r)=\max_{|z|=r}|f(z)|\big).$$

Consider, in the z-plane, the set of points

(5) $$E_R=\big\{z\colon\ R<|z|<2R,\ \log|f(z)|>\tfrac{1}{2}T(R)\big\}\quad (R>0),$$

where

(6) $$T(R)=\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta$$

is the characteristic of Nevanlinna.

Then, the open set E_R *has a 2-dimensional Lebesgue measure* $\mu(E_R)$ *which satisfies the condition*

(7) $$\mu(E_R)>R^{\delta}\quad (\delta>0,\ R>R_0(\delta)),$$

provided $\delta>0$ *has been chosen small enough.*

If (4) *is replaced by*

(8) $$\liminf_{r\to+\infty}\frac{\log\log\log M(r)}{\log r}<2,$$

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we may only assert that (7) *holds if* R *is restricted to the values* $\{R_j\}_{j=1}^{\infty}$ *of some suitable, increasing, unbounded sequence.*

As an immediate consequence of Theorem 1, we find.

COROLLARY 1.1. *Any entire function* $f(z)$ *satisfying the condition* (8) *cannot satisfy* (3) *for any fixed positive* B.

To verify that Theorem 1 is sharp, we establish the

PROPERTIES OF A SPECIAL FUNCTION. *The entire function* $\Phi(z)$, *introduced below, is such that*

$$\lim_{r\to+\infty} \frac{\log\log\log M(r)}{\log r} = 2, \quad M(r) = \max_{|z|=r} |\Phi(z)|. \tag{9}$$

It satisfies the condition

$$\mu\big(|\Phi(z)| > B\big) < +\infty, \tag{10}$$

for some suitable finite B.

Our function $\Phi(z)$ shows that the assertions of Theorem 1 no longer hold if, in (4) and (8), the symbols <2 are replaced by $\leqq 2$.

The function $\Phi(z)$ is initially introduced as an integral:

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\exp\big(\exp((\zeta \log \zeta)^2)\big)}{\zeta - z}\, d\zeta \quad (\operatorname{Re} z < e^2), \tag{11}$$

where the contour of integration Γ is the boundary of the open set

$$\Omega = \left\{z = x+iy\colon\ x > e^2,\ -\frac{\pi}{2x(\log x)^2} < y < \frac{\pi}{2x(\log x)^2}\right\}. \tag{12}$$

The orientation on Γ is the one that always leaves Ω on the right-hand side.

By modifying Γ, in (11), we verify that $\Phi(z)$ may be continued throughout the complex plane and is therefore an entire function.

The properties of $\Phi(z)$, which may have some independent interest, are summarized in our

THEOREM 2. *The entire function* $\Phi(z)$ *is real for real values of* z *and has the following properties.*

I. *There exists some constant* B_1 *such that*

$$\left(\Phi(z) - \frac{B_1}{z}\right) z^2 \quad (z \neq 0) \tag{13}$$

remains bounded for

$$z \notin S = \{z = x+iy\colon\ x > 0,\ \ -1 < y < 1\}. \tag{14}$$

II. *The expression*

$$\Phi(z) \frac{z}{(\log |z|)^2} \tag{15}$$

remains bounded for

$$|z| > e, \quad z \in S, \quad z \notin \Omega. \tag{16}$$

III. *The expression*

$$\{\Phi(z) - \exp(\exp((z \log z)^2)\} \frac{z}{(\log |z|)^2} \tag{17}$$

remains bounded for $z \in \Omega$.

Our construction of $\Phi(z)$, and our proof of Theorem 2, are straightforward adaptations of a similar construction and a similar proof given by Pólya and Szegő [3; pp. 115—116, ex. 158, 159, 160].

It follows from Theorem 2 that

$$\lim_{r \to \infty} \frac{\log \log M(r)}{(r \log r)^2} = 1, \tag{18}$$

which implies (9), and is clearly more precise. From assertions I and II of Theorem 2 we deduce the existence of a bound B $(0 < B < +\infty)$ such that $|\Phi(z)| \leq B$ $(z \notin \Omega)$. As to the area of Ω, our definition (12) implies that it is equal to

$$\pi \int_{e^2}^{+\infty} \frac{d\sigma}{\sigma (\log \sigma)^2} = \frac{\pi}{2}. \tag{19}$$

We have thus established the second property (stated above as (10)) of our special function $\Phi(z)$.

1. Proof of Theorem 1

We take for granted the following wellknown results of Nevanlinna's theory [2].

I. *The characteristic* $T(r)$, *introduced in* (6), *is a continuous, increasing function of* $r > 0$ *and*

$$\frac{T(r)}{\log r} \to +\infty \quad (r \to +\infty), \tag{1.1}$$

provided $f(z)$ *does not reduce to a polynomial.*

II. *The functions* $T(r)$ *and* $\log M(r)$ *are connected by the double inequality* [2; p. 24]

$$T(r) \leq \log M(r) \leq \frac{t+r}{t-r} T(t), \quad (0 < r < t). \tag{1.2}$$

In particular

$$\frac{1}{3} \log M\left(\frac{R}{2}\right) \leq T(R). \tag{1.3}$$

Let $U(r) > 1$ be a continuous, nondecreasing unbounded function of $r > 0$. A well-known fundamental result of E. Borel implies the following: given $\varepsilon > 0$, it is possible to find $R_0 = R_0(\varepsilon)$ such that if

$$R_0 < R \leq r \leq 2R, \quad r \notin \mathscr{E}_1(R), \tag{1.4}$$

then

$$U\left(r+\frac{r}{\{\log U(r)\}^{1+\varepsilon}}\right) < eU(r). \tag{1.5}$$

The exceptional set $\mathscr{E}_1(R)$ is a measurable subset of the interval $[R, 2R]$ and its Lebesgue linear measure $\lambda(\mathscr{E}_1(R))$ is such that

$$\frac{\lambda(\mathscr{E}_1(R))}{R} \to 0 \quad (R \to +\infty). \tag{1.6}$$

The consequences of Borel's lemma stated in (1.4), (1.5) and (1.6) are found in a paper of Edrei and Fuchs [1; p. 341].

In the following proof we apply (1.5) with $U(r)$ replaced by $T(r)$ and always take R large enough to imply

$$\lambda(\mathscr{E}_1(R)) < \frac{R}{2}, \quad \log U(R) > 1. \tag{1.7}$$

Hence, taking

$$t = \frac{r}{\{\log T(r)\}^{1+\varepsilon}},$$

we deduce from (1.2), (1.5) and (1.7)

$$\log M(r) < 3e\, T(r)\{\log T(r)\}^{1+\varepsilon}, \tag{1.8}$$

provided

$$r \in D_R = \{r\colon R < r < 2R, \quad r \notin \mathscr{E}_1(R)\} \quad (R > R_0). \tag{1.9}$$

In view of (1.7), the one-dimensional set D_R has Lebesgue measure

$$\lambda(D_R) > \frac{R}{2}. \tag{1.10}$$

Introduce the set of values of θ defined by

$$\Lambda(r) = \left\{\theta\colon \log|f(re^{i\theta})| > \tfrac{1}{2}T(R),\ 0 < \theta < 2\pi\right\}; \tag{1.11}$$

for every $r>0$, $\Lambda(r)$ is an open subset of the interval $(0, 2\pi)$. Denote by $\lambda(\Lambda(r))$ the one-dimensional Lebesgue measure of $\Lambda(r)$. The definition of $\mu(E_R)$, as a two-dimensional Lebesgue measure, and Fubini's theorem yield

$$\mu(E_R) = \iint r\,dr\,d\theta = \int_R^{2R} r\,dr \int_{\Lambda(r)} d\theta = \int_R^{2R} r\lambda(\Lambda(r))\,dr, \tag{1.12}$$

where the double integral in (1.12) is extended to all points $z=re^{i\theta}\in E_R$.

By (1.9) and (1.12)

$$\mu(E_R) \geqq \int_{\Delta_R} r\lambda(\Lambda(r))\,dr. \tag{1.13}$$

To complete the proof we note that the definition of $T(r)$ (in (6)) and (1.11) imply

$$T(r) \leqq \frac{1}{2\pi} \int_{A(r)} \log M(r)\, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} T(R)\, d\theta.$$

Hence, in view of (1.8), (1.9) and the increasing character of $T(r)$, we find

$$\frac{1}{2} T(r) < \frac{3e}{2\pi} T(r) \big(\log T(r)\big)^{1+\varepsilon} \lambda\big(A(r)\big) \quad (r \in D_R,\ r > r_0),$$

$$\lambda\big(A(r)\big) > e^{-1} \big(\log T(r)\big)^{-1-\varepsilon} \quad (r \in D_R,\ r > r_0),$$

which used in (1.13) yields

$$\mu(E_R) \geqq e^{-1} \int_{\Delta_R} r \{\log T(2R)\}^{-1-\varepsilon}\, dr \geqq e^{-1} R \{\log T(2R)\}^{-1-\varepsilon} \lambda(D_R),$$

and finally by (1.10)

$$\mu(E_R) > \tfrac{1}{2} e^{-1} R^2 \{\log T(2R)\}^{-1-\varepsilon} \quad \big(R > R_0(\varepsilon)\big). \tag{1.14}$$

Up to this point we have not selected $\varepsilon > 0$, nor have we used (4) or the weaker assumption (8).

Assume for instance that (8) holds. Then, if $\eta > 0$ is small enough,

$$\log T(r) \leqq \log \log M(r) < r^{2(1-\eta)}, \tag{1.15}$$

as $r \to +\infty$ by values of a suitable increasing, unbounded sequence which we may write as $\{2R_j\}_{j=1}^{\infty}$. Take, in (1.14), $\eta = \varepsilon$, $R = R_j$ and note that since (1.15) now implies

$$\big(\log T(2R_j)\big)^{1+\varepsilon} < (2R_j)^{2(1-\eta^2)} \quad \big(j > j_0(\eta)\big),$$

we obtain

$$\mu(E_R) > (e^{-1}/8) R^{2\eta^2} \quad \big(R = R_j,\ j > j_0(\eta)\big). \tag{1.16}$$

This proves that, under the assumption (8), (7) holds with $R = R_j$, $j > j_0$.

The validity of (7) under the assumption (4) is obvious because then (1.16) holds for all sufficiently large values of R and not only for $R = R_j$. The proof of the Theorem is now complete.

2. Contours of integration

Let σ be a positive variable and γ a positive parameter which is restricted by the conditions

$$\frac{3}{4} \leqq \gamma \leqq \frac{5}{4}. \tag{2.1}$$

Assume that γ is fixed and consider, in the complex plane, the analytic arc described by

$$\zeta(\sigma;\gamma) = \sigma + i\tau(\sigma;\gamma), \quad \tau(\sigma;\gamma) = \frac{\pi\gamma}{2\sigma(\log\sigma)^2} \quad (e \leqq \sigma < +\infty). \tag{2.2}$$

We denote by $L_+(\alpha;\gamma)$ the arc described by $\zeta(\sigma;\gamma)$ as $\alpha\leqq\sigma<+\infty$, by $L_-(\alpha;\gamma)$ the symmetrical arc described by $\sigma-i\tau$ and by $V(\alpha;\gamma)$ the vertical segment

(2.3) $$V(\alpha;\,\gamma)=\{z=x+iy\colon\, x=\alpha,\ -\tau(\alpha;\,\gamma)\leqq y\leqq\tau(\alpha;\,\gamma)\}.$$

Denoting, as usual, opposite arcs by L and $-L$, we consider systematically contours of integration

(2.4) $$C(\alpha;\,\gamma)=-L_-(\alpha;\,\gamma)+V(\alpha;\,\gamma)+L_+(\alpha;\,\gamma)\quad\left(\alpha\geqq e,\ \frac{3}{4}\leqq\gamma\leqq\frac{5}{4}\right).$$

All the points $z\notin C(\alpha;\gamma)$ fall in two disjoint open regions. One of them:

(2.5) $$\Delta(\alpha;\gamma)=\{z=x+iy,\ x>\alpha,\ -\tau(x;\gamma)<y<\tau(x,\gamma)\}$$

has a finite area. (This fact is an obvious consequence of (19)).

The other one, which contains the whole negative axis, will be denoted by $\tilde{\Delta}(\alpha;\gamma)$.

3. The function $\Phi(z)$ is entire

Consider in the half-plane $\operatorname{Re} z\geqq 2$ the analytic function

(3.1) $$F(z)=\exp(e^{(z\log z)^2})\quad(\log e=1),$$

where the branch of $\log z$ is determined by its value at e.

We shall first verify that for any $\gamma\in[3/4,\,5/4]$

(3.2) $$\int\limits_{L_+(e^2,r)}|F(\zeta)|\,|d\zeta|=\int\limits_{e^2}^{+\infty}|F(\zeta)|\left|\frac{d\zeta}{d\sigma}\right|d\sigma<+\infty.$$

This follows at once from

(3.3) $$\frac{d\zeta}{d\sigma}\to 1\quad(\sigma\to+\infty,\ \gamma\text{ fixed})$$

and from the elementary estimates contained in

LEMMA 3.1. *If* $\zeta\in L_+(e^2;\gamma)3/4\leqq\gamma\leqq 5/4)$ *then*

(3.4) $$F(\zeta)=\exp\left(e^{(\sigma\log\sigma)^2}e^{i\pi\gamma}\left\{1+\frac{A\omega}{\log\sigma}\right\}\right)\quad(\operatorname{Re}\zeta=\sigma\geqq e^2,\quad \omega=\omega(\sigma,\gamma)),$$

where, in the error term,

$0<A=$*absolute const.*, $|\omega(\sigma,\gamma)|\leqq 1$.

Moreover, if $\alpha\geqq\alpha_0>e^2$ *and if* α_0 *is large enough, then*

(3.5) $$\left|F\left(\alpha+\frac{i\pi\gamma}{2\alpha(\log\alpha)^2}\right)\right|\leqq\exp\left(-\frac{1}{2}e^{(\alpha\log\alpha)^2}\right)\quad\left(\alpha\geqq\alpha_0,\ \frac{3}{4}\leqq\gamma\leqq\frac{5}{4}\right).$$

PROOF. An elementary evaluation shows that (2.1) and (2.2) imply

$$(3.6)\qquad (\zeta \log \zeta)^2 = (\sigma \log \sigma)^2 + i\pi\gamma + \frac{A\omega}{\log \sigma} \quad (\omega = \omega(\sigma, \gamma), \quad |\omega| \leqq 1).$$

In (3.6), and throughout the paper, we denote by ω a complex quantity, which may depend on all the parameters of the problem, but is always of modulus $\leqq 1$. The symbol ω, as well as A (our symbol for positive absolute constants), may assume different values at each occurrence.

We note that, with this convention,

$$(3.7)\qquad e^u = 1 + \omega u e^{|u|}.$$

It is obvious that (3.6) and (3.7) yield (3.4). Observing that

$$\operatorname{Re} e^{i\pi\gamma}\left\{1 + \frac{A\omega}{\log \alpha}\right\} \leqq \cos(\pi\gamma) + \frac{A}{\log \alpha} < -\frac{1}{2} \quad \left(\frac{3}{4} \leqq \gamma \leqq \frac{5}{4}, \ \alpha \geqq \alpha_0\right),$$

we deduce (3.5) from (3.4).

This completes the proof of Lemma 3.1.

Now the integrals in (3.2) are clearly convergent by (3.3) and (3.4). Noticing that the contour Γ, which appears in the definition (11) of $\Phi(z)$, coincides with $C(e^2; 1)$ defined in (2.4), we may rewrite

$$(3.8)\qquad \Phi(z) = \frac{1}{2\pi i} \int\limits_{C(e^2;1)} \frac{F(\zeta)}{\zeta - z}\, d\zeta \quad (\operatorname{Re} z < e^2).$$

This shows that $\Phi(z)$ is a function holomorphic in the half-plane

$$(3.9)\qquad \operatorname{Re} z < e^2.$$

The fact that $C(e^2; 1)$ has the real axis for axis of symmetry, and that $F(z)$ is real for real z, shows that $\Phi(z)$ is real for real z.

By Cauchy's theorem, under the restriction (3.9), we may replace the representation (3.8) by

$$(3.10)\qquad \Phi(z) = \frac{1}{2\pi i} \int\limits_{C(\alpha;1)} \frac{F(\zeta)}{\zeta - z}\, d\zeta \quad (\alpha > e^2)$$

and let $\alpha \to +\infty$. This step is certainly justified because $F(z)$ is holomorphic throughout $\operatorname{Re} z \geqq 2$. The form (3.10) shows that our original function, given by (3.8), may be continued throughout $\operatorname{Re} z < \alpha$. Hence $\Phi(z)$ is in fact an entire function.

4. Proof of assertions I and II of Theorem 2

If $z \in \tilde{\Delta}(e^2; 1)$, Cauchy's theorem and (3.5) show that we may use the representation

$$(4.1)\qquad \Phi(z) = \frac{1}{2\pi i} \int\limits_{C(e^2;3/4)} \frac{F(\zeta)}{\zeta - z}\, d\zeta,$$

instead of (3.8).

Using in (4.1) the identity

$$\frac{1}{\zeta - z} = -\frac{1}{z} - \frac{\zeta}{z^2} + \frac{\zeta^2}{z^2(\zeta - z)} \quad (z \neq 0), \tag{4.2}$$

and writing

$$B_1 = -\frac{1}{2\pi i} \int_{C_1} F(\zeta)\, d\zeta, \quad B_2 = -\frac{1}{2\pi i} \int_{C_1} \zeta F(\zeta)\, d\zeta, \quad C_1 = C\left(e^2, \frac{3}{4}\right), \tag{4.3}$$

we find

$$\Phi(z) = \frac{B_1}{z} + \frac{B_2}{z^2} + \frac{1}{2\pi i z^2} \int_{C_1} \frac{\zeta^2 F(\zeta)}{\zeta - z}\, d\zeta \quad (z \in \tilde{\Delta}(e^2;\, 1)). \tag{4.4}$$

To complete the proof of assertions I and II of Theorem 2, there only remains to estimate the integral in (4.4). It is clear that its modulus cannot exceed

$$\frac{1}{\delta_1(z)} \int_{C_1} |\zeta|^2 F(\zeta)\, |d\zeta|, \tag{4.5}$$

where $\delta_1(z)$ denotes the shortest distance between z and the contour C_1.

If $z \notin S$, an inspection of (12) and (14) shows that

$$\delta_1(z) > (9/10), \tag{4.6}$$

and hence (4.4) yields

$$\left|\Phi(z) - \frac{B_1}{z} - \frac{B_2}{z^2}\right| \leq \frac{(10/9)}{2\pi |z|^2} \int_{C_1} |\zeta|^2 F(\zeta)|\, d\zeta = \frac{B_3}{|z|^2}.$$

Assertion I of Theorem 2 is now obvious. To obtain assertion II of Theorem 2 it suffices to replace, in the previous proof, the inequality (4.6) by another one, valid under the restrictions (16).

If

$$\operatorname{Re} z = x > e^2 + 1, \quad y \geq \frac{\pi}{2x(\log x)^2},$$

we have

$$\delta_1(z) \geq \frac{\pi}{2x(\log x)^2} - \max_{x-1 \leq \sigma \leq x+1} \frac{3\pi}{8\sigma \log \sigma)^2} = \tag{4.7}$$

$$= \frac{\pi}{2}\left(\frac{1}{x(\log x)^2} - \frac{3}{4(x-1)(\log(x-1))^2}\right),$$

$$\delta_1(z) \geq \frac{\pi}{10x(\log x)^2} \quad (x \geq x_0 > e^2 + 1) \tag{4.8}$$

provided x_0 is chosen large enough. Using (4.8) in (4.5) and returning to (4.4) we find, for some suitable constant $B_4 > 0$,

$$\Phi(z) = \frac{B_1}{z} + \frac{B_2}{z^2} + \frac{B_4 \omega}{z^2} x (\log x)^2 \quad (z \in \tilde{\Delta}(x_0,\, 1)).$$

Hence the expression (15) remains bounded for

$$|z| \geqq x_0+1, \quad z\in S, \quad z\notin \Omega. \tag{4.9}$$

Since $\Phi(z)$ is entire it is also bounded in the disk $|z|\leqq x_0+1$. This enables us to replace the restrictions (4.9) by the less restrictive conditions (16). The proof of assertion II of Theorem 2 is now complete.

5. Proof of assertion III of Theorem 2

We first confine z to an open rectangle

$$\mathscr{R} = \left\{z = x+iy\colon\, e^2-1 < x < e^2, -\frac{\pi}{8e^2} < y < \frac{\pi}{8e^2}\right\}. \tag{5.1}$$

Let H be the contour of integration formed by the boundary of $\mathscr{R}$, taken in the positive sense. A first application of Cauchy's theorem yields

$$\frac{1}{2\pi i}\int_H \frac{F(\zeta)}{\zeta-z}\,d\zeta = \exp\left(\exp\left((z\log z)^2\right)\right), \tag{5.2}$$

and consequently

$$\Phi(z)-\exp\left(\exp\left((z\log z)^2\right)\right) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{F(\zeta)}{\zeta-z}\,d\zeta, \tag{5.3}$$

where Γ_1 is the contour formed by the juxtaposition of $-L_{-1}(e^2;1)$, three sides of $\mathscr{R}$, and $L_+(e^2;1)$.

It is obvious that the integral in (5.3) yields the analytic continuation of the left-hand side of (5.3) throughout the open region (of finite area) enclosed by Γ_1.

In particular (5.3) is valid for all points $z\in\Omega$. A new application of Cauchy's theorem and (3.5) enable us to replace (5.3) by

(5.4)

$$\Phi(z)-\exp\left(\exp\left((z\log z)^2\right)\right) = \frac{1}{2\pi i}\int_{C_2}\frac{F(\zeta)}{\zeta-z}\,d\zeta \quad \left(C_2 = C\left(e^2;\frac{5}{4}\right),\ z\in\Omega\right).$$

We now repeat the argument in §4: from (4.2) and (5.4) we see that, instead of (4.4), we obtain

$$\Phi(z)-\exp\left(\exp\left((z\log z)^2\right)\right) = \frac{B_1}{z}+\frac{B_2}{z^2}+\frac{1}{2\pi i z^2}\int_{C_2}\frac{\zeta^2F(\zeta)}{\zeta-z}\,d\zeta \quad (z\in\Omega). \tag{5.5}$$

The constants B_1 and B_2 are again given by (4.3) because, by Cauchy's theorem and (3.5), the values of the relevant integrals are not affected when the contour of integration C_1 is replaced by C_2.

To complete the proof of assertion III of Theorem 2 we need a lower bound for the distance $\delta_2(z)$ between z and C_2. As in (4.7), we find

$$\delta_2(z) \geqq \min_{x-1\leqq\sigma\leqq x+1}\left\{\frac{(5/4)\pi}{2\sigma(\log\sigma)^2}\right\}-\frac{\pi}{2x(\log x)^2},$$

provided $x \geqq e^2+1$, $z \in \Omega$. Hence, if x_1 is choosen large enough

$$\delta_2(z) > \frac{\pi}{10x(\log x)^2} \quad (x \geqq x_1 \geqq e^2+1). \tag{5.6}$$

Using (5.6) in (5.5) we complete the proof of assertion III of Theorem 2 by the arguments which led to the proof of assertion II.

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EXTENDING COMPATIBLE PROXIMITIES

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In 1930 F. Hausdorff [4] showed that a compatible metric defined on a closed subset of a metrizable space X can be extended to a compatible metric on X. R. H. Bing [1] rediscovered this result in 1947.

In 1978 L. Úry [5] proved that for any completely regular topological space $(X, \mathcal{T})$ every compatible uniformity on each closed subset of X can be extended to a compatible uniformity on $(X, \mathcal{T})$ iff $\mathcal{T}$ is collectionwise normal.

Our aim is to prove the following theorem.

Let $(X, \mathcal{T})$ be a completely regular topological space. Then every compatible proximity on each closed subset of X can be extended to a compatible proximity on X iff $\mathcal{T}$ is normal.

First observe that a proximity $\mathcal{P}$ defined on a set X is said to be compatible with the topological space $(X, \mathcal{T})$ if $\mathcal{T}$ is induced by $\mathcal{P}$ (cf. [2] p. 124). (In the sequel we shall use the notions and notations of [2].)

For any closed subset S of a topological space $(X, \mathcal{T})$ we say that S is *strongly P-embedded* in X if for every proximity $\mathcal{P}_1$ defined on the set S and compatible with $(S, \mathcal{T}|S)$ there is a proximity $\mathcal{P}$ defined on the set X compatible with $(X, \mathcal{T})$ and such that $\mathcal{P}_1=\mathcal{P}|S$.

According to this terminology we can reformulate our theorem as follows.

For any completely regular topological space $(X, \mathcal{T})$ every closed subset of X is strongly P-embedded in X iff $\mathcal{T}$ is normal.

Proof. Suppose that $\mathcal{T}$ is not a normal topology and let A and B be disjoint closed subsets of X without disjoint neighbourhoods. Let $S=A\cup B$. Let $\mathcal{P}_1$ be the finest proximity defined on S and inducing $\mathcal{T}|S$. Then $A\overline{\mathcal{P}}_1B$ holds since A and B can be separated by a continuous function. Namely, the function $f\colon S\to\mathbf{R}$ defined by

$$f(p)=\begin{cases}0 & \text{if} \quad p\in A\\ 1 & \text{if} \quad p\in B\end{cases}$$

is continuous on $(S, \mathcal{T}|S)$.

This proximity $\mathcal{P}_1$ cannot be extended to a proximity $\mathcal{P}$ defined on X and inducing $(X, \mathcal{T})$.

In fact suppose that $\mathcal{P}$ is a proximity defined on the set X and inducing $(X, \mathcal{T})$. Then $A\mathcal{P}B$ since otherwise A and B would have disjoint neighbourhoods in $(X, \mathcal{T})$ contradicting the assumption. However $A\mathcal{P}B$ and $A\overline{\mathcal{P}}_1B$ imply $\mathcal{P}_1\neq\mathcal{P}|S$ which proves the first part of the theorem.

Now suppose that $\mathcal{T}$ is normal and let S be a closed subset of X.

Let $\mathcal{P}_1$ be a proximity defined on S and inducing the topology $\mathcal{T}|S$. Let Φ_1 be the set of all bounded $\mathcal{P}_1$ proximally continuous functions and Φ the set of all bounded continuous extensions to $(X, \mathcal{T})$ of them. Then $\mathcal{P}_{\Phi_1}=\mathcal{P}_1$ (see [2] p. 171) and $\mathcal{P}_\Phi|S=\mathcal{P}_{\Phi_1}$ (see [2] p. 168). Thus we have only to show that $\mathcal{T}_\Phi=\mathcal{T}$.

Since each element of Φ is a continuous function with respect to $\mathcal{T}$ and $\mathcal{T}_\Phi$ is the coarsest topology on X for which all $f\in\Phi$ are continuous we need only to show that $\mathcal{T}$ is coarser than $\mathcal{T}_\Phi$ i.e. that for each $p\in X$ and for each open $\mathcal{T}$ neighbourhood V of p in X, V is a $\mathcal{T}_\Phi$ neighbourhood of p as well.

We have to consider two particular cases.

1. $p\in S$. Then $V\cap S$ is a $\mathcal{T}|S$ neighbourhood of p and thus it is a $\mathcal{P}_1$-proximal neighbourhood of $\{p\}$, i.e. $\{p\}\ \overline{\mathcal{P}}_1 S-(V\cap S)$. Hence there is an $f_1\in\Phi_1$ separating $\{p\}$ and $S-(V\cap S)$, i.e. for which $f_1(p)=0$ and $f_1(S-V\cap S)\subset\{1\}$. Now let $f^*\colon (X-V)\cup S\to\mathbf{R}$ be defined by $f^*|S=f_1$ and $f^*(X-V)\subset\{1\}$. f^* is clearly well defined. Since $f^*|S=f_1$ is continuous with respect to $\mathcal{T}|S$ and $f^*|X-V$ is continuous with respect to $\mathcal{T}|X-V$; moreover since both S and $X-V$ are closed sets in $(X, \mathcal{T})$, it follows that f^* is a bounded continuous function on $((X-V)\cup S, \mathcal{T}|(X-V)\cup S)$. However $\mathcal{T}$ is a normal topology and $(X-V)\cup S$ is a closed set. Consequently according to the Tietze—Urysohn extension theorem there is an $f\in\Phi$ such that $f|(X-V)\cup S=f^*$ (see [3] p. 97). Hence $f(p)=0$ and $f(X-V)\subset\{1\}$ and thus the $\mathcal{T}$ neighbourhood

$$V'=\{q\in X\colon\ |f(q)-f(p)|<1\}$$

of p lies in V. V is a $\mathcal{T}_\Phi$ neighbourhood of p indeed.

2. $p\notin S$. Let $f\colon X\to\mathbf{I}$ be a continuous function for which $f(p)=0$ and $f((X-V)\cup S)\subset\{1\}$. Since $V-S$ is an open neighbourhood of p and $\mathcal{T}$ is completely regular, such a function exists. Moreover $f|S$ is a constant function and thus $f|S\in\Phi_1$. Hence $f\in\Phi$. However the $\mathcal{T}$ neighbourhood

$$V'=\{q\in X\colon\ |f(p)-f(q)|<1\}$$

of p is contained in V and thus V becomes a $\mathcal{T}_\Phi$ neighbourhood of p in this particular case too.

The proof of the theorem is complete.

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MULTIPLICATIVE FUNCTIONS WITH REGULARITY PROPERTIES. V

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1. Introduction This paper is a continuation of [1], [2], [3], [4]. The notations $\mathcal{M}^*$, $\mathcal{L}$ stated in [3] will be preserved here. We should like to determine all $f, g \in \mathcal{M}^*$ that satisfy

$$\sum_{n=1}^{\infty} \frac{|g(n+K)-f(n)|}{n} < \infty. \tag{1.1}$$

This problem has been solved completely in [3] for $K=2$ or $K=$ odd (see Theorem 4). Recalling the previous results we may assume that $f(p)=0$ and $g(p)=0$ if $p|K$ and $f(p)\neq 0$, $g(p)\neq 0$ if $(p, K)=1$. ([3], Theorem 1).

In [4] we determined all the solutions of (1.1) for every K under the additional condition

$$|g(n)| \geqq 1, \quad |f(n)| \geqq 1 \quad \text{if} \quad (n, K) = 1. \tag{1.2}$$

Our purpose is to give the solutions without the assumption (1.2).

THEOREM 1. *Let $f, g \in \mathcal{M}^*$ satisfy* (1.1) *with a positive integer K. Assume that $f(n)=g(n)=0$ if $(n, K)>1$, $f(n)\neq 0$, $g(n)\neq 0$ if $(n, K)=1$. Then*

(a) $$f, g \in \mathcal{L}, \quad \textit{or}$$

(b) $$f(n) = n^s F(n), \quad g(n) = n^s G(n), \quad 0 \leqq \operatorname{Re} s < 1,$$

$$G(n+K) = F(n) \quad (\forall n \in N). \tag{1.3}$$

Conversely, if (a) *or* (b) *holds with $F, G \in \mathcal{M}^*$ then* (1.1) *is true.*

REMARK. The solutions of (1.3) have been completely determined in [4] Theorem 1.

2. Proof. It is enough to prove that if $f \notin \mathcal{L}$, $g \notin \mathcal{L}$, then $|f(n)|=|g(n)|$. Indeed, using the notation $h(n)=|f(n)|=|g(n)|$, from (1.1) we get immediately

$$\sum \frac{|h(n+K)-h(n)|}{n} < \infty,$$

and this by $h \notin \mathcal{L}$ implies $h(n)=n^\sigma$, $0 \leqq \sigma < 1$. Consequently, condition (1.2) holds and this case has been treated in [4].

Since $|g(n+K)-f(n)| \leqq \| g(n+K)|-|f(n)\||$, we may assume that $g(n)\geqq 0$, $f(n)\geqq 0$. Let $H(n)=\dfrac{g(n)}{f(n)}$. $H(n)$ is well defined for $(n, K)=1$, and $H(n)>0$.

Let C be an arbitrary positive integer such that $(C, K)=1$, $C-1=\lambda\varrho$, $(\varrho, K)=1$, $(\lambda+1, K)=1$, where all the prime factors of λ divide K. We shall prove that

$$H(\lambda+1) = H(C). \tag{2.1}$$

By using (1.1) several times we get

$$\sum_N \frac{1}{N} |g(C)f((\lambda+1)\varrho N)-g(C)g((\lambda+1)\varrho N+K)| < \infty,$$

$$\sum_N \frac{1}{N} |g((\lambda+1)\varrho CN+KC)-f((\lambda+1)\varrho CN+K\lambda\varrho)| < \infty,$$

$$\sum_N \frac{1}{N} |f(\varrho)f(\lambda+1)CN+K\lambda)-f(\varrho)g((\lambda+1)CN+K(\lambda+1))| < \infty,$$

$$\sum_N \frac{1}{N} |f(\varrho)g(\lambda+1)g(CN+K)-f(\varrho)g(\lambda+1)f(CN)| < \infty.$$

Collecting these relations we get

$$\sum_N \frac{1}{N} |g(C)f(\lambda+1)f(\varrho)-f(\varrho)g(\lambda+1)f(C)||f(N)| < \infty.$$

Since $f\notin \mathcal{L}$, hence (2.1) follows immediately.

