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TOPOLOGICAL COMPLEXITY OF GRAPHS AND THEIR SPANNING TREES

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Introduction

Let X be a perfect Polish space. Let $G = (V, E)$ be a simple graph such that $V \subseteqq X$. Let $\vec{E} = \{(x,y) \mid \{x,y\} \in E\}$. We call G *analytic* if $\vec{E}$ is an analytic subset of the perfect Polish space $X \times X$. Similarly, G is *σ-analytic* if $\vec{E}$ belongs to the σ-algebra generated by all the analytic subsets of $X \times X$. Having assigned this topological complexity to G, one may ask questions concerning the topological complexity of objects related to G. In this paper we study spanning trees. The basic question is: Given the topological complexity of G, how simple can a spanning tree of G be?

By Zorn's Lemma, it is easy to show that every forest F in a connected simple graph G can be extended into a spanning tree of G, but the proof is not constructive. In Section 1 we show that if G is analytic, F is σ-analytic, $\mathrm{V}(F)$ is analytic, and F has countably many connectivity components, then F can be extended into a σ-analytic spanning tree of G (Theorem 1.2).

In Section 2 we study weighted graphs. A *weighted graph* is a triple $W = (G, \mathbf{w}, \lambda)$ such that $G = (V, E)$ is a simple graph, λ is an ordinal, and $\mathbf{w} : E \to \lambda$. For every $\beta < \lambda$, let G^β be the subgraph of G consisting of all edges $u \in E$ such that $\mathbf{w}(u) = \beta$. Here we are looking for a spanning tree of G whose "total weight" is as small as possible. This makes a clear sense in case that G and λ are finite. In the infinite case, Ron Aharoni [2] gave the definition: T is a *minimal spanning tree* of W if T is a spanning tree of G and if we replace one edge in T by a lighter edge, then T stops being a spanning tree. First, we prove a purely combinatorial fact: every connected weighted graph has a minimal spanning tree (Theorem 2.2). Then we prove a topological version of this fact: Let $W = (G, \mathbf{w}, \lambda)$ be a connected weighted graph where G is an analytic graph, λ is a countable ordinal, and for every $\beta \in \lambda$, the graph G^β is analytic and has countably many components. Then W has a σ-analytic minimal spanning tree (Theorem 2.4).

In Section 3 we show that Theorem 2.4 is optimal. First, we show that an analytic minimal spanning tree for W does not always exists. Second, we show that if G^β has uncountably many components for some $\beta \in \lambda$ or λ is uncountable then a σ-analytic minimal spanning tree for W does not always exist. Finally, we list some open problems.

0. Preliminaries

A *graph* G is a pair of sets (V,E) such that $E \subseteqq [V]^2 = \{\{x,y\} \mid x,y \in \in V\}$. Elements of V are called *vertices* and elements of E are called *edges*. We also denote V by $\mathrm{V}(G)$, and E by $\mathrm{E}(G)$. A *path* in G is a finite sequence $p=(x_0,x_1,\dots,x_n)$ of distinct vertices of G such that $\{x_i,x_{i+1}\} \in \mathrm{E}(G)$ for every $i<n$. A graph G is *connected* if for every $u,v \in \mathrm{V}(G)$ there is a path $p=(u,x_1,\dots,x_{n-1},v)$ in G. This is an equivalence relation on $\mathrm{V}(G)$, and its equivalence classes are called *connectivity components*, or *components*, for short. A *cycle* in G is a path $p=(x_0,x_1,\dots,x_n)$ such that $n \geqq 2$ and $\{x_n,x_0\} \in E(G)$. A *forest* is a graph with no cycles. A *tree* is a connected forest. A graph $G'=(V',E')$ is a *subgraph* of a graph $G=(V,E)$ if $V' \subseteqq V$ and $E' \subseteqq E$. We also say that G *extends* G'. If $V'=V$ then G' is a *spanning subgraph* of G. If, in addition, G' is a tree then G' is a *spanning tree of* G. By a simple use of Zorn's Lemma one gets the following theorem.

THEOREM 0.1. *Let F be a forest in a connected graph G. Then F can be extended into a spanning tree of G.*

Let X be a perfect Polish space, i.e., a perfect separable complete metric space. A set $B \subseteqq X$ is called *Borel* if B belongs to the σ-algebra generated by all the open subsets of X; B is *analytic* if B is a Σ^1_1 set, i.e., B is a continuous image of some Borel subset of some perfect Polish space; B is *co-analytic* if B is a Π^1_1 set, i.e., $X-B$ is a Σ^1_1 set. We call B *σ-analytic* if B belongs to the σ-algebra generated by all the analytic subsets of X.

Let R and R^* be binary relations. We say that R^* *uniformizes* R if $R^* \subseteqq R$, R^* is a function, and $\mathrm{dom}\,(R) = \mathrm{dom}\,(R^*)$. The axiom of choice implies that any binary relation can be uniformized by some function, but it does not specify the function. In particular, if $R \subseteqq X \times X$, we would like R^* to have a not much greater topological complexity than R. This is true in some cases. For example, The Kondo–Addison Uniformization Theorem asserts that if R is a Π^1_1 relation then it can be uniformized by a Π^1_1 relation (see Kondo [4] and Addison [1]). Another theorem which we use in this paper is:

THEOREM 0.2. *Let X, Y be two perfect Polish spaces.*

(i) *Every analytic relation $R \subseteqq X \times Y$ can be uniformized by a σ-analytic relation.*

(ii) *There is a Borel relation $R \subseteqq X \times Y$ such that R cannot be uniformized by an analytic relation and* $\mathrm{dom}\,(R) = X$ *(Indeed, R can be chosen to be $\mathcal{F}_\sigma$).*

For a proof of this theorem and for more extensive and detailed discussion of this subject see Kuratowski [5] and Moschovakis [8].

1. Spanning trees in analytic graphs

Let X be a perfect Polish space. Every graph $G=(V,E)$ in this paper is such that $V\subseteqq X$. Let $\vec{\mathrm{E}}=\{(x,y)\mid\{x,y\}\in\mathrm{E}\}$. We call G Borel, analytic, or σ-analytic if $\vec{\mathrm{E}}$ is a Borel, analytic, or σ-analytic subset of the perfect Polish space $X\times X$, respectively. Note that if $G=(V,E)$ is an analytic connected graph then $V=\{x\mid\exists y:\ (x,y)\in\vec{\mathrm{E}}\}$ is also an analytic subset of X.

LEMMA 1.1. *Let G be an analytic connected graph, and let T_0 be a σ-analytic tree in G such that $\mathrm{V}(T_0)$ is analytic. Then T_0 can be extended into a σ-analytic spanning tree T of G.*

PROOF. For every $n\in\mathbf{N}$, define by induction two analytic sets $V_n\subseteqq X$ and $R_n\subseteqq X\times X$. Let $V_0=\mathrm{V}(T_0)$. For every $n\in\mathbf{N}$, let

$$V_{n+1}=V_n\cup\left\{x\;\middle|\;(\exists y)\big[y\in V_n\wedge\{x,y\}\in\mathrm{E}(G)\big]\right\},$$

$$R_n=\{(x,y)\mid\{x,y\}\in\mathrm{E}(G),\ x\in V_{n+1},\ y\in V_n\}.$$

By Theorem 0.2(i), each R_n can be uniformized by a σ-analytic relation $R_n^*\subseteqq R_n$. For every n, define a σ-analytic relation E_n:

$$E_n=\{(x,y)\mid(x,y)\in R_n^*\wedge x\notin V_n\}.$$

Let T be the subgraph of G whose edges are:

$$\mathrm{E}(T)=\mathrm{E}(T_0)\cup\left(\bigcup_{n\in\mathbf{N}}\{\{x,y\}\mid(x,y)\in E_n\}\right).$$

Then T is σ-analytic and extends T_0. It is left to show that T is a spanning tree of G.

1. *T has no cycles:* Suppose that $(x_0,x_1,\dots,x_k)$ is a cycle in T. Since T_0 is a tree, one of the edges in the cycle does not belong to $\mathrm{E}(T_0)$. Therefore, without loss of generality, we may assume that $(x_0,x_k)\in E_{n_0}$ for some $n_0\in\in\mathbf{N}$. Hence, $x_0\in V_{n_0+1}-V_{n_0}$ and $x_k\in V_{n_0}$. This implies that $(x_0,x_1)\notin E_{n_0}$ (since $R_{n_0}^*$ is a function), $(x_0,x_1)\notin E_n$ for every $n\neq n_0$ (since $x_0\in V_{n_0+1}--V_{n_0}$), and $\{x_0,x_1\}\notin\mathrm{E}(T_0)$ (since $x_0\notin V_{n_0}\supseteqq V_0=\mathrm{V}(T_0)$). Hence, it must be that $(x_1,x_0)\in E_{n_1}$, for some $n_1\in\mathbf{N}$. Then $n_1>n_0$, since $n_1\leqq n_0$

implies that $x_0 \in V_{n_1} \subseteqq V_{n_0}$, in contradiction to $x_0 \notin V_{n_0}$. Continuing like this, we get $n_0 < n_1 < n_2 < \cdots < n_k$ such that $(x_j, x_{j-1}) \in E_{n_j}$, for every $0 < j \leqq k$. In particular, $(x_k, x_{k-1}) \in E_{n_k}$, which implies $x_k \notin V_{n_k}$. But $n_k > n_0$ implies that $x_k \notin V_{n_0}$. This contradicts $x_k \in V_{n_0}$.

2. *T is connected and it spans G:* From the connectivity of G it follows that $\mathrm{V}(G) = \bigcup_{n \in \mathbf{N}} V_n$. Let $x \in \mathrm{V}(G)$ and $n \in \mathbf{N}$. Call n *the rank of* x if n is the least such that $x \in V_n$. By induction on the rank of $x \in \mathrm{V}(G)$, we show that there is a path in T from x to some $r \in V_0$. This is obvious for $n = 0$. If x is of rank $n+1$ then $(x, y) \in R_n$ for some $y \in V_n$. Hence, $(x, y') \in R_n^*$ for some $y' \in V_n$. But $x \notin V_n$, hence $(x, y') \in E_n$, so $\{x, y'\} \in \mathrm{E}(T)$. The rank of y' must be n, hence by the induction hypothesis there is a path $(y', \dots, r)$ in T where $r \in V_0$. Then $(x, y', \dots, r)$ is a path in T from x to r.

This argument shows that $\mathrm{V}(T) = \mathrm{V}(G)$, and that T is connected (since $V_0 = \mathrm{V}(T_0)$ and T_0 is a tree). □

THEOREM 1.2. *Let G be an analytic connected graph. Let F be a σ-analytic forest in G such that $\mathrm{V}(F)$ is analytic and F has countably many components. Then F can be extended into a σ-analytic spanning tree of G.*

PROOF. By Theorem 0.1 G has a spanning tree $\widetilde{T}$ which extends F. Choose $r \in \mathrm{V}(\widetilde{T})$, and for every component $\mathrm{V}(T)$ of F, choose p_T to be the shortest path in $\widetilde{T}$ from r to $\mathrm{V}(T)$. There is only one such path, since $\widetilde{T}$ is a tree that extends F. Let $\mathrm{E}(P)$ and $\mathrm{V}(P)$ be the sets of all edges and vertices, respectively, that belong to one of those paths. There are at most countably many such paths and each path is finite, hence $\mathrm{E}(P)$ and $\mathrm{V}(P)$ are countable sets.

Now let T_0 be the subgraph of G whose edges are $\mathrm{E}(T_0) = \mathrm{E}(P) \cup \mathrm{E}(F)$. Then $\mathrm{V}(T_0) = \mathrm{V}(P) \cup \mathrm{V}(F)$. Obviously, T_0 is a σ-analytic tree in G such that $\mathrm{V}(T_0)$ is analytic. By Lemma 1.1, T_0 can be extended into a σ-analytic spanning tree T of G. But T_0 extends F, therefore, T also extends F. □

2. Minimal spanning trees in analytic weighted graphs

Our main goal in this paper is to prove the topological analog of the next purely combinatorial theorem. For simplicity, we divide the task to two parts. First, we prove the purely combinatorial theorem. Second, we prove the topological version thereof by going through the first proof and by taking care of the topological complexity part.

DEFINITION 2.1. (i) A *weighted graph* is a triple $W = (G, \mathbf{w}, \lambda)$, where G is a graph, λ is an ordinal, and $\mathbf{w} : \mathrm{E}(G) \to \lambda$ is a *weight function* (where λ is viewed as the set of all ordinals $\alpha < \lambda$).

(ii) Let $G = (V, E)$ be a graph, and let $u, v \in [V]^2$. We denote by $G(u/v)$ the graph whose edges are $(\mathrm{E}(G) - \{u\}) \cup \{v\}$.

(iii) Let $W = (G, \mathbf{w}, \lambda)$ be a weighted graph. A *minimal spanning tree* of W is a spanning tree T of G such that for every $v \in \mathrm{E}(G) - \mathrm{E}(T)$ and $u \in \mathrm{E}(T)$, if $T(u/v)$ is a spanning tree of G then $\mathbf{w}(u) \leqq \mathbf{w}(v)$. Especially, if G and λ are finite, then a minimal spanning tree is one whose total weight is smallest. Similarly a *maximal spanning tree* is defined.

(iv) Let $W = (G, \mathbf{w}, \lambda)$ be a weighted graph. For every $\beta \in \lambda$, we denote by G_β and G^β the subgraphs of G whose edges are:

$$\mathrm{E}(G_\beta) = \{ u \in \mathrm{E}(G) \mid \mathbf{w}(u) \leqq \beta \},$$

$$\mathrm{E}(G^\beta) = \{ u \in \mathrm{E}(G) \mid \mathbf{w}(u) = \beta \}.$$

We denote by $\mathcal{C}_\beta$ the collection of all components of G_β where each component is viewed as subgraph of G_β, and for every $\alpha \in \beta$ and $K \in \mathcal{C}_\beta$ let

$$\mathcal{C}_\alpha | K = \{ H \in \mathcal{C}_\alpha \mid H \text{ is a subgraph of } K \}.$$

THEOREM 2.2. *Let $W = (G, \mathbf{w}, \lambda)$ be a connected weighted graph. Then W has a minimal spanning tree.*

PROOF. By transfinite induction on $\beta \in \lambda$, we construct a spanning tree T_K for every $K \in \mathcal{C}_\beta$ such that if $\alpha < \beta$ and $H \in \mathcal{C}_\alpha | K$ then T_H is a subtree of T_K. First, let $\beta = 0$ and let $K \in \mathcal{C}_0$. By Theorem 0.1, K has a spanning tree T_K. Next, suppose $\beta = \alpha + 1$ and let $K \in \mathcal{C}_\beta$. By the induction hypothesis, there is a spanning tree T_H for every $H \in \mathcal{C}_\alpha | K$. Let $\widetilde{F}_\alpha = \bigcup \{ T_H \mid H \in \mathcal{C}_\alpha | K \}$. Every $H_1, H_2 \in \mathcal{C}_\alpha | K$ are vertex-disjoint, hence, T_{H_1} and T_{H_2} are vertex-disjoint. Therefore, $\widetilde{F}_\alpha$ is a forest in K. By Theorem 0.1, $\widetilde{F}_\alpha$ can be extended into a spanning tree T_K of K. Finally, suppose that β is a limit ordinal and let $K \in \mathcal{C}_\beta$. For every $\alpha < \beta$, let $\widetilde{F}_\alpha$ be as above. Then $\widetilde{F}_\alpha$ is a forest in K, and if $\alpha < \alpha' < \beta$ then the induction hypothesis implies that $\widetilde{F}_\alpha$ is a subforest of $\widetilde{F}_{\alpha'}$. Hence $F = \bigcup_{\alpha<\beta} \widetilde{F}_\alpha$ is a forest in K. By Theorem 0.1, F can be extended to a spanning tree T_K of K.

Now, for every $\beta \in \lambda$ let $F_\beta = \bigcup \{ T_K \mid K \in \mathcal{C}_\beta \}$. Then F_β is a forest in G_β, and if $\alpha < \beta$ then F_α is a subforest of F_β (it follows from the way the T_K's were constructed). Hence $F = \bigcup_{\beta<\lambda} F_\beta$ is a forest in G. By Theorem 0.1, F can be extended into a spanning tree T of G.

We claim that T is a minimal spanning tree of W. For suppose that $v \in \mathrm{E}(G) - \mathrm{E}(T)$, $u \in \mathrm{E}(T)$, and $T(u/v)$ is a spanning tree of G. We need to show that $\mathbf{w}(u) \leqq \mathbf{w}(v)$. Let $\alpha = \mathbf{w}(v)$ and $\beta = \mathbf{w}(u)$. Assume that $\alpha < \beta$ toward a contradiction. Let $H \in \mathcal{C}_\alpha$ be such that $v \in \mathrm{E}(H)$. Then $T_H \subseteqq F_\alpha \subseteqq T$. Therefore $T_H(u/v) \subseteqq F_\alpha(u/v) \subseteqq T(u/v)$. Hence $T_H(u/v)$ is a forest in H. But $u, v \notin \mathrm{E}(T_H)$ (since $\mathbf{w}(u) = \beta > \alpha$ and $v \notin T$), which implies that $T_H(u/v) = T_H \cup \{v\} \supset T_H$. A contradiction to the fact that T_H is a spanning tree of H. $\square$

Before we turn to the topological version of the last theorem, we prove a lemma that seems to be a converse of it.

LEMMA 2.3. *Let $W = (G, \mathbf{w}, \lambda)$ be a connected weighted graph and let T be a minimal spanning tree of W. Then for every $\beta \in \lambda$ and every $K \in \mathcal{C}_\beta$, $T \cap K$ is a spanning tree of K.*

PROOF. Toward a contradiction, assume that there are $\beta \in \lambda$ and $K \in \in \mathcal{C}_\beta$ such that $\widetilde{T} = T \cap K$ is not a spanning tree of K. Then there is an edge $v = \{x_0, x_1\} \in \mathrm{E}(K) - \mathrm{E}(T)$ such that $\widetilde{T} \cup \{v\}$ is still a forest. However, $T \cup \{v\}$ has a cycle that contains the edge v. Let $c = (x_0, x_1, \ldots, x_n)$ be this cycle. For convenience, denote $x_{n+1} = x_0$. Let $1 \leqq j \leqq n$ be the least such that $\{x_j, x_{j+1}\} \notin \mathrm{E}(K)$. There is such a j for otherwise c is a cycle in K and therefore c is a cycle in $(T \cup \{v\}) \cap K = \widetilde{T} \cup \{v\}$, but $\widetilde{T} \cup \{v\}$ is a forest — a contradiction.

Clearly, $T\big(\{x_j, x_{j+1}\}/v\big)$ is a spanning tree of G. Now, $\{x_j, x_{j+1}\} \notin \notin \mathrm{E}(K)$ while $x_j \, \mathrm{V}(K)$ (since $\{x_{j-1}, x_j\} \in \mathrm{E}(K)$). Therefore, $\{x_j, x_{j+1}\} \notin \notin \mathrm{E}(G_\beta)$ Hence $\mathbf{w}\big(\{x_j, x_{j+1}\}\big) > \beta$, while $\mathbf{w}(v) \leqq \beta$. A contradiction to the minimality of T. □

Now we proceed to the topological version of Theorem 2.2. The proof of Theorem 2.2 can be described in terms of a greedy algorithm. The way we constructed T was straightforward without any special obstructions. But in order to obtain a σ-analytic T, one cannot use a greedy procedure. The choice of T_K for every $K \in \mathcal{C}_\beta$ cannot be arbitrary and has to be made according to Theorem 1.2.

THEOREM 2.4. *Let $W = (G, \mathbf{w}, \lambda)$ be a connected weighted graph where G is an analytic graph, λ is a countable ordinal, and for every $\beta \in \lambda$, the graph G^β is analytic and has countably many components. Then W has a σ-analytic minimal spanning tree.*

PROOF. For every $\beta \in \lambda$ we have the following facts.

1. $G_\beta = \bigcup_{\alpha \leqq \beta} G^\alpha$.

2. G_β is an analytic graph (by Fact 1).

3. $\mathcal{C}_\beta$ is countable (by Fact 1).

4. For every $\alpha < \beta$ and $K \in \mathcal{C}_\beta$, $\mathcal{C}_\alpha | K$ is countable (by Fact 3).

5. For every $K \in \mathcal{C}_\beta$, K is an analytic graph. The proof is: For every $n \in \mathbf{N}$, define by induction a set $E_n \subseteqq X \times X$ as follows: $E_0 = \{u\}$ where $u \in \mathrm{E}(K)$ is arbitrary, and

$$E_{n+1} = \mathrm{E}(G_\beta) \cap \big\{ \{x, y\} \mid \exists z : \ \{y, z\} \in E_n \big\}.$$

Then $\mathrm{E}(K) = \bigcup_{n \in \mathbf{N}} E_n$ and, by Fact 2, K is analytic.

We follow the proof of Theorem 2.2. For every $K \in \mathcal{C}_\beta$, we construct a σ-analytic spanning tree T_K using the same transfinite induction as in Theorem 2.2. First, let $\beta = 0$ and let $K \in \mathcal{C}_0$. By Fact 5 and Theorem 1.2, K has a σ-analytic spanning tree T_K. Next, suppose that $\beta = \alpha + 1$ and let $K \in \mathcal{C}_\beta$. By the induction hypothesis, every $H \in \mathcal{C}_\alpha | K$ has a σ-analytic spanning tree T_H. Therefore, by Fact 4, $\widetilde{F}_\alpha = \bigcup \{ T_H \mid H \in \mathcal{C}_\alpha | K \}$ is a σ-analytic forest in K with countably many components and $\mathrm{V}(\widetilde{F}_\alpha)$ is analytic, since $\mathrm{V}(\widetilde{F}_\alpha) = \bigcup \{ \mathrm{V}(T_H) \mid H \in \mathcal{C}_\alpha | K \}$ and, by Fact 5, $\mathrm{V}(T_H) = \mathrm{V}(H)$ is analytic for every $H \in \mathcal{C}_\alpha | K$. Therefore, by Fact 5 and Theorem 1.2, $\widetilde{F}_\alpha$ can be extended into a σ-analytic spanning tree T_K of K.

Finally, suppose β is a limit ordinal and let $K \in \mathcal{C}_\beta$. For every $\alpha < \beta$, $\widetilde{F}_\alpha$ is a σ-analytic forest in K with countably many components and $\mathrm{V}(\widetilde{F}_\alpha)$ is analytic. Therefore, $F = \bigcup_{\alpha<\beta} \widetilde{F}_\alpha$ is a σ-analytic forest in K with countably many components and $\mathrm{V}(F)$ is analytic (since β is countable). Therefore by Fact 5 and Theorem 1.2, F can be extended into a σ-analytic spanning tree T_K of K. This finishes the construction of the σ-analytic trees T_K.

For every $\beta \in \lambda$, by Fact 3, $F_\beta = \bigcup \{ T_K \mid K \in \mathcal{C}_\beta \}$ is a σ-analytic forest in G_β with countably many components and, by Fact 5, $\mathrm{V}(F_\beta)$ is analytic. Let $F = \bigcup_{\beta\in\lambda} F_\beta$. Then F is a σ-analytic forest in G with countably many components and $\mathrm{V}(F)$ is analytic (since λ is countable). Therefore by Theorem 1.2, F can be extended into a σ-analytic spanning tree T of G. By Theorem 2.2, T is a minimal spanning tree of W. □

REMARK 2.5. Theorem 2.2 does not hold for maximal spanning trees as the following simple counterexample shows (Fig. 1). However, Theorem 2.4 is still true for a maximal spanning tree if λ is a finite ordinal: define $W' = (G, \mathbf{w}', \lambda)$ where $\mathbf{w}'(v) = \lambda - \mathbf{w}(v) - 1$ for every $v \in \mathrm{E}(G)$, then a minimal spanning tree of W' is a maximal spanning tree of W.

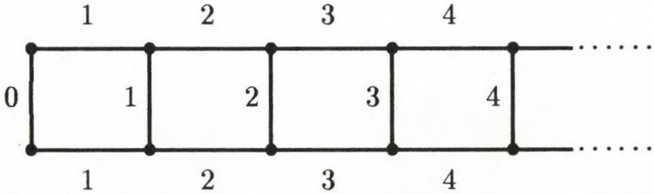

Fig. 1

3. Optimality

In this section we show that Theorem 2.4 is optimal. First, we show that an analytic minimal spanning tree of W does not always exists. Second, we show that if G^β has uncountably many components for some $\beta \in \lambda$ or λ is uncountable, then a σ-analytic minimal spanning tree of W does not always exist.

THEOREM 3.1. *Let X be a perfect Polish space. There is a Borel connected graph $G = (X, E)$ that has no analytic spanning tree.*

PROOF. Let $x_0 \in X$ and let $A, B \subseteqq X$ be two disjoint perfect Polish subspaces such that $x_0 \notin A \cup B$. Let $R \subseteqq A \times B$ be as in Theorem 0.2(ii). Define G to be the graph whose edges are:

$$\mathrm{E}(G) = \big\{ \{x, y\} \mid (x = x_0 \text{ and } y \notin A) \text{ or } (x, y) \in R \big\}.$$

Then G is a Borel connected graph (in fact, G is $\mathcal{F}_\sigma$) such that $\mathrm{V}(G) = X$ (since dom $(R) = A$). We claim that G does not have an analytic spanning tree. Suppose, for contradiction, that T is an analytic spanning tree of G. For every $n \in \mathbf{N}$, define by induction an analytic set $R_n \subseteqq X \times X$ as follows:

$$R_0 = \big\{ (x, x_0) \mid x \in B \big\},$$

$$R_{n+1} = \big\{ (x, y) \mid \exists z : \ (y, z) \in R_n \big\}.$$

Let $R^* = \bigcup_{n \in \mathbf{N}} R_{2n+1}$. Then R^* is analytic and we leave it to the reader to check that R^* uniformizes R. A contradiction to Theorem 0.2(ii). □

Now, let $W = (G, \mathbf{w}, \lambda)$ where G is the graph defined in Theorem 3.1 and $\mathbf{w}$ is a constant function. Then W satisfies the conditions of Theorem 2.4, but from Theorem 3.1, W has no analytic minimal spanning tree.

Let $\mathbf{C}$ be the Cantor space, that is: $\mathbf{C} = \big\{ x \mid x : \mathbf{N} \to \{0, 1\} \big\}$ with the metric $d(x, y) = \sum_{n=0}^{\infty} |x(n) - y(n)| \cdot 2^{-n-1}$. Clearly, $\mathbf{C}$ is a perfect Polish space. For every $x, y \in \mathbf{C}$ define $x \sim y$ iff x and y differ in at most finitely many coordinates. Clearly, this is an equivalence relation with uncountably many equivalence classes. The following two well-known theorems are needed (see Moschovakis [8]).

THEOREM 3.2 (Vitali). *If a set $A \subseteqq \mathbf{C}$ contains exactly one element from each equivalence class of $\sim$, then A is not measurable.*

THEOREM 3.3. *Every analytic subset of $\mathbf{C}$ is measurable. Therefore, every σ-analytic set is measurable.*

THEOREM 3.4. *There is a Borel graph $G = (\mathbf{C}, E)$ and a Borel forest $F = (\mathbf{C}, E')$ in G with uncountably many components such that F cannot be extended into a σ-analytic spanning tree of G.*

PROOF. Define $F = (\mathbf{C}, E')$ to be the graph such that $\{x,y\} \in E'$ iff there is $n \in \mathbf{N}$ such that

(i) For every $i < n$, $x(i) = y(i) = 0$.

(ii) $x(n) = 0$ and $y(n) = 1$.

(iii) For every $i > n$, $x(i) = y(i)$.

Then F is a Borel forest (in fact F is $\mathcal{F}_\sigma$). The components of F are the same as the equivalence classes of $\sim$. Therefore, F has uncountably many components. Now, define $G = (\mathbf{C}, E)$ to be the graph whose edges are:

$$E = E' \cup \{\{\vec{0}, y\} \mid \vec{0} \neq y \in \mathbf{C}\}.$$

Then G is a Borel connected graph (in fact G is $\mathcal{F}_\sigma$) and F is a forest in G. We claim that F cannot be extended into a σ-analytic spanning tree of G. Assume by contradiction that T is a σ-analytic spanning tree of G that extends F. Let

$$A = \{\vec{0}\} \cup \{x \in \mathbf{C} \mid \{x, \vec{0}\} \in \mathrm{E}(T) \text{ and } x \not\sim \vec{0}\}.$$

For every $x \in \mathbf{C}$, let $F[x]$ be the set of vertices of the component-tree of F to which x belongs. Clearly, $A \cap F[\vec{0}] = \{\vec{0}\}$. If $x \notin F[\vec{0}]$ then there is a path $(x_0, \dots, x_n, \vec{0})$ in T such that $x_0 = x$. Clearly, $x_i \in F[x]$ for every $i \leqq n$. Therefore, $x_n \in A \cap F[x]$ (since $\{x_n, \vec{0}\} \in \mathrm{E}(T)$). We claim that $A \cap F[x] = \{x_n\}$. For if there is $x_n \neq y \in A \cap F[x]$ then there is a path $(x_n, y_1, \dots, y_n, y)$ in F and $(x_n, y_1, \dots, y_n, y, \vec{0})$ would be a cycle in the tree T.

Therefore, A is the same as in Theorem 3.2. Hence, A is not measurable. But A is σ-analytic (since T is), therefore A is measurable. A contradiction. □

Now, let $W = (G, \mathbf{w}, \lambda)$ where G is the graph defined in Theorem 3.4 and $\mathbf{w}(v) = 0$ if $v \in \mathrm{E}(F)$, $\mathbf{w}(v) = 1$ if $v \notin \mathrm{E}(F)$. Then W satisfies all the conditions of Theorem 2.4 except the condition that G^β has countably many components for each $\beta \in \lambda$ (since G^0 has uncountably many components). But from Theorem 3.4, W has no σ-analytic minimal spanning tree, since every minimal spanning tree T must extend F: otherwise, suppose that $u \in \mathrm{E}(F) - \mathrm{E}(T)$. Then $\mathrm{E}(T) \cup \{u\}$ has a cycle. Let v be an edge in this cycle such that $v \notin \mathrm{E}(F)$ (since F is a forest). It is easy to see that $T(v/u)$ is a spanning tree. But $\mathbf{w}(u) = 0 < \mathbf{w}(v) = 1$. A contradiction to the minimality of T.

Finally, let $W = (G, \mathbf{w}, \lambda)$ where G is as in Theorem 3.4, $\lambda = 2^{\aleph_0} + 1$, $\mathbf{w}(v) = 2^{\aleph_0}$ iff $v \notin \mathrm{E}(F)$ and $\mathbf{w}(u) = \mathbf{w}(v)$ iff u and v belong to the same component-tree of F. Then W satisfies all the conditions of Theorem 2.4 except the condition that λ is countable. But from Theorem 3.4, W has no σ-analytic minimal spanning tree, since every minimal spanning tree T must extend F: let T_0 be a component-tree of F. It is easy to see that $T_0 \in \mathcal{C}_{\beta_0}$

for some $\beta_0 \in \lambda$. By Lemma 2.3 $T \cap T_0$ is a spanning tree of T_0. But T_0 is a tree, hence $T \cap T_0 = T_0$. Therefore $T_0 \subseteqq T$.

Problems. 1. Suppose $G = (X, E)$ is a co-analytic connected graph. Does G have a σ-analytic spanning tree?

2. Suppose $G = (X, E)$ is a co-analytic (or even Borel) connected graph. Does G have a co-analytic spanning tree?

3. A graph G is said to be *regular* if all of its vertices have the same degree. Let G be a regular connected graph. Andersen and Thomassen [3] used the axiom of choice to show that G has a regular spanning tree. One may ask: If $G = (X, E)$ is analytic (or even Borel), does G have a σ-analytic regular spanning tree?

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ON THE CONTROLLABILITY OF A STRING WITH RESTRAINED CONTROLS

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I. Introduction

Consider a string along the segment [0,1] fixed at the two ends 0 and 1 and controlled at some point $0 < a < 1$ with a function $u(t)$. If we denote by $y(x,t)$ the distance of the point x of the string from the equilibrium state at time t, then we have the equation

(1/a) $$\rho(x)y_{tt} = y_{xx} + \delta(x-a)u(t), \quad x \in (0,1), \quad t > 0$$

with boundary condition

(1/b) $$y(0,t) = y(1,t), \quad t > 0$$

and with initial conditions

(2) $$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x).$$

Here $\rho \in C^2[0,1]$ is the linear density of the substance, $\rho(x) > 0$ $(0 \leqq x \leqq 1)$, the control $u(\cdot) \in \Omega_T \subset L^2(0,T)$ belongs to some subset Ω_T in the space $L^2(0,T)$.

The control problem of this system was investigated by many authors; see e.g. [1], [2], [3], [4]. For the controllability (investigated by A. G. Butkovskii [2]) the question is to find conditions for the position of the point $0 < a < 1$ that for any initial conditions

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x)$$

given in some function space, the string can be relaxed in a finite time $T < \infty$, i.e. the control $u(\cdot)$ can be given such that

$$y(x,T) = y_t(x,T) = 0.$$

The reachable movement states question (investigated in [3], [5], [6] in recent years) refers to the structure of the reachability set $D_a(T) := \Big\{ \big(y(\cdot,T),$

$y_t(\cdot,T)\big): u(t)$ runs over some space of controls$\Big\}$. Many interesting results were obtained by using the method of Riesz's bases. It should be noted that until now the authors investigated this problem only in the case when the controls $u(\cdot)$ run over the full space of controls. In this paper we shall be concerned with the controllability of the above system in the case when the controls belong only to some subset of the space of controls. We emphasize that for investigating this system we will use the so called discretization method. This method can be seen in some previous papers [7], [8], [9].

II. Definitions, notations

We need the following notations and definitions.

DEFINITION 1. We say that $y \in L^2\big([0,1]\times[0,T]\big)$ is a solution of (1/a)–(1/b) if for any $z \in C^2\big([0,1]\times[0,T]\big)$ such that $z(0,t)=z(1,t)=0$ $(0 \leqq t \leqq T)$, $z(\cdot,T)=z_t(\cdot,T)=0$ the equation

$$\int_0^1\int_0^T y(\rho z_{tt} - z_{xx})\,dx\,dt = \int_0^T z(a,t)u(t)\,dt$$

holds.

As we know (see I. Joó [6]) for every $y_0 \in H^1(0,1)$, $y_1 \in L^2(0,1)$, $\rho \in C^2(0,1)$, $\rho > 0$ and any control $u(\cdot) \in L^2(0,T)$ the system (1/a), (1/b), (2) has a unique solution $y(x,t) = \sum_{n=1}^{\infty} c_n(t)v_n(x)$; this sum converges in $L^2\big((0,1)\times(0,T)\big)$, further the series can be differentiated term by term. Moreover

$$c_n(t) = c_{n,0}\cos\sqrt{\lambda_n}t + c'_{n,0}\frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} + v_n(a)\int_0^t u(\tau)\cdot\frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}}\,d\tau$$

where

$$y_0(x) = \sum_{n=1}^{\infty} c_{n,0}v_n(x), \quad y_1(x) = \sum_{n=1}^{\infty} c'_{n,0}v_n(x).$$

Here $\{v_n\}$ are the solutions of the boundary value problem

$$v''_n(x) + \lambda_n\rho(x)v_n(x) = 0, \quad v_n(0) = v_n(1) = 0.$$

Define the following reachability set:

$$D_a(T) := \Big\{ \big(y(\cdot,T),\, y_t(\cdot,T)\big) : u \in L^2(0,T) \Big\}$$

and the corresponding set in l_2:

$$B_a(t) := \Big\{ \Big(\sqrt{\lambda_n} c_n(T) + i c_n'(T)\Big) : u \in L^2(0,T) \Big\}.$$

From I. Joó [6] and M. Horváth [5] we know that there exists

$$I: H^1(0,1) \oplus L^2(0,1) \to l_2,$$

$$\big(y(\cdot,T), y_t(\cdot,T)\big) \mapsto \Big(\sqrt{\lambda_n} c_n(T) + i c_n'(T)\Big)$$

which is an isomorphism between l_2 and $H^1(0,1) \oplus L^2(0,1)$ for every a and T. So instead of investigating this system in $H^1(0,1) \oplus L^2(0,1)$ we can do it in l_2 and as we shall see later this gives us many advantages in our work.

DEFINITION 2. For a given controlled set $\Omega_T \subset L^2(0,T)$ we define

$$D_a(T,\Omega_T) := \Big\{ \big(y(\cdot,T), y_t(\cdot,T)\big) \in H^1(0,1) \oplus L^2(0,1) : u \in \Omega_T \Big\},$$

$$B_a(T,\Omega_T) := \Big\{ \big(\lambda_n c_n(T) + i c_n(T)\big)_{n=1}^{\infty} \in l_2 : u \in \Omega_t \Big\},$$

$$S_a(T,\Omega_T) := \Big\{ (y_0, y_1) \in H^1(0,1) \oplus L^2(0,1) : \quad \text{for every} \quad \varepsilon > 0$$

$$\text{there exists} \quad u \in \Omega_T \quad \text{such that} \quad \Big\| \big(y(\cdot,T), y_t(\cdot,T)\big) \Big\| < \varepsilon \Big\},$$

and

$$S_a(\Omega_\infty) := \bigcup_{T>0} S_a(T,\Omega_T).$$

REMARK. The set $S_a(T,\Omega_T)$ is an approximately null-controllability set in $H^1(0,1) \oplus L^2(0,1)$ after time T under controls $u \in \Omega_T$, $S_a(\Omega_\infty)$ is an approximately null-controllability set after unbounded time.

We say that the system (1/a)–(1/b) is approximately locally controllable (ALC) if

$$(0,0) \in \operatorname{int} S_a(\Omega_\infty)$$

and approximately globally controllable (AGC) if

$$H^1(0,1) \oplus L^2(0,1) \equiv S_a(\Omega_\infty).$$

III. Discrete-time systems

In what follows we will show some properties about the controllability of discrete-time systems (see N. K. Son [10] and N. V. Su [9]). Consider the discrete-time system

$$(A, B, \Omega)\colon x_{n+1} = Ax_n + Bu_n, \quad x_n \in X, \quad u_n \in \Omega \subset U,$$

where X, U are Hilbert spaces; $A \in L(X, X)$, $B \in L(U, X)$; Ω is an arbitrary subset in U for which $0 \in \Omega$.

Let $U^n = U \times U \times \ldots \times U$, where the direct product is taken n times, and let us consider the operator $F_n\colon U^n \to X$ defined by

$$F_n\big(u^{(n)}\big) = A^{n-1}Bu_0 + A^{n-2}Bu_1 + \ldots + Bu_{n-1}$$

where $u^{(n)} = (u_0, u_1, \ldots, u_{n-1}) \in U^n$. Clearly,

$$F_n(U^n) = A^{n-1}BU + A^{n-2}BU + \ldots + BU,$$

and

$$F_n(\Omega^n) = A^{n-1}B\Omega + A^{n-2}B\Omega + \ldots + B\Omega,$$

where $\Omega^n = \Omega \times \Omega \times \ldots \times \Omega$ (Here the direct product is also taken n times.)

We define the approximate controllability set as

$$S_n(\Omega) := \Big\{x \in X\colon -A^n x \in \overline{F_n(\Omega^n)}\Big\}$$

and

$$S(\Omega) := \bigcup_{n=1}^{\infty} S_n(\Omega).$$

We remark that

$$S_n(\Omega) := \Big\{x \in X\colon \quad \text{for every} \quad \varepsilon > 0 \quad \text{there exists} \quad u^{(n)} \in \Omega^n \quad \text{such that} \quad \Big\|A^n x + F_n\big(u^{(n)}\big)\Big\| < \varepsilon\Big\}$$

and

$$S(\Omega) := \Big\{x \in X\colon \quad \text{for every} \quad \varepsilon > 0 \quad \text{there exists} \quad n \quad \text{and} \quad u^{(n)} \in \Omega^n \quad \text{such that} \quad \Big\|A^n x + F_n\big(u^{(n)}\big)\Big\| < \varepsilon\Big\}.$$

Definition 3. We say the system (A, B, Ω) is approximately locally controllable (ALC) if

$$0 \in \operatorname{int} S(\Omega)$$

and approximately globally controllable (AGL) if

$$X \equiv S(\Omega).$$

Let further

$$\widehat{S}_n(\Omega) = \left\{ x \in X : -A^n x \in F_n(\Omega^n) \right\}$$

and

$$\widehat{S}(\Omega) = \bigcup_{n=1}^{\infty} \widehat{S}_n(\Omega).$$

We say that the system (A, B, Ω) is locally controllable (LC) if

$$0 \in \operatorname{int} \widehat{S}(\Omega)$$

and globally controllable (GC) if $X \equiv \widehat{S}(\Omega)$.

We denote the spectrum of the operator A by $\sigma(A)$.

Assumption 1. $\sigma(A) \subset \left\{ \lambda \in \mathbf{C} : |\lambda| = 1 \right\}$, $\|A\| \leqq 1$.

Now we are in position to state the following theorem.

Theorem 1. *Consider the system (A, B, Ω) with the condition that Ω is convex and $0 \in \Omega$. Assume that Assumption 1 is fulfilled. The system (A, B, Ω) is AGC if and only if*

(a) *there is no eigenvector x^* of A^*, $A^* x^* = \lambda x^*$, $\lambda > 0$, such that $\langle x^*, B\Omega \rangle \geqq 0$,*

(b) *there is no eigenvector x^* of A^*, $A^* x^* = \lambda x^*$, λ is complex, such that $\langle x^*, B\Omega \rangle = 0$.*

Proof. *Necessity.* Assume that the system (A, B, Ω) is AGC. In order to prove the conditions (a) and (b), we assume the contrary: let us suppose there exists $\lambda > 0$ such that $A^* x^* = \lambda x^*$ and

$$\langle x^*, B\Omega \rangle \geqq 0$$

or there exists complex λ such that $A^* x^* = \lambda x^*$ and

$$\langle x^*, B\Omega \rangle = 0.$$

We show that both cases lead to a contradiction. We can assume, without loss of generality, that $\|x^*\| = 1$. Let x_0 be a point in X such that $\langle x^*, x_0 \rangle =$

$= 1$. Then for any $\delta > 0$ and any trajectory $\{x_n\}$ of (A, B, Ω) steered from x_0 we have, in both cases,

$$\langle x^*, x_{n+1}\rangle = \langle x^*, A^n(\delta x_0)\rangle + \langle x^*, A^{n-1}Bu_0\rangle + \ldots + \langle x^*, Bu_{n-1}\rangle =$$

$$= \delta\lambda^n\langle x^*, x_0\rangle + \sum_{i=1}^{n} \lambda^{n-i}\langle x^*, Bu_{i-1}\rangle = \delta\lambda^n + \sum_{i=1}^{n} \lambda^{n-i}\langle x^*, Bu_{i-1}\rangle.$$

Since $|\lambda| = 1$ for all $\lambda \in \sigma(A)$ (from Assumption 1) it follows from the above equality that

$$\|x_{n+1}\| \geqq |\langle x^*, x_{n+1}\rangle| \geqq \delta|\lambda|^n = \delta \quad \text{for all} \quad n.$$

Thus, in any 2δ neighbourhood of the origin there exists a point δx_0 which cannot be steered to the $\delta/2$-neighbourhood of the origin. This means that the system (A, B, Ω) is not AGC, a contradiction.

Sufficiency. Assume that for the system (A, B, Ω) the conditions (a) and (b) are fulfilled. We will first show that

$$\overline{S(\Omega)} \equiv X.$$

Assume the contrary: there exists a point $z \in X$ for which $z \notin \overline{S(\Omega)}$. Since a point z is a compact set and $\overline{S(\Omega)}$ is a closed, convex set (because of the convexity of Ω), we have by the Separation Theorem that there exists a hiperspace which separates the point z and the set $\overline{S(\Omega)}$, that is there exist $\alpha \in R$ and $f^* \in X^*$ such that

$$S(\Omega) \subset \{x \in X : f^*(x) \leqq \alpha\} \tag{3}$$

and

$$f^*(z) > \alpha. \tag{4}$$

Consider the following space:

$$X^0 = \{x \in X : f^*(x) = 0\}.$$

As we known X^0 is a subspace of X and $\operatorname{codim} X^0 = 1$. Therefore

$$X = X^0 \oplus X^1$$

(here $\oplus$ denotes direct sum) and $\dim X^1 = 1$. Let $P: X \to X^1$ be the projection on X^1 along X^0. Let $B_1: U \to X^1$ be defined as

$$B_1 u = PBu, \quad u \in U.$$

For $x = x^0 + x^1$, where $x \in X$, $x^0 \in X^0$, $x^1 \in X^1$ we define $A_1: X^1 \to X^1$ as $PA = A_1 P$, that is $PAx = A_1 Px$ for all $x \in X$. Clearly, A_1 and B_1 are linear, bounded operators. Consider the following system:

$$(A_1, B_1, \Omega)\colon\ x^1_{n+1} = A_1 x^1_n + B_1 u_n,\ x^1_n \in X^1,\ u_n \in \Omega \subset U.$$

We show that the system (A_1, B_1, Ω) is LC. Assume the contrary. Then by Theorem 2.2 of [11], either there exists a non-zero eigenvector f of A_1^* with an eigenvalue $\lambda > 0$ supporting $B_1\Omega$, or there exists a non-zero eigenvector f of A_1^* with a complex eigenvalue $\lambda \neq 0$ orthogonal to $B_1\Omega$. We show that both cases lead to a contradiction. Note first that

$$\langle f, B_1 u\rangle = \langle f, PBu\rangle = \langle P^* f, Bu\rangle$$

and

$$A^*(P^* f) = (PA)^* f = (A_1 P)^* f = P^* A_1^* f = \lambda(P^* f).$$

Since P is onto, $\|P^* f\| \geqq \alpha\|f\|$ for some $\alpha > 0$ (see Theorem 4.15 of [12].) Consequently, under the above hypothesis, $P^* f$ is a non-zero eigenvector of A^* supporting, in the first case, or orthogonal, in the second case, to $B\Omega$. This contradicts conditions (a) and (b). So the system (A_1, B_1, Ω) is LC. Since $\|A\| < 1$ (by Assumption 1), the system (A_1, B_1, Ω) is GC, too (see Lemma 1 of [13]). It means that for this system (A_1, B_1, Ω) $\widehat{S}(\Omega) \equiv X^1$. This contradicts (3) and (4), and shows that $\overline{S(\Omega)} \equiv X$. So for an arbitrary $x \in X$ and for any $\varepsilon > 0$ there exist $n \in N$ and $y \in S_n(\Omega)$ such that

$$\|x - y\| \leqq \frac{\varepsilon}{2}.$$

Since $y \in S_n(\Omega)$ there exist $u_0, u_1, \ldots, u_{n-1} \in \Omega$ such that

$$\|A^n y + A^{n-1} B u_0 + A^{n-2} B u_1 + \ldots + B u_{n-1}\| \leqq \frac{\varepsilon}{2}.$$

By Assumption 1 we have

$$\begin{aligned}
&\|A^n x + A^{n-1} B u_0 + A^{n-2} B u_1 + \ldots + B u_{n-1}\| = \\
&= \|A^n x - A^n y + A^n y + A^{n-1} B u_0 + A^{n-2} B u_1 + \ldots + B u_{n-1}\| \leqq \\
&\leqq \big\| A^n (x - y) + A^n y + A^{n-1} B u_0 + A^{n-2} B u_1 + \ldots + B u_{n-1}\big\| \leqq \\
&\leqq \|A\|^n \|x - y\| + \|A^n y + A^{n-1} B u_0 + A^{n-2} B u_1 + \ldots + B u_{n-1}\| \leqq \\
&\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means that the system (A, B, Ω) is AGC. This completes the proof of Theorem 1.

PROPOSITION 1. *Assume that for the system* (A,B,Ω) *Assumption* 1 *is fulfilled. Then the system* (A,B,Ω) *is AGC if and only if it is ALC.*

PROOF. The necessity of the Proposition is immediate. Assume that the system (A,B,Ω) is ALC. Since $0\in \operatorname{int} S(\Omega)$, $S(\Omega)$ contains some ball B_μ. Taking a number N such that $\frac{\|x\|}{\mu\cdot N}<1$ for an arbitrary $x\in X$ and for any $\varepsilon>0$, we have that

$$A^k\Big(\frac{x}{N}\Big)\in B_\mu \quad \text{for} \quad k=0,1,2,\dots .$$

Since $\frac{x}{N}\in S(\Omega)$, we can find $u_i^1\in\Omega$, $i=1,2,\dots,n_1$ such that

$$\Big\|A^{n_1}\Big(\frac{x}{N}\Big)+A^{n_1-1}Bu_1^1+\ldots+Bu_{n_1}^1\Big\|\leqq\frac{\varepsilon}{N}.$$

Analogously, using the fact that $A^k\big(\frac{x}{N}\big)\in S(\Omega)$ for all k, we can write the inequalities

$$\Big\|A^{n_2}\Big(A^{n_1}\Big(\frac{x}{N}\Big)\Big)+A^{n_2-1}Bu_1^2+\ldots+Bu_{n_2}^2\Big\|\leqq\frac{\varepsilon}{N}$$

and

$$\Big\|A^{n_N}\Big(A^{n_1+n_2+\ldots+n_{N-1}}\Big(\frac{x}{N}\Big)\Big)+A^{n_N-1}Bu_1^N+\ldots+Bu_{n_N}^N\Big\|\leqq\frac{\varepsilon}{N},$$

where $u_i^j\in\Omega$, $j=n_2,n_3,\dots,n_N$, $i=1,2,\dots,n_k$; $k=2,3,\dots,N$. Denote $p=n_1+n_2+\ldots+n_N$. Applying the operators $A^{p-n_1},A^{p-n_1-n_2}$, $,\dots,A^{p-n_1-n_2-\ldots-n_{N-1}}$, in turn, to each of the above inequalities, except the last one, and summing then up, we obtain

$$\|A^px+A^{p-1}Bu_1^1+\ldots+A^{p-n_1}Bu_{n_1}^1+A^{p-n_1-1}Bu_1^2+\ldots+A^{p-n_1-n_2}Bu_{n_2}^2+$$
$$+\ldots+A^{n_N-1}Bu_1^N+\ldots+Bu_n^N\|\leqq\varepsilon.$$

This means that the system (A,B,Ω) is AGC. The proof is complete.

IV. Discretization

In this part we introduce the discretization method. We will show that the above system described by (1/a), (1/b) and (2) is equivalent with some discrete-time system in the sense of the controllability.

Let $\Omega \subset \bigcup_{T>0} L^2(0,T)$. An $u(\cdot) \in \Omega$ is an admissible control if Ω has the following properties:

(a) $\Omega \cap L^2(0,1)$ is convex; denote $\widetilde{\Omega} := \Omega \cap L^2(0,1)$,

(b) $0 \in \Omega$,

(c) if $u(\cdot) \in \Omega$ then $u_i(\cdot) \in L^2(0,1)$ defined by $u_i(\Theta) = u(i+\Theta)$, $\Theta \in [0,1]$ is such that $u_i \in \Omega$ for each $i \in N$.

Let us return to the system (1/a), (1/b) and (2). It is easy to see that

$$\sqrt{\lambda_n} c_n(T) + ic'_n(T) = c_{n,0}\sqrt{\lambda_n} e^{-i\sqrt{\lambda_n}\cdot T} - ic'_{n,0} e^{-i\sqrt{\lambda_n}\cdot T} -$$

$$-iv_n(a) \int_0^T u(\tau) e^{-i\sqrt{\lambda_n}(T-\tau)}\, d\tau.$$

From this we have

$$\sqrt{\lambda_n} c_n(T+1) + ic'_n(T+1) = c_{n,0}\sqrt{\lambda_n} e^{-i\sqrt{\lambda_n}(T+1)} -$$

$$-ic'_{n,0} - e^{-i\sqrt{\lambda_n}(T+1)} - iv_n(a) \int_0^{T+1} u(\tau) \cdot e^{-i\sqrt{\lambda_n}(T+1-\tau)}\, d\tau =$$

$$= e^{-i\sqrt{\lambda_n}} \Big(c_{n,0}\sqrt{\lambda_n} e^{-i\sqrt{\lambda_n}\cdot T} - ic'_{n,0} e^{-i\sqrt{\lambda_n}\cdot T} -$$

$$-iv_n(a) \int_0^T u(\tau) e^{-i\sqrt{\lambda_n}(T-\tau)}\, d\tau \Big) + iv_n(a) \int_0^1 u(T+\tau) e^{-i\sqrt{\lambda_n}(1-\tau)}\, d\tau =$$

$$= e^{-i\sqrt{\lambda_n}} \left(\sqrt{\lambda_n} c_n(T) + ic'_n(T) \right) + iv_n(a) \int_0^1 u(T+\tau) e^{-i\sqrt{\lambda_n}(1-\tau)}\, d\tau.$$

Consider the following discrete-time system in l_2 as follows. Let

$$A := \begin{pmatrix} e^{-i\sqrt{\lambda_1}} & & & \\ & e^{-i\sqrt{\lambda_2}} & 0 & \\ & \ddots & & \\ 0 & & e^{-i\sqrt{\lambda_n}} & \\ & & \ddots & \end{pmatrix}_{\infty\times\infty}$$

be an $\infty \times \infty$ diagonal matrix. Clearly, $A \in L(l_2, l_2)$. Let

$$B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \\ \vdots \end{pmatrix}_{1\times\infty}$$

be an $l \times \infty$ column vector, where $B_n\colon L^2(0,1) \to \mathbf{C}$,

$$B_n u(\cdot) = iv_n(a) \int_0^1 u(\tau) \cdot e^{-i\sqrt{\lambda_n}(1-\tau)}\, d\tau, \quad n = 1, 2, \ldots .$$

It is easy to see that $B\colon \sqrt{\lambda_n}(0,1) \to l_2$ and $B \in L\big(L^2(0,1), l_2\big)$. Consider the discrete-time system

$$(A, B, \widetilde{\Omega}) \begin{cases} x_{n+1} = Ax_n + Bu_n \\ x_n = \big(x_n^1, x_n^2, x_n^3, \ldots\big) \in l_2,\ u_n \in \widetilde{\Omega} \subset L^2(0,1), \\ A \in L(l_2, l_2),\ B \in L\big(L^2(0,1), l_2\big), \\ \widetilde{\Omega} \text{ is a convex set in } L^2(0,1), \quad 0 \in \widetilde{\Omega}. \end{cases}$$

For this system we have clearly

$$\sigma(A) \subset \{\lambda\colon \lambda \in \mathbf{C},\ |\lambda| = 1\} \quad \text{and} \quad \|A\| \leqq 1,$$

that is Assumption 1 is fulfilled for the system $(A, B, \widetilde{\Omega})$. Consider the system (1/a), (1/b) and (2). Assume that the control set Ω satisfies properties (a), (b) and (c).

THEOREM 2. *The system* (1/a), (1/b) *and* (2) *is AGC or ALC if and only if the system* $(A, B, \widetilde{\Omega})$ *is AGC or ALC.*

PROOF. For the system $(A, B, \widetilde{\Omega})$ we denote the reachability set by

$$R_0^d := \{0\}, \quad R_k^d := A^{k-1}B\widetilde{\Omega} + A^{k-2}B\widetilde{\Omega} + \ldots + B\widetilde{\Omega} \quad \text{for} \quad k \geqq 1.$$

It suffices to show that

$$R_k^d = B_a(k, \Omega_k).$$

Let $\varphi \in B_a(k, \Omega_k)$, then there exists $u(\cdot) \in \Omega \cap L^2(0, k)$ such that

$$\varphi = \left(\sqrt{\lambda_n} c_n(k) + i c_n'(k)\right)_{n=1}^{\infty}.$$

Defining a control sequence $u_i \in \widetilde{\Omega}$, $i = 1, 2, \ldots, k$ by $u_i(\Theta) = u(i - 1 + \Theta)$, $\Theta \in [0, 1]$, we can easily show that

$$\varphi = \left(e^{-i\sqrt{\lambda_n}}\left(\sqrt{\lambda_n} c_n(k-1) + i c_n'(k-1)\right)\right)_{n=1}^{\infty} +$$

$$+ \left(i v_n(a) \int_0^1 u(k - 1 + \tau) e^{-i\sqrt{\lambda_n}(1-\tau)}\, d\tau\right)_{n=1}^{\infty} =$$

$$= A^{k-1}Bu_1 + A^{k-2}Bu_2 + \ldots + Bu_k,$$

thus $\varphi \in R_k^d$. Conversely, let $\varphi \in R_k^d$, then there exists a control sequence $u_i \in \widetilde{\Omega}$, $i = 1, 2, \ldots, k$ such that

$$\varphi = A^{k-1}Bu_1 + A^{k-2}Bu_2 + \ldots + Bu_k.$$

Taking $u(t) = u_i\big(t - (i - 1)\big)$ for $i - 1 \leqq t < i$, $i = 1, 2, \ldots, k$ we have $u(\cdot) \in \Omega \cap L^2(0, k)$ and

$$\varphi = \left(\sqrt{\lambda_n} c_n(k) + i c_n'(k)\right)_{n=1}^{\infty}.$$

Hence, $\varphi \in B_a(k, \Omega_k)$. The proof is complete.

V. Main results

Applying the discretization method we obtain the following results.

THEOREM 3. *Consider the system* (1/a), (1/b) *and* (2). *Assume that the control set* Ω *satisfies properties* (a), (b) *and* (c). *Then the system* (1/a), (1/b) *and* (2) *is AGC if and only if there is no* $\sqrt{\lambda_n}$ *such that*

$$v_n(a)\int_0^1 u(\tau)\cdot e^{-i\sqrt{\lambda_n}(1-\tau)}\,d\tau=0 \quad \textit{for all} \quad u(\cdot)\in\widetilde{\Omega}.$$

PROOF. First we describe the system (1/a), (1/b) and (2) as in Section IV. We obtain an equivalent system $(A,B,\widetilde{\Omega})$. We note that the operator A has only complex eigenvalues, $e^{-i\sqrt{\lambda_n}}$, $n=1,2,\dots$. The corresponding eigenvectors are $1\times\infty$ vectors of the form

$$\begin{pmatrix}0\\0\\\vdots\\1\\0\\\vdots\end{pmatrix}.$$

Thus from Theorems 1 and 2, we obtain Theorem 3 immediately.

REMARK. The condition

$$v_n(a)\int_0^1 e^{-i\sqrt{\lambda_n}(1-\tau)}u(\tau)\,d\tau=0 \quad \text{for all} \quad u(\cdot)\in\widetilde{\Omega}$$

can be simply checked in many cases.

COROLLARY 1. *If* $\rho(x)\equiv 1$ *and* a *is rational number* $\left(a=\frac{p}{q}\right)$, *then the system* (1/a), (1/b) *and* (2) *with restrained controls* $\left(u(\cdot)\in\Omega\right)$ *is never AGC (so never ALC).*

PROOF. Since $\rho(x)=1$, $v_n(a)=\sin(n\pi a)=\sin\left(n\pi\frac{p}{q}\right)$. Therefore $v_n(a)=0$ if $q|n$. Thus Corollary 1 is immediate from Theorem 3.

COROLLARY 2 (see I. Joó [6]). *If* $\rho(x)\equiv 1$ *and the controls* $u(\cdot)\in$ $\in L^2(0,T)$ *run over the whole space* $L^2(0,T)$, *then*

(a) *if* $a = \frac{p}{q}$ *is rational, the system* (1/a), (1/b) *and* (2) *is not AGC;*
(b) *if* a *is irrational, the system* (1/a), (1/b) *and* (2) *is AGC.*

PROOF. (a) Immediate from Corollary 1.
(b) If a is irrational, then $v_n(a) \neq 0$ for all $n \in N$. Thus the condition

$$\int_0^1 e^{-i\sqrt{\lambda_n}(1-\tau)} u(\tau)\, d\tau = 0 \quad \text{for all} \quad u(\cdot) \in L^2(0,1)$$

is equivalent to the condition $e^{-i\sqrt{\lambda_n}(1-t)} \equiv 0$ for all $t \in [0,1]$. From Theorem 3 we obtain that the system (1/a), (1/b) and (2) is AGC (see I. Joó [6] and M. Horváth [5]).

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WEIGHTED (0, 2)-INTERPOLATION ON THE ROOTS OF JACOBI POLYNOMIALS

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1. Introduction

Weighted (0, 2)-interpolation means the following problem. Let (a, b) be a finite or infinite open interval,

$$(1) \qquad -\infty \leqq a < x_{n,n} < \dots < x_{1,n} < b \leqq +\infty \qquad (n \in \mathbf{N})$$

distinct fundamental points and $w \in C^2(a, b)$ a weight function. Determine a polynomial R_n of lowest possible degree satisfying the conditions

$$(2) \quad R_n(x_{k,n}) = y_{k,n}, \qquad (wR_n)''(x_{k,n}) = y''_{k,n} \qquad (k = 1, 2, \dots, n;\ n \in \mathbf{N}),$$

where $y_{k,n}$ and $y''_{k,n}$ are arbitrarily given real numbers.

P. Turán suggested to study this problem and it was investigated firstly by J. Balázs. In [1] he proved that if the fundamental points (1) are the roots of the ultraspherical polynomial $P_n^{(\alpha)}$ $(\alpha > -1)$, and the weight function is

$$w(x) = (1 - x^2)^{\frac{\alpha+1}{2}} \qquad (x \in [-1, 1]),$$

then generally there does not exist any polynomial of degree $\leqq 2n - 1$ satisfying the requirements (2). But he could show that if n is even then under the condition

$$R_n(0) = \sum_{k=1}^{n} y_{k,n} l_{k,n}^2(0),$$

where $l_{k,n}$ represent the Lagrange-fundamental polynomials corresponding to the nodal points $x_{k,n}$, there exists a unique polynomial of degree $\leqq 2n$ (if n is odd then the uniqueness is not true.) He gave the explicit form of this polynomial and proved the following convergence theorem.

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THEOREM A. *Let $\alpha > 0$. Suppose that the differentiable function $f : [-1,1] \to \mathbf{R}$ satisfies the condition $f' \in \mathrm{Lip}_\mu$, $\frac{1}{2} < \mu \leqq 1$. Further let*

$$y_{k,n} = f(x_{k,n}), \qquad y''_{k,n} = o\left(\sqrt{n}\right)\left(1 - x_{k,n}^2\right)^{\frac{\alpha-3}{2}} \qquad (k = 1,2,\dots,n).$$

Then the sequence of weighted $(0,2)$-interpolation polynomials R_n ($n = 2,4,\dots$) converges uniformly to f in $[-1+\varepsilon, 1-\varepsilon]$, where $\varepsilon \in (0,1)$ is an arbitrarily fixed number.

In [6] J. Prasad extended the result of J. Balázs to the case when the nodes of interpolation are the roots of Jacobi polynomials $P_n^{(\alpha,-\alpha)}$ $(0 < |\alpha| \leqq \frac{1}{2})$ (see also [7] and [8]). In [10] L. Szili investigated this problem in the case when $(a,b) = (-\infty, +\infty)$, the fundamental points (1) are the roots of Hermite polynomials and the weight function is $w(x) = \exp\left(-\frac{x^2}{2}\right)$ $(x \in \mathbf{R})$.

In this paper we want to study some analogous problems in the case when the fundamental points are the roots of Jacobi polynomials and the weight function is

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}} \qquad (x \in [-1,1];\ \alpha, \beta > -1).$$

Section 2 contains the results. Section 3 provides the proofs of Theorems 1 and 2. We collected the tools for proving the convergence theorem in Section 4. Finally, we prove Theorem 3 in Section 5.

2. Results

Let $P_n^{(\alpha,\beta)}$ $(\alpha, \beta > -1;\ n \in \mathbf{N})$ be the Jacobi polynomial of degree n with the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \sim n^\alpha,$$

where $a_n \sim b_n$ means that $|a_n| = O(b_n)$ and $|b_n| = O(a_n)$.

These polynomials are orthogonal over the interval $[-1,1]$ with respect to the weight function

$$\varrho(x) = (1-x)^\alpha(1+x)^\beta \qquad (x \in (-1,1)).$$

Denote by $x_{k,n}$ $(k = 1,2\dots,n; n \in \mathbf{N})$ the roots of $P_n^{(\alpha,\beta)}$ in decreasing order and let $l_{k,n}$ represent the Lagrange-fundamental polynomials corresponding to the nodal points $x_{k,n}$, i.e.

$$l_{k,n}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)'}(x_{k,n})(x - x_{k,n})} \qquad (k = 1, 2, \cdots, n; n \in \mathbf{N}). \tag{3}$$

The following theorems are true.

THEOREM 1. *If* $\alpha, \beta > -1$, *the nodal points* $x_{k,n}$ *are the roots of the Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ *and the weight function is*

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}} \qquad (x \in [-1, 1]) \tag{4}$$

then there does not exist –in general– a polynomial R_n *of degree* $\leqq 2n - 1$ *satisfying the conditions* (2).

THEOREM 2. *Let* $\alpha, \beta > -1$,

$$A_{k,n}(x) = l_{k,n}^2(x) + \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)'}(x_{k,n})} \int_0^x \frac{l_{k,n}(t)(a_{k,n}t + b_{k,n}) - l'_{k,n}(t)}{t - x_{k,n}}\, dt, \tag{5}$$

where

$$a_{k,n}x_{k,n} + b_{k,n} = l'_{k,n}(x_{k,n}), \qquad a_{k,n} = -\frac{w''(x_{k,n})}{2w(x_{k,n})}, \tag{6}$$

and

$$B_{k,n}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{2w(x_{k,n})P_n^{(\alpha,\beta)'}(x_{k,n})} \int_0^x l_{k,n}(t)\, dt \tag{7}$$

$$(k = 1, 2, \ldots, n;\ n \in \mathbf{N}).$$

If n *is a such number that* $P_n^{(\alpha,\beta)}(0) \neq 0$ *then*

$$R_n(x) = \sum_{k=1}^n y_{k,n}A_{k,n}(x) + \sum_{k=1}^n y''_{k,n}B_{k,n}(x) \tag{8}$$

is the uniquely determined polynomial of degree $\leqq 2n$ *satisfying the requirements*

$$\begin{cases} R_n(x_{k,n}) = y_{k,n}, \quad (wR_n)''(x_{k,n}) = y''_{k,n}, \\ R_n(0) = \sum_{k=1}^n y_{k,n}l_{k,n}^2(0), \end{cases} \tag{9}$$

where $y_{k,n}$ *and* $y''_{k,n}$ *are arbitrary real numbers.*

REMARK 1. If n is a such number that $P_n^{(\alpha,\beta)}(0)=0$ (for example if $\alpha=\beta$ and n is an arbitrary odd number) then there are infinitely many polynomials of degree $\leqq 2n$ satisfying (9). Indeed, in these cases for every real number c the polynomials

$$R_n(x)=\sum_{k=1}^{n} y_{k,n}A_{k,n}(x)+\sum_{k=1}^{n} y''_{k,n}B_{k,n}(x)+cP_n^{(\alpha,\beta)}(x)$$

satisfy (9).

COROLLARY 1. *Let n be a natural number satisfying the condition $P_n^{(\alpha,\beta)}(0)\neq 0$. If S is an arbitrary polynomial of degree $\leqq 2n$ then for all $x\in\mathbf{R}$*

$$S(x)\equiv\sum_{k=1}^{n} S(x_{k,n})A_{k,n}(x)+\sum_{k=1}^{n}(wS)''(x_{k,n})B_{k,n}(x)+C_nP_n^{(\alpha,\beta)}(x),$$

where

$$C_n=\frac{1}{P_n^{(\alpha,\beta)}(0)}\left(S(0)-\sum_{k=1}^{n} S(x_{k,n})l_{k,n}^2(0)\right).$$

We introduce the following notations: $n_{\alpha,\beta}$ denotes an odd number if $\alpha-\beta=4l+2$ $(l\in\mathbf{Z})$, an even number if $\alpha-\beta=4l$ $(l\in\mathbf{Z})$ and an arbitrary natural number otherwise;

$$\gamma:=\begin{cases}\min(\alpha,\beta), & \text{if } \min(\alpha,\beta)<-\frac{1}{2}\\ -\frac{1}{2}, & \text{if } \min(\alpha,\beta)\geqq-\frac{1}{2}.\end{cases}\tag{10}$$

THEOREM 3. *Let $f:[-1,1]\to\mathbf{R}$ be a continuously differentiable function, $\alpha,\beta>-1$ and $\varepsilon\in(0,1)$ a fixed number. If*

$$y_{k,n}=f(x_{k,n}),\ y''_{k,n}=O(1)\left((1-x_{k,n})^{\frac{\alpha-1}{2}}(1+x_{k,n})^{\frac{\beta-1}{2}}n\omega\left(f';\frac{1}{2n}\right)\right)$$
$$(k=1,2,\dots,n),$$

then the sequence of the weighted $(0,2)$-interpolation polynomials $R_{n_{\alpha,\beta}}(f;x)$ satisfying (9) obeys the estimate

$$|f(x)-R_{n_{\alpha,\beta}}(f;x)|=O(1)\left(\omega(f';\frac{1}{2n_{\alpha,\beta}})+\frac{\log n_{\alpha,\beta}}{n_{\alpha,\beta}^{2(\gamma+1)}}\right)$$
$$(x\in[-1+\varepsilon;1-\varepsilon]),$$

where $\omega(f';\delta)$ is the modulus of continuity of f' and O does not depend on $n_{\alpha,\beta}$ and x.

3. Existence and uniqueness of the interpolation polynomials

PROOF OF THEOREM 1. Let $n \in \mathbf{N}$ and $j \in \{1,2,\dots n\}$ be fixed natural numbers and choose $y_{k,n}, y''_{k,n}$ $(k=1,2,\dots,n)$ such that

$$(11)\qquad y_{k,n}=0,\qquad y''_{k,n}=\delta_{k,j}\qquad (k=1,2,\dots,n),$$

where $\delta_{k,j}$ denotes the Kronecker symbol.

For the proof of the theorem assume that there exists a polynonial R_n of degree $\leqq 2n-1$ satisfying the requirements (2). Then R_n has the following form

$$R_n(x)=P_n^{(\alpha,\beta)}(x)Q_{n-1}(x),$$

where Q_{n-1} is a polynomial of degree $\leqq n-1$.

It is known (see [9], (4.2.1)) that the polynomial $P_n^{(\alpha,\beta)}$ satisfies the differential equation

$$(12)\qquad (1-x^2)P_n^{(\alpha,\beta)''}(x)+\left[(\beta-\alpha)-(\alpha+\beta+2)x\right]P_n^{(\alpha,\beta)'}(x)+$$
$$+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0,$$

thus

$$\left(wP_n^{(\alpha,\beta)}\right)''(x_{k,n})=(1-x_{k,n})^{\frac{\alpha-1}{2}}(1+x_{k,n})^{\frac{\beta-1}{2}}\times$$
$$\times\left[(1-x_{k,n}^2)P_n^{(\alpha,\beta)''}(x_{k,n})+\{\beta-\alpha-(\alpha+\beta+2)x_{k,n}\}P_n^{(\alpha,\beta)'}(x_{k,n})\right]=0,$$

which proves that

$$(13)\qquad \left(wP_n^{(\alpha,\beta)}\right)''(x_{k,n})=0\qquad (k=1,2,\dots,n;\ n\in\mathbf{N}).$$

Using these relations and (11) we obtain

$$(wR_n)''(x_{k,n})=2w(x_{k,n})P_n^{(\alpha,\beta)'}(x_{k,n})Q'_{n-1}(x_{k,n})=0$$

for $k=1,2,\dots j-1,j+1,\dots,n$; from which it follows that $Q'_{n-1}(x)\equiv 0$ for all $x\in\mathbf{R}$, contradicting (11). This completes the proof of Theorem 1. □

We seek the interpolation polynomials R_n of degree $\leqq 2n$ satisfying the conditions (2) in the form

$$R_n(x)=\sum_{k=1}^{n} y_{k,n}A_{k,n}(x)+\sum_{k=1}^{n} y''_{k,n}B_{k,n}(x),$$

where the so called *fundamental polynomials of the first kind* satisfy the requirements

$$(14)\quad A_{k,n}(x_{i,n})=\delta_{k,i},\qquad (wA_{k,n})''(x_{i,n})=0\qquad (i,k=1,2\dots,n;\ n\in\mathbf{N})$$

and the so called *fundamental polynomials of the second kind* obey the requirements

$$(15)\qquad B_{k,n}(x_{i,n})=0,\qquad (wB_{k,n})''(x_{i,n})=\delta_{k,i}\quad (i,k=1,2,\dots,n).$$

LEMMA 1. *For every $\alpha,\beta>-1$ the fundamental polynomials* (5) *of first kind are of degree $2n$ and satisfy conditions* (14).

PROOF. Fix the number k. The polynomials $A_{k,n}$ are of degree $2n$, indeed, and it is easy to see that they satisfy the first requirements of (14). For the proof of the second conditions of (14) firstly suppose that $x_{i,n}\neq x_{k,n}$ $(i=1,2,\dots,n)$. In this case we have

$$(wA_{k,n})''(x_{i,n})=$$

$$=2w(x_{i,n})\left[l'_{k,n}(x_{i,n})\right]^2-2w(x_{i,n})\frac{P_n^{(\alpha,\beta)'}(x_{i,n})}{P_n^{(\alpha,\beta)'}(x_{k,n})(x_{i,n}-x_{k,n})}l'_{k,n}(x_{i,n})=0.$$

If $x_{i,n}=x_{k,n}$ then

$$(wA_{k,n})''(x_{k,n})=$$

$$=w''(x_{k,n})+4w'(x_{k,n})l'_{k,n}(x_{k,n})+4w(x_{k,n})\left[l'_{k,n}(x_{k,n})\right]^2+2w(x_{k,n})a_{k,n},$$

so from (6) we get

$$(wA_{k,n})''(x_{k,n})=4l'_{k,n}(x_{k,n})\left[w'(x_{k,n})+w(x_{k,n})l'_{k,n}(x_{k,n})\right].$$

From the differential equation (12) it follows

$$l'_{k,n}(x_{k,n})=\frac{P_n^{(\alpha,\beta)''}(x_{k,n})}{2P_n^{(\alpha,\beta)'}(x_{k,n})}=-\frac{\beta-\alpha-(\alpha+\beta+2)x_{k,n}}{2(1-x_{k,n}^2)}$$

thus

$$w'(x_{k,n}) + w(x_{k,n})l'_{k,n}(x_{k,n}) = 0$$

and this completes the proof of Lemma 1. □

LEMMA 2. *For every $\alpha, \beta > -1$ the fundamental polynomials* (7) *of second kind are of degree $2n$ and satisfy conditions* (15).

PROOF. Fix the number k. The polynomials $B_{k,n}$ are of degree $2n$, indeed, and they satisfy the first conditions of (15). Using (13) and

$$(wB_{k,n})''(x_{i,n}) = \frac{1}{2w(x_{k,n})P_n^{(\alpha,\beta)'}(x_{k,n})}\Big\{\big(wP_n^{(\alpha,\beta)}\big)''(x_{i,n})\int_0^{x_{i,n}} l_{k,n}(t)\,dt +$$

$$+2\left[w'(x_{i,n})P_n^{(\alpha,\beta)}(x_{i,n}) + w(x_{i,n})P_n^{(\alpha,\beta)'}(x_{i,n})\right] l_{k,n}(x_{i,n})\Big\}$$

we obtain that the polynomials $B_{k,n}$ satisfy the second conditions of (15), too. □

PROOF OF THEOREM 2. From Lemmas 1 and 2 it follows that the polynomial

$$R_n(x) = \sum_{k=1}^{n} y_{k,n}A_{k,n}(x) + \sum_{k=1}^{n} y''_{k,n}B_{k,n}(x)$$

satisfies the conditions (9). Suppose that the polynomial $\tilde{R}_n$ also obeys (9). Then for $k = 1, 2, \ldots, n$ we have

$$R_n(x_{k,n}) - \tilde{R}_n(x_{k,n}) = 0, \qquad \big(w(R_n - \tilde{R}_n)\big)''(x_{k,n}) = 0,$$

$$R_n(0) - \tilde{R}_n(0) = 0.$$

Hence it follows that

$$R_n(x) - \tilde{R}_n(x) = P_n^{(\alpha,\beta)}(x)H_n(x),$$

where the polynomial H_n is of degree $\leqq n$. By our condition $P_n^{(\alpha,\beta)}(0) \neq 0$ so $H_n(0) = 0$.

For the second derivative we get

$$\big(w(R_n - \tilde{R}_n)\big)''(x_{k,n}) = 2w(x_{k,n})P_n^{(\alpha,\beta)'}(x_{k,n})H'_n(x_{k,n}) = 0,$$

i.e. $H'_n(x_{k,n}) = 0$ $(k = 1, 2, \ldots, n)$ and this means that $H_n(x)$ is constant. Since $H_n(0) = 0$ thus $R_n(x) \equiv \tilde{R}_n(x)$ for all x and this completes the proof of Theorem 2. □

PROOF OF COROLLARY 1. Let $S(x)$ be an arbitrary polynomial of degree $\leqq 2n$ and consider the polynomial

$$U(x) \equiv S(x) - \sum_{k=1}^{n} S(x_{k,n}) A_{k,n}(x) - \sum_{k=1}^{n} (wS)''(x_{k,n}) B_{k,n}(x).$$

By Lemmas 1 and 2 we have

$$U(x_{k,n}) = 0 \qquad (k = 1, 2, \ldots, n),$$

i.e. the polynomial U is of the form

$$U(x) = P_n^{(\alpha,\beta)}(x) H_n(x),$$

where the polynomial H_n is of degree $\leqq n$.

Using (13) we obtain

$$(wU)''(x_{k,n}) = 0 = 2w(x_{k,n}) P_n^{(\alpha,\beta)\prime}(x_{k,n}) H_n'(x_{k,n}) \qquad (k = 1, 2, \ldots n),$$

thus $H_n(x) \equiv C_n$ (constant). Hence

$$C_n P_n^{(\alpha,\beta)}(x) \equiv S(x) - \sum_{k=1}^{n} S(x_{k,n}) A_{k,n}(x) - \sum_{k=1}^{n} (wS)''(x_{k,n}) B_{k,n}(x).$$

The value of C_n follows from the above relations. □

4. Estimates with respect to the fundamental polynomials $A_{k,n}$ and $B_{k,n}$

Firstly we mention some basic relations with respect to the Jacobi polynomials which will be used later.

For the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ we have the asymptotical relation

$$(16) \qquad 1 - x_{k,n}^2 \sim \frac{k^2}{n^2} \qquad (k = 1, 2, \ldots, n;\ n \in \mathbf{N})$$

(see [5], Lemma 2 or [9], (8.9.1));

$$(17) \qquad P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

(see [9], (4.13));

$$(18)\qquad \left| P_n^{(\alpha,\beta)\prime}(x_{k,n})\right| \sim \frac{n^{\alpha+2}}{k^{\alpha+\frac{3}{2}}} \qquad (x_{k,n} \geqq 0)$$

(see [9], (8.9.2)) and for every $\varepsilon \in (0,1)$

$$(19)\qquad |P_n^{(\alpha,\beta)}(x)| = O\left(\frac{1}{\sqrt{n}}\right) \qquad (x \in [-1+\varepsilon, 1-\varepsilon];\ \alpha,\beta > -1)$$

(see [9], (7.32.5).

The Jacobi polynomial satisfies the relation

$$(20)\qquad \left|\int_0^x P_n^{(\alpha,\beta)}(t)\,dt\right| = O\left(\frac{1}{n^{3/2}}\right) \qquad (n \in \mathbf{N};\ \alpha,\beta > -1)$$

for all $x \in [-1+\varepsilon; 1-\varepsilon]$ where $\varepsilon \in (0,1)$ is a fixed number.

For the proof of (20) we integrate the differential equation (12)

$$\int_0^x P_n^{(\alpha,\beta)}(t)\,dt = \frac{1}{n(n+\alpha+\beta+1)+(\alpha+\beta)}\Big[-(1-x^2)P_n^{(\alpha,\beta)\prime}(x)-$$

$$-\big[(\beta-\alpha)-(\alpha+\beta)x\big]\,P_n^{(\alpha,\beta)}(x) + P_n^{(\alpha,\beta)\prime}(0) + (\beta-\alpha)P_n^{(\alpha,\beta)}(0)\Big].$$

Applying this identity, (19) and

$$\left| P_n^{(\alpha,\beta)\prime}(x)\right| = O\big(\sqrt{n}\big) \qquad (x \in [-1+\varepsilon, 1-\varepsilon])$$

(see [9], (8.9.5)) we obtain the estimate (20).

We shall also use the following statement. Let $\varepsilon \in (0,1)$ be a fixed number and $x \in [-1+\varepsilon, 1-\varepsilon]$. Suppose that $x_{k,n} \notin [0,x]$ if $x > 0$ and $x_{k,n} \notin [x,0]$ if $x < 0$. Then

$$(21)\qquad \left|\int_0^x \frac{P_n^{(\alpha,\beta)}(t)}{t-x_{k,n}}\,dt\right| = O(1)\left(\frac{1}{n^{3/2}|x-x_{k,n}|}\right),$$

where O does not depend on n, k and x.

Integration by parts gives

$$\int_0^x \frac{P_n^{(\alpha,\beta)}(t)}{t-x_{k,n}}\,dt = \left[\frac{\int_0^t P_n^{(\alpha,\beta)}(u)du}{t-x_{k,n}}\right]_0^x + \int_0^x \frac{\int_0^t P_n^{(\alpha,\beta)}(u)du}{(t-x_{k,n})^2}\,dt =$$

$$= \frac{\int_0^x P_n^{(\alpha,\beta)}(u)du}{x-x_{k,n}} + \int_0^x \frac{\int_0^t P_n^{(\alpha,\beta)}(u)du}{(t-x_{k,n})^2}\,dt.$$

Using (20) we get

$$\left|\int_0^x \frac{P_n^{(\alpha,\beta)}(t)}{t-x_{k,n}}\,dt\right| = O(1)\left\{\frac{1}{n^{3/2}|x-x_{k,n}|}+\frac{1}{n^{3/2}}\int_{-\infty}^{x}\frac{1}{(t-x_{k,n})^2}\,dt\right\} =$$

$$= O(1)\left(\frac{1}{n^{3/2}|x-x_{k,n}|}\right).$$

Thus the inequality (21) is proved.

LEMMA 3. *Let $\varepsilon \in (0,1)$ be a fixed number and $\alpha,\beta > -1$. Then the Lebesgue function of the second kind polynomials satisfies the inequality*

$$\text{(22)} \qquad \sum_{k=1}^{n}(1-x_{k,n})^{\frac{\alpha-3}{2}}(1+x_{k,n})^{\frac{\beta-3}{2}}\,|B_{k,n}(x)| = O\left(\frac{1}{n^{2(\gamma+1)}}\right)$$

for all $x \in [-1+\varepsilon, 1-\varepsilon]$, where γ is the number defined in (10) *and O does not depend on x and n.*

PROOF. If $x = x_{k,n}$ for a $1 \leqq k \leqq n$ then (22) is obvious. Suppose that $x \in [0, 1-\varepsilon]$. Let $l = 1, 2, \ldots, n$ be the index for which

$$x_{l+1,n} < x < x_{l,n}.$$

From (7) we get

$$\text{(23)} \qquad \sum_{k=1}^{n}(1-x_{k,n})^{\frac{\alpha-3}{2}}(1+x_{k,n})^{\frac{\beta-3}{2}}\,|B_{k,n}(x)| =$$

$$= \frac{|P_n^{(\alpha,\beta)}(x)|}{2}\sum_{k=1}^{n}\frac{1}{(1-x_{k,n}^2)^2|P_n^{(\alpha,\beta)\prime}(x_{k,n})|}\left|\int_0^x l_{k,n}(t)\,dt\right| =$$

$$= \frac{|P_n^{(\alpha,\beta)}(x)|}{2}\left(\sum_{k\in\Delta_1}+\sum_{k\in\Delta_2}+\sum_{k\in\Delta_3}+\sum_{k\in\Delta_4}+\sum_{k\in\Delta_5}\right),$$

where

$$\Delta_1 := \left\{k \in \mathbf{N} \mid 1-\frac{\varepsilon}{2} \leqq x_{k,n} < 1\right\},$$

$$\Delta_2 := \left\{k \in \mathbf{N} \mid x_{l,n} < x_{k,n} < 1-\frac{\varepsilon}{2}\right\},$$

$$\Delta_3 := \{k \in \mathbf{N} \mid 0 \leqq x_{k,n} \leqq x_{l,n}\},$$

$$\Delta_4 := \left\{k \in \mathbf{N} \mid -1 + \frac{\varepsilon}{2} < x_{k,n} < 0\right\},$$

$$\Delta_5 := \left\{k \in \mathbf{N} \mid -1 < x_{k,n} \leqq -1 + \frac{\varepsilon}{2}\right\}.$$

Applying (16), (18), (19) and (21) we have for all $\alpha, \beta > -1$

$$(24) \qquad \frac{|P_n^{(\alpha,\beta)}(x)|}{2} \sum_{k \in \Delta_1} \frac{1}{(1-x_{k,n}^2)^2 |P_n^{(\alpha,\beta)\prime}(x_{k,n})|^2} \left| \int_0^x \frac{P_n^{(\alpha,\beta)}(t)}{t - x_{k,n}}\, dt \right| =$$

$$= O(1)\left(\frac{1}{n^{2(\alpha+1)}} \sum_{k=1}^{n} k^{2\alpha-1}\right) = O(1) \begin{cases} \frac{1}{n^2}, & \text{if } \alpha > 0 \\ \frac{\log n}{n^2}, & \text{if } \alpha = 0, \\ \frac{1}{n^{2(\alpha+1)}}, & \text{if } -1 < \alpha < 0. \end{cases}$$

It is known that

$$(25) \qquad |x_{l,n} - x_{k,n}| \sim \frac{|l^2 - k^2|}{n^2}$$

(see [5], Lemma 2), thus using (18), (19), (21), (25) and Cauchy's inequality we have for the second term of (23)

$$(26) \qquad \frac{|P_n^{(\alpha,\beta)}(x)|}{2} \sum_{k \in \Delta_2} \frac{1}{(1-x_{k,n}^2)^2 |P_n^{(\alpha,\beta)\prime}(x_{k,n})|^2} \left| \int_0^x \frac{P_n^{(\alpha,\beta)}(t)}{t - x_{k,n}}\, dt \right| =$$

$$= O(1)\frac{1}{\sqrt{n}} \sum_{k \in \Delta_2} \frac{1}{|P_n^{(\alpha,\beta)\prime}(x_{k,n})|^2} \left| \int_0^x \frac{P_n^{(\alpha,\beta)}(t)}{t - x_{k,n}}\, dt \right| =$$

$$= O(1)\left(\frac{1}{n^{2\alpha+4}} \sum_{\substack{k=1 \\ k \neq l}}^{n} \frac{k^{2\alpha+3}}{|k^2 - l^2|}\right) = O(1)\frac{1}{n^{2\alpha+4}}\left(\sum_{k=1}^{n} k^{4\alpha+4}\right)^{1/2} = O\left(\frac{1}{n^{3/2}}\right)$$

for all $\alpha, \beta > -1$.

Let us consider the third term of (23), i.e. suppose that $0 \leqq x_{k,n} \leqq x_{l,n}$. Firstly we prove that

$$(27) \qquad \left| \int_0^x l_{k,n}(t)\, dt \right| = O(1)\left(\frac{1}{n}\left(\frac{k}{n}\right)^{\alpha+\frac{1}{2}}\right) \quad (k \in \Delta_3).$$

Suppose that $k \neq l$. Then

$$\int_0^x l_{k,n}(t)\,dt = \int_0^{x_{k+1,n}} l_{k,n}(t)\,dt + \int_{x_{k+1,n}}^{x_{k-1,n}} l_{k,n}(t)\,dt + \int_{x_{k-1,n}}^{x} l_{k,n}(t)\,dt.$$

Using (18), (21) and (25) we have

$$\left|\int_0^{x_{k+1,n}} l_{k,n}(t)\,dt\right| = \tag{28}$$

$$= O(1)\left(\frac{1}{n^{3/2}\left|P_n^{(\alpha,\beta)\prime}(x_{k,n})\right|\,|x_{k+1,n}-x_{k,n}|}\right) = O(1)\left(\frac{1}{n}\left(\frac{k}{n}\right)^{\alpha+\frac{1}{2}}\right)$$

and

$$\left|\int_{x_{k-1,n}}^{x} l_{k,n}(t)\,dt\right| = O(1)\left(\frac{1}{n}\left(\frac{k}{n}\right)^{\alpha+\frac{1}{2}}\right). \tag{29}$$

Since ([9], (4.5.2)) for $x \neq y$ we have

$$\sum_{j=0}^{n-1}\frac{1}{h_j^{(\alpha,\beta)}}P_j^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(y) =$$

$$= \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta}\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha)\Gamma(n+\beta)} \times$$

$$\times \frac{P_n^{(\alpha,\beta)}(x)P_{n-1}^{(\alpha,\beta)}(y) - P_{n-1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)}{x-y},$$

where

$$h_j^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2j+\alpha+\beta+1}\frac{\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{\Gamma(j+1)\Gamma(j+\alpha+\beta+1)},$$

and

$$\sum_{j=0}^{n-1}\frac{1}{h_j^{(\alpha,\beta)}}\left[P_j^{(\alpha,\beta)}(x)\right]^2 =$$

$$= \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta}\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha)\Gamma(n+\beta)} \times$$

$$\times\left[P_n^{(\alpha,\beta)\prime}(x)P_{n-1}^{(\alpha,\beta)}(x) - P_{n-1}^{(\alpha,\beta)\prime}(x)P_n^{(\alpha,\beta)}(x)\right],$$

therefore for every $t \in [-1,1]$ we get

$$l_{k,n}(t) = \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)'}(x_{k,n})(t - x_{k,n})} =$$

$$= \frac{2n+\alpha+\beta}{2^{-\alpha-\beta}} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \frac{1}{P_{n-1}^{(\alpha,\beta)}(x_{k,n})P_n^{(\alpha,\beta)'}(x_{k,n})} \times$$

$$\times \sum_{j=0}^{n-1} \frac{1}{h_j^{(\alpha,\beta)}} P_j^{(\alpha,\beta)}(x_{k,n}) P_j^{(\alpha,\beta)}(t).$$

The Jacobi polynomials satisfy the identity

$$(2n+\alpha+\beta)(1-x^2)P_n^{(\alpha,\beta)'}(x) =$$

$$= -n\left[(2n+\alpha+\beta)x + \beta - \alpha\right] P_n^{(\alpha,\beta)}(x) + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)$$

(see [9], (4.5.7)) thus

$$P_{n-1}^{(\alpha,\beta)}(x_{k,n}) = \frac{2n+\alpha+\beta}{2(n+\alpha)(n+\beta)}(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n}),$$

so we obtain that

$$l_{k,n}(t) = \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)'}(x_{k,n})(t - x_{k,n})} =$$

$$\frac{(n+\alpha)(n+\beta)}{2^{-\alpha-\beta-1}} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \frac{1}{(1-x_{k,n}^2)\left[P_n^{(\alpha,\beta)'}(x_{k,n})\right]^2} \times$$

$$\times \sum_{j=0}^{n-1} \frac{1}{h_j^{(\alpha,\beta)}} P_j^{(\alpha,\beta)}(x_{k,n}) P_j^{(\alpha,\beta)}(t).$$

Using the relations

$$h_j^{(\alpha,\beta)} \sim \frac{1}{j} \qquad (j = 1, 2, \ldots, n-1),$$

$$\frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{\Gamma(m+1)\Gamma(m+\alpha+\beta+1)} \sim c,$$

where the constant c does not depend on m, we conclude that

$$\left|\int_{x_{k-1,n}}^{x_{k+1,n}} l_{k,n}(t)\,dt\right| =$$

$$= O(1)\frac{1}{(1-x_{k,n}^2)\left|P_n^{(\alpha,\beta)\prime}(x_{k,n})\right|^2}\sum_{j=0}^{n-1} j\left|P_j^{(\alpha,\beta)}(x_{k,n})\right| \left|\int_{x_{k-1,n}}^{x_{k+1,n}} P_j^{(\alpha,\beta)}(t)\,dt\right|.$$

Since in this case $|x_{k,n}| < 1-\frac{\varepsilon}{2}$, thus

$$\left|P_j^{(\alpha,\beta)}(x_{k,n})\right| = O\left(\frac{1}{\sqrt{j}}\right) \qquad (j=1,2,\dots,n-1).$$

By (19) and (25) we get

$$\left|\int_{x_{k-1,n}}^{x_{k+1,n}} P_j^{(\alpha,\beta)}(t)\,dt\right| = O\left(\frac{|x_{k-1,n}-x_{k+1,n}|}{\sqrt{j}}\right),$$

from which it follows that

$$(30)\qquad \left|\int_{x_{k-1,n}}^{x_{k+1,n}} l_{k,n}(t)\,dt\right| = O(1)\left(n\frac{|x_{k-1,n}-x_{k+1,n}|}{\left|P_n^{(\alpha,\beta)\prime}(x_{k,n})\right|^2}\right) =$$

$$= O(1)\left(\frac{1}{n}\left(\frac{k}{n}\right)^{2\alpha+4}\right) = O(1)\left(\frac{1}{n}\left(\frac{k}{n}\right)^{\alpha+\frac{1}{2}}\right).$$

The inequalities (28)–(30) prove (27) for $k\neq l$. It is easy to see that (27) also holds for $k=l$, too.

