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ON A PROBLEM OF A. M. ODLYZKO ON ALGEBRAIC UNITS OF BOUNDED DEGREE

K. GYŐRY* (Debrecen), corresponding member of the Academy

To Professor K. Tandori on his 70th birthday

1. Introduction

For an algebraic number field K, denote by $M(K)$ the maximal length m of a sequence $(\varepsilon_1, \dots, \varepsilon_m)$ in K such that $\varepsilon_i - \varepsilon_j$ is a unit for all i, j with $1 \leqq i < j \leqq m$. Here we may assume without loss of generality that $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ and $\varepsilon_3, \dots, \varepsilon_m$ are units. This can be achieved by translation and by multiplication by a unit. By a theorem of H. W. Lenstra Jr. [5], the number field K is Euclidean for the norm provided that $M(K)$ exceeds the square root of the discriminant of K in absolute value times a (number-geometric) coefficient which depends only on the signature of K. It was also proved in [5] that $M(K) \leqq 2^{[K:\mathbf{Q}]}$. The above-quoted theorem of Lenstra was used by Lenstra [5], A. Leutbecher and J. Martinet [6], J.–F. Mestre [8] and others to give several hundred new examples of Euclidean number fields. For related results and further references, see e.g. [7], [1] and [9].

For given positive integer n, denote by $M(n)$ the maximal number m of algebraic units $\varepsilon_1, \dots, \varepsilon_m$ of degree $\leqq n$ over $\mathbf{Q}$ (which can lie in different number fields) such that $\varepsilon_i - \varepsilon_j$ is a unit for all distinct i, j with $1 \leqq i, j \leqq \leqq m$. In a letter in February 1985, A. M. Odlyzko proposed the following problem: What is the value of $M(n)$?

Clearly, $M(n) \geqq M(K) - 1$ for all algebraic number fields K of degree n. In 1985, I was able to prove $M(n) < \infty$ only. In the proof I needed the use of the Thue–Siegel–Roth–Schmidt method.

THEOREM. *We have*

$$M(n) < \exp\exp\{39n(n^{2n+1})!\}. \tag{1}$$

We shall reduce the problem to n decomposable form equations. Then an explicit upper bound of ours for the number of solutions of such equations (cf. [2], [4] and the Lemma in Section 2) will be applied to prove the

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Theorem. We note that the proofs in [2] depend, among other things, on H. P. Schlickewei's p-adic generalization (cf. [11]) of W. M. Schmidt's quantitative Subspace Theorem [12].

Our explicit bound concerning decomposable form equations has recently been improved by J. H. Evertse (private communication). Using his improvement, one can prove that

$$M(n) < \exp\{36n^{2n+5}\}. \tag{2}$$

Finally, we mention that using Theorem 3 of the author [3], the existence of $M(n)$ can be proved in the more general situation as well, when the ground ring $\mathbf{Z}$ is replaced by an arbitrary finitely generated and integrally closed integral domain over $\mathbf{Z}$.

2. Proof of the Theorem

Let K be an algebraic number field of degree k with ring of integers O_K and unit group O_K^*. Let $F(x_0, x_1, \dots, x_q)$ $(q \geqq 1)$ be a decomposable form of degree t with coefficients in O_K, i.e. a homogeneous polynomial which factorizes into linear factors over a finite extension G of K. Two solutions $\underline{x}, \underline{x}'$ of the *decomposable form equation*

$$F(x_0, x_1, \dots, x_q) \in O_K^* \quad \text{in} \quad \underline{x} = (x_0, \dots, x_q) \in O_K^{q+1} \tag{3}$$

are called proportional if $\underline{x}' = \varepsilon \underline{x}$ for some $\varepsilon \in O_K^*$. Let d denote the degree of the normal closure of G over $\mathbf{Q}$.

To prove our Theorem, we need the following.

LEMMA. *Suppose that $t > 2q$ and that any $q+1$ linear factors in the factorization of F are linearly independent. Then equation* (3) *has at most*

$$(5kd)^{2^{37qd} \cdot k^6} \tag{4}$$

pairwise non-proportional solutions.

PROOF. This is an immediate consequence of Theorem 6 of the author [4]. Its proof involves, among other things, an estimate of Schlickewei [10] for the number of solutions of S-unit equations. □

We note that a more general but qualitative version of the Lemma was proved in [3] over an arbitrary finitely generated integral domain over $\mathbf{Z}$.

REMARK 1. On combining the above-mentioned result of Evertse with the proof of Theorem 3 of [3], our Lemma can be proved with the bound

$$\left(2^{34}t^2\right)^{q^3 \cdot k}. \tag{5}$$

PROOF OF THE THEOREM. Let $\varepsilon_1, \dots, \varepsilon_m$ be algebraic units of degree $\leqq n$ with $m = M(n)$ such that $\varepsilon_i - \varepsilon_j$ is a unit for all $i \neq j$. If $m \leqq 2n+1$ then we are done. Hence assume that $m > 2n+1$. Consider the number field $K = \mathbf{Q}(\varepsilon_1, \dots, \varepsilon_{2n+1})$. Its degree, denoted by k, is at most n^{2n+1}. Each ε_i with $i > 2n+1$ is of degree at most n over K. For a given positive integer q with $1 \leqq q \leqq n$, consider those i with $2n+1 < i \leqq m$ for which ε_i is of degree q over K, and denote by M_q the number of i under consideration.

For each unit ε_i of degree q over K, denote by $\varepsilon_i^{(1)}, \dots, \varepsilon_i^{(q)}$ the conjugates of ε_i over K. Then for each j with $1 \leqq j \leqq 2n+1$, $\varepsilon_j - \varepsilon_i^{(p)}$ is an algebraic unit for $p = 1, \dots, q$. There exist algebraic integers $x_{1i}, \dots x_{qi}$ in K such that

$$\prod_{p=1}^{q} \left(\varepsilon_j - \varepsilon_i^{(p)} \right) = \varepsilon_j^q + x_{1i} \varepsilon_j^{q-1} + \dots + x_{qi}.$$

Further, this product is a unit in K. Putting

$$F(x_0, x_1, \dots, x_q) = \prod_{j=1}^{2n+1} \left(x_0 \varepsilon_j^q + x_1 \varepsilon_j^{q-1} + \dots + x_q \right),$$

F is a decomposable form of degree $2n+1$ with coefficients in O_K. For each ε_i under consideration, the corresponding tuple $(1, x_{1i}, \dots, x_{qi})$ is a solution of the equation

$$(6) \qquad F(1, x_1, \dots, x_q) \in O_K^* \quad \text{in} \quad (1, x_1, \dots, x_q) \in O_K^{q+1}.$$

We apply now our Lemma to equation (6). Denote by d the degree of the normal closure over $\mathbf{Q}$ of the splitting field of F over K. We have $d \leqq k!$. We notice that if the units ε_i and $\varepsilon_{i'}$ are of degree q over K with $2n+1 \leqq$ $\leqq i,\ i' \leqq m,\ i \neq i'$, then they lead to the same solution $(1, x_1, \dots, x_q)$ of (6) if and only if ε_i and $\varepsilon_{i'}$ are conjugates to each other over K. Hence, by our Lemma we infer that

$$M_q \leqq q(5kd)^{2^{37qd}\cdot k^6} \leqq \exp\exp\{ 38q(n^{2n+1})! \}.$$

This implies that

$$M(n) \leqq \sum_{q=1}^{n} M_q \leqq \exp\exp\{ 39n(n^{2n+1})! \}$$

which completes the proof of the Theorem. □

REMARK 2. It is clear from the proof that using our Lemma with the bound (5) in place of (4), we obtain estimate (2) instead of (1).

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RELATION BETWEEN BERNSTEIN- AND NIKOLSKII-TYPE INEQUALITIES

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Dedicated to Professor Károly Tandori on his 70th birthday

1. Introduction

In 1911, S. N. Bernstein in his doctoral dissertation [1] proved the inequality[2]

$$\|t_n'\|_{C[2\pi]} \leqq cn\|t_n\|_{C[2\pi]}$$

where t_n is a trigonometric polynomial of order n.

Three years later, F. Riesz [14] extended the inequality for the L^p-norm, namely, he proved that

$$(1) \qquad \|t_n'\|_{L^p[2\pi]} \leqq c(p)n\|t_n\|_{L^p[2\pi]} \quad (1 \leqq p < \infty).$$

S. M. Nikolskii [12] proved that if $1 \leqq p < q \leqq \infty$, then

$$(2) \qquad \|t_n\|_{L^q[2\pi]} \leqq c(p,q)n^{\frac{1}{p}-\frac{1}{q}}\|t_n\|_{L^p[2\pi]}.$$

Bernstein and Nikolskii inequalities play an important role in Fourier analysis and approximation theory, for example in the proofs of converse and imbedding theorems.

Recently, inequalities of the same type have been established for various systems of functions.

In this paper we show a close connection between Bernstein- and Nikolskii-type inequalities. More exactly, we prove that inequalities of the second type can be deduced from that of the first type. This statement will be considered also for arbitrary function systems in general function spaces, such as symmetric spaces.

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[2] Throughout the paper, c denotes an absolute constant and $c(x,\ldots)$ will denote a constant depending only on variables specified in the brackets.

2. Main results

We use the usual notation $L^p = L^p(a,b)$ $(1 \leqq p \leqq \infty,\ -\infty \leqq a < b \leqq \infty)$ for the Banach space of functions defined on (a,b) with the norm

$$\|f\|_p = \|f\|_{L^p(a,b)} = \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}} \quad (1 \leqq p < \infty),$$

$$\|f\|_\infty = \|f\|_{L^\infty(a,b)} = \operatorname*{ess\,sup}_{x \in (a,b)} |f(x)|.$$

The L_p space of 2π-periodic functions is denoted by $L^p[2\pi]$.

Let $F = \{f_n\}_{n=0}^\infty$ be a linearly independent system of functions. Let for $n = 0, 1, \ldots$

$$F_n := \left\{ \Phi_n = \sum_{k=0}^n \alpha_k f_k \colon \quad \alpha_k \text{ are real numbers} \right\}.$$

Let $\{\lambda\} = \{\lambda_n \uparrow \infty\}$ be an increasing sequence of positive numbers tending to ∞. We say that the system F satisfies a Bernstein-type inequality in L^p of order $\{\lambda\}$, in notation $F \in B\big(L^p, \{\lambda\}\big)$, if $F \in L^p$, f_k are locally absolutely continuous on (a,b), $f_k' \in L^p$ $(k = 0, 1, \ldots)$ and the inequality

$$\|\Phi_n'\|_p \leqq c(p)\lambda_n \|\Phi_n\|_p \tag{3}$$

holds for every $\Phi_n \in F_n$ $(n = 0, 1, \ldots)$.

Suppose that for a given pair $1 \leqq p < q \leqq \infty$, $F \subset L^p \cap L^q$, and

$$\|\phi_n\|_q \leqq c(p,q)\lambda_n^{\frac{1}{p}-\frac{1}{q}} \|\Phi_n\|_p \quad (\Phi_n \in F_n,\ n = 0, 1, \ldots) \tag{4}$$

then F is said to satisfy a Nikolskii-type inequality between L^p and L^q of order $\{\lambda\}$, in notation $F \in N\big(L^p, L^q, \{\lambda\}\big)$.

In the case when the constants $c(p)$ in (3) and $c(p,q)$ in (4) cannot be replaced by γ_n with $\gamma_1 \geqq \gamma_2 \geqq \ldots \geqq \gamma_m \to 0$, we say that inequalities (3) and (4) are sharp, and then we use the notations $F \in B_{\mathbf{sharp}}\big(L^p, \{\lambda\}\big)$ and $F \in N_{\mathbf{sharp}}\big(L^p, L^q, \{\lambda\}\big)$, respectively.

One of the main results of our paper is

THEOREM 1. *Let $-\infty \leqq a < b \leqq \infty$. Let F be a system of functions on (a,b). Let $\{\lambda\} = \{\lambda_n \uparrow \infty\}$ be a sequence satisfying*

$$\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \geqq \frac{c}{n\lambda_n}, \quad \lambda_n \sim \lambda_{n+1} \tag{5}$$

and for which there exists a sequence $\{u_i\}$ *of natural numbers satisfying* $d_1 \geqq u_i/u_{i-1} \geqq d_2 > 1$*, such that for any* $n \geqq 1$

$$\frac{\lambda_{u_n}}{u_n} \leqq \frac{\lambda_{u_{n-1}}}{u_{n-1}} \tag{6.a}$$

or

$$\frac{\lambda_{u_{n-1}}}{u_{n-1}} \leqq \frac{\lambda_{u_n}}{u_n} \leqq c\frac{\lambda_{u_{n-1}}}{u_{n-1}}. \tag{6.b}$$

Then

a) *If for some* $1 \leqq p < \infty$, $F \in B(L^p, \{\lambda\})$*, then* $F \in N(L^p, L^q, \{\lambda\})$ *for every* $p < q \leqq \infty$.

b) *Assume that the sequence* $\{\lambda\}$ *satisfies Conditions* (5) *and* (6.a). *If* $F \in N_{\text{sharp}}(L^{p_0}, L^{q_0}, \{\lambda\})$ *for some pair* $1 \leqq p_0 < q_0 \leqq \infty$, *and* $F \in B(L^{p_0}, \{\lambda\})$*, then* $F \in B_{\text{sharp}}(L^{p_0}, \{\lambda\})$.

REMARK 1. It is easy to see that the sequence, $\{\lambda\} = \{n^\alpha\}$ $(\alpha > 0)$ satisfies (5). It also satisfies (6.a) (for $0 < \alpha \leqq 1$), and (6.b) (for $\alpha \geqq 1$) with $\{u_i := 2^i\}$.

The proof of our theorem will be based on two inequalities concerning the L^p-modulus of continuity of functions defined as

$$\omega(f,\delta)_p = \omega(f,\delta)_{L^p(a,b)} = \sup_{0<h\leqq\delta} \left\{ \int_a^{b-h} |f(x+h) - f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

LEMMA 1. *Let* $f \in L^p(a,b)$ $(1 \leqq p < \infty)$. *Let* f^* *be the decreasing rearrangement of* f. *Then for* $I' := (0, b-a)$

$$\omega(f^*,\delta)_{L^p(I')} \leqq 7\omega(f,\delta)_{L^p(a,b)}. \tag{7}$$

PROOF. In the case $(a,b) = (0,1)$ (7) was proved in [9] (see also [4]). The case of arbitrary finite intervals then can be deduced easily from that of (0,1). Suppose now that $I = (a,b)$ is an infinite interval. Let J be a finite interval contained in I. For $f \in L^p(I)$ let $f_J(x) := f(x)$ $(x \in J)$, 0 $(x \in I \setminus J)$. Denote by f_J^* the decreasing rearrangement of f_J. It is easy to see that for all $x \in I'$, $f_J^*(x) \to f^*(x)$ $(J \to I)$. Let $J' := (o, |J|)$,

$$J'(t) := \{x \in J' : x + t \in J'\}.$$

Denote by χ_E the characteristic function of the set E. Using inequality (7) for the (finite) interval J, by Fatou's Lemma we have for $0 < t \leqq \delta$

$$\omega(f,\delta)_{L^p(I)} \geqq \liminf_{J\to I} \omega(f,\delta)_{L^p(J)} \geqq \frac{1}{7} \liminf_{J\to I} \omega(f_J^*,\delta)_{L^p(J')} \geqq$$

$$\geqq \frac{1}{7} \liminf_{J\to I} \|\chi_{J'(t)} \Delta_t f_J^*\|_{L^p(I')} \geqq \frac{1}{7} \|\chi_{I'(t)} \Delta_t f^*\|_{L^p(I')}.$$

Hence (7) follows.

LEMMA 2. *Let* $1 \leqq p < \infty$, $1 \leqq u < \infty$. *Let* $g \in L^p(0,u)$ *be nondecreasing. Then for any* $0 < x \leqq \frac{u}{2}$

$$g(x) \leqq c(p) \int_x^u \frac{\omega(g,t)_{L^p(0,u)}}{t^{1+\frac{1}{p}}}\,dt + \frac{1}{\left(\frac{u}{2}\right)^{\frac{1}{p}}} \|g\|_{L^p(0,u)}. \tag{8}$$

PROOF. Introduce $h(y) := g(uy)$ $(y \in (0,1))$. Using [10, (3.3)] we have for $0 < y \leqq \frac{1}{2}$

$$h(y) - h\left(\frac{1}{2}\right) \leqq c(p) \int_y^1 \frac{\omega(h,z)_{L^p(0,1)}}{z^{1+\frac{1}{p}}}\,dz. \tag{9}$$

By exchanging $y = \frac{x}{u}$, $z = \frac{t}{u}$ and observing that

$$\omega\left(h,\frac{t}{u}\right)_{L^p(0,1)} = \frac{1}{u^{\frac{1}{p}}}\omega(g,t)_{L^p(0,u)}, \quad g\left(\frac{u}{2}\right) \leqq \frac{1}{\left(\frac{u}{2}\right)^{\frac{1}{p}}}\|g\|_{L^p(0,u)}$$

we have (8) from (9).

LEMMA 3. *Let* $\{u_i\}$ *be a sequence of positive numbers satisfying* $d_1 \geqq \geqq u_i/u_{i-1} \geqq d_2 > 1$. *Let* $\phi(t) \geqq 0$ *be a nonincreasing function on* $[0,\infty)$. *Let* $\psi(t) > 0$ *be a function defined on* $[1,\infty)$ *satisfying one of the following two conditions:*

(A) $\psi(t)/t$ *is nondecreasing and*

$$\frac{\psi(u_i)}{u_i} \leqq c\frac{\psi(u_{i-1})}{u_{i-1}} \quad (i \geqq 1), \tag{10}$$

(B) $\psi(t)/t$ *is nonincreasing.*

Then for any $k < l$ we have

$$\sum_{i=k}^{l} \psi(u_i)\phi(u_i) \leqq c \sum_{u_k \leqq n \leqq u_l} \frac{\psi(n+1)}{n+1}\phi(n).$$

PROOF. We prove the lemma in the case when ψ satisfies Condition A. The other case can be proved similarly.

By (10) we have

$$N := \sum_{i=k}^{l} \psi(u_i)\phi(u_i) \leqq c \sum_{i=k}^{l} \frac{\psi(u_{i-1})}{u_i - 1}\phi(u_i)u_i.$$

Then using the property of the sequence $\{u_i\}$ we get

$$N \leqq c \sum_{i=k}^{l} \frac{\psi(u_{i-1})}{u_{i-1}}\phi(u_i)(u_i - u_{i-1}).$$

Hence, by the fact that $\phi(t)$ and $t/\psi(t)$ are nonincreasing we obtain

$$N \leqq c \int_{u_k}^{u_l} \frac{\psi(t)}{t}\phi(t)\,dt \leqq c \sum_{u_k \leqq n \leqq u_l} \frac{\psi(n+1)}{n+1}\phi(n).$$

PROOF OF THEOREM 1. Let $f \in L^p := L^p(a,b)$. Without loss of generality one can assume that $d := b - a \geqq 1$. Let

$$E_n = E_n(f)_p := \inf_{\phi_n \in F_n} \|f - \phi_n\|_p \quad (n = 0, 1, \ldots).$$

Since $F \in \left(L^p, \{\lambda\}\right)$, using Lemma 3, by a well-known technique of approximation theory (see e.g. [6, p. 59]) we can prove

$$\omega\left(f, \frac{1}{\lambda_n}\right)_p \leqq c(p)\frac{1}{\lambda_n}\sum_{k=0}^{n} \frac{\lambda_{k+1}}{k+1}E_k \quad (n \geqq 1). \tag{11}$$

Now introduce the function

$$\alpha(t) = \alpha_{f,p}(t) := \begin{cases} E_{k-1} & \text{if } t \in \left(\frac{1}{\lambda_{k+1}}, \frac{1}{\lambda_k}\right] \quad (k = 1, 2, \ldots) \\ \|f\|_p & \text{for } \frac{1}{\lambda_1} < t < \infty. \end{cases}$$

Then, from (5) by (11) we have for $1 \leqq u \leqq d$

$$\omega(f,t)_p \leqq c(p)t \int_t^u \frac{\alpha(y)}{y^2}\,dy \quad \left(0 < t \leqq \frac{u}{2}\right). \tag{12}$$

(a) First we consider the case $p < q < \infty$. Since $\|f^*\|_p = \|f\|_p < \infty$ and f^* is nonincreasing, there exists $0 < v_0 < \infty$ such that $f^*(x) \leqq 1$ for $x \geqq v_0$. By this we have for any $v \geqq v_0$

$$\left\{\int_0^\infty [f^*(x)]^q\,dx\right\}^{\frac{1}{q}} = \left\{\int_0^v [f^*(x)]^q\,dx + \int_v^\infty [f^*(x)]^q\,dx\right\}^{\frac{1}{q}} \leqq \tag{13}$$

$$\leqq \left\{\int_0^v [f^*(x)]^q\,dx\right\}^{\frac{1}{q}} + \left\{\int_v^\infty [f^*(x)]^p\,dx\right\}^{\frac{1}{q}}.$$

Since the last integral tends to zero when v tends to infinity, there exists $1 \leqq u < \infty$ such that

$$\left\{\int_0^\infty [f^*(x)]^q\,dx\right\}^{\frac{1}{q}} \leqq 2\left\{\int_0^{\frac{u}{2}} [f^*(x)]^q\,dx\right\}^{\frac{1}{q}}. \tag{14}$$

We now estimate the integral on the right hand side of (14). Using (7) and (8) one has for $0 < x \leqq \frac{u}{2}$

$$f^*(x) \leqq c(p) \int_x^u \frac{\omega(f,t)_{L^p}}{t^{1+\frac{1}{p}}}\,dt + \frac{1}{(\frac{u}{2})^{\frac{1}{p}}}\|f\|_{L_p}. \tag{15}$$

By (12) and (15), using an inequality of Hardy we get (see e.g. [13, p. 186])

$$\left\{\int_0^{\frac{u}{2}} |f^*(x)]^q\right\}^{\frac{1}{q}} \leqq c(p)\left\{\int_0^u \left[t^{1-\frac{1}{p}} \int_t^u \frac{\psi(y)}{y^2}\,dy\right]^q dt\right\}^{\frac{1}{q}} + \frac{u^{\frac{1}{q}-\frac{1}{p}}}{2^{\frac{1}{p}}}\|f\|_{L_p} \leqq \tag{16}$$

$$\leqq c(p)\left\{\int_0^u [t^{-\frac{q}{p}}\psi(t)]^q\,dt\right\}^{\frac{1}{q}} + \frac{u^{\frac{1}{q}-\frac{1}{p}}}{2^{\frac{1}{p}}}\|f\|_{L_p}.$$

Let now $f = \Phi_n \in F_n$. Then by (11)

$$\psi(t) = \psi_{\Phi_n,p}(t) \leqq \begin{cases} \|\Phi_n\|_{L_p} & \left(\frac{1}{\lambda_n} \leqq t < \infty\right) \\ 0 & \left(0 < t < \frac{1}{\lambda_n}\right). \end{cases}$$

Consequently, by (14) and (16) we have

(17) $$\|\Phi\|_q = \|\Phi^*\|_q \leqq$$

$$\leqq c(p)\|\Phi_n\|_p \left\{ \int_{\frac{1}{\lambda_n}}^{u} t^{-\frac{q}{p}}\, dt \right\}^{\frac{1}{q}} + \frac{u^{\frac{1}{q}-\frac{1}{p}}}{2^{\frac{1}{p}}}\|\Phi_n\|_p \leqq c(p)\lambda_n^{\frac{1}{p}-\frac{1}{q}}\|\Phi_n\|_p.$$

Here we have used the assumption $q > p$.

(b) Let now $q = \infty$. In the case when (a,b) is a finite interval, we get $F \in N\left(L^p, L^\infty, \{\lambda\}\right)$ by taking limit $(q \to \infty)$ on both sides of (17). If (a,b) is infinite, we can argue as follows. Let $J \subset (a,b)$ be an arbitrary finite interval. By (17) we have

$$\|\Phi_n\|_{L^q(J)} \leqq c(p)\lambda^{\frac{1}{p}-\frac{1}{q}}\|\Phi_n\|_{L^p(a,b)}.$$

Hence, by taking limit $(q \to \infty)$, we have

$$\|\Phi_n\|_{L^\infty(J)} \leqq c(p)\lambda_n^{\frac{1}{p}}\|\Phi_n\|_{L^p(a,b)}.$$

Since J is an arbitrary finite sub-interval of (a,b), the last inequality implies that $F \in N\left(L^p, L^\infty, \{\lambda\}\right)$.

We turn to the proof of the sharpness part of our theorem. Suppose that $F \notin B_{\text{sharp}}\left(L^{p_0}, \{\lambda\}\right)$. This means by definition that $F \in B\left(L^{p_0}, \{\mu\}\right)$ with $\{\mu\} := \{\mu_n\}$, $\mu_n = \gamma_n\lambda_n$ and $\gamma_1 \geqq \gamma_2 \geqq \ldots \geqq \gamma_m \to 0$. It is easy to see that the sequence $\{\mu\}$ also satisfies conditions (5) and (6.a). Therefore repeating the technique used in the first part of our proof we get $F \in N\left(L^{p_0}, L^{q_0}, \{\mu\}\right)$, which is impossible, since $F \in N_{\text{sharp}}\left(L^{p_0}, L^{q_0}, \{\lambda\}\right)$.

By this we have completed the proof of Theorem 1.

In the case of finite intervals, Theorem 1 can be generalized in the sense that L^p is replaced by an arbitrary symmetric function space containing the basis function $x^{\frac{1}{p}}$. Let us describe exactly this statement. The definition of symmetric function spaces can be found for example in [5]. Let X be a symmetric function space on (0,1) having the fundamental function $\varphi(t) :=$

$:= \|\chi_{(0,t)}\|_X = t^{\frac{1}{p}}$ $(1 \leqq p < \infty)$. The Lorentz spaces $L_{p,r}$ $(1 \leqq p,\ r < \infty)$, defined as the collection of all measurable functions f on (0,1), for which

$$\|f\|_{p,r} := \left\{ \int_0^1 \left[x^{\frac{1}{p}} f^*(x) \right]^r \frac{dx}{x} \right\}^{\frac{1}{r}} < \infty$$

are typical examples for such spaces. Remark that $L^p = L_{p,p}$.

Now we can define the classes $B(X,\{\lambda\})$, $B_{\text{sharp}}(X,\{\lambda\})$, $N(X,L^q,\{\lambda\})$ and $N_{\text{sharp}}(X,L^q,\{\lambda\})$, similarly to $B(L^p,\{\lambda\})$, $B_{\text{sharp}}(L^p,\{\lambda\})$, $N(L^p,L^q,\{\lambda\})$ and $N_{\text{sharp}}(L^p,L^q,\{\lambda\})$, respectively ($L^p$ is replaced by X everywhere).

The following theorem is true.

THEOREM 2. *Let X be a symmetric function space on $(0,1)$ with the fundamental function $t^{\frac{1}{p}}$ $(1 \leqq p < \infty)$. Let $\{\lambda\} = \{\lambda_n \uparrow \infty\}$ be a sequence of positive numbers satisfying* (5) *and* (6.a) *or* (6.b). *Let F be a function system.*

(A) *If $F \in B(X,\{\lambda\})$, then for every $p < q \leqq \infty$, $F \in N(X,L^q,\{\lambda\})$, while for $q = p$ we have*

$$\text{(18)} \qquad \|\Phi\|_{L^p} \leqq c(p)(\log \lambda_n)^{\frac{1}{p}} \|\Phi_n\|_X \quad (\Phi_n \in F_n, \quad n = 1,2,\ldots).$$

(B) *Assume that the sequence $\{\lambda\}$ satisfies* (5) *and* (6.a). *Then, if for some $1 \leqq p < q_0 \leqq \infty$ F belongs to $N_{\text{sharp}}(X,L^{q_0},\{\lambda\})$ and $B(X,\{\lambda\})$ then $F \in B_{\text{sharp}}(X,\{\lambda\})$.*

PROOF. We use the fact that inequalities (7) and (8) remain true if one replaces $\omega(f,t)_{L^p}$ by

$$\omega(f,t)_X := \sup_{0<h\leqq t} \left\| \chi_{(0,1-h)}(x) \Delta_h f(x) \right\|_X$$

(see e.g. [5], [9], [10]).

With this the proof of Theorem 2 in the case $p < q \leqq \infty$ is similar to that of Theorem 1, while in the case $q = p$ inequality (17) becomes to

$$\|\Phi_n\|_p = \|\Phi_n^*\|_p \leqq$$

$$\leqq c(p)\|\Phi_n\|_X \left\{ \int_{\frac{1}{\lambda_n}}^1 t^{-1}\, dt \right\}^{\frac{1}{p}} + \frac{1}{2^{\frac{1}{p}}} \|\Phi_n\|_X \leqq c(p)(\log \lambda_n)^{\frac{1}{p}} \|\Phi_n\|_X.$$

Finally, the sharpness part of Theorem 2 is also clear from the proof.

3. Application

Although inequalities of Bernstein- and Nikolskii-type have been known for many special function systems, as an illustration to our results, we consider some examples.

1. Inequality (2) indeed can be deduced from (1) by using Theorem 1. We may obtain another inequality for trigonometric polynomials between L^p and $L_{p,r}$ spaces. Notice that for $1 < p \leqq \infty$, $1 \leqq r < p < s \leqq \infty$ we have

$$\|\cdot\|_{p,s} \leqq \|\cdot\|_p \leqq \|\cdot\|_{p,r}. \tag{19}$$

In order to get a converse inequality of (19) for trigonometric polynomials we shall apply Theorem 2. Let T be the trigonometric system. Since $T \in \left(L^p, \{n\}\right)$ $(1 \leqq p \leqq \infty)$, we have by the interpolation theorem $T \in$ $\in B\left(L_{p,r}, \{n\}\right)$ $(1 \leqq p,\ r \leqq \infty)$. Now, using Theorem 2 in the case $p = q$ we get

$$\|t_n\|_p \leqq c(\log n)^{\frac{1}{p}} \|t_n\|_{p,r} \quad (t_n \in T_n,\ 1 \leqq p < r < \infty). \tag{20}$$

2. Let $P_n^{(\alpha,\beta)}$ be the n-th orthonormal Jacobi polynomial with parameters $\alpha, \beta \geqq -\frac{1}{2}$. Consider the system

$$J := \left\{ P_n^{(\alpha,\beta)}(\cos\theta)(1-\cos\theta)^{\frac{\alpha}{2}}(1+\cos\theta)^{\frac{\beta}{2}} \sin^{\frac{1}{2}}\theta \right\}.$$

Combining Lemma 14 and Theorem 14 of Nevai [11] we have $J \in$ $\in B\left(L^p(0,\pi), \{n\}\right)$. Hence, by Theorem 1, $J \in N\left(L^p, L^q, \{n\}\right)$ $(1 \leqq p <$ $< q \leqq \infty)$.

3. Let

$$L_\alpha := \left\{ P_n^\alpha(x) x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} \right\}_{n=0}^{\infty}$$

where P_n^α is the n-th Laguerre polynomial with parameter $\alpha \geqq 0$. It follows from (6), (7) and (8) of [2] that $L_\alpha \in B\left(L^p(0,\infty), \{n\}\right)$ $(1 \leqq p < \infty)$. Hence $L_\alpha \in N\left(L^p, L^q, \{n\}\right)$ $(1 \leqq p < q \leqq \infty)$. This is a result of Markett [7].

REMARK 2. In general, there is no converse variant of our theorems. A typical example is the Walsh-system, which satisfies a Nikolskii-type inequality, but Walsh functions are not locally absolutely continuous on (0,1) (see [3]).

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ON GRÜNBAUM'S PROBLEM ABOUT INNER ILLUMINATION OF CONVEX BODIES

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Introduction and main results

A set $F \subset \operatorname{bd} K$ is called by P. Soltan [3] an (inner) illuminating set of a convex body $K \subset E^d$ (i.e. a compact convex set with nonempty interior) provided for every point $x \in \operatorname{bd} K$ there is a point $y \in F$ such that $x \neq y$ and the open line interval $]x, y[$ is contained in int K. P. Soltan (see [3], [4]) posed the problem on the least number of points in an illuminating set of a convex body in E^d, and he has proved that this minimum number is at most $d+1$, with $d+1$ characterizing simplices.

Due to Grünbaum [1], an illuminating set F of a convex body $K \subset E^d$ is called *primitive* if no proper subset of F illuminates K. Grünbaum [1] (see also [2], p. 423) suggested the question on the maximum number of points in a primitive illuminating set of convex bodies in E^d. This maximum is easily shown to be four for $d = 2$ (cf. [2], p. 423). For $d \geqq 3$, even a proof for the existence of the maximum is lacking. Grünbaum formulated the following problem.

PROBLEM 1. Prove that any primitive illuminating set of a convex body in E^d has at most 2^d points.

We show in this paper that Problem 1 has a positive answer for the case $d = 3$. More exactly, we prove the following assertion.

THEOREM 1. *Any primitive illuminating set of a convex body in E^3 has at most eight points, and only a convex polytope combinatorially equivalent to the 3-cube has a primitive illuminating set of eight points (placed at its vertices).*

Based on Theorem 1, one can sharpen Grünbaum's problem as follows.

PROBLEM 1′. Prove that any primitive illuminating set of a convex body in E^d has at most 2^d points, and that only a convex polytope combinatorially equivalent to the d-cube has a primitive illuminating set of 2^d points (placed at its vertices).

It is easily seen that a convex polytope $P \subset E^3$ combinatorially equivalent to the 3-cube has a primitive illuminating set of eight points (placed at

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its vertices), and any other primitive illuminating set of P has at most seven points (since a set $F \subset \operatorname{bd} P$ illuminates P from within if and only if it illuminates each vertex of P). Therefore Theorem 1 can be deduced from the following result.

THEOREM 2. *If for a convex body $K \subset E^3$ there is a primitive illuminating set of at least eight points, then K is a convex polytope combinatorially equivalent to the 3-cube.*

Proof of Theorem 2

We divided the proof of Theorem 2 into a sequence of lemmas. From now on, a convex body K is assumed to be three-dimensional, i.e., $K \subset E^3$. By a face of K we mean any of its two-dimensional faces, and an edge is a one-dimensional face of K. The abbreviations aff, conv, bd, int, rint, rbd, and card are used for affine hull, convex hull, boundary, interior, relative interior, relative boundary (taken in the affine hull), and cardinality, respectively. The notations $[x,y]$, $]x,y[$, $\langle x,y\rangle$, $[x,y\rangle$ mean closed line segment, open line interval, the line passing through distinct points x, y, and the ray with apex x passing through y ($\neq x$), respectively.

In the sequel F denotes a primitive illuminating set of K. For any $x \in F$ there is at least one point $y \in \operatorname{bd} K$ (depending on x) illuminated by x and by no other point of F. Every such a point y will be called simply illuminated, and the set of all simply illuminated points (relative to a given set F) will be denoted by G. Trivially, $\operatorname{card} G \geqq \operatorname{card} F$. For any point $x \in \operatorname{bd} K$, let I_x be the set of all points in $\operatorname{bd} K$ illuminated by x.

LEMMA 1. *If F has a common point with the relative interior of a face (or an edge) M of K, then M contains no other point in F.*

PROOF. Let a point $x \in F$ belong to $\operatorname{rint} M$. We claim that $I_z \subset I_x$ for any point $z \in M$. Indeed, since $x \in \operatorname{rint} M$, there is a point $w \in M$ such that $x \in]z,w[$. In this situation, for any point $v \in I_z$, one has $]x,v[\subset \operatorname{int} K$, i.e., $v \in I_x$.

Now, if M contained a point $y \in F$ distinct from x, then, by the above, $F \setminus \{y\}$ would be an illuminating set of K, which is impossible by the choice of F. Hence M contains no other point in F. □

LEMMA 2. *Any face of K contains at most four points in F. If a face M of K contains four points in F, then M is a convex quadrangle and the points lie at the vertices of M.*

PROOF. Assume that a face M of K contains at least four points in F. By Lemma 1, all these lie in $\operatorname{rbd} M$. Enumerate by x_1, x_2, x_3, x_4 some four of them according to an orientation of the relative boundary of M. By Lemma 1, no three of these four points are collinear. Denote by y_1, y_2, y_3, y_4 simply illuminated points of bd K corresponding to x_1, x_2, x_3, x_4, respectively. Trivially, none of y_1, y_2, y_3, y_4 lies in M. Since y_1 is illuminated by none of x_2, x_3, x_4, the segments $[x_2, y_1]$, $[x_3, y_1]$, $[x_4, y_1]$ lie in bd K; so do the segments $[x_1, y_2]$, $[x_3, y_2]$, $[x_4, y_2]$. If y_2 belonged to the open part N of bd K bounded by the segments $[x_2, y_1]$, $[x_4, y_1]$, and the arc $x_2x_3x_4$ of rbd M disjoint to $\{x_1\}$, then x_1 would illuminate y_2. If y_2 belonged to $[x_2, y_1] \cup [x_4, y_1]$, say to $]x_2, y_1[$, then x_1, y_1, y_2 would lie in a common face of K and $[x_1, y_1] \subset \text{bd } K$, which is impossible. Similarly, if y_2 belonged to the ray $[x_3, y_1\rangle$ with apex x_3 passing through y_1, but not to N, then $[x_2, y_2]$ would lie in bd K because of the inclusion $[x_2, y_1] \subset \text{bd } K$. Hence $y_2 \notin N \cup [x_2, y_1] \cup [x_4, y_1] \cup [x_3, y_1\rangle$.

Now consider the intervals $]x_2, y_1[$, $]x_3, y_2[$, and $]x_4, y_1[$. By the above, all these lie in bd K, and $]x_3, y_2[$ intersects one of $]x_2, y_1[$, $]x_4, y_1[$. If $]x_3, y_2[$ intersected $]x_2, y_1[$, then x_2, x_3, y_1, y_2 would lie in a face of K. Since x_2 illuminates y_2, the last is impossible. Hence $]x_3, y_2[$ intersects the interval $]x_4, y_1[$. Thus x_3, x_4, y_1, y_2 belong to a face of K. In this case the segment $[x_3, x_4]$ belongs to two faces of K, and hence lies in an edge.

Similarly, each of the sets

$$\{x_1, x_2, y_3, y_4\}, \quad \{x_1, x_4, y_2, y_3\}, \quad \{x_2, x_3, y_1, y_4\}$$

determines a face of K. Therefore M is a convex quadrangle with the vertices x_1, x_2, x_3, x_4. Note that $[x_1, y_3]$, $[x_2, y_4]$, $[x_3, y_1]$, and $[x_4, y_2]$ belong to edges of K.

Observe that M contains no other point in F. Indeed, if $z \in M$ were some other point in F, then z would belong to the relative interior of M or of one of its edges, say $[x_1, x_2]$. In both cases (see Lemma 1), the set $F \setminus \setminus \{x_1\}$ also illuminates K, which is impossible. Now the proof of Lemma 2 follows easily. □

LEMMA 3. *If there is a point in G which is not extreme for K, then* card $F \leqq 7$.

PROOF. Assume first that a simply illuminated point $y \in G$ lies in the relative interior of a face M of K. Denote by x the point in F illuminating y. Since $[y, z] \subset \text{bd } K$ for any point $z \in F \setminus \{x\}$, the set $F \setminus \{x\}$ lies in M. By Lemma 2, M contains at most four points in F, i.e. card $(F \setminus \{x\}) \leqq 4$. Thus card $F \leqq 5$.

Now assume that a simply illuminated point $y \in G$ lies in the interior of an edge $[a, b]$ of K, and let x be the point in F simply illuminating y. If $[a, b]$ does not belong to any face of K, then $F \setminus \{x\}$ lies in $[a, b]$ and, by Lemma 1, card $(F \setminus \{x\}) \leqq 2$, i.e., card $F \leqq 3$.

If $[a, b]$ is an edge of exactly one face M of K, then $F \setminus \{x\}$ lies in M and, by Lemma 2, there are at most four points in $F \setminus \{x\}$. Hence card $F \leqq 5$.

Let $[a, b]$ be in two distinct faces M_1, M_2 of K. As above, the set $F \setminus \{x\}$ lies in $M_1 \cup M_2$. If each of M_1, M_2 contains at most three points in $F \setminus \{x\}$, then card $F \leqq 7$. Let M_1 contain four points in $F \setminus \{x\}$. By Lemma 2, M_1

is a convex quadrangle, and its vertices a, b belong to $F \setminus \{x\}$. In this case $M_2 \setminus M_1$ contains at most two points in $F \setminus \{x\}$, and hence $\operatorname{card} F \leqq 7$. □

LEMMA 4. *If a face of K contains four points in F, then* $\operatorname{card} F \leqq 8$, *with* $\operatorname{card} F = 8$ *only if K is a convex polytope combinatorially equivalent to the 3-cube.*

PROOF. Let a face M of K contain four points x_1, x_2, x_3, x_4 in F, and let y_1, y_2, y_3, y_4 be points in $\operatorname{bd} K$, simply illuminated by x_1, x_2, x_3, x_4, respectively. Due to the proof of Lemma 2, the face M is a convex quadrilateral with the vertices x_1, x_2, x_3, x_4, and the line segments

$$(1) \qquad [x_1, y_3],\ [x_2, y_4],\ [x_3, y_1],\ [x_4, y_2]$$

belong to edges of K.

Furthermore, there are also four faces of K, namely,

$$(2) \qquad \begin{cases} M_1 = K \cap \operatorname{aff}(x_1, x_2, y_3, y_4), & M_2 = K \cap \operatorname{aff}(x_2, x_3, y_1, y_4), \\ M_3 = K \cap \operatorname{aff}(x_3, x_4, y_1, y_2), & M_4 = K \cap \operatorname{aff}(x_1, x_4, y_2, y_3). \end{cases}$$

Assume first that x_1, x_2, x_3, x_4 do not illuminate the whole set $\operatorname{bd} K \setminus \setminus M$. Then the set, say Z, of all points in $\operatorname{bd} K \setminus M$ which are not illuminated by $\{x_1, x_2, x_3, x_4\}$ coincides with one of the sets $M_1 \cap M_3$, $M_2 \cap M_4$. Hence Z is either a point or a line segment.

If Z is a point, and a point $x_5 \in F$ illuminates Z, then x_5 does not belong to M (by Lemma 1) and to any of the lines spanned by segments (1). In this case x_5 illuminates two of the points x_1, x_2, x_3, x_4. Since the other two vertices of M are illuminated by at most two points of F, say x_6, x_7, and M is illuminated by x_5, x_6, x_7, one has $\operatorname{card} F \leqq 7$.

Let Z be a line segment. If Z is illuminated by a point of F, then, as above, $\operatorname{card} F \leqq 7$. Suppose that Z is illuminated by two points of F, say x_5, x_6. It is easily seen that in this case x_5 and x_6 illuminate the whole M, and hence $\operatorname{card} F \leqq 6$.

Suppose now that x_1, x_2, x_3, x_4 illuminate the whole set $\operatorname{bd} K \setminus M$. Since M is illuminated by points of F if and only if each of the vertices x_1, x_2, x_3, x_4 is illuminated by a point of F, the whole face M is illuminated by at most four points of F. Thus $\operatorname{card} F \leqq 8$.

Assume that $\operatorname{card} F = 8$, and let x_5, x_6, x_7, x_8 be the other points in F. By the above considerations, these new points belong, respectively, to the lines spanned by segments (1). Let

$$x_5 \in \langle x_1, y_3 \rangle,\ x_6 \in \langle x_2, y_4 \rangle,\ x_7 \in \langle x_3, y_1 \rangle,\ x_8 \in \langle x_4, y_2 \rangle.$$

Since y_1, y_2, y_3, y_4 are simply illuminated, both segments $[x_5, y_1]$ and $[x_6, y_2]$ lie in $\operatorname{bd} K$. The last is possible only if x_5, x_6, y_1, y_2 are the respective end points of the edges

$$K \cap \langle x_1, y_3 \rangle, \; K \cap \langle x_2, y_4 \rangle, \; K \cap \langle x_3, y_1 \rangle, \; K \cap \langle x_4, y_2 \rangle,$$

and x_5, x_6, y_1, y_2 lie in a face of K. Hence

$$K \cap \langle x_1, y_3 \rangle = [x_1, x_5], \quad K \cap \langle x_2, y_4 \rangle = [x_2, x_6],$$

$$K \cap \langle x_3, y_1 \rangle = [x_3, y_1], \quad K \cap \langle x_4, y_2 \rangle = [x_4, y_2].$$

Similarly, considering $[x_7, y_3]$, $[x_8, y_4]$, we have

$$K \cap \langle x_1, y_3 \rangle = [x_1, y_3], \quad K \cap \langle x_2, y_4 \rangle = [x_2, y_4],$$

$$K \cap \langle x_3, y_1 \rangle = [x_3, x_7], \quad K \cap \langle x_4, y_2 \rangle = [x_4, x_8].$$

Hence $x_5 = y_3$, $x_6 = y_4$, $x_7 = y_1$, $x_8 = y_2$. Therefore K is a convex polytope combinatorially equivalent to the 3-cube and F is its vertex-set. □

The proofs of the following three assertions are similar to those of Lemmas 1–3.

LEMMA 5. *If a point $x \in G$ is simply illuminated by a point $z \in F$ and belongs to the relative interior of a face (or of an edge) M of K, then M contains no point in G simply illuminated by a point of $F \setminus \{z\}$.* □

LEMMA 6. *Any face of K contains at most four points in G corresponding to distinct points in F. If a face M of K contains four points in G corresponding to distinct points in F, then M is a convex quadrangle and the points of G lie at the vertices of M.* □

LEMMA 7. *If a point in F is not extreme for K, then* $\operatorname{card} F \leqq 7$. □

LEMMA 8. *If a face of K contains four points in G corresponding to distinct points in F, then* $\operatorname{card} F \leqq 8$, *with* $\operatorname{card} F = 8$ *only if K is a convex polytope combinatorially equivalent to the 3-cube.*

PROOF. Let a face M of K contain four points x_1, x_2, x_3, x_4 in G simply illuminated by different points $y_1, y_2, y_3, y_4 \in F$, respectively. Due to Lemma 6, the face M is a convex quadrilateral with the vertices x_1, x_2, x_3, x_4, and the sets (2) are faces of K (see the proof of Lemma 2). In particular, the line segments (1) belong to edges of K. If at least one of the points y_1, y_2, y_3, y_4 is not extreme for K, then $\operatorname{card} F \leqq 7$ (see Lemma 7). Assume that all y_1, y_2, y_3, y_4 are extreme points of K. Then the segments (1) are edges of K. In this case, as easily seen, any point $y \in \operatorname{bd} K$ not in

$$M \cup [x_1, y_3] \cup [x_2, y_4] \cup [x_3, y_1] \cup [x_4, y_2]$$

illuminates at least one of x_1, x_2, x_3, x_4. This implies (since x_1, x_2, x_3, x_4 are simply illuminated by y_1, y_2, y_3, y_4, respectively) that any point $y \in F \setminus \{y_1, y_2, y_3, y_4\}$, if exists, lies in M. By Lemma 2, M contains at most four points in F, and hence $\operatorname{card} F \leqq 8$.

Now consider the case $\operatorname{card} F = 8$. By the above, $\operatorname{card} F = 8$ implies that the points $y_5, y_6, y_7, y_8 \in F$ are the vertices of M, i.e.

$$\{x_1, x_2, x_3, x_4\} = \{y_5, y_6, y_7, y_8\}.$$

By Lemma 4, K is a convex polytope combinatorially equivalent to the 3-cube. □

LEMMA 9. *Let $N \subset E^3$ be a convex body and X be a set of at least seven points in* bd N *such that no four of them lie in a face of N. Then there are two points $x, y \in X$ such that $]x, y[\subset \operatorname{int} N$.*

PROOF. Proof of the lemma is based on the following result by Zamfirescu [5]: if a convex body $S \subset E^3$ is neither a bounded cone nor a convex polytope combinatorially equivalent to the triangular prism, then there are two distinct extreme points x, y of S such that $]x, y[\subset \operatorname{int} S$.

Choose in X any subset Y of seven points and put $P = \operatorname{conv} Y$. If P contains a pair x, y of vertices with $]x, y[\subset \operatorname{int} P$, then $x, y \in X$ and $]x, y[\subset \operatorname{int} N$. Assume that P has no such pair of vertices. Since P is not combinatorially equivalent to the triangular prism, it must be a bounded cone. By the hypothesis, the base of P intersects int N. Then any diagonal $[x, y]$ of this base satisfies the inclusion $]x, y[\subset \operatorname{int} N$. □

The following lemma, together with Lemmas 4 and 8, gives a final point in the proof of Theorem 2.

LEMMA 10. *If F has at least eight points, then there is a face of K containing four points in F or four points in G corresponding to distinct points in F.*

PROOF. Assume, in order to obtain a contradiction, that no face of K contains four points in F or four points in G corresponding to distinct points in F. Due to Lemmas 3 and 7, each point in $F \cup G$ is extreme for K. By Lemma 9, there is a pair of points $y_1, y_2 \in G$ such that $]y_1, y_2[\subset \operatorname{int} K$. We will consider each of the following cases: 1) both y_1, y_2 belong to F, 2) $y_1 \in F$ and $y_2 \notin F$, 3) none of y_1, y_2 belongs to F.

1) Let $y_1, y_2 \in F$. Due to the hypothesis $\operatorname{card} F \geqq 8$, there are some other six points, say $x_3, \ldots, x_8$, in F. Since y_1, y_2 illuminate each other, none of $x_3, \ldots, x_8$ illuminates any of y_1, y_2. In other words, each of $x_3, \ldots, x_8$ is connected with both y_1, y_2 by line segments lying in bd K. Let y_3 be a point in bd K simply illuminated by x_3.

Assume first that $y_3 \in F$. Without loss of generality, one can put $y_3 = x_8$. Then each of x_4, x_5, x_6, x_7 is connected with y_3 by line segments lying in bd K. Suppose that $x_4, \ldots, x_8$ are enumerated in correspondence with a bypass of the surface of K around the line $\langle y_1, y_2\rangle$. (Since each of $x_3, \dots, x_8$ is connected with both y_1, y_2 by line segments lying in $\operatorname{bd} K$, no four points of the form x_i, x_j, y_1, y_2 lie in a common half-plane with boundary line $\langle y_1, y_2\rangle$.) In this case y_3 lies in the open part of $\operatorname{bd} K$ bounded by the segments $[x_4, y_1]$, $[x_4, y_2]$, $[x_7, y_1]$, $[x_7, y_2]$ such that x_5, x_6 lie outside it. Since the segment $[x_5, y_3]$ lies in $\operatorname{bd} K$, it intersects the simple closed polygonial curve $y_1 x_4 y_2 x_7 y_1$. We know that all of x_4, x_7, y_1, y_2 are extreme points for K. Hence $[x_5, y_3]$ intersects one of the open intervals $]x_4, y_1[$, $]x_4, y_2[$, $]x_7, y_1[$, $]x_7, y_2[$. Let $[x_5, y_3]$ intersect e.g. $]x_4, y_1[$. Then x_4, x_5, y_1, y_3 are four points in F lying in a common face of K, which is impossible by the hypothesis.

Hence $y_3 \notin F$. As above, we suppose that y_3 lies in the open part of $\operatorname{bd} K$ bounded by the segments $[x_4, y_1]$, $[x_4, y_2]$, $[x_8, y_1]$, $[x_8, y_2]$ and not containing x_5, x_6, x_7, where $x_4, \dots, x_8$ are enumerated according to a bypass of $\operatorname{bd} K$ around the line $\langle y_1, y_2\rangle$. Then x_6 and y_3 lie in different open parts of $\operatorname{bd} K$ determined by the closed simple polygonal curve $\Gamma = y_1 x_5 y_2 x_7 y_1$.

Since y_3 is simply illuminated by x_3, it is connected with each of $x_4, \dots, x_8$ by line segments lying in $\operatorname{bd} K$. As above, $]x_6, y_3[$ intersects Γ and contains none of x_4, x_5, x_7, x_8. Hence $[x_6, y_3]$ intersects one of the open line intervals $]x_5, y_1[$, $]x_5, y_2[$, $]x_7, y_1[$, $]x_7, y_2[$. If $[x_6, y_3]$ intersected $]x_5, y_1[$, then it would intersect one of $]x_4, y_1[$, $]x_4, y_2[$, and four points x_4, x_5, x_6, y_1 or four points x_4, x_5, x_6, y_2 of F would lie in a common face of K, contradicting the hypothesis. Similarly, $[x_6, y_3]$ cannot intersect any of $]x_5, y_2[$, $[x_7, y_1[$, $]x_7, y_2[$. The obtained contradiction shows that the case $y_1, y_2 \in F$ is impossible.

2) $y_1 \in F$ and $y_2 \notin F$. Then there are some other points $x_2, \dots, x_8 \in \in F$. Since y_2 is illuminated by y_1, it is simply illuminated by y_1. Without loss of generality, one can assume that x_8 simply illuminates y_1. Denote by y_2' a point in $\operatorname{bd} K$ simply illuminated by x_2. Under these conditions, each of x_3, x_4, x_5, x_6, x_7 is connected with both y_1, y_2 by line segments lying in $\operatorname{bd} K$, and x_8 is connected with both y_2, y_2' by line segments in $\operatorname{bd} K$. Assume also that $x_3, \dots, x_7$ are enumerated in correspondence with a bypass of the surface of K around the line $\langle y_1, y_2\rangle$.

Suppose first that $y_2' \in F$. Since x_8 illuminates y_1, one has $y_2' \neq x_8$ (otherwise y_1 would illuminate y_2', which is impossible due to the assumption that x_2 illuminates y_2' simply). Also $y_2' \neq y_1$, x_2. Hence $y_2' \in \{x_3, \dots, x_7\}$. Let, for example, $y_2' = x_7$. Then y_2' belongs to the open part of $\operatorname{bd} K$ bounded by the segments $[x_3, y_1]$, $[x_3, y_2]$, $[x_6, y_1]$, $[x_6, y_2]$ such that x_4, x_5 lie outside it. Since no face of K contains four points in F, none of the open intervals $]x_3, y_1[$, $]x_6, y_1[$ intersects at least one of the segments $[x_4, y_2']$, $[x_5, y_2']$, and none of $]x_3, y_2[$, $]x_6, y_2[$ intersects both $[x_4, y_2']$, $[x_5, y_2']$. Let, for example, $[x_4, y_2']$ intersect $]x_3, y_2[$ and $[x_5, y_2']$ intersect $]x_6, y_2[$. Then x_3, x_4, y_2, y_2', and x_5, x_6, y_2, y_2' lie in common faces of K, respectively.

Now consider the point x_8. Since no face of K contains four points in F, none of the faces

$$M_1 = K \cap \operatorname{aff}(x_3, x_4, y_2, y_2'), \quad M_2 = K \cap \operatorname{aff}(x_5, x_6, y_2, y_2')$$

contains any of x_8, y_1. By the same arguments, the segment $[x_8, y_2']$ (which lies in $\operatorname{bd} K$) cannot intersect at least one of the open intervals $]x_3, y_1[$, $]x_4, y_1[$, $]x_5, y_1[$, $]x_6, y_1[$. Therefore the open interval $]x_8, y_2'[$ is disjoint to the simple closed polygonal curves $y_1 x_3 y_2' x_6 y_1$, $y_1 x_4 y_2' x_5 y_1$, lying in $\operatorname{bd} K$. Similarly, the open interval $]y_2, y_2'[$ is disjoint to the same polygonal curves. Moreover, the path $[x_8, y_2'] \cup [y_2', y_2]$ crosses each of these polygonal curves at y_2'. Hence x_8 and y_2 lie in different open parts of $\operatorname{bd} K$ bounded by any of these polygons. If $[x_8, y_2] \subset \operatorname{bd} K$, then $]x_8, y_2[$ intersects some open side of any of these polygons, and thus x_8 and the end points of these sides are at least four points of F lying in a face of K. If $]x_8, y_2[\subset \operatorname{int} K$, we get a contradiction, since y_1 simply illuminates y_2.

Hence $y_2' \notin F$. As above, we may suppose that y_2' lies in the open part of $\operatorname{bd} K$ bounded by the segments $[x_3, y_1]$, $[x_3, y_2]$, $[x_7, y_1]$, $[x_7, y_2]$ and not containing x_4, x_5, x_6, where $x_3, \dots, x_7$ are enumerated in correspondence with a bypass of $\operatorname{bd} K$ around the line $\langle y_1, y_2 \rangle$. Similarly to the above, $[x_4, y_2']$ intersects none of $]x_7, y_1[$, $]x_7, y_2[$, and $[x_6, y_2']$ intersects none of $]x_3, y_1[$, $]x_3, y_2[$ (otherwise either x_4, x_5, x_6, x_7 or x_3, x_4, x_5, x_6 would lie in a face of K). Hence $[x_4, y_2']$ intersects one of $]x_3, y_1[$, $]x_3, y_2[$, and $[x_6, y_2']$ intersects one of $]x_7, y_1[$, $]x_7, y_2[$. Since $[x_5, y_2']$ lies in $\operatorname{bd} K$, and since x_5, y_2' belong to distinct open parts of $\operatorname{bd} K$ determined by the closed polygonal curve $y_1 x_4 y_2 x_6 y_1$, the segment $[x_5, y_2']$ intersects one of the intervals $]x_4, y_1[$, $]x_4, y_2[$, $]x_6, y_1[$, $]x_6, y_2[$.

We observe that $[x_5, y_2']$ cannot intersect $]x_4, y_1[\cup]x_4, y_2[$ if $[x_4, y_2']$ has a common point with $]x_3, y_1[$. Indeed, if $[x_5, y_2']$ intersected either $]x_4, y_1[$ or $]x_4, y_2[$ (and thus intersected one of the intervals $]x_3, y_1[$, $]x_3, y_2[$), then four points x_3, x_4, x_5, y_1 in F would lie in a common face of K. By the same arguments, $[x_5, y_2']$ cannot intersect $]x_4, y_1[$ (and hence cannot intersect $]x_3, y_1[$) if $[x_4, y_2']$ intersects $]x_3, y_2[$. Similarly, $[x_5, y_2']$ does not intersect $]x_6, y_1[\cup x_6, y_2[$ if $[x_6, y_2']$ has a common point with $]x_7, y_1[$, and $[x_5, y_2']$ does not intersect $]x_6, y_1[$ if $[x_6, y_2']$ intersects $]x_7, y_2[$.

Summing up, one has (up to symmetry) two possible cases: a) $[x_4, y_2']$ intersects $]x_3, y_2[$, $[x_5, y_2']$ intersects $]x_4, y_2[$, and $[x_6, y_2']$ intersects $]x_7, y_1[$; b) $[x_4, y_2']$ intersects $]x_3, y_2[$, $[x_5, y_2']$ intersects $]x_4, y_2[$, and $[x_6, y_2']$ intersects $]x_7, y_2[$. Consider each of these cases separately.

a) Let y_6 be a point in $\operatorname{bd} K$ simply illuminated by x_6. We claim that there is no suitable position for y_6 in $\operatorname{bd} K$. Indeed, y_6 cannot belong to the face $N_1 = K \cap \operatorname{aff}(x, x_7, y_1, y_2')$, since otherwise $[x_6, y_6] \subset \operatorname{bd} K$. Similarly, $[x_7, y_6]$ cannot intersect $]x_6, y_2[$. If y_6 belonged to the face $N_2 = K \cap \cap \operatorname{aff}(x_3, x_4, x_5, y_2, y_2')$, then as easily seen, $[y_1, y_6]$ would intersect $]x_6, y_2[$. Hence y_6 may belong either to the open part P_1 of $\operatorname{bd} K$ bounded by the line segments $[x_6, y_2]$, $[x_7, y_2]$ and the arc x_6x_7 of rbd N_1 which does not contain y_1, or to the open part P_2 of bd K bounded by the line segments $[x_7, y_2]$, $[y_2, y_2']$ and the arc x_7y_2' on rbd N_2 which does not contain y_1. In each case y_6 is illuminated by y_1, which is impossible by the hypothesis.

b) Let y_6, y_7 be points in bd K simply illuminated by x_6, x_7, respectively. As in a), y_6 can lie only in the open part Q of bd K bounded by $[x_3, y_1]$, $[x_4, y_1]$ and the arc x_3x_4 of rbd N_2 which does not contain y_2. Similarly, y_7 can belong only to the open part of bd K bounded by $[x_4, y_1]$, $[x_5, y_1]$ and the arc x_4x_5 of rbd N_2 which does not contain y_2. Since y_7 is not illuminated by x_3, one has $[x_3, y_7] \subset$ bd K and hence $]x_3, y_7[\cap]x_4, y_1[\neq \emptyset$. Therefore x_3, x_4, y_1, y_7 lie in a common face of K, and Q coincides with the interior of the plane triangle $\Delta(x_3, x_4, y_1) \subset$ bd K. The last is impossible since the point $y_6 \in Q$ is extreme for K (see Lemma 3).

3) None of y_1, y_2 belongs to F. Let x_1, x_2 illuminate y_1, y_2, respectively, and let y_3 be a point in G simply illuminated by x_3.

Assume first that $y_3 \in F$. Without loss of generality, one can put $y_3 = x_8$. Then each of x_4, x_5, x_6, x_7 is connected with y_3 by line segments lying in bd K. Suppose that $x_4, \dots, x_8$ are enumerated in correspondence with a bypass of the surface of K around the line $\langle y_1, y_2 \rangle$. In this case y_3 lies in the open part of bd K bounded by the segments $[x_4, y_1]$, $[x_4, y_2]$, $[x_7, y_1]$, $[x_7, y_2]$, which contains none of x_5, x_6. By considerations similar to the above ones, both segments $[x_5, y_3]$, $[x_6, y_3]$ cannot intersect one of the intervals $]x_4, y_1[$, $]x_4, y_2[$, $]x_7, y_1[$, $]x_7, y_2[$, and cannot intersect either each of $]x_4, y_1[$, $]x_4, y_2[$ or each of $]x_7, y_1[$, $]x_7, y_2[$. (If, for instance, $[x_5, y_3]$ intersected $]x_4, y_1[$ and $[x_6, y_6]$ intersected $]x_4, y_2[$, then either x_5 would illuminate y_2 or x_6 would illuminate y_1.) We can consider (up to symmetry) that $[x_5, y_3]$ intersects e.g. $]x_4, y_2[$, and $[x_6, y_3]$ intersects one of $]x_7, y_1[$, $]x_7, y_2[$.

a) Let $[x_6, y_3]$ intersect $]x_7, y_2[$. Consider the point x_1. It cannot belong to any of the faces

$$L_1 = K \cap \operatorname{aff}(x_4, x_5, y_2, y_3), \quad L_2 = K \cap \operatorname{aff}(x_6, x_7, y_2, y_3),$$

because otherwise there would be four points in F lying in a common face of K. If $[x_1, y_3]$ intersected $]x_4, y_1[$, then $[x_1, y_2]$ would intersect $[x_5, y_1]$ (due to the inclusion $[x_1, y_2] \subset$ bd K), and x_1, x_5, y_1, y_2 would lie in a common face of K, which is impossible by $]y_1, y_2[\subset$ int K. Similarly, $[x_1, y_3]$ cannot intersect $]x_7, y_1[$. Thus x_1 lies in the open part of bd K bounded by the segments $[x_4, y_1]$, $[x_7, y_1]$ and the arcs x_4x_8, x_7x_8 of rbd L_1, rbd L_2, respectively, both disjoint to y_2. But in this case x_1 illuminates y_2, which is impossible by the choice of y_2.

b) Let $[x_6, y_3]$ intersect $]x_7, y_1[$. Similarly to a), x_1 does not belong to any of the faces

$$R_1 = K \cap \operatorname{aff}(x_4, x_5, y_2, y_3), \quad R_2 = K \cap \operatorname{aff}(x_6, x_7, y_1, y_3),$$

and $[x_1, y_3]$ cannot intersect one of the intervals $]x_4, y_1[$, $]x_7, y_2[$. If x_1 belonged to the open part of $\operatorname{bd} K$ bounded by the polygonal curve $y_1x_4y_3y_1$ and not containing y_2, then x_1 would illuminate y_2. Hence x_1 belongs to the open part H_1 of $\operatorname{bd} K$ bounded by the polygonal curve $y_2y_3x_7y_2$ and not containing y_1.

By the same arguments, x_2 belongs to the open part H_2 of $\operatorname{bd} K$ bounded by the polygonal curve $y_1x_4y_3y_1$ and not containing y_2. Therefore both H_1 and H_2 are not open plane triangles lying in $\operatorname{bd} K$.

Let y_5 be a point in $\operatorname{bd} K$ simply illuminated by x_5. If y_5 belonged to one of the faces R_1, R_2, then y_5 would be illuminated by both x_1, x_2. Since $[y_3, y_5]$ $(= [x_8, y_5])$ lies in $\operatorname{bd} K$, and since H_1 and H_2 are not planar triangular regions in $\operatorname{bd} K$, $[y_3, y_5]$ cannot intersect any of the intervals $]x_4, y_1[$, $]x_7, y_2[$. Let, for example, $y_5 \in H_1$. The last is possible only if $[x_2, y_5] \subset \subset \operatorname{bd} K$, because x_2 does not illuminate y_5. But in this case (since each of x_2, y_5 does not belong to one of the faces R_1, R_2), $[x_2, y_5]$ passes through y_3, which contradicts the inclusion $y_3 \in \operatorname{ext} K$.

Hence $y_3 \notin F$. As above, we suppose that the points $x_4, \dots, x_8$ are enumerated in correspondence with a bypass of $\operatorname{bd} K$ around the line $\langle y_1, y_2 \rangle$, and let y_3 be in the open part of $\operatorname{bd} K$ bounded by the segments $[x_4, y_1]$, $[x_4, y_2]$, $[x_8, y_1]$, $[x_8, y_2]$, such that x_5, x_6, x_7 lie outside it. By considerations similar to the above ones, we conclude that each of the segments $[x_5, y_3]$, $[x_6, y_3]$, $[x_7, y_3]$ intersects one of the open intervals $]x_4, y_1[$, $]x_4, y_2[$, $]x_8, y_1[$, $]x_8, y_2[$. Moreover, $[x_6, y_3]$ cannot intersect $]x_5, y_i[$ if $[x_5, y_3]$ intersects $]x_4, y_{3-i}[$, where $i = 1, 2$. (If, for example, $[x_5, y_3]$ intersected $]x_4, y_1[$ and $[x_6, y_3]$ intersected $]x_5, y_2[$, then x_4, x_5, x_6, y_1, y_2, y_3 would lie in a common face of K, which is impossible because of the inclusion $]y_1, y_2[\subset \operatorname{int} K$.) Similarly, $[x_6, y_3]$ cannot intersect $]x_7, y_i[$ if $[x_7, y_3]$ intersects $]x_8, y_{3-i}[$, where $i = 1, 2$.

Due to these arguments, we can conclude that up to symmetry there are two possibilities for positions of the segments $[x_5, y_3]$, $[x_6, y_3]$, $[x_7, y_3]$: two of them, say $[x_5, y_3]$, $[x_6, y_3]$, intersect the interval $]x_4, y_2[$ such that $[x_6, y_3]$ intersects $]x_5, y_2[$, and $[x_7, y_3]$ intersects either $]x_8, y_1[$ or $]x_8, y_2[$.

c) Let $[x_7, y_3]$ intersect $]x_8, y_1[$. We claim that a point y_7 simply illuminated by x_7 belongs to the open part N of $\operatorname{bd} K$ bounded by the polygonal curve $y_2y_3x_8y_2$, not containing y_1 and lies in the face $T_1 = = K \cap \operatorname{aff}(x_4, x_5, y_2, y_3)$. Indeed, if y_7 belonged to the face $T_2 = K \cap \cap \operatorname{aff}(x_7, x_8, y_1, y_3)$, then $[x_7, y_7] \subset \operatorname{bd} K$, which is impossible. If y_7 belonged to the open part of $\operatorname{bd} K$ bounded by the polygonal curve $y_1y_3x_4y_1$ and not containing y_2, then, due to $[x_8, y_7] \subset \operatorname{bd} K$, y_7 would belong to the face T_2, which is impossible by the above. Similarly, if y_7 belonged to the open part of $\operatorname{bd} K$ bounded by the line segments $[x_4, y_1]$, $[x_8, y_2]$ and the boundary arcs $x_4x_5x_6y_2$ and $y_1x_7x_8$ of the faces T_2 and T_1, respectively, and not containing y_3, then due to the inclusion $[x_5, y_7] \cup [x_8, y_7] \subset \operatorname{bd} K$, we would obtain $[x_7, y_7] \subset \operatorname{bd} K$. Hence $y_7 \in N$. Since $[x_5, y_7] \subset \operatorname{bd} K$, one has $y_7 \in T_1$.

Now consider the point x_1. We claim that $x_1 \in N$. Since T_1 still contains three points in F, one has $x_1 \notin T_1$. If $[x_1, y_3]$ intersected $]x_4, y_1[$, then x_1, x_4, y_1, y_2 would lie in a common face of K, which is impossible by $]x_1, y_1[\subset \operatorname{int} K$. If x_1 belonged to the face T_2, then $[x_1, y_1] \subset \operatorname{bd} K$. If x_1 belonged to the open part of bd K bounded by the polygonal curve $y_1y_3x_4y_1$ and not containing y_2, then x_1 would illuminate y_2. If $[x_1, y_3]$ intersected $]x_8, y_2[$, then x_1, x_8, y_2, y_3 would lie in a common face of K and N would be a plane open triangle, which is impossible, since y_7 is an extreme point of K lying in N. Hence $x_1 \in N$.

It remains to determine the position of y_8 in bd K. Since $[x_1, y_8]$ lies in bd K and cannot intersect any of the intervals $]x_8, y_2[$, $]x_8, y_3[$ (otherwise $[x_8, y_8] \subset \operatorname{bd} K$), and since x_1 cannot lie in T_1, y_8 belongs to N. Since $[x_5, y_8]$ lies in bd K, y_8 belongs to the face T_1. But in this case T_1 contains four points y_2, y_3, y_7, y_8 in G, which is impossible by the assumption.

d) Let $[x_7, y_3]$ intersect $]x_8, y_2[$. Consider the points x_1, x_2. Since the face $V_1 = K \cap \operatorname{aff}(x_4, x_5, x_6, y_2, y_3)$ contains three points in F, none of x_1, x_2 belongs to V_1. Similarly, the face $V_2 = K \cap \operatorname{aff}(x_7, x_8, y_2, y_3)$ contains at most one of x_1, x_2. Let, for example, $x_1 \notin V_2$. If $[x_1, y_3]$ intersected one of the intervals $]x_4, y_1[$, $]x_8, y_1[$, then either x_1, x_7, y_1, y_2 or x_1, x_6, y_1, y_2 would lie in a common face of K, which is impossible by $]y_1, y_2[\subset \operatorname{int} K$. Hence x_1 belongs to the open part of bd K bounded by the segments $[x_4, y_1]$, $[x_8, y_1]$, and by the boundary arcs x_4y_3, x_8y_3 of the faces V_1, V_2, respectively, and not containing y_2. But in this case x_1 illuminates y_2, a contradiction. □

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MINIMAX OF THE ANGLES IN A PLANE CONFIGURATION OF POINTS

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1. Introduction

More than half a century ago L. M. Blumenthal [1] has formulated the problem of finding the largest number $\alpha(N)$ such that any plane configuration of N points contains three points determining an angle $\beta \geqq \alpha(N)$; $0 \leqq \beta \leqq \pi$.

It is easy to see that:

$$\alpha(3) = \frac{1}{3}\pi, \quad \alpha(4) = \frac{1}{2}\pi, \quad \alpha(5) = \frac{3}{5}\pi, \quad \alpha(6) = \alpha(7) = \alpha(8) = \frac{2}{3}\pi.$$

In [8] it is proved that:

$$\alpha(9) = \alpha(10) = \frac{5}{7}\pi, \quad \alpha(11) = \alpha(12) = \ldots = \alpha(16) = \frac{3}{4}\pi.$$

In this paper we shall prove that:

$$\alpha(N) = \left\{1 - \frac{1}{2n+1}\right\}\pi \quad \text{for} \quad 2^n < N \leqq 2^n + 2^{n-2}, \tag{1.1}$$

and

$$\alpha(N) = \left\{1 - \frac{1}{n+1}\right\}\pi \quad \text{for} \quad 2^n + 2^{n-2} < N \leqq 2^{n+1}. \tag{1.2}$$

In G. Szekeres [9] and in P. Erdős and G. Szekeres [4] it is proved that

$$\alpha(2^n) = (1 - 1/n)\pi$$

and that

$$\alpha(2^n + 1) > (1 - 1/n)\pi.$$

P. Erdős and G. Szekeres have conjectured first in [4] that (1.2) is true for $2^n < N \leqq 2^{n+1}$. After I recently conjectured (1.1) and (1.2), P. Erdős asked me in a personal letter if the different values of $\alpha(N)$ for $2^{n-1} < N \leqq 2^n$ were at all limited in number.

To prove (1.1) and (1.2), we introduce the so called *generalized plane configurations of points.* These generalized configurations have extreme elements for which $\alpha(N)$ is achieved. This is not valid for the ordinary plane configurations of points for more than 6 points.

The technics of the generalized plane configurations of points may be successfully used also in three and more dimensional spaces.

Let $\alpha_m(N)$ be the largest number such that in any configuration of N points in the m-dimensional Euclidean space there are three points determining an angle $\beta \geqq \alpha_m(N)$; $0 \leqq \beta \leqq \pi$.

Until now very little is known for the exact values of $\alpha_m(N)$. It is trivial that

$$\alpha_m(m+1) = \pi/3.$$

Following a conjecture of P. Erdős and Szekeres [4], L. Danzer and B. Grünbaum [3] proved that

$$\alpha_m(2^m) = \pi/2.$$

The problem of determining the values of $\alpha_m(N)$ for $m > 2$ is difficult even for small N. For example, it is possible to calculate directly that

$$\pi/3 < \alpha_3(5) = \arccos\frac{1}{7} < \pi/2.$$

One has

$$\alpha_3(6) = \alpha_3(7) = \pi/2$$

by H. T. Croft [2]; a simpler proof is in K. Schütte [6]. B. Grünbaum [5] proved a more general statement. Namely, as observed by Danzer–Grünbaum [3], a set determining only acute angles is strictly antipodal, and Grünbaum [5] showed that in R^3 a strictly antipodal set has at most 5 elements. ($X \subset \subset R^d$ is strictly antipodal if for $x \neq y \in X$ the convex hull of X has two different parallel supporting hyperplanes, one intersecting X in x, the other one in y.)

2. Generalized plane configurations of points

We shall consider sets of finite number of points on the plane in general position (no three points are collinear). To emphasize this, we shall use the notion *plane configuration of points.*

DEFINITION 2.1. Let $C = \{c_1(o_1, r_1), c_2(o_2, r_2), \ldots, c_M(o_M, r_M)\}$ be a set of circles on the plane with centers o_i and radii r_i. C is a *semi ordered* set if the circles of C have disjoint circumferences and some circles may be inside another circles. A circle from C is of range 1 if it is not inside any other circle. A circle from C is of range k if the highest range of the circle in which it is contained is $k-1$. A circle which does not contain any other circle is called a *primitive circle*. A semi ordered set of circles in which no three centers are collinear is called a *plane configuration of circles.*

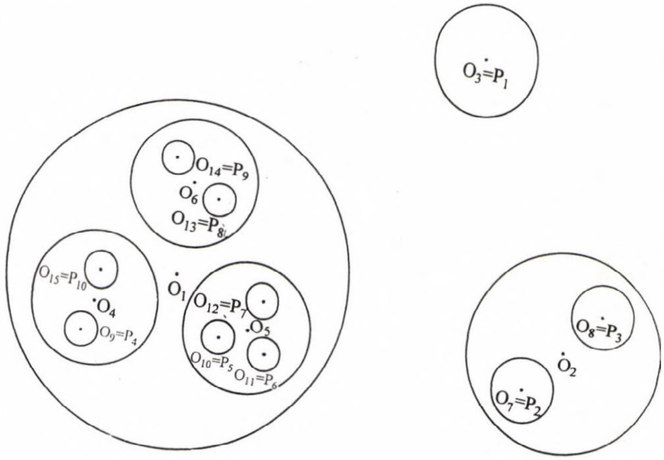

Fig. 1

In Fig. 1, a semi-ordered set of circles is given. The circles with the centers o_1, o_2, o_3 are of range 1. The circles with the centers o_4, o_5, o_6, o_7, o_8 are of range 2 and these with centers $o_3, o_7, o_8, o_9, \dots, o_{15}$ are primitive.

DEFINITION 2.2. A set $V = \{P, C\} = \{p_1, p_2, \dots, p_N;\ c_1, c_2, \dots, c_M\}$ of a plane configuration of $N = |P|$ points and a plane configuration of $M = |C| \geqq N$ circles is called a *generalized plane configuration of points*, or shortly GC, if:

a) Every point of P coincides with the center of a primitive circle of C. A point is of range k if it is a center of a primitive circle of range k.

b) The circles of range 1 in P are two or more.

c) Every non primitive circle of C contains two or more circles.

It is obvious that the content of every non primitive circle from a GC (the points from P and the circles from C inside this circle) is also a GC.

In Fig. 1 a GC with 10 points and 15 circles is represented. The content of the circle with center o_1 on Fig. 1 is a GC with 7 points and 10 circles.

We may consider every plane configuration of $N \geqq 2$ points as a GC with an equal number of points and circles, with all circles being primitive.

DEFINITION 2.3. Let p and q be two points in the GC $V = \{P, C\}$, $o(p; q)$ be the center of the circle of the lowest range containing p and not containing q, and $o(q; p)$ be the center of the circle of the lowest range containing q and not containing p. The *generalized direction* $\mathrm{GD}(pq)$ from p to q is defined as the direction of the vector $o(p;q)o(q;p)$ or

$$\mathrm{GD}(pq) = G(pq) = D\big(o(p;q)o(q;p)\big).$$

In Fig. 1 $o(p_4;p_2) = o_1$, $o(p_2;p_4) = o_2$ and $\mathrm{GD}(p_2p_4) = D(o_2o_1)$.

DEFINITION 2.4. Let $V = \{P, C\}$ and $V' = \{P', C'\}$ be two GC's with equal number of points. We call V and V' *equivalent*, $V \approx V'$ if there exists a correspondence $p_i \approx p_i'$ between the points P and P' such that for every i, j the directions $G(p_i, p_j)$ and $G(p_i', p_j')$ coincide.

For two equivalent GC's V and V' we shall use notation $p_i \approx p_i'$; $i = 1, 2, \dots, N$.

DEFINITION 2.5. Every three points p, q, r, taken in this order, define an angle $A(p,q,r) < \pi$. We define the *generalized angle* $\mathrm{GA}(p,q,r)$ between these points in a GC as the angle $\in [0, \pi)$ between the two directions $G(qp)$ and $G(qr)$. Every two lines l and l^* define an angle $0 < A(l, l^*) < \pi$, measured in positive direction from l to l^*.

In Fig. 1 $\mathrm{GA}(p_2, p_4, p_8) = A(o_1o_2, o_4o_6)$.

DEFINITION 2.6. $A(P)$ is the maximal angle in a plane configuration of points P. $\mathrm{GA}(V) = \mathrm{GA}\big(\{P, C\}\big)$ is the maximal generalized angle in a generalized plane configuration of points $V = \{P, C\}$.

The following lemma is obvious.

LEMMA 2.1. *If*

$$V = \{P, C\} \approx V' = \{P', C'\}, \quad p, q, r \in C \quad \textit{and} \quad p', q', r' \in C'$$

are the corresponding points, then

$$\mathrm{GA}(p, q, r) = \mathrm{GA}(p', q', r')$$

and

$$\mathrm{GA}\big(\{P, C\}\big) = \mathrm{GA}\big(\{P'C'\}\big).$$

DEFINITION 2.7. Let $d_k\big(\{P, C\}\big)$ be the smallest distance between two centers of the circles of range k in the GC $V = \{P, C\}$. The GC V is *h-normal*, $0 < h < \frac{1}{2}$, if $d_1\big(\{P, C\}\big) = 1$ and all circles in C of range k have radii $r(k) = hd_k\big(\{P, C\}\big)$.

LEMMA 2.2. *Let* $V = \{P, C\}$ *be a* GC. *For every* $0 < h < \frac{1}{2}$ *there exists an h-normal* GC $V(h)$ *equivalent to* V.

PROOF. It is obvious that for every three numbers $t > 0$, a, b, the transformation of the plane

$$\xi = tx + a, \quad \eta = ty + b,$$

transforms a GC into an equivalent GC. Let

$$\{P^{(1)}, C^{(1)}\} = \{P, C\} \quad \text{and} \quad d_1(\{P^{(1)}, C^{(1)}\}) = 1.$$

In the first step we take a circle $c_i(o_i, r_i)$ of range 1 from $C^{(1)}$ and apply to the content of this circle the transformation

$$\xi = h(x - x_i)/r_i + x_i, \quad \eta = h(y - y_i)/r_i + y_i,$$

where $o_i = (x_i, y_i)$. This transformation does not change any generalized direction determined by the points of $P^{(1)}$. If we apply this transformation to all circles of range 1 in $C^{(1)}$, we shall construct a GC

$$\{P^{(2)}, C^{(2)}\} = \{P^{(1)}, C^{(1)}\} = \{P, C\}$$

such that the lemma is fulfilled for $k = 1$.

In step k we transform every circle $c_j(o_j, r_j)$ of range k from $C^{(k)}$ by the transformation

$$\xi = h_k(x - x_j)/r_j + x_j, \quad \eta = h_k(y - y_j)/r_j + y_j,$$

where $o_j(x_j, y_j)$, $h_k = hd_k(\{P^{(k)}, C^{(k)}\})$.

If m is the highest range of the circles in P, then $V(h) = \{P^{(m)}, C^{(m)}\}$ is h-normal and satisfies the conditions of the Lemma.

LEMMA 2.3. *If P is a plane configuration of points and $V = \{P, C\}$ is an h-normal* GC *with the same set of points P, then for every three points p, q, r from P the inequalities*

$$|A(p, q, r) - \mathrm{GA}(p, q, r)| < 2\pi h$$

and

$$\left|A(P) - \mathrm{GA}(\{P, C\})\right| < 2\pi h$$

hold.

PROOF. If the two points p, q are of range 1, then by definition the generalized direction $\mathrm{GD}(pq)$ coincides with the direction $D(pq)$. If p, q are of the same range k and are in the same circle of range $k-1$, again $\mathrm{GD}(pq) = D(pq)$. If p, q are of different ranges, or are of the same range k but are in different circles of range $k-1$, then the angle between $D(pq)$ and $\mathrm{GD}(pq)$ is not bigger than πh, which completes the proof.

DEFINITION 2.8.

$$\alpha(N) = \inf\{ A(P) : P = \{p_1, p_2, \dots, p_N\}\},$$

$$G\alpha(N) = \inf\Big\{\mathrm{GA}\big(\{P, C\}\big) : P = \big\{p_1, p_2, \dots, p_N\big\}\Big\}.$$

The second inf is over all possible choices of the points P and the circles C.

THEOREM 2.1. *For every natural number N, the equality*

$$G\alpha(N) = \alpha(N) \tag{2.3}$$

holds.

PROOF. In a GC with circles only of range 1, the generalized angles are equal to the respective ordinary angles. From this fact it follows that

$$G\alpha(N) \leqq \alpha(N). \tag{2.4}$$

To complete the proof of (2.3), it remains to prove that

$$G\alpha(N) \geqq \alpha(N). \tag{2.5}$$

Let us assume the contrary, that for a fixed natural number N, there exists a positive number $\delta > 0$ such that

$$G\alpha(N) < \alpha(N) - \delta. \tag{2.6}$$

From (2.6) it follows that there exists a GC $\{P^*, C^*\}$ such that

$$\mathrm{GA}\big(\{P^*, C^*\}\big) < \alpha(N) - \delta,$$

and consequently, for every plane configuration of points $P = \{p_1, p_2, \dots, p_N\}$ we have the inequality

$$\mathrm{GA}\big(\{P^*, C^*\}\big) < A(P) - \delta. \tag{2.7}$$

According to Lemma 2.2 and Lemma 2.3, there exists an h-normal GC $\{P^{**}, C^{**}\}$ equivalent to $\{P^*, C^*\}$ and such that

$$\mathrm{GA}\big(\{P^*, C^*\}\big) = \mathrm{GA}\big(\{P^{**}, C^{**}\}\big) > A(P) - 2\pi h. \tag{2.8}$$

But (2.8) contradicts (2.7) for small h and (2.4) is proved.

According to Theorem 2.1 we may calculate $\alpha(N)$ considering not only plane configurations of points but the larger set of the generalized plane configurations of points. The benefit of this is that in the set of GC's there exist extreme elements $V^* = \{P^*, C^*\}$ with $|P^*| = N$, such that $\mathrm{GA}(V^*) = G\alpha(N) = \alpha(N)$. This is proved in Section 5.

3. Perfect GC's

We will need a lemma of P. Erdős and G. Szekeres [4] for the partition of a complete graph.

Let $K^{(N)}$ be a complete graph of order N (a graph with N vertices in which any two vertices are joined by an edge). An even (odd) circuit of a graph G is a closed circuit containing an even (odd) number of edges.

Following [4], we call a *partition* of G any decomposition $G = G_1 + G_2 + \ldots + G_n$ into subgraphs G_i with the following property: Each G_i consists of all vertices and some edges of G such that each edge of G appears in one and only one G_i (G_i may not contain any edge at all). A partition is called *even*, if no G_i contains an odd circuit.

In [4] and [9], the following lemma is proved:

LEMMA 3.1. *If $K^{(N)} = G_1 + G_2 + \ldots + G_n$ is an even partition of the complete graph into n parts, then*

$$N \leqq 2^n.$$

PROOF. For the sake of completeness we repeat the proof. Since G_1 contains no odd circuit we can divide the vertices of $K^{(N)}$ in classes A and B, containing N_1 and N_2 vertices respectively, such that each edge of G_1 connects a point of A with a point of B. But then $G_1 + G_2 + \ldots + G_n$ induces an even partition $G_2'' + G_3'' + \ldots + G_n''$ of $K' = K^{(N)}|A$ and since K' is a complete graph of order N_1, we conclude by induction that $N_1 \leqq 2^{n-1}$. Similarly $N_2 \leqq 2^{n-1}$, hence $N = N_1 + N_2 \leqq 2^n$.

We shall use sometimes complex numbers to represent points in the Euclidean plane E. A direction θ; $0 \leqq \theta < 2\pi$, in E is a vector from $o(0,0)$ to $e^{i\theta}$ on the unit circle.

DEFINITION 3.1. Let θ be a direction and $0 \leqq \lambda < 2\pi$. The set of points

$$T(\theta,\lambda) = \{z\colon z = ae^{i\varphi},\ a \quad \text{real}, \quad \theta \leqq \varphi < \theta + \lambda\} \setminus o(0,0)$$

is called a *double sector* with base θ and angle λ. Every double sector consists of two sectors (connected components), called *parts* of the double sector. We say that a set Q of points on the plane *belongs* to the double sector T if every two points of Q determine a vector with direction in this sector.

The following lemma is obvious.

LEMMA 3.2. *If a set of points Q belongs to a double sector T with angle λ and for three points p, q, r from Q the direction $D(pq)$ and $D(qr)$ are inside one of the parts of T, then*

$$A(p,q,r) > \pi - \lambda.$$

Lemma 3.3. *Let $V = \{P, C\}$ be a* GC, $Q \subseteqq P$ *and every generalized direction determined by two points from Q be inside one of the disjoint double sectors $T(\theta_1, \lambda), T(\theta_2, \lambda), \dots, T(\theta_k, \lambda)$. If*

$$\text{(3.9)} \qquad \mathrm{GA}(Q) \leqq \pi - \lambda, \quad \textit{then} \quad |Q| \leqq 2^k.$$

Proof. Let $K^{(k)}$ be the complete graph with vertices Q and G_i; $i = 1, 2, \dots, k$ be the graph with vertices Q and two points are joint with an edge in G_i if they determine a generalized direction inside the double sector $T(\theta_i, \lambda)$.

We assert that the decomposition $K^{(k)} = G_1 + G_2 + \ldots + G_k$ is an even partition. In fact, if a graph G_i has an odd circuit, then there shall be two consecutive vectors in this cycle, belonging to one of the parts of the double sector $T(\theta_i, \lambda)$. But this, according to Lemma 3.2 contradicts the first inequality (3.9). Then, the second inequality (3.9) follows from Lemma 3.1.

Definition 3.2. A non-primitive circle of range k in a GC is called *perfect* if it contains exactly two circles of range $k+1$ and the contents of these two circles are equivalent as GC's.

Definition 3.3. A GC is called *perfect* if all its non-primitive circles are perfect.

Let $V = \{P, C\}$ be a GC. We shall represent V with centers $o_1, o_2, \dots, o_s$ of its circles of range 1 and with lines $l_{i,1}, l_{i,2}, \dots, l_{i,k(i)}$ passing through the centers o_i of the circles c_i; $i = 1, 2, \dots, s$ and parallel to the different generalized directions defined by the points of P inside c_i.

The number of different generalized directions defined by the points of P inside c_i is $k(i)$.

In Fig. 2 a perfect GC is represented with 4 centers and respective number of lines.

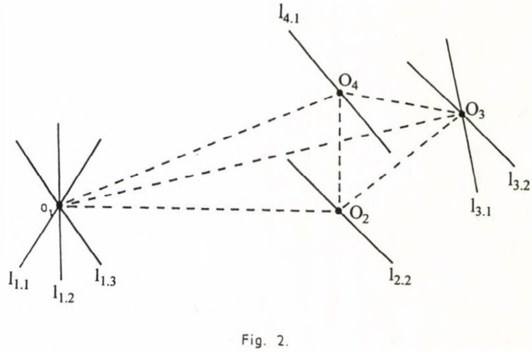

Fig. 2

LEMMA 3.4. *If*

$$V = \{P, C\} = \{o_1, o_2, \dots, o_s;\ l_{i,1}, l_{i,2}, \dots, l_{i,k(i)};\ i = 1, 2, \dots, s\}$$

is a perfect GC, *then the number of points* $N = |P|$ *of this* GC *is equal to*

$$N = 2^{k(1)} + 2^{k(2)} + \dots + 2^{k(s)}.$$

PROOF. If a circle of range 1 has one line, then there are exactly two points from V inside this circle. Let the lemma be true for $k-1$ and let a circle of range 1 have inside itself two circles of range 2 and every one of these two circles have $k-1$ lines and 2^{k-1} points inside itself. The centers of the circles of range 2 define one more line for the circle of range 1 in addition to the $k-1$ lines parallel to the lines of the circles of range 2. Hence, the circle of range 1 has k lines and 2^k points from V inside itself. The lemma is proved inductively.

LEMMA 3.5. *Let*

$$V = \{P, C\} = \{o_1, o_2, \dots, o_s;\ l_{i,1}, l_{i,2}, \dots, l_{i,k(i)};\ i = 1, 2, \dots, s\}$$

be a GC, $Q = \{o_1, o_2, \dots, o_s\}$ *be the set of centers of the circles of range* 1 *in* V, $\lambda = A(Q)$, $P_i \subset P$ *be the set of points inside the circle* c_i, $\mu_i = \mathrm{GA}(P_i)$, $\mu = \max\{\mu_1, \mu_2, \dots, \mu_s\}$, φ_i *be the smallest angle between a line from* $\{l_{i,j};\ j = 1, 2, \dots, k(i)\}$ *and a line from* $\{o_i o_j;\ i \neq j = 1, 2, \dots, s\}$, *and* $\varphi = \min\{\varphi_1, \varphi_2, \dots, \varphi_s\}$. *Then*

$$\mathrm{GA}(V) = \max\{\lambda, \mu, \pi - \varphi\}.$$

PROOF. Let p, q, r be three points from P, such that $\mathrm{GA}(p,q,r) = \mathrm{GA}(V)$. If these three points are in different circles of range 1, then $\mathrm{GA}(p,q,r) = \lambda$. If two of these three points are in one circle of range 1 and the third point is in another circle of range 1, then $\mathrm{GA}(p,q,r) = \pi - \varphi$. If the three points are in one circle of range 1, then $\mathrm{GA}(p,q,r) = \mu$. That completes the proof of the lemma.

LEMMA 3.6. *If* $l_1, l_2, \dots, l_k$ *are lines incident with the center* o^* *of the circle* c^* *and* $A(l_i, l_{i+1}) \geqq \lambda$; $i = 1, 2, \dots, k$; $l_{k+1} = l_1$, *then there exist* 2^k *points* Q *inside* c^* *and a* GC $V^* = \{Q, C\}$ *such that* $\mathrm{GA}(V^*) \leqq \pi - \lambda$.

PROOF. We may assume that the radius of c^* is equal to 1.

In the first step, let $o_{1,1}, o_{1,2}$ be the points on the line $l_1 = l_{1,1}$, such that $|o^* - o_{1,1}| = |o^* - o_{1,2}| = 2^{-1}$ and $c^*_{1,i} = c(o_{1,i}, 3^{-1})$; $i = 1, 2$.

In the second step, let $l_{2,i}$; $i = 1, 2$ be the lines incident respectively with the points $o_{1,i}$; $i = 1, 2$ and parallel to l_2. Let $o_{2,1}, o_{2,2}$ be the points on $l_{2,1}$

such that $|o_{1,1} - o_{2,1}| = |o_{1,1} - o_{2,2}| = 2^{-2}, o_{2,3}, o_{2,4}$ be the points on $l_{2,2}$ such that $|o_{1,2} - o_{2,3}| = |o_{1,2} - o_{2,4}| = 2^{-2}$ and $c^*_{2,i} = c(o_{2,i}, 3^{-2})$; $i = 1,2,3,4 = 2^2$.

In step k we shall produce 2^k points $p_i = o_{k,i}$; $i = 1,2,\dots,2^k$ and a semi ordered system of circles defining the GC V^* such that $\mathrm{GA}(V^*) \leqq \pi - \lambda$.

LEMMA 3.7. *Let*

$$V = \{P, C\} = \{o_1, o_2, \dots, o_s;\ l_{i,1}, l_{i,2}, \dots, l_{i,k(i)};\ i = 1,2,\dots,s\}$$

be a GC, N_i; $i = 1,2,\dots,s$ *be the number of points in the circle* $c_i = c(o_i, r_i)$ *of range* 1, $L_i = \{t_{i,1}, t_{i,2}, \dots, t_{i,s-1}\}$ *be the lines incident with* o_i *and another center* o_j; $j \neq i$, $\pi\varphi_{i,j} = A(t_{i,j}, t_{i,j+1})$; $j = 1,2,\dots,s-1$, $t_{i,s} = t_{i,1}$, $\varphi_{i,1} + \varphi_{i,2} + \dots + \varphi_{i,s-1} = 1$ *and* $\mathrm{GA}(V) = (1 - 2/u)\pi$. *Then*

$$N_i \leqq 2^{k'(i)}, \tag{3.10}$$

$$k'(i) = \big([u\varphi_{i,1}/2] - 1\big)_+ + \big([u\varphi_{i,2}/2] - 1\big)_+ + \dots + \big([u\varphi_{i,s-1}/2] - 1\big)_+.$$

PROOF. Let $\{\psi_{i,1}, \psi_{i,2}, \dots, \psi_{i,q}\} \subset \{\varphi_{i,1}, \varphi_{i,2}, \dots, \varphi_{i,s-1}\}$ be the subset of angles $\{\varphi_{i,l}\}$ for which the corresponding members of the sum (3.10) are not zero, or such that $\psi_{i,j} = \varphi_{i,l} \geqq 4/u$. We construct $[u\psi_{i,j}/2] - 1$ double sectors

$$T\big(\pi(\Psi_{i,j} + 2k/u); 2\pi/u\big);\ k = 1,2,\dots,[u\psi_{i,j}/2] - 1 \tag{3.11}$$

with angle $2\pi/u$, where $\Psi_{i,j} \in [0, \pi)$ is the angle of direction of that line $t_{i,m}$, for which $\psi_{i,j}$ was chosen as $\varphi_{i,m}$. According to Lemma 3.5, for every two points p, p' from P, inside the circle c_i, the direction $\mathrm{GD}(pp')$ is inside one of the sectors (3.11). Hence, from Lemma 3.3, there follows (3.10).

Let us consider the lines incident with the center of the circle c_i and parallel to the direction $\pi(\Psi_{i,j} + 2k/u)$; $j = 1,2,\dots,q$, $k = 1,2,\dots,[u\psi_{i,j}/2) - 1$. Let us rename them $l^*_{i,j}$, $1 \leqq j \leqq k'(i)$ with j increasing in the positive sense of rotation. Then, according to Lemma 3.6, the perfect GC $V^* = \{P^*, C^*\} = \{o_1, o_2, \dots, o_s;\ l^*_{i,1}, l^*_{i,2}, \dots, l^*_{i,k'(i)};\ i = 1,2,\dots,s\}$ is a GC with number of points not less than one in V and $\mathrm{GA}(V^*) = \mathrm{GA}(V)$. In this way we prove the following:

LEMMA 3.8. *For every* GC$V = \{P, C\}$ *there exists a perfect* GC

$$V^* = \{P^*, C^*\} = \{o_1, o_2, \dots, o_s;\ l^*_{i,1}, l^*_{i,2}, \dots, l^*_{i,k'(i)};\ i = 1,2,\dots,s\},$$

such that $|P^*| \geqq |P|$ *and* $\mathrm{GA}(V^*) = \mathrm{GA}(V)$.

Now we replace $k(i)$ in Lemma 3.4 with $k'(i)$ from Lemma 3.7.

Finally, from Lemma 3.8, we have:

THEOREM 3.1. *For every natural N, the equality*

$$G\alpha(N) = \alpha(N) = \inf\Big\{\mathrm{GA}\big(\{P,C\}\big) : |P| = N, \{P,C\} \text{ perfect}\Big\}$$

holds.

According to Theorem 3.1, to find $G\alpha(N) = \alpha(N)$ we have to consider only perfect GC's.

Every perfect GC

$$V = \{P,C\} = \{o_1, o_2, \dots, o_s;\ l_{i,1}, l_{i,2}, \dots, l_{i,k(i)};\ i = 1,2,\dots,s\}$$

shall be represented by the centers of the circles of range 1 and the respective lines.

4. Proof of the main result

LEMMA 4.1. *Let*

$$V = \{P,C\} = \{o_1, o_2, \dots, o_s;\ l_{i,1}, l_{i,2}, \dots, l_{i,k(i)};\ i = 1,2,\dots,s\}$$

be a perfect GC, $\mathrm{GA}(V) = (1 - 2/u)\pi$, $[u/2] = n$ *and* $\delta = u/2 - n$, *then*

$$|P| \leqq 2^n \quad \text{for} \quad 0 \leqq \delta < 1/2 \quad \text{or} \quad 2n \leqq u < 2n+1$$

and

$$|P| \leqq 2^n + 2^{n-2} \quad \text{for} \quad 1/2 \leqq \delta < 1 \quad \text{or} \quad 2n+1 \leqq u < 2n+2.$$

PROOF. According to Lemma 3.4 the number of points is

$$N = |P| = \sum_{i=1}^{s} 2^{k(i)}$$

and according to Lemma 3.7

$$k(i) = \big([u\varphi_{i,1}/2] - 1\big)_+ + \big([u\varphi_{i,2}/2] - 1\big)_+ + \dots + \big([u\varphi_{i,s-1}/2] - 1\big)_+.$$

We shall assume that the indexing is such one that

$$k(1) \geqq k(2) \geqq \dots \geqq k(s)$$

and we shall use induction on s.

a) For $s = 2$, $\varphi_{1,1} = \varphi_{2,1} = 1$, then

$$k(1) = k(2) = n - 1$$

and

$$|P| = 2^n.$$

b) For $s = 3$ the angles are

$$\varphi_{1,1},\ \varphi_{1,2} = 1 - \varphi_{1,1},\ \varphi_{2,1},\ \varphi_{2,2} = 1 - \varphi_{2,1},$$
$$\varphi_{3,1} = 1 - \varphi_{1,1} - \varphi_{2,1},\ \varphi_{3,2} = \varphi_{1,1} + \varphi_{2,1}$$

and

$$k(1) = \left([u\varphi_{1,1}/2] - 1\right)_+ + \Big(\big[u(1 - \varphi_{1,1})/2\big] - 1\Big)_+,$$
$$k(2) = \left([u\varphi_{2,1}/2] - 1\right)_+ + \Big(\big[u(1 - \varphi_{2,1})/2\big] - 1\Big)_+,$$
$$k(3) = \Big(\big[u(\varphi_{1,1} + \varphi_{2,1}/2\big] - 1\Big)_+ + \Big(\big[u(1 - \varphi_{1,1} - \varphi_{2,1}/2\big] - 1\Big)_+,$$

and

$$2/u \leqq \varphi_{1,1} + \varphi_{2,1} \leqq 1 - 2/u. \tag{4.12}$$

If

$$n - 2 \geqq k(1) \geqq k(2) \geqq k(3),$$

then

$$|P| \leqq 3 \cdot 2^{n-2} < 2^n.$$

Let $k(1) = n - 1$. This is possible only if

$$\varphi_{1,1} \leqq \frac{\delta}{n + \delta}. \tag{4.13}$$

We consider two subcases:

b.1) For $0 \leqq \delta < 1/2$ from (4.13) it follows that $\varphi_{1,1} = \frac{1}{2n+1}$. From (4.12) we have that

$$\varphi_{2,1} \geqq \frac{1}{n + \delta} - \frac{1}{2n + 1} > \frac{1}{2n + 1},$$

hence $n - 2 \geqq k(2) \geqq k(3)$ and

$$|P| \leqq 2^{n-1} + 2 \cdot 2^{n-2} = 2^n.$$

In this subcase the maximum is achieved if

$$k(1) = n - 1, \quad k(2) = k(3) = n - 2.$$

b.2) For $1/2 \leqq \delta < 1$, if $k(2) = n - 2$, then

$$|P| \leqq 2^{n-1} + 2 \cdot 2^{n-2} = 2^n.$$

Let $k(1) = k(2) = n - 1$, then according to (4.13)

$$\varphi_{1,1} < \frac{1}{n+1}, \quad \varphi_{2,1} < \frac{1}{n+1},$$

and from (4.12) we have

$$2/u \leqq \varphi_{1,1} + \varphi_{2,1} < \frac{2}{n+1}.$$

Hence

$$k(3) = \Big(\big[u/(n+1)\big] - 1\Big)_+ + \Big(\big[u(1-2/u)/2\big] - 1\Big)_+ = [u/2] - 2 = n - 2,$$

and

$$|P| \leqq 2 \cdot 2^{n-1} + 2^{n-2} = 2^n + 2^{n-2}.$$

c) For $s = 4$ we consider two subcases:
c.1) For $0 \leqq \delta < 1/2$ the maximum of $|P|$ is achieved if

$$k(1) = n - 1, \quad k(2) = n - 2, \quad k(3) = k(4) = n - 3$$

and then

$$|P| \leqq 2^n.$$

c.2) For $1/2 \leqq \delta < 1$ the maximum of $|P|$ is achieved if

$$k(1) = n - 1, \quad k(2) = k(3) = n - 2, \quad k(4) = n - 3$$

and then

$$|P| = 2^n + 2^{n-3} < 2^n + 2^{n-2}.$$

Let the lemma be proved for $s - 1$, then for s we have two cases:
s.1) For $0 \leqq \delta < 1/2$ the maximum of $|P|$ is achieved if

$$k(1) = n - 1, \quad k(2) = n - 2, \ \ldots, \ k(s-1) = k(s) = n - s + 1$$

and then

$$|P| \leqq 2^{n-1} + 2^{n-2} + \ldots + 2^{n-s+1} + 2^{n-s+1} = 2^n.$$

s.2) For $1/2 \leqq \delta < 1$ the maximum of $|P|$ is achieved if

$$k(1) = n-1,\ k(2) = n-2, \ldots, k(s-2) = k(s-1) = n-s+2,$$
$$k(s) = n-s+1$$

and then

$$|P| = 2^{n-1} + 2^{n-2} + \ldots + 2^{n-s+2} + 2^{n-s+2} + 2^{n-s+1} =$$
$$= 2^n + 2^{n-s+1} \leqq 2^n + 2^{n-2}.$$

That completes the proof of the lemma.

DEFINITION 4.1. $N(\alpha)$ is the largest natural number such that there exists a $\mathrm{GC}(V) = \{P, C\}$ with $|P| = N(\alpha)$ and $\mathrm{GA}(V) \leqq \alpha$.

Obviously both functions $\alpha(N)$ and $N(\alpha)$ are non-decreasing.

From Lemma 4.1 there follows:

LEMMA 4.2. *If* $\alpha = (1 - 2/u)\pi$, *then*

$$N(\alpha) \leqq 2^n \quad \textit{for} \quad 2n \leqq u < 2n+1$$

and

$$N(\alpha) \leqq 2^n + 2^{n-2} \quad \textit{for} \quad 2n+1 \leqq u < 2n+2.$$

Now we shall prove:

LEMMA 4.3. *If* $\alpha = (1 - 2/u)\pi$, *then*

$$N(\alpha) \geqq 2^n \quad \textit{for} \quad 2n \leqq u < 2n+1 \tag{4.14}$$

and

$$N(\alpha) \geqq 2^n + 2^{n-2} \quad \textit{for} \quad 2n+1 \leqq u < 2n+2. \tag{4.15}$$

PROOF. Let V be a GC with N points and two circles of range 1. From Lemma 3.4 and Lemma 3.7, for $s = 2$ we have (see Fig. 3)

$$N = 2^{[u/2]-1} + 2^{[u/2]-1} = 2^n.$$

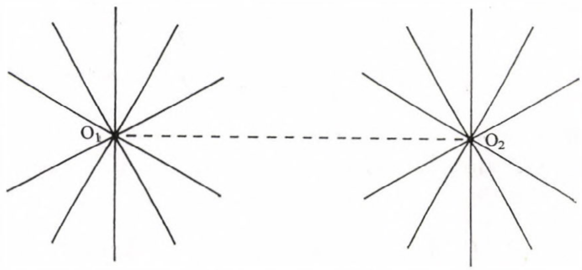

Fig. 3.

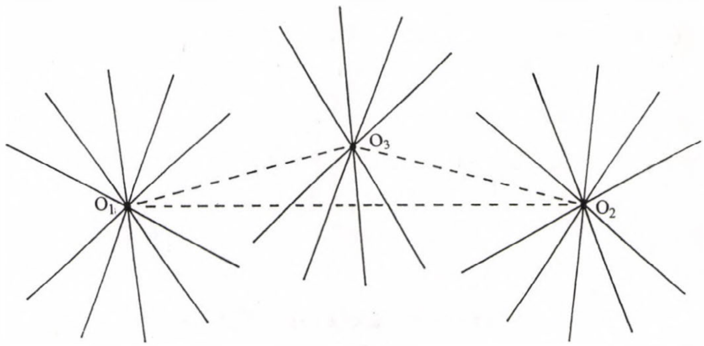

Fig. 4.

Hence (4.14) is proved.

Let V be a GC with N points and three circles of range 1. Let the centers of these three circles be $o_1 = (-1, 0)$, $o_2 = (1, 0)$ and $o_3 = \left(0, \tan\frac{\pi}{2n+1}\right)$ (see Fig. 4). Then we have

$$\varphi_{1,1} = \varphi_{2,1} = \frac{1}{2n+1}, \quad \varphi_{1,2} = \varphi_{2,2} = 1 - \frac{1}{2n+1},$$

$$\varphi_{3,1} = \frac{2}{2n+1}, \quad \varphi_{3,2} = 1 - \frac{2}{2n+1}$$

and for $2n + 1 \leqq u < 2n + 2$:

$$\left([u\varphi_{i,1}/2] - 1\right)_+ = 0 \quad \text{for} \quad i = 1, 2, 3,$$

$$\left([u\varphi_{i,2}/2] - 1\right)_+ = n - 1 \quad \text{for} \quad i = 1, 2,$$

$$\left([u\varphi_{3,2}/2] - 1\right)_+ = n - 2.$$

According to Lemma 3.4 and Lemma 3.7, from the above equalities it follows that

$$N = 2^{n-1} + 2^{n-1} + 2^{n-2} = 2^n + 2^{n-2}.$$

Hence (4.15) is also proved.

From Lemma 4.2 and Lemma 4.3, having in mind that the functions $\alpha(N)$ and $N(\alpha)$ satisfy $\alpha\big(N(\alpha)\big) = \alpha$, we obtain finally the following:

THEOREM 4.1. *If $\alpha(N)$ is the largest number such that any plane configuration of N points contains three points determining an angle $\beta \geqq \alpha(N)$; $0 \leqq \beta \leqq \pi$, then*

$$\alpha(N) = \left\{1 - \frac{2}{2n+1}\right\}\pi \quad \textit{for} \quad 2^n < N \leqq 2^n + 2^{n-2}$$

and

$$\alpha(N) = \left\{1 - \frac{1}{n+1}\right\}\pi \quad \textit{for} \quad 2^n + 2^{n-2} < N \leqq 2^{n+1}.$$

5. Existence of extreme GC's

DEFINITION 5.1. A GC$V = \{P, C\}$ with N points is *extreme* if

$$\mathrm{GA}(V) = G\alpha(N) = \alpha(N).$$

In this section we shall prove the existence of an extreme GC for every natural number N.

DEFINITION 5.2. For every plane configuration of points $P = \{p_1, p_2, \dots, p_N\}$ we define $r(P)$ as the radius of the smallest circle $c\big(o(P), r(P)\big)$ containing P, where $o(P)$ is the center of this circle.

The following lemma is obvious.

LEMMA 5.1. *Let P be a plane configuration of points. For every number $r' > 0$ and every point o' on the plane there exists a plane configuration of points $P' \approx P$ such that $o(P') = o'$ and $r(p') = r'$.*

DEFINITION 5.3. For every plane configuration of points $P = \{p_1, p_2, \dots, p_N\}$ we define a set of segments

$$S(P) = \big\{S_1(P), S_2(P), \dots, S_K(P)\big\}, \quad K = \frac{1}{2}N(N-1),$$

where the segments $S_i(P)$; $i = 1, 2, \dots, K$ are the intersections of the lines $p_1p_2, p_1p_3, \dots, p_1p_K, p_2p_3, \dots, p_{K-1}p_K$ with the circle $c\big(o(P), 2r(P)\big)$. The set $\{P, S(P)\}$ is called a *complete configuration*, corresponding to the configuration P.

DEFINITION 5.4. A set $\{P, S\}$ of points $P = \{p_1, p_2, \ldots, p_N\}$ and segments $S = S(P) \cup S'$, where $S' = \{S_{K+1}, S_{K+2}, \ldots, S_Q\}$ are segments which are intersections of some lines, passing through some of the points of P (the points depending on the line), with the circle $c(o(P), 2r(P))$, is called *extended complete configuration*. The segments S' are called *additional segments*.

We shall define a distance between two plane configurations of points using their complete configurations. This distance shall be sensitive to the angles defined by the points. If the distance between two configurations is "small", then the difference between two respective angles shall be also "small".

We shall use the Euclidean distance

$$\rho(u, v) = \sqrt{(x_u - x_v)^2 + (y_u - y_v)^2}$$

between two points u and v and the Hausdorff distance [7]

$$r(S_1, S_2) = \max\{\max_{u \in S_1} \min_{v \in S_2} \rho(u, v),\ \max_{u \in S_2} \min_{v \in S_1} \rho(u, v)\}$$

between two segments S_1 and S_2.