We are almost ready now. Let $\lambda=K^*$ be composed from the prime factors of K and contain each primefactor of K at least on the first power. Let $C=1+K^*\varrho$, $(\varrho, K)=1$. Then $(\lambda+1, K)=1$ obviously holds, so from (2.1) we get

$$H(1+K^*\varrho) = H(K^*+1). \tag{2.2}$$

Let ν be a positive integer coprime to K. Since $(1+K^*)^\nu=1+\nu K^*+\ldots=1+\varrho K^*$, $(\varrho, K)=1$, we get immediately that $H((1+K^*)^\nu)=H(1+K^*)=1$. Let $n\equiv 1 \pmod K$ be an arbitrary positive integer. Since it can be written as $n=1+K^*\varrho$, $(\varrho, K)=1$ for a suitably chosen K^*, therefore $H(n)=1$. Since for every m comprime to K, $m^{\varphi(K)}\equiv 1 \pmod K$ and H is multiplicative, we get $H(m)^{\varphi(K)}=1$. Since $H(m)>0$, we get $H(m)=1$. So we have proved that $H(n)=1$ for every n coprime to K.

By this the proof of our theorem has been completed.

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DIVERGENCE OF TRIGONOMETRIC LACUNARY INTERPOLATION

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1. Introduction and preliminary results

1.1. In their papers [1], [2] and [3] J. Balázs, J. Surányi and P. Turán investigated the so called (0, 2) algebraic interpolatory polynomials.

For the unique trigonometric polynomials $\bar{R}_n(f, x)$ of order n having the property that for a fixed integer $M \geqq 2$

$$\bar{R}_n(f, x_{kn}) = f(x_{kn}), \quad \bar{R}_n^{(M)}(f, x_{kn}) = \beta_{kn}, \quad k = 0, 1, \ldots, n-1, \tag{1.1}$$

(where $f \in \tilde{C}$ ($= f$ is continuous and 2π-periodic),

$$x_{kn} = \frac{2\pi k}{n}, \quad k = 0, 1, \ldots, n-1, \tag{1.2}$$

and β_{kn} are given real numbers), explicit formulae were found by O. Kis [4] ($M=2$) and later by A. Sharma and A. K. Varma [5] ($M \geqq 2$).

1.2. From now on we assume that $M=2, 4, 6, \ldots$ and $n=1, 3, 5, \ldots$. We quote the following result.

THEOREM 1.1 ([4], [5]). *The trigonometric polynomials* $\bar{R}_n(f, x)$ *given by*

$$\bar{R}_n(f, x) = \sum_{k=0}^{n-1} f(x_{kn}) F_n(x - x_{kn}) + \sum_{k=0}^{n-1} \beta_{kn} G_n(x - x_k), \quad n = 1, 3, 5, \ldots, \tag{1.3}$$

satisfy conditions (1.1), *provided that for a fixed even* M

$$F_n(x) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{n-1} \frac{(n-j)^M \cos jx}{(n-j)^M - j^M}\right], \quad n = 1, 3, 5, \ldots, \tag{1.4}$$

and

$$G_n(x) = (-1)^{(M-2)/2}\left[\frac{1 - \cos nx}{n^{M+1}} + \frac{2}{n}\sum_{j=1}^{n-1} \frac{\cos jx}{(n-j)^M - j^M}\right], \quad n = 1, 3, 5, \ldots. \tag{1.5}$$

1.3. Let us suppose that the function $\omega_M(t)$ satisfies the conditions

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(i) $\omega_M(t)>0$ for $t>0$, $\omega_M(0)=0$, $\omega_M(T)>\omega_M(t)$ if $T>t$, $\omega_M(t)$ is continuous for $t\geqq 0$,

(ii) $\dfrac{t^M}{\omega_M(t)}$ is monotonically increasing for $t\geqq 0$,

(iii) $\lim\limits_{t\to+0}\dfrac{t^M}{\omega_M(t)}=0$.

Sometimes we will suppose that for an arbitrary fixed $k=1,2,\ldots$,

(iv)
$$\sum_{j=1}^{k}2^{Mj}\omega_M\left(\frac{1}{2^j}\right)=O(1)2^{Mk}\omega_M\left(\frac{1}{2^k}\right).$$

For example $\omega_M(t)=t^\alpha$ $(0<\alpha<M)$ fulfils all the conditions.

Let us denote by $\tilde{C}(\omega_M)$ the class of all $f\in\tilde{C}$ for which $\omega_M(f,t)\leqq a(f)\omega_M(t)$. If all the β_{kn} are zero, i.e.

$$\bar{R}_n(f,x)=R_n(f,x):=\sum_{k=0}^{n-1}f(x_{kn})F_n(x-x_{kn}) \tag{1.6}$$

we have the following result

THEOREM 1.2 ([4], [5], [6]). *If $f\in\tilde{C}(\omega_M)$ where $\omega_M(t)$ now satisfies* (i), (ii) *and* (iv) *then*

$$\|R_n(f,x)-f(x)\|=O(n)\omega_M\left(\frac{1}{n}\right),\quad n=1,3,5,\ldots. \tag{1.7}$$

On the other hand, if $\omega_M(t)$ fulfils (i), (ii) *and* (iii) *then there exists an $f\in\tilde{C}(\omega_M)$ and a sequence $\{n_j\}$ such that*

$$|R_n(f,\pi)-f(\pi)|>n\omega_M\left(\frac{1}{n}\right),\quad n=n_1,n_2. \tag{1.8}$$

(Here $\|g\|=\max\limits_{0\leqq x<2\pi}|g(x)|$; throughout the theorem we supposed that M is a fixed even number.)

1.4. If we consider the trigonometric interpolatory polynomials $L_n(f,x)$ based on the nodes (1.2) we know that for a suitable $g\in\tilde{C}$, $L_n(g,x)$ do not converge uniformly to $g(x)$ on $[0,2\pi)$ when $n\to\infty$. On the other hand, J. Marcinkiewicz [10] proved the following proposition.

THEOREM 1.3. *For any $f\in\tilde{C}$ we have*

$$\lim_{n\to\infty}\int_0^{2\pi}|L_n(f,x)-f(x)|^p\,dx=0,\quad p>0.$$

Therefore, as P. Turán suggested in connection with the algebraic case, we may hope that considering mean convergence we obtain better convergence result. Our expectations seem to be even more reasonable if we remark that for any $f\in\tilde{C}$

$$\lim_{n\to\infty}\int_0^{2\pi}R_{2m+1}(f,x)\,dx=\int_0^{2\pi}f(x)\,dx \tag{1.8}$$

if M is even.

(In order to see (1.8), we consider (1.3) and (1.4) from where

$$\int_0^{2\pi} R_n(f,x)\,dx = \sum_{k=0}^{n-1} f(x_{kn}) \int_0^{2\pi} F(x-x_{kn})\,dx = \frac{2\pi}{n}\sum_{k=0}^{n-1} f(x_{kn}), \quad n = 1, 3, \ldots,$$

a Riemann sum, which, of course, tends to the corresponding integral. Actually, the analogous relations for the algebraic case were found very recently by A. K. Varma [7].)

2. New results

2.1. We intend to prove the following. Let $p(x)\geqq 0$ be a summable weight function, $p(x)\geqq\alpha>0$ on a set $P\subset[0, 2\pi)$ of positive measure and let ω_M satisfy (i), (ii) and (iii). As above, $M=2, 4, 6, \ldots$, $n=1, 3, 5, \ldots$.

THEOREM 2.1. *For any given* $\gamma>0$ *there exist a function* $g\in\tilde{C}(\omega_2)$ *and a sequence* $\{n_i\}$ *such that* $n_i\to\infty$ *when* $i\to\infty$ *and*

$$\text{(2.1)} \qquad \Big\{\int_0^{2\pi} |R_n(g,x)-g(x)|^\gamma p(x)\,dx\Big\}^{1/\gamma} > n\omega_M\left(\frac{1}{n}\right), \quad n = n_1, n_2, \ldots.$$

2.2. Thus, by Theorems 1.2 and 2.1, we can state the rather unusual fact, that for the function class $\tilde{C}(\omega_M)$ the necessary and sufficient condition for the uniform and mean convergence generally is the same: one has to assume that $\omega_M(t)=o(t)$. (For the algebraic case, see P. Vértesi [12]).

2.3. Let $\varphi_n:=n^M\omega_M(1/n)$. By (iii), $\lim\limits_{n\to\infty}\varphi_n=\infty$. Now Theorem 2.1 can be obtained from the next statement.

THEOREM 2.2. *Let* $\{\varepsilon_n\}$ *be any sequence of positive numbers such that* $\lim\limits_{n\to\infty}\varphi_n\varepsilon_n^2=\infty$ *and* $\lim\limits_{n\to\infty}\varepsilon_n^2 n=\infty$. *Then there exist a function* $h\in\tilde{C}(\omega_M)$, *sets* $H_n\subset[0, 2\pi)$ *and a sequence* $\{n_i\}$ *such that*

$$\text{(2.2)} \qquad |H_n| \geqq 2\pi-\varepsilon_n, \quad n = n_1, n_2, \ldots,$$

and

$$\text{(2.3)} \qquad |R_n(h,x)-h(x)| > \varepsilon_n^2 n\omega_M\left(\frac{1}{n}\right) \quad \textit{for any} \quad x\in H_n, \quad n = n_1, n_2, \ldots.$$

Now to get Theorem 2.1, let $\varepsilon_n:=|P|/2$. Again by Theorem 2.2 we have

THEOREM 2.3. *If* $\lim\limits_{n\to\infty}\varepsilon_n^2 n=\infty$, $\lim\limits_{n\to\infty}\varphi_n\varepsilon_n^2=\infty$ *and* $\lim\limits_{n\to\infty}\varepsilon_n=0$, *then*

$$\text{(2.4)} \qquad \varlimsup_{n\to\infty}\frac{|R_n(h,x)-h(x)|}{\varepsilon_n^2 n\omega_M\left(\dfrac{1}{n}\right)} > 1 \quad \textit{almost everywhere in} \quad [0, 2\pi).$$

Here we used the previous notations.

Indeed, if we consider the set $H=\bigcap_{i=1}^{\infty}\left(\bigcup_{k=i}^{\infty} H_{n_i}\right)$ then for $T_i=\bigcup_{k=i}^{\infty} H_{n_i}$ we have $T_1\supset T_2\supset\dots$ and $|T_i|\geqq 2\pi-\varepsilon_{n_i}$ so $|H|=2\pi$. Moreover, if $x\in H$ then $x\in T_i$ for any i, i.e. $x\in H_{m_j}$ for infinitely many m_j, where $\{m_j\}\subset\{n_i\}$. Then, by (2.3) we get (2.4).

3. Proofs

3.1. PROOF OF THEOREM 2.2. First we prove our main lemma which states that the polynomial $F_n(x)$ behaves like the function $\sin\frac{nx}{2}$. More precisely, if $c, c_1, c_2, \dots$ are absolute positive constants, we state

LEMMA 3.1. *There exist absolute positive constants c_2, c_3 and c_4 such that for any even M we have* (*with* $0/0=1$)

$$\frac{c_2}{M}\leqq\frac{F_n(\pm x)}{\sin n\dfrac{x}{2}}\leqq\frac{c_3}{M}\quad \textit{if}\quad \frac{c_4}{n}\leqq x\leqq 2\pi-\frac{c_4}{n},\quad n=1,3,5,\dots. \tag{3.1}$$

PROOF OF LEMMA 3.1. Using [5], (25) and (26), we have

$$F_n(x)=A_1(x)+A_2(x)+A_3(x), \tag{3.2}$$

where

$$A_1(x)=\frac{1}{M}\sum_{j=1}^{n-1}\frac{\cos jx}{n-2j}, \tag{3.3}$$

$$A_2(x)=\frac{1}{Mn}\left(1+\sum_{j=1}^{n-1}\cos jx\right)=\frac{1}{2Mn}\left(\frac{\sin\dfrac{2n-1}{2}x}{\sin\dfrac{x}{2}}+1\right), \tag{3.4}$$

(3.5)
$$A_3(x)=\frac{1}{Mn}\left[1+2\sum_{j=1}^{n-1}h\left(\frac{j}{n-j}\right)\cos jx\right]\quad\text{where}\quad 0<h(y)\leqq 1\quad\text{if}\quad y>0$$

and $h(y)$ is decreasing.

3.3. First we consider the term $A_1(x)$. By formulas (10) and (21) in [4],

$$A_1(x)=\frac{1}{M}\left(\sin n\frac{x}{2}\int_0^{x/2}\sin nt\cot t\,dt-\frac{1}{2n}+\frac{1}{2n}\cos nx\right). \tag{3.6}$$

First we claim that

$$J\left(\frac{x}{2}\right)=\int_0^{x/2}\sin nt\cot t\,dt>\frac{1}{2}\quad\text{if}\quad\frac{\pi}{n}\leqq\frac{x}{2}\leqq\pi. \tag{3.7}$$

Indeed, if $\sin n\frac{x}{2}\cot\frac{x}{2}>0$ and $\frac{l\pi}{n}<\frac{x}{2}<(l+1)\frac{\pi}{n}$, then by Lemmas 7 and 8 and formulas (32), (33) and (36) from [4]

$$J\left(\frac{x}{2}\right)>J\left(\frac{l\pi}{n}\right)>\frac{1}{2}.$$

On the other hand, if $\sin n\frac{x}{2}\cot\frac{x}{2}<0$ and $\frac{l\pi}{n}<\frac{x}{2}<(l+1)\frac{\pi}{n}$, then

$$J\left(\frac{x}{2}\right)>J\left(\frac{(l+1)\pi}{n}\right)>\frac{1}{2},$$

again by Lemmas 7 and 8 and formulas (32), (33) and (36) in [4]. If $\sin n\frac{x}{2}\cot\frac{x}{2}=0$, we can use similar arguments. Subsequently we will need the ralations

$$\left|J\left(\frac{x}{2}\right)\right|<\pi,\quad 0\leqq x\leqq 2\pi. \tag{3.8}$$

Let us remark that

$$A_1'(x)=\frac{1}{M}\left[\frac{n}{2}\cos n\frac{x}{2}J\left(\frac{x}{2}\right)+\sin^2 n\frac{x}{2}\cot\frac{x}{2}-\frac{\sin nx}{2}\right], \tag{3.9}$$

$$A_1''(x)=\frac{1}{M}\left[-\frac{n^2}{4}\sin n\frac{x}{2}J\left(\frac{x}{2}\right)+\frac{3n}{4}\sin nx\cot\frac{x}{2}-2\frac{\sin^2 n\frac{x}{2}}{\sin^2\frac{x}{2}}-\frac{n\sin nx}{2}\right]. \tag{3.10}$$

3.4 By (3.4) we have

$$A_2'(x)=\frac{1}{2Mn}\frac{(2n-1)\cos\frac{2n-1}{2}x-\sin\frac{2n-1}{2}x\cot\frac{x}{2}}{\sin\frac{x}{2}}, \tag{3.11}$$

(3.12)

$$A_2''(x)=-\frac{1}{4Mn}\frac{(2n-1)^2\sin\frac{2n-1}{2}x\sin\frac{x}{2}+2(2n-1)\cos\frac{2n-1}{2}x\cos\frac{x}{2}}{\sin^2\frac{x}{2}}+$$

$$+\frac{\dfrac{\sin\frac{2n-1}{2}x}{\sin\frac{x}{2}}-\sin\frac{2n-1}{2}x\cot\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}}.$$

3.5. Consider now $A_3'(x)$ and $A_3''(x)$. By (3.5) and the Abel's inequality we have for certain s and t, $1 \leqq s$, $t \leqq n$,

$$(3.13)\quad |A_3'(x)| = \frac{1}{Mn}\left|2\sum_{j=1}^{n-1} h\left(\frac{j}{n-j}\right) j\sin jx\right| \leqq \frac{1}{Mn}\max_{1\leqq v\leqq n-1}\left|2\sum_{j=1}^{v} j\sin jx\right| =$$

$$= \frac{1}{Mn}\left|2\sum_{j=1}^{s-1} j\sin jx\right| = \frac{1}{Mn}\left|\left(\frac{\sin\frac{2s-1}{2}x}{\sin\frac{x}{2}}\right)'\right|,$$

(3.14)

$$|A_3''(x)| = \frac{1}{Mn}\left|2\sum_{j=1}^{k-1} h\left(\frac{j}{n-j}\right) j^2\cos jx\right| \leqq \frac{1}{Mn}\max_{1\leqq v\leqq n-1}\left|2\sum_{j=1}^{v} j^2\cos jx\right| =$$

$$= \frac{1}{Mn}\left|2\sum_{j=1}^{t-1} j^2\cos jx\right| = \frac{1}{Mn}\left|\left(\frac{\sin\frac{2t-1}{2}x}{\sin\frac{x}{2}}\right)''\right|,$$

that is they have terms similar to those in (3.11) and (3.12), respectively.

3.6. Consider now the term $\frac{n}{2M}\cos n\frac{x}{2} J\left(\frac{x}{2}\right)$ in $A_1'(x)$. By (3.7) and (3.8), if $\frac{2k\pi}{n} \leqq x \leqq \frac{4k+1}{2n}\pi$, $k=1, 2, \dots n-1$, then

$$\frac{n}{4\sqrt{2}\,M} \leqq (-1)^k\frac{n}{2M}\cos\frac{nx}{2} J\left(\frac{x}{2}\right) \leqq \frac{\pi}{2M}n.$$

On the other hand, the sum of the absolute values of all the remaining terms of $A_1'(x)$, $A_2'(x)$ and $A_3''(x)$ is less than, say, $\frac{n}{5\sqrt{2}\,M}$ if $\left|\sin\frac{x}{2}\right| \geqq cn^{-1}$ with a suitable c. But this can be attained if we assume that $c_5 n^{-1} \leqq x \leqq 2\pi - c_6 n^{-1}$. Therefore we proved that for the function $F_n' = A_1' + A_2' + A_3'$ we have the relation

$$(3.15)\qquad \frac{c_7}{M}n \leqq (-1)^k F_n'(x) \leqq \frac{c_8}{M}n$$

if $\frac{2k\pi}{n} \leqq x \leqq \frac{(4k+1)\pi}{2n}$ and $c_9 \leqq k \leqq n - c_{10}$ where c_7, c_8, c_9 and c_{10} are suitable constants.

3.7. Using analogous argument for the term $-\frac{n^2}{4M}\sin n\frac{x}{2} J\left(\frac{x}{2}\right)$ in $A_1''(x)$ on the intervals $(4k+1)(2n)^{-1}\pi \leqq x \leqq (2k+1)\pi n^{-1}$, one can prove (for sake of simplicity with the above constants) that

$$(3.16)\qquad \frac{c_7}{M}n^2 \leqq (-1)^{k+1} F_n''(x) \leqq \frac{c_8}{M}n^2$$

if $\dfrac{4k+1}{2n}\pi \leqq x < \dfrac{2k+1}{n}\pi$ and $c_9 \leqq k \leqq n-c_{10}$. Using similar arguments we can also prove, again with the same constants, for sake of simplicity,

$$\frac{c_7}{M} n \leqq (-1)^{k+1} F_n'(x) \leqq \frac{c_8}{M} n \tag{3.17}$$

if $\dfrac{(4k+3)}{2n} \leqq x \leqq \dfrac{2(k+1)\pi}{n}$ and $c_9 \leqq k \leqq n-c_{10}$,

$$\frac{c_7}{M} n^2 \leqq (-1)^{k+1} F_n''(x) \leqq \frac{c_8}{M} n^2 \tag{3.18}$$

if $\dfrac{(2k+1)\pi}{n} \leqq x \leqq \dfrac{(4k+3)\pi}{2n}$ and $c_9 \leqq k \leqq n-c_{10}$. Moreover, by (3.2)—(3.5) and (3.8), obviously

$$|F_n(x)| \leqq \frac{2\pi}{M} \quad \text{for any } x. \tag{3.19}$$

3.8. Let, e.g., $c_9 \leqq k_0 \leqq n-c_{10}$ be even. Then $F_n(2k_0\pi n^{-1}) = F_n(2(k_0+1)^\pi n^{-1}) = 0$, $F_n(x)$ is monotone increasing in $[2k_0\pi n^{-1}, (4k+1)\pi(2n)^{-1}]$, is monotone decreasing in $[(4k_0+3)\pi(2n)^{-1}, 2(k_0+1)\pi n^{-1}]$, $|F_n'(x)| \sim n$ in these intervals (see (3.15) and (3.17)). Moreover, $F_n(x)$ is concave in $[(4k_0+1)\pi(2n)^{-1}, (4k_0+3)\pi(2n)^{-1}]$ (see (3.16) and (3.18)). Summarizing these facts and considering that $|F_n(x)|$ is bounded (see (3.19)), we obtain (3.1) for the interval $[2k_0\pi n^{-1}, 2(k_0+1)\pi n^{-1}]$. We can argue in this way for the other values of k to obtain Lemma 3.1, considering that the constants do not depend on k and n. Finally, remarking that $F(x) = F(-x)$ (see (3.2)—(3.4)), we can state (3.1) for $F(-x)$, too,

3.9. Now we can prove another statement.

LEMMA 3.2. *Let $\{\eta_n\}$ be a sequence of positive numbers and $[a_n, b_n] \subset [0, 2\pi]$ arbitrary intervals. Then there exist sets $S_n \subset [0, 2\pi)$ of measure $|S_n| \geqq 2\pi - \eta_n - \dfrac{4c_4}{n} - b_n + a_n$ such that*

$$\sum_{x_k \in [a_n, b_n]} |F_n(x - x_{kn})| \geqq \left[\frac{b_n - a_n}{2\pi} n\right] \frac{c_{11}\eta_n}{M} \quad \textit{whenever} \quad x \in S_n. \tag{3.20}$$

First we construct the sets S_n. Let $\delta_{kn} = \left\{x;\ |x - x_{kn}| \leqq \dfrac{\eta_n}{2n}\right\}$. If

$$|S_n = [0, 2\pi) \setminus \bigcup_{k=0}^{n} \delta_{kn} \setminus \left[0, \frac{c_4}{n}\right] \setminus \left[a_n - \frac{c_4}{n},\ b_n + \frac{c_4}{n}\right] \setminus \left[2\pi - \frac{c_4}{n},\ 2\pi\right],$$

then obviously $|S_n| \geqq 2\pi - \eta_n - 4c_4 n^{-1} - b_n + a_n$, moreover, if $x \in S_n$ and $x_k \in [a_n, b_n]$ then $c_4 n^{-1} \leqq |x - x_k| \leqq 2\pi - c_4 n^{-1}$, i.e. by (3.1) we get that

$$|F_n(x - x_k)| \geqq \frac{c_3}{M} \sin\frac{nx}{2} \geqq c_{11}\eta_n/M.$$

Considering that the interval $[a_n, b_n]$ contains at least $[(b_n-a_n)n\,2\pi^{-1}]$ nodes, we obtain (3.20).

3.10. The remaining part is a proper modification of [6], 3.12. Let us consider the intervals $[0, b_n]$ and the corresponding sets S_n. Then obviously $|S_n| \geqq 2\pi-\eta_n-\frac{4c_4}{n}-b_n$. Later we use that

$$(3.21)\qquad (-1)^k \sin n\,\frac{x-x_k}{2} = (-1)^j \sin n\,\frac{x-x_j}{2}.$$

3.11. Now we define the function $g_n(x)$ as follows (see e.g. [8]).

$$(3.22)\qquad g_n(x_k) = \begin{cases} (-1)^k & \text{if } \; x_k \in [0, b_n], \\ \;0 & \text{if } \; x_k \notin [0, b_n], \quad k = 0, 1, \dots, n-1, \\ g_n(x_n) = g_n(2\pi) = g_n(0) = 1. \end{cases}$$

If $g_n(x_k) \neq g_n(x_{k+1})$ then in $[x_k, x_{k+1}]$ let $g_n(x)$ be that Hermite interpolating polynomial of degree $2M-1$ which is equal to $g_n(x_k)$ or $g_n(x_{k+1})$ at the end-points, respectively, further $g_n'(x_i)=g_n''(x_i)=\dots=g_n^{(M-1)}(x_i)=0$, $i=k, k+1$. Then it can be proved that (O. Kis, [13], (30))

(3.23)

$$g_n(x) = g_n(x_k)\left[1-\frac{(2M-1)!!}{(2M-2)!!}\int_{-1}^{n\frac{x-x_k}{\pi}-1}(1-t^2)^{M-1}dt\right], \quad 0 \leqq x_k \leqq x \leqq x_{k+1} \leqq 2\pi.$$

Let us consider some further properties of $g_n(x)$. First, if $g_n(x_k)=g_n(x_{k+1})$, then $g_n(x) \equiv g_n(x_k)$ for $x \in [x_k, x_{k+1}]$. Moreover

$$(3.24)\qquad \omega(g_n^{(i)}, t) \leqq Dn^{1+i}t, \quad i = 0, 1, \dots, M-1, \quad D > 0$$

because

$$(3.25)\qquad d_n \overset{\text{def}}{=} \min_k (x_{k+1}-x_k) = \frac{2\pi}{n}.$$

By formula 3.3(1) in [9] we obtain from (3.24)

$$(3.26)\qquad \omega_M(g_n, t) \leqq Dn^M t^M$$

i.e. $g_n(x) \in \tilde{C}(\omega_M)$. Moreover,

$$(3.27)\qquad |g_n(x)| \leqq 1$$

which means for any x

$$|R_n(g_n, x)| \leqq \sum_{k=0}^{n-1} |F_n(x-x_k)| \leqq \frac{2\pi n}{M}$$

(see (3.19)). On the other hand, by (3.20)—(3.22) and (3.1) we get

$$|R_n(g_n, x)| = \left|\sum_{k=1}^{n} g_n(x_k) F_n(x-x_k)\right| = \left|\sum_{x_k \in [0, b_n]} (-1)^k F_n(x-x_k)\right| \geqq$$

$$\geqq \sum_{x_k \in [0, b_n)} |F_n(x-x_k)| \geqq \frac{c_{11}}{M}\left[\frac{b_n n}{2\pi}\right]\eta_n$$

if $x \in S_n$.

Now we assume that $b_n = Mc_{12}\eta_n$, $\eta_n = cM\varepsilon_n$ and $n\eta_n \geqq 1$. i.e. we have with a proper c_{12}

$$|R_n(g_n, x)| > \eta_n^2 n \tag{3.28}$$

if $x \in S_n$.

Later we shall use that for any fixed N

$$\|R_n(g_N, x) - g_N(x)\| \leqq \frac{d(N)}{n^{M-1}}. \tag{3.29}$$

Indeed, by [9], 4.8(18), $\|T_n^{(M)}\| \leqq c\ n^M \omega_M(T_n, n^{-1})$ from where we get by (3.26)

$$\|U_n^{(M)}\| \leqq cn^M \omega_M\left(U_n - g_N + g_N, \frac{1}{n}\right) \leqq cn^M\left[\|g_N - U_n\| + \omega_M\left(g_N, \frac{1}{n}\right)\right] \leqq c(N).$$

Here T_n is an arbitrary trigonometric polynomial of degree $\leqq n$, U_n is the best approximating trigonometric polynomial of degree $\leqq n-1$ to g_N in uniform norm, (c may depend on M). Then by (3.19), $U_n(x) = \bar{R}_n(U_n, x)$ (when $\beta_k = U_n^{(M)}(x_k)$) and $\sum_{k=0}^{n-1} |G_n(x-x_k)| \leqq cn^{1-M}$ (see (1.1) and [5], (27)),

$$|R_n(g_N, x) - g_N(x)| \leqq |R_n(g_N, x) - U_n(x)| + |U_n(x) - g_N(x)| \leqq$$

$$\leqq |R_n(g_N - U_n, x)| + \sum_{k=0}^{n-1} |U_n^{(M)}|\,|G_n(x-x_k)| + c(N)n^{-M} \leqq$$

$$\leqq c(N)(nn^{-M} + n^{1-M} + n^{-M}) = d(N)n^{1-M},$$

as stated.

3.12. Let us define the sequence $\{n_i\}_{i=1}^{\infty} \subset \{1, 3, \ldots\}$ as follows (see (i)—(iii) and $\lim \varphi_n \varepsilon_n^2 = \infty$).

$$\sum_{i=1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) \leqq 1, \tag{3.30}$$

$$\sum_{i=k+1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) \leqq \frac{1}{n_k}\,\omega_M\left(\frac{1}{n_k}\right), \tag{3.31}$$

$$\sum_{i=1}^{j-1} \omega_M\left(\frac{1}{n_i}\right) n_i^M \leqq \omega_M\left(\frac{1}{n_j}\right) n_j^M, \quad j = 2, 3, 4 \ldots \tag{3.32}$$

$$\omega_M\left(\frac{1}{n_1}\right) \leqq q,\ \omega_M\left(\frac{1}{n_{k+1}}\right) \leqq q\omega_M\left(\frac{1}{n_k}\right), \quad k = 1, 2, \ldots, 0 < q < 1/2, \tag{3.33}$$

$$\varphi_{n_k}\eta_{n_k}^2 > 3D(n_{k-1}) := 3 \max_{1 \leqq i \leqq k-1} d(n_i), \quad k = 2, 3, \ldots. \tag{3.34}$$

Let

$$f(x) = \sum_{i=1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) g_{n_i}(x). \tag{3.35}$$

First we prove $f \in \tilde{C}(\omega_M)$. Obviously

$$\omega_M(f, t) \leqq \sum_{i=1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) \omega_M(g_{n_i}, t) = \sum_{i=1}^{j} + \sum_{i=j+1}^{\infty},$$

i.e. if $n_{j+1}^{-1} < t \leqq n_j^{-1}$, we can write

$$\sum_{i=1}^{j} \omega_M\left(\frac{1}{n_i}\right) \omega_M(g_{n_i}, t) \leqq Dt^2 \left[\sum_{i=1}^{j-1} \omega_M\left(\frac{1}{n_i}\right) n_i^M + \omega_M\left(\frac{1}{n_j}\right) n_j^M\right] \leqq$$

$$\leqq 2Dt^M \omega_M\left(\frac{1}{n_j}\right) n_j^M \leqq 2D\omega_M(t)$$

(see (3.26), (3.32) and (ii)).
Further by (3.27) and (3.33)

$$\sum_{i=j+1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) \omega_M(g_{n_i}, t) \leqq 2^M \sum_{i=j+1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) \leqq \frac{2^M}{1-q} \omega_M\left(\frac{1}{n_{j+1}}\right) \leqq \frac{2^M}{1-q} \omega_M(t)$$

which means that $f \in \tilde{C}(\omega_M)$, indeed.

3.13. Now we prove Theorem 2.2. Let $x \in S_n$. Then

$$|R_{n_k}(f, x) - f(x)| = \left|\sum_{i=1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) [R_{n_k}(g_{n_i}, x) - g_{n_i}(x)]\right| \geqq \tag{3.36}$$

$$\geqq \omega_M\left(\frac{1}{n_k}\right) |R_{n_k}(g_{n_k}, x)| - \sum_{i=1}^{k-1} \omega_M\left(\frac{1}{n_i}\right) |R_{n_k}(g_{n_i}, x) - g_{n_i}(x)| -$$

$$- \sum_{i=k+1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) |R_{n_k}(g_{n_i}, x)| - \sum_{i=k}^{\infty} \omega_M\left(\frac{1}{n_i}\right) |g_{n_i}(x)|.$$

Here by (3.29) and (3.30)

$$\sum_{i=1}^{k-1} \omega_M\left(\frac{1}{n_i}\right) |R_{n_k}(g_{n_i}, x) - g_{n_i}(x)| \leqq \frac{D(n_{k-1})}{n_k^{M-1}} \sum_{i=1}^{k-1} \omega_M\left(\frac{1}{n_i}\right) \leqq \frac{D(n_{k-1})}{n_k^{M-1}}; \tag{3.37}$$

by (3.19), (3.27) and (3.31)

$$\sum_{i=k+1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) |R_{n_k}(g_{n_i}, x)| \leqq \frac{2\pi}{M} n_k \sum_{i=k+1}^{\infty} \omega_M\left(\frac{1}{n_i}\right) \leqq \frac{2\pi}{M} \omega_M\left(\frac{1}{n_k}\right), \tag{3.38}$$

by (3.27) and (3.33)

$$\sum_{i=k}^{\infty} \omega_M\left(\frac{1}{n_i}\right) |g_{n_i}(x)| \leqq \frac{1}{1-q} \omega_M\left(\frac{1}{n_k}\right) \leqq 2\omega_M\left(\frac{1}{n_k}\right). \tag{3.39}$$

Considering now (3.28) and (3.36)—(3.39) we get

$$|R_{n_k}(f,x)-f(x)| \geqq n_k \omega_M\left(\frac{1}{n_k}\right)\left[\eta_{n_k}^2-\frac{D(n_{k-1})}{\varphi_{n_k}}-\frac{2\pi}{Mn_k}-\frac{2}{n_k}\right].$$

Here, by $\eta_n = cM\varepsilon_n$, we get, using (3.34) and $\lim \varepsilon_n^2 n = \infty$,

$$[\ldots] \geqq \eta_{n_k}^2-\frac{1}{3}\eta_{n_k}^2-\frac{1}{3}\eta_{n_k}^2=\frac{1}{3}\eta_{n_k}^2=\frac{1}{3}c^2M^2\varepsilon_n^2.$$

Further, by 3.10 and $b_n = Mc_{12}\eta_n$,

$$|S_n| \geqq 2\pi-(2+Mc_{12})\eta_n \geqq 2\pi-\varepsilon_n$$

with a proper c. Now, if $h=3c^{-2}M^{-2}f$, and $S_n = H_n$, we obtain Theorem 2.2.