Applying (16), (18), (19) and (27) we conclude that for all $x\in[-1+\varepsilon, 1-\varepsilon]$ and $\alpha,\beta > -1$

$$(31)\quad \frac{\left|P_n^{(\alpha,\beta)}(x)\right|}{2}\sum_{k\in\Delta_3}\frac{1}{(1-x_{k,n}^2)^2\left|P_n^{(\alpha,\beta)\prime}(x_{k,n})\right|}\left|\int_0^x l_{k,n}(t)\,dt\right| = O\left(\frac{1}{n}\right).$$

Similarly to (24) and (26) we obtain by (18)

(32)

$$\frac{\left|P_n^{(\alpha,\beta)}(x)\right|}{2}\sum_{k\in\Delta_4}\frac{1}{(1-x_{k,n}^2)^2\left|P_n^{(\alpha,\beta)\prime}(x_{k,n})\right|^2}\left|\int_0^x \frac{P_n^{(\alpha,\beta)}(t)}{t-x_{k,n}}\,dt\right| = O\left(\frac{1}{n^{3/2}}\right)$$

and

$$(33)\qquad \frac{|P_n^{(\alpha,\beta)}(x)|}{2}\sum_{k\in\Delta_5}\frac{1}{(1-x_{k,n}^2)^2|P_n^{(\alpha,\beta)\prime}(x_{k,n})|^2}\left|\int_0^x\frac{P_n^{(\alpha,\beta)}(t)}{t-x_{k,n}}\,dt\right|=$$

$$=O(1)\begin{cases}\frac{1}{n^2}, & \text{if } \beta>0\\ \frac{\log n}{n^2}, & \text{if } \beta=0,\\ \frac{1}{n^{2(\beta+1)}}, & \text{if } -1<\beta<0.\end{cases}$$

Combining (23), (24), (26), (31), (32) and (33) gives (22) and this completes the proof of Lemma 3 for $x\in[0,1-\varepsilon]$. For $x\in[-1+\varepsilon,0]$ the proof is similar. □

To estimate of the Lebesgue function of the fundamental polynomials (5) we need some other representation of these polynomials.

LEMMA 4. *The fundamental polynomials of first kind* (5) *can be written in the form*

$$A_{k,n}(x)=l_{k,n}^2(x)+a_{k,n}\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt+$$

$$+\frac{[-x_{k,n}+\frac12(\beta-\alpha-(\beta+\alpha+2)x]P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}l_{k,n}(x)+$$

$$+\frac{[2x_{k,n}-(\beta-\alpha)]P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}l_{k,n}(0)+\frac{(1-x^2)P_n^{(\alpha,\beta)\prime}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}l_{k,n}(x)-$$

$$-\frac{1-x^2}{2(1-x_{k,n}^2)}l_{k,n}^2(x)-\frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}l_{k,n}'(0)+$$

$$+\frac{n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt$$

$$(k=1,2,\dots,n;\ n\in\mathbf{N};\ \alpha,\beta>-1).$$

PROOF. Using (6) we obtain that

$$(34)\qquad \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x\frac{l_{k,n}(t)(a_{k,n}t+b_{k,n})-l_{k,n}'(t)}{t-x_{k,n}}\,dt=$$

$$= a_{k,n}\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x \frac{l_{k,n}(t)(a_{k,n}x_{k,n}+b_{k,n})-l'_{k,n}(t)}{t-x_{k,n}}\,dt = J_1+J_2.$$

Since

$$l'_{k,n}(x_{k,n}) = \frac{P_n^{(\alpha,\beta)\prime\prime}(x_{k,n})}{2P_n^{(\alpha,\beta)\prime}(x_{k,n})} = -\frac{(\beta-\alpha)-(\alpha+\beta+2)x_{k,n}}{2(1-x_{k,n}^2)},$$

we have

$$J_2 = \frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\times$$

$$\times\int_0^x \frac{-\left[(\beta-\alpha)-(\alpha+\beta+2)x_{k,n}\right]l_{k,n}(t)-2(1-x_{k,n}^2)l'_{k,n}(t)}{t-x_{k,n}}\,dt =$$

$$= \frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x \frac{1}{t-x_{k,n}}\times$$

$$\times\Big\{-(\alpha+\beta+2)(t-x_{k,n})l_{k,n}(t)-2(t^2-x_{k,n}^2)l'_{k,n}(t)+$$

$$+2(t^2-1)l'_{k,n}(t)-\left[(\beta-\alpha)-(\alpha+\beta+2)t\right]l_{k,n}(t)\Big\}\,dt =$$

$$= -\frac{(\alpha+\beta+2)P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt-$$

$$-\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}(x+x_{k,n})l_{k,n}(x)+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}x_{k,n}l_{k,n}(0)+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\times$$

$$\times\int_0^x \frac{-2(1-t^2)l'_{k,n}(t)-[(\beta-\alpha)-(\alpha+\beta+2)t]l_{k,n}(t)}{t-x_{k,n}}\,dt.$$

From (3) and (12) we get

$$(1-x^2)(x-x_{k,n})l''_{k,n}(x)+2(1-x^2)l'_{k,n}(x)+$$

$$+(x-x_{k,n})\big[\beta-\alpha-(\alpha+\beta+2)x\big]\,l'_{k,n}(x)+$$

$$+\big(\beta-\alpha-(\alpha+\beta+2)x\big)\,l_{k,n}(x)+(x-x_{k,n})n(n+\alpha+\beta+1)l_{k,n}(x)=0.$$

Thus integration by parts provides

$$J_2=-\frac{(\alpha+\beta+2)P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt-$$

$$-\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}(x+x_{k,n})l_{k,n}(x)+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}x_{k,n}l_{k,n}(0)+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}(1-x^2)l'_{k,n}(x)-\frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}l'_{k,n}(0)+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}xl_{k,n}(x)-\frac{P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt+$$

$$+\frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}\big[\beta-\alpha-(\alpha+\beta+2)x\big]\,l_{k,n}(x)-$$

$$-\frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})}(\beta-\alpha)l_{k,n}(0)+$$

$$+\frac{(\alpha+\beta+2)P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt+$$

$$+\frac{n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt=$$

$$=\frac{\left[-x_{k,n}+\frac{1}{2}(\beta-\alpha-(\alpha+\beta+2)x\right]P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})}l_{k,n}(x)+$$

$$+\frac{\left(x_{k,n}-\frac{1}{2}(\beta-\alpha)\right)P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})}l_{k,n}(0)+$$

$$+\frac{(1-x^2)P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})}l'_{k,n}(x)-\frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})}l'_{k,n}(0)+$$

$$+\frac{n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})}\int_0^x l_{k,n}(t)\,dt.$$

Using

$$l'_{k,n}(x)=\frac{(x-x_{k,n})P_n^{(\alpha,\beta)'}(x)-P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)'}(x_{k,n})(x-x_{k,n})^2}$$

we obtain our statement. □

LEMMA 5. *Let $\varepsilon\in(0,1)$ be a fixed number and $\alpha,\beta>-1$. The Lebesgue function of the fundamental polynomials of first kind* (5) *obeys the estimate*

$$\sum_{k=1}^{n}\left|A_{k,n}(x)\right|=O(n)\qquad(x\in[-1+\varepsilon,1-\varepsilon]),\tag{35}$$

where O does not depend on n and x.

PROOF. It is known (see [3], (2.34)) that

$$\sum_{k=1}^{n}\left(\frac{1-x}{1-x_{k,n}}\right)^{\alpha+1}\left(\frac{1+x}{1+x_{k,n}}\right)^{\beta+1}l_{k,n}^2(x)\leqq 1\tag{36}$$

$$(|x|<1;\ \alpha,\beta>-1).$$

This implies

$$(37)\quad \sum_{k=1}^{n} l_{k,n}^2(x) = O(1)\sum_{k=1}^{n}\left(\frac{1-x}{1-x_{k,n}}\right)^{\alpha+1}\left(\frac{1+x}{1+x_{k,n}}\right)^{\beta+1} l_{k,n}^2(x) = O(1)$$

$$(x \in [-1+\varepsilon, 1-\varepsilon]).$$

An easy calculation shows that

$$|a_{k,n}| = \left|\frac{w''(x_{k,n})}{2w(x_{k,n})}\right| = O(1)\left(\frac{1}{(1-x_{k,n}^2)^2}\right),$$

thus by Lemma 3 we obtain for $\alpha, \beta > -1$

$$(38)\qquad \sum_{k=1}^{n} |a_{k,n}| \frac{|P_n^{(\alpha,\beta)}(x)|}{|P_n^{(\alpha,\beta)\prime}(x_{k,n})|}\left|\int_0^x l_{k,n}(t)\,dt\right| = O\left(\frac{1}{n^{2(\gamma+1)}}\right),$$

where γ is the number defined in (10).

From the Cauchy inequality, (16), (18) and (37) we have for $\alpha, \beta > -1$

$$(39)\qquad \sum_{k=1}^{n} \frac{|-x_{k,n} + \frac{1}{2}(\beta-\alpha) - (\alpha+\beta+2)x|\; |P_n^{(\alpha,\beta)}(x)|}{(1-x_{k,n}^2)|P_n^{(\alpha,\beta)\prime}(x_{k,n})|}|l_{k,n}(x)| =$$

$$= O(1)|P_n^{(\alpha,\beta)}(x)|\left[\sum_{k=1}^{n}\frac{1}{(1-x_{k,n}^2)^2|P_n^{(\alpha,\beta)\prime}(x_{k,n})|^2}\sum_{k=1}^{n} l_{k,n}^2(x)\right]^{\frac{1}{2}} = O(\sqrt{n}).$$

Similarly,

$$(40)\qquad \sum_{k=1}^{n}\left|\frac{(x_{k,n} - \frac{1}{2}(\beta-\alpha))\, P_n^{(\alpha,\beta)}(x)}{(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})} l_{k,n}(0)\right| = O(\sqrt{n})$$

for all $\alpha, \beta > -1$ and

$$(41)\qquad \sum_{k=1}^{n}\left|\frac{(1-x^2)P_n^{(\alpha,\beta)\prime}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)\prime}(x_{k,n})} l_{k,n}(x)\right| =$$

$$= \sum_{k=1}^{n}\frac{\left|n\left[(2n+\alpha+\beta)x+\beta-\alpha\right]P_n^{(\alpha,\beta)}(x) + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)\right|}{2(2n+\alpha+\beta)(1-x_{k,n}^2)|P_n^{(\alpha,\beta)\prime}(x_{k,n})|}.$$

$$\cdot \left| l_{k,n}(x) \right| = O\left(\sqrt{n}\right) \sum_{k=1}^{n} \frac{|l_{k,n}(x)|}{(1-x_{k,n}^2) \left| P_n^{(\alpha,\beta)\prime}(x_{k,n}) \right|} =$$

$$= O\left(\sqrt{n}\right) \left[\sum_{k=1}^{n} \frac{1}{(1-x_{k,n}^2)^2 \left| P_n^{(\alpha,\beta)\prime}(x_{k,n}) \right|^2} \sum_{k=1}^{n} l_{k,n}^2(x) \right]^{\frac{1}{2}} = O(n)$$

$$(\alpha, \beta > -1).$$

From (37), (16), (18) and (19) it follows that

$$\frac{1}{2} \sum_{k=1}^{n} \frac{1-x^2}{1-x_{k,n}^2} l_{k,n}^2(x) = \tag{41}$$

$$= O(1) \left(\sum_{|x_{k,n}| \leqq 1-\frac{\varepsilon}{2}} l_{k,n}^2(x) + \sum_{|x_{k,n}| > 1-\frac{\varepsilon}{2}} \frac{\left[P_n^{(\alpha,\beta)}(x) \right]^2}{(1-x_{k,n}^2) \left| P_n^{(\alpha,\beta)\prime}(x_{k,n}) \right|} \right) = O(1).$$

Let us consider the next term:

$$\sum_{k=1}^{n} \left| \frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2) P_n^{(\alpha,\beta)\prime}(x_{k,n})} l_{k,n}'(0) \right| = \sum_{|x_{k,n}| \leqq \frac{1}{2}} + \sum_{|x_{k,n}| > \frac{1}{2}} .$$

Using (3), (16), (17), (18), (19) and $\left| P_n^{(\alpha,\beta)\prime}(0) \right| = O\left(\sqrt{n}\right)$ we obtain

$$\sum_{|x_{k,n}| > \frac{1}{2}} \left| \frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2) P_n^{(\alpha,\beta)\prime}(x_{k,n})} \right| \left| \frac{P_n^{(\alpha,\beta)\prime}(0) x_{k,n} + P_n^{(\alpha,\beta)}(0)}{x_{k,n}^2 P_n^{(\alpha,\beta)\prime}(x_{k,n})} \right| = \tag{42}$$

$$= O(1) \sum_{k=1}^{n} \frac{1}{(1-x_{k,n}^2) \left| P_n^{(\alpha,\beta)\prime}(x_{k,n}) \right|^2} = O(1)$$

for all $x \in [-1+\varepsilon, 1-\varepsilon]$ and $\alpha, \beta > -1$.

Since $|l_{k,n}'(0)| = O(n)$, by (17), (18) and (19) we have

$$\sum_{|x_{k,n}| \leqq \frac{1}{2}} \left| \frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2) P_n^{(\alpha,\beta)\prime}(x_{k,n})} l_{k,n}'(0) \right| = \tag{43}$$

$$= O\left(\sqrt{n}\right) \sum_{k=1}^{n} \frac{1}{\left| P_n^{(\alpha,\beta)\prime}(x_{k,n}) \right|} = O(n)$$

for all $x \in [-1+\varepsilon, 1-\varepsilon]$ and $\alpha, \beta > -1$. Combining (42) and (43) we get

$$(44) \qquad \sum_{k=1}^{n} \left| \frac{P_n^{(\alpha,\beta)}(x)}{2(1-x_{k,n}^2)P_n^{(\alpha,\beta)'}(x_{k,n})} l'_{k,n}(0) \right| = O(n)$$

for all $x \in [-1+\varepsilon, 1-\varepsilon]$ and $\alpha, \beta > -1$.

Finally, similarly to the proof of (22)

$$(45) \quad \frac{n(n+\alpha+\beta+1)}{2} \sum_{k=1}^{n} \frac{|P_n^{(\alpha,\beta)}(x)|}{(1-x_{k,n}^2)|P_n^{(\alpha,\beta)'}(x_{k,n})|} \left| \int_0^x l_{k,n}(t)\,dt \right| = O(n)$$

for all $x \in [-1+\varepsilon, 1-\varepsilon]$ and $\alpha, \beta > -1$.

From the relations (37)–(45) we get the inequality (35). □

5. Proof of the convergence theorem

Firstly we prove the following statement. Denote $n_{\alpha,\beta}$ an odd number if $\alpha - \beta = 4l + 2$ ($l \in \mathbf{Z}$), an even number if $\alpha - \beta = 4l$ ($l \in \mathbf{Z}$) and an arbitrary natural number otherwise. Then there exist $n_0 \in \mathbf{N}$ and $c > 0$ such that

$$(46) \qquad |P_{n_{\alpha,\beta}}^{(\alpha,\beta)}(0)| > \frac{c}{\sqrt{n_{\alpha,\beta}}} \qquad (n_{\alpha,\beta} > n_0).$$

It is known (see [9], Theorem 8.21.8) that

$$P_n^{(\alpha,\beta)}(0) = n^{-\frac{1}{2}} k_0 \cos(N\frac{\pi}{2} + \gamma) + O(n^{-\frac{3}{2}}),$$

where

$$k_0 = \pi^{-\frac{1}{2}} \left(\sin\frac{\pi}{4}\right)^{-(\alpha+\beta)-1}, \quad N = n + \frac{\alpha+\beta+1}{2}, \quad \gamma = -\left(\alpha + \frac{1}{2}\right)\frac{\pi}{2}.$$

Since for every $n_{\alpha,\beta}$ we have

$$\cos\left(N\frac{\pi}{2} + \gamma\right) \neq 0,$$

we obtain the inequality (46).

We remark that for these $n_{\alpha,\beta}$ the interpolation polynomials $R_{n_{\alpha,\beta}}$ are uniquely determined by the conditions (9) and we can also use Corollary 1 in these cases.

In the following we denote the index $n_{\alpha,\beta}$ briefly by n.

For the proof of the Theorem 4 we also use the following result of I.E. Gopengauz (see [2]): if $f \in C^1[-1,1]$ then for every $n \geqq 9$ there exists a polynomial $p_n(x)$ of degree at most n such that

$$|f^{(j)}(x) - p_n^{(j)}(x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}\right)^{1-j} \omega\left(f', \frac{1}{2n}\right)$$

$$(x \in [-1,1];\ j = 0,1),$$

and

$$(47) \qquad |p_n''(x)| = O(1)\left(\frac{n}{\sqrt{1-x^2}}\omega\left(f'; \frac{1}{2n}\right)\right) \qquad (x \in [-1,1]).$$

For $n \geqq 6$ a fixed integer let p_{2n} be the polynomial of approximation to f guaranteed by the above theorem. Applying Lemma 5 we obtain that for every $x \in [-1+\varepsilon, 1-\varepsilon]$

$$(48) \qquad |f(x) - R_n(f;x)| \leqq |f(x) - p_{2n}(x)| + |p_{2n}(x) - R_n(f;x)| \leqq$$

$$\leqq |f(x) - p_{2n}(x)| + \sum_{k=1}^{n} |f(x_{k,n}) - p_{2n}(x_{k,n})|\ |A_{k,n}(x)| +$$

$$+ \sum_{k=1}^{n} |y_{k,n}'' - (wp_{2n})''(x_{k,n})|\ |B_{k,n}(x)| + |C_n P_n^{(\alpha,\beta)}(x)| \leqq$$

$$= O(1)\left(\omega\left(f'; \frac{1}{2n}\right)\right) + \sum_{k=1}^{n} |y_{k,n}'' - (wp_{2n})''(x_{k,n})|\ |B_{k,n}(x)| +$$

$$+ |C_n P_n^{(\alpha,\beta)}(x)|.$$

We have

$$\sum_{k=1}^{n} |y_{k,n}'' - (wp_{2n})''(x_{k,n})|\ |B_{k,n}(x)| =$$

$$= O(1)\Bigg[\sum_{k=1}^{n} |w''(x_{k,n})p_{2n}(x_{k,n})|\ |B_{k,n}(x)| +$$

$$+ \sum_{k=1}^{n} |w'(x_{k,n})p_{2n}'(x_{k,n})|\ |B_{k,n}(x)| +$$

$$+\sum_{k=1}^{n}\left|w(x_{k,n})p''_{2n}(x_{k,n})B_{k,n}(x)\right|+\sum_{k=1}^{n}\left|y''_{k,n}B_{k,n}(x)\right|\Bigg]=$$

$$+U_1+U_2+U_3+U_4.$$

Since $|p_{2n}(x)|$ is bounded and

$$w''(x)=(1-x)^{\frac{\alpha-3}{2}}(1+x)^{\frac{\beta-3}{2}}\times$$

$$\times\left[\frac{\alpha^2-1}{4}(1+x)^2-2\frac{\alpha+1}{2}\frac{\beta+1}{2}(1-x^2)-\frac{\beta^2-1}{4}(1-x)^2\right],$$

thus from Lemma 3 we obtain

$$(49)\quad U_1=O(1)\sum_{k=1}^{n}(1-x_{k,n})^{\frac{\alpha-3}{2}}(1+x_{k,n})^{\frac{\beta-3}{2}}\left|B_{k,n}(x)\right|=O\left(\frac{1}{n^{2(\gamma+1)}}\right),$$

where γ is the number defined in (10).

The polynomial $p'_{2n}(x)$ is also bounded and

$$w'(x)=-\frac{\alpha+1}{2}(1-x)^{\frac{\alpha-1}{2}}(1+x)^{\frac{\beta+1}{2}}+\frac{\beta+1}{2}(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta-1}{2}}$$

so similarly to the proof of (22) we have

$$(50)\qquad U_2=O(1)\sum_{k=1}^{n}(1-x_{k,n})^{\frac{\alpha-1}{2}}(1+x_{k,n})^{\frac{\beta-1}{2}}\left|B_{k,n}(x)\right|=$$

$$=O(1)\sum_{k=1}^{n}\frac{\left|P_n^{(\alpha,\beta)}(x)\right|}{(1-x_{k,n}^2)\left|P_n^{(\alpha,\beta)\prime}(x_{k,n})\right|}\left|\int_0^x l_{k,n}(t)\,dt\right|=O\left(\frac{1}{n}\right)$$

for all $x\in[-1+\varepsilon,1-\varepsilon]$ and $\alpha,\beta>-1$.

For U_3 we get

$$(51)\qquad U_3=O\left(n\omega\left(f';\frac{1}{2n}\right)\right)\sum_{k=1}^{n}\frac{w(x_{k,n})}{\sqrt{1-x_{k,n}^2}}\left|B_{k,n}(x)\right|=$$

$$=O\left(n\omega\left(f';\frac{1}{2n}\right)\right)\sum_{k=1}^{n}(1-x_{k,n})^{\frac{\alpha}{2}}(1+x_{k,n})^{\frac{\beta}{2}}\left|B_{k,n}(x)\right|=$$

$$= O\left(n\omega\left(f';\frac{1}{2n}\right)\right)\sum_{k=1}^{n}\frac{1}{\sqrt{1-x_{k,n}^2}}\frac{|P_n^{(\alpha,\beta)}(x)|}{|P_n^{(\alpha,\beta)\prime}(x_{k,n})|}\left|\int_0^x l_{k,n}(t)\,dt\right| =$$

$$= O\left(\omega\left(f';\frac{1}{2n}\right)\right)$$

for all $x \in [-1+\varepsilon, 1-\varepsilon]$ and $\alpha, \beta > -1$.

By the condition of the theorem we have

$$|y_{k,n}''| = O(1)\left((1-x_{k,n})^{\frac{\alpha-1}{2}}(1+x_{k,n})^{\frac{\beta-1}{2}} n\omega\left(f';\frac{1}{2n}\right)\right),$$

thus

$$U_4 = O(1)\left(\omega\left(f';\frac{1}{2n}\right)\right). \tag{52}$$

From (49)–(52) we obtain

(53)
$$\sum_{k=1}^{n}\left|y_{k,n}'' - (wp_{2n})''(x_{k,n})\right|\,\left|B_{k,n}(x)\right| = O(1)\left(\omega\left(f';\frac{1}{2n}\right) + \frac{1}{n^{2(\gamma+1)}}\right),$$

where γ is the number defined in (10).

Finally, we consider the term $|C_n P_n^{(\alpha,\beta)}(x)|$. From Corollary 1 we get

$$\left|C_n P_n^{(\alpha,\beta)}(x)\right| = \frac{|P_n^{(\alpha,\beta)}(x)|}{|P_n^{(\alpha,\beta)}(0)|}\left|p_{2n}(0) - \sum_{k=1}^{n} p_{2n}(x_{k,n}) l_{k,n}^2(0)\right|.$$

Using (19) and (46) we obtain that

$$\left|C_n P_n^{(\alpha,\beta)}(x)\right| = O(1)\left|p_{2n}(0) - \sum_{k=1}^{n} p_{2n}(x_{k,n}) l_{k,n}^2(0)\right| =$$

$$= O(1)\left|\sum_{k=1}^{n}\left[p_{2n}(0) - p_{2n}(x_{k,n})\right] l_{k,n}^2(0) + p_{2n}(0)\left(1 - \sum_{k=1}^{n} l_{k,n}^2(0)\right)\right|$$

for all $n = n_{\alpha,\beta} > n_0$.

In [4] I. Joó proved that

$$\left|1-\sum_{k=1}^{n} l_{k,n}^2(0)\right| = O(1)\sum_{k=1}^{n} \frac{|x_{k,n}|}{1-x_{k,n}^2} l_{k,n}^2(0) =$$

$$= O(1)\sum_{k=1}^{n} \frac{\left[P_n^{(\alpha,\beta)}(0)\right]^2}{(1-x_{k,n}^2)|x_{k,n}|\left|P_n^{(\alpha,\beta)\prime}(x_{k,n})\right|^2} = O(1)\left(\frac{\log n}{n^{2(\gamma+1)}}\right),$$

where γ is the number defined in (10).

From

$$\left|\sum_{k=1}^{n}\left[p_{2n}(0)-p_{2n}(x_{k,n})\right] l_{k,n}^2(0)\right| \leqq \sum_{k=1}^{n} |x_{k,n}|\left|p_{2n}'(\xi_{k,n})\right| l_{k,n}^2(0) =$$

$$= O(1)\sum_{k=1}^{n} |x_{k,n}| l_{k,n}^2(0) = O(1)\sum_{k=1}^{n} \frac{|x_{k,n}|}{1-x_{k,n}^2} l_{k,n}^2(0) = O(1)\left(\frac{\log n}{n^{2(\gamma+1)}}\right)$$

we obtain that

$$\left|C_n P_n^{(\alpha,\beta)}(x)\right| = O(1)\left(\frac{\log n}{n^{2(\gamma+1)}}\right),$$

which proves our theorem. □

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ON POWER SUMS OF COMPLEX NUMBERS WHOSE SUM IS 0

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Introduction. Let $z_1, \dots, z_n$ be complex numbers, $s_\nu = z_1^\nu + \dots + z_n^\nu$, and put $M_n = \inf \max_{1 \leqq \nu \leqq n} |s_\nu|$, where the infimum is taken over systems $\min_{1 \leqq j \leqq n} |z_j| = 1$ and $s_1 = 0$. The determination of M_n, which is clearly a minimum by Weierstrass' theorem, is raised by P. Turán in his posthumous book [1]. Simple examples show $M_{2m} \leqq 2$, $M_{3m-1} \leqq 3$, $M_{6m-3} \leqq 3$ ([2]) and by a note of [2] $M_n = O(1)$ for the outstanding case $n = 6m + 1$. M. Szalay has proved the lower bound $1 + \bigl(\log 2 - o(1)\bigr) / \log n$ for M_n ([2]). It is known that $M_2 = 2$, $M_3 = 3$, $M_4 = 2$, $1.9219 < M_5 < 2.2321$ and $1.7936 < M_6 \leqq 1.9968$ ([2]). We shall prove

THEOREM 1. $M_n \leqq 2 + \frac{3\pi^2/2 + o(1)}{n}$.

We also improve the lower bound:

THEOREM 2. $1 + \frac{1 - 55/\log n}{\log n} \leqq M_n$ $\bigl(n \geqq 10^{24}\bigr)$.

In Section 3 we obtain some numerical estimates for M_n $(6 \leqq n \leqq 19)$. The lower bounds are deduced as in [2], the upper ones are gained by direct computing of examples $|z_j| = 1$ $(j = 1, \dots, n)$. A detailed calculation is given for the cases $n = 6, 7$.

1. To prove Theorem 1 we can assume n is odd, since $M_{2m} \leqq 2$ by the result of [2]. Let $n = 2m - 1$ $(m \geqq 2)$ and consider the system

$$z_j = \begin{cases} \alpha e^{\varphi i j} & \text{for } 1 \leqq j \leqq m-1 \\ e^{\varphi i (j+1)} & \text{for } m \leqq j \leqq n \end{cases}$$

where $i^2 = -1$, $\varphi = 2\pi/(n+2)$ and $\alpha \geqq 1$ is to be chosen later. Clearly $\min_{1 \leqq j \leqq n} |z_j| = 1$ and we shall see that $s_1 = 0$ holds for a suitable α. For $1 \leqq \nu \leqq n$ we obtain

$$s_\nu = \alpha^\nu \sum_{j=1}^{m-1} e^{\varphi i j \nu} + \sum_{j=m+1}^{n+1} e^{\varphi i j \nu} =$$

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$$= (\alpha^\nu - 1)\sum_{j=1}^{m-1} e^{\varphi ij\nu} + \sum_{\substack{0\leqq j\leqq n+1\\ j\neq 0,m}} e^{\varphi ij\nu} = (\alpha^\nu - 1)\sum_{j=1}^{m-1} e^{\varphi ij\nu} - 1 - e^{\varphi im\nu} =$$

$$= e^{\varphi m\nu i/2}\left[(\alpha^\nu - 1)\frac{\sin\varphi(m-1)\nu/2}{\sin\varphi\nu/2} - 2\cos\varphi m\nu/2\right],$$

i.e.,

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\sin\varphi(m-1)\nu/2}{\sin\varphi\nu/2} - 2\cos\varphi m\nu/2\right|.$$

With the notation $\lambda = \frac{\varphi}{4} = \frac{\pi}{2(n+2)}$ we have

$$\frac{\varphi(m-1)}{2} = \frac{\pi}{2} - 3\lambda \quad\text{and}\quad \frac{\varphi m}{2} = \frac{\pi}{2} - \lambda,$$

so

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\sin\left(\frac{\pi}{2}\nu - 3\lambda\nu\right)}{\sin 2\lambda\nu} - 2\cos\left(\frac{\pi}{2}\nu - \lambda\nu\right)\right|,$$

which yields

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\pm\cos 3\lambda\nu}{\sin 2\lambda\nu} \mp 2\sin\lambda\nu\right| \text{ according as } \nu\equiv 1 \text{ or } 3 \pmod 4,$$

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\pm\sin 3\lambda\nu}{\sin 2\lambda\nu} \pm 2\cos\lambda\nu\right| \text{ according as } \nu\equiv 2 \text{ or } 4 \pmod 4.$$

Thus

$$(1)\qquad |s_\nu| = \begin{cases} \left|(\alpha^\nu - 1)\frac{\cos 3\lambda\nu}{\sin 2\lambda\nu} - 2\sin\lambda\nu\right| & \text{if } \nu \text{ is odd}\\ \left|(\alpha^\nu - 1)\frac{\sin 3\lambda\nu}{\sin 2\lambda\nu} + 2\cos\lambda\nu\right| & \text{if } \nu \text{ is even.}\end{cases}$$

Now define $\nu' = n + 2 - \nu$, then $\lambda\nu = \pi/2 - \lambda\nu'$ and (1) gives

$$(2)\qquad |s_\nu| = \begin{cases} \left|(\alpha^\nu - 1)\frac{\sin 3\lambda\nu'}{\sin 2\lambda\nu'} + 2\cos\lambda\nu'\right| & \text{if } \nu \text{ is odd}\\ \left|(\alpha^\nu - 1)\frac{\cos 3\lambda\nu'}{\sin 2\lambda\nu'} - 2\sin\lambda\nu'\right| & \text{if } \nu \text{ is even.}\end{cases}$$

Denoting $\min\{\nu, \nu'\}$ by κ and noting that ν and ν' are of opposite parity, we get by (1) and (2)

$$(3)\qquad |s_\nu| = \begin{cases} \left|(\alpha^\nu - 1)\frac{\cos 3\lambda\kappa}{\sin 2\lambda\kappa} - 2\sin\lambda\kappa\right| & \text{if } \kappa \text{ is odd}\\ \left|(\alpha^\nu - 1)\frac{\sin 3\lambda\kappa}{\sin 2\lambda\kappa} + 2\cos\lambda\kappa\right| & \text{if } \kappa \text{ is even.}\end{cases}$$

(3) shows that $s_1 = 0$ holds if α is chosen such that $\alpha - 1 = \frac{2\sin\lambda\sin 2\lambda}{\cos 3\lambda}$. Clearly

$$\alpha = 1 + \frac{2\lambda \cdot 2\lambda}{1}\big(1 + o(1)\big) = 1 + \big(\pi^2 + o(1)\big)/n^2 =$$
$$= \exp\Big\{\big(\pi^2 + o(1)\big)/n^2\Big\},$$

i.e.,

$$\alpha^n = \exp\Big\{\big(\pi^2 + o(1)\big)/n\Big\} = 1 + \big(\pi^2 + o(1)\big)/n. \tag{4}$$

Suppose first that κ is even. Since $\kappa < (n+2)/2$, $0 < \lambda\kappa < \pi/4$, we can observe that $\cos\lambda\kappa$, $\sin 2\lambda\kappa$ and $\sin 3\lambda\kappa$ are positive. Thus we can omit the sign of absolute-value in (3) and deduce by (4)

$$|s_\nu| \leqq (\alpha^n - 1)\frac{\sin 3\lambda\kappa}{\sin 2\lambda\kappa} + 2\cos\lambda\kappa < (\alpha^n - 1)\frac{3}{2} + 2 = 2 + \frac{3\pi^2/2 + o(1)}{n}.$$

Secondly, assume κ to be odd. If $\kappa = 1$ then $\nu = 1$ and $s_\nu = 0$. Therefore we only have to deal with the case $\kappa \geqq 3$. (3), (4) and $0 < \lambda\kappa < \pi/4$ yield

$$|s_\nu| \leqq \frac{\alpha^n - 1}{\sin 2\lambda\kappa} + 2\sin\lambda\kappa < \frac{\big(\pi^2 + o(1)\big)/n}{\sin 2\lambda\kappa} + 2\sin\lambda\kappa \leqq$$
$$\leqq \frac{\big(\pi^2 + o(1)\big)/n}{\frac{2}{\pi}\cdot 2\lambda\kappa} + 2\lambda\kappa = \frac{\big(\pi^2 + o(1)\big)/n}{\frac{2}{n+2}\kappa} + \frac{\pi}{n+2}\kappa = \frac{\pi^2 + o(1)}{2\kappa} + \frac{\pi}{n+2}\kappa.$$

This gives for $3 \leqq \kappa \leqq (n+2)/12$ and all large enough n

$$|s_\nu| \leqq \frac{\pi^2 + o(1)}{6} + \frac{\pi}{12} < \frac{10}{6} + \frac{4}{12} = 2,$$

and for $(n+2)/12 < \kappa$ and all large enough n

$$|s_\nu| \leqq \frac{\pi^2 + o(1)}{2(n+2)/12} + \frac{\pi}{2} < \frac{4}{2} = 2.$$

The results gained in the even and in the odd cases imply Theorem 1. □

REMARK. One would expect that a similar good or perhaps a better estimate can be obtained for M_n by the system

$$z_j = \begin{cases} \alpha + e^{\varphi ij} & \text{for } 1 \leqq j \leqq m-1 \\ e^{\varphi i(j+1)} & \text{for } m \leqq j \leqq n \end{cases}$$

where the complex α is chosen such that $s_1 = 0$. However, this is not so, since — by similar calculation as above — for any fixed and even ν

$$|s_\nu| = 2 + \frac{2\nu + o(1)}{n} \quad \text{as} \quad n \to \infty.$$

2. To prove Theorem 2 we use the fact (see [2]) that the unique positive root R_n of the polynomial (of degree $[n/2]$)

$$F_n(x) = -1 + \sum_{2j_2 + \ldots + nj_n = n} \prod_{2 \leqq \nu \leqq n} \frac{1}{j_\nu!}\left(\frac{x}{\nu}\right)^{j_\nu}$$

furnishes a lower bound for M_n (the j_ν's are nonnegative integers). Using the formula

$$\sum_{j_1 + \ldots + nj_n = n} \prod_{1 \leqq \nu \leqq n} \frac{1}{j_\nu!}\left(\frac{x}{\nu}\right)^{j_\nu} = \binom{x+n-1}{n}$$

(see [3]) we easily get for any positive integer $k < n/2$

$$F_n(x) = -1 + \sum_{r=0}^{2k-1} \binom{x+n-r-1}{n-r} \frac{(-x)^r}{r!} +$$

$$+ \frac{x^{2k}}{(2k-1)!} \sum_{j_1 + \ldots + nj_n = n-2k} \frac{1}{j_1 + 2k} \prod_{1 \leqq \nu \leqq n} \frac{1}{j_\nu!}\left(\frac{x}{\nu}\right)^{j_\nu} \leqq$$

$$\leqq -1 + \sum_{r=0}^{2k} \binom{x+n-r-1}{n-r} \frac{(-x)^r}{r!}.$$

Now let $n \geqq 10^{24}$, $\varepsilon = \frac{55}{\log n}$, $1 < x \leqq 1 + \frac{1-\varepsilon}{\log n}$ and $k = \left[\frac{n}{3}\right]$, then

$$F_n(x) \leqq -1 + \binom{x+n-2k-1}{n-2k} \sum_{r=0}^{2k} \frac{\binom{x+n-r-1}{n-r}}{\binom{x+n-2k-1}{n-2k}} \frac{(-x)^r}{r!}.$$

We have

$$\binom{x+n-2k-1}{n-2k} < \exp\left(\sum_{j=1}^{n-2k} \frac{x-1}{j}\right) <$$

$$< \exp\left\{(1-\varepsilon)\frac{\log(n-2k)+1}{\log n}\right\} < e^{1-\varepsilon},$$

and for any $0 \leqq r \leqq 2k$

$$1 \leqq \frac{\binom{x+n-r-1}{n-r}}{\binom{x+n-2k-1}{n-2k}} \leqq \frac{\binom{x+n-1}{n}}{\binom{x+n-2k-1}{n-2k}} < \exp\left(\sum_{j=n-2k+1}^{n} \frac{x-1}{j}\right) <$$

$$< \exp\left(\frac{1}{\log n} \sum_{j=n-2k+1}^{n} \frac{1}{j}\right) < \exp\left\{\frac{\log n - \log(n-2k)}{\log n}\right\} < \exp\left(\frac{2}{\log n}\right),$$

i.e.,

$$\left|\frac{\binom{x+n-r-1}{n-r}}{\binom{x+n-2k-1}{n-2k}} - 1\right| < \exp\left(\frac{2}{\log n}\right) - 1 < \frac{2}{\log n} \exp\left(\frac{2}{\log n}\right).$$

Thus

$$F_n(x) + 1 < e^{1-\varepsilon}\left\{\sum_{j=0}^{2k} \frac{(-x)^j}{j!} + \frac{2}{\log n} \exp\left(\frac{2}{\log n}\right) \sum_{j=0}^{2k} \frac{x^j}{j!}\right\} <$$

$$< e^{1-\varepsilon}\left\{e^{-x} + \frac{x^{2k+1}}{(2k+1)!} + \frac{2}{\log n} \exp\left(\frac{2}{\log n}\right) e^x\right\} <$$

$$< e^{1-\varepsilon}\left\{\frac{1}{e} + \frac{2^{2k+1}}{\left((2k+1)/3\right)^{2k+1}} + \frac{2}{\log n} \exp\left(1 + \frac{3}{\log n}\right)\right\} <$$

$$< e^{1-\varepsilon}\left(\frac{1}{e} + \frac{6}{2k+1} + \frac{6}{\log n}\right) <$$

$$< e^{1-\varepsilon}\left(\frac{1}{e} + \frac{7}{\log n}\right) < e^{-\varepsilon} + \frac{20}{\log n} < e^{-\varepsilon} + \frac{\varepsilon}{e} < e^{-\varepsilon} + \varepsilon e^{-\varepsilon},$$

since $n \geqq 10^{24}$ implies $\varepsilon < 1$. Finally

$$F_n(x) < -1 + e^{-\varepsilon}(1+\varepsilon) < -1 + e^{-\varepsilon} e^{\varepsilon} = 0.$$

Now $0 \leqq F_n(M_n)$, hence

$$1 + \frac{1 - 55/\log n}{\log n} = 1 + \frac{1-\varepsilon}{\log n} < M_n \quad (n \geqq 10^{24}). \quad \square$$

REMARK. A similar argument leads to the result

$$R_n = 1 + \frac{1 - o(1)}{\log n}.$$

3. Finally we consider systems of type

$$\left.\begin{array}{ll} z_j = \exp(\varphi_j i) & (1 \leqq j \leqq m) \\ z_{j+m} = \exp(-\varphi_j i) & (1 \leqq j \leqq m) \end{array}\right\} \quad \text{if} \quad n = 2m,$$

$$\left.\begin{array}{ll} z_1 = 1 & \\ z_{j+1} = \exp(\varphi_j i) & (1 \leqq j \leqq m-1) \\ z_{j+m} = \exp(-\varphi_j i) & (1 \leqq j \leqq m-1) \end{array}\right\} \quad \text{if} \quad n = 2m-1$$

where the φ_j are real numbers. If $s_\nu = z_1^\nu + \ldots + z_n^\nu$ and $s_1 = 0$ then $\max_{1 \leqq \nu \leqq n} |s_\nu|$ provides an upper estimate for M_n.

First we deal with the case $n = 6$ in detail. If

$$(5) \qquad (z - z_1) \ldots (z - z_6) = z^6 + a_1 z^5 + \ldots + a_5 z + a_6$$

then $a_6 = 1$, $a_2 = a_4 = \alpha$ and $a_3 = \beta$ with some real α and β and the condition $s_1 = 0$ implies $a_1 = a_5 = 0$. It is easy to verify that the numbers $\lambda_j = 2\cos\varphi_j$ $(j = 1, 2, 3)$, which lie in the interval $[-2; 2]$, are real roots of the equation

$$(6) \qquad \lambda^3 + (\alpha - 3)\lambda + \beta = 0.$$

Conversely, if we choose the real α and β such that (6) has three roots in $[-2; 2]$ and define a_6 to be 1, $a_1 = a_5$ to be 0, $a_2 = a_4$ to be α and a_3 to be β then the numbers $z_1, \ldots, z_6$ determined by (5) lie on the unit circle $|z| = 1$ and they satisfy $s_1 = 0$.

Calculating the power sums in terms of α and β by the Newton–Girard formulae we get

$$s_2 = -2\alpha,\ s_3 = -3\beta,\ s_4 = 2\alpha^2 - 4\alpha,\ s_5 = 5\alpha\beta,\ s_6 = 3\beta^2 - 2\alpha^3 + 6\alpha^2 - 6.$$

It seems to be convenient to put $\alpha = 1 - \varepsilon$ and $\beta = \frac{2}{5}(1 + \varepsilon)$, where $0 \leqq \varepsilon \leqq \frac{2}{5}$, since then

$$\max_{1 \leqq \nu \leqq 5} |s_\nu| = 2(1 - \varepsilon^2)$$

and

$$|s_6| = 2(1 - \varepsilon^2) - \frac{2}{25}(25\varepsilon^3 - 19\varepsilon^2 - 63\varepsilon + 6).$$

It can be checked that $25\varepsilon^3 - 19\varepsilon^2 - 63\varepsilon + 6$ has the only real root $\varepsilon = 0.092951\ldots$ in the interval $[0; 2/5]$ and this ε determines an $\alpha =$

$= 0.907048\dots$ and a $\beta = 0.437180\dots$ such that (6) has three roots in $[-2;\, 2]$. Thus the inequality

$$M_6 \leqq 2(1-\varepsilon^2) = 1.982720\dots$$

follows. □

Secondly, let $n = 7$. If $z_1 = 1$ and

$$(z-z_1)(z-z_2)\dots(z-z_7) = z^7 + a_1 z^6 + \dots + a_6 z + a_7 \tag{7}$$

then $a_7 = -1$, $a_2 = -\alpha$, $a_5 = \alpha$ and $a_3 = -\beta$, $a_4 = \beta$ with some real α and β and the condition $s_1 = 0$ implies $a_1 = a_6 = 0$. It is easy to verify that the numbers $\lambda_j = 2\cos\varphi_j$ $(j = 1,2,3)$, which lie in the interval $[-2;\, 2]$, are real roots of the equation

$$\lambda^3 + \lambda^2 - (\alpha+2)\lambda - (\alpha+\beta+1) = 0. \tag{8}$$

Conversely, if we choose the real α and β such that (8) has three roots in $[-2;\, 2]$ and define a_7 to be -1, $a_1 = a_6$ to be 0, a_2 as $-\alpha$, a_5 as α, a_3 as $-\beta$ and finally a_4 as β then the numbers $z_1 = 1, z_2, \dots, z_7$ determined by (7) lie on the unit circle $|z| = 1$ and they satisfy $s_1 = 0$.

It is convenient to put $\beta = 2\alpha/3$. Calculating the power sums in terms of α by the Newton–Girard formulae we get

$$s_2 = s_3 = 2\alpha, \quad s_4 = 2\alpha^2 - \frac{8}{3}\alpha, \quad s_5 = \frac{10}{3}\alpha^2 - 5\alpha,$$

$$s_6 = 2\alpha^3 - \frac{8}{3}\alpha^2, \quad s_7 = 7\left(1 - \frac{13}{9}\alpha^2 + \frac{2}{3}\alpha^3\right),$$

which yields that for $9/10 \leqq \alpha < 1$

$$\max_{1\leqq\nu\leqq 6} |s_\nu| = 2\alpha \quad \text{and} \quad |s_7| = 7\left(1 - \frac{13}{9}\alpha^2 + \frac{2}{3}\alpha^3\right).$$

It can be checked that $2\alpha = 7\left(1 - \frac{13}{9}\alpha^2 + \frac{2}{3}\alpha^3\right)$ has the only real root $\alpha = 0.947181\dots$ in the interval $[0;\, 9/10]$ and this α determines a $\beta =$ $= 0.631454\dots$ such that (8) has three roots in $[-2;\, 2]$. Thus the inequality

$$M_7 \leqq 2\alpha = 1.894363\dots$$

holds. □

Further on we indicate for comparison the lower bounds R_n of Section 2. The upper bounds are derived from systems described above. We have the following inequalities for M_n $(6 \leqq n \leqq 19)$:

$$1.793610\ldots \leqq M_6 \leqq 1.982720\ldots$$

$$1.719907\ldots \leqq M_7 \leqq 1.894363\ldots$$

$$1.662581\ldots \leqq M_8 \leqq 1.999796\ldots \qquad \begin{cases} \varphi_1 = 33.987585\ldots^\circ \\ \varphi_2 = 73.303745\ldots^\circ \\ \varphi_3 = 109.097547\ldots^\circ \\ \varphi_4 = 142.118198\ldots^\circ \end{cases}$$

$$1.618555\ldots \leqq M_9 \leqq 1.790782\ldots \qquad \begin{cases} \varphi_1 = 38.430487\ldots^\circ \\ \varphi_2 = 67.220086\ldots^\circ \\ \varphi_3 = 134.114614\ldots^\circ \\ \varphi_4 = 167.022740\ldots^\circ \end{cases}$$

$$1.583255\ldots \leqq M_{10} \leqq 1.973688\ldots \qquad \begin{cases} \varphi_1 = 32.074778\ldots^\circ \\ \varphi_2 = 57.616740\ldots^\circ \\ \varphi_3 = 91.887543\ldots^\circ \\ \varphi_4 = 117.618984\ldots^\circ \\ \varphi_5 = 152.425330\ldots^\circ \end{cases}$$

$$1.554267\ldots \leqq M_{11} \leqq 2.119011\ldots \qquad \begin{cases} \varphi_1 = 44.038349\ldots^\circ \\ \varphi_2 = 70.364417\ldots^\circ \\ \varphi_3 = 96.533487\ldots^\circ \\ \varphi_4 = 125.493345\ldots^\circ \\ \varphi_5 = 149.374899\ldots^\circ \end{cases}$$

$$1.529965\ldots \leqq M_{12} \leqq 1.998574\ldots \qquad \begin{cases} \varphi_1 = 26.280566\ldots^\circ \\ \varphi_2 = 52.020428\ldots^\circ \\ \varphi_3 = 76.396810\ldots^\circ \\ \varphi_4 = 102.127160\ldots^\circ \\ \varphi_5 = 129.177757\ldots^\circ \\ \varphi_6 = 154.877600\ldots^\circ \end{cases}$$

$$1.509245\ldots \leqq M_{13} \leqq 2.126728\ldots \qquad \begin{cases} \varphi_1 = 19.191938\ldots^\circ \\ \varphi_2 = 39.803824\ldots^\circ \\ \varphi_3 = 88.675687\ldots^\circ \\ \varphi_4 = 117.904497\ldots^\circ \\ \varphi_5 = 142.222650\ldots^\circ \\ \varphi_6 = 167.789878\ldots^\circ \end{cases}$$

$$1.491331\ldots \leqq M_{14} \leqq 1.828905\ldots \qquad \begin{cases} \varphi_1 = 9.069892\ldots^\circ \\ \varphi_2 = 31.062936\ldots^\circ \\ \varphi_3 = 52.885792\ldots^\circ \\ \varphi_4 = 98.672340\ldots^\circ \\ \varphi_5 = 121.786322\ldots^\circ \\ \varphi_6 = 142.291313\ldots^\circ \\ \varphi_7 = 168.191278\ldots^\circ \end{cases}$$

$$1.475659\ldots \leqq M_{15} \leqq 1.967363\ldots \qquad \begin{cases} \varphi_1 = 21.810490\ldots^\circ \\ \varphi_2 = 40.279274\ldots^\circ \\ \varphi_3 = 61.823098\ldots^\circ \\ \varphi_4 = 105.380928\ldots^\circ \\ \varphi_5 = 125.044087\ldots^\circ \\ \varphi_6 = 146.843332\ldots^\circ \\ \varphi_7 = 170.714120\ldots^\circ \end{cases}$$

$$1.4618007\ldots \leqq M_{16} \leqq 2$$

$$1.449458\ldots \leqq M_{17} \leqq 1.948290\ldots \qquad \begin{cases} \varphi_1 = 19.331397\ldots^\circ \\ \varphi_2 = 39.866743\ldots^\circ \\ \varphi_3 = 58.023924\ldots^\circ \\ \varphi_4 = 75.955699\ldots^\circ \\ \varphi_5 = 114.414529\ldots^\circ \\ \varphi_6 = 133.645456\ldots^\circ \\ \varphi_7 = 153.498174\ldots^\circ \\ \varphi_8 = 170.045331\ldots^\circ \end{cases}$$

$$1.438363\ldots \leqq M_{18} \leqq 2$$

$$1.428328\ldots \leqq M_{19} \leqq 1.888063\ldots \qquad \begin{cases} \varphi_1 = 18.409141\ldots^\circ \\ \varphi_2 = 37.223032\ldots^\circ \\ \varphi_3 = 52.895537\ldots^\circ \\ \varphi_4 = 70.133067\ldots^\circ \\ \varphi_5 = 88.133122\ldots^\circ \\ \varphi_6 = 123.294802\ldots^\circ \\ \varphi_7 = 139.974654\ldots^\circ \\ \varphi_8 = 156.231133\ldots^\circ \\ \varphi_9 = 172.269165\ldots^\circ \end{cases}$$

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GENERAL RESULTS ON STRONG APPROXIMATION BY CESÀRO MEANS OF NEGATIVE ORDER

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1. Introduction. Let $\{\varphi_n(x)\}$ be an orthonormal system on a finite interval (a, b). We shall consider real orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \qquad \text{with} \qquad \sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.1}$$

It is well known that the partial sums $s_n(x)$ of any such series converge in the L^2 norm to a square-integrable function $f(x)$.

The following theorem, proved in [3], provides a very good quantitative estimate for the pointwise approximation of $f(x)$ by the arithmetic means of $s_n(x)$:

Let $0 < \gamma < 1$. *If*

$$\sum_{n=0}^{\infty} c_n^2 n^{2\gamma} < \infty, \tag{1.2}$$

then

$$\frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu(x) - f(x) = o_x(n^{-\gamma})$$

almost everywhere (a.e.) in (a, b).

This result was extended by G. Sunouchi [18] to strong approximation as follows:

THEOREM A. *Let* $0 < \gamma < 1$ *and* $\alpha > 0$. *If* (1.2) *holds and* $0 < p < \gamma^{-1}$, *then*

$$\left\{ \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} |s_k(x) - f(x)|^p \right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.3}$$

a.e. in (a, b), *where* $A_n^\alpha := \binom{n+\alpha}{n}$.

In [5] we generalized this result in such a way that we replaced the partial sums in (1.3) by Cesàro means of negative order and the external

Cesàro means by a general summation. We consider a regular summation method T_n determined by a triangular matrix $\|\alpha_{nk}/A_n\|$ ($\alpha_{nk} \geqq 0$ and $A_n := \sum_{k=0}^{n} \alpha_{nk}$), i.e. if s_k tends to s, then

$$T_n := \frac{1}{A_n} \sum_{k=0}^{n} \alpha_{nk} s_k \to s.$$

In the sequel K, K_i will denote positive constants, not necessarily the same ones, furthermore $K(.)$ denotes constants depending only on those parameters indicated. Our generalization reads as follows:

THEOREM B. *Suppose that* $0 < \gamma < 1$ *and* $0 < p < \gamma^{-1}$, *furthermore that there exists a number* $\rho > 1$ *such that*

$$\frac{\rho p}{\rho - 1} \geqq 2,$$

and with this ρ *for any* $0 < \delta < 1$ *and* $2^m < n \leqq 2^{m+1}$

$$\sum_{\ell=0}^{m} \left\{ \sum_{\nu=2^\ell-1}^{\min(2^{\ell+1},n)} \alpha_{n\nu}^{\rho} (\nu+1)^{\rho(1-\delta)-1} \right\}^{1/\rho} \leqq K n^{-\delta} A_n.$$

Then, (1.2) *implies, for arbitrary*

$$d > 1 - \frac{\rho - 1}{\rho p},$$

that

$$\left\{ \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |f(x) - \sigma_\nu^{d-1}(x)|^p \right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.4}$$

holds a.e. in (a,b), *where* $\sigma_n^\alpha(x)$ *denotes the* n*-th* (C,α)*-means of series* (1.1).

After several articles have dealt with strong approximation (see e.g. [4], [6]–[10], [16]), in a joint paper with A. Meir [15], we proved a very general result which includes almost all of the theorems proved previously and gave some new consequences, as well.

In order to recall this joint result we present some definitions and notations, furthermore some assumptions to be kept throughout this paper.

Let $\alpha := \{\alpha_k(\omega)\}$, $k = 0, 1, \ldots$ denote a sequence of non-negative functions defined for $0 \leqq \omega < \infty$, satisfying

$$\sum_{k=0}^{\infty} \alpha_k(\omega) \equiv 1.$$

We shall assume that the linear transformation of real sequences $x := \{x_k\}$ given by

$$A_\omega(x) := \sum_{k=0}^{\infty} \alpha_k(\omega) x_k, \qquad \omega \to \infty$$

is regular [2, p.49]. Let $\eta := \eta(t)$ and $g(t)$ denote non-decreasing positive functions defined for $0 \leqq t < \infty$, furthermore let $\mu := \{\mu_m\}$, $m = 0, 1, \ldots$ denote a fixed, increasing sequence of integers with $\mu_0 = 0$. We shall assume that there exist positive integers N and h so that

$$\mu_{m+1} \leqq N\mu_m, \qquad m = 1, 2, \ldots, \tag{1.5}$$

$$\eta(\mu_{m+1}) \leqq N\eta(\mu_n), \qquad m = 1, 2, \ldots, \tag{1.6}$$

$$\eta(\mu_{m+h}) \geqq 2\eta(\mu_m), \qquad m = 1, 2, \ldots. \tag{1.7}$$

For $r > 1$, $\omega > 0$ and $m = 1, 2, \ldots$ we define

$$\rho_m(\omega, r) := \left\{ \frac{1}{\mu_{m+1}} \sum_{k=\mu_m}^{\mu_{m+1}-1} \big(\alpha_k(\omega)\big)^r \right\}^{1/r}. \tag{1.8}$$

In terms of the quantities introduced above we formulate our result proved in [15].

THEOREM C. *Let $p > 0$. Suppose that there exist $r > 1$ and a constant $K(r, \mu, \eta)$ such that for any $\omega > 0$*

$$\sum_{m=0}^{\infty} \mu_m \rho_m(\omega, r) \eta(\mu_m)^{-p} \leqq K(r, \mu, \eta)\big(g(\omega)/\eta(\omega)\big)^p. \tag{1.9}$$

If

$$\sum_{n=1}^{\infty} c_n^2 \eta(n)^2 < \infty \tag{1.10}$$

then

(1.11)

$$A_\omega(f,p,\nu;x) := \left\{\sum_{k=0}^{\infty} \alpha_k(\omega)\left|s_{\nu_k}(x)-f(x)\right|^p\right\}^{1/p} = O_x\big(g(\omega)/\eta(\omega)\big)$$

a.e. in (a,b) *for any increasing sequence* $\nu := \{\nu_k\}$ *of positive integers.*

If, in addition, for every fixed m,

$$\rho_m(\omega,r) = o\left(\big(g(\omega)/\eta(\omega)\big)^p\right), \qquad as \qquad \omega\to\infty,$$

then the O_x *in* (1.11) *can be replaced by* o_x.

Theorem C gives estimates for the pointwise approximation of $f(x)$ by a large family of Hausdorff transformations and $[J,f]$-transformations. Because of the generality of Theorem C it is really not easy to realize how many well-known summation methods are included in this theorem. We refer to [15] for some examples. Here we present only three known and frequently used methods, namely the Cesàro, the Riesz and the generalized Abel transformations. We also mention that the corollaries to be recalled here were proved before appearing Theorem C individually, as well. Corollaries C.1 and C.2 were proved in [16], and C.3 in [6].

COROLLARY C.1. *Let* $p>0$, $\alpha>0$. *If* $0<\gamma<p^{-1}$ *and* (1.2) *holds, then*

$$\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}\left|s_{\nu_k}(x)-f(x)\right|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.12}$$

a.e. in (a,b) *for any increasing sequence* $\{\nu_k\}$.

COROLLARY C.2. *Let* $p>0$, $\beta>0$. *If* $0<p\gamma<\beta$ *and* (1.2) *holds, then*

$$\left\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}\left|s_{\nu_k}(x)-f(x)\right|^p\right\}^{1/p} = o_x(n^{-\gamma})$$

a.e. in (a,b) *for any* $\{\nu_k\}$.

COROLLARY C.3. *Let* q *be a non-negative integer and* $p>0$. *If* $0<\gamma<$ $<p^{-1}$ *and* (1.2) *holds, then*

$$\left\{(1-t)^{q+1}\sum_{k=0}^{\infty}\binom{q+k}{k}t^k\left|s_{\nu_k}(x)-f(x)\right|^p\right\}^{1/p} = o_x\big((1-t)^\gamma\big)$$

a.e. in (a,b) *for any* $\{\nu_k\}$.

We mention that Corollary C.3 is an extension of a result of L. Rempulska [17] to strong means.

Comparing the theorem of Sunouchi and Corollary C.1, we see that among the assumptions of Corollary C.1 the restriction $\gamma < 1$ does not appear. This is a great advantage of Corollary C.1. But if we consider Theorem B in the special case $\alpha_{n\nu} = A_{n-\nu}^{\alpha-1}$, then (1.4) has the advantage regarding (1.12) that in (1.4) we can approximate the function $f(x)$ by Cesàro means of negative order, although then among the conditions the restriction $\gamma < 1$ appeared again. So it is natural to ask whether in the general case, or only in the Cesàro case, if we want to approximate the function $f(x)$ by Cesàro means of negative order, then the restriction $\gamma < 1$ can be omitted.

We have so far proved only that in the special case $\eta(t) = t^\gamma$ with the restriction $\gamma < 1$ the function $f(x)$ can be approximated by Cesàro means of negative order for the same class of summations given in Theorem C.

The main tool of the proof was the following Proposition proved recently in [11]. This reads as follows:

PROPOSITION. *If* $p > 0$, $0 < \gamma < 1$ *and* $d > \max(1/2, (p-1)/p)$, *then* (1.2) *implies*

$$\left\{\frac{1}{n}\sum_{k=n+1}^{2n} \left|\sigma_k^{d-1}(\nu;x) - f(x)\right|^p\right\}^{1/p} = o_x(n^{-\gamma})$$

a.e. in (a,b) *for any increasing sequence* $\nu := \{\nu_k\}$, *where*

$$\sigma_n^\alpha(\nu;x) := \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_{\nu_k}(x).$$

Using the notations introduced above we formulate the result proved in [12], and mention that it is the most general result in this field.

THEOREM D. *Let* $p > 0$, $d > \max(1/2, (p-1)/p)$ *and* $0 < \gamma < 1$. *Suppose that there exist* $r > 1$ *and a constant* $K(r,\mu)$ *such that for any* $\omega > 0$

$$\sum_{m=0}^{\infty} \mu_m \rho_m(\omega, r)\mu_m^{-\gamma p} \leqq K(r,\mu)\big(g(\omega)/\omega^\gamma\big)^p. \tag{1.13}$$

If (1.2) *holds, then*

(1.14)

$$A_\omega(f,p,d,\nu;x) := \left\{\sum_{k=0}^{\infty} \alpha_k(\omega)\left|\sigma_k^{d-1}(\nu;x) - f(x)\right|^p\right\}^{1/p} = O_x\big(g(\omega)/\omega^\gamma\big)$$

a.e. in (a,b) *for any increasing sequence* $\nu := \{\nu_k\}$ *of positive integers.*

If, in addition, for every fixed m,

$$(1.15) \qquad \rho_m(\omega, r) = o\left(\left(g(\omega)/\omega^\gamma\right)^p\right), \qquad \textit{as} \qquad \omega \to \infty,$$

then the O_x *in* (1.14) *can be replaced by* o_x.

We point out that the most important special case of Theorem D, in our view, is when both (1.13) and (1.15) are satisfied with $g(\omega) \equiv 1$. In this case we get that

$$(1.16) \qquad A_\omega(f, p, d, \nu; x) = o_x(\omega^{-\gamma})$$

holds a.e. in (a,b).

We want to point out again that Theorem C, contrary to Theorems A, B and D, does not claim the extra restriction $\gamma < 1$. This is a great advantage of this theorem, but it does not allow approximating with Cesàro means of negative order.

The common kernel of the proofs of Theorem A, B and D is based on a very interesting result of T. M. Flett [1] and a useful lemma of G. Sunouchi [18]. Unfortunately, Sunouchi's lemma requires the assumption $0 < \gamma < 1$, furthermore the Flett's result works only if $\eta(n)$ has the form of n^γ; e.g. if $\eta(n) = n^\gamma$, $\gamma > 0$.

Recently we ([13]) generalized Flett's result replacing the factors n^γ by more general factors $\gamma(n)$. Having this generalization of Flett's result (here Lemma 4) and after extending the lemma of Sunouchi by similar way in the present paper (here Lemma 6) we shall be ready to generalize Theorem D, and in a certain range of the functions $\eta(t)$ our new result will generalize Theorem C replacing the partial sums by Cesàro means.

2. The main result. Before formulating our result we recall a definition and define three properties of the function $\gamma(t)$ which will replace essentially the function $\eta(t)$ used above.

A sequence $\{\gamma_n\}$ of positive numbers is said to be *quasi geometrically increasing (decreasing)* if there exist natural numbers μ, ν and a real number $K \geqq 1$ such that $\gamma_{n+\mu} \geqq 2\gamma_n$ and $\gamma_n \leqq K\gamma_{n+1}$ $\left(\gamma_{n+\mu} \leqq \frac{1}{2}\gamma_n \text{ and } \gamma_{n+1} \leqq K\gamma_n\right)$ hold for all natural numbers $n \geqq \nu$.

We shall say that the function $\gamma(t)$ has the following properties:

P_1: the sequence $\{\gamma(2^n)\}$ is quasi geometrically increasing;

P_2: the sequence $\{\gamma(2^n)2^{-n}\}$ is quasi geometrically decreasing;

$P_{p,r}$: the sequence $\{\gamma(2^n)(2^{n(1-r)/rp}\}$ is quasi geometrically decreasing with some $r > 1$ and $p > 0$.

In terms of the quantities and properties introduced above we are ready to state our first theorem.

THEOREM 1. *Let $p > 0$, $d > \max\left(1/2, (p-1)/p\right)$ and let $\gamma(t)$ be a positive non-decreasing function defined for $0 \leqq t < \infty$ with properties P_1 and P_2. If there exist $r > 1$ and a constant $K(r,\mu)$ such that for any $\omega > 0$*

$$\sum_{m=0}^{\infty} \mu_m \rho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K(r,\mu)\left(g(\omega)/\gamma(\omega)\right)^p, \tag{2.1}$$

then

$$\sum_{n=1}^{\infty} c_n^2 \gamma(n)^2 < \infty \tag{2.2}$$

implies that

(2.3)

$$A_\omega(f,p,d,\nu;x) := \left\{\sum_{k=0}^{\infty} \alpha_k(\omega)\left|\sigma_k^{d-1}(\nu;x) - f(x)\right|^p\right\}^{1/p} = O_x\left(g(\omega)/\gamma(\omega)\right)$$

holds a.e. in (a,b) for any increasing sequence $\nu := \{\nu_k\}$ of positive integers.

If, in addition, for every fixed m,

$$\rho_m(\omega, r) = o\left(\left(g(\omega)/\gamma(\omega)\right)^p\right), \quad as \quad \omega \to \infty, \tag{2.4}$$

then the O_x in (2.3) can be replaced by o_x.

It is easy to see that Theorem 1 in the special case $\gamma(t) = t^\gamma$ with $0 < \gamma < 1$ reduces to Theorem D; and if $\eta(t) = \gamma(t)$ then all of the conditions of Theorem C are satisfied under the assumptions of Theorem 1. This means that Theorem 1 is a slight improvement of Theorem D; but it is only partly a generalization of Theorem C, namely we claim more about $\gamma(t)$ in Theorem 1 than what $\eta(t)$ has to satisfy in Theorem C.

In order to help the comparison of the new and known results we shall follow the structure of the paper [15].

3. Lemmas. To prove our theorems and their consequences we need five known lemmas and three new ones to be proved in this paper.

LEMMA 1 [8]. *Let $\delta > 0$ and $\{\delta_n\}$ be an arbitrary sequence of positive numbers. Suppose that for any orthonormal system the condition*

$$\sum_{n=1}^{\infty} \delta_n \left(\sum_{k=n}^{\infty} c_k^2\right)^\delta < \infty$$

implies that the sequence $\{s_n(x)\}$ *possesses a property* P*, then any subsequence* $\{s_{\nu_n}(x)\}$ *also possesses property* P.

LEMMA 2 [2]. *Let* $\{\alpha_k(n)\}$*, the coefficients of a regular Hausdorff transformation, be given by*

$$\alpha_k(n) := \int_0^1 \binom{n}{k} t^k(1-t)^{n-k}\phi(t)dt,$$

where $\phi(t) \in L^r(0,1)$ *for some* $r > 1$*. Then*

$$\sum_{k=0}^{n} |\alpha_k(n)|^r \leqq K(r)(n+1)^{1-r}. \tag{3.1}$$

LEMMA 3 [15]. *Let* $\{\alpha_k(\omega)\}$*, the coefficients of a regular* $[J,t]$*-transformation, be given by*

$$\alpha_k(\omega) := \frac{\omega^k}{k!}\int_0^1 t^\omega\big(\log(1/t)\big)^k\phi(t)dt,$$

where $\phi(t) \in L^r(0,1)$ *for some* $r > 1$*. Then for* $\ell = 0, 1, \dots$

$$\sum_{k=\ell}^{\infty} |\alpha_k(\omega)|^r \leqq K(r)\big((1+\omega)^{-1}e^{-\ell/(1+\omega)}\big)^{r-1}. \tag{3.2}$$

Before formulating the next lemma we recall some definitions and notations.

Let $k \geqq 1$, $\alpha > -1$ and $\gamma(t)$ be a positive non-decreasing function defined for $1 \leqq t < \infty$. We say that a numerical series $\sum_{n=0}^{\infty} a_n$ is summable $|C,\alpha,\gamma(t)|_k$ if the series $\sum_{n=1}^{\infty}\gamma(n)^k n^{-1}|\tau_n^\alpha|^k$ is convergent, where $\tau_n^\alpha := n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$ and σ_n^α denotes the nth Cesàro mean of order α of the series $\sum a_n$. It is well known that if $\alpha > 0$ then $\tau_n^\alpha = \alpha(\sigma_n^{\alpha-1} - \sigma_n^\alpha)$.

LEMMA 4 [13]. *Let* $r \geqq k > 1$, $\alpha > -1$, $\beta \geqq \alpha + k^{-1} - r^{-1}$*, and* $\gamma(t)$ *be a non-decreasing positive function defined for* $1 \leqq t < \infty$ *so that with some* $C > 1$

$$\limsup_{t\to\infty}\frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1}. \tag{3.3}$$

Then if the series $\sum_{n=0}^{\infty} a_n$ *is summable* $\left|C,\alpha,\gamma(t)\right|_k$, *it is summable* $\left|C,\beta,\gamma(t)\right|_r$, *furthermore*

$$\left\{\sum_{n=1}^{\infty}\gamma(n)^r n^{-1}|\tau_n^\beta|^r\right\}^{1/r} \leqq K\left\{\sum_{n=1}^{\infty}\gamma(n)^k n^{-1}|\tau_n^\alpha|^k\right\}^{1/k}. \tag{3.4}$$

LEMMA 5 [14]. *For any positive sequence* $\{\gamma_n\}$ *the inequalities*

$$\sum_{n=m}^{\infty}\gamma_n \leqq K\gamma_m \qquad (m=1,2,\ldots,K\geqq 1), \tag{3.5}$$

or

$$\sum_{n=1}^{m}\gamma_n \leqq K\gamma_m \qquad (m=1,2,\ldots,K\geqq 1) \tag{3.6}$$

hold if and only if the sequence $\{\gamma_n\}$ *is quasi geometrically decreasing or increasing, respectively.*

LEMMA 6. *Let* $\gamma(t)$ *be a positive non-decreasing function defined for* $0 \leqq$ $\leqq t < \infty$ *with property* P_2. *If* (2.2) *holds then*

$$\int_a^b\left\{\sum_{n=0}^{\infty}\gamma(n+1)^2(n+1)^{-1}\left|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)\right|^2\right\}dx \leqq K\sum_{n=1}^{\infty}c_n^2\gamma(n)^2$$

for any $\alpha > 1/2$.

This lemma in the special case $\gamma(t)=t^\gamma$ with $0<\gamma<1$ was proved by G. Sunouchi [18]. Our proof to be given below follows the line of the proof given by Sunouchi.

PROOF. Since

$$\int_a^b\left|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)\right|^2dx = \left\{\sum_{\nu=0}^{n}\nu^2(A_{n-\nu}^{\alpha-1})^2c_\nu^2\right\}\alpha^{-2}(A_n^\alpha)^{-2},$$

it follows that

$$\sum_{n=0}^{\infty}\int_a^b\gamma(n+1)^2(n+1)^{-1}\left|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)\right|^2dx \leqq$$

$$\leqq K\sum_{n=0}^{\infty}\left\{\sum_{n=0}^{\infty}\nu^2(n-\nu+1)^{2(\alpha-1)}c_\nu^2\right\}\gamma(n+1)^2(n+1)^{-2\alpha-1} =$$

$$= K\sum_{\nu=0}^{\infty}\nu^2 c_\nu^2 \sum_{n=\nu}^{\infty}\gamma(n+1)^2(n-\nu+1)^{2(\alpha-1)}(n+1)^{-2\alpha-1} =$$

$$= K\sum_{\nu=0}^{\infty}\nu^2 c_\nu^2\left(\sum_{n=\nu}^{2\nu}+\sum_{n=2\nu+1}^{\infty}\right) =: \sum\nolimits_1+\sum\nolimits_2,$$

say. On account of $\alpha > 1/2$,

$$\sum\nolimits_1 \leqq K\sum_{\nu=0}^{\infty}\nu^2 c_\nu^2\gamma(2\nu+1)^2\nu^{-2\alpha-1}\sum_{n=1}^{\nu}n^{2(\alpha-1)} \leqq$$

$$\leqq K\sum_{\nu=0}^{\infty}c_\nu^2\gamma(2\nu+1)^2 \leqq K_1\sum_{\nu=0}^{\infty}\gamma(\nu)^2 c_\nu^2 < \infty.$$

At the last estimation we used the fact that the sequence $\{\gamma(2^n)2^{-n}\}$ is quasi geometrically decreasing.

To estimate $\sum_2$ we apply statement (3.5) of Lemma 5, furthermore the obvious fact that if a sequence $\{\gamma_n\}$ is quasi geometrically decreasing then so is $\{\gamma_n^2\}$. Then if $m = m(\nu)$ satisfies the inequalities $2^{m-1} \leqq 2\nu+1 < 2^m$ we have

$$\sum\nolimits_2 \leqq K\sum_{\nu=0}^{\infty}\nu^2 c_\nu^2\sum_{n=2\nu+1}^{\infty}\gamma(n+1)^2 n^{-3} \leqq$$

$$\leqq K\sum_{\nu=1}^{\infty}\nu^2 c_\nu^2\left(\sum_{n=2\nu+1}^{2^m}+\sum_{k=m}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\right)\gamma(n+1)^2 n^{-3} \leqq$$

$$\leqq K\sum_{\nu=1}^{\infty}\nu^2 c_\nu^2\left\{\gamma(2^m)^2 2^{-2m}+\sum_{k=m}^{\infty}\left(\gamma(2^k)2^{-k}\right)^2\right\} \leqq$$

$$\leqq K_1\sum_{\nu=1}^{\infty}\nu^2 c_\nu^2\left(\gamma(2^m)2^{-m}\right)^2 \leqq K_2\sum_{\nu=1}^{\infty}c_\nu^2\gamma(\nu)^2 < \infty.$$

Summing up our estimations we obtain the statement of Lemma 6.

LEMMA 7. *Let $p > 0$, $d > \max\left(1/2,(p-1)/p\right)$ and let $\gamma(t)$ be a positive non-decreasing function defined for $1 \leqq t < \infty$ with property P_2. If (2.2) holds, then*

(3.7)

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{\gamma(n)^p}{n} \sum_{\nu=n+1}^{2n} \left| \sigma_\nu^{d-1}(x) - \sigma_\nu^d(x) \right|^p \right)^{1/p} \right\}^2 dx \leqq K \sum_{n=1}^{\infty} c_n^2 \gamma(n)^2.$$

PROOF. On account of $d > \max\left(\frac{1}{2}, 1 - \frac{1}{p}\right)$ it is easy to see that there exists a number $\rho > 1$ such that

$$(3.8) \qquad \rho p \geqq 2 \qquad \text{and} \qquad d > 1 - (\rho p)^{-1}$$

hold. Putting $\tau_n^d(x) := d\left(\sigma_n^{d-1}(x) - \sigma_n^d(x)\right)$ $\left(= n\left(\sigma_n^d(x) - \sigma_{n-1}^d(x)\right)\right.$ since $d > 0)$ and applying Hölder's inequality we obtain that

$$(3.9) \qquad \sum_{\nu=n+1}^{2n} \left| \tau_\nu^d(x) \right|^p \leqq n^{1/\rho'} \left\{ \sum_{\nu=n+1}^{2n} \left| \tau_\nu^d(x) \right|^{p\rho} \right\}^{1/\rho} \leqq$$

$$\leqq K n \gamma(n)^{-p} \left\{ \sum_{\nu=n+1}^{2n} \gamma(\nu)^{p\rho} \nu^{-1} \left| \tau_\nu^d(x) \right|^{p\rho} \right\}^{1/\rho}.$$

By the second statement of (3.8) we can choose α^* such that

$$(3.10) \qquad d - \frac{1}{2} + \frac{1}{\rho p} > \alpha^* > \frac{1}{2}$$

holds. By (3.10) the parameter conditions of Lemma 4 are fulfilled with $r = p\rho$, $k = 2$, $\alpha = \alpha^*$ and $\beta = d$. The assumption (3.3) is also fulfilled since the sequence $\{\gamma(2^n)2^{-n}\}$ is quasi geometrically decreasing. Using Lemma 4 we get

$$(3.11) \quad \left\{ \sum_{n=1}^{\infty} \gamma(n)^{p\rho} n^{-1} \left| \tau_n^d(x) \right|^{p\rho} \right\}^{1/p\rho} \leqq K \left\{ \sum_{n=1}^{\infty} \gamma(n)^2 n^{-1} \left| \tau_n^{\alpha^*}(x) \right|^2 \right\}^{1/2}.$$

Thus, by (3.9), (3.10), (3.11) and Lemma 6, we get

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{\gamma(n)^p}{n} \sum_{\nu=n+1}^{2n} \left| \tau_\nu^d(x) \right|^p \right)^{1/p} \right\}^2 dx \leqq$$

$$\leqq K\int_a^b \left\{\sum_{n=1}^{\infty}\gamma(n)^2 n^{-1}\left|\tau_n^{\alpha^*}(x)\right|^2\right\}dx \leqq K\sum_{n=1}^{\infty}c_n^2\gamma(n)^2<\infty,$$

which proves statement (3.7).

LEMMA 8. *If $p>0$, $d>\max(1/2,(p-1)/p)$, $\gamma(t)$ has the same properties as in Lemma 7, and additionally it has property P_1, too, then (2.2) implies*

$$\left\{\frac{1}{n}\sum_{\nu=n+1}^{2n}\left|f(x)-\sigma_\nu^{d-1}(x)\right|^p\right\}^{1/p}=o_x\big(\gamma(n)^{-1}\big) \tag{3.12}$$

a.e. in (a,b).

PROOF. It is clear that

$$\left\{\frac{1}{n}\sum_{\nu=n+1}^{2n}\left|f(x)-\sigma_\nu^{d-1}(x)\right|^p\right\}^{1/p}\leqq \tag{3.13}$$

$$\leqq K\left\{\frac{1}{n}\sum_{\nu=n+1}^{2n}\left|f(x)-\sigma_\nu^{d}(x)\right|^p\right\}^{1/p}+K\left\{\frac{1}{n}\sum_{\nu=n+1}^{2n}\left|\sigma_\nu^{d}(x)-\sigma_\nu^{d-1}(x)\right|^p\right\}^{1/p}.$$

First we show that the first term has the required order. Since $d>1/2$, so by Theorem C with $p=1$, $r=2$, $\omega=n$, $n=1,2,\dots$, $\mu_m=2^m$, $\eta(t)=\gamma(t)$, $g(t)\equiv 1$, $\alpha_k(n)=A_{n-k}^{d-1}/A_n^d$, $\nu_k=k$, we get that

$$f(x)-\sigma_n^d(x)=o_x\big(\gamma(n)^{-1}\big) \tag{3.14}$$

a.e. in (a,b). We admit that it is not very easy to see that all of the assumptions of Theorem C are satisfied in the case given above, but a standard and elementary consideration shows that Theorem C with $o_x\big(\gamma(n)^{-1}\big)$ works, whence (3.14) follows clearly.

Using (3.14) we see that the first term in (3.13) has the required order $o_x\big(\gamma(n)^{-1}\big)$.

Next we show that the second term in (3.13) also has the same order. Let ε be any positive number. Let us choose N so large that

$$\sum_{n=N+1}^{\infty}c_n^2\gamma(n)^2<\varepsilon^3. \tag{3.15}$$

By means of N let us split series (2.2) into

$$\sum_{n=1}^{N} c_n^2\gamma(n)^2 < \infty \qquad \text{and} \qquad \sum_{n=N+1}^{\infty} c_n^2\gamma(n)^2 < \varepsilon^3,$$

and consider the corresponding orthogonal series, i.e., let

$$(3.16) \qquad \sum_{n=0}^{\infty} a_n\varphi_n(x) \qquad \text{with} \qquad a_n = \begin{cases} c_n & \text{for } n \leqq N, \\ 0 & \text{for } n > N; \end{cases}$$

and

$$(3.17) \qquad \sum_{n=0}^{\infty} b_n\varphi_n(x) \qquad \text{with} \qquad b_n = \begin{cases} 0 & \text{for } n \leqq N, \\ c_n & \text{for } n > N. \end{cases}$$

If, in this proof, $\sigma_n^\alpha(a;x)$ and $\sigma_n^\alpha(b;x)$ denote the (C,α)-means of series (3.16) and (3.17), respectively, then

$$(3.18) \qquad \sigma_n^\alpha(x) = \sigma_n^\alpha(a;x) + \sigma_n^\alpha(b;x).$$

Since the number of the coefficient $a_n \neq 0$ is finite,

$$\sigma_\nu^{d-1}(a;x) - \sigma_\nu^d(a;x) = \frac{1}{A_\nu^d}\sum_{k=0}^{N} kA_{\nu-k}^{d-1}c_k\varphi_k(x)$$

if $\nu > N$; and for any $k \leqq N$ $A_{\nu-k}^{d-1}/A_\nu^d = O(1/\nu)$, so using Hölder's inequality, we get that

$$\sum_{\nu=n+1}^{2n} \left|\sigma_\nu^{d-1}(a;x) - \sigma_\nu^d(a;x)\right|^p \leqq$$

$$\leqq n^{1-1/\rho}\left\{\sum_{\nu=n+1}^{2n} \left|\sigma_\nu^{d-1}(a;x) - \sigma_\nu^d(a;x)\right|^{p\rho}\right\}^{1/\rho} =$$

$$= O_x(1)n^{1-1/\rho}n^{1/\rho-p} = O_x(1)n^{1-p},$$

whence

$$(3.19) \qquad \sum\nolimits_n^a := \left\{\frac{1}{n}\sum_{\nu=n+1}^{2n} \left|\sigma_\nu^{d-1}(a;x) - \sigma_\nu^d(a;x)\right|^p\right\}^{1/p} = O_x(n^{-1})$$

follows a.e. in (a,b).

Since $\gamma(n)n^{-1} \to 0$, thus (3.19) implies that

$$\sum\nolimits_n^a = o_x\big(\gamma(n)^{-1}\big) \tag{3.20}$$

also holds a.e. in (a,b).

In order to estimate the suitable terms of series (3.17) we use Lemma 7 and (3.15). Then

$$\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{\gamma(n)^p}{n} \sum_{\nu=n+1}^{2n} \left|\sigma_\nu^{d-1}(b;x) - \sigma_\nu^d(b;x)\right|^p \right)^{1/p} \right\}^2 dx \leqq K\varepsilon^3.$$

Hence

$$\operatorname{meas}\left\{ x : \limsup \left(\frac{\gamma(n)^p}{n} \sum_{\nu=n+1}^{2n} \left|\sigma_\nu^{d-1}(b;x) - \sigma_\nu^d(b;x)\right|^p \right)^{1/p} > \varepsilon \right\} \leqq K\varepsilon.$$

This, (3.13), (3.14), (3.18), (3.19) and (3.20) imply (3.12) a.e. in (a,b), so our proof is complete.

4. Proof of Theorem 1. First we show that for arbitrary positive p

$$\Delta_m(x) := \left\{ \frac{1}{\mu_{m+1}} \sum_{k=\mu_m}^{\mu_{m+1}-1} \left| f(x) - \sigma_k^{d-1}(x)\right|^p \right\}^{1/p} = o_x\big(\gamma(\mu_m)^{-1}\big) \tag{4.1}$$

holds a.e. in (a,b).

Let us assume that for a given μ_m $n = n(m)$ is the largest integer such that

$$n < \mu_m,$$

furthermore let λ be the smallest integer with $N \leqq 2^\lambda$. Then, by (1.5), it is clear that

$$n < \mu_m < \mu_{m+1} \leqq N\mu_m \leqq 2^\lambda \mu_m \leqq 2^{\lambda+1} n,$$

whence, by Lemma 8 and taking into account that the sequence $\{\gamma(2^n)2^{-n}\}$ is quasi geometrically decreasing, we get

$$\Delta_m(x) \leqq \left\{ \frac{1}{n} \sum_{k=n+1}^{2^{\lambda+1}n} \left| f(x) - \sigma_k^{d-1}(x)\right|^p \right\}^{1/p} \leqq$$

$$\leqq K(p,\lambda)\sum_{i=0}^{\lambda}\left\{\frac{1}{2^i n}\sum_{k=2^i n+1}^{2^{i+1}n}\left|f(x)-\sigma_k^{d-1}(x)\right|^p\right\}^{1/p}=$$

$$=o_x\big(\gamma(n)^{-1}\big)=o_x\big(\gamma(\mu_m)^{-1}\big)$$

and this proves (4.1).

Now we set $r' := r/(r-1)$, i.e. let $1/r+1/r'=1$. By Hölder's inequality, using the properties of $\gamma(t)$, (1.5) and (4.1) with pr' in place of p, we get

$$A_\omega(f,p,d;x)^p := \sum_{k=0}^{\infty}\alpha_k(\omega)\left|\sigma_k^{d-1}(x)-f(x)\right|^p \leqq \tag{4.2}$$

$$\leqq \sum_{m=0}^{\infty}\left\{\sum_{k=\mu_m}^{\mu_{m+1}-1}\alpha_k(\omega)^r\right\}^{1/r}\left\{\sum_{k=\mu_m}^{\mu_{m+1}-1}\left|\sigma_k^{d-1}(x)-f(x)\right|^{pr'}\right\}^{1/r'}\leqq$$

$$\leqq K\sum_{m=0}^{\infty}\mu_m\rho_m(\omega,r)\left\{\frac{1}{\mu_{m+1}}\sum_{k=\mu_m}^{\mu_{m+1}-1}\left|\sigma_k^{d-1}(x)-f(x)\right|^{pr'}\right\}^{1/r'}\leqq$$

$$\leqq K\sum_{m=0}^{\infty}\mu_m\rho_m(\omega,r)o_x\big(\gamma(\mu_m)^{-p}\big)$$

a.e. in (a,b). By (2.1), (4.2) clearly yields

$$A_\omega(f,p,d;x)=O_x\big(g(\omega)/\gamma(\omega)\big) \tag{4.3}$$

a.e. in (a,b).

If (2.4) is also satisfied, then we derive the statement

$$A_\omega(f,p,d;x)=o_x\big(g(\omega)/\gamma(\omega)\big)\qquad(\omega\to\infty) \tag{4.4}$$

as follows.

Let $\varepsilon>0$ be given. If x is a point where (4.2) holds, then let $M(x)$ be a positive integer such that for $m>M(x)$ the inequality $o_x\big(\gamma(\mu_m)^{-p}\big) < \varepsilon^p\gamma(\mu_m)^{-p}$ is valid. For such x we get from (4.2) that

$$\big(\gamma(\omega)/g(\omega)\big)^p A_\omega(f,p,d;x)^p \leqq$$

$$\leqq K(x)\left\{\sum_{m=0}^{M(x)}\mu_m\rho_m(\omega,r)\gamma(\mu_m)^{-p}\right\}\big(\gamma(\omega)/g(\omega)\big)^p+$$

$$+K\varepsilon^p\big(\gamma(\omega)/g(\omega)\big)^p \sum_{m=M(x)+1}^{\infty} \mu_m \rho_m(\omega,r)\gamma(\mu_m)^{-p}.$$

When $\omega \to \infty$, the first sum on the right converges to zero by (2.4); and the second sum remains $O\left(\big(g(\omega)/\gamma(\omega)\big)^p\right)$, by (2.1).

Thus (4.4) clearly follows. Since (4.2) holds a.e. in (a,b), it follows that (4.4) also holds a.e. in (a,b).

From statements (4.3) and (4.4) the suitable statements of Theorem 1, i.e. (2.3) and its variant with o_x, follow easily applying Lemma 1, and this completes the proof of Theorem 1.

5. Applications. First we treat those results which can be derived from Theorem 1 in the special case when $g(\omega) \equiv 1$ and both (2.1) and (2.4) are satisfied.

5.1. If

$$p_{nk}(t) := \binom{n}{k} t^k (1-t)^{n-k}, \qquad k=0,1,\dots,n;\ n=1,2,\dots \tag{5.1}$$

and $\phi(t) \in L^1(0,1)$ is a non-negative function with $\|\phi\|_1 = 1$, then the matrix $\|\alpha_k(n)\|$ defined by

$$\alpha_k(n) := \int_0^1 p_{nk}(t)\phi(t)\,dt, \qquad k=0,1,\dots,n;\ n=1,2,\dots \tag{5.2}$$

yields the coefficients of a regular Hausdorff transformation. For these transformations we have the following results.

