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ON THE INTEGRAL OF THE LEBESGUE FUNCTION OF INTERPOLATION. II

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To Professor K. Tandori on his seventieth birthday

Let

$$x_k = \cos t_k \quad (k = 1, \dots, n+1;\ 0 \leqq t_1 < \dots < t_{n+1} \leqq \pi) \tag{1}$$

be an arbitrary system of nodes of interpolation, and let

$$\lambda_n(x) := \sum_{k=1}^{n+1} |l_k(x)|$$

(where $l_k(x)$ are the fundamental functions of Lagrange interpolation) be the corresponding Lebesgue function. In [1], we gave a lower estimate for the integral of the Lebesgue function with respect to an arbitrary set of nodes (1) over a fixed interval $[a, b] \subset [-1, 1]$, for n's sufficiently large depending on the interval $[a, b]$. In this paper we extend this result to intervals depending on n, and for all n's (Theorem 1). The method of proof is the same as in [1], with a slight modification. We also prove that, apart form a multiplicative constant, our result is best possible. (In fact, Theorem 2 is slightly stronger than that, since it estimates the maximum of the Lebesgue function in the interval in question.)

THEOREM 1. *There exists an absolute constant $c > 0$ such that for an arbitrary system of nodes* (1) *and arbitrary intervals $[a_n, b_n] \subseteqq [-1, 1]$ we have*

$$\int_{a_n}^{b_n} \lambda_n(x)\,dx \geqq c(b_n - a_n) \log\big(n(\alpha_n - \beta_n) + 2\big) \quad (a_n = \cos a_n,\ b_n = \cos \beta_n).$$

PROOF. Assume first that

$$\frac{\log\big(n(\alpha_n - \beta_n) + 2\big)}{n(\alpha_n - \beta_n)} > \frac{\log 2}{20\pi}.$$

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Then $n(\alpha_n - \beta_n) < c_1$ with an absolute constant $c_1 > 0$, and the statement of the theorem trivially holds, since $\lambda_n(x) \geqq 1$ ($|x| \leqq 1$). So from now on we assume that

$$\frac{\log\big(n(\alpha_n - \beta_n) + 2\big)}{n(\alpha_n - \beta_n)} \leqq \frac{\log 2}{20\pi}. \tag{2}$$

Also, without loss of generality we may assume that

$$\frac{\alpha_n}{2} \leqq \beta_n \leqq \alpha_n \leqq \frac{\pi}{2}, \tag{3}$$

namely

$$\cos\frac{\alpha_n}{2} - \cos\alpha_n \geqq b_n - a_n.$$

$[\beta_n, \alpha_n]$ can be considered as the union of subintervals obtained by considering the partition points $t_k \in [\beta_n, \alpha_n]$. Among these subintervals, let $[\delta_n, \gamma_n]$ be of maximal length. (If there are no t_k's in $[\beta_n, \alpha_n]$, then let $\gamma_n = \alpha_n$, $\delta_n = \beta_n$.) We distinguish two cases.

Case 1: $\gamma_n - \delta_n > \frac{10\pi}{\log 2} \cdot \frac{\log(n(\alpha_n - \beta_n)+2)}{n}$. Then let

$$c_n := \cos\frac{3\gamma_n + 2\delta_n}{5}, \qquad d_n := \cos\frac{2\gamma_n + 3\delta_n}{5},$$

and

$$p_n(x) := \prod_{z_k \notin [c_n, d_n]} (x - z_k),$$

a polynomial formed from the roots of the Chebyshev polynomial

$$T_n(x) := \cos(n \arccos x) = 2^{n-1} \prod_{k=1}^{n} (x - z_k).$$

Since in this case $\frac{\gamma_n - \delta_n}{5} > \frac{2\pi}{n}$, there exists an absolute constant $c_2 > 0$ and a set $H_n \subset [c_n, d_n]$ such that

$$\big|T_n(y)\big| \geqq \frac{1}{2} \qquad (y \in H_n)$$

and

$$|\tilde{H}_n| \geqq c_2(\gamma_n - \delta_n),$$

where $|\tilde{H}_n|$ denotes the measure of the set obtained from H_n by projecting it to the unit circle. Thus, if $|H_n|$ denotes the ordinary measure of the set H_n, then we get

(4)

$$|H_n| \geqq \frac{2}{\pi}|\tilde{H}_n|\sin\beta_n \geqq c_2 \sin\frac{\alpha_n}{2}(\gamma_n-\delta_n) \geqq \frac{10c_2\sin\frac{\alpha_n}{2}}{\log 2}\cdot\frac{\log\left(n(\alpha_n-\beta_n)+2\right)}{n}.$$

Now denoting $C_n = \cos\gamma_n$, $D_n = \cos\delta_n$ we obtain for $x \in [-1,1] \setminus \setminus (C_n, D_n)$, $z_k \in [c_n, d_n]$ and $y \in H_n$

$$\left|\frac{y-z_k}{x-z_k}\right| \leqq \frac{d_n-c_n}{D_n-c_n} = \frac{\sin\frac{\gamma_n+\delta_n}{2}}{2\cos\frac{\gamma_n-\delta_n}{2}\sin\left(\frac{\gamma_n+\delta_n}{2}-\frac{3(\gamma_n-\delta_n)}{10}\right)} \leqq \frac{1}{\sqrt{2}},$$

provided $\gamma_n - \delta_n$ is smaller than a properly chosen absolute constant. (If this fails to hold, then the statement of the theorem reduces to that of [1].) Hence

$$\left|p_n(x)\right| = \frac{|T_n(x)|}{\prod\limits_{z_k\in[c_n,d_n]}|x-z_k|}\left|p_n(y)\right|\cdot\left|\frac{T_n(x)}{T_n(y)}\right|\cdot\prod_{z_k\in[c_n,d_n]}\left|\frac{y-z_k}{x-z_k}\right| \leqq$$

$$\leqq 2\left|p_n(y)\right|\cdot\prod_{z_k\in[c_n,d_n]}\frac{1}{\sqrt{2}} \leqq \left|p_n(y)\right|\cdot 2^{1-\frac{n(\gamma_n-\delta_n)}{10\pi}} \leqq$$

$$\leqq \left|p_n(y)\right|\cdot 2^{1-\frac{\log(n(\alpha_n-\beta_n)+2)}{\log 2}} = \frac{2}{n(\alpha_n-\beta_n)+2}\left|p_n(y)\right|,$$

since evidently, there are at least $\frac{n(\gamma_n-\delta_n)}{5\pi}$ z_k's in the interval $[c_n, d_n]$. Hence using the reproducing property of Lagrange interpolation we get

$$\left|p_n(y)\right| \leqq \sum_{k=1}^{n+1}\left|p_n(x_k)\right|\cdot\left|l_k(y)\right| \leqq \frac{2}{n(\alpha_n-\beta_n)}\left|p_n(y)\right|\lambda_n(y) \quad (y\in H_n),$$

since by construction, there are no x_k's in the interval (C_n, D_n). Thus

$$\lambda_n(y) \geqq n(\alpha_n-\beta_n)/2 \geqq \frac{n(b_n-a_n)}{2\sin\alpha_n} \quad (y\in H_n).$$

Hence and by (4)

$$\int_{a_n}^{b_n}\lambda_n(y)\,dy \geqq \int_{H_n}\lambda_n(y)\,dy \geqq \frac{|H_n|n(b_n-a_n)}{2\sin\alpha_n} \geqq$$

$$\geqq \frac{5c_2}{2\cos\frac{\alpha_n}{2}\log 2}(b_n - a_n)\log\big(n(a_n - \beta_n) + 2\big) \geqq$$

$$\geqq c(b_n - a_n)\log\big(n(\alpha_n - \beta_n) + 2\big).$$

Case 2: $\gamma_n - \delta_n \leqq \frac{10\pi}{\log 2} \cdot \frac{\log(n(\alpha_n-\beta_n)+2)}{n}$. Then by (2), $\frac{\gamma_n-\delta_n}{\alpha_n-\beta_n} \leqq \frac{1}{4}$, which means that there are at least two x_k's in $[a_n, b_n]$, one of them in $\left[a_n, \frac{a_n+b_n}{2}\right]$, say. Thus the sums appearing in the inequality

$$\int_{a_n}^{b_n} \lambda_n(x)\,dx \geqq \frac{1}{8} \sum_{a_n \leqq x_m \leqq \frac{a_n+b_n}{2}} \Delta x_m \sum_{x_m < x_k \leqq b_n} \frac{\Delta x_k}{x_k - x_m}$$

(where $\Delta x_j := x_j - x_{j+1}$) are certainly not empty. This inequality is taken from [1]; its proof is the same as therein. Now let

$$I_{tm} := \big[x_m + t(D_n - C_n), x_m + (t+1)(D_n - C_n)\big) \quad (t = 0, \dots, s_n)$$

where by (3)

$$s_n := \left[\frac{b_n - a_n}{2(D_n - C_n)}\right] - 1 \geqq c_3 \frac{\sin\beta_n}{\sin\alpha_n} \cdot \frac{\alpha_n - \beta_n}{\gamma_n - \delta_n} \geqq c_4 \frac{n(\alpha_n - \beta_n)}{\log\big(n(\alpha_n - \beta_n) + 2\big)}.$$

Then $I_{tm} \subset [a_n, b_n]$ $(t = 0, \dots, s_n)$, and each I_{tm} contains at least one x_k. Thus

$$\sum_{x_m < x_k \leqq b_n} \frac{\Delta x_k}{x_k - x_m} = \sum_{t=0}^{s_n} \sum_{x_k \in I_{tm}} \frac{\Delta x_k}{x_k - x_m} \geqq \frac{1}{D_n - C_n} \sum_{t=0}^{s_n} \frac{1}{t+1} \sum_{x_k \in I_{tm}} \Delta x_k \geqq$$

$$\geqq \frac{1}{2(D_n - C_n)} \sum_{t=0}^{[s_n/2]} \frac{1}{t+1} \sum_{x_k \in I_{tm} \cup I_{t+1,m}} \Delta x_k \geqq \frac{1}{2} \sum_{t=0}^{[s_n/2]} \frac{1}{t+1} \geqq c_5 \log s_n \geqq$$

$$\geqq c_6 \log \frac{n(\alpha_n - \beta_n)}{\log\big(n(\alpha_n - \beta_n) + 2\big)} \geqq c_7 \log\big(n(\alpha_n - \beta_n) + 2\big).$$

Therefore

$$\int_{a_n}^{b_n} \lambda_n(x)\,dx \geqq \frac{c_7}{8} \log\big(n(\alpha_n - \beta_n)\big) \sum_{a_n \leqq x_m \leqq \frac{a_n+b_n}{2}} \Delta x_m \geqq$$

$$\geqq c(b_n - a_n)\log\big(n(\alpha_n - \beta_n) + 2\big),$$

and Theorem 1 is proved.

THEOREM 2. *Given an arbitrary interval* $[a_n, b_n] \subseteqq [-1,1]$, *there exists system of nodes* (1) *such that*

$$\max_{a_n \leqq x \leqq b_n} \lambda_n(x) = O\big(\log(n(\alpha_n - \beta_n) + 2)\big).$$

PROOF. Since the proof is a routine calculation, we give only a sketch. Let (1) be the roots of the polynomial $(x - z_0)T_n(x)$, where $z_0 = \cos t_0$ is a nearest point to the interval $[a_n, b_n]$ such that $|T_n(z_0)| = 1$. (If $\alpha_n - \beta_n \geqq \pi/n$, then $z_0 \in [a_n, b_n]$; otherwise only $\left|z_0 - \frac{a_n+b_n}{2}\right| \leqq \frac{\pi}{2n}$ is guaranteed.) Denoting by $l_0(x)$ the fundamental function of Lagrange interpolation associated with the point z_0, evidently $|l_0(x)| = |T_n(x)| \leqq 1$, so this part can be omitted when estimating the Lebesgue function. For the fundamental polynomials belonging to the Chebyshev abscissas $z_k = \cos t_k$ $(k = 1, \dots, n)$ we have

$$|l_k(x)| = \left|\frac{(x - z_0)T_n(x)}{T_n'(z_k)(z_k - z_0)(x - z_k)}\right|.$$

Without loss of generality we may assume that, with the notation $x = \cos t$, $0 \leqq t \leqq t_0 \leqq \pi/2$. Then an easy calculation yields

$$|l_k(x)| = \begin{cases} O\left(\dfrac{\sin\frac{t_0-t}{2}}{n\sin\frac{|t_k-t_0|}{2}\sin\frac{|t_k-t|}{2}}\right) & \text{if } |t - t_k| \geqq \frac{1}{n}, \\ O(1) & \text{if } |t - t_k| \leqq \frac{1}{n}. \end{cases} \qquad (k = 1, \dots, n).$$

Now

$$\sum_{t_k \leqq \frac{3t-t_0}{2}} |l_k(x)| = O\left(\frac{1}{n}\sin\frac{1_0 - t}{2}\right) \sum_{t_k \leqq \frac{3t-t_0}{2}} \frac{1}{\sin^2\frac{t-t_k}{2}} + O(1) = O(1)$$

(if $t \geqq t_0/3$; otherwise this sum does not appear at all). Further

$$\sum_{|t-t_k| \leqq \frac{1}{2}(t_0-t)} |l_k(x)| = O(n^{-1}) \sum_{|t-t_k| \leqq \frac{1}{2}(t_0-t)} \frac{1}{\sin\frac{|t-t_k|}{2}} + O(1) =$$

$$= O\big(\log(n(t_0 - t))\big) = O\big(\log(n(\alpha_n - \beta_n)) + 2\big),$$

by $t \in [\beta_n, \alpha_n]$ and the definition of t_0.

The remaining two sums, namely those extended for $|t_0 - t_k| \leqq \frac{1}{2}(t_0 - t_k)$ and $t_k \geqq \frac{3t_0-t}{2}$ are entirely similar to the ones considered above.

Reference

[1] P. Erdős and J. Szabados, On the integral of the Lebesgue function of interpolation, *Acta Math. Acad. Sci. Hungar.*, **32** (1978), 191–195.

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ON THE CONNECTION BETWEEN QUASI POWER-MONOTONE AND QUASI GEOMETRICAL SEQUENCES WITH APPLICATION TO INTEGRABILITY THEOREMS FOR POWER SERIES

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Dedicated to Professor Károly Tandori on his seventieth birthday

1. Introduction

Recently Mazhar [5] generalized partly and slightly one of our theorems [4]. The purpose of the present note is to continue the generalization by one step further, and expose the features of the sequences appearing in our new theorem.

It is well known that several papers have been devoted to studying integrability theorems for power series; we refer to the references in [4].

In [4] we proved a theorem generalizing most of the results known up to that time. Before recalling our theorem we present some notions and notations:

$\mathbf{\Phi} = \mathbf{\Phi}(p)$ $(p \geqq 1)$ denotes the set of all nonnegative functions $\varphi(u)$ having the properties: $\varphi(u)/u$ is nondecreasing and $\varphi(u)/u^p$ is nonincreasing on $(0, \infty)$.

$\mathbf{\Psi} = \mathbf{\Psi}(p)$ denotes the set of all functions $\psi(u)$ whose inverse functions belong to $\mathbf{\Phi}$.

$\mathbf{P} = \mathbf{P}(R)$ denotes the set of all nonnegative nondecreasing functions $\rho(u)$ with $\rho(u^2) \leqq R\rho(u)$ $(u \in (0, \infty))$.

$\mathbf{\Lambda} = \mathbf{\Lambda}(\delta_1, \delta_2)$ denotes the collection of all nonnegative functions $\lambda(t)$ defined on [0,1] having the following properties:

$$\delta_1 \mathbf{\Lambda}_m^{(k)} \leqq \lambda(t) \leqq \delta_2 \mathbf{\Lambda}_M^{(k)}, \quad 0 < \delta_1 < \delta_2 < \infty \tag{1.1}$$

holds for any $\frac{1}{k+1} < t < \frac{1}{k}$, $k = 1, 2, \ldots$, where

$$\mathbf{\Lambda}_m^{(k)} := \min \left[\lambda\left(\frac{1}{k}\right), \lambda\left(\frac{1}{k+1}\right) \right]$$

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and

$$\mathbf{\Lambda}_M^{(k)} := \max\left[\lambda\left(\frac{1}{k}\right), \lambda\left(\frac{1}{k+1}\right)\right].$$

(It clearly holds for any quasi-monotone function.)

We shall say that a sequence $\gamma = \{\gamma_n\}$ of positive terms is *quasi β-power-monotone increasing (decreasing)* if there exists a constant $K = K(\beta, \gamma) \geqq 1$ such that

$$Kn^\beta \gamma_n \geqq m^\beta \gamma_m \quad (n^\beta \gamma_n \leqq Km^\beta \gamma_m) \tag{1.2}$$

holds for any $n \geqq m$. Here and in the sequel, K and K_i denote positive constants, not necessarily the same at each occurrence. If (1.2) holds with $\beta = 0$ then we omit the attribute "β-power"; and if (1.2) holds with $K = 1$ then we neglect the attribute "quasi". In brief, sometimes, we shall call these sequences *quasi β-increasing (quasi β-decreasing)*, etc.

Furthermore we shall say that a sequence $\gamma = \{\gamma_n\}$ of positive terms is *quasi geometrically increasing (decreasing)* if there exist a natural number μ and a constant $K = K(\gamma) \geqq 1$ such that

$$\gamma_{n+\mu} \geqq 2\gamma_n \quad \text{and} \quad \gamma_n \leqq K\gamma_{n+1} \quad \left(\gamma_{n+\mu} \leqq \frac{1}{2}\gamma_n \quad \text{and} \quad \gamma_{n+1} \leqq K\gamma_n\right) \tag{1.3}$$

hold for all natural numbers n.

Finally a sequence $\{\gamma_n\}$ will be called *bounded by blocks* if the inequalities

$$\alpha_1 \mathbf{\Gamma}_m^{(k)} \leqq \gamma_n \leqq \alpha_2 \mathbf{\Gamma}_M^{(k)}, \quad 0 < \alpha_1 \leqq \alpha_2 < \infty \tag{1.4}$$

hold for any $2^k \leqq n \leqq 2^{k+1}$, $k = 1, 2 \dots$, where

$$\mathbf{\Gamma}_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad \text{and} \quad \mathbf{\Gamma}_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

Our theorem mentioned above reads as follows:

THEOREM A. *Let $\lambda(t)$ be a positive nonincreasing function on the interval $0 < t \leqq 1$ such that*

$$\sum_{n=k}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \leqq K\lambda\left(\frac{1}{k}\right) k^{-1} \tag{1.5}$$

and let $\{\alpha_n\}$ be a positive increasing sequence with

$$\sum_{n=1}^{\infty} \frac{1}{n\alpha_n} < \infty. \tag{1.6}$$

Suppose that $\rho(u) \in \mathbf{P}$, *that* $\eta(u)$ *denotes either a function of* $\mathbf{\Phi}$ *or a function of* $\mathbf{\Psi}$, *and that*

$$F(x) = \sum_{n=0}^{\infty} c_n x^n, \quad 0 \leqq x < 1. \tag{1.7}$$

Then, under the condition

$$c_n > -Mn^{-1}\overline{\eta}\left(\frac{n}{\alpha_n \lambda(1/n)\rho(n)}\right) \quad (M > 0), \tag{1.8}$$

we have

$$\lambda(1-x)\eta\big(|F(x)|\big)\rho\big(|F(x)|\big) \in L(0,1)$$

if and only if

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\rho(n)\eta\Big(\sum_{k=0}^{n}|c_k|\Big) < \infty. \tag{1.9}$$

This theorem was generalized by Mazhar [5] in the following form:

THEOREM B. *Let* $\lambda(t)$ *be a positive function such that* $t^{-\delta}\lambda(t)$ *is nonincreasing for some* $\delta \in (0,1]$ *and*

$$\sum_{n=k}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\rho(n) \leqq K\lambda\left(\frac{1}{k}\right) k^{-1}\rho(k), \tag{1.10}$$

where $\rho \in P$. *Then under conditions* (1.6), (1.7) *and* (1.8),

$$\lambda(1-x)\eta\big(|F(x)|\big)\rho\big(|F(x)|\big) \in L(0,1)$$

if and only if

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\eta\Big(\sum_{k=0}^{n}|c_k|\Big)\rho\Big(\sum_{k=0}^{n}|c_k|\Big) < \infty, \tag{1.11}$$

for $\eta(u) = \varphi(u)$ *or* $\eta(u) = \psi(u)$ *with* $\rho(u) \equiv 1$.

Comparing our theorem to that of Mazhar it is easy to see that $t^{-\delta}\lambda(t)$ is nonincreasing for any $\delta > 0$, whenever $\lambda(t)$ is nonincreasing, but the converse is not necessarily true; thus Theorem B slightly generalizes Theorem A, but only if $\eta(u) = \varphi(u)$ or $\rho(x) \equiv 1$ and $\eta(u) = \psi(u)$.

2. Theorems

First we prove the following theorem which slightly generalizes both Theorems A and B, but in this proof we shall use some results of Theorems 2 and 3 to be proved later in this work.

THEOREM 1. *Let* $\rho(u)$ *be a function belonging to the class* $\mathbf{P}$ *and let* $\{\alpha_n\}$ *be a positive increasing sequence with* (1.6). *Let* $\lambda(t)$ *be a function of* $\mathbf{\Lambda}$ *such that the sequence*

$$\gamma_n := \lambda\left(\frac{1}{n}\right) n^{-1}\rho(n) \tag{2.1}$$

is quasi β*-power-monotone decreasing with some positive* β. *Furthermore, let* $\eta(u)$ *denote either a function of* $\mathbf{\Phi}$ *or a function of* $\mathbf{\Psi}$, *and let* $F(x)$ *be given in* (1.7). *Then, under condition* (1.8), $\lambda(1-x)\eta\big(|F(x)|\big)\rho\big(|F(x)|\big) \in L(0,1)$ *if and only if* (1.9) *holds.*

Recently in [2] and [3] it turned out that the quasi power-monotone sequences appearing in Theorem 1 and the quasi geometrically monotone sequences are closely interlinked; furthermore that these sequences have been appearing in the generalizations of several classical results, sometimes only implicitly.

We shall prove shortly that if a sequence γ is, e.g., quasi β-power monotone increasing with a negative β, then the sequence $\{\gamma_{2^n}\}$ is quasi geometrically increasing. It is clear that the converse of this assertion cannot be true since the 2^n-th terms of the sequence do not determine the behavior of the other terms of the sequence $\{\gamma_n\}$. Therefore a relevant question is the following: What additional assumption on the sequence $\{\gamma_n\}$ along with the assertion that $\{\gamma_{2^n}\}$ is quasi geometrically increasing will imply that the whole sequence $\{\gamma_n\}$ should be quasi β-power-monotone increasing with some negative β? We shall verify that a fitting very mild sufficient condition is the boundedness by blocks.

We would like to point out that if the sequence $\{\gamma_n\}$ is either quasi β-power-monotone increasing or decreasing, then condition (1.4), i.e. the boundedness by blocks, is always fulfilled.

Taking into consideration all of these remarks we can formulate two further results as follows.

THEOREM 2. *If a positive sequence* $\{\gamma_n\}$ *is quasi* β*-power-monotone increasing (decreasing) with a certain negative (positive) exponent* β, *then the sequence* $\{\gamma_{2^n}\}$ *is quasi geometrically increasing (decreasing).*

THEOREM 3. *If a positive sequence* $\{\gamma_n\}$ *is bounded by blocks and its partial sequence* $\{\gamma_{2^n}\}$ *is quasi geometrically increasing (decreasing), then the whole sequence* $\{\gamma_n\}$ *is quasi* β*-power-monotone increasing (decreasing) with a certain negative (positive) exponent* β.

The following corollary is an immediate consequence of Theorems 2 and 3.

COROLLARY 1. *A positive sequence* $\{\gamma_n\}$ *bounded by blocks is* β*-power-monotone increasing (decreasing) with a certain negative (positive) exponent* β *if and only if the sequence* $\{\gamma_{2^n}\}$ *is quasi geometrically increasing (decreasing).*

Corollary 1 and Lemma 3, by (1.4), easily imply Corollary 2.

COROLLARY 2. *A positive sequence* $\{\gamma_n\}$ *bounded by blocks is* β*-power-monotone increasing (decreasing) with a certain negative (positive) exponent* β *if and only if the inequality*

$$\sum_{n=1}^{m} \gamma_n n^{-1} \leqq K\gamma_m \quad \left(\sum_{n=m}^{\infty} \gamma_n n^{-1} \leqq K\gamma_m\right)$$

holds for any natural number m.

Consult S. Aljančič [1] for related results.

It is quite obvious that if we extended the given definitions from sequences to functions according to the sense, then analogous theorems and corollaries for functions would be also valid.

It is also easy to see that similar results with $\beta = 0$ do not hold, see e.g. the constant sequence $\gamma \equiv 1$.

According to these results, it is easy to see that the assumptions of Theorem A, or B, one by one, imply the presumptions of Theorem 1. Namely if $t^{-\delta}\lambda(t)$ $(\delta > 0)$ is nonicreasing, which implies boundedness by blocks and property (1.1), and the sequence $\{\gamma_n\}$ given in (2.1) fulfils condition (1.10), then the sequence $\{\gamma_n\}$ is quasi β-power-monotone decreasing with some positive β. Thus, our Theorem 1 generalizes both Theorems A and B.

3. Lemmas

We require the following lemmas.

LEMMA 1. *If* $\varphi \in \mathbf{\Phi}$ *and* $\rho \in \mathbf{P}$, *then*

$$\varphi\left(\frac{\sum_{i=1}^{\infty} a_i b_i}{\sum_{i=1}^{\infty} a_i}\right)\rho\left(\frac{\sum_{i=1}^{\infty} a_i b_i}{\sum_{i=1}^{\infty} a_i}\right) \leqq K\frac{\sum_{i=1}^{\infty} a_i \varphi(b_i)\rho(b_i)}{\sum_{i=1}^{\infty} a_i} \tag{3.1}$$

holds.

Furthermore if $\psi \in \mathbf{\Psi}$, *then the following inequalities are valid:*

$$\psi(a+b) \leqq \psi(a) + \psi(b), \tag{3.2}$$

and for any $0 < k \leqq 1$

$$\psi(kx) \leqq k^{1/p}\psi(x). \tag{3.3}$$

Inequality (3.1) immediately follows from results of Mulholland [6] (see Theorem 1 and Remark (2.34)) and from the properties of the functions $\varphi(u)$ and $\rho(u)$. The validity of inequalities (3.2) and (3.3) can easily be derived from the definition of $\psi(u)$.

LEMMA 2. *If* $\rho(u) \in \mathbf{P}$ *then for any integer* r *there exists a constant* C_r *such that for any numbers* $\alpha > 0$, $\beta > 0$

$$\alpha\rho(\beta) \leqq C_r\alpha\rho(\alpha) + \beta^{\frac{1}{r}}\rho(\beta) \tag{3.4}$$

holds.

This inequality is implicitly proved in Lemma 13 of [8] (see statement (2.38)).

LEMMA 3 ([2]). *For any positive sequence* $\gamma = \{\gamma_n\}$ *the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \leqq K\gamma_m \quad (m = 1, 2, \ldots; K \geqq 1), \tag{3.5}$$

or

$$\sum_{n=1}^{m} \gamma_n \leqq K\gamma_m \quad (m = 1, 2, \ldots; K \geqq 1) \tag{3.6}$$

hold if and only if the sequence γ *is quasi geometrically decreasing or increasing, respectively.*

LEMMA 4. *If a positive sequence* $\{c_n\}$ *is quasi geometrically increasing (decreasing), then there exists a positive* ε *such that the sequence* $\{c_n 2^{-n\varepsilon}\}$ $(\{c_n 2^{n\varepsilon}\})$ *is also quasi geometrically increasing (decreasing).*

PROOF. Assuming that the sequence $\{c_n\}$ is quasi geometrically increasing, according to the definition there exist a natural μ and a constant $K = K(c) \geqq 1$ such that

$$c_{n+\mu} \geqq 2c_n \quad \text{and} \quad c_n \leqq Kc_{n+1} \tag{3.7}$$

hold for all n.

Now let $\varepsilon := (2\mu)^{-1}$ and $\mu_1 := 2\mu$. Then, by (3.7),

$$c_{n+\mu_1} = c_{n+2\mu} \geqq 4c_n$$

and thus

$$c_{n+\mu_1} 2^{-(n+\mu_1)\varepsilon} \geqq 4c_n 2^{-n\varepsilon} 2^{-1} = 2c_n 2^{-n\varepsilon}, \tag{3.8}$$

furthermore

$$c_n 2^{-n\varepsilon} \leqq K c_{n+1} 2^{-n\varepsilon} = K 2^{\varepsilon} c_{n+1} 2^{-(n+1)\varepsilon} \tag{3.9}$$

hold. Inequalities (3.8) and (3.9) imply that the sequence $\{c_n 2^{-n\varepsilon}\}$ is also quasi geometrically increasing, since the requirements of the definition given in (1.3) are satisfied with μ_1 and $K2^{\varepsilon}$ in place of μ and K, respectively.

The decreasing case could be proved similarly.

The following lemma is a slight generalization of Lemma of [4].

LEMMA 5. *Let $\lambda(t)$, $\rho(u)$ and $\eta(u)$ be defined as in our Theorem* 1 *and let*

$$f(x) := \sum_{k=0}^{\infty} a_k x^k \quad \textit{with} \quad a_k \geqq 0,\ 0 \leqq x < 1. \tag{3.10}$$

Then

$$\lambda(1-x)\eta\big(f(x)\big)\rho\big(f(x)\big) \in L(0,1) \tag{3.11}$$

if and only if

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \eta(A_n)\rho(n) < \infty, \tag{3.12}$$

or equivalently

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \eta(A_n)\rho(A_n) < \infty, \tag{3.13}$$

where

$$A_n = \sum_{k=0}^{n} a_k.$$

PROOF. We follow the line of the proof of the Lemma in [4] with the required changes.

First we show that (3.12) and (3.13) are equivalent. It is easy to verify that (3.12) implies (3.13). Namely, (3.12) implies the existence of a natural number t such that for all $n (\geqq 2)$

$$A_n \leqq n^t, \tag{3.14}$$

so the implication (3.12) $\Rightarrow$ (3.13) is obvious. In order to show that (3.13) also implies (3.12) we use Lemma 2. Using (3.4) and considering $\eta(u)/u^p \downarrow$ and $\rho(u) \in \mathbf{P}$ we obtain for any integer r that

$$(3.15) \qquad \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\eta(A_n)\rho(n) \leqq$$

$$\leqq C_r \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\eta(A_n)\rho\big(\eta(A_n)\big) + \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}n^{1/r}\rho(n) \leqq$$

$$\leqq K \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\eta(A_n)\rho(A_n) + \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}n^{\frac{1}{r}}\rho(n).$$

By (3.13) and Lemma 4 it is easy to verify that

$$(3.16) \qquad \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}n^{1/r}\rho(n) < \infty$$

for all sufficiently large values of r. So, by (3.13) (3.15) and (3.16) we have (3.12).

Now we prove the equivalence of (3.11) and (3.13). Set $y = 1 - x$. Since $\left(1 - \frac{1}{n}\right)^n$ is an increasing sequence, we have for $\frac{1}{n+1} \leqq y \leqq \frac{1}{n}$ $(n \geqq 2)$:

$$f(1-y) \geqq \sum_{k=0}^{n} a_k(1-y)^k \geqq \sum_{k=0}^{n} a_k\left(1 - \frac{1}{n}\right)^k \geqq \left(1 - \frac{1}{n}\right)^n \sum_{k=0}^{n} a_k \geqq \frac{1}{4}A_n.$$

Using this and (1.1) we obtain

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\eta(A_n)\rho(A_n) \leqq K \sum_{n=1}^{\infty} \eta(A_n)\rho(A_n) \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)dy \leqq$$

$$\leqq K + K \sum_{n=2}^{\infty} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\big(f(1-y)\big)\rho\big(f(1-y)\big)\,dy \leqq$$

$$\leqq K + K \int_0^1 \lambda(1-x)\eta\big(f(x)\big)\rho\big(f(x)\big)\,dx.$$

This proves that (3.13) follows from (3.10). To prove the inverse statement of the equivalence we consider the following estimations:

$$(3.17) \qquad \int_0^1 \lambda(1-x)\eta\big(f(x)\big)\rho\big(f(x)\big)\,dx =$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\big(f(1-y)\big)\,\rho\big(f(1-y)\big)\,dy =$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\Big(\sum_{k=0}^{\infty} a_k(1-y)^k\Big)\rho\Big(\sum_{k=0}^{\infty} a_k(1-y)^k\Big)dy \leqq$$

$$\leqq \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\Big(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\Big)\rho\Big(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\Big)dy \leqq$$

$$\leqq K\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\eta\Big(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\Big)\rho\Big(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\Big).$$

Since $\frac{1}{2} \geqq \left(1-\frac{1}{n+1}\right)^n \geqq \left(1-\frac{1}{n+2}\right)^{n+1}$ for $n = 1, 2\dots$, we have

$$\text{(3.18)} \qquad \sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k \leqq \sum_{j=0}^{\infty}\sum_{k=nj}^{n(j+1)} a_k\left(1-\frac{1}{n+1}\right)^k \leqq$$

$$\leqq \sum_{j=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{nj}\sum_{k=nj}^{n(j+1)} a_k \leqq 2\sum_{i=1}^{\infty} 2^{-i}A_{ni}.$$

Henceforth we split the proof into two parts. If $\eta(u) = \varphi(u)$ then we use Lemma 1. By (3.1) we get:

$$\text{(3.19)} \qquad \varphi\Big(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\Big)\rho\Big(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\Big) \leqq K\sum_{i=1}^{\infty} 2^{-i}\varphi(A_{ni})\rho(A_{ni}).$$

Hence and from (3.17), (3.18) and (3.19) we get that

$$\int_0^1 \lambda(1-x)\varphi\big(f(x)\big)\,\rho\big(f(x)\big)\,dx \leqq K\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\sum_{i=1}^{\infty}2^{-i}\varphi(A_{ni})\rho(A_{ni}) \leqq$$

$$\leqq K\sum_{i=1}^{\infty}2^{-i}\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi(A_{ni})\rho(A_{ni}) \leqq$$

$$\leqq K\sum_{i=1}^{\infty}2^{-i}i^2\sum_{n=1}^{\infty}\lambda\left(\frac{1}{ni}\right)(ni)^{-2}\varphi(A_{ni})\rho(A_{ni}) \leqq$$

$$\leqq K\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi(A_n)\rho(A_n).$$

If $\eta(u) = \psi(u)$ the proof runs similarly but we use the following inequality

$$\psi\Big(\sum_{i=1}^{\infty} 2^{-i} A_{ni}\Big)\rho\Big(\sum_{i=1}^{\infty} 2^{-i} A_{ni}\Big) \leqq \tag{3.20}$$

$$\leqq K\sum_{i=1}^{\infty} 2^{-\frac{i}{p}}\psi(A_{ni})\rho\Big(\sum_{i=1}^{\infty}\frac{(ni)^t}{2^i}\Big) \leqq K\sum_{i=1}^{\infty} 2^{-\frac{i}{p}}\psi(A_{ni})\rho(n)$$

instead of (3.19). Inequality (3.20) is just an easy consequence of Lemma 1 (see (3.2) and (3.3)) and (3.14). Thus the proof of Lemma 5 is complete.

4. Proof of the theorems

Proof of Theorem 1. Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$ for $0 \leqq x < 1$ with $a_0 = 0$ and

$$a_k = Mk^{-1}\overline{\eta}\Big(\frac{k}{\alpha_k\lambda(1/k)\rho(k)}\Big), \quad k \geqq 1.$$

First we consider the case $\eta(u) = \varphi(u)$. We show that the coefficients a_k satisfy condition (3.12). Using the inequality

$$\sum_{n=1}^{\infty} \lambda_n\varphi(A_n) \leqq K_1\sum_{n=1}^{\infty}\lambda_n\varphi\Big(\frac{a_n}{\lambda_n}\sum_{k=n}^{\infty}\lambda_k\Big), \tag{4.1}$$

which holds for any $\lambda_n > 0$ and $a_n \geqq 0$ (see inequality (8) of [7]), with $\lambda_n = \lambda(1/n)n^{-2}\rho(n)$ and the inequality

$$\sum_{n=k}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\rho(n) \leqq K\lambda\left(\frac{1}{k}\right)k^{-1}\rho(k),$$

which follows from the assumption that the sequence $\{\gamma_n\}$ given in (2.1) is quasi β-power-monotone decreasing with some positive β (see e.g. Corollary 2), we have

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi(A_n)\rho(n) \leqq K_1\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\rho(n)\varphi(Kna_n) \leqq$$

$$\leqq K_1\big((KM)^p + 1\big)\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\rho(n)n\frac{1}{\alpha_n\lambda(1/n)\rho(n)} \leqq K_2\sum_{n=1}^{\infty}\frac{1}{n\alpha_n} < \infty.$$

Hereby we proved that the coefficients of the function $A(x)$ satisfy condition (3.12), so by Lemma 5

$$\lambda(1-x)\varphi\big(A(x)\big)\,\rho\big(A(x)\big) \in L(0,1). \tag{4.2}$$

By (1.8) the coefficients $a_n + c_n$ are positive, thus the function

$$A(x)+F(x)=\sum_{n=0}^{\infty}(a_n+c_n)x^n$$

has the property

$$\lambda(1-x)\varphi\big(A(x)+F(x)\big)\,\rho\big(A(x)+F(x)\big) \in L(0,1) \tag{4.3}$$

if and only if

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi\Big(\sum_{k=0}^{n}(a_k+c_k)\Big)\rho(n)<\infty. \tag{4.4}$$

If $\lambda(1-x)\varphi\big(|F(x)|\big)\,\rho\big(|F(x)|\big) \in L(0,1)$, then this and (4.2) imply (4.3), (see e.g. (4.5) below) which implies (4.4). But by (1.8) we have

$$|c_n| \leqq 2a_n + c_n,$$

whence, by (3.12) and (4.4), (1.9) follows.

If (1.9) holds, then this implies (4.4), because from (3.1) immediately follows that

$$\varphi(a+b)\rho(a+b) \leqq K\big(\varphi(a)\rho(a)+\varphi(b)\rho(b)\big),\ a>0,\ b>0. \tag{4.5}$$

But (4.4) yields (4.3). By (4.2) and (4.3)

$$\lambda(1-x)\varphi\big(|F(x)|\big)\,\rho\big(|F(x)|\big) \in L(0,1)$$

follows obviously.

Thus Theorem 1 is proved for $\eta(u)=\varphi(u)$. The proof for $\eta(u)=\psi(u)$ runs similarly. To prove (3.12) we use Lemma 1 (see (3.2)), thus

$$\sum_{n=2}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\rho(n)\psi\Big(\sum_{k=1}^{n}a_k\Big) \leqq \sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\lambda\left(\frac{1}{n}\right)n^{-2}\rho(n)\psi\Big(\sum_{k=1}^{2^{m+1}}a_k\Big) \leqq$$

$$\leqq K\sum_{m=0}^{\infty}\lambda\left(\frac{1}{2^{m+1}}\right)2^{-m}\rho(2^{m+1})\psi\Big(\sum_{k=1}^{m+1}\overline{\psi}\Big(\frac{2^k}{\lambda(1/2^k)\alpha_{2^k}\rho(2^k)}\Big)\Big) \leqq$$

$$\leqq K\sum_{k=1}^{\infty}\frac{2^k}{\lambda(1/2^k)\alpha_{2^k}\rho(2^k)}\sum_{m=k}^{\infty}\lambda\left(\frac{1}{2^{m+1}}\right)2^{-m}\rho(2^m)\leqq K\sum_{k=1}^{\infty}\frac{1}{\alpha_{2^k}}<\infty.$$

From this point the proof runs on the same line as before. The proof of Theorem 1 is thus complete.

We shall detail only the proofs of the decreasing cases of Theorems 2 and 3, the increasing cases would run likewise.

PROOF OF THEOREM 2. Since the sequence is quasi β-power-monotone decreasing with a positive β, therefore, by (1.2),

$$2^{(m+\mu)\beta}\gamma_{2^{m+\mu}}\leqq K2^{m\beta}\gamma_{2^m} \tag{4.6}$$

holds for arbitrary natural numbers m and μ. If μ is large enough, e.g. if $2^{\mu\beta}>2K$, then (4.6) implies that

$$\gamma_{2^{m+\mu}}\leqq\frac{1}{2}\gamma_{2^m} \tag{4.7}$$

holds for any m. The inequality

$$\gamma_{2^{m+1}}\leqq K\gamma_{2^m} \tag{4.8}$$

obviously follows from (4.6). Thus, (4.7) and (4.8) show that the sequence $\{\gamma_{2^n}\}$ is quasi geometrically decreasing.

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Since the sequence $\{\gamma_{2^n}\}$ is quasi geometrically decreasing, therefore, by Lemmas 3 and 4 there exists a positive ε such that

$$\sum_{n=m}^{\infty}2^{n\varepsilon}\gamma_{2^n}\leqq K_0 2^{m\varepsilon}\gamma_{2^m}$$

holds for any m. Hence, since the sequence γ is bounded by blocks, thus, by (1.4) an elementary calculation shows that if $n\geqq m$ then

$$n^{\varepsilon}\gamma_n\leqq K(\alpha_1,\alpha_2,K_0)m^{\varepsilon}\gamma_m \tag{4.9}$$

holds with a constant K depending on α_1,α_2 and K_0.

Finally, setting $\beta:=\varepsilon$ and $K:=K(\alpha_1,\alpha_2,K_0)$, (4.9) verifies that the sequence $\{\gamma_n\}$ is quasi β-power-monotone decreasing with a positive β.

The proof of Theorem 3 is complete.

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ASYMPTOTICS FOR DIRECTED RANDOM WALKS IN RANDOM ENVIRONMENTS

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Dedicated to Professor Károly Tandori for his seventieth birthday

1. Introduction

Random walks in random environments have received much attention from mathematical physicists as well as probabilists in recent years. Buffet and Hannigan [4] points out that directed random walks in random environments are mainly used as "models for the motion of electrons in crystals with impurities. The presence of the defects perturbs the normal hopping behaviour of the electrons from one ion of the crystal to the next, thus modifying the transport properties of the medium. Because the nature and location of the defects can only be controlled in a statistical sense, their effect is best taken into account by treating the *transition rates* of the walk as *random variables*." Let $\{X(t), 0 \leqq t < \infty\}$ denote the position of the integer-valued random walk at time t. Following Aslangul et al. [1]–[3] and Buffet and Hannigan [4], we assume that $X(t)$ is a pure birth process and $P_n(t)$, the probability that at time t the random walk is in state n, satisfies

$$P_n'(t) = -w_n P_n(t) + w_{n-1}P_{n-1}(t), \quad 1 \leqq n < \infty, \tag{1.1}$$

$$P_0'(t) = -w_0 P_0(t), \tag{1.2}$$

where $\mathbf{w} = \{w_i,\ 0 \leqq i < \infty\}$ are independent, identically distributed nonnegative random variables. We also assume that the process $X(t)$ starts its random walk from state 0.

Using holding times, Buffet and Hannigan [4] gave a simple representation for $X(t)$. Let $S(0) = 0$ and $S(n)$ denote the time of the n^{th} jump of the random walk. The holding times are defined by $T_j = S(j+1) - S(j)$, $0 \leqq j < \infty$. It is well known (cf. Feller [10], Cinlar [6] and Buffet and Hannigan [4]) that $\{T_j, 0 \leqq j < \infty\}$ are conditionally independent, exponential r.v.'s with parameters $\{w_j, 0 \leqq j < \infty\}$, i.e. we have

$$P_{\mathbf{w}}\{T_{n_j} > x_j, 1 \leqq j \leqq k\} = \exp\left(-\sum_{1 \leqq j \leqq k} w_{n_j} x_j\right), \tag{1.3}$$

for all $1 \leqq k < \infty$ and $0 \leqq n_1 < n_2 < \ldots < n_k$. It follows from the definition of the holding times that

$$X(t) = \max\{n : S(n) \leqq t\}. \tag{1.4}$$

Since $X(t)$ is the inverse of $S(n)$, the properties of the partial sums can be used to get asymptotic properties of $X(t)$.

In Section 2 we study the rate of convergence in the strong law for $X(t)$ and in Section 3 we obtain approximations for $X(t)$ using suitable constructed Wiener processes.

2. The rate of convergence in the strong laws of large numbers

Horváth and Shao [13] obtained the necessary and sufficient condition for the law of large numbers.

THEOREM 2.1. (i) *If* $Ew_0^{-1} < \infty$, *then*

$$P_{\mathbf{w}}\left\{\lim_{t\to\infty} X(t)/t = (Ew_0^{-1})^{-1}\right\} = 1 \tag{2.1}$$

for almost all realizations of $\mathbf{w}$.

(ii) *If there is a positive constant* c *such that*

$$P_{\mathbf{w}}\left\{\lim_{t\to\infty} X(t)/t = c\right\} = 1 \tag{2.2}$$

for almost all realizations of $\mathbf{w}$, *then* $Ew_0^{-1} < \infty$ *and* $c = Ew_0^{-1}$.

The main result of this section is the rate of convergence in (2.1).

THEOREM 2.2 (i) LET $1 \leqq \nu < 2$.
If $Ew_0^{-\nu} < \infty$, *then*

$$P_{\mathbf{w}}\left\{\lim_{t\to\infty} \frac{X(t) - t(Ew_0^{-1})^{-1}}{t^{1/\nu}} = 0\right\} = 1 \tag{2.3}$$

for almost all realizations of $\mathbf{w}$.

If there is a positive constant c *such that*

$$P_{\mathbf{w}}\left\{\lim_{t\to\infty} \frac{X(t) - ct}{t^{1/\nu}} = 0\right\} = 1 \tag{2.4}$$

for almost all realizations of $\mathbf{w}$, *then* $Ew_0^{-\nu} < \infty$ *and* $c = (Ew_0^{-1})^{-1}$.

(ii) *If* $Ew_0^{-2} < \infty$, *then*

$$P_{\mathbf{w}}\left\{\limsup_{t\to\infty} \frac{|X(t) - t/Ew_0^{-1}|}{(t\log\log t)^{1/2}} < \infty\right\} = 1 \tag{2.5}$$

for almost all realizations of $\mathbf{w}$.

(iii) *If there is a constant* b *such that*

$$P_{\mathbf{w}}\left\{\limsup_{t\to\infty} \frac{|X(t) - bt|}{(t\log\log t)^{1/2}} < \infty\right\} = 1 \tag{2.6}$$

for almost all realizations of $\mathbf{w}$, *then* $Ew_0^{-2} < \infty$ *and* $b = Ew_0^{-1}$.

The first part of Theorem 2.2 generalizes the Marcinkiewicz–Zygmund strong law of large numbers to directed random walks in random environments. Assuming that $1 \leqq \nu < 2$, then $X(t)/t$ goes to $(Ew_0^{-1})^{-1}$ $P_{\mathbf{w}}$-a.s. and the rate of convergence is $o(t^{1/\nu-1})$ if and only if $Ew_0^{-\nu} < \infty$. The second part says that the law of the iterated logarithm holds for $X(t)$ if and only if $Ew_0^{-2} < \infty$.

The proof of Theorem 2.2 is based on the following lemma.

LEMMA 2.1. (i) *We assume that* $Ew_0^{-\nu} < \infty$ *with some* $0 < \nu < 2$. *Let* $b = 0$, *if* $0 < \nu < 1$ *and* $b = Ew_0^{-1}$, *if* $1 \leqq \nu < 2$. *Then*

$$P_{\mathbf{w}}\left\{\lim_{n\to\infty} \frac{S(n) - nb}{n^{1/\nu}} = 0\right\} = 1 \tag{2.7}$$

for almost all realizations of $\mathbf{w}$.

(ii) *Let* $0 < \nu < 2$. *If there is a constant* $0 \leqq c < \infty$ *such that*

$$P_{\mathbf{w}}\left\{\lim_{n\to\infty} \frac{S(n) - nc}{n^{1/\nu}} = 0\right\} = 1 \tag{2.8}$$

for almost all realizations of $\mathbf{w}$, *then* $Ew_0^{-\nu} < \infty$. *Also,* $c = Ew_0^{-1}$, *if* $1 \leqq v < 2$.

(iii) *If* $Ew_0^{-2} < \infty$, *then there is a constant* c *such that*

$$P_{\mathbf{w}}\left\{\limsup_{n\to\infty} \frac{|S(n) - nEw_0^{-1}|}{(n\log\log n)^{1/2}} \leqq c\right\} = 1 \tag{2.9}$$

for almost all realizations of $\mathbf{w}$.

(iv) *If there is a r.v.* $c(\mathbf{w})$, *depending only on the environment* $\mathbf{w}$ *and a constant* b *such that*

$$P_{\mathbf{w}}\left\{\limsup_{n\to\infty}\frac{|S(n)-nb|}{(n\log\log n)^{1/2}}\leqq c(\mathbf{w})\right\}=1 \tag{2.10}$$

for almost all realizations of $\mathbf{w}$, *then* $Ew_0^{-2}<\infty$ *and* $b=Ew_0^{-1}$.

PROOF. (i) *and* (iii). It is proven in Horváth and Shao [13].

(iii) The strong approximation of $S(n)$ in Section 3 (cf. Lemma 3.1) implies that

$$P_{\mathbf{w}}\left\{\limsup_{n\to\infty}\frac{\left|S(n)-\sum_{0\leqq i\leqq n-1}w_i^{-1}\right|}{\left(\sum_{0\leqq i\leqq n-1}w_i^{-2}\log\log\sum_{0\leqq i\leqq n-1}w_i^{-2}\right)^{1/2}}=2^{1/2}\right\}=1 \quad P_{\mathbf{w}}\text{-a.s.} \tag{2.11}$$

The law of the iterated logarithm for partial sums of i.i.d.r.v.'s yields

$$P\left\{\limsup_{n\to\infty}\frac{\left|\sum_{0\leqq i\leqq n-1}w_i^{-1}-nEw_0^{-1}\right|}{(2n\log\log n)^{1/2}}=(var\ w_0^{-1})^{1/2}\right\}=1. \tag{2.12}$$

Putting together (2.11) and (2.12) we get immediately (2.9).

(iv) It is easy to see that (2.10) implies

$$P\left\{\limsup_{n\to\infty}\frac{|S(n)-nb|}{(n\log\log n)^{1/2}}=d\right\}=1 \tag{2.13}$$

with some constant d. Since $S(n)$ is a sum of i.i.d.r.v.'s, (2.13) holds if and only if $ET_0^2<\infty$ and $b=ET_0$. If F denotes the distribution function of w_0, then we have

$$P\{T_0>t\}=\int_0^\infty e^{-tx}dF(x), \tag{2.14}$$

and therefore elementary calculations show that $ET_0=Ew_0^{-1}$ and $ET_0^2<\infty$ if and only if $Ew_0^{-2}<\infty$.

PROOF OF THEOREM 2.2. (i) It follows from (1.4) that

$$X(t) = n, \quad \text{if} \quad S(n) \leqq t < S(n+1), \tag{2.15}$$

and therefore we have

$$\frac{n - S(n+1)/Ew_0^{-1}}{S^{1/\nu}(n+1)} \leqq \frac{X(t) - t/Ew_0^{-1}}{t^{1/\nu}} \leqq \frac{n - S(n)/Ew_0^{-1}}{S^{1/\nu}(n)}, \tag{2.16}$$

if $S(n) \leqq t < S(n+1)$. Lemma 2.1 gives

$$\lim_{n\to\infty} S(n)/n = Ew_0^{-1} \qquad P_{\mathbf{w}}\text{-a.s.}, \tag{2.17}$$

and therefore (2.3) follows from (2.7) and (2.16).

Since $1 \leqq \nu < 2$, (2.14) implies that

$$\lim_{t\to\infty} X(t)/t = c \qquad P_{\mathbf{w}}\text{-a.s.}, \tag{2.18}$$

and therefore Lemma 1 of Buffet and Hannigan [4] gives

$$\lim_{n\to\infty} S(n)/n = \frac{1}{c} \qquad P_{\mathbf{w}}\text{-a.s.} \tag{2.19}$$

According to Lemma 2.1, if (2.19) holds, then $c = 1/Ew_0^{-1}$. Using again (2.15) we obtain

$$\limsup_{n\to\infty} \left| S(n) - nEw_0^{-1} \right| / n^{1/\nu} \leqq \tag{2.20}$$

$$\leqq \limsup_{n\to\infty} \frac{S^{1/\nu}(n)}{n^{1/\nu}} \limsup_{t\to\infty} Ew_0^{-1} \frac{\left| X(t) - t/Ew_0^{-1} \right|}{t^{1/\nu}} = 0 \qquad P_{\mathbf{w}}\text{-a.s.}$$

Now Lemma 2.1 implies that $Ew_0^{-\nu} < \infty$.

(ii) *and* (iii). Putting together (2.15) and Lemma 2.1(iii) and (iv) we get the last two statements in Theorem 2.2.

3. Gaussian approximations for $X(t)$

Let

$$\mu(i)=\sum_{0\leqq j\leqq i-1} w_j^{-1} \tag{3.1}$$

and

$$\mu(t)=\mu(i)+w_i^{-1}(t-i),\quad i\leqq t\leqq i+1 \tag{3.2}$$

be the broken line connecting the points $\{(i,\mu(i)),1\leqq i<\infty\}$. We also define

$$S(x)=\sum_{0\leqq j\leqq x-1} T_j,\qquad 1\leqq x<\infty, \tag{3.3}$$

$$\tau^2(t)=\sum_{0\leqq j\leqq t-1} w_j^{-2} \tag{3.4}$$

and

$$\gamma^2=Ew_0^{-2}. \tag{3.5}$$

We say that $\{\Gamma(t),0\leqq t<\infty\}$ is a Wiener process, if Γ is an almost surely continuous Gaussian process with covariance $E\Gamma(t)=0$ and $E\Gamma(t)\Gamma(s)=$ $=\min(t,s)$.

LEMMA 3.1. *Assume that $Ew_0^{-\nu}<\infty$ for some $\nu\geqq 2$. Then for almost all realizations of* $\mathbf{w}$ *we can define a Wiener process $\{\Gamma_{\mathbf{w}}(t),0\leqq t<\infty\}$ such that*

$$P_{\mathbf{w}}\left\{\lim_{n\to\infty}\sup_{1\leqq x\leqq n}\left|S(x)-\mu(x)-\Gamma_{\mathbf{w}}\left(\tau^2(x)\right)\right|/n^{1/\nu}=0\right\}=1 \tag{3.6}$$

for almost all realizations of $\mathbf{w}$.

PROOF. Let

$$\Omega_0=\left\{\mathbf{w}:\sum_{1\leqq i<\infty}\frac{w_i^{-2\nu}}{i^2}<\infty\right\}. \tag{3.7}$$

We show that

$$P(\Omega_0)=1. \tag{3.8}$$

The moment condition $Ew_0^{-\nu} < \infty$ implies

$$P\{w_i^{-1} > i^{1/\nu} \quad i.o.\} = 0 \tag{3.9}$$

and therefore it suffices to establish that

$$\sum_{1 \leqq i < \infty} \frac{w_i^{-2\nu}}{i^2} I\{w_i^{-1} \leqq i^{1/\nu}\} < \infty \quad \text{a.s.} \tag{3.10}$$

It is easy to see that

$$\sum_{1 \leqq i < \infty} E\left(\frac{w_i^{-2\nu}}{i^2} I\{w_i^{-1} \leqq i^{1/\nu}\}\right) = \tag{3.11}$$

$$= \sum_{1 \leqq i < \infty} \frac{1}{i^2} E\left(w_0^{-2\nu} I\{w_0^{-1} \leqq i^{1/\nu}\}\right) =$$

$$= \sum_{1 \leqq i < \infty} \frac{1}{i^2} \sum_{1 \leqq j \leqq i} E\left(w_0^{-2\nu} I\{(j-1)^{1/\nu} < w_0^{-1} \leqq j^{1/\nu}\}\right) =$$

$$= \sum_{1 \leqq j < \infty} \sum_{j \leqq i < \infty} \frac{1}{i^2} E\left(w_0^{-2\nu} I\{(j-1)^{1/\nu} < w_0^{-1} \leqq j^{1/\nu}\}\right) \leqq$$

$$\leqq 2 \sum_{1 \leqq j < \infty} \frac{1}{j} E\left(w_0^{-2\nu} I\{(j-1)^{1/\nu} < w_0^{-1} \leqq j^{1/\nu}\}\right) \leqq$$

$$\leqq 2 \sum_{1 \leqq i < \infty} E\left(w_0^{-\nu} I\{(j-1)^{1/\nu} < w_0^{-1} \leqq j^{1/\nu}\}\right) \leqq 2Ew_0^{-\nu},$$

which implies (3.10).

Now (3.8) yields

$$\sum_{1 \leqq i < \infty} E_{\mathbf{w}} T_i^{2\nu} / i^2 < \infty, \quad \text{if} \quad \mathbf{w} \in \Omega_0. \tag{3.12}$$

We can write (3.12) as

$$\sum_{1 \leqq i < \infty} E_{\mathbf{w}} \left| T_i - \frac{1}{w_i} \right|^{2\nu} / (i^{1/\nu})^{2\nu} < \infty, \quad \text{if} \quad \mathbf{w} \in \Omega_0. \tag{3.13}$$

Hence by Theorem 1.4 of Shao [18] (cf. also Einmahl [9]) for all $\mathbf{w} \in \Omega_0$ we can find a Wiener process $\{\Gamma_{\mathbf{w}}(t), 0 \leqq t < \infty\}$ such that

$$(3.14) \qquad P_{\mathbf{w}}\left\{\lim_{n\to\infty} \max_{1\leqq k\leqq n} |S(k) - \mu(k) - \Gamma(\tau^2(k))|/n^{1/\nu} = 0\right\} = 1,$$

if $\mathbf{w} \in \Omega_0$. Observing that $S(x) = S(k), \mu(x) = \mu(k)$, if $k \leqq x < k+1$ and

$$\max_{1\leqq k\leqq n} \sup_{k\leqq x<k+1} \left|\mu(k) - \mu(x)\right|/n^{1/\nu} = \max_{1\leqq k\leqq n} \frac{1}{w_k}/n^{1/\nu} = o(1) \text{ a.s.},$$

Lemma 3.1 follows immediately from (3.14).

Hanson and Russo [11] obtained the following result for the increments of a Wiener process $\{\Gamma(t), 0 \leqq t < \infty\}$.

LEMMA 3.2. *Assume that* $b(T) \geqq 0, a(T) > 0$ *and* $a(T) + b(T) \to \infty$ *as* $T \to \infty$. *Then*

(3.15)

$$\limsup_{T\to\infty} \sup_{0\leqq t\leqq b(T)} \sup_{0\leqq s\leqq a(T)} \frac{\left|\Gamma(t+s) - \Gamma(t)\right|}{\left(2a(T)\left(\log\frac{a(T)+b(T)}{a(T)} + \log\log\left(b(T)+a(T)\right)\right)\right)^{1/2}} \leqq$$

$$\leqq 1 \qquad \text{a.s.}$$

The proofs of the approximations of $X(t)$ require some elementary results on $\mu(t)$. Let

$$\Delta(T) = \sup_{0\leqq t\leqq T} \sup_{0\leqq s\leqq u(T)} \left|\mu(t+s) - \mu(t) - sEw_0^{-1}\right|.$$

LEMMA 3.3. (i) *If* $Ew_0^{-2} < \infty$ *and* $0 < u(T) \leqq T$, *then*

$$(3.16) \quad \Delta(T) \overset{\text{a.s.}}{=} o(T^{1/2}) + O\left(\left(u(T)\left(\log\frac{T+u(T)}{u(T)} + \log\log T\right)\right)^{1/2}\right).$$

(ii) *If* $Ew_0^{-\nu} < \infty$ *for some* $\nu > 2$ *and* $0 < u(T) \leqq T$, *then*

$$(3.17) \quad \Delta(T) \overset{\text{a.s.}}{=} o(T^{1/\nu}) + O\left(\left(u(T)\left(\log\frac{T+u(T)}{u(T)} + \log\log T\right)\right)^{1/2}\right).$$

PROOF. Let

$$\sigma_k^2 = \int_0^{2^{n/2}} x^2 dF^*(x) - \left(\int_0^{2^{n/2}} x dF^*(x) \right)^2, \quad \text{if} \quad 2^n \leqq k < 2^{n+1},$$

where F^* is the distribution function of w_0^{-1}. Major [16] constructed a Wiener process $\{\Gamma(t), 0 \leqq t < \infty\}$ such that

$$\max_{1 \leqq k \leqq n} \left| \sum_{0 \leqq i \leqq k-1} w_i^{-1} - kEw_0^{-1} - \Gamma\big(q^2(k)\big) \right| \overset{\text{a.s.}}{=} o(n^{1/2}), \tag{3.18}$$

where

$$q^2(k) = \sum_{0 \leqq i \leqq k-1} \sigma_i^2. \tag{3.19}$$

It follows from the definition of σ_k^2 that

$$\lim_{k \to \infty} \sigma_k^2 = var\ w_0^{-1}. \tag{3.20}$$

It is easy to see that

$$\sup_{0 \leqq t \leqq T} \sup_{0 \leqq s \leqq u(T)} \left| \Gamma\big(q^2(t+s)\big) - \Gamma\big(q^2(t)\big) \right| \leqq \tag{3.21}$$

$$\leqq \sup_{0 \leqq t \leqq q^2(2T)} \sup_{0 \leqq s \leqq Cu(T)} \left| \Gamma(t+s) - \Gamma(t) \right|,$$

where $C = \sup_{1 \leqq i < \infty} \sigma_i^2$. Now (3.16) follows immediately from Lemma 3.2.

If $Ew_0^{-\nu} < \infty$ for some $\nu > 2$, then by Komlós, Major and Tusnády [14], [15] and Major [17], there is a Wiener process $\{\Gamma(t), 0 \leqq t < \infty\}$ such that

$$\max_{1 \leqq k \leqq n} \left| \sum_{0 \leqq i \leqq k-1} w_i^{-1} - kEw_0^{-1} - (var\ w_0^{-1})^{1/2} \Gamma(k) \right| \overset{\text{a.s.}}{=} o(n^{1/\nu}). \tag{3.22}$$

Thus Lemma 3.2 implies (3.17).

After these preliminary results we are ready to get some approximations for $X(t)$. Let $\mu^{-1}(t)$ denote the inverse of $\mu(t)$ and

$$\sigma^2 = Ew_0^{-2}/(Ew_0^{-1})^3.$$

THEOREM 3.1. *Assume that* $Ew_0^{-2} < \infty$. *Then for almost all realizations of* $\mathbf{w}$ *we can define a Wiener process* $\{\hat{\Gamma}_{\mathbf{w}}(t), 0 \leqq t < \infty\}$ *such that*

(3.23)
$$P_{\mathbf{w}}\Big\{\lim_{T\to\infty}(T\log\log T)^{-1/2}\sup_{0\leqq t\leqq T}\Big|\big(X(t)-\mu^{-1}(t)\big)/\sigma-\hat{\Gamma}_{\mathbf{w}}(t)\Big|=0\Big\}=1$$

for almost all realizations of $\mathbf{w}$ *and for all* $\varepsilon > 0$ *we have*

$$\lim_{T\to\infty}P_{\mathbf{w}}\Big\{T^{-1/2}\sup_{0\leqq t\leqq T}\Big|\big(X(t)-\mu^{-1}(t)\big)/\sigma-\hat{\Gamma}_{\mathbf{w}}(t)\Big|>\varepsilon\Big\}=0 \tag{3.24}$$

for almost all realizations of $\mathbf{w}$.

PROOF. The strong law of large numbers gives

$$\tau^2(x)-\gamma^2 x \overset{\text{a.s.}}{=} o(x), \qquad \text{as} \quad x\to\infty. \tag{3.25}$$

If $\Gamma_{\mathbf{w}}$ is the Wiener process of Lemma 3.1, then by Lemma 3.2 and (3.25) we have

$$P_{\mathbf{w}}\left\{\lim_{T\to\infty}(T\log\log T)^{-1/2}\sup_{0\leqq t\leqq T}\left|\Gamma_{\mathbf{w}}\big(\tau^2(t)\big)-\Gamma_{\mathbf{w}}(\gamma^2 t)\right|=0\right\}= \tag{3.26}$$
$$=1 \qquad P_{\mathbf{w}}\text{-a.s.}$$

Using Lemma 3.1, (2.1) of Theorem 2.1 and (3.26) we get

(3.27)
$$P_{\mathbf{w}}\Big\{\lim_{T\to\infty}(T\log\log T)^{-1/2}\sup_{0\leqq t\leqq T}\big|S\big(\mu^{-1}(t)\big)-t-\Gamma_{\mathbf{w}}\big(\gamma^2\mu^{-1}(t)\big)\big|=0\Big\}=$$
$$=1 \quad P_{\mathbf{w}}\text{-a.s.}$$

By the law of the iterated logarithm we have

$$\sup_{0\leqq t\leqq T}\big|\mu(t)-tEw_0^{-1}\big| \overset{\text{a.s.}}{=} O\big((T\log\log T)^{1/2}\big), \tag{3.28}$$

and therefore Lemma of Horváth [12] implies

$$\sup_{0\leqq t\leqq T}\big|\mu^{-1}(t)-t/Ew_0^{-1}\big| \overset{\text{a.s.}}{=} O\big((T\log\log T)^{1/2}\big). \tag{3.29}$$

Lemma 3.2 yields that

(3.30) $$P_{\mathbf{w}}\Big\{\limsup_{T\to\infty}(T\log\log T)^{-1/4}(\log T)^{-1/2}\sup_{0\leqq t\leqq T}\big|\Gamma_{\mathbf{w}}\big(\gamma^2\mu^{-1}(t)\big)- -\Gamma_{\mathbf{w}}(\gamma^2 t/Ew_0^{-1})\big|<\infty\Big\}=1 \qquad P_{\mathbf{w}}\text{-a.s.}$$

Let

(3.31) $$\tilde{\Gamma}_{\mathbf{w}}(t)=\frac{1}{\gamma}(Ew_0^{-1})^{1/2}\Gamma_{\mathbf{w}}(\gamma^2 t/Ew_0^{-1}).$$

It is easy to see that $\tilde{\Gamma}_{\mathbf{w}}$ is a Wiener process. Putting together (3.27) and (3.30) we obtain

(3.32) $$P_{\mathbf{w}}\left\{\lim_{T\to\infty}(T\log\log T)^{-1/2}\sup_{0\leqq t\leqq T}\left|S\big(\mu^{-1}(t)\big)-t- -\frac{\gamma}{(Ew_0^{-1})^{1/2}}\tilde{\Gamma}_{\mathbf{w}}(t)\right|=0\right\}=1 \qquad P_{\mathbf{w}}\text{-a.s.}$$

Hence by Theorem 3.1 of Csörgő, Horváth and Steinebach [8] (cf. also Chapter 2 in Csörgő and Horváth [7]), there is a Wiener process $\{\hat{\Gamma}_{\mathbf{w}}(t), 0\leqq t< <\infty\}$ such that

(3.33)
$$P_{\mathbf{w}}\left\{\lim_{T\to\infty}(T\log\log T)^{-1/2}\sup_{0\leqq t\leqq T}\left|\mu(X(t))-t\frac{\gamma}{(Ew_0^{-1})^{1/2}}\hat{\Gamma}_{\mathbf{w}}(t)\right|=0\right\}= =1 \qquad P_{\mathbf{w}}\text{-a.s.}$$

The law of the iterated logarithm yields

(3.34)
$$P_{\mathbf{w}}\left\{\limsup_{T\to\infty}(2T\log\log T)^{-1/2}\sup_{0\leqq t\leqq T}|\hat{\Gamma}_{\mathbf{w}}(t)|=1\right\}=1 \qquad P_{\mathbf{w}}\text{-a.s.},$$

and therefore (3.29) and (3.33) imply

(3.35)
$$P_{\mathbf{w}}\left\{\limsup_{T\to\infty}(T\log\log T)^{-1/2}\sup_{0\leqq t\leqq T}|X(t)-\mu^{-1}(t)|<\infty\right\}=1 \qquad P_{\mathbf{w}}\text{-a.s.}$$

Lemma 3.3 and (3.35) yield that

$$P_{\mathbf{w}}\left\{\lim_{T\to\infty} T^{-1/2}\sup_{0\leqq t\leqq T}\left|\mu(X(t)-\mu^{-1}(t))-\right.\right.$$
$$\left.\left.-(X(t)-\mu^{-1}(t))\,Ew_0^{-1}\right|=0\right\}=1 \qquad P_{\mathbf{w}}\text{-a.s.} \tag{3.36}$$

Combining (3.33) and (3.36) we get immediately (3.23).

Since $\Gamma_{\mathbf{w}}$ is almost surely continuous, (3.25) and the scale transformation of the Wiener process imply

$$\lim_{T\to\infty} P_{\mathbf{w}}\left\{T^{-1/2}\sup_{0\leqq t\leqq T}\left|\Gamma_{\mathbf{w}}(\tau^2(t)-\Gamma_{\mathbf{w}}(\gamma^2 t)\right|>\varepsilon\right\}=$$
$$=0 \qquad P_{\mathbf{w}}\text{-a.s.} \tag{3.37}$$

for all $\varepsilon>0$. Hence similarly to (3.27) we have

$$\lim_{T\to\infty} P_{\mathbf{w}}\left\{T^{-1/2}\sup_{0\leqq t\leqq T}\left|S(\mu^{-1}(t))-t-\Gamma_{\mathbf{w}}(\gamma^2\mu^{-1}(t))\right|>\varepsilon\right\}=0$$
$$P_{\mathbf{w}}\text{-a.s.} \tag{3.38}$$

for all $\varepsilon>0$. Putting together (3.37), (3.38) and (3.30) we get that

$$\lim_{T\to\infty} P_{\mathbf{w}}\left\{T^{-1/2}\sup_{0\leqq t\leqq T}\left|S(\mu^{-1}(t))-t-\frac{\gamma}{(Ew_0^{-1})^{1/2}}\tilde{\Gamma}_{\mathbf{w}}(t)\right|>\varepsilon\right\}=$$
$$=0 \qquad P_{\mathbf{w}}\text{-a.s.} \tag{3.39}$$

for all $\varepsilon>0$, where $\tilde{\Gamma}_{\mathbf{w}}$ is defined by (3.31). Theorem 4.1 of Csörgő, Horváth, and Steinebach [8] gives that the Wiener process $\{\hat{\Gamma}_{\mathbf{w}}(t), 0\leqq t<\infty\}$ of (3.33) also satisfies

$$\lim_{T\to\infty} P_{\mathbf{w}}\left\{T^{-1/2}\sup_{0\leqq t\leqq T}\left|\mu(X(t))-t-\frac{\gamma}{(Ew_0^{-1})^{1/2}}\hat{\Gamma}_{\mathbf{w}}(t)\right|>\varepsilon\right\}=$$
$$=0 \qquad P_{\mathbf{w}}\text{-a.s.} \tag{3.40}$$

for all $\varepsilon>0$. Now (3.24) follows from (3.40) and (3.36).

The law of the iterated logarithm for directed random walks follows immediately from Theorem 3.1. If $Ew_0^{-2} < \infty$, then

$$P_{\mathbf{w}}\Big\{\limsup_{T\to\infty} (2T\log\log T)^{-1/2} \sup_{0\leqq t\leqq T} \big|X(t)-\mu^{-1}(t)\big| = \sigma\Big\} = 1 \tag{3.41}$$

for almost all realizations of $\mathbf{w}$. Also, it is clear from Theorem 3.1 that $\{T^{-1/2}\big(X(Tt)-\mu^{-1}(Tt)\big)/\sigma, 0 \leqq t \leqq 1\}$ converges weakly to a Wiener process for almost all realizations of $\mathbf{w}$. Thus we have

$$\lim_{t\to\infty} P_{\mathbf{w}}\left\{\frac{X(t)-\mu^{-1}(t)}{\sigma t^{1/2}} \leqq x\right\} = \Phi(x), \tag{3.42}$$

where Φ is the standard normal distribution function and

$$\lim_{T\to\infty} P_{\mathbf{w}}\left\{T^{-1/2}\sup_{0\leqq t\leqq T}\big|X(t)-\mu^{-1}(t)\big|/\sigma \leqq x\right\} = H(x),$$

for almost all realizations of $\mathbf{w}$, where

$$H(x) = \frac{4}{\pi}\sum_{0\leqq k<\infty}\frac{(-1)^k}{2k+1}\exp\big(-\pi^2(2k+1)^2/(8x^2)\big).$$

We note that (3.42) was also proven by Buffet and Hannigan [4] under much stronger moment condition than $Ew_0^{-2} < \infty$.

Assuming stronger moment condition one can improve on the rates of approximation in Theorem 3.1.

THEOREM 3.2. *Assume that* $Ew_0^{-\nu} < \infty$ *for some* $2 < \nu < \infty$. *Then for almost all realizations of* $\mathbf{w}$ *we can define a Wiener process* $\{\hat{\Gamma}_{\mathbf{w}}(t), 0 \leqq t < < \infty\}$ *such that*

(3.43)

$$P_{\mathbf{w}}\Big\{\lim_{T\to\infty} T^{-1/\nu}(\log T)^{-1/2}\sup_{0\leqq t\leqq T}\Big|\big(X(t)-\mu^{-1}(t)\big)/\sigma - \hat{\Gamma}_{\mathbf{w}}(t)\Big| = 0\Big\} = 1$$

for almost all realizations of $\mathbf{w}$, *if* $2 < \nu < 4$ *and*

$$P_{\mathbf{w}}\Big\{\limsup_{T\to\infty}(T\log\log T)^{-1/4}(\log T)^{-1/2} \sup_{0\leqq t\leqq T}\Big|\big(X(t)-\mu^{-1}(t)\big)/\sigma - \hat{\Gamma}_{\mathbf{w}}(t)\Big| < \infty\Big\} = 1$$

for almost all realizations of $\mathbf{w}$, *if* $4 \leqq \nu < \infty$.

PROOF. We follow the proof of (3.23). The Marcinkiewicz–Zygmund law of large numbers (cf. Chow and Teicher [5], p. 125) and the law of iterated logarithm yield

$$(3.44)\qquad \tau^2(x)-\gamma^2 x \overset{\text{a.s.}}{=} \begin{cases} o(x^{2/\nu}), & \text{if} \quad 2<\nu<4 \\ O\big((x\log\log x)^{1/2}\big), & \text{if} \quad 4\leqq \nu<\infty, \end{cases}$$

as $x\to\infty$. Let

$$(3.45)\qquad r_\nu(T)=\begin{cases} T^{1/\nu}(\log T)^{1/2}, & \text{if} \quad 2<\nu<4 \\ (T\log\log T)^{1/4}(\log T)^{1/2}, & \text{if} \quad 4\leqq\nu<\infty. \end{cases}$$

First we assume that $2<\nu<4$. If $\Gamma_{\mathbf{w}}$ is the Wiener process defined in Lemma 3.1, then by Lemma 3.2 and (3.45) we have

$$(3.46)\quad P_{\mathbf{w}}\left\{\lim_{T\to\infty}\sup_{0\leqq t\leqq T}\left|\Gamma_{\mathbf{w}}\big(\tau^2(t)\big)-\Gamma_{\mathbf{w}}(\gamma^2 t)\right|/r_\nu(T)=0\right\}=1\quad P_{\mathbf{w}}\text{-a.s.}$$

Using Lemma 3.1, (3.30) and (3.47) we obtain that

$$(3.47)\, P_{\mathbf{w}}\left\{\lim_{T\to\infty}\sup_{0\leqq t\leqq T}\left|S\big(\mu^{-1}(t)\big)-t-\frac{\gamma}{(Ew_0^{-1})^{1/2}}\,\tilde{\Gamma}_{\mathbf{w}}(t)\right|/r_\nu(T)=0\right\}= \\ =1\quad P_{\mathbf{w}}\text{-a.s.},$$

where $\tilde{\Gamma}_{\mathbf{w}}$ is defined in (3.31). Applying again Theorem 3.1 of Csörgő, Horváth, and Steinebach [8] (cf. also Chapter 2 in Csörgő and Horváth [7]) we can define a Wiener process $\{\hat{\Gamma}_{\mathbf{w}}(t), 0\leqq t<\infty\}$ such that

$$(3.48)\; P_{\mathbf{w}}\left\{\lim_{T\to\infty}\sup_{0\leqq t\leqq T}\left|\mu\big(X(t)\big)-t-\frac{\gamma}{(Ew_0^{-1})^{1/2}}\,\hat{\Gamma}_{\mathbf{w}}(t)\right|/r_\nu(T)=0\right\}= \\ =1\qquad P_{\mathbf{w}}\text{-a.s.}$$

Similarly to (3.36) one can easily establish that

$$(3.49) \\ P_{\mathbf{w}}\left\{\lim_{T\to\infty}T^{-1/\nu}\sup_{0\leqq t\leqq T}\left|\mu\big(X(t)\big)-\mu\big(\mu^{-1}(t)\big)-\big(X(t)-\mu^{-1}(t)\big)\,Ew_0^{-1}\right|=0\right\}= \\ =1\qquad P_{\mathbf{w}}\text{-a.s.}$$

Now (3.43) follows immediately from (3.49) and (3.50).

Similar arguments work, if $4 \leqq \nu < \infty$. The probabilities in (3.47)–(3.50) should be replaced by

$$P_{\mathbf{w}}\left\{\limsup_{T\to\infty}\sup_{0\leqq t\leqq T}\left|\Gamma_{\mathbf{w}}\left(\tau^2(t)\right)-\Gamma_{\mathbf{w}}(\gamma^2 t)\right|/r_4(T)<\infty\right\}=1\quad P_{\mathbf{w}}\text{-a.s.},$$

$$P_{\mathbf{w}}\left\{\limsup_{T\to\infty}\sup_{0\leqq t\leqq T}\left|S\left(\mu^{-1}(t)\right)-t-\frac{\gamma}{(Ew_0^{-1})^{1/2}}\tilde{\Gamma}_{\mathbf{w}}(t)\right|/r_4(T)<\infty\right\}=$$

$$=1\qquad P_{\mathbf{w}}\text{-a.s.},$$

$$P_{\mathbf{w}}\left\{\limsup_{T\to\infty}\sup_{0\leqq t\leqq T}\left|\mu\left(X(t)\right)-t-\frac{\gamma}{(Ew_0^{-1})^{1/2}}\hat{\Gamma}_{\mathbf{w}}(t)\right|/r_4(T)<\infty\right\}=$$

$$=1\qquad P_{\mathbf{w}}\text{-a.s.}$$

and

$$P_{\mathbf{w}}\left\{\limsup_{T\to\infty}\sup_{0\leqq t\leqq T}\left|\mu\left(X(t)\right)-\mu(\mu^{-1}(t))-\left(X(t)-\mu^{-1}(t)\right)Ew_0^{-1}\right|/r_4(T)<\infty\right\}=$$

$$=1\qquad P_{\mathbf{w}}\text{-a.s.},$$

and we get immediately (3.44).

Theorem 3.2 implies, for example, Chung's law of iterated logarithm for directed random walks in random environments. If $Ew_0^{-\nu}<\infty$ for some $2<\nu<\infty$, then

$$P_{\mathbf{w}}\left\{\liminf_{T\to\infty}T^{-1/2}(\log\log T)^{1/2}\sup_{0\leqq t\leqq T}|X(t)-\mu^{-1}(t)|=\sigma\frac{\pi}{8^{1/2}}\right\}=$$

$$=1\qquad P_{\mathbf{w}}\text{-a.s.}$$

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INTERPOLATION BETWEEN CONTINUOUS PARAMETER MARTINGALE SPACES: THE REAL METHOD

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To Professor K. Tandori on his seventieth birthday

1. Introduction

It is well known that interpolation spaces of L_p spaces are Lorentz spaces and that interpolation spaces of Lorentz spaces are Lorentz spaces, too. More exactly (see e.g. Bergh, Löfström [2]),

$$(L_{p_0,q_0}, L_{p_1,q_1})_{\theta,q} = L_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

where $0 < \theta < 1$, $0 < p_0 < p_1 \leqq \infty$ and $0 < q_i, q \leqq \infty$. In [10] and [19] Fefferman, Riviere and Sagher have identified the intermediate spaces between the classical Hardy spaces; they have shown that

$$(1) \qquad (\mathcal{H}_{p_0,q_0}, \mathcal{H}_{p_1,q_1})_{\theta,q} = \mathcal{H}_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

where $0 < \theta < 1$, $0 < p_0 < p_1 \leqq \infty$, $0 < q_i, q \leqq \infty$, and $\mathcal{H}_{p,q} = L_{p,q}$ if $1 < p \leqq \infty$. (1) is proved between the classical $\mathcal{H}_{p_0,q_0}$ and $\mathcal{BMO}$ in Hanks [13] and Bennett, Sharpley [1]:

$$(2) \qquad (\mathcal{H}_{p_0,q_0}, \mathcal{BMO})_{\theta,q} = \mathcal{H}_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}$$

where, again, $0 < \theta < 1$, $0 < p_0 < \infty$ and $0 < q_0, q \leqq \infty$.

These results were shown for the discrete parameter martingale H_p^* spaces defined by the maximal function by Janson and Jones [14] and Milman [17], but for $1 \leqq p_0$, only. This will be generalized for martingale $H_p^{[]}$ spaces generated by quadratic variation. Recall that $H_\infty^* \neq H_\infty^{[]}$. (1) and (2) were also verified in the discrete time for martingale $H_p^{\langle\rangle}$ spaces defined by conditional

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quadratic variation (see Weisz [21], [22]). In this paper these results will be extended to continuous parameter martingale spaces.

2. Basic facts of interpolation theory

For a measurable function f the *non-increasing rearrangement* is introduced by

$$\tilde{f}(t) := \inf\{y : P(\{x : |f(x)| > y\}) \leqq t\}.$$

The Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty, 0 < q < \infty$

$$\|f\|_{p,q} := \left(\int_0^\infty \tilde{f}(t)^q t^{q/p} \frac{dt}{t} \right)^{1/q}$$

while for $0 < p \leqq \infty$

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p} \tilde{f}(t).$$

Set

$$L_{p,q} := L_{p,q}(\Omega, \mathcal{A}, P) := \left\{ f : \|f\|_{p,q} < \infty \right\}.$$

One can show that $L_{p,p} = L_p$ and $L_{p,\infty}$ is the weak L_p space $(0 < p \leqq \infty)$.

The basic definitions and theorems of interpolation theory in the real method are given shortly. For the details see Bennett, Sharpley [1] or Bergh, Löfström [2]. Suppose that A_0 and A_1 are quasi-normed spaces embedded continuously in a topological vector space A. The *interpolation spaces* between A_0 and A_1 are defined by the means of an interpolating function $K(t, f, A_0, A_1)$. If $f \in A_0 + A_1$, define

$$K(t, f, A_0, A_1) := \inf_{f=f_0+f_1} \left\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1} \right\}$$

where the infimum is taken over all choices of f_0 and f_1 such that $f_0 \in A_0$, $f_1 \in A_1$ and $f = f_0 + f_1$. The interpolation space $(A_0, A_1)_{\theta,q}$ is defined as the space of all functions f in $A_0 + A_1$ such that

$$\|f\|_{(A_0,A_1)_{\theta,q}} := \left(\int_0^\infty \left[t^{-\theta} K(t, f, A_0, A_1) \right]^q \frac{dt}{t} \right)^{1/q} < \infty$$

where $0 < \theta < 1$ and $0 < q \leqq \infty$. We use the conventions $(A_0, A_1)_{0,q} = A_0$ and $(A_0, A_1)_{1,q} = A_1$ for each $0 < q \leqq \infty$.

Suppose that B_0 and B_1 are also quasi-normed spaces embedded continuously in a topological vector space B. A map

$$T : A_0 + A_1 \longrightarrow B_0 + B_1$$

is said to be *quasilinear* from (A_0, A_1) to (B_0, B_1) if for given $a \in A_0 + A_1$ and $a_i \in A_i$ with $a_0 + a_1 = a$ there exist $b_i \in B_i$ satisfying

$$Ta = b_0 + b_1$$

and

$$\|b_i\|_{B_i} \leqq K_i \|a_i\|_{A_i} \qquad (K_i > 0, \ i = 0, 1).$$

THEOREM A (Riviere, Sagher [19]). *If* $0 < q \leqq \infty$, $0 \leqq \theta \leqq 1$ *and* T *is a quasilinear map from* (A_0, A_1) *to* (B_0, B_1) *then*

$$T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$$

and

$$\|Ta\|_{(B_0,B_1)_{\theta,q}} \leqq K_0^{1-\theta} K_1^{\theta} \|a\|_{(A_0,A_1)_{\theta,q}}.$$

The reiteration theorem below is one of the most important general results in interpolation theory. It says that the interpolation space of two interpolation spaces is also an interpolation space of the original spaces.

THEOREM B (Reiteration theorem; Bergh, Löfström [2]). *Suppose that* $0 \leqq \theta_0 < \theta_1 \leqq 1$, $0 < q_0, q_1 \leqq \infty$ *and* $X_i = (A_0, A_1)_{\theta_i, q_i}$ $(i = 0, 1)$. *If* $0 < < \eta < 1$ *and* $0 < q \leqq \infty$ *then*

$$(X_0, X_1)_{\eta,q} = (A_0, A_1)_{\theta,q}$$

where

$$\theta = (1 - \eta)\theta_0 + \eta\theta_1.$$

If, in addition, A_0 *and* A_1 *are complete and* $0 < \theta_0 = \theta_1 = \rho < 1$ *then*

$$\Big((A_0, A_1)_{\rho,q_0}, (A_0, A_1)_{\rho,q_1}\Big)_{\eta,q} = (A_0, A_1)_{\rho,q}$$

where

$$\frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}.$$

THEOREM C (Wolff [27]). *Let* A_1, A_2, A_3 *and* A_4 *be quasi-Banach spaces satisfying* $A_1 \cap A_4 \subset A_2 \cap A_3$. *Suppose that*

$$A_2 = (A_1, A_3)_{\phi,q}, \qquad A_3 = (A_2, A_4)_{\psi,r}$$

for any $0 < \phi, \psi < 1$ *and* $0 < q, r \leqq \infty$. *Then*

$$A_2 = (A_1, A_4)_{\rho,q}, \qquad A_3 = (A_1, A_4)_{\theta,r}$$

where

$$\rho = \frac{\phi\psi}{1 - \phi + \phi\psi}, \qquad \theta = \frac{\psi}{1 - \phi + \phi\psi}.$$

The result about Lorentz spaces mentioned in the Introduction is stated as follows.

THEOREM D (Bergh, Löfström [2]). *Suppose that* $0 < \eta < 1$ *and* $0 < p_0, p_1, q_0, q_1, q \leqq \infty$. *If* $p_0 \neq p_1$ *then*

$$(L_{p_0,q_0}, L_{p_1,q_1})_{\eta,q} = L_{p,q}, \qquad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

In particular,

$$(L_{p_0}, L_{p_1})_{\eta,p} = L_p, \qquad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

Furthermore, for $0 < p < \infty$,

$$(L_{p,q_0}, L_{p,q_1})_{\eta,q} = L_{p,q}, \qquad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}.$$

3. Preliminaries and notations of martingale theory

Let $(\Omega, \mathcal{A}, P)$ *be a probability measure space and* $\mathcal{F} = (\mathcal{F}_t, t \in \mathbf{R}^+)$ *a non-decreasing family of sub-*σ*-algebras of* $\mathcal{A}$. *The* σ*-algebra* $\vee_{t \in \mathbf{R}^+} \mathcal{F}_t$ *is denoted by* $\mathcal{F}_\infty$ *and it is supposed that* $\mathcal{F}_\infty = \mathcal{A}$.

With the family $(\mathcal{F}_t, t \in \mathbf{R}^+)$ the following families $(\mathcal{F}_{t+}, t \in \mathbf{R}^+)$ and $(\mathcal{F}_{t-}, t \in \mathbf{R}^+)$ of σ-algebras are associated:

$$\mathcal{F}_{t+} := \cap_{s>t} \mathcal{F}_s, \qquad \mathcal{F}_{t-} := \vee_{s<t} \mathcal{F}_s.$$

For $t = 0$ we set $\mathcal{F}_{0-} := \mathcal{F}_0$. In this paper it is assumed that the family $(\mathcal{F}_t, t \in \mathbf{R}^+)$ is *right-continuous* (i.e. $\mathcal{F}_t = \mathcal{F}_{t+}$ for every t) and that every set F which belongs to the P-completion of the σ-algebra $\mathcal{F}_\infty$ with $P(F) = 0$ belongs to $\mathcal{F}_0$.

A real *stochastic process* X is a mapping from $(\mathbf{R}^+ \times \Omega)$ into $\mathbf{R}$ such that, for every $t \in \mathbf{R}^+$, $\omega \mapsto X_t(\omega) = X(t, \omega)$ is $\mathcal{A}$-measurable. A stochastic process X is *adapted* if the preceding mapping is $\mathcal{F}_t$ measurable for every

t. A process X is *regular* if X is adapted and all functions $t \mapsto X(t,\omega)$ have left and right limit for every $\omega \in \Omega$. We use the notation X_{t-} (X_{t+}) for the left (right) limit at a point t. If these functions are right-continuous (left-continuous) then X is called *right-continuous* (*left-continuous*).

A subset of $\mathbf{R}^+ \times \Omega$ is called *well-measurable* (or *optional*) if it belongs to the σ-algebra $\mathcal{G}$ which is generated by the real regular right-continuous processes. We say that a real process is well-measurable or optional when it is $\mathcal{G}$-measurable. The σ-algebra of subsets of $\mathbf{R}^+ \times \Omega$, which is generated by the adapted continuous real processes is called the σ-algebra of *predictable* sets and will be denoted by $\mathcal{P}$. A process which is $\mathcal{P}$-measurable is called predictable.

A mapping $\nu : \Omega \longrightarrow \mathbf{R}^+ \cup \{\infty\}$ is a *stopping time* if, for every $t \in \in \mathbf{R}^+$, the subset $\{\nu \leqq t\}$ of Ω belongs to $\mathcal{F}_t$. It is well-known (see e.g. Metivier [15]) that ν is a stopping time if and only if $1_{[0,\nu]}$ is predictable or, equivalently, if and only if $\{\nu < t\} \in \mathcal{F}_t$ for every t. The graph of ν is defined by

$$[\nu] := \{(t,\omega) : t = \nu(\omega) < \infty, t \in \mathbf{R}^+, \omega \in \Omega\}.$$

For a stopping time ν one has $[\nu] \in \mathcal{G}$. If $[\nu] \in \mathcal{P}$ then the stopping time is called *predictable.*

To every stopping time ν we associate two σ-algebras $\mathcal{F}_\nu$ and $\mathcal{F}_{\nu-}$:

$$\mathcal{F}_\nu := \{F \in \mathcal{F}_\infty : F \cap \{\nu \leqq t\} \in \mathcal{F}_t, t \in \mathbf{R}^+\}$$

and $\mathcal{F}_{\nu-}$ is generated by

$$\{F \cap \{\nu < t\} : F \in \mathcal{F}_t, t \in \mathbf{R}^+\} \cup \mathcal{F}_0.$$

Note that for a constant stopping time $\nu(\omega) = t$ we have $\mathcal{F}_\nu = \mathcal{F}_t$ and $\mathcal{F}_{\nu-} = = \mathcal{F}_{t-}$.

Let us denote the expectation operator by E, the conditional expectation operator relative to $\mathcal{F}_t$, $\mathcal{F}_{t-}$, $\mathcal{F}_\nu$ and $\mathcal{F}_{\nu-}$ by E_t, E_{t-}, E_ν and $E_{\nu-}$, respectively. For the space $L_p(\Omega, \mathcal{A}, P)$ let us use the shorter notation L_p and suppose that for $f \in L_1$ one has $E_0 f = 0$.

A stochastic process X is a *martingale* when X is an adapted process with $E|X_t| < \infty$ ($t \in \mathbf{R}^+$) and $E_s X_t = X_s$ for every $s < t$. For simplicity we always suppose for a martingale X that $X_0 = 0$. Of course, the theorems that are to be proved later hold without this condition, too. A martingale X is said to be L_p-*bounded* if

$$\sup_{t \in \mathbf{R}^+} \|X_t\|_p < \infty.$$

An adapted process X is a *local martingale* if there exists an increasing sequence of stopping times ν_n such that $\lim_{n \to \infty} \nu_n = \infty$ a.e. and the processes $X_{t\wedge\nu_n}$ are all uniformly integrable martingales ($n \in \mathbf{N}$). We can suppose that a martingale or a local martingale is regular and right-continuous (see Meyer [16] p. 291). A local martingale is said to be *locally L_p-bounded* if there exists an increasing sequence of stopping times ν_n such that $\lim_{n\to\infty} \nu_n = \infty$ a.e. and the processes $X_{t\wedge\nu_n}$ are L_p-bounded martingales ($n \in \mathbf{N}$).

The following definitions will be used for a local martingale X:

$$\Delta X_t := X_t - X_{t-} \quad (X_0 = 0),$$

$$X_s^* := \sup_{t \leqq s} |X_t|, \qquad X_\infty^* := \sup_{t \in \mathbf{R}^+} |X_t|.$$

If X is a locally L_2-bounded local martingale then there exists a unique predictable, right-continuous and increasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a local martingale vanishing at 0 (see Dellacherie and Meyer [9]). $\langle X \rangle$ is called the *sharp bracket* or the *conditional quadratic variation* of X. Moreover, if X is a local martingale then there exists a unique right-continuous and increasing process $[X]$ such that $X^2 - [X]$ is a local martingale and $\Delta[X]_t = |\Delta X_t|^2$ ($[X]_0 = 0$). This process is called the *square bracket* or the *quadratic variation* of X.

Let us introduce *Hardy–Lorentz spaces* for $0 < p, q \leqq \infty$; denote by $H_{p,q}^{[]}$, $H_{p,q}^{\langle\rangle}$ and $H_{p,q}^*$ the spaces of local martingales for which

$$\|X\|_{H_{p,q}^{[]}} := \|[X]_\infty^{1/2}\|_{p,q} < \infty,$$

$$\|X\|_{H_{p,q}^{\langle\rangle}} := \left\|\langle X \rangle_\infty^{1/2}\right\|_{p,q} < \infty$$

and

$$\|X\|_{H_{p,q}^*} := \|X_\infty^*\|_{p,q} < \infty,$$

respectively.

A local martingale X is in the space $\mathcal{P}_{p,q}$ if and only if there exists an adapted, left-continuous and increasing process A such that

$$|X_t| \leqq A_t, \qquad A_\infty := \sup_{t \in \mathbf{R}^+} A_t \in L_{p,q}.$$

Endow this space with the following quasi-norm:

$$\|X\|_{\mathcal{P}_{p,q}} := \inf \|A_\infty\|_{p,q} \qquad (0 < p \leqq \infty)$$

where the infimum is taken over all predictable processes having the property above.

If, in the previous definition, we replace the inequality $|X_t| \leqq A_t$ by

$$[X]_t^{1/2} \leqq B_t$$

then the local martingale is in $\mathcal{Q}_{p,q}$. We define the $\mathcal{Q}_{p,q}$ quasi-norm by

$$\|X\|_{\mathcal{Q}_{p,q}} := \inf \|B_\infty\|_{p,q} \qquad (0 < p \leqq \infty)$$

where the infimum is taken over all predictable processes again.

It is clear that the infima taken in $\mathcal{P}_{p,q}$ and $\mathcal{Q}_{p,q}$ norms can be attained. Indeed, for every $k \in \mathbf{N}$ let $A^{(k)}$ be a left-continuous process having the above property such that $\|A_\infty^{(k)}\|_{p,q} \to \|X\|_{\mathcal{P}_{p,q}}$ as $k \to \infty$. Set

$$A_t := \inf_{k \in \mathbf{N}} A_t^{(k)} \qquad (t \in \mathbf{R}^+).$$

It is obvious that the process A is adapted, left-continuous, increasing and a majorant of X and

$$\|X\|_{\mathcal{P}_{p,q}} = \|A_\infty\|_{p,q}.$$

The proof is similar for $\mathcal{Q}_{p,q}$ spaces.

Note that in case $p = q$ the usual definitions of Hardy spaces $H_{p,p}^{\langle\rangle} = H_p^{\langle\rangle}$, $H_{p,p}^{[]} = H_p^{[]}$, $H_{p,p}^* = H_p^*$, $\mathcal{P}_{p,p} = \mathcal{P}_p$ and $\mathcal{Q}_{p,p} = \mathcal{Q}_p$ are obtained.

It is a well-known statement in the martingale theory that if $X \in H_p^*, H_p^{[]}, H_p^{\langle\rangle}$ $(p \geqq 1)$ then there exists X_∞ such that $X_t \to X_\infty$ a.e. and in L_1 as $t \to \infty$. Moreover, the Burkholder–Davis–Gundy inequality says that $H_1^* \sim H_1^{[]}$ and $H_p^* \sim H_p^{[]} \sim L_p$ for $p > 1$ where $\sim$ denotes the equivalence of the spaces (see e.g. Dellacherie and Meyer [9] or Weisz [25]). Moreover, by Doob's inequality, $H_\infty^* = L_\infty$.

It was proved by Dellacherie, Meyer [9] and Pratelli [18] that the dual of $H_1^{[]}$ resp. $H_1^{\langle\rangle}$ is BMO_2^- resp. BMO_2 (see also Weisz [25]), where BMO_p^- resp. BMO_p denote those martingales X closed on the right by $X_\infty \in L_p$ for which

$$\|X\|_{\mathrm{BMO}_p^-} := \sup_{t \in \mathbf{R}^+} \left\| \left(E_t |X_\infty - X_{t-}|^p \right)^{1/p} \right\|_\infty < \infty \qquad (1 \leqq p < \infty)$$

resp.

$$\|X\|_{\mathrm{BMO}_p} := \sup_{t \in \mathbf{R}^+} \left\| \left(E_t |X_\infty - X_t|^p \right)^{1/p} \right\|_\infty < \infty \qquad (1 \leqq p < \infty).$$

In [23] we introduced the sharp functions

$$X_r^{\langle\rangle} := \sup_{t\in\mathbf{R}^+}\left[E_t\big(\langle X\rangle_\infty - \langle X\rangle_t\big)^{r/2}\right]^{1/r} \qquad (0<r<\infty)$$

and

$$X_r^{[]} := \sup_{t\in\mathbf{R}^+}\left[E_t\big([X]_\infty - [X]_{t-}\big)^{r/2}\right]^{1/r} \qquad (0<r<\infty).$$

Denote the operators $X \mapsto X_r^{\langle\rangle}$ and $X \mapsto X_r^{[]}$ by $T_r^{\langle\rangle}$ and $T_r^{[]}$, respectively.

The following result, which is a generalization of an inequality due to Fefferman, Stein [11] and Garsia [12], is verified in Weisz [23].

THEOREM E. *The L_∞ resp. L_r norm of $T_u^{\langle\rangle}(X)$ is equivalent to the* BMO_2 *resp. $H_r^{\langle\rangle}$ norm of X and, moreover, the L_∞ resp. L_r norm of $T_u^{[]}(X)$ is equivalent to the* BMO_2^- *resp. $H_r^{[]}$ norm of X.*

4. Interpolation of martingale spaces

In this section the interpolation spaces between martingale Hardy spaces, between Hardy and BMO and, moreover, between L_p and BMO spaces are identified.

First a new decomposition theorem for martingales is given. The proofs of the following two theorems are based on the atomic decomposition given in Weisz [25].

THEOREM 1. *Let $X \in H_p^{\langle\rangle}$, $y>0$ and fix $0<p\leqq 1$. Then X can be decomposed into the sum of two martingales Y and Z such that*

$$\|Y\|_{H_\infty^{\langle\rangle}} \leqq 4y$$

and

$$\|Z\|_{H_p^{\langle\rangle}} \leqq C_p\left(\int_{\{\langle X\rangle_\infty^{1/2}\geqq y\}} \langle X\rangle_\infty^{p/2}\,dP\right)^{1/p}.$$

PROOF. Choose $N\in\mathbf{Z}$ such that $2^{N-1}<y\leqq 2^N$. Let us consider the following predictable stopping times for all $k\in\mathbf{Z}$:

$$\nu_k := \inf\left\{t\in\mathbf{R}^+ : \langle X\rangle_t^{1/2} \geqq 2^k\right\}.$$

Note that $\inf \emptyset = \infty$. Set

$$Y_t := \sum_{k=-\infty}^{N} \mu_k a_t^k$$

and

$$Z_t := \sum_{k=N+1}^{\infty} \mu_k a_t^k$$

where

$$\mu_k := 2^k 3 P(\nu_k \neq \infty)^{1/p}$$

and

$$a_t^k := \frac{1}{\mu_k}\left(X_t^{\nu_{k+1}^-} - X_t^{\nu_k^-}\right).$$

It was proved in Theorem 1 of [25] that $X_t = Y_t + Z_t$ for all $t \in \mathbf{R}^+$ and $Y = X^{\nu_{N+1}^-}$. By the definition of ν_{N+1} we get that

$$\langle Y\rangle_\infty^{1/2} = \langle X^{\nu_{N+1}^-}\rangle_\infty^{1/2} \leqq 2^{N+1} \leqq 4y$$

which proves the first inequality of the theorem.

On the other hand, the inequality

$$\|Z\|_{H_p^{\langle\rangle}}^p \leqq \sum_{k=N+1}^{\infty} |\mu_k|^p = C_p \sum_{k=N+1}^{\infty} (2^k)^p P\left(\langle X\rangle_\infty^{1/2} \geqq 2^k\right)$$

follows from Theorem 1 in [25]. By Abel rearrangement, we obtain

$$\|Z\|_{H_p^{\langle\rangle}}^p \leqq C_p \int_{\{\langle X\rangle_\infty^{1/2} \geqq 2^N\}} \langle X\rangle_\infty^{p/2}\, dP \leqq C_p \int_{\{\langle X\rangle_\infty^{1/2} \geqq y\}} \langle X\rangle_\infty^{p/2}\, dP.$$

The proof of the theorem is complete. □

By the help of Theorem 2 in [25], the following result can be proved similarly.

THEOREM 2. *A result analogous to Theorem 1 holds if we replace $H_p^{\langle\rangle}$ resp. $\langle X\rangle$ by $\mathcal{P}_p$ resp. A or by $\mathcal{Q}_p$ resp. B where A resp. B is the adapted, left-continuous and non-decreasing least majorant process of X resp. of $[X]^{1/2}$.*

The interpolation spaces between these three martingale Hardy spaces can be identified in the following way.

THEOREM 3. *If* $0 < \theta < 1$, $0 < p_0 \leqq 1$ *and* $0 < q \leqq \infty$ *then*

$$(H_{p_0}, H_\infty)_{\theta,q} = H_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}$$

where H *denotes one of the spaces* $H^{\langle\rangle}$, $\mathcal{P}$ *and* $\mathcal{Q}$. *(Note that* $\mathcal{P}_\infty = L_\infty$ *and* $\mathcal{Q}_\infty = H^{[]}_\infty$.*)*

We are going to show the theorem for $H^{\langle\rangle}$ spaces only. The main step in the proof is the following result.

LEMMA 1. *If* $0 < p_0 \leqq 1$ *then*

$$K(t, X, H^{\langle\rangle}_{p_0}, H^{\langle\rangle}_\infty) \leqq C\left(\int_0^{t^{p_0}} \tilde{M}(x)^{p_0}\, dx\right)^{1/p_0}$$

where $M := \langle X\rangle_\infty^{1/2}$.

PROOF. Choose y in Theorem 1 such that, for a fixed $t \in [0,1]$, $y = \tilde{M}(t^{p_0})$. For this y let us denote the two martingales in Theorem 1 by Y^t and Z^t. By the definition of the functional K,

$$K(t, X, H^{\langle\rangle}_{p_0}, H^{\langle\rangle}_\infty) \leqq \|Z^t\|_{H^{\langle\rangle}_{p_0}} + t\|Y^t\|_{H^{\langle\rangle}_\infty}.$$

By Theorem 1 we get that

$$\|Z^t\|_{H^{\langle\rangle}_{p_0}} \leqq C\left(\int_{\{M \geqq \tilde{M}(t^{p_0})\}} M^{p_0}\, dP\right)^{1/p_0} = C\left(\int_0^{t^{p_0}} \tilde{M}(x)^{p_0}\, dx\right)^{1/p_0}.$$

On the other hand,

$$t\|Y^t\|_{H^{\langle\rangle}_\infty} \leqq Ct\tilde{M}(t^{p_0}) \leqq C\left(\int_0^{t^{p_0}} \tilde{M}(x)^{p_0}\, dx\right)^{1/p_0}$$

which shows the lemma. □

The next lemma, which is due to Riviere and Sagher ([19] Theorem 8) will also be used in the proof of Theorem 3.

LEMMA 2. *Let* $f \geqq 0$ *be a non-increasing function on* $(0,\infty)$ *and* $0 < q \leqq \infty$, $0 < s < q$. *Then*

$$\left(\int_0^\infty \left(1/t \int_0^t f(u)\, du\right)^q t^s \frac{dt}{t}\right)^{1/q} \leqq C_{q,s}\left(\int_0^\infty f(t)^q t^s \frac{dt}{t}\right)^{1/q}.$$

PROOF OF THEOREM 3. By the definition of interpolation spaces and by Lemma 1

$$\|X\|^q_{(H^{\langle\rangle}_{p_0},H^{\langle\rangle}_\infty)_{\theta,q}} \leqq C\int_0^1 t^{-\theta q}\left(\int_0^{t^{p_0}} \tilde M(x)^{p_0}\,dx\right)^{q/p_0}\frac{dt}{t} \leqq$$

$$\leqq C\int_0^1 t^{(1-\theta)q/p_0}\left(\frac{1}{t}\int_0^{t} \tilde M(x)^{p_0}\,dx\right)^{q/p_0}\frac{dt}{t}.$$

Applying now Lemma 2 we conclude

$$\|X\|^q_{(H^{\langle\rangle}_{p_0},H^{\langle\rangle}_\infty)_{\theta,q}} \leqq C\int_0^1 t^{(1-\theta)q/p_0}\tilde M(t)^q\,\frac{dt}{t} = C\|M\|^q_{p,q}.$$

To prove the converse consider the sublinear operator $T : X \mapsto M$. Observe that $T : H^{\langle\rangle}_\infty \longrightarrow L_\infty$ and $T : H^{\langle\rangle}_{p_0} \longrightarrow L_{p_0}$ are bounded. Therefore, by Theorems A and D,

$$T : (H^{\langle\rangle}_{p_0}, H^{\langle\rangle}_\infty)_{\theta,q} \longrightarrow (L_{p_0}, L_\infty)_{\theta,q} = L_{p,q}$$

is bounded, too, that is to say $X \in (H^{\langle\rangle}_{p_0}, H^{\langle\rangle}_\infty)_{\theta,q}$ implies

$$\|X\|_{H^{\langle\rangle}_{p,q}} = \|M\|_{p,q} \leqq C\|X\|_{(H^{\langle\rangle}_{p_0},H^{\langle\rangle}_\infty)_{\theta,q}}.$$

The proof of Theorem 3 is complete if $0 < q < \infty$. With a fine modification of the previous proof the theorem can be shown in case $q = \infty$, too. □

Applying the reiteration theorem we get the following result, which can be found in Weisz [21] for discrete time.

COROLLARY 1. *Suppose that* $0 < \eta < 1$ *and* $0 < p_0, p_1, q_0, q_1, q \leqq \infty$. *If* $p_0 \neq p_1$ *then*

$$(H_{p_0,q_0}, H_{p_1,q_1})_{\eta,q} = H_{p,q}, \qquad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

In particular,

$$(H_{p_0}, H_{p_1})_{\eta,p} = H_p, \qquad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

Furthermore, for $0 < p < \infty$,

$$(H_{p,q_0}, H_{p,q_1})_{\eta,q} = H_{p,q}, \qquad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$$

where H denotes again one of the spaces $H^{\langle\rangle}$, $\mathcal{P}$ and $\mathcal{Q}$.

This result was proved by Fefferman, Riviere, Sagher [10] and by Janson, Jones [14] in the classical case.

The interpolation spaces between $H_p^{\langle\rangle}$ and BMO_2 are to be identified. In the classical case this is due to Hanks [13].

THEOREM 4. *If $0 < \theta < 1$, $0 < q \leqq \infty$ and $0 < r < \infty$ then*

$$(H_r^{\langle\rangle}, \mathrm{BMO}_2)_{\theta,q} = H_{p,q}^{\langle\rangle}, \qquad \frac{1}{p} = \frac{1-\theta}{r}.$$

PROOF. It is simple to see that

$$\|X\|_{\mathrm{BMO}_2} = \sup_{t\in\mathbf{R}^+} \left\| \left(E_t[\langle X\rangle_\infty - \langle X\rangle_t] \right)^{1/2} \right\|_\infty \leqq$$

$$\leqq \sup_{t\in\mathbf{R}^+} \left\| \left(E_t[\langle X\rangle_\infty] \right)^{1/2} \right\|_\infty \leqq \| \langle X\rangle_\infty^{1/2} \|_\infty .$$

Thus

$$\|X\|_{(H_r^{\langle\rangle}, \mathrm{BMO}_2)_{\theta,q}} \leqq C\|X\|_{(H_r^{\langle\rangle}, H_\infty^{\langle\rangle})_{\theta,q}} = C\|X\|_{H_{p,q}^{\langle\rangle}} .$$

To see the converse consider the operator $T_u^{\langle\rangle}$ for a fixed $0 < u < r$. By Theorem E the operator $T_u^{\langle\rangle}$ is bounded from $H_r^{\langle\rangle}$ to L_r and from BMO_2 to L_∞. Using Theorem A we get that

$$T_u^{\langle\rangle} : (H_r^{\langle\rangle}, \mathrm{BMO}_2)_{\theta,q} \longrightarrow (L_r, L_\infty)_{\theta,q} = L_{p,q}$$

is bounded as well. Henceforth, by Theorem E, one can see that $X \in (H_r^{\langle\rangle}, \mathrm{BMO}_2)_{\theta,p}$ implies

$$\|X\|_{H_p^{\langle\rangle}} \leqq C_p \|T_u^{\langle\rangle}(X)\|_p \leqq C_p \|X\|_{(H_r^{\langle\rangle}, \mathrm{BMO}_2)_{\theta,p}}$$

which proves the theorem for $p = q$, namely,

$$(H_r^{\langle\rangle}, \mathrm{BMO}_2)_{\theta,p} = H_p^{\langle\rangle}, \qquad \frac{1}{p} = \frac{1-\theta}{r}.$$

Applying the reiteration theorem we can prove the theorem with a usual argument (cf. Hanks [13] or Weisz [21]). □

Considering continuous parameter continuous martingales, only, we proved in [25] that $H_p^{\langle\rangle} \sim H_p^* \ (0 < p < \infty)$ and $\mathrm{BMO}_2 \sim \mathrm{BMO}_2^-$, so, in this case,

$$(H_r^*, \mathrm{BMO}_2^-)_{\theta,p} = H_p^*, \qquad \frac{1}{p} = \frac{1-\theta}{r}$$

where $0 < \theta < 1$ and $0 < r < \infty$. This result will be extended later.

As a further application of the reiteration theorem we get the following

Corollary 2. *If $0 < \theta < 1$, $0 < p_0 < \infty$ and $0 < q_0, q \leqq \infty$ then*

$$(H_{p_0,q_0}^{\langle\rangle}, \mathrm{BMO}_2)_{\theta,q} = H_{p,q}^{\langle\rangle}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

The following result can be proved with the duality theorem (cf. Bergh, Löfström [2]) in the same way as in the discrete case (see Weisz [21]). For $p = q$ it is due to Pratelli [18].

Theorem 5. *The dual of $H_{p,q}^{\langle\rangle}$ is $H_{p',q'}^{\langle\rangle}$, where $1 < p < \infty$, $1 \leqq q < \infty$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.*

Let us turn to martingale Hardy spaces defined by quadratic variation and maximal function and prove similar interpolation theorems for them.

Theorem 6. *If $0 < \theta < 1$ and $0 < q \leqq \infty$ then*

$$(H_1^{[]}, H_\infty^{[]})_{\theta,q} = H_{p,q}^{[]}, \qquad \frac{1}{p} = 1 - \theta \tag{3}$$

and

$$(H_1^*, H_\infty^*)_{\theta,q} = H_{p,q}^*, \qquad \frac{1}{p} = 1 - \theta. \tag{4}$$

We remark that $H_\infty^* = L_\infty$ and that H_∞^* is not equivalent to $H_\infty^{[]}$. So the results of Theorem 6 are two different statements. (4) was proved by Fefferman, Riviere, Sagher [10] in the classical case and by Milman [17] and Janson, Jones [14] in the discrete martingale case.

Since the proofs of (3) and (4) are similar, we verify the first one only. As we have seen in Theorem 3, this statement follows from the next lemma.

Lemma 3. *One has*

$$K(t, X, H_1^{[]}, H_\infty^{[]}) \leqq C \int_0^t \tilde{S}(x)\, dx$$

where $S := [X]_\infty^{1/2}$.

PROOF. First we need some new definitions. Considering a regular process X we use the notation $V_{X,t}$ for the *variation* of the mapping $s \mapsto X_s$ on the interval $[0,t]$ ($t \in \mathbf{R}^+ \cup \infty$). To be locally integrable, for an optional process A of finite variation it is necessary and sufficient that there exists a unique predictable process B of finite variation (and which is increasing if A is increasing) such that $A - B$ is a local martingale vanishing at 0. B is called the the *predictable compensator* of A (see Dellacherie, Meyer [9]).