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ON THE MODULUS OF CONTINUITY WITH RESPECT TO FUNCTIONS DEFINED ON VILENKIN GROUPS

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Introduction

In this paper we generalize a theorem of Rubinstein [2]. By the usual definition the modulus of continuity of a function defined on a Vilenkin group (G_m) can be represented by a sequence of real numbers. Rubinstein [2] characterized the sequences which turn to be the modulus of continuity of a function in $C(G_m)$, $L^1(G_m)$, $L^2(G_m)$. Now we prove his conjecture, namely the theorem is true for $L^p(G_m)$ $(1 \leqq p < \infty)$ too.

§ 1.

Let $m := (m_k, k \in \mathbf{N})$ be a sequence for which $m_k \in \mathbf{N} := \{0, 1, \dots\}$ and $m_k \geqq 2$ $(k \in \mathbf{N})$ hold and denote by Z_{m_k} $(k \in \mathbf{N})$ the cyclic group of order m_k. Define the group G_m as the direct product of Z_{m_k}'s $(k \in \mathbf{N})$. Then G_m is a compact Abelian group with the Haar measure μ satisfying $\mu(G_m) = 1$.

Further we need the following subsets of G_m:

$$I_{n+1} := \{x = (x_0, x_1, \dots, x_k, \dots) \in G_m | x_i = 0, \quad i \leqq n\} \quad (n \in \mathbf{N},\ I_0 := G_m).$$

I_n form a basis for neighbourhoods of the zero element of G_m, therefore the topology of G_m is completely determined by I_n's. Obviously I_n are subgroups of G_m and

$$I_0 \supset I_1 \supset \dots \supset I_n \supset \dots \quad (n \in \mathbf{N}).$$

If we introduce the notations

$$M_{n+1} := \prod_{i=0}^{n} m_i \quad (n \in \mathbf{N}),\ M_0 := 1,$$

then it is clear that $\mu(I_n) = \dfrac{1}{M_n}$.

Let us denote by $C(G_m)$ and $L^p(G_m)$ $(1 \leqq p < \infty)$ the usual set of complex valued functions defined on G_m, i.e. $f \in C(G_m)$ iff f is continuous on G_m and thus

$$\|f\| := \sup_{x \in G_m} |f(x)| < \infty,$$

furthermore $f \in L^p(G_m)$ iff f is measurable with respect to the Haar measure μ and

$$\|f\|_p := \Big(\int_{G_m} |f|^p \, d\mu \Big)^{1/p} < \infty.$$

The modulus of continuity is usually defined as the sequence of numbers

$$\omega_n(f) := \sup_{h\in I_n}\sup_{x\in G_m} |f(x+h)-f(x)| \quad (f\in C(G_m)),$$

$$\omega_n^p(f) := \sup_{h\in I_n} \|f(\cdot+h)-f(\cdot)\|_p \quad (f\in L^p(G_m)) \quad (n\in\mathbf{N}),$$

(where $+$ denotes the group operation on G_m). It is clear that $(\omega_n(g),\ n\in\mathbf{N})$ $(g\in C(G_m))$ and $(\omega_n^p(f),\ n\in\mathbf{N})$ $(f\in L^p(G_m),\ 1\le p<\infty)$ monotonously decrease to zero. The following theorem shows that this property of a sequence characterizes the modulus of continuity.

THEOREM. *Let* $(\omega_n,\ n\in\mathbf{N})$ *be a sequence of real numbers monotonously decreasing to zero. Then there exist* $g\in C(G_m)$ *and for all* $p\in[1,\infty)$ *an* $f\in L^p(G_m)$ *such that* $\omega_n^p(f)=\omega_n(g)=\omega_n$ $(n\in\mathbf{N})$.

First we remark that the cases $C(G_m)$ and $L^1(G_m)$, $L^2(G_m)$ were proved by Rubinstein in [2]. In this paper we give a complete proof for $L^p(G_m)$ $(1\le p<\infty)$. We observe that in the meantime Rubinstein announced the above Theorem in [4] and proved it for the dyadic case.

Let us denote by $(\psi_n,\ n\in\mathbf{N})$ the character-system of G_m endowed with the so called Walsh—Paley order (see e.g. [3]), and define $E_n(f)$ $(n\in\mathbf{N},\ f\in C(G_m))$ and $E_n^{(p)}(f)$ $(f\in L^p(G_m),\ n\in\mathbf{N})$ as the distance in $C(G_m)$ and in $L^p(G_m)$, respectively, between f and the subspace generated by $\{\psi_i,\ i<M_n\}$ $(n\in\mathbf{N})$.

By a known result of Efimov [2]

$$E_n(f) \le \omega_n(f) \le 2E_n(f) \quad (f\in C(G_m));$$

$$E_n^{(p)}(f) \le \omega_n^{(p)}(f) \le 2E_n^{(p)}(f) \quad (f\in L^p(G_m)) \quad (n\in\mathbf{N}),$$

and thus we have from Theorem the following

COROLLARY. *For all* $(E_n,\ n\in\mathbf{N})$ *monotonously decreasing to zero there exist* $g\in C(G_m)$ *and for all* $p\in[1,\infty)$ *an* $f\in L^p(G_m)$ *for which*

$$\tfrac{1}{2}E_n \le E_n(g) \le E_n \quad \textit{and} \quad \tfrac{1}{2}E_n \le E_n^{(p)}(f) \le E_n \quad (n\in\mathbf{N})$$

hold.

§ 2.

PROOF OF THEOREM. Let $1\le p<\infty$ be fixed and introduce the notations

$$L_*^p(G_m) := \{g\in L^p(G_m) \mid g|_{I_i\setminus I_{i+1}} = \text{const.},\quad i\in\mathbf{N}\},$$

$$F_i(g) := g(x) \quad (g\in L_*^p(G_m),\quad x\in I_i\setminus I_{i+1},\ i\in\mathbf{N})$$

(where $g|_A$ is the restriction of g to the set A). The following identity is easy to see, but it will be very useful for us. If $g\in L_*^p(G_m)$ such that $(F_i(g),\ i\in\mathbf{N})$ is increasing, then

$$(1) \qquad \omega_k^{(p)}(g) = \omega_k^{(p)}(g|\chi_{I_k}) = \left[2 \sum_{i=k+1}^{\infty} (F_i(g)-F_k(g))^p \left(\frac{1}{M_i}-\frac{1}{M_{i+1}}\right)\right]^{1/p} = $$
$$= 2^{1/p} \|(g-F_k(g))\,|\,\chi_{I_k}\|_p \quad (k\in\mathbf{N}).$$

Here χ_{I_k} denotes the characteristic function of the set I_k.

In order to prove the theorem we need the following lemma.

LEMMA. *There exists a sequence of functions* $f_n \in L_*^p(G_m)$ $(n\in\mathbf{N})$ *having the following properties:*

(i) $(F_i(f_n),\ i\in\mathbf{N})$ *is increasing and* $F_0(f_n)=0$,

$$\text{(ii)} \qquad \omega_k^{(p)}(f_n) = \begin{cases} \omega_k & k \leqq n \\ 0 & k > n \end{cases} \quad (k\in\mathbf{N}).$$

PROOF OF LEMMA. Let $n\in\mathbf{N}$ be given and x_{n+1} an arbitrary real number. Define $x_i := x_{n+1}$ $(i>n)$. We shall prove the existence of $x_i\in\mathbf{R}$ $(n\geqq i\in\mathbf{N})$, for which $x_{n+1} \geqq x_n \geqq \ldots \geqq x_0$ and

$$(2) \qquad \left[2 \sum_{i=k+1}^{\infty} (x_i-x_k)^p \left(\frac{1}{M_i}-\frac{1}{M_{i+1}}\right)\right]^{1/p} = \omega_k \quad (k \leqq n).$$

This statement is trivial for $k=n$, further it will be verified inductively. Assume that the numbers

$$x_{n+1} \geqq x_n \geqq \ldots \geqq x_{j+1}$$

satisfy (2) for $j+1 \leqq k \leqq n$, and let

$$r_j(x) := \left[2 \sum_{i=j+1}^{\infty} (x_i-x)^p \left(\frac{1}{M_i}-\frac{1}{M_{i+1}}\right)\right]^{1/p} \quad (x_{j+1} \geqq x\in\mathbf{R}).$$

From the assumption we have that $r_j(x_{j+1})=\omega_{j+1}$. Furthermore, the continuity and monotonicity of r_j and $\lim\limits_{x\to-\infty} r_j(x)=+\infty$ imply the existence of $x_j \leqq x_{j+1}$ such that $r_j(x_j)=\omega_j$, which proves our statement.

If we define $f_n\in L_*^p(G_m)$ by means of $F_i(f_n):=x_i-x_0$ $(i\in\mathbf{N})$, then in view of (1) f_n satisfies (i), and (ii), as stated.

Let $(f_n,\ n\in\mathbf{N})$ be a sequence in $L_*^p(G_m)$ having the properties (i), (ii) of Lemma for any $n\in\mathbf{N}$. We observe that by $\omega_0^{(p)}(f_n)=\omega_0$ $(n\in\mathbf{N})$ and by (1) we have

$$(3) \qquad \|f_n\|_p = 2^{-1/p}\omega_0 \quad \text{and} \quad \omega_k^{(p)}(f_n) \leqq \omega_k \quad (k,\, n\in\mathbf{N}).$$

Since the well-known M. Riesz condition concerning the compactness in $L^p([0, 1])$ can be transferred to $L^p(G_m)$ $(1\leqq p<\infty)$, therefore it follows from (3), that there exist a sequence of indices ν and a function $f\in L^p(G_m)$ such that $\lim\limits_n \|f-f_{\nu(n)}\|_p=0$. By $\lim\limits_n \omega_k^{(p)}(f_n)=\omega_k$ we have that $\omega_k^{(p)}(f)=\omega_k$ $(k\in\mathbf{N})$, consequently f is the desired function. The proof of Theorem is complete.

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ÜBER DIE MITTEL VON ORTHOGONALEN FUNKTIONEN. II

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1. In einer vorigen Arbeit [3] haben wir die Konvergenzverhältnisse der Mittel

$$\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\varphi_k(x) \quad (n=1,2,\ldots)$$

betrachtet, wobei $\lambda=\{\lambda_n\}_1^\infty$ eine Folge mit

$$0<\lambda_n\leqq\lambda_{n+1} \quad (n=1,2,\ldots), \quad \lambda_n\to\infty \quad (n\to\infty),$$

$a=\{a_k\}_1^\infty$ eine reelle Zahlenfolge und $\varphi=\{\varphi_k(x)\}_1^\infty$ ein im Intervall $(0,1)$ orthonormiertes Funktionensystem sind. In dieser Arbeit werden wir diese Betrachtungen weiterführen.

Die folgenden Sätze bleiben gültig für in einem beliebigen nichtatomischen Maßraum vom Maß 1 orthonormierte Funktionensysteme; nur einfachkeitshalber beschränken wir uns auf das Intervall $(0,1)$.

Für $1\leqq K\leqq\infty$ bezeichnen wir mit $\Omega(K)$ die Klasse der in $(0,1)$ orthonormierten Funktionensysteme φ, für die

$$|\varphi_k(x)|\leqq K \quad (x\in(0,1);\ k=1,2,\ldots)$$

besteht. $\Omega(\infty)$ ist also die Klasse aller in $(0,1)$ orthonormierten Funktionensysteme; im Falle $\varphi\in\Omega(1)$ gilt aber

$$|\varphi_k(x)|=1 \quad (x\in(0,1) \text{ fast überall}; \ k=1,2,\ldots).$$

Es ist klar, daß

$$\Omega(K_1)\subseteqq\Omega(K_2) \quad (1\leqq K_1<K_2\leqq\infty).$$

Wir werden die folgenden Sätze beweisen.

SATZ I. *Für jedes K $(1\leqq K\leqq\infty)$ gibt es eine Klasse $M(K,\lambda)$ von Folgen a mit folgenden Eigenschaften. Ist $a\in M(K,\lambda)$, so gilt*

$$\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\varphi_k(x)\to 0 \quad (n\to\infty) \tag{1}$$

für jedes $\varphi\in\Omega(K)$ fast überall in $(0,1)$. Ist aber $a\notin M(K,\lambda)$, so gibt es ein $\Phi\in\Omega(K)$ derart, daß die Folge

$$\left\{\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\Phi_k(x)\right\}_{n=1}^{\infty} \tag{2}$$

in $(0,1)$ fast überall divergiert.

Offensichtlich haben wir

$$M(\lambda_1, K_1) \supseteqq M(\lambda_1, K_2) \quad (1 \leqq K_1 < K_2 \leqq \infty).$$

Es gilt aber der folgende Satz.

SATZ II. *Für jedes* $1<K<\infty$ *gilt* $M(K, \lambda)=M(1, \lambda)$.

BEMERKUNG I. Man kann zeigen, daß für jede Folge λ mit den besagten Eigenschaften $M(\infty, \lambda)\neq M(1, \lambda)$ $\big(M(\infty, \lambda)\subset M(1, \lambda)\big)$ ist.

BEMERKUNG II. Aus den Sätzen I—II erhalten wir folgendes. Gibt es ein $\varphi\in\Omega(K)$ mit einem $K\,(1<K<\infty)$, für welches (1) in einem Menge von positivem Maß nicht erfüllt ist, so gibt es ein $\Phi\in\Omega(1)$ derart, daß die Folge (2) in (0, 1) fast überall divergiert. Gilt aber (1) für jedes $\varphi\in\Omega(1)$ in einer Menge von positivem Maß (diese Menge kann von φ abhängen), so besteht (1) für jedes $\varphi\in\bigcup_{K<\infty}\Omega(K)$ in (0, 1) fast überall.

Für ein K $(1\leqq K\leqq\infty)$ und für eine Folge a setzen wir

$$\|a;\, K;\, \lambda\| = \sup_{\varphi\in\Omega(K)}\left\{\int_0^1 \sup_{n\geqq 1}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\varphi_k(x)\right)^2 dx\right\}^{1/2} \quad (\leqq\infty).$$

Nach den Definitionen gilt

$$\|a;\, K_1;\, \lambda\| \leqq \|a;\, K_2;\, \lambda\| \quad (1 \leqq K_1 < K_2 \leqq \infty)$$

für jede Folge a. Man kann aber auch die umgekehrte Ungleichung zeigen.

SATZ III. *Für jedes* K $(1<K<\infty)$ *gibt es eine nur von* K *abhängige positive Zahl* $C(K)$ *derart, daß*

$$\|a;\, K;\, \lambda\| \leqq C(K)\|a;\, 1;\, \lambda\|$$

für jede Folge a *besteht. Weiterhin gilt*

$$C(K)=O(K).$$

BEMERKUNG III. Für $K=\infty$ ist diese Behauptung nicht richtig.

SATZ IV. *Es seien* $1\leqq K\leqq\infty$ *und* $a\in M(K, \lambda)$. *Gilt für eine Folge* b

$$|b_k| \leqq |a_k| \quad (k=1, 2, \ldots),$$

so ist $b\in M(K, \lambda)$.

BEMERKUNG IV. Aus dem Satz IV ergibt sich folgendes. Es sei $|b_k| \leqq |a_k|$ $(k=1, 2, \ldots)$. Gilt (1) für jedes $\varphi \in \Omega(K)$ (für ein $1 \leqq K \leqq \infty$) in einer Menge von positivem Mass (die Menge kann von φ abhängen), so gilt

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} b_k \varphi_k(x) \to 0 \quad (n \to \infty)$$

für jedes $\varphi \in \Omega(K)$ fast überall in $(0, 1)$. Gibt es aber ein $\varphi \in \Omega(K)$ derart, daß die obige Relation in einer Menge von positivem Mass nicht erfüllt ist, so gibt es ein $\Phi \in \Omega(K)$ derart, daß (2) in $(0, 1)$ fast überall divergiert.

SATZ V. *Es sei* $1 \leqq K \leqq \infty$. *Ist* $a \in M(K, \lambda)$, *so gibt es eine monoton wachsende, ins Unendliche strebende Folge* $\mu = \{\mu_k\}_1^\infty$ *von positiven Zahlen mit* $\mu a \in M(K, \lambda)$. *Ist aber* $a \notin M(\infty, \lambda)$, *so gibt es eine monoton abnehmende, zu* 0 *strebende Folge* $\mu = \{\mu_k\}_1^\infty$ *von positiven Zahlen mit* $\mu a \notin M(\infty, \lambda)$.

BEMERKUNG V. Die zweite Behauptung des Satzes V ist im Falle $1 \leqq K < \infty$ im allgemeinen nicht richtig.

2. Zum Beweis der Sätze haben wir gewisse Bezeichnungen vorauszuschicken.

Für eine in $(0, 1)$ definierte Funktion $f(x)$ und für ein Intervall $I=(a, b)$ setzen wir

$$f(I;\, x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & (x \in I), \\ 0 & \text{sonst,} \end{cases}$$

und für eine Menge $H(\subseteqq (0, 1))$ sei $H(I)$ jene Menge, die aus H unter der Transformation $y=(b-a)x+a$ hervorgeht.

Die Menge H nennen wir einfach, wenn sie die Vereinigung endlichvieler Intervalle ist.

Für eine Folge a und für positive ganze Zahlen N_1, N_2 $(N_1 \leqq N_2)$ setzen wir

$$a(N_1, N_2) = \{0, \ldots, 0, a_{N_1}, \ldots, a_{N_2}, 0, \ldots\}, \quad a(N_1, \infty) = \{0, \ldots, 0, a_{N_1}, a_{N_1+1}, \ldots\}.$$

Weiterhin seien für ein $K\,(1 \leqq K \leqq \infty)$, für eine Folge a und für positive ganze Zahlen N_1, N_2 $(N_1 \leqq N_2)$

$$\|a;\, K;\, \lambda;\, N_1, N_2\| = \sup_{\varphi \in \Omega(K)} \left\{ \int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} a_k \varphi_k(x) \right)^2 dx \right\}^{1/2},$$

$$\|a;\, K;\, \lambda;\, N_1, \infty\| = \sup_{\varphi \in \Omega(K)} \left\{ \int_0^1 \sup_{n \geqq N_1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} a_k \varphi_k(x) \right)^2 dx \right\}^{1/2} \quad (\leqq \infty).$$

Offensichtlich gilt folgendes für beliebige Folgen a, b, für beliebige positive ganze Zahlen N_1, N_2 $(N_1 \leqq N_2)$, für jede reelle Zahl c und für alle K, K_1, K_2 $(1 \leqq K, K_1, K_2 \leqq$

$\leqq\infty$; $K_1<K_2$):

$$\|a;\ K;\ \lambda;\ 1,\ \infty\| = \|a;\ K;\ \lambda\|,$$

$$\|ca;\ K;\ \lambda;\ N_1,\ N_2\| = |c|\,\|a;\ K;\ \lambda;\ N_1,\ N_2\|,$$

$$\|a+b;\ K;\ \lambda;\ N_1,\ \infty\| \leqq \|a;\ K;\ \lambda;\ N_1,\ \infty\| + \|b;\ K;\ \lambda;\ N_1,\ \infty\|,$$

$$\|a(N_1,\ N_2);\ K;\ \lambda;\ N_1,\ \infty\| = \|a(N_1,\ N_2);\ K;\ \lambda;\ N_1,\ N_2\|,$$

$$\|a;\ K;\ \lambda;\ N_1,\ N_2\| \leqq \|a(1,\ N_2);\ K;\ \lambda;\ N_1,\ N_2\|,$$

$$\|a;\ K_1;\ \lambda;\ N_1,\ N_2\| \leqq \|a;\ K_2;\ \lambda;\ N_1,\ N_2\|,$$

$$\|a;\ K;\ \lambda;\ N_1,\ N_2\| \leqq \|a;\ K;\ \lambda;\ N_1,\ N_2+1\|,$$

$$\|a;\ K;\ \lambda;\ N_1-1,\ N_2\| \geqq \|a;\ K;\ \lambda;\ N_1,\ N_2\|,$$

$$\lim_{N_2\to\infty} \|a;\ K;\ \lambda;\ N_1,\ N_2\| = \|a;\ K;\ \lambda;\ N_1,\ \infty\|,$$

$$\|a;\ K;\ \lambda;\ N_1,\ \infty\| \leqq \|a;\ K;\ \lambda;\ N_1,\ N_2\| + \|a;\ K;\ \lambda;\ N_2+1,\ \infty\|,$$

$$\left\{\frac{1}{\lambda_{N_2}^2}\sum_{k=1}^{N_2} a_k^2\right\}^{1/2} \leqq \|a;\ K;\ \lambda;\ 1,\ N_2\| \leqq \frac{1}{\lambda_1}\sum_{k=1}^{N_2}|a_k|.$$

3. Zum Beweis der Sätze benötigen wir gewisse Hilfssätze. Es sei $r_k(x)=$ $=\operatorname{sign}\sin 2^k\pi x$ die k-te Rademachersche Funktion ($k=1, 2, \ldots$).

HILFSSATZ I. *Gilt für die Folge a*

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k r_k(x) = 0$$

in einer Menge $E(\subseteqq(0, 1))$ *von positivem Maß, so besteht*

$$\lim_{n\to\infty}\frac{1}{\lambda_n^2}\sum_{k=1}^{n} a_k^2 = 0.$$

BEWEIS DES HILFSSATZES I. Es seien

$$g_N(x) = \sup_{n\geqq N}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k r_k(x)\right)^2 \quad (N = 1, 2, \ldots),$$

$\varepsilon>0$ beliebig. Dann gibt es eine positive ganze Zahl N_0 und eine meßbare Menge $E_1(\subseteqq E)$ von positivem Maß derart, daß

$$g_{N_0}(x) \leqq \varepsilon^2 \quad (x\in E_1)$$

ist. In diesem Fall gilt aber

$$\max_{N_0\leqq n\leqq N}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k r_k(x)\right)^2 \leqq g_{N_0}(x) \leqq \varepsilon^2 \quad (x\in E_1;\ N = N_0,\ N_0+1, \ldots),$$

und daraus folgt

$$\int_{E_1}\left(\frac{1}{\lambda_N}\sum_{k=1}^{N} a_k r_k(x)\right)^2 dx \le \varepsilon^2 \operatorname{mes} E_1 \quad (N = N_0, N_0+1, \dots). \tag{3}$$

Auf Grund eines bekannten Satzes (s. zB. [4], S. 213) gibt es einen Index $k_0 = k_0(E_1)$ mit

$$\int_{E_1}\left(\sum_{k=k_0}^{N} a_k r_k(x)\right)^2 dx \ge \frac{1}{2}\operatorname{mes} E_1 \sum_{k=k_0}^{N} a_k^2 \quad (N = k_0, k_0+1, \dots). \tag{4}$$

Es sei weiterhin $\overline{N}_0$ ein Index mit

$$\frac{1}{\lambda_{\overline{N}_0}}\sum_{k=1}^{k_0-1} |a_k| \le \varepsilon. \tag{5}$$

Aus (3), (4) und (5) ergibt sich:

$$\sqrt{\frac{1}{2}\operatorname{mes} E_1 \frac{1}{\lambda_N^2}\sum_{k=k_0}^{N} a_k^2} \le \sqrt{\int_{E_1}\left(\frac{1}{\lambda_N}\sum_{k=k_0}^{N} a_k r_k(x)\right)^2 dx} \le$$

$$\le \sqrt{\int_{E_1}\left(\frac{1}{\lambda_N}\sum_{k=1}^{n} a_k r_k(x)\right)^2 dx} + \sqrt{\int_{E_1}\left(\frac{1}{\lambda_N}\sum_{k=1}^{k_0-1} a_k r_k(x)\right)^2 dx} < 2\varepsilon\sqrt{\operatorname{mes} E_1}$$

für $N \ge \max(N_0, k_0, \overline{N}_0)$; dh. es ist

$$\frac{1}{\lambda_N^2}\sum_{k=k_0}^{N} a_k^2 \le 8\varepsilon^2 \quad (N \ge \max(N_0, k_0, \overline{N}_0)).$$

Da

$$\frac{1}{\lambda_N^2}\sum_{k=1}^{k_0-1} a_k^2 \le \varepsilon^2 \quad (N \ge \overline{\overline{N}}_0)$$

mit einem Index $\overline{\overline{N}}_0$ gilt, besteht auch

$$\frac{1}{\lambda_N^2}\sum_{k=1}^{N} a_k^2 \le 9\varepsilon^2 \quad (N \ge \max(N_0, k_0, \overline{N}_0, \overline{\overline{N}}_0))$$

für ein beliebiges $\varepsilon > 0$.

Damit haben wir Hilfssatz I bewiesen.

Aus dem Hilfssatz I folgt unmittelbar:

HILFSSATZ II. *Gilt für die Folge a*

$$\frac{1}{\lambda_n^2}\sum_{k=1}^{n} a_k^2 \nrightarrow 0 \quad (n \to \infty),$$

so besteht

$$\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k r_k(x) \nrightarrow 0 \quad (n \to \infty)$$

fast überall in (0, 1).

HILFSSATZ III. *Es sei* $\Phi=\{\Phi_k(x)\}_1^\infty$ *eine Folge von meßbaren und fast überall endlichen Funktionen. Gilt*

$$F_n(x)=\frac{1}{\lambda_n}\sum_{k=1}^{n}\Phi_k(x)\nrightarrow 0\quad(n\to\infty)$$

fast überall in (0, 1), *so ist die Folge*

$$\left\{\frac{1}{\lambda_n}\sum_{k=1}^{n}\pm\Phi_n(x)\right\}_{n=1}^{\infty}$$

mit gewissen Vorzeichen in (0, 1) *fast überall divergent.*

BEMERKUNG VI. Diesen Hilfssatz werden wir nur im Falle anwenden, daß Φ in (0, 1) gleichmäßig beschränkt ist, und diese spezielle Aussage ließe sich auch einfacher beweisen; vollständigkeitshalber beweisen wir den Hilfssatz dennoch in der allgemeinen Fassung.

BEWEIS DES HILFSSATZES III. Ist die Folge $\{F_n(x)\}_1^\infty$ in (0, 1) fast überall divergent, so trifft die Behauptung zu.

Im entgegengesetzten Fall gibt es meßbare, disjunkte Mengen E_1, E_2 $(\subseteq(0,1))$ derart, daß

$$\operatorname{mes} E_1+\operatorname{mes} E_2=1,$$

$$\lim_{n\to\infty}F_n(x)=f_1(x)\quad(x\in E_1),\qquad \overline{\lim_{n_1,n_2\to\infty}}\,|F_{n_2}(x)-F_{n_1}(x)|=f_2(x)\quad(x\in E_2)$$

mit gewissen Funktionen $f_1(x)\neq 0$ $(x\in E_1)$, $f_2(x)>0$ $(x\in E_2)$ erfüllt sind. Es seien

$$E_1(1)=\{x\in E_1\colon |f_1(x)|>1\},\; E_1(m)=\left\{x\in E_1\colon \frac{1}{m-1}\geqq|f_1(x)|>\frac{1}{m}\right\}$$

$$(m=2,3,\dots),$$

$$E_2(1)=\{x\in E_2\colon f_2(x)>1\},\; E_2(m)=\left\{x\in E_2\colon \frac{1}{m-1}\geqq f_2(x)>\frac{1}{m}\right\}\quad(m=2,3,\dots).$$

Wir werden eine Indexfolge $\{N_m\}_1^\infty$ $(1=N_1<\dots<N_m<\dots)$ und eine Mengenfolge $\{H_m\}_1^\infty$ meßbarer Teilmengen von (0, 1) derart angeben, daß für jedes m $(=1,2,\dots)$

(i) $$H_m\subseteq\bigcup_{\mu=1}^{m}\left(E_1(\mu)\cup E_2(\mu)\right),\quad \operatorname{mes} H_m\leqq\frac{1}{m},$$

(ii) $$|F_{N_{m+1}}(x)|\geqq 1/2\mu\quad(x\in E_1(\mu)\setminus H_m;\ \mu=1,\dots,m),$$

(iii) $$\frac{1}{\lambda_{N_{m+1}}}\sum_{k=1}^{N_m}|\Phi_k(x)|\leqq 1/8\mu\quad(x\in(E_1(\mu)\cup E_2(\mu))\setminus H_m;\ \mu=1,\dots,m),$$

(iv) $$\max_{N_m<n_1\leqq n_2\leqq N_{m+1}}|F_{n_2}(x)-F_{n_1}(x)|\geqq 1/2\mu\quad(x\in E_2(\mu)\setminus H_m;\ \mu=1,2,\dots,m)$$

erfüllt werden.

Es sei $N_1=1$. Da

$$\sup_{N_1<n_1\leqq n_2} |F_{n_2}(x)-F_{n_1}(x)|>1 \quad (x\in E_2(1))$$

ist, gibt es eine ganze Zahl N_2 $(>N_1)$ und meßbare Mengen $H_1^*(\subseteqq E_1(1))$, $H_1^{**}(\subseteqq E_2(1))$ mit

$$\operatorname{mes} H_1^* \leqq 1/2, \quad \operatorname{mes} H_1^{**} \leqq 1/2, \quad |F_{N_2}(x)| \geqq 1/2 \quad (x\in E_1(1)\setminus H_1^*),$$

$$\frac{1}{\lambda_{N_2}} \sum_{k=1}^{N_1} |\Phi_k(x)| \leqq 1/8\cdot 2 \quad (x\in (E_1(1)\setminus H_1^*)\cup (E_2(1)\setminus H_1^{**})),$$

$$\max_{N_1<n_1\leqq n_2\leqq N_2} |F_{n_2}(x)-F_{n_1}(x)| \geqq 1/2 \quad (x\in E_2(1)\setminus H_1^{**}).$$

Es sei $H_1=H_1^*\cup H_1^{**}$. Dann sind (i)—(iv) für N_1, N_2, H_1 im Falle $m=1$ offensichtlich erfüllt.

Es sei m_0 eine positive ganze Zahl. Wir nehmen an, daß die Indizes $1=N_1<\dots$ $\dots<N_{m_0+1}$ und die meßbaren Mengen $H_1, \dots, H_{m_0}$ schon derart definiert wurden, daß (i)—(iv) im Falle $m=1, \dots, m_0$ erfüllt sind.

Da

$$\sup_{N_{m_0+1}<n_1\leqq n_2} |F_{n_2}(x)-F_{n_1}(x)| \geqq 1/\mu \quad (x\in E_2(\mu);\quad \mu=1, \dots, m_0+1)$$

gilt, gibt es eine ganze Zahl $N_{m_0+2}(>N_{m_0+1})$ und meßbare Mengen

$$H_{m_0+1}^*(\mu)\subseteqq E_1(\mu),\ H_{m_0+1}^{**}(\mu)\subseteqq E_2(\mu) \quad (\mu=1, \dots, m_0+1)$$

mit

$$\operatorname{mes} H_{m_0+1}^*(\mu) \leqq 1/2(m_0+1)^3, \quad \operatorname{mes} H_{m_0+1}^{**}(\mu) \leqq 1/2(m_0+1)^3 \quad (\mu=1, \dots, m_0+1),$$

$$|F_{N_{m_0+2}}(x)| \geqq 1/2\mu \quad (x\in E_1(\mu)\setminus H_{m_0+1}^*(\mu);\ \mu=1, \dots, m_0+1),$$

$$\frac{1}{\lambda_{N_{m_0+2}}} \sum_{k=1}^{Nm_0+1} |\Phi_k(x)| \leqq 1/8\mu$$

$$(x\in ((E_1(\mu)\setminus H_{m_0+1}^*(\mu))\cup (E_2(\mu)\setminus H_{m_0+1}^{**}(\mu));\ \mu=1, \dots, m_0+1),$$

$$\max_{N_{m_0+1}<n_1\leqq n_2\leqq N_{m_0+2}} |F_{n_2}(x)-F_{n_1}(x)| \geqq 1/2\mu \quad (x\in E_2(\mu)\setminus H_{m_0+1}^{**}(\mu);\ \mu=1, \dots, m_0+1).$$

Es sei

$$H_{m_0+1} = \left(\bigcup_{\mu=1}^{m_0+1} H_{m_0+1}^*(\mu)\right)\cup\left(\bigcup_{\mu=1}^{m_0+1} H_{m_0+1}^{**}(\mu)\right).$$

Es ist offensichtlich, dass (i)—(iv) für $N_{m_0+1}, N_{m_0+2}, H_{m_0+1}$ auch im Falle $m=$ $=m_0+1$ erfüllt werden.

Die Folgen $\{H_m\}_1^\infty$, $\{N_m\}_1^\infty$ mit den geforderten Eigenschaften erhalten wir durch Induktion.