THEOREM 2. *Let $p > 0$, $d > \max(1/2,(p-1)/p)$ and let $\gamma(t)$ be a positive non-decreasing function defined for $0 \leqq t < \infty$. Suppose that $\alpha_k(n)$ are given by* (5.1) *and* (5.2), *where $\phi(t) \in L^r(0,1)$ with some $r > 1$. If $\gamma(t)$ has properties P_1, P_2 and $P_{p,r}$ with these p and r, and* (2.2) *holds, then*

$$\left\{\sum_{k=0}^{n} \alpha(n)\left|\sigma_k^{d-1}(\nu;x) - f(x)\right|^p\right\}^{1/p} = o_x\big(\gamma(n)^{-1}\big) \tag{5.3}$$

a.e. in (a,b) for any increasing sequence $\nu := \{\nu_i\}$ of positive integers.

COROLLARY 2.1. *Let $p > 0$, $d > \max(1/2,(p-1)/p)$, $\alpha > 0$ and $\alpha^* :=$ $:= \min(1,\alpha)$. Suppose that $\|\alpha_k(n)\|$ is the matrix of a Cesàro (C,α) or a Hölder (H,α) transformation, and that the function $\gamma(t)$ has properties P_1 and P_2, furthermore $\left\{\gamma(2^n)2^{-n\alpha^*/p}\right\}$ is quasi geometrically decreasing. Then* (5.3) *also holds a.e. in (a,b) under condition* (2.2).

PROOF OF THEOREM 2. First we show that conditions (2.1) and (2.4) of Theorem 1 are satisfied if $g(\omega) \equiv 1$ and $n := [\omega]$, where $[\omega]$ denotes the integral part of ω. By inequality (3.1) of Lemma 2 we get

$$(5.4) \qquad \rho_m(\omega, r) \leqq K(r)\mu_m^{-1/r}\omega^{1/r-1},$$

whence (2.4) follows on account of the property $P_{p,r}$ of $\gamma(t)$, and hence if a sequence $\{\gamma_n\}$ is quasi geometrically decreasing then for any $p > 0$ $\{\gamma_n^p\}$ is also quasi geometrically decreasing.

Since $\rho_m(\omega, r) = 0$ if $\mu_m > \omega$, thus, again from (3.1), we get

$$(5.5) \qquad \sum_{m=0}^{\infty} \mu_m \rho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K\omega^{1/r-1} \sum \mu_m^{1-1/r}\gamma(\mu_m)^{-p},$$

where the summation on the right is for $\mu_m \leqq \omega$. Because of the assumptions made on the sequences $\{\mu_m\}$ and $\{\gamma(2^n)2^{n(1-r)/rp}\}$ the last sum, by Lemma 5, is $O\big(\omega^{1-1/r}\gamma(\omega)^{-p}\big)$, thus the previous inequality proves (2.1).

The conclusion (5.3) of Theorem 2 follows from Theorem 1, and this completes the proof.

PROOF OF COROLLARY 2.1. Both the (C, α) and (H, α) transformations are Hausdorff ones with $\phi_1(t) := \alpha(1-t)^{\alpha-1}$ and $\phi_2(t) := \Gamma(\alpha)^{-1} \cdot$ $\cdot (\log 1/t)^{\alpha-1}$, respectively, If $\alpha \geqq 1$ $(\alpha^* = 1)$, then $\phi_i(t) \in L^r(0,1)$ for arbitrary large r. Since then $\{\gamma(2^n)2^{-n/p}\}$ is quasi geometrically decreasing, and it will keep this property after multiplying its terms by $2^{n\varepsilon}$ if $\varepsilon(>0)$ is small enough (and if $1/r < \varepsilon p$), then the sequence $\{\gamma(2^n)2^{n(1-r)/rp}\}$ will be quasi geometrically decreasing as well. If $0 < \alpha < 1$ $(\alpha^* = \alpha)$, then $\phi_i(t) \in L^r(0,1)$ if $1/r > 1 - \alpha$. Applying the previous consideration with $1/r - 1 + \alpha < \varepsilon p$, then we get that the sequence $\{\gamma(2^n)2^{n(1-r)/rp}\}$ is also quasi geometrically decreasing, namely $\{\gamma(2^n)2^{-n\alpha/p}\}$ has this property.

Consequently Theorem 2 is applicable and it yields Corollary 2.1.

5.2. If

$$(5.6) \qquad \lambda_k(\omega, t) := \frac{\big(\omega \log(1/t)\big)^k}{k!} t^{\omega}, \qquad k = 0, 1, \ldots,$$

and $\phi(t) \in L^1(0,1)$ is a non-negative function with $\|\phi\|_1 = 1$, then the function-sequence $\{\alpha_k(\omega)\}$ defined by

$$(5.7) \qquad \alpha_k(\omega) := \int_0^1 \lambda_k(\omega, t)\phi(t)dt, \qquad k = 0, 1, \ldots$$

yields the coefficients of a regular $[J, f]$-transformation. For such transformations we have the following result.

THEOREM 3. *Let $p, d, r, \gamma(t)$ and $\phi(t)$ have the same meaning and properties as in Theorem 2. Suppose that $\alpha_k(\omega)$ are given by (5.6) and (5.7). If (2.2) holds then*

$$\left\{\sum_{k=0}^{\infty}\alpha_k(\omega)\left|\sigma_k^{d-1}(\nu;x)-f(x)\right|^p\right\}^{1/p}=o_x\left(\gamma(\omega)^{-1}\right) \tag{5.8}$$

a.e. in (a,b) for any increasing sequence $\nu:=\{\nu_i\}$ of positive integers.

COROLLARY 3.1. *If (2.2) holds and $\{\alpha_k(\omega)\}$ is the coefficient-sequence of the Abel transformation, then (5.8) holds whenever the sequence $\left\{\gamma(2^n)2^{-n/p}\right\}$ is quasi geometrically decreasing; assuming that $p, d, \left\{\gamma(2^n)\right\}$ and $\left\{\gamma(2^n)2^{-n}\right\}$ have the same properties as in Theorem 1.*

PROOF OF THEOREM 3. We again show that (2.1) and (2.4) are satisfied with $g(\omega)\equiv 1$ under the assumptions of Theorem 3. Now we obtain (5.4) from (3.2), whence (2.4) follows by the same reasoning as in the proof of Theorem 2.

Furthermore, by (3.2), we get

$$\begin{aligned}\sum\nolimits_1 &:= \sum_{\mu_m\leqq\omega}\mu_m\rho_m(\omega,r)\gamma(\mu_m)^{-p}\leqq\\ &\leqq K(r)(1+\omega)^{1/r-1}\sum_{\mu_m\leqq\omega}\mu_m^{1-1/r}\gamma(\mu_m)^{-p}.\end{aligned}\tag{5.9}$$

The last sum, by reasoning made after (5.5) in the proof of Theorem 2, is $O\left(\omega^{1-1/r}\gamma(\omega)^{-p}\right)$, thus (5.9) yields

$$\sum\nolimits_1\leqq K(r)\gamma(\omega)^{-p}. \tag{5.10}$$

On the other hand, by (3.2) and (5.4), we obtain that

$$\begin{aligned}\sum\nolimits_2 &:= \sum_{\mu_m>\omega}\mu_m\rho_m(\omega,r)\gamma(\mu_m)^{-p}\leqq\\ &\leqq K(r)\sum_{\mu_m>\omega}\left(\frac{\mu_m}{1+\omega}\right)^{1-1/r}e^{-\mu_m(1-1/r)/(1+\omega)}\gamma(\mu_m)^{-p}\leqq\\ &\leqq K(r)\sum_{\mu_m>\omega}\gamma(\mu_m)^{-p}\end{aligned}$$

due to the fact $xe^{-x} < 1$ for any $x > 0$. Since the sequence $\{\gamma(2^n)\}$ is quasi geometrically increasing, thus, by virtue of Lemma 5, the last sum is $O\big(\gamma(\omega)^{-p}\big)$, whence

$$\sum\nolimits_2 \leqq K(r)\gamma(\omega)^{-p} \tag{5.11}$$

follows. Inequalities (5.9), (5.10) and (5.11) prove (2.1), therefore Theorem 3 is also a consequence of Theorem 1, as desired.

Proof of Corollary 3.1. If $\phi(t) \equiv 1$ in (5.7), then $\alpha_k(\omega) := := \omega^k/(1+\omega)^{k+1}$ for $k = 0, 1, \ldots$, which yields the classical Abel transformation. In this case, clearly, $\phi(t) \in L^r(0,1)$ for any $r > 1$, thus the assumption, that the sequence $\{\gamma(2^n)2^{-n/p}\}$ is quasi geometrically decreasing, implies that if r is large enough then $\{\gamma(2^n)2^{n(1-r)/rp}\}$ has the same property (see the reasoning given in the proof of Corollary 2.1), consequently all of the conditions of Theorem 3 are satisfied. Therefore the statement of Corollary 3.1 follows from Theorem 3 immediately.

5.3. If the function $\phi(t)$ in (5.2) satisfies

$$0 \leqq \phi(t) \leqq K(\beta)t^{\beta-1}$$

with $\beta > 0$, then it is easy to show that

$$\alpha_k(\omega) \leqq K(\beta)\frac{(k+1)^{\beta-1}}{(n+1)^\beta} \tag{5.12}$$

for $0 \leqq k \leqq n$, $n = 1, 2, \ldots$. Using (5.12) we can verify by easy calculations that in these cases (2.1) and (2.4) hold whenever $g(t) \equiv 1$ and the sequence $\{\gamma(2^n)2^{-n\beta/p}\}$ is quasi geometrically decreasing. For example if $\phi(t) = = \beta t^{\beta-1}$, then the matrix $\|\alpha_k(n)\|$ yields, essentially, the Riesz transformation of order β. Consequently Theorem 1 with $d = 1$ gives a slight improvement of Corollary C.2.

5.4. If the function $\phi(t)$ in (5.7) satisfies

$$0 \leqq \phi(t) \leqq K(q)\left(\log\frac{1}{t}\right)^q$$

with $q \geqq 0$, then an easy calculation yields that

$$\alpha_k(\omega) \leqq K(q)\frac{(k+1)^q}{(\omega+1)^{q+1}}\left(\frac{\omega}{\omega+1}\right)^k$$

for $k=0,1,\ldots$. Using this we can show easily that in this case (2.1) and (2.4) hold whenever the sequence $\{\gamma(2^n)2^{-n(q+1)/p}\}$ is quasi geometrically decreasing. For example, if $\phi(t):=\big(\Gamma(q+1)\big)^{-1}\left(\log\frac{1}{t}\right)^q$, $q\geqq 0$, then

$$\alpha_k(\omega)=(\omega+1)^{-q-1}\binom{k+q}{k}\left(\frac{\omega}{\omega+1}\right)^k, \qquad k=0,1,\ldots,$$

which yields the generalized Abel transformation of order $q+1$. In view of this, it is easy to see that Theorem 1 with $d=1$ gives a generalization of Corollary C.3 under a slightly relaxed condition.

5.5. Next we mention two further applications of Theorem 1 with $g(\omega):=$ $:=\big(\log(1+\omega)\big)^{1/p}$. The proofs would run as in the previous cases, therefore we shall detail only one of them. These special cases of Theorem 1 include some of the so called "limit-case" theorems (see e.g. [6] and [10]).

THEOREM 2*. *Under the assumptions of Theorem* 2 *with*

$$\sum_{n=0}^{m}2^{n(1-1/r)}\gamma(2^n)^{-p}\leqq K(\log m)2^{m(1-1/r)}\gamma(2^m)^{-p} \tag{5.13}$$

in place of property $P_{p,r}$, *but assuming that the terms of the sum are non-decreasing, we get*

$$\left\{\sum_{k=0}^{n}\alpha_k(n)\big|\sigma_k^{d-1}(\nu;x)-f(x)\big|^p\right\}^{1/p}=o_x\big((\log n)^{1/p}\gamma(n)^{-1}\big)$$

a.e. in (a,b) *for any increasing sequence* $\nu:=\{\nu_i\}$ *of positive integers.*

THEOREM 3*. *Under the assumptions of Theorem* 3 *with* (5.13) *in place of property* $P_{p,r}$ *we get*

$$\left\{\sum_{k=0}^{\infty}\alpha_k(\omega)\big|\sigma_k^{d-1}(\nu;x)-f(x)\big|^p\right\}^{1/p}=o_x\left(\big(\log(1+\omega)\big)^{1/p}\gamma(\omega)^{-1}\right)$$

a.e. in (a,b) *for any increasing sequence* $\nu:=\{\nu_i\}$ *of positive integers.*

REMARK. Theorems 2* and 3*, like Theorems 2 and 3 above, because of their generality, do not include the limit-cases theorems proved for the Cesàro, the Riesz and the generalized Abel summation methods (see e.g. [6] and [10]), but our main result, Theorem 1, yields the results for the above mentioned classical summation methods as well.

PROOF OF THEOREM 2*. We show that conditions (2.1) and (2.4) of Theorem 1 are satisfied if $n := [\omega]$ and $g(\omega) := \big(\log(1+\omega)\big)^{1/p}$. From (3.1) we derive (5.4), whence (2.4) follows. Namely, by condition (5.13),

$$\rho_m(\omega, r) \leqq K(r)\mu_m^{-1/r}\omega^{1/r-1} =$$

$$= K(r)\mu_m^{-1/r}O\big(\gamma(\omega)^{-p}\big) = o\Big(\big(\log(1+\omega)\big)\gamma(\omega)^{-p}\Big)$$

clearly holds.

To show (2.1) we take into account that $\rho_m(\omega, r) = 0$ if $\mu_m > \omega$. Therefore, by (1.5) and (5.13), (3.1) implies that

$$\sum_{m=0}^{\infty} \mu_m \rho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K\omega^{1/r-1} \sum_{\mu_m \leqq \omega} \mu_m^{1-1/r}\gamma(\mu_m)^{-p} \leqq$$

$$\leqq K\big(\log(1+\omega)\big)\gamma(\omega)^{-p},$$

what is the required inequality (2.1).

Consequently we can apply Theorem 1 and this completes the proof of Theorem 2*.

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FIRST RETURN PATH SYSTEMS: DIFFERENTIABILITY, CONTINUITY, AND ORDERINGS

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1. Introduction

In this paper we continue the study of first return path systems, which were introduced in [15] as examples of minimally thin path systems sufficiently rich to generate many of the properties of more standard systems, such as the one used to study approximate differentiability and continuity. (The name and structure of these paths are derived from the use of a dense trajectory and the Poincaré first return map of dynamics. This will become more apparent as we continue through this section and the next.)

With regard to differentiability, a question of immediate concern is the determination of whether various established derivatives, such as approximate, Peano, and approximate Peano, can be realized among the class of first return derivatives. Surprisingly, this seems hard to establish directly in each case. (For example, finding the trajectory to establish the approximate result seems nontrivial.) Fortunately, all of the above types have been shown to be composite derivatives of a special type. (See [13], [5], and [6].) In this paper we establish that all such composite derivatives are first return derivatives. In fact, there is a non-apparent universality underlying the trajectory which will be shown.

A similar universality is found with respect to first return continuity and the familiar class of Baire* 1, Darboux functions. Furthermore, the basic concept of first return continuity is shown to be equivalent to the Baire 1, Darboux property. We also establish via examples that converses of two of our major theorems are not valid.

Lastly, we examine the fundamental idea of trajectory. As mentioned in [15], two objects are necessary to create a trajectory: a countable dense set D and an ordering of D into a sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points. For any such D it is easy to create orderings of D having the property that there is no continuous function g such that $g^n(x_0) = x_n$ for all n, where for each n, $g^n(x_0) = g\big(g^{n-1}(x_0)\big)$. Therefore, it would be logical to anticipate that those first return path systems generated by trajectories of transitive continuous functions would possess nicer properties than general first return path systems. However, this is *not* the case. As a final result we show that each first return path system can be generated from a trajectory of a continuous transitive mapping, and, again a certain amount of unanticipated universality or flexibility is exhibited.

2. Definitions and notation

By a *trajectory* we simply mean any sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points in $(0,1)$, which is dense in $[0,1]$. (Note that this is a slight deviation from the definition used in [15], but the alteration is both cosmetic and notationally beneficial.) One way to produce a trajectory is to begin with a transitive continuous function $g\colon[0,1]\to[0,1]$, *i.e.*, a function having the property that for some y_0 the sequence of iterates $\{y_0, g(y_0), g^2(y_0), \dots\}$ is dense in $[0,1]$. Such a sequence is called the *trajectory of* y_0 *under* g. In this paper the most common method of specifying a trajectory will be that of assigning an enumeration or *ordering* to a given countable dense subset D of $(0,1)$. Throughout this work we shall refer to such a set D as a *support* set and will only use the symbol D to denote such sets.

Let $\{x_n\}$ be a fixed trajectory. For a given interval $(a,b)\subset[0,1]$, $r(a,b)$ will be the first element of the trajectory in (a,b). For $0\leqq y<1$, the *right first return path to* y, R_y^+, is defined recursively via

$$y_1^+ = 1, \quad \text{and} \quad y_{k+1}^+ = r(y, y_k^+).$$

For $0<y\leqq 1$, the *left first return path to* y, R_y^-, is defined similarly. For $0<y<1$, we set $R_y = R_y^+ \cup R_y^- \cup \{y\}$, and $R_0 = \{0\}\cup R_0^+$, $R_1 = R_1^- \cup \{1\}$. The collection $\mathcal{R} \equiv \{R_y \colon y\in[0,1]\}$ satisfies the definition of a *path system* as defined in [3] and we shall refer to it as the *first return path system determined by the trajectory* $\{x_n\}$. (It should be noted that for a fixed support set D, the nature of the first return path system will clearly depend on the ordering of D which defines the trajectory $\{x_n\}$, and this is why we have emphasized this concept in our title.) Let $f\colon[0,1]\to\mathbf{R}$. If the

$$\lim_{\substack{t\to y\\ t\in R_y\setminus\{y\}}} \frac{f(t)-f(y)}{t-y} = f'_{\mathcal{R}}(y)$$

exists and is finite, then we say that f is $\mathcal{R}$-differentiable at y, or is *first return differentiable at* y *with respect to the trajectory* $\{x_n\}$. If the above equality holds for all $y\in[0,1]$, we say that f is $\mathcal{R}$-differentiable to the function $f'_{\mathcal{R}}$ on $[0,1]$, or f is *first return differentiable to* $f'_{\mathcal{R}}$ *on* $[0,1]$ *with respect to the trajectory* $\{x_n\}$. If f and g are functions on $[0,1]$ and there exists some trajectory $\{x_n\}$ for which f is first return differentiable to g on $[0,1]$ with respect to $\{x_n\}$, we simply say that f is *first return differentiable to g on* $[0,1]$, and call g a *first return derivative of f on* $[0,1]$.

Now suppose that f and g are functions with the property that for every support set D there is an ordering $\{x_n\}$ of D, such that f is first return differentiable to g on $[0,1]$ with respect to $\{x_n\}$. In this situation we say that f is *universally* first return differentiable to g on $[0,1]$, and call g a *universal* first return derivative of f on $[0,1]$. As we shall observe in the Examples section of this paper, it is possible for a function to have more than one universal first return derivative on $[0,1]$.

The concepts of a function being

(i) *first return continuous on* $[0,1]$ *with respect to a trajectory* $\{x_n\}$,

(ii) *first return continuous on* $[0,1]$,

(iii) *universally first return continuous on* $[0,1]$,

are all defined in the analogous manner.

Next, we need to review the notion of composite differentiation as defined in [16]. A *decomposition* of $[0,1]$ is a collection of closed sets E_n, $n = 1,2,\ldots$ such that $\bigcup_{n=1}^{\infty} E_n = [0,1]$. A function $f\colon[0,1] \to \mathbf{R}$ is said to have a function $g\colon[0,1] \to \mathbf{R}$ as a *composite derivative relative to the decomposition* $\{E_n\}$ if for each n and each $y \in E_n$

$$\lim_{\substack{t \to y \\ t \in E_n}} \frac{f(t) - f(y)}{t - y} = g(y).$$

The function f is said to be *compositely differentiable to a function g* if there exists a decomposition such that f has g as a composite derivative with respect to that decomposition. Similarly, we could say that a function f is *compositely continuous* if there exists a decomposition such that the restriction of f to each set in the decomposition is continuous. It is known (see [1] or [12]) that this property is equivalent to the Baire* 1 property. Recall that a function $f\colon[0,1] \to \mathbf{R}$ has the Baire* 1 property if for each perfect set $P \subseteqq [0,1]$ there is an open interval I such that $P \cap I \neq \emptyset$, and the restriction of f to $P \cap I$ is continuous.

Finally, as an aid to concisely stating our results, we wish to alert the reader to the following caveat: Whenever we make a statement referring to bilateral behavior at each point in $[0,1]$, we wish to have this mean behavior from the right at 0 and behavior from the left at 1; that is, bilateral should be interpreted relative to $[0,1]$.

3. First return differentiation

We begin this work by exploring the relationship between composite differentiation and first return differentiation. First, note that not every composite derivative is a first return derivative. For example, consider the function

$$f(x) = \left| x - \frac{1}{2} \right|,$$

which is compositely differentiable to the function

$$g(x) = \begin{cases} -1 & \text{if } 0 \leqq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leqq x \leqq 1 \end{cases}$$

relative to the decomposition of $[0,1]$ given by the perfect sets $E_1 = \left[\frac{1}{2}, 1\right]$, and $E_n = E_1 \cup \left[0, \frac{1}{2} - \frac{1}{n+1}\right]$ for $n = 2, 3, \ldots$. Note that f clearly cannot be first return differentiable to any function since it has different left and right ordinary derivatives at $\frac{1}{2}$. However, if we not only require that a function f be compositely differentiable to a function g on $[0,1]$, but further require that $g(x)$ be a bilateral derived number of f at each x, then f will be universally first return differentiable to g. We shall prove this result in two stages. First, we will show that under these hypotheses, the decomposition can be assumed to have a very nice structure relative to any given support set D. Then we shall utilize this structure to prove the main theorem. Before stating the first result, we recall that if f is compositely differentiable to g on $[0,1]$, then the Baire category theorem shows the existence of a dense open set $U(f)$ on which f is differentiable to g.

LEMMA 1. *Let $f\colon [0,1] \to \mathbf{R}$ be compositely differentiable to $g\colon [0,1] \to \to \mathbf{R}$, and suppose that for each $x \in [0,1]$ $g(x)$is a bilateral derived number of f at x. Let D be any support set. Then there exists a nondecreasing sequence $\{H_n\}$ of perfect sets whose union is $[0,1]$ and such that for each natural number n*

A) *the restriction of f to H_n is differentiable to g at each point of H_n,*

B) *each point of H_n is a bilateral limit point of H_{n+1},*

C) *each component of $[0,1] \setminus H_n$ has both endpoints in $U(f) \cap D \cup \{0,1\}$.*

PROOF. From Proposition 1 in [14] there exists a nondecreasing sequence $\{E_n\}$ of perfect sets whose union is $[0,1]$ and such that for each natural number n the restriction of f to E_n is differentiable to g at each point of E_n. Let $U \equiv U(f)$ and set

$$T = U \cap D \cup \{0,1\}.$$

We shall construct a sequence of perfect sets $\{H_n\}$ with $E_n \subseteqq H_n$ for each natural number n and with f being compositely differentiable to g with respect to $\{H_n\}$ such that each H_n has the desired additional properties.

The sequence of sets $\{H_n\}$ is constructed inductively. A procedure for enlarging a certain type of perfect set to another perfect set containing it will be described. We begin by enlarging the perfect set E_1 to a perfect set H_1 as follows. We know that the function f restricted to the perfect set

E_1 is differentiable to g restricted to E_1. Thus, by a result of Laczkovich and Petruska [7], there is a differentiable function F such that F agrees with f on E_1 and so that F' agrees with g on E_1 as well. To temporarily normalize the situation, we introduce the function $f_1 \equiv F - f$. Then f_1 is compositely differentiable with respect to the sequence $\{E_n\}$, f_1 is zero on E_1, f_1 restricted to E_1 has derivative 0 at each point of E_1, and f_1 has an ordinary derivative at each point of U. Enumerate the components of $[0,1]\setminus \setminus E_1$ in a finite or denumerable sequence $\{(a_n,b_n)\}$. We shall show that for each n there is a sequence $\{L_{n,k}: k = 1,2,\dots\}$ of closed intervals converging to a_n from the right such that for each k

(1) $L_{n,k} \subset U \cap (a_n, (a_n + b_n)/2)$,

(2) the endpoints of $L_{n,k}$ are in T,

(3) for each $x \in L_{n,k}$ we have $\left|\frac{f_1(x)}{x-a_n}\right| < \frac{1}{n+k}$.

To see this, first note that since 0 is a bilateral derived number of f_1 at a_n, we know that there is a sequence of points $\{x_{n,k}: k = 1,2,\dots\}$ in $\big(a_n, (a_n + + b_n)/2\big)$ converging to a_n from the right and for which

$$\left|\frac{f_1(x_{n,k})}{x_{n,k} - a_n}\right| < \frac{1}{n+k}.$$

For each k let $W_{n,k}$ denote the open region enclosed by the rhombus formed by the four lines $y = \frac{1}{n+k}(x - a_n)$, $y = -\frac{1}{n+k}(x - a_n)$, $y = \frac{1}{n+k}(x - b_n)$, and $y = -\frac{1}{n+k}(x - b_n)$. Then for each k we select positive numbers $\varepsilon_{n,k}$ and $\delta_{n,k}$ such that $x_{n,k} + \delta_{n,k} < (a_n + b_n)/2$ and such that the the rectangular region $(x_{n,k} - \delta_{n,k}, x_{n,k} + \delta_{n,k}) \times \big(f_1(x_{n,k}) - \varepsilon_{n,k}, f_1(x_{n,k}) + \varepsilon_{n,k}\big)$ lies in $W_{n,k}$. Utilizing Theorem 3.2 in [16], we have that f_1 has $F' - g$ as a selective derivative and, hence, by Theorem 11 in [11] f_1 is a Baire*1, Darboux function. Utilizing this Baire*1, Darboux property we know that for each k there is a point $y_{n,k} \in (x_{n,k} - \delta_{n,k}, x_{n,k} + \delta_{n,k}) \cap U$ such that $f_1(y_{n,k}) \in \in \big(f_1(x_{n,k}) - \varepsilon_{n,k}, f_1(x_{n,k}) + \varepsilon_{n,k}\big)$ (e.g., see Theorem 2 in [10]). It is now an easy matter to select an appropriate closed interval $L_{n,k}$ containing $y_{n,k}$ that will have all of the listed properties. Likewise, for each n there is a sequence $\{R_{n,k}: k = 1,2,\dots\}$ of closed intervals converging to b_n from the left such that for each k

(1) $R_{n,k} \subset U \cap \big((a_n + b_n)/2, b_n\big)$,

(2) the endpoints of $R_{n,k}$ are in T,

(3) for each $x \in R_{n,k}$ we have $\left|\frac{f_1(x)}{x-b_n}\right| < \frac{1}{n+k}$.

Then we set

$$H_1 = E_1 \bigcup \left[\bigcup_n \bigcup_k (L_{n,k} \cup R_{n,k})\right].$$

Note that H_1 is perfect, $E_1 \subset H_1$, f_1 restricted to H_1 is differentiable at each point of H_1; in particular, this derivative is 0 at each point of E_1 and at each point of $H_1 \setminus E_1$ f_1 has an ordinary derivative since such points belong to U. Furthermore, each endpoint of every component of $[0,1] \setminus H_1$ is an endpoint of an $L_{n,k}$ or an $R_{n,k}$ and, hence, is in T. Returning our attention to the original function f, we have that f restricted to H_1 is differentiable at each point of H_1 to g. Next, we form the union of E_2 and H_1 and repeat the previous procedure to enlarge this perfect set to a perfect set H_2. Then each point of H_1 is clearly a bilateral limit point of H_2.

In general, having formed the perfect set H_n, we enlarge the perfect set $H_n \cup E_{n+1}$ by this procedure to obtain the perfect set H_{n+1}. The sequence of sets $\{H_n : n = 1, 2, \ldots\}$ will then satisfy our requirements.

THEOREM 1. *Let $f:[0,1] \to \mathbf{R}$ be compositely differentiable to $g:[0,1] \to \mathbf{R}$, and suppose that for each $x \in [0,1]$ $g(x)$ is a bilateral derived number of f at x. Then f is universally first return differentiable to g on $[0,1]$.*

PROOF. Let D be a support set and let $\{H_n\}$ be the sequence of perfect sets obtained from Lemma 1, and for convenience let $H_0 = \{0,1\}$. Let $\{d_s\}_{s=0}^{\infty}$ be an ordering of D. We shall utilize this ordering and the sequence of sets $\{H_n : n = 0, 1, \ldots\}$ to construct the desired ordering $\{x_n\}$ of D. As a mechanism to assist in this endeavor we shall construct a sequence of partitions $\{\mathcal{P}_k\}$, where each $\mathcal{P}_k$ consists of points chosen from $D \cup \{0,1\}$ and each $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$.

For each $y \in [0,1]$ we shall find it convenient to adopt the notation $n(y)$ for the smallest integer n for which $y \in H_n$. For each $y \in (0,1]$ and each non-negative integer k we shall let $\lambda^k(y)$ denote the closest element of the partition $\mathcal{P}_k$ lying strictly to the left of y; and for each $y \in [0,1)$ and each non-negative integer k we shall let $\rho^k(y)$ denote the closest element of the partition $\mathcal{P}_k$ lying strictly to the right of y. For convenience, we let $\lambda^{-1}(y) = -1$ and $\rho^{-1}(y) = 2$ for all $y \in [0,1]$. We shall construct the partitions inductively in such a manner that for each $k = 0, 1, 2, \ldots$, we have

A. For each $y \in (0,1]$, $\left[\lambda^{k-1}(y), \lambda^k(y)\right] \cap \mathcal{P}_k \subset H_{n(y)+1}$.

B. For each $y \in [0,1)$, $\left[\rho^k(y), \rho^{k-1}(y)\right] \cap \mathcal{P}_k \subset H_{n(y)+1}$.

We start by setting $\mathcal{P}_0 = \{p_0^0 = 0, p_1^0 = 1\}$. We further set $x_{-2} = 0$ and $x_{-1} = 1$. (We will want to continue this dual labelling scheme, wherein we label the points in each $\mathcal{P}_k$ in the natural order for each k and label the points in $\bigcup_{k=0}^{\infty} \mathcal{P}_k$ lexicographically.) Note that for $k = 0$ conditions A and B are trivially satisfied.

Proceeding inductively, we assume that a partition,

$$\mathcal{P}_k = \{p_0^k = 0 < p_1^k < \ldots < p_{l_k}^k = 1\},$$

has been chosen; that each point of $\mathcal{P}_k$ belongs to $D \cup \{0,1\}$ and has been labelled as an x_i, where $-2 \leqq i \leqq l_k - 2$; and that conditions A and B are satisfied. We let s_k denote the largest integer s such that $d_s \in \mathcal{P}_k$. To construct $\mathcal{P}_{k+1}$ we proceed as follows. Naturally, since $\mathcal{P}_{k+1}$ is to be a refinement of $\mathcal{P}_k$, we first put all points of $\mathcal{P}_k$ into $\mathcal{P}_{k+1}$. For each $0 \leqq i \leqq \leqq l_k - 1$, we shall select points to properly refine $\{p_i^k, p_{i+1}^k\}$. Fix such an i and let $n(i,k)$ be the smallest n such that $(p_i^k, p_{i+1}^k) \cap H_n \neq \emptyset$. Now consider the set $[p_i^k, p_{i+1}^k] \setminus H_{n(i,k)}$. If this set is empty, then we select any point from T which belongs to the middle third of the interval $[p_i^k, p_{i+1}^k]$ and put it in the collection of points which will form $\mathcal{P}_{k+1}$. Then we put all elements of $\{d_s: s \leqq s_k\} \cap (p_i^k, p_{i+1}^k) \cap H_{n(i,k)}$ in the collection of points which will form $\mathcal{P}_{k+1}$ and move on to the next i. On the other hand, if $[p_i^k, p_i^k] \setminus H_{n(i,k)}$ is not empty, then we proceed differently. In this case we let $V_{i,k} = [p_i^k, p_{i+1}^k] \setminus \setminus H_{n(i,k)}$ and set β_i^k equal to the length of the longest component(s) of $V_{i,k}$. We select the endpoints of all components of length β_i^k and put them in $\mathcal{P}_{k+1}$. Then we put all elements of $\{d_s: s \leqq s_k\} \cap (p_i^k, p_{i+1}^k) \cap H_{n(i,k)}$ in the collection of points which will form $\mathcal{P}_{k+1}$. Now we pause and look at the partition of $[p_i^k, p_{i+1}^k]$ formed so far. If the norm of this partition is less than or equal to β_i^k, then we move on to the next i; if not, then we add additional points from $H_{n(i,k)} \cap [p_i^k, p_{i+1}^k] \cap D$ to refine the partition until its norm is less than or equal to β_i^k; specifically, we may select endpoints of components of $V_{i,k}$ other than those of length β_i^k and/or points from D lying in the interior of $[p_i^k, p_{i+1}^k] \cap H_{n(i,k)}$ to accomplish this. Then we move on to the next i. Once we have done this for each interval $[p_i^k, p_{i+1}^k]$, $0 \leqq i \leqq l_k - 1$, we have constructed our partition $\mathcal{P}_{k+1} = \{p_0^{k+1} = 0 < p_1^{k+1} < \ldots < p_{l_{k+1}}^{k+1} = 1\}$ and we label the points in $\mathcal{P}_{k+1} \setminus \mathcal{P}_k$ from left to right as $x_{l_k-1}, x_{l_k}, \ldots, x_{l_{k+1}-2}$.

We need to verify that conditions A and B are satisfied with k replaced by $k+1$. Let us first consider condition A. So let $y \in (0,1]$. We must show that $[\lambda^k(y), \lambda^{k+1}(y)] \cap \mathcal{P}_{k+1} \subset H_{n(y)+1}$. By the inductive hypothesis we know that $\lambda^k(y) \in H_{n(y)+1}$. If $\lambda^k(y) = \lambda^{k+1}(y)$, then we are done. Suppose $\lambda^k(y) \neq \lambda^{k+1}(y)$ and let $t \in (\lambda^k(y), \lambda^{k+1}(y)] \cap \mathcal{P}_{k+1}$. Then $t \in \mathcal{P}_{k+1} \setminus \mathcal{P}_k$. There is an i such that $\lambda^k(y) = p_i^k$. Then $t \in (p_i^k, y)$ and $y \in (t, p_{i+1}^k]$. We know that $t \in H_{n(i,k)}$. Since $y \in (p_i^k, p_{i+1}^k]$, we must have $n(i,k) \leqq n(y) + 1$; indeed, if $y \in (p_k^k, p_{i+1}^k)$, then $n(i,k) \leqq n(y)$, and if $y = p_{i+1}^k$, then we still have $n(i,k) \leqq n(y) + 1$ since every point of $H_{n(y)}$ is a bilateral limit point of $H_{n(y)+1}$. Consequently $t \in H_{n(y)+1}$. Thus, condition A holds. Condition B is verified by a symmetric argument.

Next we show that $\lim_{k\to\infty} \operatorname{mesh}(\mathcal{P}_k) = 0$. This is equivalent to showing that the closure of $\bigcup_{i=0}^{\infty} \mathcal{P}_i$, $\operatorname{cl}(\bigcup_{i=0}^{\infty} \mathcal{P}_i)$, is $[0,1]$. To the contrary, assume that $\operatorname{cl}(\bigcup_{i=0}^{\infty} \mathcal{P}_i) \neq [0,1]$ and let (a,b) be a component of $[0,1] \setminus \operatorname{cl}(\bigcup_{i=0}^{\infty} \mathcal{P}_i)$. Either a or b does not belong to $\bigcup_{i=0}^{\infty} \mathcal{P}_i$ because we have that $\mathcal{P}_{i+1}$ is a proper refinement of $\mathcal{P}_i$ for each i. Without loss of generality assume that $a \notin \bigcup_{i=0}^{\infty} \mathcal{P}_i$. For each $k \geqq 0$, let j_k be such that $(a,b) \subset [p_{j_k}^k, p_{j_k+1}^k]$. Observe that $\{p_{j_k}^k\}_{k=0}^{\infty}$ and $\{p_{j_k+1}^k\}_{k=0}^{\infty}$ are non-decreasing and non-increasing sequences which converge to a and b, respectively. Note that $\{p_{j_k}^k\}_{k=0}^{\infty} \subset$ $\subset H_{n(a)}$ and $\{p_{j_k+1}^k\}_{k=0}^{\infty} \subset H_{n(a)}$ because $a \in \bigcap_{k=0}^{\infty}(p_{j_k}^k, p_{j_k+1}^k)$. Thus we have $b \in H_{n(a)}$. As $a \notin \bigcup_{i=0}^{\infty} \mathcal{P}_i$ and $H_{n(a)-1}$ is closed, $\{p_{j_k}^k\}_{k=r}^{\infty} \subset H_{n(a)} \setminus H_{n(a)-1}$ for some r. Since $\{p_{j_k}^k\}_{k=r}^{\infty}$ is a non-decreasing sequence converging to a, whose range is contained in $H_{n(a)} \setminus H_{n(a)-1}$ but does not contain a, we have that $n(j_s, s) = n(a)$ for some s. (Specifically, choose $s > r$ such that $p_{j_{s+1}}^{s+1} >$ $> p_{j_s}^s$.) For any $t > s$ we have that $n(j_t, t) = n(a)$ because $\{n(j_k, k)\}_{k=0}^{\infty}$ is non-decreasing sequence and $a \in \bigcap_{k=0}^{\infty}(p_{j_k}^k, p_{j_k+1}^k)$. As $a, b \in H_{n(a)}$, we have that (a,b) is the largest component of $[p_{j_k}^k, p_{j_k+1}^k] \setminus H_{n(j_k,k)}$ for large enough k. However, this contradicts our method of defining $\mathcal{P}_{k+1}$ as a is not in $\bigcup_{i=0}^{\infty} \mathcal{P}_i$, and completing the proof of the fact that our trajectory is dense in $[0,1]$.

We next want to show that the range of the sequence $\{x_n\}_{n=0}^{\infty}$ is all of D. Let $d_s \in D$, and let $\varepsilon = \operatorname{dist}\left(d_s, H_{n(d_s)-1}\right)$. Let $k > s$ be such that $\operatorname{mesh}(\mathcal{P}_k) < \varepsilon$. Then, if $d_s \notin \mathcal{P}_k$, then $d_s \in (p_i^k, p_{i+1}^k) \subset (d_s - \varepsilon, d_s + \varepsilon)$ for some i, and hence $n(d_s) = n(i,k)$. This implies that $d_s \in \mathcal{P}_{k+1}$, proving that D is the range of $\{x_n\}$.

To complete the proof, we need to show that for each $y \in [0,1]$, $f'_{\mathcal{R}}(y)$, the $\mathcal{R}$-first return path derivative of f at y exists and equals $g(y)$, the composite derivative of f at y based on the sequence of sets $\{H_n\}$. First, we shall show that the right $\mathcal{R}$-first return path derivative of f at y, $f'_{\mathcal{R}}(y)$, exists and equals $g(y)$.

Based on the enumeration scheme used for $\{x_n\}$, it is clear that the right first return path to y is simply

$$R_y^+ = \bigcup_{k=0}^{\infty} \{\rho^k(y)\},$$

and condition B guarantees that $R_y^+ \subset H_{n(y)+1}$. Thus

$$f'_{+\mathcal{R}}(y) \equiv \lim_{\substack{t \to y \\ t \in R_y^+}} \frac{f(t) - f(y)}{t - y} = \lim_{\substack{t \to y^+ \\ t \in H_{n(y)+1}}} \frac{f(t) - f(y)}{t - y} = g(y).$$

Similarly, the enumeration scheme used for $\{x_n\}$ yields the conclusion that the left first return path to y is

$$R_y^- = \bigcup_{k=0}^{\infty} \left\{ [\lambda^{k-1}(y), \lambda^k(y)] \cap \mathcal{P}_k \right\},$$

and condition A guarantees that $R_y^- \subset H_{n(y)+1}$. Thus

$$f'_{-\mathcal{R}}(y) \equiv \lim_{\substack{t \to y \\ t \in R_y^-}} \frac{f(t)-f(y)}{t-y} = \lim_{\substack{t \to y^- \\ t \in H_{n(y)+1}}} \frac{f(t)-f(y)}{t-y} = g(y),$$

completing the proof that

$$f'_{\mathcal{R}}(y) = g(y).$$

COROLLARY 1. *Every approximate derivative, every Peano derivative of every order, and, indeed, every approximate Peano derivative of every order is a universal first return derivative.*

PROOF. O'Malley has shown than an approximate derivative is both a composite derivative [13] and a selective derivative [11]. The latter implies that the bilateral condition of the hypotheses for Theorem 1 will be satisfied. Likewise, Fejzić [5] has shown that every k^{th} Peano derivative of a function f is both a composite derivative and a selective derivative of the $(k-1)^{\text{th}}$ Peano derivative of f. Superseding both of these results, Fejzić [6] has recently established that that every approximate k^{th} Peano derivative of a function f is both a composite derivative and a selective derivative of the $(k-1)^{\text{th}}$ approximate Peano derivative of f.

4. First return continuity

Here we shall show that first return continuity is equivalent to the Baire 1, Darboux property. We begin by showing that derivatives are first return continuous. In proving this result, we shall utilize the following lemma. In its statement we use the symbol $d\big(y,[a,b]\big)$ to denote the distance from a point y to an interval $[a,b]$. The proof is elementary and is left to the reader.

LEMMA. *Suppose F is differentiable at y and $\varepsilon > 0$ and y is contained in some closed interval I such that if $t \in I \setminus \{y\}$ then* $\left|\frac{F(t)-F(y)}{t-y} - F'(y)\right| \leqq \varepsilon$. *If $[a,b] \subset I$ and $y \notin [a,b]$, then* $\left|\frac{F(a)-F(b)}{a-b} - F'(y)\right| \leqq \varepsilon \cdot \left(1 + 2 \cdot \frac{d\left(y,[a,b]\right)}{|a-b|}\right)$.

THEOREM 2. *Let $f\colon [0,1] \to \mathbf{R}$ be a derivative. Then, f is first return continuous.*

PROOF. We will construct a sequence of partitions $\{\mathcal{P}_k\}$ where each $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$ by induction. At the same time, we will define a trajectory $\{x_n\}$ using $\{\mathcal{P}_k\}$. For $k \geqq -1$, we let $\lambda^k(y)$ and $\rho^k(y)$ be the same as in the proof of Theorem 1. If F has a derivative at x and x is contained in the interior of a closed interval I, then we will let

$$\Delta(F,x,I)=\sup\left\{\left|\frac{F(x)-F(y)}{x-y}-F'(x)\right|:y\in I\setminus\{x\}\right\}.$$

Let F be such that $F'=f$. Let $\mathcal{P}_0=\{p_0^0=0,p_1^0=1\}$, and as before set $x_{-2}=0$ and $x_{-1}=1$. For each non-negative integer n, we want $\mathcal{P}_n$ to satisfy the following conditions:

(1) If $y\in(0,1]$, $t\in\left(\lambda^{n-1}(y),\lambda^n(y)\right]\cap\mathcal{P}_n$, and $t>0$, then $\left|f(t)-f(y)\right|\leqq 5\cdot\Delta\left(F,y,\left[\lambda^{n-1}(y),\rho^{n-1}(y)\right]\right)$.

(2) If $y\in[0,1)$, $t\in[\rho^n(y),\rho^{n-1}(y))\cap\mathcal{P}_n$, and $t<1$, then $\left|f(t)-f(y)\right|\leqq 5\cdot\Delta\left(F,y,\left[\lambda^{n-1}(y),\rho^{n-1}(y)\right]\right)$.

(3) And, $\operatorname{mesh}(\mathcal{P}_n)\leqq\left(\frac{2}{3}\right)^n$.

Note that $\mathcal{P}_0$ satisfies conditions 1–3. (It satisfies 1 and 2 vacuously.) Suppose that $\mathcal{P}_k$ has been defined and it satisfies conditions 1–3. Let $\mathcal{P}_k=\{p_0^k=0<p_1^k<\ldots<p_{l_k}^k=1\}$. Since $\mathcal{P}_{k+1}$ has to be a refinement of $\mathcal{P}_k$, put all points of $\mathcal{P}_k$ in $\mathcal{P}_{k+1}$. For each $0\leqq i\leqq l_k-1$, we will select points which properly refine $\{p_i^k,p_{i+1}^k\}$. Fix such an i and let $l_i^k=\frac{2}{3}p_i^k+\frac{1}{3}p_{i+1}^k$ and $r_i^k=\frac{1}{3}p_i^k+\frac{2}{3}p_{i+1}^k$. Using the mean value theorem, obtain $p_{i,l}^k\in[p_i^k,l_i^k]$, $p_{i,m}^k\in[l_i^k,r_i^k]$, and $p_{i,r}^k\in[r_i^k,p_{i+1}^k]$ such that

$$f(p_{i,l}^k)=F'(p_{i,l}^k)=\frac{F(p_i^k)-F(l_i^k)}{p_i^k-l_i^k},$$

$$f(p_{i,m}^k)=F'(p_{i,m}^k)=\frac{F(l_i^k)-F(r_i^k)}{l_i^k-r_i^k},$$

and

$$f(p_{i,r}^k)=F'(p_{i,r}^k)=\frac{F(r_i^k)-F(p_{i+1}^k)}{r_i^k-p_{i+1}^k}.$$

We put all points of form $p_{i,l}^k$, $p_{i,m}^k$, and $p_{i,r}^k$, $0\leqq i\leqq l_{k-1}$ in the partition $\mathcal{P}_{k+1}$ and order it in the natural increasing fashion as $\mathcal{P}_k=\{p_0^{k+1}=0<p_1^{k+1}<\ldots<p_{l_{k+1}}^{k+1}=1\}$. We also label points in $\mathcal{P}_{k+1}\setminus\mathcal{P}_k$ from left to right as $x_{l_k-1},x_{l_k},\ldots,x_{l_{k+1}-2}$.

We need to show that conditions 1-3 are satisfied by $\mathcal{P}_{k+1}$. Though the partitions are defined by induction, conditions 1 and 2 are directly verified. Let us first consider condition 1. Let $y\in(0,1]$, $t\in\left(\lambda^k(y),\lambda^{k+1}(y)\right]\cap\mathcal{P}_{k+1}$, and $t>0$. Let i be such that $p_i^k<y\leqq p_{i+1}^k$. Then, t has to be one of $p_{i,l}^k$,

$p^k_{i,m}$, or $p^k_{i,r}$. Let us first assume that $t = p^k_{i,l}$. Now we have two cases to consider: $y \in [p^k_i, l^k_i]$ or $y \notin [p^k_i, l^k_i]$. If $y \in [p^k_i, l^k_i]$, then

$$|f(t) - f(y)| = |f(p^k_{i,l}) - f(y)| = \left|\frac{F(p^k_i) - F(l^k_i)}{p^k_i - l^k_i} - f(y)\right| \leqq$$

$$\leqq \Delta\left(F, y, \left[\lambda^k(y), \rho^k(y)\right]\right),$$

where this estimate holds because the difference quotient $\frac{F(p^k_i)-F(l^k_i)}{p^k_i - l^k_i}$ lies between $\frac{F(p^k_i)-F(y)}{p^k_i - y}$ and $\frac{F(y)-F(l^k_i)}{y - l^k_i}$. If $y \notin [p^k_i, l^k_i]$, then by Lemma 1, we have that

$$|f(t) - f(y)| = |f(p^k_{i,l}) - f(y)| = \left|\frac{F(p^k_i) - F(l^k_i)}{p^k_i - l^k_i} - f(y)\right| \leqq$$

$$\leqq \Delta\left(F, y, \left[\lambda^k(y), \rho^k(y)\right]\right) \cdot \left(1 + 2 \cdot \frac{d\left(y, [p^k_i, l^k_i]\right)}{|p^k_i - l^k_i|}\right).$$

Since $d\left(y, [p^k_i, l^k_i]\right) \leqq \frac{2}{3}|p^k_i - p^k_{i+1}|$ and $|p^k_i - l^k_i| = \frac{1}{3}|p^k_i - p^k_{i+1}|$, we have that

$$|f(t) - f(y)| \leqq 5 \cdot \Delta\left(F, y, \left[\lambda^k(y), \rho^k(y)\right]\right).$$

If $t = p^k_{i,m}$ or $t = p^k_{i,r}$, we may also obtain by an argument similar to the above that $\left|f(t) - f(y)\right| \leqq 5 \cdot \Delta\left(F, y, \left[\lambda^k(y), \rho^k(y)\right]\right)$. We just consider the interval $[l^k_i, r^k_i]$ if $t = p^k_{i,m}$, and the interval $[r^k_i, p^k_{i+1}]$ if $t = p^k_{i,r}$. Thus, condition 1 holds. Condition 2 may be verified by a symmetric argument.

That mesh $(\mathcal{P}_{k+1}) \leqq \left(\frac{2}{3}\right)^{k+1}$ easily follows from the induction hypothesis and the facts that for every $0 \leqq i \leqq l_{k-1}$, each of $|p^k_i - p^k_{i,l}|$ and $|p^k_{i+1} - p^k_{i,r}|$ is less than $\frac{1}{3} \cdot |p^k_i - p^k_{i+1}|$, and each of $|p^k_{i,l} - p^k_{i,m}|$ and $|p^k_{i,m} - p^k_{i,r}|$ is less than $\frac{2}{3} \cdot |p^k_i - p^k_{i+1}|$. Thus, condition 3 is satisfied.

It follows from condition 3 that $\{x_n\}$ is a trajectory. Now we want to show that f is first return continuous with respect to this trajectory at each point. Let $y \in (0, 1]$. First, observe that the left first return path to y is

$$R^-_y = \bigcup_{k=0}^{\infty} \left\{\left(\lambda^{k-1}(y), \lambda^k(y)\right] \cap \mathcal{P}_k\right\}.$$

Let $\varepsilon > 0$. Let $\delta > 0$ such that $y - \delta > 0$, and if $0 < |t - y| < \delta$, then $\left|\frac{F(t)-F(y)}{t-y} - f(y)\right| < \frac{\varepsilon}{6}$. Let n be a positive integer such that $\left(\frac{2}{3}\right)^n < \frac{\delta}{2}$.

Then, by condition 1 of the induction hypothesis, we have that if $k > n$, and $t \in \{(\lambda^{k-1}(y), \lambda^k(y)] \cap \mathcal{P}_k\}$, then

$$|f(t) - f(y)| \leqq 5 \cdot \Delta\left(F, y, [\lambda^{k-1}(y), \rho^{k-1}(y)]\right) \leqq 5 \cdot \frac{\varepsilon}{6} < \varepsilon.$$

Consequently, $|f(t) - f(y)| < \varepsilon$ for all $t \in \bigcup_{k=n+1}^{\infty} \{(\lambda^{k-1}(y), \lambda^k(y)] \cap \mathcal{P}_k\}$. Thus, we have that f is left first return continuous. A symmetric argument also shows that f is right first return continuous at each point of $[0,1)$.

THEOREM 3. *A function $f:[0,1] \to \mathbf{R}$ is Darboux and of Baire class 1 if and only if f is first return continuous.*

PROOF. ($\Leftarrow$) Suppose f is first return continuous. For each positive integer $n > 2$, let h_n be the natural piecewise linear continuous function that is obtained by connecting the first n points of the trajectory. Then, $\{h_n\}$ converges pointwise to f. Therefore, f is Baire 1. To see that f is Darboux, recall that a Baire 1 function is Darboux iff each $x \in [0,1]$ has a bilateral road [2]. Since f is first return continuous, the first return path at each point is a bilateral road for that point. Therefore, f is Darboux.

($\Rightarrow$) Suppose f is Darboux, and of Baire class 1. By the Maximoff–Preiss theorem ([8], [17]), there exists a derivative $g:[0,1] \to \mathbf{R}$ and a homeomorphism $h:[0,1] \to [0,1]$ such that $f(x) = g(h(x))$ for all x. By Theorem 2, let $\{x_n\}$ be a trajectory such that g is first return continuous with respect to $\{x_n\}$. Then, f is first return continuous with respect to the trajectory $\{h^{-1}(x_n)\}$.

We note that the "if" portion of Theorem 3 is an immediate consequence of Theorem 4 in [9], but since the proof for this direction is short, we have included it for completeness.

COROLLARY 2. *First return derivatives are first return continuous.*

PROOF. This follows immediately from Theorem 2 in [15] and Theorem 3.

There are Baire 1, Darboux functions which are not universally first return continuous. For example, consider Croft's [4] familiar example of a Baire 1, Darboux function which is not identically zero, but is zero on a set T of full measure in $[0,1]$. If one selects a support set $D \subset T$, then clearly there is no ordering of D with respect to which this function will be first return continuous. However, if we strengthen the Baire 1 condition to Baire* 1, then the situation changes and we may obtain a universal first return continuity result. To show this we begin with a result analogous to Lemma 1. Before stating this result, we recall that if $f:[0,1] \to \mathbf{R}$ is a Baire* 1 function, then the interior $V(f)$ of the set of points of continuity of f is dense in $[0,1]$.

LEMMA 3. *Let $f:[0,1] \to \mathbf{R}$ be a Baire* 1, Darboux function. Let D be any support set. Then there exists a nondecreasing sequence $\{H_n\}$ of perfect* *sets whose union is* $[0,1]$ *and such that for each natural number* n

A) *the restriction of* f *to* H_n *is continuous,*

B) *each point of* H_n *is a bilateral limit point of* H_{n+1},

C) *each component of* $[0,1] \setminus H_n$ *has both endpoints in* $V(f) \cap D \cup \{0,1\}$.

PROOF. The proof this lemma is rather similar to, but somewhat simpler than, the proof of Lemma 1. Let $U \equiv V(f)$ and set

$$T = D \cap U \cup \{0,1\}.$$

Since f is a Baire* 1 function, there is a sequence of perfect sets $\{E_n\}$ whose union is $[0,1]$ such that the restriction of f to E_n is continuous. (See, for example, Lemma 5 in [1] or Theorem 2.1 in [12].)

Proceeding exactly as in Lemma 1, we construct the sequence $\{H_n\}$ inductively, but this time letting F be a continuous extension of $f|E_1$. Define f_1 as before, and it follows that f_1 is Darboux Baire* 1. We define $L_{n,k}$'s and $R_{n,k}$'s as before except replacing Condition 3 with

(3) for each $x \in L_{n,k}$ (and $R_{n,k}$ as well) we have $|f_1(x)| < \frac{1}{n+k}$.

It is possible to construct $L_{n,k}$'s and $R_{n,k}$'s which satisfy this new condition 3 because every Darboux Baire 1 function has a bilateral perfect road at each point [2] and the points of continuity of a Darboux Baire* 1 function is dense in the graph [10]. Then, H_1 is defined as in Lemma 1 and an analogous argument shows that $f|H_1$ is continuous. The induction may be carried on as previously, yielding a decompostion $\{H_n: n = 1,2,3,\ldots\}$ satisfying the required properties.

THEOREM 4. *If* $f:[0,1] \to \mathbf{R}$ *is a Baire* 1, Darboux function, then* f *is universally first return continuous.*

PROOF. The construction of $\{x_n\}$ and the resulting first return path system $\mathcal{R}$, proceeds exactly as in the proof of Theorem 1, with the sets U and T having the definitions supplied in the proof of Lemma 3. That f is first return continuous with respect to $\mathcal{R}$, then follows along the same lines as the final part of the proof of Theorem 1, but, of course, is somewhat simpler.

COROLLARY 3. *First return differentiable functions are universally first return continuous.*

PROOF. First return differentiable functions were observed to be Baire* 1, Darboux in Theorem 2 of [15].

5. Examples

Here we present two examples to show that the converses of Theorems 4 and 1 are false. The second example further demonstrates that a function can have more than one universal first return derivative on $[0,1]$.

EXAMPLE. There exists a function $f\colon [0,1] \to \mathbf{R}$ which is universally first return continuous, but is not Baire* 1.

PROOF. Let $C \subset (0,1)$ be a Cantor set, and $\{c_i\}_{i=1}^{\infty}$ be a dense subset of C. Let $f\colon [0,1] \to \mathbf{R}$ be such that

(1) $f(c_n) = \frac{1}{2^n}$, and if $x \in C \setminus \{c_1, c_2, \dots\}$ then $f(x) = 0$,

(2) f is continuous on $[0,1] \setminus C$, and

(3) if $x \in C$ and I is any interval containing x, then $f(I) = [0,1]$.

Note f is Darboux and Baire 1, but f is not Baire* 1 because $f|C$ is continuous only on $C \setminus \{c_1, c_2, \dots\}$, a set which does not contain an open set relative to C.

We now want to show that given any support set D there exists an ordering of D such that f is first return continuous with respect to this ordering. Let $U = [0,1] \setminus C$, and $G = C \setminus \{c_1, c_2, c_3, \dots\}$. Enumerate $D \cap U$ as $\{a_i\}_{i=1}^{\infty}$, and $D \cap G$ as $\{b_i\}_{i=1}^{\infty}$. Note that $D \cap G$ may be finite or even empty. However, we will consider the worst possible case and assume that $D \cap G$ is infinite.

As we have done several times before, we will construct a sequence of partitions $\{\mathcal{P}_k\}$ such that each $\mathcal{P}_k \subset D$, and each $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$. At the same time, we will define a trajectory $\{x_n\}$ using $\{\mathcal{P}_k\}$. For $k \geqq -1$, we let $\lambda^k(y)$ and $\rho^k(y)$ be the same as in the proof of Theorem 1.

Let $\mathcal{P}_0 = \{p_0^0 = 0, p_1^0 = 1\}$, $x_{-2} = 0$ and $x_{-1} = 1$. For each non-negative integer n, we want $\mathcal{P}_n$ and its labelling $\{x_i\}_{i=-2}^{m_n-2}$ to satisfy the following conditions:

I. If $1 \leqq i, j \leqq n$ and $i \neq j$, then $\left[\lambda^n(c_i), \rho^n(c_i)\right] \cap \left[\lambda^n(c_j), \rho^n(c_j)\right] = \emptyset$.

II. If $1 \leqq j < n$, and t belongs to both $\left(\lambda^{n-1}(c_j), \rho^{n-1}(c_j)\right) \cap \mathcal{P}_n$ as well as either the left or right first return sequence to c_j restricted to $\{x_i\}_{i=-2}^{m_n-2}$, then $\left|f(c_j) - f(t)\right| \leqq \frac{1}{2^n}$.

III. Let $y \in G$, $1 \leqq j < n$, and $\left[\lambda^{n-1}(y), \rho^{n-1}(y)\right] \cap \cap \bigcup_{i=1}^{j} \left[\lambda^n(c_i), \rho^n(c_i)\right] = \emptyset$. If t belongs to both $\left(\lambda^{n-1}(y), \rho^{n-1}(y)\right) \cap \mathcal{P}_n$ as well as either the left or right first return sequence to y restricted to $\{x_i\}_{i=-2}^{m_n-2}$, then $\left|f(t) - f(y)\right| \leqq \frac{1}{2^{j-1}}$.

IV. $\{a_1, a_2, \dots, a_n\} \cup \{b_1, b_2, \dots, b_n\} \cup \left(\{c_1, c_2, \dots, c_n\} \cap D\right) \subset \mathcal{P}_n$.

Note that $\mathcal{P}_0$ satisfies conditions I–IV vacuously. Suppose that $\mathcal{P}_k$ has been defined and it satisfies conditions I–IV. Since $\mathcal{P}_{k+1}$ has to be a refinement of $\mathcal{P}_k$, put all points of $\mathcal{P}_k$ in $\mathcal{P}_{k+1}$. Now we pick some more points in the following fashion: For each $1 \leqq j \leqq k+1$, let l_{c_j} and r_{c_j} be points of $\{a_i\} \setminus \setminus \mathcal{P}_k$ such that

a. $l_{c_j} < c_j < r_{c_j}$,

b. $\left|f(c_j) - f(l_{c_j})\right| < 2^{-(k+1)}$, $\left|f(c_j) - f(r_{c_j})\right| < 2^{-(k+1)}$,

c. $[l_{c_j}, r_{c_j}] \subset \left[\lambda^k(c_j), \rho^k(c_j)\right]$,

d. if $i \neq j$, then $[l_{c_i}, r_{c_i}] \cap [l_{c_j}, r_{c_j}] = \emptyset$, and

e. $b_{k+1} \notin \bigcup_{i=1}^{k+1}[l_{c_j}, r_{c_j}]$ and $a_{k+1} \notin \bigcup_{i=1}^{k+1}[l_{c_j}, r_{c_j}]$ as well.

Let $y_1 < z_1 < y_2 < z_2 < \ldots < y_{n_{k+1}} < z_{n_{k+1}}$ be points of $\{a_i\} \setminus \mathcal{P}_k$ such that

A. $f(y_1) = f(z_1) = \ldots = f(y_{n_{k+1}}) = f(z_{n_{k+1}}) = 0$,

B. $\bigcup_{i=1}^{k+1}[l_{c_i}, r_{c_i}] \cap \bigcup_{i=1}^{n_{k+1}}[y_i, z_i] = \emptyset$,

C. $C \setminus \bigcup_{i=1}^{k+1}[l_{c_i}, r_{c_i}] \subset \bigcup_{i=1}^{n_{k+1}}[y_i, z_i]$, and

D. $a_{k+1} \notin \bigcup_{i=1}^{n_{k+1}}[y_i, z_i]$.

Let

$$\mathcal{P}_{k+1} = \mathcal{P}_k \cup \{l_{c_1}, l_{c_2}, \ldots, l_{c_{k+1}}\} \cup \{r_{c_1}, r_{c_2}, \ldots, r_{c_{k+1}}\} \cup$$

$$\cup \{y_1, y_2, \ldots, y_{n_{k+1}}\} \cup \{z_1, z_2, \ldots, z_{n_{k+1}}\} \cup$$

$$\cup \{b_{k+1}\} \cup \{a_{k+1}\} \cup (\{c_{k+1}\} \cap D).$$

We label points of $\mathcal{P}_{k+1} \setminus \mathcal{P}_k$ as x_n's in the following order:

Step 1. Label $l_{c_1}, r_{c_1}, \ldots, l_{c_{k+1}}, r_{c_{k+1}}$ as listed if c_{k+1} does not belong D. If $c_{k+1} \in D$, then put c_{k+1} between $l_{c_{k+1}}$ and $r_{c_{k+1}}$ in the listing.

Step 2. Next label $y_1, z_1, y_2, z_2, \ldots, y_{n_{k+1}}, z_{n_{k+1}}$ as listed. Then label b_{k+1} and, finally, label a_{k+1} unless if it has not already been labelled.

We now want to show that $\mathcal{P}_{k+1}$ satisfies conditions I–IV. Condition I follows from the construction of $\mathcal{P}_{k+1}$ and the facts that for each $1 \leqq i \leqq \leqq k+1$, $\lambda^{k+1}(c_i) = l_{c_i}$ and $\rho^{k+1}(c_i) = r_{c_i}$.

To show condition II, assume that $1 \leqq j < k+1$, and t belongs to both $(\lambda^k(c_j), \rho^k(c_j)) \cap \mathcal{P}_{k+1}$ as well as either the left or right first return sequence to c_j restricted to $\{x_i\}_{i=-2}^{m_{k+1}-2}$. From the induction hypothesis I for k and the way the l_{c_i}'s and r_{c_i}'s were constructed, we have that $t = l_{c_j}$ or $t = r_{c_j}$. Therefore, from b it follows that $|f(t) - f(c_j)| \leqq 2^{-(k+1)}$.

To show condition III, assume that $y \in G$, $1 \leqq j < k+1$, and that

$$[\lambda^k(y), \rho^k(y)] \cap \left(\bigcup_{i=1}^{j} [\lambda^{k+1}(c_i), \rho^{k+1}(c_i)] \right) = \emptyset.$$

Furthermore, assume that t belongs to both $(\lambda^k(y), \rho^k(y)) \cap \mathcal{P}_{k+1}$ as well as either the left or right first return sequence to y restricted to $\{x_i\}_{i=-2}^{m_{k+1}-2}$. Note that t cannot be a_{k+1}. To see this, note that $y \in \bigcup_{i=j+1}^{k+1}[l_{c_i}, r_{c_i}]$ or $y \in \in \bigcup_{i=1}^{n_{k+1}}[y_i, z_i]$. Now, if $t = a_{k+1}$, then a_{k+1} was labelled last. Hence, in either of these cases an application of condition e or D produces a contradiction to the assumption that t was in the left or right first return path to y, and, consequently, $t \neq a_{k+1}$. Therefore t has to be one of l_{c_i}, r_{c_i}, c_i for some $i > j$, or t has to be one of $y_1, z_1, y_2, z_2, \ldots, y_{n_{k+1}}, z_{n_{k+1}}$, or $t = b_{k+1}$. If t is one of $y_1, z_1, y_2, z_2, \ldots, y_{n_{k+1}}, z_{n_{k+1}}$, then by A we have that

$$|f(y) - f(t)| = |f(t)| = 0.$$

If $t = b_{k+1}$ then $f(t) = f(y) = 0$. If t is one of l_{c_i}, r_{c_i}, c_i for some $i > j$, then

$$|f(t) - f(y)| \leqq |f(t) - f(c_i)| + |f(c_i) - f(y)| =$$

$$= |f(t) - f(c_i)| + |f(c_i)| < 2^{-i} + 2^{-i} < 2^{-j+1}.$$

Thus, condition III holds. Condition IV follows from the construction of $\mathcal{P}_{k+1}$. That trajectory $\{x_i\}_{i=0}^{\infty}$ is a well-ordering of D follows from condition IV. We now want to show that f is first return continuous with respect to $\{x_i\}_{i=0}^{\infty}$. Let $x \in [0,1]$. We have three cases to consider: $x \in U$, $x = c_i$ for some i, or $x \in G$. If $x \in U$, then f is first return continuous at x because f is continuous at x.

Consider next the case $x = c_i$ for some i. Let $\varepsilon > 0$. Let N be a positive integer such that $N > i$ and $2^{-N} < \varepsilon$. We want to show that if $t \notin \mathcal{P}_N$ and t is in either the left or right first return sequence to x, then $|f(t) - f(x)| < \varepsilon$. Let t be as described. Then, there is $m > N$ such that $t \in \mathcal{P}_m \setminus \mathcal{P}_{m-1}$. Since t is in either the left or right first return sequence to x, $t \in (\lambda^{m-1}(x), \rho^{m-1}(x)) \cap \mathcal{P}_m$. We have by condition II that $|f(t) - f(x)| \leqq$ $\leqq 2^{-m} < 2^{-N} < \varepsilon$. Thus, f is first return continuous at x with respect to $\{x_n\}$.

Finally, consider the case where $x \in G$. Let $\varepsilon > 0$ and j be a positive integer such that $2^{-j+1} < \varepsilon$. Utilizing the fact that $\{x_i\}_{i=0}^{\infty}$ is a trajectory, we may obtain a positive integer $N > j$ such that for all $m > N$ we have

$$[\lambda^{m-1}(x), \rho^{m-1}(x)] \cap \bigcup_{i=1}^{j} [\lambda^m(c_i), \rho^m(c_i)] = \emptyset.$$

We must show that if $t \notin \mathcal{P}_N$ but t is in either the left or right first return sequence to x, then we have $|f(t) - f(x)| < \varepsilon$. Let t be as described. Then, there is $m > N$ such that $t \in \mathcal{P}_m \setminus \mathcal{P}_{m-1}$. Since t is in either the left or right first return sequence to x, $t \in (\lambda^{m-1}(x), \rho^{m-1}(x)) \cap \mathcal{P}_m$. We have by condition III that $|f(t) - f(x)| \leqq 2^{-j+1} < \varepsilon$. Thus, f is first return continuous at x with respect to $\{x_n\}$ and the proof is complete.

EXAMPLE 2. There are functions $f\colon [0,1] \to \mathbf{R}$, $g\colon [0,1] \to \mathbf{R}$, and $h\colon [0,1] \to \mathbf{R}$ such that f is compositely differentiable to h, f is universally first return differentiable to both g and h, and yet f is not compositely differentiable to g.

PROOF. Let $C \subset [0,1]$ be the standard middle third Cantor set constructed in the standard fashion. Let $\{(u_i, v_i)\}_{i=1}^{\infty}$ an enumeration of the contiguous intervals to C, listed in such a way that both of the sequences $\{u_{2j}\}_{j=1}^{\infty}$ and $\{u_{2j-1}\}_{j=1}^{\infty}$ are dense in C

Let $f\colon [0,1] \to \mathbf{R}$ be such that

A. $f(C) = \{0\}$,

B. f is differentiable on $[0,1] \setminus C$, and

C. for each even natural number i, f is zero on $\left(a_i, \frac{3a_i+b_i}{4}\right] \cup \left[\frac{a_i+3b_i}{4}, b_i\right)$, $f\left(\frac{a_i+b_i}{2}\right) = 1$, f is increasing on $\left[\frac{3a_i+b_i}{4}, \frac{a_i+b_i}{2}\right]$, and f is decreasing on $\left[\frac{a_i+b_i}{2}, \frac{a_i+3b_i}{4}\right]$; likewise, for each odd i, f is zero on $\left(a_i, \frac{3a_i+b_i}{4}\right] \cup \left[\frac{a_i+3b_i}{4}, b_i\right)$, $f\left(\frac{a_i+b_i}{2}\right) = -1$, f is decreasing on $\left[\frac{3a_i+b_i}{4}, \frac{a_i+b_i}{2}\right]$ and f is increasing on $\left[\frac{a_i+b_i}{2}, \frac{a_i+3b_i}{4}\right]$.

Let h be zero on C and be the derivative of f on $[0,1] \setminus C$. Let $\{c_i\}_{i=1}^{\infty}$ be a dense subset of C, containing no endpoint of a contiguous interval, and neither 0 nor 1. Let $g\colon [0,1] \to \mathbf{R}$ be such that $g(c_i) = \frac{1}{2^i}$, g is the derivative of f on $[0,1] \setminus C$, and g is zero on $C \setminus \{c_1, c_2, \ldots\}$.

It is easily seen that f is compositely differentiable to h on $[0,1]$ and that for each x, $h(x)$ is a bilateral derived number of f at x. Consequently, Theorem 1 shows that f is universally first return differentiable to h on $[0,1]$. Let us now show that no composite derivative of f equals g. Let v be a composite derivative of f obtained by a sequence of closed sets $\{E_n\}_{n=1}^{\infty}$. By the Baire category theorem, there is a positive integer n such that $E_n \cap C$ is a non-empty set which is open relative to C. Since f is zero on C and differentiable on $E_n \cap C$, v has to be zero on $E_n \cap C$. Since g is positive on a dense subset of C, v does not equal g.

We now want to show that given any support set D, there exists an ordering of D such that the first return derivative of f is g with respect to this ordering. Let U, G, $\{a_i\}_{i=1}^{\infty}$, and $\{b_i\}_{i=1}^{\infty}$ be the same as in Example 1. We will also construct $\{\mathcal{P}_n\}$ and $\{x_n\}$ in a fashion similar to that of Example 1. We let $\lambda^k(y)$ and $\rho^k(y)$ have the same meaning as in Example 1.

Let $\mathcal{P}_0 = \{p_0^0 = 0, p_1^0 = 1\}$, $x_{-2} = 0$ and $x_{-1} = 1$. For each non-negative integer n, we want $\mathcal{P}_n$ and its labelling $\{x_i\}_{i=-2}^{m_n-2}$ to satisfy the conditions I and IV of Example 1 as well as the following replacements for conditions II and III.

II. If $1 \leqq j < n$, and t belongs to both $\left(\lambda^{n-1}(c_j), \rho^{n-1}(c_j)\right) \cap \mathcal{P}_n$ as well as either the left or right first return sequence to c_j restricted to $\{x_i\}_{i=-2}^{m_n-2}$, then we have $\left|\frac{f(c_j)-f(t)}{c_j-t} - g(c_j)\right| \leqq \frac{1}{2^n}$.

III. Let $y \in G$, $1 \leqq j < n$, and

$$[\lambda^{n-1}(y), \rho^{n-1}(y)] \cap \bigcup_{i=1}^{j} [\lambda^n(c_i), \rho^n(c_i)] = \emptyset.$$

If t belongs to both $(\lambda^{n-1}(y), \rho^{n-1}(y)) \cap \mathcal{P}_n$ as well as either the left or right first return sequence to y restricted to $\{x_i\}_{i=-2}^{m_n-2}$, then we have $\left|\frac{f(t)-f(y)}{t-y} - g(y)\right| \leqq \frac{6}{2^{j-1}}$.

Now we proceed as in Example 1 up to the point of defining $\mathcal{P}_{k+1}$. We want l_{c_j} and r_{c_j} to satisfy the same conditions a, c, d, e of the previous example, as well as the following conditions b and f.

b. $\left|\frac{f(c_j)-f(l_{c_j})}{c_j-l_{c_j}} - g(c_j)\right| < 2^{-(k+1)}$, $\left|\frac{f(c_j)-f(r_{c_j})}{c_j-r_{c_j}} - g(c_j)\right| < 2^{-(k+1)}$,

f. $\left|\frac{c_j-l_{c_j}}{l_{c_j}-y}\right| \leqq 6$, $\left|\frac{c_j-r_{c_j}}{r_{c_j}-y}\right| \leqq 6$ for every $y \in C$.

We may readily obtain l_{c_j}'s and r_{c_j}'s which will satisfy all of conditions a–f by using the fact that for every c_j there are contiguous intervals $I = (u_s, v_s)$, and $J = (u_t, v_t)$, with s even and t odd, arbitrarily close to c_j such that $v_s < c_j < u_t$ and

$$\frac{|I|}{d(c_j, I) + |I|} \geqq \frac{1}{2}, \quad \text{and} \quad \frac{|J|}{d(c_j, J) + |J|} \geqq \frac{1}{2}.$$

Again we proceed as in Example 1, defining y_i's, z_i's satisfying conditions A through D, and we order the points of $\mathcal{P}_{k+1} \setminus \mathcal{P}_k$ as $\{x_n\}$ as we did before.

Next, we want to show that $\mathcal{P}_{k+1}$ satisfies conditions I–IV. Conditions I, II and IV hold for reasons similar to those of Example 1. To see the validity of condition III, assume that $y \in G$, $1 \leqq j < k+1$, and

$$[\lambda^k(y), \rho^k(y)] \cap \left(\bigcup_{i=1}^{j} [\lambda^{k+1}(c_i), \rho^{k+1}(c_i)]\right) = \emptyset.$$

Assume that t belongs to both $(\lambda^k(y), \rho^k(y)) \cap \mathcal{P}_{k+1}$ as well as either the left or right first return sequence to y restricted to $\{x_i\}_{i=-2}^{m_{k+1}-2}$. For reasons similar to those in Example 1, $t \neq a_{k+1}$. Therefore, t has to be one of l_{c_i}, r_{c_i}, c_i for some $i > j$, or t has to be one of $y_1, z_1, y_2, z_2, \ldots, y_{n_{k+1}}, z_{n_{k+1}}$, or $t = b_{k+1}$. If t is one of $y_1, z_1, y_2, z_2, \ldots, y_{n_{k+1}}, z_{n_{k+1}}, b_{k+1}$, then condition III is satisfied because $f(t) = f(y) = 0$. If t is one of l_{c_i}, r_{c_i}, c_i for some $i > j$, then

$$\left|\frac{f(t)-f(y)}{t-y} - g(y)\right| = \left|\frac{f(t)-f(y)}{t-y}\right| \leqq$$

$$\leqq \left|\frac{f(t)-f(c_i)}{t-c_i}\right|\left|\frac{t-c_i}{t-y}\right| + \left|\frac{f(c_i)-f(y)}{c_i-y}\right|\left|\frac{c_i-y}{t-y}\right| \leqq$$

$$\leqq \left|\frac{f(t)-f(c_i)}{t-c_i} - g(c_i)\right|\left|\frac{t-c_i}{t-y}\right| + \left|g(c_i)\right|\left|\frac{t-c_i}{t-y}\right| + 0 \leqq$$

$$\leqq 6\cdot 2^{-i} + 6\cdot 2^{-i} < 6\cdot 2^{-j+1}.$$

Thus, condition III holds.

That f is first return differentiable to g with respect to $\{x_i\}_{i=0}^{\infty}$ can now be shown by following the same reasoning utilized at the end of the proof of Example 1 to show that f was first return continuous.

6. Orderings and trajectories

Here we explore a relationship between what we have been calling trajectories or orderings of support sets and trajectories of continuous mappings of the unit interval. The following theorem in some sense justifies the use of the term trajectory in the definition of first return path systems. Recall that in the terminology of dynamics two mappings f and g of the unit interval are said to be topologically conjugate if there is a homeomorphism h of the unit interval such that $f = h \circ g \circ h^{-1}$.

THEOREM 5. *Let D be a support set and let $\{x_n\}_{n=0}^{\infty}$ be an enumeration of D. Let $g\colon[0,1]\to[0,1]$ be a transitive continuous map. Then there is a function $f\colon[0,1]\to[0,1]$, topologically conjugate to g, such that*

A. *the range of the trajectory of x_0 under f is D; i.e., the range of the sequence $\{x_0, f(x_0), f^2(x_0), f^3(x_0),\dots\}$ is D.*

B. *the first return path system determined by the ordering $\{x_0,x_1,\dots\}$ is identical to that determined by the ordering $\{x_0, f(x_0), f^2(x_0),\dots\}$.*

PROOF. Let y_0 be such that the trajectory of y_0 under g is dense in $[0,1]$. For each $m = 0,1,\dots$, let $y_m = g^m(y_0)$. We inductively define an increasing homeomorphism h from $[0,1]$ onto itself such that $h\left(\bigcup_{m=0}^{\infty}\{y_m\}\right) = D$. Start by letting $h(0)=0$, $h(1)=1$, and $h(y_0)=x_0$. For each non-negative integer m let $\mathcal{Q}_m$ denote the partition of $[0,1]$ generated by $\{0,1,y_0,y_1,\dots,y_m\}$, and assume that h has been defined on $\mathcal{Q}_m$. Let $\mathcal{P}_m$ denote the partition of $[0,1]$ formed by the points $\{0,1,h(y_0),h(y_1),\dots,h(y_m)\}$. Let a and b be neighboring nodes of $\mathcal{Q}_m$ such that $a < y_{m+1} < b$. Define $h(y_{m+1})$ to be that x_n having the property that $x_n \in (h(a),h(b))$ and no x_j with $j < n$ belongs to that interval. In this manner we have now defined h on $\mathcal{Q}_{m+1}$ and it is increasing on that set. Thus h is defined on $\{y_0,\dots y_m,\dots\}$ so as to be increasing and its range is contained in D.

Let us next show by induction that $h(\{y_0,y_1,\dots\}) = D$. We have that $x_0 \in h(\{y_0,y_1,\dots\})$. Suppose that $x_i = h(y_{k_i})$ is in the range for $0 \leqq i \leqq$

$\leqq N-1$. Let $m = \max\{k_i : i = 0, 1, \ldots, N-1\}$. If $x_N = h(y_i)$ for $0 \leqq i \leqq m$, then we are done. Otherwise, let a and b be neighboring nodes of $\mathcal{P}_m$ with $a < x_N < b$. Let $j > m$ be the least integer such that $h^{-1}(a) < y_j < h^{-1}(b)$. For this j we have that $h(y_j) = x_N$.

Now extend h to be an homeomorphism of $[0,1]$ onto itself and define the function $f : [0,1] \to [0,1]$ by

$$f = h \circ g \circ h^{-1}.$$

Observe that for each $i \geqq 1$, $f^i(x_0) = h(y_i)$ and condition A of the theorem holds.

Next, we show that condition B also holds. We will just show that for $x \in [0,1)$, the right first return path to x generated by ordering $\{x_0, x_1, \ldots\}$ and the right first return path to x generated by ordering $\{x_0, f(x_0), f^2(x_0), \ldots\}$ are the same. Fix an $x \in [0,1)$. Let $\{x_{n_k}\}_{k=-1}^{\infty}$ be the right first return path to x determined by the ordering $\{x_0, x_1, \ldots\}$. As $\{x_0, f(x_0), f^2(x_0), \ldots\} = \{x_0, h(y_0), \ldots\}$, we may denote the right first return path to x determined by the the ordering $\{x_0, f(x_0), f^2(x_0), \ldots\}$ as $\{h(y_{m_k})\}_{k=-1}^{\infty}$. (For sake of notational convienience, we are letting $n_{-1} = m_{-1} = -1$ and $x_{-1} = y_{-1} = 1$. We are also assuming that $\{x_{n_k}\}_{k=-1}^{\infty}$ and $\{h(y_{m_k})\}_{k=-1}^{\infty}$ are labelled in the natural fashion, i.e. $\{n_k\}_{i=-1}^{\infty}$ and $\{m_k\}_{i=-1}^{\infty}$ are increasing sequences of integers.)

We now use induction to show that $x_{n_k} = h(y_{m_k})$ for all $k \geqq -1$. Suppose that we have that $x_{n_j} = h(y_{m_j})$ for $-1 \leqq j \leqq N-1$. Put $\nu = m_N - 1$. Let α and β be neighboring nodes of $\mathcal{Q}_\nu$ such that $\alpha < y_{m_N} < \beta$. By the fashion in which h was constructed, $h(\alpha)$ and $h(\beta)$ are neighboring nodes of $\mathcal{P}_\nu$. As $h(\alpha) < h(y_{m_N}) < h(\beta)$ and $h(y_{m_N})$ is in the right first return sequence to x generated by $\{h(y_0), h(y_1), \ldots,\}$ we have that $h(\alpha) \leqq x < h(\beta)$. Consequently, $h(\beta) = h(y_{m_{N-1}})$. By definition, $h(y_{m_N}) = x_t$ where t is the least integer such that $h(\alpha) < x_t < h(\beta) = h(y_{m_{N-1}})$. We also have that $x_t > x$ as $x_t = h(y_{m_N})$ is in the right first return path to x. Thus, we have that t is the least integer such that $x_t > x$ and $x_t < h(y_{m_{N-1}}) = x_{n_{N-1}}$, forcing $h(y_{m_N}) = x_{n_N}$ and completing the proof of the theorem.

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NECESSARY AND SUFFICIENT TAUBERIAN CONDITIONS FOR CERTAIN WEIGHTED MEAN METHODS OF SUMMABILITY

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1. Introduction

Let $(s_k : k = 0, 1, \dots)$ be a real or complex sequence. Let $p := (p_k)$ be a nonnegative sequence with $p_0 > 0$,

$$(1.1) \qquad P_n := \sum_{k=0}^{n} p_k \to \infty \qquad (n \to \infty).$$

The weighted means of the sequence (s_k) are defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \qquad (n = 0, 1, \dots),$$

and (s_k) is said to be summable $(\overline{N}, p)$ if the limit

$$(1.2) \qquad \lim_{n \to \infty} t_n \qquad \text{exists and finite.}$$

It is well-known that condition (1.1) is necessary and sufficient that every convergent sequence (s_k) be summable $(\overline{N}, p)$ to the same limit.

2. Main results

Define $\lambda_n := [\lambda n]$ for a positive number λ, where [.] denotes the integral part. By C we shall denote a positive constant not necessarily the same at different occurrences.

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We will prove the following one-sided Tauberian theorem.

THEOREM 1. *Let (s_k) be a real sequence, (p_k) a nonnegative sequence satisfying condition* (1.1) *and such that for each $\lambda > 1$,*

$$(2.1) \qquad 1 < \liminf_{n\to\infty} \frac{P_{\lambda_n}}{P_n} \leqq \limsup_{n\to\infty} \frac{P_{\lambda_n}}{P_n} < \infty,$$

and for each $0 < \lambda < 1$,

$$(2.2) \qquad 1 < \liminf_{n\to\infty} \frac{P_n}{P_{\lambda_n}} \leqq \limsup_{n\to\infty} \frac{P_n}{P_{\lambda_n}} < \infty.$$

If (s_k) is summable $(\overline{N}, p)$ to a finite limit s, then the limit

$$(2.3) \qquad \lim_{k\to\infty} s_k = s \qquad \textit{exists}$$

if and only if

$$(2.4) \qquad \limsup_{\lambda\downarrow 1} \liminf_{n\to\infty} \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(s_k - s_n) \geqq 0$$

and

$$(2.5) \qquad \limsup_{\lambda\uparrow 1} \liminf_{n\to\infty} \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k(s_n - s_k) \geqq 0;$$

in which case we necessarily have for each $\lambda > 1$,

$$(2.6) \qquad \lim_{n\to\infty} \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(s_k - s_n) = 0,$$

and for each $0 < \lambda < 1$,

$$(2.7) \qquad \lim_{n\to\infty} \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k(s_n - s_k) = 0.$$

REMARK 1. According to [4] (see also [1, pp. 124-125]) a real sequence (s_k) is said to be slowly decreasing if

$$(2.8) \qquad \lim_{\lambda\downarrow 1} \liminf_{n\to\infty} \min_{n<k\leqq\lambda_n} (s_k - s_n) \geqq 0,$$

or equivalently,

$$\lim_{\lambda\uparrow 1}\liminf_{n\to\infty}\min_{\lambda_n<k\leqq n}(s_n-s_k)\geqq 0.$$

Conditions (2.4) and (2.5) are obviously satisfied if (s_k) is slowly decreasing.

REMARK 2. The classical one-sided Tauberian condition

$$k(s_k-s_{k-1})\geqq -C \qquad (k=1,2,\dots) \tag{2.9}$$

of Landau [2] is sufficient for (2.8).

REMARK 3. The symmetric counterparts of conditions (2.4) and (2.5) are the following:

$$\liminf_{\lambda\downarrow 1}\limsup_{n\to\infty}\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_k(s_k-s_n)\leqq 0 \tag{2.10}$$

and

$$\liminf_{\lambda\uparrow 1}\limsup_{n\to\infty}\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_k(s_n-s_k)\leqq 0. \tag{2.11}$$

Assume that conditions (1.1), (1.2), (2.1), and (2.2) are satisfied. Analogously to Theorem 1, one can prove that condition (2.3) is satisfied if and only if (2.10) and (2.11) are satisfied. As a by-product, we may state that if conditions (2.4) and (2.5) are satisfied, then conditions (2.10) and (2.11) are also satisfied, and vice versa.

We extend Theorem 1 for complex sequences as follows.

THEOREM 2. *Let* (s_k) *be a complex sequence and* (p_k) *a nonnegative sequence satisfying condition* (1.1). *If* (s_k) *is summable* $(\overline{N},p)$ *to a finit limit* s *and*

(i) *if condition* (2.1) *is satisfied, then* (s_k) *converges to* s *if and only if*

$$\liminf_{\lambda\downarrow 1}\limsup_{n\to\infty}\left|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_k(s_k-s_n)\right|=0; \quad \textit{or} \tag{2.12}$$

(ii) *if condition* (2.2) *is satisfied, then* (s_k) *converges to* s *if and only if*

$$\liminf_{\lambda\uparrow 1}\limsup_{n\to\infty}\left|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_k(s_n-s_k)\right|=0. \tag{2.13}$$

In either case, we necessarily have (2.6) *for all* $\lambda > 1$, *or* (2.7) *for all* $0 < \lambda < 1$, *respectively.*

REMARK 4. In the complex case, the classical Tauberian condition

$$k|s_k - s_{k-1}| \leqq C \qquad (k = 1, 2, \dots)$$

is sufficient for (2.12) and (2.13). (Cf. (2.9) in the real case.)

REMARK 5. If $p_k = 1$ for all k, then the t_n are the $(C, 1)$-means (i.e., the first arithmetic means) of the sequence (s_k). In this case, Theorems 1 and 2 were proved in [3].

3. Auxiliary result: representation of the difference $s_n - t_n$

LEMMA. (i) *Let* $\lambda > 1$. *For each* n *such that* $P_{\lambda_n} > P_n$,

$$(3.1) \qquad s_n - t_n = \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n}(t_{\lambda_n} - t_n) - \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(s_k - s_n).$$

(ii) *Let* $0 < \lambda < 1$. *For each* n *such that* $P_n > P_{\lambda_n}$,

$$(3.2) \qquad s_n - t_n = \frac{P_{\lambda_n}}{P_n - P_{\lambda_n}}(t_n - t_{\lambda_n}) + \frac{1}{P_n - P_{\lambda_n}} \sum_{k=n+1}^{\lambda_n} p_k(s_n - s_k).$$

PROOF. (i) By definition,

$$t_{\lambda_n} - t_n = \frac{1}{P_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k s_k - \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k =$$

$$= \frac{P_n - P_{\lambda_n}}{P_n P_{\lambda_n}} \sum_{k=0}^{n} p_k s_k + \frac{1}{P_{\lambda_n}} \sum_{k=n+1}^{\lambda_n} p_k s_k.$$

Hence

$$\frac{P_{\lambda_n}}{P_{\lambda_n} - P_n}(t_{\lambda_n} - t_n) - \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k s_k = -t_n,$$

which is equivalent to (3.1).

(ii) The proof of (3.2) is similar.

4. Proofs of the theorems

PROOF OF THEOREM 1. *Necessity.* By (1.2) and (2.3),

$$\lim_{n\to\infty}(s_n - t_n) = 0. \tag{4.1}$$

Let $\lambda > 1$. Using (2.1),

$$\left|\frac{P_{\lambda_n}}{P_{\lambda_n} - P_n}(t_{\lambda_n} - t_n)\right| = \frac{P_{\lambda_n}/P_n}{P_{\lambda_n}/P_n - 1}|t_{\lambda_n} - t_n| \leqq \tag{4.2}$$

$$\leqq \frac{1+\delta}{(1+\gamma)/2 - 1}|t_{\lambda_n} - t_n|$$

if n is large enough, where

$$\gamma := \liminf_{n\to\infty} P_{\lambda_n}/P_n \quad \text{and} \quad \delta := \limsup_{n\to\infty} P_{\lambda_n}/P_n.$$

By (1.2),

$$\lim_{n\to\infty}\frac{P_{\lambda_n}}{P_{\lambda_n} - P_n}(t_{\lambda_n} - t_n) = 0. \tag{4.3}$$

The same is true in the case when $0 < \lambda < 1$.

Now, (2.6) (respectively (2.7)) follows from (3.1) (respectively (3.2)), (4.1), and (4.3).