DEFINITION 5.5. Let

$$\{P, S\} = \{p_1, p_2, \ldots, p_N;\ S_1, S_2, \ldots, S_Q\}$$

and

$$\{P', S'\} = \{p'_1, p'_2, \ldots, p'_N;\ S'_1, S'_2, \ldots, S'_Q\}$$

be two extended complete configurations. We define the distance

$$R(\{P, S\}, \{P', S'\})$$

between these configurations in the following way

$$R(\{P, S\}, \{P', S'\}) =$$

$$= \max\left\{ \max_{1 \leqq i \leqq N} \min_{1 \leqq j \leqq N'} \rho(p_i, p'_j),\ \max_{1 \leqq i \leqq N'} \min_{1 \leqq j \leqq N} \rho(p'_i, p_j) \right\} +$$

$$+ \max\left\{ \max_{1 \leqq i \leqq Q} \min_{1 \leqq j \leqq Q'} r(S_i, S'_j),\ \max_{1 \leqq i \leqq Q'} \min_{1 \leqq j \leqq Q} r(S'_i, S_j) \right\}.$$

DEFINITION 5.6. Let P be a plane configuration of points and $\{P', S'\}$ be an extended complete configuration of points. We define the distance

$$R(P, \{P', S'\}) = R\Big(\{P, S(P)\}, \{P', S'\}\Big).$$

From the definition of the distance R between two plane configurations of points, there immediately follows:

LEMMA 5.2. *If the distance R between two plane configurations of points is less than ε then for every point from the first configuration there exists a corresponding point from the second configuration (possibly nor unique) such that the distance between these two points is less than ε, and for every segment from the first configuration there exists a corresponding segment from the second configuration (possibly not unique) such that the distance between these two segments is less than ε.*

COROLLARY 5.1. *If the distance R between two plane configurations of points P', P'' is less than ε and $r(P'), r(P'') \geqq 1$, then for every angle from the first configuration there exists a corresponding angle from the second configuration (possibly not unique) such that the difference between these two angles is less than 4ε.*

PROOF. As the considered segments are longer than 2, if the Hausdorff distance between two segments is less than ε, then the angle between these segments shall be less than $\arcsin \varepsilon \leqq 2\varepsilon/\pi$. Hence, the difference between two corresponding angles is less than 4ε.

From the argument for compactness and the finite number of points and segments, we obtain the following.

LEMMA 5.3. *Let $H(N)$ be the space of all extended complete configurations $\{P, S\} = \{p_1, p_2, \dots, p_k;\ S_1, S_2, \dots, S_l\}$ satisfying the conditions:*

$$r(P) = 1, \quad o(P) = (0,0), \quad k \leqq N \quad \textit{and} \quad l \leqq \frac{1}{2}N(N-1),$$

with metric R. Then $H(N)$ is a compact metric space.

LEMMA 5.4. *From every sequence $\{P^{(m)}, S^{(m)}\}$; $m = 1, 2, 3, \dots$ of elements of $H(N)$ it is a possible to choose a convergent subsequence.*

We are ready to prove the existence of extreme GC's.

THEOREM 5.1. *For every natural N there exists an extreme* GC $V =$ $= \{P, C\}$ *with* $|P| = N$.

PROOF. For the natural number N and every natural number m there exists a plane configuration of N points $P^{(m)} = \{p_1^{(m)}, p_2^{(m)}, \dots, p_N^{(m)}\}$ such that

$$A\big(P^{(m)}\big) < \alpha(N) + \frac{1}{m}. \tag{5.16}$$

According to Lemma 5.1 we may suppose that

$$o\big(P^{(m)}\big) = (0,0), \quad r\big(P^{(m)}\big) = 1.$$

Let $\{P^{(m)}, S(P^{(m)})\}$ be the complete configuration corresponding to $P^{(m)}$. According to Lemma 5.4 we may suppose that the sequence

$$\Big\{\{P^{(m)}, S(P^{(m)})\}\Big\}$$

is convergent to the extended complete configuration

$$\{P', S'\} = \{p'_1, p'_2, \dots, p'_k;\ S'_1, S'_2, \dots, S'_Q\},$$

where $K \leqq N$, $Q \leqq \frac{1}{2}N(N-1)$.

Let d be the smallest distance between two points in P'. We define the circles $C^{(1)} = \{c_i;\ i = 1, 2, \dots, K\}$ of range 1 with centers $o_i = p'_i$ and radii $\frac{1}{3}d$.

If $K = N$, then the GC $V^* = \{P', C^{(1)}\}$ is extreme. In this case the extreme GC is an ordinary configuration of points.

If $K < N$, then the configuration $\{P', S'\}$ is extended and groups of points from $\{P^{(m)}\}$ converge to one point p'_i, producing segments passing through the point p'_i. Let the sequences of points

$$P_i^{(m)} = \{p_{i,1}^{(m)}, p_{i,2}^{(m)}, \dots, p_{i,k(i)}^{(m)}\}$$

converge to p'_i, or

$$\lim_{m\to\infty} p_{i,j}^{(m)} = p'_i; \quad j = 1, 2, \dots, k(i).$$

We replace the plane configuration of points $P_i^{(m)}$ with an equivalent one (according to Lemma 5.1), such that

$$o\big(P_i^{(m)}\big) = p'_i, \quad r\big(P_i^{(m)}\big) = d/4.$$

For simplicity we do not change the notations.

According to Lemma 5.4 we may suppose that the sequence

$$\{P_i^{(m)}, S(P_i^{(m)})\}$$

is convergent. In such a way we repeat the procedure with $P_i^{(m)}$ as with $P^{(m)}$ and produce the circles $C^{(2)}$ of range 2.

Here, since $r(P_i^{(m)}) = d/4 > 0$, the limiting points of $P_i^{(m)}$ are at least two. Repeating this procedure, say t times, we shall produce a GC V^* with N points and circles $C = C^{(1)} \cup C^{(2)} \cup \ldots \cup C^{(t)}$.

We assert that V^* is extreme. In fact, the directions determined by every two points in $P^{(m)}$ have a limit as a generalized direction in V^*. That means that the generalized angles in V^* are limits of angles in $P^{(m)}$, and according to (5.16), we have

$$\mathrm{GA}(V^*) = \alpha(N).$$

That completes the proof.

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QUASI-UNIFORM COMPLETENESS IN TERMS OF CAUCHY NETS

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Introduction

In [10] and [11], a theory of completion and completeness for quasi-uniform spaces is developed. This paper presents completeness in terms of nets rather than filters. The usual way to pass from filters to nets by choosing arbitrary elements from the sets in the filter does not work in the non-symmetric case of quasi-uniformities. The reason for this is that our Cauchy filters do not contain sets which are arbitrarily small in an absolute sense, hence the resulting net need not be Cauchy.

The completion of [10] and [11] is performed in a larger category than the category of quasi-uniform spaces and uniformly continuous functions. Those quasi-uniform spaces for which the construction of the completion gives a quasi-uniform space again are called completable. We are able to give a characterisation of completability in terms of Cauchy nets. This enables us to give an easy proof of the fact that products of completable spaces are completable and products of complete spaces are complete. As a byproduct we get the result that for completable spaces our notion of completeness coincides with the well-known concept of bicompleteness developed in [5]. Moreover, the completion coincides with the bicompletion in this case. (The results concerning bicompleteness and bicompletion may also be found in [4].) All totally bounded spaces are shown to be completable.

1. Preliminaries and notations

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ of binary relations (called *entourages*) on X such that

(a) Each element of $\mathcal{U}$ contains the diagonal Δ_X of $X \times X$.

(b) For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ satisfying $V^2 \subseteqq U$.

Here V^2 is an abbreviation for VV, where $UV := \{(x,y) \mid \exists z \in X \,.\, x\,U\,z\,V\,y\}$ is the usual relational product. (We use the notation $x\,U\,y$ for $(x,y) \in U$.) If $\mathcal{U}$ is a quasi-uniformity on X, then so is $\mathcal{U}^{-1} := \{U^{-1} \mid U \in \mathcal{U}\}$, where $U^{-1} := \{(x,y) \mid y\,U\,x\}$. The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*.

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The coarsest uniformity (i.e. quasi-uniformity having a base of symmetric entourages) finer than $\mathcal{U}$ is denoted by $\mathcal{U}^*$. It is generated by the entourages $U^* := U \cap U^{-1}$, where U runs over $\mathcal{U}$.

We employ the notation $[x]U$ for $\{y \in X \mid x\, U\, y\}$ and analogously $[A]U$ to mean $\bigcup_{a \in A}[a]U = \{y \in X \mid \exists x \in A \,.\, x\, U\, y\}$ for any subset A of X. This unusual notation is chosen because it fits nicely with the relational product:

$$\big[[x]U\big]\, V = [x]UV.$$

A quasi-uniformity $\mathcal{U}$ on a set X induces a topology $\mathcal{T}(\mathcal{U})$ on X having as neighbourhood filter of x the set $\mathcal{N}(x) := \big\{[x]U \mid U \in \mathcal{U}\big\}$. From now on we will only consider *separated* spaces, i.e. spaces where the topology $\mathcal{T}(\mathcal{U})$ satisfies the T_0-axiom. This is the case if and only if the relation $\bigcap_{U \in \mathcal{U}} U$ is a partial order.

A function $f : X \to Y$ between quasi-uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is *(quasi-) uniformly continuous* if for any given entourage $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that the relation $x\, U\, x'$ always implies $f(x)\, V\, f(x')$. The quasi-uniform spaces form a category **QUS** with the uniformly continuous functions as morphisms.

A quasi-uniform space $(X, \mathcal{U})$ is *totally bounded* if for any entourage $U \in \mathcal{U}$ there are finitely many sets $A_1, \dots, A_n \subseteqq X$ such that $A_1 \cup \dots \cup A_n = X$ and $A_i \times A_i \subseteqq U$ for all $i \in \{1, \dots, n\}$. An equivalent condition is that for any entourage U there is a finite set $F \subseteqq X$ such that $X = [F]U^*$.

Further information on the basic theory of quasi-uniformities may be found in [5].

2. Completeness in terms of filters

In [10] and [11] a theory of completeness and completion for quasi-uniform spaces is established. In this section we summarise some of the results obtained there.

To be able to give a completion for all quasi-uniform spaces, the category **TQUS** is introduced. Its objects are *topological quasi-uniform spaces*. These are triples $(X, \mathcal{U}, \mathcal{T})$, where $(X, \mathcal{U})$ is a quasi-uniform space and $\mathcal{T}$ is an additional topology, which has to satisfy certain axioms. These axioms ensure that $\mathcal{T}$ is contained in and shares many properties with $\mathcal{T}(\mathcal{U})$. If, for example, the quasi-uniformity is totally bounded, then $\mathcal{T}$ has to agree with $\mathcal{T}(\mathcal{U})$. The morphisms of **TQUS** are those uniformly continuous functions, which are continuous with respect to the additional topologies, too.

Any quasi-uniform space $(X, \mathcal{U})$ together with the induced topology $\mathcal{T}(\mathcal{U})$ yields a topological quasi-uniform space $\big(X, \mathcal{U}, \mathcal{T}(\mathcal{U})\big)$, hence the category **QUS** may be regarded as a full subcategory of **TQUS**. A suitable notion of completeness for topological quasi-uniform spaces has been introduced in

[10] and [11]. Moreover, a completion for all spaces has been constructed there.

For the case of quasi-uniform spaces, the relevant definitions are shown to simplify to (see also [9]):

DEFINITION. A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is a *round Cauchy filter* if
(rd) $\forall A \in \mathcal{F} . \exists B \in \mathcal{F} . \exists U \in \mathcal{U} . [B]U \subseteqq A$ and
(Cy) $\forall U \in \mathcal{U} . \forall A \in \mathcal{F} . \exists x \in A . [x]U \in \mathcal{F}$.
The space is *complete* if every round Cauchy filter is the neighbourhood filter of a unique point.

A quasi-uniform space is called *completable* if the construction of its completion in **TQUS** stays inside the subcategory **QUS**. In [11] the following characterisation of completability is given:

DEFINITION. A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is *stable* [3] if for any entourage $U \in \mathcal{U}$, the set $\bigcap_{A\in\mathcal{F}}[A]U$ is in the filter $\mathcal{F}$.

PROPOSITION 1. *A quasi-uniform space* $(X,\mathcal{U})$ *is completable if and only if every round Cauchy filter on* X *is stable.* □

3. Completeness in terms of nets

In the present paper, however, we show how to use nets rather than filters. First, we generalise a definition of [9] for Cauchy sequences. This notion of Cauchy sequence does also appear in [1] where it is traced back to [7].

DEFINITION. (See also [8].) A *Cauchy net* on a quasi-uniform space $(X,\mathcal{U})$ is a net $(x_\lambda)_{\lambda\in\Lambda}$ with the property that for any entourage U there is an index $\lambda \in \Lambda$ such that for all indices μ and ν with $\lambda \leqq \mu \leqq \nu$ the relation $x_\mu \, U \, x_\nu$ holds.

The net is said to *converge strongly* to a point $x \in X$ if it converges to x with respect to the topology $\mathcal{T}(\mathcal{U}^*)$.

The net is *bi-Cauchy,* if it is Cauchy with respect to the uniformity $\mathcal{U}^*$. This is the case if and only if for all entourages $U \in \mathcal{U}$ there exists some $\lambda \in \Lambda$ such that $x_\mu \, U \, x_\nu$ holds for all indices $\mu, \nu \geqq \lambda$ (regardless of their order).

REMARK. Let us give an intuitive explanation why $\mathcal{T}(\mathcal{U}^*)$-convergence is the appropriate concept here. Convergence with respect to the topology $\mathcal{T}(\mathcal{U})$ is clearly too weak in general, since we do want unique limits and we do not want all spaces with a least element to be complete. Therefore, a stronger notion has to be found. The idea is that a Cauchy net together with its limit point has to satisfy the Cauchy condition, too. This means that if x is the strong limit of a Cauchy net $(x_\lambda)_{\lambda\in\Lambda}$ and we enlarge the index set Λ

by a greatest element ∞ and define $x_\infty := x$, we want the resulting net to remain a Cauchy net. That is the case if and only if the condition

$$\text{for all entourages } U \text{ there is an index } \lambda \in \Lambda$$

$$\text{such that } x_\mu\, U\, x_\infty \text{ whenever } \lambda \leqq \mu$$

holds, i.e. the net converges to x_∞ with respect to the topology $\mathcal{T}(\mathcal{U}^{-1})$. This together with $\mathcal{T}(\mathcal{U})$-convergence is exactly the strong convergence defined above.

LEMMA 2. *Let $(x_\lambda)_\Lambda$ be a Cauchy net on $(X,\mathcal{U})$. If $x \in X$ is a $\mathcal{T}(\mathcal{U}^*)$-cluster point of $(x_\lambda)_\Lambda$, then the net converges strongly to x.*

PROOF. Suppose an entourage U is given. Choose an entourage V such that $V^2 \subseteqq U$. By Cauchyness, there is some λ such that for all $\nu \geqq \mu \geqq \lambda$ the relation $x_\mu\, V\, x_\nu$ holds. As x is supposed to be a $\mathcal{T}(\mathcal{U}^*)$-cluster point of the net, there exists an index $\mu \geqq \lambda$ satisfying $x\, V^*\, x_\mu$. For any $\nu \geqq \mu$ one has now $x\, V\, x_\mu\, V\, x_\nu$, whence $x\, U\, x_\nu$. To establish the converse relation we choose $\kappa \geqq \nu$ with $x\, V^*\, x_\kappa$ and get $x_\nu\, V\, x_\kappa\, V\, x$. Therefore we have $\{x_\nu \mid \nu \geqq \mu\} \subseteqq [x]U^*$, i.e. x is indeed the strong limit of $(x_\lambda)_\Lambda$. □

Now we prove a generalisation of Proposition 2.7 of [9].

THEOREM 3. *For any Cauchy net $(x_\lambda)_\Lambda$ on a quasi-uniform space $(X,\mathcal{U})$ there exists a round Cauchy filter $\mathcal{F}(x_\lambda)$ on X such that the net converges strongly to a point $x \in X$ if and only if $\mathcal{F}(x_\lambda)$ is the neighbourhood filter of x. Conversely, if $\mathcal{F}$ is a round Cauchy filter on X then there exists a Cauchy net $(x_\lambda)_\Lambda$ such that $\mathcal{F}(x_\lambda) = \mathcal{F}$.*

PROOF. We define the filter $\mathcal{F}(x_\lambda)$ to be generated by the sets

$$[E_\lambda]U \qquad \text{with} \qquad E_\lambda := \{x_\mu \mid \mu \geqq \lambda\},$$

where U ranges over $\mathcal{U}$ and λ over the index set Λ. This is obviously a base generating a round filter. Property (Cy) is an immediate consequence of the net being Cauchy: Given $[E_\lambda]U$ in $\mathcal{F}(x_\lambda)$ and $V \in \mathcal{U}$, choose an entourage W with $W^2 \subseteqq V$ and an index $\mu \geqq \lambda$ such that $\mu \leqq \nu$ always implies $x_\mu\, W\, x_\nu$. Then $E_\mu \subseteqq [x_\mu]W$ hence $[E_\mu]W \subseteqq [x_\mu]W^2 \subseteqq [x_\mu]V$. Therefore $[x_\mu]V \in \mathcal{F}$.

Now suppose that $\mathcal{F}(x_\lambda) = \mathcal{N}(x)$ holds for some point $x \in X$ and $U \in \mathcal{U}$ and $\lambda \in \Lambda$ are given. Then $[x]U \in \mathcal{N}(x) \subseteqq \mathcal{F}(x_\lambda)$, hence there is an index $\mu \geqq \lambda$ such that $E_\mu \subseteqq [x]U$ holds. On the other hand, $[E_\mu]U \in \mathcal{F}(x_\lambda) \subseteqq \mathcal{N}(x)$, thus we certainly have $x \in [E_\mu]U$. Combining these arguments, we get some $\nu \geqq \mu$ with $x\, U^*\, x_\nu$. Therefore, x is the strong limit of the net by Lemma 2. If, conversely, $(x_\lambda)_\Lambda$ converges strongly to x we show that $\mathcal{N}(x)$ coincides with $\mathcal{F}(x_\lambda)$. Suppose some neighbourhood $[x]U^2 \in \mathcal{N}(x)$ is given. Then — by the $\mathcal{T}(\mathcal{U})$-part of the strong convergence — there is an index λ such that $E_\lambda \subseteqq [x]U$ holds. Now $[E_\lambda]U \subseteqq [x]U^2$, hence the given neighbourhood of x is an element of $\mathcal{F}(x_\lambda)$. If on the other hand $[E_\lambda]U^2 \in \mathcal{F}(x_\lambda)$ is given, then — by the $\mathcal{T}(\mathcal{U}^{-1})$-part of the strong convergence — there exists an index $\mu \geqq \lambda$ with $E_\mu \subseteqq [x]U^{-1}$. This implies $x \in [E_\mu]U$, hence $[x]U \subseteqq [E_\mu]U^2 \subseteqq [E_\lambda]U^2$, therefore the set $[E_\lambda]U^2$ is an element of $\mathcal{N}(x)$.

For the second assertion of the theorem, suppose $\mathcal{F}$ is some round Cauchy-filter. We define the index set

$$\Lambda = \mathcal{U} \times \mathcal{F}$$

and, for each $(U, A) \in \Lambda$, we choose an element $x_{(U,A)} \in A$ such that

$$[x_{(U,A)}]U \in \mathcal{F}$$

holds. This is possible by Cauchyness of $\mathcal{F}$.

Now the main step of the proof: We have to find an order on Λ such that the net (x_λ) is Cauchy. The naïve order $(U, A) \leqq (V, B)$ if and only if $V \subseteqq U$ and $B \subseteqq A$ is not sufficient to prove Cauchyness of the resulting net. This is because, given an entourage, there is no way to determine an index (U, A) to satisfy the Cauchy condition. The problem is to find a set $A \in \mathcal{F}$ which is small enough to ensure that the elements with greater index are close together. The only sets that we actually know to be elements of the filter are those of the form $[x_{(U,X)}]U$. However, these need not be small as it might happen that for every entourage U the set $[x_{(U,X)}]U$ is the whole space X.

The solution is to encode the desired property of points being 'better' for larger indices in the order on Λ. Hence we order the set Λ by

$$(U, A) \leqq (V, B) \quad \overset{\text{def}}{\iff} \quad V \subseteqq U \quad \text{and} \quad B \subseteqq [x_{(U,A)}]U \cap A.$$

This relation is clearly transitive, we have to prove directedness. For given indices (U, A) and (V, B) choose the entourage $W = U \cap V \in \mathcal{U}$ and the set $C = A \cap B \cap [x_{(U,A)}]U \cap [x_{(V,B)}]V \in \mathcal{F}$ to obtain the common upper bound (C, W). Cauchyness is established easily: If an entourage U is given, we choose the index $\lambda = (U, X)$ and observe that for larger indices $(V, B) \leqq (W, C)$ we have $x_{(W,C)} \in C \subseteqq [x_{(V,B)}]V \subseteqq [x_{(V,B)}]U$, i.e. the relation $x_{(V,B)}\, U\, x_{(W,C)}$ holds.

It remains to prove the equality $\mathcal{F} = \mathcal{F}(x_\lambda)$. If $A \in \mathcal{F}$, then by roundness there is an entourage $U \in \mathcal{U}$ and a set $B \in \mathcal{F}$ such that $[B]U \subseteqq A$. Then for $\lambda = (X \times X, B)$ we have $x_\mu \in B$ whenever $\mu \geqq \lambda$. Thus $E_\lambda \subseteqq B$, hence the set $[E_\lambda]U$, which belongs to the filter $\mathcal{F}(x_\lambda)$, is a subset of A. Therefore $A \in \mathcal{F}(x_\lambda)$. If conversely a set $A = [E_\lambda]U \in \mathcal{F}(x_\lambda)$ is given, we choose some index $\mu \geqq \lambda$ such that $\mu \geqq (U, X)$. Then $A \supseteqq [x_\mu]U \in \mathcal{F}$, hence $A \in \mathcal{F}$. □

COROLLARY 4. *A quasi-uniform space is complete if and only if every Cauchy net strongly converges to a unique point.* □

THEOREM 5. *A quasi-uniform space $(X,\mathcal{U})$ is completable if and only if every Cauchy net on X is bi-Cauchy.*

PROOF. We use the same notations as in the proof of Theorem 3. Suppose that $(X,\mathcal{U})$ is completable and $(x_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net on X. Then $\mathcal{F}(x_\lambda)$ is a round Cauchy filter on X which is stable by assumption and Proposition 1. This means that for any $U\in\mathcal{U}$ we have that $\bigcap_{\lambda\in\Lambda}\bigcap_{V\in\mathcal{U}}[E_\lambda]VU$ is contained in $\mathcal{F}(x_\lambda)$. But as this set is a subset of $\bigcap_{\lambda\in\Lambda}[E_\lambda]U^2$ we have also that

$$(*)\qquad \bigcap_{\lambda\in\Lambda}[E_\lambda]U\in\mathcal{F}(x_\lambda)$$

for all entourages U. Now let us establish that the net is bi-Cauchy. Suppose an entourage U is given. We choose $V^2\subseteqq U$ and an index $\lambda\in\Lambda$ such that $\lambda\leqq\mu\leqq\nu$ always implies $x_\mu\,V\,x_\nu$ (Cauchyness) and with $E_\lambda\subseteqq\bigcap_{\kappa\in\Lambda}[E_\kappa]V$ which is possible by $(*)$. Suppose $\lambda\leqq\mu,\nu$. Then $x_\nu\in E_\lambda\subseteqq[E_\mu]V$, hence there exists $\kappa\geqq\mu$ satisfying $x_\kappa\,V\,x_\nu$. As $x_\mu\,V\,x_\kappa$ holds, this gives us $x_\mu\,V^2\,x_\nu$ implying our goal $x_\mu\,U\,x_\nu$. Therefore the net is bi-Cauchy.

For the converse, suppose that every Cauchy net on X is bi-Cauchy and that $\mathcal{F}$ is a given round Cauchy filter which we have to prove to be stable. Then there exists by Theorem 3 some Cauchy net $(x_\lambda)_{\lambda\in\Lambda}$ such that $\mathcal{F}=\mathcal{F}(x_\lambda)$. This net is bi-Cauchy by assumption. This implies that for any given entourage U there is some index $\mu\in\Lambda$ such that $E_\mu\subseteqq[E_\lambda]U$ for all $\lambda\in\Lambda$. But from that we deduce $[E_\mu]U\subseteqq\bigcap_{\lambda\in\Lambda}[E_\lambda]U^2\subseteqq\bigcap_{\lambda\in\Lambda}\bigcap_{V\in\mathcal{U}}[E_\lambda]VU^2$, which means that the filter $\mathcal{F}(x_\lambda)$ is stable. Thus the space is completable by Proposition 1. □

REMARK. In light of Corollary 4, it is not too surprising that Cauchy nets on completable spaces are bi-Cauchy. The reason for this is that any Cauchy net on a completable space will remain Cauchy when regarded as a net on the completion. Thus the net is a strongly converging net. Since strong convergence is convergence with respect to the symmetrised topology, the net must be bi-Cauchy on the completion and also on the original space.

COROLLARY 6. *Arbitrary products of completable spaces are completable; arbitrary products of complete spaces are complete.*

PROOF. A net on a product space is Cauchy if and only if all the coordinate nets are Cauchy. Moreover, a net in the product converges strongly if and only if all the coordinate nets converge strongly. This implies the assertions. □

COROLLARY 7. *Every uniform space is a completable quasi-uniform space.*

PROOF. It is immediate that Cauchyness and bi-Cauchyness of nets coincide for uniform spaces. □

These results give us also the tools to prove (cf. [4])

COROLLARY 8. *For completable quasi-uniform spaces the completion coincides with the bicompletion of* [5].

PROOF. It suffices to prove that a completable quasi-uniform space is complete if and only if it is bicomplete, then the universal properties of both completions give us the result. It is easy to see (and proved in [Propositon 22][11]) that the Cauchy filters in the sense of [2] and [5], which are used to characterise bicompleteness, coincide with the $\mathcal{U}^*$-Cauchy filters. Hence a completable space is bicomplete if and only if any $\mathcal{U}^*$-Cauchy filter converges if and only if any bi-Cauchy net converges if and only if (by completability) any Cauchy net converges if and only if the space is complete. □

REMARK. This result does *not* mean that the notions completeness and bicompleteness coincide. The set of natural numbers $\mathbf{N}$ may serve as a counterexample. We choose the collection of all relations containing the usual order on $\mathbf{N}$ as a quasi-uniformity $\mathcal{U}$. Then the sequence $1, 2, 3, \ldots$ is a Cauchy sequence which is not bi-Cauchy, since $\mathcal{U}^*$ is the discrete uniformity. Hence this space is not completable, although it is bicomplete. Note that $(\mathbf{N}, \mathcal{U}^{-1})$ is completable and hence complete by Corollary 8. The reason for this is that all Cauchy nets on this space are eventually constant. The completion of $(\mathbf{N}, \mathcal{U}, \mathcal{T}(\mathcal{U}))$ in the category **TQUS** consists of the natural numbers with a top element, the quasi-uniformity generated by the order and the Scott-topology on this poset.

COROLLARY 9. *Completion of uniform spaces is a special case of completion of quasi-uniform spaces.*

PROOF. Completion of uniform spaces is a special case of bicompletion of quasi-uniform spaces. Hence Corollaries 7 and 8 give the result. □

The proof of the following proposition is a generalisation of Theorem 2.3 of [9]. For the result cf. also Lemma 4.5 of [3].

PROPOSITION 10. *Any totally bounded space is completable.*

PROOF. Suppose $(x_\lambda)_\Lambda$ is a Cauchy net and U a given entourage. Choose $V \in \mathcal{U}$ such that $V^2 \subseteqq U$ and — by total boundedness of X — sets $A_1, \ldots, A_n \subseteqq X$ with $A_1 \cup \cdots \cup A_n = X$ and $A_i \times A_i \subseteqq V$ for all i. Now we can choose an index $\lambda \in \Lambda$ such that

(1) $\lambda \leqq \mu \leqq \nu$ always implies $x_\mu \, V \, x_\nu$ (Cauchyness) and

(2) if A_i meets $\{x_\mu \mid \mu \geqq \lambda\}$, then $(x_\lambda)_\Lambda$ is frequently in A_i for $i = 1, \ldots, n$.

If we are now given indices $\mu, \nu \geqq \lambda$ then there is some i with $x_\nu \in A_i$. By (2) there is an index $\kappa \geqq \mu$ with $x_\kappa \in A_i$. Now (1) implies $x_\mu \, V \, x_\kappa$ and as the set A_i is V-small, we have $x_\mu \, V \, x_\kappa \, V \, x_\nu$, hence $x_\mu \, U \, x_\nu$. The converse relation may be proved analogously; therefore, the net is bi-Cauchy. □

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LAGRANGE INTERPOLATION POLYNOMIALS IN $E^p(D)$ WITH $1 < p < +\infty$

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Dedicated to the memory of my supervisor Prof. Shen Xie-Chang

§1. Introduction

Let D be a Jordan domain in the complex plane $\mathbf{C}$ bounded by Γ. For $0 < p < \infty$, a kind of extension of Hardy spaces can be defined by

$$E^p(D) = \Big\{ f\colon f \text{ analytic on } D \text{ and there exists a series of curves } \Gamma_n \subset D$$

$$\text{tending to } \Gamma, \quad \text{such that} \quad \sup_n \int_{\Gamma_n} |f(z)|^p |dz| < +\infty \Big\}.$$

If Γ is rectifiable and $f \in E^p(D)$, then f has a non-tangential limit almost everywhere on Γ. We define

$$\|f\|_p = \left\{ \int_\Gamma |f(z)|^p |dz| \right\}^{\frac{1}{p}}.$$

It is well known that Faber expansion is an effective tool to construct approximation polynomials in $E^p(D)$ [1]. Comparing with it, we can see that interpolation polynomials may be constructed more directly. In 1989, X.C. Shen and L. Zhong [2] took the Fejér points of interior level curves as interpolation nodes. Under the assumption of $\Gamma \in C^{2+\delta}$, it is shown that the interpolation polynomials have the same order of convergence as the best approximation polynomials in $E^p(D)$ for $1 < p < +\infty$. For $0 < p < 1$, L. Zhong [3] proved Jackson's theorem in $E^p(D)$ by means of interpolation polynomials. When the boundary Γ has some corners, L. Zhong and L. Y. Zhu [4] recently showed that the interpolation polynomials based on the roots of Faber polynomials converge in $E^p(D)$ for $1 < p < +\infty$.

In this paper, the boundary Γ is assumed piecewise C^2 smooth with no cusps. The interpolation nodes consist of geometric reflections of rotated

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Fejér points on the exterior level curves. It is proved that the Lagrange interpolation polynomials converge in $E^p(D)$ for $1 < p < +\infty$. Comparing with [2], we substantially weaken the restriction on the boundary, and we do not need to calculate conformal maps of the domains outside the interior level curves. Comparing with [4], we avoid finding the roots of the Faber polynomials, which may be unstable for high degree. We also do not need to interpolate derivatives of the functions, which may happen in [4] if the Faber polynomials have multiple roots.

In the following part of this paper, we always assume that $1 < p < +\infty$ and Γ is piecewise C^2 smooth with no cusps. We denote by c positive constant depending only on p and Γ, which may represent different values at different places. The notation $A \asymp B$ means $cB \leqq A \leqq cB$.

§2. Construction of interpolation nodes

Let U be the unit disk, and let $\Psi: \mathbf{C} \setminus U \to \mathbf{C} \setminus D$ be the conformal map satisfying $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$.

Let $\zeta_i = \Psi(e^{i\theta_j})$ $\big(\theta_j \in [0, 2\pi), j = 1, 2, \ldots, l\big)$ be the corners of Γ, with exterior angles $\alpha_j\pi$ $(0 < \alpha_j < 2, j = 1, 2, \ldots, l)$ respectively.

Then for $1 \leqq |u|, |w| \leqq 2$, we have [5]

$$\big|\Psi(u) - \Psi(w)\big| \asymp |u - w|\big(|u - e^{i\theta_k}| + |u - w|\big)^{\alpha_k - 1} \tag{2.1}$$

where $e^{i\theta_k}$ is the nearest point to u among $\{e^{i\theta_j}, j = 1, 2, \ldots, l\}$.

From elementary mathematics we know that there is a series $\big\{\phi_n \in \big[0, \frac{2\pi}{n}\big)\big\}$ such that

$$\min_{j,k} \left|\phi_n + \frac{2k\pi}{n} - \theta_j\right| \geqq \frac{\pi}{(l+1)n}. \tag{2.2}$$

Actually, for each $n > 0$, ϕ_n can be chosen among $\left\{\frac{2m\pi}{(l+1)n}, m = 0, 1, \ldots, l\right\}$.

By (2.1), and (2.2) for $0 < r < 1$, we have

$$\left|\Psi\left[\left(1 + \frac{r}{n}\right)e^{i\left(\phi_n + \frac{2k\pi}{n}\right)}\right] - \Psi\left(e^{i\left(\phi_n + \frac{2k\pi}{n}\right)}\right)\right| \asymp \tag{2.3}$$

$$\asymp \frac{r}{n}\left[\left|e^{i\left(\phi_n + \frac{2k\pi}{n}\right)} - e^{i\theta_{j0}}\right| + \frac{r}{n}\right]^{\alpha_{j0} - 1} \asymp \frac{r}{n}\left|e^{i\left(\phi_n + \frac{2k\pi}{n}\right)} - e^{i\theta_{j0}}\right|^{\alpha_{j0} - 1}$$

where $e^{i\theta_{j0}}$ is the nearest point to $e^{i\left(\phi_n + \frac{2k\pi}{n}\right)}$ among $\{e^{i\theta_j}\, j = 1, 2, \ldots, l\}$.

Similarly

$$\left|\Psi\left[\left(1 + \frac{r}{n}\right)e^{i\left(\phi_n + \frac{2k\pi}{n}\right)}\right] - \zeta_{j0}\right| = \tag{2.4}$$

$$= \left|\Psi\left[\left(1+\frac{r}{n}\right)e^{i\left(\phi_n+\frac{2k\pi}{n}\right)}\right]-\Psi\left(e^{i\theta_{j0}}\right)\right| \asymp \left|e^{i\left(\phi_n+\frac{2k\pi}{n}\right)}-e^{i\theta_{j0}}\right|^{\alpha_{j0}}.$$

Together with (2.2) and (2.3) we can find a positive constant r_0 sufficiently small and depending only on Γ, such that

$$(2.5)\qquad \left|\psi\left[\left(1+\frac{r_0}{n}\right)e^{i\left(\phi_n+\frac{2k\pi}{n}\right)}\right]-\Psi\left(e^{i\left(\phi_n+\frac{2k\pi}{n}\right)}\right)\right| \leqq$$
$$\leqq \frac{1}{4\rho}\left|\Psi\left[\left(1+\frac{r_0}{n}\right)e^{i\left(\phi_n+\frac{2k\pi}{n}\right)}\right]-\zeta_{j0}\right|$$

where $1<\rho<\infty$ is the maximum of the ratio of the local arclength of Γ to the chord.

We set

$$(2.6)\qquad z_{k,n}^{*}=\Psi\left[\left(1+\frac{r_0}{n}\right)e^{i\left(\phi_n+\frac{2k\pi}{n}\right)}\right],\quad k=0,1,\dots,n-1.$$

These are the so-called rotated Fejér points on the exterior level curve $\Psi\left(|w|=1+\frac{r_0}{n}\right)$.

Let $z_{k,n}^{**}$ be the nearest point on Γ to $z_{k,n}^{*}$. It follows from (2.5) that $z_{k,n}^{**}$ is not a corner. Set

$$(2.7)\qquad z_{k,n}=2z_{k,n}^{**}-z_{k,n}^{*}.$$

Then we obtain the geometric reflection of $z_{k,n}^{*}$ through Γ. In this paper, the points $\{z_{k,n},\ k=0,1,\dots,n-1\}$ are the interpolation nodes.

LEMMA 1. *There exists a constant $c>0$ such that for any $z\in\mathbf{C}\setminus\overline{D}$, $d(z,\Gamma)<c$ and*

$$(2.8)\qquad d(z,\Gamma)\leqq\frac{1}{4\rho}\min_{1\leqq j\leqq l}|z-\zeta_j|,$$

the geometric reflection of z is in D.

The proof of this lemma consists of elementary calculus, we leave it to the end of the paper.

By Lemma 1, $\{z_{k,n}\}\subset D$ if n is sufficiently large. For $f\in E^p(D)$, let $L_{n-1}(f,z)$ denote the $n-1$-th Lagrange interpolation polynomial to $f(z)$ based on $\{z_{k,n},k=0,1,\dots,n-1\}$.

The main result of this paper is the following.

THEOREM. *Suppose $1<p<\infty$, and Γ is piecewise C^2 smooth with no cusps. For any $f\in E^p(D)$, we have*

$$(2.9)\qquad \lim_{n\to+\infty}\left\|f(z)-L_{n-1}(f,z)\right\|_p=0.$$

§3. Integral representation of interpolation polynomials

It is well known that for $f \in E^p(D)$

$$L_{n-1}(f,z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega_n(\zeta)-\omega_n(z)}{\omega_n(\zeta)}\,\frac{f(\zeta)}{\zeta - z}\,d\zeta$$

where

$$\omega_n(z) = \prod_{k=0}^{n-1}(z - z_{k,n}).$$

By the Cauchy's formula, for $z \in D$, we have

$$f(z) - L_{n-1}(f,z) = \frac{\omega_n(z)}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\omega_n(\zeta)}\,\frac{d\zeta}{\zeta - z}.$$

Since the Cauchy's singular integral operator is $p-p$ type [6] for $p > 1$, we have

$$(3.1)\quad \left\| f(z) - L_{n-1}(f,z)\right\|_p \leqq \frac{1}{2\pi}\max_{z\in\Gamma}\left|\omega_n(z)\right| \left\| \int_\Gamma \frac{f(\zeta)}{\omega_n(\zeta)}\,\frac{d\zeta}{\zeta - z}\right\|_p \leqq$$

$$\leqq c\max_{z\in\Gamma}\left|\omega_n(z)\right| \left\|\frac{f(\zeta}{\omega_n(\zeta)}\right\|_p \leqq c\max_{z,\zeta\in\Gamma}\left|\frac{\omega_n(z)}{\omega_n(\zeta)}\right| \|f\|_p.$$

Lemma 2. *If Γ is piecewise C^2 smooth with no cusps, then for $z \in \Gamma$ we have*

$$\left|\omega_n(z)\right| \asymp d^n$$

where $d = \Psi'(\infty)$.

Notice that the theorem follows easily from Lemma 2. Actually, Lemma 2 and (3.1) imply that the interpolation polynomial operators are bounded uniformly. Then (2.9) holds. Furthermore, we have [4]

$$\left\| f(z) - L_{n-1}(f,z)\right\|_p \leqq c \min_{\deg D \leqq n-1} \|f - Q\|_p.$$

§4. Proof of the lemmas

There is no loss of generality, and the computations are easier, if we assume only one corner on Γ. Furthermore, we may assume $\zeta_1 = 1 = \Psi(1)$, and set $\alpha = \alpha_1$.

PROOF OF LEMMA 2. Let

$$\omega_n^*(z) = \prod_{k=0}^{n-1}(z - z_{k,n}^*).$$

There are two steps in the proof of Lemma 2. In Step I, we show

$$|\omega_n^*(z)| \asymp d^n, \quad z \in \Gamma. \tag{4.1}$$

In Step II, we show

$$|\omega_n(z)| \asymp |\omega_n^*(z)|, \quad z \in \Gamma. \tag{4.2}$$

Step I. As in [7], let

$$\chi(w,u) = \begin{cases} \frac{\Psi(w)-\Psi(u)}{d(w-u)} & u \neq w \\ \frac{\Psi'(w)}{d} & u = w. \end{cases}$$

Suppose

$$\log \chi(w,u) = \sum_{k=1}^{\infty} \frac{a_k(w)}{u^k}.$$

Then for $|w| = 1$

$$a_k(w) = \frac{1}{2\pi i}\int_{|u|=1+\varepsilon} u^{k-1}\log\chi(w,u)\,du =$$

$$= \frac{1}{2\pi k i}\int_{|u|=1+\varepsilon} u^k\left[\frac{\Psi'(u)}{\Psi(w)-\Psi(u)} - \frac{1}{w-u}\right]du = \frac{1}{k}\left[w^k - F_k \circ \Psi(w)\right]$$

where $F_k(z)$ is the k-th Faber polynomial with respect to D.

Since D is a bounded rotation domain, $F_k(z)$ $(k = 1, 2, \ldots)$ are bounded uniformly on Γ [8]. Therefore

$$|a_k(w)| \leqq \frac{c}{k}, \quad |w| = 1. \tag{4.3}$$

For $z=\Psi(w)\in\Gamma$,

$$\log \frac{\omega_n^*(z)}{d^n\left(w^n-\left(1+\frac{r_0}{n}\right)^n e^{in\phi_n}\right)}=$$

$$=\sum_{k=0}^{n-1}\log\chi\left(w,\left(1+\frac{r_0}{n}\right)e^{i(\phi_n+\frac{2k\pi}{n})}\right)=n\sum_{m=1}^{\infty}a_{nm}(w)\Big(1+\frac{r_0}{n}\Big)^{-nm}e^{-inm\phi_n}.$$

By (4.3) we have

$$\left|\log \frac{\omega_n^*(z)}{d^n\left(w^n-\left(1+\frac{r_0}{n}\right)^n e^{-in\phi_n}\right)}\right|\leqq c.$$

Hence

$$\left|\frac{\omega_n^*(w)}{d^n}\right|\asymp\left|w^n-\left(1+\frac{r_0}{n}\right)^n e^{in\phi_n}\right|.$$

Since $|w|=1$, we have

$$1\asymp\left(1+\frac{r_0}{n}\right)^n-1\leqq\left|w^n-\left(1+\frac{r_0}{n}\right)^n e^{in\phi_n}\right|\leqq\left(1+\frac{r_0}{n}\right)^n+1\asymp 1.$$

Then we have (4.1).

Step II. Since $z_{k,n}^{**}$ is the nearest point on Γ to $z_{k,n}^*$, it follows from (2.5) that $z_{k,n}^{**}$ is not a corner. Then the tangent of Γ at $z_{k,n}^{**}$ is perpendicular to the segment $z_{k,n}z_{k,n}^*$. Let $\beta(z;k,n)$ denote the angle between the segment $zz_{k,n}^{**}$ and the tangent of Γ at $z_{k,n}^{**}$. By the cosine rule

$$|z-z_{k,n}|^2=|z-z_{k,n}^{**}|^2+|z_{k,n}-z_{k,n}^{**}|^2\mp 2|z-z_{k,n}^{**}|\,|z_{k,n}-z_{k,n}^{**}|\sin\beta(z;k,n)$$

and

$$|z-z_{k,n}^*|^2=|z-z_{k,n}^{**}|^2+|z_{k,n}^*-z_{k,n}^{**}|^2\pm 2|z-z_{k,n}^{**}|\,|z_{k,n}^*-z_{k,n}^{**}|\sin\beta(z;k,n).$$

Then we have

$$\left|\frac{|z-z_{k,n}|^2}{|z-z_{k,n}^*|^2}-1\right|=\frac{4|z_{k,n}^*-z_{k,n}^{**}|\,|z-z_{k,n}^{**}|}{|z-z_{k,n}^*|^2}\big|\sin\beta(z;k,n)\big|. \tag{4.4}$$

Without loss of generality we prove (4.2) only for $z=\Psi(e^{it})$, $0<t\leqq\pi$, since for $z=\Psi(e^{it})$, $\pi<t\leqq 2\pi$, (4.2) can be shown in the same way, and $z=1$ is the limit of both cases.

For $0 \leqq k \leqq \left[\frac{n}{2}\right]$, z and $z_{k,n}$ are on the C^2 smooth arc $\left\{\Psi(e^{i\theta}): : 0 \leqq \theta \leqq \frac{3\pi}{2}\right\}$ and we know that $\beta(z;k,n)$ has the same order as $|z - z_{k,n}^{**}|$, which implies

$$(4.5) \qquad \left|\sin\beta(z;k,n)\right| \leqq c|z - z_{k,n}^{**}|, \quad k = 0, 1, \dots, \left[\frac{n}{2}\right].$$

We also have

$$(4.6) \qquad |z - z_{k,n}^{**}| \leqq |z - z_{k,n}^{*}| + |z_{k,n}^{*} - z_{k,n}^{**}| =$$

$$= |z - z_{k,n}^{*}| + \min_{\zeta\in\Gamma}|\zeta - z_{k,n}^{*}| \leqq 2|z - z_{k,n}^{*}|$$

and by (2.3)

$$(4.7) \quad |z_{k,n}^{*} - z_{k,n}^{**}| \leqq \left|\Psi\left[\left(1 + \frac{r_0}{n}\right)e^{i\left(\phi_n + \frac{2k\pi}{n}\right)}\right] - \Psi\left(e^{i\left(\phi_n + \frac{2k\pi}{n}\right)}\right)\right| \leqq$$

$$\leqq \frac{c}{n}\left|e^{i\left(\phi_n + \frac{2k\pi}{n}\right)} - 1\right|^{\alpha - 1}.$$

Hence

$$(4.8) \qquad \left|\frac{|z - z_{k,n}|^2}{|z - z_{k,n}^{*}|^2} - 1\right| \leqq \frac{c}{n}\left|e^{i\left(\phi_n + \frac{2k\pi}{n}\right)} - 1\right|^{\alpha - 1}.$$

By (2.2) we have

$$\frac{c}{n}\left|e^{i\left(\phi_n + \frac{2k\pi}{n}\right)} - 1\right|^{\alpha - 1} \leqq c\frac{(1+k)^{\alpha - 1}}{n^{\alpha}}.$$

Therefore

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\left|\frac{z - z_{k,n}|^2}{|z - z_{k,n}^{*}|^2} - 1\right| \leqq c.$$

For $z = \Psi(e^{it})$, $0 < t \leqq \pi$ and $\left[\frac{n}{2}\right] < k < n$, (4.5) may not be true, but we have

$$\left|\sin\beta(z;k,n)\right| \leqq \min\left\{c\left|z_{k,n}^{**} - 1\right| + \frac{|z - 1|}{\left|z_{k,n}^{**} - 1\right|}, 1\right\}.$$

This is because the angle between the segment $z_{k,n}^{**}1$ and the tangent of Γ at $z_{k,n}^{**}$ is not larger than $c|z_{k,n}^{**} - 1|$, and the angle between the segments $z_{k,n}^{**}1$

and $z_{k,n}^{**}z$ is not larger than $\frac{|z-1|}{|z_{k,n}^{**}-1|}$. These imply that $\beta(z;k,n)$ is not larger than

$$c|z_{k,n}^{**}-1|+\frac{|z-1|}{|z_{k,n}^{**}-1|}.$$

By (4.4) and (4.6)

$$\left|\frac{|z-z_{k,n}|^2}{|z-z_{k,n}^*|^2}-1\right| \leqq c\frac{|z_{k,n}^*-z_{k,n}^{**}|}{|z-z_{k,n}^*|}\left(|z_{k,n}^{**}-1|+\min\left\{\frac{|z-1|}{|z_{k,n}^{**}-1|},1\right\}\right).$$

By (2.1) and (4.7)

$$\frac{|z_{k,n}^*-z_{k,n}^{**}|}{|z-z_{k,n}^*|}\leqq c\frac{(n-k)^{\alpha-1}}{\big(n-k+n|t|\big)^{\alpha}}.$$

By (2.1) $|z-1|\asymp|t|^{\alpha}$. By (2.5)

$$|z_{k,n}^{**}-1|\leqq|z_{k,n}^{**}-z_{k,n}^*|+|z_{k,n}^*-1|\leqq\frac{5}{4}|z_{k,n}^*-1| \tag{4.10}$$

and by (2.5)

$$|z_{k,n}^{**}-1|\geqq|z_{k,n}^*-1|-|z_{k,n}^*-z_{k,n}^{**}|\geqq\frac{3}{4}|z_{k,n}^*-1|. \tag{4.11}$$

These imply

$$|z_{k,n}^{**}-1|\asymp|z_{k,n}^*-1|\asymp\left(\frac{n-k}{n}\right)^{\alpha}.$$

Hence

$$\left|\frac{|z-z_{k,n}|^2}{|z-z_{k,n}^*|^2}-1\right|\leqq c\frac{(n-k)^{2\alpha-1}}{n^{\alpha}\big(n-k+|nt|\big)^{\alpha}}+$$

$$+c\frac{(n-k)^{\alpha-1}}{\big(n-k+|nt|\big)^{\alpha}}\min\left\{\frac{|nt|^{\alpha}}{(n-k)^{\alpha}},1\right\}.$$

Evidently

$$\sum_{[\frac{n}{2}]<k<n}\frac{(n-k)^{2\alpha-1}}{n^{\alpha}\big(n-k+|nt|\big)^{\alpha}}\leqq\sum_{[\frac{n}{2}]<k<n}\frac{(n-k)^{\alpha-1}}{n^{\alpha}}\leqq c$$

and

$$\sum_{[\frac{n}{2}]<k<n} \frac{(n-k)^{\alpha-1}}{\left(n-k+|nt|\right)^{\alpha}} \min\left\{\frac{|nt|^{\alpha}}{(n-k)^{\alpha}},1\right\} \leqq$$

$$\leqq 2 \sum_{[\frac{n}{2}]<k<n} \frac{(n-k)^{\alpha-1}}{\left(n-k+|nt|\right)^{\alpha}} \frac{|nt|^{\alpha}}{(n-k)^{\alpha}+|nt|^{\alpha}} \asymp$$

$$\asymp \sum_{j=1}^{[\frac{n}{2}]-1} \frac{|nt|^{\alpha} j^{\alpha-1}}{j^{2\alpha}+|nt|^{2\alpha}} \asymp \int_1^{[\frac{n}{2}]} \frac{|nt|^{\alpha} x^{\alpha-1}}{x^{2\alpha}+|nt|^{2\alpha}}\, dx \leqq$$

$$\leqq \frac{1}{\alpha} \int_0^{+\infty} \frac{|nt|^{\alpha}}{y^2+|nt|^{2\alpha}}\, dy = \begin{cases} 0 & t=0 \\ \frac{\pi}{2\alpha} & t\neq 0. \end{cases}$$

Then we have

$$\sum_{[\frac{n}{2}]<k<n} \left| \frac{|z-z_{k,n}|^2}{|z-z_{k,n}^*|^2} - 1 \right| \leqq c.$$

Together with (4.9), we conclude that

$$\left| \frac{\omega_n(z)}{\omega_n^*(z)} \right|^2 \leqq c.$$

The proof of the inverse inequality

$$\left| \frac{\omega_n^*(z)}{\omega_n(z)} \right|^2 \leqq c$$

is similar to the above process. But it needs the estimation of the lower bound of $|z-z_{k,n}|$. For n sufficiently large, we shall show

$$|z-z_{k,n}^*| \leqq c|z-z_{k,n}|. \tag{4.12}$$

For $0 \leqq k \leqq \left[\frac{n}{2}\right]$ we can see that the right side of (4.8) is not larger than 0.5 if n is sufficiently large, and we have (4.12).

For $\left[\frac{n}{2}\right] < k < n-1$, if the subarc $\widehat{zz_{k,n}^{**}}$ connecting z and $z_{k,n}$ with shorter arclength crosses $\zeta = 1$, then by (2.5) and (4.11) we have

$$|z_{k,n}^{**} - z_{k,n}^*| \leqq \frac{1}{3\rho}|1 - z_{k,n}^{**}|.$$

Since

$$|z - z^{**}_{k,n}| \leqq |z - z_{k,n}| + |z^{**}_{k,n} - z^{*}_{k,n}| \leqq$$

$$\leqq |z - z_{k,n}| + \frac{1}{3\rho}|1 - z^{**}_{k,n}| \leqq |z - z_{k,n}| + \frac{1}{3\rho}\left|\overset{\frown}{zz^{**}_{k,n}}\right| \leqq |z - z_{k,n}| + \frac{1}{3}|z - z^{**}_{k,n}|,$$

then

$$|z - z^{**}_{k,n}| \leqq \frac{3}{2}|z - z_{k,n}|.$$

We have

$$|z - z^{*}_{k,n}| \leqq |z - z^{**}_{k,n}| + |z^{**}_{k,n} - z^{*}_{k,n}| \leqq |z - z^{**}_{k,n}| + \frac{1}{3\rho}|1 - z^{**}_{k,n}| \leqq$$

$$\leqq |z - z^{**}_{k,n}| + \frac{1}{3\rho}\left|\overset{\frown}{zz^{**}_{k,n}}\right| \leqq |z - z^{**}_{k,n}| + \frac{1}{3}|z - z^{**}_{k,n}| = \frac{4}{3}|z - z^{**}_{k,n}| \leqq 2|z - z_{k,n}|.$$

When $\left[\frac{n}{2}\right] < k < n - 1$ and the subarc $\overset{\frown}{zz^{**}_{k,n}}$ connecting z and $z_{k,n}$ with shorter arclength does not cross $\zeta = 1$, then

$$\left|\overset{\frown}{z1}\right| + \left|\overset{\frown}{1z^{**}_{k,n}}\right| > \frac{|\Gamma|}{2}.$$

Let Γ_1 be the subarc of Γ which begins at 1 and whose arclength equals $\frac{3|\Gamma|}{4}$ and let Γ_2 be the subarc of Γ which ends at 1 and whose arclength also equals $\frac{3|\Gamma|}{4}$. Then z and $z^{**}_{k,n}$ are both in one of these two subarcs at the same time. Since both subarcs are C^2 smooth, (4.5) and (4.8) are also true as both z and $z^{**}_{k,n}$ are on a C^2 smooth arc. We have (4.12) for n sufficiently large.

PROOF OF LEMMA 1. Let z^{**} be the nearest point on Γ to z^*. By (2.8) we know that $z^{**} \neq 1$. Then the segment z^*z^{**} is perpendicular to the tangent of Γ at z^{**}. Let z denote the geometric reflection of z^{**}.

Let Γ_1 and Γ_2 be two subarcs of Γ defined in the proof of Lemma 2. We shall show Lemma 1 for $z^{**} \in \Gamma_1$.

There exists a sufficiently small positive constant η, such that for any $z1 \in \Gamma_1$, $|z1 - 1| \leqq \eta$, the angle between the two tangents of Γ_1 at $z1$ and 1 is not larger than $\frac{\pi}{4}$.

If $|z^{**} - 1| \geqq \frac{\eta}{2}$, similar to (4.10) we know that z^* keeps at least $\frac{2\eta}{5}$ away from the corner. It is easy to see that the reflection of z^* across the smooth subarc is in D when z^* is very close to Γ.

If $|z^{**} - 1| \leqq \frac{\eta}{2}$ and $z \notin D$, then the segment of $z^{**}z$ must meet Γ at $\widetilde{z} \neq$ $\neq z^{**}$. If $\widetilde{z} \in \Gamma_1$, then there is a point $\widehat{z}$ on the subarc $z^{**}\widetilde{z}$, such that the tangent at $\widehat{z}$ is parallel to the segment of $z^{**}z$. Then the angle between the two tangents of Γ_1 at $\widehat{z}$ and 1 is larger than $\frac{\pi}{4}$.

On the other hand, similarly to (4.11)

$$|z^* - 1| \leqq \frac{4}{3}|z^{**} - 1|.$$

Hence

$$|\widehat{z} - 1| \leqq |\widehat{z} - z^{**}| + |z^{**} - 1| \leqq |\widetilde{z}z^{**}| + |z^{**} - 1| \leqq$$

$$\leqq \rho|\widetilde{z} - z^{**}| + |z^{**} - 1| \leqq \rho d(z^*, \Gamma) + |z^{**} - 1| \leqq \frac{4}{3}|z^{**} - 1| < \eta,$$

a contradiction.

Finally, if $\widetilde{z} \in \Gamma_2 \setminus \Gamma_1$, we have

$$|\widetilde{z} - z^{**}| \geqq \rho^{-1} \left|\widehat{\widetilde{z}\, z^{**}}\right| \geqq \rho^{-1} \left|\widehat{1z^{**}}\right| \geqq \rho^{-1}|1 - z^{**}| \geqq$$

$$\geqq \frac{3}{4\rho}|1 - z^*| \geqq 3|z^{**} - z^*| \geqq 3|\widetilde{z} - z^{**}|,$$

again a contradiction.

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CHARACTERIZATION OF SOME PECULIAR TOPOLOGICAL SPACES VIA $\mathcal{A}$- AND $\mathcal{B}$-SETS

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1. Preliminaries

Throughout this paper we consider spaces on which no separation axioms are assumed unless explicitly stated. A "space" will always mean a topological space. The topology of a space is denoted by τ and (X,τ) will be replaced by X if there is no chance for confusion. For $A \subset X$ the closure, the interior and the boundary of A in X are denoted by $\overline{A}$, $\operatorname{Int} A$ and $\operatorname{Fr} A$ respectively.

Next we recall some definitions.

DEFINITION 1. A subset A of a space (X,τ) is called:
(1) a preopen set [6] if $A \subset \operatorname{Int} \overline{A}$,
(2) a semi-open set [5] if $A \subset \overline{\operatorname{Int} A}$,
(3) a semi-preopen set [1] if $A \subset \overline{\operatorname{Int} \overline{A}}$,
(4) locally closed [2] if $A = U \cap F$, where U is open and F is closed,
(5) an $\mathcal{A}$-set [8] if $A = U \cap F$, where U is open and F is regular closed,
(6) a $\mathcal{B}$-set [9] if $A = U \cap F$, where U is open and F is semi-closed,
(7) a t-set [9] if $\operatorname{Int} \overline{A} = \operatorname{Int} A$,
(8) nowhere dense if $\operatorname{Int} \overline{A} = \emptyset$,
(9) an NDB-set if A has nowhere dense boundary.

The complements of preopen, semi-open and semi-preopen sets are called preclosed, semi-closed and semi-preclosed sets, respectively. A set A is called regular closed if $A = \overline{\operatorname{Int} A}$.

A space X is called submaximal [2] if every dense subset of X is open. Recall that a space X is called a partition space (or locally indiscrete) [7, 3] if every open subset of X is closed. A space X is extremally disconnected (or extremal) if the closure of each open subset of X is open or equivalently iff every semi-open set is preopen.

REMARK 1.1. This paper is based on [10].

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2. $\mathcal{A}$-sets, $\mathcal{B}$-sets and t-sets

THEOREM 2.1. *Every $\mathcal{B}$-set is an NDB-set.*

PROOF. It is trivial to see that the intersection of two NDB-sets is an NDB-set. Since a $\mathcal{B}$-set is the intersection of a (semi-)open and a semi-closed set, it is enough to show that every semi-open and every semi-closed set is an NDB-set. If A is semi-open, then for some open U we have $U \subset A \subset \subset \overline{U}$. Since $\operatorname{Fr} A = \overline{A} \cap \overline{X \setminus A} = \overline{U} \cap \overline{X \setminus A} \subset \overline{U} \cap \overline{X \setminus U} = \operatorname{Fr} U$, clearly $\operatorname{Fr} A$ is nowhere dense being a subset of the nowhere dense set $\operatorname{Fr} U$. In fact it is obvious that every open set has nowhere dense boundary. Thus every semi-open (and hence every semi-closed) set is an NDB-set. □

REMARK 2.2. The converse is not true. For consider the space $X = \{a, b, c\}$ with the only non-trivial open set $\{a\}$. The subset $\{a, b\}$ is an NDB-set but not a $\mathcal{B}$-set.

In [9] Tong defines the notion of t-sets in topological spaces. The following result shows that the defined property coincides with the class of semi-closed sets.

THEOREM 2.3. *For a subset A of a space X the following are equivalent:*
(1) *A is a t-set.*
(2) *A is semi-closed.*
(3) *A is a semi-preclosed $\mathcal{B}$-set.*
(4) *A is a semi-preclosed NDB-set.*

PROOF. (1) $\Rightarrow$ (2). Since $\operatorname{Int} \overline{A} = \operatorname{Int} A$, then $\operatorname{Int} \overline{A} = \operatorname{Int} A \subset A \Leftrightarrow X \setminus \setminus A \subset \overline{\operatorname{Int}(X \setminus A)}$. Thus $X \setminus A$ is semi-open, hence A is semi-closed.

(2) $\Rightarrow$ (3). Every semi-closed set is trivially semi-preclosed. Since $A = A \cap X$, where A is semi-closed and X is open, then A is a $\mathcal{B}$-set.

(3) $\Rightarrow$ (4). Theorem 2.1.

(4) $\Rightarrow$ (1). Since A is an NDB-set, then $B = X \setminus A$ is also an NDB-set. It is easy to see that from the identity

$$\operatorname{Int}(\operatorname{Fr} B) = \operatorname{Int} \overline{B} \cap \operatorname{Int}(\overline{X \setminus B}) = \operatorname{Int} \overline{B} \cap (X \setminus \overline{\operatorname{Int} B}) = \operatorname{Int} \overline{B} \setminus \overline{\operatorname{Int} B},$$

it follows that $\operatorname{Int} \overline{B} \subset \overline{\operatorname{Int} B}$. Since B is semi-preopen, $B \subseteq \overline{\operatorname{Int} \overline{B}}$. Thus $B \subset \subset \overline{\operatorname{Int} B}$ or equivalently $\overline{B} = \overline{\operatorname{Int} B}$. Since $B = X \setminus A$, $\operatorname{Int} \overline{A} = \operatorname{Int} A$. Thus A is a t-set. □

THEOREM 2.4. *For a subset A of a space X the following are equivalent:*
(1) *A is an $\mathcal{A}$-set.*
(2) *A is semi-open and locally closed.*
(3) *A is semi-preopen and locally closed.*

PROOF. (1) $\Rightarrow$ (2). Every $\mathcal{A}$-set is trivially locally closed. The second part is Theorem 3.1 in [8].

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Since A is locally closed, $A = U \cap \overline{A}$, where U is an open set. Since A is semi-preopen and since trivially $\overline{\operatorname{Int} \overline{A}} \subset \overline{A}$, $\overline{A}$ is regular closed. Thus A is the intersection of an open and a regular closed set, i.e., it is an $\mathcal{A}$-set. □

The equivalence of conditions (1) and (2) in the previous theorem was first proved by Ganster and Reilly in [4].

The results in the next theorem were proved by Tong [9], Ganster and Reilly [4] and we list them without proof for further citation.

THEOREM 2.5. *For a subset A of a space X the following are equivalent:*
(1) *A is open.*
(2) *A is preopen and locally closed.*
(3) *A is a preopen $\mathcal{A}$-set.*
(4) *A is a preopen $\mathcal{B}$-set.* □

3. Some peculiar spaces

THEOREM 3.1. *For a space X the following are equivalent:*
(1) *X is submaximal.*
(2) *Every subset of X is a $\mathcal{B}$-set.*
(3) *Every dense subset of X is a $\mathcal{B}$-set.*

PROOF. (1) $\Rightarrow$ (2). Let $A \subset X$. Since every subspace of a submaximal space is submaximal, then $\overline{A}$ is submaximal. Since A is dense in $\overline{A}$, A is open in $\overline{A}$. Thus $A = U \cap \overline{A}$, where U is open in X and $\overline{A}$ is closed in X. Thus A is locally closed and hence a $\mathcal{B}$-set, since every closed set is semi-closed.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $A \subset X$ be dense. By (3) $A = U \cap B$, where U is open and B is semi-closed. Since $A \subset B$, then B is dense. Thus $\operatorname{Int} B = \operatorname{Int} \overline{B} = \operatorname{Int} X = X$ and hence $B = X$. Thus $A = U$ is open and so X is submaximal. □

THEOREM 3.2. *For a space X the following are equivalent:*
(1) *X is a partition space.*
(2) *Every $\mathcal{B}$-subset is clopen.*
(3) *Every $\mathcal{B}$-subset is closed.*

PROOF. (1) $\Rightarrow$ (2). If A is a $\mathcal{B}$-set, $A = U \cap B$, where U is open and B is semi-closed. By (1) U is clopen. On the other hand $\overline{B}$ is open by (1) and thus $\operatorname{Int} \overline{B} \subset B \subset \overline{B}$ implies $B = \operatorname{Int} B = \overline{B}$ and thus B is clopen. Thus A is clopen being the intersection of two clopen sets.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Every open set is a $\mathcal{B}$-set by Theorem 2.5 and thus by (3) closed. □

THEOREM 3.3. *For a space X the following are equivalent:*
(1) *X is indiscrete.*
(2) *The only $\mathcal{B}$-sets in the X are the trivial ones.*
(3) *The only $\mathcal{A}$-sets in the X are the trivial ones.*

PROOF. (1) $\Rightarrow$ (2). If A is a $\mathcal{B}$-set, then $A = U \cap B$, where U is open and B is semi-closed. If $A \neq \emptyset$, then $U \neq \emptyset$ and by (1) $U = X$. Thus $A = B$ and so $\operatorname{Int} A = \operatorname{Int} \overline{A} = \operatorname{Int} X = X$. Hence $A = X$.

(2) $\Rightarrow$ (3). Every $\mathcal{A}$-set is a $\mathcal{B}$-set.

(3) $\Rightarrow$ (1). Since by Theorem 2.5 every open set is an $\mathcal{A}$-set, by (3) the only open sets in X are the trivial ones, i.e., X is indiscrete. □

THEOREM 3.4. *For a space X the following are equivalent:*
(1) *X is discrete.*
(2) *Every subset of X is an $\mathcal{A}$-set.*

PROOF. (1) $\Rightarrow$ (2). By (1) every set $A \subset X$ is open and regular closed. Hence A is an $\mathcal{A}$-set.

(2) $\Rightarrow$ (1). By (2) every singleton $\{x\} \in X$ is an $\mathcal{A}$-set and by Theorem 3.1 in [8] semi-open. If $\operatorname{Int}\{x\} = \emptyset$, then we have the contradiction $\{x\} \subset \subset \overline{\operatorname{Int}\{x\}} = \emptyset$. Thus $\{x\} = \operatorname{Int}\{x\}$ or equivalently every singleton in X is open. □

The following result was first proved by Ganster and Reilly in [4] and thus we present it without proof.

THEOREM 3.5. *For a space X the following are equivalent:*
(1) *X is extremally disconnected.*
(2) *Every $\mathcal{A}$-subset of X is open.*
(3) *The collection of all $\mathcal{A}$-subsets of X is a topology for X.*
(4) *The intersection of two $\mathcal{A}$-sets is an $\mathcal{A}$-set.* □

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JACKSON ORDER OF APPROXIMATION BY LAGRANGE INTERPOLATION. II

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In [3], we considered the problem of finding classes of functions where the order of Lagrange interpolation is the same as the Jackson order of approximation, for some suitably chosen systems of nodes. (For the origin of the problem see also the references in [3].) It turned out that for the nodes

$$x_{kn} = \cos t_{kn}, \quad t_{kn} = \frac{k-1}{n-1}\pi, \qquad k = 1, \ldots, n \tag{1}$$

which are the roots of the polynomial

$$\omega_n(x) = \sin(n-1)t \sin t, \quad x = \cos t \tag{2}$$

(this is the Chebyshev polynomial of second kind of degree $n-2$ multiplied by $1-x^2$), this phenomenon occurs, provided that the function to be interpolated is a piecewise polynomial. Motivated by this paper, Xin Li [4] generalized our result for finitely differentiable functions which are piecewise analytic functions, if these pieces are analytically extended to the whole interval $[-1,1]$.

It is the purpose of this paper to show that Xin Li's condition of analyticity can be essentially relaxed and at the same time the proof can be drastically simplified.

To begin with, let

$$-1 = a_s < a_{s-1} < \ldots < a_0 = 1 \qquad (s \geqq 1) \tag{3}$$

be a fixed partition of the interval $[-1,1]$. Further denote

$$I_j = [a_{j+1}, a_j], \qquad j = 0, 1, \ldots, s-1.$$

The restriction of a function $f \in C[-1,1]$ to the interval I_j will be denoted by $f|_{I_j}$. The Lagrange interpolation polynomial of $f \in C[-1,1]$ with respect to the nodes (1) is $L_n(f,x)$, and the error of interpolation is

$$\Delta_n(f,x) = f(x) - L_n(f,x).$$

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THEOREM. *Let $r \geqq 0$ be a fixed integer, and (3) a fixed partition of the interval $[-1,1]$. If $f(x) \in C^r[-1,1]$ and for $f_j(x) := f|_{I_j} \in C^{r+2}(I_j)$, $j = 0,\dots,s-1$ we have*

$$\omega(f_j^{(r+2)}, h) = O\left(\frac{1}{\log\frac{1}{h}}\right) \qquad (h > 0,\ j = 0,\dots,s-1) \tag{4}$$

(where ω is the modulus of continuity of the corresponding function on the corresponding interval), then

$$|\Delta_n(f,x)| = O\left(\frac{|\omega_n(x)|}{n^{r+1}}\right)\min\left(1, \frac{1}{n\min_{1\leqq j\leqq s-1}|x-a_j|}\right) \qquad (|x| \leqq 1).$$

Introducing the usual notations

$$x_+ = \begin{cases} x & \text{if} \quad x \geqq 0, \\ 0 & \text{if} \quad x < 0 \end{cases} \quad \text{and} \quad x_- = \begin{cases} 0 & \text{if} \quad x \geqq 0, \\ x & \text{if} \quad x < 0 \end{cases},$$

a typical example of a function satisfying the conditions of the theorem is

$$f(x) = g(x) + \sum_{j=1}^{s-1}(x-a_j)_+^{r+1} \qquad (s \geqq 2)$$

where $g \in C^{r+2}[-1,1]$ is such that $\omega(g^{(r+2)}, h) = O(1/\log 1/h)$. If g is not analytic in $[-1,1]$ then the above mentioned result of Xin Li does not apply to such functions.

The proof is divided into a series of lemmas.

LEMMA 1. *For the polynomial (2) we have the following estimate in the complex plane:*

(5)

$$|\omega_n(x+iy)| \geqq \begin{cases} \frac{1}{2}\sqrt{1-x^2}\left[\left(1+|y|\sqrt{1-x^2}\right)^{n/2}-1\right] & \text{if } |x|,|y| \leqq 1, \\ \frac{1}{2}(2^{n/2}-1) & \text{if } |x| \leqq 1, y = \pm 1. \end{cases}$$

PROOF. We use the formula

$$\omega_n(z) = \frac{1}{2}\sqrt{z^2-1}\left[\left(z+\sqrt{z^2-1}\right)^{-n} - (z+\sqrt{z^2-1})^n\right] \quad (z \in \mathbf{C} \setminus [-1,1])$$

where the square root is that value for which $\left|z+\sqrt{z^2-1}\right|>1$ for the z's in question (see Xin Li [4]). By symmetry, without loss of generality we may assume that $0\leqq x,y\leqq 1$. Then an easy calculation yields

$$\sqrt{z^2-1}=\sqrt{\frac{\sqrt{(1-x^2+y^2)^2+4x^2y^2}-(1-x^2+y^2)}{2}}+$$

$$+i\sqrt{\frac{\sqrt{(1-x^2+y^2)^2+4x^2y^2}+(1-x^2+y^2)}{2}},$$

whence

$$\left|\sqrt{z^2-1}\right|\geqq\left|\operatorname{Im}\sqrt{z^2-1}\right|\geqq\sqrt{1-x^2},$$

$$\left|z+\sqrt{z^2-1}\right|^2\geqq x^2+\left(y+\sqrt{1-x^2}\right)^2\geqq 1+y\sqrt{1-x^2}+y^2.$$

Substituting these estimates into (5), we get the statement of the lemma.

LEMMA 2. *We have*

$$(6)\quad \left|\Delta_n\big((\cdot-y)_+^l,x\big)\right|=O\left(\frac{|\omega_n(x)|}{n^l\left(\sqrt{1-y^2}\right)^{l+1}}\right)\min\left\{1,\frac{1}{|x-y|n\sqrt{1-y^2}}\right\}$$

$$(|x|\leqq 1,|y|<1,l=1,2,\ldots).$$

The special case $y=0$ was proved in [4], Theorem 2.1. Our proof is along the same lines, but since it is much simpler, we give it in full details. Of course, a similar estimate holds for the functions $(x-y)_-^l$.

PROOF. Evidently

$$L_n\big((\cdot-y)_+^l,x\big)=\sum_{x_k>y}\frac{\omega_n(x)(x_k-y)^l}{\omega_n'(x_k)(x-x_k)},$$

whence using the residue theorem of complex integration

$$\Delta_n\big((\cdot-y)_+^l,x\big)=\frac{\omega_n(x)}{2\pi i}\oint_\Gamma\frac{(z-y)^l}{\omega_n(z)(z-x)}\,dz\qquad(|x|,|y|\leqq 1),$$

where $\Gamma = \bigcup_{j=1}^4 \Gamma_j$,

$$\Gamma_1 = \{z = u - i:\ y \leqq u \leqq 2\}, \quad \Gamma_2 = \{z = 2 + iv:\ |v| \leqq 1\},$$
$$\Gamma_3 = \{z = u + i:\ 2 \geqq u \geqq y\}, \quad \Gamma_4 = \{z = y + iv:\ |v| \leqq 1\}.$$

Using Lemma 1 we get

$$|\omega_n(z)| \geqq \begin{cases} \frac{1}{2}(2^{n/2} - 1) & \text{if} \quad z \in \bigcup_{j=1}^3 \Gamma_j, \\ \frac{1}{2}\sqrt{1-y^2}\left[\left(1 + |v|\sqrt{1-y^2}\right)^{n/2} - 1\right] & \text{if} \quad z = y + iv \in \Gamma_4. \end{cases}$$

Thus

$$\frac{1}{2\pi i}\left|\int_{\bigcup_{j=1}^3 \Gamma_j} \frac{(z-y)^l}{\omega_n(z)(z-x)}\,dz\right| = O\left(\max_{z\in\bigcup_{j=1}^3 \Gamma_j} |\omega_n(z)|^{-1}\right) = O(2^{-n/2})$$
$$(|x|, |y| \leqq 1),$$

and the contribution resulting from this is much less than the right hand side of (6).

Now we estimate the remaining integral. Using Lemma 1 again,

$$(7) \qquad \left|\int_{\Gamma_4} \frac{(z-y)^l}{\omega_n(z)(z-x)}\,dz\right| \leqq$$
$$\leqq \frac{4}{\sqrt{1-y^2}} \int_0^1 \frac{v^l}{\left[\left(1 + v\sqrt{1-y^2}\right)^{n/2} - 1\right] |y - x + iv|}\,dv.$$

In order to estimate the integral

$$I = \int_0^1 \frac{v^l}{\left(1 + v\sqrt{1-y^2}\right)^{n/2} - 1}\,dv,$$

we substitute $v = \frac{u}{n\sqrt{1-y^2}}$ to get

$$(8) \qquad I = \frac{1}{\left(n\sqrt{1-y^2}\right)^{l+1}} \int_0^{n\sqrt{1-y^2}} \frac{u^l}{(1 + \frac{u}{n})^{n/2} - 1}\,du \leqq$$

$$\leqq \frac{1}{\left(n\sqrt{1-y^2}\right)^{l+1}}(I_1+I_2),$$

where

$$I_1=\int_0^1 \frac{u^l}{(1+\frac{u}{n})^{n/2}-1}\,du \leqq 2\int_0^1 u^{l-1}\,du=\frac{2}{l},$$

and

$$I_2=\int_1^\infty \frac{u^l}{(1+\frac{u}{n})^{n/2}-1}\,du \leqq 3\int_0^\infty u^l\left(1+\frac{u}{n}\right)^{-n/2}du=$$

$$=3l!\frac{(2n)^l}{\prod_{j=1}^l(n-2j)}=O(1)\qquad(n>2l),$$

where the last integral was calculated by integration by parts. Hence (8) gives

$$I=O\left(\frac{1}{\left(n\sqrt{1-y^2}\right)^{l+1}}\right).$$

Using this estimate, as well as the inequality $|y-x+iv|\geqq\max\{|x-y|,|v|\}$, we obtain from (7)

$$\left|\int_{\Gamma_4}\frac{(z-y)^l}{\omega_n(z)(z-x)}\,dz\right|=O\left(\frac{1}{n^l\left(\sqrt{1-y^2}\right)^{l+1}}\right)\min\left\{1,\frac{1}{n|x-y|\sqrt{1-y^2}}\right\}.$$

Collecting all these estimates, we get the statement of the lemma.

The next lemma can be of independent interest, since it provides an estimate of the error of interpolation which reflects the interpolatory property.

LEMMA 3. *If* $g\in C^l[-1,1]$ $(l\geqq 1)$ *then we have*

$$|\Delta_n(g,x)|=O\left(\frac{|\omega_n(x)|}{n^l}\omega\left(g^{(l)},\frac{1}{n}\right)\log n\right)\qquad(|x|\leqq 1).$$

PROOF. For an arbitrary $x\in[-1,1]$ let

$$|x-x_j|:=\min_{1\leqq k\leqq n}|x-x_k|.$$

(In case this definition of the index j is not unique, take either one of the two possibilities.) Using mean value theorem we get

$$(9) \qquad \left|\Delta_n(g,x)\right| \leqq |x-x_j| \cdot \left\|\Delta_n'(g,x)\right\| \qquad (|x| \leqq 1)$$

where the notation $\|\cdot\|$ indicates supremum norm over the interval $[-1,1]$. We estimate the two factors appearing on the right hand side here separately.

In estimating $|x-x_j|$, we distinguish two cases.

Case 1: $|t-t_j| \leqq \frac{\pi}{4(n-1)}$ (see the notations (1)–(2)). Then, using mean value theorem again,

$$\left|\omega_n(x)\right| = \left|\omega_n(x) - \omega_n(x_j)\right| = |x-x_j| \cdot \left|\omega_n'(\cos\xi_j)\right|$$
$$\left((\xi_j \in (t,t_j),\ |\xi_j - t_j| \leqq \frac{\pi}{4(n-1)}\right).$$

Hence

$$(10) \qquad |x-x_j| = O\left(\frac{\left|\omega_n(x)\right|}{n}\right).$$

Case 2: $\frac{\pi}{4(n-1)} < |t-t_j| \leqq \frac{\pi}{2(n-1)}$. Then

$$|x-x_j| = O\left(\frac{\sin t}{n}\right) = O\left(\frac{\left|\omega_n(x)\right|}{n}\right),$$

which is the same as (10).

In estimating the second factor on the right hand side of (9), we use a well-known theorem of Gopengauz [2] which says that under the conditions of Lemma 3 there exist polynomials $p_n(x)$ of degree at most n such that

$$(11) \qquad \left|g^{(m)}(x) - p_n^{(m)}(x)\right| = O\left(\left(\frac{\sqrt{1-x^2}}{n}\right)^{l-m}\right)\omega\left(g^{(l)}, \frac{\sqrt{1-x^2}}{n}\right)$$
$$(|x| \leqq 1,\ m = 0, \dots, l).$$

Using this with $m=1$ (and without the pointwise estimate) we obtain

$$\left\|\Delta_n'(g)\right\| \leqq \|g' - p_n'\| + \left\|L_n'(p_n - g)\right\| =$$
$$= O(n^{1-l})\omega\left(g^{(l)}, \frac{1}{n}\right) + \left\|L_n'(p_n - g)\right\|.$$

Here applying (11) with $m = 0$,

$$|L_n(p_n - g, x)| \leqq \sum_{k=2}^{n-1} |p_n(x_k) - g(x_k)| \frac{|\omega_n(x)|}{|\omega_n'(x_k)| \cdot |x - x_k|} =$$

$$= O(n^{-l-1})\omega\left(g^{(l)}, \frac{1}{n}\right) |\omega_n(x)| \sum_{k=2}^{n-1} \frac{\sin t_k}{|\cos t - \cos t_k|} =$$

$$= O(n^{-l}\sqrt{1-x^2})\omega\left(g^{(l)}, \frac{1}{n}\right) \log n \qquad (|x| \leqq 1).$$

Hence by a result of V. K. Dzyadyk [1, Theorem 2′] we get

$$\| L_n'(p_n - g)\| = O(n^{1-l})\omega\left(g^{(l)}, \frac{1}{n}\right) \log n.$$

Substituting this as well as (10) into (9) we obtain the statement of the lemma.

Proof of the Theorem. Let $H(f, x)$ be the unique Hermite interpolating polynomial of degree at most $(r+1)(s-1)-1$ satisfying the conditions

$$H^{(m)}(f, a_j) = f^{(m)}(a_j) \qquad (m = 0, \ldots, r;\ j = 1, \ldots, s-1).$$

Then for the function

$$F(x) := f(x) - H(f, x) \in C^r[-1, 1]$$

we evidently have

(12)
$$\Delta_n(F, x) = \Delta_n(f, x) \qquad \left(n \geqq (r+1)(s-1)\right),$$

$$F|_{I_j} \in C^{r+2}(I_j) \qquad (j = 0, \ldots, s-1)$$

and

$$F^{(m)}(a_j) = 0 \qquad (m = 0, \ldots, r;\ j = 1, \ldots, s-1).$$

Thus if we define

$$F_j(x) := \begin{cases} F(x) & \text{if} \quad x \in I_j, \\ 0 & \text{if} \quad x \in [-1, 1] \setminus I_j \end{cases} \qquad (j = 0, \ldots, s-1),$$

then $F_j \in C^r[-1,1] \quad (j=0,\dots,s-1)$ and

$$F(x)=\sum_{j=0}^{s-1}F_j(x). \tag{13}$$

Moreover, define

(14)

$$g_j(x):=F_j(x)+\sum_{m=r+1}^{r+2}\frac{F^{(m)}(a_j-0)(x-a_j)_+^m+F^{(m)}(a_{j+1}+0)(x-a_{j+1})_-^m}{m!}$$

$$(j=0,\dots,s-1).$$

Then by calculating the left and right derivatives of g_j at a_j and a_{j+1} it is easy to see that

$$g_j^{(m)}(a_j)=F^{(m)}(a_j-0),\quad g_j^{(m)}(a_{j+1})=F^{(m)}(a_{j+1}+0)$$

$$(m=r+1,r+2;j=0,\dots,s-1),$$

and therefore $g_j \in C^{r+2}[-1,1] \quad (j=0,\dots,s-1)$. By (4)

$$\omega(g_j^{(r+2)},h)=\omega(F_j^{(r+2)},h)=O\big(\omega(f_j^{(r+2)},h)\big)=O(1/\log 1/h) \tag{15}$$

$$(j=0,\dots,s-1),$$

since $g_j^{(r+2)}(x)$ is constant in $[-1,1]\setminus I_j \quad (j=0,\dots,s-1)$. Now (14) yields

$$\big|\Delta_n(F_j,x)\big|\leqq\big|\Delta_n(g_j,x)\big|+ \tag{16}$$

$$+\sum_{m=r+1}^{r+2}\frac{\big|F^{(m)}(a_j-0)\Delta_n\big((\cdot-a_j)_+^m,x\big)\big|+\big|F^{(m)}(a_{j+1}+0)\Delta_n\big((\cdot-a_{j+1})_-^m,x\big)\big|}{m!}$$

$$(|x|\leqq 1,\ j=0,\dots,s-1).$$

Here applying Lemma 3 with g_j and $r+2$ for g and l, respectively, we get from (15)

(17)

$$\big|\Delta_n(g_j,x)\big|=O\left(\big|\omega_n(x)\big|n^{-r-2}\right)\omega\left(g_j^{(r+2)},\frac{1}{n}\right)\log n=O\left(\big|\omega_n(x)\big|n^{-r-2}\right)$$

$$(j=0,\dots,s-1;\ |x|\leqq 1).$$

On the other hand, applying Lemma 2 (and its analogue for $(x-y)_-^m$) with m and a_j, a_{j+1} for l and y, respectively, we get

(18)
$$\sum_{m=r+1}^{r+2} \frac{\left|F^{(m)}(a_j-0)\Delta_n\big((\cdot-a_j)_+^m,x\big)\right|+\left|F^{(m)}(a_{j+1}+0)\Delta_n\big((\cdot-a_{j+1})_-^m,x\big)\right|}{m!}=$$

$$=O\left(\frac{|\omega_n(x)|}{n^{r+1}}\right)\min\left\{1,\frac{1}{n\max_{1\leqq j\leqq s-1}|x-a_j|}\right\}$$

$$(j=0,\dots,s-1;\ |x|\leqq 1).$$

Finally, (12), (13), (16)–(18) yield

$$\left|\Delta_n(f,x)\right|=\left|\Delta_n(F,x)\right|\leqq$$

$$\leqq\sum_{j=0}^{s-1}\left|\Delta_n(F_j,x)\right|=O\left(\frac{|\omega_n(x)|}{n^{r+1}}\right)\min\left\{1,\frac{1}{n\max_{1\leqq j\leqq s-1}|x-a_j|}\right\}$$

$$(|x|\leqq 1).$$

We do not know if the condition on the order of piecewise differentiability of the function can be further weakened. Nevertheless, if we do not insist on the second estimate in the theorem which gives a better estimate away from the singularities a_j, then the following corollary holds true:

COROLLARY 1. *Let $r\geqq 0$ be a fixed integer, and (3) a fixed partition of the interval $[-1,1]$. If $f(x)\in C^r[-1,1]$ and for $f_j(x):=f|_{I_j}\in C^{r+1}(I_j)$, $j=0,\dots,s-1$ we have*

$$\omega(f_j^{(r+1)},h)=O\left(\frac{1}{\log\frac{1}{h}}\right)\qquad(h>0,\ j=0,\dots,s-1)$$

then

$$\left|\Delta_n(f,x)\right|=O\left(\frac{|\omega_n(x)|}{n^{r+1}}\right)\qquad(|x|\leqq 1).$$

This can be easily seen from the proof of the theorem.

Another consequence (of Lemma 2) is the following

COROLLARY 2. *If $f^{(r)}(x)$ is of bounded variation in* $[-1,1]$ *then*

$$\| \Delta_n(f,x) \| = O(n^{-r}) \int_{-1}^{1} \left| df^{(r)}(y) \right|.$$

This easily follows from the obvious formula

$$\Delta_n(f,x) = \int_{-1}^{1} \Delta_n\big((\cdot - y)_+^r, x\big)\, df^{(r)}(y).$$

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AN EXPLICIT ESTIMATE OF EXPONENTIAL SUMS ASSOCIATED WITH A CUBIC POLYNOMIAL

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1. Introduction

Let $\mathbf{x} = (x_1, \ldots, x_n)$ denote a vector in the space $\mathbf{Z}^n$ where $\mathbf{Z}$ denotes as usual the ring of integers. Let q be a positive integer and f a polynomial in $\mathbf{Z}[\mathbf{x}]$. The exponential sum associated with this polynomial is defined as

$$S(f;q) = \sum e^{\frac{2\pi i f(\mathbf{x})}{q}}$$

where the sum is over a complete set of residues $\mathbf{x}$ modulo q.

As a result of his proof of the Weil conjectures, Deligne [2] showed that if p is a prime, then

$$\left| S(f;p) \right| \leqq (m-1)^n p^{n/2}$$

where m denotes the total degree of the associated polynomial f, under the condition that the homogeneous part of f having the highest degree is non-singular modulo p. Deligne's work opens the way to estimates of the sum associated with any positive integer q. Loxton and Vaughan [6] for example found very precise estimate for the sum in terms of invariants associated with a one-variable polynomial f. However, the general results for polynomials of several variables are less complete.

It can be shown that $S(f;q)$ has a multiplicative property with respect to q (see for example Loxton and Smith [5]). That is if q_1, q_2 have no common factors then there exist integers m_1 and m_2 such that

$$S(f; q_1 q_2) = S(m_2 f, q_1)\, S(m_1 f, q_2).$$

Consequently it suffices to examine exponential sums of the form $S(f;p^\alpha)$.

In this paper we give an estimate for such an exponential sum with f a cubic polynomial with coefficients in the ring $\mathbf{Z}$.

2. Some preliminary results

In the following discussion we will denote $e^{2\pi it/p^\alpha}$ by $e_{p^\alpha}(t)$ for any integer t. Let $f(x,y)$ be a polynomial with integer coefficients. Atan [8] adapted the results of Loxton and Smith [5], to show that the estimate for $S(f;p^\alpha)$ is dependent on $N(f;p^\alpha)$, the number of common solutions to the congruences

$$f_x(x,y)\equiv 0,\quad f_y(x,y)\equiv 0 \mod p^\alpha.$$

Here f_x and f_y denote the usual partial derivatives of f with respect to the variables x and y respectively. We rewrite Atan's assertion as follows.

THEOREM 2.1. *Let p be a prime and $f(x,y)$ be a polynomial in $Z[x,y]$. For $\alpha>1$, let*

$$S(f;p^\alpha)=\sum_{x,y \bmod p^\alpha} e_{p^\alpha}\big(f(x,y)\big)$$

and $\Theta=[\alpha/2]$. Then

$$\left|S(f;p^\alpha)\right| \leqq p^{2(\alpha-\Theta)}N(f;p^\Theta).$$

PROOF. Define $\gamma=\alpha-\Theta$ so that $2\gamma\geqq\alpha$ and $\gamma\geqq\Theta\geqq 1$. Let $\mathbf{z}$ denote the pair (z,z') in Z^2 and $\mathbf{z}=\mathbf{u}+p^\gamma\mathbf{v}$, so that $\mathbf{x}$ runs through the residue classes modulo p^α as $\mathbf{u}$ runs through the residue classes modulo p^γ and $\mathbf{v}$ runs through the residue classes modulo p^Θ. By using the Taylor expansion of $f(\mathbf{x})=f(\mathbf{u}+p^\gamma\mathbf{v})$ we can rewrite $S(f;p^\alpha)$ as follows:

$$S(f;p^\alpha)=\sum_{\mathbf{u} \bmod p^\gamma} e_{p^\alpha}\big(f(\mathbf{u})\big)\sum_{\mathbf{v} \bmod p^\Theta} e_{p^\alpha}\big(p^\gamma \operatorname{grad} f(\mathbf{u})\cdot\mathbf{v}\big).$$

The inner sum clearly vanishes unless both $f_x(\mathbf{u})$ and $f_y(\mathbf{u})$ are congruent to 0 modulo p^α. Under this condition each term in the inner sum is equal to 1. It follows then that the inner sum is equal to $p^{2\Theta}$, and hence

$$S(f;p^\alpha)=p^{2\Theta}\sum e_{p^\alpha}\big(f(\mathbf{u})\big),$$

where the sum is taken over all $\mathbf{x}$ modulo p^α such that

$$\operatorname{grad} f(\mathbf{u})\equiv 0 \mod p^\Theta.$$

Since there are $p^{2(\gamma-\Theta)}$ points $\mathbf{u}$ modulo p^γ corresponding to each solution of the above congruences modulo p^Θ, we have

$$\left|S(f;p^\alpha)\right| \leqq p^{2\Theta+2(\gamma-\Theta)}N(f;p^\Theta)$$

as required.

If α is odd a slightly sharper estimate than the one in Theorem 2.1 can be obtained. Towards this end we define the set

$$K_f(\mathbf{u}) = \{\mathbf{v} = (v, v') \bmod p \colon \mathbf{v}J_f(\mathbf{u}) \equiv 0 \bmod p\}$$

where $\mathbf{u} = (u, u')$ and J_f is the Jacobian matrix

$$J_f(\mathbf{u}) = \begin{bmatrix} f_{xx}(\mathbf{u}) & f_{xy}(\mathbf{u}) \\ f_{xy}(\mathbf{u}) & f_{yy}(\mathbf{u}) \end{bmatrix}.$$

Our result for this category of α is as follows.

THEOREM 2.2. *Let p be a prime and $f(x, y)$ be a polynomial in $Z[x, y]$. Let $\alpha = 2\Theta + 1$ with $\Theta \geqq 1$. Then*

$$|S(f; p^\alpha)| \leqq p^\alpha \sum |K_f(\mathbf{u})|^{1/2}$$

where the sum is taken over all $\mathbf{u} = (u, u')$ modulo p^Θ such that grad $f(\mathbf{u}) \equiv \equiv 0 \bmod p^\Theta$ *and, in addition, when p is odd* grad $f(\mathbf{u}) \cdot \mathbf{v} \equiv 0 \bmod p^{\Theta+1}$ *for all $\mathbf{v}$ in $K_f(\mathbf{u})$.*

PROOF. From the proof of Theorem 2.1

$$S(f; p^\alpha) = p^{2\Theta} \sum e_{p^\alpha}(f(\mathbf{x})),$$

where the sum is taken over all $\mathbf{x} = (x, x')$ modulo p^γ at which $f_x(\mathbf{x})$ and $f_y(\mathbf{x})$ vanish modulo p^Θ and $\gamma = \Theta + 1$. Let

$$\mathbf{x} = \mathbf{u} + p^\Theta \mathbf{v}$$

so that $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$ run through the residue classes modulo p^γ, p^Θ and p respectively. By a Taylor expansion

$$f(\mathbf{x}) = f(\mathbf{u}) + p^\Theta \operatorname{grad} f(\mathbf{u}) \cdot \mathbf{v} + \frac{1}{2} p^{2\Theta} \mathbf{v} J_f(\mathbf{u}) \mathbf{v}^t \pmod{p^\alpha}$$

we obtain

$$S(f; p^\alpha) = p^{2\Theta} \sum e_{p^\alpha}(f(\mathbf{u}))\, G_f(\mathbf{u}),$$

where the sum is taken over all $\mathbf{u}$ modulo p^Θ such that grad $f(\mathbf{u}) \equiv 0 \bmod p^\Theta$ and $G_f(\mathbf{u})$ denotes the Gaussian sum

$$G_f(\mathbf{u}) = \sum_{\mathbf{v} \bmod p} e_p\left(\frac{1}{2} \mathbf{v} J_f(\mathbf{u}) \mathbf{v}^t + p^{-\Theta} \operatorname{grad} f(\mathbf{u}) \cdot \mathbf{v}\right).$$

Now consider

$$\left|G_f(\mathbf{u})\right|^2 = \sum_{\mathbf{v},\mathbf{w}} e_p\left(\frac{1}{2}\mathbf{v}J_f(\mathbf{u})\mathbf{v}^t - \mathbf{w}J_f(\mathbf{u})\mathbf{w}^t + p^{-\Theta}\operatorname{grad} f\cdot(\mathbf{v}-\mathbf{w})\right).$$

Write $\mathbf{v} = \mathbf{w} + \mathbf{z}$ and carry out the summation over $\mathbf{w}$. This gives

$$\left|G_f(\mathbf{u})\right|^2 = p^2 \sum_{\mathbf{z}J_f(\mathbf{u})=0 \bmod p} e_p\left(\frac{1}{2}\mathbf{z}J_f(\mathbf{u})\mathbf{z}^t + p^{-\Theta}\operatorname{grad} f(\mathbf{u})\cdot\mathbf{z}\right).$$

If we replace here $\mathbf{z}$ by $\mathbf{z}+\mathbf{v}$ where $\mathbf{v}$ is any point in $K_f(\mathbf{u})$, we get

$$\left|G_f(\mathbf{u})\right|^2 = e_p\left(\frac{1}{2}\mathbf{v}J_f(\mathbf{u})\mathbf{v}^t + p^{-\Theta}\operatorname{grad} f(\mathbf{u})\cdot\mathbf{v}\right)\left|G_f(\mathbf{u})\right|^2.$$

Hence, $G_f(\mathbf{u})$ is 0 unless $\frac{1}{2}\mathbf{v}J_f(\mathbf{u})\mathbf{v}^t + p^{-\Theta}\operatorname{grad} f(\mathbf{u})\cdot\mathbf{v} \equiv 0 \pmod p$ for all $\mathbf{v}$ in $K_f(\mathbf{u})$. If p is odd, this condition is equivalent to $p^{-\Theta}\operatorname{grad} f(\mathbf{u})\cdot\mathbf{v} \equiv 0$ $(\operatorname{mod} p)$ for all $\mathbf{v}$ in $K_f(\mathbf{u})$ and we have

$$\left|G_f(\mathbf{u})\right|^2 = p^2\left|K_f(\mathbf{u})\right|.$$

From this, we get the estimate in the theorem. If $p=2$, the condition for $G_f(\mathbf{u})$ to be non-zero does not simplify, but we still have $\left|G_f(\mathbf{u})\right|^2 \leqq$ $\leqq p^2\left|K_f(\mathbf{u})\right|$ and the required estimate follows.

From the above it is seen that the estimate for $S(f;p^\alpha)$ is dependent on the estimates of $N(f;p^\Theta)$ and $K_f(\mathbf{u})$. In the following section we will examine these two quantities. In the ensuing discussion p will always denote a prime and for a rational number x, $\operatorname{ord}_p x$ will indicate the highest power of p dividing x. We will set $\operatorname{ord}_p x = \infty$ if $x = 0$.

3. Estimation of $N(\mathbf{f};p^\alpha)$

Let p be a prime and $\mathbf{f} = (f_1,\dots,f_n)$ be an n-tuple of polynomials in $\mathbf{x} = (x_1,\dots,x_n)$ with coefficients in $\mathbf{Z}$. Let $N(\mathbf{f};p^\alpha)$ denote the cardinality of the set

$$V(\mathbf{f};p^\alpha) = \left\{\mathbf{u} \bmod p^\alpha : \mathbf{f}(\mathbf{u}) \equiv 0 \bmod p^\alpha\right\}$$

where $\alpha > 0$ and each component of $\mathbf{u}$ runs through a complete set of residues modulo p^α. The estimation of $N(\mathbf{f};p^\alpha)$ has been the topic of research of many authors. For example Loxton and Smith [5] showed that for $\alpha > 0$ and a one-variable polynomial $f(x)$ in $\mathbf{Z}[x]$, $N(f;p^\alpha) \leqq mp^{\alpha-(\alpha-\delta)/e}$ if $\alpha > \delta$, where m is the number of distinct zeros ξ_i of f that generate its associated algebraic number field K, and δ is the highest power of p such that $D(f) \equiv 0 \pmod{p^\delta}$ where $D(f)$ represents the intersections of the fractional ideals of K generated by the numbers $\frac{f^{(e_i)}(\xi_i)}{e_i!}$, $i \geqq 1$ and $e = \max_j e_j$ with e_j denoting the multiplicity of ξ_j. A similar result was obtained by Chalk and Smith [1] by employing Hensel's Lemma.

In their work Loxton and Smith [5] showed that for $\mathbf{f} = (f_1, \ldots, f_n)$,

$$N(\mathbf{f}; p^\alpha) \leqq \begin{cases} p^{n\alpha} & \text{if } \alpha \geqq n\delta \\ (\operatorname{Deg} \mathbf{f}) p^{n\delta} & \text{if } \alpha < n\delta \end{cases}$$

where δ is the highest power of p dividing the discriminant of f. Atan [9] considered linear polynomials $\mathbf{f} = (f_1, \ldots, f_n)$ in $(x_1, \ldots, x_n)$ with coefficients in the p-adic ring $\mathbf{Z}_p$. He showed that

$$N(\mathbf{f}; p^\alpha) \leqq \min\left\{ p^{n\alpha}, p^{(n-r)\alpha + r\delta} \right\}$$

where δ is the minimum of the p-adic orders of $r \times r$ non-singular submatrices of the reduced coefficient matrix of $\mathbf{f}$.

Let $\mathbf{f} = (f_1, \ldots, f_n)$ be an n-tuple of polynomials in $Z_p[\mathbf{x}]$ where $\mathbf{x} = (x_1, \ldots, x_n)$, $\boldsymbol{\xi}_i$ common zeros of $\mathbf{f}$ and $H_i(\alpha) = \left\{ \mathbf{x} \in \Omega_p^n : \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_i) = \max_j \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_j) \text{ and } \operatorname{ord}_p \mathbf{f}(\mathbf{x}) \geqq \alpha \right\}$ where Ω_p is a complete and algebraically closed p-adic field. Following the method of Loxton and Smith [5] we show below that the cardinality $N(\mathbf{f}; p^\alpha)$ of the set $V(\mathbf{f}; p^\alpha) = \left\{ \mathbf{x} \bmod p^\alpha : \mathbf{f}(\mathbf{x}) \equiv 0 \bmod p^\alpha \right\}$ is dependent on the p-adic distance between a common zero $\boldsymbol{\xi}_i$ of $\mathbf{f}$ and elements $\mathbf{x}$ in $H_i(\alpha)$.

LEMMA 3.1. *Let p be a prime, and $\mathbf{f}$ an n-tuple of polynomials in $\mathbf{x} = (x_1, \ldots, x_n)$ with coefficients in $\mathbf{Z}_p$. Let $\boldsymbol{\xi}_i$ be a common zero of $\mathbf{f}$. Then*

$$N(\mathbf{f}; p^\alpha) \leqq \sum_i p^{n\left(\alpha - \gamma_i(\alpha)\right)}$$

where

$$\gamma_i(\alpha) = \inf_{\mathbf{x} \in H_i(\alpha)} \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_i).$$

PROOF. Consider the set $V_i(\mathbf{f}; p^\alpha)$ of points in $V(\mathbf{f}; p^\alpha)$ that are close p-adically to a common zero $\boldsymbol{\xi}_i$ of $\mathbf{f}$. That is

$$V_i(\mathbf{f}; p^\alpha) - \left\{ \mathbf{x} \in V(\mathbf{f}; p^\alpha) : \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_i) = \max_j \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_j) \right\}.$$

Then,

$$N(\mathbf{f};p^{\alpha}) \leqq \sum_i \operatorname{card} V_i(\mathbf{f};p^{\alpha}). \tag{1}$$

Consider the set

$$H_i(\alpha) = \left\{ \mathbf{x} \in \Omega_p^n : \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_i) = \max_j \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_j) \text{ and } \operatorname{ord}_p \mathbf{f}(\mathbf{x}) \geqq \alpha \right\}.$$

Let

$$\gamma_i(\alpha) = \inf_{\mathbf{x}\in H_i(\alpha)} \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_i) \tag{2}$$

for all i. Now, for every $\alpha \geqq 1$, $V_i(\mathbf{f};p^{\alpha}) \subseteqq H_i(\alpha)$ for all i. Hence,

$$\operatorname{card} V_i(\mathbf{f};p^{\alpha}) \leqq \operatorname{card}\left\{ \mathbf{x} \bmod p^{\alpha} : \operatorname{ord}_p(\mathbf{x} - \boldsymbol{\xi}_i) \geqq \gamma_i(\alpha) \right\}.$$

That is,

$$\operatorname{card} V_i(\mathbf{f};p^{\alpha}) \leqq p^{n\alpha - n\gamma_i(\alpha)} \tag{3}$$

with $\alpha \geqq \gamma_i(\alpha)$ for all i. Our assertion then follows from (1), (2) and (3).

The determination of the size of $\gamma_i(\alpha)$ in the estimate above has been the subject of scrutiny of several researchers (see for example Loxton and Smith [5]). Atan [9] employed the technique of Atan and Loxton [7] as an extension of the p-adic technique of Koblitz [4] called the Newton polyhedron method described briefly below to arrive at the estimates of $\gamma_i(\alpha)$ for certain polynomials of lower degrees.

If p is a prime and $f(x,y) = \sum a_{ij}x^iy^i$ a polynomial with coefficients in Ω_p the completion of the algebraic closure of the field Q_p of p-adic numbers, then the Newton polyhedron of f is defined to be the lower convex hull of the set of points $(i, j, \operatorname{ord}_p a_{ij})$ in the Euclidean space where ord_p denotes the extension of the usual additive p-adic valuation from Q_p to Ω_p, with the convention $\operatorname{ord}_p 0 = \infty$.

Associated with each Newton polyhedron is an indicator diagram which is defined as the plane graph consisting of vertices $(\lambda/\nu,\ \mu/\nu)$ and edges with the former corresponding to the normals (λ, μ, ν) of faces in the polyhedron and the latter joining adjacent vertices corresponding to normals whose faces share a common edge in the Newton polyhedron.

Atan and Loxton showed that (ξ, η) is a zero of f if and only if $(\operatorname{ord}_p \xi,$ $\operatorname{ord}_p \eta)$ is a point on the indicator diagram associated with the Newton polyhedron of f. Suppose (λ, μ) is a simple point of intersection of non-coincident edges of the indicator diagrams associated with polynomials f

and g in $Q_p[x,y]$. Atan [9] showed that if (λ,μ) is not a vertex then there is a point (ξ,η) in Ω_p^2 at which f and g vanish with $\operatorname{ord}_p \xi = \lambda$ and $\operatorname{ord}_p \eta = \mu$.

Using this information as our tool we arrive at the estimate $\gamma_i(\alpha)$ associated with the two-variable polynomial

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex + my + n \tag{0}$$

with coefficients in the ring of p-adic integers $\mathbf{Z}_p$, with the property that $c^2 - 3bd$, $bc - 9ad$ and $b^2 - 3ac$ are non-zero, in the following lemma.

LEMMA 3.2. *Let p be a prime and* (0) *a polynomial in $\mathbf{Z}_p[x,y]$ with non-vanishing coefficients in the homogeneous part of degree* 3. *Let $\delta = \max\{\operatorname{ord}_p 3a, \operatorname{ord}_p b, \operatorname{ord}_p c, \operatorname{ord}_p 3d\}$. Suppose (x_0,y_0) is in Ω_p^2 with $\operatorname{ord}_p f_x(x_0,y_0)$, $\operatorname{ord}_p f_y(x_0,y_0) \geqq \alpha$. If $\alpha > \delta$, then there is a zero (ξ,η) of f_x and f_y in Ω_p^2 such that $\operatorname{ord}_p f_x(\xi - x_0, \eta - y_0)$, $\operatorname{ord}_p f_y(\xi - x_0, \eta - y_0) \geqq \frac{1}{2}(\alpha - \delta)$.*

PROOF. Let $X = x - x_0$, $Y = y - y_0$, and $h = f_x$, $g = f_y$. Then

$$h(X,Y) = 3aX^2 + 2bXY + cY^2 + h_xX + h_yY + h_0,$$

$$g(X,Y) = bX^2 + 2cXY + 3dY^2 + g_xX + g_yY + g_0$$

where l_z denotes the partial derivatives of the polynomial l with respect to z defined at (x_0,y_0) and $l_0 = l(x_0,y_0)$.

Let α,β be the roots of the quadratic polynomial

$$u(x) = (c^2 - 3bd)x^2 + (bc - 9ad)x + b^2 - 3ac.$$

If $\alpha \neq \beta$, then it can be shown that the polynomials

$$H(U,V) = (3a + b\alpha)(h + \alpha g), \quad G(U,V) = (3a + b\beta)(h + \beta g)$$

with

$$U = (3a + b\alpha)X + (b + c\alpha)Y, \quad V = (3a + b\beta)X + (b + c\beta)Y$$

will have a simple intersection in the indicator diagrams associated with their respective Newton polyhedrons. By a theorem of Atan and Loxton [7] and resubstitution of variables there is a common zero (ξ,η) of h and g with $\operatorname{ord}_p(\xi - x_0)$, $\operatorname{ord}_p(\eta - y_0) \geqq \frac{1}{2}(\alpha - \delta_0)$ where $\delta_0 = \max\{\operatorname{ord}_p 3a, \operatorname{ord}_p b\}$. We obtain the required estimate from our hypothesis since clearly $\delta_0 \leqq \delta$.

Suppose now that $\alpha = \beta$. Consider the linear combinations of h and g as follows:

$$H(X,Y) = (3a + b\alpha)(h + \alpha g), \tag{1}$$

$$G(X,Y) = (3a + b\alpha)^2(cf - bg). \tag{2}$$

Set

$$U = (3a + b\alpha)X + (b + c\alpha)Y, \quad V = (3a + b\alpha)X - (b + c\alpha)Y.$$

Then (1) and (2) will become

$$H(U,V) = U^2 + AU + (3a + b\alpha)(h_0 + \alpha g_0), \tag{3}$$

$$G(U,V) = BUV + CU + DV + (3a + b\alpha)^2(ch_0 - bg_0) \tag{4}$$

where

$$A = 2\big[(3a + b\alpha)x_0 + (b + c\alpha)y_0\big], \quad B = b^2 - 3ac,$$
$$C = (b^2 - 3ac)\big[(3a + b\alpha)x_0 - (b + c\alpha)y_0\big],$$
$$D = (b^2 - 3ac)\big[(3a + b\alpha)x_0 + (b + c\alpha)y_0\big].$$

Now let

$$W = BV + C, \quad T = U + D/B.$$

Since α is a double root of $u(x)$ we would have

$$(c^2 - 3bd)(3a + b\alpha)^2 = (b^2 - 3ac)(b + c\alpha)^2$$

Hence with the substitution of T and W in (3) and (4) we will have

$$H(T,W) = T^2 - (3a + b\alpha)(h_0 + \alpha g_0), \tag{5}$$

$$G(T,W) = TW. \tag{6}$$

Consider the indicator diagrams associated with the polynomials (5) and (6). By a method of Atan and Abdullah [10] the indicator diagrams associated with the Newton polyhedrons of these polynomials will have a simple intersection at the point

$$(\lambda, \mu) = \left(\frac{1}{2}\operatorname{ord}_p (3a + b\alpha)(h_0 + \alpha g_0), \infty\right).$$

Hence by a theorem of Atan and Loxton [7], H and G will have a common zero (T_0, W_0) such that

$$\operatorname{ord}_p T_0 = \lambda, \quad \operatorname{ord}_p W_0 = \mu.$$

This remark and resubstitution of variables together with our hypothesis will lead us to our assertion as above.

The following theorem gives the estimate for $N(f_x, f_y; p^\alpha)$ where f is as in the above lemma. The proof follows from the results of Lemmas 3.1 and 3.2, and the fact that by a theorem of Bezout (see for example [3]) f_x and f_y have at most 4 common zeros.

THEOREM 3.1. *Let p be a prime and* (0) *a polynomial in $Z[x,y]$ with non-vanishing terms in its homogeneous part of degree* 3. *Let $\alpha > 0$ and $\delta = \max\{\operatorname{ord}_p 3a, \operatorname{ord}_p b, \operatorname{ord}_p c, \operatorname{ord}_p 3d\}$. Then*

$$N(f_x, f_y; p^\alpha) \leqq \min\{p^{2\alpha}, 4p^{\alpha+\delta}\}.$$

4. Estimation of $S(f; p^\alpha)$

Let f be a two-variable polynomial with integer coefficients of total degree m and p a prime. From the work of Deligne on Weyl's conjecture it can be shown that

$$|S(f; p^\alpha)| \leqq (m-1)^2 p$$

under suitable conditions on f.

Let p be an odd prime and $\alpha > 1$. If $f(x,y) = ax^3 + bx^2y + cx + dy + e$ is a polynomial in $\mathbf{Z}[x,y]$, and $\delta = \max\left\{\operatorname{ord}_p 3a, \frac{3}{2}\operatorname{ord}_p b\right\}$, Atan [8] showed that for this cubic polynomial

$$|S(f; p^\alpha)| \leqq \min\left\{p^{2\alpha}, 4p^{\frac{3\alpha}{2}+\delta}\right\}.$$

In the following theorem we will consider a more general cubic polynomial than the one above of the form (0) with coefficients in $\mathbf{Z}$ and we will show that δ is in fact dependent on the coefficients of the dominant terms of f, provided that each term in this homogeneous portion of highest degree of f is non-zero. The assertion generalizes and improves the result as stated immediately above especially in the determination of the value of δ.

THEOREM 4.1. *Let p be an odd prime and $\alpha > 1$. Let* (0) *be a polynomial in $\mathbf{Z}[x,y]$, with non-zero coefficients in its cubic segment. Let $\delta = \max\{\operatorname{ord}_p 3a, \operatorname{ord}_p b, \operatorname{ord}_p c, \operatorname{ord}_p 3d\}$. Then*

$$|S(f; p^\alpha)| \leqq \min\left\{p^{2\alpha}, 4p^{\frac{3\alpha}{2}+\delta}\right\}.$$

PROOF. In Theorem 3.1 it is shown that

$$N(f_x, f_y; p^\alpha) \leqq \min\{p^{2\Theta}, 4p^{\Theta+\delta}\}$$

where $\Theta = [\alpha/2]$. If $\alpha = 2\Theta$, it follows from Theorem 2.1 that $|S(f;p^\alpha)| \leqq \leqq p^{2(\alpha-\Theta)} \min\{p^{2\Theta}, 4p^{\Theta+\delta}\}$ which lead us to the required estimate.