Wir setzen

$$\psi_1(x)=\Phi_1(x), \quad \psi_k(x)=(-1)^r\Phi_k(x) \quad (x\in(0, 1);\ N_{2r+1}<k\leqq N_{2r+3};\ r=0, 1, \dots).$$

Es sei

$$g_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} \psi_k(x) \quad (n = 1, 2, \ldots),$$

und

$$M_m^{(1)} = \left(\bigcup_{\mu=1}^{m} E_1(\mu)\right)\setminus H_m, \quad M_m^{(2)} = \left(\bigcup_{\mu=1}^{m} E_2(\mu)\right)\setminus H_m \quad (m = 1, 2, \ldots).$$

Auf Grund der Definitionen sind

$$\operatorname{mes}\left(\overline{\lim_{r\to\infty}}\, M_{2r+2}^{(1)}\right) = \operatorname{mes} E_1, \quad \operatorname{mes}\left(\overline{\lim_{r\to\infty}}\, M_{2r+2}^{(2)}\right) = \operatorname{mes} E_2.$$

Für jedes $r(=0, 1, \ldots)$ und für jedes $x\in(0, 1)$ besteht

$$g_{N_{2r+3}}(x) = \frac{1}{\lambda_{N_{2r+3}}} \sum_{k=N_{2r+2}+1}^{N_{2r+3}} \psi_k(x) + \frac{1}{\lambda_{N_{2r+3}}} \sum_{k=1}^{N_{2r+2}} \psi_k(x) =$$

$$= (-1)^r \frac{1}{\lambda_{N_{2r+3}}} \sum_{k=N_{2r+2}+1}^{N_{2r+3}} \Phi_k(x) + \frac{1}{\lambda_{N_{2r+3}}} \sum_{k=1}^{N_{2r+2}} \psi_k(x) =$$

$$= (-1)^r F_{N_{2r+3}}(x) + \frac{1}{\lambda_{N_{2r+3}}} \sum_{k=1}^{N_{2r+2}} \psi_k(x) - \frac{(-1)^r}{\lambda_{N_{2r+3}}} \sum_{k=1}^{N_{2r+2}} \Phi_k(x).$$

So erhalten wir, auf Grund von (ii), (iii)

$$|g_{N_{2r+3}}(x)| \geqq \frac{1}{2\mu} - 2\frac{1}{8\mu} \geqq \frac{1}{4\mu} \quad (x\in E_1(\mu)\setminus H_{2r+2};\ \mu = 1, \ldots, 2r+2),$$

$$\operatorname{sign} g_{N_{2r+3}}(x) = (-1)^r \operatorname{sign} F_{N_{2r+3}}(x) \quad (x\in E_1(\mu)\setminus H_{2r+2};\ \mu = 1, \ldots, 2r+2)$$

für jedes $r(=0, 1, \ldots)$. Daraus folgt, daß die Folge $\{g_n(x)\}_1^\infty$ in der Menge $\overline{\lim_{r\to\infty}}\, M_{2r+2}^{(1)}$ divergiert. Nach obigen divergiert also die Folge $\{g_n(x)\}_1^\infty$ fast überall in E_1.

Für jedes $r(=0, 1, \ldots)$ und für jedes $x\in(0, 1)$ besteht weiterhin

$$\max_{N_{2r+2}<n_1\leqq n_2\leqq N_{2r+3}} |g_{n_2}(x) - g_{n_1}(x)| \geqq$$

$$\geqq \max_{N_{2r+2}<n_1\leqq n_2\leqq N_{2r+3}} \left| \frac{1}{\lambda_{n_2}} \sum_{k=N_{2r+1}+1}^{n_2} \psi_k(x) - \frac{1}{\lambda_{n_1}} \sum_{k=N_{2r+1}+1}^{n_1} \psi_k(x) \right| - \frac{1}{\lambda_{N_{2r+2}}} \sum_{k=1}^{N_{2r+1}} |\psi_k(x)| =$$

$$= \max_{N_{2r+2}<n_1\leqq n_2\leqq N_{2r+3}} \left| \frac{1}{\lambda_{n_2}} \sum_{k=N_{2r+1}+1}^{n_2} \Phi_k(x) - \frac{1}{\lambda_{n_1}} \sum_{k=N_{2r+1}+1}^{n_1} \Phi_k(x) \right| - \frac{1}{\lambda_{N_{2r+2}}} \sum_{k=1}^{N_{2r+1}} |\Phi_k(x)| \geqq$$

$$\geqq \max_{N_{2r+2}<n_1\leqq n_2\leqq N_{2r+3}} |F_{n_2}(x) - F_{n_1}(x)| - \frac{2}{\lambda_{N_{2r+2}}} \sum_{k=1}^{N_{2r+1}} |\Phi_k(x)|.$$

Daraus ergibt sich, auf Grund von (iii)—(iv),

$$\max_{N_{2r+2}<n_1\leqq n_2\leqq N_{2r+3}} |g_{n_2}(x) - g_{n_1}(x)| \geqq \frac{1}{2\mu} - \frac{2}{8\mu} = \frac{1}{4\mu}$$

$$(x\in E_2(\mu)\setminus H_{2r+2};\ \mu = 1, \ldots, 2r+2),$$

was nach sich zieht, daß die Folge $\{g_n(x)\}_1^\infty$ in der Menge $\overline{\lim}_{r\to\infty} M_{2r+2}^{(2)}$ divergiert. Nach obigen divergiert die Folge $\{g_n(x)\}_1^\infty$ fast überall in E_2.

Damit haben wir Hilfssatz III bewiesen.

Aus den Hilfssätze II und III folgt der

HILFSSATZ IV. *Gilt für eine Folge* a

$$\frac{1}{\lambda_n^2}\sum_{k=1}^{n} a_k^2 \not\to 0 \quad (n\to\infty),$$

so gibt es ein System $\Phi\in\Omega(1)$ *derart, daß die Folge* (2) *in* $(0, 1)$ *fast überall divergiert.*

HILFSSATZ V. *Es sei* $1\leqq K\leqq\infty$. *Gilt für eine Folge* a $\lim_{N\to\infty}\|a;\ K;\ \lambda;\ N;\ \infty\|=0$, *dann besteht* (1) *für jedes* $\varphi\in\Omega(K)$ *in* $(0, 1)$ *fast überall.*

BEWEIS DES HILFSSATZES V. Es sei $\varphi\in\Omega(K)$. Wir setzen

$$F_N(x)=\sup_{n\geqq N}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\varphi_k(x)\right)^2 \quad (N=1, 2, \ldots).$$

Dann gilt

(6) $$F_N(x)\geqq F_{N+1}(x) \quad (x\in(0, 1);\ N=1, 2, \ldots).$$

Auf Grund der Definition von $\|\cdot;\ K;\ \lambda;\ N,\ \infty\|$ haben wir

$$\left\{\int_0^1 F_N(x)\,dx\right\}^{1/2}\leqq\|a;\ K;\ \lambda;\ N,\ \infty\|.$$

Daraus und aus (6) folgt $\lim_{N\to\infty} F_N(x)=0$ fast überall in $(0, 1)$, also gilt (1) in $(0, 1)$ fast überall.

HILFSSATZ VI. *Es seien* $1\leqq K<\infty$, a *eine Folge und* N_1, N_2 $(N_1\leqq N_2)$ *positive ganze Zahlen. Dann gibt es ein System* $\psi\in\Omega(K)$ *von Treppenfunktionen* $\psi_1(x), \ldots$ $\ldots, \psi_{N_2}(x)$ *mit*

$$\int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\psi_k(x)\right)^2 dx\geqq\frac{1}{2}\|a(1, N_2);\ K;\ \lambda;\ N_1, N_2\|^2.$$

BEWEIS DES HILFSSATZES VI. Der Hilfssatz soll nur im Falle $\|a(1, N_2);\ K;\ \lambda;\ N_1, N_2\|>0$ bewiesen werden.

Fall $K=1$. Auf Grund der Definition von $\|\cdot;\ K;\ \lambda;\ N_1, N_2\|$ gibt es System $\psi\in\Omega(1)$ von Funktionen $\psi_1(x), \ldots, \psi_{N_2}(x)$ mit

$$\int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\psi_k(x)\right)^2 dx>\frac{1}{2}\|a(1, N_2);\ K;\ \lambda;\ N_1, N_2\|^2.$$

Man kann leicht erreichen, daß diese Funktionen $\psi_1(x), \ldots, \psi_{N_2}(x)$ Treppenfunktionen seien.

Fall $1<K<\infty$. Auf Grund der Definition von $\|\cdot;K;\lambda;N_1,N_2\|$ gibt es ein System $\bar\varphi\in\Omega(K)$ von Funktionen $\bar\varphi_1(x),\dots,\bar\varphi_{N_2}(x)$ mit

$$\text{(7)}\qquad \int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\bar\varphi_k(x)\right)^2 dx > \|a(1,N_2);K;\lambda;N_1,N_2\|^2-\frac{\varepsilon}{3}$$

$$(\varepsilon=\|a(1,N_2);K;\lambda;N_1,N_2\|^2/4).$$

Es seien $\varphi_k(x)$ $(k=1,\dots,N_2)$ Treppenfunktionen mit

$$\int_0^1 (\bar\varphi_k(x)-\varphi_k(x))^2\,dx<\eta\quad(k=1,\dots,N_2)$$

und

$$\text{(8)}\qquad |\varphi_k(x)|\leqq|\bar\varphi_k(x)|\quad(x\in(0,1);\ k=1,\dots,N_2).$$

Wir setzen

$$\alpha_{i,j}=\int_0^1\varphi_i(x)\varphi_j(x)\,dx\quad(i,j=1,\dots,N_2).$$

Es werde a $(0<a<1)$ so gewählt, daß die Ungleichungen

$$\text{(9)}\qquad \bigl(1-(1-a^2)^2(1-a)\bigr)/a\leqq K^2,$$

$$\text{(10)}\qquad (1-a)(1-a^2)\int_0^1\max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\bar\varphi_k(x)\right)^2dx\geqq$$

$$\geqq\int_0^1\max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\varphi_k(x)\right)^2dx-\frac{\varepsilon}{3}$$

erfüllt werden. Wir teilen das Intervall $(1-a^2,1)$ in $N_2(N_2-1)$ paarweise disjunkte Intervalle $I_{i,j}$ $(i,j=1,\dots,N_2;\ i\neq j)$ gleicher Länge. Ist η genügend klein $(\eta\leqq\eta_1)$, so gelten die Abschätzungen

$$\text{(11)}\qquad \int_0^1\max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\varphi_k(x)\right)^2dx>\int_0^1\max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\bar\varphi_k(x)\right)^2dx-\frac{\varepsilon}{3},$$

$$\text{(12)}\qquad \frac{N_2(N_2-1)}{2}\max_{\substack{1\leqq i,j\leqq N_2\\ i\neq j}}|\alpha_{i,j}|\leqq K^2,$$

$$\text{(13)}\qquad \alpha_{k,k}\geqq 1-a^2\quad(k=1,\dots,N_2).$$

Wir setzen

$$\bar{\psi}_k(x) = \begin{cases} \varphi_k\left(\dfrac{x}{1-a^2}\right) & (x\in(0, 1-a^2)), \\ \sqrt{\dfrac{N_2(N_2-1)}{2a^2}}\,|\alpha_{k,i}|\,\sqrt{1-a^2} & (x\in I_{k,i};\ 1\leqq i\leqq N_2;\ i\neq k), \\ -\sqrt{\dfrac{N_2(N_2-1)}{2a^2}}\,|\alpha_{k,i}|\,\operatorname{sign}\alpha_{k,i}\sqrt{1-a^2} & (x\in I_{i,k};\ 1\leqq i\leqq N_2;\ i\neq k), \\ 0 & \text{sonst} \end{cases}$$

$(k=1, \ldots, N_2)$.

Offensichtlich bilden die Treppenfunktionen $\bar{\psi}_k(x)$ $(k=1, \ldots, N_2)$ ein orthogonales System in $(0, 1)$, weiterhin gelten auf Grund von (8), (12), (13) und der Definition der Funktionen $\bar{\psi}_k(x)$ die folgenden Ungleichungen:

$$|\bar{\psi}_k(x)| \leqq K \quad (x\in(0, 1);\ k=1, \ldots, N_2), \tag{14}$$

$$\int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\bar{\psi}_k(x)\right)^2 dx \geqq \int_0^{1-a^2} \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\bar{\psi}_k(x)\right)^2 dx = \tag{15}$$

$$= (1-a^2)\int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\varphi_k(x)\right)^2 dx,$$

$$\int_0^1 \bar{\psi}_k^2(x)\,dx \geqq \int_0^{1-a^2} \bar{\psi}_k^2(x)\,dx = (1-a^2)\alpha_{k,k} \geqq (1-a^2)^2 \quad (k=1, \ldots, N_2). \tag{16}$$

Es sei endlich

$$\psi_k(x) = \begin{cases} \bar{\psi}_k\left(\dfrac{x}{1-a}\right) & (x\in(0, 1-a)) \\ m_k r_k\left(\dfrac{x-1+a}{a}\right) & (x\in(1-a, 1)) \end{cases} \quad (k=1, \ldots, N_2),$$

wobei m_k derart bestimmt ist, daß die Funktionen $\psi_k(x)$ normiert sind; dh. es gilt

$$(1-a)\int_0^1 \bar{\psi}_k^2(x)\,dx + m_k^2 a = 1 \quad (k=1, \ldots, N_2). \tag{17}$$

Auf Grund von (9), (16) und (17) folgt:

$$m_k \leqq \sqrt{\frac{1-(1-a^2)^2(1-a)}{a}} \leqq K \quad (k=1, \ldots, N_2).$$

Die Funktionen $\psi_1(x), \ldots, \psi_{N_2}(x)$ sind offensichtlich Treppenfunktionen, und aus (14) erhalten wir $\psi=\{\psi_k(x)\}_1 \in \Omega(K)$; weiterhin gilt

$$\int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=1}^n a_k \psi_k(x)\right)^2 dx \geqq \int_0^{1-a} \max_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=1}^n a_k \psi_k(x)\right)^2 dx =$$

$$= (1-a) \int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=1}^n a_k \bar{\psi}_k(x)\right)^2 dx.$$

Daraus erhalten wir

$$\int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=1}^n a_k \psi_k(x)\right)^2 dx \geqq \|a(1, N_2);\ K;\ \lambda;\ N_1, N_2\|^2 - \varepsilon =$$

$$= \frac{1}{2} \|a(1, N_2);\ K;\ \lambda;\ N_1, N_2\|^2,$$

auf Grund von (7), (10), (11) und (15).

Damit haben wir Hilfssatz VI bewiesen.

HILFSSATZ VII. *Es seien* $1 < K < \infty$, a *eine Folge und* N *eine positive ganze Zahl mit*

$$\|a(1, N);\ K;\ \lambda;\ 1, N\|^2 \geqq 128K^2 \max_{1 \leqq n \leqq N} \frac{1}{\lambda_n^2} \sum_{k=1}^n a_k^2.$$

Dann gibt es ein System $\psi=\{\psi_k(x)\}_1^N \in \Omega(1)$ *von Treppenfunktionen mit folgender Eigenschaft. Es gilt*

$$\max_{1 \leqq n \leqq N} \frac{1}{\lambda_n} \left(\sum_{k=1}^n a_k \psi_k(x)\right) \geqq \frac{1}{4K} \|a(1, N);\ K;\ \lambda;\ 1, N\| \quad (x \in E),$$

wobei $E(\subseteqq(0, 1))$ *eine einfache Menge ist, für die* mes $E \geqq 1/10$ *besteht.*

BEWEIS DES HILFSSATZES VII. Wir gebrauchen eine Idee von B. S. Kašin [1] (s. noch [2]). Ohne Beschränkung der Allgemeinheit können wir

$$\|a(1, N);\ K;\ \lambda;\ 1, N\|^2 = 4$$

voraussetzen. Durch Anwendung des Hilfssatzes VI bekommen wir ein System $\bar{\psi}=\{\bar{\psi}_k(x)\}_1^N \in \Omega(K)$ von Treppenfunktionen mit

$$(18) \qquad \int_0^1 \max_{1 \leqq n \leqq N} \left(\frac{1}{\lambda_n} \sum_{k=1}^n a_k \bar{\psi}_k(x)\right)^2 dx > \frac{1}{2} \|a(1, N);\ K;\ \lambda;\ 1, N\|^2 (=2).$$

Es sei $I_r=(a_r, b_r)$ $(r=1, \ldots, \varrho)$ eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Teilintervalle derart, daß jede Funktion $\bar{\psi}_k(x)$ in jedem I_r konstant ist. Den Wert der Funktion

$$\max_{1 \leqq n \leqq N} \frac{1}{\lambda_n} \left|\sum_{k=1}^n a_k \bar{\psi}_k(x)\right|$$

im Intervall I_r bezeichnen wir mit w_r. Nach (18) gilt

(19)

$$4 = \|a(1, N);\ K;\ \lambda;\ 1, N\|^2 \geqq \sum_{r=1}^{\varrho} w_r^2 \operatorname{mes} I_r > \frac{1}{2} \|a(1, N);\ K;\ \lambda;\ 1, N\|^2 = 2.$$

Es seien $1 \leqq r_1 < \ldots < r_\lambda \leqq \varrho$ diejenigen Indizes, für die $w_{r_l} \geqq 1$ $(l=1, \ldots, \lambda)$ ist; die Indizes r $(1 \leqq r \leqq \varrho)$, die von $r_1, \ldots, r_\lambda$ verschieden sind, bezeichnen wir der Reihe nach mit $s_1, \ldots, s_{\varrho-\lambda}$. Aus (19) folgt

$$4 \geqq \sum_{l=1}^{\lambda} w_{r_l}^2 \operatorname{mes} I_{r_l} > 1.$$

Wir setzen

$$a = \sum_{l=1}^{\lambda} w_{r_l}^2 \operatorname{mes} I_{r_l}, \quad b = \sum_{l=1}^{\varrho-\lambda} \operatorname{mes} I_{s_l}$$

wobei dann offensichtlich

(20) $$a \leqq 4, \quad b \leqq 1$$

gelten. Es seien $J_l' = (a_l', b_l')$ $(l=1, \ldots, \lambda)$ disjunkte Intervalle in $(0, a)$ mit $\operatorname{mes} J_l' = w_{r_l}^2 \operatorname{mes} I_{r_l}$, und $J_l'' = (a_l'', b_l'')$ $(l=1, \ldots, \varrho-\lambda)$ disjunkte Intervalle in $(a, a+b)$ mit $\operatorname{mes} J_l'' = \operatorname{mes} I_{s_l}$. Wir setzen

$$\tilde{\psi}_k(x) = \begin{cases} \bar{\psi}_k\left(\dfrac{x-a_l''}{b_l''-a_l''}(b_{s_l}-a_{s_l})+a_{s_l}\right) & (x \in J_l'';\ l = 1, \ldots, \varrho-\lambda), \\ \dfrac{1}{w_{r_l}} \bar{\psi}_k\left(\dfrac{x-a_l'}{b_l'-a_l'}(b_{r_l}-a_{r_l})+a_{r_l}\right) & (x \in J_l';\ l = 1, \ldots, \lambda) \end{cases} \quad (k = 1, \ldots, N).$$

Es sei

$$\psi_k^*(x) = \tilde{\psi}_k((a+b)x)/K \quad (x \in (0, 1);\ k = 1, \ldots, N).$$

Offensichtlich bilden die Treppenfunktionen $\psi_k^*(x)$ $(k=1, \ldots, N)$ in $(0, 1)$ ein orthogonales System, und es gilt

(21) $$|\psi_k^*(x)| \leqq 1 \quad (x \in (0, 1);\ k = 1, \ldots, N).$$

Es sei $\bar{E}$ die Bildmenge des Intervalls $(0, a)$ durch die lineare Transformation $y = x/(a+b)$. Aus (20) folgt:

(22) $$\operatorname{mes} \bar{E} \geqq 1/5.$$

Weiterhin gilt auf Grund der Definition der Funktionen $\psi_k^*(x)$

(23) $$\max_{1 \leqq n \leqq N} \frac{1}{\lambda_n} \left| \sum_{k=1}^{n} a_k \psi_k^*(x) \right| = \frac{1}{K} = \frac{1}{2K} \|a(1, N);\ K;\ \lambda;\ 1, N\| \quad (x \in \bar{E}).$$

Es sei J_s $(s=1, \ldots, \sigma)$ eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle derart, daß jede Funktion $\psi_k^*(x)$ in jedem J_s konstant ist und $\bar{E}$ die Vereinigung einiger J_s ist. Den Wert von $\psi_k^*(x)$ im Intervall J_s bezeichnen wir mit $\varrho_s^{(k)}$. Für jeden Index s $(1 \leqq s \leqq \sigma)$ sei $\{\chi_s^{(k)}(x)\}_{k=1}^N$ ein orthogonales System von Treppenfunktionen derart, daß

$$\int_0^1 \chi_s^{(k)}(x)\,dx=0 \quad (k=1, \dots, N)$$

gilt und jede Funktion $\chi_s^{(k)}(x)$ den Wertbereich $\{1-\varrho_s^{(k)}, -1-\varrho_s^{(k)}\}$ besitzt. (Im Falle $\varrho_s^{(k)}=1$ soll man $\chi_s^{(k)}(x)\equiv 0$ setzen.) Aus (21) folgt

$$|\chi_s^{(k)}(x)|\leqq 2 \quad (x\in(0,2);\ k=1, \dots, N;\ s=1, \dots, \sigma). \tag{24}$$

Es sei

$$\psi_k^{**}(x)=\psi_k^*(x)+\sum_{s=1}^{\sigma}\chi_s^{(k)}(J_s;\ x) \quad (k=1, \dots, N).$$

Man kann leicht einsehen, daß $\psi^{**}=\{\psi_k^{**}(x)\}_1^N\in\Omega(1)$ ist. Für jeden Index s $(1\leqq s\leqq\sigma)$ sei n_s eine positive ganze Zahl mit $n_s\leqq N$ und

$$\frac{1}{\lambda_{n_s}}\left|\sum_{k=1}^{n_s}a_k\psi_k^*(x)\right|=\max_{1\leqq n\leqq N}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\psi_k^*(x)\right| \quad (x\in J_s).$$

Daraus folgt

$$\begin{aligned}
&\operatorname{mes}\left\{x\in J_s:\ \max_{1\leqq n\leqq N}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\psi_k^{**}(x)\right|\geqq\frac{1}{4K}\|a(1, N);\ K;\ \lambda;\ 1, N\|\right\}\geqq\\
&\geqq\operatorname{mes}\left\{x\in J_s:\ \frac{1}{\lambda_{n_s}}\left|\sum_{k=1}^{n_s}a_k\psi_k^{**}(x)\right|\geqq\frac{1}{4K}\|a(1, N);\ K;\ \lambda;\ 1, N\|\right\}\geqq\\
&\geqq\operatorname{mes}\left\{x\in J_s:\ \frac{1}{\lambda_{n_s}}\left|\sum_{k=1}^{n_s}a_k\psi_k^{*}(x)\right|\geqq\frac{1}{2K}\|a(1, N);\ K;\ \lambda;\ 1, N\|\right\}-\\
&-\operatorname{mes}\left\{x\in J_s:\ \frac{1}{\lambda_{n_s}}\left|\sum_{k=1}^{n_s}a_k\chi_s^{(k)}(J_s;\ x)\right|\geqq\frac{1}{4K}\|a(1, N);\ K;\ \lambda;\ 1, N\|\right\}=\\
&=\operatorname{mes}\left\{x\in J_s:\ \max_{1\leqq n\leqq N}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\psi_k^{*}(x)\right|\geqq\frac{1}{2K}\|a(1, N);\ K;\ \lambda;\ 1, N\|\right\}-\\
&-\operatorname{mes}\left\{x\in J_s:\ \frac{1}{\lambda_{n_s}}\left|\sum_{k=1}^{n_s}a_k\chi_s^{(k)}(J_s;\ x)\right|\geqq\frac{1}{4K}\|a(1, N);\ K;\ \lambda;\ 1, N\|\right\}.
\end{aligned}\tag{25}$$

Nach (24) ergibt sich durch Anwendung der Tschebyscheffschen Ungleichung

$$\begin{aligned}
&\operatorname{mes}\left\{x\in J_s:\ \frac{1}{\lambda_{n_s}}\left|\sum_{k=1}^{n_s}a_k\chi_s^{(k)}(J_s;\ x)\right|\geqq\frac{1}{4K}\|a(1, N);\ K;\ \lambda;\ 1, N\|\right\}\leqq\\
&\leqq\operatorname{mes}J_s.16K^2\frac{1}{\lambda_{n_s}^2}\sum_{k=1}^{n_s}a_k^2\int_0^1(\chi_s^{(k)}(x))^2\,dx/\|a(1, N);\ K;\ \lambda;\ 1, N\|^2\leqq\\
&\leqq\operatorname{mes}J_s.64K^2\frac{1}{\lambda_{n_s}^2}\sum_{k=1}^{n_s}a_k^2/\|a(1, N);\ K;\ \lambda;\ 1, N\|^2\leqq\operatorname{mes}J_s/2,
\end{aligned}$$

auf Grund der Voraussetzung des Hilfssatzes VII. Daraus und aus (23), (25) folgert man

(26)

$$\operatorname{mes}\left\{x\in J_s:\ \max_{1\le n\le N}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\psi_k^{**}(x)\right|\ge\frac{1}{4K}\|a(1,N);\ K;\ \lambda;\ 1,N\|\right\}\ge\operatorname{mes}J_s/2.$$

Es sei

$$E=\bigcup_{\substack{s\\ J_s\subseteq E}}\left\{x\in J_s:\ \max_{1\le n\le N}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\psi_k^{**}(x)\right|\ge\frac{1}{4K}\|a(1,N);\ K;\ \lambda;\ 1,N\|\right\}.$$

Für jedes s $(s=1,\ldots,\sigma)$ sei m_s die kleinste ganze Zahl mit $m_s\le N$ und

$$\frac{1}{\lambda_{m_s}}\left|\sum_{k=1}^{m_s}a_k\psi_k^{**}(x)\right|=\max_{1\le n\le N}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\psi_k^{**}(x)\right|\quad(x\in J_s).$$

Es seien

$$E_1(s)=\left\{x\in J_s:\ \frac{1}{\lambda_{m_s}}\sum_{k=1}^{m_s}a_k\psi_k^{**}(x)\ge 0\right\}\quad(s=1,\ldots,\sigma),$$

$$E_2(s)=\left\{x\in J_s:\ \frac{1}{\lambda_{m_s}}\sum_{k=1}^{m_s}a_k\psi_k^{**}(x)< 0\right\}\quad(s=1,\ldots,\sigma).$$

Wir setzen

$$\psi_k(x)=\begin{cases}\psi_k^{**}(x) & \left(x\in\bigcup_{s=1}^{\sigma}E_1(s)\right),\\ -\psi_k^{**}(x) & \left(x\in\bigcup_{s=1}^{\sigma}E_2(s)\right)\end{cases}\quad(k=1,\ldots,N).$$

Die Funktionen $\psi_k(x)$ $(k=1,\ldots,N)$ sind Treppenfunktionen und es gilt $\psi=\{\psi_k(x)\}_1^N\in\Omega(1)$. Weiterhin ist E einfach, und es besteht

$$\max_{1\le n\le N}\frac{1}{\lambda_n}\sum_{k=1}^{n}a_k\psi_k(x)\ge\frac{1}{4K}\|a(1,N);\ K;\ \lambda;\ 1,N\|\quad(x\in E).\tag{27}$$

Aus (26) und (27) ergibt sich, daß für das System ψ und für die Menge E alle Erforderungen des Hilfssatzes VII erfüllt sind.

Damit haben wir Hilfssatz VII bewiesen.

HILFSSATZ VIII. *Es seien* $1<K<\infty$ *und* a *eine Folge mit*

$$\lim_{N\to\infty}\|a;\ K;\ \lambda;\ N,\infty\|\ne 0.$$

Dann gibt es eine positive ganze Zahl ϱ *mit folgender Eigenschaft. Für jede positive ganze Zahl* N_1 *gibt es eine positive ganze Zahl* $N_2(\ge N_1)$ *mit*

$$\|a(N_1,N_2);\ K;\ \lambda;\ N_1,N_2\|>\varrho.$$

BEWEIS DES HILFSSATZES VIII. Auf Grund der Voraussetzung gibt es eine positive Zahl ϱ mit

$$\|a;\ K;\ \lambda;\ N,\infty\|>2\varrho\quad(N=1,2,\ldots).\tag{28}$$

Wegen $\lambda_n \to \infty$ $(n\to\infty)$ gibt es eine ganze Zahl $\bar{N}_1(\geqq N_1)$ mit

$$\|a(1, N_1-1);\ K;\ \lambda;\ \bar{N}_1, \infty\| \leqq \frac{K}{\lambda_{\bar{N}_1}} \sum_{k=1}^{N_1-1} |a_k| < \varrho.$$

Daraus und aus (28) erhalten wir

(29)

$$\|a(N_1, \infty);\ K;\ \lambda;\ \bar{N}_1, \infty\| \geqq \|a;\ K;\ \lambda;\ \bar{N}_1, \infty\| - \|a(1, N_1-1);\ K;\ \lambda;\ \bar{N}_1, \infty\| > \varrho.$$

Aus (29) ergibt sich

$$\varrho < \|a(N_1, \infty);\ K;\ \lambda;\ \bar{N}_1, N_2\| \leqq \|a(N_1, N_2);\ K;\ \lambda;\ \bar{N}_1, N_2\| \leqq$$
$$\leqq \|a(N_1, N_2);\ K;\ \lambda;\ N_1, N_2\|$$

für genügend grosse $N_2(\geqq N_1)$.

Hilfssatz IX. *Es seien* $1<K<\infty$ *und* a *eine Folge, für welche eine streng wachsende Indexfolge* $\{N_m\}_1^\infty$ *mit*

$$\|a(N_m+1, N_{m+1});\ K;\ \lambda;\ N_m+1, N_{m+1}\| > \varrho(>0),$$

$$\|a(N_m+1, N_{m+1});\ K;\ \lambda;\ N_m+1, N_{m+1}\|^2 \geqq 128K^2 \max_{N_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n^2}\sum_{k=1}^{n} a_k^2$$

$(m=1, 2, \ldots)$ *existiert. Dann gibt es ein* $\Phi\in\Omega(1)$ *derart, daß die Folge* (2) *in* $(0, 1)$ *fast überall divergiert.*

Beweis des Hilfssatzes IX. Auf Grund des Hilfssatzes VII gibt es für jedes $m(=1, 2, \ldots)$ ein System $\Phi^{(m)}=\{\Phi_k^{(m)}(x)\}_{k=N_m+1}^{N_{m+1}}\in\Omega(1)$ von Treppenfunktionen und eine einfache Menge $E_m(\subseteqq(0, 1))$ mit

$$\text{mes}\, E_m \geqq 1/10, \tag{30}$$

$$\max_{N_m<n\leqq N_{m+1}} \frac{1}{\lambda_n}\sum_{k=N_m+1}^{N_{m+1}} a_k\Phi_k^{(m)}(x) \geqq \frac{1}{4K}\varrho \quad (x\in E_m). \tag{31}$$

Durch Induktion werden wir ein System $\tilde{\Phi}=\{\tilde{\Phi}_k(x)\}_1^\infty\in\Omega(1)$ der Treppenfunktionen und eine Folge $\{F_m\}_1^\infty$ von stochastisch unabhängigen, einfachen Teilmengen von $(0, 1)$ darart definieren, daß für jedes $m(=1, 2, \ldots)$

$$\text{mes}\, F_m \geqq 1/20, \tag{i}$$

und

$$\max_{N_m<n\leqq N_{m+1}} \frac{1}{\lambda_n}\left|\sum_{k=1}^{n} a_k\tilde{\Phi}_k(x)\right| \geqq \frac{1}{4K}\varrho \quad (x\in F_m) \tag{ii}$$

erfüllt sind.

Es sei $\tilde{\Phi}_k(x)=r_k(x)$ $(k=1, \ldots, N_1)$. Dann gibt es eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle $I_1, \ldots, I_\varrho$ derart, daß jede Funktion

$\tilde{\Phi}_k(x)$ $(k=1, \ldots, N_1)$ in jedem I_r konstant ist. Es seien

$$\alpha_r^{(1)} = \operatorname{sign} \frac{1}{\lambda_{N_1}} \sum_{k=1}^{N_1} a_k \tilde{\Phi}_k(x) \quad (x \in I_r;\ r = 1, \ldots, \varrho),$$

$$\tilde{\Phi}_k^{(1)}(x) = \Phi_k^{(1)}\left(\left(0, \tfrac{1}{2}\right);\ x\right) - \Phi_k^{(1)}\left(\left(\tfrac{1}{2}, 1\right);\ x\right) \quad (k = N_1+1, \ldots, N_2),$$

$$E_1^{(r)} = \begin{cases} E_1\left(\left(0, \tfrac{1}{2}\right)\right), & \alpha_r^{(1)} = 1{,}0 \quad (r = 1, \ldots, \varrho). \\ E_1\left(\left(\tfrac{1}{2}, 1\right)\right), & \alpha_r^{(1)} = -1. \end{cases}$$

Wir setzen

$$\tilde{\Phi}_k(x) = \sum_{r=1}^{\varrho} \tilde{\Phi}_k^{(1)}(I_r;\ x) \quad (k = N_1+1, \ldots, N_2),$$

und $F_1 = \bigcup_{r=1}^{\varrho} E_1^{(r)}(I_r)$.

Offensichtlich sind die Funktionen $\tilde{\Phi}_k(x)$ $(k=1, \ldots, N_2)$ Treppenfunktionen, F_1 ist einfach, und es gilt $\{\tilde{\Phi}_k(x)\}_1^{N_2} \in \Omega(1)$. Aus (30) und (31) ergibt sich weiterhin, dass (i) und (ii) für $m=1$ erfüllt sind.