Sufficiency. Assume the fulfillment of (1.1), (1.2), (2.4), and (2.5). From (2.4) there exists a sequence $\{\lambda_j\}$ monotone decreasing to 1 such that

$$\lim_{j\to\infty}\liminf_{n\to\infty}\frac{1}{P_{\lambda_{jn}} - P_n}\sum_{k=n+1}^{\lambda_{jn}} p_k(s_k - s_n) \geqq 0, \tag{4.4}$$

where $\lambda_{jn} := [\lambda_j n]$. From (3.1),

$$\limsup_{n\to\infty}(s_n - t_n) \leqq \lim_{j\to\infty}\limsup_{n\to\infty}\frac{P_{\lambda_{jn}}}{P_{\lambda_{jn}} - P_n}(t_{\lambda_{jn}} - t_n) + \tag{4.5}$$

$$+\lim_{j\to\infty}\limsup_{n\to\infty}\left(-\frac{1}{P_{\lambda_{jn}} - P_n}\sum_{k=n+1}^{\lambda_{jn}} p_k(s_k - s_n)\right).$$

From (1.2) (cf. (4.2)), for each j,

$$\lim_{n\to\infty} \frac{P_{\lambda_{jn}}}{P_{\lambda_{jn}} - P_n}(t_{\lambda_{jn}} - t_n) = 0.$$

Thus, (4.4) and (4.5) yield

$$\limsup_{n\to\infty}(s_n - t_n) = -\lim_{j\to\infty}\liminf_{n\to\infty} \frac{1}{P_{\lambda_{jn}} - P_n} \sum_{k=n+1}^{\lambda_{jn}} p_k(s_k - s_n) \leqq 0. \tag{4.6}$$

From (2.5) there exists a sequence $\{\lambda_j\}$ monotone increasing to 1 such that

$$\lim_{j\to\infty}\liminf_{n\to\infty} \frac{1}{P_n - P_{\lambda_{jn}}} \sum_{k=\lambda_{jn}+1}^{n} p_k(s_n - s_k) \geqq 0. \tag{4.7}$$

Using (3.2),

$$\liminf_{n\to\infty}(s_n - t_n) \geqq \lim_{j\to\infty}\liminf_{n\to\infty} \frac{P_{\lambda_{jn}}}{P_n - P_{\lambda_{jn}}}(t_n - t_{\lambda_{jn}}) + \tag{4.8}$$

$$+ \lim_{j\to\infty}\liminf_{n\to\infty} \frac{1}{P_n - P_{\lambda_{jn}}} \sum_{k=\lambda_{jn}+1}^{n} p_k(s_n - s_k).$$

But for each j,

$$\lim_{n\to\infty} \frac{P_{\lambda_{jn}}}{P_n - P_{\lambda_{jn}}}(t_n - t_{\lambda_{jn}}) = 0.$$

Thus, (4.7) and (4.8) yield

$$\liminf_{n\to\infty}(s_n - t_n) \geqq 0. \tag{4.9}$$

Combining (4.6) and (4.9) provides (4.1), which is equivalent to (2.3), due to (1.2).

PROOF OF THEOREM 2. The proof of this theorem also relies on representations (3.1) and (3.2), and is modeled after that of Theorem 1. We omit the details.

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NUMBERS WITH COMPLICATED DECIMAL EXPANSIONS

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1. Introduction

One of the most basic results in the theory of distribution modulo 1 is that, if α is an irrational, then the sequence $(n\alpha)_{n=1}^{\infty}$ is dense, and even uniformly distributed, modulo 1. In particular, given any digits $a_1, a_2, \dots, a_k$, there exists a positive integer m for which the decimal expansion of $m\alpha$ contains this block of digits. A considerable strengthening of this result was obtained by Mahler [13] who proved that, moreover, there necessarily exists an m for which the decimal expansion of $m\alpha$ contains the given block infinitely often. Mahler also established an upper bound for the minimal value M of the number m with that property; $M = M(k)$ depends only on the number k of digits, but not on α.

It was noted by Furstenberg that, employing a certain result of Glasner [10], one can provide a very short proof of the finiteness of $M(k)$ (see [1, Corollary 7.2]). Motivated by his approach, the authors [4] gave another short proof of Mahler's result, which at the same time yielded a better upper bound, best possible up to a constant factor.

The density modulo 1 of the sequence $(n\alpha)$ is a special case of a result which asserts that, given any polynomial P with real coefficients, at least one of which (besides the free term) is irrational, the sequence $P(n)$ is dense modulo 1. (Better known is Weyl's even stronger result by which this sequence is uniformly distributed modulo 1 [15].) It was shown in [4, Theorem 1.2] that Mahler's result is true in this more general setting as well. A few other sequences besides polynomial sequences, for example $(\log n)$ and (n^{θ}) for positive rational non-integer θ, were shown to satisfy the same property.

In this paper we present a general framework for the study of sequences in which there exist terms whose expansions tend to be complicated in the sense that they contain "numerous" blocks, perhaps appearing "many" times. In Section 2 we introduce the required definitions and show that some sequences, and families of sequences, possess these properties. Section 3 deals with

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the question as to what extent the number m in Mahler's theorem may be restricted to various sets of integers. The methods developed by Alon and Peres [1] enabled them to show that m may be chosen to be a square, a prime, or to belong to any prescribed set of positive density, etc. We obtain other results in this direction, which cannot be obtained with their methods. In Section 4 it is shown that the analogue of Mahler's theorem with continued fraction expansion instead of decimal expansion is false.

Obviously, the base 10 plays no special role. The abovementioned results, as well as the results of this paper, are valid in any base.

2. Block-complete and block-saturated sequences

Let $g \geqq 2$ be an integer. The set $D_g = \{0, 1, \ldots, g-1\}$ is the set of *digits* in base g. A *k-block* (in base g) is an element of D_g^k, namely a sequence of length k of digits. Denote by $\mathcal{B}_g$ the set of all blocks in base g:

$$\mathcal{B}_g = \bigcup_{k=1}^{\infty} D_g^k .$$

Given a real number α and a block $B \in \mathcal{B}_g$, let $\#(\alpha, B)$ denote the number of occurrences of B in the g-adic expansion of α (thus we may well have $\#(\alpha, B) = \infty$).

REMARK 2.1. To avoid ambiguity, we agree that the g-adic expansion of a rational g-adic number is the one containing only 0's from some place on. Also, we implicitly assume all numbers whose g-adic expansions are considered to be non-negative. These conventions have no effect on the results of the paper.

A number α is *block-complete* (or BC for brevity) if $\#(\alpha, B) \geqq 1$ for every $B \in \mathcal{B}_g$. As is well known, the set of all numbers which are BC in base g is large both metrically (i.e., its complement is of 0 Lebesgue measure) and topologically (i.e., it contains an intersection of countably many open dense sets). Clearly, if α is BC in base g then it is even *block-saturated* (or BS) in base g, namely $\#(\alpha, B) = \infty$ for every $B \in \mathcal{B}_g$.

We now generalize these two concepts, of block-completeness and of block-saturatedness, to sequences of real numbers. A sequence (α_i) is *block-complete* (resp. *block-saturated*) if the set $\{i \geqq 1 : \#(\alpha_i, B) \geqq 1\}$ (resp. $\{i \geqq 1 : \#(\alpha_i, B) = \infty\}$) is infinite for every $B \in \mathcal{B}_g$.

These two properties admit stronger versions. We shall be interested in cases where the sets in question are not just infinite, but have some density properties. Thus, (α_i) is *block-complete in density* if the set $\{i \geqq 1 : \#(\alpha_i, B) \geqq 1\}$ is of density 1 for every $B \in \mathcal{B}_g$. The notions of a sequence which is *BC in Banach density* is analogously defined. (Recall that the *Banach density* of a set $A \subseteqq \mathbf{N}$ is given by

$$\mathrm{BD}(A) = \lim_{N-M\to\infty} \frac{A \cap [M, N-1]}{N-M}$$

if the limit exists.) Finally, (α_i) is *eventually BC* if for every $B \in \mathcal{B}_g$ the set $\{i \geqq 1 : \#(\alpha_i, B) \geqq 1\}$ contains all sufficiently large integers. Again, all these admit straightforward analogues for BS sequences.

Note that a number α is BC if and only if the constant sequence formed by it is a BC (or an eventually BC) sequence. However, a general sequence may be eventually BC while none of its terms is.

EXAMPLE 2.1. The sequence of all positive integers in ascending order, $1, 2, \ldots$, is BC, and even BC in Banach density, in every base, but it is not eventually BC. The sequence $1, 12, 123, \ldots$, given by the recurrence

$$\alpha_1 = 1, \qquad \alpha_{n+1} = n + 10^{1+[\log n]}\alpha_n, \qquad n \geqq 1,$$

is eventually BC in base 10.

EXAMPLE 2.2. No sequence of integers is BS.

The following two lemmas are immediate.

LEMMA 2.1. *Let $\boldsymbol{\alpha} = (\alpha_i)$ be a sequence of real numbers and $g \geqq 2$ an integer. Then:*

(1) *$\boldsymbol{\alpha}$ is a BC sequence if and only if it has an eventually BC subsequence.*

(2) *$\boldsymbol{\alpha}$ is BC in density if and only if it has a subsequence of density 1 which is eventually BC.*

(3) *$\boldsymbol{\alpha}$ is BC in Banach density if and only if it has a subsequence of Banach density 1 which is eventually BC.*

LEMMA 2.2. *Let $\boldsymbol{\alpha} = (\alpha_i)$ be a sequence of real numbers, $n_1, n_2, \ldots$ any integers and $g \geqq 2$ an integer. Then $\boldsymbol{\alpha}$ is a BC (or BS) sequence (in density, in Banach density, eventually) if and only if the sequence $(g^{n_1}\alpha_1, g^{n_2}\alpha_2, \ldots)$ is.*

THEOREM 2.1. *Let $\boldsymbol{\alpha}$ be a sequence of positive real numbers. Then:*

(1) *If the sequence $(\log_g \alpha_i)$ is dense modulo 1 then $\boldsymbol{\alpha}$ is a BC sequence.*

(2) *If the sequence $(\log_g \alpha_i)$ is uniformly distributed modulo 1 then $\boldsymbol{\alpha}$ is BC in density.*

(3) *If the sequence $(\log_g \alpha_i)$ is well-distributed modulo 1 then $\boldsymbol{\alpha}$ is BC in Banach density.*

(4) *If the sequence (α_i) is strictly monotonic and $\frac{\alpha_{i+1}-\alpha_i}{\alpha_i-\alpha_{i-1}} \longrightarrow 1$, then $\boldsymbol{\alpha}$ is BC in Banach density.*

REMARK 2.2. One may considerably weaken condition (2) (and, in an analogous fashion, condition (3)) in the theorem while obtaining the same result. In fact, denote by μ_n the normalized counting measure concentrated on the first n terms of the sequence $(\{\log_g \alpha_i\})$, for $n \in \mathbf{N}$. Uniform distribution of the sequence in question is equivalent to the sequence (μ_n) converging weakly to the Lebesgue measure on $[0,1]$. It can be shown that, if this sequence of measures converges to any absolutely continuous measure, or even if it is only known that every limit measure of the sequence is absolutely continuous, then $\boldsymbol{\alpha}$ is BC in density.

PROOF OF THEOREM 2.1. (1) By Lemma 2.2 we may assume that $1 \leqq \alpha_i < g$ for each i. The condition in the theorem then amounts to the assertion that the sequence $(\log_g \alpha_i)$ is dense in the interval $[0,1]$. Therefore the sequence $\boldsymbol{\alpha}$ is dense in the interval $[1,g]$. In particular, given any block B, there exist infinitely many terms in the sequence $\boldsymbol{\alpha}$ whose g-adic expansion starts with B.

(2) Similarly to the preceding part, we may assume the sequence $(\log_g \alpha_i)$ to lie in the interval $[-1,0]$ and to be uniformly distributed there. Then the sequence $\boldsymbol{\alpha}$ defines an absolutely continuous measure on the interval $[0,1]$. Now take an arbitrary g-block B. Divide $[0,1]$ into g^l subintervals of equal lengths. Since almost every number (with respect to the Lebesgue measure) is normal in base g, given any $\varepsilon > 0$, as l becomes large enough, at least $(1-\varepsilon)g^l$ of these subintervals have the property that the initial block of length l given by the g-adic expansion of any (interior) point contains the block B. Since α defines an absolutely continuous measure, this implies that the lower density of the subsequence of $\boldsymbol{\alpha}$, consisting of those terms containing B within their initial block of length l, becomes arbitrarily close to 1 as l increases. It follows that the subsequence of $\boldsymbol{\alpha}$, consisting of those terms containing B within their g-adic expansion is of density 1, so that $\boldsymbol{\alpha}$ is BC in density.

(3) The proof follows that of part (2).

(4) We outline the proof. The condition means that large chunks of the sequence, placed in far enough places, look "almost" as arithmetic progressions. Then we are basically in the situation of Example 2.1, and the continuation is routine.

This completes the proof.

EXAMPLE 2.3. Using part (4) of the theorem, we easily verify that the sequences $\alpha_i = F(i)$ are BC in Banach density for the following functions F:

(1) F is a non-constant rational function.

(2) $F(x) = x^\theta$ for any $\theta \neq 0$. (Note that by [4, Example 1.1] this sequence is also BS for positive rational non-integer $\boldsymbol{\theta}$.)

(3) $F(x) = \log\log(x+1)$. (In view of [4, Example 1.1] this sequence is also BS.)

(4) $F(x) = 2^{\sqrt{x}}$.

(5) Any function $F(x)$ belonging to a Hardy field (see [3] for a definition,

examples and references on Hardy fields), which approaches ∞ slower than c^x for every $c > 1$, say $F(x) = x^{7+\log^2 x}$ or $F(x) = \int_1^x t^{\sqrt{t}}dt$.

EXAMPLE 2.4. Let g, h be multiplicatively independent positive integers (that is, they are not integer powers of the same integer; equivalently, $\frac{\log h}{\log g}$ is irrational). In view of Theorem 2.1.(4) (see also [6]), the sequence (h^i) is BC in Banach density. We mention that according to a certain conjecture of Furstenberg [8, Conjecture 2'], the sequence is, moreover, eventually BC. (Strictly speaking, the conjecture relates directly only to the case where h divides some power of g.) It is possible to deal similarly with other recurrence sequences. Thus, for example, the Fibonacci sequence F_n and the sequence $n^3 2^n$ are BC in Banach density in every base g. (Note that the proof of this fact for the latter sequence involves considerations similar to those discussed in Remark 2.2. For $g = 2$ this sequence is certainly not eventually BC.)

EXAMPLE 2.5. Let r be a rational number such that $r \notin \mathbf{Z}\left[\frac{1}{g}\right]$. It follows easily from [1, Corollary 7.1] or [5, Lemma 5.1] that the sequence (r^i) is eventually BS. The same holds, more generally, for any sequence of the form (sr^i), where $s > 0$ is rational. We do not know whether this is true for every real $s > 0$.

EXAMPLE 2.6. By [6], the sequence $(\log_g n!)$ is uniformly distributed modulo 1 for every g, so that the sequence $n!$ is BC in density in every base. Note that in [3] necessary and sufficient conditions are provided for density, uniform distribution and well-distribution of a large class of sequences defined by certain formulae and recurrent relations. These criteria show also that $(\log_g n!)$ is uniformly distributed modulo 1, but it is not well-distributed, so that Theorem 2.1 does not imply that $n!$ is BC in Banach density. Of course, one would expect it to be, moreover, eventually BC in every base.

EXAMPLE 2.7. The sequence $\binom{2i}{i}$ is BC in Banach density for every base g. In fact, suppose first that g is not a power of 2. Since

$$\frac{\binom{2i}{i}}{\binom{2(i-1)}{i-1}} = 4 - \frac{2}{i}$$

and $\log_g 4$ is irrational, the sequence $\log_g \binom{2i}{i}$ is well-distributed modulo 1, and by Theorem 2.1(3) our sequence is BC in Banach density. It remains to deal with $g = 2$. Obviously, we may replace the given sequence by $\alpha_i = \frac{\binom{2i}{i}}{4^i}$. Now

$$\alpha_{i+1} - \alpha_i = \frac{1}{4^{i+1}}\left(\left(4 - \frac{2}{i+1}\right)\binom{2i}{i} - 4\binom{2i}{i}\right) = -\frac{2\binom{2i}{i}}{4^{i+1}(i+1)},$$

so that the sequence (α_i) is strictly decreasing, and

$$\frac{\alpha_{i+1}-\alpha_i}{\alpha_i-\alpha_{i-1}}=\frac{2i-1}{2i+2}\longrightarrow 1\,.$$

Theorem 2.1(4) shows that the sequence is BC in Banach density.

EXAMPLE 2.8. While there are many examples of sequences of integers which are BC in (Banach) density, it is not easy to prove that a specific sequence is eventually BC, even though from heuristic probabilistic arguments one would expect this to be the case. Thus it is interesting to note that the sequence $\frac{10^{3^n}-1}{3^{n+2}}$, defined by a natural formula, is eventually BC in base 10. In fact, the decimal expansion of the general term of the sequence is (essentially – except for an additional block of 0's) the recurring part in the decimal expansion of $\frac{1}{3^{n+2}}$, so our claim follows from Example 2.6.

3. Saturating sets

Let $g \geqq 2$ be an integer.

DEFINITION 3.1. A set $\Delta \subseteqq \mathbf{N}$ is an *M_g-set* if for every irrational α and every g-block B there exists an $m \in \Delta$ such that the number $m\alpha$ contains the block B infinitely often in its base g expansion.

REMARK 3.1. Since there are only finitely many blocks of each particular length, if Δ is an M_g-set, then for every irrational α and positive integer k there exists an $m \in \Delta$ such that every block of length k appears infinitely often in the g-adic expansion of $m\alpha$.

REMARK 3.2. If Δ is an M_g-set, then there are infinitely many numbers m in Δ having the property in the definition (see Corollary 3.3). Consequently, we may rephrase Definition 3.1 by defining Δ to be an M_g-set if $\Delta\alpha$ is a BS sequence for every irrational α.

The following lemma follows from Lemma 2.2.

LEMMA 3.1. *Let $\Delta = \{m_1, m_2, \ldots\} \subseteqq \mathbf{N}$ and $n_1, n_2, \ldots$ be any non-negative integers. Then Δ is an M_g-set if and only if $\{g^{n_1}m_1, g^{n_2}m_2, \ldots\}$ is.*

COROLLARY 3.1. *There exist M_g-sequences growing arbitrarily fast.*

COROLLARY 3.2. *It suffices to study the M_g property for sets of integers not divisible by g.*

For closed subsets (E_k) and E of $\mathbf{T}$, we denote $E_k \longrightarrow E$ if the sequence E_k converges to E in the Hausdorff metric.

LEMMA 3.2. *Let E_k be a sequence of g-invariant subsets of* $\mathbf{T}$ *and r a non-zero integer. Then $rE_k \longrightarrow \mathbf{T}$ if and only if $E_k \longrightarrow \mathbf{T}$.*

PROOF. The "if" part is trivial. For the "only if" part, assume that $rE_k \longrightarrow \mathbf{T}$. Let E_0 be any limit point (in the Hausdorff metric) of the sequence of sets E_k. Then $rE_0 = \mathbf{T}$. Hence E_0 has a non-empty interior. Being invariant under multiplication by g, the set E must be the whole of $\mathbf{T}$. This proves the lemma.

LEMMA 3.3. $\Delta \subseteqq \mathbf{N}$ *is an M_g-set if and only if for every irrational α there exists a sequence m_k in Δ such that for every sequence n_k of non-negative integers $\{m_k g^n \alpha : n \geqq n_k\} \longrightarrow \mathbf{T}$.*

PROOF. Let Δ be an M_g-set. Let $(B_k)_{k=1}^{\infty}$ be a sequence of g-blocks so that B_k contains every block of length k. For each k, take an $m_k \in \Delta$ such that the base g expansion of $m_k\alpha$ contains the block B infinitely often. Clearly, for any n_k, the set $\{m_k g^n \alpha : n \geqq n_k\}$ is $\frac{2}{g^k}$-dense in $\mathbf{T}$, and therefore $\{m_k g^n \alpha : n \geqq n_k\} \longrightarrow \mathbf{T}$.

The converse direction is similarly proved.

The two preceding lemmas give

LEMMA 3.4. *Let $\Delta \subseteqq \mathbf{N}$ and $r \in \mathbf{N}$. Then $r\Delta$ is an M_g-set if and only if Δ is.*

THEOREM 3.1. *The following conditions are equivalent:*

(1) *Δ is an M_g-set.*

(2) *$\overline{\Delta E} = \mathbf{T}$ for every infinite g-invariant set $E \subseteqq \mathbf{T}$.*

(3) *For every infinite g-invariant set $E \subseteqq \mathbf{T}$ and $\varepsilon > 0$ there exists an $m \in \Delta$ such that mE is ε-dense in $\mathbf{T}$.*

PROOF. (2) $\Longrightarrow$ (1): Given an irrational α, let $E \subseteqq \mathbf{T}$ denote the set of all limit points of the sequence $(g^n\alpha)$. It is readily verified that E is infinite. Hence $\overline{\Delta E} = \mathbf{T}$, so that if B is any g-block there exist $m \in \Delta$ and $x \in E$ such that the base g expansion of mx starts with $0.B01$. It follows that if $g^n\alpha$ is sufficiently close to x, which happens for infinitely many numbers n, then the base g expansion of $mg^n\alpha$ starts with the block B. Consequently, the base g expansion of α contains B infinitely often.

(3) $\Longrightarrow$ (2): Trivial.

(1) $\Longrightarrow$ (3): We have to show that, if Δ is an M_g-set, then any infinite g-invariant set E has arbitrarily dense dilations by elements of Δ. Let $\varepsilon > 0$. Suppose first that E contains an irrational point α. Take a positive integer k such that $\frac{1}{g^k} < \frac{\varepsilon}{2}$. Since Δ is an M_g-set there exists an $m \in \Delta$ such that the g-base expansion of $m\alpha$ contains every g-block of length k infinitely often. Therefore, the set mE is ε-dense.

It remains to deal with the case where E consists of rational points only. We deal first with the special case in which 0 is an accumulation point of E. In this case there exists a real number $x \neq 0$ such that $\frac{x}{g^n} \in E$ for every non-negative integer n. (This follows, for example, from [2, Lemma 4.3].) Of course, x is rational, say $x = \frac{p}{q}$, and, replacing E by $-E = \{-x : x \in E\}$ if necessary, we may assume that $p, q > 0$. Consider the number

$$\alpha = x \sum_{j=1}^{\infty} g^{-s_j} ,$$

where (s_j) is an arbitrary sequence satisfying $s_{j+1} - s_j \longrightarrow \infty$. Clearly, α is irrational. Since, by Lemma 3.4, $q\Delta$ is an M_g-set, there exists a sequence m_k in Δ such that for every sequence n_k of positive integers we have $\{qm_k g^n \alpha : n \geqq n_k\} \longrightarrow \mathbf{T}$. Now

$$qm_k\alpha = pm_k \sum_{j=1}^{\infty} g^{-s_j} .$$

Thus $\left\{pm_k g^n \sum_{j=1}^{\infty} g^{-s_j} : n \geqq n_k\right\} \longrightarrow \mathbf{T}$. Consequently $pm_k E \longrightarrow \mathbf{T}$, and by Lemma 3.2 we obtain $m_k E \longrightarrow \mathbf{T}$ as well. Hence E has arbitrarily dense dilations by elements of Δ.

In general, given any accumulation point β of E, take a non-zero integer l with $l\beta = 0$. In view of the preceding case, there exists a sequence m_n in Δ with $m_n l E \longrightarrow \mathbf{T}$, so that by Lemma 3.2 we have $m_n E \longrightarrow \mathbf{T}$ as well.

This completes the proof.

COROLLARY 3.3. *If Δ is an M_g-set, then so is $\Delta - F$ for every finite set F.*

In fact, this follows from the equivalence of conditions (1) and (3) in the theorem and the fact that there exist g-invariant nowhere dense sets.

The theorem gives several general examples of M_g-sets. In [5], motivated by a result of Glasner [10], the notion of a Glasner set was defined. A set $\Delta \subseteqq \mathbf{N}$ is a *Glasner set* if, given any infinite subset A of $\mathbf{T}$ and $\varepsilon > 0$, there exists an $m \in \Delta$ such that mA is ε-dense in $\mathbf{T}$. Since condition (3) of the theorem is the same as this condition, but with the requirement applying only to infinite g-invariant sets, we immediately get the following result of Alon and Peres, based on an idea of Furstenberg (see Corollary 7.2 and the remark following it in [1]).

COROLLARY 3.4. *Any Glasner set is an M_g-set.*

Thus, (the non-quantitative version of) Mahler's theorem follows from Glasner's result, which may be paraphrased as stating that $\mathbf{N}$ is a Glasner set. Moreover, by Theorem 1.3 in [5], every set of positive upper (Banach) density is a Glasner set. Hence we obtain

COROLLARY 3.5. *For any irrational α and g-block B, the set of those integers n for which the block B does not appear infinitely often in the base g expansion of $n\alpha$ is of* 0 *(Banach) density.*

However, Theorem 3.1 provides also examples of M_g-sets which are not Glasner sets. In fact, according to [5, Theorem 1.4] if (n_j) is a *lacunary sequence,* i.e., satisfies

$$\frac{n_{j+1}}{n_j} \geqq \lambda > 1, \qquad j \geqq 1,$$

then it is not a Glasner set. In particular, any "one-parameter" multiplicative semigroup $\Delta = \{h^i : i \in \mathbf{N}\}$ is not a Glasner set. Now if g and h are multiplicatively independent, then it follows easily from the results of [7, Chapter IV] that condition (2) of Theorem 3.1 is satisfied. Consequently we obtain

COROLLARY 3.6. *If g and h are multiplicatively independent, then the semigroup $\{h^i : i \in \mathbf{N}\}$ is an M_g-set.*

It is worthwhile to note that another conjecture of Furstenberg [9] states that, under the conditions of Corollary 3.6 we have $h^i E \longrightarrow \mathbf{T}$ for every infinite g-invariant set $E \subsetneqq \mathbf{T}$. According to this conjecture the set $\{h^{i_j} : j \in \in \mathbf{N}\}$ is an M_g-set for any increasing sequence (i_j). Thus, if true, the conjecture would provide examples of arbitrarily fast growing M_g-sequences more interesting than those of Corollary 3.1.

PROPOSITION 3.1. *If Δ is an M_g-set then for every finite g-block there exists an $m \in \Delta$ whose base g expansion contains this block.*

PROOF. Let B be a g-block. Extending B if necessary we may assume it to neither start nor end with a 0. Consider the following g-invariant infinite subset of $\mathbf{T}$:

$$E = \left\{ \frac{1}{g^n} : n \in \mathbf{N} \right\}.$$

According to the implication (1) $\Longrightarrow$ (3) in Theorem 3.1, there exists an $m \in \Delta$ and a point $x \in E$ such that the base g expansion of mx is $0.B\ldots$ modulo 1. This easily implies that the block B appears in the base g expansion of m. This proves the proposition.

EXAMPLE 3.1. The proposition provides examples of "pretty large" sets of integers which are not M_g-sets. In view of Corollary 3.5, such sets must have density 0. However, the "Hausdorff dimension" of the set of all integers not containing a certain g-block in their expansion can be made arbitrarily close to 1 by taking this block sufficiently long, and such a set cannot be an M_g-set by Proposition 3.1.

EXAMPLE 3.2. Let Δ be the set of all positive integers whose g-adic expansion does not contain the block consisting of l_g consecutive $(g-1)$'s for each $g \geqq 2$, where (l_g) is an arbitrary sequence of positive integers. Selecing (l_g) to grow sufficiently fast, we can make the "dimension" of the resulting set arbitrarily close to 1, while the set is not M_g in any base $g \geqq 2$.

We conclude this section with a few

QUESTIONS. (1) Suppose Δ is an M_g-set. Is the set $\Delta + 1 = \{a+1 : a \in \in \Delta\}$ necessarily an M_g-set as well?

(2) Is the condition in Proposition 3.1 sufficient for Δ to be an M_g-set?

(3) Suppose $\Delta_1 \cup \Delta_2$ is an M_g-set. Is (at least) one of the sets Δ_1 and Δ_2 necessarily an M_g-set? In other words, does the property of being an M_g-set have the Ramsey property?

4. The failure of the analogue for continued fraction expansions

The analogue of Mahler's result for continued fraction expansions would assert that, given any irrational α and finite block of positive integers, there exists a positive integer m such that the continued fraction expansion of $m\alpha$ contains this block infinitely often. Our main result in this section shows in particular that this analogue is not true.

THEOREM 4.1. *There exist uncountably many irrationals α, having the property that the sequence of partial quotients of each multiple $m\alpha$ diverges to ∞.*

The proof will be carried out in a series of steps. We start with a few notations and definitions. Given an irrational α, denote by $c_n(\alpha) = = \frac{p_n(\alpha)}{q_n(\alpha)}$, $n \geqq 1$, the sequence of convergents of α.

DEFINITION 4.1. A sequence of rationals $r_k = \frac{p_k}{q_k}$, $k \geqq 1$, is an

(1) *ASC-sequence for an irrational* α if there exists an integer m such that $r_k = c_{k+m}(\alpha)$ for all sufficiently large k. (ASC – Asymptotic Sequence of Convergents.)

(2) *ASC-sequence* – if it is ASC for some irrational α.

REMARK 4.1. When writing $c_n(\alpha) = \frac{p_n(\alpha)}{q_n(\alpha)}$ or $r_k = \frac{p_k}{q_k}$ we shall implicitly assume that $(p, q) = 1$ and $q \geqq 1$.

LEMMA 4.1. *A sequence of rationals $r_k = \frac{p_k}{q_k}$, $k \geqq 1$, is ASC if and only if there exists a k_0 such that either*

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^k, \quad q_{k+1} > q_k, \qquad k \geqq k_0$$

or

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^{k+1}, \quad q_{k+1} > q_k, \qquad k \geqq k_0 .$$

The lemma follows routinely from the basic theory of continued fractions. In fact, the conditions of the lemma and the formulas connecting the sequences of numerators and denominators of the convergents with the sequence of partial quotients of a number enable a direct calculation of the partial quotients of a number α whose convergents are eventually the rationals r_k.

DEFINITION 4.2. A sequence of positive integers q_k is *well-divisible* if for every positive integer n we have $n \mid q_k q_{k+1}$ for all k large enough.

DEFINITION 4.3. A sequence of positive integers q_k is *super-lacunary* if

$$\frac{q_{k+1}}{q_k} \underset{k\to\infty}{\longrightarrow} \infty \; .$$

PROPOSITION 4.1. *There exist uncountably many ASC-sequences of rationals* $r_k = \frac{p_k}{q_k}$ *such that the sequence* q_k *of denominators is well-divisible and super-lacunary.*

PROOF. We shall construct an irrational $\alpha = [a_0; a_1, a_2, \ldots]$ whose sequence of convergents $\frac{p_k}{q_k}$ satisfies the required conditions. Start in an arbitrary way, say $a_0 = 0$ and $a_1 = 1$, so that $\frac{p_0}{q_0} = \frac{0}{1}$ and $\frac{p_1}{q_1} = \frac{1}{1}$. The sequence of partial quotients a_k, $k \geqq 2$, will be defined inductively and have the following properties:

(1) If $s \leqq k$ is prime and $s \nmid q_{k-1}$, then $q_{k-1}a_k \equiv -q_{k-2} \pmod{s^k}$.

(2) $a_k > k$.

It is easily verified that these properties can be fulfilled, and in fact each a_k can be chosen in infinitely many ways. Now the first of these conditions ensures that the sequence q_k is well-divisible, and the second – that it is super-lacunary. This proves the proposition.

THEOREM 4.2. *Let* $r_k = \frac{p_k}{q_k}$, $k \geqq 1$, *be an ASC-sequence, with the sequence of denominators* q_k *well-divisible and super-lacunary. Then for every integer* $m \geqq 1$ *the sequence* $mr_k = \frac{p_k'}{q_k'}$ *is ASC and its sequence of denominators* q_k' *is both well-divisible and super-lacunary.*

PROOF. Obviously, it suffices to deal with the case of prime m. Employing the information regarding the sequence q_k and Lemma 4.1, and omitting finitely many terms from the sequence r_k if necessary, we may assume that, say:

$$m \mid q_{2l-1}, \; m \nmid q_{2l}, \qquad l \geqq 1,$$

$$q_{k+1} > mq_k, \qquad k \geqq 1,$$

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^k, \qquad k \geqq 1 \; .$$

Then

$$(4.1)\qquad p'_{2l-1}=p_{2l-1},\quad q'_{2l-1}=\frac{q_{2l-1}}{m},\quad p'_{2l}=mp_{2l},\quad q'_{2l}=p_{2l},\quad l\geqq 1,$$

and consequently:

$$p'_k q'_{k+1}-p'_{k+1}q'_k=(-1)^k,\quad q'_{k+1}>q'_k,\qquad k\geqq 1\,.$$

Using again Lemma 4.1, this implies that mr_k is indeed an ASC-sequence. From (4.1) and the fact that q_k is well-divisible and super-lacunary we infer that q'_k possesses the same properties as well, which completes the proof.

PROPOSITION 4.2. *If a sequence of rationals $r_k=\frac{p_k}{q_k}$, $k\geqq 1$, is ASC with q_k super-lacunary, then r_k converges to an irrational α whose partial quotients diverge to ∞.*

The proof is straightforward.

PROOF OF THEOREM 4.1. Follows from Proposition 4.1, Theorem 4.2 and Proposition 4.2.

As is well known, the continued fraction expansion of almost every real number (with respect to the Lebesgue measure) contains every finite block infinitely often (and even in some prescribed positive density). Thus there is no wonder that to construct a counter-example one needs some special numbers. The numbers used in Theorem 4.1 have good approximation properties. For example, they may be Liouville numbers, but we require that the sequence of partial quotients will satisfy some extra conditions. It seems as if these conditions are indispensable. Namely, one can show that no rate of growth of the partial quotients of a number can ensure that it behaves as in Theorem 4.1. For example, Petruska studies in a very recent paper [14], the set of *strong Liouville numbers* (i.e., numbers α whose sequence of convergents $\frac{p_n(\alpha)}{q_n(\alpha)}$ eventually satifies $q_{n+1}>q_n^M$ for any $M>0$; equivalently, the sequence of partial quotients satisfies an analogous condition). He proves that, usually, if α is a strong Liouville number, then 2α is not. The conditions specified in Theorem 4.2 are very natural in view of [14, Theorem 3], and it may be that Petruska's techniques can provide a version of our construction.

THEOREM 4.3. *If α is a quadratic irrational, then for any block B of positive integers $b_1,b_2,\dots,b_k$ there exist infinitely many positive integers n for which the block B appears infinitely often in the continued fraction expansion of $n\alpha$.*

PROOF. We may clearly assume that α is of the form $a\pm\sqrt{d}$, where a and d are integers, d not a square. If $\alpha=a+\sqrt{d}$, then the continued fraction expansion of α, as well as of any multiple thereof, is (disregarding the integer part) purely periodic (see, for example, [12]). Since the sequence $(n\alpha)$ is dense modulo 1, given an arbitrary block B, we can find infinitely many n's for which the expansion of $n\alpha$ starts (upon omitting the integer part) with B. But then for each such n the block B occurs infinitely often in the expansion of $n\alpha$. In the remaining case, namely $\alpha = a - \sqrt{d}$, since the numbers α and $\beta = a + \sqrt{d}$ are equivalent, as are $n\alpha$ and $n\beta$ for any n, if some n works for β and the block B, it works for α with the same block, and we are done by the preceding part. This proves the theorem.

REMARK 4.2. We believe that Theorem 4.2 is valid for any badly approximable number α.

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ON A GALLAI-TYPE PROBLEM FOR LATTICES

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1. Introduction

Motivated by the well-known Helly-theorem [2], Gallai [1] raised the following problem in the Euclidean plane $\mathbf{E}^2$. Let $\mathcal{D}$ denote a finite collection of closed disks in $\mathbf{E}^2$ such that every two disks of $\mathcal{D}$ intersect. Find the minimum integer n with the property that for an arbitrary $\mathcal{D}$ there are n points in $\mathbf{E}^2$ such that every disk of $\mathcal{D}$ contains at least one of the points. Independently from each other, Danzer (unpublished) and Stachó [3] proved that $n \leqq 4$ i.e. any $\mathcal{D}$ can be pinned down by 4 needles. An analogous problem arises if the needles can be chosen from a rather regular subset of $\mathbf{E}^2$ only. Let $\mathbf{L}$ be the lattice of $\mathbf{E}^2$, i.e. the set of points of $\mathbf{E}^2$ which have integer coordinates.

It is easy to prove the following Helly-type theorem (see [4]). If $\mathcal{F}$ is a finite collection of convex sets in $\mathbf{E}^2$ such that any four of the sets of $\mathcal{F}$ have a lattice point in common, then there exists a lattice point common to every set of $\mathcal{F}$. Moreover this theorem can be extended to the d-dimensional Euclidean space $\mathbf{E}^d$ replacing 4 by 2^d. Thus it is very natural to ask the following Gallai-type problem for planar lattices. Let $\mathcal{F}$ denote a finite collection of convex sets in $\mathbf{E}^2$ such that any three of the sets of $\mathcal{F}$ have a lattice point in common. Find the least integer n such that for an arbitrary $\mathcal{F}$ there exist n lattice points (i.e. n needles positioned at the lattice points) with the property that every set of $\mathcal{F}$ contains (i.e. is pinned down) by at least one of the n lattice points (i.e. needles).

We prove the following

THEOREM 1. *If $\mathcal{F}$ is a finite family of convex sets in $\mathbf{E}^2$ such that any three of them have a lattice point in common, then there exist two lattice points which pin down $\mathcal{F}$.*

REMARK. It is easy to see that 2, i.e. the number of needles cannot be reduced to 1. Moreover, if we replace 3 (the number which guarantees that so many convex sets always intersect in a common lattice point) by 2, then the problem has a trivial negative answer.

2. Proof of Theorem 1

First we introduce some simple notations. The points of the plane will be denoted by $A, B, \ldots$. The segment with endpoints A and B is denoted by AB, and the line passing through the points A and B is denoted by $\overline{AB}$. We fix a so-called negative orientation of the plane. A convex polygon will be described with the sequence of its vertices according to the given negative orientation.

The line $\overline{AB}$ splits the plane into two open half-planes $\mathbf{F}_{A,B}$ and $\mathbf{F}_{B,A}$. In this notation the order of the subscripts is important, namely, for any point C (D, resp.) of $\mathbf{F}_{A,B}$ ($\mathbf{F}_{B,A}$, resp.) the sequence ABC (BAD, resp.) determines the negative orientation of the plane. For the closed half-plane determined by the open half-plane $\mathbf{F}_{A,B}$ we use the notation $\overline{\mathbf{F}}_{A,B}$ (i.e. $\overline{F}_{A,B} = \mathbf{F}_{A,B} \cup \overline{AB}$).

To each convex pentagon $ABCDE$ we assign the convex pentagon

$$\overline{ABCDE} = \overline{\mathbf{F}}_{A,C} \cap \overline{\mathbf{F}}_{B,D} \cap \overline{\mathbf{F}}_{C,E} \cap \overline{\mathbf{F}}_{D,A} \cap \overline{\mathbf{F}}_{E,B}.$$

(In other words $\overline{ABCDE}$ is enclosed by the diagonals of $ABCDE$.) The following two concepts are basically important for our proof.

Definition 1. Let $\mathbf{L}$ be the set of points of $\mathbf{E}^2$ which have integer coordinates. A point of $\mathbf{L}$ is called lattice point. A lattice point P is called a fixed lattice point (shortly an fl-point) if there are three sets of $\mathcal{F}$ the intersection of which contains P as the only lattice point.

Definition 2. We define the following fixed lattice-point algorithm (FLP-algorithm). For each $\mathbf{K} \in \mathcal{F}$ we proceed as follows. Let $\mathbf{K}^{(1)}$ be the convex hull of the lattice points which are points in common of $\mathbf{K}$ with two more sets of $\mathcal{F}$. Note that $\mathbf{K}^{(1)}$ is a convex lattice-polygon. Let $\mathcal{F}^{(1)}$ be the family arising from $\mathcal{F}$ when we replace $\mathbf{K}$ in it by $\mathbf{K}^{(1)}$. In general, suppose that $\mathbf{K}^{(i)}$ as well as $\mathcal{F}^{(i)}$ have already been defined. Then take a vertex of $\mathbf{K}^{(i)}$ which is not an fl-point with respect to a triplet of $\mathcal{F}^{(i)}$ containing $\mathbf{K}^{(i)}$. Remove this vertex from the vertices of $\mathbf{K}^{(i)}$. Obviously, this algorithm terminates after finitely many steps, say n. Then it is easy to see that every vertex of $\mathbf{K}^{(n)}$ is an fl-point with respect to a triplet of $\mathcal{F}^{(n)}$ containing $\mathbf{K}^{(n)}$. Observe that $\mathcal{F}^{(n)}$ satisfies the conditions of the theorem.

After this for the next $\mathbf{K}$ we use $\mathcal{F}^{(n)}$ instead of $\mathcal{F}$. Finally (after finitely many steps), the above FLP-algorithm yields a "new" $\mathcal{F}$ such that every vertex of any $\mathbf{K}$ of $\mathcal{F}$ is an fl-point with respect to a triplet of $\mathcal{F}$ containing $\mathbf{K}$. Then we say that $\mathcal{F}$ is fixed.

We shall make use of the following

Lemma 1. *If $ABCDE$ is a convex lattice-pentagon, then $\overline{ABCDE}$ contains a lattice point.*

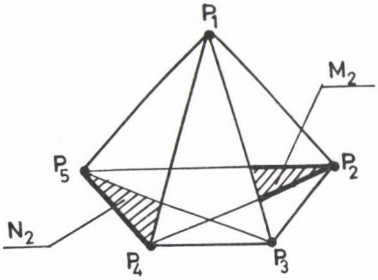

Fig. 1

PROOF. (Indirect.) Let $P_1P_2P_3P_4P_5$ be the convex lattice-pentagon with minimum number of lattice points for which the claim is false. Let $\mathbf{M}_2$ denote the region $\overline{\mathbf{F}}_{5,2} \cap \overline{\mathbf{F}}_{2,4} \cap \mathbf{F}_{3,1}$ (see Fig. 1).

Similarly we get $\mathbf{M}_1, \mathbf{M}_3, \mathbf{M}_4$ and $\mathbf{M}_5$. Furthermore, let $\mathbf{N}_2$ be the region $\mathbf{F}_{5,3} \cap \mathbf{F}_{1,4} \cap \overline{\mathbf{F}}_{4,5}$. In the same way we define the regions $\mathbf{N}_1, \mathbf{N}_3, \mathbf{N}_4$ and $\mathbf{N}_5$. It is easy to see that the convex lattice-pentagon $P_1P_2P_3P_4P_5$ contains a lattice point different from its vertices. Let P_6 be one of these lattice-points. By assumption, $P_6 \notin \overline{P_1P_2P_3P_4P_5}$. Suppose that $P_6 \in \mathbf{M}_2$. Then for the convex lattice-pentagon $P_1P_6P_3P_4P_5$ we have $\overline{P_1P_6P_3P_4P_5} \subset \subset \overline{P_1P_2P_3P_4P_5}$, a contradiction by the indirect assumption. This implies that the regions $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ and $\mathbf{M}_5$ do not contain a lattice point different from P_1, P_2, P_3, P_4 and P_5. Thus we may suppose that $P_6 \in \mathbf{N}_i$ for some $i \in \{1,2,3,4,5\}$. Let $i = 2$. As the convex lattice-pentagon $P_1P_2P_3P_6P_5$ contains less lattice points than $P_1P_2P_3P_4P_5$ the indirect assumption implies the existence of a lattice-point $P_7 \in \overline{P_1P_2P_3P_6P_5}$. Then it is easy to prove that either $P_7 \in \mathbf{M}_5$ or $P_7 \in \overline{P_1P_2P_3P_4P_5}$. In both cases we get a contradiction. This completes the proof of Lemma 1. Q.E.D.

THEOREM 2. *Consider five convex sets in* $\mathbf{E}^2$ *such that any three of them have a point of* $\mathbf{L}$ *in common. Then for each convex set there are three others such that the intersection of these four sets contains a point of* $\mathbf{L}$.

PROOF. Let the five convex sets be denoted by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. We are going to prove our claim for the set $\mathbf{K}_1$. We shall make use of the following special notation. $P_{i_1,i_2,\dots,i_k}$ ($\overline{P}_{i_1,i_2,\dots,i_k}$ resp.) stands for a lattice-point in $\mathbf{K}_1 \cap \mathbf{K}_{i_1} \cap \ldots \cap \mathbf{K}_{i_k}$ ($(\mathbf{E}^2 \setminus \mathbf{K}_1) \cap \mathbf{K}_{i_1} \cap \ldots \cap \mathbf{K}_{i_k}$ resp.), $2 \leqq i_1 < < i_2 < \ldots < i_k \leqq 5$.

The following rather technical lemma reduces the number of cases we have to investigate in the proofs of many statements.

LEMMA 2. *Let P_{23} be a fixed lattice point with respect to the convex sets $\mathbf{K}_1, \mathbf{K}_2$, and $\mathbf{K}_3$, and let $P_{23}P_2P_3P_{2'}$ be a convex lattice-quadrangle where P_2 and $P_{2'}$ are distinct lattice-points in $\mathbf{K}_1 \cap \mathbf{K}_2$. Then $P^*_{23} \in \overline{\mathbf{F}}_{2,23} \cap \overline{F}_{23,2'} \cap$ $\cap\, \mathbf{F}_{3,2'} \cap \mathbf{F}_{2,3}$.*

PROOF. If $P^*_{23} \in \overline{\mathbf{F}}_{2,23} \cap \overline{\mathbf{F}}_{3,2}$ then $P_2 \in P_3P_{23}P^*_{23}$ i.e. $P_2 \in \mathbf{K}_3$, but $P_2 \not\equiv$ $\not\equiv P_{23}$ in contradiction with the fl-point property of P_{23}. Similarly, we get a contradiction if $P^*_{23} \in \overline{\mathbf{F}}_{32,2'} \cap \overline{\mathbf{F}}_{2',3}$ (Fig. 2.).

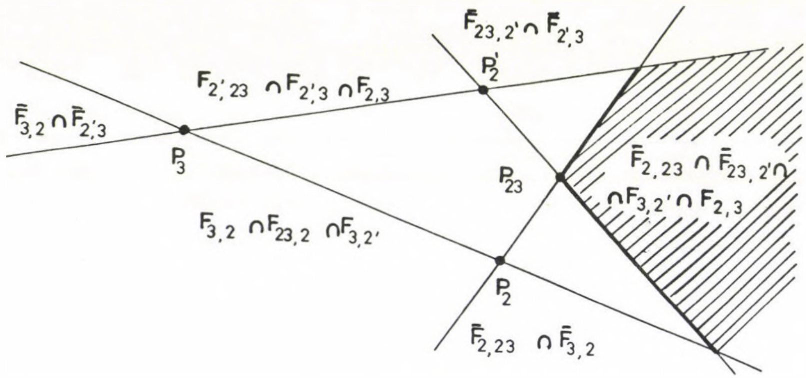

Fig. 2

If $P^*_{23} \in \mathbf{F}_{3,2} \cap \mathbf{F}_{23,2} \cap \mathbf{F}_{3,2'}$, then $P_{23}P_2P^*_{23}P_3P_{2'}$ is a convex lattice pentagon. By Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P_2P^*_{23}P_3P_{2'}} \subset P_{23}P^*_{23}P_{2'} \cap P_{23}P^*_{23}P_3 \cap P_{23}P_2P_3 \subset \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_1,$$

but $A \not\equiv P_{23}$ in contradiction with the fl-point property of P_{23}. The case $P^*_{23} \in \mathbf{F}_{2',3} \cap \mathbf{F}_{2',23} \cap \mathbf{F}_{2,3}$ can be disproved similarly.

If $P^*_{23} \in \overline{\mathbf{F}}_{3,2} \cap \overline{\mathbf{F}}_{2',3}$ then $P_3 \in P_{2'}P_2P^*_{23} \subset \mathbf{K}_2$ but $P_3 \not\equiv P_{23}$, a contradiction.

If $P^*_{23} \in \mathbf{F}_{2',23} \cap \mathbf{F}_{2,3} \cap \mathbf{F}_{2,23}$, then $P_{23}P^*_{23}P_2P_3P_{2'}$ is a convex lattice pentagon. By Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P^*_{23}P_2P_3P_{2'}} \subset P_{23}P_2P_{2'} \cap P_{23}P^*_{23}P_3 \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3,$$

but $A \not\equiv P_{23}$, a contradiction. Similarly, we get a contradiction if $P^*_{23} \in$ $\in \mathbf{F}_{23,2'} \cap \mathbf{F}_{23,2} \cap \mathbf{F}_{3,2}$. □

Let $\mathcal{C} = \{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_5\}$ and apply the FLP-algorithm to $\mathcal{C}$. Then we take $\mathbf{K}_1$ which is convex lattice-polygon with the property that each vertex is an fl-point P_{ij} for some i and j with respect to $\mathbf{K}_1$, furthermore we take $\mathbf{K}_i$ and $\mathbf{K}_j$. Obviously, two vertices cannot have the same "name" P_{ij}. As the number of sides of $\mathbf{K}_1$ is at most 6 we distinguish 5 cases. Each of them has some further subcases depending on the positions of the P_{ij}'s. We prove Theorem 2 as well as the fact that $\mathbf{K}_1$ is either a triangle or a point. The rough idea of the proof is the following: we take a point P^*_{ijk} and show that independently from its position the above claim is true. However, there are some cases where we have to consider the positions of two P^*_{ijk}'s.

I. $\mathbf{K}_1$ *is a convex hexagon.* The vertices of $\mathbf{K}_1$ are the points P_{ij}. Suppose that a vertex of $\mathbf{K}_1$, say P_{23}, belongs to more than three convex sets, say $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap K_3 \cap \mathbf{K}_4$. But then P_{24} is not an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_4$, a contradiction. Thus every vertex of $\mathbf{K}_1$ belongs to exactly three convex sets. Next we prove that any two opposite vertices of $\mathbf{K}_1$ cannot be covered by $\mathbf{K}_i$, where $i > 1$. Namely, assume that $\mathbf{K}_1 = A_1A_2A_3A_4A_5A_6$ with $A_1 = P_{23}$ and $A_4 = P_{24}$. Without loss of generality we may assume that $A_3 = P_{25}$. First we consider the case $A_2 = P_{34}$. As $P_{23}P_{34}P_{25}P_{24}P_{45}$ is a convex pentagon, Lemma 1 implies that there exists a lattice point B such that

$$B \in \overline{P_{23}P_{34}P_{25}P_{24}P_{45}} \subset P_{34}P_{24}P_{45} \cap P_{23}P_{25}P_{24} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4.$$

Finally, $B \not\equiv P_{24}$, a contradiction since P_{24} must be an fl-point.

Now assume that $A_2 = P_{35}$. Since $P_{23}P_{35}P_{25}P_{24}P_{45}$ is a convex lattice pentagon, hence there exists a lattice point B such that

$$B \in \overline{P_{23}P_{35}P_{25}P_{24}P_{45}} \subset P_{23}P_{25}P_{24} \cap P_{35}P_{25}P_{45} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

but $B \not\equiv P_{24}$ so we get a contradiction since P_{24} is an fl-point. Finally, if $A_2 \equiv P_{45}$, then a similar argument yields a contradiction.

Thus it is sufficient to consider the convex hexagon $P_{23}P_{25}P_{35}P_{45}P_{34}P_{24}$ (see Fig. 3).

If P_{345} exists, then $P_{345} \not\equiv P_{34}$ which we proved above, and this is contradiction since P_{34} is an fl-point. Hence P^*_{345} exists. As P_{35} is an fl-point and $P_{35}P_{45}P_{23}P_{25}$ is a convex quadrangle, by Lemma 2 we get $P^*_{345} \in \mathbf{F}_{45,23}$. On the other hand P_{34} is an fl-point and $P_{34}P_{24}P_{23}P_{45}$ is a convex quadrangle so by Lemma 2 we get $P^*_{345} \in \mathbf{F}_{23,45}$, a contradiction. □

II. K_1 *is a convex pentagon.* We may assume that the vertices of $\mathbf{K}_1$ are $P_{23}, P_{24}, P_{25}, P_{34}$ and P_{35}. It is easy to prove that we have to investigate four cases only.

(a) $\mathbf{K}_1$ *is the pentagon* $P_{23}P_{35}P_{25}P_{34}P_{24}$. By Lemma 1 there is a lattice point A such that

$$A \in \overline{P_{23}P_{35}P_{25}P_{34}P_{24}} \subset P_{23}P_{25}P_{24} \cap P_{23}P_{35}P_{34} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3.$$

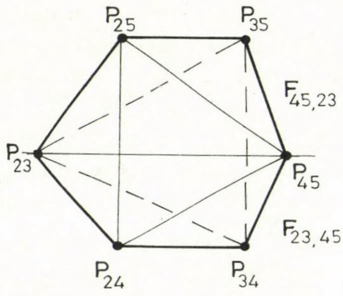

Fig. 3

Since $A \not\equiv P_{23}$, this contradicts the fl-point property of P_{23}.

(b) $\mathbf{K}_1$ *is the pentagon* $P_{25}P_{35}P_{23}P_{34}P_{24}$. If $P_{45} \in P_{23}P_{24}P_{25}$, then $P_{45} \in \mathbf{K}_2$, but $P_{45} \not\equiv P_{24}$, a contradiction.

If $P_{45} \in P_{23}P_{34}P_{24}$, then $P_{23}P_{45}P_{24}P_{25}P_{35}$ is a convex pentagon, so by Lemma 1 we have a lattice point A, such that

$$A \in \overline{P_{23}P_{45}P_{24}P_{25}P_{35}} \subset P_{23}P_{24}P_{25} \cap P_{45}P_{25}P_{35} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

but $A \not\equiv P_{25}$, a contradiction. Similarly we get a contradiction if $P_{45} \in \in P_{25}P_{35}P_{23}$.

Notice that if $\mathbf{K}_1$ is a $P_{25}P_{34}P_{23}P_{35}P_{24}$ pentagon we can proceed similarly.

(c) $\mathbf{K}_1$ *is the pentagon* $P_{34}P_{35}P_{23}P_{25}P_{24}$. We may assume that $P_{45} \in \in P_{23}P_{24}P_{34}$ (Fig. 4). Namely, if $P_{45} \in P_{23}P_{34}P_{35}$, then $P_{45} \in \mathbf{K}_3$. As $P_{45} \not\equiv P_{34}$, this contradicts the fl-point property of P_{34}.

Since P_{25} is an fl-point, $P_{25}P_{24}P_{45}P_{23}$ is a convex quadrangle. Then Lemma 2 implies that $P^*_{235} \in \mathbf{F}_{45,23}$. If P_{235} exists, then P_{23} and P_{25} are fl-points. As P_{35} is an fl-point and $P_{45}P_{34}P_{35}P_{23}$ is a convex quadrangle by Lemma 2 we get $P^*_{235} \in \mathbf{F}_{23,45}$, a contradiction.

(d) $\mathbf{K}_1$ *is the pentagon* $P_{34}P_{23}P_{25}P_{24}P_{35}$. As P_{34} and P_{35} are fl-points, P^*_{345} does exist (Fig. 5). Since P_{34} is an fl-point and $P_{34}P_{23}P_{24}P_{35}$ is a convex quadrangle, we get by Lemma 2 that $P^*_{345} \in \mathbf{F}_{24,35} \cap \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{23,34} \cap \mathbf{F}_{23,24}$.

If $P^*_{345} \in \overline{\mathbf{F}}_{25,34} \cap \overline{\mathbf{F}}_{34,35}$, then $P_{34} \in P^*_{345}P_{25}P_{35} \subset \mathbf{K}_5$, which contradicts the fl-point property of P_{35}. Hence we may suppose that $P^*_{345} \in \overline{\mathbf{F}}_{23,34} \cap \cap \mathbf{F}_{34,25} \cap \mathbf{F}_{24,35}$.

If P_{235} exists, then we get a contradiction since P_{23} and P_{25} are fl-points. Thus P^*_{235} exists.

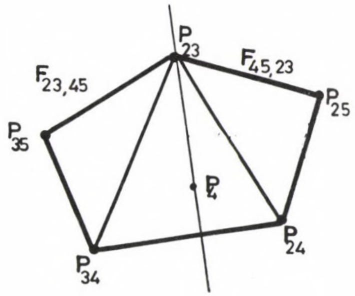

Fig. 4

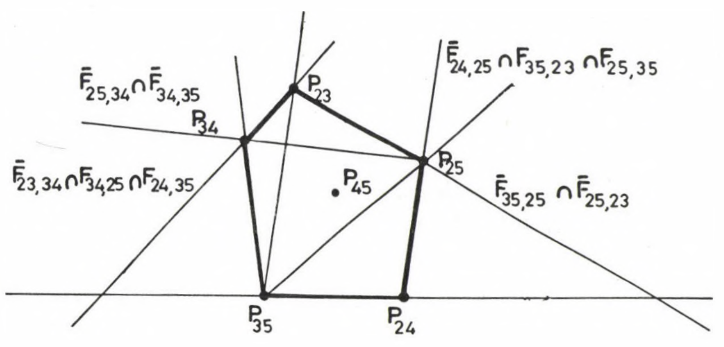

Fig. 5

Since P_{25} is an fl-point, $P_{25}P_{24}P_{35}P_{23}$ is a convex quadrangle thus Lemma 2 implies that

$$P^*_{235} \in \mathbf{F}_{35,23} \cap \overline{\mathbf{F}}_{25,23} \cap \overline{\mathbf{F}}_{24,25} \cap \mathbf{F}_{24,35}.$$

If $P^*_{235} \in \overline{\mathbf{F}}_{35,25} \cap \overline{\mathbf{F}}_{25,23}$, then $P_{25} \in P^*_{235}P_{35}P_{23} \subset \mathbf{K}_3$ which contradicts the fl-point property of P_{23}. Hence we may assume that $P^*_{235} \in \mathbf{F}_{24,25} \cap \mathbf{F}_{35,23} \cap \cap \mathbf{F}_{25,35}$.

Since $P^*_{345} \in \mathbf{F}_{25,34} \cap \mathbf{F}_{235^*,25} \cap \mathbf{F}_{235^*,35}$ we get that $P^*_{345}P^*_{235}P_{25}P_{35}$ is a convex quadrangle. As P_{25} is an fl-point $P_{23} \in P^*_{345}P^*_{235}P_{25}P_{35} \subset \mathbf{K}_5$ cannot occur. Thus $P_{23} \notin P^*_{345}P^*_{235}P_{25}P_{35}$ so

$$P_{23} \in \mathbf{F}_{235^*,25} \cap \mathbf{F}_{35,345^*} \cap \mathbf{F}_{235^*,345^*}.$$

It follows from the foregoing that $P_{23}P_{235}P_{25}P_{35}P^*_{345}$ is a convex pentagon. Hence by Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P^*_{235}P_{15}P_{35}P^*_{345}} \subset P_{23}P_{25}P_{35} \cap P^*_{345}P^*_{235}P_{35} \subset \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_5.$$

Since $A \not\equiv P_{35}$ and P_{35} is an fl-point, this is a contradiction. □

III. $\mathbf{K}_1$ *is a quadrangle.* It is easy to prove that we have to investigate four cases only.

(a) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{24}P_{45}P_{35}$. If $P_{34} \in P_{23}P_{24}P_{25} \subset \mathbf{K}_2$ or $P_{34} \in P_{25}P_{35}P_{45} \subset \mathbf{K}_5$, then this contradicts the fl-point property of P_{23} and P_{24} or P_{35} and P_{45}. Thus we may assume that $P_{34} \in P_{24}P_{45}P_{25}$ (Fig. 6).

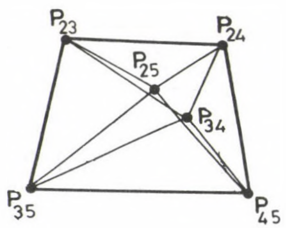

Fig. 6

Similarly we may assume that $P_{25} \in P_{23}P_{24}P_{34}$. Then $P_{23}P_{25}P_{34}P_{45}P_{35}$ is a convex pentagon, and according to Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P_{25}P_{34}P_{45}P_{35}} \subset P_{23}P_{34}P_{35} \cap P_{25}P_{45}P_{35} \subset \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_5$$

but $A \not\equiv P_{35}$, a contradiction.

(b) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{24}P_{35}P_{45}$. If $P_{25} \in P_{23}P_{35}$, then $P_{25} \in \mathbf{K}_3$, but this contradicts the fl-point property of P_{23} and P_{35} (Fig. 7).

If $P_{25} \in P_{23}P_{35}P_{45}$ then $P_{23}P_{24}P_{35}P_{25}$ is a convex quadrangle and since P_{23} is an fl-point, applying Lemma 2 we get that $P^*_{234} \in \overline{\mathbf{F}}_{23,25} \cap \overline{\mathbf{F}}_{24,23}$. (If P_{234} exists we get a contradiction since P_{23} and P_{24} are fl-points.) Then $P_{23} \in P^*_{234}P_{24}P_{45} \subset \mathbf{K}_4$, but this contradicts the fl-point property of P_{23} and P_{24}. Similarly, we get a contradiction if $P_{25} \in P_{35}P_{23}P_{24}$.

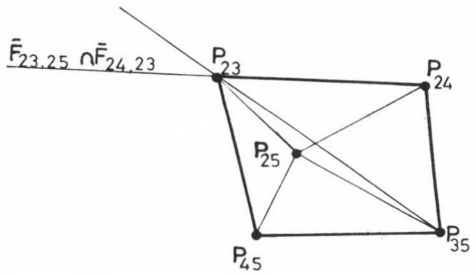

Fig. 7

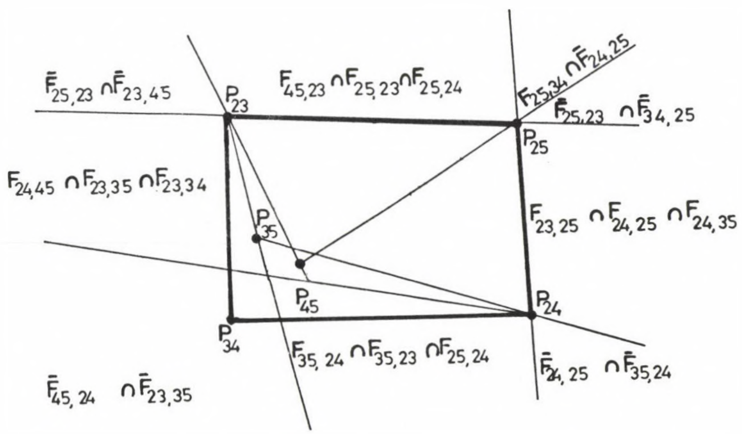

Fig. 8

(c) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{25}P_{24}P_{34}$. If P_{35} or $P_{45} \in P_{23}P_{25}P_{24} \subset \subset \mathbf{K}_2$, then we have a contradiction since P_{23} and P_{25} or P_{24} and P_{25} are fl-points. Thus we may assume that P_{35} and $P_{45} \in P_{23}P_{24}P_{34}$ (Fig. 8).

If P_{235} exists, then we get a contradiction as P_{23} and P_{25} are fl-points. Thus we may suppose that P^*_{235} exists.

If $P^*_{235} \in \overline{\mathbf{F}}_{25,23} \cap \overline{\mathbf{F}}_{34,25}$, then $P_{25} \in P^*_{235}P_{34}P_{23} \subset \mathbf{K}_3$, but $P_{25} \not\equiv P_{23}$, a contradiction. Our proof is similar if $P^*_{235} \in \overline{\mathbf{F}}_{25,34} \cap \overline{\mathbf{F}}_{24,25}$.

If $P^*_{235} \in \mathbf{F}_{23,25} \cap \mathbf{F}_{24,25} \cap \mathbf{F}_{24,35}$, then $P^*_{235}P_{24}P_{35}P_{23}P_{25}$ is a convex pentagon. Applying Lemma 2 we have a lattice point A for which

$$A \in \overline{P^*_{235}P_{24}P_{35}P_{23}P_{25}} \subset P_{24}P_{23}P_{25} \cap P^*_{235}P_{35}P_{23} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3.$$

As $A \not\equiv P_{23}$ this is a contradiction. We can settle the case $P^*_{235} \in \mathbf{F}_{45,23} \cap \cap \mathbf{F}_{25,23} \cap \mathbf{F}_{25,24}$ similarly.

If $P^*_{235} \in \mathbf{F}_{24,25} \cap \mathbf{F}_{35,24}$, then $P_{24} \in P^*_{235}P_{35}P_{25} \subset \mathbf{K}_5$, a contradiction. If $P^*_{235} \in \overline{\mathbf{F}}_{25,35} \cap \overline{\mathbf{F}}_{35,24}$, then the reasoning is similar.

If $P^*_{235} \in \mathbf{F}_{35,24} \cap \mathbf{F}_{35,23}\mathbf{F}_{25,24}$, then $P^*_{235}P_{35}P_{23}P_{25}P_{24}$ is a convex pentagon thus according to Lemma 1 we have a lattice point A such that

$$A \in \overline{P^*_{235}P_{35}P_{23}P_{25}P_{24}} \subset P^*_{235}P_{35}P_{25} \cap P^*_{235}P_{23}P_{25} \cap P_{35}P_{25}P_{24} \in \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5.$$

As $A \not\equiv P_{25}$ we get a contradiction. The reasoning in the case $P^*_{235} \in \mathbf{F}_{24,25} \cap \cap \mathbf{F}_{23,35} \cap \mathbf{F}_{23,34}$ follows word for word the previous reasoning.

If $P^*_{235} \in \overline{\mathbf{F}}_{45,25} \cap \overline{\mathbf{F}}_{23,35}$, then P_{35} or $P_{45} \in P^*_{235}P_{23}P_{24} \subset \mathbf{K}_2$, but this is a contradiction since P_{23} and P_{25} or P_{24} and P_{25} are fl-points.

(d) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{34}P_{25}P_{24}$. If P_{35} or $P_{45} \in P_{25}P_{24}P_{23}$, then we get a contradiction as in the case (c). Hence we may assume that P_{35} and $P_{45} \in P_{34}P_{25}P_{23}$ (Fig. 9).

P^*_{235} does exist. (The proof is the same as in the case (c).)

If $P^*_{235} \in \mathbf{F}_{45,25} \cap \mathbf{F}_{25,35} \cap \mathbf{F}_{34,23}$, then $P_{35} \in P^*_{235}P_{25}P_{23} \subset \mathbf{K}_2$, but this contradicts the fl-point property of P_{23} and P_{25}.

If $P^*_{235} \in \overline{\mathbf{F}}_{45,23} \cap \overline{\mathbf{F}}_{25,45}$, then $P_{45} \in P^*_{235}P_{25}P_{23} \subset K_2$, but this is a contradiction since P_{24} and P_{25} are fl-points.

If $P^*_{235} \in \overline{\mathbf{F}}_{23,45} \cap \overline{\mathbf{F}}_{25,34}$, then $P_{45} \in P^*_{235}P_{23}P_{34} \subset \mathbf{K}_3$. This is possible only in case $P_{45} \equiv P_{34}$. But then this vertex is a P_{35} vertex and changing $\mathbf{K}_4$ and $\mathbf{K}_5$ we get case (c). (Notice that we have not utilized the fl-point property of P_{34} in the reasoning of case (c).) Hence we get a contradiction just like in case (c).

If $P^*_{235} \in \mathbf{F}_{24,25} \cap \mathbf{F}_{34,25} \cap \mathbf{F}_{24,23}$, then $P^*_{235}P_{24}P_{23}P_{25}$ is a convex pentagon. According to Lemma 1 we have a lattice point A such that

$$A \in \overline{P^*_{235}P_{24}P_{23}P_{34}P_{25}} \subset P^*_{235}P_{23}P_{34} \cap P_{24}P_{23}P_{25} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3.$$

Since $A \not\equiv P_{23}$ this is a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{23,34} \cap \overline{\mathbf{F}}_{24,35}$, then $P_{24} \in P^*_{235}P_{23}P_{25} \subset \mathbf{K}_2$. As $P_{24} \not\equiv P_{34}$ this is a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{24,25} \cap \overline{\mathbf{F}}_{35,24}$, then $P_{24} \in P^*_{235}P_{35}P_{25} \subset \mathbf{K}_5$. Since $P_{24} \not\equiv P_{25}$ this is a contradiction.

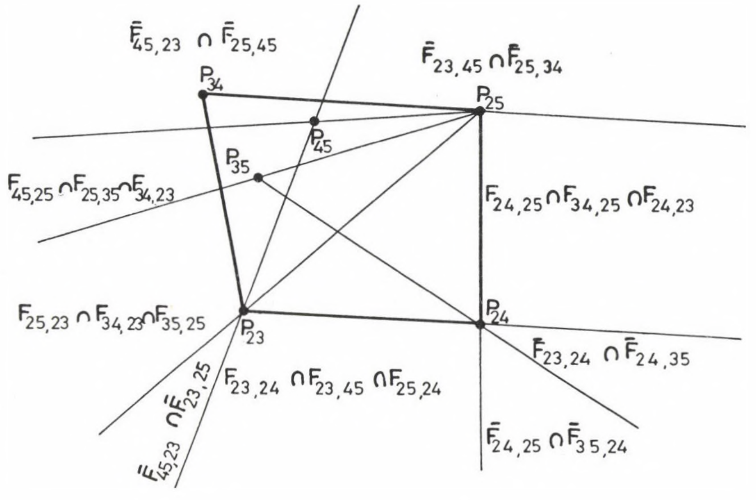

Fig. 9

If $P^*_{235} \in \mathbf{F}_{23,24} \cap \mathbf{F}_{23,45} \cap \mathbf{F}_{25,24}$, then $P^*_{235}P_{23}P_{45}P_{25}P_{24}$ is a convex pentagon, hence by Lemma 1 we get a lattice point A such that

$$A \in \overline{P^*_{235}P_{23}P_{45}P_{25}P_{24}} \subset P_{24}P_{23}P_{25} \cap P^*_{235}P_{45}P_{25} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5.$$

As $A \not\equiv P_{25}$ we get a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{45,23} \cap \overline{\mathbf{F}}_{23,25}$, then $P_{23} \in P^*_{235}P_{45}P_{25} \subset \mathbf{K}_5$. Since $P_{25} \not\equiv P_{23}$ we get a contradiction.

Thus we may suppose that $P^*_{235} \in \mathbf{F}_{25,23} \cap \mathbf{F}_{34,23} \cap \mathbf{F}_{35,25}$.

If P_{245} exists, then we have a contradiction as P_{24} and P_{25} are fl-points. Hence we may assume that P^*_{245} exists.

Since P_{24} is an fl-point and $P_{23}P_{34}P_{25}P_{24}$ is a convex quadrangle hence applying Lemma 2 we get that

$$P^*_{245} \in \overline{\mathbf{F}}_{24,25} \cap \overline{\mathbf{F}}_{23,24} \cap \mathbf{F}_{34,25} \cap \mathbf{F}_{23,34}.$$

Since $P^*_{245} \in \mathbf{F}_{35,25} \cap \mathbf{F}_{235^*,35} \cap \mathbf{F}_{235^*,25}$, $P^*_{245}P^*_{235}P_{35}P_{25}$ is a convex quadrangle.

If $P_{24} \in P^*_{245}P^*_{235}P_{35}P_{25} \subset \mathbf{K}_5$, then since $P_{24} \not\equiv P_{25}$ we get a contradiction.

If $P_{24} \notin P^*_{245}P^*_{235}P_{35}P_{25}$, then $P_{24} \in \mathbf{F}_{235^*,35} \cap \mathbf{F}_{235^*,245^*} \cap \mathbf{F}_{25,245^*}$, thus $P_{24}P^*_{235}P_{35}P_{25}P_{245}$ is a convex pentagon. By Lemma 1 we get a lattice point A such that

$$A \in \overline{P_{24}P^*_{235}P_{35}P_{25}P^*_{245}} \subset P_{34}P_{35}P_{25} \cap P_{24}P^*_{235}P_{25} \cap P^*_{235}P_{25}P^*_{245} \subset$$
$$\subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5.$$

But this is a contradiction since $A \not\equiv P_{25}$. □

IV. $\mathbf{K}_1$ *is a triangle.* It is easy to prove that we have to investigate three cases only.

(a) $\mathbf{K}_1$ *is the triangle* $P_{23}P_{24}P_{25}$. Then $P_{34} \in \mathbf{K}_1 \cap \mathbf{K}_2$, which is a contradiction since P_{23} and P_{24} are fl-points.

(b) $\mathbf{K}_1$ *is the triangle* $P_{23}P_{24}P_{34}$. If P_{234} exists, then we get a contradiction as P_{23} and P_{24} are fl-points. Thus we may suppose that P^*_{234} exists (Fig. 10).

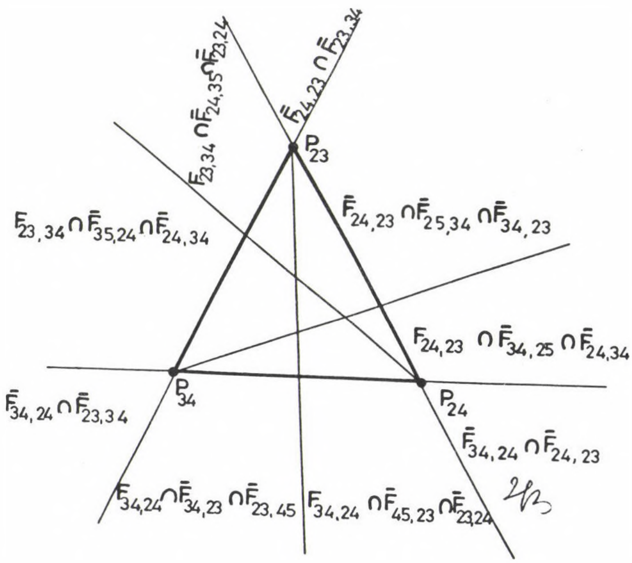

Fig. 10

It is easy to prove that we have to investigate the following two cases.

If $P^*_{234} \in \overline{\mathbf{F}}_{34,24} \cap \overline{\mathbf{F}}_{24,23}$, then $P_{24} \in P^*_{234}P_{34}P_{23} \subset \mathbf{K}_3$ but this contradicts the fl-point property of P_{23}.

If $P^*_{234} \in \mathbf{F}_{24,23} \cap \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{24,34}$, then $P_{24} \in P^*_{234}P_{34}P_{23} \subset \mathbf{K}_3$ i.e. $P_{25} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$. Thus in this case Theorem 2 is true.

(c) $\mathbf{K}_1$ *is the triangle* $P_{24}P_{25}P_{34}$. If $P_{45} \in P_{23}P_{24}P_{25} \subset \mathbf{K}_2$, then $P_{45} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$ which proves Theorem 2 in this case.

If $P_{23} \in P_{45}P_{34}P_{24} \subset \mathbf{K}_4$, then $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4$. Hence we may assume that $P_{45} \in \mathbf{F}_{24,23}$.

If $P_{45} \in P_{23}P_{34} \subset \mathbf{K}_3$, then $P_{45} \in \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_4 \cap \mathbf{K}_5$ and we are done. We may assume that $P_{45} \notin P_{23}P_{34}$.

It follows from the foregoing that we have to investigate the following two cases:

(α) $P_{23}P_{45}P_{34}P_{24}$ *is a convex quadrangle.* If P_{234} exists, then we get a contradiction since P_{24} and P_{34} are fl-points. Thus P^*_{234} exists (Fig. 11).

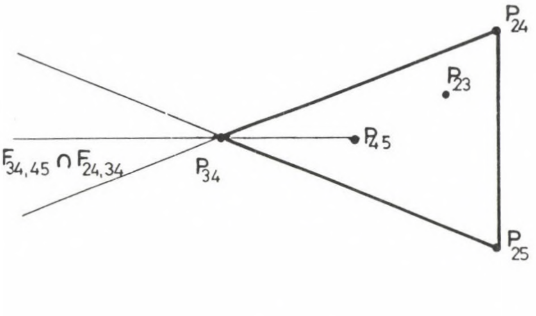

Fig. 11

Since P_{34} is an fl-point and $P_{34}P_{24}P_{23}P_{45}$ is a convex quadrangle, applying Lemma 2 we get that $P^*_{234} \in \overline{\mathbf{F}}_{34,45} \cap \overline{\mathbf{F}}_{24,34}$. Then we have that $P_{34} \in P^*_{234}P_{24}P_{25} \subset \mathbf{K}_2$ thus $P_{34} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4$ which is our claim.

(β) $P_{34}P_{45}P_{23}P_{25}$ *is a convex quadrangle.* If P_{245} exists we get a contradiction since P_{24} and P_{25} are fl-points. Thus P^*_{245} exists (Fig. 12).

If $P^*_{245} \in \overline{\mathbf{F}}_{23,45} \cap \overline{\mathbf{F}}_{25,23}$, then $P_{23} \in P^*_{245}P_{25}P_{45} \subset \mathbf{K}_5$. Thus $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P^*_{245} \in \mathbf{F}_{25,24} \cap \mathbf{F}_{25,34} \cap \mathbf{F}_{45,23}$, then $P_{23} \in P^*_{245}P_{34}P_{24} \subset \mathbf{K}_4$. Thus $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4$.

If $P^*_{245} \in \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{25,24}$, then $P_{25} \in P^*_{245}P_{34}P_{24} \subset \mathbf{K}_4$. Thus $P_{25} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

If $P^*_{245} \in \mathbf{F}_{23,25} \cap \mathbf{F}_{34,24}$, then we have two cases since $\mathbf{F}_{23,25} \cap \mathbf{F}_{34,24} = (\mathbf{F}_{23,25} \cap \mathbf{F}_{35,24}) \cup (\mathbf{F}_{24,35} \cap \mathbf{F}_{34,24})$ wherever P_{35} is. If $P^*_{245} \in \mathbf{F}_{23,25} \cap \mathbf{F}_{35,24}$,

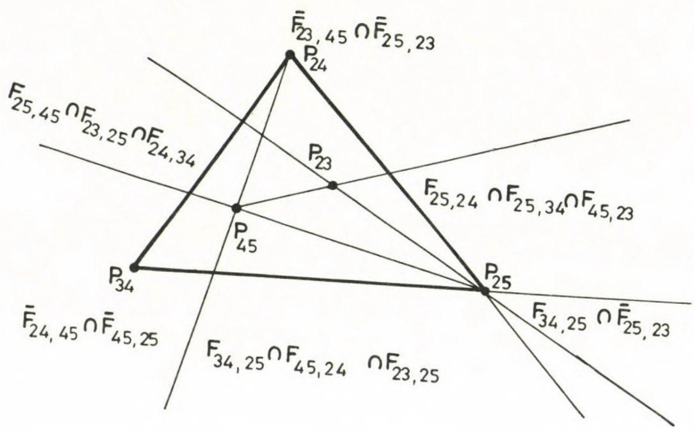

Fig. 12

then $P_{35} \in P^*_{245}P_{34}P_{24} \subset \mathbf{K}_4$. Thus $P_{35} \in \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_5$. If $P^*_{245} \in \mathbf{F}_{24,35} \cap \cap \mathbf{F}_{34,24}$, then $P_{35} \in P^*_{245}P_{24}P_{25} \subset \mathbf{K}_2$. Thus $P_{35} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P^*_{245} \in \mathbf{F}_{24,45} \cap \mathbf{F}_{45,25}$, then $P_{45} \in P^*_{245}P_{24}P_{25} \subset \mathbf{K}_2$. Thus $P_{45} \in \mathbf{K}_1 \cap \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

Thus we may assume that $P^*_{245} \in \mathbf{F}_{25,45} \cap \mathbf{F}_{23,25} \cap \mathbf{F}_{24,34}$.

If P_{235} exists then Theorem 2 is true. Hence we may suppose that P^*_{235} exists.

If $P^*_{235} \in \overline{\mathbf{F}}_{24,25} \cap \overline{\mathbf{F}}_{45,25}$, then the proof is similar to the previous one.

If $P^*_{235} \in \mathbf{F}_{25,45} \cap \mathbf{F}_{23,25} \cap \mathbf{F}_{24,34}$, then $P_{45} \in P^*_{235}P_{23}P_{34} \subset \mathbf{K}_3$. Thus $P_{45} \in \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

If $P^*_{235} \in \overline{\mathbf{F}}_{23,45} \cap \overline{\mathbf{F}}_{25,23}$, then the proof is similar to the proof of the case P^*_{245}.

If $P^*_{235} \in \mathbf{F}_{25,24} \cap \mathbf{F}_{25,34} \cap \mathbf{F}_{45,23}$, then $P^*_{235}P_{25}P_{45}P^*_{245}$ is a convex quadrangle. Namely, $P^*_{245} \in \mathbf{F}_{235,45} \cap \mathbf{F}_{25,235} \cap \mathbf{F}_{25,45}$.

If $P_{23} \in P^*_{235}P_{25}P_{45}P^*_{245}$, then $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P_{23} \notin P^*_{235}P_{25}P_{45}P^*_{245}$, then $P_{23} \in \mathbf{F}_{25,235} \cap \mathbf{F}_{45,245} \cap \mathbf{F}_{235,245}$. Thus $P_{23}P^*_{235}P_{25}P_{45}P^*_{245}$ is a convex pentagon. By Lemma 1 we have a lattice point A such that

$$A \in \overline{P_{23}P^*_{235}P_{25}P_{45}P^*_{245}} \subset P_{23}P_{25}P_{45} \cap P^*_{235}P_{25}P^*_{245} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{25,23}$, then $P_{25} \in P^*_{235}P_{34}P_{23} \subset \mathbf{K}_3$. Thus $P_{25} \in \mathbf{K}_1 \cap \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P^*_{235} \in \mathbf{F}_{34,25} \cap \mathbf{F}_{45,24} \cap \mathbf{F}_{23,25}$, then $P^*_{245}P_{23}P_{25}P^*_{235}$ is a convex quadrangle. Namely, $P^*_{245} \in \mathbf{F}_{23,35} \cap \mathbf{F}_{25,235} \cap \mathbf{F}_{23,235}$.

If $P_{45} \in P^*_{245}P_{23}P_{25}P^*_{235}$, then $P_{45} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

If $P_{45} \notin P^*_{245}P_{23}P_{25}P^*_{235}$, then $P_{45} \in \mathbf{F}_{245,23} \cap \mathbf{F}_{25,235} \cap \mathbf{F}_{245,235}$. Thus $P_{45}P^*_{245}P_{23}P_{25}P^*_{235}$ is a convex pentagon. By Lemma 1 we have a lattice point A such that

$$A \in \overline{P_{45}P^*_{245}P_{23}P_{25}P^*_{235}} \subset P_{45}P_{23}P_{25} \cap P^*_{245}P_{25}P^*_{235} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

a contradiction. □

V. $\mathbf{K}_1$ *is a segment.* Then $\mathbf{K}_i \cap \mathbf{K}_j \cap \mathbf{K}_1$ contains a lattice point in common. Thus applying Helly's theorem to the segment $\mathbf{K}_i \cap \mathbf{K}_1$ we get that they have a lattice point in common. Hence, we have proved that in this case the convex sets have a lattice point in common, which proves Theorem 2.

In fact, we have proved more. Namely, we have shown that the fixed system of five convex sets of Theorem 2 either have a lattice point in common or each of them is a triangle. □

Now we are able to prove Theorem 1, though we still need a few definitions and several lemmas to do so.

We need the following

DEFINITION 3. Let $\mathcal{F}$ be a fixed system of at least four sets such that any three of them have a lattice point in common. We say that $\mathcal{F}$ is good if the convex hull of $\mathcal{F}$ possesses a vertex S which belongs to exactly three sets. Let us denote these sets by $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ and call them the main configurations of $\mathcal{F}$. If a set of $\mathcal{F}$ is not a main configuration then we call it an ordinary configuration.

THEOREM 3. *Let $\mathcal{F}$ be a good system of convex sets. Then one of the three main configurations of $\mathcal{F}$ is such that removing it from $\mathcal{F}$ the remaining convex sets have a lattice point in common.*

In the following proof step by step we discover more. We are going to characterize the good systems of convex sets. Notice that applying the FLP-algorithm we get lattice-polygons.

LEMMA 3. *Each vertex of a main configuration is included in another one.*

PROOF. Let A be a vertex of $\mathbf{K}_1$. Suppose that $A \notin \mathbf{K}_2$ and $A \notin \mathbf{K}_3$. This entails a contradiction. As A is a vertex of $\mathbf{K}_1$ we can find $\mathbf{K}_4$ and $\mathbf{K}_5$ such that A is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_4$ and $\mathbf{K}_5$. It follows from the foregoing that $\mathbf{K}_1, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_2$; $\mathbf{K}_1, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_3$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_5$ groups of four sets do not contain a lattice point in common. So we cannot choose further three sets from $\mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$ to $\mathbf{K}_1$ such that this four sets have a lattice point in common. Thus it is a contradiction with Theorem 2. □

Let us denote the convex hull of $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ by $\mathbf{M}$. Let $\mathbf{M}$ be the convex lattice-polygon $A_1A_2 \dots A_kS$, where S is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$. A_i is naturally a vertex of some main configuration of $\mathcal{F}$. Hence according to Lemma 3 it is included in another one, too. Then we say A_i is a type B_{12} vertex, if $A_i \notin \mathbf{K}_3$ and $A_i \in \mathbf{K}_1 \cap \mathbf{K}_2$. We define type B_{13} and type B_{23} vertices similarly.

LEMMA 4. $\mathbf{M}$ *has got type* B_{12}, B_{13} *and* B_{23} *vertices.*

PROOF. Assume that there is no type B_{12} vertex. Then $A_i \in \mathbf{K}_3$ for each i. Since $S \in \mathbf{K}_3$ we get that $\mathbf{M} \subset \mathbf{K}_3$. But $\mathbf{K}_3 \subset \mathbf{M}$ thus $\mathbf{K}_3 \equiv \mathbf{M}$. We show that there is only one lattice point in $\mathbf{K}_1 \cap \mathbf{K}_2$. Suppose that there is a lattice point S_1 such that $S_1 \not\equiv S$ and $S_1 \in \mathbf{K}_1 \cap \mathbf{K}_2$. In this way we get that $S_1 \in \mathbf{K}_1 \cap \mathbf{K}_2 \subset \mathbf{K}_3$, that is, $S_1 \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3$ which contradicts the fl-point property of S. Thus the only lattice point of $\mathbf{K}_1 \cap \mathbf{K}_2$ is S. Since any three sets of $\mathcal{F}$ have a lattice point in common, hence any set of $\mathcal{F}$ contains S, which is a contradiction. □

LEMMA 5. $\mathbf{M}$ *has got exactly one type* B_{12}, B_{13} *and* B_{23} *vertex.*

PROOF. (Indirect.) Let n be the least number with the following property: There exists a system $\mathcal{C}$ of n convex sets such that any three sets of $\mathcal{C}$ have a lattice point in common, moreover the claim is false for $\mathcal{C}$. Let us consider such a $\mathcal{C}$. Then we may assume that there are two type B_{12} vertices, say A_1 and A_2.

It is trivial that $n \geqq 5$. We show that $n \geqq 6$. Namely, if $n = 5$ then among the vertices of $\mathbf{K}_1$ we have S, A_1, A_2 and a type B_{13} vertex. But that is impossible since we have already proved that $\mathbf{K}_1$ is a triangle or a point. Thus $n \geqq 6$.

We need the following

LEMMA 6. *There exists at most one ordinary configuration of* $\mathcal{C}$ *with the following property: Removing this configuration from* $\mathcal{C}$ *then* A_1 *will not be an fl-point with respect to any triplet of* $\mathcal{C}$ *containing a main configuration.*

PROOF. Suppose that this statement is false. Then there are two sets $\mathbf{K}_4$ and $\mathbf{K}_5$ with the previous property. It is easy to see that A_1 is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_4$ and $\mathbf{K}_5$; and similarly with respect to $\mathbf{K}_2, \mathbf{K}_4$ and $\mathbf{K}_5$. Then the sets of groups $\mathbf{K}_1, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_3$; $\mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_3$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_5$ do not contain a lattice point in common. But this contradicts Theorem 2. □

If there exists a convex set of $\mathcal{C}$ that satisfies the conditions of Lemma 6 then let us call it $\mathbf{K}_4$. Similarly we define $\mathbf{K}_5$ by replacing A_1 by A_2. Since $n \geqq 6$ there exists a convex set of $\mathcal{C}$, say $\mathbf{K}_i$, different from $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. Removing $\mathbf{K}_i$ from $\mathcal{C}$ we get a convex set system $\mathcal{C}'$, containing

$n-1$ sets. Let us apply the FLP-algorithm to $\mathcal{C}'$. Notice that $\mathcal{C}'$ is good with respect to S. We prove that the claim is false for $\mathcal{C}'$. By Lemma 6 we get a triplet of $\mathcal{C}'$ containing $\mathbf{K}_1$, in which A_1 is an fl-point with respect to it. According to Lemma 6 we have that A_1 or A_2 is an fl-point with respect to a triplet of $\mathcal{C}$ containing $\mathbf{K}_1$ or $\mathbf{K}_2$ (all the variations are allowed).

In this way, applying the FLP-Algorithm we cannot eliminate A_1 or A_2 from neither $\mathbf{K}_1$ nor $\mathbf{K}_2$. Thus for $\mathcal{C}'$ the claim is false, a contradiction. □

In the following part of our proof we will describe all the good $\mathcal{C}$ systems containing five sets.

Let the five sets be denoted by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. Let $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ be the main configuration of $\mathcal{C}$ with respect to S.

Let $\mathbf{M}'$ be the convex hull of $\mathcal{C}$. Then $\mathbf{M} \equiv \mathbf{M}'$. Namely, each triplet of $\mathcal{C}$ contains a main configuration. Let A_1, A_2 and A_3 be the type B_{23}, B_{13} and B_{12} vertex of $\mathbf{M}$, resp. Let $\mathbf{M}$ be the convex quadrangle $SA_1A_2A_3$. As each set of $\mathcal{C}$ is a triangle, $\mathbf{K}_1$ is the triangle SA_2A_3, $\mathbf{K}_2$ the triangle SA_1A_3 and $\mathbf{K}_3$ the triangle SA_1A_2. We prove that $A_1A_2A_3$ is a member of $\mathcal{C}$.