For a fixed t consider the following two stopping times:

$$\nu := \inf\left\{ s \in \mathbf{R}^+ : [X]_s^{1/2} > \tilde{S}(t)\right\},$$

$$\tau := \inf\left\{ s \in \mathbf{R}^+ : V_{R,s} \geqq \tilde{S}(t)\right\}$$

where the process R denotes the predictable compensator of $\Delta X_\nu 1_{[\nu]}$. Recall that the graph of a stopping time is introduced in Section 3. So $\Delta X_\nu 1_{[\nu]} - R$ is a local martingale. Set

$$W := X^\nu - (\Delta X_\nu 1_{[\nu]} - R)$$

and

$$Y := W^{\tau^-}, \qquad Z := X - Y.$$

Since R is predictable, τ is a predictable stopping time and so Y is indeed a local martingale (see Dellacherie, Meyer [9] or Weisz [25]).

It is easy to see that

$$V_{\Delta X_\nu 1_{[\nu]},\infty} = 2|\Delta X_\nu| 1_{\nu<\infty}. \tag{5}$$

By inequality (12) in Weisz [25] we obtain

$$V_{R,\infty} \leqq 2B_\infty \tag{6}$$

where B is the predictable compensator of $|\Delta X_\nu| 1_{[\nu]}$. This yields that the local martingale $\Delta X_\nu 1_{[\nu]} - R$ has finite variation. Consequently, the continuous part of $\Delta X_\nu 1_{[\nu]} - R$ is zero (for the definition and result see Metivier [15]). The continuous part of X is denoted by X^c. It is proved in Metivier [15] (p. 122) that, in this case,

$$[W]_u = [X^c]_{u\wedge\nu} + \sum_{s\leqq u} (\Delta X_s 1_{s\leqq\nu} - \Delta X_s 1_{\nu=s} + \Delta R_s)^2 \qquad (u \in \mathbf{R}^+).$$

Since $[X^c]_u$ is continuous in u, we have

$$[W]_u^{1/2} \leqq \Big([X^c]_{u\wedge\nu-} + \sum_{\substack{s\leqq u\\ s<\nu}} |\Delta X_s|^2\Big)^{1/2} + \Big(\sum_{s\leqq u} |\Delta R_s|^2\Big)^{1/2} \leqq$$

$$\leqq [X]_{u\wedge\nu-}^{1/2} + \sum_{s\leqq u} |\Delta R_s| \leqq [X]_{u\wedge\nu-}^{1/2} + V_{R,u}.$$

As $[Y]_\infty^{1/2} = [W]_{\tau-}^{1/2}$, by the definitions of ν and τ we get that

$$[Y]_\infty^{1/2} \leqq [X]_{\nu-\wedge\tau-}^{1/2} + V_{R,\tau-} \leqq 2\tilde{S}(t).$$

Hence

$$t\big\| [Y]_\infty^{1/2}\big\|_\infty \leqq 2t\tilde{S}(t) \leqq 2\int_0^t \tilde{S}(x)\,dx.$$

We have for the square bracket of Z that

$$(7) \qquad [Z]_\infty^{1/2} \leqq [X - X^{\nu\wedge\tau-}]_\infty^{1/2} + \big[\Delta X_\nu 1_{[\nu]} - R\big]_\infty^{1/2}.$$

Since $\Delta X_\nu 1_{[\nu]} - R$ is of finite variation, we have

$$\big[\Delta X_\nu 1_{[\nu]} - R\big]_\infty^{1/2} \leqq X_{\Delta X_\nu 1_{[\nu]}-R,\infty} \leqq V_{\Delta X_\nu 1_{[\nu]},\infty} + V_{R,\infty}.$$

By the convexity lemma (see Dellacherie, Meyer [9]) and (5) and (6),

$$\|V_{R,\infty}\|_1 \leqq 2\|B_\infty\|_1 \leqq 2\big\| |\Delta X_\nu| 1_{\nu<\infty}\big\|_1 = \big\| V_{\Delta X_\nu 1_{[\nu]},\infty}\big\|_1.$$

Henceforth

$$(8) \qquad \big\| [\Delta X_\nu 1_{[\nu]} - R]_\infty^{1/2}\big\|_1 \leqq 4\|\Delta X_\nu 1_{\nu<\infty}\|_1 \leqq$$

$$\leqq 4\int_{\{[X]_\infty^{1/2} > \tilde{S}(t)\}} [X]_\infty^{1/2}\,dP \leqq 4\int_0^t \tilde{S}(x)\,dx.$$

On the other hand, by the definitions of ν and τ,

$$\big\| [X - X^{\nu\wedge\tau-}]_\infty^{1/2}\big\|_1 \leqq \int_{\{\nu<\infty\}} [X]_\infty^{1/2}\,dP + \int_{\{\nu=\infty\}\cap\{\tau<\infty\}} [X]_\infty^{1/2}\,dP \leqq$$

$$\leqq \int_{\{[X]_\infty^{1/2} > \tilde{S}(t)\}} [X]_\infty^{1/2}\,dP + P(\{\nu=\infty\}\cap\{\tau<\infty\})\tilde{S}(t).$$

By Markov's inequality

$$P(\tau < \infty)\tilde{S}(t) = \tilde{S}(t)P(V_{R,\infty} \geqq \tilde{S}(t)) \leqq \|V_{R,\infty}\|_1$$

and this is estimated in (8) by $\int_0^t \tilde{S}(x)\,dx$. Taking into account (7) and (8) we finished the proof of the lemma. □

Applying the reiteration theorem we get

COROLLARY 3. *Suppose that* $0 < \theta < 1$, $1 < p_0 < p_1 \leqq \infty$ *and* $0 < q_0, q_1, q \leqq \infty$ *or* $p_0 = q_0 = 1$. *Then*

$$(H_{p_0,q_0}, H_{p_1,q_1})_{\theta,q} = H_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

In particular,

$$(H_{p_0}, H_{p_1})_{\eta,p} = H_p, \qquad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

Furthermore, for $1 < p < \infty$,

$$(H_{p,q_0}, H_{p,q_1})_{\eta,q} = H_{p,q}, \qquad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$$

where H *denotes one of the spaces* $H^{[]}$ *and* H^*.

Observe that

$$(H^{[]}_{p_0,q_0}, H^{[]}_{p_1,q_1})_{\theta,q} \subset H^{[]}_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and

$$(H^{*}_{p_0,q_0}, H^{*}_{p_1,q_1})_{\theta,q} \subset H^{*}_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

follow for all $0 < p_0 < p_1 \leqq \infty$ in this case, too.

We remark that $H_1^* \sim H_1^{[]}$ and $H_p^* \sim H_p^{[]} \sim L_p$ for $1 < p < \infty$. From Corollary 3 and Theorem D we get immediately the following result.

COROLLARY 4. *For* $1 < p < \infty$ *and* $0 < q \leqq \infty$ *we have the equivalence* $H^{[]}_{p,q} \sim H^*_{p,q} \sim L_{p,q}$.

Analogously to Theorem 4 and Corollary 2 the following result can be formulated.

THEOREM 7. *Suppose that* $0 < \theta < 1$, $1 < p_0 < \infty$ *and* $0 < q_0, q \leqq \infty$ *or* $p_0 = q_0 = 1$. *Then*

$$(H^{[]}_{p_0,q_0}, \mathrm{BMO}_2^-)_{\theta,q} = H^{[]}_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

An analogous result was proved for the discrete time by Janson and Jones [14] with the complex method.

Observe again that

$$(H^{[]}_{p_0,q_0}, \mathrm{BMO}_2^-)_{\theta,q} \subset H^{[]}_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}$$

holds for every $0 < p_0 < \infty$.

Note that Theorems 6 and 7 cannot be extended to $p_0 < 1$ (cf. Janson, Jones [14]).

Applying Theorems 7 and C we obtain

COROLLARY 5. *If* $0 < \theta < 1$, $0 < p_0 < \infty$ *and* $0 < q_0, q \leqq \infty$ *then*

$$(L_{p_0,q_0}, \mathrm{BMO}_2^-)_{\theta,q} = L_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}. \tag{9}$$

PROOF. By Theorem 7 and Corollary 4,

$$(L_{p_0,q_0}, \mathrm{BMO}_2^-)_{\theta,q} = L_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}$$

where $1 < p_0 < \infty$ and $0 < q_0 \leqq \infty$. We are going to apply Wolff's theorem. Set $A_1 = L_{p_0,q_0}$ for any $0 < p_0 \leqq 1$, $A_2 = L_{p_1,q_1}$ for any $1 < p_1 < \infty$, $A_3 = L_{p,q}$ for any $p_1 < p < \infty$ and $A_4 = \mathrm{BMO}_2^-$. By Theorem D we can apply Theorem C and so we get (9) for $1 < p < \infty$. Let us apply again Wolff's theorem. Now set $A_1 = L_{p_0,q_0}$ for any $0 < p_0 < 1$, $A_2 = L_{p,q}$ for any $p_0 < p \leqq 1$, $A_3 = L_{p_1,q_1}$ for any $1 < p_1 < \infty$ and $A_4 = \mathrm{BMO}_2^-$. Applying (9) to $1 < p < \infty$ together with Theorems C and D we obtain (9) for all $0 < p < \infty$. The proof is complete. □

This theorem can be found in the dyadic case in Schipp, Wade, Simon, Pál [20] for L_2 and BMO_2^- with the complex method.

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THE RATE OF u-CONVERGENCE OF MULTIPLE FOURIER SERIES

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Dedicated to Professor Károly Tandori on his 70th birthday

§1. Introduction

Let $m \geqq 2$ be an integer, Z^m the set of lattice points in R^m and

$$\sum_{n \in Z^m} a_n \tag{1}$$

an m-dimensional series. There exist many different definitions of convergence of the series (1). For instance, let

$$S_\nu = \sum_{|n_1| \leqq \nu} \dots \sum_{|n_m| \leqq \nu} a_n$$

for natural ν,

$$S_r = \sum_{|n| \leqq r} a_n$$

for $r > 0$ and

$$S_n = \sum_{|k_1| \leqq n_1} \dots \sum_{|k_m| \leqq n_m} a_k$$

for $n \in Z^m \cap [0, +\infty)^m$. If the finite limits $\lim_{\nu \to \infty} S_\nu = \alpha$, $\lim_{r \to \infty} S_r = \alpha$ or $\lim_{\min_{1 \leqq j \leqq m} n_j \to \infty} S_\nu = \alpha$ exist then the series (1) is called convergent to α cubically, spherically or in Pringsheim sense, respectively. One can also define the convergence with respect to hyperbolic crosses, simplexes, etc. All these definitions have one thing in common. Namely, the partial sums are taken with respect to sets $U \subset Z^m$ which possess the following properties:

1. U is symmetric with respect to each coordinate hyperplane.
2. If $k \in U$ then all lattice parallelepipeds

$$\prod_{j=1}^{m} \left[\min(0, k_j), \max(0, k_j)\right] \cap Z^m \subseteqq U.$$

So, let us introduce the following

DEFINITION 1. Let $U \subset Z^m$ be bounded. Then $U \in A_3$ iff for every $k \in U$ we have $\prod_{j=1}^{m} \left[-|k_j|, |k_j|\right] \cap Z^m \subseteqq U$.

DEFINITION 2. We will say that the series (1) u-converges to α iff for every $\varepsilon > 0$ there exists N such that for every set $U \in A_3$ which contains the ball $\left\{ n \in Z^m : |n| \leqq N \right\}$ we have

$$\left| \sum_{n \in U} a_n - \alpha \right| < \varepsilon.$$

It is clear, that if the series (1) u-converges to α then it converges to the same number in Pringsheim's sense, cubically, spherically, etc. At the same time the double numerical series $\sum_{n \in Z^2} a_n$ with $a_{0,k} = k^{-1}$, $a_{1,k} = -k^{-1}$, for $k = 1, 2, \dots$ and $a_n = 0$ otherwise converges in Pringsheim's sense, but does not u-converge.

Definition 2 was introduced by F. G. Arutunian [1], [2] in connection with the problems of representation of functions by multiple trigonometric series. In [3] the use of the notion of u-convergence in the theory of multiple Fourier series was observed. Namely, it was stated there that the Fourier series of functions of m variables with bounded variation in the sense of Hardy u-converge at every point. So the results of G. Hardy [4], K. Chandrasekharan and S. Minakhisundaram [5], B. I. Golubov [6], V. N. Temlyakov [7] about different types of convergence of those series were generalized. In [8] the smoothness conditions in L_p-spaces ($1 \leqq p \leqq \infty$) sufficient for the u-convergence of Fourier series in L_p-metric were found and it was proved that those results cannot be improved (the corresponding theorems will be formulated below).

The purpose of the present paper is to give the best possible estimate for the rate of u-convergence of Fourier series in L_p-metric where $1 \leqq p \leqq \infty$.

We need some notations. All the functions below are supposed to be 2π-periodical with respect to each variable. Let $T^m = [-\pi, \pi)^m$, $1 \leqq p \leqq \infty$ and $f(x) \in L_p(T^m)$, where $L_\infty(T^m) \equiv C(T^m)$. Then

$$\|f(x)\|_p = \|f\|_p = \left(\frac{1}{(2\pi)^m} \int_{T^m} |f(x)|^p, dx \right)^{1/p} \quad \text{if} \quad 1 \leqq p < \infty,$$

$$\|f\|_\infty = \max_{x \in T^m} |f(x)|,$$

and if k is a natural number then define the k-th difference of the function f at a point x with step t as

$$\Delta_k(f,x,t) = \sum_{r=0}^{k} (-1)^r C_k^r f(x+rt),$$

and the modulus of smoothness of order k in the space $L_p(T^m)$ as

$$\omega_k(f,\delta)_p = \sup_{t\in R^m:|t|\leqq\delta} \left\| \Delta_k(f,x,t) \right\|_p$$

DEFINITION 3. Let $1 \leqq p \leqq \infty$, $\omega(\delta)$ the modulus of smoothness of order ν, where ν is a positive integer, and $H_p^\omega(T^m) = \{f(x) \in L_p(T^m)$: for every $k > \nu$ we have $\omega_k(f,\delta)_p = O\big(\omega(\delta)\big)$ when $\delta \to {}_+0\}$. If we change O to o in this definition we will denote the corresponding class by $h_p^\omega(T^m)$. For $\omega(\delta) = \delta^\alpha$ the classes $H_p^\omega(T^m)$ and $h_p^\omega(T^m)$ are denoted by $H_p^\alpha(T^m)$ and $h_p^\alpha(T^m)$, respectively.

More information about the Nikolskii classes $H_p^\alpha(T^m)$ can be found in [9]. In [8] there is a brief survey of the results about the convergence in Pringsheim's sense and spherical convergence in L_p-metrics. We have also proved there the following results. Here and below we denote

$$r(m,p) = (m-1)|1/2 - 1/p|.$$

THEOREM A. *Let $m \geqq 2$, $1 \leqq p \leqq \infty$, $p \neq 2$, and the function $f(x) \in h_p^{r(m,p)}(T^m)$. Then the Fourier series of $f(x)$ u-converges in $L_p(T^m)$-metric.*

THEOREM B. *Let $m \geqq 2$ and $1 \leqq p \leqq \infty$, $p \neq 2$. Then there exists a function $f(x) \in H_p^{r(m,p)}(T^m)$ such that its Fourier series u-diverges in $L_p(T^m)$-metric.*

In Section 2 the following result will be proved.

THEOREM 1. *Let $m \geqq 2$, $1 \leqq p \leqq \infty$, $p \neq 2$, and assume that the modulus of smoothness $\omega(\delta)$ of order q is such that $\lim_{\delta\to+0} \frac{\omega(\delta)}{\delta^{r(m,p)}} = 0$. Further let $f(x) \in H_p^\omega(T^m)$. Then for every set $U \in A_3$ we have the estimate*

$$\left\| f - S_U(f) \right\|_p \leqq C(p,m,q) \sup_{N\geqq N(U)} \omega(N^{-1})N^{r(m,p)},$$

where $N(U) = \max\{\nu\colon (\nu,\dots,\nu) \in U\}$,

$$S_U(f) = S_U(f,x) = \sum_{k\in U} a_k(f) e^{ikx}$$

and $\{a_k(f)\}_{k\in Z^m}$ *is the sequence of Fourier coefficients of* f *with respect to the multiple trigonometric system.*

Here and below we will denote by C positive absolute constants (which need not be the same in different cases), by $C(m)$ positive constants which depend only on the dimension m, etc.

In Section 3 the final character of Theorem 1 will be established. Namely, the following theorem is true.

THEOREM 2. *Let* $m \geqq 2$, $1 \leqq p \leqq \infty$, $p \neq 2$, *and assume that the modulus of smoothness* $\omega(\delta)$ *is such that* $\lim\limits_{\delta\to+0} \frac{\omega(\delta)}{\delta^{r(m,p)}} = 0$. *Then there exist an* $f(x) \in H_p^\omega(T^m)$ *and a sequence of* A_3*-sets* U_n *such that* $N(U_n) \to \infty$ *when* $n \to \infty$ *and*

$$\left\| f - S_{U_n}(f)\right\|_p \geqq C(f) \sup_{N \geqq N(U_n)} \omega(N^{-1})N^{r(m,p)}$$

for every n.

We will give the proof only for $p = \infty$ and $p = 1$. The proof for other p uses practically the same considerations. The proof required the refinement of the methods of [8].

This article was written while the author visited Uppsala University. He expresses his gratitude to Swedish colleagues for their help.

§2. Proof of Theorem 1

At first we introduce some more notations.

If $x, y \in R^m$ then we will say that $x \geqq y$ ($x > y$), iff $x_j \geqq y_j$ ($x_j > y_j$) for $j = 1, \dots, m$. Furthermore, let $xy = \sum\limits_{j=1}^{m} x_j y_j$, and if $1 \leqq j \leqq m$, let $x(\widehat{j}) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \in R^{m-1}$. $[a]$ will denote the integer part of a. By a we will also denote the vector with all coordinates equal to a.

For functions $f(x) \in L(T^m)$, $1 \leqq j \leqq m$ and $\nu = 1, 2, \dots$ denote by $W(\nu, j, f)$ the de la Vallée-Poussin's mean of the Fourier series of $f(x)$ of order 2^ν with respect to the j-th variable, that is

$$W(\nu, j, f) = \frac{S_{2^{\nu-1}}(j, f) + \dots + S_{2^\nu - 1}(j, f)}{2^{\nu-1}} = 2\sigma_{2^\nu - 1}(j, f) - \sigma_{2^{\nu-1}-1}(j, f),$$

where

$$\sigma_r(j, f)(x) = \frac{1}{\pi} \int_T f\big(x + (0, \dots, 0, t_j, 0, \dots, 0)\big) K_r(t_j)\, dt_j$$

and

$$K_r(\tau) = \frac{1}{r+1} \frac{\sin^2 \frac{r+1}{2}\tau}{2\sin^2 \frac{1}{2}\tau}$$

is the Fejér kernel. Then let $\Gamma(1,j,f) = W(1,j,f)$ and

$$\Gamma(r,j,f) = W(r,j,f) - W(r-1,j,f) \quad \text{for} \quad r = 2,3,\dots.$$

For $n = (n_1,\dots,n_m) \in Z^m \cap (0,\infty)^m$ denote

$$G(n,f) = \Gamma\big(n_m, m, \Gamma(n_{m-1}, m-1, \dots, \Gamma(n,1,f)\dots\big).$$

Further, let

$$B_n = \prod_{j=1}^{m} \big[2_-^{n_j-1}, 2^{n_j}\big), \quad \text{where} \quad 2_-^{\alpha} = \begin{cases} 2^{\alpha}, & \text{if } \alpha \geqq 0, \\ 0, & \text{if } \alpha < 0. \end{cases}$$

For every set $u \in A_3$ denote

$$\tau_U = \tau_U(f,x) = \sum_{n \geqq 1:\, B_{n-1} \cap U \neq \emptyset} G(n,f)(x)$$

where the quantities $S_U(f) = S_U(f,x)$ were defined in the introduction (see Theorem 1).

We need some auxiliary results.

LEMMA 1. *Under the previous notations for every $k \in U$ we have $a_k(\tau_U) = a_k(f)$.*

PROOF. As U is symmetric it is sufficient to consider the case $k \geqq 0$. Let $n_1,\dots,n_m$ be the least natural numbers for which $k_j < 2_-^{n_j-1}$ for $j = 1,\dots,m$. Then $k \in B_{n-1} \cap U \neq \emptyset$. So we have

$$a_k(\tau_U) = a_k\left(\sum_{\gamma \leqq n} G(\gamma,f)\right) + a_k\left(\sum_{\substack{\gamma:\max\left(\gamma_j-(n_j+1)\right) \geqq 0 \\ 1 \leqq j \leqq m}} G(\gamma,f)\right) =$$

$$= a_k\big(W(n_m, m, W(n_{m-1}, m-1, \dots, W(n_1,1,f)\dots + 0 = a_k(f),$$

and the lemma is proved.

COROLLARY 1. *If $p \in [1,\infty]$, $f(x) \in L_p(T^m)$ and $U \in A_3$ then the estimate*

$$\left\|f - S_U(f)\right\|_p \leqq \|f-\tau_U\|_p + \sum_{n\geqq 1:\, B_{n-1}\cap U\neq\emptyset} \left\|G(n,f) - S_U\big(G(n,f)\big)\right\|_p$$

holds.

PROOF. From Lemma 1 we have $S_U(f-\tau_U)\equiv 0$. So

$$\left\|f - S_U(f)\right\|_p = \left\|f-\tau_U+\tau_U - S_U(f-\tau_U) - S_U(\tau_U)\right\|_p \leqq$$

$$\leqq \|f-\tau_U\|_p + \left\|\tau_U - S_U(\tau_U)\right\|_p \leqq \|f-\tau_U\|_p +$$

$$+ \sum_{n\geqq 1:\, B_{n-1}\cap U\neq\emptyset} \left\|G(n,f) - S_U\big(G(n,f)\big)\right\|_p .$$

LEMMA 2. *Let $m \geqq 2$, $p\neq 2$, $\omega(\delta)$ the modulus of smoothness of some order q and $f(x)\in H_p^\omega(T^m)$. Then for every $n\in Z^m\cap(0,+\infty)^m$ we have*

$$\left\|G(n,f)\right\|_p \leqq C(m,p,q)\omega\big(2^{-\nu(n)}\big), \quad \textit{where} \quad \nu(n) = \max_{1\leqq j\leqq m} n_j .$$

PROOF. We point out that the norms of the one-dimensional de la Vallée-Poussin's operators are uniformly bounded and that these operators with respect to different variables can be transposed. Therefore for $k\neq j$ we have

$$\Gamma\big(n_j, j, \Gamma(n_k,k,f)\big) = \Gamma\big(n_k,k,\Gamma(n_j,j,f)\big). \tag{2}$$

Moreover [9, p. 192–193] for $n_j>1$ we have

$$\left\|\Gamma(n_j,j,f)\right\|_p = \left\|W(n_j,j,f) - W(n_j-1,j,f)\right\|_p \leqq \tag{3}$$

$$\leqq \left\|W(n_j,j,f)-f\right\|_p + \left\|W(n_j-1,j,f)-f\right\|_p \leqq C(p)E_{2^{n_j-2},j}(f)_p \leqq$$

$$\leqq C(m,p,q)\omega\big(2^{-n_j}\big)$$

where $E_{k,j}(f)_p$ is the best approximation of $f(x)$ by trigonometric polynomials of order at most k with respect to the j-th variable in the metric of $L_p(T^m)$. If $n_j=1$ then the estimate (3) is evident. Now the statement of Lemma 2 follows from (2) and (3).

Further for $U\in A_3$ let

$$F_U = \{n\geqq 1\colon B_{n-1}\cap U\neq 0\} \quad \text{and} \quad D(U) = \max\big\{r\colon (r,\dots,r)\in F_U\big\}.$$

COROLLARY 2. *Let $f(x)$ satisfy the conditions of Theorem* 1. *Then we have*

$$\|f-\tau_U\|_p \leqq C(p,m,q) \sup_{N\geqq N(U)} \omega(N^{-1})N^{r(m,p)} \quad \textit{for every} \quad U\in A_3.$$

PROOF. Using Lemma 2 and the above mentioned theorem from the book of Nikolskii [9, pp. 192–193] we get (for brevity $D(U)$ is denoted by D)

$$\|f-\tau_U\|_p \leqq \Big\| f - \sum_{\gamma:\gamma\leqq D-1} G(\gamma,f)\Big\|_p + \sum_{\substack{\gamma:\max(\gamma_j-D)\geqq 0\\ 1\leqq j\leqq m}} \|G(\gamma,f)\|_p \leqq$$

$$\leqq \Big\| f - W\big(D-1,m,W(D-1,m-1,\dots,W(D-1,1,f)\dots)\Big\|_p +$$

$$+C(m,p,q)\sum_{\substack{\gamma:\max\gamma_j\geqq D\\ 1\leqq j\leqq m}} \omega\big(2^{-\nu(\gamma)}\big) \leqq$$

$$\leqq C(m,p,q)\left(\omega(2^{-D}) + \sup_{s>D}\omega(2^{-s})2^{sr(m,p)} \sum_{\substack{\gamma:\max\gamma_j\geqq D\\ 1\leqq j\leqq m}} 2^{-\nu(\gamma)r(m,p)}\right) \leqq$$

$$\leqq C(m,p,q) \sup_{N\geqq N(U)} \omega(N^{-1})N^{r(m,p)}$$

and this was to be proved.

LEMMA 3. *Let $1\leqq p\leqq\infty$, $U\in A_3$ and for some $n\in Z^m$, $U\subseteq \subseteq \prod_{j=1}^{m}\left[-2^{n_j},2^{n_j}\right]$. Then*

$$\|S_U\|_{L_p\to L_p} \leqq C(m,p,q)2^{(n_1+\ldots+n_m-\nu(n))\left|\frac{1}{p}-\frac{1}{2}\right|}\big(\nu(n)-\xi(n)+1\big)^{\left|\frac{2}{p}-1\right|},$$

where $\xi(n)=\min_{1\leqq j\leqq m} n_j$.

PROOF. Without loss of generality we may suppose that $\nu(n)=n_m$ and $\xi(n)=n_1$. For $p=1$ and $p=\infty$ the norm $\|S_U\|_{L_p\to L_p}$ coincides with the norm of the Dirichlet kernel $D_U(x)=\sum_{n\in U} e^{inx}$ in the space $L(T^m)$. Further, in [10] the author deduced from a result of A. A. Judin and V. A. Judin [11] that if the bounded set $U\subseteq Z^m\cap(0,\infty)^m$, the intersection of U with every line parallel to the coordinate axes is either an integer interval or empty; P_1

and P_2 are the numbers of points of the projections of U on two different coordinate hyperplanes and $P_1 \leqq P$ then we have

$$\|D_U\|_1 \leqq CP_1^{\frac{1}{2}}\left(\ln\frac{P_2}{P_1}+1\right).$$

This is the estimate of our lemma for $p=1$ and $p=\infty$. For $p=2$ the uniform boundedness of the norms of S_U is trivial and for other p we should only apply the interpolation theorem of M. Riesz [12, p. 144].

Now we begin the direct proof of Theorem 1. Let $U \in A_3$. For every transportation $\sigma=(\sigma_1,\dots,\sigma_m)$ of the numbers $\{1,\dots,m\}$ we denote $F_\sigma = \left\{n \in F_U : n_{\sigma_1} \leqq n_{\sigma_2} \leqq \dots \leqq n_{\sigma_m}\right\}$. Then

$$(4)\qquad \sum_{n\in F_U}\left\|G(n,f)-S_U\big(G(n,f)\big)\right\|_p \leqq$$
$$\leqq \sum_\sigma\sum_{n\in F_\sigma}\left\|G(n,f)-S_U\big(G(n,f)\big)\right\|_p = \sum_\sigma V_\sigma.$$

We can estimate all V_σ in the same way, so we deal only with $V_1 = V_{\{1,\dots,m\}}$. We have

$$(5)\qquad V_1 \leqq \sum_{n_1=1}^{N(U)}\sum_{n_2=n_1}^{\nu_2(U,n_1)}\cdots\sum_{n_m=n_{m-1}}^{\nu_m(U,n_1,\dots,n_{m-1})}\left\|G(n,f)-S_U\big(G(n,f)\big)\right\|_p.$$

Let

$$\lambda_m = \lambda_m(U,n_1,\dots,n_{m-1}) =$$
$$=\min\left\{n_m \geqq n_{m-1}: S_U\Big(G\big((n_1,\dots,n_{m-1},n_m),f\big)\Big) \neq \right.$$
$$\left. \neq G\big((n_1,\dots,n_{m-1},n_m),f\big)\right\}.$$

It is clear that if $(n_1,\dots,n_{m-1}) \leqq (n_1',\dots,n_{m-1}')$, then

$$(6)\qquad \lambda_m(U,n_1,\dots,n_{m-1}) \geqq \lambda_m(U,n_1',\dots,n_{m-1}') \geqq N(U)-1.$$

Taking under consideration that $\lambda_m(U,n_1,\dots,n_{m-1}) \geqq n_{m-1}$ we get from the estimate (6) that for fixed $n_1,\dots,n_{m-2}$ and for every $n_{m-1} \in \big[n_{m-2}, \nu_{m-1}(U,n_1,\dots,n_{m-2})\big]$ the following estimates hold:

$$(7)\qquad \lambda_m(U,n_1,\dots,n_{m-1}) \geqq \lambda_m\big(U,n_1,\dots,n_{m-2},\nu_{m-1}(n_1,\dots,n_{m-2})\big) \geqq$$
$$\geqq \nu_{m-1}(n_1,\dots,n_{m-2}).$$

In many cases below we will write λ_m instead of $\lambda_m(U, n_1, \dots, n_{m-1})$, ν_m instead of $\nu_m(U, n_1, \dots, n_{m-1})$ etc., for brevity. Sometimes we will omit U in subsequent formulas. Taking into account that for fixed n we have

$$S_U\big(G(n,f)\big) = S_{U_n}\big(G(n,f)\big), \quad \text{where} \quad U_n = U \cap \prod_{j=1}^{m} \left[-2^{n_j}, 2^{n_j}\right],$$

and using Lemmas 2 and 3 we get for fixed $n_1, \dots, n_{m-1}$

$$\sum_{n_m = n_{m-1}}^{\nu_m} \left\| G(n,f) - S_U\big(G(n,f)\big) \right\|_p \leqq$$

$$\leqq \sum_{n_m=\lambda_m}^{\nu_m} \left\| G(n,f) \right\|_p \left(\|S_u\|_{L_p \to L_p} + 1 \right) \leqq C(p,m,q)\varepsilon_{N-1} \times$$

$$\times \sum_{n_m=\lambda_m}^{\nu_m} 2^{\frac{1}{2}(n_1+\dots+n_{m-1})\left|\frac{1}{2}-\frac{1}{p}\right|} 2^{-n_m \frac{m-1}{2}\left|\frac{1}{2}-\frac{1}{p}\right|} (n_m - n_1 + 1)^{\left|1-\frac{2}{p}\right|} \leqq$$

$$\leqq C(p,m,q)\varepsilon_{N-1} 2^{\frac{1}{2}\left(n_1+\dots+n_{m-1}-\lambda_m(m-1)\right)\left|\frac{1}{2}-\frac{1}{p}\right|} (\lambda_m - n_1 + 1)^{\left|1-\frac{2}{p}\right|},$$

where $\varepsilon_{N-1} = \varepsilon_{N(U)-1} = \sup\limits_{k \geqq N(U)-1} \omega(k^{-1}) k^{(m-1)\left|\frac{1}{2}-\frac{1}{p}\right|}$. Then (see (5))

$$(8) \quad V_1 \leqq C(p,m,q)\varepsilon_{N-1} \sum_{n_1=1}^{N} 2^{\frac{1}{2}n_1} \dots \sum_{n_{m-1}=n_{m-2}}^{\nu_{m-1}} 2^{\frac{1}{2}\left(n_{m-1}-\lambda_m(m-1)\right)\left|\frac{1}{2}-\frac{1}{p}\right|} \cdot$$

$$\cdot (\lambda_m - n_1 + 1)^{\left|1-\frac{2}{p}\right|}.$$

Using (6) and (7) and taking into account that for fixed $n_1, \dots, n_{m-2}$ and $n_{m-1} < n'_{m-1}$ we have

$$k = n_{m-1} - (m-1)\lambda_m(n_1, \dots, n_{m-2}, n_{m-1}) <$$
$$< n'_{m-1} - \lambda_m(n_1, \dots, n_{m-2}, n'_{m-1}) = k'$$

and that for $m > 2$

$$\lambda_m - n_1 + 1 \leqq (m-1)\lambda_m - n_{m-1} - (m-3)n_{m-1} - n_1 + 1 \leqq$$
$$\leqq (m-1)\lambda_m - n_{m-1} - (m-2)n_1 + m - 2,$$

we get the following estimate (for $m = 2$ see (11) below with λ_2 instead of ν_2)

$$(9) \qquad \sum_{n_{m-1}=n_{m-2}}^{\nu_{m-1}} 2^{\frac{1}{2}\left(n_{m-1}-\lambda_m(m-1)\right)\left|\frac{1}{2}-\frac{1}{p}\right|}(\lambda_m - n_1 + 1)^{\left|1-\frac{2}{p}\right|} \leqq$$

$$\leqq \sum_{k=k_0}^{\infty} 2^{-\frac{1}{2}\left|\frac{1}{2}-\frac{1}{p}\right|k}\big(k-(m-2)n_1+(m-2)\big)^{\left|1-\frac{2}{p}\right|} \leqq$$

$$\leqq C(m,p,q)2^{-\frac{1}{2}\left|\frac{1}{2}-\frac{1}{p}\right|k_0}\big(k_0-(m-2)n_1+(m-2)\big)^{\left|1-\frac{2}{p}\right|},$$

where (see (7))

$$k_0 \geqq (m-1)\nu_{m-1} - \nu_{m-1} = (m-2)\nu_{m-1}.$$

So (see (8))

$$(10)\, V_1 \leqq C(p,m,q)\varepsilon_{N-1}\sum_{n_1=1}^{N} 2^{\frac{1}{2}n_1}\dots\sum_{n_{m-2}=n_{m-3}}^{\nu_{m-2}} 2^{\frac{1}{2}(n_{m-2}-(m-2)\nu_{m-1})\left|\frac{1}{2}-\frac{1}{p}\right|}\cdot$$

$$\cdot(\nu_{m-1}-n_1+1)^{\left|1-\frac{2}{p}\right|}.$$

As for $(n_1,\dots,n_{m-2}) \leqq (n'_1,\dots,n'_{m-3})$ we have $\nu_{m-1}(n_1,\dots,n_{m-2}) \geqq \geqq \nu_{m-1}(n'_1,\dots,n'_{m-2})$ and as for every fixed $n_1,\dots,n_{m-3}$ and every $n_{m-2} \in \in \left[n_{m-3},\nu_{m-2}(n_1,\dots,n_{m-3})\right]$ we have

$$\nu_{m-1}(n_1,\dots,n_{m-2}) \geqq \nu_{m-1}\big(n_1,\dots,n_{m-3},\nu_{m-2}(n_1,\dots,n_{m-3})\big) \geqq$$

$$\geqq \nu_{m-2}(n_1,\dots,n_{m-3}),$$

we can estimate the inner sum in (10) in the same way as in (9). If we repeat the previous considerations we finally get

$$(11) \qquad V_1 \leqq C(p,m,q)\varepsilon_{N-1}\sum_{n_1=1}^{N} 2^{\frac{1}{2}(n_1-\nu_2)\left|\frac{1}{2}-\frac{1}{p}\right|}(\nu_2-n_1+1)^{\left|1-\frac{2}{p}\right|} =$$

$$= C(m,p,q)\varepsilon_{N-1}.$$

From Corollaries 1,2 and from (4) and (11) we get the statement of Theorem 1.

§3. The proof of Theorem 2 for $p=\infty$ and $p=1$.

At first we fix $m \geqq 2$. Let $\{\varepsilon_r\}_{r=0}^{\infty}$ be the sequence of Rudin–Shapiro signs [13] and let $\nu_k = \left[\frac{1}{m}2^k\right]$ for $k = 1,2,\dots$. Then let

$$Q_k(x) = \sum_{n(\widehat{m})=0}^{\nu_k} e^{i\left(n(\widehat{m})x(\widehat{m})+\left(2^k-(n_1+\ldots+n_m)\right)x_m\right)} \prod_{j=1}^{m-1} \varepsilon_{n_j} =$$

$$= e^{i2^k x_m} \prod_{j=1}^{m-1} \left(\sum_{n_j=0}^{\nu_k} \varepsilon_{n_j} e^{in_j(x_j-x_m)} \right) = e^{i2^k x_m} \prod_{j=1}^{m-1} P_{\nu_k}(x_j - x_m),$$

where $P_\gamma(t)$ is the corresponding Rudin–Shapiro polynomial. It is well known [13] that

$$\|P_\gamma\|_\infty = 5\sqrt{\gamma+1} \quad \text{for} \quad \gamma = 0,1,\dots.$$

So for all k we have

$$\|Q_k\|_\infty \leqq C(m) 2^{k\frac{m-1}{2}}. \tag{12}$$

Now define the function

$$f(x) = \sum_{r=1}^{\infty} \omega\left(2^{-k_r}\right) 2^{-k_r \frac{m-1}{2}} Q_{k_r}(x), \tag{13}$$

where $k_r < k_{r+1}$ for $r = 1,2$ are such that

$$\omega\left(2^{-k_{r+1}}\right) \leqq \frac{1}{2}\omega\left(2^{-k_r}\right)$$

and

$$\omega\left(2^{-k_r}\right) 2^{k_r \frac{m-1}{2}} \geqq C(\omega) \sup_{k \geqq 2^{k_r}} \omega(k^{-1}) k^{\frac{m-1}{2}}.$$

Then (see (12)) the series (13) converges uniformly. The spectra of the polynomials $Q_k(x)$ with different k's do not intersect, so for every $n \in Z^m$ the Fourier coefficient $a_n(f)$ either coincides with the corresponding coefficient of the polynomial $\omega\left(2^{-k_{r(n)}}\right) 2^{-k_{r(n)}\frac{m-1}{2}} Q_{k_{r(n)}}(x)$, or is equal to zero. Now we check that $f(x) \in H_\infty^\omega(T^m)$. Let k be a natural number and $k > k_1$. Then there exists r such that $k_r < k \leqq k_{r+1}$. Since $\omega(\delta)$ is the modulus of smoothness we get for sufficiently large ν by usual arguments that

$$\omega_\nu\left(f, 2^{-k}\right)_\infty \leqq C(m) \left(2^{-k\nu} \sum_{\gamma=1}^{r} \omega\left(2^{-k_\gamma}\right) 2^{k_\gamma \nu} + \sum_{\gamma=r+1}^{\infty} \omega\left(2^{-k_\gamma}\right) \right) \leqq$$

$$\leqq C(m,\omega)\left(2^{-k\nu}\omega\left(2^{-k_r}\right)2^{k_r\nu}+\omega\left(2^{-k_{r+1}}\right)\right)\leqq C(m,\omega)\omega(2^{-k}).$$

So $f(x)\in H^{\omega}_{\infty}(T^m)$.

Further let

$$\tau_N(t)=\sum_{\gamma=0}^{N}\varepsilon_\gamma^+e^{i\gamma t}\quad\text{for}\quad N=0,1,\dots,\quad\text{where}\quad\varepsilon_\gamma^+=\max(\varepsilon_\gamma,0),$$

then $\tau_N(t)=\frac{1}{2}\left(P_N(t)+D_N(t)\right)$, where $P_N(t)$ is the Rudin–Shapiro polynomial and $D_N(t)$ is the Dirichlet kernel. Now we have

$$\|\tau_N\|_{C(T)}\geqq\frac{1}{2}\left(\|D_N\|_{C(T)}-\|P_N\|_{C(T)}\right)\geqq C\left(N-N^{\frac{1}{2}}\right)\geqq CN. \tag{14}$$

Define for $r=1,2,\dots$ the sets

$$V_r=\left\{n\in Z^m\colon|n_1|+\dots+|n_m|<2^{k_r}\right\}$$

and

$$V_r'=V_r\cup\left\{n\in Z^m\colon|n_1|+\dots+|n_m|=2^{k_r}\quad\text{and}\quad \varepsilon_{|n_j|}=1\quad\text{for}\quad j=1,\dots,m-1\right\}.$$

Then $V_r,V_r'\in A_3$ for every r, $N(V_r)\simeq N(V_r')\simeq 2^{k_r}$ and (see (14))

$$\left\|S_{V_r}(f)-S_{V_r'}(f)\right\|_\infty=$$

$$=\omega\left(2^{-k_r}\right)2^{-k_r\frac{m-1}{2}}\left\|\sum_{\substack{n(\widehat{m})=0:\\ \varepsilon_{n_j}=1,1\leqq j\leqq m-1}}e^{i(n_1(x_1-x_m)+\dots+n_{m-1}(x_{m-1}-x_m))}\right\|_\infty\geqq$$

$$\geqq C(m)\omega\left(2^{-k_r}\right)2^{k_r\frac{m-1}{2}}\geqq C(m,\omega)\sup_{k\geqq 2^{k_r}}\omega(k^{-1})k^{\frac{m-1}{2}}.$$

This is equivalent to the statement of Theorem 2 for $p=\infty$.

Now we construct an example for $p=1$. The Fejér's kernel is

$$K_r(t)=\frac{D_0(t)+\dots+D_r(t)}{r+1}=\sum_{k=-r}^{r}\left(1-\frac{|k|}{r+1}\right)e^{ikx}.$$

It is well known that $\|K_r\|_{L(T)} = 1$ for all r. Further, let

$$(15) \qquad f(x) = \sum_{r=1}^{\infty} \omega\left(2^{-k_r}\right) F_k(x),$$

where

$$F_k(x) = \sum_{n(\widehat{m})=0}^{2\nu_k} \left(\prod_{j=1}^{m-1} \left(1 - \frac{|n_j - \nu_k|}{\nu_k + 1}\right) e^{in_j x_j} \right) e^{i\left(2^{k+1} - (n_1 + \ldots + n_{m-1})\right)x_m} =$$

$$= e^{i2^{k+1}x_m} \prod_{j=1}^{m-1} e^{i\nu_k(x_j - x_m)} \prod_{j=1}^{m-1} K_{\nu_k}(x_j - x_m),$$

$\nu_k = \left[\frac{1}{m}2^k\right]$ and the numbers k_r were defined above. It is easy to check that the series (15) converges in $L_1(T^m)$ metric and $f(x) \in H_p^{\omega}(T^m)$.

Now denote

$$V_r = \left\{ n \in Z^m : 0 \leqq n(\widehat{m}) \leqq 2\nu_{k_r},\ n_m = 2^{k_r+1} - (n_1 + \ldots + n_{m-1}) \quad \text{and} \right.$$
$$\left. \varepsilon_{|n_j - \nu_k|} = 1, \quad 1 \leqq j \leqq m-1 \right\},$$

where $\{\varepsilon_r\}_{r=0}^{\infty}$ is the sequence of Rudin–Shapiro signs. Let

$$R_r(x) = \sum_{n \in V_r} a_n\left(F_{k_r}\right) e^{inx} \quad \text{for} \quad r = 1, 2, \ldots .$$

Notice that (for brevity we denote ν_{k_r} by $\nu(r)$)

$$(16) \qquad R_r(x) = e^{i2^{k_r+1}x_m} e^{i\nu(r)(x_1 + \cdots + x_{m-1} - (m-1)x_m)} \times$$

$$\times \prod_{j=1}^{m-1} \left(\frac{1}{2} K_{\nu(r)} \left(\sum_{n_j = -\nu(r)}^{\nu(r)} \varepsilon_{|n_j|} e^{in_j(x_j - x_m)} + \sum_{n_j = -\nu(r)}^{\nu(r)} e^{in_j(x_j - x_m)} \right) \right) =$$

$$= e^{i\xi(x)} \prod_{j=1}^{m-1} \left(\frac{1}{2} \eta_r(x_j - x_m) \right),$$

where $\xi(x)$ is a real function and $K_{\nu(r)}$ is the corresponding Fejér operator. Then we have

$$(17) \qquad \|\eta_r\|_{L(T)} \geqq \left\| K_{\nu(r)} \left(\operatorname{Re} \sum_{n_j=0}^{\nu(r)} \varepsilon_{n_j} e^{in_j t} \right) \right\|_{L(T)} -$$

$$- \left\| \frac{1}{2} \right\|_{L(T)} - \| K_{\nu(r)}(t) \|_{L(T)} \geqq$$

$$\geqq \left\| K_{\nu(r)} \left(\sum_{n_j=0}^{\nu(r)} \varepsilon_{n_j} \cos n_j t \right) \right\|_{L(T)} - C.$$

Now let

$$\Theta_r(t) = \sum_{n_j=0}^{\nu(r)} \varepsilon_n \cos nt,$$

then since $\|\Theta_r\|_{C(T)} \leqq C 2^{\frac{1}{2}k_r}$, we get

$$(18) \qquad \| K_{\nu(r)}(\Theta_r) \|_{C(T)} \leqq C 2^{\frac{1}{2}k_r}.$$

Moreover,

$$(19) \qquad \| K_{\nu(r)}(\Theta_r) \|_{L_2(T)} \geqq C 2^{\frac{1}{2}k_r}.$$

So (see (18) and (19)) we have

$$C_1 2^{\frac{1}{2}k_r} \leqq \| K_{\nu(r)}(\Theta_r) \|_{L_2(T)} \leqq \| K_{\nu(r)}(\Theta_r) \|_{L(T)}^{\frac{1}{2}} \| K_{\nu(r)}(\Theta_r) \|_{C(T)}^{\frac{1}{2}} \leqq$$

$$\leqq C \| K_{\nu(r)}(\Theta_r) \|_{L(T)}^{\frac{1}{2}} 2^{\frac{1}{4}k_r}$$

and

$$(20) \qquad \| K_{\nu(r)}(\Theta_r) \|_{L(T)} \geqq C 2^{\frac{1}{2}k_r}.$$

From (17) and (20) we get that

$$\|\eta_r\|_{L(T)} \geqq C 2^{\frac{1}{2}k_r},$$

and so (see (16))

$$\|R_r\|_1 \geqq C(m) 2^{\frac{1}{2}k_r(m-1)}. \tag{21}$$

Finally, let

$$W_r = \{ n \in Z^m : |n_1| + \ldots + |n_m| < 2^{k_r+1} \},$$

$$W_r' = W_r \cup \{ n \in Z^m : |n_1| + \ldots + |n_m| = 2^{k_r+1} \text{ and } (|n_1|, \ldots, |n_m|) \in V_r \}.$$

Now the required statement follows from (15) and (21) the same manner as for $p = \infty$.

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THE CESÀRO OPERATOR ON THE BANACH ALGEBRA OF $L(\mathbf{R}^2)$ MULTIPLIERS. III (EVEN-ODD CASE)

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Dedicated to Károly Tandori on his 70th birthday

1. Introduction

We make a consistent use of the notations occurring in [3,4], where we studied the case of odd and even multipliers for $L(\mathbf{R}^2)$. We summarize briefly the basic concepts.

Let f be a complex-valued function defined on $\mathbf{R}^2 := (-\infty, \infty) \times \times(-\infty, \infty)$. If $f \in L(\mathbf{R}^2)$, then its double Fourier transform is defined by

$$(1.1) \qquad \widehat{f}(x,y) := \int\int_{\mathbf{R}^2} f(\xi,\eta) e^{-i(x\xi+y\eta)}\, d\xi\, d\eta, \quad (x,y) \in \mathbf{R}^2.$$

For simplicity, we omitted the norming factor $(4\pi^2)^{-1}$.

If $f(x,y)$ is even in x and odd in y:

$$f(x,y) = -f(x,-y) = f(-x,y) = -f(-x,-y), \quad (x,y) \in \mathbf{R}^2_+,$$

where $\mathbf{R}^2_+ := [0,\infty) \times [0,\infty)$, then (1.1) becomes a cosine-sine Fourier transform:

$$(1.2) \qquad \widehat{f}(x,y) = -4i \int\int_{\mathbf{R}^2} f(\xi,\eta) \cos x\xi \sin y\eta\, d\xi\, d\eta.$$

Let λ be a measurable function on $\mathbf{R}^2$. We say that λ is a multiplier for $L(\mathbf{R}^2)$ (or simply, an $L(\mathbf{R}^2)$ multiplier) if for every $f \in L(\mathbf{R}^2)$ there exists a function $g \in L(\mathbf{R}^2)$ such that

$$\lambda(x,y)\widehat{f}(x,y) = \widehat{g}(x,y), \quad (x,y) \in \mathbf{R}^2.$$

As it is well known (see, e.g. [9, p. 94]), a necessary and sufficient condition for a measurable function λ to be a multiplier for $L(\mathbf{R}^2)$ is that there exists

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a finite Borel measure μ on $\mathbf{R}^2$ such that λ is the Fourier–Stieltjes transform of μ:

$$\lambda(x,y)=\int\int_{\mathbf{R}^2} e^{-i(x\xi+y\eta)}\,d\mu(\xi,\eta),\quad (x,y)\in\mathbf{R}^2. \tag{1.3}$$

Hence it follows immediately that if λ is a multiplier for $L(\mathbf{R}^2)$, then we may assume without loss of generality that λ is bounded and continuous (even uniformly) on $\mathbf{R}^2$.

If $\lambda(x,y)$ is even in x and odd in y, then the measure μ associated with λ according to (1.3) is also even in the first and odd in the second component, i.e. for any Borel sets $D,E\subseteqq\mathbf{R}_+$ we have

$$\mu(D\times E)=-\mu\big(D\times(-E)\big)=\mu\big((-D)\times E\big)=-\mu\big((-D)\times(-E)\big).$$

In particular, μ vanishes along the x-axis:

$$\mu\big(\{x,0):x\in D\}\big)=0,\quad D\subseteqq\mathbf{R}_+. \tag{1.4}$$

Furthermore, in this case (1.3) goes into

$$\lambda(x,y)=-4i\int\int_{\mathbf{R}^2_+}\cos x\xi\sin y\eta\,d\mu(\xi,\eta). \tag{1.5}$$

R. Fefferman [5] introduced a new kind of Hardy space as follows

$$\mathcal{H}(\mathbf{R}\times\mathbf{R}):=\big\{f\in L(\mathbf{R}^2): H_1f, H_2f,\quad\text{and}\quad H_1H_2f\in L(\mathbf{R}^2)\big\},$$

where the Hilbert transforms are defined by

$$(H_1f)^\wedge(x,y):=-i(\operatorname{sign}x)\widehat{f}(x,y),$$
$$(H_2f)^\wedge(x,y):=-i(\operatorname{sign}y)\widehat{f}(x,y),$$

and

$$(H_1H_2f)^\wedge(x,y):=-\operatorname{sign}(xy)\widehat{f}(x,y),\quad (x,y)\in\mathbf{R}^2.$$

As it is well known, for almost all $(x,y)\in\mathbf{R}^2$ we have

$$H_1f(x,y)=-\frac{1}{\pi}\int_{\to 0}^{\infty}\big\{f(x+u,y)-f(x-u,y)\big\}\,\frac{du}{u}, \tag{1.6}$$

$$H_2f(x,y)=-\frac{1}{\pi}\int_{\to 0}^{\infty}\big\{f(x,y+v)-f(x,y-v)\big\}\,\frac{dv}{v} \tag{1.7}$$

and

$$(1.8)\quad H_1H_2f(x,y)=\frac{1}{\pi^2}\int_{\to 0}^{\infty}\int_{\to 0}^{\infty}\{f(x+u,y+v)-f(x+u,y-v)-$$
$$-f(x-u,y+v)+f(x-u,y-v)\}\,\frac{du}{u}\,\frac{dv}{v}.$$

We note that definition of the ordinary Hardy space $\mathcal{H}(\mathbf{R}^2)$ relies on the notion of Riesz transforms. (See, e.g., [10, pp. 223–224].) Among others, in [2] we proved the strict inclusion $\mathcal{H}(\mathbf{R}\times\mathbf{R})\subset\mathcal{H}(\mathbf{R}^2)$.

Finally, we agree to write $F\in L^\wedge(\mathbf{R}^2)$ if there exists a function $f\in$ $\in L(\mathbf{R}^2)$ such that

$$(1.9)\qquad F(x,y)=\widehat{f}(x,y),\quad (x,y)\in\mathbf{R}^2.$$

In other words, $L^\wedge(\mathbf{R}^2)$ is the space of the double Fourier transforms on $\mathbf{R}^2$. If we have $f\in\mathcal{H}(\mathbf{R}\times\mathbf{R})$ in (1.9), then we write $F\in\widehat{\mathcal{H}}(\mathbf{R}\times\mathbf{R})$.

2. Main results

Let λ be a locally integrable function on $\mathbf{R}^2$. As usual, we define the Cesàro mean of λ by

$$(2.1)\qquad \sigma\lambda(u,v):=\frac{1}{uv}\int_0^u\int_0^v\lambda(x,y)\,dxdy,\quad u,v\neq 0.$$

If λ is continuous on $\mathbf{R}^2$, then we define additionally

$$\sigma\lambda(u,0):=\frac{1}{u}\int_0^u\lambda(x,0)\,dx,\quad u\neq 0,$$

$$\sigma\lambda(0,v):=\frac{1}{v}\int_0^v\lambda(0,y)\,dy,\quad v\neq 0,$$

$$\sigma\lambda(0,0):=\lambda(0,0);$$

in which case $\sigma\lambda$ is also continuous on $\mathbf{R}^2$.

The main results of the present paper are summarized in Theorems 1, 2 and Corollary 1 below.

THEOREM 1. *If $\lambda(x,y)$ is a multiplier for $L(\mathbf{R}^2)$, even in x and odd in y, then*

(i) $\sigma\lambda$ *is also a multiplier for* $L(\mathbf{R}^2)$,
(ii) $\sigma\lambda \in L^\wedge(\mathbf{R}^2)$ *if and only if*

$$\mu\Big(\{(0,y)\colon y \in E\}\Big) = 0, \qquad E \subseteqq \mathbf{R}_+, \tag{2.2}$$

where μ *is the finite Borel measure on* $\mathbf{R}^2$ *associated with* λ *by* (1.3).

We note that condition (2.2) can be rephrased to say that the measure μ is continuous on the y-axis.

THEOREM 2. *Let* $\lambda(x,y)$ *be a multiplier for* $L(\mathbf{R}^2)$, *even in* x *and odd in* y, *and let*

$$f(x,y) := \int_x^\infty \int_y^\infty \frac{d\mu(\xi,\eta)}{\xi\eta}, \tag{2.3}$$

$$f(x,-y) = -f(-x,y) = f(-x,-y) := -f(x,y), \quad (x,y) \in \mathbf{R}^2_+,$$

where μ *is the finite Borel measure on* $\mathbf{R}^2$ *associated with* λ *by* (1.3). *Then*

$$\int\!\!\int_{\mathbf{R}^2_+} \left| \pi H_1 f(x,y) - \frac{2}{x} \int_0^{x/2} \int_y^\infty \frac{d\mu(\xi,\eta)}{\eta} \right| dx\,dy \leqq C\|\mu\|, \tag{2.4}$$

$$\int\!\!\int_{\mathbf{R}^2_+} |H_2 f(x,y)|\,dx\,dy \leqq C\|\mu\|, \tag{2.5}$$

and

$$\int\!\!\int_{\mathbf{R}^2_+} \left| \pi^2 H_1 H_2 f(x,y) - \frac{2}{x} \int_0^{x/2} \int_{y/2}^\infty \ln\left(\frac{y}{|y-\eta|}\right) \frac{d\mu(\xi,\eta)}{\eta} - \right. \tag{2.6}$$

$$\left. -\frac{2}{x} \int_0^{x/2} \int_0^{y/2} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\eta} \right| dx\,dy \leqq C\|\mu\|,$$

where $\|\mu\|$ *denotes the total variation of the measure* μ *over* $\mathbf{R}^2_+$.

Here and in the sequel, by $C, C_1, C_2, \ldots$ we denote positive absolute constants.

It is plain that condition (2.7) below implies the fulfillment of condition (2.2). Now, it turns out from the proof of Theorem 1 that then we have $\sigma f = \widehat{f}$ (cf. (4.6)). The following corollary hinges on this equality.

COROLLARY 1. *If $\lambda(x,y)$ is a multiplier for $L(\mathbf{R}^2)$, even in x and odd in y, then $\sigma\lambda \in \widehat{\mathcal{H}}(\mathbf{R}\times\mathbf{R})$ if and only if*

$$\frac{1}{x}\int_0^x \int_{\mathbf{R}_+} d\mu(\xi,\eta) \in L(\mathbf{R}_+) \quad (in\ x). \tag{2.7}$$

We note that in the case of multipliers for $L(\mathbf{R})$, analogous results were proved in [1] and [6]. The above Theorems 1, 2 and Corollary 1 can be considered as a kind of mixture of those results.

3. Auxiliary results

The first two of them are related to the (improper) Riemann–Stieltjes integral.

LEMMA 1 (see [1]). *If $f(x)$ is a continuous and bounded function, while μ is a finite positive Borel measure on $\mathbf{R}_+$, then*

$$\int_{\mathbf{R}_+} f(x)\,d\mu(x) = \int_{\to 0}^{\infty} f(x)\,d\mu(x) + f(0)\mu(\{0\}). \tag{3.1}$$

LEMMA 2 (see [3]). *If $f(x,y)$ is a continuous and bounded function, while μ is a finite positive Borel measure on $\mathbf{R}^2_+$, then*

$$\int\int_{\mathbf{R}^2_+} f(x,y)\,d\mu(x,y) = \int_{\to 0}^{\infty}\int_{\to 0}^{\infty} f(x,y)\,d\mu(x,y) + \tag{3.2}$$

$$+\int_{\to 0}^{\infty} f(x,0)\,d\mu(x,0) + \int_{\to 0}^{\infty} f(0,y)\,d\mu(0,y) + f(0,0)\mu\Big(\{(0,0)\}\Big).$$

We emphasize that, under the conditions of Lemmas 1 and 2, the integrals on the left-hand side of (3.1) and (3.2) can be equally considered to be Lebesgue–Stieltjes integrals and (improper) Riemann–Stieltjes integrals, since in these cases they coincide. (See, e.g., [8, Chs. 1 and 2].)

The third lemma is due to de Leeuw [7] and states that if a function λ is a multiplier for $L(\mathbf{R}^2)$, then λ restricted to $\mathbf{R}$ is a multiplier for $L(\mathbf{R})$. In particular, what we need is the following

LEMMA 3. *If $\lambda(x,y)$ is a multiplier for $L(\mathbf{R}^2)$, then the marginal functions $\lambda(\cdot,0)$ and $\lambda(0,\cdot)$ are multipliers for $L(\mathbf{R})$.*

Analysing the proof of [1, Theorem 1] yields the following

LEMMA 4. *If μ is an odd finite Borel measure on $\mathbf{R}$, then there exists a function $h \in \mathcal{H}(\mathbf{R})$ such that*

$$\int_0^\infty \frac{1-\cos v\eta}{v\nu}\, d\mu(\eta) = \widehat{h}(v), \quad v \in \mathbf{R}.$$

4. Proof of Theorem 1

Without loss of generality, we may assume that the finite Borel measure μ associated with λ by (1.3) is positive on $\mathbf{R}^2_+$. We claim that the function f defined in (2.4) belongs to $L(\mathbf{R}^2)$. In fact, $f(x,y) \geqq 0$ and by Fubini's theorem

$$(4.1) \qquad \int\int_{\mathbf{R}^2_+} f(x,y)\,dx\,dy = \int_{\to 0}^\infty \int_{\to 0}^\infty \frac{d\mu(\xi,\eta)}{\xi\eta} \int_0^\xi \int_0^\eta dx\,dy =$$

$$= \int_{\to 0}^\infty \int_{\to 0}^\infty d\mu(\xi,\eta) \leqq \mu(\mathbf{R}^2_+).$$

By the evenness of $f(x,y)$ in x and oddness in y, hence we get $f \in L(\mathbf{R}^2)$.

In order to reveal the connection between $\widehat{f}$ and $\sigma\lambda$, we start with the representation (1.2). Fix $u, v, \delta, \varepsilon > 0$. By (2.4) and Fubini's theorem,

$$(4.2) \qquad \int_\delta^\infty \int_\varepsilon^\infty f(x,y) \cos ux \sin vy\,dx\,dy =$$

$$= \int_\delta^\infty \int_\varepsilon^\infty \frac{d\mu(\xi,\eta)}{\xi\eta} \int_\delta^\xi \cos ux\,dx \int_\varepsilon^\eta \sin vy\,dy =$$

$$= -\int_\delta^\infty \int_\varepsilon^\infty \frac{(\sin u\xi - \sin u\delta)(\cos v\eta - \cos v\varepsilon)}{u\xi v\eta}\, d\mu(\xi,\eta).$$

By Lebesgue's dominated convergence theorem, hence it follows that

$$\widehat{f}(u,v) = 4i \int_{\to 0}^\infty \int_{\to 0}^\infty \frac{\sin u\xi}{u\xi} \frac{1-\cos v\eta}{v\eta}\, d\mu(\xi,\eta).$$

Now, by Lemma 2 and (1.4),

$$(4.3) \qquad \widehat{f}(u,v) = 4i \left\{ \int\int_{\mathbf{R}^2_+} \frac{\sin u\xi}{u\xi} \frac{1-\cos v\eta}{v\eta}\, d\mu(\xi,\eta) - \int_{\to 0}^\infty \frac{1-\cos v\eta}{v\eta}\, d\mu(0,\eta) \right\}.$$

On the other hand, by (1.5) and (2.1),

$$\sigma\lambda(u,v) = \frac{-4i}{uv}\int_0^u dx \int_0^v dy \int\int_{\mathbf{R}^2_+} \cos x\xi \sin y\eta \, d\mu(\xi,\eta) = \tag{4.4}$$

$$= \frac{-4i}{uv}\int\int_{\mathbf{R}^2_+} d\mu(\xi,\eta) \int_0^u \cos x\xi \, dx \int_0^v \sin y\eta \, dy =$$

$$= \frac{-4i}{uv}\int\int_{\mathbf{R}^2_+} \frac{\sin u\xi}{\xi} \frac{\cos v\eta - 1}{\eta} \, d\mu(\xi,\eta).$$

Comparing (4.3) and (4.4) yields

$$\sigma\lambda(u,v) = \widehat{f}(u,v) + \int_{\to 0}^{\infty} \frac{1-\cos v\eta}{v\eta} \, d\mu(0,\eta). \tag{4.5}$$

This and Lemma 4 give that $\sigma\lambda(u,v)$ is a multiplier for $L(\mathbf{R}^2)$, and statement (i) is proved.

It is plain that if condition (2.2) is satisfied, then we have

$$\sigma\lambda(u,v) = \widehat{f}(u,v), \quad (u,v) \in \mathbf{R}^2, \tag{4.6}$$

which is the sufficiency part in statement (ii).

Fix v and let $u \to \infty$ in (4.5). As a result, we get

$$0 = \int_{\to 0}^{\infty} \frac{1-\cos v\eta}{v\eta} \, d\mu(0,\eta).$$

Since this is true for all $v \in \mathbf{R}_+$, we conclude that condition (2.2) must be satisfied. (Remember that μ is positive.) This completes the proof of the necessity part in statement (ii).

5. Proof of Theorem 2

The proof is a combination of certain parts from the proofs of [3, Theorem 1] and [4, Theorem 2]. If the reader feels that the presentation below is too concise, we suggest to consult the corresponding parts of [3,4].

Again, we may assume that the finite Borel measure μ associated with λ by (1.3) is positive. Then the function f defined by (2.3) is nonnegative and nonincreasing in each variable on $\mathbf{R}^2_+$.

PROOF OF (2.4). By (1.6),

(5.1)
$$\pi H_1 f(x,y) = \left\{ \int_{\to 0}^{x/2} + \int_{x/2}^{\infty} \right\} \frac{f(x-u,y) - f(x+u,y)}{u}\, du := I_1 + I_2, \text{ say.}$$

By (2.3),

$$I_1 = \int_{\to 0}^{x/2} \frac{du}{u} \int_{x-u}^{x+u} \int_y^\infty \frac{d\mu(\xi,\eta)}{\xi\eta},$$

whence, by Fubini's theorem,

(5.2)
$$\int\int_{\mathbf{R}^2_+} |I_1(x,y)|\, dx\, dy =$$

$$= \int_{\to 0}^\infty \frac{du}{u} \int_{2u}^\infty dx \int_0^\infty dy \int_{x-u}^{x+u} \int_y^\infty \frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= \int_{\to 0}^\infty \frac{du}{u} \int_u^\infty \int_0^\infty \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{\max(\xi-u,2u)}^{\xi+u} dx \int_0^\eta dy \leqq$$

$$\leqq 2 \int_{\to 0}^\infty du \int_u^\infty \int_0^\infty \frac{d\mu(\xi,\eta)}{\xi} = 2 \int_{\to 0}^\infty \int_0^\infty \frac{d\mu(\xi,\eta)}{\xi} \int_0^\xi du \leqq 2\mu(\mathbf{R}^2_+).$$

Keeping (2.3) in mind, we decompose I_2 as follows

(5.3)
$$I_2 =$$

$$= -\int_{x/2}^\infty \frac{f(x+u,y)}{u}\, du + \int_{x/2}^x \frac{f(x-u,y)}{u}\, du + \int_x^\infty \frac{f(u-x,y)}{u}\, du =:$$

$$=: \alpha_1 + \int_{\to 0}^\infty \int_y^\infty \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{\max(x-\xi,x/2)}^x \frac{du}{u} + \int_{\to 0}^\infty \int_y^\infty \frac{d\mu(\xi,\eta)}{\xi\eta} \int_x^{x+\xi} \frac{du}{u} =$$

$$= \alpha_1 + \int_{\to 0}^{x/2} \int_y^\infty \ln\left(\frac{x+\xi}{x-\xi}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} +$$

$$+ \int_{x/2}^\infty \int_y^\infty \ln\left(\frac{2(x+\xi)}{x}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} =: \alpha_1 + \alpha_2 + \alpha_3, \quad \text{say.}$$

By (4.1),

(5.4)
$$\int\int_{\mathbf{R}^2_+} |\alpha_1(x,y)|\, dx\, dy = \int\int_{\mathbf{R}^2_+} dx\, dy \int_{3x/2}^\infty \frac{f(u,y)}{u-x}\, du \leqq (\ln 3)\mu(\mathbf{R}^2_+)$$

and, by the substitutions $x := \xi t$ and $s := 1/t$,

$$(5.5)\qquad \int\int_{\mathbf{R}^2_+} |\alpha_3(x,y)|\,dx\,dy =$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta}\int_0^{2\xi}\ln\left(\frac{2(x+\xi)}{x}\right)dx\int_0^{\eta}dy =$$

$$= \int\int_{\mathbf{R}^2_+} d\mu(\xi,\eta)\int_0^2 \ln\left(\frac{2(t+1)}{t}\right)dt \leqq \mu(\mathbf{R}^2_+)\int_{1/2}^{\infty}\frac{\ln\big(2(s+1)\big)}{s^2}\,ds.$$

Collecting (5.1)–(5.5) yields

$$(5.6)\qquad \int\int_{\mathbf{R}^2_+} |\pi H_1 f(x,y) - \alpha_2(x,y)|\,dx\,dy \leqq C_1\mu(\mathbf{R}^2_+).$$

Next, we rely on the estimates

$$(5.7)\qquad 0 \leqq \frac{2}{x-\xi} - \frac{1}{\xi}\ln\left(\frac{x+\xi}{x-\xi}\right) \leqq \frac{4\xi}{(x-\xi)^2},\quad 0 \leqq \xi \leqq x/2.$$

Setting

$$(5.8)\qquad \alpha_4 := \frac{2}{x}\int_0^{x/2}\int_y^{\infty}\frac{d\mu(\xi,\eta)}{\eta},$$

it follows easily that

$$|\alpha_2 - \alpha_4| \leqq \int_0^{x/2}\int_y^{\infty}\left\{\frac{4\xi}{(x-\xi)^2} + \frac{2\xi}{x(x-\xi)}\right\}\frac{d\mu(\xi,\eta)}{\eta}.$$

By Fubini's theorem,

$$(5.9)\qquad \int\int_{\mathbf{R}^2_+} |\alpha_2(x,y) - \alpha_4(x,y)|\,dx\,dy \leqq$$

$$\leqq \int\int_{\mathbf{R}^2_+}\frac{d\mu(\xi,\eta)}{\eta}\int_{2\xi}^{\infty}\left\{\frac{4\xi}{(x-\xi)^2} + \frac{2\xi}{x(x-\xi)}\right\}dx\int_0^{\eta}dy \leqq$$

$$\leqq \mu(\mathbf{R}^2_+)\left\{4 + 2\int_2^{\infty}\frac{dt}{t(t-1)}\right\},\quad x := \xi t.$$

Combining (5.6), and (5.9) results in (2.4).

PROOF OF (2.5). By (1.7),

$$\pi H_2 f(x,y) = \tag{5.10}$$

$$= \left\{ \int_{\to 0}^{y/2} + \int_{y/2}^{\infty} \right\} \frac{f(x,y-v) - f(x,y+v)}{v}\, dv := I_3 + I_4, \quad \text{say.}$$

By (2.3),

$$I_3 = \int_{\to 0}^{y/2} \frac{dv}{v} \int_x^{\infty} \int_{y-v}^{y+v} \frac{d\mu(\xi,\eta)}{\xi\eta},$$

whence, similarly to (5.2), we get

$$\int\int_{\mathbf{R}_+^2} |I_3(x,y)|\, dx\, dy \leqq 2\mu(\mathbf{R}_+^2). \tag{5.11}$$

We proceed to decompose I_4 as follows (see (2.3))

$$I_4 = \tag{5.12}$$

$$= -\int_{y/2}^{\infty} \frac{f(x,y+v)}{v}\, dv + \int_{y/2}^{y} \frac{f(x,y-v)}{v}\, dv - \int_y^{\infty} \frac{f(x,v-y)}{v}\, dv =:$$

$$=: \alpha_5 + \int_x^{\infty} \int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{\max(y-\eta, y/2)}^{y} \frac{dv}{v} - \int_x^{\infty} \int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_y^{y+\eta} \frac{dv}{v} =$$

$$= \alpha_5 + \int_x^{\infty} \int_{\to 0}^{y/2} \ln\left(\frac{y}{y-\eta}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} + (\ln 2) \int_x^{\infty} \int_{y/2}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} -$$

$$- \int_x^{\infty} \int_{\to 0}^{\infty} \ln\left(\frac{y+n}{y}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= \alpha_5 + \int_x^{\infty} \int_{\to 0}^{y/2} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} -$$

$$- \int_x^{\infty} \int_{y/2}^{\infty} \ln\left(\frac{y+\eta}{2y}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} =:$$

$$=: \alpha_5 + \alpha_6 + \alpha_7, \quad \text{say.}$$

Similarly to (5.4) and (5.5), we derive that

$$\int\int_{\mathbf{R}^2_+} |\alpha_5(x,y)|\,dx\,dy \leqq (\ln 3)\mu(\mathbf{R}^2_+) \tag{5.13}$$

and

$$\int\int_{\mathbf{R}^2_+} |\alpha_7(x,y)|\,dx\,dy \leqq \mu(\mathbf{R}^2_+)\int_{1/2}^{\infty} \frac{\ln\big((s+1)/2\big)}{s^2}\,ds. \tag{5.14}$$

Finally, it is not difficult to check that

$$\int\int_{\mathbf{R}^2_+} |\alpha_6(x,y)|\,dx\,dy = \tag{5.15}$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta}\int_0^{\xi} dx \int_{2\eta}^{\infty} \ln\left(\frac{y^2}{y^2-\eta^2}\right) dy =$$

$$= \int\int_{\mathbf{R}^2_+} d\mu(\xi,\eta)\int_2^{\infty} \ln\left(\frac{t^2}{t^2-1}\right) dt \leqq \mu(\mathbf{R}^2_+)\int_2^{\infty}\frac{dt}{t^2-1}.$$

Combining (5.10)–(5.15) gives (2.5).

Proof of (2.6). By (1.8),

$$\pi^2 H_1 H_2 f(x,y) = \tag{5.16}$$

$$= \left\{\int_{\to 0}^{x/2}\int_{\to 0}^{y/2} + \int_{\to 0}^{x/2}\int_{y/2}^{\infty} + \int_{x/2}^{\infty}\int_{\to 0}^{y/2} + \int_{x/2}^{\infty}\int_{y/2}^{\infty}\right\}$$

$$\big\{f(x+u,y+v) - f(x+u,y-v) - f(x-u,y+v) + f(x-u,y-v)\big\}\frac{du}{u}\frac{dv}{v} =:$$

$$=: I_{00} + I_{01} + I_{10} + I_{11}, \quad \text{say}.$$

Step (i). By (2.3),

$$I_{00} = \int_{\to 0}^{x/2}\frac{du}{u}\int_{\to 0}^{y/2}\frac{dv}{v}\int_{x-u}^{x+u}\int_{y-v}^{y+v}\frac{d\mu(\xi,\eta)}{\xi\eta},$$

whence (cf. (5.2))

$$\int\int_{\mathbf{R}^2_+} |I_{00}(x,y)|\,dx\,dy = \tag{5.17}$$

$$= \int_{\to 0}^{\infty} \frac{du}{u} \int_{\to 0}^{\infty} \frac{dv}{v} \int_{2u}^{\infty} dx \int_{2v}^{\infty} dy \int_{x-u}^{x+y} \int_{y-u}^{y+u} \frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= \int_{\to 0}^{\infty} \frac{du}{u} \int_{\to 0}^{\infty} \frac{dv}{v} \int_{u}^{\infty} \int_{v}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{\max(\xi-u,2u)}^{\xi+u} dx \int_{\max(\eta-v,2v)}^{\eta+v} dy =$$

$$= 4 \int_{\to 0}^{\infty} \int_{\to 0}^{\infty} du\, dv \int_{u}^{\infty} \int_{v}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \leqq 4\mu(\mathbf{R}_+^2).$$

Step (ii). We decompose I_{01} as follows (cf. (5.12))

$$(5.18) \qquad I_{01} = \int_{\to 0}^{x/2} \int_{y/2}^{\infty} \{ f(x+u, y+u) - f(x-u, y+v) \} \frac{du}{u} \frac{dv}{v} -$$

$$- \left\{ \int_{\to 0}^{x/2} \int_{y/2}^{y} + \int_{\to 0}^{x/2} \int_{y}^{\infty} \right\} \{ f(x+u, y-v) - f(x-u, y-v) \} \frac{du}{u} \frac{dv}{v} =:$$

$$=: \beta_1 + \int_{\to 0}^{x/2} \frac{du}{u} \int_{y/2}^{y} \frac{dv}{v} \int_{x-u}^{x+u} \int_{y-v}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} -$$

$$- \int_{\to 0}^{x/2} \frac{du}{u} \int_{y}^{\infty} \frac{dv}{v} \int_{x-u}^{x+u} \int_{v-y}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= \beta_1 + \int_{x/2}^{3x/2} \int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{|x-\xi|}^{x/2} \frac{du}{u} \int_{\max(y-\eta, y/2)}^{y} \frac{dv}{v} -$$

$$- \int_{x/2}^{3x/2} \int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{|x-\xi|}^{x/2} \frac{du}{u} \int_{y}^{y+\eta} \frac{dv}{v} =$$

$$= \beta_1 + \int_{x/2}^{3x/2} \int_{\to 0}^{\infty} \ln\left(\frac{x}{2|x-\xi|}\right) \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} -$$

$$- \int_{x/2}^{3x/2} \int_{y/2}^{\infty} \ln\left(\frac{x}{2|x-\xi|}\right) \ln\left(\frac{y+\eta}{2y}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} =: \beta_1 + \beta_2 + \beta_3, \quad \text{say.}$$

It is easy to see that

$$\beta_1 = - \int_{\to 0}^{x/2} \frac{du}{u} \int_{y/2}^{\infty} \frac{dv}{v} \int_{x-u}^{x+u} \int_{y+v}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= - \int_{x/2}^{3x/2} \int_{3y/2}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{|x-\xi|}^{x/2} \frac{du}{u} \int_{y/2}^{\eta-y} \frac{dv}{v} =$$

$$= -\int_{x/2}^{3x/2}\int_{3y/2}^{\infty}\ln\left(\frac{x}{2|x-\xi|}\right)\ln\left(\frac{2(\eta-y)}{y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}.$$

Hence, by integration by substitutions,

$$\int\int_{\mathbf{R}^2_+}|\beta_1(x,y)|\,dx\,dy = \tag{5.19}$$

$$= \int\int_{\mathbf{R}^2_+}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{2\xi/3}^{2\xi}\ln\left(\frac{x}{2|x-\xi|}\right)dx\int_0^{2\eta/3}\ln\left(\frac{2(\eta-y)}{y}\right)dy =$$

$$= \int\int_{\mathbf{R}^2_+}d\mu(\xi,\eta)\int_{2/3}^{2}\ln\left(\frac{s}{2|s-1|}\right)ds\int_0^{2/3}\ln\left(\frac{2(1-t)}{t}\right)dt \leqq C_2\mu(\mathbf{R}^2_+).$$

Similarly to (5.14) and (5.15), we obtain

$$\int\int_{\mathbf{R}^2_+}|\beta_2(x,y)|\,dx\,dy = \tag{5.20}$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{2\xi/3}^{2\xi}\ln\left(\frac{x}{2|x-\xi|}\right)dx\int_{2\eta}^{\infty}\ln\left(\frac{y^2}{y^2-\eta^2}\right)dy =$$

$$= \int\int_{\mathbf{R}^2_+}d\mu(\xi,\eta)\int_{2/3}^{2}\ln\left(\frac{s}{2|s-1|}\right)ds\int_2^{\infty}\ln\left(\frac{t^2}{t^2-1}\right)dt \leqq C_3\mu(\mathbf{R}^2_+)$$

and

$$\int\int_{\mathbf{R}^2_+}|\beta_3(x,y)|\,dx\,dy = \tag{5.21}$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{2\xi/3}^{2\xi}\ln\left(\frac{x}{2|x-\xi|}\right)dx\int_0^{2\eta}\ln\left(\frac{y+\eta}{y}\right)dy =$$

$$= \int\int_{\mathbf{R}^2_+}d\mu(\xi,\eta)\int_{2/3}^{2}\ln\left(\frac{s}{2|s-1|}\right)ds\int_0^{2}\ln\left(\frac{t+1}{t}\right)dt \leqq C_4\mu(\mathbf{R}^2_+).$$

Combining (5.18)–(5.21) yields

$$\int\int_{\mathbf{R}^2_+}|I_{01}(x,y)|\,dx\,dy \leqq C_5\mu(\mathbf{R}^2_+). \tag{5.22}$$

Step (iii). We decompose I_{10} as follows (cf. (5.18))

$$(5.23)\qquad I_{10} = \int_{x/2}^{\infty}\int_{\to 0}^{y/2} \{ f(x+u, y+v) - f(x+u, y-v)\} \frac{du}{u}\frac{dv}{v} -$$

$$-\left\{\int_{x/2}^{x}\int_{\to 0}^{y/2} + \int_{x}^{\infty}\int_{\to 0}^{y/2}\right\} \{ f(x-u, y+v) - f(x-u, y-v)\} \frac{du}{u}\frac{dv}{v} =:$$

$$=: \beta_4 + \int_{x/2}^{x} \frac{du}{u}\int_{\to 0}^{y/2}\frac{dv}{v}\int_{x-u}^{\infty}\int_{y-v}^{y+v}\frac{d\mu(\xi,\eta)}{\xi\eta} +$$

$$+\int_{x}^{\infty} \frac{du}{u}\int_{\to 0}^{y/2}\frac{dv}{v}\int_{u-x}^{\infty}\int_{y-v}^{y+v}\frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= \beta_4 + \int_{\to 0}^{\infty}\int_{y/2}^{3y/2}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{\max(x-\xi, x/2)}^{x}\frac{du}{u}\int_{|y-\eta|}^{y/2}\frac{dv}{v} +$$

$$+\int_{\to 0}^{\infty}\int_{y/2}^{3y/2}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{x}^{x+\xi}\frac{du}{u}\int_{|y-\eta|}^{y/2}\frac{dv}{v} =$$

$$= \beta_4 + \int_{\to 0}^{x/2}\int_{y/2}^{3y/2}\ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y}{2|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} +$$

$$+\int_{x/2}^{\infty}\int_{y/2}^{3y/2}\ln\left(\frac{2(x+\xi)}{x}\right)\ln\left(\frac{y}{2|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} =: \beta_4+\beta_5+\beta_6, \text{ say.}$$

It is easy to see that

$$\beta_4 = -\int_{x/2}^{\infty}\frac{du}{u}\int_{\to 0}^{y/2}\frac{dv}{v}\int_{x+u}^{\infty}\int_{y-v}^{y+v}\frac{d\mu(\xi,\eta)}{\xi\eta}.$$

Similarly to (5.19) and (5.21), we have

$$(5.24)\qquad \int\int_{\mathbf{R}^2_+} |\beta_4(x,y)|\, dx\, dy \leqq C_6\mu(\mathbf{R}^2_+),$$

$$(5.25)\qquad \int\int_{\mathbf{R}^2_+} |\beta_6(x,y)|\, dx\, dy \leqq C_7\mu(\mathbf{R}^2_+),$$

while the quantity

$$(5.26)\qquad \beta_5 := \int_{\to 0}^{x/2}\int_{y/2}^{3y/2}\ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y}{2|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}$$

will be estimated later on.

Step (iv). By definition,

$$(5.27)\qquad I_{11} =$$

$$= \int_{x/2}^{\infty}\int_{y/2}^{\infty}\{f(x+u,y+v)-f(x+u,y-v)-f(x-u,y+v)+$$

$$+f(x-y,y-v)\}\,\frac{du}{u}\,\frac{dv}{v} =: \gamma_1+\gamma_2+\gamma_3+\gamma_4,\}, \quad \text{say.}$$

First, similarly to (5.4), we have

$$(5.28)\qquad \int\int_{\mathbf{R}^2_+} |\gamma_1(x,y)|\,dx\,dy \leqq (\ln 3)^2\mu(\mathbf{R}^2_+).$$

Second, we decompose as follows (cf. (5.12))

$$(5.29)\qquad \gamma_2 = -\left\{\int_{x/2}^{\infty}\int_{y/2}^{y}+\int_{x/2}^{\infty}\int_{y}^{\infty}\right\} f(x+u,y-v)\,\frac{du}{u}\,\frac{dv}{v} =$$

$$= -\int_{x/2}^{\infty}\frac{du}{u}\int_{y/2}^{y}\frac{dv}{v}\int_{x+u}^{\infty}\int_{y-v}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}+$$

$$+\int_{x/2}^{\infty}\frac{du}{u}\int_{y}^{\infty}\frac{dv}{v}\int_{x+u}^{\infty}\int_{v-y}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= -\int_{3x/2}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{x/2}^{\xi-x}\frac{du}{u}\int_{\max(y-\eta,y/2)}^{y}\frac{dv}{v}+$$

$$+\int_{3x/2}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{x/2}^{\xi-x}\frac{du}{u}\int_{y}^{y+\eta}\frac{dv}{v} =$$

$$= -\int_{3x/2}^{\infty}\int_{\to 0}^{y/2}\ln\left(\frac{2(\xi-x)}{x}\right)\ln\left(\frac{y^2}{y^2-\eta^2}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}+$$

$$+\int_{3x/2}^{\infty}\int_{y/2}^{\infty}\ln\left(\frac{2(\xi-x)}{x}\right)\ln\left(\frac{y+\eta}{2y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} =: \gamma_{21}+\gamma_{22}, \quad \text{say.}$$

Similarly to (5.14) and (5.20), we derive that

$$(5.30)\qquad \int\int_{\mathbf{R}^2_+} |\gamma_{21}(x,y)|\,dx\,dy =$$

$$= \int_{\to 0}^{\infty} \int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_0^{2\xi/3} \ln\left(\frac{2(\xi-x)}{x}\right) dx \int_{2\eta}^{\infty} \ln\left(\frac{y^2}{y^2-\eta^2}\right) dy =$$

$$= \iint_{\mathbf{R}^2_+} d\mu(\xi,\eta) \int_0^{2/3} \ln\left(\frac{2(1-s)}{s}\right) ds \int_2^{\infty} \ln\left(\frac{t^2}{t^2-1}\right) dt \leqq C_8 \mu(\mathbf{R}^2_+)$$

and

$$\iint_{\mathbf{R}^2_+} |\gamma_{22}(x,y)|\, dx\, dy = \tag{5.31}$$

$$= \int_{\to 0}^{\infty} \int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_0^{2\xi/3} \ln\left(\frac{2(\xi-x)}{x}\right) dx \int_0^{2\eta} \ln\left(\frac{y+\eta}{2y}\right) dy =$$

$$= \iint_{\mathbf{R}^2_+} d\mu(\xi,\eta) \int_0^{2/3} \ln\left(\frac{2(1-s)}{s}\right) ds \int_0^{2} \ln\left(\frac{t+1}{2t}\right) dt \leqq C_9 \mu(\mathbf{R}^2_+).$$

Collecting (5.29)–(5.31) gives

$$\iint_{\mathbf{R}^2_+} |\gamma_2(x,y)|\, dx\, dy \leqq C_{10} \mu(\mathbf{R}^2_+). \tag{5.32}$$

Third, we decompose as follows (cf. (5.3))

$$\gamma_3 = -\left\{ \int_{x/2}^{x} \int_{y/2}^{\infty} + \int_x^{\infty} \int_{y/2}^{\infty} \right\} f(x-u, y+v) \frac{du}{u} \frac{dv}{v} = \tag{5.33}$$

$$= -\int_{x/2}^{x} \frac{du}{u} \int_{y/2}^{\infty} \frac{dv}{v} \int_{x-u}^{\infty} \int_{y+v}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} -$$

$$- \int_x^{\infty} \frac{du}{u} \int_{y/2}^{\infty} \frac{dv}{v} \int_{u-x}^{\infty} \int_{y+v}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} =$$

$$= -\int_{\to 0}^{\infty} \int_{3y/2}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{\max(x-\xi, x/2)}^{x} \frac{du}{u} \int_{y/2}^{\eta-y} \frac{dv}{v} -$$

$$- \int_{\to 0}^{\infty} \int_{3y/2}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_x^{x+\xi} \frac{du}{u} \int_{y/2}^{\eta-y} \frac{dv}{v} =$$

$$= -\int_{\to 0}^{x/2} \int_{3y/2}^{\infty} \ln\left(\frac{x+\xi}{x-\xi}\right) \ln\left(\frac{2(\eta-y)}{y}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} -$$

$$- \int_{x/2}^{\infty} \int_{3y/2}^{\infty} \ln\left(\frac{2(x+\xi)}{x}\right) \ln\left(\frac{2(\eta-y)}{y}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} =: \gamma_{31} + \gamma_{32}, \quad \text{say.}$$

Similarly to (5.5) and (5.30), we get

$$\int\int_{\mathbf{R}^2_+} \left|\gamma_{32}(x,y)\right| \, dx\, dy = \tag{5.34}$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_0^{2\xi} \ln\left(\frac{2(x+\xi)}{x}\right) dx \int_0^{2\eta/3} \ln\left(\frac{2(\eta-y)}{y}\right) dy =$$

$$= \int\int_{\mathbf{R}^2_+} d\mu(\xi,\eta) \int_0^2 \ln\left(\frac{2(s+1)}{s}\right) ds \int_0^{2/3} \ln\left(\frac{2(t-1)}{t}\right) dt \leqq C_{11}\mu(\mathbf{R}^2_+),$$

while the quantity

$$\gamma_{31} := -\int_{\to 0}^{x/2}\int_{3y/2}^{\infty} \ln\left(\frac{x+\xi}{x-\xi}\right) \ln\left(\frac{2(\eta-y)}{y}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} \tag{5.35}$$

will be estimated later on.

Fourth, we decompose as follows

$$\gamma_4 = \left\{ \int_{x/2}^{x}\int_{y/2}^{y} + \int_{x/2}^{x}\int_{y}^{\infty} + \int_{x}^{\infty}\int_{y/2}^{y} + \int_{x}^{\infty}\int_{y}^{\infty} \right\} \tag{5.36}$$

$$f(x-y, y-v)\frac{du}{u}\frac{dv}{v} =: \gamma_{41}+\gamma_{42}+\gamma_{43}+\gamma_{44},$$

$$\gamma_{41} = \int_{x/2}^{x} \frac{du}{u} \int_{y/2}^{y} \frac{dv}{v} \int_{x-u}^{\infty}\int_{y-v}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} = \tag{5.37}$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{\max(x-\xi,x/2)}^{x} \frac{du}{u} \int_{\max(y-\eta,y/2)}^{y} \frac{dv}{v} =$$

$$= \int_{\to 0}^{x/2}\int_{\to 0}^{y/2} \ln\left(\frac{x}{x-\xi}\right) \ln\left(\frac{y}{y-\eta}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} +$$

$$+(\ln 2)\int_{\to 0}^{x/2}\int_{y/2}^{\infty} \ln\left(\frac{x}{x-\xi}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} +$$

$$+(\ln 2)\int_{x/2}^{\infty}\int_{\to 0}^{y/2} \ln\left(\frac{y}{y-\eta}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} +$$

$$+(\ln 2)^2\int_{x/2}^{\infty}\int_{y/2}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} =: \gamma_{411}+\gamma_{412}+\gamma_{413}+\gamma_{414},$$

$$(5.38)\qquad \gamma_{42}=-\int_{x/2}^{x}\frac{du}{u}\int_{y}^{\infty}\frac{dv}{v}\int_{x-u}^{\infty}\int_{v-y}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}=$$

$$=-\int_{\to 0}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{\max(x-\xi,x/2)}^{x}\frac{du}{u}\int_{y}^{y+\eta}\frac{dv}{v}=$$

$$=-\int_{\to 0}^{x/2}\int_{\to 0}^{y/2}\ln\left(\frac{x}{x-\xi}\right)\ln\left(\frac{y+\eta}{y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}-$$

$$=-\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\ln\left(\frac{x}{x-\xi}\right)\ln\left(\frac{y+\eta}{y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}-$$

$$-(\ln 2)\int_{x/2}^{\infty}\int_{\to 0}^{y/2}\ln\left(\frac{y+\eta}{y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}-$$

$$-(\ln 2)\int_{x/2}^{\infty}\int_{y/2}^{\infty}\ln\left(\frac{y+\eta}{y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}=:\gamma_{421}+\gamma_{422}+\gamma_{423}+\gamma_{424},$$

the symmetric counterpart of (5.38):

$$(5.39)\qquad \gamma_{43}=\int_{x}^{\infty}\frac{du}{u}\int_{y/2}^{y}\frac{dv}{v}\int_{u-x}^{\infty}\int_{u-x}^{\infty}\int_{y-v}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}=$$

$$=\int_{\to 0}^{x/2}\int_{\to 0}^{y/2}\ln\left(\frac{x+\xi}{x}\right)\ln\left(\frac{y}{y-\eta}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}+$$

$$+(\ln 2)\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\ln\left(\frac{x+\xi}{x}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}+$$

$$+\int_{x/2}^{\infty}\int_{\to 0}^{y/2}\ln\left(\frac{x+\xi}{x}\right)\ln\left(\frac{y}{y-\eta}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}+$$

$$+(\ln 2)\int_{x/2}^{\infty}\int_{y/2}^{\infty}\ln\left(\frac{x+\xi}{x}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}=:\gamma_{431}+\gamma_{432}+\gamma_{433}+\gamma_{434},$$

and

$$(5.40)\qquad \gamma_{44}=-\int_{x}^{\infty}\frac{du}{u}\int_{y}^{\infty}\frac{dv}{v}\int_{u-x}^{\infty}\int_{v-y}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}=$$

$$=-\int_{\to 0}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_{x}^{x+\xi}\frac{du}{u}\int_{y}^{y+\eta}\frac{dv}{v}=$$

$$= -\left\{\int_0^{x/2}\int_{\to 0}^{y/2} + \int_{\to 0}^{x/2}\int_{y/2}^{\infty} + \int_{x/2}^{\infty}\int_{\to 0}^{y/2} + \int_{x/2}^{\infty}\int_{y/2}^{\infty}\right\}$$

$$\ln\left(\frac{x+\xi}{x}\right)\ln\left(\frac{y+\eta}{y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} =: \gamma_{441}+\gamma_{442}+\gamma_{443}+\gamma_{444}, \quad \text{say.}$$

It is easy to see that

$$\gamma_{414}+\gamma_{424}+\gamma_{434}+\gamma_{444} = -\int_{x/2}^{\infty}\int_{y/2}^{\infty}\ln\left(\frac{2(x+\xi)}{x}\right)\ln\left(\frac{y+\eta}{2y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta},$$

whence

$$(5.41)\quad \iint_{\mathbf{R}^2_+}\left|\gamma_{414}(x,y)+\gamma_{424}(x,y)+\gamma_{434}(x,y)+\gamma_{444}(x,y)\right|dx\,dy =$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_0^{2\xi}\ln\left(\frac{2(x+\xi)}{x}\right)dx\int_0^{2\eta}\ln\left(\frac{y+\eta}{2y}\right)dy =$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty}d\mu(\xi,\eta)\int_0^2\ln\left(\frac{2(s+1)}{s}\right)ds\int_0^2\ln\left(\frac{t+1}{2t}\right)dt \leqq C_{12}\mu(\mathbf{R}^2_+).$$

Analogously,

$$\gamma_{413}+\gamma_{423}+\gamma_{433}+\gamma_{443} = \int_{x/2}^{\infty}\int_{\to 0}^{y/2}\ln\left(\frac{2(x+\xi)}{x}\right)\ln\left(\frac{y^2}{y^2-\eta^2}\right)\frac{d\mu(\xi,\eta)}{\xi\eta},$$

whence

$$(5.42)\quad \iint_{\mathbf{R}^2_+}\left|\gamma_{413}(x,y)+\gamma_{423}(x,y)+\gamma_{433}(x,y)+\gamma_{434}(x,y)\right|dx\,dy =$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\xi\eta}\int_0^{2\xi}\ln\left(\frac{2(x+\xi)}{x}\right)dx\int_{2\eta}^{\infty}\ln\left(\frac{y^2}{y^2-\eta^2}\right)dy =$$

$$= \iint_{\mathbf{R}^2_+}d\mu(\xi,\eta)\int_0^2\ln\left(\frac{2(s+1)}{s}\right)ds\int_2^{\infty}\ln\left(\frac{t^2}{t^2-1}\right)dt \leqq C_{13}\mu(\mathbf{R}^2_+);$$

and

$$\gamma_{422}+\gamma_{442} = -\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\ln\left(\frac{x^2}{x^2-\xi^2}\right)\ln\left(\frac{y+\eta}{y}\right)\frac{d\mu(\xi,\eta)}{\xi\eta},$$

whence

$$\int\int_{\mathbf{R}^2_+} \left|\gamma_{422}(x,y)+\gamma_{442}(x,y)\right| \, dx\, dy = \tag{5.43}$$

$$= \int_{\to 0}^{\infty}\int_{\to 0}^{\infty} \frac{d\mu(\xi,\eta)}{\xi\eta} \int_{2\xi}^{\infty} \ln\left(\frac{x^2}{x^2-\xi^2}\right) dx \int_0^{2\eta} \ln\left(\frac{y+\eta}{y}\right) dy =$$

$$= \int\int_{\mathbf{R}^2_+} d\mu(\xi,\eta) \int_2^{\infty} \ln\left(\frac{s^2}{s^2-1}\right) ds \int_0^2 \ln\left(\frac{t+1}{t}\right) dt \leqq C_{14}(\mathbf{R}^2_+).$$

Finally, the quantities

$$\gamma_{412}+\gamma_{432} = (\ln 2)\int_{\to 0}^{x/2}\int_{y/2}^{\infty} \ln\left(\frac{x+\xi}{x-\xi}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} \tag{5.44}$$

and

$$\gamma_{411}+\gamma_{421}+\gamma_{431}+\gamma_{441} = \int_{\to 0}^{x/2}\int_{\to 0}^{y/2} \ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y^2}{y^2-\eta^2}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} \tag{5.45}$$

will be estimated in the next step.

Step (v). Combining (5.26), (5.35), and (5.44) yields

$$\beta_5+\gamma_{31}+\gamma_{412}+\gamma_{432} = \int_{\to 0}^{x/2}\int_{y/2}^{\infty} \ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}. \tag{5.46}$$

Taking into account (5.16), (5.17), (5.22)–(5.28), (5.32)–(5.46) results into the following:

$$\int\int_{\mathbf{R}^2_+} \left|\pi^2 H_1 H_2 f(x,y) - \int_{\to 0}^{x/2}\int_{y/2}^{\infty} \ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y}{|y-\xi|}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} - \right. \tag{5.47}$$

$$\left. - \int_{\to 0}^{x/2}\int_{\to 0}^{y/2} \ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y^2}{y^2-\eta^2}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}\right| dx\, dy \leqq C_{15}\mu(\mathbf{R}^2_+).$$

We rely again on estimate (5.7) and similarly to (5.8) and (5.9), to obtain the following:

$$\int\int_{\mathbf{R}^2_+} \left|\int_{\to 0}^{x/2}\int_{y/2}^{\infty} \ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\xi\eta} - \right.$$

$$-\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\frac{2}{x-\xi}\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\eta}\Bigg|\,dx\,dy \leqq$$

$$\leqq \int\int_{\mathbf{R}_+^2} dx\,dy\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\frac{4\xi}{(x-\xi)^2}\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\eta}=$$

$$=\int\int_{\mathbf{R}_+^2}\frac{d\mu(\xi,\eta)}{\eta}\int_{2\xi}^{\infty}\frac{4\xi}{(x-\xi)^2}\,dx\int_0^{2\eta}\ln\left(\frac{y}{|y-\eta|}\right)dy=$$

$$=\int\int_{\mathbf{R}_+^2} d\mu(\xi,\eta)\int_2^{\infty}\frac{4}{(s-1)^2}\,ds\int_0^2\ln\left(\frac{t}{|t-1|}\right)dt\leqq C_{16}\mu(\mathbf{R}_+^2)$$

and

$$\int\int_{\mathbf{R}_+^2}\Bigg|\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\frac{1}{x-\xi}\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\eta}-$$

$$-\frac{1}{x}\int_0^{x/2}\int_{y/2}^{\infty}\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\eta}\Bigg|\,dx\,dy=$$

$$=\int\int_{\mathbf{R}_+^2} dx\,dy\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\frac{2\xi}{x(x-\xi)}\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\eta}=$$

$$=\int\int_{\mathbf{R}_+^2}\frac{d\mu(\xi,\eta)}{\eta}\int_{2\xi}^{\infty}\frac{2\xi}{x(x-\xi)}\,dx\int_0^{2\eta}\ln\left(\frac{y}{|y-\eta|}\right)dy=$$

$$=\int\int_{\mathbf{R}_+^2} d\mu(\xi,\eta)\int_2^{\infty}\frac{ds}{s(s-1)}\int_0^2\ln\left(\frac{t}{|t-1|}\right)dt\leqq C_{17}(\mathbf{R}_+^2).$$

From the last two inequalities it follows immediately that

$$(5.48)\qquad \int\int_{\mathbf{R}_+^2}\Bigg|\int_{\to 0}^{x/2}\int_{y/2}^{\infty}\ln\left(\frac{x+\xi}{x-\xi}\right)\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\xi\eta}-$$

$$-\frac{2}{x}\int_0^{x/2}\int_{y/2}^{\infty}\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\eta}\Bigg|\leqq C_{18}\mu(\mathbf{R}_+^2).$$

We proceed analogously in the case of the last term occurring in the integrand in (5.47):

$$\int\int_{\mathbf{R}^2_+} \left| \int_{\to 0}^{x/2} \int_{\to 0}^{y/2} \ln\left(\frac{x+\xi}{x-\xi}\right) \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} - \right.$$

$$\left. - \int_{\to 0}^{x/2} \int_{\to 0}^{y/2} \frac{2}{x-\xi} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\eta} \right| dx\, dy \leqq$$

$$\leqq \int\int_{\mathbf{R}^2_+} dx\, dy \int_{\to 0}^{x/2} \int_{\to 0}^{y/2} \frac{4\xi}{(x-\xi)^2} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\eta} =$$

$$= \int\int_{\mathbf{R}^2_+} \frac{d\mu(\xi,\eta)}{\eta} \int_{2\xi}^{\infty} \frac{4\xi}{(x-\xi)^2}\, dx \int_{2\eta}^{\infty} \ln\left(\frac{y^2}{y^2-\eta^2}\right) dy =$$

$$= \int\int_{\mathbf{R}^2_+} d\mu(\xi,\eta) \int_2^{\infty} \frac{4}{(s-1)^2}\, ds \int_2^{\infty} \ln\left(\frac{t^2}{t^2-1}\right) dt \leqq C_{19}\mu(\mathbf{R}^2_+)$$

and

$$\int\int_{\mathbf{R}^2_+} \left| \int_{\to 0}^{x/2} \int_{\to 0}^{y/2} \frac{1}{x-\xi} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\eta} - \right.$$

$$\left. - \frac{1}{x} \int_{0}^{x/2} \int_{\to 0}^{y/2} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\eta} \right| dx\, dy =$$

$$= \int\int_{\mathbf{R}^2_+} dx\, dy \int_{\to 0}^{x/2} \int_{\to 0}^{y/2} \frac{2\xi}{x(x-\xi)} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\eta} =$$

$$= \int\int_{\mathbf{R}^2_+} \frac{d\mu(\xi,\eta)}{\eta} \int_{2\xi}^{\infty} \frac{2\xi}{x(x-\xi)}\, dx \int_{2\eta}^{\infty} \ln\left(\frac{y^2}{y^2-\eta^2}\right) dy =$$

$$= \int\int_{\mathbf{R}^2_+} d\mu(\xi,\eta) \int_2^{\infty} \frac{2}{s(s-1)}\, ds \int_2^{\infty} \ln\left(\frac{t^2}{t^2-1}\right) dt \leqq C_{20}\mu(\mathbf{R}^2_+).$$

Collecting the last two inequalities yields

$$(5.49) \quad \int\int_{\mathbf{R}^2_+} \left| \int_{\to 0}^{x/2} \int_{\to 0}^{y/2} \ln\left(\frac{x+\xi}{x-\xi}\right) \ln\left(\frac{y^2}{y^2-\xi^2}\right) \frac{d\mu(\xi,\eta)}{\xi\eta} - \right.$$

$$-\frac{2}{x}\int_0^{x/2}\int_{\to 0}^{y/2}\ln\left(\frac{y^2}{y^2-\eta^2}\right)\frac{d\mu(\xi,\eta)}{\eta}\Bigg|\,dx\,dy \leqq C_{21}\mu(\mathbf{R}_+^2).$$

Combining (5.47)–(5.49) gives (2.6). This completes the proof of Theorem 2.

6. Proof of Corollary 1

By (2.4), we have $H_1 f \in L(\mathbf{R}^2)$ if and only if

$$\frac{2}{x}\int_0^{x/2}\int_0^{\infty}\frac{d\mu(\xi,\eta)}{\eta} \in L(\mathbf{R}_+^2) \quad (\text{in } x \text{ and } y),$$

or equivalently, by Fubini's theorem

$$\int_{\mathbf{R}_+}\left\{\frac{2}{x}\int_0^{x/2}\int_y^{\infty}\frac{d\mu(\xi,\eta)}{\eta}\right\}dy = \frac{2}{x}\int_0^{x/2}\int_{\mathbf{R}_+}\frac{d\mu(\xi,\eta)}{\eta}\int_0^{\eta}dy =$$

$$= \frac{2}{x}\int_0^{x/2}\int_{\mathbf{R}_+}d\mu(\xi,\eta) \in L(\mathbf{R}_+) \quad (\text{in } x),$$

which is equivalent to (2.7).

By (2.5), we have $H_2 f \in L(\mathbf{R}^2)$ without any additional condition.

Now, we prove that under condition (2.7) we also have $H_1 H_2 f \in L(\mathbf{R}^2)$. Indeed, by Fubini's theorem,

$$\int_{\mathbf{R}_+}\left\{\frac{2}{x}\int_0^{x/2}\int_{y/2}^{\infty}\ln\left(\frac{y}{|y-\eta|}\right)\frac{d\mu(\xi,\eta)}{\eta}\right\}dy = \tag{6.1}$$

$$= \frac{2}{x}\int_0^{x/2}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\eta}\int_0^{2\eta}\ln\left(\frac{y}{|y-\eta|}\right)dy =$$

$$= \frac{2}{x}\int_0^{x/2}\int_{\mathbf{R}_+}d\mu(\xi,\eta)\int_0^{2}\ln\left(\frac{t}{|t-1|}\right)dt$$

belongs to $L(\mathbf{R}_+)$ in x, provided condition (2.7) is satisfied. Furthermore,

$$\int_{\mathbf{R}_+}\left\{\frac{2}{x}\int_0^{x/2}\int_{\to 0}^{y/2}\ln\left(\frac{y^2}{y^2-\eta^2}\right)\frac{d\mu(\xi,\eta)}{\eta}\right\}dy = \tag{6.2}$$

$$= \frac{2}{x}\int_0^{x/2}\int_{\to 0}^{\infty}\frac{d\mu(\xi,\eta)}{\eta}\int_{2\eta}^{\infty}\ln\left(\frac{y^2}{y^2-\eta^2}\right)dy =$$

$$= \frac{2}{x} \int_0^{x/2} \int_{\mathbf{R}_+} d\mu(\xi, \eta) \int_2^\infty \ln \left(\frac{t^2}{t^2 - 1} \right) dt$$

also belongs to $L(\mathbf{R}_+)$ in x, provided (2.7) holds. Combining (6.1) and (6.2) with (2.6) shows that we also have $H_1 H_2 f \in L(\mathbf{R}^2)$. This completes the proof of Corollary 1.

7. Extension to $L(\mathbf{T}^2)$ multipliers

For a complex-valued function $f \in L(\mathbf{T}^2)$, periodic in each variable, its double Fourier series

$$\sum_{(j,k) \in \mathbf{Z}^2} \sum \widehat{f}(j,k) e^{i(jx+ky)} \tag{7.1}$$

is defined by

$$\widehat{f}(j,k) := \frac{1}{4\pi^2} \int \int_{\mathbf{T}^2} f(\xi, \eta) e^{-i(j\xi + k\eta)} \, d\xi \, d\eta, \quad (j,k) \in \mathbf{Z}^2.$$

Here and in the sequel,

$$\mathbf{T}^2 := \mathbf{T} \times \mathbf{T}, \quad \mathbf{T} := (-\pi, \pi], \quad Z^2 := \mathbf{Z} \times \mathbf{Z}, \quad \mathbf{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

If $f(x,y)$ is even in x and odd in y, then (7.1) becomes a cosine-sine series:

$$2 \sum_{k=1}^\infty \widehat{f}(0,k) \sin ky + 4 \sum_{j=1}^\infty \sum_{k=1}^\infty \widehat{f}(j,k) \cos jx \sin ky =:$$

$$=: 2 \sum_{(j,k) \in \mathbf{Z}_+^2} \sum \rho_j \widehat{f}(j,k) \cos jx \sin ky,$$

where $\rho_0 = 1$ and $\rho_j = 2$ for $j = 1, 2, \ldots$; furthermore,

$$\widehat{f}(j,k) = \frac{1}{\pi^2} \int \int_{\mathbf{T}_+^2} f(\xi, \eta) \cos j\xi \sin k\eta \, d\xi \, d\eta, \quad (j,k) \in \mathbf{Z}_+^2, \tag{7.2}$$

$$\mathbf{T}_+ := [0, \pi] \text{ and } \mathbf{Z}_+ := \{0, 1, 2, \ldots\}.$$

The definition of the three conjugate series associated with (7.1) is the following:

$$\sum\sum_{(j,k)\in\mathbf{Z}^2}(-i\operatorname{sign} j)\widehat{f}(j,k)e^{i(jx+ky)} \tag{7.3}$$

(conjugate with respect to x),

$$\sum\sum_{(j,k)\in\mathbf{Z}^2}(-i\operatorname{sign} k)\widehat{f}(j,k)e^{i(jx+ky)} \tag{7.4}$$

(conjugate with respect to y), and

$$\sum\sum_{(j,k)\in\mathbf{Z}^2}(-i\operatorname{sign} j)(-i\operatorname{sign} k)\widehat{f}(j,k)e^{i(jx+ky)} \tag{7.5}$$

(conjugate with respect to x and y). The corresponding conjugate functions are defined by

$$\widetilde{f}^{(1,0)}(x,y) := \frac{1}{\pi}(\text{P.V.})\int_{\mathbf{T}}\frac{f(x-u,y)}{2\tan(u/2)}\,du = \tag{7.6}$$

$$= -\frac{1}{\pi}\int_{\to 0}^{\pi}\frac{f(x+u,y)-f(x-u,y)}{2\tan(u/2)}\,du,$$

$$\widetilde{f}^{(0,1)}(x,y) := \frac{1}{\pi}(\text{P.V.})\int_{\mathbf{T}}\frac{f(x,y-v)}{2\tan(v/2)}\,dv, \tag{7.7}$$

$$\widetilde{f}^{(1,1)}(x,y) =: \frac{1}{\pi^2}(\text{P.V.})\int\int_{\mathbf{T}^2}\frac{f(x-u,y-v)}{4\tan(u/2)\tan(v/2)}\,du\,dv = \tag{7.8}$$

$$= \frac{1}{\pi^2}\int_{\to 0}^{\pi}\int_{\to 0}^{\pi}$$

$$\frac{f(x+u,y+v)-f(x+u,y-v)-f(x-u,y+v)+f(x-u,y-v)}{4\tan(u/2)\tan(v/2)}\,du\,dv.$$

R. Fefferman [5] introduced the following new kind of Hardy space:

$$\mathcal{H}(\mathbf{T}\times\mathbf{T}) := \left\{f\in L(\mathbf{T}^2)\colon \widetilde{f}^{(1,0)}, \widetilde{f}^{(0,1)}, \quad\text{and}\quad \widetilde{f}^{(1,1)}\in L(\mathbf{T}^2)\right\}.$$

It is well known that if $f \in \mathcal{H}(\mathbf{T} \times \mathbf{T})$, then the conjugate series (7.3)–(7.5) are the Fourier series of the conjugate functions (7.6)–(7.8), respectively.

The definition of the customary Hardy space $\mathcal{H}(\mathbf{T}^2)$ relies on the notion of the periodic Riesz transforms (see, e.g. [10, p. 283]). Again we have the strict inclusion $\mathcal{H}(\mathbf{T} \times \mathbf{T}) \subset \mathcal{H}(\mathbf{T}^2)$ (see [2]).

Let $\lambda := \{\lambda(j,k): (j,k) \in \mathbf{Z}^2\}$ be a double sequence of complex numbers. We say that λ is a multiplier for $L(\mathbf{T}^2)$ if for every $f \in L(\mathbf{T}^2)$ there exists a function $g \in L(\mathbf{T}^2)$ such that

$$\lambda(j,k)\widehat{f}(j,k) = \widehat{g}(j,k), \quad (j,k) \in \mathbf{Z}^2.$$

As it is well known (see, e.g. [10, p. 259]), a double sequence λ is a multiplier for $L(\mathbf{T}^2)$ if and only if there exists a finite Borel measure μ on $\mathbf{T}^2$ such that

$$\lambda(j,k) = \frac{1}{4\pi^2} \int\int_{\mathbf{T}^2} e^{-i(j\xi + k\eta)}\, d\mu(\xi,\eta), \quad (j,k) \in \mathbf{Z}^2. \tag{7.9}$$

It follows immediately that if λ is a multiplier for $L(\mathbf{T}^2)$, then λ is bounded; furthermore, if $\lambda(j,k)$ is even in j and odd in k, then the periodic measure μ associated with λ by (7.9) is also even in the first and odd in the second component, and

$$\lambda(j,k) = \frac{1}{\pi^2} \int\int_{\mathbf{T}^2_+} \cos j\xi \sin k\eta \, d\mu(\xi,\eta), \quad (j,k) \in \mathbf{Z}^2_+ \tag{7.10}$$

(cf. (7.2)). Motivated by the double Fourier–Stieltjes series

$$\sum\sum_{(j,k)\in\mathbf{Z}^2} \lambda(j,k) e^{i(jx+ky)} = \sum\sum_{(j,k)\in\mathbf{Z}^2_+} \rho\lambda(j,k) \cos jx \sin ky$$

of the measure μ, we form the arithmetic means

$$T\lambda(m,n) := \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} \rho_j \lambda(j,k), \quad (m,n) \in \mathbf{Z}^2_+.$$

The following results are the counterparts of Theorems 1, 2 and Corollary 1 for $L(\mathbf{T}^2)$ multipliers, or equivalently, for double Fourier–Stieltjes series.

THEOREM 3. *If $\lambda(j,k)$ is a multiplier for $L(\mathbf{T}^2)$, even in j and odd in k, then the double series*

$$\sum\sum_{(m,n)\in\mathbf{Z}^2_+} T\lambda(m,n) \cos mx \sin ny \tag{7.11}$$

(i) *is also a Fourier–Stieltjes series of some finite Borel measure on* $\mathbf{T}^2$*;*
(ii) *is a Fourier series of some function in* $L(\mathbf{T}^2)$ *if and only if*

$$\mu\Big(\big\{(0,y): y \in E\big\}\Big) = 0, \quad E \subseteqq T_+, \tag{7.12}$$

where μ *is the finite Borel measure on* $\mathbf{T}^2$ *associated with* λ *by* (7.9).