Es sei m_0 $(\geqq 2)$ eine ganze Zahl. Wir nehmen an, daß die Treppenfunktionen $\tilde{\Phi}_k(x)$ $(k=1, \ldots, N_{m_0})$ und die einfachen Mengen $F_m(\subseteqq (0, 1))$ $(m=1, \ldots, m_0-1)$ schon derart definiert sind, daß $\{\tilde{\Phi}_k(x)\}_1^{N_{m_0}} \in \Omega(1)$ ist, die Mengen F_m $(m=1, \ldots, m_0-1)$ stochastisch unabhängig sind, weiterhin (i), (ii) für $m=1, \ldots, m_0-1$ erfüllt werden. Dann gibt es eine Einteilung des Intervalls (0, 1) in paarweise disjunkte Intervalle $J_1, \ldots, J_\sigma$ derart, daß jede Funktion $\tilde{\Phi}_k(x)$ $(k=1, \ldots, N_{m_0})$ in jedem J_s konstant ist, und jede Menge F_m $(m=1, \ldots, m_0-1)$ die Vereinigung gewisser J_s ist. Es seien

$$\alpha_s^{(m_0)} = \operatorname{sign} \frac{1}{\lambda_{N_{m_0}}} \sum_{k=1}^{N_{m_0}} \tilde{\Phi}_k(x) \quad (x \in J_s;\ s = 1, \ldots, \sigma),$$

$$\tilde{\Phi}_k^{(m_0)}(x) = \Phi_k^{(m_0)}\left(\left(0, \tfrac{1}{2}\right);\ x\right) - \Phi_k^{(m_0)}\left(\left(\tfrac{1}{2}, 1\right);\ x\right) \quad (k = N_{m_0}+1, \ldots, N_{m_0+1}),$$

$$E_{m_0}^{(s)} = \begin{cases} E_{m_0}\left(\left(0, \tfrac{1}{2}\right)\right), & \alpha_s^{(m_0)} = 1{,}0, \\ E_{m_0}\left(\left(\tfrac{1}{2}, 1\right)\right), & \alpha_s^{(m_0)} = -1 \end{cases} \quad (s = 1, \ldots, \sigma).$$

Wir setzen

$$\tilde{\Phi}_k(x) = \sum_{s=1}^{\sigma} \tilde{\Phi}_k^{(m_0)}(J_s;\ x) \quad (k = N_{m_0}+1, \ldots, N_{m_0+1}),$$

und $F_{m_0} = \bigcup_{s=1}^{\sigma} E_{m_0}^{(s)}(I_s)$.

Es ist klar, daß die Funktionen $\tilde{\Phi}_k(x)$ $(k=N_{m_0}+1, \ldots, N_{m_0+1})$ Treppenfunktionen, F_{m_0} einfach und $\{\tilde{\Phi}_k(x)\}_1^{N_{m_0+1}} \in \Omega(1)$ sind. Die Mengen $F_1, \ldots, F_{m_0}$ sind stochastisch unabhängig. Auf Grund von (30), (31) besteht weiterhin (i), (ii) auch im Falle $m=m_0$. Das Funktionensystem $\tilde{\Phi}$ und die Mengenfolge $\{F_m\}_1^\infty$ mit den geforderten Eigenschaften erhalten wir durch Induktion.

Es sei $F = \overline{\lim_{m\to\infty}} F_m$. Aus (i) erhalten wir durch Anwendung des Borel—Cantellischen Lemmas mes $F=1$. Aus (ii) folgt

$$\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k \tilde{\Phi}_k(x) \nrightarrow 0 \quad (n\to\infty)\ (x\in F).$$

Daraus ergibt sich durch Anwendung des Hilfssatzes III, die Existenz eines $\Phi\in\Omega(1)$ derart, daß die Folge (2) in (0, 1) fast überall divergiert.

Damit haben wir Hilfssatz IX bewiesen.

Hilfssatz X. *Es seien* $1<K<\infty$ *und* a *eine Folge mit* $\lim_{N\to\infty}\|a;K;\lambda;N,\infty\|\neq 0$ *und*

$$\lim_{n\to\infty}\frac{1}{\lambda_n^2}\sum_{k=1}^{n} a_k^2 = 0.$$

Dann gibt es ein $\Phi\in\Omega(1)$ *derart, daß die Folge* (2) *in* (0, 1) *fast überall divergiert.*

Beweis des Hilfssatzes X. Durch Anwendung des Hilfssatzes VIII kann man eine streng wachsende Indexfolge $\{N_m\}_1^\infty$ derart angeben, daß die Voraussetzungen des Hilfssatzes IX erfüllt werden. Durch Anwendung des Hilfssatzes IX erhalten wir dann die Behauptung.

Hilfssatz XI. *Es sei* $1<K_1<K_2<\infty$. *Dann gibt es eine nur von* K_1 *und* K_2 *abhängige positive Zahl* $C(K_1, K_2)$ *derart, daß für jede Folge* a *und für beliebige positive ganze Zahlen* N_1, N_2 $(N_1\leqq N_2)$

$$\|a(1,N_2);K_2;\lambda;N_1,N_2\| \leqq C(K_1,K_2)\|a(1,N_2);K_1;\lambda;N_1,N_2\|$$

besteht. Weiterhin gilt

$$C(K_1,K_2) = 2\frac{K_2^2-1}{K_1^2-1}.$$

Beweis des Hilfssatzes XI. Es seien a eine beliebige Folge, und N_1, N_2 $(N_1\leqq N_2)$ beliebige positive ganze Zahlen. Durch Anwendung des Hilfssatzes VI ergibt sich ein System $\{\psi_k(x)\}_1^N\in\Omega(K_2)$ mit

$$\int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\psi_k(x)\right)^2 dx \geqq \frac{1}{2}\|a(1,N_2);K_2;\lambda;N_1,N_2\|^2.$$

Es seien

$$a = (K_1^2-1)\Big/\left(K_1^2-\frac{K_1^2}{K_2^2}\right)$$

und

$$\varphi_k(x) = \begin{cases} \dfrac{K_1}{K_2}\psi_k\left(\dfrac{x}{a}\right) & (x\in(0,a)), \\[2ex] K_1 r_k\left(\dfrac{x-a}{1-a}\right) & (x\in(a,1)) \end{cases} \qquad (k=1,\dots,N_2).$$

Offensichtlich gilt $\varphi=\{\varphi_k(x)\}_1^{N_2}\in\Omega(K_1)$. Weiterhin ist

$$\|a(1, N_2);\ K_1;\ \lambda;\ N_1, N_2\|^2 \geqq \int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\varphi_k(x)\right)^2 dx \geqq$$

$$\geqq \int_0^a \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\varphi_k(x)\right)^2 dx = \frac{K_1^2}{K_2^2}\int_0^a \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\psi_k\left(\frac{x}{a}\right)\right)^2 dx =$$

$$= K_1^2\frac{K_1^2-1}{K_1^2K_2^2-K_1^2}\int_0^1 \max_{N_1\leqq n\leqq N_2}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\psi_k(x)\right)^2 dx.$$

Hieraus folgt nach dem obigen

$$\|a(1, N_2);\ K_2;\ \lambda;\ N_1, N_2\|^2 \leqq \frac{2}{K_1^2}\frac{K_1^2K_2^2-K_1^2}{K_1^2-1}\|a(1, N_2);\ K_1;\ \lambda;\ N_1, N_2\|^2,$$

also gilt die Behauptung mit

$$C_1^2(K_1, K_2) = 2\frac{K_2^2-1}{K_1^2-1}.$$

4. Beweis des Satzes I. Für $1\leqq K_0\leqq\infty$ sei

$$M(K_0, \lambda) = \left\{a:\ \lim_{N\to\infty}\|a;\ K_0;\ \lambda;\ N, \infty\| = 0\right\}.$$

a) Im Falle $a\in M(K_0, \lambda)$ folgt aus dem Hilfssatz V, daß (1) für jedes $\varphi\in\Omega(K_0)$ in (0, 1) fast überall besteht.

b) Es sei nun $a\notin M(K_0, \lambda)$.

Im Falle $K_0=\infty$ haben wir bewiesen, daß ein $\Phi\in\Omega(\infty)$ derart existiert, daß die Folge (2) in (0, 1) fast überall divergiert [3].

Es sei nun $1\leqq K_0<\infty$. Dann besteht $\lim\limits_{N\to\infty}\|a;\ K;\ \lambda;\ N;\ \infty\|\neq 0$.

(i) Ist

$$\frac{1}{\lambda_n^2}\sum_{k=1}^n a_k^2 \nrightarrow 0 \quad (n\to\infty),$$

so zieht der Hilfssatz IV die Existenz eines $\Phi\in\Omega(1)$ nach sich, derart daß die Folge (2) in (0, 1) fast überall divergiert.

(ii) Es sei

$$\frac{1}{\lambda_n^2}\sum_{k=1}^n a_k^2 \to 0 \quad (n\to\infty).$$

Ist $1<K_0$, so gibt es auf Grund des Hilfssatzes X ein $\Phi\in\Omega(1)\subseteqq\Omega(K_0)$, derart, daß die Folge (2) in (0, 1) fast überall divergiert.

Es sei nun $K_0=1$. Wegen $a\notin M(1, \lambda)$ gilt

$$\lim_{N\to\infty}\|a;\ 1;\ \lambda;\ N, \infty\| \neq 0. \tag{32}$$

Da wir

$$\|a;\ 1;\ \lambda;\ N,\ \infty\| \leqq \|a;\ 2;\ \lambda;\ N,\ \infty\| \quad (N = 1, 2, \dots)$$

haben, erhalten wir aus (32) $\lim\limits_{N\to\infty} \|a;\ 2;\ \lambda;\ N,\ \infty\| \neq 0$.

Durch Anwendung des Hilfssatzes X im Falle $K=2$ ergibt sich ein $\Phi\in\Omega(1)$ derart, daß die Folge (2) in (0, 1) fast überall divergiert.

Damit haben wir Satz I bewiesen.

BEWEIS DES SATZES II. Da $M(1, \lambda) \supseteqq M(K;\ \lambda)$ $(1<K<\infty)$ offensichtlich ist, genügt es

$$M(1, \lambda) \subseteqq M(K;\ \lambda) \quad (1 < K < \infty) \tag{33}$$

zu beweisen.

Es seien nun $1<K<\infty$ und $a\notin M(K, \lambda)$.

Aus den Hilfssätzen IV, bzw. X bekommen wir ein $\Phi\in\Omega(1)$ derart, daß die Folge (2) in (0, 1) fast überall divergiert, d.h. $a\notin M(1, \lambda)$. Damit haben wir (33) und somit Satz II bewiesen.

BEWEIS DER BEMERKUNG I. Wegen $0<\lambda_n\leqq\lambda_{n+1}\to\infty$ $(n\to\infty)$ gibt es eine Indexfolge $\{n_k\}_1^\infty$ $(1=)n_1<\dots<n_k<\dots$ mit

$$(\lambda_{n_1}+\dots+\lambda_{n_{k-1}})(k-1)^{1/4} \leqq 2\lambda_{n_k} k^{1/4} \quad (k = 2, 3, \dots). \tag{34}$$

Es sei

$$a_n = \begin{cases} \dfrac{\lambda_{n_k}}{k^{1/4}}, & n = n_k;\quad k = 1, 2, \dots, \\ 0 & \text{sonst.} \end{cases}$$

Es sei $\varphi\in\Omega(1)$ und n positive ganze Zahl. Dann gibt es einen Index $l=l(n)$ mit $\lim\limits_{n\to\infty} l(n)=\infty$ und $n_l\leqq n<n_{l+1}$. Durch einfache Rechnung ergibt sich auf Grund von (34)

$$\frac{1}{\lambda_n}\left|\sum_{k=1}^{n} a_k\varphi_k(x)\right| \leqq \frac{1}{\lambda_n}\sum_{k=1}^{n}|a_k| \leqq \frac{1}{\lambda_n}\sum_{i=1}^{l}|a_{n_i}| \leqq \frac{1}{\lambda_{n_l}}\sum_{i=1}^{l}\frac{\lambda_{n_i}}{i^{1/4}} \to 0 \quad (x\in(0, 1);\ n\to\infty).$$

D. h. es gilt $a\in M(1, \lambda)$.

Es sei $\{I_k\}_1^\infty$ eine Folge von einfachen Teilmengen von (0, 1) mit folgenden Eigenschaften. Es gelte

$$\operatorname{mes} I_k = 1/k \quad (k = 1, 2, \dots), \tag{35}$$

und für jedes $x\in(0, 1)$ bestehe $x\in I_k$ für unendlich viele k. (Solche Mengenfolge kann man leicht angeben.)

Wir definieren ein System $\Phi\in\Omega(\infty)$ von Treppenfunktionen mit folgenden Eigenschaften:

$$|\Phi_{n_k}(x)| = \begin{cases} 1/\sqrt{\operatorname{mes} I_k} & (x\in I_k), \\ 0 & \text{sonst} \end{cases} \quad (k = 1, 2, \dots), \tag{36}$$

$$|\Phi_n(x)| = 1 \quad (x\in(0, 1);\ n = 1, 2, \dots;\ n \neq n_k;\ k = 1, 2, \dots). \tag{37}$$

Es sei $\Phi_1(x)\equiv 1$ $(x\in(0,1))$. Es sei n_0 eine positive ganze Zahl. Wir nehmen an, daß die Treppenfunktionen $\Phi_n(x)$ $(n=1, \dots, n_0)$ schon derart definiert sind, daß $\{\Phi_k(x)\}_1^{n_0}\in\Omega(\infty)$ ist, und (36), (37) für $n_k\leqq n_0$, bzw. für $n\leqq n_0$ erfüllt sind.

Es sei nun $n_0+1\neq n_k$ $(k=1, 2, \dots)$. Dann gibt es eine Einteilung des Intervalls (0, 1) in paarweise disjunkte Interwalle $J_1, \dots, J_\sigma$ derart, daß jede Funktion $\Phi_n(x)$ $(n=1, \dots, n_0)$ in jedem Intervall J_s konstant ist. Die zwei Hälften von J_s bezeichnen wir mit J_s'' bzw. mit J_s'' $(s=1, \dots, \sigma)$. Dann setzen wir

$$\Phi_{n_0+1}(x)=\begin{cases} 1 & (x\in J_s';\ s=1, \dots, \sigma),\\ -1 & (x\in J_s'';\ s=1, \dots, \sigma).\end{cases}$$

Es sei nun $n_0+1=n_{k_0}$ (für ein gewisses k_0). Dann gibt es eine Einteilung des Intervalls (0, 1) in paarweise disjunkte Intervalle $J_1, \dots, J_\sigma$ derart, daß jede Funktion $\Phi_n(x)$ $(n=1, \dots, n_0)$ in jedem J_s konstant ist und I_{k_0} die Vereinigung gewisser J_s wird. Die zwei Hälften von J_s bezeichnen wir mit J_s', bzw. mit J_s'' $(s=1, \dots, \sigma)$. Dann setzen wir

$$\Phi_{n_0+1}(x)=\begin{cases} 1/\sqrt{\operatorname{mes} I_{k_0}} & (x\in J_s';\ s=1, \dots, \sigma;\ J_s\subseteqq I_{k_0}),\\ -1/\sqrt{\operatorname{mes} I_{k_0}} & (x\in J_s'';\ s=1, \dots, \sigma;\ J_s\subseteqq I_{k_0}),\\ 0 & \text{sonst.}\end{cases}$$

Offensichtlich ist $\Phi_{n_0+1}(x)$ einfach, gilt $\{\Phi_n(x)\}_1^{n_0+1}\in\Omega(\infty)$, weiterhin bestehen (36), (37) für $n_k\leqq n_0+1$, bzw. für $n\leqq n_0+1$. Das System Φ mit den geforderten Eigenschaften erhalten wir durch Induktion.

Es seien $x\in(0,1)$ beliebig und k_0 ein Index mit $x\in I_{k_0}$. Dann gilt

$$(38)\qquad \frac{1}{\lambda_{n_{k_0}}}\left|\sum_{l=1}^{n_{k_0}} a_l\Phi_l(x)\right| \geqq \frac{1}{\lambda_{n_{k_0}}}|a_{n_{k_0}}\Phi_{n_{k_0}}(x)|-\frac{1}{\lambda_{n_{k_0}}}\sum_{l=1}^{n_{k_0}-1}|a_l\Phi_l(x)| =$$

$$=\frac{1}{k_0^{1/4}}\sqrt{k_0}-\frac{1}{\lambda_{n_{k_0}}}\sum_{i=1}^{k_0-1}|a_{n_{k_i}}\Phi_{n_{k_i}}(x)|\geqq k_0^{1/4}-\frac{1}{\lambda_{n_{k_0}}}\sum_{i=1}^{k_0-1}\frac{\lambda_{n_{k_i}}}{i^{1/4}}\sqrt{i}\geqq\frac{1}{2}k_0^{1/4}$$

auf Grund von (34). Da $x\in I_{k_0}$ für unendlich viele k_0 erfüllt ist, gilt

$$\lim_{k\to\infty}\frac{1}{\lambda_{n_k}}\left|\sum_{l=1}^{n_k} a_l\Phi_l(x)\right|=\infty.$$

Da diese Relation für jedes $x\in(0,1)$ besteht, folgt $a\notin M(\lambda,\infty)$.

Damit haben wir Bemerkung I bewiesen.

BEMERKUNG VII. In der Arbeit [3] haben wir gezeigt, dass aus $\|a;\infty;\lambda\|<\infty$ die Relation $\lim\limits_{N\to\infty}\|a;\infty;\lambda;N,\infty\|=0$ folgt.

Diese Implikation gilt aber im Falle $1\leqq K<\infty$ im allgemeinen nicht.

Es sei nämlich $1\leqq K_0<\infty$. Wegen $\lambda_n\to\infty$ $(n\to\infty)$ gibt es eine Indexfolge $\{n_k\}_1^\infty$ $(n_1<\dots<n_k<\dots)$ mit

$$2(\lambda_{n_1}+\dots+\lambda_{n_{k-1}})\leqq\lambda_{n_k}\quad (k=2, 3, \dots).$$

Es sei

$$a_n = \begin{cases} \lambda_{n_k} & (n = n_k;\ k = 1, 2, \ldots), \\ 0 & (n \neq n_k;\ k = 1, 2, \ldots). \end{cases}$$

Dann gilt

$$\frac{1}{\lambda_n} \left| \sum_{k=1}^{n} a_k \varphi_k(x) \right| \leqq K_0 \frac{1}{\lambda_n} \sum_{k=1}^{n} |a_k| \leqq 2K_0 \quad (x \in (0, 1);\ n = 1, 2, \ldots)$$

für jedes System $\varphi \in \Omega(K_0)$, und so gilt $\|a;\, K_0;\, \lambda\| \leqq 2K_0 (<\infty)$. Es sei N eine positive ganze Zahl. Dann gilt für jedes k_0 mit $n_{k_0} \geqq N$

$$\frac{1}{\lambda_{n_{k_0}}} \left| \sum_{k=1}^{n_{k_0}} a_k r_k(x) \right| \geqq \frac{|a_{n_{k_0}}|}{\lambda_{n_{k_0}}} - \frac{1}{\lambda_{n_{k_0}}} \sum_{k=1}^{n_{k_0}-1} |a_k| \geqq 1 - \frac{1}{\lambda_{n_{k_0}}} \sum_{k=1}^{k_0-1} |a_{n_k}| \geqq \frac{1}{2} \quad (x \in (0, 1)),$$

woraus sich

$$\|a;\, K_0;\, \lambda;\, N, \infty\| \geqq \|a;\, 1;\, \lambda;\, N, \infty\| \geqq 1/2 \quad (N = 1, 2, \ldots)$$

ergibt. Also besteht $\lim\limits_{N \to \infty} \|a;\, K_0;\, \lambda;\, \infty\| \neq 0$ für diese Folge a.

Beweis des Satzes III. Wir nehmen an, daß die Behauptung des Satzes nicht zutrifft. Dann gibt es eine Zahl K_0, $1 \leqq K_0 < \infty$, mit folgender Eigenschaft:

a) für jede positive Zahl M existiert eine Folge a mit $\|a;\, K_0;\, \lambda\| > M \|a;\, 1;\, \lambda\|$. Wir zeigen, daß in diesem Falle auch die folgende Behauptung gilt:

b) für jede positive ganze Zahl N und für jede positive Zahl M gibt es eine Folge a mit

$$\|a;\, K_0;\, \lambda;\, N, \infty\| > M \|a;\, 1;\, \lambda;\, N;\, \infty\|.$$

Im entgegengesetzten Falle gäbe es nämlich eine positive ganze Zahl N_0 und eine positive Zahl M_0 derart, daß für jede Folge a

$$\|a;\, K_0;\, \lambda;\, N_0, \infty\| \leqq M_0 \|a;\, 1;\, \lambda;\, N_0, \infty\|$$

besteht. Dann würde aber für jede Folge a

$$\|a;\, K_0;\, \lambda\| \leqq \|a;\, K_0;\, \lambda;\, 1, N_0 - 1\| + \|a;\, K_0;\, \lambda;\, N_0, \infty\| \leqq$$

$$\leqq \frac{1}{\lambda_1} \sum_{k=1}^{N_0-1} |a_k| + M_0 \|a;\, 1;\, \lambda;\, N_0, \infty\| \leqq$$

$$\leqq \frac{1}{\lambda_1} \sqrt{N_0} \sqrt{\sum_{k=1}^{N_0-1} a_k^2} + M_0 \|a;\, 1;\, \lambda\| \leqq \sqrt{N_0} \frac{\lambda_{N_0}}{\lambda_1} \|a;\, 1;\, \lambda;\, 1, N_0 - 1\| + M_0 \|a;\, 1;\, \lambda\| \leqq$$

$$\leqq \left(\sqrt{N_0} \frac{\lambda_{N_0}}{\lambda_1} + M_0 \right) \|a;\, 1;\, \lambda\|$$

gelten, was a) widerspricht.

Aus b) folgt aber folgendes:

c) für jede positive ganze Zahlen N_1, N_2 und jede positive Zahl M gibt es eine Folge a mit

$$\|a(N_2, \infty);\, K_0;\, \lambda;\, N_1, \infty\| > M \|a(N_2, \infty);\, 1;\, \lambda;\, N_1, \infty\|. \tag{39}$$

Im entgegengesetzten Falle gäbe es nämlich positive ganze Zahlen $\bar{N}_1, \bar{N}_2$ und eine positive Zahl $\bar{M}$ derart, daß für jede Folge a

$$\|a(\bar{N}_2, \infty);\ K_0;\ \lambda;\ \bar{N}_1, \infty\| \leqq \bar{M}\|a(\bar{N}_2, \infty);\ 1;\ \lambda;\ \bar{N}_1, \infty\|$$

besteht. Dann würde aber für jede Folge a

$$\|a;\ K_0;\ \lambda;\ \bar{N}_1, \infty\| \leqq \|a(1, \bar{N}_2-1);\ K_0;\ \lambda;\ \bar{N}_1, \infty\| + \|a(N_2, \infty);\ K_0;\ \lambda;\ \bar{N}_1, \infty\| \leqq$$
$$\leqq \|a(1, \bar{N}_2-1);\ K_0;\ \lambda;\ \bar{N}_1, \infty\| + \bar{M}\|a(\bar{N}_2, \infty);\ 1;\ \lambda;\ \bar{N}_1, \infty\| \leqq$$
$$\leqq \|a(1, \bar{N}_2-1);\ K_0;\ \lambda;\ \bar{N}_1, \infty\| + \bar{M}(\|a(1, \bar{N}_2-1);\ 1;\ \lambda;\ \bar{N}_1, \infty)\| +$$
$$+\|a;\ 1;\ \lambda;\ \bar{N}_1, \infty\|) \leqq (1+\bar{M})\|a(1, \bar{N}_2-1);\ K_0;\ \lambda;\ \bar{N}_1, \infty\| + \bar{M}\|a;\ 1;\ \lambda\|$$

und deswegen für jede Folge a

$$\|a;\ K_0;\ \lambda\| \leqq \|a;\ K_0;\ \lambda;\ 1, \bar{N}_1-1\| + \|a;\ K_0;\ \lambda;\ \bar{N}_1, \infty\| \leqq$$
$$\leqq \|a(1, \bar{N}_1-1);\ K_0;\ \lambda\| + (1+\bar{M})\|a(1, N_2-1);\ K_0;\ \lambda;\ \bar{N}_1, \infty\| + \bar{M}\|a;\ 1;\ \lambda\| \leqq$$
$$\leqq \frac{\lambda_{\bar{N}_1}}{\lambda_1}\sqrt{\bar{N}_1}\,\|a;\ 1;\ \lambda;\ 1, \bar{N}_1-1\| +$$
$$+\frac{\lambda_{\bar{N}_2}}{\lambda_1}\sqrt{\bar{N}_2}\,(1+\bar{M})\|a;\ 1;\ \lambda;\ 1, \bar{N}_2-1\| + \bar{M}\|a;\ 1;\ \lambda\| \leqq$$
$$\leqq (2+\bar{M})\frac{\max(\lambda_{\bar{N}_1}, \lambda_{\bar{N}_2})}{\lambda_1}\left(\sqrt{\max(\bar{N}_1, \bar{N}_2)} + \bar{M}\right)\|a;\ 1;\ \lambda\|$$

gelten, was aber a) widerspricht.

Aus (39) folgt, daß für genügend große ganze Zahlen $N_3(\geqq N_1, N_2)$

$$\|a(N_2, N_3);\ K_0;\ \lambda;\ N_1, \infty\| = \|a(N_2, \infty);\ K_0;\ \lambda;\ N_1, N_3\| >$$
$$> M\|a(N_2, \infty);\ 1;\ \lambda;\ N_1, N_3\| = M\|a(N_2, N_3);\ 1;\ \lambda;\ N_1, \infty\|$$

besteht.

Aus a) bekommen wir also die folgende Behauptung:

d) im Falle a) gibt es für beliebige positive ganze Zahlen N_1, N_1 und für jede positive Zahl M eine Folge a und eine ganze Zahl $N_3(\geqq N_1, N_2)$ mit

$$\|a(N_2, N_3);\ K_0;\ \lambda;\ N_1, \infty\| > M\|a(N_2, N_3);\ 1;\ \lambda;\ N_1, \infty\|.$$

Auf Grund von d) können wir eine Folge von ganzen Zahlen $\{N_m\}_1^\infty$ $(0=N_1<\dots$ $\dots<N_m<\dots)$ und für jedes $m(=1, 2, \dots)$ eine Folge $a^{(m)}$ derart angeben, daß für jedes $m(=1, 2, \dots)$

(40)
$$\|a^{(m)}(N_m+1, N_{m+1});\ K_0;\ \lambda;\ N_m+1, \infty\| > m^2\|a^{(m)}(N_m+1, N_{m+1});\ 1;\ \lambda;\ N_m+1, \infty\|$$

besteht; ohne Beschränkung der Allgemeinheit können wir dabei

(41) $$\|a^{(m)}(N_m+1, N_{m+1});\ 1;\ \lambda;\ N_m+1, \infty\| = \frac{1}{m^2} \quad (m = 1, 2, \dots)$$

annehmen.

Wir setzen

$$a_k = a_k^{(m)} \quad (k = N_m+1, \dots, N_{m+1};\ m = 1, 2, \dots).$$

Es sei N eine positive ganze Zahl. Dann gibt es eine positive ganze Zahl $m=m(N)$ mit $N_m < N \leqq N_{m+1}$; offensichtlich gilt $\lim_{N\to\infty} m(N)=\infty$. Aus (41) folgt für eine positive ganze Zahl $m_0(\leqq m(N))$

$$\|a;\ 1;\ \lambda;\ N,\ \infty\| \leqq \|a(1, N_{m_0});\ 1;\ \lambda;\ N,\ \infty\| + \|a(N_{m_0}+1, \infty);\ 1;\ \lambda;\ N,\ \infty\| \leqq$$

$$\leqq \|a(1, N_{m_0});\ 1;\ \lambda;\ N,\ \infty\| + \|a(N_{m_0}+1, \infty);\ 1;\ \lambda;\ N_m+1,\ \infty\| \leqq$$

$$\leqq \|a(1, N_{m_0});\ 1;\ \lambda;\ N,\ \infty\| + \sum_{\mu=m_0}^{\infty} \|a(N_\mu+1, N_{\mu+1});\ 1;\ \lambda;\ N_m+1,\ \infty\| \leqq$$

$$\leqq \|a(1, N_{m_0});\ 1;\ \lambda;\ N,\ \infty\| + \sum_{\mu=m_0}^{\infty} \|a(N_\mu+1, N_{\mu+1});\ 1;\ \lambda;\ N_\mu+1,\ N_{\mu+1}\| \leqq$$

$$\leqq \|a(1, N_{m_0});\ 1;\ \lambda;\ N,\ \infty\| + \sum_{\mu=m_0}^{\infty} \frac{1}{\mu^2} \leqq \|a(1, N_{m_0});\ 1;\ \lambda;\ N,\ \infty\| + \frac{1}{m_0}.$$

Es seien $\varepsilon > 0$ beliebig und m_0 so groß, daß $1/m_0 < \varepsilon/2$ besteht. Dann ist

$$\|a;\ 1;\ \lambda;\ N,\ \infty\| \leqq \|a(1, N_{m_0});\ 1;\ \lambda;\ N,\ \infty\| + \frac{\varepsilon}{2} < \varepsilon,$$

für genügend großes N. Also gilt $\lim_{N\to\infty} \|a;\ 1;\ \lambda;\ N,\ \infty\| = 0$. Daraus folgt

$$a \in M(1, \lambda). \tag{42}$$

Nach Hilfssatz IV ergibt sich daraus

$$\frac{1}{\lambda_n^2} \sum_{k=1}^{n} a_k^2 \to 0 \quad (n \to \infty).$$

Hieraus und aus (40), (41) erhalten wir durch Anwendung des Hilfssatzes X, daß ein System $\Phi \in \Omega(1)$ derart existiert, daß die Folge (2) in (0, 1) fast überall divergiert, und so ist $a \notin M(1, \lambda)$, was (42) widerspricht.

Damit haben wir gezeigt, daß es für jedes $K(1<K<\infty)$ eine nur von K abhängige positive Zahl $C(K)$ derart gibt, daß $\|a; K; \lambda\| \leqq C(K)\|a; 1; \lambda\|$ für jede Folge a besteht.

Zum Beweis des Satzes III wollen wir nur $C(K)$ abschätzen.

Es seien $K>2$ und a eine beliebige Folge. Durch Anwendung des Hilfssatzes XI ergibt sich

$$\|a;\ K;\ \lambda;\ 1,\ N\| \leqq C(2, K)\|a;\ 2;\ \lambda;\ 1,\ N\|$$

mit

$$C(2, K) = 2\sqrt{\frac{K^2-1}{2^2-1}} \leqq 2K,$$

woraus

$$\|a;\ K;\ \lambda\| \leqq 2K\|a;\ 2;\ \lambda\| \tag{43}$$

folgt. Andererseits gilt nach dem Satz III $\|a;\ 2;\ \lambda\| \leqq C(2)\|a;\ 1;\ \lambda\|$ mit einer von a unabhängigen Konstante $C(2)$. Daraus und aus (43) erhalten wir, daß $\|a;\ K;\ \lambda\| \leqq C(K)\|a;\ 1;\ \lambda\|$ für jede Folge a besteht, wobei $C(K) \leqq 2K + C(2) = O(K)$ ist.

Damit haben wir Satz III vollständig bewiesen.

BEWEIS DES SATZES IV. Für den Fall $K=\infty$ haben wir die Behauptung in [3] bewiesen.

Es seien nun $1 \leqq K < \infty$ und a, b Folgen mit $a \in M(K, \lambda)$ und $|b_k| \leqq |a_k|$ $(k = 1, 2, \ldots)$. Dann gilt

$$\lim_{N\to\infty} \|a;\ K;\ \lambda;\ N, \infty\| = 0, \tag{44}$$

wegen der Definition von $M(K, \lambda)$. Es seien N, N_1 $(N \leqq N_1)$ positive ganze Zahlen. Nach dem Hilfssatz VI gibt es ein System $\varphi = \{\varphi_k(x)\}_1^{N_1} \in \Omega(K)$ von Treppenfunktionen mit

$$\frac{1}{2}\|b;\ K;\ \lambda;\ N, N_1\|^2 = \frac{1}{2}\|b(1, N_1);\ K;\ \lambda;\ N, N_1\|^2 \leqq \tag{45}$$

$$\leqq \int_0^1 \max_{N \leqq n \leqq N_1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} b_k \varphi_k(x)\right)^2 dx.$$

Es seien $1 \leqq k_1 < \ldots < k_\varrho \leqq N_1$ diejenigen Indizes, für die $a_{k_r} \neq 0$ ist; die anderen Indizes k, die von $k_1, \ldots, k_\varrho$ verschieden sind, bezeichnen wir der Reihe nach mit l_s $(s = 1, \ldots, N_1 - \varrho)$. Wir setzen

$$\psi_{k_r}(x) = \begin{cases} \dfrac{b_{k_r}}{a_{k_r}} \sqrt{2}\, \varphi_{k_r}(2x) & \left(x \in \left(0, \dfrac{1}{2}\right)\right), \\ \left(1 - \dfrac{b_{k_r}^2}{a_{k_r}^2}\right)^{1/2} \sqrt{2}\, \varphi_{k_r}(2x-1) & \left(x \in \left(\dfrac{1}{2}, 1\right)\right) \end{cases} \qquad (r = 1, \ldots, \varrho),$$

$$\psi_{l_s}(x) = \sqrt{2}\, \varphi_{l_s}(2x-1) \quad \left(x \in \left(\tfrac{1}{2}, 1\right);\ s = 1, \ldots, N_1 - \varrho\right).$$

Dann gilt $\psi = \{\psi_k(x)\}_1^{N_1} \in \Omega(\sqrt{2}K)$. Weiterhin bekommen wir aus (45)

$$\|a;\ \sqrt{2}K;\ \lambda;\ N, N_1\|^2 = \|a(1, N);\ \sqrt{2}K;\ \lambda;\ N, N_1\|^2 \geqq \tag{46}$$

$$\geqq \int_0^1 \max_{N \leqq n \leqq N_1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} a_k \psi_k(x)\right)^2 dx \geqq \int_0^{1/2} \max_{N \leqq n \leqq N_1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} a_k \psi_k(x)\right)^2 dx =$$

$$= 2\int_0^{1/2} \max_{N \leqq n \leqq N_1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} b_k \varphi_k(2x)\right)^2 dx = \int_0^1 \max_{N \leqq n \leqq N_1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} b_k \varphi_k(x)\right)^2 dx \geqq$$

$$\geqq \frac{1}{2}\|b;\ K;\ \lambda;\ N;\ N_1\|^2.$$

Durch Anwendung des Hilfssatzes XI ergibt sich, daß für die Folge a

$$\|a;\ \sqrt{2}K;\ \lambda;\ N, N_1\|^2 = \|a(1, N_1);\ \sqrt{2}K;\ \lambda;\ N, N_1\|^2 \leqq$$
$$\leqq C(K, \sqrt{2}K)\|a(1, N_1);\ K;\ \lambda;\ N, N_1\|^2 = C(K, \sqrt{2}K)\|a;\ K;\ \lambda;\ N, N_1\|^2$$

besteht. Daraus und aus (46) bekommen wir

$$\|b;\ K;\ \lambda;\ N, N_1\|^2 \leqq 2C(K, \sqrt{2}K)\|a;\ K;\ \lambda;\ N, N_1\|^2$$

für alle N und N_1 $(N \leqq N_1)$. Daraus folgt

$$\|b;\ K;\ \lambda;\ N, \infty\| \leqq \sqrt{2}\sqrt{C(K, \sqrt{2}K)}\|a;\ K;\ \lambda;\ N, \infty\| \quad (N = 1, 2, \ldots).$$

Auf Grund von (44) bekommen wir $\lim\limits_{N\to\infty} \|b;\ K;\ \lambda;\ N, \infty\| = 0$; also ist $b \in M(K, \lambda)$.