If each of the points A_1, A_2 and A_3 is covered by four sets of $\mathcal{C}$, then $\mathbf{K}_4$ and $\mathbf{K}_5$ will contain A_1, A_2 and A_3. Since $\mathbf{K}_4$ and $\mathbf{K}_5$ are triangles we get that $A_1A_2A_3 \equiv \mathbf{K}_4 \equiv \mathbf{K}_5$.

If some A_i is covered by exactly three sets of $\mathcal{C}$, then $\mathcal{C}$ will also be good with respect to A_i. Thus it follows from this that $A_1A_2A_3$ is a member of $\mathcal{C}$. Let us call it $\mathbf{K}_4$.

We show that SA_2 and A_1A_3 do not contain any lattice point except the endpoints.

Let N be the intersection of the diagonals of $\mathbf{M}$. Notice that any three sets of $\mathcal{C}$ have a point in common, hence it follows from the Helly-theorem that there exists a point common to every set of $\mathcal{C}$. As the intersection of $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$ is a point N we get that $N \in \mathbf{K}_5$.

Let D be one of S, A_1, A_2 and A_3. If DN contains a lattice point different from D, say E, then E is covered by all sets $\mathbf{K}_i$ covering D. But D is an fl-point with respect to some triplet of $\mathcal{C}$, thus we are led to a contradiction. Hence the diagonals of $\mathbf{M}$ do not contain a lattice point except the endpoints. Since $\mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{L} \equiv S \cup A_2$ and $\mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{L} \equiv A_1 \cup A_3$, $\mathbf{K}_5$ contains two neighbouring vertices of $\mathbf{M}$. Let these two neighbouring vertices be A_1 and A_2. As $\mathbf{K}_5$ is a triangle, its third vertex is A_5 where $A_5 \in \mathbf{K}_1 \cap \mathbf{K}_2$. This way we described all good $\mathcal{C}$ containing five sets (see Fig. 13). □

Let $\mathcal{C}$ be a good system of convex sets, and let A_1, A_2 and A_3 be the type B_{23}, B_{13} and B_{12} vertex of $\mathbf{M}$, resp.

LEMMA 7. *There exists an ordinary configuration of* $\mathcal{C}$, $\mathbf{K}_j$ *such that* $A_2 \in \mathbf{K}_j$ *and* A_2 *is an fl-point with respect to* $\mathbf{K}_1$, $\mathbf{K}_3$ *and* $\mathbf{K}_j$.

PROOF. Suppose that the claim is false. A_2 is an fl-point with respect to a triplet containing $\mathbf{K}_1$. Let this triplet be $\mathbf{K}_1$, $\mathbf{K}_4$ and $\mathbf{K}_5$. Let us consider $\mathcal{G} = \{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_5\}$. Apply the FLP-algorithm to $\mathcal{G}$ as follows: Let us consider $\mathbf{K}_3$. A_2 is not an fl-point with respect to a triplet containing $\mathbf{K}_3$,

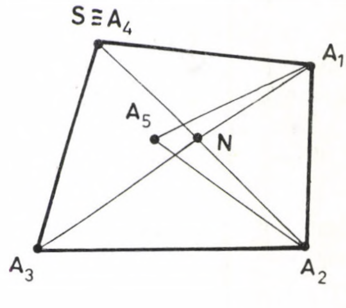

Fig. 13

otherwise A_2 would be an fl-point with respect to $\mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. Then we could get a contradiction in the same way as in the proof of Lemma 6. Thus applying the FLP-algorithm we can remove A_2 from $\mathbf{K}_3$. Hence we get a good $\mathcal{G}'$ with the property that one of the main configurations of $\mathcal{G}'$, $\mathbf{K}_1$, has got a vertex A_2 which is not included in another main configuration, and this contradicts Lemma 3. □

LEMMA 8. *A_2 is covered by all the ordinary configurations of $\mathcal{C}$.*

PROOF. According to Lemma 7 there exists an ordinary configuration of $\mathcal{C}$; $\mathbf{K}_4$ such that A_2 is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_3$ and $\mathbf{K}_4$. Assume that there exists an ordinary configuration $\mathbf{K}_5$ not containing A_2. Let $\mathcal{G} = \{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_5\}$. Applying the FLP-algorithm to $\mathcal{G}$ we get a good $\mathcal{G}'$. Let $\mathbf{M}$ be the convex hull of $\mathcal{G}'$. Obviously, A_2 and S are vertices of $\mathbf{M}$. Let A_3' be a type B_{12} vertex and A_1' be a type B_{23} vertex of $\mathbf{M}$. We prove that $\mathbf{M}$ is the quadrangle $SA_1'A_2A_3'$. Consider $\mathcal{C}$. If H is a type B_{23} lattice point, then $H \in \overline{\mathbf{F}}_{SA}$; otherwise we get a contradiction since S is an fl-point with respect to $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$. Similarly if G is a type B_{12} lattice point of $\mathbf{M}$, then $G \in \overline{\mathbf{F}}_{AS}$. Thus it follows that $\mathbf{M}$ is the quadrangle $SA_1'A_2A_3'$. Notice that A_2 is not covered by any set of $\mathcal{C}$ different from $\mathbf{K}_1, \mathbf{K}_3$ and $\mathbf{K}_4$. Thus $\mathcal{G}$ has got two opposite vertices S and A_2 with the following property: S and A_2 are included in exactly three sets of $\mathcal{C}$. But this is impossible. Thus we get a contradiction. □

Notice that Theorem 3 follows from Lemma 8. □

Let us consider a convex set system $\mathcal{F}$ satisfying the conditions of Theorem 1. Applying the FLP-algorithm to $\mathcal{F}$ we get a fixed $\mathcal{F}'$. Let $\mathbf{M}$ be the convex hull of $\mathcal{F}'$. Let R be one of its vertices. Obviously R is an fl-point. Suppose that R is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$. Removing all sets of $\mathcal{F}'$ containing R and different from $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ we get a convex set system $\mathcal{C}$. Applying the FLP-algorithm to $\mathcal{C}$ we get $\mathcal{C}'$. Obviously $\mathcal{C}'$ is good. According to Theorem 3 there exists a lattice point J covered by all ordinary configurations of $\mathcal{C}'$. It is easy to see that J and R pin down $\mathcal{F}$. The proof of Theorem 1 is complete. □

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PROPERTIES OF HYPERCONNECTED SPACES

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1. Introduction

A topological space X is said to be *hyperconnected* [21] if every pair of nonempty open sets of X has nonempty intersection. Several notions which are equivalent to hyperconnectedness were defined and investigated in the literature. Levine [11] called a topological space X a *D-space* if every nonempty open set of X is dense in X and showed that X is a D-space if and only if it is hyperconnected. Pipitone and Russo [19] defined a topological space X to be *semi-connected* if X is not the union of two disjoint nonempty semi-open sets of X and showed that X is semi-connected if and only if it is a D-space. Maheshwari and Tapi [12] defined a topological space X to be *s-connected* if X is not the union of two nonempty semiseparated sets and showed the equivalence of s-connectedness and semi-connectedness. Hyperconnected spaces are also called *irreducible* in [22]. Recently, Ajmal and Kohli [2] have investigated the further properties of hyperconnected spaces.

In the present paper, we shall use the terminology "hyperconnected" to express the equivalent notions previously stated. In Section 3, we obtain several characterizations of hyperconnected spaces by using semi-preopen sets [3] and almost feebly continuous functions. The main result of the last section is that hyperconnectedness is preserved under almost feebly continuous surjections. This is an improvement of the result that hyperconnected spaces are preserved by feebly continuous surjections [2, 15].

2. Preliminaries

Throughout the present paper, (X,τ) and (X,σ) (or simply X and Y) will denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *semi-open* [10] (resp. *preopen* [13], *β-open* [1]) if $A \subset \mathrm{Cl}\big(\mathrm{Int}(A)\big)$ (resp. $A \subset \mathrm{Int}\big(\mathrm{Cl}(A)\big)$, $A \subset \mathrm{Cl}\big(\mathrm{Int}(\mathrm{Cl}(A))\big)$.

Andrijević [3] defined a subset A to be *semi-preopen* if there exists a preopen set V in X such that $V \subset A \subset \mathrm{Cl}(V)$ and showed the equivalence of β-openness and semi-preopenness. The complement of a semi-open (resp. preopen, semi-preopen) set is said to be *semi-closed* (resp. *preclosed, semi-preclosed*). The *semi-closure* [5] (resp. *preclosure* [6], *semi-preclosure* [3]) of A, denoted by $\mathrm{s\,Cl}(A)$ (resp. $\mathrm{p\,Cl}(A)$, $\mathrm{sp\,Cl}(A)$), is defined by the intersection of all semi-closed (resp. preclosed, semi-preclosed) sets of X containing A. The union of all semi-open sets contained in A is called the *semi-interior* of A and is denoted by $\mathrm{s\,Int}(A)$. A subset A is said to be *regular open* (resp. *regular closed*) if $A = \mathrm{Int}\,(\mathrm{Cl}(A))$ (resp. $A = \mathrm{Cl}\,(\mathrm{Int}\,(A))$). The family of all semi-open (resp. preopen, semi-preopen, regular open, regular closed) sets of X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\mathrm{SPO}(X)$, $\mathrm{RO}(X)$, $\mathrm{RC}(X)$).

LEMMA 2.1. *The following properties hold for a topological space* (X,τ):
(a) $\tau \subset \mathrm{SO}(X) \cap \mathrm{PO}(X)$,
(b) $\mathrm{SO}(X) \cup \mathrm{PO}(X) \subset \mathrm{SPO}(X)$.

LEMMA 2.2 (Andrijević [3]). *Let A be a subset of a topological space X. Then the following properties hold:*
(a) $\mathrm{s\,Cl}(A) = A \cup \mathrm{Int}\,(\mathrm{Cl}(A))$,
(b) $\mathrm{p\,Cl}(A) = A \cup \mathrm{Cl}\,(\mathrm{Int}\,(A))$, *and*
(c) $\mathrm{sp\,Cl}(A) = A \cup \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(A)))$.

LEMMA 2.3 (Noiri [16]). *A topological space X is hyperconnected if and only if $U \cap V \neq \emptyset$ for any nonempty sets $U, V \in \mathrm{SO}(X)$.*

3. Characterizations of hyperconnected spaces

In Theorem 3.1 of [16], the author showed that a topological space X is hyperconnected if and only if $\mathrm{p\,Cl}(U) = X$ for every nonempty set $U \in$ $\in \mathrm{SO}(X)$. This type of characterizations of hyperconnected spaces are completely clarified by statements (a)–(e) in the following theorem and Example 3.2 (below).

THEOREM 3.1. *The following are equivalent for a topological space X:*
(a) *X is hyperconnected;*
(b) *$\mathrm{Cl}(W) = X$ for every nonempty set $W \in SPO(X)$;*
(c) *$\mathrm{s\,Cl}(W) = X$ for every nonempty set $W \in \mathrm{SPO}(X)$;*
(d) *$\mathrm{p\,Cl}(U) = X$ for every nonempty set $U \in \mathrm{SO}(X)$;*
(e) *$\mathrm{sp\,Cl}(U) = X$ for every nonempty set $U \in \mathrm{SO}(X)$;*
(f) *$U \cap W \neq \emptyset$ for any nonempty sets $U \in \mathrm{SO}(X)$ and $W \in \mathrm{SPO}(X)$.*

PROOF. (a)$\rightarrow$(b): Let W be any nonempty semi-preopen set of X. Then, we have $\mathrm{Int}\,(\mathrm{Cl}(W)) \neq \emptyset$ and hence $X = \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}(W))) =$ $= \mathrm{Cl}\,(W)$.

(b)→(c): Let W be any nonempty semi-preopen set of X. By Lemma 2.2, we have $\operatorname{s Cl}(W) = W \cup \operatorname{Int}(\operatorname{Cl}(W)) = W \cup \operatorname{Int}(X) = X$.

(c)→(e): Let U be any nonempty semi-open set of X. By Lemma 2.2, we have $\operatorname{sp Cl}(U) = U \cup \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(U))) = U \cup \operatorname{Int}(\operatorname{Cl}(U)) = \operatorname{s Cl}(U) = X$ since $\mathrm{SO}(X) \subset \mathrm{SPO}(X)$.

(e)→(f): Suppose that there exist nonempty sets $U \in \mathrm{SO}(X)$ and $W \in \mathrm{SPO}(X)$ such that $U \cap W = \emptyset$. Since $W \in \mathrm{SPO}(X)$, we have $\emptyset = \operatorname{sp Cl}(U) \cap W = X \cap W = W$. This is a contradiction.

(f)→(a): By Lemma 2.1, we have $\tau \subset \mathrm{SO}(X) \subset \mathrm{SPO}(X)$ and hence $U \cap V \neq \emptyset$ for any nonempty open sets $U, V \in \tau$. Therefore, (X, τ) is hyperconnected.

The equivalence of (a) and (d) is shown in [16, Theorem 3.1].

In [14, Theorem 3.1], the author showed that a topological space X is hyperconnected if and only if $\operatorname{s Cl}(U) = X$ for every nonempty set $U \in \mathrm{SO}(X)$. Now, we consider the following properties:

(p) $\operatorname{p Cl}(V) = X$ for every nonempty set $V \in \mathrm{PO}(X)$ and

(β) $\operatorname{sp Cl}(W) = X$ for every nonempty set $W \in \mathrm{SPO}(X)$.

It follows from Lemma 2.1 that (β) implies both (p) and (c) in Theorem 3.1 and also that (p) implies connectedness. It is well known that hyperconnectedness is strictly stronger than connectedness. Example 3.2 (below) shows that (c) in Theorem 3.1 can be replaced by neither (p) nor (β). Moreover, it also shows that (c) in Theorem 3.1 cannot be replaced by the following property:

(β') $\operatorname{sp Cl}(V) = X$ for every nonempty set $V \in \mathrm{PO}(X)$.

EXAMPLE 3.2. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ *and* $A = \{a\}$. *Then* (X, τ) *is a hyperconnected space and* $A \in \mathrm{PO}(X, \tau) \subset \mathrm{SPO}(X, \tau)$. *By Lemma* 2.2, *we have* $\operatorname{p Cl}(A) = \operatorname{sp Cl}(A) = A \neq X$.

To obtain another type of characterizations of hyperconnected spaces, we shall first recall the definition of feebly continuous functions. A function $f: X \to Y$ is said to be *feebly continuous* [4] if, for every nonempty open set V of Y, $f^{-1}(V) \neq \emptyset$ implies $\operatorname{Int}(f^{-1}(V)) \neq \emptyset$. This definition is different from the one in the sense of Frolík [7] because f need not be surjective. A function $f: X \to Y$ is said to be *semi-continuous* [10] if $f^{-1}(V) \in \mathrm{SO}(X)$ for every open set V of Y. It is shown in [2] that every semi-continuous function is feebly continuous but not conversely.

DEFINITION 3.3. A function $f: X \to Y$ is said to be *almost feebly continuous* if, for every nonempty $V \in \mathrm{RO}(Y)$, $f^{-1}(V) \neq \emptyset$ implies $\operatorname{s Int}(f^{-1}(V)) \neq \emptyset$.

REMARK 3.4. Every feebly continuous function is obviously almost feebly continuous. However, the converse is false even if the function is bijective as the following example shows.

EXAMPLE 3.5. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Let $f: (X,\tau) \to (X,\sigma)$ be a function defined as follows: $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is an almost feebly continuous bijection but it is not feebly continuous. For, we have $\mathrm{RO}(X,\sigma) = \{\emptyset, X, \{b\}, \{a,c\}\}$, $\mathrm{s\,Int}\left(f^{-1}(\{b\})\right) = \{a\}$, $\mathrm{s\,Int}\left(f^{-1}(\{a,c\})\right) = \{b,c\}$ and $\mathrm{Int}\left(f^{-1}(\{a\})\right) = \emptyset$.

The topological space consisting of two points with the discrete topology is usually denoted by 2. Ajmal and Kohli [2] obtained some characterizations of hyperconnected spaces by using feebly continuous functions.

THEOREM 3.6. *The following are equivalent for a topological space* X*:*
(a) X *is hyperconnected;*
(b) *every almost feebly continuous function of* X *into a Hausdorff space is constant;*
(c) *every almost feebly continuous function* $f: X \to 2$ *is constant;*
(d) *no semi-continuous function* $f: X \to 2$ *is surjective.*

PROOF. (a)$\to$(b): Suppose that there exist a Hausdorff space Y and an almost feebly continuous function $f: X \to Y$ such that f is not constant. Then, there exist two points x and y of X such that $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open sets G and H in Y such that $f(x) \in G$, $f(y) \in H$ and $G \cap H = \emptyset$. Put $U = \mathrm{Int}\left(\mathrm{Cl}(G)\right)$ and $V = \mathrm{Int}\left(\mathrm{Cl}(H)\right)$, then we have $\emptyset \neq U \in \mathrm{RO}(Y)$, $\emptyset \neq V \in \mathrm{RO}(Y)$ and $U \cap V = \emptyset$. Since f is almost feebly continuous, $\mathrm{s\,Int}\left(f^{-1}(U)\right) \neq \emptyset$ and $\mathrm{s\,Int}\left(f^{-1}(V)\right) \neq \emptyset$. However, we have $\mathrm{s\,Int}\left(f^{-1}(U)\right) \cap \mathrm{s\,Int}\left(f^{-1}(V)\right) \subset f^{-1}(U \cap V) = \emptyset$. It follows from Lemma 2.3 that X is not hyperconnected.

The proofs of the implications (b)$\to$(c) and (c)$\to$(d) are obvious. The equivalence of (d) and (a) is shown in [14, Theorem 3.1].

Janković and Long [9] introduced a weak form of hyperconnectedness which is called θ-irreducible and showed that an almost-regular space is hyperconnected if and only if it is θ-irreducible. We shall slightly improve this result. For this purpose we shall recall some definitions.

DEFINITION 3.7. A topological space (X,τ) is said to be
(a) *almost-regular* [20] if for each $F \in \mathrm{RC}(X)$ and each point $x \in X - F$ there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$;
(b) *strongly s-regular* [8] if for each closed set A of X and each point $x \in X - A$ there exists an $F \in \mathrm{RC}(X)$ such that $x \in F$ and $F \cap A = \emptyset$;
(c) *weakly* P_Σ [17] (resp. P_Σ [23]) if every $V \in \mathrm{RO}(X)$ (resp. $V \in \tau$) is the union of regular closed sets of X.

It is shown in [8, Theorem 1] that a topological space X is strongly s-regular if and only if it is P_Σ. In Examples 3 and 4 of [8], it is shown that almost-regularity and strong s-regularity are independent of each other even if the space is Hausdorff. On the other hand, it is pointed out that every almost-regular space is weakly P_Σ and every P_Σ is weakly P_Σ but not conversely [17, Example 3.2]. Therefore, the notion of "weakly P_Σ" is strictly weaker than both almost-regularity and strong s-regularity (or P_Σ).

DEFINITION 3.8. A topological space X is said to be *θ-irreducible* [9] if every pair of nonempty regular closed sets of X has nonempty intersection.

It is pointed out in [9] that every hyperconnected (or irreducible) space is θ-irreducible but not conversely. The following theorem is a slight improvement of [9, Theorem 2].

THEOREM 3.9. *A weakly P_Σ space X is hyperconnected if and only if it is θ-irreducible.*

PROOF. Suppose that X is not hyperconnected. There exist nonempty open sets U and V of X such that $U \cap V = \emptyset$. Since U and V are disjoint, we obtain $\mathrm{Int}\,(\mathrm{Cl}\,(U)) \cap V = \emptyset$ and $\mathrm{Int}\,(\mathrm{Cl}\,(U)) \cap \mathrm{Cl}\,(V) = \emptyset$. For a point $x \in \mathrm{Int}\,(\mathrm{Cl}\,(U))$, there exist $F \in \mathrm{RC}\,(X)$ such that $x \in F \subset \mathrm{Int}\,(\mathrm{Cl}\,(U))$. Therefore, we have $F \cap \mathrm{Cl}\,(V) = \emptyset$ and $\mathrm{Cl}\,(V) \in \mathrm{RC}\,(X)$. This shows that X is not θ-irreducible.

COROLLARY 3.10 (Janković and Long [9]). *An almost-regular space is irreducible if and only if it is θ-irreducible.*

COROLLARY 3.11. *A strongly s-regular space is hyperconnected if and only if it is θ-irreducible.*

4. Hyperconnected spaces and functions

For a function $f: X \to Y$, the subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and is denoted by $G(f)$. A function $f: X \to Y$ is said to be *somewhat nearly continuous* [18] if, for every nonempty open set V of Y, $f^{-1}(V) \neq \emptyset$ implies $\mathrm{Int}\,(\mathrm{Cl}\,(f^{-1}(V))) \neq \emptyset$. Every feebly continuous function is obviously somewhat nearly continuous but the converse is false as the following example shows.

EXAMPLE 4.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the identity function $f: (X, \tau) \to (X, \sigma)$ is somewhat nearly continuous but it is not feebly continuous since $\mathrm{Int}\,(\mathrm{Cl}\,(f^{-1}(\{b\}))) = \{b, c\}$ and $\mathrm{Int}\,(f^{-1}(\{b\})) = \emptyset$.

THEOREM 4.2. *If X is a hyperconnected space, $f: X \to Y$ is somewhat nearly continuous and $G(f)$ is closed in $X \times Y$, then f is constant.*

PROOF. Suppose that f is not constant. There exist two points x and y of X such that $f(x) \neq f(y)$. Then, we have $(x, f(y)) \in X \times Y - G(f)$. Since $G(f)$ is closed, there exist open neighborhoods U and V of x and $f(y)$, respectively, such that $(U \times V) \cap G(f) = \emptyset$; hence $f(U) \cap V = \emptyset$. Therefore, we have $U \cap f^{-1}(V) = \emptyset$ and hence $U \cap \operatorname{Int}\left(\operatorname{Cl}(f^{-1}(V))\right) = \emptyset$. Since f is somewhat nearly continuous, $\operatorname{Int}\left(\operatorname{Cl}(f^{-1}(V))\right) \neq \emptyset$. This contradicts that X is hyperconnected.

COROLLARY 4.3 (Thompson [22]). *Let X be a hyperconnected space. If $f: X \to Y$ is a continuous function with a closed graph, then f is constant.*

In [15, Theorem 3.1] and [2, Theorem 2.7], it is shown that hyperconnectedness is preserved under feebly continuous surjections. This result is improved as follows:

THEOREM 4.4. *If X is a hyperconnected space and $f: X \to Y$ is an almost feebly continuous surjection, then Y is hyperconnected.*

PROOF. Suppose that Y is not hyperconnected. There exist disjoint nonempty open sets G and H of Y. Put $U = \operatorname{Int}\left(\operatorname{Cl}(G)\right)$ and $V = \operatorname{Int}\left(\operatorname{Cl}(H)\right)$, then we have $\emptyset \neq U \in \operatorname{RO}(Y)$, $\emptyset \neq V \in \operatorname{RO}(Y)$ and $U \cap V = \emptyset$. Therefore, we obtain $\emptyset = f^{-1}(U) \cap f^{-1}(V) \supset \operatorname{s\,Int}\left(f^{-1}(U)\right) \cap \operatorname{s\,Int}\left(f^{-1}(V)\right)$. Since f is an almost feebly continuous surjection, $\operatorname{s\,Int}\left(f^{-1}(U)\right) \neq \emptyset$ and $\operatorname{s\,Int}\left(f^{-1}(V)\right) \neq \emptyset$. It follows from Lemma 2.3 that X is not hyperconnected.

COROLLARY 4.5 (Ajmal and Kohli [2], Noiri [15]). *If X is hyperconnected and $f: X \to Y$ is a feebly continuous surjection, then Y is hyperconnected.*

DEFINITION 4.6. A function $f: X \to Y$ is said to be

(a) *feebly open* [7] if $\operatorname{Int}\left(f(U)\right) \neq \emptyset$ for any nonempty open set U of X;

(b) *almost feebly open* if $\operatorname{s\,Int}\left(f(U)\right) \neq \emptyset$ for any nonempty $U \in \operatorname{RO}(X)$.

Every feebly open function is obviously almost feebly open but the converse is false. For, in Example 3.5, f is bijective and hence $f^{-1}: (X, \sigma) \to (X, \tau)$ is almost feebly open but it is not feebly open.

THEOREM 4.7. *If Y is a hyperconnected space and $f: X \to Y$ is an almost feebly open injection, then X is hyperconnected.*

PROOF. Let U and V be any nonempty open sets of X. Put $G = \operatorname{Int}\left(\operatorname{Cl}(U)\right)$ and $H = \operatorname{Int}\left(\operatorname{Cl}(U)\right)$, then we have $\emptyset \neq G \in \operatorname{RO}(X)$ and $\emptyset \neq H \in \operatorname{RO}(X)$. Since f is almost feebly open, $\operatorname{s\,Int}\left(f(G)\right) \neq \emptyset$ and $\operatorname{s\,Int}\left(f(H)\right) \neq \emptyset$. Since Y is hyperconnected, by Lemma 2.3 we have

$$\emptyset \neq \operatorname{Int}\left(f(G)\right) \cap \operatorname{s\,Int}\left(f(H)\right) \subset f(G) \cap f(H).$$

Moreover, since f is injective, we obtain $G \cap H \neq \emptyset$ and hence $U \cap V \neq \emptyset$. This shows that X is hyperconnected.

COROLLARY 4.8 (Ajmal and Kohli [2]). *If Y is hyperconnected and $f: X \to Y$ is a feebly open injection, then X is hyperconnected.*

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A WEIGHTED L^2 MARKOFF TYPE INEQUALITY FOR CLASSICAL WEIGHTS

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1. Introduction

Let $\mathcal{P}_n$ be the class of real algebraic polynomials P of degree at most n, such that $|P(t)| \leqq \varphi(t)$ $(-1 \leqq t \leqq 1)$, where $\varphi(t)$ is a non-negative function. Turán's problem is: how large $\|P'\|_\infty$ can be, where $\|.\|_\infty$ is the supremum norm on $[-1,1]$. Such problems first appeared in approximation theory notably in the work of Dzyadyk [4] and Pierre and Rahman [10].

The most interesting cases are those where φ vanishes at ± 1. In the case of parabolic majorant ($\varphi(t) = \sqrt{1-t^2}$) the answer was given by Rahman [11].

THEOREM 1.1. *Let $P \in \mathcal{P}_n$ and $|P(t)| \leqq \sqrt{1-t^2}$ $(-1 \leqq t \leqq 1)$. Then*

$$\|P'\|_\infty \leqq 2(n-1).$$

Equality is attained at the points $t = \pm 1$, if and only if $P(t) = (1-t^2)U_{n-2}(t)$, where U_{n-2} is the $(n-2)$-th Chebyshev polynomial of the second kind.

This result can be stated in the following form: If $P \in \mathcal{P}_n$ is such that $P(\pm 1) = 0$, then

$$\sup_{P \in P_n - \{0\}} \frac{\|P'\|_\infty}{\|P/\varphi\|_\infty} = 2(n-1), \tag{1}$$

or in an equivalent form, if $Q \in \mathcal{P}_{n-2}$, then

$$\sup_{Q \in \mathcal{P}_{n-2} - \{0\}} \frac{\left\| [\varphi^2 Q]' \right\|_\infty}{\|\varphi Q\|_\infty} = 2(n-1). \tag{2}$$

The classical weight functions ($w \in CW$) correspond to special orthogonal polynomials and intervals as follows (cf [15]):

Interval (a,b)	Weight function	Symbol	Name
$[-1,1]$	$(1-t)^{\alpha}(1+t)^{\beta}$ $(\alpha,\beta>-1)$	$P_n^{(\alpha,\beta)}$	Jacobi polynomials
$(0,+\infty)$	$t^{s}e^{-t}$ $(s>0)$	$L_n^{(s)}$	Generalized Laguerre
$(-\infty,\infty)$	e^{-t^2}	H_n	Hermite

Let $w \in CW$ on (a,b) and set

$$\|f\|_{w_m} = \left(\int_a^b w_m(t)f^2(t)\,dt\right)^{1/2}, \tag{3}$$

where $w_m = A^m w$ and

$$A(t) = \begin{cases} 1 & \text{(Hermite case)}, \\ t & \text{(generalized Laguerre case)}, \\ 1-t^2 & \text{(Jacobi case)}. \end{cases} \tag{4}$$

Let $w \in CW$. In this paper the extremal problem

$$\max_{P\in\mathcal{P}_n;\|P\|_{w_m}\leqq 1} \left\|\left(\sqrt{A}/w_m\right)\left(w_m P^{(m)}\right)'\right\|_{w_m} \tag{5}$$

is considered.

Concerning this problem many important contributions were made with different definitions of the norm, by Agarwal and Milovanović [2], Milne [8], Milovanović [9], Rahman [11], Rahman and Watt [12], Varma [13, 14], and Zalik [16, 17].

We note that in the Jacobi case $w(t) = 1-t^2$ our problem for $m=0$ is the same as (2) in L^2-norm.

2. The main result

With the notation of Section 1, we state the solution of our extremal problem and its corollaries in this section with respect to the Jacobi, generalized Laguerre, and Hermite weight functions on $(-1,1)$, $(0,+\infty)$, and $(-\infty,+\infty)$, respectively. Let $w \in CW$, let A be given by (4) and $B(t)$ the polynomials defined by (cf [15])

$$B(t) = \begin{cases} -2t & \text{(Hermite case)}, \\ s+1-t & \text{(generalized Laguerre case)}, \\ \beta-\alpha-(\alpha+\beta+2)t & \text{(Jacobi case)}. \end{cases} \tag{6}$$

We now formulate our main result.

THEOREM 2.1. *Let $P \in \mathcal{P}_n$ be such that $\|P\|_{w_m} \leqq 1$. Then we have*

$$\left\| \left(\sqrt{A}/w_m\right)\left(w_m P^{(m)}\right)' \right\|_{w_m} \leqq \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}\beta_{n,m}} \tag{7}$$

where

$$\lambda_{n,\nu} = -(n-\nu)\left(\frac{1}{2}(n+\nu-1)A''(0)+B'(0)\right) \quad (\nu = 0,\ldots,m-1), \tag{8}$$

and

$$\beta_{n,m} = \lambda_{n,m} + B'(0) + (k-1)A''(0). \tag{9}$$

Equality is attained if and only if P is a constant multiple of the classical polynomial Q_n orthogonal with respect to the weight function $w \in CW$.

COROLLARY 2.1. *Let $w_m(t) = (1-t)^{\alpha+m}(1+t)^{\beta+m}$ $(\alpha,\beta > -1)$ and $P \in \mathcal{P}_n$ such that $\|P\|_{w_m} \leqq 1$. Then we have*

$$\left\| \left(\sqrt{1-t^2}/w_m\right)\left(w_m P^{(m)}\right)' \right\|_{w_m} \leqq \sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}\beta_{n,m}}, \tag{10}$$

where

$$\beta_{n,m} = (n-m)(n-m+\alpha+\beta+1)+\alpha+\beta+2m. \tag{11}$$

The supremum is attained if and only if $P(t) = \gamma P_n^{(\alpha,\beta)}(t)$, where γ is an arbitrary real constant.

REMARK 3.1. Daugavet and Rafal'son [3] and Konjagin [5] considered extremal problems of the form

$$\left\| P^{(m)} \right\|_{p,\mu} \leqq A_{n,m}(r,\mu;p,\nu)\,\|P\|_{r,\mu} \qquad (P \in \mathcal{P}_n), \tag{12}$$

where

$$\|f\|_{r,\nu} = \begin{cases} \left(\displaystyle\int_{-1}^{1} \left| f(t)(1-t^2)^{\mu}\right|^r dt\right)^{1/r}, & 0 \leqq r < +\infty, \\ \operatorname*{ess\,sup}_{-1\leqq t\leqq 1} \left|f(t)\right|(1-t^2)^{\mu}, & r = +\infty. \end{cases}$$

The case when $p = r \geqq 1$, $\mu = \nu = 0$, and $m = 1$, was considered by Hille, Szegő, and Tamarkin [6]. The exact constant $A_{n,m}(r,\mu;p,\nu)$ is known in a few cases, for example, $A_{n,1}(\infty,0;\infty,0) = n^2$ is the best constant in Markov's inequality [7], and $A_{n,1}(\infty,0;\infty,1/2) = n$ is the best constant in Bernstein's inequality [1]. Also, we have

$$A_{n,m}(2,\mu;2,\mu+m/2) = \sqrt{\frac{n!\Gamma(n+4\mu+m+1)}{(n-m)!\Gamma(n+4\mu+1)}}.$$

The last case, in fact, is the result of Lemma 2.1 with the Gegenbauer weight $(\alpha = \beta = 2\mu)$.

REMARK 3.2. In the case $\alpha = \beta = 1$, $m = 0$, we have the following extension to L^2 of the Rahman inequality (2) in L^∞:

$$\int_{-1}^{1} \left[(1-t^2)P(t)\right]'^2\,dt \leqq (n+1)(n+2)\int_{-1}^{1}(1-t^2)P(t)^2\,dt,$$

or in an equivalent form: If $Q \in \mathcal{P}_{n+2}$ is such that $Q(\pm 1) = 0$, then

$$\text{(13)} \qquad \int_{-1}^{1} Q'(t)^2\,dt \leqq (n+1)(n+2)\int_{-1}^{1}\frac{Q^2(t)}{1-t^2}\,dt,$$

with equality if and only if $Q = c(1-t^2)P^{(1,1)}$ $(c \in \mathbf{R})$.

Inequality (13) can be represented in the form

$$\|Q'\|_{2,0} \leqq A_{n,1}(2,-1;2,0)\,\|Q\|_{2,-1/2}\,.$$

This formula extends (12) to the case when the weight function has a non-integrable singularity.

In the generalized Laguerre case, Theorem 2.1 reduces to:

COROLLARY 2.2. *Let $w_m(t) = t^{s+m}e^{-t}$ $(s > -1)$ on $(0,+\infty)$. Then for every $P \in \mathcal{P}_n$ such that $\|P\|_{w_m} \leqq 1$, we have*

$$\text{(14)} \qquad \left\|\left(\sqrt{A}/w_m\right)\left(w_m P^{(m)}\right)'\right\|_{w_m} \leqq \sqrt{\frac{n!}{(n-m)!}(n-m-1)}\;\|P\|_w\,,$$

with equality if and only if $P(t) = cL_n^s(t)$.

For the normal weight function $w(t) = e^{-t^2}$ associated with the Hermite polynomials, we get

COROLLARY 2.3. *Let $w_m(t) = e^{-t^2}$ on $(-\infty, +\infty)$. Then for every $P \in \mathcal{P}_n$ such that $\|P\|_{w_m} \leqq 1$, we have*

$$\left\|(1/w_m)\left(w_m P^{(m)}\right)'\right\|_{w_m} \leqq 2^{(m+1)/2}\sqrt{n!/(n-m-1)!} \tag{15}$$

with equality if and only if $P(t) = cH_n(t)$.

3. Proof of the theorem

We prove in this section two lemmas which will be needed for the proof of the main result.

The starting point of our investigations is the differential equation

$$\frac{d}{dt}\left(A(t)w(t)\frac{dy}{dt}\right) + \lambda_n w(t)y = 0, \tag{16}$$

satisfied by the classical orthogonal polynomials, where the spectral parameter λ_n is given by

$$\lambda_n = \begin{cases} 2n & \text{(Hermite case)}, \\ n & \text{(generalized Laguerre case)}, \\ n(n+\alpha+\beta+1) & \text{(Jacobi case)}, \end{cases}$$

The solution of (16) has the remarkable property that derivatives of these solutions of any order m also satisfy an equation of this type:

$$\frac{d}{dt}\left(A(t)w_m(t)\frac{dy}{dt}\right) + \lambda_{n,m}w_m(t)y = 0, \tag{17}$$

where

$$t \mapsto w_m(t) = A(t)^m w(t), \tag{18}$$

and

$$\lambda_{n,m} = -(n-m)\left(\frac{1}{2}(n+m-1)A''(0) + B'(0)\right). \tag{19}$$

LEMMA 3.1. *For all $P \in \mathcal{P}_n$ the inequality*

$$\left\|A^{m/2}P^{(m)}\right\|_w \leqq \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}}\,\|P\|_w \tag{20}$$

holds, where

$$\|f\|_w^2 = \int_a^b w(t)f(t)^2\,dt. \tag{21}$$

Equality is attained if and only if P is a constant multiple of the classical polynomial Q_n orthogonal with respect to the weight function $w \in CW$.

PROOF. Suppose that $P \in \mathcal{P}_n$, and let $w \in CW$. Integration by parts gives

$$\left\|\sqrt{A}P'\right\|_w^2 = \int_a^b w(t)A(t)P'(t)^2\,dt = -\int_a^b P(t)\big(w(t)A(t)P'(t)\big)'\,dt.$$

Cauchy–Schwarz inequality yields

$$\left\|\sqrt{A}P'\right\|_w^2 \leqq \|P\|_w \left\|1/\sqrt{w(t)}\big(w(t)A(t)P'(t)\big)'\right\|_w. \tag{22}$$

Equality is achieved if and only if

$$\big(w(t)A(t)P'(t)\big)' = \lambda w(t)P(t) \qquad (\lambda \in \mathbf{R}). \tag{23}$$

This has a polynomial solution if $\lambda = \lambda_n$ where λ_n is defined by (16). Since $\lambda_\nu \leqq \lambda_n$ ($\nu = 0,\dots,n$), from the eigenvalue problem (16) and inequality (22), we can determine the best constant in the extremal problem (20) for $m = 1$. Namely,

$$\left\|\sqrt{A}P'\right\|_w^2 \leqq \sqrt{\lambda_n}\,\|P\|_w. \tag{24}$$

Then the extremal polynomial is the eigenfunction $Q_n(t)$ corresponding to the maximal eigenvalue.

If we use the differential equation (17) instead of (16), we get the inequality

$$\left\|A^{k/2}P^{(k)}\right\|_w \leqq \sqrt{\lambda_{n,k-1}}\left\|A^{(k-1)/2}P^{(k-1)}\right\|_w \qquad (P \in \mathcal{P}_n),$$

with equality if and only if $P(t) = cQ_n(t), c \in \mathbf{R}$. Finally, iterating this inequality for $k = 1,\dots,m$, we finish the proof. □

LEMMA 3.2. *Let P be any real algebraic polynomial of degree n. Then we have*

$$\left\|\big(\sqrt{A}/w_m\big)\big(w_mP^{(m)}\big)'\right\|_{w_m} \leqq \sqrt{\beta_{n,m}}\left\|P^{(m)}\right\|_{w_m} \tag{25}$$

where

$$\beta_{n,m} = \lambda_{n,m} + B'(0) + (k-1)A''(0). \tag{26}$$

Equality is attained if and only if P is a constant multiple of the classical polynomial Q_n orthogonal with respect to the weight function $w \in CW$.

PROOF. Let Q_n be the classical polynomial orthogonal with respect to the weight function $w \in CW$. Let $U(t) = w_m(t)Q_n(t)$, where $w_m(t) = A(t)^m w(t)$. Then a direct calculation gives that $U(t)$ is a particular solution of the differential equation

$$w_m\big(A/w_m U'\big)' + \beta_{n,m}U = 0 \tag{27}$$

where $\beta_{n,m}$ is given in (26).

Similarly to the proof of Lemma 3.1, we can see

$$\Big\|\big(\sqrt{A}/w_m\big)\big(w_m P^{(m)}\big)'\Big\|^2_{w_m} = \int_a^b (A/w_m)\big(w_m P^{(m)}\big)'^2\,dt =$$

$$= -\int_a^b w_m P^{(m)}\Big((A/w_m)\big(w_m P^{(m)}\big)'\Big)',\ dt$$

From this, Cauchy–Schwarz inequality and from the eigenvalue problem (27), inequality (25) follows.

We now turn to proof of our main result.

PROOF OF THEOREM 2.1. The proof of (7) can be obtained immediately from (20) and (25).

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GRADED RADICAL GRADED SEMISIMPLE CLASSES

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1. Introduction

In [7] Stewart has given a characterization of radical semisimple classes of associative rings. He shows that if $\mathcal{C}$ is a proper subclass of all associative rings, then $\mathcal{C}$ is a radical semisimple class if and only if there is a strongly hereditary finite set $\mathcal{C}(\mathcal{F})$ of finite fields such that $R \in \mathcal{C}$ if and only if R is isomorphic to a subdirect sum of fields in $\mathcal{C}(\mathcal{F})$ or equivalently $R \in \mathcal{C}$ if and only if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathcal{C}(\mathcal{F})$. In [4] Fang and Stewart give some examples of graded radical graded semisimple classes and mention that it remains an open question how to characterize such classes. We answer their question in this paper.

Let G be a multiplicative group with identity element e. A *G-graded ring* R is a ring together with a direct sum decomposition $R = \bigoplus_{g\in G} R_g$, where R_g, $g \in G$, is an additive subgroup of R such that $R_g R_h \subseteqq R_{gh}$ for all $g, h \in G$. The abelian subgroup R_g is called *the homogeneous g-component of R*. It is to be noted that the e-component R_e is a subring of R. A subring S of a graded ring R is said to be a *homogeneous subring of* R if $S = \bigoplus_{g\in G} R_g \cap S$. We call an ideal I of R which is a homogeneous subring of R a *homogeneous ideal* of R and write $I \trianglelefteq_h R$. By $I \trianglelefteq_{hl} R$ we mean that I is a homogeneous left ideal of R. If I is a homogeneous ideal of R, then the quotient ring R/I has a natural G-gradation given by $R/I = \bigoplus_{g\in G}(I + R_g)/I$. We denote by $h(R)$ the set of all homogeneous elements of R, so $h(R) = \bigcup_{g\in G} R_g$. Throughout the paper we have considered graded rings, graded by a finite group G of order n. It can be seen easily that some of our results hold even if G is not finite. By a *graded homomorphism f of degree* (h, k) between two graded rings R and S we mean a ring homomorphism $f: R \to S$ such that $f(R_g) \subseteqq S_{hgk}$ for all $g \in G$ and $h, k \in G$. A graded isomorphism is denoted by $\cong$, and a complete direct sum (direct product) by $\sum^*$. The symbols Z, Z_+, $|S|$ respectively denote the set of integers, the set of positive integers, and the cardinality of the set S. For most of the undefined terms in graded rings we refer to [5] and for those in radical theory for graded rings we refer to [4].

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By $[x]$, where $x \in R$, we denote the subring of a graded ring R generated by x. In Section 2 we study a class $\mathcal{D}^g$ of graded rings R, called $\mathcal{D}^g$-rings, for which the subring $[x] = [x]^2$, where x is any homogeneous element of R. We show that R is a $\mathcal{D}^g$-ring if and only if for all $x \in h(R)$, there exists $n(x) > 1$ such that $x = x^{n(x)}$. A graded ring R is said to be a *graded division ring* if every nonzero homogeneous element of R is invertible. It is clear that a graded ring R with identity is a graded division ring if and only if it has no nontrivial homogeneous left (right) ideals. We prove that if R is a $\mathcal{D}^g$-ring, then R is a graded subdirect sum of graded division rings in $\mathcal{D}^g$. Finally we show that $\mathcal{D}^g$ is a graded radical class.

In Section 3 we show that if a graded radical graded semisimple class $\mathcal{K}$ does not consist of all the graded rings, then $\mathcal{K} \subseteqq \mathcal{D}^g$. A class $\mathcal{K}$ of graded rings is called *graded strongly hereditary* if every homogeneous subring of a ring in $\mathcal{K}$ is also in $\mathcal{K}$. In Theorem 3.10 we obtain characterizations of a graded radical graded semisimple class $\mathcal{K}$ in terms of a graded strongly hereditary finite set of finite graded division rings.

We also consider in this section the class $\mathcal{K}_m$ of graded rings whose homogeneous elements satisfy the relation $x^m = x$, where m is a positive integer $\geqq 2$, and show that $\mathcal{K}_m$ is a graded radical graded semisimple class.

In Section 4 we give graded versions of some results of Andrunakievic [2] and get another characterization of a graded radical graded semisimple class in terms of a graded special radical and its dual graded radical.

2. $\mathcal{D}^g$-rings

We shall say that a G-graded ring R is a *$\mathcal{D}^g$-ring* if for each $x \in h(R)$ we have $[x] = [x]^2$. Clearly a homogeneous subring of a $\mathcal{D}^g$-ring is a $\mathcal{D}^g$-ring and a graded homomorphic image of a $\mathcal{D}^g$-ring is a $\mathcal{D}^g$-ring. If $\mathcal{D}^g$ denotes the class of all G-graded $\mathcal{D}^g$-rings, then the class $\mathcal{D} = \{R_e \mid R \in \mathcal{D}^g\}$ is a radical class and every ring in $\mathcal{D}$ is commutative (see [7]).

LEMMA 2.1. *Let $R \in \mathcal{D}^g$ and let $0 \neq a \in h(R)$. Then a is not nilpotent, $[a]$ is finite and there are positive integers k and $m > 1$ depending on a such that $ka = 0$ and $a^m = a$.*

PROOF. First, we will show that if $0 \neq a \in h(R)$, then a is not nilpotent. Suppose $a^s = 0$ for some integer $s > 1$. Then $[a] = [a]^2 = \ldots = [a]^s = 0$ and we get $a = 0$, which is a contradiction. Since $[a] = [a]^2$, $a = \sum_{i=2}^{r} k_i a^i$, $k_i \in Z$. Hence $1_a = \sum_{i=2}^{r} k_i a^{i-1}$ is the identity of $[a]$. Moreover, $[1_a] = Z1_a \not\cong Z$, for Z does not satisfy $[x] = [x]^2$ for all $x \in Z$. Therefore, $Z1_a \cong Z/kZ$ and $k1_a = 0$. Hence $ka = k1_a a = 0$. Thus R is a torsion ring, and so $R = \bigoplus_p R_p$, where R_p denotes the p-component of R. Let $0 \neq x \in h(R_p)$. Then $px = 0$, for if the additive order of x is p^m, $m > 1$, then

$px \neq 0$ but $(px)^m = p^m x^m = 0$, which is a contradiction as R has no nonzero homogeneous nilpotent elements. Since $[x] = [x]^2$, we have $x = \sum_{i=2}^r \lambda_i x^i$ for some index r with $0 \leqq \lambda_i < p$, and $\lambda_r \neq 0$. Since $(\lambda_r, p) = 1$, there exist integers u, v such that $1 = u\lambda_r + vp$. Hence $x^r = u\lambda_r x^r + vpx^r = u\lambda_r x^r = ux - u\lambda_2 x^2 - \ldots - u\lambda_{r-1}x^{r-1}$ and for $s \geqq r$, $x^s = \mu_{r-1}x^{r-1} + \ldots + \mu_1 x$, $0 \leqq \mu_i < p$, $1 \leqq i \leqq r-1$. Thus $[x]$ is finite, so the powers $x, x^2, x^3, \ldots$ are not distinct. Suppose $x^s = x^t$ and $l = s - t > 0$ is minimal. Then $x^t(1_x - x^l) = 0$. Let $1_x - x^l = b$. Then $b \in h(R)$ and $x^t b = 0$. If $t > 1$, then $(x^{t-1}bx^{t-1})^2 = 0$. But $x^{t-1}bx^{t-1} \in h(R)$ so $x^{t-1}bx^{t-1} = 0$. Then $(x^{t-1}b)^2 = 0$, and hence $x^{t-1}b = 0$. Continuing in this way, we get $xb = 0$. If $t = 1$, then $xb = 0$. In any case $xb = 0$, whence $x = x^{l+1}$. Let $0 \neq a \in h(R)$. Then $a = x_1 + \ldots + x_k$, $x_i \in R_{p_i}$, $1 \leqq i \leqq k$. Hence $[a] \subseteqq \bigoplus_{i=1}^k [x_i]$ is finite and $a^m = a$ for some $m > 1$ as above.

Corollary 2.2. *A graded ring R is a $\mathcal{D}^g$-ring if and only if for each $a \in h(R)$, there exists an integer $n(a) \geqq 2$ such that $a^{n(a)} = a$.*

It is to be noted that in a $\mathcal{D}^g$-ring if $a = a^{n(a)}$, then we can take $n(a) \geqq 3$, for if $a = a^2$, then also $a = a^3$.

Lemma 2.3. *Let $R \in \mathcal{D}^g$, $a \in h(R)$, and $I \trianglelefteq_{hl} R$. Then $aR = Ra$, $I = RI_e = I_e R$ and $I \trianglelefteq_h R$. In particular, $R = RR_e = R_e R$.*

Proof. If $a = 0$, then clearly $aR = Ra$. Suppose $a \neq 0$. Then there exists an integer $n(a) \geqq 3$ such that $a^{n(a)} = a$. Now $Ra = Ra^{n(a)} = (Ra)a^{n(a)-1} \subseteqq Ra^{n(a)-1} = (Ra^{n(a)-2})a \subseteqq Ra$. Hence $Ra = Ra^{n(a)-1}$. Similarly $aR = a^{n(a)-1}R$. It can be seen easily that $a^{n(a)-1}$ is a nonzero homogeneous idempotent. Let $x = a^{n(a)-1}$ and $y \in h(R)$. Then $(xy - xyx)^2 = 0$. By Lemma 2.1, R has no nonzero nilpotent homogeneous elements. Hence $xy = xyx$. Similarly $yx = xyx$, so $x = a^{n(a)-1}$ is central. Hence $Ra = Ra^{n(a)-1} = a^{n(a)-1}R = aR$. Since $I \trianglelefteq_l R$, $RI_e \subseteqq RI \subseteqq I$. Let $x \in h(I)$. Then there exists an integer $n(x) \geqq 2$ such that $x^{n(x)} = x$. Hence $x \in RI_e$, for $x = xx^{n(x)-1}$ and $x^{n(x)-1} \in R_e \cap I = I_e$. Thus $I = RI_e$. Since $aR = Ra$ for all $a \in h(R)$, it follows that $RI_e = I_e R$ and $I \trianglelefteq_h R$.

Lemma 2.4. *A $\mathcal{D}^g$-ring R is a graded division ring of characteristic p if and only if R_e is a field of characteristic p, where p is a prime.*

Proof. Let R be a graded division ring of characteristic p. Then R_e is a field of characteristic p, for $1 \in R_e$. Conversely, let R_e be a field of characteristic p. Let $0 \neq I \trianglelefteq_{hl} R$. Then $I = RI_e$ by Lemma 2.3 and $0 \neq I_e \trianglelefteq_l R_e$. Since R_e is a field, $I_e = R_e$. Hence $I = RR_e = R$, for R is a $\mathcal{D}^g$-ring. Hence R has no proper homogeneous left (right) ideals. Let 1 denote the identity element of R_e and let $x \in h(R)$. Then $x = x^{n(x)}$ for some positive integer $n(x) > 1$. Since $x^{n(x)-1} \in R_e$, $1x = \left(1x^{n(x)-1}\right)x =$

$= x^{n(x)-1}x = x$. Similarly $x1 = x$, so 1 is the identity of R. Hence R is a graded division ring and the characteristic of R is p.

THEOREM 2.5. *Let R be a finitely generated $\mathcal{D}^g$-ring. Then R_e is finitely generated.*

PROOF. Since R is finitely generated, R is generated also by a finite set X of homogeneous elements. Let $0 \neq x \in h(R)$. If x is a product of elements in X, let $l(x)$ denote the minimum number of homogeneous generators (which need not be distinct) of which x is a product. We call $l(x)$ the *length of* x, and prove the theorem by induction on the lengths of such homogeneous elements. Put $l(0) = 0$. We note that every homogeneous element of R is a finite sum of homogeneous elements which have lengths.

Let S be the finite set of homogeneous elements of R of length $\leqq n$, where $n = |G|$, and let T be the subset of S consisting of elements in R_e. Let U be the finite set of elements u of the form $u = aba^{m(a)-2}$, where $a \in S$, $b \in T$ and $m(a)$ is a positive integer $\geqq 3$ such that $a^{m(a)} = a$. Clearly, if $a \in R_g$, then $a^{m(a)-2} \in R_{g^{-1}}$ and $a^{m(a)-1} \in R_e$. Note that $T \subseteqq U$, for if we take $a = b$, then $u = b$. We claim that R_e is generated by U. Since the assertion is true for elements of length $\leqq n$ in R_e, we apply induction and suppose it is true for all elements x in R_e such that $l(x) \leqq m$, $m \geqq n$. Let $y \in R_e$ with $l(y) = m + 1$, and let $y = x_1x_2\cdots x_{m+1}$, $x_i \in X \cap R_{g_i}$, $1 \leqq i \leqq m+1$. Consider elements $g_1, g_1g_2, \ldots, g_1g_2\cdots g_{n+1}$ in G. They are not distinct, for $|G| = n$, so $g_1g_2\cdots g_r = g_1g_2\cdots g_k$ for some r, k such that $1 \leqq r < k \leqq n+1 \leqq m+1$. Hence $g_{r+1}\cdots g_k = e$. If $k = m+1 = n+1$, then we can write $y = y_1y_2$, where $y_1 = x_1x_2\cdots x_r$ and $y_2 = x_{r+1}\cdots x_{m+1} \in R_e$. Hence $y_1 \in R_e$. Also $1 \leqq l(y_1) \leqq n$, $1 \leqq l(y_2) \leqq n$ so $y_1, y_2 \in U$. If $k \neq m+1$, we can write $y = y_1y_2y_3$ where $y_1 = x_1x_2\cdots x_r$, $y_2 = x_{r+1}\cdots x_k$, $y_3 = x_{k+1}\cdots x_{m+1}$. Also $1 \leqq l(y_1) \leqq n$, $1 \leqq l(y_2) \leqq n$, and $y_2 \in R_e$, $y_1y_3 \in R_e$, $l(y_1y_3) \leqq m$. Since R is a $\mathcal{D}^g$-ring, there is a positive integer $m(y_1) \geqq 3$ such that $y_1^{m(y_1)} = y_1$. Also $y = y_1y_2y_3 = y_1^{m(y_1)}y_2y_3 = y_1y_2y_1^{m(y_1)-1}y_3$, for $y_1^{m(y_1)-1} \in R_e$, $y_2 \in R_e$ and R_e is commutative. Thus $y = y_1y_2y_1^{m(y_1)-2}y_1y_3$. Now $y_1y_2y_1^{m(y_1)-2} \in U$ and since $l(y_1y_3) \leqq m$, y_1y_3 is a finite product of u's by the inductive hypothesis. Thus y is a finite product of u's. The theorem follows.

REMARK 2.6. We note that, in general, if R is a finitely generated graded ring, which is not a $\mathcal{D}^g$-ring, then R_e may not be finitely generated. We illustrate it by an example. Let H be a multiplicative free group on two generators and let K be its commutator subgroup. Then K is of infinite rank (see [6], Theorem 11.11) and $G = H/K$ is free abelian with two generators. Let R be the group ring $Z[H]$. Then $R = \bigoplus_{h\in H} Rh = \bigoplus_{h\in H} Zh$. Define $R_{\bar h} = \bigoplus_{k\in K} R_{kh}$, where $\bar h = Kh \in H/K$. Then R is H/K-graded (see [5], page 1) and $R_{\bar e} = Z[K]$. Moreover, R is a finitely generated H/K-graded ring but its e-component $R_{\bar e}$ is not a finitely generated ring.

In the sequel finite graded division rings play a dominant role. Hence we would like to describe some of their properties here. A finite graded division ring need not be commutative. For example, the group ring $Z_2[S_3]$, where Z_2 is a field of two elements and S_3 denotes the symmetric group on three symbols, is a graded division ring (graded on S_3), but it is not commutative. However, if D is a finite graded division ring, then D_e is a finite field of order p^k, where p is a prime and k a positive integer. For G finite, a G-graded division ring is finite if and only if D_e is finite, for any two nonzero homogeneous components of D are isomorphic as abelian groups. The support of $D = \{g \in G \mid D_g \neq 0\} \subseteqq G$ is a subgroup of G. If two graded division rings D_1, D_2 are graded isomorphic, then $D_{1_e} \cong D_{2_e}$ and $H_1 \cong H_2$, where H_1, H_2 are supports of D_1, D_2 respectively. However, the converse may not be true. For instance, consider the graded division rings D_1, D_2 on the same support $H = \{e, h\}$, where $h^2 = e$, such that $D_i = D_{i_e} \oplus D_{i_h}$, $i = 1, 2$, where $D_{i_e} = \{0, 1, x, x^2\}$ and $D_{i_h} = \{0, y, xy, x^2y\}$. For D_1 we have the relations: $x^3 = 1$, $y^2 = 1$, $xy = yx$, $1 + x = x^2$, $1 + x^2 = x$, $x + x^2 = 1$, $21 = 0$, and for D_2 we have the same relations except that $yx = x^2y$. Then $D_1 \not\cong D_2$, although they have the same support and the same e-component. However, for G finite, there are only a finite number of nonisomorphic finite graded division rings with isomorphic e-components. We note that a graded division ring has no nonzero homogeneous zero divisors, so a homogeneous subring of a finite graded division ring is again a finite graded division ring.

THEOREM 2.7. *A $\mathcal{D}^g$-ring is a graded subdirect sum of graded division rings of prime characteristic belonging to $\mathcal{D}^g$.*

PROOF. Let $R \in \mathcal{D}^g$. Then $R_e \in \mathcal{D}$. Hence, by [7], R_e is a subdirect sum of a family $\{F_\lambda : \lambda \in \Lambda\}$ of algebraic fields F_λ of characteristic p_λ. Hence there exists a family $\{I_\lambda : \lambda \in \Lambda\}$ of ideals of R_e such that $F_\lambda \cong R_e/I_\lambda$ and $\bigcap_{\lambda \in \Lambda} I_\lambda = 0$. By Lemma 2.3, $RI_\lambda \trianglelefteq_h R$, and $\bigcap_{\lambda \in \Lambda} RI_\lambda = R\big(\bigcap_{\lambda \in \Lambda} I_\lambda\big) = 0$, for $\big(\bigcap_{\lambda \in \Lambda} RI_\lambda\big)_e = \bigcap_{\lambda \in \Lambda} I_\lambda$. Hence R is a graded subdirect sum of graded rings R/RI_λ, $\lambda \in \Lambda$. Since $(R/RI_\lambda)_e \cong R_e/I_\lambda \cong F_\lambda$, and $R/RI_\lambda \in \mathcal{D}^g$, R/RI_λ is a graded division ring of characteristic p_λ by Lemma 2.4. The theorem follows.

THEOREM 2.8. *A $\mathcal{D}^g$-ring R is a finite direct sum of finite graded division rings if and only if R_e is a finite direct sum of finite fields.*

PROOF. Let R be a finite direct sum of finite graded division rings D_i, where $1 \leqq i \leqq k$. Then R_e is a finite direct sum of finite fields D_{i_e}, $1 \leqq i \leqq k$. Conversely, let R_e be a finite direct sum of finite fields F_i, $1 \leqq i \leqq k$. Since $R \in \mathcal{D}^g$, by Lemma 2.3, $R = RR_e = R\big(\bigoplus_{i=1}^k F_i\big) = \sum_{i=1}^k RF_i$. By Lemma 2.4, RF_i, $1 \leqq i \leqq k$, is a graded division ring and so is finite as F_i is finite. It remains to shows that $\sum_{i=1}^k RF_i = \bigoplus_{i=1}^k RF_i$. Now, by Lemma 2.3, $RF_i \trianglelefteq_h R$, $\sum_{j \neq i} RF_j \trianglelefteq_h R$, so $I = RF_i \cap \big(\sum_{j \neq i} RF_j\big)$ is a homogeneous ideal of R, and $I_e = F_i \cap \sum_{j \neq i} F_j = 0$. By Lemma 2.3, $I = RI_e = 0$. Hence the sum of RF_i is direct and $R = RR_e = \bigoplus_{i=1}^{k} RF_i$. This completes the proof of the theorem.

COROLLARY 2.9. *A finitely generated $\mathcal{D}^g$-ring is a finite direct sum of finite graded division rings, and so is finite.*

PROOF. By Theorem 2.5, R_e is a finitely generated $\mathcal{D}$-ring. Hence, by [7], R_e is a finite direct sum of finite fields. The result follows from Theorem 2.8.

LEMMA 2.10. *Let R be a graded ring without nonzero homogeneous nilpotent elements. If R_e is a $\mathcal{D}$-ring, then R is a $\mathcal{D}^g$-ring.*

PROOF. Let R_e be a $\mathcal{D}$-ring and let $0 \neq x \in R_g$, $g \neq e$. Then $g^n = e$ and $0 \neq x^n \in R_e$, for R has no nonzero homogeneous nilpotent elements. Since R_e is a $\mathcal{D}$-ring, there exists an integer $m > 1$ such that $(x^n)^m = x^n$. Then $x^{n-1}(x^{nm-n+1} - x) = 0$. Let $b = x^{nm-n+1} - x$. Then $b \in h(R)$ and $x^{n-1}b = 0$. If $n = 2$, we get $xb = 0$. If $n > 2$, we have $(x^{n-2}bx^{n-2})^2 = 0$ and since R has no nonzero homogeneous nilpotent elements, $x^{n-2}bx^{n-2} = 0$. Thus $(x^{n-2}b)^2 = 0$ and so $x^{n-2}b = 0$. Continuing in this way, we get $xb = 0$. In any case $x(x^{nm-n+1} - x) = 0$. Consequently, $(x^{nm-n+1} - x)^2 = 0$. Hence $x^{nm-n+1} = x$, proving that $R \in \mathcal{D}^g$.

COROLLARY 2.11. *Let R be a graded ring without nonzero homogeneous nilpotent elements. If $a^m = a$ for all $a \in R_e$, then $x^{n(m-1)+1} = x$ for all $x \in h(R)$.*

COROLLARY 2.12. *Let R be a finite graded division ring such that $|R_e| = m$. Then $x^{n(m-1)+1} = x$ for all $x \in h(R)$.*

LEMMA 2.13. *Let $\mathcal{F} = \{D_i : 1 \leqq i \leqq k\}$ be a finite set of finite graded division rings. Then we have the following results.*

(i) *There exists an integer N such that $x^N = x$ for all $x \in h(D_i)$, $1 \leqq i \leqq k$.*

(ii) *Let R be a graded subdirect sum of rings from $\mathcal{F}$. Then $x^N = x$ for all $x \in h(R)$. The same holds for a homogeneous subring and a graded homomorphic image of R.*

(iii) *Let D be a graded division ring which is a graded homomorphic image of a graded subdirect sum R of rings from $\mathcal{F}$. Then D is finite.*

(iv) *A graded division ring D is finite if $x^N = x$ for all $x \in h(D)$, where $N > 1$ is a fixed integer.*

PROOF. (i) By Corollary 2.12, there exist integers $n_i > 1$ such that $x^{n_i} = x$ for all $x \in h(D_i)$, $1 \leqq i \leqq k$. Let $N = \prod_{i=1}^{k}(n_i - 1) + 1$. Then $x^N = x$ for all $x \in h(D_i)$, $1 \leqq i \leqq k$.

(ii) Let $x = \langle x_j \rangle \in h(R)$. Then each x_j is a homogeneous element of some D_i, so $x_j^N = x_j$. Hence $x^N = \langle x_j^N \rangle = \langle x_j \rangle = x$.

(iii) Let p_i be the characteristic of D_i. Let $x \in h(R)$. Then $x = \langle x_j \rangle$, where x_j is a homogeneous element of some D_i. Let $q = p_1p_2 \cdots p_k$. Then $qx = 0$. Hence $qx = 0$ for all $x \in D$, so D is of finite characteristic. Also, since $x^N = x$ for all $x \in h(R)$, we have $x^N = x$ for all $x \in h(D)$. Hence D is a $\mathcal{D}^g$-ring, and D_e is a $\mathcal{D}$-ring. Hence D_e is a field of finite characteristic such that each element of D_e is algebraic over its prime subfield, satisfying $x^N = x$, so D_e is finite. Hence D is finite.

(iv) D is a $\mathcal{D}^g$-ring, so it is of prime characteristic by Lemma 2.4. Hence D is finite as in (iii).

LEMMA 2.14. *A graded ring R is a $\mathcal{D}^g$-ring if and only if every finitely generated homogeneous subring of R is a finite direct sum of finite graded division rings.*

PROOF. Let S be a finitely generated homogeneous subring of a $\mathcal{D}^g$-ring R. Then S is also a $\mathcal{D}^g$-ring, so by Corollary 2.9, S is a finite direct sum of finite graded division rings. Conversely, if every finitely generated homogeneous subring of R is a finite direct sum of finite graded division rings, then so is $[x]$, where $0 \neq x \in h(R)$. Hence, by Lemma 2.13(ii), there exists an integer $n(x) \geqq 2$ such that $x^{n(x)} = x$. Thus $R \in \mathcal{D}^g$.

We recall that the class $\mathcal{D} = \{R_e \mid R \in \mathcal{D}^g\}$ is a radical class. We now show that, in fact, the class $\mathcal{D}^g$ is a graded radical class.

THEOREM 2.15. *The class $\mathcal{D}^g$ forms a graded strongly hereditary graded radical class.*

PROOF. Clearly, $\mathcal{D}^g$ is closed under graded homomorphisms. Let $I_1 \subseteqq \subseteqq I_2 \subseteqq \ldots \subseteqq I_\lambda \subseteqq \ldots$, $\lambda \in \Lambda$, be an ascending chain of homogeneous $\mathcal{D}^g$-ideals in a graded ring R. Then $I = \bigcup_{\lambda\in\Lambda} I_\lambda$ is also a homogeneous $\mathcal{D}^g$-ideal. It remains to verify the graded extension property. Let R be a graded ring and let $J \trianglelefteq_h R$ such that both J and R/J are in $\mathcal{D}^g$. Let $0 \neq x \in R_g$. Since $R/J \in \mathcal{D}^g$, $x + J = (x+J)^m$ for some positive integer $m \geqq 1$. Hence $x - x^m \in J$. If $g \neq g^m$, then $x \in J_g$, and so $[x] = [x]^2$. If $g = g^m$, then $x - x^m \in J_g$, so $[x - x^m] = [x - x^m]^2$. Hence $x - x^m = \sum_{i=2}^k n_i(x - x^m)^i$ for some positive integer $k \geqq 2$, $n_i \in Z$, so again $[x] = [x]^2$. Hence R is a $\mathcal{D}^g$-ring. Since a homogeneous subring of a $\mathcal{D}^g$-ring is a $\mathcal{D}^g$-ring, the class $\mathcal{D}^g$ is graded strongly hereditary.

3. Graded radical graded semisimple classes

A graded radical α is called *graded strict* if every homogeneous α-subring of a graded ring R is contained in $\alpha(R)$.

THEOREM 3.1. *If a class $\mathcal{K}$ of graded rings is a graded semisimple class, closed under graded homomorphisms, then $\mathcal{K}$ is also a graded radical class and $\mathcal{K}$ is graded strongly hereditary.*

PROOF. Let $I_1 \subseteqq I_2 \subseteqq \ldots \subseteqq I_\lambda \subseteqq \ldots$, $\lambda \in \Lambda$, be an ascending chain of homogeneous ideals of a graded ring R such that each $I_\lambda \in \mathcal{K}$. We shall show that $I = \bigcup_{\lambda \in \Lambda} I_\lambda \in \mathcal{K}$. Let J be the subset of the complete direct sum $\sum^* I_\lambda$ defined by

$$J = \left\{ \langle x_1, x_2, \ldots, x_k, x_{k+1}, \ldots \rangle : x_i \in I_i,\ x_i = x_k,\ i \geqq k, \text{ some } k \in Z_+ \right\}.$$

Then J is a graded subdirect sum of $\{I_\lambda\}$ so $J \in \mathcal{K}$. Define $\theta: J \to I$ by setting $\theta(x) = x_k \in I_k \subseteqq I$, where $x = \langle x_1, x_2, \ldots, x_k, x_k, \ldots \rangle$. Then θ is a graded epimorphism. Hence $I \in \mathcal{K}$. Therefore $\mathcal{K}$ is a graded radical class.

Let S be a homogeneous subring of $R \in \mathcal{K}$. Let $R_i = R$, $i \in Z_+$, and consider $\sum^*_{i \in Z_+} R_i$. Let $D_S = \left\{ \langle x, x, \ldots \rangle : x \in S \right\}$. Then D_S is a homogeneous subring of $\sum^*_{i \in Z_+} R_i$ and $D_S \cong S$. Since $\bigoplus_{i \in Z_+} R_i$ is a graded subdirect sum of $\{R_i\}$, so is $\bigoplus_{i \in Z_+} R_i + D_S$. Hence $\bigoplus_{i \in Z_+} R_i + D_S \in \mathcal{K}$, and $S \cong D_S \cong \left(\bigoplus_{i \in Z_+} R_i + D_S \right) / \bigoplus_{i \in Z_+} R_i \in \mathcal{K}$. Therefore $\mathcal{K}$ is graded strongly hereditary.

COROLLARY 3.2. *A graded semisimple class is a graded radical graded semisimple class if and only if it is graded homomorphically closed, and then it is graded strongly hereditary.*

As in the ungraded case (see [8]) one can show that

PROPOSITION 3.3. *A graded radical is graded strict if and only if its graded semisimple class is graded strongly hereditary.*

COROLLARY 3.4. *Let $\mathcal{K}$ be a graded radical graded semisimple class. Then its upper graded radical is graded strict.*

PROOF. By Corollary 3.2 and Proposition 3.3 the proof follows.

Let $\{X_g : g \in G\}$ be a family of sets. We call $X = \bigcup_{g \in G} X_g$ a *G-graded set.* Let X be a G-graded set such that the X_g's are mutually disjoint. Let $R = Z[X]$ be the free ring generated by X. An element of R is a finite sum of elements of the form $n x_{g_1} x_{g_2} \cdots x_{g_r}$, where $n \in Z$ and $x_{g_i} \in X_{g_i}$. We say that the element $n x_{g_1} x_{g_2} \cdots x_{g_r}$ is of *degree* $k = g_1 g_2 \cdots g_r \in G$. Also a finite sum of elements of degree k is defined to be of degree k. Then R_k, the set of all elements of R of degree k, is an additive subgroup of R and $R = \bigoplus_{k \in G} R_k$ with $R_k R_{k'} \subseteqq R_{kk'}$. Thus R becomes a graded ring over G. We say that R is a *graded free ring* with the set X as its set of generators. Any graded ring is a graded homomorphic image of a graded free ring. By a theorem of Amitsur (see 1, Corollary 3) the free ring $Z[X]$ is a subdirect sum of full matrix rings of finite order over Z. Let $M_n(R)$ denote the ring of $n \times n$ matrices over a ring R. If R is a graded ring, then $M_n(R)$ is a graded ring with gradation given by $M_n(R) = \bigoplus_{g \in G} M_n(R)_g$ where $M_n(R)_g = M_n(R_g)$. Then the group ring $M_n(Z)[G]$ is graded isomorphic to $M_n\left(Z[G]\right)$. We thus get the following graded version of Amitsur's theorem.

THEOREM 3.5. *A graded free ring $R = Z[X]$ is graded isomorphic to a homogeneous subring of a complete direct sum of full matrix rings of finite order over $Z[G]$.*

PROOF. Let $R = \bigoplus_{k\in G} R_k$ be a graded free ring and $x \in R$. Then $x = \sum_{k\in G} x_k$ uniquely, where $x_k \in R_k$. By Amitsur's theorem, R is a subdirect sum of a family of rings $\{R_\lambda : \lambda \in \Lambda\}$, where $R_\lambda = M_n(Z)$ for some $n \in Z_+$ depending on λ. Hence we can write $x_k = \langle r_\lambda \rangle$, where $r_\lambda \in R_\lambda$. We define a mapping $\theta: R \to \sum_{\lambda\in\Lambda} R_\lambda[G]$ by setting $\theta(x_k) = \langle r_\lambda k \rangle$ and $\theta(x) = \sum_{k\in G} \theta(x_k)$. Then θ is a graded monomorphism. But $R_\lambda[G] = M_n(Z)[G] \cong M_n(Z[G])$, whence the theorem follows.

THEOREM 3.6. *Let $\mathcal{K}$ be a graded radical graded semisimple class. If $\mathcal{K}$ contains a graded ring R in which there is an $x \in h(R)$ such that $[x] \neq [x]^2$, then $\mathcal{K}$ contains all the graded nilpotent rings.*

PROOF. By Theorem 3.1, $[x] \in \mathcal{K}$. Hence also $0 \neq S = [x]/[x]^2 \in \mathcal{K}$. Let Z^0 be the zero ring on Z, considered trivially graded. Then S is a graded homomorphic image of Z^0 and also that of any nonzero homogeneous ideal of Z^0. It follows that $Z^0 \in \mathcal{K}$ for $S \in \mathcal{K}$. Hence $\mathcal{K}$ contains all the graded nilpotent rings (see [4]).

COROLLARY 3.7. *If $\mathcal{K}$ contains a nonzero graded nilpotent ring, then it contains all the graded nilpotent rings.*

PROOF. Let $R \in \mathcal{K}$ such that $R^k = 0$, $k \geqq 2$, but $S = R^{k-1} \neq 0$. Then $S \in \mathcal{K}$ and $S^2 = 0$. Let $0 \neq x \in h(S)$. Then $0 = [x]^2 \neq [x]$, so the result follows from Theorem 3.6.

We now give a graded version of a theorem of Armendariz (see [3], Theorem 4.4).

THEOREM 3.8. *Let $\mathcal{K}$ be a graded radical graded semisimple class. If $\mathcal{K}$ contains a nilpotent graded ring, then $\mathcal{K}$ consists of all the graded rings.*

PROOF. Let $\mathcal{K}$ contain a graded nilpotent ring. Then it contains all the graded nilpotent rings by Corollary 3.7. Let us consider $M_n(Z[G])$. Now the graded ring $(2)[G]/(2^k)[G]$, $k \geqq 1$, is nilpotent, so also is $M_n((2)[G]/(2^k)[G])$ and therefore it belongs to $\mathcal{K}$. But

$$M_n((2)[G]/(2^k)[G]) \cong M_n((2)[G])/M_n((2^k)[G]),$$

so the right hand side belongs to $\mathcal{K}$. Now

$$\bigcap_{k\in Z_+} M_n((2^k)[G]) = M_n\Big(\bigcap_{k\in Z_+} (2^k)[G]\Big) = M_n(0) = 0.$$

Hence $M_n\big((2)[G]\big)$ is a graded subdirect sum of graded rings in $\mathcal{K}$, and so is itself in $\mathcal{K}$. Let $p \geqq 3$ be a prime. Then $(2)/(2p) = (2)/(2) \cap (p) \cong \big((2) + (p)\big)/(p) = Z/(p)$. Hence

$$M_n\big(Z[G]\big)/M_n\big((p)[G]\big) \cong M_n\big(Z[G]/(p)[G]\big) \cong M_n\big(Z/(p)[G]\big) \cong$$
$$\cong M_n\big((2)/(2p)[G]\big) \cong M_n\big((2)[G]/(2p)[G]\big) \cong$$
$$\cong M_n\big((2)[G]\big)/M_n\big((2p)[G]\big) \in \mathcal{K}.$$

But $\bigcap_p M_n\big((p)[G]\big) = M_n\big(\bigcap_p (p)[G]\big) = M_n(0) = 0$. Hence $M_n\big(Z[G]\big)$, being a graded subdirect sum of graded rings in $\mathcal{K}$, is also in $\mathcal{K}$. Since G is finite, $\sum^* M_n\big(Z[G]\big)$ is a graded subdirect sum of rings $M_n\big(Z[G]\big)$ and so it belongs to $\mathcal{K}$. Hence by Theorem 3.1, all its homogeneous subrings are in $\mathcal{K}$. By Theorem 3.5, every graded free ring belongs to $\mathcal{K}$. But every graded ring is a graded homomorphic image of a graded free ring. Hence every graded ring is in $\mathcal{K}$.

THEOREM 3.9. *Let $\mathcal{K}$ be a graded strongly hereditary finite set of finite graded division rings. Then a graded ring R is graded isomorphic to a graded subdirect sum of graded division rings in $\mathcal{K}$ if and only if every finitely generated homogeneous subring of R is a finite direct sum of graded division rings in $\mathcal{K}$.*

PROOF. By Lemma 2.13(i), there exists an integer N such that $x^N = x$ for all $x \in h(D_i)$ for all $D_i \in \mathcal{K}$. Let R be graded isomorphic to a graded subdirect sum of graded division rings $D_\lambda \in \mathcal{K}$. Then there exists a family $\{I_\lambda \colon \lambda \in \Lambda\}$ of maximal homogeneous ideals of R such that $\bigcap_{\lambda \in \Lambda} I_\lambda = 0$ and $D_\lambda \cong R/I_\lambda$. Let S be a finitely generated homogeneous subring of R. By Lemma 2.13(ii), S is a $\mathcal{D}^g$-ring, so by Corollary 2.9, $S = \bigoplus_{i=1}^k D'_i$ where D'_i, $1 \leqq i \leqq k$, is a graded division ring. Consider D'_i as a homogeneous subring of R. Since $\bigcap_{\lambda \in \Lambda} I_\lambda = 0$, D'_i is not contained in all I_λ, so for some $\mu \in \Lambda$, $D'_i \nsubseteqq I_\mu$. Hence $I_\mu \cap D'_i \neq D'_i$, so $I_\mu \cap D'_i = 0$. Hence $D'_i \cong (D'_i + I_\mu)/I_\mu \subseteqq R/I_\mu \cong D_\mu$, so $D'_i \in \mathcal{K}$, for $\mathcal{K}$ is graded strongly hereditary. Hence S is graded isomorphic to a finite direct sum of graded division rings in $\mathcal{K}$.

Conversely, let every finitely generated homogeneous subring of R be graded isomorphic to a finite direct sum of graded division rings in $\mathcal{K}$. Then for every $x \in h(R)$, $[x]$ is a finitely generated homogeneous subring of R, so by Lemma 2.13(ii), there exists an integer $N > 1$ such that $x^N = x$ for all $x \in h(R)$. Hence $R \in \mathcal{D}^g$ and by Theorem 2.7, R is a graded subdirect sum of graded division rings D_α. Thus there exist homogeneous ideals I_α such that $\bigcap I_\alpha = 0$ and $D_\alpha \cong R/I_\alpha$. Each D_α satisfies the relation $x^N = x$ for all $x \in h(D_\alpha)$. Hence by Lemma 2.13(iv), D_α is finite. Therefore D_α is a graded homomorphic image of a finitely generated homogeneous subring S_α of R. By assumption, $S_\alpha \cong \bigoplus_{k=1}^t D_k$, $D_k \in \mathcal{K}$, and so D_α is graded isomorphic to one of the graded division rings in $\{D_k : 1 \leqq k \leqq t\}$. Hence R is a graded subdirect sum of graded division rings D_α, $D_\alpha \in \mathcal{K}$.

We are now able to answer the question of Fang and Stewart concerning graded radical graded semisimple classes mentioned in the introduction in the following theorem.

THEOREM 3.10. *Let $\mathcal{K}$ be a proper subclass of all G-graded rings. Then the following are equivalent.*

(i) *$\mathcal{K}$ is a graded radical graded semisimple class.*

(ii) *There is a graded strongly hereditary finite set $\mathcal{F}$ of finite graded division rings such that a graded ring $R \in \mathcal{K}$ if and only if R is graded isomorphic to a graded subdirect sum of graded division rings in $\mathcal{F}$.*

(iii) *There is a graded strongly hereditary finite set $\mathcal{F}$ of finite graded division rings such that a graded ring $R \in \mathcal{K}$ if and only if every finitely generated homogeneous subring of R is graded isomorphic to a finite direct sum of graded division rings in $\mathcal{F}$.*

PROOF. (ii) and (iii) are equivalent by Theorem 3.9.

(i) *implies* (ii). Let $\mathcal{K}$ be a graded radical graded semisimple class. Then $\mathcal{K}_e = \{R_e \mid R \in \mathcal{K}\}$ is a proper radical semisimple class. By Theorems 3.6 and 3.8, $\mathcal{K} \subseteqq \mathcal{D}^g$. Hence each $R \in \mathcal{K}$ is a graded subdirect sum of graded division rings in $\mathcal{K}$ (see Theorem 2.7). Let $\{D_i : i \in I\}$ be the class of all graded division rings in $\mathcal{K}$. Then $\{D_{i_e} : i \in I\}$ is the class of all fields in $\mathcal{K}_e$. Since $\mathcal{K}_e$ is a radical semisimple class, $\{D_{i_e}\}$ is a strongly hereditary finite set of finite fields. Since G is finite, there are only a finite number of graded division rings in $\{D_i\}$ and they are all finite. Thus $\{D_i\}$ is a graded strongly hereditary finite set of finite graded division rings, for $\mathcal{K}$ is a graded strongly hereditary class of rings. Thus if $R \in \mathcal{K}$, then R is a graded subdirect sum of rings in $\{D_i\}$. Conversely, any graded ring graded isomorphic to a graded subdirect sum of rings in $\{D_i\}$ is in $\mathcal{K}$, for $\mathcal{K}$ is a graded semisimple class. Thus $\mathcal{K}$ satisfies (ii).

(ii) *implies* (i). Assume that $\mathcal{K}$ satisfies (ii). It can be easily shown that $\mathcal{F}$ is a graded special class, so $\mathcal{K}$ is a graded semisimple class. We will show that $\mathcal{K}$ is graded homomorphically closed. Let $R \in \mathcal{K}$ and let R' be a graded homomorphic image of R. Since $R \in \mathcal{K}$, R is a graded subdirect sum of graded division rings in $\mathcal{F}$. By Lemma 2.13(ii), there exists an integer $N > 1$ such that $x^N = x$ for all $x \in h(R)$. Now R' is a graded homomorphic image of R, so it satisfies the same relation. Hence $R' \in \mathcal{D}^g$. By Theorem 2.7, R' is a graded subdirect sum of graded division rings, say $\{D'_m\}$. Each D'_m is a graded homomorphic image of R', so also of R. By Lemma 2.13(iii), D'_m is finite. Hence D'_m is a graded homomorphic image of a finitely generated homogeneous subring S of R. By Theorem 3.9, S is a finite direct sum of graded division rings in $\mathcal{F}$, say D_k, $1 \leqq k \leqq m$. Hence $D'_m \cong D_k$ for some k, and so $D'_m \in \mathcal{F}$. Therefore, it follows that $R' \in \mathcal{K}$. By Theorem 3.1, $\mathcal{K}$ is a graded radical graded semisimple class.

We now introduce an important class of graded rings. Let $\mathcal{K}_m$ be the class of all graded rings R for which $x^m = x$, where m is a fixed positive integer > 1 and x is any homogeneous element of R. We have then the following theorem.

THEOREM 3.11. *$\mathcal{K}_m$ is a graded radical graded semisimple class.*

PROOF. Clearly, $\mathcal{K}_m$ is closed under graded homomorphisms and graded subdirect sums. Let $R \in \mathcal{K}_m$. Then $R \in \mathcal{D}^g$, so by Theorem 2.7, R is a graded subdirect sum of graded division rings in $\mathcal{K}_m$. The class $\mathcal{F}$ of all graded division rings in $\mathcal{K}_m$ is a graded special class. Hence the class $\mathcal{K}'$ of all graded subdirect sums of rings from $\mathcal{F}$ is a graded semisimple class. Thus $R \in \mathcal{K}_m$ if and only if $R \in \mathcal{K}'$. Hence $\mathcal{K}_m = \mathcal{K}'$, and $\mathcal{K}_m$ is a graded radical graded semisimple class.

COROLLARY 3.12. *All the graded division rings in $\mathcal{K}_m$ are finite and their number is also finite.*

PROOF. By Theorem 3.10 the proof follows.

4. Dual graded radical

We give here graded versions of some results of Andrunakievic [2]. The proofs of these results, being straightforward adaptations of those given by him, are omitted.

We call a graded ring R *graded subdirectly irreducible* if the graded heart of R (i.e, the intersection of all nonzero homogeneous ideals of R) is not zero. We call a graded ring R *graded strongly α-semisimple* if every graded homomorphic image of R is α-semisimple, where α is a graded radical. A graded radical α' is called *graded complementary to α* if α' is the largest graded radical such that $\alpha(R) \cap \alpha'(R) = 0$ for all graded rings R. If α is graded hereditary, then there always exists a graded radical α' graded complementary to α, where α' is the upper graded radical determined by the class of all graded subdirectly irreducible rings with α-radical graded hearts and the α'-radical rings are just the graded strongly α-semisimple rings.

Two graded radicals α and γ will be called *mutually graded complementary* if γ is graded complementary to α and α is graded complementary to γ. If α and γ are mutually graded complementary, then $\alpha = \gamma'$, $\gamma = \alpha'$, and thus $\alpha = (\alpha')' = \alpha''$ and $\gamma = (\gamma')' = \gamma''$. A graded radical α will be called a *dual graded radical* if there exist α' and α'' such that $\alpha = \alpha''$, that is, if α and α' are mutually graded complementary. We then say that α, α' are *duals of each other or they form a dual pair.* We then have the following theorem.

THEOREM 4.1. *If α is a graded supernilpotent radical, then there exist graded radicals α' and α'' such that α' is graded complementary to α, α'' is graded complementary to α', and α' and α'' form a dual pair. Moreover,*

$\alpha = \alpha''$ if and only if α is the graded special radical determined by the class of graded subdirectly irreducible rings with α-semisimple graded hearts. The α'-radical rings are precisely the graded strongly α-semisimple rings.

We therefore get the following characterization of a graded radical graded semisimple class $\mathcal{K}$.

THEOREM 4.2. *A class $\mathcal{K}$ of graded rings is a graded radical graded semisimple class if and only if $\mathcal{K}$ is the graded semisimple class of a graded special radical α and the graded radical class of α', the dual graded radical of α.*

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A MONTGOMERY–HOOLEY TYPE THEOREM FOR PRIME k-TUPLETS

K. KAWADA (Tsukuba)

§ 1. Introduction and notation

Let a_j, b_j $(j = 0, \dots, k-1)$ be $2k$ integers. If all the numbers $a_j n + b_j$ $(j = 0, \dots, k-1)$ are primes, then we call $(a_0 n + b_0, \dots, a_{k-1} n + b_{k-1})$ a prime k-tuplet. When we choose $a_0 = a_1 = 1$, $b_0 = 0$ and $b_1 = 2$, the prime 2-tuplet is the "prime twins".

As for the distribution of primes in arithmetic progressions, Barban [1], in 1966, considered the sum

$$E_0(x,Q) = \sum_{q \leqq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Bigg(\sum_{\substack{n \leqq x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{x}{\phi(q)} \Bigg)^2,$$

where Λ and ϕ denote the von Mangoldt function and Euler's totient function, respectively. Gallagher [4] showed that, for any $A > 0$,

$$E_0(x,Q) \ll x^2 (\log x)^{-A} \quad \text{provided that} \quad Q \leqq x(\log x)^{-A-1}, \tag{1.1}$$

which is an improvement of Barban [1] and Davenport–Halberstam [3]. These results should be compared with the well-known Bombieri–Vinogradov theorem which states that, for any $A > 0$, there exists $B > 0$ such that

$$\sum_{q \leqq x^{1/2} (\log x)^{-B}} \max_{(a,q)=1} \Bigg| \sum_{\substack{n \leqq x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{x}{\phi(q)} \Bigg| \ll x (\log x)^{-A}.$$

Further, Montgomery [8] obtained an asymptotic formula for $E_0(x,Q)$, and Hooley [5] sharpened Montgomery's formula when $Q < x$.

In our previous paper [6] we treated a Bombieri–Vinogradov type theorem for prime k-tuplets. In this paper we generalize the function $E_0(x,Q)$ for prime k-tuplets, and obtain a result similar to the Montgomery–Hooley's asymptotic formula.

In order to state our result, here we repeat the notation of [6]. For an integer $k \geqq 2$, we take k non-zero integers $a_0, a_1, \ldots, a_{k-1}$ and take an integer b_0 with $(a_0, b_0) = 1$. Let b_j $(1 \leqq j \leqq k-1)$ be arbitrary integers and put

$$\mathbf{b} = (b_0, \ldots, b_{k-1}).$$

To count the number of n in an arithmetic progression for which all $a_j n + b_j$ $(0 \leqq j \leqq k-1)$ are primes and $\leqq x$, we introduce the function

$$\Psi(x; \mathbf{b}, a, q) = \sum_{\substack{n \in N(\mathbf{b}) \\ n \equiv a \pmod q}} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j),$$

where

$$N(\mathbf{b}) = N(x; \mathbf{b}) = \{t;\ 1 \leqq a_j t + b_j \leqq x \text{ for all } 0 \leqq j \leqq k-1\}.$$

On the other hand, for any prime p (in the sequel p always denotes a prime number), let $\rho(p) = \rho(p, \mathbf{b})$ be the number of solutions of the congruence

$$\prod_{j=0}^{k-1} (a_j n + b_j) \equiv 0 \pmod p,$$

and, making use of this number, put

$$\sigma(\mathbf{b}; q) = \begin{cases} \dfrac{1}{q} \prod_{p|q} \left(1 - \dfrac{\rho(p)}{p}\right)^{-1} \prod_{p} \left(1 - \dfrac{\rho(p)}{p}\right) \left(1 - \dfrac{1}{p}\right)^{-k} & \\ \qquad\qquad \text{if } \rho(p) < p \text{ for all } p \text{ and} & \\ \qquad\qquad a_i b_j \neq a_j b_i \text{ for all } 1 \leqq i < j \leqq k-1, & \\ 0 \qquad\qquad \text{otherwise} & \end{cases}$$

and

$$\sigma(\mathbf{b}; a, q) = \begin{cases} \sigma(\mathbf{b}; q) & \text{if } (a_j a + b_j, q) = 1 \text{ for all } 0 \leqq j \leqq k-1, \\ 0 & \text{otherwise.} \end{cases}$$

By a heuristic evidence (see Bateman–Horn [2]), when $\sigma(\mathbf{b}; a, q) > 0$, it is expected that

$$\Psi(x; \mathbf{b}, a, q) \sim \sigma(\mathbf{b}; a, q) \big| N(\mathbf{b}) \big|, \tag{1.2}$$

where $|N(\mathbf{b})|$ denotes the length of the interval $N(\mathbf{b})$. Now we consider the average of dispersion of (1.2). We define the set

$$Z = Z(x) = \{\mathbf{b}; |N(\mathbf{b})| \neq 0\},$$

and evaluate the function

$$E(x,Q) = \sum_{q \leqq Q} \sum_{a=1}^{q} \sum_{\mathbf{b} \in Z} \left| \Psi(x;\mathbf{b},a,q) - \sigma(\mathbf{b};a,q)|N(\mathbf{b})| \right|^2, \tag{1.3}$$

which is a generalization of $E_0(x,Q)$. The purpose of this paper is to obtain a precise asymptotic formula for $E(x,Q)$, according to the methods of Montgomery [8] and Hooley [5]. We need more definitions. Put

$$a_* = \max_{0 \leqq j \leqq k-1} |a_j|, \quad \Omega = \int_0^{1/a_*} \prod_{j=0}^{k-1} (1 - |a_j|u)\, du,$$

and denote by $g(p)$ the number of a_j's such that $p|a_j$.