THEOREM 4. *Let* $\lambda(j,k)$ *be a multiplier for* $L(\mathbf{T}^2)$*, even in* j *and odd in* k*, and let*

$$f(x,y) := \int_x^\pi \int_y^\pi \frac{d\mu(\xi,\eta)}{\xi\eta}, \tag{7.13}$$

$$f(x,-y) = -f(-x,y) = f(-x,-y) := -f(x,y), \quad (x,y) \in \mathbf{T}_+^2,$$

where μ *is the finite Borel measure on* $\mathbf{T}^2$ *associated with* λ *by* (7.9)*. Then*

$$\int\int_{\mathbf{T}_+^2} \left| \pi \widetilde{f}^{(1,0)}(x,y) - \frac{2}{x} \int_0^{x/2} \int_y^\pi \frac{d\mu(\xi,\eta)}{\eta} \right| dx\, dy \leqq C\|\mu\|,$$

$$\int\int_{\mathbf{T}_+^2} \left| \widetilde{f}^{(0,1)}(x,y) \right| dx\, dy \leqq C\|\mu\|,$$

and

$$\int\int_{\mathbf{T}_+^2} \left| \pi^2 \widetilde{f}^{(1,1)}(x,y) - \frac{2}{x} \int_0^{x/2} \int_{y/2}^\pi \ln\left(\frac{y}{|y-\eta|}\right) \frac{d\mu(\xi,\eta)}{\eta} - \right.$$
$$\left. - \frac{2}{x} \int_0^{x/2} \int_0^{y/2} \ln\left(\frac{y^2}{y^2-\eta^2}\right) \frac{d\mu(\xi,\eta)}{\eta} \right| dx\, dy \leqq C\|\mu\|,$$

where $\|\mu\|$ *denotes the total variation of the measure* μ *over* $\mathbf{T}_+^2$.

COROLLARY 2. *If* $\lambda(j,k)$ *is a multiplier for* $L(\mathbf{T}^2)$*, even in* j *and odd in* k*, then the double series* (7.11) *and its conjugate series*

$$\sum_{(m,n)\in \mathbf{Z}_+^2}\sum T\lambda(m,n) \sin mx \sin ny,$$

$$-\sum_{(m,n)\in \mathbf{Z}_+^2}\sum T\lambda(m,n) \cos mx \cos ny,$$

$$-\sum_{(m,n)\in \mathbf{Z}_+^2}\sum T\lambda(m,n) \sin mx \cos ny$$

are all Fourier series of some functions in $L(\mathbf{T}^2)$ *if and only if*

$$\frac{1}{x}\int_0^x \int_{\mathbf{T}_+} d\mu(\xi,\eta) \in L(\mathbf{T}_+) \quad (in\ x), \tag{7.14}$$

where μ *is the finite Borel measure on* $\mathbf{T}^2$ *associated with* λ *by* (7.9).

We point out that condition (7.14) implies (7.12), under which we have

$$T\lambda(m,n) = \widehat{f}(m,n), \quad (m,n) \in \mathbf{Z}_+^2,$$

where f is defined in (7.13).

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REPRESENTATION OF FUNCTIONS OF BESOV CLASS ON MANIFOLDS BY ALGEBRAIC POLYNOMIALS

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Devoted to Professor Károly Tandori on his 70th birthday with great respect and pleasure

This note is a generalization of my recent results [4], [5], [6].

Results concerning the class $H_p^r(\Gamma)$ are carried over to the Besov class $B_{p\Theta}^r(\Gamma)$.

We consider the n-dimensional Euclidean space $R^n \ni x = (x_1, \dots, x_n)$ and an m-dimensional $(1 \leqq m \leqq n)$ manifold $\Gamma \subset R^n$.

By definition, a point $x^0 \in \Gamma$ if it is possible to cover x^0 by a rectangle Δ which, after some renumbering of coordinates is of the form

$$\Delta = \left\{x : \left|x_i - x_i^0\right| < \alpha,\ i = 1, \dots, m;\ \left|x_j - x_j^0\right| < \beta,\ j = m+1, \dots, n\right\}$$

with projection

$$\Delta' = \left\{x' : \left|x_i - x_i^0\right| < \alpha,\ i = 1, \dots, m\right\}, \quad x' = (x_1, \dots, x_m)$$

where

$$(1) \quad x_j = \psi_j(x') = \psi_j(x_1, \dots, x_m), \quad x' \subset \Omega \subset \Delta' \qquad (j = m+1, \dots, n)$$

are continuously differentiable functions which describe the intersection $\gamma = \Gamma \cap \Delta$ (γ is part of Γ).

If it is possible to achieve $\Omega = \Delta'$, then x^0 is an inner point of Γ. Otherwise Ω can only be an essential part of Δ'. Then x^0 is a boundary point of Γ. In such a case we assume Ω to have Lipschitz boundary in $R^m \ni (x_1, \dots, x_m)$ or (for $m = 1$) Ω can be a segment. Such conditions on Γ are called B-conditions (see [4], [6]).

We assume Γ is bounded and closed. Therefore it is possible to represent it as a finite sum:

$$(2) \qquad \Gamma = \bigcup \gamma = \bigcup_{i=1}^{\nu} \gamma_i.$$

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We write $\Gamma \in C^k$, if the functions $\psi_j(x')$ have continuous derivatives of order k.

We consider functions f defined on Γ or on $G \supset \Gamma$, belonging to the class $B^r_{p\Theta}(G)$ $(r > 0,\ 1 \leqq p,\ \Theta \leqq \infty,\ B^r_{p\infty} = H^r_p)$, and approximate them by algebraic polynomials

$$P_N(x) = \sum_{|k| \leqq N} a_k x_k \quad \left(k = (k_1, \dots, k_n),\ |k| = \sum_{j=1}^n k_j,\ x^k = x_1^{k_1}, \dots, x_n^{k_n}\right)$$

of degree N.

Note the following assertion.

ASSERTION A. *Let $\Gamma \in C^k$ $(k > r)$ be bounded, closed, with boundary points satisfying Condition* B *and $G \supset \Gamma$ an open set in R^n. Then any function $f \in B^r_{p\Theta}(\Gamma)$ can be continued to a function $f(x) \in B^{r+\frac{n-m}{p}}_{p\Theta}(G)$ finite in G such that*

$$\|f(x)\|_{B^{r+\frac{n-m}{p}}_{p\Theta}(G)} \leqq c\|f\|_{B^r_{p\Theta}(\Gamma)},$$

where c is a constant independent (here and later) of the sets indicated (see [5], [6]).

For functions $f(x)$ given on Γ we introduce the $L_p(\Gamma)$ norm:

$$\|f(x)\|_{L_p(\Gamma)} = \left(\int_\Gamma |f(x)|^p d\,\Gamma\right)^{1/p}, \quad 1 \leqq p \leqq \infty$$

and

$$\|f(x)\|_{L_\infty(\Gamma)} = \sup_{x \in \Gamma} \operatorname{vrai} |f(x)|,$$

where $d\,\Gamma$ is the element of Γ.

If $f(x)$ is defined on γ, then

$$f|_\gamma = f\left(x_1, \dots, x_m, \psi_{m+1}(x'), \dots, \psi_n(x')\right) = F(x'), \quad x' \in \Omega, \tag{3}$$

$$\|f(x)\|_{L_p(\gamma)} = \left(\int_\Omega |F(x')|^p X(x') dx'\right)^{1/p},$$

$$X(x') = \left(\sum_{1 \leqq j_1 \leqq \dots \leqq j_m \leqq n} \left(\frac{D(x_{j_1}, \dots, x_{j_m})}{D(x_1, \dots, x_m)}\right)^2\right)^{1/2}.$$

We consider an open cube in R^n:

$$\Delta_a = \{x : |x_j| \leqq a,\ j = 1, \dots, n\} \quad (\Gamma = \overline{\Gamma} \subset \Delta_a)$$

and its transform defined by the equations $x_j = a\cos\varphi_j$, $j = 1, \dots, n$, or, shortly

$$x = a\cos\varphi, \quad x \in \Delta_a \rightleftarrows \Delta_* \ni \varphi,$$

$$x' \in \Omega \rightleftarrows \omega \ni (\varphi_1, \dots, \varphi_m) = \varphi', \quad \Delta_* = \underbrace{(0,\pi) \times \cdots \times (0,\pi)}_{n \text{ times}}.$$

The $L_p(\gamma)$ norm in terms of φ can be written as follows:

$$\|f(x)\|_{L_p(\gamma)} = \left(\int_\gamma |f(a\cos\varphi)|^p \, d\gamma\right)^{1/p} =$$

$$= \left(\int_N \left|f\left(a\cos\varphi_1, \dots, a\cos\varphi_n, \psi_{m+1}(a\cos\varphi'), \dots, \psi_n(a\cos\varphi')\right)\right|^p \times \right.$$

$$\left. \times x_1(\varphi')\, d\varphi'\right)^{1/p'}.$$

The differential $d\Gamma$ is of the form

$$d\Gamma = x_1(\varphi')\, d\varphi'. \tag{4}$$

We do not go into details.

For functions $f(x)$ defined on Δ_a we introduce a new norm:

$$|||f(a\cos\varphi)|||_{L_p(\Gamma)} = \sup_\alpha \|f(a\cos(\varphi+\alpha)\|_{L_p(\Gamma)} \tag{5}$$

where the supremum is taken over all vectors $\alpha = (\alpha_1, \dots, \alpha_n)$. An important property of this form is:

$$|||f(a\cos(\varphi+\alpha)|||_{L_p(\Gamma)} = |||f(a\cos\varphi)|||_{L_p(\Gamma)}$$

for any $\alpha = (\alpha_1, \dots, \alpha_n)$.

Let us write

$$\overline{E_N(f)}_{L_p(\Gamma)} = \inf_{P_N} |||f(a\cos\varphi) - P_N(a\cos\varphi)|||_{L_p(\Gamma)}, \tag{6}$$

where the inf is taken over all algebraic polynomials of a given degree N.

THEOREM 1. *Let the manifold satisfy the conditions of Assertion* A *and let* $f \in B^r_{p\Theta}(\Gamma)$ $(1 \leqq p, \Theta \leqq \infty)$. *It is possible to continue* f *from* Γ *to* Δ_a *with the following property of the extended function* $f(x)$, $x \in \Delta_a$:

(7)
$$\|f\|_{{}_1B^r_{p\Theta}(\Gamma)} = \|f(a\cos\varphi)\|^*_p + \left(\sum_{s=0}^{\infty} 2^{sr\Theta}\overline{E_{2^s}}(f)^{\Theta}_{L_p(\Gamma)}\right)^{1/\Theta} \leqq c_1\|f\|_{B^r_{p\Theta}(\Gamma)},$$

$$\|\cdot\|_{p^*} = \|\cdot\|_{L_p(\Delta^n_*)}, \quad \Delta^n_* = \underbrace{[-\pi,\pi]\times\cdots\times[-\pi,\pi]}_{n \text{ times}},$$

$$\text{(8)} \qquad \|f\|_2B^r_{p\Theta}(\Gamma) = \inf\left(\sum_{s=0}^{\infty} 2^{sr\Theta}|||q_s(a\cos\varphi)|||^{\Theta}_{L_p(\Gamma)}\right)^{1/\Theta} \leqq c_2\|f\|_{B^r_{p\Theta}(\Gamma)},$$

where the inf *is taken over all*

$$\text{(9)} \qquad f(x) = \sum_{s=0}^{\infty} q_s(x),$$

where $q_s(x)$ *are algebraic polynomials of degree* 2^s, *and the convergence of the infinite series is meant in the metric* $|||\cdot|||_{L_p(\Gamma)}$.

Moreover, we have the equivalency

$$\text{(10)} \qquad {}_1B^r_{p\Theta}(\Gamma) \sim {}_2B^r_{p\Theta}(\Gamma),$$

i.e.

$$c_1\|f\|_{{}_1B^r_{p\Theta}(\Gamma)} \leqq \|f\|_{{}_2B^r_{p\Theta}(\Gamma)} \leqq c_2\|f\|_{{}_1B^r_{p\Theta}(\Gamma)}$$

(see (32) *below).*

PROOF. We write

$$\cos\Omega^* = \Omega, \quad \Omega^* \subset \Delta_*, \quad \Omega \subset \Delta_a.$$

The set Δ_a is open and $\Gamma = \overline{\Gamma} \subset \Delta_a$, therefore there are open sets $\Omega_1, \Omega_2, \Omega_3$ such that

$$\Gamma \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_3 \subset \overline{\Omega}_3 \subset \Delta_a.$$

Using Assertion A one can continue every $f \in B^r_{p\Theta}(\Gamma)$ from f to Δ_a such that the extended function $f(x)$, $x \in \Delta_a$ satisfies the following properties:

$$(11) \qquad f(x) \in B_{p\Theta}^{r+\frac{n-m}{p}}(\Delta_a), \quad \|f(x)\|_{B_{p\Theta}^{r+\frac{n-m}{p}}(\Delta_a)} \leqq c\|f\|_{B_{p\Theta}^r(\Gamma)},$$

$$f(x) = 0, \quad x \in \Delta_a \setminus \overline{\Omega}_1.$$

The transformation $x = a\cos\varphi$ is differentiable on both sides of $\overline{\Omega}_3$, infinitely many times, therefore (see [1, §21]) the function $f(a\cos\varphi)$ of φ belongs to $B_{p\Theta}^{r+\frac{n-m}{p}}(\Omega_2^*)$ and

$$(12) \qquad \|f(a\cos\varphi)\|_{B_{p\Theta}^{r+\frac{n-m}{p}}(\Omega_2^*)} \leqq c_1\|f(x)\|_{B_{p\Theta}^{r+\frac{n-m}{p}}(\Omega_3)}.$$

But

$$f(a\cos\varphi) = 0, \qquad \varphi \in \Delta_a \setminus \overline{\Omega}_1^*,$$

$$f(x) = 0, \quad x \in \Delta_a \setminus \overline{\Omega}_1,$$

therefore we can write

$$\|f(a\cos\varphi)\|_{B_{p\Theta}^{r+\frac{n-m}{p}}(\Delta_*)} \leqq c_2\|f\|_{B_{p\Theta}^{r+\frac{n-m}{p}}(\Delta_a)}$$

instead of (12).

Since

$$f(a\cos\varphi) = 0, \quad \varphi \in \Delta_* \setminus \overline{\Omega}_1^*$$

we can consider $f(a\cos\varphi)$ to belong to the class $B_{p^*\Theta}^{r+\frac{n-m}{p}}$ of periodic (of period 2π) functions with the estimate

$$(13) \qquad \|f(a\cos\varphi)\|_{B_{p^*\Theta}^{r+\frac{n-m}{p}}} \leqq c\|f\|_{B_{p\Theta}^r(\Gamma)}.$$

It is known that the norm of the function $f(a\cos\varphi)$ of the class $B_{p^*\Theta}^{r+\frac{n-m}{p}}$ can be written in the following equivalent form:

$$(14) \qquad \|f\|_{B_{p^*\Theta}^{r+\frac{n-m}{p}}} = \inf\left(\sum_{s=0}^{\infty} 2^{s(r+\frac{n-m}{p})\Theta}\|q_s(a\cos\varphi)\|_{p^*}^{\Theta}\right)^{1/\Theta}$$

where the inf is taken over all representations of (9) of f (converging in $\|\cdot\|_p^*$). (See [2, 5.6(6)] for the class $B_{p\Theta}^s(R^n)$ determined by functions of exponential type.) For the periodic one dimensional case see [7].

Now we have to apply an inequality for trigonometric polynomials (see [5], [6,2.1])

$$(15)\qquad |||q_s(a\cos\varphi)|||_{L_p(\Gamma)} \leqq c2^{s\frac{n-m}{p}}\|q_s(a\cos\varphi)\|_{p^*},\quad \|\cdot\|_{p^*}=\|\cdot\|_{L_p(\Delta_*^n)}.$$

Now (13), (14), (15) imply

$$\inf\left(\sum_{s=0}^{\infty}2^{sr\Theta}|||q_s(a\cos\varphi)|||^{\Theta}_{L_p(\Gamma)}\right)^{1/\Theta} \leqq c\|f\|_{B^r_{p\Theta}(\Gamma)}$$

and (8) is obtained.

Let us now determine

$$P_{2^s}(a\cos\varphi)=\sum_{i\leqq s}q_i(a\cos\varphi).$$

Then $\left(|||\cdot|||_{l_p(\Gamma)}=|||\cdot|||\right)$

$$|||f(a\cos\varphi)-P_{2^s}(a\cos\varphi)|||_{L_p(\Gamma)}\leqq \sum_{i>s}|||q_s|||.$$

Therefore (see (6))

$$(16)\qquad \left(\sum_{s=0}^{\infty}2^{sr\Theta}\overline{E_{2^s}}(f)^{\Theta}_{L_p(\Gamma)}\right)^{1/\Theta} \leqq \left(\sum_{s=0}^{\infty}2^{sr\Theta}|||f-P_{2^s}|||^{\Theta}\right)^{1/\Theta} \leqq$$

$$\leqq \left(\sum_{s=0}^{\infty}2^{sr\Theta}\left(\sum_{i>s}|||q_s|||\right)^{\Theta}\right)^{1/\Theta} \leqq c\left(\sum_{s=0}^{\infty}2^{sr\Theta}|||q_s(a\cos\varphi)|||^{\Theta}_{L_p(\Gamma)}\right)^{1/\Theta}$$

(see [2, (5.6) (20)]).

Note now the inequality [6, 3.27]

$$\|f(a\cos\varphi)\|_{p^*}\leqq c|||f(a\cos\varphi)|||_{L_p(\Gamma)},$$

which is used below:

$$(17)\qquad \|f(a\cos\varphi)\|_{p^*}\leqq c|||f(a\cos\varphi)|||_{L_p(\Gamma)}\leqq c\sum_{s=0}^{\infty}|||q_s|||=$$

$$= c_1 \sum_{s=0}^{\infty} 2^{-sr} 2^{sr} |||q_s||| \leqq c_1 \left(\sum_{s=0}^{\infty} 2^{-sr\Theta'} \right)^{1/\Theta'} \left(\sum_{s=0}^{\infty} 2^{sr\Theta} |||q_s|||^{\Theta} \right)^{1/\Theta} =$$

$$= c \left(\sum_{s=0}^{\infty} 2^{sr\Theta} |||q_s(a \cos \varphi)|||^{\Theta}_{L_p(\Gamma)} \right)^{1/\Theta}, \quad \frac{1}{\Theta} + \frac{1}{\Theta'} = 1.$$

Finally from (16), (17) we shall have

$$\|f(a \cos \varphi)\|_{p^*} + \left(\sum_{s=0}^{\infty} 2^{sr\Theta} \overline{E_{2^s}}(f)^{\Theta}_{L_p(\Gamma)} \right)^{1/\Theta} \leqq$$

$$\leqq c \left(\sum_{s=0}^{\infty} 2^{sr\Theta} |||q_s(a \cos \varphi)|||^{\Theta}_{L_p(\Gamma)} \right)^{1/\Theta},$$

or

(18) $$\|f\|_{{}_1 B^r_{p\Theta}(\Gamma)} \leqq c \|f\|_{{}_2 B^r_{p\Theta}(\Gamma)}.$$

To prove the converse inequality let us take a function f with the finite sum

(19) $$\|f(a \cos \varphi)\|_{p^*} + \left(\sum_{s=0}^{\infty} 2^{sr\Theta} \overline{E_{s^s}}(f)^{\Theta}_{L_p(\Gamma)} \right)^{1/\Theta} < \infty$$

and let P_{2^s} be an algebraic polynomial of degree 2^s for which

$$|||f - P_{2^s}|||_{L_p(\Gamma)} \leqq 2\overline{E_{2^s}}(f)_{L_p(\Gamma)}.$$

Then (19) implies

$$|||f - P_{2^s}|||_{L_p(\Gamma)} \to 0, \quad s \to \infty.$$

Therefore the series

$$f = \sum_{s=0}^{\infty} q_s, \qquad q_0 = P_{2^0},\ q_s = P_{2^s} - P_{2^{s-1}}, \quad s = 1, 2, \ldots$$

converges to f in the metric $|||\cdot|||_{L_p(\Gamma)}$.

We have

$$\left(\sum_{s=0}^{\infty} 2^{sr\Theta} |||q_s(a\cos\varphi)|||_{L_p(\Gamma)}^{\Theta}\right)^{1/\Theta} \leqq$$

$$\leqq |||q_0||| + \left(\sum_{s=1}^{\infty} 2^{sr\Theta} \left(|||P_{2^s} - f||| + |||f - P_{2^{s-1}}|||\right)^{\Theta}\right)^{1/\Theta} \leqq$$

$$\leqq c\left(|||q_0(a\cos\varphi)|||_{L_p(\Gamma)} + \left(\sum_{s=0}^{\infty} 2^{sr\Theta} \overline{E_{2^s}}(f)_{L_p(\Gamma)}^{\Theta}\right)\right)^{1/\Theta}.$$

Note that q_0 belongs to the finite dimensional space of polynomials of first degree, therefore the norms

$$\|q_0\|_{p^*} \sim |||q_0|||_{L_p(\Gamma)}$$

are equivalent.

Inequality (19) says

$$\|f\|_{{}_2B^r_{p\Theta}(\Gamma)} \leqq c\|f\|_{{}_1B^r_{p\Theta}(\Gamma)}. \tag{20}$$

(18) and (20) imply (10).

THEOREM 2 (converse). *Let* $\Gamma \in C^k$ $(k > r)$,

$$f(x) = \sum_{s=0}^{\infty} q_s(x), \quad x \in \Delta_a, \tag{21}$$

$$K = \inf\left(\sum_{s=0}^{\infty} 2^{sr\Theta} |||q_s(a\cos\varphi)|||_{L_p(\Gamma)}^{\Theta}\right)^{1/\Theta} \tag{22}$$

where the inf *is taken over all representations* (9) *of* $f(x)$ *by algebraic polynomials* $q_s(x)$ *of degree* 2^s.

Then $f \in B^r_{p\Theta}(\Gamma)$ *and*

$$\|f\|_{B^r_{p\Theta}(\Gamma)} \leqq cK. \tag{23}$$

PROOF. Let us take a function $f(x)$, $x \in \Delta_a$ which has the representation (9), for which

$$\sum_{s=0}^{\infty} 2^{sr\Theta} |||q_s(a\cos\varphi)|||_{L_p(\Gamma)}^{\Theta} < \infty. \tag{24}$$

Let also γ be one of the pieces determined in (1). We wish to use (see (2), (3)) the definition

$$|||\Delta^k_{x_i,h}(f)_\gamma|||_{L_p(\gamma)} = \left(\int_{\Omega_{x_i,kh}} \left|\Delta^k_{x_i,h}F(x')\right|^p X(x')\,dx'\right)^{1/p}$$

of the $L_p(\gamma)$-norm of the difference of f of order k on x_i along γ with step $h > 0$. Here $\Omega_{x_i,h}$ is the set of points $x' \in \Omega$ whose distance to the boundary Ω along x_i is greater than kh.

For any algebraic polynomial of degree N the following Bernstein type inequality holds (see [4], [6, 4.14]):

$$\|\Delta^k_{x_i,h}(P_N)_\gamma\|_{L_p(\gamma)} \leqq c\frac{(hN)^k|||P_N(a\cos\varphi)|||_{L_p(\gamma)}}{(\sin\delta)^{\lambda_k}}, \tag{25}$$
$$\lambda_k = 1 + 2 + \cdots + k,$$

where δ $(0 < \delta < \pi/2)$ depends on γ. Such a number exists because $\Gamma = \overline{\Gamma} \subset \subset \Delta_a$.

We also have the inequality

$$\|\Delta^k_{x_i,h}(f)_\gamma\|_{L_p(\gamma)} \leqq c|||f(a\cos\varphi)|||_{L_p(\gamma)}. \tag{26}$$

Now let $f(x)$ be a function from Theorem 2,

$$\lambda_s = |||q_s(a\cos\varphi)|||_{L_p(\Gamma)}, \tag{27}$$

$$\Omega^k(f,\delta) = \sup_{|h|\leqq\delta, 1\leqq i\leqq m} |||\Delta^k_{x_i,h}(f)_\gamma|||_{L_p(\gamma)}. \tag{28}$$

Then using (25) where $P_N = q_s$, $N = 2^s$ and (26) where $f = q_s$, we obtain (see (17))

$$\|f(x)\|_{L_p(\gamma)} \leqq \sum_{s=0}^{\infty}\|q_s\|_{L_p(\gamma)} = \tag{29}$$

$$= \sum_{s=0}^{\infty}|||q_s(a\cos\varphi)|||_{L_p(\Gamma)} \leqq \left(\sum_{s=0}^{\infty}2^{sr\Theta}\lambda_s^{\Theta}\right)^{1/\Theta},$$

$$\|\Delta^k_{x_i,h}(f)_\gamma\|_{L_p(\gamma)} \leqq \sum_{h2^s<1}\|\Delta^k_{x_ih}(f)_\gamma\|_{L_p(\gamma)} + \sum_{h2^s\geqq 1}\lambda_s,$$

$$\Omega^k(f,2^{-N}) \leqq c2^{-kN}\sum_{s=0}^{N}2^{sr}\lambda_s + c\sum_{s=N}^{\infty}\lambda_s.$$

Further (see [2,5.6 (16), (17), (18)]) for $k > r$

$$(30)\quad \|f\|^{\Theta}_{b^r_{p\Theta}(\gamma)} = \int_0^1 t^{-1-\Theta r}\Omega^k(f,t)^{\Theta}\,dt \leqq c_1\sum_{N=0}^{\infty}2^{rN\Theta}\Omega^k\left(f,2^{-N}\right)^{\Theta} \leqq$$

$$\leqq c_2\sum_{N=0}^{\infty}2^{(r-k)N\Theta}\left(\sum_{s=0}^{N}2^{sk}\lambda_s\right)^{\Theta} + c_2\sum_{N=0}^{\infty}2^{rN\Theta}\left(\sum_{s=N}^{\infty}\lambda_s\right)^{\Theta} \leqq c\sum_{s=0}^{\infty}2^{sr\Theta}\lambda_s^{\Theta}.$$

It follows from (29), (30) that

$$\|f\|_{B^r_{p\Theta}(\gamma)} = \|f\|_{L_p(\gamma)} + \|f\|_{b^r_{p\Theta}(\gamma)} \leqq c_3\left(\sum_{s=0}^{\infty}2^{sr\Theta}\lambda_s^{\Theta}\right)^{1/\Theta}$$

and (see (21))

$$\|f\|_{B^r_{p\Theta}(\Gamma)} = \sum_{i=1}^{\nu}\|f\|_{B^r_{p\Theta}(\gamma_i)} \leqq c\left(\sum_{s=0}^{\infty}2^{sr\Theta}\lambda_s^{\Theta}\right)^{1/\Theta}.$$

Therefore

$$(31)\qquad \|f\|_{B^r_{p\Theta}(\Gamma)} \leqq c\inf\left(\sum_{s=0}^{\infty}2^{sr\Theta}\|\|q_s(a\cos\varphi)\|\|^{\Theta}_{L_p(\Gamma)}\right)^{1/\Theta} = cK.$$

Theorem 2 is proved. Since $K = \|f\|_{{}_2B^r_{p\Theta}(\Gamma)}$ the inequalities (7), (8), (31) and the equivalence (10) imply

$$(32)\qquad {}_1B^r_{p\Theta}(\Gamma) \sim {}_2B^r_{p\Theta}(\Gamma) \sim B^r_{p\Theta}(\Gamma) \qquad (\Gamma \in C^k,\ k > r).$$

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ON HERMITE FUNCTIONS. II

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Dedicated to Professor Károly Tandori on his 70th birthday

The aim of this paper is to improve the results of [1], [2] in some sense. This will be applied later to obtain Alexits type saturation theorems like [10]–[12]. First we introduce some notations. Denote

$$x_1 > x_2 > \ldots > x_n$$

the zeros of the Hermite-polynomial

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

and let

$$l_k(x) := \frac{H_n(x)}{H_n'(x_k)(x - x_k)} \quad (k = 1, \ldots, n)$$

be the fundamental polynomials of Lagrange interpolation. If f is a continuous function on $\mathbf{R}$ and bounded on $\mathbf{R}$ then in [1] a sharp estimation for $\left| f(x) - L_n(f,x) \right|$ was proved, where

$$L_n(f,x) := \sum_{k=1}^{n} f(x_k) l_k(x).$$

If f is a uniformly continuous function on $\mathbf{R}$ then in [2], Theorem 9, Theorem 10 and Theorem 11 a sharp estimation was proved for $\left| f(x) - L_n(f,x) \right|$.

For a function $f : \mathbf{R} \to \mathbf{R}$ we shall use a special form of modulus of continuity $w(f,\delta)$ (see e.g. [3], [4]):

$$w(f,\delta) := \sup_{0 \leqq t \leqq \delta} \left\| e^{-(x+t)^2/2} f(x+t) - e^{-x^2/2} f(x) \right\| + \left\| \tau(\delta x) e^{-x^2/2} f(x) \right\|,$$

where

$$\tau(x) := \begin{cases} |x|, & \text{if } |x| \leqq 1 \\ 1, & \text{if } |x| > 1 \end{cases}$$

and $\|\cdot\|$ denotes the sup-norm in $C(\mathbf{R})$. If $f \in C(\mathbf{R})$ and

$$\lim_{|x|\to\infty} e^{-x^2/2} f(x) = 0$$

then

$$\lim_{\delta\to 0+} w(f,\delta) = 0.$$

We prove the following

THEOREM. *If $f \in C(\mathbf{R})$ and $e^{-x^2/2}f(x)$ is uniformly bounded on $\mathbf{R}$, then*

$$e^{-x^2/2}\big|f(x) - L_{n+1}(f,x)\big| = O(1)n^{1/6} w\left(f, \frac{1}{\sqrt{n}}\right) \quad (x \in \mathbf{R}),$$

where $O(1)$ does not depend on x, n and f.

For the proof we need a lemma.

LEMMA. *If $f \in C(\mathbf{R})$ and $e^{-x^2/2}f(x)$ is uniformly bounded on $\mathbf{R}$, then there exist polynomials p_n of degree $\leqq n$ such that*

$$e^{-x^2/2}\big|f(x) - p_n(x)\big| = O(1) w\left(f, \frac{1}{\sqrt{n}}\right).$$

PROOF. See [3], Theorem 1 and [4], Theorem A.

PROOF OF THE THEOREM. Using the Lemma we obtain

$$\tag{1} e^{-x^2/2}\big|f(x) - L_{n+1}(f,x)\big| \leqq$$

$$\leqq e^{-x^2/2}\big|f(x) - p_n(x)\big| + e^{-x^2/2}\big|L_{n+1}(p_n - f, x)\big| =$$

$$= O(1)w\left(f, \frac{1}{\sqrt{n}}\right) + O(1)w\left(f, \frac{1}{\sqrt{n}}\right) e^{-x^2/2} \sum_{k=1}^{n} e^{x_k^2/2}\big|l_k(x)\big|.$$

Without loss of generality we may assume that $x \geqq 0$. Using [8], (6.1) and (6.2) we obtain

$$\tag{2} e^{-x^2/2} \sum_{k=1}^{n} e^{x_k^2/2}\big|l_k(x)\big| \asymp \frac{e^{-x^2/2}\cdot\big|H_n(x)\big|}{\sqrt{2^n n!}} \sum_{k=1}^{n} \sqrt{\phi_n(x_k)} \cdot \frac{1}{|x - x_k|},$$

where $\phi_n(x_k) = x_k - x_{k+1}$. Since $x \geqq 0$ and the zeros are symmetrical with respect to the origin, $x_k = -x_{n-k}$, therefore $\phi_n(x_k) \asymp \phi_n(x_{n-k})$, $\frac{1}{|x-x_k|} \geqq \frac{1}{|x-x_{n-k}|}$ and then it is enough to estimate

$$(3) \qquad \frac{e^{-x^2/2}|H_n(x)|}{\sqrt{2^n n!}} \sum_{1 \leqq k \leqq \frac{n}{2}} \frac{\sqrt{x_k - x_{k+1}}}{|x - x_k|}.$$

We know [9],

$$(4) \qquad x_k - x_{k+1} \asymp n^{-1/6} \cdot k^{-1/3}, \quad 1 \leqq k \leqq \frac{n}{2}.$$

Using this we obtain from (3)

$$\frac{e^{-x^2/2} \cdot |H_n(x)|}{\sqrt{2^n n!}} n^{-1/12} \sum_{1 \leqq k \leqq \frac{n}{2}} \frac{k^{-1/6}}{|x - x_k|}.$$

Denote k_0 the index $1 \leqq k_0 \leqq \frac{n}{2}$ for which $|x - x_k|$ is minimal. By the known inequality [7]

$$e^{-x^2} \sum_{k=1}^{n} e^{x_k^2} l_k^2(x) \leqq 1 \quad (x \in \mathbf{R}),$$

finitely many members of the sum (1) can be estimated by $O(1)$. Hence it is enough to estimate in (5)

$$(6) \qquad \frac{e^{-x^2/2} \cdot |H_n(x)|}{\sqrt{2^n n!}} \cdot n^{-1/12} \sum_{\substack{1 \leqq k \leqq \frac{n}{2} \\ k \neq k_0, k_0 \pm 1}} \frac{k^{-1/6}}{|x - x_k|}.$$

Here

$$|x - x_k| \geqq c|x_{k_0} - x_k| \asymp n^{-1/6}(k_0^{-1/3} + \ldots + k^{-1/3}) \asymp \frac{|k^{2/3} - k_0^{2/3}|}{n^{1/6}}.$$

We know from [5] that $e^{-x^2/2}|H_n(x)|\sqrt{2^n n!} = O(1)n^{-1/12}$. Thus we obtain from (6)

$$(7) \qquad \sum_{\substack{1 \leqq k \leqq n/2 \\ k \neq k_0, k_0 \pm 1}} \frac{k^{-1/6}}{|k^{2/3} - k_0^{2/3}|} = \sum_{1 \leqq k \leqq k_0/2} + \sum_{\substack{k_0/2 \leqq k \leqq 3k_0/2 \\ k \neq k_0, k_0 \pm 1}} + \sum_{3k_0/2 \leqq k \leqq n/2}$$

Here

$$\sum_{1 \leqq k \leqq k_0/2} \frac{k^{-1/6}}{|k^{2/3} - k_0^{2/3}|} \asymp \sum_{1 \leqq k \leqq k_0/2} \frac{k^{-1/6}}{k_0^{2/3}} \asymp k_0^{-1/6} = O(1) n^{1/6},$$

and

$$\sum_{3k_0/2 \leqq k \leqq n/2} \frac{k^{-1/6}}{|k^{2/3} - k_0^{2/3}|} \asymp \sum_{3k_0/2 \leqq k \leqq n/2} \frac{k^{-1/6}}{k_0^{2/3}} = O(1) n^{1/6}.$$

Finally

$$\sum_{\substack{k_0/2 \leqq k \leqq 3k_0/2 \\ k \neq k_0, k_0 \pm 1}} \frac{k^{-1/6}}{|k^{2/3} - k_0^{2/3}|} \asymp \sum_{\substack{k_0/2 \leqq k \leqq 3k_0/2 \\ k \neq k_0, k_0 \pm 1}} \frac{k_0^{-1/6}}{k_0^{-1/3} |k - k_0|} \asymp$$

$$\asymp k_0^{-1/6} \log(k_0 + 1).$$

Here the logarithmic term can be eliminated by two ways in cases $x \leqq \sqrt{2n+1} - (2n+1)^{-1/6+\delta}$ and $x \geqq \sqrt{2n+1} - (2n+1)^{-1/6+\delta}$, where $0 < \delta < 2/3$ is fixed.

a) $x \leqq \sqrt{2n+1} - (2n+1)^{-1/6+\delta}$. Then by [5]

$$\frac{e^{-x^2/2} \cdot |H_n(x)|}{\sqrt{2^n n!}} = O(1) n^{-1/8} (2n+1)^{(-1/6+\delta)(-1/4)} = O(1) n^{-1/12} \cdot n^{-\delta/4}$$

and then $n^{-\delta/4} \log(k_0 + 1) = O(1)$ eliminates the logarithm.

b) $x \geqq \sqrt{2n+1} - (2n+1)^{-1/6+\delta}$. We know from [6] that $\sqrt{2n+1} - x_1 \asymp n^{-1/6}$, hence by (4)

$$\text{(8)} \qquad \sqrt{2n+1} - x_k \asymp \sum_{i=1}^{k} n^{-1/6} i^{-1/3} \asymp k^{2/3} n^{-1/6} \quad \left(1 \leqq k \leqq \frac{n}{2}\right).$$

Consequently for $x \leqq \sqrt{2n+1} - \varepsilon_0 \cdot n^{-1/6}$ ($\varepsilon_0 > 0$ is small)

$$n^{-1/6+\delta} \geqq c(\sqrt{2n+1} - x) \asymp \sqrt{2n+1} - x_{k_0} \asymp k_0^{2/3} \cdot n^{-1/6},$$

i. e., $k_0^{2/3} \leqq c n^{\delta}$, $k_0 \leqq c n^{3\delta/2}$, and this implies by $3\delta/2 < 1$ that $k_0^{-1/6} \log(k_0 + 1) \leqq c n^{1/6}$. If $x \geqq \sqrt{2n+1} - \varepsilon_0 n^{-1/6}$ then $k_0 = 1$, hence $k_0^{-1/6} \log(k_0 + 1) = O(1)$.

Summarizing our estimates we obtain

$$e^{-x^2/2}\sum_{k=1}^{n} e^{x_k^2/2}\bigl|l_k(x)\bigr| = O(1)n^{1/6}. \tag{9}$$

Now we prove the sharpness of the upper estimate in (9). Let x^* be such that $x^* > 0$ and $\bigl|H_n(x^*)\bigr| \asymp e^{(x^*)^2/2}\frac{\sqrt{2^n n!}}{n^{1/12}}$. Then

$$\sum_{k=1}^{n} e^{x_k^2/2}\bigl|l_k(x^*)\bigr| \asymp \frac{1}{\sqrt{2^n n!}}\sum_{k=1}^{n}\sqrt{\phi_n(x_k)}\left|\frac{H_n(x^*)}{x^*-x_k}\right| \geqq$$

$$\geqq c\cdot\frac{1}{\sqrt{2^n n!}}\cdot\frac{1}{\sqrt{n}}\sum_{\substack{k=1\\ x_k\leqq-\sqrt{n}}}^{n}\sqrt{\phi_n(x_k)}\bigl|H_n(x^*)\bigr| \geqq$$

$$\geqq c\cdot\frac{1}{\sqrt{n}}\cdot\frac{1}{n^{1/12}}\cdot e^{(x^*)^2/2}\sum_{k=c_0\cdot n}^{n}\frac{1}{n^{1/12}k^{1/6}} \asymp e^{(x^*)^2/2}n^{1/6},$$

where $1/2 < c_0 < 1$ is an absolute constant.

Using (9) we obtain from (1)

$$e^{-x^2/2}\bigl|f(x)-L_{n+1}(x)\bigr| = O(1)n^{1/6}w\left(f,\frac{1}{\sqrt{n}}\right).$$

The Theorem is proved.

REMARK. The estimate in the Theorem is sharp, namely there exist functions $\{f_n\}$ and real numbers $\{y_n\}$ such that $f_n \in C(\mathbf{R})$, $e^{-x^2/2}f_n(x)$ is uniformly bounded on **R** and

$$e^{-y_n^2/2}\bigl|f_n(y_n)-L_{n+1}(f_n,y_n)\bigr| \geqq cn^{1/6}w\left(f_n,\frac{1}{\sqrt{n}}\right).$$

This can be proved similarly as in [1], Satz 4, but we have to make some simple modifications, e. g. $f_n(x_k) := e^{x_k^2/2}\operatorname{sign} l_k(y_n)$, $y_n := x^*$, and we can extend the definition of f_n such that $\bigl|f_n(x)\bigr| \leqq e^{x^2/2}$ for all x.

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THE DIMENSION OF A CLOSED SUBSET OF R^n AND RELATED FUNCTION SPACES

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Dedicated to Professor Károly Tandori on his seventieth birthday

1. Introduction

Let $F \neq \emptyset$ be a closed subset of $\mathbf{R}^n$ with empty interior. There are several proposals what should be called the dimension of F, globally and locally. Besides the classical Hausdorff dimension there exist nearby but, in general, not identical definitions, better adapted to the needs of measure theory, see [14] and also [9] and [7]: Ch.II,1. One aim of our paper is to contribute to this field of research by introducing two types of a dimension of F, the distributional dimension and the cascade dimension. The first notion is connected with the question whether there exist non-trivial singular distributions with a support on F and belonging to some function spaces of Besov type on $\mathbf{R}^n$. The second notion is related to the ε-entropy and ε-capacity of F and its neighbourhood and is connected with atomic representations of function spaces. Our approach is intimately linked with function spaces on $\mathbf{R}^n$ and on F. Hence, the second aim of this paper is to introduce some function spaces of Besov type on F. In that sense this note might be considered as a direct continuation of our paper [13]. On the other hand function spaces on (closed) subsets of $\mathbf{R}^n$ have been studied extensively by A. Jonsson and H. Wallin, see [5], [6], [7], [8] and [15]. Our approach is closely related to this work and should also be seen in the context of the theory developed there.

The paper has two sections. Section 2 deals with function spaces on $\mathbf{R}^n$ and on F and related problems: dimension, extension, duality. In Section 3 we introduce two notion of dimensions. Our main results are Theorems 2.3 and 3.3.

All unimportant positive constants are denoted by c, occasionally with additional subscripts within the same formula or the same step of the proof. Furthermore, (k.l/m) refers to formula (m) in subsection k.l, whereas (j) means formula (j) in the same subsection. Similarly we refer to remarks, theorems etc.

2. Function spaces

2.1. Besov spaces on $\mathbf{R}^n$. Since this paper is a direct continuation of [13], we restrict the definitions to the bare minimum, necessary to make this paper self-contained, independently readable. We take over some material from [13].

Let $\mathbf{R}^n$ be the euclidean n-space. The Schwartz space $S(\mathbf{R}^n)$ and its dual space $S'(\mathbf{R}^n)$ of all complex-valued tempered distributions have the usual meaning here. Furthermore, $L_p(\mathbf{R}^n)$ with $0 < p \leqq \infty$, is the usual quasi-Banach space with respect to Lebesgue measure, quasi-normed by $\| \cdot \mid L_p(\mathbf{R}^n)\|$. Let $\varphi \in S(\mathbf{R}^n)$ be such that

$$\text{(1)} \qquad \operatorname{supp}\varphi \subset \{y \in \mathbf{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \text{ if } |x| \leqq 1;$$

let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for each $j \in \mathbf{N}$ (natural numbers) and put $\varphi_0 = \varphi$. Then, since $1 = \sum_{j=0}^{\infty} \varphi_j(x)$ for all $x \in \mathbf{R}^n$, the φ_j form a dyadic resolution of unity. Let $\hat{f}$ and $\check{f}$ be the Fourier transform and its inverse, respectively, of $f \in S'(\mathbf{R}^n)$. Then for any $f \in S'(\mathbf{R}^n)$, $(\varphi_j \hat{f})\check{}$ is an entire analytic function on $\mathbf{R}^n$.

Definition 1. Let $s \in \mathbf{R}$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Then $B^2_{pq}(\mathbf{R}^n)$ is the collection of all $f \in S'(\mathbf{R}^n)$ such that

$$\text{(2)} \qquad \|f \mid B^s_{pq}(\mathbf{R}^n)\|_{\varphi} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})\check{} \mid L_p(\mathbf{R}^n)\|^q\right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$) is finite.

Remark 1. Systematic treatments of the theory of these spaces may be found in [11] and [12]. In particular, $B^s_{pq}(\mathbf{R}^n)$ is a quasi-Banach space which is independent of the function $\varphi \in S(\mathbf{R}^n)$ chosen according to (1), in the sense of equivalent quasi-norms. This justifies our omission of the subscript φ in (2) in what follows. If $p \geqq 1$ and $q \geqq 1$, $B^s_{pq}(\mathbf{R}^n)$ is a Banach space.

Remark 2. Of peculiar interest for us will be the Hölder spaces $C^s(\mathbf{R}^n) = B^s_{\infty\infty}(\mathbf{R}^n)$ with $0 < s = [s] + \{s\}$, where $[s] \in \mathbf{N}_0 = \{0\} \cup \mathbf{N}$ and $0 < \{s\} < 1$, with the equivalent norm

$$\text{(3)} \qquad \|f \mid C^s(\mathbf{R}^n)\| =$$

$$= \sum_{|\alpha| \leqq [s]} \|D^\alpha f \mid L_\infty(\mathbf{R}^n)\| + \sum_{|\alpha| = [s]} \sup \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{s\}}},$$

where the supremum is taken over all $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$ with $x \neq y$.

Atoms in $\mathbf{R}^n$. We follow [13]. We adapt the atoms introduced by Frazier and Jawerth in [2], [3] and [4] to our later purposes. Let $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$, and let $\mathbf{Z}^n$ be the lattice of all points in $\mathbf{R}^n$ with integer-valued components. Let $b > 0$ be given, $\nu \in \mathbf{N}_0$ and $k \in \mathbf{Z}^n$. Then $Q_{\nu k}$ denotes a cube in $\mathbf{R}^n$ with sides parallel to the axes, centered at $x^{\nu,k} \in \mathbf{R}^n$ with

$$|x^{\nu,k} - 2^{-\nu}k| \leqq b2^{-\nu} \tag{4}$$

and with side-length $2^{-\nu}$. If Q is a cube in $\mathbf{R}^n$ and if $r > 0$, then rQ is the cube in $\mathbf{R}^n$ concentric with Q and with side-length r times the side-length of Q. We always tacitly assume in the sequel, that $d > 0$ is chosen in dependence on b such that for all choices $\nu \in \mathbf{N}_0$ and all choices of the centres $x^{\nu,k}$ in (4)

$$\bigcup_{k \in \mathbf{Z}^n} dQ_{\nu k} = \mathbf{R}^n. \tag{5}$$

Recall that $C^\sigma(\mathbf{R}^n)$ with $0 < \sigma \notin \mathbf{N}$ may be normed by (3). Let $C^0(\mathbf{R}^n)$ be the space of all complex-valued bounded continuous functions on $\mathbf{R}^n$ equipped with the L_∞-norm. If $c \in \mathbf{R}$ then we put $c_+ = \max(c, 0)$. Furthermore, we use the abbreviation

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+, \tag{6}$$

with $0 < p \leqq \infty$.

DEFINITION 2. Let $0 < p \leqq \infty$ and $s > \sigma_p$. Let $0 < \sigma \notin \mathbf{N}$. Then $a(x)$ is called an (s,p)-atom (or more precisely $(s,p)_\sigma$-atom) if

$$\operatorname{supp} a \subset dQ_{\nu k} \quad \text{for some } \nu \in \mathbf{N}_0 \text{ and some } k \in \mathbf{Z}^n \tag{7}$$

and

$$a \in C^\sigma(\mathbf{R}^n) \quad \text{with} \quad \left\| a(2^{-\nu}\cdot) \mid C^\sigma(\mathbf{R}^n)\right\| \leqq 2^{-\nu(s-\frac{n}{p})}. \tag{8}$$

REMARK 3. The number d has the above meaning, see (5) and is assumed to be fixed throughout this paper. The reason for the normalizing factor in (8) is that there exists a constant c such that for all these atoms

$$\left\| a \mid B^s_{pq}(\mathbf{R}^n)\right\| \leqq c. \tag{9}$$

In other words, atoms are normalized smooth building blocks.

REMARK 4. The above definition is adapted to our later needs, where we carry over this definition from $\mathbf{R}^n$ to closed sets in $\mathbf{R}^n$. Then the Whitney extension plays a decisive role. This explains why we used C^σ with $0 < \sigma \notin \notin \mathbf{N}$. In case of $\mathbf{R}^n$ one would otherwise prefer C^K, the space of all $f \in C^0$ with $D^\alpha f \in C^0$ if $|\alpha| \leqq K$. Doing so the normalizing condition in (8) can be re-written as

$$|D^\alpha a(x)| \leqq 2^{-\nu(s-\frac{n}{p})} 2^{\nu|\alpha|}, \quad |\alpha| \leqq K. \tag{10}$$

This is the usual way to introduce atoms, see the above mentioned papers by Frazier and Jawerth or [12]: 1.9.2, 3.2.2.

DEFINITION 3. Let $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Then b_{pq} is the collection of all sequences $\lambda = \{\lambda_{\nu k} : \lambda_{\nu k} \in \mathbf{C},\, \nu \in \mathbf{N}_0 \text{ and } k \in \mathbf{Z}^n\}$ such that

$$\|\lambda \mid b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{k \in \mathbf{Z}^n} |\lambda_{\nu k}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{11}$$

(with the usual modifications if p and/or q is infinite) is finite.

If the atom $a(x)$ is supported by $dQ_{\nu k}$ in the sense of (7) then we write $a_{\nu k}(x)$, hence

$$\operatorname{supp} a_{\nu k} \subset dQ_{\nu k}; \quad \nu \in \mathbf{N}_0 \quad \text{and} \quad k \in \mathbf{Z}^n. \tag{12}$$

Recall the abbreviation (6).

THEOREM. *Let $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $s > \sigma_p$. Let $\sigma > s$ and $\sigma \notin \notin \mathbf{N}$.*

(i) *Let $\lambda \in b_{pq}$ and $a_{\nu k}(x)$ with $\nu \in \mathbf{N}_0$, $k \in \mathbf{Z}^n$ be $(s,p)_\sigma$-atoms in the sense of Definition 2 with (12). Then*

$$\sum_{\nu=0}^{\infty} \sum_{k \in \mathbf{Z}^n} \lambda_{\nu k} a_{\nu k}(x) \tag{13}$$

converges in $S'(\mathbf{R}^n)$.

(ii) *$f \in S'(\mathbf{R}^n)$ belongs to $B^s_{pq}(\mathbf{R}^n)$ if and only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbf{Z}^n} \lambda_{\nu k} a_{\nu k}(x), \quad \textit{convergence in } S'(\mathbf{R}^n), \tag{14}$$

where $a_{\nu k}(x)$ are $(s,p)_\sigma$-atoms in the sense of Definition 2 with (12) and $\lambda \in b_{pq}$. Furthermore,

$$\inf \|\lambda \mid b_{pq}\|, \tag{15}$$

where the infimum is taken over all admissible representations (14), *is an equivalent quasi-norm in* $B^s_{pq}(\mathbf{R}^n)$.

REMARK 5. This theorem is at least in principle covered by the work of Frazier and Jawerth, see [2], [3] and [4]. Our formulation is different and, as we hope, more handsome, even on $\mathbf{R}^n$, and we switched from requirements like (10) to their counterparts (8). We used this modification in [13] to study intrinsic atomic characterizations of B^s_{pq} spaces in non-smooth domains. In the present paper we are interested in corresponding characterizations for spaces of Besov type on closed subsets of $\mathbf{R}^n$.

REMARK 6. In [13] we gave atomic characterizations of all spaces B^s_{pq}, $s \in \mathbf{R}$, on $\mathbf{R}^n$ and on domains. If $s \leqq \sigma_p$ (see (6)), then the atoms $a_{\nu k}$ on $\mathbf{R}^n$ with $\nu \in \mathbf{N}$ are required to have vanishing moments up to order $[\sigma_p - - s]$. In the same paper one can also find analogous atomic characterizations of the spaces F^s_{pq}, $0 < p < \infty$, $0 < q \leqq \infty$, $s \in \mathbf{R}$, on $\mathbf{R}^n$ and on bounded non-smooth domains.

2.2. The spaces $\mathcal{C}^\sigma(F)$. Throughout this paper $F \neq \emptyset$ is a closed subset of $\mathbf{R}^n$ with empty interior:

$$(1) \qquad F \neq \emptyset, \quad \operatorname{int} F = \emptyset \quad (\text{that means } \overline{\mathbf{R}^n \setminus F} = \mathbf{R}^n).$$

Let $\mathcal{C}^0(F)$ be the space of all complex-valued bounded continuous functions on F equipped with the L_∞-norm. Of course, $\mathcal{C}^0(F)$ is the restriction to F of $\mathcal{C}^0(\mathbf{R}^n)$ introduced preceding Definition 2.1/2. For the definition of the Lipschitz spaces $\mathrm{Lip}(\sigma, F)$ with $\sigma > 0$ we refer to [10]: pp.173 and 176 and [7]: 2.3. If $\sigma > 1$ then one needs in general the jet-version characterized by $\{f^{(j)}\}_{|j| \leqq k}$ where $k \in \mathbf{N}$ with $k < \sigma \leqq k + 1$, normed in a way described there. The advantage of this definition is that Whitney's extension method leads to linear extension operators.

We prefer here the following modification. For $c \in \mathbf{R}$, $[c]$ denotes the largest integer not greater than c.

DEFINITION 1. Let $0 < \sigma \notin \mathbf{N}$. Then $\mathcal{C}^\sigma(F)$ is the space of all functions $f \in \mathcal{C}^0(F)$ for which there exists a corresponding jet $\{f^{(j)}\}_{|j| \leqq [\sigma]} \in \mathrm{Lip}(\sigma, F)$ with $f^0 = f$.

REMARK 1. Of course $\mathcal{C}^\sigma(F)$ is normed via $\mathrm{Lip}(\sigma, F)$. Now we have the advantage that the so-defined spaces $\mathcal{C}^\sigma(F)$ are continuously embedded into $\mathcal{C}^0(F)$. Of course they are restrictions of the corresponding spaces on $\mathbf{R}^n$,

$$(2) \qquad \mathcal{C}^\sigma(F) = \mathcal{C}^\sigma(\mathbf{R}^n) \mid F = B^\sigma_{\infty\infty}(\mathbf{R}^n) \mid F,$$

where the latter equality holds since $\sigma \notin \mathbf{N}$, see Remark 2.1/2. This modification paves the way to an atomic characterization of function spaces on F.

Atoms on F. To introduce atoms on F with (1) we again rely on the cubes $Q_{\nu k}$, see 2.1. Of course only cubes with $dQ_{\nu k} \cap F \neq \emptyset$ are of interest. If $Q_{\nu k}$ is such a cube we additionally assume

$$x^{\nu,k} \in F, \tag{3}$$

that means $Q_{\nu k}$ is centered at F. Let, for brevity,

$$F^{\nu} = \{x \in \mathbf{R}^n : 2^{-\nu} x \in F\}, \quad \nu \in \mathbf{N}_0. \tag{4}$$

DEFINITION 2. Let $\sigma > 0$, $0 < \tau \notin \mathbf{N}$. Then $a(x)$ is called a σ_τ-atom on F if

$$a \in \mathcal{C}^\tau(F) \quad \text{with} \quad \left\| a(2^{-\nu}\, \cdot\,) \mid \mathcal{C}^\tau(F^\nu) \right\| \leqq 2^{-\nu\sigma} \tag{5}$$

and

$$\operatorname{supp} a \subset F \cap dQ_{\nu k} \text{ for some } \nu \in \mathbf{N}_0 \text{ and } k \in \mathbf{Z}^n \text{ with (3).} \tag{6}$$

REMARK 2. The analogue of this definition in [13] looks more complicated. But our situation here is somewhat simpler. The same holds for Definition 2.1/2 and its analogue in the mentioned paper.

Next we modify Definition 2.1/3 in an obvious way. Let now

$$\lambda = \{\lambda_{\nu k} : \lambda_{\nu k} \in \mathbf{C},\ \nu \in \mathbf{N}_0,\ k \in \mathbf{Z}^n,\ x^{\nu,k} \in F\}, \tag{7}$$

and let $\sum_{k \in \mathbf{Z}^n}^{\nu,F}$ for fixed $\nu \in \mathbf{N}_0$ be the sum over $k \in \mathbf{Z}^n$ with $x^{\nu,k} \in F$.

DEFINITION 3. Let $0 < p \leqq \infty$, $0 < q \leqq \infty$. Then $b_{pq}(F)$ is the collection of all sequences λ given by (7) such that

$$\left\| \lambda \mid b_{pq}(F) \right\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{k \in \mathbf{Z}^n}^{\nu,F} |\lambda_{\nu k}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{8}$$

(with the usual modification if p and/or q is infinite) is finite.

REMARK 3. See also [13]: Definition 3.4/1 for a similar construction for bounded (non-smooth) domains instead of F.

We are interested in atomic representations

$$f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbf{Z}^n}^{\nu,F} \lambda_{\nu k} a_{\nu k}(x) \tag{9}$$

of function spaces on F of Besov type, where in analogy to (2.1/12) the subscripts ν, k indicate that $a_{\nu k}$ has a support in $F \cap dQ_{\nu k}$ in the sense of (6).

Let $a_{\nu k}(x)$ be σ_τ-atoms with $\tau > \sigma$, $\tau \notin \mathbf{N}$ and with (6). Let $\lambda \in$ $\in b_{\infty\infty}(F)$. Then it follows easily that (9) converges in $C^0(F)$.

PROPOSITION. *Let $0 < \sigma \notin \mathbf{N}$ and let $\sigma < \tau \notin \mathbf{N}$. Then $f \in C^0(F)$ belongs to $C^\sigma(F)$ if and only if it can be represented by* (9), *convergence in $C^0(F)$, with $\lambda \in b_{\infty\infty}(F)$ and σ_τ-atoms $a_{\nu k}(x)$. Furthermore,*

$$\inf \left\| \lambda \mid b_{\infty\infty}(F) \right\|, \tag{10}$$

where the infimum is taken over all admissible representations (9), *is an equivalent norm in $C^\sigma(F)$.*

PROOF. *Step* 1. Let f be given by (9) with $\lambda \in b_{\infty\infty}(F)$ and σ_τ-atoms $a_{\nu k}(x)$. By the above mentioned Whitney extension method and the multiplication with an appropriate cut-off function each atom $a_{\nu k}(x)$ can be extended individually to a corresponding atom $b_{\nu k}(x)$ on $\mathbf{R}^n$ in the sense of Definition 2.1/2. See [13]: 3.5 for the details of this construction. Let for $\nu \in \mathbf{N}_0$, $k \in \mathbf{Z}^n$ with $dQ_{\nu k} \cap F = \emptyset$, $\lambda_{\nu k} = 0$ and $b_{\nu k} = 0$. By Theorem 2.1 the (non-linearly) extended function

$$\operatorname{ext} f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbf{Z}^n} \lambda_{\nu k} b_{\nu k}(x)$$

belongs to $B^\sigma_{\infty\infty}(\mathbf{R}^n) = C^\sigma(\mathbf{R}^n)$. Hence, by restriction, $f \in C^\sigma(F)$.

Step 2. Let $f \in C^\sigma(F)$. Then by Whitney's extension method we find an $\operatorname{ext} f \in C^\sigma(\mathbf{R}^n) = B^\sigma_{\infty\infty}(\mathbf{R}^n)$ with $\operatorname{ext} f \mid F = f$. Using again Theorem 2.1 we obtain an atomic representation of $\operatorname{ext} f$ which, by restriction, yields the representation (9) with $\lambda \in b_{\infty\infty}(F)$. Since all these procedures are norm-preserving (besides unimportant constants) we have that (10) is an equivalent norm.

REMARK 4. The proof is surprisingly simple. But we used two rather deep ingredients: Whitney's extension method, now non-linear because of our definition of $C^\sigma(F)$, and the knowledge of the atomic representations in $\mathbf{R}^n$ covered by Theorem 2.1. In particular, the main assertion of the above proposition, the independence of the chosen value of τ, is induced by that theorem.

2.3. The spaces $\mathcal{B}^\sigma_{pq}(F)$. Again $F \neq \emptyset$ denotes a closed subset of $\mathbf{R}^n$ with empty interior, see (2.2/1). Encouraged by Proposition 2.2 and in analogy to [13] we are going to introduce function spaces of Besov type on F, always considered as subspaces of $C^0(F)$. We have to circumvent the problem of the (local or global) dimension of F which is quite clear if one looks at Definition 2.1/2 and Theorem 2.1 on the one hand and Definition 2.2/2 on the other hand. In 2.4 we add a discussion about this subject.

We rely again on atomic representations

$$f = \sum_{\nu=0}^{\infty} \sum_{k \in \mathbf{Z}^n}^{\nu,F} \lambda_{\nu k} a_{\nu k}(x), \tag{1}$$

now with σ_τ-atoms $a_{\nu k}(x)$ and $\lambda \in b_{pq}(F)$.

DEFINITION. Let $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $\sigma > 0$. Let $\sigma + \frac{n}{p} < \tau \notin \mathbf{N}$. Then $\mathcal{B}^\sigma_{pq}(F)$ is the collection of all $f \in \mathcal{C}^0(F)$ which can be represented by (1) with $\lambda \in b_{pq}(F)$ and σ_τ-atoms $a_{\nu k}(x)$ on F.

REMARK 1. As we mentioned above, under the conditions of the definition, (1) converges always in $\mathcal{C}^0(F)$. Furthermore, it turns out that $\mathcal{B}^\sigma_{pq}(F)$ is independent of the chosen τ in the sense of equivalent quasi-norms. This justifies our omission of an additional index τ in the definition of $\mathcal{B}^\sigma_{pq}(F)$.

REMARK 2. By Proposition 2.2 we have

$$\mathcal{C}^\sigma(F) = \mathcal{B}^\sigma_{\infty\infty}(F) \tag{2}$$

where $0 < \sigma \notin \mathbf{N}$. Now we extend (2) to $\sigma \in \mathbf{N}$. Then we have the full scale of Hölder–Zygmund spaces on F.

THEOREM. *Let* $0 < p \leqq \infty$, $0 < q \leqq \infty$ *and* $\sigma > 0$.
(i) *Let* $\sigma + \frac{n}{p} < \tau \notin \mathbf{N}$. *Let*

$$\| f \mid \mathcal{B}^\sigma_{pq}(F) \| = \inf \| \lambda \mid b_{pq}(F) \|, \tag{3}$$

where the infimum is taken over all representations (1) *of* $f \in \mathcal{C}^0(F)$ *with* $\lambda \in b_{pq}(F)$. *Then* (3) *is a quasi-norm (norm if* $p \geqq 1$ *and* $q \geqq 1$*) and* $\mathcal{B}^\sigma_{pq}(F)$ *equipped with* (3) *is a quasi-Banach space (Banach space if* $p \geqq 1$ *and* $q \geqq 1$*). Furthermore,* $\mathcal{B}^\sigma_{pq}(F)$ *is independent of* τ *in the sense of equivalent quasi-norms.*
(ii) *Let* $s = \sigma + \frac{n}{p}$. *Then* $\mathcal{B}^\sigma_{pq}(F)$ *is the restriction of* $B^s_{pq}(\mathbf{R}^n)$ *to* F,

$$\mathcal{B}^\sigma_{pq}(F) = B^s_{pq}(\mathbf{R}^n) \mid F. \tag{4}$$

(iii) *Let* $0 < \sigma \notin \mathbf{N}$. *Then*

$$\mathcal{C}^\sigma(F) = \mathcal{B}^\sigma_{\infty\infty}(F), \tag{5}$$

where $\mathcal{C}^\sigma(F)$ *has the same meaning as in* 2.2.

PROOF. Part (iii) is covered by the above remarks.

Let $a_{\nu k}(x)$ be a σ_τ-atom on F in the sense of Definition 2.2/2. Using again Whitney's extension method combined with the multiplication by an appropriate cut-off function we can extend $a_{\nu k}(x)$ to an $(s,p)_\tau$-atom on $\mathbf{R}^n$. For the details of this individual extension we refer to [13]: 3.5. Now both (i) and (ii) follow in the same way as in [13]: 3.5 and in the proof of Proposition 2.2.

REMARK 3. The proof makes clear that this part of our paper is the direct continuation of the relevant parts of [13].

2.4. The dimension problem. Let $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $s > \frac{n}{p}$, and let $F = \mathbf{R}^d$ with $0 \leqq d < n$, interpreted in the usual way as a subspace of $\mathbf{R}^n$. By the known trace theorem, see [11]: 2.7.2, and the above considerations we have

$$B^s_{pq}(\mathbf{R}^n) \mid F = B^{s_d}_{pq}(\mathbf{R}^d) = \mathcal{B}^\sigma_{pq}(F) \tag{1}$$

with

$$0 < \sigma = s - \frac{n}{p} = s_d - \frac{d}{p}. \tag{2}$$

In other words, σ, sometimes called differential dimension, is invariant, but neither are the smoothness s, nor, of course, the dimension. This sheds some light on our approach in 2.3 and makes clear why we used the letter $\mathcal{B}$ instead of B so far. Only in the case $p = \infty$ the above problem does not occur. If one wishes to introduce Besov spaces B^s_{pq} on arbitrary closed subsets F of $\mathbf{R}^n$ with (2.2/1) then one must first clarify what is meant by its, maybe fractional, dimension d, locally or globally, and define s_d via (2). We return to this problem later on looking for adequate notions of dimensions. For that purpose duality taken as a guide is shortly discussed in 2.6.

2.5. The extension problem. Let again $F \neq \emptyset$ be a closed subset of $\mathbf{R}^n$ with empty interior. With $\sigma > 0$ and $s = \sigma + \frac{n}{p}$ we have (2.3/4). Of course, the restriction operator from $B^s_{pq}(\mathbf{R}^n)$ onto $\mathcal{B}^\sigma_{pq}(F)$ is linear and bounded. On the other hand, the extensions constructed so far are non-linear for at least two reasons. Firstly our definition of $\mathcal{C}^\sigma(F)$ destroys the linearity of Whitney's extension method, at least in general. Secondly to ensure that atoms on F are extended to atoms on $\mathbf{R}^n$ one needs cut-off functions which are individual, see [13]: 3.5 for details. But it would be desirable to have linear and bounded extension operators in connection with (2.3/4). We start with a preparation.

PROPOSITION. *Let $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $s > \frac{n}{p}$.*

(i)

$$B^s_{pq,F}(\mathbf{R}^n) = \left\{ g \in B^s_{pq}(\mathbf{R}^n) : g|F = 0 \right\} \tag{1}$$

is a closed subspace of $B^s_{pq}(\mathbf{R}^n)$.

(ii) *Let* $\sigma = s - \frac{n}{p}$. *Then* $\mathcal{B}^\sigma_{pq}(F)$ *is isomorphic to the factor space*

$$B^s_{pq}(\mathbf{R}^n) \mid B^s_{pq,F}(\mathbf{R}^n). \tag{2}$$

PROOF. Part (i) follows immediately from $B^s_{pq}(\mathbf{R}^n) \subset C^0(\mathbf{R}^n)$, see [11]: 2.7.1. Now (ii) is a consequence of Theorem 2.3.

Extension operator. Assume that $B^s_{pq,F}(\mathbf{R}^n)$ is a complemented subspace, that means that there exists a linear and bounded projection operator P from $B^s_{pq}(\mathbf{R}^n)$ onto $B^s_{pq,F}(\mathbf{R}^n)$. Then

$$\mathrm{ext} : f \in \mathcal{B}^\sigma_{pq}(F) \mapsto (\mathrm{id} - P)g, \quad g \in B^s_{pq}(\mathbf{R}^n),\ g|F = f \tag{3}$$

may serve as a linear bounded extension operator we are looking for. Unfortunately it is unlikely that, in general, $B^s_{pq,F}(\mathbf{R}^n)$ is a complemented subspace. See [1]: VII, p.157, where the Lindenstrauss–Tzafriri result is quoted. It states that Hilbert spaces are the only Banach spaces such that any closed subspace is complemented. On the other hand in case of Hilbert spaces we have even orthogonal projections.

THEOREM. *Let* $\sigma > 0$. *Then*

$$\mathcal{H}^\sigma(F) = \mathcal{B}^\sigma_{2,2}(F) \tag{4}$$

is a Hilbert space (equivalent norm) and there exists a linear and bounded extension operator from $\mathcal{H}^\sigma(F)$ *into*

$$H^s(\mathbf{R}^n) = B^s_{2,2}(\mathbf{R}^n) \quad \text{with} \quad s = \sigma + \frac{n}{p}. \tag{5}$$

PROOF. The proof is obvious by what has been said preceding the theorem.

2.6. Duality. Interpreted as the usual $S - S'$ pairing we have

$$\left(B^s_{pq}(\mathbf{R}^n)\right)' = B^{-s}_{p'q'}(\mathbf{R}^n), \tag{1}$$

where $s \in \mathbf{R}$, $1 \leqq p < \infty$, $0 < q < \infty$ with $q' = \infty$ if $0 < q \leqq 1$ and otherwise

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1. \tag{2}$$

This can be complemented if $p = \infty$ and/or $q = \infty$, or $p < 1$, see [11]: 2.11. There might be a temptation to introduce spaces on the above closed set F

(see (2.2/1)) with negative smoothness s or negative differential dimension σ via duality. However, at least at the first glance there is no natural counterpart of $S'(\mathbf{R}^n)$. We refer in this context to the papers by Jonsson and Wallin, especially [8]. We shall not follow this path.

We are going to use duality as a vehicle to deal with the dimension problem. For that purpose we assume that $B^s_{pq,F}(\mathbf{R}^n)$ is defined via (2.5/1) whenever the trace on F makes sense. For example, if $1 \leqq p \leqq \infty$ and $F = \mathbf{R}^d$ with $0 \leqq d < n$, then this is the case if $s > (n-d)/p$. Let, in addition, $p \geqq 1$, such that (1) is applicable. Then by standard arguments the dual space of the factor space in (2.5/2) may be identified with

$$\{f \in B^{-s}_{p'q'}(\mathbf{R}^n) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbf{R}^n) \text{ with } \varphi|F = 0\}. \tag{3}$$

Since F has an empty interior it is clear that the space in (3) consists exclusively of singular distributions (with the exception of 0-distribution). Looking for the largest non-trivial spaces one arrives via embedding arguments at $p' = q' = \infty$, that means the spaces

$$\mathcal{C}^{\tau,F}(\mathbf{R}^n) = \{f \in \mathcal{C}^{\tau}(\mathbf{R}^n) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbf{R}^n) \text{ with } \varphi|F = 0\}, \tag{4}$$

where $\mathcal{C}^\tau(\mathbf{R}^n) = B^\tau_{\infty\infty}(\mathbf{R}^n)$, now with $\tau \leqq 0$. Searching for the largest τ for which $\mathcal{C}^{\tau,F}(\mathbf{R}^n)$ is non-trivial means that one looks for the most massive parts of F, being able to carry such a non-trivial singular distribution. To illustrate the situation we deal first with the case $F = \mathbf{R}^d$. Here δ^{n-d} is the δ-distribution in $\mathbf{R}^{n-d}$. Furthermore, $x = (x^d, x^{n-d})$ with $x^d \in \mathbf{R}^d$, $x_{n-d} \in \mathbf{R}^{n-d}$.

PROPOSITION. *Let $F = \mathbf{R}^d$ with $0 \leqq d < n$.*
(i) *Let $f \in S'(\mathbf{R}^n)$. Then*

$$f(\varphi) = 0 \quad \textit{for all} \quad \varphi \in S(\mathbf{R}^n) \quad \textit{with} \quad \varphi|F = 0 \tag{5}$$

if and only if

$$f = f_d \otimes \delta^{n-d} \quad \textit{with} \quad f_d \in S'(\mathbf{R}^d). \tag{6}$$

(ii) *$\mathcal{C}^{-n+\gamma,F}(\mathbf{R}^n)$ is non-trivial if and only if*

$$\gamma \leqq d. \tag{7}$$

PROOF. *Step* 1. We prove (i). The "only if" part is clear. Assume that we have $f \in S'(\mathbf{R}^n)$ with (5). Let $\varphi(x) = \psi(x^d)\eta(x^{n-d})$. For fixed ψ

$$\eta \mapsto f_\psi(\eta) = f(\psi\eta) \tag{8}$$

belongs to $S'(\mathbf{R}^{n-d})$ and it holds

$$\operatorname{supp} f_\psi = \{0\} \text{ in } \mathbf{R}^{n-d}. \tag{9}$$

Hence, f_ψ is a finite linear combination of δ^{n-d} and some of its derivatives. Since $f_\psi(\eta) = 0$ if $\eta(0) = 0$ it follows easily

$$f_\psi = c(\psi)\delta^{n-d}, \tag{10}$$

and $f_d(\psi) = c(\psi)$ belongs to $S'(\mathbf{R}^d)$.

Step 2. We prove (ii). Let f be given by (6). Let φ^d and φ_j^{n-d}, $j \in \mathbf{N}$, be the same functions as in 2.1 with $\mathbf{R}^d$ and $\mathbf{R}^{n-d}$ respectively. We have

$$\varphi^d(2^{-j}\,\cdot)f_d \to f_d \text{ in } S'(\mathbf{R}^d) \text{ if } j \to \infty \tag{11}$$

and

(12)
$$\left\| \left(\varphi^d(2^{-j}\,\cdot)\varphi_j^{n-d}\hat{f}\right)^\vee \,|\, L_\infty(\mathbf{R}^n)\right\| = c2^{(n-d)j} \left\| \left(\varphi^d(2^{-j}\,\cdot)\hat{f}\right)^\vee \,|\, L_\infty(\mathbf{R}^d)\right\|$$

with $c \neq 0$. Her we used $\varphi_j^{n-d} = \varphi_1^{n-d}(2^{-j+1}\,\cdot)$. Now (7) follows from the counterpart of (2.1/2) with $p = q = \infty$. We tacitly used a replacement of the annuli in 2.1 by cubes, see [11]: 2.5.4, which, in particular, covers also the "if" part of (ii).

3. Dimensions

3.1. The distributional dimension. By (2.6/7) the dimension d of $F = \mathbf{R}^d$ can be characterized by asking under which conditions $\mathcal{C}^{-n+\gamma,F}(\mathbf{R}^n)$ is non-trivial. We take this observation as a starting point of a corresponding definition. However, for more general sets F a structural result of type (2.6/6) cannot be expected.

DEFINITION. Let $F \neq \emptyset$ be a closed subset of $\mathbf{R}^n$ with empty interior. Then the distributional dimension of F is

$$\dim_D(F) = \sup\left\{\gamma : \mathcal{C}^{-n+\gamma,F}(\mathbf{R}^n) \text{ is non-trivial}\right\}. \tag{1}$$

REMARK 1. We defined $\mathcal{C}^{\tau,F}(\mathbf{R}^n)$ in (2.6/4). Of course

$$0 \leqq \dim_D(F) \leqq n \tag{2}$$

since $\delta \in \mathcal{C}^{-n}(\mathbf{R}^n)$ and since $\mathcal{C}^{\varepsilon}(|\mathbf{R}^n)$ with $\varepsilon > 0$ consists of continuous functions. Furthermore,

$$\dim_D(\mathbf{R}^d) = d \tag{3}$$

by Proposition 2.6. Thus, any (smooth) d-dimensional surface in $\mathbf{R}^n$ has also the distributional dimension d, as it should be.

REMARK 2. The distributional dimension is local by nature: One has to look for the most massive part of F; the other parts are without any influence. This is in contrast to the ε-entropy or ε-capacity.

REMARK 3. If F is sufficiently regular then $\dim_D(F)$ coincides with the Hausdorff dimension and also with the uniform metric dimension, see [14].

3.2. The cascade dimension. The question is how to calculate $\dim_D(F)$, which may be fractional. For that purpose we introduce a more geometrical dimension which is closely connected with the ε-entropy and ε-capacity of sets, but now in an adapted localized version. See also [9].

DEFINITION. Let $F \neq \emptyset$ be a closed subset of $\mathbf{R}^n$ with empty interior.
(i) Let $\gamma > 0$ and $2^\gamma \in \mathbf{N}$. Then a sequence of points

$$\left\{ x^{j,k} \in \mathbf{R}^n \quad \text{with} \quad j = J, \dots \quad \text{and} \quad k = 1, \dots, 2^{\gamma(j-J)} \right\}$$

is called a γ-cascade (with respect to F) if

$$2^{-j-1} \leqq \operatorname{dist}(x^{j,k}, F) \leqq 2^{-j} \tag{1}$$

for all admissible j and k;

$$\operatorname{dist}(x^{j,k}, x^{j+1,m}) \geqq c_1 2^{-j} \tag{2}$$

and

$$\operatorname{dist}(x^{j,k_1}, x^{j,k_2}) \geqq c_1 2^{-j} \tag{3}$$

for some $c_1 > 0$ and all admissible j, k, m, k_1 and k_2 with $k_1 \neq k_2$;

$$\operatorname{dist}(x^{j,k}, x^{j+1,l}) \leqq c_2 2^{-j} \tag{4}$$

for some $c_2 > 0$, all admissible j, k and

$$l = 2^\gamma(k-1) + 1, \dots, 2^\gamma k. \tag{5}$$

(ii) The cascade dimension of F is

$$\dim_C(F) = \sup\{\gamma : \text{there exists a } \gamma\text{-cascade}\}. \tag{6}$$

REMARK 1. One can replace the supremum in (6) by the maximum. In each layer

$$\{x \in \mathbf{R}^n : 2^{-j-1} \leqq \operatorname{dist}(x, F) \leqq 2^{-j}\} \tag{7}$$

we have by (3) the usual procedure of the ε-capacity with $\varepsilon \sim 2^{-j}$. But the allowed density of the points is getting larger and larger if $j \to \infty$. By (4) the 2^γ points $x^{j+1,l}$ with (5) are subordinated to $x^{j,k}$. In other words, any point $x^{j,k}$ is the spring of 2^γ points $x^{j+1,l}$ with (5), where all these points keep maximal distances by (2) and (3).

REMARK 2. To get a feeling what is going on one can again consider $F = \mathbf{R}^d$, $0 \leqq d < n$. Looking for γ-cascades there is no contribution "orthogonal" to $\mathbf{R}^d$ and there are $\gamma = d$ contributions "parallel" to $\mathbf{R}^d$. In other words whether one finds a γ-cascade depends on the question whether one finds in each layer (7) "parallel" to F sufficiently many points satisfying (3) and (4). In any case

$$0 \leqq \dim_C(F) \leqq n. \tag{8}$$

Indeed: choose a point $x \notin F$ and connect it with a point on F that minimizes its distance from F. On that line one constructs easily a 0-cascade. This proves the left-hand side of (8).As for the right-hand side we obtain a much sharper result in the next subsection.

REMARK 3. Instead of points with controlled distances one might use disjoint balls of a given radius ε and ask how densely they can be packed. Then one is near to constructions suggested in [9]: p. 179.

3.3. The main theorem. All notations have the previous meaning, in particular the two types of dimensions we introduced in the preceding two subsections.

THEOREM. *Let $F \neq \emptyset$ be a closed set in $\mathbf{R}^n$ with empty interior. Then*

$$0 \leqq \dim_C(F) \leqq \dim_D(F) \leqq n. \tag{1}$$

PROOF. *Step* 1. By (3.1/2) and the left-hand side of (3.2/8) we must prove the middle part of (1).

Step 2. Let $\gamma = \dim_C(F)$ and let $\{x^{j,k}\}$ be a corresponding γ-cascade. We are going to construct a non-trivial distribution $f \in \mathcal{C}^{-n+\gamma,F}(\mathbf{R}^n)$ in the sense of Definition 3.1 by means of an atomic representation. Since we are now concerned with negative smoothness indices some of the atoms in this representation are required to have vanishing moments up to a certain order, say $L \geqq 0$. See also Remark 2.1/5, Remark 2.1/6 and the papers mentioned there. Details will be given in due course.

Let $\{Q_{j,k}\}$ be a related sequence of disjoint cubes centered at $x^{j,k}$ and with side-length $\varrho 2^{-j}$ for some sufficiently small $\varrho > 0$. Let φ be a C^∞ function in $\mathbf{R}^n$ which has a support in a cube centered at the origin with side-length ϱ such that

$$\int \varphi(x)\,dx = 1, \quad \int x^\beta \varphi(x)\,dx = 0 \ \text{ if } \ 0 < |\beta| \leqq L \tag{2}$$

for some $L \in \mathbf{N}$. Then

$$\varphi_{j,k}(x) = 2^{(n-\gamma)j}\varphi\big(2^j(x - x^{j,k})\big) \tag{3}$$

are located in the above cubes $Q_{j,k}$. This corresponds to (2.1/12) and (2.1/10) with $j = \nu$, $p = \infty$ and $s = -n + \gamma$ besides some constants. Let $x^{j+1,l}$ be the 2^γ subordinated points with respect to $x^{j,k}$ in the sense of (3.2/4) and (3.2/5). We put

$$a_{j,k}(x) = -\varphi_{j,k}(x) + \sum_{l=2^\gamma(k-1)+1}^{2^\gamma k} \varphi_{j+1,l}(x). \tag{4}$$

Thus we have

$$\int x^\beta a_{j,k}(x)\,dx = 0 \ \text{ if } \ 0 \leqq |\beta| \leqq L, \tag{5}$$

now with $\beta = 0$ included. We claim that if we chose L sufficiently large, $L \geqq n - \gamma$,

$$f = \varphi_{J,1}(x) + \sum_{j,k} a_{j,k}(x) \tag{6}$$

is the distribution we are looking for. Firstly, by (3)–(6) and the theory of atomic representations of distributions in $\mathcal{C}^{-n+\gamma}(\mathbf{R}^n)$ developed in [2], [3], [4], see also [13], we have

$$f \in \mathcal{C}^{-n+\gamma}(\mathbf{R}^n). \tag{7}$$

In particular, (6) converges in $S'(\mathbf{R}^n)$. Let $j_0 > J$ and

$$f^{j_0} = \varphi_{J,1}(x) + \sum_{j \leqq j_0} \sum_k a_{j,k}(x). \tag{8}$$

All terms cancel each other with the exception of terms with $j = j_0$. This proves

$$\text{(9)} \qquad \operatorname{supp} f \subset F \quad \text{and} \quad \int f^{j_0}(x)\,dx = 2^{-\gamma J},$$

the first by $f^{j_0} \to f$ in $S'(\mathbf{R}^n)$, the second by (2)–(4). By the last assertion, f is not trivial. Finally, let $\psi \in S(\mathbf{R}^n)$ with $\psi(x) = 0$ on F. Then we have

$$\text{(10)} \qquad f(\psi) = \lim_{j_0 \to \infty} f^{j_0}(\psi) = \lim_{j_0 \to \infty} \int f^{j_0}(x)\psi(x)\,dx = 0,$$

the latter again by (2)–(4) and $\psi(x) = O\big(\operatorname{dist}(x, F)\big)$. Hence, $f \in \mathcal{C}^{-n+\gamma,F}(\mathbf{R}^n)$ in the sense of Definition 3.1. The proof is complete.

3.4. Besov spaces $B^s_{pq}(F)$. If s stands for smoothness then (2.4/1) and (2.4/2) suggest how $B^s_{pq}(F)$ and $\mathcal{B}^\sigma_{pq}(F)$ might be related. However the, say, distributional dimension of F reflects the most massive part of F. So it seems to be reasonable to introduce a dimension at each point $x \in F$.

DEFINITION. (local distributional dimension). Let $F \neq \emptyset$ be a closed set in $\mathbf{R}^n$ with empty interior. Then

$$\text{(1)} \qquad d(x) = \inf \dim_D(F \cap B), \quad x \in F,$$

where the infimum is taken over all balls B centered at x.

REMARK. We have

$$\text{(2)} \qquad d(x) = \lim_{\varepsilon \to 0} \dim_D \{y : y \in F,\ |x - y| \leqq \varepsilon\}.$$

Besov spaces. Let $\sigma > 0$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Then

$$\text{(3)} \qquad B^{s(x)}_{pq}(F) = \mathcal{B}^\sigma_{pq}(F) \ \text{ with } \ s(x) = \sigma + \frac{d(x)}{p}$$

is at least reasonable. It coincides with $B^s_{pq}(F)$, defined in the usual way if F is a smooth d-dimensional surface, see also (2.4/1) and (2.4/2). But now the smoothness $s(x)$ may vary from point to point.

Added in proof (December 1, 1994). Let $\dim_H F$ be the Hausdorff dimension of an arbitrary set F in $\mathbf{R}^n$. We obtained recently the following result:

THEOREM. *Let F be a Borel set (or, more generally, a Suslin set) in $\mathbf{R}^n$ with empty interior. Then* $\dim_H F = \dim_D F$.

A proof will be published elsewhere.

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ON A RESTRICTION PROBLEM OF DE LEEUW TYPE FOR LAGUERRE MULTIPLIERS

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Dedicated to Professor Károly Tandori on the occasion of his 70th birthday

1. Introduction

In 1965 K. de Leeuw [3] proved among other things in the Fourier transform setting:

If a continuous function $m(\xi_1, \ldots, \xi_n)$ on $\mathbf{R}^n$ generates a bounded transformation on $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, then its trace $\tilde{m}(\xi_1, \ldots, \xi_k) = m(\xi_1, \ldots, \xi_k, 0, \ldots, 0)$, $k < n$, generates a bounded transformation on $L^p(\mathbf{R}^k)$.

The purpose of this paper is to discuss the analogous question: suppose $\{m_k\}_{k \in \mathbf{N}_0}$ generates a bounded transformation with respect to a Laguerre function expansion of order α on some L^p-space, does it also generate a corresponding bounded map with respect to a Laguerre function expansion of order β? To become more precise let us first introduce some notation. Consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \left\{ f : \|f\|_{L^p_{w(\gamma)}} = \left(\int_0^\infty \left| f(x) e^{-x/2} \right|^p x^\gamma \, dx \right)^{1/p} < \infty \right\}, \quad 1 \leqq p < \infty,$$

$$L^\infty_{w(\gamma)} = \left\{ f : \|f\|_{L^\infty_{w(\gamma)}} = \operatorname*{ess\,sup}_{x>0} \left| f(x) e^{-x/2} \right| < \infty \right\}, \quad p = \infty,$$

where $\gamma > -1$. Let $L_n^\alpha(x)$, $\alpha > -1$, $n \in \mathbf{N}_0$, denote the classical Laguerre polynomials (see Szegö [15, p. 100]) and set

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0), \qquad L_n^\alpha(0) = A_n^\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

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Associate to f its formal Laguerre series

$$f(x) \sim \left(\Gamma(\alpha+1)\right)^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(x),$$

where the Fourier–Laguerre coefficients of f are defined by

$$\text{(1)} \qquad \hat{f}_\alpha(n) = \int_0^\infty f(x) R_n^\alpha(x) x^\alpha e^{-x}\, dx$$

(if the integrals exist). A sequence $m = \{m_k\}_{k\in\mathbf{N}_0}$ is called a (bounded) multiplier from $L^p_{w(\gamma)}$ into $L^q_{w(\delta)}$, notation $m \in M^{p,q}_{\alpha;\gamma,\delta}$, if

$$\left\| \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha \right\|_{L^q_{w(\delta)}} \leqq C \left\| \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha \right\|_{L^p_{w(\gamma)}}$$

for all polynomials f; the smallest constant C for which this holds is called the multiplier norm $\|m\|_{M^{p,q}_{\alpha;\gamma,\delta}}$. For the sake of simplicity we write $M^{p,q}_{\alpha;\gamma} :=$ $:= M^{p,q}_{\alpha;\gamma,\gamma}$ if $\gamma = \delta$ and, if additionally $p = q$, $M^p_{\alpha;\gamma} := M^{p,p}_{\alpha;\gamma}$.

We are *mainly* interested in the question: when is $M^{p,q}_{\alpha;\alpha}$ continuously embedded in $M^{p,q}_{\beta;\beta}$:

$$M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta}, \quad 1 \leqq p \leqq q \leqq \infty \;?$$

The Plancherel theory immediately yields

$$l^\infty = M^2_{\alpha;\alpha} = M^2_{\beta;\beta}, \quad \alpha, \beta > -1.$$

A combination of sufficient multiplier conditions with necessary ones indicates which results are to be expected. To this end, define the fractional difference operator Δ^δ of order δ by

$$\Delta^\delta m_k = \sum_{j=0}^{\infty} A_j^{-\delta-1} m_{k+j}$$

(whenever the series converges), the classes $wbv_{q,\delta}$, $1 \leqq q \leqq \infty$, $\delta > 0$, of weak bounded variation (see [5]) of bounded sequences which have finite norm $\|m\|_{q,\delta}$, where

$$\|m\|_{q,\delta} := \sup_k |m_k| + \sup_{N\in\mathbf{N}_0} \left(\sum_{k=N}^{2N} \left| (k+1)^\delta \Delta^\delta m_k \right|^q \frac{1}{k+1} \right)^{1/q}, \quad q < \infty,$$

$$\|m\|_{\infty,\delta} := \sup_k |m_k| + \sup_{N\in\mathbf{N}_0} \left|(k+1)^\delta \Delta^\delta m_k\right|, \quad q=\infty.$$

Observing the duality (see [14])

$$(2) \qquad M^p_{\alpha;\gamma} = M^{p'}_{\alpha;\alpha p'-\gamma p'/p}, \quad -1<\gamma<p(\alpha+1)-1, \quad 1<p<\infty,$$

where $1/p+1/p'=1$, we may restrict ourselves to the case $1<p<2$. The Corollary 1.2 b) in [14] gives the embedding

$$(3) \qquad M^p_{\alpha;\alpha} \hookrightarrow wbv_{p',s}, \quad s=(2\alpha+2/3)(1/p-1/2), \quad \alpha>-1/3,$$

when $(2\alpha+2)(1/p-1/2)>1/2$. Theorem 5 in [5] gives the first embedding in

$$wbv_{p',s} \hookrightarrow wbv_{2,s} \hookrightarrow M^p_{\beta;\beta},$$

whereas the last one follows from Corollaries 1.2 and 4.5 in [14] provided $s>$ $>\max\{(2\beta+2)(1/p-1/2),\,1\}$, $\beta>-1$. Hence, choosing $\gamma=\alpha$ in (2), we obtain

PROPOSITION 1.1. *Let $1<p<\infty$ and α be such that $(2\alpha+2/3)|1/p-$ $-1/2|>1$. Then*

$$M^p_{\alpha;\alpha} \hookrightarrow M^p_{\beta;\beta}, \quad -1<\beta<\alpha-2/3.$$

In the same way we can derive a result for $M^{p,q}$-multipliers. The necessary condition in [6, Corollary 1.3] can easily be extended in the sense of [6, Corollary 2.5 b)] to

$$\sup_k \left|(k+1)^\sigma m_k\right| + \sup_n \left(\sum_{k=n}^{2n} \left|(k+1)^{\sigma+s}\Delta^s m_k\right|^{q'}/k\right)^{1/q'} \leqq C\|m\|_{M^{p,q}_{\alpha;\alpha}},$$

where $\alpha>-1/3$, $1/q=1/p-\sigma/(\alpha+1)$, $1<p<q<2$, $(\alpha+1)(1/q-$ $-1/2)>1/4$, and $s=(2\alpha+2/3)(1/q-1/2)>0$. Using this and the sufficient condition for $M^{p,q}_{\beta;\beta}$-multipliers given in [4, Corollary 1.2], which is proved only for $\beta\geqq 0$, we obtain

$$M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta},$$

$$0\leqq\beta<\alpha-2/3, \quad (2\alpha+2/3)(1/q-1/2)>1,\ 1<p<q<2.$$

In this context let us mention that the same technique yields for $1<$ $<p,q<2$

(4)

$$M^p_{\alpha;\alpha} \hookrightarrow M^q_{\beta;\beta}, \quad (2\alpha+2/3)(1/p-1/2)>\max\{(2\beta+2)(1/q-1/2),\,1\}.$$

This embedding is in so far interesting as it allows to go from p, $1 < p < 2$, to $q \neq p$, $1 < q < 2$, connected with a loss in the size of β if $q < p$ or a gain in β if $1 < p < q < 2$; e.g.

$$M^{4/3}_{10;10} \hookrightarrow M^q_{5;5}, \quad 1.08 \leqq q \leqq 2, \quad \text{or} \quad M^{8/7}_{2;2} \hookrightarrow M^q_{4;4}, \quad 3/2 \leqq q \leqq 2.$$

Improvements of (4) can be expected by better necessary conditions and/or better sufficient conditions; but this technique *cannot* give something like

$$M^p_{\alpha;\alpha} \hookrightarrow M^q_{\beta;\beta}, \quad (\alpha+1)(1/p-1/2) > (\beta+1)(1/q-1/2), \; 1 < p, q < 2,$$

which is suggested by (4) when choosing "large" α with p near 2 since then the number $4(1/p - 1/2)/3$, which describes the smoothness gap between the necessary conditions and the sufficient conditions in [14, Corollary 1.2], is "negligible".

Concerning the general problem "When does $M^{p,q}_{\alpha;\gamma_1,\delta_1} \hookrightarrow M^{p,q}_{\beta;\gamma_2,\delta_2}$ hold?", we mention results in Stempak and Trebels [14, Corollary 4.3]: For $1 < p < \infty$ there holds

$$M^p_{\beta;\beta p/2+\delta} = M^p_{0;\delta} \quad \text{if} \begin{cases} -1 - \beta p/2 < \delta < p - 1 + \beta p/2, & -1 < \beta < 0, \\ -1 < \delta < p - 1, & 0 \leqq \beta, \end{cases}$$

which for $\delta = 0$ contains half of Kanjin's [9] result and for $\delta = p/4 - 1/2$ Thangavelu's [16]. In particular, there holds for $-1 < \beta < \alpha$, $1 < p < \infty$,

$$(5) \qquad M^p_{\beta;\beta} = M^p_{\beta;\beta p/2+\beta p(1/p-1/2)} = M^p_{\alpha;\alpha p/2+\beta p(1/p-1/2)}, \qquad (2\beta+2)|1/p - 1/2| < 1.$$

These results are based on Kanjin's [9] transplantation theorem and its weighted version in [14]. The latter gives further insight into our problem in so far as it implies that the restriction $\beta < \alpha - 2/3$ in Proposition 1.1 is not sharp.

To this end we first note that the following extension of Corollary 4.4 in [14] holds

$$wbv_{2,s} \hookrightarrow M^p_{\alpha;\alpha p/2+\eta(p/2-1)}, \quad 0 \leqq \eta \leqq 1, \quad 1 < p \leqq 2, \quad s > 1/p.$$

(For the proof observe that for $\alpha = 0$ the parameter $\gamma = \eta(p/2 - 1)$, $0 \leqq \eta \leqq$ $\leqq 1$, is admissible in [14, Theorem 1.1] and then follow the argumentation of

[14, Corollary 4.4].) This combined with (3) yields for $s = (2\alpha + 2/3)(1/p - 1/2) > 1/p$

$$M^p_{\alpha;\alpha} \hookrightarrow wbv_{2,s} \hookrightarrow M^p_{\alpha;\alpha p/2+p/2-1}, \quad 1 < p \leqq 2, \quad \alpha > (p+1)/(6-3p).$$

Thus, by interpolation with change of measure,

$$M^p_{\alpha;\alpha} \hookrightarrow M^p_{\alpha;\alpha p/2+\delta}, \quad p/2 - 1 \leqq \delta \leqq \alpha - \alpha p/2, \quad \alpha > (p+1)/(6-3p).$$

Since (5) gives

$$M^p_{\alpha;\alpha p/2+\beta p(1/p-1/2)} = M^p_{\beta;\beta}$$

we arrive at

PROSOSITION 1.2. *Let* $1 < p \leqq 2$ *and* $\alpha > (p+1)/(6-3p)$. *Then*

$$M^p_{\alpha;\alpha} \hookrightarrow M^p_{\beta;\beta}, \quad (2\beta + 2)(1/p - 1/2) < 1, \quad -1 < \beta < \alpha.$$

The first restriction on β is equivalent to $\beta < (2p-2)/(2-p)$. This combined with the restriction on α gives $\alpha - \beta > (7-5p)/(6-3p)$, the latter being decreasing in p and taking the value $2/3$ at $p = 1$. Hence Proposition 1.2 is an improvement of the previous one for all $1 < p < 2$ provided $(p+1)/(6-3p) < \alpha \leqq (2p-2)/(2-p)$. For big α's, Proposition 1.1 is certainly better. If in the transplantation theorem in [14] higher exponents could be allowed in the power weight — which is possible in the Jacobi expansion case as shown by Muckenhoupt [12] — the technique just used would give the embedding when $-1 < \beta < \alpha$, $1 < p < 2$, and $\alpha > (p+1)/(6-3p)$.

Summarizing, it is reasonable to

CONJECTURE. $M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta}$, $\quad -1 < \beta < \alpha, \quad 1 \leqq p \leqq q \leqq \infty$.

Apart from the above fragmentary results, so far we can only prove the conjecture in the extreme case when $q = \infty$ and $\beta \geqq 0$; the latter restriction arises from the fact that we have to make use of the twisted Laguerre convolution (see [7]) which is proved till now only for Laguerre polynomials $L^\alpha_n(x)$ with $\alpha \geqq 0$. Our main result is

THEOREM 1.3. *If* $1 \leqq p \leqq \infty$, *then*

$$M^{p,\infty}_{\alpha;\alpha} \hookrightarrow M^{p,\infty}_{\beta;\beta}, \quad 0 \leqq \beta < \alpha.$$

REMARKS. 1) One could speculate that an interpolation argument applied to

$$M^2_{\alpha;\alpha} = M^2_{\beta;\beta}, \quad M^\infty_{\alpha;\alpha} = M^1_{\alpha;\alpha} \hookrightarrow M^1_{\beta;\beta} = M^\infty_{\beta;\beta}, \quad \beta < \alpha,$$

could give the open case $M^p_{\alpha;\alpha} \hookrightarrow M^p_{\beta;\beta}$, $1 < p < 2$. In this respect we mention a result of Zafran [17, p. 1412] for the Fourier transform pointed out to us by A. Seeger:

Denote by $M^p(\mathbf{R})$ the set of bounded Fourier multipliers on $L^p(\mathbf{R})$ and by $M^\wedge(\mathbf{R})$ the set of Fourier transforms of bounded measures on **R**. *Then $M^p(\mathbf{R})$, $1 < p < 2$, is* **not** *an interpolation space with respect to the pair $\left(M^\wedge(\mathbf{R}), L^\infty(\mathbf{R})\right)$.*

Thus de Leeuw's result mentioned at the beginning cannot be proved by interpolation.

2) It is perhaps amazing to note that the *wbv*-classes do not play only an auxiliary role in dealing with the above formulated general problem. In the framework of one-dimensional Fourier transforms/series this was shown by Muckenhoupt, Wheeden, and Wo-Sang Young [13]. That this phenomenon also occurs in the framework of Laguerre expansions can be seen from the following two theorems.

THEOREM 1.4. *If $\alpha > -1$, $\alpha \neq 0$, then*

$$wbv_{2,1} \hookrightarrow M^2_{\alpha;\alpha+1}.$$

In the case $-1 < \alpha < 0$ the multiplier operator is defined only on the subspace $\left\{ f \in L^2_{w(\alpha+1)} : \hat{f}_\alpha(0) = 0 \right\}$.

THEOREM 1.5. *If $\alpha > -1$, then*

$$M^2_{\alpha;\alpha+1} \hookrightarrow wbv_{2,1}.$$

A combination of these two results leads to

$$M^2_{\alpha;\alpha+1} = M^2_{\beta;\beta+1} = wbv_{2,1}, \quad \alpha,\beta > -1, \quad \alpha,\beta \neq 0, \tag{6}$$

and a combination with [14, (19)] gives

$$M^2_{\alpha;\alpha+1} \hookrightarrow M^p_{\alpha;\alpha}, \quad \alpha \geqq 0, \quad (2\alpha+2)/(\alpha+1) < p \leqq 2.$$

2. Proof of Theorem 1.3

Theorem 1.3 is an immediate consequence of the combination of the following two theorems.

THEOREM 2.1. *Let $f \in L^p_{w(\alpha)}$ with $\alpha > -1$ when $1 \leqq p < \infty$ and $\alpha \geqq 0$ when $p = \infty$. Then there exists a function $g \in L^p_{w(\beta)}$, $-1 < \beta < \alpha$, with*

$$g(x) \sim \big(\Gamma(\beta+1)\big)^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\beta(x), \quad \|g\|_{L^p_{w(\beta)}} \leqq C \|f\|_{L^p_{w(\alpha)}}.$$

PROOF. First let $1 \leqq p < \infty$ and, without loss of generality, let f be a polynomial (these are dense in $L^p_{w(\alpha)}$). We recall the projection formula (3.31) in Askey and Fitch [2]

$$e^{-x} L_n^\beta(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_x^\infty (y-x)^{\alpha-\beta-1} e^{-y} L_n^\alpha(y)\, dy, \quad -1 < \beta < \alpha.$$

Then the following computations are justified.

$$\|g\|_{L^p_{w(\beta)}} = C \left(\int_0^\infty \left| \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\beta(x) e^{-x/2} \right|^p x^\beta dx \right)^{1/p} =$$

$$= C \left(\int_0^\infty \left| \int_x^\infty (y-x)^{\alpha-\beta-1} e^{-y} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(y)\, dy \right|^p x^\beta e^{xp/2} dx \right)^{1/p} \leqq$$

$$\leqq C \int_1^\infty (t-1)^{\alpha-\beta-1} \left(\int_0^\infty \left| \sum_k \hat{f}_\alpha(k) L_k^\alpha(xt) x^{\alpha-\beta+\beta/p} e^{-x(t-1/2)} \right|^p dx \right)^{1/p} dt$$

after a substitution and application of the integral Minkowski inequality. Additional substitutions lead to

$$\|g\|_{L^p_{w(\beta)}} \leqq C \int_0^\infty s^{\alpha-\beta-1} (s+1)^{\beta/p'-\alpha-1/p} \times$$

$$\times \left(\int_0^\infty \left| \sum_k \hat{f}_\alpha(k) L_k^\alpha(y) e^{-y/2} y^{(\alpha-\beta)/p'} e^{-ys/2(s+1)} \right|^p y^\alpha dy \right)^{1/p} ds \leqq$$

$$\leqq C \int_0^\infty s^{(\alpha-\beta)/p-1} (s+1)^{-(\alpha+1)/p} \left(\int_0^\infty \left| \sum_k \hat{f}_\alpha(k) L_k^\alpha(y) e^{-y/2} \right|^p y^\alpha dy \right)^{1/p} ds,$$

where we used the inequality $y^{(\alpha-\beta)/p'} e^{-ys/2(s+1)} \leqq C\big((s+1)/s\big)^{(\alpha-\beta)/p'}$. Since $-1 < \beta < \alpha$ it is easily seen that the outer integration only gives a bounded contribution.

If $f \in L^\infty_{w(\alpha)}$ then $\left|(k+1)^{-1/2}\hat{f}_\alpha(k)\right| \leqq C\|f\|_{L^\infty_{w(\alpha)}}$ by [10, Lemma 1] and, therefore, the Abel–Poisson means of an arbitrary $f \in L^\infty_{w(\alpha)}$ can be represented by

$$P_r f(x) = \left(\Gamma(\alpha+1)\right)^{-1} \sum_k r^k \hat{f}_\alpha(k) L_k^\alpha(x), \quad 0 \leqq r < 1, \quad x \geqq 0,$$

and, by the convolution theorem in Görlich and Markett [7, p. 169],

$$\|P_r f\|_{L^\infty_{w(\alpha)}} \leqq C\|f\|_{L^\infty_{w(\alpha)}}, \quad 0 \leqq r < 1, \quad \alpha \geqq 0.$$

A slight modification of the argument in the case $1 \leqq p < \infty$ shows that

$$\|g_r\|_{L^\infty_{w(\beta)}} := \left\| \left(\Gamma(\beta+1)\right)^{-1} \sum_k r^k \hat{f}_\alpha(k) L_k^\beta \right\|_{L^\infty_{w(\beta)}} \leqq$$

$$\leqq C\|P_r f\|_{L^\infty_{w(\alpha)}} \leqq C\|f\|_{L^\infty_{w(\alpha)}}.$$

By the weak* compactness there exists a function $g \in L^\infty_{w(\beta)}$ with $\hat{g}_\beta(k) = \hat{f}_\alpha(k)$ and $\|g\|_{L^\infty_{w(\beta)}} \leqq \liminf_{k\to\infty} \|g_{r_k}\|_{L^\infty_{w(\beta)}}$ for a suitable sequence $r_k \to 1^-$; hence also the assertion in the case $p = \infty$.

THEOREM 2.2. *For $\alpha \geqq 0$ there holds*

i) $M^{1,p}_{\alpha;\alpha} = M^{p',\infty}_{\alpha;\alpha} = \left(L^p_{w(\alpha)}\right)^\wedge, \quad 1 < p \leqq \infty,$

ii) $M^{1,1}_{\alpha;\alpha} = M^{\infty,\infty}_{\alpha;\alpha} = \left\{ m = \{m_k\}_{k\in\mathbf{N}_0} : \left\| P_r(m) \right\|_{L^1_{w(\alpha)}} = O(1),\ r \to 1^- \right\},$

where $P_r(m)(x) = \left(\Gamma(\alpha+1)\right)^{-1} \sum_k r^k m_k L_k^\alpha(x)$.

PROOF. The first equalities in i) and ii) are the standard duality statements. Let us briefly indicate the second equalities (which are also more or less standard).

If $m = \{m_k\}_{k\in\mathbf{N}_0}$ are the Fourier–Laguerre coefficients of an $L^p_{w(\alpha)}$-function, $1 < p \leqq \infty$, or in the case $p = 1$ of a bounded measure with respect to the weight $e^{-x/2}x^\alpha$, then Young's inequality in Görlich and Markett [7] (or a slight extension of it to measures in the case $p = 1$) shows that $m \in M^{p',\infty}_{\alpha;\alpha}$.

Conversely, associate formally to a sequence $m = \{m_k\}$ an operator T_m by

$$T_m f(x) \sim \left(\Gamma(\alpha+1)\right)^{-1} \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha(x). \tag{7}$$

Then, essentially in the notation of Görlich and Markett [7],

$$T_m(P_rf)(x) = P_r(m) * f(x) = \int_0^\infty T_x^\alpha\big(P_r(m)(y)\big)\, f(y)e^{-y}y^\alpha\, dy,$$

where T_x^α is the Laguerre translation operator. If $\|f\|_{L^{p'}_{w(\alpha)}} = 1$ then

$$\big\|T_m(P_rf)\big\|_{L^\infty_{w(\alpha)}} \leqq \|m\|_{M^{p',\infty}_{\alpha;\,\alpha}} \|P_rf\|_{L^{p'}_{w(\alpha)}} \leqq C\|m\|_{M^{p',\infty}_{\alpha;\,\alpha}},$$

and hence, by the converse of Hölder's inequality,

$$\sup_{\|f\|_{L^{p'}_{w(\alpha)}}=1} \left|\int_0^\infty T_x^\alpha\big(P_r(m)(y)\big)\, e^{-y/2}y^{\alpha/p} f(y)e^{-y/2}y^{\alpha/p'}\, dy\right| =$$

$$= \big\|T_x^\alpha\big(P_r(m)\big)\big\|_{L^p_{w(\alpha)}} \leqq C\|m\|_{M^{p',\infty}_{\alpha;\,\alpha}}$$

for $x \geqq 0$, $0 \leqq r < 1$. In particular, for $x = 0$ we obtain

$$\big\|P_r(m)\big\|_{L^p_{w(\alpha)}} \leqq C\|m\|_{M^{p',\infty}_{\alpha;\,\alpha}}, \quad 0 \leqq r < 1.$$

Now weak* compactness gives the desired converse embedding.

3. Proof of Theorems 1.4 and 1.5

The proof relies heavily on the Parseval formula

$$\frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^\infty A_k^\alpha \big|\hat f_\alpha(k)\big|^2 = \int_0^\infty \big|f(x)e^{-x/2}\big|^2 x^\alpha\, dx \tag{8}$$

and its extension

$$\sum_{k=0}^\infty A_k^{\alpha+\lambda} \big|\Delta^\lambda \hat f_\alpha(k)\big|^2 \approx \int_0^\infty \big|f(x)e^{-x/2}\big|^2 x^{\alpha+\lambda}\, dx, \quad \lambda \geqq 0, \tag{9}$$

which is a consequence of the formula

$$\Delta^\lambda \hat f_\alpha(k) = C_{\alpha,\lambda} \hat f_{\alpha+\lambda}(k) \tag{10}$$

(see e.g. the proof of Lemma 2.1 in [6]). For the proof of Theorem 1.4 we further need the following discrete analog of the $p = 2$ case of a weighted

Hardy inequality in Muckenhoupt [11] whose proof can at once be read off from [11] by replacing the integrals there by sums and using the fact that

$$a \leqq 2(a+b)^{1/2}\left[(a+b)^{1/2}-b^{1/2}\right]$$

when $a,\, b \geqq 0$; see also the extensions in [1, Section 4].

LEMMA 3.1. *Let* $\{u_k\}_{k\in\mathbf{N}_0}, \{v_k\}_{k\in\mathbf{N}_0}$ *be non-negative sequences (if* $v_k = 0$ *we set* $v_k^{-1}=0$*). Then*

a) $$\sum_{k=0}^{\infty}\Big|\sum_{j=0}^{k} a_j\Big|^2 u_k \leqq C \sup_N \Big(\sum_{k=N}^{\infty} u_k \sum_{k=0}^{N} v_k^{-1}\Big)\sum_{j=0}^{\infty}|a_j|^2 v_j.$$

b) $$\sum_{k=0}^{\infty}\Big|\sum_{j=k}^{\infty} a_j\Big|^2 u_k \leqq C \sup_N \Big(\sum_{k=0}^{N} u_k \sum_{k=N}^{\infty} v_k^{-1}\Big)\sum_{j=0}^{\infty}|a_j|^2 v_j.$$

PROOF OF THEOREM 1.4. Using (9) and the operator T_m defined in (7), we obtain

$$\int_0^\infty \left|T_m f(x)e^{-x/2}\right|^2 x^{\alpha+1}dx \approx \sum_{k=0}^{\infty} A_k^{\alpha+1}\left|\Delta\big(m_k\hat f_\alpha(k)\big)\right|^2.$$

Since

$$\Delta\big(m_k\hat f_\alpha(k)\big) = m_k\Delta\hat f_\alpha(k) + \hat f_\alpha(k+1)\Delta m_k \tag{11}$$

we first observe that

$$\sum_{k=0}^{\infty} A_k^{\alpha+1}|m_k|^2\left|\Delta\hat f_\alpha(k)\right|^2 \leqq \|m\|_\infty^2 \sum_{k=0}^{\infty} A_k^{\alpha+1}\left|\Delta\hat f_\alpha(k)\right|^2 \leqq C\|m\|_\infty^2\|f\|^2_{L^2_{w(\alpha+1)}}.$$

To dominate the term containing Δm_k we deduce from (8) that for $\alpha \geqq 0$ the Fourier–Laguerre coefficients tend to zero as $k\to\infty$. Hence

$$\sum_{k=0}^{\infty} A_k^{\alpha+1}\left|\hat f_\alpha(k+1)\Delta m_k\right|^2 = \sum_{k=0}^{\infty} A_k^{\alpha+1}|\Delta m_k|^2\Big|\sum_{j=k+1}^{\infty}\Delta\hat f_\alpha(j)\Big|^2 =: I.$$

In order to apply Lemma 3.1 b), we choose $u_k = A_k^{\alpha+1}|\Delta m_k|^2$ and $v_k = A_k^{\alpha+1}$, and observe that when $M\in\mathbf{N}$, $2^{M-1}\leqq N < 2^M$, we have that

$$\Big(\sum_{k=0}^{N} u_k\sum_{k=N}^{\infty} v_k^{-1}\Big) \leqq C(N+1)^{-\alpha}\sum_{j=0}^{M}\sum_{k=2^j-1}^{2^{j+1}-2}(k+1)|\Delta m_k|^2\frac{A_k^{\alpha+1}}{k+1} \leqq$$

$$\leqq C(N+1)^{-\alpha}\sum_{j=0}^{M}\left(2^j\right)^{\alpha}\|m\|_{2,1}^2 \leqq C\|m\|_{2,1}^2$$

uniformly in N if $\alpha > 0$. Then Lemma 3.1 b) gives

$$I \leqq C\|m\|_{2,1}^2\sum_{j=0}^{\infty}A_j^{\alpha+1}\left|\Delta\hat{f}_\alpha(j)\right|^2 \leqq C\|m\|_{2,1}^2\|f\|_{L^2_{w(\alpha+1)}}^2$$

by (9). Thus there remains to consider the case $-1 < \alpha < 0$. For the same choice of u_k and v_k one easily obtains

$$\left(\sum_{k=N}^{\infty}u_k\sum_{k=0}^{N}v_k^{-1}\right) \leqq C\|m\|_{2,1}^2 .$$

Now assume that $\hat{f}(0) = 0$. Then we have

$$\sum_{k=0}^{\infty}A_k^{\alpha+1}\left|\hat{f}_\alpha(k+1)\Delta m_k\right|^2 = \sum_{k=0}^{\infty}A_k^{\alpha+1}|\Delta m_k|^2\left|\sum_{j=0}^{k}\Delta\hat{f}_\alpha(j)\right|^2 \leqq$$

$$\leqq C\|m\|_{2,1}^2\|f\|_{L^2_{w(\alpha+1)}}^2 ,$$

where the last estimate follows by Lemma 3.1 a); thus Theorem 1.4 is established.