Damit haben wir Satz IV bewiesen.

Beweis des Satzes V. Im Falle $a \in M(K, \lambda)$ gilt $\lim\limits_{N\to\infty} \|a;\ K;\ \lambda;\ N, \infty\| = 0$ auf Grund der Definition von $M(K, \lambda)$. Es sei $(0=)N_1 < \ldots < N_m < \ldots$ eine Indexfolge mit

$$\|a;\ K;\ \lambda;\ N_m+1, \infty\| \leqq \frac{1}{m^3} \quad (m = 1, 2, \ldots).$$

Dann sei

$$\mu_k = m \quad (k = N_m+1, \ldots, N_{m+1};\ m = 1, 2, \ldots).$$

Für jede positive ganze Zahl N sei $m = m(N)$ derjenige Index, für welchen $N_m < N \leqq N_{m+1}$ gilt. Offensichtlich ist $\lim\limits_{N\to\infty} m(N) = \infty$. Dann gilt

$$\|\lambda a;\ K;\ \lambda;\ N, \infty\| \leqq \sum_{m=m(N)}^{\infty} \|\mu a;\ K;\ \lambda;\ N_m+1,\ N_{m+1}\| \leqq$$
$$\leqq \sum_{m=m(N)}^{\infty} m\, \|a;\ K;\ \lambda;\ N_m+1, \infty\| \leqq \sum_{m=m(N)}^{\infty} \frac{1}{m^2} = O\left(\frac{1}{m(N)}\right),$$

woraus sich $\lim\limits_{N\to\infty} \|\mu a;\ K;\ \lambda;\ N, \infty\| = 0$, d. h. $\mu a \in M(K, \lambda)$ ergibt.

Ist $a \notin M(\infty, \lambda)$, so gilt $\|a;\ \infty;\ \lambda\| = \infty$, auf Grund eines bekannten Satzes [3]. Daraus folgt, daß für jede positive ganze Zahl M $\lim\limits_{N\to\infty} \|a;\ \infty;\ \lambda;\ M, N\| = \infty$.

So kann man eine Indexfolge $(0=)N_1 < \ldots < N_m < \ldots$ mit der Eigenschaft

$$\|a;\ \infty;\ \lambda;\ N_m+1, N_{m+1}\| \geqq m^2 \ (m = 1, 2, \ldots)$$

definieren. Es sei

$$\mu_k = \frac{1}{m} \quad (k = N_m+1, \ldots, N_{m+1};\ m = 1, 2, \ldots).$$

Dann gilt

$$\|\mu a;\ \infty;\ \lambda;\ N_m+1, \infty\| \geqq \|\mu a;\ \infty;\ \lambda;\ N_m+1, N_{m+1}\| \geqq m \quad (m = 1, 2, \ldots),$$

woraus $\lim\limits_{N\to\infty} \|\mu a;\ \infty;\ \lambda;\ N, \infty\| = \infty$ und so $\mu a \in M(\infty, \lambda)$ sich ergibt.

BEWEIS DER BEMERKUNG V. Es sei $1 \leqq K < \infty$. Wegen $\lim_{k\to\infty} \lambda_k = \infty$ gibt es eine Indexfolge $(0=)n_1 < \ldots < n_k < \ldots$ mit $2(\lambda_{n_1} + \ldots + \lambda_{n_{k-1}}) \leqq \lambda_{n_k}$ $(k=2, 3, \ldots)$.

Wir setzen

$$a_n = \begin{cases} \lambda_{n_k} & (n = n_k;\ k = 2, 3, \ldots), \\ 0 & (n \neq n_k;\ k = 2, 3, \ldots). \end{cases}$$

Dann gilt

$$\left| \frac{1}{\lambda_{n_k}} \sum_{l=1}^{n_k} a_l r_l(x) \right| \geqq 1 - \frac{1}{\lambda_{n_k}} \sum_{s=1}^{k-1} \lambda_{n_s} \geqq \frac{1}{2} \quad (k = 2, 3, \ldots)$$

fast überall in $(0, 1)$. So gilt aber für das System $\{r_k(x)\}_1^\infty \in \Omega(1) \leqq \Omega(K)$

$$\frac{1}{\lambda_n} \sum_{l=1}^{n} a_l r_l(x) \not\to 0 \quad (n \to \infty)$$

fast überall in $(0, 1)$, woraus $a \notin M(K, \lambda)$ folgt.

Es sei $\mu = \{\mu_k\}_1^\infty$ eine beliebige abnehmende, gegen 0 strebende Folge von positiven Zahlen. Dann gilt für jedes $\varphi \in \Omega(K)$

$$\frac{1}{\lambda_n} \left| \sum_{k=1}^{n} \mu_k a_k \varphi_k(x) \right| \leqq K \frac{1}{\lambda_n} \sum_{k=1}^{n} |\mu_k a_k| \quad (x \in (0, 1);\ n = 1, 2, \ldots).$$

Durch einfache Rechnung folgt

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k |a_k| \to 0 \quad (n \to \infty).$$

Nach dem obigen gilt aber

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k a_k \varphi_k(x) = 0$$

für jedes System $\varphi \in \Omega(K)$ überall in $(0, 1)$; d. h. es gilt $\mu a \in M(K, \lambda)$ für jede monoton abnehmende, gegen 0 strebende Folge μ von positiven Zahlen.

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ON THE POROUS MEDIUM EQUATIONS WITH LOWER ORDER SINGULAR NONLINEAR TERMS

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I. Introduction. In this paper we shall study the Cauchy problem

$$Lu \equiv -u_t + (u^{(m)})_{xx} - b(u^\lambda)_x - cu^p = 0 \quad \text{in} \quad \mathbf{R}\times(0, \infty), \tag{1}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathbf{R}, \tag{2}$$

where $m \geqq 1$, $\lambda > 0$, $p > 0$, $b > 0$, $c > 0$ are constants. The function $u_0(x)$ is supposed to be bounded, continuous and nonnegative, satisfying $u_0(x) > 0$ on I and $u_0(x) \equiv 0$ in $\mathbf{R} \setminus I$ for a bounded interval $I = (a_1, a_2) \subset \mathbf{R}$.

Equation (1) is parabolic when $u > 0$ and degenerates when $u = 0$. In the case $b = c = 0$ (1) is called the equation of nonlinear heat conductivity or filtration equation. The terms cu^p and $b(u^\lambda)_x$ in (1) correspond to the absorption and to the transport of heat or matter, respectively.

The Cauchy problem (1), (2) usually has no classical solution having continuous derivatives which appear in (1) even in the case $b = c = 0$ (see Zeldovich and Komponeets [7]). Oleinik, Kalashnikov and Yui-lin [6] proved that the Cauchy problem (1), (2) has a unique generalized solution in the case $b = c = 0$.

DEFINITION. A continuous nonnegative function $u(x, t)$ is said to be a generalized solution of (1), (2) if u satisfies (2) and if the integral identity

$$I(u, f, P) \equiv \int_{t_0}^{t_1} \int_{x_0}^{x_1} (u^m f_{xx} + bu^\lambda f_x - cu^p f + uf_t)\, dx\, dt - \int_{t_0}^{t_1} u^m f_x|_{x_0}^{x_1}\, dt - \int_{x_0}^{x_1} uf|_{t_0}^{t_1}\, dx = 0$$

holds whenever $P \equiv [x_0, x_1] \times [t_0, t_1] \subset \mathbf{R} \times (0, \infty)$ and for all $f \in C^{2;1}_{x;t}(P)$ such that $f(x_0, t) = f(x_1, t) = 0$.

Throughout this paper solution means generalized solution.

In the case $m > 1$, $\lambda \geqq 1$, $p \geqq 1$, Kalashnikov [4] proved that (1)—(2) has a unique solution which is classical on the open set $p[u] \equiv \{(x, t) \in \mathbf{R} \times (0, \infty) | u(x, t) > 0\}$. The domain $p[u]$ is bounded by two continuous curves, that is there exist continuous functions $\zeta_i(t)$ $(i = 1, 2)$ such that

$$p[u] = \{(x, t) \in \mathbf{R} \times (0, \infty) | \zeta_1(t) < x < \zeta_2(t)\}.$$

These curves are called the interface curves or the front of heat perturbation or simply the front. Kalashnikov investigated the behaviour of the interface curves, too: in the case $1 < \lambda < m$, $p \geqq 1$ he proved that there exists a constant $x_0 \in \mathbf{R}$ such that $u(x, t) = 0$ for all $x \leqq x_0$ and for all $t \geqq 0$. This phenomenon is called the localization on the left side. If $1=\lambda<m,\ p\geqq 1$, then there exists a backfront, i.e. for any $x_0\in\mathbf{R}$ there exist $t_0, t_1>0$ such that $u(x_0, t)=0$ whenever $t\notin[t_0, t_1]$.

In the case $m\geqq 1, \lambda\geqq 1,\ 0<p<1$, the term corresponding to the absorption alters the character of interface curves strongly: there exist $T_0, L_0>0$ such that $u(x, t)=0$ outside $[-L_0, L_0]\times[0, T_0]$. ("total extinction in finite time plus localization on both sides").

Diaz and Kersner [2] established that there is no right side front when $c=0$ and $m\geqq 1>\lambda>0$ and for the left hand side front the inequality $\zeta_1(t)\geqq c_1t-c_2$ is valid (here $t\geqq 0$ and c_1, c_2 are constants).

In this paper we shall prove that even in the case when $c>0,\ m\geqq 1>\lambda>0$ and $m-\lambda>1-p$ there is no right side front. This is one of the main results of this paper. The second main result is concerned with the regularity of the solution. The third theorem gives a sufficient condition for the right hand side localization.

Let S and $S(\tau)$ denote the strips $\mathbf{R}\times[\tau, T]$ and $\mathbf{R}=[\tau, T]$ respectively, for some constants $T>\tau>0$. Let

$$\sigma=\begin{cases} m-\lambda & \text{if } m-\lambda\geqq 1-p \\ \max\left(m-\lambda, \dfrac{m-p}{2}\right) & \text{if } m-\lambda<1-p. \end{cases} \tag{3}$$

THEOREM 1. *If $u(x, t)$ is the solution of* (1)—(2) *in the strip S, then*

$$|(u^\sigma)_x|\leqq M \tag{4}$$

in the strip $S(\tau)$, where $(u^\sigma)_x$ is a distribution derivative, M is a constant. If $\sup|(u_0^\sigma)_x|<\infty$ *then the same estimate is valid in the strip S, too.*

Let $M_0=\sup u_0,\ v=u^\sigma$, where σ is as defined by (3).

THEOREM 2. *Suppose $u(x, t)$ is the solution of* (1)—(2) *in the case*

$$m\geqq 1>\lambda>0,\quad p\geqq\lambda \tag{5}$$

and

$$v_x(x, 0)\geqq-\alpha \tag{6}$$

where $v_x(x, 0)$ is a classical derivative and α is an arbitrary constant such that

$$\alpha<b\lambda(1-\lambda)/\big(m(m+2\lambda-1)\big). \tag{7}$$

Then

a) *if $p\geqq 1$ then we have*

$$v_x\geqq-\varrho\frac{v}{t}\quad\text{in}\quad\mathbf{R}\times(0, \infty) \tag{8}$$

for some constant $\varrho>\varrho_0(m, \lambda, b, M_1, \alpha)$;

b) *if $1>p\geqq\lambda$ then for all $T>0$ we get*

$$v_x\geqq-\varrho\frac{v}{t}\quad\text{in}\quad\mathbf{R}\times(0, T]$$

where $\varrho>\varrho_0(m, \lambda, p, b, c, M_1, \alpha, T)$.

COROLLARY 1. *If condition* (6) *of Theorem* 2 *holds then both in the cases* a) *and* b) *there is no front on the right side, i.e. if* $u(x_0, t_0)>0$ *for some* $(x_0, t_0)\in \mathbf{R}\times(0, \infty)$, *then* $u(x, t_0)>0$ *for all* $x\geqq x_0$.

COROLLARY 2. *If condition* (6) *does not hold then we can prove the nonexistence of the right side front only. Indeed, if* $u_0(x)$ *is an arbitrary initial value of* (1)—(2) *then we can construct a function* $\bar{u}_0(x)$ *satistying* $u_0(x)\geqq\bar{u}_0(x)$ *and* $(\bar{u}_0^\sigma)_x(x)\geqq -\alpha$.

Applying Theorem 2 for the solution (1)—(2) with the initial value $\bar{u}_0(x)$ and using comparison argument we get that there is no right side front in this case, too.

COROLLARY 3. *Case* b) *of Theorem* 2 *may also be valid in the case* $p<1$, *i.e. the following phenomenon may appear: no front exists on the right side and there exists a* $T_0>0$ *such that* $u(x, t)=0$ *if* $t\geqq T_0$.

THEOREM 3 (Localization). *Suppose that* $u(x, t)$ *is a solution of* (1)—(2) *and*

$$m \geqq 1 > \lambda > 0, \quad \lambda > p. \tag{9}$$

Then there exists a constant $L_0>0$ *such that* $u(x, t)=0$ *for* $x\geqq L_0$ *and all* $t>0$.

REMARK 1. We have seen that there always exists localization on the left side in this case. From assumption (9) it follows that $1>\lambda>p$ hence there exists a $T_0>0$ such that $u(x_0, t_0)=0$ if $t\geqq T_0$ ("total extinction in finite time").

II. Preliminaries. To prove the theorems we need the following facts.

1. The Cauchy problem (1)—(2) has a unique solution.

2. The solution $u(x, t)$ of (1)—(2) is the pointwise limit of a monotonically decreasing sequence $\{u_n(x, t)\}$ of positive functions, which are classical solutions of the initial-boundary value problem

$$\begin{cases} u_t = (u^m)_{xx} - b\,(u^\lambda)_x - cu^p + c\left(\dfrac{1}{n}\right)^p & \text{in} \quad (-n, n)\times(0, n), \\ u(x, 0) = u_0(x;\, n) \quad \text{for} \quad x\in(-n, n), \\ u(\pm n, t) = M_1 \quad \text{for} \quad t\in(-n, n). \end{cases} \tag{10}$$

Here $M_1=\max u_0+1$ and $u_0(x; n)\in C^\infty$ satisfy the following relations:

(i) $\dfrac{1}{n}<u_0(x;\, n)<M_1$, where $x\in(-n, n)$,

(ii) $u_0(\pm n;\, n)=M_1$,

(iii) $u_0(x;\, n)$ is strictly monotone decreasing when n increases and tends to $u_0(x)$, when $n\to\infty$.

3. It follows from the maximum principle that

$$\frac{1}{n} \leqq u_n(x, t) \leqq M_1.$$

4. DEFINITION. A bounded continuous nonnegative function $v(x, t)$ is said to be a (generalized) supersolution of (1) if the integral inequality $I(v, f;\, P)\leqq 0$ holds for any $P\equiv[x_0, x_1]\times[t_0, t_1]\subset\mathbf{R}\times(0, \infty)$ and any nonnegative function $f\in C^{2,1}_{x,t}(P)$ such that $f(x_0, t)=f(x_1, t)=0$.

5. Lemma. *Let $u(x, t)$ be a solution of (1)—(2) and $v(x, t)$ be a (generalized) supersolution of (1) in $G\equiv\{(x, t)|s<x<\infty,\ 0<t<\infty\}$ where S is an arbitrary number. Let $u_0(x)\leqq v(x, 0)$ for $s<x<\infty$ and let $u(s, t)\leqq v(s, t)$ for $t\geqq 0$. Then $u(x, t)\leqq$ $\leqq v(x, t)$ everywhere in G.*

The above mentioned statements are easy to verify by means of a slight modification of the proofs of the corresponding theorems in [5].

III. Proof of Theorem 1. The idea of proof of the Theorem belongs to Bernstein. It was perfected by Aronson (for equation $u_t=(u^m)_{xx}$) in [1] and by Kalashnikov (for equation $u_t=(u^m)_{xx}-cu^p$) in [3].

The solution $u(x, t)$ of the problem (1), (2) is a pointwise limit of a monotone decreasing sequence $\{u_n(x, t)\}$ of positive functions, which are classical solutions of (10) in $Q_n\equiv(-n, n)\times(0, n)$. To prove the first assertion of Theorem 1 it suffices to establish the validity of (4) for each $u_n(x, t)$ in $Q_n\cap S(\tau)$ with a constant M independent of n.

Let $v:=u_n^\sigma$, then

$$(u_n)_t=\frac{1}{\sigma}v^{\frac{1}{\sigma}-1}v_t;\ (u_n^\lambda)_x=\frac{\lambda}{\sigma}v^{\frac{\lambda}{\sigma}-1}v_x,\quad (u_n^m)_{xx}=\frac{m}{\sigma}\left(\frac{m}{\sigma}-1\right)v^{\frac{m}{\sigma}-2}v_x^2+\frac{m}{\sigma}v^{\frac{m}{\sigma}-1}v_{xx}.$$

Thus by (10), $v(x, t)$ satisfies the equation

$$v_t=m\left(\frac{m}{\sigma}-1\right)v^{\frac{m-1}{\sigma}-1}v_x^2+mv^{\frac{m-1}{\sigma}}v_{xx}-b\lambda v^{\frac{\lambda-1}{\sigma}}v_x-c\sigma v^{\frac{p-1}{\sigma}+1}+c\sigma n^{-p}v^{1-\frac{1}{\sigma}}. \tag{11}$$

Denote by f the Aronson's function: $f(w):=\frac{N_0}{3}w(4-w)$, where $N_0:=M_0^\sigma$, $M_0=$ $=\sup u_0$. Thus we have $f'(w)=\frac{4N_0}{3}-\frac{2N_0}{3}w$ and

$$\frac{2N_0}{3}\leqq f'(w)\leqq\frac{4N_0}{3};\quad f''(w)\equiv-\frac{2N}{3}<0;\quad \left(\frac{f''}{f'}\right)'=-\frac{1}{(2-w)^2}\leqq-\frac{1}{4}, \tag{12}$$

when $0\leqq w\leqq 1$. Define $w(x, t)$ by $v=f(w)$. We get

$$v_t=f'(w)w_t,\quad v_x=f'(w)w_x,\quad v_{xx}=f''(w)w_x^2+f'(w)w_{xx}.$$

Rewriting (11) for w, we obtain

$$\begin{aligned} w_t=m\left(\frac{m}{\sigma}-1\right)f^{\frac{m-1}{\sigma}-1}f'w_x^2+mf^{\frac{m-1}{\sigma}}\frac{f''}{f'}w_x^2+mf^{\frac{m-1}{\sigma}}w_{xx}-\\ -b\lambda f^{\frac{\lambda-1}{\sigma}}w_x-c\sigma\frac{f^{\frac{p-1}{\sigma}+1}}{f'}+c\sigma n^{-p}\frac{f^{1-\frac{1}{\sigma}}}{f'}.\end{aligned} \tag{13}$$

Provided the notation $p=w_x$ is introduced, (13) differentiated with respect to x

and multiplied by p yields

$$pp_t - mf^{\frac{m-1}{\sigma}} pp_{xx} = \tag{14}$$

$$= \left\{m\left(\frac{m}{\sigma}-1\right)\left(\frac{m-1}{\sigma}-1\right) f^{\frac{m-1}{\sigma}-2}(f')^2 + m\left(\frac{2m-1}{\sigma}-1\right) f^{\frac{m-1}{\sigma}-1} f'' + \right.$$

$$\left. + mf^{\frac{m-1}{\sigma}}\left(\frac{f''}{f'}\right)'\right\} p^4 + \left\{m\left(\frac{3m-1}{\sigma}-2\right) f^{\frac{m-1}{\sigma}-1} f' + 2mf^{\frac{m-1}{\sigma}}\frac{f''}{f'}\right\} p^2 p_x -$$

$$-b\lambda\frac{\lambda-1}{\sigma} f^{\frac{\lambda-1}{\sigma}-1} f' p^3 - b\lambda f^{\frac{\lambda-1}{\sigma}} pp_x - c\sigma\left(\frac{f^{\frac{p-1}{\sigma}+1}}{f'}\right)' p^2 + c\sigma n^{-p}\left(\frac{f^{1-\frac{1}{\sigma}}}{f'}\right)' p^2.$$

Let $(x_0, t_0) \in Q_n \cap S(\tau)$ be an arbitrary point such that $|x_0| < n-2$. Let $\zeta(x, t)$ denote a smooth function satisfying the properties: $0 \leqq \zeta \leqq 1$, $\zeta = 1$ in $p_1 \equiv \equiv (x_0-1, x_0+1) \times (\tau, T)$ and $\zeta = 0$ in the neighbourhood of the lines $t=0$, $x = x_0 \pm 2$. Set $P_2 \equiv (x_0-2, x_0+2) \times (\tau, T)$, and consider the function $z(x, t) = \zeta^2 p^2$ in $\bar{P}_2$. At the maximum point of $z(x, t)$, we have $z_t > 0$, $z_{xx} \leqq 0$, $z_x = 0$ that is

$$\zeta^2 pp_x = -\zeta\zeta_x p^2 \tag{15}$$

and

$$\tfrac{1}{2}(z_t - mf^{\frac{m-1}{\sigma}} z_{xx}) \geqq 0. \tag{16}$$

More explicitly, the last inequality takes the form

$$\zeta^2(pp_t - mf^{\frac{m-1}{\sigma}} pp_{xx}) \geqq \{(\zeta_x^2 + \zeta_{xx}) p^2 + 4\zeta\zeta_x pp_x + \zeta^2 p_x\} mf^{\frac{m-1}{\sigma}} - \zeta\zeta_t p^2.$$

Note that

$$|4\zeta\zeta_x pp_x| \leqq \zeta^2 p_x^2 + 4\zeta_x^2 p^2$$

and

$$\zeta^2(pp_t - mf^{\frac{m-1}{\sigma}} pp_{xx}) \geqq \{(\zeta_{xx} - 3\zeta_x^2) mf^{\frac{m-1}{\sigma}} - \zeta\zeta_t\} p^4.$$

Multiplying the equation (14) by ζ^2, we can use (15) and (16). Taking into account the above remark, we gain the following inequality:

$$-\zeta^2 m\left(\frac{m}{\sigma}-1\right)\left(\frac{m-1}{\sigma}-1\right) f^{\frac{m-1}{\sigma}-2}(f')^2 p^4 + b\lambda\frac{\lambda-1}{\sigma} f^{\frac{\lambda-1}{\sigma}-1} f'\zeta^2 p^3 -$$

$$-b\lambda f^{\frac{\lambda-1}{\sigma}}\zeta\zeta_x p^2 + c\sigma\left(\frac{f^{\frac{p-1}{\sigma}+1}}{f'}\right)'\zeta^2 p^2 - c\sigma n^{-p}\left(\frac{f^{1-\frac{1}{\sigma}}}{f'}\right)'\zeta^2 p^2 -$$

$$-\zeta^2\left\{m\left(\frac{2m-1}{\sigma}-1\right) f^{\frac{m-1}{\sigma}-1} f'' + mf^{\frac{m-1}{\sigma}}\left(\frac{f''}{f'}\right)'\right\} p^4 \leqq$$

$$\leqq \{\zeta\zeta_t - mf^{\frac{m-1}{\sigma}}\zeta\zeta_{xx} + 3\zeta_x^2 mf^{\frac{m-1}{\sigma}}\} p^2 -$$

$$-\zeta\zeta_x\left\{m\left(\frac{3m-1}{\sigma}-2\right) f^{\frac{m-1}{\sigma}-1} f' + 2mf^{\frac{m-1}{\sigma}}\frac{f''}{f'}\right) p^3.$$

Multiplying this by $f^{2-\frac{m-1}{\sigma}}$, we obtain

(17)
$$\{\zeta\zeta_t f^{2-\frac{m-1}{\sigma}}-mf^2\zeta\zeta_{xx}+3mf^2\zeta_x^2\}p^2-\zeta\zeta_x\left\{m\left(\frac{3m-1}{\sigma}-2\right)ff'+2mf^2\frac{f''}{f'}\right\}p^3\geqq$$
$$\geqq-\zeta^2 m\left(\frac{m}{\sigma}-1\right)\left(\frac{m-1}{\sigma}-1\right)(f')^2p^4+b\lambda\frac{\lambda-1}{\sigma}f^{1-\frac{m-\lambda}{\sigma}}f'\zeta^2p^3-$$
$$-b\lambda f^{2-\frac{m-\lambda}{\sigma}}\zeta\zeta_x p^2+c\sigma f^{2-\frac{m-1}{\sigma}}\left(\frac{f^{\frac{p-1}{\sigma}+1}}{f'}\right)'\zeta^2p^2-c\sigma n^{-p}f^{2-\frac{m-1}{\sigma}}\left(\frac{f^{1-\frac{1}{\sigma}}}{f'}\right)'\zeta^2p^2-$$
$$-\zeta^2\left\{m\left(\frac{2m-1}{\sigma}-1\right)ff''+mf^2\left(\frac{f''}{f'}\right)'\right\}p^4\equiv I_1+\ldots+I_7.$$

Case 1. If $m-\lambda\geqq 1-p$ then $\sigma=m-\lambda$ by virtue of (3). Then the coefficient of ζ^2p^4 in the term I_1 on the right side of (17) is strictly positive, $|I_2|\leqq c_1\zeta|p|^3$, $|I_3|\leqq c_2p^2$ and the coefficient of ζ^2p^4 in I_6 and I_7 is nonnegative. We shall estimate I_4 and I_5 together:
$$I_4+I_5=c\sigma f^{2-\frac{m-1}{\sigma}}\zeta^2p^2\left\{\left(\frac{p-1}{\sigma}+1\right)f^{\frac{p-1}{\sigma}}-f^{\frac{p-1}{\sigma}+1}\frac{f''}{(f')^2}-\right.$$
$$\left.-n^{-p}\left(1-\frac{1}{\sigma}\right)f^{-\frac{1}{\sigma}}+n^{-p}f^{1-\frac{1}{\sigma}}\frac{f''}{(f')^2}\right\}=$$
$$=c\sigma f^{2-\frac{m}{\sigma}}\zeta^2p^2\left\{\left(\frac{p-1}{\sigma}+1\right)f^{\frac{p}{\sigma}}-\left(1-\frac{1}{\sigma}\right)n^{-p}-(f^{\frac{p}{\sigma}}-n^{-p})\frac{ff''}{(f')^2}\right\}.$$

Here $u_n^\sigma=v=f(w)$ and using that $\frac{1}{n}\leqq u_n$ in Q_n (see the Preliminaries), we get $f^{\frac{p}{\sigma}}(w)-n^{-p}\geqq 0$, and (12) yields
$$-(f^{\frac{p}{\sigma}}(w)-n^{-p})\frac{ff''}{(f')^2}\geqq 0.$$
Since $\sigma=m-\lambda\geqq 1-p$ we have $\frac{p-1}{\sigma}+1\geqq 0$ and thus
$$\left(\frac{p-1}{\sigma}+1\right)f^{\frac{p}{\sigma}}-\left(1-\frac{1}{\sigma}\right)n^{-p}\geqq\frac{p}{\sigma}n^{-p}>0,$$
i.e. $I_4+I_5>0$. The left side of (17) is less than $c_3p^2+c_4\zeta|p|^3$ because $2-\frac{m-1}{\sigma}=$ $=2-\frac{m-1}{m-\lambda}\geqq 1$. (These constants c_i do not depend on n). So we may drop I_4, I_5, I_6, I_7 on the right hand side of (17) and divide by the coefficient of ζ^2p^4 in I_1. We obtain
$$2\zeta^2p^4\leqq c_5p^2+c_6\zeta|p|^3.$$

Since $c_6\zeta|p|^3 \leqq \zeta^2 p^4 + \left(\frac{c_6}{2}\right)^2 p^2$ it follows that $\zeta^2 p^2 \leqq c_7$. Therefore

$$\max_{\bar{P}_1} |w_x| = \max_{\bar{P}_1} |\zeta w_x| \leqq \max_{\bar{P}_2} |\zeta w_x| \leqq \max_{\bar{P}_2} \sqrt{\zeta^2 p^2} \leqq c_8.$$

Finally, $v_x = f'(w) w_x$ and (12) imply that $\max_{\bar{P}_1} |v_x| \leqq c_9$ which completes the proof for Case 1.

Case 2. If $m-\lambda < 1-p$ then $\sigma = \max\left(m-\lambda, \frac{m-p}{2}\right)$ by virtue of (3). Here $\sigma<1$ in both cases: when $\sigma = m-\lambda$ then we have $\sigma = m-\lambda < 1-p < 1$ and for $\sigma = \frac{m-p}{2}$ we have $m-p < 1+\lambda-2p < 1+\lambda < 2$ since $m-\lambda < 1-p$.

Thus the coefficient of $\zeta^2 p^4$ in I_1 on the right side of (17) is strictly positive, $|I_2| \leqq c_1\zeta|p|^3$, $|I_3| \leqq c_2 p^2$ because $1-\frac{m-\lambda}{\sigma} \geqq 0$. We shall estimate I_4 and I_5 together:

$$I_4 + I_5 = c\sigma f^{2-\frac{m-1}{\sigma}} \left(\frac{p-1}{\sigma}+1\right) f^{\frac{p-1}{\sigma}} \zeta^2 p^2 -$$

$$-c\sigma n^{-p} f^{2-\frac{m-1}{\sigma}} \zeta^2 p^2 \left(1-\frac{1}{\sigma}\right) f^{-\frac{1}{\sigma}} - c\sigma f^{2-\frac{m-1}{\sigma}} f^{\frac{p-1}{\sigma}+1} \frac{f''}{(f')^2} \zeta^2 p^2 +$$

$$+c\sigma n^{-p} f^{2-\frac{m-1}{\sigma}} f^{1-\frac{1}{\sigma}} \frac{f''}{(f')^2} \zeta^2 p^2 = c\sigma \left(\frac{p-1}{\sigma}+1\right) f^{2-\frac{m-p}{\sigma}} \zeta^2 p^2 +$$

$$+c(1-\sigma) n^{-p} f^{2-\frac{m}{\sigma}} \zeta^2 p^2 - c\sigma f^{3-\frac{m}{\sigma}} \frac{f''}{(f')^2} \left(f^{\frac{p}{\sigma}} - n^{-p}\right) \zeta^2 p^2.$$

Here the absolute value of the first term is less than $c_3\zeta^2 p^2$ since $\sigma \geqq \frac{m-p}{2}$. The second term is positive since $\sigma<1$, the third term is also positive because of (12) and $f^{p/\sigma}(w) - n^{-p} \geqq 0$. Thus both terms on the right side of (17) can be omitted. The coefficients of $\zeta^2 p^4$ in I_6 and I_7 are nonnegative, since $2m-1 \geqq m > \sigma$, thus these terms may also be omitted.

The assumption $0 < m-\lambda < 1-p$ of Case 2 yields $\sigma \geqq \frac{m-p}{2} > \frac{m-1}{2}$, therefore $2-\frac{m-1}{\sigma} \geqq 0$. This shows the left side of (17) to be less than $c_4 p^2 + c_5\zeta|p|^3$.

Thus similarly to Case 1, the following inequality holds: $2\zeta^2 p^4 \leqq c_6 p^2 + c_7\zeta|p|^3$. So, to finish the proof, one may proceed as in Case 1.

The proof of the second assertion of Theorem 1 is similar, the only difference is that instead of $\zeta(x, t)$, now a function $\zeta(x)$ must be taken with $\zeta=1$ on $[x_0-1, x_0+1]$.