THEOREM 1. *Let $A > 0$ and $B > 1$ be arbitrary constants. Then there exist $k+3$ numbers $\alpha_0, \alpha_1, \beta_0, \dots, \beta_k$ depending only on $a_0, \dots, a_{k-1}$ which satisfy the following relations.*

(I) *For $Q < x/a_*$, we have*

$$E(x,Q) = \frac{1}{\phi(|a_0|)} x^k Q\big((\log x) - 1\big)^k - x^k Q \sum_{m=0}^{k} \beta_m \left(\log \frac{x}{Qa_*}\right)^m + \tag{1.4}$$
$$+ O\left(x^{k-\frac{1}{k+2}} Q^{\frac{k+3}{k+2}} + x^{k+1}(\log x)^{-A}\right).$$

(II) *For $x/a_* \leqq Q \leqq x^B$, we have*

$$E(x,Q) = \frac{1}{\phi(|a_0|)} x^k Q\big((\log x) - 1\big)^k - \alpha_0 x^{k+1} \left(\log \frac{Qa_*}{x}\right) + \tag{1.5}$$
$$+ \alpha_1 x^{k+1} + O\left(x^k Q (\log x)^{-A}\right).$$

Moreover, $f_1(p)$ being $\left(1-\frac{1}{p}\right)^{-k} \left\{ \left(1-\frac{1}{p}\right)^2 \left(\frac{p-2}{p-1}\right)^{k-g(p)} + \frac{2}{p} - \frac{1}{p^2} \right\}$, α_0 and β_k are given by

$$\alpha_0 = \frac{2}{\phi(|a_0|)} \Omega \prod_p f_1(p), \tag{1.6}$$

and

$$\beta_k = \frac{1}{\phi(|a_0|)\,k!}. \tag{1.7}$$

REMARK. From (1.4), we have a non-trivial bound for $E(x,Q)$, namely

$$E(x,Q) \ll x^{k+1}(\log x)^{-A},$$

provided that $Q \ll x(\log x)^{-A-k}$.

Secondly, we consider the same problem for the distribution of prime k-tuplets in a short interval. For $0 < y \leqq x$, instead of $N(\mathbf{b})$, we define

$$N(x,y;\mathbf{b}) = \{t;\, x-y < a_j t + b_j \leqq x \text{ for all } 0 \leqq j \leqq k-1\},$$
$$Z(x,y) = \left\{\mathbf{b};\ |N(x,y;\mathbf{b})| \neq 0\right\},$$

and evaluate

$$E(x,y;Q) = \sum_{q \leqq Q} \sum_{a=1}^{q} \sum_{\mathbf{b} \in Z(x,y)} \big| \Psi(x;\mathbf{b},a,q) - \Psi(x-y;\mathbf{b},a,q) -$$
$$-\sigma(\mathbf{b};a,q)|N(x,y;\mathbf{b})|\big|^2.$$

Then we can prove asymptotic formulas for $E(x,y;Q)$:

THEOREM 2. *Let $A, B, \alpha_0, \alpha_1, \beta_0, \dots, \beta_k$ be the same as in Theorem 1, and let D be some positive constant depending only on k and A. We take y satisfying*

$$x^{\frac{2}{3}}(\log x)^D < y \leqq x.$$

(I) *For $Q \leqq y/a_*$, we have*

$$E(x,y;Q) = \frac{1}{\phi(|a_0|)} y^k Q\big((\log x) - 1\big)^k + y^k Q \sum_{m=0}^{k} \beta_m \left(\log \frac{y}{Qa_*}\right)^m +$$
$$+ O\left(y^{k-\frac{1}{k+2}} Q^{\frac{k+3}{k+2}} + y^{k+1}(\log x)^{-A}\right).$$

(II) *For $y/a_* \leqq Q \leqq x^B$, we have*

$$E(x,y;Q) = \frac{1}{\phi(|a_0|)} y^k Q\big((\log x) - 1\big)^k - \alpha_0 y^{k+1}\left(\log \frac{Qa_*}{y}\right) +$$
$$+ \alpha_1 y^{k+1} + O\left(y^k Q(\log x)^{-A}\right).$$

We omit the proof of Theorem 2, because it is almost the same as that of Theorem 1.

REMARK. For the case $k=1$, that is the case of primes in arithmetic progressions, it is possible to extend, by the large sieve method, the range of validity of the above formulas to $x^{7/12+\varepsilon} \leqq y \leqq x$ with any $\varepsilon > 0$.

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§ 2. Proof of Theorem 1

We start from the identity

$$E(x,Q) = T + \sum_{q \leqq Q} U_1 - 2\sum_{q \leqq Q} U_2 - \sum_{q \leqq Q} U_3, \tag{2.1}$$

where

$$T = Q \sum_{\mathbf{b} \in Z} \sum_{n \in N(\mathbf{b})} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j)^2,$$

$$U_1 = \sum_{\mathbf{b} \in Z} \sum\sum_{\substack{n,m \in N(\mathbf{b}) \\ n \neq m \\ n \equiv m \pmod q}} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j)\Lambda(a_j m + b_j),$$

$$U_2 = \sum_{a=1}^{q} \sum_{\mathbf{b} \in Z} \sigma(\mathbf{b}; a, q) \big| N(\mathbf{b}) \big| \times \big\{ \Psi(x; \mathbf{b}; a, q) - \sigma(\mathbf{b}; a, q) |N(\mathbf{b})| \big\},$$

$$U_3 = \sum_{a=1}^{q} \sum_{\mathbf{b} \in Z} \sigma(\mathbf{b}; a, q)^2 \big| N(\mathbf{b}) \big|^2.$$

Making use of the prime number theorem for arithmetic progressions, we obtain

$$T = Q \sum_{1 \leqq a_0 n + b_0 \leqq x} \Lambda(a_0 n + b_0)^2 \prod_{j=1}^{k-1} \Bigg(\sum_{\substack{b_j \\ 1 \leqq a_j n + b_j \leqq x}} \Lambda(a_j n + b_j)^2 \Bigg) = \tag{2.2}$$

$$= Q\Bigg(\sum_{\substack{m \leqq x \\ m \equiv b_0 \pmod{a_0}}} \Lambda(m)^2 \Bigg) \Bigg(\sum_{m \leqq x} \Lambda(m)^2 \Bigg)^{k-1} =$$

$$= \frac{1}{\phi(|a_0|)} x^k Q\big((\log x) - 1\big)^k + O\left(x^k Q(\log x)^{-A}\right).$$

Next we estimate $\sum U_2$. Noticing that

$$\sum_{\substack{a=1 \\ \left(\prod_{j=0}^{k-1}(a_j a + b_j), q\right)=1}}^{q} 1 = q \prod_{p|q} \left(1 - \frac{\rho(p)}{p}\right)$$

and that

$$\sum_{\substack{a=1 \\ \left(\prod_{j=0}^{k-1}(a_j a + b_j), q\right)=1}}^{q} \left\{ \Psi(x; \mathbf{b}; a, q) - \sigma(\mathbf{b}; a, q)|N(\mathbf{b})| \right\} =$$

$$= \Psi(x; \mathbf{b}; 1, 1) - \sigma(\mathbf{b}; 1, 1)\big|N(\mathbf{b})| + O\left((\log x)^{k+1}\right),$$

we have

$$\sum_{q \leqq Q} U_2 = \sum_{q \leqq Q} \sum_{\mathbf{b} \in Z} \sigma(\mathbf{b}; q)\big|N(\mathbf{b})\big| \times$$

$$\times \left\{ \Psi(x; \mathbf{b}; 1, 1) - \sigma(\mathbf{b}; 1, 1)\big|N(\mathbf{b})| + O\left((\log x)^{k+1}\right) \right\} \ll$$

$$\ll x(\log x)^2 \sum_{\mathbf{b} \in Z} \big|\Psi(x; \mathbf{b}; 1, 1) - \sigma(\mathbf{b}; 1, 1)\big|N(\mathbf{b})\big|\big| + x^k (\log x)^{k+3}.$$

Then, applying Theorem 1 of [6] with $Q = 1$, we get

$$\sum_{q \leqq Q} U_2 \ll x^{k+1} (\log x)^{-A}. \tag{2.3}$$

As for $\sum U_1$ and $\sum U_3$, we shall prove the following lemmas in later sections. For simplicity we write

$$f_2(p) = \left(1 - \frac{1}{p}\right)^{-k} \left(\frac{1}{p} + \left(1 - \frac{1}{p}\right)\left(\frac{p-2}{p-1}\right)^{k-g(p)}\right),$$

$$f_3(p) = \left(\frac{1}{p} + \left(1 - \frac{1}{p}\right)\left(\frac{p-2}{p-1}\right)^{k-g(p)}\right)^{-1}.$$

Then we have

LEMMA 1.

$$\sum_{q \leqq Q} U_3 = \frac{2}{\phi(|a_0|)} x^{k+1} \Omega \prod_p f_2(p) \sum_{q \leqq Q} \frac{1}{q} \prod_{p|q} f_3(p) + \tag{2.4}$$
$$+ O\left(x^{k+1}(\log x)^{-A}\right),$$

and

$$\sum_{q \leqq Q} U_3 = \frac{2}{\phi(|a_0|)} x^{k+1} \Omega(\log Q) \prod_q f_1(p) + C_1 x^{k+1} + \tag{2.5}$$
$$+ O\left(x^{k+1}(\log x)^{-A} + x^{k+1} Q^{-1} (\log Q)^{k+1}\right),$$

where C_1 *is a constant depending only on* $a_0, \dots, a_{k-1}$.

LEMMA 2. *We have, for* $Q \leqq x/a_*$,

$$\sum_{q \leqq Q} U_1 = \frac{2}{\phi(|a_0|)} x^{k+1} \Omega \prod_p f_2(p) \sum_{q \leqq Q} \frac{1}{q} \prod_{p|q} f_3(p) + \tag{2.6}$$
$$+ O\left(x^k Q \left(\log\left(2\frac{x}{Qa_*}\right)\right)^k + x^{k+1}(\log x)^{-A}\right),$$

and, for $Q \geqq x/a_*$,

$$\sum_{q \leqq Q} U_1 = \frac{2}{\phi(|a_0|)} x^{k+1} \Omega(\log(x/a_*)) \prod_p f_1(p) + C_2 x^{k+1} + \tag{2.7}$$
$$+ O\left(x^{k+1}(\log x)^{-A}\right),$$

where C_2 *is a constant depending only on* $a_0, \dots, a_{k-1}$.

LEMMA 3. *Let* $Q_0 = (\log x)^{-A-k}$. *For* $Q_0 \leqq Q \leqq x/a_*$, *we have*

$$\sum_{Q_0 < q \leqq Q} U_1 = \frac{2}{\phi(|a_0|)} x^{k+1} \Omega \left(\log \frac{Q}{Q_0}\right) \prod_q f_1(p) - \tag{2.8}$$

$$-x^kQ\sum_{m=0}^{k}\beta_m\left(\log\frac{x}{Qa_*}\right)^m+O\left(x^{k-\frac{1}{k+2}}Q^{\frac{k+3}{k+2}}+x^{k+1}(\log x)^{-A}\right),$$

where $\beta_0,\dots,\beta_k$ *are constants depending only on* $a_0,\dots,a_{k-1}$.

Then, for $Q\geqq x/a_*$, (1.6) and (1.8) follow at once from (2.1), (2.2), (2.3), (2.5) and (2.7).

For $Q\leqq x/a_*$, using (2.1), (2.2), (2.3), (2.4) and (2.6), we see that

$$E(x,Q)=$$

$$=\frac{1}{\phi(|a_0|)}x^kQ\left((\log x)-1\right)^k+O\left(x^{k+1}(\log x)^{-A}+x^kQ\left(\log\left(2\frac{x}{Qa_*}\right)\right)^k\right),$$

which proves that (1.5) is true for $Q\leqq Q_0=x(\log x)^{-A-k}$.

Finally, let $Q_0\leqq Q\leqq x/a_*$. Applying (2.4) and (2.6) with $Q=Q_0$, we get

$$\sum_{q\leqq Q_0}U_1-\sum_{q\leqq Q_0}U_3\ll x^{k+1}(\log x)^{-A}.$$

Thus, it follows from (2.1), (2.2) and (2.3) that

$$E(x,Q)=\frac{1}{\phi(|a_0|)}x^kQ\left((\log x)-1\right)^k+ \tag{2.9}$$

$$+\sum_{Q_0<q\leqq Q}U_1-\sum_{Q_0<q\leqq Q}U_3+O\left(x^{k+1}(\log x)^{-A}\right).$$

As for the third term on the right hand side of (2.9), we apply (2.5). We have

(2.10)

$$\sum_{Q_0<q\leqq Q}U_3=\frac{2}{\phi(|a_0|)}x^{k+1}\Omega\left(\log\frac{Q}{Q_0}\right)\prod_p f_1(p)+O\left(x^{k+1}(\log x)^{-A}\right).$$

Then (1.5) follows at once from (2.9), (2.10) and Lemma 3.

Consequently, we proved Theorem 1 on the validity of Lemmas 1, 2 and 3.

§ 3. Proof of Lemma 1

In what follows, we use following notation:

$$e(x) = e^{2\pi i x},$$

$$c_n(m) = \sum_{\substack{h=1\\(h,n)=1}}^{n} e\left(\frac{h}{n}m\right) \quad \text{(the Ramanujan sum)},$$

$\tau_m(n) =$ the number of ways of writing n as a product of m factors (the order of the factors being taken account),

$\tau(n) = \tau_2(n)$ (the divisor function),

$\mu(n) =$ the Möbius function,

$\gamma =$ the Euler constant,

$$R(\mathbf{b}) = \prod_{j=0}^{k-1} a_j \cdot \prod_{0 \leqq i < j \leqq k-1} (a_i b_j - a_j b_i),$$

and for a $(k-1)$-dimensional vector $\mathbf{q} = (q_1, \ldots, q_{k-1}) \in \mathbf{Z}^{k-1}$, we define

$[\mathbf{q}] =$ the least common multiple of all q_j's.

First we derive (2.5) from (2.4). For a square-free natural number d, we define $h(d) = \prod_{p|d} \left(f_3(p) - 1\right)$. Then

$$\sum_{q \leqq Q} \frac{1}{q} \prod_{p|q} f_3(p) = \sum_{q \leqq Q} \frac{1}{q} \sum_{d|q} \mu(d)^2 h(d) = \sum_{d \leqq Q} \frac{\mu(d)^2 h(d)}{d} \sum_{m \leqq Q/d} \frac{1}{m}.$$

Since the last innermost sum equals $\log Q - \log d + \gamma + O(d/Q)$, we have

$$\text{(3.1)} \qquad \sum_{q \leqq Q} \frac{1}{q} \prod_{p|q} f_3(p) = (\log Q + \gamma) \prod_{p} \left(1 - \frac{1}{p} + \frac{1}{p} f_3(p)\right) -$$

$$- \sum_{d=1}^{\infty} \frac{\mu(d)^2 h(d)}{d} (\log d) + O\left(Q^{-1} (\log Q)^{k+1}\right).$$

Now, it is easily seen that (2.4) yields (2.5).

In [6, §6], we observed that, if $R(\mathbf{b}) \neq 0$ and $(a_0a + b_0, q) = 1$ then $\sigma(\mathbf{b}; a, q)$ is a "singular series". For $Q_1 = (\log x)^C$ with $C > 0$, we put

$$S_1 = \sum_{q \leqq Q_1} \frac{\mu(r)^2}{\phi(|a_0|[q,r])} \sum_{[\mathbf{q}]=r} \prod_{j=1}^{k-1} \frac{\mu(q_j)}{\phi(q_j)} \sum_{\mathbf{d}}^{\mathbf{q}}{}^{\#} \prod_{j=1}^{k-1} c_{q_j}(a_j d_j + b_j),$$

where $\mathbf{q} = (q_1, \dots, q_{k-1})$ and $\sum_{\mathbf{d}}^{\mathbf{q}}{}^{\#}$ denotes the summation over all vectors $\mathbf{d} = (d_1, \dots, d_{k-1})$ that satisfy the following four conditions:

1. $1 \leqq d_j \leqq q_j$ for all $1 \leqq j \leqq k-1$,
2. $d_j \equiv a \pmod{(q_j, q)}$ for all $1 \leqq j \leqq k-1$,
3. $d_i \equiv d_j \pmod{(q_i, q_j)}$ for all $1 \leqq i < j \leqq k-1$,
4. $(a_0 d_j + b_0, q_j) = 1$ for all $1 \leqq j \leqq k-1$.

Now we use our results in [6, §6]. By (6.3) of [6], we get an estimate

$$S_1 \ll \frac{\tau(q)}{q}(\log Q_1)^L,$$

where L is a constant depending only on k. If $R(\mathbf{b}) \neq 0$ and $(a_0a + b_0, q) = 1$, then we have

$$\sigma(\mathbf{b}; a, q) = |a_0| S_1 + O\left(\frac{\tau_K(q)}{q}\tau_K\big(R(\mathbf{b})\big)(\log x)^{-C+1}\right),$$

where K is a constant depending only on k. Since the number of $\mathbf{b}$'s with $R(\mathbf{b}) = 0$ is $O(x^{k-2})$, we obtain, for a sufficiently large constant C,

$$\sum_{q \leqq Q} U_3 = |a_0|^2 \sum_{q \leqq Q} \sum_{\substack{a=1 \\ (a_0a+b_0,q)=1}}^{q} \sum_{\mathbf{b} \in Z} S_1^2 \left|N(\mathbf{b})\right|^2 + O\left(x^{k+1}(\log x)^{-A}\right). \tag{3.2}$$

Next, we calculate $\sum_{\mathbf{b}} S_1^2 \left|N(\mathbf{b})\right|^2$. We substitute S_1 by its definition, and $\left|N(\mathbf{b})\right|^2$ by $\displaystyle\int\int_{\substack{1 \leqq a_j t_i + b_j \leqq x \\ \text{for all } 0 \leqq j \leqq k-1 \\ \text{and for } i=1,2}} dt_1\, dt_2$, then the calculation of $\sum_{\mathbf{b}} S_1^2 \left|N(\mathbf{b})\right|^2$ can be reduced to that of the sum

$$\sum_{\substack{b_j \\ 1 \leqq a_j t_i + b_j \leqq x \\ \text{for } i=1,2}} e\left(b_j\left(\frac{h}{q_j}+\frac{h'}{q'_j}\right)\right) \quad \text{with } (h,q_j)=(h',q'_j) \text{ and } q_j, q'_j \leqq Q_1.$$

As far as $\left\|\frac{h}{q_j}+\frac{h'}{q'_j}\right\| \neq 0$, we have an estimate

$$\left\|\frac{h}{q_j}+\frac{h'}{q'_j}\right\| \geqq \frac{1}{q_j q'_j} \geqq Q_1^{-2} = (\log x)^{-2C},$$

and

$$\sum_{\substack{1 \leqq a_j t_i + b_j \leqq x \\ \text{for } i=1,2}} e\left(b_j\left(\frac{h}{q_j}+\frac{h'}{q'_j}\right)\right) \ll (\log x)^{2C}.$$

If $\left\|\frac{h}{q_j}+\frac{h'}{q'_j}\right\| = 0$, then $q_j = q'_j$, $h \equiv h' \pmod{q_j}$ and

$$\sum_{\substack{1 \leqq a_j t_i + b_j \leqq x \\ \text{for } i=1,2}} e\left(b_j\left(\frac{h}{q_j}+\frac{h'}{q'_j}\right)\right) = \max\left\{x - |a_j(t_1-t_2)|, 0\right\} + O(1).$$

Taking account of these results, we have

$$\sum_{\mathbf{b}\in Z} S_1^2 \left| N(\mathbf{b})\right|^2 = S(a,q)\cdot J + O\left(x^k \phi(q)^{-2} (\log x)^{2(k+1)C}\right), \tag{3.3}$$

where

$$J = \iint_{\substack{1 \leqq a_0 t_i + b_0 \leqq x \\ \text{for } i=1,2 \\ a_*|t_1-t_2| \leqq x}} \prod_{j=1}^{k-1} \left(x - |a_j(t_1-t_2)|\right) dt_1\, dt_2,$$

$$S(a,q) = \sum_{r \leqq Q_1} \frac{\mu(r)^2}{\phi\left(|a_0|[q,r]\right)^2} S_0(r),$$

with

$$S_0(r)=\sum_{[\mathbf{q}]=r}\prod_{j=1}^{k-1}\frac{\mu(q_j)^2}{\phi(q_j)^2}\sum_{\mathbf{d}_1}^{\mathbf{q}}{}^{\#}\sum_{\mathbf{d}_2}^{\mathbf{q}}{}^{\#}\prod_{j=1}^{k-1}c_{q_j}\left(a_j\left(d_j^{(1)}-d_j^{(2)}\right)\right).$$

Here we use the notation $\mathbf{d}_i=\left(d_1^{(i)},\dots,d_{k-1}^{(i)}\right)$ for $i=1,2$.

It is easily seen that

$$J=\int\limits_{a_*|t|\leqq x}\prod_{j=1}^{k-1}\left(x-|a_jt|\right)\left(\int\limits_{\substack{1\leqq a_0t_1+b_0\leqq x\\ 1\leqq a_0(t_1-t)+b_0\leqq x}}dt_1\right)dt= \tag{3.4}$$

$$=2|a_0|^{-1}\int\limits_{0\leqq t\leqq x/a_*}\prod_{j=0}^{k-1}\left(x-|a_j|t\right)dt+O(x^k)=$$

$$=2|a_0|^{-1}x^{k+1}\Omega+O(x^k).$$

Simple calculation shows that $S_0(r)$ is a multiplicative function in r. So it is sufficient to calculate $S_0(p)$ only for a prime number p. The condition $[\mathbf{q}]=p$ holds if and only if $q_j=1$ or p for all $1\leqq j\leqq k-1$, and at least one q_j equals p. We denote by M the set of subscripts of q_j's such that $q_j=p$ and by $\#M$ its cardinality. Then we get

$$S_0(p)=\sum_{\substack{M\subset\{1,\dots,k-1\}\\ \#M\geqq 1}}\left(\frac{1}{(p-1)^2}\right)^{\#M}\sum_{\substack{d_1=1\\ d_i\equiv a\ (\mathrm{mod}\,(p,q))\\ (a_0d_i+b_0,p)=1\\ \text{for } i=1,2}}^{p}\sum_{d_2=1}^{p}\prod_{j\in M}c_p\big(a_j(d_1-d_2)\big)=$$

$$=\sum_{d_1}\sum_{d_2}\left(\left\{\sum_{M\subset\{1,\dots,k-1\}}\prod_{j\in M}\left(\frac{c_p\big(a_j(d_1-d_2)\big)}{(p-1)^2}\right)\right\}-1\right)=$$

$$=\sum_{\substack{1\leqq d_1,d_2\leqq p\\ d_i\equiv a\ (\mathrm{mod}\,(p,q))\\ (a_0d_i+b_0,p)=1\\ \text{for } i=1,2}}\sum\left(\prod_{j=1}^{k-1}\left(1+\frac{c_p\big(a_j(d_1-d_2)\big)}{(p-1)^2}\right)-1\right)=$$

$$
= \begin{cases}
\left(1-\frac{1}{p}\right)^{-k+1} - 1 & \text{if } p|q \\
p\left(1-\frac{1}{p}\right)^{-k+1} + & \\
+(p^2-p)\left(1-\frac{1}{(p-1)^2}\right)^{k-g(p)}\left(1-\frac{1}{p-1}\right)^{g(p)-1} - p^2 & \\
 & \text{if } p\nmid q \text{ and } p|a_0 \\
(p-1)\left(1-\frac{1}{p}\right)^{-k+1} + & \\
+\left((p-1)^2-(p-1)\right)\left(1-\frac{1}{(p-1)^2}\right)^{k-1-g(p)}\left(1+\frac{1}{p-1}\right)^{g(p)} - & \\
-(p-1)^2 & \\
 & \text{if } p\nmid qa_0.
\end{cases}
$$

For a square-free r, we have

$$
\phi(\,|a_0|[q,r]) = |a_0|q \prod_{p|a_0 q}\left(1-\frac{1}{p}\right) \prod_{\substack{p|r\\ p|a_0\\ p\nmid q}} p \prod_{\substack{p|r\\ p\nmid a_0 q}} (p-1)
$$

and

$$
\frac{\mu(r)^2 S_0(r)}{\phi(\,|a_0|[q,r])^2} = \frac{1}{|a_0|^2 q^2} \prod_{p|a_0 q}\left(1-\frac{1}{p}\right)^{-2} \prod_{\substack{p|r\\ p|q}} \left(\left(1-\frac{1}{p}\right)^{-k+1} - 1\right) \times
$$

$$
\times \prod_{\substack{p|r\\ p|a_0\\ p\nmid q}} \left(\left(1-\frac{1}{p}\right) f_2(p) - 1\right) \prod_{\substack{p|r\\ p\nmid a_0 q}} \left(f_2(p)-1\right).
$$

Now it is clear that

$$
\frac{\mu(r)^2 S_0(r)}{\phi(\,|a_0|[q,r])^2} \ll q^{-2} \prod_{p|q}\left(1-\frac{1}{p}\right)^{-2} \cdot \left(r, q\prod_{j=0}^{k-1} a_j\right) \tau_{K_1}(r) r^{-2},
$$

hence

$$(3.5)\qquad S(a,q)=\sum_{r=1}^{\infty}\frac{\mu(r)^2S_0(r)}{\phi(|a_0|[q,r])^2}+O\left(\sum_{r>Q_1}\frac{\mu(r)^2S_0(r)}{\phi(|a_0|[q,r])^2}\right)=$$

$$=\frac{1}{q^2}\prod_{p|a_0q}\left(1-\frac{1}{p}\right)^{-1}\prod_{p|q}f_3(p)\prod_{p}f_2(p)+$$

$$+O\left(q^{-2}\tau_{K_2}(q)(\log x)^{-C+1}\right),$$

where K_1 and K_2 are constants depending only on k.

Now, from (3.2), (3.3), (3.4), (3.5) and the fact

$$\sum_{\substack{a=1\\(a_0a+b_0,q)=1}}^{q}1=q\prod_{\substack{p|q\\p\nmid a_0}}\left(1-\frac{1}{p}\right),$$

(2.4) follows immediately, which completes the proof of Lemma 1.

§ 4. Proof of Lemma 2

According to Montgomery [8], we shall show (2.6) and (2.7). Montgomery's argument is based on the following lemma due to Lavrik [7]:

LEMMA 4. *Let*

$$F(x;a,q;h)=$$

$$=\begin{cases}\displaystyle\sum_{\substack{1\leqq n\leqq x\\1\leqq n+h\leqq x\\n\equiv a\ (\mathrm{mod}\,q)}}\Lambda(n)\Lambda(n+h)-\\[2ex] \displaystyle -2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|qh}\left(\frac{p-1}{p-2}\right)\cdot\frac{(x-|h|)}{\phi(q)}\\[2ex] \qquad\qquad\qquad\text{if } h\equiv 0\ (\mathrm{mod}\,2) \text{ and } (a,q)=(a+h,q)=1\\[2ex] \displaystyle\sum_{\substack{1\leqq n\leqq x\\1\leqq n+h\leqq x\\n\equiv a\ (\mathrm{mod}\,q)}}\Lambda(n)\Lambda(n+h)\quad \text{otherwise.}\end{cases}$$

Then, for any $A, B > 0$,

$$\sum_{0<|h|\leqq x} \tau\big(|h|\big)\,\big|F(x;a,q;h)\big| \ll x^2(\log x)^{-A},$$

uniformly for $q \leqq (\log x)^B$.

In the definition of U_1 we put $r = m - n$ and obtain

$$U_1 = \sum_{\substack{0<|r|\leqq x/a_* \\ r\equiv 0 \pmod q}} \sum_{\substack{n \\ 1\leqq a_0 n+b_0\leqq x \\ 1\leqq a_0(n+r)+b_0\leqq x}} \Lambda(a_0 n + b_0)\Lambda\big(a_0(n+r)+b_0\big) \times$$

$$\times \prod_{j=1}^{k-1}\Bigg(\sum_{\substack{b_j \\ 1\leqq a_j n+b_j\leqq x \\ 1\leqq a_j(n+r)+b_j\leqq x}} \Lambda(a_j n + b_j)\Lambda\big(a_j(n+r)+b_j\big)\Bigg).$$

By virtue of Lemma 4 we have

$$\sum_{q\leqq Q} U_1 = \sum_{q\leqq Q} \sum_{\substack{0<|r|\leqq x/a_* \\ r\equiv 0 \pmod q}} \frac{2^k}{\phi\big(|a_0|\big)} \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)^k \times \tag{4.1}$$

$$\times \prod_{j=0}^{k-1}\left\{ \prod_{\substack{p|a_j r \\ p>2}} \left(\frac{p-1}{p-2}\right)\cdot\big(x-|a_j r|\big)\right\}+$$

$$+O\Bigg(\big(x(\log x)\big)^{k-1} \sum_{0<|r|\leqq x/a_*} \tau\big(|h|\big)\,\cdot$$

$$\cdot\Bigg(F(x;b_0,a_0;a_0 r) + \sum_{j=1}^{k-1} F(x;1,1,a_j r)\Bigg)\Bigg) =$$

$$= \frac{2^{k+1}}{\phi\big(|a_0|\big)} \prod_{p>2}\left\{\left(1-\frac{1}{(p-1)^2}\right)^k \left(\frac{p-1}{p-2}\right)^{g(p)}\right\}\cdot H_0(Q,x)+$$

$$+O\left(x^{k+1}(\log x)^{-A}\right),$$

where

$$H_0(Q,x)=\sum_{q\leqq Q}\sum_{\substack{0<m\leqq\frac{x}{a_*q}\\ a_jqm\equiv 0\pmod 2\\ \text{for all }0\leqq j\leqq k-1}}\prod_{\substack{p|mq\\ p>2}}\left(\frac{p-1}{p-2}\right)^{k-g(p)}\prod_{j=0}^{k-1}\left(x-|a_jqm|\right).$$

We decompose $H_0(Q,x)$ into

$$H(Q,t)=\sum_{q\leqq Q}\sum_{0<m\leqq\frac{t}{a_*q}}\prod_{\substack{p|mq\\ p>2}}\left(\frac{p-1}{p-2}\right)^{k-g(p)}\prod_{j=0}^{k-1}\left(t-|a_jqm|\right).$$

If $g(2)=k$, that is, all a_j's are even, then the conditions $a_jqm\equiv 0\pmod 2$ for all $0\leqq j\leqq k-1$ are always true, so we have

$$H_0(Q,x)=H(Q,x), \tag{4.2}$$

and if $g(2)<k$, that is, at least one a_j is odd, then we have

$$H_0(Q,x)=2^kH(Q,x/2)+2^kH(Q/2,x/2)-2^{2k}H(Q/2,x/4). \tag{4.3}$$

As for $H(Q,t)$, we need the following Lemma 5. Let

$$f_4(p)=1-\frac{1}{p}+\frac{1}{p}\left(\frac{p-1}{p-2}\right)^{k-g(p)}$$

and

$$f_5(p)=\left(1-\frac{1}{p}\right)^2+\frac{1}{p}\left(2-\frac{1}{p}\right)\left(\frac{p-1}{p-2}\right)^{k-g(p)}.$$

LEMMA 5. *We have, for* $Q\leqq 2t/a_*$,

$$H(Q,t)=t^{k+1}\Omega\prod_{p>2}f_4(p)\sum_{q\leqq Q}\frac{1}{q}\prod_{\substack{p|q\\ p>2}}f_3(p)+O\left(t^kQ\left(\log 3\frac{t}{Qa_*}\right)^k\right), \tag{4.4}$$

and, for $Q\geqq t/a_*$,

$$H(Q,t)=t^{k+1}\Omega\log(t/a_*)\prod_{p>2}f_5(p)+C_3t^{k+1}+O\left(t^{k+\frac{1}{2}}(\log t)^{k+2}\right), \tag{4.5}$$

where C_3 is a constant depending only on $a_0, \dots, a_{k-1}$.

Before proving Lemma 5, we derive Lemma 2 from Lemma 5. For $Q \geqq x/a_*$, since $f_1(2) = 2^k$ or $(3/4)2^k$ according as $g(2) = k$ or not, (2.7) follows from (4.1), (4.2), (4.3) and (4.5).

Next we assume $Q \leqq x/a_*$. If $g(2) < k$, then $f_3(2) = 2$ and, by (4.3) and (4.4), we have

$$(4.6) \qquad H_0(Q,x) =$$

$$= x^{k+1}\Omega \prod_{p>2} f_4(p) \cdot \left\{ \frac{1}{2} \sum_{\substack{q \leqq Q \\ 2\nmid q}} \frac{1}{q} \prod_{\substack{p|q \\ p>2}} f_3(p) + \sum_{\substack{q \leqq Q \\ 2|q}} \frac{1}{q} \prod_{\substack{p|q \\ p>2}} f_3(p) \right\} +$$

$$+ O\left(x^k Q \left(\log \left(2\frac{x}{Qa_*} \right) \right)^k \right) =$$

$$= \frac{1}{2} x^{k+1}\Omega \prod_{p>2} f_4(p) \cdot \sum_{q \leqq Q} \frac{1}{q} \prod_{p|q} f_3(p) + O\left(x^k Q \left(\log \left(2\frac{x}{Qa_*} \right) \right)^k \right).$$

If $g(2) = k$, then $f_3(2) = 1$ and, by (4.2) and (4.4), we have

$$(4.7) \qquad H_0(Q,x) = x^{k+1}\Omega \prod_{p>2} f_4(p) \cdot \sum_{q \leqq Q} \frac{1}{q} \prod_{p|q} f_3(p) + O\left(x^k Q \left(\log \left(2\frac{x}{Qa_*} \right) \right)^k \right).$$

Since $f_2(2) = 2^k$ or 2^{k-1} according as $g(2) = k$ or not, (2.6) follows from (4.6), (4.7) and (4.1), which completes our proof of Lemma 2.

We now prove Lemma 5. Let $w(d)$ be a completely multiplicative function defined by

$$w(p) = \begin{cases} \left(\dfrac{p-1}{p-2} \right)^{k-g(p)} - 1 & p \nmid 2q \\ 0 & p | 2q \end{cases}$$

for a prime p. For $y \geqq 1$, we have

$$(4.8) \qquad \sum_{m \leqq y} \prod_{\substack{p|m \\ p\nmid 2q}} \left(\frac{p-1}{p-2} \right)^{k-g(p)} = \sum_{m \leqq y} \sum_{d|m} \mu(d)^2 w(d) =$$

$$= y\sum_{d=1}^{\infty}\frac{\mu(d)^2 w(d)}{d} + O\Big(y\sum_{d>y}\frac{\mu(d)^2 w(d)}{d} + \sum_{d\leqq y}\mu(d)^2 w(d)\Big) =$$

$$= y\prod_{p>2} f_4(p)\prod_{\substack{p|q\\ p>2}} f_4(p)^{-1} + O\left((\log 2y)^k\right).$$

Then for $Q \leqq t/a_*$, (4.4) follows from (4.8) by partial summation. For $t/a_* < Q \leqq 2t/a_*$, (4.4) is still valid, because $\sum_{t/a_*<q\leqq Q}\frac{1}{q}\prod_{\substack{p|q\\ p>2}} f_3(p) \ll 1$.

Next we assume $Q \geqq t/a_*$. We have

$$H(Q,t) = H(t/a_*,t) = \tag{4.9}$$

$$= \sum_{qm\leqq t/a_*}\prod_{\substack{p|qm\\ p>2}}\left(\frac{p-1}{p-2}\right)^{k-g(p)}\prod_{j=0}^{k-1}\left(t-|a_j|qm\right) =$$

$$= \sum_{q\leqq\left(\frac{t}{a_*}\right)^{1/2}}\ \sum_{m\leqq\frac{t}{a_* q}} + \sum_{m\leqq\left(\frac{t}{a_*}\right)^{1/2}}\ \sum_{q\leqq\frac{t}{a_* m}} - \sum_{q\leqq\left(\frac{t}{a_*}\right)^{1/2}}\ \sum_{m\leqq\left(\frac{t}{a_*}\right)^{1/2}} =$$

$$= 2H\left((t/a_*)^{1/2},t\right) - H_1, \quad \text{say.}$$

Since $(t/a_*)^{1/2} \leqq 2t/a_*$, we can apply (4.4) with (3.1) for the first term of (4.9). For the second one, we use (4.8) to obtain

$$H_1 = \sum_{q\leqq(t/a_*)^{1/2}}\ \sum_{m\leqq(t/a_*)^{1/2}}\ \prod_{\substack{p|qm\\ p>2}}\left(\frac{p-1}{p-2}\right)^{k-g(p)}\prod_{j=0}^{k-1}\left(t-|a_j|qm\right) =$$

$$= t^{k+1}\prod_{p>2} f_4(p)\int_{(ta_*)^{-1/2}}^{1/a_*}\ \sum_{u(ta_*)^{1/2}\leqq q\leqq\left(\frac{t}{a_*}\right)^{1/2}}(1/q)\prod_{\substack{p|q\\ p>2}} f_3(p)\prod_{j=0}^{k-1}\left(1-|a_j|u\right)du +$$

$$+ O\left(t^{k+\frac{1}{2}}(\log t)^k\right).$$

We calculate the integrand by (3.1), then we get, by partial integration, that

(4.10)

$$H_1 = t^{k+1} \prod_{p>2} f_5(p) \int_0^{1/a_*} \frac{1}{u} \int_0^1 \prod_{j=0}^{k-1} \left(1 - |a_j| v\right) \, dv \, du + O\left(t^{k+\frac{1}{2}} (\log t)^{k+2}\right).$$

Since the double integral in (4.10) depends only on $a_0, \ldots, a_{k-1}$, (4.5) follows from (4.9), (4.10), (3.1) and (4.4).

§ 5. Proof of Lemma 3

In this section we shall prove Lemma 3 along Hooley's way [5]. For $Q_0 \leqq Q \leqq x/a_*$ we put

$$V(Q) = \sum_{Q<q\leqq x/a_*} U_1.$$

Since $\sum_{Q_0<q\leqq Q} U_1 = V(Q_0) - V(Q)$, it suffices to show that

(5.1)
$$V(Q) = \frac{2}{\phi(|a_0|)} x^{k+1} \Omega\left(\log \frac{x}{Qa_*}\right) \prod_p f_1(p) + \alpha_1 x^{k+1} +$$

$$+ x^k Q \sum_{m=0}^{k} \beta_m \left(\log \frac{x}{Qa_*}\right)^m + O\left(x^{k-\frac{1}{k+2}} Q^{\frac{k+3}{k+2}} + x^{k+1} (\log x)^{-A}\right).$$

To prove (5.1), we use the following Lemma 6 which is a slight modification of Theorem 1 of [6].

LEMMA 6. *Let T be any interval. Then for any positive number A, there exists a positive number B depending on A and k such that*

$$\sum_{q\leqq x^{1/2}(\log x)^{-B}} \max_a \sum_{\mathbf{b}\in Z} \left| \Psi(x; \mathbf{b}; a, q; T) - \sigma(\mathbf{b}; a, q) I \right| \ll x^k (\log x)^{-A},$$

where

$$\Psi(x; \mathbf{b}; a, q; T) = \sum_{\substack{n\in N(\mathbf{b}) \\ n\in T \\ n\equiv a \pmod q}} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j),$$

and I is the length of the interval $N(\mathbf{b})\cap T$, that is,

$$I=\int_{\substack{t\in N(\mathbf{b})\\ t\in T}} dt.$$

The proof of Lemma 6 proceeds exactly on the same lines as that of our Theorem 1 of [6], only by replacing $P_{aq}(\alpha)$ with $P_{aq}(\alpha;T)$ which is defined by

$$P_{aq}(\alpha;T)=\sum_{\substack{1\leqq a_0n+b_0\leqq x\\ n\in T\\ n\equiv a\pmod q}}\Lambda(a_0n+b_0)e(n\alpha).$$

Putting $m=n+hq$,

$$V(Q)=2\sum_{\mathbf{b}\in Z}\sum_{Q<q\leqq x/a_*}\sum_{\substack{n,m\in N(\mathbf{b})\\ n\equiv m\pmod q\\ n<m}}\prod_{j=0}^{k-1}\Lambda(a_jn+b_j)\Lambda(a_jm+b_j)=$$

$$=2\sum_{h\leqq\frac{x}{Qa_*}}\sum_{\mathbf{b}\in Z}\sum_{n\in N(\mathbf{b})}\prod_{j=0}^{k-1}\Lambda(a_jn+b_j)\sum_{\substack{m\in N(\mathbf{b})\\ m>n+hQ\\ M\equiv n\pmod h}}\prod_{j=0}^{k-1}\Lambda(a_jm+b_j).$$

Noticing that $Q\geqq Q_0$ and $x/(Qa_*)\ll(\log x)^{A+k}$, we apply Lemma 3 and obtain

$$V(Q)=2\sum_{h\leqq\frac{x}{Qa_*}}\sum_{\mathbf{b}}\sigma(\mathbf{b};h)\sigma(\mathbf{b};1)\int\int_{\substack{t_1,t_2\in N(\mathbf{b})\\ t_2<t_1-hQ}}dt_1\,dt_2+O\left(x^{k+1}(\log x)^{-A}\right)=$$

$$=2\sum_{h\leqq\frac{x}{Qa_*}}\sum_{\substack{a=1\\ (a_0a+b_0,h)=1}}^{h}\int\int_{\substack{1\leqq a_0t_i+b_0\leqq x\\ \text{for } i=1,2\\ t_2<t_1-hQ}}\cdot$$

$$\cdot\sum_{\substack{1\leqq a_jt_i+b_j\leqq x\\ \text{for } 1\leqq j\leqq k-1\\ \text{and for } i=1,2}}\sigma(\mathbf{b};a,h)^2\,dt_1\,dt_2+O\left(x^{k+1}(\log x)^{-A}\right).$$

By the same argument we used for the calculation of $\sum_{\mathbf{b}} S_1^2 |N(\mathbf{b})|^2$ in Section 3, we can obtain

$$V(Q) = \frac{2}{\phi(|a_0|)} x^{k+1} \prod_p f_2(p) \cdot V_1, \tag{5.2}$$

where

$$V_1 = \sum_{h \leqq \frac{x}{Qa_*}} h^{-1} \prod_{p|h} f_3(p) \int_{hQ/x}^{1/a_*} \prod_{j=0}^{k-1} (1 - |a_j|u)\, du.$$

We put $v = 1 - a_* u$, and define $r_1, \ldots, r_k$ as

$$\prod_{j=0}^{k-1} (1 - |a_j|u) = \prod_{j=0}^{k-1} \left(1 - \frac{|a_j|}{a_*} + \frac{|a_j|}{a_*} v\right) = \sum_{m=1}^{k} r_m v^m. \tag{5.3}$$

Then a simple calculation shows

$$V_1 = \frac{1}{a_*} \sum_{m=1}^{k} \frac{r_m}{m+1} \sum_{h \leqq \frac{x}{Qa_*}} \left(1 - \frac{hQa_*}{x}\right)^{m+1} \frac{1}{h} \prod_{p|h} f_3(p) = \tag{5.4}$$

$$= \frac{1}{a_*} \sum_{m=1}^{k} \frac{r_m}{m+1} \Xi\left(\frac{x}{a_*}, m+1\right),$$

where

$$\Xi(z, m) = \sum_{h \leqq z} \left(1 - \frac{h}{z}\right)^m \frac{1}{h} \prod_{p|h} f_3(p).$$

Next we examine $\Xi(z, m)$ for $m \geqq 2$. Let $s = \sigma + it$ be a complex variable. For $\sigma > 1$, we put

$$\xi(s) = \sum_{h=1}^{\infty} h^{-s} \prod_{p|h} f_3(p) = \prod_p \left(1 + p^{-s}(1 - p^{-s})^{-1} f_3(p)\right),$$

and define $\eta(s)$ by

$$\xi(s) = \zeta(s)\zeta(s+1)^k \eta(s), \tag{5.5}$$

where ζ denotes the Riemann zeta function. In the half plane $\sigma \geqq -\frac{1}{2} + \varepsilon$ with any fixed $\varepsilon > 0$, it is easily seen that η is analytic and $\eta(s) \ll 1$. Thus

(5.5) gives the analytic continuation of $\xi(s)$ over $\sigma \geqq -\frac{1}{2}+\varepsilon$. For $|t|>1$ and for any $\varepsilon>0$, we have

$$\left|\zeta(s)\right| \ll \begin{cases} |t|^{\frac{1}{2}-\sigma+\varepsilon} & \text{if } \sigma \leqq 0, \\ |t|^{\frac{1}{2}(1-\sigma)+\varepsilon} & \text{if } 0 \leqq \sigma \leqq 1 \end{cases}$$

(see Tichmarsh's book [9], for example). Therefore, for $-\frac{1}{2}+\varepsilon \leqq \sigma \leqq 0$,

$$\left|\xi(s)\right| \ll |t|^{-\left(\frac{k}{2}+1\right)\sigma+\frac{1}{2}+\varepsilon}. \tag{5.6}$$

We note here that the exponent of $|t|$ is less than 2 provided that $\sigma > > -3/(k+2)$.

It is known that, for $c>0$, $u>0$ and $m \geqq 1$,

$$(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} u^{-s}\left(\prod_{j=0}^{m}(s+j)\right)^{-1} ds = \begin{cases} \dfrac{1}{m!}(1-u)^m & \text{if } 0<u\leqq 1, \\ 0 & \text{if } u>1. \end{cases}$$

Making use of this formula, we have

$$\Xi(z,m) = m!(2\pi i)^{-1} \int_{1-i\infty}^{1+i\infty} \xi(s+1)z^s\left(\prod_{j=0}^{m}(s+j)\right)^{-1} ds =$$

$$= m!R_{0,m} + m!R_{1,m} + m!(2\pi i)^{-1} \int_{-\frac{k+3}{k+2}-i\infty}^{-\frac{k+3}{k+2}+i\infty} \xi(s+1)z^2\left(\prod_{j=0}^{m}(s+j)\right)^{-1} ds,$$

where $R_{0,m}$ and $R_{1,m}$ are the residues of the integrand at $s=0$ and $s= = -1$, respectively. (5.6) shows, for $m \geqq 2$, that the last integral converges absolutely and is bounded by $O\left(z^{-\frac{k+3}{k+2}}\right)$.

On the other hand, by (5.5), we see that the integrand has poles of order 2 and $k+1$ at $s=0$ and $s=-1$, respectively. We find that

$$m!R_{0,m} = \zeta(2)^k\eta(1)(\log z) + \alpha_2$$

and

$$m!R_{1,m} = z^{-1}\sum_{j=0}^{k}\gamma_{j,m}(\log z)^j,$$

where α_2 and the $\gamma_{j,m}$'s are constants depending only on $a_0, \dots, a_{k-1}$. We note that

$$\zeta(2)^k \eta(1) = \lim_{s\to 1} \frac{\xi(s)}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} f_3(p)\right),$$

and

$$(5.7) \qquad \gamma_{k,m} = -\frac{m}{k!}\zeta(0)\eta(0) = \frac{m}{2k!}\prod_p f_2(p)^{-1}.$$

Taking into account these results, we have

$$(5.8) \qquad \Xi(z,m) = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} f_3(p)\right) \cdot (\log z) + \alpha_2 +$$

$$+ z^{-1} \sum_{j=0}^{k} \gamma_{j,m} (\log z)^j + O\left(z^{-\frac{k+3}{k+2}}\right).$$

Since

$$\sum_{m=1}^{k} r_m/(m+1) = \int_0^1 \sum_{m=1}^{k} r_m v^m \, dv = a_* \int_0^1 \prod_{j=0}^{k-1} \left(1 - |a_j| u\right) du = a_* \Omega,$$

(5.2), (5.4) and (5.8) yield (5.1). Noticing (5.7) and $\sum\limits_{m=1}^{k} r_m = 1$, we also get (1.7), and our proof is complete.

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CAUCHY STRUCTURES IN CLOSURE AND PROXIMITY SPACES

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0. Introduction

It is well-known (see e.g. [10]) that a Cauchy structure $\mathfrak{S}$ on a set X induces a limitation $\lambda = \lambda(\mathfrak{S})$ defined by

$$(0.1) \qquad \mathfrak{s} \in \lambda(x) \quad \text{iff} \quad \mathfrak{s} \cap \dot{x} \in \mathfrak{S} \quad (x \in X,\ \mathfrak{s} \in \operatorname{Fil} X)$$

and, through it, a closure $c = c(\mathfrak{S})$ for which the neighbourhood filter $\mathfrak{v}_c(x)$ of $x \in X$ coincides with $\cap\lambda(x)$. The same closure $c(\mathfrak{S})$ may be also obtained through a proximity $\delta = \delta(\mathfrak{S})$ defined by

$$(0.2) \qquad A\delta B \quad \text{iff there is} \quad \mathfrak{s} \in \mathfrak{S} \quad \text{satisfying} \quad A, B \in \sec \mathfrak{s};$$

then $x \in c(A)$ for $c = c(\mathfrak{S})$, $A \subset X$ iff $\{x\}\delta A$, i.e. iff there is $\mathfrak{s} \in \mathfrak{S}$ satisfying $x \in \cap\mathfrak{s}$, $A \in \sec \mathfrak{s}$. We say that $\mathfrak{S}$ induces $\delta(\mathfrak{S})$ and $c(\mathfrak{S})$.

The purpose of the present paper is to look for necessary and/or sufficient conditions under which a given closure or proximity can be deduced from a Cauchy structure or, in other words, under which a closure or a proximity admits a compatible Cauchy structure.

Some results of this kind are contained in [6]. Terminology and notation concerning Cauchy structures will be taken from that paper; in particular, we shall use, in what follows, the expression *Cauchy screen* (or *C-screen*) for a Cauchy structure. For generalities on closures, proximities and screens, the reader may consult [7], 0.1–0.2 and [5].

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1. Preliminaries

We shall need some separation conditions for closures. It is well-known that a closure c on X is said to be *symmetric* iff

$$(\mathrm{S}_0) \qquad y \in c(\{x\}) \quad \text{implies} \quad x \in c(\{y\}) \quad (x, y \in X).$$

A stronger condition is given by

$$(\mathrm{S}_1) \qquad y \in c(\{x\}) \quad \text{implies} \quad \mathfrak{v}_c(x) = \mathfrak{v}_c(y) \quad (x, y \in X).$$

By [4], 4.3, (S_1) holds iff $x \notin c(A)$ implies $c(\{x\}) \cap c(A) = \emptyset$, i.e. iff c is *weakly separated* in the sense of [7], 0.1. A still stronger condition is

$$(\mathrm{S}_2) \qquad \mathfrak{v}_c(x) \Delta \mathfrak{v}_c(y) \quad \text{implies} \quad \mathfrak{v}_c(x) = \mathfrak{v}_c(y) \quad (x, y \in X)$$

(see [6]) ($\mathfrak{a} \Delta \mathfrak{b}$ means $A \cap B \neq \emptyset$ for $A \in \mathfrak{a}$, $B \in \mathfrak{b}$).

The closure c is *regular* (see [1], 27B.1, 27B.2) iff

(R) for $V \in \mathfrak{v}_c(x)$ there is $W \in \mathfrak{v}_c(x)$ such that $c(W) \subset V$ $(x \in X)$.

LEMMA 1.1. *If a regular closure is* S_1 *then it is* S_2.

PROOF. By (S_1), $\mathfrak{v}_c(x) \neq \mathfrak{v}_c(y)$ implies the existence of $V \in \mathfrak{v}_c(x)$ such that $y \notin V$; for a W such as in (R), we have $W \in \mathfrak{v}_c(x)$, $X - W \in \mathfrak{v}_c(y)$. □

Here (S_1) cannot be replaced by (S_0):

EXAMPLE 1.2. Let $X = \mathbf{R} \times \{0, 1\}$, $\mathfrak{v}_c(x, y)$ be the trace on X of the Euclidean neighbourhood filter $\mathfrak{v}(x, y)$ of (x, y) whenever $x \neq 0$; for $p = (0, 0)$ and $q = (0, 1)$, let $\mathfrak{v}_c(p) = \mathfrak{v}(p) \cap \dot{q}$, $\mathfrak{v}_c(q) = \mathfrak{v}(q) \cap \dot{p}$. □

On the other hand, a regular closure is necessarily S_0 ([1], 27B.4, 23B.3).

We shall say that a closure is S_3 iff it is regular and S_1, and T_i $(i = 1, 2, 3)$ iff it is S_i and T_0 where

$$(\mathrm{T}_0) \qquad \mathfrak{v}_c(x) = \mathfrak{v}_c(y) \quad \text{implies} \quad x = y \quad (x, y \in X).$$

Clearly, (S_i) and (T_i) $(i = 1, 2, 3)$ coincide with the usual separation axioms (see [2], Chapter 2.5) if c is a topology. T_1-closures are often called *separated* ([7], [4]).

In a closure space (X, c), a set $K \subset X$ is said to be *compact* iff the subspace $(K, c|K)$ is compact in the sense of [1], 29B.4 or [4], p. 286, i.e. iff each proper filter base (i.e. one composed of non-empty sets) in K possesses a cluster point in K. Let us say that $A \subset X$ is *semi-compact* iff each proper filter base in A admits a cluster point in X, and it is *weakly semi-compact* iff, whenever $\mathfrak{r}$ is a proper filter base in A, the filter

$$\mathfrak{v}_c(\mathfrak{r}) = \{S \subset X : \operatorname{int}_c S \in \operatorname{fil}_X \mathfrak{r}\}$$

has a cluster point in X. (Where $\mathrm{int}_c S = \{x \in X : S \in \mathfrak{v}_c(x)\}$.) If A is contained in a compact set then it is clearly semi-compact, and a semi-compact set is weakly semi-compact.

In order to establish a converse, let us first observe:

LEMMA 1.3. *If c is a regular closure on X and, for a filter base $\mathfrak{r}$ in X, the point $p \in X$ is a cluster point of the filter $\mathfrak{v}_c(\mathfrak{r})$, then p is a cluster point of $\mathfrak{r}$ itself.*

PROOF. Assume $V \in \mathfrak{v}_c(p)$, $R \in \mathfrak{r}$, $V \cap R = \emptyset$. For $W \in \mathfrak{v}_c(p)$, $c(W) \subset V$, we would have $X - W \in \mathfrak{v}_c(\mathfrak{r})$. □

Now we can prove:

LEMMA 1.4. *If A is weakly semi-compact in a regular closure space (X, c) then $c(A)$ is compact.*

PROOF. Let $\mathfrak{r}$ be a filter base composed of non-empty subsets of $c(A)$. Then $A \in \sec \mathfrak{v}_c(\mathfrak{r})$, hence $\mathfrak{r}_0 = \mathfrak{v}_c(\mathfrak{r})|A$ is a proper filter base in A. By hypothesis there exists a cluster point $p \in X$ of $\mathfrak{v}_c(\mathfrak{r}_0)$; by 1.3 it is a cluster point of $\mathfrak{r}_0$, hence of $\mathfrak{v}_c(\mathfrak{r})$ and of $\mathfrak{r}$, too. Clearly $p \in c(A)$. □

COROLLARY 1.5. *In a regular closure space, A is semi-compact iff it is weakly semi-compact iff $c(A)$ is compact.* □

Regularity cannot be replaced by T_2:

EXAMPLE 1.6. For $X = \mathbf{R}$, let $\mathfrak{v}_c(x)$ coincide with the Euclidean neighbourhood filter $\mathfrak{v}(x)$ whenever $x \neq 0$, and set

$$\mathfrak{v}_c(0) = \mathrm{fil}_X\{V - N : V \in \mathfrak{v}(0)\}, \quad N = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}.$$

Then c is a T_2-topology. The set $A = [-1, 1] - N$ is semi-compact in (X, c) while $c(A) = [-1, 1]$ is not compact. On the other hand, N is a closed, weakly semi-compact set without $c(N) = N$ being semi-compact. □

We also need some operations on screens. In [6], 2.3, one of them is described: for a screen $\mathfrak{S}$ on X, let $\mathfrak{S}^e$ be composed of the elementary filters (i.e. intersections of finitely many ultrafilters) contained in $\mathfrak{S}$ and of $\exp X$. If $\mathfrak{S}$ is a Cauchy screen then so is $\mathfrak{S}^e$. It is shown in [6], 2.3 that $c(\mathfrak{S}^e) = c(\mathfrak{S})$; in fact, a stronger result is valid:

LEMMA 1.7. *For any screen $\mathfrak{S}$, we have $\delta(\mathfrak{S}^e) = \delta(\mathfrak{S})$.*

PROOF. $\mathfrak{S}^e \subset \mathfrak{S}$ implies that $\delta(\mathfrak{S}^e)$ is finer than $\delta(\mathfrak{S})$. On the other hand, if $A, B \in \sec \mathfrak{s}$, $\mathfrak{s} \in \mathfrak{S}$, then $(\mathfrak{s}|A)\,\delta(\mathfrak{s}|B)$ for $\delta = \delta(\mathfrak{S})$ (i.e. $M \delta N$ for $M \in \mathfrak{s}|A$, $N \in \mathfrak{s}|B$), hence, by [5] 3.6, there are ultrafilters $\mathfrak{u} \supset \mathfrak{s}|A$, $\mathfrak{v} \supset \mathfrak{s}|B$ such that $\mathfrak{u}\delta\mathfrak{v}$, and then, by [5], 3.7, $\mathfrak{u} \cap \mathfrak{v} \in \mathfrak{S}^e$ fulfils $A, B \in \sec(\mathfrak{u} \cap \mathfrak{v})$. □

If $\mathfrak{S}$ is a screen then the fixed elements of $\mathfrak{S}$ constitute a base for a screen $\mathfrak{S}^f$. Clearly $c(\mathfrak{S}^f) = c(\mathfrak{S})$ ([4], 3.6). If $\mathfrak{S}$ is a Cauchy screen then so is $\mathfrak{S}^f$.

2. CR- and CL-screens

It is easy to solve our fundamental problem for CR- or CL-screens in closure spaces:

THEOREM 2.1. *A closure can be induced by a CR-screen iff it is* S_2, *and by a CL-screen iff it is an* S_2*-topology.*

PROOF. If $\mathfrak{S}$ is a CR-screen and $c = c(\mathfrak{S})$ then c is S_2 by [6], 2.1; if $\mathfrak{S}$ is Lodato then c is a topology ([6]). If c is an S_2-closure then the filters $\mathfrak{v}_c(x)$ $(x \in X)$ constitute by [6], 2.2 a base for a CR-screen $\mathfrak{S}$ satisfying $c = c(\mathfrak{S})$; $\mathfrak{S}$ is Lodato provided c is a topology. □

COROLLARY 2.2. *If c is an* S_2*-closure then the finest CR-screen compatible with c is generated by the screen base composed of the filters* $\mathfrak{v}_c(x)$. *It coincides with the finest Riesz screen* $\mathfrak{S}^1_R(c)$ *compatible with c. If c is an* S_2*-topology then* $\mathfrak{S}^1_R(c)$ *is Lodato and, therefore, it is the finest CL-screen compatible with c.*

PROOF. [5], 2.8. □

In order to treat the question of existence of coarsest compatible CR-screens, let us say that a filter $\mathfrak{s}$ is *strongly c-compressed* in a closure space (X, c) iff $\mathfrak{s} \to x$ for c whenever x is a c-cluster point of $\mathfrak{s}$; such a filter is c-compressed ([4]), 3.3).

LEMMA 2.3. *If $\mathfrak{S}$ is a CR-screen, $c = c(\mathfrak{S})$, then every element of $\mathfrak{S}$ is strongly c-compressed.*

PROOF. If x is a cluster point of $\mathfrak{s} \in \mathfrak{S}$ then $\mathfrak{v}_c(x)\Delta\mathfrak{s}$ and $\mathfrak{v}_c(x) \in \mathfrak{S}$, hence $\mathfrak{s}' = \mathfrak{v}_c(x) \cap \mathfrak{s} \in \mathfrak{S}$ is fixed at x, hence $\mathfrak{s}' \to x$ by [4], 3.1 and 3.3. *A fortiori*, $\mathfrak{s} \to x$. □

THEOREM 2.4. *In an arbitrary closure space (X, c), the strongly c-compressed filters constitute a Cauchy screen $\widehat{\mathfrak{S}} = \widehat{\mathfrak{S}}(c)$ such that $\widehat{c} = c(\widehat{\mathfrak{S}})$ is finer than c. $\widehat{\mathfrak{S}}$ contains all filters $\mathfrak{v}_c(x)$ iff c is* S_2 *and then $\widehat{\mathfrak{S}}$ is the coarsest CR-screen compatible with c.*

PROOF. For $x \in X$, $\dot{x}$ is strongly c-compressed. A filter finer than a strongly c-compressed filter is strongly c-compressed. If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are strongly c-compressed and $\mathfrak{s}_1\Delta\mathfrak{s}_2$ then $\mathfrak{s} = \mathfrak{s}_1 \cap \mathfrak{s}_2$ is strongly c-compressed. In fact, $\mathfrak{v}_c(x)\Delta\mathfrak{s}$ implies $\mathfrak{v}_c(x)\Delta\mathfrak{s}_i$ for $i = 1$ or 2, say for $i = 1$. Then $\mathfrak{s}_1 \to x$, hence $\mathfrak{v}_c(x)\Delta\mathfrak{s}_2$, $\mathfrak{s}_2 \to x$, and $\mathfrak{s} \to x$.

If $x \in \widehat{c}(A)$ then there is $\mathfrak{s} \in \widehat{\mathfrak{S}}$ such that $\{x\}$, $A \in \sec \mathfrak{s}$. Since x is a c-cluster point of $\mathfrak{s}$, necessarily $\mathfrak{s} \to x$ for c and $x \in c(A)$.

A point y is a c-cluster point of $\mathfrak{v}_c(x)$ iff $\mathfrak{v}_c(x)\Delta\mathfrak{v}_c(y)$. So if c is S_2 then $\mathfrak{v}_c(x)$ is strongly c-compressed for $x \in X$ and $\widehat{\mathfrak{S}}$ contains all c-neighbourhood filters. Conversely if $\mathfrak{v}_c(x) \in \widehat{\mathfrak{S}}$ for $x \in X$ then $\mathfrak{v}_c(x)\Delta\mathfrak{v}_c(y)$ implies $\mathfrak{v}_c(x) \to$ $\to y$, $\mathfrak{v}_c(x) \supset \mathfrak{v}_c(y)$ and similarly $\mathfrak{v}_c(x) \subset \mathfrak{v}_c(y)$, so that c is S_2. If so, then

$x \in c(A)$ implies $\{x\}$, $A \in \sec \mathfrak{v}_c(x)$, $\mathfrak{v}_c(x) \in \widehat{\mathfrak{S}}$ and $x \in \widehat{c}(A)$. Thus $\widehat{\mathfrak{S}}$ is a compatible CR-screen; it is the coarsest one by 2.3. □

Therefore we shall write $\mathfrak{S}^0_{CR} = \mathfrak{S}^0_{CR}(c)$ for the screen $\widehat{\mathfrak{S}}$ described in 2.4 in the case when c is S_2. However, even if c is a T_1-topology, $\widehat{\mathfrak{S}}$ need not be a Riesz screen in general.

EXAMPLE 2.5. For an infinite set M, let $X = M \cup \{p, q\}$, $\mathfrak{v}_c(x) = \dot{x}$ for $x \in M$, $\mathfrak{v}_c(p) = \dot{p} \cap \operatorname{fil}_X \mathfrak{s}_0$ where $\mathfrak{s}_0$ is composed of all cofinite subsets of M, $\mathfrak{v}_c(q) = \dot{q} \cap \operatorname{fil}_X \mathfrak{u}_0$ where $\mathfrak{u}_0$ is a free ultrafilter in M. Clearly c is a T_1-topology.

If $\mathfrak{u} \neq \mathfrak{u}_0$ is a free ultrafilter in M, then $\mathfrak{s} = \dot{p} \cap \operatorname{fil}_X \mathfrak{u}$ is strongly c-compressed, so $\mathfrak{v}_c(p) \subset \mathfrak{v}_{\widehat{c}}(p) \subset \mathfrak{s}$. Now $\mathfrak{v}_c(p)$ is the intersection of all such filters $\mathfrak{s}$; in fact, if $A \subset M$ is not cofinite in M, then $X - A$ is contained in at least one free ultrafilter $\mathfrak{u}$ in M distinct from $\mathfrak{u}_0$, so $\{p\} \cup A \notin \dot{p} \cap \operatorname{fil}_X \mathfrak{u}$. Therefore $\mathfrak{v}_{\widehat{c}}(p) = \mathfrak{v}_c(p)$, but this filter does not belong to $\widehat{\mathfrak{S}}$ since q is a cluster point of it without being its limit point. □

Observe that the only strongly c-compressed filter fixed at q is $\dot{q}$, so $\mathfrak{v}_{\widehat{c}}(q) = \dot{q}$ and $\widehat{c}$ is strictly finer than c.

On the other hand, $\widehat{c} = c$ can occur even if c is not S_2:

EXAMPLE 2.6 (J. Deák). Let $X = D \cup \{p, q\}$, $D = \omega_1 \times \omega$, $\mathfrak{v}_c(x) = \dot{x}$ for $x \in D$, $\mathfrak{v}_c(p) = \mathfrak{p} \cap \dot{p}$, $\mathfrak{v}_c(q) = \mathfrak{q} \cap \dot{q}$, where the filters $\mathfrak{p}, \mathfrak{q}$ are defined as follows. Set

$$P(\alpha, m) = \{(\alpha, n) \in D : m \leqq n\},$$

$$Q(\alpha, m) = \{(\beta, m) \in D : \alpha \leqq \beta\},$$

$$\mathfrak{p}(\alpha) = \operatorname{fil}_X \{P(\alpha, m) : m \in \omega\},$$

$$\mathfrak{q}(m) = \operatorname{fil}_X \{Q(\alpha, m) : \alpha \in \omega_1\},$$

$$\mathfrak{p} = \bigcap \{\mathfrak{p}(\alpha) : \alpha \in \omega_1\},$$

$$\mathfrak{q} = \bigcap \{\mathfrak{q}(m) : m \in \omega\}.$$

Now c is a T_1-topology (since $\mathfrak{p}(\alpha)$, $\mathfrak{q}(m)$, $\mathfrak{p}$ and $\mathfrak{q}$ are free) but it fails to be T_2. In fact, each $Q \in \mathfrak{q}$ contains a subset of the form $\bigcup\{Q(m) : m \in \omega\}$, $Q(m) = Q(\alpha_m, m)$, and by choosing $\alpha \in \omega_1$ such that $\alpha_m \leqq \alpha$ for $m \in \omega$, we have $Q(\alpha, m) \subset Q$ for every $m \in \omega$. Now any $P \in \mathfrak{p}$ contains $P(\alpha, m)$ for some $m \in \omega$ so that $(\alpha, m) \in P(\alpha, m) \cap Q(\alpha, m) \subset P \cap Q \neq \emptyset$.

Clearly $\mathfrak{v}_{\widehat{c}}(x) = \dot{x}$ for $x \in D$ since $\widehat{c}$ is finer than c. Further both $\mathfrak{p}(\alpha) \cap \cap \dot{p}$ and $\mathfrak{q}(m) \cap \dot{q}$ are strongly c-compressed since they have no cluster points other than p or q, respectively. Therefore

$$\mathfrak{v}_c(p) \subset \mathfrak{v}_{\widehat{c}}(p) \subset \mathfrak{p}(\alpha) \cap \dot{p} \quad (\alpha \in \omega_1),$$

$$\mathfrak{v}_c(q) \subset \mathfrak{v}_{\hat{c}}(q) \subset \mathfrak{q}(m) \cap \dot{q} \quad (m \in \omega)$$

implying

$$\mathfrak{v}_c(p) \subset \mathfrak{v}_{\hat{c}}(p) \subset \mathfrak{p} \cap \dot{p}, \qquad v_c(q) \subset \mathfrak{v}_{\hat{c}}(q) \subset \mathfrak{q} \cap \dot{q},$$

i.e.

$$\mathfrak{v}_c(p) = \mathfrak{v}_{\hat{c}}(p), \quad \mathfrak{v}_c(q) = \mathfrak{v}_{\hat{c}}(q). \quad \square$$

COROLLARY 2.7. *If c is an S_2-topology then the coarsest CL-screen $\mathfrak{S}^0_{CL} = \mathfrak{S}^0_{CL}(c)$ compatible with c is generated by the c-open elements of $\mathfrak{S}^0_{CR}$.*

PROOF. The filters in question clearly constitute a screen base for a Cauchy screen $\mathfrak{S}$. Obviously $\mathfrak{S}^1_R \subset \mathfrak{S} \subset \mathfrak{S}^0_{CR}$, hence $c(\mathfrak{S}) = c$ and $\mathfrak{S}$ is a CL-screen compatible with c. It is the coarsest one because any compatible CL-screen $\mathfrak{S}'$ must be contained in $\mathfrak{S}^0_{CR}$ and generated by a screen base composed of c-open filters. $\square$

$\mathfrak{S}^0_{CL} \neq \mathfrak{S}^0_{CR}$ may happen: consider (X, c) in 1.6 in which the Fréchet filter $\mathfrak{s}$ corresponding to the sequence $\left(\frac{1}{n}\right)$ does not have any cluster points (hence $\mathfrak{s} \in \mathfrak{S}^0_{CR}$) while $\mathfrak{v}_c(\mathfrak{s})$ clusters at 0 without c-converging to 0.

On the other hand:

LEMMA 2.8. *If c is a regular topology then $\mathfrak{S}^0_{CR} = \mathfrak{S}^0_{CL}$.*

PROOF. For $\mathfrak{s} \in \mathfrak{S}^0_{CR}$ we have $\mathfrak{v}_c(\mathfrak{s}) \in \mathfrak{S}^0_{CR}$. In fact, if $x \in X$ is a cluster point of $\mathfrak{v}_c(\mathfrak{s})$, then it is a cluster point of $\mathfrak{s}$ by 1.3, so $\mathfrak{s} \to x$, and, c being a topology, $\mathfrak{v}_c(\mathfrak{s}) \to x$. Hence $\mathfrak{S}^0_{CR}$ is Lodato. $\square$

Observe that 2.4 and 2.7 show a certain contrast to [6], 2.17 (according to which a compatible CL-screen given on a subspace of a T_2-topological space may possess compatible CL-extensions without having a coarsest CR- or CL-extension).

Our previous results permit to formulate some sufficient conditions for the existence of compatible CR- or CL-screens in proximity spaces (X, δ):

THEOREM 2.9. *If a proximity δ admits a compatible CR-screen then $c(\delta)$ is S_2. Conversely, if c is an S_2-closure on X, then the proximities*

$$\delta^1_R = \delta^1_R(c) = \delta\big(\mathfrak{S}^1_R(c)\big)$$

and

$$\delta^* = \delta^*(c) = \delta\big(\mathfrak{S}^0_{CR}(c)\big)$$

admit the compatible CR-screens $\mathfrak{S}^1_R(c)$ and $\mathfrak{S}^0_{CR}(c)$, respectively.

PROOF. 2.1, 2.2, 2.4. $\square$

THEOREM 2.10. *If δ admits a compatible CL-screen then $c(\delta)$ is an S_2-topology. Conversely, if c is an S_2-topology, then the proximities $\delta^1_R(c)$ and*

$$\delta^{**} = \delta^{**}(c) = \delta\big(\mathfrak{S}^0_{CL}(c)\big)$$

admit the compatible CL-screens $\mathfrak{S}^1_R(c)$ and $\mathfrak{S}^0_{CL}(c)$, respectively.

PROOF. 2.1, 2.2, 2.7. □

COROLLARY 2.11. *For an* S_2*-closure* c, $\delta^1_R(c)$ *and* $\delta^*(c)$ *are the finest and the coarsest proximities, respectively, compatible with* c *and admitting a compatible* CR*-screen.* □

Concerning proximities between $\delta^1_R(c)$ and $\delta^*(c)$, see 3.17.

COROLLARY 2.12. *For an* S_2*-topology* c, $\delta^1_R(c)$ *and* $\delta^{**}(c)$ *are the finest and the coarsest proximities, respectively, compatible with* c *and admitting a compatible* CL*-screen.* □

According to 2.11 and 2.12, it is reasonable to write

$$\delta^*(c) = \delta^0_{CR}(c), \quad \delta^{**}(c) = \delta^0_{CL}(c).$$

It is not difficult to find direct constructions for $\delta^1_R(c)$, $\delta^0_{CR}(c)$ and $\delta^0_{CL}(c)$. As to $\delta = \delta^1_R(c)$, it is obvious (cf. [4], 4.1) that $A\delta B$ iff $c(A) \cap c(B) \neq \emptyset$.

THEOREM 2.13. *For an* S_2*-closure* $c, \delta^* = \delta^0_{CR}(c)$, *we have* $A\delta^* B$ *iff* $c(A) \cap c(B) \neq \emptyset$ *or neither* A *nor* B *is semi-compact.*

PROOF. Suppose there is a strongly c-compressed filter $\mathfrak{s}$ such that $A, B \in \in \sec \mathfrak{s}$. If $\mathfrak{s}$ has a cluster point $x \in X$, then $\mathfrak{s} \to x$, hence $c(A) \cap c(B) \neq \emptyset$. If $\mathfrak{s}$ does not have any cluster points, then $\mathfrak{s}|A$ and $\mathfrak{s}|B$ are filter bases without cluster points, implying that neither A nor B is semi-compact.

Conversely, if $x \in c(A) \cap c(B)$, then $\mathfrak{v}_c(x) \in \mathfrak{S}^0_{CR}(c)$, $A, B \in \sec \mathfrak{v}_c(x)$. If there are proper filter bases $\mathfrak{r}_A$ and $\mathfrak{r}_B$ in A and B, respectively, without cluster points in X, then $\mathfrak{s} = \operatorname{fil}_X \mathfrak{r}_A \cap \operatorname{fil}_X \mathfrak{r}_B \in \mathfrak{S}^0_{CR}(c)$ since $\mathfrak{s}$ has no cluster points, and $A, B \in \sec \mathfrak{s}$. □

THEOREM 2.14. *For an* S_2*-topology* $c, \delta^{**} = \delta^0_{CL}(c)$, *we have* $A\delta^{**}B$ *iff* $c(A) \cap c(B) \neq \emptyset$ *or neither* A *nor* B *is weakly semi-compact.*

PROOF. If $\mathfrak{s}$ is c-open and strongly c-compressed, $A, B \in \sec \mathfrak{s}$, then either $c(A) \cap c(B) \neq \emptyset$ or $\mathfrak{s}$ has no cluster points, and then the same holds for the finer filters $\mathfrak{v}_c(\mathfrak{s}|A)$ and $\mathfrak{v}_c(\mathfrak{s}|B)$. Conversely, if $\mathfrak{r}_A$ and $\mathfrak{r}_B$ are proper filter bases in A and B, respectively, and neither $\mathfrak{v}_c(\mathfrak{r}_A)$ nor $\mathfrak{v}_c(\mathfrak{r}_B)$ has cluster points, then $\mathfrak{s} = \mathfrak{v}_c(\mathfrak{r}_A) \cap \mathfrak{v}_c(\mathfrak{r}_B)$ is a c-open filter without cluster points such that $A, B \in \sec \mathfrak{s}$. □

By 2.8, $\delta^0_{CR}(c) = \delta^0_{CL}(c)$ if c is a regular topology. On the other hand, these two proximities are distinct for the T_2-topology c in 1.6, since P and Q are proximal for the first one and far for the second one provided

$$P = \left\{\frac{1}{2n} : n \in \mathbf{N}\right\}, \quad Q = \left\{\frac{1}{2n+1} : n \in \mathbf{N}\right\}$$

(P and Q are weakly semi-compact without being semi-compact).

It is possible that $c(\delta)$ is a T_2-topology but δ does not admit a compatible CR-screen or a compatible Cauchy screen at all (see 3.12, 3.13).

3. Cauchy screens

It is not difficult to establish necessary conditions for the existence of compatible C-screens in general.

THEOREM 3.1. *If $\mathfrak{S}$ is a Cauchy screen then $\delta(\mathfrak{S})$ is a Riesz proximity, hence $c(\mathfrak{S})$ is an S_1-closure.*

PROOF. By putting $\delta = \delta(\mathfrak{S})$, $c = c(\delta) = c(\mathfrak{S})$, suppose $x \in c(A) \cap c(B)$. Then there are $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathfrak{S}$ such that $\{x\}, A \in \sec \mathfrak{s}_1$, $\{x\}, B \in \sec \mathfrak{s}_2$. Now $\mathfrak{s}_1 \Delta \mathfrak{s}_2$, so $\mathfrak{s} = \mathfrak{s}_1 \cap \mathfrak{s}_2 \in \mathfrak{S}$, and $A, B \in \sec \mathfrak{s}$, $A \delta B$.

The rest follows from [4], 5.9 (there Riesz proximities are called weakly Lodato, S_1-closures weakly separated). □

THEOREM 3.2. *Let $\mathfrak{S}$ be a Cauchy screen, $\delta = \delta(\mathfrak{S})$, $A \delta B_i$ for $i \in I$, $B_i \bar{\delta} B_j$ if $i, j \in I$, $i \neq j$. Then*

$$|I| \leqq 2^{2^{|A|}}. \tag{3.2.1}$$

PROOF. Let $\mathfrak{s}_i \in \mathfrak{S}$ be chosen such that $A, B_i \in \sec \mathfrak{s}_i$, and suppose that (3.2.1) is false. Then there are $i, j \in I$, $i \neq j$ satisfying $\mathfrak{s}_i | A = \mathfrak{s}_j | A$, consequently $\mathfrak{s}_i \Delta \mathfrak{s}_j$. So $\mathfrak{s} = \mathfrak{s}_i \cap \mathfrak{s}_j \in \mathfrak{S}$ and $B_i, B_j \in \sec \mathfrak{s}$, which fact would imply $B_i \delta B_j$. □

In order to deduce from 3.2 a necessary condition concerning closures, let us denote, for a closure c on X, by $\mathfrak{P}(c)$ the partition of X corresponding to the equivalence relation

$$x \sim y \quad \text{iff} \quad \mathfrak{v}_c(x) = \mathfrak{v}_c(y) \quad (x, y \in X).$$

If c is S_1 then $x \sim y$ iff $y \in c(\{x\})$.

COROLLARY 3.3. *If $\mathfrak{S}$ is a Cauchy screen on X, $c = c(\mathfrak{S})$, $A \subset X$, then*

$$\Big|\{P \in \mathfrak{P}(c): P \cap c(A) \neq \emptyset\}\Big| \leqq 2^{2^{|A|}}. \tag{3.3.1}$$

In particular, if c is T_1, then

$$\big|c(A)\big| \leqq 2^{2^{|A|}}. \tag{3.3.2}$$

PROOF. Set

$$\{P \in \mathfrak{P}(c): P \cap c(A) \neq \emptyset\} = \{P_i: i \in I\},$$
$$P_i \neq P_j \quad \text{for} \quad i, j \in I,\ i \neq j.$$

Choose $x_i \in P_i \cap c(A)$ $(i \in I)$. Then $\{x_i\}\delta A$ for $\delta = \delta(\mathfrak{S})$, and $\{x_i\}\overline{\delta}\{x_j\}$ for $i, j \in I$, $i \neq j$ since c is S_1 by 3.1. Thus (3.2.1) holds by 3.2. If c is T_1 then each element of $\mathfrak{P}(c)$ is a singleton. □

A further necessary condition concerns elementary neighbourhood filters in closure spaces.

LEMMA 3.4. *Let $\mathfrak{S}$ be a screen on $X, c = c(\mathfrak{S})$. If $x \in X$, $\mathfrak{v}_c(x)$ is an elementary filter, $\mathfrak{u}$ is an ultrafilter, and $\mathfrak{u} \to x$, then $\mathfrak{u} \cap \dot{x} \in \mathfrak{S}$.*

PROOF. The statement is obvious if $\mathfrak{u} = \dot{x}$. Assume $\mathfrak{u} \neq \dot{x}$, $\mathfrak{v}_c(x) = \dot{x} \cap \cap \bigcap_0^n \mathfrak{u}_i$ where each $\mathfrak{u}_i$ is an ultrafilter and $\mathfrak{u}_i \neq \dot{x}$, $\mathfrak{u} = \mathfrak{u}_0$, $\mathfrak{u}_i \neq \mathfrak{u}_j$ if $i \neq j$. Choose $A \in \mathfrak{u}_0$ satisfying $A \notin \mathfrak{u}_i$ for $i \neq 0$. We have $x \in c(A)$, hence there is a filter $\mathfrak{s} \in \mathfrak{S}$ such that $x \in \bigcap \mathfrak{s}$, $A \in \sec \mathfrak{s}$, consequently $\mathfrak{s} \to x$ by [4], 3.1 and 3.3. Now $\mathfrak{v}_c(x) \subset \mathfrak{s}$ implies

$$\mathfrak{s} = \dot{x} \cap \bigcap_{i \in I} \mathfrak{u}_i, \quad I \subset \{0, \ldots, n\}.$$

Clearly $A \in \sec \mathfrak{u}_i$ for some $i \in I$ so that $A \in \mathfrak{u}_i$, $i = 0$, and $\mathfrak{s} \subset \mathfrak{u}_0 \cap \dot{x} \in \mathfrak{S}$. □

COROLLARY 3.5. *Under the hypotheses of 3.4, if $\mathfrak{S}$ is a Cauchy screen, then $\mathfrak{v}_c(x) \in \mathfrak{S}$.* □

COROLLARY 3.6. *If $\mathfrak{S}$ is a Cauchy screen, $c = c(\mathfrak{S})$, $x, y \in X$, $\mathfrak{v}_c(x)$ and $\mathfrak{v}_c(y)$ are elementary filters, and $\mathfrak{v}_c(x) \Delta \mathfrak{v}_c(y)$, then $\mathfrak{v}_c(x) = \mathfrak{v}_c(y)$.*

PROOF. 3.5 and 3.1. □

For a special class of closure spaces, we can now prove a necessary and sufficient condition:

THEOREM 3.7. *Let (X, c) be a closure space such that each neighbourhood filter $\mathfrak{v}_c(x)$ is elementary. There is a Cauchy screen compatible with c iff c is S_2.*

PROOF. By 3.5, a compatible C-screen has to be CR, and 2.1 can be applied. □

Unfortunately, the collection of the necessary conditions 3.1, 3.3 and 3.6 is not sufficient in the general case.

EXAMPLE 3.8. Let M be a countably infinite set, $M \cap N = \emptyset$, and suppose that p is a bijection onto N from the set of all free ultrafilters in M. Define $X = M \cup N \cup \{q\}$ where $q \notin M \cup N$. Set $\mathfrak{v}_c(x) = \dot{x}$ for $x \in M$, $\mathfrak{v}_c(x) = \dot{x} \cap \mathrm{fil}_X \mathfrak{u}$ for $x = p(\mathfrak{u})$, $\mathfrak{u}$ a free ultrafilter in M, and $\mathfrak{v}_c(q) = \dot{q} \cap \mathrm{fil}_X \mathfrak{s}_0$ where $\mathfrak{s}_0$ is composed of all cofinite subsets of M. Then c is a T_1-topology on X, $|X| \leqq 2^{2^\omega}$ implies that (3.3.2) is fulfilled and the same is true for 3.6. However, if $\mathfrak{S}$ were a Cauchy screen compatible with c then $q \in c(M)$ would imply the existence of $\mathfrak{s} \in \mathfrak{S}$ such that $q \in \bigcap \mathfrak{s}$, $M \in \sec \mathfrak{s}$. Take an ultrafilter

$\mathfrak{u}$ in M such that $\mathfrak{s}|M \subset \mathfrak{u}$, then $\dot{p}(\mathfrak{u}) \cap \mathrm{fil}_X\, \mathfrak{u} \in \mathfrak{S}$ by 3.5, $\mathfrak{s}\Delta\mathfrak{v}_c(p(\mathfrak{u}))$ and $\mathfrak{s} \cap \mathfrak{v}_c(p(\mathfrak{u})) \in \mathfrak{S}$ would imply $p(\mathfrak{u}) \in c(\{q\})$: a contradiction. □

For the next example, we recall that the coarsest Riesz proximity $\delta = \delta_R^0(c)$ compatible with an S_1-closure c is defined by $A\delta B$ iff $c(A) \cap c(B) \neq \emptyset$ or both A and B are infinite (see [7], Theorem 1.5, but a direct proof is also straightforward).

EXAMPLE 3.9. Let M, N, p denote the same as in 3.8, $X = M \cup N$, $\mathfrak{v}_c(x) = \dot{x}$ for $x \in M$, $\mathfrak{v}_c(x) = \dot{x} \cap \mathrm{fil}_X\, \mathfrak{u}$ for $x = p(\mathfrak{u})$. Now c is a T_2-topology and, by 3.5, each Cauchy screen compatible with c must be Riesz.

Now $\delta = \delta_R^0(c)$ is strictly coarser than $\delta^* = \delta_{CR}^0(c)$ because $A\delta B$ if A and B are infinite, $A, B \subset M$, $A \cap B = \emptyset$, but then both A and B are semi-compact and $c(A) \cap c(B) = \emptyset$, hence $A\overline{\delta^*}B$ (see 2.13). By 2.11, there is no Cauchy screen compatible with δ (although $c = c(\delta)$ admits, by 2.1, a compatible CL-screen).

Observe that δ is Riesz and every family of pairwise disjoint non-empty subsets in X has cardinality $\leqq 2^{2^\omega}$ so that the condition in 3.2 is fulfilled. □

The situation that $c(\delta)$ admits a compatible Cauchy screen without the Riesz proximity δ doing so cannot occur if $\delta = \delta_R^1(c)$.

LEMMA 3.10. *If a screen $\mathfrak{S}$ is generated by a screen base composed of fixed filters and $\delta = \delta(\mathfrak{S})$ is a Riesz proximity, then $\delta = \delta_R^1(c)$ for $c = c(\mathfrak{S})$.*

PROOF. It suffices to show that δ is finer than $\delta_R^1(c)$. Now if $A\delta B$, there is a fixed filter $\mathfrak{s} \in \mathfrak{S}$ such that $A, B \in \sec \mathfrak{s}$. If $\mathfrak{s}$ is fixed at x, clearly $x \in c(A) \cap c(B)$. □

COROLLARY 3.11. *If $\mathfrak{S}$ is a Cauchy screen, $c = c(\mathfrak{S})$, then $\delta(\mathfrak{S}^f) = \delta_R^1(c)$.*

PROOF. $\mathfrak{S}^f$ is a Cauchy screen, so $\delta(\mathfrak{S}^f)$ is Riesz by 3.1 and $c(\mathfrak{S}^f) = c$. □

COROLLARY 3.12. *An S_1-closure c admits a compatible Cauchy screen iff $\delta_R^1(c)$ does so.* □

In order to formulate a partial analogue for the proximity $\delta_R^0(c)$, let us agree in saying that a closure c is *Fréchet* iff $x \in c(A)$ implies the existence of a sequence (x_n) such that $x_n \in A$, $x_n \to x$ for c. If c is a topology, this terminology coincides with the usual one (see e.g. [8], p. 53). If every neighbourhood filter $\mathfrak{v}_c(x)$ has a countable base, c is clearly Fréchet.

We also recall that an infinite set A contains $2^{2^{|A|}}$ free ultrafilters (see e.g. [9], 9.2); we shall need this fact in the case $|A| = \omega$.

THEOREM 3.13. *Suppose c is a Fréchet S_1-closure on X and $|X| \leqq 2^{2^\omega}$. Then $\delta = \delta_R^0(c)$ admits a compatible Cauchy screen.*

PROOF. Consider the pairs (A,B) where $A,B \subset X$, $A \cap B = \emptyset$, and either $|A| = |B| = \omega$ or $A = \{a\}$, $|B| = \omega$ and the filter base composed of the subsets cofinite in B c-converges to a. Since the cardinality of the family of the countable subsets of X is $\leqq (2^{2^\omega})^\omega = 2^{2^\omega}$, we can enumerate these pairs in the form (A_α, B_α), $\alpha \in 2^{2^\omega}$.

If (A_α, B_α) is of the first type, we choose free ultrafilters $\mathfrak{v}_\alpha$, $\mathfrak{w}_\alpha$ such that $A \in \mathfrak{v}_\alpha$, $B \in \mathfrak{w}_\alpha$; if (A_α, B_α) is of the second kind, we choose a free ultrafilter $\mathfrak{u}_\alpha$ such that $B_\alpha \in \mathfrak{u}_\alpha$. This can be done in the manner that all ultrafilters $\mathfrak{u}_\alpha$, $\mathfrak{v}_\alpha$, $\mathfrak{w}_\alpha$ are distinct; in fact, there are 2^{2^ω} free ultrafilters in a countably infinite set and, for a given $\alpha \in 2^{2^\omega}$, the family of the traces on A_α or B_α of the ultrafilters $\mathfrak{u}_\beta$, $\mathfrak{v}_\beta$, $\mathfrak{w}_\beta$ $(\beta < \alpha)$ has cardinality less than 2^{2^ω}.

Let $\mathfrak{B}$ be composed of the filters $\mathfrak{v}_\alpha \cap \mathfrak{w}_\alpha$, $\dot{a}_\alpha \cap \mathfrak{u}_\alpha$ (where $A_\alpha = \{a_\alpha\}$) and $\dot{F}$ where F is a finite subset contained in one of the elements of the partition $\mathfrak{P}(c)$. Consider the intersections $\bigcap_0^n \mathfrak{r}_i$ where $\mathfrak{r}_i \in \mathfrak{B}$ and $(\mathfrak{r}_0, \dots, \mathfrak{r}_n)$ is a Cauchy chain ([6]) (i.e. $\mathfrak{r}_{i-1} \Delta \mathfrak{r}_i$ for $i = 1, \dots, n$). It is easy to see that these intersections constitute a base for a Cauchy screen $\mathfrak{S}$.