Suppose now that $\alpha = 2\Theta + 1$ with $\Theta \geqq 1$. We will apply the result of Theorem 2.2 in this case. If $D = (bm - ce)^2 - (3am - be)(cm - 3de)$ is not divisible by p, then the congruences

$$f_x = 3ax^2 + 2bxy + cy^2 + e, \quad f_y = bx^2 + 2cxy + 3dy^2 + m$$

and

$$|J_f| = (12ac - 4b^2)x^2 + (36ad - 4bc)xy + (12bd - 4c^2)y^2 \equiv 0 (\mathrm{mod}\ p)$$

do not have a common solution. Thus, in this case each term in the sum $\sum |K_f(\mathbf{u})|^{1/2}$ is 1. Consequently,

$$|S(f;p^\alpha)| \leqq p^\alpha N(f;p^\Theta)$$

and the required estimate follows. If D is divisible by p then there are two possibilities in $\sum |K_f(\mathbf{u})|^{1/2}$. If $|K_f(\mathbf{u})| \leqq 1$ then the term corresponding to $\mathbf{u}$ is counted with weight at most $p^{\alpha+1/2}$. If $|K_f(\mathbf{u})| = 2$ then the term corresponding to $\mathbf{u}$ must satisfy the stronger congruence grad $f(\mathbf{u}) \equiv \equiv 0\ (\mathrm{mod}\ p^{\Theta+1})$ and hence must be counted with weight at most p^α. As a result we would have

$$|S(f;p^\alpha)| \leqq p^{\alpha+1/2} N(f;p^\Theta) \leqq p^{\alpha+1/2} \min\{p^{2\Theta}, 4p^{\Theta+\delta}\}$$

and the estimate as required follows.

5. Conclusion

In this paper we obtained explicit estimate for $S(f;p^\alpha)$ for a more general cubic polynomial f than the one considered in an earlier work. The result generalizes and improves that in the previous work and give some indications on how the general case should be examined especially in the search for the most suitable discriminant analogous to the one-variable case.

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ADDITIVE GROUPS OF TRIVIAL NEAR-RINGS

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Groups G in this note are not necessarily abelian, but the group operation will nevertheless be denoted by $+$. In order to remain consistent with this notation, direct products of groups will be called direct sums. The semigroup of endomorphisms of G will be denoted by $E(G)$. All near-rings are left near-rings. The additive group of a near-ring R will be denoted by R^+. A near-ring R, with $R^+ = G$, is said to be trivial if there exists a subset $S \subseteqq G$ such that multiplication in R is defined by

$$x \cdot y = \begin{cases} y, & \text{if } x \in S \\ 0, & \text{otherwise.} \end{cases}$$

Clearly if $|G| \leqq 2$ then every near-ring R with $R^+ = G$ is trivial. In [1, Problem 2.16] Clay posed the problem whether or not there are other groups satisfying this property. Some partial results concerning this problem are obtained in this note, including an answer for abelian groups, and finite groups.

DEFINITION. A group G is said to be a *TNR-group* (trivial near-ring), if every near-ring R, with $R^+ = G$, is trivial.

THEOREM 1. *Let G be a group, and let $\phi \in E(G)$ such that $\phi \neq 1_G$, $\phi \neq$ $\neq 0$, and either* (1) $\phi^2 = \phi$, (2) $\phi^2 = 1_G$, *or* (3) $\phi^2 = 0$. *Then G is not a TNR-group.*

PROOF. Suppose that ϕ satisfies (1). Let $R = (G, +, \cdot)$, with multiplication in R defined by

$$x \cdot y = \begin{cases} 0, & \text{if } x \in \ker \phi \\ \phi(y), & \text{otherwise.} \end{cases}$$

Let $x, y, z \in R$. If $(x \in \ker \phi) \vee (y \in \ker \phi)$ then it is readily seen that $(xy)z =$ $= x(yz) = 0$. If $(x \notin \ker \phi) \wedge (y \notin \ker \phi)$ then $(xy)z = x(yz) = \phi(z)$, and so multiplication in R is associative. It is readily seen that $x(y + z) = xy + xz$ for all $x, y, z \in R$.

CLAIM. *There exists $a \in R$ which satisfies $\phi(a) \neq 0$, and $\phi(a) \neq a$.*

PROOF OF CLAIM. Since $\phi \neq 0$ there exists $a_1 \in R$ such that $\phi(a_1) \neq$ $\neq 0$. It may be assumed that $\phi(a_1) = a_1$. Since $\phi \neq 1_G$ there exists $a_2 \in R$

such that $\phi(a_2) \neq a_2$. It may be assumed that $\phi(a_2) = 0$. Let $a = a_1 + a_2$. Then $\phi(a) = a_1$ with $a_1 \neq 0$, and $a_1 \neq a$. Let a be as in the Claim. Then $a \cdot a = \phi(a)$, so $a \cdot a \neq 0$, and $a \cdot a \neq a$, i.e, R is not trivial.

Suppose that ϕ satisfies (2). Let $a \in G$ such that $\phi(a) \neq 0$, and $\phi(a) \neq a$. Let $R = (G, +, \cdot)$ with multiplication in R defined by

$$x \cdot y = \begin{cases} \phi(y) & \text{if } x = a \\ y & \text{if } x = \phi(a) \\ 0 & \text{otherwise.} \end{cases}$$

Let $x, y, z \in R$. If $\big(x \notin \{a, \phi(a)\}\big) \vee \big(y \notin \{a, \phi(a)\}\big)$ then it is readily seen that $(xy)z = x(yz) = 0$. If $\big(x \in \{a, \phi(a)\}\big) \wedge \big(y \in \{a, \phi(a)\}\big)$ then a direct computation shows that $(xy)z = x(yz)$ in all four cases. Therefore multiplication is associative in R. Again $x(y + z) = xy + xz$ for all $x, y, z \in \in R$. Since $a \cdot a = \phi(a)$, and $\phi(a) \neq 0$, or a, it follows that R is not trivial.

Suppose that ϕ satisfies (3). Let $R = (G, +, \cdot)$ with multiplication defined by

$$x \cdot y = \begin{cases} 0 & \text{if } x \in \operatorname{im} \phi \\ \phi(y) & \text{if } x \notin \operatorname{im} \phi. \end{cases}$$

It is easy to see that R is a near-ring. Let $a \in R$ such that $\phi(a) \neq 0$. Since $\phi^2 = 0$, it follows that $a \notin \operatorname{im} \phi$. Therefore $a \cdot a = \phi(a)$ with $\phi(a) \neq a$, and $\phi(a) \neq 0$, which yields that R is not trivial.

COROLLARY 2. *Let G be a non-trivial semi-direct product of a group K by a group H. Then G is not a TNR-group.*

PROOF. The natural projection of G onto H along K satisfies condition (1) of Theorem 1.

COROLLARY 3. *An abelian group G is a TNR-group if and only if $|G| \leqq \leqq 2$.*

PROOF. Let G be an abelian group with $|G| > 2$. The map $\phi : G \to G$ defined by $\phi(x) = -x$ for all $x \in G$, either satisfies condition (2) of Theorem 1, or $2 \cdot x = 0$ for all $x \in R$, in which case G is the direct sum of α copies of $Z(2)$, with $\alpha > 1$. Corollary 2 implies that G is not a TNR-group. Actually there exists a field F with $F^+ = G$; choose F to be a field extension of degree α of the prime field of order 2. Let $a \in F$, $a \neq 0$, $a \neq 1$. Then $a \cdot a \neq 0$, or a so F is not a trivial near-ring.

COROLLARY 4. *Let G be a group such that $G/Z(G)$ possesses an element of even order ($Z(G) =$ the center of G). Then G is not a TNR-group.*

PROOF. There exists an element $a \in G$, such that $a \notin Z(G)$, but $2a \in \in Z(G)$. Conjugation by a satisfies condition (2) of Theorem 1.

COROLLARY 5. *Let G be a finite TNR-group with $|G| > 2$. Then G is an odd order group.*

PROOF. Let $G_2 = \{a \in G \mid |a| = \text{ a power of } 2\}$. Corollary 4 yields that G_2 is a subgroup of G, and that $G_2 \leqq Z(G)$. Since $(|G_2|, |G/G_2|) = 1$, it follows that G is a splitting extension of G_2, [2, Theorem 15.2.2], and so $G_2 = \{0\}$ by Corollaries 2 and 3.

THEOREM 6. *Let G be a finite group. Then G is a TNR-group if and only if $|G| \leqq 2$.*

PROOF. Suppose that G is a finite TNR-group, with $|G| > 2$. Let G' be the commutator subgroup of G, and let p be a prime dividing $|G/G'|$. Since G is solvable by Corollary 5 and the Feit–Thompson Odd Order Theorem, G/G' is non-trivial, and has a cyclic direct summand of order p^k generated by $\overline{a}$, with $k \geqq 1$. Suppose there exists $b \in G'$, with $|b| = p$. Let $\vartheta : G \to \to G/G'$ be the canonical epimorphism, let π be a projection of G/G' onto $(\overline{a})$, and let $\rho : (\overline{a}) \to G$ be the homomorphism induced by the map $\overline{a} \mapsto \mapsto b$. Then $\phi = \rho \circ \pi \circ \vartheta$ is an endomorphism of G with $\phi \neq 0$, but $\phi^2 = 0$, a contradiction. Therefore $p \nmid |G'|$ for every prime p dividing the order of G/G', and so $(|G'|, |G/G'|) = 1$. By, [2, Theorem 15.2.2], it follows that G is a semi-direct sum of G' by a subgroup H of G, contradicting Corollary 2.

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ON THE INTEGRABILITY AND L^1-CONVERGENCE OF DOUBLE TRIGONOMETRIC SERIES. II

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1. Introduction

We consider double cosine series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k a_{jk}\cos jx\cos ky, \tag{1.1}$$

double sine series

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}\sin jx\sin ky, \tag{1.2}$$

and cosine-sine series

$$\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\lambda_j a_{jk}\cos jx\sin ky \tag{1.3}$$

on the positive quadrant $\mathbf{T}^2 := [0,\pi]\times[0,\pi]$ of the two-dimensional torus, where $\lambda_0 = 1/2$ and $\lambda_j = 1$ if $j \geqq 1$. Our basic assumption is that the real coefficients a_{jk} form a null sequence of bounded variation, that is,

$$a_{jk}\to 0 \quad \text{as} \quad j+k\to\infty \tag{1.4}$$

and

$$\sum_j\sum_k|\Delta_{11}a_{jk}|<\infty. \tag{1.5}$$

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We remind the reader that the differences $\Delta_{pq}a_{jk}$ are defined for all pairs of nonnegative integers p and q as follows:

$$\Delta_{pq}a_{jk} := \sum_{i=0}^{p}\sum_{l=0}^{q}(-1)^{i+l}\binom{p}{i}\binom{q}{l}a_{j+i,k+l} \quad (j,k \geqq 0).$$

Then the following recurrence relations hold: $\Delta_{00}a_{jk} = a_{jk}$,

$$\Delta_{pq}a_{jk} = \Delta_{p-1,q}a_{jk} - \Delta_{p-1,q}a_{j+1,k} \quad (p \geqq 1),$$

$$\Delta_{pq}a_{jk} = \Delta_{p,q-1}a_{jk} - \Delta_{p,q-1}a_{j,k+1} \quad (q \geqq 1).$$

The next simple observation will be useful on many occasions: If (1.4) is satisfied and we have $\Delta_{pq}a_{jk} \geqq 0$ $(j,k \geqq 0)$ for some $p,q \geqq 0$, then we also have $\Delta_{p_1,q_1}a_{jk} \geqq 0$ $(j,k \geqq 0)$ for all p_1, q_1, where $0 \leqq p_1 \leqq p$ and $0 \leqq q_1 \leqq q$. Consequently, the sequences $\{\Delta_{p_1,q_1}a_{jk}\}$ are nonincreasing both in j and in k whenever either $p_1 < p$ or $q_1 < q$.

Now, it is proved in [3] that, under conditions (1.4) and (1.5), series (1.1) converges to a function $f(x,y)$, say, for all $(x,y) \in \mathbf{T}^2$ with $x \neq 0$, $y \neq 0$, in Pringsheim's sense:

$$(1.6) \quad s_{mn}(x,y) := \sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k a_{jk}\cos jx \cos ky \to f(x,y) \quad \text{as} \quad m,n \to \infty.$$

Analogously, under conditions (1.4) and (1.5), series (1.2) converges to a function $g(x,y)$ for all $(x,y) \in \mathbf{T}^2$, and series (1.3) converges to a function $h(x,y)$ for all $(x,y) \in \mathbf{T}^2$ with $x \neq 0$; in Pringsheim's sense in each case. We denote by $s_{mn}(x,y)$ the rectangular partial sums of each series in (1.1)–(1.3).

We will be concerned with the following problems:

(i) the sum of one of the series (1.1)–(1.3) is integrable on $\mathbf{T}^2$ in Lebesgue's sense, in sign: $\in L^1(\mathbf{T}^2)$;

(ii) the series converges in L^1-norm.

We denote by $\|\cdot\|$ the two-dimensional $L^1(\mathbf{T}^2)$-norm. Occasionally, $\|\cdot\|$ may stand for the one-dimensional $L^1(\mathbf{T})$-norm, $\mathbf{T} := [0,\pi]$. However, it will be clear from the context what the case is.

2. Double cosine series

We will consider double sequences $\{a_{jk}\}$ satisfying either condition (2.1) or (2.2), where

$$(2.1) \qquad \Delta_{21}a_{jk} \geqq 0 \quad \text{and} \quad \Delta_{12}a_{jk} \geqq 0 \quad (j,k \geqq 0)$$

(may be called convex case) and

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(j+1)(k+1)|\Delta_{22}a_{jk}| < \infty \tag{2.2}$$

(may be called quasiconvex case).

REMARKS. (a) Condition (2.1) is equivalent to the fact that the sequence $\{\Delta_{11}a_{jk}\}$ is nonincreasing both in j and in k.

(b) Condition (1.4) together with (2.1) or (2.2) imply (1.5).

(c) Condition (1.4) and

$$\Delta_{22}a_{jk} \geqq 0 \quad (j,k \geqq 0),$$

which is stronger than (2.1), imply (2.2). It is an open problem whether (1.4) and (2.1) imply (2.2) or not.

The following theorem was essentially proved in [3, pp. 206–211].

THEOREM 1. (i) *If* $\{a_{jk}\}$ *satisfies* (1.4) *and* (2.1), *then the sum* f *of series* (1.1) *is in* $L^1(\mathbf{T}^2)$ *and* (1.1) *is the Fourier series of* f.

(ii) *If, in addition,*

$$a_{m0}\ln m \to 0 \quad \text{and} \quad a_{0n}\ln n \to 0 \quad \text{as} \quad m,n \to \infty, \tag{2.3}$$

then

$$\|s_{mn} - f\| \to 0 \quad \text{as} \quad m,n \to \infty \tag{2.4}$$

if and only if

$$a_{mn}\ln m \ln n \to 0 \quad \text{as} \quad m,n \to \infty. \tag{2.5}$$

The first part of the next theorem is also contained in [3, Corollary 3].

THEOREM 2. (i) *If* $\{a_{jk}\}$ *satisfies* (1.4) *and* (2.2), *then the sum* f *of series* (1.1) *is in* $L^1(\mathbf{T}^2)$ *and* (1.1) *is the Fourier series of* f.

(ii) *If, in addition,*

$$(\ln n)\sum_{j=0}^{\infty}(j+1)|\Delta_{20}a_{jn}| \to 0 \quad \text{and} \quad n \to \infty \tag{2.6}$$

and

$$(\ln m)\sum_{k=0}^{\infty}(k+1)|\Delta_{02}a_{mk}| \to 0 \quad \text{as} \quad m \to \infty, \tag{2.7}$$

then statements (2.4) *and* (2.5) *are equivalent.*

REMARKS. (a) It follows from (2.2) that the single sequences $\{a_{jn}: j \geqq 0\}$ are also quasiconvex for each $n \geqq 0$ and

$$\sum_{j=0}^{\infty}(j+1)|\Delta_{20}a_{jn}| \to 0 \quad \text{as} \quad n \to \infty.$$

But (2.6) requires more than this. Analogous observations hold for the single sequences $\{a_{mk}: k \geqq 0\}$, too.

(b) If

$$\Delta_{20}a_{jk} \geqq 0 \quad \text{and} \quad \Delta_{02}a_{jk} \geqq 0 \quad (j, k \geqq 0),$$

then (2.6) and (2.7) are equivalent to (2.3), respectively. So, [3, Corollary 3] is a particular case of Theorem 2.

(c) Theorems 1 and 2 extend Kolmogorov's results [1] (see also [7, pp. 109–110]) from single to double cosine series.

PROOF OF PART (ii) IN THEOREM 2. Let

$$I_m := \{2^m, 2^m + 1, \dots, 2^{m+1} - 1\} \quad (m \geqq 0). \tag{2.8}$$

Since

$$\sum_{m=0}^{\infty} 2^m \max_{j \in I_m} |\Delta_{10}a_{jn}| \leqq \sum_{m=0}^{\infty} 2^m \sum_{j=2^m}^{\infty} |\Delta_{20}a_{jn}| \leqq$$

$$\leqq \sum_{j=1}^{\infty} |\Delta_{20}a_{jn}| + 2\sum_{i=2}^{\infty}\sum_{j=i}^{\infty} |\Delta_{20}a_{jn}| \leqq 2\sum_{j=1}^{\infty} j|\Delta_{20}a_{jn}|$$

and

$$|\Delta_{10}a_{0n}| \leqq \sum_{j=0}^{\infty} |\Delta_{20}a_{jn}|,$$

condition (2.6) implies [3, condition (1.13)] for $p = \infty$. Analogously, (2.7) implies [3, condition (1.14)] for $p = \infty$. Thus, [3, Corollary 2] applies and gives the wanted equivalence (2.4)$\Leftrightarrow$ (2.5).

3. Double sine series

The following theorem was partly proved in [3, pp. 214–217].

THEOREM 3. *If* $\{a_{jk}\}$ *satisfies* (1.4) *and*

$$\Delta_{11} a_{jk} \geqq 0 \quad (j, k \geqq 1), \tag{3.1}$$

then the sum g *of series* (1.2) *is in* $L^1(\mathbf{T}^2)$ *if and only if*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\Delta_{11} a_{jk}) \ln(j+1) \ln(k+1) < \infty, \tag{3.2}$$

in which case

$$\|s_{mn} - g\| \to 0 \quad as \quad m, n \to \infty;$$

consequently, (1.2) *is the Fourier series of* g.

REMARKS. (a) Under (1.4) and (3.1), condition (3.2) is equivalent to

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} < \infty.$$

(b) Conditions (1.4), (3.1) and (3.2) imply (2.5).

(c) Theorem 3 extends W. H. Young's result [6] (see also [7, pp. 111–112]) from single to double sine series.

PROOF OF THEOREM 3. The proofs of sufficiency and L^1-convergence are already contained in the proof of [3, Theorem 5].

Necessity. By summation by parts,

$$s_{m,n}(x, y) := \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \sin jx \sin ky = \tag{3.3}$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{n} \widetilde{D}_j(x) \widetilde{D}_k(y) \Delta_{11} a_{jk} + \sum_{j=1}^{m} \widetilde{D}_j(x) \widetilde{D}_n(y) \Delta_{10} a_{j,n+1} +$$

$$+ \sum_{k=1}^{n} \widetilde{D}_m(x) \widetilde{D}_k(y) \Delta_{01} a_{m+1,k} + a_{m+1,n+1} \widetilde{D}_m(x) \widetilde{D}_n(y),$$

where

$$\widetilde{D}_m(x) := \sum_{j=1}^{m} \sin jx = \frac{\cos(x/2) - \cos(m+1/2)x}{2\sin(x/2)} \quad (m \geqq 1)$$

is the conjugate Dirichlet kernel. Following an argument similar to the one in [3, pp. 205–206] for the case of double cosine series, we obtain that series (1.2) converges in Pringsheim's sense to the function g defined by

$$g(x,y) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \widetilde{D}_j(x)\widetilde{D}_k(y)\Delta_{11}a_{jk}$$

for all $(x,y) \in \mathbf{T}^2$.

We shall use the representation

$$\widetilde{D}_m(x) = \widetilde{D}^*_m(x) + \frac{1}{2}\sin mx, \quad \text{where} \quad \widetilde{D}^*_m(x) := \frac{1-\cos mx}{2\tan(x/2)}$$

is nonnegative. It is easy to see that there exist positive constants C_1 and C_2 such that

$$C_1 \ln(m+1) \leqq \|\widetilde{D}^*_m\| \leqq C_2 \ln(m+1) \quad (m \geqq 1). \tag{3.4}$$

Accordingly, we may write

$$g(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \widetilde{D}^*_j(x)\widetilde{D}^*_k(y)\Delta_{11}a_{jk} + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \widetilde{D}^*_j(x)(\sin ky)\Delta_{11}a_{jk} +$$

$$+\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\sin jx)\widetilde{D}^*_k(y)\Delta_{11}a_{jk} + \frac{1}{4}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\sin jx)(\sin ky)\Delta_{11}a_{jk} :=$$

$$:= g^*(x,y) + g_1(x,y) + g_2(x,y) + g_3(x,y), \quad \text{say.}$$

By (3.2), g_3 is continuous. Clearly,

$$|g_1(x,y)| \leqq \frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \widetilde{D}^*_j(x)\Delta_{11}a_{jk}.$$

By (3.4) and (3.2),

$$\|g_1\| \leqq C_2 \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\Delta_{11}a_{jk})\ln(j+1) < \infty.$$

This means $g_1 \in L^1(\mathbf{T}^2)$. Similarly, $g_2 \in L^1(\mathbf{T}^2)$.

To sum up, $g \in L^1(\mathbf{T}^2)$ if and only if $g^* \in L^1(\mathbf{T}^2)$. By (3.4) again,

$$\|g^*\| \geqq C_1 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\Delta_{11} a_{jk}) \ln(j+1) \ln(k+1).$$

This completes the proof of necessity.

4. Cosine-sine series

This type of series was not considered in [3]. We shall prove the following

THEOREM 4. (i) *If* $\{a_{jk}\}$ *satisfies* (1.4),

$$\Delta_{21} a_{jk} \geqq 0 \quad (j \geqq 0,\ k \geqq 1), \tag{4.1}$$

and

$$\sum_{k=1}^{\infty} (\Delta_{01} a_{0k}) \ln(k+1) < \infty, \tag{4.2}$$

then the sum h *of series* (1.3) *is in* $L^1(\mathbf{T}^2)$ *and* (1.3) *is the Fourier series of* h.

(ii) *If, in addition,*

$$(\ln m) \sum_{k=1}^{\infty} (\Delta_{01} a_{mk}) \ln(k+1) \to 0 \quad \textit{as} \quad m \to \infty, \tag{4.3}$$

then

$$\|s_{mn} - h\| \to 0 \quad \textit{as} \quad m, n \to \infty. \tag{4.4}$$

REMARKS. (a) Condition (4.1) is equivalent to the fact that the single sequences $\{\Delta_{11} a_{jk} \colon j \geqq 0\}$ are nonincreasing for each $k \geqq 1$.

(b) Under (1.4) and (4.1), condition (4.2) is equivalent to

$$\sum_{j=1}^{\infty} \frac{a_{0k}}{k} < \infty,$$

and also equivalent to

$$\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} (\Delta_{11} a_{jk}) \ln(k+1) < \infty;$$

while (4.3) is equivalent to

$$(\ln m)\sum_{j=1}^{\infty}\frac{a_{mk}}{k}\to 0 \quad \text{as} \quad m\to\infty,$$

and also equivalent to

$$(\ln m)\sum_{j=m}^{\infty}\sum_{k=1}^{\infty}(\Delta_{11}a_{jk})\ln(k+1)\to\infty \quad \text{and} \quad m\to\infty.$$

(c) It follows from (1.4), (4.1) and (4.2) that

$$\sum_{k=1}^{\infty}(\Delta_{01}a_{mk})\ln(k+1)\to 0 \quad \text{as} \quad m\to\infty, \tag{4.5}$$

in a nonincreasing way. But (4.3) requires more than this.

(d) Furthermore, under (1.4), (4.1) and (4.2), we have (1.5), the second limit relation in (2.3), and (2.5).

To prove Theorem 4, we begin with an inequality due to Sidon [4] (see also [5]).

LEMMA. *For every sequence $\{b_j\}$ of real numbers and integer $m>0$,*

$$\left\|\sum_{j=0}^{m}b_jD_j\right\|\leqq 2(m+1)\max_{0\leqq j\leqq m}|b_j|.$$

Hence it follows immediately that

$$\left\|\sum_{j=0}^{\infty}b_jD_j\right\|\leqq 4\left\{|b_0|+\sum_{m=0}^{\infty}2^m\max_{j\in I_m}|b_j|\right\}. \tag{4.6}$$

First, we prove a more general result than Theorem 4.

THEOREM 5. *If $\{a_{jk}\}$ satisfies (1.4) and*

$$\sum_{k=1}^{\infty}\ln(k+1)\left\{|\Delta_{11}a_{0k}|+\sum_{m=0}^{\infty}2^m\max_{j\in I_m}|\Delta_{11}a_{jk}|\right\}<\infty, \tag{4.7}$$

then the sum h of series (1.3) is in $L^1(\mathbf{T}^2)$ and (1.3) is the Fourier series of h.

PROOF OF THEOREM 5. By summation by parts,

$$s_{mn}(x,y) := \sum_{j=0}^{m}\sum_{k=1}^{n} \lambda_j a_{jk} \cos jx \sin ky = \tag{4.8}$$

$$= \sum_{j=0}^{m}\sum_{k=1}^{n} D_j(x)\widetilde{D}_k(y)\Delta_{11}a_{jk} + \sum_{j=0}^{m} D_j(x)\widetilde{D}_n(y)\Delta_{10}a_{j,n+1}+$$

$$+\sum_{k=1}^{n} D_m(x)\widetilde{D}_k(y)\Delta_{01}a_{m+1,k} + a_{m+1,n+1}D_m(x)\widetilde{D}_n(y)$$

(cf. (3.3)). Hence it follows that series (1.3) converges in Pringsheim's sense to the function

$$h(x,y) := \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} D_j(x)\widetilde{D}_k(y)\Delta_{11}a_{jk} \tag{4.9}$$

for all $(x,y) \in \mathbf{T}^2$ with $x \neq 0$. Analogously to (3.4), there exists a constant C such that

$$\|D_m\|, \|\widetilde{D}_m\| \leqq C\ln(m+1) \quad (m \geqq 1). \tag{4.10}$$

Now, it is plain that

$$\|h\| \leqq C\sum_{k=1}^{\infty} \ln(k+1)\left\|\sum_{j=0}^{\infty} D_j\Delta_{11}a_{jk}\right\|.$$

Making use of (4.6) yields the first part of Theorem 5.

As to the second part, we refer to [2], where we proved that if the sum h is a cosine-sine series, with coefficients $\{a_{jk}\}$ satisfying conditions (1.4) and (1.5), is in $L^1(\mathbf{T}^2)$, then it is the Fourier series of h.

PROOF OF THEOREM 4. By (4.1), the left-hand side in (4.7) does not exceed the following sums:

$$\sum_{k=1}^{\infty} \ln(k+1)\left\{\Delta_{11}a_{0k} + \Delta_{11}a_{1k} + \sum_{m=1}^{\infty} 2^m\Delta_{11}a_{2^m,k}\right\} \leqq \tag{4.11}$$

$$\leqq \sum_{k=1}^{\infty} \ln(k+1)\left\{\Delta_{11}a_{0k} + \Delta_{11}a_{1k} + 2\sum_{j=1}^{\infty} \Delta_{11}a_{jk}\right\} \leqq$$

$$\leqq 3\sum_{k=1}^{\infty}(\Delta_{01}a_{0k})\ln(k+1)<\infty,$$

due to (4.2). So, Theorem 5 applies and gives Part (i) as a corollary.

It remains to prove Part (ii). By (4.8) and (4.9),

$$h(x,y)-s_{mn}(x,y)=\left\{\sum_{j=0}^{m}\sum_{k=n+1}^{\infty}+\sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty}\right\}D_j(x)\widetilde{D}_k(y)\Delta_{11}a_{jk}-$$

$$-\sum_{j=0}^{m}D_j(x)\widetilde{D}_n(y)\Delta_{10}a_{j,n+1}-$$

$$-\sum_{k=1}^{n}D_m(x)\widetilde{D}_k(y)\Delta_{01}a_{m+1,k}-a_{m+1,n+1}D_m(x)\widetilde{D}_n(y).$$

First, by (4.1), (4.6) and (4.10), we deduce that

$$\left\|\left\{\sum_{j=0}^{m}\sum_{k=n+1}^{\infty}+\sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty}\right\}D_j\widetilde{D}_k\Delta_{11}a_{jk}\right\|\leqq \tag{4.12}$$

$$\leqq C\sum_{k=n+1}^{\infty}\ln(k+1)\left\|\sum_{j=0}^{m}D_j\Delta_{11}a_{jk}\right\|+C\sum_{k=1}^{\infty}\ln(k+1)\left\|\sum_{j=m+1}^{\infty}D_j\Delta_{11}a_{jk}\right\|\leqq$$

$$\leqq 2C\sum_{k=n+1}^{\infty}\ln(k+1)\left\{|\Delta_{11}a_{0k}|+\sum_{l=0}^{\infty}2^l\max_{j\in I_l}|\Delta_{11}a_{jk}|\right\}+$$

$$+2C\sum_{k=1}^{\infty}\ln(k+1)\sum_{l=l_0}^{\infty}2^l\max_{j\in I_l}|\Delta_{11}a_{jk}|\leqq$$

$$\leqq 6C\sum_{k=n+1}^{\infty}(\Delta_{01}a_{0k})\ln(k+1)+4C\sum_{k=1}^{\infty}(\Delta_{01}a_{2^{l_0-1},k}\ln(k+1),$$

provided $m\geqq 3$, where the integer l_0 is defined by the condition $2^{l_0}\leqq m+1<2^{l_0+1}$. Since $2^{l_0-1}>m/4$, by (4.5), we conclude that the right-most side in (4.12) tends to zero as $m,n\to\infty$.

Second, by (4.1), (4.2), (4.6) and (4.10),

$$\left\| \sum_{j=0}^{m} D_j \widetilde{D}_n \Delta_{10} a_{j,n+1} \right\| \leqq$$

$$\leqq 2C \ln(n+1) \left\{ |\Delta_{10} a_{0,n+1}| + \sum_{l=0}^{\infty} 2^l \max_{j \in I_l} |\Delta_{10} a_{j,n+1}| \right\} \leqq$$

$$\leqq 6C a_{0,n+1} \ln(n+1) \to 0 \quad \text{as} \quad n \to \infty$$

(cf. (4.11) and (4.12)).

Third, analogously to the above estimates, a termwise approach gives

$$\left\| \sum_{k=1}^{n} D_m \widetilde{D}_k \Delta_{01} a_{m+1,k} \right\| \leqq$$

$$\leqq C^2 \ln(m+1) \sum_{k=1}^{n} (\Delta_{01} a_{m+1,k}) \ln(k+1) \to 0 \quad \text{as} \quad m \to \infty,$$

uniformly in n. Here we used (4.3).

Fourth, again by (4.3) and (4.10),

$$\|a_{m+1,n+1} D_m \widetilde{D}_n\| \leqq C^2 a_{m+1,n+1} \ln(m+1) \ln(n+1) \to 0 \quad \text{as} \quad m, n \to \infty.$$

This completes the proof of Part (ii) in Theorem 4.

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TRIANGULAR MEAN VALUE THEOREMS AND FRÉCHET'S EQUATION

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1. Fréchet's functional equation

Suppose that A is an additive Abelian semigroup, B is an additive Abelian group and $f : A \to B$. For $y \in A$ define $\tau_y f : A \to B$ by

$$\tau_y f(x) = f(x+y) \quad \text{for} \quad x \in A$$

and define $\Delta_y f : A \to B$ by

$$\Delta_y f(x) = f(x+y) - f(x) \qquad \text{for} \quad x \in A.$$

With addition defined "pointwise", B^A — the set of all functions from A into B — is an Abelian group and, for each $y \in A, \tau_y$ and Δ_y are homomorphisms of B^A into itself and

$$\Delta_y f = \tau_y f - f \quad \text{for all} \quad f \in B^A.$$

If B is a vector space then so is B^A and in this case, for every $y \in A, \tau_y$ and Δ_y are linear operators on B^A. If $A = \mathbf{Z}^+ := \{0,1,2,\ldots\}$ we will write Δ instead of Δ_1 (in conformity with standard notation in the calculus of finite differences).

Let $\mathbf{N} = \{1,2,3,\ldots\}$, let $\mathbf{Z}$ denote the set of all integers, let $\mathbf{R}$ denote the set of all real numbers and let $\mathbf{C}$ denote the set of all complex numbers. Also let $\mathbf{Z}^+ = \{0\} \cup \mathbf{N}$.

For $m \in \mathbf{N}$ and $y \in A, \Delta_y^m$ — the m-th iterate of Δ_y — has the property that, for all $f \in B^A$, and for all $x \in A$,

$$(1) \quad \Delta_y^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+ky) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \tau_{ky} f(x).$$

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The Fréchet functional equation

$$\Delta_y^m f(x) = 0 \tag{2}$$

has been studied by numerous authors going back to Fréchet [9] and including Mazur and Orlicz [15] and McKiernan [16]. In case B satisfies a weak divisibility condition, the solutions of (2) are "generalized polynomials" in the sense of the following theorem of Djoković [7].

THEOREM A. *Suppose that A is an Abelian semigroup, B is an Abelian group, $m \in \mathbf{Z}^+$ and B has the property that the map $b \to (m!)b$ is an automorphism of B. Then a function $f : A \to B$ has the property that*

$$\Delta_y^{m+1} f(x) = 0 \qquad \text{for all} \quad x, y \in A$$

if and only if there exists $a_0 \in B$ and $a_k : A^k \to B$ for $1 \leqq k \leqq m$ such that, if $m \geqq 1$ then a_1 is additive $(a_1(x+y) = a_1(x) + a_1(y)$ for all $x, y \in A)$, if $m \geqq 2$ then a_k is symmetric and multiadditive for $2 \leqq k \leqq m$ and

$$f(x) = a_0 + a_1(x) + \cdots + a_m(x, x, \ldots, x) \quad \text{for all} \quad x \in A.$$

If $A = \mathbf{R}^n$ and $B = \mathbf{R}$ then, under mild regularity assumptions, the solutions of (2) are genuine polynomials. For example Kemperman [13] has proved some general results which imply the following.

THEOREM B. *If $m \in \mathbf{N}$, $f : \mathbf{R}^n \to \mathbf{R}$, C is a subset of $\mathbf{R}^n$ with positive inner Lebesgue measure, (2) holds for all $x \in \mathbf{R}^n$ and all $y \in C$ and f is bounded on some set of positive Lebesgue measure then f is a polynomial of degree at most $m - 1$.*

Our first aim is to generalize the following result from [2].

THEOREM C. *If $0 < a < b$, a/b is irrational, $m \in \mathbf{N}$, $f : \mathbf{R} \to \mathbf{C}$, f is Lebesgue integrable on an interval of length ma and*

$$\Delta_a^m f = \Delta_b^m f = 0$$

then f is almost everywhere equal to a polynomial function of degree at most $m - 1$.

In order to obtain our desired generalization (Theorem 1 below) we will need an estimate (Proposition 1 below) for which, in turn, we require several lemmas.

LEMMA 1. *If A and B are additive Abelian groups, $m \in \mathbf{N}$, $f : A \to B$, $y \in A$ and $\Delta_y^m f = 0$ then $\Delta_{-y}^m f = 0$ and $\Delta_y^m f^- = 0$ where $f^-(x) = f(-x)$ for all $x \in A$.*

PROOF. For every $x \in A$,

$$0 = \Delta_y^m f(x - my) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f\big((x - my) + ky\big) =$$

$$= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f\big(x + (m-k)(-y)\big) =$$

$$= \sum_{j=0}^{m} (-1)^{m-(m-j)} \binom{m}{m-j} f\big(x + j(-y)\big) =$$

$$= (-1)^m \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f\big(x + j(-y)\big) = (-1)^m \Delta_{-y}^m f(x).$$

Hence, for all $x \in A$, we also have

$$0 = \Delta_{-y}^m f(-x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(-x - ky) = \Delta_y^m f^-(x). \quad \square$$

The next lemma is a well known assertion from the calculus of finite differences (see e.g. [5], page 51).

LEMMA 2. *Suppose that* $\sigma : \mathbf{Z}^+ \to \mathbf{C}$ *(or any rational vector space) and* $m \in \mathbf{Z}^+$. *Then* $\Delta^{m+1}\sigma = 0$, *i.e.*

$$\sum_{\nu=0}^{m+1} (-1)^{m+1-\nu} \binom{m+1}{\nu} f(k+\nu) = 0 \quad \textit{for all} \quad k \in \mathbf{Z}^+,$$

if and only if σ *is a polynomial function of degree at most* m; *in fact*

$$\sigma(k) = \sum_{j=0}^{m} \frac{\Delta^j \sigma(0)}{j!} \mu_j(k) \quad \textit{for all} \quad k \in \mathbf{Z}^+$$

where $\mu_0(k) = 1$ *and* $\mu_j(k) = \prod_{\nu=0}^{j-1}(k - \nu)$ *for* $j \in \mathbf{N}$ *and* $k \in \mathbf{Z}^+$.

LEMMA 3. *If* $\sigma : \mathbf{Z}^+ \to \mathbf{C}$, $m \in \mathbf{Z}^+$, $\Delta^{m+1}\sigma = 0$ *and* $|\sigma(k)| \leqq M$ *for* $0 \leqq \leqq k \leqq m$ *then*

$$|\sigma(k)| \leqq e^2 M (k+m)^m \quad \textit{for all} \quad k \in \mathbf{Z}^+.$$

PROOF. If $j \in \mathbf{Z}^+$ then, by (1),

$$\left|\Delta^j \sigma(0)\right| \leqq \sum_{k=0}^{j} \binom{j}{k} M = 2^j M.$$

If $1 \leqq j \leqq m$ and $k \in \mathbf{Z}^+$ then

$$\left|\mu_j(k)\right| \leqq \prod_{\nu=0}^{j-1} (k+\nu) \leqq (k+m-1)^j \leqq (k+m)^m.$$

Thus, for all $k \in \mathbf{Z}^+$,

$$\left|\sigma(k)\right| \leqq \sum_{j=0}^{m} \frac{2^j M}{j!}(k+m)^m \leqq e^2 M(k+m)^m. \quad \square$$

LEMMA 4. *Suppose that* $F : \mathbf{R} \to \mathbf{C}$, $m \in \mathbf{Z}^+$, $\Delta_1^{m+1} F = 0$ *and* $\left|F(x)\right| \leqq M$ *for* $|x| < m+1$. *Then* $\left|F(x)\right| \leqq e^2 M\big(|x|+m\big)^m$ *for all* $x \in \mathbf{R}$.

PROOF. Fix t temporarily in $[0,1)$ and define $\sigma(k) = F(t+k)$ for $k \in \mathbf{Z}^+$. Then

$$\Delta^{m+1}\sigma(k) = \Delta_1^{m+1} F(t+k) = 0 \quad \text{for all} \quad k \in \mathbf{Z}^+$$

and $\left|\sigma(k)\right| \leqq M$ for $0 \leqq k \leqq m$. By Lemma 3,

$$\left|\sigma(k)\right| \leqq e^2 M(k+m)^m \quad \text{for all} \quad k \in \mathbf{Z}^+.$$

It follows that

$$\left|F(t+k)\right| \leqq e^2 M(k+m)^m \quad \text{for} \quad 0 \leqq t < 1 \quad \text{and} \quad k \in \mathbf{Z}^+.$$

For $0 \leqq x \in \mathbf{R}$, if $[x]$ denotes the integer part of x, $\left|F(x)\right| = \left|F\big((x-[x]) + [x]\big)\right| \leqq e^2 M\big([x]+m\big)^m \leqq e^2 M(x+m)^m$. It then follows from the second part of Lemma 1 that $\left|F(x)\right| \leqq e^2 M\big(|x|+m\big)^m$ for all $x \in \mathbf{R}$. $\square$

For $T \subseteqq \mathbf{R}^n$ let

$$\langle T \rangle = \left\{ \sum_{j=1}^{k} m_j t_j : \ k \in \mathbf{N},\ m_j \in \mathbf{Z} \text{ and } t_j \in T \text{ for } 1 \leqq j \leqq k \right\}$$

be the subgroup of $\mathbf{R}^n$ generated by T; we say that T is *substantial* provided $\langle T \rangle$ is dense in $\mathbf{R}^n$. For example, if $0 \neq a, b \in \mathbf{R}$ and a/b is irrational then

$\{a, b\}$ is a substantial subset of $\mathbf{R}$. It is clear that a substantial subset of $\mathbf{R}^n$ must contain a basis for $\mathbf{R}^n$ (as a real vector space) and must have at least $n+1$ members. It follows from a theorem of Kronecker (see [4]) that there are substantial subsets of $\mathbf{R}^n$ containing exactly $n+1$ members. If $n \geqq 2$ then $S^{n-1} := \{x \in \mathbf{R}^n : |x| = 1\}$ is clearly a substantial subset of $\mathbf{R}^n$: for $x = (x_1, \dots x_n)$ and $y = (y_1, \dots, y_n) \in \mathbf{R}^n$ we let $x \cdot y = \sum_{k=1}^{n} x_k y_k$ and $|x| = (x \cdot x)^{\frac{1}{2}}$.

PROPOSITION 1. *Suppose that $f : \mathbf{R}^n \to \mathbf{C}$, f is continuous, $m \in \mathbf{Z}^+$, T is a substantial subset of $\mathbf{R}^n$ and $\Delta_y^{m+1} f = 0$ for all $y \in T$. Then f has polynomial growth; in fact there exists $A > 0$ such that*

$$\left| f(x) \right| \leqq A\big(|x| + 1\big)^{mn} \quad \textit{for all} \quad x \in \mathbf{R}^n.$$

PROOF. Choose $y_\gamma \in T$ for $1 \leqq \gamma \leqq n$ such that $\{y_1, \dots, y_n\}$ is a basis for $\mathbf{R}^n$. Let L be that invertible linear transformation of $\mathbf{R}^n$ onto itself such that $y_\gamma = Lb_\gamma$ for $1 \leqq \gamma \leqq n$ where $\{b_1, \dots, b_n\}$ is the standard basis for $\mathbf{R}^n$ and let $g(\xi) = f(L\xi)$ for $\xi \in \mathbf{R}^n$. Then for $1 \leqq \gamma \leqq n$ and all $\xi \in \mathbf{R}^n$ we have

$$\Delta_{b_\gamma}^{m+1} g(\xi) = \sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} g(\xi + kb_\gamma) =$$

$$= \sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} f(L\xi + kLb_\gamma) = \Delta_{y_\gamma}^{m+1} f(L\xi) = 0.$$

Since f is continuous on $\mathbf{R}^n$, so is g. Let

$$M = \max\left\{ \left| g(x_1, \dots, x_n) \right| : x_j \in \mathbf{R} \text{ and } |x_j| \leqq m+1 \text{ for } 1 \leqq j \leqq n \right\}.$$

If $n = 1$ the desired conclusion follows directly from Lemma 4. So suppose that $n \geqq 2$.

Fix $x_2, \dots, x_n$ temporarily in $[-m-1, m+1]$ and let $F(x) = g(x, x_2, \dots, x_n)$ for $x \in \mathbf{R}$. Then $\Delta_1^{m+1} F(x) = \Delta_{b_1}^{m+1} g(x, x_2, \dots, x_n) = 0$ for $x \in \mathbf{R}$ and $\left| F(x) \right| \leqq M$ for $|x| \leqq m+1$ so that, by Lemma 4, $\left| F(x) \right| \leqq e^2 M\big(|x| + m\big)^m$ for all $x \in \mathbf{R}$. We have thus shown that $\left| g(x_1, x_2, \dots, x_n) \right| \leqq e^2 M\big(|x_1| + m\big)^m$ for $x_1 \in \mathbf{R}$ and $x_2, \dots, x_n \in [-m-1, m+1]$.

Next fix x_1 temporarily in $\mathbf{R}$. If $n = 2$ let $F(x) = g(x_1, x)$ for $x \in \mathbf{R}$ and observe that $\Delta_1^{m+1} F(x) = \Delta_{b_2}^{m+1} g(x_1, x) = 0$ for all $x \in \mathbf{R}$ and $\left| F(x) \right| \leqq e^2 M\big(|x_1| + m\big)^m$ for $|x| \leqq m+1$; by Lemma 4,

$$\left| F(x) \right| \leqq e^2 (e^2 M\big(|x_1| + m\big)^m)\big(|x| + m\big)^m \quad \text{for} \quad x \in \mathbf{R}.$$

We have shown that if $n = 2$ then

$$\left|g(x_1, x_2)\right| \leqq e^4 M\big(|x_1| + m\big)^m \big(|x_2| + m\big)^m \quad \text{for all} \quad x_1, x_2 \in \mathbf{R}.$$

If $n > 2$ then by temporarily fixing x_1 in $\mathbf{R}$ and $x_3, \ldots, x_n$ in $[-m - 1, m + 1]$ we find, by reasoning as in the last paragraph, that

$$\left|g(x_1, x_2, x_3, \ldots, x_n)\right| \leqq e^4 M\big(|x_1| + m\big)^m \big(|x_2| + m\big)^m$$

for all $x_1, x_2 \in \mathbf{R}$ and all $x_3, \ldots, x_n$ in $[-m - 1, m + 1]$.

By induction it follows that

$$\left|g(x_1, \ldots, x_n)\right| \leqq e^{2n} M \prod_{j=1}^{n} \big(|x_j| + m\big)^m \leqq e^{2n} M\big(|x| + m\big)^{mn}$$

for all $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Thus

$$\left|f(x)\right| = \left|g(L^{-1}x)\right| \leqq e^{2n} M\big(|L^{-1}x| + m\big)^{mn} \quad \text{for all} \quad x \in \mathbf{R}^n$$

and the desired conclusion is therefore evident. □

We aim to prove that, under the hypothesis of Proposition 1, f is a polynomial function of degree at most m. In addition to Proposition 1 we will use some distributional ideas from [2].

2. A distributional interlude

Our notation and terminology is, for the most part, that of Rudin [18]. Let $\mathcal{D}_n$ denote the space of all Schwartz test functions on $\mathbf{R}^n$ and let $\mathcal{D}'_n$ be the space of Schwartz distribution on $\mathbf{R}^n$. If $u \in \mathcal{D}'_n$ then the support of u is denoted by $\operatorname{supp} u$.

A function $f : \mathbf{R}^n \to \mathbf{C}$ is said to be *locally integrable* provided it is Lebesgue measurable and $\int_K \left|f(x)\right| dx < +\infty$ for every compact $K \subseteqq \mathbf{R}^n$. The space of all such functions is denoted by $L^1_{\mathrm{Loc}}(\mathbf{R}^n)$. For $f \in L^1_{\mathrm{Loc}}(\mathbf{R}^n)$ let Λ_f denote the *regular distribution* corresponding to f; thus $\Lambda_f(\varphi) = \int_{\mathbf{R}^n} f(x)\varphi(x)dx$ for all $\varphi \in \mathcal{D}_n$.

Clearly $\Lambda_{\tau_y f}(\varphi) = \Lambda_f(\tau_{-y}\varphi)$ for all $f \in L^1_{\mathrm{Loc}}(\mathbf{R}^n)$, all $y \in \mathbf{R}^n$ and all $\varphi \in \mathcal{D}_n$. For $u \in \mathcal{D}'_n$ and $y \in \mathbf{R}^n$ it is therefore natural to define $\tau_y u : \mathcal{D}_n \to \mathbf{C}$ by

$$(\tau_y u)(\varphi) = u(\tau_{-y}\varphi) \quad \text{for} \quad \varphi \in \mathcal{D}_n.$$

(Note that Rudin would write τ_{-y} instead of τ_y.)

It follows that $\tau_y u \in \mathcal{D}'_n$ whenever $u \in \mathcal{D}'_n$ and $y \in \mathbf{R}^n$ and $\Lambda_{\tau_y f} = \tau_y \Lambda_f$ whenever $f \in L^1_{\mathrm{Loc}}(\mathbf{R}^n)$ and $y \in \mathbf{R}^n$. Moreover, for each $y \in \mathbf{R}^n, \tau_y$ is a topological automorphism of $\mathcal{D}'_n$. For $u \in \mathcal{D}'_n$ and $y \in \mathbf{R}^n$ define $\Delta_y u \in \mathcal{D}'_n$ by $\Delta_y u = \tau_y u - u$.

Let $\mathcal{S}_n$ denote the Fréchet space of rapidly decreasing C^∞ functions on $\mathbf{R}^n$ and let $\mathcal{S}'_n$ denote its dual, the space of all tempered distributions on $\mathbf{R}^n$ (see [18], Chapter 7). Since $\mathcal{D}_n$ is a linear topological subspace of $\mathcal{S}_n$, it follows that $\mathcal{S}'_n$ is a linear topological subspace of $\mathcal{D}'_n$. Moreover, for each $y \in \mathbf{R}^n, \tau_y u \in \mathcal{S}'_n$ whenever $u \in \mathcal{S}'_n$ and the map $u \to \tau_y u$ is a topological automorphism of $\mathcal{S}'_n$. A function f in $L^1_{\mathrm{Loc}}(\mathbf{R}^n)$ will be called *temperate* provided $\Lambda_f \in S'_n$; for this to hold it suffices that f be Lebesgue measurable and have polynomial growth, i.e. there exist $A > 0$ and $N \in \mathbf{N}$ such that $|f(x)| \leqq A(1+|x|)^N$ for all $x \in \mathbf{R}^n$.

Observe that if (1) holds for some $y \in \mathbf{R}^n$, some $m \in \mathbf{N}$ and some $f \in$ $\in L^1_{\mathrm{Loc}}(\mathbf{R}^n)$ and if $u = \Lambda_f$ then

$$(1)' \qquad \Delta_y^m u = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \tau_{ky} u = 0.$$

For $\varphi \in \mathcal{S}_n$ denote the Fourier transform of φ by $\hat{\varphi}$, i.e.

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(x) \exp(-i\xi \cdot x) dx \quad \text{for} \quad \xi \in \mathbf{R}^n.$$

The delightful behaviour of the Fourier transform on $\mathcal{S}_n$ (see [18] Chapter 7) allows one to, quite naturally, define the Fourier transform on $\mathcal{S}'_n$ by duality; for $u \in \mathcal{S}'_n$ one defines $\mathcal{F}u \in \mathcal{S}'_n$ by $\mathcal{F}u(\varphi) = u(\hat{\varphi})$ for $\varphi \in \mathcal{S}_n$ and $\mathcal{F}u$ is called the *Fourier transform* of u. It follows that $\mathcal{F}$ is a topological automorphism of $\mathcal{S}'_n$.

For $y \in \mathbf{R}^n$ let $e_y(x) = \exp(ix \cdot y)$ for all $x \in \mathbf{R}^n$. The following lemma (see [18], page 167) is crucial.

LEMMA 5. *If $u \in \mathcal{S}'_n$ and $y \in \mathbf{R}^n$ then $\mathcal{F}(\tau_y u) = e_y \mathcal{F}u$.*

The following two lemmas may be found in [2].

LEMMA 6. *If $u, v \in \mathcal{D}'_n$, $F : \mathbf{R}^n \to \mathbf{C}$ is C^∞ and $Fu = v$ then* $\operatorname{supp} u \subseteqq$ $\subseteqq \{x \in \mathbf{R}^n : F(x) = 0\} \cup \operatorname{supp} v$.

LEMMA 7. *If $u \in \mathcal{S}'_n$ then* $\operatorname{supp} \mathcal{F}u \subseteqq \{0\}$ *if and only if $u = \Lambda_p$ for some polynomial function $p : \mathbf{R}^n \to \mathbf{C}$.*

3. Back to Fréchet's equation

We can now prove one of the two main results of this paper. It will be used to deduce the mean value theorems of the last section.

THEOREM 1. *Suppose that* $f : \mathbf{R}^n \to \mathbf{C}$, f *is continuous*, $m \in \mathbf{N}$, T *is a substantial subset of* $\mathbf{R}^n$ *and* $\Delta_y^m f = 0$ *for each* $y \in T$. *Then* f *is a polynomial function of degree at most* $(m-1)n$. *The degree of* f *is at most* $m-1$ *if* $m = 1$, $n = 1$ *or* $n \geqq 2$ *and* $T = S^{n-1}$, *where* S^{n-1} *is the unit sphere in* $\mathbf{R}^n$.

REMARK. If $m = 1$ we have the well known fact that if a continuous function has sufficiently many periods then it must be constant.

PROOF. By Proposition 1, f is temperate. If $u = \Lambda_f$ it follows that $u \in \mathcal{S}'_n$ and

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \tau_{ky} u = 0 \quad \text{for} \quad y \in T.$$

By Lemma 5, for each $y \in T$,

$$0 = \mathcal{F}\left(\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \tau_{ky} u \right) = (-1)^m \left(\sum_{k=0}^{m} \binom{m}{k} (-1)^k e_{ky} \right) \mathcal{F}u.$$

If for $y \in T$ we let

$$F_y(x) = \sum_{k=0}^{m} \binom{m}{k} \big[-\exp(iy \cdot x) \big]^k = \big[1 - \exp(iy \cdot x) \big]^m \qquad \text{for} \quad x \in \mathbf{R}^n$$

it follows that $F_y \mathcal{F} u = 0$ for all $y \in T$.

Suppose that $x \in \operatorname{supp} \mathcal{F}u$. By Lemma 6,

$$0 = F_y(x) = \big[1 - \exp(iy \cdot x) \big]^m = 0 \quad \text{for all} \quad y \in T.$$

Hence $y \cdot x \in 2\pi\mathbf{Z}$ for all $y \in T$. Since T is substantial, $z \cdot x \in 2\pi\mathbf{Z}$ for all $z \in \mathbf{R}^n$. Thus $t|x|^2 = (tx) \cdot x \in 2\pi\mathbf{Z}$ for all $t \in \mathbf{R}$ and therefore $x = 0$. We have proved that $\operatorname{supp} \mathcal{F}u \subseteqq \{0\}$. Lemma 7 and the continuity of f imply that f is a polynomial function; by Proposition 1 its degree is at most $(m-1)n$.

Using multi-index notation ([18], page 142) we may write

$$f(x) = \sum_{k=0}^{N} \left(\sum_{|\alpha|=k} c_\alpha x^\alpha \right), \quad x \in \mathbf{R}^n$$

where $c_\alpha \in \mathbf{C}$ for each multi index α of order $|\alpha| \leqq N$. We may, and do, assume that $N \geqq m$.

To prove the assertions concerning the degree of f, let us begin by temporarily fixing $y \in T$ such that $y \neq 0$ and let $p(t) = f(ty)$ for $t \in \mathbf{R}$. Then p is a polynomial function (in one real variable) of degree at most N and $\Delta_1^m P(t) = \Delta_y^m f(ty) = 0$ for all $t \in \mathbf{R}$. It follows from Theorem 2.7.3 on page 51 of [5] that

$$p(t) = p(0) + \Delta_1 p(0)t + \cdots + \frac{\Delta_1^N p(0)}{N!} t(t-1)\cdots(t-N+1) \qquad \text{for all} \quad t \in \mathbf{R}.$$

But $\Delta_1^k p(t) = 0$ for $t \in \mathbf{R}$ and $m \leqq k \in \mathbf{N}$. Hence the degree of p is at most $m-1$. If $n = 1$ it follows that the degree of f is at most $m-1$.

Suppose $m = 1$. Then $\Delta_y f(x) = f(x+y) - f(x) = 0$ for all $x \in \mathbf{R}^n$ and all $y \in T$, i.e.

$$f(x+y) = f(x) \quad \text{for all} \quad x \in \mathbf{R}^n \quad \text{and all} \quad y \in T.$$

Thus $f(x+z) = f(x)$ for all $x \in \mathbf{R}^n$ and all $z \in \langle T \rangle$. But $\langle T \rangle$ is dense in $\mathbf{R}^n$ and f is continuous so

$$f(x+z) = f(x) \quad \text{for all} \quad x, z \in \mathbf{R}^n$$

Thus f is constant, i.e. f has degree $m-1$.

Now suppose that $n \geqq 2$ and $T = S^{n-1}$. We know that, for each $y \in S^{n-1}$, the map $t \to f(ty), t \in \mathbf{R}$, is a real polynomial function of degree at most $m-1$. But, for $y \in S^{n-1}$ and $t \in \mathbf{R}$,

$$f(ty) = \sum_{k=0}^{N} \Bigl(\sum_{|\alpha|=k} c_\alpha (ty)^\alpha \Bigr) = \sum_{k=0}^{N} \Bigl(\sum_{|\alpha|=k} c_\alpha y^\alpha \Bigr) t^k.$$

Hence $\sum\limits_{|\alpha|=k} c_\alpha y^\alpha = 0$ for $m \leqq k \leqq N$ and $y \in S^{n-1}$. Thus $\sum\limits_{|\alpha|=k} c_\alpha x^\alpha = 0$ for $m \leqq k \leqq \mathbf{N}$ and $x \in \mathbf{R}^n$ so that $f(x) = \sum\limits_{k=0}^{m-1} \sum\limits_{|\alpha|=k} c_\alpha x^\alpha$ for all $x \in \mathbf{R}$ and the degree of f is therefore at most $m-1$. □

It seems plausible, although we have been unable to prove, that the degree of f is at most $m-1$ in any case.

For distributions we have the following analogue.

THEOREM 2. *Suppose that $m \in \mathbf{N}$, $u \in \mathcal{D}'_n$, T is a substantial subset of $\mathbf{R}^n$ and $\Delta_y^m u = 0$ for all $y \in T$. Then there exists a polynomial function $p : \mathbf{R}^n \to \mathbf{C}$ such that $u = \Lambda_p$ and $\Delta_y^m p = 0$ for all $y \in T$.*

PROOF. Choose $\varphi \in \mathcal{D}_n$ such that $\varphi(t) \geqq 0$ for all $t \in \mathbf{R}^n$ and $\int_{\mathbf{R}^n} \varphi(t)dt = 1$. For $k \in \mathbf{N}$ let $\varphi_k(t) = k^n \varphi(kt)$ for $t \in \mathbf{R}^n$ and let $u_k = u * \varphi_k$ — the convolution of u with φ_k. It follows from [3] that $\Delta_y^m u_k = 0$ for all $k \in \mathbf{N}$ and all $y \in T$. But each u_k is a C^∞ function and hence, by Theorem 1, a polynomial function of degree at most $(m-1)n$. Also $\Lambda_{u_k} \to u$ in $\mathcal{D}'_n$ as $k \to +\infty$.

Thus u is the limit of a sequence belonging to a subspace of $\mathcal{D}'_n$ of dimension at most $(m-1)n$. Since every finite dimensional subspace of $\mathcal{D}'_n$ is closed, u must belong to this subspace, i.e. $u = \Lambda_p$ for some polynomial function $p : \mathbf{R}^n \to \mathbf{C}$ of degree at most $(m-1)n$. Note finally that, for all $y \in T, 0 = \Delta_y^m \Lambda_p = \Lambda_{\Delta_y^m p}$ so that $\Delta_y^m p \doteq 0$. □

Theorem 1 can also be generalized as follows.

THEOREM 3. *Suppose $m \in \mathbf{N}$, $f \in L^1_{\mathrm{Loc}}(\mathbf{R}^n)$, T is a substantial subset of $\mathbf{R}^n$ and, for each $y \in T$,*

$$\Delta_y^m f(x) = 0 \quad \textit{for} \quad \textit{a.e.} \quad x \in \mathbf{R}^n.$$

Then there exists a polynomial function $p : \mathbf{R}^n \to \mathbf{C}$ of degree at most $(m-1)n$ such that

$$f(x) = p(x) \quad \textit{for} \quad \textit{a.e.} \quad x \in \mathbf{R}^n$$

and $\Delta_y^m p(x) = 0$ for all $x \in \mathbf{R}^n$ and all $y \in T$.

PROOF. Let $u = \Lambda_f$. Then $\Delta_y^m u = 0$ for all $y \in T$ and, by Theorem 2, there exists a polynomial function $p : \mathbf{R}^n \to C$ such that $u = \Lambda_p$. □

4. Some mean value theorems and questions

In [19] Walsh proved a mean value theorem which, in geometric language, may be phrased as follows.

THEOREM D. *If $2 \leqq N \in \mathbf{N}$, $f : \mathbf{R}^2 \to \mathbf{R}$ and f is continuous then the following are equivalent:*

(i) *the value of f at the center of any regular N-gon is the arithmetic mean of its values on the vertices,*

(ii) *f is a harmonic polynomial of degree at most $N-1$.*

For triangles this assertion was improved by Djoković [6] as follows.

THEOREM E. *Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ and f is bounded on a subset of $\mathbf{R}^2$ having positive Lebesgue measure. Then the following are equivalent:*

(i) $f(x-t,y) + f(x+t,y) + f\left(x, y+t\sqrt{3}\right) = 3f\left(x, y+t/\sqrt{3}\right)$ *for all* $x, y, t \in \mathbf{R}$,

(ii) *there exist $a, b, c, \alpha, \beta, \gamma \in \mathbf{R}$ such that*

$$f(x,y) = a + bx + cy + \alpha(x^2 - y^2) + \beta xy + \gamma(x^3 - 3xy^2) \quad \textit{for all} \quad x, y \in \mathbf{R}.$$

In geometric language, (i) says that for any equilaterial triangle having a side parallel to the "x-axis" (but of arbitary "size"), the value of f at its center is the arithmetic mean of its values on the vertices. Property (ii) asserts that f is a special type of harmonic polynomial of degree at most 3.

Several results related to Theorem D were proved in [1]. In particular, for squares with diagonals parallel to the coordinate axes we have the following.

THEOREM F. *If* $f:\mathbf{R}^2\to\mathbf{R}$ *and* f *is bounded on a subset of* $\mathbf{R}^2$ *having positive Lebesgue measure then the following are equivalent:*

(i) $f(x+t,y+t)+f(x+t,y-t)+f(x-t,y+t)+f(x-t,y-t)=4f(x,y)$ *for all* $x,y,t\in\mathbf{R}$,

(ii) $f(x,y)=p(x,y)+\alpha(x^3y-xy^3)$ *for all* $x,y\in\mathbf{R}$ *where* p *is a harmonic polynomial of degree at most* 3 *and* $\alpha\in\mathbf{R}$.

This result was (partially) generalized to higher dimensions in [2].

Theorems E and F address particular cases of the following problem. Given $a_k\in\mathbf{R}^n$ and $c_k\in\mathbf{C}$ for $0\leqq k\leqq N$, which functions $f:\mathbf{R}^n\to\mathbf{C}$ satisfy the functional equation

$$\sum_{k=0}^{N}c_kf(x+ta_k)=0\qquad\text{for}\quad x\in\mathbf{R}^n\quad\text{and}\quad t\in\mathbf{R}? \tag{3}$$

Many papers have been written concerning variants of this problem; see e.g. [1], [2], [3], [6], [8], [10], [11], [12], [13], [14] and [17]. The solutions, under mild regularily assumptions, are typically polynomials and, for "mean value equations", the solutions are usually harmonic polynomials.

It is clear that the solution set of (3) is closed under translations (if f is a solution and $x_0\in\mathbf{R}^n$ then $x\to f(x+x_0)$ is a solution, and dilations (if f is a solution and $\rho>0$ then $x\to f(\rho x)$ is a solution) but not, a priori, closed under all isometries. In general terms, figures of arbitrary "size" and "position" are admitted but only those having certain "orientations". On the other hand, the set of functions satisfying the mean value property (i) of Theorem D is obviously closed under all translations, dilations, and isometries. In particular, if f is a solution and if $U:\mathbf{R}^2\to\mathbf{R}^2$ is an orthogonal linear map then $x\to f(Ux)$ is also a solution.

The remainder of this paper is motivated by the following problem: Given $\rho>0$, which functions $f:\mathbf{R}^2\to\mathbf{C}$ have the property that for every equilateral triangle of radius ρ, the value of f at its center is the arithmetic mean of its values on the vertices? The set of all such f is clearly closed under all isometries but not, a priori, under dilations.

This question is a particular case of the following general problem about which little appears to be known: Given $a_k\in\mathbf{R}^n$ and $c_k\in\mathbf{C}$ for $0\leqq k\leqq N$, which functions $f : \mathbf{R}^n \to \mathbf{C}$ satisfy

$$(4) \qquad \sum_{k=0}^{N} c_k f(x + Ua_k) = 0 \quad \text{for} \quad x \in \mathbf{R}^n \quad \text{and} \quad U \in \mathcal{O}_n?$$

Here $\mathcal{O}_n$ denotes the group of all orthogonal linear maps of $\mathbf{R}^n$ onto itself. Note how (3) and (4) are related. When $c_0 = -N$, $c_k = 1$ for $1 \leqq k \leqq N$, $a_0 = 0$ and $a_1, \ldots, a_N$ constitute the vertices of some regular geometric object centered at 0, then (4) may be thought of as a mean value property of f.

5. Triangular mean value theorems

Our other main result is the following.

THEOREM 4. *Suppose that $\rho > 0$, $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous and, for every equivalent triangle in $\mathbf{R}^2$ of radius ρ, the value of f at its center is the arithmetic mean of its values on the vertices. Then f is a harmonic polynomial of degree at most* 2.

Note that the relevant mean value property can be expressed as follows if we identify $\mathbf{R}^2$ with $\mathbf{C}$:

$$(*) \; f(z+e^{it}) + f(z+e^{it}\omega) + f(z+e^{it}\overline{\omega}) = 3f(z) \quad \text{for} \quad z \in \mathbf{C} \quad \text{and} \quad t \in \mathbf{R}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Also note that the converse follows from Theorem D. Related results can be found in [14].

PROOF. We may assume that $\rho = 1$; otherwise consider, instead of f, the map $z \to f(\rho z)$, $z \in \mathbf{C}$.

Suppose that $z \in \mathbf{R}^2$ and $y \in S^1$. Consider the equilateral triangle with z as a vertex and with center $z + y$. Let v_1 and v_2 be the other two vertices of this triangle. The triangle obtained from it by reflexion in the line through v_1 and v_2 has center $z + 2y$ and vertices $z + 3y, v_1$ and v_2 (a picture is convincing). Thus

$$f(z + 3y) + f(v_1) + f(v_2) = 3f(z + 2y)$$

and

$$f(z) + f(v_1) + f(v_2) = 3f(z + y).$$

It follows that

$$f(z + 3y) - 3f(z + 2y) + 3f(z + y) - f(z) = 0$$

or

$$\Delta_y^3 f(z) = 0 \quad \text{for all} \quad z \in \mathbf{R}^2 \quad \text{and all} \quad y \in S^1.$$

By Theorem 1, f is a polynomial function of degree at most 2 so there exist $a, b, c, \alpha, \beta, \gamma \in \mathbf{R}$ such that

$$f(s,t) = a + bs + ct + \alpha s^2 + 2\beta st + \gamma t^2 \quad \text{for all} \quad (s,t) \in \mathbf{R}^2.$$

According to Theorem D, the map $(s,t) \to a + bs + ct + 2\beta st$, being a harmonic polynomial of degree $\leqq 2$, satisfies our (linear) mean value property. Hence, if $h(s,t) = \alpha s^2 + \gamma t^2$ for $(s,t) \in \mathbf{R}^2$ then h also has our mean value property. In particular

$$0 = 3h(0,0) = h(1,0) + h\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + h\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{3}{2}(\alpha + \gamma).$$

Thus $\gamma = -\alpha$ and hence f is harmonic. □

COROLLARY. *If $f \in L^1_{\text{Loc}}(\mathbf{R}^2)$ and, for a.e. $z \in \mathbf{R}^2$, $(*)$ holds for all $t \in \mathbf{R}$ then there exists a harmonic polynomial $p : \mathbf{R}^2 \to \mathbf{C}$ of degree at most 2 such that $f(z) = p(z)$ for a.e. $z \in \mathbf{R}^2$.*

This can be deduced from Theorem 4 by using the same ideas that were used to derive Theorem 3 from Theorem 1. Distributional variants of Theorem 4 (in the spirit of Theorem 2) are also clearly possible.

The mean value property of Theorem 4 can also be expressed as follows:

$$(5) \qquad f\left(\left(\frac{1}{3}\right)(x+y+z)\right) = \left(\frac{1}{3}\right)[f(x) + f(y) + f(z)]$$

whenever $x, y, z \in \mathbf{R}^2$ and $|x-y| = |y-z| = |z-x| = l$ where $l = \sqrt{3}\rho$. This observation leads to

THEOREM 5. *Suppose that V is a real inner product space of dimension at least 3, $l > 0$, $f : V \to \mathbf{R}$, f is continuous and (5) holds whenever $x, y, z \in V$ and $\|x-y\| = \|y-z\| = \|z-x\| = l$. Then f is affine, i.e. $x \to f(x) - f(0)$ is linear.*

PROOF. Suppose that the theorem is true when the dimension of V is 3. Let $g(x) = f(x) - f(0)$ for $x \in V$. Given $x, y \in V$ and $t \in \mathbf{R}$, choose a 3 dimensional subspace, W, of V containing both x and y. Since the restriction of g to W is linear, $g(tx+y) = tg(x) + g(y)$. Since x and y are chosen arbitrarily, g is linear on V. It thus suffices to assume that $V = \mathbf{R}^3$.

We may apply Theorem 4 to the restriction of f to any two dimensional linear manifold in $\mathbf{R}^3$. In particular, for each $x \in \mathbf{R}$ there exist $a_1(x)$, $b_1(x)$, $c_1(x)$, $\alpha_1(x)$ and $\beta_1(x)$ in $\mathbf{R}$ such that

$$(6) \qquad f(x,y,z) = a_1(x) + b_1(x)y + c_1(x)z + \alpha_1(x)(y^2 - z^2) + z\beta_1(x)yz$$

for all $y, z \in \mathbf{R}$. Similarily, for each $y \in \mathbf{R}$ there exist real numbers $a_2(y)$, $b_2(y)$, $c_2(y)$, $\alpha_2(y)$ and $\beta_2(y)$ such that

$$(7) \quad f(x,y,z) = a_2(y) + b_2(y)x + c_2(y)z + \alpha_2(y)(x^2 - z^2) + 2\beta_2(y)xz$$

for all $x, z \in \mathbf{R}$.

Put $z = 0$ in (6) and (7) and compare the resulting equations to deduce that, for all $x, y \in \mathbf{R}$,

$$(8) \qquad a_1(x) + b_1(x)y + \alpha_1(x)y^2 = a_2(y) + b_2(y)x + \alpha_2(y)x^2.$$

By considering the three equations obtained from (8) by letting $y = 0$, $y = 1$ and $y = -1$ we find that a_1, b_1 and α_1 are polynomials of degree at most 2. The same is, of course, true of a_2, b_2 and α_2.

Compare (6) and (7) in light of (8) to deduce that, for all $x, y, z \in \mathbf{R}$,

$$(9) \qquad c_1(x)z - \alpha_1(x)z^2 + 2\beta_1(x)yz = c_2(y)z - \alpha_2(y)z^2 + 2\beta_2(y)xz.$$

When $z = 1$, (9) asserts that

$$(10) \quad c_1(x) - \alpha_1(x) + 2\beta_1(x)y = c_2(y) - \alpha_2(y) + 2\beta_2(y)x \quad \text{for} \quad x, y \in \mathbf{R}.$$

With $z = -1$, (9) says that

$$(11) \quad c_1(x) + \alpha_1(x) + 2\beta_1(x)y = c_1(y) + \alpha_2(y) + \beta_2(y)x \quad \text{for} \quad x, y \in \mathbf{R}.$$

From (10) and (11) it follows (by subtraction) that $2\alpha_1(x) = 2\alpha_2(y)$ for all $x, y \in \mathbf{R}$. Thus α_1 and α_2 are constant functions and $\alpha_1 = \alpha_2$, say $\alpha_1(x) = \alpha_2(y) = \alpha$ for all $x, y \in \mathbf{R}$.

It now follows from (10) that

$$c_1(x) + 2\beta_1(x)y = c_2(y) + 2\beta_2(y)x \quad \text{for all} \quad x, y \in \mathbf{R}$$

and hence

$$c_1(x) = c_2(0) + 2\beta_2(0)x \quad \text{and} \quad c_2(y) = c_1(0) + 2\beta_1(0)y \quad \text{for all} \quad x, y \in \mathbf{R}.$$

That is, c_1 and c_2 are affine.

Now put $y = z = 1$ in (6) and (7) to deduce that

$$a_1(x) + b_1(x) + c_1(x) + 2\beta_1(x) = a_2(1) + b_2(1)x + c_2(1) + 2\beta_2(1)x$$

for all $x \in \mathbf{R}$. But a_1 and b_1 are polynomials of degree at most 2 and c_1 is affine. Hence β_1 is a polynomial function of degree at most 2. Similarly, the same is true of β_2. It now follows from (6) that f is a polynomial function of degree at most 4.

But f is quadratic on each two dimensional subspace of $\mathbf{R}^3$ so f must have degree at most 2. Choose $a, b, c, d, \alpha, \beta, \gamma, \rho, \sigma, \tau \in \mathbf{R}$ such that

$$(12)\quad f(x,y,z) = a + bx + cy + dz + \alpha x^2 + \beta y^2 + \gamma z^2 + 2\rho xy + 2\sigma yz + 2\tau xz$$

for all $x, y, z \in \mathbf{R}$.

By applying Theorem 4 to the map $(x,y) \to f(x,y,0)$ we find, from (12), that $\alpha + \beta = 0$. Similarly, $\alpha + \gamma = 0$ and $\beta + \gamma = 0$ so that $\alpha = \beta = \gamma = 0$.

Since our mean value property is linear and every affine function satisfies it, we deduce from (12) that if $g(x,y,z) = 2\rho xy + 2\sigma yz + 2\tau xz$ for $(x,y,z) \in \in \mathbf{R}^3$ then g has our mean value property. But the quadratic function g can be orthogonally diagonalized. That is, there exist an orthonormal basis $\{b_1', b_2', b_3'\}$ for $\mathbf{R}^3$ and there exists $\alpha', \beta', \gamma' \in \mathbf{R}$ such that

$$g(rb_1' + sb_2' + tb_3') = \alpha' r^2 + \beta' s^2 + \gamma' t^2$$

for all $r, s, t \in \mathbf{R}$. But the map $(r,s,t) \to g(rb_1' + sb_2' + tb_3')$ has the mean value property since g does and since this property is invariant under isometries. From the last paragraph it follows that $\alpha' = \beta' = \gamma' = 0$. Thus $\rho = \sigma = \tau = 0$ and therefore, by (12),

$$f(x,y,z) = a + bx + cy + dz \text{ for all } x, y, z \in \mathbf{R}$$

as desired. $\square$

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STABILITY OF SOLUTIONS OF GENERALIZED LOGISTIC DIFFERENCE EQUATIONS

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1. Introduction

In this paper we investigate the stability character of the generalized version of the logistic difference equation

$$x_{n+1} = ax_n^k\left(1 - x_n^r\right), \quad n = 0, 1, 2, \ldots \tag{1}$$

where a, k, and r are positive constants.

If we set

$$f(u) = au^k(1 - u^r),$$

then it is easily seen that when $0 < a < (k+r)^{k/r+1}/\left(rk^{k/r}\right) \equiv A$, f maps (0,1) to (0,1).

Clearly $x_0 \in (0,1)$ implies $x_n \in (0,1)$ for $n = 1, 2, \ldots$, and so the solutions are always positive and well-defined.

We shall consider the three cases:

(I) $k > 1$.

(II) $k = 1$.

(III) $0 < k < 1$.

In Section 2 we shall establish and prove three theorems which encompass these three cases. For some related results see [2] and [4].

For the difference equations of the general form

$$x_{n+1} = f(x_n) \tag{$*$}$$

where $f(u)$ is any function of u, we state the following definitions and lemmas which are needed in our discussion.

DEFINITION 1. $x = c$ is an equilibrium of equation $(*)$ if $c = f(c)$.

DEFINITION 2. Let $x = c$ be an equilibrium of equation $(*)$. $x = c$ is said to be stable if given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|x_0 - c| < \delta \quad \text{implies} \quad |x_n - c| < \varepsilon \quad \text{for all} \quad n \geqq 0.$$

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$x = c$ is said to be asymptotically stable if it is stable and there exists a $\delta_0 > 0$ such that

$$|x_0 - c| < \delta_0 \quad \text{implies} \quad x_n \to c \quad \text{as} \quad n \to \infty.$$

REMARK. In what follows, as a convention, when we say an equilibrium $x = c$ being stable we always mean that it is asymptotically stable.

DEFINITION 3. An equilibrium $x = c$ of equation $(*)$ is said to be unstable if it is not stable.

DEFINITION 4. An equilibrium $x = c$ is said to be semistable from above (from below, respectively) if there exists $\delta_0 > 0$ such that $c < x_0 < c + \delta_0$ ($c - \delta_0 < x_0 < c$, respectively) implies $x_n \to c$ as $n \to \infty$; while $c - \delta_0 < x_0 < c$ ($c < x_0 < c + \delta_0$, respectively) implies $|x_n - c| > \varepsilon_0$ for some $\varepsilon_0 > 0$ and some values of n.

DEFINITION 5. An equilibrium $x = c$ is said to be a global attractor if for a given $x_0, x_n \to c$ as $n \to \infty$.

LEMMA 1. *For equation $(*)$ the equilibrium $x = c$ is stable if $|f'(c)| < 1$, and is unstable if $|f'(c)| > 1$ (including the case of $|f'(c) = \infty$).*

LEMMA 2. *Let $x = c$ be an equilibrium of equation $(*)$. If $f'(c) = 1$ and $f''(c) \neq 0$, then $x = c$ is semistable. In particular, if $f''(c) < 0$ (including the case of $f''(c) = -\infty$) then it is semistable from above, while if $f''(c) > 0$ (including the case of $f''(c) = +\infty$) then it is semistable from below.*

LEMMA 3. *Let $x = c$ be an equilibrium of equation $(*)$, $f'(c) = 1$, and $f''(c) = 0$. Then $x = c$ is stable if $f'''(c) < 0$ (including the case of $f'''(c) = -\infty$) while is unstable if $f'''(c) > 0$ (including the case of $f'''(c) = +\infty$).*

LEMMA 4. *Let $x = c$ be an equilibrium of equation $(*)$ and $f'(c) = -1$. Compute*

$$D(c) = -2f'''(c) - 3\left[f''(c)\right]^2.$$

If $D(c) < 0$ then $x = c$ is stable, and if $D(c) > 0$ then it is unstable.

For the proofs of Lemmas 1–4 we refer to Theorems 1.18, 4.1, 4.2, and 4.6 in [5], respectively.

2. The main results

Clearly, equation (1) always has $\overline{x} = 0$ as an equilibrium. We look for the other equilibria $\overline{x}$ by setting

$$\overline{x} - a\overline{x}^k(1 - \overline{x}^r), \quad \text{or} \quad a\overline{x}^{k+r-1} - a\overline{x}^{-k-1} + 1 = 0.$$

Now consider the function

$$g(u) = au^{k+r-1} - au^{k-1} + 1.$$

Then

$$g'(u) = au^{k-2}\left[(k+r-1)u^r - (k-1)\right],$$

and

$$g''(u) = au^{k-3}\left[(k+r-1)(k+r-2)u^r - (k-1)(k-2)\right].$$

Clearly, $g(u)$ has only one nonzero critical point in [0,1], namely

$$u = \left[(k-1)/(k+r-1)\right]^{1/r} \equiv u^*,$$

and since $g''(u^*) > 0$ and $g(0) = g(1) = 1$, $g(u^*)$ is the minimum value of g in [0,1]. Therefore, if $g(u^*) > 0$ there are no positive equilibria of equation (1); if $g(u^*) = 0$ then u^* is the only positive equilibrium of equation (1); while if $g(u^*) < 0$ there are two positive equilibria of equation (1).

We are now in a position to establish the following results.

THEOREM 1. *Let $k > 1$ in equation* (1). *Then*

(i) *The equilibrium $\overline{x} = 0$ is stable.*

(ii) *For $0 < a < A_1 \equiv (k+r-1)^{(k-1)/(r+1)}/\left[r(k-1)^{(k-1)/r}\right]$, $\overline{x} = 0$ is the only equilibrium and is a global attractor.*

(iii) *For $a = A_1$, equation* (1) *has two equilibria: 0 and u^*, and there exists $\widetilde{x} \in \left(\left[k/(k+r)\right]^{1/r}, 1\right)$ such that $x_0 \in (0, u^*) \cup (\widetilde{x}, 1)$ implies $x_n \to 0$ (as $n \to \infty$) while if $x_0 \in [u^*, \widetilde{x}]$, then x_n decreases to $\overline{x} = u^*$ (as $n \to \infty$). Hence, $\overline{x} = u^*$ is semistable.*

(iv) *For $A_1 < a < A$, equation* (1) *has three equilibria: $\overline{x} = 0$, $\overline{x}_1$ and $\overline{x}_2$ with $0 < \overline{x}_1 < u^* < \overline{x}_2$; moreover, $\overline{x}_1$ is always unstable while $\overline{x}_2$ is stable if $A_1 < a \leqq A_3 \equiv (k+r+1)^{(k-1)/r+1}/\left[r(k+1)^{(k-1)/r}\right]$ and unstable if $A_3 < a < A$.*

PROOF. (i) It follows immediately from Lemma 1.

(ii) For $0 < a < A_1$, it can be shown that $\overline{x} = 0$ is the only equilibrium and that for any positive solution (x_n) of equation (1), $g(x_n) > 0$ so

$$x_n > ax_n^k\left(1 - x_n^r\right) = x_{n+1}, \quad n = 1, 2, \ldots.$$

Hence, $\{x_n\}$ is decreasing with nonnegative lower bound. Taking limits in equation (1) yields that $x_n \to 0$.

(iii) In this case equation (1) has two equilibria: $\overline{x} = 0$ and $\overline{x} = u^*$. Since $u^* = \left[(k-1)/(k+r-1)\right]^{1/r} < \left[k/(k+r)\right]^{1/r}$, the maximum point of $f(u)$,

$$f\left(\left[k/(k+r)\right]^{1/r}\right) > f(u^*) = u^*.$$

On the other hand, $f(1) = 0 < u^*$, so by the continuity of f there exists an $\widetilde{x}$ in $\left(\left[k/(k+r)\right]^{1/r}, 1\right)$ such that $f(\widetilde{x}) = u^*$. Now

(1) If $x_0 \in (0, u^*)$, then $g(x_0) > g(u^*) = 0$, so

$$x_0 > a x_0^k\left(1 - x_0^r\right) = x_1,$$

and by iteration,

$$0 < \ldots < x_{n+1} < x_n < \ldots < x_3 < x_2 < x_1 < u^*.$$

Hence $x_n \to 0$ as in (i).

(2) If $x_0 = u^*$, then $x_n = x_0 = u^*$, $n = 1, 2, \ldots$

(3) If $x_0 \in (u^*, \widetilde{u})$, then $u^* < x_1 < x_0 < \widetilde{x}$, and in general,

$$u^* < \ldots < x_{n+1} < x_n < \ldots < x_3 < x_2 < x_1 < x_0 < \widetilde{x}.$$

Now taking limits in equation (1) yields that x_n decreases to u^*.

(4) If $x_0 = \widetilde{x}$, then $x_1 = f(x_0) = f(\widetilde{x}) = u^*$, and thus $x_n = u^*$ for $n = 1, 2, 3, \ldots$.

(5) If $x_0 \in (\widetilde{x}, 1)$, then $x_1 = f(x_0) < f(\widetilde{x}) = u^*$ and $x_n \to 0$ as in (i). That is, $\overline{x} = u^*$ is semistable from above.

(iv) Since in this case $g(u^*) < 0$, $g(u)$ has two positive roots $\overline{x}_1$ and $\overline{x}_2$, $0 < \overline{x}_1 < u^* < \overline{x}_2$ where $g(\overline{x}_1) = g(\overline{x}_2) = 0$. Thus equation (1) has three equilibria: 0, $\overline{x}_1$, and $\overline{x}_2$.

We note that $g'(u) < 0$ for $u \in (0, u^*)$. Hence if $x_n \in (0, \overline{x}_1)$ then $g(x_n) > g(\overline{x}_1) = 0$, i.e., $a x_n^{k+r-1} - a x_n^{k-1} + 1 > 0$ which implies $x_{n+1} < x_n$; whereas if $x_n \in (\overline{x}_1, u^*)$ then $g(x_n) < g(x_1) = 0$ which implies $x_{n+1} > x_n$. Therefore, $\overline{x}_1$ is unstable.

Now with regard to $\overline{x}_2$ we suppose that $\overline{x}_2 = \left[k/(k+r)\right]^{1/r}$, then we have

$$g\left(\left[k/(k+r)\right]^{1/r}\right) = a\left[k/(k+r)\right]^{(k-1)/r}\left[k/(k+r) - 1\right] + 1 = 0,$$

or

$$a = (k+r)^{(k-1)/r+1} / \left[r k^{(k-1)/r}\right] \equiv A_2.$$

From $g(\overline{x}_2) = 0$ we can derive that

$$a(\overline{x}_2) = 1/\left[\overline{x}_2^{k-1}(1 - \overline{x}_2^r)\right].$$

Since for $\overline{x}_2 > u^* = \left[(k-1)/(k+r-1)\right]^{1/r}$,

$$a'(\overline{x}_2) = \overline{x}_2^{k-2}\left[(k+r-1)\overline{x}_2^r - (k-1)\right] / \left[\overline{x}_2^{k-1}(1 - \overline{x}_2^r)\right]^2 > 0,$$

and thus $a(\overline{x}_2)$ is increasing in $\overline{x}_2$ and vice versa.

Next, we consider the following two situations:

(1) $A_1 < a \leqq A_2$. It can be shown using an argument similar to that in the proof of (iii) that $\overline{x}_2$ is stable.

(2) $A_2 < a < A$. Then we have

$$\left| f'(\overline{x}_2) \right| = \left| \left[k - (k+r)\overline{x}_2^r \right] / (1 - \overline{x}_2^r) \right|,$$

and it is easily seen that $\left| f'(\overline{x}_2) \right| < 1$ if and only if

$$u^* = \left[(k-1)/(k+r-1) \right]^{1/r} < \overline{x}_2 < \left[(k+1)/(k+r+1) \right]^{1/r}.$$

But since $a(\overline{x}_2)$ is increasing in $\overline{x}_2$ for $\overline{x}_2 > u^*$, $\left| f'(\overline{x}_2) \right| < 1$ if and only if

$$a(u^*) < a < a\Big(\left[(k+1)/(k+r+1) \right]^{1/r} \Big),$$

or

$$A_1 = (k+r-1)^{(k-1)/r+1} / \left[r(k-1)^{(k-1)/r} \right] < a <$$

$$< (k+r+1)^{(k-1)/r+1} / \left[r(k+1)^{(k-1)/r} \right] = A_3.$$

Therefore, for $A_2 < a < A_3$, $\overline{x}_2$ is stable; while for $A_3 < a < A$, $\left| f'(\overline{x}_2) \right| > 1$, and $\overline{x}_2$ is unstable by Lemma 1.

For $a = A_3$, $f'(\overline{x}_2) = -1$, and it can be verified that $D(\overline{x}_2) < 0$ and so $\overline{x}_2$ is stable by Lemma 4. □

THEOREM 2. *Let $k = 1$ in equation* (1). *Then*

For $0 < a < 1$, $\overline{x} = 0$ is the only equilibrium and it is stable.

(ii) *For $a = 1$, $\overline{x} = 0$ is the only equilibrium and it is stable if $r > 1$ and is semistable from above if $0 < r \leqq 1$.*

(iii) *For $a > 1$, equation* (1) *has two equilibria:* 0 *and $\overline{x} \in (0,1)$. The* 0 *equilibrium is always unstable while $\overline{x}$ is stable if $1 < a \leqq (2+r)/r$ and is unstable if $(2+r)/r < a < (1+r)^{1/r+1}/r$.*

PROOF. (i) For $0 < a < 1$, $f'(0) = a < 1$ and by Lemma 1, $\overline{x} = 0$ is stable.

(ii) For $a = 1$, if $r \leqq 1$ then $\overline{x} = 0$ is semistable from above by Lemma 2: while if $r > 1$ then it is easy to see that $f''(0) = 0$ and $f'''(0) < 0$ and so $\overline{x} = 0$ is stable by Lemma 3.

(iii) For $a > 1$, $\overline{x} = 0$ is an unstable equilibrium of equation (1) since $\left| f'(0) \right| = a > 1$.

Now set

$$\overline{x} = a\overline{x}(1 - \overline{x}^r).$$

This implies that $\overline{x} = \left[(a-1)/a\right]^{1/r}$ and obviously $\overline{x} \in (0,1)$. Then we have three cases:

(1) For $1 < a < A_3 = (2+r)/r$ it is easily verified that $\left|f'(\overline{x})\right| < 1$ and so $\overline{x}$ is stable by Lemma 1.

(2) For $A_3 = (2+r)/r < a < (1+r)^{1/r+1}/r \equiv A, \overline{x} > \left[(2/(2+r)\right]^{1/r}$ and so $\left|f'(\overline{x})\right| > 1$. Hence $\overline{x}$ is unstable by Lemma 1.

(3) For $\overline{x} = A_3 = (2+r)/r$, $f'(x) = -1$ and it can be verified that $D(\overline{x}) <$ < 0. Thus $\overline{x}$ is stable by Lemma 4. □

THEOREM 3. *Let* $0 < k < 1$ *in equation* (1). *Then*

(i) *Equation* (1) *has two equilibria:* 0, *and* $\overline{x} \in (0,1)$, *and the* 0 *equilibrium is always unstable.*

(ii) $\overline{x}$ *ix stable if* $0 < a < A_3$, *unstable if* $A_3 < a < A$; *and at* $a = A_3$, $\overline{x}$ *is stable if* $r \geqq 1$.

PROOF. (i) Clearly, 0 is always an equilibrium of equation (1). In order to find nonzero equilibria, we solve

$$\overline{x} = a\overline{x}(1-\overline{x}^r), \quad \text{or} \quad h(\overline{x}) \equiv a\overline{x}^r + \overline{x}^{1-k} - a = 0.$$

Now $h(0) = -a < 0$, $h(1) = 1 > 0$, and $h'(u) = aru^{r-1} + (1-k)u^{-k} > 0$ for $u \in (0,1)$, so there exists a unique $\overline{x} \in (0,1)$ such that $h(\overline{x}) = 0$; i.e., equation (1) has a unique equilibrium $\overline{x} \in (0,1)$.

The 0 equilibrium is unstable by Lemma 1 since $\left|f'(0)\right|$ is unbounded.

(ii) If $0 < a < A_3$ then $\overline{x} < \left[(k+1)/(k+r+1)\right]^{1/r}$ and it can be shown that $\left|f'(\overline{x})\right| < 1$, hence $\overline{x}$ is stable by Lemma 1; while if $A_3 < a < A$, then $\overline{x} > \left[(k+1)/(k+r+1)\right]^{1/r}$ which implies $\left|f'(\overline{x})\right| > 1$ and so $\overline{x}$ is unstable by Lemma 1. If $a = A_3$, then $\overline{x} = \left[(k+1)/(k+r+1)\right]^{1/r}$ and so $f'(\overline{x}) =$ $= -1$. Thus we use $D(\overline{x})$ and Lemma 4 to determine the stability at $\overline{x}$.

Now some computation yields:

$$D(\overline{x}) = \left[(k+r+1)/(k+1)\right]^{2/r}[-8k^3 - 12k^2r - 4kr^2 - r^2 -$$
$$-6kr + 1 - 3k^4 - 3k^2r^2 - 6k^2 - 6k^3r],$$

from which it is easy to see that $D(\overline{x}) < 0$ if $r \geqq 1$; while if $r < 1$ no definite conclusion on the sign of $D(\overline{x})$ can be reached since that sign depends on the special values of k and r. □

3. Summary and discussion

In the previous sections we have completely characterized the stability behaviour of equation (1) when it is considered as a mapping of the unit interval $[0,1]$ onto itself. As the parameter a goes from 0 to A the graph of equation (1) increases in height.

In short, we conclude as follows:

(A) For the case of $k > 1$: $a < A_1$ implies 0 is the only equilibrium at $a = A_1$, a bifurcation occurs and a second equilibrium exists at this parameter value; when $a > A_1$ there are two positive equilibria. These subcases are covered by Theorem 1.

The method used for $k > 1$ has been found to apply to the more general equation

$$x_{n+1} = ax_n^k\left(1 - x_n^r\right)^p, \quad k > 1, \quad r, p > 0, \tag{2}$$

when the mapping is from $[0,1]$ onto itself.

The stability behaviour is the same as in Theorem 1 for $0 < a < A$. The values of A_1, A_2, A_3, and A are as follows:

$$A_1 = (k + pr - 1)^{(k+pr-1)/r}\left[(pr)^p(k-1)^{(k-1)/r}\right],$$

$$A_2 = (k + pr)^{(k+pr-1)/r}/\left[(pr)^p k^{(k-1)/r}\right],$$

$$A_3 = (k + pr + 1)^{(k+pr-1)/r}/\left[(pr)^p(k+1)^{(k-1)/r}\right],$$

$$A = (k + pr)^{(k+pr)/r}/\left[(pr)^p k^{k/r}\right].$$

Both equation (1) and the more general equation (2) are of the form $x_{n+1} = f(x_n)$ with its graph starting below the line $f(x) = x$ and this accounts for the presence of two positive equilibria after bifurcation.

(B) For the case of $0 < k \leqq 1$: equation (1) is of similar form but with its graph starting above the line $f(x) = x$ and this accounts for the single positive equilibrium with its stability nature fully described in Theorems 2 and 3. We believe that the method we used there can also be applied to equation (2) for the $0 < k \leqq 1$ case.

For $k = r = 1$, equation (1) is the well known discrete logistic equation and chaos exists in the sense of Li and Yorke [1] when a is sufficiently large.

For $k > 1$ equation (1) also seems to exhibit chaos for large enough a.

One criterion for a continuous mapping $f : J \to J$ to exhibit chaos is that there exists a point $y^* \in J$ such that $f(y^*) = a$, $f^2(y^*) = b$, $f^3(y^*) = c$ and $c \leqq y^* < a < b$ (see [3], p. 28).

We have found such a point y^* for several values of $k > 1$ with a large enough. For instance for $k = 3/2$, $r = 5/2$, $y^* = .4$ is such a point when $a = 2.8$.

We believe that for all $k > 1$ and a large enough, equation (1) will exhibit chaos.

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ON SIMULTANEOUS APPROXIMATION TO A DIFFERENTIABLE FUNCTION AND ITS DERIVATIVE BY PÁL-TYPE INTERPOLATION POLYNOMIALS

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1. Introduction

Let $p_n(x)$ be the Legendre polynomial of degree n with the usual normalization $p_n(1) = 1$, write

$$W_n(x) = -n(n-1)\int_{-1}^{x} p_{n-1}(t)\,dt = (1-x^2)p'_{n-1}(x).$$

It is clear that $p_{n-1}(x)$ has $n-1$ distinct zeros (note that they are also all distinct zeros of $W'_n(x)$)

$$-1 < x^*_{n-1} < x^*_{n-2} < \ldots < x^*_1 < 1$$

on the interval $(-1,1)$, hence $W_n(x)$ has n zeros

$$(1) \qquad -1 = x_n < x_{n-1} < \ldots < x_1 = 1$$

which interlace the zeros of $p_{n-1}(x)$. In what follows let $r \geqq 1$. For an r times differentiable function $f(x)$ on $[-1,1]$ (in symbol $f \in C^r_{[-1,1]}$), the Pál-type interpolating polynomial is the algebraic polynomial $Q_n(f,x)$ of degree $2n-1$ satisfying

$$Q_n(f,x_k) = f(x_k), \quad Q'_n(f,x^*_k) = f'(x^*_k)$$

for $k = 1,2,\ldots,n$ where $x^*_n = -1$. It is not difficult to verify that $Q_n(f,x)$ is uniquely determined by $f(x)$. Furthermore, for any polynomial $q(x)$ of degree $\leqq 2n-1$ one has $Q_n(q,x) = q(x)$. Taking a further look at $Q_n(f,x)$, we note that the Pál-type interpolating polynomial interpolates $f(x)$ at the zeros of $W_n(x)$, while $Q'_n(x)$ interpolates $f'(x)$ at the zeros of $W'_n(x)$. In this

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sense, this kind of interpolation may have its new significance and applications.

On the convergence of Pál-type interpolating approximation, Eneduanya [2] showed that for $f \in C^r_{[-1,1]}$,

$$f(x) - Q_n(f,x) = O(1)n^{-r+3/2}\log n\omega\big(f^{(r)}, n^{-1}\big)$$

holds uniformly for all $x \in [-1,1]$, where $\omega(f,t)$ is the usual modulus of continuity of a continuous function f. Therefore, if $f' \in \operatorname{Lip}\alpha$, $\alpha > 1/2$, it follows that $Q_n(x)$ converges to $f(x)$ uniformly on $[-1,1]$ as $n \to \infty$. By employing some new ideas, Xie [7] gave the above result an essential improvement, that is, for $f \in C^r_{[-1,1]}$,

$$f(x) - Q_n(f,x) = O(1)\frac{|W_n(x)|}{\sqrt{n}}n^{-r+1}\omega\big(f^{(r)}, n^{-1}\big) \tag{2}$$

holds uniformly for all $x \in [-1,1]$. Since $W_n(x) = O(\sqrt{n})$, the above estimate implies that $Q_n(f,x)$ converges to $f(x) \in C^1_{[-1,1]}$ uniformly on $[-1,1]$ as $n \to \infty$.

Because the structure of $Q_n(x)$ is also related to $f'(x)$, it is natural to ask under what conditions will $Q'_n(f,x)$ converge to $f'(x)$ uniformly on $[-1,1]$. Exactly, one could ask if the following inequality

$$f'(x) - Q'_n(f,x) = O(1)\frac{|W'_n(x)|}{\sqrt{n}}n^{-r+1}\omega\big(f^{(r)}, n^{-1}\big)$$

holds for $f \in C^r_{[-1,1]}$ and for all $x \in [-1,1]$ (corresponding to (2)). Indeed, Xie raised this problem to some people two years ago, but no conclusion has been achieved since. Very recently, in a personal communication B. Z. Li told Xie that he established that

$$f'(x) - Q'_n(f,x) = O(1)n^{-r+5/2}\omega\big(f^{(r)}, n^{-1}\big), \quad x \in [-1,1],$$

holds for $f \in C^r_{[-1,1]}$, $r \geqq 2$. This is surely a new development in this direction, however, it is not a satisfactory answer that we expected.

Let $E_n(f)$ denote, as usual, the best approximation of a continuous function $f(x)$ by polynomials of degree n. In this paper we will establish that the following stronger estimate

$$f'(x) - Q'_n(f,x) = O(1)\frac{|W'_n(x)|}{\sqrt{n}}n^{-r+1}E_{2n-r-1}\big(f^{(r)}\big), \quad n \geqq \frac{r+1}{2}$$

holds for $f \in C^r_{[-1,1]}$ and for all $x \in [-1,1]$, therefore according to Jackson theorem we have a complete positive answer to the above mentioned problem.

2. Preliminaries

Although Pál gave a general representation of $Q_n(f,x)$ in [5], we now prefer to use the following more explicit formula here, which was established recently in [8].

For given distinct nodes

$$-1 \leqq \widetilde{x}_n < \widetilde{x}_{n-1} < \ldots < \widetilde{x}_1 \leqq 1,$$

write $\omega_n(x) = \prod_{k=1}^n (x - \widetilde{x}_k)$. Also, let $-1 \leqq \widetilde{x}^*_{n-1} < \ldots < \widetilde{x}^*_1 \leqq 1$ denote $n-1$ distinct zeros of $\omega'_n(x)$. Let

$$A_k(x) = l_k^2(x)\big(1 - 2l'_k(\widetilde{x}_k)(x - \widetilde{x}_k)\big) - 2\frac{\omega_n(x)}{\omega'_n(\widetilde{x}_k)} \times$$

$$\times \int_{-1}^{x} \frac{l'_k(t)\big(1 - 2l'_k(\widetilde{x}_k)(t - \widetilde{x}_k)\big) - l'_k(\widetilde{x}_k)l_k(t)}{t - \widetilde{x}_k}\,dt, \quad k = 1, 2, \ldots, n,$$

$$B_k(x) = \frac{\omega_n(x)}{\omega_n(\widetilde{x}^*_k)} \int_{-1}^{x} \frac{\omega'_n(t)}{\omega''_n(\widetilde{x}^*_k)(t - \widetilde{x}^*_k)}\,dt, \quad k = 1, 2, \ldots, n-1,$$

and

$$B_n(x) = \frac{\omega_n(x)}{\omega'_n(-1)},$$

where

$$l_k(x) = \frac{\omega_n(x)}{\omega'_n(\widetilde{x}_k)(x - \widetilde{x}_k)}.$$

Define

$$\widetilde{Q}_n(f,x) = \sum_{k=1}^{n} f(\widetilde{x}_k)A_k(x) + \sum_{k=1}^{n} f'(\widetilde{x}^*_k)B_k(x),$$

where $\widetilde{x}^*_n = -1$. Then $\widetilde{Q}_n(f,x)$ is a polynomial of degree $2n-1$ satisfying the following conditions

$$\widetilde{Q}_n(f, \widetilde{x}_k) = f(\widetilde{x}_k), \quad k = 1, 2, \ldots, n,$$

$$\widetilde{Q}'_n(f, \widetilde{x}^*_k) = f'(\widetilde{x}^*_k), \quad k = 1, \ldots, n-1,$$

and

$$\widetilde{Q}'_n(f,-1)=\sum_{k=1}^{n-1} f'(\widetilde{x}_k^*)B'_k(-1)+f'(-1).$$

In the *special case* when the nodes x_k, $k=1,2,\ldots,n$, are the roots of the integrated Legendre polynomial $W_n(x)$ we have $W_n(-1)=0$ and from this it follows that

$$B'_k(-1)=0 \quad \text{for} \quad k=1,\ldots,n-1.$$

So (only!) in this case the polynomial

$$Q_n(f,x)=\sum_{k=1}^{n} f(x_k)A_k(x)+\sum_{k=1}^{n} f'(x_k^*)B_k(x),$$

where $x_n^*=-1$ and A_k, B_k, $k=1,2,\ldots,n$ are given above, is the unique polynomial of degree $2n-1$ satisfying

$$Q_n(f,x_k)=f(x_k),\quad Q'_n(f,x_k^*)=f'(x_k^*),\quad k=1,\ldots,n.$$

We now list some estimates of the Legendre polynomial $p_{n-1}(x)$ as follows, whose proof could be found in [6]:

$$p_{n-1}(x)=O(1)\frac{1}{n\Delta_n^{1/2}(x)}, \tag{3}$$

where $\Delta_n(x)=\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}$;

$$1-x_k^2\sim \sin^2\frac{k\pi}{n},\quad k=2,3,\ldots,n-1, \tag{4}$$

where the notation $a_{nk}\sim b_{nk}$ means that there is a positive constant M independent of n and k such that $M^{-1}\leqq a_{nk}/b_{nk}\leqq M$;

$$\left|p_{n-1}(x_k)\right|\sim\left(n\sin\frac{k\pi}{n+1}\right)^{-1/2},\quad k=2,3,\ldots,n-1; \tag{5}$$

$$\left|p'_{n-1}(x_k^*)\right|\sim n^2\left(n\sin\frac{k\pi}{n+1}\right)^{-3/2},\quad k=1,2,\ldots,n-1; \tag{6}$$

on writing $x_k^* = \cos\theta_k^*$,

$$\frac{2k-1}{2n-1}\pi \leqq \theta_k^* \leqq \frac{2k}{2n-1}\pi, \quad k = 1, 2, \ldots, n-1. \tag{7}$$

3. Lemmas

First we establish some lemmas.

Lemma 1. *For $k = 2, 3, \ldots, n-1$ we have*

$$|A_k'(x)| \leqq \frac{|W_n'(x)|}{n(n-1)|p_{n-1}(x_k)|}\left(\frac{1}{|x-x_k|} + \frac{2}{(1-x_k^2)p_{n-1}^2(x_k)}\right). \tag{8}$$

Proof. Clearly for $k = 2, 3, \ldots, n-1$, $l_k'(x_k) = 0$. Therefore

$$A_k'(x) = -\frac{2W_n'(x)}{W_n'(x_k)}\int_{-1}^{x}\frac{l_k'(t)}{t-x_k}\,dt. \tag{9}$$

When $x < x_k$, we see that

$$\int_{-1}^{x}\frac{l_k'(t)}{t-x_k}\,dt = \frac{l_k(x)}{x-x_k} + \int_{-1}^{x}\frac{l_k(t)}{(t-x_k)^2}\,dt,$$

therefore

$$\left|\int_{-1}^{x}\frac{l_k'(t)}{t-x_k}\,dt\right| \leqq \frac{2}{|x-x_k|} \tag{10}$$

follows from $|l_k(x)| \leqq 1$ (see [2]). In case $x > x_k$, by applying the known equality (see [1]: Lemma 9.1)

$$\int_{-1}^{1}\frac{l_k'(t)}{t-x_k}\,dt = -\frac{1}{(1-x_k^2)p_{n-1}^2(x_k)},$$

in a similar way we get

$$\left|\int_{-1}^{x}\frac{l_k'(t)}{t-x_k}\,dt\right| \leqq \frac{2}{|x-x_k|} + \frac{1}{(1-x_k^2)p_{n-1}^2(x_k)}. \tag{11}$$

Thus (8) follows from combining (9)–(11) and $W_n'(x_k) = -n(n-1)p_{n-1}(x_k)$. Lemma 1 is proved. □

LEMMA 2. *We have*

$$\left|A_1'(x)\right| \leqq 24\left|W_n'(x)\right| \log n.$$

PROOF. Direct calculations lead to

$$W_n'(1) = -n(n-1), \quad \left|l_1'(x)\right| \leqq n^2, \quad \left|l_1''(x)\right| \leqq n^4.$$

So that

$$\left|A_1'(x)\right| \leqq \frac{2\left|W_n'(x)\right|}{n(n-1)} \left| \int_{-1}^{x} \frac{l_1'(t) - l_1'(1) + l_1'(1)\big(1 - l_1(t)\big)}{t-1} - 2l_1'(1)l_1'(t)\,dt \right| \leqq$$

$$\leqq \frac{2\left|W_n'(x)\right|}{n(n-1)} \left(\int_{-1}^{1-1/n^2} \frac{n^2}{|t-1|}\,dt + \int_{1-1/n^2}^{1} 2n^4\,dt \right) + 4\left|W_n'(x)\right| \leqq$$

$$\leqq 24\left|W_n'(x)\right| \log n. \quad \square$$

Similarly

LEMMA 3. *We have*

$$\left|A_n'(x)\right| \leqq 24\left|W_n'(x)\right| \log n.$$

LEMMA 4. *For $k = 1, 2, \ldots, n-1$ we have*

$$\left|B_k'(x) \leqq \left|W_n'(x)\right| \frac{1}{\left(1 - (x_k^*)^2\right)\left(p_{n-1}'(x_k^*)\right)^2} \cdot\right.$$

$$\cdot \left(\frac{2\|W_n(x)\|}{n(n-1)|x - x_k^*|} + \frac{1}{\left|1 - (x_k^*)^2\right| \left|p_{n-1}'(x_k^*)\right|} \right),$$

where $\|\cdot\| = \max_{-1 \leqq x \leqq 1} |\cdot|$.

PROOF. Obviously,

$$B_k'(x) = \frac{W_n'(x)}{W_n(x_k^*)} \int_{-1}^{x} \frac{W_n'(t)}{W_n''(x_k^*)(t - x_k^*)}\,dt + \frac{W_n(x)W_n'(x)}{W_n(x_k^*)W_n''(x_k^*)(x - x_k^*)}.$$

In view of that (see [4])

$$\int_{-1}^{1} \frac{W_n'(t)}{t - x_k}\,dt = -n(n-1)\frac{1}{W_n(x_k^*)},$$

we have, in a similar way to the proof of Lemma 1, that

$$\left|\int_{-1}^{x} \frac{W_n'(t)}{t-x_k}\,dt\right| \leqq \frac{n(n-1)}{|W_n(x_k^*)|} + \frac{2\|W_n\|}{|x-x_k|},$$

from which the required inequality follows if we note that $W_n'(x) = -(n-1)p_{n-1}(x)$ and $W_n(x) = (1-x^2)p_{n-1}'(x)$. □

Lemma 5. *Let $f \in C^r_{[-1,1]}$. Then for any given $x_0 \in [-1,1]$ there is a polynomial $q_n(x)$ of degree $2n-1$ and a positive constant M depending only upon r such that*

$$|f^{(k)}(x) - q_n^{(k)}(x)| \leqq Mn^{-r+k}E_{2n-r-1}(f^{(r)})$$

for $k = 0,1,\dots,r$ and $|x| \leqq 1$, and

$$f'(x_0) = q_{2n-1}'(x_0).$$

Proof. According to [3], there is a polynomial $\widetilde{q}_n(x)$ of degree $2n-1$ so that

$$|f^{(k)}(x) - \widetilde{q}_n^{(k)}(x)| \leqq M_1 n^{-r+k}E_{2n-r-1}(f^{(r)}) \tag{12}$$

for $k = 0,1,\dots,r$ and $|x| \leqq 1$, where M_1 is a positive constant depending only upon r. Construct that

$$q_n(x) = \widetilde{q}_n(x) + \left(f'(x_0) - \widetilde{q}_n'(x_0)\right)\frac{4T_{2[n/2]+1}\left((x-x_0)/4\right)}{2[n/2]+1}\operatorname{sign} T_{2[n/2]+1}'(0),$$

where $T_m(x)$ is the Chebyshev polynomial of degree m of the first kind. Evidently,

$$\frac{4T_{2[n/2]+1}^{(s)}\left((x-x_0)/4\right)}{2[n/2]+1} = O(n^{s-1}) \tag{13}$$

for $s = 0,1,2\dots$ and $|x| \leqq 1$. At the same time by a direct calculation (see the following Lemma 6) we get

$$\frac{d}{dx}\left(4T_{2[n/2]+1}\left((x-x_0)/4\right)\operatorname{sign} T_{2[n/2]+1}'(0)\right)\Big|_{x=x_0} = 2[n/2]+1, \tag{14}$$

therefore

$$q_n'(x_0) = f'(x_0).$$

Furthermore from (12),

$$\left| f'(x_0) - \widetilde{q}_n'(x_0) \right| \leqq M_1 n^{-r+1} E_{2n-r-1}\left(f^{(r)}\right),$$

so by the definition of $q_n(x)$ and (13) we obtain that

$$\left| f^{(k)}(x) - q_n^{(k)}(x) \right| \leqq M n^{-r+k} E_{2n-r-1}\left(f^{(r)}\right). \quad \square$$

We write the proof of (14) as the following

LEMMA 6. *For any natural number m, we have*

$$T'_{2m-1}(0) = (-1)^{m-1}(2m-1).$$

PROOF. It is well-known that $T_0(x) = 1$ and $T_1(x) = 1$. Applying the recurrence formula

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

for $n = 3, 4, \ldots$, by induction we can get

$$T_{2m-1}(x) = a_0 x^{2m-1} + a_1 x^{2m-3} + \ldots + a_{m-1}x^3 + (-1)^{m-1}(2m-1)x.$$

Consequently,

$$T'_{2m-1}(0) = (-1)^{m-1}(2m-1).$$

We have completed the proof of Lemma 6 as well as Lemma 5. $\square$

4. New result and proof

THEOREM 1. *Let $f \in C^r_{[-1,1]}$, $r \geqq 2$, $n \geqq (r+1)/2$. Then*

$$f'(x) - Q_n'(f,x) = O(1)\frac{|W_n'(x)|}{\sqrt{n}} n^{-r+1} E_{2n-r-1}\left(f^{(r)}\right) \tag{15}$$

holds uniformly for all $x \in [-1,1]$, where $O(1)$ depends only on r.