The proof of Theorem 1.5 is essentially contained in [6]. As in [6], consider a monotone decreasing C^∞-function $\phi(x)$ with

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leqq x \leqq 2 \\ 0 & \text{if } x \geqq 4 \end{cases} , \quad \phi_i(x) = \phi(x/2^i).$$

Then the $\phi_i(k)$ are the Fourier–Laguerre coefficients of an $L^2_{w(\alpha+1)}$-function $\Phi^{(i)}$ with norm $\|\Phi^{(i)}\|_{L^2_{w(\alpha+1)}} \leqq C\left(2^i\right)^{\alpha/2}$ and

$$\sum_{k=2^i}^{2^{i+1}}A_k^{\alpha+1}|\Delta m_k|^2 = \sum_{k=2^i}^{2^{i+1}}A_k^{\alpha+1}\left|\Delta\left(m_k\phi_i(k)\right)\right|^2 \leqq$$

$$\leqq \sum_{k=0}^{2^{i+2}}A_k^{\alpha+1}\left|\Delta\left(m_k\phi_i(k)\right)\right|^2 \leqq C\left\|T_m\Phi^{(i)}\right\|_{L^2_{w(\alpha+1)}}^2 \leqq$$

$$\leqq C\|m\|^2_{M^2_{\alpha;\,\alpha+1}} \|\Phi^{(i)}\|^2_{L^2_{w(\alpha+1)}} \leqq C 2^{i\alpha} \|m\|^2_{M^2_{\alpha;\,\alpha+1}}.$$

This immediately leads to

$$\|m\|_\infty + \left(\sum_{2^i}^{2^{i+1}} \left| (k+1)\Delta m_k \right|^2 \frac{1}{k+1} \right)^{1/2} \leqq C\|m\|_{M^2_{\alpha;\,\alpha+1}},$$

uniformly in i, since by [6, (10)] there holds $\|m\|_\infty \leqq C\|m\|_{M^2_{\alpha;\,\alpha+1}}$; thus Theorem 1.5 is established.

REMARK. 3) (*Added on August 10, 1994.*) The characterization (6) can easily be extended to

$$M^2_{\alpha,\alpha+l} = wbv_{2,l}, \quad \alpha > -1, \quad \alpha \neq 0, \dots, l-1,\ l \in \mathbf{N}. \tag{12}$$

In the case $\alpha < l-1$ the multiplier operator is defined only on the subspace $\{f \in L^2_{w(\alpha+l)} : \hat{f}_\alpha(k) = 0,\ 0 \leqq k < (l-1-\alpha)/2\}$.

The necessity part carries over immediately (see also [6]). The sufficiency part will be proved by induction. Thus suppose that (12) is true for $l = 1, \dots, n$ and α's as indicated. Then, as in the case $n=1$, by (9)

$$\int_0^\infty \left| T_m f(x) e^{-x/2} \right|^2 x^{\alpha+n+1} dx \approx \sum_{k=0}^\infty A_k^{\alpha+n+1} \left| \Delta^n \Delta \big(m_k \hat{f}_\alpha(k) \big) \right|^2 \leqq$$

$$\leqq C \sum_{k=0}^\infty A_k^{\alpha+n+1} \left| \Delta^n \big(m_k \Delta \hat{f}_\alpha(k) \big) \right|^2 +$$

$$+ C \sum_{k=0}^\infty A_k^{\alpha+n+1} \left| \Delta^n \big(\hat{f}_\alpha(k+1) \Delta m_k \big) \right|^2 =: I + II$$

By the assumption and (10)

$$I \leqq C\|m\|^2_{wbv_{2,n}} \sum_{k=0}^\infty A_k^{\alpha+n+1} \left| \Delta^n \hat{f}_{\alpha+1}(k) \big) \right|^2 \leqq$$

$$\leqq C\|m\|^2_{wbv_{2,n+1}} \int_0^\infty \left| f(x) e^{-x/2} \right|^2 x^{\alpha+n+1} dx$$

on account of the embedding properties of the *wbv*-spaces [5]. Analogously II can be estimated by

$$II \leqq C\big\|\{(k+1)\Delta m_k\}\big\|^2_{wbv_{2,n}} \sum_{k=0}^{\infty} A_k^{\alpha+n+1}\Big|\Delta^n\Big(\frac{\hat f_\alpha(k+1)}{k+1}\Big)\Big|^2.$$

By the Leibniz formula for differences there holds

$$\Delta^n\Big(\frac{\hat f_\alpha(k+1)}{k+1}\Big) \leqq C\sum_{j=0}^{n}\big|\Delta^j \hat f_\alpha(k+1)\big|\,\Big|\Delta^{n-j}\frac{1}{j+k+1}\Big| \leqq$$

$$\leqq C\sum_{j=0}^{n}(j+k+1)^{j-n-1}\big|\Delta^j \hat f_\alpha(k+1)\big|.$$

Hence we have to dominate for $j=0,\dots,n$

$$II_j := \sum_{k=0}^{\infty} A_k^{\alpha-n-1+2j}\big|\Delta^j \hat f_\alpha(k+1)\big|^2.$$

If $\alpha > n$ then $c_j := -\alpha - 2j + n + 1 < 1$ for all $j = 0,\dots,n$, $\Delta^j \hat f_\alpha(k+1) = \sum_{i=k+1}^{\infty}\Delta^{j+1}\hat f_\alpha(i)$, and we can apply [8, Theorem 346] repeatedly to obtain

$$II_j \leqq C\sum_{k=0}^{\infty} A_k^{\alpha-n-1+2j}\big|(k+1)\Delta^{j+1}\hat f_\alpha(k+1)\big|^2 \approx$$

$$\approx \sum_{k=0}^{\infty} A_k^{\alpha-n+2j+1}\big|\Delta^{j+1}\hat f_\alpha(k+1)\big|^2 \leqq$$

$$\leqq \dots \leqq C\sum_{k=0}^{\infty} A_k^{\alpha+n+1}\big|\Delta^{n+1}\hat f_\alpha(k+1)\big|^2 \leqq C\int_0^\infty \big|f(x)e^{-x/2}\big|^2 x^{\alpha+n+1}\,dx\,.$$

Since $\big\|\{(k+1)\Delta m_k\}\big\|_{wbv_{2,n}} \leqq C\|m\|_{wbv_{2,n+1}}$, this gives the assertion for the weight x^{n+1} in the case $\alpha > n$.

If $\alpha < n$, $\alpha \neq 0,\dots,n$, then some $c_j > 1$. For the application of [8, Theorem 346] one needs $c_j \neq 1$; this is guaranteed by the hypothesis $\alpha \neq 0,\dots,n$ (in the case of an additional weight x^{n+1}). For the j for which $c_j > 1$ we have to use the representation

$$\Delta^j \hat f_\alpha(k+1) = -\sum_{i=0}^{k}\Delta^{j+1}\hat f_\alpha(i), \quad \text{if } \Delta^j \hat f_\alpha(0) = 0,$$

i.e., the first $(j+1)$ Fourier–Laguerre coefficients have to vanish to ensure this representation. But $0 \leqq j \leqq j_0$, where j_0 is choosen in such a way that $c_{j_0} > 1$ and $c_{j_0+1} < 1$, hence $j_0 = \left[(n-\alpha)/2\right]$ (with respect to the additional weight x^{n+1}); here we used the standard notation for $[a]$, $a \in \mathbf{R}$, to be the greatest integer $\leqq a$. Hence the condition that the first $\left[(n-\alpha)/2\right] + 1$ Fourier–Laguerre coefficients have to vanish is needed if the additional weight is x^{n+1}. A repeated application of [8, Theorem 346] with appropriate $c > 1$ or $c < 1$ now gives the assertion.

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ON COMPLETENESS OF NONDETERMINISTIC AUTOMATA

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To Professor K. Tandori on his 70th birthday

1. Introduction

Representations of automata by products have been intensively studied since the beginning of the sixties. Two types of representations have been in the foreground of research, namely, homomorphic and isomorphic representations. A central problem in the area of products is to characterize those systems of automata which are complete in the sense that every automaton is a homomorphic or isomorphic image of a subautomaton of a product of automata from them. Such systems are called homomorphically, respectively, isomorphically complete. The first necessary and sufficient conditions for homomorphic completeness were given in [7]. In [1], it is shown that from the point of view of homomorphic representation the product is equivalent to one of its special forms in which the feed-back length is at most two. Isomorphic completeness was first studied in [3]. Isomorphic representations of special classes of automata are investigated in [5] and [6]. The monograph [2] gives a systematic summary of results concerning a special product hierarchy. All studies mentioned above concern deterministic automata. Together with the spread of parallel computation the practical importance of nondeterministic automata is increasing. This is our main motivation to introduce the concept of the product of nondeterministic automata and study homomorphic and isomorphic representations by such products.

2. Notions and notations

First we introduce some basic concepts on the line of relational systems (cf. [4]).

By a *nondeterministic automaton* we mean a couple $\mathfrak{A} = (X, A)$ where X, A are nonempty finite sets and for every $x \in X$, x is realized as a binary

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relation $x^{\mathfrak{A}}$ on A. The elements of A are also called *states*. For any $a \in A$, $x \in X$, we denote by $ax^{\mathfrak{A}}$ the set $\{\bar{a} : \bar{a} \in A \,\&\, ax^{\mathfrak{A}}\bar{a}\}$. If $ax^{\mathfrak{A}}$ is a one-element set $\{\bar{a}\}$, then we simply write $ax^{\mathfrak{A}} = \bar{a}$. Let $\mathfrak{A} = (X, A)$ and $\mathfrak{B} = (X, B)$ be nondeterministic automata. $\mathfrak{B}$ is called a *subautomaton* of $\mathfrak{A}$ if $B \subseteqq A$ and $x^{\mathfrak{B}}$ is the restriction of $x^{\mathfrak{A}}$ to B for all $x \in X$. A mapping μ of A into B is called a *homomorphism* of $\mathfrak{A}$ into $\mathfrak{B}$ if $\mu(ax^{\mathfrak{A}}) = \mu(a)x^{\mathfrak{B}}$ holds for all $a \in A$ and $x \in X$. If, in addition, μ is an onto mapping, then $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$. In particular, if $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$ under a homomorphism μ and μ is one-to-one, then we call μ an *isomorphism* and we also say that $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*.

Now let us consider the nondeterministic automata $\mathfrak{A} = (X, A)$, $\mathfrak{A}_j = (X_j, A_j)$, $j = 1, \ldots, n$, and let Φ be a family of mappings below

$$\varphi_j : A_1 \times \ldots \times A_n \times X \to X_1 \times \ldots \times X_n, \quad j = 1, \ldots, n.$$

It is said that $\mathfrak{A}$ is the *general product* of $\mathfrak{A}_j$ with respect to Φ if the following conditions are satisfied:

(1) $A = \prod_{j=1}^{n} A_j$,

(2) for any $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in A$, $x \in X$, $(a_1, \ldots, a_n)x^{\mathfrak{A}}(b_1, \ldots, b_n)$ if and only if $a_j x_j^{\mathfrak{A}_j} b_j$ holds with $x_j = \varphi_j(a_1, \ldots, a_n, x)$ for all $j \in \{1, \ldots, n\}$.

For the general product above we use the notation

$$\mathfrak{A} = \prod_{j=1}^{n} \mathfrak{A}_j(X, \Phi).$$

Let $\mathcal{K}$ be a system of nondeterministic automata. $\mathcal{K}$ is *isomorphically complete* with respect to the general product if for any nondeterministic automaton $\mathfrak{A}$, there exist automata $\mathfrak{A}_j \in \mathcal{K}$, $j = 1, \ldots, n$, such that $\mathfrak{A}$ is isomorphic to a subautomaton of a general product of $\mathfrak{A}_j$, $j = 1, \ldots, n$. It is said that $\mathcal{K}$ is *homomorphically complete* with respect to the general product if for any nondeterministic automaton $\mathfrak{A}$, there are $\mathfrak{A}_j \in \mathcal{K}$, $j = 1, \ldots, n$, such that $\mathfrak{A}$ is a homomorphic image of a subautomaton of a general product of $\mathfrak{A}_j$, $j = 1, \ldots, n$.

The proofs of the following results can be given in a straightforward way.

STATEMENT 1. *If* $\mathfrak{A} = \prod_{i=1}^{n} \mathfrak{A}_i(X, \Phi)$ *and* $\mathfrak{A}_i = \prod_{j=i_1}^{i_l} \mathfrak{A}_j(X_j, \Phi_j)$, $i = 1, \ldots, n$, *then* $\mathfrak{A}$ *is isomorphic to a general product* $\mathfrak{A}_{1_1} \times \ldots \times \mathfrak{A}_{1_l} \times \ldots \times \mathfrak{A}_{n_1} \times \ldots \times \mathfrak{A}_{n_l}(X, \Phi')$.

STATEMENT 2. *If* $\mathfrak{A}_j$ *is a homomorphic (isomorphic) image of a subautomaton of* $\mathfrak{B}_j$, $j = 1, \ldots, n$, *then every general product* $\prod_{j=1}^{n} \mathfrak{A}_j(X, \Phi)$ *is* *a homomorphic (isomorphic) image of a subautomaton of a general product* $\prod_{i=1}^{n} \mathfrak{B}_j(X, \Phi')$.

STATEMENT 3. *If* $\mathfrak{C}$ *is a homomorphic (isomorphic) image of a subautomaton of* $\mathfrak{B}$ *and* $\mathfrak{B}$ *is a homomorphic (isomorphic) image of a subautomaton of* $\mathfrak{A}$, *then* $\mathfrak{C}$ *is a homomorphic (isomorphic) image of a subautomaton of* $\mathfrak{A}$.

3. Isomorphic completeness

We need a special two-state nondeterministic automaton. To each binary relation on $\{0,1\}$ let us assign a symbol and let R denote the set of these symbols. Define the nondeterministic automaton $\mathfrak{D} = (R, \{0,1\})$ such that for any $\rho \in R$, $\rho^{\mathfrak{D}}$ is the corresponding binary relation.

Now we are ready to characterize isomorphically complete systems. The next theorem gives necessary and sufficient conditions for a system of nondeterministic automata to be isomorphically complete with respect to the general product.

THEOREM 1. *A system* $\mathcal{K}$ *of nondeterministic automata is isomorphically complete with respect to the general product if and only if* $\mathcal{K}$ *contains (not necessarily distinct) nondeterministic automata* $\mathfrak{A}_r = (X_r, A_r)$, $\mathfrak{A}_s = (X_s, A_s)$ *for which there exist* $a_r \neq b_r \in A_r$, $x_r, y_r, z_r \in X_r$ *and* $\bar{a}_s \neq \bar{b}_s \in A_s$, $\bar{x}_s, \bar{y}_s, \bar{z}_s \in X_s$ *such that*

$$\{a_r, b_r\} \subseteqq a_r x_r^{\mathfrak{A}_r}, \quad \{a_r, b_r\} \subseteqq b_r y_r^{\mathfrak{A}_r}, \quad \{a_r, b_r\} \cap a_r z_r^{\mathfrak{A}_r} = \{b_r\}$$

and

$$\{\bar{a}_s, \bar{b}_s\} \subseteqq \bar{a}_s \bar{x}_s^{\mathfrak{A}_s}, \quad \{\bar{a}_s, \bar{b}_s\} \subseteqq \bar{b}_s \bar{y}_s^{\mathfrak{A}_s}, \quad \{\bar{a}_s, \bar{b}_s\} \cap \bar{a}_s \bar{z}_s^{\mathfrak{A}_s} = \{\bar{a}_s\}.$$

PROOF. Let us assume that $\mathcal{K}$ is isomorphically complete with respect to the general product. Then there exist $\mathfrak{A}_j \in \mathcal{K}$, $j = 1, \ldots, n$, such that $\mathfrak{D}$ is isomorphic to a subautomaton of a general product $\mathfrak{A} = \prod_{j=1}^{n} \mathfrak{A}_j(R, \Phi)$. Let μ denote a suitable isomorphism and let

$$\mu(0) = (a_{01}, \ldots, a_{0n}) \quad \text{and} \quad \mu(1) = (a_{11}, \ldots, a_{1n}).$$

Denote by M the set $\{m : 1 \leqq m \leqq n \,\&\, a_{0m} \neq a_{1m}\}$. Obviously, $M \neq \emptyset$. By the definition of $\mathfrak{D}$, there is an $x \in R$ with $0x^{\mathfrak{D}} = \{0,1\}$. Since μ is an isomorphism,

$$(a_{01}, \ldots, a_{0n})x^{\mathfrak{A}} \supseteqq \{(a_{01}, \ldots, a_{0n}), (a_{11}, \ldots, a_{1n})\}.$$

But then $\{a_{0m}, a_{1m}\} \subseteqq a_{0m} x_m^{\mathfrak{A}_m}$ holds with $x_m = \varphi_m(a_{01}, \ldots, a_{0n}, x)$ for all $m \in M$. In a similar way we obtain that for any $m \in M$, there exists a $y_m \in$

$\in X_m$ with $\{a_{0m}, a_{1m}\} \subseteqq a_{1m} y_m^{\mathfrak{A}_m}$. By the definition of $\mathfrak{D}$, there is a $z \in R$ with $0z^{\mathfrak{D}} = 1$. Since μ is an isomorphism,

$$\{\mu(0), \mu(1)\} \cap \mu(0) z^{\mathfrak{A}} = \mu(1).$$

Thus $a_{1m} \in a_{0m} z_m^{\mathfrak{A}_m}$ with $z_m = \varphi_m(a_{01}, \ldots, a_{0n}, z)$ for all $m \in M$. Then there exists at least one index $r \in M$ such that $\{a_{0r}, a_{1r}\} \cap a_{0r} z_r^{\mathfrak{A}_r} = \{a_{1r}\}$. Similarly, we obtain that there is an index $s \in M$ with $\{a_{0s}, a_{1s}\} \cap a_{0s} \bar{z}_s^{\mathfrak{A}_s} = \{a_{0s}\}$. But then the conditions of Theorem 1 are satisfied by $\mathfrak{A}_r$ and $\mathfrak{A}_s$, and so, the necessity is proved.

In order to prove the sufficiency let $\mathfrak{A}_r = (X_r, A_r)$, $\mathfrak{A}_s = (X_s, A_s) \in \mathcal{K}$ satisfy the conditions with a_r, b_r, x_r, y_r, z_r and $\bar{a}_s, \bar{b}_s, \bar{x}_s, \bar{y}_s, \bar{z}_s$, respectively. Let $X = \{u_0, \ldots, u_3\}$ and define the general product $(\mathfrak{A}_r \times \mathfrak{A}_r \times \mathfrak{A}_s \times \mathfrak{A}_s \times \mathfrak{A}_r)(X, \Phi)$ as follows. For any $(a_1, \ldots, a_5) \in A_r \times A_r \times A_s \times A_s \times A_r$, let

$$\varphi_i(a_1, \ldots, a_5, u_1) = \varphi_i(a_1, \ldots, a_5, u_3) = \begin{cases} x_r & \text{if } a_i = a_r, \\ y_r & \text{otherwise} \end{cases} \qquad (i = 1, 2),$$

$$\varphi_i(a_1, \ldots, a_5, u_2) = \begin{cases} z_r & \text{if } a_i = a_r, \\ y_r & \text{otherwise} \end{cases} \qquad (i = 1, 2),$$

$$\varphi_j(a_1, \ldots, a_5, u_2) = \varphi_j(a_1, \ldots, a_5, u_3) = \begin{cases} \bar{x}_s & \text{if } a_j = \bar{a}_s, \\ \bar{y}_s & \text{otherwise} \end{cases} \qquad (j = 3, 4),$$

$$\varphi_j(a_1, \ldots, a_5, u_1) = \begin{cases} \bar{z}_s & \text{if } a_j = \bar{a}_s, \\ \bar{y}_s & \text{otherwise} \end{cases} \qquad (j = 3, 4),$$

$$\varphi_5(a_1, \ldots, a_5, u_0) = z_r, \quad \varphi_5(a_1, \ldots, a_5, u_t) = x_r \quad (t = 1, 2, 3),$$

and define Φ arbitrarily in all other cases.

Now let us consider the subautomaton $\mathfrak{A}$ of the above defined general product which is determined by

$$A = \{(a_r, b_r, \bar{a}_s, \bar{b}_s, a_r), (b_r, a_r, \bar{b}_s, \bar{a}_s, a_r)\}.$$

It is easy to prove that for any $\mathbf{a} \in A$ and $V \subseteqq A$, there exists an $x \in X$ with $V = \mathbf{a} x^{\mathfrak{A}}$.

For the sake of simplicity, let us denote the elements of A by 0 and 1. Now let $\mathfrak{C} = (\bar{X}, \{c_1, \ldots, c_m\})$ be an arbitrary nondeterministic automaton. We show that $\mathfrak{C}$ is isomorphic to a subautomaton of a general product $\mathfrak{A}^{2^m-1}(\bar{X}, \Phi)$. For this reason we define a matrix. Consider all m-dimensional column vectors with components 0, 1. Leave out that one for which each component is 1 and order the rest in lexicographically increasing order. Let $\mathbf{Q}$

denote the matrix formed by these column vectors. Then $\mathbf{Q}$ is a matrix of type $m \times (2^m - 1)$ over $\{0, 1\}$. Since each m-dimensional unit vector occurs in $\mathbf{Q}$ as a column vector, the row vectors of $\mathbf{Q}$ are pairwise different. Moreover, let us observe that for any nonempty subset V of the set $\{1, \dots, m\}$, there exists an index $k \in \{1, \dots, 2^m - 1\}$ such that for all $r \in \{1, \dots, m\}$, $r \in V$ if and only if $q_{rk} = 0$. Let us define the one-to-one mapping μ of $\{c_1, \dots, c_m\}$ onto the set of the row vectors of $\mathbf{Q}$ by $\mu(c_i) = (q_{i1}, \dots, q_{i2^m-1})$, $i = 1, \dots, m$. Let B denote the set $\{\mu(c_i) : i = 1, \dots, m\}$. Then $B \subseteqq A^{2^m-1}$.

Now define the general product $\mathfrak{A}^{2^m-1}(\bar{X}, \Phi)$ as follows. Let $(a_1, \dots, a_{2^m-1}) \in A^{2^m-1}$, $\bar{x} \in \bar{X}$, $j \in \{1, \dots, 2^m - 1\}$ be arbitrary elements.

If $(a_1, \dots, a_{2^m-1}) \notin B$ then let $\varphi_j(a_1, \dots, a_{2^m-1}, \bar{x})$ be an arbitrarily fixed element of X.

If $(a_1, \dots, a_{2^m-1}) \in B$ then there is an $i \in \{1, \dots, m\}$ with $\mu(c_i) = (a_1, \dots, a_{2^m-1})$. Let $c_i\bar{x}^{\mathfrak{C}} = \{c_{i_1}, \dots, c_{i_s}\}$. Then $0 \leqq s \leqq m$. For each $j = 1, \dots, 2^m - 1$, let $V_j = \{q_{i_1 j}, \dots, q_{i_s j}\}$. From our assumption on $\mathfrak{A}$, it follows that there exists an $x_j \in X$ with $V_j = q_{ij}x_j^{\mathfrak{A}}$. Let

$$\varphi_j(q_{i1}, \dots, q_{i2^m-1}, \bar{x}) = x_j.$$

Since the vectors $\mu(c_i)$, $i = 1, \dots, m$, are pairwise different, the mappings φ_j, $j = 1, \dots, 2^m - 1$, are well-defined.

Let us consider the system $\mathfrak{B} = (\bar{X}, B)$ where $\bar{x}^{\mathfrak{B}}$ is the restriction of $\bar{x}^{\mathfrak{A}^{2^m-1}(\bar{X},\Psi)}$ to B for all $\bar{x} \in \bar{X}$. Then $\mathfrak{B}$ is a subautomaton of $\mathfrak{A}^{2^m-1}(\bar{X}, \Phi)$. We prove that μ is an isomorphism of $\mathfrak{C}$ onto $\mathfrak{B}$. Let $i, r \in \{1, \dots, m\}$ and $\bar{x} \in \bar{X}$ be arbitrary. Let us suppose that $c_i\bar{x}^{\mathfrak{C}}c_r$. Then, by the definition of the mappings φ_j, $j = 1, \dots, 2^m - 1$, $q_{ij}x_j^{\mathfrak{A}}q_{rj}$ is valid for all $j \in \{1, \dots, 2^m - 1\}$. By the definition of the general product, this yields $\mu(c_i)\bar{x}^{\mathfrak{A}^{2^m-1}(\bar{X},\Psi)}\mu(c_r)$, and so, $\mu(c_i)\bar{x}^{\mathfrak{B}}\mu(c_r)$.

Conversely, let us assume that $\mu(c_i)\bar{x}^{\mathfrak{B}}\mu(c_r)$ is valid for some $i, r \in \{1, \dots, m\}$ and $\bar{x} \in \bar{X}$. Then $q_{ij}x_j^{\mathfrak{A}}q_{rj}$ holds with $x_j = \varphi_j(q_{i1}, \dots, q_{i2^m-1}, \bar{x})$ for all $j \in \{1, \dots, 2^m - 1\}$. Let $W = c_i\bar{x}^{\mathfrak{C}}$. If $W = \emptyset$ then, by the definition of φ_j, we get $q_{ij}x_j^{\mathfrak{A}} = \emptyset$ contradicting $q_{ij}x_j^{\mathfrak{A}}q_{rj}$. Now let us suppose that $W = \{c_{i_1}, \dots, c_{i_s}\}$ where $1 \leqq s \leqq m$. By the definition of the mappings φ_j, $j = 1, \dots, 2^m - 1$, $q_{ij}x_j^{\mathfrak{A}}q_{i_t j}$ is valid for all $t \in \{1, \dots, s\}$, $j \in \{1, \dots, 2^m - 1\}$. On the other hand, by the observation on $\mathbf{Q}$, there exists a $k \in \{1, \dots, 2^m - 1\}$ such that $q_{lk} = 0$ if $l \in \{i_1, \dots, i_s\}$ and $q_{lk} = 1$ otherwise. Thus, by the definition of φ_k, $q_{ik}x_k^{\mathfrak{A}}q_{lk}$ if and only if $l \in \{i_1, \dots, i_s\}$. But then, by $q_{ik}x_k^{\mathfrak{A}}q_{rk}$, we have $r \in \{i_1, \dots, i_s\}$. Therefore, $c_i\bar{x}^{\mathfrak{C}}c_r$, and so, μ is an isomorphism of $\mathfrak{C}$ onto $\mathfrak{B}$. This, by Statements 1, 2 and 3, ends the proof of Theorem 1.

4. Homomorphic completeness

In order to give a sufficient condition for a system of nondeterministic automata to be homomorphically complete, we need some preparation.

LEMMA 1. *Let* $\mathfrak{A} = (X, A)$ *be a nondeterministic automaton which has three (not necessarily different) states* a_0, a_1, a_1' *and two input words* $p = x_2 \dots x_{k_1}$ *and* $q = y_2 \dots y_{k_2}$ $(x_2, \dots, x_{k_1}, y_2, \dots, y_{k_2} \in X)$ *such that the following conditions are fulfilled:*

(1) $a_1 \neq a_1'$,

(2) *for each subset* V *of* $\{a_1, a_1'\}$, *there is an* $x \in X$ *satisfying* $a_0 x^{\mathfrak{A}} = V$,

(3) $a_i x_{i+1}^{\mathfrak{A}} = a_{i+1}, i = 1, \dots, k_1 - 2$, $a_{k_1-1} x_{k_1}^{\mathfrak{A}} = a_0$, $a_j' y_{j+1}^{\mathfrak{A}} = a_{j+1}'$, $j = 1, \dots, k_2 - 2$ *and* $a_{k_2-1}' y_{k_2}^{\mathfrak{A}} = a_0$, *for some* $a_2, \dots, a_{k_1-1}, a_2', \dots, a_{k_2-1}' \in A$.

Then $a_0, a_1, \dots, a_{k_1-1}, a_1', \dots, a_{k_2-1}'$ *and* p, q *can be given in such a way that they satisfy conditions* (1)–(3), *the elements of the sequences* $a_0, a_1, \dots, a_{k_1-1}$ *and* $a_0, a_1', \dots, a_{k_2-1}'$ *are pairwise distinct and one of the following three conditions holds:*

(a) $k_1, k_2 > 1$ *and* $\{a_1, \dots, a_{k_1-1}\} \cap \{a_1', \dots, a_{k_2-1}'\} = \emptyset$,

(b) $k_1 > 1$ *and* $k_2 = 1$,

(c) $k_1, k_2 > 1$ *and for some* i *and* j $(1 < i < k_1; 1 \leqq j < k_2)$, $\{a_1, \dots, a_{i-1}\} \cap \{a_1', \dots, a_{j-1}'\} = \emptyset$, $a_i = a_j'$, $a_{i+1} = a_{j+1}', \dots, a_{k_1-1} = a_{k_2-1}'$ *and* $q = y_2 \dots y_j x_{i+1} \dots x_{k_1}$.

PROOF. At least $a_1 \neq a_0$ or $a_1' \neq a_0$ holds. Without loss of generality we may assume that $a_1 \neq a_0$. Then $k_1 > 1$. If $a_j = a_l$ for some j and l with $1 \leqq j < l \leqq k_1$, where $a_{k_1} = a_0$, then let us take the word $p' = x_2 \dots x_j x_{l+1} \dots x_{k_1}$ for p and the states $a_0, a_1, \dots, a_j, a_{l+1}, \dots, a_{k_1-1}$ for $a_0, a_1, \dots, a_{k_1-1}$. If $a_0, a_1, \dots, a_j, a_{l+1}, \dots, a_{k_1-1}$ has two elements which are equal, then repeat the above process for this sequence and p'. Finally we arrive at a sequence with pairwise different members. Thus we may suppose that the elements of $\{a_0, a_1, \dots, a_{k_1-1}\}$ are pairwise distinct. If $a_1' = a_0$ then we may suppose that $q = e$ where e denotes the empty word over X. So we have case (b).

Next let $a_1' \neq a_0$. Apply the above process to $a_0, a_1', \dots, a_{k_2-1}'$ and q. The members of the resulting sequence are different. Therefore, we may assume that the elements of $a_0, a_1', \dots, a_{k_2-1}'$ are pairwise distinct. If $\{a_1, \dots, a_{k_1-1}\} \cap \{a_1', \dots, a_{k_2-1}'\} = \emptyset$, then we have case (a). In the opposite case there are integers i and j with $1 < i < k_1$ and $1 \leqq j < k_2$ or $1 \leqq i < k_1$ and $1 < j < k_2$ such that $a_i = a_j'$ and $\{a_1, \dots, a_{i-1}\} \cap \{a_1', \dots, a_{j-1}'\} = \emptyset$. If $i > 1$ then taking $a_1', \dots, a_{j-1}', a_i, \dots, a_{k_1-1}$ and $q' = y_2 \dots y_j x_{i+1} \dots x_{k_1}$ for $a_1', \dots, a_{k_2-1}'$ and q we have case (c). If $i = 1$ then let us take $a_1', \dots, a_{k_2-1}'$ and $p' = y_2 \dots y_{k_2-1}$ for $a_1, \dots, a_{k_1-1}$ and p. Moreover, let us take $a_j', \dots, a_{k_2-1}'$ and $y_{j+1} \dots y_{k_2}$ for $a_1', \dots, a_{k_2-1}'$ and q. Then we again have case (c), which ends the proof of Lemma 1.

LEMMA 2. *Let $\mathfrak{A} = (X, A)$ be a nondeterministic automaton satisfying the conditions of Lemma 1. Then there exists a general power of $\mathfrak{A}$ which has a subautomaton $\mathfrak{B} = (Y, B)$ such that the following conditions are satisfied:*

(i) $B = \{b_0, b_1, \ldots, b_{k-1}\} \cup \{b'_1, \ldots, b'_{k-1}\}$ *for an integer* $k > 1$,

(ii) $b_1 \neq b'_1$,

(iii) *for each subset* V *of* $\{b_1, b'_1\}$, *there exists some* $y \in Y$ *such that* $b_0 y^{\mathfrak{B}} = V$,

(iv) $b_i \bar{y}^{\mathfrak{B}} = b_{i+1}$, $i = 1, \ldots, k-2$, $b_{k-1}\bar{y}^{\mathfrak{B}} = b_0$, $b'_i \bar{y}^{\mathfrak{B}} = b'_{i+1}$, $i = 1, \ldots, k-2$ *and* $b'_{k-1}\bar{y}^{\mathfrak{B}} = b_0$ *under a fixed* $\bar{y} \in Y$.

PROOF. Take states $a_0, a_1, \ldots, a_{k_1-1}, a'_1, \ldots, a'_{k_2-1}$ and words $p = x_2 \ldots x_{k_1-1}$, $q = y_2 \ldots y_{k_2-1}$ satisfying the conclusions of Lemma 1. First form the single-factor power $\mathfrak{A}_1 = (\{\bar{x}\}, A) = \mathfrak{A}(\{\bar{x}\}, \Phi^*)$ given in the following way: $\varphi_1^*(a_i, \bar{x}) = x_{i+1}$, $i = 1, \ldots, k_1 - 1$, $\varphi_1^*(a_0, \bar{x}) = x_1$, where x_1 is an input signal of $\mathfrak{A}$ with $a_0 x_1^{\mathfrak{A}} = a_1$. (By (2) of Lemma 1, there is such an x_1.) Finally, in all other cases φ_1^* is given arbitrarily. It is obvious that $\{a_0, a_1, \ldots, a_{k_1-1}\}$ forms a k_1-state cycle of $\mathfrak{A}_1$. Let us denote by $\mathfrak{A}'_1 = (\bar{x}, \{a_0, \ldots, a_{k_1-1}\})$ this cyclic subautomaton. Similarly, taking $a_0, a'_1, \ldots, a'_{k_2-1}$ we can define a single-factor power $\mathfrak{A}_2 = (\{\bar{x}\}, A) = \mathfrak{A}(\{\bar{x}\}, \Phi')$ which has a k_2-state cyclic subautomaton $\mathfrak{A}'_2 = (\{\bar{x}\}, \{a_0, a'_1, \ldots, a'_{k_2-1}\})$ with $a'_i \bar{x}^{\mathfrak{A}'_2} = a'_{i+1 \,(\mathrm{mod}\, k_2)}$, $i = 0, \ldots, k_2 - 1$, where $a'_0 = a_0$. (In case (b) $\mathfrak{A}'_2$ is a single-state automaton.) Then the direct product of $\mathfrak{A}'_1$ and $\mathfrak{A}'_2$ has a subautomaton which is isomorphic to the k-state cyclic automaton $\mathfrak{C} = (\{\bar{x}\}, C)$ with $C = \{0, 1, \ldots, k-1\}$ and $r\bar{x}^{\mathfrak{C}} = r + 1 \,(\mathrm{mod}\, k)$ $(0 \leqq r < k)$, where k is the least common multiple of k_1 and k_2.

Finally, take the product $\bar{\mathfrak{B}} = (Y, \bar{B}) = (\mathfrak{C} \times \mathfrak{A})(Y, \Phi)$ with $Y = X \cup \{\bar{y}\}$ where $\bar{y}$ is a new symbol. Let $x_1, y_1 \in X$ be inputs with $a_0 x_1^{\mathfrak{A}} = a_1$ and $a_0 y_1^{\mathfrak{A}} = a'_1$. By (2) of Lemma 1, there exist such x_1 and y_1. To define Φ we distinguish three cases depending on $\mathfrak{A}$.

First let us assume that $\mathfrak{A}$ satisfies (a) of Lemma 1. Then for all $(r, a) \in C \times A$ let

(1) $\varphi_1\big((r, a), x\big) = \bar{x}$, for all $x \in Y$,

$$(2)\ \varphi_2\big((r,a),\bar{y}\big) = \begin{cases} x_1 & \text{if } a = a_0,\ 0 < r \text{ and } r \equiv 0 \ (\mathrm{mod}\ k_1), \\ x_{s+1} & \text{if } a = a_s \text{ and } 1 \leqq s \leqq k_1 - 1, \\ y_1 & \text{if } a = a_0,\ 0 < r \text{ and } r \equiv 0 \ (\mathrm{mod}\ k_2), \\ y_{s+1} & \text{if } a = a'_s \text{ and } 1 \leqq s \leqq k_2 - 1, \end{cases}$$

(3) $\varphi_2\big((r, a), x\big) = x$, for all $x \in X$,

(4) φ_2 is given arbitrarily in all other cases.

Let $b_r = (r, a_{r \,(\mathrm{mod}\, k_1)})$, $r = 0, 1, \ldots, k-1$ and $b'_r = (r, a'_{r \,(\mathrm{mod}\, k_2)})$, $r = 1, \ldots, k-1$, where $a'_0 = a_0$.

Suppose that (b) of Lemma 1 holds for $\mathfrak{A}$. Then $k = k_1 > 1$. Now for all $(r, a) \in C \times A$ let

(1) $\varphi_1\big((r,a),x\big) = \bar{x}$, for all $x \in Y$,

(2) $\varphi_2\big((r,a),\bar{y}\big) = \begin{cases} x_{s+1} & \text{if } a = a_s \text{ and } 1 \leqq s < k, \\ y_1 & \text{if } a = a_0, \end{cases}$

(3) $\varphi_2\big((r,a),x\big) = x$, for all $x \in X$,

(4) φ_2 is given arbitrarily in all other cases.

Let $b_r = (r, a_r)$, $r = 0, 1, \dots, k-1$ and $b'_r = (r, a_0)$, $r = 1, \dots, k-1$.

Finally let us assume that $\mathfrak{A}$ satisfies (c) of Lemma 1 with $1 < i \leqq k_1$ and $1 \leqq j < k_2$. In this case for all $(r, a) \in C \times A$, let

(1) $\varphi_1\big((r,a),x\big) = \bar{x}$, for all $x \in Y$,

(2) $\varphi_2\big((r,a),\bar{y}\big) = \begin{cases} x_1 & \text{if } a = a_0,\ 0 < r \text{ and } r \equiv 0 \ (\mathrm{mod}\ k_1), \\ x_{s+1} & \text{if } a = a_s \text{ and } 1 \leqq s \leqq k_1 - 1, \\ y_1 & \text{if } a = a_0,\ 0 < r \text{ and } r \equiv 0 \ (\mathrm{mod}\ k_2), \\ y_{s+1} & \text{if } a = a'_s \text{ and } 1 \leqq s < j, \end{cases}$

(3) $\varphi_2\big((r,a),x\big) = x$, for all $x \in X$,

(4) φ_2 is given arbitrarily in all other cases.

Let $b_r = (r, a_{r\ (\mathrm{mod}\ k_1)})$, $r = 0, 1, \dots, k-1$ and $b'_r = (r, a'_{r\ (\mathrm{mod}\ k_2)})$, $r = 1, \dots, k-1$.

For all three cases above, let us denote by $\mathfrak{B}$ the subautomaton of $\bar{\mathfrak{B}}$ formed by $\{b_0, b_1, \dots, b_{k-1}\} \cup \{b'_1, \dots, b'_{k-1}\}$. Obviously, $\mathfrak{B}$ satisfies the conditions of Lemma 2. Finally, by Statements 1, 2 and 3, $\mathfrak{B}$ is isomorphic to a subautomaton of a power of $\mathfrak{A}$.

LEMMA 3. *Every two-state nondeterministic automaton is a homomorphic image of a subautomaton of a general power of $\mathfrak{B}$, where $\mathfrak{B}$ is the automaton given in Lemma 2.*

PROOF. Let $\bar{\mathfrak{A}} = (X, \{0,1\})$ be an arbitrary nondeterministic two-state automaton. Take the k-th general power $\mathfrak{C} = (X, C) = (\mathfrak{B} \times \dots \times \mathfrak{B})(X, \Phi)$ of $\mathfrak{B}$ given in the following way. Let C' be the subset of C consisting of all elements $\mathbf{c} = (c_0, \dots, c_{k-1})$ for which there exists an i $(0 \leqq i \leqq k-1)$ such that c_j is equal to b_l or b'_l if $j - i \equiv l \ (\mathrm{mod}\ k)$. Observe that $c_i = b_0$. Let $\mu : C' \to \{0,1\}$ be the mapping given as follows: $\mu(c_0, \dots, c_{k-1}) = 1$ if $c_j = b_1$ for $j \equiv i+1 \ (\mathrm{mod}\ k)$, where i is the integer given for $(c_0, \dots, c_{k-1})$ in the definition of C', and $\mu(c_0, \dots, c_{k-1}) = 0$ in all other cases. (Observe, that $\mu(c_0, \dots, c_{k-1}) = 0$ if and only if $c_j = b'_1$ for $j \equiv i+1 \ (\mathrm{mod}\ k)$.) Moreover, let $\tau : \{b_1, b'_1\} \to \{0,1\}$ be the mapping given by $\tau(b_1) = 1$ and $\tau(b'_1) = 0$. Now take an arbitrary $\mathbf{c} = (c_0, \dots, c_{k-1}) \in C'$ and an $x \in X$. Let $\mu(\mathbf{c})x^{\bar{\mathfrak{A}}} = V'$ and $\tau^{-1}(V') = V$. Let i be the integer for which $c_i = b_0$. Then $\varphi_i(\mathbf{c}, x) = y$ where y is the symbol of $\mathfrak{B}$ for which $b_0 y^{\mathfrak{B}} = V$, and $\varphi_j(\mathbf{c}, x) = \bar{y}$ for all $j \neq i$. In all other cases Φ is defined arbitrarily. Denote by $\mathfrak{C}'$ the subautomaton of $\mathfrak{C}$ determined by C'.

We show that under the above choice of Φ, μ is a homomorphism of $\mathfrak{C}'$ onto $\bar{\mathfrak{A}}$. For this take an arbitrary $r \in \{0,1\}$ and $\mathbf{c} = (c_0, \ldots, c_{k-1}) \in C'$ with $\mu(\mathbf{c}) = r$. Let i be the integer for which $c_i = b_0$. Moreover, take a symbol $x \in X$. Assume that $s \in rx^{\bar{\mathfrak{A}}}$. Then there is a $\mathbf{c}' = (c'_0, \ldots, c'_{k-1}) \in \mathbf{c}x^{\mathfrak{C}'}$ satisfying the following conditions:

(1) $c'_i = \tau^{-1}(s)$,

$$(2)\ c'_j = \begin{cases} b_{l+1\ (\mathrm{mod}\, k)} & \text{if } j \neq i \text{ and } c_j = b_l, \\ b'_{l+1\ (\mathrm{mod}\, k)} & \text{if } j \neq i \text{ and } c_j = b'_l, \end{cases}$$

where $b'_0 = b_0$. Therefore, $c'_{(i-1)\ (\mathrm{mod}\, k)} = b_0$, $c'_i = \tau^{-1}(s) \in \{b_1, b'_1\}$ and $\mathbf{c}' \in$ $\in C'$. Thus, $\mathbf{c}' \in \mu^{-1}(s)$. Furthermore, it can be seen in a similar way that for every $\mathbf{c}' \in \mathbf{c}x^{\mathfrak{C}'}$, we have $\mu(\mathbf{c}') \in rx^{\bar{\mathfrak{A}}}$, which ends the proof of Lemma 3.

Using Statements 1, 2 and 3 and Theorem 1, from Lemmas 1, 2 and 3 we obtain

THEOREM 2. *Assume that a system $\mathcal{K}$ of nondeterministic automata has an automaton $\mathfrak{A}$ which satisfies the conditions of Lemma 1. Then $\mathcal{K}$ is homomorphically complete with respect to the general product.*

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GENERALIZED PROJECTIONS FOR NON-NEGATIVE FUNCTIONS

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Dedicated to Károly Tandori on his 70th birthday

1. Introduction

In this paper, a *distance* d on a set S means any non-negative valued function defined on $S \times S$ such that $d(s,t) = 0$ iff $s = t$. For $E \subset S$ we write

$$d(E,t) = \inf_{s \in E} d(s,t). \tag{1.1}$$

If there is a unique $s^* \in E$ satisfying $d(s^*,t) = d(E,t)$, it is called the *d-projection* of t onto E. Even if the d-projection does not exist, it may happen that every sequence $\{s_n\} \subset E$ with $d(s_n,t) \to d(E,t)$ converges to a unique $s^* \in S$, in a suitable topology; this s^* will be called the *generalized d-projection* of t to E.

Minimization problems as above occur in many different contexts. This author's main motivation has been inference via maximum entropy and related methods. Such methods have been widely and successfully applied in quite diverse areas, cf., e.g., the collection [5].

A typical example is a linear inverse problem when an unknown positive valued function $s(x)$ defined on a set X has to be inferred from the knowledge of certain integrals

$$\int a_i(x)s(x)\mu(dx) = b_i, \quad i = 1,\dots,k, \tag{1.2}$$

where μ is a given measure; it is also assumed that in the absence of the information (1.2), $s = t$ would be inferred, where $t(x)$ is a given function on X. E.g. s may be a probability density function on the real line, the available knowledge consisting in its first $k-1$ moments. Then μ is the Lebesgue measure, and the equations (1.2) hold with $a_i(x) = x^{i-1}$, $i = 1,\dots,k$, $b_1 = 1$, and b_i is the known moment of order $i-1$, $i = 2,\dots,k$. The "prior guess"

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may be the standard Gaussian density $t(x) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{t^2}{2}\right)$. Another example, in X-ray tomography, is when X is the examined two-dimensional domain, and (assuming, as an approximation, that there are k distinct X-rays whose paths are represented by thin rectangles) a_i is the characteristic function of the path of the i'th ray, μ is the Lebesgue measure, and a possible choice for the "prior guess" t is a constant. For such problems, a natural method of inference is to take the d-projection of t onto the set of functions satisfying the constraints (1.2), for a suitable distance d.

To motivate what distances we will be interested in for positive valued functions, consider first the discrete version of the above problem. Then an unknown vector $\mathbf{s} = (s_1, \ldots, s_n)$ with positive components should be inferred from the knowledge that

$$\mathbf{a}_i\mathbf{s} = b_i, \quad i = 1, \ldots, k, \tag{1.3}$$

when a vector $\mathbf{t}$ is also given as "prior guess". A favored method is to take the *I-projection* of $\mathbf{t}$ onto the set E of vectors satisfying the constraints (1.3), i.e., d-projection with $d(\mathbf{s}, \mathbf{t}) = I(\mathbf{s}\|\mathbf{t})$, where

$$I(\mathbf{s}\|\mathbf{t}) = \sum_{j=1}^{n} \left(s_j \log \frac{s_j}{t_j} - s_j + t_j \right) \tag{1.4}$$

is the Kullback–Leibler distance or *I-divergence*. Some other distances have also been used in this context. In the most common case when the components of $\mathbf{t}$ are equal, and the constraints (1.3) include $\sum s_j = 1$, minimizing (1.4) is equivalent to maximizing $H(\mathbf{s}) = -\sum s_j \log s_j$, the *entropy* of $\mathbf{s}$. This is why inference by d-projection is sometimes referred to as maximum entropy type method, even if d is different from (1.4).

This author has studied inference methods of this type axiomatically in [10]. It was shown that the I-projection is distinguished in several respects. Two classes of distances were also characterized as possible alternatives, by postulating (different) natural desiderata on the resulting projections. The distances in both families are determined by strictly convex, differentiable functions f on the open interval $(0, \infty)$, satisfying

$$f(1) = f'(1) = 0, \quad \lim_{u \to 0} f'(u) = -\infty, \tag{1.5}$$

as follows:

$$D_f(\mathbf{s}, \mathbf{t}) = \sum_{j=1}^{n} t_j f\left(\frac{s_j}{t_j}\right), \tag{1.6}$$

$$B_f(\mathbf{s},\mathbf{t}) = \sum_{j=1}^{n} \Delta_f(s_j, t_j) \tag{1.7}$$

where

$$\Delta_f(u,v) = f(u) - f(v) - f'(v)(u-v) \geqq 0. \tag{1.8}$$

The only common member of these two classes is the I-divergence (1.4) (up to a constant factor), with $f(u) = u \log u - u + 1$.

While in [10] attention was restricted to vectors with strictly positive components, here we will permit zero components, too. Also, we will not insist on the condition $\lim_{u\to 0} f'(u) = -\infty$, the condition making sure that projections onto sets determined by linear constraints (cf. (1.3)) always be vectors with positive components, if the set contains any such vector. To deal with zeros, we adopt the understandings

$$f(0) = \lim_{u\to 0} f(u), \quad f'(0) = \lim_{u\to 0} f'(u), \quad 0f\left(\frac{u}{0}\right) = u \lim_{v\to\infty} f'(v), \tag{1.9}$$

and if $f(0) = \infty$, we set $\Delta_f(u,0) = \infty$ for $u > 0$ and 0 for $u = 0$.

REMARK. The conditions $f(1) = f'(1) = 0$ are essential for (1.6) only, to make D_f a distance, but not for (1.7). On the other hand, adopting those conditions also for (1.7) does not restrict generality, since to any differentiable convex function f there exists another, say $\tilde{f}$, satisfying $\tilde{f}(1) = \tilde{f}'(1) = 0$ and such that always $B_f(\mathbf{s},\mathbf{t}) = B_{\tilde{f}}(\mathbf{s},\mathbf{t})$. This follows from the fact that if f and $\tilde{f}$ differ only by a linear function of u then $\Delta_f(u,v) = \Delta_{\tilde{f}}(u,v)$ for every u and v. Notice also that if the distance D_f were considered for probability distributions only (i.e., vectors whose components have sum 1) then the condition $f'(1) = 0$ could be dropped also for (1.7).

The class of distances D_f, called *f-divergences*, was first introduced by Csiszár [6], for probability distributions (without any assumptions on f other than convexity), and independently by Ali and Silvey [1]; cf. also Csiszár [7]. It includes many distances used in statistics, such as (in addition to I-divergence) the reversed I-divergence $d(\mathbf{s},\mathbf{t}) = I(\mathbf{t}\|\mathbf{s})$ (with $f(u) = u - \log u - 1$), the Hellinger distance

$$He(\mathbf{s},\mathbf{t}) = \sum_{j=1}^{n} \left(\sqrt{s_j} - \sqrt{t_j}\right)^2 \quad \left(f(u) = \left(\sqrt{u} - 1\right)^2\right) \tag{1.10}$$

and the χ^2 distance

$$\chi^2(\mathbf{s},\mathbf{t}) = \sum_{j=1}^{n} (s_j - t_j)^2/t_j \quad \left(f(u) = (u-1)^2\right). \tag{1.11}$$

For a thorough treatment of f-divergences and their applications in statistics cf. Liese and Vajda [15].

The distances B_f will be referred to as *Bregman distances*. They represent a subclass of the distances introduced by Bregman [4]. Among them, particularly those defined by the following functions f_α deserve interest:

$$(1.12) \qquad f_\alpha(u) = \begin{cases} u^\alpha - \alpha u + \alpha - 1 & \text{if } \alpha > 1 \text{ or } \alpha < 0 \\ -u^\alpha + \alpha u - \alpha + 1 & \text{if } 0 < \alpha < 1 \\ u \log u - u + 1 & \text{if } \alpha = 1 \\ u - \log u - 1 & \text{if } \alpha = 0. \end{cases}$$

Notice that $\alpha = 2$ gives squared Euclidean distance, $\alpha = 1$ gives I-divergence, and $\alpha = 0$ the distance of Itakura and Saito [11]. Those Bregman distances with f satisfying (1.5) that lead to scale-invariant inference for linear inverse problems were characterized in [10] as those defined by the functions f_α with $\alpha \leqq 1$, up to a constant factor.

It is interesting to note that the distances D_f and B_f have applications also in the theory of means. Namely Ben-Tal, Charnes and Teboulle [2] suggested to consider (1.6) resp. (1.7) with $s_1 = \ldots = s_n = u$, and to minimize for u; they called the minimizer an *entropic mean* of the numbers $t_1, \ldots, t_n$. Thus entropic means are defined by D_f- resp. B_f-projections onto the half-line $\{s : s_1 = \ldots = s_n > 0\}$. In [2], a large variety of means known in the literature was shown to be entropic; in particular, arithmetic mean arises from reversed I-divergence and geometric mean from I-divergence.

In this paper, we will be interested primarily in projections onto convex sets of non-negative valued functions. Motivated by the previous discussion, we will concentrate on projections with respect to distances which are generalized versions of those in (1.6), (1.7). The results of Jones [12] and Jones and Byrne [13] provide important theoretical support for preferring the distances analogous to (1.7), that they called *projective distortions*. These were axiomatically characterized by an orthogonality property of projections, analogous to that of Euclidean projections.

Given a σ-finite measure space $(X, \mathcal{X}, \mu)$, the f-divergences and Bregman distances of non-negative ($\mathcal{X}$-measurable) functions $s(x)$ and $t(x)$ are defined by

$$(1.13) \qquad D_{f,\mu}(s,t) = \int t(x) f\left(\frac{s(x)}{t(x)}\right) \mu(dx),$$

$$(1.14) \qquad B_{f,\mu}(s,t) = \int \Delta_f\big(s(x), t(x)\big)\, \mu(dx)$$

where Δ_f is defined by (1.8). Here f is any differentiable, strictly convex function on $(0, \infty)$ satisfying

$$f(1) = f'(1) = 0. \tag{1.15}$$

Then $D_{f,\mu}$ and $B_{f,\mu}$ are distances, considering μ-a.e. equal functions as identical (cf. the remark after (1.9)). The I-divergence $I_\mu(s\|t)$ is the special case of both (1.13) and (1.14) when $f(u) = u \log u - u + 1$. Let us notice that, introducing a measure $\nu \ll \mu$ with $\frac{d\nu}{d\mu} = t(x)$, the f-divergence (1.13) can be represented as

$$D_{f,\mu}(s,t) = B_{f,\nu}\left(\frac{s}{t}, 1\right) + \left(\lim_{v\to\infty} f'(v)\right) \int_{\{x: t(x)=0\}} s(x)\mu(dx). \tag{1.16}$$

This identity follows using (1.15) and (1.9). $B_{f,\nu}\left(\frac{s}{t}, 1\right)$ is well defined because $\frac{s(x)}{t(x)}$ is well defined ν-a.e.

Usually, f-divergences (and, in particular, I-divergence) are defined as distances between measures rather than functions. The f-divergence of arbitrary finite measures ν, μ on $(\mathrm{X}, \mathcal{X})$ can be defined by taking any σ-finite measure λ with $\nu \ll \lambda$, $\mu \ll \lambda$ and letting

$$D_f(\nu, \mu) = D_{f,\lambda}(s,t), \quad s = \frac{d\nu}{d\lambda}, t = \frac{d\mu}{d\lambda},$$

or, equivalently, by

$$D_f(\nu, \mu) = \int f\left(\frac{d\nu_a}{d\mu}\right) d\mu + \left(\lim_{v\to\infty} f'(v)\right)\nu_s(X) \tag{1.17}$$

where $\nu = \nu_a + \nu_s$ is the decomposition of μ into absolutely continuous and singular components with respect to μ, cf. Csiszár [6], [7], or Liese and Vajda [15]. In particular, if $\nu \ll \mu$ then $D_f(\nu, \mu)$ is given by setting $s = \frac{d\nu}{d\mu}$, $t = 1$ in (1.13), i.e., by

$$J_{f,\mu}(s) = \int f(s(x))\, \mu(dx), \tag{1.18}$$

with $s = \frac{d\nu}{d\mu}$. In this paper, we will consider the integral (1.18) also when μ is not a finite measure and/or (1.15) does not hold, although in that case it does not represent a distance. The integral (1.18) is sometimes called the f-*entropy* of s (with respect to μ); if its minimum subject to given constraints, such as in (1.2), is attained for a function s^*, this s^* is called the "best entropy estimate", cf. Borwein and Lewis [3], Teboulle and Vajda [17].

As in the discrete case, among distances between non-negative functions or between measures, the I-divergence is of primary importance. I-projections for probability measures were studied in Csiszár [8]. Unlike in the discrete case, the I-projection may fail to exist even onto a set of measures defined by specified values of a finite number of integrals; for sets of functions, this means that the I-projection may fail to exist even onto a set of functions defined by constraints of form (1.2). On the other hand, it was shown in [8] that the generalized I-projection to a convex set of measures always exists (in non-trivial cases), without attaching any particular significance to that result. The significance of the concept of generalized I-projection was first recognized by Topsøe [18] (who called it "center of attraction"). Csiszár [9] proved a conditional limit theorem associated with large deviations, showing that generalized I-projection (in the context of probability measures) has the same probabilistic significance as regular I-projection.

Our aim here is to extend results available for I-projections to D_f- and B_f-projections, in the general setting. One of the main results will be that generalized projections onto convex sets of functions exist under very general conditions.

2. Existence of generalized projections

Let f be a strictly convex, differentiable function on $(0, \infty)$. Recall the understandings (1.9). In most but not all cases the condition (1.15) will also be assumed, cf. below.

Given a σ-finite measure space $(X, \mathcal{X}, \mu)$, let S denote the set of non-negative finite valued $\mathcal{X}$-measurable functions on X. We will consider the distances $D_{f,\mu}(s,t)$, $B_{f,\mu}(s,t)$ and the integrals $J_{f,\mu}(s)$, defined by (1.13), (1.14), (1.18), for functions s, t in S. In particular, at this point we do not require integrability of these functions. The indices f and μ will be omitted when this does not cause ambiguity. The condition (1.15) is assumed when dealing with the f-divergences $D(s,t) = D_{f,\mu}(s,t)$ and the Bregman distances $B(s,t) = B_{f,\mu}(s,t)$ (although for the latter it is irrelevant, cf. the remark after (1.9)) but not when dealing with $J(s) = J_{f,\mu}(s)$. In particular, the integrand of (1.18) need not be positive, and the integral may be undefined for some $s \in S$; in that case we set $J(s) = +\infty$. Notice that when $\mu(X) = \infty$ and $s \in L_1(\mu)$, a necessary condition for the finiteness of $J(s)$ is $f(0) = 0$; thus, to obtain meaningful results for that case, the condition (1.15) has to be abandoned.

In the sequel, for subsets of X defined by relations involving functions on X we will use a shorthand notation such as $\{s \leqq K\}$ or $\{|s-t| \leqq \varepsilon\}$, etc., meaning $\{x: s(x) \leqq K\}$ or $\left\{x: |s(x) - t(x)| \leqq \varepsilon\right\}$, etc.

Convergence in measure and a weakened version of this concept will be useful tools in this section. A sequence $\{s_n\}$ of (measurable) functions on X is said to converge in μ-measure to a function t, denoted by $s_n \overset{\mu}{\to} t$, if

$$\lim_{n\to\infty} \mu\Big(\big\{\,|s_n - t| > \varepsilon\big\}\Big) = 0 \quad \text{for all} \quad \varepsilon > 0. \tag{2.1}$$

We will say that $\{s_n\}$ converges *loosely in μ-measure* to t, denoted by $s_n \overset{\mu}{\rightsquigarrow} t$, if for every $A \in \mathcal{X}$ with $\mu(A) < \infty$

$$\lim_{n\to\infty} \mu\Big(A \cap \big\{\,|s_n - t| > \varepsilon\big\}\Big) = 0 \quad \text{for all} \quad \varepsilon > 0. \tag{2.2}$$

Further, $\{s_n\}$ will be said to be a Cauchy sequence loosely in μ-measure if for every $A \in \mathcal{X}$ with $\mu(A) < \infty$

$$\lim_{m,n\to\infty} \mu\Big(A \cap \big\{\,|s_m - s_n| > \varepsilon\big\}\Big) = 0 \quad \text{for all} \quad \varepsilon > 0. \tag{2.3}$$

Notice that such a sequence has a subsequence converging μ-a.e. to some function t (and then it follows that $s_n \overset{\mu}{\rightsquigarrow} t$).

LEMMA 1. *To every $\varepsilon > 0$ and $K > 0$ there exists $\gamma > 0$ (depending on f but not on μ) such that for every s, t in S and every set $C \in \mathcal{X}$ on which either s or t is upper bounded by K, we have*

$$\mu\Big(C \cap \big\{\,|s - t| \geqq \varepsilon\big\}\Big) \leqq \gamma B(s,t). \tag{2.4}$$

Further, to every $K > 0$ there exists $\beta > 0$ (depending on f but not on μ) such that for every $C \subset \{t \leqq K\}$

$$\mu\big(C \cap \{s \geqq L\}\big) \leqq \frac{\beta}{L} B(s,t) \quad \textit{if} \quad L \geqq 3K. \tag{2.5}$$

COROLLARY. *For a sequence $\{s_n\} \subset S$ such that either $B(s_n,t) \to 0$ or $B(t,s_n) \to 0$, we have $s_n \overset{\mu}{\rightsquigarrow} t$, and if t satisfies*

$$\lim_{K\to\infty} \mu\big(\{t \geqq K\}\big) = 0, \tag{2.6}$$

then $s_n \overset{\mu}{\to} t$, as well. Moreover, if t satisfies (2.6) *and $B(s,t) < \infty$ for some $s \in S$ then s also satisfies* (2.6).

REMARKS. (i) In general, $s_n \overset{\mu}{\rightsquigarrow} t$ does not imply $s_n \overset{\mu}{\to} t$, even if (2.6) holds (supposing, of course, that $\mu(X) = \infty$). A trivial counterexample is obtained by taking a sequence of sets $A_1 \subset A_2 \subset \ldots$, each of finite measure and with $\cup A_n = X$, letting s_n be the characteristic function of A_n, and $t = 1$.

(ii) In general, $B(s_n, t) \to 0$ does not imply $s_n \xrightarrow{\mu} t$. To see this, take s, t with $B(s,t) < \infty$, and let s_n be equal to t on A_n and to s on the complement of A_n, where the sets A_n are as in (i). Then $B(s,t) \to 0$, but $s_n \xrightarrow{\mu} t$ does not hold if $\mu\Big(\big\{|s-t| > \varepsilon\big\}\Big) = \infty$ for some $\varepsilon > 0$. It depends on the choice of f whether the last condition is compatible wit $B(s,t) < \infty$. One easily sees that it is, e.g., in the cases $f = f_\alpha$, $\alpha \leqq 1$, cf. (1.12).

PROOF. Since f is differentiable and strictly convex, f' is continuous and strictly increasing. From the identity

$$\Delta_f(u,v) = \int_v^u \big(f'(\xi) - f'(v)\big)\, d\xi, \tag{2.7}$$

or by looking at the graph of f, one sees that $\Delta_f(u,v)$ decreases if the larger of u and v is decreased. Hence the minimum of the two positive numbers $\min_{u \leqq K} \Delta_f(u, u+\varepsilon)$ and $\min_{v \leqq K} \Delta_f(v+\varepsilon, v)$ is a lower bound to $\Delta_f(u,v)$ subject to $|u-v| \geqq \varepsilon$, $\min(u,v) \leqq K$. Denoting this lower bound by $\frac{1}{\gamma}$, it follows for C as in the hypothesis that

$$B(s,t) \geqq \int_{C \cap \{|s-t| \geqq \varepsilon\}} \Delta_f\big(s(x), t(x)\big)\, \mu(dx) \geqq \frac{1}{\gamma}\mu\Big(C \cap \big\{|s-t| \geqq \varepsilon\big\}\Big).$$

this proves (2.4).

If $L \geqq 3K$, i.e., $\frac{2}{3}L \geqq 2K$, (2.7) and the monotonicity of f' imply for $u \geqq L$, $v \leqq K$ that

$$\Delta_f(u,v) \geqq \int_{\frac{2}{3}L}^{L} \big(f'(\xi) - f'(v)\big)\, d\xi \geqq \frac{1}{3}L\big(f'(2K) - f'(K)\big).$$

It follows that

$$B(s,t) \geqq \int_{C \cap \{s \geqq L\}} \Delta_f\big(s(x), t(x)\big)\, \mu(dx) \geqq$$

$$\geqq \frac{L}{3}\big(f'(2K) - f'(K)\big)\mu\big(C \cap \{s \geqq L\}\big)$$

proving (2.5).

To prove the Corollary, take any $A \in \mathcal{X}$ with $\mu(A) < \infty$. Given any $\eta > 0$, let K be so large that $\mu(A \cap \{t > K\}) < \eta$, and apply (2.4) to $C = A \cap \{t \leqq K\}$. It follows that

$$\mu\Big(A \cap \big\{|s_n - t| \geqq \varepsilon\big\}\Big) \leqq \mu\Big(C \cap \big\{|s_n - t| \geqq \varepsilon\big\}\Big) + \mu(A \cap \{t > K\}) \leqq$$
$$\leqq \gamma B(s_n, t) + \eta.$$

This proves that $B(s_n, t) \to 0$ implies (2.2), i.e., $s_n \overset{\mu}{\rightsquigarrow} t$. Since in the last step $B(s_n, t)$ could be replaced by $B(t, s_n)$, the same result follows also when $B(t, s_n) \to 0$. Under hypothesis (2.6), the last argument works also for $A = X$. It gives then (2.1), i.e., $s_n \overset{\mu}{\to} t$.

The last assertion of the Corollary follows from (2.5).

We will write for any $E \subset S$, in accordance with (1.1),

$$B(E,t) = \inf_{s \in E} B(s,t), \quad D(E,t) = \inf_{s \in E} D(s,t), \quad J(E) = \inf_{s \in E} J(s). \tag{2.8}$$

If there exists $s^* \in E$ with $B(s^*, t) = B(E, t)$ resp. $D(s^*, t) = D(E, t)$, and this s^* is unique (considering μ-a.e. functions as identical), it will be called the B- resp. D-projection of t onto E. Even for nice convex sets E, such as those defined by constraints of the type (1.2), these projections may not exist. We will show, however, that the generalized projections in the sense of the following definition always exist, under very weak hypotheses. The related problem of minimizing $J(s)$ subject to $s \in E$ also be considered, but no specific terminology will be introduced for that problem. Notice that the latter differs from the D-projection problem with $t = 1$ only in the lack of assuming (1.15).

DEFINITION 1. Given $E \subset S$ and $t \in S$, a sequence $\{s_n\} \subset E$ is B-, D- or J-minimizing sequence if $B(s_n, t)$, $D(s_n, t)$ or $J(s_n)$ converges to the corresponding infimum in (2.8). If there is an $s^* \in S$ such that every B-resp. D-minimizing sequence converges to s^* loosely in μ-measure, this s^* is called the generalized B- resp. D-projection of t to E. Here B, D, J always means $B_{f,\mu}, D_{f,\mu}, J_{f,\mu}$; when the short notation may cause ambiguity, the full notation will be used.

LEMMA 2. *If the generalized B- or D-projection of t to E exists, it is unique (up to μ-a.e. equality), satisfies*

$$B(s^*, t) \leqq B(E, t) \quad \textit{resp.} \quad D(s^*, t) \leqq D(E, t), \tag{2.9}$$

and the B- resp. D-projection of t onto E exists iff $s^* \in E$ (in which case the projection equals the generalized projection s^*).

PROOF. The uniqueness of s^* is obvious from the definition. As a minimizing sequence $\{s_n\}$ converging to s^* loosely in μ-measure has a subsequence that converges to s^* μ-a.e., (2.9) follows by applying Fatou's lemma to that subsequence. If $s^* \in E$, it follows from (2.8) and (2.9) that the minimum of $B(s,t)$ subject to $s \in E$ is attained for $s = s^*$. On the other hand, if that minimum is attained for some $s \in E$ then, applying the definition of generalized projections to the trivial minimizing sequence $\{s_n\}$ with $s_n = s$, $n = 1, 2, \ldots$, we obtain that $s^* = s$.

REMARK. The J-analogue of (2.9) can not be asserted in general: from the convergence of a J-minimizing sequence $\{s_n\} \subset E$ to some $s^* \in S$ one can not conclude $J(s^*) \leqq J(E)$, lacking the non-negativity of the integrand needed for Fatou's lemma. Of course, this problem does not occur if μ is a finite measure and $f(u)$ is bounded below (as, e.g., the most commonly used $f(u) = u \log u$).

THEOREM 1. *Let E be a convex subset of S, and $t \in S$.*
(a) *If $B(E,t)$ is finite, there exists $s^* \in S$ such that*

$$B(s,t) \geqq B(E,t) + B(s,s^*) \quad \text{for every} \quad s \in E. \tag{2.10}$$

(b) *If $D(E,t)$ is finite, there exists $\widetilde{s}^* \in S$ such that*

$$D(s,t) \geqq D(E,t) + B_{f,\nu}\left(\frac{s}{t}, \widetilde{s}^*\right) \quad \text{for every} \quad s \in E, \tag{2.11}$$

where the measure $\nu \ll \mu$ is defined by $\frac{d\nu}{d\mu} = t$.
(c) *If $J(E)$ is finite, there exists $s^* \in S$ such that*

$$J(s) \geqq J(E) + B(s,s^*) \quad \text{for every} \quad s \in E. \tag{2.12}$$

COROLLARY. *Under the hypotheses of the Theorem,*
(a) *The generalized B-projection of t to E exists and equals the s^* in* (2.10).
(b) *Every D-minimizing sequence satisfies $s_n 1_{\{t>0\}} \overset{\mu}{\rightsquigarrow} s^*$, where $s^* = t\widetilde{s}^*$ with $\widetilde{s}^*$ in* (2.11), *and $1_{\{t>0\}}$ is the characteristic function of the set $\{t > 0\}$. In particular, if $t > 0$ μ-a.e., or if f satisfies the condition*

$$\lim_{v \to \infty} f'(v) = \infty, \tag{2.13}$$

the generalized D-projection of t to E exists, and equals s^*.
(c) *Every J-minimizing sequence satisfies $s_n \overset{\mu}{\rightsquigarrow} s^*$, for s^* in* (2.12).

REMARKS. For the I-divergence $I(s\|t)$ in the role of $B(s,t)$ and $D(s,t)$ (corresponding to $f(u) = u \log u - u + 1$) resp. for the negative entropy

$I(s) = \int s \log s \, d\mu$ in the role of $J(s)$ (corresponding to $f(u) = u \log u$), the results of Theorem 1 were established by Topsøe [18]. Previously, Csiszár [8] proved the I-divergence special case of (2.10) when $s^* \in E$, i.e., when the I-projection existed, and gave sufficient conditions for that existence. Both in [8] and [18], the terminology of measures rather than functions was used, and attention was restricted to probability measures, i.e., in the present terminology, to probability density rather than arbitrary non-negative functions. The proof below uses the ideas of [8] and [18]. A similar approach was used by Schroeder [16] to establish the existence of B-projection under certain conditions and to prove the inequality (2.10) in that case. An inequality equivalent to (2.12) appears in Teboulle and Vajda [17], again for the case when the minimum of $J(s)$ subject to $s \in E$ is attained.

PROOF. (a) The proof relies upon the identity

(2.14)

$$\alpha B(s,t) + (1-\alpha)B(s',t) = B\big(\alpha s + (1-\alpha)s', t\big) + \alpha B\big(s, \alpha s + (1-\alpha)s'\big) + $$
$$+ (1-\alpha)B\big(s', \alpha s + (1-\alpha)s'\big).$$

This holds for every s, s' in S and $0 < \alpha < 1$, as can be checked by simple algebra using the definitions (1.14) and (1.8).

Now let $\{s_n\}$ be a B-minimizing sequence such that $B(s_n, t)$ is finite for each n. Applying (2.14) to $s = s_m$, $s' = s_n$, $\alpha = \frac{1}{2}$ yields

$$B(s_m, t) + B(s_n, t) =$$
$$= 2B\left(\frac{s_m + s_n}{2}, t\right) + B\left(s_m, \frac{s_m + s_n}{2}\right) + B\left(s_n, \frac{s_m + s_n}{2}\right).$$

This implies, in particular, that $B\left(\frac{s_m+s_n}{2}, t\right) \leqq \max_n B(s_n, t)$ for all m, n and that

$$\lim_{m,n\to\infty} B\left(s_m, \frac{s_m + s_n}{2}\right) = 0 \tag{2.15}$$

(the latter because $B\left(\frac{s_m+s_n}{2}, t\right) \geqq B(E, t)$, since $\frac{s_m+s_n}{2} \in E$ by the convexity of E).

We claim that $\{s_n\}$ is a Cauchy sequence loosely in μ-measure. To prove this, take any $A \in \mathcal{X}$ with $\mu(A) < \infty$. Given any $\eta > 0$, first pick K so large that $\mu\big(A \cap \{t > K\}\big) < \eta$ then use (2.5) with $s = \frac{s_m+s_n}{2}$ and $C = A \cap \{t \leqq K\}$ to get that for a sufficiently large L we have for every m, n $\mu\left(A \cap \left\{t \leqq K, \frac{s_m+s_n}{2} \geqq L\right\}\right) \leqq \eta$, and consequently

$$\mu\left(A \cap \left\{\frac{s_m + s_n}{2} \geqq L\right\}\right) \leqq 2\eta. \tag{2.16}$$

Next we use the bound (2.4) with $s = s_m, t = \frac{s_m+s_n}{2}$, and $C = A \cap \cap \left\{\frac{s_m+s_n}{2} \leqq L\right\}$, now L playing the role of K. It follows that for a suitable $\gamma > 0$ (depending on ε and L) $\mu\left(A \cap \left\{\frac{s_m+s_n}{2} \leqq L, \left|\frac{s_m-s_n}{2}\right| \geqq \varepsilon\right\}\right) \leqq \leqq \gamma B\left(s_m, \frac{s_m+s_n}{2}\right)$, and thus, on account of (2.16),

$$\text{(2.17)} \qquad \mu\Big(A \cap \left\{|s_m - s_n| \geqq 2\varepsilon\right\}\Big) \leqq \gamma B\left(s_m, \frac{s_m + s_n}{2}\right) + 2\eta.$$

Since $\eta > 0$ has been arbitrary, (2.15) and (2.17) give (2.3), establishing our claim.

Since $\{s_n\}$ is a Cauchy sequence loosely in μ-measure, it has an a.e.-convergent subsequence, say $s_{n_k} \to s^*$, μ-a.e. To prove that this s^* satisfies (2.10), pick any $s \in E$ such that $B(s,t)$ is finite, and apply (2.14) to this s and $s' = s_{n_k}$. Bound the right hand side replacing the first term by $B(E,t)$ and the last term by 0. Thus, after rearranging, we obtain that

(2.18)
$$B(s,t) \geqq B(E,t) - \frac{1-\alpha}{\alpha}\left(B(s_{n_k},t) - B(E,t)\right) + B\left(s, \alpha s + (1-\alpha)s_{n_k}\right).$$

Letting $\alpha = \alpha_k$ go to 0 sufficiently slowly, the second term on the right will go to 0. As $s_{n_k} \to s^*$ a.e. implies, by Fatou's lemma, that $\liminf_{k\to\infty} B\left(s, \alpha_k s + + (1-\alpha_k)s_{n_k}\right) \geqq B(s, s^*)$, the inequality (2.10) follows from (2.18).

(b) Using the representation (1.16) of $D(s,t)$ and the identity (2.14), one gets the following analogue of (2.14) for f-divergences:

$$\text{(2.19)} \qquad hfill \alpha D(s,t) + (1-\alpha)D(s',t) = D\left(\alpha s + (1-\alpha)s', t\right) +$$
$$+\alpha B_{f,\nu}\left(\frac{s}{t}, \frac{\alpha s + (1-\alpha)s'}{t}\right) + (1-\alpha)B_{f,\nu}\left(\frac{s'}{t}, \frac{\alpha s + (1-\alpha)s'}{t}\right)$$

(notice that dividing by t is permissible when the underlying measure is ν since, by definition, $t > 0$ ν-a.e.). Starting with (2.19) instead of (2.14), one sees exactly as in (a) above that for a D-minimizing sequence such that $D(s_n,t)$ is finite for all n (i) the sequence $\left\{\frac{s_n}{t}\right\}$ is Cauchy loosely in ν-measure and (ii) if $\widetilde{s}^*$ is the limit of a ν-a.e. convergent subsequence $\left\{\frac{s_{n_k}}{t}\right\}$ then this $\widetilde{s}^*$ satisfies (2.11).

(c) The proof is identical to that of (a), starting now with the following (easily checked) analogue of (2.14):

(2.20)
$$\alpha J(s) + (1-\alpha)J(s') = J\left(\alpha s + (1-\alpha)s'\right) + \alpha B\left(s, \alpha s + (1-\alpha)s'\right) +$$
$$+(1-\alpha)B\left(s', \alpha s + (1-\alpha)s'\right),$$

which holds whenever $J(s)$ and $J(s')$ are finite.

Parts (a) and (c) of the Corollary follow immediately from (2.10) and (2.12), by the Corollary of Lemma 1. For part (b), notice that (2.11) implies $B_{f,\nu}\left(\frac{s_n}{t},\widetilde{s}^*\right)\to 0$, and hence $\frac{s_n}{t}\overset{\nu}{\rightsquigarrow}\widetilde{s}^*$ by the Corollary of Lemma 1. To prove that $s_n 1_{\{t>0\}}\overset{\mu}{\rightsquigarrow} s^*$, where $s^*=t\widetilde{s}^*$, we have to verify (2.2) for every $A\in\in\mathcal{X}$ with $\mu(A)<\infty$, for $s_n 1_{\{t>0\}}$ and s^* in the roles of s_n and t. Clearly, attention may be restricted to sets $A\subset\{t>0\}$. Given such an A, let $A_K==A\cap\{t\leqq K\}$. Then $\nu(A_K)\leqq K\mu(A)<\infty$, thus on account of $\frac{s_n}{t}\overset{\nu}{\rightsquigarrow}\widetilde{s}^*$ we have $\left(A_K\cap\left\{\left|\frac{s_n}{t}-\widetilde{s}^*\right|>\frac{\varepsilon}{K}\right\}\right)\to 0$ and consequently

$$\nu\Big(A_K\cap\big\{|s_n-s^*|>\varepsilon\big\}\Big)\to 0\quad\text{as}\quad n\to\infty.$$

Since $\nu(A_K)<\infty$ and $\frac{d\nu}{d\mu}=t$ is positive on A_K, it follows that also

$$\mu\Big(A_K\cap\big\{|s_n-s^*|>\varepsilon\big\}\Big)\to 0\quad\text{as}\quad n\to\infty. \tag{2.21}$$

As this holds for every K, and $\mu(A\setminus A_K)\to 0$ as $K\to\infty$, (2.21) implies the assertion $s_n 1_{\{t>0\}}\overset{\mu}{\rightsquigarrow} s^*$. If $t>0$ μ-a.e., the latter means the same as $s_n\overset{\mu}{\rightsquigarrow} s^*$. Under the condition (2.13), $D(s_n,t)<\infty$ implies $s_n=0$ μ-a.e. on $\{t=0\}$ cf. (1.16). Since $s^*=t\widetilde{s}^*$ also vanishes on $\{t=0\}$, in this case $s_n 1_{\{t>0\}}\overset{\mu}{\rightsquigarrow} s^*$ is again equivalent to $s_n\overset{\mu}{\rightsquigarrow} s^*$. Thus, in both cases, s^* is the generalized D-projection of t onto E.

REMARK. The generalized D-projection to a convex set $E\subset S$ with $D(E,t)<\infty$ does not always exist if $\lim_{v\to\infty}f'(v)<\infty$, even if E is defined by constraints of the type (1.2). A trivial counterexample when several (not μ-a.e. equal) functions $s\in E$ achieve $D(s,t)=D(E,t)$ is given by $E==\left\{s\colon\int_{\{t=0\}}s(x)\mu(dx)=b\right\}$, where b is a positive constant. Then $D(s,t)==D(E,t)$ for every $s\in E$ such that $s=t$ on $\{t>0\}$.

In our definition of generalized projections, the convergence of minimizing sequences has been required in a very weak sense, viz. loosely in μ-measure. This enabled us to prove the existence of generalized projections under the weakest concievable hypotheses. Still, in many cases of interest, convergence in a stronger sense can be established. Some simple general results in that direction will be given in Theorem 2, also including the convergence of J-minimizing sequences. We send forward

LEMMA 3. *Suppose that μ is a finite measure.*
(a) *If f satisfies the condition*

$$\inf_{v\geqq 1}\big(f'(Kv)-f'(v)\big)>0\quad\textit{for some}\quad K>1 \tag{2.22}$$

then for every $t \in L_1(\mu)$ and $r > 0$, the functions $s \in S$ in the "Bregman ball" $\{s\colon B(s,t) \leqq r\}$ are uniformly integrable.

(b) *If f does not satisfy (2.22), and μ is not atomic, there exists $t \in \in L_1(\mu)$ to which non-integrable functions $s_n \in S$ can be found with $B(s_n, t) \to 0$.*

REMARK. Condition (2.22) is stronger than (2.13) but among the functions f_α, cf. (1.12), either condition holds for those with $\alpha \geqq 1$.

PROOF. (a) Given $t \in L_1(\mu)$, define $\bar{t} \in L_1(\mu)$ by $\bar{t}(x) = \max(t(x), 1)$. It suffices to prove that to any $\varepsilon > 0$ there exists $M > 0$ such that

$$\int_{\{s>M\bar{t}\}} s\, d\mu < \varepsilon \quad \text{if} \quad B(s,t) \leqq r. \tag{2.23}$$

Now, notice that (2.22) implies that for M equal to a sufficiently large power of K, $\inf_{v \geqq 1}\big(f'(Mv) - f'(v)\big)$ will be as large as desired. In particular, there exists $M > 2$ such that

$$f'\left(\frac{Mv}{2}\right) - f'(v) > \frac{2r}{\varepsilon} \quad \text{for all} \quad v \geqq 1.$$

With such an M, (2.7) implies that for $u > Mv$, $v \geqq 1$

$$\Delta_f(u,v) \geqq \int_{\frac{Mv}{2}}^{u} \big(f'(\xi) - f'(v)\big)\, d\xi > \frac{2r}{\varepsilon}\left(u - \frac{Mv}{2}\right) > \frac{r}{\varepsilon} u.$$

It follows that

$$B(s,t) > \int_{\{s>M\bar{t}\}} \Delta_f\big(s(x), t(x)\big)\, \mu(dx) \geqq$$

$$\geqq \int_{\{s>M\bar{t}\}} \Delta_f\big(s(x), \bar{t}(x)\big)\, \mu(dx) > \frac{r}{\varepsilon} \int_{\{s>M\bar{t}\}} s(x)\mu(dx)$$

establishing (2.23).

(b) If (2.22) does not hold, for arbitrarily large K there exists $v \geqq 1$ such that $f'(Kv) - f'(v)$ is arbitrarily small. Thus for some sequence of numbers $v_n \geqq 1$ (and then necessarily $v_n \to \infty$)

$$f'(nv_n) - f'(v_n) \leqq \frac{1}{n}, \quad n = 1, 2, \ldots$$

Then from (2.7)

$$\Delta_f(nv_n, v_n) = \int_{v_n}^{nv_n} \big(f'(\xi) - f'(v_n)\big)\, d\xi \leqq v_n. \tag{2.24}$$

Let $t \in S$ be a function whose range is a subset of $\{v_1, v_2, \ldots\}$, and let $s(x) = = nv_n$ if $t(x) = v_n$. Since μ is non-atomic, such t and s exist with $t \in L_1(\mu)$, $s \notin L_1(\mu)$, and then (2.24) shows that $B(s,t) < \infty$. Finally, let $s_n(x)$ equal $t(x)$ or $s(x)$ according as $t(x)$ equals v_m with $m \leqq n$ or with $m > n$, respectively. Then $B(s_n, t) \to 0$, while neither s_n is integrable.

THEOREM 2. *Under the hypotheses of Theorem* 1:

(a) *If t satisfies condition* (2.6) *then every B-minimizing sequence converges in μ-measure. If μ is a finite measure, $t \in L_1(\mu)$, and f satisfies condition* (2.22), *then every B-minimizing sequence converges in $L_1(\mu)$ norm.*

(b) *If $t \in L_1(\mu)$ and f satisfies condition* (2.13) *then every D-minimizing sequence converges in $L_1(\mu)$ norm.*

(c) *If μ is a finite measure and f satisfies condition* (2.13) *then every J-minimizing sequence converges in $L_1(\mu)$ norm.*

PROOF. (a) A B-minimizing sequence satisfies $B(s_n, s^*) \to 0$, by (2.10). Since $B(s^*, t) \leqq B(E, t) < \infty$ by Lemma 2, the Corollary of Lemma 1 gives that if t satisfies (2.6) then so does also s^*. But then $B(s_n, s^*) \to 0$ implies $s_n \xrightarrow{\mu} s^*$, again by the Corollary of Lemma 1. The last assertion follows from Lemma 3, since convergence in measure plus uniform integrability imply convergence in $L_1(\mu)$ norm.

(b) Without any loss of generality, a D-minimizing sequence can be assumed to satisfy $D(s_n, t) < \infty$ for all n. Then (2.13) implies that $s_n = 0$ μ-a.e. on the set $\{t = 0\}$, and

$$D(s_n, t) = \int f\left(\frac{s_n}{t}\right) d\nu, \tag{2.25}$$

with $\nu \ll \mu$, $\frac{d\nu}{d\mu} = t$, by the definition (1.13) of $D(s,t)$. Under the hypothesis $t \in L_1(\mu)$, ν is a finite measure. Thus $B_{f,\nu}\left(\frac{s_n}{t}, s^*\right) \to 0$ (implied by (2.11)) gives by the Corollary of Lemma 1 that $\frac{s_n}{t} \xrightarrow{\nu} \widetilde{s}^*$. As ν is a finite measure, it follows from (2.13) and the uniform boundedness of the integrals (2.25) that $\frac{s_n}{t}$ is uniformly ν-integrable. Hence $\frac{s_n}{t} \to \widetilde{s}^*$ also in $L_1(\nu)$-norm. Since $\int \left|\frac{s_n}{t} - \widetilde{s}^*\right| d\nu = \int |s_n - s^*|\, d\mu$, where $s^* = t\widetilde{s}^*$, this means the same as $s_n \to \to s^*$ in $L_1(\mu)$ norm.

(c) If μ is a finite measure, each J-minimizing sequence satisfies $s_n \xrightarrow{\mu} s^*$, by the Corollary of Theorem 1. As in (b), the finiteness of μ, the condition (2.13), and the uniform boundedness of the integrals $J(s_n) = \int f(s_n)\, d\mu$ imply that the functions s_n are uniformly integrable (although f is now not assumed to satisfy (1.15), by convexity and the condition (2.13) it must be bounded from below). Hence it follows that $s_n \to s^*$ also in $L_1(\mu)$ norm.

REMARKS. In assertion (a) of Theorem 2, the condition (2.22) for L_1-convergence of B-minimizing sequences can not be relaxed in general. As Lemma 3 shows, in the trivial case $E = S$ there exist $t \in L_1(\mu)$ and a B-

minimizing sequence $\{s_n\}$ (in this case, simply a sequence with $B(s_n, t) \to 0$) such that neither s_n is integrable. In assertions (b) and (c), the condition (2.13) can not be relaxed either. Notice finally that if t is a bounded function and μ a finite measure, then already condition (2.13) suffices for the uniform integrability of "Bregman balls" and hence for the L_1-convergence of B-minimizing sequences.

3. Projections and generalized projections

We proceed to use the terminology introduced before. In particular, S denotes the set of non-negative finite valued $\mathcal{X}$-measurable functions on X, and the symbols B and D are shorthands for $B_{f,\mu}$ and $D_{f,\mu}$, cf. (1.13), (1.14). In this section, the convex function f is always assumed to satisfy (1.15), and E will always denote a convex subset of S, unless stated otherwise.

We have seen in Section 2 that the generalized B- resp. D-projection of $t \in S$ to the convex set $E \subset S$ always exists, except for trivial cases. A question of the interest is, however, whether the B- resp. D-projection exists. By Lemma 2, this is the same question whether the generalized projection s^* belongs to E. A sufficient condition for the latter, by the very definition of generalized projections, is the closedness of E for loose convergence in μ-measure. If the minimizing sequences are known to converge in some stronger sense, closedness of E (or of its intersection with a B-ball resp. D-ball of center t and radius larger than $B(E,t)$ resp., $D(E,t)$) suffices. In particular, using Theorem 2, the closedness of $E \cap L_1(\mu)$ in $L_1(\mu)$ norm is sufficient for the existence of D-projection if f satisfies (2.13), resp. for the existence of B-projection if μ is a finite measure and f satisfies (2.22), whenever $t \in L_1(\mu)$ and $D(E,t)$, resp. $B(E,t)$ is finite.

In the literature, the usual way to derive sufficient conditions for the existence of projections onto a set E is by compactness arguments, introducing a topology in which E is compact and the distance to be minimized is a lower semicontinuous function on E. The last mentioned sufficient conditions for the existence of D- and B-projections can be obtained also by this method, cf. Teboulle and Vajda [17] for D-projections (they actually considered integrals $J(s)$, with f satisfying (2.13), but rewriting their result for D-divergences is just a matter of translation). In Theorem 3 we will also offer a sufficient condition for the existence of D-projection that does not require $L_1(\mu)$ closedness; we will give a proof that relies on Theorem 1 and some Orlicz space theory.

As hinted to in the Introduction, the projection problem occurs in practice mostly for sets of functions defined by equality constraints on certain integrals, cf. (1.2). Sometimes one has inequality rather than equality constraints. Thus, consider subsets of S of form

$$E = \left\{ s\colon \int a_\gamma(x)s(x)\mu(dx) \leqq b_\gamma, \quad \gamma \in \Gamma \right\} \tag{3.1}$$

where Γ is any index set, the s_γ are given (measurable) functions on X, and the b_γ are given constants. The set (3.1) may contain functions $s \in S$ with $\int a_\gamma s\, d\mu = -\infty$, but none with $\int a_\gamma s\, d\mu$ undefined. Of course, sets defined by equality constraints can also be represented in the form (3.1), since a constraint $\int a_\gamma s\, d\mu = b_\gamma$ is equivalent to a pair of constraint $\int a_\gamma s\, d\mu \leqq b_\gamma$, $\int (-a_\gamma) s\, d\mu \leqq -b_\gamma$.

THEOREM 3. *Let E be a given by* (3.1) *and let $t \in S$ be such that $B(E,t)$ resp. $D(E,t)$ is finite.*

(i) *If each $a_\gamma \in S$, the B-projection of t onto E always exists, and the D-projection exists if $t > 0$ μ-a.e., or if f satisfies* (2.13).

(ii) *If each a_γ is bounded below, and $t \in L_1(\mu)$, the D-projection of t onto E exists if f satisfies* (2.13), *and the B-projection exists if μ is a finite measure and f satisfies* (2.22).

(iii) *If each a_γ satisfies*

$$\int f^*(\lambda a_\gamma^-) t\, d\mu < \infty \quad \textit{for every} \quad \lambda > 0 \tag{3.2}$$

where f^* denotes the convex conjugate of f and $a_\gamma^- = \max(0, -a_\gamma)$, the D-projection of t onto E exists provided $t \in L_1(\mu)$ and f satisfies (2.13).

COROLLARY. *For a set E defined as in* (3.1) *but with equality (rather than inequality) constraints, assertion* (ii) *will hold if the hypothesis of lower boundedness of the functions a_γ is replaced by boundedness, and* (iii) *will hold if in the hypothesis* (3.2), *a_γ^- is changed to $|a_\gamma|$.*

REMARK. The convex conjugate of a convex function f is the convex function $f^*(v) = \sup_u (uv - f(u))$. Under our assumptions on f (including (2.13)), f^* is finite valued and differentiable on $(0, \infty)$ (we are not interested in $f^*(v)$ for $v < 0$) and is given by

$$f^*(v) = \int_0^v (f')^{-1}(\xi)\, d\xi. \tag{3.3}$$

In particular,

$$f^*(v) = e^v - 1 \quad \text{if} \quad f(u) = u \log u - u + 1. \tag{3.4}$$

PROOF. (i) If $\{s_n\} \subset E$ converges loosely in μ-measure, it has a subsequence such that $s_{n_k} \to s'$ μ-a.e. The hypothesis $a_\gamma \geqq 0$ implies by Fatou's lemma that

$$\int a_\gamma s' \, d\mu \leqq \liminf_{k\to\infty} \int a_\gamma s_{n_k} \, d\mu,$$

thus $s' \in E$. This means that E is closed for loose convergence in μ-measure, and the assertion follows by the previous discussion.

(ii) If $\{s_n\} \subset E \cap L_1(\mu)$ converges to some s' in $L_1(\mu)$ norm, writing $a_\gamma = a_\gamma^+ - a_\gamma^-$ we obtain, since a_γ^- is bounded by hypothesis, that $\int s_n a_\gamma^- \, d\mu \to \int s' a_\gamma^- \, d\mu$. On the other hand, $\int a_\gamma^+ s' \, d\mu \leqq \liminf_{n\to\infty} \int a_\gamma^+ s_n \, d\mu$ as above, therefore $s' \in E \cap L_1(\mu)$. Thus $E \cap L_1(\mu)$ is closed in $L_1(\mu)$ norm, and the assertion follows by the previous discussion.

(iii) We assume, without restricting generality, that μ is a finite measure and $t = 1$; the case of arbitrary μ and $t \in L_1(\mu)$ reduces to this replacing μ by ν, where $\frac{d\nu}{d\mu} = t$, and giving the role of s to $\frac{s}{t}$. Thus we consider a sequence $\{s_n\} \subset E$ with

$$D(s_n, 1) = \int f(s_n) \, d\mu \to D(E, 1), \tag{3.5}$$

known to converge in μ-measure (and even in $L_1(\mu)$ norm, by Theorem 2) to the generalized D-projection s^* of $t = 1$ to E; our goal is to show that $s^* \in E$.

We need the following facts from the theory of Orlicz spaces, cf. Krasnoselskii and Rutitskii [14]. Given a strictly convex function $M(u)$ on $(0, \infty)$ with

$$\lim_{u\to 0} \frac{M(u)}{u} = 0 \quad \lim_{u\to\infty} \frac{M(u)}{u} = \infty, \tag{3.6}$$

the Orlicz class $L_M = L_M(\mu)$ is the set of all (measurable) functions $u(x)$ on X such that

$$\rho(\mu, M) = \int M\big(|u(x)|\big)\mu(dx) < \infty. \tag{3.7}$$

The Orlicz space L_M^* and its (not necessarily proper) subspace E_M consist of those functions u for which $\lambda u \in L_M$ for some $\lambda > 0$, respectively for all $\lambda > 0$. L_M^* is a Banach space with the Orlicz norm

$$\|u\|_M = \inf_{\lambda>0} \frac{1}{\lambda}\big(1 + \rho(\lambda u, M)\big). \tag{3.8}$$

Moreover, setting $N = M^*$, one can consider on L_M^* the E_N-weak topology: $u_n \to u$ E_N-weakly iff $\int u_n v\,d\mu \to \int uv\,d\mu$ for all $v \in E_N$. The result we need is the following:

[14], Theorem 14.6: If a sequence $\{u_n\} \subset L_M^*$ is bounded in Orlicz norm and converges in μ-measure to some function u, then $u \in L_M^*$ and $u_n \to u$ E_N-weakly.

We will apply this theorem to the sequence $\{s_n\} \subset E$ in (3.5), with the choice $M(u) = f(1+u)$. Notice that this M satisfies the conditions (3.6). Further, the sequence $\{s_n\}$ is bounded in Orlicz norm, since by (3.7) and (3.8)

$$\|s_n\|_M \leqq 2\left(1+\rho\left(\frac{s_n}{2}, M\right)\right) = 2 + 2\int f\left(1+\frac{s_n}{2}\right)\,d\mu \leqq$$

$$\leqq 2 + f(2)\mu(X) + \int f(s_n)\,d\mu$$

(in the last step we have used the convexity of f). It follows by the above theorem that $s_n \to s^*$ E_N-weakly.

Since $M(u) = f(1+u)$, its convex conjugate $N(v)$ equals $f^*(v) - v$. Notice that $N = M^*$ always satisfies the conditions (3.6) if M does. Thus for any function $v(x)$ in S, the integrability of $N(v) = f^*(v) - v$ is equivalent to that of $f^*(v)$. It follows that the hypothesis (3.2) (with $t = 1$) means exactly that $a_\gamma^- \in E_N$. Hence the E_N-weak convergence $s_n \to s^*$ implies $\int s_n a_\gamma^-\,d\mu \to \int s^* a_\gamma^-\,d\mu$; after having established this, the proof is completed as previously.

The Corollary is immediate since an equality constraint can be regarded as two inequality constraints, one with a_γ and the other with $-a_\gamma$.

COMMENTS. Part (i) of Theorem 3 is very general as far as f, μ and t are concerned, but it involves a very restrictive hypothesis on the functions a_γ. Since that hypothesis can not be non-trivially met by both a_γ and $-a_\gamma$, part (i) has no counterpart for sets defined by equality constraints as in the Corollary. Part (ii) involves a considerable restriction on the underlying f, but only a weaker hypothesis on the functions a_γ, which is often met in practice. As hinted to before, this part of Theorem 3 is not new, although for B-projections, our hypothesis on f is weaker than under which a similar result had been proved previously (Schroeder [16]). Part (iii) further relaxes the hypothesis on the functions a_γ, but at present it is available for D-projection only. Of course, a B-projection counterpart of (iii) does hold for certain functions f. E.g., for $f(u) = (u-1)^2$, cf. (1.12) with $\alpha = 2$, we have $f^*(v) = \frac{v^2}{4} + v$. In this case $B(s,t)$ equals the squared $L_2(\mu)$ distance of s and t, and the conditions (3.8) are obviously sufficient for the existence of the B-projection of any $t \in L_2(\mu)$ onto E. The difficulty we encountered when trying to prove the B counterpart of assertion (iii) was that, without an additional condition on f, or on t (such as its boundedness), we were unable to prove the Orlicz norm boundedness of B-minimizing sequences, and suspect that this may not be true in general.

For the case of I-divergence, i.e., $f(u) = u \log u - u + 1$, when $f^*(v)$ is given by (3.4), condition (3.2) requires (omitting the irrelevant constant) that for each a_γ, the integral of $te^{\lambda a_\gamma^-}$ be finite for every $\lambda > 0$. The corresponding condition for equality constraints (cf. the Corollary) is the finiteness of the integrals $\int te^{\lambda |a_\gamma|}\, d\mu$, for every $\lambda > 0$, which means the same as the finiteness of $\int te^{\lambda a_\gamma}\, d\mu$ for every $\lambda \in \mathbf{R}$. In the perhaps most important case when $t(x)$ is a probability density function with respect to μ, the last integral as a function of λ is known as the *moment generating function* of a_γ, for the probability distribution with μ-density $t(x)$. It has been well known that for a set E of probability densities determined by a finite number of equality constraints as in (1.2), the I-projection onto E of the density t surely exists if the functions involved in the constraints have everywhere finite moment generating functions for the distribution with density t. It appears new that the same is true also for any number of constraints (providing there exists any $s \in E$ with $I(s\|t) < \infty$).

As an example, consider probability distributions on a Banach space X, with a common dominating measure μ, and let E be the set of the density functions of those among them whose expectation (in the Pettis integral sense) exists and equals a given $b \in X$. Then E is determined by the linear constraints

$$\int s(x)\Theta(x)\mu(dx) = \Theta(b), \quad \Theta \in X^*,$$

in addition to $\int s(x)\mu(dx) = 1$, where X^* denotes the dual of X. Our sufficient condition for the existence of the I-projection onto this set E of a given density $t(x)$ with $I(E,t) < \infty$ is that $\int e^{\lambda\Theta(x)} t(x)\mu(dx)$ be finite for all $\Theta \in X^*$ and $\lambda \in \mathbf{R}$. Since with $\Theta \in X^*$ also $\lambda\Theta \in X^*$, the sufficient condition is actually the finiteness of $\int e^{\Theta(x)} t(x)\mu(dx)$ for all $\Theta \in X^*$. Notice that this is weaker than the finiteness of the moment generating function of $\|x\|$ for all $\lambda > 0$, which is the well known condition often used in such problems.

Although generalized B- and D-projections (to convex sets $E \subset S$) practically always exist, satisfactory conditions for their belonging to E, i.e., for the existence of projections, were obtained only subject to the condition (2.13) on f (for D-projection) or the stronger condition (2.22) (for B-projections). Without those, good sufficient conditions for the existence of projections, of the general flavor as above (i.e., not involving topological hypotheses on X an on the functions a_γ in the constraints) apparently can not be given. A very thorough study of a class of minimization problems that includes D- and B-projections onto sets determined by finitely many equality constraints, has been carried out by Borwein and Lewis [3], under compactness and continuity hypotheses. Their approach very much differs from ours, heavily relying upon convex analysis, particularly on advanced duality techniques.

Theorem 4 below will shed some light on the problem, and it fits nicely into our framework, not requiring tools beyond standard measure theory. This theorem is not stated explicitly in Borwein and Lewis [3] but related results are (though inequality constraints are not considered). The author could not determine to what extent can the assertion be considered new; it will be offered here at least for the simple proof.

We consider the D-projection problem for $E \subset S$ defined by (3.1), with $t \in L_1(\mu), t > 0$ μ-a.e. As in the proof of Theorem 3, we assume without restricting generality that μ is a finite measure and $t = 1$. Recall from the Introduction that f-divergences are most naturally looked at as distances of measures rather than of functions. Thus $D_{f,\mu}(s, 1)$ — equal to the integral (1.18) — is the f-divergence $D_f(\nu, \mu)$ of the measure $\nu \ll \mu$ with $\frac{d\nu}{d\mu} = s$ from the measure μ. Hence, minimizing $D_{f,\mu}(s, 1)$ subject to $s \in E$ is equivalent to minimizing $D_f(\nu, \mu)$ subject to $\nu \in F$, $\nu \ll \mu$, where F is the set of finite measures ν on $(X, \mathcal{X})$ satisfying

$$\int a_\gamma(x)\nu(dx) \leqq b_\gamma, \quad \gamma \in \Gamma. \tag{3.9}$$

Dropping the constraint $\nu \ll \mu$ would not change the problem if (2.13) held, since then $D_f(\nu, \mu)$ would be infinite when $\nu \not\ll \mu$, but now we are dealing with the "bad" case, $\lim_{v \to \infty} f'(v) < \infty$. Then it often happens that the minimum without the condition $\nu \ll \mu$ is attained for some $\nu \not\ll \mu$. Interesting examples, involving the function $f_0(u)$ of (1.12), are given in [3]. It should be noted that our restriction to finite measures, meaning in our case restriction to functions $s \in L_1(\mu)$, is harmless: although the set E may contain functions $s \notin L_1(\mu)$, for them $D_f(s, 1) = \infty$ thus they can be disregarded.

More valuable information can be obtained under topological regularity hypotheses only. We send forward the simple

LEMMA 4. *Let X be a compact metric space, and let the set of measures F be defined by continuous functions a_γ, cf. (3.9). Then, providing there exists $\nu \in F$ with $D_f(\nu, \mu) < \infty$, the minimum of $D_f(\nu, \mu)$ subject to $\nu \in F$ is always attained, and the μ-absolutely continuous component ν_a^* of a minimizing ν^* is uniquely determined. If Γ is finite, there exists a minimizing measure ν^* whose μ-singular component is concentrated on a finite subset of X, consisting of no more points than the cardinality of Γ plus 1.*

PROOF. The hypothesis implies that for any finite M, the set $F \cap \cap \{\nu : \nu(X) \leqq M\}$ is compact in the weak* topology of measures. Since $D_f(\nu, \mu)$ is a lower semicontinuous function of ν in that topology (cf., e.g., [15]), the existence of a minimizing ν^* follows. With the decomposition $\nu^* = = \nu_a^* + \nu_s^*$ into absolutely continuous and singular components with respect to μ, it is seen from the definition (1.17) of $D_f(\nu,\mu)$, using the strict convexity of f, that ν_a^* must be unique. We henceforth assume that $\nu_s^*(X) > 0$, else there is nothing left to prove.

If ν^* minimizes $D_f(\nu,\mu)$ subject to $\nu \in F$, and ν_s' is any measure satisfying

$$(3.10) \qquad \nu_s'(X) = \nu_s^*(X), \quad \int a_\gamma \, d\nu_s' = \int a_\gamma \, d\nu_s^*, \quad \gamma \in \Gamma$$

then $\nu' = \nu_a^* + \nu_s'$ also belongs to F, and if ν_s' is μ-singular then (1.17) gives that $D_f(\nu',\mu) = D_f(\nu^*,\mu)$, thus ν' also minimizes $D_f(\nu,\mu)$ subject to $\nu \in$ $\in F$. Actually, if (3.10) holds then ν_s' must be μ-singular. Otherwise, using the inequality

$$(3.11) \qquad f(u+t) < f(u) + \left(\lim_{v\to\infty} f'(v)\right)t \quad \text{if} \quad t > 0,$$

an obvious consequence of the strict convexity of f, from (1.17) the contradiction $D_f(\nu',\mu) < D_f(\nu^*,\mu)$ would follow.

Suppose now that $\Gamma = \{1,\dots,k\}$. Consider the continuous map $T\colon X \to$ $\to \mathbf{R}^k$ defined by

$$(3.12) \qquad T(x) = \left(f_1(x),\dots,f_k(x)\right).$$

The vector with components $\left(\int a_\gamma \, d\nu_s^*\right)/\nu_s^*(X)$, $\gamma = 1,\dots,k$ belongs to the convex hull of $T(X)$, hence by Caratheodory's theorem it can be represented as the convex combination $\sum \alpha_j T(x_j)$ of at most $k+1$ points $T(x_1),\dots,T(x_l)$, $l \leqq k+1$. It follows that the measure $\nu_s' = \sum_{j=1}^l \alpha_j \nu_s^*(X)\delta_{x_j}$ satisfies (3.10), where δ_x denotes the unit mass at x. Thus, by the previous paragraph, $\nu' = \nu_a^* + \nu_s^*$ attains the minimum of $D_f(\nu,\mu)$ subject to $\nu \in F$, completing the proof of the lemma.

For Theorem 4 we need the following hypotheses (required also in Borwein and Lewis [3]):

(i) X is a compact metric space, and $a_\gamma, \gamma \in \Gamma$ are continuous functions on X,

(ii) Γ is finite: $\Gamma = \{1,\dots,k\}$,

(iii) there exists $s_0 \in E$ which is bounded and bounded away from 0,

(iv) $\mathrm{Supp}(\mu) = X$.

Here E denotes the subset of S defined by (3.1); also, F will denote as before the set of (finite) measures on X satisfying the constraints (3.9). The *support* $\mathrm{Supp}(\nu)$ of a (finite) measure ν on the compact set X is the smallest compact $K \subset X$ with $\nu(K) = \nu(X)$. Notice that Hypothesis (iv) does not restrict generality, as one could always redefine X to equal $\mathrm{Supp}(\mu)$. Instead of Hypothesis (iii), it would suffice to postulate the existence of a μ-a.e.

positive $s_0 \in E$. It is not hard to see that this formally weaker assumption, which is the "primal constraint qualification" in [3], is actually equivalent to Hypothesis (iii).

THEOREM 4. *Under Hypotheses* (i)–(iv), $D_{f,\mu}(E,1) = \inf_{s\in E} D_{f,\mu}(s,1)$ *is equal to the minimum of* $D_f(\nu,\mu)$ *subject to* $\nu \in F$. *Denoting by* ν_a^* *the* μ*-absolutely continuous component of a* ν^* *attaining that minimum, the generalized* $D_{f,\mu}$*-projection of* $t=1$ *to* E *is equal to* $\frac{d\nu_a^*}{d\mu}$.

PROOF. We know by Lemma 4 that the minimum of $D_f(\nu,\mu)$ subject to $\nu \in F$ is attained for some ν^*. As discussed in the paragraph containing (3.9), then

$$D_f(\nu^*,\mu) \leqq D_f(E,1). \tag{3.13}$$

To show that here the equality holds, and $\frac{d\nu_a^*}{d\mu}$ equals the generalized $D_{f,\mu}$-projection of $t=1$ to E (which we know to exist, by the Corollary of Theorem 1), it suffices to find a sequence $\{s_n\} \subset E$ such that

$$D_{f,\mu}(s_n,1) \to D_f(\nu^*,\mu), \quad s_n \xrightarrow{\mu} \frac{d\nu_a^*}{d\mu}. \tag{3.14}$$

These functions s_n will be obtained as densities of measure $\nu_n \ll \mu$, satisfying (3.9) and approximating ν^*. By Lemma 4, we may assume that $\text{Supp}(\nu_s^*) = \{x_1,\dots,x_l\}$. We first smooth ν^*, replacing the point masses at $x_1,\dots,x_l$ by measures having constant μ-density in the balls B_{jn} of radius $\rho_n \to 0$ about the points x_j, $j=1,\dots,l$. The resulting smoothed version of $\nu^* \ll \mu$ is $\nu_n^* \ll \mu$ with

$$\frac{d\nu_n^*}{d\mu} = \frac{d\nu_a^*}{d\mu} + \sum_{j=1}^{l} \frac{\nu_s^*(\{x_j\})}{\mu(B_{jn})} 1_{B_{jn}}; \tag{3.15}$$

this is well defined, since the balls B_{jn} have positive μ-measure, by Hypothesis (iv). Since ν_n^* does not necessarily satisfy the constraints (3.9), it will be replaced by

$$\nu_n = (1-\varepsilon_n)\nu_n^* + \varepsilon_n \nu_{0n}, \tag{3.16}$$

with $\varepsilon_n \to 0$ specified later, and with measures $\nu_{0n} \ll \mu$ having the following two properties:

(i) for a suitable constant M, the functions $s_{n0} = \frac{d\nu_{n0}}{d\mu}$ satisfy $\frac{1}{M} < s_{n0} < M$, for all sufficiently large n;

(ii) denoting by ν_0 the measure with $\frac{d\nu_0}{d\mu} = s_0$, for s_0 in Hypothesis (iii), we have

$$T\big((1-\varepsilon_n)\nu_n^* + \varepsilon_n \nu_{0n}\big) = T\big((1-\varepsilon_n)\nu^* + \varepsilon_n \nu_0\big); \tag{3.17}$$

here T is the natural extension of the point map (3.12) to measures on X. Notice that (3.17) guarantees that ν_n defined by (3.16) satisfies the constraints (3.9), whereas (i) is a technical condition.

Assuming that measures ν_{0n} with properties (i), (ii) can be found, the proof is completed as follows: Set $s_n = \frac{d\nu_n}{d\mu}$; then, as ν_n satisfies (3.9), we have $s_n \in E$. It is clear from (3.15), (3.16) that $s_n \overset{\mu}{\to} \frac{d\nu_a^*}{d\mu}$ (actually, $s_n \to \to \frac{d\nu_a^*}{d\mu}$ pointwise, except for the set $\{x_1, \dots, x_j\} = \mathrm{Supp}(\nu_s^*)$). To prove the remaining part of our claim (3.14), notice that by Jensen's inequality and (3.11) we have

$$D_{f,\mu}(s_n, 1) = \int f(s_n)\, d\mu \leqq \tag{3.18}$$

$$\leqq (1-\varepsilon_n) \int f\left(\frac{d\nu_a^*}{d\mu} + \sum_{j=1}^{l} \frac{\nu_s^*(\{x_j\})}{\mu(B_{jn})} 1_{B_{jn}} \right) d\mu + \varepsilon_n \int f(s_{n0})\, d\mu \leqq$$

$$\leqq (1-\varepsilon_n) \left[\int f\left(\frac{d\nu_a^*}{d\mu}\right) d\mu + \left(\lim_{v\to\infty} f'(v)\right) \sum_{j=1}^{l} \nu_s^*(\{x_j\}) \right] +$$

$$+\varepsilon_n \int f(s_{n0})\, d\mu.$$

The last bracket is equal to $D_f(\nu^*, \mu)$, and $\int f(s_{n0})\, d\mu$ is bounded, by property (i) above. Thus, (3.18), compared with (3.13), establishes the desired result.

To complete the proof of Theorem 4, we have to verify that measures ν_{0n} with the properties (i) and (ii) do exist. Now, (3.17) can be written as

$$T(\nu_{0n}) - T(\nu_0) = \frac{1-\varepsilon_n}{\varepsilon_n}\big(T(\nu^*) - T(\nu_n^*)\big), \tag{3.19}$$

where $T(\nu^*) - T(\nu_n^*) \to 0$ as $n \to \infty$, because the radius of the balls B_{jn} about the points x_j, $j = 1, \dots, l$ goes to 0 as $n \to \infty$, cf. (3.15). Thus, letting $\varepsilon_n \to 0$ sufficiently slowly, the right hand side of (3.19) will go to 0. This means that $T(\nu_{0n})$ is required to equal an element of an arbitrarily small neighbourhood of $T(\nu_0)$, within the affine hull of $C \subset \mathbf{R}^k$, the image under

T of the set of all measures on X. Thus the possibility of finding ν_{0n} with the desired properties is an immediate consequence of the following facts: (i) denoting by C_M the image under T of the set of measures $\nu \ll \mu$ with $\frac{1}{M} < \frac{d\nu}{d\mu} < M$, C is contained in the closure of the union of these sets C_M (this follows by Hypothesis (iv)) and (ii) each C_M is a relatively open subset of the affine hull of C (because to any $\mathbf{u} \in C_M$ and $\mathbf{v} \in C_M$, there exists $\varepsilon > 0$ such that $\mathbf{u} + \varepsilon(\mathbf{u} - \mathbf{v}) \in C_M$).

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FUNCTIONAL EQUATIONS ON CONVEX SETS

Z. DARÓCZY, member of the Academy and GY. MAKSA (Debrecen)*

Dedicated to Professor K. Tandori on the occasion of his seventieth birthday

1. Throughout the paper X denotes a real or complex linear space. For a convex absorbing balanced set $K \subset X$ let p denote the Minkowski functional of K, i.e.

$$p(x) = \inf\left\{t > 0 \,:\, \frac{1}{t}x \in K\right\}, \quad x \in X.$$

It is well-known that the function $p : X \to [0, +\infty[$ is a seminorm on X and

$$\{x \in X \,:\, p(x) < 1\} \subset K \subset \{x \in X \,:\, p(x) \leqq 1\} \tag{1.1}$$

(Rudin [6], Yosida [10], Larsen [5]).