If $M \delta N$, there is a filter $\mathfrak{s} \in \mathfrak{S}$ such that $M, N \in \sec \mathfrak{s}$. In the case $x \in$ $\in M \cap N$ we can take $\mathfrak{s} = \dot{x} \in \mathfrak{B} \subset \mathfrak{S}$. If $M \cap N = \emptyset$, M and N are infinite, there are $A \subset M$, $B \subset N$, $|A| = |B| = \omega$, thus $(A,B) = (A_\alpha, B_\alpha)$ for some α and $\mathfrak{s} = \mathfrak{v}_\alpha \cap \mathfrak{w}_\alpha \in \mathfrak{B}$ can be chosen.

Suppose $M \cap N = \emptyset$, $x \in c(M) \cap c(N)$ and, say, $|M| < \omega$. Then there are $p \in M$ such that $x \in c(\{p\})$ and a sequence (x_n) such that $x_n \in N$ and $x_n \to x$. If $x_n = q$ for infinitely many indices n, then $x \in c(\{q\})$, hence $p, q \in$ $\in P$ for some $P \in \mathfrak{P}(c)$, and then $\mathfrak{s} = \dot{F} \in \mathfrak{B}$ can be taken for $F = \{p, q\}$. If there is no such q then $B = \{x_n : n \in \mathbf{N}\}$ is infinite and $(\{p\}, B) = (A_\alpha, B_\alpha)$ for some α ($p, x \in P \in \mathfrak{P}(c)$, so $x_n \to p$).

Assume now that there is $\mathfrak{s} \in \mathfrak{S}$ such that $M, N \in \sec \mathfrak{s}$. We can suppose that $\mathfrak{s} = \bigcap_0^n \mathfrak{r}_i$, $\mathfrak{r}_i \in \mathfrak{B}$ and $(\mathfrak{r}_0, \dots, \mathfrak{r}_n)$ is a Cauchy chain. We show $M \delta N$. This is clear if M and N are infinite. Suppose one of them, say M, is finite.

Assume $\mathfrak{r}_j = \mathfrak{v}_\alpha \cap \mathfrak{w}_\alpha$ for some j; then $\mathfrak{r}_i = \mathfrak{r}_j$ for all i (since $\mathfrak{v}_\alpha$ and $\mathfrak{w}_\alpha$ are free and distinct from $\mathfrak{u}_\beta$, $\mathfrak{v}_\beta$, $\mathfrak{w}_\beta$ $(\beta \neq \alpha)$). Hence $\mathfrak{s} = \mathfrak{r}_j$ and both M and N would be infinite. Thus each $\mathfrak{r}_i$ is either of the form $\dot{a}_\alpha \cap \mathfrak{u}_\alpha$ or of the form $\dot{F}$, so that all points a_α and all sets F occurring are contained in the same $P \in \mathfrak{P}(c)$ (since each $\mathfrak{u}_\alpha$ is free and distinct from $\mathfrak{u}_\beta$ $(\beta \neq \alpha)$). As $M \in \sec \mathfrak{r}_i$, $N \in \sec \mathfrak{r}_j$ for suitable i and j, necessarily $M \cap P \neq \emptyset$ and $P \cap c(N) \neq \emptyset$ in all possible cases, hence $P \subset c(N)$ implies $M \cap c(N) \neq \emptyset$, $M \delta N$. □

COROLLARY 3.14. *If c is a Fréchet* S_1*-closure on X and $|X| \leqq 2^{2^\omega}$ then c admits a compatible Cauchy screen.* □

COROLLARY 3.15. *The cofinite topology on a set X admits a compatible Cauchy screen iff $|X| \leqq 2^{2^\omega}$.*

PROOF. For this T_1-topology c, we have $c(A) = X$ whenever A is infinite. Thus 3.3 and 3.14 can be applied (c is obviously Fréchet). □

It can be happen, for a Riesz proximity δ, that $c(\delta)$ is a T_2-topology, δ does not have a compatible CR-screen but admits a compatible Cauchy screen:

EXAMPLE 3.16. Let (X, c) be the Euclidean line, $\delta = \delta_R^0(c)$, and apply 3.13 and 2.11 (δ is strictly coarser than $\delta_{CR}^0(c)$ since $[0,1]\delta[2,3]$). □

It can also happen that $\delta_1 \subset \delta_2 \subset \delta_3$, $c(\delta_i) = c$ for $i = 1, 2, 3$, δ_1 and δ_3 admit compatible CL-screens, but the Riesz proximity δ_2 does not have any compatible Cauchy screen:

EXAMPLE 3.17. Let $X = A \cup \bigcup_{i \in I} B_i$ where A and the sets B_i are countably infinite and pairwise disjoint. For the discrete topology c on X, let $\mathfrak{S}_1$ consist of the filters $\dot{x}$ $(x \in X)$, $\mathfrak{S}_3$ be generated by a screen base composed of the filters $\dot{x}$ and of all free filters. Clearly $\mathfrak{S}_1$ and $\mathfrak{S}_3$ are CL-screens inducing c, $\delta_1 = \delta(\mathfrak{S}_1) = \delta_R^1(c)$, $\delta_3 = \delta(\mathfrak{S}_3) = \delta_R^0(c)$.

Define $M\delta_2 N$ iff $M \cap N \neq \emptyset$ or there is an i such that $M \cap A$ and $N \cap B_i$ are infinite or there is an i such that $M \cap B_i$ and $N \cap A$ are infinite. Then δ_2 is a Riesz proximity and $\delta_1 \subset \delta_2 \subset \delta_3$. However, $A\delta_2 B_i$ $(i \in I)$, $B_i \overline{\delta}_2 B_j$ $(i, j \in I,\ i \neq j)$ so that, by 3.2, δ_2 does not admit any compatible Cauchy screen if $|I| > 2^{2^\omega}$. □

We have seen that if a closure c admits a compatible CR- or CL-screen then there are among them a coarsest one and a finest one (2.4, 2.7 and [6], 2.8, 3.1 for $I = \emptyset$). The situation is completely different in the case of Cauchy screens: both a coarsest compatible C-screen and a finest compatible C-screen fail to exist in general. The next two examples show this fact in a stronger form, i.e. for proximities as well as for closures.

EXAMPLE 3.18 (cf. [4], 3.15). Let X be an infinite set, $p \in X$, $\mathfrak{u}_0$ a free ultrafilter in X. Let a screen base for $\mathfrak{S}$ be composed of all filters $\dot{x}$ $(x \in X)$ and of the filters $\dot{p} \cap \bigcap_1^n \mathfrak{u}_i$ where $n \in \mathbf{N}$ and $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ are free ultrafilters distinct from $\mathfrak{u}_0$. Clearly $\mathfrak{S}$ is a Cauchy-screen and $\delta(\mathfrak{S}) = \delta$ can be described in the following way: $A\delta B$ iff

$$A \cap B \neq \emptyset$$

or

$$p \in A, \quad B \text{ is infinite}$$

or

$$A \text{ is infinite}, \quad p \in B$$

or

$$\text{both } A \text{ and } B \text{ are infinite}$$

(since any infinite subset of X is contained in a free ultrafilter distinct from $\mathfrak{u}_0$). Thus $c(\delta)$ is a T_2-topology.

Now a screen finer than all these screens $\mathfrak{S}$ (corresponding to the possible choices of $\mathfrak{u}_0$) necessarily coincides with $\{\dot{x}: x \in X\}$, hence it generates the discrete proximity and the discrete topology of X, and then it is compatible with neither δ nor $c(\delta)$. □

EXAMPLE 3.19. Let $X = \mathbf{R}$, c be the Euclidean topology on X and $\delta = \delta_R^0(c)$. Consider the construction contained in the proof of 3.13 by beginning the transfinite sequence (A_α, B_α) with the pair $(\{0\}, B_0)$, $B_0 = \left\{\frac{1}{n+1}: n \in \mathbf{N}\right\}$. Choose a free ultrafilter $\mathfrak{u}_0$ satisfying $B_0 \in \mathfrak{u}_0$. By this, we obtain a Cauchy screen $\mathfrak{S}_1$ such that $\delta(\mathfrak{S}_1) = \delta$.

Apply now the construction another time, now beginning the enumeration with (A_0, B_0), where $A_0 = \mathbf{N}$, B_0 has the same meaning as above, and choose a free ultrafilter $\mathfrak{v}_0$ such that $A_0 \in \mathfrak{v}_0$, finally put $\mathfrak{w}_0 = \mathfrak{u}_0$. We obtain a Cauchy screen $\mathfrak{S}_2$ with $\delta(\mathfrak{S}_2) = \delta$.

If a Cauchy screen $\mathfrak{S}$ is coarser than both $\mathfrak{S}_1$ and $\mathfrak{S}_2$ then it contains $\mathfrak{s} = \dot{0} \cap \mathfrak{u}_0 \cap \mathfrak{v}_0$ so that $\mathfrak{s}$ has to $c(\mathfrak{S})$-converge to 0. Therefore $c(\mathfrak{S}) \neq c$ (since $\mathfrak{v}_0$ does not c converge to 0), implying $\delta(\mathfrak{S}) \neq \delta$. □

4. Coarsest screens as Cauchy screens

It is well-known that, if c is an S_0-closure, then the coarsest screen $\mathfrak{S}^0(c)$ compatible with c is composed of all c-compressed filters ([4], 3.2). Similarly, for a proximity δ, the coarsest screen $\mathfrak{S}^0(\delta)$ compatible with δ is composed of all δ-compressed filters ([3], (6.9) and (6.11)). We examine the question: when is $\mathfrak{S}^0(c)$ or $\mathfrak{S}^0(\delta)$ Cauchy?

The case of $\mathfrak{S}^0(c)$ is rather simple:

THEOREM 4.1. *Let (X, c) be a symmetric closure space, $\mathfrak{s}_0$ the filter composed of all cofinite subsets of X. The following statements are equivalent:*

(a) *$\mathfrak{S}_0(c)$ is a Cauchy screen,*

(b) *c is S_2 and every c-compressed filter is strongly c-compressed,*

(c) *all sets $P \in \mathfrak{P}(c)$ are finite and $\mathfrak{v}_c(x) = \dot{P}$ for $x \in P \in \mathfrak{P}(c)$, with one possible exception P_0 for which $\mathfrak{v}_c(x) = \dot{P}_0 \cap \mathfrak{s}_0$ for $x \in P_0$.*

PROOF. (a) ⇒ (b): If $\mathfrak{S}^0(c)$ is Cauchy then c is S_1 by 3.1. Thus every neighbourhood filter $\mathfrak{v}_c(x)$ is c-compressed, so that $\mathfrak{S}^0(c)$ is a CR-screen, c is S_2, and $\mathfrak{S}^0(c)$ is finer than $\mathfrak{S}_{CR}^0(c)$.

(b) ⇒ (c): Now $\mathfrak{S}^0(c) = \mathfrak{S}_{CR}^0(c)$. If $x \in X$ and $\mathfrak{v}_c(x)$ has a finite element then $\mathfrak{v}_c(x)$ has a smallest (finite) element P, $\mathfrak{v}_c(x) = \dot{P}$, and $P \in \mathfrak{P}(c)$ as c is S_1.

Suppose now that there is $x \in X$ such that every element of $\mathfrak{v}_c(x)$ is infinite. Then x is a cluster point of $\mathfrak{s}_0$ and, since a free filter is c-compressed, hence strongly c-compressed by the hypothesis, necessarily $\mathfrak{s}_0 \to x$, $\mathfrak{v}_c(x) \subset \subset \mathfrak{s}_0$. As c is S_2, two such points x must belong to the same element P_0 of $\mathfrak{P}(c)$, and then $\mathfrak{v}_c(x) \subset \dot{P}_0$, $\mathfrak{v}_c(x) \subset \dot{P}_0 \cap \mathfrak{s}_0$. If $x \in P_0$, $y \in X - P_0$, (S_1) implies that there is $V \in \mathfrak{v}_c(x)$ such that $y \notin V$, so $\dot{P}_0 \cap \mathfrak{s}_0 \subset \mathfrak{v}_c(x)$ and $\mathfrak{v}_c(x) = \dot{P}_0 \cap \mathfrak{s}_0$ for $x \in P_0$.

(c) $\Rightarrow$ (b): Suppose c has the structure described in (c). Then clearly c is S_2. Let $\mathfrak{s}$ be a c-compressed filter that has $x \in X$ for cluster point. If $y \in \bigcap \mathfrak{s}$ then $\mathfrak{s} \to y$, hence $\mathfrak{v}_c(x) = \mathfrak{v}_c(y)$ by (S_2) and $\mathfrak{s} \to x$. If $\mathfrak{s}$ is free then $x \in P_0$ and $\mathfrak{s}_0 \subset \mathfrak{s}$ implies $\mathfrak{s} \to x$ again.

(b) $\Rightarrow$ (a): Obvious by $\mathfrak{S}^0(c) = \mathfrak{S}^0_{CR}(c)$. □

While the answer to the question concerning $\mathfrak{S}^0(c)$ leads to a rather peculiar class of spaces, the question concerning $\mathfrak{S}^0(\delta)$ is more interesting. For this purpose, we recall that a proximity δ on X is said to be *Efremovich* iff $A\bar{\delta}B$ implies the existence of $U, V \subset X$ such that $U \cap V = \emptyset$, $A\bar{\delta}X - U$, $B\bar{\delta}X - V$.

Now we can prove:

THEOREM 4.2. *Let (X, δ) be a proximity space. The following statements are equivalent:*

(a) $\mathfrak{S}^0 = \mathfrak{S}^0(\delta)$ *is a Cauchy screen,*

(b) $\mathfrak{S}^{0e}$ *is a Cauchy screen,*

(c) *for ultrafilters in X, the relation δ is transitive,*

(d) δ *is an Efremovich proximity.*

PROOF. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Let $\mathfrak{u}_i$ $(i = 1, 2, 3)$ be ultrafilters in X, $\mathfrak{u}_1\delta\mathfrak{u}_2$, $\mathfrak{u}_2\delta\mathfrak{u}_3$. By [5], 3.7, $\mathfrak{u}_1 \cap \mathfrak{u}_2$ and $\mathfrak{u}_2 \cap \mathfrak{u}_3$ are δ-compressed, hence they belong to $\mathfrak{S}^{0e}$, further $(\mathfrak{u}_1 \cap \mathfrak{u}_2)\Delta(\mathfrak{u}_2 \cap \mathfrak{u}_3)$, so that $\mathfrak{s} = \mathfrak{u}_1 \cap \mathfrak{u}_2 \cap \mathfrak{u}_3 \in \mathfrak{S}^{0e}$. Both $\mathfrak{u}_1$ and $\mathfrak{u}_3$ being finer than the δ-compressed filter $\mathfrak{s}$, $A_1 \in \mathfrak{u}_1$, $A_3 \in \mathfrak{u}_3$ imply $A_1, A_3 \in \sec \mathfrak{s}$, hence $A_1\delta A_3$, and $\mathfrak{u}_1\delta\mathfrak{u}_3$.

(c) $\Rightarrow$ (d): Suppose $A, B \subset X$ and $U \cap V \neq \emptyset$ whenever $A\bar{\delta}X - U$, $B\bar{\delta}X - - V$. The sets U and V in question constitute filters $\mathfrak{s}$ and $\mathfrak{t}$, respectively, that fulfil $\mathfrak{s}\Delta\mathfrak{t}$. Let $\mathfrak{w}$ be an ultrafilter finer than $\mathfrak{s}(\cap)\mathfrak{t}$. For $W \in \mathfrak{w}$, $A\bar{\delta}W$ would imply $X - W \in \mathfrak{s} \subset \mathfrak{w}$ which is impossible. Hence $\dot{A}\delta\mathfrak{w}$, and similarly $\dot{B}\delta\mathfrak{w}$. By [5], 3.6, there are ultrafilters $\mathfrak{u}$ and $\mathfrak{v}$ such that $\dot{A} \subset \mathfrak{u}$, $\dot{B} \subset \mathfrak{v}$, $\mathfrak{u}\delta\mathfrak{w}$, $\mathfrak{v}\delta\mathfrak{w}$. By hypothesis $\mathfrak{u}\delta\mathfrak{v}$, hence $A\delta B$.

(d) $\Rightarrow$ (a): Let $\mathfrak{s}$ and $\mathfrak{t}$ be δ-compressed filters such that $\mathfrak{s}\Delta\mathfrak{t}$. We show that $\mathfrak{s} \cap \mathfrak{t}$ is δ-compressed. In fact, if $A, B \subset X$, $A, B \in \sec(\mathfrak{s} \cap \mathfrak{t})$ and, say, $A, B \in \sec \mathfrak{s}$, then $A\delta B$. The same is true if $A, B \in \sec \mathfrak{t}$. If $A \in \sec \mathfrak{s}$, $B \in \in \sec \mathfrak{t}$ and $A\bar{\delta}B$ were true, then two sets U and V would exist such that $U \cap V = \emptyset$, $A\bar{\delta}X - U$, $B\bar{\delta}X - V$. The δ-compressedness of $\mathfrak{s}$ and $\mathfrak{t}$ implies $U \in \in \mathfrak{s}$, $V \in \mathfrak{t}$ which would contradict $\mathfrak{s}\Delta\mathfrak{t}$. If $A \in \sec \mathfrak{t}$, $B \in \sec \mathfrak{s}$, the reasoning is similar. □

The essential content of the implication (d) $\Rightarrow$ (a) lies in the fact that the δ-compressed filters coincide with the Cauchy filters of the totally bounded uniformity compatible wit δ (see e.g. [2], (4.2.26), (5.2.8) and (5.2.9)).

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INTERPOLATION BETWEEN HARDY–LORENTZ–ORLICZ SPACES

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1. Preliminaries

In what follows L^p will denote $L^p[0,1]$. For any set A we shall denote the characteristic function of A by $\chi(A)$. For each $f \in L^1$, we put

$$\varepsilon_n f := s_{2^n} f \qquad (n \in \mathbf{N});$$

where $s_{2^n} f$ is the 2^n-th partial sum of the Walsh–Fourier series of f.

By Jensen's inequality we have

$$\|\varepsilon_n f\|_p \leqq \|f\|_p. \tag{1}$$

The dyadic maximal operator for $f \in L^1$ is defined by

$$\varepsilon f := \sup_{n \in \mathbf{N}} |\varepsilon_n f|.$$

For each $f \in L^1$ and $0 < p < \infty$ set

$$\|f\|_{\mathbf{H}^p} := \|\varepsilon f\|_p.$$

On the set of Walsh polynomials $\mathcal{P}$, the map $f \to \|f\|_{\mathbf{H}^p}$ is a norm for $1 \leqq p < \infty$ and a quasi-norm for $0 < p < 1$.

The dyadic Hardy spaces $\mathbf{H}^p$ are defined to be the closure of $\mathcal{P}$ in the quasi-norm $\|\ldots\|_{\mathbf{H}^p}$ for $0 < p < \infty$.

Definition 1 (Kalugina). We shall say that f is a parameter function or $f \in B_k$ if it is a positive, increasing and continuous function on $(0,\infty)$ such that

$$C_f = \int_0^\infty \min(1, 1/t)\overline{f}(t)\,dt/t < \infty,$$

where

$$\overline{f}(t) = \sup_{s>0} \frac{f(t)}{f(s)}.$$

For $f \in B_k$ and an Orlicz function F we introduced in [2] the function norm $\Phi_{f,F}$ by

$$\Phi_{f,F}(u) = \inf\left\{r > 0\colon \int_0^\infty F\left(\frac{u(t)}{rf(t)}\right)\,dt/t \leqq 1\right\}$$

where u is a non-negative measurable function on $(0,\infty)$.

For a Banach couple of interpolation $\overline{A} = (A_0, A_1)$ we define the space

$$\overline{A}_{f,F} = (A_0, A_1)_{f,F}$$

as the space of all $a \in \sum(\overline{A})$ such that $\Phi_{f,F}\left[K(t,a)\right] < \infty$, where $K(t,a)$ is the Peetre functional.

By construction, $\overline{A}_{f,F}$ is an interpolation space between A_0 and A_1 (see [6]). For a detailed study of these spaces see [2].

Definition 2. Let (Ω, μ) be a σ-finite measure space. Suppose ϕ is a non-decreasing concave function on $[0,\infty)$ such that $\phi(0) = 0$ and F is an Orlicz function. The Lorentz–Orlicz space $L_{\phi F} = L_{\phi F}(\mu)$ is defined to be the space of all (classes of) μ-measurable functions x on Ω such that the functional

$$\|x\|^*_{\phi F} := \inf\left\{r > 0\colon \int_0^\infty F\left(\frac{\phi(t)x^*(t)}{r}\right)\,dt/t \leqq 1\right\}$$

is finite, where x^* is the non-increasing rearrangement of x with respect to the measure μ.

It is usual to define a norm $\|x\|_{\phi F}$ on $L_{\phi F}$ as follows:

$$\|x\|_{\phi F} := \inf\left\{r > 0\colon \int_0^\infty F\left(\frac{\phi(t)x^{**}(t)}{r}\right)\,dt/t \leqq 1\right\},$$

where

$$x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds.$$

The following result has been proved in [2].

Proposition 1. *If* $\phi \in B_k$ *and* $f(t) = \frac{t}{\phi(t)}$ *then*

$$(L^1, L^\infty)_{f,F} = L_{\phi F}.$$

2. Interpolation between dyadic Hardy–Lorentz–Orlicz spaces

In this section we introduce the dyadic Hardy–Lorentz–Orlicz space $H_{\phi F}$, and obtain some results about interpolation between them.

DEFINITION 3. Suppose ϕ is a non-decreasing, concave function on $[0,\infty)$ such that $\phi(0)=0$, and F is an Orlicz function. We shall say that a function $f\in L^1$ is in the dyadic Hardy–Lorentz–Orlicz space $H_{\phi F}$ if and only if $\varepsilon f\in$ $\in L_{\phi F}$. We provide $H_{\phi F}$ with the norm

$$\|f\|_{H\phi F} := \|\varepsilon f\|_{\phi F}.$$

For the spaces $H_{\phi F}$ we have the following result.

THEOREM 1. *If $\phi\in B_k$ and $f(t)=\frac{t}{\phi(t)}$ then*

$$(H^1,L^\infty)_{f,F}=H_{\phi F}.$$

PROOF. We define the operator $T(a)=\varepsilon a$ on L^1. $T(a)$ is sublinear, and using inequality (1) of the preliminaries we get

(i) $\|T(a)\|_1=\|a\|_{H^1}$, and

(ii) $\|T(a)\|_\infty\leqq C\|a\|_\infty$.

These inequalities imply that T is bounded from H^1 into L^1 and from L^∞ into L^∞. By interpolation we conclude that T is bounded from $(H^1,L^\infty)_{f,F}$ into $(L^1,L^\infty)_{f,F}=L_{\phi F}$. Thus there exists $C_1>0$ such that

$$\|a\|_{H_{\phi F}}=\|\varepsilon a\|_{\phi F}\leqq C_1\|a\|_{f,F}.$$

Therefore $(H^1,L^\infty)_{f,F}\subset H_{\phi F}$.

From Lemma 2 in [1] we have that, given $a\in L^1$ and $t>0$, there are functions h_t, g_t belonging to H^1 and L^∞ respectively such that $a=h_t+g_t$ and

$$\|h_t\|_{H^1}\leqq\int_{\{\varepsilon a>(\varepsilon a)^*(t)\}}\varepsilon a(S)\,ds,\quad \|g_t\|_\infty\leqq C(\varepsilon a)^*(t).$$

The inequality

$$K(t,a;H^1,L^\infty)\leqq\|h_t\|_{H^1}+t\|g_t\|_{L^\infty}$$

implies

$$F\left(\frac{1}{2}C^{-1}\frac{k(t,a)}{f(t)}\right)\leqq F\left(\frac{1}{2}\left[t(\varepsilon a)^*(t)+\int_{\{\varepsilon a>(\varepsilon a)^*(t)\}}\varepsilon a(s)\,ds\right]\frac{1}{f(t)}\right)\leqq$$

$$\leqq F\left(\frac{1}{2}\left[t(\varepsilon a)^*(t)+\int_0^t(\varepsilon a)^*(s)\,ds\right]\frac{1}{f(t)}\right)\leqq$$

$$\leqq F\left(\frac{1}{f(t)}\int_0^1 (\varepsilon a)^*(s)\,ds\right) = F\left(\frac{1}{f(t)}(\varepsilon a)^{**}(t)\,ds\right) = F\big(\phi(t)(\varepsilon a)^{**}(t)\big).$$

Therefore we conclude that $\|a\|_{f,F} \leqq 2C\|a\|_{H_{\phi F}}$. The theorem is proved.
Applying the reiteration theorem given in [2] we get the following result.

THEOREM 2. *Let the functions* ϕ_0, ϕ_1, f *and*

$$\phi_{01}(t) = \frac{\phi_0(t)}{\phi_1(t)}$$

belong to B_k. *Set*

$$\phi_2 = \frac{\phi_0(t)}{f\big(\phi_{01}(t)\big)}.$$

Then

$$(H_{\phi_0 F_0}, H_{\phi_1 F_1})_{f,F} = H_{\phi_2 F}.$$

PROOF. Let $f_i(t) = \frac{t}{\phi_i(t)}$, $i = 0, 1$ and $\tau(t) = \frac{f_0(t)}{f_1(t)}$. From Theorem 1 and the reiteration theorem in [2] we have

$$(H_{\phi_0 F_0}, H_{\phi_1 F_1})_{f,F} = \Big((H^1, L^\infty)_{f_0,F_0}, (H^1, L^\infty)_{f_1,F_1}\Big)_{f,F} = (H^1, L^\infty)_{f_2,F}$$

where

$$f_2(t) = f_0(t) f\big(\tau(t)\big) = \frac{t}{\phi_0(t)} f\big(\phi_{01}(t)\big).$$

Using Theorem 1 again we obtain

$$(H_{\phi_0 F_0}, H_{\phi_1 F_1})_{f,F} = (H^1, L^\infty)_{f_2,F} = H_{\phi_2 F}$$

with $\phi_2(t) = \frac{t}{f_2(t)}$. The proof is complete.

In the classical case, each Hardy space H^p consists of functions G analytic on the unit disc and satisfying

$$\|G\|_{H^p} := \sup_{0\leqq r<1} \left(\frac{1}{2\pi}\int_{2\pi}^0 \big|G(re^{i\theta})\big|^p\,d\theta\right)^{1/p} < \infty.$$

This condition is equivalent to the L^p-integrability of the non-tangential maximal function

$$G_s^*(e^{i\theta}) := \sup_{Z\in\Gamma_s(\theta)} \big(G(z)\big|$$

where for each $0 < s < 1$ and $\theta \in [0, 2\pi]$, $\Gamma_s(\theta)$ represents the convex hull of the set $\{e^{i\theta}\}$ and $\{z \in C : |z| \leqq \}$.

By taking real parts of the boundary functions

$$\lim_{r \uparrow 1} G(re^{i\theta}) \qquad (0 \leqq \theta \leqq 2\pi)$$

and identifying the boundary of the unit disc with the interval $[0, 1]$, one generates the classical real Hardy spaces $\mathcal{H}^p$ for $0 < p < \infty$. $\mathcal{H}^p$ is endowed with the quasi-norm

$$\|g\|_{\mathcal{H}^p} = \|G_s^*\|_p$$

where G is the analytic function associated to g.

In the classical case we shall say that a function g is in the Hardy–Lorentz–Orlicz space $\mathcal{H}_{\phi F}$ if and only if the non-tangential maximal function $G_s^* \in L_{\phi F}$.

Using a "canonical decomposition" for functions g in $\mathcal{H}^1$ given in [5] (p. 192) we get an analogous result to Lemma 2 for g in $\mathcal{H}^1$. Using that result we obtain results for $\mathcal{H}_{\phi F}$ analogous to those proved above for $\mathcal{H}_{\phi F}$.

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ON THE AVERAGE VALUE FOR THE NUMBER OF DIVISORS OF NUMBERS OF FORM $ab+1$

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1. The number of distinct prime divisors of the integer n is denoted by $\omega(n)$. The number of positive divisors of n is denoted by $\tau(n)$. $\varphi(n)$ denotes Euler's function. $\mu(n)$ is the Möbius function. $P(n)$ denotes the greatest prime factor of n.

Let $\mathcal{A}$, $\mathcal{B}$ be two sets of distinct integers with

$$\mathcal{A},\mathcal{B} \subset [1,x], \quad |\mathcal{A}|,|\mathcal{B}| > \varepsilon x. \tag{1}$$

In the last 15 years several authors have studied the prime factor structure and arithmetic properties of the sums $a+b$ with $a \in \mathcal{A}$, $b \in \mathcal{B}$; see the survey paper [7]. In particular, Erdős, Maier and Sárközy [4] showed that assuming (1), an Erdős–Kac [3] type theorem holds for the sums $a+b$, i.e., the frequency amongst all sums $a+b$ with $a \in \mathcal{A}$, $b \in \mathcal{B}$ of those for which

$$\omega(a+b) - \log\log x \leqq z(\log\log x)^{1/2}$$

is approximately Gaussian for large x. This result has been extended and sharpened in various directions by Elliott and Sárközy [1] and Tenenbaum [9]. Moreover, Sárközy and Stewart [8] gave a lower bound for

$$\big(|\mathcal{A}||B|\big)^{-1} \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \tau(a+b). \tag{2}$$

While many results have been proved on arithmetic properties of sums $a+b$, much less is known on products ab. Recently, Elliott and Sárközy [2] have proved the multiplicative analogue of the above mentioned result of Erdős, Maier and Sárközy by showing that assuming (1), an Erdős–Kac type theorem holds for the numbers $ab+1$. The goal of this paper is to study the multiplicative analogue of the mean (2), i.e., to study the sum

$$T = \big(|\mathcal{A}||\mathcal{B}|\big)^{-1} \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \tau(ab+1). \tag{3}$$

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One might like to show that assuming (1), the mean (3) is $\sim \log x^2 = 2\log x$ (note that this is certainly not so taking the average of the values of $\tau(ab)$ instead of $\tau(ab+1)$). This could be considered as the "sequences analogue" of the Titchmarsh divisor problem. However, no asymptotics can be given for the mean (3) as the following examples show:

Assume that $x \to +\infty$, $H \to +\infty$, $H < \frac{1}{2}\log x$, and let

$$\mathcal{A} = \left\{ a : a \leqq x,\ a \equiv 1 \quad \left(\operatorname{mod} \prod_{p \leqq H} p \right) \right\},$$

$$\mathcal{B} = \left\{ b : b \leqq x,\ b \equiv -1 \quad \left(\operatorname{mod} \prod_{p \leqq H} p \right) \right\}.$$

Then writing

$$U = \frac{x^2}{|\mathcal{A}||\mathcal{B}|}, \tag{4}$$

we have $U = \exp\Big(\big(2 + o(1)\big) H\Big)$ and

$$T = \exp\left(\big(\log 2 + o(1)\big) \frac{H}{\log H} \right) \log x$$

(where T is defined by (3)). On the other hand, let

$$\mathcal{A} = \left\{ a : a \leqq x,\ 0 < \text{ index of } a \text{ modulo } p < \frac{p-1}{2} \text{ for all } p \leqq H \right\},$$

$$\mathcal{B} = \left\{ b : b \leqq x,\ \frac{p-1}{2} < \text{ index of } b \text{ modulo } p < p - 1 \text{ for all } p \leqq H \right\}.$$

Then defining T and U by (3) and (4), respectively, we have

$$U = \exp\left(\big(2\log 2 + o(1)\big) \frac{H}{\log H} \right)$$

and

$$T = \big(c + o(1)\big) (\log H)^{-1} \log x$$

(where c is a positive absolute constant) so that T can be both much greater and much smaller, than $\log x$.

As these examples show, the mean (3) may depend considerably on the contribution of the small prime factors of the numbers $ab+1$. Thus we may expect asymptotics for the average value of the number of divisors of the numbers $ab+1$ only if we restrict ourselves to the divisors all of whose prime factors are large enough depending on $U = X^2\big(|\mathcal{A}||\mathcal{B}|\big)^{-1}$.

We shall use the following notations:

If K is a positive real number, then we write $P_K = \prod_{p \leqq K} p$ and $\mathcal{N}_K = \{n\colon (n, P_K) = 1\}$. If $K > 0$ is a fixed parameter, then $\sum'$ will denote summation over $\mathcal{N}_K$ so that, e.g., $\sum'_{d|n}$ means summation over the numbers d with $d|n$, $d \in \mathcal{N}_K$. Moreover, we define the positive integers $u_K(n)$ and $v_K(n)$ by

$$n = u_K(n)v_K(n), \quad P\big(u_K(n)\big) \leqq K, \quad v_K(n) \in \mathcal{N}_K. \tag{5}$$

We write

$$\tau_K(n) = \tau\big(v_K(n)\big) = \sum_{d|n}{}' 1. \tag{6}$$

Our goal is to show that if $\mathcal{A}$, $\mathcal{B}$ are "dense" sets, K is large enough in terms of U (defined by (4)), but it is not "very large" in terms of x, then, writing

$$T_K = \big(|\mathcal{A}||\mathcal{B}|\big)^{-1} \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \tau_K(ab+1), \tag{7}$$

we have

$$T_K \sim \prod_{p\leqq K}\left(1-\frac{1}{p}\right)\log x^2 \quad \left(= 2\prod_{p\leqq K}\left(1-\frac{1}{p}\right)\log x\right) \tag{8}$$

as expected. In fact, we will prove

THEOREM. *Assume that x is an integer with $x \geqq 3$, K is a real number with*

$$2 \leqq K \leqq \exp\big((\log x)^{1/2}\big) \tag{9}$$

and $\mathcal{A}, \mathcal{B} \subset \{1, 2, \dots, x\}$. Then there is an absolute constant c_1 such that

$$\left|T_K - 2\prod_{p\leqq K}\left(1-\frac{1}{p}\right)\log x\right| < \tag{10}$$

$$< c_1\big(|\mathcal{A}||\mathcal{B}|\big)^{-1}x^2\left(\frac{\log x}{K^{1/2}\log^2 K}+\log\log x+\log K\right)$$

where T_K is defined by (7).

It follows that if

$$U=\frac{x^2}{|\mathcal{A}||\mathcal{B}|}=o\left(\min\left(\frac{\log x}{\log\log x\log K},\ K^{1/2}\log K,\ \frac{\log x}{\log^2 K}\right)\right),$$

then (8) holds.

We remark that in a similar way, one could sharpen the result in [8] by proving a theorem analogous to the one above with sums $a+b$ in place of the numbers $ab+1$.

2. Preliminary lemmas. Write

$$f(m)=\Big|\big\{(a,b)\colon a\in\mathcal{A},\ b\in\mathcal{B},\ m|(ab+1)\big\}\Big|. \tag{11}$$

LEMMA 1. *If m is a positive integer, $x\geqq m$ and $\mathcal{A},\mathcal{B}\subset\{1,2,\dots,[x]\}$, then we have*

$$f(m)\leqq 2\min\big(|\mathcal{A}|,|\mathcal{B}|\big)\frac{x}{m}. \tag{12}$$

PROOF. We may assume that

$$|\mathcal{A}|\leqq|\mathcal{B}|. \tag{13}$$

Then we have

$$f(m)=\sum_{a\in\mathcal{A}}\Big|\big\{b\colon b\in\mathcal{B},\ ab+1\equiv 0 \ (\mathrm{mod}\ m)\big\}\Big|\leqq$$

$$\leqq\sum_{a\in\mathcal{A}}\Big|\big\{n\colon n\leqq x,\ an+1\equiv 0\ (\mathrm{mod}\ m)\big\}\Big|\leqq\sum_{a\in\mathcal{A}}\left(\frac{x}{m}+1\right)\leqq$$

$$\leqq\sum_{a\in\mathcal{A}}\frac{2x}{m}=2|\mathcal{A}|\frac{x}{m}$$

which, by (13), proves (12).

LEMMA 2. *There is an absolute constant c_2 such that for $K \geqq 2$ we have*

$$\left|\sum_{d|P_k} \frac{\mu(d)\log d}{d}\right| < c_2.$$

PROOF. We have

$$\sum_{d|P_k} \frac{\mu(d)\log d}{d} = \sum_{d|P_k} \frac{\mu(d)}{d} \sum_{p|d} \log p =$$

$$= \sum_{p\leqq K} \log p \sum_{p|d,\, d|P_k} \frac{\mu(d)}{d} = \sum_{p\leqq K} \log p \sum_{t|(P_K/p)} \frac{\mu(pt)}{pt} =$$

$$= \sum_{p\leqq K} \log p \left(-\frac{1}{p} \sum_{t|(P_K/p)} \frac{\mu(t)}{t}\right) = -\sum_{p\leqq K} \frac{\log p}{p} \prod_{q\leqq K,\, q\neq p} \left(1-\frac{1}{q}\right) =$$

$$= O\left(\left(\sum_{p\leqq K} \frac{\log p}{p}\right) \prod_{q\leqq K} \left(1-\frac{1}{q}\right)\right) = O\big((\log K)(\log K)^{-1}\big) = O(1).$$

LEMMA 3. *There is an absolute constant c_3 such that for $K \geqq 2$, $z \geqq 1$ we have*

$$\left|{\sum_{n\leqq z}}' \frac{1}{n} - (\log z) \prod_{p\leqq K} \left(1-\frac{1}{p}\right)\right| < c_3 \log K.$$

PROOF. By Lemma 2 we have

$$\sum_{n\leqq z}{}' \frac{1}{n} = \sum_{\substack{n\leqq z\\ (n,P_K)=1}} \frac{1}{n} = \sum_{n\leqq z} \left(\sum_{d|(n,P_K)} \mu(d)\right) \frac{1}{n} =$$

$$= \sum_{d|P_K} \mu(d) \sum_{\substack{n\leqq z\\ d|n}} \frac{1}{n} = \sum_{d|P_K} \mu(d) \sum_{t\leqq z/d} \frac{1}{td} =$$

$$= \sum_{d|P_K} \frac{\mu(d)}{d} \sum_{t\leqq z/d} \frac{1}{t} = \sum_{d|P_K} \frac{\mu(d)}{d} \log\frac{z}{d} + O\left(\sum_{d|P_K} \frac{1}{d}\right) =$$

$$= (\log z) \sum_{d|P_K} \frac{\mu(d)}{d} - \sum_{d|P_K} \frac{\mu(d)\log d}{d} + O\left(\prod_{p \leqq K}\left(1+\frac{1}{p}\right)\right) =$$

$$= (\log z) \prod_{p \leqq K}\left(1-\frac{1}{p}\right) + O(\log K).$$

LEMMA 4. *There is an absolute constant c_4 such that for $K \geqq 2$, $z \geqq 1$ we have*

$$\sum_{n \leqq z}{}' \left(\frac{1}{\varphi(n)} - \frac{1}{n}\right) < c_4 \frac{\log z}{K \log^2 K}. \tag{14}$$

PROOF. Let $\mathcal{G}$ denote the set of positive integers composed of powers of primes p with $K < p \leqq z$ so that $\mathcal{G}$ contains every integer n with $n \leqq z$, $n \in \mathcal{N}_K$. Then we have

$$\sum_{n \leqq z}{}' \left(\frac{1}{\varphi(n)} - \frac{1}{n}\right) < \sum_{n \in \mathcal{G}} \left(\frac{1}{\varphi(n)} - \frac{1}{n}\right) = \tag{15}$$

$$= \prod_{K<p\leqq z} \sum_{r=0}^{+\infty} \frac{1}{\varphi(p^r)} - \prod_{K<p\leqq z} \sum_{r=0}^{+\infty} \frac{1}{p^r} =$$

$$= \left(\prod_{K<p\leqq z} \sum_{r=0}^{+\infty} \frac{1}{p^r}\right)\left(\prod_{K<p\leqq z}\left(1+\sum_{r=1}^{+\infty}\frac{1}{\varphi(p^r)}\right)\left(\sum_{r=0}^{+\infty}\frac{1}{p^r}\right)^{-1} - 1\right) =$$

$$= \prod_{K<p\leqq z}\left(1-\frac{1}{p}\right)^{-1}\left(\prod_{K<p\leqq z}\left(1+\frac{p}{(p-1)^2}\right)\left(1-\frac{1}{p}\right) - 1\right) =$$

$$= \prod_{K<p\leqq z}\left(1-\frac{1}{p}\right)^{-1}\left(\prod_{K<p\leqq z}\left(1+\frac{1}{(p-1)p}\right) - 1\right) \ll$$

$$\ll \frac{\log z}{\log K}\left(\prod_{K<p\leqq z}\left(1+\frac{2}{p^2}\right) - 1\right) \ll \frac{\log z}{\log K} \sum_{K<p\leqq z} \frac{1}{p^2} <$$

$$< \frac{\log z}{\log K} \sum_{K<p} \frac{1}{p^2}.$$

Using the prime number theorem, we obtain by partial summation that

$$\sum_{K<p} \frac{1}{p^2} \ll \frac{1}{K \log K}. \tag{16}$$

Combining (15) and (16), we obtain (14).

LEMMA 5. *There is an absolute constant c_5 such that for $K \geqq 2$, $z \geqq 1$ we have*

$$\sum_{n \leqq z}{}' \frac{1}{\varphi(n)} < c_5 \left(\frac{\log z}{\log K} + \log K \right).$$

PROOF. This follows from Lemma 3, Lemma 4 and

$$\prod_{p \leqq K} \left(1 - \frac{1}{p}\right) \ll \frac{1}{\log K}.$$

LEMMA 6. *If $z \geqq 1$ and m is a positive integer, then we have*

$$\left| \sum_{\substack{i \leqq z \\ (i,m)>1}} 1 - z\left(1 - \frac{\varphi(m)}{m}\right) \right| \leqq 2\tau(m).$$

PROOF. By

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}$$

and

$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d},$$

we have

$$\left| \sum_{\substack{i \leqq z \\ (i,m)>1}} 1 - z\left(1 - \frac{\varphi(m)}{m}\right) \right| =$$

$$= \left| \left(\sum_{i \leqq z} 1 - \sum_{\substack{i \leqq z \\ (i,m)=1}} 1 \right) - z\left(1 - \frac{\varphi(m)}{m}\right) \right| \leqq$$

$$\leqq \left| \sum_{i \leqq z} 1 - z \right| + \left| z\frac{\varphi(m)}{m} - \sum_{\substack{i \leqq z \\ (i,m)=1}} 1 \right| \leqq$$

$$\leqq 1 + \left| z \sum_{d|m} \frac{\mu(d)}{d} - \sum_{i \leqq z} \sum_{d|(i,m)} \mu(d) \right| = 1 + \left| \sum_{d|m} \mu(d)\frac{z}{d} - \sum_{d|m} \mu(d) \sum_{\substack{i \leqq z \\ d|i}} 1 \right| =$$

$$= 1 + \left| \sum_{d|m} \mu(d) \left(\frac{z}{d} - \left[\frac{z}{d}\right] \right) \right| \leqq 1 + \sum_{d|m} \left| \frac{z}{d} - \left[\frac{z}{d}\right] \right| \leqq$$

$$\leqq 1 + \sum_{d|m} 1 = 1 + \tau(m) \leqq 2\tau(m).$$

LEMMA 7. *There is a positive constant c_6 such that uniformly for $2 \leqq$ $\leqq K < L$ we have*

$$\sum_{\substack{P(n) \leqq K \\ n > L}} \frac{1}{n} < c_6 \log K \exp\left(-\frac{\log L}{4 \log K}\right). \tag{17}$$

PROOF. Set $\sigma = (4 \log K)^{-1}$. Then we have

$$\sum_{\substack{P(n) \leqq K \\ n > L}} \frac{1}{n} \leqq \sum_{P(n) \leqq K} \frac{(n/L)^\sigma}{n} = L^{-\sigma} \sum_{P(n) \leqq K} n^{\sigma - 1} = \tag{18}$$

$$= \exp\left(-\frac{\log L}{4 \log K}\right) \prod_{p \leqq K} \left(1 - \frac{1}{p^{1-\sigma}}\right)^{-1}.$$

If $K \geqq 10$, then

$$\prod_{p\leqq K}\left(1-\frac{1}{p^{1-\sigma}}\right)^{-1} = \exp\left(-\sum_{p\leqq K}\log\left(1-\frac{1}{p^{1-\sigma}}\right)\right) \ll \tag{19}$$

$$\ll \exp\left(\sum_{p\leqq K}\frac{1}{p^{1-\sigma}}\right) = \exp\left(\sum_{p\leqq K}\frac{1}{p}+\sum_{p\leqq K}\frac{p^{\sigma}-1}{p}\right) =$$

$$= \exp\left(\left(\log\log K + O(1)\right) + O\left(\sum_{p\leqq K}\frac{\sigma\log p}{p}\right)\right) \ll \log K$$

(and the product estimated in (19) is bounded for $2 \leqq K < 10$).

(17) follows from (18) and (19).

Write

$$h(m) = \sum_{d|m} d^{-1/2}$$

and

$$\mathcal{M}(z) = \left\{m: m \leqq z,\ m \in \mathcal{N}_K,\ h(m) > 2\right\}.$$

LEMMA 8. *For $z > 1$ we have*

$$\sum_{m\in\mathcal{M}(z)}\frac{1}{m} < c_7 K^{-1/2}\left((\log K)^{-2}\log z + 1\right). \tag{20}$$

PROOF. By Lemma 3 we have

$$\sum_{m\in\mathcal{M}(z)}\frac{1}{m} \leqq {\sum_{m\leqq z}}'\frac{h(m)-1}{m} = {\sum_{m\leqq z}}'\frac{1}{m}\sum_{\substack{d|m\\ d>1}} d^{-1/2} = \tag{21}$$

$$= {\sum_{1<d\leqq z}}' d^{-1/2}{\sum_{\substack{m\leqq z\\ d|m}}}'\frac{1}{m} = {\sum_{1<d\leqq z}}' d^{-1/2}{\sum_{t\leqq z/d}}'\frac{1}{dt} \leqq$$

$$\leqq {\sum_{1<d\leqq z}}' d^{-3/2}{\sum_{t\leqq z}}'\frac{1}{t} \ll \left(\frac{\log z}{\log K}+\log K\right){\sum_{1<d\leqq z}}' d^{-3/2}.$$

Here we have

$$(22)\qquad \sum_{1<d\leqq z}{}' d^{-3/2} \leqq \prod_{K<p\leqq z}\left(1-p^{-3/2}\right)^{-1} - 1 \ll \sum_{K<p} p^{-3/2} \ll$$

$$\ll K^{-1/2}(\log K)^{-1}.$$

(20) follows from (21) and (22).

LEMMA 9. *If M, N, q are positive integers, $a_{M+1}, a_{M+2}, \dots, a_{M+N}$ are complex numbers and we write*

$$S(\chi) = \sum_{n=M+1}^{M+N} a_n \chi(n),$$

then we have

$$\sum_{\chi (\operatorname{mod} q)} \left|S(\chi)\right|^2 \leqq \varphi(q)\left(1 + \left[\frac{N-1}{q}\right]\right) \sum_{\substack{M<n\leqq M+N \\ (n,q)=1}} |a_n|^2.$$

This is a well-known inequality; see, e.g. [6, p. 51].

Gallagher's character version of the large sieve [5] (see also [6, p. 15]) will play a crucial role in the proof:

LEMMA 10. *If M, N, $a_{M+1}, a_{M+2}, \dots, a_{M+N}$ and $S(\chi)$ are defined as in Lemma 9 then for $Q \geqq 1$ we have*

$$\sum_{q\leqq Q} \frac{q}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)}{}^{*} \left|S(\chi)\right|^2 \leqq (Q^2 + \pi N) \sum_{n=M+1}^{M+N} |a_n|^2$$

where the asterisk indicates summation over the primitive characters χ modulo q.

3. Throughout the rest of the proof, we assume that (9) holds, and use the following notations: we write

$$L = K^{20\log\log x} = (\log x)^{20\log K} \quad \text{and} \quad y = xL^{-2},$$

and define $f(m)$ by (11).

In this section, we will reduce the problem to the estimate of $\sum_{m\leqq y}{}' f(m)$ by proving

LEMMA 11. *There is an absolute constant c_8 such that if x is an integer with $x \gg 3$, $\mathcal{A}, \mathcal{B} \subset \{1, 2, \dots, x\}$ and K satisfies (9), then we have*

$$\left| \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \tau_K(ab+1) - 2 \sum_{m \leqq y}{}' f(m) \right| < c_8 x^2 (\log\log x + \log K). \tag{23}$$

PROOF. Clearly, we have

$$\left| \tau_K(n) - 2 \sum_{\substack{m | v_K(n) \\ m^2 \leqq v_K(n)}} 1 \right| = \left| \tau\big(v_K(n)\big) - 2 \sum_{\substack{m | v_K(n) \\ m^2 \leqq v_K(n)}} 1 \right| \leqq 1. \tag{24}$$

Let

$$\mathcal{R}_1 = \big\{ (a,b,m) : a \in \mathcal{A},\ b \in \mathcal{B},\ m \in \mathcal{N}_K,\ m | (ab+1),\ m^2 \leqq v_K(ab+1) \big\},$$

$$\mathcal{R}_2 = \big\{ (a,b,m) : a \in \mathcal{A},\ b \in \mathcal{B},\ m \in \mathcal{N}_K,\ m | (ab+1),\ m \leqq y \big\},$$

$$\mathcal{R}_3 = \big\{ (a,b,m) : a \in \mathcal{A},\ b \in \mathcal{B},\ m \in \mathcal{N}_K,\ m | (ab+1),\ y^2 < v_K(ab+1),$$
$$y^2 < m^2 \leqq v_K(ab+1) \big\},$$

$$\mathcal{R}_4 = \big\{ (a,b,m) : a \in \mathcal{A},\ b \in \mathcal{B},\ m \in \mathcal{N}_K,\ m | (ab+1),\ v_K(ab+1) < y^2,$$
$$v_K(ab+1) < m^2 \leqq y^2 \big\},$$

$$\mathcal{R}_5 = \big\{ (a,b,m) : a \in \mathcal{A},\ b \in \mathcal{B},\ m \in \mathcal{N}_K,\ m | (ab+1),\ ab+1 < y^2 L,\ m \leqq y \big\}$$

and

$$\mathcal{R}_6 = \big\{ (a,b,m) : a \in \mathcal{A},\ b \in \mathcal{B},\ m \in \mathcal{N}_K,\ m | (ab+1),\ ab+1 \geqq y^2 L,$$
$$v_K(ab+1) < y^2,\ m \leqq y \big\}.$$

By (24), we have

$$\left| \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \tau_K(ab+1) - 2|\mathcal{R}_1| \right| \leqq \Big| \big\{ (a,b) : a \in \mathcal{A},\ b \in \mathcal{B} \big\} \Big| = |\mathcal{A}||\mathcal{B}| \leqq x^2. \tag{25}$$

Moreover, clearly we have $|\mathcal{R}_2| = \sum'_{m \leqq y} f(m)$, $\mathcal{R}_3 = \mathcal{R}_1 \setminus \mathcal{R}_2$, $\mathcal{R}_4 = \mathcal{R}_2 \setminus \mathcal{R}_1$ and $\mathcal{R}_4 \subset \mathcal{R}_5 \cup \mathcal{R}_6$, so that

$$(26) \qquad \Big| |\mathcal{R}_1| - \sum_{m \leqq y}{}' f(m) \Big| = \big| |\mathcal{R}_1| - |\mathcal{R}_2| \big| \leqq$$

$$\leqq |\mathcal{R}_1 \setminus \mathcal{R}_2| + |\mathcal{R}_2 \setminus \mathcal{R}_1| = |\mathcal{R}_3| + |\mathcal{R}_4| \leqq |\mathcal{R}_3| + |\mathcal{R}_5 \cup \mathcal{R}_6| \leqq$$

$$\leqq |\mathcal{R}_3| + |\mathcal{R}_5| + |\mathcal{R}_6|$$

so that it remains to estimate $|\mathcal{R}_3|$, $|\mathcal{R}_5|$ and $|\mathcal{R}_6|$.

It follows from $y^2 < m^2 \leqq v_K(ab+1)$ that $y^2 < m^2 \leqq ab + 1 \leqq x^2 + 1$ so that $y < m \leqq x$. Thus by Lemmas 1 and 3 we have

$$(27) \quad |\mathcal{R}_3| \leqq \Big| \big\{ (a,b,m) : a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{N}_K, m|(ab+1), y < m \leqq x \big\} \Big| =$$

$$= \sum_{y<m\leqq x}{}' f(m) \leqq 2 \min\big(|\mathcal{A}|, |\mathcal{B}| \big) x \sum_{y<m\leqq x}{}' \frac{1}{m} \ll$$

$$\ll x^2 \left(\frac{\log(x/y)}{\log K} + \log K \right) \ll x^2(\log L + \log K) \ll x^2(\log\log x + \log K).$$

To estimate $|\mathcal{R}_5|$, note that $ab + 1 < y^2 L$ implies $\min(a,b) \leqq yL^{1/2}$. If, say, $a \leqq yL^{1/2}$ holds, a is fixed and $m \in \mathcal{N}_K$, $m \leqq y < x$, then the number of integers b with $b \leqq x$, $ab + 1 \equiv 0 (\operatorname{mod} m)$ is at most $\left[\frac{x}{m}\right] + 1 \leqq 2\frac{x}{m}$. Thus by (9) and Lemma 3 we have

$$(28) \qquad |\mathcal{R}_5| \leqq 2 \Big| \big\{ a : a \leqq yL^{1/2} \big\} \Big| \sum_{m \leqq y}{}' 2\frac{x}{m} \ll xyL^{1/2} \sum_{m\leqq y}{}' \frac{1}{m} \ll$$

$$\ll xyL^{1/2} \left(\frac{\log y}{\log K} + \log K \right) = x^2 L^{-3/2} \left(\frac{\log y}{\log K} + \log K \right) \ll x^2.$$

Assume now that $(a,b,m) \in \mathcal{R}_6$. Then we have

$$u_K(ab+1) = \frac{ab+1}{v_K(ab+1)} > \frac{y^2 L}{y^2} = L.$$

Let m_1 denote the least positive integer with $m_1 | u_K(ab+1)$, $m_1 > L$. All the prime factors of $u_K(ab+1)$ are $\leqq K$, thus m_1 satisfies

$$(29) \qquad L < m_1 \leqq LK, \quad P(m_1) \leqq K.$$

By $m \in \mathcal{N}_K$ and $P(m_1) \leqq K$ we have $(m, m_1) = 1$. Thus $m|(ab+1)$, $m_1 \big| u_K(ab+1) \big| (ab+1)$ imply

$$ab + 1 \equiv 0 \pmod{mm_1}. \tag{30}$$

Moreover, we have

$$mm_1 \leqq yLK = x\frac{K}{L} \ll x. \tag{31}$$

Without loss of generality we may assume that $|\mathcal{A}| \leqq |\mathcal{B}|$. Then a in (30) can be chosen in $|\mathcal{A}|$ ways, and if a, m, m_1 are fixed, then, by (31), an integer b satisfying (30) and $b \leqq x$ can be chosen in at most

$$\left[\frac{x}{mm_1}\right] + 1 \leqq \frac{2x}{mm_1}$$

ways. Thus by (29), for $|\mathcal{A}| \leqq |\mathcal{B}|$ the number of solutions of (30) in a, b, m, m_1 is

$$\leqq |\mathcal{A}| \sideset{}{'}\sum_{m \leqq y} \sum_{\substack{L < m_1 \leqq LK \\ P(m_1) \leqq K}} \frac{2x}{mm_1} = 2|\mathcal{A}|x \sideset{}{'}\sum_{m \leqq y} \frac{1}{m} \sum_{\substack{L < m_1 \leqq LK \\ P(m_1) \leqq K}} \frac{1}{m_1}$$

so that, by Lemmas 3 and 7, and in view of (9),

$$|\mathcal{R}_6| \ll \min\big(|\mathcal{A}|, |\mathcal{B}|\big)\, x \left(\frac{\log y}{\log K} + \log K\right) \log K \exp\left(-\frac{\log L}{4 \log K}\right) \ll \tag{32}$$

$$\ll x^2 \log x \exp(-5 \log\log x) \ll x^2.$$

(23) follows from (25), (26), (27), (28) and (32), and this completes the proof of the lemma.

4. In this section, we will estimate $\sideset{}{'}\sum\limits_{m \leqq y} f(m)$.

LEMMA 12. *There is an absolute constant c_9 such that if $x \geqq 3$, $\mathcal{A}, \mathcal{B} \subset \subset \{1, 2, \dots, x\}$ and K satisfies (9), then*

$$\left| \sideset{}{'}\sum_{m \leqq y} f(m) - |\mathcal{A}||\mathcal{B}| \sideset{}{'}\sum_{m \leqq y} \frac{1}{m} \right| < c_9 \frac{x^2 \log x}{K^{1/2} \log^2 K}. \tag{33}$$

PROOF. Define $\mathcal{M}(z)$ as in Lemma 8, and write $\mathcal{M}=\mathcal{M}(y)$, $\overline{\mathcal{M}}=\left(\{1,2,\dots,[y]\}\cap\mathcal{N}_K\right)\setminus\mathcal{M}$ so that

$$\mathcal{M}=\left\{m\colon m\leqq y,\ m\in\mathcal{N}_K,\ h(m)=\sum_{d\mid m}d^{-1/2}>2\right\},$$

$$\overline{M}=\left\{m\colon m\leqq y,\ m\in\mathcal{N}_K,\ h(m)=\sum_{d\mid m}d^{-1/2}\leqq 2\right\}$$

and

$$\mathcal{M}\cup\overline{\mathcal{M}}=\{m\colon m\leqq y,\ m\in\mathcal{N}_K\},\ \mathcal{M}\cap\overline{\mathcal{M}}=\emptyset.$$

First we estimate $\sum\limits_{m\in\overline{\mathcal{M}}}f(m)$. Clearly we have

$$f(m)=\frac{1}{\varphi(m)}\sum_{\chi(\bmod m)}\overline{\chi}(-1)\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\chi(ab)$$

for all m.

By Lemma 6, the contribution of the principal character χ_0 can be estimated in the following way:

$$\left|\frac{1}{\varphi(m)}\overline{\chi}_0(-1)\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\chi_0(ab)-\frac{|\mathcal{A}||\mathcal{B}|}{m}\right|\leqq$$

$$\leqq\frac{1}{\varphi(m)}\left|\sum_{\substack{a\in\mathcal{A}\\(ab,m)=1}}\sum_{b\in\mathcal{B}}1-|\mathcal{A}||\mathcal{B}|\right|+\left|\frac{1}{\varphi(m)}-\frac{1}{m}\right||\mathcal{A}||\mathcal{B}|=$$

$$=\frac{1}{\varphi(m)}\left|\sum_{\substack{a\in\mathcal{A}\\(ab,m)=1}}\sum_{b\in\mathcal{B}}1-\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}1\right|+\left(\frac{1}{\varphi(m)}-\frac{1}{m}\right)|\mathcal{A}||\mathcal{B}|=$$

$$=\frac{1}{\varphi(m)}\sum_{\substack{a\in\mathcal{A}\\(ab,m)>1}}\sum_{b\in\mathcal{B}}1+\left(\frac{1}{\varphi(m)}-\frac{1}{m}\right)|\mathcal{A}||\mathcal{B}|\leqq$$

$$\leqq\frac{1}{\varphi(m)}\left(\sum_{\substack{a\in\mathcal{A}\\(a,m)>1}}|\mathcal{B}|+\sum_{\substack{b\in\mathcal{B}\\(b,m)>1}}|\mathcal{A}|\right)+\left(\frac{1}{\varphi(m)}-\frac{1}{m}\right)|\mathcal{A}||\mathcal{B}|\leqq$$

$$\leqq \frac{2x}{\varphi(m)} \sum_{\substack{i \leqq x \\ (i,m)>1}} 1 + \left(\frac{1}{\varphi(m)} - \frac{1}{m}\right) x^2 \leqq$$

$$\leqq \frac{2x}{\varphi(m)} \left(x \left(1 - \frac{\varphi(m)}{m}\right) + 2\tau(m) \right) + \left(\frac{1}{\varphi(m)} - \frac{1}{m}\right) x^2 \ll$$

$$\ll x^2 \left(\frac{1}{\varphi(m)} - \frac{1}{m}\right) + x\frac{\tau(m)}{\varphi(m)}.$$

Thus we have

$$\left| f(m) - \frac{|\mathcal{A}||\mathcal{B}|}{m} \right| \ll \left(x^2 \left(\frac{1}{\varphi(m)} - \frac{1}{m}\right) + x\frac{\tau(m)}{\varphi(m)} \right) + \tag{34}$$

$$+\frac{1}{\varphi(m)} \left| \sum_{\substack{\chi (\operatorname{mod} m) \\ \chi \neq \chi_0}} \overline{\chi}(-1) \sum_{a \in \mathcal{A}} \chi(a) \sum_{b \in \mathcal{B}} \chi(b) \right| \leqq$$

$$\leqq \left(x^2 \left(\frac{1}{\varphi(m)} - \frac{1}{m}\right) + x\frac{\tau(m)}{\varphi(m)} \right) + \frac{1}{\varphi(m)} \sum_{\substack{\chi (\operatorname{mod} m) \\ \chi \neq \chi_0}} \left| \sum_{a \in \mathcal{A}} \chi(a) \right| \left| \sum_{b \in \mathcal{B}} \chi(b) \right| .$$

If the modulo m character χ is induced by the modulo q primitive character χ_1 (so that $q|m$), then $\chi(n) = \chi_1(n)$ for all $(n,m) = 1$. Thus writing

$$T_1(\chi, d) = \sum_{\substack{a \in \mathcal{A} \\ d|a}} \chi(a), \quad T_2(\chi, d) = \sum_{\substack{b \in \mathcal{B} \\ d|b}} \chi(b)$$

and using

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n > 1, \end{cases}$$

we have

$$\sum_{\substack{\chi (\operatorname{mod} m) \\ \chi \neq \chi_0}} \left| \sum_{a \in \mathcal{A}} \chi(a) \right| \left| \sum_{b \in \mathcal{B}} \chi(b) \right| = \tag{35}$$

$$= \sum_{\substack{\chi (\operatorname{mod} m) \\ \chi \neq \chi_0}} \left| \sum_{a \in \mathcal{A}} \left(\sum_{d|(a,m)} \mu(d) \right) \chi(a) \right| \left| \sum_{b \in \mathcal{B}} \left(\sum_{d|(b,m)} \mu(d) \right) \chi(b) \right| =$$

$$= \sum_{\substack{q|m\\ q>1}} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \left|\sum_{d|m} \mu(d) T_1(\chi,d)\right| \left|\sum_{d|m} \mu(d) T_2(\chi,d)\right| \leqq$$

$$\leqq \sum_{\substack{q|m\\ q>1}} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \left(\sum_{d|m} |T_1(\chi,d)|\right) \left(\sum_{d|m} |T_2(\chi,d)|\right)$$

where the asterisk indicates summation over the primitive characters χ modulo q. Using the inequalities $2|uv| \leqq u^2 + v^2$, $(u+v)^2 \leqq 2(u^2+v^2)$ (both for real u, v) and the Cauchy–Schwarz inequality repeatedly, for $m \in \overline{\mathcal{M}}$ we obtain

$$(36) \qquad \left(\sum_{d|m} |T_1(\chi,d)|\right) \left(\sum_{d|m} |T_2(\chi,d)|\right) \ll$$

$$\ll \left(\sum_{d|m} |T_1(\chi,d)|\right)^2 + \left(\sum_{d|m} |T_2(\chi,d)|\right)^2 \ll$$

$$\ll \left(\sum_{\substack{d|m\\ d<L}} |T_1(\chi,d)|\right)^2 + \left(\sum_{\substack{d|m\\ d\geqq L}} |T_1(\chi,d)|\right)^2 +$$

$$+ \left(\sum_{\substack{d|m\\ d<L}} |T_2(\chi,d)|\right)^2 + \left(\sum_{\substack{d|m\\ d\geqq L}} |T_2(\chi,d)|\right)^2 \leqq$$

$$\leqq \left(\sum_{\substack{d|m\\ d<L}} d^{1/2}\left(|T_1(\chi,d)|^2 + |T_2(\chi,d)|^2\right)\right) \left(\sum_{d|m} d^{-1/2}\right) +$$

$$+ \left(\sum_{\substack{d|m\\ d\geqq L}} \left(|T_1(\chi,d)|^2 + |T_2(\chi,d)|^2\right)\right) \sum_{d|m} 1 \ll$$

$$\ll \sum_{\substack{d|m\\ d<L}} d^{1/2}\left(\left|T_1(\chi,d)\right)^2+\left|T_2(\chi,d)\right|^2\right)+$$

$$+\tau(m)\sum_{\substack{d|m\\ d\geqq L}}\left(\left|T_1(\chi,d)\right|^2+\left|T_2(\chi,d)\right|^2\right).$$

It follows from (34), (35) and (36) that

$$\left|\sum_{m\in\overline{\mathcal{M}}} f(m)-|\mathcal{A}||\mathcal{B}|\sum_{m\in\overline{\mathcal{M}}}\frac{1}{m}\right| \ll \sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3+\sum\nolimits_4+\sum\nolimits_5 \tag{37}$$

where

$$\sum\nolimits_1=\sum_{m\leqq y}{}'\left(x^2\left(\frac{1}{\varphi(m)}-\frac{1}{m}\right)+x\frac{\tau(m)}{\varphi(m)}\right),$$

$$\sum\nolimits_2=\sum_{m\in\overline{\mathcal{M}}}{}'\frac{1}{\varphi(m)}\sum_{\substack{q|m\\ q>1}}\sum_{\chi(\mathrm{mod}\,q)}{}^{*}\sum_{\substack{d|m\\ d<L}} d^{1/2}\left|T_1(\chi,d)\right|^2,$$

$$\sum\nolimits_3=\sum_{m\in\overline{\mathcal{M}}}{}'\frac{\tau(m)}{\varphi(m)}\sum_{\substack{q|m\\ q>1}}\sum_{\chi(\mathrm{mod}\,q)}{}^{*}\sum_{\substack{d|M\\ d\geqq L}}\left|T_1(\chi,d)\right|^2$$

and $\sum_4$, resp. $\sum_5$ are the sums analogous to $\sum_2$ and $\sum_3$ with T_2 in place of T_1.

Clearly, we have

$$\sum_{m\leqq y}{}'\frac{\tau(m)}{\varphi(m)}<\prod_{K<p\leqq y}\left(1+\sum_{r=1}^{+\infty}\frac{\tau(p^r)}{\varphi(p^r)}\right)\ll\prod_{K<p\leqq y}\left(1+\frac{2}{p}\right)\ll\left(\frac{\log y}{\log K}\right)^2.$$

Thus by Lemma 4 and (9) we have

$$\sum\nolimits_1=\frac{2}{x}\sum_{m\leqq y}{}'\left(\frac{1}{\varphi(m)}-\frac{1}{m}\right)+x\sum_{m\leqq y}{}'\frac{\tau(m)}{\varphi(m)}\ll$$

$$\ll x^2\frac{\log y}{K\log^2 K}+x\left(\frac{\log y}{\log K}\right)^2\ll x^2\frac{\log x}{K\log^2 K}.$$

In $\sum_2$ we have $q|m$, $d|m$, whence $[q,d]|m$. Write $m=[q,d]r$. It is easy to see that

$$\varphi(uv) \geqq \varphi(u)\varphi(v) \tag{38}$$

for all u, v. Thus by Lemma 5 and in view of (9) we have

$$\sum\nolimits_2 \leqq \sum_{d<L}{}' d^{1/2} \sum_{\substack{1<q\leqq y\\ [q,d]\leqq y}} \frac{1}{\varphi([q,d])} \sum_{\chi (\mathrm{mod}\, q)}^{*} \left|T_1(\chi,d)\right|^2 \sum_{r\leqq y/[q,d]}{}' \frac{1}{\varphi(r)} \ll$$

$$\ll \left(\frac{\log y}{\log K}+\log K\right) \sum_{d<L}{}' d^{1/2} \sum_{\substack{1<q\leqq y\\ [q,d]\leqq y}} \frac{1}{\varphi([d,q])} \sum_{\chi (\mathrm{mod}\, q)}^{*} \left|T_1(\chi,d)\right|^2 \ll$$

$$\ll \frac{\log x}{\log K} \sum_{d<L}{}' d^{1/2} \sum_{\substack{1<q\leqq y\\ [q,d]\leqq y}} \frac{1}{\varphi([q,d])} \sum_{\chi (\mathrm{mod}\, q)}^{*} \left|T_1(\chi,d)\right|^2.$$

Write $(d,q)=s$, $d=(d,q)t=st$. By (38), we have

$$\varphi([q,d]) = \varphi\left(q\cdot\frac{d}{(d,q)}\right) = \varphi(qt) \geqq \varphi(q)\varphi(t).$$

Thus we have

$$\sum\nolimits_2 \ll \frac{\log x}{\log K} \sum_{s<L}{}' \sum_{t<L/s}{}' (st)^{1/2} \sum_{\substack{1<q\leqq y\\ s|q}}{}' \frac{1}{\varphi(q)\varphi(t)} \sum_{\chi (\mathrm{mod}\, q)}^{*} \left|T_1(\chi,st)\right|^2.$$

Let $s_0=K$ for $s=1$ and let $s_0=s$ for $s\geqq 2$. Moreover, let $\mathcal{J}(s)$ denote the set of positive integers j with $2^{j-1}s_0\leqq y$, and for $j\in\mathcal{J}(s)$, let $\mathcal{J}_j(s)$ denote the set of integers n with $2^{j-1}s_0<n\leqq 2^j s_0$. It follows from $1<q\leqq y$, $s|q$, $q\in\mathcal{N}_K$ that $q\in\mathcal{J}_j(s)$ for some $j\in\mathcal{J}(s)$. Thus we have

$$\sum\nolimits_2 \ll \frac{\log x}{\log K} \sum_{s<L}{}' \sum_{t<L/s}{}' \frac{(st)^{1/2}}{\varphi(t)} \sum_{j\in\mathcal{J}(s)} \sum_{q\in\mathcal{J}_j(s)} \frac{1}{\varphi(q)} \sum_{\chi (\mathrm{mod}\, q)}^{*} \left|T_1(\chi,st)\right|^2. \tag{39}$$

By Lemma 10, for $st < L$ we have

$$\sum_{j\in\mathcal{J}(s)}\sum_{q\in\mathcal{J}_j(s)}\frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}^{*}\left|T_1(\chi,st)\right|^2<$$

$$<\sum_{j\in\mathcal{J}(s)}\sum_{q\in\mathcal{J}_j(s)}\frac{q}{2^{j-1}s_0}\cdot\frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}^{*}\left|T_1(\chi,st)\right|^2\leqq$$

$$\leqq\sum_{j\in\mathcal{J}(s)}\frac{1}{2^{j-1}s_0}\sum_{q\leqq 2^j s_0}\frac{q}{\varphi(q)}\sum_{\chi(\bmod q)}^{*}\left|T_1(\chi,st)\right|^2\ll$$

$$\ll\sum_{j\in\mathcal{J}(s)}\frac{1}{2^j s_0}(2^{2j}s_0^2+x)\sum_{\substack{a\in\mathcal{A}\\ st\mid a}}1\leqq$$

$$\leqq\left(s_0\sum_{j\in\mathcal{J}(s)}2^j+\frac{x}{s_0}\sum_{j\in\mathcal{J}(s)}\frac{1}{2^j}\right)\sum_{\substack{a\leqq x\\ st\mid a}}1\ll$$

$$\ll\left(y+\frac{x}{s_0}\right)\frac{x}{st}\ll\begin{cases}\frac{x^2}{Kt} & \text{for } s=1,\\ \left(y+\frac{x}{s}\right)\frac{x}{st} & \text{for } s>1.\end{cases}$$

Then separating the $s=1$ term in (39), we obtain that

$$\sum\nolimits_2\ll\frac{\log x}{\log K}\left(\sum_{t<L}{}'\frac{t^{1/2}}{\varphi(t)}\cdot\frac{x^2}{Kt}+\sum_{1<s<L}{}'\sum_{t<L}{}'\frac{(st)^{1/2}}{\varphi(t)}\left(y+\frac{x}{s}\right)\frac{x}{st}\right)\leqq$$

$$\leqq\frac{\log x}{\log K}\left(\frac{x^2}{K}+xy\sum_{1<s<L}\frac{1}{s^{1/2}}+x^2\sum_{1<s<L}{}'\frac{1}{s^{3/2}}\right)\sum_{t<L}\frac{1}{t^{1/2}\varphi(t)}\ll$$

$$\ll\frac{\log x}{\log K}\left(\frac{x^2}{K}+xy\sum_{s<L}\frac{1}{s^{1/2}}+x^2\sum_{1<s<L}{}'\frac{1}{s^{3/2}}\right).$$

Here we have

$$xy\sum_{s<L}\frac{1}{s^{1/2}}\ll xyL^{1/2}=x^2L^{-3/2}$$

and

$$\sum_{1<s<L}{}'\frac{1}{s^{3/2}}=\sum_{s<L}{}'\frac{1}{s^{3/2}}-1<\prod_{K<p<L}\left(1-\frac{1}{p^{3/2}}\right)^{-1}-1\ll$$

$$\ll \sum_{K<p<L} \frac{1}{p^{3/2}} \ll \frac{1}{K^{1/2}\log K}$$

so that

$$\sum\nolimits_2 \ll x^2 \frac{\log x}{\log K}\left(\frac{1}{K}+\frac{1}{L^{3/2}}+\frac{1}{K^{1/2}\log K}\right) \ll x^2 \frac{\log x}{K^{1/2}\log^2 K}.$$

To estimate $\sum_3$, we use Lemma 9. We obtain that

$$\sum\nolimits_3 \ll \sum_{m\leqq y}{}' \frac{\tau(m)}{\varphi(m)} \sum_{q|m} \sum_{\substack{d|m\\ d\geqq L}} \sum_{\chi(\operatorname{mod} q)}{}^{*} \left|T_1(\chi,d)\right|^2 \ll$$

$$\ll \sum_{m\leqq y}{}' \frac{\tau(m)}{\varphi(m)} \sum_{q|m} \sum_{\substack{d|m\\ d\geqq L}} x \sum_{\substack{a\in\mathcal{A}\\ d|a}} 1 \leqq \sum_{m\leqq y}{}' \frac{\tau(m)}{\varphi(m)} \sum_{q|m} \sum_{\substack{d|m\\ d\geqq L}} x\cdot\frac{x}{d}.$$

By $q|m$ and $d|m$ we have $[q,d]|m$. Write $(d,q)=r$, $q=(d,q)s=rs$ and $m=[q,d]t=dst$. Then using (38) and $\tau(uv)\leqq\tau(u)\tau(v)$ (for all u, v) we obtain that

$$\sum\nolimits_3 \ll x^2 \sum_{L\leqq d\leqq y}{}' \frac{1}{d} \sum_{r|d} \sum_{s\leqq y/r}{}' \sum_{t\leqq y/ds}{}' \frac{\tau(dst)}{\varphi(dst)} \leqq$$

$$\leqq x^2 \sum_{L\leqq d\leqq y}{}' \frac{\tau(d)}{d\varphi(d)} \sum_{r|d} \left(\sum_{s\leqq y/r}{}' \frac{\tau(s)}{\varphi(s)}\right)\left(\sum_{t\leqq y/ds}{}' \frac{\tau(t)}{\varphi(t)}\right) \leqq$$

$$\leqq x^2 \sum_{L\leqq d\leqq y}{}' \frac{\tau(d)}{d\varphi(d)} \sum_{r|d} \left(\sum_{n\leqq y}{}' \frac{\tau(n)}{\varphi(n)}\right)^2 = x^2 \sum_{L\leqq d\leqq y}{}' \frac{\tau^2(d)}{d\varphi(d)} \left(\sum_{n\leqq y}{}' \frac{\tau(n)}{\varphi(n)}\right)^2.$$

Here we have

$$\sum_{n\leqq y}{}' \frac{\tau(n)}{\varphi(n)} \leqq \prod_{K<p\leqq y}\left(1+\sum_{i=1}^{+\infty}\frac{\tau(p^i)}{\varphi(p^i)}\right) \ll \prod_{K<p\leqq x}\left(1+\frac{2}{p}\right) \ll \frac{\log^2 x}{\log^2 K}$$

so that

$$\sum\nolimits_3 \ll \frac{x^2\log^4 x}{\log^4 K} \sum_{L\leqq d\leqq y} \frac{\tau^2(d)}{d\varphi(d)}.$$

Here the innermost sum is

$$\ll \sum_{L \leqq d} \frac{1}{d^{3/2}} \ll L^{-1/2}$$

whence

$$\sum\nolimits_3 \ll \frac{x^2 \log^4 x}{L^{1/2} \log^4 K} \ll \frac{x^2}{K}.$$

$\sum_4$ and $\sum_5$ can be estimated in the same way as $\sum_2$ and $\sum_3$. Combining the estimates above, we obtain from (37) that

$$\left| \sum_{m \in \overline{\mathcal{M}}} f(m) - |\mathcal{A}||\mathcal{B}| \sum_{m \in \overline{\mathcal{M}}} \frac{1}{m} \right| \ll \tag{40}$$

$$\ll \frac{x^2 \log x}{K \log^2 K} + \frac{x^2 \log x}{K^{1/2} \log^2 K} + \frac{x^2}{K} \ll \frac{X^2 \log x}{K^{1/2} \log^2 K}.$$

Finally, by Lemmas 1 and 8, and in view of (9), it follows from (40) that

$$\left| \sum_{m \leqq y}{}' f(m) - |\mathcal{A}||\mathcal{B}| \sum_{m \leqq y}{}' \frac{1}{m} \right| =$$

$$= \left| \left(\sum_{m \in \overline{\mathcal{M}}} f(m) - |\mathcal{A}||\mathcal{B}| \sum_{m \in \overline{\mathcal{M}}} \frac{1}{m} \right) + \left(\sum_{m \in M} \left(f(m) - \frac{|\mathcal{A}||\mathcal{B}|}{m} \right) \right) \right| \leqq$$

$$\leqq \left| \sum_{m \in \overline{\mathcal{M}}} f(m) - |\mathcal{A}||\mathcal{B}| \sum_{m \in \overline{\mathcal{M}}} \frac{1}{m} \right| + \sum_{m \in M} \left(f(m) + \frac{|\mathcal{A}||\mathcal{B}|}{m} \right) \ll$$

$$\ll \frac{x^2 \log x}{K^{1/2} \log^2 K} + \sum_{m \in \mathcal{M}} \left(\frac{x^2}{m} + \frac{x^2}{m} \right) \ll \frac{x^2 \log x}{K^{1/2} \log^2 K} + x^2 \sum_{m \in M} \frac{1}{m} \ll$$

$$\ll \frac{x^2 \log x}{K^{1/2} \log K} + \frac{x^2}{K^{1/2}} \left(\frac{\log y}{(\log K)^2} + 1 \right) \ll \frac{x^2 \log x}{K^{1/2} \log^2 K}$$

and this completes the proof of Lemma 12.

5. Completion of the proof of the theorem. By Lemmas 11 and 12, we have

$$\left|\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\tau_K(ab+1)-2|\mathcal{A}||\mathcal{B}|{\sum_{m\leqq y}}'\frac{1}{m}\right|\leqq \tag{41}$$

$$\leqq\left|\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\tau_K(ab+1)-2{\sum_{m\leqq y}}'f(m)\right|+2\left|{\sum_{m\leqq y}}'f(m)-|\mathcal{A}||\mathcal{B}|{\sum_{m\leqq y}}'\frac{1}{m}\right|\ll$$

$$\ll x^2\left(\log\log x+\log K+\frac{\log x}{K^{1/2}\log^2 K}\right).$$

Moreover, by Lemma 3 we have

$$\left|{\sum_{m\leqq y}}'\frac{1}{m}-\log x\prod_{p\leqq K}\left(1-\frac{1}{p}\right)\right|\leqq \tag{42}$$

$$\leqq\left|{\sum_{m\leqq y}}'\frac{1}{m}-{\sum_{m\leqq x}}'\frac{1}{m}\right|+\left|{\sum_{m\leqq x}}'\frac{1}{m}-\log x\prod_{p\leqq K}\left(1-\frac{1}{p}\right)\right|\ll$$

$$\ll\left(\log\frac{x}{y}\right)\prod_{p\leqq K}\left(1-\frac{1}{p}\right)+\log K\ll\frac{\log L}{\log K}+\log K\ll\log\log x+\log K.$$

(10) follows from (41) and (42), and this completes the proof of the theorem.

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A DIRECT APPROACH TO KOEKOEK'S DIFFERENTIAL EQUATION FOR GENERALIZED LAGUERRE POLYNOMIALS

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1. Introduction

In [4] Koorwinder studied some classes of orthogonal polynomials. One class was given by the polynomials $\left\{L_n^{\alpha,N}(x)\right\}_{n=0}^{\infty}$ which are orthogonal on the interval $[0,\infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha}e^{-x}+N\delta(x),\quad \alpha>-1,\ N\geqq 0.$$

They can be written as $L_0^{\alpha,N}(x)=1$, and for $n\geqq 1$,

$$L_n^{\alpha,N}(x)=\left[1+N\frac{(\alpha+2)_{n-1}}{(n-1)!}\right]L_n^{(\alpha)}(x)+N\frac{(\alpha+1)_n}{n!}\frac{d}{dx}L_n^{(\alpha)}(x), \tag{1.1}$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomial defined by

$$L_n^{(\alpha)}(x)=\frac{1}{n!}\sum_{k=0}^{n}(-n)_k(\alpha+k+1)_{n-k}\frac{x^k}{k!}= \tag{1.2}$$

$$=\frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\left.\begin{matrix}-n\\ \alpha+1\end{matrix}\right|x\right),\quad n=0,1,2,\dots. \tag{1.3}$$

The representation (1.2) is valid for all real α; (1.3) is not defined if α is a negative integer. Note that $L_n^{\alpha,0}(x)=L_n^{(\alpha)}(x)$.

For these polynomials J. Koekoek and R. Koekoek [3] found a differential equation of the form

$$N\sum_{i=0}^{\infty}a_i(x)y^{(i)}(x)+xy''(x)+(\alpha+1-x)y'(x)+ny(x)=0, \tag{1.4}$$

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where the coefficients $a_i(x)$, $i \in \{1,2,3,\ldots\}$, are independent of n and $a_0(x) = a_0(n,\alpha)$ depends on n but is independent of x.

This differential equation is of infinite order in general, but for nonnegative integer values of the parameter α the order reduces to $2\alpha + 4$. For some special integer values of α such a differential equation was already known (see [3] for references). From Koornwinder's representation of the polynomials J. and R. Koekoek derived two systems of equations for the coefficients $a_i(x)$ and computed a number of the $a_i(x)$. Then they guessed what the general formula for the coefficients might be and they showed that it actually satisfies the systems of equations. Furthermore they proved that the solution is unique.

At a conference in Erice (May 1990) R. A. Askey [1] posed the problem of finding difference equations of similar form for generalizations of the discrete orthogonal polynomials which are orthogonal with respect to the classical weight function at which a point mass at the point $x = 0$ is added. In [2] a solution for this problem for Charlier polynomials is given and a method of finding the coefficients is introduced. In the present note this method is used to derive a formula for the coefficients $a_i(x)$ in the Laguerre case and to give a new direct proof of the results in [3].

2. The systems of equations

Inserting (1.1) into (1.4) and using the second order differential equation for the Laguerre polynomials, J. and R. Koekoek obtained the following systems of equations for the coefficients $a_i(x)$:

$$\sum_{i=0}^{\infty} a_i(x) D^i L_n^{(\alpha)}(x) + \frac{(\alpha+1)_n}{n!} D L_n^{(\alpha)}(x) - \frac{(\alpha+1)_n}{n!} D^2 L_n^{(\alpha)}(x) = 0, \tag{2.1}$$

$$n \sum_{i=0}^{\infty} a_i(x) D^i L_n^{(\alpha)}(x) + (\alpha+1) \sum_{i=0}^{\infty} a_i(x) D^{i+1} L_n^{(\alpha)}(x) = 0 \tag{2.2}$$

for all real $x, \alpha > -1$, $N \geqq 0$ and $n = 0,1,2\ldots$. Here $D = \frac{d}{dx}$. For $n = 0$ the systems reduce to $a_0(0,\alpha) = 0$. In the sequel we take $n \geqq 1$. By using the well-known formulae for Laguerre polynomials, which are valid for all real x, for all real α and for all $n \in \{1,2,\ldots\}$,

$$D L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x) \tag{2.3}$$

and

$$L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha+1)}(x) = L_n^{(\alpha+1)}(x) \tag{2.4}$$

the equations (2.1) and (2.2) can be written in the following form:

$$\sum_{i=0}^{\infty}(-1)^i a_i(x)L_{n-i}^{(\alpha+i)}(x)=\frac{(\alpha+1)_n}{n!}L_{n-1}^{(\alpha+2)}(x), \tag{2.5}$$

$$\sum_{i=0}^{\infty}(-1)^i a_i(x)L_{n-i-1}^{(\alpha+i+1)}(x)=\frac{(\alpha+2)_{n-1}}{(n-1)!}L_{n-1}^{(\alpha+2)}(x). \tag{2.6}$$

Here $L_{-k}^{(\alpha)}(x)=0$ for all real α, all real x and all $k\in\{1,2,3,\dots\}$. If we multiply equation (2.5) by $L_{n-1}^{(\alpha+1)}(x)$ and equation (2.6) by $L_n^{(\alpha)}(x)$ and subtract we obtain

$$\sum_{i=1}^{\infty}(-1)^i a_i(x)\left[L_{n-1}^{(\alpha+i)}(x)L_{n-1}^{(\alpha+1)}(x)-L_{n-i-1}^{(\alpha+i+1)}(x)L_n^{(\alpha)}(x)\right]=$$

$$=L_{n-1}^{(\alpha+2)}(x)\left[\frac{(\alpha+1)_n}{n!}L_{n-1}^{(\alpha+1)}(x)-\frac{(\alpha+2)_{n-1}}{(n-1)!}L_n^{(\alpha)}(x)\right].$$

The right-hand side vanishes for $x=0$, whereas the expression between the square brackets at the left-hand side is different from 0 when $x=0$ (provided that $\alpha>-1$). Since this equation holds for all n we may conclude step by step that $a_i(0)=0$ for all $i\in\{1,2,\dots\}$. If we substitute $x=0$ into (2.5) we conclude that

$$a_0(n,\alpha)=L_{n-1}^{(\alpha+2)}(0)=\frac{(\alpha+3)_{n-1}}{(n-1)!}=\binom{n+\alpha+1}{n-1}.$$

Hence the systems of equations (2.5) and (2.6) can be written in the form

$$\sum_{i=1}^{\infty}(-1)^i a_i(x)L_{n-i}^{(\alpha+i)}(x)=L_n^{(\alpha)}(0)L_{n-1}^{(\alpha+2)}(x)-L_n^{(\alpha)}(x)L_{n-1}^{(\alpha+2)}(0), \tag{2.7}$$

$$\sum_{i=1}^{\infty}(-1)^i a_i(x)L_{n-i-1}^{(\alpha+i+1)}(x)= \tag{2.8}$$

$$=L_{n-1}^{(\alpha+1)}(0)L_{n-1}^{(\alpha+2)}(x)-L_{n-1}^{(\alpha+1)}(x)L_{n-1}^{(\alpha+2)}(0),$$

for $n=1,2,3,\dots,\alpha-1$ and all real x.

PROPOSITION 2.1. *Each solution $a_i(x)$ $(i=1,2,\ldots)$ of the system* (2.8) *is also a solution of* (2.7).