PROOF. For a given point $x_0 \in [-1,1]$, according to Lemma 5, there is a polynomial $q_n(x)$ of degree $2n-1$ such that

$$\left| f^{(k)}(x) - q_n^{(k)}(x) \right| \leqq M n^{-r+k} E_{2n-r-1}\left(f^{(r)}\right) \tag{16}$$

for $k = 0, 1, \dots, r$ and $|x| \leqq 1$, and

$$f'(x_0) = q_n'(x_0). \tag{17}$$

Obviously, we see from the uniqueness of $Q_n(f,x)$ that $Q_n(1,x) = \sum_{k=1}^{n} A_k(x) \equiv 1$, hence

$$\sum_{k=1}^{n} A_k'(x) = 0. \tag{18}$$

Then (18) implies that

$$Q_n'(f,x) = \sum_{k=1}^{n} \big(f(x_k) - f(x) \big) A_k'(x) + \sum_{k=1}^{n} f'(x_k^*) B_k'(x). \tag{19}$$

Because $Q_n(q_n, x) = q_n(x)$, we deduce that

$$Q_n'(f, x_0) - f'(x_0) = Q_n'(f - q_n, x_0) + q_n'(x_0) - f'(x_0) = Q_n'(f - q_n, x_0)$$

together with (17). Combining it with (19) we then have

$$Q_n'(f, x_0) - f'(x_0) = \sum_{k=1}^{n} \big(f(x_k) - q_n(x_k) - f(x_0) + q_n(x_0) \big) A_k'(x_0) + \\ + \sum_{k=1}^{n} \big(f'(x_k^*) - q_n'(x_k^*) \big) B_k'(x_0).$$

On writing

$$I_1 = \big(f(1) - q_n(1) - f(x_0) + q_n(x_0) \big) A_1'(x_0),$$

$$I_2 = \big(f(-1) - q_n(-1) - f(x_0) + q_n(x_0) \big) A_n'(x_0),$$

$$I_3 = \sum_{k=2}^{n-1} \big(f(x_k) - q_n(x_k) - f(x_0) + q_n(x_0) \big) A_k'(x_0),$$

$$I_4 = \sum_{k=1}^{n-1} \big(f'(x_k^*) - q_n'(x_k^*) \big) B_k'(x_0),$$

$$I_5 = \big(f'(-1) - q_n'(-1) \big) B_n'(x_0),$$

we have

$$Q_n'(f, x_0) - f'(x_0) = \sum_{j=1}^{5} I_j. \tag{20}$$

We are going to estimate I_j, $j = 1, 2, \dots, 5$, separately.

From Lemma 2 and (16) it follows immediately that

$$|I_1| \leqq 24M \log n \big| W_n'(x_0) \big| n^{-r} E_{2n-r-1}\big(f^{(r)}\big), \tag{21}$$

and similarly from Lemma 3 and (16),

$$|I_2| \leqq 24M \log n \big| W_n'(x_0) \big| n^{-r} E_{2n-r-1}\big(f^{(r)}\big) \tag{22}$$

holds. Meanwhile, the combination of Lemma 1 and (16) yields that

$$|I_3| \leqq \sum_{k=2}^{n-1} \big| W_n'(x_0) \big| \left(\frac{M}{n(n-1)\big|p_{n-1}(x_k)\big|} n^{-r+1} E_{2n-r-1}\big(f^{(r)}\big) + \right.$$

$$\left. + \frac{2M}{n(n-1)(1-x_k^2)\big|p_{n-1}(x_k)\big|^3} n^{-r} E_{2n-r-1}\big(f^{(r)}\big) \right).$$

By using inequalities (4) and (5) we see that

$$\sum_{k=2}^{n-1} \frac{1}{\big|p_{n-1}(x_k)\big|} = O\big(n^{3/2}\big),$$

and

$$\sum_{k=2}^{n-1} \frac{1}{(1-x_k^2)\big|p_{n-1}(x_k)\big|^3} = O\big(n^{5/2}\big),$$

consequently,

$$I_3 = O(1) \frac{\big| W_n'(x_0) \big|}{\sqrt{n}} n^{-r+1} E_{2n-r-1}\big(f^{(r)}\big). \tag{23}$$

On the other hand, applying Bernstein's inequality together with (3) we get

$$p_{n-1}'(x) = O(1) n^{-1} \Delta_n^{-3/2}(x).$$

So

$$\|W_n\| = O\left(\sqrt{n}\right) \tag{24}$$

due to the definition $W_n(x) = (1-x^2)p'_{n-1}(x)$. Then it is deduced from Lemma 4, (16) and (24) that

$$|I_4| \leqq O(1)\sum_{k=1}^{n-1}\left|W'_n(x_0)\right|\left(\frac{\sqrt{n}n^{-r+2}E_{2n-r-1}\left(f^{(r)}\right)}{n(n-1)(\left(1-(x_k^*)^2\right)\left|p'_{n-1}(x_k^*)\right|^2}+\right.$$
$$\left.+\frac{n^{-r+1}E_{2n-r-1}\left(f^{(r)}\right)}{\left(1-(x_k^*)^2\right)\left|p'_{n-1}(x_k^*)\right|^3}\right),$$

while applying (6) and (7) yields that

$$\sum_{k=1}^{n-1}\frac{1}{\left(1-(x_k^*)^2\right)\left|p'_{n-1}(x_k^*)\right|^2} = O(1)\sum_{k=1}^{n-1}\frac{\sin(k\pi/n)}{n} = O(1),$$

$$\sum_{k=1}^{n-1}\frac{1}{\left(1-(x_k^*)^2\right)\left|p'_{n-1}(x_k^*)\right|^3} = O(1)\sum_{k=1}^{n-1}n^{-3/2}\sin^{5/2}\frac{k\pi}{n} = O\left(n^{-1/2}\right),$$

altogether,

$$I_4 = O(1)\frac{\left|W'_n(x_0)\right|}{\sqrt{n}}n^{-r+1}E_{2n-r-1}\left(f^{(r)}\right). \tag{25}$$

As for I_5 we have

$$I_5 = O(1)\frac{\left|W'_n(x_0)\right|}{n^2}n^{-r+1}E_{2n-r-1}\left(f^{(r)}\right) \tag{26}$$

since $\left|W'_n(-1)\right| \sim n^2$.

Combining the estimates (20)–(23) and (25)–(26), we finally establish that

$$Q'_n(f,x_0) - f'(x_0) = O(1)\frac{\left|W'_n(x_0)\right|}{\sqrt{n}}n^{-r+1}E_{2n-r-1}\left(f^{(r)}\right).$$

Furthermore we note that in the above proof, $O(1)$ is independent of any given $x_0 \in [-1,1]$ so we have the desired inequality (15). Theorem 1 is completed. □

Evidently, according to Jackson's Theorem,

$$E_{2n-r-1}\big(f^{(r)}\big) = O\big(\omega(f^{(r)}, 1/n)\big),$$

we thus have the following

THEOREM 2. *Let $f \in C^r_{[-1,1]}$, $r \geqq 2$. Then*

$$f'(x) - Q_n'(f, x) = O(1)\frac{|W_n'(x)|}{\sqrt{n}} n^{-r+1}\omega\big(f^{(r)}, 1/n\big)$$

holds uniformly for all $x \in [-1, 1]$, where $O(1)$ depends only on r.

With $W_n'(x) = O(n^2)$, from Theorems 1 and 2 we obtain immediately the following corollaries.

COROLLARY 1. *Let $f \in C^r_{[-1,1]}$, $r \geqq 2$, $n \geqq (r+1)/2$. Then*

$$f'(x) - Q_n'(f, x) = O(1)n^{-r+5/2}E_{2n-r-1}\big(f^{(r)}\big)$$

holds uniformly for all $x \in [-1, 1]$.

COROLLARY 2. *Let $f \in C^r_{[-1,1]}$, $r \geqq 2$. Then*

$$f'(x) - Q_n'(f, x) = O(1)n^{-r+5/2}\omega\big(f^{(r)}, 1/n\big)$$

holds uniformly for all $x \in [-1, 1]$.

COROLLARY 3. *Let $f \in C^2_{[-1,1]}$, $f'' \in$ Lip α, $\alpha > 1/2$. Then $Q_n'(f, x)$ converges to $f'(x)$ uniformly on $[-1, 1]$.*

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ON THE SUMS OF NON-NEGATIVE QUASI-CONTINUOUS FUNCTIONS

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The notion of quasi-continuity for real functions of several real variables was introduced over sixty years ago by S. Kempisty [2], as a generalization of the notion of continuity, and it has been intensively studied since then. A comprehensive survey on this topic can be found in [4]. In [3] I proved the following theorem.

THEOREM 1. *Given a cliquish function* $f: \mathbf{R}^m \to \mathbf{R}$ *and an* $\eta > 0$ *we can find a Lebesgue function* α *such that the functions* $f/2 + \alpha$ *and* $f/2 - \alpha$ *are quasi-continuous,* $\mathcal{D}(\alpha) \subset \mathcal{D}(f)$ *and* $\|\alpha\| \leqq \|f\| + \eta$ [3, Theorem 4.2].

It follows that each cliquish function $f: \mathbf{R}^m \to \mathbf{R}$ can be written as the sum of two quasi-continuous functions and we may require the summands to be bounded provided that f is bounded. However it turns out we cannot require the summands to be non-negative or even bounded below in case f is non-negative. In this paper I characterize functions which are expressible as the sum of at most k non-negative quasi-continuous functions for each positive integer k and study an analogous problem concerning the sums of quasi-continuous functions bounded below.

First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbf{R}$ and the set of positive integers by $\mathbf{N}$. To the end of this article m is a fixed positive integer. The word *function* means mapping from $\mathbf{R}^m$ into $\mathbf{R}$ unless otherwise explicitly stated. The Euclidean metric in $\mathbf{R}^m$ will be denoted by ϱ. For every set $A \subset \mathbf{R}^m$, let $\operatorname{int} A$ be its interior, $\operatorname{cl} A$ its closure, $\operatorname{fr} A$ its boundary, $\operatorname{diam} A$ its diameter $\big($i.e., $\operatorname{diam} A = \sup\{\varrho(x,y) : x, y \in \in A\}\big)$, and χ_A its characteristic function. For any function f we write $\|f\|$ for $\sup\{|f(t)| : t \in \mathbf{R}^m\}$ (f need not be bounded), we denote by $\mathcal{C}(f)$ the set of points of continuity of f, and we set $\mathcal{D}(f) = \mathbf{R}^m \setminus \mathcal{C}(f)$.

The *oscillation of a function* f *on a non-empty set* $A \subset \mathbf{R}^m$ will be denoted by $\omega(f, A)$ $\big($i.e., $\omega(f,A) = \sup\{|f(x) - f(y)| : x, y \in A\}\big)$. Similarly, the *oscillation of a function* f *at a point* $x \in \mathbf{R}^m$ will be denoted by $\omega(f,x)$ $\big($i.e., $\omega(f,x) = \lim_{r\to 0^+} \omega\big(f, \{y \in \mathbf{R}^m : \varrho(x,y) < r\}\big)\big)$. We will write $M(f,x)$ for $\max\Big\{\limsup_{t\to x} f(t), f(x)\Big\}$ and $m(f,x)$ for $\min\Big\{\liminf_{t\to x} f(t), f(x)\Big\}$. Observe that for any function f, $\omega(f,\cdot) = M(f,\cdot) - m(f,\cdot)$, the functions

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$M(f,\cdot)$ and $\omega(f,\cdot)$ are upper semicontinuous, and $m(f,\cdot)$ is lower semicontinuous.

We say that a function f is *quasi-continuous* (resp. *cliquish*) at a point $x \in \mathbf{R}^m$ if for each $\varepsilon > 0$ and each open set $U \ni x$ we can find a non-empty open set $V \subset U$ such that $\omega\bigl(f,\{x\} \cup V\bigr) < \varepsilon$ $\bigl($resp. $\omega(f,V) < \varepsilon\bigr)$. We say that f is *quasi-continuous* (resp. *cliquish*) if it is quasi-continuous (resp. cliquish) at each point $x \in \mathbf{R}^m$. Cliquish functions are also known as *pointwise discontinuous*.

We will use the following well-known (and easy to prove) facts.

LEMMA 2. (1) *A function f is quasi-continuous at a point $x \in \mathbf{R}^m$ iff there exists an open set $H \subset \mathbf{R}^m$ such that $x \in \operatorname{cl} H$ and $f|\bigl(\{x\} \cup H\bigr)$ is continuous at x.*

(2) *The limit of a uniformly convergent sequence of quasi-continuous functions is quasi-continuous.*

(3) *For an arbitrary function f and $x \in \mathbf{R}^m$, the existence of a sequence $x_1, x_2, \ldots \in \mathcal{C}(f)$ such that $x_n \overset{n\to\infty}{\longrightarrow} x$ and $f(x_n) \overset{n\to\infty}{\longrightarrow} f(x)$ implies quasi-continuity of f at x.*

(4) *Each quasi-continuous function is cliquish.*

(5) *The sum of two cliquish functions is cliquish.*

(6) *A function f is cliquish iff $\mathcal{D}(f)$ is of the first category.*

(7) *Let f be a cliquish function which is quasi-continuous at a point $x \in \mathbf{R}^m$ and let $A \subset \mathbf{R}^m$ be a set of first category. Then we can find a sequence $x_1, x_2, \ldots \notin A$ such that $x_n \overset{n\to\infty}{\longrightarrow} x$ and $f(x_n) \overset{n\to\infty}{\longrightarrow} f(x)$.* □

To simplify notation, for a given cliquish function f and a point $x \in \mathbf{R}^m$ we will write $\overline{\mathrm{LIM}}(f,x)$ for $\limsup\limits_{y\to x, y \in \mathcal{C}(f)} f(y)$. $\bigl($By Lemma 2 (6) this notion is always reasonable.$\bigr)$ For each $k > 1$ we will denote by $\mathcal{S}_k$ the family of all non-negative cliquish functions f such that $\overline{\mathrm{LIM}}(f,x) \geqq f(x)/k$ whenever $x \in \mathcal{D}(f)$.

The word *interval* will always mean non-degenerate compact interval in $\mathbf{R}^m$, i.e., the Cartesian product of m non-degenerate compact intervals in $\mathbf{R}$.

In the proof of the main results we will need a few lemmas. The first of them can be proved actually in the same way as [3, Lemma 3.4] (for $k = 2$).

LEMMA 3. *Assume that A is a nowhere dense closed set which contains all points of discontinuity of the quasi-continuous functions $\overline{h}^{(1)}, \ldots, \overline{h}^{(k)}$. Then there exists a family of non-overlapping intervals $\mathcal{I} = \bigl\{I_n : n \in \mathbf{N}\bigr\}$ such that*

i) *each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{I}$,*

ii) $\operatorname{diam} I_n \leqq \varrho(I_n, A)$ *for each $n \in \mathbf{N}$,*

iii) *for each $i \in \{1,\ldots,k\}$ and each $x \in A$ there is a subfamily $\bigl\{I_{n_l} : l \in$*

$\in \mathbf{N}\big\}$ *such that* $x \in \operatorname{cl} \bigcup_{l \in \mathbf{N}} I_{n_l}$ *and* $\overline{h}^{(i)} \big| \big(\{x\} \cup \bigcup_{l \in \mathbf{N}} \operatorname{int} I_{n_l}\big)$ *is continuous at* x. □

LEMMA 4. *Given a* $k > 1$, *a function* $g \in \mathcal{S}_k$ *and an* $\varepsilon > 0$ *we can find a function* $\overline{g} \in \mathcal{S}_k$ *such that* $B = \operatorname{cl} \mathcal{D}(\overline{g}) \subset \mathcal{D}(g)$, $g = \overline{g}$ *on* B, $0 \leqq g - \overline{g} \leqq \varepsilon$ *on* $\mathbf{R}^m \setminus B$ *and* $g - \overline{g} \in \mathcal{S}_k$.

PROOF. Use Lemma 3 with $A = \big\{x \in \mathbf{R}^m : \omega(g, x) \geqq \varepsilon \cdot (1 - 1/k)\big\}$ to find a family of non-overlapping intervals $\big\{I_n : n \in \mathbf{N}\big\}$ satisfying conditions i) and ii) of that lemma.

For each $n \in \mathbf{N}$ do the following. If $g = 0$ on I_n, then define $\alpha_n = 0$ on $\mathbf{R}^m$. Otherwise use that g is bounded on I_n and choose an $x_n \in \operatorname{int} I_n \cap \cap \mathcal{C}(g)$ such that $g(x_n) > \max\big\{\sup\{g(x) : x \in I_n \cap \mathcal{C}(g)\} - 1/n, 0\big\}$. Let $U_n \subset \subset \operatorname{cl} U_n \subset \operatorname{int} I_n$ be an open neighborhood of x_n such that $\big|g(x) - g(x_n)\big| < < \tau_n = \min\big\{g(x_n)/5, 1/n\big\}$ for $x \in U_n$. Let α_n be a continuous function such that $\alpha_n(x_n) = g(x_n) - 5\tau_n$, $\alpha_n = 0$ outside U_n and $0 \leqq \alpha_n \leqq g(x_n) - - 5\tau_n$ on U_n.

Observe that for each $x \notin A$

$$M(g, x) - m(g, x) < \varepsilon \cdot (1 - 1/k),$$

$$\frac{M(g, x)}{1 - 1/k} - \varepsilon < \frac{m(g, x)}{1 - 1/k} \leqq \frac{m\big(\overline{\mathrm{LIM}}(g, \cdot), x\big)}{1 - 1/k},$$

$$M(g, x) - \varepsilon < \min\left\{\frac{m\big(\overline{\mathrm{LIM}}(g, \cdot), x\big) - M(g, x)/k}{1 - 1/k}, m(g, x) - \varepsilon/k\right\},$$

$$M(g - \varepsilon, x) < m\left(\min\left\{\frac{\overline{\mathrm{LIM}}(g, \cdot) - g/k}{1 - 1/k}, g\right\}, x\right),$$

so, since $g \in \mathcal{S}_k$,

$$M\big(\max\{g - \varepsilon, 0\}, x\big) \leqq m\left(\min\left\{\frac{\overline{\mathrm{LIM}}(g, \cdot) - g/k}{1 - 1/k}, g\right\}, x\right).$$

By a well-known theorem $\big($see [1, Theorem 1.7.15 (b)]$\big)$, there exists a continuous function $\beta \colon \mathbf{R}^m \setminus A \longrightarrow \mathbf{R}$ such that

$$\max\big\{g(x) - \varepsilon, 0\big\} \leqq \beta(x) \leqq \min\left\{\frac{\overline{\mathrm{LIM}}(g, x) - g(x)/k}{1 - 1/k}, g(x)\right\}$$

for each $x \in \mathbf{R}^m \setminus A$.

Now define the function $\overline{g}$ by the formula

$$\overline{g}(x)=\begin{cases}\max\{\alpha_n(x),\beta(x)\} & \text{if } x\in I_n,\ n\in\mathbf{N},\\ g(x) & \text{otherwise.}\end{cases}$$

Fix an $x\in\mathbf{R}^m$. First suppose $x\in A$. Take a sequence $y_1,y_2,\ldots\in\mathcal{C}(g)$ with $y_n\xrightarrow{n\to\infty}x$ and $\lim_{n\to\infty}g(y_n)\geqq g(y)/k$. For each $n\in\mathbf{N}$ there is an $l_n\in\in\mathbf{N}$ with $y_n\in I_{l_n}$. Consider the sequence (x_{l_n}). Then

$$\varrho(x,x_{l_n})\leqq\varrho(x,y_n)+\operatorname{diam}I_{l_n}\leqq 2\cdot\varrho(x,y_n)\xrightarrow{n\to\infty}0,$$

so

$$\overline{\mathrm{LIM}}(\overline{g},x)\geqq\limsup_{n\to\infty}\overline{g}(x_{l_n})\geqq\limsup_{n\to\infty}\alpha_{l_n}(x_{l_n})\geqq$$

$$\geqq\limsup_{n\to\infty}\big(g(y_n)-6/l_n\big)\geqq g(x)/k=\overline{g}(x)/k.$$

Meanwhile $\overline{\mathrm{LIM}}(g-\overline{g},x)\geqq 0=\big(g(x)-\overline{g}(x)\big)/k$.

Now let $x\notin A$. Then $x\in\mathcal{C}(\overline{g})$, so by the above, $\overline{g}\in\mathcal{S}_k$. On the other hand,

$$\overline{\mathrm{LIM}}(g-\beta,x)=\overline{\mathrm{LIM}}(g,x)-\beta(x)\geqq$$

$$\geqq\beta(x)\cdot(1-1/k)+g(x)/k-\beta(x)=\big(g(x)-\beta(x)\big)/k,$$

and, if $x\in\operatorname{cl}U_n$ for some $n\in\mathbf{N}$, then

$$\overline{\mathrm{LIM}}(g-\alpha_n,x)=\overline{\mathrm{LIM}}(g,x)-\alpha_n(x)\geqq g(x_n)-\tau_n-\alpha_n(x)>$$

$$>g(x)-\alpha_n(x)-2\tau_n>\frac{g(x)-\alpha_n(x)}{2}\geqq\frac{g(x)-\alpha_n(x)}{k}.$$

Hence $g-\overline{g}\in\mathcal{S}_k$. The other requirements are easy to prove. □

LEMMA 5. *Let g be a non-negative function which is continuous on some interval I and $k>1$. Then for each $\varepsilon\in\big(0,\|g\cdot\chi_I\|/2\big)$ we can find non-negative continuous functions $\alpha^{(1)},\ldots,\alpha^{(k)}$ such that $\alpha^{(1)}+\ldots+\alpha^{(k)}=g$ on I and for $i\in\{1,\ldots,k\}$*

a) *$\alpha^{(i)}(x)>0$ if $g(x)>0$, $x\in I$,*

b) *$\alpha^{(i)}=g/k$ on $\operatorname{fr}I$,*

c) *$\alpha^{(i)}(I)\supset\big[\varepsilon,\|g\cdot\chi_I\|-\varepsilon\big]$.*

PROOF. First find pairwise disjoint non-empty open sets $V_1,\ldots,V_k\subset\subset I$ such that $g>\|g\cdot\chi_I\|-\varepsilon/k$ on $V_1\cup\ldots\cup V_k$. Then choose arbitrary

$x_i \in V_i$ $\big(i \in \{1,\dots,k\}\big)$ and use the Tietze extension theorem to construct a continuous function β such that $\beta = g/k$ on $I \setminus (V_1 \cup \ldots \cup V_k)$, $\beta(x_i) = \varepsilon/k$ for $i \in \{1,\dots,k\}$ and $\varepsilon/k \leqq \beta \leqq g/k$ on $V_1 \cup \ldots \cup V_k$. Finally, for $i \in \{1,\dots,k\}$ define

$$\alpha^{(i)}(x) = \begin{cases} g(x) - (k-1)\cdot\beta(x) & \text{if } x \in V_i, \\ \beta(x) & \text{otherwise.} \end{cases}$$

It is easy to see that then a)–c) are fulfilled. □

LEMMA 6. *Assume that $k > 1$, the nowhere dense closed sets $B \subset A$, a function $g \in \mathcal{S}_k$ and non-negative quasi-continuous functions $\overline{h}^{(1)},\dots,\overline{h}^{(k)}$ are such that $g = 0$ on B, $\mathcal{D}(g) \subset A$ and $\mathcal{D}\big(\overline{h}^{(1)}\big) \cup \ldots \cup \mathcal{D}\big(\overline{h}^{(k)}\big) \subset B$. Then we can find quasi-continuous functions $h^{(1)},\dots,h^{(k)}$ such that $h^{(1)} + \ldots + h^{(k)} = \overline{h}^{(1)} + \ldots + \overline{h}^{(k)} + g$, $\mathcal{D}\big(h^{(1)}\big) \cup \ldots \cup \mathcal{D}\big(h^{(k)}\big) \subset A$ and for $i \in \{1,\dots,k\}$*

i) $h^{(i)} - \overline{h}^{(i)} \geqq 0$ *on* $\mathbf{R}^m$,

ii) $h^{(i)}(x) - \overline{h}^{(i)}(x) > 0$ *if* $g(x) > 0$, $x \in \mathbf{R}^m$,

iii) $h^{(i)} - \overline{h}^{(i)} = g/k$ *on* A.

PROOF. First find a family of non-overlapping intervals $\{I_n : n \in \mathbf{N}\}$ according to Lemma 3. Fix an $n \in \mathbf{N}$.

If $g = 0$ on I_n then define $\alpha_n^{(1)} = \ldots = \alpha_n^{(k)} = 0$ on $\mathbf{R}^m$.

Otherwise apply Lemma 5 with $\varepsilon = \varepsilon_n = \min\{\|g\cdot\chi_{I_n}\|/3, 1/n\}$ and $I = I_n$, and find non-negative continuous functions $\alpha_n^{(1)},\dots,\alpha_n^{(k)}$ such that $\alpha_n^{(1)} + \ldots + \alpha_n^{(k)} = g$ on I_n and for $i \in \{1,\dots,k\}$: $\alpha_n^{(i)}(x) > 0$ if $g(x) > 0$ $(x \in I_n)$, $\alpha_n^{(i)} = g/k$ on $\operatorname{fr} I_n$ and

$$(*) \qquad \alpha_n^{(i)}(I_n) \supset \big[\varepsilon_n, \|g\cdot\chi_{I_n}\| - \varepsilon_n\big].$$

Define for $i \in \{1,\dots,k\}$

$$h^{(i)}(x) = \overline{h}^{(i)}(x) + \begin{cases} \alpha_n^{(i)}(x) & \text{if } x \in I_n,\ n \in \mathbf{N}, \\ g(x)/k & \text{otherwise.} \end{cases}$$

Then conditions i)–iii) are clearly satisfied and, by condition i) of Lemma 3, the functions $h^{(1)},\dots,h^{(k)}$ are continuous on $\mathbf{R}^m \setminus A$, so we need only to show that they are quasi-continuous on A to complete the proof.

Take an $i \in \{1,\dots,k\}$ and an $x \in A$. First suppose $x \in B$. By condition iii) of Lemma 3, there exists a subfamily $\{I_{n_l} : l \in \mathbf{N}\}$ such that $x \in$ $\in \operatorname{cl} \bigcup_{l\in\mathbf{N}} I_{n_l}$ and $\overline{h}^{(i)} | \left(\{x\} \cup \bigcup_{l\in\mathbf{N}} \operatorname{int} I_{n_l}\right)$ is continuous at x. Put

$$H = \bigcup_{l\in\mathbf{N}} \Bigl(\operatorname{int} I_{n_l} \cap \bigl(\alpha_{n_l}^{(i)}\bigr)^{-1}\bigl((-2/n_l, 2/n_l)\bigr)\Bigr).$$

Then H is open and by $(*)$, $x \in \operatorname{cl} H$. Since $g(x) = 0$, so $h^{(i)} | \bigl(\{x\} \cup H\bigr)$ is continuous at x. Hence and by Lemma 2 (1), $h^{(i)}$ is quasi-continuous at x.

Now let $x \notin B$. By the assumptions on g there is a sequence $x_1, x_2, \dots \in$ $\in \mathcal{C}(g)$ such that $x_n \stackrel{n\to\infty}{\longrightarrow} x$ and $g(x_n) > g(x)/k - 1/n$ for each $n \in \mathbf{N}$. Since the set A is nowhere dense, we may assume that each x_n belongs to some I_{l_n}. Use the condition $(*)$ to find a $y_n \in I_{l_n}$ with $\bigl|\alpha_{l_n}^{(i)}(y_n) - g(x)/k\bigr| \leqq \varepsilon_{l_n} +$ $+ 1/n$. Then $y_n \stackrel{n\to\infty}{\longrightarrow} x$ (by condition i) of Lemma 3), $h^{(i)}(y_n) \stackrel{n\to\infty}{\longrightarrow} h^{(i)}(x)$ (since $\overline{h}^{(i)}$ is continuous at x) and $y_1, y_2, \dots \in \mathcal{C}\bigl(h^{(i)}\bigr)$. So by Lemma 2 (3), $h^{(i)}$ is quasi-continuous at x. □

To the end of the article let $\mathfrak{A}$ be a vector space of functions which is closed with respect to uniform limits and such that each function w which coincides with some function $v \in \mathfrak{A}$ on $\operatorname{cl} \mathcal{D}(w)$ is an element of $\mathfrak{A}$, too.

THEOREM 7. *For each $k > 1$ and each function $f \in \mathfrak{A}$ the following three conditions are equivalent:*

A) *there exist non-negative quasi-continuous functions $h^{(1)}, \dots, h^{(k)}$ such that $f = h^{(1)} + \dots + h^{(k)}$,*

B) *f belongs to $\mathcal{S}_k$,*

C) *there exist non-negative quasi-continuous functions $h^{(1)}, \dots, h^{(k)} \in \mathfrak{A}$ such that $f = h^{(1)} + \dots + h^{(k)}$ and for $i \in \{1,\dots,k\}$: $\mathcal{D}\bigl(h^{(i)}\bigr) \subset \mathcal{D}(f)$ and $h^{(i)}(x) > 0$ whenever $f(x) > 0$, $x \in \mathbf{R}^m$.*

PROOF. A) $\Rightarrow$ B). The function f is clearly non-negative and, by Lemmas 2 (4) and 2 (5), cliquish. Fix an $x \in \mathcal{D}(f)$. Let $i \in \{1,\dots,n\}$ be such that $h^{(i)}(x) \geqq f(x)/k$. By Lemmas 2 (6) and 2 (7), there exists a sequence $x_1, x_2, \dots \in \mathcal{C}(f)$ such that $x_n \stackrel{n\to\infty}{\longrightarrow} x$ and $h^{(i)}(x_n) \stackrel{n\to\infty}{\longrightarrow} h^{(i)}(x)$. Then

$$\overline{\mathrm{LIM}}(f,x) \geqq \limsup_{n\to\infty} f(x_n) \geqq \limsup_{n\to\infty} h^{(i)}(x_n) = h^{(i)}(x) \geqq f(x)/k.$$

B) $\Rightarrow$ C). Let $f_0 = 0$ on $\mathbf{R}^m$ and $B_0 = \emptyset$. For each $n \in \mathbf{N}$ use Lemma 4 with $\varepsilon = 2^{-n}$ and $g = f - (f_0 + \dots + f_{n-1})$ to find a function $f_n \in \mathcal{S}_k$ such that $B_n = \operatorname{cl} \mathcal{D}(f_n) \subset \mathcal{D}(f)$ (so B_n is nowhere dense), $f = f_0 + \dots + f_n$ on B_n (so, by assumptions on $\mathfrak{A}$, $f_n \in \mathfrak{A}$), $0 \leqq f - (f_0 + \dots + f_n) \leqq 2^{-n}$ on $\mathbf{R}^m \setminus B_n$ and $f - (f_0 + \dots + f_n) \in \mathcal{S}_k$.

Define $h_0^{(1)} = \ldots = h_0^{(k)} = 0$ on $\mathbf{R}^m$. For each $n \in \mathbf{N}$ use Lemma 6 with $B = B_0 \cup \ldots \cup B_{n-1}$, $A = B_0 \cup \ldots \cup B_n$, $\overline{h}^{(i)} = h_{n-1}^{(i)}$ $\big(i \in \{1,\ldots,k\}\big)$ and $g = f_n$, and find quasi-continuous functions $h_n^{(1)},\ldots,h_n^{(k)}$ such that $h_n^{(1)} + \ldots + h_n^{(k)} = f_0 + \ldots + f_n$, $\mathcal{D}\big(h_n^{(1)}\big) \cup \ldots \cup \mathcal{D}\big(h_n^{(k)}\big) \subset B_0 \cup \ldots \cup B_n$ and for $i \in \{1,\ldots,k\}$: $h_n^{(i)} - h_{n-1}^{(i)} \geqq 0$ on $\mathbf{R}^m$, $h_n^{(i)}(x) - h_{n-1}^{(i)}(x) > 0$ whenever $f_n(x) > 0$ $(x \in \mathbf{R}^m)$ and $h_n^{(i)} - h_{n-1}^{(i)} = f_n/k$ on $B_0 \cup \ldots \cup B_n$. Then by assumptions on $\mathfrak{A}$, $h_n^{(1)},\ldots,h_n^{(k)} \in \mathfrak{A}$.

In this way we constructed k sequences of non-negative quasi-continuous functions. For $i \in \{1,\ldots,k\}$, if $s > n$, then

$$0 \leqq h_s^{(i)} - h_n^{(i)} \leqq \sum_{l=n+1}^{s} f_l \leqq \sum_{l=n+1}^{s} 2^{1-l} < 2^{1-n}$$

on $\mathbf{R}^m$, so these sequences are uniformly convergent. Hence for $i \in \{1,\ldots,k\}$ the function $h^{(i)} = \lim_{n\to\infty} h_n^{(i)}$ is quasi-continuous (by Lemma 2 (2)), $h^{(i)} \in \mathfrak{A}$ (by assumptions on $\mathfrak{A}$) and $\mathcal{D}\big(h^{(i)}\big) \subset \mathcal{D}(f)$. Moreover, $h^{(1)} + \ldots + h^{(k)} = f$.

Let $x \in \mathbf{R}^m$ be such that $f(x) > 0$. Then there is an $n \in \mathbf{N}$ such that $f_n(x) > 0$, so $h^{(i)}(x) \geqq h_n^{(i)}(x) - h_{n-1}^{(i)}(x) > 0$ for $i \in \{1,\ldots,k\}$.

C) $\Rightarrow$ A). This implication is obvious. □

Now we will study the sums of quasi-continuous functions bounded below. We get the main result as a corollary from the above theorem.

COROLLARY 8. *For each $k > 1$ and each function $f \in \mathfrak{A}$ the following three conditions are equivalent:*

a) *there exist quasi-continuous functions $h^{(1)},\ldots,h^{(k)}$ bounded below such that $f = h^{(1)} + \ldots + h^{(k)}$,*

b) *f is a cliquish function bounded below and*

$$\inf\big\{\,\overline{\mathrm{LIM}}(f,x) - f(x)/k : x \in \mathcal{D}(f)\big\} > -\infty,$$

c) *there exist quasi-continuous functions $h^{(1)},\ldots,h^{(k)} \in \mathfrak{A}$ bounded below such that $f = h^{(1)} + \ldots + h^{(k)}$ and $\mathcal{D}\big(h^{(i)}\big) \subset \mathcal{D}(f)$ for $i \in \{1,\ldots,k\}$.*

PROOF. a) $\Rightarrow$ b). Let $s \in \mathbf{R}$ be such that $h^{(i)} \geqq s$ on $\mathbf{R}^m$ for $i \in \{1,\ldots,k\}$. Then the function $\overline{f} = f - k\cdot s$ can be expressed as the sum of k non-negative quasi-continuous functions, so for each $x \in \mathcal{D}(f)$, by condition B) of Theorem 7,

$$\overline{\mathrm{LIM}}(f,x) - f(x)/k = \overline{\mathrm{LIM}}\big(\overline{f},x\big) - \overline{f}(x)/k + (k-1)\cdot s \geqq (k-1)\cdot s.$$

b) ⇒ c). Let $s \leqq \inf\{\overline{\mathrm{LIM}}(f,x) - f(x)/k : x \in \mathcal{D}(f)\} / (1 - 1/k)$ be such that $f \geqq s$ on $\mathbf{R}^m$. Then the function $\overline{f} = f - s$ is non-negative and

$$\overline{\mathrm{LIM}}(\overline{f}, x) = \overline{\mathrm{LIM}}(f,x) - s \geqq f(x)/k - s/k = \overline{f}(x)/k$$

for each $x \in \mathcal{D}(f)$, so by condition C) of Theorem 7, there are non-negative quasi-continuous functions $\overline{h}^{(1)}, \dots, \overline{h}^{(k)} \in \mathfrak{A}$ with $\overline{f} = \overline{h}^{(1)} + \ldots + \overline{h}^{(k)}$ and $\mathcal{D}(\overline{h}^{(i)}) \subset \mathcal{D}(f)$ for $i \in \{1, \dots, k\}$. Clearly the functions $h^{(i)} = \overline{h}^{(i)} + s/k$ fulfil our requirements.

c) ⇒ a). This implication is obvious. □

EXAMPLE 1. For each $k \in \mathbf{N}$ the function $u_k \colon \mathbf{R} \to \mathbf{R}$ defined by

$$u_k(x) = \begin{cases} 1/k & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$$

is bounded and positive, it is continuous except one point, it can be written as the sum of k positive quasi-continuous functions but cannot be expressed as the sum of less than k non-negative quasi-continuous functions.

EXAMPLE 2. The function $u \colon \mathbf{R} \to \mathbf{R}$,

$$u(x) = \begin{cases} \min\{|x|, 1\} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

is bounded and positive, it is continuous except one point (so it is cliquish), and it cannot be written as the sum of finitely many non-negative quasi-continuous functions. (Observe that $\inf\{u(x) : x \in \mathbf{R}\} = 0$.)

EXAMPLE 3. For each $k \in \mathbf{N}$ the function $v_k \colon \mathbf{R} \to \mathbf{R}$ defined by

$$v_k(x) = \begin{cases} x + 1 & \text{if } x \in \mathbf{N}, \\ |x|/k + 1 & \text{otherwise} \end{cases}$$

is positive and discontinuous on a discrete set, it can be written as the sum of k quasi-continuous functions bounded below but cannot be expressed as the sum of less than k quasi-continuous functions bounded below.

EXAMPLE 4. The function $v \colon \mathbf{R} \to \mathbf{R}$,

$$v(x) = \begin{cases} x + 1 & \text{if } x \in \mathbf{N}, \\ 1 & \text{otherwise,} \end{cases}$$

is positive and discontinuous on a discrete set (so it is cliquish), and it cannot be written as the sum of finitely many quasi-continuous functions bounded below. (Though $\inf\{u(x) : x \in \mathbf{R}\} > 0$.)

Finally I would like to present a query. Theorem 1 implies that each approximately continuous function (resp. each derivative) can be written as the sum of two approximately continuous and quasi-continuous functions (resp. of two quasi-continuous derivatives). It would be of interest to know whether we can find summands satisfying these additional requirements in Theorem 7 and Corollary 8, i.e.:

Can every approximately continuous function from $\mathcal{S}_k$ be represented as the sum of k non-negative, quasi-continuous, approximately continuous functions?

Can every derivative from $\mathcal{S}_k$ be represented as the sum of k non-negative quasi-continuous derivatives?

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ON A LIMIT THEOREM FOR SOME MODIFIED OPERATORS

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1. Introduction

Let f be a function defined on [0,1]. The Bernstein polynomial of order n of $f(x)$ is defined by

$$B_n(f,x)=\sum_{j=0}^{n} f\left(\frac{j}{n}\right)p_{n,j}(x), \tag{1.1}$$

where

$$p_{n,j}(x)=\binom{n}{j}x^j(1-x)^{n-j}.$$

A well-known result about the approximation of $B_n(f,x)$ to $f(x)$ is that (see [7]) if $f^{(2k)}(x)$ exists at x, then

(1.2)

$$\lim_{n\to\infty} n^k\left[B_n(f,x)-f(x)-\sum_{i=2}^{2k-1}\frac{f^{(i)}(x)}{i!}T_{ni}(x)\right]=\left(\frac{x(1-x)}{2}\right)^k\frac{f^{(2k)}(x)}{k!},$$

where $T_{ni}(x)=\sum_{j=0}^{n}\left(\frac{j}{n}-x\right)^i p_{n,j}(x)$. $T_{ni}(x)=B_n\left((\cdot-x)^i,x\right)$ is called the ith moment of Bernstein polynomial of order n. Hence we may say that (1.2) is the moment expansion of $B_n(f,x)$. Khan [6] has shown that the moment expansion holds for a class of Feller operators. Consider the modified Bernstein polynomials

$$P_n(f,x)=(n+1)\sum_{j=0}^{n}\left(\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(t)\,dt\right)p_{n,j}(x). \tag{1.3}$$

The approximate properties of $P_n(f,x)$ have been studied by many authors; Kantorovich [5], Hoeffding [4], Bojanic and Shisha [2], Ditzian and Totik [3] among others. But the limit result of (1.2) for $P_n(f,x)$ has not been seen in the literature except a wrong result in [10]. The goal of this article is to establish (1.2) for $P_n(f,x)$ and for the Szász–Kantorovich operator. We also establish a non-moment expansion for the Baskakov–Kantorovich operator. The Baskakov–Kantorovich operator and Szász–Kantorovich operator are defined, respectively, by (see [3, p.115])

$$B_n^*(f,x) = n\sum_{j=0}^{\infty}\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t)\,dt\right) b_{n,j}(x) \tag{1.4}$$

and

$$S_n^*(f,x) = n\sum_{j=0}^{\infty}\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t)\,dt\right) s_{n,j}(x), \tag{1.5}$$

where

$$b_{n,j}(x) = \binom{n+j-1}{j}\frac{x^j}{(1+x)^{n+j}} \tag{1.6}$$

and

$$s_{n,j}(x) = \frac{(nx)^j}{j!}e^{-nx}, \tag{1.7}$$

$j = 0,1,2,\dots$, and $0 \leqq x \leqq a < \infty$. A probabilistic method has been used to prove our results. It is not only convenient but also powerful. Without using probabilistic methods, the following Theorem 2.3 could be difficult to prove by the usual analytic method.

Let $Y_1,\dots,Y_n$ be iid (independent and identically distributed) random variables on a probability space $(\Omega,\mathcal{F},P_x)$ with $P_x(Y_1=1)=x$, $P_x(Y_1 = 0) = 1-x$ and $0 \leqq x \leqq 1$. Y_1 is called the Bernoulli random variable. Let $S_n = \sum_{i=1}^n Y_i$ and E_xY denote the expectation of the random variable Y under the probability measure P_x. By using probabilistic notations, (1.3) can be written as

$$P_n(f,x) = (n+1)E_x\left(\int_{\frac{S_n}{n+1}}^{\frac{S_n+1}{n+1}} f(t)\,dt\right). \tag{1.8}$$

Similarly, (1.4) is

$$B_n^*(f,x) = nE_x\left(\int_{\frac{S_n}{n}}^{\frac{S_n+1}{n}} f(t)\,dt\right), \tag{1.9}$$

with $S_n = \sum_{i=1}^n Y_i$, $Y_1, \dots, Y_n$ are iid random variables with the distribution $P_x(Y_1 = j) = pq^j$, $j = 0, 1, 2, \dots$, where $0 < p = \frac{1}{x+1}$ with $x \geqq 0$ and $p + q = = 1$.

Finally, (1.5) can be written as

$$S_n^*(f, x) = nE_x \left(\int_{\frac{S_n}{n}}^{\frac{S_n+1}{n}} f(t)\,dt \right), \tag{1.10}$$

with $S_n = \sum_{i=1}^n Y_i$, where $Y_1, \dots, Y_n$ are iid random variables with the distribution $P_x(Y_1 = j) = \frac{x^j}{j!}e^{-x}$, $j = 0, 1, 2, \dots$.

2. Main results

The first result is about the modified Bernstein polynomial.

THEOREM 2.1. *If the derivative $f^{(2k)}(x)$ exists at x with $x \in (0, 1)$, then*

$$\lim_{n\to\infty} n^k \left\{ P_{n-1}(f, x) - f(x) - \frac{n}{x(1-x)} \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{(i+1)!} T_{n,i+2}(x) \right\} = \tag{2.1}$$

$$= \left[x(1-x) \right]^k \frac{f^{(2k)}(x)}{2^k k!}.$$

REMARK. Comparing (2.1) with (1.2), $T_{n,i}(x)$ in (1.2) are replaced by $\frac{n}{x(1-x)} T_{n,i+2}(x)$ in (2.1). Moreover, the term with $f'(x)$ is included in (2.1) (see the result in [10]), but not in (1.2) because $T_{n,1}(x) = 0$.

The proof of the above result is based on the following lemma due to Bojanic and Shisha [2].

LEMMA 2.2. *Let $f(x)$ be a Lebesgue integrable function on $[0, 1]$. Then, for $x \in [0, 1]$,*

$$x(1-x)\left[P_{n-1}(f, x) - f(x) \right] =$$

$$= nE_x \left\{ \left(\frac{S_n}{n} - x \right) \int_0^{\frac{S_n}{n} - x} \left(f(x+t) - f(x) \right) dt \right\}.$$

PROOF OF THEOREM 2.1. From Taylor's formula and Lemma 2.2,

$$x(1-x)\left[P_{n-1}(f, x) - f(x) \right] = n \sum_{i=1}^{2k} \frac{f^{(i)}(x)}{(i+1)!} E \left(\frac{S_n}{n} - x \right)^{i+2} + R_n,$$

where

$$R_n = nE_x\left(\frac{S_n}{n} - x\right)\int_0^{\frac{S_n}{n}-x} g(\xi)t^{2k}\,dt,$$

$|\xi| < \left|\frac{S_n}{n} - x\right|$ and $g(\xi)$ is uniformly bounded, $g(\xi) \to 0$ as $|\xi| \to 0$. To complete the proof, it suffices to show

$$\lim_{n\to\infty} \frac{1}{(2k+1)!} n^{k+1} E\left(\frac{S_n}{n} - x\right)^{2k+2} = \frac{1}{2^k k!}\left[x(1-x)\right]^{k+1} \tag{2.2}$$

and

$$\lim_{n\to\infty} n^k R_n = 0. \tag{2.3}$$

(2.2) is a consequence of Lemma 5 in [6]. (2.3) follows by using the argument of the proof of Theorem 2 in [6].

For the Baskakov–Kantorovich operator, we have the following result.

THEOREM 2.3. *If the derivative $f^{(2k)}(x)$ exists at x with $x > 0$,*

$$\lim_{n\to\infty} n^k \left\{ B_n^*(f,x) - f(x) - \frac{n(x+1)}{x}\sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{(i+1)!} M_{n,i+2}(x) \right\} =$$

$$= \left[x(x+1)\right]^k \frac{f^{(2k)}(x)}{2^k k!}$$

where $M_{n,i+2}(x) = E_x\left\{\left(\frac{S_n}{n+S_n-1} - \frac{x}{x+1}\right)\left(\frac{S_n}{n} - x\right)^{i+1}\right\}$.

REMARK. We note that the above theorem is an expansion of $B_n^*(f,x)$ in terms of $M_{n,i+2}(x)$, which are not usual moments. To establish the above expansion, a crucial step is to find $\lim_{n\to\infty} n^{k+1} M_{n,2k+2}(x)$. By using the probabilistic method, we avoid the complicated calculation of $M_{n,2k+2}(x)$.

The proof of Theorem 2.3 relies on the following lemma.

LEMMA 2.4. *Let $f(x)$ be a Lebesgue integrable function on any finite interval. Then, for each $x > 0$,*

$$\frac{x}{1+x}\left[B_n^*(f,x) - f(x)\right] =$$

$$= n\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n+k-1} - \frac{x}{1+x}\right)\int_0^{\frac{k}{n}-x} \left(f(x+t) - f(x)\right)dt.$$

PROOF. It can be shown that

$$\frac{x}{1+x}\left[b_{n,k-1}(x)-b_{n,k}(x)\right]=b_{n,k}(x)\left(\frac{k}{n+k-1}-\frac{x}{1+x}\right), \tag{2.4}$$

$$\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n+k-1}-\frac{x}{1+x}\right)=0, \tag{2.5}$$

and

$$n\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n+k-1}-\frac{x}{1+x}\right)\left(\frac{k}{n}-x\right)=\frac{x}{1+x}. \tag{2.6}$$

Write

$$B_n^*(f,x)=\int_0^{\infty} K_n(x,t)f(t)\,dt,$$

where

$$K_n(x,t)=n\sum_{k=0}^{\infty} b_{n,k}(x)\chi_{(\frac{k}{n},\frac{k+1}{n}]}(t),$$

$\chi_{(\frac{k}{n},\frac{k+1}{n}]}(t)$ being the indicator function of $\left(\frac{k}{n},\frac{k+1}{n}\right]$. By

$$K_n(x,t)=-nb_{n,0}(x)\chi_{[0,0]}(t)+n\sum_{k=1}^{\infty}\left(b_{n,k-1}(x)-b_{n,k}(x)\right)\chi_{[0,\frac{k}{n}]},$$

(2.4) and (2.5) imply that

$$\frac{x}{1+x}B_n^*(f,x)=n\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n+k-1}-\frac{x}{1+x}\right)\int_0^{\frac{k}{n}} f(t)\,dt= \tag{2.7}$$

$$=n\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n+k-1}-\frac{x}{1+x}\right)\int_0^{\frac{k}{n}-x} f(x+t)\,dt.$$

Combining (2.6) and (2.7), the lemma follows.

PROOF OF THEOREM 2.3. From Taylor's formula and Lemma 2.4,

$$\frac{x}{x+1}\left[B_n^*(f,x)-f(x)\right]=n\sum_{i=1}^{2k}\frac{f^{(i)}(x)}{(i+1)!}M_{n,i+2}(x)+R_n,$$

where

$$R_n = nE_x\left(\frac{S_n}{n+S_n-1}-\frac{x}{x+1}\right)\int_0^{\frac{S_n}{n}-x} g(\xi)t^{2k}\,dt,$$

$|\xi| < \left|\frac{S_n}{n}-x\right|$ and $g(\xi)$ is uniformly bounded, $g(\xi)\to 0$ as $|\xi|\to 0$. Let $|g(\xi)| \leqq C$ for any ξ. To complete the proof, it remains to show

$$\text{(2.8)}\quad \lim_{n\to\infty}\frac{1}{(2k+1)!}n^{k+1}E_x\left\{\left(\frac{S_n}{n+S_n-1}-\frac{x}{x+1}\right)\left(\frac{S_n}{n}-x\right)^{2k+1}\right\}=$$

$$=\frac{1}{2^k k!}x^{k+1}(x+1)^{k-1}$$

and

$$\text{(2.9)}\qquad \lim_{n\to\infty} n^k R_n = 0.$$

Write

$$\text{(2.10)}\qquad \frac{S_n}{n+S_n-1}-\frac{x}{x+1}=\frac{\left(\frac{S_n}{n}-x\right)+\frac{x}{n}}{(x+1)\left(\frac{S_n}{n}+1-\frac{1}{n}\right)}.$$

By strong law of large number [1, p.290], $(x+1)\left(\frac{S_n}{n}+1-\frac{1}{n}\right)\to(x+1)^2$ with probability one. Let

$$Z_n=\frac{\sqrt{n}\left(\frac{S_n}{n}-x\right)}{\left[(x+1)\left(\frac{S_n}{n}+1-\frac{1}{n}\right)\right]^{\frac{1}{2k+2}}}.$$

From the central limit theorem, $\sqrt{n}\left(\frac{S_n}{n}-x\right)$ converges in distribution to a normal random variable with mean zero and variance $x(x+1)$. It follows that, by Slutsky's theorem [9, p.19], Z_n converges in distribution to a normal random variable, say Z, with mean zero and variance $x(x+1)^{\frac{k-1}{k+1}}$. From (2.10), neglecting the higher order term, we have

$$\lim_{n\to\infty}\frac{1}{(2k+1)!}n^{k+1}E_x\left\{\left(\frac{S_n}{n+S_n-1}-\frac{x}{x+1}\right)\left(\frac{S_n}{n}-x\right)^{2k+1}\right\}=$$

$$=\lim_{n\to\infty}\frac{1}{(2k+1)!}E_xZ_n^{2k+2}.$$

This and $\frac{1}{(2k+1)!}E_x Z^{2k+2} = \frac{1}{2^k k!}x^{k+1}(x+1)^{k-1}$ imply (2.8) if we can show

$$\lim_{n\to\infty} E_x Z_n^{2k+2} = E_x Z^{2k+2}. \tag{2.11}$$

To this end, from the corollary in [1, p.348], it suffices to show

$$\sup_n E_x|Z_n|^r < \infty, \quad \text{for some } r > 2k+2. \tag{2.12}$$

Since $Y_1 - x$ has finite moment generating function (Y_1 is a geometric random variable), Lemma 5 in [8, p.54] implies, for some $T > 0$ and $a > 0$, $E_x e^{t(Y_1-x)} \leqq e^{\frac{1}{2}at^2}$ for $|t| \leqq T$. From Theorem 15 in [8, p.52],

$$P\left(\sqrt{n}\left|\frac{S_n}{n} - x\right| \geqq t\right) \leqq 2e^{-\frac{at^2}{2}}, \quad \text{if} \quad 0 \leqq t \leqq T,$$

and

$$P\left(\sqrt{n}\left|\frac{S_n}{n} - x\right| \geqq t\right) \leqq 2e^{-\frac{tT}{2}}, \quad \text{if} \quad t \geqq T.$$

It follows that, from (2.19) in [1, p.282],

$$E\left|\sqrt{n}\left(\frac{S_n}{n} - x\right)\right|^r = \int_0^\infty P\left(\left|\sqrt{n}\left(\frac{S_n}{n} - x\right)\right|^r > t\right) dt \leqq \tag{2.13}$$

$$\leqq 2\int_0^{T^r} e^{-\frac{at^{\frac{2}{r}}}{2}} dt + 2\int_{T^r}^\infty e^{-\frac{T}{2}t^{\frac{1}{r}}} dt < \infty.$$

Since the right hand side of (2.13) is independent of n, $\sup_n E_x\big|\sqrt{n}\big(\frac{S_n}{n} - - x\big)\big|^r < \infty$. Hence, from $(x+1)\left(\frac{S_n}{n} + 1 - \frac{1}{n}\right) \geqq \frac{n-1}{n}$,

$$\sup_n E_x|Z_n|^r \leqq 2^{\frac{r}{2k+2}} E_x\left|\sqrt{n}\left(\frac{S_n}{n} - x\right)\right|^r < \infty.$$

To prove (2.9), for any $\varepsilon > 0$, there exists a $\delta > 0$, such that $|g(\xi)| < \varepsilon$ if $\left|\frac{S_n}{n} - x\right| < \delta$. From (2.10) for $n \geqq 2$,

$$E_x\left|\left(\frac{S_n}{n+S_n-1} - \frac{x}{x+1}\right)\left(\frac{S_n}{n} - x\right)^{2k+1}\right| \leqq$$

$$\leqq 2E_x\left|\frac{S_n}{n} - x\right|^{2k+2} + \frac{2x}{n}E_x\left|\frac{S_n}{n} - x\right|^{2k+1}.$$

This implies

$$n^k|R_n| \leqq 2\varepsilon n^{k+1} E_x \left(\frac{S_n}{n} - x\right)^{2k+2} + 2x\varepsilon n^k E_x \left|\frac{S_n}{n} - x\right|^{2k+1} +$$

$$+2Cn^{k+1} \left\{ E_x \left\{ \left(\frac{S_n}{n} - x\right)^{2k+2} I\left(\left|\frac{S_n}{n} - x\right| > \delta\right) \right\} + \right.$$

$$\left. + \frac{2x}{n} E_x \left\{ \left|\frac{S_n}{n} - x\right|^{2k+1} I\left(\left|\frac{S_n}{n} - x\right| > \delta\right) \right\} \right\},$$

where $I(A)$ denotes the indicator function of the random event A. Using the argument of the proof of Theorem 2 in [6], we can show

$$\lim_{n\to\infty} \sup n^k |R_n| \leqq \frac{\varepsilon(2k+1)!}{2^{k-1}k!} \left[x(x+1)\right]^{k+1}.$$

Since ε is arbitrary, (2.9) follows.

Our final result for the Szász–Kantorovich operator is the following.

THEOREM 2.5. *If the derivative $f^{(2k)}(x)$ exists at x with $x > 0$, then*

$$\lim_{n\to\infty} n^k \left\{ S_n^*(f,x) - f(x) - \frac{n}{x} \sum_{i=1}^{2k-1} \frac{f^{(i)}(x)}{(i+1)!} Q_{n,i+2}(x) \right\} = \frac{f^{(2k)}(x) x^k}{2^k k!} \tag{2.14}$$

where $Q_{n,i+2}(x) = E_x\left(\frac{S_n}{n} - x\right)^{i+2}$.

The proof of Theorem 2.5 is based on the following lemma.

LEMMA 2.6. *Let $f(x)$ be a Lebesgue integrable function on any finite interval. Then, for each $x > 0$,*

(2.15)

$$x\left[S_n^*(f,x) - f(x)\right] = n \sum_{k=0}^{\infty} s_{n,k}(x) \left(\frac{k}{n} - x\right) \int_0^{\frac{k}{n}-x} \left(f(x+t) - f(x)\right) dt.$$

PROOF. It is easy to check that

$$x\left[s_{n,k-1}(x) - s_{n,k}(x)\right] = s_{n,k}(x)\left(\frac{k}{n} - x\right), \tag{2.16}$$

$$\sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k}{n} - x\right) = 0, \tag{2.17}$$

and

$$n \sum_{k=0}^{\infty} s_{n,k}(x) \left(\frac{k}{n} - x \right)^2 = x. \tag{2.18}$$

The rest of the proof, which is the same as that of Lemma 2.4, is omitted.

The proof of Theorem 2.5 is omitted.

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A NOTE ON STRONGLY PRIME RADICALS

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1. Introduction

The main aim of this note is to present some of the details of the author's example cited in [7], where it was used to show that the strongly prime radical is not left-right symmetric and given along with another example due to the authors for the same purpose. At the same time the opportunity is taken to discuss the relationship between strongly prime and the concept of normality, as defined for radicals in [4] and for classes of prime rings in [5]. It is shown that the class of strongly prime rings is not a normal class of prime rings, although it is known to be a special class of rings [2], and that the strongly prime radical is not normal, although it is special [2]. Parallel remarks can be made for the uniformly strongly prime rings and the uniformly strongly prime radical, the upper radical determined by this class. These observations vis-a-vis normality do not seem to have been made before. On the other hand, it is shown that in the category of rings with identity, strongly prime and uniformly strongly prime (see [6]) are Morita invariant properties as is semisimplicity with respect to either the strongly prime radical or the uniformly strongly prime radical.

The concept of a strongly prime ring was defined by Handelman and Lawrence [3]. A *right insulator* in a ring R is a finite subset F of R such that $Fr = 0, r \in R$, implies $r = 0$. The ring R is said to be *right strongly prime* if every non-zero ideal (two-sided) contains an insulator and the *right strongly prime radical*, $s_r(R)$, is defined to be the intersection of all the ideals I of R for which R/I is right strongly prime. The left strongly prime radical, $s_l(R)$, is defined analogously. Groenewald and Heyman [2] have shown that s_r and s_l are radicals in the Kurosh–Amitsur sense. In the original formulation R is said to be *right strongly prime* if for each non-zero $a \in R$ there is a finite subset F_a of R such that the right annihilator of aF_a is zero, in which case F_a is called a *right insulator of a*. If a common insulator can be found for all the non-zero elements of R, then R is said to be *uniformly strongly prime* and the common insulator is referred to as a *uniform insulator*. It is known that R is uniformly strongly prime if and only if there is a finite subset F of R such that $xFy = 0$ implies $x = 0$ or $y = 0$, from which it is seen that the

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concept is left-right symmetric. The upper radical $us(R)$ determined by the class of uniformly strongly prime rings is called the us-prime radical. The class of uniformly strongly prime rings is known to be special and so us is a special radical. Proofs of these results on uniformly strongly prime rings can be found in [6].

2. The example

EXAMPLE 1. Let K be a field, $X = \{X_1, X_2, \ldots\}$ a set of non-commuting indeterminates, and $A = K\langle X \rangle$, the free associative algebra with 1 on X. Let B be the ideal of A generated by $\{X_iX_j, X_iX_{i+1}X_i - X_i \mid 1 \leqq i, i+2 \leqq j\}$ and put $R = A/B$. Write $y_i = X_i + B, i \geqq 1$, so that R is generated as a K-algebra by $\{y_i \mid i \geqq 1\}$ with $y_iy_j = 0$ and $y_iy_{i+1}y_i = y_i$ for $1 \leqq i$ and $i+2 \leqq j$.

A monomial in the y's will be called *irreducible* if it cannot be reduced in degree using these relations. A number of properties of R will now be given and then used to establish the (negative) results already described.

PROPERTY 1. *The only right strongly prime non-zero factor ring of R is $R/M \cong K$, where $M = (y_1, y_2, \ldots)$.*

PROOF. Suppose P is an ideal of R and that $0 \neq R/P$ is right strongly prime. Let $F = \{f_1 + P, \ldots, f_n + P\}$ be an insulator in R/P with $f_1, \ldots, f_n \in$ $\in R$. Let m be the largest subscript to appear on y's in $f_1, \ldots, f_n$. Then $f_ky_j = 0$ for all $1 \leqq k \leqq n$, $j \geqq m+2$ and so $(f_k + P)(y_j + P) = 0$, that is $F(y_j + P) = 0$. Hence $y_j \in P$ for all $j \geqq m+2$. However, it is clear from the relations on R, if $y_{i+1} \in I$, an ideal of R, then $y_i \in I$ also. Hence $\{y_1, y_2, \ldots\} \subseteqq P$ and $P \supseteqq M = (y_1, y_2, \ldots)$. Therefore $P = M$ as required. □

PROPERTY 2. *The set $\{y_1, y_2\}$ is a left insulator in R.*

PROOF. Suppose $r \in R$ is such that $ry_1 = ry_2 = 0$. If $r \neq 0$ then we can write $r = \sum \alpha_\mu \mu$ where $\alpha_\mu \in K$ and the μ's are distinct irreducible monomials in the y's. Writing $\mu = y_{i_1} \ldots y_{i_k}$, μy_1 is irreducible unless $i_k = 2$ and $i_{k-1} = 1$, in which case $\mu y_1 = y_{i_1} \ldots y_{i_{k-1}} = \lambda y_1$ where $\lambda = y_{i_1} \ldots y_{i_{k-2}}$ in irreducible form and λy_1 is irreducible. Thus working down by degrees, we see that $ry_1 = 0$ implies $r = \sum \alpha_\lambda \lambda(1 - y_1y_2)$. In this format á monomial of largest degree will be of the form λy_1y_2 so that $\lambda y_1y_2y_2$ is both irreducible and uncancellable within ry_2, so $ry_2 \neq 0$. Hence $\{y_1, y_2\}$ is a left insulator in R. □

PROPERTY 3. *The element $e = y_1y_2$ is an idempotent in R and eRe is a domain.*

PROOF. It is clear that e is an idempotent. Let $r, s \in R$ and write ere and ese as linear combinations of irreducible monomials. Since e must appear in each monomial, the monomials will have degree $\geqq 2$ and the product of monomials of degree k and l from ere and ese respectively will have degree $k+l-2$ since the only reduction comes from $e^2 = e$. The resulting product when k and l are maximal cannot arise in any other way in $erese$ and so $ere \neq 0$ and $ese \neq 0$ implies $erese \neq 0$. Thus eRe is a domain. □

PROPERTY 4. *The ring R is a prime ring.*

PROOF. Let r and s be non-zero elements written as linear combinations of irreducible monomials. Let $\lambda = y_{i_1}y_{i_2}\dots y_{i_k}$ and $\mu = y_{j_1}y_{j_2}\dots y_{j_l}$ be monomials of largest degree in r and s respectively. If $i_k > j_1 + 1$ then $\lambda\mu$ is irreducible of degree $k+l$ so $rs \neq 0$. If $i_k = j_1 + 1$ then $\lambda y_{i_k}\mu$ is irreducible of degree $k+l+1$ so $ry_{i_k}s \neq 0$. Finally, if $i_k \leqq j_1$ then $\lambda y_{i_k}y_{i_k+1}\dots y_{j_1}\mu$ is irreducible of degree $k+l+(j_1 - i_k + 1)$ so $ry_{i_k}y_{i_k+1}\dots y_{j_1}s \neq 0$. Hence $rRs \neq 0$ and R is a prime ring. □

Recall that a radical ρ is called a *normal radical* if whenever (S,V,W,T) is a Morita context $V\rho(T)W \subseteqq \rho(S)$. Jaegermann ([4], Theorem 1.9) has shown that if ρ is a normal radical and e is an idempotent in a ring R, then $\rho(eRe) = e\rho(R)e$. Normal special radicals are upper radicals determined by *normal classes of prime rings,* a concept introduced in [5] where it is shown that if $\mathcal{P}$ is a normal class of prime rings and L is a left ideal of a right ideal of a prime ring S, then $L \in \mathcal{P}$ implies $S \in \mathcal{P}$.

THEOREM 1. (a) *The radicals s_l and s_r are different* [2].

(b) *The class of right (left) strongly prime rings is not a normal class of prime rings.*

(c) *The radicals s_l and s_r are not normal radicals.*

(d) *The class of uniformly strongly prime rings is not a normal class of prime rings.*

(e) *The uniformly strongly prime radical is not normal.*

PROOF. (a) In the example $s_r(R) = (y_1, y_2, \dots)$ from Property 1. It is shown in ([7], Corollary 2.2) that $s_l(R)$ does not contain any insulator in R and so from Property 2 $\{y_1, y_2\}$ is not contained in $s_l(R)$. Hence $s_l(R) \neq s_r(R)$.

(b) From Property 3 eRe is a right and left strongly prime ring. It is also a left ideal of the right ideal eR of R. From Property 1 R is not a right strongly prime ring whilst from Property 4 R is a prime ring and so the class of right strongly prime rings is not normal. Dually, the result holds for left strongly prime.

(c) Again from Property 3, $s_l(eRe) = s_r(eRe) = 0$. From Property 1 $s_r(R) = (y_1, y_2, \dots)$ so $es_r(R)e \neq 0$, in particular $es_r(R)e$ is not contained in $s_r(eRe)$, so the radical s_r is not normal. The same can be said of s_l.

(d) and (e) As in (b) and (c). □

The observation (c) may also be seen from the fact that Jaegermann [4] shows that a supernilpotent radical is normal if and only if it is left stable and right hereditary whilst in [8] it is noted that the strongly prime radical is neither right (left) hereditary nor left (right) stable. I am indebted to Patrick Stewart for pointing this out to me as well as for suggesting that uniformly strongly prime could be considered in parallel with strongly prime. Similarly, (b) follows from this observation together with the fact that in [5] it is shown that the upper radical determined by a normal class is a normal (and special) radical, so the class of strongly prime rings cannot be normal.

In contrast, it is shown in Theorem 2 below that, when working in the category of rings with identity, strongly prime and uniformly strongly prime are Morita invariant, as indeed is being semisimple with respect to the strongly prime radical. In this case it is not possible for a ring to be strongly prime radical since all rings have simple images with identity, but the same cannot be said about the us radical in the light of the example in [1] of a simple ring with 1 which is not a us-prime ring. Note that in [6] it is stated that us-primeness is a Morita invariant property.

THEOREM 2. *Let (S,V,W,T) be a Morita context with $VW=S$ and $WV=T$, where S and T are rings with 1. If I is an ideal of S such that S/I is a (right, uniformly) strongly prime ring, then T/J, where $J=$ $=WIV$, is also a (right, uniformly) strongly prime ring. Furthermore, $s_r(T)=Ws_r(S)V$ and $us(T)=Wus(S)V$.*

PROOF. Write $1_S=\sum_{i=1}^n v_iw_i$ and $1_T=\sum_{j=1}^m w'_jv'_j$. Let $t\in T$ with $t\notin J$. Put $\bar{t}=t+J\neq 0$. Then VtW is not contained in I, so there are $v_0\in\{v'_j\mid 1\leqq j\leqq m\}$ and $w_0\in\{w_i\mid 1\leqq i\leqq n\}$ with $s=v_0tw_0\notin I$. Put $\bar{s}=s+I\neq 0$. Then there is a finite subset $\overline{G}_s$ of S/I for which the right annihilator in S/I is zero. Write $\overline{G}_s=\{g+I\mid g\in G_s\subseteqq S\}$ so that $|G_s|=|\overline{G}_s|<\infty$. Then $sG_sx\subseteqq I, x\in S$ implies $x\in I$. Put $F_t=\cup_j w_0G_sv'_j\subseteqq$ $\subseteqq T$ and suppose $tF_ty\subseteqq J$ with $y\in T$. Then, for all $1\leqq j\leqq m$, $tw_0G_sv'_jy\subseteqq$ $\subseteqq J$, and so $sG_sv'_jy\subseteqq v_0J$, from which we find that $sG_sv'_jyW\subseteqq I$. But then, from the annihilator condition, $v'_jyW\subseteqq I$ for all j. Hence $w'_jv'_jyWV\subseteqq J$ for all j and so, on summation over j, $y\in J$. Thus F_t is a right insulator of t. This deals with the right strongly prime part of the result. For the uniformly strongly prime part we replace $\overline{G}_s$ by $\overline{G}$ and G_s by G, and observe that since the choice of w_0 can be restricted to the finite set $\{w_i\mid 1\leqq i\leqq n\}$ we can take, as uniform insulator, the finite set $F=\cup_{i,j} w_iGv'_j$.

Now let

$$\mathcal{S}=\{I\mid S/I \text{ is a right strongly prime ring}\}.$$

Then $s_r(S)=\cap I$, $I\in\mathcal{S}$ and so $s_r(T)\subseteqq\cap(WIV)$. However $\cap(WIV)=$ $=W(\cap I)V$; one inclusion is clear, whilst if $t\in WIV$ for all $I\in\mathcal{S}$, then $vtw\in I$ for all $v\in V$, $w\in W$, $I\in\mathcal{S}$, so that $VtW\subseteqq\cap I$ and $t\in W(\cap$ $\cap I)V$. Therefore $s_r(T)\subseteqq Ws_r(S)V$. By symmetry, $s_r(S)\subseteqq Vs_r(T)W$ and so $Ws_r(S)V\subseteqq s_r(T)$, giving the stated equality. A similar argument can be used to show that $Wus(S)V=us(T)$. □

COROLLARY 1. *In the notation of the Theorem:*

(a) *if S is a right strongly prime ring, then T is also a right strongly prime ring;*

(b) *if S is a uniformly strongly prime ring, then T is also a uniformly strongly prime ring;*

(c) *if $s_r(S) = 0$, then $s_r(T) = 0$;*

(d) *if $us(S) = 0$, then $us(T) = 0$.*

COROLLARY 2. *For any ring S,*

(a) $s_r\big(M_n(S)\big) = M_n\big(s_r(S)\big)$ [2];

(b) $us\big(M_n(S)\big) = M_n\big(us(S)\big)$ [6].

PROOF. When S has a 1 we use the context $\big(S, V, W, M_n(S)\big)$ where V and W are the modules of row and column vectors with n components from S. If S does not contain 1, then it can be embedded as an ideal in a ring S' with 1. Then, for (a), $s_r(S) = S \cap s_r(S')$, so that

$$M_n\big(s_r(S)\big) = M_n(S) \cap M_n\big(s_r(S')\big) = M_n(S) \cap \big(s_r\big(M_n(S')\big)\big) = s_r\big(M_n(S)\big)$$

since $M_n(S)$ is an ideal of $M_n(S')$ and special radicals are hereditary. Analogous statements can be made for us. $\square$

We remark that the presence of identities in both rings is crucial for Theorem 2 and Corollary 1, since in our example the context (eRe, eR, Re, ReR) has all the properties except that ReR does not have an identity element. The ring eRe is uniformly strongly prime (and so right strongly prime), but ReR is right strongly prime radical (and so uniformly strongly prime radical). To see this note $s_r(ReR) = s_r(R) \cap ReR$, since s_r being special is hereditary, and $e \in M = s_r(R)$ so $ReR \subseteqq s_r(R)$, whence $s_r(ReR) = ReR$.

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SHORT NOTES ON QUASI-UNIFORM SPACES I. UNIFORM LOCAL SYMMETRY

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A quasi-uniformity $\mathcal{U}$ on X is *locally symmetric* [18] if for any $U \in \mathcal{U}$ and $x \in X$ there is a $V \in \mathcal{U}$ with $V^{-1}[Vx] \subset Ux$ (Ux denotes $U[\{x\}]$). Equivalently, there are $V, W \in \mathcal{U}$ with $W^{-1}[Vx] \subset Ux$. This is a localized version of the following characterization of symmetry: $\mathcal{U}$ is a uniformity iff for any $U \in \mathcal{U}$ there are $V, W \in \mathcal{U}$ such that $W^{-1} \circ V \subset U$. Allowing W, but not V, depend on x, we obtain a new notion: $\mathcal{U}$ is *uniformly locally symmetric* if for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$ there is a $W \in \mathcal{U}$ with $W^{-1}[Vx] \subset Ux$.

In §1, we shall recall some other notions of quasi-uniform symmetry, and compare them with uniform local symmetry. It will be proved that a mixed-symmetric, uniformly regular quasi-uniformity is uniformly locally symmetric, while a uniformly locally symmetric quasi-uniformity is quiet. Some counterexamples will also be constructed, e.g. an open-symmetric, not point-symmetric T_2 quasi-uniformity. We shall also consider the category of the uniformly locally symmetric quasi-uniformities. It will be shown in §2 that some symmetry properties are not hereditary. In §3, we shall show that a quasi-uniform space is uniformly locally symmetric provided that it has a uniformly locally symmetric sup-dense subspace. (The analogous statement for local symmetry is known to be false [21].)

§1. Some symmetry properties

1.1. Throughout this paper, $\mathcal{U}$ is a quasi-uniformity, and its fundamental set is denoted by X; $\delta_{\mathcal{U}}$ is the quasi-proximity, and $\mathcal{T}_{\mathcal{U}}$ the topology induced by $\mathcal{U}$. Topological properties (open, dense, etc.) are to be understood with respect to $\mathcal{T}_{\mathcal{U}}$. The adjective *doubly* means "for $\mathcal{U}$ as well as for $\mathcal{U}^{-1}$"; a subset of X is *sup-dense* if it is dense in the topology $\sup\{\mathcal{T}_{\mathcal{U}}, \mathcal{T}_{\mathcal{U}^{-1}}\}$.

$\mathcal{U}$ is *proximally symmetric* (also known as "Smyth symmetric"; our terminology has been taken from [24]) if $\delta_{\mathcal{U}}$ is symmetric; it is *closed-symmetric* (introduced in [6]; the equivalence of the definitions shown in [17]; present

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terminology from [9]), *open-symmetric*[2] [17], respectively *mixed-symmetric* [9] if $A\,\delta_{\mathcal{U}}\,B$ implies $B\,\delta_{\mathcal{U}}\,A$ for closed sets A and B, for open sets A and B, respectively for A open and B closed. $\mathcal{U}$ is *point-symmetric* [18] if for any $U \in \mathcal{U}$ and $x \in X$ there is a $V \in \mathcal{U}$ with $V^{-1}x \subset Ux$. (Equivalently: $\mathcal{T}_{\mathcal{U}^{-1}}$ is finer than $\mathcal{T}_{\mathcal{U}}$.)[3] Doubly point-symmetric = doubly locally symmetric. (Assume that $\mathcal{U}$ is doubly point-symmetric, and, for $U \in \mathcal{U}$ and $x \in X$, take $W, V \in \mathcal{U}$ such that $W^{-2}x \subset Ux$, $Vx \subset W^{-1}x$.)

Most of the notions of symmetry are special cases of a more general one: Given systems $\mathfrak{a}, \mathfrak{b} \subset \exp X$ (which are in some way determined by $\mathcal{U}$, e.g. the open or the closed sets), we say that $\mathcal{U}$ is $(\mathfrak{a},\mathfrak{b})$-*symmetric* if $A\,\delta_{\mathcal{U}}\,B$ implies $B\,\delta_{\mathcal{U}}\,A$ whenever $A \in \mathfrak{a}$, $B \in \mathfrak{b}$. Denoting by $\mathfrak{o}$, $\mathfrak{c}$, and $\mathfrak{s}$ the systems of all open sets, closed sets, and singletons, respectively and putting $\mathfrak{p} = \exp X$: proximally symmetric = $(\mathfrak{p},\mathfrak{p})$-symmetric, closed-symmetric = $=(\mathfrak{c},\mathfrak{c})$-symmetric, open symmetric = $(\mathfrak{o},\mathfrak{o})$-symmetric, mixed-symmetric = $=(\mathfrak{o},\mathfrak{c})$-symmetric, point-symmetric = $(\mathfrak{p},\mathfrak{s})$-symmetric. It was observed in [9] (although not using the above terminology) that proximally symmetric = $(\mathfrak{c},\mathfrak{o})$-symmetric,[4] and also that a closed- or open-symmetric quasi-uniformity is mixed-symmetric. The reason for these simple facts will be more clear from:

LEMMA. *For an arbitrary* $\mathfrak{a} \subset \exp X$, $(\mathfrak{a},\mathfrak{p})$-*symmetric* = $(\mathfrak{a},\mathfrak{o})$-*symmetric and* $(\mathfrak{p},\mathfrak{a})$-*symmetric* = $(\mathfrak{c},\mathfrak{a})$-*symmetric.*

PROOF. 1° Assume that $\mathcal{U}$ is $(\mathfrak{a},\mathfrak{o})$-symmetric, $B\,\overline{\delta}_{\mathcal{U}}\,A$, $A \in \mathfrak{a}$. Take a $U \in \mathcal{U}$ with $U^2[B] \cap A = \emptyset$; we can assume that $Ux \in \mathfrak{o}$ $(x \in X)$; then $U[B] \in \in \mathfrak{o}$, $U[B]\overline{\delta}_{\mathcal{U}}\,A$, $A\,\overline{\delta}_{\mathcal{U}}\,U[B]$, $A\,\overline{\delta}_{\mathcal{U}}\,B$. Hence $\mathcal{U}$ is $(\mathfrak{a},\mathfrak{p})$-symmetric. The converse is evident.

2° To prove the other statement, use that $A\overline{\delta}_{\mathcal{U}}\,B$ implies $A\overline{\delta}_{\mathcal{U}}\,\overline{B}$. □

In particular, open-symmetric = $(\mathfrak{o},\mathfrak{p})$-symmetric and closed-symmetric = $(\mathfrak{p},\mathfrak{c})$-symmetric; hence both properties are stronger than mixed-symmetry. The statement proximally symmetric = $(\mathfrak{c},\mathfrak{o})$-symmetric follows applying both parts of the lemma.

1.2. Mixed-symmetric regular quasi-uniformities are locally symmetric ([9] Remark b)); we are going to prove a uniform version of this statement. Recall that $\mathcal{U}$ is *uniformly regular* ([1], [15]) if for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ with $\overline{Vx} \subset Ux$ $(x \in X)$. (This property evidently implies regularity.)

[2] The notion *open-set symmetric* introduced in [16] is a much stronger version of open-symmetry.

[3] Although not apparent from the definition, *small-set symmetry* defined in [16], [17] is equivalent to the point-symmetry of the conjugate quasi-uniformity, see [22] Lemma 4.

[4] This follows also from [23] Proposition 2, which states that a quasi-proximity is determined by the strong inclusion between open sets.

LEMMA. *$\mathcal{U}$ is mixed-symmetric iff for any $U \in \mathcal{U}$ and $F \in \mathfrak{c}$ there is a $V \in \mathcal{U}$ with $V^{-1}[F] \subset \overline{U[F]}$.*

PROOF. *Necessity.* $F \bar{\delta}_{\mathcal{U}} X \setminus U[F]$, thus $F \bar{\delta}_{\mathcal{U}} \operatorname{int}(X \setminus U[F])$, implying $\operatorname{int}(X \setminus U[F]) \bar{\delta}_{\mathcal{U}} F$ by the mixed-symmetry, and so $V\left[\operatorname{int}(X \setminus U[F])\right] \subset$ $\subset X \setminus F$ for some $V \in \mathcal{U}$, i.e. $V^{-1}[F] \subset \overline{U[F]}$.

Sufficiency. Assume $F \bar{\delta}_{\mathcal{U}} G$, $F \in \mathfrak{c}$, $G \in \mathfrak{o}$. Then $U[F] \subset X \setminus G$ with some $U \in \mathcal{U}$. Take $V \in \mathcal{U}$ such that $V^{-1}[F] \subset \overline{U[F]}$. Now $V^{-1}[F] \subset X \setminus G$, $V[G] \subset X \setminus F$, $G \bar{\delta}_{\mathcal{U}} F$. □

REMARK. A similar characterization of closed- or open-symmetry can be obtained replacing $\overline{U[F]}$ by $U[F]$ (almost the same as the original definition in [6]), respectively $\mathfrak{c}$ by $\mathfrak{p}$ (similar to [17] Proposition 4.1).

PROPOSITION. *Any mixed-symmetric uniformly regular quasi-uniformity is uniformly locally symmetric.*

PROOF. Let $U \in \mathcal{U}$, and take $Z, V \in \mathcal{U}$ such that $\overline{Z^2x} \subset Ux$, $\overline{Vx} \subset Zx$ $(x \in X)$. For $x \in X$ fixed, $Z\left[\overline{Vx}\right] \subset Z^2x$, thus the lemma applied to Z and $F = \overline{Vx}$ yields a $W \in \mathcal{U}$ with $W^{-1}\left[\overline{Vx}\right] \subset \overline{Z^2x}$, implying $W^{-1}[Vx] \subset Ux$. □

A uniformly locally symmetric quasi-uniformity is evidently uniformly regular (a stronger statement will be proved in 1.3), but not necessarily mixed-symmetric:

EXAMPLE. The restriction of Sorgenfrey quasi-uniformity $\mathcal{U}_{\mathrm{so}}$ (see $\mathcal{Z}$ in [18] 1.1) to $X = \{1/n, -1/n \colon n \in \mathbf{N}\}$ is doubly uniformly locally symmetric, but not mixed-symmetric. □

1.3. $\mathcal{U}$ is *quiet* [13] if for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $Vx \in \mathfrak{f}^1, V^{-1}y \in \mathfrak{f}^{-1}$ imply xUy whenever $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is a Cauchy filter pair. (*Cauchy* means that for each $U \in \mathcal{U}$ there are $S_i \in \mathfrak{f}^i$ with $S_{-1} \times S_1 \subset U$.) Such a V is said to be *quiet* for U. Quiet quasi-uniformities are uniformly regular [15] (in fact they are doubly uniformly regular, since if $\mathcal{U}$ is quiet then so is $\mathcal{U}^{-1}$, see [13]).

PROPOSITION. *Any uniformly locally symmetric quasi-uniformity is quiet.*

PROOF. Given $U \in \mathcal{U}$, take $U_0 \in \mathcal{U}$ with $U_0^2 \subset U$, and then $V, W_z \in \mathcal{U}$ such that $W_z^{-1}[Vz] \subset U_0 z$ $(z \in X)$. We claim that V is quiet for U.

Assume that $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is a Cauchy filter pair, $Vx \in \mathfrak{f}^1$, $V^{-1}y \in \mathfrak{f}^{-1}$. Take $S_i \in \mathfrak{f}^i$ such that $S_{-1} \times S_1 \subset Wx$. Now $Vx \cap S_1 \in \mathfrak{f}^1$, $V^{-1}y \cap S_{-1} \in \mathfrak{f}^{-1}$, so we can pick points $z_1 \in Vx \cap S_1$ and $z_{-1} \in V^{-1}y \cap S_{-1}$. Then $z_{-1} \, W_x \, z_1$, thus $x \, V \, z_1 \, W_x^{-1} \, z_{-1}$, implying $x \, U_0 \, z_{-1}$. Moreover, $z_{-1} \, V \, y$, thus $z_{-1} \, U_0 \, y$, $x \, U \, y$. □

In view of Proposition 1.2, the above result generalizes the proposition of [9], which states that each mixed-symmetric uniformly regular quasi-uniformity is quiet. Double local symmetry implies double uniform regularity (see e.g. [8] Remark 1.3 or [17] 3.6), but it does not imply quietness (see e.g. [8] Example 1.3 b)). Hence a (doubly) locally symmetric quasi-uniformity may not be uniformly locally symmetric. It can also occur that $\mathcal{U}^{-1}$ is uniformly locally symmetric, $\mathcal{U}$ is locally symmetric, but not uniformly so:

EXAMPLE. Let $X = \big(\{(0,0)\} \cup \mathbf{N}^2\big) \times \mathbf{N}$, and define $\mathcal{U}$ by the following quasi-metric:[5]

$$d\big((0,0,n),\ (1,k,n)\big) = 1/n \quad (k,n \in \mathbf{N}),$$

$$d\big((0,k,n),\ (1,k,n)\big) = 1/k \quad (k,n \in \mathbf{N}).$$

The bitopology of $\mathcal{U}$ is discrete, thus $\mathcal{U}$ is doubly locally symmetric. $\mathcal{U}$ is not uniformly locally symmetric, since there is no $W \in \mathcal{U}$ with $W^{-1}[Vx] \subset Ux$ if $U = U_{(1)}$, $V = U_{(\varepsilon)}$, $1/n < \varepsilon$, $x = (0,0,n)$. $\mathcal{U}^{-1}$ is, however, uniformly locally symmetric: for any $0 < \varepsilon \leqq 1$, $W\Big[U_{(\varepsilon)}^{-1}x\Big] \subset U_{(\varepsilon)}^{-1}x$ holds with suitable $W \in \mathcal{U}$ depending on x, since $U_{(\varepsilon)}^{-1}x$ contains at most two points, and $\mathcal{T}_{\mathcal{U}}$ is discrete. □

1.4. A mixed-symmetric regular quasi-uniformity is locally symmetric ([9] Remark b)). But there are open-symmetric, closed-symmetric regular quasi-uniformities that are not uniformly locally symmetric: The Pervin quasi-uniformity ([18] 2.2) of a non-discrete T_3 space is not uniformly regular ([1] 8.2), hence not uniformly locally symmetric, although it is (as all the Pervin quasi-uniformities are) open-symmetric and closed-symmetric ([17] §4). Without separation axioms, open-symmetry and closed-symmetry together do not even imply point-symmetry: take a non-symmetric quasi-uniformity on a two-point set; local symmetry does not even follow with T_2: take the Pervin quasi-uniformity of a non-regular T_2 space. A closed-symmetric T_1 quasi-uniformity is evidently point-symmetric; but there is an open-symmetric, not point-symmetric T_2 quasi-uniformity:

EXAMPLE. Let $(X, \mathcal{T})$ be a non-regular T_2 space such that, except for a single point p, all the points have a neighbourhood base consisting of closed sets. (E.g. modify the usual topology on $\mathbf{R}$ by deleting the points $1/n$ ($n \in \mathbf{N}$) from the basic neighbourhoods of 0.) Put

$$\mathfrak{b} = \{G\colon\ G \text{ is } \mathcal{T}\text{-open},\ p \notin \overline{G} \setminus G\}.$$

[5] When defining a quasi-metric d, it will be understood that $d(x,y) = 1$ for pairs $x \neq y$ not mentioned in the definition. Notation: $U_{(\varepsilon)} = U_{(\varepsilon)}(d) = \{(x,y) : d(x,y) < \varepsilon\}$.

$\mathfrak{b}$ is a base for $\mathcal{T}$. It is easy to check that $\mathfrak{b}$ is closed for finite unions and finite intersections. Moreover, if $G \in \mathfrak{b}$ then $X \setminus \overline{G} \in \mathfrak{b}$. Now we define $\mathcal{U}$ similarly to the Pervin quasi-uniformity, substituting $\mathfrak{b}$ for the system of all open sets (such quasi-uniformities were considered in [25] and [3]), i.e. let $\{U_G : G \in \mathfrak{b}\}$ be a subbase for $\mathcal{U}$, where

$$U_G = G \times G \cup (X \setminus G) \times X.$$

$\mathcal{U}$ is a quasi-uniformity compatible with $\mathcal{T}$ ([24] 5.2 or [3] §1). As $\mathfrak{b}$ is a lattice, $U[A] \in \mathfrak{b}$ whenever $A \subset X$ and U is a basic entourage, i.e. $U = \bigcap_{k=1}^{n} U_{G_k}$, $G_k \in \mathfrak{b}$ ([3] Lemma 2). Hence $A \,\overline{\delta}_{\mathcal{U}}\, B$ iff there is a $C \in \mathfrak{b}$ with $A \subset C$, $B \cap \cap C = \emptyset$.

$\mathcal{U}$ is open-symmetric. Take open sets G, H such that $G \,\overline{\delta}_{\mathcal{U}}\, H$, and pick $G' \in \mathfrak{b}$ with $G \subset G'$, $G' \cap H = \emptyset$. Now $H' = X \setminus \overline{G'} \in \mathfrak{b}$, $H \subset H'$, $H' \cap G = \emptyset$, thus $H \,\overline{\delta}_{\mathcal{U}}\, G$.

$\mathcal{U}$ is not point-symmetric. If $p \in G \in \mathfrak{b}$ then $U_G^{-1} p = X$; if $p \notin G \in \mathfrak{b}$ then $p \notin \overline{G}$, and $U_G^{-1} p = X \setminus G$ is a $\mathcal{T}_{\mathcal{U}}$-closed $\mathcal{T}_{\mathcal{U}}$-neighbourhood of p. Thus the $\mathcal{T}_{\mathcal{U}^{-1}}$-neighbourhood filter of p has a subbase, hence a base, consisting of $\mathcal{T}_{\mathcal{U}}$-closed $\mathcal{T}_{\mathcal{U}}$-neighbourhoods of p, i.e. $\mathcal{T}_{\mathcal{U}^{-1}}$ cannot be finer than $\mathcal{T}_{\mathcal{U}}$. □

Let us also note that an open-symmetric, closed-symmetric uniformly regular quasi-uniformity is not necessarily proximally symmetric: restrict the Sorgenfrey quasi-uniformity to $\{0\} \cup \{1/n : n \in \mathbf{N}\}$.

1.5. Generalizing a result from [6], it is proved in [20] Proposition 6(c) that (with the terminology used there) a mixed-symmetric uniformly regular quasi-uniformity is D-complete iff it is bicomplete. These two notions of completeness do not coincide in the more general class of uniformly locally symmetric spaces: $\mathcal{U}$ from Example 1.2 is uniformly locally symmetric, bicomplete (since $\sup\{\mathcal{U}^{-1}, \mathcal{U}\}$ is the discrete uniformity), but not D-complete (the filter generated by the sequence $\langle 1/n \rangle_{n \in \mathbf{N}}$ is D-Cauchy, but not convergent).

1.6. Some of the symmetry properties were defined in terms of $\delta_{\mathcal{U}}$; point-symmetry only depends on the induced bitopology. Local symmetry is also a property of $\delta_{\mathcal{U}}$: $\mathcal{U}$ is locally symmetric iff $\{x\} \,\overline{\delta}_{\mathcal{U}}\, A$ implies that $A \,\overline{\delta}_{\mathcal{U}}\, G$ for a suitable $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\delta_{\mathcal{U}}}$-neighbourhood of x (see [18] 2.23). Uniform local symmetry behaves differently: If $\mathcal{U}$ is a uniformly locally symmetric, not proximally symmetric quasi-uniformity then the totally bounded quasi-uniformity compatible with $\delta_{\mathcal{U}}$ is not uniformly locally symmetric. (Assume it is; then it is quiet, a contradiction, since any quiet totally bounded quasi-uniformity is a uniformity, see [14] Proposition 3 or [19].

1.7. If $W_i^{-1}[V_i x] \subset U_i x$ $(1 \leqq i \leqq n)$, $W = \bigcap_{i=1}^n W_i$, $V = \bigcap_{i=1}^n V_i$, $U = \bigcap_{i=1}^n U_i$ then $W^{-1}[Vx] \subset Ux$; thus it is enough to take U from a subbase in the definition of uniform local symmetry. Moreover, if $f: Y \to X$ is a function, $y \in Y$, and U, V, W are entourages on X such that $W^{-1}\left[Vf(y)\right] \subset Uf(y)$ then $W_0^{-1}[V_0 y] \subset U_0 y$ holds with $W_0 = f^{-1}W$, $V_0 = f^{-1}V$, $U_0 = f^{-1}U$; thus if $(X, \mathcal{U})$ is uniformly locally symmetric then so is $(Y, f^{-1}\mathcal{U})$. This means that the uniformly locally symmetric quasi-uniformities form a concretely reflective subcategory in the category of quasi-uniformities. In particular, uniform local symmetry is productive.

§2. Heredity

Proximal symmetry, (uniform) local symmetry and point-symmetry are evidently hereditary properties. It is also straightforward that a closed (respectively open) subspace of a closed-symmetric (respectively open-symmetric) space has the same property. Moreover, a dense subspace of an open-symmetric space is open-symmetric $\big($[21] Proposition 8(b)$\big)$. In the next example, we have a dense open subspace in a closed-symmetric space, a closed subspace in an open-symmetric space, and subspaces of both types in mixed-symmetric spaces such that the subspaces do not possess the property in question; the spaces are also uniformly regular.

EXAMPLE (cf. the examples in [9]). Let $\mathcal{U}$ be the trace of the Sorgenfrey quasi-uniformity of $\mathbf{R}^2$ on

$$X = \left\{(0,0\right\} \cup \left\{(0,1/n): n \in \mathbf{N}\right\} \cup \left\{(1/k,1/n): k,n \in \mathbf{N}, \quad k \geqq n\right\};$$

in other words, $\mathcal{U} = \mathcal{U}(d)$, where

$$d\big((x',x''),(y',y'')\big) = \max\{y'-x', y''-x''\} \quad \text{if} \quad x' \leqq y',\ x'' \leqq y''.$$

Consider the following subsets of X:

$$X_0 = X \setminus \left\{(0,0)\right\},\ X_1 = X \setminus \left\{(1/k,1/n): k \neq n\right\}, \quad X_2 = X_0 \cap X_1.$$

X_0 is dense open, X_1 is closed, X_2 is closed in X_0, and it is dense open in X_1. Put $\mathcal{U}_i = \mathcal{U}|X_i$.

$\mathcal{U}$ and $\mathcal{U}_0$ are open-symmetric. It was shown in [9] Example b) that $\mathcal{U}_0$ is open-symmetric, so if G, H are open in X, $G\,\overline{\delta}_{\mathcal{U}}\,H$ then $H \setminus \{p\}\,\overline{\delta}_{\mathcal{U}}\,G \setminus \setminus \{p\}$, where $p = (0,0)$. This means $H\,\overline{\delta}_{\mathcal{U}}\,G$ if $p \notin G \cup H$. If $p \in G$ then $p \notin H$, $H \setminus \{p\} = H$, and $H\,\overline{\delta}_{\mathcal{U}}\,\{p\}$, so $H\,\overline{\delta}_{\mathcal{U}}\,G$ again. Finally, if $p \in H$ then

$G \setminus \{p\} = G$, and $\{p\} \overline{\delta}_{\mathcal{U}} G$, since H (which is disjoint from G) contains a neighbourhood of p; hence $H \overline{\delta}_{\mathcal{U}} G$.

$\mathcal{U}$ is closed-symmetric. Assume that A and B are closed, $A \delta_U B$. Take points $x_k = (x'_k, x''_k) \in A$ and $y_k \in B$ such that $d(x_k, y_k) < 1/k$. Now either there is a subsequence of $\langle x_k \rangle_{k \in \mathbf{N}}$ for which $x''_k = 1/n$ with the same n, implying $(0, 1/n) \in A \cap B$, or there is a subsequence with $x''_k \to 0$, and then $p \in A \cap B$. This means that $A \overline{\delta}_{\mathcal{U}} B$ for any pair of disjoint closed sets. (Quasi-uniformities with this property are called equinormal [18].)

$\mathcal{U}_1$ is closed-symmetric, because X_1 is closed (or see [9] Example a)).

$\mathcal{U}_0$ is not closed-symmetric, $\mathcal{U}_1$ is not open-symmetric, see [9] Examples b) and a).

$\mathcal{U}_2$ is not mixed-symmetric. The sets $A = \{0\} \times \{1/n : n \in \mathbf{N}\}$ and $B = X_2 \setminus A$ are open-closed in X_2, $A \delta_{\mathcal{U}_2} B$, but $B \overline{\delta}_{\mathcal{U}_2} A$. □

§3. Symmetry properties preserved by extensions

3.1. Let $(X, \mathcal{U})$ be a subspace of the quasi-uniform space $(Y, \mathcal{V})$. We say that $\mathcal{V}$ is a *half-extension*, an *extension*, and a *firm extension* of $\mathcal{U}$ if X is dense ($= \mathcal{T}_{\mathcal{V}}$ dense), doubly dense, and sup-dense in Y, respectively. The following general question was investigated in [21]: which properties of quasi-uniformities are preserved by half-extensions or (firm) extensions? According to [21] Example 1(a), point-symmetry and local symmetry are not even preserved by firm extensions. We give a different example, with $\mathcal{V}$ quiet:

EXAMPLE. On $Y = \{(0,0)\} \cup \{(\pm 1/n, 0), (0, 1/n) : n \in \mathbf{N}\}$, let $\mathcal{V}$ be the trace of $\mathcal{U}_{\text{so}} \times \mathcal{U}_{\text{eu}}$, where $\mathcal{U}_{\text{so}}$ is the Sorgenfrey and $\mathcal{U}_{\text{eu}}$ the Euclidean quasi-uniformity on $\mathbf{R}$. $\mathcal{V}$ is quiet, since quietness is hereditary and productive [13]. $X = Y \setminus \{(0,0)\}$ is sup-dense in Y, $\mathcal{V}|X$ is doubly locally symmetric (as its bitopology is discrete), but $\mathcal{V}$ is not point-symmetric (the two neighbourhood filters of $(0,0)$ are incomparable). □

3.2. [21] Proposition 8(a) states that open-symmetry is preserved by firm extensions. We are going to prove a somewhat stronger result. Let us first recall some definitions and notations:

Let $(X, \mathcal{U})$ be a quasi-uniform space, $Y \supset X$, with filters $\mathfrak{f}(a)$ in X (called *trace filters*) prescribed for each $a \in Y$. We say that $\mathcal{V}$ is a half-extension inducing these trace filters if $\mathcal{V}$ is a half-extension, and $\mathfrak{f}(a)$ is the trace on X of the $\mathcal{T}_{\mathcal{V}}$-neighbourhood filter of a ($a \in Y$). For each $U \in \mathcal{U}$, define a relation ${}^{5)}U$ on Y as follows:

$$a \, {}^{5)}U \, b \quad \text{iff} \quad U[A] \in \mathfrak{f}(b) \quad \text{whenever} \quad A \in \mathfrak{f}(a).$$

Under certain assumptions, $\left\{{}^{5)}U : U \in \mathcal{U}\right\}$ will be a base for a half-extension ${}^{5)}\mathcal{U}$ inducing the prescribed trace filters. The necessary and sufficient conditions (which will not be needed here) together with historical references can be found in [11] 6.2. Unlike in [11], ${}^{5)}\mathcal{U}$ will be regarded as not defined when the conditions are not satisfied.) Firm extensions are always of the form ${}^{5)}\mathcal{U}$ ([7] 3.13).

PROPOSITION. *${}^{5)}\mathcal{U}$ preserves open-symmetry.*

PROOF. Put $\mathcal{V} = {}^{5)}\mathcal{U}$. It is enough to check that if G and H are $\mathcal{T}_{\mathcal{V}}$-open and $G\,\delta_{\mathcal{V}}H$ then $G \cap X\,\delta_{\mathcal{U}}\,H \cap X$, since then $H \cap X\,\delta_{\mathcal{U}}\,G \cap X$ by the open-symmetry of $\mathcal{U}$, and so $H\,\delta_{\mathcal{V}}\,G$.

For $U \in \mathcal{U}$ fixed, we have to find $x \in G \cap X$ and $y \in H \cap X$ with $x\,U\,y$. Pick $a \in G$, $b \in H$ such that $a\,{}^{5)}U\,b$. As ${}^{5)}\mathcal{U}$ is a half-extension inducing the prescribed trace filters, we have $A = {}^{5)}Ua \cap G \cap X \in \mathfrak{f}(a)$, and, by the definition of ${}^{5)}U$, $U[A] \in \mathfrak{f}(b)$, so $B = U\{A\} \cap H \in \mathfrak{f}(b)$. Now any $y \in B$ and some $x \in A$ will do. □

3.3. PROPOSITION. *Uniform local symmetry is preserved by firm extensions.*

PROOF. Let $\mathcal{V}$ on Y be a firm extension of $\mathcal{U}$ on X. For $U \in \mathcal{V}$ fixed, take $V \in \mathcal{V}$ with $V^3 \subset U$, and put $V_0 = V|X$. By the uniform local symmetry of $\mathcal{U}$, there are $W_0, Z_0 \in \mathcal{U}$ (the latter depending on x) such that

$$(1) \qquad Z_0^{-1}[W_0x] \subset V_0x.$$

Choose $W \in \mathcal{V}$ satisfying $W^3|X \subset W_0$ and $W \subset V$. For $a \in Y$ fixed, pick

$$(2) \qquad x \in W^{-1}a \cap Wa \cap X,$$

and choose $Z \in \mathcal{V}$ with $Z^3|X \subset Z_0$ (where Z_0 belongs to x from (2)). We claim that

$$(3) \qquad Z^{-1}[Wa] \subset Ua;$$

hence $\mathcal{V}$ is uniformly locally symmetric.

Assume $c \in Z^{-1}[Wa]$. Then there is a $b \in Y$ with $a\,W\,b\,Z^{-1}\,c$. Now $x\,W\,a$ by (2), $a\,W\,b$, and there is a $y \in X$ with $b\,W\,y$, $b\,Z\,y$. So $x\,W^3\,y$, $x\,W_0\,y\,Z^{-1}\,b\,Z^{-1}\,c$. Pick $z \in X$ such that $z\,V\,c$, $z\,Z\,c$. Then $c\,Z^{-1}\,z$, $y\,Z^{-3}\,z$, $y\,Z_0^{-1}\,z$, i.e. $x\,W_0\,y\,Z_0^{-1}\,z$, implying $x\,V_0\,z$ by (1), thus $x\,V\,z$. Moreover, $z\,V\,c$ (see the choice of z) and $a\,V\,x$ (by (2) and $W \subset V$), so $A\,V^3\,c$, $a\,U\,c$, $c \in Ua$, proving (3). □

Uniform localsymmetry is not preserved by (half-)extensions, not even by any of the constructions in [7] 3.13: Let $Y = \{0\} \cup \{\pm 1/n : n \in \mathbf{N}\}$, $\mathcal{V} = \mathcal{U}_{so}|Y$, $X = Y \setminus \{0\}$, $\mathcal{U} = \mathcal{V}|X$; then any of the constructions applied to $\mathcal{U}$ yields $\mathcal{V}$, $\mathcal{U}$ is uniformly locally symmetric (Example 1.2), but $\mathcal{V}$ is not.

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MEAN CONVERGENCE OF HERMITE INTERPOLATION REVISITED

P. VÉRTESI (Budapest) and Y. XU (Eugene)*

1. Introduction

Following the solution of mean convergence of Lagrange interpolation based on the zeros of generalized Jacobi polynomials [6], mean convergence of Hermite interpolation has been studied in several recent papers (cf. [1,7,9,12–15 and the references therein]). However, most of these papers concentrated on the sufficient conditions for the convergence, except perhaps [6, 7] where both necessary and sufficient conditions are established for Lagrange interpolation and Hermite interpolation of the second order, respectively, and [13] where conditions that are almost necessary and sufficient are provided for the Hermite interpolation of higher order. In this paper we shall investigate the mean convergence of Hermite interpolation of higher order with extended nodes. We shall establish conditions that are both necessary and sufficient. Since the results in [6, 7] are the prototypes of what we shall present in this paper, we state them in the following (cf. Theorem 1.1–1.3).

We need a few notations; their exact definitions are given in the next section. If w is a Jacobi weight function, we write $w \in J$. For a real valued function f, let $L_n(w,f)$ denote the Lagrange interpolating polynomial which interpolates f at the zeros of Jacobi polynomials $p_n(w)$, $w \in J$. Let $H_{n,2}(w,f)$ be the Hermite interpolating polynomial which interpolates both f and its first order derivative at the zeros of $p_n(w)$. Throughout this paper we let $\varphi(x) = \sqrt{1-x^2}$. Then for the weighted mean convergence of $L_n(w,f)$ and $H_{n,2}(w,f)$, we have

THEOREM 1.1 [6, Theorem 6]. *Let* $0 < p < \infty$, $u, w \in J$. *Then*

$$\text{(i)} \quad \lim_{n\to\infty} \left\| L_n(w,f) - f \right\|_{u,p} = 0, \quad \forall f \in C \quad \Longleftrightarrow \quad \text{(ii)} \quad u(w\phi)^{-p/2} \in L^1.$$

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THEOREM 1.2 [7, Theorem 1]. *Let* $0 < p < \infty$, $u, w \in J$. *Then*

$$\text{(i)} \quad \lim_{n\to\infty} \left\| H_{n,2}(w,f) - f \right\|_{u,p} = 0, \quad \forall f \in C^1 \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \text{(ii)} \quad \varphi^2(x)u(x) \leqq K w^p(x)$$

for some constant $K > 0$ *depending only on* p, w, *and* u.

THEOREM 1.3 [7, Theorems 2 and 3]. *Let* $i = 0$ *or* 1 *be fixed. Let* $0 < p < < \infty$, $u, w \in J$, *and* $u\varphi^{-ip} \in L^1$. *Then there is a constant* $K_i > 0$ *depending only on* p, u, *and* w *such that*

$$\text{(i)} \quad \left\| H_{n,2}^{(i)}(w,f) - f^{(i)} \right\|_{u,p} \leqq \frac{K_i}{n^{1-i}} E_{2n-2}(f'), \quad \forall f \in C^1 \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \text{(ii)} \quad u(w\varphi^{1+i})^{-p} \in L^1,$$

where $E_n(f)$ *is the rate of best uniform approximation of* f *by polynomials of degree at most* n.

We should mention that these theorems are stated and proved in [6,7] for weight functions that are more general than Jacobi weight. The extension of these theorems to interpolation of higher order is not straightforward, there are new phenomena and several essential difficulties have to be overcome. By now, the proof for the sufficient part is more or less standard, it uses asymptotic estimates of the fundamental polynomials which can be established using the method developed in [12,13], and uses the weighted L^p boundedness of $L_n(w,f)$. An alternative method is through the proof of Marcinkiewicz–Zygmund inequalities [15], which can be used on more general weight functions but in cases prototyped by Theorem 1.2 it yields slightly weaker results. The necessary part, on the other hand, is more difficult. One of the essential ingredients requires that the weighted L^p norm of higher order derivatives of orthogonal polynomials be bounded below by the weighted L^p norm of the weight functions. We shall establish this lower bound for the Jacobi polynomials in this paper. At this point, though, it is not clear how to extend our proof to the generalized Jacobi weight functions. It is for this reason, we restrict our consideration to Jacobi weight functions here. The Hermite interpolation that we shall consider in this paper is defined more generally than $H_{n,2}$, not only in higher degree but also in that we include two end points of $[-1,1]$ as possible points of interpolation. Thus, even when the sufficient conditions are concerned, our theorems are more general than many previous results.

The paper is organized as follows. In the following section we give preliminaries and notations. In Section 3 we state the main results and include the discussions and remarks. The lemmas are stated and proved in Section 4, and the theorems are proved in Section 5.

2. Notations and preliminaries

Throughout this paper we denote by Π_n the space of polynomials of degree at most n, and by Π the space of all polynomials.

2.1. *Weight functions.* The function w is called a Jacobi weight function if $w(x) = (1-x)^\alpha(1+x)^\beta$, $|x| \leqq 1$, and $w(x) = 0$, $|x| > 1$, where $\alpha, \beta > -1$. We let J denote the class of Jacobi weight functions. Sometimes we write $w = w^{(\alpha,\beta)}$ to emphasize the parameters α and β. We call w a generalized Jacobi weight function ($w \in GJ$), if it can be written as

(2.1)

$$w(x) = \psi(x) \prod_{i=0}^{T+1} |x - t_i|^{\Gamma_i}, \quad \Gamma_i > -1, \quad -1 = t_0 < t_1 < \ldots < t_T < t_{T+1} = 1$$

for $x \in [-1, 1]$, where ψ is a positive continuous function in $[-1, 1]$ and the modulus of continuity ω of ψ satisfies $\int_0^1 \big(\omega(t)/t\big)\,dt < +\infty$. Furthermore, if $w \in GJ$ and $\Gamma_i \geqq 0$ for $1 \leqq i \leqq T$, we write $w \in GPJ$. When $\Gamma_i = 0$ for $1 \leqq i \leqq k$ and $\psi = 1$, the generalized Jacobi weight reduces to a Jacobi one.

2.2. *Space of functions.* For $0 < p < +\infty$ and a non-negative measurable function u, the space L_u^p is defined to be the set of measurable functions f such that

$$(2.2) \qquad \|f\|_{p,u} = \left(\int_{-1}^{1} |f(t)|^p u(t)\,dt\right)^{1/p}, \qquad 0 < p < +\infty\,,$$

is finite. Of course, when $0 < p < 1$, $\|\cdot\|_p$ is not a norm; nevertheless, we keep this notation for convenience. For $u = 1$ this is the ordinary L^p space. We use the usual notation $\|f\|_\infty = \text{ess sup}_{t\in[-1,1]} \big|f(t)\big|$ for the uniform norm of f on $[-1, 1]$, and C for the space of continuous functions on $[-1, 1]$. For $d \in \mathbf{N}$, we write C^d for the space of functions that have dth continuous derivative on $[-1, 1]$.

2.3. *Orthogonal polynomials.* We consider only Jacobi polynomials. Let $w \in J$, $w = w^{(\alpha,\beta)}$. The Jacobi polynomials $p_n(w)$ are orthonormal polynomials with respect to the weight function w, i.e.,

$$\int_{-1}^{1} p_n(w,x) p_m(w,x) w(x) dx = \delta_{n,m}.$$

We should mention that our $p_n(w)$ is different from $P_n^{(\alpha,\beta)}$ in books such as [8] and [10] by a normalizing constant of order exactly $\sqrt{n}$. It is well known that $p_n(w)$ has n distinct zeros in $(-1,1)$. These zeros are denoted by $x_{kn}(w)$ and the following order is assumed:

$$(2.3) \qquad 1 > x_{1n}(w) > x_{2n}(w) > \cdots > x_{nn}(w) > -1.$$

Later, when we fix w, we shall write x_{kn} instead of $x_{kn}(w)$.

2.4. *Hermite interpolation.* All interpolations considered in this paper are based on the zeros of the Jacobi polynomial (2.3) and sometimes two end points of the interval $[-1,1]$.

For $w \in J$ and bounded function f, the Lagrange interpolating polynomial, denoted by $L_n(w,f)$, is defined by

(2.4.1)

$$L_n(w,f) = \sum_{k=1}^{n} f\big(x_{kn}(w)\big)\,\ell_{k,n}(w), \quad \ell_{kn}(w,x) = \frac{p_n(w,x)}{p_n'(w,x_{kn})(x - x_{kn})}.$$

For a given integer $r \geqq 0$, $s \geqq 0$ and $m \geqq 1$, the Hermite interpolation that we shall investigate is defined to be the unique polynomial of degree $N = mn + r + s - 1$, denoted by $H_{n,m,r,s}(w,f)$, satisfying

$$(2.4.2) \qquad \begin{cases} H_{n,m,r,s}^{(t)}(w,f,x_{kn}) = f^{(t)}(x_{kn}), & 0 \leqq t \leqq m-1, \quad 1 \leqq k \leqq n, \\ H_{n,m,r,s}^{(t)}(w,f,1) = f^{(t)}(1), & 0 \leqq t \leqq r-1, \\ H_{n,m,r,s}^{(t)}(w,f,-1) = f^{(t)}(-1), & 0 \leqq t \leqq s-1, \end{cases}$$

for $f \in C^M$, where $M = \max\{m-1, r-1, s-1\}$. If $r = 0$ or $s = 0$ then we have no interpolation at 1 or -1, respectively. We shall fix the integers m, r, and s for the rest of the paper, and omit them from the notations. Thus, for example, *we shall write $H_n(w,f)$ instead of $H_{n,m,r,s}(w,f)$ from now on.* From the definition of $H_n(w,f)$ it is easy to see that

$$(2.4.3) \qquad H_n(w,f) = \sum_{t=0}^{m-1}\sum_{k=1}^{n} f^{(t)}(x_{kn})h_{t,k}(x) +$$

$$+ \sum_{t=0}^{r-1} f^{(t)}(1)h_{t,0}(x) + \sum_{t=0}^{s-1} f^{(t)}(-1)h_{t,n+1}(x)$$

where $h_{t,k} = h_{t,k,n,m,r,s}$ (again, they depend on n, m, r, and s) are the fundamental polynomials of interpolation, they are determined uniquely by the conditions

$$(2.4.4)\qquad \begin{cases} h_{t,k}^{(l)}(x_{jn}) = \delta_{t,l}\delta_{k,j}, & 0 \leqq t,l \leqq m-1, \quad 1 \leqq k,j \leqq n, \\ h_{t,0}^{(l)}(1) = \delta_{t,l}, & 0 \leqq t,l \leqq r-1, \\ h_{t,n+1}^{(l)}(-1) = \delta_{t,l}, & 0 \leqq t,l \leqq s-1. \end{cases}$$

Using the fundamental polynomials of Lagrange interpolation, these functions can be expressed as follows:

(2.4.5)

$$\begin{cases} h_{t,k}(x) = \left(\dfrac{1-x}{1-x_{kn}}\right)^r \left(\dfrac{1+x}{1+x_{kn}}\right)^s \dfrac{(x-x_{kn})^t}{t!} \ell_{k,n}^m(x) \displaystyle\sum_{i=0}^{m-1-t} e_{i,t,k}(x-x_{kn})^i \\ \qquad\qquad 1 \leqq k \leqq n, \\ h_{t,0}(x) = \left(\dfrac{1+x}{2}\right)^s \left(\dfrac{p_n(w,x)}{p_n(w,1)}\right)^m \dfrac{(x-1)^t}{t!} \displaystyle\sum_{i=0}^{r-1-t} e_{i,t,0}(x-1)^i, \\ h_{t,n+1}(x) = \left(\dfrac{1-x}{2}\right)^r \left(\dfrac{p_n(w,x)}{p_n(w,-1)}\right)^m \dfrac{(x+1)^t}{t!} \displaystyle\sum_{i=0}^{s-1-t} e_{i,t,n+1}(x+1)^i, \end{cases}$$

where $e_{i,t,k} = e_{i,t,k,n,m,r,s}$ are constants that can be expressed using derivative values of $p_n(w)$ at x_{kn} and ± 1. Although it is difficult to find the exact formulae of these constants, they can be estimated from both above and below (cf. [2, Lemma 4.3], [12]). We have, for example,

$$(2.4.6)\qquad e_{0,t,k} = 1, \quad e_{i,t,k} \leqq c\left(\frac{n}{\varphi(x_{kn}(w))}\right)^i,$$

where c is independent of i, k and n. These estimates are useful in our discussion below.