If $\alpha \in]0, 1[$ is fixed, K is a convex subset of X and $f : K \to \mathbf{C}$ then define the Jensen difference on K generated by f with the weight α by

$$J_{\alpha,f}(x,y) = f\big(\alpha x + (1-\alpha)y\big) - \alpha f(x) - (1-\alpha)f(y), \quad x, y \in K. \tag{1.2}$$

In the case $\alpha = \frac{1}{2}$ the function $J_{\alpha,f} : K \times K \to \mathbf{C}$ is the well-known symmetric Jensen difference (Kuczma [4], Aczél [1], Hardy–Littlewood–Pólya [3]).

In the following let $\alpha \neq \frac{1}{2}$. In this case one can ask the following:

PROBLEM. Which are those functions $f : K \to \mathbf{C}$ for which the Jensen difference $J_{\alpha,f}$ is symmetric, i.e.,

$$J_{\alpha,f}(x,y) = J_{\alpha,f}(y,x) \tag{1.3}$$

for all $x, y \in K$. Since $\alpha \neq \frac{1}{2}$, (1.3) can be written in the form

$$f(x) + \frac{1}{1-2\alpha} f\big(\alpha x + (1-\alpha)y\big) - \frac{1}{1-2\alpha} f\big((1-\alpha)x + \alpha y\big) - f(y) = 0 \tag{1.4}$$

for all $x, y \in K$. This provides the reason for introducing the following notion.

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DEFINITION. Let $K \subset X$ be convex. The function $f : K \to \mathbf{C}$ is of degree n on K if there exist functions $f_i : K \to \mathbf{R}$ and different numbers $0 \leqq \lambda_i < 1$, $i = 1, \dots, n+1$ such that

$$(1.5) \qquad f(x) + \sum_{i=1}^{n+1} f_i\big(\lambda_i x + (1-\lambda_i)y\big) = 0$$

holds for all $x, y \in K$. The vector $(\lambda_1, \dots, \lambda_{n+1})$ is called the parameter of f.

It is clear that the function $f : K \to \mathbf{C}$ satisfying the functional equation (1.4) (or (1.3)) is a function of degree 2 on K with the parameter $(\alpha, 1-\alpha, 0)$.

The main purpose of this paper is to find all solutions of the functional equation (1.4) and, in addition, to present some general results on functions of degree n.

2. In this section let K denote a convex absorbing balanced subset of X and let p be the Minkowski functional of K.

THEOREM 1. *Let $f : K \to \mathbf{C}$ be a function of degree n on $K_0 := K$ with the parameter $(\lambda_1, \dots, \lambda_{n+1})$ (The λ_i's are different numbers from the interval $[0, 1[$, $i = 1, \dots, n+1$.) Let*

$$(2.1) \qquad \lambda = \max\left\{1, \frac{\lambda_i}{1-\lambda_i} : i = 1, 2, \dots, n+1\right\}$$

and $K_1 = \left\{x \in X : p(x) < \frac{1}{2\lambda}\right\}$. Then for all $s \in K_1$ the difference function

$$x \to \Delta_s f(x) := f(x+s) - f(x) \quad (x \in K_1)$$

is of degree $(n-1)$ on the (convex absorbing balanced) set $K_1 \subset K_0$ with the parameter $(\lambda_1, \dots, \lambda_n)$.

PROOF. Since f is of degree n, (1.5) holds with some functions $f_1, \dots, f_{n+1} : K \to \mathbf{C}$. Let $x, y, s \in K_1$. Then, because of the inequalities

$$p(x+s) \leqq p(x) + p(s) < \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{1}{\lambda} \leqq 1,$$

we get $x + s \in K_0$. Write $x + s$ instead of x in (1.5).

Thus we have

$$(2.2) \qquad f(x+s) + \sum_{i=1}^{n+1} f_i\big(\lambda_i x + (a-\lambda_i)y + \lambda_i s\big) = 0.$$

On the other hand

$$p\left(y+\frac{\lambda_{n+1}}{1-\lambda_{n+1}}s\right) \leqq p(y)+\frac{\lambda_{n+1}}{1-\lambda_{n+1}}p(s) < \frac{1}{2\lambda}+\lambda\frac{1}{2\lambda} \leqq 1,$$

whence by (1.1), $y+\frac{\lambda_{n+1}}{1-\lambda_{n+1}}s \in K_0$. Write $y+\frac{\lambda_{n+1}}{1-\lambda_{n+1}}s$ instead of y in (1.5). We obtain

$$f(x)+\sum_{i=1}^{n+1} f_i\left(\lambda_i x+(1-\lambda_i)y+(1-\lambda_i)\frac{\lambda_{n+1}}{1-\lambda_{n+1}}s\right)=0. \tag{2.3}$$

Subtracting (2.3) from (2.2) we have

$$f(x+s)-f(x)+\sum_{i=1}^{n}\Bigg[f_i\big(\lambda_i x+(1-\lambda_i)y+\lambda_i s\big)-$$
$$-f_i\left(\lambda_i x+(1-\lambda_i)y+(1-\lambda_i)\frac{\lambda_{n+1}}{1-\lambda_{n+1}}s\right)\Bigg]=0.$$

This can be written in the form

$$\Delta_s f(x)+\sum_{i=1}^{n}\left(\Delta_{\lambda_i s}-\Delta_{(1-\lambda_i)\frac{\lambda_{n+1}}{1-\lambda_{n+1}}s}\right)f_i\big(\lambda_i x+(1-\lambda_i)y\big)=0,$$

that is, with the notation

$$g_i^{(s)} := \left(\Delta_{\lambda_i s}-\Delta_{(1-\lambda_i)\frac{\lambda_{n+1}}{1-\lambda_{n+1}}s}\right)f_i \quad \text{on} \quad K_1 \quad (i=1,\dots,n)$$

we get

$$\Delta_s f(x)+\sum_{i=1}^{n} g_i^{(s)}\big(\lambda_i x+(1-\lambda_i)y\big)=0.$$

This implies that $\Delta_s f$ is a function of degree $(n-1)$ on K_1 with the parameter $(\lambda_1,\dots,\lambda_n)$ for all $s \in K_1$. □

The next two corollaries follow easily from Theorem 1.

COROLLARY 1. *Let f be a function of degree n on $K_0=K$ with the parameter $(\lambda_1,\dots,\lambda_{n+1})$, let λ be the number defined by* (2.1) *and*

$$K_\ell=\left\{x\in X : p(x)<\frac{1}{2^\ell\lambda^\ell}\right\} \quad (\ell=1,2,\dots,n+1).$$

Then $K_{n+1} \subset K_n \subset \cdots \subset K_1 \subset K$ *and*

$$\Delta_s^{n+1} f(x) = 0$$

for all $x, s \in K_{n+1}$.

Throughout the remainder of this paper for all positive homogeneous functions $p : X \to [0, +\infty[$ (i.e. $p(tx) = tp(x)$ holds for all $t \in [0, +\infty[$ and $x \in X$) and for all $0 < r \leqq +\infty$ we shall use the following notations:

$$B(r) = \{x \in X : p(x) < r\}$$

and

$$\Delta_n(r) = \{(x, y) \in X \times X : x, x + y, \dots, x + (n+1)y \in B(r)\} \quad (n \in \mathbf{N}).$$

COROLLARY 2. *Let f be a function of degree n on K. Then there exists* $0 < r < 1$ *such that* $B(r) \subset K$ *and*

$$\Delta_y^{n+1} f(x) = 0 \tag{2.4}$$

for all $(x, y) \in \Delta_n(r)$.

3. Corollary 2 provides the reason for introducing the following

DEFINITION. Let $0 < r \leqq +\infty$, and $p : X \to [0, +\infty[$ a positive homogeneous function. The function $f : B(r) \to \mathbf{C}$ is a local polynomial of degree n on $B(r)$ if (2.4) holds for all $(x, y) \in \Delta_n(r)$.

A similar concept was introduced by Székelyhidi in [9].

Following the ideas of Székelyhidi [7] we prove the following extension theorem.

THEOREM 2. *Let $p : X \to [0, +\infty[$ be a positive homogeneous function,* $p \neq 0$, $0 < r \leqq +\infty$ *and* $f : B(r) \to \mathbf{C}$ *a local polynomial of degree n on* $B(r)$. *Then there exists a function* $F : X \to \mathbf{C}$ *such that*

$$\Delta_y^{n+1} F(x) = 0 \tag{3.1}$$

for all $x, y \in X$ *and*

$$F|B(r) = f.$$

PROOF. The proof is based on Zorn's lemma. Let $\mathcal{F}$ denote the set of all functions φ with the following properties:

a) $f \subset \varphi$,

b) the domain of φ is the set $B(r)$ for some $0 < r \leqq +\infty$,

c) the range of φ is a subset of $\mathbf{C}$,
d) $\Delta_y^{n+1}\varphi(x) = 0$ for all $(x, y) \in \Delta_n(r)$.

Since $f \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. The set $\mathcal{F}$ is partially ordered with the obvious inclusion of functions: for $\varphi, \psi \in \mathcal{F}$, $\varphi \subset \psi$ if ψ is an extension of φ. Moreover, if $\mathcal{F}_0 \subset \mathcal{F}$ is an ordered subset then $\bigcup \mathcal{F}_0 \in \mathcal{F}$. Thus, by Zorn's lemma, there exists a maximal element $F \in \mathcal{F}$. Let $B(R)$ be the domain of F and suppose that $R < +\infty$. For $z \in B\left(\frac{n+1}{n}R\right)$ define

$$\tilde{F}(z) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} F\left(\frac{n+1-k}{n+1} z\right).$$

Since $p\left(\frac{n+1-k}{n+1} z\right) = \frac{n+1-k}{n+1} p(z) < \frac{n+1-k}{n+1} \frac{n+1}{n} R \leqq R$ therefore $\frac{n+1-k}{n+1} z \in \in B(R)$, $k = 1, \ldots, n+1$, thus $\tilde{F}$ is well defined. Furthermore, $z \in B(R)$ implies that $\left(0, \frac{z}{n+1}\right) \in \Delta_n(R)$ hence the identity $\tilde{F}(z) = F(z) - \Delta^{n+1}_{\frac{z}{n+1}} F(0)$ which is proved in [7] and $F \in \mathcal{F}$ provide that $F \subset \tilde{F}$. If $(x, y) \in \Delta_n\left(\frac{n+1}{n}R\right)$ then $\left(\frac{n+1-k}{n+1}x, \frac{n+1-k}{n+1}y\right) \in \Delta_n(R)$, $k = 1, \ldots, n+1$ and $x + (n+1)y \in \in B\left(\frac{n+1}{n}R\right)$ thus, as it is proved in [7], we have that $\Delta_y^{n+1}\tilde{F}(x) = 0$. Consequently, $\tilde{F} \in \mathcal{F}$ and $F \subset \tilde{F}$. Since $p \neq 0$, $B(R) \subsetneqq B\left(\frac{n+1}{n}R\right)$ therefore $F \subsetneqq \tilde{F}$ which is a contradiction. □

4. In connection with the functional equation (3.9) it is well-known (Djoković [2], Székelyhidi [8]) that a function $F : X \to \mathbf{C}$ satisfies (3.1) if and only if there exist k-additive symmetric functions

$$A_k : X^k \to \mathbf{C} \qquad (k = 0, 1, \ldots, n; \quad X^0 := X)$$

such that

$$F(x) = \sum_{k=0}^{n} A_k^*(x), \qquad x \in X$$

where A_k^* is the diagonalization of A_k (A_0 is a constant function). This implies the following

THEOREM 3. *Let $p : X \to [0, +\infty[$ be a positive homogeneous function, $0 < r < +\infty$ and $f : B(r) \to \mathbf{C}$ a local polynomial of degree n on $B(r)$. Then there exist k-additive symmetric functions, $A_k : X^k \to \mathbf{C}$ ($k = = 0, 1, \ldots, n$) such that*

$$f(x) = \sum_{k=0}^{n} A_k^*(x)$$

for all $x \in B(r)$ *(*A_k^* *is the diagonalization of* A_k*,* $k = 0, \ldots, n$*).*

5. Now we return to the investigation of the functional equation (1.4). First we prove the following

THEOREM 4. *Let* $p : X \to [0, +\infty[$ *be a positive homogeneous function and* $0 < r \leqq +\infty$*. Suppose that* $A_\ell : X^\ell \to \mathbf{C}$ *is an* ℓ*-additive and symmetric function* ($\ell = 0, 1, 2$) *and define the function* f *on* $B(r)$ *by*

$$F(x) := A_2(x,x) + A_2(x) + A_0, \quad x \in B(r). \tag{5.1}$$

Then f *is a solution of* (1.4) *on* $B(r)$ *if and only if*

$$\begin{cases} \text{(a) } A_2(\alpha x, y) = \alpha A_2(x,y) & \text{for all } x, y \in X, \text{ and} \\ \text{(b) } A_1(\alpha x) = \alpha A_1(x) & \text{for all } x \in X. \end{cases} \tag{5.2}$$

PROOF. Substituting f in (1.4) an easy calculation shows that

$$\begin{aligned} &A_2(\alpha x, y) - A_2(\alpha y, y) - \alpha A_2(x,x) + \alpha A_2(y,y) + \\ &+ A_1(\alpha x) - \alpha A_1(x) - \left[A_1(\alpha y) - \alpha A_1(y)\right] = A_2(\alpha y, x) - A_2(\alpha x, x) \end{aligned} \tag{5.3}$$

for all $x, y \in B(r)$. With the substitution $y = 0$ this implies

$$A_2(\alpha x, x) - \alpha A_2(x,x) + A_1(\alpha x) - \alpha A_1(x) = 0 \tag{5.4}$$

for all $x \in B(r)$. Writing $\frac{x}{2}$ instead of x in (5.4), multiplying by 4 the equation so obtained and subtracting this equation from (5.4) we get that

$$A_1(\alpha x) = \alpha A_1(x), \qquad x \in B(r) \tag{5.5}$$

and

$$A_2(\alpha x, x) = \alpha A_2(x,x), \qquad x \in B(r). \tag{5.6}$$

Let $x \in X$. If $p(x) = 0$ then, because of $x \in B(r)$, (5.5) and (5.6) hold. If $p(x) > 0$ then there exists a rational number $\varrho > 0$ such that $\varrho < \frac{r}{p(x)}$. Therefore

$$p(\varrho x) = \varrho p(x) < \frac{r}{p(x)} p(x) = r$$

thus $\varrho x \in B(r)$ which implies that $A_1(\alpha \varrho x) = \alpha A_1(\varrho x)$ and $A_2(\alpha \varrho x, \varrho x) = \alpha A_2(\varrho x, \varrho x)$, whence (5.2) (b) and

$$A_2(\alpha x, x) = \alpha A_2(x,x), \quad x \in X \tag{5.7}$$

follow. According to the equations (5.5), (5.6) and (5.3) we have

$$A_2(\alpha x, y) = A_2(\alpha y, x) = A_2(x, \alpha y) \tag{5.8}$$

for all $x, y \in B(r)$. Let $x, y \in X$ for which $p(x) > 0$ and $p(y) > 0$ and let $\varrho > 0$ be a rational number so that

$$\varrho < \min\left\{\frac{r}{p(x)}, \frac{r}{p(y)}\right\}.$$

Then $p(\varrho x) = \varrho p(x) < r$ and $p(\varrho y) = \varrho p(y) < r$ therefore

$$A_2(\alpha\varrho x, \varrho y) = A_2(\varrho x, \alpha\varrho y)$$

which implies (5.8). If $p(x) = p(y) = 0$ then (5.8) holds since $x, y \in B(r)$. If $p(x) = 0$, $p(y) > 0$ (or $p(x) > 0$, $p(y) = 0$) then $A_2(\alpha x, \varrho y) = A_2(x, \alpha\varrho y)$ (or $A_2(\alpha\varrho x, y) = A_2(\varrho x, \alpha y)$) implies (5.8). Finally, define the functions a and b on X by

$$a(x) = A_2(\alpha x, x) \quad \text{and} \quad b(x) = A_2(x, x),$$

respectively. Then, by (5.8) and (5.7),

$$2A_2(\alpha x, y) = a(x+y) - a(x) - a(y) = \alpha\big(b(x+y) - b(x) - b(y)\big) = \alpha 2A_2(x, y)$$

for all $x, y \in X$. □

THEOREM 5. *Let $K \subset X$ be a convex absorbing balanced set, p its Minkowski functional and suppose that $g : K \to \mathbf{C}$ is a solution of the functional equation* (1.4). *If there exists $0 < r < 1$ such that $g(x) = 0$ for all $x \in B(r) \subset K$ then $g(x) = 0$ for all $x \in K$.*

PROOF. Without loss of generality we may assume that $0 < \alpha < \frac{1}{2}$. Let $r_k = \min\left\{\frac{r}{(1-\alpha)^{k-1}}, 1\right\}$, $k \in \mathbf{N}$. We show that $g(x) = 0$ for all $x \in B(r_k)$ and for all $k \in \mathbf{N}$. This will be proved by induction on k. For $k = 1$ the statement is obviously true. Suppose that the statement is true for k and let $x \in B(r_{k+1})$. Then

$$p(\alpha x) \leqq p\big((1-\alpha)x\big) = (1-\alpha)p(x) < (1-\alpha)r_{k+1} =$$

$$= \min\left\{\frac{r}{(1-\alpha)^{k-1}}, 1-\alpha\right\} \leqq r_k.$$

Therefore $g(\alpha x) = g\big((1-\alpha)x\big) = 0$. This and (1.4) with $y = 0$ imply that $(2\alpha - 1)g(x) = 0$, that is, $g(x) = 0$. Thus $g(x) = 0$ for all $x \in B(r_k)$ and

$k \in \mathbf{N}$. Since $\lim_{k\to\infty} r_k = 1$ we have that $g(x) = 0$ for all $x \in B(1) \cap K$. According to the inclusions (1.1) now it is enough to prove that $g(x) = 0$ if $x \in K$ and $p(x) = 1$. Let $y = 0$ in (1.4). Then $g(\alpha x) - \alpha g(x) = g\big((1-\alpha)x\big) - (1-\alpha)g(x)$, $p(\alpha x) = \alpha p(x) = \alpha < 1$ and $p\big((1-\alpha)x\big) = (1-\alpha)p(x) = 1 - \alpha < 1$ imply that $(2\alpha - 1)g(x) = 0$, that is, $g(x) = 0$. □

6. Our previous results allow us to formulate the following theorem which gives the solution of the problem raised in the first section.

THEOREM 6. *Let $\alpha \in \,]0,1[$ be fixed, $\alpha \neq \frac{1}{2}$. Suppose that $K \subset X$ is a convex absorbing balanced set. For the function $f : K \to \mathbf{C}$, $J_{\alpha,f}$ is a symmetric Jensen difference if and only if there exist $A_\ell : X^\ell \to \mathbf{C}$ ($\ell = 0,1,2$) ℓ-additive functions for which A_2 is symmetric,*

$$(6.1) \qquad \begin{cases} \text{(a) } A_2(\alpha x, y) = \alpha A_2(x,y) & \textit{for all } x, y \in X \\ \text{(b) } A_1(\alpha x) = \alpha A_1(x) & \textit{for all } x \in X \end{cases}$$

and

$$(6.2) \qquad f(x) = A_2(x,x) + A_1(x) + A_0$$

for all $x \in K$.

PROOF. If the Jensen difference $J_{\alpha,f}$ is symmetric then f satisfies (1.4). This implies that f is a function of degree 2. Therefore, by Corollary 2 of Theorem 1, there exists $0 < r < 1$ so that $B(r) \subset K$ and f is a local polynomial of degree 2 on $B(r)$. Thus Theorem 3 implies the existence of ℓ-additive functions $A_\ell : X^\ell \to \mathbf{C}$ ($\ell = 0,1,2$) such that A_2 is symmetric and

$$(6.3) \qquad f(x) = A_2(x,x) + A_1(x) + A_0$$

for all $x \in B(r)$. It follows from Theorem 4 that f satisfies (1.4) on $B(r)$ if and only if (6.1) (a) and (b) hold. Define the function g on K by

$$g(x) = f(x) - A_2(x,x) - A_1(x) - A_0, \quad x \in K.$$

Obviously g is a solution of (1.4) on K and $g(x) = 0$ if $x \in B(r) \subset K$. Therefore Theorem 5 implies that $g(x) = 0$ for all $x \in K$, that is, f has the form (6.2).

The converse follows from Theorem 4.

REMARK. (6.1) (a) and (b) are obviously satisfied by A_1 and A_2 if α is a rational number.

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ON IMBEDDING OF FUNCTION CLASSES $H^{\omega}_{\psi_1,q_1}$ INTO CLASSES $E_{\psi_2,q_2}(\lambda)$

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To Professor K. Tandori on his seventieth birthday

1. The imbedding theory of function classes starts from the articles of E. Titchmarsh, G. Hardy and J. Littlewood. In 1927 the latter obtained the first theorem of imbedding of Lipschitz classes from Lebesgue spaces L_p into L_q, $1 \leqq p < q < \infty$. The general theory of imbedding of spaces of differentiable functions of several variables was initiated by Sobolev. Further development of this theory is due to Nikol'skiĭ, who created the theory of imbedding H-classes and applied methods of approximation theory. Ul'ianov found necessary and sufficient conditions for the imbedding $H^{\omega}_p \subset L_q$ as well as sufficient conditions for the imbedding $H^{\omega_1}_p \subset H^{\omega_2}_q$, $1 \leqq p < q < \infty$. These results were further developed by others.

The present paper deals with imbedding theorems for periodic functions of a single variable. We give necessary and sufficient conditions for imbedding $H^{\omega}_{\psi_1,q_1}$ classes into $E_{\psi_2,q_2}(\lambda)$ classes.

We give the necessary definitions. A function $\psi(t)$ is called a φ-function, if $\psi(t)$ is continuous, nondecreasing, and concave on $[0;2\pi]$, $\psi(0)=0$. We define the indices of a φ-function $\psi(t)$ as

$$\alpha_\psi = \underline{\lim}_{t\to+0} \frac{\psi(2t)}{\varphi(t)}, \quad \beta_\psi = \overline{\lim}_{t\to+0} \frac{\psi(2t)}{\psi(t)}.$$

It is clear that the inequalities $1 \leqq \alpha_\psi \leqq \beta_\psi \leqq 2$ hold for all φ-functions $\psi(t)$.

The quasi-normed Lorentz space $\Lambda_{\psi,q}$ $(0<q<\infty)$ is defined as the set of 2π-periodic measurable functions $f(x)$ for which the quasi-norm

$$\|f\|_{\psi,q} = \left\{ \int_0^{2\pi} \left[\psi(t)\frac{1}{t}\int_0^t f^*(x)\,dx \right]^q \frac{dt}{t} \right\}^{\frac{1}{q}}$$

is finite, where $f^*(x)$ is a nonincreasing function on $(0;2\pi]$ equimeasurable with $|f(x)|$ (for the definition of $f^*(x)$ see [1], p. 213).

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The quasi-normed Lorentz space $\Lambda_{p,q}$ $(0 < p, q < \infty)$ is defined as the set of 2π-periodic measurable functions $f(x)$ for which the quasi-norm

$$\|f\|^*_{p,q} = \left\{ \int_0^{2\pi} t^{q/p} f^{*q}(t) \frac{dt}{t} \right\}^{\frac{1}{q}}$$

is finite.

Notice that if $q > 1$ and the φ-function $\psi(t) = t^{\frac{1}{q}}$, then the Lorentz space $\Lambda_{\psi,q}$ coincides with Lebesgue space L_q, and if $p > 1$, $q > 0$ and the φ-function $\psi(t) = t^{\frac{1}{p}}$, then $\Lambda_{\psi,q}$ coincides with $L_{p,q}$ (see [2]).

If $f(x) \in \Lambda_{\psi,q}$, $0 < q < \infty$, $\psi(t)$ is a φ-function, then

$$\omega_{\psi,q}(\delta; f) = \sup_{0 \leqq h \leqq \delta} \left\| f(x+h) - f(x) \right\|_{\psi,q}, \quad \delta \in [0; 2\pi]$$

is called the modulus of continuity of $f(x)$.

Let $\omega(\delta)$ be a nondecreasing, continuous function on $[0; 2\pi]$ such that $\omega(0) = 0$, and

$$\omega(\delta + \eta) \leqq \omega(\delta) + \omega(\eta) \quad \text{as} \quad 0 \leqq \delta \leqq \eta \leqq \delta + \eta \leqq 2\pi.$$

Such functions are called moduli of continuity.

If we have a modulus of continuity $\omega(\delta)$, a φ-function $\psi(t)$ and $0 < q < \infty$, then let $H^{\omega}_{\psi,q}$ be the set of functions $f(x) \in \Lambda_{\psi,q}$ for which

$$\omega_{\psi,q}(\delta; f) = O\{\omega(\delta)\}, \quad \text{as} \quad \delta \to +0.$$

If $f(x) \in \Lambda_{\psi,q}$, $0 < q < \infty$ and $\psi(t)$ is a φ-function, then

$$E_n(f)_{\psi,q} = \inf_{T_n} \|f - T_n\|_{\psi,q}$$

is called the best approximation of $f(x)$ by trigonometric polinomials $T_n(x)$ of degree at most $n-1$, $n \in \mathbf{N}$, in the space $\Lambda_{\psi,q}$.

If we have a φ-function $\psi(t)$, a sequence of positive numbers $\lambda \equiv \{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_n \downarrow 0$ when $n \uparrow \infty$ (i.e. $\lambda_n \geqq \lambda_{n+1}$, $n \in \mathbf{N}$, $\lim_{n \to +\infty} \lambda_n = 0$), and $0 < q < \infty$, then $E_{\psi,q}(\lambda)$ is the set of functions $f(x) \in \Lambda_{\psi,q}$ for which

$$E_n(f)_{\psi,q} = O\{\lambda_n\}.$$

Our main result is the following

PROPOSITION. *Let $0 < q_1 < \infty$, $1 \leqq q_2 < \infty$, and let the φ-functions $\psi_1(t)$ and $\psi_2(t)$ be such that $1 < \alpha_{\psi_2} \leqq \beta_{\psi_2} < \alpha_{\psi_1} \leqq \beta_{\psi_1} < 2$. Let $\omega(\delta)$ be a modulus of continuity, $\lambda \equiv \{\lambda_n\}_{n=1}^{\infty}$ a sequence of positive numbers such that $\lambda_n \downarrow 0$ when $n \uparrow \infty$. Then for the imbedding $H^{\omega}_{\psi_1,q_1} \subset E_{\psi_2,q_2}(\lambda)$ it is necessary and sufficient that*

$$\left\{\sum_{k=n}^{\infty}\left[\frac{\psi_2\left(\frac{1}{k}\right)\omega\left(\frac{1}{k}\right)}{\psi_1\left(\frac{1}{k}\right)}\right]^{q_2}\cdot\frac{1}{k}\right\}^{\frac{1}{q_2}} = O\{\lambda_n\}.$$

This proposition is a consequence of Theorems 1 and 2 below.

We note that earlier Zhainibekova [7] proved a theorem which is a special case of our proposition, when $\psi_1(t) = t^{\frac{1}{p}}$, $\psi_2(t) = t^{\frac{1}{q}}$, $q_1 = p$, $q_2 = q$, $1 < p < q < \infty$.

2. In the proof of the proposition we need the following lemmas.

LEMMA 1 [8]. *For any function $f(t) \in L_1(0;2\pi)$ we have*

$$\int_0^{\Theta} f^*(t)\,dt = \sup_{|E|=\Theta}\int_E |f(t)|\,dt, \quad \Theta \in (0;2\pi].$$

LEMMA 2 [8]. *If $\psi(t)$ is a φ-function then $\psi(t)/t$ is nondecreasing on $(0;2\pi]$ and*

$$\psi(t_1+t_2) \leqq \psi(t_1)+\psi(t_2), \quad 0 \leqq t_1 \leqq t_2 \leqq t_1+t_2 \leqq 2\pi.$$

LEMMA 3 [4]. *If $\alpha_\psi > 1$ for a φ-function $\psi(t)$ then for all $p > 0$*

$$\sum_{n=k}^{\infty}\psi^p\left(\frac{1}{n}\right)\cdot\frac{1}{n} = O\left(\psi^p\left(\frac{1}{k}\right)\right).$$

LEMMA 4 [6]. *If $\alpha_\psi > 1$ for a φ-function $\psi(t)$ then for all $p > 0$*

$$\sum_{n=1}^{k}\frac{1}{\psi^p\left(\frac{1}{n}\right)\cdot n} = O\left(\frac{1}{\psi^p\left(\frac{1}{k}\right)}\right).$$

LEMMA 5 [4]. *Let $\varphi(t)$ and $\psi(t)$ be φ-functions and let $\alpha_\varphi > \beta_\psi$. Then*

$$\sum_{n=1}^{k}\frac{\psi^p\left(\frac{1}{n}\right)}{n\varphi^p\left(\frac{1}{n}\right)} = O\left(\frac{\psi^p\left(\frac{1}{k}\right)}{\varphi^p\left(\frac{1}{k}\right)}\right).$$

LEMMA 6 [3]. *Let $\psi(t)$ be a φ-function, $\beta_\psi < 2$, $0 < q < \infty$. Then for all functions $f(t) \in \Lambda_{\psi,q}$*

$$C_1(\psi,q)\|f\|^*_{\psi,q} \leqq \|f\|_{\psi,q} \leqq C_2(\psi,q)\|f\|^*_{\psi,q},$$

where the functional $\|f\|^*_{\psi,q}$ is defined as

$$\|f\|^*_{\psi,q} = \left\{\int_0^{2\pi} [\psi(t)f^*(t)]^q \,\frac{dt}{t}\right\}^{\frac{1}{q}}.$$

LEMMA 7 [4]. *Let $\varphi(t)$ and $\psi(t)$ be φ-functions and let $\alpha_\varphi > \beta_\psi$. Then for the function*

$$\Theta(t) = \begin{cases} 0, & t = 0 \\ \frac{\varphi(t)}{\psi(t)}, & t \in (0; 2\pi], \end{cases}$$

there exists a φ-function $\Theta_1(t)$ such that $\alpha_{\Theta_1} > 1$ and

$$C_1(\varphi,\psi)\Theta_1(t) \leqq \Theta(t) \leqq C_2(\varphi,\psi)\Theta_1(t).$$

LEMMA 8 [4]. *Let $\psi(t)$ be a φ-function and $1 \leqq q < \infty$. Then for any $f(x) \in \Lambda_{\psi,q}$*

$$E_n(f)_{\psi,q} \leqq C(\psi,q)\omega_{\psi,q}\left(f;\frac{1}{n}\right), \quad n \in \mathbf{N}.$$

LEMMA 9 [5]. *Let $\psi_1(t)$ and $\psi_2(t)$ be φ-functions such that $1 < \alpha_{\psi_2} \leqq \leqq \beta_{\psi_2} < \alpha_{\psi_1} \leqq \beta_{\psi_1} \leqq 2$, $q_1 > 0$ and $q_2 > 0$ and let $\omega_1(\delta)$ and $\omega_2(\delta)$ be moduli of continuity.*

a) *If*

$$\sum_{n=1}^\infty \left[\frac{\psi_2\left(\frac{1}{n}\right)\omega_1\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2} \cdot \frac{1}{n} < \infty$$

then $H^{\omega_1}_{\psi_1,q_1} \subset \Lambda_{\psi_2,q_2}$ and for any $f(x) \in \Lambda_{\psi_1,q_1}$

$$\|f\|_{\psi_2,q_2} \leqq$$

$$\leqq C(\psi_1,\psi_2,q_1,q_2)\left\{\|f\|_{\psi_1,q_1} + \left(\sum_{n=1}^\infty \left[\frac{\psi_2\left(\frac{1}{n}\right)\omega_{\psi_1,q_1}\left(f;\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2} \cdot \frac{1}{n}\right)^{\frac{1}{q_2}}\right\}.$$

b) *If*

$$\left(\sum_{n=k}^{\infty}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega_1\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}\cdot\frac{1}{n}\right)^{\frac{1}{q_2}}=O\left\{\omega_2\left(\frac{1}{k}\right)\right\},$$

then $H^{\omega_1}_{\psi_1,q_1}\subset H^{\omega_2}_{\psi_2,q_2}$ *and for any* $f(x)\in\Lambda_{\psi_1,q_1}$

$$\omega_{\psi_2,q_2}\left(f;\frac{1}{k}\right)\leqq C(\psi_1,\psi_2,q_1,q_2)\left\{\sum_{n=k}^{\infty}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega_{\psi_1,q_1}\left(f;\frac{1}{n}\right)}{\varphi_1\left(\frac{1}{n}\right)}\right]^{q_2}\cdot\frac{1}{n}\right\}^{\frac{1}{q_2}}.$$

c) *Moreover if* $\beta_{\psi_1}<2$, *then conditions* a) *and* b) *are necessary.*

LEMMA 10 [9]. *Let* $\omega(\delta)$ *be a modulus of continuity. Then there exists a concave modulus of continuity* $\omega_1(\delta)$ *such that*

$$\omega(\delta)\leqq\omega_1(\delta)\leqq 2\omega(\delta),\quad 0\leqq\delta\leqq 2\pi.$$

LEMMA 11 [6]. *Let* $0<q<\infty$ *and let* $\psi(t)$ *be a* φ*-function such that* $1<\alpha_\psi\leqq\beta_\psi<2$. *Let* $\nu\geqq\mu\geqq 0$ *be integers,* $D_{\nu,\mu}(x)=\sum\limits_{k=\mu}^{\nu}\cos kx$. *Then*

a) $\left\| D_{\nu,\mu}\right\|_{\psi,q}\leqq C(\psi,q)(\nu-\mu+1)\psi\left(\frac{1}{\nu-\mu+1}\right)$;

b) $\sup\limits_{t\in(0;2\pi]}\frac{t}{\psi(t)}D^*_{\nu,\mu}(t)\leqq\frac{C(\psi)}{\psi\left(\frac{1}{\nu-\mu+1}\right)}$.

LEMMA 12 [6]. *Let* $q>0$,

$$p=\begin{cases}1,\ 1\leqq q<\infty;\\ q,\ 0<q\leqq 1,\end{cases}$$

and let $\psi(t)$ *be a* φ*-function. Then for any* $f(x)\in\Lambda_{\psi,q}$

$$\omega_{\psi,q}\left(f;\frac{1}{n}\right)\leqq\frac{C(\psi,q)}{n}\left\{\sum_{k=1}^{n}k^{p-1}E_k^p(f)_{\psi,q}\right\}^{\frac{1}{p}},\quad n\in\mathbf{N}.$$

LEMMA 13 [8]. *Let* $f(x)$ *and* $g(x)$ *be* 2π*-periodic measurable functions. Then*

$$\int_0^{2\pi}f(t)g(t)\,dt\leqq\int_0^{2\pi}f^*(t)g^*(t)\,dt.$$

LEMMA 14 [6]. *Let* $0 < q_1 \leqq q_2 < 0$ *and let* $\psi(t)$ *be a* φ*-function. Then for any* $f(x) \in \Lambda_{\psi,q_1}$

$$\|f\|_{\psi,q_2} \leqq C(\psi, q_1, q_2)\|f\|_{\psi,q_1}$$

holds true. In particular, it follows that $\Lambda_{\psi,q_1} \subset \Lambda_{\psi,q_2}$.

The following two lemmas are proved by standard methods, therefore we will formulate them without the proofs.

LEMMA 15. *Let* $q > 0$ *and let* $\psi(t)$ *be a* φ*-function such that* $1 < \alpha_\psi \leqq \leqq \beta_\psi < 2$. *Let* $f(x) \in \Lambda_{\psi,q}$, $f(x) \sim \sum\limits_{k=1}^{\infty} a_k \cos kx$, *where* $a_k \geqq 0$, $k \in \mathbf{N}$. *Then*

$$E_n(f)_{\psi,q} \geqq C(\psi, q)\psi\left(\frac{1}{n}\right)\sum_{m=n}^{2n} a_m, \quad n \in \mathbf{N}.$$

LEMMA 16. *Let* $q > 0$ *and let* $\psi(t)$ *be a* φ*-function such that* $1 < \alpha_\psi \leqq \leqq \beta_\psi < 2$. *Let* $f(x) \in \Lambda_{\psi,q}$, $f(x) \sim \sum\limits_{k=1}^{\infty} a_k \cos kx$, *where* $a_k \geqq 0$, $k \in \mathbf{N}$. *If there exists* $\tau > 0$ *such that* $a_n \cdot n^{-\tau} \downarrow$, *then*

$$E_n(f)_{\psi,q} \geqq C(\psi, q)n\psi\left(\frac{1}{n}\right)a_{2n}, \quad n \in \mathbf{N}.$$

The proofs of Lemmas 15 and 16 repeat almost word for word the proofs of the analogous lemmas for the space L_p, $1 < p < \infty$, and, therefore, are omitted.

LEMMA 17 [8]. *Let* $q > 0$ *and let* $\psi(t)$ *be a* φ*-function such that* $1 < \alpha_\psi \leqq \leqq \beta_\psi < 2$. *Let* $f(x) \in \Lambda_{\psi,q}$, $f(x) \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$, $S_n(x) = \frac{a_0}{2} + \sum\limits_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx)$. *Then*

$$\|f - S_n\|_{\psi,q} \leqq C(\psi, q)E_n(f)_{\psi,q}, \quad n \in \mathbf{N}.$$

LEMMA 18 [8]. *Let* $q > 0$ *and let* $\psi(t)$ *be a* φ*-function such that* $1 < < \alpha_\psi \leqq \beta_\psi < 2$. *Let* $f(x) \in \Lambda_{\psi,q}$, $f(x) \sim \sum\limits_{k=1}^{\infty} a_k \cos kx$, *where* $a_k \downarrow 0$, *when* $k \uparrow \infty$. *Then*

$$\|f\|_{\psi,q} \asymp \left\{\sum_{k=1}^{\infty} a_k^q \cdot \psi^q\left(\frac{1}{k}\right) \cdot k^{q-1}\right\}^{\frac{1}{q}}.$$

LEMMA 19. *Let $q>0$ and let $\psi(t)$ be a φ-function such that $1<\alpha_\psi \leqq \beta_\psi<2$. Let $f(x)\in\Lambda_{\psi,q}$, $f(x)\sim\sum_{k=1}^{\infty} a_k\cos kx$, where $a_k\downarrow 0$, when $k\uparrow\infty$. Then*

$$E_{[\frac{n}{2}]}(f)_{\psi,q} \geqq C(\psi,q)\left\{\sum_{m=n}^{\infty} a_m^q\cdot\psi^q\left(\frac{1}{m}\right)\cdot m^{q-1}\right\}^{\frac{1}{q}}.$$

PROOF. Using Lemmas 18, 17, 6 and part a) of Lemma 11, we obtain

$$\sum_{m=n}^{\infty} a_m^q\psi^q\left(\frac{1}{m}\right)m^{q-1} \leqq a_n^q\sum_{m=1}^{n-1}\psi^q\left(\frac{1}{m}\right)m^{q-1}+\sum_{m=n}^{\infty}a_m^q\psi^q\left(\frac{1}{m}\right)m^{q-1}\leqq$$

$$\leqq C(\psi,q)\left\|f(x)-S_n(x)+a_n\sum_{m=1}^{n-1}\cos mx\right\|_{\psi,q}^q \leqq C(\psi,q)\Bigg(\|f-S_n\|_{\psi,q}^q+$$

$$+a_n^q\left\|\sum_{m=1}^{n-1}\cos mx\right\|_{\psi,q}^q\Bigg) \leqq C(\psi,q)\left(E_n^q(f)_{\psi,q}+a_n^q\cdot n^q\psi^q\left(\frac{1}{n}\right)\right).$$

Applying Lemma 16,

$$E_{[\frac{n}{2}]}(f)_{\psi,q}\geqq C(\psi,q)\left[\frac{n}{2}\right]\psi\left(\frac{1}{\left[\frac{n}{2}\right]}\right)a_{2[\frac{n}{2}]}\geqq C(\psi,q)n\psi\left(\frac{1}{n}\right)a_n.$$

Hence

$$E_{[\frac{n}{2}]}(f)_{\psi,q} \geqq C(\psi,q)\left\{\sum_{m=n}^{\infty} a_m^q\psi^q\left(\frac{1}{m}\right)\cdot m^{q-1}\right\}^{\frac{1}{q}}.$$

Lemma is proved.

LEMMA 20. *Let $\omega(\delta)$ be a modulus of continuity such that $\omega(\delta)\neq O(\delta)$, let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\lambda_n\downarrow 0$ when $n\uparrow\infty$, let $\psi_1(t)$ and $\psi_2(t)$ be φ-functions such that $1<\alpha_{\psi_2}\leqq\beta_{\psi_2}<\alpha_{\psi_1}\leqq\beta_{\psi_1}\leqq 2$, and let $q>0$. If*

$$\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\leqq C\lambda_n,$$

then there exists a nondecreasing sequence of integers $\{n_k\}_{k=1}^{\infty}$, and a sequence of positive numbers $\{B_n\}_{n=1}^{\infty}$ such that

1) *$B_n\downarrow 0$ as $n\uparrow\infty$;*

2) *$B_n\leqq\omega\left(\frac{1}{n}\right)$, $n\in\mathbf{N}$;*

3) $\left\{ \sum_{n=1}^{N} n^{p-1} B_n^p \right\}^{\frac{1}{p}} \leqq CN\omega\left(\frac{1}{N}\right),\ N \in \mathbf{N},\ p = \begin{cases} 1, & \textit{if } 1 \leqq q < \infty; \\ q, & \textit{if } 0 < q \leqq 1; \end{cases}$

4) $B_{n_{k+1}} \leqq \frac{1}{2} B_{n_k},\ n_{k+1} > 2n_k,\ k \in \mathbf{N};$

5) $\sum_{n=n_k}^{\infty} \left(\frac{B_n \psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)} \right)^q \cdot \frac{1}{n} \geqq C_1(\psi_1, \psi_2, q) \sum_{n=n_k}^{\infty} \left(\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)} \right)^q \cdot \frac{1}{n} -$
$- C_2(\psi_1, \psi_2, q)\lambda_{n_k}^q,\ k \in \mathbf{N};$

6) $\sum_{n=N}^{n_{k+1}-1} \left(\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)} \right)^q \cdot \frac{1}{n} \leqq C(\psi_1, \psi_2, q)\lambda_N^q,\ n_k \leqq N < n_{k+1}.$

Proof. Using Lemma 10, we will suppose further that $\omega(\delta)$ is concave. Then $\frac{\omega(\delta)}{\delta} \uparrow$ when $\delta \downarrow +0$. If $\frac{\omega(\delta)}{\delta} \leqq C$ when $\delta \downarrow +0$, then we have $\omega(\delta) = O(\delta)$, but this contradicts the conditions of the lemma. Hence $\frac{\omega(\delta)}{\delta} \uparrow \infty$ when $\delta \downarrow +0$. Let $n_0 = 0$, $n_1 = 1$. If $n_1 < n_2 < \ldots < n_k$ have been chosen, we define m_{k+1} to be the smallest integer N for which $N\omega\left(\frac{1}{N}\right) > 2n_k\omega\left(\frac{1}{n_k}\right)$, i.e. $n\omega\left(\frac{1}{n}\right) \leqq 2n_k\left(\frac{1}{n_k}\right)$, when $n_k \leqq n < m_{k+1}$ and $m_{k+1}\omega\left(\frac{1}{m_{k+1}}\right) > 2n_k\omega\left(\frac{1}{n_k}\right)$. Owing to the fact that

$$\frac{\omega(\delta)}{\delta} \uparrow \infty \quad (\delta \downarrow +0)$$

such an integer m_{k+1} exists. We note that $m_{k+1} > 2n_k$. Indeed, if $n_k \leqq n \leqq 2n_k$, then due to the monotonicity of $\omega(\delta)$, we have

$$n\omega\left(\frac{1}{n}\right) \leqq 2n_k\omega\left(\frac{1}{n_k}\right)$$

and, consequently, $n \neq m_{k+1}$. If

$$\omega\left(\frac{1}{m_{k+1}}\right) \leqq \frac{1}{2}\omega\left(\frac{1}{n_k}\right),$$

then we set $n_{k+1} = m_{k+1}$. If $\omega\left(\frac{1}{m_{k+1}}\right) > \frac{1}{2}\omega\left(\frac{1}{n_k}\right)$, then we take n_{k+1} to be the smallest integer N for which

$$\omega\left(\frac{1}{N}\right) \leqq \frac{1}{2}\omega\left(\frac{1}{n_k}\right).$$

It is clear that in this case

$$n_{k+1} > m_{k+1} > 2n_k,\ \omega\left(\frac{1}{n_{k+1}}\right) \leqq \frac{1}{2}\omega\left(\frac{1}{n_k}\right),$$

$$\omega\left(\frac{1}{n}\right) > \frac{1}{2}\omega\left(\frac{1}{n_k}\right),\ n_k \leqq n < n_{k+1}.$$

We set

$$(*)\qquad B_1=\omega(1),\ B_n=\omega\left(\frac{1}{n_{k+1}}\right),\ n_k<n\leqq n_{k+1},\ k\in\mathbf{N}.$$

Since $\omega(\delta)\downarrow +0$, as $\delta\downarrow +0$, $(*)$ implies that $B_n\downarrow 0$ as $n\uparrow\infty$. Therefore, parts 1), 2) and 4) of the lemma are proved. Notice that $n_{k+1}\omega\left(\frac{1}{n_{k+1}}\right)>$ $>2n_k\omega\left(\frac{1}{n_k}\right)$, $k\in\mathbf{N}$. Hence for $n_{l-1}<N\leqq n_l$, $l\in\mathbf{N}$ we have

$$\sum_{n=1}^{N}n^{p-1}B_n^p=\sum_{k=1}^{l-1}B_{n_k}^p\sum_{n=n_{k-1}+1}^{n_k}n^{p-1}+B_N^p\sum_{n=n_{l-1}+1}^{N}n^{p-1}\leqq$$

$$\leqq\sum_{k=1}^{l-1}\omega^p\left(\frac{1}{n_k}\right)\sum_{n=1}^{n_k}n^{p-1}+\omega^p\left(\frac{1}{N}\right)\sum_{n=1}^{N}n^{p-1}\leqq$$

$$\leqq C(p)\left(\sum_{k=1}^{l-1}\left[n_k\omega\left(\frac{1}{n_k}\right)\right]^p+\left(N\omega\left(\frac{1}{N}\right)\right)^p\right)\leqq$$

$$\leqq C(p)\left(\left[n_{l-1}\omega\left(\frac{1}{n_{l-1}}\right)\right]^p\sum_{S=0}^{\infty}\left(\frac{1}{2^S}\right)^p+\left[N\omega\left(\frac{1}{N}\right)\right]^p\right)\leqq$$

$$\leqq C(q)\left[N\omega\left(\frac{1}{N}\right)\right]^p.$$

Thus part 3) is proved. Let $n_k\leqq N<n_{k+1}$, $k\in\mathbf{N}$. If $n_{k+1}=m_{k+1}$, then, using Lemmas 2 and 3 we obtain

$$(\mathrm{A})\qquad \sum_{n=N}^{n_{k+1}-1}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^q\cdot\frac{1}{n}=$$

$$=\sum_{n=N}^{n_{k+1}-1}\left[\frac{n\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{n\psi_1\left(\frac{1}{n}\right)}\right]^q\cdot\frac{1}{n}\leqq\left[\frac{(n_{k+1}-1)\omega\left(\frac{1}{n_{k+1}-1}\right)}{N\psi_1\left(\frac{1}{N}\right)}\right]^q\times$$

$$\times\sum_{n=N}^{\infty}\psi_2^q\left(\frac{1}{n}\right)\cdot\frac{1}{n}\leqq\left[\frac{2N\omega\left(\frac{1}{N}\right)}{N\psi_1\left(\frac{1}{N}\right)}C(\psi_2,q)\psi_2\left(\frac{1}{N}\right)\right]^q=$$

$$=C(\psi_2,q)\left[\frac{\omega\left(\frac{1}{N}\right)\psi_2\left(\frac{1}{N}\right)}{\psi_1\left(\frac{1}{N}\right)}\right]^q\leqq C(\psi_2,q)\lambda_N^q.$$

If $n_{k+1} > m_{k+1}$ then using Lemma 5 we obtain

$$\text{(B)} \qquad \sum_{n=N}^{n_{k+1}-1} \left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^q \cdot \frac{1}{n} \leqq \omega^q\left(\frac{1}{N}\right)\sum_{n=1}^{n_{k+1}} \frac{\psi_2^q\left(\frac{1}{n}\right)}{n\psi_1^q\left(\frac{1}{n}\right)} \leqq$$

$$\leqq C(\psi_1,\psi_2,q)\left[\frac{\omega\left(\frac{1}{n_{k+1}}\right)\psi_2\left(\frac{1}{n_{k+1}}\right)}{\psi_1\left(\frac{1}{n_{k+1}}\right)}\right]^q \leqq$$

$$\leqq C(\psi_1,\psi_2,q)\lambda_{n_{k+1}}^q \leqq C(\psi_1,\psi_2,q)\lambda_N^q.$$

Hence we proved that part 6) of the lemma is true. Using (A) and (B) we obtain

$$\sum_{n=n_k}^{\infty}\left(\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^q \cdot \frac{1}{n} = \sum_{l=k}^{\infty}\sum_{n=n_l}^{n_{l+1}-1}\left(\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^q \cdot \frac{1}{n} \leqq$$

$$\leqq C(\psi_1,\psi_2,q)\sum_{l=k}^{\infty}\left[\frac{\omega\left(\frac{1}{n_l}\right)\psi_2\left(\frac{1}{n_l}\right)}{\psi_1\left(\frac{1}{n_l}\right)}\right]^q.$$

Applying Lemma 2 we have

$$\sum_{n=n_k}^{\infty}\left(\frac{B_n\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^q \cdot \frac{1}{n} \geqq \sum_{l=k+1}^{\infty} B_{n_l}^q \sum_{n=n_{l-1}+1}^{n_l}\left(\frac{\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^q \cdot \frac{1}{n} \geqq$$

$$\geqq \sum_{l=k+1}^{\infty}\left[B_{n_l}\cdot\psi_2\left(\frac{1}{n_l}\right)\right]^q \cdot \sum_{n=\left[\frac{n_l}{2}\right]}^{n_l}\frac{1}{n\psi_1^q\left(\frac{1}{n}\right)} \geqq$$

$$\geqq C_1(\psi_1,q)\sum_{l=k+1}^{\infty}\left[\frac{\omega\left(\frac{1}{n_l}\right)\psi_2\left(\frac{1}{n_l}\right)}{\psi_1\left(\frac{1}{n_l}\right)}\right]^q =$$

$$= C_1(\psi_1,q)\sum_{l=k}^{\infty}\left[\frac{\omega\left(\frac{1}{n_l}\right)\psi_2\left(\frac{1}{n_l}\right)}{\psi_1\left(\frac{1}{n_l}\right)}\right]^q - C_1(\psi_1,q)\left[\frac{\omega\left(\frac{1}{n_k}\right)\psi_2\left(\frac{1}{n_k}\right)}{\psi_1\left(\frac{1}{n_k}\right)}\right]^q \geqq$$

$$\geqq C_2(\psi_1,\psi_2,q)\sum_{n=n_k}^{\infty}\left(\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^q \cdot \frac{1}{n} - C_3(\psi_1,q)\lambda_{n_k}^q.$$

Thus part 5) of the lemma is proved. Lemma 20 is completely proved.

3. THEOREM 1. *Let* $0 < q_1 < \infty$, $1 \leqq q_2 < \infty$, *and let* $\psi_1(t)$ *and* $\psi_2(t)$ *be* φ-*functions such that* $1 < \alpha_{\psi_2} \leqq \beta_{\psi_2} < \alpha_{\psi_1} \leqq \beta_{\psi_1} \leqq 2$. *Let* $\omega(\delta)$ *be a modulus of continuity,* $\lambda \equiv \{\lambda_n\}_{n=1}^{\infty}$ *a sequence of positive numbers such that* $\lambda_n \downarrow 0$ *as* $n \uparrow \infty$. *If*

$$\left\{\sum_{n=k}^{\infty}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}\cdot\frac{1}{n}\right\}^{\frac{1}{q_2}} = O\{\lambda_k\}, \tag{1}$$

then $H^{\omega}_{\psi_1,q_1} \subset E_{\psi_2,q_2}(\lambda)$, *and for any* $f(x) \in \Lambda_{\psi_1,q_1}$,

$$E_k(f)_{\psi_2,q_2} \leqq C(\psi_1,\psi_2,q_1,q_2)\left\{\sum_{n=k}^{\infty}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega_{\psi_1,q_1}\left(f;\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}\cdot\frac{1}{n}\right\}^{\frac{1}{q_2}}. \tag{2}$$

PROOF. Let $f(x) \in H^{\omega}_{\psi_1,q_1}$, and let condition (1) be satisfied. Using Lemmas 8 and 9 we obtain

$$E_k(f)_{\psi_2,q_2} \leqq C(\psi_2,q_2)\omega_{\psi_2,q_2}\left(f;\frac{1}{k}\right) \leqq$$

$$\leqq C(\psi_1,\psi_2,q_1,q_2)\left\{\sum_{n=k}^{\infty}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega_{\psi_1,q_1}\left(f;\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}\cdot\frac{1}{n}\right\}^{\frac{1}{q_2}} \leqq$$

$$\leqq C(\psi_1,\psi_2,q_1,q_2)\left\{\sum_{n=k}^{\infty}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}\cdot\frac{1}{n}\right\}^{\frac{1}{q_2}} \leqq C(\psi_1,\psi_2,q_1,q_2)\lambda_k.$$

Hence we see that $f(x) \in E_{\psi_2,q_2}(\lambda)$, and (2) holds. Theorem 1 is proved.

4. THEOREM 2. *Let* $0 < q_1, q_2 < \infty$, *and let* $\psi_1(t)$ *and* $\psi_2(t)$ *be* φ-*functions such that* $1 < \alpha_{\psi_2} \leqq \beta_{\psi_2} < \alpha_{\psi_1} \leqq \beta_{\psi_1} < 2$. *Let* $\omega(\delta)$ *be a modulus of continuity,* $\lambda \equiv \{\lambda_n\}_{n=1}^{\infty}$ *a sequence of positive numbers such that* $\lambda_n \downarrow 0$ *as* $n \uparrow \infty$, *and let* $H^{\omega}_{\psi_1,q_1} \subset E_{\psi_2,q_2}(\lambda)$. *Then*

$$\left\{\sum_{n=k}^{\infty}\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}\cdot\frac{1}{n}\right\}^{\frac{1}{q_2}} = O\{\lambda_k\}. \tag{3}$$

PROOF. We suppose that $H^{\omega}_{\psi_1,q_1} \subset E_{\psi_2,q_2}(\lambda)$ holds but condition (3) is not satisfied. Then there exists an increasing sequence of integers $\{r_i\}_{i=1}^{\infty}$ such that

$$\left\{\sum_{k=r_i}^{\infty}\left[\frac{\psi_2\left(\frac{1}{k}\right)\omega\left(\frac{1}{k}\right)}{\psi_1\left(\frac{1}{k}\right)}\right]^{q_2}\cdot\frac{1}{k}\right\}^{\frac{1}{q_2}} \geqq A_i\lambda_{r_i}, \tag{4}$$

where $A_i \uparrow \infty$ as $i \uparrow \infty$.

Using Lemma 10, without loss of generality we assume that $\omega(\delta)$ is a concave modulus of continuity. Notice that without loss of generality we may assume that

$$\sum_{k=1}^{\infty}\left[\frac{\psi_2\left(\frac{1}{k}\right)\omega\left(\frac{1}{k}\right)}{\psi_1\left(\frac{1}{k}\right)}\right]^{q_2}\cdot\frac{1}{k} < \infty, \tag{4'}$$

since otherwise, using Lemma 9, there exists a function $g(x)$ in $H^{\omega}_{\psi_1,q_1}$ which does not belong to Λ_{ψ_2,q_2} and to $E_{\psi_2,q_2}(\lambda) \in \Lambda_{\psi_2,q_2}$.

Since $\psi_1(t)$ and $\psi_2(t)$ are φ-functions,

$$\sum_{k=n}^{\infty}\left[\frac{\psi_2\left(\frac{1}{k}\right)\omega\left(\frac{1}{k}\right)}{\psi_1\left(\frac{1}{k}\right)}\right]^{q_2}\cdot\frac{1}{k} \geqq \tag{5}$$

$$\geqq \sum_{k=n}^{2n}\left[\frac{\psi_2\left(\frac{1}{k}\right)\omega\left(\frac{1}{k}\right)}{\psi_1\left(\frac{1}{k}\right)}\right]^{q_2}\cdot\frac{1}{k} \geqq C(q_2)\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}.$$

Due to (4) and (5), we distinguish the following two cases:

a) There exists a sequence of positive integers such that

$$\frac{\omega\left(\frac{1}{n_i}\right)\psi_2\left(\frac{1}{n_i}\right)}{\psi_1\left(\frac{1}{n_i}\right)} \geqq D_i\lambda_{n_i} \tag{6}$$

where $D_i \uparrow \infty$ as $i \uparrow \infty$.

b) For all $n \geqq 1$

$$\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)} \leqq C\lambda_n. \tag{7}$$

Let the modulus of continuity $\omega(\delta)$ satisfy the condition

$$\omega(\delta) = O(\delta) \quad \text{as} \quad \delta \downarrow +0. \tag{8}$$

Then we have Case a). Indeed, as $\omega(\delta) \not\equiv 0$, $\frac{\omega(\delta)}{\delta}$ is decreasing due to Lemma 10, therefore $C_1\delta \leqq \omega(\delta) \leqq C_2\delta$. From here, applying Lemmas 2 and 3 we obtain

$$\sum_{k=n}^{\infty} \left[\frac{\psi_2\left(\frac{1}{k}\right)\omega\left(\frac{1}{k}\right)}{\psi_1\left(\frac{1}{k}\right)}\right]^{q_2} \cdot \frac{1}{k} \leqq$$

$$\leqq C(q_2)\sum_{k=n}^{\infty}\left[\frac{\psi_2\left(\frac{1}{k}\right)}{k\psi_1\left(\frac{1}{k}\right)}\right]^{q_2} \cdot \frac{1}{k} \leqq C(q_2)\left[\frac{1}{n\psi_1\left(\frac{1}{n}\right)}\right]^{q_2} \cdot \sum_{k=n}^{\infty}\psi_2^{q_2}\left(\frac{1}{k}\right)\cdot\frac{1}{k} \leqq$$

$$\leqq C(\psi_2, q_2)\left[\frac{\psi_2\left(\frac{1}{n}\right)}{n\psi_1\left(\frac{1}{n}\right)}\right]^{q_2} \leqq C(\psi_2, q_2)\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}.$$

Hence and from (5),

$$C(q_2)\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2} \leqq$$

$$\leqq \sum_{k=n}^{\infty}\left[\frac{\psi_2\left(\frac{1}{k}\right)\omega\left(\frac{1}{k}\right)}{\psi_1\left(\frac{1}{k}\right)}\right]^{q_2} \cdot \frac{1}{k} \leqq C(\psi_2, q_2)\left[\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right]^{q_2}.$$

We obtained that condition (4) is equivalent to condition (6) in the case (8).

We will show that in case (8), if condition (4) or (6) is satisfied then $H^{\delta}_{\psi_1,q_1} \not\subset E_{\psi_2,q_2}(\lambda)$, i.e. there exists an increasing sequence of positive integers $\{n_i\}_{i=1}^{\infty}$, $n_1 = 1$ such that

$$\frac{\psi_2\left(\frac{1}{n_i}\right)}{n_i\psi_1\left(\frac{1}{n_i}\right)} \geqq D_i\lambda_{n_i}$$

where $D_i \uparrow \infty$ as $i \uparrow \infty$, then there exists $f(x) \in H^{\delta}_{\psi_1,q_1}$, which does not belong to the class $E_{\psi_2,q_2}(\lambda)$. Without loss of generality we assume that

$$n_1 = 1,\ n_{i+1} > 2n_i, \quad \text{and} \quad D_{i+1} > 2D_i,\ i \in \mathbf{N}.$$

We will consider the series

$$f(x) \overset{\Lambda_{\psi_1,q_1}}{=\!=} \sum_{i=1}^{\infty} \frac{1}{n_i^2\psi_1\left(\frac{1}{n_i}\right)} D_i^{-\frac{1}{2}}\left[\cos n_i x + \ldots + \cos(2n_i - 1)x\right].$$

We obtain

$$\sum_{i=1}^{\infty} \frac{1}{n_i^2 \psi_1\left(\frac{1}{n_i}\right)} D_i^{-\frac{1}{2}} \left| \cos n_i x + \ldots + \cos(2n_i - 1)x \right| \leqq$$

$$\leqq \sum_{i=1}^{\infty} \frac{1}{n_i^2 \psi_1\left(\frac{1}{n_i}\right)} D_i^{-\frac{1}{2}} \cdot n_i =$$

$$= \sum_{i=1}^{\infty} \frac{1}{n_i \psi_1\left(\frac{1}{n_i}\right)} D_i^{-\frac{1}{2}} \leqq \frac{1}{n_1 \psi_1\left(\frac{1}{n_1}\right)} \sum_{i=1}^{\infty} D_i^{-\frac{1}{2}} < \infty,$$

i.e. our series is finite, therefore the sum of the series is a continuous function $f(x)$. Then $f(x) \in \Lambda_{\psi_1,q_1}$. We will show that $f(x) \in H^{\delta}_{\psi_1,q_1}$. Let

$$p = \begin{cases} 1, & 1 \leqq q_1 < \infty; \\ q_1, & 0 < q_1 \leqq 1. \end{cases}$$

Let $n_i \leqq n < 2n_i$, $i \in \mathbf{N}$. Then using Lemma 11 we have

$$E_n^p(f)_{\psi_1,q_1} \leqq E_{n_i}^p(f)_{\psi_1,q_1} \leqq \tag{9}$$

$$\leqq \sum_{j=i}^{\infty} \left(\frac{1}{n_j^2 \psi_1\left(\frac{1}{n_j}\right)} D_j^{-\frac{1}{2}} \right)^p \left\| \cos n_j x + \ldots + \cos(2n_j - 1)x \right\|^p_{\psi_1,q_1} \leqq$$

$$\leqq C(\psi_1, q_1) \sum_{j=i}^{\infty} \left(\frac{1}{n_j^2 \psi_1\left(\frac{1}{n_j}\right)} D_j^{-\frac{1}{2}} \cdot n_j \psi_1\left(\frac{1}{n_j}\right) \right)^p =$$

$$= C(\psi_1, q_1) \sum_{j=i}^{\infty} \left(\frac{D_j^{-\frac{1}{2}}}{n_j} \right)^p \leqq \frac{C(\psi_1, q_1)}{n_i^p} D_i^{-\frac{p}{4}} \times$$

$$\times \sum_{j=i}^{\infty} D_j^{-\frac{p}{4}} \leqq \frac{C(\psi_1, q_1)}{n_i^p} D_i^{-\frac{p}{4}}.$$

Let $2n_i \leqq n < n_{i+1}$, $i \in \mathbf{N}$. Then using Lemma 11 we have

$$E_n^p(f)_{\psi_1,q_1} \leqq E_{2n_i}^p(f)_{\psi_1,q_1} \leqq \tag{10}$$

$$\leqq \sum_{j=i+1}^{\infty}\left(\frac{1}{n_j^2\psi_1\left(\frac{1}{n_j}\right)}D_j^{-\frac{1}{2}}\right)^p \|\cos n_jx+\ldots+\cos(2n_j-1)x\|_{\psi_1,q_1}^p \leqq$$

$$\leqq C(\psi_1,q_1)\sum_{j=i+1}^{\infty}\left(\frac{1}{n_j^2\psi_1\left(\frac{1}{n_j}\right)}D_j^{-\frac{1}{2}}n_j\psi_1\left(\frac{1}{n_j}\right)\right)^p=$$

$$=C(\psi_1,q_1)\sum_{j=i+1}^{\infty}\left(\frac{D_j^{-\frac{1}{2}}}{n_j}\right)^p \leqq$$

$$\leqq \frac{C(\psi_1,q_1)}{n_{i+1}^p}D_{i+1}^{-\frac{p}{4}}\cdot\sum_{j=i+1}^{\infty}D_j^{-\frac{p}{4}} \leqq \frac{C(\psi_1,q_1)}{n_{i+1}^p}D_i^{-\frac{p}{4}}.$$

Using Lemma 12 and the estimates (9) and (10), as $n_i \leqq n < 2n_i$, $i \in \mathbf{N}$, we have

$$\omega_{\psi_1,q_1}\left(f;\frac{1}{n}\right) \leqq \frac{C(\psi_1,q_1)}{n}\left\{\sum_{k=1}^{n}k^{p-1}E_k^p(f)_{\psi_1,q_1}\right\}^{\frac{1}{p}}=$$

$$=\frac{C(\psi_1,q_1)}{n}\left\{\sum_{j=1}^{i-1}\left(\sum_{k=n_j}^{2n_j-1}k^{p-1}E_k^p(f)_{\psi_1,q_1}+\sum_{k=2n_j}^{n_{j+1}-1}k^{p-1}E_k^p(f)_{\psi_1,q_1}\right)+\right.$$

$$\left.+\sum_{k=n_i}^{n}k^{p-1}E_k^p(f)_{\psi_1,q_1}\right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1,q_1)}{n}\left\{\sum_{j=1}^{i-1}\left(\frac{D_j^{-\frac{p}{4}}}{n_j^p}\sum_{k=n_j}^{2n_j-1}k^{p-1}+\right.\right.$$

$$\left.\left.+\frac{D_{j+1}^{-\frac{p}{4}}}{n_{j+1}^p}\sum_{k=2n_j}^{n_{j+1}-1}k^{p-1}\right)+\frac{D_i^{-\frac{p}{4}}}{n_i^p}\sum_{k=n_i}^{n}k^{p-1}\right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1,q_1)}{n}\left\{\sum_{j=1}^{i-1}\left(D_j^{-\frac{p}{4}}+D_{j+1}^{-\frac{p}{4}}\right)+D_i^{-\frac{p}{4}}\right\}^{\frac{1}{p}} \leqq \frac{C(\psi_1,q_1)}{n}.$$

Using Lemma 12 and the estimates (9) and (1), as $2n_i \leqq n < n_{i+1}$, $i \in \mathbf{N}$, we have

$$\omega_{\psi_1,q_1}\left(f;\frac{1}{n}\right) \leqq \frac{C(\psi_1,q_1)}{n}\left\{\sum_{k=1}^{n} k^{p-1}E_k^p(f)_{\psi_1,q_1}\right\}^{\frac{1}{p}} =$$

$$= \frac{C(\psi_1,q_1)}{n}\left\{\sum_{j=1}^{i-1}\left(\sum_{k=n_j}^{2n_j-1} k^{p-1}E_k^p(f)_{\psi_1,q_1}+\right.\right.$$

$$\left.+\sum_{k=2n_j}^{n_{j+1}-1} k^{p-1}E_k^p(f)_{\psi_1,q_1}\right)+\sum_{k=n_i}^{2n_i-1} k^{p-1}E_k^p(f)_{\psi_1,q_1}+$$

$$\left.+\sum_{k=2n_i}^{n} k^{p-1}E_k^p(f)_{\psi_1,q_1}\right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1,q_1)}{n}\cdot\left\{\sum_{j=1}^{i-1}\left(\frac{D_j^{-\frac{p}{4}}}{n_j^p}\sum_{k=n_j}^{2n_j-1} k^{p-1}+\frac{D_{j+1}^{-\frac{p}{4}}}{n_{j+1}^p}\sum_{k=2n_j}^{n_{j+1}-1} k^{p-1}\right)+\right.$$

$$\left.+\frac{D_i^{-\frac{p}{4}}}{n_i^p}\sum_{k=n_i}^{2n_i-1} k^{p-1}+\frac{D_{i+1}^{-\frac{p}{4}}}{n_{i+1}^p}\sum_{k=2n_i}^{n} k^{p-1}\right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1,q_1)}{n}\left\{\sum_{j=1}^{i-1}\left(D_j^{-\frac{p}{4}}+D_{j+1}^{-\frac{p}{4}}\right)+D_i^{-\frac{p}{4}}+D_{i+1}^{-\frac{p}{4}}\right\}^{\frac{1}{p}} \leqq \frac{C(\psi_1,q_1)}{4}.$$

Hence $f(x) \in H^{\delta}_{\psi_1,q_1}$. Using Lemma 9 and (4′) we have $f(x) \in \Lambda_{\psi_2,q_2}$. We will show that $f(x) \notin E_{\psi_2,q_2}(\lambda)$. By virtue of Lemma 15,

$$E_{n_i}(f)_{\psi_2,q_2} \geqq C(\psi_2,q_2)\psi_2\left(\frac{1}{n_i}\right)\sum_{k=n_i}^{2n_i-1} a_k =$$

$$= C(\psi_2,q_2)\psi_2\left(\frac{1}{n_i}\right)\cdot\frac{1}{n_i^2\psi_1\left(\frac{1}{n_i}\right)}D_i^{-\frac{1}{2}}\cdot n_i =$$

$$= C(\psi_2,q_2)\frac{\psi_2\left(\frac{1}{n_i}\right)}{n_i\psi_1\left(\frac{1}{n_i}\right)}D_i^{-\frac{1}{2}} \geqq C(\psi_2,q_2)D_i^{\frac{1}{2}}\lambda_{n_i}, \quad D_i^{\frac{1}{2}}\uparrow\infty \text{ as } i\uparrow\infty.$$

This contradicts $H^{\delta}_{\psi_1,q_1} \subset E_{\psi_2,q_2}(\lambda)$. Hence (4) is not true, therefore in case (8) the theorem is proved.

Consider case (6), when $\omega(\delta) \neq O(\delta)$. Without loss of generality we assume that $\omega(\delta)$ is a concave modulus of continuity. Hence $\frac{\omega(\delta)}{\delta} \uparrow \infty$ as $\delta \downarrow +0$. From the sequence $\{n_i\}_{i=1}^{\infty}$ we will choose a subsequence $\{m_i\}_{i=1}^{\infty}$ so that $m_1 = 1$, and if $m_1, m_2, \ldots, m_k$ have been chosen, then we define m'_{k+1} to be the smallest of the integers $N \in \{n_i\}_{i=1}^{\infty}$ for which $N\omega\left(\frac{1}{N}\right) > 2m_k\omega\left(\frac{1}{m_k}\right)$. Then $m\omega\left(\frac{1}{m}\right) \leqq 2m_k\omega\left(\frac{1}{m_k}\right)$, $m \in \{n_i\}$, $m_k \leqq m < m'_{k+1}$ and $m'_{k+1}\omega\left(\frac{1}{m'_{k+1}}\right) > 2m_k\omega\left(\frac{1}{m_k}\right)$. Owing to the fact that $\frac{\omega(\delta)}{\delta} \uparrow \infty$, $\delta \downarrow 0$ such an integer m'_{k+1} exists. Since $\omega(\delta) \downarrow 0$ as $\delta \downarrow +0$, then $m\omega\left(\frac{1}{m}\right) \leqq 2m_k\omega\left(\frac{1}{m_k}\right)$ as $m_k \leqq m \leqq 2m_k$, therefore from the obtained inequalities it follows that $m'_{k+1} > 2m_k$. If $\omega\left(\frac{1}{m'_{k+1}}\right) < \frac{1}{2}\omega\left(\frac{1}{m_k}\right)$, then we set $m_{k+1} = m'_{k+1}$. If $\omega\left(\frac{1}{m'_{k+1}}\right) \geqq \frac{1}{2}\omega\left(\frac{1}{m_k}\right)$, then we define m_{k+1} to be the smallest number N such that $N \in \{n_i\}$, $N > m'_{k+1}$, for which $\omega\left(\frac{1}{N}\right) < \frac{1}{2}\omega\left(\frac{1}{m_k}\right)$. Thus in this case $\omega\left(\frac{1}{m}\right) \geqq \frac{1}{2}\omega\left(\frac{1}{n_k}\right)$ as $m \in \{n_i\}$, $m_k \leqq m < m_{k+1}$ and $\omega\left(\frac{1}{m_{k+1}}\right) < \frac{1}{2}\omega\left(\frac{1}{m_k}\right)$. As $\omega(\delta) \downarrow 0$ when $\delta \downarrow 0$, such m_{k+1} exists. Besides, we obtain that $m_{k+1}\omega\left(\frac{1}{m_{k+1}}\right) > 2m_k\omega\left(\frac{1}{m_k}\right)$ and $m_{k+1} > 2m_k$. We set $B_1 = \omega(1)$, $B_n = \omega\left(\frac{1}{m_{k+1}}\right)$, $m_k < n \leqq m_{k+1}$, $k \in \mathbf{N}$. Hence $B \downarrow 0$ as $n \uparrow \infty$, $B_{m_{k+1}} < \frac{1}{2}B_{m_k}$, $k \in \mathbf{N}$. We will consider the series

$$f(x) \overset{\Lambda_{\psi_1,q_1}}{=\!=\!=} \sum_{k=1}^{\infty} \frac{1}{m_k\psi_1\left(\frac{1}{m_k}\right)} B_{m_k}\left[\cos m_k x + \ldots + \cos(2m_k - 1)x\right].$$

By virtue of Lemma 11, we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{m_k\psi_1\left(\frac{1}{m_k}\right)} B_{m_k}\left\|\cos m_k x + \ldots + \cos(2m_k - 1)x\right\|_{\psi_1,q_1}\right)^p \leqq$$

$$\leqq C(\psi_1, q_1) \cdot \sum_{k=1}^{\infty}\left(\frac{1}{m_k\psi_1\left(\frac{1}{m_k}\right)} B_{m_k} \cdot m_k\psi_1\left(\frac{1}{m_k}\right)\right)^p =$$

$$= C(\psi_1, q_1)\sum_{k=1}^{\infty} B_{m_k}^p \leqq C(\psi_1, q_1) \cdot B_1^p \cdot \sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}}\right)^p \leqq C(\psi_1, q_1)B_1^p.$$

Hence the series converges in norm to a function $f(x) \in \Lambda_{\psi_1,q_1}$. We will show that $f(x) \in H^{\omega}_{\psi_1,q_1}$. Let $m_i \leqq n < 2m_i$, $i \in \mathbf{N}$. Using Lemma 11 we have

$$E_n^p(f)_{\psi_1,q_1} \leqq E_{m_i}^p(f)_{\psi_1,q_1} \leqq$$

$$\leqq \sum_{j=i}^{\infty} \left(\frac{1}{m_j \psi_1\left(\frac{1}{m_j}\right)} B_{m_j} \left\| \cos m_j x + \ldots + \cos(2m_j - 1)x \right\|_{\psi_1,q_1} \right)^p \leqq$$

$$\leqq C(\psi_1, q_1) \sum_{j=1}^{\infty} \left(\frac{1}{m_j \psi_1\left(\frac{1}{m_j}\right)} B_{m_j} \cdot m_j \psi_1\left(\frac{1}{m_j}\right) \right)^p \leqq$$

$$\leqq C(\psi_1, q_1) B_{m_i}^p \sum_{S=0}^{\infty} \left(\frac{1}{2^S}\right)^p \leqq C(\psi_1, q_1) B_{m_i}^p.$$

Let $2m_i \leqq n < m_{i+1}$, $i \in \mathbf{N}$. Using Lemma 11 we have

$$E_n^p(f)_{\psi_1,q_1} \leqq E_{2m_i}^p(f)_{\psi_1,q_1} \leqq$$

$$\leqq \sum_{j=i+1}^{\infty} \left(\frac{1}{m_j \psi_1\left(\frac{1}{m_j}\right)} B_{mj} \left\| \cos m_j x + \ldots + \cos(2m_j - 1)x \right\|_{\psi_1,q_1} \right)^p \leqq$$

$$\leqq C(\psi_1, q_1) \sum_{j=i+1}^{\infty} \left(\frac{1}{m_j \psi_1\left(\frac{1}{m_j}\right)} B_{m_j} \cdot m_j \psi_1\left(\frac{1}{m_j}\right) \right)^p \leqq$$

$$\leqq C(\psi_1, q_1) B_{m_{i+1}}^p \sum_{S=0}^{\infty} \left(\frac{1}{2^S}\right)^p \leqq C(\psi_1, q_1) B_{m_{i+1}}^p.$$

Let $m_i \leqq n < 2m_i$, $i \in \mathbf{N}$. According to Lemma 12 we obtain

$$\omega_{\psi_1,q_1}\left(f; \frac{1}{n}\right) \leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{k=1}^{n} k^{p-1} E_k^p(f)_{\psi_1,q_1} \right\}^{\frac{1}{p}} =$$

$$= \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{j=1}^{i-1} \left(\sum_{k=m_j}^{2m_j - 1} k^{p-1} E_k^p(f)_{\psi_1,q_1} + \sum_{k=2m_j}^{m_{j+1}-1} k^{p-1} E_k^p(f)_{\psi_1,q_1} \right) + \right.$$

$$\left. + \sum_{k=m_i}^{n} k^{p-1} E_k^p(f)_{\psi_1,q_1} \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{j=1}^{i-1} \left(B_{m_j}^p \cdot m_j^p + B_{m_{j+1}}^p \cdot m_{j+1}^p \right) + (m_i \cdot B_{m_i})^p \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{j=1}^{i} \left[m_j \omega \left(\frac{1}{m_j} \right) \right]^p \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \omega \left(\frac{1}{m_i} \right) \cdot m_i \cdot \left(\sum_{s=0}^{\infty} \left(\frac{1}{2^s} \right)^p \right)^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \omega \left(\frac{1}{n} \right) \cdot n = C(\psi_1, q_1) \omega \left(\frac{1}{n} \right).$$

Let $2m_i \leqq n < m_{i+1}$, $i \in \mathbf{N}$. Applying Lemma 12 we obtain

$$\omega_{\psi_1, q_1} \left(f; \frac{1}{n} \right) \leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{k=1}^{n} k^{p-1} E_k^p(f)_{\psi_1, q_1} \right\}^{\frac{1}{p}} =$$

$$= \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{j=1}^{i-1} \left(\sum_{k=m_j}^{2m_j-1} k^{p-1} E_k^p(f)_{\psi_1, q_1} + \sum_{k=2m_j}^{m_{j+1}-1} k^{p-1} E_k^p(f)_{\psi_1, q_1} \right) + \right.$$

$$\left. + \sum_{k=m_i}^{2m_i-1} k^{p-1} E_k^p(f)_{\psi_1, q_1} + \sum_{k=2m_i}^{n} k^{p-1} E_k^p(f)_{\psi_1, q_1} \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{j=1}^{i-1} \left(B_{m_j}^p \cdot m_j^p + B_{m_j+1}^p \cdot m_{j+1}^p \right) + m_i^p \cdot B_{m_i}^p + n^p B_{m_j+1}^p \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{j=1}^{i} \left[m_j \omega \left(\frac{1}{m_j} \right) \right]^p + n^p \omega^p \left(\frac{1}{m_{j+1}} \right) \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \left\{ \left[m_i \omega \left(\frac{1}{m_i} \right) \right]^p \sum_{s=0}^{\infty} \left(\frac{1}{2^s} \right)^p + n^p \omega^p \left(\frac{1}{n} \right) \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \omega \left(\frac{1}{n} \right) \cdot n = C(\psi_1, q_1) \cdot \omega \left(\frac{1}{n} \right).$$

Hence $f(x) \in H^{\omega}_{\psi_1,q_1}$. Due to Lemma 9 and condition (4′), we obtain that $f(x) \in \Lambda_{\psi_2,q_2}$. We will show that $f(x) \notin E_{\psi_2,q_2}(\lambda)$. By virtue to Lemma 15 we have

$$E_{m_i}(f)_{\psi_2,q_2} \geqq C(\psi_2,q_2)\psi_2\left(\frac{1}{m_i}\right)\sum_{k=m_i}^{2m_i-1} a_k =$$

$$= C(\psi_2,q_2)\psi_2\left(\frac{1}{m_i}\right)\cdot m_i \frac{1}{m_i\psi_1\left(\frac{1}{m_i}\right)} B_{m_i} =$$

$$= C(\psi_2,q_2)\cdot\frac{\psi_2\left(\frac{1}{m_i}\right)\cdot\omega\left(\frac{1}{m_i}\right)}{\psi_1\left(\frac{1}{m_i}\right)} \geqq D_{j_i}\lambda_{m_i},$$

if we suppose $m_i = n_{j_i}$, but $\{D_{j_i}\} \subset \{D_i\}$ and therefore $D_{j_i} \uparrow \infty$ as $i \uparrow \infty$. Hence $f(x) \notin E_{\psi_2,q_2}(\lambda)$, and $H^{\omega}_{\psi_1,q_1} \not\subset E_{\psi_2,q_2}(\lambda)$, i.e. (4) is not true, therefore in this case the theorem is proved.

Consider case (7). As we noticed in this case $\omega(\delta) \neq O(\delta)$. By virtue of Lemma 20, there exists an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that $n_k \uparrow \infty$, and a decreasing sequence of positive numbers $\{B_n\}_{n=1}^{\infty}$ which possesses all properties of Lemma 20. Then $n_{k+1} > 2n_k$, $k \in \mathbf{N}$. We will show that there exists a subsequence $\{s_k\} \subset \{n_k\}$ such that

$$(11) \qquad \sum_{n=s_k}^{\infty}\left(\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n} \geqq T_k\cdot\lambda_{s_k}^{q_2}, \quad T_k\uparrow\infty \quad \text{as} \quad k\uparrow\infty.$$

We suppose that this condition is not satisfied, i.e. there exists a constant C such that for all $n_k, k \in \mathbf{N}$,

$$\sum_{n=n_k}^{\infty}\left(\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n} \leqq C\lambda_{n_k}^{q_2}.$$

Then due to (6) there exists a set of pairs of positive integers $\left(n_{k_i}, n_{k_i+1}\right)$ and a subsequence $\{r_i'\} \subset \{r_i\}$ such that $n_{k_i} < r_i' < n_{k_i+1}$ and

$$(12) \qquad \sum_{n=r_i'}^{\infty}\left(\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n} \geqq A_i'\cdot\lambda_{r_i'}^{q_2}, \quad A_i'\uparrow\infty \quad \text{as} \quad i\uparrow\infty$$

where $\{A_i'\} \subset \{A_i\}$.

On the other hand we have

$$\sum_{n=r_i'}^{\infty}\left(\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n}=\sum_{n=r_i'}^{n_{k_i+1}-1}\left(\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n}+$$

$$+\sum_{n=n_{k_i+1}}^{\infty}\left(\frac{\psi_2\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n}\leqq C_1\lambda_{r_i'}^{q_2}+C\lambda_{n_{k_i+1}}\leqq C_2\lambda_{r_i'}^{q_2}.$$

This contradicts (12). Hence (11) is proved.

Consider the series ($n_0=0$)

$$f(x)\overset{\Lambda_{\psi_1,q_1}}{=\!=}\sum_{k=1}^{\infty}\frac{1}{n_k\psi_1\left(\frac{1}{n_k}\right)}B_{n_k}\left[\cos(n_{k-1}+1)x+\ldots+\cos n_kx\right].$$

We will show that the series converges in norm to a function $f(x)\in\Lambda_{\psi_1,q_1}$. Using Lemma 11,

$$\sum_{k=1}^{\infty}\left(\frac{1}{n_k\psi_1\left(\frac{1}{n_k}\right)}B_{n_k}\left\|\cos(n_{k-1}+1)x+\ldots+\cos n_kx\right\|_{\psi_1,q_1}\right)^p\leqq$$

$$\leqq C(\psi_1,q_1)\sum_{k=1}^{\infty}\left(\frac{1}{n_k\psi_1\left(\frac{1}{n_k}\right)}B_{n_k}\cdot n_k\psi_1\left(\frac{1}{n_k}\right)\right)^p\leqq$$

$$\leqq C(\psi_1,q_1)B_1^p\sum_{s=0}^{\infty}\left(\frac{1}{2^s}\right)^p\leqq C(\psi_1,q_1)B_1^p.$$

We will show that this function belongs to the class $H^{\omega}_{\psi_1,q_1}$. Let $n_{k-1}<n\leqq$ $\leqq n_k$. Then, applying Lemma 11 we obtain

$$E_n^p(f)_{\psi_1,q_1}\leqq E^p_{n_{k-1}+1}(f)_{\psi_1,q_1}\leqq$$

$$\leqq\sum_{j=k}^{\infty}\left(\frac{1}{n_j\psi_1\left(\frac{1}{n_j}\right)}B_{n_j}\left\|\cos(n_{j-1}+1)x+\ldots+\cos n_jx\right\|_{\psi_1,q_1}\right)^p\leqq$$

$$\leqq C(\psi_1,q_1)\sum_{j=k}^{\infty}\left(\frac{1}{n_j\psi_1\left(\frac{1}{n_j}\right)}B_{n_j}\cdot n_j\psi_1\left(\frac{1}{n_j}\right)\right)^p\leqq$$

$$\leqq C(\psi_1, q_1) B^p_{n_k} \sum_{s=0}^{\infty} \left(\frac{1}{2^s}\right)^p \leqq C(\psi_1, q_1) B^p_{n_k} = C(\psi_1, q_1) B^p_n.$$

Further, according to Lemma 12, we have

$$\omega_{\psi_1, q_1}\left(f; \frac{1}{n}\right) \leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{k=1}^{n} k^{p-1} E^p_k(f)_{\psi_1, q_1} \right\}^{\frac{1}{p}} \leqq$$

$$\leqq \frac{C(\psi_1, q_1)}{n} \left\{ \sum_{k=1}^{n} k^{p-1} B^p_k \right\}^{\frac{1}{p}} \leqq \frac{C(\psi_1, q_1)}{n} B_n \cdot n =$$

$$= C(\psi_1, q_1) B_n \leqq C(\psi_1, q_1) \omega\left(\frac{1}{n}\right).$$

Hence we proved that $f(x) \in H^{\omega}_{\psi_1, q_1}$. By virtue of Lemma 9 and condition (4′) we have $f(x) \in \Lambda_{\psi_2, q_2}$. We will prove that $f(x) \notin E_{\psi_2, q_2}(\lambda)$. Applying Lemma 19 to the function $f(x)$ we obtain

$$E_{s_k}(f)_{\psi_2, q_2} \geqq C(\psi_2, q_2) \left\{ \sum_{n=2s_k}^{\infty} a_n^{q_2} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2 - 1} \right\}^{\frac{1}{q_2}}, \tag{13}$$

where a_n are coefficients of the Fourier series $f(x)$. Further,

$$\sum_{n=n_{k-1}+1}^{n_k} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} \geqq \sum_{n=\left[\frac{n_k+1}{2}\right]}^{n_k} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} \geqq$$

$$\geqq \psi_2^{q_2}\left(\frac{1}{\left[\frac{n_k+1}{2}\right]}\right) \left[\frac{n_k+1}{2}\right]^{q_2} \cdot \frac{1}{n_k} \cdot \frac{n_k+1}{2} \geqq \frac{1}{2^{q_2+1}} \psi_2^{q_2}\left(\frac{1}{n_k}\right) \cdot n_k^{q_2}.$$

Applying Lemma 4 we obtain

$$\sum_{n=n_{k-1}+1}^{n_k} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} \leqq C(\psi_2, q_2) \psi_2^{q_2}\left(\frac{1}{n_k}\right) \cdot n_k^{q_2}.$$

Using Lemma 20 and the obtained estimates, we can write

$$\sum_{n=2s_k}^{\infty} a_n^{q_2} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} =$$

$$= \sum_{n=s_k+1}^{\infty} a_n^{q_2} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} - \sum_{n=s_k+1}^{2s_k-1} a_n^{q_2} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} \geqq$$

$$\geqq \sum_{m=j_k+1}^{\infty} \sum_{n=n_{m-1}+1}^{n_m} a_n^{q_2} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} - \sum_{n=n_{j_k}+1}^{n_{j_k+1}} a_n^{q_2} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} =$$

$$= \sum_{m=j_k+1}^{\infty} \left[\frac{B_{n_m}}{n_m \psi_1\left(\frac{1}{n_m}\right)}\right]^{q_2} \sum_{n=n_{m-1}+1}^{n_m} \psi_2^{q_2}\left(\frac{1}{n}\right) n^{q_2-1} -$$

$$- \left[\frac{B_{n_{j_k+1}}}{n_{j_k+1} \psi_1\left(\frac{1}{n_{j_k+1}}\right)}\right]^{q_2} \sum_{n=n_{j_k}+1}^{n_{j_k+1}} \psi_2^{q_2}\left(\frac{1}{n}\right) \cdot n^{q_2-1} \geqq$$

$$\geqq C(q_2) \sum_{m=j_k+1}^{\infty} \left(\frac{B_{n_m} \psi_2\left(\frac{1}{n_m}\right)}{\psi_1\left(\frac{1}{n_m}\right)}\right)^{q_2} - C(\psi_2, q_2)\left[\frac{B n_{j_k+1} \psi_2\left(\frac{1}{n_{j_k+1}}\right)}{\psi_1\left(\frac{1}{n_{j_k}+1}\right)}\right]^{q_2} =$$

$$= C(q_2) \sum_{m=j_k}^{\infty} \left(\frac{B_{n_m} \psi_2\left(\frac{1}{n_m}\right)}{\psi_1\left(\frac{1}{n_m}\right)}\right)^{q_2} - C(q_2)\left[\frac{B_{n_{j_k}} \psi_2\left(\frac{1}{n_{j_k}}\right)}{\psi_1\left(\frac{1}{n_{j_k}}\right)}\right]^{q_2} -$$

$$-C(\psi_2, q_2)\left[\frac{B_{n_{j_k+1}} \psi_2\left(\frac{1}{n_{j_k+1}}\right)}{\psi_1\left(\frac{1}{n_{j_k+1}}\right)}\right]^{q_2} \geqq$$

$$\geqq C(\psi_1, \psi_2, q_2) \sum_{m=j_k}^{\infty} \sum_{n=n_{m-1}+1}^{n_m} \left(\frac{B_n \psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2} \cdot \frac{1}{n} -$$

$$-C(q_2)\left[\frac{\psi_2\left(\frac{1}{n_{j_k}}\right)\omega\left(\frac{1}{n_{j_k}}\right)}{\psi_1\left(\frac{1}{n_{j_k}}\right)}\right]^{q_2} - C(\psi_2, q_2)\left[\frac{\psi_2\left(\frac{1}{n_{j_k+1}}\right)\omega\left(\frac{1}{n_{j_k+1}}\right)}{\psi_1\left(\frac{1}{n_{j_k+1}}\right)}\right]^{q_2} \geqq$$

$$\geqq C(\psi_1, \psi_2, q_2) \cdot \sum_{n=n_{j_k-1}+1}^{\infty} \left(\frac{B_n \psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2} \cdot \frac{1}{n} -$$

$$-C(q_2)\lambda_{n_{j_k}}^{q_2} - C(\psi_2, q_2)\lambda_{n_{j_k+1}}^{q_2} \geqq C(\psi_1, \psi_2, q_2) \cdot \sum_{n=S_K}^{\infty} \left(\frac{B_n \psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2} \cdot \frac{1}{n} +$$

$$+C(\psi_1,\psi_2,q_2)\sum_{n=n_{j_k-1}+1}^{n_{j_k}-1}\left(\frac{B_n\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n}-$$

$$-C(\psi_2,q_2)\lambda_{S_k}^{q_2}\geqq C(\psi_1,\psi_2,q_2)\sum_{n=S_k}^{\infty}\left(\frac{B_n\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n}-$$

$$-C(\psi_2,q_2)\lambda_{S_k}^{q_2}\geqq C(\psi_1,\psi_2,q_2)\cdot\sum_{n=S_k}^{\infty}\left(\frac{\omega\left(\frac{1}{n}\right)\psi_2\left(\frac{1}{n}\right)}{\psi_1\left(\frac{1}{n}\right)}\right)^{q_2}\cdot\frac{1}{n}-$$

$$-C(\psi_1,\psi_2,q_2)\lambda_{S_k}^{q_2}-C(\psi_2,q_2)\lambda_{S_k}^{q_2}\geqq C(\psi_1,\psi_2,q_2)T_k\lambda_{S_k}^{q_2}-C(\psi_1,\psi_2,q_2)\lambda_{S_k}^{q_2}=$$

$$=\left(C(\psi_1,\psi_2,q_2)T_k-C(\psi_1,\psi_2,q_2)\right)\lambda_{S_k}^{q_2}.$$

Using this estimate and (13) we obtain

$$E_{S_k}^{q_2}(f)_{\psi_2,q_2}\geqq\left(C(\psi_1,\psi_2,q_2)T_k-C(\psi_1,\psi_2,q_2)\right)\lambda_{S_k}^{q_2},$$

where $T_k\uparrow\infty$ as $k\uparrow\infty$. This contradicts the assumption $H_{\psi_1,q_1}\subset E_{\psi_2,q_2}(\lambda)$. Thus Theorem 2 is completely proved.

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AN APPROXIMATION TO INFINITELY DIVISIBLE LAWS

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1. The approximation

One question Professor Tandori asked at my doctoral defense on February 2, 1972, was about infinite divisibility. Since he was satisfied, my answer probably included that, according to Lévy's formula ([9], p. 84), a distribution on the real line $\mathbf{R}$ is infinitely divisible if and only if its characteristic function $\varphi(t)$, $t \in \mathbf{R}$, is given by

$$\varphi(t) = \exp\left\{ i\theta t - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dL(x) + \right.$$

$$\left. + \int_{0}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dR(x) \right\},$$

where i is the imaginary unit, $\theta \in \mathbf{R}$ and $\sigma \geqq 0$ are constants, the function $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and the function $R(\cdot)$ is right-continuous and non-decreasing on $(0, \infty)$ with $R(\infty) = 0$, such that

$$\int_{-\varepsilon}^{0} x^2 dL(x) + \int_{0}^{\varepsilon} x^2 dR(x) < \infty \quad \text{for every} \quad \varepsilon > 0.$$

Little did I think at the time that I should be able to answer the question somewhat more thoroughly twenty-three years later. I hope he will like a few late details here.

For a given quadruple $\big(\theta, \sigma, L(\cdot), R(\cdot)\big)$ with the described properties, let $F_{\theta,\sigma,L,R}(\cdot)$ denote the corresponding distribution function, so that $\varphi(t) =$

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$= \int_{-\infty}^{\infty} e^{itx} dF_{\theta,\sigma,L,R}(x)$, $t \in \mathbf{R}$. Consider the inverse functions

$$\psi_L(u) := \inf\left\{x < 0 : L(x) > u\right\}$$

and

$$\psi_R(u) := \inf\{x < 0 : -R(-x) > u\}, \quad 0 < u < \infty,$$

where the infimum of the empty set is taken to be zero. These are non-decreasing, non-positive, right-continuous functions on the half-line $(0, \infty)$ such that

$$\text{(1.1)} \qquad \int_\varepsilon^\infty \psi_L^2(u)\,du + \int_\varepsilon^\infty \psi_R^2(u)\,du < \infty \quad \text{for every} \quad \varepsilon > 0.$$

Let $Y_1, Y_2, \ldots$ be independent exponentially distributed random variables with mean 1, so that $P\{Y_k > x\} = e^{-x}$, $x > 0$, $k \in \mathbf{N}$, and consider the corresponding partial sums $S_n := Y_1 + \cdots + Y_n$, $n \in \mathbf{N}$. Let Z be a standard normal random variable, let $\left\{S_n^{(L)}\right\}_{n=1}^\infty$ and $\left\{S_n^{(R)}\right\}_{n=1}^\infty$ be distributionally equivalent copies of the sequence $\{S_n\}_{n=1}^\infty$ such that the sequences $\left\{S_n^{(L)}\right\}_{n=1}^\infty$, $\left\{S_n^{(R)}\right\}_{n=1}^\infty$ and Z are independent. Now consider

$$\text{(1.2)} \quad V_n^{(M)} := \sum_{j=1}^n \psi_M\left(S_j^{(M)}\right) - \int_1^{S_n^{(M)}} \psi_M(u)\,du, \quad n \in \mathbf{N}, \quad M = L, R,$$

and the problem of approximating $F_{\theta,\sigma,L,R}(\cdot)$ by the distribution functions

$$F_{\theta,\sigma,L,R}^{n,m}(x) := P\left\{V_n^{(L)} + \sigma Z - V_m^{(R)} + \theta - \theta_L + \theta_R \leqq x\right\}, \quad x \in \mathbf{R}, \quad n, m \in \mathbf{N},$$

where

$$\text{(1.3)} \qquad \theta_M := \int_0^1 \frac{\psi_M(s)}{1 + \psi_M^2(s)}\,ds - \int_1^\infty \frac{\psi_M^3(s)}{1 + \psi_M^2(s)}\,ds, \quad M = L, R.$$

More precisely, we are interested in seeing how fast the Lévy distances $D_{n,m}(L, R)$ between $F_{\theta,\sigma,L,R}^{n,m}(\cdot)$ and $F_{\theta,\sigma,L,R}(\cdot)$, defined as

$$\inf\Big\{\varepsilon > 0 : F_{\theta,\sigma,L,R}^{n,m}(x - \varepsilon) - \varepsilon \leqq F_{\theta,\sigma,L,R}(x) \leqq F_{\theta,\sigma,L,R}^{n,m}(x + \varepsilon) + \varepsilon \quad \text{for all} \quad x \in \mathbf{R}\Big\},$$

go to zero as $n, m \to \infty$. As it turns out, this depends upon how fast the functions $\psi_L(u)$ and $\psi_R(u)$ approach zero as $u \to \infty$.

For $M = L$ and $M = R$ and any $a > 0$ consider

$$(1.4)\qquad \begin{cases} v_M(a) := \sqrt{\int_a^\infty \psi_M^2(u)\,du}, & \text{so that } \left|\psi_M(a)\right| \downarrow 0 \quad \text{and} \\ v_M(a) \downarrow 0 \quad \text{as} \quad a \uparrow \infty, \end{cases}$$

and for a fixed $1/2 \leqq d_M \leqq 2e^{(e-2)/2}$ choose a finite $a_M^* > 0$ so that

(1.5)

$$\begin{cases} \psi_M(a_M^*) = 0 \quad \text{if } \psi_M(a) = 0 \text{ for some } a > 0 \quad \text{and} \\ v_M(a_M^*) < \dfrac{1}{e^{2/e}} \text{ and } \left|\psi_M(a_M^*)\right| \leqq \dfrac{1}{e^{e^{e/2}/d_M}} \text{ if } \psi_M(a) < 0 \text{ for all } a > 0, \end{cases}$$

and, with log standing for the natural logarithm, for all $a \geqq a_M^*$ define

$$w_M(a) :=$$

$$:= \begin{cases} w_1^{(M)}(a) := \sqrt{\frac{e}{2}}\, v_M(a)\sqrt{\log \frac{1}{v_M(a)}}, & \text{if } \sqrt{\frac{2}{e}\log\frac{1}{v_M(a)}} \leqq \frac{v_M(a)}{|\psi_M(a)|}, \\ w_2^{(M)}(a) := \frac{1}{2}\left|\psi_M(a)\right| \log \frac{1}{v_M(a)|\psi_M(a)|}, & \text{if } \sqrt{\frac{2}{e}\log\frac{1}{v_M(a)}} > \frac{v_M(a)}{|\psi_M(a)|} \geqq 1, \\ w_3^{(M)}(a) := d_M\left|\psi_M(a)\right| \log \frac{1}{|\psi_M(a)|}, & \text{if } \frac{v_M(a)}{|\psi_M(a)|} < 1 < \sqrt{\frac{2}{e}\log\frac{1}{v_M(a)}}, \end{cases}$$

where, since $d_M \leqq 2e^{(e-2)/2}$, the second inequality in the specification of $w_3^{(M)}(a)$ is satisfied because $v_M(a) < \left|\psi_M(a)\right| \leqq e^{-e^{e/2}/d_M} \leqq e^{-e/2}$. While it is understood that $w_M(a) := 0$ if $\psi_M(a) = 0$, since otherwise $2w_2^{(M)}(a) \leqq \leqq \left|\psi_M(a)\right| \log\left(1/\left|\psi_M(a)\right|\right) + v_M(a)\log\left(1/v_M(a)\right)$, it is clear that $w_M(a) \to \to 0$ as $a \to \infty$, $M = L, R$. Finally, setting

$$(1.6)\qquad r_n^{(M)}(a) := \begin{cases} P\{S_n \leqq a\} + 2w_M(a), & \text{if } \psi_M(\cdot) \not\equiv 0 \text{ on } (0,\infty), \\ 0, & \text{if } \psi_M(\cdot) \equiv 0 \text{ on } (0,\infty) \end{cases}$$

for $M = L, R$ and $a \geqq a_M^*$, the main result is the following.

THEOREM. *If $a_L \geqq a_L^*$ and $a_R \geqq a_R^*$, then $D_{n,m}(L,R) \leqq r_n^{(L)}(a_L) + +r_m^{(R)}(a_R)$ for every $n, m \in \mathbf{N}$.*

It will be also clear from the proof (and will be followed in bracketed phrases) that in the case when $\psi_M(u) < 0$ for all $u > 0$, if $w_M(a) = w_1^{(M)}(a)$ for all $a \geqq \tilde{a}_M$ and $v_M(\bar{a}_M) \leqq e^{-2/e}$ for some $\tilde{a}_M, \bar{a}_M > 0$, then the choice

$a^*_M = \max(\tilde a_M, \bar a_M)$ is permissible, while if $w_M(a) = w_2^{(M)}(a)$ for all $a \geqq \tilde a_M$ and $|\psi_M(\bar a_M)| < 1$, $v_M(\bar a_M) < 1$ and the product $v_M(\bar a_M)|\psi_M(\bar a_M)| \leqq e^{-2}$ for some $\tilde a_M, \bar a_M > 0$, then again we may take $a^*_M = \max(\tilde a_M, \bar a_M)$, $M = L, R$. The constant d_M enters the threshold a^*_M as in (1.5) only if the case $w_M(a) = w_3^{(M)}(a)$ cannot be excluded for $M = L$ or $M = R$.

To use the theorem, one will choose two positive sequences $\{a_n^{(L)} : n \in \mathbf{N}\}$ and $\{a_n^{(R)} : n \in \mathbf{N}\}$ such that $\limsup_{n\to\infty} a_n^{(M)}/n < 1$, $M = L, R$, and obtain $D_{n,m}(L,R) \leqq r_n^{(L)}(a_n^{(L)}) + r_m^{(R)}(a_m^{(R)})$ for all n and m such that $a_n^{(L)} \geqq a^*_L$ and $a_m^{(R)} \geqq a^*_R$. For $a_n \equiv a_n^{(L)}$ or $a_n \equiv a_n^{(R)}$, the limsup condition is to force the gamma probabilities

$$P\{S_n \leqq a_n\} = \int_0^{a_n} \frac{x^{n-1}}{(n-1)!} e^{-x}\, dx$$

go to zero as $n \to \infty$. This convergence is the fastest if $a_n \equiv a$ for some $a \geqq a^*_L$ or $a \geqq a^*_R$, in which case an expansion of the incomplete gamma function ([8], p. 135) yields

$$(1.7)\quad P\{S_n \leqq a\} = \frac{a^n}{n!} e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{(n+1)(n+2)\cdots(n+k)} \leqq \frac{a^n}{n!}, \quad n \in \mathbf{N}.$$

On the other hand, $w_M(a_n) \to 0$ fast for $M = L$ or $M = R$ if $a_n \to \infty$ fast as $n \to \infty$. For the fastest possible sequence $a_n \equiv \tau n$, the elementary Lemma 3.1 in [7] gives

$$(1.8)\qquad P\{S_n \leqq \tau n\} \leqq e^{-(1-\tau)^2 n/2} \quad \text{whenever} \quad 0 < \tau < 1, \quad n \in \mathbf{N}.$$

In a concrete situation a trade-off between the opposing tendencies has to be found.

If the limiting infinitely divisible distribution function $F_{\theta,\sigma,L,R}(\cdot)$ is absolutely continuous with density $f_{\theta,\sigma,L,R}(\cdot)$ for which $K_{\theta,\sigma,L,R} := \sup\{f_{\theta,\sigma,L,R}(x) : x \in \mathbf{R}\} < \infty$, then by the theorem and a well-known inequality connecting the Kolmogorov and Lévy distances, for any two positive sequences $\{a_n^{(L)} : n \in \mathbf{N}\}$ and $\{a_n^{(R)} : n \in \mathbf{N}\}$ as above,

$$(1.9)\qquad \sup_{x\in\mathbf{R}} \left| F^{n,m}_{\theta,\sigma,L,R}(x) - F_{\theta,\sigma,L,R}(x)\right| \leqq$$
$$\leqq \left[1 + K_{\theta,\sigma,L,R}\right]\left[r_n^{(L)}(a_n^{(L)}) + r_m^{(R)}(a_m^{(R)})\right]$$

for all n and m such that $a_n^{(L)} \geqq a^*_L$ and $a_m^{(R)} \geqq a^*_R$.

Another general corollary is for the case when the Lévy measure of the underlying infinitely divisible distribution is finite, i.e. in our terminology, both $L(0-) < \infty$ and $R(0+) > -\infty$. In this case, $\psi_L(\cdot)$ is zero on the half-line $\big[L(0-), \infty\big)$ and $\psi_R(\cdot)$ is zero on the half-line $\big[-R(0+), \infty\big)$, and the theorem and (1.7) together yield

$$(1.10)\qquad D_{n,m}(L,R) \leqq \frac{\big[L(0-)\big]^n}{n!} + \frac{\big[-R(0+)\big]^m}{m!} \quad \text{for all} \quad n, m \in \mathbf{R}.$$

A result of the type of the theorem, though somewhat different in nature, was first proved by Hall [10] for the approximation of stable laws. A closer version was derived among other results in [2]. Stable laws are considered among the illustrative examples in Section 3, following the proof. The theorem above improves the main result in [3], where a special integrability condition was assumed on the functions ψ_L and ψ_R, restricting (1.1). The approach here differs from that in [3] in the realization that there is no point insisting on the deterministic centering $\int_1^n \psi_M(u)\,du$ instead of the present $\int_1^{S_n^{(M)}} \psi_M(u)\,du$ in $V_n^{(M)}$ in (1.2), $M = L, R$, and in the associated use of moment generating functions, rather than just moments, resulting in faster rates of approximation and no restriction on $L(\cdot)$ and $R(\cdot)$. As explained in [3], these approximations are made possible by a probabilistic representation of a random variable with a given, arbitrary infinitely divisible distribution, obtained in [1]. The sums $\sum_{j=1}^n \psi_L\big(S_j^{(L)}\big) \leqq 0$ in $V_n^{(L)}$ and $-\sum_{j=1}^m \psi_R\big(S_j^{(R)}\big) \geqq 0$ in $-V_m^{(R)}$ are to be viewed as the asymptotic contributions of fixed numbers, n and m, of the smallest and the largest terms in a sum of independent and identically distributed random variables in the domain of partial attraction of the infinitely divisible law given by the quadruple $\big(\theta, \sigma, L(\cdot), R(\cdot)\big)$. (For a recent discussion of such domains the reader is referred to [4].) Thus $V_n^{(L)}$ and $-V_m^{(R)}$ themselves are centered versions of these asymptotic contributions, presently with random centerings. This is why such approximations were called "extreme-sum approximations" in [3].

2. Proof of the theorem

On the same probability space $(\Omega, \mathcal{A}, P)$ where the random variables $V_{n,m} := V_{n,m}(\theta, \sigma, L, R) := V_n^{(L)} + \sigma Z - V_m^{(R)} + \theta - \theta_L + \theta_R$ are defined, and expressed in terms of the same independent sequences $\big\{S_n^{(L)}\big\}_{n=1}^\infty$, $\big\{S_n^{(R)}\big\}_{n=1}^\infty$ and Z, for a given quadruple (θ, σ, L, R) let $V := V(\theta, \sigma, L, R) := V_L + \sigma Z -$

$V_R + \theta - \theta_L + \theta_R$, where for the independent left-continuous Poisson processes

$$N_M(u) := \sum_{k=1}^{\infty} I\{\, S_k^{(M)} < u\}, \quad 0 \leqq u < \infty, \quad M = L, R,$$

with unit intensity, where $I\{\cdot\}$ is the indicator function,

$$V_M := \int_{S_1^{(M)}}^{\infty} [u - N_M(u)]\, d\psi_M(u) + \int_1^{S_1^{(M)}} u\, d\psi_M(u) + \psi_M(1), \quad M = L, R.$$

Then by Theorem 3 in [1], the distribution function of the variable $V = V(\theta, \sigma, L, R)$ is the function $F_{\theta,\sigma,L,R}(\cdot)$ to be estimated. Since $D_{n,m}(L,R) \leqq r_n^{(L)}(a_L) + r_m^{(R)}(a_R)$ if $P\{\, |V - V_{n,m}| > r_n^{(L)}(a_L) + r_m^{(R)}(a_R)\} \leqq \leqq r_n^{(L)}(a_L) + r_m^{(R)}(a_R)$, the inequality claimed in the theorem will follow if we show that $P\{\, \big| V_M - V_n^{(M)} \big| > r_n^{(M)}(a)\} \leqq r_n^{(M)}(a)$ holds for all $n \in \mathbf{N}$ and $a \geqq \geqq a_M^*$, for both $M = L$ and $M = R$. Dropping the indices in (1.4)–(1.6), i.e. setting $v^2(a) := \int_a^\infty \psi^2(u)\, du$, for some $1/2 \leqq d \leqq 2e^{(e-2)/2}$ choosing $a^* > 0$ so that

$$\text{(2.1)} \quad \begin{cases} \psi(a^*) = 0 \quad \text{if } \psi(a) = 0 \text{ for some } a > 0 \quad \text{and} \\[2ex] v(a^*) \leqq \dfrac{1}{e^{2/e}} \quad \text{and} \quad \big|\psi(a^*)\big| \leqq \dfrac{1}{e^{e^{e/2}/d}} \quad \text{if } \psi(a) < 0 \text{ for all } a > 0, \end{cases}$$

and for $a \geqq a^*$, with the same convention that $w(a) = 0$ if $\psi(a) = 0$, defining

(2.2)

$$w(a) := \begin{cases} w_1(a) := \sqrt{\frac{e}{2}}\, v(a) \sqrt{\log \frac{1}{v(a)}}, & \text{if } \sqrt{\frac{2}{e} \log \frac{1}{v(a)}} \leqq \frac{v(a)}{|\psi(a)|}, \\[1ex] w_2(a) := \frac{1}{2} \big|\psi(a)\big| \log \frac{1}{v(a)|\psi(a)|}, & \text{if } \sqrt{\frac{2}{e} \log \frac{1}{v(a)}} > \frac{v(a)}{|\psi(a)|} \geqq 1, \\[1ex] w_3(a) := d \big|\psi(a)\big| \log \frac{1}{|\psi(a)|}, & \text{if } \sqrt{\frac{2}{e} \log \frac{1}{v(a)}} > 1 > \frac{v(a)}{|\psi(a)|}, \end{cases}$$

for a non-decreasing, non-positive, right-continuous function $\psi(\cdot)$ on $(0, \infty)$, for which $v(a) < \infty$ for all $a > 0$, we have to show that for all $n \in \mathbf{N}$ and $a \geqq a^*$,

$$\text{(2.3)} \qquad P\{\, |\Delta_n| > r_n(a)\} \leqq r_n(a),$$

where

$$r_n(a) := \begin{cases} P\{S_n \leqq a\} + 2w(a), & \text{if } \psi(\cdot) \not\equiv 0, \\ 0, & \text{if } \psi(\cdot) \equiv 0, \end{cases}$$

and, using that the jump-points of the Poisson process $N(u) := \sum_{k=1}^{\infty} I\{S_k < u\}$, $0 \leqq u < \infty$, hit the possible discontinuity points of $\psi(\cdot)$ with probability zero,

$$\Delta_n := \int_{S_1}^{\infty} [u - N(u)]\, d\psi(u) + \int_1^{S_1} u\, d\psi(u) +$$

$$+\psi(1) - \sum_{j=1}^{n} \psi(S_j) + \int_1^{S_n} \psi(u)\, du =$$

$$= \int_{S_n}^{\infty} [u - N(u)]\, d\psi(u) + \sum_{j=1}^{n-1} \left\{ \int_{S_j}^{S_{j+1}} u\, d\psi(u) - j\left[\psi(S_{j+1}) - \psi(S_j)\right] \right\} +$$

$$+ \int_1^{S_1} u\, d\psi(u) + \psi(1) - \sum_{j=1}^{n} \psi(S_j) + \int_1^{S_n} \psi(u)\, du =$$

$$= \int_{S_n}^{\infty} [u - N(u)]\, d\psi(u) + \int_1^{S_n} u\, d\psi(u) + \int_1^{S_n} \psi(u)\, du - n\psi(S_n) + \psi(1) =$$

$$= \int_{S_n}^{\infty} [u - N(u)]\, d\psi(u) + \psi(S_n)\left[S_n - n\right]$$

almost surely. (Throughout the usual convention $\int_c^d \cdots d\psi := \int_{(c,d]} \cdots d\psi$ applies for all $0 < c < d < \infty$. The integral on the half-line (S_1, ∞) exists almost surely as an improper Riemann integral by (1.1), i.e. by the fact that $v(a) < \infty$ for all $a > 0$.)

If $\psi(u) = 0$ for all $u > 0$, there is in fact nothing to prove. (And here we have $P\{|\Delta_n| > 0\} = 0$ since $\Delta_n \equiv 0$.) If $\psi(\cdot) \not\equiv 0$ on $(0, \infty)$, two cases are distinguished. The trivial case is when $\psi(a^*) = 0$ and hence $\psi(v) = 0$ for all $v \geqq a^*$. In this case, $P\{|\Delta_n| > r_n(a)\} \leqq P\{S_n \leqq a\} + P\{|\Delta_n| > r_n(a), S_n > > a\} = P\{S_n \leqq a\}$ and $w(a) = 0$ for all $a \geqq a^*$, and so (2.3) follows with $r_n(a) = P\{S_n \leqq a\}$.