REMARK 2. Theorem 1 is sharp, in general. This may be established by the explicit solution

$$u(x, t) = \begin{cases} A(x_0-x)^\beta & \text{if} \quad a_1 \leqq x \leqq x_0 \\ 0 & \text{otherwise} \end{cases}$$

where $m-\lambda=\lambda-p$, $\beta:=\dfrac{2}{m-p}=\dfrac{1}{\lambda-p}$ and

$$A=\left(\frac{-b\lambda(m-\lambda)+(m-\lambda)\sqrt{b^2\lambda^2+4cm\lambda}}{2m\lambda}\right)^{2/m-p}.$$

Proof of Theorem 2. Using the notation of the previous Theorem and differentiating (11) with respect to x, we get

$$-v_{xt}+m\left(\frac{m}{\sigma}-1\right)\left(\frac{m-1}{\sigma}-1\right)v^{\frac{m-1}{\sigma}-2}v_x^3+$$
$$+m\left[2\left(\frac{m}{\sigma}-1\right)+\frac{m-1}{\sigma}\right]v^{\frac{m-1}{\sigma}-1}v_xv_{xx}+mv^{\frac{m-1}{\sigma}}v_{xxx}-$$
$$-b\lambda\frac{\lambda-1}{\sigma}v^{\frac{\lambda-1}{\sigma}-1}v_x^2-b\lambda v^{\frac{\lambda-1}{\sigma}}v_{xx}-c(p-1+\sigma)v^{\frac{p-1}{\sigma}}v^{\frac{p-1}{\sigma}}v_x+c(\sigma-1)n^{-p}v^{-\frac{1}{\sigma}}v_x=0.$$

Let $\mathscr{L}$ denote the nonlinear differential operator given by

$$\mathscr{L}w\equiv-w_t+m\left(\frac{m}{\sigma}-1\right)\left(\frac{m-1}{\sigma}-1\right)v^{\frac{m-1}{\sigma}-2}w^3+$$
$$+m\left[2\left(\frac{m}{\sigma}-1\right)+\frac{m-1}{\sigma}\right]v^{\frac{m-1}{\sigma}-1}ww_x+$$
$$+mv^{\frac{m-1}{\sigma}}w_{xx}-b\lambda\,\frac{\lambda-1}{\sigma}v^{\frac{\lambda-1}{\sigma}-1}w^2-b\lambda v^{\frac{\lambda-1}{\sigma}}w_x-$$
$$-c(p-1+\sigma)v^{\frac{p-1}{\sigma}}w+c(\sigma-1)n^{-p}v^{-\frac{1}{\sigma}}w$$

which occurred in the previous equation, i.e., if $w=v_x$ then $\mathscr{L}w=0$.

First we show that

(18) $$v_x\geqq-\alpha \quad\text{in}\quad \mathbf{R}\times(0,\infty).$$

Due to the maximum principle and condition (6) it is sufficient to show that $\mathscr{L}(-\alpha)\geqq0$ in $\mathbf{R}\times(0,\infty)$. By direct computation we get

(19) $$\mathscr{L}(-\alpha)=-m\left(\frac{m}{\sigma}-1\right)\left(\frac{m-1}{\sigma}-1\right)v^{\frac{m-1}{\sigma}-2}\alpha^3+$$
$$+b\lambda\frac{1-\lambda}{\sigma}v^{\frac{\lambda-1}{\sigma}-1}\alpha^2+c\alpha(p-1+\sigma)v^{\frac{p-1}{\sigma}}-c\alpha(\sigma-1)n^{-p}v^{-\frac{1}{\sigma}}.$$

Both in the cases *a)* and *b)* by virtue of (3), $\sigma=m-\lambda$, thus the first and the second term on the right side in (19) is positive. Therefore

$$\mathscr{L}(-\alpha)\geqq c\alpha\left[(p-1+\sigma)v^{\frac{p-1}{\sigma}}-(\sigma-1)n^{-p}v^{-\frac{1}{\sigma}}\right].$$

Since $\sigma=m-\lambda$, we have $\sigma+p-1\geqq 0$, thus using that $\frac{1}{n}\leqq u_n=v^{\frac{1}{\sigma}}$ in Q_n (see the Preliminaries) it follows that

$$(p-1+\sigma)v^{\frac{p-1}{\sigma}}\geqq (p-1+\sigma)n^{-p}v^{-\frac{1}{\sigma}}.$$

Then we obtain

$$\mathscr{L}(-\alpha)\geqq c\alpha p n^{-p}v^{-\frac{1}{\sigma}}\geqq 0.$$

To prove Theorem 2 we need only show that $\mathscr{L}\left(-\varrho\frac{v}{t}\right)>0$ in $\mathbf{R}\times(0,\infty)$ and in case *b)* in $\mathbf{R}\times(0,T]$, respectively. By direct computation we get

$$\mathscr{L}\left(-\varrho\frac{v}{t}\right)\equiv \varrho\frac{v_t}{t}-\varrho\frac{v}{t^2}-m\left(\frac{m}{\sigma}-1\right)\left(\frac{m-1}{\sigma}-1\right)v^{\frac{m-1}{\sigma}+1}\frac{\varrho^3}{t^3}+$$

$$+\varrho^2 m\left[2\left(\frac{m}{\sigma}-1\right)+\frac{m-1}{\sigma}\right]v^{\frac{m-1}{\sigma}}\frac{v_x}{t}-\varrho m v^{\frac{m-1}{\sigma}}\frac{v_{xx}}{t}+$$

$$+b\lambda\frac{1-\lambda}{\sigma}v^{\frac{\lambda-1}{\sigma}+1}\frac{\varrho^2}{t^2}+\varrho b\lambda v^{\frac{\lambda-1}{\sigma}}\frac{v_x}{t}+c(p-1+\sigma)v^{\frac{p-1}{\sigma}+1}\frac{\varrho}{t}-$$

$$-c(\sigma-1)\varrho n^{-p}v^{1-\frac{1}{\sigma}}\frac{1}{t}\equiv I_1+\ldots+I_9.$$

We shall estimate I_1, I_5, I_7 on the basis of (11):

$$I_1+I_5+I_7=\frac{\varrho}{t}\left\{vt-mv^{\frac{m-1}{\sigma}}v_{xx}+b\lambda v^{\frac{\lambda-1}{\sigma}}v_x\right\}=$$

$$=\frac{\varrho}{t}\left\{m\left(\frac{m}{\sigma}-1\right)v^{\frac{m-1}{\sigma}-1}v_x^2-c\sigma v^{\frac{p-1}{\sigma}+1}+c\sigma n^{-p}v^{1-\frac{1}{\sigma}}\right\}.$$

Combining this with I_8 and I_9 we get

$$I_1+I_5+I_7+I_8+I_9=\varrho m\left(\frac{m}{\sigma}-1\right)v^{\frac{m-1}{\sigma}-1}\frac{v_x^2}{t}+ \tag{20}$$

$$+\frac{\varrho}{t}\left\{-c\sigma v^{\frac{p-1}{\sigma}+1}+c\sigma n^{-p}v^{1-\frac{1}{\sigma}}+c(p-1+\sigma)v^{\frac{p-1}{\sigma}+1}-\right.$$

$$\left.-c(\sigma-1)n^{-p}v^{1-\frac{1}{\sigma}}\right\}\geqq\frac{c\varrho}{t}\left\{(p-1)v^{\frac{p-1}{\sigma}+1}+n^{-p}v^{1-\frac{1}{\sigma}}\right\}\geqq\frac{c\varrho}{t}(p-1)v^{\frac{p-1}{\sigma}+1}.$$

Now we distinguish two cases.

Case a):

$$m\geqq 1>\lambda>0\quad\text{and}\quad p\geqq 1. \tag{21}$$

By (20) we have

$$I_1+I_5+I_7+I_8+I_9 \geqq \frac{c\varrho}{t}(p-1)v^{\frac{p-1}{\sigma}+1} \geqq 0.$$

From (21), (3) it follows that $\sigma=m-\lambda$ and thus $I_3 \geqq 0$. Using (18) and $\frac{m-1}{\sigma}=$ $=1+\frac{\lambda-1}{\sigma}$, $v=u_n^\sigma \leqq M_1^\sigma$ in the estimate of I_2, I_4, I_6 we get

$$I_2+I_4+I_6 = \frac{\varrho}{t^2}\left\{-v+\varrho m\left[2\left(\frac{m}{\sigma}-1\right)+\frac{m-1}{\sigma}\right]v^{\frac{m-1}{\sigma}}v_x+b\lambda\frac{1-\lambda}{\sigma}v^{\frac{\lambda-1}{\sigma}+1}\right\} \geqq$$

$$\geqq \frac{\varrho v^{\frac{m-1}{\sigma}}}{t^2}\left\{-v^{\frac{1-\lambda}{\sigma}}-\alpha\varrho m\frac{m+2\lambda-1}{\sigma}+\varrho b\lambda\frac{1-\lambda}{\sigma}\right\} \geqq$$

$$\geqq \frac{\varrho v^{\frac{m-1}{\sigma}}}{t^2}\left\{-M_1^{1-\lambda}-\alpha\varrho m\frac{m+2\lambda-1}{\sigma}+\varrho b\lambda\frac{1-\lambda}{\sigma}\right\}.$$

If α satisfies (7), i.e.,

$$b\lambda\frac{1-\lambda}{\sigma} > \alpha m\frac{m+2\lambda-1}{\sigma}$$

then we can choose ϱ so large that

$$\varrho\left[\frac{b\lambda(1-2)}{\sigma}-\alpha m\frac{m+2\lambda-1}{\sigma}\right]-M_1^{1-\lambda} > 0.$$

Thus we have $\mathscr{L}\left(-\varrho\frac{v}{t}\right)>0$ which completes the proof of case a).

Case b):

(22) $$m \geqq 1 > \lambda > 0 \quad \text{and} \quad 1 > p \geqq \lambda.$$

From (20) it follows that

$$I_1+I_5+I_7+I_8+I_9 \geqq \frac{c\varrho}{t}(p-1)v^{\frac{p-1}{\sigma}+1}$$

but now the right side is negative. Similarly to the Case a) from (22), (3) it follows that $\sigma=m-\lambda$ and $I_3 \geqq 0$. Using (18) and $\frac{m-1}{\sigma}=1+\frac{\lambda-1}{\sigma}$ $v=u_n^\sigma \leqq M_1^\sigma$ and the fact that $\frac{1}{t^2} \geqq \frac{1}{tT}$ for $t\in(0, T]$ in the estimate of I_2, I_4, I_6 we get

$$I_2+I_4+I_6 \geqq \frac{\varrho v^{\frac{m-1}{\sigma}}}{t^2}\left\{-M_1^{1-\lambda}-\alpha\varrho m\frac{m+2\lambda-1}{\sigma}+\varrho b\lambda\frac{1-\lambda}{\sigma}\right\} \geqq$$

$$\geqq \frac{\varrho}{tT}\left\{-M_1^{1-\lambda}-\alpha\varrho m\frac{m+2\lambda-1}{\sigma}+\varrho b\lambda\frac{1-\lambda}{\sigma}\right\}v^{\frac{m-1}{\sigma}}.$$

Due to $(p-\lambda)/\sigma=1-(m-p)/\sigma$, $v=u_n^\sigma \leqq M_1^\sigma$ we can give an estimate for the sum

$$\mathscr{L}\left(-\varrho\frac{v}{t}\right)=I_1+\ldots+I_9 \geqq$$

$$\geqq \frac{\varrho}{t}\left\{-c(1-p)M_1^{p-\lambda}-\frac{M_1^{1-\lambda}}{T}-\alpha\frac{\varrho}{T}\frac{m(m+2\lambda-1)}{\sigma}+\frac{\varrho}{T}b\lambda\frac{1-\lambda}{\sigma}\right\}v^{\frac{m-1}{\sigma}}.$$

If α satisfies (7), then we can choose ϱ sufficiently large, that is

$$\frac{\varrho}{T}\left\{b\lambda\frac{1-\lambda}{\sigma}-\alpha\frac{m(m+2\lambda-1)}{\sigma}\right\}-c(1-p)M_1^{p-\lambda}-\frac{M_1^{1-\lambda}}{T}>0.$$

Then we have $\mathscr{L}\left(-\varrho\frac{v}{t}\right)>0$ which completes the proof of Case b) and the proof of Theorem 2.

Proof of Theorem 3. Let $\mathscr{L}$ denote the nonlinear differential operator given by

$$\mathscr{L}u \equiv -u_t+(u^m)_{xx}-b(u^\lambda)_x-cu^p.$$

We also define a function $v(x, t)$ by

$$v(x,t)=\begin{cases} A(x_0-x)^\beta & \text{if} \quad a_1 \leqq x \leqq x_0 \\ 0 & \text{in} \quad (a_1,\infty)\times(0,\infty)\end{cases} \tag{23}$$

where A, x_0, β are positive constants which will be specified later. Recall that $u_0(x)>0$ on $I=(a_1, a_2)$ and $u_0(x)\equiv 0$ in $\mathbf{R}\setminus I$.

To prove Theorem 3 we have to show only that the constants A, x_0, β can be chosen in such a way that the function $v(x, t)$ is a supersolution of (1)—(2) in $(a_1, \infty)\times(0, \infty)$ for which $u_0(x)\leqq v(x, 0)$ when $x \geqq a_1$ and $u(a_1, t)\leqq v(a_1, t)$ when $t\geqq 0$. (See the Lemma in the Preliminaries.) It can be easily checked that

$$\mathscr{L}v = A^m\beta m(\beta m-1)(x_0-x)^{\beta m-2}+bA^2\beta\lambda(x_0-x)^{\beta\lambda-1}-cA^p(x_0-x)^{\beta p}= \tag{24}$$

$$=A^p(x_0-x)^{\beta p}\{A^{m-p}\beta m(\beta m-1)(x_0-x)^{\beta(m-p)-2}+bA^{\lambda-p}\beta\lambda(x_0-x)^{\beta(\lambda-p)-1}-c\}.$$

To prove that $v(x, t)$ is a supersolution of (1)—(2) in $(a_1, \infty)\times(0, \infty)$ it is sufficient to verify that $\mathscr{L}v\leqq 0$ if $a_1\leqq x\leqq x_0$.

From (24) it follows that

$$\mathscr{L}v \leqq A^p(x_0-a_1)^{\beta p}[A^{m-p}\beta m(\beta m-1)(x_0-a_1)^{\beta(m-p)-2}+ \tag{25}$$

$$+bA^{\lambda-p}\beta\lambda(x_0-a_1)^{\beta(\lambda-p)-1}-c].$$

First we choose β

$$\beta := \max\left\{\frac{2}{m-p}, \frac{1}{\lambda-p}\right\}$$

and after that we define A by

$$A := \frac{M_0}{(x_0-a_2)^\beta}. \tag{26}$$

From (9) it follows that β is finite. It is not difficult to see that from (26) it follows that $u_0(x) \leqq v(x, 0)$ for $x \geqq a_1$ and $u(a_1, t) \leqq v(a_1, t)$ for $t \geqq 0$. The expression in the square brackets in (25) turns into

$$M_0^{m-p}\beta m(\beta m-1)\left(\frac{x_0-a_1}{x_0-a_2}\right)^{\beta(m-p)}\frac{1}{(x_0-a_1)^2}+$$

$$+bM_0^{\lambda-p}\beta\lambda\left(\frac{x_0-a_1}{x_0-a_2}\right)^{\beta(\lambda-p)}\frac{1}{x_0-a_1}-c.$$

Here the terms

$$M_0^{m-p}\beta m(\beta m-1)\left(\frac{x_0-a_1}{x_0-a_2}\right)^{\beta(m-p)}$$

and

$$bM_0^{\lambda-p}\beta\lambda\left(\frac{x_0-a_1}{x_0-a_2}\right)^{\beta(\lambda-p)}$$

are bounded from above by a positive constant independent of x_0, thus we can choose x_0 so large that $\mathscr{L}v \leqq 0$ if $a_1 \leqq x \leqq x_0$. This concludes the proof of the Theorem 3.

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DENSEST PACKING OF TRANSLATES OF A DOMAIN

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Let s be an open set of points in the Euclidean plane. Let $d(s)$ be the density of the densest packing of translates of s. Let $d^*(s)$ be the density of the densest lattice-packing of translates of s. We recall a theorem of Rogers [1]: If s is convex then $d(s)=d^*(s)$.

We shall prove a similar theorem for a special class of not necessarily convex domains. This suggests the problem of trying to extend Rogers' theorem to some other classes of domains. We start with mentioning some negative results.

Let h be a regular hexagon, O the image of its centre reflected in a side of h, h' the image of h reflected in O, and $u=h\cup h'$ the union of h and h'. Obviously, the plane can be packed with translates of u with density 1. On the other hand, the density of the densest lattice-packing of translates of u (Fig. 1) is equal to 3/4.

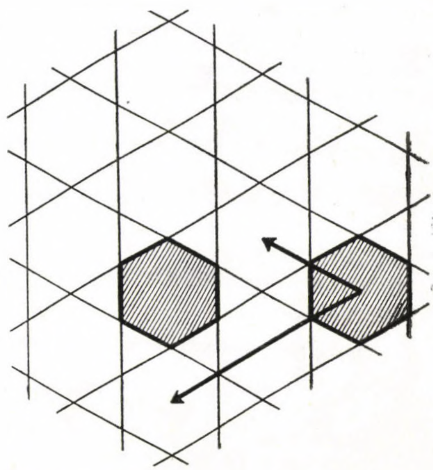

Fig. 1

Motivated by this remark, due jointly to J. Pach and the author, we shall restrict ourselves to open, connected sets, in short to domains.

A comparatively simple domain u such that $d(u)>d^*(u)$ was constructed by A. Bezdek [2]. His insectlike domain, consists of the union of five rectangles (Fig. 2). It is easy to show that in a densest lattice-packing of translates of u either a part of the "abdomen" of an "insect" lies between the "feelers" of another insect, or the abdomens of two insects touch one another. In both kinds of packings it is not difficult to find the densest lattice. The maximal density is attained in the second case (Fig. 2) but this density is superpassed by the packing shown in Fig. 3.

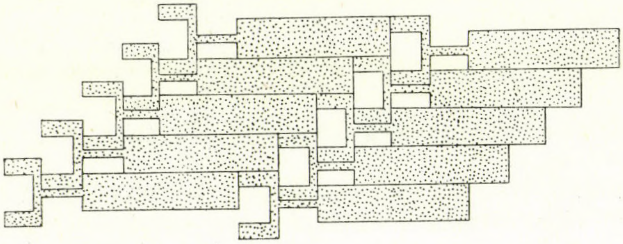

Fig. 2

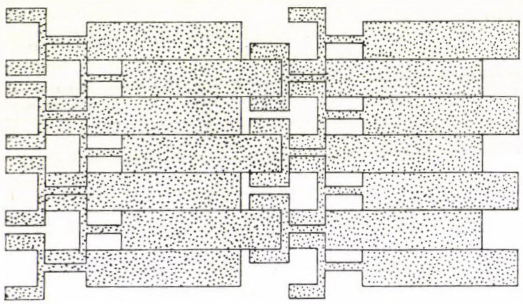

Fig. 3

Bezdek succeeded also in constructing a direction-convex domain u with $d(u) > d^*(u)$ defined by the property that there is a line l such that any line parallel to l intersecting u intersects u in one segment.

A further negative result concerns semi-convex domains defined as follows. On the boundary of a convex domain u let A and B be two points lying on opposite parallel support-lines of u. Let $\widehat{AB}$ be one of the two arcs into which A and B divide the boundary of u. A semi-convex domain s is bounded by the convex arc $\widehat{AB}$ and an arbitrary Jordan-arc $\widehat{BA}$ lying in u. By distorting the insect of Bezdek, G. Kertész constructed a semi-convex domain s such that $d(s) > d^*(s)$. However the densest packing of translates of s has a strange feature: It consists of pairs of translates of s which are interlocked so that they cannot be taken apart by a continuous motion without overlapping one another.

The theorem we referred to above concerns a subclass of semi-convex domains. Choose an arbitrary point C on the arc $\widehat{AB}$ and translate $\widehat{AB}$ through the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The original arc $\widehat{AB}$ and its translates enclose a region v. The Jordan-arc $\widehat{BA}$ which, along with $\widehat{AB}$, bounds the domain is now allowed to lie in v instead of u. We shall call such a domain limited semi-convex.

THEOREM. *If s is a limited semi-convex domain then $d(s)=d^*(s)$.*

The proof is based on an idea used in a new proof of Rogers' theorem [3]. Let $\{s_1, s_2, ...\}$ be a packing of translates of a limited semi-convex domain s. In the above definition of s let the support-lines of u containing A and B be vertical, and let $\widehat{AB}$ be the lower part of the boundary of u. We associate with each s_i a region r_i consisting of the point-set union of all vertical line-segments whose lower endpoints lie in s_i and which do not intersect any other domain s_j.

We shall denote a domain and its area with the same symbol. We shall prove the theorem by showing that the density s_i/r_i of each domain s_i in the region r_i associated with it is less than or equal to the density of a lattice-packing of translates of s.

Let s be identical with s_i. If apart from s_i no other domain s_j overlaps the region v defined above then $r_i \supset v$ so that $s_i/r_i \leqq s/v$. Note that v is the region associated with s in the lattice-packing generated by the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. Since v is a unit-cell of the lattice, s/v is the density of this lattice-packing.

We now assume that a domain s_j other than s_i overlaps v. Then there are translates s_A and s_B of s touching both s and s_j (Fig. 4). If s_A and s_B are not uniquely determined then let s_B be the domain arising from s_j by the translation $s_A \to s$. Observe that s_j belongs to the lattice-packing generated by the translations $s \to s_A$ and $s \to s_B$. Let u be the region associated with s in this lattice-packing. Since no translate of s can overlap u without overlapping s or s_j, we have $r_i \supset u$ which implies $s_i/r_i \leqq s/u$. On the other hand, s/u is the density of the lattice-packing under consideration.

This completes the proof of the theorem.

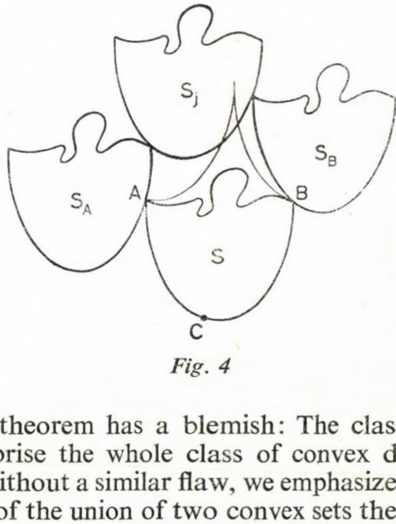

Fig. 4

Unfortunately, the theorem has a blemish: The class of limited semi-convex domains does not comprise the whole class of convex domains. With a view to obtain a positive result without a similar flaw, we emphasize the following conjecture: If the domain s consists of the union of two convex sets then $d(s)=d^*(s)$.

Finally we list some further classes of domains for which the problem is still undecided.

1. Domains consisting of the union of less than five convex domains.

2. Domains which are direction-convex in two or more bundles of direction. This is meant as follows. If u is direction-convex with respect to the line l then u is

direction-convex also with respect to all lines arising from l by some continuous rotations in both directions. We say that these lines represent one bundle of direction.

3. Star-shaped domains defined by containing a point O such that along with any point P contained in the domain the whole segment OP is contained in the domain.

4. Domains defined analogously as semi-convex domains by letting the part of the "opposite" points A and B play by points halving the perimeter of u.

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ON APPROXIMATION OF CONTINUOUS FUNCTIONS IN LIPSCHITZ NORMS

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Let f be a continuous and 2π-periodic function, briefly denoted by $f\in C_{2\pi}$, and let

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

be its Fourier series. We denote by $s_n(x)=s_n(f)(x)$ the n-th partial sum of (1) and the usual supremum norm by $\|\cdot\|_C$. If φ is an increasing positive function on $(0, \infty)$ we define the φ-norm by

$$\|f\|_\varphi=\|f\|_C+\sup_{x\neq y}\frac{|f(x)-f(y)|}{\varphi(|x-y|)}=\|f\|_C+\sup_{\delta>0}\frac{\|f(\cdot)-f(\cdot+\delta)\|_C}{\varphi(\delta)}.$$

Recently several authors have considered various problems of approximation in φ-norms, $\|\cdot\|_\varphi$, with specific functions φ. E.g. S. Prössdorf [5] proved that if $f\in\operatorname{Lip}\alpha\ (0<\alpha\leqq 1)$, $\varphi(\delta)=\delta^\beta$, $\|\cdot\|_\varphi=\|\cdot\|_\beta$, then

$$\|\sigma_n(f)-f\|_\beta=\begin{cases}O(n^{\beta-\alpha}) & \text{if}\quad \alpha<1,\\ O\big(n^{\beta-\alpha}(1+\log n)\big) & \text{if}\quad \alpha=1,\end{cases} \tag{2}$$

where $\sigma_n(f)=\sigma_n(f)(x)=(n+1)^{-1}\sum_{k=0}^{n}s_k(f)(x)$ are the Fejér means of (1). For the more general de la Vallée Poussin-type means

$$V_n(\lambda, f)(x)=\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n+1}^{n}s_\nu(f)(x), \tag{3}$$

where $\lambda=\{\lambda_n\}$ is a monotone nondecreasing sequence of integers such that $\lambda_1=1$, $\lambda_{n+1}-\lambda_n\leqq 1$, L. Leindler [1] obtained, among others, the following result: if $f\in\operatorname{Lip}\alpha$, then

$$\|V_n(\lambda, f)=f\|_\beta=\begin{cases}O(\lambda_n^{\beta-\alpha}) & \text{if}\quad \alpha<1\quad\text{or}\quad \beta>0,\\ O\left(\dfrac{1+\log\lambda_n}{\lambda_n}\right) & \text{if}\quad \alpha=1\quad\text{and}\quad \beta=0.\end{cases} \tag{4}$$

For further relevant results see [1], [2], [6] and [7].

In this note we prove an inequality which, in many cases, reduces the problems

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associated with similar estimations to the relatively simple problem of approximation in the C-norm.

Let $\{A_n\}$ be a sequence of linear convolution operators from $C_{2\pi}$ into $C_{2\pi}$ with operator norms $\|A_n\|$ and let $\omega(f;\delta)$ denote the modulus of continuity of $f\in C_{2\pi}$. Then we have

$$\|A_n(f)-f\|_\varphi \leqq \|A_n(f)-f\|_C\left(1+2/\varphi\left(\frac{1}{n}\right)\right)+\sup_{0<\delta\leqq 1/n}\frac{2\omega(f;\delta)}{\varphi(\delta)}(1+\|A_n\|). \tag{5}$$

EXAMPLES. Let $\varphi(\delta)=\delta^\beta$, $\|\cdot\|_\varphi=\|\cdot\|_\beta$, $f\in \operatorname{Lip}\alpha$, $0\leqq\beta<\alpha\leqq 1$.

1. If $A_n=s_n$, then (5) yields that

$$\|s_n(f)-f\|_\beta = O(n^{\beta-\alpha}\log n). \tag{6}$$

This, clearly, also implies that the same estimate holds for the Borel, Euler, Meyer—König and Zeller means (see [4]) as well as for a wider class of means introduced by A. Meir [3].

2. For the generalized de la Vallée Poussin means (3), we obtain from (5) that under the assumptions made above

$$\|V_n(\lambda, f)-f\|_\beta = \begin{cases} O\left(n^{\beta-\alpha}\log\dfrac{2n}{\lambda_n}\right) & \text{if} \quad \alpha<1,\\[2ex] O\left(\left(n^{\beta-1}+\dfrac{n^\beta}{\lambda_n}\right)\log\dfrac{n}{n-\lambda_n+1}\right)\log\dfrac{2n}{\lambda_n} & \text{if} \quad \alpha=1\end{cases}$$

(compare with (4)). Here we have used the estimates $\|V_n(\lambda)\|\leqq K\log(2n/\lambda_n)$ and (see [7, Theorem 5])

$$\|V_n(\lambda, f)-f\|_C \leqq \frac{K}{\lambda_n}\left(\sum_{k=n-\lambda_n+1}^{n}\omega\left(f,\frac{1}{k}\right)\right)\log\frac{2n}{\lambda_n}.$$

In the special case $\lambda_n=n$ we obtain (2).

PROOF OF (5). For $n=1, 2, \dots$ and $\delta\geqq 1/n$,

$$|(A_n(f)-f)(x)-(A_n(f)-f)(x+\delta)|/\varphi(\delta) \leqq 2\|A_n(f)-f\|_C/\varphi(1/n),$$

while for $0<\delta\leqq 1/n$ we have

$$|(A_n(f)-f)(x)-(A_n(f)-f)(x+\delta)|/\varphi(\delta) \leqq |A_n(f)(x)-A_n(f)(x+\delta)|/\varphi(\delta)+$$
$$+|f(x)-f(x+\delta)|/\varphi(\delta) \leqq (|A_n(f(\cdot)-f(\cdot+\delta))(x)|+\omega(f;\delta))/\varphi(\delta)\leqq$$
$$\leqq (\|A_n\|\,\|f(\cdot)-f(\cdot+\delta)\|_C+\omega(f;\delta))/\varphi(\delta)\leqq \frac{2\omega(f;\delta)}{\varphi(\delta)}(1+\|A_n\|).$$

These inequalities establish (5).

Remarks. 1. It may happen that in some cases a similarly provable estimate of the form

$$\|A_n(f)-f\|_\varphi \leqq \|A_n(f)-f\|_C\left(1+2/\varphi\left(\frac{1}{m}\right)\right)+\sup_{0<\delta\leqq 1/m}\frac{2\omega(f;\delta)}{\varphi(\delta)}(1+\|A_n\|)$$

gives a better approximation order with suitably chosen $m\geqq 1$.

2. The same result clearly holds if the sup norm is replaced by any "reasonable" norm (e.g. by L^p-norm $1 \leqq p < \infty$).

3. The estimate (5) holds without any ado for higher order moduli of smoothness.

4. In general, (5) cannot be improved. For, if $A_n = s_n$ and $f \in \operatorname{Lip} \alpha$, $\varphi(\delta) = \delta^\beta$, then the estimate (6) is the best possible. Indeed, let

$$f_n(t) = \begin{cases} n^{-\alpha} \sin(n+1/2)t & \text{if } \quad t \in \left(\pi/(n+1/2);\ \left[\sqrt{n}\right]\pi/(n+1/2)\right) \\ 0 & \text{for other } t \text{ from } \quad [-\pi, \pi). \end{cases}$$

It is easy to check that $\left|s_n(f_n)(0) - s_n(f_n)\left(1/(n+1/2)\right)\right| \geqq cn^{-\alpha} \log n$ ($c > 0$ fixed) and so the function $f(x) = \sum_{k=1}^{\infty} f_{n_k}(x)$ with sufficiently rapidly increasing n_k will belong to $\operatorname{Lip} \alpha$ but for every k we will have

$$\|s_{n_k}(f) - f\|_\beta \geqq \frac{c}{2}(n_k^{\beta-\alpha} \log n_k).$$

5. The preceding remark can be further justified by the fact that in certain cases (5) can be reversed. Let us consider e.g. (2) in the case $\alpha < 1$. We show that $\|\sigma_n(f) - f\|_\beta \leqq Kn^{\beta-\alpha}$ implies that $f \in \operatorname{Lip} \alpha$. From the assumption we obtain easily that

$$\left|\left(\sigma_n(f) - f\right)(x) - \left(\sigma_n(f) - f\right)\left(x + 2\pi/(n+1)\right)\right| \leqq Kn^{-\alpha}.$$

Since $\sigma_n(f)$ is periodic with period $2\pi/(n+1)$ it follows that $\left|f(x) - f(x + 2\pi/(n+1))\right| \leqq$ $\leqq Kn^{-\alpha}$. Now, every δ, $0 < \delta < 1$, can be represented in the form $\delta = 2\pi/(n_1+1) +$ $+ 2\pi/(n_2+1) + \dots$ with $n_{k+1} > 2n_k$ and so

$$|f(x) - f(x+\delta)| \leqq K(n_1^{-\alpha} + n_2^{-\alpha} + \dots) \leqq K\delta^\alpha.$$

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ON THE CONVERGENCE OF SERIES OF PAIRWISE INDEPENDENT RANDOM VARIABLES

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1. In [2] we proved that *if the pairwise independent random variables* $X_1, X_2, \dots$ *satisfy the conditions*

$$\sum_{m=1}^{\infty} \frac{D^2(X_m)}{m^2} < \infty$$

and

$$\frac{1}{n} \sum_{m=1}^{n} E|X_m - EX_m| = O(1), \quad as \quad n \to \infty, \tag{1}$$

then almost surely we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} (X_m - EX_m) = 0.$$

Here EX and $D^2(X) = E(X-EX)^2$ denote the expectation and the variance of the random variable X.

This result is a strong law of large numbers for pairwise independent random variables under the assumption (1) and may be considered as a variant of Kolmogorov's law of large numbers for non-identically distributed totally independent random variables. See the discussion in [2]. The aim of the present note is to prove a convergence theorem for series of pairwise independent random variables which has the flavour of the result above. It will turn out that our theorem cannot be extended to uncorrelated random variables and the result is best possible in a certain sense.