3. Main results

3.1. Statement of the theorems. Let $m \geqq 1$, $r \geqq 0$, $s \geqq 0$ be fixed. For $w \in J$ and $f \in C^M$, we define the Hermite interpolating polynomial

$H_n(w,f) = H_{n,m,r,s}(w,f)$ as in Section 2. The following weight function is important for the characterization of the mean convergence of $H_n(w,f)$:

$$(3.1.1) \qquad w_m^{(r,s)}(x) := \left[(1-x)^{\alpha-\frac{2r}{m}+\frac{1}{2}}(1+x)^{\beta-\frac{2s}{m}+\frac{1}{2}}\right]^{\frac{m}{2}}.$$

When $r = s = 0$ we write w_m instead of $w_m^{(0,0)}$. Throughout this paper we let $N = mn + r + s - 1$ and $M = \max\{m-1, r-1, s-1\}$.

First we state two fairly general theorems, then we show some corollaries.

THEOREM 3.1. *Let $m \geqq 1$, $r \geqq 0$, $s \geqq 0$, $L \geqq M$ and t, $0 \leqq t \leqq m-1$, be fixed integers, $\tau > 0$ a fixed real number, $0 < p < \infty$, $u \in GJ$, $w \in J$ with*

$$(3.1.2) \qquad u\varphi^{-tp} \in L^1, \qquad w_m^{(r,s)}\varphi^L \in L^1.$$

Then

(i) $\left\| H_n^{(t)}(w,f) - f^{(t)} \right\|_{u,p} \leqq \text{const.} \dfrac{E_{N-L}(f^{(L)})}{n^{L-t-\tau}}, \quad \forall f \in C^L$

if with a fixed constant $K > 0$

(ii) $F_t(x) := \dfrac{u(x)\varphi^{p\tau+2}(x)}{\left(w_m^{(r,s)}(x)\varphi^t(x)\right)^p} \leqq K$ *in* $[-1,1]$.

If, moreover, $1 \leqq p < \infty$, $u \in GPJ$ and $L = m-1$, then

(i*) $\left\| H_n^{(t)}(w,f) - f^{(t)} \right\|_{u,p} \leqq \text{const.} \dfrac{\|f^{(L)}\|_\infty}{n^{L-t-\tau}}, \quad \forall f \in C^L$

implies (ii).

In Theorem 3.1 we had to suppose that $\tau > 0$. The result corresponding to the case $\tau = 0$ has a different character.

THEOREM 3.2. *Let $m \geqq 1$, $r \geqq 0$, $s \geqq 0$, $L \geqq M$ and t, $0 \leqq t \leqq m-1$, be fixed integers, $0 < p < \infty$, $u \in GJ$, $w \in J$ satisfying* (3.1.2). *Then*

$(\tilde{\text{i}})$ $\left\| H_n^{(t)}(w,f) - f^{(t)} \right\|_{u,p} \leqq \text{const.} \dfrac{E_{N-L}(f^{(L)})}{n^{L-t}}, \quad \forall f \in C^L$

if

$(\tilde{\text{ii}})$ $\dfrac{u}{\left(w_m^{(r,s)}\varphi^t\right)^p} \in L^1$.

If, moreover, $u \in GPJ$ and $L = m-1$, then

$(\tilde{\text{i}}^*)$ $\left\| H_n^{(T)}(w,f) - f^{(t)} \right\|_{u,p} \leqq \text{const.} \dfrac{\|f^{(L)}\|}{n^{L-t}}, \quad \forall f \in C^L$

implies $(\tilde{\text{ii}})$.

Now we formulate some special cases which are, hopefully, more illuminating. All consequences are "iff" statements, moreover, they deal with the case $L = m-1$ (whence, by $L \geqq M$, $m \geqq r, s$).

Theorem 3.1 immediately yields

THEOREM 3.3. *Let $m \geqq r, s$, $L = m - 1$ and $0 \leqq t \leqq m - 1$ be fixed integers, $\tau > 0$ a fixed real number, $1 \leqq p < \infty$, $u, w \in J$ satisfying* (3.1.2). *Then* (i) $\Longleftrightarrow$ (ii).

From Theorem 3.2 one can deduce

THEROEM 3.4. *Let $m \geqq r, s$, $L = m - 1$ and $0 \leqq t \leqq m - 1$ be fixed integers, $0 < p < \infty$, $u, w \in J$ satisfying* (3.1.2). *Then* ($\tilde{\text{i}}$) $\Longleftrightarrow$ ($\widetilde{\text{ii}}$).

The following consequence of Theorem 3.1 is not so obvious.

THEOREM 3.5. *Let us restrict m, r, s, L, p, u and w as in Theorem* 3.2. *Then, for every fixed $0 \leqq t \leqq m - 2$,*

(a) $\lim_{n\to\infty} \left\| H_n^{(t)}(w, f) - f^{(t)} \right\|_{u,p} = 0 \quad \forall f \in C^{m-1}$

iff

(b) $\dfrac{u\varphi^{p(m-1-t)+2}}{\left(w_m^{(r,s)}\varphi^t\right)^p} \leqq K$ *in* $[-1, 1]$.

First let us prove (b) $\Longrightarrow$ (a). Let in Theorem 3.1 $\tau = \tau_t = m - 1 - t (\geqq 1$, by $t \leqq m - 2)$. Then (b) $\equiv$ (ii), which implies (i), where, using $L - t - \tau_t = 0$, the denominator is one (the exponent of n is zero). Then (i) implies (a).

Now we verify (a) $\Longrightarrow$ (b). Indeed, (a) means that (i*) holds with $\tau = m - 1 - t$ and $L = m - 1$. But then we have (ii), too, which is exactly (b). □

Using similar argument, Theorem 3.2 yields for $t = m - 1$

THEOREM 3.6. *Let us restrict m, r, s, L, p, u and w as in Theorem* 3.4. *Then*

($\tilde{\text{a}}$) $\lim_{n\to\infty} \left\| H_n^{(m-1)}(w, f) - f^{(m-1)} \right\|_{u,p} = 0 \quad \forall f \in C^{m-1}$

iff

($\tilde{\text{b}}$) $\dfrac{u}{\left(w_m^{(r,s)}\varphi^{m-1}\right)^p} \in L^1$.

3.2. Remarks. We prove Theorems 3.1 and 3.2 mainly because of their generality, but since our main interest in this paper is the necessary part, we shall not try to give a complete list of all previous results in this field that may have been covered by these general theorems. Instead, in the remarks that follow we shall discuss the possible extensions and the things that are left open.

3.2.1. The careful reader may have already noticed that in Theorem 3.1 at the part (i*) $\Rightarrow$ (i) we have $1 \leqq p < \infty$ (while in Theorem 3.2 we have $0 < p < \infty$). The problem occurs when we use Hölder inequality (see the proof of (5.2.7)). For $t = 0$ and $t = 1$ we do have Theorem 3.1 for all $p > 0$. Moreover, the Theorem 3.1 is stated as (i*) $\Rightarrow$ (ii) for every fixed t. If we change this a little, then we can state a theorem for $0 < p < 1$ as well. For example, if we replace (i) in Theorem 3.1 by

$$\text{(i}'\text{)} \quad \left\| H_{n,M}^{(j)}(w,f) - f^{(j)} \right\|_{u,p} \leqq \text{const.} \frac{E_{N-m-1}(f^{(m-1)})}{n^{m-t-\tau-1}},$$

$$\forall f \in C^{m-1} \quad 0 \leqq j \leqq t,$$

then we have (i′) $\Leftrightarrow$ (ii) for $0 < p < \infty$.

3.2.2. When $r = s = 0$ and $m = 2$ Theorem 3.5 becomes Theorem 1.2 and Theorem 3.4 becomes Theorem 1.3. When $r = s = 0$ and $m = 1$, Theorem 3.6 becomes Theorem 1.1. However, our Theorem 3.1 has a new character. This is due to the presence of τ in the theorem. It should be noted that this number τ does not need to be an integer. This offers us a variety of choices. In particular, Theorem 3.5 corresponds to the choice $\tau = m - t - 1$. The closer τ gets to 0, the more restrictive condition (ii) becomes. When $\tau = 0$, the condition (ii) in Theorem 3.1 is no longer enough, we need (ii) in Theorem 3.2.

3.2.3. Comparing Theorems 3.1 and 3.2 we see that $(\widetilde{\text{ii}})$ is also the limiting case of (ii). Actually, another way to describe the condition $(\widetilde{\text{ii}})$ is to say that the weight function $\varphi^{-tp}\left(w_m^{(r,s)}\right)^{-p} u$ is bounded by $1/(1-x^2)^{\rho}$ for some $\rho < 1$, which is clearly stronger than the $\tau = 0$ case of (ii).

3.2.4. The second condition of (3.1.2), $w_m^{(r,s)}\varphi^L \in L^1$, means

$$\text{(3.2.2)} \qquad \alpha > -\frac{1}{2} - \frac{2+L-2r}{m}, \quad \text{and} \quad \beta > -\frac{1}{2} - \frac{2+L-2s}{m},$$

which put restrictions on w. For example, if $r = s = 2$ and $L = m - 1 = 1$, then we have to have $\alpha > 0$. However, if $r = s = 0$, i.e., there is no interpolation at the end points of $[-1, 1]$, then these conditions become

$$\text{(3.2.2}'\text{)} \qquad \alpha, \beta > -\frac{1}{2} - \frac{2+L}{m}.$$

Since $\alpha, \beta > -1$ and $L \geqq m - 1$, the above conditions are automatically satisfied for all m. The restriction (3.2.2) or (3.2.2′) is weaker than those known previously (cf. [1]).

3.2.5. The condition (ii) in Theorem 3.1 does not appear for Lagrange interpolation (cf. Theorem 1.1), it is observed first in [7] for $m = 2$ (cf. Theorem 1.2). It is interesting to notice that for $m \geqq 2$ all derivatives of the Hermite interpolation obey this type of conditions, except the highest one which behaves like the Lagrange interpolation.

3.2.6. Our theorems are proved for the interpolation based on the zeros of $p_n(w)$, $w \in J$. A natural question is whether similar theorems can be proved for $w \in GJ$. The main problem is in Lemma 4.3 below, while the proof of this lemma for $w \in GJ$ seems to be merely a technical problem, we can prove it only for $w \in J$ at present. Another question that may be of some interests is to consider the necessary conditions for the interpolation process which has the interpolation conditions at ± 1 distributed on points near ± 1. We hope to return to these questions in the near future.

4. Lemmas

Throughout this paper, we will use the symbols "const." (sometimes c) to denote a generic positive constant, its value may be different at different occurrences, even in subsequent formulae. The value of this constant depends on weight functions and other fixed parameters involved. Constants that depend on other parameters will be denoted differently and defined locally. The notation $A \sim B$ means $|A^{-1}B| \leqq$ const. and $|AB^{-1}| \leqq$ const. uniformly in the variables under consideration.

We prove the essential lemmas in this section. The most important one is perhaps Lemma 4.3, others have had their predecessors for less completed interpolation processes. We begin with properties of orthogonal polynomials.

4.1. For the various estimates in this subsection we refer to [10] and [8]. For $x \in [-1,1]$ we write as usual $x = \cos\theta$, $0 \leqq \theta \leqq \pi$. For $w \in J$ we write $x_{kn}(w) = \cos\theta_{kn}(w)$, $0 \leqq k \leqq n+1$, where $x_{0,n} = 1$ and $x_{n+1,n} = -1$. Then it is known that

$$\theta_{k+1,n}(w) - \theta_{k,n}(w) \sim \frac{1}{n} \tag{4.1.1}$$

uniformly for $n \in \mathbf{N}_0$ and $0 \leqq k \leqq n$. For $w \in J$ we need estimates of $p_n(w)$. For $x \in [-1,1]$, we let $x_{jn}(w)$ be one of the closest zeros of $p_n(w)$ to x, that is,

$$\left|x - x_{jn}(w)\right| = \left|x - x_{j(n),n}(w)\right| = \min_{1 \leqq k \leqq n} \left|x - x_{kn}(w)\right|. \tag{4.1.2}$$

Then the following properties of $p_n(w)$ are known:

$$p_n(w,-x) = (-1)^n p_n(w,x), \tag{4.1.3}$$

and

$$\left|p_n(w,x)\right| \sim \left(n|\theta - \theta_{j,n}|\right) \frac{n^{\alpha+1/2}}{j^{\alpha+1/2}}, \quad -1+\delta \leqq x \leqq 1, \tag{4.1.4}$$

where $0 < \delta < 2$. A similar estimate holds true for $-1 \leqq x \leqq 1 - \delta$ (cf. (4.1.3)). Also,

$$p_n'(w,x) = h_n p_n(w^{(\alpha+1,\beta+1)}, x), \quad h_n \sim n \tag{4.1.5}$$

and

$$\left| p_n'\big(w, x_{kn}(w)\big) \right| \sim \frac{n^{\alpha+5/2}}{j^{\alpha+3/2}}, \tag{4.1.6}$$

uniformly for $-1 + \delta \leqq x_{kn}(w) < 1$.

If $u \in GJ$, for a fixed $d \geqq 0$, we define $\Delta_n(d)$ by

$$\Delta_n(d) = [-1 + dn^{-2}, 1 - dn^{-2}] \Big\backslash \bigcup_{i=1}^{T} [t_i - dn^{-1}, t_i + dn^{-1}].$$

Let χ_E denote the characteristic function of a set E.

LEMMA 4.1 [4, Lemma 2.2, p. 105]. *Let $u \in GJ$. Then for each $0 < p < + \infty$ and $\ell > 0$ there exists a $\gamma_0 = \gamma_0(p) > 0$ such that for every $R \in \Pi_{\ell n}$ and $0 \leqq \gamma \leqq \gamma_0$*

$$\|R\|_{u,p} \leqq \text{const.}_{\gamma_0} \left\| R\chi_{\Delta_n(\gamma)} \right\|_{u,p}.$$

4.2. Let m, r, s be fixed integers, and $w \in J$. Let $H_n(w, f)$ be the Hermite interpolation defined by (2.4.2). For later use, we separate the last terms in the formulae (2.4.5). We define

(4.2.1)

$$\begin{cases} q_{t,k}(x) = \left(\dfrac{1-x}{1-x_{kn}}\right)^r \left(\dfrac{1+x}{1+x_{kn}}\right)^s \dfrac{(x-x_{kn})^t}{t!} \cdot \\ \qquad \cdot \ell_{kn}^m(x) e_{m-1-t,t,k}(x - x_{kn})^{m-1-t}, \quad 1 \leqq k \leqq n, \\ q_{t,0}(x) = \left(\dfrac{1+x}{2}\right)^s \left(\dfrac{p_n(w,x)}{p_n(w,1)}\right)^m \dfrac{(x-1)^t}{t!} e_{r-1-t,t,0}(x-1)^{r-1-t}, \\ q_{t,n+1}(x) = \left(\dfrac{1-x}{2}\right)^r \left(\dfrac{p_n(w,x)}{p_n(w,-1)}\right)^m \dfrac{(x+1)^t}{t!} e_{s-1-t,t,n+1}(x+1)^{s-1-t}, \end{cases}$$

and

$$h_{t,k}^* = h_{t,k} - q_{t,k}, \quad 0 \leqq k \leqq n+1. \tag{4.2.2}$$

Our next lemma estimates the summation of the absolute value of the fundamental polynomials. We define

$$\mathcal{I}_{t,L}(x) := \sum_{k=0}^{n+1} \left(\frac{\varphi(x_{kn})}{n}\right)^{L-t} \left|h_{t,k}(x)\right| \tag{4.2.3}$$

and

$$\mathcal{I}^*_{t,L}(x) := \sum_{k=0}^{n+1} \left(\frac{\varphi(x_{kn})}{n}\right)^{L-t} \left|h^*_{t,k}(x)\right|, \tag{4.2.4}$$

where if (t,k) belongs to neither $\{0 \leqq t \leqq m-1,\ 1 \leqq k \leqq m\}$, nor $\{0 \leqq t \leqq \leqq r-1,\ k=0\}$, nor $\{0 \leqq t \leqq s-1,\ k=n+1\}$, then we take $h_{t,k}$ and $h^*_{t,k}$ as zero functions.

LEMMA 4.2. *Let $L \geqq M$, and let $w \in J$ be such that $w_m^{(r,s)}\varphi^L \in L^1$. Then for $-1 \leqq x \leqq 1$ and $0 \leqq t \leqq m-1$, with $2y_{kn} = x_{kn} + x_{k+1,n}$, $k = 1,2,\dots,n-{}-1$,*

$$\mathcal{I}_{t,L}(x) \leqq \text{const.}\,\mathcal{I}_{t,L}(y_{jn}) \leqq \tag{4.2.5}$$

$$\leqq \begin{cases} \text{const.}\, n^{-L}\left[\left(\varphi(x_{jn})\right)^L \log n + \left(w_m^{(r,s)}(x_{jn})\right)^{-1}\right] & \text{if } m-t = \text{odd} \\ \text{const.}\, n^{-L}\left[\left(\varphi(x_{jn})\right)^L + \frac{\log n}{n}\left(w_m^{(r,s)}(x_{jn})\right)^{-1}\right] & \text{if } m-t = \text{even}, \end{cases}$$

and

$$\mathcal{I}^*_{t,L}(x) \leqq \text{const.}\,\mathcal{I}^*_{t,L}(y_{jn}) \leqq \frac{\text{const.}}{n^L}\left[\varphi(x_{jn})^L + \frac{\log n}{n}\left(w_m^{(r,s)}(x_{jn})^{-1}\right)\right], \tag{4.2.6}$$

$$0 \leqq k \leqq M.$$

PROOF. The first inequalities come from (2.4.5) and relations (4.1.1)–(4.1.6). To verify the other ones, we remark that apart from the terms $k=0$ and $n+1$, they involve tedious but mainly routine calculations detailed in [13] and [14]. Therefore here we show how to estimate the zero- and $(n+1)$th terms. First let us consider the zero term in $\mathcal{I}_{t,L}$:

$$A_t(x) := \left(\frac{\varphi(x_{0,n})}{n}\right)^{L-t} \left|h_{t,0}(x)\right|.$$

If $r < m$, by $L-t \geqq L-(r-1) > L-m+1 \geqq 0$, relation $\varphi^{L-t}(x_{0,n}) = 0$ yields that $A_t(x) = 0$. Let now $r \geqq m$. Again if $L-t > 0$, then $A_t(x) = 0$.

So we only have to consider the case $L = t$ which by $L \geqq r-1 \geqq m-1 \geqq t$ gives that $L = r-1 = m-1 = t$. Then, by (2.4.5) and $e_{0,t,k} = 1$, if $x \geqq 0$, say,

$$A_t(x) = \left|h_{m-1,0}(x)\right| \leqq \text{const.} \left|h_{m-1,0}(y_{jn})\right| \sim \left|\frac{p_n(w,y_{jn})}{p_n(w,1)}\right|^m (1-y_{jn})^{m-1}.$$

Further, using (2.4.5) and Part 4.1, for the first term in $\mathcal{I}_{m-1,m-1}(y_{jn})$, by $\left|p_n'(w,x_{1n})\right| \sim \left|p_n(w,1)\right| n^2$, we can write $\left|h_{m-1,1}(y_{jn})\right| \sim \left|h_{m-1,0}(y_{jn})\right|$.

The same equivalence holds true for $x \leqq 0$, too. So the zero-term of the corresponding sum can be estimated by the first one. Finally if $r > m$, by $L \geqq r-1 > m-1 \geqq t$, $L-1>0$ whence $A_t(x) = 0$ again.

The $(n+1)$th term can be paired with the nth one. We omit the details. The proof of (4.2.6) is similar. □

4.3. To prove the part (i) ⇒ (ii) in our theorems, we have to know the size of the derivatives of $p_n(w)$. The difficult part is to estimate the weighted L^p norm of such derivatives from below, which we overcome in Lemma 4.3. For $a \geqq 0$, let $\mathcal{E}_n(a) = [-1+an^{-2}, 1-an^{-2}]$.

LEMMA 4.3. *Let $0 < p < \infty$, $0 \leqq t \leqq m$. Let $w \in J$ and $u \in GJ$ such that $\varphi^{-tp}u \in L^1$. Then for any fixed non-negative numbers d_1 and d_2,*

$$\text{(4.3.1)} \qquad \left\|\left[p_n^m(w)\right]^{(t)}\chi_{\Delta_n(d_1)}\right\|_{u,p} \sim \left\|\left[p_n^m(w)\right]^{(t)}\chi_{\mathcal{E}_n(d_2)}\right\|_{u,p} \sim$$

$$\sim \left\|\frac{n^t}{\varphi^t}p_n^m(w)\chi_{\Delta_n(d_1)}\right\|_{u,p} \sim \left\|\frac{n^t}{\varphi^t w_m}\chi_{\mathcal{E}_n(d_2)}\right\|_{u,p},$$

where the equivalences "$\sim$" depend on d_1, d_2, and p, but they are independent of n.

PROOF. By Leibniz's rule

$$\text{(4.3.2)} \qquad \left(p_n^m(w)\right)^{(t)} = \sum \frac{t!}{i_1!\dots i_m!} p_n^{(i_1)}(w)\dots p_n^{(i_m)}(w)$$

where the summation is over all nonnegative integers $i_1, i_2, \dots, i_m$ such that $i_1 + i_2 + \dots + i_m = t$. For $w^{(\alpha,\beta)} \in J$ we denote by $w_{(t)} = w_{(t)}^{(\alpha,\beta)}$ the weight function

$$w_{(t)}(x) = w(x)(1-x^2)^t = (1-x)^{\alpha+t}(1+x)^{\beta+t}.$$

From (4.1.5)

$$\text{(4.3.3)} \qquad p_n^{(t)}(w) = C_n p_{n-t}(w_{(t)}), \quad C_n \sim n^t.$$

From the estimate of $p_n(w)$ at (4.1.4) and the above formula, we derive the estimate

$$\left|\left(p_n^m(w,x)\right)^{(t)}\right| \leqq c\left(\frac{n^2}{j}\right)^t\left(\frac{n}{j}\right)^{m(\alpha+1/2)}, \quad -1+\delta \leqq x \leqq 1,$$

for a fixed δ, $0<\delta<2$, where j is defined as in (4.1.2). From Lemma 4.1, (4.1.1), (4.1.3) and (4.1.4) we have that

$$(4.3.4) \qquad \left\|\left[p_n^m(w)\right]^{(t)}\right\|_{u,p} \leqq \text{const.}\left\|\frac{n^t}{\varphi^t}p_n^m(w)\right\|_{u,p} \sim$$

$$\sim \left\|\frac{n^t}{\varphi^t}p_n^m(w)\chi_{\Delta_n(d_1)}\right\|_{u,p} \sim \left\|\frac{n^t}{\varphi^t}p_n^m(w)\chi_{\mathcal{E}_n(d_1)}\right\|_{u,p}.$$

The proof of the opposite inequality is more complicated. From (4.3.2) we have

$$\left[p_n^m(w)\right]^{(t)}(x) = \frac{m!}{(m-t)!}\left(p_n'(w,x)\right)^t\left(p_n(w,x)\right)^{m-t}+$$

$$+\left(p_n(w,x)\right)^{m-t+1}\sum{}' A(t,m)p_n^{(i_1)}(w,x)\dots p_n^{(i_{t-1})}(w,x)$$

where the summation $\sum'$ is over all nonnegative integers $i_1, i_2, \dots, i_{t-1}$ such that $i_1+i_2+\dots i_{t-1}=t$; here $A(t,m)$ are integers depending on m and t only. Using the estimate of $p_n(w)$ at (4.1.4) and (4.3.3) we conclude that

$$\left|\left[p_n^m(w)\right]^{(t)}(x)\right| \geqq \frac{m!}{(m-t)!}\left|\left(p_n'(w,x)\right)^t\left(p_n(w,x)\right)^{m-t}\right| -$$

$$-\text{const.}\left[\left(n\left|\theta-\theta_{k,n}(w)\right|\right)\left(\frac{n}{j}\right)^{\alpha+1/2}\right]^{m-t+1}\left[\left(\frac{n}{j}\right)^{(\alpha+1/2)(t-1)}\left(\frac{n^2}{j}\right)^t\right] =$$

$$= \frac{m!}{(m-t)!}\left|\left(p_n'(w,x)\right)^t\left(p_n(w,x)\right)^{m-t}\right| -$$

$$-\text{const.}\left(n\left|\theta-\theta_{k,n}(w)\right|\right)^{m-t+1}\left(\frac{n}{j}\right)^{(\alpha+1/2)m}\left(\frac{n^2}{j}\right)^t.$$

Using (4.1.4) and (4.3.3) again we get then

(4.3.5)

$$\left|\left[p_n^m(w)\right]^{(t)}(x)\right| \geqq \left[c_0(x)\left(n\left|\theta-\theta_{k,n}(w)\right|\right)\right]^{m-t}\left(\frac{n}{j}\right)^{(\alpha+1/2)m}\left(\frac{n^2}{j}\right)^t \times$$
$$\times\left[\frac{m!}{(m-t)!}\left(c_1(x)n\left|\theta-\theta_{j,n-1}(w_{(1)})\right|\right)^t-\text{const.}\,n\left|\theta-\theta_{j,n}(w)\right|\right],$$

where $c_0(x)$ and $c_1(x)$ are functions with $c_0(x)\sim 1\sim c_1(x)$ uniformly. We need the estimate on the zeros of $p_n^{(t)}(w)$. By [8, (9.7) and (9.12)] we have with $0<\eta<1$

$$\theta_{k,n-t}(w_{(t)})=\frac{2k+\alpha+t-1/2}{2n+\alpha+\beta+1}\pi+\rho_{k,n}(w_{(t)}),\quad 1\leqq k\leqq(1-\eta)n,$$

where $\left|\rho_{k,n}(w_{(t)})\right|=O\big(1/(kn)\big)$ uniformly in n and k. From this it readily follows that

$$\text{(4.3.6)}\qquad \frac{\theta_{k,n}(w)+\theta_{k+1,n}(w)}{2}=\theta_{k,n-1}(w_{(1)})+\varepsilon_{k,n},$$

where $|\varepsilon_{k,n}|=O\big(1/(kn)\big)$ and

$$\text{(4.3.7)}\qquad \frac{\theta_{k,n-1}(w_{(1)})+\theta_{k+1,n-1}(w_{(1)})}{2}=\theta_{k+1,n}(w)+\varepsilon'_{k,n}$$

where $|\varepsilon'_{k,n}|=O\big(1/(kn)\big)$. For a given $\varepsilon>0$, we define a set of θ as follows:

$$\eta_{j,n}(\varepsilon):=\left\{\theta:\frac{\varepsilon}{2n}\leqq\left|\theta-\theta_{j,n}(w)\right|\leqq\frac{\varepsilon}{n}\right\}.$$

By (4.3.6) and (4.3.7), $\left|\theta-\theta_{k,n-1}(w_{(1)})\right|>1/n$, say, if $\theta\in\eta_{j,n}(\varepsilon)$, $k\geqq k_0$ and $n\geqq n_0$. It follows from (4.3.5) that for $\theta\in\eta_{j,n}(\varepsilon)$ we have

(4.3.8)

$$\left|\left[p_n^m(w)\right]^{(t)}(x)\right|\geqq\text{const.}\left(\frac{n}{j}\right)^{(\alpha+1/2)m}\left(\frac{n^2}{j}\right)^t\left(\frac{\varepsilon}{2}\right)^{m-t}\left[\frac{m!}{(m-t)!}-\text{const.}\,\varepsilon\right].$$

Therefore, if ε is small enough, say, $\varepsilon\leqq\varepsilon_0$, then we have for $x\in[-1+\delta,1]$,

$$\text{(4.3.9)}\quad \left|\left[p_n^m(w)\right]^{(t)}(x)\right|\geqq\text{const.}\,\varepsilon\left(\frac{n}{j}\right)^{(\alpha+1/2)m}\left(\frac{n^2}{j}\right)^t,\quad \theta\in\bigcup_{j=k_0}^{c(\delta)n}\eta_{j,n}(\varepsilon),$$

where $c(\delta) < 1$. One may prove the relation (4.3.9) for $\theta \in \eta_{j,n}(\varepsilon)$ with $1 \leqq k \leqq 2k_0$ as follows. By [10, (8.1.3)], we have that $\lim_{n\to\infty}(n-t)\theta_{k,n-t}(w_{(t)}) = j_k^{(\alpha+t)}$ where $j_k^{(\alpha)}$ is the k-th positive root of the Bessel function of parameter α. Using [10, (1.71.5)], we have $j_{k+1}^{(\alpha)} < j_k^{(\alpha+1)} < j_k^{(\alpha)}$. Therefore, we can show that relations analogous to (4.3.6) and (4.3.7) are valid, whence (4.3.9) holds true for $1 \leqq k \leqq 2k_0$ with possibly another $\varepsilon > 0$. The interval $[-1, 1-\delta]$ can be treated similarly (cf. (4.1.3)). Let

$$\mathcal{S}_n(\varepsilon) = \bigcup_{j=k_0}^{c(\delta)n} \eta_{j,n}(\varepsilon), \quad \mathcal{R}_n(\varepsilon) = \{x : x = \cos\theta, \theta \in \mathcal{S}_n(\varepsilon)\}.$$

Since $\varphi(x) \sim j/n$ and $\left|p_n(w,x)\right| \sim \big(w(x)\varphi(x)\big)^{-1/2}$ for $x \in \mathcal{R}_n(\varepsilon)$, from the proof of (4.3.4) and (4.3.9) we have

$$\text{(4.3.10)} \quad \left|\left[p_n^m(w)\right]^{(t)}(x)\right| \sim \frac{n^t}{\varphi^t(x)w_m(x)} \sim \frac{n^t}{\varphi^t(x)}\left|p_n^m(w,x)\right|, \quad x \in \mathcal{R}_n(\varepsilon),$$

where the equivalences depend on ε. By Lemma 4.1, (4.1.4), (4.1.3) and (4.1.10) we have for any given $d > 0$

$$\left\|\left[p_n^m(w)\right]^{(t)}\right\|_{u,p} \sim \left\|\left[p_n^m(w)\right]^{(t)}\chi_{\Delta_n(\gamma_0)}\right\|_{u,p} \leqq \text{const.}\left\|\frac{n^t}{\varphi^t w_m}\chi_{\Delta_n(\gamma_0)}\right\|_{u,p} \sim$$

$$\sim \left\|\frac{n^t}{\varphi^t w_m}\chi_{\Delta_n(d)}\right\|_{u,p} \sim \left\|\frac{n^t}{\varphi^t(x)w_m(x)}\chi_{\Delta_n(d)\cap R_n(\varepsilon)}\right\|_{u,p} \sim$$

$$\sim \left\|\left[p_n^m(w)\right]^{(t)}\chi_{\Delta_n(d)\cap R_n(\varepsilon)}\right\|_{u,p} \leqq \text{const.}\left\|\left[p_n^m(w)\right]^{(t)}\chi_{\Delta_n(d)}\right\|_{u,p} \leqq$$

$$\leqq \text{const.}\left\|\left[p_n^m(w)\right]^{(t)}\chi_{\mathcal{E}_n(d)}\right\|_{u,p} \sim \left\|\left[p_n^m(w)\right]^{(t)}\right\|_{u,p}.$$

The same argument works for the equivalence involving $\left\|n^t\varphi^{-t}p_n^m(w)\right\|_{u,p}$, which proves the lemma. □

From the proof of this lemma, it is not hard to see that the following corollary also holds.

COROLLARY 4.3.1. *Let the condition be the same as in Lemma 4.3. Then for $s \leqq t$ and $d \geqq 0$,*

$$\left\|\left[p_n^m(w)\right]^{(t)}\chi_{\Delta_n(d)}\right\|_{u,p} \sim \left\|\left[p_n^\prime(w)\right]^{(t)}\left[p_n(w)\right]^{m-t}\chi_{\Delta_n(d)}\right\|_{u,p} \sim$$

$$\sim \left\| \frac{n^{t-s}}{\varphi^{t-s}} \left[p_n^m(w) \right]^{(s)} \chi_{\Delta_n(d)} \right\|_{u,p}.$$

4.4. Lemma 4.3 will also help us to estimate the derivatives of the function Ω_n^m, where

$$(4.4) \qquad \Omega_n(x) = (1-x)^{\frac{r}{m}}(1+x)^{\frac{s}{m}} p_n(w,x) = w^{(\frac{r}{m},\frac{s}{m})}(x) p_n(w,x).$$

This function will be used in the following section to establish the necessary conditions of mean convergence.

LEMMA 4.4. *Let $0 < p < \infty$ and $u \in GJ$ be such that $\left(w^{(r,s)}\varphi^{-t}\right)^p u \in L^1$. Then for A sufficiently large,*

$$\left\| (\Omega_n^m)^{(t)} \chi_{\mathcal{E}_n(A)} \right\|_{u,p} \geqq \text{const.} \left\| w^{(r,s)} \left(p_n^m(w) \right)^{(t)} \chi_{\mathcal{E}_n(A)} \right\|_{u,p} \sim$$

$$\sim \left\| \frac{n^t}{\varphi^t} w^{(r,s)} p_n^m(w) \chi_{\mathcal{E}_n(A)} \right\|_{u,p}.$$

PROOF. By Leibniz's rule, we have

$$(\Omega^m)^{(t)} = \sum_{j=0}^{t} \binom{t}{j} \left(w^{(r,s)} \right)^{(j)} \left(p_n^m(w) \right)^{(t-j)}.$$

For $x \in [0,1]$, we have $w^{(r,s)}(x) \leqq \text{const.} (1-x)^r \sim \varphi^{2r}(x)$. Therefore, for $x \in [0,1]$ we have

$$\left| (\Omega_n^m)^{(t)}(x) \right| \geqq$$

$$\geqq \text{const.} \left[\left| \varphi^{2r} (p_n^m)^{(t)}(x) \right| - \sum_{j=1}^{t} \binom{t}{j} \left| (1-x)^{r-j} \left(p_n^m(w) \right)^{(t-j)} \right| \right].$$

Using the fact that on $[0, 1 - An^{-2}]$ we have $n(1-x)^{1/2} \geqq \sqrt{A}$, we get upon using Corollary 4.3.1,

$$\left\| (\Omega_n^m)^{(t)} \chi_{\mathcal{E}_n(A) \cap [0,1]} \right\|_{u,p} \geqq \text{const.} \left[\left\| \varphi^{2r} (p_n^m)^{(t)} \chi_{\mathcal{E}_n(A) \cap [0,1]} \right\|_{u,p} - \right.$$

$$\left. - \sum_{j=1}^{t} \frac{1}{\left(\sqrt{A}\right)^j} \left\| \varphi^{2r} \left(p_n^m(w) \right)^{(t)} \chi_{\mathcal{E}_n(A) \cap [0,1]} \right\|_{u,p} \right].$$

Similar inequality can be established for $[-1+A/n^2,0]$. Therefore, if A is sufficiently large, then we obtain

$$\left\| (\Omega_n^m)^{(t)} \chi_{\mathcal{E}_n(A)} \right\|_{u,p} \geqq \text{const.} \left\| w^{(r,s)} (p_n^m)^{(t)} \chi_{\mathcal{E}_n(A)} \right\|_{u,p}.$$

By Lemma 4.3 this implies immediately the desired result. □

4.5. Another ingredient in proving the part (i) ⇒ (ii) is the following spline function. Let $x_{k,n} = x_{k,n}(w)$. For $0 \leqq k \leqq n$ we define

$$s_{k,n}(x) = \frac{1}{(m-1)!} \left(\frac{x_{k+1,n} - x_{k,n}}{\pi} \right)^{m-1} \left(\sin \frac{x - x_{k,n}}{x_{k+1,n} - x_{k,n}} \pi \right)^{m-1} \cdot$$

$$\cdot \cos \frac{x - x_{k,n}}{x_{k+1,n} - x_{k,n}} \pi, \qquad x \in [x_{k+1,n}, x_{k,n}].$$

It is easy to verify that $s_{k,n}$ satisfies the following properties:

$$s_{k,n}^{(t)}(x_{k,n}) = s_{k,n}^{(t)}(x_{k+1,n}) = 0, \quad 0 \leqq t \leqq m-2,$$

$$s_{k,n}^{(m-1)}(x_{k,n}) = 1, \quad s_{k,n}^{(m-1)}(x_{k+1,n}) = (-1)^m.$$

Moreover,

$$\left| s_{k,n}^{(t)}(x) \right| \leqq \frac{\text{const.}}{n^{m-1-t}}, \quad 0 \leqq t \leqq m-1, \quad x \in [x_{k+1,n}, x_{k,n}].$$

Next, we define the function S_n on $[-1,1]$:

$$S_n(x) = \sum_{k=0}^{n} (-1)^{km} s_{kn}(x).$$

From the properties of $s_{k,n}$ we can easily derive properties of S_n; some of them are collected in the following lemma.

LEMMA 4.5. *The function S_n belongs to C^{m-1}, and it satisfies*

$$S_n^{(t)}(x_{k,n}) = 0, \quad 0 \leqq k \leqq n+1, \quad 0 \leqq t \leqq m-2,$$

$$S_n^{(m-1)}(x_{k,n}) = (-1)^{km}, \quad 0 \leqq k \leqq n+1,$$

and

$$\left\| S_n^{(t)} \right\|_\infty \leqq \frac{\text{const.}}{n^{m-1-t}}, \quad 0 \leqq t \leqq m-1.$$

4.6. We need two more lemmas. The first one is a special case of [6, Theorem 1].

LEMMA 4.6. *Let $0 < p < \infty$, and $W \in GJ$. Let $U \in GJ$ and V be a Jacobi weight, not necessarily integrable, such that $U \in L^p$, $UV \in L^p$, $U/\sqrt{W\varphi} \in L^p$, and $V\sqrt{W\varphi} \in L^1$. Then for every bounded function f*

$$\left\| L_n(W, Vf)U \right\|_p \leqq \text{const.} \|f\|_\infty, \quad n \geqq 1.$$

The second one is a much simplified version of a theorem proved in [7, Theorem 4],

LEMMA 4.7. *Let $0 < p < \infty$, $r \geqq 0$. Let $U \in GJ$ and $W \in J$. Then there is a constant $C > 0$ such that*

$$\int_{-1}^{1} \frac{\chi_\Delta(x)U(x)dx}{W(x)^{\frac{r+p}{2}}(1-x^2)^{\frac{r+3p}{4}}} \leqq$$

$$\leqq C \liminf_{n\to\infty} \frac{1}{n^p} \int_{-1}^{1} \left|p_n(W,x)\right|^r \left|p_n'(W,x)\right|^p \chi_\Delta(x)U(x)dx$$

for every interval $\Delta \subseteqq [-1,1]$.

5. Proof of the theorems

5.1. PROOF OF THEOREM 3.1. (ii) $\Rightarrow$ (i). Let $R \in \Pi_N$ satisfy

$$(5.1.1)\quad \left|f^{(i)}(x) - R^{(i)}(x)\right| \leqq \text{const.} \left(\frac{\varphi(x)}{n}\right)^{L-i} E_{N-L}(f^{(L)}), \quad 0 \leqq i \leqq L,$$

(cf. [1] or [3]). Using triangle inequality we have

$$(5.1.2) \left\| H_n^{(t)}(f) - f^{(t)} \right\|_{u,p} \leqq \text{const.} \left[\left\| H_n^{(t)}(f-R) \right\|_{u,p} + \left\| R^{(t)} - f^{(t)} \right\|_{u,p} \right] \leqq$$

$$\leqq \text{const.}\, E_{N-L}\big(f^{(L)}\big)\, n^{-(L-t)} + \text{const.} \left\| H_n^{(t)}(f-R) \right\|_{u,p}.$$

Using Bernstein inequality (cf. [4, (1.10)]) repeatedly, which is permitted by the first condition of (3.1.2), we obtain

$$(5.1.3)\qquad \left\| H_n^{(t)}(f-R) \right\|_{u,p} \leqq \text{const.}\, n^t \left\| \varphi^{-t} H_n(f-R) \right\|_{u,p}.$$

By (2.4.3), (5.1.1), Lemma 4.1, and Lemma 4.2 we have

$$\left\| H_n^{(t)}(f-R) \right\|_{u,p} \leqq \text{const.}\, n^t \left\| \varphi^{-t} H_n(f-R)\chi_{\Delta_n(\sigma)} \right\|_{u,p} \leqq$$

$$\leqq \text{const.}\, n^{-(L-t)} E_{N-L}(f^{(L)}) \left\| \varphi^{-t} \sum_{i=0}^{m-1} \mathcal{I}_{i,m-1}\chi_{\Delta_n(\sigma)} \right\|_{u,p} \leqq$$

$$\leqq \text{const.}\, n^{-(L-t)} E_{N-L}(f^{(L)}) \left[\log n \|\varphi^{L-t}\|_{u,p} + \left\| \varphi^{-t} \left(w_m^{(r,s)} \right)^{-1} \chi_{\Delta_n(\sigma)} \right\|_{u,p} \right].$$

To evaluate the second term in the last expression we use (ii), which yields

$$\left(\int_{\Delta_n(\sigma)} \frac{u}{\left(\varphi^t w_m^{(r,s)}\right)^p} dx \right)^{1/p} \leqq K \left(\int_{\Delta_n(\sigma)} \varphi^{-p\tau-2} dx \right)^{1/p} \leqq \text{const.}\, n^{\tau},$$

since $\tau > 0$. Therefore, we obtain that

$$\left\| H_n^{(t)}(f-R) \right\|_{u,p} \leqq \text{const.}\, E_{N-L}(f^{(L)})/n^{L-t-\tau}. \quad \square$$

5.2. Proof of Theorem 3.1. (i*) ⇒ (ii). First let $r, s < m$, i.e. $r + s \leqq 2m - 2$.

Let $f_0(x) = x$, Ω_n be defined as in 4.4, and S_n be defined as in 4.5. Then it is not hard to see that

$$(5.2.1)\; \mathcal{T}_n(x) := x H_n(w, S_n; x) - H_n(w, f_0 S_n; x) = \sum_{k=1}^{n} (x - x_{k,n}) h_{m-1,k}(x) =$$

$$= \frac{\Omega_n^m(x)}{(m-1)!} \sum_{k=1}^{n} \frac{(-1)^{km}}{\left[\Omega_n'(x_{kn})\right]^m},$$

where we have used the fact that $e_{t0} = 1$. Since it is easy to verify that

$$(5.2.2) \qquad \Omega'(x_{k,n}) = w^{\left(\frac{r}{m}, \frac{s}{m}\right)}(x_{k,n}) p_n'(w, x_{k,n}) =$$

$$= (-1)^{k+1} \left| w^{\left(\frac{r}{m}, \frac{s}{m}\right)}(x_{k,n}) p_n'(w, x_{k,n}) \right|, \quad 1 \leqq k \leqq n,$$

we conclude that

$$(5.2.3) \qquad \mathcal{T}_n(x) = \frac{(-1)^m \Omega_n^m(x)}{(m-1)!} \sum_{k=1}^{n} \frac{1}{\left| \Omega_n'(x_{k,n}) \right|^m}.$$

The second condition in (3.1.2) implies that $(\alpha + 3/2)m - 2r > -1$ and a similar inequality with β in place of α. Using this condition, (4.1.6) and (5.2.1), it readily follows that

$$(5.2.4) \qquad \sum_{k=1}^{n} \frac{1}{\left| \Omega_n'(x_{kn}) \right|^m} \sim n^m \sum_{k=1}^{n} \left(\frac{k}{n} \right)^{(\min\{\alpha,\beta\}+3/2)m-2r} \sim n^{1-m}.$$

Therefore, by Lemma 4.4

$$(5.2.5)\qquad \left\| \mathcal{T}_n^{(t)} \right\|_{u,p} \geqq \text{const.}\, n^{1-m} \left\| (\Omega_n^m)^{(t)} \right\|_{u,p} \geqq$$

$$\geqq \text{const.}\, n^{1-m+t} \left\| w^{(r,s)} \varphi^{-t} p_n^m(w) \chi_{\mathcal{E}_n(A)} \right\|_{u,p}.$$

On the other hand, from Lemma 4.5 it follows that $E_n(g) \leqq$ const. for $g = S_n^{(m-1)}$ and $g = f_0 S_n^{(m-1)}$. Therefore, with $L = m-1$ by Lemma 4.1, Lemma 4.5, and (i*) in Theorem 3.1 we have

$$(5.2.6)\qquad \left\| \mathcal{T}_n^{(t)} \right\|_{u,p} \leqq \text{const.} \left\| \mathcal{T}_n^{(t)} \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} \leqq$$

$$\leqq \text{const.} \left\| \left[f_0 \big(H_n(w, S_n) - S_n \big) + \big(f_0 S_n - H_n(w, f_0 S_n) \big) \right]^{(t)} \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} \leqq$$

$$\leqq \text{const.} \left[\left\| \big(H_n^{(t-1)}(w, S_n) - S_n^{(t-1)} \big) \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} + \left\| H_n^{(t)}(w, S_n) - S_n^{(t)} \right\|_{u,p} + \right.$$

$$\left. + \left\| H_n^{(t)}(w, f_0 S_n) - (f_0 S_n)^{(t)} \right\|_{u,p} \right] \leqq$$

$$\leqq \text{const.} \left\| \big(H_n^{(t-1)}(w, S_n) - S_n^{(t-1)} \big) \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} + \text{const.}\, n^{1-m+t+\tau}.$$

Let $G_n = H_n^{(t-1)}(w, S_n) - S_n^{(t-1)}$. For a fixed $x \in \Delta_n(\gamma_0)$, we define t_ρ by $|x - t_\rho| = \min_i |x - t_i|$. Let us assume, say, $x_{j,n}(w) \leqq x \leqq x_{j-1,n}(w) < t_\rho$. By Hölder inequality

$$G_n(x) = \int_{x_{j,n}}^x G_n'(y)dy \leqq$$

$$\leqq \left(\int_{x_{j,n}}^x |G_n'(y)|^p u(y) dy \right)^{1/p} \left(\int_{x_{j,n}}^x u^{-q/p}(y) dy \right)^{1/q} \leqq$$

$$\leqq \text{const.}\, \|G_n'\|_{u,p} u^{-1/p}(x) / n^{1/q},$$

where in the second inequality we have used the fact that for $y \in [x_{j,n}, x]$, $u(y) \sim u(x)$ uniformly in n and x. Using Hölder inequality again, we have

$$\left\| G_n \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} \leqq \text{const.} \frac{\|G_n'\|_{u,p}}{n^{1/q}} \left(\int_{-1}^1 [u(x)]^{1-1/p} dx \right)^{1/p} \leqq$$

$$\leqq \text{const.} \frac{\|G_n'\|_{u,p}}{n^{1/q}} \left(\int_{-1}^1 u(x) dx \right)^{q/p} 2^{1/p} \leqq \text{const.}\, n^{-1/q} \|G_n'\|_{u,p},$$

whence, using the definition of G_n,

$$\left\| G_n \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} \leqq \text{const.}\, n^{-1/q} \left\| H_n^{(t)}(w, S_n) - S_n^{(t)} \right\|_{u,p}.$$

Thus, by (i*) and (5.2.6) we immediately obtain

$$\left\| \mathcal{T}_n^{(t)} \right\|_{u,p} \leqq \text{const.}\, n^{1-m+t+\tau}. \tag{5.2.7}$$

If we define $e_n(A) = [-1 + An^{-2}, -1 + 2An^{-2}]$ or $[1 - 2An^{-2}, 1 - An^{-2}]$, then formulae (5.2.5), (5.2.7), (3.1.1), and (4.1.1) yield that

$$\text{const.} \geqq \text{const.}\, n^{-\tau} \left\| w^{(r,s)} \varphi^{-t} p_n^m(w) \chi_{\mathcal{E}_n(A)} \right\|_{u,p} \geqq$$

$$\geqq \text{const.}\, n^{-\tau} \left\| w^{(r,s)} \varphi^{-t} p_n^m(w) \chi_{e_n(A)} \right\|_{u,p} \sim \varphi^{\tilde{s}}(x_{\varrho,n}) \frac{\left(u(x_{\varrho,n}) \varphi^2(x_{\varrho,n}) \right)^{1/p}}{w_m^{(r,s)}(x_{\varrho,n}) \varphi^t(x_{\varrho,n})},$$

where $\varrho = 1$ or n. Therefore, we obtain

$$u(x_{\varrho,n}) \varphi^{(\tau-t)p}(x_{\varrho,n}) \left[w_m^{(r,s)}(x_{\varrho,n}) \right]^{-p} \leqq \frac{K}{1 - x_{\varrho,n}^2} \tag{5.2.8}$$

for some constant K.

For simplicity, let $T = 1$ and $t_1 = 0$. With proper C and D, (5.2.8) can be written as

$$\sum_{k=1,n} (1 - x_{kn})^C (1 + x_{kn})^D |x_{kn}|^{\Gamma_1} \leqq K,$$

whence $C \geqq 0$ and $D \geqq 0$. Clearly, if $1/2 \leqq x \leqq 1$ we have $F_t(x) \sim (1-x)^C$ (cf. (ii)) whence by $D \geqq 0$, $F_t(x) \leqq K$ $(1/2 \leqq x \leqq 1)$. For $-1 \leqq x \leqq -1/2$, the argument is similar. Finally, let $|x| < 1/2$. Then $F_t(x) \sim |x|^{\Gamma_1}$ whence by $\Gamma_1 \geqq 0$, $F_t(x) \leqq K$, again.

If $r = m$, say, then by $\left| h_{m-1,0}(y_{jn}) \right| \sim \left| h_{m-1,1}(y_{jn}) \right|$ we can argue analogously. We omit the further details. □

5.3. Proof of Theorem 3.6. $(\tilde{\text{ii}}) \Rightarrow (\tilde{\text{i}})$. Let R be defined as in (5.1.1). Again we have (5.1.2) and (5.1.3). Using (2.4.3) and (4.2.2) we have

$$\left\| H_n^{(t)}(f - R) \right\|_{u,p} \leqq$$

$$\leqq \text{const.} \left[n^t \left\| \varphi^{-t} \sum_{m-i=\text{even}} \sum_{k=0}^{n+1} \left| (f-R)^{(i)}(x_{kn}) h_{ik} \right| \chi_{\Delta_n(\sigma)} \right\|_{u,p} \right] +$$

$$+n^t \left\| \varphi^{-t} \sum_{m-i=\text{odd}} \sum_{k=0}^{n+1} |(f-R)^{(i)}(x_{kn}) h_{ik}^*| \chi_{\Delta_n(\sigma)} \right\|_{u,p} +$$

$$+n^t \left\| \varphi^{-t} \sum_{m-i=\text{odd}} \left| \sum_{k=0}^{n+1} (f-R)^{(i)}(x_{kn}) q_{ik} \right| \chi_{\Delta_n(\sigma)} \right\|_{u,p} := \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3.$$

From Lemma 4.2, it is clear that $\mathcal{S}_1$ and $\mathcal{S}_2$ could be estimated similarly as done in 5.1, where by using $(\widetilde{\text{ii}})$, we have

$$\mathcal{S}_1 + \mathcal{S}_2 \leqq$$

$$\leqq \text{const.} \frac{E_n(f^{(L)})}{n^{L-t}} \left[\|\varphi^{m-t-1}\|_{u,p} + \frac{\log n}{n} \left\| \varphi^{-t} \left[w_m^{(r,s)} \right]^{-1} \chi_{\Delta_n(\sigma)} \right\|_{u,p} \right] \leqq$$

$$\leqq \text{const.} \frac{E_n(f^{(L)})}{n^{L-t}}.$$

To estimate $\mathcal{S}_3$ we need to use Lemma 4.6. First, we write

$$\mathcal{S}_3(x) = n^t \left\| \varphi^{-t} \sum_{m-i=\text{odd}} |B_i| \chi_{\Delta_n(\sigma)} \right\|_{u,p} \tag{5.3}$$

where

$$B_i = B_i(x) := \sum_{k=0}^{n+1} (f-R)^{(i)}(x_{kn}) q_{ik}(x).$$

For simplicity, let $\max\{r,s\} \leqq m$. Then by (5.1.1) $(f^{(i)} - R^{(i)})(\pm 1) = 0$. By the definition of q_{tk}, (5.1.1), (4.2.1) and (2.4.6), there exist functions $C_i(x)$, such that

$$B_i(x) = \frac{p_n^{m-1}(w,x) w^{(r,s)}(x)}{i! n^L} E_{N-L}(f^{(L)}) \cdot$$

$$\cdot \sum_{k=1}^{n} C_i(x_{kn}) \frac{\varphi^{(L)}(x_{kn}) w_{m-1}(x_{kn})}{w^{(r,s)}(x_{kn})} \ell_{kn}(w,x),$$

and $|C_t(x)| \leqq$ const. uniformly. Using (4.1.4) we then obtain

$$|B_i(x)| \leqq \text{const.} \frac{E_{N-L}(f^{(L)})}{n^L} \frac{w^{(r,s)}(x)}{w_{m-1}(x)}.$$

$$\cdot \sum_{k=1}^{n} C_i(x_{kn}) \frac{\varphi^L(x_{kn}) w_{m-1}(x_{kn})}{w^{(r,s)}(x_{kn})} \ell_{kn}(w,x).$$

We now apply Lemma 4.6 on the right hand side of the above inequality with $W = w$, $U = w^{(r,s)} u^{1/p} \varphi^{-t} w_{m-1}^{-1}$, $V = \varphi^L w_{m-1} \left[w_m^{(r,s)} \right]^{-1}$, and $f = C_i$ to conclude that

$$\| \varphi^{-t} B_i \|_{u,p} \leqq \text{const.} \frac{E_{N-L}(f^{(L)})}{n^L}.$$

We check the condition of Lemma 4.6: $UV = \varphi^{L-t} u^{1/p} \in L^p$ is clear, $\left(U / \sqrt{W\varphi} \right)^p = u \left(w_m^{(r,s)} \varphi^t \right)^{-p} \in L^1$ is the assumption $(\tilde{\text{ii}})$, $V\sqrt{w\varphi} = w_m^{(r,s)} \varphi^L \in L^1$ is the second condition of (3.1.2), and $U \in L^p$ follows from the first condition of (3.1.2) if $\alpha, \beta \leqq -1/2$ and from $(\tilde{\text{ii}})$ if $\min\{\alpha, \beta\} \geqq -1/2$. Therefore, by (5.3) we have

$$S_3 \leqq \text{const.} \frac{E_{N-L}(f^{(L)})}{n^L},$$

which concludes the proof of this theorem. □

5.4. PROOF OF THEOREM 3.2. $(\tilde{\text{i}}^*) \Rightarrow (\tilde{\text{ii}})$. We notice that this theorem corresponds to the case $\tau = 0$ in Theorem 3.1. Similarly to the proof in 5.2 we have (again, let $r, s < m$)

$$\| \mathcal{T}_n^{(t)} \|_{u,p} \leqq \text{const.} \Big(\| H_n^{(t-1)}(w, S_n) - S_n^{(t-1)} \|_{u,p} +$$

$$+ \| H_n^{(t)}(w, S_n) - S_n^{(t)} \|_{u,p} + \| H_n^{(t)}(w, f_0 S_n) - (f_0 S_n)^{(t)} \|_{u,p} \Big).$$

By $(\tilde{\text{i}}^*)$ and Lemma 4.5, we have

$$\| H_n^{(t)}(w, S_n) - S_n^{(t)} \|_{u,p} \leqq \text{const.}\, n^{1+t-m}$$

and

$$\| H_n^{(t)}(w, f_0 S_n) - (f_0 S_n)^{(t)} \|_{u,p} \leqq \text{const.}\, n^{1+t-m}.$$

Moreover, let Δ be a fixed interval inside $(-1, 1)$. By Lemma 4.2 with $L = m - 1$ we have

$$| H_n(w, S_n; x) | = \left| \sum_{k=0}^{n+1} h_{m-1,k}(x) \right| \leqq$$

$$\leqq \text{const.} \frac{1}{n^{m-1}} \left[\left(\varphi(x_{jn}) \right)^{m-1} + \frac{\log n}{n} \left(w_m^{(r,s)}(x_{jn}) \right)^{-1} \right].$$

Therefore, by a theorem in [11, 4.8.72] we derive that

$$\left| H_n^{(t-1)}(w,S_n,x)\right| \leqq$$

$$\leqq \text{const.}\,\frac{1}{n^{m-1}}\frac{n^{t-1}}{\varphi^{t-1}(x_{jn})}\left[\left(\varphi(x_{jn})\right)^{m-1}+\frac{\log n}{n}\left(w_m^{(r,s)}(x_{jn})\right)^{-1}\right].$$

Hence, for $x \in \Delta$, we obtain the inequality

$$\left| H_n^{(t-1)}(w,S_n;x)\right| \leqq \text{const.}\,\frac{1}{n^{m-t}}, \quad x\in\Delta.$$

Thus, by Lemma 4.5, we get

$$\left\| \left[H_n^{(t-1)}(w,S_n)-S_n^{(t-1)}\right]\chi_\Delta\right\|_{u,p} \leqq$$

$$\leqq \left\| H_n^{(t-1)}(w,S_n)\chi_\Delta\right\|_{u,p}+\left\| S_n^{(t-1)}\chi_\Delta\right\|_{u,p} \leqq$$

$$\leqq \text{const.}\left[\frac{1}{n^{m-t}}+\frac{1}{n^{m-t}}\right].$$

Putting these estimates together, we have proved that (cf. (5.2.6))

$$\limsup_{n\to\infty} n^{m-t-1}\left\| \mathcal{T}_n^{(t)}\chi_\Delta\right\|_{u,p} \leqq \text{const.}$$

By (5.2.1) and (5.2.4) we then conclude that

$$\limsup_{n\to\infty} n^{-t}\left\| (\Omega_n^m)^{(t)}\chi_\Delta\right\|_{u,p} \leqq \text{const.},$$

which implies, by the argument of the proofs in Lemma 4.4 and Corollary 4.3.1,

$$\limsup_{n\to\infty} n^{-t}\left\| w^{(r,s)}p_n^{m-t}(w)\left[p_n'(w)\right]^{t}\chi_\Delta\right\|_{u,p} \leqq \text{const.}$$

If $t>0$, we apply Lemma 4.7 with $tp>0$ in place of p, $s=(m-t)p$, and $U=w^{(r,s)}u$ to conclude

$$\left\| \left(w_m^{(r,s)}\right)^{-1}\varphi^{-1}\chi_\Delta\right\|_{u,p} \leqq \text{const.}$$

with a constant independent of Δ. Since this is true for every fixed $\Delta \in \in (-1,1)$, we have proved $(\widetilde{\text{ii}})$ for $t>0$. However, the case $t=0$ is even easier, since in this case there is no derivative and $(\widetilde{\text{i}^*})$ implies

$$\left\| f_0H_n(w,S_n)-H_n(f,S_n)\right\|_{u,p} \leqq \text{const.}\,n^{-m+1},$$

whence, by (5.2.1) and (5.2.4)

$$\limsup_{n\to\infty} \left\| w^{(r,s)} p_n^m \right\|_{u,p} \leqq \text{const.}$$

By (4.1.4) and (4.1.3), we then have

$$\text{const.} \geqq \left\| w^{(r,s)} p_n^m(w) \right\|_{u,p} \geqq \left\| w^{(r,s)} p_n^m(w) \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} \sim$$

$$\sim \left\| w^{(r,s)} w_1^{-m} \chi_{\Delta_n(\gamma_0)} \right\|_{u,p} \sim \left\| 1/w^{(r,s)} \right\|_{u,p}.$$

This concludes the proof. □

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ON NOETHERIAN MODULES GRADED BY G-SETS

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Introduction

Let R be a G-graded ring and M be a graded R-module. One of the most important problems in module theory is to see whether a graded module having a certain property has a similar property when regarded without grading. The idea of graded R-modules has been extended by Năstăsescu in [6,7], to graded R-modules over G-sets. That is for the case where $R = \bigoplus_{g \in G} R_g$ is a G-graded ring, A is a left G-set and $M = \bigoplus_{a \in A} M_a$ where M_a is an additive subgroup of M and $R_g M_a \subseteq M_{ga}$ for all $g \in G$ and $a \in A$. Since any group G is itself a G-set (G acts on itself by left translation), then any result on A-graded R-modules where A is an arbitrary G-set can be applied directly for the elements of R-gr (the category of all graded R-modules).

Now, we have the following classical problem: If M has a certain property in (G, A, R)-gr (the category of all left R-modules graded by a G-set A), then does M have the same property in R-mod (the category of all R-modules). In this paper, we discuss this question for the Noetherian property. In other words, is the condition M is A-graded R-Noetherian sufficient to say that M is R-Noetherian.

In general, the answer is no because it is not true for graded R-modules. But if we add some extra conditions on A and G then the answer will be yes.

In Section 1, we recall a series of notations which are used in group action and give some results on modules graded by G-sets. For these notations and facts one can look in [2].

In Section 2, we give a positive answer for this question to the following cases:

1. G is a finite group and A is any G-set.
2. G is an abelian group and A is a finite G-set.
3. $G = \mathbf{Z} \times F$ where F is a finite abelian group and A is a denumerable G-set.

The question remains open for the case where G is a finitely generated abelian group and A is a denumerable G-set.

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1. Modules graded by G-sets

Let G be a group with identity e. A G-set A is a nonempty set, together with a left action $G \times A \to A$ of G on A given by $(g,a) \to ga$, such that $ea = a$ for all $a \in A$ and $(gh)a = g(ha)$ for all $g,h \in G$, $a \in A$. For $t \in A$, we denote $G_t = \{g \in G | \, gt = t\}$ and $Gt = \{gt | \, g \in G\}$. Obviously, G_t is a subgroup of G and $|Gt| = [G:G_t]$ (see [2], page 152).

LEMMA 1.1. *Let G be an abelian group and A be a G-set. If $t \in A$ then $G_s = G_t$ for each $s \in Gt$.*

PROOF. Let $s = gt$, where $g \in G$, and let $h \in G_t$. Then $hs = g(ht) = gt = s$ and hence $G_t \subseteqq G_s$. On the other hand, if $r \in G_s$ then $rt = rg^{-1}s = g^{-1}s = t$, i.e., $G_s \subseteqq G_t$.

Throughout this paper, A will be a G-set and R will be an associative G-graded ring with unity 1, i.e., $R = \bigoplus_{g \in G} R_g$ where each R_g is an additive subgroup of R and $R_g R_h \subseteqq R_{gh}$ for all $g,h \in G$.

A (left) R-module M is said to be an A-graded R-module if $M = \bigoplus_{a \in A} M_a$ where each M_a is an additive subgroup of M and $R_g M_a \subseteqq M_{ga}$ for all $g \in G$, $a \in A$. The elements of R_g (resp. M_a) are called homogeneous of dimension g (resp. a). For $x \in M$, we write x_a for the component of x in M_a. An R-submodule N of M is said to be an A-graded R-submodule if $N = \bigoplus_{a \in A} N_a$ where $N_a = N \cap M_a$.

Clearly, if R is a strongly G-graded ring ($R_g R_h = R_{gh}$ for all $g,h \in G$) then $R_g M_a = M_{ga}$ for all $g \in G$, $a \in A$.

PROPOSITION 1.2. *Let M be an A-graded R-module and $x \in A$. Then $R^{(G_x)} = \sum_{g \in G_x} R_g$ is a graded subring of R and M_x is an $R^{(G_x)}$-submodule of M.*

PROOF. Obvious, because $gx = x$ for all $g \in G_x$, i.e., $R^{(G_x)} M_x \subseteqq M_x$.

PROPOSITION 1.3. *For each $t \in A$, $M^{(Gt)} = \sum_{s \in Gt} M_s$ is an A-graded R-submodule of M.*

PROOF. The relation $\sim$ defined on G by $g \sim h$ iff $gt = ht$ is an equivalence relation. The equivalence class determined by the element g is gG_t. Let $\{g_i\}_{i \in \Delta_t}$ be a set of representatives for the equivalence classes in G. Then one can easily show $M^{(Gt)} = \bigoplus_{i \in \Delta_t} M_{g_i t}$ and $RM^{(Gt)} \subseteqq M^{(Gt)}$.

Now for each $x, y \in A$ we have $Gx = Gy$ or $Gx \cap Gy = \emptyset$. So, the relation $\mathcal{E}$ defined on A by $x \mathcal{E} y$ iff $Gx = Gy$ is an equivalence relation. The equivalence class determined by the element $x \in A$ is Gx. Let $\{x_i\}_{i \in \Delta}$ be a set of representatives for the equivalence classes in A. Then $M = \bigoplus_{i \in \Delta} M^{(Gx_i)}$.

2. Noetherian A-graded R-modules

DEFINITION 2.1. An A-graded R-module M is said to be A-graded R-Noetherian if M satisfies the ascending chain condition on A-graded R-submodules of M.

In the rest of this section, we assume *M is A-graded R-Noetherian.*

PROPOSITION 2.2. *For each $t \in A$, $M^{(Gt)}$ is A-graded R-Noetherian and M_t is $R^{(G_t)}$-Noetherian.*

PROOF. The first part follows directly from Proposition 1.3. For the second part, let $X_1 \subseteqq X_2 \subseteqq \dots$ be a chain of $R^{(G_t)}$-submodules of M_t. Then $RX_1 \subseteqq RX_2 \subseteqq \dots$ is a chain of A-graded R-submodules of M. So, there exists $n \in \mathbf{N}$ such that $RX_n = RX_{n+1} = \dots$, and hence $(RX_n)_t = (RX_{n+1})_t = = \dots$. But $(RX_n)_t = R^{(G_t)}X_n = X_n$ implies $X_n = X_{n+1} = \dots$.

PROPOSITION 2.3. *For Δ given at the end of Section 1, there exists a finite subset I of Δ such that $M = \bigoplus_{i \in I} M^{(Gx_i)}$.*

PROOF. Suppose not. Then there exists a denumerable subset $\mathbf{N}$ of Δ such that $M^{(Gx_j)} \neq 0$ for all $j \in \mathbf{N}$. But then $M^{(Gx_1)} \subseteqq M^{(Gx_1)} + M^{(Gx_2)} \subseteqq \subseteqq \dots$ is a chain of A-graded R-submodules of M without maximum.

COROLLARY 2.4. *M is R-Noetherian iff $M^{(Gx)}$ is R-Noetherian for all $x \in A$.*

In the remainder of this section, we add conditions on G and A so that M is R-Noetherian.

THEOREM 2.5 (Theorem 4 of [3]). *Let G be a finite group with identity e, and M be a graded R-module. Then M is R-Noetherian iff M is R_e-Noetherian.*

PROPOSITION 2.6. *Let G be a finite group. Then M is R_e-Noetherian. Consequently, M is R-Noetherian.*

PROOF. Let $x \in A$. Since G is a finite group, Δ_x is finite (see Proposition 1.3, for notations). Suppose $\Delta_x = \{1, 2, \dots, n\}$, then $M^{(Gx)} = \bigoplus_{i=1}^{n} M_{g_i x}$. By Proposition 2.2, $M_{g_i x}$ is $R^{(G_{g_i x})}$-Noetherian for $i = 1, \dots, n$.

Since $R^{(G_{g_i x})}$ is a graded ring of type $G_{g_i x}$ and $G_{g_i x}$ is a finite group, then by Theorem 2.5, $M_{g_i x}$ is $\left(R^{(G_{g_i x})}\right)_e$-Noetherian for each $i \in \Delta_x$. But $\left(R^{(G_{g_i x})}\right)_e = R_e$ for all $i \in \Delta_x$, i.e., $M_{g_i x}$ is R_e-Noetherian and hence $M^{(Gx)}$ is R_e-Noetherian. Therefore, M is R_e-Noetherian by Proposition 2.3.

One can notice that if A is a finite G-set and $G_x = G_y$ for all $x, y \in A$ then M is R-Noetherian. In the following proposition, we show that if A is a finite G-set and G is an abelian group then M is R-Noetherian. We leave the case where A is a finite G-set and G is an arbitrary group as an open question.

PROPOSITION 2.7. *Let G be an abelian group and A be a finite G-set. Then M is R-Noetherian.*

PROOF. Assume $A=\{1,2,\dots,n\}$ and $t\in A$. By Corollary 2.4 it is enough to show $M^{(Gt)}$ is R-Noetherian.

Step 1. Consider an arbitrary R-submodule X of $M^{(Gt)}$. For each $i=1,\dots,n$, let $X^{(i)}$ be the R-submodule of $M^{(Gt)}$ generated by the elements x_i where $x\in X$ and $x_j=0$ for all $j>i$.

We show that if $X\subseteqq Y$ are two R-submodules of $M^{(Gt)}$ such that $X^{(i)}=Y^{(i)}$ for all $i=1,\dots,n$, then $X=Y$. Let $y\in Y-0$, then $y=y_1+\ldots+y_m$ where $y_i\in M_i$ and $y_m\neq 0$. By induction on m, we show that $y\in X$. If $m=1$, then $y=y_1\in Y^{(1)}=X^{(1)}\subseteqq X$. Assume $m>1$, then $y_m\in Y^{(m)}-0$ and hence $m\in Gt$. By assumption $y_m\in X^{(m)}$, write $y_m=r_1x_m^{(1)}+\ldots+r_px_m^{(p)}$ where $x^{(i)}\in X$ and $x_j^{(i)}=0$ for all $j>m$. Let $r_i=(r_i)_{c_1}+\ldots+(r_i)_{c_s}$ where $(r_i)_{c_j}\in R_{c_j}$, $(r_i)_{c_j}x_m^{(i)}\neq 0$ and $c_jm=m$ for all $j=1,\dots,s$. Then $c_j\in G_m$ for all $j=1,\dots,s$ and $r_ix^{(i)}=r_ix_1^{(i)}+\ldots+r_ix_m^{(i)}$. If $x_j^{(i)}\neq 0$ for some $j<m$ then $j\in Gt$ and hence by Lemma 1.1, $G_j=G_t=G_m$. Thus $r_ix_j^{(i)}\in M_j$ for all $j<m$. Since $\sum_{j=1}^{p}r_jx^{(j)}\in X$, then $z=y-\sum_{j=1}^{p}r_jx^{(j)}\in Y$ and $z_j=0$ for all $j\geqq m$. So, by induction $z\in X$ and hence $y\in X$. Therefore, $X=Y$.

Step 2. To show that $M^{(Gt)}$ is R-Noetherian, let $X_1\subseteqq X_2\subseteqq\ldots$ be a chain of R-submodules of $M^{(Gt)}$. Then for each $i=1,\dots,n$, $X_1^{(i)}\subseteqq X_2^{(i)}\subseteqq\ldots$ is a chain of A-graded R-submodules of $M^{(Gt)}$. Hence there exists $m_i\in\mathbf{N}$ such that $X_{m_i}^{(i)}=X_{m_i+1}^{(i)}=\ldots$. Let $m_0=\max\{m_i\colon i=1,\dots,n\}$. Then $X_{m_0}^{(i)}=X_{m_0+1}^{(i)}=\ldots$ for each $i=1,\dots,n$ and hence $X_{m_0}=X_{m_0+1}=\ldots$, i.e., $M^{(Gt)}$ is R-Noetherian.

Let G be an abelian group and A be a denumerable G-set. Let $t\in A$ such that $G=\langle c\rangle G_t$ where $\langle c\rangle$ is an infinite cyclic subgroup of G. We denote $G^+=\{c^i\,g\colon i\geqq 0\text{ and }g\in G_t\}$ and $G^-=\{c^i\,g\colon i\leqq 0\text{ and }g\in G_t\}$.

PROPOSITION 2.8. *With the above notations, we have*

1. $R^{(G^+)}=\sum_{r\in G^+}R_r$ *is a graded subring of R.*
2. $M^{(G^+t)}=\sum_{i=0}^{\infty}M_{c^i\cdot t}$ *is an A-graded $R^{(G^+)}$-submodule of $M^{(Gt)}$.*

PROOF. Since $(c^ig)(c^jh)=c^{i+j}gh\in G^+$ for all $c^ig,\ c^jh\in G^+$, then $R^{(G^+)}R^{(G^+)}\subseteqq R^{(G^+)}$, and hence $R^{(G^+)}$ is a graded subring of R. Similarly, if $c^i g \in G^+$ and $j \geqq 0$, then $(c^i g)(c^j t) = c^{i+j} t$. Therefore, $R^{(G^+)} M^{(G^+t)} \subseteqq \subseteqq M^{(G^+t)}$, i.e., $M^{(G^+t)}$ is an A-graded $R^{(G^+)}$-submodule of $M^{(Gt)}$.

PROPOSITION 2.9. *$M^{(G^+t)}$ is A-graded $R^{(G^+)}$-Noetherian.*

PROOF. The relation $\sim$ defined on $\{c^i : i \geqq 0\}$ by $c^i \sim c^j$ if $c^i t = c^j t$ is an equivalence relation. Let $\{r_i = c^{s_i}\}_{i \in I}$ be a set of representatives for the equivalence classes in $\{c^i : i \geqq 0\}$. Then $M^{(G^+t)} = \bigoplus_{i \in I} M_{r_i \cdot t}$.

If I is a finite set, then $M^{(G^+t)}$ is $R^{(G_t)}$-Noetherian and hence it is A-graded $R^{(G^+)}$-Noetherian ($G_t \subseteqq G^+$). Let $I = \{1, 2, \ldots\}$ and $0 \leqq s_1 < s_2 < < \ldots$. Suppose N is an A-graded $R^{(G^+)}$-submodule of $M^{(G^+t)}$, then $N = = \bigoplus_{i \in I} N_{r_i \cdot t}$. Since RN is an A-graded R-submodule of $M^{(Gt)}$, it is finitely generated over R. Let $x_1, \ldots, x_k$ be a homogeneous elements of N which generates RN over R. Suppose $x_i \in M_{r_{p_i} \cdot t}$ for $i = 1, \ldots, k$ and $p_1 \leqq \ldots \leqq p_k$. Then $\sum_{i=1}^{p_k} M_{r_i \cdot t}$ is $R^{(G_t)}$-Noetherian and hence $\sum_{i=1}^{p_k} N_{r_i \cdot t}$ is finitely generated over $R^{(G_t)}$. Let $y_1, \ldots, y_s$ be generators of $\sum_{i=1}^{p_k} N_{r_i \cdot t}$ over $R^{(G_t)}$. We show $\{x_1, \ldots, x_k, y_1, \ldots, y_s\}$ generates N over $R^{(G^+)}$.

Let $Z \in N_{r_m \cdot t}$ be a homogeneous element of N. If $m \leqq p_k$, then Z is an $R^{(G_t)}$-linear sum of $y_1, \ldots, y_s$. Suppose $m \geqq p_k$. Since $Z \in RN$, $Z = a_1 x_1 + \ldots + a_k x_k$ where $a_i \in R$. If $a_i x_i \neq 0$, let $a_i = (a_i)_{b_1 \cdot g_1} + \ldots + (a_i)_{b_\ell \cdot g_\ell}$ where $b_j = c^{n_j}$, $(a_i)_{b_j \cdot g_j} \in R_{b_j \cdot g_j}$, $n_j \in \mathbf{Z}$, $g_j \in G_t$ and $b_j \cdot g_j r_{p_i} \cdot t = = c^{s_m} \cdot t$. Then $c^{(-s_m + s_{p_i} + n_j)} \in G_t$, i.e., $b_j \in c^{(s_m - s_{p_i})} \cdot G_t$ and hence $b_j \cdot g_j \in G^+$ $(s_m - s_{p_i} > 0)$ for each $j = 1, \ldots, \ell$. Therefore, Z is an $R^{(G^+)}$-linear sum of $x_1, \ldots, x_k$ and hence $M^{(G^+t)}$ is finitely generated, i.e., $M^{(G^+t)}$ is A-graded $R^{(G^+)}$-Noetherian.

By using the same arguments as in Propositions 2.7 and 2.8, one can easily show that if $R^{(G^-)} = \sum_{r \in G^-} R_r$ and $M^{(G^-t)} = \sum_{i=0}^{\infty} M_{\bar{c}^i \cdot t}$, then $M^{(G^-t)}$ is A-graded $R^{(G^-)}$-Noetherian.

Now, consider an R-submodule X of $M^{(Gt)}$. For $i \in \mathbf{N}$, let $X^{(i)}$ be the A-graded $R^{(G^+)}$-submodule of $M^{(G^+t)}$ generated by the elements $x_{c^i \cdot t}$ where $x \in X$, which satisfies the following conditions:

1. $x \notin M^{(G^-t)}$ and
2. x can be written as $x = x_{r_1 \cdot t} + \ldots + x_{r_p \cdot t} + x_{c^i \cdot t}$ where $r_j = c^{s_j}$, $s_j \in \mathbf{Z}$ and $s_1 < \ldots < s_p < i$.

Suppose $X^* = \sum_{i=1}^{\infty} X^{(i)}$. Then $X^{(1)} \subseteqq X^{(1)} + X^{(2)} \subseteqq \ldots \subseteqq X^*$ is a chain of A-graded $R^{(G^+)}$-submodules of $M^{(G^+t)}$, and hence by Proposition 2.9, there exists $n \in \mathbf{N}$ such that $X^{(1)} + \ldots + X^{(n)} = X^*$. Now, for each $i = 1, \ldots, n$ let $\{x^{(is)}_{c^i \cdot t}: s = 1, \ldots, k_i$ where $x^{(is)} \in X$ satisfies the above two conditions$\}$ be a finite subset of $X^{(i)}$ that generates it over $R^{(G^+)}$. With these notations we have the following lemma.

LEMMA 2.10. *Let $X \subseteqq Y$ be R-submodules of $M^{(Gt)}$ such that $X \cap \cap M^{(G^-t)} = Y \cap M^{(G^-t)}$, $X^* = Y^*$ and $X^{(i)} = Y^{(i)}$ for all $i = 1, \ldots, n$. Then $X = Y$.*

PROOF. Let $y \in Y$. If $y \in M^{(G^-t)}$, then $y \in X$. So let $y = y_{r_1 \cdot t} + \ldots + y_{r_m \cdot t}$ where $r_j = c^{s_j}$, $s_1 < s_2 < \ldots < s_m$ and $s_m > 0$. We proceed by induction on s_m.

If $s_m = 1$, then $y_{r_m \cdot t} \in Y^{(1)} = X^{(1)}$. Let $y_{r_m \cdot t} = a_1 x^{(11)}_{ct} + \ldots + a_{k_1} x^{(1k_1)}_{ct}$, $a_i \in R^{(G^+)}$. If $a_i x^{(1i)}_{ct} \neq 0$, then let $a_i = (a_i)_{w_1 \cdot g_1} + \ldots + (a_i)_{w_s \cdot g_s}$ where $w_j = c^{m_j}$, $m_j \geqq 0$, $g_j \in G_t$ and $w_j \cdot g_j ct = c^{s_m} t = ct$ for all $j = 1, \ldots, s$. Then $w_j \in G_{ct} = G_t$. So, $y - \sum_{i=1}^{k_1} a_i x^{(1i)} \in M^{(G^-t)}$ and hence $y \in X$. Assume it is true for all $s_m \leqq n$. Let $s_m > n$. Since $y_{r_m \cdot t} \in X^* = X^{(1)} + \ldots + X^{(n)}$, then

$$y_{r_m \cdot t} = \sum_{i=1}^{n} \sum_{s=1}^{k_i} a_{is} x^{(is)}_{c^i \cdot t}; \quad a_{is} \in R^{(G^+)}.$$

If $a_{is} x^{(is)}_{c^i \cdot t} \neq 0$, then let $a_{is} = (a_{is})_{v_1 \cdot b_1} + \ldots + (a_{is})_{v_q \cdot b_q}$ where $v_j = c^{\ell_j}$, $\ell_j \geqq \geqq 0$, $b_j \in G_t$ and $v_j \cdot b_j \cdot c^i \cdot t = c^{s_m} \cdot t$ for all $j = 1, \ldots, q$. Then $v_j = c^{\ell_j} \in \in c^{s_m - i} G_t$, let $v_j = c^{s_m - i} d_j$; $d_j \in G_t$, $j = 1, \ldots, q$. Now, if $a_{is} x^{(is)}_{c^k \cdot t} \neq 0$ for some $k < i$ then $a_{is} x^{(is)}_{c^k \cdot t} = \sum_{j=1}^{q} (a_{is})_{v_j \cdot b_j} x^{(is)}_{c^k \cdot t}$. Since $v_j \cdot b_j c^k \cdot t = c^{s_m - 1 + k} \cdot t$ and $\sum_{i=1}^{n} \sum_{s=1}^{k_i} a_{is} x^{(is)} \in X$ then

$$Z = y - \sum_{i=1}^{n} \sum_{s=1}^{k_i} a_{is} x^{(is)} \in Y$$

and $Z = Z_{c^{e_1} \cdot t} + \ldots + Z_{c^{e_p} \cdot t}$ where $e_1 < \ldots < e_p < s_m$. By induction $y \in X$.

Let X be an $R^{(G^+)}$-submodule of $M^{(G^+t)}$. For $i \in \mathbf{N} \cup \{0\}$, let $X^{(i)}$ be the A-graded $R^{(G^+)}$-submodule of $M^{(G^+t)}$ generated by the elements $x_{c^i \cdot t}$ where $x \in X$ can be written as $x = x_{r_1 \cdot t} + \ldots + x_{r_p \cdot t} + x_{c^i \cdot t}$ where $r_j = c^{s_j}$, $s_j \in \mathbf{N} \cup \{0\}$ and $s_1 < \ldots < s_p < i$. As before there exists $n \in \mathbf{N}$ such that $X^{(0)} + \ldots + X^{(n)} = X^* = \sum_{i=0}^{\infty} X^{(i)}$.

By using these notations and similar argument to that used in Lemma 2.10, one can easily show the following: If $X \subseteqq Y$ are $R^{(G^+)}$-submodules of $M^{(G^+t)}$ such that $X^* = Y^*$ and $X^{(i)} = Y^{(i)}$ for all $i = 0, 1, \ldots, n$ then $X = Y$.

PROPOSITION 2.11. *$M^{(G^+t)}$ is $R^{(G^+)}$-Noetherian.*

PROOF. Let $X_1 \subseteqq X_2 \subseteqq \ldots$ be a chain of $R^{(G^+)}$-submodules of $M^{(G^+t)}$. Then $X_1^* \subseteqq X_2^* \subseteqq \ldots$ is a chain of A-graded $R^{(G^+)}$-submodules of $M^{(G^+t)}$. So, by Proposition 2.9, there exists $n \in \mathbf{N}$ such that $X_n^* = X_{n+1}^* = \ldots$. Let $m \in \mathbf{N}$ such that $X_n^* = X_n^{(0)} + \ldots + X_n^{(m)}$. Then $X_{n+i}^* = X_{n+i}^{(0)} + \ldots + X_{n+i}^{(m)}$ for each $i \in \mathbf{N}$. Now, for each $j = 0, \ldots, m$, $X_n^{(j)} \subseteqq X_{n+1}^{(j)} \subseteqq \ldots$ is a chain of A-graded $R^{(G^+)}$-submodules of $M^{(G^+t)}$. Hence there exists $n_j \in \mathbf{N}$ such that $X_{n+n_j}^{(j)} = X_{n+n_j+1}^{(j)} = \ldots$. Let $p = \max\{n + n_j\}_{j=0}^m$. Then $X_p^* = X_{p+1}^* = \ldots$ and $X_p^{(i)} = X_{p+1}^{(i)} = \ldots$, for all $i = 0, \ldots, m$. Therefore, $X_p = X_{p+1} = \ldots$, i.e., $M^{(G^+t)}$ is $R^{(G^+)}$-Noetherian.

Similarly, $M^{(G^-t)}$ is an $R^{(G^-)}$-Noetherian.

PROPOSITION 2.12. *$M^{(Gt)}$ is R-Noetherian.*

PROOF. Let $X_1 \subseteqq X_2 \subseteqq \ldots$ be a chain of R-submodules of $M^{(Gt)}$. Then $X_1 \cap M^{(G^-t)} \subseteqq X_2 \cap M^{(G^-t)} \subseteqq \ldots$ is a chain of $R^{(G^-)}$-submodules of $M^{(G^-t)}$. So there exists $n_0 \in \mathbf{N}$ such that $X_{n_0} \cap M^{(G^-t)} = X_{n_0+1} \cap M^{(G^-t)} = \ldots$.

Since $X_{n_0}^* \subseteqq X_{n_0+1}^* \subseteqq \ldots$ is a chain of A-graded $R^{(G^+)}$-submodules of $M^{(G^+t)}$ there exists $s \in \mathbf{N}$ such that $X_{n_0+s}^* = X_{n_0+s+1}^* = \ldots$. Let $X_{n_0+s}^{(1)} + \ldots + X_{n_0+s}^{(m)} = X_{n_0+s}^{(i)}$. For $i = 1, \ldots, m$ choose $n_i \in \mathbf{N}$ such that $X_{n_0+s+n_i}^{(i)} = X_{n_0+s+n_i+1}^{(i)} = \ldots$. Let $n = \max\{n_0 + s + n_j\}_{j=1}^m$, then $X_n \cap M^{(G^-t)} = X_{n+1} \cap M^{(G^-t)} = \ldots$, $X_n^* = X_{n+1}^* = \ldots$ and $X_n^{(i)} = X_{n+1}^{(i)} = \ldots$ for all $i = 1, \ldots, m$. By Lemma 2.10, $X_n = X_{n+1} = \ldots$, i.e., $M^{(Gt)}$ is R-Noetherian.

PROPOSITION 2.13. *Suppose A is a denumerable G-set such that $G = \langle c\rangle G_t$ where $\langle c\rangle$ is a cyclic subgroup of G, for each $t \in A$. Then M is R-Noetherian.*

PROOF. Follows directly from Proposition 2.12 and Corollary 2.4.

COROLLARY 2.14. *Suppose R is a $\mathbf{Z}$-graded ring and M is A-graded R-Noetherian where A is a denumerable $\mathbf{Z}$-set. Then M is R-Noetherian.*

In the rest of this section, G will be an abelian group and $H \leqq G$ such that $G/H = \langle cH\rangle$ is an infinite cyclic group. Also, M is A-graded R-Noetherian where A is a denumerable G-set.

For $t \in A$, let $S = \langle c\rangle G_t$ be a subgroup of G. Then $R^{(S)} = \sum_{s\in S} R_s$ is a graded subring of R. For each $h \in H$, the relation $\sim$ defined on $\langle c\rangle h =$

$= \{c^i h : i \in \mathbf{Z}\}$ by $c^i h \sim c^j h$ iff $c^i ht = c^j ht$ is an equivalence relation. Let $\{c^{s_i} h\}_{i\in\Delta_h}$ be a set of representatives for the equivalence classes in $\langle c\rangle h$. Clearly, $A_h = \{c^{s_i} h : i \in \Delta_h\}$ is an S-set with an action given by $(c^i g)(c^{s_i} ht) = c^{s_r} ht$ where $c^{i+s_i} \cdot ht = c^{s_r} \cdot ht$. Also, one can easily show that $M_{(h)}^{(Gt)} = \sum_{a\in A_h} M_a$ is an A_h-graded $R^{(S)}$-module.

PROPOSITION 2.15. *With the above notations, $M_{(h)}^{(Gt)}$ is A_h-graded $R^{(S)}$-Noetherian.*

PROOF. First, we show that, if X is an A_h-graded $R^{(S)}$-submodule of $M_{(h)}^{(Gt)}$, then $RX \cap M_{(h)}^{(Gt)} = X$. Let $r \in R$ and x be a homogeneous element of X. Then $r = (r)_{w_1\cdot b_1} + \ldots + (r)_{w_p\cdot b_p}$; $w_i = c^{m_i}$, $m_i \in \mathbf{Z}$, $b_i \in H$ and $x \in M_{c^{s_q}\cdot ht}$ for some $q \in \Delta_h$. Now, if rx is a homogeneous element in $M_{(h)}^{(Gt)}$ then $rx \in M_{c^{s_\ell}\cdot ht}$ for some $\ell \in \Delta_h$. Assume $(r)_{w_i\cdot b_i} x \neq 0$, then $w_i \cdot b_i c^{s_q} \cdot ht = c^{s_\ell} \cdot ht$, i.e., $w_i \cdot b_i \in S$. Hence $rx \in X$.

Let $X_1 \subseteqq X_2 \subseteqq \ldots$ be a chain of A_h-graded $R^{(S)}$-submodules of $M_{(h)}^{(Gt)}$. Then $RX_1 \subseteqq RX_2 \subseteqq \ldots$ is a chain of A-graded R-submodules of $M^{(Gt)}$. Hence there exists $n \in \mathbf{N}$ such that $RX_n = RX_{n+1} = \ldots$, and then $RX_n \cap M_{(h)}^{(Gt)} = RX_{n+1} \cap M_{h}^{(Gt)} = \ldots$. Therefore, $X_n = X_{n+1} = \ldots$.

PROPOSITION 2.16. *$M_{(h)}^{(Gt)}$ is $R^{(S)}$-Noetherian.*

PROOF. Clearly, $M_{(h)}^{(Gt)} = \left(M_{(h)}^{(Gt)}\right)^{(S_{ht})}$. But $S_{ht} = G_t$ by Lemma 1.1. Thus $S = \langle c\rangle S_{ht}$ and hence by Propositions 2.13 and 2.15, $M_{(h)}^{(Gt)}$ is $R^{(S)}$-Noetherian.

For $h_1, h_2 \in H$, we have $M_{(h_1)}^{(Gt)} = M_{(h_2)}^{(Gt)}$ or $M_{(h_1)}^{(Gt)} \cap M_{(h_2)}^{(Gt)} = 0$. So the relation $\mathcal{E}$ defined on H by $h_1 \mathcal{E} h_2$ iff $c^i h_1 t = c^j h_2 t$ for some $i, j \in \mathbf{Z}$ is an equivalence relation. Let $A^{(H)}$ be a set of representatives for the equivalence classes in H. Then $M^{(Gt)} = \bigoplus_{u\in A^{(H)}} M_{(u)}^{(Gt)}$. So, if $A^{(H)}$ is a finite subset of H, then $M^{(Gt)}$ is $R^{(S)}$-Noetherian and hence is R-Noetherian.

COROLLARY 2.17. *Let F be a finite abelian group and $G = \mathbf{Z} \times F$. Then M is R-Noetherian.*

PROOF. Let $H = F$, then $G/H = \mathbf{Z}$. By the previous discussion, $M^{(Gt)}$ is R-Noetherian for each $t \in A$. The result follows from Corollary 2.4.

Finally, the above techniques may be extended to show that if M is A-graded R-Noetherian where G is a finitely generated abelian group, then M is R-Noetherian. We leave this case as an open problem.

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THE VARIETY OF CH-ALGEBRAS

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In a compact T_2 space, every ultrafilter converges to exactly one point. This allows us to introduce into any such space certain "limit" operations which really are operations (albeit infinitary) in the universal algebraic sense. Throwing away the topology then leaves an abstract algebra which we call a "CH-algebra" (for "compact Hausdorff").

The algebraic properties of these CH-algebras relate nicely to the topological properties of the compact T_2 spaces from which they derive: homomorphisms, subalgebras, and direct products correspond to continuous maps, closed subspaces, and topological products. Thus it is perhaps not surprising that the class of CH-algebras is a variety, i.e. is closed under these operations.

This result is well-known to category theorists and appears in somewhat different language in [12], [13], [17]. However, the purely universal-algebraic approach offers several advantages. First of all, we are able to write down an explicit scheme of equations which axiomatizes the variety of CH-algebras; this yields a characterization of topologies presented in terms of ultrafilters. This result is analogous to Birkhoff's characterization of topologies presented in terms of nets ([2], Theorem 7 or [10], p. 74); our theorem is narrower, covering only compact T_2 spaces, but our axioms are also a little simpler.

Our approach also sheds new light on some well-known theorems from general topology. For instance, the existence of the Stone–Čech compactification becomes a straightforward consequence of standard facts about free algebras in a variety. Thus we obtain yet another construction of this important object; our construction has the advantage that, with trivial modifications, it also produces the Bohr compactification, etc. Finally, the universal-algebraic point of view advertized here also illuminates the relationship between compact T_2 spaces and Stone spaces (i.e. totally disconnected compact T_2 spaces).

(A universal-algebraic approach to general topology was attempted in [6], but that paper was complicated by the introduction of certain operations λ_D which have arity a proper class and which remain partial even in the compact

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T_2 case. None of our applications follow from the result of [6]. Operations were also introduced into topological spaces in [5].)

This paper continues the general program, begun in [18] and [19], of applying universal algebraic ideas to sets equipped with nonalgebraic structure (in this case, topology). It has benefited from comments made by John Coleman, Melvin Henriksen, Walter Taylor, and the referee.

Basic ideas of universal algebra, used by us with brief explanations, are given in full detail in [4]. Readers with a background in category theory may also find the chapters on algebraic and varietal categories in [8] and [1] of interest.

I.

We begin by defining CH-algebras. Let $\mathcal{A} = \langle A, \mathcal{T} \rangle$ be a compact T_2 space with underlying set A and topology $\mathcal{T}$, and let I be any nonempty set and $\mathcal{U}$ an ultrafilter over I. Then any function $x: I \to A$ induces an ultrafilter $\mathcal{U}_{\mathbf{x}}$ over A defined by

$$\mathcal{U}_{\mathbf{x}} = \{ U \subset A : \mathbf{x}^{-1}(U) \in \mathcal{U} \} .$$

We then define an operation $f_{I,\mathcal{U}}: A^I \to A$ by setting $f_{I,\mathcal{U}}(\mathbf{x})$ equal to the unique point all of whose neighbourhoods belong to $\mathcal{U}_{\mathbf{x}}$. (Such a point exists since $\mathcal{A}$ is compact, and it is unique since $\mathcal{A}$ is T_2.) The point $f_{I,\mathcal{U}}(\mathbf{x})$ may be described as "the limit of $\mathbf{x}$ with respect to $\mathcal{U}$." Of course, $f_{I,\mathcal{U}}$ is an infinitary operation if I is infinite.

Let $\mathbf{A} = \langle A, \{f_{I,\mathcal{U}}\} \rangle$ denote the abstract algebra with underlying set A together with all of these operations $f_{I,\mathcal{U}}$, for any nonempty set I and any ultrafilter $\mathcal{U}$ over I. (In our notation, $\mathbf{B}$ will be the algebra arising from the compact T_2 space $\mathcal{B}$, etc.) We define *CH-algebras* to be precisely those abstract algebras which are derivable from compact T_2 spaces in this manner.

The notions of homomorphism, isomorphism, subalgebra, and direct product are fairly self-evident and can be summarized as: a map which commutes with the operations $f_{I,\mathcal{U}}$; a 1–1 homomorphism; a subset closed under the operations $f_{I,\mathcal{U}}$; and the cartesian product with the operations $f_{I,\mathcal{U}}$ defined coordinatewise. (Strictly speaking, we have just defined the underlying set of a subalgebra; the actual subalgebra is this set endowed with the restricted operations.)

For the reader's convenience we include proofs of the following rather standard propositions.

PROPOSITION 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be compact T_2 spaces and* $\mathbf{A}$ *and* $\mathbf{B}$ *the corresponding CH-algebras. Then a map $\phi: A \to B$ is continuous from $\mathcal{A}$ to $\mathcal{B}$ iff it is a homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$, *and ϕ is a homeomorphism of $\mathcal{A}$ into $\mathcal{B}$ iff it is an isomorphism of* $\mathbf{A}$ *into* $\mathbf{B}$.

PROOF. Suppose $\phi: A \to B$ is continuous from $\mathcal{A}$ to $\mathcal{B}$ and $f_{I,\mathcal{U}}(\mathbf{x}) = y$; to show ϕ is a homomorphism we need $f_{I,\mathcal{U}}(\phi(\mathbf{x})) = \phi(y)$. But for any neighbourhood U of $\phi(y)$, $\phi^{-1}(U)$ is a neighbourhood of y, hence $\phi^{-1}(U) \in \mathcal{U}_{\mathbf{x}}$. Thus $\mathbf{x}^{-1}(\phi^{-1}(U)) \in \mathcal{U}$ and so $U \in \mathcal{U}_{\phi\circ\mathbf{x}}$. Thus every neighbourhood of $\phi(y)$ is in $\mathcal{U}_{\phi\circ\mathbf{x}}$, so by the definition of the operation $f_{I,\mathcal{U}}$ indeed $f_{I,\mathcal{U}}(\phi(\mathbf{x})) = \phi(y)$.

Conversely, suppose ϕ is not continuous; we now must show ϕ is not a homomorphism. Let U be an open subset of $\mathcal{B}$ such that $\phi^{-1}(U)$ is not open. Then there exists $y \in \phi^{-1}(U)$ such that $\phi^{-1}(U)$ is not a neighbourhood of y. Now let $I = A$; let $\mathcal{U}$ be any ultrafilter over A containing every neighbourhood of y and also the set $A - \phi^{-1}(U)$; and let $\mathbf{x}: A \to A$ be the identity map. Then $f_{I,\mathcal{U}}(\mathbf{x}) = y$ but $f_{I,\mathcal{U}}(\phi(\mathbf{x})) \neq \phi(y)$ since $U \notin \mathcal{U}_{\phi\circ\mathbf{x}}$. Thus ϕ is not a homomorphism from $\mathbf{A}$ to $\mathbf{B}$.

The second statement follows from the observation that, for compact T_2 spaces, a homeomorphism is just a 1–1 continuous map, while an isomorphism of CH-algebras is by definition a 1–1 homomorphism. □

PROPOSITION 2. *Let $\mathcal{A}$ be a compact T_2 space with corresponding CH-algebra* $\mathbf{A}$. *Let* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$ *with underlying set B. Then B is a closed (hence compact) subset of $\mathcal{A}$ and* $\mathbf{B}$ *is the CH-algebra corresponding to the space $\mathcal{B}$ which is B endowed with the induced topology. Conversely, every closed subset of $\mathcal{A}$ underlies a subalgebra of* $\mathbf{A}$. *(It follows that the space $\mathcal{A}$ can be recovered from the algebraic structure of* $\mathbf{A}$: *the closed sets of $\mathcal{A}$ are precisely those subsets which underlie subalgebras of* $\mathbf{A}$.*)*

PROOF. Let y be any element in the closure of B. Then let $I = B$; let $\mathcal{U}$ be an ultrafilter over B which contains $B \cap U$ for every neighbourhood U of y; and let $\mathbf{x}: B \to B$ be the identity map. Then $f_{I,\mathcal{U}}(\mathbf{x}) = y$, and since B is closed under operations $f_{I,\mathcal{U}}$ we conclude that $y \in B$. Thus B is a closed subset of $\mathcal{A}$.

To prove that $\mathbf{B}$ derives from $\mathcal{B}$ we must show that the operations of $\mathbf{B}$ (which are the restrictions of the operations of $\mathbf{A}$) agree with the operations derived from $\mathcal{B}$. Thus suppose that $f_{I,\mathcal{U}}(\mathbf{x}) = y$ in the sense of $\mathbf{B}$, where the range of $\mathbf{x}$ is in B and $y \in B$; we must show that $f_{I,\mathcal{U}}(\mathbf{x}) = y$ in the sense of $\mathcal{B}$, i.e. every $\mathcal{B}$-neighbourhood of y is in $\mathcal{U}_{\mathbf{x}}$. Let U_B be such a neighbourhood, so $U_B = U \cap B$ for some $\mathcal{A}$-neighbourhood U of y. Then $U \in \mathcal{U}_{\mathbf{x}}$, i.e. $\mathbf{x}^{-1}(U) \in \mathcal{U}$. Since the range of $\mathbf{x}$ is contained in B, we have $\mathbf{x}^{-1}(U_B) = \mathbf{x}^{-1}(U)$, hence $\mathbf{x}^{-1}(U_B) \in \mathcal{U}$ and thus $U_B \in \mathcal{U}_{\mathbf{x}}$, as desired.

Finally, let B be any closed subset of $\mathcal{A}$; we want to show that B is closed under the operations $f_{I,\mathcal{U}}$. Thus fix $I, \mathcal{U}$, and $\mathbf{x}: I \to B$ and suppose $f_{I,\mathcal{U}}(\mathbf{x}) = y$. If $y \notin B$ then $A - B$ is a neighbourhood of y, hence $\mathbf{x}^{-1}(A - B) \in \mathcal{U}$. But since $\mathbf{x}(I) \subset B, \mathbf{x}^{-1}(A - B) = \emptyset$, a contradiction. This completes the proof. □

PROPOSITION 3. *If $\{\mathcal{A}_\lambda\}$ is a collection of compact T_2 spaces with corresponding CH-algebras $\{\mathbf{A}_\lambda\}$, then $\prod \mathcal{A}_\lambda$ has corresponding CH-algebra $\prod \mathbf{A}_\lambda$.*

PROOF. Fix $I, \mathcal{U}$, and $\mathbf{x}: I \to \prod \mathbf{A}_\lambda$ and suppose $f_{I,\mathcal{U}}(\mathbf{x}) = y$. (Recall that the operations $f_{I,\mathcal{U}}$ are defined on $\prod \mathbf{A}_\lambda$ coordinatewise.) We wish to show that the product topology on $\prod \mathcal{A}_\lambda$ verifies that $f_{I,\mathcal{U}}(\mathbf{x}) = y$, i.e. if U is any neighbourhood of y then $\mathbf{x}^{-1}(U) \in \mathcal{U}$. Since $\mathcal{U}$ contains every superset of each of its elements, it is enough to show this when U is a basic open set; and since $\mathcal{U}$ is closed under finite intersections, we may in fact assume U is a subbasic open set, say $U = \pi_\lambda^{-1}(V)$ where π_λ is the projection onto A_λ and $V \subset \mathcal{A}_\lambda$ is open. In this case $\mathbf{x}^{-1}(U) = (\pi_\lambda \circ \mathbf{x})^{-1}(V)$. Furthermore, $f_{I,\mathcal{U}}(\pi_\lambda \circ \mathbf{x}) = \pi_\lambda(y)$ by the coordinatewise definition of the operations $f_{I,\mathcal{U}}$ on $\prod \mathbf{A}_\lambda$. But V is a neighbourhood of $\pi_\lambda(y)$, so we conclude that $(\pi_\lambda \circ \mathbf{x})^{-1}(V) \in \mathcal{U}$, hence $\mathbf{x}^{-1}(U) \in \mathcal{U}$, as desired. □

For the next proposition, recall that a *congruence relation* (or *congruence*) θ on a CH-algebra $\mathbf{A}$ is by definition an equivalence relation on A which is compatible with the operations $f_{I,\mathcal{U}}$ in the following way:

$$\langle \mathbf{x}(i), \mathbf{y}(i) \rangle \in \theta \text{ for all } i \in I \text{ implies } \langle f_{I,\mathcal{U}}(\mathbf{x}), f_{I,\mathcal{U}}(\mathbf{y}) \rangle \in \theta.$$

It is a general and easily verified fact that every congruence relation determines a quotient structure which is a homomorphic image of $\mathbf{A}$, and conversely every surjective homomorphism arises in this manner (up to isomorphism of the range space). An equivalent formulation of congruences is: θ is an equivalence relation which underlies a subalgebra of $\mathbf{A} \times \mathbf{A}$, i.e. (by Propositions 2 and 3) an equivalence relation which is a closed subset of $\mathcal{A} \times \mathcal{A}$.

PROPOSITION 4. *Any intersection of congruences is a congruence, hence the set of all congruences is a complete lattice when ordered by inclusion.*

PROOF. The first statement follows from the last characterization of congruences, since any intersection of equivalence relations is an equivalence relation and any intersection of closed subsets of $\mathcal{A} \times \mathcal{A}$ is a closed subset of $\mathcal{A} \times \mathcal{A}$. The second statement is an immediate consequence of the first. □

We can now give a fairly painless proof of the following theorem. Recall that a *variety* is a class of algebras closed under the formation of homomorphic images, subalgebras, and direct products.

THEOREM A. (Linton [12], p. 90; see also [13], p. 153 and [17]). *The class of CH-algebras is a variety.*

PROOF. We have already observed in Propositions 2 and 3 above that the class of CH-algebras is closed under the formation of subalgebras and direct products, and it is trivially closed under the formation of isomorphic images. Thus, we need only show that if $\mathbf{A}$ is a CH-algebra and θ is a congruence relation on $\mathbf{A}$, then $\mathbf{A}/\theta$ is a CH-algebra.

Suppose $\mathbf{A}$ corresponds to $\mathcal{A} = \langle A, \mathcal{T} \rangle$. The quotient topology $\mathcal{T}/\theta$ is obviously compact, and as θ is a closed subset of $\mathcal{A} \times \mathcal{A}$, it follows from a theorem of Bourbaki ([3], § I.10.4, Proposition 8) that $\mathcal{T}/\theta$ is T_2. We claim that $\mathbf{A}/\theta$ is the CH-algebra corresponding to $\mathcal{A}/\theta = \langle A/\theta, \mathcal{T}/\theta\rangle$. To see this, fix $I, \mathcal{U}$, and $\mathbf{x}: I \to A/\theta$ and suppose $f_{I,\mathcal{U}}(\mathbf{x}) = y$ according to the quotient CH-algebra structure of $\mathbf{A}/\theta$. Now define a function $\mathbf{x}_0: I \to A$ in such a way that for each $i \in I$, $\mathbf{x}_0(i)$ is a representative of the class $\mathbf{x}(i)$; it follows from the definition of the quotient structure that if $f_{I,\mathcal{U}}(\mathbf{x}_0) = y_0$, then y_0 is a representative of the class y.

Now we want to show that for any $\mathcal{A}/\theta$-neighbourhood U of $y, \mathbf{x}^{-1}(U) \in \in \mathcal{U}$. However U lifts to an $\mathcal{A}$-neighbourhood U_0 of y_0, and we know $\mathbf{x}_0^{-1}(U_0) \in \mathcal{U}$. But $\mathbf{x}^{-1}(U) = \mathbf{x}_0^{-1}(U_0)$, so we are done. □

II.

We begin this section with a brief review of terms and equations.

Let X be any set; we think of its elements as variables. Then the *terms* over X are all of the "words" which can be written using the elements of X and the operations $f_{I,\mathcal{U}}$. More precisely, the terms over X form the smallest class $\mathcal{C}$ such that $X \subset \mathcal{C}$ and such that whenever I is a set, $\mathcal{U}$ is an ultrafilter over I, and $\mathbf{x}$ is a function from I into $\mathcal{C}$, the formal expression $f_{I,\mathcal{U}}(\mathbf{x})$ is also an element of $\mathcal{C}$.

The class $\mathcal{C}$ can also be characterized "internally" via a definition by transfinite induction. Here we construct $\mathcal{C}$ in stages. At stage zero we put in the elements of X; at stage α for α an ordinal > 0, we put in all the formal expressions $f_{I,\mathcal{U}}(\mathbf{x})$ where the range of $\mathbf{x}$ consists of terms already constructed. An advantage of this method of defining $\mathcal{C}$ is that it allows us to define the *order* of a term as the ordinal at which it is first constructed. In other words, the order of an element of X is zero, and the order of the term $f_{I,\mathcal{U}}(\mathbf{x})$ is the sup, over $i \in I$, of the order of $\mathbf{x}(i)$ plus 1.

Let $\mathbf{A} = \langle A, \{f_{I,\mathcal{U}}\}\rangle$ be a set together with some collection of operations $f_{I,\mathcal{U}}$; do not assume $\mathbf{A}$ is a CH-algebra. Given any function $\phi: X \to A$ (i.e. an assignment of values to the variables), the terms over X can be evaluated in an obvious way. Thus, we define the evaluation $\widetilde{\phi}(t)$ of a term t by induction on the order of t, setting $\widetilde{\phi}(v) = \phi(v)$ for $v \in X$ and $\widetilde{\phi}(f_{I,\mathcal{U}}(\mathbf{x})) = f_{I,\mathcal{U}}(\widetilde{\phi} \circ \circ \mathbf{x})$.

An *equation* is a formal expression $t_1 = t_2$ where t_2 and t_2 are terms. We say that $\mathbf{A}$ *satisfies* the equation $t_1 = t_2$ if $\widetilde{\phi}(t_1) = \widetilde{\phi}(t_2)$ for every function $\phi: X \to A$. Birkhoff's fundamental theorem on varieties implies that the class of CH-algebras is definable by equations. We will now show it is in fact definable by the following three equational schemes. In these, I, J and X are nonempty sets, $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters over I and J respectively, and $\mathbf{x}$ and $\mathbf{y}$ are functions $\mathbf{x}: I \to X$ and $\mathbf{y}: J \to X$.

α) (triviality) If $I = \{p\}$ and $\mathbf{x}(p) = v$ then $f_{I,\mathcal{U}}(\mathbf{x}) = v$.

β) (restriction) Suppose $J \in \mathcal{U}$, $\mathcal{V} = \mathcal{U}|_J$, and $\mathbf{y} = \mathbf{x}|_J$. Then $f_{I,\mathcal{U}}(\mathbf{x}) = f_{J,\mathcal{V}}(\mathbf{y})$.

γ) (iteration) For each $i \in I$ let I_i be a nonempty set with an ultrafilter $\mathcal{U}_i$. Suppose $J = \cup I_i$ and $\mathcal{V} \subset 2^J$ satisfies

$$K \in \mathcal{V} \quad \text{iff} \quad \{i: K \cap I_i \in \mathcal{U}_i\} \in \mathcal{U}.$$

Then

$$f_{I,\mathcal{U}}\big(f_{I_i,\mathcal{U}_i}(\mathbf{y}|_{I_i})\big) = f_{J,\mathcal{V}}(\mathbf{y}).$$

In the last axiom scheme, the union $\cup I_i$ need not be disjoint.

THEOREM B (cf. [2], Theorem 8 or [10], p. 74). *An algebra* $\mathbf{A} = \langle A, \{f_{I,\mathcal{U}}\}\rangle$ *is a CH-algebra iff it satisfies the axiom schemes* α)–γ).