For the non-trivial case, suppose that $\psi(v) < 0$ for all $v > 0$. Fix $n \in \in \mathbf{N}$ and $a \geqq a^*$, and put $g_n(x) := x^{n-1}e^{-x}/(n-1)!$, $x > 0$, for the density function of S_n. By the definition of $r_n(a)$ in (2.3) and by Markov's inequality we have

$$(2.4) \qquad P\{\,|\Delta_n| > r_n(a)\} \leqq P\{S_n \leqq a\} + P\{\,\Delta_n \geqq 2w(a), S_n > a\} +$$

$$+P\{\,-\Delta_n \geqq 2w(a), S_n > a\} \leqq$$

$$\leqq P\{S_n \leqq a\} + e^{-2sw(a)} E\left(e^{s\Delta_n} I\{S_n > a\}\right) + e^{-2tw(a)} E\left(e^{-t\Delta_n} I\{S_n > a\}\right) =$$

$$= P\{S_n \leqq a\} + e^{-2sw(a)} \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{s\psi(v)} - 1 - s\psi(v) \right] dv \right\} g_n(x)\, dx +$$

$$+ e^{-2tw(a)} \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{-t\psi(v)} - 1 + t\psi(v) \right] dv \right\} g_n(x)\, dx$$

for every $s > 0$ and every $t > 0$, where the last equation for the restricted moment generating functions follows by a slight modification of the first part of the proof of Theorem 4 in [1]. Actually, the slight modification is just the trivial one to account for the restrictive presence of the indicators. Indeed, that taken for granted and setting

$$\Delta_n^* := \int_{S_n}^\infty [v - N(v)]\, d\psi(v) + \int_1^{S_n} v\, d\psi(v) +$$

$$+ \int_1^n \psi(v)\, dv - (n-1)\psi(S_n) + \psi(1),$$

Theorem 4 in [1] directly gives (replacing the *it* there by u) that for all $u \in \mathbf{R}$,

$$E\left(e^{u\Delta_n^*} I\{S_n > a\}\right) = \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{u\psi(v)} - 1 - \frac{u\psi(v)}{1 + \psi^2(v)} \right] dv + u\psi(v) + \right.$$

$$\left. + u \int_{1+x}^n \psi(v)\, dv + u \int_x^{1+x} \frac{\psi(v)}{1+\psi^2(v)}\, dv - u \int_{1+x}^\infty \frac{\psi^3(v)}{1+\psi^2(v)}\, dv \right\} g_n(x)\, dx =$$

$$= \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{u\psi(v)} - 1 - u\psi(v) \right] dv + u\psi(x) + u \int_x^n \psi(v)\, dv \right\} g_n(x)\, dx,$$

where the second equation is by straightforward algebra. Hence for

$$\Delta_n^* - \psi(S_n) - \int_{S_n}^n \psi(v)\, dv = \int_{S_n}^\infty [v - N(v)]\, d\psi(v) + \int_1^{S_n} v\, d\psi(v) +$$

$$+ \int_1^{S_n} \psi(v)\, dv - n\psi(S_n) + \psi(1) =$$

$$= \int_{S_n}^\infty [v - N(v)]\, d\psi(v) + S_n\psi(S_n) - n\psi(S_n) = \Delta_n$$

we clearly obtain

$$E\left(e^{u\Delta_n} I\{S_n > a\}\right) = \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{u\psi(v)} - 1 - u\psi(v) \right] dv \right\} g_n(x)\, dx,$$

$$u \in \mathbf{R},$$

proving (2.4), where the integrals on the right may or may not be finite at this stage.

To estimate the integrands there, we use the inequality that if $c \geqq 0$ is a constant, then $e^u - 1 - u \leqq e^c u^2/2$ for all $-\infty < u \leqq c$. For the first integral in (2.4), we have $-\infty < s\psi(v) < 0$ for all $v \geqq a$, so that

$$e^{s\psi(v)} - 1 - s\psi(v) \leqq \frac{s^2}{2}\psi^2(v), \quad v \geqq a, \quad \text{for every} \quad s > 0.$$

For the second, since the negative function $\psi(\cdot)$ is non-decreasing, we obviously have $0 < -t\psi(v) = t|\psi(v)| \leqq |\psi(v)|/|\psi(a)| \leqq 1$ whenever $0 < t \leqq \leqq 1/|\psi(a)|$ and $v \geqq a$, so that

$$e^{-t\psi(v)} - 1 + t\psi(v) \leqq \frac{e\,t^2}{2}\psi^2(v), \quad v \geqq a, \quad \text{for every} \quad 0 < t \leqq \frac{1}{|\psi(a)|}.$$

Hence, moving down x to a in the integrals in both exponents, from (2.4) we obtain

$$\text{(2.5)} \qquad P\{|\Delta_n| > r_n(a)\} \leqq$$

$$\leqq P\{S_n \leqq a\} + \exp\left\{\frac{s^2}{2}v^2(a) - 2sw(a)\right\} + \exp\left\{\frac{et^2}{2}v^2(a) - 2tw(a)\right\}$$

for all $s > 0$ and $0 < t \leqq 1/|\psi(a)|$.

Using (2.2), for all choices of $a \geqq a^*$ for which $w(a) = w_2(a)$ we have

$$w_2(a) > \frac{1}{2}\sqrt{\frac{e}{2}}\,\frac{v(a)}{\sqrt{\log\frac{1}{v(a)}}}\log\frac{1}{v^2(a)} = \sqrt{\frac{e}{2}}\,v(a)\sqrt{\log\frac{1}{v(a)}} = w_1(a).$$

Also, for all $a \geqq a^*$ for which $w(a) = w_3(a)$ the choice of a^* in (2.1) forces

$$x := \frac{1}{v(a)} > \frac{1}{|\psi(a)|} \geqq e^{e^{e/2}/d} > e^{e/d} \geqq e^{\frac{e}{2d^2}} \quad \text{since}$$

$$v(a) < |\psi(a)| \quad \text{and} \quad d \geqq \frac{1}{2}.$$

This implies that $\sqrt{\log x} > \sqrt{e/2}/d$ or, what is of course the same, $\sqrt{e/2}/ / (d\sqrt{\log x}) < 1$ and, consequently, $\sqrt{e/2}\,x/(d\log x) < x/\sqrt{\log x}$. So,

$$\frac{1}{d}\sqrt{\frac{e}{2}}\,\frac{y}{\log y} < \frac{1}{d}\sqrt{\frac{e}{2}}\,\frac{x}{\log x} \leqq \frac{x}{\sqrt{\log x}}$$

whenever $y := 1/|\psi(a)| < x$, since the function $y/\log y$, $y > 0$, is increasing on the half-line $[e, \infty)$ and $y = 1/|\psi(a)| > e$ by the choice of a^* and the upper bound on d. But by (2.2) the inequality $y = 1/|\psi(a)| < 1/v(a) = x$ is equivalent to $w(a) = w_3(a)$. Thus, if $a \geqq a^*$ and $w(a) = w_3(a)$, then $\sqrt{e/2}\, y/\log y < d\, x/\sqrt{\log x}$ or, what is the same,

$$\sqrt{\frac{e}{2}}\,\frac{1}{|\psi(a)|\log\frac{1}{|\psi(a)|}} < d\,\frac{1}{v(a)\sqrt{\log\frac{1}{v(a)}}}, \quad \text{that is,} \quad w_1(a) < w_3(a).$$

(We see that $w_2(a) > w_1(a)$ whenever $v(a) < 1$, $|\psi(a)| < 1$ and $w(a) = w_2(a)$.) For reference purposes the foregoing may be summarized by saying that whenever $a \geqq a^*$,

$$\text{(2.6)} \qquad \text{if} \quad w(a) = w_j(a), \quad \text{then} \quad w_j(a) > w_1(a), \quad j = 2, 3.$$

Consider the convex function $f_a(s) := \frac{s^2}{2}\, v^2(a) - 2sw(a)$, $s > 0$. Then $f_a(\cdot)$ is negative on the interval $\big(0, 4w(a)/v^2(a)\big)$ and takes its minimum at $s_* = 2w(a)/v^2(a)$. Hence, choosing $s = s_*$ and using (2.6) twice, the second term of the bound in (2.5) is

$$\exp\{f_a(s_*)\} = \exp\left\{-\frac{2w^2(a)}{v^2(a)}\right\} \leqq \exp\left\{-\frac{2w_1^2(a)}{v^2(a)}\right\} = \exp\left\{-e\log\frac{1}{v(a)}\right\} <$$
$$< v(a) \leqq w_1(a) \leqq w(a).$$

The inequality before the last holds since $v(a) \leqq e^{-2/e}$ for all $a \geqq a^*$ by (2.1).

The convex function $h_a(t) := \frac{et^2}{2}\, v^2(a) - 2tw(a)$, $t > 0$, is also negative on the interval $\big(0, [4w(a)]/[ev^2(a)]\big)$ and takes its minimum at the point $t_* := [2w(a)]/[ev^2(a)]$. However, here we also have to satisfy the constraint $0 < t \leqq 1/|\psi(a)|$. So, choosing $t_\diamond := \min\{1/|\psi(a)|, t_*\}$, the third term of the bound in (2.5) becomes $\exp\{h_a(t_\diamond)\}$. Let $a \geqq a^*$. If $w(a) = w_1(a)$, so that

$$0 < t_* = \frac{2\,w_1(a)}{e\,v^2(a)} = \sqrt{\frac{2}{e}}\,\frac{\sqrt{\log\frac{1}{v(a)}}}{v(a)} \leqq \frac{1}{|\psi(a)|},$$

we have (whenever $v(a) \leqq e^{-2/e}$ as above)

$$\exp\{h_a(t_\diamond)\} = \exp\{h_a(t_*)\} = \exp\left\{\log\frac{1}{v(a)} - 2t_*w_1(a)\right\} =$$
$$= \exp\left\{\log\frac{1}{v(a)} - 2\log\frac{1}{v(a)}\right\} = v(a) \leqq w_1(a) = w(a).$$

If $w(a)=w_j(a)$, $j=2,3$, then by (2.6) again,

$$t_* = \frac{2\,w_j(a)}{e\,v^2(a)} > \frac{2\,w_1(a)}{e\,v^2(a)} = \sqrt{\frac{2}{e}}\;\frac{\sqrt{\log\frac{1}{v(a)}}}{v(a)} > \frac{1}{|\psi(a)|}, \qquad j=2,3.$$

Hence if $w(a)=w_2(a)$, then (whenever $v(a)<1$, $|\psi(a)|<1$ and $v(a)|\psi(a)| \leqq \leqq e^{-2}$)

$$\exp\{h_a(t_\diamond)\} = \exp\{h_a(1/|\psi(a)|)\} =$$
$$= \exp\left\{\frac{e}{2}\,\frac{v^2(a)}{\psi^2(a)} - \frac{2w_2(a)}{|\psi(a)|}\right\} < \frac{1}{v(a)}\exp\left\{-\frac{2w_2(a)}{|\psi(a)|}\right\} =$$
$$= \frac{1}{v(a)}\exp\left\{-\log\frac{1}{v(a)|\psi(a)|}\right\} = |\psi(a)| \leqq w_2(a) = w(a)$$

by the choice of a^*, while if $w(a)=w_3(a)$, then

$$\exp\{h_a(t_\diamond)\} = \exp\{h_a(1/|\psi(a)|)\} =$$
$$= \exp\left\{\frac{e}{2}\,\frac{v^2(a)}{\psi^2(a)} - \frac{2w_3(a)}{|\psi(a)|}\right\} < e^{e/2}\exp\left\{-\frac{2w_3(a)}{|\psi(a)|}\right\} =$$
$$= e^{e/2}\,|\psi(a)|^{2d} \leqq e^{e/2}\,|\psi(a)| \leqq d\,|\psi(a)|\log\frac{1}{|\psi(a)|} = w_3(a) = w(a)$$

since $2d \geqq 1$ and $|\psi(a)| \leqq 1/e^{e^{e/2}/d}$ by the choice of a^*. Therefore, the inequality $\exp\{h_a(t_\diamond)\} < w(a)$ holds for all $a \geqq a^*$.

Thus if $a \geqq a^*$, then the bound in (2.5) is less than $P\{S_n \leqq a\} + 2w(a) = r_n(a)$. This fact establishes (2.3) in the non-trivial case, and hence the theorem. (The collection of bracketed phrases also establishes the remark concerning the choice of the thresholds.)

3. Examples

The first three examples show, in particular, that all three versions of the rate function provided by the three branches of $w_M(\cdot)$, $M=L,R$, defined between (1.5) and (1.6), may in fact occur. For simplicity of exposition, we deal with spectrally one-sided infinitely divisible distributions, that is, we choose $L(\cdot)\equiv 0$, with the exception of the stable and compound Poisson examples. The last four examples are of interest in their own right, the negative binomial being weird enough to deserve attention in any case. In Examples 2–4, the threshold remark beneath the theorem is used without further notice.

EXAMPLE 1. If $L \equiv 0$ and $\psi_R(u) = -e^{-u}$, $u > 0$, then $w_L(a) = 0$, $v_R(a) = e^{-a}/\sqrt{2}$ and $w_R(a) = w_3^{(R)}(a) = d\,a\,e^{-a}$ for all $a > 0$ for $v_R(a)/ /|\psi_R(a)| = 1/\sqrt{2} < 1$. Regardless of $\theta \in \mathbf{R}$ and $\sigma \geqq 0$, for the corresponding Lévy distance the theorem and (1.8), with $a_n \equiv (2-\sqrt{3})\,n$, give

$$D_{n,n}(0,R) \leqq \frac{3d(2-\sqrt{3})\,n}{\exp\{(2-\sqrt{3})n\}} \quad \text{for all} \quad n \geqq \frac{\max\left(\frac{e^{e/2}}{d}, \frac{2}{e} - \log\sqrt{2}\right)}{2-\sqrt{3}}$$

and each fixed $1/2 \leqq d \leqq 2e^{(e-2)/2} = 2.86419\ldots$. For $d = 2e^{(e-2)/2}$ this holds for all $n \geqq 6$ and for $d = 1/2$ the inequality is true for all $n \geqq 30$.

EXAMPLE 2. If $L \equiv 0$ and $\psi_R(u) = -\sqrt{u}\,e^{-u/2}$, then $w_L(a) = 0$, $v_R(a) = \sqrt{a+1}\,e^{-a/2}$ and $w_R(a) = w_2^{(R)}(a) = 2^{-1}\sqrt{a}\,e^{-a/2}\log\left(e^a[a^2+a]^{-1/2}\right)$ for all $a \geqq 2/(e-2) = 2.78442\ldots$, say. Regardless of $\theta \in \mathbf{R}$ and $\sigma \geqq 0$, for the corresponding Lévy distance the theorem and (1.8), with $a_n \equiv (3-\sqrt{5})\,n/2$, give

$$D_{n,n}(0,R) \leqq \frac{3\sqrt{(3-\sqrt{5})\,n}}{2^{3/2}\exp\left\{\frac{3-\sqrt{5}}{4}n\right\}} \log \frac{\exp\left\{\frac{3-\sqrt{5}}{2}n\right\}}{\sqrt{\frac{3-\sqrt{5}}{2}n}\,\sqrt{1+\frac{3-\sqrt{5}}{2}n}}$$

$$\text{for all} \quad n \geqq 5 > \frac{3.35}{3-\sqrt{5}}.$$

EXAMPLE 3: Stable laws. Let $F_{\alpha,\beta,\eta,\zeta}(\cdot)$ be the distribution function of a non-normal stable law with exponent $0 < \alpha < 2$, given by its characteristic function

$$\int_{-\infty}^{\infty} e^{itx}\,dF_{\alpha,\beta,\eta,\zeta}(x) =$$

$$= \begin{cases} \exp\{i\zeta t - \eta|t|^\alpha[1 - i\beta\,\mathrm{sgn}(t)\tan(\alpha\pi/2)]\}, & \text{if } \alpha \neq 1, \\ \exp\{i\zeta t - \eta|t|^\alpha[1 + i\beta\,\mathrm{sgn}(t)\frac{2}{\pi}\log|t|]\}, & \text{if } \alpha = 1, \end{cases}$$

with skewness, scale and location parameters $-1 \leqq \beta \leqq 1$, $\eta > 0$ and $\zeta \in \mathbf{R}$, where $\mathrm{sgn}(t)$ is the sign function, $t \in \mathbf{R}$. In Lévy's canonical form at the beginning of the paper, this is given by some $\theta = \theta(\alpha,\beta,\eta,\zeta)$, $\sigma = 0$ and $L(\cdot)$ and $R(\cdot)$ functions such that $\psi_M(u) = -c_M u^{-1/\alpha}$, $u > 0$, where $c_M = c_M(\alpha,\beta,\eta,\zeta) \geqq 0$ are some constants, $M = L, R$, such that $c_L(\alpha,1,\eta,\zeta) = 0$ and $c_L(\alpha,\beta,\eta,\zeta) > 0$ for every $-1 \leqq \beta < 1$, while $c_R(\alpha,-1,\eta,\zeta) = 0$ and $c_R(\alpha,\beta,\eta,\zeta) > 0$ for every $-1 < \beta \leqq 1$; cf. [9], [1], [4]. Setting $K_{\alpha,\beta,\eta,\zeta} := \sup\{f_{\alpha,\beta,\eta,\zeta}(x) : x \in \mathbf{R}\} < \infty$ for the corresponding density function

$f_{\alpha,\beta,\eta,\zeta}(\cdot) := F'_{\alpha,\beta,\eta,\zeta}(\cdot)$ and $\vartheta = \vartheta(\alpha,\beta,\eta,\zeta) := \theta - \theta_L + \theta_R$, where θ_L and θ_R are given through (1.3), let $F^{n,m}_{\alpha,\beta,\eta,\zeta}(x) := P\{V_n^{(L)} - V_m^{(R)} + \vartheta \leqq x\}$, $x \in \mathbf{R}$, where, in the present situation $V_n^{(M)}$ of (1.2), for $M = L, R$ and $n \in \mathbf{N}$, is given by

$$V_n^{(M)} = \begin{cases} -c_M \sum_{j=1}^n \left(S_j^{(M)}\right)^{-1/\alpha} + \frac{\alpha c_M}{\alpha-1}\left(S_n^{(M)}\right)^{\frac{\alpha-1}{\alpha}} - \frac{\alpha c_M}{\alpha-1}, & \text{if } \alpha \neq 1, \\ -c_M \sum_{j=1}^n \left(S_j^{(M)}\right)^{-1} + c_M \log S_n^{(M)}, & \text{if } \alpha = 1. \end{cases}$$

Elementary calculation shows that $v_M(a) = \sqrt{\alpha/(2-\alpha)}\, c_M\, a^{-(2-\alpha)/(2\alpha)}$, $a >$ > 0, and

$$w_M(a) = w_1^{(M)}(a) = \frac{\sqrt{e}}{2}\,\frac{c_M}{a^{\frac{1}{\alpha}-\frac{1}{2}}}\sqrt{\log\left(a\left[c_M\sqrt{\alpha/(2-\alpha)}\right]^{-2\alpha/(2-\alpha)}\right)}$$
$$\text{for all} \quad a \geqq a_M^*$$

if $c_M > 0$, where, putting $\rho := \alpha/(2-\alpha)$, $u_M := \big(2/(\rho e)\big)\log\big(1/(c_M\sqrt{\rho})\big)$ and $v_M := 1/(2\rho^2)$, the threshold a_M^* may be chosen as $a_M^* =$ $= \max\big(\rho^\rho c_M^{2\rho} e^{4\rho/e}, a_M^\circ\big)$, where a_M° is the smallest positive number such that $a \geqq u_M + v_M \log a$ for all $a \geqq a_M^\circ$, $M = L, R$. Picking now any $\tau \in (0,1)$ in (1.8) and letting $n_M^* := \max\big(a_M^*/\tau, n_M^\circ\big)$, where n_M° is the smallest $n \in \mathbf{N}$ for which $\exp\{-(1-\tau)^2 n/2\} \leqq w_1^{(M)}(\tau n)$, $M = L, R$, the inequality in (1.9) gives that for all $n \geqq n_L^*$ and $m \geqq n_R^*$,

$$\sup_{x\in\mathbf{R}} \left| F^{n,m}_{\alpha,\beta,\eta,\zeta}(x) - F_{\alpha,\beta,\eta,\zeta}(x)\right| \leqq 3\left[1 + K_{\alpha,\beta,\eta,\zeta}\right]\left[w_1^{(L)}(\tau n) + w_1^{(R)}(\tau m)\right].$$

Neglecting thresholds and constants, the qualitative meaning of this is that

$$\sup_{x\in\mathbf{R}} \left| F^{n,m}_{\alpha,\beta,\eta,\zeta}(x) - F_{\alpha,\beta,\eta,\zeta}(x)\right| = O\left(c_L\frac{\sqrt{\log n}}{n^{\frac{1}{\alpha}-\frac{1}{2}}} + c_R\frac{\sqrt{\log m}}{m^{\frac{1}{\alpha}-\frac{1}{2}}}\right)$$
$$\text{as} \quad n, m \to \infty.$$

Improving Theorem 2.2 in [2], the latter rate has also been established in Remark 1.3 of Janssen and Mason [11] by completely different methods.

EXAMPLE 4: Limiting St. Petersburg distributions. In a classical St. Petersburg game, a player gains 2^k ducats with probability 2^{-k}, $k \in \mathbf{N}$. As determined in [5], the class $\{G_\gamma(\cdot) : 1/2 < \gamma \leqq 1\}$ of all possible non-degenerate subsequential limiting types of distribution functions for the cumulative gains of a player in a sequence of independent St. Petersburg games, under any deterministic centering and norming, is described by the family of infinitely divisible characteristic functions

$$\int_{-\infty}^{\infty} e^{itx} dG_\gamma(x) = \exp\left\{ i\theta_\gamma t + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dR_\gamma(x) \right\}, \quad t \in \mathbf{R},$$

where, with Log standing for the logarithm to the base 2 and, for any $y \in \mathbf{R}$, with $\lfloor y \rfloor$ denoting the greatest integer not greater than y, having a fractional part $\langle y \rangle = y - \lfloor y \rfloor$,

$$\theta_\gamma := \sum_{k=1}^{\infty} \frac{\gamma^2}{\gamma^2 + 4^k} - \sum_{k=0}^{\infty} \frac{1}{1 + \gamma^2 4^k} - \operatorname{Log} \gamma$$

and

$$R_\gamma(x) = -\gamma\, 2^{-\lfloor \operatorname{Log}(\gamma x) \rfloor}, \quad x > 0,$$

so that $\psi_\gamma(u) := \psi_{R_\gamma}(u) = -2^{-\lfloor \operatorname{Log}(u/\gamma) \rfloor}/\gamma$, $u > 0$. Hence by lengthier but elementary and quite delicate computation through (1.2) and (1.3),

$$W_n^{(\gamma)} := -V_n^{(R_\gamma)} + \theta_\gamma + \theta_{R_\gamma} = \frac{1}{\gamma} \sum_{j=1}^{n} \frac{1}{2^{\lfloor \operatorname{Log}(S_j/\gamma) \rfloor}} - \operatorname{Log} S_n + \delta\left(\frac{S_n}{\gamma} \right),$$

where $\delta(s) = 1 + \langle \operatorname{Log} s \rangle - 2^{\langle \operatorname{Log} s \rangle}$, $s > 0$, and it can be seen in similarly elementary fashion that $0 \leqq \delta(s) \leqq 1 - (1 + \log\log 2)/\log 2 = 0.08607\ldots$ for all $s > 0$. (The function $\delta(\cdot)$ plays a special role in the theory of the St. Petersburg game, described in [6], and the present example has some motivational value at some point there.) Also, since $1/u \leqq |\psi_\gamma(u)| < 2/u$ for all $u > 0$, for the corresponding $v_\gamma^2(a) := \int_a^\infty \psi_\gamma^2(u)\, du$ we obtain $1/\sqrt{a} \leqq v_\gamma(a) < \sqrt{2}/\sqrt{a}$ for every $a > 0$ and $1/2 < \gamma \leqq 1$. Thus we have $v_\gamma(a) < e^{-2/e}$ if $a \geqq a^* := := 2e^{4/e} = 8.71168\ldots$ and $\sqrt{a}/2 < v_\gamma(a)/|\psi_\gamma(u)| \leqq \sqrt{2a}$, so

$$w_{R_\gamma}(a) = w_1^{(R_\gamma)}(a) = \sqrt{\frac{e}{2}}\, v_\gamma(a) \sqrt{\log\left[1/v_\gamma(a)\right]} < \sqrt{\frac{e}{2}}\, \frac{\sqrt{\log a}}{\sqrt{a}} =: w_1(a)$$

for all $a > 1$ and all $1/2 < \gamma \leqq 1$. Since the densities $g_\gamma(\cdot) = G_\gamma'(\cdot)$ exist and it can be shown that $\sup_{1/2 < \gamma \leqq 1} \sup\{ g_\gamma(x) : x \in \mathbf{R}\} \leqq 1/2$, for any $0 < \tau < 1$ in (1.8), finally (1.9) yields

$$\sup_{\frac{1}{2} < \gamma \leqq 1} \sup_{x \in \mathbf{R}} \left| P\{ W_n^{(\gamma)} \leqq x \} - G_\gamma(x) \right| < C(\tau) \frac{\sqrt{\log n}}{\sqrt{n}} \quad \text{for all} \quad n \geqq n^*(\tau),$$

where the bound is a trivial upper bound for $3w_1(\tau n)$ with $C(\tau) := 9\sqrt{e/(2\tau)}/2$ and $n^*(\tau) := \lceil \max\left(2e^{4/e}/\tau, n_*(\tau)\right) \rceil$, where $\lceil x \rceil := \min\{k \in \mathbf{N} : k \geqq x\}$, $x > 0$, and $n_*(\tau) := \min\{k \in \mathbf{N} : \exp\{-(1-\tau)^2 k\} \leqq w_1^2(\tau k)\}$. Here, rounding up, $C(1) = 5.24620$ is unachievable, and we get $C(0.707) = 6.23929$, $n^*(0.707) = 13$; $C(0.8) = 5.86543$, $n^*(0.8) = 53$; $C(0.9) = 5.52998$, $n^*(0.9) = 376$; $C(0.95) = 5.38249$, $n^*(0.95) = 2\,107$; $C(0.99) = 5.27263$, $n^*(0.99) = 86\,177$ and $C(0.999) = 5.24883$, $n^*(0.999) = 13\,297\,850$.

EXAMPLE 5: Compound Poisson laws. Let $N_\lambda, X_1, X_2, \ldots$ be independent random variables such that N_λ has the Poisson distribution on the integers $\{0,1,2,\ldots\}$ with mean $\lambda > 0$ and $X_1, X_2, \ldots$ have the same distribution function $G(x) := P\{X \leqq x\}$, $x \in \mathbf{R}$. Then Lévy's canonical form of the characteristic function of the infinitely divisible compound Poisson distribution function $F_{\lambda,G}(x) := P\{\sum_{k=1}^{N_\lambda} X_k \leqq x\}$, $x \in \mathbf{R}$, is given by $\sigma = 0$ and, with $G_-(\cdot)$ denoting the left-continuous version of $G(\cdot)$,

$$\theta = \theta_{\lambda,G} = \lambda \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dG(x), \quad L(x) = \lambda G_-(x),\ x < 0,$$

$$R(x) = \lambda\left[G(x) - 1\right],\ x > 0.$$

Hence, letting $G_+^{-1}(\cdot)$ denote the right-continuous version of the left-continuous generalized inverse $G^{-1}(s) := \inf\{x \in \mathbf{R} : G(x) \geqq s\}$, $0 < s < 1$, the usual quantile function, pertaining to $G(\cdot)$, we have

$$\psi_L(u) = \begin{cases} G_+^{-1}\left(\frac{u}{\lambda}\right), & \text{if } 0 < u < \lambda G_-(0), \\ 0, & \text{if } u \geqq \lambda G_-(0), \end{cases}$$

and

$$\psi_R(u) = \begin{cases} -G^{-1}\left(1 - \frac{u}{\lambda}\right), & \text{if } 0 < u < \lambda[1 - G(0)], \\ 0, & \text{if } u \geqq \lambda[1 - G(0)]. \end{cases}$$

For the Lévy distance $D_{n,m}(\lambda, G)$ between $F_{\lambda,G}(\cdot)$ and its approximation $F_{\lambda,G}^{n,m}(x) := P\{V_n^{(L)} - V_m^{(R)} + \theta_{\lambda,G} - \theta_L + \theta_R \leqq x\}$, $x \in \mathbf{R}$, given by the present $\psi_L(\cdot)$ and $\psi_R(\cdot)$ through (1.2) and (1.3), by (1.10) we obtain

$$D_{n,m}(\lambda, G) \leqq \frac{\left[\lambda G_-(0)\right]^n}{n!} + \frac{\left[\lambda\{1 - G(0)\}\right]^m}{m!} \quad \text{for all} \quad n, m \in \mathbf{N}.$$

The Poisson law itself, with mean λ, is the special case when $G(x)$, $x \in \mathbf{R}$, degenerates at the point $x = 1$ and $\theta_\lambda := \theta_{\lambda,G} = \lambda/2$ for the corresponding quantity. In this case, $\psi_L(u) = 0$ and $\psi_R(u) = -I\{u < \lambda\}$ for all $u > 0$, and enjoyable calculation shows that $-V_n^{(R)} + \theta_\lambda + \theta_R = \sum_{j=1}^n I\{S_j < \lambda\} + (\lambda - S_n)I\{S_n < \lambda\}$ for every $n \in \mathbf{N}$. If $D_n(\lambda)$ denotes the Lévy distance between the distribution function $F_{n,\lambda}(\cdot)$ of the latter random variable and the Poisson distribution function $F_\lambda(x) := e^{-\lambda} \sum_{k=1}^{\lfloor x \rfloor} \lambda^k/k!$, $x \in \mathbf{R}$, of N_λ, with an empty sum understood as zero as above, then the result reduces to the inequality $D_n(\lambda) \leqq \lambda^n/n!$ for all $n \in \mathbf{N}$. Furthermore, if $D_n^*(\lambda)$ is the Lévy distance between $F_\lambda(\cdot)$ and $F_{n,\lambda}^*(x) := P\{\sum_{j=1}^n I\{S_j < \lambda\} \leqq x\}$, $x \in \mathbf{R}$, then a trivial extra step based on the triangle inequality for a Lévy distance yields $D_n^*(\lambda) \leqq 2\,\lambda^n/n!$ for all $n \in \mathbf{N}$.

EXAMPLE 6: Negative binomial distributions. For a fixed order $\ell \in \mathbf{N}$ and success probability $0 < p < 1$, consider the negative binomial distribution function

$$F_{\ell,p}(x) := P\{V_\ell(p) \leqq x\} := p^\ell \sum_{k=\ell}^{\lfloor x \rfloor} \binom{k-1}{\ell-1} q^{k-\ell}, \quad x \in \mathbf{R},$$

where $q := 1 - p$. As is well known, it is infinitely divisible and it is a routine exercise to show that the Lévy form of the characteristic function is

$$\int_{-\infty}^{\infty} e^{itx} dF_{\ell,p}(x) = \exp\left\{ i\theta_{\ell,p} t + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dR_{\ell,p}(x) \right\}, \quad t \in \mathbf{R},$$

where

$$\theta_{\ell,p} = \ell + \ell \sum_{m=1}^{\infty} \frac{q^m}{1+m^2} \quad \text{and} \quad R_{\ell,p}(x) = \ell \sum_{m=1}^{\lfloor x \rfloor} \frac{q^m}{m} + \ell \log p, \quad x > 0.$$

So, noting that $-\ell \log p = \ell \sum_{m=1}^\infty q^m/m$,

$$\psi_{\ell,p}(u) := \psi_{R_{\ell,p}}(u) = -\sum_{k=1}^{\infty} k\, I\left\{ \log \frac{1}{p^\ell} - \ell \sum_{m=1}^{k} \frac{q^m}{m} \leqq u < \log \frac{1}{p^\ell} - \ell \sum_{m=1}^{k-1} \frac{q^m}{m} \right\},$$
$$u > 0,$$

thus $\psi_{\ell,p}(u) = 0$ for all $u \geqq -\ell \log p$. Evaluating (1.2) and (1.3) with this, the result is

$$W_n^{\ell,p} := -V_n^{(R_{\ell,p})} + \theta_{\ell,p} + \theta_{R_{\ell,p}} =$$

$$= \ell + \sum_{j=1}^{n} \sum_{k=1}^{\infty} k\, I\Big\{ \ell \sum_{m=k+1}^{\infty} \frac{q^m}{m} \leqq S_j < \ell \sum_{m=k}^{\infty} \frac{q^m}{m} \Big\} +$$

$$+ \sum_{k=1}^{\infty} k \Big[\ell \sum_{m=k}^{\infty} \frac{q^m}{m} - S_n \Big] I\Big\{ \ell \sum_{m=k+1}^{\infty} \frac{q^m}{m} \leqq S_n < \ell \sum_{m=k}^{\infty} \frac{q^m}{m} \Big\} =: \ell + T_n^{\ell,p} + R_n^{\ell,p}.$$

(Note that $P\{W_n^{\ell,p} = \ell\} = P\{\ell + T_n^{\ell,p} = \ell\} = p^\ell = P\{V_\ell(p) = \ell\}$.) If now $D_n(\ell,p)$ is the Lévy distance between $F_{\ell,p}(\cdot)$ and $P\{W_n^{\ell,p} \leqq \cdot\}$ and $D_n^*(\ell,p)$ is the Lévy distance between $F_{\ell,p}(\cdot)$ and $P\{\ell + T_n^{\ell,p} \leqq \cdot\}$, then (1.10) and an extra step as above yield

$$D_n(\ell,p) \leqq \frac{[-\ell \log p]^n}{n!} \quad \text{and} \quad D_n^*(\ell,p) \leqq 2\, \frac{[-\ell \log p]^n}{n!} \quad \text{for all} \quad n \in \mathbf{N}.$$

If $\ell = 1$, this is of course a result for the approximation of the geometric distribution function $F_{1,p}(x) = p \sum_{k=1}^{\lfloor x \rfloor} q^{k-1}$, $x \in \mathbf{R}$.

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ON THE RESULTANT OF FORCES

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To Professor K. Tandori on his seventieth birthday

1. Introduction. Let $\oplus$ be a binary operation on $\mathbf{R}^n$ satisfying the following conditions:

(1) $\oplus$ is commutative and associative;
(2) $a \oplus b = a + b$ whenever $a, b \in \mathbf{R}^n$ and b is a scalar multiple of a;
(3) $(Aa) \oplus (Ab) = A(a \oplus b)$ whenever $a, b \in \mathbf{R}^n$ and A is a rotation of $\mathbf{R}^n$ (that is, an orthogonal transformation with determinant 1).

D'Alembert proved in 1769 that if $n = 3$ and $\oplus$ is continuous, then (1), (2) and (3) imply that $a \oplus b = a + b$ holds for every $a, b \in \mathbf{R}^3$. D'Alembert used this result to demonstrate that the resultant of forces is obtained by the vectorial sum of the components. (See [1], Chapters 1 and 8. For more on the history of the problem or on d'Alembert's work in particular, we refer to [3] and [5].)

In this note our first aim is to give a complete description of all operations on $\mathbf{R}^n$ ($n \geqq 3$) that satisfy (1), (2) and (3). As a corollary we show that, for $n \geqq 3$, any operation satisfying (1), (2) and (3) must also satisfy the following condition:

(3*) $(Aa) \oplus (Ab) = A(a \oplus b)$ whenever $a, b \in \mathbf{R}^n$ and A is an orthogonal transformation of $\mathbf{R}^n$.

We shall also prove that, again for $n \geqq 3$, the condition of continuity in d'Alembert's theorem can be replaced by the following weaker condition:

(4) there is a $\delta > 0$ such that the set $\{a \oplus b : |a| = |b| < \delta\}$ is not everywhere dense in $\mathbf{R}^n$.

In the plane the situation is different. We shall prove that for $n = 2$ the conditions (1), (2) and (3) do not imply (3*). Moreover, on $\mathbf{R}^2$ there are noncontinuous operations satisfying (1), (2), (3*) and (4). As we shall see, among those operations on $\mathbf{R}^2$ that satisfy (1), (2) and (3*), condition (4) is equivalent to the triangle inequality $|a \oplus b| \leqq |a| + |b|$, and we shall describe all operations satisfying these conditions.

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2. The case $n \geqq 3$. Let f be an automorphism of $\mathbf{R}$ as an additive group; that is, let f be a bijection of $\mathbf{R}$ onto itself satisfying Cauchy's functional equation $f(x+y)=f(x)+f(y)$ $(x,y\in\mathbf{R})$. For every unit vector $u\in\mathbf{R}^n$ and $x\in\mathbf{R}$ we define $\phi(xu)=f(x)\cdot u$. Since f is odd, we have $f(-x)\cdot(-u)=f(x)\cdot u$, and thus the definition of ϕ makes sense. Clearly, ϕ is a bijection of $\mathbf{R}^n$ onto itself. We shall say that the operation

$$a\oplus b=\phi^{-1}\big(\phi(a)+\phi(b)\big) \tag{5}$$

is generated by the automorphism f.

THEOREM 1. *Every operation generated by an automorphism of* $\mathbf{R}$ *satisfies conditions* (1), (2), *and* (3*). *Conversely, if* $n\geqq 3$ *and the operation* $\oplus$ *satisfies conditions* (1), (2), (3), *then* $\oplus$ *is generated by an automorphism of* $\mathbf{R}$.

PROOF. Let $\oplus$ be an operation on $\mathbf{R}^n$ generated by an automorphism f of $\mathbf{R}$. It is clear that $\oplus$ satisfies condition (1). If $a=xu$, $b=yu$ $(u\in\mathbf{R}^n$, $|u|=1$, $x,y\in\mathbf{R})$, then

$$a\oplus b=\phi^{-1}\big(f(x)u+f(y)u\big)=\phi^{-1}\big(f(x+y)u\big)=$$
$$=\phi^{-1}\big(\phi((x+y)u)\big)=(x+y)u=a+b,$$

proving (2). Let A be an orthogonal transformation of $\mathbf{R}^n$. We prove that ϕ and A commute. Indeed, if $a=xu$ $(u\in\mathbf{R}^n$, $|u|=1$, $x\in\mathbf{R})$, then $|Au|=1$ and $Aa=x(Au)$. Thus

$$A\big(\phi(a)\big)=A\big(f(x)u\big)=f(x)\cdot Au=\phi(Aa).$$

Therefore

$$(Aa)\oplus(Ab)=\phi^{-1}\big(\phi(Aa)+\phi(Ab)\big)=$$
$$=\phi^{-1}\big(A(\phi(a))+A(\phi(b))\big)=\phi^{-1}\big(A(\phi(a)+\phi(b))\big)=$$
$$=\phi^{-1}\big(A(\phi(a\oplus b))\big)=\phi^{-1}\big(\phi(A(a\oplus b))\big)=A(a\oplus b),$$

and hence (3*) holds.

Now suppose $n\geqq 3$ and let $\oplus$ be an operation on $\mathbf{R}^n$ satisfying (1), (2) and (3). First we show that $a\oplus b$ is always a linear combination of a and b. We may assume that a and b are linearly independent, since otherwise the statement follows from (2). We may also suppose that a and b belong to the subspace $V=\{(x_1,\dots,x_n): x_i=0\ (3\leqq i\leqq n)\}$. Let $A(x_1,\dots,x_n)=(-x_1,-x_2,x_3,\dots,x_n)$; then A is a rotation such that $Ax=-x$ if and only if $x\in V$. Let $c=a\oplus b$. Then, by (3), $(-a)\oplus(-b)=Ac$ and hence it follows from (1) and (2) that $c \oplus Ac = a \oplus b \oplus (-a) \oplus (-b) = 0$. Thus, using (1) and (2) again, we obtain $Ac = -c$. Thus $c \in V$, so that c is a linear combination of a and b, as we stated.

Let the perpendicular unit vectors u_0 and v_0 be fixed. Then $u_0 \oplus (xv_0)$ is a linear combination of u_0 and v_0 for every $x \in \mathbf{R}$, and hence there are functions $\alpha, \beta : \mathbf{R} \to \mathbf{R}$ such that $u_0 \oplus (xv_0) = \alpha(x)u_0 + \beta(x)v_0$ for every $x \in \mathbf{R}$. Then $\alpha(x) \neq 0$, because $u_0 \oplus (xv_0) = \beta v_0$ implies, by (2), $u_0 = (\beta - x)v_0$, which is impossible. We define $f(x) = \beta(x)/\alpha(x)$ $(x \in \mathbf{R})$. Then, by (3), $u \oplus (xv) = \alpha(x)(u + f(x)v)$ holds whenever u and v are perpendicular unit vectors and $x \in \mathbf{R}$. (We remark that the definition of the function f is due to G. Darboux, who gave an independent proof of d'Alembert's theorem in [4]. Darboux also proved, using a geometric argument, that f satisfies Cauchy's functional equation. See also [6], pp. 4–6.) The function f is odd. Indeed,

$$\alpha(-x)(u + f(-x)v) = u \oplus \big((-x)v\big) = u \oplus \big(x(-v)\big) = \alpha(x)\big(u + f(x)(-v)\big),$$

and hence $\alpha(-x) = \alpha(x)$ and $f(-x) = -f(x)$.

We show that f is injective. First note that, by (1) and (2), $-(a \oplus b) = (-a) \oplus (-b)$ for every $a, b \in \mathbf{R}^n$. Suppose that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$. If u and v are perpendicular unit vectors, then $u \oplus (x_i v) = \alpha(x_i)(u + yv)$ $(i = 1, 2)$ and thus, by (2),

$$(x_1 - x_2)v = \big(u \oplus (x_1 v)\big) \oplus \big(-(u \oplus (x_2 v))\big) =$$

$$= \big(u \oplus (x_1 v)\big) + \big(-(u \oplus (x_2 v))\big) = \big(\alpha(x_1) - \alpha(x_2)\big)\big(u \oplus (yv)\big).$$

Since $x_1 \neq x_2$, we have $(x_1 - x_2)v \neq 0$ and hence $\alpha(x_1) - \alpha(x_2) \neq 0$. Let $\beta = (x_1 - x_2)/\big(\alpha(x_1) - \alpha(x_2)\big)$, then $\beta v = u \oplus (yv)$, $(\beta - y)v = u$, which is impossible.

Let u and v be unit vectors with $u \neq \pm v$, and let $x, y \in \mathbf{R}$. Since $n \geqq 3$, there is a unit vector w that is perpendicular to both u and v. Then $w \oplus (xu) = \alpha(x)(w + f(x)u)$, and hence $\big(w \oplus (xu)\big) \oplus (yv)$ is a linear combination of $w + f(x)u$ and v. Let

$$\big(w \oplus (xu)\big) \oplus (yv) = \lambda(w + f(x)u) + \mu v$$

and, similarly,

$$\big(w \oplus (yv)\big) \oplus (xu) = \lambda'(w + f(y)v) + \mu' u,$$

where $\lambda, \lambda', \mu, \mu' \in \mathbf{R}$. Since the left hand sides are equal, this implies $\lambda' = \lambda$, $\mu' = \lambda f(x)$, and thus

$$w \oplus \big((xu) \oplus (yv)\big) = \lambda(w + f(x)u + f(y)v).$$

On the other hand, if $(xu) \oplus (yv) = rz$, where $r \in \mathbf{R}$, $r \geqq 0$, $z \in \mathbf{R}^n$ and $|z| = 1$, then z is perpendicular to w and hence

$$w \oplus \big((xu) \oplus (yv)\big) = \alpha(r)(w + f(r)z).$$

This gives $\alpha(r) = \lambda$ and $f(r)z = f(x)u + f(y)v$. That is, for every $x, y \in \mathbf{R}$ and linearly independent unit vectors $u, v \in \mathbf{R}^n$, we have

$$(6) \qquad f(r)z = f(x)u + f(y)v \quad \big((xu) \oplus (yv) = rz,\ r \geqq 0,\ |z| = 1\big).$$

We show that $f(\mathbf{R}) = \mathbf{R}$. Let u and v be as above. Then, by (6), $\big|f\big(|(xu) \oplus (yv)|\big)\big| = \big|f(x)u + f(y)v\big|$ for every $x, y \in \mathbf{R}$. With $x = y$ this gives $\big|f(x)\big| \cdot |u + v| = \big|f\big(|(xu) \oplus (xv)|\big)\big|$, and hence $\big|f(x)\big| \cdot |u + v|$ belongs to $R\big(|f|\big)$, the range of $|f|$. If v runs through all unit vectors different from $\pm u$, then $|u + v|$ runs through the interval $(0, 2)$, and hence $\big(0, 2|f(x)|\big) \subset \subset R\big(|f|\big)$ for every $x \in \mathbf{R}$. This gives $R\big(|f|\big) = [0, \infty)$, since $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$. As f is an odd function, we have $f(\mathbf{R}) = R\big(|f|\big) \cup \cup \big(-R\big(|f|\big)\big) = \mathbf{R}$. Thus f is a bijection of $\mathbf{R}$ onto itself.

Let $\phi(xu) = f(x)u$ ($x \in \mathbf{R}$, $u \in \mathbf{R}^n$, $|u| = 1$). Since f is odd, this definition makes sense. Also, ϕ is a bijection of $\mathbf{R}^n$ onto itself. If $|u| = |v| = 1$, $v \neq \pm u$, and $x, y \in \mathbf{R}$, then we have, by (6),

$$\phi\big((xu) \oplus (yv)\big) = f(x)u + f(y)v = \phi(xu) + \phi(yv).$$

This proves (5) if a and b are linearly independent.

Now we prove that f is additive. Let $x_1, x_2, y \in \mathbf{R} \setminus \{0\}$ and let $u, v \in \mathbf{R}^n$, $|u| = |v| = 1$, $v \neq \pm u$. Then $x_1 u$ and $(x_2 u) \oplus (yv)$ are linearly independent, and hence

$$(x_1 u) \oplus \big((x_2 u) \oplus (yv)\big) = \phi^{-1}\big(\phi(x_1 u) + \phi((x_2 u) \oplus (yv))\big) =$$
$$= \phi^{-1}\big(\phi(x_1 u) + \phi(x_2 u) + \phi(yv)\big) = \phi^{-1}\big(f(x_1)u + f(x_2)u + f(y)v\big).$$

On the other hand,

$$\big((x_1 + x_2)u\big) \oplus (yv) = \phi^{-1}\big(\phi\big((x_1 + x_2)u\big) + \phi(yv)\big) =$$
$$= \phi^{-1}\big(f(x_1 + x_2)u + f(y)v\big).$$

This gives $f(x_1 + x_2) = f(x_1) + f(x_2)$ for every $x_1, x_2 \neq 0$, and thus f is additive.

If a and b are scalar multiples of each other, say $a = xu$, $b = yu$, then

$$\phi\big((xu) \oplus (yv)\big) = \phi\big((x + y)u\big) = f(x + y)u =$$
$$= \big(f(x) + f(y)\big)u = \phi(xu) + \phi(yu),$$

and hence (5) holds in this case as well. Therefore, $\oplus$ is generated by f, and the proof is complete. □

COROLLARY 2. *If* $\oplus$ *is a binary operation on* $\mathbf{R}^n$ $(n \geqq 3)$ *and if* $\oplus$ *satisfies* (1), (2), *and* (3), *then* $\oplus$ *also satisfies* (3*).

THEOREM 3. *If* $n \geqq 3$ *and* $\oplus$ *is a binary operation on* $\mathbf{R}^n$ *that satisfies* (1), (2), (3) *and* (4), *then* $a \oplus b = a + b$ *holds for every* $a, b \in \mathbf{R}^n$.

PROOF. By Theorem 1, $\oplus$ is generated by an automorphism f of $\mathbf{R}$. We show that f is linear. Let e and u be unit vectors enclosing an acute angle, and let v denote the reflection of u about the line of e. Then for every $x > 0$ we have

$$(xu) \oplus (xv) = \phi^{-1}\big(f(x)u + f(x)v\big) =$$

$$= \phi^{-1}\big(f(x)|u+v| \cdot e\big) = f^{-1}\big(f(x)|u+v|\big) \cdot e.$$

The function f^{-1} is also additive. If f^{-1} is not linear, then its range over any interval is dense in $\mathbf{R}$. Let $x > 0$ be fixed, and let u run through all unit vectors enclosing an acute angle with e. Then $f(x)|u+v|$ runs through a non-degenerate interval, and hence the set of numbers $f^{-1}\big(f(x)|u+v|\big)$ is dense in $\mathbf{R}$. Therefore the set of vectors $(xu) \oplus (xv)$ is dense in the line $\{\lambda e : \lambda \in \mathbf{R}\}$, and, rotating the vector e it follows that the set $\{a \oplus b : |a| = |b| = x\}$ is everywhere dense in $\mathbf{R}^n$. Since $x > 0$ was arbitrary, this contradicts (4), and hence f^{-1} and f must be linear. If $f(x) = \lambda x$ for every $x \in \mathbf{R}$ for some $0 \neq \lambda \in \mathbf{R}$, then $\phi(u) = \lambda u$ for every $u \in \mathbf{R}^n$, and $a \oplus b = (\lambda a + \lambda b)/\lambda = a + b$ for every $a, b \in \mathbf{R}^n$. □

3. The case $n = 2$. In the sequel we shall identify $\mathbf{R}^2$ with the set $\mathbf{C}$ of complex numbers. We shall use the notation $e(x) = e^{2\pi i x}$ $(x \in \mathbf{R})$. Let U denote the circle group $\{u \in \mathbf{C} : |u| = 1\}$, and let χ be a bijection of U onto itself. If $r \in \mathbf{R}$, $r \geqq 0$ and $u \in U$ then we define

$$\phi(ru) = r\chi(u).$$

Clearly, ϕ is a bijection of $\mathbf{C}$ onto iteself. If the binary operation $\oplus$ is defined by

$$a \oplus b = \phi^{-1}\big(\phi(a) + \phi(b)\big) \quad (a, b \in \mathbf{C}),$$

then we say that $\oplus$ is generated by the bijection χ.

LEMMA 4. *If the operation* $\oplus$ *is generated by the bijection* χ *then* $\oplus$ *satisfies* (1) *and the triangle inequality* $|a \oplus b| \leqq |a| + |b|$. *If* χ *is odd, that is* $\chi(-u) = -\chi(u)$ *for every* $u \in U$, *then* $\oplus$ *also satisfies* (2).

PROOF. It is clear that $\oplus$ is commutative and associative. The triangle inequality follows from $|\phi(z)| = |z|$ $(z \in \mathbf{C})$. If χ is odd then $\phi(ru) = r\chi(u)$

holds for every $r \in \mathbf{R}$ and $u \in U$. Indeed, if $r \geqq 0$ then this is the definition of $\phi(ru)$. If $r < 0$ then we have $\phi(ru) = \phi\big((-r)(-u)\big) = (-r)\chi(-u) = r\chi(u)$, as we stated. Now, if b is a scalar multiple of a then $a = ru$ and $b = su$ where $r, s \in \mathbf{R}$ and $u \in U$. Therefore

$$\phi(a+b) = \phi\big((r+s)u\big) = (r+s)\chi(u) = r\chi(u) + s\chi(u) = \phi(a) + \phi(b),$$

and hence (2) holds. □

THEOREM 5. *There exists an operation on* $\mathbf{R}^2$ *that satisfies* (1), (2), *and* (3), *but does not satisfy* (3*).

PROOF. Let

$$\chi\big(e(x)\big) = \begin{cases} e(x) & \text{if } x \in [0,1/4) \cup [1/2,3/4) \\ -e(x) & \text{if } x \in [1/4,1/2) \cup [3/4,1); \end{cases}$$

then χ is a bijection of U onto itself such that $\chi(-u) = -\chi(u)$ $(u \in U)$. If $\otimes$ denotes the operation generated by χ, then $\otimes$ satisfies (1) and (2) by Lemma 4.

Let μ denote the (unique) rotation invariant normalized Borel measure on U. It is clear that for every $a, b \in \mathbf{C}$ the function $u \mapsto u^{-1}\big((ua) \otimes (ub)\big)$ is bounded and measurable on U. We define

$$a \oplus b = \int_U u^{-1}\big((ua) \otimes (ub)\big)\, d\mu(u) \qquad (a, b \in \mathbf{C}).$$

It is easy to check that $\oplus$ also satisfies (1) and (2). By the invariance of μ we have

$$(va) \oplus (vb) = \int_U u^{-1}\big((uva) \otimes (uvb)\big)\, d\mu(u) =$$

$$= v \int_U (uv)^{-1}\big((uva) \otimes (uvb)\big)\, d\mu(u) =$$

$$= v \int_U u^{-1}\big((ua) \otimes (ub)\big)\, d\mu(u) = v(a \oplus b)$$

for every $v \in U$ and hence $\oplus$ satisfies (3), too.

We show that (3*) fails for $\oplus$. Let $\phi(ru) = r\chi(u)$ for $r \in \mathbf{R}$ and $u \in U$. Let a and b be perpendicular vectors. Then, by the definition of χ, we have either $\phi(a) = a$ and $\phi(b) = -b$ or $\phi(a) = -a$ and $\phi(b) = b$, and hence

$$a \otimes b = \phi^{-1}\big(\phi(a) + \phi(b)\big) = \phi^{-1}\big(\pm(b-a)\big) = \pm(b-a).$$

Since ua and ub are also perpendicular for every $u \in U$, this implies

$$u^{-1}\big((ua)\otimes(ub)\big) = u^{-1}\big(\pm(ub-ua)\big) = \pm(b-a).$$

If $\{u \in U : u^{-1}\big((ua)\otimes(ub)\big) = b-a\} = V$, then

$$a \oplus b = \mu(V)(b-a) + \mu(U\setminus V)\big(-(b-a)\big) = \lambda(b-a),$$

where λ is real. Now suppose that (3*) holds for $\oplus$. Let a and b be perpendicular unit vectors and let A denote the reflection about the line of $a+b$. Then $Aa = b$ and $Ab = a$, so that $Aa \oplus Ab = b \oplus a = \lambda(b-a)$ and $A(a\oplus b) = A\big(\lambda(b-a)\big) = \lambda(a-b)$. If these vectors are equal then $\lambda = 0$, $a \oplus b = 0$ and $b = -a$, which is impossible. □

The rest of the paper will be devoted to the characterization of those operations that are generated by an automorphism of U. A map $\chi : U \to U$ is said to be an automorphism of U if χ is a bijection of U onto itself and $\chi(uv) = \chi(u)\chi(v)$ holds for every $u, v \in U$.

THEOREM 6. *If the operation $\oplus$ is generated by an automorphism of U, then $\oplus$ satisfies* (1), (2), (3*) *with $n = 2$, as well as the triangle inequality* $|a \oplus b| \leqq |a| + |b|$ $(a, b \in \mathbf{R}^2)$.

PROOF. Let $\oplus$ be generated by the automorphism χ. Then χ is odd and hence, by Lemma 4, $\oplus$ satisfies (1), (2) and the triangle inequality.

Let O_2 denote the set of isometries of the plane leaving the origin fixed. A map belongs to O_2 if it is a rotation about the origin, or a reflection about a line going through the origin. We prove that if $A \in O_2$ then $\phi \circ A \circ \phi^{-1} \in O_2$. Indeed, if A is rotation, then $Az = cz$ $(z \in \mathbf{C})$ for some $c \in U$ and hence

$$\phi\big(A(\phi^{-1}(ru))\big) = \phi\big(A(r\chi^{-1}(u))\big) = \phi\big(rc\chi^{-1}(u)\big) =$$
$$= r\chi\big(c\chi^{-1}(u)\big) = \chi(c)ru.$$

Therefore $\phi \circ A \circ \phi^{-1}$ is the rotation $z \mapsto \chi(c)z$. If A is a reflection then $Az = c\overline{z}$ $(z \in \mathbf{C})$, where $c \in U$ and $\overline{z}$ denotes the complex conjugate of z. Since $\chi(\overline{u}) = \chi(1/u) = 1/\chi(u) = \overline{\chi(u)}$ for every $u \in U$, we have

$$\phi\big(A(\phi^{-1}(ru))\big) = \phi\big(A(r\chi^{-1}(u))\big) = \phi\big(rc\overline{\chi^{-1}(u)}\big) =$$
$$= \phi\big(rc\chi^{-1}(\overline{u})\big) = r\chi\big(c\chi^{-1}(\overline{u})\big) = \chi(c)r\overline{u}.$$

Therefore, in this case, $\phi \circ A \circ \phi^{-1}$ is also a reflection.

Now let $A \in O_2$ be arbitrary, and put $B = \phi \circ A \circ \phi^{-1} \in O_2$. Then for every $a, b \in \mathbf{C}$ we have

$$(Aa) \oplus (Ab) = \phi^{-1}\big(\phi(Aa) + \phi(Ab)\big) = \phi^{-1}\left(B\big(\phi(a)\big) + B\big(\phi(b)\big)\right) =$$
$$= \phi^{-1}\left(B\big(\phi(a) + \phi(b)\big)\right) = \phi^{-1}\left(B\big(\phi(a \oplus b)\big)\right) = A(a \oplus b)$$

which proves (3*). □

We remark that if the automorphism χ is not continuous, then the generated operation $\oplus$ is not continuous either. Indeed, if $u \in U$ then $1 \oplus u = \phi^{-1}\big(\chi(1) + \chi(u)\big) = \phi^{-1}\big(1 + \chi(u)\big)$ and hence $|1 \oplus u| = \big|1 + \chi(u)\big|$. This implies that $|1 \oplus u|$ is not a continuous function of $u \in U$. In this way we have constructed noncontinuous operations satisfying (1), (2), (3*), (4) and even the triangle inequality.

Our next theorem shows that the automorphisms of U generate all operations satisfying (1), (2), (3*) and (4).

THEOREM 7. *Let $\oplus$ be a binary operation defined on $\mathbf{R}^2$ and suppose that $\oplus$ satisfies* (1), (2), *and* (3*) *with $n = 2$. Then the following are equivalent.*

(i) *$\oplus$ is generated by an automorphism of U;*

(ii) *$|a \oplus b| \leqq |a| + |b|$ for every $a, b \in \mathbf{R}^2$;*

(iii) *there is a $\delta > 0$ such that the set $\{a \oplus b : |a| = |b| < \delta\}$ is not everywhere dense in $\mathbf{R}^2$.*

PROOF. We have already proved (i) $\Longrightarrow$ (ii). Since (ii) $\Longrightarrow$ (iii) is obvious, we only have to prove (iii) $\Longrightarrow$ (i).

Let $\oplus$ be a binary operation defined on $\mathbf{R}^2$ and satisfying (1), (2), (3*), and (iii). Let $a, b, c \in \mathbf{R}^2$, $|a| = |b| = |c| = 1$ and $a + b = rc$, where $r \geqq 0$. Let A denote the reflection about the line of c. Then it follows from (3*) that, for every $t \in \mathbf{R}$, $(ta) \oplus (tb)$ is parallel to c; moreover, $(ta) \oplus (tb) = sc$, where $s \in \mathbf{R}$ only depends on t and on the angle of a and b. If the angle between a and b is $2x$ $(0 \leqq x \leqq \pi/2)$, then we denote $H(t, x) = s/2$, where $(ta) \oplus (tb) = sc$, $s \in \mathbf{R}$. It easily follows from (1) and (2) that, for every fixed x, the function $t \mapsto H(t, x)$ is additive. If this function is not linear then its range on the interval $(0, \delta)$ is dense in $\mathbf{R}$ for every $\delta > 0$. This implies that the set of points $\{(ta) \oplus (tb) : 0 < t < \delta\}$ is dense in the line $\{\lambda c : \lambda \in \mathbf{R}\}$. Thus, rotating a and b simultaneously, it follows that the set $\{u \oplus v : |u| = |v| < \delta\}$ is everywhere dense in $\mathbf{R}^2$, contradicting (iii). Therefore $t \mapsto H(t, x)$ is linear for every $x \in [0, \pi/2]$; that is, there exists a function $h : [0, \pi/2] \to \mathbf{R}$ such that $H(t, x) = h(x) \cdot t$ for every $t \in \mathbf{R}$ and $x \in [0, \pi/2]$. Clearly, $h(0) = 1$ and $h(\pi/2) = 0$. We extend h to $\mathbf{R}$ as follows: we put $h(\pi - x) = -h(x)$ $(x \in [0, \pi/2])$, $h(-x) = h(x)$ $(x \in [0, \pi])$, and from $[-\pi, \pi]$ we extend h periodically. Following a well-known argument due to d'Alembert (see [1] pp. 4–5), we find

$$h(x+y)+h(x-y)=2h(x)h(y) \quad (x,y\in \mathbf{R}). \tag{7}$$

Strictly speaking, this argument only proves (7) if $0\leqq y\leqq x\leqq \pi/2$; however, using the extension of h, it is easy to check that it holds also for every $x,y\in \in \mathbf{R}$. This implies $h(x)=2h^2(x/2)-1\geqq -1$ and hence $h\geqq -1$ everywhere. On the other hand, $h\big(x+(\pi/2)\big)+h\big(x-(\pi/2)\big)=0$, and thus $\big|h(x)\big|\leqq 1$ for every x.

Next we prove that $\oplus$ satisfies (ii). If $|a|=|b|=t$, then it follows from the definition of H and h that $|a\oplus b|=\big|2h(x)t\big|\leqq 2t=|a|+|b|$ (where the angle between a and b is $2x$). Now let $a,b\in \mathbf{R}^2$ be arbitrary, and put $c=a\oplus \oplus b$. To prove (ii) we may assume $c\neq 0$. Let a' and b' denote the reflections of a and b about the line of c. Then, as we saw above, $|a\oplus a'|\leqq 2|a|$ and $|b\oplus b'|\leqq 2|b|$. Also, we have $a'\oplus b'=c$ by (3*) and thus, using (2) and (1) we obtain

$$2c=(a\oplus b)+(a'\oplus b')=(a\oplus b)\oplus(a'\oplus b')=$$
$$=(a\oplus a')\oplus(b\oplus b')=(a\oplus a')+(b\oplus b'),$$

since $(a\oplus a')$ and $(b\oplus b')$ are also scalar multiples of c. This gives $2|c|\leqq |a\oplus \oplus a'|+|b\oplus b'|\leqq 2|a|+2|b|$, proving (ii).

Since the function h satisfies the functional equation (7), and not every value of h is ± 1, it follows from a theorem of J. A. Baker that there are constants $p\in \mathbf{R}$ and $k\in \mathbf{C}$ such that the function $G(x)=h(x)+k\big(h(x+ +p)-h(x-p)\big)$ has the following properties: $G(x+y)=G(x)G(y)$ for every $x,y\in \mathbf{R}$, and $h(x)=\big(G(x)+G(-x)\big)/2$ for every $x\in \mathbf{R}$ (see Theorem 2 and its proof in [2], pp. 412-413, or Theorem 16 and its proof in [1], Chapter 13, pp. 220-222). As h is bounded and periodic mod 2π, so is G. Since $h\not\equiv 0$, we have $G\not\equiv 0$ and hence, taking into account that G is bounded, it follows that $|G|\equiv 1$. Thus, there is a function $g:\mathbf{R}\to\mathbf{R}$ such that $G(2\pi x)=e\big(g(x)\big)$ for every $x\in \mathbf{R}$. Since G is periodic mod 2π, g may be taken as periodic mod 1. Also, $G(x+y)=G(x)G(y)$ implies $g(x+y)\equiv g(x)+g(y)$ (mod 1); that is, we may consider g as a group homomorphism from the additive group $\mathbf{R}$ into the torus $T=\mathbf{R}/\mathbf{Z}$. Since $g(-x)=-g(x)$, we have

$$h(2\pi x)=\frac{G(2\pi x)+G(-2\pi x)}{2}=\frac{e\big(g(x)\big)+e\big(-g(x)\big)}{2}=\cos\big(2\pi g(x)\big)$$
$$(x\in \mathbf{R}).$$

Since $-1=h(\pi)=\cos\big(2\pi g(1/2)\big)$, we have $g(1/2)=1/2$, and hence $g(1/4)=1/4$ or $3/4$. We may assume that $g(1/4)=1/4$, since otherwise we replace g by $1-g$. We define $\chi\big(e(x)\big) = e\big(g(x)\big)$ $(x \in \mathbf{R})$. Our aim is to prove that χ is an automorphism of U and that $\oplus$ is generated by χ.

First we show that $g(x) = 0$ implies $x \in \mathbf{Z}$. Since $a \oplus b = 0 \iff b = -a$, it follows from the definition of H and h that $h(x) = 0$ if and only if $x = (\pi/2) + k\pi$ $(k \in \mathbf{Z})$. This implies that if $h(x) = -1$ then $2h^2(x/2) = h(x) + 1 = 0$ and thus $x/2 = (\pi/2) + k\pi$, $x = (2k+1)\pi$. Now, if $g(x) = 0$ then $g\big(x + (1/2)\big) = 1/2$, and $h\big(2\pi(x + (1/2))\big) = \cos\big(2\pi g(x + (1/2))\big) = \cos\pi = -1$. Thus $2\pi\big(x + (1/2)\big) = (2k+1)\pi$ and hence $x \in \mathbf{Z}$. Since g is additive, this implies that g is injective on $[0,1)$.

Our next aim is to show that if c and d are perpendicular vectors with $|c| = |d|$, then

$$(8)\qquad \big(\cos\big(2\pi g(x)\big)\cdot c\big) \oplus \big(\sin\big(2\pi g(x)\big)\cdot d\big) = (\cos 2\pi x)\cdot c + (\sin 2\pi x)\cdot d$$

for every $x \in \mathbf{R}$. Suppose first $0 \leqq x \leqq 1/4$, and let u and v be unit vectors such that $u + v = rc$ and $v - u = sd$ with $r, s > 0$, and the angle between u and v is $4\pi x$. It is easy to see that

$$(9)\qquad v = \cos 2\pi x \cdot (c/|c|) + \sin 2\pi x \cdot (d/|d|).$$

Now

$$\big(|c|u/2\big) \oplus \big(|c|v/2\big) = |c|h(2\pi x)\cdot(c/|c|) = \cos\big(2\pi g(x)\big)\cdot c,$$

and, using $g(1/4) = 1/4$,

$$(-|c|u/2) \oplus \big(|c|v/2\big) = |c|h\big(2\pi((1/4) - x)\big)\cdot(d/|d|) =$$

$$= \cos\Big(2\pi\big(g(1/4) - g(x)\big)\Big)\cdot d = \sin\big(2\pi g(x)\big)\cdot d.$$

This gives, by (1) and (2),

$$\big(\cos\big(2\pi g(x)\big)\cdot c\big) \oplus \big(\sin\big(2\pi g(x)\big)\cdot d\big) = |c|v,$$

and thus (8) follows from (9).

Next let $1/4 \leqq x \leqq 1/2$, $x = (1/2) - x'$. Then we have $\cos\big(2\pi g(x)\big) = -\cos\big(2\pi g(x')\big)$ and $\sin\big(2\pi g(x)\big) = \sin\big(2\pi g(x')\big)$. Since $0 \leqq x' \leqq 1/4$, we may apply (8) with x' in place of x and $-c$ in place of c. This proves (8) for $1/4 \leqq x \leqq 1/2$. If $1/2 \leqq x < 1$, $x = x' + (1/2)$, then $\cos\big(2\pi g(x)\big) = -\cos\big(2\pi g(x')\big)$ and $\sin\big(2\pi g(x)\big) = -\sin\big(2\pi g(x')\big)$. Since $0 \leqq x' \leqq 1/2$, we may apply (8) with x' in place of x, and obtain that (8) is valid for every $x \in [0,1)$. Since g is periodic mod 1, (8) is true for every x.

Let $x \in \mathbf{R}$, $x \neq n, n+(1/2)$ $(n \in \mathbf{Z})$, and put $y = \cot\big(2\pi g(x)\big)$ and $z = \operatorname{cosec}\big(2\pi g(x)\big)$. Then we have

$$(10) \qquad (ya) \oplus b = z\big((\cos 2\pi x)a + (\sin 2\pi x)b\big)$$

and

$$(11) \qquad \big|(ya) \oplus b\big| = |z| \cdot |a|$$

whenever a and b are perpendicular vectors with $|a| = |b|$. Indeed, applying (8) for $c = za$ and $d = zb$ we obtain (10), while (11) is obvious from (10).

The set $\{\cot\big(2\pi g(x)\big) : x \in \mathbf{R}\}$ is dense in $\mathbf{R}$. Indeed, as g is injective on $[0,1)$, there is an x for which $g(x)$ is irrational (that is, an element of T of infinite order). Then $\{g(nx) : n \in \mathbf{Z}\}$ is dense in T and hence $\{\cot\big(2\pi g(nx)\big) : n \in \mathbf{Z}\}$ is dense in $\mathbf{R}$. By (11) this implies that if a and b are perpendicular vectors with $|a| = |b|$ then $\big|(ya) \oplus b\big| = \sqrt{y^2+1} \cdot |a|$ holds for a set of y's everywhere dense in $\mathbf{R}$. On the other hand, the function $y \mapsto \big|(ya) \oplus b\big|$ is continuous. Indeed, if $y_1, y_2 \in \mathbf{R}$ then

$$\big|(y_2 a) \oplus b\big| = \big|(y_2 - y_1)a \oplus \big((y_1 a) \oplus b\big)\big| \leqq \big|(y_2 - y_1)a\big| + \big|(y_1 a) \oplus b\big|$$

by (ii), and hence

$$\big|(y_2 a) \oplus b\big| - \big|(y_1 a) \oplus b)\big| \leqq |y_2 - y_1| \cdot |a|.$$

This implies that $\big|(ya) \oplus b\big| = \sqrt{y^2+1} \cdot |a|$ holds for every $y \in \mathbf{R}$ whenever a and b are perpendicular vectors with $|a| = |b|$.

Next we show that g is surjective. Let $w \in T$ be arbitrary; we prove that $w \in g(\mathbf{R})$. We may assume that $w \neq 0, 1/2, 1/4, 3/4$, as these values belong to the range of g. We fix the perpendicular unit vectors a, b, and put $y = \cot(2\pi w)$ and $z = \operatorname{cosec}(2\pi w)$. Let $z^{-1}\big((ya) \oplus b\big) = \lambda a + \mu b$. Since $\big|(ya) \oplus b\big| = |z||a| = |z|$, we have $\lambda^2 + \mu^2 = 1$ and hence $\lambda = \cos 2\pi x$, $\mu = \sin 2\pi x$ for some $x \in \mathbf{R}$. Since $w \neq 0, 1/2, 1/4, 3/4$, it follows that $y \neq 0$, $\lambda \neq 0$, $\mu \neq 0$, and thus $x \neq n$, $n + (1/2)$ $(n \in \mathbf{Z})$. Let $g(x) = w_1$ and $\cot(2\pi w_1) = y_1$, $\operatorname{cosec}(2\pi w_1) = z_1$. Then, by (10),

$$y_1 a \oplus b = z_1 \cdot \big((\cos 2\pi x)a + (\sin 2\pi x)b\big) = z_1 \cdot (\lambda a + \mu b).$$

Thus we may apply (2) and obtain

$$(y_1 - y)a = \big((y_1 a) \oplus b\big) \oplus \big(-((ya) \oplus b)\big) =$$

$$= \big((y_1 a) \oplus b\big) + \big(-((ya) \oplus b)\big) = (z_1 - z) \cdot (\lambda a + \mu b).$$

Since a and b are linearly independent and $\lambda \neq 0$, $\mu \neq 0$, this gives $y_1 = y$ and $z_1 = z$; that is, $\cot(2\pi w_1) = \cot(2\pi w)$ and $\operatorname{cosec}(2\pi w_1) = \operatorname{cosec}(2\pi w)$. This implies $w = w_1 = g(x) \in g(\mathbf{R})$, and this is what we wanted to show.

We have proved that g is a bijection between $[0,1)$ and T. This implies that $\chi\big(e(x)\big) = e\big(g(x)\big)$ $(x \in \mathbf{R})$ defines an automorphism of U. In order to complete the proof of the theorem, we have to show that $\oplus$ is generated by χ. Let $\phi\big(r \cdot e(x)\big) = r \cdot e\big(g(x)\big)$ $(r, x \in \mathbf{R})$. We prove first that (5) holds if a and b are perpendicular. Let $a = r \cdot e(w)$ and $b = s \cdot e\big(w + (1/4)\big)$. Since g is surjective, there exists an $x \in \mathbf{R}$ such that

$$\frac{r}{\sqrt{r^2+s^2}} = \cos\big(2\pi g(x)\big), \quad \frac{s}{\sqrt{r^2+s^2}} = \sin\big(2\pi g(x)\big).$$

Let $c = \sqrt{r^2+s^2} \cdot e(w)$ and $d = \sqrt{r^2+s^2} \cdot e\big(w + (1/4)\big)$. Then, by (8), we have

$$a \oplus b = \sqrt{r^2+s^2}\big(\cos 2\pi x + i \sin 2\pi x\big)\, e(w) = \sqrt{r^2+s^2} \cdot e(w+x),$$

and hence

$$\phi(a \oplus b) = \sqrt{r^2+s^2} \cdot e\big(g(w+x)\big) = (r+is)e\big(g(w)\big).$$

On the other hand,

$$\phi(a) + \phi(b) = r \cdot e\big(g(w)\big) + s \cdot e\big(g(w) + g(1/4)\big) = (r+is)e\big(g(w)\big),$$

and hence (5) holds.

Finally, if the vectors a and b are not perpendicular, then we can choose a vector c perpendicular to a such that $a \oplus c$ is parallel to b. Indeed, there is $x \in \mathbf{R}$ such that $\cos(2\pi x) \cdot a + \sin(2\pi x) \cdot (ia)$ is parallel to b. Since a and b are not perpendicular, we have $\cos(2\pi x) \neq 0$, and hence $\cos\big(2\pi g(x)\big) \neq 0$. Now, applying (8) with $c_1 = a / \cos\big(2\pi g(x)\big)$ and $d_1 = (ia)/\cos\big(2\pi g(x)\big)$, we obtain that $a \oplus c$ is parallel to b, where $c = \tan\big(2\pi g(x)\big) \cdot (ia)$.

If $b = y(a \oplus c)$ then, using the fact that a and c are perpendicular, we obtain

$$\phi(b) = y\phi(a \oplus c) = y\big(\phi(a) + \phi(c)\big) = \phi(ya) + \phi(yc) = \phi\big((ya) \oplus (yc)\big).$$

This gives

$$a \oplus b = a \oplus (ya) \oplus (yc) = \big((1+y)a\big) \oplus (yc) = \phi^{-1}\big(\phi((1+y)a) + \phi(yc)\big) =$$
$$= \phi^{-1}\big(\phi(a) + \phi(ya) + \phi(yc)\big) = \phi^{-1}\big(\phi(a) + \phi(b)\big).$$

Therefore (5) holds for every $a, b \in \mathbf{R}^2$, and the proof is complete. □

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SIDON-TYPE INEQUALITIES FOR LEGENDRE POLYNOMIALS

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Dedicated to Professor Károly Tandori on his 70-th birthday

1. Introduction

S. Sidon [17] proved an inequality for the linear combinations of trigonometric Dirichlet kernels in 1939, named after him. Let D_n denote the n-th Dirichlet kernel, i.e.

$$(1)\qquad D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos kx \qquad (n\in\mathbf{N}:=\{0,1,2,\dots\},\ x\in[-\pi,\pi]).$$

Then the inequality in question is

$$(2)\qquad \frac{1}{N}\left\|\sum_{n=0}^{N}a_nD_n\right\|_1 \leqq C\max_{0\leqq n\leqq N}|a_k| \quad (N\in\mathbf{P}:=\{1,2,\dots\}),$$

where $(a_n, n\in\mathbf{N})$ is an arbitrary sequence of real numbers, $C>0$ is an absolute constant and $\|\cdot\|_1$ denotes the $L^1(0,\pi)$-norm. (Throughout this paper, C will denote absolute and C_p positive constants depending only on p, not necessarily the same in different occurrences.)

A generalization of (2) was given by R. Bojanič and V. Stanojevič [3] in the form

$$(3)\qquad \frac{1}{N}\left\|\sum_{n=0}^{N}a_nD_n\right\|_1 \leqq C_p\left(\frac{1}{N}\sum_{n=0}^{N}|a_n|^p\right)^{1/p} \qquad (N\in\mathbf{P}),$$

where $1<p\leqq\infty$ and $C_p=O\left(\frac{1}{p-1}\right)$ as $p\to1$, and $C_p=O(1)$ as $p\to\infty$.

It is easy to see that (3) does not hold for $p=1$. Indeed, if $a_N=1$ and $a_n=0$ $(n\leqq N-1)$ then the left side of (3) is of order $(\log N)/N$ while the right side is of order $1/N$ as $N\to\infty$.

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It is known that the trigonometric system is a basis in $L^p(-\pi,\pi)$ if $1 < p < \infty$. If $p=1$ then there exists a function in $L^1(-\pi,\pi)$ such that its Fourier series does not converge in $L^1(-\pi,\pi)$-norm. Sidon-type inequalities can be used to investigate L^1-convergence of trigonometric series. For example, if the difference sequence $\triangle a_k = a_k - a_{k-1}$ $(k \in \mathbf{N},\ a_{-1}=0)$ of $(a_n,\ n \in \mathbf{N})$ satisfies

$$\sum_{m=0}^{\infty} 2^m \left(2^{-m} \sum_{k=2^m}^{2^{m+1}-1} |\triangle a_k|^p \right)^{1/p} < \infty \tag{4}$$

for some $p > 1$ then the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{5}$$

is a Fourier series of an even function $f \in L^1(-\pi,\pi)$. Moreover, if in addition $a_k \log k \to 0$ as $k \to \infty$, then the series (5) converges in $L^1(-\pi,\pi)$-norm to f.

In order to obtain a more general condition for L^1-convergence, we need a sharper upper estimation in (3). In this direction compare the results of M. Buntinas and N. Tanovič-Miller [4] and F. Schipp [14]. The best possible rearrangement invariant upper and lower estimation was given by S. Fridli [6].

A similar inequality was proved for the Walsh system by F. Móricz and F. Schipp [7] and for some other systems (UDMD systems, Ciesielski system) by F. Schipp [15].

The aim of this paper is to give similar inequalities and conditions for norm-convergence with respect to the Legendre system.

2. Results

The mean convergence of Jacobi–Fourier series with respect to several weight functions has been investigated by H. Pollard [10], [11], [12], B. Muckenhoupt [8], R. R. Askey [1], L. Colzani [4], G. M. Wing [20], and others. B. Muckenhoupt proved a general theorem for the orthonormal Jacobi polynomials $p_n^{(\alpha,\beta)}$ $(n \in \mathbf{N})$ with $\alpha > -1$, $\beta > -1$ and with respect to the weight function $\varrho_{a,b}(x) := (1-x)^a(1+x)^b$ $(x \in (-1,1))$. His result (see [8], Theorem 1, 2) for the Legendre polynomials $p_n := p_n^{(0,0)}$ and for $a = b$ is of the following form.

Let $a \in \mathbf{R}$ be fixed,

$$\varrho_a(x) := (1 - x^2)^a \qquad (x \in (-1, 1)),$$

and for $1 \leqq p < \infty$ denote $L^p_{\varrho a}$ the set of measurable functions in $(-1, 1)$ for which

$$\|f\|_{p,a} := \left(\int_{-1}^{1} |f(x)|^p \varrho_a(x)\, dx \right)^{1/p} < \infty.$$

Denote $S_n f$ $(n \in \mathbf{N})$ the n-th partial sum of the Legendre–Fourier series of the function $f \in L^p_{\varrho a}$.

THEOREM A. *If $1 < p < \infty$ and $\frac{1}{4}p - 1 < a < \frac{3}{4}p - 1$ then*

$$\lim_{n \to \infty} \|f - S_n f\|_{p,a} = 0.$$

Moreover, if a does not satisfy this condition, then there exists a function in $L^p_{\varrho a}$ such that $S_n f$ does not converge in $L^p_{\varrho a}$-norm.

Following the method of Muckenhoupt (see [8]) one can show that such a positive result is not true for $p = 1$. In this paper we give a sufficient condition for the $L^p_{\varrho a}$-norm convergence of Legendre series in the critical case $p = 1$. To this end we prove a Sidon-type inequality for the Legendre–Dirichlet kernels

$$D^t_n(x) := \sum_{k=0}^{n} p_k(t) p_k(x) \qquad (-1 \leqq x, t \leqq 1, \ \ n \in \mathbf{N}). \tag{6}$$

We prove the following inequality.

THEOREM 1. *Let $\delta \in (0, 1)$, $t \in (-1 + \delta, 1 - \delta)$, $1 < p < \infty$, and $a > -\frac{3}{4}$. Then for any sequence of real numbers $(a_n, n \in \mathbf{N})$ we have*

$$\frac{1}{N} \left\| \sum_{n=0}^{N} a_n D^t_n \right\|_{1,a} \leqq \frac{C_{p,a}}{\delta^{5/4}} \left(\frac{1}{N} \sum_{n=0}^{N} |a_n|^p \right)^{1/p} \qquad (N \in \mathbf{P}), \tag{7}$$

where $C_{p,a} > 0$ depends only on p and a.

From (7) we get a Sidon-type inequality for the even kernels

$$D^*_{2n}(x) := \sum_{k=0}^{n} (-1)^k p_{2k}(x) \qquad (x \in [-1, 1], \ n \in \mathbf{N}). \tag{8}$$

THEOREM 2. *For any sequence* $(a_n, n \in \mathbf{N})$ *of real numbers, and for* $1 < p \leqq \infty$, $a > -\frac{3}{4}$, $N \in \mathbf{P}$ *we have*

$$\frac{1}{N}\left\|\sum_{n=0}^{N} a_n D_{2n}^*\right\|_{1,a} \leqq C_{p,a}\left(\frac{1}{N}\sum_{n=0}^{N}|a_n|^p\right)^{1/p}, \tag{9}$$

where the constant $C_{p,a} > 0$ *depends only on* p *and* a.

This theorem can be used to deduce a coefficient condition for the convergence in $L^1_{\varrho_a}$-norm of the even Legendre series

$$\sum_{n=0}^{\infty} a_n p_{2n}(x) \tag{10}$$

which is the analogue of the corresponding result with respect to the cosine series mentioned above.

THEOREM 3. *Let* $1 < p \leqq \infty$, $a > -\frac{3}{4}$ *and suppose* $a_k \log k \to 0$ *as* $k \to \infty$ *and*

$$\sum_{m=0}^{\infty} 2^m \left(2^{-m} \sum_{k=2^m}^{2^{m+1}-1} |a_k + a_{k+1}|^p\right)^{1/p} < \infty. \tag{11}$$

Then the series (10) *converges in* $L^1_{\varrho_a}$*-norm. Moreover, if* $a \leqq 0$, *then the series in* (10) *is a Legendre–Fourier series of an even function* $f \in L^1_{\varrho_a}$.

3. Proofs

PROOF OF THEOREM 1. Since the $L^1_{\varrho_a}$-norm decreases in a, we may assume that

$$-\frac{3}{4} < a < -\frac{1}{4}.$$

This implies $4a + 1 < 0$ and $\frac{2}{-4a-1} > 1$. The p-adic mean increases in p, therefore it can be assumed that

$$1 < p < \frac{2}{-4a-1}.$$

To prove Theorem 1 we need the following weighted form of the Hausdorff–Young inequality. Denote

$$\langle f\varrho_a, p_n\rangle := \int_{-1}^{1} f(x)p_n(x)\varrho_a(x)\,dx \qquad (n \in \mathbf{N})$$

the n-th Legendre–Fourier coefficient of $f\varrho_a$. If $1 \leqq p \leqq 2$, and q is the conjugate index to p, i.e. $1/p + 1/q = 1$, then

$$\left(\sum_{n=0}^{\infty} \left|\langle f\varrho_a, p_n\rangle\right|^q\right)^{1/q} \leqq C\|f\|_{p,\alpha}, \tag{12}$$

where $\alpha = pa - \frac{2-p}{4}$.

Indeed, let the operator T be defined by

$$Tf := \big(\langle f\varrho_a, p_n\rangle, n \in \mathbf{N}\big).$$

Since by the well-known estimation

$$\left|p_n(x)\right| \leqq \frac{1}{(1-x^2)^{1/4}} = \varrho_{-1/4}(x) \qquad (x \in (-1,1),\ n \in \mathbf{N}) \tag{13}$$

(see [19], pp. 131 and 128) we have

$$\|Tf\|_{l^\infty} := \sup_{n\in\mathbf{N}} \left|\langle f\varrho_a, p_n\rangle\right| \leqq \int_{-1}^{1} \left|f(x)\right| \varrho_{a-1/4}(x)\,dx = \|f\|_{1,a-1/4}.$$

By the Parseval formula we get

$$\|Tf\|_{l^2} = \left(\sum_{n=0}^{\infty} \left|\langle f\varrho_a, p_n\rangle\right|^2\right)^2 = \left(\int_{-1}^{1} \left|f(x)\varrho_a(x)\right|^2 dx\right)^{1/2} = \|f\|_{2,2a}.$$

Applying the Riesz–Thorin interpolation theorem with weight functions (see [2], Corollary 5.5.2) we obtain

$$\|Tf\|_{l^q} \leqq C\left(\int_{-1}^{1} \left|f(x)\right|^p w(x)\,dx\right)^{1/p} = \|f\|_{p,\alpha} \qquad (1 \leqq p \leqq 2),$$

where

$$\frac{1}{p} = \frac{1-\Theta}{2} + \frac{\Theta}{1} = \frac{1+\Theta}{2},$$

$$\frac{1}{q}=\frac{1-\Theta}{2}+\frac{\Theta}{\infty}=\frac{1-\Theta}{2}\qquad(0<\Theta<1),$$

$$w=\varrho_{2a}^{\frac{p}{2}(1-\Theta)}\varrho_{a-1/4}^{p\Theta}=\varrho_\alpha,$$

and

$$\alpha=2a\frac{p}{2}(1-\Theta)+p\Theta\left(a-\frac{1}{4}\right)=pa-\frac{2-p}{4},$$

i.e. (12) is true.

Fix $N\in\mathbf{P}$, $\delta\in(0,1)$, $t\in(-1+\delta,1-\delta)$, and the sequence $(a_n,n\in\mathbf{N})$. First let $N\leqq 2/\delta$. For $t\in(-1+\delta,1-\delta)$ we have by (13) that

$$(1-x^2)^{1/4}|D_n^t(x)|\leqq(1-x^2)^{1/4}\sum_{k=0}^{n}|p_k(t)|\,|p_k(x)|\leqq\frac{n}{\delta^{1/4}}\leqq\frac{N}{\delta^{1/4}}$$

for all $n\leqq N$ and $-1\leqq x\leqq 1$. Consequently,

$$\frac{1}{N}\left\|\sum_{n=0}^{N}a_nD_n^t\right\|_{1,a}=\frac{1}{N}\int_{-1}^{1}\left|\sum_{n=0}^{N}a_nD_n^t(x)\right|\varrho_a(x)\,dx\leqq$$

$$\leqq\frac{1}{N}\sum_{n=0}^{N}|a_n|\int_{-1}^{1}|D_n^t(x)|(1-x^2)^{1/4}\varrho_{a-1/4}(x)\,dx\leqq$$

$$\leqq\frac{N}{\delta^{1/4}}\left(\frac{1}{N}\sum_{n=0}^{N}|a_n|\right)\int_{-1}^{1}\varrho_{a-1/4}(x)\,dx\leqq\frac{C_a}{\delta^{5/4}}\frac{1}{N}\sum_{n=0}^{N}|a_n|\leqq$$

$$\leqq\frac{C_a}{\delta^{5/4}}\left(\frac{1}{N}\sum_{n=0}^{N}|a_n|^p\right)^{1/p}$$

for all $p\geqq 1$ and $a>-\frac{3}{4}$. Thus (7) is proved for $N\leqq 2/\delta$.

Let now $N>2/\delta$. Set $g:=\operatorname{sign}(\sum_{n=0}^{N}a_nD_n^t)$. Then

$$\left\|\sum_{n=0}^{N}a_nD_n^t\right\|_{1,a}=\int_{-1}^{1}\left(\sum_{n=0}^{N}a_nD_n^t(x)\right)g(x)\varrho_a(x)\,dx.\tag{14}$$

We shall use the notations

$$I_N:=I_N(t):=[-1,1]\cap[t-N^{-1},t+N^{-1}],$$

$$\widetilde{I}_N := \widetilde{I}_N(t) := [-1,1] \setminus I_N,$$

$$G_N(x) := G_{N,t}(x) := \begin{cases} g(x), & \text{for } x \in I_N \\ 0, & \text{for } x \in \widetilde{I}_N, \end{cases}$$

$$\widetilde{G}_N(x) := \widetilde{G}_{N,t}(x) := g(x) - G_N(x) \qquad (x \in (-1,1)).$$

By (14) we have

$$(15) \qquad \frac{1}{N}\left\|\sum_{n=0}^{N} a_n D_n^t\right\|_{1,a} \leqq \frac{1}{N}\sum_{n=0}^{N}\left|a_n \int_{-1}^{1} D_n^t(x) G_N(x) \varrho_a(x)\,dx\right| +$$

$$+\frac{1}{N}\sum_{n=0}^{N}\left|a_n \int_{-1}^{1} D_n^t(x)\widetilde{G}_N(x)\varrho_a(x)\,dx\right| =: \sum\nolimits_1 + \sum\nolimits_2 .$$

To estimate $\sum_1$, first we write

$$(16) \qquad \sum\nolimits_1 \leqq \frac{1}{N}\left(\sum_{n=0}^{N}|a_n|\right)\max_{0\leqq n\leqq N}\left|\int_{-1}^{1} D_n^t(x) G_N(x)\varrho_a(x)\,dx\right|.$$

Denote χ_A the characteristic function of the set $A \subset \mathbf{R}$. Since $N > 2/\delta$, we have $I_N \subset (-1+\delta/2, 1-\delta/2)$, therefore

$$\varrho_{a-1/4}(x) < \frac{C}{\delta^{1/4-a}} \qquad (x \in I_N,\ n \in \mathbf{N}).$$

Consequently, we have by (13) that

$$\left|\int_{-1}^{1} D_n^t(x) G_N(x)\varrho_a(x)\,dx\right| \leqq$$

$$\leqq \int_{-1}^{1}\left(\sum_{k=0}^{n}|p_k(t)|(1-x^2)^{1/4}|p_k(x)|\right)\chi_{I_N}(x)\varrho_{a-1/4}(x)\,dx \leqq$$

$$\leqq \frac{n}{\delta^{1/4}}\int_{-1}^{1}\chi_{I_N}(x)\varrho_{a-1/4}(x)\,dx \leqq \frac{n}{\delta^{1/4}}\frac{C}{\delta^{1/4-a}}\int_{-1}^{1}\chi_{I_N}(x)\,dx \leqq$$

$$\leqq \frac{C}{\delta^{1/2-a}} \leqq \frac{C}{\delta^{5/4}}$$

for all $n \leqq N$ and $a > -\frac{3}{4}$. Thus from (16) we have

$$(17)\qquad \sum\nolimits_1 \leqq \frac{C}{\delta^{5/4}}\frac{1}{N}\left(\sum_{n=0}^{N}|a_n|\right) \leqq \frac{C}{\delta^{5/4}}\left(\frac{1}{N}\sum_{n=0}^{N}|a_n|^p\right)^{1/p}$$

for all $p \geqq 1$ and $a > -\frac{3}{4}$.

Now we turn to the estimation of $\sum_2$ in (15). We need the Christoffel–Darboux formula with respect to the Legendre polynomials (see [19], p. 122)

$$D_n^t(x) = \sum_{k=0}^{n} p_k(t)p_k(x) = \frac{n+1}{\sqrt{(2n+1)(2n+3)}}\frac{p_n(t)p_{n+1}(x) - p_{n+1}(t)p_n(x)}{x-t}$$

$$(x \in (-1,1),\ n \in \mathbf{N}).$$

Set

$$h_N(x) := \begin{cases} \dfrac{\widetilde{G}_N(x)}{x-t}, & \text{for } x \in \widetilde{I}_N \\ 0, & \text{for } x \in I_N. \end{cases}$$

Using these notations we have

$$(18)\qquad \int_{-1}^{1} D_n^t(x)\widetilde{G}_N(x)\varrho_a(x)\,dx =$$

$$= \frac{n+1}{\sqrt{(2n+1)(2n+3)}}\big(p_n(t)\langle h_N\varrho_a, p_{n+1}\rangle - p_{n+1}(t)\langle h_N\varrho_a, p_n\rangle\big).$$

By (13) and Hölder's inequality we obtain from (18) that

$$\sum\nolimits_2 = \frac{1}{N}\sum_{n=0}^{N}\left|a_n\int_{-1}^{1} D_n^t(x)\widetilde{G}_N(x)\varrho_a(x)\,dx\right| \leqq$$

$$\leqq \frac{2}{\delta^{1/4}}\frac{1}{N}\left(\sum_{n=0}^{N}|a_n|^p\right)^{1/p}\left(\sum_{n=0}^{\infty}|\langle h_N\varrho_a, p_n\rangle|^q\right)^{1/q}.$$

Applying inequality (12) we get

$$\left(\sum_{n=0}^{N}|\langle h_N\varrho_a, p_n\rangle|^q\right)^{1/q} \leqq \|h_N\|_{p,\alpha},$$

where $\alpha = pa - \frac{2-p}{4} > -1$. Let

$$I_N^\delta := \widetilde{I}_N \cap \left(-1+\frac{\delta}{2}, 1-\frac{\delta}{2}\right),$$

$$J_N^\delta := \left[-1, -1+\frac{\delta}{2}\right] \cup \left[1-\frac{\delta}{2}, 1\right].$$

By definition

$$\|h_N\|_{p,\alpha} \leqq \left(\int_{\widetilde{I}_N} \frac{\varrho_\alpha(x)\,dx}{|x-t|^p}\right)^{1/p} \leqq$$

$$\leqq \left(\int_{I_N^\delta} \frac{\varrho_\alpha(x)\,dx}{|x-t|^p}\right)^{1/p} + \left(\int_{J_N^\delta} \frac{\varrho_\alpha(x)\,dx}{|x-t|^p}\right)^{1/p} \leqq$$

$$\leqq \frac{C}{\delta}\left(\int_{\widetilde{I}_N} \frac{dx}{|x-t|^p}\right)^{1/p} + \frac{C}{\delta}\left(\int_{-1}^{1} \varrho_\alpha(x)\,dx\right)^{1/p}.$$

Since

$$\left(\int_{\widetilde{I}_N} \frac{dx}{|x-t|^p}\right)^{1/p} \leqq 2\left(\int_{1/N}^{\infty} \frac{du}{u^p}\right)^{1/p} =$$

$$= \frac{2}{(p-1)^{1/p}}\left(\frac{1}{N}\right)^{(1-p)/p} \leqq \frac{2}{p-1} N^{1-1/p},$$

and $\alpha > -1$ we have

$$\|h_N\|_{p,\alpha} \leqq \frac{C_{p,a}}{\delta} N^{1-1/p}.$$

Consequently,

$$\sum\nolimits_2 \leqq \frac{C_{p,a}}{\delta^{5/4}}\left(\frac{1}{N}\sum_{n=0}^{N} |a_n|^p\right)^{1/p}.$$

The proof of Theorem 1 is complete. □

Proof Theorem 2. It is well known that the orthonormal Legendre polynomials $(p_n, n \in \mathbf{N})$ satisfy the relations

$$\begin{cases} p_{2k+1}(0) = 0, \\ p_{2k}(0) = (-1)^k \sqrt{\dfrac{4k+1}{2}}\, \dfrac{(2k)!}{2^{2k}[k!]^2} \qquad (k \in \mathbf{N}) \end{cases} \tag{19}$$

(see [19], p. 120). We will prove that

$$\lim_{k\to\infty}(-1)^k p_{2k}(0)=\sqrt{\frac{2}{\pi}}.$$

More precisely, there exists a positive real number C such that for every $k\in\mathbf{P}$

$$\left|\sqrt{\frac{2}{\pi}}-(-1)^k p_{2k}(0)\right|\leqq\frac{C}{k}. \tag{20}$$

For the proof of (20) we use Stirling's formula, which states that for all $k\in\mathbf{P}$ there exists a number $\Theta_k\in(0,1)$ such that

$$k!=\sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{\Theta_k}{12k}}$$

(see [13] p. 392). From this and (19) there follows that

$$(-1)^k p_{2k}(0)=\sqrt{\frac{4k+1}{2}\frac{(2k)!}{2^{2k}[k!]^2}}=\sqrt{\frac{2}{\pi}}\sqrt{1+\frac{1}{4k}}e^{\frac{1}{6k}\left(\frac{\Theta_{2k}}{4}-\Theta_k\right)}.$$

It is easy to see that

$$\frac{1}{6}\left|\frac{\Theta_{2k}}{4}-\Theta_k\right|\leqq\frac{1}{4}$$

and there exists a positive real number C such that for all $|\beta|\leqq 1/4$ we have

$$\left|\sqrt{1+\frac{x}{4}e^{\beta x}}-1\right|\leqq C\,x\qquad(0\leqq x\leqq 1).$$

Estimate (20) is proved.

Consider the Legendre–Dirichlet kernels at the point $t=0$, i.e.

$$D_n^0(x)=\sum_{k=0}^{n}p_k(0)p_k(x).$$

Using Theorem 1 we have that for every sequence $(a_n, n\in\mathbf{N})$ of real numbers, $p>1$, and $N\in\mathbf{P}$

$$\frac{1}{N}\left\|\sum_{n=0}^{N}a_n D_{2n}^0\right\|_{1,a}\leqq C_{p,a}\left(\frac{1}{N}\sum_{n=0}^{N}|a_n|^p\right)^{1/p}, \tag{21}$$

where the constant $C_{p,a}>0$ depends only on p and a. Let

$$\widetilde{D}_{2n}(x):=D_{2n}^0(x)-\sqrt{\frac{2}{\pi}}D_{2n}^*(x)= \tag{22}$$

$$=\sum_{k=0}^{n}\left(p_{2k}(0)-(-1)^k\sqrt{\frac{2}{\pi}}\right)p_{2k}(x)=:\sum_{k=0}^{n}\alpha_{2k}p_{2k}(x)$$

$$(x\in[-1,1],\ n\in\mathbf{N}).$$

Set $\widetilde{g}:=\operatorname{sign}\widetilde{D}_{2n}$. Applying Hölder's inequality, (22), (20), and (12) we get by $\alpha>-1$ that

$$\|\widetilde{D}_{2n}\|_{1,a}=\int_{-1}^{1}|\widetilde{D}_{2n}(x)|\,\varrho_a(x)dx=\int_{-1}^{1}\left(\sum_{k=0}^{n}\alpha_{2k}p_{2k}(x)\right)\widetilde{g}(x)\varrho_a(x)=$$

$$=\sum_{k=0}^{n}\alpha_{2k}\langle\widetilde{g}\varrho_a,p_{2k}\rangle\leqq\left(\sum_{k=0}^{n}|\alpha_{2k}|^p\right)^{1/p}\left(\sum_{k=0}^{n}|\langle\widetilde{g}\varrho_a,p_{2k}\rangle|^q\right)^{1/q}\leqq C_{p,a}.$$

Consequently,

$$\left\|\sum_{n=0}^{N}a_n\widetilde{D}_{2n}\right\|_{1,a}\leqq\sum_{n=0}^{N}|a_n|\,\|\widetilde{D}_{2n}\|_{1,a}\leqq C_{p,a}\sum_{n=0}^{N}|a_n|. \tag{23}$$

From (21), (22), and (23) we have

$$\frac{1}{N}\left\|\sum_{n=0}^{N}a_nD_{2n}^*\right\|_{1,a}\leqq\frac{C}{N}\left\|\sum_{n=0}^{N}a_nD_{2n}^0\right\|_{1,a}+$$

$$+\frac{C}{N}\left\|\sum_{n=0}^{N}a_n\widetilde{D}_{2n}\right\|_{1,a}\leqq C_{p,a}\left(\frac{1}{N}\sum_{n=0}^{N}|a_n|^p\right)^{1/p}.$$

The proof of Theorem 2 is complete. □

Proof of Theorem 3. First we prove that

$$\|D_{2n}^*\|_{1,a}\leqq C_a\log(n+2)\qquad(n\in\mathbf{N}). \tag{24}$$

Observe that by (13) and by the Christoffel–Darboux formula we have

$$\left| D_{2n}^0(x)\right| \leqq \frac{Cn}{(1-x^2)^{1/4}} \qquad (-1<x<1),$$

$$\left| D_{2n}^0(x)\right| \leqq \frac{C}{x(1-x^2)^{1/4}} \qquad (-1<x<1,\ x\neq 0).$$

Thus

$$(25)\quad \|D_{2n}^0\|_{1,a} \leqq C\left(\int_0^{1/n} \left| D_{2n}^0(x)\right| \varrho_a(x)\,dx + \int_{1/n}^{1/2} \left| D_{2n}^0(x)\right| \varrho_a(x)\,dx + \right.$$

$$\left. + \int_{1/2}^{1} \left| D_{2n}^0(x)\right| \varrho_a(x)\,dx\right) \leqq C_a\left(1+\int_{1/n}^{1/2}\frac{dx}{x}+\int_0^1 \varrho_{a-1/4}(x)\,dx\right) \leqq$$

$$\leqq C_a \log(n+2).$$

Since by (22) and (23)

$$\left\| D_{2n}^0 - \sqrt{\frac{2}{\pi}} D_{2n}^* \right\|_{1,a} \leqq C_a,$$

we have that (25) implies (24).

To prove the convergence of (10) we use Abel transformation:

$$\sum\nolimits_{N,M} := \sum_{n=N}^{M} a_n p_{2n} = \sum_{n=N}^{M} (-1)^n a_n (-1)^n p_{2n} =$$

$$= \sum_{n=N}^{M} (-1)^n a_n \left(D_{2n}^* - D_{2n-2}^*\right) =$$

$$= \sum_{n=N}^{M-1} (-1)^n (a_n + a_{n+1}) D_{2n}^* + (-1)^M a_M D_{2M}^* - (-1)^N a_N D_{2N-2}^*.$$

Denote

$$A_{N,M} := \sum_{n=N}^{M-1} (-1)^n (a_n + a_{n+1}) D_{2n}^*.$$

Define $m_0, m_1 \in \mathbf{N}$ by the relations

$$2^{m_0-1} \leqq N < 2^{m_0}, \qquad 2^{m_1} < M \leqq 2^{m_1+1}$$

and take the decomposition

$$A_{N,M} = A_{N,2^{m_0}} + A_{2^{m_0},2^{m_1}} + A_{2^{m_1},M}.$$

Then

$$\left\|\sum\nolimits_{N,M}\right\|_{1,a} \leqq \|A_{N,2^{m_0}}\|_{1,a} + \|A_{2^{m_0},2^{m_1}}\|_{1,a} +$$

$$+\|A_{2^{m_1},M}\|_{1,a} + |a_M|\,\|D^*_{2M}\|_{1,a} + |a_N|\,\|D^*_{2N-2}\|_{1,a}.$$

By our assumption and by (24) the last two terms tend to 0 if $N, M \to \infty$. To estimate the remainder terms we use inequality (9). Thus

$$\|A_{2^{m_0},2^{m_1}}\|_{1,a} \leqq \sum_{m=m_0}^{m_1-1} \left\| \sum_{n=2^m}^{2^{m+1}-1} (-1)^n (a_n + a_{n+1}) D^*_{2n} \right\|_{1,a} \leqq$$

$$\leqq C \sum_{m=m_0}^{\infty} 2^m \left(2^{-m} \sum_{n=2^m}^{2^{m+1}-1} |a_n + a_{n+1}|^p \right)^{1/p} =: r_{m_0}$$

and by (11) we get that $r_{m_0} \to 0$, as $m_0 \to \infty$. Similarly, applying (9) again for the first term we have

$$\|A_{N,2^{m_0}}\|_{1,a} \leqq C 2^{m_0} \left(2^{-m_0} \sum_{n=2^{m_0-1}}^{2^{m_0}-1} |a_n|^p \right)^{1/p}$$

which tends to 0 if $N \to \infty$. The same argument shows that the third term tends to 0.

Denote the $L^1_{\varrho_a}$-norm limit of (10) by f, i.e.

$$\lim_{N\to\infty} \left\| \sum_{n=0}^{N} a_n p_{2n} - f \right\|_{1,a} = 0. \tag{26}$$

Since $a \leqq 0$ we get for $k \leqq N$ that

$$\left| a_k - \langle f, p_{2k} \rangle \right| = \left| \left\langle \sum_{n=0}^{N} a_n p_{2n} - f, p_{2k} \right\rangle \right| \leqq$$

$$\leqq \|p_k\|_\infty \left\| \sum_{n=0}^{N} a_n p_{2n} - f \right\|_{1,a} .$$

The right side tends to 0 by (26) as $N \to \infty$. Consequently we have

$$\langle f, p_{2k} \rangle = a_k \qquad (k \in \mathbf{N}).$$

In a similar way we get

$$\langle f, p_{2k+1} \rangle = 0 \qquad (k \in \mathbf{N}).$$

The proof of Theorem 3 is complete. □

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RESONANCE PRINCIPLES WITH APPLICATIONS TO MEAN ERGODIC THEOREMS AND PROJECTION OPERATORS

D. NASRI-ROUDSARI, R. J. NESSEL and R. ZELER (Aachen)

Dedicated to Professor Tandori on the occasion of his seventieth birthday, in high esteem.

It is the purpose of this paper to discuss several extensions of previous (cf. [7]) quantitative uniform boundedness principles. Indeed, motivated by recent work of O. V. Davydov [5; 6] and S. P. Zhou [16; 17], the present treatment particularly contributes to situations where the (abstract) moduli of continuity, determining generalized Lipschitz classes, do not possess any additional properties. While the main resonance principle is given in Section 1, Section 2 then reconsiders and extends some of the previous results on the condensation of singularities. In Section 3 further applications are worked out in connection with the mean ergodic theorem and the approximation by trigonometric projection operators.

1. Resonance in b-complete spaces

Let Y be a linear space over, e. g., the field $\mathbf{R}$ (set of real numbers), equipped with a family of seminorms $\{|\cdot|_p : p \in [0,\infty)\}$ where $|f|_p = \infty$ for some $f \in Y$, $p \in [0,\infty)$ may still be possible. The space $(Y, |\cdot|_p)$ is called b-complete (boundedly complete), if for each sequence $(f_n)_{n\in\mathbf{N}} \subset Y$ with ($\mathbf{N}$ being the set of natural numbers)

$$\sup_{n\in\mathbf{N}, p\in[0,\infty)} |f_n|_p < \infty \quad \text{and} \quad \lim_{m,n\to\infty} |f_m - f_n|_p = 0 \quad \text{for each } p \in [0,\infty)$$

there exists an element $f \in Y$ such that

$$\lim_{n\to\infty} |f_n - f|_p = 0 \quad \text{for each } p \in [0,\infty).$$

Let Y^* be the class of non-negative functionals T on Y which are sublinear (i. e., $T(f+g) \leqq Tf + Tg$ and $T(\alpha f) = |\alpha| Tf$ for all $f, g \in Y, \alpha \in \mathbf{R}$) and bounded, i. e., there exist $K = K_T < \infty$ and $r = r_T \in [0,\infty)$ such that for all $f \in Y$

$$(1.1) \qquad Tf \leqq K \sup_{q\in[0,r]} |f|_q .$$

THEOREM 1. *For a b-complete space* $\left(Y, |\cdot|_p\right)$ *let* $(T_n)_{n\in\mathbf{N}} \subset Y^*$, $(h_n)_{n\in\mathbf{N}} \subset Y$ *be such that (case* $M = \infty$ *being included)*

$$\sup_{n\in\mathbf{N}, p\in[0,\infty)} |h_n|_p \leqq C < \infty, \tag{1.2}$$

$$\lim_{n\to\infty}\left[\sup_{q\in[0,p]} |h_n|_q\right] = 0 \qquad \textit{for each } p\in[0,\infty), \tag{1.3}$$

$$\limsup_{p\to\infty} |h_n|_p \leqq \delta_n \qquad \textit{for each } n\in\mathbf{N}, \tag{1.4}$$

$$\limsup_{n\to\infty} T_n h_n \geqq M > 0, \tag{1.5}$$

where $(\delta_n)_{n\in\mathbf{N}} \subset \mathbf{R}$ *is a strictly positive sequence tending to zero. Then for each* $\varepsilon > 0$ *there exists an element* $f_0 = f_{0,\varepsilon} \in Y$ *such that*

$$\sup_{p\in[0,\infty)} |f_0|_p \leqq C + \varepsilon, \tag{1.6}$$

$$\limsup_{n\to\infty} T_n f_0 \geqq M. \tag{1.7}$$

PROOF. Since each T_n is bounded, there exist constants $1 \leqq K_n \leqq K_{n+1}$ and $r_n \leqq r_{n+1} \in [0,\infty)$ such that (cf. (1.1))

$$T_n f \leqq K_n \sup_{q\in[0,r_n]} |f|_q. \tag{1.8}$$

Now one may successively construct sequences $(n_k)_{k\in\mathbf{N}} \subset \mathbf{N}, (p_k)_{k\in\mathbf{N}} \subset \subset [0,\infty)$ and $(\theta_k)_{k\in\mathbf{N}} \subset \{-1,1\}$ as follows: After having determined $0 = p_1 < < p_2 < \ldots < p_{k-1}$ as well as $n_1 < n_2 < \ldots < n_{k-1}$ (for a suitable $n_1 \in \mathbf{N}$, cf. (1.11; 12)) and $1 = \theta_1, \theta_2, \ldots, \theta_{k-1}$, one may choose $p_k > \max\left\{k, p_{k-1}, r_{n_{k-1}}\right\}$ such that (cf. (1.4))

$$|h_{n_j}|_p \leqq 2\delta_{n_j} \quad \text{for all } 1 \leqq j \leqq k-1, p \geqq p_k. \tag{1.9}$$

Having p_k and using (1.3; 5) as well as the fact that (δ_n) tends to zero, for a given $\varepsilon > 0$ one may now select $n_k > n_{k-1}$ such that

$$K_{n_{k-1}} \sup_{q\in[0,p_k]} |h_{n_k}|_q \leqq 2^{-k}\varepsilon, \tag{1.10}$$

$$\delta_{n_k} \leqq 2^{-(k+1)}\varepsilon, \tag{1.11}$$

$$T_{n_k} h_{n_k} \geqq M_k := \begin{cases} M - \dfrac{1}{k} & \text{if } M < \infty, \\ k & \text{if } M = \infty. \end{cases} \tag{1.12}$$

Since each T_n is sublinear, one also has that for all $f, g \in Y, n \in \mathbf{N}$

$$\text{(1.13)} \qquad \max\{T_n(f+g), T_n(f-g)\} \geqq \frac{1}{2}[T_n(f+g) + T_n(f-g)] \geqq$$
$$\geqq \frac{1}{2} T_n\big((f+g)-(f-g)\big) = T_n g.$$

Hence it is possible to choose $\theta_k \in \{-1, 1\}$ such that

$$T_{n_k}\left(\sum_{j=1}^{k-1} \theta_j h_{n_j} + \theta_k h_{n_k}\right) \geqq T_{n_k} h_{n_k},$$

thus in view of (1.12)

$$\text{(1.14)} \qquad T_{n_k}\left(\sum_{j=1}^{k} \theta_j h_{n_j}\right) \geqq M_k.$$

Given $p \in [0, \infty)$, let $k \in \mathbf{N}$ be such that $p_k \leqq p < p_{k+1}$. Then for $k \leqq l \leqq m$ (cf. (1.10))

$$\left|\sum_{j=1}^{m} \theta_j h_{n_j} - \sum_{j=1}^{l} \theta_j h_{n_j}\right|_p \leqq \sum_{j=l+1}^{\infty} |\theta_j h_{n_j}|_p \leqq \sum_{j=l+1}^{\infty} \sup_{q\in[0,p_j]} |h_{n_j}|_q \leqq$$
$$\leqq \sum_{j=l+1}^{\infty} K_{n_{j-1}}^{-1} 2^{-j}\varepsilon \leqq \sum_{j=l+1}^{\infty} 2^{-j}\varepsilon = 2^{-l}\varepsilon$$

so that $\left(\sum_{j=1}^{m} \theta_j h_{n_j}\right)_{m\in\mathbf{N}}$ is a Cauchy sequence. Moreover, by (1.2; 9–11) it follows that for each $m \in \mathbf{N}$

$$\left|\sum_{j=1}^{m} \theta_j h_{n_j}\right|_p \leqq \sum_{j=1}^{k-1} |h_{n_j}|_p + |h_{n_k}|_p + \sum_{j=k+1}^{\infty} |h_{n_j}|_p \leqq$$
$$\leqq \sum_{j=1}^{k-1} 2\delta_{n_j} + C + \sum_{j=k+1}^{\infty} \sup_{q\in[0,p_j]} |h_{n_j}|_q \leqq$$
$$\leqq \sum_{j=1}^{k-1} 2^{-j}\varepsilon + C + \sum_{j=k+1}^{\infty} K_{n_{j-1}}^{-1} 2^{-j}\varepsilon \leqq C + \varepsilon.$$

Therefore, since $(Y, |\cdot|_p)$ is b-complete, the element $f_0 := \sum_{j=1}^{\infty} \theta_j h_{n_j}$ is well-defined in Y and satisfies (1.6). Furthermore, $T_n \in Y^*$ and (1.8; 10; 14) imply (1.7) since (note that $r_{n_k} < p_{k+1}$)

$$T_{n_k} f_0 \geqq T_{n_k}\left(\sum_{j=1}^{k} \theta_j h_{n_j}\right) - \sum_{j=k+1}^{\infty} T_{n_k} h_{n_j} \geqq$$

$$\geqq M_k - \sum_{j=k+1}^{\infty} K_{n_{j-1}} \sup_{q\in[0,p_j]} |h_{n_j}|_q \geqq M_k - \sum_{j=k+1}^{\infty} 2^{-j}\varepsilon =$$

$$= \begin{cases} M - \dfrac{1}{k} - 2^{-k}\varepsilon & \text{if } M < \infty, \\ k - 2^{-k}\varepsilon & \text{if } M = \infty. \end{cases} \qquad \square$$

Let us mention that the elegant argument around (1.13) is due to Davydov (cf. [5]). Note that Theorem 1 for the particular case $\delta_n = 0, n \in \mathbf{N}$, subsumes the result already given in [7] (see also the literature cited there). The present more general condition (1.4) was suggested by work of Zhou (see [16; 17]), but one should also consult the material presented in [13, p. 15ff] (and the literature cited there).