2. The convergence theorem mentioned is the following: *Let* $X_1, X_2, \dots$ *be pairwise independent random variables and* $(0=)N_0 < \dots < N_m < \dots$ *be integers. We assume that*

$$\sum_{m=0}^{n} \sqrt{\sum_{n=N_m+1}^{N_{m+1}} D^2(X_n)} < \infty \tag{2}$$

and

$$\sum_{n=N_m+1}^{N_{m+1}} E|X_n - EX_n| = O(1), \quad as \quad m \to \infty, \tag{3}$$

are satisfied. Then the series

$$\sum_{n=1}^{\infty} (X_n - EX_n)$$

converges almost surely.

PROOF. By (2) there are integers $1 \leqq l_0 \leqq l_1 \leqq l_2 \leqq \ldots$ tending to infinity such that

$$\sum_{m=0}^{\infty} l_m \sqrt{\sum_{n=N_m+1}^{N_{m+1}} D^2(X_n)} < \infty \tag{4}$$

is satisfied. Now (3) and (4) easily yield the existence of a sequence $0 = M_0 < M_1 < M_2 < \ldots$ such that $\{N_k\}$ is a subsequence of $\{M_k\}$, for every m the number of the M_k's between N_m+1 and N_{m+1} is at most l_m and

$$\sum_{n=M_m+1}^{M_{m+1}} E|X_n - EX_n| = o(1) \tag{5}$$

as $m \to \infty$. (For the latter take also into account that by (4) $E|X_n - EX_n|$ tends to zero as $n \to \infty$.) By (4) we have

$$\sum_{m=0}^{\infty} \sqrt{\sum_{n=M_m+1}^{M_{m+1}} D^2(X_n)} < \infty. \tag{6}$$

In the rest of the proof we shall rely only on (5) and (6).

We may assume $EX_n = 0$, $n = 1, 2, \ldots$. Let $X_n^+ (X_n^-)$ be the positive (negative) part of X_n. Then, clearly, $D(X_n^+) \leqq D(X_n)$, $n = 1, 2, \ldots$, and so, by (6), we have

$$\sum_{m=0}^{\infty} \sqrt{\sum_{n=M_m+1}^{M_{m+1}} D^2(X_n^+)} < \infty. \tag{7}$$

Put $S_m = \sum_{n=1}^{m} X_n$, $S_m^+ = \sum_{n=1}^{m} (X_m^+ - EX_m^+)$, $S_m^- = \sum_{n=1}^{m} (X_n^- - EX_n^-)$, $m = 1, 2, \ldots$. By the pairwise independence of X_n we have the same for X_n^+ and so (7) gives that

$$\sum_{m=0}^{\infty} E|S_{M_{m+1}}^+ - S_{M_m}^+| \leqq \sum_{m=0}^{\infty} \sqrt{E(S_{M_{m+1}}^+ - S_{M_m}^+)^2} =$$

$$= \sum_{m=0}^{\infty} D(X_{M_m+1}^+ + \ldots + X_{M_{m+1}}^+) = \sum_{m=0}^{\infty} \sqrt{\sum_{n=M_m+1}^{M_{m+1}} D^2(X_n^+)} < \infty,$$

and this in turn implies that the sequence $\{S_{M_m}^+\}_{m=0}^{\infty}$ converges almost surely. In the same way $\{S_{M_m}^-\}_{m=0}^{\infty}$ converges with probability 1.

Let now $M_m < n \leqq M_{m+1}$. We have

$$S_{M_m}^+ \leqq S_n^+ + (EX_{M_m+1}^+ + \ldots + EX_n^+) \leqq S_{M_m+1}^+ + (EX_{M_{m+1}}^+ + \ldots + EX_{M_{m+1}}^+)$$

and so

$$S_{M_m}^+ - (EX_{M_m+1}^+ + \ldots + EX_{M_{m+1}}^+) \leqq S_n^+ \leqq S_{M_{m+1}}^+ + (EX_{M_m+1}^+ + \ldots + EX_{M_{m+1}}^+).$$

Similarly,

$$-S_{M_{m+1}}^- - (EX_{M_m+1}^- + \ldots + EX_{M_{m+1}}^-) \leqq -S_n^- \leqq -S_{M_m}^- + (EX_{M_m+1}^- + \ldots + EX_{M_{m+1}}^-),$$

and adding our last two inequalities we obtain

(8)
$$S_{M_m}-(S^-_{M_{m+1}}-S^-_{M_m})-\sum_{n=M_m+1}^{M_{m+1}} E|X_n| \leqq S_n \leqq S_{M_m}+(S^+_{M_{m+1}}-S^+_{M_m})+\sum_{n=M_m+1}^{M_{m+1}} E|X_n|.$$

Since the sequences $\{S^{\pm}_{M_m}\}$ converge almost surely, we get the desired result from (8) and (5).

3. Now we show that the assumption in the preceding section that the random variables be pairwise independent is essential, it cannot be relaxed to orthogonality. Namely we have the following result: *There are pairwise uncorrelated random variables $X_1, X_2, \ldots$ such that*

$$\sum_{m=0}^{\infty} \sqrt{\sum_{n=2^m+1}^{2^{m+1}} D^2(X_n)} < \infty \tag{9}$$

and

$$\sum_{n=2^m+1}^{2^{m+1}} E|X_n-EX_n| = O(1), \quad as \quad m \to \infty, \tag{10}$$

are satisfied, nevertheless the series $\sum_{n=1}^{\infty} (X_n-EX_n)$ *diverges with probability* 1.

PROOF. We shall use the following lemma from [2].

LEMMA. *If p is a positive integer then there are step functions $f_1, \ldots, f_p$ defined on* (0, 1) *such that they constitute an orthonormal system and for some interval I_p we have the relations*

$$\max_{n \leqq 1 \leqq p} \left| \sum_{k=1}^{n} f_k(x) \right| \geqq C_1 \sqrt{p} \log 2p \quad (x \in I_p), \quad \text{meas}\, I_p \geqq C_2,$$

$$\int_0^1 f_k(x)\,dx = 0, \quad k = 1, \ldots, p, \quad \int_0^1 |f_k(x)|\,dx \leqq C_3 \frac{\log 2p}{\sqrt{p}}, \quad k = 1, \ldots, p,$$

with some positive absolute constants C_1, C_2, C_3.

Let the measure space $(\Omega, \mathcal{A}, P)$ be the Lebesgue measure space over (0, 1). We define in an inductive way a sequence $\{X_n\}$ of pairwise uncorrelated random variables and a sequence $\{E_m\}$ of stochastically independent simple sets* such that every X_n is a step function defined on (0, 1) and for every $m=2, 3, \ldots$ the following

* A set is simple if it is the union of finitely many intervals.

relations are satisfied:

(i) $$P(E_m) \geqq C_2 \frac{1}{m},$$

(ii) $$\max_{2^m < n \leqq 2^{m+1}} \left| \sum_{k=2^m+1}^{n} X_k(\omega) \right| \geqq C_1 m^{1/4}, \quad \omega \in E_m,$$

(iii) $$EX_n = 0 \quad (n = 2^m+1, \ldots, 2^{m+1}),$$

(iv) $$E|X_n| \leqq C_3 2^{-m} m^{-3/4} \quad (n = 2^m+1, \ldots, 2^{m+1}),$$

(v) $$E|X_n|^2 = 2^{-m} m^{-5/2} \quad (n = 2^m+1, \ldots, 2^{m+1}).$$

Let $X_1 \equiv \ldots X_4 \equiv 0$, $E_1=(0, 1)$, and let us suppose that for some $m_0(\geqq 2)$, $X_1, \ldots, X_{2^{m_0}}$ and $E_1, \ldots, E_{m_0-1}$ have already been defined, $X_1, \ldots, X_{2^{m_0}}$ are orthogonal step functions, $E_1, \ldots, E_{m_0-1}$ are stochastically independent simple sets and for every $m=2, \ldots, m_0-1$ the relations (i)—(v) are satisfied. We apply the lemma for $p=2^{m_0}$, the functions and interval obtained are $f_1, \ldots, f_{2^{m_0}}$ and I, resp. Let

$$g_i(x) = \begin{cases} \sqrt{m_0}\, f_i(m_0 x) & \text{if } \ 0 < x < 1/m_0, \\ 0 & \text{otherwise,} \end{cases}$$

$(i=1, \ldots, 2^{m_0})$ and let J be the image of I at the mapping $y=x/m_0$. Since $X_1, \ldots, X_{2^{m_0}}$ are step functions and $E_1, \ldots, E_{m_0-1}$ are simple sets, there exists a disjoint partition of $(0, 1)$ into intervals $I_1, \ldots, I_\varrho$ such that on every interval I_r every function $X_i\,(i=1, \ldots, 2^{m_0})$ is constant and every $E_m\,(m=1, \ldots, m_0-1)$ is the union of certain I_r's.

Let us introduce the notations: if $G=(a, b)\subseteqq(0, 1)$ is an arbitrary interval, $H\subseteqq(0, 1)$ and f is a function defined on $(0, 1)$ then let

$$H(G) = \{y \mid y = (b-a)x+a \quad \text{for some } x\in H\}$$

and

$$f(G;\, x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & \text{if } \ a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Now let

$$X_{2^{m_0}+i}(\omega) = 2^{-m_0/2} m_0^{-1-1/4} \sum_{r=1}^{\varrho} g_i(I_r;\, \omega), \ \omega\in(0, 1); \quad i = 1, \ldots, 2^{m_0},$$

and $E_{m_0} = \bigcup_{r=1}^{\varrho} J(I_r)$. Evidently, these random variables and sets satisfy (i)—(v) and so the definition of our two sequences is complete.

Since the sets E_m are stochastically independent and we have by (i) $\sum_{m=1}^{\infty} P(E_m) = \infty$, we obtain from the second Borel—Cantelly lemma that $P\left(\varlimsup_{m\to\infty} E_m\right) = 1$.

However, for every $\omega \in \overline{\lim}_{m\to\infty} E_m$ (ii) is satisfied for infinitely many m, so the series $\sum_{n=1}^{\infty} X_n$ diverges with probability 1.

On the other hand we obtain from (v) that

$$\sum_{m=2}^{\infty} \sqrt{\sum_{n=2^m+1}^{2^{m+1}} D^2(X_n)} = \sum_{m+2}^{\infty} \sqrt{2^m 2^{-m} m^{-5/2}} = \sum_{m=2}^{\infty} m^{-5/4} < \infty,$$

and from (iv) that

$$\sum_{n=2^m+1}^{2^{m+1}} E|X_n| \leqq C_3 2^{-m} m^{-3/4} 2^m = C_3 m^{-3/4} \to 0 \quad \text{as} \quad m \to \infty.$$

Hence, for this sequence $\{X_n\}$, (9) and (10) are satisfied as well, since $EX_n=0$, $n=1, 2, \ldots$. The proof is complete.

4. Let us turn back to the result of Section 2. If $\sum_{n=1}^{\infty} D^2(X_n) < \infty$, then there exists a sequence $\{N_m\}$ such that (2) is satisfied but if we want to apply our theorem then we have to verify (3) for this sequence $\{N_m\}$. Indeed, it is quite obvious that

$$\sum_{n=1}^{\infty} D^2(X_n) < \infty$$

and condition (3) with any *fixed* sequence $\{N_m\}$ do not ensure the almost sure convergence of $\sum_{n=1}^{\infty} (X_n - EX_n)$. Thus, our theorem is not a "generalisation" of Kolmogorov's three series theorem along the same route which was sketched in Section 1 and it seems that the result mentioned in Section 1 can not be derived from it by the technique applied in the case of total independence.

Nevertheless, our theorem is the best possible one in the following sense:

If $\gamma_m \downarrow 0$ then in general, neither

$$\sum_{m=1}^{\infty} \gamma_m \sqrt{\sum_{n=N_m+1}^{N_{m+1}} D^2(X_n)} < \infty$$

and condition (3), *nor*

$$\sum_{n=N_m+1}^{N_{m+1}} E|X_n - EX_n| = O(\gamma_m^{-1}) \tag{11}$$

and condition (2) *imply the almost sure convergence of* $\sum_{n=1}^{\infty} (X_n - EX_n)$.

These facts can be justified by certain rearrangements of any divergent Walsh series with a coefficient sequence belonging to l^2.

5. Taking into account the Rademacher—Menshow theorem for orthogonal series, the proof in Section 2 proves the following result as well:

If $X_1, X_2, \ldots$ *are pairwise independent random variables such that for some sequence*

$$\sum_{m=1}^{\infty}\left(\sum_{n=N_m+1}^{N_{m+1}} D^2(X_n)\right)\log^2 m < \infty \tag{12}$$

and (3) *are satisfied then* $\sum_{n=0}^{\infty}(X_n - EX_n)$ *converges almost surely.*

Using the lemma from Section 2 one can easily prove that here, again, pairwise independence cannot be relaxed to orthogonality (take e.g. $N_m = m^m$). Also, a result of Bočkarev [1] and Nakata [3] can be used to show that *if* $\gamma_m \downarrow 0$ *then in general neither*

$$\sum_{m=1}^{\infty}\gamma_m\left(\sum_{n=N_m+1}^{N_{m+1}} D^2(X_n)\right)\log m < \infty$$

and condition (3), *nor conditions* (11) *and* (12) *imply the almost sure convergence of* $\sum_{n=1}^{\infty}(X_n - EX_n)$. We omit the details.

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SOME NEW ESTIMATES FOR THE EIGENFUNCTIONS OF HIGHER ORDER OF A LINEAR DIFFERENTIAL OPERATOR

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In the spectral theory of non-selfadjoint differential operators it is important to have different estimates for the eigenfunctions of higher order (cf. [2]—[8]). In the present paper we obtain some new results of this type.

The first theorem of this paper contains a more general formula than the formula proved in [5], Theorem 1. The second theorem contains the same estimates as in [5], Theorem 2, but under more general conditions. For this, we had to change the proof in several places. The third theorem shows that the estimates for the derivatives of an eigenfunction, proved in the second theorem, are exact. Its proof is based on the extended formula of the first theorem. Finally, the fourth theorem shows that the estimates, proved in the second theorem for the relation between the different norms of an eigenfunction, remain also valid for the derivatives of the eigenfunctions.

Let $G\subset\mathbf{R}$ be an arbitrary open interval and consider the formal differential operator

$$Lu = u^{(n)}+q_1u^{(n-1)}+\ldots+q_nu, \tag{1}$$

where $n\in\mathbf{N}$ and $q_1, \ldots, q_n$ are arbitrary complex functions from the class $L^1_{\mathrm{loc}}(G)$.

Given a complex number λ, the function $u\colon G\to\mathbf{C}$, $u\equiv 0$ is called an eigenfunction of order (-1) of the operator L with eigenvalue λ. More generally, as usual, a function $u\colon G\to\mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction of order m $(m=0, 1, \ldots)$ of the operator L with the eigenvalue λ, if the following two conditions are satisfied:

(i) u, together with its first $n-1$ derivatives is absolute continuous on every compact subinterval of G;

(ii) there exists an eigenfunction u^* of order $m-1$ of the operator L with eigenvalue λ such that for almost all $x\in G$,

$$(Lu)(x) = \lambda u(x)+u^*(x). \tag{2}$$

For $r=0, 1, \ldots$, let us introduce the functions $K_{nr}\colon \mathbf{C}\times\mathbf{R}\to\mathbf{C}$ as follows:

$$K_{n,0}(\mu, x) = \begin{cases} \displaystyle\sum_{p=1}^{n} \frac{\omega_p}{n\mu^{n-1}}\, e^{\mu\omega_p x} & \text{if } \mu\neq 0,\\[2mm] \displaystyle\frac{x^{n-1}}{(n-1)!} & \text{if } \mu = 0\end{cases}$$

(here $\omega_1, \ldots, \omega_n$ denote the n-th roots of unity),

$$K_{n,r}(\mu, x) = \int_0^x K_{n,0}(\mu, x-t) K_{n,r-1}(\mu, t)\, dt \quad \text{for} \quad r > 0.$$

THEOREM 1. *Given arbitrary* $n \in \mathbf{N}$ *and* $m \in \{0, 1, \ldots\}$, *there exist holomorphic entire functions* $f_{nm}, f_{nmjik} = f_{n,m,j,i,k}$ $(j = 0, 1, \ldots, m,\ i = 0, 1, \ldots, n-1,\ k = 1, 2, \ldots, \ldots, (m+1)n := N)$ *such that*

$$f_{nm}(z) \neq 0 \quad \text{if} \quad |z| < \pi, \tag{3}$$

and the following formulas are valid:

Given any eigenfunction u *of order* $\leqq m$ *of the operator* L *with some eigenvalue* $\lambda = \mu^n$, *introducing the notations*

$$\begin{cases} u_m = u, \quad u_{m-j-1}\colon G \to \mathbf{C} \quad \text{continuous,} \\ u_{m-j-1} = Lu_{m-j} - \lambda u_{m-j} \quad \text{a.e. in } G \quad (j = 0, \ldots, m), \end{cases} \tag{4}$$

we have for any $j \in \{0, 1, \ldots, m\}$, $i \in \{0, 1, \ldots, n-1\}$, $i_0 \in \{0, 1, \ldots, n-1\}$, $x \in G$ *and* $x + Nt \in G$,

$$\begin{aligned} f_{nm}(\mu t)\, t^{jn+i-i_0} u_{m-j}^{(i)}(x) = \sum_{k=1}^{N} f_{n,m,j,i-i_0,k}(\mu t) \Big[u_m^{(i_0)}(x+kt) + \\ + \sum_{r=0}^{m} \sum_{s=1}^{n} \int_x^{x+kt} D_2^{i_0} K_{n,r}(\mu, x+kt-\tau)\, q_s(\tau)\, u_{m-r}^{(n-s)}(\tau)\, d\tau \Big] \end{aligned} \tag{5}$$

if $i_0 \leqq i$,

$$\begin{aligned} & f_{nm}(\mu t)(\mu t)^{(m-j+1)n} t^{jn+i-i_0} u_{m-j}^{(i)}(x) = \\ & = \sum_{k=1}^{N} \left[\sum_{\alpha=j}^{m} (-1)^{\alpha-j} (\mu t)^{(m-\alpha)n} f_{n,m,\alpha,i+n-i_0,k}(\mu t) \right] \times \\ & \times \left[u_m^{(i_0)}(x+kt) + \sum_{r=0}^{m} \sum_{s=1}^{n} \int_x^{x+kt} D_2^{i_0} K_{n,r}(\mu, x+kt-\tau)\, q_s(\tau)\, u_{m-r}^{(n-s)}(\tau)\, d\tau \right] \end{aligned} \tag{6}$$

if $i_0 > i$.

Furthermore in case $j = i - i_0 = 0$, (5) *can be simplified by* $f_{nm}(\mu t)$.

Finally, there exist holomorphic entire functions h_{n,r,i_0} *such that*

$$D_2^{i_0} K_{n,r}(\mu, x) \equiv x^{(r+1)n-1-i_0} h_{n,r,i_0}(\mu x). \tag{7}$$

PROOF. It suffices to consider the case $q_s \equiv 0$, $s = 1, \ldots, n$, because the general case follows by the method of the paper [5]. The representations (7) were also shown in [5].

In this special case, it follows from [5], Theorem 1 that

$$f_{nm}(\mu t)\, t^{jn+i-i_0} u_{m-j}^{(i-i_0)}(x) = \sum_{k=1}^{N} f_{n,m,j,i-i_0,k}(\mu t)\ u_m(x+kt)$$

if $i_0 \leqq i$. Differentiating by x i_0 times, hence (5) follows.

In case $i_0 > i$ we have by [5], Theorem 1

$$f_{nm}(\mu t)\, t^{jn+i-i_0+n}\, u_{m-j}^{(i-i_0+n)}(x) = \sum_{k=1}^{N} f_{n,m,j,i-i_0+n,k}(\mu t)\, u_m(x+kt).$$

Differentiating by x i_0 times and taking into account (4), we obtain

$$(8) \qquad f_{nm}(\mu t)\, t^{jn+i-i_0+n}[\mu^n u_{m-j}^{(i)}(x) + u_{m-j-1}^{(i)}(x)] = \sum_{k=1}^{N} f_{n,m,j,i-i_0+n,k}(\mu t)\, u_m^{(i_0)}(x+kt)$$

for $j=0, \dots, m$. Now the formulas (6) are linear combinations of the formulas (8).

The assertion of the simplification follows from the fact proved in [5] that $f_{n,m,0,0,k}$ can be simplified by f_{nm}. $\square$

THEOREM 2. *Assume that*

(9) G *is bounded and* $q_s \in L^p(G)$, $s=1, \dots, n$, *for some* $p \in [1, \infty]$.

Then for all eigenfunctions u *of* $L, u, u', \dots, u^{(n-1)}$ *and* u^* *can be extended to absolute continuous functions* $\bar{G} \to \mathbf{C}$.

Moreover, there exist constants $C_0, C_1, \dots$ *such that for any eigenfunction* u *of order* $\leqq m$ *of the operator* L *with some eigenvalue* $\lambda = \mu^n$ *we have*

$$(10) \qquad \|u^{(i)}\|_{L^{p'}(G)} \leqq C_m(1+|\mu|)^i \|u\|_{L^{p'}(G)} \quad (i = 0, \dots, n-1),$$

$$(11) \qquad \|u^*\|_{L^{p'}(G)} \leqq C_m(1+|\lambda|) \|u\|_{L^{p'}(G)},$$

$$(12) \qquad \|u\|_{L^\infty(G)} \leqq C_m(1+|\mu|)^{1/\varrho} \|u\|_{L^\varrho(G)} \quad (\varrho \in [1, \infty]).$$

REMARKS. The assertion concerning the absolute continuity follows also from more general results (see [1]) but the following proof is quite elementary.

The constants C_m depend also on n, $|G|$ (= the length of G), $q_1, q_2, \dots, q_n$. The case $q_1 \equiv 0$ was settled in [5] but the proof in [5] does not work immediately in this situation. However, in that special case we obtain also some supplementary information on C_m: C_m depends on n, $|G|$, $\|q_2\|_{L^p(G)}, \dots, \|q_n\|_{L^p(G)}$.

PROOF. We shall use the notations of Theorem 1. Let $G=(a, b)$ and let u be an eigenfunction of order $\leqq m$ of the operator L with some eigenvalue $\lambda = \mu^n$. Set

$$(13) \qquad Q = \max\{\|q_s\|_{L^p(G)},\ \|q_s\|_{L^1(G)}\colon\ s = 1, \dots, n\},$$

$$(14) \qquad R = \min\left\{\frac{|G|}{2N+2},\ \frac{1}{|\mu|}\right\} \quad \left(\frac{1}{0} := \infty\right),$$

$$(15) \qquad M_{\varepsilon, p'} = \max\{R^{jn+i}\|u_{m-j}^{(i)}\|_{L^{p'}(a+\varepsilon, b-\varepsilon)}\colon\ j = 0, 1, \dots, m, \quad i = 0, 1, \dots, n-1\}$$
$$(\varepsilon \in [0, (b-a)/2)).$$

First we prove the theorem for Q sufficiently small, hence the general case will follow easily.

It follows from the case $i_0=0$ of Theorem 1 that there exists an absolute constant A_m such that

$$|t^{jn+i}u_{m-j}^{(i)}(x)| \leqq \tag{16}$$

$$\leqq A_m \sum_{k=1}^{N} |u_m(x+kt)| + A_m \sum_{r=0}^{m} \sum_{s=1}^{n} |t|^{rn+n-1} \|q_s\|_{L^p(x,\, x+Nt)} \|u_{m-r}^{(n-s)}\|_{L^{p'}(x,\, x+Nt)}$$

whenever $x\in G$, $x+Nt\in G$, $|\mu t|\leqq 1$, $j\in\{0, 1, \dots, m\}$, $i\in\{0, 1, \dots, n-1\}$.

To prove the absolute continuity, it suffices to show that

$$u_{m-j}^{(i)}\in L^\infty(G) \tag{17}$$

for all admissible j and i, i.e. for all $j\in\{0, 1, \dots, m\}$ and $i\in\{0, 1, \dots, n-1\}$. Indeed, then

$$u_{m-j}^{(n)} = \lambda u_{m-j} + u_{m-j-1} - \sum_{s=1}^{n} q_s u_{m-j}^{(n-s)} \in L^1(G)$$

and G is bounded.

Being G bounded, we can assume that $p=1$ (and then $p'=\infty$). Applying (16) with $x\in\left[a+\varepsilon, \dfrac{a+b}{2}\right]$, $\varepsilon\in(0, R)$, $t=R$, we obtain

$$|R^{jn+i}u_{m-j}^{(i)}(x)| \leqq NA_m\|u_m\|_{L^\infty(a+R,\, b-R)} +$$

$$+A_m(1+R^{n-1})Q \sum_{r=0}^{m} \sum_{s=1}^{n} R^{rn+n-s} \|u_{m-r}^{(n-s)}\|_{L^\infty(a+\varepsilon,\, b-\varepsilon)} \leqq$$

$$\leqq NA_m\|u_m\|_{L^\infty(a+R,\, b-R)} + NA_m(1+R^{n-1})QM_{\varepsilon,\infty}$$

for all admissible i and j. The same estimate can be shown similarly if $x\in\left[\dfrac{a+b}{2}, b-\varepsilon\right]$ hence

$$M_{\varepsilon,\infty} \leqq NA_m\|u_m\|_{L^\infty(a+R,\, b-R)} + NA_m(1+R^{n-1})QM_{\varepsilon,\infty}.$$

Assume now that

$$2NA_m(1+|G|^{n-1})Q \leqq 1, \tag{18}$$

then

$$M_{\varepsilon,\infty} \leqq 2NA_m\|u_m\|_{L^\infty(a+R,\, b-R)}$$

and taking the limit $\varepsilon\to 0$,

$$M_{0,\infty} \leqq 2NA_m\|u_m\|_{L^\infty(a+R,\, b-R)}. \tag{19}$$

Now (17) follows at once from (19).

To prove (10) and (11), let us apply (16) with $x\in\left(a, \dfrac{a+b}{2}\right]$ and $t=R$ for any admissible i and j:

$$R^{jn+i}|u_{m-j}^{(i)}(x)| \leqq A_m \sum_{k=1}^{N} |u_m(x+kR)| + NA_m(1+R^{n-1})QM_{0,p'}.$$

Taking the $L^{p'}\left(a, \dfrac{a+b}{2}\right)$ norm of both sides,

$$R^{jn+i}\|u_{m-j}^{(i)}\|_{L^{p'}\left(a,\frac{a+b}{2}\right)} \leqq NA_m\|u_m\|_{L^{p'}(G)}+NA_m(1+R^{n-1})(b-a)^{1/p'}QM_{0,p'};$$

and then by symmetry

$$M_{0,p'} \leqq 2NA_m\|u_m\|_{L^{p'}(G)}+2NA_m(1+R^{n-1})(b-a)^{1/p'}QM_{0,p'}.$$

Assume now that (18) and

(20) $$4NA_m(1+|G|^{n-1})|G|^{1/p'}Q \leqq 1$$

are satisfied. Then $M_{0,p'}$ is finite and therefore

(21) $$M_{0,p'} \leqq 4NA_m\|u_m\|_{L^{p'}(G)};$$

(10) and (11) follow from (21).

Finally we prove (12). Let us apply (16) with $x\in\left(a, \dfrac{a+b}{2}\right]$ and $t\in[0, R]$ for any admissible i and j. Hence

$$t^{jn+i}|u_{m-j}^{(i)}(x)| \leqq A_m \sum_{k=1}^{N} |u_m(x+kt)|+NA_m(1+R^{n-1})QM_{0,\infty}.$$

Applying the transformation $NR^{-1}\int\limits_0^R dt$ and taking into account that

$$\int\limits_0^R |u_m(x+kt)|\,dt \leqq R^{1-1/\varrho}\|u_m\|_{L^\varrho(G)},$$

we obtain for all $x\in\left(a, \dfrac{a+b}{2}\right]$ the inequality

$$R^{jn+i}|u_{m-j}^{(i)}(x)| \leqq N^2A_mR^{-1/\varrho}\|u_m\|_{L^\varrho(G)}+N^2A_m(1+R^{n-1})QM_{0,\infty};$$

hence by symmetry

$$M_{0,\infty} \leqq N^2A_mR^{-1/\varrho}\|u_m\|_{L^\varrho(G)}+N^2A_m(1+R^{n-1})QM_{0,\infty}.$$

Assume now that

(22) $$2N^2A_m(1+|G|^{n-1})Q \leqq 1$$

(this implies (18)). Then we conclude from the above inequality that

(23) $$M_{0,\infty} \leqq 2N^2A_mR^{-1/\varrho}\|u_m\|_{L^\varrho(G)},$$

and hence (12) follows.

Let us now turn to the proof of the theorem in the general case. Let us divide G to the union of a finite number of subintervals, each of which satisfies the (suitably modified) conditions (20) and (22). Then the theorem is valid for each of these subintervals, but then also for their union i.e. for G. $\square$

REMARK. The estimate (19) contains some supplementary information to the theorem which is sometimes very useful: an estimate by the norm of the function on a smaller interval.

THEOREM 3. *Assume that* (9) *is satisfied. Then there exist positive constants* $B_0, B_1, \ldots$ *such that for any eigenfunction* u *of order* $\leqq m$ *of the operator* L *with eigenvalue* $\lambda=\mu^n$, *we have*

$$\|u^{(i_0)}\|_{L^{p'}(G)} \geqq B_m(1+|\mu|)^{i_0}\|u\|_{L^{p'}(G)} \quad (i_0=0, \ldots, n-1) \tag{24}$$

if $|\mu|$ *is sufficiently large.*

PROOF. We use the notations of the preceding theorems. Applying (5) and (6) with $x\in\left(a, \frac{a+b}{2}\right]$ and $t=R$, we have for all admissible i and j

$$|\mu R|^{(m-j+1)n} R^{jn+i}|u_{m-j}^{(i)}(x)| \leqq A_m^* \sum_{k=1}^{N} R^{i_0}|u_m^{(i_0)}(x+kR)|+$$
$$+A_m^* \sum_{r=0}^{m}\sum_{s=1}^{n} R^{rn+n-1} Q\, \|u_{m-r}^{(n-s)}\|_{L^{p'}(G)},$$

where A_m^* is an absolute constant. Taking the $L^{p'}\left(a, \frac{a+b}{2}\right)$ norm of both sides,

$$|\mu R|^{(m-j+1)n} R^{jn+i}\|u_{m-j}^{(i)}\|_{L^{p'}\left(a, \frac{a+b}{2}\right)} \leqq$$
$$\leqq NA_m^* R^{i_0}\|u_m^{(i_0)}\|_{L^{p'}(G)} + NA_m^*(1+R^{n-1})QM_{0,p}|G|^{\frac{1}{p'}}.$$

Hence by symmetry

$$|\mu R|^{(m+1)n} M_{0,p'} \leqq 2NA_m^* R^{i_0}\|u_m^{(i_0)}\|_{L^{p'}(G)}+2NA_m^*(1+R^{n-1})QM_{0,p'}|G|^{\frac{1}{p'}}.$$

Assume now that

$$4NA_m^*(1+|G|^{n-1})Q|G|^{\frac{1}{p'}} \leqq 1. \tag{25}$$

Then

$$|\mu R|^{(m+1)n} M_{0,p'} \leqq 2NA_m^* R^{i_0}\|u_m^{(i_0)}\|_{L^{p'}(G)} + \tfrac{1}{2} M_{0,p'}.$$

In the case

$$|\mu| \geqq \frac{2N+2}{|G|} \tag{26}$$

we have $|\mu R|=1$ and therefore

$$M_{0,p'} \leqq 4NA_m^* R^{i_0}\|u_m^{(i_0)}\|_{L^{p'}(G)}.$$

Hence (24) follows and the theorem is proved for Q sufficiently small. The general case follows as in Theorem 2. □

THEOREM 4. *Assume that* (9) *is satisfied. Then there exist constants* $D_0, D_1, \ldots$ *such that for any eigenfunction* u *of order* $\leqq m$ *of the operator* L *with the eigenvalue*

$\lambda=\mu^n$, *we have*

(27) $\|u^{(i_0)}\|_{L^\infty(G)} \leqq D_m(1+|\mu|)^{1/\varrho}\|u^{(i_0)}\|_{L^\varrho(G)}, \quad i_0=0, 1, \ldots, n-1, \quad \varrho\in[1, \infty],$

if $|\mu|$ *is sufficiently large.*

Proof. Using the notations as before and applying Theorem 1, we have for all $x\in\left(a, \dfrac{a+b}{2}\right)$, $t\in(0, R)$ and for all admissible i and j,

$$|\mu t|^{(m-j+1)n} t^{jn+i}|u_{m-j}^{(i)}(x)| \leqq A_m^* R^{i_0}\sum_{k=1}^N |u_m^{(i_0)}(x+kt)|+NA_m^*(1+R^{n-1})QM_{0,\infty}.$$

Applying the transformation $2NR^{-1}\int\limits_0^R dt$ and repeating the argument of Theorem 2 to prove (12), we obtain

$$|\mu R|^{(m-j+1)n}M_{0,\infty} \leqq 2N^2A_m^*R^{i_0-1/\varrho}\|u_m^{(i_0)}\|_{L^\varrho(G)}+2N^2A_m^*(1+R^{n-1})QM_{0,\infty}.$$

Assume that (26) and

(28) $$4N^2A_m^*(1+|G|^{n-1})Q \leqq 1$$

are satisfied. Then we obtain

$$M_{0,\infty} \leqq 4N^2A_m^*R^{i_0-1/\varrho}\|u_m^{(i_0)}\|_{L^\varrho(G)}$$

which implies (27). Thus the theorem is proved under the hypothesis (28). The general case hence follows as before. □

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[1] G. Szegő, *Orthogonal polynomials,* AMS Coll, Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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