PROOF. Consider the forward direction. Suppose $\mathbf{A}$ is a CH-algebra and let it derive from the compact T_2 space $\mathcal{A}$. Choose $\phi: X \to A$. To verify triviality, we must show that $\tilde{\phi}\big(f_{I,\mathcal{U}}(\mathbf{x})\big) = \tilde{\phi}(v)$; as $\tilde{\phi}\big(f_{I,\mathcal{U}}(\mathbf{x})\big) = f_{I,\mathcal{U}}\big(\tilde{\phi} \circ \mathbf{x}\big)$, it is enough to show that $U \in \mathcal{U}_{\tilde{\phi}\circ\mathbf{x}}$ for any neighbourhood U of $\phi(v) = \tilde{\phi}(v)$. However, since $\phi(v) \in U$, $\big(\tilde{\phi} \circ \mathbf{x}\big)^{-1}(U) = I \in \mathcal{U}$, hence $U \in \mathcal{U}_{\tilde{\phi}\circ\mathbf{x}}$ as desired.

To verify restriction, let U be a neighbourhood of $f_{J,\mathcal{V}}\big(\tilde{\phi} \circ \mathbf{y}\big)$. It is enough to show that $U \in \mathcal{U}_{\tilde{\phi}\circ\mathbf{x}}$; this will establish the center equality of

$$\tilde{\phi}\big(f_{I,\mathcal{U}}(\mathbf{x})\big) = f_{I,\mathcal{U}}\big(\tilde{\phi} \circ \mathbf{x}\big) = f_{J,\mathcal{V}}\big(\tilde{\phi} \circ \mathbf{y}\big) = \tilde{\phi}\big(f_{J,\mathcal{V}}(\mathbf{y})\big),$$

thus verifying the restriction equation. But we know that $\big(\tilde{\phi} \circ \mathbf{x}\big)^{-1}(U)$ contains $\big(\tilde{\phi} \circ \mathbf{y}\big)^{-1}(U)$, which is in $\mathcal{V}$ and hence in $\mathcal{U}$. Thus $\big(\tilde{\phi} \circ \mathbf{x}\big)^{-1}(U) \in \mathcal{U}$, i.e. $U \in \mathcal{U}_{\tilde{\phi}\circ\mathbf{x}}$, as desired.

To verify iteration, let U be a closed neighbourhood of $f_{J,\mathcal{V}}\big(\tilde{\phi} \circ \mathbf{y}\big)$. Now $\big(\tilde{\phi} \circ \mathbf{y}\big)^{-1}(U) \in \mathcal{V}$, hence the set of $i \in I$ such that $\big(\tilde{\phi} \circ \mathbf{y}\big)^{-1}(U) \cap I_i \in \mathcal{U}_i$ is a member of $\mathcal{U}$. By the restriction axiom, and since U is a closed subset, we conclude that for each such i, $f_{I_i,\mathcal{U}_i}\big(\tilde{\phi} \circ \mathbf{y}|_{I_i}\big) \in U$. Then another application of restriction yields that

$$f_{I,\mathcal{U}}\Big(f_{I_i,\mathcal{U}_i}\big(\tilde{\phi} \circ \mathbf{y}|_{I_i}\big)\Big) \in U.$$

To conclude we observe that in a T_2 space, the intersection of all closed neighbourhoods of a point is precisely that point. Thus

$$f_{I,\mathcal{U}}\Big(f_{I_i,\mathcal{U}_i}\big(\tilde{\phi} \circ \mathbf{y}|_{I_i}\big)\Big) = f_{J,\mathcal{V}}\big(\tilde{\phi} \circ \mathbf{y}\big),$$

which is enough. We have now seen that a CH-algebra must satisfy the axioms $\alpha) - \gamma)$.

For the reserve direction, suppose **A** satisfies axioms $\alpha) - \gamma)$. We begin by showing that the members of Sub(**A**) are precisely the closed sets of a compact topology on **A**.

Sub(**A**) is obviously closed under intersections. To show it is closed under finite unions, let $U, V \in$ Sub(**A**) and choose $I, \mathcal{U}$, and $\mathbf{x}: I \to U \cup V$; we must show that $f_{I,\mathcal{U}}(\mathbf{x}) \in U \cup V$. Now either $\mathbf{x}^{-1}(U) \in \mathcal{U}$ or $\mathbf{x}^{-1}(V) \in \mathcal{U}$. In the first case, by axiom $\beta)$ with $J = x^{-1}(U)$, we have

$$f_{I,\mathcal{U}}(\mathbf{x}) = f_{J,\mathcal{U}}(\mathbf{y}) \in U,$$

where $f_{J,\mathcal{V}}(\mathbf{y}) \in U$ since $U \in$ Sub(**A**) and $\mathbf{y}(J) \subset U$. Thus in any case $f_{I,\mathcal{U}}(\mathbf{x}) \in U \cup V$, hence $U \cup V \in$ Sub(**A**). Thus Sub(**A**) consists of the closed sets of a topology $\mathcal{T}$.

Now let $\mathcal{U}$ be an ultrafilter over A and let $x = f_{A,\mathcal{U}}(1_A)$ where 1_A is the identity map. By the axioms $\beta)$, any member of $\mathcal{U}$ must contain x in its closure, so x is a cluster point of $\mathcal{U}$. Thus every ultrafilter over A has a cluster point with respect to $\mathcal{T}$, so $\langle A, \mathcal{T} \rangle$ is compact.

For the second part of the proof, let $\mathbf{x}: I \to A$ and let $\mathcal{U}$ be an ultrafilter over I. Suppose x is a cluster point of $\mathcal{U}_\mathbf{x}$ with respect to $\mathcal{T}$; we will show that $x = f_{I,\mathcal{U}}(\mathbf{x})$. This will establish that $\mathcal{T}$ is T_2 (ultrafilters have unique limits) and that **A** derives from $\langle A, \mathcal{T} \rangle$.

For each $K \in \mathcal{U}$, x is in the closure of $\mathbf{x}(K)$, i.e. the subalgebra $\mathbf{Sg}(\mathbf{x}(K))$ generated by $\mathbf{x}(K)$. However $\mathbf{Sg}(\mathbf{x}(K))$ has underlying set

$$\left\{ f_{K,\mathcal{V}}(\mathbf{x}|_K) : \mathcal{V} \text{ is an ultrafilter over } K \right\}.$$

(For principal ultrafilters $\mathcal{V}$, the axioms $\beta)$ and $\alpha)$ imply that this set contains $\mathbf{x}(K)$; and by $\gamma)$ it is closed under the operations $f_{I,\mathcal{U}}$.) It follows that for each $K \in \mathcal{U}$ we can choose an ultrafilter $\mathcal{V}_K$ over K such that $f_{K,\mathcal{V}_K}(\mathbf{x}|_K) = x$. Let $\mathcal{U}'$ be an ultrafilter over $\mathcal{U}$ containing the sets

$$\{K : K_0 \cap K \in \mathcal{V}_K\}$$

for all $K_0 \in \mathcal{U}$. Then by the axioms $\gamma)$ and $\alpha)$ we have

$$f_{I,\mathcal{U}}(\mathbf{x}) = f_{\mathcal{U},\mathcal{U}'}\big(f_{K,\mathcal{V}_K}(\mathbf{x}|_K)\big) = x$$

as desired. □

See [16] for a very different axiomatization of the class of compact T_2 spaces.

Theorem B can be used to define compact T_2 topologies in situations where the filter point of view is natural. Thus, for each ultrafilter $\mathcal{U}$ over a set A one may simply define its point of convergence $c(\mathcal{U})$; by Theorem B this describes the convergence structure of a compact T_2 topology iff the operations

$$f_{I,\mathcal{U}}(\mathbf{x}) = c(\mathcal{U}_{\mathbf{x}})$$

satisfy axioms $\alpha) - \gamma)$.

Theorem B also provides us with important information on terms in the variety of CH-algebras. Let us say two terms t_1 and t_2 are equivalent if $\widetilde{\phi}(t_1) = \widetilde{\phi}(t_2)$ for any CH-algebra $\mathbf{A}$ and any map $\phi: X \to A$.

COROLLARY. *Every term over X is equivalent to a term of the form $f_{I,\mathcal{U}}(1_X)$, where $\mathcal{U}$ is an ultrafilter over X and 1_X is the identity map.*

PROOF. As in the proof of Theorem B, the axioms $\beta)$ and $\alpha)$ imply that every element of X is equivalent to a term of the given form; and using the axioms $\gamma)$, a trivial transfinite induction on order establishes the same for any term. □

III.

A key observation in the following is that arbitrary T_2 spaces correspond to "partial CH-algebras," meaning one still has the operation $f_{I,\mathcal{U}}$ but they are no longer defined everywhere; $f_{I,\mathcal{U}}(\mathbf{x})$ exists precisely if the ultrafilter $\mathcal{U}_{\mathbf{x}}$ converges. One still has versions of Propositions 1–3. In particular, a map $\phi: \mathcal{A} \to \mathcal{B}$ from one T_2 space to another is continuous iff whenever $f_{I,\mathcal{U}}(\mathbf{x})$ is defined in $\mathcal{A}$ and equals y, then $f_{I,\mathcal{U}}(\phi \circ \mathbf{x})$ is defined in $\mathcal{B}$ and equals $\phi(y)$.

The *free CH-algebra* F_X over a set X is defined as follows. Its elements are the terms over X, with two terms t_1 and t_2 identified if they are equivalent (i.e. $\widetilde{\phi}(t_1) = \widetilde{\phi}(t_2)$ for any CH-algebra $\mathbf{A}$ and any map $\phi: X \to \mathcal{A}$). The operation $f_{I,\mathcal{U}}$ on F_X is defined in the obvious way, and simply maps the term $\mathbf{x}$ to the term $f_{I,\mathcal{U}}(\mathbf{x})$. F_X is a CH-algebra because it automatically satisfies any equation which is satisfied by all CH-algebras, and the class of CH-algebras is definable by equations.

It is a general and easy universal algebraic fact that free algebras possess a universal mapping property: any function from X into a CH-algebra $\mathbf{A}$ extends uniquely to a homomorphism from F_X into $\mathbf{A}$. Moving over to the topological space point of view, we see that F_X corresponds to the Stone–Čech compactification βX of the discrete set X. That is to say, if we let βX be the compact T_2 space corresponding to F_X, then every map from X into a compact T_2 space $\mathcal{A}$ extends uniquely to a continuous map from βX into $\mathcal{A}$.

(A subtlety here is the fact that F_X is a set and not a proper class. This follows immediately from the Corollary to Theorem B. Alternatively, we can observe directly that F_X is a set as follows. The closure of X in βX corresponds to a subalgebra of F_X; this subalgebra contains X and hence must be F_X itself. Thus X is dense in βX, i.e. every element of βX is the limit of an ultrafilter over X. Hence the cardinality of βX is no greater than the cardinality of the set of ultrafilters over X.)

Once we have βX for X discrete, it is easy to construct $\beta\mathcal{X}$ for any T_2 space $\mathcal{X} = \langle X, \mathcal{T}\rangle$. Namely, let θ be the least congruence relation on F_X which contains all pairs $\langle f_{I,\mathcal{U}}(\mathbf{x}), v\rangle$ for which $f_{I,\mathcal{U}}(\mathbf{x})$ exists and equals v according to the topology $\mathcal{T}$ (here $\mathbf{x}$ maps I into X). Such a congruence exists because the lattice of congruences is complete (Proposition 4). Now we define $\beta\mathcal{X}$ to be the compact T_2 space corresponding to F_X/θ.

THEOREM C. *Let $\mathcal{X}, \theta$, and $\beta\mathcal{X}$ be as above. Then the natural map $\rho\colon \mathcal{X} \to \beta\mathcal{X}$ is continuous, and if $\phi\colon \mathcal{X} \to \mathcal{A}$ is any continuous map into a compact T_2 space $\mathcal{A}$ then there exists a unique continuous map $\tilde{\phi}\colon \beta\mathcal{X} \to \mathcal{A}$ such that $\phi = \tilde{\phi} \circ \rho$. If $\mathcal{X}$ is completely regular then ρ is a homeomorphism.*

PROOF. ρ is defined as $\rho = \pi \circ \sigma$ where $\pi\colon F_X \to F_X/\theta$ is the natural projection and σ is the natural embedding of X into F_X. Then the fact that ρ is continuous is an immediate consequence of the algebraic criterion for continuity described at the beginning of the section, together with the definition of θ which insures that the desired conclusion $f_{I,\mathcal{U}}(\rho \circ \mathbf{x}) = \rho(y)$ always holds.

To see that ϕ lifts uniquely to $\beta\mathcal{X}$, recall that we already know ϕ lifts uniquely to a continuous map $\phi_0\colon \beta X \to \mathcal{A}$. Since $\beta\mathcal{X}$ is a quotient of βX, this settles the question of uniqueness and leaves only the question of whether ϕ_0 is compatible with the congruence θ. For this, it suffices to observe that $\phi_0\big(f_{I,\mathcal{U}}(\mathbf{x})\big) = \phi_0(v)$ whenever $\big\langle f_{I,\mathcal{U}}(\mathbf{x}), v\big\rangle$ is one of the pairs which generate θ. Again, this is an immediate consequence of our algebraic criterion for continuity.

Finally, suppose $\mathcal{X}$ is completely regular. Then if $v \in X$ and $I, \mathcal{U}$, and $\mathbf{x}\colon I \to X$ are such that some open set U containing v is not in $\mathcal{U}_{\mathbf{x}}$, it follows that there is a continuous map $\phi\colon \mathcal{X} \to \mathcal{I}$ into the unit interval such that $\phi(v) = 0$ and $\phi(X - U) = 1$: so that

$$(\phi \circ \mathbf{x})^{-1}(1) = (\phi \circ \mathbf{x})^{-1}\big(\phi(X - U)\big) \supset \mathbf{x}^{-1}(X - U) \in \mathcal{U},$$

hence $f_{I,\mathcal{U}}(\phi \circ \mathbf{x}) = 1 \neq \phi(v)$. This shows that if $f_{I,\mathcal{U}}(\mathbf{x}) \neq v$ (perhaps because the left side is not defined), then there is a continuous map $\phi\colon \mathcal{X} \to \mathcal{I}$ such that $f_{I,\mathcal{U}}(\phi \circ \mathbf{x}) \neq \phi(v)$. Now every such ϕ lifts to $\tilde{\phi}\colon \beta\mathcal{X} \to \mathcal{I}$, such that $\phi = \tilde{\phi} \circ \rho$; it follows that $f_{I,\mathcal{U}}(\rho \circ \mathbf{x}) \neq \rho(v)$ in every such case — otherwise, applying $\tilde{\phi}$ would yield a contradiction. Thus in general $f_{I,\mathcal{U}}(\mathbf{x})$ exists and equals v iff $f_{I,\mathcal{U}}(\rho \circ \mathbf{x})$ exists and equals $\rho(v)$, hence ρ is a homeomorphism. $\square$

It is interesting that the Stone–Čech compactification exists as a consequence of these universal algebraic arguments; see [7] for other constructions and [15] for a category-theoretic approach. An advantage to our approach is that it also shows the existence of compactifications in any variety of topological algebras.

For instance, consider the class of compact T_2 abelian groups. Given any such topological group $\mathcal{A}$ we may replace the topology with a CH-algebra structure, yielding a CH-algebra $\mathbf{A}$ which is also an abelian group. We call $\mathbf{A}$ an *abelian CH-group*.

Now the group structure on $\mathbf{A}$ is related to its CH-algebra structure: the group operations are continuous. This can be asserted by means of the following equations:

$$f_{I,\mathcal{U}}(x) + f_{I,\mathcal{U}}(\mathbf{y}) = f_{I,\mathcal{U}}(\mathbf{x}+\mathbf{y}),$$

$$-f_{I,\mathcal{U}}(\mathbf{x}) = f_{I,\mathcal{U}}(-\mathbf{x}).$$

Thus the class of abelian CH-groups is definable by equations (the above equations plus the equations for abelian groups plus the equations α)–γ)), and therefore it is a variety. This conclusion is not at all special to abelian groups, as the continuity of any algebraic operation can be asserted in the same manner as the above. Thus the class of CH-groups, the class of CH-rings, the class of CH-lattices, etc. are all also varieties.

Consequently, for any set X there exists a free abelian CH-group over X with the usual universal mapping property. It consists of all terms over X modulo equivalence, where now terms are built up using the group operations as well as the operations $f_{I,\mathcal{U}}$. And the arguments of Theorem C can be mimicked down to the last detail to show that for any T_2 abelian topological group $\mathcal{X}$, there exists a compact T_2 abelian group $\beta\mathcal{X}$ and a continuous homomorphism $\rho\colon \mathcal{X} \to \beta\mathcal{X}$, such that if $\phi\colon \mathcal{X} \to \mathcal{A}$ is any continuous homomorphism into a compact T_2 abelian group $\mathcal{A}$, then there exists a unique continuous homomorphism $\tilde{\phi}\colon \beta\mathcal{X} \to \mathcal{A}$ such that $\phi = \tilde{\phi} \circ \rho$. $\beta\mathcal{X}$ is in fact the so-called Bohr compactification and is usually constructed quite differently [9]. We emphasize that this construction is also suitable for any other varieties of topological algebras.

(Again, we mention the problem of showing that $\beta\mathcal{X}$ is a set and not a proper class. It is now easiest to mimic the second proof of the corresponding fact for F_X. Thus, simply observe that the closure of $\rho(\mathcal{X})$ in $\beta\mathcal{X}$ is a compact group containing $\rho(\mathcal{X})$, hence it must be all of $\beta\mathcal{X}$ by construction. As before this bounds the cardinality of $\beta\mathcal{X}$.)

Let us observe now that the variety of CH-algebras contains no proper subvarieties. (This argument was given in [14].) For, any nontrivial subvariety $\mathcal{V}$ must contain a CH-algebra with more than one element. Then, by closure under subalgebras, $\mathcal{V}$ must contain a CH-algebra with exactly two elements. By closure under products it then contains the CH-algebra corresponding to the Cantor set $\mathcal{K}$, and by closure under homomorphic images it contains the CH-algebra corresponding to the unit interval $\mathcal{I}$. Finally, every compact T_2 space embeds in a power of $\mathcal{I}$, hence by closure under products and subalgebras $\mathcal{V}$ must contain an isomorphic copy (and, by closure under isomorphic images, an exact copy) of every CH-algebra.

The above conclusion can be summarized as: the CH-algebra 2 which corresponds to the two-element discrete space generates the variety of CH-algebras. However, it is a standard fact from universal algebra that the variety generated by an algebra is precisely the class of all homomorphic images of subalgebras of powers of that algebra; symbolically, in this case, it is $HSP(2)$. (Indeed, it is essentially trivial to check that this class is a variety.) It is well-known that a compact T_2 space is totally disconnected, i.e. Stonean, iff it embeds in a power of the two-element discrete space. Thus from the above we conclude that every compact T_2 space is a continuous image of a Stone space ([11] § 41.IX).

If the compact T_2 space is metrizable, it embeds in $\mathcal{I}^\omega$ and hence by the above line of reasoning is a continuous image of a closed subspace of $(2^\omega)^\omega = 2^\omega = \mathcal{K}$. This nearly proves the well-known fact that every compact metric space is a continuous image of the Cantor set; to complete the proof we merely observe that every nonempty closed subspace of $\mathcal{K}$ is a continuous image, in fact a retract, of $\mathcal{K}$. This is done as follows. Consider $\mathcal{K}$ as lying in $\mathcal{I}$; then any nonempty closed subspace $\mathcal{K}_0$ of $\mathcal{K}$ is a closed subspace of $\mathcal{I}$ and hence is the complement of countably many open intervals of $\mathcal{I}$. From each such interval J choose a distinguished element x_J which is not in $\mathcal{K}$, and define a map $\mathcal{K} \to \mathcal{K}_0$ by fixing each element of $\mathcal{K}_0$, and sending $x \in \mathcal{K} \cap$ $\cap J$ to the closest element of $\mathcal{K}_0$ to its right if $x > x_J$, to its left if $x < x_J$. We leave the reader to check that this map is continuous.

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MEASURES OF NONCOMPACTNESS, DARBO MAPS AND DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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1. Introduction

This paper presents existence theory for initial and boundary value problems in Banach spaces. Specifically we examine the initial value problem

$$(1.1)\qquad \begin{cases} y'(t) = f\big(t, y(t)\big), & t \in (0, T] \\ y(0) = a \in B \end{cases}$$

and the Dirichlet boundary value problem

$$(1.2)\qquad \begin{cases} y'' + \beta y' - \varepsilon y = f(t, y, y'), & 0 < t < 1 \\ y(0) = a \in B,\ y(1) = b \in B\,;\ \beta, \varepsilon \in \mathbf{R}. \end{cases}$$

Throughout B is a real Banach space with norm $|\cdot|$. In case $B = H$, a real Hilbert space, we denote its inner product by $\langle\cdot,\cdot\rangle$ and then $|x|^2 = \langle x, x\rangle$ for $x \in H$. $C^m\big([c,d], B\big)$ is the Banach space of functions $u : [c,d] \to B$ such that $u^{(m)}$ is continuous with norm

$$|u|_m = \max\Big\{|u|_0, |u'|_0, \dots, \big|u^{(m)}\big|_0\Big\}$$

where $|v|_0 = \max\big\{\big|v(t)\big| : t \in [c,d]\big\}$.

The boundary value problem (1.2) has been extensively examined; see [2, 5, 8, 9, 14, 16] and their references. It is well known [3, 8] that, in contrast to systems in $\mathbf{R}^n$, even the initial value problem may have no solution (local) in the Banach space case when the nonlinearity f is continuous. Various additional compactness conditions are needed to assure existence in the infinite dimensional setting. However in [12, 13] the case when f is continuous and satisfies a monotonicity type condition is discussed. This paper examines the situation when the nonlinearity f has a splitting of the form $g + h$, with h continuous and g satisfying some compactness condition. Our results improve, compliment and extend the existing theory in [9, 10, 13, 19, 21]. We remark as well that many other boundary conditions of Sturm–Liouville

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type could be treated; however since the strategy and ideas are similar, we choose as a result to omit the details. Also in section 2 the initial value problem (1.1) will be discussed with f having the above type splitting. Again the results extend and compliment the theory in [3, 8, 11, 15]. An extra feature of the technique here is that it yields a global as well as a local result.

The existence theory in this paper is based on a Leray–Schauder type nonlinear alternative [4, 7, 17]. Before we state the theorem let us recall some facts on measures of noncompactness [1, 7, 11, 14, 18]. Let E be a Banach space and Ω_E the bounded subsets of E. The *Kuratowski measure of noncompactness* is the map $\alpha : \Omega_E \to [0,\infty)$ defined by

$$\alpha(X) = \inf\big\{\delta : X \subseteqq \cup_{i=1}^{n} X_i \text{ and } \operatorname{diam}(X_i) \leqq \delta\big\}, \text{ here } X \subseteqq \Omega_E.$$

REMARK. This paper only uses Kuratowski's measure of noncompactness; the results however are valid for other measures of noncompactness.

For convenience we recall some properties [1, 14, 18] of α:

Let $S, T \in \Omega_E$. Then

(i) $\alpha(S) = 0$ iff $\overline{S}$ is compact;

(ii) $\alpha(\overline{S}) = \alpha(S)$;

(iii) if $S \subseteqq T$ then $\alpha(S) \leqq \alpha(T)$;

(iv) $\alpha(S \cup T) = \max\big\{\alpha(s), \alpha(T)\big\}$;

(v) $\alpha(rS) = |r|\alpha(S)$, $r \in \mathbf{R}$;

(vi) $\alpha(S+T) \leqq \alpha(S) + \alpha(T)$.

Let E_1 and E_2 be two Banach spaces and let $T : Y \subseteqq E_1 \to E_2$ map bounded sets onto bounded sets. We call T a *Darbo map* if T is continuous and $\alpha\big(T(X)\big) \leqq k_0\alpha(X)$, for some $0 \leqq k_0 < 1$, for all bounded sets $X \subseteqq Y$. We now state a nonlinear alternative which combines the classical Leray–Schauder fixed point theory [4, 6] with the fixed point theory of Krasnoselski and Sadovskii [18, 20].

THEOREM 1.1 (nonlinear alternative [4, 7, 17]). *Let C be a convex subset of a Banach space E. Suppose U is a nonempty bounded open set in C and $N : \overline{U} \to C$ is a Darbo map with $p \in U$. Then either*

(i) *N has a fixed point in $\overline{U}$; or*

(ii) *there is point $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda N u + (1-\lambda)p$.*

To conclude the introduction we gather together two standard results which will be used in this paper.

THEOREM 1.2 (Arzela–Ascoli theorem [11]). *Suppose M is a subset of $C\big([c,d], B\big)$. Then M is relatively compact in $C\big([c,d], B\big)$ iff M is bounded, equicontinuous and the set $\{f(t) : f \in M\}$ is relatively compact for each $t \in$ $\in [c,d]$.*

THEOREM 1.3 (Wirtinger's inequality [10]). (i) *Let H be a real Hilbert space and suppose $u : [c,d] \to H$ has a continuous derivative together with*

$u(c) = u(d) = 0$. *Then*

$$\pi^2 \int_c^d |u(t)|^2 \, dt \leqq (d-c)^2 \int_c^d |u'(t)|^2 \, dt.$$

(ii) *Let H be a real Hilbert space and suppose $u : [c,d] \to H$ has a continuous derivative together with $u(c) = 0$ or $u(d) = 0$. Then*

$$\pi^2 \int_c^d |u(t)|^2 \, dt \leqq 4(d-c)^2 \int_c^d |u'(t)|^2 \, dt.$$

2. Initial value problems

Consider the initial value problem

$$\begin{cases} y' = q(t)f(t,y), & t \in (0,T] \\ y(0) = a \in B \end{cases} \tag{2.1}$$

with

$$f : [0,T] \times \mathbf{B} \to \mathbf{B} \text{ continuous} \tag{2.2}$$

and

$$q \in C(0,T] \text{ with } q > 0 \text{ on } (0,T] \text{ and } \int_0^T q(s)\, ds < \infty. \tag{2.3}$$

REMARK. By a solution to (2.1) we mean a function $y \in C\big([0,T],B\big) \cap \cap C^1\big((0,T],B\big)$ which satisfies the differential equation on $(0,T]$ and the stated initial condition.

Associated with (2.1) we have the family of problems

$$\begin{cases} y' = \lambda q(t)f(t,y), & t \in (0,T] \\ y(0) = a \in B \end{cases} \tag{2.4$_\lambda$}$$

for $0 < \lambda < 1$.

REMARK. For notational purposes let

$$C_{\mathcal{B}}\big([0,T],B\big) = \big\{u \in C\big([0,T],B\big) \, : \, u(0) = a\big\}.$$

We begin this section by establishing an existence principle for problem (2.1).

THEOREM 2.1. *Suppose (2.2) and (2.3) are satisfied. In addition assume f has the splitting $f(t,u)=g(t,u)+h(t,u)$ with $g,h:[0,T]\times B\to B$ continuous together with*

$$(2.5)\qquad \begin{cases} \text{for each bounded (with respect to the supremum norm)} \\ \Omega \subseteqq C_B\big([0,T],B\big) \text{ and for each } t\in[0,T] \text{ the set} \\ \left\{ \int_0^t q(s)g\big(s,u(s)\big)\,ds : u\in\Omega \right\} \text{ is relatively compact,} \end{cases}$$

$$(2.6)\qquad \begin{cases} \text{for each } r>0 \text{ there exists } \phi\in C(0,T] \text{ with } \phi>0 \text{ on } (0,T] \text{ and} \\ \int_0^T q(s)\phi(s)\,ds<\infty \text{ such that } |z|\leqq r \text{ implies } \big|g(t,z)\big|\leqq\phi(t) \\ \text{for } t\in(0,T] \end{cases}$$

and

$$(2.7)\qquad \big|h(t,u)-h(t,v)\big|\leqq K|u-v|,\ t\in[0,T] \text{ and } u,v\in B.$$

Now suppose there is a constant M, independent of λ, with $|y|_0\leqq M$ for each solution y to $(2.4)_\lambda$. Then (2.1) has at least one solution $y\in C\big([0,T],B\big)\cap$ $\cap\, C^1\big((0,T],B\big)$.

REMARK. (i) If B is finite dimensional then (2.5) and (2.6) are automatically satisfied.

(ii) Note (2.5) and (2.6) are satisfied if $qg:[0,T]\times B\to B$ is completely continuous. To see this let $\Omega\subseteqq C_B\big([0,T],B\big)$ be bounded. Then there exists a compact set $A_0\subseteqq B$ such that $q(s)g\big(s,y(s)\big)\subseteqq A_0$ for all $s\in[0,T]$ and $y\in\Omega$. Fix $t\in(0,T]$ and notice

$$\frac{1}{t}\int_0^t q(s)g\big(s,y(s)\big)\,ds\in\overline{\mathrm{co}}(A_0)$$

which is compact; here $\mathrm{co}(A_0)$ is the convex hull of A_0. Thus (2.5) is true. Fix $r>0$. Then there exists a compact set $A_1\subseteqq B$ such that $q(s)g\big(s,y(s)\big)\subseteqq A_1$ for all $s\in[0,T]$ and y with $|y|\leqq r$. Now since A_1 is bounded we immediately have (2.6).

PROOF. Throughout let $C\big([0,T],B\big)$ be the Banach space of functions $u\in C\big([0,T],B\big)$ with norm

$$|u|_K=\max_{[0,T]}\big|e^{-KQ(t)}u(t)\big|$$

where $Q(t)=\int_0^t q(s)\,ds$.

REMARK. $|u|_0$ and $|u|_K$ are equivalent since $e^{-KQ(T)}|u|_0 \leqq |u|_K \leqq |u|_0$.

Solving $(2.4)_\lambda$ is equivalent to finding a $y \in C_{\mathcal{B}}([0,T],B)$ which satisfies

$$(2.8) \qquad y(t) = a + \lambda \int_0^t q(s)f(s, y(s))\, ds =$$

$$= (1-\lambda)a + \lambda \left[a + \int_0^t q(s)f(s,y(s))\, ds\right].$$

Define a mapping $N : C_{\mathcal{B}}([0,T],B) \to C_{\mathcal{B}}([0,T],B)$ by

$$Ny(t) = a + \int_0^t q(s)f(s,y(s))\, ds.$$

Thus $(2.4)_\lambda$ (i.e (2.8)) is equivalent to the fixed point problem $y = (1-\lambda)a + \lambda Ny$. We *claim* that $N : C_{\mathcal{B}}([0,T],B) \to C_{\mathcal{B}}([0,T],B)$ is a Darbo map. Suppose the claim is true, then set

$$U = \{u \in C_{\mathcal{B}}([0,T],B) : |u|_K < M_0 + 1\}, \quad C = C_{\mathcal{B}}([0,T],B),$$

$$E = \big(C([0,T],B), |\cdot|_K\big)$$

with $M_0 = \max\{|a|, M\}$ and apply Theorem 1.1 to deduce that N has a fixed point i.e. (2.1) has a solution $y \in C([0,T],B)$. In addition we have $y \in C^1((0,T],B)$ from (2.8) with $\lambda = 1$. Consequently the theorem is proven once we show $N : C_{\mathcal{B}}([0,T],B) \to C_{\mathcal{B}}([0,T],B)$ is a Darbo map. Let $Nu(t) = \big(a + N_1u(t)\big) + N_2u(t) - a \equiv N_3u(t) + N_2u(t) - a$ where

$$N_1u(t) = \int_0^t q(s)g(s,u(s))\, ds \quad \text{and} \quad N_2u(t) = a + \int_0^t q(s)h(s,u(s))\, ds.$$

Certainly $N : C_{\mathcal{B}}([0,T],B) \to C_{\mathcal{B}}([0,T],B)$ is continuous. Now let $\Omega \subseteqq C_{\mathcal{B}}([0,T],B)$ be bounded (with respect to $|\cdot|_K$). Then

$$N(\Omega) \subseteqq N_3(\Omega) + N_2(\Omega) - a$$

and this together with the properties of the measure of noncompactness (stated in the introduction) yields

$$\alpha\big(N(\Omega)\big) \leqq \alpha\big(N_3(\Omega) + N_2(\Omega) - a\big) \leqq \alpha\big(N_3(\Omega)\big) + \alpha\big(N_2(\Omega)\big) + \alpha(-a).$$

Thus

$$\alpha\big(N(\Omega)\big) \leqq \alpha\big(N_3(\Omega)\big) + \alpha\big(N_2(\Omega)\big). \tag{2.9}$$

We next show that $N_3 : C_{\mathcal{B}}\big([0,T],B\big) \to C_{\mathcal{B}}\big([0,T],B\big)$ is completely continuous. This is immediate once we show that $N_1 : C_{\mathcal{B}}\big([0,T],B\big) \to C_0\big([0,T],B\big)$ is completely continuous; here $C_0\big([0,T],B\big) = \{u \in C\big([0,T],B\big)\,;\, u(0) = 0\}$. To see this we apply the Arzela–Ascoli theorem. The boundedness of $N_1(\Omega)$ is easy and the equicontinuity on $[0,T]$ follows from (2.6) and the following inequality (here $u \in \Omega$ and $0 \leqq s \leqq t \leqq T$):

$$\big|\,N_1u(t) - N_1u(s)\big| \leqq \int_s^t q(z)\,\big|g\big(z,u(z)\big)\,\big|\,dz.$$

The above together with (2.5) and the Arzela–Ascoli theorem (Theorem 1.2) implies that N_1 is completely continuous. Consequently $N_3 : C_{\mathcal{B}}\big([0,T],B\big) \to$ $\to C_{\mathcal{B}}\big([0,T],B\big)$ is completely continuous and so $\alpha\big(N_3(\Omega)\big) = 0$. Thus (2.9) reduces to

$$\alpha\big(N(\Omega)\big) \leqq \alpha\big(N_2(\Omega)\big). \tag{2.10}$$

Also for $u, v \in \Omega$ and $t \in [0,T]$ we have

$$|N_2u - N_2v|_K = \max_{[0,T]}\left|e^{-KQ(t)}\int_0^t q(s)\left[h\big(s,u(s)\big) - h\big(s,v(s)\big)\right]\,ds\right| \leqq$$

$$\leqq K\max_{[0,T]}\left|e^{-KQ(t)}\int_0^t q(s)e^{KQ(s)}e^{-KQ(s)}\big|u(s) - v(s)\big|\,ds\right| \leqq$$

$$\leqq |u - v|_K \max_{[0,T]}\left|e^{-KQ(t)}\left[e^{KQ(t)} - 1\right]\right| =$$

$$= |u - v|_K\left[1 - e^{-KQ(T)}\right] \equiv k_0|u - v|_K,$$

with $k_0 < 1$. This together with (2.10) implies

$$\alpha\big(N(\Omega)\big) \leqq k_0\alpha(\Omega).$$

Thus N is a Darbo map and we are finished. □

THEOREM 2.2. *Suppose* (2.2) *and* (2.3) *are satisfied. In addition assume f has the splitting $f(t,u)=g(t,u)+h(t,u)$ with $g,h:[0,T]\times\mathbf{B}\to\mathbf{B}$ continuous together with* (2.5) *and* (2.7) *holding. Also suppose*

$$(2.11)\quad \begin{cases} \text{there is a continuous nondecreasing function } \psi:[0,\infty)\to(0,\infty) \\ \text{and a function } \phi\in C(0,T],\ \phi>0 \text{ on } (0,T] \text{ and} \\ \displaystyle\int_0^T \phi(s)q(s)\,ds<\infty, \text{ with } \left|g(t,u)\right| \leqq \phi(t)\psi\big(|u|\big) \\ \text{for } t\in(0,T] \text{ and } u\in B \end{cases}$$

and

$$(2.12)\quad \begin{cases} \displaystyle\int_0^T r(s)\,ds<\int_{|a|}^{\infty}\frac{dx}{Kx+\psi(x)+L_0} \text{ where } L_0=\sup_{[0,T]}\left|h(t,0)\right| \\ \text{and } r(s)=\max\left\{q(s)\phi(s),q(s)\right\}. \end{cases}$$

Then (2.1) *has at least one solution $y\in C\big([0,T],B\big)\cap C^1\big((0,T],B\big)$.*

REMARK. Notice that (2.11) implies (2.6).

PROOF. Let y be a solution to $(2.4)_\lambda$. Now $y(t)-y(0)=\int_0^t y'(s)\,ds$ which yields

$$(2.13)\qquad \left|y(t)\right| \leqq |a|+\int_0^t \left|y'(s)\right| ds \equiv \theta(t).$$

The fact that $\theta'(t)=\left|y'(t)\right|$ together with the differential equation yields

$$(2.14)\qquad \theta'(t) \leqq q(t)\phi(t)\psi\big(|y(t)|\big)+q(t)\left|h\big(t,y(t)\big)\right|.$$

In addition (2.7) yields

$$\left|h(t,u)\right| \leqq K|u|+\left|h(t,0)\right| \leqq K|u|+L_0 \quad \text{where} \quad L_0=\sup_{[0,T]}\left|h(t,0)\right|.$$

This together with (2.14) will give

$$\theta'(t) \leqq q(t)\phi(t)\psi\big(|y(t)|\big)+Kq(t)\left|y(t)\right|+q(t)L_0 \leqq$$

$$\leqq q(t)\phi(t)\psi\big(\theta(t)\big)+Kq(t)\theta(t)+q(t)L_0 \leqq r(t)\left[\psi\big(\theta(t)\big)+K\theta(t)+L_0\right]$$

since ψ is nondecreasing. Consequently

$$\int_{|a|}^{\theta(t)} \frac{dx}{Kx+\psi(x)+L_0} = \int_0^t \frac{\theta'(s)\,ds}{K\theta(s)+\psi\big(\theta(s)\big)+L_0} \leqq$$

$$\leqq \int_0^t r(s)\,ds \leqq \int_0^T r(s)\,ds.$$

Let $J(z) = \int_{|a|}^{z} \frac{dx}{Kx+\psi(x)+L_0}$ and so $\theta(t) \leqq J^{-1}\left(\int_0^T r(s)\,ds\right) \equiv M$. This together with (2.13) yields $\left|y(t)\right| \leqq M$ for $t \in [0,T]$. Existence of a solution to (2.1) now follows from Theorem 2.1. □

REMARK. (i) If $B = H$, a real Hilbert space, then the assumption that ψ is nondecreasing can be deleted in Theorem 2.2. To see this suppose $\left|y(t)\right| >$ $> |a|$ for some $t \in (0,T]$. Then there exists $(\eta,t) \subseteqq (0,T)$ with $\left|y(t)\right| > |a|$ on (η,t) and $\left|y(\eta)\right| = |a|$. In addition $\left|y(s)\right|' \leqq \left|y'(s)\right|$ for $s \in (\eta,t)$ and this together with the differential equation yields

$$\left|y(s)\right|' \leqq \left|y'(s)\right| \leqq q(s)\phi(s)\psi\big(|y(s)|\big) + Kq(s)\left|y(s)\right| + q(s)L_0.$$

Consequently

$$\int_{|a|}^{|y(t)|} \frac{dx}{Kx+\psi(x)+L_0} = \int_0^t \frac{\left|y(s)\right|'\,ds}{K\left|y(s)\right|+\psi\big(|y(s)|\big)+L_0} \leqq$$

$$\leqq \int_\eta^t r(s)\,ds \leqq \int_0^T r(s)\,ds$$

and the result follows as before.

(ii) In fact if $B = H$, a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, then (2.11) and (2.12) can be replaced by (2.6) and

$$(2.11)^\star \quad \begin{cases} \text{there is a continuous function } \psi:[0,\infty)\to(0,\infty) \text{ and} \\ \text{a function } \phi \in C(0,T],\ \phi>0 \text{ on } (0,T] \text{ and} \\ \displaystyle\int_0^T \phi(s)q(s)\,ds < \infty, \text{ with } \big\langle u, g(t,u)\big\rangle \leqq \phi(t)\psi\big(|u|\big) \\ \text{for } t \in (0,T] \text{ and } u \in B \end{cases}$$

and

$$(2.12)^\star \quad \begin{cases} \displaystyle\int_0^T r(s)\,ds < \int_{|a|}^{\infty} \frac{x\,dx}{Kx^2+\psi(x)+L_0x} \\ \text{where } r \text{ is as defined in (2.12)} \end{cases}$$

in Theorem 2.2, and existence of a solution to (2.1) is again guaranteed. To see this let (η, t) be as in Remark (i). Then the differential equation and $(2.11)^{\star}$ yield

$$\langle y, y' \rangle \leqq q(t)\phi(t)\psi(|y|) + q(t)|y||h(t,y)| \leqq r(t)\left(\psi(|y|) + K|y|^2 + L_0|y|\right).$$

Consequently

$$\int_{|a|}^{|y(t)|} \frac{x\,dx}{Kx^2 + \psi(x) + L_0 x} \leqq \int_{\eta}^{t} r(s)\,ds$$

and the result follows.

3. Boundary value problems

Consider the Dirichlet boundary value problem

$$\text{(3.1)} \qquad \begin{cases} y'' + \beta y' - \varepsilon y = q(t) f(t, y, y'), & 0 < t < 1 \\ y(0) = a \in B,\ y(1) = b \in B;\ \beta, \varepsilon \in \mathbf{R} \end{cases}$$

with

(3.2) $f : [0,1] \times \mathbf{B}^2 \to \mathbf{B}$ continuous and $4\varepsilon + \beta^2 \neq 4(n\pi)^2$, $n = -1, -2, \ldots$

and

$$\text{(3.3)} \qquad q \in C(0,1) \text{ with } q > 0 \text{ on } (0,1) \text{ and } \int_0^1 q(s)\,ds < \infty.$$

Throughout this section let $w, \tau : [0,1] \to [0, \infty)$ be continuous functions with $\tau > 0$ on $[0,1]$ and $w > 0$ on $(0,1)$. In addition assume there exists a constant $N_0 > 0$ with $|u|_\star \leqq N_0 |u|_1$ where

$$|u|_\star = \max\left\{ \sup_{[0,1]} \frac{|u(x)|}{w(x)},\ \sup_{[0,1]} \frac{|u'(x)|}{\tau(x)} \right\}.$$

Also we let $\left(C^1([0,1], B), |.|_\star\right)$ denote the Banach space of functions $u \in C^1([0,1], B)$ with norm $|u|_\star$.

REMARKS. (i) Note $\frac{|u|_1}{m} \leqq |u|_\star \leqq N_0 |u|_1$ where $m = \max\left\{ \max_{[0,1]} w(x), \max_{[0,1]} \tau(x) \right\}$.

(ii) We could of course take $w = \tau = 1$ throughout this section. However in some cases it may be beneficial to take $w(x) = \sin(\pi x)$, $\tau(x) = 2w(x) + (1-2x)w'(x)$. In this case $N_0 = \frac{1}{2}$. To see this notice for $t \in [0, \frac{1}{2}]$ we have $|u(t)| = |\int_0^t u'(s)\,ds| \leqq t|u'|_0$ and consequently $\frac{|u(t)|}{w(t)} \leqq \frac{t|u'|_0}{\sin(\pi t)} \leqq \frac{1}{2}|u'|_0$ for $t \in [0, \frac{1}{2}]$. On the other hand for $t \in [\frac{1}{2}, 1]$ we have $|u(t)| \leqq (1-t)|u'|_0$ so $\frac{|u(t)|}{w(t)} \leqq \frac{(1-t)|u'|_0}{\sin(\pi t)} \leqq \frac{1}{2}|u'|_0$ for $t \in [\frac{1}{2}, 1]$. Thus $\sup_{[0,1]} \frac{|u(t)|}{w(t)} \leqq \frac{1}{2}|u'|_0$ and it is easy to check that $\sup_{[0,1]} \frac{|u'(t)|}{\tau(t)} \leqq \frac{1}{2}|u'|_0$ since $\min_{[0,1]} \tau(t) = 2$. Thus $N_0 = \frac{1}{2}$.

(iii) The ideas in this section can be extended to the situation when we have Sturm–Liouville boundary data i.e. $-\alpha_1 y(0) + \beta_1 y'(0) = c$, $a_1 y(1) + b_1 y'(1) = d$. Here $\alpha_1^2 + \beta^2 > 0$, $a_1^2 + b_1^2 > 0$, $\alpha_1, \beta_1, a_1, b_1 \geqq 0$ and $a_1\beta_1 + b_1\alpha_1 + a_1\alpha_1 > 0$.

By a solution to (3.1) we mean a function $y \in C^1([0,1], B) \cap C^2((0,1), B)$ which satisfies the differential equation on $(0,1)$ and the above stated boundary condition. Associated with (3.1) we have the family of problems

$$(3.4)_\lambda \qquad \begin{cases} y'' + \beta y' - \varepsilon y = \lambda q(t) f(t, y, y'), & 0 < t < 1 \\ y(0) = a \in B,\ y(1) = b \in B\,;\ \beta, \varepsilon \in \mathbf{R} \end{cases}$$

with $0 < \lambda < 1$.

REMARK. For notational purposes let $C_B^1([0,1], B) = \{u \in C^1([0,1], B) : u(0) = a, u(1) = b\}$ and $C_0([0,1], B) = \{u \in C([0,1], B) : u(0) = 0\}$.

THEOREM 3.1. *Suppose* (3.2) *and* (3.3) *are satisfied. In addition assume f has the splitting $f(t,u,v) = g(t,u,v) + h(t,u,v)$ with $g, h : [0,1] \times B^2 \to B$ continuous together with*

$$(3.5) \qquad \begin{cases} \text{for each bounded (with respect to the supremum norm)} \\ \Omega \subseteqq C_B^1([0,1], B) \text{ and for each } t \in [0,1] \text{ the set} \\ \left\{ \int_0^t q(s) g(s, u(s), u'(s))\, ds : u \in \Omega \right\} \text{ is relatively compact,} \end{cases}$$

$$(3.6) \qquad \begin{cases} \text{for each } r > 0 \text{ there exists } \phi \in C(0,1) \text{ with } \phi > 0 \text{ on } (0,1) \text{ and} \\ \int_0^1 q(s)\phi(s)\, ds < \infty \text{ such that } |z| \leqq r,\ |v| \leqq r \text{ imply} \\ |g(t,z,v)| \leqq \phi(t) \text{ for } t \in (0,1) \end{cases}$$

and

$$(3.7)\quad \begin{cases} q(t)\big|h(t,u_1,v_1)-h(t,u_2,v_2)\big| \leqq K_1p_1(t)|u_1-u_2|+ \\ +K_2p_2(t)|v_1-v_2|, \quad t\in[0,1] \text{ with } u_1,u_2,v_1,v_2\in B. \\ \text{Here } p_1,p_2\in C(0,1) \text{ with } p_1,p_2>0 \text{ on } (0,1) \text{ and} \\ \int_0^1 p_i(s)\,ds<\infty,\ i=1,2. \text{ Also there exist} \\ \text{constants } k_0<1 \text{ and } k_1<1 \text{ with} \\ \int_0^1 \big|G(t,s)\big|\big\{K_1p_1(s)w(s)+K_2p_2(s)\tau(s)\big\}\,ds \leqq k_0w(t) \text{ and} \\ \int_0^1 \big|G_t(t,s)\big|\big\{K_1p_1(s)w(s)+K_2p_2(s)\tau(s)\big\}\,ds \leqq k_1\tau(t). \\ \text{Here } G(t,s) \text{ is the Green's function associated with the problem} \\ y''+\beta y'-\varepsilon y=0,\ y(0)=y(1)=0. \end{cases}$$

Now suppose there is a constant M, independent of λ, with $|y|_1 \leqq M$ for each solution y to $(3.4)_\lambda$. Then (3.1) has at least one solution $y\in C^1\big([0,1],B\big)\cap$ $\cap\, C^2\big((0,1),B\big)$.

REMARKS. (i) Of course we could take $w=1$ and $\tau=1$ and so (3.7) gives a condition for K_1 and K_2 to satisfy. However consider the case $\beta=0$, $\varepsilon=0$ and $p_1=p_2=1$. In this case a less restrictive condition on K_1 and K_2 can be obtained if we take $w(x)=\sin(\pi x)$ and $\tau(x)=2w(x)+(1-2x)w'(x)$. Here $G(t,s)=(1-t)s$ if $0\leqq s\leqq t$ whereas $G(t,s)=t(1-s)$ if $t\leqq s\leqq 1$. It is easy to check using integration by parts with the fact that $w''=-\pi^2w$ and $\tau''+4w''=-\pi^2\tau$ that $\int_0^1\big|G(t,s)\big|w(s)\,ds=\frac{w(t)}{\pi^2}$ and $\int_0^1\big|G_t(t,s)\big|w(s)\,ds=$ $=\frac{\tau(t)}{\pi^2}$ together with

$$\int_0^1\big|G(t,s)\big|\tau(s)\,ds\leqq\frac{4w(t)}{\pi^2} \quad\text{and}\quad \int_0^1\big|G_t(t,s)\big|\tau(s)\,ds\leqq\frac{4\tau(t)}{\pi^2}.$$

Thus (3.7) reduces to assuming that $K_1+4K_2<\pi^2$.

(ii) If $qg:[0,1]\times B^2\to B$ is completely continuous then (3.5) and (3.6) are satisfied.

PROOF. Solving $(3.4)_\lambda$ is equivalent to finding a $y\in C_B^1\big([0,1],B\big)$ which satisfies

$$(3.8)\quad y'(t)-y'(0)+\beta y(t)-\beta a-\varepsilon\int_0^t y(s)\,ds=\lambda\int_0^t q(s)f\big(s,y(s),y'(s)\big)\,ds.$$

Define mappings $F,T : C^1_B([0,1],B) \to C_0([0,1],B)$ by

$$Ty(t) = y'(t) - y'(0) + \beta y(t) - \beta a - \varepsilon \int_0^t y(s)\,ds$$

and

$$Fy(t) = \int_0^t q(s)f\bigl(s, y(s), y'(s)\bigr)\,ds.$$

We first *claim* that T^{-1} is continuous. Let $k \in C_0([0,1],B)$. The difference y of two solutions to $Tz = k$ satisfies $y'' + \beta y' - \varepsilon y = 0$, $y(0) = y(1) = 0$ and consequently $y \equiv 0$ since $4\varepsilon + \beta^2 \neq 4(n\pi)^2$, $n = -1, -2, \dots$. Hence T is one to one. To see that T is onto there are five cases to consider.

Case (i): $4\varepsilon + \beta^2 < 0$. Then the equation $Ty(t) = k(t)$ has the solution

$$(3.9)\qquad y(t) = \frac{be^{\frac{\beta}{2}} - a\cos\left(\frac{\mu}{2}\right)}{\sin\left(\frac{\mu}{2}\right)} e^{\frac{-\beta t}{2}} \sin\left(\frac{\mu t}{2}\right) + ae^{\frac{-\beta t}{2}}\cos\left(\frac{\mu t}{2}\right) -$$

$$- \left(\frac{\beta^2+\mu^2}{2\gamma}\right) e^{\frac{-\beta t}{2}} \sin\left(\frac{\mu(1-t)}{2}\right) \int_0^t e^{\frac{\beta s}{2}} \left[\mu\cos\left(\frac{\mu s}{2}\right) + \right.$$

$$\left. +\beta \sin\left(\frac{\mu s}{2}\right)\right] k(s)\,ds -$$

$$- \left(\frac{\beta^2+\mu^2}{2\gamma}\right) e^{\frac{-\beta t}{2}} \sin\left(\frac{\mu t}{2}\right) \int_t^1 e^{\frac{\beta s}{2}} \left[-\mu\cos\left(\frac{\mu(1-s)}{2}\right) + \right.$$

$$\left. +\beta \sin\left(\frac{\mu(1-s)}{2}\right)\right] k(s)\,ds$$

where $\mu = \sqrt{-(4\varepsilon + \beta^2)}$ and $\gamma = -\left(\frac{\beta^2+\mu^2}{2}\right)\mu \sin\left(\frac{\mu}{2}\right)$.

REMARK. One can check directly that the y given in (3.9) satisfies $(Ty)(t) = k(t)$. One way to construct the solution is to notice that $y = y_1 + y_2$ where $y_1 = \frac{be^{\frac{\beta}{2}} - a\cos\left(\frac{\mu}{2}\right)}{\sin\left(\frac{\mu}{2}\right)} e^{\frac{-\beta t}{2}} \sin\left(\frac{\mu t}{2}\right) + ae^{\frac{-\beta t}{2}}\cos\left(\frac{\mu t}{2}\right)$ satisfies $y'' + \beta y' - \varepsilon y = 0$, $y(0) = a$, $y(1) = b$, and $y_2 = y_3'$ where y_3 satisfies $y'' + \beta y' - \varepsilon y = k$, $y'(0) = y'(1) = 0$ (of course the construction of y_3 is easy using Green's functions). Consequently $y(0) = 0 + a = a$, $y(1) = 0 + b = b$ and

$$y'(t) - y'(0) + \beta y(t) - \beta a - \varepsilon \int_0^t y(s)\,ds = \left[y_3''(t) + \beta y_3'(t) - \varepsilon y_3(t)\right] -$$

$$-\left[y_3''(0)+\beta y_3'(0)-\varepsilon y_3(0)\right]+y_1'(t)-y_1'(0)+\beta\big(y_1(t)-a\big)-\varepsilon\int_0^t y_1(s)\,ds=$$

$$=k(t)-k(0)+0=k(t).$$

Thus T is invertible and the continuity of T^{-1} is immediate. Note as well that $(T^{-1}k)(s)$ is equal to the right hand side of (3.9) for any $k\in C_0\big([0,1],B\big)$.

Case (ii): $4\varepsilon+\beta^2>0$ and $\varepsilon\neq 0$. Then the equation $Ty(t)=k(t)$ has the solution

$$(3.10)\quad y(t)=\frac{be^{\frac{\beta}{2}}-a\cosh\left(\frac{\delta}{2}\right)}{\sinh\left(\frac{\delta}{2}\right)}e^{\frac{-\beta t}{2}}\sinh\left(\frac{\delta t}{2}\right)+ae^{\frac{-\beta t}{2}}\cosh\left(\frac{\delta t}{2}\right)+$$

$$+\left(\frac{\delta^2-\beta^2}{2\eta}\right)e^{\frac{-\beta t}{2}}\sinh\left(\frac{\delta(1-t)}{2}\right)\int_0^t e^{\frac{\beta s}{2}}\left[-\delta\cosh\left(\frac{\delta s}{2}\right)+\right.$$

$$\left.+\beta\sinh\left(\frac{\delta s}{2}\right)\right]k(s)\,ds+$$

$$+\left(\frac{\delta^2-\beta^2}{2\eta}\right)e^{\frac{-\beta t}{2}}\sinh\left(\frac{\delta t}{2}\right)\int_t^1 e^{\frac{\beta s}{2}}\left[-\delta\cosh\left(\frac{\delta(1-s)}{2}\right)+\right.$$

$$\left.+\beta\sinh\left(\frac{\delta(1-s)}{2}\right)\right]k(s)\,ds$$

where $\delta=\sqrt{4\varepsilon+\beta^2}$ and $\eta=\left(\frac{\delta^2-\beta^2}{2}\right)\delta\sinh\left(\frac{\delta}{2}\right)=2\varepsilon\delta\sinh\left(\frac{\delta}{2}\right)$.

REMARK. Again $y=y_1+y_2$ where y_1 satisfies $y''+\beta y'-\varepsilon y=0$, $y(0)=$ $=a$, $y(1)=b$, and $y_2=y_3'$ where y_3 satisfies $y''+\beta y'-\varepsilon y=k$, $y'(0)=y'(1)=$ $=0$ (note the Green's function exists since $\varepsilon\neq 0$ and $4\varepsilon+\beta^2\neq 4(n\pi)^2$, $n=$ $=-1,-2,\dots$).

Thus T^{-1} is continuous and $(T^{-1}k)(s)$ is equal to the right hand side of (3.10) for any $k\in C_0\big([0,1],B\big)$.

Case (iii): $\varepsilon=0$ and $\beta\neq 0$. In this case $Ty(t)=y'(t)-y'(0)+\beta y(t)-\beta a$. Then the equation $Ty(t)=k(t)$ has the solution

$$(3.11)\qquad y(t)=\frac{b-ae^{-\beta}}{1-e^{-\beta}}+\frac{(a-b)e^{-\beta t}}{1-e^{-\beta}}+$$

$$+\frac{e^{-\beta}(e^{-\beta t}-1)}{1-e^{-\beta}}\int_0^1 e^{\beta s}k(s)\,ds+e^{-\beta t}\int_0^t e^{\beta s}k(s)\,ds.$$

REMARK. Note $y_1 = \frac{b-ae^{-\beta}}{1-e^{-\beta}} + \frac{(a-b)e^{-\beta t}}{1-e^{-\beta}}$ satisfies $y'' + \beta y' = 0$, $y(0) = a$, $y(1) = b$.

Thus T^{-1} is continuous and $(T^{-1}k)(s)$ is equal to the right hand side of (3.11).

Case (iv): $4\varepsilon + \beta^2 = 0$ with $\varepsilon \neq 0$. In this case $Ty(t) = k(t)$ has the solution

$$y(t) = ae^{\frac{-\beta t}{2}} + (be^{\frac{\beta t}{2}} - a)te^{\frac{-\beta t}{2}} + \tag{3.12}$$

$$+\frac{e^{\frac{-\beta(t-1)}{2}}(1-t)}{2e^{\frac{\beta}{2}}} \int_0^t e^{\frac{\beta s}{2}}(2+\beta s)k(s)\,ds - \frac{te^{\frac{-\beta t}{2}}}{2} \int_t^1 e^{\frac{\beta s}{2}}\left(2-\beta(1-s)\right)k(s)\,ds.$$

REMARK. Here $y = y_1 + y_2$ where $y_1 = ae^{\frac{-\beta t}{2}} + (be^{\frac{\beta t}{2}} - a)te^{\frac{-\beta t}{2}}$ satisfies $y'' + \beta y' - \varepsilon y = 0$, $y(0) = a$, $y(1) = b$, and $y_2 = y_3'$ where y_3 satisfies $y'' + \beta y' - \varepsilon y = k$, $y'(0) = y'(1) = 0$.

Thus T^{-1} is continuous and $(T^{-1}k)(s)$ is equal to the right hand side of (3.12).

Case (v): $\varepsilon = 0$ with $\beta = 0$. In this case $Ty(t) = y'(t) - y'(0)$. Then the equation $Ty(t) = k(t)$ has the solution

$$y(t) = a + (b-a)t - t\int_0^1 k(s)\,ds + \int_0^t k(s)\,ds. \tag{3.13}$$

REMARK. Note $y_1 = a + (b-a)t$ satisfies $y'' = 0$, $y(0) = a, y(1) = b$.

Thus T^{-1} is continuous and $(T^{-1}k)(s)$ is equal to the right hand side of (3.13).

Thus $(3.4)_\lambda$ is equivalent (see (3.9), (3.10), (3.11), (3.12), (3.13)) to the fixed point problem

$$y = T^{-1}(\lambda Fy) = (1-\lambda)y_1 + \lambda T^{-1}Fy \equiv (1-\lambda)p + \lambda Ny \tag{3.14}$$

where y_1 is as described in cases (i) to (v) above and $N = T^{-1}F$. We now *claim* that $N : C^1_B([0,1], B) \to C^1_B([0,1], B)$ is a Darbo map. Suppose the claim is true, then set

$$U = \{u \in C^1_B([0,1], B) : |u|_\star < M_0 + 1\},$$

$$C = C^1_B([0,1], B), \quad E = (C^1([0,1], B), |\cdot|_\star)$$

with $M_0 = \max\left\{\sup_{[0,1]}|y_1(t)|, MN_0\right\}$, where N_0 is as described at the beginning of this section. Now apply Theorem 1.1 to deduce that N has a fixed point i.e. (3.1) has a solution $y \in C^1([0,1],B)$. In addition we have $y \in C^2((0,1),B)$ from (3.8) with $\lambda = 1$. So it remains to show that N is a Darbo map. Firstly $N : C^1_B([0,1],B) \to C^1_B([0,1],B)$ is continuous. Let $\Omega \subseteqq C^1_B([0,1],B)$ be bounded (with respect to $|\cdot|_\star$). Also let $Fu(t) = F_1u(t) + F_2u(t)$ where $F_1, F_2 : C^1_B([0,1],B) \to C_0([0,1],B)$ are defined by

$$F_1u(t) = \int_0^t q(s)g\big(s,u(s),u'(s)\big)\,ds \text{ and } F_2u(t) = \int_0^t q(s)h\big(s,u(s),u'(s)\big)\,ds.$$

Then

$$F(\Omega) \subseteqq F_1(\Omega) + F_2(\Omega)$$

and consequently

$$\text{(3.15)}\qquad N(\Omega) = T^{-1}\big(F(\Omega)\big) \subseteqq T^{-1}\big(F_1(\Omega) + F_2(\Omega)\big) \subseteqq$$
$$\subseteqq T^{-1}\big(F_1(\Omega)\big) + T^{-1}\big(F_2(\Omega)\big) + (-y_1)$$

since if $x \in T^{-1}\big(F_1(\Omega) + F_2(\Omega)\big)$ there exist $z_1, z_2 \in \Omega$ with $x(t) = T^{-1}\left(F_1\big(z_1(t)\big) + F_2\big(z_2(t)\big)\right)$ and consequently $x(t) = T^{-1}\left(F_1\big(z_1(t)\big)\right) + T^{-1}\left(F_2\big(z_2(t)\big)\right) - y_1(t)$ from (3.9), (3.10), (3.11), (3.12) and (3.13). Now (3.15) together with the properties of the measure of noncompactness yields

$$\alpha\big(N(\Omega)\big) \leqq \alpha\big(T^{-1}F_1(\Omega) + T^{-1}F_2(\Omega) + (-y_1)\big) \leqq$$
$$\leqq \alpha\big(T^{-1}F_1(\Omega)\big) + \alpha\big(T^{-1}F_2(\Omega)\big) + \alpha(-y_1).$$

Thus

$$\text{(3.16)}\qquad \alpha\big(N(\Omega)\big) \leqq \alpha\big(T^{-1}F_1(\Omega)\big) + \alpha\big(T^{-1}F_2(\Omega)\big).$$

We next show that $F_1 : C^1_B([0,1],B) \to C_0([0,1],B)$ is completely continuous. To see this we apply the Arzela–Ascoli theorem. Clearly $F_1(\Omega)$ is bounded and the equicontinuity on $[0,1]$ follows from (3.6) and the following inequality (here $u \in \Omega$ and $0 \leqq s \leqq t \leqq 1$),

$$|F_1u(t) - F_1u(s)| \leqq \int_s^t q(z)\,\big|g\big(z,u(z),u'(z)\big)\big|\,dz.$$

The above together with (3.5) and the Arzela–Ascoli theorem (Theorem 1.2) implies that F_1 is completely continuous. In addition since T^{-1} is continuous we have $T^{-1}F_1 : C^1_B([0,1],B) \to C^1_B([0,1],B)$ completely continuous. Consequently $\alpha(T^{-1}F_1(\Omega)) = 0$ and so (3.16) reduces to

$$\alpha(N(\Omega)) \leqq \alpha(T^{-1}F_2(\Omega)). \tag{3.17}$$

Now for $u, v \in \Omega$ and $t \in [0,1]$ we have

$$\left|T^{-1}F_2u(t) - T^{-1}F_2v(t)\right| =$$

$$= \left|T^{-1}\left(\int_0^t q(s)h(s,u,u')\,ds\right) - T^{-1}\left(\int_0^t q(s)h(s,v,v')\,ds\right)\right| =$$

$$= \left|\int_0^1 G(t,s)q(s)h(s,u,u')\,ds - \int_0^1 G(t,s)q(s)h(s,v,v')\,ds\right| =$$

$$= \left|\int_0^1 G(t,s)q(s)\left[h(s,u(s),u'(s)) - h(s,v(s),v'(s))\right]\,ds\right|$$

using (3.9), (3.10), (3.11), (3.12), (3.13) and changing the order of integration once; here $G(t,s)$ is the Green's function associated with $y'' + \beta y' - \varepsilon y = 0$, $y(0) = y(1) = 0$. Also for $u, v \in \Omega$ and $t \in [0,1]$ we have

$$\left|(T^{-1}F_2)'u(t) - (T^{-1}F_2)'v(t)\right| =$$

$$= \left|\int_0^1 G_t(t,s)q(s)\left[h(s,u(s),u'(s)) - h(s,v(s),v'(s))\right]\,ds\right|.$$

The above together with (3.7) implies

$$\left|\frac{T^{-1}F_2u(t)}{w(t)} - \frac{T^{-1}F_2v(t)}{w(t)}\right| \leqq$$

$$\leqq \frac{1}{w(t)}\int_0^1 |G(t,s)|\left\{K_1p_1(s)|u(s)-v(s)| + K_2p_2(s)|u'(s)-v'(s)|\right\}ds \leqq$$

$$\leqq \frac{1}{w(t)}\int_0^1 |G(t,s)|\left\{K_1p_1(s)\frac{|u(s)-v(s)|}{w(s)}w(s) + \right.$$

$$\left. + K_2p_2(s)\frac{|u'(s)-v'(s)|}{\tau(s)}\tau(s)\right\}ds \leqq$$

$$\leqq |u-v|_\star\left(\frac{1}{w(t)}\int_0^1 |G(t,s)|\left\{K_1p_1(s)w(s)+\right.\right.$$

$$\left.\left.+K_2p_2(s)\tau(s)\right\}ds\right) \leqq k_0|u-v|_\star$$

where k_0 is as described in (3.7). Thus

$$\sup_{[0,1]}\left|\frac{T^{-1}F_2u(t)}{w(t)}-\frac{T^{-1}F_2v(t)}{w(t)}\right| \leqq k_0|u-v|_\star. \tag{3.18}$$

Also

$$\left|\frac{(T^{-1}F_2)'u(t)}{\tau(t)}-\frac{(T^{-1}F_2)'v(t)}{\tau(t)}\right| \leqq$$

$$\leqq \frac{1}{\tau(t)}\int_0^1 |G_t(t,s)|\left\{K_1p_1(s)\frac{|u(s)-v(s)|}{w(s)}w(s)+\right.$$

$$\left.+K_2p_2(s)\frac{|u'(s)-v'(s)|}{\tau(s)}\tau(s)\right\}ds \leqq$$

$$\leqq |u-v|_\star\left(\frac{1}{\tau(t)}\int_0^1 |G_t(t,s)|\left\{K_1p_1(s)w(s)+\right.\right.$$

$$\left.\left.+K_2p_2(s)\tau(s)\right\}ds\right) \leqq k_1|u-v|_\star.$$

Thus

$$\sup_{[0,1]}\left|\frac{(T^{-1}F_2)'u(t)}{\tau(t)}-\frac{(T^{-1}F_2)'v(t)}{\tau(t)}\right| \leqq k_1|u-v|_\star. \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$|T^{-1}F_2u-T^{-1}F_2v|_\star \leqq \max\{k_0,k_1\}|u-v|_\star \equiv k_3|u-v|_\star$$

with $k_3 < 1$ and this together with (3.17) yields

$$\alpha(N(\Omega)) \leqq k_3\alpha(\Omega).$$

Thus N is a Darbo map and we are finished. □

REMARK. If q is continuous on $[0,1]$, then in fact $y \in C^2([0,1],B)$ in Theorem 3.1. This follows immediately from (3.8) with $\lambda = 1$.

The above existence principle is now used to obtain two existence theorems for differential equations in Banach spaces. The first considers the case $B = H$, a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$.

REMARK. For the remainder of this section let $\|u\|^2 = \int_0^1 |u|^2\,dt$ for appropriate functions $u : [0,1] \to H$.

We first must extend Theorem 1.3 for the problems considered in this section.

THEOREM 3.2. *Let H be a real Hilbert space and suppose $u : [0,1] \to H$ has a continuous derivative together with $u(0) = a \in H$ and $u(1) = b \in H$. Then*

$$\|u\|^2 \leqq \frac{1}{\pi^2}\|u'\|^2 + \frac{\sqrt{2}(|a| + |b|)}{\pi}\|u'\| + \frac{1}{2}(|a|^2 + |b|^2).$$

PROOF. Notice first that

$$|u|^2 = \langle (u-a)+a, (u-a)+a \rangle = |u-a|^2 + |a|^2 + 2\langle (u-a)+a, a\rangle \leqq$$
$$\leqq |u-a|^2 + |a|^2 + 2|a||u-a|.$$

Thus Hölder's inequality together with Theorem 1.3(ii), since $u(0) - a = 0$, implies

$$\int_0^{\frac12} |u|^2\,dt \leqq \int_0^{\frac12} |u-a|^2\,dt + 2|a|\int_0^{\frac12}|u-a|\,dt + \frac{1}{2}|a|^2 \leqq$$
$$\leqq \int_0^{\frac12} |u-a|^2\,dt + \sqrt{2}|a|\left(\int_0^{\frac12}|u-a|^2\,dt\right)^{\frac12} + \frac{1}{2}|a|^2 \leqq$$
$$\leqq \frac{4\left(\frac12 - 0\right)^2}{\pi^2}\int_0^{\frac12}|u'|^2\,dt + \sqrt{2}|a|\left(\frac{4\left(\frac12 - 0\right)^2}{\pi^2}\int_0^{\frac12}|u'|^2\,dt\right)^{\frac12} + \frac{1}{2}|a|^2.$$

Thus

$$\int_0^{\frac12}|u|^2\,dt \leqq \frac{1}{\pi^2}\int_0^{\frac12}|u'|^2\,dt + \frac{\sqrt{2}|a|}{\pi}\|u'\| + \frac{1}{2}|a|^2. \tag{3.20}$$

On the other hand since $|u|^2 \leqq |u-b|^2 + |b|^2 + 2|b||u-b|$ we have

$$\int_{\frac12}^1 |u|^2\,dt \leqq \frac{1}{\pi^2}\int_{\frac12}^1 |u'|^2\,dt + \sqrt{2}|b|\left(\frac{1}{\pi^2}\int_{\frac12}^1|u'|^2\,dt\right)^{\frac12} + \frac{1}{2}|b|^2.$$

Thus

$$\int_{\frac{1}{2}}^{1} |u|^2\,dt \leqq \frac{1}{\pi^2}\int_{\frac{1}{2}}^{1} |u'|^2\,dt + \frac{\sqrt{2}|b|}{\pi}\|u'\| + \frac{1}{2}|b|^2. \tag{3.21}$$

Adding (3.20) and (3.21) gives the result. □

THEOREM 3.3. *Let $B = H$ and suppose* (3.2) *and* (3.3) *are satisfied. In addition assume f has the splitting $f(t,u,v) = g(t,u,v) + h(t,u,v)$ with $g, h : [0,1] \times H^2 \to H$ continuous together with* (3.5) *and* (3.7) *being satisfied. Also assume $g(t,u,v) = g_1(t,u,v) + g_2(t,u,v)$ with*

$$q, p_1, p_2 \text{ are continuous on } [0,1], \tag{3.22}$$

$$\begin{cases} |g_1(t,u,p)| \leqq A_0\{|u|^\delta + |p|^\sigma + 1\} \text{ for } 0 \leqq \delta, \sigma < 1 \\ \text{and some constant } A_0, \end{cases} \tag{3.23}$$

$$\langle u, g_2(t,u,p)\rangle \geqq c|u|^2 + d|u||p| \text{ for constants } c \text{ and } d, \tag{3.24}$$

$$\begin{cases} \text{there is a function } \psi : [0,\infty) \to (0,\infty) \text{ such that } \dfrac{1}{\psi} \text{ is} \\ \text{integrable and } |f(t,u,p)| \leqq \psi(|p|) \text{ for } (t,u) \text{ in bounded sets} \end{cases} \tag{3.25}$$

and

$$\int_{q_0}^{\infty} \frac{u\,du}{\psi(u) + u + 1} = \infty \text{ for any constant } q_0 > 0. \tag{3.26}$$

Then (3.1) *has at least one solution $y \in C^2([0,1], H))$ if*

$$\delta_0|\varepsilon| + \delta_1|c|N_0 + \delta_2|d|\pi N_0 + K_1\Big(\sup_{[0,1]} p_1(t)\Big) + K_2\pi\Big(\sup_{[0,1]} p_2(t)\Big) < \pi^2 \tag{3.27}$$

where $N_0 = \sup_{[0,1]} q(t)$ and $\delta_0 = 0$ if $\varepsilon \geqq 0$, $\delta_0 = 1$ if $\varepsilon < 0$, $\delta_1 = 0$ if $c \geqq 0$, $\delta_1 = 1$ if $c < 0$ and $\delta_2 = 0$ if $d \geqq 0$, $\delta_2 = 1$ if $d < 0$.

REMARK. If in addition we assume

$$\langle u, h(t,u,v)\rangle \geqq c_0|u|^2 + d_0|u||p| + e_0|u| \quad \text{for} \quad c_0, d_0, e_0 \in \mathbf{R}$$

then minor adjustments in the analysis below will show that (3.1) will again have a $C^2([0,1], H))$ solution if (3.27) is replaced by the less restrictive condition

$$\delta_0|\varepsilon| + \delta_1|c|N_0 + \delta_2|d|\pi N_0 + \delta_3|c_0|N_0 + \delta_4|d_0|\pi N_0 < \pi^2$$

where $\delta_3 = 0$ if $c_0 \geqq 0$, $\delta_3 = 1$ if $c_0 < 0$ and $\delta_4 = 0$ if $d_0 \geqq 0$, $\delta_4 = 1$ if $d_0 < 0$.

REMARK. If (3.22), (3.23) and

$$\left|g_2(t,u,p)\right| \leqq C(t,u)|p|^2 + D(t,u)$$

where C, D are bounded on bounded sets are satisfied, then in fact (3.25) and (3.26) hold with $\psi(z) = L_0 + L_1|z|^2$ for some constants, $L_0, L_1 > 0$.

PROOF. Let y be a solution to $(3.4)_\lambda$. Also let $V\big(y(t)\big) = y(t) - a(1 - t) - bt$ and note that $V\big(y(0)\big) = V\big(y(1)\big) = 0$. Integration by parts yields

$$\int_0^1 \big\langle V\big(y(t)\big), y''(t)\big\rangle\, dt = -\int_0^1 \big\langle y'(t) + a - b, y'(t)\big\rangle\, dt =$$

$$= -\|y'\|^2 + \int_0^1 \big\langle b - a, y'(t)\big\rangle\, dt$$

and

$$\int_0^1 \big\langle V\big(y(t)\big), y'(t)\big\rangle\, dt = \frac{|a|^2}{2} - \frac{|b|^2}{2} + \int_0^1 \big\langle b - a, y(t)\big\rangle\, dt.$$

These together with the differential equation in $(3.4)_\lambda$ yield

$$-\|y'\|^2 + \int_0^1 \big\langle b - a, y'(t)\big\rangle\, dt +$$

$$+\beta\left(\frac{|a|^2}{2} - \frac{|b|^2}{2} + \int_0^1 \big\langle b - a, y(t)\big\rangle\, dt\right) - \varepsilon\|y\|^2 =$$

$$= \lambda \int_0^1 q(t)\big\langle y(t), h(t,y,y') + g(t,y,y')\big\rangle\, dt.$$

Thus

$$-\|y'\|^2 \leqq -\varepsilon\|y\|^2 + |a - b| \int_0^1 |y'|\, dt +$$

$$+|\beta|\left(\frac{\big||a|^2 - |b|^2\big|}{2} + |b - a| \int_0^1 |y|\, dt\right) +$$

$$+\sup_{[0,1]} q(t) \int_0^1 |y|\big|g_1(t,y,y')\big|\, dt + \int_0^1 |y| q(t)\big|h(t,y,y')\big|\, dt -$$

$$-\lambda \int_0^1 q(t)\big\langle y(t), g_2(t,y,y')\big\rangle\, dt.$$

Also assumption (3.7) implies

$$q(t)\big|h\big(t,y(t),y'(t)\big)\big| \leqq K_1 p_1(t)\big|y(t)\big| + K_2 p_2(t)\big|y'(t)\big| + L_0$$

where $L_0 = \sup_{[0,1]} q(t)\big|h(t,0,0)\big|$. The above together with Hölder's inequality implies that there exist constants P_0, P_1, P_2 and P_3 with

(3.28)

$$\|y'\|^2 \leqq -\varepsilon\|y\|^2 + P_0 + P_1\|y'\| + P_2\|y\| + P_3\int_0^1 |y|\,|g_1(t,y,y')|\,dt +$$

$$+K_1\|y\|^2 \sup_{[0,1]} p_1 + K_2\|y\|\|y'\| \sup_{[0,1]} p_2 - \lambda\int_0^1 q\big\langle y, g_2(t,y,y')\big\rangle\,dt.$$

Now Theorem 3.2 implies that there are constants P_4 and P_5 with

$$\|y\| \leqq P_4\|y'\| + P_5. \tag{3.29}$$

Also (3.23) implies

(3.30)

$$\int_0^1 |y|\,|g_1(t,y,y')|\,dt \leqq A_0\left(\int_0^1 |y|^{\delta+1}\,dt + \int_0^1 |y|\,|y'|^\sigma\,dt + \int_0^1 |y|\,dt\right).$$

Now Hölder's inequality and Theorem 3.2 imply

$$\int_0^1 |y|^{\delta+1}\,dt \leqq \|y\|^{\delta+1} \leqq P_6\|y'\|^{\delta+1} + P_7 \text{ and } \int_0^1 |y|\,dt \leqq P_8\|y'\| + P_9 \tag{3.31}$$

for some constants P_6, P_7, P_8 and P_9. Also note since $y(t) = a + \int_0^t y'(s)\,ds$ we have using Hölder's inequality that

$$\big|y(t)\big| \leqq |a| + \int_0^t \big|y'(s)\big|\,ds \leqq |a| + \|y'\|. \tag{3.32}$$

In addition Hölder's inequality and (3.32) yield

$$\int_0^1 |y|\,|y'|^\sigma\,dt \leqq \|y'\|^\sigma\left(\int_0^1 |y|^{\frac{2}{2-\beta}}\,dt\right)^{\frac{2-\beta}{2}} \leqq \|y'\|^\sigma\big(P_{10}\|y'\| + P_{11}\big) \tag{3.33}$$

for some constants P_{10} and P_{11}. Putting (3.31) and (3.32) into (3.29) and then putting the result with (3.29) into (3.28) yields

(3.34)
$$\|y'\|^2 \leqq -\varepsilon\|y\|^2 + P_{12} + P_{13}\|y'\| + P_{14}\|y'\|^{\delta+1} + P_{15}\|y'\|^{\sigma+1} + P_{16}\|y'\|^{\sigma} +$$
$$+K_1\|y\|^2 \sup_{[0,1]} p_1 + K_2\|y\|\|y'\| \sup_{[0,1]} p_2 - \lambda \int_0^1 q\langle y, g_2(t,y,y')\rangle\, dt$$

for some constants P_{12}, P_{13}, P_{14}, P_{15} and P_{16}. Another application of Theorem 3.2 yields

$$\left(1 - \frac{K_1 \sup_{[0,1]} p_1(t)}{\pi^2} - \frac{K_2 \sup_{[0,1]} p_2(t)}{\pi}\right)\|y'\|^2 \leqq \tag{3.35}$$
$$\leqq -\varepsilon\|y\|^2 + P_{17} + P_{18}\|y'\| + P_{14}\|y'\|^{\delta+1} +$$
$$+P_{15}\|y'\|^{\sigma+1} + P_{16}\|y'\|^{\sigma} + P_{19}\|y'\|^{\frac{3}{2}} - \lambda \int_0^1 q(t)\langle y(t), g_2(t,y,y')\rangle\, dt$$

for some constants P_{17}, P_{18} and P_{19}. There are eight cases to consider.

Case (i): $c \geqq 0, d \geqq 0$ and $\varepsilon \geqq 0$. Then $\langle y(t), g_2(t,y,y')\rangle \geqq 0$ and so (3.35), since $\varepsilon \geqq 0$, becomes

$$\left(1 - \frac{K_1 \sup_{[0,1]} p_1(t)}{\pi^2} - \frac{K_2 \sup_{[0,1]} p_2(t)}{\pi}\right)\|y'\|^2 \leqq$$
$$\leqq P_{17} + P_{18}\|y'\| + P_{14}\|y'\|^{\delta+1} + P_{15}\|y'\|^{\sigma+1} + P_{16}\|y'\|^{\sigma} + P_{19}\|y'\|^{\frac{3}{2}}.$$

Now since $0 \leqq \delta, \sigma < 1$ and $K_1 \sup_{[0,1]} p_1(t) + \pi K_2 \sup_{[0,1]} p_2(t) < \pi^2$, then there exists a constant M_0 independent of λ with

$$\|y'\| \leqq M_0 \tag{3.36}$$

for each solution y to $(3.4)_\lambda$.

Case (ii): $c \geqq 0, d \geqq 0$ and $\varepsilon < 0$. Notice first that since $\varepsilon < 0$ Theorem 3.2 implies

$$(-\varepsilon)\|y\|^2 \leqq \frac{(-\varepsilon)}{\pi^2}\|y'\|^2 + P_{20}\|y'\| + P_{21} \tag{3.37}$$

for some constants P_{20} and P_{21}. Now put (3.37) into (3.35) and use the fact that $\langle y(t), g_2(t,y,y')\rangle \geqq 0$ to obtain

$$\left(1+\frac{\varepsilon}{\pi^2}-\frac{K_1\sup_{[0,1]}p_1}{\pi^2}-\frac{K_2\sup_{[0,1]}p_2}{\pi}\right)\|y'\|^2 \leqq$$

$$\leqq P_{22}+P_{23}\|y'\|+P_{14}\|y'\|^{\delta+1}+P_{15}\|y'\|^{\sigma+1}+P_{16}\|y'\|^{\sigma}+P_{19}\|y'\|^{\frac{3}{2}}$$

for some constants P_{22} and P_{23}. Consequently (3.36) is again true since $|\varepsilon| + K_1\sup_{[0,1]}p_1(t)+\pi K_2\sup_{[0,1]}p_2(t)<\pi^2$.

Case (iii): $c<0, d\geqq 0$ and $\varepsilon<0$. Then (3.35) and (3.37) yield

$$\left(1+\frac{\varepsilon}{\pi^2}-\frac{K_1\sup_{[0,1]}p_1(t)}{\pi^2}-\frac{K_2\sup_{[0,1]}p_2(t)}{\pi}\right)\|y'\|^2 \leqq \tag{3.38}$$

$$\leqq P_{22}+P_{23}\|y'\|+P_{14}\|y'\|^{\delta+1}+P_{15}\|y'\|^{\sigma+1}+P_{16}\|y'\|^{\sigma}+P_{19}\|y'\|^{\frac{3}{2}}-$$

$$-\lambda\int_0^1 q(t)\langle y(t), g_2(t,y,y')\rangle\,dt.$$

Also since $\langle y(t), g_2(t,y,y')\rangle \geqq c|y|^2$ we have from Theorem 3.2 that

$$-\lambda\int_0^1 q(t)\langle y(t), g_2(t,y,y')\rangle\,dt \leqq (-c)\sup_{[0,1]}q(t)\int_0^1|y|^2\,dt \leqq$$

$$\leqq \frac{(-c)N_0}{\pi^2}\|y'\|^2+P_{24}\|y'\|+P_{25}$$

for some constants P_{24} and P_{25}. Putting this into (3.38) yields

$$\left(1+\frac{\varepsilon}{\pi^2}+\frac{cN_0}{\pi^2}-\frac{K_1\sup_{[0,1]}p_1(t)}{\pi^2}-\frac{K_2\sup_{[0,1]}p_2(t)}{\pi}\right)\|y'\|^2 \leqq$$

$$\leqq P_{26}+P_{27}\|y'\|+P_{14}\|y'\|^{\delta+1}+P_{15}\|y'\|^{\sigma+1}+P_{16}\|y'\|^{\sigma}+P_{19}\|y'\|^{\frac{3}{2}}$$

for some constants P_{26} and P_{27}. Consequently (3.36) is again true.

Case (iv): $c\geqq 0, d<0$ and $\varepsilon<0$. In this case (3.38) is again true. Also since $\langle y(t), g_2(t,y,y')\rangle \geqq d|y||y'|$ we have from Theorem 3.2 and Hölder's inequality that

$$-\lambda\int_0^1 q(t)\langle y(t), g_2(t,y,y')\rangle\,dt \leqq (-d)N_0\int_0^1|y||y'|\,dt \leqq (-d)N_0\|y\|\|y'\| \leqq$$

$$\leqq \frac{(-d)N_0}{\pi}\|y'\|^2+P_{28}\|y'\|^{\frac{3}{2}}+P_{29}$$

for some constants P_{28} and P_{29}. Putting this into (3.38) yields

$$\left(1+\frac{\varepsilon}{\pi^2}+\frac{dN_0}{\pi}-\frac{K_1\sup_{[0,1]}p_1(t)}{\pi^2}-\frac{K_2\sup_{[0,1]}p_2(t)}{\pi}\right)\|y'\|^2\leqq$$

$$\leqq P_{30}+P_{23}\|y'\|+P_{14}\|y'\|^{\delta+1}+P_{15}\|y'\|^{\sigma+1}+P_{16}\|y'\|^{\sigma}+P_{31}\|y'\|^{\frac{3}{2}}$$

for some constants P_{30} and P_{31}. Consequently (3.36) is again true.

Case (v): $c<0, d<0$ and $\varepsilon<0$. Combining cases (iii) and (iv) will again yield (3.36).

Case (vi): $c<0, d\geqq 0$ and $\varepsilon\geqq 0$. Then (3.35) yields, since $\varepsilon\geqq 0$, that

$$\left(1-\frac{K_1\sup_{[0,1]}p_1(t)}{\pi^2}-\frac{K_2\sup_{[0,1]}p_2(t)}{\pi}\right)\|y'\|^2\leqq \tag{3.39}$$

$$\leqq P_{17}+P_{18}\|y'\|+P_{14}\|y'\|^{\delta+1}+P_{15}\|y'\|^{\sigma+1}+P_{16}\|y'\|^{\sigma}+P_{19}\|y'\|^{\frac{3}{2}}-$$

$$-\lambda\int_0^1 q(t)\langle y(t),g_2(t,y,y')\rangle\,dt.$$

Now using the ideas of case (iii) will again yield (3.36).

Case (vii): $c\geqq 0, d<0$ and $\varepsilon\geqq 0$. Then (3.39) is satisfied and the ideas of case (iv) will yield (3.36).

Case (viii): $c<0, d<0$ and $\varepsilon\geqq 0$. Then (3.39) is satisfied and then combine the ideas of case (iii) and (iv) to again yield (3.36).

Thus in all cases there exists a constant M_0 independent of λ with (3.36) holding i.e. $\|y'\|\leqq M_0$. In addition (3.32) yields

$$\sup_{[0,1]}|y(t)|\leqq|a|+M_0\equiv M_1 \tag{3.40}$$

for any solution y to $(3.4)_\lambda$.

Now fix $z\in H$ with norm 1 and set $r(t)=\langle y(t),z\rangle$. Notice $|r(t)|\leqq$ $\leqq|y(t)|$ so $|r(0)|\leqq|a|$ and $|r(1)|\leqq|b|$ and also there exists t_0 (dependent on y and z) in $[0,1]$ with $|r'(t_0)|=|r(1)-r(0)|\leqq M_2$ where $M_2=|b|+$ $+|a|$. That is $|\langle y'(t_0),z\rangle|\leqq M_2$ for all $z\in H$ of norm 1. If $y'(t_0)\neq 0$ set $z=\frac{y'(t_0)}{|y'(t_0)|}$ to obtain $|y'(t_0)|\leqq M_2$, which also holds if $y'(t_0)=0$. Thus there exists $t_0\in[0,1]$ with

$$|y'(t_0)|\leqq M_2. \tag{3.41}$$

For $t \in [0,1]$ we have

$$\pm\left|y'(t)\right|' = \pm\frac{\langle y'(t), y''(t)\rangle}{\left|y'(t)\right|} \leqq \left|y''(t)\right|$$

whenever $y'(t) \neq 0$. This estimate, (3.25), (3.40) and $(3.4)_\lambda$ yield

$$\begin{aligned}(3.42)\qquad \pm\left|y'(t)\right|' &\leqq |\beta|\left|y'(t)\right| + |\varepsilon| M_1 + \psi\left(|y'(t)|\right) N_0 \leqq \\ &\leqq J_0\left(\psi\left(|y'(t)|\right) + \left|y'(t)\right| + 1\right)\end{aligned}$$

at any point $t \in [0,1]$ where $y'(t) \neq 0$; here $J_0 = \max\{|\beta|, |\varepsilon|, N_0\}$ and $N_0 = \sup_{[0,1]} q(t)$. Suppose $\left|y'(t)\right| > M_2$ for some $t \in [0,1]$. From (3.41) we can deduce that there exists an interval (μ,ν) containing t with $\left|y'(s)\right| > M_2$ on (μ,ν) and $\left|y'(\mu)\right|$ and/or $\left|y'(\nu)\right|$ equals M_2. Without loss of generality assume $\left|y'(\mu)\right| = M_2$. By (3.42) and Hölder's inequality we have

$$\int_\mu^t \frac{\left|y'(t)\right|\left|y'(t)\right|' dt}{\psi\left(|y'(t)|\right) + \left|y'(t)\right| + 1} \leqq J_0 \int_\mu^t \left|y'(s)\right| ds \leqq J_0\|y'\| \leqq J_0 M_0$$

using (3.36). Making the change of variables $u = \left|y'(s)\right|$ we obtain

$$\int_{M_2}^{|y'(t)|} \frac{u\,du}{\psi(u) + u + 1} \leqq J_0 M_0.$$

Let $I_0(z) = \int_{M_2}^z \frac{u\,du}{\psi(u)+u+1}$ and so $\left|y'(t)\right| \leqq I_0^{-1}(J_0 M_0) \equiv M_3$. Consequently

$$(3.43)\qquad \sup_{[0,1]}\left|y'(t)\right| \leqq \max\{M_2, M_3\}$$

for any solution y to $(3.4)_\lambda$. The result now follows from Theorem 3.1 and the estimates (3.40) and (3.43). □

The final existence theorem concerns differential equations in a real Banach space.

THEOREM 3.4. *Suppose* (3.2) *and* (3.3) *are satisfied. In addition assume* f *has the splitting* $f(t,u,v) = g(t,u,v) + h(t,u,v)$ *with* $g, h : [0,1] \times B^2 \to B$ *continuous together with* (3.5) *and* (3.7) *being satisfied. Also assume*

$$(3.44)\qquad \begin{cases} \left|g(t,u,p)\right| \leqq A_0\{|u|^\delta + |p|^\theta + 1\} \text{ for } 0 \leqq \delta, \theta < 1 \\ \text{and some constant } A_0. \end{cases}$$

(i) *If w chosen in* (3.7) *satisfies $w > 0$ on $[0,1]$ then* (3.1) *has a solution* $y \in C^2([0,1], B)$.

(ii) *If w chosen in* (3.7) *satisfies $w(0) = 0$ and/or $w(1) = 0$ then* (3.1) *has a solution $y \in C^2([0,1], B)$ if we assume*

$$(3.45) \qquad \begin{cases} \text{there exist constants } k_2 < 1 \text{ and } k_3 < 1 \text{ with} \\ \int_0^1 |G(t,s)| \{K_1 p_1(s) + K_2 p_2(s)\}\, ds \leqq k_2 \text{ and} \\ \int_0^1 |G_t(t,s)| \{K_1 p_1(s) + K_2 p_2(s)\}\, ds \leqq k_3. \text{ Here} \\ G(t,s) \text{ is the Green's function associated with the} \\ \text{problem } y'' + \beta y' - \varepsilon y = 0,\ y(0) = y(1) = 0. \end{cases}$$

PROOF. Let y be a solution to $(3.4)_\lambda$.Then

$$(3.46) \qquad y(t) = \lambda \int_0^1 G(t,s) q(s) f\big(s, y(s), y'(s)\big)\, ds + y_1(t)$$

where y_1 is the solution to $y'' + \beta y' - \varepsilon y = 0$, $y(0) = a, y(1) = b$ as described in Theorem 3.1. Also assumption (3.7) implies

$$(3.47) \qquad q(t)\,\big|h\big(t, y(t), y'(t)\big)\big| \leqq K_1 p_1(t)\big|y(t)\big| + K_2 p_2(t)\big|y'(t)\big| + L_0$$

where $L_0 = \sup_{[0,1]} q(t)\big|h(t,0,0)\big|$. Putting (3.44) and (3.47) into (3.46) yields

$$(3.48) \qquad \big|y(t)\big| \leqq \int_0^1 \big|G(t,s)\big| \left[K_1 p_1(s)\big|y(s)\big| + K_2 p_2(s)\big|y'(s)\big|\right] ds +$$

$$+ A_0 \int_0^1 \big|G(t,s)\big| q(s) \left[\big|y(s)\big|^\delta + \big|y'(s)\big|^\theta + 1\right] ds + L_1$$

where $L_1 = \sup_{[0,1]} \left\{\big|y_1(t)\big| + L_0 \int_0^1 \big|G(t,s)\big| q(s)\, ds\right\}$. Also (3.46) implies

$$y'(t) = \lambda \int_0^1 G_t(t,s) q(s) f\big(s, y(s), y'(s)\big)\, ds + y_1'(t)$$

and this together with (3.44) and (3.47) yields

$$(3.49) \qquad \big|y'(t)\big| \leqq \int_0^1 \big|G_t(t,s)\big| \left[K_1 p_1(s)\big|y(s)\big| + K_2 p_2(s)\big|y'(s)\big|\right] ds +$$

$$+A_0 \int_0^1 \left|G_t(t,s)\right| q(s) \left[\left|y(s)\right|^{\delta} + \left|y'(s)\right|^{\theta} + 1\right] ds + L_2$$

where $L_2 = \sup_{[0,1]} \left\{ \left| y_1'(t) \right| + L_0 \int_0^1 \left| G_t(t,s) \right| q(s)\, ds \right\}$.

Case (i): $w > 0$ on $[0,1]$. Now (3.48) implies

$$\left|y(t)\right| \leqq |y|_\star \int_0^1 \left|G(t,s)\right| \left[K_1 p_1(s) w(s) + K_2 p_2(s) \tau(s)\right] ds + L_1 +$$

$$+A_0 \int_0^1 \left|G(t,s)\right| q(s) \left[|y|_\star^{\delta} \big(w(s)\big)^{\delta} + |y|_\star^{\theta} \big(\tau(s)\big)^{\theta} + 1\right] ds \leqq$$

$$\leqq k_0 w(t) |y|_\star + R_0 |y|_\star^{\delta} + R_1 |y|_\star^{\theta} + R_2$$

for some constants R_0, R_1 and R_2; here k_0 is as in (3.7). Consequently

$$\sup_{[0,1]} \frac{\left|y(t)\right|}{w(t)} \leqq k_0 |y|_\star + R_3 |y|_\star^{\delta} + R_4 |y|_\star^{\theta} + R_5 \tag{3.50}$$

where $R_3 = \frac{R_0}{\mu}, R_4 = \frac{R_1}{\mu}, R_5 = \frac{R_2}{\mu}$ with $\mu = \min_{[0,1]} w(t) > 0$. Also (3.49) yields

$$\left|y'(t)\right| \leqq |y|_\star \int_0^1 \left|G_t(t,s)\right| \left[K_1 p_1(s) w(s) + K_2 p_2(s) \tau(s)\right] ds + L_2 +$$

$$+A_0 \int_0^1 \left|G_t(t,s)\right| q(s) \left[|y|_\star^{\delta} \big(w(s)\big)^{\delta} + |y|_\star^{\theta} \big(\tau(s)\big)^{\theta} + 1\right] ds \leqq$$

$$\leqq k_1 \tau(t) |y|_\star + R_6 |y|_\star^{\delta} + R_7 |y|_\star^{\theta} + R_8$$

for some constants R_6, R_7 and R_8. Consequently

$$\sup_{[0,1]} \frac{\left|y'(t)\right|}{\tau(t)} \leqq k_1 |y|_\star + R_9 |y|_\star^{\delta} + R_{10} |y|_\star^{\theta} + R_{11} \tag{3.51}$$

for some constants R_9, R_{10} and R_{11}. Combining (3.50) and (3.51) yields

$$|y|_\star \leqq \max\{k_0, k_1\} |y|_\star + R_{12} |y|_\star^{\delta} + R_{13} |y|_\star^{\theta} + R_{14}$$

for some constants R_{12}, R_{13} and R_{14}. Thus there exists a constant M_0 independent of λ with $|y|_\star \leqq M_0$ since $\max\{k_0, k_1\} < 1$. Hence

$$|y|_1 \leqq M_0 \nu \quad \text{where} \quad \nu = \max\{\max_{[0,1]} w(t), \max_{[0,1]} \tau(t)\}$$

for each solution y to $(3.4)_\lambda$, and the result follows from Theorem 3.1.

Case (ii): $w(0) = 0$ and/or $w(1) = 0$. Then (3.48) yields

$$|y(t)| \leqq |y|_1 \int_0^1 |G(t,s)| \left[K_1 p_1(s) + K_2 p_2(s)\right] ds + L_1 +$$

$$+ A_0 \int_0^1 |G(t,s)| q(s) \left[|y|_1^\delta + |y|_1^\theta + 1\right] ds$$

and so with k_2 as in (3.45) we have

$$|y|_0 \leqq k_2 |y|_1 + R_{15} |y|_1^\delta + R_{16} |y|_1^\theta + R_{17} \tag{3.52}$$

for some constants R_{15}, R_{16} and R_{17}. Similarly (3.49) yields

$$|y'|_0 \leqq k_3 |y|_1 + R_{18} |y|_1^\delta + R_{19} |y|_1^\theta + R_{20} \tag{3.53}$$

for some constants R_{18}, R_{19} and R_{20}. Combining (3.52) and (3.53) yields

$$|y|_1 \leqq \max\{k_2, k_3\} |y|_1 + R_{21} |y|_1^\delta + R_{22} |y|_1^\theta + R_{23}$$

for some constants R_{21}, R_{22} and R_{23}. Thus there exists a constant M_0 independent of λ with $|y|_1 \leqq M_0$ for each solution y to $(3.4)_\lambda$, and the result now follows from Theorem 3.1. □

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ORDER OF ORBITS IN HOMOGENEOUS SPACES

Y. VILLARROEL (Caracas)

The object of this paper is to study the order of orbits [4], in homogeneous spaces, using contact theory.

Let G be a compact, connected Lie group and $H \subset G$ a closed subgroup. Consider the homogeneous space $M = G/H$, the canonical projection $\pi : G \longrightarrow G/H$, the canonical action $\alpha : G \times M \longrightarrow M$ and the unique analytic manifold structure on M under which both π and α are analytic. Let $K \subset G$ be a closed subgroup and $K(o)$ the orbit of $o = \pi(H)$ under the restriction of α to K. Put $\dim K(o) = n$.

Let $C^{s,n}M$ be the contact bundle of order s of n-dimensional submanifolds in M and $C^s_x N$ the contact element of order s at $x \in N$, of an n-submanifold $N \subset M$. The canonical action α induces an action α^s of G on $C^{s,n}M$. Using the isotropy subgroup G^s of G at $C^s_o K(o)$, i.e.:

$$G^s = \{g \in G : \alpha^s(g, C^s_o K(o)) = C^s_o K(o)\},$$

we will construct the decreasing sequence

$$H = G^0 \supset \ldots \supset G^s \supset G^{s+1} \supset \ldots$$

and a corresponding decreasing sequence of Lie subalgebras

$$\mathbf{h} = \mathbf{g}^0 \supset \ldots \supset \mathbf{g}^s \supset \mathbf{g}^{s+1} \supset \ldots .$$

We will prove that the first index r such that $\mathbf{g}^r = \mathbf{g}^{r+1}$, is the order of the orbit $K(o)$ [4].

In consequence the order of the orbit $K(o)$ depends only on the contact element $C^r_o K(o)$. If $K(o)$ has order r and $K_1 \subset G$ is another Lie subgroup with $\dim K_1(o) = n$, and $C^{r+1}_o K_1(o) = C^{r+1}_o K(o)$, then the order of the orbit $K_1(o)$ is r.

Moreover, we will prove that the Lie subalgebras $\mathbf{g}^i$ coincide with the Lie subalgebras $\mathbf{q}^i$ defined in [4]. This gives a geometric meaning to such subalgebras.

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1. Orbits of the contact elements

Let M be a smooth $(n+m)$-dimensional manifold. Two imbedded submanifolds $N_1, N_2 \subset M$, of dimension n, with $n \leqq m$, have contact of order s at $x \in N_1 \cap N_2$ if there exist local parametrizations of N_1 and N_2 , given by the imbeddings

$$f_1, f_2 : U \subset R^n \longrightarrow M,$$

and a local coordinate system $\left(V,(x^i,y^j)\right)$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$, about $x \in M$ such that $f_1(o) = f_2(o) = x$, $x^i \circ f_l^j = x^i$, $l = 1,2$ and the partial derivatives at o of $(y^j \circ f_1)$ and $(y^j \circ f_2)$, are equal up to the order s. The contact element of order s at $x \in N_1$ is denoted by $C^s_x N_1$, and $C^{s,n}M$ denotes the set of all contact elements $C^s_x N$, with $x \in N$ and $N \subset M$ an imbedded n-dimensional submanifold.

Let $j \leqq s$ and consider the canonical projection

$$\pi^s_j : C^{s,n}M \longrightarrow C^{j,n}M$$

given by $C^s_x N \longmapsto C^j_x N$. Consider also for any submanifold $N \subset M$ the canonical inmersion

$$i^s : N \longrightarrow C^{s,n}M$$

given by $x \in N \longmapsto C^s_x N$.

Moreover, there is a canonical inmersion

$$i^{1,s} : C^{s+1,n}M \longrightarrow C^{1,n}(C^{s,n}M),$$

given by

$$C^{s+1}_x N \longmapsto C^1_{C^s_x N} C^s N.$$

Consider the manifold structure on $C^{s,n}M$ under which $\pi^s_o : C^{s,n}M \longrightarrow M$ is smooth.

Two submanifolds N_1, N_2, with $\dim N_l \leqq \dim M$, $l = 1,2$, have contact of order $s+1$ at $x \in N_1$ iff they have contact of order s at x and $T_{C^s_x N_1} C^s N_1 = T_{C^s_x N_2} C^s N_2$. Then we can identify the contact element $C^{s+1}_x N$ with $\left(C^s_x N, T_{C^s_x N} C^s N\right)$ belonging to the Grassmann bundle of n planes on $C^{s,n}M$ [3].

Now we consider $M = G/H$, a homogeneous space with G, a connected Lie group and $H \subset G$ a closed subgroup. The action $\alpha : G \times M \longrightarrow M$ induces an action

$$\alpha^s : G \times C^{s,n}M \longrightarrow C^{s,n}M,$$

given by $\alpha^s(g, C^s_x N) = C^s_{g\cdot x} g \cdot N$, where $g \cdot x$ denotes $\alpha(g,x)$ and $g \cdot N$ the image of N by $\alpha_g : x \in M \longmapsto \alpha(g,x) \in M$.

Let $K \subset G$ be a Lie subgroup and $K(o)$ the orbit of the point $o = \pi(H)$ under the action α restricted to K. The natural action of G on the Grassmann bundle $Gr^n M$, allows the identification

$$C^1_{g\cdot o} g \cdot K(o) = (g \cdot o, T_o\alpha_g(T_o K(o)) .$$

Suppose now that $K(o)$ is an imbedded submanifold, which always holds if K is a closed subgroup, and put $\dim K(o) = n$. The following propositions serve to establish the fact that the orbit $K(C^s_o K(o))$, under the action of α^s restricted to K, is the submanifold $C^s K(o)$ obtained by the inmersion $i^s : K(o) \longrightarrow C^{s,n} M$.

PROPOSITION 1. *Let $K \subset G$ be a Lie group such that $K(o)$ is an imbedded submanifold of dimension n of M. Then $i^s((K(o))$ is the orbit of the contact element $C^s_o K(o)$ under the action of α^s restricted to K.*

PROOF. By induction: for $s = 1$, using the identification given above, we have

$$i^1(K(o)) = \{C^1_{\tilde{k}} K(o) | \tilde{k} \in K(o)\} = \{(T_{\tilde{k}} K(o) | \tilde{k} \in K(o)\}.$$

But if $k \in K$, then $T_{k\cdot o} K(o) = T_o\alpha_k(T_o K(o))$, and

$$(k \cdot o, T_o\alpha_k(T_o K(o)) = k \cdot (o, T_o K(o)) = k \cdot C^1_o K(o),$$

in consequence,

$$i^1(K(o)) = C^1 K(o) = \{k \cdot C^1_o K(o) | k \in K\} = K(C^1_o K(o)).$$

For $s \geqq 1$, the proof is similar, using the identification

$$C^s_o K(o) = (T_{C^{s-1}_o K(o)} C^{s-1} K(o)) . \quad \square$$

2. Isotropy subalgebras and the order of the orbit

Let G^s be the isotropy subgroup of G at $C^s_o K(o)$, i.e.

$$G^s = \{g \in G | \alpha^s(g, C^s_o K(o)) = C^s_o K(o)\} .$$

Since G^o is closed, it is a Lie subgroup. It can be seen that π^s_j is equivariant and $C^s_o K(o)$ is projected on $C^{s-1}_o K(o)$. We have the decreasing sequence of Lie groups

$$H = G^0 \supset \ldots \supset G^s \supset G^{s+1} \supset \ldots$$

and the corresponding decreasing sequence of Lie algebras,

$$\mathbf{h} = \mathbf{g}^0 \supset \ldots \supset \mathbf{g}^s \supset \mathbf{g}^{s+1} \supset \ldots .$$

PROPOSITION 2. *Let $K \subset G$ be a Lie subgroup and $K(o)$ an imbedded submanifold of dimension n in $M = G/H$. Then $g \in G^{s+1}$ if and only if*

$$g \in G^s \quad \text{and} \quad T_o\alpha_g\big(T_{C_o^sK(o)}C^sK(o)\big) = T_{C_o^sK(o)}C^sK(o).$$

PROOF. We consider the contact element $C_o^{s+1}K(o)$, identified with its image

$$i^{1,s}\big(C^{s+1}K(o)\big) = C^1_{C_o^sK(o)}C^sK(o).$$

The last term can be identified with $\big(T_{C_o^sK(o)}C^sK(o)\big) \in Gr^n(C^{s,n}M)$, then $g \in G^{s+1}$ if and only if

$$g \cdot \big(T_{C_o^sK(o)}C^sK(o)\big) = \big(T_{C_o^sK(o)}C^sK(o)\big),$$

i.e., $g \in G^s$ and $T_{C_o^sK(o)}\alpha_g^s$ leaves invariant the subspace $T_{C_o^sK(o))}C^sK(o)$. □

Let $\mathbf{f}^s \subset \mathbf{g}$ be the inverse image of the subspace $T_{C_o^sK(o)}C^sK(o)$ under the tangent linear map $T_{C_o^sK(o)}\alpha_g^s$ given by $g \in G \longmapsto g \cdot C_o^sK(o)$. The following proposition yields a characterization of the Lie subalgebras $\mathbf{g}^s$.

PROPOSITION 3. *Let G be a compact Lie group and $K \subset G$ a closed Lie subgroup. Consider the decreasing sequence*

$$\mathbf{h} = \mathbf{a}^0 \supset \ldots \supset \mathbf{a}^s \supset \mathbf{a}^{s+1} \supset \ldots .$$

of subsets defined successively as follows: $\mathbf{Z} \in \mathbf{a}^{s+1}$ if given any $X \in \mathbf{f}^s$ there is a Y belonging to the Lie algebra $\mathbf{k}$ of K and a $Z_* \in \mathbf{a}^s$ such that

$$[Z, X] = Y + Z_*.$$

Then $\mathbf{a}^s$ is a Lie subalgebra of $\mathbf{g}$ and it is the Lie algebra of G^s.

PROOF. We will prove by induction that $\mathbf{a}^s$ is a subalgebra. Obviously $\mathbf{a}^0 = \mathbf{h}$ is a subalgebra. Assume now that $\mathbf{a}^s$ is a subalgebra. Let $Z_1, Z_2 \in \mathbf{a}^{s+1}$, then for any $X \in \mathbf{f}^s$ there are $Y_1, Y_2 \in \mathbf{k}$ and $Z_{1*}, Z_{2*} \in \mathbf{a}^s$ such that

$$[Z_1, X] = Y_1 + Z_{1*} \quad \text{and} \quad [Z_2, X] = Y_2 + Z_{2*}.$$

Consequently

$$[\xi Z_1 + \eta Z_2, X] = \xi Y_1 + \eta Y_2 + \xi Z_{1*} + \eta Z_{2*}$$

belong to $\mathbf{k} + \mathbf{a}^s$ for any $\eta, \xi \in \Re$, since $\xi Z_{1*} + \eta Z_{2*} \in \mathbf{a}^s$ holds, by the inductive assumption. Moreover,

$$\big[[Z_1, Z_2], X\big] = \big[Z_1, [Z_2, X]\big] - \big[Z_2, [Z_1, X]\big] = [Z_1, Y_2 + Z_{2*}] - [Z_2, Y_1 + Z_{1*}]$$

and the inductive assumption implies that $[Z_i, Y_i] \in \mathbf{k} + \mathbf{a}^s$ and $[Z_1, Z_{1*}] \in \in \mathbf{a}^s$, in consequence $\big[[Z_1, Z_2], X\big] \in \mathbf{k} + \mathbf{a}^s$.

Now, also using induction, we will see that $\mathbf{a}^s = \mathbf{g}^s$. For $s = 1$, this is proved in [4].

Since G is a compact group and $K \subset G$ is a closed subgroup, we have the canonical identification

$$\tilde{\alpha}^s : G/G^s \longrightarrow G\big(C^s_o K(o)\big),$$

and the orbit $K(o)$ is an imbedded submanifold of M [1]. Moreover, by Proposition 1, the orbit $K\big(C^s K(o)\big)$ is equal to the submanifold $C^s K(o)$, which is an imbedded submanifold of $G\big(C^s K(o)\big)$. Then, using the identification of the subspace $T_{C^s_o K(o)} C^s K(o)$ with the corresponding subspace in $T_o G/G^s$ given by $T_e \tilde{\alpha}^s$, we have

$$\mathbf{f}^s = \big(T_e \alpha^s_{C^s_o K(o)}\big)^{-1} \big(T_{C^s_o K(o)} C^s K(o)\big) = (T_e \pi_s)^{-1} \big(T_{C^s_o K(o)} C^s K(o)\big),$$

with $\pi_s : G \longrightarrow G/G^s$ the canonical projection.

Asume now that $\mathbf{a}^s = \mathbf{g}^s$. Let $Z \in \mathbf{g}^{s+1}$, then $Z \in \mathbf{g}^s$ and for any $X \in \mathbf{f}^s$ we have

$$T_e \pi_s [Z, X] = T_e \pi_s \left.\frac{d}{dt}\right|_{t=o} (Ad_{\exp tZ} X) = \left.\frac{d}{dt}\right|_{t=o} T_e \pi_s (Ad_{\exp tZ} X),$$

and the last term is equal to

$$\left.\frac{d}{dt}\right|_{t=o} \big(T_{C^s_o K(o)} \alpha_{\exp tZ} \big(T_e \pi_s(X)\big)\big),$$

with $T_e \pi_s(X)$ belonging to $T_{C^s_o K(o)} K(o)$ and $\exp tZ$ is contained in G^s. Then $T_{C^s_o K(o)} \alpha^s_{\exp tZ} \big(T_e \pi(X)\big)$ is a curve $\Gamma(t)$ in $T_{C^s_o K(o)} K(o)$ and there exists a curve $\gamma(t)$ in $\mathbf{k}$ such that

$$T_e \pi_s|_e [Z, X] = \left.\frac{d}{dt}\right|_{t=o} \pi_s\big(\gamma(t)\big) = T_e \pi_s \left.\frac{d}{dt}\right|_{t=o} \gamma(t) = T_e \pi_s(Y),$$

where $Y \in \mathbf{k}$. In consequence

$$T_e \pi_s [Z, X] = T_e \pi_s(Y),$$

thus $[Z,X] = Y + Z_*$, with $Z_* \in \mathbf{g}^s$ but, by the inductive assumption, $\mathbf{g}^s = \mathbf{a}^s$. Hence $[Z,X] \in \mathbf{k} + \mathbf{a}^s$.