It would be nice to have Theorem 1 for $\varepsilon = 0$ as well (compare with the treatment given in [5; 6]). Obviously, it holds true if $M = \infty$ (simply substitute f_0 by $Cf_0/(C+\varepsilon)$). Moreover, it is also valid if $\delta_n = 0$ for all $n \in \mathbf{N}$. Indeed, one may then select p_k, n_k such that

$$(1.9^*) \qquad \sum_{j=1}^{k-1} |h_{n_j}|_p \leqq \frac{C}{k+2} \qquad \text{for all } p \geqq p_k,$$

$$(1.10^*) \qquad K_{n_{k-1}} \sup_{q\in[0,p_k]} |h_{n_k}|_q \leqq 2^{-k}\frac{C}{k+2}.$$

Now it is readily verified that the candidate

$$f_0 = \sum_{j=1}^{\infty} \frac{j}{j+2}\theta_j h_{n_j}$$

satisfies (1.6; 7) for $\varepsilon = 0$.

2. General applications

As already mentioned, the aim of this section is to reconsider and extend some of the previous quantitative resonance principles. Indeed, when Theorem 1 is applied to questions concerning the sharpness of error bounds, assertion (1.6) guarantees the counterexample f_0 to belong to certain smoothness (Lipschitz) classes. In concrete applications (e. g., in approximation theory, numerical analysis) the smoothness of elements is often measured in terms of a given family of bounded functionals (e. g., moduli of continuity of functions, K-functionals). In these cases it is therefore useful to work with resonance principles which in fact reflect this particular structure. Several such quantitative resonance principles have been developed during the last years. However, since the smoothness classes were described by abstract moduli of continuity which had to satisfy some additional condition (cf. (2.9)), not all classes could be examined. In this connection Theorem 1 now enables one to extend these results to all cases.

To this end, for a Banach space X let X^* be the corresponding class of non-negative, sublinear functionals T on X which are bounded, thus (cf. (1.1))

$$\|T\|_{X^*} := \sup_{0\neq f\in X} \frac{Tf}{\|f\|_X} < \infty. \tag{2.1}$$

Let ω be an abstract modulus of continuity, i. e., a function, continuous on $[0,\infty)$ such that

$$0=\omega(0)<\omega(s)\leqq\omega(s+t)\leqq\omega(s)+\omega(t) \qquad \text{for all } s,t>0.$$

Then the following inequality holds true (cf. [14, p. 99]):

$$\frac{\omega(t)}{t}\leqq 2\frac{\omega(s)}{s} \qquad \text{for all } 0<s\leqq t. \tag{2.2}$$

THEOREM 2. *Let X be a Banach space and $(T_n)_{n\in\mathbf{N}}, \{S_t : t>0\}\subset X^*$. Suppose that $(\Phi_n)_{n\in\mathbf{N}}$ is a sequence of strictly positive numbers tending to zero, $\sigma(t)$ a strictly positive function on $(0,\infty)$ and $\omega(t)$ an (arbitrary) abstract modulus of continuity. Moreover, assume that there exist $(h_n)_{n\in\mathbf{N}}\subset X$ such that*

$$\|h_n\|_X\leqq C_1 \qquad \textit{for all } n\in\mathbf{N}, \tag{2.3}$$

$$S_t h_n\leqq C_2\min\left\{1,\frac{\sigma(t)}{\Phi_n}\right\} \qquad \textit{for all } n\in\mathbf{N}, t>0, \tag{2.4}$$

$$S_t h_n\leqq \delta_n\frac{\omega(\sigma(t))}{\omega(\Phi_n)} \qquad \textit{for all } n\in\mathbf{N}, t>0 \textit{ with } \sigma(t)\leqq\varrho_n, \tag{2.5}$$

$$(2.6)\qquad \limsup_{n\to\infty} T_n h_n \geqq M > 0,$$

where $(\varrho_n)_{n\in\mathbf{N}}$ *and* $(\delta_n)_{n\in\mathbf{N}}$ *are (suitable) strictly positive sequences with* (δ_n) *tending to zero. Then for each* $\varepsilon > 0$ *there exists an element* $f_\omega = f_{\omega,\varepsilon} \in X$ *such that*

$$(2.7)\qquad S_t f_\omega \leqq (2C_2 + \varepsilon)\omega\big(\sigma(t)\big) \qquad \text{for all } t > 0,$$

$$(2.8)\qquad \limsup_{n\to\infty} \frac{T_n f_\omega}{\omega(\Phi_n)} \geqq M.$$

If ω *additionally satisfies (e. g.,* $\omega(t) = t$ *or* ω *concave* (cf. [14, p. 97]))

$$(2.2^*)\qquad \frac{\omega(t)}{t} \leqq \frac{\omega(s)}{s} \qquad \text{for all } 0 < s \leqq t,$$

then one may indeed replace (2.7) *by*

$$(2.7^*)\qquad S_t f_\omega \leqq (C_2 + \varepsilon)\omega\big(\sigma(t)\big) \qquad \text{for all } t > 0.$$

Proof. To apply Theorem 1, set $Y = X$ and ($[\![p]\!]$ being the largest integer strictly smaller than p)

$$|f|_p = \begin{cases} \|f\|_X & p = 0,\\ \sup\left\{\dfrac{S_t f}{\omega\big(\sigma(t)\big)} : \sigma(t) \in I_p := [\Phi_1, \infty)\right\} \quad \text{for} & p \in (0,1],\\ \sup\left\{\dfrac{S_t f}{\omega\big(\sigma(t)\big)} : \sigma(t) \in I_p := \big[\Phi_{[\![p]\!]+1}, \Phi_{[\![p]\!]}\big)\right\} & p > 1,\end{cases}$$

where $|\cdot|_p := 0$, if there does not exist any $t > 0$ with $\sigma(t) \in I_p$, or if $I_p = \emptyset$. Obviously, $(Y, |\cdot|_p)$ is well-defined (in the sense, considered in Section 1) and $X^* \subset Y^*$. To show that $(Y, |\cdot|_p)$ is b-complete, given a Cauchy sequence (f_n) with $\sup\{|f_n|_p : n \in \mathbf{N}, p \in [0,\infty)\} < \infty$, there exists an element $f \in X$ such that $|f - f_n|_0 = \|f - f_n\|_X$ tends to zero, since X is a Banach space. Moreover, for each $p > 0$ and $\eta > 0$ there exists $n_0 = n_0(\eta, p) \in \mathbf{N}$ such that $|f_m - f_n|_p < \eta$ for all $m, n \geqq n_0$. Consequently, since $S_t \in X^*$ (cf. (2.1)), one obtains that for $n \geqq n_0$ and for each $t > 0$ with $\sigma(t) \in I_p$

$$\frac{S_t(f - f_n)}{\omega\big(\sigma(t)\big)} \leqq \limsup_{m\to\infty}\left[\frac{S_t(f - f_m)}{\omega\big(\sigma(t)\big)} + \frac{S_t(f_m - f_n)}{\omega\big(\sigma(t)\big)}\right] \leqq$$

$$\leqq \limsup_{m\to\infty}\left[\frac{\|S_t\|_{X^*}\|f-f_m\|_X}{\omega\big(\sigma(t)\big)}+|f_m-f_n|_p\right]\leqq \eta.$$

Hence $|f-f_n|_p \leqq \eta$, and $(Y,|\cdot|_p)$ is b-complete. Let us apply Theorem 1 to

$$\tilde{h}_n := \omega(\Phi_n)h_n, \qquad \tilde{T}_n := \frac{T_n}{\omega(\Phi_n)}.$$

Then (2.6) in fact coincides with (1.5). In view of (2.2) and (2.4) it follows that

$$\frac{S_t\tilde{h}_n}{\omega\big(\sigma(t)\big)} \leqq C_2 \begin{cases} \dfrac{\omega(\Phi_n)}{\omega\big(\sigma(t)\big)} \leqq 1 & \text{if } \Phi_n \leqq \sigma(t), \\[2ex] \dfrac{\omega(\Phi_n)}{\omega\big(\sigma(t)\big)}\cdot\dfrac{\sigma(t)}{\Phi_n} \leqq 2 & \text{if } \sigma(t) \leqq \Phi_n, \end{cases}$$

and indeed $S_t\tilde{h}_n/\omega\big(\sigma(t)\big) \leqq C_2$, if ω satisfies (2.2*). Therefore, since $\big(\omega(\Phi_n)\big) \subset \mathbf{R}$ is bounded, (2.3) first of all implies (1.2) with $C = \max\{2C_2, \max_{n\in\mathbf{N}} C_1\omega(\Phi_n)\}$. However, since $\omega(\Phi_n)$ tends to zero, one may assume $\max_{n\in\mathbf{N}} C_1\omega(\Phi_n) \leqq 2C_2$ (without loss of generality). Thus (1.2) with constant $C = 2C_2$, and by the same reasoning the constant even reduces to $C = C_2$, if ω satisfies (2.2*). Concerning (1.3), for $p = 0$ one has by (2.3) that

$$|\tilde{h}_n|_0 = \omega(\Phi_n)\|h_n\|_X \leqq C_1\omega(\Phi_n) = o(1) \qquad (n\to\infty).$$

Moreover, in view of (2.4) for each $\sigma(t) \in I_p$

$$\frac{S_t\tilde{h}_n}{\omega\big(\sigma(t)\big)} \leqq C_2\frac{\omega(\Phi_n)}{\omega\big(\sigma(t)\big)} \leqq C_2\frac{\omega(\Phi_n)}{\omega(\Phi_{[\![p]\!]+1})} = o(1) \qquad (n\to\infty),$$

thus (1.3). Finally, (1.4) follows from (2.5) and the fact that $(\Phi_p)_{p\in\mathbf{N}}$ tends to zero. Indeed,

$$|\tilde{h}_n|_p = \sup_{\sigma(t)\in I_p}\frac{S_t\tilde{h}_n}{\omega\big(\sigma(t)\big)} \leqq \delta_n$$

for p large enough, thus $\sigma(t)$ small enough such that $\sigma(t) \leqq \varrho_n$. Obviously, the present assertions (2.7; 7*; 8) then follow by Theorem 1. □

To deal with arbitrary moduli of continuity, condition (2.5) (see [16; 17]) is added in comparison with the treatment in [7]. On the other hand, if ω is a modulus of continuity additionally satisfying

$$\lim_{t\to0+}\frac{\omega(t)}{t} = \infty, \tag{2.9}$$

then Theorem 2 indeed reduces to the corresponding result, already given in [7]. This is a consequence of the fact that (2.4) together with (2.9) implies (2.5). To this end, in view of (2.9), thus $t/\omega(t) = o(1)$, for each $n \in \mathbf{N}$ there exists $\varrho_n > 0$ such that

$$\frac{t}{\omega(t)} \leqq \frac{\Phi_n}{\sqrt{\omega(\Phi_n)}} \qquad \text{for all } t \leqq \varrho_n.$$

By (2.4) this implies that for $\sigma(t) \leqq \varrho_n$

$$S_t h_n \leqq C_2 \frac{\sigma(t)}{\omega(\sigma(t))} \frac{\omega(\Phi_n)}{\Phi_n} \frac{\omega(\sigma(t))}{\omega(\Phi_n)} \leqq C_2 \sqrt{\omega(\Phi_n)} \frac{\omega(\sigma(t))}{\omega(\Phi_n)},$$

thus (2.5) with $\delta_n = C_2\sqrt{\omega(\Phi_n)}$. Note that if there exists $a > 0$ such that $\sigma(t) > a$ for all $t > 0$, then (2.5) is trivially satisfied for $\varrho_n = a, n \in \mathbf{N}$.

As a consequence of Theorem 2 one may now establish the following (extended version of a) theorem by Davydov (see [5]).

THEOREM 3. *Let X be a Banach space and $(T_n)_{n\in\mathbf{N}}, \{S_t : t > 0\} \subset \subset X^*$. Suppose that for each $r > 0$ there exists a constant $B_r < \infty$ such that $\|S_t\|_{X^*} \leqq B_r$ for all $t \geqq r$. Moreover, assume that there exist $(h_n)_{n\in\mathbf{N}} \subset X$ such that $(C > 0)$*

$$S_t h_n \leqq C < \infty \qquad \textit{for all } n \in \mathbf{N}, t > 0, \tag{2.10}$$

$$\lim_{n\to\infty} \|h_n\|_X = 0, \tag{2.11}$$

$$\limsup_{t\to 0+} S_t h_n \leqq \delta_n \qquad \textit{for all } n \in \mathbf{N}, \tag{2.12}$$

$$\limsup_{n\to\infty} T_n h_n \geqq M > 0, \tag{2.13}$$

where $(\delta_n)_{n\in\mathbf{N}} \subset \mathbf{R}$ is a strictly positive sequence tending to zero. Then for each $\varepsilon > 0$ there exists an element $f_0 = f_{0,\varepsilon} \in X$ such that

$$S_t f_0 \leqq C + \varepsilon \qquad \textit{for all } t > 0, \tag{2.14}$$

$$\limsup_{n\to\infty} T_n f_0 \geqq M. \tag{2.15}$$

PROOF. Let us apply Theorem 2 to (cf. (2.11))

$$\Phi_n = \|h_n\|_X, \qquad \sigma(t) = C/\|S_t\|_{X^*}, \qquad \omega(t) = t,$$

$$\tilde{h}_n = h_n/\|h_n\|_X, \qquad \tilde{T}_n = \|h_n\|_X T_n, \qquad \tilde{S}_t = S_t/\|S_t\|_{X^*},$$

where, without loss of generality, it is assumed that $h_n \neq 0, S_t \neq 0$ for all $n \in \mathbf{N}, t > 0$, respectively. Then (2.13) in fact coincides with (2.6). Obviously, one has (2.3) with $C_1 = 1$, and (2.4) follows with $C_2 = 1$ since (cf. (2.10))

$$\tilde{S}_t\tilde{h}_n = \frac{S_t h_n}{\|S_t\|_{X^*}\|h_n\|_X} \leqq \begin{cases} \dfrac{\|S_t\|_{X^*}\|h_n\|_X}{\|S_t\|_{X^*}\|h_n\|_X} = 1, \\ \dfrac{C}{\|S_t\|_{X^*}\|h_n\|_X} = \dfrac{\sigma(t)}{\Phi_n}. \end{cases}$$

Concerning (2.5), in view of (2.12), for each $n \in \mathbf{N}$ there exists $r_n > 0$ such that $S_t h_n \leqq 2\delta_n$ for all $0 < t \leqq r_n$. On the other hand, by assumption, there exists B_{r_n} such that $\|S_t\|_{X^*} \leqq B_{r_n}$ for all $t \geqq r_n$. Therefore, setting $\varrho_n = C/2B_{r_n}$, if $t > 0$ is such that $\sigma(t) \leqq \varrho_n$, thus $\|S_t\|_{X^*} \geqq 2B_{r_n}$, one necessarily concludes $t < r_n$, hence $S_t h_n \leqq 2\delta_n$. Consequently, for all $t > 0$ with $\sigma(t) \leqq \varrho_n$

$$\tilde{S}_t\tilde{h}_n = \frac{S_t h_n}{\|S_t\|_{X^*}\|h_n\|_X} = \frac{1}{C}\frac{\sigma(t)}{\Phi_n}S_t h_n \leqq \frac{1}{C}\frac{\omega\big(\sigma(t)\big)}{\omega(\Phi_n)}2\delta_n,$$

and (2.5) is established. An application of Theorem 2 to the present quantities then shows that for each $\tilde{\varepsilon} > 0$ there exists $f_0 = f_{0,\tilde{\varepsilon}} \in X$ such that (cf. (2.7*))

$$\frac{S_t f_0}{\|S_t\|_{X^*}} = \tilde{S}_t f_0 \leqq (1+\tilde{\varepsilon})\omega\big(\sigma(t)\big) = (1+\tilde{\varepsilon})\frac{C}{\|S_t\|_{X^*}},$$

thus (2.14) for $\tilde{\varepsilon} = \varepsilon/C$, as well as (cf. (2.8))

$$\limsup_{n\to\infty} T_n f_0 = \limsup_{n\to\infty} \frac{\tilde{T}_n f_0}{\omega(\Phi_n)} \geqq M. \quad \square$$

Let us mention that, instead of using Theorem 2, a proof of Theorem 3 could also be given directly via Theorem 1 as applied to $Y = X$ and

$$|f|_p = \begin{cases} \|f\|_X & \text{for } p = 0, \\ S_{1/p}f & \text{for } p > 0. \end{cases}$$

Therefore the remark at the end of Section 1 shows that Theorem 3 holds true even for $\varepsilon = 0$, if (2.12) is satisfied for $\delta_n = 0, n \in \mathbf{N}$. In that case Theorem 3 indeed reduces to the result given in [5].

Let us conclude this section with a consequence (see [6]) of Theorem 3, useful in connection with concrete applications (cf. Section 3.1).

COROLLARY 1. *Let X be a Banach space and $(T_n)_{n\in\mathbf{N}} \subset X^*$ be such that*

$$\limsup_{n\to\infty} \|T_n\|_{X^*} = \infty. \tag{2.16}$$

Moreover, let $Z := \{f \in X : \limsup_{n\to\infty} T_n f < \infty\}$ be dense in X. Then there exists an element $f_0 \in X$ such that

$$\sup_{n\in\mathbf{N}} T_n f_0 < \infty, \tag{2.17}$$

$$\limsup_{n\to\infty} T_n f_0 > 0. \tag{2.18}$$

PROOF. Once Theorem 3 is established, one may follow the argument given in [6]. Nevertheless, let us outline the proof for the sake of completeness. To this end, if there exists $f_0 \in Z$ such that $\limsup_{n\to\infty} T_n f_0 > 0$, then the assertion follows in view of the definition of Z. Thus suppose that

$$\lim_{n\to\infty} T_n f = 0 \qquad \text{for all } f \in Z. \tag{2.19}$$

In view of (2.16) and the definition of $\|T_n\|_{X^*}$ (cf. (2.1)), for each $n \in \mathbf{N}$ there exist $n \leqq j_n \in \mathbf{N}, f_{j_n} \in X$ satisfying

$$\|f_{j_n}\|_X \leqq 1, \qquad T_{j_n} f_{j_n} \geqq n.$$

Since Z is dense in X, one may assume $f_{j_n} \in Z$. For $g_n := f_{j_n}/n \in Z$ it then follows that

$$\|g_n\|_X \leqq \frac{1}{n}, \qquad T_{j_n} g_n \geqq 1. \tag{2.20}$$

On the other hand, by (2.19) for each $n \in \mathbf{N}$ there exists $N := N_n \in \mathbf{N}$ such that $T_k g_n \leqq 1$ for all $k \geqq N$. For the remaining $k < N$, however, one either has $T_k g_n \leqq 1$, if $\|T_k\|_{X^*} \leqq n$, or

$$T_k g_n \leqq \kappa_n := \max\{T_j g_n : j < N, \|T_j\|_{X^*} > n\}.$$

Thus for $h_n := g_n / \max\{1, \kappa_n\} \in Z$ one obtains

$$T_k h_n \leqq 1 \qquad \text{for all } k, n \in \mathbf{N}. \tag{2.21}$$

With the aid of $(h_n)_{n\in\mathbf{N}}$ one can now select a sequence (T_{k_n}) as follows: If $\kappa_n \leqq 1$, set $k_n = j_n$ (cf. (2.20)); otherwise there exists $k_n \in \mathbf{N}$ with $\|T_{k_n}\| > n$ and $T_{k_n} g_n = \kappa_n$. Thus, in both cases it follows that

$$T_{k_n} h_n \geqq 1. \tag{2.22}$$

An application of Theorem 3 to $S_t = T_n$ for $1/n \leqq t < 1/(n-1)$ $(t > 0, n \in \mathbf{N})$ and to $\tilde{T}_n = T_{k_n}$ for $n \in \mathbf{N}$ then delivers (2.17; 18), since (2.10–13) are satisfied for $C = M = 1$ (cf. (2.19–22) and note that $h_n \in Z$). □

3. Applications to mean ergodic theorems and projection operators

3.1. *Mean ergodic theorems.* Let X be a (real) Banach space and T a mapping of X into itself which is linear and bounded, thus $T \in [X]$ with $\|T\|_{[X]} := \sup\{\|Tf\|_X : \|f\|_X \leqq 1\} < \infty$ (cf. (2.1)). Let I be the identity operator and $\mathbf{P} := \mathbf{N} \cup \{0\}$. For $\alpha > 0$ consider the Cesàro (C, α)-means of the sequence $(T^n)_{n \in \mathbf{P}} \subset [X]$, given by

$$\sigma_n^\alpha(T) := \frac{1}{C_n^{\alpha+1}} \sum_{j=0}^{n} C_{n-j}^\alpha T^j, \qquad C_n^\alpha := \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!}.$$

For $\alpha \geqq 1$ it is a well-known fact (see [4], cf. [3] and the literature cited there) that for operators $T \in [X]$ with the property

$$\|T^n\|_{[X]} \leqq K < \infty \qquad \text{for all } n \in \mathbf{P} \tag{3.1}$$

there holds true the mean ergodic theorem

$$\lim_{n \to \infty} \|\sigma_n^\alpha(T)f - Pf\|_X = 0 \qquad \text{for all } f \in X_0, \tag{3.2}$$

where $X_0 := N(I-T) \oplus \overline{R(I-T)}$ is the direct sum of the kernel $N(I-T)$ and the closure $\overline{R(I-T)}$ of the range of $I - T$ and where P is the bounded linear projection of X_0 onto $N(I-T)$, parallel to $\overline{R(I-T)}$. In [2; 3] the convergence assertion (3.2) was then equipped with rates. Indeed, with the aid of the projection P and the restriction T_0 of T to X_0 one may introduce the operator A via

$$Af = g \qquad \text{on } D(A) := N(I-T_0) \oplus R(I-T_0), \tag{3.3}$$

where $g \in X_0$ is uniquely determined by $(I-P)f = (I-T_0)g$ and $Pg = 0$. It follows that $D(A)$ is a linear subspace which is dense in X_0, and that A is a closed linear operator of $D(A)$ into X_0. Consider the K-functional

$$K(f,t) := K\big(f,t;X_0,D(A)\big) := \inf_{g \in D(A)} \big[\|f-g\|_X + t\|Ag\|_X\big].$$

Then for $\alpha \geqq 1$ and for each $T \in [X]$ satisfying (3.1) there holds true the direct approximation theorem

$$(3.4) \qquad \left\| \sigma_n^\alpha(T)f - Pf \right\|_X \leqq B_\alpha K\left(f, \frac{1}{n+1}; X_0, D(A)\right),$$

where the constant B_α is independent of $f \in X_0, n \in \mathbf{P}$ (see [3]). Moreover, it was essentially shown in [3; 9] that for $f \in X_0 \quad (n \to \infty)$

$$(3.5) \quad \left\| \sigma_n^\alpha(T)f - Pf \right\|_X = O\left(\omega\left(\frac{1}{n}\right)\right) \Leftrightarrow K\left(f, \frac{1}{n+1}\right) = O\left(\omega\left(\frac{1}{n}\right)\right),$$

where ω is an arbitrary modulus of continuity. In this connection an application of Corollary 1 now delivers the following result on the sharpness of (3.4), quite parallel to the treatment given in [6] for the approximation by semigroups of operators.

COROLLARY 2. *Let X be a Banach space and $T \in [X]$ satisfy* (3.1). *Suppose A as defined via* (3.3) *is unbounded. Then for $\alpha \geqq 1$ and for each modulus of continuity ω there exists a counterexample $f_\omega \in X_0$ such that* $(n \to \infty)$

$$\left\| \sigma_n^\alpha(T)f_\omega - Pf_\omega \right\|_X \begin{cases} = O\left(\omega\left(\frac{1}{n}\right)\right), \\ \neq o\left(\omega\left(\frac{1}{n}\right)\right). \end{cases}$$

PROOF. Since $X_0 \subset X$ is a closed linear subspace, X_0 is a Banach space, too. Setting

$$T_n f = \frac{\left\| \sigma_n^\alpha(T)f - Pf \right\|_X}{\omega\left(\frac{1}{n}\right)} \qquad \text{for } f \in X_0, n \in \mathbf{N},$$

one obviously has $T_n \in X_0^*$. It is essentially shown in [8; 10] that (even under the weaker condition $\left\| \sigma_n^{\alpha-1}(T) \right\|_{[X]} = o(n)$) the assumption, A being unbounded, is equivalent to

$$\limsup_{n\to\infty} \left\| \sigma_n^\alpha(T) - P \right\|_{[X_0]} > 0,$$

which implies

$$\limsup_{n\to\infty} \|T_n\|_{X_0^*} = \limsup_{n\to\infty} \frac{\left\| \sigma_n^\alpha(T) - P \right\|_{[X_0]}}{\omega\left(\frac{1}{n}\right)} = \infty.$$

Moreover, in view of (3.4), for each $f \in D(A)$

$$\left\| \sigma_n^\alpha(T)f - Pf \right\|_X \leqq B_\alpha \frac{\|Af\|_X}{n+1},$$

and therefore by (2.2) for each $f \in D(A)$

$$\limsup_{n\to\infty} T_n f \leqq \limsup_{n\to\infty} B_\alpha \|Af\|_X \frac{\frac{1}{n}}{\omega\left(\frac{1}{n}\right)} \leqq 2B_\alpha \|Af\|_X \frac{1}{\omega(1)} < \infty.$$

Since $D(A)$ is dense in X_0, it follows that $Z = \{f \in X_0 : \limsup_{n\to\infty} T_n f < < \infty\}$ is dense in X_0. Hence an application of Corollary 1 establishes the assertion. □

The sharpness of the error bound (3.4) is discussed in Corollary 2 under the assumption that A is unbounded. If, however, A is bounded, then indeed $D(A) = X$ and therefore (cf. [2])

$$\left\| \sigma_n^\alpha(T)f - Pf \right\|_X = O\left(\frac{1}{n}\right) \qquad \text{for each } f \in X$$

as well as

$$\left\| \sigma_n^\alpha(T)f - Pf \right\|_X \neq o\left(\frac{1}{n}\right) \qquad \text{if and only if} \quad f \notin N(I-T).$$

Let us finally mention that analoguous results also hold true for the Abel means $(1-r)\sum_{j=0}^{\infty} r^j T^j$ (cf. [2; 3; 8; 10]).

3.2. *Trigonometric projection operators.* The material of this subsection originates in work of P. O. Runck, J. Szabados and P. Vértesi [12] on the approximation by sequences $(P_n)_{n\in\mathbf{P}} \subset [C_{2\pi}]$ of trigonometric projection operators, thus in particular $P_n t_n = t_n$ for all $t_n \in \Pi_n$, the set of trigonometric polynomials of degree less than or equal to n. Here $C_{2\pi}$ is the space of all continuous, 2π–periodic functions f, endowed with the usual norm $\|f\|_{C_{2\pi}} := \max_{x\in\mathbf{R}} |f(x)|$. Moreover, for $r \in \mathbf{P}$ let $C_{2\pi}^{(r)} \subset C_{2\pi}$ be the space of r-times continuously differentiable functions f with norm $\|f\|_{C_{2\pi}^{(r)}} := := \max_{0\leqq j\leqq r} \left\| f^{(j)} \right\|_{C_{2\pi}}$. Consider the Lipschitz classes $(0 < \alpha \leqq 1)$

$$\operatorname{Lip}\alpha := \left\{ f \in C_{2\pi} : \omega(f,t) = O(t^\alpha), t \to 0+ \right\},$$

$$\omega(f,t) := \sup\left\{ \left\| f(x+u) - f(x) \right\|_{C_{2\pi}} : |u| \leqq t \right\}.$$

In these terms it was shown in [12] (see also [15]) that for each (abstract) modulus of continuity ω satisfying (2.9) there exists a counterexample $f_\omega \in C_{2\pi}^{(r)}$ such that

$$\omega\left(f_\omega^{(r)}, t\right) = O\big(\omega(t)\big),$$

$$\limsup_{n\to\infty} \frac{\left\| P_n^{(r)} f_\omega - f_\omega^{(r)} \right\|_{C_{2\pi}}}{\omega\left(\frac{1}{n}\right)\log n} > 0.$$

This result was then extended to the case $\omega(t) = t$ by S. P. Zhou [16; 17]. Here we would like to reconsider the latter case in the light of Theorem 1.

COROLLARY 3. *Let $r \in \mathbf{P}$ and $(P_n)_{n\in\mathbf{P}} \subset [C_{2\pi}]$ be a sequence of trigonometric projection operators such that for all $f \in C_{2\pi}, u \in \mathbf{R}, n \in \mathbf{P}$*

$$\left\| P_n^{(r)}\big(f(\cdot)\big)(x) \right\|_{C_{2\pi}} = \left\| P_n^{(r)}\big(f(\cdot + u)\big)(x) \right\|_{C_{2\pi}}. \tag{3.6}$$

Then there exists a counterexample $f_0 \in C_{2\pi}^{(r)}$ such that

$$f_0^{(r)} \in \operatorname{Lip} 1, \tag{3.7}$$

$$\limsup_{n\to\infty} \frac{n}{\log n} \left\| P_n^{(r)} f_0 - f_0^{(r)} \right\|_{C_{2\pi}} > 0. \tag{3.8}$$

PROOF. Setting $Y = \{f \in C_{2\pi}^{(r)} : f^{(r)} \in \operatorname{Lip} 1\}$, one has by a theorem of Hardy–Littlewood (cf. [1, p. 366]) that

$$Y = \Big\{ f \in C_{2\pi}^{(r)} : \text{there exists } g\,(= f^{(r+1)}) \in L_{2\pi}^{\infty} \text{ such that}$$

$$f^{(r)}(x) = f^{(r)}(-\pi) + \int_{-\pi}^{x} g(u)du \text{ for all } x \in \mathbf{R} \Big\},$$

where $L_{2\pi}^{\infty}$ is the space of measurable, 2π-periodic functions, essentially bounded on $\mathbf{R}$. Introducing the seminorms

$$|f|_p := \begin{cases} \|f\|_{C_{2\pi}^{(r)}} & \text{if } p = 0, \\ \operatorname{ess\,sup}\left\{ \left| f^{(r+1)}(x) \right| : |x| \in I_p := \left[\frac{\pi}{[\![p]\!]+2}, \frac{\pi}{[\![p]\!]+1}\right] \right\} & \text{if } p > 0, \end{cases}$$

it follows that $(Y, |\cdot|_p)$ is b-complete (cf. [7]). Indeed, consider a Cauchy sequence $(f_n) \subset Y$ with $|f_n|_p \leqq B$ for all $n \in \mathbf{N}, p \in [0, \infty)$. Since $C_{2\pi}^{(r)}$ and $L^\infty(I_p)$ are Banach spaces, there exist $f \in C_{2\pi}^{(r)}$ and a measurable function g on $[-\pi, \pi]$, continued 2π-periodically, such that

$$\lim_{n\to\infty} \|f - f_n\|_{C_{2\pi}^{(r)}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \left[\operatorname*{ess\,sup}_{x\in I_p} \left| f_n^{(r+1)}(x) - g(x) \right| \right] = 0.$$

Using $\operatorname{ess\,sup}_{x\in I_p} |f_n^{(r+1)}(x)| \leqq B$ for all $p > 0$ and $n \in \mathbf{N}$, one also obtains $\operatorname{ess\,sup}_{x\in I_p} |g(x)| \leqq B$ for all $p > 0$, and thus $g \in L_{2\pi}^\infty$. Since $f_n \in Y$, it then follows by Lebesgue's dominated convergence theorem that

$$f^{(r)}(-\pi) + \int_{-\pi}^{x} g(u)du = \lim_{n\to\infty} f_n^{(r)}(-\pi) + \lim_{n\to\infty} \int_{-\pi}^{x} f_n^{(r+1)}(u)du =$$

$$= \lim_{n\to\infty} f_n^{(r)}(x) = f^{(r)}(x),$$

thus $f \in Y$, and Y is b-complete. To apply Theorem 1, consider the (normalized) error functionals $T_n \in Y^*$, given by

$$T_n f = \frac{n}{\log n} \left\| P_n^{(r)} f - f^{(r)} \right\|_{C_{2\pi}},$$

in connection with the testelements ($r \in \mathbf{P}, 2 \leqq n \in \mathbf{N}$, see [16; 17])

$$h_n(x) = n^{-1} \sum_{k=[\![n^{1/2}]\!]}^{[\![n^{2/3}]\!]} \left[\frac{\cos\left((n-k)x - \frac{r\pi}{2}\right)}{k(n-k)^r} - \frac{\cos\left((n+k)x - \frac{r\pi}{2}\right)}{k(n+k)^r} \right].$$

It was already shown in [16] that these trigonometric polynomials indeed possess the properties ($M > 0$)

$$\left\| h_n^{(j)} \right\|_{C_{2\pi}} \leqq C_1 n^{-1} \qquad \text{for } 0 \leqq j \leqq r, \tag{3.9}$$

$$\left\| h_n^{(r+1)} \right\|_{C_{2\pi}} \leqq C_2, \tag{3.10}$$

$$\left| h_n^{(r+1)}(x) \right| \leqq \begin{cases} A_1 n^{-1/3} & \text{if } \quad |x| \leqq n^{-1}, \\ A_2 n^{-1/4} & \text{if } n^{-1/4} \leqq |x| \leqq \pi, \end{cases} \tag{3.11}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h_n^{(r)}(x) D_n(x) dx \geqq M n^{-1} \log n, \tag{3.12}$$

where $D_n(x) := 1 + 2\sum_{k=1}^{n} \cos kx$ is the n-th Dirichlet kernel. Therefore conditions (1.2–4) are satisfied in view of (3.9–11). Concerning (1.5) we follow the argument, given in [12], and use the Faber–Marcinkiewicz–Berman identity (cf. [11, p. 97]) to deduce

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^{(r)}\big(h_n(\cdot+u)\big)(-u)du = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_n^{(r)}(u)D_n(u)du.$$

Since $P_n^{(r)}\big(h_n(\cdot+u)\big)(-u)$ is continuous, there exists a point u_n where $|P_n^{(r)}\big(h_n(\cdot+u)\big)(-u)|$ attains its maximum so that (3.12) implies

$$\left\|P_n^{(r)}\big(h_n(\cdot+u_n)\big)\right\|_{C_{2\pi}} \geqq \left|P_n^{(r)}\big(h_n(\cdot+u_n)\big)(-u_n)\right| \geqq$$

$$\geqq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^{(r)}\big(h_n(\cdot+u)\big)(-u)du = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_n^{(r)}(u)D_n(u)du \geqq M\frac{\log n}{n}.$$

Therefore in view of (3.6; 9)

$$T_n h_n \geqq \frac{n}{\log n}\left[\left\|P_n^{(r)}\big(h_n(\cdot+u_n)\big)(x)\right\|_{C_{2\pi}} - \left\|h_n^{(r)}\right\|_{C_{2\pi}}\right] \geqq M - \frac{C_1}{\log n},$$

thus (1.5). An application of Theorem 1 then delivers the assertions (3.7; 8). □

Let us mention that, in contrast to [12], one cannot directly work with $h_n(\cdot+u_n)$ as resonance elements, since on the one hand the estimate (3.11) is used to establish (1.3; 4), on the other hand nothing is known about the location of the points u_n. Therefore it seems that condition (3.6) has to be added to the argument in [16; 17], too.

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SPLINE ORTHOGONAL SYSTEMS AND FRACTAL FUNCTIONS

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To Professor K. Tandori on his 70th birthday

1. Introduction. In recent years more and more attention is paid in mathematical papers to *fractal functions* and to *fractal sets.* There are various definitions of those objects. We assume that a compact set $K \subset R^{d+1}$ is fractal, by definition, if its *box (entropy) dimension* $\dim_b(K) \neq \neq j$ for $j = 0, 1, \ldots, d+1$ and $0 < \dim_b(K) < d+1$. At the same time the function $f : I^d \to R$, $I = [0,1]$, is *fractal,* by definition, if its graph $\Gamma_f = \{(\mathbf{t}, f(\mathbf{t})) : \mathbf{t} \in I^d\}$ has box dimension satisfying the inequalities $d < \dim_b(\Gamma_f) < d+1$. For the definitions and properties of lower $\underline{\dim}_b(K)$ and upper $\overline{\dim}_b(K)$ box (-counting) dimension we refer to [6]. In case $\underline{\dim}_b(K) = \overline{\dim}_b(K)$ by definition $\dim_b(K)$ is the common value. However for the sake of completeness we recall one of the equivalent definitions of the box dimension of a given $K \subset R^d$. For given $\delta > 0$ we consider the family $\mathcal{Q}(\delta)$ of closed cubes generated by the d-fold Cartesian product of the mesh $\{i\delta, i \in \mathbf{Z}\}$. Denote the number of cubes from $\mathcal{Q}(\delta)$ intersecting K by $N(K;\delta)$. Now,

$$\overline{\dim}_b(K) = \limsup_{\delta \to 0_+} \frac{\log N(K;\delta)}{\log \frac{1}{\delta}}, \quad \underline{\dim}_b(K) = \liminf_{\delta \to 0_+} \frac{\log N(K;\delta)}{\log \frac{1}{\delta}}.$$

In these definitions one can restrict considerations to $\delta = \frac{1}{2^j}$ as $j \to \infty$. The relation between box dimension and the graph of a function satisfying *Hölder condition* is known for years. In particular, it is not hard to see that the Hölder condition with some α, $0 < \alpha \leqq 1$, i.e.

$$(1.1) \qquad \left| f(\mathbf{t}) - f(\mathbf{t}') \right| \leqq C \cdot |\mathbf{t} - \mathbf{t}'|^\alpha \quad \text{for} \quad \mathbf{t}, \mathbf{t}' \in I^d,$$

implies that

$$(1.2) \qquad \overline{\dim}_b(\Gamma_f) \leqq d + 1 - \alpha.$$

Our aim is to describe some classes of functions f, e.g. subclasses satisfying (1.1), for which the box dimension exists and the equality takes place in (1.2). The Hölder classes, as it was shown in [2], can be characterized by means of the coefficients of the spline orthogonal expansions, and it seems natural to apply this tool to solve our problem. For simplicity, the result is presented here in case $d = 1$. The extension to $d > 1$ is possible with essentially no new ideas (cf. [3]).

In Section 2 we describe the construction of the orthogonal spline system on $I = [0,1]$ and recall some of its essential properties. Senction 3 contains the main result on Hölder subclasses for which we have (1.2) and

$$\underline{\dim}_b(\Gamma_f) \geqq d + 1 - \beta. \tag{1.3}$$

2. The orthogonal spline systems. The orthogonal Haar functions over I, normalized in the maximum norm, can be defined by means of the function sign(t). Define

$$h_0(t) = \frac{\operatorname{sign}\left(t + \frac{1}{2}\right) - \operatorname{sign}\left(t - \frac{1}{2}\right)}{2},$$

$$h_1(t) = \frac{\operatorname{sign}\left(t + \frac{1}{2}\right) + \operatorname{sign}\left(t - \frac{1}{2}\right)}{2} - \operatorname{sign}(t) \quad \text{for} \quad t \in R$$

and

$$h_{j,k}(t) = h_1\left(2^k\left(t - \frac{2j-1}{2^{k+1}}\right)\right) \quad \text{where} \quad j = 1, \dots, 2^k; \quad k = 0, 1, \dots.$$

The *Haar orthogonal system* on I with respect to the Lebesgue measure is complete and it is denoted by

$$\mathcal{H} := \{1, h_{j,k},\ j = 1, \dots, 2^k;\ k = 0, 1, \dots\}. \tag{2.1}$$

This is an orthogonal system of spline functions of order 1 i.e. of degree 0. We note also that

$$\operatorname{supp} h_{j,k} = \left[\frac{(j-1)}{2^k}, \frac{j}{2^k}\right].$$

The Haar functions are not continuous. Performing integration on (2.1) over $[0,t]$ and then completing the result with a constant function, after normalization in the max norm, we obtain the *Faber–Schauder basis*

$$\mathcal{S} := \{1, t, s_{j,k},\ j = 1, \dots, 2^k;\ k = 0, 1, \dots\}. \tag{2.2}$$

Again

$$\operatorname{supp} s_{j,k} = \left[\frac{(j-1)}{2^k}, \frac{j}{2^k}\right].$$

The *Franklin orthogonal system* is obtained now from (2.2) by applying the Gramm–Schmidt orthogonalization procedure and then normalizing them in the max norm

$$\mathcal{F} := \{f_0, f_1, f_{j,k},\ j = 1, \dots, 2^k;\ k = 0, 1, \dots\}, \tag{2.3}$$

where $f_0 = 1$ and $f_1(t) = 2t - 1$. This time $\operatorname{supp} f_{j,k} = I$ but instead we have exponential estmimates for the Franklin functions [1]. The Franklin functions are splines of order 2.

The three step procedure of integration, orthogonalization and normalization used above to the Haar system can now be applied to the Franklin system to obtain an orthogonal spline system of order 3. Repeating this procedure $r-1$ times we get the *orthogonal system of splines of order* r

$$\mathcal{F}^{(r)} := \left\{ f_{2-r}^{(r)}, \dots, f_1^{(r)}, f_{j,k}^{(r)},\ j = 1, \dots, 2^k;\ k = 0, 1, \dots \right\}, \tag{2.4}$$

where the first r functions $\left\{ f_{2-r}^{(r)}, \dots, f_1^{(r)} \right\}$ are the Legendre orthogonal polynomials on I normalized in the max norm. Clearly $\mathcal{F}^{(1)} = \mathcal{H}$ and $\mathcal{F}^{(2)} = \mathcal{F}$.

It is convenient to introduce the following notation for the partial sums of the Fourier expansion of $f \in L^p(I)$ with respect to $\mathcal{F}^{(r)}$ for $j \geqq 0$:

$$P_j f = \sum_{i=-1}^{j} Q_i f, \tag{2.5}$$

with

$$Q_{-1} f = \sum_{i=2-r}^{1} \left(f, f_i^{(r)}\right) f_i^{(r)} \quad \text{and} \quad Q_j f = \sum_{k=1}^{2^j} \left(f, f_{j,k}^{(r)}\right) f_{j,k}^{(r)}, \quad j \geqq 0. \tag{2.6}$$

The family of projection operators $\{P_j, j = -1, 0, 1, \dots\}$ acting on $L^p(I)$, $1 \leqq p \leqq \infty$, is bounded uniformly in j and p. For this and other properties of $\mathcal{F}^{(r)}$ we refer to [2] and [4].

3. Box dimension of functions given by spline series. The first theorem is related to the charcterization by spline basis expansions of functions satisfying Hölder condition.

THEOREM 3.1. *Let* $r \geqq 1$, $0 < \alpha \leqq 1$,

$$\limsup_j \left(2^{\alpha j} \max_k |a_{j,k}|\right) < \infty, \tag{3.2}$$

and let

$$f := \sum_j \sum_k a_{j,k} f_{j,k}^{(r)}. \tag{3.3}$$

Then

$$\overline{\dim}_b(\Gamma_f) \leqq 2 - \alpha. \tag{3.4}$$

PROOF. For $r > 1$ and $\alpha < 1$ condition (3.2) implies that f satisfies (1.1) and consequently (1.2). In case $r > 1$ and $\alpha = 1$ condition (3.2) implies that for each $0 < \epsilon < 1$

$$\left| f(t) - f(t') \right| \leqq C_\epsilon \cdot |t - t'|^{1-\epsilon} \quad \text{for} \quad t, t' \in I,$$

whence (3.4) follows as well. It remains the Haar case $r = 1$. Now,

$$\|f - P_{j-1} f\|_{L^\infty(I)} \leqq \sum_{i=j}^{\infty} \|Q_i f\|_{L^\infty(I)} \leqq \sum_{i=j}^{\infty} \max_{1 \leqq k \leqq 2^i} |a_{i,k}| \leqq C \cdot 2^{-\alpha j}.$$

On the other hand introducing $I_{j,k} = \operatorname{supp} h_{j,k}$ we note that

$$\|f - P_{j-1} f\|_{L^\infty(I)} = \max_k \|f - P_{j-1} f\|_{L^\infty(I_{j,k})},$$

and for each k, $1 \leqq k \leqq 2^j$, we have

$$\|f - P_{j-1} f\|_{L^\infty(I_{j,k})} = \left\| f - \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f \right\|_{L^\infty(I_{j,k})} \geqq$$

$$\geqq \frac{1}{2} \operatorname{Osc}(f; I_{j,k}) \geqq \frac{1}{2^{j+1}} N\left(\Gamma_f, I_{j,k}; \frac{1}{2^j} \right),$$

where $\operatorname{Osc}(f; J) = \left\{ \sup \left| f(t) - f(s) \right| : t, s \in J \right\}$ and $N(\Gamma_f, J; \delta)$ is the number of squares, with sides of size δ, over J which cover the part of the graph Γ_f lying over this interval. It now follows that

$$\|f - P_{j-1} f\|_{L^\infty(I)} \geqq \frac{1}{2^{2j+1}} N\left(\Gamma_f, I; \frac{1}{2^j} \right).$$

This in combination with the previous inequality completes the proof.

The next theorem gives sufficient conditions on the coefficients $a_{j,k}$ in (3.3) for the estimate from below for the lower box dimension of Γ_f with f given by (3.3).

THEOREM 3.5. *Let $r \geqq 1$, $0 < \beta \leqq 1$ and let*

$$\liminf_j \left(2^{\beta j} \frac{1}{2^j} \sum_k |a_{j,k}| \right) > 0. \tag{3.6}$$

Then for f for which (3.3) converges uniformly on I we have

$$\underline{\dim}_b(\Gamma_f) \geqq 2 - \beta. \tag{3.7}$$

PROOF. For the proof let us introduce the local spline operators

$$T_j f(t) = \int_I K_j(t,s) f(s) ds, \quad t \in I,$$

$$K_j(t,s) = \sum_m 2^j N_{j,m}(t) N_{j,m}(s),$$

where

$$N_{j,m}(t) = \frac{r}{2^j} \cdot \left[\frac{m}{2^j}, \dots, \frac{(m+r)}{2^j}; (\cdot - t)_+^{r-1} \right].$$

Here $[t_0, \dots, t_r; f(\cdot)]$ denotes the divided difference of order r of the function f taken at the point $t_0, \dots, t_r$ and $t_+ = \max(t, 0)$. The functions $(N_{j,m})$ form a partition of unity. The kernel K_j is positive and the $L^p(I)$ norms of the cerresponding operators T_j are all equal to 1. Moreover, for $1 \leqq j \leqq 2^k$ we have

$$\int_{I_{j,k}} |f(t) - T_j f(t)| \, dt \leqq \int_{I_{j,k}} \sum_{k-r \leqq m < k} N_{j,m}(t) \int_I |f(s) - f(t)| N_{j,m}(s) \, ds \, dt \leqq$$

$$\leqq \frac{1}{2^j} \sup \left\{ |f(s) - f(t)| : t, s \in \left(\frac{k-r}{2^j}, \frac{k+r}{2^j} \right) \right\} \leqq \frac{1}{2^j} \sum_{|m-k|<r} \operatorname{Osc}(f; I_{j,m}).$$

Now,

$$\operatorname{Osc}(f; I_{j,m}) \leqq \frac{1}{2^j} N\left(\Gamma_f, I_{j,m}; \frac{1}{2^j} \right).$$

Consequently,

$$\int_I |f(t) - T_j f(t)| \, dt \leqq \frac{2r}{2^{2j}} \sum_k N\left(\Gamma_f, I_{j,k}; \frac{1}{2^j} \right) = \frac{2r}{2^{2j}} N\left(\Gamma_f, \frac{1}{2^j} \right).$$

Applying the basic properties of the system $\mathcal{F}^{(r)}$ we obtain

$$\frac{1}{2^j}\sum_k |a_{j,k}| \leqq C\|Q_j f\|_{L^1(I)} \leqq C'\|f - Q_{j-1}f\|_{L^1(I)} \leqq$$

$$\leqq C''\|f - T_j f\|_{L^1(I)} \leqq C'''\frac{1}{2^{2j}} N\left(\Gamma_f, \frac{1}{2^j}\right),$$

and this completes the proof.

REMARKS. It follows from the proofs that Theorems 3.1 and 3.5 hold for the Faber–Schauder $\mathcal{S}$ as well.

COMMENTS. For discussion on related topics we refer to the recent papers [7] and [5]. The results presented here can be used to calculate the box dimension for some particular functions: Using the Haar system one can find the box dimention $2-\alpha$ of the graph of the particular Weierstrass function

$$\sum_{j=0}^{\infty} \frac{1}{2^{j\alpha}} \sin(2\pi 2^j t), \quad 0 < \alpha < 1.$$

The Haar system can be applied to calculate the box dimension of the graph of functions given by similar Rademacher series. With the help of the Franklin system one can prove that the graph of almost every trajectory of the fractional Brownian motion with exponent $0 < \beta \leqq 2$ has the box dimension equal to $2 - \frac{\beta}{2}$. The fractional Brownian motion is the Gaussian stochastic process $(X(t), t \in I)$ with mean zero and such that $E\big|X(t) - X(s)\big|^2 = |t-s|^\beta$.

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ON THE NEWLY GENERALIZED ABSOLUTE CESÀRO SUMMABILITY OF ORTHOGONAL SERIES

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Dedicated to Professor Károly Tandori on the occasion of his 70th birthday with admiration and friendship

1. Introduction. In [2] T. M. Flett defined a very useful extension of absolute Cesàro summability. According to his definition we shall say that a series $\sum a_n$ is summable $|C,\alpha,\gamma|_k$, where $k \geqq 1$, $\alpha > -1$, $\gamma \geqq 0$, if the series $\sum n^{\gamma k+k-1}|\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k$ is convergent, σ_n^α being the nth Cesàro mean of order α of the series $\sum_0^\infty a_n$.

Among others, he proved the following result.

THEOREM. *Let $r \geqq k > 1$, $\gamma \geqq 0$, $\alpha > \gamma - 1$, $\beta \geqq \alpha + 1/k - 1/r$. Then if $\sum_0^\infty a_n$ is summable $|C,\alpha,\gamma|_k$ it is summable $|C,\beta,\gamma|_r$ and with $\tau_n^\alpha := n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$*

$$\left\{\sum n^{r\gamma-1}|\tau_n^\beta|^r\right\}^{1/r} \leqq \left\{\sum n^{\gamma k-1}|\tau_n^\alpha|^k\right\}^{1/k}. \tag{1.1}$$

If $k = 1$, (1.1) holds when $r \geqq 1, \gamma \geqq 0$, $\alpha > \gamma - 1$, $\beta > \alpha + 1/k - 1/r$.

This theorem is a very important result, also in itself, moreover it has turned out that inequality (1.1) is crucial in the proofs of theorems concerning strong approximation of orthogonal series having approximation order $o_x(1/n^\gamma)$ (see e.g. G. Sunouchi [12], and [5], [6], [7]). Recently we intended to generalize that, this can be done, in our view and experience, only if previously we can generalize the Theorem by the same way, that is, if in the Theorem we can replace the factor n^γ by a suitable factor $\gamma(n)$.

This was our motivation for generalizing this important result of Flett. Naturally, having found the method for such an extension, we used it for generalizing some further interesting theorems of Flett [2], see e.g. [8] and [9]. In [8] we introduced the newly generalized notion of absolute Cesàro summability, i.e. the definition of $\left|C,\alpha,\gamma(t)\right|_k$-summability, where $\gamma(t)$ is a positive nondecreasing function defined for $1 \leqq t < \infty$. We say that the

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series $\sum_0^\infty a_n$ is summable $\left|C, \alpha, \gamma(t)\right|_k$ if the series

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k,$$

or briefly

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{-1} |\tau_n^\alpha|^k$$

is convergent.

Among others in [8] we proved the following theorems which will be used in the course of the proofs of our results to be presented in this work.

THEOREM A. *Let $r \geqq k > 1$, $\alpha > -1$, $\beta \geqq \alpha + 1/k - 1/r$, and $\gamma(t)$ be a non-decreasing positive function defined for $1 \leqq t < \infty$ so that with some $C > 1$*

$$\limsup_{t \to \infty} \frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1}. \tag{1.2}$$

If the series

$$\sum_{n=0}^{\infty} a_n \tag{1.3}$$

is summable $\left|C, \alpha, \gamma(t)\right|_k$, then it is summable $\left|C, \beta, \gamma(t)\right|_r$ and

$$\left\{ \sum \gamma(n)^r n^{-1} |\tau_n^\beta|^r \right\}^{1/r} \leqq K \left\{ \sum \gamma(n)^k n^{-1} |\tau_n^\alpha|^k \right\}^{1/k}. \tag{1.4}$$

If $k = 1$, the result holds when $r \geqq 1$, $\beta > \alpha + 1 - 1/r$ and (1.2) is satisfied.

If we keep $r = k$, then the factor $\gamma(n)$ on the left-hand side of (1.4) can be replaced by another factor $\mu(n)$ as follows.

THEOREM B. *Let $k \geqq 1$, $\alpha > -1$, $\delta > 0$, $\beta \geqq \alpha - \delta$, and $\beta > -1$, furthermore let $\mu(t)$ be a positive monotone, and $\gamma(t)$ a non-decreasing positive function defined for $1 \leqq t < \infty$, so that*

$$C^\delta \limsup_{t \to \infty} \frac{\mu(Ct)}{\mu(t)} < \liminf_{t \to \infty} \frac{\gamma(Ct)}{\gamma(t)} \leqq \limsup_{t \to \infty} \frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1} \tag{1.5}$$

with some $C>1$. *If series* (1.3) *is summable* $\left|C,\alpha,\gamma(t)\right|_k$, *then it is summable* $\left|C,\beta,\mu(t)\right|_k$ *and*

$$\sum \mu(n)^k n^{-1}|\tau_n^{\beta}|^k \leqq K \sum \gamma(n)^k n^{-1}|\tau_n^{\alpha}|^k. \tag{1.6}$$

In the case of strict inequality $\beta > \alpha - \delta$ we can prove a consistency result for $r<k$, too.

THEOREM C. *Let* $k>r\geqq 1$, $\alpha>-1$, $\delta>0$, *and* $\beta>\max(\alpha-\delta,-1)$. *If* $\mu(t)$ *is a positive monotone, and* $\gamma(t)$ *is a positive non-decreasing function defined for* $1\leqq t<\infty$, *furthermore they satisfy* (1.5) *and series* (1.3) *is summable* $\left|C,\alpha,\gamma(t)\right|_k$, *then* (1.3) *is also summable* $\left|C,\beta,\mu(t)\right|_r$ *and*

$$\left\{\sum \mu(n)^r n^{-1}|\tau_n^{\beta}|^r\right\}^{1/r} \leqq K\left\{\sum \gamma(n)^k n^{-1}|\tau_n^{\alpha}|^k\right\}^{1/k}. \tag{1.7}$$

Using these results we intend to generalize some theorems pertaining to generalized absolute Cesàro summability of orthogonal series. One of the first results of this type is due to K. Tandori [14] who investigated the absolute $|C,1|$-summability. His result was extended to $|C,\alpha|$-summability by us [4], and a certain part of ours to $\left|C,\alpha,\gamma\right|_k$ by I. Szalay [13].

We shall not cite these theorems to be generalized here, because Theorem 1 and the sufficiency part of Theorem 2 to be proved in our present paper in the special case $\gamma(t)=t^{\gamma}$ with appropriate γ will reduce to the relevant theorems of I. Szalay, which in turn contain our results in the special case $\gamma = 0$ and $k=1$. Furthermore one of our theorems in the case $\alpha=1$ embodies Tandori's theorem. The necessity part of Theorem 2 and Theorem 3 were proved only in the special case $k=1$ and $\gamma(t)=1$ by us [4].

Before formulating our new theorems we recall some further notations and definitions.

Let $\{\varphi_n(x)\}$ be an orthonormal system on the interval $(0,1)$. We shall consider real orthogonal series

$$\sum_{n=0}^{\infty} c_n\varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.8}$$

It is well known that the partial sums $s_n(x)$ of (1.8) converge in the L^2 norm to a square-integrable function $f(x)$. The (C,α)-means of (1.8) will be denoted by $\sigma_n^{\alpha}(x)$, i.e.

$$\sigma_n^{\alpha}(x) := \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} s_k(x),$$

where $A_n^{(\alpha)} := \binom{n+\alpha}{n}$.

A sequence $\gamma = \{\gamma_n\}$ of positive terms will be called *quasi β-power-monotone increasing (decreasing)* if there exists a constant $K = K(\beta, \gamma) \geqq 1$ such that

$$(1.9) \qquad K n^\beta \gamma_n \geqq m^\beta \gamma_m \quad (n^\beta \gamma_n \leqq K m^\beta \gamma_m)$$

holds for any $n \geqq m$.

Furthermore we shall say that a sequence $\gamma = \{\gamma_n\}$ of positive terms is *quasi geometrically increasing (decreasing)* if there exist a natural number μ and a constant $K = K(\gamma) \geqq 1$ such that

$$(1.10) \qquad \gamma_{n+\mu} \geqq 2\gamma_n \text{ and } \gamma_n \leqq K\gamma_{n+1} \quad \left(\gamma_{n+\mu} \leqq \frac{1}{2}\gamma_n \text{ and } \gamma_{n+1} \leqq K\gamma_n\right)$$

hold for all natural numbers n.

An orthonormal system $\{\chi_n(x)\}$ will be called *Haar-type*, or briefly *H-type*, if for every $x \in (0,1)$

$$\chi_n(x)\chi_m(x) = 0 \text{ with } 2^k < m,\ n \leqq 2^{k+1},\ m \neq n,\ k = 0, 1, \dots$$

holds true.

2. Theorems. Using the notations and definitions introduced above we can formulate our results.

THEOREM 1. *Let us assume that $\alpha > \frac{1}{2}$, $1 \leqq k \leqq 2$ and $\gamma(t)$ is a positive nondecreasing function on $[1, \infty)$ such that the sequence $\{\gamma(n)\}$ is quasi η-power-monotone decreasing with some $\eta > -1$. Then the condition*

$$(2.1) \qquad \sum_{m=0}^{\infty} \gamma(2^m)^k \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{k/2} < \infty$$

is necessary and sufficient for series (1.8) *to be summable $|C, \alpha, \gamma(t)|_k$ for every orthonormal system $\{\varphi_n(x)\}$ almost everywhere (a.e.) in* $(0,1)$.

The following corollaries can be proved using Theorem 1 and Theorems A, B and C, respectively.

COROLLARY 1.A. *Under the assumptions of Theorem 1 with the additional premises: $r \geqq k > 1$ and $\beta \geqq \alpha + 1/k - 1/r$; or $r \geqq k = 1$ and $\beta > \alpha + 1 - 1/r$; series* (1.8) *is also summable $|C, \beta, \gamma(t)|_r$ a.e. in* $(0,1)$.

Corollary 1.B. *Let* $\alpha > 1/2$, $1 \leqq k \leqq 2$, $0 < \delta < \gamma < 1$, $\beta \geqq \alpha - \delta$ *and* $\beta > -1$, *furthermore let* $\gamma(t)$ *and* $\mu(t)$ *be positive nondecreasing functions on* $[1, \infty)$. *If the function* $\gamma(t)$ *has the added property*

$$\lim_{t\to\infty} \frac{\gamma(Ct)}{\gamma(t)} = C^{\gamma} \quad \textit{for some} \quad C \geqq 1, \tag{2.2}$$

and the sequence $\{\mu(n)\}$ *is quasi* η*-power-monotone decreasing with some* $\eta > \delta - \gamma$, *then condition* (2.1) *implies the* $|C, \beta, \mu(t)|_k$*-summability of series* (1.8) *a.e. in* $(0, 1)$.

Corollary 1.C. *Let* $\alpha > 1/2$, $1 \leqq r < k \leqq 2$, $0 < \delta < \gamma < 1$ *and* $\beta > \max(\alpha - \delta, -1)$. *If the functions* $\gamma(t)$ *and* $\mu(t)$ *have the same properties as in Corollary* 1.B, *then condition* (2.1) *implies the* $|C, \beta, \mu(t)|_r$*-summability of series* (1.8) *a.e. in* $(0, 1)$.

Theorem 2. *Let* $1 \leqq k \leqq 2$ *and let* $\gamma(t)$ *be the same function as in Theorem* 1. *Then the conditions*

$$\sum_{m=0}^{\infty} \gamma(2^m)^k m^{k/2} \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{k/2} < \infty, \quad \textit{if } \alpha = 1/2; \tag{2.3}$$

and

$$\sum_{m=0}^{\infty} \gamma(2^m)^k 2^{km(1/2-\alpha)} \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{k/2} < \infty, \quad \textit{if } -1 < \alpha < 1/2; \tag{2.4}$$

are sufficient for series (1.8) *to be summable* $|C, \alpha, \gamma(t)|_k$ *a.e. in* $(0, 1)$. *If the sequence of the coefficients* $|c_n|$ *is monotone then conditions* (2.3) *and* (2.4) *are also necessary that series* (1.8) *for every orthonormal system* $\{\varphi_n(x)\}$ *be summable* $|C, \alpha, \gamma(t)|_k$ *a.e. in* $(0, 1)$.

We remark that the necessity of conditions (2.3) and (2.4) in the special case $\gamma(t) = t^{\gamma}$ was not proved in the paper of I. Szalay [13]. Now we shall prove this part by using an almost hidden lemma of L. Csernyák and L. Leindler [1].

From this theorem and Theorems A, B and C we can deduce the following corollaries.

Corollary 2.A. *Under the assumptions of Theorem* 2 *with the additional premises:* $r \geqq k > 1$ *and* $\beta \geqq \alpha + 1/k - 1/r$; *or* $r \geqq k = 1$ *and* $\beta > \alpha + 1 - 1/r$; *series* (1.8) *is summable* $|C, \beta, \gamma(t)|_r$ *a.e. in* $(0, 1)$.

Corollary 2.B. *Let* $1 \leqq k \leqq 2$, $\delta > 0$, $0 \leqq \gamma < 1$, $-1 < \alpha \leqq 1/2$, $\beta > \alpha - \delta$ *and* $\beta > -1$. *If the functions* $\gamma(t)$ *and* $\mu(t)$ *have the same properties* *as in Corollary* 1.B, *then if* $\alpha = 1/2$ *condition* (2.3), *and if* $-1 < \alpha < 1/2$ *conditon* (2.4) *imply the* $\left|C, \beta, \mu(t)\right|_k$*-summability of series* (1.8) *a.e. in* $(0,1)$.

COROLLARY 2.C. *Let* $1 \leqq r < k \leqq 2$, $\delta > 0$, $0 \leqq \gamma < 1$, $-1 < \alpha \leqq 1/2$ *and* $\beta > \max(\alpha - \delta, -1)$. *If the functions* $\gamma(t)$ *and* $\mu(t)$ *have the same properties as in Corollary* 1.B, *then in the case* $\alpha = 1/2$ *condition* (2.3), *and if* $-1 < \alpha < 1/2$ *then condition* (2.4) *imply the* $\left|C, \beta, \mu(t)\right|_r$*-summability of series* (1.8) *a.e. in* $(0,1)$.

In [4], among others, we also proved a further theorem pertaining to Haar-type systems in connection with $|C,\alpha|$-summability. As far as we know nobody extended our H-type result to $|C,\alpha,\gamma|_k$-summability. Now we shall generalize this theorem to $\left|C,\alpha,\gamma(t)\right|_k$-summability directly. Our general H-type result reads as follows:

THEOREM 3. *Let* $1 \leqq k \leqq 2$ *and let* $\gamma(t)$ *be the same function as in Theorem* 1. *Then the conditions*

$$\sum_{m=0}^{\infty} \gamma(2^m)^k \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{k/2} < \infty \tag{2.5}$$

and

$$\sum_{m=0}^{\infty} \gamma(2^m)^k 2^{m(k(1-\alpha)-1)} \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{k/2} < \infty \tag{2.6}$$

are necessary and sufficient that series (1.8) *for every Haar-type system* $\{\chi_n(x)\}$ *be summable* $\left|C,\alpha,\gamma(t)\right|_k$ *a.e. in* $(0,1)$ *for any* $\alpha \geqq 1 - \frac{1}{k}$ *and for any* $-1 < \alpha < 1 - \frac{1}{k}$, *respectively.*

Theorem 3 in the special case $\gamma(t) \equiv 1$ and $k = 1$ reduces to our theorem proved in [4].

Finally we mention that analogous corollaries as in connection with Theorems 1 and 2 were stated can be proved using Theorem 3 and Theorems A, B and C, as well.

3. Lemmas. Before formulating the lemmas required in the proofs we present some further notions, notations and some elementary facts.

A function $\psi(x)$ defined on $(0,1)$ is called a step function if there is a partition of this interval by a finite set of points $0 = x_0 < x_1 < \ldots < x_n = 1$ such that $\psi(x)$ has a constant value on each subinterval (x_{k-1}, x_k). At the points x_k the function may be defined arbitrarily.

The nth Rademacher function is $r_n(x) := \operatorname{sgn} \sin 2^n \pi x$.

$\mu(E)$ will denote the Lebesgue measure of E.

If $I := (u, v)$ is a finite interval and $h(x)$ is a function defined on $(0, 1)$, then

$$h(x, I) := \begin{cases} h(\frac{x-u}{v-u}), & \text{if } u < x < v, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\int_u^v h(x, I)dx = \mu(I) \int_0^1 h(x)dx$.

A set E is called a simple set if it is the union of finite intervals.

We can find the following facts in A. Zygmund [15] (see p. 77):

$$(3.1)\quad 0 < c_1(\alpha) \leqq \frac{A_m^{(\alpha)}}{m^\alpha} \leqq c_2(\alpha) \quad \left(m \geqq 1,\ \alpha > -1,\ A_m^{(\alpha)} = \binom{m+\alpha}{m}\right),$$

$$(3.2)\qquad A_m^{(\alpha)} > 0 \quad (m \geqq 0,\ \alpha > -1),$$

and

$$(3.3)\qquad A_{m+1}^{(\alpha)} > A_m^{(\alpha)} \quad (m \geqq 0,\ \alpha > 0),$$

where $c_1(\alpha)$ and $c_2(\alpha)$ are independent of m.

We define

$$L_{n,\nu}^{(\alpha)} := \frac{A_{n+1-\nu}^{(\alpha)}}{A_{n+1}^{(\alpha)}} - \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} = \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} \cdot \frac{\nu\alpha}{(n+1-\nu)(n+1+\alpha)}.$$

From (3.1), (3.2) and (3.3) it clearly follows

$$(3.4)\qquad 0 < d_1(\alpha)\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} \leqq |L_{n,\nu}^{(\alpha)}| \leqq d_2(\alpha)\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}}$$

and

$$\operatorname{sgn} L_{n,\nu}^{(\alpha)} = \operatorname{sgn} \alpha$$

for any $n = 1, 2, \ldots;\ \nu = 0, 1, \ldots;\ \alpha > -1$, where $d_1(\alpha)$ and $d_2(\alpha)$ are independent of n.

$K, K_1, K_2, \ldots$ will denote positive constants depending only on the parameters concerned in the particular problem in which it appears. The constants are not necessarily the same at any two occurrences.

LEMMA 1 ([4]). *Let $\{R_n(x)\}$ be a system of step functions defined on $(0, 1)$. Denote $J_s(n)$ $(n = 1, 2, \ldots;\ s = 1, 2, \ldots, s_n)$ the intervals on which $R_n(x)$ is constant. If for every $m > n$*

$$\int_{J_s(n)} \operatorname{sgn} R_m(x)dx = 0 \quad (s = 1, \ldots, s_n),$$

then for any sequence of numbers $d_1,\dots,d_N$ there exists a set E_k of subintervals such that for any $x\in E_k$

$$\left|\sum_{\ell=1}^{N} d_\ell R_\ell(x)\right| \geqq |d_{N-k}R_{N-k}(x)|$$

and

$$\mu\big(E_k\cap J_s(N-k-1)\big)=\frac{\mu\big(J_s(N-k-1)\big)}{2^{k+1}}$$

hold for any $k=0,1,\dots,N-1; s=1,2,\dots,s_{N-k-1}$; and $J_1(0):=(0,1)$.

LEMMA 2 ([15], p. 213). *If $\sum_{n=0}^\infty c_n^2<\infty$ and $f(x)\sim\sum_{n=0}^\infty c_n r_n(x)$ is given by the Riesz–Fischer theorem, then*

$$A\Big\{\sum_{n=0}^{\infty}c_n^2\Big\}^{1/2}\leqq\int_0^1|f(x)|\,dx\leqq B\Big\{\sum_{n=0}^{\infty}c_n^2\Big\}^{1/2}, \tag{3.5}$$

where A and B are absolute constants.

LEMMA 3 ([1]). *For an arbitrary sequence $\{c_n\}$ let the sets $E_{n,m}$ be defined by*

$$E_{n,m}:=\left\{x:\left|\sum_{\nu=n}^{n+m}c_\nu r_\nu(x)\right|\right\}>\frac{A}{2}\Big\{\sum_{\nu=n}^{n+m}c_\nu^2\Big\}^{1/2}.$$

Then $E_{n,m}$ are simple sets with $\mu(E_{n,m})\geqq A^2/4$, where A is given by (3.5) in Lemma 2.

LEMMA 4. *By means of the coefficients c_n of series (1.8) we can construct an orthonormal Haar-type system $\{\chi_n(x)\}$ of step functions with the following properties: For any natural number s the interval $(0,1)$ can be partitioned into subintervals J_ρ $(1\leqq\rho\leqq\rho_s)$ such that on any J_ρ every $\chi_n(x)$ $(n=0,1,\dots,2^s)$ is constant. These intervals $J_\rho:=(u_\rho,v_\rho)$ are decomposed into 2^s subintervals $I_k(s,J_\rho):=(u_\rho^{(k)},v_\rho^{(k)})$ $(k=1,2,\dots,2^s)$, using the quantities $C_s:=\Big\{\sum_{n=2^s+1}^{2^{s+1}}c_n^2\Big\}^{1/2}$,*

$$\rho_0^{(s)}:=0\quad\text{and}\quad\rho_k^{(s)}:=C_s^{-2}\sum_{n=1}^{k}c_{2^s+n}^2\quad(k=1,2,\dots,2^s),$$

by the following definitions:

$$u_\rho^{(k)}:=u_\rho+\mu(J_\rho)\rho_{k-1}^{(s)}\quad\text{and}\quad v_\rho^{(k)}:=u_\rho+\mu(J_\rho)\rho_k^{(s)}. \tag{3.7}$$

The functions $\chi_{2^s+k}(x)$ $(s = 1, 2, \dots$ *and* $k = 1, 2, \dots, 2^s)$ *are defined as follows:*

$$\chi_{2^s+k}(x) := \frac{C_s}{|c_{2^s+k}|} \sum_{\rho=1}^{\rho_s} r_s\big(x; I_k(s, J_\rho)\big). \tag{3.8}$$

This lemma is proved in [4] implicitly (see pp. 247–249).

LEMMA 5 ([10]). *For any positive sequence* $\gamma := \{\gamma_n\}$ *the inequality*

$$\sum_{n=m}^{\infty} \gamma_n \leqq K\gamma_m \quad (m = 1, 2, \dots, K \geqq 1)$$

holds if and only if the sequence γ *is quasi geometrically decreasing.*

LEMMA 6 ([11]). *If a positive sequence* $\{\gamma_n\}$ *is quasi* β*-power-monotone decreasing with a certain positive exponent* β *then the sequence* $\{\gamma_{2^n}\}$ *is quasi geometrically decreasing.*

This lemma is just a part of a theorem proved in [11], and Lemma 5 is also just a part of a lemma of [10].

4. Proofs. PROOF OF THEOREM 1. First we prove the sufficiency of condition (2.1). We may suppose, without loss of generality, that $c_0 = c_1 = 0$. Taking into account (3.1) and (3.4), furthermore using Hölder's inequality, we get

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} \int_0^1 \left|\sigma_{n+1}^\alpha(x) - \sigma_n^\alpha(x)\right|^k dx \leqq \tag{4.1}$$

$$\leqq K_1 \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{m(k-1)} \sum_{n=2^m+1}^{2^{m+1}} \left\{\int_0^1 \left|\sigma_n^\alpha(x) - \sigma_{n+1}^\alpha(x)\right|^2 dx\right\}^{k/2} \leqq$$

$$\leqq K_1 \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{mk/2} \left\{\sum_{n=2^m+1}^{2^{m+1}} \int_0^1 \left|\sigma_{n+1}^\alpha(x) - \sigma_n^\alpha(x)\right|^2 dx\right\}^{k/2} \leqq$$

$$\leqq K_2 \sum_{m=1}^{\infty} \gamma(2^m)^k 2^{mk/2} \left\{\sum_{n=2^m+1}^{2^{m+1}} \left(\sum_{\nu=0}^{n} \big(L_{n,\nu}^{(\alpha)}\big)^2 c_\nu^2 + \big(A_{n+1}^{(\alpha)}\big)^{-2} c_{n+1}^2\right)\right\}^{k/2} \leqq$$

$$\leqq K_3 \sum_{m=1}^{\infty} \gamma(2^m)^k 2^{mk/2} \left\{\sum_{n=2^m+1}^{2^{m+1}} \sum_{\nu=0}^{n} n^{2\alpha-2}(n+1-\nu)^{2\alpha-2}\nu^2 c_\nu^2\right\}^{k/2} +$$

$$+K_3\sum_{m=1}^{\infty}\gamma(2^m)^k 2^{mk/2}\Bigg\{\sum_{n=2^m+2}^{2^{m+1}+1} n^{-2\alpha}c_n^2\Bigg\}^{k/2} =: \sum\nolimits_1 + \sum\nolimits_2, \text{ say.}$$

This estimation holds for any $\alpha > -1$.

If $\alpha > 1/2$ and $n(\ell) := \min(2^{\ell+1}, n)$, $m(\nu) := \max(2^{m+1}+1, \nu)$, then an elementary consideration shows that

$$\sum\nolimits_1 \leqq \tag{4.2}$$

$$\leqq K_4\sum_{m=1}^{\infty}\gamma(2^m)^k 2^{mk/2}\Bigg\{\sum_{n=2^m+1}^{2^{m+1}}\sum_{\ell=0}^{m}\sum_{\nu=2^\ell+1}^{n(\ell)} n^{-2\alpha-2}(n+1-\nu)^{2\alpha-2}\nu^2 c_\nu^2\Bigg\}^{k/2} \leqq$$

$$\leqq K_5\sum_{m=1}^{\infty}\Bigg\{\gamma(2^m)^2 2^{-m(1+2\alpha)}\sum_{n=2^m+1}^{2^{m+1}}\sum_{\ell=0}^{m}\sum_{\nu=2^\ell+1}^{n(\ell)}(n+1-\nu)^{2\alpha-2}\nu^2 c_\nu^2\Bigg\}^{k/2} \leqq$$

$$\leqq K_5\sum_{m=1}^{\infty}\Bigg\{\gamma(2^m)^2 2^{-m(1+2\alpha)}\sum_{\ell=0}^{m}\sum_{\nu=2^\ell+1}^{2^{\ell+1}}\nu^2 c_\nu^2\sum_{n=m(\nu)}^{2^{m+1}}(n+1-\nu)^{2\alpha-2}\Bigg\}^{k/2} \leqq$$

$$\leqq K_6\sum_{m=1}^{\infty}\Bigg\{\gamma(2^m)^2 2^{-2m}\sum_{\ell=0}^{m}\sum_{\nu=2^\ell+1}^{2^{\ell+1}}\nu^2 c_\nu^2\Bigg\}^{k/2} \leqq$$

$$\leqq K_7\sum_{\ell=0}^{\infty}2^{\ell k}\Bigg(\sum_{\nu=2^\ell+1}^{2^{\ell+1}}c_\nu^2\Bigg)^{k/2}\sum_{m=\ell}^{\infty}\gamma(2^m)^k 2^{-km}.$$

Since the sequence $\{\gamma(n)\}$ is quasi η-power-monotone decreasing with $\eta > -1$, the sequence $\{\gamma(n)n^{-1}\}$ is ε-power-monotone decreasing with a positive ε. Namely if $\eta = \varepsilon - 1$ $(\varepsilon > 0)$, then $n^\eta\gamma(n) = n^\varepsilon\gamma(n)n^{-1}$, whence everything is clear. Now if we apply Lemma 6 with ε and $\gamma(n)n^{-1}$ in places of β and γ_n, respectively, we get that the sequence $\{\gamma(2^n)2^{-n}\}$ is quasi geometrically decreasing and therefore the sequence $\{(\gamma(2^n)2^{-n})^k\}$ is also quasi geometrically decreasing (see definition (1.10)). Thus, by Lemma 5,

$$\sum_{m=\ell}^{\infty}\gamma(2^m)^k 2^{-km} \leqq K\gamma(2^\ell)^k 2^{-k\ell}. \tag{4.3}$$

Using this and (4.2) we have

$$\sum_1 \leqq K_8 \sum_{\ell=0}^{\infty} \gamma(2^\ell)^k \Big(\sum_{\nu=2^\ell+1}^{2^{\ell+1}} c_\nu^2 \Big)^{k/2}. \tag{4.4}$$

The estimation of $\sum_2$ is very easy. Namely, $\alpha > 1/2$, thus

$$\sum_2 \leqq K_9 \sum_{m=1}^{\infty} \gamma(2^{m+1})^k 2^{mk(1/2-\alpha)} \Big\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \Big\}^{k/2} \leqq \tag{4.5}$$

$$\leqq K_{10} \sum_{m=1}^{\infty} \gamma(2^m)^k \Big\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \Big\}^{k/2}.$$

Now collecting the results (4.1), (4.4) and (4.5), by Beppo Levi's theorem, we have proved the sufficiency of condition (2.1).

To prove the necessity of (2.1) let us consider the series

$$\sum_{n=0}^{\infty} c_n \chi_n(x), \tag{4.6}$$

where the functions $\chi_n(x)$ are given in Lemma 4 by (3.8). Denote $\overline{\sigma}_n^\alpha(x)$ the nth (C,α)-means of series (4.6). Let us assume that series (1.8) for any orthonormal system is summable $|C,\alpha,\gamma(t)|_k$ a.e. in $(0,1)$. Then series (4.6) is also summable $|C,\alpha,\gamma(t)|_k$ a.e. in $(0,1)$, consequently

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\overline{\sigma}_{n+1}^\alpha(x) - \overline{\sigma}_n^\alpha(x)|^k < \infty$$

a.e. in $(0,1)$.

Let $\varepsilon > 0$ to be given later. Owing to the previous statement we can apply Egorov's theorem which conveys that there exist a measurable set E with $\mu(E) \geqq 1-\varepsilon$ and a positive constant M such that for every $x \in E$

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\overline{\sigma}_n^\alpha(x) - \overline{\sigma}_{n+1}^\alpha(x)|^k < M.$$

Therefore

$$\sum_{n=2}^{\infty} \int_E \gamma(n)^k n^{k-1} |\overline{\sigma}_{n+1}^\alpha(x) - \overline{\sigma}_n^\alpha(x)|^k dx \leqq M\mu(E). \tag{4.7}$$

Let m and n be integers such that $2^m < n \leqq 2^{m+1}$. Then we define

$$R_\ell(x;m,n) := \sum_{\nu=2^\ell+1}^{2^{\ell+1}} L_{n,\nu}^{(\alpha)} c_\nu \chi_\nu(x) \quad (\ell = 0,1,\dots,m-1),$$

$$R_m(x;m,n) := \sum_{\nu=2^m+1}^{n} L_{n,\nu}^{(\alpha)} c_\nu \chi_\nu(x),$$

$$R_{m+1}(x;m,n) := \left(A_{n+1}^{(\alpha)}\right)^{-1} c_{n+1}\chi_{n+1}(x).$$

These functions $R_\ell(x;m,n)$ $(\ell = 0,1,\dots,m+1)$ satisfy all of the assumptions of Lemma 1. Thus we can use Lemma 1 with $N = m+1$ and $k = 3$. The compatible set E_k will be denoted by $E_3(m,n)$. Then we have

$$\text{(4.8)} \qquad \sum_{n=2^3+1}^{\infty} \int_E \gamma(n)^k n^{k-1} \left|\overline{\sigma}_{n+1}^\alpha(x) - \overline{\sigma}_n^\alpha(x)\right|^k dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \gamma(n)^k n^{k-1} \int_E \left| \sum_{\nu=0}^{n} L_{n,\nu}^{(\alpha)} c_\nu \chi_\nu(x) + \left(A_{n+1}^{(\alpha)}\right)^{-1} c_{n+1}\chi_{n+1}(x)\right|^k dx =$$

$$= \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \gamma(n)^k n^{k-1} \int_E \left| \sum_{\ell=0}^{m+1} R_\ell(x;m,n)\right|^k dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \gamma(n)^k n^{k-1} \int_{E\cap E_3(m,n)} \left| R_{m-2}(x;m,n)\right|^k dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \gamma(n)^k n^{k-1} \Bigg(\int_{E_3(m,n)} -$$

$$- \int_{E_3(m,n)\setminus E\cap E_3(m,n)} \Bigg) \left| R_{m-2}(x;m,n)\right|^k dx =: J.$$

By Lemmas 1 and 4, keeping in mind (3.6), (3.7), (3.8) and that $\{\chi_n(x)\}$ is of H-type, we get

$$\text{(4.9)} \qquad \int_{E_3(m,n)} \left| R_{m-2}(x;m,n)\right|^k dx =$$

$$= \int_{E_3(m,n)} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} \big(L_{n,\nu}^{(\alpha)}\big)^k |c_\nu|^k \big|\chi_\nu(x)\big|^k dx \geqq$$

$$\geqq \sum_{i=1}^{2^{m-2}} \big(L_{n,2^{m-2}+i}^{(\alpha)}\big)^k |c_{2^{m-2}+i}|^k \sum_{\rho=1}^{\rho_{m-2}} \int_{E_3(m,n)\cap I_i(m-2,J_s)} \frac{C_{m-2}^k}{|c_{2^{m-2}+i}|^k} dx \geqq$$

$$\geqq \sum_{i=1}^{2^{m-2}} \big(L_{n,2^{m-2}+i}^{(\alpha)}\big)^k C_{m-2}^k \sum_{\rho=1}^{\rho_{m-2}} 2^{-4}\mu\big(I_i(m-2,J_s)\big) =$$

$$= 2^{-4} \sum_{i=1}^{2^{m-2}} \big(L_{n,2^{m-2}+i}^{(\alpha)}\big)^k C_{m-2}^k \sum_{\rho=1}^{\rho_{m-2}} \mu(J_\rho) c_{2^{m-2}+i}^2 C_{m-2}^{-2} =$$

$$= 2^{-4} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} \big(L_{n,\nu}^{(\alpha)}\big)^k C_{m-2}^{k-2} c_\nu^2 \geqq \delta_1(\alpha) 2^{-mk} C_{m-2}^k,$$

where $\delta_1(\alpha)(>0)$ depends only on α. If $k<2$ then the second integral in (4.8) can be estimated easily using Hölder's inequality. Namely

$$J_1 := \int_{E_3(m,n)\setminus E_3(m,n)\cap E} \big|R_{m-2}(x;m,n)\big|^k dx \leqq \tag{4.10}$$

$$\leqq \varepsilon^{1-\frac{k}{2}} \left(\int_0^1 \left| \sum_{\nu=2^{m-2}+1}^{2^{m-1}} L_{n,\nu}^{(\alpha)} c_\nu \chi_\nu(x) \right|^2 dx \right)^{k/2} \leqq \delta_2(\alpha) \varepsilon^{1-\frac{k}{2}} 2^{-mk} C_{m-2}^k,$$

where $\delta_2(\alpha)(>0)$ also depends only on α.

By (4.9) and (4.10), choosing $\varepsilon < \left(\frac{\delta_1(\alpha)}{2\delta_2(\alpha)}\right)^{2/(2-k)}$, we have

$$J \geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \gamma(n)^k n^{k-1} 2^{-mk} C_{m-2}^k \big(\delta_1(\alpha) - \delta_2(\alpha)\varepsilon^{1-\frac{k}{2}}\big) \geqq$$

$$\geqq \sum_{m=3}^{\infty} \gamma(2^m)^k C_{m-2}^k 2^{-k-1} \delta_1(\alpha) =: J_2,$$

which, by (4.7) and (4.8), implies (2.1) for $1 \leqq k < 2$. If $k=2$ we can estimate J_1 in (4.10) using the definitions of the functions $\chi_\nu(x)$. Namely if

$2^{m-2} < \nu \leqq 2^{m-1}$ then $\left|c_\nu \chi_\nu(x)\right| \leqq C_{m-2}$ and $|L_{n,\nu}^{(\alpha)}| \leqq K(\alpha)2^{-m}$ for $n > 2^m$ and $2^{m-2} < \nu \leqq 2^{m-1}$, which is the case in our case. Thus, since the system $\{\chi_n(x)\}$ is of H-type,

$$\left| R_{m-2}(x; m, n)\right| \leqq K(\alpha)2^{-m}C_{m-2},$$

whence

$$J_1 \leqq K_2(\alpha)2^{-2m}C_{m-2}^2\mu(E_3(m,n)\setminus E_3(m,n)\cap E) \leqq$$

$$\leqq K^2(\alpha)2^{-2m}C_{m-2}^2\varepsilon.$$

Hence and from (4.9), choosing $\varepsilon < \frac{\delta_1(\alpha)}{2K^2(\alpha)}$, we obtain that $J \geqq J_2$ for $k = 2$ also holds, and this shows, by (4.7) and (4.8), that (2.1) is fulfilled in the case $k = 2$, too.

Herewith we have verified the necessity of condition (2.1) for any $1 \leqq k \leqq 2$, and this completes the proof of Theorem 1.

Proof of Corollary 1.A. By Theorem A and Theorem 1 we have only to show that the presumption of Theorem 1 that the sequence $\{\gamma(n)\}$ is quasi η-power-monotone decreasing with some $\eta > -1$, implies assumption (1.2) of Theorem A. As we have proved in the course of the proof of Theorem 1, the cited property of the sequence $\{\gamma(n)\}$ dictates that the sequence $\{\gamma(2^n)2^{-n}\}$ is quasi geometrically decreasing. Therefore, by definition, there exist a natural number μ and a constant $K \geqq 1$ such that

$$\text{(4.11)}\quad \gamma(2^{n+\mu})2^{-(n+\mu)} \leqq \frac{1}{2}\gamma(2^n)2^{-n} \quad\text{and}\quad \gamma(2^{n+1})2^{-(n+1)} \leqq K\gamma(2^n)2^{-n}.$$

It is easy to see that we may assume without loss of generality that μ is as large as we want. Let us assume that $2^\mu > K^{1/\alpha}$. Then, using (4.11) and the monotonicity of the function $\gamma(t)$, we have for any $t \in [2^{n-1}, 2^n]$

$$\text{(4.12)}\qquad \frac{\gamma(2^\mu t)}{\gamma(t)} \leqq \frac{\gamma(2^{\mu+n})}{\gamma(2^{n-1})} \leqq \frac{2K\gamma(2^{\mu+n})}{\gamma(2^n)} \leqq K2^\mu < 2^{\mu(1+\alpha)}.$$

Setting $C := 2^\mu$, (4.12) clearly yields (1.2). Therefore we can apply both Theorem 1 and Theorem A, and thus Corollary 1.A is proved.

Proof of Corollary 1.B. In the proof the crucial point is to verify that under the assumptions of Corollary 1.B conditions (1.5) of Theorem B are satisfied. Since $\alpha > 1/2$ and $0 < \gamma < 1$ thus $C^\gamma < C^{\alpha+1}$ is trivial. It remains to prove that there exists $C > 1$ such that

$$\text{(4.13)}\qquad C^\delta \limsup_{t\to\infty} \frac{\mu(Ct)}{\mu(t)} < C^\gamma$$

holds.

If $\eta = 2\varepsilon + \delta - \gamma$ ($\varepsilon > 0$) then $n^\eta \mu(n) = n^\varepsilon \mu(n) n^{\varepsilon+\delta-\gamma}$. i.e. the sequence $\{\mu(n)n^{\varepsilon+\delta-\gamma}\}$ is ε-power-monotone decreasing with a positive ε. Thus, by Lemma 6, the sequence $\{\mu(2^n)2^{n(\varepsilon+\delta-\gamma)}\}$ is quasi geometrically decreasing. Following a consideration made in the proof of Corollary 1.A we get that

$$\mu(2^{n+1}) \leqq K 2^{\gamma-\delta-\varepsilon} \mu(2^n)$$

and

$$\mu(2^{n+\mu}) \leqq \frac{1}{2} 2^{\mu(\gamma-\delta-\varepsilon)} \mu(2^n),$$

whence for any $t \in [2^{n-1}, 2^n]$

$$\frac{\mu(2^\mu t)}{\mu(t)} \leqq \frac{\mu(2^{\mu+n})}{\mu(2^{n-1})} \leqq \frac{K 2^{\gamma-\delta-\varepsilon}}{2} \cdot 2^{\mu(\gamma-\delta-\varepsilon)} \tag{4.14}$$

follows. If μ is so large that $2^{\mu\varepsilon} > K 2^{\gamma-1-\delta-\varepsilon}$, then (4.14) with $C := 2^\mu$ implies (4.13). This shows that we can apply again both Theorem 1 and Theorem B, and then the statement of Corollary 1.B is an immediate consequence of (1.6).

PROOF OF COROLLARY 1.C. Since we have already verified in the proof of Corollary 1.B that under the assumptions of Corollary 1.C conditions of (1.5) hold, therefore the statement of Corollary 1.C is an obvious consequence of (1.7), namely both Theorem C and Theorem 1 are applicable.

PROOF OF THEOREM 2. The proofs of the cases $\alpha = 1/2$ and $-1 < \alpha < < 1/2$ are alike therefore we shall only prove the sufficiency of condition (2.3); and show that if the coefficients $|c_n|$ are monotone, then condition (2.4) is also necessary. But we suggest the reader to consult page 254 of our paper [4] for a little trick appearing in the proof of sufficiency of (2.4).

In the proof of the sufficiency of (2.3) we can follow the argument of Theorem 1 to the end of estimation (4.1), the difference will appear in the estimations of $\sum_1$ and $\sum_2$.

If $\alpha = 1/2$, furthermore $n(\ell)$ and $m(\nu)$ have the same meaning as in (4.2), we get, by (4.2), that

$$\sum\nolimits_1 \leqq \tag{4.15}$$

$$\leqq K_5 \sum_{m=1}^{\infty} \left\{ \gamma(2^m)^2 2^{-m(1+2\alpha)} \sum_{\ell=0}^{m} \sum_{\nu=2^\ell+1}^{2^{\ell+1}} \nu^2 c_\nu^2 \sum_{n=m(\nu)}^{2^{m+1}} (n+1-\nu)^{2\alpha-2} \right\}^{k/2} \leqq$$

$$\leqq K_6 \sum_{m=1}^{\infty} \left\{ \gamma(2^m)^2 2^{-2m} m \sum_{\ell=0}^{m} \sum_{\nu=2^\ell+1}^{2^{\ell+1}} \nu^2 c_\nu^2 \right\}^{k/2} \leqq$$

$$\leqq K_7 \sum_{\ell=0}^{\infty} 2^{\ell k} \left(\sum_{\nu=2^\ell+1}^{2^{\ell+1}} c_\nu^2 \right)^{k/2} \sum_{m=\ell}^{\infty} m^{k/2} \gamma(2^m)^k 2^{-km}.$$

In the proof of Theorem 1 we have already verified that the sequence $\{\gamma(2^m)^k 2^{-km}\}$ is quasi geometrically decreasing. This, by definition (1.10), clearly implies that the sequence $\{m^{k/2}\gamma(2^m)^k 2^{-km}\}$ is also quasi geometrically decreasing. Therefore, by Lemma 5,

$$\sum_{m=\ell}^{\infty} m^{k/2} \gamma(2^m)^k 2^{-km} \leqq K \ell^{k/2} \gamma(2^\ell)^k 2^{-\ell k},$$

thus, by (4.15),

$$\sum\nolimits_1 \leqq K_8 \sum_{\ell=0}^{\infty} \gamma(2^\ell)^k \ell^{k/2} \left(\sum_{\nu=2^\ell+1}^{2^{\ell+1}} c_\nu^2 \right)^{k/2}. \tag{4.16}$$

The estimation of $\sum_2$ runs exactly as in (4.5) and we get for $\alpha = 1/2$ as well

$$\sum\nolimits_2 \leqq K_{10} \sum_{m=1}^{\infty} \gamma(2^m)^k \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{k/2}.$$

This, furthermore (4.1) and (4.16) convey the sufficiency of condition (2.3) regarding Beppo Levi's theorem.

In order to prove the necessity of condition (2.4) for monotone coefficients we assume that $\{c_n\}$ is a positive nonincreasing sequence and consider the following Rademacher series:

$$\sum_{n=0}^{\infty} c_n r_n(x). \tag{4.17}$$

Let $\tilde{\sigma}_n^\alpha(x)$ denote the nth (C, α) mean of series (4.17).

Let α be an arbitrary number with $-1 < \alpha < 1/2$ and $\varepsilon < A^2 \cdot 2^{-3}$, where A denotes the constant appearing in Lemma 3. Let us assume that series (4.17) is summable $\left|C, \alpha, \gamma(t)\right|_k$ a.e. in $(0,1)$. Thus, by Egorov's theorem, there exist a measurable set E with $\mu(E) \geqq 1 - \varepsilon$ and a positive constant M such that for every $x \in E$

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} \left| \tilde{\sigma}_{n+1}^\alpha(x) - \tilde{\sigma}_n^\alpha(x) \right|^k < M.$$

Hence, using Lemma 3 and (3.1), we obtain that

$$(4.18)\qquad M \geqq \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} \int_E \left|\tilde\sigma_{n+1}^{\alpha}(x) - \tilde\sigma_n^{\alpha}(x)\right|^k dx \geqq$$

$$\geqq \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} A^2 2^{-3} A^k 2^{-k} \left\{ \sum_{\nu=0}^{n} \left(L_{n,\nu}^{(\alpha)}\right)^2 c_\nu^2 + \left(A_{n+1}^{(\alpha)}\right)^{-2} c_{n+1}^2 \right\}^{k/2} \geqq$$

$$\geqq K(k,A) \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} \{n^{-2\alpha} c_{n+1}^2\}^{k/2} \geqq$$

$$\geqq K(k,A,\alpha,\gamma) \sum_{n=2}^{\infty} \gamma(n)^k n^{k-1-k\alpha} c_n^k \geqq$$

$$\geqq K(k,A,\alpha,\gamma) \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{mk(\frac12-\alpha)} \sum_{n=2^m+1}^{2^{m+1}} n^{\frac{k}{2}-1} c_n^k \geqq$$

$$\geqq K(k,A,\alpha,\gamma) \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{mk(\frac12-\alpha)} 2^{(m+1)(\frac{k}{2}-1)} 2^m c_{2^{m+1}}^k \geqq$$

$$\geqq \frac12 K(k,A,\alpha,\gamma) \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{mk(\frac12-\alpha)} \left\{ \sum_{n=2^{m+1}+1}^{2^{m+2}} c_n^2 \right\}^{k/2},$$

where $K(k,A)$ and $K(k,A,\alpha,\gamma)$ are positive constants depending only on the parameters in the brackets.

The result given by (4.18) clearly shows the necessity of condition (2.4) for monotone coefficients.

Herewith we have finished the proof of Theorem 2.

The proofs of Corollaries 2.A, 2.B and 2.C run alike as those of Corollaries 1.A, 1.B and 1.C, therefore we omit them.

Proof of Theorem 3. First we prove the sufficiency of conditions (2.5) and (2.6). We may assume again that $c_0 = c_1 = 0$. Furthermore let $n(\ell) := := \min(2^{\ell+1}, n)$ and $m(\nu) := \max(2^m + 1, \nu)$. An elementary calculation shows that

$$(4.19)\qquad \sum_{n=2}^{\infty} \gamma(n)^k n^{k-1} \int_0^1 \left|\sigma_{n+1}^{\alpha}(x) - \sigma_n^{\alpha}(x)\right|^k dx \leqq$$

$$\leqq K_1 \sum_{m=0}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \gamma(n)^k n^{k-1} \left\{ \sum_{\ell=0}^{m} \int_0^1 \left| \sum_{\nu=2^\ell+1}^{n(\ell)} L_{n,\nu}^{(\alpha)} c_\nu \chi_\nu(x) \right|^k dx + \right.$$

$$+\frac{1}{A_{n+1}^{(\alpha)k}}|c_{n+1}|^k\int_0^1|\chi_{n+1}(x)|^k dx\Big\} =: \sum\nolimits_1 + \sum\nolimits_2 .$$

If $k=2$ then we easily get that

$$\sum\nolimits_2 \leqq K_2\sum_{m=0}^{\infty}\gamma(2^m)^2 2^m\sum_{n=2^m+1}^{2^{m+1}} n^{-\alpha 2}|c_n|^2\int_0^1\chi_n(x)^2dx \leqq \tag{4.20}$$

$$\leqq K_3\sum_{m=0}^{\infty}\gamma(2^m)^2 2^{m(1-2\alpha)}\sum_{n=2^m+1}^{2^{m+1}} c_n^2,$$

what is exactly the same sum which appears in (2.5) for $\alpha=\frac{1}{2}$, and in (2.6) for $-1<\alpha<\frac{1}{2}(k=2)$. If $1\leqq k<2$ then we set $J_n:=\{x|\chi_n(x)\neq 0\}$. Since the system $\{\chi_n(x)\}$ is of H-type we have

$$\int_0^1|\chi_n(x)|^k dx = \int_{J_n}|\chi_n(x)|^k dx \leqq \tag{4.21}$$

$$\leqq\left(\int_{J_n}dx\right)^{1-k/2}\left(\int_{J_n}\chi_n^2(x)dx\right)^{k/2} = \mu(J_n)^{1-k/2}.$$

Using this, Hölders's inequality, (3.1) and (3.4), we obtain

$$\sum\nolimits_2 \leqq K_4\sum_{m=0}^{\infty}\gamma(2^m)^k 2^{m(k(1-\alpha)-1)}\sum_{n=2^m+1}^{2^{m+1}}|c_n|^k\mu(J_n)^{(2-k)/2} \leqq \tag{4.22}$$

$$\leqq K_4\sum_{m=0}^{\infty}\gamma(2^m)^k 2^{m(k(1-\alpha)-1)}\left\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2\right\}^{k/2}\left\{\sum_{n=2^m+1}^{2^{m+1}}\mu(J_n)\right\}^{1-k/2} \leqq$$

$$\leqq K_4\sum_{m=0}^{\infty}\gamma(2^m)^k 2^{m(k(1-\alpha)-1)}\left\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2\right\}^{k/2}.$$

It is easy to see that for $\alpha=1-\frac{1}{k}$ this last sum is the same which appears in (2.5), and if $\alpha<1-\frac{1}{k}$ then it is the sum emerging in (2.6).

Next we estimate the sum $\sum_1$, too. Since the system $\{\chi_n(x)\}$ is of H-type and (4.21) holds, thus we easily get for any $1\leqq k<2$ that

$$\sum\nolimits_1 \leqq \tag{4.23}$$

$$\leqq K_2 \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{m(k-1)} \sum_{n=2^m+1}^{2^{m+1}} \sum_{\ell=0}^{m} \sum_{\nu=2^\ell+1}^{n(\ell)} \left|L_{n,\nu}^{(\alpha)}\right|^k |c_\nu|^k \int_0^1 \left|\chi_\nu(x)\right|^k dx \leqq$$

$$\leqq K_2 \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{m(k-1)} \sum_{\ell=0}^{m} \sum_{\nu=2^\ell+1}^{2^{\ell+1}} |c_\nu|^k \mu(J_\nu)^{(2-k)/k} \sum_{n=m(\nu)}^{2^{m+1}} \left|L_{n,\nu}^{(\alpha)}\right|^k \leqq$$

$$\leqq K_2 \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{m(k-1)} \sum_{\ell=0}^{m} \left\{ \sum_{\nu=2^\ell+1}^{2^{\ell+1}} c_\nu^2 \right\}^{k/2} \sum_{n=m(\nu)}^{2^{m+1}} \left|L_{n,\nu}^{(\alpha)}\right|^k.$$

This estimation holds for $k = 2$, too, namely then $\int_0^1 \chi_\nu(x)^2 dx = 1$, therefore the first line of the estimation gives the final inequality after changing the order of the summations. If $2^\ell < \nu \leqq 2^{\ell+1} (\ell = 0, 1, \ldots, m-2)$ then by (3.4)

$$\text{(4.24)} \qquad \sum_{n=m(\nu)}^{2^{m+1}} \left|L_{n,\nu}^{(\alpha)}\right|^k \leqq K_1 \sum_{n=2^m+1}^{2^{m+1}} n^{-(\alpha+1)k} (n-\nu)^{(\alpha-1)k} \nu^k \leqq$$

$$\leqq K_2 2^{\ell k} \cdot 2^{m(1-2k)},$$

and if $2^{m-1} < \nu \leqq n \leqq 2^{m+1}$ then

$$\text{(4.25)} \qquad \sum_{n=m(\nu)}^{2^{m+1}} \left|L_{n,\nu}^{(\alpha)}\right|^k \leqq K_1 2^{-m\alpha k} \sum_{n=m(\nu)}^{2^{m+1}} (n+1-\nu)^{(\alpha-1)k} \leqq K_2 2^{-\alpha m k}.$$

Thus, (4.3), (4.23), (4.24) and (4.25) yield

$$\sum\nolimits_1 \leqq$$

$$\leqq K_3 \sum_{m=0}^{\infty} \gamma(2^m)^k 2^{m(k-1)} \left\{ \sum_{\ell=0}^{m-2} C_\ell^{k/2} \cdot 2^{\ell k} \cdot 2^{m(1-2k)} + \sum_{\ell=m-1}^{m} C_\ell^{k/2} 2^{-\alpha m k} \right\} \leqq$$

$$\leqq K_3 \left\{ \sum_{\ell=0}^{\infty} C_\ell^{k/2} 2^{\ell k} \sum_{m=\ell}^{\infty} \gamma(2^m)^k 2^{-mk} + \sum_{m=0}^{\infty} C_m^{k/2} \gamma(2^m)^k 2^{m(k(1-\alpha)-1)} \right\} \leqq$$

$$\leqq K_3 \left\{ \sum_{\ell=0}^{\infty} C_\ell^{k/2} \gamma(2^\ell)^k + \sum_{m=0}^{\infty} C_m^{k/2} \gamma(2^m)^k 2^{m(k(1-\alpha)-1)} \right\}.$$

Hence, by (4.19), (4.20) and (4.22), we can see that conditions (2.5) and (2.6) are indeed sufficient that series (1.8) for every Haar-type system be summable $\left|C,\left(1-\frac{1}{k}\right),\gamma(t)\right|_k$ and $\left|C,\alpha,\gamma(t)\right|_k$ with $-1<\alpha<1-\frac{1}{k}$, respectively.

In order to show that condition (2.5) implies the $\left|C,\alpha,\gamma(t)\right|_k$-summability for any $\alpha\geqq 1-\frac{1}{k}$ we can apply Theorem A with $r=k$ and $\beta=\alpha\geqq\geqq 1-\frac{1}{k}$. Thus we get that series (1.8) for any H-type system $\{\chi_n(x)\}$ is also summable $\left|C,\alpha,\gamma(t)\right|_k$ if $\alpha\geqq 1-\frac{1}{k}$ and condition (2.5) is fulfilled.

Next we prove the necessity of (2.5). Let us consider series (4.6) where the functions $\chi_n(x)$ are given in Lemma 4 by (3.8). This system is of H-type. Therefore if we repeat the proof of necessity of (2.1) word for word, the necessity of condition (2.5) will be proved, namely the assumption $\alpha>\frac{1}{2}$ is not used in the proof. Another argument for the proof is the following one: If series (4.6) is summable $\left|C,\alpha,\gamma(t)\right|_k$ $\left(\alpha\geqq 1-\frac{1}{k}\right)$, then by Theorem A it is summable $\left|C,\beta,\gamma(t)\right|_k$ for any $\beta\geqq\alpha$, and if $\beta\geqq\frac{1}{2}$, then the sum

$$\sum_{m=1}^{\infty}\gamma(2^m)^k\left\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2\right\}^{k/2}$$

is finite by Theorem 1, i.e. condition (2.5) is necessary for any $\alpha\geqq 1-\frac{1}{k}$.

Finally we prove the necessity of (2.6). Consider again series (4.6) and use the notations introduced in the proof of necessity of (2.1). Now we use Lemma 1 with $N=m+1$ and $k=0$, furthermore the result of (4.7). The compatible set E_0 will be denoted by $E_0(m,n)$. Then we have

$$S:=\sum_{n=2}^{\infty}\gamma(n)^k n^{k-1}\int_E\left|\overline{\sigma}_{n+1}(x)-\overline{\sigma}_n^{\alpha}(x)\right|^k dx= \tag{4.26}$$

$$=\sum_{m=1}^{\infty}\sum_{n=2^m}^{2^{m+1}-1}\gamma(n)^k n^{k-1}\int_E\left|\sum_{\nu=0}^{n}L_{n,\nu}^{(\alpha)}c_\nu\chi_\nu(x)+\frac{1}{A_{n+1}^{(\alpha)}}c_{n+1}\chi_{n+1}(x)\right|^k dx\geqq$$

$$\geqq\sum_{m=1}^{\infty}\gamma(2^m)^k 2^{m(k-1)}\sum_{n=2^m}^{2^{m+1}-1}\int_{E_0(n,m)\cap E}\left|\frac{1}{A_{n+1}^{(\alpha)}}c_{n+1}\chi_{n+1}(x)\right|^k dx\geqq$$

$$\geqq\sum_{m=1}^{\infty}\gamma(2^m)^k 2^{m(k-1)}\sum_{n=2^m}^{2^{m+1}-1}\left(\int_{E_0(n,m)}-\right.$$

$$\left.-\int_{E_0(n,m)-E_0(n,m)\cap E}\right)\left|\frac{1}{A_{n+1}^{(\alpha)}}c_{n+1}\chi_{n+1}(x)\right|^k dx\geqq$$

$$\geqq \sum_{m=1}^{\infty} \gamma(2^m)^k 2^{m(k-1)} \left\{ \sum_{n=2^m}^{2^{m+1}-1} \int_{E_0(n,m)} \left| \left(A_{n+1}^{(\alpha)}\right)^{-1} c_{n+1}\chi_{n+1}(x) \right|^k dx - \right.$$

$$\left. - \int_{E_0(n,m)-E_0(n,m)\cap E} \left| \sum_{n=2^m}^{2^{m+1}-1} \left(A_{n+1}^{(\alpha)}\right)^{-1} c_{n+1}\chi_{n+1}(x) \right|^k dx \right\},$$

where at the final step we used the fact that the system $\{\chi_n(x)\}$ is of H-type. Similar reasoning as we have made in (4.9) and (4.10) gives that

$$\int_{E_0(n,m)} \left| \left(A_{n+1}^{(\alpha)}\right)^{-1} c_{n+1}\chi_{n+1}(x) \right|^k dx \geqq \tag{4.27}$$

$$\geqq \int_{E_0(n,m)} e_1(\alpha)^k (n+1)^{-\alpha k} |c_{n+1}|^k \left|\chi_{n+1}(x)\right|^k dx \geqq$$

$$\geqq e_1(\alpha)^k (n+1)^{-\alpha k} |c_{n+1}|^k \sum_{\rho=1}^{\rho_m} \int_{E_0(m,n)\cap I_{n+1-2^m}(m,J_\rho)} \frac{C_m^k}{|c_{n+1}|^k} dx \geqq$$

$$\geqq e_1(\alpha)^k (n+1)^{-\alpha k} C_m^k \sum_{\rho=1}^{\rho_m} 2^{-1} \mu\left(I_{n+1-2^m}(m, J_\rho)\right) =$$

$$= 2^{-1} e_1(\alpha)^k (n+1)^{-\alpha k} C_m^k \sum_{\rho=1}^{\rho_m} \mu(J_\rho) c_{n+1}^2 C_m^{-2} =$$

$$= 2^{-1} e_1(\alpha)^k (n+1)^{-\alpha k} C_m^{k-2} c_{n+1}^2$$

and

(4.28)

$$\int_{E_0(n,m)-E_0(n,m)\cap E} \left| \sum_{n=2^m}^{2^{m+1}-1} \left(A_{n+1}^{(\alpha)}\right)^{-1} c_{n+1}\chi_{n+1}(x) \right|^k dx \leqq e_2^k(\alpha) 2^{-mk\alpha} C_m^k \cdot \varepsilon,$$

where $e_1(\alpha)$ and $e_2(\alpha)$ depend only on α.

Collecting inequalities (4.26), (4.27) and (4.28) we have

$$S \geqq \sum_{m=1}^{\infty} \gamma(2^m)^k 2^{m(k-1)} \left\{ \sum_{n=2^m}^{2^{m+1}-1} 2^{-1} e_1(\alpha)^k (n+1)^{-\alpha k} C_m^{k-2} c_{n+1}^2 - \right.$$

$$-\varepsilon e_2^k(\alpha)2^{-mk\alpha}C_m^k\Big\} \geqq$$

$$\geqq \sum_{m=1}^{\infty} \gamma(2^m)^k 2^{m(k-1)} 2^{-mk\alpha} C_m^k \left(2^{-1}e_1^k(\alpha) - \varepsilon e_2^k(\alpha)\right),$$

whence the necessity of (2.6) follows clearly if $\varepsilon \leqq 2^{-2}e_1(\alpha)^k e_2(\alpha)^{-k}$.

The proof of Theorem 3 is thus complete.

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THE MEAN ERGODIC THEOREM FOR COSINE OPERATOR FUNCTIONS WITH OPTIMAL AND NON-OPTIMAL RATES

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Dedicated to Professor Károly Tandori on the occasion of his 70 th birthday, in great respect

1. Introduction and statements of main results

Let $\mathcal{C} = \{C(t), t \in \mathbf{R}\}$ be a cosine operator function, thus a family of bounded, linear operators mapping the Banach space X (with norm $\|.\|_X$) into itself, satisfying the d'Alembert functional equation $C(s+t)+C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbf{R}$ with $C(0) = I$, the identity operator on X, together with the strong continuity property $\lim_{h\to 0} \|C(t+h)f - C(t)f\|_X = 0$ all $f \in X, t \in \mathbf{R}$. The (infinitesimal) generator A of $\mathcal{C}$ is defined by $Af = \underset{h\to 0}{\text{s-lim}}\, 2h^{-2}[C(h)f - f]$ for those elements $f \in X$ for which this strong limit exists, namely for $f \in D(A)$. The cosine operator function with $C(t) = I$, all $t \in \mathbf{R}$, is called trivial.

The following result was essentially established by J.A. Goldstein et al [15]:

Let $\{C(t); t \in \mathbf{R}\}$ be an equibounded family of cosine operator functions on the Banach space X. Then $\mathcal{C}$ is strongly Cesàro-ergodic, i.e.,

$$(C,1)\text{-}\lim_{t\to\infty} C(t)f \equiv \underset{t\to\infty}{\text{s-lim}}\, E(t)f := \underset{t\to\infty}{\text{s-lim}}\, \frac{2}{t^2}\int_0^t\int_0^u C(v)f\,dv\,du = Pf$$

for each $f \in X_0$, where P is the linear, bounded projection of X_0 onto the kernel $N(A) := \{f \in D(A); Af = \theta\}$ parallel to $\overline{R(A)}$, the closure of the range $R(A)$. Here, $X_0 = \overline{R(A)} \oplus N(A)$, the direct sum being well defined as $\overline{R(A)} \cap N(A) = \{\theta\}$. X_0 is a closed subspace of X. If X is reflexive, then $X_0 = X$.