Assume, conversely, that $Z \in \mathbf{a}^{s+1}$ and consider $\widetilde{X} \in T_{C_o^s K(o)} C^s K(o)$. We will see that

$$T_{C_c^s K(o)} \alpha^s_{\exp tZ} \widetilde{X} \in T_{C_o^s K(o)} C^s K(o),$$

thus $Z \in \mathbf{g}^{s+1}$. Indeed, let $Z \in \mathbf{a}^{s+1}$ and $X \in \mathbf{f}^{s+1}$, such that $T_e\pi(X) = \widetilde{X}$, then

$$T\alpha^s_{\exp tZ} \widetilde{X} = T\alpha^s_{\exp tZ}\big(T_e\pi(X)\big) = T_e\pi_s\big(Ad_{\exp tZ}(X)\big).$$

Now

$$\left.\frac{d}{dt}\right|_{t=o} T_e\pi_s\big(Ad_{\exp tZ}(X)\big) = T_e\pi_s \lim_{u \to o} \frac{Ad(\exp(t+u)Z) - Ad(\exp tZ)}{u} X,$$

but this last term is equal to

$$T_e\pi_s\big(Ad(\exp tZ)\big) \cdot \lim_{u \to o} \frac{Ad(\exp uZ - Id)}{u} X = T_e\pi_s(Ad_{\exp tZ})[Z,X],$$

and by the inductive assumption this is equal to $T_e\pi_s Ad_{\exp tZ}(Y + Z_*)$, with $Y \in \mathbf{k}$ and $Z_* \in \mathbf{a}^s = \mathbf{g}^s$. Then there exists a curve $\gamma(t) \in \mathbf{k}$ such that

$$\left.\frac{d}{dt}\right|_{t=o} T_e\pi_s = \big(Ad_{\exp tZ}(X)\big) = T_e\pi_s\gamma(t) \in T_{C_o^s K(o)} C^s K(o). \quad \square$$

THEOREM. *Let G be a compact Lie group, $K \subset G$ a closed Lie subgroup and $C_o^s K(o)$ the contact element of order s of the orbit $K(o)$ of the point $o \in G/H$ under G. Consider the decreasing sequence*

$$\mathbf{h} = \mathbf{g}^0 \supset \ldots \supset \mathbf{g}^s \supset \mathbf{g}^{s+1} \supset \ldots .$$

of the Lie algebras of the isotropy groups G^s at $C_o^s K(o)$. Then the order of the orbit $K(o)$ is the first index r such that $\mathbf{g}^r = \mathbf{g}^{r+1}$.

PROOF. Let $\mathbf{f} \subset \mathbf{g}$ be defined by

$$\mathbf{f} = (T_e\pi)^{-1} = \big(T_o K(o)\big)$$

and consider the decreasing sequence

$$\mathbf{h} = \mathbf{q}^0 \supset \ldots \supset \mathbf{q}^s \supset \mathbf{q}^{s+1} \supset \ldots$$

defined as follows: $Z \in \mathbf{q}^{s+1}$ if given $X \in \mathbf{f}$ there is a $Y \in \mathbf{k}$ and $Z_* \in \mathbf{q}^{s-1}$ such that $[Z,X] = Y + Z_*$. Then, the first index r such that $\mathbf{q}^r = \mathbf{q}^{r+1}$ is the order of the orbit $K(o)$ [4].

Now, $K \subset G$ is a closed Lie subgroup and, since G is compact, $K(o)$ is an imbedded submanifold of $M = G/G^o$. Hence $C^s K(o)$ is an imbedded submanifold of G/G^s. Therefore the subspace $\mathbf{f}^s$ defined in Proposition 4 is equal to the subspace $\mathbf{f}$ defined above. In consequence, the Lie algebra $\mathbf{a}^s$ is equal to the Lie algebra $\mathbf{g}^s$ of the isotropy subgroup G^s, so the first index r such that $\mathbf{g}^r = \mathbf{g}^{r+1}$ is the order of the orbit. □

COROLLARY. *The order r of the orbit of a subgroup K depends only on the contact element of order $r+1$ of $K(o)$. If K_1 is another closed subgroup of G with dimension of $K_1(o)$ equal to the dimension of $K(o)$, and $C_o^{r+1} K_1(o) = C_o^{r+1} K(o)$, then the order of the orbit $K_1(o)$ is r.* □

Now we will find the order of the orbit for some examples, and finally we will give a scheme to calculate the Lie algebra of the isotropy group G^s, using the action of the Lie algebra $\mathbf{g}^s$ on $T^*(C^1_{C^s_o K(o)} G/G^s)$.

EXAMPLES. 1. Let G be group of rigid motions of the 3-dimensional euclidean space $\Re^3$ and $H = SO(3)$ the isotropy subgroup at $o \in \Re^3$.

We represent G by $\{(A, x) : A \in SO(3),\ x \in \Re^3\}$, and its action on $\Re^3$ is given by $(A, x) \cdot y = Ay + x$, for $(A, x) \in G$, $y \in \Re^3$.

Let $\{(\theta^i), (\omega^i_j)\}$ be the Maurer–Cartan forms of G.

Consider the involutive 3-dimensional left invariant distribution D on G defined by the equations

$$\theta^3 = o, \quad \omega^3_1 = o, \quad \omega^3_2 = o,$$

and K the analytic subgroup of G with Lie algebra D.

The group K is given by

$$K = \left\{ \left((a^i_j), (x_1, x_2, x_3) \right) : \quad a^1_3 = a^2_3 = o, a^3_3 = 1, x_3 = o \right\},$$

and $\pi(K) = \Re^2 \subset \Re^3$ is the xy-plane in $\Re^3$ ([3], p.50).

Since the Lie algebra $\mathbf{g}^0$ of the isotropy group G^1 of the induced action on the space of contact elements at $C^1_o K(o)$ is equal to the Lie algebra $\mathbf{g}^2$ therefore the order of the orbit $K(o)$ is equal to 1. □

2. Let G be the rigid motions group of $\Re^3$ as above and $H = SO(3)$.

Consider the involutive 3-dimensional left-invariant distribution D on G defined by the equations

$$\theta^3 = o, \quad \omega^1_3 = k\theta^1, \quad \omega^2_3 = k\theta^2, \quad k \in \Re, \quad k > o.$$

Let $K \subset G$ be the analytic subgroup of G whose Lie algebra is D. Then $\pi(K)$ is the sphere in $\Re^3$ of radius r centered at $(o, o, -r)$ ([3], p.51).

The Lie algebra $\mathbf{g}^0$ of the isotropy group G^1 of the induced action on the space of contact elements at $C^1_o K(o)$ is equal to the Lie algebra $\mathbf{g}^2$, therefore the order of the orbit $K(o)$ is equal to 1. □

3. Let $G = O(4)$. Consider the canonical action of G on $C_o^{1,2}\Re^4$. Given a coordinate system x, y, z, v, let $r \in C_o^{1,2}\Re^4$ be given by the xy-plane and $H = O(3) \subset O(4)$ the isotropy subgroup at r. There is an involutive left invariant distribution D on G defined by the equations

$$\theta^3 = \theta^4 = o, \quad \omega_3^1 = \omega_4^1 = \omega_3^2 = \omega_4^2 = o,$$

see e.g. [6], [7]. Let $K \subset G$ be the maximal integral manifold of this distribution through the identity element. It can be shown (see next paragraph) that $\mathbf{g}^1 = \mathbf{g}^2$, consequently the order of the orbit $K(r)$ is equal to 1. □

To calculate the Lie algebra of the isotropy group G^s of a contact element $X^s \in C^{s,n}M$, we consider the action of the isotropy subgroup G^{s-1} of the element $X^{s-1} = \pi^s_{s-1}(X^s)$ on the fiber $H^s \subset C^{s,n}M$ which projects onto X^{s-1}. Also we consider the action of the Lie algebra $\mathbf{g}^s$ on $T^*_{X^s}H^s$ (see [2]).

We shall study the integral curves of the fundamental vector field defined by $\mathbf{g}^s$ on the manifold H^s.

Consider the following scheme which will be detailed for Example 2 and can be similarly used for the other examples.

Let G be the rigid motions group of $\Re^3$ and $H = SO(3)$,

$$\{\theta^i, \omega_j^i; \quad \omega_j^i + \omega_i^j = o\}$$

the Maurer–Cartan forms. Consider the isomorphism $T_oG/H \simeq T_o\Re^3$ induced by the map $\alpha^o : g \in G \longmapsto g \cdot o \in \Re^3$.

Identify $\mathbf{g}$ with T_eG ($\mathbf{h}$ with T_eH).

The forms θ^i allow us to define a basis of $T_o^*\Re^3$ as follows:

Given $\tilde{v} \in T_o\Re^3$, consider $v \in \mathbf{g}$ such that $T_e(\alpha^o)(v_e) = \tilde{v}$ and define

$$\tilde{\theta}^i(\tilde{v}) = \theta_e(v_e). \tag{1}$$

It is clear that this definition is independent of the choice of v, since if $v, u \in \mathbf{g}$ and $T_e\alpha^o(v_e) = T_e\alpha^o(u_e)$ then $T_e\alpha^o(u_e - v_e) = o$, and $u - v \in \mathbf{h}$. In consequence, $\theta_e^i(v_e) = \theta_e^i(u_e)$.

The set $\{\tilde{\theta}^i\}$ defines a basis of $T_o^*\Re^3$.

Consider a coordinate system in $C_o^{1,2}\Re^3$ defined on the following open set:

$$U^{i,j} = \left\{C_o^1S \in C_o^{1,2}\Re^3 : \ \tilde{\theta}^i|T_oS, \tilde{\theta}^j|T_oS \text{ are linearly independent}\right\},$$

where the coordinates (p_s^i, p_s^j), $s \neq i, j$, are given by the relations:

$$\tilde{\theta}^s|T_oS = p_s^i\tilde{\theta}^i|T_oS + p_s^j\tilde{\theta}^j|T_oS.$$

The group H acts on $C_o^{1,2}\Re^3$ and we have the map

$$F^o : \mathbf{h} \longrightarrow \mathcal{X}(C_o^1\Re^3),$$

given by

$$\varepsilon \in \mathbf{h} \longmapsto F_\varepsilon^o; \quad \text{where} \quad F_\varepsilon^o(X^1) = \left.\frac{d}{dt}\right|_{t=o} (\exp \varepsilon \cdot X^1).$$

Using the coordinates defined above and expressing ε in coordinates as $\varepsilon = (\varepsilon_j^i)$, it is possible to prove that F_ε^o is given by

$$F_\varepsilon^o(p^1, p^2) = (-\varepsilon_3^1 + p^2\varepsilon_2^1)\frac{\partial}{\partial p^1} + (-\varepsilon_3^2 - p^1\varepsilon_2^1)\frac{\partial}{\partial p^2}.$$

It is clear that the map F_ε^o is surjective, thus the action of H on $C_o^{1,2}\Re^3$ is transitive; but the action of G on $\Re^3$ is transitive and in consequence the action of G on $C^{1,2}\Re^3$ is transitive.

Consider the contact element $X_o^1 = C_o^1 S_o$ given in coordinates as

$$p^1 = p^2 = o, \quad \text{i.e.} \quad X_o^1 = (o, o), \quad (\theta^3 | T_o S_o = o).$$

The Lie algebra of the isotropy group G^1 of X_o^1 is given by

$$\mathbf{g}^1 = \{\varepsilon \in \mathbf{h} : \quad \varepsilon_3^1 = \varepsilon_3^2 = o\}.$$

Considering the isomorphism $T_o G/G^1 \simeq T_{X_o^1}(C^{1,2}\Re^3)$ defined by

$$\alpha^1 : g \in G \longmapsto g \cdot X_o^1 \in C^{1,2}\Re^3,$$

and using a similar argument to (1), we can define a basis of forms

$$\{\tilde\theta^1, \tilde\theta^2, \tilde\theta^3, \tilde\omega_3^1, \tilde\omega_3^2\} \in T_{X_o^1} C^{1,2}\Re^3.$$

Also we can consider a coordinate system of $C_{X_o^1}^{1,2}(C^{1,2}\Re^3)$ defined on the following open set:

$$V^{1,2} = \left\{X^2 \in C_{X_o^1}^{1,2} C^{1,2}\Re^3 : \ \tilde\theta^1 | X^2, \ \tilde\theta^2 | X^2 \text{ are linearly independent}\right\},$$

with coordinate functions $(p^1, p^2, b_1, b_2, b_4, b_3)$, defined by the relations

$$\tilde\theta^3 | X^2 = p^1 \tilde\theta^1 | X^2 + p^2 \tilde\theta^2 | X^2,$$

$$\tilde\omega_3^1 | X^2 = b_1 \tilde\theta^1 | X^2 + b_2 \tilde\theta^2 | X^2,$$

$$\tilde\omega_3^2 | X^2 = b_4 \tilde\theta^1 | X^2 + b_3 \tilde\theta^2 | X^2.$$

Using coordinates, we can show that the contact elements

$$C^{2,2}\Re^3 \hookrightarrow C^{1,2}(C^{1,2}\Re^3)$$

are characterized by $b_2 = b_4$.

Let $H^2 \subset C^{2,2}\Re^3$ be the fiber of contact elements which project onto X_o^1, then G^1 acts on H^2 and we have, as above, the map

$$F^1 : \mathbf{g}^1 \longrightarrow \mathcal{X}(\mathbf{H}^2).$$

Using coordinates, we can see that this map is given by

$$F^1_{(\varepsilon^i_j)}(b_1, b_2, b_3) = \varepsilon^1_2(2b_2\frac{\partial}{\partial b_1} + (b_3 - b_1)\frac{\partial}{\partial b_2} - 2b_2\frac{\partial}{\partial b_3}).$$

If $b_i = o$, i.e. $\omega^1_3 = \omega^2_3 = o$ then

$$F^1_{\varepsilon^i_j}(b_1, b_2, b_3) = o \quad \text{and} \quad \mathbf{g}^2 = \mathbf{g}^1.$$

Moreover, if $p^i = o$, $b_2 = o$, $b_3 = b_1 = k \in \Re$, then $F^1_{\varepsilon^i_j}(b^l) = o$, i.e.

$$\theta^3 = o, \quad \omega^1_3 = k\theta^1, \quad \omega^2_3 = k\theta^2 \quad \text{and} \quad \mathbf{g}^2 = \mathbf{g}^1.$$

This is the case of Example 2.

With a similar procedure, we obtain Examples 1 and 3.

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ADDITIVE COMPLETION AND DISJOINT TRANSLATIONS

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1. Introduction

Let $A \subset [0, N]$ be a nonempty set of integers. We define the *covering number* L of A as

$$L = \min \{ |B| : B \subset \mathbf{Z}, A + B \supset \{0, 1, \dots, N\} \}.$$

(The implicit dependence on N is not indicated.) Such a set B is called an *additive complement* of A. Bounds for L for arithmetically important sets were given in numerous papers. Obviously $L \geqq (N+1)/|A|$. For the classical results on additive completion see Halberstam–Roth [3].

Let now P be a polynomial of degree $d \geqq 2$, with integral coefficients and positive leading coefficient, and consider (for varying N) the sets

$$A = A_N = \{ P(n) : n \in \mathbf{N}, 0 \leqq P(n) \leqq N \}. \tag{1}$$

The best known lower bound of L for these sets is due to Cilleruelo [1] (where only $P(n) = n^k$ is considered, and the formulation is slightly different) and Habsieger [2] and it sounds as follows.

THEOREM 1. *For a polynomial set* (1.1) *we have (for a fixed polynomial P and $N \to \infty$)*

$$L \geqq \big(1 + o(1)\big) s(1/d) \frac{N}{|A|},$$

where

$$s(t) = \frac{\sin \pi t}{\pi t(1-t)} = 1 + t + \Big(1 - \frac{\pi^2}{6}\Big) t^2 + \dots .$$

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Here we consider the covering number together with a "dual" problem. We define the *disjoint translation number* D as

$$D = \sup \{ |B| : B \subset [0,N] \cap \mathbf{Z}, A + b_1 \cap A + b_2 = \emptyset$$

$$\text{for all } b_1, b_2 \in B, b_1 \neq b_2 \}.$$

With $r(n) = \#\{(a,b) : a \in A, b \in B, a + b = n\}$, D is defined by the requirement $r(n) \leqq 1$ and L by $r(n) \geqq 1$ for $0 \leqq n \leqq N$. By an obvious counting argument we have $D \leqq (2N+1)/|A|$. For polynomial sets we have an improvement, similar to Theorem 1.

THEOREM 2. *For a polynomial set* (1.1) *we have (for a fixed polynomial P and $N \to \infty$)*

$$D \leqq \big(1 + o(1)\big)\, s(1/d) \frac{N}{|A|}.$$

The similarity of the bounds suggests that there is a closer connection between L and D than this formal analogy; this will be explored in the next section.

For certain polynomials the bound of Theorem 2 is rather tight.

THEOREM 3. *Let $P(x) = x^d$ with an odd integer $d \geqq 3$. We have*

$$D \geqq \big(1 + o(1)\big) \frac{N}{|A|}.$$

We can prove a somewhat weaker bound for even powers. For a general polynomial we cannot decide whether $D \gg N/|A|$ holds.

2. Generalizations

We define the *fractional covering number* as follows:

$$\Lambda = \min \sum_{k=-N}^{N} \lambda_k,$$

where the numbers λ_k are subject to the conditions $\lambda_k \geqq 0$ and

$$\sum_{a \in A} \lambda_{n-a} \geqq 1$$

for all $0 \leqq n \leqq N$. With the additional restriction $\lambda_k = 0$ or 1 we get the definition of L.

We define a fractional analog of D similarly:

$$\Delta = \max \sum_{k=0}^{N} \delta_k,$$

where $\delta_k \geqq 0$ and

$$\sum_{a\in A, n-N\leqq a\leqq n} \delta_{n-a} \leqq 1$$

for all $0 \leqq n \leqq 2N$. Again, imposing $\delta_k = 0$ or 1 we get D.

The quantities L, D, Λ, Δ are connected in the following way.

THEOREM 4. *We have always*

$$D \leqq \Delta = \Lambda \leqq L.$$

PROOF. The inequalities are obvious. To prove the middle equality, consider the matrix (α_{ij}), $0 \leqq i \leqq 2N$, $0 \leqq j \leqq N$ with the entries

$$\alpha_{ij} = \begin{cases} 1 & \text{if } i-j \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Δ is the maximum of $\sum_{j=0}^{N} y_j$ under the constraints $y_j \geqq 0$,

$$\sum_{j=0}^{N} \alpha_{ij} y_j \leqq 1 \quad (i = 0, \dots, 2N).$$

A familiar result on linear duality yields that this is the same as the minimum of $\sum_{i=0}^{2N} x_i$ under the assumptions $x_i \geqq 0$,

$$\sum_{i=0}^{2N} \alpha_{ij} x_i \geqq 1 \quad (j = 0, \dots, N).$$

This reduces to the definition of Λ with the transformation $\lambda_k = x_{N-k}$. □

In the light of this result, Theorems 1 and 2 have the following common generalization.

THEOREM 5. *For a polynomial set* (1.1) *we have (for a fixed polynomial P and $N \to \infty$)*

$$\Lambda = \big(1 + o(1)\big)\, s(1/d) \frac{N}{|A|}.$$

We shall determine the asymptotic behaviour of Λ for a wider class of sequences. Let A be an infinite set of nonnegative integers. We use the same letter A to denote its counting function, so that

$$A(x) = \#\{a \in A, a \leqq x\}.$$

Let Λ_N denote the fractional covering number of the set $A \cap [0, N]$ for the interval $[0, N]$.

THEOREM 6. *Suppose that with some $\alpha \in (0,1)$ we have $A(tN)/A(N) \to \to t^\alpha$ for all $t \in [0,1]$ as $N \to \infty$. Then we have*

$$\Lambda_N = \big(1 + o(1)\big)\, s(\alpha) \frac{N}{A(N)}.$$

We prove this result in the next section. It implies Theorems 1, 2, 5.

3. Proof of Theorem 6

As a motivation for the following argument, we mention the continuous analog of the problem, which concerns the convolution

$$h(t) = f * g(t) = \int f(x) g(t-x) dx.$$

In the first case, we want to have $h(t) \geqq 1$ for $0 \leqq t \leqq 1$ and minimize $\int g$; in the second case we want $h(t) \leqq 1$, $g(t) = 0$ for $t \notin [0,1]$ and maximize $\int g$ (and we assume $g \geqq 0$ in both cases). Now for the function

$$f(x) = \begin{cases} \alpha x^{\alpha-1} & \text{for } x \in [0,1], \\ 0 & \text{otherwise} \end{cases}$$

which is the density corresponding to distribution x^α, not only the results are equal but there is a common g that solves both problems, namely

$$g(t) = s(\alpha)(1-\alpha)t^{-\alpha}.$$

Indeed, a change of variable yields

$$\int_0^t x^{\alpha-1}(t-x)^{-\alpha}\, dx = \int_0^1 y^{\alpha-1}(1-y)^{-\alpha}\, dy = \frac{\pi}{\sin \pi\alpha}. \tag{3.1}$$

Now we start the proof. Take an $\varepsilon > 0$. We shall find an N_0 such that

$$(3.2) \qquad (1-\varepsilon)s(\alpha)\frac{N}{A(N)} < \Lambda_N < (1+\varepsilon)s(\alpha)\frac{N}{A(N)}$$

for $N > N_0$.

For $0 \leqq x \leqq N$, write

$$A(x) = (x/N)^\alpha A(N) + R_N(x), \quad R_N = \max\left|R_N(x)\right|.$$

The convergence of $A(tN)/A(N)$ to t^α, as any convergence of monotonic functions to a continuous monotonic function, must be uniform, that is, $R_N = o\big(A(N)\big)$ as $N \to \infty$.

First we construct nonnegative reals λ_n such that

$$(3.3) \qquad \sum_a \lambda_{n-a} \geqq 1$$

for all n, while $\sum \lambda_n \leqq (1+\varepsilon)s(\alpha)N/A(N)$. To this end we select a number $\eta > 0$, depending on ε (this dependence will be made explicit later). We set $\varrho = \eta(1-\eta)$ and define

$$\lambda_n = \beta(n+\eta N)^{-\alpha}, \quad -\varrho N \leqq n \leqq N,$$

where

$$\beta = (1+\varepsilon/2)s(\alpha)(1-\alpha)\frac{N^\alpha}{A(N)}.$$

Since

$$\sum_{-\varrho N}^{N}(n+\eta N)^{-\alpha} < \int_{-\varrho N-1}^{N}(t+\eta N)^{-\alpha}\,dt < \int_{-\eta N}^{N}(t+\eta N)^{-\alpha}\,dt =$$

$$= \frac{1}{1-\alpha}N^{1-\alpha}(1+\eta)^{1-\alpha},$$

we have

$$\sum \lambda_n < (1+\varepsilon/2)(1+\eta)^{1-\alpha}s(\alpha)\frac{N}{A(N)} < (1+\varepsilon)s(\alpha)\frac{N}{A(N)}$$

if η is so small that

$$(3.4) \qquad (1+\varepsilon/2)(1+\eta)^{1-\alpha} < 1+\varepsilon.$$

Now we prove (3.3). We have

$$\sum \lambda_{n-a} = \beta \sum_{0 \leqq a \leqq \min(N, n+\varrho N)} (n+\eta N - a)^{-\alpha} \geqq \beta \sum_{0 \leqq a \leqq M} (n+\eta N - a)^{-\alpha},$$

where

$$M = (n+\eta N)(1-\eta) \leqq \min(N, n+\varrho N)$$

will be more comfortable for the following computations.

We turn this sum into a Stieltjes integral to obtain

$$\beta \sum_{0 \leqq a \leqq M} (n+\eta N - a)^{-\alpha} = \beta \int_{0-}^{M} (n+\eta N - t)^{-\alpha}\, dA(t) =$$

$$= \beta A(N) N^{-\alpha} \int_{0}^{M} (n+\eta N - t)^{-\alpha}\, dt^{\alpha} + \beta \int_{0-}^{M} (n+\eta N - t)^{-\alpha}\, dR_N(t).$$

After a change of variable $t = (n+\eta N)u$, the main term becomes

$$\beta A(N) N^{-\alpha} \alpha \int_{0}^{1-\eta} u^{\alpha-1}(1-u)^{-\alpha}\, du > 1 + \varepsilon/3,$$

if η is so small that

$$\int_{0}^{1-\eta} u^{\alpha-1}(1-u)^{-\alpha}\, du > \frac{1+\varepsilon/3}{1+\varepsilon/2} \frac{\pi}{\sin \pi\alpha}.$$

(We remind that $\pi/\sin\pi\alpha$ is the value of the integral from 0 to 1, as mentioned in (3.1).)

We use integration by parts to estimate the remainder term:

$$\left| \beta \int_{0-}^{M} (n+\eta N - t)^{-\alpha}\, dR_N(t) \right| =$$

$$= \beta \left| R_N(M)(n+\eta N - M)^{-\alpha} + \alpha \int_{0}^{M} R_N(t)(n+\eta N - t)^{-\alpha-1}\, dt \right| \leqq$$

$$\leqq 2\beta R_N (n+\eta N - M)^{-\alpha} = 2\beta R_N(\eta)^{-\alpha}(n+\eta N)^{-\alpha} \leqq$$

$$\leqq 2\beta R_N \eta^{-3\alpha} N^{-\alpha} < \varepsilon/3$$

if N is so large that

$$R_N < \mu A(N), \qquad \mu = \frac{\varepsilon \eta^{3\alpha}}{6(1+\varepsilon)s(\alpha)(1-\alpha)}.$$

Hence for sufficiently large N, the required property (3.3) is established.

To prove $\Lambda_N > (1-\varepsilon)s(\alpha)N/A(N)$, we construct nonnegative numbers δ_n for $0 \leqq n \leqq N$, such that $\sum_a \delta_{n-a} \leqq 1$ for all n, while $\sum \delta_n \geqq (1-\varepsilon)s(\alpha)N/A(N)$. To this end we select a number $\eta > 0$, depending on ε and define

$$\delta_n = \beta(n+\eta N)^{-\alpha}, \quad 0 \leqq n \leqq N,$$

where now

$$\beta = (1-\varepsilon/2)s(\alpha)(1-\alpha)\frac{N^\alpha}{A(N)}.$$

We suppress the details of the estimates of the calculations, which are very similar to the previous ones.

4. Proof of Theorem 3

In this section $P(x) = x^k$, with k odd.

First we observe that if p is a prime number such that $p-1$ and k are relatively prime, then we have

$$m^k \equiv n^k \bmod p \Longrightarrow m \equiv n \bmod p.$$

Indeed, if p divides m (or n) this claim is obvious. Let us assume that $m^k \equiv n^k \bmod p$, with $p \nmid mn$. Then we have $m^{p-1} \equiv n^{p-1} \equiv 1 \bmod p$. Moreover there exist integers u and v such that $(p-1)u + kv = 1$. Thus we get

$$m \equiv m^{(p-1)u+kv} \equiv (m^k)^v \equiv (n^k)^v \equiv n^{(p-1)u+kv} \equiv n \bmod p,$$

and the claim is proved.

Let N be a sufficiently large number. Let us choose a prime p with $p > N^{\frac{1}{k}}$ and $p \sim N^{\frac{1}{k}}$ as N goes to infinity, and such that $p-1$ and k are relatively prime. Let us define

$$B := \{ap : 1 \leqq a \leqq N/p\}.$$

We clearly have

$$|B| \sim N/p \sim N^{1-\frac{1}{k}}.$$

We prove that the sets $P(\lambda) + B$ are disjoint for $0 \leqq P(\lambda) \leqq N$. Assume that $\lambda_1^k + b_1 = \lambda_2^k + b_2$, with $0 \leqq P(\lambda_1), P(\lambda_2) \leqq N$ and $b_1, b_2 \in B$. Then $\lambda_1^k \equiv \lambda_2^k \bmod p$, hence we have $\lambda_1 \equiv \lambda_2 \bmod p$. Moreover we know that $|\lambda_1 - \lambda_2| \leqq P^{-1}(N) = N^{\frac{1}{k}} < p$, thus $\lambda_1 = \lambda_2$.

5. Remarks

With some change in the proof, Theorem 6 can be extended to the cases $\alpha = 0$ and $\alpha = 1$ (we define $s(\alpha)$ by continuity at these numbers, so that $s(0) = s(1) = 1$).

Assume that $0 \in A$. In this case there are additive complements consisting exclusively of nonnegative integers; let L^* be the minimal cardinality of such a set. We can also define a fractional analog Λ^* of this number. We have obviously $L \leqq L^*$ and $\Lambda \leqq \Lambda^*$. We think that under the conditions of Theorem 6 we have $\Lambda \sim \Lambda^*$, though the proof in Section 3 does not immediately yield this.

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ON TWO VERSIONS OF L^2-DISCREPANCY AND GEOMETRICAL INTERPRETATION OF DIAPHONY

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This paper concentrates on a comparison of two versions of L^2-discrepancy: the "usual" one and the one on parallelepipeds modulo the unit cube. We also show that the last variant of discrepancy is closely connected with diaphony, and in the most important case these two characteristics of distribution coincide up to a multiplicative constant.

Part of the results of this paper was earlier published in preliminary form in [3] (Russian).

1. Notation and definitions

Let

$$\mathbf{Q}^s = \left\{ x \in \mathbf{R}^s \mid 0 \leqq x_j < 1; \quad j = 1, \dots, s \right\}$$

be the s-dimensional unit cube, or, equally,

$$\mathbf{Q}^s = \{ x \in \mathbf{R}^s \mid \mathbf{0} \leqq x < \mathbf{1} \}$$

(we reserve lower indices for coordinates of vectors and put $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$).

By a net $S = (X, \rho)$ we will mean a finite weighted set of points in $\mathbf{Q}^s$, that is, a set of points in the unit cube

$$X = \{ x^{(k)} \in \mathbf{Q}^s \mid k = 1, \dots, N \}$$

and a set of non-negative real weights

$$\rho = \{ \rho_k \geqq 0 \mid k = 1, \dots, N \}$$

corresponding to these points. The pair $(x^{(k)}, \rho_k)$ is said to be the k-th node of the net S, and $\rho_0 = \rho_1 + \dots + \rho_N$ will denote the sum of the weights of the nets.

Numeration of the nodes of nets is not essential: we will not distinguish nets differing only by order of following their nodes.

Generally speaking, we do not impose any additional restrictions on the points (some of them may coincide) or on the weights (their sum may differ from 1). A net having the sum of its weights equal to 1 (that is, a net with $\rho_0 = 1$) we will refer to as a *normed* one; in such a case, the system of weights will also be called normed. Therefore, for the normed net with N nodes and with equal weights we have

$$\rho_1 = \cdots = \rho_N = 1/N.$$

It should be pointed out that those "traditional" nets are included, as a particular case, in all of the results of this paper. In other words, none of the results are based on using some "exotic" weights.

By a *shift of a system of points* X by a vector x modulo $\mathbf{Q}^s$ we will mean the new system of points

$$X + x = \{\{x^{(k)} + x\} \mid k = 1, \dots, N\}$$

in $\mathbf{Q}^s$ (the inner braces denote fractional part of the vector, that is, the vector of fractional parts of coordinates).

By a *shift of a net* S by a vector x modulo $\mathbf{Q}^s$ we will mean the new net $S + x = (X + x, \rho)$ with the nodes

$$\left(\{x^{(k)} + x\}, \rho_k\right), \qquad k = 1, \dots, N.$$

For $1 \leqq t \leqq s$ let $\mathbf{Q}^t$ be some t-dimensional face of $\mathbf{Q}^s$, and let π be the operator of orthogonal projection of $\mathbf{Q}^s$ to $\mathbf{Q}^t$. By the *projection of* S *to* $\mathbf{Q}^t$ we will mean the new net $S' = (X', \rho)$ in $\mathbf{Q}^t$, whose nodes $(x'^{(k)}, \rho_k)$ are defined by $x'^{(k)} = \pi\left(x^{(k)}\right)$.

Let $I \subset \{1, \dots, s\}$ be a system of indices, Σ the corresponding system of s-dimensional hyperplanes

$$x_i = 1/2 \quad (i \in I)$$

and $\sigma : \mathbf{Q}^s \to \mathbf{Q}^s$ the transformation of the unit cube to itself, involving symmetries relative to all hyperplanes from Σ, followed by obtaining of fractional parts (order of executing these operations obviously does not influence the result). By the *image of* S *under the transformation* σ (or under symmetry relative to the hyperplanes of the system Σ) we will mean the net $S' = (X', \rho)$ in $\mathbf{Q}^s$ whose nodes $(x'^{(k)}, \rho_k)$ are defined by

$$x'^{(k)} = \sigma(x^{(k)}),$$

so that

$$x_i'^{(k)} = \begin{cases} \left\{1 - x_i^{(k)}\right\}; & i \in I, \\ x_i^{(k)}; & i \notin I. \end{cases}$$

We call the net symmetric if it coincides with its image under any transformation of the described type.

For $u, v \in \mathbf{R}^s$ consider the parallelepiped

$$\Pi(u,v) = \{x \in \mathbf{R}^s \mid u \leqq x < v\}$$

(the parallelepiped is non-empty only if $u < v$). Let us denote its volume by $|\Pi(u,v)|$ and introduce the *local discrepancy of the net* S *in* $\Pi(u,v)$ by

$$R(u,v) = \sum_{x^{(k)} \in \Pi(u,v) \pmod{\mathbf{Q}^s}} \rho_k - |\Pi(u,v)| \tag{1}$$

(where the summation is extended over all points $x^{(k)}$ of the net S belonging to $\Pi(u,v)$ "modulo the unit cube", that is, over those points $x^{(k)}$ some integral shift of which belongs to $\Pi(u,v)$). Clearly, the local discrepancy $R(u,v)$ is invariant relative to shifts of $\Pi(u,v)$ by any integer vector.

For $\kappa \in [1;\infty]$ define the L^κ*-discrepancy* of a net S by

$$D_\kappa(S) = \|R(\mathbf{0},\cdot)\|_\kappa, \tag{2}$$

that is

$$D_\kappa(S) = \left(\int_{\mathbf{Q}^s} |R(\mathbf{0},\gamma)|^\kappa d\gamma\right)^{1/\kappa}, \qquad \kappa < \infty,$$

and

$$D_\infty(S) = \sup_{\gamma \in \mathbf{Q}^S} |R(\mathbf{0},\gamma)|;$$

we set also

$$\begin{cases} \widetilde{D}_\kappa(S) = \left(\int_{\mathbf{Q}^s \times \mathbf{Q}^s} |R(\alpha,\alpha+\gamma)|^\kappa d\alpha d\gamma\right)^{1/\kappa}, & \kappa < \infty, \\ \widetilde{D}_\infty(S) = \sup_{\alpha,\gamma \in \mathbf{Q}^s} |R(\alpha,\alpha+\gamma)|. \end{cases} \tag{3}$$

The value, defined by (3), is called the *Weyl* L^κ*-discrepancy* of the net S.

In particular, in the case of $\kappa = 2$ we obtain the L^2*-discrepancy* $D_2(S)$ and the *Weyl* L^2*-discrepancy* $\widetilde{D}_2(S)$, and in the case of $\kappa = \infty$ — the *supreme discrepancy* $D_\infty(S)$ and the *Weyl supreme discrepancy* $\widetilde{D}_\infty(S)$. The supreme discrepancy $D_\infty(S)$ is often called just *discrepancy* and is denoted by $D(S)$; similarly, Weyl supreme discrepancy $\widetilde{D}_\infty(S)$ will be denoted simply by $\widetilde{D}(S)$.

Let $S+\alpha$ be the shift of S by a vector $\alpha \in \mathbf{Q}^s$, and R_α be the local discrepancy of $S+\alpha$. It is obvious from (1) that $R_\alpha(\mathbf{0},\gamma) = R(-\alpha,-\alpha+\gamma)$ and therefore, according to definitions (2) and (3),

$$\widetilde{D}_\kappa(S) = \left(\int_{\mathbf{Q}^s} \left| D_\kappa(S+\alpha)\right|^\kappa d\alpha\right)^{1/\kappa};$$

that is, the Weyl L^κ-discrepancy $\widetilde{D}_\kappa(S)$ may be considered as the L^κ-average of the "usual" L^κ-discrepancies $D_\kappa(S+\alpha)$ of all shifts of S by vectors α from $\mathbf{Q}^s$.

2. The diaphony $F_2(S)$ and its connection with Weyl L^2-discrepancy $\widetilde{D}_2(S)$

One more characteristic of the multidimensional net distribution uniformity $S=(X,\rho)$ — diaphony, denoted by $F_2(S)$ — was introduced first by P. Zinterhof in 1976 (see [5]) and then considered by a number of authors (see, for instance, [1], [3], [4]). There are two well-known "classical" definitions of diaphony. We will reproduce them with some changes, caused, mainly, by our intention to consider (unlike the "classical" approach) the general case of nets with arbitrary weights.

One of the definitions uses the representation by infinite series:

$$F_2(S) = \left(\sum_{m\in\mathbf{Z}^s}{}' \frac{\left|T(m)\right|^2}{\overline{m}^2}\right)^{1/2},$$

where

$$\overline{m} = \overline{m}_1 \cdots \overline{m}_s, \quad \overline{m}_j = \max\left(1, |m_j|\right),$$

$T(m)$ is a trigonometric sum of the net S, defined by

$$T(m) = \sum_{k=1}^{N} \rho_k \cdot e^{2\pi i \langle m\cdot x^{(k)}\rangle}$$

($\langle m\cdot x^{(k)}\rangle$ is the scalar product of the vectors m and $x^{(k)}$), and, finally, the dash in the sum means that the zero vector $m=\mathbf{0}$ should be excluded from the sum.

Another definition is

$$F_2(S) = \left(\sum_{k,l=1}^{N} \rho_k\rho_l \prod_{j=1}^{s}\left(1+2\pi^2 B_2\left(\{x_j^{(k)} - x_j^{(l)}\}\right)\right) - \rho_0^2\right)^{1/2},$$

where $B_2(\xi) = \xi^2 - \xi + 1/6$ is the Bernoulli polynomial of order 2 (we recall, that $\rho_0 = \rho_1 + \cdots + \rho_N$).

The equivalence of the two definitions mentioned above can be easily verified in view of the well-known expansion of $B_2(\xi)$ into a Fourier series:

$$B_2(\xi) = \frac{1}{2\pi^2} \sum_{m\in\mathbf{Z}}{}' \frac{1}{m^2} e^{2\pi i m \xi}.$$

We state below three exciting properties of diaphony, following almost immediately from the above definitions.

STATEMENT 1. *Let $S + \alpha$ be the shift of S by a vector $\alpha \in \mathbf{Q}$. Then $F_2(S + \alpha) = F_2(S)$.*

PROOF. This follows from the "second definition" in view of $\{\{x_j^{(k)} + \alpha\} - \{x_j^{(l)} + \alpha\}\} = \{x_j^{(k)} - x_j^{(l)}\}$.

STATEMENT 2. *Let $S' = \pi(S)$ be a projection of S. Then $F_2(S') \leqq F_2(S)$.*

PROOF. We can assume that the projection π is defined by

$$\pi\big((x_1, x_2, \ldots, x_s)\big) = (0, x_2, \ldots, x_s).$$

Let T' be the trigonometric sum of the net S'. Then, obviously,

$$T\big((0, m_2, \ldots, m_s)\big) = T'\big((m_2, \ldots, m_s)\big),$$

and according to the "first definition",

$$F_2^2(S) = \sum_{m\in\mathbf{Z}^s}{}' \frac{|T(m)|^2}{\overline{m}^2} \geqq \sum_{m\in\mathbf{Z}^s,\ m_1=0}{}' \frac{|T(m)|^2}{\overline{m}^2} = F_2^2(S').$$

STATEMENT 3. *Let $S' = \sigma(S)$ be the image of S under some symmetry σ. Then $F_2(S') = F_2(S)$.*

PROOF. Again, we can consider only the particular case of the symmetry σ defined by

$$\sigma\big((x_1, x_2, \ldots, x_s)\big) = \big(\{1 - x_1\},\ x_2, \ldots, x_s\big).$$

Let T' be the trigonometric sum of the net S'. Then, obviously,

$$T'\big((m_1, m_2, \ldots, m_s)\big) = T\big((-m_1, m_2, \ldots, m_s)\big),$$

and we use the first definition to complete our proof.

Do Statements 1–3 hold for the discrepancies $\widetilde{D}_2(S)$ and $D_2(S)$ instead of the diaphony $F_2(S)$? As we show later, the answer is affirmative for $\widetilde{D}_2(S)$ and negative for $D_2(S)$.

Note that, unlike discrepancies, diaphony has pure analytic definitions which do not explain its "geometrical nature". The following theorem[1] clarifies this nature by establishing a direct connection between diaphony and Weyl L^2-discrepancy $\widetilde{D}_2(S)$.

THEOREM 1. *We have*

$$\widetilde{D}_2^2(S) = 3^{-s}\left(\sum_{m\in\mathbf{Z}^s}{}' \frac{|T(m)|^2}{\overline{m}^2}\cdot\left(\frac{3}{2\pi^2}\right)^{\nu(m)} + |T(\mathbf{0})-1|^2\right),$$

where the sum is extended over all non-zero s-dimensional integer vectors, and $\nu(m)$ is the number of non-zero coordinates of m.

PROOF. Let us consider the local discrepancy $R(\alpha,\alpha+\gamma)$ as a function of α with fixed γ, and evaluate its Fourier coefficients. If $m\neq\mathbf{0}$, then the corresponding coefficient is

$$\widehat{R}(m) = \int_{\mathbf{Q}^s} R(\alpha,\alpha+\gamma)e^{-2\pi i\langle m\cdot\alpha\rangle}d\alpha =$$

$$= \sum_{k=1}^{N}\rho_k \int_{x^{(k)}-\gamma<\alpha\leqq x^{(k)}} e^{-2\pi i\langle m\cdot\alpha\rangle}d\alpha =$$

$$= \sum_{k=1}^{N}\rho_k e^{-2\pi i\langle m\cdot x^{(k)}\rangle}\int_{-\gamma<\alpha\leqq\mathbf{0}} e^{-2\pi i\langle m\cdot\alpha\rangle}d\alpha =$$

$$= T(-m)\int_{\mathbf{0}\leqq\alpha<\gamma} e^{2\pi i\langle m\cdot\alpha\rangle}d\alpha;$$

and if $m=\mathbf{0}$, then

$$\widehat{R}(m) = \widehat{R}(\mathbf{0}) = T(\mathbf{0})\int_{\mathbf{0}\leqq\alpha<\gamma}d\alpha - \int_{\mathbf{Q}^s}\gamma_1\cdots\gamma_s d\alpha = (T(\mathbf{0})-1)\gamma_1\cdots\gamma_s.$$

[1] This theorem was first proved by the author in [3] (Russian). However, we reproduce it here for the sake of completeness.

Hence, using Parseval's equality, from definition (3) we obtain:

$$\widetilde{D}_2^2(S) = \int_{\mathbf{Q}^s} \sum_m{}' \left|T(m)\right|^2 \cdot \left| \int_{0 \leqq \alpha < \gamma} e^{2\pi i \langle m \cdot \alpha \rangle} d\alpha \right|^2 d\gamma +$$

$$+ \int_{\mathbf{Q}^s} \left|T(\mathbf{0}) - 1\right|^2 \gamma_1^2 \cdots \gamma_s^2 d\gamma =$$

$$= \sum_m{}' \left|T(m)\right|^2 \prod_{j=1}^{s} \int_0^1 \left| \int_0^{\gamma_j} e^{2\pi i m_j \alpha_j} d\alpha_j \right|^2 d\gamma_j + 3^{-s} \left|T(\mathbf{0}) - 1\right|^2,$$

and it remains to observe that the integral over α_j is equal to $\frac{1}{3m_j^2}$ if $m_j = 0$, and is equal to $\frac{1}{3m_j^2} \cdot \frac{3}{2\pi^2}$ otherwise.

The theorem just proved is a direct analog of the well-known Erdős-Turán inequality for the metrics of L^2. It shows that up to a multiplicative constant and the summand measuring "unnormness" of the net, the diaphony of a net coincides with its Weyl L^2-discrepancy. In particular, for normed nets we immediately obtain the following

COROLLARY 1. *Let S be normed: $\rho_0 = 1$; then*

$$\left(2\pi^2 \cdot 3^{s-1}\right)^{1/2} \widetilde{D}_2(S) \leqq F_2(S) \leqq \left(2\pi^2\right)^{s/2} \widetilde{D}_2(S).$$

PROOF. It is sufficient to use in Theorem 1 the obvious inequality $1 \leqq \leqq \nu(m) \leqq s$ $(m \neq \mathbf{0})$.

COROLLARY 2. *Statements* 1–3 *hold for Weyl L^2-discrepancy $\widetilde{D}_2(S)$ instead of the diaphony $F_2(S)$.*

PROOF. Follows the proof of Statements 1–3 with Theorem 1 instead of the "first definition" of diaphony, and (3) instead of the "second definition".

3. $\widetilde{D}_2$ versus D_2: one-dimensional nets

The rest of the paper investigates the correspondence between Weyl L^2-discrepancy $\widetilde{D}_2(S)$ and the diaphony $F_2(S)$, on the one hand, and "usual" L^2-discrepancy $D_2(S)$, on the other hand. We will see that the question is not a trivial one, and even for normed nets allows a direct solution only in a few particular cases.

We start from the simplest of them, the case of *one-dimensional normed nets*.

In this case one of the estimates (of type $\widetilde{D}_2(S) \ll D_2(S)$) follows easily from

LEMMA 1. *Let S be a one-dimensional normed net: $s=1$, $\rho_0=1$. Then*

$$D_2^2(S)=\frac{1}{2\pi^2}F_2^2(S)+\left(\sum_{k=1}^{N}\rho_k x^{(k)}-\frac{1}{2}\right)^2.$$

PROOF. The identity of the lemma is a "weighted variant" of a well-known identity (see, for instance, [2], equality (2.27)). It can be obtained by means of expansion of $R(\mathbf{0},\gamma)$ into a Fourier series and then applying Parseval's equality — the method, used in Theorem 1.

COROLLARY. *Let S be a one-dimensional normed net: $s=1$, $\rho_0=1$. Then*

$$\widetilde{D}_2(S)\leqq D_2(S).$$

PROOF. According to Theorem 1, in our case

$$\widetilde{D}_2^2(S)=3^{-1}F_2^2(S)\cdot(3/2\pi^2)=F_2^2(S)/2\pi^2,$$

and in view of Lemma 1,

$$D_2^2(S)\geqq F_2^2(S)/2\pi^2.$$

Is it possible to obtain an inverse estimate, that is, an estimate of type $D_2(S)\ll\widetilde{D}_2(S)$? The following example, suitable for the general s-dimensional case, shows that it is not the case.

EXAMPLE 1. Let $S=(X,\rho)$ vary over a sequence of s-dimensional nets with the number of nodes increasing to infinity and the *Weyl supreme discrepancy* decreasing to zero: $N\to\infty$, $\widetilde{D}(S)\to 0$. Set

$$\delta=\left(\widetilde{D}(S)\right)^{1/(s+0.5)}\to 0,$$

$$P=\left\{x\in\mathbf{Q}^s\mid x\geqq(1-\delta)\mathbf{1}\right\}$$

(so that δ and P depend on S) and consider the new net $S'=(Y,\rho)$ with the same weights as S and the system of points Y, defined by

$$y^{(k)}=\begin{cases}x^{(k)}; & x^{(k)}\notin P,\\ \mathbf{0}; & x^{(k)}\in P.\end{cases}$$

Denote the local discrepancy of S' by R'. It is clear that for $\gamma<(1-\delta)\mathbf{1}$

$$R'(\mathbf{0},\gamma)\geqq\sum_{x^{(k)}\in P}\rho_k-\left|R(\mathbf{0},\gamma)\right|\geqq$$

$$\geqq|P|-\left|R\big((1-\delta)\mathbf{1},\mathbf{1}\big)\right|-\left|R(\mathbf{0},\gamma)\right|\geqq\delta^s-2\widetilde{D}(S),$$

and so for sufficiently large N

$$(4) \qquad D_2^2(S') \geqq (1-\delta)^s\big(\delta^s - 2\widetilde{D}(S)\big)^2 \gg \delta^{2s}.$$

On the other hand, if $\alpha > \mathbf{0}$ it is easily seen that

$$R'(\alpha, \alpha+\gamma) \leqq \begin{cases} \widetilde{D}(S); & \alpha_i, \alpha_i+\gamma_i \notin [1-\delta; 1] \quad (i = 1, \dots, s), \\ 2\widetilde{D}(S) + \delta^s; & \text{in any case.} \end{cases}$$

It follows that

$$(5) \qquad \widetilde{D}_2^2(S') \ll \widetilde{D}^2(S) + \delta \cdot \delta^{2s} \ll \delta^{2s+1}$$

and from (4) and (5) we conclude that for no constant C is $D_2(S') \leqq C\widetilde{D}_2(S')$ for all the nets S'.

4. $\widetilde{D}_2$ versus D_2: symmetric nets. The estimate $\widetilde{D}_2(S) \ll D_2(S)$

Another case of interest is one of symmetric nets. The subject was first investigated by P. Proinov [4], who proved an estimate of type $D_2(S) \ll \ll \widetilde{D}_2(S)$ for normed symmetric nets with equal weights. Then in [3] the author obtained an *inverse* estimate $\widetilde{D}_2(S) \ll D_2(S)$ for *all normed symmetric nets* (with not necessarily equal weights), and also generalized Proinov's proof for such nets. Below we generalize both results for arbitrary (not necessarily normed) symmetric nets and also make some refinements in formulations and proofs; this section contains a proof of the author's estimate $\widetilde{D}_2(S) \ll D_2(S)$, and the next one that of the generalized Proinov's estimate $D_2(S) \ll \widetilde{D}_2(S)$. By means of combining the two estimates with appropriate constants one obtains

THEOREM 2. *Let S be a symmetric net having all its points strictly inside the unit cube $\mathbf{Q}^s$. Then*

$$(4/5)^{s/2} D_2(S) \leqq \widetilde{D}_2(S) \leqq 8^{s/2} D_2(S).$$

Note that we have to require that S has no points on the faces of $\mathbf{Q}^s$. This requirement is not a casual one, and arises from the fact that in some sense a net having a number of its points on the faces of $\mathbf{Q}^s$ is not actually symmetric "relative to D_2" (at the same time, it may be symmetric "relative to $\widetilde{D}_2$").

We see, therefore, that in the case of symmetric nets the two L^2-discrepancies $D_2(S)$ and $\widetilde{D}_2(S)$ coincide up to a multiplicative constant.

Let $\varepsilon \in \{0,1\}^s$, and denote by S_ε the image of the net S under the symmetry σ_ε relative to all the hyperplanes $x_j = 1/2$ for which the index j satisfies the condition $\varepsilon_j = 0$:

$$\sigma_\varepsilon : x \mapsto x', \quad x'_j = \begin{cases} \{1 - x_j\}; & \varepsilon_j = 0, \\ x_j; & \varepsilon_j = 1. \end{cases} \tag{6}$$

Note that from Corollary 2 of Theorem 1 it follows that

$$\widetilde{D}_2(S_\varepsilon) = \widetilde{D}_2(S).$$

The main result of this section (which easily implies the discussed estimate of type $\widetilde{D}_2(S) \ll D_2(S)$) is

THEOREM 3. *Let S be a net having all its points strictly inside $\mathbf{Q}^s$. Then*

$$\widetilde{D}_2^2(S) \leqq 6^s \sum_{\varepsilon \in \{0,1\}^s} 3^{-\nu(\varepsilon)} D_2^2(S_\varepsilon)$$

(here S_ε varies over all nets obtained by symmetric transformations of S).

PROOF. Our idea is to divide the parallelepiped $\Pi(\alpha, \alpha+\gamma)$ into "elementary parallelepipeds", each of which entirely lies in a shift of $\mathbf{Q}^s$ by some vector $\varepsilon \in \{0,1\}^s$; then the local discrepancy of S in such elementary parallelepiped will be replaced by the local discrepancy of S_ε in some parallelepiped with the "minimal" vertex at $\mathbf{0}$.

Now perform the reasoning in detail.

Let $\overline{\Pi}(u,v) = \Pi(u,v) \cap \mathbf{Q}^s$, and let ε vary over all vectors from $\{0,1\}^s$. Denote the local discrepancies of the net S_ε in $\Pi(u,v)$ and $\overline{\Pi}(u,v)$ by $R_\varepsilon(u,v)$ and $\overline{R}_\varepsilon(u,v)$, respectively. It is clear that

$$\Pi(\alpha, \alpha+\gamma) = \cup_\varepsilon \big(\Pi(\alpha, \alpha+\gamma) \cap (\mathbf{Q}^s + \varepsilon)\big) = \\ = \cup_\varepsilon \big(\varepsilon + \overline{\Pi}(\alpha - \varepsilon,\ \alpha + \gamma - \varepsilon)\big).$$

Since ε is an integer and the parallelepipeds $\varepsilon + \overline{\Pi}(\alpha - \varepsilon, \alpha + \gamma - \varepsilon)$ are pairwise disjoint, we have

$$R(\alpha, \alpha+\gamma) = \sum_{\varepsilon \in \{0,1\}^s} \overline{R}(\alpha - \varepsilon, \alpha + \gamma - \varepsilon).$$

For $\delta \in \{0,1\}^s$ let $\Omega(\delta) \subset \mathbf{Q}^s \times \mathbf{Q}^s$ be the set of all pairs (α, γ) with the given integral part $[\alpha + \gamma] = \delta$, that is

$$\alpha_j + \gamma_j \quad \begin{cases} < 1; & \delta_j = 0, \\ \geqq 1; & \delta_j = 1. \end{cases}$$

Obviously, if $(\alpha,\gamma)\in\Omega(\delta)$ but $\varepsilon\leqq\delta$ does *not* hold, then $\overline{\Pi}(\alpha-\varepsilon,\alpha+\gamma-\varepsilon)$ is empty; hence

$$(7)\qquad \widetilde{D}_2^2(S)=\int_{\alpha,\gamma\in\mathbf{Q}^s}\left(\sum_{\varepsilon\in\{0,1\}^s}\overline{R}(\alpha-\varepsilon,\alpha+\gamma-\varepsilon)\right)^2 d\alpha d\gamma=$$

$$=\sum_{\delta\in\{0,1\}^s}\int_{\Omega(\delta)}\left(\sum_{\varepsilon\leqq\delta}\overline{R}(\alpha-\varepsilon,\alpha+\gamma-\varepsilon)\right)^2 d\alpha d\gamma\leqq\sum_{\delta\in\{0,1\}^s}2^{\nu(\delta)}\sum_{\varepsilon\leqq\delta}I(\delta,\varepsilon),$$

where

$$I(\delta,\varepsilon)=\int_{\Omega(\delta)}\overline{R}^2(\alpha-\varepsilon,\ \alpha+\gamma-\varepsilon)\,d\alpha d\gamma.$$

But one can easily see that the symmetry (6), mapping S to S_ε, also maps the parallelepiped $\overline{\Pi}(\alpha-\varepsilon,\alpha+\gamma-\varepsilon)$ into another parallelepiped $\Pi(u,v)$ (possibly, up to inclusion of some parts of the surfaces of $\Pi(u,v)$ and $\mathbf{Q}^s$), such that for $(\alpha,\gamma)\in\Omega(\delta)$, $\delta\geqq\varepsilon$ we have

$$(8)\qquad u_j=\begin{cases}1-\alpha_j-\gamma_j; & \delta_j=0,\\ 0; & \delta_j=1,\end{cases}\qquad v_j=\begin{cases}1-\alpha_j; & \varepsilon_j=0,\\ \alpha_j+\gamma_j-1; & \varepsilon_j=1.\end{cases}$$

So for all α and γ such that the surface of $\Pi(u,v)$ does not contain points of S_ε (that is, for all α and γ, except possibly a set of measure zero), we obtain

$$\overline{R}(\alpha-\varepsilon,\alpha+\gamma-\varepsilon)=R_\varepsilon(u,v),$$

and

$$(9)\qquad I(\delta,\varepsilon)=\int_{\Omega(\delta)}R_\varepsilon^2(u,v)d\alpha d\gamma,$$

where u,v depend on α,γ as shown in (8). Now set

$$(10)\qquad z_j=\begin{cases}1-\alpha_j-\gamma_j; & \delta_j=0,\\ 1-\gamma_j; & \delta_j=1,\end{cases}$$

and change in (9) the integration variables α,γ to the new variables z,v. The region of change of these new variables is contained (as follows from (8) and (10)) in the closed unit cube $\overline{\mathbf{Q}}^s$, and the old variables α,γ may be expressed in terms of the new ones by

$$\alpha_j=\begin{cases}1-v_j; & \varepsilon_j=0,\\ v_j+z_j; & \varepsilon_j=1,\end{cases}\qquad \gamma_j=\begin{cases}v_j-z_j; & \delta_j=0,\\ 1-z_j; & \delta_j=1.\end{cases}$$

It is an easy technical exercise to verify that the Jacobian of our change of variables is

$$\frac{D(\alpha,\gamma)}{D(z,v)} = (-1)^{s-\nu(\varepsilon)}.$$

Therefore

$$I(\delta,\varepsilon) \leqq \int_{z,v\in \mathbf{Q}^s} R_\varepsilon^2(u,v)dzdv,$$

where

$$u_j = \begin{cases} z_j; & \delta_j = 0, \\ 0; & \delta_j = 1. \end{cases}$$

The function $R_\varepsilon^2(u,v)$ in the last integral does not depend on the variables z_j with indices j under $\delta_j = 1$. Thus after integrating over these variables we obtain

$$I(\delta,\varepsilon) \leqq \int R_\varepsilon^2(u,v)\, dudv$$

where we set

$$du = \prod_{\delta_j=0} du_j,$$

and the region of integration is defined by $u \in \mathbf{Q}^{s-\nu(\delta)}$, $v \in \mathbf{Q}^s$ (while the coordinates u_j for the indices j under $\delta_j = 1$ are constant and equal to 0).

For $\mu \in \{0,1\}^s$, $\mu \geqq \delta$ set

$$t_j = \begin{cases} u_j; & \mu_j = 0 \\ v_j; & \mu_j = 1 \end{cases} \qquad (j = 1,\dots,s).$$

Then it is easily seen that

$$R_\varepsilon(u,v) = \sum_{\mu\geqq\delta} (-1)^{s-\nu(\mu)} R_\varepsilon(\mathbf{0},t)$$

(one can consider this equality as a variant of the inclusion-exclusion formula), and so

$$I(\delta,\varepsilon) \leqq 2^{s-\nu(\delta)} \sum_{\mu\geqq\delta} \int_{\mathbf{Q}^{s-\nu(\delta)}\times\mathbf{Q}^s} R_\varepsilon^2(\mathbf{0},t) dudv.$$

Now performing integration by variables u_j for the indices j under $\mu_j = 1$, $\delta_j = 0$, and for the variables v_j for the indices j under $\mu_j = 0$, we obtain

$$I(\delta,\varepsilon) \leqq 2^{s-\nu(\delta)} \sum_{\mu\geqq\delta} \int_{\mathbf{Q}^s} R_\varepsilon^2(\mathbf{0},t)dt = 2^{2(s-\nu(\delta))} D_2^2(S_\varepsilon),$$

so that from (7)

$$\widetilde{D}_2^2(S) \leqq \sum_{\delta\in\{0,1\}^s} 2^{\nu(\delta)} \sum_{\varepsilon\leqq\delta} 2^{2(s-\nu(\delta))} D_2^2(S_\varepsilon) = \sum_{\varepsilon\in\{0,1\}^s} D_2^2(S_\varepsilon) \sum_{\delta\geqq\varepsilon} 2^{2s} 2^{-\nu(\delta)} =$$

$$= 4^s \sum_{\varepsilon\in\{0,1\}^s} (3/2)^{s-\nu(\varepsilon)} (1/2)^{\nu(\varepsilon)} D_2^2(S_\varepsilon) = 6^s \sum_{\varepsilon\in\{0,1\}^s} 3^{-\nu(\varepsilon)} D_2^2(S_\varepsilon),$$

which was to be proved.

As a direct corollary of the theorem we obtain the required estimate of type $\widetilde{D}_2(S) \ll D_2(S)$:

COROLLARY. *Let S be a symmetric net having all its points strictly inside $\mathbf{Q}^s$. Then*

$$\widetilde{D}_2(S) \leqq 8^{s/2} D_2(S).$$

PROOF. For a net S of the considered type all the discrepancies $D_2(S_\varepsilon)$ pairwise coincide and are equal to $D_2(S)$; therefore, in view of the theorem,

$$\widetilde{D}_2^2(S) \leqq 6^s D_2^2(S) \sum_{\varepsilon\in\{0,1\}^s} 3^{-\nu(\varepsilon)} = 6^s (4/3)^s D_2^2(S) = 8^s D_2^2(S).$$

NOTE. A refinement of the proof allows us to write the inequality of the theorem in the form

$$\widetilde{D}_s^2(S) \leqq 3^s \sum_{\varepsilon\in\{0,1\}^s} D_2^2(S_\varepsilon).$$

In turn, this makes it possible to improve the constant of the corollary up to $6^{s/2}$.

5. $\widetilde{D}_2$ versus D_2: symmetric nets. The estimate $D_2(S) \ll \widetilde{D}_2(S)$

To complete our comparison of L^2-discrepancies of symmetric nets, we are going to obtain an inverse estimate of type $D_2(S) \ll \widetilde{D}_2(S)$. The historical aspect of the estimate was described in the previous section.

We start from the following lemma.

LEMMA 2. *We have*

$$D_2^2(S) = 4^{-s} \sum_{m\in\mathbf{Z}^s} \frac{\pi^{-2\nu(m)}}{\overline{m}^2} \Big| \sum_{k=1}^{N} \rho_k \prod_{j=1}^{s} c_{m_j}(x_j^{(k)}) - 1 \Big|^2,$$

where

$$c_{m_j}(x_j^{(k)}) = \begin{cases} 2(1-x_j^{(k)}); & m_j = 0, \\ 1 - e^{-2\pi i m_j x_j^{(k)}}; & m_j \neq 0. \end{cases}$$

PROOF. The assertion of the lemma is obtained by applying Parseval's equality to the function $R(\mathbf{0}, \gamma)$. If we set

$$b_{m_j} = \begin{cases} -i/\pi; & m_j < 0, \\ 1; & m_j = 0, \\ i/\pi; & m_j > 0, \end{cases}$$

then the Fourier coefficients of the function are

$$\widehat{R}(m) = \int_{\mathbf{Q}^s} R(\mathbf{0}, \gamma) e^{-2\pi i \langle m \cdot \gamma \rangle} d\gamma =$$

$$= \sum_{k=1}^{N} \rho_k \int_{\gamma > x^{(k)}} e^{-2\pi i \langle m \cdot \gamma \rangle} d\gamma - \int_{\mathbf{Q}^s} \gamma_1 \cdots \gamma_s e^{-2\pi i \langle m \cdot \gamma \rangle} d\gamma =$$

$$= \sum_{k=1}^{N} \rho_k \prod_{j=1}^{s} \int_{x_j^{(k)}}^{1} e^{-2\pi i m_j \gamma_j} d\gamma_j - \prod_{j=1}^{s} \int_0^1 \gamma_j e^{-2\pi i m_j \gamma_j} d\gamma_j =$$

$$= \sum_{k=1}^{N} \rho_k \prod_{j=1}^{s} \frac{1}{2\overline{m}_j} c_{m_j}(x_j^{(k)}) b_{m_j} - \prod_{j=1}^{s} \frac{1}{2\overline{m}_j} b_{m_j} =$$

$$= \frac{1}{2^s \overline{m}} \left(\sum_{k=1}^{N} \rho_k \prod_{j=1}^{s} c_{m_j}(x_j^{(k)}) - 1 \right) \cdot \prod_{j=1}^{s} b_{m_j},$$

and the rest is obvious.

THEOREM 4. *Let S be a symmetric net having all its points strictly inside $\mathbf{Q}^s$. Then*

$$D_2(S) \leqq (5/4)^{s/2} \widetilde{D}_2(S).$$

PROOF. We use the notions of Lemma 2, and also set $|m| = \big(|m_1|, \ldots, |m_s|\big)$, so that, for instance, the symbol $\sum_{\varepsilon \leqq |m|}$ will denote the sum over all vectors $\varepsilon \in \{0,1\}^s$ with $\varepsilon_j = 0$ for all j under $m_j = 0$. In view of the symmetry of S we have

$$\sum_{k=1}^{N} \rho_k \prod_{j=1}^{s} c_{m_j}(x_j^{(k)}) =$$

$$= \sum_{k=1}^{N} \rho_k \prod_{m_j \neq 0} c_{m_j}(x_j^{(k)}) \prod_{m_j=0} \frac{1}{2} \left(c_{m_j}(x_j^{(k)}) + c_{m_j}(1 - x_j^{(k)}) \right) =$$

$$= \sum_{k=1}^{N} \rho_k \prod_{m_j \neq 0} c_{m_j}(x_j^{(k)}),$$

and by Lemma 2

$$D_2^2(S) = 4^{-s} \sum_{m} \frac{\pi^{-2\nu(m)}}{\overline{m}^2} \Big| \sum_{k=1}^{N} \rho_k \prod_{m_j \neq 0} c_{m_j}(x_j^{(k)}) - 1 \Big|^2 =$$

$$= 4^{-s} \sum_{m} \frac{\pi^{-2\nu(m)}}{\overline{m}^2} \Big| \sum_{k=1}^{N} \rho_k \sum_{\varepsilon \leqq |m|} (-1)^{\nu(\varepsilon)} e^{-2\pi i (m_1 \varepsilon_1 x_1^{(k)} + \dots + m_s \varepsilon_s x_s^{(k)})} - 1 \Big|^2 =$$

$$= 4^{-s} \sum_{m} \frac{\pi^{-2\nu(m)}}{\overline{m}^2} \Big| \sum_{\varepsilon \leqq |m|}' (-1)^{\nu(\varepsilon)} T\big((m_1\varepsilon_1, \dots, m_s\varepsilon_s)\big) + (T(\mathbf{0}) - 1) \Big|^2 \leqq$$

$$\leqq 4^{-s} \sum_{m} \frac{\pi^{-2\nu(m)}}{\overline{m}^2} 2^{\nu(m)} \Big(\sum_{\varepsilon \leqq |m|}' \big| T\big((m_1\varepsilon_1, \dots, m_s\varepsilon_s)\big) \big|^2 + \big| T(\mathbf{0}) - 1 \big|^2 \Big) =$$

$$= 4^{-s} \sum_{\varepsilon \in \{0,1\}^s}' \sum_{|m| \geqq \varepsilon} (2/\pi^2)^{\nu(m)} \big| T\big((m_1\varepsilon_1, \dots, m_s\varepsilon_s)\big) \big|^2 / \overline{m}^2 +$$

$$+ 4^{-s} \big| T(\mathbf{0}) - 1 \big|^2 \sum_{m} \frac{1}{\overline{m}^2} (2/\pi^2)^{\nu(m)}.$$

The second summand on the right hand side may be calculated directly, while in the inner sum of the first summand one can execute direct summing over the variables m_j for the indices j under $\varepsilon_j = 0$. We obtain

(11) $$D_2^2(S) \leqq 4^{-s} \sum_{\varepsilon}' \prod_{\varepsilon_j = 0} \left(1 + 2 \frac{2}{\pi^2} \frac{\pi^2}{6} \right) \cdot$$

$$\cdot\sum_m{}^{*}(2/\pi^2)^{\nu(m)}\left|T\big((m_1\varepsilon_1,\dots,m_s\varepsilon_s)\big)\right|^2/\overline{m}^2+(5/12)^s\left|T(\mathbf{0})-1\right|^2=$$

$$=4^{-s}\sum_\varepsilon{}'(5/3)^{s-\nu(\varepsilon)}\sum_m{}^{*}(2/\pi^2)^{\nu(m)}\frac{\left|T(m)\right|^2}{\overline{m}^2}+(5/12)^s\left|T(\mathbf{0})-1\right|^2,$$

where the symbol $\sum_m^*$ means summing over all vectors m for which m_j takes the single integer value 0 if $\varepsilon_j=0$, and m_j takes all non-zero integer values otherwise.

Furthermore, in (11) each summand of the form $\left|T(m)\right|^2/\overline{m}^2$ with $m\neq\mathbf{0}$ appears precisely for one value of ε, and for this value $\nu(m)=\nu(\varepsilon)$. Hence

$$D_2^2(S)\leqq 4^{-s}(5/3)^s\sum_m{}'\frac{\left|T(m)\right|^2}{\overline{m}^2}\cdot(5/3)^{-\nu(m)}(2/\pi^2)^{\nu(m)}+$$

$$+(5/12)^s\left|T(\mathbf{0})-1\right|^2\leqq$$

$$\leqq 3^{-s}(5/4)^s\sum_m{}'\frac{\left|T(m)\right|^2}{\overline{m}^2}\cdot(6/5\pi^2)^{\nu(m)}+3^{-s}(5/4)^s\left|T(\mathbf{0})-1\right|^2\leqq$$

$$\leqq 3^{-s}(5/4)^s\Big(\sum_m{}'\frac{\left|T(m)\right|^2}{\overline{m}^2}\cdot(3/2\pi^2)^{\nu(m)}(4/5)^{\nu(m)}+\left|T(\mathbf{0})-1\right|^2\Big),$$

and it remains to apply Theorem 1.

It is seen from our proof that for a *normed* net S the constant of Theorem 4 may be slightly improved: namely, if S is a normed symmetric net having all its points strictly inside $\mathbf{Q}^s$, then $D_2(S)\leqq(5/4)^{(s-1)/2}\widetilde{D}_2(S)$.

6. Nets with $D_2(S)=o\big(\widetilde{D}_2(S)\big)$

We have seen so far that both one-dimensional and symmetric nets satisfy $\widetilde{D}_2(S)\ll D_2(S)$. The same clearly applies to the nets S having L^2-discrepancies $D_2(S+\alpha)$ of the same order of value for all $\alpha\in\mathbf{Q}^s$ (see the note at the end of Section 1). We now add one more example to this collection of nets with $\widetilde{D}_2(S)\ll D_2(S)$.

As in Section 4, denote by S_ε the net obtained from S by means of the symmetry σ_ε relative to all hyperplanes $x_j=1/2$ with indices j under the condition $\varepsilon_j=0$, and also denote by S'_ε the $\nu(\varepsilon)$-dimensional net[2] obtained

[2] Strictly speaking, we have no definition of "0-dimensional" net. To avoid problems arising for $\varepsilon=0$ we put directly $D_2^2(S'_0)=(\rho_0-1)^2$ in the formulae below.

from S by means of the projection π_ε, defined by

$$\pi_\varepsilon : x \mapsto x', \quad x'_j = \begin{cases} x_j; & \varepsilon_j = 1, \\ 0; & \varepsilon_j = 0. \end{cases}$$

LEMMA 3. *Let S be a net having all its points strictly inside $\mathbf{Q}^s$. Then the Weyl L^2-discrepancy $\widetilde{D}_2(S)$ may be estimated by "usual" L^2-discrepancies $D_2(S'_\varepsilon)$ of projections of S as follows:*

$$\widetilde{D}_2^2(S) \leqq 12^s \sum_{\varepsilon \in \{0,1\}^s} (7/6)^{\nu(\varepsilon)} D_2^2(S'_\varepsilon).$$

PROOF. Our lemma will follow immediately from Theorem 3 and the inequality

$$D_2^2(S_\varepsilon) \leqq 2^{s-\nu(\varepsilon)} \sum_{\delta \in \{0,1\}^s,\ \delta \geqq \varepsilon} D_2^2(S'_\delta), \tag{12}$$

since, assuming the inequality to be true, we obtain:

$$\widetilde{D}_2^2(S) \leqq 6^s \sum_{\varepsilon \in \{0,1\}^s} 3^{-\nu(\varepsilon)} \cdot 2^{s-\nu(\varepsilon)} \sum_{\delta \geqq \varepsilon} D_2^2(S'_\delta) \leqq$$

$$\leqq 12^s \sum_{\delta \in \{0,1\}^s} D_2^2(S'_\delta) \sum_{\varepsilon \leqq \delta} 6^{-\nu(\varepsilon)} = 12^s \sum_{\delta \in \{0,1\}^s} (7/6)^{\nu(\delta)} D_2^2(S'_\delta).$$

Therefore, we need only to prove (12).

To this end, denote by R_ε and R'_ε, respectively, the local discrepancies of S_ε and S'_ε, and for $\varepsilon, \delta \in \{0,1\}^s$ set also

$$u_j = \begin{cases} 1 - \gamma_j; & \varepsilon_j = 0 \\ 0; & \varepsilon_j = 1 \end{cases}, \qquad v_j = \begin{cases} 1; & \varepsilon_j = 0 \\ \gamma_j; & \varepsilon_j = 1 \end{cases},$$

$$z_j = \begin{cases} 1; & \varepsilon_j = 0,\ \delta_j = 0 \\ 1 - \gamma_j; & \varepsilon_j = 0,\ \delta_j = 1 \\ \gamma_j; & \varepsilon_j = 1,\ \delta_j = 1 \end{cases}.$$

Combinatorial considerations show that for almost all γ

$$R_\varepsilon(\mathbf{0}, \gamma) = R(u, v) = \sum_{\delta \geqq \varepsilon} (-1)^{\nu(\delta) - \nu(\varepsilon)} R(\mathbf{0}, z),$$

and to complete the proof, it is sufficient to observe that

$$\int_{\mathbf{Q}^s} R^2(\mathbf{0}, z)d\gamma = D_2^2(S'_\delta).$$

COROLLARY. *Let S be a net with all the points strictly inside $\mathbf{Q}^s$ and satisfying for some constant C*

$$D_2(S'_\varepsilon) \leqq C D_2(S) \qquad \text{for all} \quad \varepsilon \in \{0,1\}^s$$

(that is, the L^2-discrepancies of the projections of S are at most of the same order of value as the L^2-discrepancy of S itself). Then

$$\widetilde{D}_2(S) \leqq 26^{s/2} C D_2(S).$$

As one can see, we have now a large number of nets with $\widetilde{D}_2(S) \ll D_2(S)$ (one-dimensional ones and those with "good" shifts, symmetries or projections). Does the estimate hold *for all nets*? Here is a counterexample.

EXAMPLE 2. Let $s \geqq 2$, and let $S = (X, \rho)$ vary over a sequence of s-dimensional nets with the number of nodes increasing to infinity and the *Weyl supreme discrepancy* decreasing to zero: $N \to \infty$, $\widetilde{D}(S) \to 0$. Set

$$\delta = \big(\widetilde{D}(S)\big)^{1/2s} \to 0, \quad P = \big\{x \in \mathbf{Q}^s \mid x \geqq (1-\delta)\mathbf{1}\big\}$$

(so that δ and P depend on S), and consider a new net $S' = (Y, \rho)$ with the same weights as S and the system of points Y, defined by

$$y^{(k)} = \begin{cases} x^{(k)}; & x^{(k)} \notin P, \\ (1-\delta)\mathbf{1}; & x^{(k)} \in P \end{cases}$$

(compare with Example 1). Denote the local discrepancy of S' by R'. It is easily seen that

$$\left| R'(\mathbf{0}, \gamma) \right| \leqq \begin{cases} \widetilde{D}(S); & \gamma \notin P, \\ 2\widetilde{D}(S) + \delta^s; & \gamma \in P \end{cases}$$

and therefore

$$(13) \qquad D_2^2(S') \leqq \int_{\gamma \notin P} \widetilde{D}^2(S) d\gamma + 8 \int_{\gamma \in P} (\widetilde{D}^2(S) + \delta^{2s}) d\gamma \leqq$$

$$\leqq 8(\widetilde{D}^2(S) + \delta^{3s}) \ll \delta^{3s}.$$

On the other hand if, for instance,

$$(14)\qquad \begin{cases} 1-\delta < \alpha_1 < 1-\delta/2, \\ 1/2 < \alpha_2, \dots, \alpha_s < 1-\delta, \ 1/2 < \gamma_1, \dots, \gamma_s > 3/4, \end{cases}$$

then $\Pi(\alpha, \alpha+\gamma)$ does not contain the point $(1-\delta)\mathbf{1}$, but contains the parallelepiped

$$\Pi\big((1-\delta/2, 1-\delta, \dots, 1-\delta), \mathbf{1}\big),$$

of the value $0.5\delta^s$, free of points of S'. Then local discrepancy of S' in $\Pi(\alpha, \alpha+\gamma)$ is at least $0.5\delta^s - 2\widetilde{D}(S) \gg \delta^s$, and the volume of the region (14) is $\gg \delta$; therefore

$$(15)\qquad \widetilde{D}_2^2(S') \gg \delta \cdot \delta^{2s} = \delta^{2s+1}.$$

From (13) and (15) we conclude that for no constant C is $\widetilde{D}_2(S') \leqq CD_2(S')$ for all the nets S'.

7. Behavior of $D_2(S)$ under shifts, projections and symmetries of S

Finally, we consider once again Statements 1–3 from Section 2 and Corollary 2 of Theorem 1, this time from the following point of view: are the analogous properties satisfied for the "usual" L^2-discrepancy $D_2(S)$ instead of the Weyl L^2-discrepancy $\widetilde{D}_2(S)$ and the diaphony $F_2(S)$? In other words, is the "usual" L^2-discrepancy $D_2(S)$ invariant relative to shifts and symmetries and non-increasing under projections of net, at least, by the order of value?

STATEMENT 1'. *For any constant C there exists an s-dimensional net S and a vector $\alpha \in \mathbf{Q}^s$ satisfying*

$$D_2(S+\alpha) > C \cdot D_2(S).$$

PROOF. Suppose, on the contrary, $D_2(S+\alpha) \leqq CD_2(S)$ for all S, α. Then also $D_2(S) \leqq CD_2(S+\alpha)$, so

$$D_2(S) \leqq C\left(\int_{\mathbf{Q}^s} D_2^2(S+\alpha)\,d\alpha\right)^{1/2} = C\widetilde{D}_2(S),$$

in contradiction to Example 1.

STATEMENT 2′. *For any constant C and $s \geqq 2$ there exist an s-dimensional net S and a projection π such that $S' = \pi(S)$ satisfies*

$$D_2(S') > C \cdot D_2(S).$$

PROOF. Suppose, on the contrary, $D_2(S') \leqq CD_2(S)$ for all $S' = \pi(S)$. Then, in view of the corollary to Lemma 3, the following inequality is satisfied:

$$\widetilde{D}_2(S) \leqq 26^{s/2} C D_2(S),$$

in contradiction to Example 2.

STATEMENT 3′. *For any constant C there exist an s-dimensional net S and a symmetry σ such that $S' = \sigma(S)$ satisfies*

$$D_2(S') > C \cdot D_2(S).$$

PROOF. Suppose, on the contrary, $D_2(S') \leqq CD_2(S)$ for all $S' = \sigma(S)$. Then, if S has no points on the surface of $\mathbf{Q}^s$, in view of Theorem 3

$$\widetilde{D}_2^2 \leqq 6^s \cdot C^2 D_2^2(S) \sum_{\varepsilon \in \{0,1\}^s} 3^{-\nu(\varepsilon)} \leqq C^2 8^s D_2^2(S),$$

in contradiction to Example 2.

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JORDAN–VON NEUMANN THEOREM FOR SAWOROTNOW'S GENERALIZED HILBERT SPACE

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Introduction

Jordan–von Neumann theorem was proved in 1935. The original paper is [10]. The precise statement is the following:

THEOREM J-N. *Let $\mathcal{X}$ be a real or complex normed space and suppose that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ holds for all $x, y \in \mathcal{X}$. Then there exists a real (respectively complex) inner product $\langle x, y\rangle$ on $\mathcal{X}$ such that $\langle x, x\rangle = \|x\|^2$ for all $x \in \mathcal{X}$.*

The identity $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ is called the parallelogram identity. If we have a Hilbert space $\mathcal{H}$ with the inner product $\langle x, y\rangle$, then an easy calculation shows that the norm $\|x\| = \sqrt{\langle x, x\rangle}$ satisfies the parallelogram identity. This means that the Jordan–von Neumann theorem establishes a characterization of inner product spaces among normed spaces.

This result initiated a lot of subsequent research and still there are some open questions. One line of research was concerned with the problem which other properties of normed spaces characterize inner product spaces. An extensive collection of such results, in various directions, can be found in [39]. Another line, which preceded our present paper, started with the following observation:

If $Q(x) = \|x\|^2$, then Q is a quadratic functional while the inner product is a sesquilinear form. The Jordan–von Neumann theorem then tells us that every positive definite quadratic functional on a real or complex vector space can be represented by a hermitian positive definite sesquilinear form. The question now arises what can be said about quadratic functionals which are not positive definite. More precisely, can such functional be represented by (in general not even hermitian) sesquilinear form? It turned out that in the real case the answer is no while in the complex case the answer is yes.

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Investigations of modules over more general involutive rings, such as C^*-algebras, quaternions and function algebras for example, led several authors to investigate Jordan–von Neumann type theorems, usually using the algebraic approach. Among the positive results we mention that the representation theorem is true for modules over the following algebras: quaternions, octonions, any complex algebra with identity and bounded linear operators on a real Hilbert space. For more information the reader should consult [1], [8], [12], [28–31], [33–36] and [38].

In the present paper we investigate Jordan–von Neumann type theorems for Saworotnow's generalized Hilbert space which is a module over a so called H^*-algebra. The difference between this investigation and previous papers lies in the fact that H^*-algebras in general do not possess an identity element nor can the identity be added to them within their category. This forced us to develop a technique which uses the presence of sufficient number of projections in H^*-algebras.

General ideas for this investigation were set in Debrecen in August 1993 during the conference on functional equations. In a very relaxing and stimulating atmosphere, created by the members of Debrecen's chair of analysis, I get acquainted with the work of Lajos Molnár on generalized Hilbert spaces which together with my earlier paper [38] suggested the possibility of the result presented. Proofs were carried out during the winter of 1993/94 when I was partially supported by the grant from the Slovenian government.

Proper H^*-algebras

H^*-algebras were introduced in 1945 by Ambrose. In his pioneering paper [2] the intention was to provide an abstract framework for the class of Hilbert–Schmidt operators. Later other authors studied trace-class, centralizers, representations, characterizations and generalizations of H^*-algebras. Some papers on this subject are [4], [7], [11], [14] and [26–27] where further references are available. In this section we recall facts about H^*-algebras we use later.

DEFINITION 1. Let $\mathcal{A}$ be a complex associative algebra with an involution $*$ and a complex Hilbert space with respect to the inner product $\langle\ ,\ \rangle$. Then $\mathcal{A}$ is called an *H^*-algebra* if the following identity, which connects the product, the inner product and the involution of $\mathcal{A}$,

$$\langle xy, z\rangle = \langle x, zy^*\rangle = \langle y, x^*z\rangle \tag{1}$$

holds for all $x, y, z \in \mathcal{A}$.

$\mathcal{A}$ is called *proper* if $a\mathcal{A} = (0)$ implies $a = 0$. According to [2], this is equivalent to the fact that $\mathcal{A}a = (0)$ implies $a = 0$. From now on all H^*-

algebras will be proper. The following result was proved in part by Ambrose and in part by Saworotnow.

PROPOSITION 2. *Let $\mathcal{A}$ be a proper H^*-algebra. Then the following holds:*
(1) *The multiplication of $\mathcal{A}$ is jointly continuous.*
(2) *The involution of $\mathcal{A}$ is isometric.*

From [26–27] we know that the set

$$\mathcal{T}(\mathcal{A}) = \{xy\ ;\ x, y \in \mathcal{A}\}$$

is a selfadjoint ideal of $\mathcal{A}$ which is dense in $\mathcal{A}$ with respect to the Hilbert space topology. This ideal is called a *trace class* of $\mathcal{A}$. One can define a trace $\mathrm{Tr} : \mathcal{T}(\mathcal{A}) \to \mathbf{C}$ by $\mathrm{Tr}(xy) = \langle x, y^* \rangle$.

An element $a \in \mathcal{A}$ is *positive* if $\langle ax, x \rangle \geqq 0$ for all $x \in \mathcal{A}$. Because of the H^*-identity (1), this is equivalent to the fact that $\langle xa, x \rangle \geqq 0$ for all $x \in \mathcal{A}$. It is an elementary exercise to verify that every positive element a is selfadjoint, i.e. $a^* = a$. The following result from [26] is important in the study of the trace class.

PROPOSITION 3. *Let $a \in \mathcal{T}(\mathcal{A})$ be positive. Then there is a unique element $b \in \mathcal{A}$ such that b is positive and $b^2 = a$ holds.*

The element b is called a *square root* of a. Note that b may not be a trace class element. From the above we have immediately

COROLLARY 4. *Let $a, b \in \mathcal{A}$ be positive. If $a^2 = b^2$, then $a = b$.*

Another useful application of Proposition 3 is the following

COROLLARY 5. *There is an absolute value $|.| : \mathcal{A} \to \mathcal{A}^+$ defined by $|a| = \sqrt{aa^*}$.*

PROOF. From the H^*-identity (1) it follows

$$\langle aa^*x, x \rangle = \langle a^*x, a^*x \rangle = \|a^*x\|^2 \geqq 0$$

for all $x \in \mathcal{A}$ and so aa^* is positive. Since $aa^* \in \mathcal{T}(\mathcal{A})$, we can now apply Proposition 3. □

Now we can define a norm on the trace class (see [26–27]) by $\tau(a) = \mathrm{Tr}\left(|a|\right)$. The following is then true:

PROPOSITION 6. (1) *$\|a\| \leqq \tau(a)$ for every a in the trace class.*
(2) *$\tau(a^2) = \|a\|^2$ for every selfadjoint $a \in \mathcal{A}$.*
(3) *$\mathcal{T}(\mathcal{A})$ is complete with respect to the norm τ.*
(4) *$\left|\mathrm{Tr}(a)\right| \leqq \tau(a)$ for every $a \in \mathcal{A}$.*

A nonzero element $p \in \mathcal{A}$ is called a *projection* if $p = p^* = p^2$ holds. It is further called *minimal* if $p\mathcal{A}p = \mathbf{C}p$. From [2] and [11] we need the following facts about projections in H^*-algebras:

PROPOSITION 7. (1) *There exists an approximate identity* $\{p_\alpha\}$ *for* $\mathcal{A}$ *consisting of projections.*

(2) *Every projection is a finite sum of minimal projections* p_i *which are pairwise orthogonal in the sense that* $p_i p_j = 0$ *for* $i \neq j$.

Saworotnow's pre-Hilbert $\mathcal{A}$-modules and normed $\mathcal{A}$-modules

Let $\mathcal{A}$ be a proper H^*-algebra. Let $\mathcal{H}$ be an additive group and a module over $\mathcal{A}$. Some authors prefer left, some prefer right modules. Both theories are of course equivalent. We use the left module concept. Therefore we assume that a biadditive mapping $\circ : \mathcal{A} \times \mathcal{H} \to \mathcal{H}$ is given satisfying $(ab) \circ \circ x = a \circ (b \circ x)$. This mapping is called a *module multiplication.*

DEFINITION 8. $\mathcal{H}$ is called a *Saworotnow's pre-Hilbert* $\mathcal{A}$*-module* if there exists a mapping, called a *generalized inner product,* $[\ ,\ ] : \mathcal{H} \times \mathcal{H} \to \mathcal{T}(\mathcal{A})$ satisfying the following axioms:

(i) $[x, x]$ is positive for every $x \in \mathcal{H}$.

(ii) $[x, x] = 0$ implies $x = 0$.

(iii) $[y, x] = [x, y]^*$ holds for all $x, y \in \mathcal{H}$.

(iv) $[a \circ x, y] = a[x, y]$ holds for all $x, y \in \mathcal{H}$ and $a \in \mathcal{A}$.

If we take the H^*-algebra $\mathcal{A} = \mathbf{C}$, we get the usual inner product space. In general however this generalized Hilbert space has noncommutative and infinite-dimensional 'scalars'. Hilbert $\mathcal{A}$-modules were first defined in [21]. Bases for their theory were set in [9], [22–25] and [32]. Some recent papers on this subject are [5–6], [13] and [15–19].

REMARK. The original set of axioms was richer for two more axioms. One was the weak form of the Cauchy–Schwarz inequality which was proved to be redundant and even improved to a strong form by Lajos Molnár. The second one required that $\mathcal{H}$ should be complete with respect to a suitable metric. We omit this axiom because we are also interested in noncomplete pre-Hilbert modules.

Let $\mathcal{X}$ be another left module over $\mathcal{A}$. Now we give a set of axioms for a generalized normed space over $\mathcal{A}$.

DEFINITION 9. Let $N : \mathcal{X} \to \mathcal{A}$ be a mapping with the following properties:

(i) $N(x)$ is positive for every $x \in \mathcal{X}$.

(ii) $N(x) = 0$ implies $x = 0$.

(iii) $N(a \circ x) = |aN(x)|$ holds for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$.

(iv) $\|N(x + y)\| \leqq \|N(x)\| + \|N(y)\|$ holds for all $x, y \in \mathcal{X}$.

(v) If $\{x_\alpha\} \subset \mathcal{X}$ is a generalized sequence such that for all $\varepsilon > 0$ there exists α_0 such that for all $\alpha, \beta \geqq \alpha_0$ we have $\|N(x_\alpha - x_\beta)\| < \varepsilon$, then $N(x_\alpha)$ is a Cauchy sequence in $\mathcal{A}$.

Then $(\mathcal{X}, N)$ is called a generalized normed space.

REMARKS. If $\mathcal{A} = \mathbf{C}$ is the H^*-algebra of complex numbers with the usual involution and absolute value, then $\mathcal{X}$ is a usual complex normed space.

The notation $|.|$ stands for the absolute value in $\mathcal{A}$ defined by Corollary 5. Note that $|aN(x)|$ is in general not equal to $|a|N(x)$ because a and $N(x)$ may not commute. The notation $\|a\| = \sqrt{\langle a, a\rangle}$ stands for the Hilbert space norm in $\mathcal{A}$. Axiom (v) represents some sort of continuity property for N. We could actually define a metric topology on $\mathcal{X}$ by $\operatorname{dist}(x, y) = \|N(x - y)\|$. From axioms (i)–(iv) it follows that this is in fact a metric on $\mathcal{X}$. Then axiom (v) tells us that N maps Cauchy sequences in $(\mathcal{X}, \operatorname{dist})$ to Cauchy sequences in $\mathcal{A}$. In the classical normed space this is a consequence of the triangle inequality. Here we have the triangle inequality only for the composition of the norm in $\mathcal{A}$ and N rather than for N itself and so (v) must be stated as a separate axiom. It is easy to see that if $\mathcal{A}$ is not commutative, then the triangle inequality for N itself does not hold.

Our first goal is to prove that every pre-Hilbert $\mathcal{A}$-module $\mathcal{H}$ is a normed $\mathcal{A}$-module. This is not obvious because one must incorporate in the proof special properties of H^*-algebras. Since $[x, x]$ is a positive trace class element, we can use Corollary 5 in order to define $N : \mathcal{H} \to \mathcal{A}$ by $N(x) = \sqrt{[x, x]}$. From Definition 8 it easily follows that $N(x) = 0$ implies $x = 0$. Therefore it remains to verify (iii), (iv) and (v) from Definition 9.

Part (1) of the following lemma has a very similar proof as one of the results in [16]. Since the proof is short, we repeat some of the arguments for the sake of completeness and to illustrate a technique one often uses when dealing with H^*-algebras without identity element.

LEMMA 10. *Let $\mathcal{H}$ be a pre-Hilbert module over a proper H^*-algebra $\mathcal{A}$. Then we have*

(1) *If $x, y \in \mathcal{H}$ and $p \in \mathcal{A}$ is a projection, then the inequality*

$$\langle p[x, y], p\rangle + \langle p[y, x], p\rangle \leqq 2\|pN(x)\| \cdot \|pN(y)\|$$

holds.

(2) *Let $a = a^* \in \mathcal{A}$. Then for every $x \in \mathcal{A}$ we have $|\langle ax, x\rangle| \leqq \langle |a|x, x\rangle$.*

(3) *If $a, b \in \mathcal{A}$ are positive, then $\|a - b\|^2 \leqq \tau(a^2 - b^2)$.*

PROOF. (1) Take any real number t. Since the element $[x + (tp) \circ y, x + (tp) \circ y]$ is positive in $\mathcal{A}$, we have

$$0 \leqq \langle p[x + (tp) \circ y, x + (tp) \circ y], p\rangle.$$

Applying axioms from Definition 8, we have

$$0 \leqq \langle p[x, x], p\rangle + t\langle p^2[y, x], p\rangle + t\langle p[x, y]p^*, p\rangle + t^2\langle p[y, y]p^*, p\rangle.$$

Next we use that p is a projection and the H^*-identity (1) which results in

$$0 \leqq \langle p[x,x],p\rangle + t\left(\langle p[x,y],p\rangle + \langle p[y,x],p\rangle\right) + t^2\langle p[y,y],p\rangle$$

for all real t. Since $[x,x]$ and $[y,y]$ are positive, it follows that the middle term is real. Thus

$$\langle p[x,y],p\rangle + \langle p[y,x],p\rangle \leqq |\langle p[x,y],p\rangle + \langle p[y,x],p\rangle| \leqq$$
$$\leqq 2\sqrt{\langle p[x,x],p\rangle}\sqrt{\langle p[y,y],p\rangle}.$$

If we write $N(x) = \sqrt{[x,x]}$, then $N(x)^* = N(x)$ since $N(x)$ is positive. Hence

$$\langle p[x,y],p\rangle + \langle p[y,x],p\rangle \leqq 2\sqrt{\langle pN(x)^2,p\rangle}\sqrt{\langle pN(y)^2,p\rangle} =$$
$$= 2\sqrt{\langle pN(x),pN(x)\rangle}\sqrt{\langle pN(y),pN(y)\rangle} = 2\|pN(x)\|\cdot\|pN(y)\|.$$

(2) From [26, Lemma 1] we know that a can be expressed as $a = \sum \lambda_n e_n$ where e_n are pairwise orthogonal projections and λ_n nonzero scalars. Since $a = a^*$, all λ_n are real numbers. The above series converges in the Hilbert space topology. It is also easy to verify that λ_n tends to zero. If we set $b = \sum |\lambda_n| e_n$, then this series also converges so b is well-defined. Since $b^2 = a^2 = \sum \lambda_n^2 e_n$ and b is positive, it follows $b = |a|$.

Now we can verify in a direct way

$$\langle |a|x,x\rangle = \sum |\lambda_n|\langle e_n x,x\rangle = \sum |\lambda_n|\,\|e_n x\|^2,$$
$$|\langle ax,x\rangle| = \left|\sum \lambda_n\langle e_n x,x\rangle\right| \leqq \sum |\lambda_n||\langle e_n x,x\rangle| = \langle |a|x,x\rangle.$$

(3) From [26] we know that the trace can be represented also as $\mathrm{Tr}(a) = \sum\langle ae_\alpha, e_\alpha\rangle$ where $\{e_\alpha\}$ is (any) maximal family of pairwise orthogonal projections and $a \in \mathcal{T}(\mathcal{A})$.

The element $a - b$ is self-adjoint so we have $a - b = \sum \lambda_n e_n$ where $0 \neq \lambda_n \in \mathbf{R}$. Using the Zorn lemma we can extend the family $\{e_n\}$ to a maximal family $\{e_\alpha\}$. If we set $\lambda_\alpha = 0$ for all e_α which do not belong to the original family we still have $a - b = \sum \lambda_\alpha e_\alpha$. According to the above paragraph and (2), we have

$$\tau(a^2 - b^2) = \mathrm{Tr}\left(|a^2 - b^2|\right) = \sum\langle |a^2 - b^2|e_\alpha, e_\alpha\rangle \geqq \sum |\langle (a^2 - b^2)e_\alpha, e_\alpha\rangle|.$$

If we write $2(a^2 - b^2) = (a+b)(a-b) + (a-b)(a+b)$, we obtain

$$\tau(a^2 - b^2) \geqq \frac{1}{2} \sum |\langle (a-b)e_\alpha, (a+b)e_\alpha \rangle + \langle (a+b)e_\alpha, (a-b)e_\alpha \rangle |.$$

However, $(a-b)e_\alpha = \lambda_\alpha e_\alpha$ and so

$$\tau(a^2 - b^2) \geqq \sum |\lambda_\alpha \langle (a+b)e_\alpha, e_\alpha \rangle | .$$

Since b is positive, we have $a - b \leqq a + b$ and therefore

$$\tau(a^2 - b^2) \geqq \sum |\lambda_\alpha \langle (a-b)e_\alpha, e_\alpha \rangle | = \sum \lambda_\alpha^2 \langle e_\alpha, e_\alpha \rangle.$$

On the other hand, using the fact that e_α are pairwise orthogonal as elements of the Hilbert space, we get

$$\|a-b\|^2 = \left\| \sum \lambda_\alpha e_\alpha \right\|^2 = \sum \lambda_\alpha^2 \|e_\alpha\|^2 \leqq \tau(a^2 - b^2). \quad \square$$

REMARKS. The idea to express $2(a^2 - b^2)$ as $(a+b)(a-b) + (a-b)(a+ + b)$ is due to Powers and Stormer. It was pointed out to me by Bojan Magajna.

At this point the author wishes to express his gratitute to the referee who discovered a mistake in the proof of the above lemma in the first version of the manuscript. Note also that in the sequel we use a weaker statement than the inequality proved in Lemma 10(3) which says that for positive elements a_n and a $a_n^2 \xrightarrow{\tau} a^2$ implies $a_n \xrightarrow{\|\,\|} a$. As noted by the referee, this weaker claim can be proved with more standard methods thus avoiding structure theory of H^*-algebras. Its proof goes as follows:

Since $\mathrm{Tr}(p^2) = \|p\|^2$ for positive p, it follows that $\|a_n\|^2 = \mathrm{Tr}(a_n^2) \to \to \mathrm{Tr}(a^2) = \|a\|^2$. As a consequence of this we also obtain that the set of left multiplication operators L_{a_n} is bounded in the operator norm. According to Proposition 6 we have $a_n^2 \xrightarrow{\|\,\|} a^2$ and so $L_{a_n}^2$ converge to L_a^2 in the operator norm. Since $L_{a_n}^2$ form a norm-bounded set and continuous function calculus $T \mapsto f(T)$ for selfadjoint operators is strongly continuous on compact subsets of $\mathbf{R}$ (see for example Pedersen, Analysis Now, E 4.6.5), we obtain that L_{a_n} tends strongly to L_a. Therefore we have

$$\langle a_n, xy \rangle = \langle a_n y^*, x \rangle \to \langle a y^*, x \rangle = \langle a, xy \rangle.$$

Since elements of the form xy are dense in $\mathcal{A}$, it follows that a_n tends weakly to a. Finally we have

$$\|a_n - a\|^2 = \|a_n\|^2 + \|a\|^2 - \langle a_n, a \rangle -$$

$$-\langle a, a_n\rangle \to \|a\|^2 + \|a\|^2 - \langle a, a\rangle - \langle a, a\rangle = 0.$$

Next we recall the following result from [16]. By τ we again denote the trace-class norm. Compare also Proposition 6.

PROPOSITION 11. *Let $\mathcal{H}$ be a pre-Hilbert $\mathcal{A}$-module. Then the strong Cauchy–Schwarz inequality*

$$\tau\big([x, y]\big)^2 \leqq \tau\big([x, x]\big)\,\tau\big([y, y]\big)$$

holds for all $x, y \in \mathcal{H}$.

THEOREM 12. *Let $\mathcal{H}$ be a pre-Hilbert $\mathcal{A}$-module. If we define $N(x) = \sqrt{[x, x]}$, then $(\mathcal{H}, N)$ is a normed $\mathcal{A}$-module.*

PROOF. Take $a \in \mathcal{A}$ and $x \in \mathcal{H}$. Then

$$N(a \circ x)^2 = [a \circ x, a \circ x] = a[x, a \circ x] = a[a \circ x, x]^* =$$

$$= a\big(a[x, x]\big)^* = a[x, x]a^* = aN(x)^2 a^* =$$

$$= \big(aN(x)\big) \cdot \big(aN(x)\big)^* = \big|aN(x)\big|^2$$

where the last equality follows from Corollary 5. Since $N(a \circ x)$ and $\big|aN(x)\big|$ are positive, we can apply Corollary 4 in order to obtain $N(a \circ x) = \big|aN(x)\big|$ which proves (iii) from Definition 9.

Now fix $x, y \in \mathcal{H}$. If $p \in \mathcal{A}$ is an arbitrary projection, then we have, using the H^*-identity (1),

$$\big\|pN(x + y)\big\|^2 = \big\langle pN(x + y), pN(x + y)\big\rangle =$$

$$= \big\langle pN(x + y)^2, p\big\rangle = \big\langle p[x + y, x + y], p\big\rangle =$$

$$= \big\langle p[x, x], p\big\rangle + \big\langle p[y, y], p\big\rangle + \big\langle p[x, y], p\big\rangle + \big\langle p[y, x], p\big\rangle.$$

Apply Lemma 10 in order to prove

$$\big\|p(N(x + y)\big\|^2 \leqq \big\langle pN(x)^2, p\big\rangle + \big\langle pN(y)^2, p\big\rangle + 2\big\|pN(x)\big\| \cdot \big\|pN(y)\big\| =$$

$$= \big\langle pN(x), pN(x)\big\rangle + \big\langle pN(y), pN(y)\big\rangle + 2\big\|pN(x)\big\| \cdot \big\|pN(y)\big\| =$$

$$= \big(\big\|pN(x)\big\| + \big\|pN(y)\big\|\big)^2.$$

Thus, for an arbitrary projection $p \in \mathcal{A}$,

$$\big\|pN(x + y)\big\| \leqq \big\|pN(x)\big\| + \big\|pN(y)\big\|$$

follows. Take the approximate identity $\{p_\alpha\} \subset \mathcal{A}$ consisting of projections. Its existence is granted by Proposition 7. Hence

$$\lim_\alpha p_\alpha N(x+y) = N(x+y), \quad \lim_\alpha p_\alpha N(x) = N(x), \quad \lim_\alpha p_\alpha N(y) = N(y).$$

If we take limits on both sides of the inequality

$$\|p_\alpha N(x+y)\| \leqq \|p_\alpha N(x)\| + \|p_\alpha N(y)\|,$$

we finally obtain (iv).

Now suppose that a sequence x_α is such that for every $\varepsilon > 0$ there exists α_0 such that for all $\alpha, \beta \geqq \alpha_0$ we have $\|N(x_\alpha - x_\beta)\| < \varepsilon$. If $\alpha \geqq \alpha_0$, then

$$\|N(x_\alpha)\| = \|N(x_\alpha - x_{\alpha_0} + x_{\alpha_0})\| \leqq \varepsilon + \|N(x_{\alpha_0})\|$$

implies that there is a constant M such that $\|N(x_\alpha)\| \leqq M$ for all $\alpha \geqq \alpha_0$. Then we have for all $\alpha, \beta \geqq \alpha_0$, using Lemma 10 and Proposition 11,

$$\begin{aligned} &\|N(x_\alpha) - N(x_\beta)\|^2 \leqq \tau\big(N(x_\alpha)^2 - N(x_\beta)^2\big) = \\ &= \tau\big([x_\alpha, x_\alpha] - [x_\beta, x_\beta]\big) = \tau\big([x_\alpha, x_\alpha - x_\beta] + [x_\alpha - x_\beta, x_\beta]\big) \leqq \\ &\leqq \tau\big([x_\alpha, x_\alpha - x_\beta]\big) + \tau\big([x_\alpha - x_\beta, x_\beta]\big) \leqq \\ &\leqq \sqrt{\tau\big([x_\alpha, x_\alpha]\big)}\sqrt{\tau\big([x_\alpha - x_\beta, x_\alpha - x_\beta]\big)} + \\ &+ \sqrt{\tau\big([x_\alpha - x_\beta, x_\alpha - x_\beta]\big)}\sqrt{\tau\big([x_\beta, x_\beta]\big)} = \\ &= \|N(x_\alpha)\| \cdot \|N(x_\alpha - x_\beta)\| + \|N(x_\beta)\| \cdot \|N(x_\alpha - x_\beta)\| \leqq 2M\varepsilon. \qquad \square \end{aligned}$$

Now we can formulate the Jordan–von Neumann type theorem for Saworotnow's pre-Hilbert $\mathcal{A}$-modules. If $(\mathcal{X}, N)$ is a normed $\mathcal{A}$-module, then the parallelogram law is the identity

$$\text{(PL)} \qquad N(x+y)^2 + N(x-y)^2 = 2N(x)^2 + 2N(y)^2$$

which must hold for all $x, y \in \mathcal{X}$. If $\mathcal{X} = \mathcal{H}$ is a Saworotnow's module, then this reduces to

$$[x+y, x+y] + [x-y, x-y] = 2[x,x] + 2[y,y]$$

which is trivial to verify. The rest of the paper is devoted to prove the following

MAIN THEOREM. *Let $\mathcal{A}$ be a proper H^*-algebra and $(\mathcal{X}, N)$ a normed $\mathcal{A}$-module over $\mathcal{A}$. Then $\mathcal{X}$ satisfies the parallelogram law (PL) if and only if $\mathcal{X}$ is a Saworotnow's pre-Hilbert module with respect to the generalized inner product $[x, y]$ such that $N(x)^2 = [x, x]$ holds for all $x \in \mathcal{X}$.*

A certain functional equation on H^*-algebras

Let $\mathcal{A}$ be a proper H^*-algebra and $T, S : \mathcal{A} \to \mathcal{A}$ a pair of mappings satisfying the identity $xT(y) = S(x)y$ for all $x, y \in \mathcal{A}$. Such a pair is called a *double centralizer* of $\mathcal{A}$.

OBSERVATION 13. *Let (T, S) be a double centralizer on $\mathcal{A}$. Then T and S are bounded and linear. Moreover, $T(xy) = T(x)y$ and $S(xy) = xS(y)$ holds for all $x, y \in \mathcal{A}$.*

PROOF. Fix $x, y \in \mathcal{A}$. Then we have, for all $z \in \mathcal{A}$,

$$\big(S(x+y) - S(x) - S(y)\big)z = (x+y)T(z) - xT(z) - yT(z) = 0.$$

Since $\mathcal{A}$ is proper, the additivity of S follows. In a similar way we prove that S is homogeneous and therefore linear. Obviously the same is true for T. In order to prove that T and S are bounded, we use the closed graph theorem. Suppose that $x_n \to 0$ and $T(x_n) \to y_0$. Then we have for all $y \in \mathcal{A}$, using Proposition 2,

$$yy_0 = y \lim_n T(x_n) = \lim_n \big(yT(x_n)\big) = \lim_n (S(y)x_n) = S(y) \lim_n x_n = 0.$$

As above, $y_0 = 0$ follows and by the closed graph theorem T is bounded. In a similar way we prove that S is bounded. Finally

$$z(T(xy) - T(x)y) = zT(xy) - zT(x) \cdot y = S(z)xy - S(z)x \cdot y = 0$$

implies $T(xy) = T(x)y$. In a similar way we prove $S(xy) = xS(y)$. □

Given $a \in \mathcal{A}$, we define a left and a right multiplication operator by $L_a(b) = ab$ and $R_a(b) = ba$ respectively. It is easy to compute that (L_a, R_a) is a double centralizer of $\mathcal{A}$. The converse is not true in general, because H^*-algebras which are infinite dimensional do not have an identity element and there are double centralizers which are not of the form (L_a, R_a).

OBSERVATION 14. *Every proper H^*-algebra is semiprime.*

PROOF. Recall that a ring $\mathcal{K}$ is called semiprime if for $a \in \mathcal{K}$ condition $a\mathcal{K}a = (0)$ implies $a = 0$. Take $a \in \mathcal{A}$ such that $a\mathcal{A}a = (0)$ holds. Then $aa^*a = 0$ is also true and so the H^*-identity (1) implies

$$\|a^*a\|^2 = \langle a^*a, a^*a \rangle = \langle aa^*a, a \rangle = 0$$

and consequently $a^*a = 0$. Finally

$$\|ax\|^2 = \langle ax, ax \rangle = \langle a^*ax, x \rangle = 0$$

implies $a = 0$ since $\mathcal{A}$ is proper. □

LEMMA 15. *Let $S, T : \mathcal{A} \to \mathcal{A}$ be additive mappings. Then*
(1) *$S(xyx) = xyS(x)$ for all $x, y \in \mathcal{A}$ implies that S is linear.*
(2) *$T(xyx) = T(x)yx$ for all $x, y \in \mathcal{A}$ implies that T is linear.*
(3) *If S is bounded and satisfies (1), then $S(xy) = xS(y)$ for all $x, y \in \mathcal{A}$.*
(4) *If T is bounded and satisfies (2), then $T(xy) = T(x)y$ for all $x, y \in \mathcal{A}$.*

PROOF. Clearly it is sufficient to treat only (1) and (3). Take $\lambda \in \mathbf{C}$. Then

$$S(\lambda^2 xyx) = S\big((\lambda x)y(\lambda x)\big) = \lambda xyS(\lambda x).$$

On the other hand

$$S(\lambda^2 xyx) = S(x(\lambda^2 y)x) = \lambda^2 xyS(x)$$

and so $xy\big(S(\lambda x) - \lambda S(x)\big) = 0$. By replacing x with $x + z$, we obtain

$$xy\big(S(\lambda z) - \lambda S(z)\big) = -zy\big(S(\lambda x) - \lambda S(x)\big).$$

If we denote $A(x) = S(\lambda x) - \lambda S(x)$, then

$$\big(xyA(z)\big)\, w\big(xyA(z)\big) = -\big(zyA(x)\big)\, w\big(xyA(z)\big) =$$

$$= -z\big(yA(x)wxy\big)\, A(z) = 0.$$

Since $\mathcal{A}$ is semiprime, it follows $xyA(z) = 0$ for all $x, y, z \in \mathcal{A}$. Hence $A(z)\mathcal{A}A(z) = (0)$ implies $A = 0$ so S is linear.

If S is bounded, then we can use the approximate identity p_α from Proposition 7. First we have

$$xS(x) = \lim_\alpha xp_\alpha S(x) = \lim_\alpha S(xp_\alpha x) = S(\lim_\alpha xp_\alpha x) = S(x^2).$$

By means of linearization we obtain $xS(y) + yS(x) = S(xy + yx)$. Again using the approximate identity, we obtain

$$2S(x) = S(2x) = S(\lim_\alpha p_\alpha x + xp_\alpha) = \lim_\alpha S(p_\alpha x + xp_\alpha) =$$

$$= \lim_\alpha p_\alpha S(x) + \lim_\alpha xS(p_\alpha) = S(x) + \lim_\alpha xS(p_\alpha)$$

and therefore $S(x) = \lim_\alpha xS(p_\alpha)$. This finally gives

$$S(xy) = \lim_\alpha xyS(p_\alpha) = x \lim_\alpha yS(p_\alpha) = xS(y). \qquad \square$$

LEMMA 16. *For each nonzero $y \in \mathcal{A}$ there exists a minimal projection p such that $pyp \neq 0$.*

PROOF. Suppose that $pyp = 0$ for all minimal projections p. By taking adjoints we obtain $py^*p = 0$. If we decompose $y = h + ik$ where h and k are selfadjoint, then $php = pkp = 0$ follows. Consider the spectral decomposition $h = \sum_n \lambda_n e_n$ where λ_n are nonzero real numbers and $\{e_n\}$ pairwise orthogonal spectral projections. If $h \neq 0$, then $e_1 \neq 0$. Since e_1 is a finite sum of pairwise orthogonal minimal projections (see Proposition 7), there exists a minimal projection p satisfying $pe_1p = p$ and $pe_n = 0$ for all $n > 2$ (if there are any). Hence $php = \lambda_1 p \neq 0$ gives a contradiction. This shows that $h = 0$. In a similar way we establish $k = 0$ and finally $y = 0 + 0i = 0$ concludes the proof. $\square$

Let $E, F : \mathcal{A} \to \mathcal{A}$ be additive and suppose that

$$E(xyx) = E(x)y^*x^* + xF(y)x^* + xyE(x),$$

$$F(xyx) = F(x)y^*x^* + xE(y)x^* + xyF(x)$$

hold for all $x, y \in \mathcal{A}$. Then (E, F) is called a *Jordan $*$-derivation pair.* This is a generalization of Jordan $*$-derivations which were considered in [3], [29–31] and [38].

EXAMPLE 17. Let (T_1, S_1) and (T_2, S_2) be double centralizers. If we define $E(x) = T_1(x^*) + S_2(x)$ and $F(x) = -T_2(x^*) - S_1(x)$, then (E, F) is a Jordan $*$-derivation pair.

The converse is given in the following

PROPOSITION 18. *Let $\mathcal{A}$ be a proper H^*-algebra and (E, F) a Jordan $*$-derivation pair acting on $\mathcal{A}$. Then there exist double centralizers (T_1, S_1) and (T_2, S_2) of $\mathcal{A}$ such that*

$$E(x) = T_1(x^*) + S_2(x), \qquad F(x) = -T_2(x^*) - S_1(x).$$

PROOF. From the desired representation it easily follows that we must define the above mentioned mappings by

$$T_1(x) = \frac{1}{2i}\big(iE(x^*) - E(ix^*)\big), \qquad T_2(x) = \frac{1}{2i}\big(-iF(x^*) + F(ix^*)\big),$$

$$S_1(x) = \frac{1}{2i}\big(-F(ix) - iF(x)\big), \qquad S_2(x) = \frac{1}{2i}\big(E(ix) + iE(x)\big).$$

Then it is straightforward that $E(x)=T_1(x^*)+S_2(x)$ and $F(x)=-T_2(x^*)-$ $-S_1(x)$. In a few steps we conclude the proof by showing that (T_1,S_1) and (T_2,S_2) are in fact double centralizers.

STEP 1. *The above four mappings satisfy the identities* $T_1(xyx)=$ $=T_1(x)yx$, $T_2(xyx)=T_2(x)yx$, $S_1(xyx)=xyS_1(x)$ *and* $S_2(xyx)=xyS_2(x)$ *for all* $x,y\in\mathcal{A}$.

The proof of all four identities is similar so we give it only for S_2. In order to simplify the writing of constants we shall consider $S(x)=2E(ix)+$ $+2iE(x)=4iS_2(x)$. First we observe the equality $-E(xyx)=E(-xyx)=$ $=E\big((ix)y(ix)\big)$ which results in

$$(2)\qquad 0=2xF(y)x^*+E(x)y^*x^*+xyE(x)+ixyE(ix)-iE(ix)y^*x^*$$

for all $x,y\in\mathcal{A}$ after the expansions of both sides. Next we compute

$$2E(ixyx)=E\big((1+i)xy(1+i)x\big)=E(x)y^*x^*+E(ix)y^*x^*-$$
$$-iE(x)y^*x^*-iE(ix)y^*x^*+2xF(y)x^*+xyE(x)+$$
$$+ixyE(x)+ixyE(ix)+xyE(ix).$$

By (2) this reduces to

$$2E(ixyx)=E(ix)y^*x^*-iE(x)y^*x^*+ixyE(x)+xyE(ix)=$$
$$=i\big(-iE(ix)y^*x^*-E(x)y^*x^*+xyE(x)-ixyE(ix)\big)$$

and again using (2) this further reduces to

$$2E(ixyx)=i\big(-2xF(y)x^*-2E(x)y^*x^*-2ixyE(ix)\big).$$

This enables us to finally obtain

$$S(xyx)=2E(ixyx)+2iE(xyx)=$$
$$=i\big(-2xF(y)x^*-2E(x)y^*x^*-2ixyE(ix)+2E(xyx)\big)=$$
$$=i\big(2xyE(x)-2ixyE(ix)\big)=2xyE(ix)+2ixyE(x)=xyS(x).$$

STEP 2. *E and F are real linear.*

This follows immediately from Step 1, Lemma 15 and the fact that the involution of $\mathcal{A}$ is real linear.

STEP 3. *E and F are bounded (real linear) operators.*

The closed graph theorem is also true for real linear operators so take a sequence $\{x_n\} \subset \mathcal{A}$ which converges to zero such that $E(x_n) \to y \neq 0$. Because of Lemma 16 there exists a minimal projection p such that $pyp \neq 0$. Since p is minimal, there exist two sequences of real numbers α_n and β_n such that $px_np = (\alpha_n + i\beta_n)p$. Since $x_n \to 0$, we have $\alpha_n \to 0$ and $\beta_n \to 0$. Since F is real linear, we have

$$F(px_np) = \alpha_n F(p) + \beta_n F(ip) \to 0.$$

Therefore

$$0 = \lim_n F(px_np) = \lim_n \left(F(p)x_n^* p + pE(x_n)p + px_nE(p) \right) =$$
$$= \lim_n pE(x_n)p = pyp \neq 0$$

where the third equality follows from Proposition 2 which implies that $x_n^* \to \to 0$. This contradiction tells us that the graph of E is closed and the boundedness of E follows. In a similar way we prove that F is bounded.

STEP 4. *(T_1, S_1) and (T_2, S_2) are double centralizers.*

From Step 3 and Lemma 15 it follows that $T_1(xy) = T_1(x)y$ and similarly for T_2, S_1 and S_2. Applying this to the identity

$$E(xyx) = E(x)y^*x^* + xF(y)x^* + xyE(x)$$

we obtain

$$S_2(x)y^*x^* + xyT_1(x^*) = xT_2(y^*)x^* + xS_1(y)x^*.$$

If we replace y by iy, we obtain

$$xyT_1(x^*) = xS_1(y)x^* = S_1(xy)x^*,$$
$$S_2(x)y^*x^* = xT_2(y^*)x^* = xT_2(y^*x^*).$$

If we use the approximate identity p_α, we obtain

$$xT_1(x^*) = \lim_\alpha xp_\alpha T_1(x^*) = \lim_\alpha S_1(xp_\alpha)x^* = S_1(x)x^*.$$

Inserting $x + y$ and $x + iy$ instead of x, we easily obtain $xT_1(y) = S_1(x)y$ which means that (T_1, S_1) is a double centralizer. In a similar way we prove that (T_2, S_2) is also a double centralizer. □

In the proof of the main theorem we shall use a special situation of the above result.

PROPOSITION 19. *Let $\mathcal{A}$, E and F be as above and $\{p_\alpha\}$ the approximate identity of $\mathcal{A}$ from Proposition 7. If $\lim_\alpha E(p_\alpha)$ and $\lim_\alpha E(ip_\alpha)$ exist, then there are unique elements $a, b \in \mathcal{A}$ such that*

$$E(x) = ax^* + xb, \qquad F(x) = -bx^* - xa.$$

PROOF. Denote by (T_1, S_1) and (T_2, S_2) double centralizers which represent the Jordan $*$-derivation pair (E, F). Denote

$$c = \lim_\alpha E(p_\alpha), \qquad d = \lim_\alpha E(ip_\alpha).$$

Take any $x \in \mathcal{A}$. Then $xp_\alpha \to x$ and since E is bounded $E(xp_\alpha) \to E(x)$ follows. Hence, using Observation 13,

$$E(x) = \lim_\alpha E(xp_\alpha) = \lim_\alpha \big(T_1(p_\alpha x^*) + S_2(xp_\alpha)\big) =$$

$$= \lim_\alpha \big(T_1(p_\alpha)x^* + xS_2(p_\alpha)\big) = \lim_\alpha \Big(\big(T_1(p_\alpha)x^* + S_2(p_\alpha)x^*\big) +$$

$$+ \big(xT_1(p_\alpha) + xS_2(p_\alpha)\big) - \big(xT_1(p_\alpha) + S_2(p_\alpha)x^*\big)\Big) =$$

$$= \big(\lim_\alpha T_1(p_\alpha) + S_2(p_\alpha)\big)x^* + x\lim_\alpha \big(T_1(p_\alpha) + S_2(p_\alpha)\big) -$$

$$- \lim_\alpha \big(S_2(p_\alpha)x^* + xT_1(p_\alpha)\big) =$$

$$= cx^* + xc - \lim_\alpha p_\alpha T_2(x^*) - \lim_\alpha S_1(x)p_\alpha =$$

$$= cx^* + xc - T_2(x^*) - S_1(x) = cx^* + xc + F(x).$$

On the other hand we also have $x = \lim_\alpha(-ix)(ip_\alpha)$ which results in

$$E(x) = \lim_\alpha E\big((-ix)(ip_\alpha)\big) = \lim_\alpha \big(T_1\big((ip_\alpha)^*(-ix)^*\big) + S_2\big((-ix)(ip_\alpha)\big)\big) =$$

$$= \lim_\alpha \big(T_1\big((ip_\alpha)^*\big)(ix^*) - ixS_2(ip_\alpha)\big) =$$

$$= \lim_\alpha \big(iE(ip_\alpha)x^* - ixE(ip_\alpha) + ixT_1\big((ip_\alpha)^*\big) - iS_2(ip_\alpha)x^*\big) =$$

$$= idx^* - ixd + i\lim_\alpha S_2(x)(ip_\alpha)^* - i\lim_\alpha ip_\alpha T_2(x^*) =$$

$$= idx^* - ixd + S_2(x) + T_2(x^*) = idx^* - ixd - F(x).$$

The above calculations give us

$$E(x) + F(x) = idx^* - ixd, \qquad E(x) - F(x) = cx^* + xc.$$

If we define $a = \frac{-d+ic}{2i}$, $b = \frac{d+ic}{2i}$, then the desired representation follows. Now we have to prove the uniqueness. If a_1, b_1 and a_2, b_2 are two pairs representing (E, F) in the above sense, then $E(p_\alpha) \to a_1 + b_1$ and $E(p_\alpha) \to a_2 + b_2$ which shows that $a_1 + b_1 = a_2 + b_2$. On the other hand $E(ip_\alpha) \to i(b_1 - a_1)$ and $E(ip_\alpha) \to i(b_2 - a_2)$ imply $a_1 - b_1 = a_2 - b_2$ and hence $a_1 = a_2$, $b_1 = b_2$ concludes the proof. □

Proof of the Main Theorem

Let $\mathcal{A}$ be a proper H^*-algebra, $(\mathcal{X}, N)$ a normed $\mathcal{A}$-module satisfying the parallelogram law (PL) and $a \circ x$ the module multiplication. Define a mapping $Q : X \to A$ by $Q(x) = N(x)^2$. Then we have

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \tag{3}$$

$$Q(a \circ x) = aQ(x)a^* \tag{4}$$

for all $x, y \in \mathcal{X}$ and $a \in \mathcal{A}$. Identity (3) follows directly from (PL) while (4) follows from Definition 9(iii) since

$$Q(a \circ x) = N(a \circ x)^2 = \left|aN(x)\right|^2 = aN(x)N(x)a^* = aQ(x)a^*.$$

Let $\{p_\alpha\}$ be the approximate identity from Proposition 7 consisting of projections.

PROOF OF THE MAIN THEOREM. STEP 1. *$Q(-x) = Q(x)$ for all $x \in \mathcal{X}$.*

PROOF. This follows from (3), since we have

$$Q(x+x) + Q(x-x) = 4Q(x),$$

$$Q\big(x + (-x)\big) + Q\big(x - (-x)\big) = 2Q(x) + 2Q(-x).$$

STEP 2. *For every $x \in \mathcal{X}$ the limit $\lim_\alpha Q(p_\alpha \circ x)$ exists and is equal to $Q(x)$.*

PROOF. Since $\{p_\alpha\}$ is the approximate identity, (4) implies

$$\lim_\alpha Q(p_\alpha \circ x) = \lim_\alpha p_\alpha Q(x) p_\alpha =$$

$$= \lim_\alpha p_\alpha N(x) \cdot \lim_\alpha N(x) p_\alpha = N(x)^2 = Q(x).$$

STEP 3. *For every $a \in \mathcal{A}$ and $x \in \mathcal{X}$ the equality*

$$Q(x + a \circ x) = Q(x) + Q(a \circ x) + aQ(x) + Q(x)a^*$$

holds.

PROOF. From Step 2 and (4) it follows

$$Q(x + a \circ x) = \lim_\alpha Q\big(p_\alpha \circ (x + a \circ x)\big) = \lim_\alpha Q\big(p_\alpha \circ x + p_\alpha \circ (a \circ x)\big) =$$

$$= \lim_\alpha Q(p_\alpha \circ x + (p_\alpha a) \circ x) = \lim_\alpha Q\big((p_\alpha + p_\alpha a) \circ x\big) =$$

$$= \lim_\alpha (p_\alpha + p_\alpha a) Q(x) (p_\alpha + p_\alpha a)^* =$$

$$= \lim_\alpha p_\alpha Q(x) p_\alpha + \lim_\alpha p_\alpha a Q(x) p_\alpha + \lim_\alpha p_\alpha Q(x) a^* p_\alpha + \lim_\alpha p_\alpha a Q(x) a^* p_\alpha =$$

$$= Q(x) + \lim_\alpha p_\alpha a \cdot \lim_\alpha Q(x) p_\alpha + \lim_\alpha p_\alpha Q(x) \cdot \lim_\alpha a^* p_\alpha +$$

$$+ \lim_\alpha p_\alpha a Q(x) \cdot \lim_\alpha a^* p_\alpha =$$

$$= Q(x) + aQ(x) + Q(x)a^* + aQ(x)a^*.$$

STEP 4. *For all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ the following holds:*

(1) $Q(x + p_\alpha \circ x) \overset{\|\,\|}{\to} 4Q(x)$.

(2) $Q(x - p_\alpha \circ x) \overset{\tau}{\to} 0$.

(3) $Q(x - p_\alpha \circ x) \overset{\|\,\|}{\to} 0$.

PROOF. (1) From Step 3 we obtain

$$Q(x + p_\alpha \circ x) = Q(x) + p_\alpha Q(x) + Q(x) p_\alpha + p_\alpha Q(x) p_\alpha.$$

Since $\{p_\alpha\}$ is the approximate identity and taking into account Step 2, the result follows.

(2) Since $Q(x - p_\alpha \circ x)$ is positive, we have $\tau\big(Q(x - p_\alpha \circ x)\big) = \operatorname{Tr}\big(Q(x - - p_\alpha \circ x)\big)$. Using Step 3, we have

$$\tau\big(Q(x - p_\alpha \circ x)\big) = \operatorname{Tr}\big(N(x)N(x)\big) - \operatorname{Tr}(N(x)N(x)p_\alpha) -$$

$$- \operatorname{Tr}\big(p_\alpha N(x)N(x)\big) + \operatorname{Tr}(p_\alpha N(x)N(x)p_\alpha) = \big\langle N(x), N(x)\big\rangle -$$

$$- \big\langle p_\alpha N(x), N(x)\big\rangle - \big\langle N(x), p_\alpha N(x)\big\rangle + \big\langle p_\alpha N(x), p_\alpha N(x)\big\rangle.$$

Since $\{p_\alpha\}$ is the approximate identity and the inner product of $\mathcal{A}$ is continuous, the result follows.

(3) This follows easily from what we just proved and Proposition 6(1).

STEP 5. *For all $x, y \in \mathcal{X}$ and $\lambda \in \mathbf{C}$ a sequence $Q(x + (\lambda p_\alpha) \circ y)$ converges in the inner product topology.*

PROOF. If $\lambda = 0$ there is nothing to prove so we may assume in the sequel that λ is nonzero. Let $x_\alpha = x + (\lambda p_\alpha) \circ y$. Since $\{p_\alpha\}$ is the approximate identity, $p_\alpha N(y) \overset{\|\,\|}{\to} N(y)$.

Take $\varepsilon > 0$. There exists α_0 such that for all $\alpha, \beta \geqq \alpha_0$ we have $\| p_\alpha N(y) - p_\beta N(y) \| \leqq \frac{\varepsilon}{|\lambda|}$. This implies, for α, β as above,

$$\| N(x_\alpha - x_\beta) \|^2 = \| N\big((\lambda p_\alpha - \lambda p_\beta) \circ y\big) \|^2 =$$

$$= |\lambda|^2 \| |(p_\alpha - p_\beta) N(y)| \|^2 = |\lambda|^2 \| p_\alpha N(y) - p_\beta N(y) \|^2 \leqq \varepsilon^2.$$

Therefore, by axiom (v) from Definition 9, it follows that $N(x_\alpha)$ is a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is complete, it follows that $N(x_\alpha)$ is convergent and hence $Q(x_\alpha) = N(x_\alpha)^2$ also converges in the Hilbert space norm.

STEP 6. *Fix $x, y \in \mathcal{X}$. If we define $E, F : \mathcal{A} \to \mathcal{A}$ by*

$$E(a) = Q(x + a \circ y) - Q(x - a \circ y), \quad F(a) = Q(y - a \circ x) - Q(y + a \circ x),$$

then (E, F) is a Jordan $*$-derivation pair.

PROOF. Take $a, b \in \mathcal{A}$ and consider

$$E(a)b^*a^* + aF(b)a^* + abE(a) = (ab)\big(Q(x + a \circ y) - Q(x - a \circ y)\big) +$$

$$+ \big(Q(x + a \circ y) - Q(x - a \circ y)\big)(ab)^* + Q(a \circ y - ab \circ x) - Q(a \circ y + ab \circ x).$$

If we apply Step 3, we obtain

$$abQ(x + a \circ y) + Q(x + a \circ y)(ab)^* =$$

$$= Q(x + a \circ y + ab \circ x + aba \circ y) - Q(x + a \circ y) - Q(ab \circ x + aba \circ y)$$

and in a similar way

$$abQ(x - a \circ y) + Q(x - a \circ y)(ab)^* =$$

$$= Q(x - a \circ y + ab \circ x - aba \circ y) - Q(x - a \circ y) - Q(ab \circ x - aba \circ y).$$

This yields

$$E(a)b^*a^* + aF(b)a^* + abE(a) = Q(a \circ y - ab \circ x) - \tag{5}$$

$$-Q(a \circ y + ab \circ x) + Q(ab \circ x + aba \circ y + x + a \circ y) - Q(ab \circ x + aba \circ y) -$$

$$-Q(x + a \circ y) - Q(ab \circ x - a \circ y + x - aba \circ y) +$$

$$+Q(x - a \circ y) + Q(ab \circ x - aba \circ y).$$

By the parallelogram law (PL) we have

$$(6) \qquad Q(ab \circ x + aba \circ y + x + a \circ y) =$$

$$= 2Q(ab \circ x + aba \circ y) + 2Q(x + a \circ y) - Q(ab \circ x + aba \circ y - x - a \circ y)$$

and

$$(7) \qquad Q(ab \circ x - a \circ y + x - aba \circ y) =$$

$$= 2Q(ab \circ x - a \circ y) + 2Q(x - aba \circ y) - Q(ab \circ x - a \circ y - x + aba \circ y).$$

By Step 1 we have $Q(ab \circ x - a \circ y) = Q(a \circ y - ab \circ x)$ and so the application of (6) and (7) to (5) gives

$$(8) \qquad E(a)b^*a^* + aF(b)a^* + abE(a) = Q(ab \circ x + aba \circ y) +$$

$$+Q(x + a \circ y) - Q(a \circ y - ab \circ x) - Q(a \circ y + ab \circ x) +$$

$$+Q(x - a \circ y) + Q(ab \circ x - aba \circ y) - 2Q(x - aba \circ y).$$

Another application of (PL) tells us that

$$Q(a \circ y + ab \circ x) + Q(a \circ y - ab \circ x) = 2Q(a \circ y) + 2Q(ab \circ x),$$

$$Q(ab \circ x + aba \circ y) + Q(ab \circ x - aba \circ y) = 2Q(ab \circ x) + 2Q(aba \circ y),$$

$$Q(x + a \circ y) + Q(x - a \circ y) = 2Q(x) + 2Q(a \circ y),$$

holds and so (8) can be rewritten as

$$E(a)b^*a^* + aF(b)a^* + abE(a) =$$

$$= 2Q(x) + 2Q(aba \circ y) - 2Q(x - aba \circ y) =$$

$$= Q(x + aba \circ y) - Q(x - aba \circ y) = E(aba).$$

In a similar way we prove that $F(a)b^*a^* + aE(b)a^* + abF(a) = F(aba)$. Now it remains to prove that E and F are additive. This can be done by the technique discovered by Aczél a long time ago and since then used by Rätz, Szabó, Šemrl, Vukman, Zalar and probably many others. We repeat this argument for the sake of completeness.

Take $a, b \in \mathcal{A}$. Then, using (PL), we have

$$E(a + b) = Q(x + a \circ y + b \circ y) - Q(x - a \circ y - b \circ y) =$$

$$= Q(x + a \circ y + b \circ y) + Q(x + a \circ y - b \circ y) - Q(x + a \circ y - b \circ y) -$$

$$-Q(x-a\circ y-b\circ y)=2Q(x+a\circ y)+2Q(b\circ y)-$$
$$-2Q(x-b\circ y)-2Q(a\circ y)=Q(x+a\circ y)+$$
$$+\big(Q(x+a\circ y)-2Q(a\circ y)\big)+\big(2Q(b\circ y)-Q(x-b\circ y)\big)-$$
$$-Q(x-b\circ y)=Q(x+a\circ y)+2Q(x)-Q(x-a\circ y)-2Q(x)+$$
$$+Q(x+b\circ y)-Q(x-b\circ y)=E(a)+E(b)$$

and in a similar way we prove that F is additive as well.

STEP 7. *For each pair* $(x,y)\in\mathcal{X}\times\mathcal{X}$ *there exist unique elements* $a_{x,y}, b_{x,y}\in\mathcal{A}$ *such that*

$$Q(x+a\circ y)-Q(x-a\circ y)=a_{x,y}a^*+ab_{x,y},$$
$$Q(y-a\circ x)-Q(y+a\circ x)=-b_{x,y}a^*-aa_{x,y}.$$

PROOF. This follows immediately from Proposition 19 and Step 6.

STEP 8. *If we define* $[\ ,\ ]:\mathcal{X}\times\mathcal{X}\to\mathcal{A}$ *by* $[x,y]=\frac{1}{2}a_{x,y}$, *then* $[x,y]$ *is a generalized inner product on* X *in the sense of Definition* 8 *and so* $\mathcal{X}$ *is a Saworotnow's pre-Hilbert module. Moreover,* $[x,x]=N(x)^2$ *holds for all* $x\in\mathcal{X}$.

PROOF. The additivity of $[x,y]$ is easy to prove in a similar way as the additivity of E and F in Step 6. From the definition of E and F it follows first that $a_{x,x}=b_{x,x}$. Hence

$$4[x,x]=2a_{x,x}=a_{x,x}+b_{x,x}=\lim_\alpha a_{x,x}p_\alpha^*+p_\alpha b_{x,x}=$$
$$=\lim_\alpha Q(x+p_\alpha\circ x)-\lim_\alpha Q(x-p_\alpha\circ x)=4Q(x)=4N(x)^2$$

where the last equality follows from Step 4. Thus $[x,x]=N(x)^2$ for all $x\in\in\mathcal{X}$ and so $[x,x]$ is positive and nonzero if x is nonzero. This verifies axioms (i) and (ii) from Definition 8.

Since Q maps into positive elements of $\mathcal{A}$, E and F map into selfadjoint elements of $\mathcal{A}$. Therefore for all $a\in\mathcal{A}$

$$a_{x,y}a^*+ab_{x,y}=(a_{x,y}a^*+ab_{x,y})^*=b_{x,y}^*a^*+aa_{x,y}^*$$

and from the uniqueness of the elements $a_{x,y}$ and $b_{x,y}$ it follows that $b_{x,y}==a_{x,y}^*$. If we consider the relations

$$Q(x+a\circ y)-Q(x-a\circ y)=a_{x,y}a^*+ab_{x,y},$$
$$Q(x-a\circ y)-Q(x+a\circ y)=-b_{y,x}a^*-aa_{y,x},$$

we obtain

$$2[x,y] = a_{x,y} = b_{y,x} = a^*_{y,x} = 2[y,x]^*$$

which verifies axiom (iii) from Definition 8.

Take $c \in \mathcal{A}$ and consider

$$ca_{x,y}a^* + ab_{x,y}c^* = c(a_{x,y}a^* + ab_{x,y}) + (a_{x,y}a^* + ab_{x,y})c^* -$$
$$-(a_{x,y}(ca)^* + (ca)b_{x,y}) = cQ(x + a \circ y) + Q(x + a \circ y)c^* -$$
$$-cQ(x - a \circ y) - Q(x - a \circ y)c^* - Q(x + ca \circ y) + Q(x - ca \circ y).$$

By Step 3 this equals

$$Q(x + a \circ y + c \circ x + ca \circ y) - Q(x + a \circ y) - Q(c \circ x + ca \circ y) -$$
$$-Q(x - a \circ y + c \circ x - ca \circ y) + Q(x - a \circ y) + Q(c \circ x - ca \circ y) - Q(x + ca \circ y) +$$
$$+Q(x - ca \circ y)$$

and further, using the parallelogram law (PL),

$$Q(x + a \circ y) + Q(c \circ x + ca \circ y) - Q(x + a \circ y - c \circ x - ca \circ y) -$$
$$-Q(x - a \circ y + c \circ x - ca \circ y) + Q(x - a \circ y) + Q(c \circ x - ca \circ y) - Q(x + ca \circ y) +$$
$$+Q(x - ca \circ y) = 2Q(x) + 2Q(a \circ y) + 2Q(c \circ x) + 2Q(ca \circ y) -$$
$$-Q(x - ca \circ y + a \circ y - c \circ x) - Q\big(x - ca \circ y - (a \circ y - c \circ x)\big) -$$
$$-Q(x + ca \circ y) + Q(x - ca \circ y) =$$
$$= 2Q(x) + 2Q(a \circ y) + 2Q(c \circ x) + 2Q(ca \circ y) -$$
$$-2Q(x - ca \circ y) - 2Q(a \circ y - c \circ x) - Q(x + ca \circ y) + Q(x - ca \circ y) =$$
$$= Q(x + ca \circ y) + Q(x - ca \circ y) + 2Q(a \circ y) + 2Q(c \circ x) -$$
$$-Q(x - ca \circ y) - Q(x + ca \circ y) - 2Q(a \circ y - c \circ x) =$$
$$= Q(c \circ x + a \circ y) + Q(c \circ x - a \circ y) - 2Q(c \circ x - a \circ y) =$$
$$= Q(c \circ x + a \circ y) - Q(c \circ x - a \circ y) = a_{cx,y}a^* + ab_{cx,y}.$$

Since the representation of Jordan $*$-derivation pairs is unique, it follows $ca_{x,y} = a_{cx,y}$ and finally $2c[x,y] = 2[cx,y]$ concludes the proof of Main Theorem.

Open problems

Since we failed to produce an example of 'discontinuous' $\mathcal{A}$-norm N, it is possible that there is none. Hence we have the following

PROBLEM 20. *Is axiom* (v) *in Definition* 9 *redundant?*

If the answer is positive, then it may be interesting to study

PROBLEM 21. *Let Q be an arbitrary quadratic functional (not necessarily positive definite) on a Saworotnow's pre-Hilbert module. Can Q be represented by a sesquilinear form?*

On a Hilbert module we can define a relation $\perp$ by $x \perp y$ if and only if $[x,y]=0$. Then we have

PROBLEM 22. *What axioms of the the abstract orthogonality in the sense of Rätz and Szabó this relation satisfy?*

It is obvious that $\perp$ is symmetric, i.e. $x \perp y$ implies $y \perp x$. It is also obvious that $\perp$ is module homogeneous, i.e. $x \perp y$ implies $(a \circ x) \perp (b \circ y)$ for all $a, b \in \mathcal{A}$. More difficult question is already if $x \perp y$ and $x, y \neq 0$ imply that x, y are independent. This heavily depends on how we define the independence in a Hilbert module. If we define x, y to be dependent if there exist $a, b \in \mathcal{A}$ such that $a \circ x + b \circ y = 0$ and at least one of a, b is nonzero, then it may happen that very few vectors would be independent. Another definition, much more appealing at first glimpse, arises from Molnár's inequality described in Proposition 11.

DEFINITION 23. Let $\mathcal{H}$ be a pre-Hilbert module and $x, y \in \mathcal{H}$. Then x, y are *Molnár-dependent* if and only if $\tau\big([x,y]\big)^2 = \tau\big([x,x]\big)\tau\big([y,y]\big)$.

This definition is interesting for the Rätz–Szabó theory of abstract orthogonality because one has

OBSERVATION 24. *If x, y are nonzero and $x \perp y$ then x and y are Molnár-independent.*

PROOF. If x, y were Molnár-dependent, then

$$\tau\big([x,x]\big)\tau\big([y,y]\big) = \tau(0) = 0.$$

Since $\tau\big([x,x]\big)$ and $\tau\big([y,y]\big)$ are real numbers, this implies $\tau\big([x,x]\big) = 0$ or $\tau\big([y,y]\big) = 0$. By Proposition 6, τ is a norm and therefore $[x,x]=0$ or $[y,y]=0$. By Definition 8 axiom (ii) this is a contradiction. □

If n is an integer, we can define nx by $2x = x + x$, $3x = x + x + x$ and so on. Then it is elementary that x and nx are Molnár-dependent. There are however at least two problems with Definition 23.

PROBLEM 25. *Is Molnár-dependence a transitive relation as the usual linear dependence?*

PROBLEM 26. *Can one describe all elements which are Molnár-dependent with given* $x \in \mathcal{H}$?

Note that isometry of the involution in the Hilbert space norm (see Proposition 2) implies isometry of the involution in the trace norm and so (since $[y,x] = [x,y]^*$) Molnár-dependence is symmetric. Considering the paper [20] of Rätz and Szabó we can observe that given two additive mappings $A : \mathcal{A} \to \mathcal{A}$ and $B : \mathcal{H} \to \mathcal{A}$, the function $F : \mathcal{H} \to \mathcal{A}$ defined by $F(x) = A([x,x]) + B(x)$ is orthogonally additive in the sense that $x \perp y$ implies $F(x+y) = F(x) + F(y)$. This motivates our last

PROBLEM 27. *Is every orthogonally additive mapping* $F : \mathcal{H} \to \mathcal{A}$ *of the form described above?*

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r-CONVEX SEQUENCES AND MATRIX TRANSFORMATIONS

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1. Introduction

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup_k |x_k|$, and let v be the space of sequences of bounded variation, i.e.,

$$v = \left\{ x \colon \|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, \quad (x_{-1} = 0) \right\}.$$

Suppose that $\mathcal{B} = (B_i)$ is a sequence of infinite complex matrices with $B_i = \big(b_{np}(i)\big)$. Then $x \in l_\infty$ is said to be $F_\mathcal{B}$-convergent [7] to the value $\operatorname{Lim} \mathcal{B}x$, if

$$\lim_{n\to\infty} (B_i x)_n = \lim_{n\to\infty} \sum_{p=0}^{\infty} b_{np}(i) x_p = \operatorname{Lim} \mathcal{B}x,$$

uniformly in $i = 0, 1, 2, \ldots$.

The space $F_\mathcal{B}$ of $F_\mathcal{B}$-convergent sequences depends on the fixed chosen sequence $\mathcal{B} = (B_i)$. In case $\mathcal{B} = \mathcal{B}_0 = (I)$ (unit matrix), the space $F_\mathcal{B}$ is the same as the space c and for $\mathcal{B} = \mathcal{B}_1 = \left(B_i^{(1)}\right)$, it is same as the space $\widehat{c}$ of almost convergent sequences [1], where $B_i^{(1)} = \left(b_{np}^{(1)}(i)\right)$ with

$$b_{np}^{(1)}(i) = \begin{cases} \frac{1}{n+1}, & i \leqq p \leqq i+n \\ 0, & \text{otherwise.} \end{cases}$$

Let s be the space of all sequences, real or complex; and define

$$e_k = \left(0, 0, \ldots, 0, 1 \left(k^{\text{th}} \text{ place}\right), 0, 0, \ldots\right),$$

$$e = (1, 1, 1, \ldots),$$

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$$d_{\mathcal{B}} = \left\{ x \in s : \mathcal{B}x = \left((B_i x)_n \right) \text{ exists} \right\},$$

and

$$F_{\mathcal{B}} = \Big\{ x \in \left(d_{\mathcal{B}} \cap l_\infty \right) : \lim_{n \to \infty} t_n(i, x) \text{ exists uniformly in } i \geqq 0$$
$$\text{and is independent of } i \Big\},$$

where

$$t_n(i, x) = \sum_{p=0}^{\infty} b_{np}(i) x_p. \tag{1.1}$$

Pati and Sinha [4] defined r-convex sequences in the following manner:

A real sequence (x_k) is said to be r-convex, $r \in N$, if $\overset{r}{\Delta} x_k \geqq 0$ for all $k \in N$, where $\overset{r}{\Delta} x_k$ is defined by

$$\overset{0}{\Delta} x_k = x_k, \qquad \overset{1}{\Delta} x_k = \Delta x_k = x_k - x_{k+1}$$
$$\vdots$$
$$\overset{r}{\Delta} x_k = \Delta \left(\overset{r-1}{\Delta} x_k \right), \quad \text{for} \quad r \in N.$$

The space of all bounded r-convex sequences with $r \geqq 2$ is denoted by SC^r, i.e.

$$SC^r = \left\{ x = (x_k) \in l_\infty : \overset{r}{\Delta} x_k \geqq 0 \quad \text{for all} \quad n \in N \right\}$$

and

$$SC^1 = \{ x \in l_\infty : x_k - x_{k+1} \geqq 0 \}.$$

It is clear that $SC^1 \subseteqq c$.

It is well known (Zygmund [7]) that a bounded convex sequence (x_k) is non-increasing. It is easy to prove the identity

$$\overset{r+s}{\Delta} x_k = \overset{r}{\Delta} \left(\overset{s}{\Delta} x_k \right), \; r \geqq 0 \quad \text{and} \quad s \geqq 0,$$

which shows that $SC^r \subset SC^{r-1}$, when $r \geqq 2$. Properties of bounded r-convex sequences have been investigated by Rath [5]. Also $SC^r \subset v$.

Let X and Y be any two sequence spaces. Let $A = (a_{nk})$ $(n, k = 0, 1, 2, \dots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ converges for each n. If $x \in X \Rightarrow Ax \in Y$, we say that A is an (X, Y)-matrix or $A \in (X, Y)$.

In [6], Stieglitz has characterized $(c, F_{\mathcal{B}})$-, $(l_\infty, F_{\mathcal{B}})$-, and $(\widehat{c}, F_{\mathcal{B}})$-matrices. These classes of matrices give directly the known characterizations as special cases depending upon the choice of the sequence of matrices $\mathcal{B} = (B_i)$.

In the present paper, we establish some necessary and sufficient conditions to characterize $(l_1, F_{\mathcal{B}})$-, and $(SC^r, F_{\mathcal{B}})$-matrices, where

$$l_1 = \left\{ x: \sum_{k=0}^{\infty} |x_k| < \infty \right\}.$$

2. Main results

We write

$$A_p x = \sum_{k=0}^{\infty} a_{pk}x_k \quad \text{and} \quad g_{nk}(i) = \sum_{p=0}^{\infty} b_{np}(i)a_{pk}.$$

Using (1.1), we get

$$(2.1) \qquad t_n(i, Ax) = \sum_{p=0}^{\infty} b_{pn}(i)A_p x = \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk}x_k = \sum_{k=0}^{\infty} g_{nk}(i)x_k$$

where the change of order of summation is justified by the following lemma (see [3]). We also give the proof of the lemma for completeness.

We denote

$$\left[B_i(Ax)\right]_n = \sum_p b_{np}(i)A_p(x) = \sum_p b_{np}(i) \sum_k a_{pk}x_k,$$

also

$$\left[(B_iA)x\right]_n = \sum_k \sum_p b_{np}(i)a_{pk}x_k.$$

LEMMA 2.1. *If* $\sum_p |b_{np}(i)| < \infty$ *for each* n *and* i, *and if*

$$\sup_{p,k} \left| \sum_{j=k}^{\infty} a_{pj} \right| < \infty,$$

then for every $x \in v$

$$B_i(Ax) = (B_i A)x.$$

PROOF. By partial summation, for $x \in v$

$$(*) \qquad \sum_k a_{pk} x_k = \sum_k d_{pk}(x_k - x_{k-1}) \quad (x_{-1} = 0),$$

where $d_{pk} = \sum\limits_{l=k}^{\infty} a_{pl}$. By the hypothesis, d_{pk} is bounded for all p, k. Thus

$$[B_i(Ax)]_n = \sum_p b_{np}(i) \sum_k d_{pk}(x_k - x_{k-1}) =$$

$$= \sum_k (x_k - x_{k-1}) \sum_p b_{np}(i) d_{pk} = [(B_i A)x]_n$$

(where the inversion is justified by absolute convergence), since

$$\lim_{k \to \infty} x_k \sum_p b_{np}(i) d_{pk} = 0.$$

THEOREM 2.2. *If* $\mathcal{B} = (b_{np}(i))$ *is a sequence of infinite matrices such that* $\sum_p |b_{np}(i)| < \infty$ *for each* n *and* i, *and* $A = (a_{nk})$ *is another matrix such that* $b_{np}(i) a_{pk}$ *is of the same sign for each* n, p, k *and* i, *then* $A \in (l_1, F_{\mathcal{B}})$ *if and only if*

(i) $\sup\limits_{l,p} \left| \sum\limits_{k=p}^{\infty} a_{lk} \right| < \infty$,

(ii) *there is a constant* K *such that for* $m, i \geqq 0$; $\sup\limits_n \left| \sum\limits_{k=m}^{\infty} g_{nk}(i) \right| \leqq K$,

(iii) $\alpha_k = \lim\limits_{n \to \infty} g_{nk}(i)$ *exists uniformly in* i *for each fixed* k.

PROOF. *Sufficiency.* It is enough to show that under conditions (i), (ii) and (iii),

$$(2.2) \qquad \lim_{n \to \infty} \sum_k g_{nk}(i) x_k = \sum_k \alpha_k x_k, \quad \text{uniformly in } i,$$

whenever $x = (x_k) \in l_1$. From the conditions, we observe that $\alpha_k \in l_\infty$, so that $\sum_k \alpha_k x_k$ converges absolutely for $(x_k) \in l_1$. Similarly, the series $\sum_k g_{nk}(i) x_k$ also converges absolutely for each fixed i and n.

For a given $\varepsilon > 0$, let $k_0 \in N$ be such that

$$\sum_{k>k_0} |x_k| < \varepsilon. \tag{2.3}$$

By (iii), we can find $n_0 \in N$ such that

$$\left| \sum_{k \leqq k_0} \left[g_{nk}(i) - \alpha_k \right] x_k \right| < \varepsilon, \tag{2.4}$$

for all $n > n_0$ and uniformly in i. Then

$$\left| \sum_k \left[g_{nk}(i) - \alpha_k \right] x_k \right| \leqq \left| \sum_{k \leqq k_0} \left[g_{nk}(i) - \alpha_k \right] x_k \right| + \sum_{k>k_0} \left| g_{nk}(i) - \alpha_k \right| |x_k| \leqq$$

$$\leqq (2K+1)\varepsilon,$$

for all $n > n_0$ and uniformly in i, by (2.4), (2.3) and (ii). This proves (2.2) and hence the sufficiency.

Necessity. Condition (i) follows from the fact that $A: l_1 \to l_\infty$. Since $e_k \in l_1$, the necessity of (iii) is obvious.

For fixed p and j, it is clear that

$$x \to \sum_{k=0}^{j} a_{pk} x_k$$

is a continuous linear functional on l_1. We are given that, for all $x \in l_1$, it tends to a limit as $j \to \infty$ (for fixed p) and hence, by the Banach–Steinhaus Theorem [2], this limit $A_p x$ is also a continuous linear functional on l_1.

Put for $i \geqq 0$

$$q_i(x) = \sup_n \left| t_n(i, Ax) \right|,$$

then q_i is a continuous seminorm on l_1, and (q_i) is pointwise bounded on l_1. Therefore, by another application of Banach–Steinhaus theorem, there exists a constant K such that

$$q_i(x) \leqq K \|x\|. \tag{2.5}$$

Apply (2.5) with $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k \geqq m, \\ 0, & k < m. \end{cases}$$

Hence (ii) holds.

This completes the proof the theorem.

Now, write

$$f_{nk}^{(r)}(i) = \sum_{p=0}^{\infty} b_{np}(i) \frac{1}{k^{r-1}} \sum_{j=1}^{k} A_{k-j}^{r-1} a_{pj} = \tag{2.6}$$

$$= \sum_{j=1}^{k} \frac{A_{k-j}^{r-1}}{k^{r-1}} \sum_{p=0}^{\infty} b_{np}(i) a_{pj} = \sum_{j=1}^{k} \frac{A_{k-j}}{k^{r-1}} g_{nj}(i),$$

by Lemma 2.1, where A_{k-j}^{r-1} denotes the binomial coefficients.

THEOREM 2.3. *If $\mathcal{B} = \big(b_{np}(i)\big)$ is a sequence of infinite matrices such that $\sum_p \big| b_{np}(i) \big| < \infty$ for each n and i, then $A \in (SC^r, F_{\mathcal{B}})$ if and only if*

(i) $\sup_{l,p} \left| \sum_{k=p}^{\infty} a_{lk} \right| < \infty$,

(ii) *there exists a constant M such that for $m, i = 0, 1, 2, \dots$*

$$\sup_n \left| \sum_{k=m}^{\infty} f_{nk}^{(r)}(i) \right| \leqq M \qquad (r \geqq 2),$$

(iii) $\beta_k^{(r)} = \lim_{n\to\infty} f_{nk}^{(r)}(i)$ *uniformly in i, for each k;*

(iv) $\beta^{(r)} = \lim_{n\to\infty} \sum_{k=0}^{\infty} f_{nk}^{(r)}(i)$ *uniformly in i.*

For our convenience we will write β_k and β for $\beta_k^{(r)}$ and $\beta^{(r)}$ respectively.

PROOF. *Sufficiency*. Suppose that conditions (i)–(iv) hold and $x \in SC^r$. By virtue of condition (i), it is clear that Ax is bounded. Now conditions (iv) and (i) imply that

$$\sum_{k=0}^{\infty} f_{nk}^{(r)}(i) \quad (r \geqq 2)$$

converges for all i, n. Hence if we write

$$h_{nk}^{(r)}(i) = \sum_{l=k}^{\infty} f_{nl}^{(r)}(i),$$

then $h_{nk}^{(r)}(i)$ exists, also for fixed i, n, we have

$$h_{nk}^{(r)} \to 0 \quad \text{as} \quad k \to \infty \quad (r \geqq 2). \tag{2.6}$$

Since

$$h_{nk}^{(r)}(i) = h_{n0}^{(r)}(i) - \sum_{l=0}^{k-1} f_{nl}^{(r)}(i). \tag{2.7}$$

Now

$$\sum_{k=0}^{\infty} f_{nk}^{(r)}(i) x_k = \sum_{k=0}^{\infty} \left[h_{nk}^{(r)}(i) - h_{n,k+1}^{(r)}(i) \right] x_k = \tag{2.8}$$

$$= \sum_{k=0}^{\infty} h_{nk}^{(r)}(i)(x_k - x_{k-1})$$

by (2.6) and boundedness of x_k. Therefore,

$$\left| \sum_{k=0}^{\infty} f_{nk}^{(r)}(i) x_k \right| \leqq \sum_{k=0}^{\infty} \left| h_{nk}^{(r)}(i) \right| \cdot |x_k - x_{k-1}| \leqq M \|x\|$$

(by condition (ii)) for $x \in SC^r$. Also, by (2.8)

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} f_{nk}^{(r)}(i) x_k = \sum_{k=0}^{\infty} (x_k - x_{k-1}) \lim_{n\to\infty} h_{nk}^{(r)}(i). \tag{2.9}$$

By (2.7) and conditions (iii) and (iv), we have

$$\lim_{n\to\infty} h_{nk}^{(r)}(i) = \lim_{n\to\infty} h_{n0}^{(r)}(i) - \sum_{l=0}^{k-1} \lim_{n\to\infty} f_{nl}^{(r)}(i) = \beta - \sum_{l=0}^{k-1} \beta_l. \tag{2.10}$$

Therefore, (2.9) and (2.10) give

$$(2.11)\qquad \lim_{n\to\infty}\sum_k f_{nk}^{(r)}(i)x_k=\sum_k(x_k-x_{k-1})\left(\beta-\sum_{l=0}^{k-1}\beta_l\right)=$$
$$=\beta\lim_{k\to\infty}x_k+\sum_k x_k\beta_k.$$

Thus $\lim_n \sum_{k=0}^{\infty} f_{nk}^{(r)}(i)x_k$ exists, and hence by (2.6)

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{k}\frac{A_{k-j}^{r-1}}{k^{r-1}}g_{nj}(i)x_k$$

exists. Therefore

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}g_{nk}(i)x_k$$

exists and this implies that $Ax\in F_{\mathcal{B}}$ for $x\in SC^r$.

Necessity. Condition (i) follows from the fact that $A\colon SC^r\to l_\infty$. Since $e_k, e\in SC^r$, the necessity of (iii) and (iv) follows immediately.

For fixed p and j, it is clear that

$$x\to\sum_{k=0}^{j}a_{pk}x_k$$

is a continuous linear functional on SC^r. We are given that for all $x\in SC^r$, it tends to a limit as $j\to\infty$ (for fixed p) as in (2.11) and hence, by the Banach–Steinhaus Theorem [2], this limit A_px is also a continuous linear functional on SC^r.

Fix r, and write, for $i\geqq 0$

$$Q_i(x)=\sup_n\left|\sum_{k=0}^{\infty}f_{nk}^{(r)}(i)x_k\right|,$$

then Q_i is a continuous seminorm on SC^r, and (Q_i) is pointwise bounded on SC^r. Therefore, by another application of Banach–Steinhaus theorem, there exists a constant M, such that

$$(2.12)\qquad Q_i(x)\leqq M\|x\|.$$

Now apply (2.12) with $x = (x_k)$ defined by $x_k = 1\ (k \geqq m), 0\ (k < m)$. Hence (ii) must hold.

This completes the proof of the theorem.

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MINIMAX THEOREMS WITH ONE–SIDED RANDOMIZATION

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0. Introduction

In [9] the following generalization of a minimax theorem of Peck–Dulmage [18] and König [14] was derived:

THEOREM A. *Let Y be a countably compact topological space and F a nonvoid set of lower semicontinuous functions $f : Y \to \mathbf{R} \cup \{\infty\}$ such that for the arithmetic mean $\varphi(\alpha, \beta) = \frac{1}{2}(\alpha + \beta)$ we have*

(1)

$$\forall f_1, f_2 \in F \; \exists f_0 \in F : \; f_0 \geqq \varphi(f_1, f_2) \quad \Big(i.e., \; f_0(y) \geqq \varphi\big(f_1(y), f_2(y)\big), \, y \in Y \Big).$$

Then there exists a probability measure ν on the Borel σ-algebra $\mathcal{B}(Y)$ such that

$$\sup_{f \in F} \int f d\nu = \sup_{f \in F} \inf_{y \in Y} f(y). \tag{2}$$

Theorem A can be proved by applying an appropriate integral representation theorem to König's generalization [13] of Ky Fan's minimax theorem [4]:

THEOREM B (Ky Fan–König). *Let Y be a compact topological space and F a nonvoid set of lower semicontinuous functions $f : Y \to \mathbf{R} \cup \{\infty\}$ such that for $\varphi(\alpha, \beta) = \psi(\alpha, \beta) = \frac{1}{2}(\alpha + \beta)$ we have* (1) *as above and*

$$\forall y_1, y_2 \in Y \; \exists y_0 \in Y \; \forall f \in F : f(y_0) \leqq \psi\big(f(y_1), f(y_2)\big). \tag{3}$$

Then there exists a $z \in Y$ such that

$$\sup_{f \in F} f(z) = \sup_{f \in F} \inf_{y \in Y} f(y). \tag{4}$$

In recent years quite a lot of generalizations of Theorem B, where the arithmetic means in (1) and (3) were replaced by "generalized means", have

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been published [8], [15], [16], [21], [22]. We shall use here a modified version of the main result in [12] to derive a fairly abstract version of Theorem A.

1. A general (ψ, φ)-minimax theorem

For the rest of the paper let Y be a nonvoid set, D an infinite convex subset of $\mathbf{R} \cup \{\infty\}$, and F a nonvoid set of functions $f : Y \to D$ with

$$\inf_{y \in Y} f(y) \in D, \quad f \in F. \tag{5}$$

We set $D^0 = D \cap \mathbf{R} - \{\inf D\}$ and $D_0 = D - \{\sup D\}$.

For functions $\xi : D \times D \to D$ we consider the following properties:

(a) ξ is nondecreasing in each variable,

(b) $\xi(\alpha, \alpha) = \alpha$, $\alpha \in D$,

$(c)^+$ The functions $\xi(\cdot, \alpha)$ and $\xi(\alpha, \cdot)$, $\alpha \in D_0$, are continuous from the right on $D^0 \cap D_0$.

$(c)^-$ The functions $\xi(\cdot, \alpha)$ and $\xi(\alpha, \cdot)$, $\alpha \in D^0$, are continuous from the left on $D^0 \cap D_0$.

$(d)^+$ $\alpha, \beta \in D \cap \mathbf{R}, \alpha \neq \beta \Rightarrow \xi(\alpha, \beta) < \alpha \vee \beta$,

$(d)^-$ $\alpha, \beta \in D \cap \mathbf{R}, \alpha \neq \beta \Rightarrow \xi(\alpha, \beta) > \alpha \wedge \beta$,

(e) $\alpha, \infty \in D \Rightarrow \xi(\alpha, \infty) = \xi(\infty, \alpha) = \infty$,

$(f)^+$ $\xi(\cdot, \beta)^n(\alpha) \to \beta$ and $\xi(\beta, \cdot)^n(\alpha) \to \beta$ $(n \to \infty)$ for all $\alpha, \beta \in D \cap \mathbf{R}$ with $\alpha > \beta$,

$(f)^-$ $\xi(\cdot, \beta)^n(\alpha) \to \beta$ and $\xi(\beta, \cdot)^n(\alpha) \to \beta$ $(n \to \infty)$ for all $\alpha, \beta \in D \cap \mathbf{R}$ with $\alpha < \beta$.

PROPOSITION 1. *Let $\varphi, \psi : D \times D \to D$ be given such that conditions* (1), (a), (b), $(f)^-$, *and* (e) *are satisfied for $\xi = \varphi$ and* (3), (a), (b), *and* $(f)^+$ *hold for $\xi = \psi$. Then*

$$\inf_{y \in Y} \max_{g \in G} g(y) \leqq \sup_{f \in F} \inf_{y \in Y} f(y) \quad \textit{for all finite } G \subset F. \tag{6}$$

This result has been proved in [12]. Here we need the following modification which is also closely related to the (ψ, φ)-minimax theorems of Simons [21]:

PROPOSITION 2. *Let $\varphi, \psi : D \times D \to D$ be given such that conditions* (1), (a), $(c)^-$, $(d)^-$, *and* (e) *are satisfied for $\xi = \varphi$ and conditions* (3), (a), $(c)^+$, *and* $(d)^+$ *hold for $\xi = \psi$. Then condition* (6) *holds.*

PROOF. 1. For $\alpha \in D - \{\inf D\}$ choose a strictly increasing sequence (β_n) in $D \cap \mathbf{R}$ with $\beta_n \uparrow \alpha$. Then $\varphi(\alpha, \alpha) \geqq \lim_{n \to \infty} \varphi(\beta_{n+1}, \beta_n) \geqq \lim_{n \to \infty} \beta_n = \alpha$. Hence, $\varphi(\alpha, \alpha) \geqq \alpha$, $\alpha \in D$, and similarly $\psi(\alpha, \alpha) \leqq \alpha$, $\alpha \in D$. Now define $\varphi_*(\alpha, \beta) = \varphi(\alpha, \beta) \wedge (\alpha \vee \beta)$ and $\psi^*(\alpha, \beta) = \psi(\alpha, \beta) \vee (\alpha \wedge \beta)$, $\alpha \in D$,

$\beta \in D$. Then we have $\varphi_*(\alpha,\alpha) = \psi^*(\alpha,\alpha) = \alpha$, $\alpha \in D$, and all assumptions remain valid when φ and ψ are replaced by φ_* and ψ^*. Hence, without loss of generality we may assume that condition (b) is satisfied for $\xi = \varphi$ and $\xi = \psi$.

2. Now, as in the proof of Example 2 in [12] it follows that $\xi = \varphi$ satisfies (f)$^-$ and $\xi = \psi$ satisfies (f)$^+$. Hence, Proposition 1 can be applied.

2. Main result: finitely additive version

In the sequel $M(Y)$ will denote the set of all finitely additive probability measures on 2^Y, and $P(Y)$ is the set of all $\nu \in M(Y)$ with finite support.

PROPOSITION 3. *Let $\varphi : D \times D \to D$ be a convex function such that conditions* (1), (a), (d)$^-$, *and* (e) *with $\xi = \varphi$ are satisfied. Then*

$$\inf_{q \in P(Y)} \max_{g \in G} \int g\, dq \leqq \sup_{f \in F} \inf_{y \in Y} f(y) \quad \text{for all finite } G \subset F. \tag{7}$$

PROOF. We set $\bar{f}(q) = \int f dq$ for $f \in F$, $q \in P(Y)$ and $\bar{F} = \{\bar{f} : f \in \in F\}$. For $f_1, f_2 \in F$ choose $f_0 \in F$ according to (1). Then by the convexity of φ

$$\bar{f}_0(q) \geqq \varphi\big(\bar{f}_1(q), \bar{f}_2(q)\big), \quad q \in P(Y). \tag{8}$$

Of course, for $q_1, q_2 \in P(Y)$, $q_0 = \frac{1}{2}(q_1 + q_2)$ and $\psi(\alpha,\beta) = \frac{1}{2}(\alpha + \beta)$ we have

$$\bar{f}(q_0) = \psi\big(\bar{f}(q_1), \bar{f}(q_2)\big), \quad f \in F. \tag{9}$$

Now, as every convex nondecreasing function $\eta : D \to D$ is continuous on D_0, the assumptions of Proposition 2 with F replaced by $\bar{F}$ are satisfied.

Now we are in the position to prove the first version of our main result.

THEOREM 1. *Let $\xi : D \times D \to D$ be a convex function such that conditions* (a), (d)$^-$, *and* (e) *are satisfied and let $\eta : D \to D$ be a strictly increasing convex function with inverse η^{-1}. Suppose that condition* (1) *is satisfied for*

$$\varphi(\alpha,\beta) := \eta^{-1}\Big(\xi\big(\eta(\alpha), \eta(\beta)\big)\Big), \quad \alpha \in D,\ \beta \in D.$$

Then condition (2) *is satisfied for some $\nu \in M(Y)$.*

The special case $\xi(\alpha,\beta) = \frac{1}{2}(\alpha + \beta)$ and $\eta(\alpha) = \alpha$ gives Korollar 3.2 in [9].

PROOF. *Step 1.* The assumptions of Proposition 3 are satisfied with φ and F replaced by ξ and $\tilde{F} := \{\tilde{f} := \eta \circ f : f \in F\}$. Hence, for finite $G \subset F$ we have

$$\alpha := \inf_{q \in P(Y)} \max_{g \in G} \int \tilde{g} dq \leqq \sup_{f \in F} \inf_{y \in Y} \tilde{f} =: \beta.$$

From the convexity of η we infer $\int \tilde{g} dq \geqq \eta(\int g dq)$, $q \in P(Y)$, $g \in G$. Setting $\eta(\sup D) = \sup \eta(D)$ in case $\sup D \notin D$ we obtain

$$\eta\left(\inf_{q \in P(Y)} \max_{g \in G} \int g\, dq\right) = \inf_{q \in P(Y)} \max_{g \in G} \eta\left(\int g\, dq\right) \leqq$$

$$\leqq \alpha \leqq \beta \leqq \eta\Big(\sup_{f \in F} \inf_{y \in Y} f(y)\Big),$$

i.e., condition (7) holds.

Step 2. Set $\bar{f}(\nu) = \int f d\nu$, $f \in F$, $\nu \in M(Y)$, and $\bar{F} = \{\bar{f} : f \in F\}$. Then $M(Y)$ is a compact subset of $[0,1]^{2^Y}$ and every $\bar{f} \in \bar{F}$ is lower semicontinuous. (Let H be the set of all functions $h : Y \to \mathbf{R}$ with finite range. Then, by definition, $\bar{h}(\nu) = \sum_{t \in \mathbf{R}} t\nu(\{h = t\})$, $h \in H$, and $\bar{f}(\nu) = \sup\{\bar{h}(\nu) : f \geqq h \in H\}$, $f \in F$.) By Step 1 the system of closed subsets

$$M(f, \delta) := \{\nu \in M(Y) : \bar{f}(\nu) \leqq \delta\}, \quad f \in F,\ \delta > \gamma := \sup_{f \in F} \inf_{y \in Y} f(y)$$

has the finite intersection property. Hence, there exists a $\nu \in \bigcap \{M(f, \delta) : f \in F,\ \delta > \gamma\}$, and we obtain $\sup_{f \in F} \int f d\nu \leqq \gamma$. The converse inequality is obvious.

The following example shall protect against possible misinterpretation:

EXAMPLE 1. Suppose that $D = (0, \infty)$ and $f \in F \Rightarrow 2f \in F$. Then condition (2) is satisfied for some $\nu \in M(Y)$.

This follows from Theorem 1 with $\xi(\alpha, \beta) = 2\alpha$ and $\eta(\alpha) = \alpha$.

Unfortunately, this result is trivial, because our general assumption (5) together with "$f \in F \Rightarrow 2f \in F$" implies $\sup_{f \in F} \inf_{y \in Y} f(y) = \infty$, so condition (2) holds for *every* $\nu \in M(Y)$. On the other hand, condition (5) is indispensable. (For $Y = \mathbf{R}$, $g(y) = e^y$, $h(y) = e^{-y}$, and $F = \{n \cdot g : n \in \mathbf{N}\} \cup \cup \{n \cdot h : n \in \mathbf{N}\}$ we have $\sup_{f \in F} \inf_{y \in Y} f(y) = 0 < \infty = \sup_{f \in F} \int f d\nu$, $\nu \in \in M(Y)$.)

If we try to circumvent this dilemma by taking $D = [0, \infty)$, then the above proof is not applicable any more because now condition $(\mathrm{d})^-$ is violated.

EXAMPLE 2. Let $MC^-(D)$ denote the set of all convex functions $\xi : D \times \times D \to D$ with properties (a), (b), and (d)$^-$. (It is easy to verify that every "convex mean" $\xi \in MC^-(D)$ automatically satisfies condition (e).) We consider some examples:

For $\lambda \in (0,1)$ the *weighted arithmetic means*

$$\mu_\lambda(\alpha,\beta) = \lambda\alpha + (1-\lambda)\beta, \quad \alpha \in D,\ \beta \in D$$

are affine, hence $\mu_\lambda \in MC^-(D)$. Also the projections $\pi_1(\alpha,\beta) = \alpha$ and $\pi_2(\alpha,\beta) = \beta$ are affine, thus their maximum

$$M(\alpha,\beta) := \alpha \vee \beta = \pi_1(\alpha,\beta) \vee \pi_2(\alpha,\beta), \quad \alpha \in D,\ \beta \in D$$

is convex, and we have $M \in CM^-(D)$. Closely related are the *weighted min-max means*

$$\tau_\lambda(\alpha,\beta) = \lambda\alpha \vee \beta + (1-\lambda)\alpha \wedge \beta, \quad \alpha \in D,\ \beta \in D$$

introduced by Geraghty and Lin [7]. From the identity $\tau_\lambda(\alpha,\beta) = (1-\lambda)(\alpha + + \beta) + (2\lambda - 1)\alpha \vee \beta$ we infer that τ_λ is convex, affine, concave for $1 \geqq \lambda > \frac{1}{2}$, $\lambda = \frac{1}{2}$, $\frac{1}{2} > \lambda \geqq 0$ respectively. Further examples can be found in [2].

New "convex means" can also be constructed from old ones. Suppose that $\varphi_0, \varphi_1, \varphi_2 \in MC^-(D)$ and set $\varphi(\alpha,\beta) = \varphi_0\big(\varphi_1(\alpha,\beta), \varphi_2(\alpha,\beta)\big)$, $\alpha \in D$, $\beta \in D$. Then $\varphi \in MC^-(D)$. In particular, the set $MC^-(D)$ is convex and maximum-stable.

3. Main result: countably additive version

A triplet (X,Y,a), where X and Y are nonvoid sets and a is a function $a : X \times Y \to \mathbf{R} \cup \{\infty\}$, can be interpreted as a (two-person zero-sum) *game*, where X and Y are the sets of ("pure") strategies of player 1 and 2 and a is the pay-off function. (If player 1 chooses a strategy $x \in X$ and player 2 chooses $y \in Y$ then player 1 receives the amount $a(x,y)$ from player 2.)

There are many instances in game theory and, especially, in statistical decision theory, where player 2, say, uses mixed strategies whereas player 1 only applies pure strategies. If one is willing to admit *finitely additive* probability measures as mixed strategies, then one obtains the *right-sided finitely-additive mixed extension* $\big(X, M(Y), A\big)$ with expected pay-off

$$A(x,\nu) = \int a(x,y)\nu(dy), \quad x \in X,\ \nu \in M(Y). \tag{10}$$

Here A is well-defined if every function $a(x,\cdot)$, $x \in X$, is bounded from below (compare Step 2 of the proof of Theorem 1), and the minimax relation

$$\min_{\nu\in M(Y)} \sup_{x\in X} A(x,\nu) = \sup_{x\in X} \min_{\nu\in M(Y)} A(x,\nu)$$

holds iff condition (2) is satisfied for $F = \{a(x,\cdot) : x \in X\}$ and for some $\nu \in \in M(Y)$.

In classical game theory, however, mixed strategies for player 2 are *countably additive* measures defined on some σ-algebra $\mathcal{B}$ on Y. In order that (10) remains well-defined, we must assume that the functions $a(x,\cdot)$, $x \in X$, are $\mathcal{B}$-measurable. Moreover, $\mathcal{B}$ should contain the singletons because one wants to embed the pure strategies into the set of mixed strategies.

In the following, let a paving $\mathcal{P}$ in Y (i.e., a nonvoid $\mathcal{P} \subset 2^Y$) be given which contains

$$\mathcal{R}(F) := \Big\{\{f \leqq \alpha\} : \ f \in F, \alpha \in \mathbf{R}\Big\},$$

the system of *level sets* of F. Let $\mathcal{B} = \mathcal{B}\big(\mathcal{P} \cup \mathcal{S}(Y)\big)$ be the σ-algebra generated by $\mathcal{P}$ and the set $\mathcal{S}(Y) = \Big\{\{y\} : y \in Y\Big\}$ of singletons and let $P(Y,\mathcal{B})$ denote the set of all (countably additive) probability measures on $\mathcal{B}$.

LEMMA 1. *Let the paving $\mathcal{P} \supset \mathcal{R}(F)$ be countably compact (i.e., every countable $\mathcal{Q} \subset \mathcal{P}$ with the finite intersection property has nonvoid intersection). Then for every $\nu \in M(Y)$ there exists a $\tau \in P(Y,\mathcal{B})$ with $\int f d\tau \leqq \leqq \int f d\nu$, $f \in F$.*

PROOF. Of course, $\mathcal{P}_0 := \mathcal{P} \cup \mathcal{S}(Y) \cup \{\emptyset, Y\}$ and hence the lattice generated by $\mathcal{P}_0$ is countably compact as well (cf. [19], Lemma 1.3, 1.4 or [10], §3). By Theorem 5 in [11] there exists a $\tau \in P(Y,\mathcal{B})$ with $\tau(A) \geqq \geqq \nu(A)$, $A \in \mathcal{P}$. As $f \in F$ is bounded from below, we may assume $f(y) > 0$, $y \in Y$. Then $f_n := 2^{-n} \sum_{i=1}^{n2^n} \mathbf{1}_{\{f > i2^{-n}\}} \uparrow f$. From $\mathcal{R}(F) \subset \mathcal{P}$ we infer $\int f d\tau = = \lim_{n\to\infty} \int f_n d\tau \leqq \lim_{n\to\infty} \int f_n d\nu \leqq \int f d\nu$.

Now, by combining Theorem 1 with Lemma 1, we obtain:

THEOREM 2. *Let the assumptions of Theorem* 1 *and Lemma* 1 *be satisfied. Then condition* (2) *holds for some $\nu \in P(Y,\mathcal{B})$.*

If we want to apply Theorem 2 we have to find a countably compact paving $\mathcal{P} \supset \mathcal{R}(F)$. Sometimes, a candidate for $\mathcal{P}$ is

$$\mathcal{R}_0(G) := \big\{\{g \leqq 0\} : g \in G\big\}$$

with appropriate $G \subset E^Y$, $E \subset \mathbf{R} \cup \{\infty\}$.

COROLLARY 1. *Let the assumptions of Theorem* 1 *be satisfied and let* $E \subset \mathbf{R} \cup \{\infty\}$ *and* $G \subset E^Y$ *be given such that* $\mathcal{R}_0(G)$ *is countably compact and contains* $\mathcal{R}(F)$. *Then condition* (2) *is satisfied for some* $\nu \in P(Y, \mathcal{B})$ *with* $\mathcal{B} = \mathcal{B}(\mathcal{R}_0(G) \cup \mathcal{S}(Y))$.

LEMMA 2 (cf. [9], Satz 4.4). *Let* $G \subset [0,1]^Y$ *such that for every sequence* (g_n) *in* G *there exists a sequence* (γ_n) *in* $(0, \infty)$ *with* $\sum_{n=1}^{\infty} \gamma_n < \infty$ *such that* $g := \sum_{n=1}^{\infty} \gamma_n g_n$ *attains its infimum. Then* $\mathcal{R}_0(G)$ *is countably compact.*

PROOF. Let $g_n \in G$ and $y_n \in Y$ with $g_i(y_n) = 0$, $i \leqq n \in \mathbf{N}$. For g as above choose $y^* \in Y$ with $g(y^*) = \min_{y \in Y} g(y)$. From $g(y_n) \leqq \sum_{i=n+1}^{\infty} \gamma_i$, $n \in \mathbf{N}$, we infer $g(y^*) = 0$, hence $g_n(y^*) = 0$, $n \in \mathbf{N}$.

EXAMPLE 3 (the standard situation). Let Y be a topological space, $C(Y)$ the set of all continuous functions $f : Y \to \mathbf{R}$ and $LSC(Y)$ the set of all lower semicontinuous $f : Y \to \mathbf{R} \cup \{\infty\}$. Here

a) $\mathcal{R}_0(C(Y)) = \mathcal{R}(C(Y))$ is countably compact iff Y is pseudocompact, and

b) $\mathcal{R}(LCS(Y)) = \mathcal{R}_0(LSC(Y))$ (=system of closed subsets) is countably compact iff Y is countably compact.

PROOF. a) Apply Lemma 2 to $G = C(Y) \cap [0,1]^Y$, b) is obvious.

EXAMPLE 3.1. Let Y be endowed with a countably compact topology such that every $f \in F$ is lower semicontinuous, and let $\eta : D \to D$ be a strictly increasing convex function. Suppose that for some $\lambda \in (0,1)$ condition (1) is satisfied for

$$\varphi(\alpha, \beta) = \eta^{-1}(\lambda \eta(\alpha) + (1-\lambda)\eta(\beta)), \quad \alpha, \beta \in \mathbf{R} \cup \{\infty\}.$$

Then there exists a probability measure ν on the Borel σ-algebra $\mathcal{B}(Y)$ (resp. on $\mathcal{B}(\mathcal{B}(Y) \cup \mathcal{S}(Y))$) such that condition (2) holds.

The special case $D = \mathbf{R} \cup \{\infty\}$, $\eta(\alpha) = \alpha$, and $\lambda = \frac{1}{2}$ gives Theorem A.

PROOF. Let $\xi = \mu_\lambda$ as in Example 2. Then the assumptions of Theorem 1 are satisfied. By Example 3b) and Corollary 1 the assertion follows.

LEMMA 3. *Let nonvoid sets* $E \subset \mathbf{R} \cup \{\infty\}$ *and* $G \subset E^Y$ *be given such that*

$$(11) \qquad \begin{cases} \textit{for every nondecreasing sequence } (g_n) \textit{ in } G \textit{ there} \\ \textit{exists a } y^* \in Y \textit{ with } \sup_{n \in \mathbf{N}} g_n(y^*) = \sup_{n \in \mathbf{N}} \inf_{y \in Y} g_n(y). \end{cases}$$

Suppose that there exists a function $\Phi : E \times E \to E$ *with the following properties:*

(i) $\alpha \in E, \beta \in E \Rightarrow \Phi(\alpha, \beta) \geqq \alpha \vee \beta$,

(ii) $\alpha \in E, \beta \in E, \alpha \vee \beta \leqq 0 \Rightarrow \Phi(\alpha,\beta) \leqq 0$, *and*
(iii) $g_1, g_2 \in G \Rightarrow \Phi(g_1, g_2) \in G$.

Then $\mathcal{R}_0(G)$ is countably compact.

PROOF. For $n \in \mathbf{N}$ let $g_n \in G$ and $y_n \in Y$ with $g_i(y_n) \leqq 0$, $i \leqq n$, be given. Set $h_1 = g_1$ and, inductively, $h_n = \Phi(h_{n-1}, g_n)$, $n \geqq 2$. Then $h_1 \leqq \leqq h_2 \leqq \ldots \leqq h_n \in G$, $n \in \mathbf{N}$. Hence there is a $y^* \in Y$ with $\sup_{n\in\mathbf{N}} h_n(y^*) = = \sup_{n\in\mathbf{N}} \inf_{y\in Y} h_n(y) \leqq \sup_{n\in\mathbf{N}} h_n(y_n) \leqq 0$. Therefore, $g_n(y^*) \leqq h_n(y^*) \leqq \leqq 0$, $n \in \mathbf{N}$.

EXAMPLE 4 (compare [9], Satz 4.5). Suppose that the assumptions of Theorem 1 are satisfied. Let $G \subset E^Y$ be given such that the Dini-relation (11) and one of the following conditions (i), (ii) or (iii) hold:

(i) a) $E = \mathbf{R} \cup \{\infty\}$, b) $g_1 \in G, g_2 \in G \Rightarrow g_1 \vee g_2 \in G$, and c) $f \in F, \alpha \in \in \mathbf{R} \Rightarrow f - \alpha \in G$,

(ii) a) $E = [0,\infty]$, b) $g_1 \in G, g_2 \in G \Rightarrow g_1 + g_2 \in G$, and c) $f \in F, \alpha \in \in \mathbf{R} \Rightarrow (f - \alpha) \vee 0 \in G$,

(iii) a) $E = [0,1]$, b) $g_1 \in G, g_2 \in G \Rightarrow g_1 + g_2 - g_1 \cdot g_2 \in G$, and c) $f \in \in F, \alpha \in \mathbf{R} \Rightarrow \big[(f - \alpha) \vee 0\big] \wedge 1 \in G$.

Then condition (2) is satisfied for some $\nu \in P(Y, \mathcal{B})$ with $\mathcal{B} = \mathcal{B}\big(\mathcal{R}_0(G) \cup \cup \mathcal{S}(Y)\big)$.

PROOF. By Lemma 3, applied to $\Phi(\alpha,\beta) = \alpha \vee \beta$, $\alpha + \beta$, or $\alpha + \beta - \alpha\beta$, respectively, conditions a) and b) imply that $\mathcal{R}_0(G)$ is countably compact, and $\mathcal{R}(F) \subset \mathcal{R}_0(G)$ follows from c). Hence, the assertion follows with Corollary 1.

4. Concluding remark

We derived our Theorem 2 with the aid of Lemma 1. We could as well have used Fuchssteiner's integral representation theorem [5], [6], say, as in the proof of Satz 4.4 in [9] combined with the well-known fact [1], [3], [17] that every $\nu_0 \in P(Y, \mathcal{B}_0)$, $\mathcal{B}_0$ a σ-algebra on Y, can be extended to a $\nu \in \in P(Y, \mathcal{B})$, $\mathcal{B} = \mathcal{B}\big(\mathcal{B}_0 \cup \mathcal{S}(Y)\big)$. Other possible substitutes for our Lemma 1 can be found in the papers [20], [23] of Pollard and Topsoe.

Note added in proof (May 23, 1995). Several applications of the present results can be found in my recent paper Minimax theorems with applications to convex metric spaces, *Colloq. Math.*, **68** (1995), 179–186.

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SHORT NOTES ON QUASI-UNIFORM SPACES II. DOUBLY UNIFORMLY STRICT EXTENSIONS

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Uniformly strict extensions were introduced in [1], called there "strict extensions" (see §0 for the definition). It is the aim of this note to give a complete description of the doubly uniformly strict extensions (cf. [6] Problem 58A): $\mathcal{V}$ is called a *doubly uniformly strict extension* of $\mathcal{U}$ provided that $\mathcal{V}$ is a uniformly strict extension of $\mathcal{U}$, and $\mathcal{V}^{-1}$ of $\mathcal{U}^{-1}$. We shall also consider the smaller class of doubly uniformly regular extensions (see §2 for the definition). Formally, there is only a slight difference between the two results: the filters figuring in the first one are replaced by the associated grills in the second one.

§0. Uniformly strict extensions

0.1. Throughout this paper X will denote a non-empty set, $\mathcal{U}$ a quasi-uniformity on X and $Y \supset X$. A quasi-uniformity $\mathcal{V}$ on Y is an *extension* of $\mathcal{U}$ if $\mathcal{V}|X = \mathcal{U}$ and x is $\mathcal{T}_{\mathcal{V}}$-dense in Y; $\mathcal{V}$ is a *double extension* if, in addition, X is $\mathcal{T}_{\mathcal{V}^{-1}}$-dense in Y. (The terminology was different in [4] to [9].) For $a \in Y$, $\mathfrak{f}^1(a)$ denotes the trace on X of the $\mathcal{T}_{\mathcal{V}}$-neighbourhood filter of a, called the *trace filter* of a; in the case of double extensions, $\mathfrak{f}^{-1}(a)$ denotes the trace on X of the $\mathcal{T}_{\mathcal{V}^{-1}}$-neighbourhood filter of a, and $\big(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a)\big)$ is called the *trace filter pair* of a. In other words:

$$\mathfrak{f}^i(a) = \{V^i a \cap X : V \in \mathcal{V}\} \quad (i = \pm 1).$$

$\mathcal{V}$ is said to be an extension (double extension) for the filters $\mathfrak{f}^1(a)$ (for the filter pairs $\big(f^{-1}(a), \mathfrak{f}^1(a)\big)$). We shall also use other self-explanatory expressions: $\mathcal{V}$ induces the trace filters (filter pairs), $\mathcal{V}$ is compatible with them, etc.

Assume now that trace filters $\mathfrak{f}^1(a)$ are prescribed, and we are looking for extensions of $\mathcal{U}$ inducing them. The following conditions are necessary and sufficient for the existence of such an extension (cf. [1]): (i) for $x \in X$, $\mathfrak{f}^1(x)$ is the $\mathcal{T}_{\mathcal{U}}$-neighbourhood filter of x; (ii) each trace filter is round. (A filter

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$\mathfrak{f}$ is *round* if for each $S \in \mathfrak{f}$ there are $T \in \mathfrak{f}$ and $U \in \mathcal{U}$ such that $U[T] \subset S$.) There is a double extension for prescribed trace filter pairs $\left(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a)\right)$ iff the following conditions hold ([5] 6.1): (i) for $x \in X$, $\left(\mathfrak{f}^{-1}(x), \mathfrak{f}^1(x)\right)$ is the neighbourhood filter pair of x (i.e. $\mathfrak{f}^i(x)$ is the $\mathcal{T}_{\mathcal{U}^i}$-neighbourhood filter); (ii) each trace filter pair is round and Cauchy. (A filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is *round* if $\mathfrak{f}^i$ is $\mathcal{U}^i$-round ($i = \pm 1$), and *Cauchy* if for any $U \in \mathcal{U}$, $S_{-1} \times S_1 \subset U$ holds with suitable $S_i \in \mathfrak{f}^i$.)

When trace filters (filter pairs) are prescribed, we shall always assume that the above conditions are satisfied, and that the trace filters and the trace filter pairs are denoted by $\mathfrak{f}^1(a)$ and by $\left(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a)\right)$ $(a \in Y)$, respectively. The same notations will be used when the trace filters (filter pairs) are not prescribed, but induced by a given (double) extension.

0.2. The quasi-uniformity $\mathcal{V}$ on Y is a *uniformly strict extension* of $\mathcal{U}$ ([1] §7; see also [6] 1.9) if for each $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that

$$s^1(Wa \cap X) \subset Va \quad (a \in Y), \tag{1}$$

where

$$s^i(A) = \left\{a \in Y : A \in \mathfrak{f}^i(a)\right\} \quad (A \subset X,\ i = \pm 1).$$

(Note that the points in (1) are taken from Y, not from $Y \setminus X$.) This is a uniform version of the well-known notion of a strict extension of a topology (cf. [6] 1.9). A doubly uniformly strict double extension $\mathcal{V}$ (meaning that $\mathcal{V}^{-1}$ is also a uniformly strict extension of $\mathcal{U}^{-1}$) will be called shortly a *doubly uniformly strict extension*. $\mathcal{V}$ is a doubly uniformly strict extension of $\mathcal{U}$ iff for each $V \in \mathcal{V}$ there is a $W \in V$ such that

$$s^i(W^i a \cap X) \subset V^i a \quad (a \in Y,\ i = \pm 1). \tag{2}$$

Unlike in [1] to [6], we do not consider (double) extensions compatible with a (bi)topological extension of the induced (bi)topology, since if $\mathcal{V}$ is a uniformly strict extension of $\mathcal{U}$ then $\mathcal{T}_{\mathcal{V}}$ is a strict extension of $\mathcal{T}_{\mathcal{U}}$ ([1] 7.1), and a strict extension of a topology is determined by the trace filters; hence the problem of looking for (doubly) uniformly strict extensions in a (bi)topological space is equivalent to the same problem with prescribed trace filters or filter pairs.

0.3. Let some trace filters be prescribed in $(X, \mathcal{U})$. [1] 7.4 gives a complicated necessary and sufficient condition for the existence of a uniformly strict extension inducing these trace filters. It is not known whether the following much simpler necessary condition is also sufficient ([6] Problem 13): for any $U \in \mathcal{U}$ there are $U_0 \in \mathcal{U}$ and sets $S(a) \in \mathfrak{f}^1(a)$ $(a \in Y)$ such that $S(a) \subset Ux$ whenever $x \in X$ and $U_0 x \in \mathfrak{f}^1(a)$. (It does not change this condition if we replace Y by $Y \setminus X$.)

Unlike the case of strict extensions of topologies, there can exist several uniformly strict extensions inducing the same trace filters ([1] 7.4). A strict extension of a topology is always coarser than any other extension compatible with the same trace filters. Uniformly strict extensions behave differently: in the example below, there are a uniformly strict extension and a not uniformly strict one inducing the same trace filters such that the latter is coarser than the former.

EXAMPLE. Let $X = \mathbf{N}^2$, $Y = \omega \times \mathbf{N}$, and take the discrete uniformity $\mathcal{U}$ on X. Consider the trace filters

$$(1)\qquad \mathfrak{f}^1\big((0,k)\big) = \mathrm{fil}\Big\{\big(\mathbf{N}\times\{k\}\big)\setminus F\colon F \ \text{ is finite}\Big\} \quad (k\in\mathbf{N}).$$

Define the quasi-uniformities $\mathcal{V}$ and $\mathcal{V}'$ on Y by the following quasi-metrics:

$$d\big((0,k),\,(n,k)\big) = 1/n \quad (k,n\in\mathbf{N}),$$

$$d'\big((0,k),\,(n,m)\big) = \begin{cases} 1/n & \text{if } k,n\in\mathbf{N},\ m=k,\\ 1/k & \text{if } k,m,n\in\mathbf{N},\ m\neq k,\end{cases}$$

and $d(x,y)=1$, $d'(x,y)=1$ for pairs $x\neq y$ not mentioned in the formulas. $d'\leqq d$, thus $\mathcal{V}'$ is coarser than $\mathcal{V}$. Both are extensions compatible with the trace filters (1). $\mathcal{V}$ is a uniformly strict extension, since

$$s^1\left(U_{(\varepsilon)}(d)a\cap X\right) = U_{(\varepsilon)}(d)a \quad (a\in Y,\ 0<\varepsilon<1).$$

(Here $U_{(\varepsilon)}(d) = \big\{(a,b)\colon d(a,b)<\varepsilon\big\}$.) But $\mathcal{V}'$ is not uniformly strict:

$$s^1\big(U_{(\varepsilon)}(d')(0,k)\cap X\big) \subset U_{(1)}(d')(0,k)$$

does not hold if $0<\varepsilon<1$ and $k>1/\varepsilon$. (The left hand side contains $Y\setminus X$.) □

0.4. [1] 6.3 gives a sufficient condition for the existence of a uniformly strict extension: assume that the trace filters are stable. (A filter $\mathfrak{f}$ is *stable* if for each $U\in\mathcal{U}$, $\bigcap_{S\in\mathfrak{f}} U[S]\in\mathfrak{f}$.) The construction in [1] 6.3 is equivalent to the following one (cf. [6] 1.7, [8] 6.2, [3] 1.6): to each $U\in\mathcal{U}$ assign an entourage ${}^{5)}U$ on Y defined by

$$a\,{}^{5)}U\,b \quad\text{iff}\quad U[S]\in\mathfrak{f}^1(b) \quad\text{whenever}\quad S\in\mathfrak{f}^1(a);$$

then $\big\{{}^{5)}U\colon U\in\mathcal{U}\big\}$ is a base for a uniformly strict extension ${}^{5)}\mathcal{U}$. The following simple statement will be needed in §2:

PROPOSITION. *For stable trace filters, ${}^{5)}\mathcal{U}$ is the finest uniformly strict extension.*

PROOF. Let $\mathcal{V}$ be a uniformly strict extension for the given trace filters. For $V \in \mathcal{V}$, take $W \in \mathcal{V}$ such that $s^1(W^2 a \cap X) \subset Va$ $(a \in Y)$, and put $U = W|X$. We claim that ${}^{5)}U \subset V$, implying $V \in {}^{5)}\mathcal{U}$, $\mathcal{V} \subset {}^{5)}\mathcal{U}$.

Assume $a\,{}^{5)}U\,b$. Then $U[S] \in \mathfrak{f}^1(b)$ holds with $S = Wa \cap X \in \mathfrak{f}^1(a)$. Thus $b \in s^1\big(U[Wa \cap X]\big)$, and $U[Wa \cap X] \subset W^2 a \cap X$, hence $b \in Va$. □

§1. Doubly uniformly strict extensions

1.1. We are going to show that the doubly uniformly strict extensions are exactly those that can be obtained through a certain construction described in [5]. It will then follow that any doubly uniformly strict extension is coarser than any other double extension inducing the same trace filter pairs; in particular, there can exist only one doubly uniformly strict extension for prescribed trace filter pairs.

Let us first recall some definitions and results from [5]. A Cauchy filter pair $(\mathfrak{f}^{-1}, f^1)$ in $(X,\mathcal{U})$ is *weakly concentrated* if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $x\,U\,y$ whenever $Vx \in \mathfrak{f}^1$ and $V^{-1}y \in \mathfrak{f}^{-1}$; a family of Cauchy filter pairs is *uniformly weakly concentrated* if the above condition holds for each filter pair, with V depending only on U, but not on the filter pair (cf. [5] 7.1, 7.7 and 7.15). If trace filter pairs are prescribed then to each $U \in \mathcal{U}$ we assign an entourage ${}^4U$ on Y as follows:

$$a\,{}^4U\,b \quad \text{iff there are} \quad A \in \mathfrak{f}^{-1}(a),\, B \in \mathfrak{f}^1(b) \quad \text{with} \quad A \times B \subset U.$$

$\{{}^4U : U \in \mathcal{U}\}$ is always a base for a quasi-semiuniformity (=a filter consisting of entourages) ${}^4\mathcal{U}$ on Y such that ${}^4\mathcal{U}|X = \mathcal{U}$ $\big([5]$ Lemma 8.5 b)$\big)$. ${}^4\mathcal{U}$ induces trace filter pairs coarser than the original ones $\big([5]$ Lemma 8.9 b); even when ${}^4\mathcal{U}$ is not a quasi-uniformity, we can define neighbourhood filters, hence trace filters, in the usual way, although the neighbourhood filters will not always generate a topology$\big)$. If $\mathcal{V}$ is a double extension for the prescribed trace filter pairs then ${}^4\mathcal{U} \subset \mathcal{V}$ $\big([5]$ Lemma 8.12; (double) extensions are assumed to be quasi-uniformities$\big)$. ${}^4\mathcal{U}$ is a quasi-uniformity iff the prescribed trace filter pairs are uniformly weakly concentrated; it is a double extension compatible with the given trace filter pairs iff, in addition, the trace filter pairs are minimal Cauchy; if so then ${}^4\mathcal{U}$ is the coarsest double extension for these trace filter pairs ([5] Theorem 8.13 and Lemma 7.13). A family of filter pairs satisfying both conditions is called *uniformly concentrated*. It is enough to know that the filter pairs $\big(\mathfrak{f}^{-1}(p), \mathfrak{f}^1(p)\big)$ $(p \in Y)$ are uniformly concentrated. See [5], §7 for some equivalent formulations of the above conditions.

LEMMA. *A quasi-uniformity* $^4\mathcal{U}$ *belonging to uniformly concentrated trace filter pairs is always a doubly uniformly strict extension.*

PROOF. For reasons of symmetry, it is enough to show that $^4\mathcal{U}$ is uniformly strict. Given $V \in {}^4\mathcal{U}$, take $U \in \mathcal{U}$ with $^4U \subset V$, and then pick $W \in {}^4\mathcal{U}$ such that $W^2|X \subset U$. We claim that $s^1(Wa \cap X) \subset Va$ for each $a \in Y$.

Assume $b \in s^1(Wa \cap X)$; this means that $Wa \cap X \in \mathfrak{f}^1(b)$. Evidently, $W^{-1}a \cap X \in \mathfrak{f}^{-1}(a)$. Now $(W^{-1}a \cap X) \times (Wa \cap X) \subset W^2|X \subset U$, thus $a\,{}^4U\,b$, $a\,V\,b$, $b \in Va$. □

1.2. LEMMA. *If* $\mathcal{V}$ *is a doubly uniformly strict extension of* $\mathcal{U}$ *then* $\mathcal{V} = {}^4\mathcal{U}$, *where* $^4\mathcal{U}$ *is taken with the trace filter pairs induced by* $\mathcal{V}$.

PROOF*. It is enough to show that $\mathcal{V} \subset {}^4\mathcal{U}$, since the converse always holds (see in 1.1). Take a $V \in \mathcal{V}$; we need a $U \in \mathcal{U}$ with $^4U \subset V$. Pick $W, Z \in \mathcal{V}$ such that

$$s^1(Wc \cap X) \subset Vc, \quad s^{-1}(Z^{-1}c \cap X) \subset W^{-1}c \quad (c \in Y),$$

We claim that $^4U \subset V$ holds with $U = Z|X$.

Assume $a\,{}^4U\,b$, and take $A \in \mathfrak{f}^{-1}(a)$, $B \in \mathfrak{f}^1(b)$ with $A \times B \subset U$. For each $y \in B$, $Z^{-1}y \cap X = U^{-1}y \supset A \in \mathfrak{f}^{-1}(a)$, thus $a \in s^{-1}(Z^{-1}y \cap X) \subset W^{-1}y$, i.e. $y \in Wa$, $B \subset Wa$, and so $Wa \cap X \in \mathfrak{f}^1(b)$, $b \in s^1(Wa \cap X) \subset Va$, $a\,V\,b$. □

REMARK. The proof of the above lemma, together with Lemma 1.1, yields that a double extension $\mathcal{V}$ is doubly uniformly strict iff it is uniformly strict, and for each $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that $s^{-1}(W^{-1}x \cap X) \subset V^{-1}x$ $(x \in X)$.

THEOREM. *There exists a doubly uniformly strict extension of a quasi-uniformity* $\mathcal{U}$ *for prescribed trace filter pairs iff they are uniformly concentrated; if so then* $^4\mathcal{U}$ *is the only doubly uniformly strict extension, and it is the coarsest double extension compatible with the trace filter pairs.*

PROOF. Lemmas 1.1 and 1.2, together with the results cited preceding Lemma 1.1. □

$w({}^4\mathcal{U}) = w(\mathcal{U})$ ([5] Lemma 8.2), thus the doubly uniformly strict extensions preserve the weight. The same is false for uniformly strict extensions: Let X be infinite, $Y \setminus X = \{p\}$, and $\mathfrak{f}^1$ a filter that has no countable base. Let $\{U_S : S \in \mathfrak{f}\}$ be a base for $\mathcal{V}$ on Y, where $U_S p = S \cup \{p\}$ and $U_S x = \{x\}$ otherwise. Now $\mathcal{V}$ is a uniformly strict extension of the discrete uniformity $\mathcal{U}$ on X, but $w(\mathcal{V}) > \omega = w(\mathcal{U})$.

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* Contrary to what the reader might possibly expect, we do not have to begin the proof with showing that the trace filter pairs are uniformly concentrated. But, once the lemma is proved, it is clear that the trace filter pairs are uniformly concentrated, since they are induced by an extension of the form $^4\mathcal{U}$.

§2. Uniformly regular extensions

2.1. A quasi-uniformity $\mathcal{U}$ is *uniformly regular* ([1], [11]) if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $\mathrm{cl}^1\, Vx \subset Ux$ $(x \in X)$; it is *doubly uniformly regular* if both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are uniformly regular, i.e. if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $\mathrm{cl}^i\, Vx \subset Ux$ $(i = \pm 1,\ x \in X)$. Here cl^i is the $\mathcal{T}_{\mathcal{U}^i}$-closure; if an extension $\mathcal{V}$ is given then Cl^i denotes the $\mathcal{T}_{\mathcal{V}^i}$-closure. By a *(doubly) uniformly regular extension* we mean a (double) extension which is (doubly) uniformly regular. (Double) uniform regularity is evidently a hereditary property ([1] 8.5), hence only (doubly) uniformly regular quasi-uniformities can have (doubly) uniformly regular extensions. Note that, unlike in the case of uniformly strict extensions, given an extension $\mathcal{V}$ of $\mathcal{U}$, the statement that "$\mathcal{V}$ is a uniformly regular extension of $\mathcal{U}$" is a property of $\mathcal{V}$, and not of the pair $(\mathcal{V},\mathcal{U})$. Uniformly regular extensions are uniformly strict ([1] 8.7).

For a filter $\mathfrak{f}$ in X $\sec \mathfrak{f} = \{A \subset X : A \cap S \neq \emptyset\ (S \in \mathfrak{f})\}$ is a *grill*, i.e. a union of ultrafilters; more precisely, it is the union of the ultrafilters finer than $\mathfrak{f}$. (See e.g. [17] for more about grills.) If trace filters (filter pairs) are prescribed in $(X,\mathcal{U})$ then $\mathfrak{g}^1(a)$ is the *trace grill*, respectively $(\mathfrak{g}^{-1}(a), \mathfrak{g}^1(a))$ the *trace grill pair*, of $a \in Y$, where $\mathfrak{g}^i(a) = \sec \mathfrak{f}^i(a)$. Clearly, $A \in \mathfrak{g}^i(a)$ iff $A \subset X$ and $a \in \mathrm{Cl}^i A$. Hence $\mathrm{Cl}^i A$ can be described analogously to $s^i(A)$:

$$\mathrm{Cl}^i A = \{a \in Y : A \in \mathfrak{g}^i(a)\} \quad (A \subset X,\ i = \pm 1).$$

For $x \in X$ (when $\mathfrak{f}^i(x)$ is the $\mathcal{T}_{\mathcal{U}^i}$-neighbourhood filter of x), $\mathfrak{g}^i(x)$ will also be called the *adherence grill*, and $(\mathfrak{g}^{-1}(x), \mathfrak{g}^1(x))$ the *adherence grill pair*, of x. Compare the following assertion with the definition of uniformly strict extension:

LEMMA. *An extension $\mathcal{V}$ is uniformly regular iff for any $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that*

$$\mathrm{Cl}^1 (Wa \cap X) \subset Va \quad (a \in Y). \tag{1}$$

PROOF. The necessity is clear from $\mathrm{Cl}^1 (Wa \cap X) \subset \mathrm{Cl}^1\, Wa$. Conversely, assume that (1) holds, and pick $Z \in \mathcal{V}$ with $Z^2 \subset W$. Now $\mathrm{Cl}^1\, Za \subset \subset Va$.

Indeed, if $b \in \mathrm{Cl}^1\, Za$ and $Q \in \mathcal{V}$, $Q \subset Z$ then there are $c \in Qb \cap Za$, $x \in \in Qc \cap X$. Now $x \in Q^2 b \cap Z^2 a \subset Q^2 b \cap Wa$, thus $Q^2 b \cap Wa \cap X \neq \emptyset$. The sets $Q^2 b$ form a $\mathcal{T}_{\mathcal{V}}$-neighbourhood base of b, thus $b \in \mathrm{Cl}^1 (Wa \cap X) \subset Va$. □

Consequently, the doubly uniformly regular extensions can be characterized similarly to 0.2 (2), with s^i replaced by Cl^i.

2.2. [2] §1 gives a complicated necessary and sufficient condition for the existence of a uniformly regular extension compatible with prescribed trace filters. It is not known whether the following simple necessary condition is also sufficient ([6] Problem 13): for any $U \in \mathcal{U}$ there are $U_0 \in \mathcal{U}$ and sets $S(a) \in \mathfrak{f}^1(a)$ $(a \in Y)$ such that $S(a) \subset Ux$ whenever $x \in X$ and $U_0x \in \mathfrak{g}^1(a)$ (cf. the similar condition in 0.3). There can exist different uniformly regular extensions inducing the same trace filters ([2] 4.7). Observe also that $\mathcal{V}$ in Example 0.3 is uniformly regular.

Assume that there are uniformly regular extensions for prescribed trace filters. Then there is a finest uniformly regular extension $\mathcal{V}$ ([1] 8.9) as well as a finest uniformly strict extension $\mathcal{W}$ ([1] 7.11). In the example below, $\mathcal{V} \neq \mathcal{W}$ (this answers [6] Problems 16 and 17).

EXAMPLE (cf. [8] Example 6.2). Let $Y = \mathbf{R}^2$, $X = Y \setminus \big(\{0\} \times (\mathbf{R} \setminus \setminus \{0\})\big)$, $\mathcal{Z} = \mathcal{U}_{\text{so}} \times \mathcal{U}_{\text{eu}}$, where $\mathcal{U}_{\text{so}}$ is the Sorgenfrey, and $\mathcal{U}_{\text{eu}}$ the Euclidean quasi-uniformity on $\mathbf{R}$. $\mathcal{Z}$ is a product of uniformly regular quasi-uniformities, so it has the same property ([1] 8.6). Thus $\mathcal{U} = \mathcal{Z}|X$ has uniformly regular extensions if we consider the trace filters induced by $\mathcal{Z}$. But the finest uniformly strict extension $\mathcal{W}$ is not uniformly regular:

The trace filters are stable, thus $\mathcal{W} = {}^{5)}\mathcal{U}$ by Proposition 0.4. An entourage $U \in \mathcal{U}$ is defined by

$$Ux = \big([x', x'+1] \times [x''-1, x''+1]\big) \cap X \quad \big(x = (x', x'') \in X\big).$$

If $\mathcal{W}$ were uniformly regular then there would exist a $U_0 \in \mathcal{U}$ such that

$$\text{(1)} \qquad \operatorname{Cl}^1\big({}^{5)}U_0 a \cap X\big) \subset {}^{5)}Ua \quad (a \in Y).$$

This is, however, impossible, since if $a = (0, a'') \in Y \setminus X$ and $|a''|$ is small enough then ${}^{5)}U_0 a \cap X$ contains $]0, \varepsilon[\times \{0\}$ for some $\varepsilon > 0$, thus $x = (0,0)$ belongs to the left hand side of (1), although $a\, {}^{5)}U\, x$ does not hold. □

2.3. It should be found out which of the results known for uniformly regular extensions hold more generally for uniformly strict extensions. As an illustration, let us consider the following two very similar assertions: (i) ([16] 6.4) a uniformly regular extension of a totally bounded quasi-uniformity is totally bounded; (ii) ([14] Lemma 1) a uniformly regular extension of a Cauchy bounded quasi-uniformity is Cauchy bounded. (A quasi-uniformity is *Cauchy bounded* [13] if any ultrafilter is the second member of a Cauchy filter pair. Recall for comparison that a quasi-uniformity is totally bounded iff each ultrafilter forms a Cauchy filter pair with itself, see [12] Proposition 3.14 (b).) We are going to show that (i) remains valid for uniformly strict extensions, while (ii) does not.

PROPOSITION. *A uniformly strict extension of a totally bounded quasi-uniformity is totally bounded.*

PROOF. All the filters are stable in a totally bounded space ([1] 4.5), thus ${}^{5)}\mathcal{U}$ is the finest uniformly strict extension (Proposition 0.4). If $\mathcal{U}$ is totally bounded then so is ${}^{5)}\mathcal{U}$ ([8] Proposition 6.5), hence so are all the extensions coarser than ${}^{5)}\mathcal{U}$. □

EXAMPLE. Let $S = \{-1/n: n \in \mathbf{N}\}$, $T = \{1/n: n \in \mathbf{N}\}$,

$$X = \big(S \times \{0,2\}\big) \cup T^2, \quad Y = X \cup \{p_n: n \in \mathbf{N}\},$$

and let $\mathfrak{f}^1(p_n)$ be generated by the cofinite subsets of $T \times \{1/(2n-1), 1/2n\}$. With the quasi-metric d on X defined by

$$d\big((x',x''),(y',y'')\big) = \begin{cases} |y'-x'| & \text{if } x''=y'', \\ -x' & \text{if either } x''=0,\ y''=1/2n,\ n \in \mathbf{N}, \\ & \quad \text{or } x''=2,\ y''=1/(2n-1),\ n \in \mathbf{N}, \end{cases}$$

consider the quasi-uniformity $\mathcal{U} = \mathcal{U}(d)$, which is Cauchy bounded, since each ultrafilter forms a Cauchy filter pair with a filter containing either $S \times \{0\}$ or $S \times \{2\}$. The trace filters are round and stable, so ${}^{5)}\mathcal{U}$ is a uniformly strict extension. But it is not even precompact: $a\, {}^{5)}U_{(1)} p_n$ does not hold for any $a \neq p_n$. □

§3. Doubly uniformly regular extensions

3.1. A doubly uniformly regular extension is doubly uniformly strict, so, in order to obtain a complete description of them, it is enough to find a necessary and sufficient condition for ${}^4\mathcal{U}$ to be doubly uniformly regular (cf. Theorem 1.2). We begin with some definitions, and characterization of double uniform regularity.

A filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ in $(X,\mathcal{U})$ is *convergent* if there is an $x \in X$ such that $\mathfrak{f}^i$ $\mathcal{T}_{\mathcal{U}^i}$-converges to x $(i = \pm 1)$; it is *fixed* if one of the filters is fixed, i.e. if $\bigcap \mathfrak{f}^{-1} \cup \bigcap \mathfrak{f}^1 \neq \emptyset$. The ultrafilter fixed at x will be denoted by $\dot{x}$. Extending the definition given earlier for filter pairs, we say that a family $\mathfrak{K}$ of pairs of systems of subsets of X is *uniformly weakly concentrated* if for any $U \in \mathcal{U}$ there is a $U_0 \in \mathcal{U}$ such that $x, y \in X$, $(\mathfrak{a}^{-1}, \mathfrak{a}^1) \in \mathfrak{K}$, $U_0 x \in \mathfrak{a}^1$, $U_0^{-1} y \in \mathfrak{a}^{-1}$ imply $x\, U\, y$.

PROPOSITION. *The following conditions are equivalent for a quasi-uniformity:*

(i) *it is doubly uniformly regular;*

(ii) *the fixed Cauchy filter pairs are uniformly weakly concentrated;*

(iii) *the convergent filter pairs are uniformly weakly concentrated;*

(iv) *the convergent ultrafilter pairs are uniformly weakly concentrated;*
(v) *the adherence grill pairs are uniformly weakly concentrated.*

PROOF. (i) $\Leftrightarrow$ (ii). According to [7] Proposition 1.1, a quasi-uniformity is uniformly regular iff those Cauchy filter pairs are uniformly weakly concentrated in which the first member is fixed. So we have only to observe that the union of two uniformly weakly concentrated families has the same property.

(ii) $\Rightarrow$ (iii). Given $U \in \mathcal{U}$, take $V \in \mathcal{U}$ with $V^2 \subset U$, and then U_0 for V according to (ii). Assume that $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ converges to z, and $U_0 x \in \mathfrak{f}^1$, $U_0^{-1} y \in \mathfrak{f}^{-1}$. Now $(\mathfrak{f}^{-1}, \dot{z})$ is a fixed Cauchy filter pair, $U_0 z \in \dot{z}$, $U_0^{-1} y \in \mathfrak{f}^{-1}$, thus $z\, V\, y$. Analogously, $x\, V\, z$, and so $x\, U\, y$.

(iii) $\Rightarrow$ (ii). Let $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ be a fixed Cauchy filter pair, $x \in \bigcap \mathfrak{f}^{-1} \cup \bigcap \mathfrak{f}^1$. Then either $(\dot{x}, \mathfrak{f}^1)$ or $(\mathfrak{f}^{-1}, \dot{x})$ is a convergent filter pair finer than $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$.

(iii) $\Rightarrow$ (iv). Evident.

(iv) $\Rightarrow$ (v). Given $U \in \mathcal{U}$, the entourage U_0 furnished by (iv) will also do in (v), since if $U_0 x \in \mathfrak{g}^1(z)$, $U_0^{-1} y \in \mathfrak{g}^{-1}(z)$ then there is an ultrafilter pair $(\mathfrak{u}^{-1}, \mathfrak{u}^1)$ converging to z such that $U_0 x \in \mathfrak{u}^1$, $U_0^{-1} y \in \mathfrak{u}^{-1}$.

(v) $\Rightarrow$ (iii). If $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ converges to z then $\mathfrak{f}^i \subset \mathfrak{g}^i(z)$. □

3.2. THEOREM. *There is a doubly uniformly regular extension for prescribed trace filter pairs iff they are minimal Cauchy, and the trace grill pairs are uniformly weakly concentrated (equivalently: the (ultra)filter pairs finer than trace filter pairs are uniformly weakly concentrated); if so then* ${}^4\mathcal{U}$ *is the only doubly uniformly regular extension.*

PROOF. The equivalence of the conditions given with grills, filters, and ultrafilters follows easily, in the same way as in the proof of Proposition 3.1. The last assertion is a consequence of Theorem 1.2.

Necessity. Assume that $\mathcal{V}$ is a doubly uniformly regular extension. Given $U \in \mathcal{U}$, take a $V \in \mathcal{V}$ with $V|X = U$, and then a $V_0 \in \mathcal{V}$ (for V) according to (iii) in Proposition 3.1. Put $U_0 = V_0|X$. Let $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ be a filter pair finer than a trace filter pair, and denote by $\mathfrak{h}^i$ the filter in Y generated by $\mathfrak{f}^i$. Now $(\mathfrak{h}^{-1}, \mathfrak{h}^1)$ is convergent in $(Y, \mathcal{V})$. Assume that $x, y \in X$, $U_0 x \in \mathfrak{f}^1$, $U_0^{-1} y \in \mathfrak{f}^{-1}$. Then $V_0 x \in \mathfrak{h}^1$, $V_0^{-1} y \in \mathfrak{h}^{-1}$, thus $x\, V\, y$, $x\, U\, y$. The trace filter pairs are minimal Cauchy by Theorem 1.2.

Sufficiency. By Theorem 1.2, ${}^4\mathcal{U}$ is a double extension for the prescribed trace filter pairs. We show that (iii) from Proposition 3.1 holds for ${}^4\mathcal{U}$. Let $V \in {}^4\mathcal{U}$. Take $U \in \mathcal{U}$ with ${}^4U \subset V$, and then $U_0 \in \mathcal{U}$ such that $x\, U\, y$ whenever $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is a filter pair finer than a trace filter pair, $U_0 x \in \mathfrak{f}^1$, $U_0^{-1} y \in \mathfrak{f}^{-1}$. Choose $W \in {}^4\mathcal{U}$ such that $W^3|X \subset U_0$. We claim that if $(\mathfrak{h}^{-1}, \mathfrak{h}^1)$ is a convergent filter pair in $(Y, \mathcal{V})$, $Wa \in \mathfrak{h}^1$, $W^{-1} b \in \mathfrak{h}^{-1}$ then $a\, V\, b$.

$A = W^{-1}a \cap X \in \mathfrak{f}^{-1}(a)$, $B = Wb \cap X \in \mathfrak{f}^1(b)$, so it is enough to check that $A \times B \subset U$, since then $a\,{}^4U\,b$. Let $(\mathfrak{k}^{-1}, \mathfrak{k}^1)$ be the ${}^4\mathcal{U}$-envelope of $(\mathfrak{h}^{-1}, \mathfrak{h}^1)$, i.e. the finest one of the round filter pairs coarser than $(\mathfrak{h}^{-1}, \mathfrak{h}^1)$. If $(\mathfrak{h}^{-1}, \mathfrak{h}^1)$ converges to c then so does $(\mathfrak{k}^{-1}, \mathfrak{k}^1)$, thus $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ defined by $\mathfrak{f}^i = \mathfrak{k}^i | X$ is finer than a trace filter pair. ($\mathfrak{f}^i$ is a proper filter in X, since X is doubly dense.) $Wa \in \mathfrak{h}^1$ implies $W^2a \in \mathfrak{k}^1$ so for each $x \in A$ and $z \in W^2a \cap$ $\cap X \in \mathfrak{f}^1$ we have $x\,W^3\,x$, $x\,U_0\,z$, i.e. $U_0x \in \mathfrak{f}^1$. Analogously, $U_0^{-1}y \in \mathfrak{f}^{-1}$ $(y \in B)$, thus $x\,U\,y$, $A \times B \subset U$. □

REMARKS. a) It follows from Proposition 3.1 that if $\mathcal{U}$ is known to be doubly uniformly regular then in the theorem it is enough to consider the trace grill pairs of the points in $Y \setminus X$.

b) ${}^4\mathcal{U}$ is uniformly regular iff the trace filter pairs are minimal Cauchy and the pairs $(\mathfrak{f}^{-1}(a), \mathfrak{g}^1(a))$ are uniformly weakly concentrated.

c) The part of [6] Problem 58A concerning doubly uniformly regular compatible extensions in a bitopological space remains open in the case when the subspace is not doubly dense.

3.3. A quasi-uniformity is *quiet* [10] if the Cauchy filter pairs are uniformly weakly concentrated. (Quiet spaces are doubly uniformly regular.) It was mentioned in [15] §5 that a doubly uniformly regular extension of a quiet quasi-uniformity is quiet. More generally, a doubly strict extension of a quiet quasi-uniformity is quiet, since if $\mathcal{U}$ is quiet then so is ${}^4\mathcal{U}$ by [7] Theorem 2.2.

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ADDENDUM TO "A FLEXIBLE MINIMAX THEOREM"

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All references are to the author's paper [1] "A flexible minimax theorem", which appeared recently in this journal. The author is very grateful to Professor Heinz König for pointing out a weakness in Theorem 1. In fact, Theorem 1 is true even if condition (1.2) is not assumed. Furthermore, an examination of the later parts of the paper reveals that the inductive argument used in Theorem 1 is repeated almost verbatim, in both Theorem 8 and Theorem 9. This is clearly uneconomical. On the other hand, by changing *one word* in Theorem 1 it is possible to strengthen the result tremendously, and the strengthened result enables much shorter proofs of Theorem 8 and Theorem 9. Here are the details — the only change between the original and the new statement of Theorem 1 is the replacement of the word "finite" by "good" in just one place. We follow the same notation as in [1].

DEFINITION. We shall write f_* for $\sup_X \inf_Y f$. We say that a subset W of X is *good* if W is finite and, for all $x \in X$, $\underline{f_*}|x \cap LE(W, f_*) \neq \emptyset$.

NEW THEOREM 1. *Let Y be a topological space, and $\mathcal{B}$ be a nonempty subset of* $\mathbf{R}$ *such that* $\inf \mathcal{B} = f_*$. *Suppose that, for all $\beta \in \mathcal{B}$ and good subsets W of X (with the convention $LE(\emptyset, f_*) = Y$),*

(1.1) for all $x \in X$, $\underline{\beta}|x$ *is closed and compact,*

(1.2) $\{\underline{\beta}|x \cap LE(W, \beta)\}_{x \in X}$ *is pseudoconnected*

and,

(1.3)
$\forall x_0, x_1 \in X, \exists x \in X$ *such that* $\underline{\beta}|x_0$ and $\underline{\beta}|x_1$ *are joined by* $\underline{\beta}|x \cap LE(W, \beta)$.

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

PROOF. Let $x \in X$. If $\mu \in \mathbf{R}$ and $\mu > f_*$ then $\mu > \min f(x, Y)$, from which $\underline{\mu}|x \neq \emptyset$. From (1.1) and the finite intersection property, $\underline{f_*}|x \neq \emptyset$. Thus $\emptyset$ is good. We now prove by induction that all finite subsets of X are good. So suppose that $n \geqq 1$ and

(1.4) $W \subset X$ and $\operatorname{card} W \leqq n - 1 \Longrightarrow W$ is good.

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Let $V \subset X$ and $\operatorname{card} V = n$. Let $x_0 \in V$ and set $W := V \setminus \{x_0\}$. From the induction hypothesis (1.4), W is good. Let $x_1 \in X$ be arbitrary. Let $\beta \in \mathcal{B}$ be arbitrary. From (1.3), there exists $x \in X$ such that $\underline{\beta|} x_0$ and $\underline{\beta|} x_1$ are joined by $\underline{\beta|} x \cap LE(W, \beta)$. Equivalently,

$$\underline{\beta|} x_0 \cap LE(W,\beta) \text{ and } \underline{\beta|} x_1 \cap LE(W,\beta) \text{ are joined by } \underline{\beta|} x \cap LE(W,\beta).$$

From (1.2), $\underline{\beta|} x_0 \cap \underline{\beta|} x_1 \cap LE(W,\beta) \neq \emptyset$, that is to say, $\underline{\beta|} x_1 \cap LE(V,\beta) \neq \emptyset$. Since this holds for all $\beta \in \mathcal{B}$, from (1.1) and the finite intersection property again, $\underline{f_*|} x_1 \cap LE(V, f_*) \neq \emptyset$. Since this is valid for all $x_1 \in X$, V is good. This completes the inductive step of the proof that all finite subsets of X are good. It now follows from (1.1) and the finite intersection property for a third time that $LE(X, f_*) \neq \emptyset$. This completes the proof of the New Theorem 1.

New Proof of Theorem 8. If W is good, $x \in X$ and $\beta \in \mathcal{B}$ then

$$\underline{\beta|} x \cap LE(W,\beta) \supset \underline{f_*|} x \cap LE(W, f_*) \neq \emptyset.$$

Thus (5.2) is satisfied with $Z := LE(W,\beta)$. From Lemma 5, (1.3) is satisfied. The result follows from the New Theorem 1.

New Proof of Theorem 9–(9.1). If W is good, $x \in X$ and $\beta \in \mathcal{B}$ then there exists $y \in \underline{f_*|} x \cap LE(W, f_*)$. Since $y \in LE(W, f_*) \subset LE(W,\beta)$ and $f(x,y) \leqq f_* < \beta$, (4.3) is satisfied with $Z := LE(W,\beta)$. From Lemma 4, (1.3) is satisfied. The result follows from the New Theorem 1.

New Proof of Theorem 9–(9.2). If W is good, $x \in X$ and $\beta \in \mathcal{B}$ then

$$\underline{f_*|} x \cap LE(W,\beta) \supset \underline{f_*|} x \cap LE(W, f_*) \neq \emptyset,$$

thus (6.2) is satisfied with $Z := LE(W,\beta)$ and $\alpha := f_* < \beta$. From Lemma 6, (1.3) is satisfied. The result follows from the New Theorem 1.

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