In this paper we shall also be concerned with the strong Abel-limit of $\mathcal{C}$, i.e.,

$$(A)\text{-}\lim_{t\to\infty} C(t)f \equiv \underset{\lambda\to 0+}{\text{s-lim}}\, \lambda^2 R(\lambda^2, A)f = \underset{\lambda\to 0+}{\text{s-lim}}\, \lambda \int_0^\infty e^{-\lambda u}C(u)f\,du.$$

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A particular case of our general results will turn out to be the mean ergodic theorem for the Cesàro and Abel limits of $\{C(t); t \in \mathbf{R}\}$ below. It will turn out that our results are also valid for not-necessarily equibounded operators $C(t)$. In fact, we shall work with the class of those $C(t)$ which have a certain rate of growth at infinity for which their means $E(t)f :=$ $:= 2t^{-2} \int_0^t \int_0^u C(v)f\,dv\,du$ are equibounded. Thus

DEFINITION 1.1. Let X be an arbitrary Banach space. Let $\mathcal{C}_M, \mathcal{C}_M^2$ and $\mathcal{C}_M^\beta$ be those sets of cosine operator functions given by, M being some constant,

$$\mathcal{C}_M^\beta := \left\{\mathcal{C}; \|C(t)\|_{[X]} = O\big(|t|^\beta\big), |t| \to \infty, \|E(t)\|_{[X]} \leqq M\right\},$$

$$\mathcal{C}_M^2 := \left\{\mathcal{C}; \|C(t)\|_{[X]} = o\big(|t|^2\big), |t| \to \infty, \|E(t)\|_{[X]} \leqq M\right\},$$

$$\mathcal{C}_M := \left\{\mathcal{C}; (0,\infty) \subset \rho(A), \left\|\lambda^2 R(\lambda^2, A)\right\|_{[X]} \leqq M\right\},$$

where $\rho(A)$ is the resolvent set of the generator A.

It is clear that $\mathcal{C}_M^\beta \subset \mathcal{C}_M^2$ for $\beta < 2$. It will be shown in Lemma 4.2 that also $\mathcal{C}_M^2 \subset \mathcal{C}_M$ is valid.

It will be seen that strong Cesàro (respectively Abel) ergodicity is valid on the set $\mathcal{C}_M^2$(respectively $\mathcal{C}_M$). The set $\mathcal{C}_M^\beta$will be needed for the discussion of rates of convergence for the Cesàro operator.

THEOREM 1.1. *Let $\mathcal{C} \in \mathcal{C}_M^2$, having generator A, with $X_0 = \overline{R(A)} \oplus$ $\oplus N(A)$. The following three assertions are equivalent for any $f \in X$:*

(i) $(C,1)\text{-}\lim_{t\to\infty} C(t)f = g_1$,

(ii) $(A)\text{-}\lim_{t\to\infty} C(t)f = g_2$,

(iii) $f \in X_0$.

In this instance, the strong limits are equal with $g_1 = g_2 = Pf$, where P is a bounded linear projector of X_0 onto $N(A)$.

If, in addition, for any $f \in X$ the set $\{\lambda^2 R(\lambda^2, A)f; \lambda > 0\}$ is conditionally weakly sequentially compact or, more strongly, if X is reflexive, then $X_0 = X$, so that convergence in (i) *and* (ii) *is valid on all of X.*

This theorem will be deduced as a particular case of the following one dealing with the optimal and non-optimal rates of approximation of the Cesàro and Abel means to Pf. For this purpose, let us denote the restrictions of $C(t), A$, and $D(A)$ to X_0 by $C_0(t) := C(t)|_{X_0}$, $A_0 := A|_{X_0}$, and $D(A_0) := D(A) \cap X_0$. Then $\mathcal{C}_0 := \{C_0(t); t \in \mathbf{R}\}$ forms a cosine operator function having generator A_0 with $N(A_0) = N(A)$. Further, the resolvent $R(\lambda, A_0)$ is given by $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$.

Further, an operator B, roughly the inverse of the generator A, will be needed.

DEFINITION 1.2. Let $\mathcal{C}_0, X_0, A_0, D(A_0)$ and P be given as above. The operator B with domain $D(B) := R(A_0) \oplus N(A) \subset X_0$ and range $R(B) :=$ $:= D(A_0) \cap N(P)$ is defined by $Bf = g$, where $g \in D(A_0)$ is uniquely determined by $f = Ag + Pf$ and the side condition $Pg = \theta$.

The operator B is linear, closed with $\overline{D(B)} = X_0$, and $B = A^{-1}$ on $R(A_0) \cap D(A_0)$ (see Remark 3.1). Whereas the generator A is the second derivative of $C(t)f$ at $t = 0$, the operator B will turn out to be the right-hand derivative of $\lambda R(\lambda^2, A)[P - I]f$ at $\lambda = 0$ (see Corollary 4.1).

THEOREM 1.2. *Let $\mathcal{C} \in \mathcal{C}_M^2$ satisfy the assumptions of Theorem* 1.1.

a) *The following three assertions are equivalent for any $f \in X$:*

(i) $\|E(t)f - Pf\|_X = o(t^{-2}) \qquad (|t| \to \infty)$,

(ii) $\left\|\lambda^2 R(\lambda^2, A)f - Pf\right\|_X = o(\lambda^2) \qquad (\lambda \to 0^+)$,

(iii) $f \in N(B) = N(A)$, *i.e.,* $Pf = f$.

b) *The following three assertions are equivalent for* $0 < \alpha \leqq 2$ *and any* $f \in X$:

(i) $\|E(t)f - Pf\|_X = O\big(|t|^{-\alpha}\big)$ $(|t| \to \infty)$ *if in addition* $\mathcal{C} \in \mathcal{C}_M^0$, *i.e.,* $\mathcal{C}$ *is equibounded.*

(ii) $\left\|\lambda^2 R(\lambda^2, A)f - Pf\right\|_X = O(\lambda^\alpha) \qquad (\lambda \to 0^+)$,

(iii) $K\big(\lambda^2, f; X_0, D(B)\big) = O(\lambda^\alpha) \qquad (\lambda \to 0^+)$.

c) *If, in particular,* $\alpha = 2$, *then the assertions* (i)–(iii) *of* b) *are also equivalent to:*

(iv) $f \in \widetilde{D(B)}^{X_0} := \{f \in X_0; \exists \{f_n\}_{n \in \mathbf{N}} \subset D(B)$ *with* $\|f_n\|_{D(B)} \leqq$ $\leqq M$ *and* $\lim_{n \to \infty} \|f_n - f\|_X = 0\}$, *the relative completion of* $D(B)$ *with respect to* X_0. *Here* $\|f\|_{D(B)} := \|f\|_X + \|Bf\|_X$.

(iv)* $f \in D(B)$ *when* X_0 *is reflexive.*

d) *The assertions of part* b) *are sharp provided* $\mathcal{C} \in \mathcal{C}_M^\beta$ *with* $\beta = 0$, B *is unbounded, and* $\alpha \in (0, 2)$, *i.e., there exist elements* $f_\alpha, f_\alpha^* \in X_0$ *satisfying*

$$\text{(i)}\quad \|E(t)f_\alpha - Pf_\alpha\|_X \begin{cases} = O\big(|t|^{-\alpha}\big) \\ \neq o\big(|t|^{-\alpha}\big) \end{cases} \qquad (|t| \to \infty),$$

$$\text{(ii)}\quad \left\|\lambda^2 R(\lambda^2, A)f_\alpha^* - Pf_\alpha^*\right\|_X \begin{cases} = O(\lambda^\alpha) \\ \neq o(\lambda^\alpha) \end{cases} \qquad (\lambda \to 0^+).$$

Above, the K-functional, which is a measure of smoothness in a Banach space setting, is defined in this special case for $f \in X, \lambda \in \mathbf{R}$ by

$$K\big(\lambda, f; X, D(B)\big) := \inf_{h \in D(B)} \{\|f - h\|_X + \lambda \|Bh\|_X\}. \tag{1.1}$$

This is known to be a bounded, continuous, monotone, sublinear functional on X for $\lambda \in \mathbf{R}$; it tends to zero for $\lambda \to 0$ iff $f \in \overline{D(B)}^X$.

Parts a) to c) of Theorem 1.2 reveal that the processes $E(t)f$ and $\lambda^2 R(\lambda^2, A)f$ are saturated on X_0 with orders $O(t^{-2})$ for $|t| \to \infty$ and $O(\lambda^2)$ for $\lambda \to 0^+$, and that their saturation (or Favard) classes are both characterized as the completion of $D(B)$ relative to X_0. These are Theorems 3.2 and 4.2. In fact, it is also true, that there exist elements $f_2, f_2^* \in \widetilde{D(B)}^{X_0}$ for which the critical orders $O(t^{-2})$ and $O(\lambda^2)$ are actually attained, together with $\|E(t)f_2 - Pf_2\|_X \neq o(t^{-2})$ for $|t| \to \infty$, and $\|\lambda^2 R(\lambda^2, A)f_2^* - Pf_2^*\|_X \neq$ $\neq o(\lambda^2)$ for $\lambda \to 0^+$. This is the substance of parts a) of Theorems 3.4 and 4.4.

Assertion b) of Theorem 1.2 is concerned with non-saturated (or non-optimal) approximation; The processes $E(t)f$ and $\lambda^2 R(\lambda^2, A)f$ approximate Pf with the rates $O(|t|^{-\alpha})$ and $O(\lambda^\alpha)$ for $0 < \alpha < 2$ iff the associated K-functional is of order $O(\lambda^\alpha)$ for $\lambda \to 0^+$. Parts a)–c) of Theorem 1.2 are the counterparts of the corresponding results for classical semigroup operators due to Butzer–Dickmeis [4]. Observe that the mean ergodic theorem with rates for the classical discrete case, thus for $\{T^n\}_{n \in \mathbf{N}}$, was first considered in Butzer–Westphal [8], as is also mentioned by Krengel [18] p. 84.

S.-Y. Shaw [28] recently established parts a) and c) of Theorem 1.2 for equibounded cosine operator functions as an application of rather general theorems [26] on nets of operators (which also cover, as applications, tensor product semigroups as well as n-times integrated semigroups). He did not consider non-optimal approximation (i.e., part b)), nor the sharpness of the results (i.e., part d)), nor did he consider any concrete examples.

In Section 5, the various results are applied to the particular cosine operator function which is the solution of the wave equation $(\partial^2/\partial t^2)w(t,x) =$ $= (\partial^2/\partial x^2)w(t,x)$ under the initial conditions

$$w(0,x) = f(x), \quad (\partial/\partial t)w(t,x)|_{t=0} = 0$$

on several typical Banach spaces. In this respect, let $X(\mathbf{R})$ be the space $UCB(\mathbf{R}) := \{f : \mathbf{R} \to \mathbf{C}; f$ uniformly continuous and bounded on $\mathbf{R}\}$ with norm $\|f\|_{UCB(\mathbf{R})} = \sup_{x\in\mathbf{R}} |f(x)|$. For the solution $C(t)f(x) = (1/2)[f(x+$ $+t) + f(x-t)]$, $f \in X(\mathbf{R})$, it will turn out that the generator $A = (d/dx)^2$, with domain $D(A) := X^2(\mathbf{R}) := \{f \in UCB(\mathbf{R}); f', f'' \in UCB(\mathbf{R})\}$, and for B, which is roughly A^{-1}, we have $D(B) := X^{-2}(\mathbf{R}) := \{f \in UCB(\mathbf{R}); f =$ $= g'' + c, g \in X^2(\mathbf{R}), c \in \mathbf{C}\}$.

In this respect, one of the basic results of approximation theory is the characterization of the K-functional for the concrete couple $(X(\mathbf{R}), X^2(\mathbf{R}))$ (or more generally for $(X(\mathbf{R}), X^r(\mathbf{R}))$, $r \in \mathbf{N}$) in terms of the modulus of continuity. It reads (cf. [3], p. 192 f. or [2]) that for $\alpha \in (\theta, 2]$ (see also

Lemma 2.5 here),

$$
(1.2)\qquad \begin{cases} K(t^2, f; X(\mathbf{R}), X^2(\mathbf{R})) = O(t^\alpha) \qquad (t \to 0^+) \Leftrightarrow \\ \Leftrightarrow \|f(x+t) - 2f(x) + f(x-t)\|_{X(\mathbf{R})} = O(t^\alpha) \Leftrightarrow \\ \Leftrightarrow f \in \widetilde{X^2(\mathbf{R})}^{X(\mathbf{R})}, \text{ provided additionally } \alpha = 2. \end{cases}
$$

One of our specific ergodic applications is the following theorem. It is the counterpart of the foregoing characterization of the K-functional, now for the couple $\big(X_0(\mathbf{R}), D(B)\big) = (X_0(\mathbf{R}), X^{-2}(\mathbf{R}))$, with $t \to \infty$. It answers in part a conjecture raised in [7], [8] (see also [2]).

THEOREM 1.3. *For any $f \in X_0(\mathbf{R}), \alpha \in (0, 2]$, one has for $t \to \infty$,*

(1.3)

$$
\begin{cases} K(t^{-2}, f; X_0(\mathbf{R}), X^{-2}(\mathbf{R})) = O(t^{-\alpha}) \Leftrightarrow \\ \Leftrightarrow \left\| \dfrac{1}{t^2} \displaystyle\int_0^t \int_0^u \big[f(x+v) + f(x-v)\big]\, dv\, du - \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(u) du \right\|_{X(\mathbf{R})} = \\ = O(t^{-\alpha}) \Leftrightarrow \\ \Leftrightarrow f \in \widetilde{X^{-2}(\mathbf{R})}^{X_0(\mathbf{R})}, \textit{ provided additionally } \alpha = 2. \end{cases}
$$

Now let us look at the famous Gauss–Weierstrass semigroup (see Section 2)

$$
(1.4)\qquad [W(t)f](x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbf{R}} f(x-u) \exp(-u^2/4t)\, du,
$$

which solves the heat equation $(\partial/\partial t) w(t,x) = (\partial^2/\partial x^2) w(t,x)$ for $w(0,x) = f(x) \in X(\mathbf{R})$. It is known to have the same (semigroup) generator A. Hence $D(A) = X^2(\mathbf{R})$ and also $D(B) = X^{-2}(\mathbf{R})$. Using the mean ergodic theory for semigroup operators as developed in [4], a further application of our general theorems will be the following particular mean ergodic theorem for $t \to \infty$ of the semigroup $\{W(t), t > 0\}$:

THEOREM 1.4. *For any $f \in X_0(\mathbf{R}), \alpha \in (0,2]$ one has for $t \to \infty$,*

$$\left\| \frac{1}{t}\int_0^t W(u)f(x)\,du - \lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R} f(u)du \right\|_{X(\mathbf{R})} = O(t^{-\alpha/2}) \Leftrightarrow$$

$$\Leftrightarrow \left\| \frac{1}{t^2}\int_0^t\int_0^u \big[f(x+v)+f(x-v)\big]\,dv\,du - \lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R} f(u)du \right\|_{X(\mathbf{R})} =$$

$$= O(t^{-\alpha}).$$

Thus (1.3) turns out to be a substitute for the classical modulus of continuity $\omega_2(t,f;X(\mathbf{R})) = \sup_{0<h\leqq t}\|f(x+h)-2f(x)+f(x-h)\|_{X(\mathbf{R})}$ in the instance of ergodic theory for specific examples.

Note that all the integrals appearing in this paper, except those of the definition of the L^p-norms, may be interpreted as strong Riemann integrals (cf. [16] p. 62 ff.).

2. Preliminary results

The follwing well-known preliminary results will be needed:

(2.1) The generator A is closed, with $\overline{D(A)} = X$;

(2.2) $C(t)f \in D(A)$ with $AC(t)f = C(t)Af \qquad (f \in D(A); t \in \mathbf{R})$;

For $g := E(t)f := (2/t^2)\int_0^t\int_0^u C(v)f dv du = (2/t^2)\int_0^t (t-u)C(u)f\,du$ one has

(2.3) $g \in D(A), Ag = (2/t^2)\big[C(t)f - f\big] \qquad (f \in X, t \in \mathbf{R}, t \neq 0)$.

The theorem of Da Prato-Giusti–Fattorini–Sova (cf. [24], pp. 356–362; [13] II, pp. 63–67; [29], pp. 27–36) reads:

The operator U generates a cosine operator function $\mathcal{C}$ on a Banach space X iff U is closed, with $\overline{D(U)} = X$, and there exist constants $M \geqq 1$, $\omega \geqq 0$ such that for each $\lambda > \omega$, $\lambda^2 \in \rho(U)$, the resolvent set of U,

(2.4) $$\left\| \frac{d^m}{d\lambda^m}\big(\lambda R(\lambda^2, U)\big) \right\|_{[X]} \leqq M\frac{m!}{(\lambda-\omega)^{m+1}} \qquad (m \in \mathbf{N}_0).$$

Thus the resolvent $R(\lambda^2, A) = (\lambda^2 I - A)^{-1}$ of A exists for all $\lambda > \omega$.

In this event,

$$\left\| C(t) \right\|_{[X]} \leqq M e^{\omega |t|} \qquad (t \in \mathbf{R}), \tag{2.5}$$

$$\lambda R(\lambda^2, A) f = \int_0^\infty e^{-\lambda u} C(u) f \, du \qquad (\lambda > \omega; f \in X), \tag{2.6}$$

i.e., $\lambda R(\lambda^2, A) f$ is the Laplace transform of $C(t)f$ and so, in particular, a holomorphic function of $\lambda > 0$.

$$\operatorname*{s-lim}_{t \to 0} E(t) f = f \qquad (f \in X). \tag{2.7}$$

$$\operatorname*{s-lim}_{\lambda \to \infty} \lambda^2 R(\lambda^2, A) f = f \qquad (f \in X). \tag{2.8}$$

$$C(-t) = C(t), \qquad E(-t) = E(t) \qquad (t > 0), \tag{2.9}$$

$$R(\lambda^2, A) A f = \lambda^2 R(\lambda^2, A) f - f \qquad \left(f \in D(A)\right), \tag{2.10}$$

$$A R(\lambda^2, A) f = \lambda^2 R(\lambda^2, A) f - f \qquad (f \in X). \tag{2.11}$$

Property (2.9) allows one to prove all results concerning $E(t)$ only for the case $t > 0$. The negative case follows directly by the substitution $t \to -t$.

Further, we need a basic connection between cosine operator functions and (C_0)-semigroups, which are families of strongly continuous bounded operators $\mathcal{T} = \{T(t), t \geqq 0\}$, satisfying the functional equation $T(t+s) = T(t)T(s)$, $t, s \geqq 0$ with $T(0) = I$. Their (infinitesimal) generator A' is defined by $A'f = \operatorname*{s-lim}_{h \to 0+} h^{-1}[T(h)f - f]$, for those elements $f \in X$ for which this strong limit exists, namely for $f \in D(A')$ (see e.g. [3] p. 9).

LEMMA 2.1. *Let $\mathcal{C}$ be a cosine operator function with generator A. Then A generates a (C_0)-semigroup $\mathcal{T} = \{T(t), t \geqq 0\}$ given by*

$$T(t) f = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-s^2/4t\right) C(s) f \, ds.$$

For a proof see e.g. [13] I, Remark 5.11. Note that the converse is not generally true, i.e., there exists a Banach space and an operator A such that A is the generator of a (C_0)-semigroup, but of no cosine operator function (see [21]).

The operator norms of $\mathcal{C}$ and $\mathcal{T}$ of Lemma 2.1 are also connected, i.e., LEMMA 2.2. *Let A be the generator of a cosine operator function C.*

a) *If $\|C(t)\|_{[X]} = o(|t|^{2\gamma})$ for $|t| \to \infty$, $\gamma > 0$, then A generates a (C_0)-semigroup T with $\|T(t)\|_{[X]} = o(t^\gamma)$.*

b) *If $\|C(t)\|_{[X]} = O(|t|^{2\gamma})$ for $|t| \to \infty$, $\gamma \geqq 0$, then A generates a (C_0)-semigroup T with $\|T(t)\|_{[X]} = O(t^\gamma)$.*

PROOF. a) First of all, by the definition of the Gamma function,

$$\int_0^\infty s^{2\gamma} \exp(-s^2/4t)ds = \Gamma\left(\gamma + \frac{1}{2}\right) 4^\gamma t^{\gamma+\frac{1}{2}}. \tag{2.12}$$

Now let $\varepsilon > 0$ be arbitrary. We have to show that there exists a $\tau = \tau(\varepsilon) < < \infty$ such that $t^{-\gamma}\|T(t)\|_{[X]} < \varepsilon$ for all $t \geqq \tau$. By hypothesis there exists for $\bar{\varepsilon} := 4^{-\gamma-1/2}\varepsilon\sqrt{\pi}/\Gamma(\gamma + 1/2) > 0$ a $\bar{\tau} < \infty$ such that $t^{-2\gamma}\|C(t)\|_{[X]} < < \bar{\varepsilon}$ for all $t \geqq \bar{\tau}$. On the other hand, by (2.5) there exist $M \geqq 1, \omega \geqq 0$ such that $\|C(t)\| \leqq Me^{\omega\bar{\tau}}$ for $t < \bar{\tau}$. Hence, by Lemma 2.1 and (2.12), for $t \geqq \tau := \left(2Me^{\omega\bar{\tau}}/\varepsilon\right)^{1/\gamma} < \infty$,

$$\frac{\|T(t)f\|_X}{t^\gamma} \leqq \frac{Me^{\omega\bar{\tau}}}{t^{\gamma+\frac{1}{2}}\sqrt{\pi}} \int_0^{\bar{\tau}} \exp(-s^2/4t)ds\, \|f\|_X +$$

$$+\frac{\bar{\varepsilon}}{t^{\gamma+\frac{1}{2}}\sqrt{\pi}} \int_{\bar{\tau}}^\infty s^{2\gamma} \exp(-s^2/4t)\, ds\, \|f\|_X \leqq$$

$$\leqq \frac{Me^{\omega\bar{\tau}}}{t^\gamma} \|f\|_X + \frac{\varepsilon}{2}\|f\|_X \leqq \frac{\varepsilon}{2}\|f\|_X + \frac{\varepsilon}{2}\|f\|_X = \varepsilon\|f\|_X .$$

This proves part a). The proof of part b) is now clear. □

LEMMA 2.3. *Let C be a cosine operator function as in Lemma 2.2 b), A being its generator. Then $(0, \infty) \subset \rho(A)$, i.e., $\lambda^2 R(\lambda^2, A)$ exists for $\lambda > 0$.*

PROOF. In view of Lemma 2.2 b), A generates a (C_0)-semigroup with $\|T(t)\|_{[X]} \leqq Mt^\gamma$, for $t \geqq \tau$. Hence, $\omega_0 := \lim_{t\to\infty} \log\|T(t)\|_{[X]}/t = 0$. So $(0, \infty) \subset \rho(A)$ (see e.g. [12] Theorem VIII.1.11 p. 622). □

LEMMA 2.4. *For $f \in X, t \in \mathbf{R}, t \neq 0$ one has*

$$E(t)f - f = (2/t^2)\int_0^t \int_0^u A \int_0^v \int_0^w C(x)f\, dx\, dw\, dv\, du =$$

$$= A \int_0^t \int_0^u (1 - u/t)^2 C(v) f \, dv \, du,$$

i.e., $E(t)f - f \in R(A)$.

PROOF. The right hand side of this equation follows by a double partial integration, the other by (2.3). □

LEMMA 2.5. *For the K-functional one has for $\lambda \to 0^+$*

$$K\big(\lambda, f; X, D(B)\big) = \begin{cases} O(\lambda) & \Leftrightarrow f \in \widetilde{D(B)}^X \\ o(\lambda) & \Leftrightarrow f \in N(B). \end{cases}$$

Further, $\widetilde{D(B)}^X = D(B)$ provided X is reflexive.

For a proof see [1], p. 15. Thus the K-functional is saturated.

Since we also want to study the sharpness of approximation processes, we need the following recent theorem of O.V. Davydov (c.f. [9], [10]).

THEOREM 2.1. *Let X be a Banach space and X^+ be the set of all non-negative, sublinear, real-valued functionals S on X for which the norm $\|S\|_{X^+} := \sup\{Sf; f \in X, \|f\|_X \leqq 1\}$ is bounded. Further, let $\{S_n\}_{n\in\mathbf{N}} \subset \subset X^+$ with $\limsup_{n\to\infty} \|S_n\|_{X^+} = \infty$, and let $\left\{f \in X; \lim_{n\to\infty} S_n f = 0\right\}$ be dense in X. Then there exists an element $f_0 \in X$ satisfying $\sup_{n\in\mathbf{N}} S_n f_0 \leqq 1$ and $\limsup_{n\to\infty} S_n f_0 = 1$.*

This theorem answers a conjecture of Butzer–Dickmeis raised in [5] concerning the sharpness of non-saturated approximation of semigroup operators. Davydov makes use of deep results on the uniform boundedness principle with rates due to Dickmeis–Nessel–van Wickeren [11] (see also the literature cited there). For very recent extensions of these results see [23].

3. Ergodic theorems for the Cesàro operator

LEMMA 3.1. *Let $C \in \mathcal{C}_M$, having the generator A. Then $N(A) \cap \overline{R(A)} = \{\theta\}$.*

PROOF. For any $f \in N(A)$, $Af = \theta$. Hence, by (2.9), $\lambda^2 R(\lambda^2, A) f - f = \theta$. On the other hand, for $f \in R(A)$ there exists $g \in D(A)$ such that $f = Ag$. Hence, by (2.9), $\left\|\lambda^2 R(\lambda^2, A) f\right\|_X = \left\|\lambda^2 R(\lambda^2, A) A g\right\|_X = \lambda^2 \left\|\lambda^2 R(\lambda^2, A) g - g\right\|_X \leqq \lambda^2 (M + 1) \|g\|_X \to 0$, for $\lambda \to 0^+$. Moreover, as the operator $\lambda^2 R(\lambda^2, A)$ is equibounded by M, then $\text{s-lim}_{\lambda\to 0+} \lambda^2 R(\lambda^2, A)f = \theta$ for all $f \in \overline{R(A)}$ by the Banach–Steinhaus theorem. Hence $f = \theta$ for all $f \in N(A) \cap \overline{R(A)}$. □

Lemma 3.1 will enable us to define the direct sum of $N(A)$ and $\overline{R(A)}$ of a $\mathcal{C} \in \mathcal{C}_M$ having generator A as well as the projection P of $X_0 = \overline{R(A)} \oplus N(A)$ onto $N(A)$.

LEMMA 3.2. *Let* $\mathcal{C} \in \mathcal{C}_M$, *and* $\mathcal{C}_0$, X_0, A_0, $D(A_0)$ *and* P *be given as in Definition* 1.2. *Then*

a) X_0 *is a Banach space with norm* $\|.\|_{X_0} = \|.\|_X$; $X = X_0$ *if, in addition, for any* $f \in X$ *the set* $\{\lambda^2 R(\lambda^2, A)f; \lambda > 0\}$ *is conditionally weakly seqentially compact (valid if* X *is reflexive).*

b) P *is linear, closed and bounded.*

c) $PA_0 f = \theta$, *all* $f \in D(A_0)$, $A_0 P f = \theta$, *all* $f \in X_0$.

d) $P\mathcal{C}_0(t)f = \mathcal{C}_0(t)Pf = Pf$, *all* $f \in X_0$, $t \in \mathbf{R}$.

e) $PE(t)f = E(t)Pf = Pf$, *all* $f \in X_0, t \in \mathbf{R}$.

PROOF. Parts a) and c) are easy, see [16], p. 520 and Lemma 2.2. Part b) follows by the closed graph theorem. Regarding d), for $f \in X_0$ and $t \in \mathbf{R}$ arbitrary, by (2.3), $P\mathcal{C}_0(t)f - Pf = P[A_0 t^2 E(t)f] = \theta$. Again by (2.3) and (2.2), $\mathcal{C}_0(t)Pf - Pf = t^2 E(t)[A_0 Pf] = \theta$ for $f \in X_0$. Part e) is a consequence of d). □

Since $\mathcal{C}_0$ is a cosine operator function on X_0 having generator A_0, and $N(A_0) = N(A)$, $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$, one can write $\mathcal{C}$ instead of $\mathcal{C}_0$ and A instead of A_0 if it is clear one works with the space X_0, and no misunderstandings are possible.

Now to the mean ergodic theorem for the Cesàro operator $E(t)$.

THEOREM 3.1. *Let* $\mathcal{C} \in \mathcal{C}_M^2$ *with generator* A, X_0 *and* P *being defined as in Lemma* 3.2. *The limit* $\text{s-lim}_{|t|\to\infty} E(t)f$ *exists iff* $f \in X_0$. *If so, this limit equals* Pf.

PROOF. Let $f \in N(A) \oplus R(A)$. Then $f = Ag + Pf$, some $g \in D(A)$. Hence, by (2.2), (2.3) and Lemma 3.2 b), $\|E(t)f - Pf\|_X = \|E(t)Af\|_X = 2t^{-2}\|C(t)g - g\|_X$, which tends to zero for $t \to \infty$ by the hypotheses. Now by the Banach–Steinhaus theorem this holds for every $f \in X_0$, noting $E(t)$ is equibounded. Conversely, let the limit exist for $f \in X$, denoting it by g. By (2.3), $\|AE(t)f\| \leqq 2t^{-2}(\|C(t)\|_{[X]} + 1)\|f\|_X$, which tends to zero for $t \to \infty$. Hence $Ag = \theta$ and $g \in N(A)$, since A is closed. On the other hand, $E(t)f - f \in R(A)$ by Lemma 2.4, so that $\text{s-lim}_{t\to\infty}\{E(t)f - f\} = g - f =: h \in \overline{R(A)}$. Thus $f = g - h \in N(A) \oplus \overline{R(A)} = X_0$, and $Pf = g$. □

LEMMA 3.3. *The operator* B *of Definition* 1.2 *has for any* $n \in \mathbf{N}$ *the properties*

a) *B is linear, closed,* $\overline{D(B)} = X_0$;
b) $N(B) = N(A)$;
c) i) $PB^n f = \theta \qquad (f \in D(B^n))$;
ii) $B^n P f = \theta \qquad (f \in X_0)$;
d) i) $A^n B^n f = f - Pf \qquad (f \in D(B^n))$;
ii) $B^n A^n f = f - Pf \qquad (f \in D(A_0^n))$.

PROOF. a) That B is linear and closed follows readily. Further, $\overline{D(B)} = \overline{N(A) \oplus R(A_0)} \subset N(A) \oplus \overline{R(A)} = X_0$. Conversely, let $f \in X_0$. By Lemma 2.4, $f - E(t)f + Pf \in R(A_0) \oplus N(A)$. So, by Theorem 3.1, $f \in \overline{R(A_0) \oplus N(A)} = \overline{D(B)}$.

b) If $f \in N(B)$ has the representation $f = Ag + Pf$ with $g \in D(A) \cap N(P)$, then $\theta = Bf = g$; thus $f = A\theta + Pf = Pf$, and so $f \in N(A)$. Conversely if $f \in N(A)$, then $f = Pf = A\theta + Pf$ and we have $f \in D(B)$ with $Bf = \theta$.

c) i). Let $f \in D(B)$; then $PBf = PB(Ag + Pf) = Pg = \theta$. Now let i) be valid for all $m \leqq n$, *i.e.* $PB^m f' = \theta$ for all $f' \in D(B^m)$. Let $f \in D(B^{n+1})$, then $PB^{n+1} f = PB(B^n f) = \theta$, the assertion for $n+1$. As to ii), let $f \in X_0$, so $Pf \in N(A)$. Hence $B^n Pf = B^n(A\theta + Pf) = B^{n-1}\big(B(A\theta + Pf)\big) = B^{n-1}\theta = \theta$.

Concerning d), i), again by induction let $f \in D(B)$. So $ABf = AB(Ag + Pf) = Ag = f - Pf$. Now let i) be valid for n, and take $f \in D(B^{n+1}) \subset D(B)$. Then $A^{n+1}B^{n+1}f = AA^nB^nB(Ag + Pf) = AA^nB^n g = A(g - Pg) = Ag = f - Pf$ as $g \in D(B^n)$. The proof of d) ii) also follows by induction. □

REMARK 3.1. The operators A and B are connected. Indeed, let $f \in N(P)\big(= \overline{R(A)}\big)$. Take $n = 1$ in part c) of Lemma 3.3. Then $ABf = f$, all $f \in D(B) \cap N(P) = R(A_0)$, and $BAf = f$, all $f \in D(A_0) \cap N(P) = R(B)$. Thus $B = A^{-1}$ on $R(A_0) \cap D(A_0)$.

LEMMA 3.4. *Let $C \in C_M^\beta$. Then there exist constants M_1, $M_2 \geqq 0$ such that*
a) $\|C(t)\|_{[X]} \leqq M_1\big(|t|^\beta + 1\big) \qquad (t \in \mathbf{R})$;
b) $\|C(t)\|_{[X]} \leqq M_2\big(|t|^\beta\big) \qquad (|t| \geqq 1)$.

PROOF. a) For such a given C there are $\tau > 0$ and $N_1 > 0$ with $\|C(t)\|_{[X]} \leqq N_1|t|^\beta$ for all $|t| \geqq \tau$. On the other hand, in view of (2.5), there are $\omega \geqq 0$ and $N_2 \geqq 1$ such that, for all $|t| \leqq \tau$, $\|C(t)\|_{[X]} \leqq N_2 \exp(\omega\tau) := N_3$. This yields $\|C(t))\|_{[X]} \leqq N_1|t|^\beta + N_3$, $t \in \mathbf{R}$, establishing part a) with $M_1 := \max\{N_1, N_3\}$. b) If $|t| \geqq 1$, $|t|^\beta \geqq 1$, so by a), $\|C(t)\|_{[X]} \leqq M_1\big(|t|^\beta + 1\big) \leqq 2M_1|t|^\beta =: M_2|t|^\beta$. □

LEMMA 3.5. *Let $\mathcal{C} \in \mathcal{C}^2_M$. One has*

a) $\|E(t)f - Pf\|_X \leqq \left(M + \|P\|_{[X_0]}\right) \|f\|_X \qquad (f \in X_0);$

b) $\|E(t)f - Pf\|_X \leqq 2|t|^{\beta-2}(M' + 1)\,\|Bf\|_X \qquad \left(|t| > 1,\ f \in D(B)\right)$

if, in addition, $\mathcal{C} \in \mathcal{C}^\beta_M$, $0 \leqq \beta < 2$;

c) $$\|B\,G(t)f\|_X \leqq \frac{1}{t^2}\int_0^t\int_0^u w^2\,\|G(w)f - f\|_X\,dw\,du \qquad (f \in X_0),$$

where $G(t)f := f - E(t)f + Pf$, for $t \in \mathbf{R}$, $f \in X_0$, is an approximation process on X_0 with range in $D(B)$;

d) *If $\operatorname{s-lim}_{t\to\infty} t^2[E(t)f - Pf]$ exists, then $f \in D(B)$, and the limit equals $-2Bf$.*

PROOF. Part a) is trivial. Concerning b), let $f \in D(B)$, $|t| > 1$. Then $f = Ag + Pf$ with $g \in D(A_0) \cap N(P)$. Hence by Lemma 3.2 d), b), as $Bf = g$,

$$\|E(t)f - Pf\|_X = \|E(t)Ag\|_X = 2t^{-2}\|C(t)g - g\|_X \leqq$$

$$\leqq 2t^{-2}|t|^\beta(M' + 1)\,\|Bf\|_X\,.$$

As to c), by Lemma 2.4, if $f \in X_0$, $t \neq 0$, $G(t)f = Pf - A\int_0^t\int_0^u (1 - u/t)^2$ $C(v)f\,dv\,du$, so that $G(t)f \in N(A) \oplus R(A_0) = D(B)$.

Hence by Lemmas 3.3 and 3.2, noting that P is closed,

$$\|B\,G(t)f\|_X = \left\|B\left[Pf - A\int_0^t\int_0^u \left(1 - \frac{u}{t}\right)^2 C(v)f\,dv\,du\right]\right\|_X =$$

$$= \left\|\int_0^t\int_0^u \left(1 - \frac{u}{t}\right)^2 (C(v)f - PC(v)f)\,dv\,du\right\|_X =$$

$$= \frac{2}{t^2}\left\|\int_0^t \left[\frac{(t-u)^2}{2}\int_0^u (C(v)f - Pf)\,dv\right]du\right\|_X =$$

$$= \frac{2}{t^2}\left\|\int_0^t\int_0^u\int_0^w\int_0^x (C(v)f - Pf)\,dv\,dx\,dw\,du\right\|_X,$$

where the last step followed by a double integration by parts. A further computation gives, noting the definition of $G(t)$,

$$\|BG(t)f\|_X = \frac{2}{t^2}\left\| \int_0^t \int_0^u \frac{w^2}{2}\left[\frac{2}{w^2}\int_0^w \int_0^x C(v)f\,dv\,dx - Pf\right] dw\,du \right\|_X \leqq$$

$$\leqq \frac{1}{t^2}\int_0^t \int_0^u w^2 \left\| \frac{2}{w^2}\int_0^w \int_0^x C(v)f\,dv\,dx - Pf \right\|_X dw\,du =$$

$$= \frac{1}{t^2}\int_0^t \int_0^u w^2 \|G(w)f - f\|_X\,dw\,du.$$

Concerning d), let $f \in X_0$. Then,

$$t^2[E(t)f - Pf] = 2\int_0^t \int_0^u C(v)(f - Pf)dv\,du \in X_0,$$

by Lemma 3.2 d). As the limit in part d) exists,

$$\text{(3.1)} \qquad \underset{t\to\infty}{\text{s-lim}}\, 2\int_0^t \int_0^u C(v)(f - Pf)\,dv\,du := g$$

say, where $g \in X_0$, X_0 being closed. Let us show that $g \in D(A)$ with $Ag = 2Pf - 2f$ and $Pg = \theta$. Set $\varphi := f - Pf$. Then, in view of d'Alembert's functional equation, and partial integration,

$$2h^{-2}[C(h) - I]\int_0^t \int_0^u C(v)\varphi\,dv\,du =$$

$$= h^{-2}\int_0^t \int_0^u [2\,C(h)C(v) - 2\,C(v)]\,\varphi\,dv\,du =$$

$$= h^{-2}\left[\int_h^{t+h} (t - u + h)C(u)\varphi\,du + \int_{-h}^{t-h} (t - u - h)C(u)\varphi\,du - \right.$$

$$-2\int_0^t (t-u)C(u)\varphi\,du\Bigg] =$$

$$= h^{-2}\left[\int_t^{t+h}(t-u+h)C(u)\varphi\,du - \int_{t-h}^{t}(t-u-h)C(u)\varphi\,du\right] +$$

$$+h^{-2}\left[\int_{-h}^{0} - \int_0^h\right](t-u)C(u)\varphi\,du + h^{-2}\left[h\int_h^t - h\int_{-h}^t\right]C(u)\varphi\,du =:$$

$$=: I_1(t) + I_2(t) + I_3(t),$$

say. Integrating both integrals of $I_1(t)$ by parts,

$$h^2 I_1(t) = -h\int_0^t C(v)\varphi\,dv + \int_t^{t+h}\int_0^u C(v)\varphi\,dv\,du +$$

$$+h\int_0^t C(v)\varphi\,dv - \int_{t-h}^{t}\int_0^u C(v)\varphi\,dv\,du.$$

The two remaining integrals vanish for $t \to \infty$ as they are the Cauchy conditions for $\operatorname{s-lim}_{t\to\infty} t^2 E(t)\varphi$, which exists. Thus $I_1(t) \to \theta$ as $t \to \infty$, all $h > 0$.

As to $I_2(t)$, since $C(u) = C(-u)$, and by partial integration, $h^2 I_2(t) = = 2\int_0^h uC(u)\varphi\,du = 2\left[h\int_0^h C(v)\varphi\,dv - \int_0^h\int_0^u C(v)\varphi\,dv\,du\right]$, which is independent of t.

Finally to $I_3(t)$. Here $h^2 I_3(t) = -h\int_{-h}^h C(u)\varphi\,du = -2h\int_0^h C(u)\varphi\,du$, also independent of t.

Returning to (3.1), it therefore follows that $2h^{-2}[C(h) - I]g/2$ equals

$$2h^{-1}\int_0^h C(v)\varphi\,dv - 2h^{-2}\int_0^h\int_0^u C(v)\varphi\,dv\,du - 2h^{-1}\int_0^h C(u)\varphi\,du =$$

$$= -E(h)\varphi \to -C(0)\varphi = -\varphi,$$

for $h \to 0$ by (2.7). Thus $g \in D(A)$ with $Ag = -2\varphi = 2(Pf - f)$. Further, P being closed, and noting $P^2 f = Pf$, $2\int_u^t\int_0^u C(v)P(f - Pf)\,dv du = \theta$. Hence $Pg = \theta$ and $f = Pf - Ag/2$, so that $f \in D(B)$, $Bf = -g/2$. This completes the proof. □

THEOREM 3.2. *Let $\mathcal{C} \in \mathcal{C}_M^2$, having generator A. Further, let B, P and X_0 be defined as in Definition 1.2 and let $\alpha \in (0, 2]$. There hold for $f \in X_0$:*

a) *If, in addition $\mathcal{C} \in \mathcal{C}_M^\beta, \beta \in [0, 2)$, then there exists a constant $c_1 \in \mathbf{R}$ such that*

$$\left\| E(t)f - Pf \right\|_X \leqq c_1 K\left(|t|^{\beta-2}, f; X_0, D(B)\right) \leqq c_1 t^\beta K\left(t^{-2}, f; X_0, D(B)\right)$$
$$(|t| \geqq 1);$$

b) *If $\|E(t)f - Pf\|_X = O\left(|t|^{-\alpha}\right)$, then*

$$K\left(t^{-2}, f; X_0, D(B)\right) == O\left(|t|^{-\alpha}\right) \qquad (|t| \to \infty);$$

c) $\|E(t)f - Pf\|_X = o(t^{-2}) \quad (t \to \infty)$ *iff* $f \in N(B) = N(A)$.

PROOF. a) For $f \in X_0$, arbitrary $g \in D(B)$, one has for $t > 1$,

$$\|E(t)f - Pf\|_X \leqq \|E(t)[f - g] - P[f - g]\|_X + \|E(t)g - Pg\|_X ,$$

and the standard K-functional methods (cf. [2]) then imply by both inequalities of Lemma 3.5 a) b), the assertion of a) with $c_1 := \max\{M' + \|P\|_{[X_0]}, 2M' + 2\}$.

b) Let $f \in X_0$, $E(t)f$ having the given rate. By Lemma 3.5 c),

$$K\left(t^{-2}, f; X_0, D(B)\right) \leqq \left\| f - G(t)f \right\|_X + t^{-2} \left\| BG(t)f \right\|_X \leqq$$

$$\leqq \left\| E(t)f - Pf \right\|_X + t^{-4} \int_0^t \int_0^u v^2 \left\| G(v)f - f \right\|_X dv\, du = I_1(t) + I_2(t),$$

say. Now $I_1(t) = O(t^{-\alpha})$, for $t \to \infty$. As to $I_2(t)$, one has by definition of $G(t)$ and partial integration, for $t \geqq \tau \geqq 1$, and τ chosen so that

$$\left\| E(t)f - Pf \right\|_X \leqq M \cdot t^{-\alpha} \qquad (t \geqq \tau), \tag{3.2}$$

$$I_2(t) = t^{-4} \int_0^t \int_0^u v^2 \left\|E(v)f - Pf\right\|_X dv\, du =$$

$$= t^{-4} \left(\int_0^\tau + \int_\tau^t \right) (t - u)u^2 \left\| E(u)f - Pf \right\|_X du = I_2^1(t) + I_2^2(t),$$

say. By Lemma 3.5 a), noting that $t^{-3} \leqq t^{-\alpha}$ and $t^{-4} \leqq t^{-\alpha}$ for $t \geqq 1$,

$$I_2^1(t) \leqq t^{-4} \int_0^{\tau} (t-u)u^2 \left[M + \|P\|_{[X_0]}\right] \|f\|_X \, du \leqq$$

$$\leqq t^{-4} \left(M + \|P\|_{[X_0]}\right) \|f\|_X \cdot \left(t\frac{\tau^3}{3} - \frac{\tau^4}{4}\right) \leqq$$

$$\leqq t^{-\alpha} \left(M + \|P\|_{[X_0]}\right) \cdot \left(\frac{\tau^3}{3} - \frac{\tau^4}{4}\right) \|f\|_X = O(t^{-\alpha}).$$

Part b) now follows from (3.2), noting

$$I_2^2(t) \leqq t^{-4} M \int_0^{t} (t-u)u^{2-\alpha} \, du = O(t^{-\alpha}).$$

Now to part c). The inverse part follows for $f \in N(B)$ by Lemma 3.5 b). The direct part follows by Lemma 3.5 d), with $f \in D(B)$ and $-2Bf = \theta$, i.e., $f \in N(B)$. □

COROLLARY 3.1. *In particular, if* $\beta = 0$, *i.e.,* $\mathcal{C}$ *is equibounded, then*

$$\|E(t)f - Pf\|_X = O\left(|t|^{-\alpha}\right) \Leftrightarrow K\left(t^{-2}, f; X_0, D(B)\right) = O\left(|t|^{-\alpha}\right)$$
$$\left(|t| \to \infty\right),$$

i.e., assertions (i) *and* (iii) *of Theorem* 1.2 b) *are equivalent.*

REMARK. Note that the limit of Lemma 3.5 is connected with the Voronovskaja-type condition for the process $\{E(t); t \in \mathbf{R}\}$. Thus by Lemma 3.3 d)i) and (2.3),

$$t^2(E(t)f - Pf) + 2Bf = 2\int_0^t \int_0^u C(v)ABf \, dv \, du + 2Bf = 2C(t)Bf$$
$$(f \in D(B);\ t \in \mathbf{R}).$$

Thus such a condition would hold iff $C(t)g \to \theta$ for $t \to \infty$ for all $g \in R(B) = D(A_0) \cap N(P)$, a fact which is not true for the cosine operator of translations. So this is an example of a process that saturates but does not satisfy a true Voronovskaja-type condition.

Now we wish to discuss the sharpness of the approximation processes above, for which we need an additional theorem. The analogue for semigroups $\mathcal{T}$ is due to M. Lin (cf. [19],[20], see also [27]).

THEOREM 3.3. *Let $\mathcal{C}$ be a cosine operator function, having generator A, satisfying $\lim_{|t|\to\infty} t^{-2}\|C(t)\|_{[X]} = 0$. Then the following five assertions are equivalent:*

a) *There exists a bounded linear operator $P : X_0 \to X_0$ such that $\lim_{|t|\to\infty} \| E(t) - P \|_{[X_0]} = 0$, i.e., $\mathcal{C}$ is uniformly Cesàro-ergodic;*

b) *There exists a projection P on $N(A)$ such that $\lim_{\lambda\to 0+} \| \lambda^2 R(\lambda^2, A) - P \|_{[X_0]} = 0$, i.e., $\mathcal{C}$ is uniformly Abel-ergodic;*

c) *$\overline{R(A)} = R(A)$, i.e., $R(A)$ is closed;*

d) *$X = R(A) \oplus N(A)$, hence $X_0 = X$;*

e) *The operator B is defined on X and bounded.*

PROOF. By Lemma 2.2, A generates a (C_0)-semigroup $\mathcal{T} = \{T(t); t \geqq 0\}$ with $\lim_{t\to\infty} t^{-1}\|T(t)\|_{[X]} = 0$. Now we can use the results of M. Lin for $\mathcal{T}$, which deliver the equivalence of b), c) and d). That e) implies d) is clear by definition of B. The other direction is a conseqence of the closed graph theorem, noting B is closed. We now prove the equivalence of a) and e). Let $\mathcal{C}$ be uniformly Cesàro ergodic. Then there exists a $t < \infty$ such that $\|E(t) - P\|_{[X_0]} < 1$. Hence $\left(E(t) - P - I\right)^{-1} = -\sum_{k=0}^{\infty} \left(E(t) - P\right)^k$ exists in $[X_0]$. Now for any $f = Ag + Pf \in D(B), Bf = g$ with $Pg = \theta$, we have by Lemma 2.4,

$$\left(E(t) - P - I\right) g = E(t)g - g = 2t^{-2} \int_0^t \int_0^u \int_0^v \int_0^w C(x)Ag\,dx\,dw\,dv\,du =$$

$$= 2t^{-2} \int_0^t \int_0^u \int_0^v \int_0^w C(x)(f - Pf)dx\,dw\,dv\,du.$$

Thus

$$\|Bf\|_X = \|g\|_X \leqq$$

$$\leqq \left\| \left(E(t) - P - I\right)^{-1} \right\|_{[X_0]} \cdot \frac{2}{t^2} \int_0^t \int_0^u \int_0^v \int_0^w C(x)\,\|f - Pf\|_X\,dx\,dw\,dv\,du \leqq$$

$$\leqq \left\| \left(E(t) - P - I\right)^{-1} \right\|_{[X_0]} \cdot \frac{t^2}{12} \left(1 + \|P\|_{[X_0]}\right) \|f\|_X ,$$

i.e., B is bounded.

Conversely, if B is bounded, then by the closedness we have $D(B) = \overline{D(B)} = X_0$. So for arbitrary $f = Ag + Pf \in D(B) = X_0$, with $\|f\|_X = 1$, by Lemma 3.2 e),

$$\|E(t)f - Pf\|_X = \|E(t)(f - Pf)\|_X \|E(t)Ag\|_X = \frac{2}{t^2}\|C(t)g - g\|_X \leqq$$

$$\leqq \frac{2}{t^2}\left(\|C(t)\|_{[X_0]} + 1\right)\|B\|_{[X_0]},$$

which tends to zero as $t \to \infty$. So we have uniform convergence in X_0. □

Now we can prove the sharpness of the processes above.

THEOREM 3.4. *Let $\mathcal{C} \in \mathcal{C}_M^\beta$ be non-trivial, B as in Definition* 1.2. *Then*

a) *There exists an element $f_2 \in \widetilde{D(B)}^{X_0}$ such that*

$$\|E(t)f_2 - Pf_2\|_X \begin{cases} = O(t^{-2}) \\ \neq o(t^{-2}) \end{cases} \qquad (t \to \infty).$$

b) *If $\beta = 0$, B is unbounded and $\alpha \in (0,2)$, then there exists $f_\alpha \in X_0$ satisfying*

$$\|E(t)f_\alpha - Pf_\alpha\|_X \begin{cases} = O(t^{-\alpha}) \\ \neq o(t^{-\alpha}) \end{cases} \qquad (t \to \infty).$$

PROOF. a) Let us assume that in particular for every $f \in D(B)$, $\|E(t)f - Pf\|_X = o(t^{-2})$ for $t \to \infty$. Thus by Theorem 3.2 c) we have $f \in N(B) = N(A)$, so that $D(B) = N(B)$. Hence $B = \Theta$, the null operator, which is bounded. By Theorem 3.3 we have $X = D(B) = N(A)$ and so $A = \Theta$. This yields $C(t) = I$, for all $t \in \mathbf{R}$. This is a contradiction, $\mathcal{C}$ being non-trivial.

b) This case will be proved with Theorem 2.1. Since B is unbounded we have, in view of Theorem 3.3, $\limsup_{|t|\to\infty} \|E(t) - P\|_{[X_0]} > 0$. Now we define for $f \in X_0, n \in \mathbf{N}$ $S_n f := \sup_{n-1<t\leqq n} \|E(t)f - Pf\|_{[X_0]}|t|^\alpha \in {X_0}^+$, Hence, $\limsup_{n\to\infty} \|S_n\|_{{X_0}^+} = \infty$. Further, if $f \in D(B)$, we have, by Lemma 3.5 b), $S_n f \leqq 2(M' + 1)\|Bf\|_X \sup_{n-1<t\leqq n} |t|^{\alpha-2}$, which tends to zero, as $n \to \infty$. So $D(B) \subset Z := \left\{f \in X_0; \lim_{n\to\infty} S_n f = 0\right\}$, i.e., Z is dense in X_0, as $\overline{D(B)} = X_0$. Now we obtain, by Theorem 2.1: There exists $f_\alpha \in X_0$ with $\sup_{n\in\mathbf{N}} S_n f_\alpha \leqq 1$ and $\limsup_{n\to\infty} S_n f_\alpha = 1$. Thus we have for arbitrary $t \in \mathbf{R}$,

$\| E(t)f_\alpha - Pf_\alpha \|_X |t|^\alpha \leqq \sup_{n\in\mathbf{N}} S_n f_\alpha \leqq 1$. It remains to show that $c :=$ $:= \limsup_{t\to\infty} \| E(t)f_\alpha - Pf_\alpha \|_X |t|^\alpha = 1$. Therefore, let us assume that $c < 1$. Then for any $\varepsilon > 0$ there exists a $\tau = \tau(\varepsilon)$ such that $\| E(t)f_\alpha - Pf_\alpha \|_X |t|^\alpha \leqq$ $\leqq c + \varepsilon$ all $t \geqq \tau$. The particular choice $\varepsilon_0 = \frac{1-c}{2}$ delivers for any $n \in \mathbf{N}$ with $n > \tau(\varepsilon_0) + 1$, $S_n f_\alpha \leqq c + \varepsilon_0 = \frac{c+1}{2} < 1$. This is a contradiction to $\limsup_{n\to\infty} \| E(t)f_\alpha - Pf_\alpha \|_X |t|^\alpha = 1$. Hence, $\| E(t)f_\alpha - Pf_\alpha \|_X \neq o\big(|t|^{-\alpha}\big)$. □

4. Ergodic theorems for the resolvent operator

LEMMA 4.1. *Let $\mathcal{C} \in \mathcal{C}_M$ with generator A, $\lambda^2 \in \rho(A)$, and P, X_0 be given as in Definition* 1.2. *Then*

$$\lambda^2 R(\lambda^2, A)f - Pf = \lambda^2 R(\lambda^2, A)[f - Pf] \qquad (f \in X_0). \tag{4.1}$$

PROOF. Since $\lambda \int_0^\infty e^{-\lambda t} dt = 1$, (2.6) and Lemma 3.2 d) yield

$$\lambda^2 R(\lambda^2, A)f - Pf = \lambda \int_0^\infty e^{-\lambda t}\big[C(t)f - Pf\big]\, dt =$$

$$= \lambda \int_0^\infty e^{-\lambda t} C(t)[f - Pf]\, dt. \quad \square$$

LEMMA 4.2. *Let $\mathcal{C}$ be a cosine operator function with generator A and $(0,\infty) \subset \rho(A)$. If $E(t)$ is equibounded by M, so is $\lambda^2 R(\lambda^2, A)$ for $\lambda > 0$ with the same bound M. In particular, $\mathcal{C}_M^2 \subset \mathcal{C}_M$.*

PROOF. Let $\|E(t)\|_{[X]} \leqq M$. By (2.6) and double partial integration we obtain

$$\lambda^2 R(\lambda^2, A)f = \lambda \int_0^\infty e^{-\lambda t} C(t)f\, dt = \lambda \operatorname*{s-lim}_{R\to\infty} \int_0^R e^{-\lambda t} C(t) f\, dt =$$

$$= \operatorname*{s-lim}_{R\to\infty} \left\{ \left[\lambda e^{-\lambda t} \int_0^t C(v) f\, dv + \frac{\lambda^2 e^{-\lambda t} t^2}{2} E(t) f \right]_{t=0}^{t=R} \right\} +$$

$$+\lambda^3 \int_0^\infty e^{-\lambda t} \frac{t^2}{2} E(t) f \, dt.$$

The first term vanishes in view of Lemma 3.4 and the equiboundedness of $E(t)$, since $\lim_{R\to\infty} e^{-\lambda R} R^\gamma = 0$ for $\lambda, \gamma > 0$. Hence $\|\lambda^2 R(\lambda^2, A) f\|_X \leqq \leqq \frac{M\lambda^3}{2} \|f\|_X \int_0^\infty e^{-\lambda t} t^2 dt = \frac{M}{2}\Gamma(3) \|f\|_X = M \|f\|_X$. That $\mathcal{C}_M^2 \subset \mathcal{C}_M$ now follows by Lemma 2.3. □

THEOREM 4.1. *Let $\mathcal{C} \in \mathcal{C}_M$, A being the generator of $\mathcal{C}$ with resolvent $R(\lambda, A)$, $\lambda > 0$. Further, let B, P, X_0 be given as in Definition* 1.2. *Then*

a) $\operatorname*{s-lim}_{\lambda\to 0+} \lambda^2 R(\lambda^2, A) f = g$ *exists iff $f \in X_0$; if so, the limit equals Pf.*

b) $\lim_{\lambda\to 0+} \|R(\lambda^2, A) f - \lambda^{-2} Pf + Bf\|_X = 0 \qquad (f \in D(B))$.

PROOF. a) The first implication follows by an argument similar to that applied in Theorem 3.1 a), using the Banach–Steinhaus theorem.

As to the other direction, by (2.11), $\lambda^2 R(\lambda^2, A) f \in D(A)$, with

$$\| A\lambda^2 R(\lambda^2, A) f\|_X = \|\lambda^4 R(\lambda^2, A) f - \lambda^2 f\|_X \leqq (M+1)\lambda^2 \|f\|_X \quad (f \in X),$$

which tends to zero for $\lambda \to 0^+$. Hence $Ag = \theta$ and $g \in N(A)$. On the other hand, by (2.11); $\lambda^2 R(\lambda^2, A) f - f \in R(A)$, so that $\operatorname*{s-lim}_{\lambda\to 0+} [\lambda^2 R(\lambda^2, A) f - f] = = g - f =: h \in \overline{R(A)}$. Thus $f \in X_0$ and $Pf = g$.

b) Let $f = Ag + Pf \in D(B)$ with $g \in D(A_0) \cap N(P)$. Then, in view of (4.1), (2.9), noting $Bf = g$ and $Pg = \theta$, $R(\lambda^2, A) f - \lambda^{-2} Pf + Bf = = R(\lambda^2, A)[f - Pf] + Bf = R(\lambda^2, A) Ag + g = \lambda^2 R(\lambda^2, A) g - Pg$, which tends to zero for $\lambda \to 0^+$ by part a). □

Now to the counterpart of Lemma 3.5 for the resolvent operator $\lambda^2 R(\lambda^2, A)$.

LEMMA 4.3. *Under the assumptions of Theorem* 4.1 *one has for $\lambda > 0$:*

a) $\|\lambda^2 R(\lambda^2, A) f - Pf\|_X \leqq (M + \|P\|_{[X_0]}) \cdot \|f\|_X \quad (f \in X_0)$;

b) $\|\lambda^2 R(\lambda^2, A) f - Pf\|_X \leqq \lambda^2 (M+1) \cdot \|Bf\|_X \quad (f \in D(B))$;

c) $\|BG(\lambda) f\|_X \leqq \lambda^{-2} \|G(\lambda) f - f\|_X \quad (f \in X_0)$,

where $G(\lambda) f := f - \lambda^2 R(\lambda^2, A) f + Pf$, $f \in X_0$, is an approximation process on X_0 with range in $D(B)$;

d) *If* $\operatorname*{s-lim}_{\lambda\to 0+} (R(\lambda^2, A) f - \lambda^{-2} Pf)$ *exists, then $f \in D(B)$, and the limit equals $-Bf$.*

PROOF. Part a) is trivial. Concerning b), if $f \in D(B)$, then $f - Pf = Ag$ with $g \in D(A_0) \cap N(A)$. So by (4.1), (2.9),

$$\|\lambda^2 R(\lambda^2, A) f - Pf\|_X = \|\lambda^2 R(\lambda^2, A) Ag\|_X = \|\lambda^2 [\lambda^2 R(\lambda^2, A) g - g]\|_X \leqq$$

$$\leqq \lambda^2(M+1)\,\|Bf\|_X\,.$$

As to c), (2.11) gives $G(\lambda)f = -AR(\lambda^2, A)f + Pf \in R(A_0) \oplus N(A)$, and in view of Lemma 3.3 c)ii) and d)ii), (2.11) again,

$$\|BG(\lambda)f\|_X = \|BAR(\lambda^2, A)f - BPf\|_X =$$

$$= \|R(\lambda^2, A)f - PR(\lambda^2, A)f\|_X \leqq$$

$$\leqq \lambda^{-2}\|\lambda^2 R(\lambda^2, A)f - Pf\|_X + \lambda^{-2}\,\|P[f - \lambda^2 R(\lambda^2, A)f]\,\|_X =$$

$$= \lambda^{-2}\|\lambda^2 R(\lambda^2, A)f - Pf\|_X,$$

which reduces to assertion c). Concerning d), assume that the limit equals g, and show that $g \in D(A)$ with $Ag = -f + Pf$ and $Pg = \theta$. Setting $\varphi = f - Pf$, then, by (4.1),

$$A_h\left[R(\lambda^2, A)f - \lambda^{-2}Pf\right] =$$

$$= \lambda^{-1}h^{-2}\int_0^\infty e^{-\lambda t}\big(C(t+h) - 2C(t) + C(t-h)\big)\varphi\,dt =$$

$$= \frac{e^{\lambda h}}{\lambda h^2}\int_h^\infty e^{-\lambda u}C(u)\varphi\,du - \frac{2}{\lambda h^2}\int_0^\infty e^{-\lambda u}C(u)\varphi\,du + \frac{e^{-\lambda h}}{\lambda h^2}\int_{-h}^\infty e^{-\lambda u}C(u)\varphi\,du =$$

$$= \frac{e^{\lambda h} - 2 + e^{-\lambda h}}{h^2}\left\{\frac{1}{\lambda}\int_0^\infty e^{-\lambda u}C(u)\varphi\,du\right\} +$$

$$+\frac{e^{-\lambda h}}{\lambda h^2}\int_{-h}^0 e^{-\lambda u}C(u)\varphi\,du - \frac{e^{\lambda h}}{\lambda h^2}\int_0^h e^{-\lambda u}C(u)\varphi\,du,$$

where $A_h = 2h^{-2}\left[C(h) - I\right]$. Now the term with the curly brackets tends strongly to $0 \cdot g = \theta$ for $\lambda \to 0^+$ by assumption. The last two terms can be combined as, which in turn

$$\frac{1}{h^2}\int_0^h \frac{e^{-\lambda(h-u)} - e^{\lambda(h-u)}}{\lambda}C(u)\varphi\,du \to \frac{1}{h^2}\int_0^h \left[\,-2(h-u)\right]C(u)\varphi\,du$$

for $\lambda \to 0^+$ by L'Hospital's rule. By (2.3) the latter term equals $-E(h)\varphi$, which tends to $-\varphi$ for $h \to 0$ by (2.7). Thus

$$A \operatorname*{s-lim}_{\lambda\to 0} \left[R(\lambda^2, A)f - \lambda^{-2}Pf \right] = -\varphi,$$

so that $Ag = -\varphi = Pf - f$ with $g \in D(A_0)$. Since P is closed, (4.1) now yields

$$Pg = \operatorname*{s-lim}_{\lambda\to 0^+} P\left[R(\lambda^2, A)f - \lambda^{-2}f \right] = \operatorname*{s-lim}_{\lambda\to 0^+} \frac{1}{\lambda^2} PAR(\lambda^2, A)f = \theta.$$

The proof now follows as in Lemma 3.5 d). □

Note that part d) is the converse of Theorem 4.1 b), so that we now have

COROLLARY 4.1. *For $f \in X_0$ one has*

$$\operatorname*{s-lim}_{\lambda\to 0^+} \left(\lambda^{-2}Pf - R(\lambda^2; A)f \right) \text{ exists iff } f \in D(B).$$

In this event, the limit equals Bf.

REMARK 4.1. Now, for $f \in D(B)$, the limit above may be interpreted as the strong right hand derivative of the operator $\lambda R(\lambda^2, A)[P - I]f$ at $\lambda = 0$, namely by (4.1),

$$\operatorname*{s-lim}_{\lambda\to 0^+} \left(\lambda^{-2}Pf - R(\lambda^2; A)f \right) =$$

$$= \operatorname*{s-lim}_{\lambda\to 0^+} \frac{\lambda R(\lambda^2; A)[P - I]f}{\lambda} = \frac{d}{d\lambda} \lambda R(\lambda^2, A)[P - I]f \Big|_{\lambda=0},$$

since $\lambda R(\lambda^2, A)[P - I]f \to \theta$ for $\lambda \to 0^+$ in view of Lemma 4.3 b). Hence,

$$Bf = \frac{d}{d\lambda} \lambda R(\lambda^2, A)(Pf - f) \Big|_{\lambda=0} \qquad \left(f \in D(B) \right). \tag{4.2}$$

If, on the other hand, $\operatorname*{s-lim}_{\lambda\to 0^+} \lambda R(\lambda^2, A)(Pf - f)$ exists, then by the same argumentation as in the proof of Lemma 4.3 d) we obtain that this limit belongs to $N(A)$. Thus, if $N(A) = \{\theta\}$, it tends to zero. This indicates the following

COROLLARY 4.2. *If $N(A) = \{\theta\}$, then*

$$\frac{d}{d\lambda} \lambda R(\lambda^2, A)f \Big|_{\lambda=0} = \operatorname*{s-lim}_{\lambda\to 0^+} \int_0^\infty -te^{-\lambda t} C(t)f \, dt \text{ exists iff } f \in D(B). \tag{4.3}$$

In this event the derivate equals $-Bf$, *and, if the integral below belongs to* X_0,

$$Bf = \int_0^\infty tC(t)f\,dt. \tag{4.4}$$

PROOF. Since $N(A) = \{\theta\}$, $Pf = \theta$. Thus, if $\frac{d}{d\lambda}\lambda R(\lambda^2, A)f\big|_{\lambda=0}$ exists, so does also $\operatorname{s-lim}_{\lambda\to 0+} \lambda R(\lambda^2, A)f$, and this limit equals θ, by Remark 4.1. Hence, $\frac{d}{d\lambda}\lambda R(\lambda^2, A)f\big|_{\lambda=0} = \operatorname{s-lim}_{\lambda\to 0+} R(\lambda^2, A)f$, and the existence of this limit implies, by Corollary 4.1, $f \in D(B)$. Now by the holomorphy of the Laplace transform, we obtain by (2.6),

$$\frac{d}{d\lambda}\lambda R(\lambda^2, A)f\bigg|_{\lambda=0} = \operatorname{s-lim}_{\lambda\to 0+} \frac{d}{d\lambda}\left[\lambda R(\lambda^2, A)f\right] =$$

$$= \operatorname{s-lim}_{\lambda\to 0+} \frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}C(t)f\,dt = \operatorname{s-lim}_{\lambda\to 0+}\int_0^\infty -te^{-\lambda t}C(t)f\,dt =$$

$$= -\int_0^\infty tC(t)f\,dt. \qquad \square$$

Lemma 4.3 yields the following counterpart of Theorem 3.2, the proof being fully analogous to the latter.

THEOREM 4.2. *Let* $C \in \mathcal{C}_M$, *the generator of* C *being* A, *and its resolvent* $R(\lambda, A)$ *for* $\lambda > 0$. *Further, let* B, P *and* X_0 *be defined as in Definition* 1.2. *Let* $\alpha \in (0, 2]$. *There hold for any* $f \in X_0$:

a) $\left\|\lambda^2 R(\lambda^2, A)f - Pf\right\|_X = O(\lambda^\alpha) \Leftrightarrow K\left(\lambda^2, f; X_0, D(B)\right) = O(\lambda^\alpha)$ $(\lambda \to 0^+)$;

b) $\left\|\lambda^2 R(\lambda^2, A)f - Pf\right\|_X = o(\lambda^2)$ $\quad(\lambda \to 0^+) \Leftrightarrow f \in N(A)$.

Comparing the assertions of Theorems 3.2 c) and 4.2 b) one obtains Theorem 1.2 a).

THEOREM 4.3. *Under the assumptions of Theorem* 1.2 *the following assertions are equivalent for any* $f \in X_0$:

i) $\|E(t)f - Pf\|_X = o(t^{-2})$ $\quad(|t| \to \infty)$;

ii) $\left\|\lambda^2 R(\lambda^2, A)f - Pf\right\|_X = o(\lambda^2)$ $\quad(\lambda \to 0^+)$;

iii) $Pf = f$;

iv) $f \in N(A)$.

Now we can prove the sharpness of the error bounds of the process $\lambda^2 R(\lambda^2, A)f$ for $\lambda \to 0^+$.

THEOREM 4.4. *Let $C \in \mathcal{C}_M$ be non-trivial, B as in Definition 1.2. Then,*

a) *There exists an element $f_2 \in \widetilde{D(B)}^{X_0}$ such that*

$$\left\|\lambda^2 R(\lambda^2, A)f_2 - Pf_2\right\|_X \begin{cases} = O(\lambda^2) \\ \neq o(\lambda^2) \end{cases} \qquad (\lambda \to 0^+).$$

b) *If B is unbounded and $\alpha \in (0,2)$, then there exists $f_\alpha \in X_0$ satisfying*

$$\left\|\lambda^2 R(\lambda^2, A)f_\alpha - Pf_\alpha\right\|_X \begin{cases} = O(\lambda^\alpha) \\ \neq o(\lambda^\alpha) \end{cases} \qquad (\lambda \to 0^+).$$

PROOF. Part a) is proved similarly to that of Theorem 3.4 a). Part b) will again be established with Theorem 2.1. Since B is unbounded, in view of Theorem 3.3, $\limsup_{t\to\infty} \left\|\lambda^2 R(\lambda^2, A) - P\right\|_{[X_0]} > 0$. Now for any $f \in X_0, n \in \mathbf{N}$ we define

$$S_n f := \sup_{\frac{1}{n+1} < \lambda \leqq \frac{1}{n}} \left\|\lambda^2 R(\lambda^2, A)f - Pf\right\|_{[X_0]} \lambda^{-\alpha} \in {X_0}^+.$$

Hence, $\limsup_{n\to\infty} \|S_n\|_{{X_0}^+} = \infty$. The result now follows by an argumentation similar to that in the proof of Theorem 3.4 b). □

Thus assertion d) ii) of Theorem 1.2 is valid provided just $C \in \mathcal{C}_M$.

Observe that the condition "B unbounded" is no restriction concerning sharpness of optimal rates.

5. Applications

5.1. The wave equation in $X_{2\pi}$. Let $X_{2\pi}$ be one of the Banach spaces $C_{2\pi} := \{f : \mathbf{R} \to \mathbf{C}; f \text{ continuous, } 2\pi\text{-periodic}\}$ with norm $\|f\|_{C_{2\pi}} := \max_{x\in[0,2\pi]} \left|f(x)\right|$ or $L^p_{2\pi} := \left\{f : \mathbf{R} \to \mathbf{C}; f \text{ is } 2\pi\text{-periodic}, \|f\|_{L^p_{2\pi}} < \infty\right\}$, $1 \leqq p < \infty$, with norm $\|f\|_{L^p_{2\pi}} := \left\{(1/2\pi) \int_{-\pi}^{\pi} \left|f(x)\right|^p dx\right\}^{1/p}$; and let $C(t)$ be defined by

$$(5.1) \qquad \big(C(t)f\big)(x) := \frac{1}{2}\left[f(x+t) + f(x-t)\right] \qquad (f \in X_{2\pi};\ x,t \in \mathbf{R}).$$

Then it is known that (cf. [30]) $\mathcal{C} = \{C(t); t \in \mathbf{R}\} \in \mathcal{C}_1^0$ with $\|C(t)\|_{[X_{2\pi}]} = 1$, and generator $A = d^2/dx^2$ having domain

$$D(A) = X_{2\pi}^2 := \begin{cases} \{f \in C_{2\pi}; f'' \in C_{2\pi}\} & (X_{2\pi} = C_{2\pi}) \\ \{f \in L_{2\pi}^p; f, f' \in AC_{2\pi}, f'' \in L_{2\pi}^p\} & (X_{2\pi} = L_{2\pi}^p), \end{cases}$$

where $AC_{2\pi} := \{f : \mathbf{R} \to \mathbf{C}; f$ absolutely continuous, 2π periodic$\}$. Further, the kernel $N(A) = \{f \in X_{2\pi}; f$ constant$\}$, the point spectrum $P_\sigma(A) = \{-k^2; k \in \mathbf{N}\}$, the eigenfunctions associated with $-k^2 \in P_\sigma(A)$ are given by $c_1 \sin(kx) + c_2 \cos(kx)$, $k \in \mathbf{N}, c_1, c_2 \in \mathbf{C}, |c_1| + |c_2| \neq 0$; they build a fundamental set in $X_{2\pi}$.

Observe that $C(t)$ given by (5.1) is the unique solution of the wave equation

$$\frac{\partial^2}{\partial t^2} w(t,x) = \frac{\partial^2}{\partial x^2} w(t,x), \tag{5.2}$$

with initial conditions $w(0,x) = f(x) \in X_{2\pi}^2$, $(\partial/\partial t) w(t,x)|_{t=0} = 0$.

By Fejér's theorem,

$$f(x) = \operatorname*{s-lim}_{n\to\infty} \sum_{\substack{k=-n \\ k\neq 0}}^{n} \left(1 - \frac{|k|}{n+1}\right) \widehat{f}(k) e^{ikx} + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\,du \qquad (f \in X_{2\pi}),$$

where $\widehat{f}(k) := (1/2\pi) \int_{-\pi}^{\pi} f(u) e^{-iku} du$. Since the integral belongs to $N(A)$, the sum to $R(A)$, $X_{2\pi} = \overline{R(A)} \oplus N(A)$, valid here also for the nonreflexive $C_{2\pi}$, $L_{2\pi}^1$. Thus $X_{2\pi} = X_0$, and $Pf = (1/2\pi) \int_{-\pi}^{\pi} f(u)\,du$ for $f \in X_{2\pi}$. It is easy to check that the resolvent $R(\lambda^2, A)$ is given by, noting $(\lambda^2 - A) R(\lambda^2, A) f = f$ and $R(\lambda^2, A)(\lambda^2 - A) f = f$,

$$\lambda R(\lambda^2, A) f(x) = \sum_{k \in \mathbf{Z}} \frac{\lambda}{\lambda^2 + k^2} \left(1 - \frac{|k|}{n+1}\right) \widehat{f}(k) e^{ikx}, \tag{5.3}$$

or

$$\lambda R(\lambda^2, A) f(x) = \tag{5.4}$$

$$= \frac{1}{2} \int_x^{\infty} e^{-\lambda(u-x)} f(u)\,du + \frac{1}{2} \int_{-\infty}^{x} e^{-\lambda(x-u)} f(u)\,du = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\lambda|u-x|} f(u)\,du,$$

the singular integral of Picard.

The operator B of Definition 1.2 may be represented via

$$(5.5)\qquad Bf(x) = \operatorname*{s-lim}_{n\to\infty} \sum_{\substack{k=-n\\k\neq 0}}^{n} \left(\frac{1}{(n+1)|k|} - \frac{1}{k^2}\right) \widehat{f}(k)e^{ikx}.$$

The right hand side exists for every $f \in X_{2\pi}$, and we obtain $\|Bf\|_{X_{2\pi}} \leqq \|f\|_{X_{2\pi}}$, i.e., B is bounded. Thus, by Theorem 3.3 we have even $X_{2\pi} = R(A) \oplus N(A)$, i.e., every $f \in X_{2\pi}$ has a representation of the form $f(x) = g''(x) + c$, where $g \in X^2_{2\pi}$ and $c = \widehat{f}(0) = (1/2\pi)\int_{-\pi}^{\pi} f(u)\,du$. So this cosine operator function is uniformly Cesàro and Abel ergodic, i.e., the limits in the following theorems also hold in the uniform operator topology.

As applications of Theorems 1.1 and 1.2 we have

THEOREM 5.1. *For every $f \in X_{2\pi}$ one has*

$$\operatorname*{s-lim}_{|t|\to\infty} \frac{1}{t^2}\int_0^t\int_0^u [f(x+v)+f(x-v)]\,dv\,du =$$

$$= \operatorname*{s-lim}_{\lambda\to 0+} \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|u-x|}f(u)\,du = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du.$$

The ergodic theorems with rates for this example read

THEOREM 5.2. *For every $f \in X_{2\pi}$ one has throughout,*

a) *The following four assertions are equivalent:*

$$\text{(i) } N_t(f) := \left\| \frac{1}{t^2}\int_0^t\int_0^u [f(x+v)+f(x-v)]\,dv\,du - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du \right\|_{X_{2\pi}} = o(t^{-2}) \quad (|t|\to\infty),$$

$$\text{(ii) } N_\lambda(If) := \left\| \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|u-x|}f(u)\,du - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du \right\|_{X_{2\pi}} = o(\lambda^2)$$

$(\lambda \to 0^+)$,

(iii) $f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du$ *for (almost) all* $x \in [-\pi, \pi]$,

(iv) $f(x) = c$ *for (almost) all* $x \in [-\pi, \pi]$.

b) *The following assertions are equivalent*
(i) $N_t(f) = O(t^{-2}) \qquad (|t| \to \infty)$,
(ii) $N_\lambda(If) = O(\lambda^2) \qquad (\lambda \to 0^+)$.

Theorem 5.2 tells us that we have saturation for the cosine operator function (5.1) for every $f \in X_{2\pi}$, so that the saturation class is the whole space $X_{2\pi}$, not just a subspace of $X_{2\pi}$.

In order to give an explicit representation of the operator B, let us first consider the operator B^* defined on $X_{2\pi}$by

$$B^* f(x) := \int_0^x \int_0^u f(v)\,dv\,du - \frac{x^2}{4\pi} \int_{-\pi}^{\pi} f(u)\,du - \frac{x}{2\pi} \int_{-\pi}^{\pi} \int_0^u f(v)\,dv\,du.$$

It can readily be shown that $B^* f(x)$ is 2π-periodic, and that

$$\frac{d^2}{dx^2} B^* f(x) = f(x) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\,du \qquad (f \in X_{2\pi}).$$

Further, $B^* f''(x) = f(x) - f(0)$, all $f \in X_{2\pi}^2 = D(A)$. Thus also $B^* f \in X_{2\pi}^2$. Let us now see that the operator B, defined as follows, is the right form:

$$\text{(5.6)} \quad Bf(x) := B^*\big[f(x) - Pf(x)\big] - PB^*\big[f(x) - Pf(x)\big] \qquad (f \in X_{2\pi}).$$

Indeed, so that $f(x) = Ag^*(x) + Pf(x)$ holds for $g^* \in X_{2\pi}^2$ but not necessarily $\int_{-\pi}^{\pi} g^*(u)\,du = 0$, one has $f(x) - Pf(x) = g^{*\prime\prime}(x)$, yielding

$$Bf(x) = B^* g^{*\prime\prime}(x) - PB^* g^{*\prime\prime}(x) =$$
$$= g^*(x) - g^*(0) - P\big[g^*(x) - g^*(0)\big] = g^*(x) - Pg^*(x),$$

since $Pg^*(0) = g^*(0)$ as $g^*(0) \in N(A)$. Further, $Pg^* = P(Bf)(x) = Pg^*(x) - P^2 g^*(x) = \theta$. Thus the correct g is $g(x) := g^*(x) - Pg^*(x)$. The operator B now takes on the concrete form

$$\text{(5.7)} \quad Bf(x) = \int_0^x \int_0^u f(v)\,dv\,du - \frac{x^2}{4\pi} \int_{-\pi}^{\pi} f(v)\,dv - \frac{x}{2\pi} \int_{-\pi}^{\pi} \int_0^u f(v)\,dv\,du -$$
$$-\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^x \int_0^u f(v)\,dv\,du\,dx + \frac{\pi}{12} \int_{-\pi}^{\pi} f(v)\,dv.$$

The operator B can also be evaluted in terms of another limit, noting Theorem 4.1 and (5.4).

THEOREM 5.3. *For all* $f \in X_{2\pi}$ *we have*

$$Bf(x) = -\operatorname*{s-lim}_{\lambda\to 0+}\left[\frac{1}{2\lambda}\int_{-\infty}^{\infty} e^{-\lambda|u-x|}f(u)\,du - \frac{1}{2\pi\lambda^2}\int_{-\pi}^{\pi} f(u)\,du\right].$$

5.2. The wave equation in $X(\mathbf{R})$. Let $X(\mathbf{R})$ be one of the Banach spaces $UCB(\mathbf{R}) := \{f : \mathbf{R} \to \mathbf{C};\ f$ uniformly continuous and bounded on $\mathbf{R}\}$ with norm $\|f\|_{UCB(\mathbf{R})} := \sup_{x\in\mathbf{R}} |f(x)|$, or

$$L^p(\mathbf{R}) := \left\{f : \mathbf{R} \to \mathbf{C}; \|f\|_{L^p(\mathbf{R})} < \infty\right\}, \quad 1 \leqq p < \infty,$$

with norm

$$\|f\|_{L^p(\mathbf{R})} := \left\{\int_{-\infty}^{\infty} |f(x)|^p dx\right\}^{1/p};$$

and let $C(t)$ be defined by

$$(5.8) \quad \left(C(t)f\right)(x) := \frac{1}{2}\left[f(x+t) + f(x-t)\right] \qquad (f \in X(\mathbf{R});\ x, t \in \mathbf{R}).$$

Then it is known that (cf.[22]) $\mathcal{C} = \{C(t); t \in \mathbf{R}\} \in \mathcal{C}_1^0$ with $\|C(t)\|_{[UCB(\mathbf{R})]} = \|C(t)\|_{[L^1(\mathbf{R})]} = 1$, and generator $A = d^2/dx^2$, having domain

$$D(A) = X^2(\mathbf{R}) := \begin{cases} \{f \in UCB(\mathbf{R}); f', f'' \in UCB(\mathbf{R})\}, & X(\mathbf{R}) = UCB(\mathbf{R}) \\ \{f \in L^p(\mathbf{R}); f, f' \in AC_{\text{loc}}(\mathbf{R}), f'' \in L^p(\mathbf{R})\}, & X(\mathbf{R}) = L^p(\mathbf{R}), \end{cases}$$

where $AC_{\text{loc}}(\mathbf{R}) := \{f : \mathbf{R} \to \mathbf{C};\ f$ locally absolutely continuous$\}$. Further, the kernel

$$N(A) = \begin{cases} \{f \in X(\mathbf{R}); f \text{ constant}\}, & X(\mathbf{R}) = UCB(\mathbf{R}) \\ \{\theta\}, & X(\mathbf{R}) = L^p(\mathbf{R}). \end{cases}$$

Thus we have $P = \Theta$ if $X(\mathbf{R}) = L^p(\mathbf{R}), 1 \leqq p < \infty$.

Note that (5.8) is now the solution of the wave equation (5.2) for $w(0,x) = f(x) \in X^2(\mathbf{R})$ and $(\partial/\partial t)w(t,x)|_{t=0} = 0$.

To represent P in the $UCB(\mathbf{R})$-case, we define $P_R f := \frac{1}{2R}\int_{-R}^{R} f(u)du$ for $R > 0$. These operators are linear and bounded by 1, with $P_R f = f$ for

$f \in N(A)$. Now let $f \in R(A)$, i.e., there exists a $g \in UCB^2(\mathbf{R})$ such that $f = g''$. Hence,

$$P_R f = P_R g'' = \frac{1}{2R} \int_{-R}^{R} g''(u)\, du = \frac{g'(R) - g'(-R)}{2R} \leqq \frac{2\, \|g'\|_{UCB(\mathbf{R})}}{2R},$$

and so $\operatorname{s-lim}_{R\to\infty} P_R f = 0$ for any $f \in R(A)$. Thus for any $f = g'' + h \in R(A) \oplus \oplus N(A)$ we have $\operatorname{s-lim}_{R\to\infty} P_R f = h$. Now, by the the Banach–Steinhaus theorem, this convergence holds for all $f \in \overline{R(A)} \oplus N(A) = X_0$. If we now define the operator P by

$$(5.9) \qquad Pf = \operatorname{s-lim}_{R\to\infty} P_R f = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(u)\, du,$$

we obtain the linear projector from X_0 onto $N(A)$.

In these cases, the operator B of Definition 1.2 is unbounded, because for the particular function $f(x) = \frac{1}{1+x^2}$ we have $f \in X(\mathbf{R})$ and there exists no $g \in X^2(\mathbf{R})$ such that $f = g'' - c_1$, $c_1 \in \mathbf{C}$ ($c_1 = 0$, if $X(\mathbf{R}) = L^p(\mathbf{R})$). In fact, such a g would look like $g(x) = x \arctan x - \frac{1}{2}\log(1 + x^2) + c_1 \frac{x^2}{2} + c_2 x + c_3$, c_1 as above, c_2, c_3 arbitrary constants in $\mathbf{C}$, but for no choice of c_1, c_2, c_3 would g belong to $X(\mathbf{R})$. Hence $f \notin R(A) \oplus N(A)$, which equals X iff B is bounded, by Theorem 3.3.

Again the resolvent of A is given as in (5.4), i.e.,

$$(5.10) \qquad \lambda R(\lambda^2, A) f(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\lambda|u-x|} f(u)\, du.$$

Since the operator B in this case is the inverse of the second derivate on $R(A) \cap D(A)$, one could write

$$D(B) := X^{-2}(\mathbf{R}) := \begin{cases} \{f \in X(\mathbf{R}); f = g'' + c, g \in X_0^2(\mathbf{R}), c \in \mathbf{C}\}, & X(\mathbf{R}) = UCB(\mathbf{R}) \\ \{f \in X(\mathbf{R}); f = g'', g \in X_0^2(\mathbf{R})\}, & X(\mathbf{R}) = L^p(\mathbf{R}). \end{cases}$$

Further let,

$$X_0(\mathbf{R}) = \begin{cases} \left\{f \in X(\mathbf{R}); \exists \{g_n\}_{n\in\mathbf{N}} \subset X^2(\mathbf{R}), f = \operatorname*{s\text{-}lim}_{n\to\infty} g_n + c\right\}, & X(\mathbf{R}) = UCB(\mathbf{R}) \\ \left\{f \in X(\mathbf{R}); \exists \{g_n\}_{n\in\mathbf{N}} \subset X^2(\mathbf{R}), f = \operatorname*{s\text{-}lim}_{n\to\infty} g_n\right\}, & X(\mathbf{R}) = L^1(\mathbf{R}) \\ X(\mathbf{R}), & X(\mathbf{R}) = L^p(\mathbf{R}),\ 1 < p < \infty. \end{cases}$$

As an application of Theorem 1.1 we now have

THEOREM 5.4. *Let $f \in X_0(\mathbf{R})$ be arbitrary. Then*

$$\operatorname*{s\text{-}lim}_{|t|\to\infty} \frac{1}{t^2} \int_0^t \int_0^u \left[f(x+v) + f(x-v)\right] dv\, du = \operatorname*{s\text{-}lim}_{\lambda\to 0+} \frac{\lambda}{2} \int_{-\infty}^{\infty} e^{-\lambda|u-x|} f(u)\, du.$$

This limit vanishes if $X(\mathbf{R}) = L^p(\mathbf{R}), 1 \leqq p < \infty$, and equals

$$\lim_{R\to\infty} \tfrac{1}{2R} \int_{-R}^{R} f(u) du$$

provided $X(\mathbf{R}) = UCB(\mathbf{R})$.

Introducing the notations

$$N_t(f) :=$$

$$:= \begin{cases} \left\| \frac{1}{t^2} \int_0^t \int_0^u \left[f(x+v) + f(x-v)\right] dv\, du \right\|_{L^p(\mathbf{R})}, & X(\mathbf{R}) = L^p(\mathbf{R}), 1 \leqq p < \infty \\ \left\| \frac{1}{t^2} \int_0^t \int_0^u \left[f(x+v) + f(x-v)\right] dv\, du - \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(u)\, du \right\|_{UCB(\mathbf{R})}, & X(\mathbf{R}) = UCB(\mathbf{R}), \end{cases}$$

$$N_\lambda(If) := \begin{cases} \left\| \dfrac{\lambda}{2} \displaystyle\int_{-\infty}^{\infty} e^{-\lambda|u-x|} f(u)\,du \right\|_{L^p(\mathbf{R})}, & X(\mathbf{R}) = L^p(\mathbf{R}), 1 \leqq p < \infty \\ \left\| \dfrac{\lambda}{2} \displaystyle\int_{-\infty}^{\infty} e^{-\lambda|u-x|} f(u)\,du - \lim_{R\to\infty} \dfrac{1}{2R} \int_{-R}^{R} f(u)\,du \right\|_{UCB(\mathbf{R})}, & \\ & X(\mathbf{R}) = UCB(\mathbf{R}), \end{cases}$$

we have as an application of Theorem 1.2,

THEOREM 5.5. *Let $f \in X(\mathbf{R})$ be arbitrary.*

a) *The following three assertions are equivalent:*

(i) $N_t(f) = o(t^{-2}) \qquad (|t| \to \infty)$,

(ii) $N_\lambda(If) = o(\lambda^2) \qquad (\lambda \to 0^+)$,

(iii) $f(x) = c$ *for (almost) all* $x \in \mathbf{R}$ *and* $c = 0$ *if* $X(\mathbf{R}) = L^p(\mathbf{R}), 1 \leqq$ $\leqq p < \infty$.

b) *The following three assertions are equivalent for* $\alpha \in (0, 2]$:

(i) $N_t(f) = O(|t|^{-\alpha}) \qquad (|t| \to \infty)$,

(ii) $N_\lambda(If) = O(\lambda^\alpha) \qquad (\lambda \to 0^+)$,

(iii) $K(t^{-2}, f; X_0(\mathbf{R}), X^{-2}(\mathbf{R})) = O(|t|^{-\alpha}) \qquad (|t| \to \infty)$.

c) *If, in particular,* $\alpha = 2$, *then the assertions* (i)–(iii) *of* b) *are also equivalent to*

(iv) $f \in \widetilde{X^{-2}(\mathbf{R})}^{X_0(\mathbf{R})}$, *if* $X(\mathbf{R}) = UCB(\mathbf{R})$ *or* $L^1(\mathbf{R})$,

(iv)* $f \in X^{-2}(\mathbf{R})$, *i.e.,* $f = g''$ *with* $g \in X^2(\mathbf{R})$, *if* $X(\mathbf{R}) = L^p(\mathbf{R}), 1 <$ $< p < \infty$.

By Theorems 3.4 and 4.4 we have, concerning the sharpness,

THEOREM 5.6. *For every* $\alpha \in (0, 2]$ *there exist elements* $f_\alpha, f^*{}_\alpha \in X(\mathbf{R})$ *such that*

a) $N_t(f_\alpha) \begin{cases} = O(|t|^{-\alpha}) \\ \neq o(|t|^{-\alpha}) \end{cases} \qquad (|t| \to \infty)$

b) $N_\lambda(If^*{}_\alpha) \begin{cases} = O(\lambda^\alpha) \\ \neq o(\lambda^\alpha) \end{cases} \qquad (\lambda \to 0^+)$.

Further, in the $L^p(\mathbf{R})$ cases we can use Corollary 4.2 to compute B. We have

THEOREM 5.7. *For any* $f \in L^p(\mathbf{R}), 1 \leqq p < \infty$ *can the operator* B *of Definition 1.2 be represented as the one dimensional Newtonian potential (cf.* [17],[31]*)*

$$Bf(x) = \frac{1}{2} \int_{-\infty}^{\infty} |x - u| f(u)\,du,$$

where this integral exists in $L^p(\mathbf{R})$ *if* $f \in \mathcal{D} := \{f \in L^p(\mathbf{R}); f = g'', g, g' \in \in AC(\mathbf{R}), g'' \in L^p(\mathbf{R})\} \subset D(B)$ *(with* $\mathcal{D} = D(B)$ *in the case* $p = 1$*).*

PROOF. Since $N(A) = \{\theta\}$, by Corollary 4.2, $f \in D(B)$ iff

$$\operatorname*{s-lim}_{\lambda\to 0+} \int_0^\infty t e^{-\lambda t} C(t) f \, dt$$

exists. Further, if $f \in \mathcal{D}, f = g''$ a.e., we have by partial integration,

$$\int_0^\infty t\, C(t) f\, dt = \int_0^\infty t \frac{f(x+t) + f(x-t)}{2} dt = \tag{5.11}$$

$$= \frac{1}{2}\int_x^\infty (u-x) f(u)\, du + \frac{1}{2}\int_{-\infty}^x (x-u) f(u)\, du =$$

$$= \frac{1}{2}\int_x^\infty (u-x) g''(u)\, du + \frac{1}{2}\int_{-\infty}^x (x-u) g''(u)\, du =$$

$$= \frac{1}{2}\lim_{R\to\infty} Rg'(R) - \frac{1}{2}\int_x^\infty g'(u) du + \frac{1}{2}\lim_{R\to\infty} Rg'(-R) + \frac{1}{2}\int_{-\infty}^x g'(u)\, du = g(x),$$

since $g, g', g'' \in L^p(\mathbf{R})$ and $g, g' \in AC(\mathbf{R})$, so that $xg'(x)$ tends to zero for $x \to \infty$. If not, we would have, by L'Hospitals rule, $0 = \lim_{x\to\infty} xg(x)/x = = \lim_{x\to\infty} xg'(x) + g(x) \neq 0$, which is a contradiction. On the other hand, (5.11) equals $\frac{1}{2}\int_{-\infty}^{\infty} |u-x| f(u)\, du$. □

In his well known paper (see [25], cf. also [6], pp. 397 ff.) M. Riesz introduced the integrals of fractional order $\alpha > 2$ (actually in the space $\mathbf{R}^m$)

$$I^\alpha f(x) \equiv R_\alpha f(x) := \frac{\Gamma(\frac{1-\alpha}{2})}{\sqrt{\pi}2^\alpha \Gamma(\frac{\alpha}{2})} \int_{-\infty}^\infty f(u)|x-u|^{\alpha-1}\, du.$$

For these integrals he stated that $(d^2/dx^2) I^\alpha f(x) = -I^{\alpha-2} f(x)$, and $I^0 f(x) = f(x)$, without giving precise conditions upon f for the formula's validity. Now in Theorem 5.7 we found a necessary (and sufficient in the $L^1(\mathbf{R})$-case) condition upon $f \in L^p(\mathbf{R})$ for which this assertion holds in the case $\alpha = 2$, namely $f = g''$ with $g, g' \in AC(\mathbf{R})$, noting $g = -I^2 f$.

Theorem 5.7 was also established by K. Yosida [31] using methods applied to the Gauss–Weierstrass semigroup operator $W(t)f$ of (1.4) in the Banach space $C_\infty(\mathbf{R})$, which is the completion of the set of all continuous functions $f : \mathbf{R} \to \mathbf{R}$, having compact support. This space may also be allowed here. Our proof of Theorem 5.7 is a direct application of Corollary 4.2 to the cosine operator function of translations, which is much more simple.

Now to some final facts. The proof of Theorem 1.3 is a direct consequence of Corollary 3.1. In fact, it is even possible to show that one also has saturation in Theorem 1.3 with saturation class $\widetilde{X^{-2}(\mathbf{R})}^{X_0(\mathbf{R})}$. Theorem 1.4 follows when one combines Corollary 3.1 with an application of the mean ergodic theory for the semigroup $\{W(t); t > 0\}$ found in [4], namely that

$$\left\| \frac{1}{t} \int_0^t W(u)f\,du - \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(u)\,du \right\|_{X(\mathbf{R})} = O(t^{-\alpha}) \qquad (t \to \infty)$$

$$\Leftrightarrow K(t^{-2}, f; X_0(\mathbf{R}), X^{-2}(\mathbf{R})) = O(t^{-\alpha}).$$

Here again it is also possible to show that the saturation phenomenon holds for the Cesàro means $\{(1/t)\int_0^t W(u)f\,du; t > 0\}$ as well as the Abel means $\{\lambda \int_0^\infty e^{-\lambda u} W(u)f\,du; \lambda > 0\}$.

Ultimately, can one characterize the relative completion $\widetilde{X^{-2}(\mathbf{R})}^{X_0(\mathbf{R})}$ for $X(\mathbf{R}) = UCB(\mathbf{R})$? This would put Theorem 1.3 in a more concrete form. In this respect one has that $\widetilde{X^2(\mathbf{R})}^{X(\mathbf{R})} = \{f \in UCB(\mathbf{R}); f, f' \in AC_{\text{loc}}(\mathbf{R}), f'' \in L^\infty(\mathbf{R})\}$ if $X(\mathbf{R}) = UCB(\mathbf{R})$, and $\widetilde{X^2(\mathbf{R})}^{X(\mathbf{R})} = \{f \in L^1(\mathbf{R}); f \in AC(\mathbf{R}), f' \in NBV(\mathbf{R})\}$, if $X(\mathbf{R}) = L^1(\mathbf{R})$, a result which is known at least for the spaces $C_{2\pi}, L^1_{2\pi}, UCB(\mathbf{R}^+), L^1(\mathbf{R}^+)$ (cf. [6], pp. 373,386 ff., [3], p. 110). Our conjecture is that $\widetilde{X^{-2}(\mathbf{R})}^{X_0(\mathbf{R})} = \{f \in UCB_0(\mathbf{R}); f = g'' + c, g, g' \in L^\infty(\mathbf{R}), c \in CZ\}$ for $X(\mathbf{R}) = UCB(\mathbf{R})$. Further, probably $\left[\widetilde{(L^1(\mathbf{R}))^{-2}}\right]^{L^1_0(\mathbf{R})} = \{f \in L^1(\mathbf{R}); f = g'', g \in L^1(\mathbf{R})\}$.

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DISTRIBUTION OF THE VALUES OF q-ADDITIVE FUNCTIONS ON POLYNOMIAL SEQUENCES

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To Professor K. Tandori on his seventieth birthday

1. Introduction

1.1. As usual N, R, C denote the set of natural, real, complex numbers, respectively, $N_0 = N \cup \{0\}$. $\mathcal{P}$ is the set of primes, a general element of which is denoted by p. $\pi(x)$ denotes the number of primes up to x.

1.2. Let $q \in N$, $q \geqq 2$ be fixed, $E = \{0, 1, \ldots, q-1\}$. The q-ary expansion of $n \in N_0$ is defined by

$$n = \sum_{j=0}^{\infty} a_j(n) q^j, \quad a_j(n) \in E \tag{1.1}$$

The right hand side of (1.1) is clearly a finite sum, since $a_j(n) = 0$ for $q^j > n$.

A function $f : N_0 \to R$ is said to be q-additive if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f\big(a_j(n) q^j\big). \tag{1.2}$$

A special q-additive function is $\alpha(n) := \sum a_j(n)$, the sum of digit function. Let $\mathcal{A}_q$ be the set of q-additive functions.

1.3. The letters N, L are preserved for denoting $N = \left[\frac{\log x}{\log q}\right]$, $L = \log x$. We shall write furthermore $e(y)$ instead of $e^{2\pi i y}$.

1.4. Let $P(x)$ be an arbitrary polynomial with integer coefficients, the leading term of which is positive. Let $r = \deg P(x)$.

1.5. Let

$$m_k := \frac{1}{q} \sum_{b \in E} f(bq^k), \qquad \sigma_k^2 := \frac{1}{q} \sum_{b \in E} f^2(bq^k) - m_k^2,$$

$$M(x) := \sum_{k=0}^{N} m_k, \quad D^2(x) = \sum_{k=0}^{N} \sigma_k^2.$$

THEOREM. *Let $f \in \mathcal{A}_q$ such that $f(bq^j) = O(1)$ as $j \to \infty$, $b \in E$, furthermore that $\frac{D(x)}{(\log x)^{1/3}} \to \infty$ $(x \to \infty)$. Let $P(x)$ be a polynomial defined as in* 1.4. *Then, as $x \to \infty$,*

$$\frac{1}{x}\sharp\left\{n < x \;\middle|\; \frac{f(P(n)) - M(x^r)}{D(x^r)} < y\right\} \to \Phi(y)$$

and

$$\frac{1}{\pi(x)}\sharp\left\{p < x \;\middle|\; \frac{f(P(n)) - M(x^r)}{D(x^r)} < y\right\} \to \Phi(y),$$

where Φ is the normal distribution function.

The proof is based upon theorems of I. M. Vinogradov and L. K. Hua for trigonometric sums. We shall use furthermore a known theorem due to Erdős and Turán for the discrepancy of sequences mod 1.

A simplified version of our argument was used earlier in the paper [3] of the first named author.

2. Lemmata

2.1. LEMMA 1 (Hua [1], Theorem 10). *Let $0 < Q \leqq c_1(k)L^{\tau_1}$ and*

$$S = \sum_{\substack{p \leqq x \\ p \equiv t \pmod{Q}}} e\big(f(p)\big)$$

in which

$$f(y) = \frac{h}{q}y^k + \alpha_1 y^{k-1} + \ldots + \alpha_k, \; (h,q) = 1.$$

Suppose that $L^\tau < q \leqq x^k L^{-\tau}$. For arbitrary $\tau_0 > 0$, when $\tau \geqq 2^{6k}(\tau_0 + \tau_1 + 1)$, we always have

$$|\, S \,| \leqq c_2(k) x L^{-\tau_0} Q^{-1}.$$

where $c_2(k)$ is a suitable constant which depends only on k.

LEMMA 2 ([1], Lemma 6.2). *Let*

$$S_1 = \sum_{n \leqq x} e\big(f(n)\big),$$

and let f be the same polynomial as in Lemma 1. Let τ_0, τ_3, τ_4 be arbitrary positive numbers,

$$\tau \geqq 2^k(\tau_0 + \tau_3) + 2k\tau_4 + 2^{3(k-2)}.$$

Suppose that

$$L^\tau < q \leqq x^k \cdot L^{-\tau}.$$

Then, we have

$$S_1 \ll x \cdot L^{-\tau}.$$

The constant standing implicity in $\ll$ may depend on τ_3, τ_4.

Lemma 2 is due to I. M. Vinogradov [4].

2.2. The discrepancy D_M of the real numbers $x_1, \ldots, x_M$ mod 1 is defined by

$$\sup \left| \frac{1}{M} \sum_{\substack{n=1 \\ \{x_n\} \in [\alpha, \beta)}}^{M} 1 - (\beta - \alpha) \right|$$

where the supremum is taken for all intervals $[\alpha, \beta) \subseteqq [0, 1]$. Let $\Psi_m :=$ $:= \sum_{l=1}^{M} e(mx_l)$.

LEMMA 3 (P. Erdős–P. Turán [2]). *We have*

$$D_M \leqq c \left(\sum_{0 < h \leqq K} \frac{|\Psi_h|}{h} + \frac{M}{K} \right),$$

for any positive integer K. c is an absolute constant.

2.3. Let $0 < \xi < 1$ and

$$U := [1 - \xi, 1] \cup \bigcup_{b=1}^{q-1} \left[\frac{b}{q} - \xi, \frac{b}{q} + \xi \right] \cup [0, \xi].$$

Let P be the polynomial defined in the theorem,

$$E_j := \sharp\left\{p \leqq x \;\middle|\; \left\{\frac{P(p)}{q^{j+1}}\right\} \in U\right\}, \qquad F_j := \sharp\left\{n \leqq x \;\middle|\; \left\{\frac{P(n)}{q^{j+1}}\right\} \in U\right\}.$$

LEMMA 4. *Let $\varepsilon > 0$ be fixed, $N^\varepsilon < j < rN - N^\varepsilon$, λ an arbitrary positive constant. Then, uniformly in j, we have*

$$E_j \ll \xi\pi(x) + xL^{-\lambda}, \quad F_j \ll \xi x + xL^{-\lambda}.$$

PROOF. This follows easily from the Lemmas 1, 2 and 3. U is the union of $q+1$ subintervals, its measure is $2q\xi$. Let $K = L^{\lambda+1}$ and apply Lemma 3 for the sequences $x_n = \frac{P(n)}{q^j}$, $\left(n = 1, \ldots, [x]\right)$. The conditions of Lemmas 1, 2 clearly hold for the polynomials $f(n) = \frac{mP(n)}{q^j}$. Using the appropriate estimates we obtain the inequalities stated in Lemma 4.

3. For an arbitrary sequence of integers $(1 \leqq) l_1 < \ldots < l_h$ and $b_1, \ldots, b_h \in E$, let

$$(3.1)\ \Sigma_1 := A\left(x \;\middle|\; \begin{matrix} l_1, \ldots, l_h \\ b_1, \ldots b_h \end{matrix}\right) = \sharp\left\{n \leqq x \mid a_{l_j}\left(P(n)\right) = b_j, \quad j = 1, \ldots, h\right\},$$

$$(3.2)\ \Sigma_2 := \prod\left(x \;\middle|\; \begin{matrix} l_1, \ldots, l_h \\ b_1, \ldots b_h \end{matrix}\right) = \sharp\left\{p \leqq x \mid a_{l_j}\left(P(p)\right) = b_j, \quad j = 1, \ldots, h\right\}$$

LEMMA 5. *Assume that*

$$N^{1/3} \leqq l_1 < l_2 < \ldots < l_h \leqq rN - N^{1/3}. \tag{3.3}$$

Let λ be an abitrary constant. Then

$$\Sigma_1 = \frac{x}{q^h} + O(x \cdot L^{-\lambda}) \qquad (x \to \infty)$$

and

$$\Sigma_2 = \frac{\pi(x)}{q^h} + O(x \cdot L^{-\lambda}) \qquad (x \to \infty),$$

uniformly in l_j subject to (3.3) and $b_j \in E$. The implicit constants in the error terms may depend on the polynomial P, on h, and on λ.

4. Proof of Lemma 5

For $b \in E$ let $\varphi_b(x)$ be a function periodic mod 1, defined explicitly in $[0,1]$ by

$$\varphi_b(x) := \begin{cases} 1 & \text{if } x \in \left(\frac{b}{q}, \frac{b+1}{q}\right) \\ 1/2 & \text{if } x = \frac{b}{q}, \frac{b+1}{q} \\ 0 & \text{otherwise.} \end{cases}$$

Its Fourier expansion $\sum c_m(b)e(mx)$ can be given explicitly:

$$c_0(b) = 1/q, \qquad c_m(b) = -\frac{e\left(-\frac{mb}{q}\right)}{2\pi i m}\left(e\left(-\frac{m}{q}\right) - 1\right).$$

Let $0 < \Delta < \frac{1}{2q}$, and consider the function

$$f_b(x) := \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \varphi_b(x+z)dz = \sum_{m=-\infty}^{\infty} d_m(b)e(mx).$$

By simple computations we obtain:

$$d_0(b) = \frac{1}{q}, \qquad d_m(b) = c_m(b) \cdot \frac{e\left(\frac{m\Delta}{2}\right) - e\left(-\frac{m\Delta}{2}\right)}{2\pi i m\Delta}, \tag{4.1}$$

$$d_m(b) = 0 \text{ if } m \equiv 0 \pmod q \text{ and } m \neq 0, \tag{4.2}$$

furthermore

$$\left| d_m(b) \right| \leqq \min\left(\frac{1}{\pi \mid m \mid}, \frac{1}{\Delta\pi m^2}\right). \tag{4.3}$$

It is clear that $0 \leqq f_b(x) \leqq 1$ for every x, and that

$$f_b(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{b}{q} + \Delta, \frac{b+1}{q} - \Delta\right] \\ 0 & \text{if } x \in [0,1] \cap \left[\frac{b}{q} - \Delta, \frac{b+1}{q} + \Delta\right]. \end{cases} \tag{4.4}$$

Let $b_1, \dots, b_h \in E$, $(1 \leqq) l_1 < l_2 < \dots < l_h$ be arbitrary integers,

$$F(x_1, \dots, x_h) = f_{b_1}(x_1) \dots f_{b_h}(x_h), \tag{4.5}$$

$$t(y) = F\left(\frac{y}{ql_1 + 1}, \dots, \frac{y}{ql_h + 1}\right). \tag{4.6}$$

Let $V = \left[\frac{1}{ql_1+1}, \ldots, \frac{1}{ql_h+1}\right]$ be a vector, $\mathcal{M}$ the whole set of vectors $M = [m_1, \ldots, m_h]$ with integer entries m_j. Let $VM = \frac{A_M}{H_M}$, $(A_M, H_M) = 1$.

It is clear that

$$t(y) = \sum_M T_M e(MVy), \tag{4.7}$$

where

$$T_M = d_{m_1}(b_1) \ldots d_{m_h}(b_h). \tag{4.8}$$

Let $\Delta = \xi$ (see Lemma 4). It is clear that

$$\left|\Sigma_1 - \sum_{n \leqq x} t(P(n))\right| \leqq F_{l_1} + \ldots + F_{l_h} \tag{4.9}$$

and

$$\left|\Sigma_2 - \sum_{p \leqq x} t(P(p))\right| \leqq E_{l_1} + \ldots + E_{l_h}. \tag{4.9}$$

Furthermore,

$$\sum_{n \leqq x} t(P(n)) = \sum_M T_M \sum_{n \leqq x} e\left(\frac{A_M}{H_M} P(n)\right) \tag{4.10}$$

and

$$\sum_{p \leqq x} t(P(p)) = \sum_M T_M \sum_{p \leqq x} e\left(\frac{A_M}{H_M} P(p)\right). \tag{4.10}$$

We shall check that Lemmas 1 and 2 can be applied to the polynomials $\frac{A_M}{H_M} P(y)$ on the right hand side of (4.10), (4.11).

We can omit those M for which there is a j such that $q | m_j$, $m_j \neq 0$, since $d_{m_j}(b_j) = 0$ implies $T_M = 0$. Let $q = p_1^{e_1} \ldots p_s^{e_s}$. Assume that $q \nmid m_h$. Then $p_t^{e_t} \nmid m_h$ for some t. Thus

$$H_M(m_h + q^{l_h - l_{h-1}} m_{h-1} + \ldots + m_1 q^{l_h - l_1}) = A_M q^{l_h + 1},$$

and $p_t^{l_h e_t} \mid H_M$. Thus there exists an $\eta > 0$ depending only on q, such that $H_M \geqq q^{\eta l_h}$. We can prove similarly that $H_M \geqq q^{\eta l_s}$ holds if $q \nmid m_s$ and $m_{s+1} = \ldots = m_h = 0$.

Then Lemmas 1 and 2 can be applied to the polynomials $f(y) = \frac{A_M}{H_M} P(y)$ if $M \neq 0$. The leading coefficient of f is a rational number, the denominator of which belongs to the interval $(L^\tau, x^r/L^\tau)$ if $x > x_0(\tau)$, for every choice of τ. Consequently the exponential sums are bounded by $O(x \cdot L^{-\tau_0})$; uniformly for all choices of $l_1, \dots, l_h$ under (3.3) and for $b_1, \dots, b_h$. τ_0 is an arbitrary constant. Hence we obtain that

$$\Sigma_1 = \frac{x}{q^h} + O\Big(x \cdot L^{-\tau_0} \sum_{M \neq 0} |T_M|\Big) + O\Big(\sum_{j=1}^{h} F_{l_j}\Big)$$

and

$$\Sigma_2 = \frac{\pi(x)}{q^h} + O\Big(x \cdot L^{-\tau_0} \sum_{M \neq 0} |T_M|\Big) + O\Big(\sum_{j=1}^{h} E_{l_j}\Big).$$

The main terms $\frac{x}{q^h}, \frac{\pi(x)}{q^h}$ come from the choice $M = 0$. From (4.3) we obtain

$$\sum | T_M | \leqq \left(\frac{1}{q} + 2 \sum_{m=1}^{\infty} \min\left(\frac{1}{\pi m}. \frac{1}{\pi \Delta m^2}\right)\right)^h \leqq (2 + 2 \log 1/\Delta)^h.$$

Let $\xi = \Delta = L^{-\lambda_1}$ where λ_1 is a large constant. From the above relations and from Lemma 4 the assertion of Lemma 5 immediately follows.

5. Completion of the proof of the Theorem

Let $A = [N^{1/3}]$, $B = rN - A$,

$$f_1(P(n)) = \sum_{j=A}^{B} f\,(a_j(P(n))\,q^j)\,.$$

Then $f_1(F(n)) = f(P(n)) + O(N^{1/3})$. Let furthermore

$$M_1(x^r) = \sum_{k=A}^{B} M_k; \quad D_1^2(x^r) = \sum_{k=A}^{B} \sigma_k^2.$$

Thus

$$M_1(x^r) - M(x^r) = O(N^{1/3}), \quad D^2(x^r) - D_1^2(x^r) = O(N^{1/3}).$$

From the condition stated for f, we obtain that

$$\max_{n \leqq x} \left| \frac{f(P(n)) - M(x^r)}{D(x^r)} - \frac{f_1(P(n)) - M_1(x^r)}{D_1(x^r)} \right| \to 0 \quad \text{as} \quad x \to \infty.$$

Consequently, it is enough to prove that

$$x^{-1} \sharp \left\{ n < x \,\middle|\, \frac{f_1(P(n)) - M_1(x^r)}{D_1(x^r)} < y \right\} \to \Phi(y), \tag{5.1}$$

$$\frac{1}{\pi(x)} \sharp \left\{ p < x \,\middle|\, \frac{f_1(P(p)) - M_1(x^r)}{D_1(x^r)} < y \right\} \to \Phi(y), \tag{5.2}$$

as $x \to \infty$.

By using the Frechet–Shohat theorem, (5.1), (5.2) are valid, if all the moments

$$a_k(x) = \frac{1}{x} \sum_{n \leqq x} \left(\frac{f_1(P(n)) - M_1(x^r)}{D_1(x^r)} \right)^k$$

and

$$b_k(x) = \frac{1}{x} \sum_{p \leqq x} \left(\frac{f_1(P(p)) - M_1(x^r)}{D_1(x^r)} \right)^k$$

converge to the corresponding moment of the normal law. Instead of computing the moments we compare them with

$$c_k(x) := \frac{1}{x^r} \sum_{n \leqq x^r} \left(\frac{f_1(n) - M_1(x^r)}{D_1(x^r)} \right)^k .$$

From Lemma 5 immediately follows that for each fixed integer $k (\geqq 0)$, $a_k(x) - c_k(x) \to 0$, $b_k(x) - c_k(x) \to 0$ as $x \to \infty$.

The quantities

$$\frac{f_1(n) - M_1(x^r)}{D_1(x^r)}, \quad n = 1, \dots, [x^r]$$

are distributed in limit according to Φ. This directly follows from known theorems for the sums of independent random variables. Moreover the moment $c_k(x)$ converges to the kth moment of Φ, for every $k = 0, 1, \dots$.

Since $\lim_{x \to \infty} a_k(x) = \lim_{x \to \infty} b_k(x) = \lim_{x \to \infty} c_k(x) = \int x^k d\Phi$, our theorem is true.

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