PROOF. By means of (2.4) the left-hand side of (2.7) can be written as

$$\sum_{i=1}^{\infty}(-1)^i a_i(x)L_{n-1}^{(\alpha+i+1)}(x)-\sum_{i=1}^{\infty}(-1)^i a_i(x)L_{n-i-1}^{(\alpha+i+1)}(x).$$

If the coefficients $a_i(x)$ satisfy the system (2.8) for all n, then they also satisfy the system (2.7) if we show that

$$L_n^{(\alpha)}(0)L_{n-1}^{(\alpha+2)}(x)-L_n^{(\alpha)}(x)L_{n-1}^{(\alpha+2)}(0)=$$

$$=\left[L_n^{(\alpha+1)}(0)L_n^{(\alpha+2)}(x)-L_n^{(\alpha+1)}(x)L_n^{(\alpha+2)}(0)\right]-$$

$$-\left[L_{n-1}^{(\alpha+1)}(0)L_{n-1}^{(\alpha+2)}(x)-L_{n-1}^{(\alpha+1)}(x)L_{n-1}^{(\alpha+2)}(0)\right].$$

By combining terms and using (2.4) this is easily verified. □

3. Solution of the system

We now proceed to solve the system (2.8). We first rewrite the right-hand side by using the following formula for Laguerre polynomials:

$$\frac{L_n^{(\alpha+1)}(x)}{L_n^{(\alpha+1)}(0)}-\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}=\frac{nx}{(\alpha+1)(\alpha+2)}\frac{L_{n-1}^{(\alpha+2)}(x)}{L_{n-1}^{(\alpha+2)}(0)},$$

a direct consequence of (1.3). Equation (2.8) becomes, if we write n instead of $n-1$:

$$\text{(3.1)}\qquad \sum_{i=1}^{\infty}(-1)^i a_i(x)L_{n-i}^{(\alpha+i+1)}(x)=\frac{x}{\alpha+2}L_n^{(\alpha+1)}(0)L_{n-1}^{(\alpha+3)}(x).$$

If we consider $(-1)^i a_i(x)$ as unknown, the matrix T of the system is triangular with entries t_{ni} for which we have

$$t_{ni}=L_{n-i}^{(\alpha+i+1)}(x)\quad\text{for}\quad n,\,i=1,2,3,\ldots.$$

We will show that the entries u_{ni} of the inverse matrix U are

$$\text{(3.2)}\qquad u_{ni}=L_{n-i}^{(-\alpha-n-2)}(-x)\quad\text{for}\quad n,\,i=1,2,3,\ldots.$$

In order to prove (3.2) we use the generating function for Laguerre polynomials

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-\alpha-1}e^{\frac{xt}{t-1}}.$$

It follows that

$$\sum_{j=0}^{\infty} L_j^{(-\alpha-n-2)}(-x)t^j \sum_{m=0}^{\infty} L_m^{(\alpha+i+1)}(x)t^m = (1-t)^{n-i-1},$$

and therefore the power of t^{n-i} at both sides must be equal. Hence

$$\sum_{k=i}^{n} L_{n-k}^{(-\alpha-n-2)}(-x)L_{k-i}^{(\alpha+i+1)}(x) = \delta_{ni}.$$

We may conclude that the unique solution of the system (3.1) is given by

$$(3.3)\qquad (-1)^i a_i(x) = \frac{x}{\alpha+2}\sum_{k=1}^{i} L_{i-k}^{(-\alpha-i-2)}(-x)L_k^{(\alpha+1)}(0)L_{k-1}^{(\alpha+3)}(x),$$

$$i = 1, 2, 3, \ldots.$$

From this formula we now derive the result in [3]:

$$(3.4)\qquad a_i(x) = \frac{1}{i!}\sum_{j=1}^{i}(-1)^{i+j+1}\binom{\alpha+1}{j-1}\binom{\alpha+2}{i-j}(\alpha+3)_{i-j}x^j,$$

$$i = 1, 2, 3, \ldots.$$

(3.3) and (1.2) show that $a_i(x)$ is a polynomial in α, since $L_k^{(\alpha+1)}(0)$ contains a factor $\alpha+2$ for all $k \geqq 1$. Moreover by (1.2) the Laguerre polynomial $L_n^{(\alpha)}(x)$ and its derivatives are polynomials in α. Hence (2.1) and (2.2) are relations between polynomials in α. In the proof of (3.4) we use (1.3) and therefore we have to assume that α is not a negative integer, but by analytic continuation the result remains true for all real α.

$$(-1)^i a_i(x) =$$

$$= \frac{x}{\alpha+2}\sum_{k=0}^{i-1}\frac{(-\alpha-i-1)_{i-k-1}}{(i-k-1)!}\frac{(\alpha+4)_k}{k!}\frac{(\alpha+2)_{k+1}}{(k+1)!}\cdot$$

$$\cdot\sum_{m=0}^{i-k-1}\frac{(-i+k+1)_m}{m!(-\alpha-i-1)_m}(-x)^m\sum_{n=0}^{k}\frac{(-k)_n}{n!(\alpha+4)_n}x^n=$$

$$=x\sum_{k=0}^{i-1}\frac{(-\alpha-i-1)_{i-k-1}}{(i-k-1)!}\frac{(\alpha+4)_k}{k!}\frac{(\alpha+3)_k}{(k+1)!}\cdot$$

$$\cdot\sum_{j=0}^{i-1}x^j\sum_{m=0}^{j}\frac{(-i+k+1)_m(-1)^m(-k)_{j-m}}{m!(-\alpha-i-1)_m(j-m)!(\alpha+4)_{j-m}}=$$

$$=\sum_{j=0}^{i-1}x^{j+1}\sum_{m=0}^{j}\frac{(\alpha+3)_{i-1}(-1)^{i-1+m}}{m!(-\alpha-i-1)_m(j-m)!(\alpha+4)_{j-m}}\cdot$$

$$\cdot\sum_{k=j-m}^{i-m-1}\frac{(-1)^k(\alpha+4)_k(-i+k+1)_m(-k)_{j-m}}{k!(k+1)!(i-k-1)!}=$$

$$=\sum_{j=0}^{i-1}x^{j+1}\sum_{m=0}^{j}\frac{(\alpha+3)_{i-1}(-1)^{i-1}(i-j)_m}{m!(-\alpha-i-1)_m(j-m)!(j-m+1)!(i-j+m-1)!}\cdot$$

$$\cdot{}_2F_1\left(\begin{matrix}-i+j+1,\ \alpha+4+j-m\\ j-m+2\end{matrix}\middle|\,1\right)=$$

$$=\sum_{j=0}^{i-1}x^{j+1}\,.$$

$$\cdot\sum_{m=0}^{j}\frac{(\alpha+3)_{i-1}(-1)^{i-1}(-\alpha-2)_{i-j-1}}{m!(-\alpha-i-1)_m(j-m)!(j-m+1)!(i-j-1)!(j-m+2)_{i-j-1}}=$$

$$=\sum_{j=0}^{i-1}x^{j+1}\frac{(\alpha+3)_{i-1}(-1)^{i-1}(-\alpha-2)_{i-j-1}}{j!(j+1)!(i-j-1)!(j+2)_{i-j-1}}{}_2F_1\left(\begin{matrix}-j,\ -i\\ -\alpha-i-1\end{matrix}\middle|\,1\right)=$$

$$=\sum_{j=0}^{i-1}x^{j+1}\frac{(\alpha+3)_{i-1}(-1)^{i-1}(-\alpha-2)_{i-j-1}(-\alpha-1)_j}{j!i!(i-j-1)!(-\alpha-i-1)_j}=$$

$$=\frac{1}{i!}\sum_{j=1}^{i}x^j(-1)^{j+1}\binom{\alpha+1}{j-1}\binom{\alpha+2}{i-j}(\alpha+3)_{i-j}.$$

As is pointed out in [3], from formula (3.4) it easily follows that for nonnegative integer values of α the coefficients $a_i(x)$ vanish if $i>2\alpha+4$.

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ON DETERMINANTAL AND PERMANENTAL INEQUALITIES

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1. Introduction

The aim of this paper is to prove four inequalities concerning determinants and permanents of matrices. The first inequality is related to the results of the author [1] and is an extension of the van der Waerden–Egorychev theorem [4], [9].

The second inequality is the determinantal correspondence of the first one. The third and fourth inequalities are inequalities of Szász type and are related to the results [2]. The four inequalities are connected by the common source (Lemma 1.1) of their proofs.

Let $n \geqq 2$ be a fixed positive integer and let

$$\Gamma_k := \left\{ (i_1, \dots, i_k) \mid 1 \leqq i_1 < \dots < i_k \leqq n \right\}$$

be the set of all combinations of order k of the elements $1, \dots, n$ without repetitions and without permutations. We define an ordering of the elements of Γ_k as follows. The combination $(i_1, \dots, i_k)$ is said to precede the combination $(j_1, \dots, j_k)$ if the first nonzero difference in the sequence $j_1 - i_1, \dots, j_k - i_k$ is positive. Thus, to each element of Γ_k there corresponds an integer s, $1 \leqq s \leqq \binom{m}{k}$, such that this element is at the sth place in the above ordering (s is the ordinal number of this element in Γ_k).

The complement of $(i_1, \dots, i_k) \in \Gamma_k$ with respect to the set $\{1, \dots, n\}$ will be denoted by $(\tilde{i}, \dots, \tilde{i}_{n-k}) \in \Gamma_{n-k}$.

Let $\mathcal{M}$ be the set of all $n \times n$ matrices with complex entries and let $A = (a_{jk}) \in \mathcal{M}$. Matrices

$$A_{i_1 \dots i_k}^{j_1 \dots j_k} := \begin{pmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_k} \\ \cdot & \dots & \cdot \\ a_{i_k j_1} & \dots & a_{i_k j_k} \end{pmatrix}$$

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with

$$(i_1, \dots, i_k),\ (j_1, \dots, j_k) \in \Gamma_k \tag{1.1}$$

are called k rowed matrices of A $(k = 1, \dots, n)$.

The permanent of $A \in \mathcal{M}$ will be denoted by $\operatorname{Per} A$. Properties of permanents used in this paper can be found e.g. in [6].

The permanent (determinant) of a k rowed matrix of $A \in \mathcal{M}$ will be called a k rowed permanental (determinantal) minor of A, respectively.

The common source of our proofs is

LEMMA 1.1. *If $A \in \mathcal{M}$ then for $k = 1, \dots, n$,*

$$\sum \operatorname{Per} A^{j_1 \dots j_k}_{i_1 \dots i_k} \operatorname{Per} A^{\bar{j}_1 \dots \bar{j}_{n-k}}_{\bar{i}_1 \dots \bar{i}_{n-k}} = \binom{n}{k} \operatorname{Per} A, \tag{1.2}$$

$$\sum (-1)^{\sum_{\alpha=1}^{k} (i_\alpha + j_\alpha)} \operatorname{Det} A^{j_1 \dots j_k}_{i_1 \dots i_k} \operatorname{Det} A^{\bar{j}_1 \dots \bar{j}_{n-k}}_{\bar{i}_1 \dots \bar{i}_{n-k}} = \binom{n}{k} \operatorname{Det} A \tag{1.3}$$

where the summations run over (1.1).

PROOF. The statement of the Lemma 1.1 is a consequence of the Laplace expansion formula, which is applicable in the case of permanents too.

2. On inequalities of van der Waerden type

(a) The matrix $A \in \mathcal{M}$ with non-negative entries is said to be a doubly stochastic matrix, if all of its row and column sums are equal to one.

Let $A_0 \in \mathcal{M}$ be the doubly stochastic matrix, whose all entries are $\frac{1}{n}$.

The following conjecture was published by van der Waerden [9] in 1926.

Let $A \in \mathcal{M}$ be a doubly stochastic matrix. Then

$$\operatorname{Per} A \geqq \frac{n!}{n^n} \tag{2.1}$$

with equality if and only if $A = A_0$.

This conjecture was proved first by G. P. Egorychev [4] in 1980.

The proof of the following inequality is based on the theorem of van der Waerden–Egorychev.

Let $T_k(A)$ be the sum of all k rowed permanental minors of $A \in \mathcal{M}$, where $k = 1, \dots, n$, and let $T_0(A) = 1$.

Moreover let

$$\frac{1}{\binom{n}{k}^2} T_k(A) := t_k(A) \quad (k = 0, 1, \dots, n).$$

THEOREM 2.1. *If $A \in \mathcal{M}$ is a doubly stochastic matrix, then*

$$t_k(A)t_{n-k}(A) \geqq \frac{n!}{n^n}\frac{1}{\binom{n}{k}} \quad (k = 0, 1, \dots, n) \tag{2.2}$$

with equality if and only if $A = A_0$.

REMARK. For $k = 0$, or $k = n$ Theorem 2.1 gives the theorem of van der Waerden–Egorychev.

PROOF OF THEOREM 2.1. Applying the Schwarz inequality in (1.2), we get

$$\left(\sum \operatorname{Per}^2 A_{i_1 \dots i_k}^{j_1 \dots j_k}\right)^{\frac{1}{2}} \left(\sum \operatorname{Per}^2 A_{\bar{i}_1 \dots \bar{i}_{n-k}}^{\bar{j}_1 \dots \bar{j}_{n-k}}\right)^{\frac{1}{2}} \geqq \binom{n}{k} \operatorname{Per} A. \tag{2.3}$$

Here and in the next formula summations are extended over (1.1). Since the entries of A are non-negative numbers, we obtain

$$\sum \operatorname{Per}^2 A_{i_1 \dots i_k}^{j_1 \dots j_k} \leqq T_k^2(A) \quad (k = 0, 1, \dots, n),$$

which gives us the inequality

$$t_k(A)t_{n-k}(A) \geqq \frac{1}{\binom{n}{k}} \operatorname{Per} A \geqq \frac{1}{\binom{n}{k}} \frac{n!}{n^n} \tag{2.4}$$

by (2.3), and by inequality (2.1). Since, by the van der Waerden–Egorychev theorem, $\operatorname{Per} A = \frac{n!}{n^n}$ if and only if $A = A_0$, and since

$$t_k(A_0) = \frac{k!}{n^k}, \quad t_{n-k}(A_0) = \frac{(n-k)!}{n^{n-k}},$$

i.e.

$$t_k(A_0)t_{n-k}(A_0) = \frac{1}{\binom{n}{k}} \frac{n!}{n^n} \quad (k = 0, 1, \dots, n),$$

we get that equality holds in (2.4), consequently in (2.2) too if and only if $A = A_0$. This completes the proof of Theorem 2.1.

(b) Let E be the unit matrix. The matrix cE, where c is a positive number, is said to be a positive scalar matrix. Let $A^* \in \mathcal{M}$ denote the conjugate transpose of $A \in \mathcal{M}$, and let $\operatorname{tr} A$ denote the sum of the diagonal elements of the matrix A.

For an $A \in \mathcal{M}$ let $C_k(A)$ be the $\binom{n}{k} \times \binom{n}{k}$ matrix defined by

$$C_k(A) := \left(C_{pq}^{(k)}\right) \quad \left(1 \leqq p, q \leqq \binom{n}{k},\ k = 1, \dots, n\right),$$

where

$$C_{pq}^{(k)} = \text{Det}\, A_{i_1\ldots i_k}^{j_1\ldots j_k},$$

and the ordinal numbers of the combinations $(j_1,\ldots,j_k)$ and $(i_1,\ldots,i_k)$ in the ordering of Γ_k are p and q, respectively. Further let

$$D^2(A) := \text{tr}\, C_k(A)C_k^*(A) \quad (k = 1,\ldots,n).$$

THEOREM 3.2. *Let n be a positive integer. Let $A \in \mathcal{M}$. Then*

$$(2.5) \qquad D_k(A)D_{n-k}(A) \geqq \binom{n}{k} \text{Det}\, A \quad (k = 1,\ldots,n-1)$$

with equality if and only if

(a) *$k \neq n-k$ and A is an orthogonal matrix with determinant one, or*

(b) *$k = n-k$ and A is an orthogonal matrix with determinant one multiplied by an arbitrary positive scalar, or*

(c) *A is the zero matrix.*

PROOF. Using Schwarz inequality on the left side of (1.3), we get (2.5). We remark that to get equality in (2.5) the condition $\text{Det}\, A \geqq 0$ is obviously necessary.

First, let us suppose that $\text{Det}\, A > 0$. In this case equality holds in (2.5) if and only if a constant $\lambda_k \neq 0$ exists such that the equations

$$(2.6) \qquad (-1)^{\sum_{\alpha=1}^{k}(i_\alpha + j_\alpha)} \text{Det}\, A_{i_1\ldots i_k}^{j_1\ldots j_k} = \lambda_k \; \text{Det}\, A_{\bar{i}_1\ldots \bar{i}_{n-k}}^{\bar{j}_1\ldots \bar{j}_{n-k}}$$

are satisfied for all $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_k)$ of the set (1.1). Condition (2.6) can be written in the form

$$\mathcal{E}C_k(A)\mathcal{E} = \lambda_k \left(\text{Det}\, A_{\bar{i}_1\ldots \bar{i}_{n-k}}^{\bar{j}_1\ldots \bar{j}_{n-k}}\right) = \lambda_k P C_{n-k}(A) P \quad (k = 1,\ldots,n),$$

where $\mathcal{E} = (\mathcal{E}_{ij})$ is the matrix with elements

$$\mathcal{E}_{ij} = 0,\ i \neq j;\ \mathcal{E}_{ii} = (-1)^{i+1} \quad \left(i,j = 1,\ldots,\binom{n}{k}\right),$$

and P is a permutation matrix. By the theorem of Franke ([5], p. 104, Satz 31) we have

$$(2.7) \qquad (\text{Det}\, A)^{\binom{n-1}{k-1}} = \lambda_k^{\binom{n}{k}} (\text{Det}\, A)^{\binom{n-1}{k}}.$$

Using the identity

$$\frac{1}{\binom{n}{k}}\left[\binom{n-1}{k-1}-\binom{n-1}{k}\right]=\frac{2k}{n}-1,$$

we get by (2.7) that

$$\lambda_k=(\operatorname{Det}A)^{\frac{2k}{n}-1}\quad(k=1,\dots,n-1).$$

Since equality holds in (2.5) if condition (2.6) is satisfied, we get

$$D_k^2(A)=\lambda_k\binom{n}{k}\operatorname{Det}A=\binom{n}{k}(\operatorname{Det}A)^{\frac{2k}{n}}.$$

We obtain from here

$$(2.8)\qquad (\operatorname{Det}A)^{\frac{k}{n}}=\left(\frac{1}{\binom{n}{k}}\sum_{(j_1,\dots,j_k)\in\Gamma_k}\ \sum_{(i_1,\dots,i_k)\in\Gamma_k}\operatorname{Det}^2 A_{i_1\dots i_k}^{j_1\dots j_k}\right)^{\frac{1}{2}}\geqq$$

$$\geqq\left[\left(\prod_{(j_1,\dots,j_k)\in\Gamma_k}\ \sum_{(i_1\dots i_k)\in\Gamma_k}\operatorname{Det}^2 A_{i_1\dots i_k}^{j_1\dots j_k}\right)^{\frac{1}{2}}\right]^{\frac{1}{\binom{n}{k}}}\geqq$$

$$\geqq\left(\operatorname{Det}C_k(A)\right)^{\frac{1}{\binom{n}{k}}}=(\operatorname{Det}A)^{\frac{k}{n}}$$

by using again the theorem of Franke, and the equality

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}}=\frac{k}{n}.$$

The first inequality in (2.8) is valid by the well-known inequality between the arithmetic and geometric means, and equality holds here if and only if all the quantities

$$\sum_{(i_1,\dots,i_k)\in\Gamma_k}\operatorname{Det}^2 A_{i_1,\dots i_k}^{j_1\dots j_k},\quad (j_1,\dots,j_k)\in\Gamma_k$$

are equal. The second inequality of (2.8) holds by a theorem of Hadamard, with equality if and only if $C_k(A)$ is a diagonal matrix. Since equality should hold in both inequalities of (2.8) it follows that $C_k(A)C_k^*(A)$ is a positive scalar matrix i.e.

$$C_k(A)C_k^*(A) = C_k(AA^*) = d^2E, \quad d > 0.$$

In this case

$$\text{(2.9)} \qquad \operatorname{Det} A = d^{\frac{n}{k}},\ D_k^2(A) = D_{n-k}^2(A) = \operatorname{tr} C_k(AA^*) = \binom{n}{k} d^2,$$

thus equality holds in (2.5) if and only if $d^n = d^{2k}$. If $n \neq 2k$, then the only positive solution of the system (2.9) is $d = 1$. If $n = 2k$, then (2.9) is satisfied by an arbitrary $d > 0$. Thus $C_k(A)$ is an orthogonal matrix, and by Theorem 2 of [3], A is an orthogonal matrix with determinant 1, if $m \neq 2k$, while in the case $m = 2k$ A it is a matrix which can be obtained from an orthogonal matrix with determinant 1, by multiplying it by a positive scalar.

If $\operatorname{Det} A = 0$, and equality holds in (2.5), then the matrix

$$C_k(A)C_k^*(A) = C_k(AA^*)$$

is the zero matrix, consequently AA^*, and thus A is the zero matrix too ([3], Satz 1).

It is easy to verify that if either condition (a), or (b), or (c) is satisfied, then equality holds in (2.5).

This completes the proof of Theorem 2.2.

3. On inequalities of O. Szász type

(a) Let $R_k(A)$ be the product of all k rowed permanental minors of A, $k = 1, \dots, n$, and let $R_0(A) = 1$.

As usual, matrices

$$A(i_1, \dots, i_k) := A_{i_1 \dots i_k}^{i_1 \dots i_k}, \quad (i_1, \dots, i_k) \in \Gamma_k$$

are said to be principal k rowed matrices of $A \in \mathcal{M}$, where $k = 1, \dots, n$. It is obvious that $A(1, \dots, n) = A$.

The permanent (determinant) of a principal k rowed matrix of $A \in \mathcal{M}$ is said to be a principal k rowed permanental (determinantal) minor of A. $P_k(A)$ $\big(Q_k(A)\big)$ denotes the product of all principal k rowed permanental (determinantal) minors of A, where $k = 1, \dots, n$.

For brevity let us set

$$q_k(A) := Q_k^{\frac{1}{\binom{n-1}{k-1}}}(A) \quad (k = 1, \dots, n).$$

In [8] O. Szász refined Hadamard's celebrated inequality in the following way.

If $A \in \mathcal{M}$ is a positive definite Hermitian matrix, then

$$q_1(A) \geqq q_2(A) \geqq \cdots \geqq q_{n-1}(A) \geqq q_n(A)$$

with equality if and only if A is a diagonal matrix.

Hadamard's determinantal theorem states that if $A \in \mathcal{M}$ is a positive definite Hermitian matrix, then

$$Q_n(A) \leqq Q_1(A) \tag{3.1}$$

with equality if and only if A is a diagonal matrix.

Using Hadamard's theorem we give a short and simple proof of Szász's theorem (another short proof can be found in [7]).

Let $2 \leqq k \leqq n$ and let $(i_1, \ldots, i_k) \in \Gamma_k$. If A is a positive definite Hermitian matrix, then the matrices $A(i_1, \ldots, i_k)$, $\operatorname{adj} A(i_1, \ldots, i_k)$ have this property too. Applying the Hadamard's thorem, we get

$$\operatorname{Det}\big(\operatorname{adj} A(i_1, \ldots, i_k)\big) = \big(\operatorname{Det} A(i_1, \ldots, i_k)\big)^{k-1} \leqq Q_{k-1}\big(A(i_1, \ldots, i_k)\big).$$

Consequently,

$$Q_k(A) \leqq \Big(\prod Q_{k-1}\big(A(i_1, \ldots, i_k)\big)\Big)^{\frac{1}{k-1}}, \tag{3.2}$$

where the product is extended to Γ_k. Since the factors of $Q_{k-1}(A)$ have the common multiplicity

$$\frac{\binom{n}{k}\binom{k}{k-1}}{\binom{n}{k-1}}$$

in the expression

$$\prod Q_{k-1}\big(A(i_1, \ldots, i_k)\big)$$

and since the identity

$$\frac{\binom{n}{k}\binom{k}{k-1}}{\binom{n}{k-1}} \frac{1}{k-1} = \frac{\binom{n-1}{k-1}}{\binom{n-1}{k-2}}$$

holds, we get by (3.2) that

$$q_k(A) \leqq q_{k-1}(A) \quad (k = 2, \ldots, n) \tag{3.3}$$

which is exactly the inequality of O. Szász. Equalities hold simultaneously in all inequalities (3.3) if and only if equality holds in (3.1), i. e. if A is a diagonal matrix.

(b) Let $p_k(A)$ be defined by

$$p_k(A) := P_k^{\frac{1}{\binom{n}{k}}}(A) \quad (k = 1, \dots, n).$$

It is obvious that $p_k(A)$ is the geometric mean of all principal k rowed permanental minors of A.

THEOREM 3.1. *Suppose that $A = (a_{jk}) \in \mathcal{M}$ is a matrix with non-negative entries whose diagonal elements are positive. Then*

$$\text{(3.1.1)} \qquad \operatorname{Per} A \geqq p_1(A)p_{n-1}(A) \geqq \dots \geqq p_1^k(A)p_{n-k}^{(A)} \geqq$$
$$\geqq \dots \geqq p_1^{n-2}(A)p_2(A) \geqq P_1(A)$$

with equality if and only if all principal permanental minors of the matrix

$$\text{(3.1.2)} \qquad A - \begin{pmatrix} a_{11} & & (0) \\ & \ddots & \\ (0) & & a_{nn} \end{pmatrix} = B(A)$$

are equal to zero.

For brevity let us set

$$(m)_k := k!\binom{n}{k} \quad (k = 0, 1, \dots, n).$$

A matrix $A \in \mathcal{M}$ is a said to be a dyad if the representation

$$A = ab^*$$

holds, where a and b are n dimensional column vectors, and b^* is the transpose of b. This representation is said to be positive if the components of both vectors are positive numbers.

It is obvious that

$$r_k(A) := R_k^{\frac{1}{\binom{n}{k}^2}}(A) \quad (k = 1, \dots, n)$$

is the geometric mean of all k rowed permanental minors of $A \in \mathcal{M}$ with positive entries.

THEOREM 3.2. *Suppose that the entries of $A \in \mathcal{M}$ are positive numbers. Then*

$$\text{Per}\, A \geqq (n)_1 r_1(A) r_{n-1}(A) \geqq \cdots \geqq \geqq (n)_k r_1^k(A) r_{n-k}(A) \geqq \cdots \geqq (n)_n R_1^{\frac{1}{n}}(A) \tag{3.2.1}$$

with equality if and only if A is a dyad with positive representation.

PROOFS. We prove Theorems 3.1 and 3.2 parallel. The numbers (3.1...) and (3.2...) refer to Theorems 3.1 and 3.2, respectively.

We need the following

LEMMA 3.1. *Under the assumptions of Theorems* 3.1 *and* 3.2, *the inequalities*

$$p_k(A) \geqq p_1(A) p_{k-1}(A) \quad (k = 1, \ldots, n;\ \ p_0(A) = 1) \tag{3.1.3}$$

and

$$r_k(A) \geqq k r_1(A) r_{k-1}(A) \quad (k = 1, \ldots, n;\ \ r_0(A) = 1) \tag{3.2.2}$$

hold, respectively.

PROOF OF LEMMA 3.1. Since the entries of A are non-negative, and the principal k rowed permanental minors of A (permanental minors of A) are positive, we get by Lemma 1.1

$$\binom{n}{k} \text{Per}\, A \geqq \sum_{(i_1, \ldots, i_k) \in \Gamma_k} \text{Per}\, A(i_1, \ldots, i_k)\ \text{Per}\, A(\tilde{i}_1, \ldots, \tilde{i}_{n-k}), \tag{3.1.4}$$

$$\text{Per}\, A = \binom{n}{k} \frac{1}{\binom{n}{k}^2} \sum \text{Per}\, A_{i_1 \ldots i_k}^{j_1 \ldots j_k}\ \text{Per}\, A_{\tilde{i}_1 \ldots \tilde{i}_{n-k}}^{\tilde{j}_1 \ldots \tilde{j}_{n-k}} \tag{3.2.3}$$

where in (3.2.3) the summation is extended over (1.1). Using the well-known inequality between arithmetic and geometric means, we have

$$p_k(A) p_{n-k}(A) \leqq \text{Per}\, A, \tag{3.1.5}$$

$$\binom{n}{k} r_k(A) r_{n-k}(A) \leqq \text{Per}\, A \tag{3.2.4}$$

by (3.1.4) (by (3.2.3)). If $k = 1$ then

$$p_1(A) p_{n-1}(A) \leqq \text{Per}\, A, \tag{3.1.6}$$

$$n r_1(A) r_{n-1}(A) \leqq \text{Per}\, A \tag{3.2.5}$$

by (3.1.4) (by (3.2.4)), which gives us (3.1.3) ((3.2.2)) in the case of $k = 1$.

We can justify (3.1.3) ((3.2.2)) using (3.1.6) ((3.2.5)).

Namely, the entries of $A \in \mathcal{M}$ are non-negative numbers, and the diagonal elements of $A(i_1, \dots, i_k)$ (the entries of $A^{j_1 \dots j_k}_{i_1 \dots i_k}$) are positive, thus (3.1.6) ((3.2.5)) is applicable. Therefore

$$\operatorname{Per} A(i_1, \dots, i_k) \geqq \left(P_1 A(i_1, \dots, i_k) P_{k-1} A(i_1, \dots, i_k) \right)^{\frac{1}{k}},$$

and

$$\operatorname{Per} A^{j_1 \dots k_k}_{i_1 \dots i_k} \geqq k \left(R_1 \left(A^{j_1 \dots k_k}_{i_1 \dots i_k} \right) R_{k-1} \left(A^{j_1 \dots j_k}_{i_1, \dots i_k} \right) \right)^{\frac{1}{k^2}},$$

thus

$$P_k(A) \geqq \left(\prod P_1 \left(A(i_1, \dots, i_k) \right) P_{k-1} \left(A(i_1, \dots, i_k) \right) \right)^{\frac{1}{k}}, \tag{3.1.7}$$

where $(i_1, \dots, i_k)$ runs over Γ_k, and

$$R_k(A) \geqq \left(\prod R_1 \left(A^{j_1 \dots j_k}_{i_1 \dots i_k} \right) R_{k-1} \left(A^{j_1 \dots j_k}_{i_1 \dots i_k} \right) \right)^{\frac{1}{k^2}} k^{\binom{n}{k}^2} \tag{3.2.6}$$

where the product is extended over (1.1).

It is not difficult to see that

$$\prod P_1 \left(A(i_1, \dots, i_k) \right) = P_1^{\binom{n-1}{k-1}}(A),$$

and

$$\prod R_1 \left(A^{j_1 \dots j_k}_{i_1 \dots i_k} \right) = R_1^{\binom{n-1}{k-1}^2}(A),$$

moreover that all factors of $P_{k-1}(A)$ have the same multiplicity

$$\frac{\binom{n}{k}\binom{k}{k-1}}{\binom{n}{k-1}} = n - k + 1 \tag{3.1.8}$$

in the expression

$$\prod P_{k-1} \left(A(i_1, \dots, i_k) \right),$$

and that all factors of $R_{k-1}(A)$ have the same multiplicity

$$\left(\frac{\binom{n}{k}\binom{k}{k-1}}{\binom{n}{k-1}} \right)^2 = (n - k + 1)^2 \tag{3.2.7}$$

in the expression

$$\prod R_{k-1}\left(A^{j_1\dots j_k}_{i_1\dots i_k}\right).$$

Thus (3.1.7) ((3.2.6)) can be reduced to the form

$$P_k(A) \geqq P_1(A)^{\binom{n-1}{k-1}\frac{1}{k}} P_{k-1}(A)^{(n-k+1)\frac{1}{k}} \tag{3.1.9}$$

and

$$R_k(A) \geqq R_1(A)^{\binom{n-1}{k-1}^2\frac{1}{k^2}} R_{k-1}(A)^{(n-k+1)^2\frac{1}{k^2}k\binom{n}{k}^2}, \tag{3.2.8}$$

respectively.

Taking the $\binom{n}{k}$th ($\binom{n}{k}^2$th) root of both sides of (3.1.9) ((3.2.8)), and taking identities (3.1.8) and (3.2.7), and the identity

$$\frac{\binom{n-1}{k-1}\frac{1}{k}}{\binom{n}{k}} = \frac{1}{n}$$

into consideration, we get that (3.1.3) and (3.2.2), i.e. (3.1.1) and (3.2.1) hold.

The statements concerning equality can be proved as follows.

It is evident that equality holds simultaneously in (3.1.1) if and only if the equation

$$\operatorname{Per} A = P_1(A) \tag{3.1.10}$$

is satisfied by a matrix A with non-negative entries, where all principal permanental minors are positive. Accordingly, our aim is to find all such matrices.

Let a_{jj} $(j = 1, \dots, n)$ be the diagonal elements of A. Using the well-known Cauchy expansion formula, we get

$$\operatorname{Per} A = P_1(A) + \sum_{k=2}^{n} \mathcal{S}_k\big(B(A)\big) \tag{3.1.11}$$

by (3.1.2), where

$$\mathcal{S}_k\big(B(A)\big) := \sum_{(i_1,\dots,i_k)\in\Gamma_k} a_{\bar{i}_1\bar{i}_1}\dots a_{\bar{i}_{n-k}\bar{i}_{n-k}} \operatorname{Per} B^{i_1\dots i_k}_{i_1,\dots i_k}(A)$$

$$(k = 2, \dots, n).$$

Hence equation (3.1.10) is satisfied by A if and only if

$$\sum_{k=2}^{m} \mathcal{S}_k\big(B(A)\big) = 0. \tag{3.1.12}$$

Since all terms of the sum $\mathcal{S}_k\big(B(A)\big)$ are non-negative and the diagonal elements of A are positive, (3.1.12) is satisfied if any only if

$$\operatorname{Per} B^{i_1\dots i_k}_{i_1\dots i_k}(A) = 0, \quad (i_1,\dots,i_k) \in \Gamma_k, \quad k = 2,\dots,n, \tag{3.1.13}$$

i. e. all principal determinantal minors of $B(A)$ are zero.

Conversely, if conditions (3.1.13) are satisfied then (3.1.10) holds by (3.1.11).

The statement that equality holds simultaneously in (3.2.1) if and only if A is a dyad with positive representation can be proved as follows.

It is evident that equality holds simultaneously in (3.2.1) if and only if the equation

$$\operatorname{Per} A = n! R_1(A)^{\frac{1}{n}} \tag{3.2.9}$$

is satisfied by a matrix with positive entries.

Let $A = (a_{jk}) \in \mathcal{M}$ be a matrix with positive entries. In this case the quantities

$$\frac{1}{n!} \operatorname{Per} A, \quad R_1(A)^{\frac{1}{n}}$$

are the arithmetic and geometric means of the positive numbers

$$a_{1i_1} \dots a_{ni_n}, \quad (i_1,\dots i_n) \in \prod, \tag{3.2.10}$$

where $\prod$ denotes the set of all permutations of elements $1,\dots,n$ without repetitions. Since by (3.2.9) these means are equal, we have

$$a_{1i_1} \dots a_{ni_n} = a > 0 \quad \text{for all} \quad (i_1,\dots,i_n) \in \prod. \tag{3.2.11}$$

From (3.2.11) for the permutations

$$(i_1,\dots,i_k,i_{k+1},\dots,i_1) \quad \text{and} \quad (i_1,\dots,i_{k+1},i_k,\dots,i_n)$$

we get

$$\frac{a_{ki_{k+1}}}{a_{ki_k}} = \frac{a_{k+1,i_{k+1}}}{a_{k+1,i_k}},$$

hence

$$\frac{a_{kj+1}}{a_{kj}} = a_{j+1} \quad (j = 1, \ldots, n-1)$$

by the substitutions $i_k = j$, $i_{k+1} = j+1$, where a_{j+1} is a positive constant. Using this result we get that

$$a_{kj} = \lambda_j a_{k1} \quad (j, k = 1, \ldots, n),$$

where

$$\lambda_1 = 1, \ \lambda_j = a_2 \ldots a_j \quad (j = 2, \ldots, n),$$

i. e. A is a dyad with positive representation.

It is easy to verify that equation (3.2.9) is satisfied by such a dyad with positive representation.

This completes the proof of Theorem 3.2.

COROLLARY 3.1. *If all principal permanental minors of the symmetric matrix $A \in \mathcal{M}$ with non-negative entries are positive, then inequalities* (3.1.1) *hold with equality if and only if A is a diagonal matrix.*

PROOF. Since the conditions of Theorem 3.1 are consequences of the conditions of Corollary 3.1, inequalities (3.1.1) hold. In order that in these inequalities simultaneous equalities hold, it is necessary and sufficient that $\mathcal{S}_2\big(B(A)\big) = 0$, i. e.

$$a_{jk}a_{kj} = a_{jk}^2 = 0, \quad j \neq k \quad (j, k = 1, \ldots, n)$$

with $A = (a_{jk})$. Hence A is a diagonal matrix.

Since the principal permanental minors of a positive definite symmetric matrix with non-negative entries are positive, we get the following result (by Corollary 3.1).

COROLLARY 3.2. *Let $A \in \mathcal{M}$ be a symmetric, positive definite matrix with non-negative entries. Then inequality* (3.1.1) *holds with equality if and only if A is a diaginal matrix.*

In [2] the author formulated two conjectures (Conjectures 3.1, 3.2). Corollaries 3.1 and 3.2 are related to these conjectures.

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ON THE NORMS OF CONJUGATE TRIGONOMETRIC POLYNOMIALS

R. GÜNTTNER (Osnabrück)

1. Introduction. If

$$t_n(x)=\frac{1}{2}\alpha_0+\sum_{k=1}^{n}(\alpha_k\cos kx+\beta_k\sin kx),\qquad x\in\mathbf{R},$$

denotes a trigonometric polynomial of order at most n with real coefficients then

$$\tilde{t}_n(x):=\sum_{k=1}^{n}(\alpha_k\sin kx-\beta_k\cos kx)$$

is called the polynomial "conjugate to $t_n(x)$". Using the maximum norm $\|f\|=\max_{x\in\mathbf{R}}|f(x)|$ we define

$$C_n:=\sup_{\|t_n\|\leqq 1}\|\tilde{t}_n\|\qquad(n\geqq 1),$$

which means that for any t_n satisfying $\|t_n\|\leqq 1$ we have $\|\tilde{t}_n\|\leqq C_n$, and this bound cannot be replaced by a smaller one.

Another formulation is as follows: Let $f(z), z\in\mathbf{C}$, be a polynomial of degree at most n, $f(0)$ real, $|z|\leqq 1$ and $|\operatorname{Re} f(z)|\leqq 1$, then $|\operatorname{Im} f(z)|\leqq C_n$, where the constant C_n is the smallest possible one independent from f.

2. Results. Taikov [7] showed that

$$C_n=\frac{2}{n+1}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\cot\frac{(2k+1)\pi}{2(n+1)}. \tag{1}$$

Let us notice that this result has already been proved by Szegö [6].

It is derived here from that (cf. [6])

$$C_n=\frac{2}{\pi}\log n+O(1) \tag{2}$$

and (cf. [7])

$$(3)\qquad C_n = \frac{2}{\pi}\log\left(\sin\frac{\pi}{2(n+1)}\right)^{-1} + r_n, \qquad 0 < r_n < \frac{4}{\pi}.$$

The purpose of this paper is to get an improvement of (2) and (3). As a consequence of the following theorem we have for instance

$$(4)\qquad C_n = \frac{2}{\pi}\log n + a_0 + \varepsilon_n, \qquad 0 < \varepsilon_n < \frac{4}{3n},$$

$$a_0 := \frac{2}{\pi}(\gamma + \log\frac{4}{\pi}) = 0.5212\ldots$$

($\gamma = 0.5772\ldots$ is Euler's constant). More precisely we prove the following

THEOREM. *The sequence C_n is strictly increasing, i.e. we have $C_1 < < C_2 < C_3 < \ldots$. Further, if n is odd, then*

$$(5)\qquad C_n = \frac{2}{\pi}\log(n+1) + a_0 + \frac{a_2}{(n+1)^2} + \ldots + \frac{a_{2k}}{(n+1)^{2k}} + r_n^{(2k)},$$

$$a_{2i} := (-1)^{i-1}\frac{8\pi^{2i-1}}{2i\cdot(2i)!}\cdot\left[(2^{2i-1}-1)\cdot B_{2i}\right]^2 \qquad (i>0),$$

$$0 < (-1)^k r_n^{(2k)} < (-1)^k\frac{a_{2k+2}}{(n+1)^{2k+2}}.$$

(Here B_{2k} denotes the Bernoulli numbers, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$).

The constant a_0 coincides with the constant χ well known from optimal norms in algebraic interpolation [cf. 8 and 3].

3. Proof. We first prove $C_{2m-1} < C_{2m}$, $m = 1,2,3,\ldots$. From (1) we have

$$C_{2m-1} = \frac{2}{2m}\sum_{k=0}^{m-1}\cot\frac{(2k+1)\pi}{2\cdot 2m}, \qquad C_{2m} = \frac{2}{2m+1}\sum_{k=0}^{m-1}\cot\frac{(2k+1)\pi}{2(2m+1)},$$

Therefore it suffices to prove that

$$\frac{1}{2m}\cot\frac{(2k+1)\pi}{2\cdot 2m} < \frac{1}{2m+1}\cot\frac{(2k+1)\pi}{2(2m+1)},$$

or, equivalently,

$$\frac{(2k+1)\pi}{2\cdot 2m}\cot\frac{(2k+1)\pi}{2\cdot 2m} < \frac{(2k+1)\pi}{2(2m+1)}\cot\frac{(2k+1)\pi}{2(2m+1)}.$$

But the last inequality is a consequence of the fact that $g(x) := x \cot x$ is strictly monotone decreasing, $0 < x \leqq \frac{\pi}{2}$, which can be easily seen by differentiating g, observing that

$$g' < 0 \iff \cot x < \frac{x}{\sin^2 x} \iff \cos x < \frac{x}{\sin x},$$

and $\cos x < 1 < x/\sin x$.

Now to prove $C_{2m} < C_{2m+1}$ we only have to note that $\cot\frac{\pi}{2} = 0$ therefore by (1)

$$C_{2m} = \frac{2}{2m+1}\sum_{k=0}^{m} \cot\frac{(2k+1)\pi}{2(2m+1)},$$

and of course

$$C_{2m+1} = \frac{2}{2m+2}\sum_{k=0}^{m} \cot\frac{(2k+1)\pi}{2(2m+2)}. \tag{6}$$

By similar arguments as before we get the first statement of the theorem.

Suppose now that n is odd, $n = 2m+1$. From (6) we get

$$C_{2m+1} = \frac{1}{m+1}\sum_{k=0}^{m} \cot\frac{(2k+1)\pi}{4(m+1)}. \tag{7}$$

The asymptotic expansion of the right hand side of (7) is well known, first proved by Günttner (cf. [4]) followed by Shivakumar and Wong [5], Dzjadyk and Ivanov [1], Feng [2]. Using [4] and substituting there n by $m+1$ in formula (2) and Theorem 1 we easily get

$$C_{2m+1} = \frac{2}{\pi}\log(m+1) + A_0 + \frac{A_2}{(m+1)^2} + \ldots + \frac{A_{2k}}{(m+1)^{2k}} + R_{m+1}^{(2k)},$$

$$A_0 := \frac{2}{\pi}\left(\gamma + \log\frac{8}{\pi}\right),$$

$$A_{2i} := (-1)^{i-1}\frac{8\pi^{2i-1}}{2i\cdot 2^{2i}\cdot(2i)!}\cdot\left[(2^{2i-1}-1)\cdot B_{2i}\right]^2 \quad (i>0),$$

$$0 < (-1)^k\cdot R_{m+1}^{(2k)} < (-1)^k\cdot\frac{A_{2k+2}}{(m+1)^{2k+2}}.$$

Since here we have $2m+1 = n$, n odd, we may replace $m+1$ by $\frac{n+1}{2}$ which immediately leads to the second statement (5) of the theorem.

Finally, to prove (4) observe that for n odd the theorem implies

$$(8)\quad \frac{2}{\pi}\log n + a_0 < \frac{2}{\pi}\log(n+1) + a_0 < C_n < \frac{2}{\pi}\log(n+1) + a_0 + \frac{\pi}{18(n+1)^2}.$$

Taking into account that $\log(n+1) = \log n + \log(1+\frac{1}{n}) < \log n + \frac{1}{n}$ we easily derive

$$0 < r_n < \frac{2}{\pi n} + \frac{\pi}{18(n+1)^2} < \frac{4}{3n} \qquad (n \text{ odd}).$$

If n is even then $n-1$ and $n+1$ being odd we get from (8)

$$\frac{2}{\pi}\log n + a_0 < C_{n-1} < C_n < C_{n+1} < \frac{2}{\pi}\log(n+2) + a_0 + \frac{\pi}{18(n+2)^2}$$

which yields by analogous arguments

$$0 < r_n < \frac{4}{\pi n} + \frac{\pi}{18(n+2)^2} < \frac{4}{3n} \qquad (n \text{ even}).$$

This completes the proof.

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A LATTICE CONSTRUCTION AND CONGRUENCE-PRESERVING EXTENSIONS

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1. Introduction

To find a simple proof of the congruence lattice characterization theorem of finite lattices, H. Lakser and the first author (see [1]) introduced a special type of finite partial lattices: a meet-semilattice in which any two elements with a common upper bound have a join. If M is such a finite partial lattice, then the ideal lattice of M is a congruence-preserving extension of M; that is, every congruence of M has exactly one extension to the ideal lattice.

In [2], we introduced the name *chopped lattice* for such partial lattices, no longer necessarily finite. Of course, if M is no longer finite, we cannot expect the ideal lattice to be a congruence-preserving extension. It is natural to consider, instead, finitely generated ideals; unfortunately, they do not, in general, form a lattice. In Section 2 we introduce Condition (FG) under which the finitely generated ideals form a lattice.

Given two lattices A and B, sharing the sublattice $C = A \cap B$, we obtain the lattice $M(A, B)$ by amalgamation. If C is a principal ideal of both A and B, then $M(A, B)$ is a chopped lattice.

In Section 3, we introduce (see Definition 3) a set of sufficient conditions under which $M(A, B)$ is a chopped lattice. If A and B satisfy the conditions of Definition 3, we shall call A, B a *chopped pair*. Theorem 1 states that if A, B is a chopped pair, then $M(A, B)$ is a chopped lattice. The concept of a chopped pair does not seem strong enough to compute with it. In Section 4, we introduce two stronger versions: *sharp* and *full* chopped pairs.

In Section 5 we investigate finitely generated ideals in $M(A, B)$ for a chopped pair A, B. For a sharp chopped pair A and B, if $C = A \cap B$ satisfies the Ascending Chain Condition, then we obtain Condition (FG) (which guarantees that the finitely generated ideals form a lattice) for $M(A, B)$.

In Section 6 we investigate modular lattices. If A, B is a sharp chopped pair and both A and B are modular, then $M(A, B)$ satisfies Condition (FG)

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(Theorem 3). If A, B is a full chopped pair, then it is enough to assume that one of them is modular to obtain the same conclusion (Theorem 4).

In Section 7 we deal with the problem whether every lattice has a proper congruence-preserving extension. We apply Theorem 4 to prove that if there exists a nontrivial distributive interval in a lattice, then it has a proper congruence-preserving extension.

A modular example of a congruence-preserving extension is outlined in Section 6.

1.1. Notation. We refer the reader to [1] for the basic concepts and notation.

In a lattice L, $[x,y]_L$ denotes the interval in L, and $(a]_L$ the principal ideal generated by a. If there is no confusion, the subscript is dropped.

If L is a sublattice of K, then we call K an *extension* of L. If L has a zero, and it is also the zero of K, then K is $\{0\}$-*extension* of L.

2. Chopped lattices

A *chopped lattice* M is a lattice L with zero, 0, and unit, 1, with the unit removed: $M = L - \{1\}$; on M, 0 is a nullary operation, $\wedge$ is an operation, and $\vee$ is a partial operation. Equivalently, a chopped lattice M is a meet-semilattice with zero, 0, in which any two elements having an upper bound have a join. M will be regarded as a partial algebra $\langle M; \wedge, \vee, 0\rangle$.

We shall use the concept of *extension* for chopped lattices; observe that, by definition, an extension of a chopped lattice is a $\{0\}$-extension.

An *ideal* I of M is a subset of M containing 0 with the following two properties for x, $y \in M$:

$x \in I$ and $y \leqq x$ imply that $y \in I$.

If x, $y \in I$ and $x \vee y$ exists, then $x \vee y \in I$.

For $H \subseteq M$, there is a smallest ideal $(H]$ of M containing H. If an ideal I can be represented in the form $(H]$ for some finite set H, then the ideal I is called *finitely generated.* In particular, for $a \in M$, we let $(a] = (\{a\}]$ be the *principal ideal* generated by a in M, that is,

$$(a] = \{x \mid x \in M \text{ and } x \leqq a\}.$$

$\operatorname{Id} M$ denotes the *lattice of ideals* of M. Obviously, $\operatorname{Id} M$ is a lattice. $\operatorname{Id}_{\mathrm{fg}} M$, the *finitely generated ideals* of M, form a join-sublattice of $\operatorname{Id} M$.

By identifying $a \in M$ with $(a]$, we regard $\operatorname{Id} M$ an extension of M.

DEFINITION 1. A chopped lattice M satisfies Condition (FG) if every finitely generated ideal is a finite union of principal ideals.

If M satisfies Condition (FG), then $\mathrm{Id}_{\mathrm{fg}}\, M$ is a sublattice of $\mathrm{Id}\, M$. Indeed, if

$$I = (a_1] \cup \ldots \cup (a_n], \quad J = (b_1] \cup \ldots \cup (b_m],$$

then

$$I \cap J = \bigcup((a_i \wedge b_j] \mid 1 \leqq i \leqq n,\ 1 \leqq j \leqq m).$$

Lemma II.3.19 in [1] states the following:

LEMMA 1. *Let M be a finite chopped lattice. Then* $\mathrm{Id}\, M$ *is a congruence-preserving extension of M.*

The proof of this lemma implicitly contains the following two lemmas.

LEMMA 2. *Let M be a chopped lattice. Then every congruence relation of M has an extension to* $\mathrm{Id}\, M$.

PROOF. Let Θ be a congruence of M; define a relation $\bar{\Theta}$ on $\mathrm{Id}\, M$ as follows:

$$I \equiv J \pmod{\bar{\Theta}}$$

if for every $i \in I$ there exists a $j \in J$ such that $i \equiv j \pmod{\Theta}$, and symmetrically. The proof is the same as in [1]. □

LEMMA 3. *Let M be a chopped lattice, and let $S \supseteqq M$ be a sublattice of* $\mathrm{Id}\, M$. *Let us assume that in S every ideal $I \in S$ is a finite union of principal ideals. Then every congruence relation of M has a unique extension to S.*

PROOF. First observe that if $a \in M$ and $I \in S$, then $(a] \cap I$ is principal. Indeed,

$$I = (a_1] \cup \ldots \cup (a_n],$$

and so $(a] \cap I$ is generated by $\{a \wedge a_1, \ldots, a \wedge a_n\}$. Since this set has an upper bound (namely a), it has a join b (since M is a chopped lattice), and b obviously generates $(a] \cap I$.

Let Φ be an extension of Θ from M to S. Let $I, J \in S$, $I \equiv J \pmod{\Phi}$, and $a \in I$. Then $I \wedge (a] \equiv J \wedge (a] \pmod{\Phi}$. By the statement in the previous paragraph, there is a $b \in J$ such that $(a] \wedge J = (b]$; obviously, $a \equiv b \pmod{\Theta}$. We conclude that $I \equiv J \pmod{\bar{\Theta}}$. So $\Phi \subseteqq \bar{\Theta}$.

Conversely, let $I, J \in S$ with $I \equiv J \pmod{\bar{\Theta}}$. By the assumption on S, we can represent these ideals as

$$I = (a_1] \cup \ldots \cup (a_n], \quad J = (b_1] \cup \ldots \cup (b_m].$$

By the definition of $\bar{\Theta}$, for every a_i there is a c_i in J with $a_i \equiv c_i \pmod{\Theta}$. Symmetrically, for every b_j there is a d_j in I with $d_j \equiv b_j \pmod{\Theta}$. Since Φ is an extension of Θ, these congruences hold for Φ. The join of these $n+m$ congruences yields $I \equiv J \pmod{\Phi}$, proving that $\bar{\Theta} \subseteqq \Phi$. Thus $\bar{\Theta} = \Phi$, and so every congruence of M has a unique extension to S. □

Therefore, the following is true:

LEMMA 4. *Let M be a chopped lattice satisfying Condition* (FG). *Then* $\mathrm{Id_{fg}}\, M$ *is a congruence-preserving extension of M.*

In fact, a congruence-preserving $\{0\}$-extension.

3. Chopped pairs

Let A and B be lattice, let $C = A \cap B \neq \varnothing$. Then we can form the amalgamation $M = M(A, B)$ of A and B over C. It is well-known that on M we can define a partial ordering:

DEFINITION 2. The partial ordering $\leqq_M$ is defined on M as follows:
(1) For $x, y \in A$, let $x \leqq_M y$ iff $x \leqq_A y$.
(2) For $x, y \in B$, let $x \leqq_M y$ iff $x \leqq_B y$.
(3) For $x \in A$ and $y \in B$, let $x \leqq_M y$ iff there exists a $c \in C$ such that $x \leqq_A c$ and $c \leqq_B y$; and symmetrically, for $x \in B$ and $y \in A$.

The subscripts of $\leqq$ will be dropped whenever there is no danger of confusion.

We shall use the following notation: $M(A, B) = A \cup B$ is the *poset* obtained by amalgamating A and B over C. In A we form the ideal I_A generated by C; we set $C_A = I_A - C$; symmetrically, we define I_B and C_B. Note that the ideal C_M generated by C in M is the disjoint union of C, C_A, and C_B.

Sometimes, the poset $M(A, B)$ is a chopped lattice. The next definition formulates some natural conditions under which this is the case.

DEFINITION 3. A pair of lattices A and B is called a *chopped pair* iff the following conditions are satisfied:
(1) The lattices A and B have a common zero, 0.
(2) Let C denote the lattice $A \cap B$. Then C has a largest element i.
(3) For $x \in C_M$, there is a smallest $\overline{x} \in C$ satisfying $x \leqq \overline{x}$.
(4) For $x \in M(A, B)$, there is a largest $\underline{x} \in C$ satisfying $\underline{x} \leqq x$.
(5) For $x \in C_A$ and $y \in C_B$, the two elements: $x \vee \overline{y}$ (formed in A) and $\overline{x} \vee y$ (formed in B) are comparable (in $M(A, B)$).
(6) For $x \in A - B$ and $y \in B - A$, the two elements: $x \wedge \underline{y}$ (formed in A) and $\underline{x} \wedge y$ (formed in B) are comparable (in $M(A, B)$).

THEOREM 1. *Let A, B be a chopped pair. Then $M(A, B)$ is a chopped lattice.*

PROOF. There are two claims to verify.

Claim 1: $M(A, B)$ is a meet-semilattice. Let $x, y \in M(A, B)$. We have to find $u = \inf_{M(A,B)}\{x, y\}$. We shall distinguish several cases.

Case 1.1: $x, y \in A$. Let $u = x \wedge y$ be formed in A. Obviously in $M(A, B)$, $u \leqq x$ and $u \leqq y$. Now let $v \in M(A, B)$ be a common lower bound of x and y. There are two subcases to consider.

Case 1.1a: $v \in A$. By Definition 2.1, v is a common lower bound of x and y in A, hence, $v \leqq u$.

Case 1.1b: $v \in B$. By Definition 2.3, there are elements c_x and c_y in C such that $v \leqq_B c_x \leqq_A x$ and $v \leqq_B c_y \leqq_A y$. Then $c_x \wedge c_y \in C$, and $v \leqq_B c_x \wedge c_y \leqq_A u$. So indeed, $u = \inf_{M(A,B)}\{x, y\}$.

Case 1.2: $x, y \in B$. Proceed as in Case 1.1.

Case 1.3: $x \in A$, $y \in B$. In view of the previous cases, we can assume that $x \in A - B$ and $y \in B - A$. Since by Definition 2.3, any common lower bound must be in C_M, we can replace x by $x \wedge i$ and y by $y \wedge i$. So again referring to the previous cases, we can assume that $x \in C_A$ and $y \in C_B$. Now take a common lower bound v of x and y.

Now we claim that of the common lower bounds $v \in A$, there is a largest one, $x \wedge \underline{y}$. Indeed, $x \wedge \underline{y}$ is a lower bound. If $t \in A$ is also a lower bound, then $t \leqq y$ in $M(A, B)$, hence by Definition 2.3, there is a $c \in C$ satisfying $t \leqq_A c \leqq_B y$. Obviously, $c \leqq \underline{y}$, and so $t \leqq_A x \wedge \underline{y}$, as claimed.

Now we claim that of the common lower bounds $v \in B$, there is a largest one, $\underline{x} \wedge y$. To prove this, proceed as in the previous paragraph.

Finally, by Definition 3.6, $x \wedge \underline{y}$ and $\underline{x} \wedge y$ are comparable, hence $\inf_{M(A,B)}\{x, y\}$ exists and it equals $\sup\{x \wedge \underline{y}, \underline{x} \wedge y\}$.

Case 1.4: $x \in B$, $y \in A$. Proceed as in Case 1.3.

This completes the proof of Claim 1.

Claim 2: In $M(A, B)$, any two elements, x and y, having a common upper bound, v, have a join. Let $x, y \in M(A, B)$, and let v be an upper bound of x and y. We have to find $u = \sup_{M(A,B)}\{x, y\}$. We shall distinguish several cases.

Case 2.1: $x, y \in A$. Form $u = x \vee y$ in A. We have to show that if t is any upper bound of x and y in $M(A, B)$, then $u \leqq t$.

Case 2.1a: $t \in A$. This case is obvious.

Case 2.1b: $t \in B$. By Definition 2.3, there are c_x, $c_y \in C$ so that $x \leqq_A c_x \leqq_B t$ and $y \leqq_A c_y \leqq_B t$. Therefore, $u = x \vee y \leqq_A c_x \vee c_y \leqq_B t$; so again, by Definition 2.3, $u \leqq_{M(A,B)} t$, completing Case 2.1.

Case 2.2: $x, y \in B$. Proceed as in Case 2.1.

Case 2.3: $x \in A$ and $y \in B$. In view of Cases 2.1–2.2, we can assume that $x \in A - B$ and $y \in B - A$. Without loss of generality, we can assume that $t \in A$. It follows that $y \in C_B$. Again, we distinguish two subcases.

Case 2.3a: $x \in C_A$. If $t \in A$ is an upper bound of x and y, then $x \vee \vee \overline{y} \leqq t$. Similarly, if $t \in B$ is an upper bound of x and y, then $\overline{x} \vee y \leqq t$. By Definition 3.5, the elements $x \vee \overline{y}$ and $\overline{x} \vee y$ are comparable, hence,

$$\sup\{x, y\} = \inf\{x \vee \overline{y}, \overline{x} \vee y\}.$$

Case 2.3b: $x \notin C_A$. In this case, no upper bound of x is in B, hence, $\sup\{x, y\} = x \vee \overline{y}$ formed in A.

Case 2.4: $x \in B$ and $y \in A$. Proceed as in Case 2.3.

This completes the proof of Claim 2 and of the lemma. □

4. Some examples and special cases

It is easy to give examples that the last two strange conditions of Definition 3 do not follow from the others. Here is one: let $A = B$ be the direct product of the two element chain $\{0,1\}$ with the three element chain $\{0,a,1\}$. The elements are of the form $\langle x,y\rangle$, where $x \in \{0,1\}$ and $y \in \{0,a,1\}$. We make A and B disjoint (we shall denote $\langle x,y\rangle \in A$ by $\langle x,y\rangle_A$, and the same for B), then we identify elements as follows:
$\langle 0,0\rangle_A$ with $\langle 0,0\rangle_B$;
$\langle 1,0\rangle_A$ with $\langle 0,1\rangle_B$;
$\langle 0,1\rangle_A$ with $\langle 1,0\rangle_B$;
$\langle 1,1\rangle_A$ with $\langle 1,1\rangle_B$.
So $C = \{\langle 0,0\rangle_A, \langle 1,0\rangle_A, \langle 0,1\rangle_A, \langle 1,1\rangle_A\}$ is a four-element Boolean lattice. It is easy to see that Definitions 3.1–3.4 hold, but both Definitions 3.5 and 3.6 fail. Indeed, let $x = \langle a,0\rangle_A \in C_A$ and $y = \langle a,0\rangle_B \in C_B$. Then $\overline{x} = \langle 1,0\rangle_A$ and $\overline{y} = \langle 1,0\rangle_B = \langle 0,1\rangle_B$. Hence,

$$x \vee \overline{y} = \langle a,1\rangle_A \text{ and } \overline{x} \vee y = \langle a,1\rangle_B,$$

and these two elements are not comparable.

If A, B is a chopped pair, then we know that in $M(A,B)$ any pair of elements with a common upper bound has a join. To perform computations we need more; we must have a formula for the join we can work with.

DEFINITION 4. A chopped pair of lattices, A and B, is called *sharp* iff

$$x \vee \overline{y} = \overline{x} \vee y,$$

for $x \in C_A$ and $y \in C_B$, and

$$x \wedge \underline{y} = \underline{x} \wedge y,$$

for $x \in A - B$ and $y \in B - A$.

There are many equivalent forms of these conditions; for instance, the first is equivalent to

$$x \vee \overline{y} \in C,$$

for $x \in C_A$ and $y \in C_B$; or to

$$x \vee y = \overline{x} \vee \overline{y}.$$

Observe that if A and B form a sharp chopped pair, then in $M(A,B)$, we have $x \wedge y \in C$, for $x \in C_A$ and $y \in C_B$; and $x \vee y \in C$, for $x \in C_A$ and $y \in C_B$.

Two important examples of chopped pairs follow in which C is largest and smallest possible:

EXAMPLE 1. $C = (i]$ is a principal ideal of both A and B.

We considered this special case for finite lattices in a previous paper [2]. In this case, $C_A = C_B = \varnothing$; for every $x \in M(A, B)$, $\underline{x} = x \wedge i$; and for every $x \in C = C_M$, $\overline{x} = x$. The conditions of Definition 3 and Definition 4 are trivially satisfied — in fact,

$$x \vee \overline{y} = \overline{x} \vee y = x \vee y \quad \text{and} \quad x \wedge \underline{y} = \underline{x} \wedge y = x \wedge y \wedge i.$$

EXAMPLE 2. $C = \{0, i\}$.

In this case, again, the conditions of Definition 3 are trivially satisfied — in fact,

$$x \vee \overline{y} = \overline{x} \vee y = i \quad \text{and} \quad x \wedge \underline{y} = \underline{x} \wedge y = 0.$$

In these two examples, the conditions of Definition 3 and Definition 4 hold in a much stronger form.

We name the first example:

DEFINITION 5. A chopped pair of lattices, A and B, is called *full* if $C = (i]_A = (i]_B$.

5. Finitely generated ideals

In this section, we shall investigate conditions under which $M(A, B)$ satisfies Condition (FG). The following two lemmas are easy to verify, but they are crucial to our investigations. First some definitions.

DEFINITION 6. Let A, B be a chopped pair, $C = A \cap B$. Let $a \in A - C$ and $b \in B - C$. We define the elements:

$$
\begin{aligned}
a_0 &= a, && \\
b_0 &= b, && \\
b_1 &= b_0 \vee \overline{a_0 \wedge i} && \text{(formed in } B\text{)},\\
a_1 &= a_0 \vee \overline{b_1 \wedge i} && \text{(formed in } A\text{)},\\
b_2 &= b_1 \vee \overline{a_1 \wedge i}\ (= b \vee \overline{a_1 \wedge i}) && \text{(formed in } B\text{)},\\
a_2 &= a_2 \vee \overline{b_2 \wedge i}\ (= a \vee \overline{b_1 \wedge i}) && \text{(formed in } A\text{)},\\
&\dots && \\
b_{n+1} &= b_n \vee \overline{a_n \wedge i}\ (= b \vee \overline{a_n \wedge i}) && \text{(formed in } B\text{)},\\
a_{n+1} &= a_n \vee \overline{b_{n+1} \wedge i}\ (= a \vee \overline{b_{n+1} \wedge i}) && \text{(formed in } A\text{)},\\
&\dots &&
\end{aligned}
$$

See Figure 1 — the white filled elements are in A (and maybe in C); the shaded elements are in B (and maybe in C), and the black filled elements are in C.

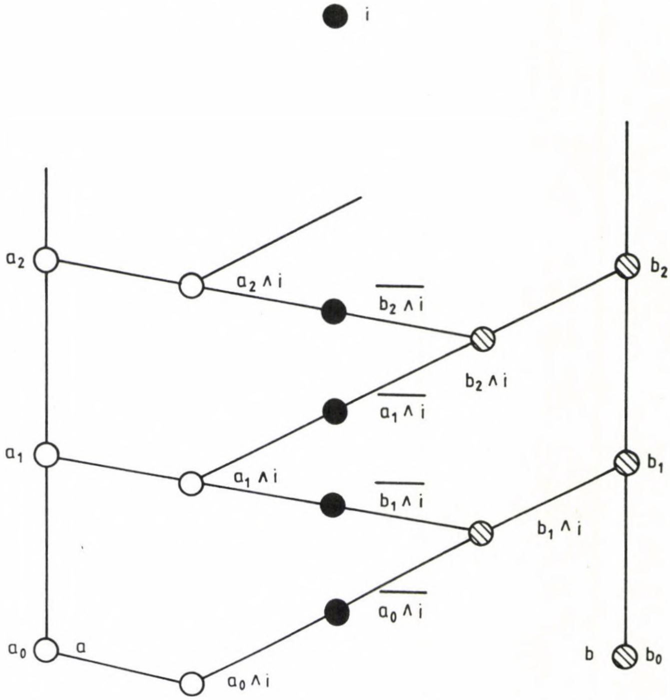

Fig. 1

LEMMA 5. *Let A and B be a sharp chopped pair. Then in $M(A,B)$, the following inequalities hold:*

$$a = a_0 \leqq a_1 \leqq a_2 \leqq \dots \text{ (in } A), \tag{1}$$

$$b = b_0 \leqq b_1 \leqq b_2 \leqq \dots \text{ (in } B), \tag{2}$$

and

$$\overline{a_0 \wedge i} \leqq \overline{b_1 \wedge i} \leqq \overline{a_1 \wedge i} \leqq \overline{b_2 \wedge i} \leqq \overline{a_2 \wedge i} \leqq \dots \text{ (in } C). \tag{3}$$

If, for some n, $a_n = a_{n+1}$, then (1) terminates at n, and (2) terminates at $n+1$; and symmetrically, for (2). If (3) does not terminate, neither do (1) and (2).

So either all three sequences terminate or none terminate.

PROOF. Let $a_n = a_{n+1}$; then $\overline{a_n \wedge i} = \overline{a_{n+1} \wedge i}$. Therefore,

$$b_{n+2} = b \vee \overline{a_{n+1} \wedge i} = b \vee \overline{a_n \wedge i} = b_{n+1};$$

and so $\overline{b_{n+1} \wedge i} = \overline{b_{n+2} \wedge i}$. By the definition of a_{n+1} and a_{n+2}, it follows that $a_{n+1} = a_{n+2}$. Hence, $\overline{a_{n+1} \wedge i} = \overline{a_{n+2} \wedge i}$, so $b_{n+2} = b_{n+3}$. It is now clear that

$$a_n = a_{n+1} = a_{n+2} = \dots,$$

and

$$b_{n+1} = b_{n+2} = \dots .$$

Finally,

$$\overline{a_n \wedge i} \leqq \overline{b_{n+1} \wedge i} \leqq \overline{a_{n+1} \wedge i} \leqq \overline{b_{n+2} \wedge i},$$

$\overline{a_n \wedge i} = \overline{a_{n+1} \wedge i}$ and $\overline{b_{n+1} \wedge i} = \overline{b_{n+2} \wedge i}$; therefore,

$$\overline{a_n \wedge i} = \overline{b_{n+1} \wedge i} = \overline{a_{n+1} \wedge i} = \overline{b_{n+2} \wedge i}, \ldots,$$

so sequence (3) also terminates. Conversely, if sequence (3) terminates, then sequences (1) and (2) terminate by the definitions of a_{n+1} and b_{n+1} in Definition 6. □

LEMMA 6. *Let A and B be a sharp chopped pair; let $a \in A - C$, $b \in \in B - C$. The ideal $(a,b]$ of $M(A,B)$ generated by $\{a,b\}$ can be described as follows:*

$$(a,b] = \bigcup\left((a_n]_A \mid n < \omega\right) \cup \bigcup\left((b_n]_B \mid n < \omega\right).$$

This is not a finitely generated ideal if, and only if, none of the sequences of Lemma 5 terminate. If $(a,b]$ is a finitely generated ideal, then $(a,b] = (a_n] \cup \cup (b_n]$ for some $n < \omega$.

PROOF. Let $R = \bigcup((a_n]_A \mid n < \omega) \cup \bigcup((b_n]_B \mid n < \omega)$. If we know that R is an ideal of $M(A,B)$, then it is straightforward to verify that R is the ideal of $M(A,B)$ generated by $\{a,b\}$, and the rest follows from Lemma 5.

So we verify that R is an ideal of $M(A,B)$.

Firstly, let $x \in R$ and $y \leqq x$ in $M(A,B)$. Without loss of generality we can assume that $x \leqq a_n$ for some n and $y \leqq x$. If $y \in A$, then $y \leqq a_n$; therefore $y \leqq a_n$ in A, and so $y \in R$. If $y \in B$, then $\overline{y} \leqq a_n$, and so $\overline{y} \leqq a_n \wedge \wedge i \leqq b_n$. This implies that $y \leqq b_n$ in B, therefore $y \in R$; completing the proof of $y \in R$.

Secondly, let x, $y \in R$, and let x and y have a common upper bound z in $M(A,B)$. Without loss of generality we can assume that $z \in A$. We want to show that $x \vee y \in R$. We shall distinguish several cases.

Case 1: x, $y \in A$.

Case 1.1: $x \leqq a_n$ and $y \leqq a_m$ for some n and m. In this case, as in all the subsequent cases, we can assume without loss of generality that $n = m$. Then $x \vee y \leqq a_n$, so $x \vee y \in R$.

Case 1.2: $x \leqq a_n$ and $y \leqq b_n$. Since $y \in A$ and $b_n \in B$, the condition $y \leqq b_n$ implies that $y \leqq i$. Hence, $y \leqq b_n \wedge i \leqq a_n$, and so $x \vee y \leqq a_n$, yielding $x \vee y \in R$.

Case 1.3: $x \leqq b_n$ and $y \leqq a_n$. Proceed as in Case 1.2.

Case 1.4: $x \leqq b_n$ and $y \leqq b_n$. As in Case 1.2, we can verify that $x \leqq a_n$ and $y \leqq a_n$, so Case 1.1 completes this case.

Case 2: $x \in A$, $y \in B$. Observe that $y \leqq i$ since $y \leqq z$, $y \in B$ and $z \in A$.

Case 2.1: $x \leqq a_n$ and $y \leqq a_n$. So $x \vee y = x \vee \overline{y} \leqq a_n$, hence $x \vee y \in R$.

Case 2.2: $x \leqq a_n$ and $y \leqq b_n$. Since $y \leqq i$, it follows that $y \leqq b_n \wedge i$, so $y \leqq a_n$; hence $x \vee y \leqq a_n$, yielding $x \vee y \in R$.

Case 2.3: $x \leqq b_n$ and $y \leqq a_n$. Proceed as in Case 2.2.

Case 2.4: $x \leqq b_n$ and $y \leqq b_n$. Then as in Case 2.2, $x \leqq a_n$ and $y \leqq a_n$, so we can proceed as in Case 1.

Case 3: $x \in B$, $y \in A$. This is symmetric to Case 2.

Case 4: $x, y \in B$.

Case 4.1: $x \leqq a_n$ and $y \leqq a_n$. Using the argument of Case 2.2, we obtain that $x \leqq b_{n+1}$ and $y \leqq b_{n+1}$, which is symmetric to Case 1.1. Hence $x \vee y \in R$.

Case 4.2: $x \leqq a_n$ and $y \leqq b_n$. Again, $x \in B$ and $x \leqq a_n$ imply that $x \leqq b_{n+1}$, which is symmetric to Case 1.1.

Case 4.3: $x \leqq b_n$ and $y \leqq a_n$. Proceed as in Case 4.2.

Case 4.4: $x \leqq b_n$ and $y \leqq b_n$. This is symmetric to Case 1.1. □

Observe that this lemma fully describes all finitely generated ideals, since a finitely generated ideal of $M(A, B)$ is obviously one- or two-generated.

Now we prove:

THEOREM 2. *Let A and B form a sharp chopped pair, and let $C = A \cap \cap B$. Let us assume that C satisfies the Ascending Chain Condition. Then $M(A, B)$ satisfies condition* (FG), *and* $\mathrm{Id_{fg}}\, M(A, B)$ *is a congruence-preserving extension of $M(A, B)$ (in fact, a congruence-preserving $\{0\}$-extension).*

PROOF. If C satisfies the Ascending Chain Condition, then sequence (3) of Lemma 5 must terminate. By Lemma 5, the sequences (1) and (2) terminate, and so the statement of the Theorem follows from Lemma 6.

Finally, the statement concerning congruence-preserving extension follows from Lemma 4. □

For full chopped pairs, Definition 6, Lemma 5, and Lemma 6 take on a much simpler form:

DEFINITION 7. Let A, B be a full chopped pair, $C = A \cap B$. Let $a \in \in A - C$ and $b \in B - C$. Then we define the elements:

$$
\begin{aligned}
a_0 &= a,\\
b_0 &= b,\\
b_1 &= b_0 \vee (a_0 \wedge i),\\
a_1 &= a_0 \vee (b_1 \wedge i),\\
b_2 &= b_1 \vee (a_1 \wedge i)\ (= b \vee (a_1 \wedge i)),\\
a_2 &= a_1 \vee (b_2 \wedge i)\ (= a \vee (b_1 \wedge i)),\\
&\dots\\
b_{n+1} &= b_n \vee (a_n \wedge i)\ (= b \vee (a_n \wedge i)),\\
a_{n+1} &= a_n \vee (b_{n+1} \wedge i)\ (= a \vee (b_{n+1} \wedge i)),\\
&\dots
\end{aligned}
$$

See Figure 2 — the white filled elements are in A (and maybe in C); the shaded elements are in B (and maybe in C), and the black filled elements are in C.

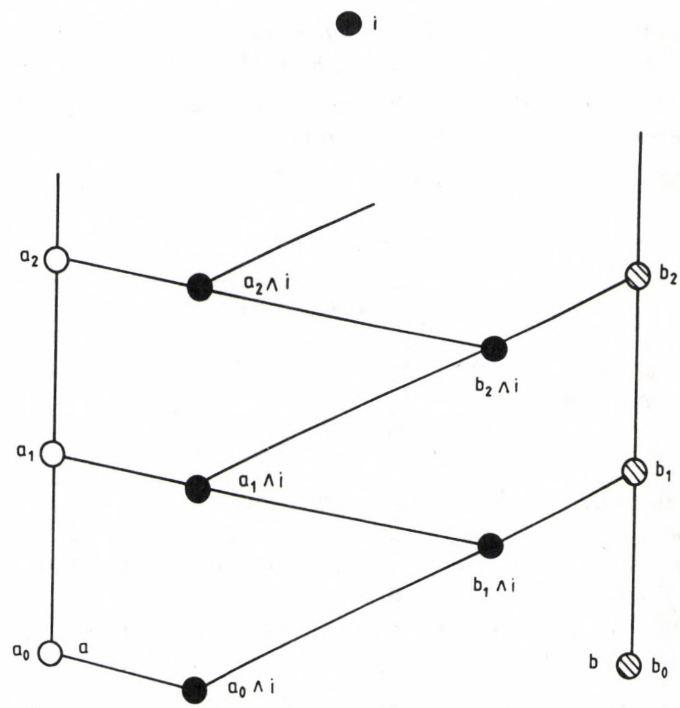

Fig. 2

LEMMA 7. *Let A and B be a full chopped pair. Then in $M(A, B)$, the following inequalities hold:*

$$a = a_0 \leqq a_1 \leqq a_2 \leqq \dots \text{ (in } A), \tag{4}$$

$$b = b_0 \leqq b_1 \leqq b_2 \leqq \dots \text{ (in } B), \tag{5}$$

and

$$a_0 \wedge i \leqq b_1 \wedge i \leqq a_1 \wedge i \leqq b_2 \wedge i \leqq a_2 \wedge i \leqq \dots \text{ (in } C). \tag{6}$$

If, for some n, $a_n = a_{n+1}$, then (4) terminates at n, and (5) terminates at $n+1$; and symmetrically, for (5). If (6) does not terminate, neither do (4) and (5).

The proof of this lemma is a simplified version of the proof of Lemma 5. Lemma 6 remains valid for full chopped pairs; in this case, the sequences a_n and b_n will be the ones defined in Definition 7.

6. Modular lattices

By inspecting Figure 1, we can see that if A and B are modular, then a lot of elements must collapse. In fact, we have the following result:

THEOREM 3. *Let A and B form a sharp chopped pair. Let us assume that both A and B are modular. Then $M(A,B)$ satisfies condition* (FG), *and* $\mathrm{Id}_{\mathrm{fg}}\, M(A,B)$ *is a congruence-preserving extension of $M(A,B)$ (in fact, a congruence-preserving $\{0\}$-extension).*

PROOF. Let A and B be modular. The equations (see Figure 1)

$$a_0 \wedge \overline{b_1 \wedge i} = a_0 \wedge (a_1 \wedge i) = a_0 \wedge i,$$

$$a_0 \vee \overline{b_1 \wedge i} = a_0 \vee (a_1 \wedge i) = a_1$$

hold in $M(A,B)$. By the modularity of A, the two equations imply that $\overline{b_1 \wedge i} = a_1 \wedge i$. So

$$\overline{a_1 \wedge i} = \overline{\overline{b_1 \wedge i}} = \overline{b_1 \wedge i}.$$

By the modularity of B, a similar argument yields that $\overline{b_2 \wedge i} = \overline{a_1 \wedge i}$, and so on. So the sequence (3) has only one or two members; it terminates. By Lemma 5, the sequences (1) and (2) terminate. So the statement of the Theorem follows from Lemma 6.

Finally, the statement concerning congruence-preserving extension follows from Lemma 4. □

We can prove a stronger statement for full chopped pairs.

LEMMA 8. *Let A, B be a full chopped pair. If A is a modular lattice, then*

$$(a, b] = (a_1] \cup (b_1].$$

PROOF. As in Theorem 3, the modularity of A implies that $b_1 \wedge i = a_1 \wedge i$. Hence $b_2 = b_1 \vee (a_1 \wedge i) = b_1 \vee (b_1 \wedge i) = b_1$, and $a_2 = a_1 \vee (b_2 \wedge i) = a_1 \vee$ $\vee (b_1 \wedge i) = a_1 \vee (a_1 \wedge i) = a_1$. So the statement of the Lemma follows from Lemma 6. □

So now we can conclude a stronger form of Theorem 3 for full chopped pairs:

THEOREM 4. *Let A, B be a full chopped pair. If A is a modular lattice, then $M(A,B)$ satisfies condition* (FG).

7. Congruence-preserving extensions

In [2] we raised the following question:

PROBLEM. Is it true that every lattice with more than one element has a proper congruence-preserving extension?

We proved in [2] that in the finite case this is true. This result is generalized by the following theorem:

THEOREM 5. *Let L be a lattice with zero, 0. If there exists an element $\alpha > 0$ in L such that the interval $[0, \alpha]$ is distributive, then L has a proper congruence-preserving extension K.*

PROOF. To prove this result, we need a construction due to the second author. Let M_3 denote the five-element modular nondistributive lattice on the set $\{0, a, b, c, 1\}$, and let D be a bounded distributive lattice. Let

$$M_3[D] = \{\langle x, y, z\rangle \in D^3 \mid x \wedge y = x \wedge z = y \wedge z\}.$$

Then $M_3[D]$ is a modular lattice; it contains M_3 as a $\{0,1\}$-sublattice (on the set $\{\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle, \langle 1,1,1\rangle\}$), and each prime interval of this M_3 contains (in $M_3[D]$) a copy of D; for instance, the interval $[\langle 0,0,0\rangle, \langle 1,0,0\rangle]$ can be described as $\{\langle d,0,0\rangle \mid d \in D\}$. If we identify D with $\{\langle d,0,0\rangle \mid d \in D\}$, we find that the lattice $M_3[D]$ is a congruence-preserving $\{0\}$-extension of D.

Now let $D = [0, \alpha]$, and let $A = M_3[D]$. Then A has a spanning M_3; let $i = \langle a, 0, 0\rangle$. Let $B = L$, and define $i = \alpha$ in B. Then $A \cap B = (i]$, and A, B form a full chopped pair in which A is modular. So we can form the chopped lattice $M(A, B)$. Obviously, $M(A, B)$ is a proper congruence-preserving $\{0\}$-extension of L. By Theorem 4, (FG) holds for $M(A, B)$. Therefore, by Lemma 4, $\mathrm{Id}_{\mathrm{fg}}\, M(A, B)$ is a congruence-preserving $\{0\}$-extension of $M(A, B)$. We conclude that $\mathrm{Id}_{\mathrm{fg}}\, M(A, B)$ is a proper congruence-preserving $\{0\}$-extension of L. □

The following result is a generalization of Theorem 5.

THEOREM 6. *Let L be a lattice. If there exist a nontrivial distributive interval in L, then L has a proper congruence-preserving extension K.*

PROOF. Let $[\alpha, \beta]$ be a nontrivial distributive interval in L. Let us form the lattice $B = [\alpha)$ in L. Obviously, B satisfies the conditions of Theorem 5; therefore, B has a congruence-preserving $\{0\}$-extension K_1. Clearly, B is an ideal of K_1 and a dual ideal of L; hence we can glue L and K_1 over B; let K be the resulting lattice.

Let Θ be a congruence relation on L. Let Θ_B be the restriction of Θ to B. Since K_1 is a congruence-preserving extension of B, there is a unique extension Φ of Θ_B to K_1. It is easy to see that $\overline{\Theta} = \Theta \cup \Phi$ is the unique extension of Θ to K. Hence K is a congruence-preserving extension of L. Obviously, it is a proper extension. □

8. A modular example

It is easy to give examples of classes of lattices that have proper congruence-preserving extensions that have nothing to do with distributivity.

For instance, every simple lattice with more than one element has a proper simple extension; this is obviously a proper congruence-preserving extension.

In this section we outline a modular example with no proper distributive sublattice.

Let C be a continuous geometry with zero, 0, and unit, 1. Then C has the following properties:

(1) For $a < b$, the interval $[a, b]$ is isomorphic to C.

(2) C is a simple lattice.

Let I be a nonprincipal ideal of C and F a nonprincipal dual ideal of C satisfying $I \cap F = \varnothing$. Let L be the sublattice $I \cup F$. The congruence lattice of L is the three element chain.

We choose in C a spanning $M_3 = \{0 < a,\ b,\ c < 1\}$. The interval $[0, a]$ is isomorphic to C. Therefore, we find in $[0, a]$ a copy I_a of I and a copy F_a of F. The projectivities in the spanning M_3 define the ideals and dual ideals, I_b, I_c, F_b, F_c in the intervals $[0, b]$ and $[0, c]$. Similarly, we obtain the ideal I_a^u and dual ideal F_a^u in $[a, 1]$, I_b^u and F_b^u in $[b, 1]$, I_c^u and F_c^u in $[c, 1]$.

Let I be the ideal of C generated by the three "small" ideals, I_a, I_b, I_c. Similarly, the three dual ideals F_a^u, F_b^u,F_c^u generate a dual ideal F. We consider the sublattice

$$K = I \cup F \cup F_a \cup F_b \cup F_c \cup I_a^u \cup I_b^u \cup I_c^u.$$

It is easy to see that K is a sublattice of C, and it is a congruence-preserving extension of the sublattice $L \subsetneqq [0, a]$.

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BAIRE 1 FUNCTIONS WHICH ARE NOT COUNTABLE UNIONS OF CONTINUOUS FUNCTIONS

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1. Introduction

A real-valued function f on a space X is *countably continuous* provided that X can be partitioned into countably many sets $E_1, E_2, \dots$ such that for every i, the restriction $f \restriction E_i$ is continuous. Adjan and Novikov [1] constructed (answering a question of Lusin, cf. also Keldyš [6]) an upper semicontinuous function on $[0,1]$ that is not countably continuous (we discuss their construction in Lemma 4.1 and Comment 6.1(B) below). A similar construction was used also by Sierpiński [10] (who did not address Lusin's question directly, but the solution is implicit in his reasoning). We thank the referee for pointing out this fact to us.

Jackson and Mauldin [5] proved recently, using some notions from recursion theory, that Lebesgue measure λ considered on the space of nonempty closed subsets of the unit interval is not countably continuous (being upper semicontinuous). They conjectured [5, Questions 5 and 6] that in the Banach spaces of bounded Baire 1 functions and of bounded derivatives, respectively, the countably continuous functions form meager sets.

In this note we prove these conjectures. We also establish a universal property of the map λ on the space of nonempty closed subsets of the unit interval, which gives in particular a direct proof of the result of Jackson and Mauldin mentioned above.

2. Terminology

As usual, $\mathbf{I}$ denotes the interval $[0,1]$ and Q the infinite product $\mathbf{I}^\infty$. By a *space* we mean a metrizable topological space. If $X = \prod_{n=1}^{\infty} X_n$ is an infinite

[1] This note was partly written during the second author's visit to Vrije Universiteit (Amsterdam). He would like to thank the Department of Mathematics of this university for its hospitality.

product of spaces then for every $x \in X$ and $n \in \mathbf{N}$ the n-th coordinate of x is denoted by x_n.

Let X be a compact space. The collection of all nonempty closed subsets of X is denoted by $\mathcal{K}(X)$. It can be topologized as follows. Let d be an arbitrary admissible metric for X. If $A \subseteqq X$ and $\varepsilon > 0$ then $U_\varepsilon(A)$ denotes the open ε-ball of radius ε about A. The formula

$$d_H(A,B) = \inf\left\{\varepsilon : A \subseteqq U_\varepsilon(B) \text{ and } B \subseteqq U_\varepsilon(A)\right\}$$

defines a metric on $\mathcal{K}(X)$, the so-called *Hausdorff metric*, and $\mathcal{K}(X)$ endowed with the topology derived from this metric is called the ***hyperspace of*** X. One can show that the topology of $\mathcal{K}(X)$ is independent of the choice of the admissible metric d. Also, $\mathcal{K}(X)$ is a compact space. For details, see Engelking [4] and [9, §4.7].

Let X and (Y,d) be spaces. For functions $f,g: X \to Y$ we define their *distance* $\hat{d}(f,g) \in [0,\infty]$ as follows:

$$\hat{d}(f,g) = \sup\left\{d\big(f(x),g(x)\big) : x \in X\right\}.$$

Let X be a space. A function $f: X \to \mathbf{R}$ is called *lower (upper) semicontinuous* if for every $r \in \mathbf{R}$ the set $f^{-1}(r,\infty)$ (the set $f^{-1}(-\infty,r)$) is open. It is clear that a function $f: X \to \mathbf{R}$ is continuous if and only if it is both lower and upper semicontinuous. We will use the well-known fact that for every lower (upper) semicontinuous function f on X there exists a sequence $\{f_i\}_i$ of continuous real-valued functions on X such that for every $x \in X$ we have $f_i(x) \nearrow f(x)$ ($f_i(x) \searrow f(x)$). We will also use the fact that the functions $\inf: Q \to \mathbf{I}$ and $\sup: Q \to \mathbf{I}$ defined by

$$\inf(x) = \inf\{x_n : n \in \mathbf{N}\}$$

and

$$\sup(x) = \sup\{x_n : n \in \mathbf{N}\}$$

are upper semicontinuous and lower semicontinuous, respectively. For details and references concerning these facts, see Engelking [4, pp. 61–62].

We finish this section by establishing the following easy results which are probably well-known.

2.1. THEOREM. *Let $r \in [0,1)$. In addition, let X be a compact space and let $f: X \to [0,r]$ be upper semicontinuous. Then there is an embedding $e: X \to Q$ such that for each $x \in X$ we have*

$$\inf\big(e(x)\big) = f(x).$$

PROOF. Write $\mathbf{N}$ as the union of two disjoint infinite sets, say E_1 and E_2. Since Q is universal for separable metrizable spaces ([9, Theorem 1.4.18]), there is an embedding $\xi: X \to [r,1]^{E_1}$. Since f is upper semicontinuous there is a sequence $\{f_i\}_{i\in E_2}$ of continuous functions from X to $[0,r]$ such that for every $x \in X$ we have $\{f_i(x)\}_{i\in E_2} \searrow f(x)$. Now define $e: X \to Q$ by

$$e(x)_i = \begin{cases} \xi(x)_i & (i \in E_1), \\ f_i(x) & (i \in E_2). \end{cases}$$

Then e is clearly as required. □

We conclude that in a sense the pair $(Q, \inf)$ is "universal" for upper semicontinuous functions. Similarly one derives that the pair $(Q, \sup)$ is "universal" for lower semicontinuous functions.

2.2. THEOREM. *Let $r \in (0,1]$. In addition, let X be a compact space and let $f: X \to [r,1]$ be lower semicontinuous. Then there is an embedding $e: X \to Q$ such that for each $x \in X$ we have*

$$\sup\big(e(x)\big) = f(x).$$

3. A universal property of Lebesgue measure

In this section we formulate and prove that the pair $\big(\mathcal{K}([-1,1]), \lambda\big)$ is "universal" for upper semicontinuous functions. In §6.1 we will present several "explicit" examples of upper semicontinuous functions that are not countably continuous. In view of Theorem 3.1 below this implies that λ is not countably continuous.

3.1. THEOREM. *Let X be a compact space and let $f: X \to \mathbf{I}$ be upper semicontinuous. Then there is a topological embedding $e: X \to \mathcal{K}([-1,1])$ such that for every $x \in X$ we have*

$$\lambda\big(e(x)\big) = f(x).$$

PROOF. We will construct a function $\alpha: X \to \mathcal{K}([-1,0])$ and a function $\beta: X \to \mathcal{K}([0,1])$. The desired embedding e will then be defined by the formula $e(x) = \alpha(x) \cup \beta(x)$ $(x \in X)$.

Claim 1. There is an embedding $\alpha: X \to \mathcal{K}([-1,0])$ such that for every $x \in X$ we have $\lambda\big(\alpha(x)\big) = 0$.

This is easy. Pick points a_n and b_n in $[-1,0]$ such that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \cdots \nearrow 0.$$

Let $\hat{Q} = \prod_{n=1}^{\infty}[a_n, b_n]$. Define an embedding $\varphi: \hat{Q} \to \mathcal{K}([-1,0])$ by $\varphi(x) = \{0\} \cup \{x_n: n \in \mathbf{N}\}$. Clearly, $\lambda(\varphi(x)) = 0$ for every $x \in \hat{Q}$. The desired result now easily follows because $\hat{Q} \approx Q$ is universal for separable metrizable spaces ([9, Theorem 1.4.18]).

We now come to the interesting part of the proof.

Claim 2. There is a continuous function $\beta: X \to \mathcal{K}(\mathbf{I})$ such that for every $x \in X$ we have $\lambda(\beta(x)) = f(x)$.

Since f is upper semicontinuous we may pick a sequence $\{f_i\}_i$ of continuous functions from X to $\mathbf{I}$ such that for every $x \in X$, $f_i(x) \searrow f(x)$. Define $\xi_1: X \to \mathcal{K}(\mathbf{I})$ by $\xi_1(x) = [0, f_1(x)]$. Then ξ_1 is clearly a continuous function and has the property that $\lambda(\xi_1(x)) = f_1(x)$ for every $x \in X$. Define $\xi_2: X \to \mathcal{K}(\mathbf{I})$ as follows:

$$\xi_2(x) = \left[0, \frac{1}{2}f_2(x)\right] \cup \left[\frac{1}{2}f_1(x), \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x)\right].$$

Then ξ_2 is clearly a continuous function. Observe the following:

(1) If $x \in X$ then the intervals $\left[0, \frac{1}{2}f_2(x)\right]$ and $\left[\frac{1}{2}f_1(x), \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x)\right]$ overlap in at most one point because $f_2(x) \leqq f_1(x)$, so that

$$\lambda(\xi_2(x)) = \frac{1}{2}f_2(x) + \frac{1}{2}f_2(x) = f_2(x).$$

(2) If $x \in X$ then $\xi_2(x) \subseteqq \xi_1(x)$. (Again because $f_2(x) \leqq f_1(x)$.)

(3) If $x \in X$ then

$$d_H(\xi_1(x), \xi_2(x)) = \frac{1}{2}(f_1(x) - f_2(x)) \leqq \frac{1}{2}.$$

(Here d is the euclidean metric on $\mathbf{I}$.)

We now continue in the obvious way and obtain a sequence of continuous functions $\xi_n: X \to \mathcal{K}(\mathbf{I})$ having the following properties:

(1) For every $x \in X$, $\xi_1(x) \supseteqq \xi_2(x) \supseteqq \cdots$.

(2) For every $x \in X$ and $n \in \mathbf{N}$, $\lambda(\xi_n(x)) = f_n(x)$.

(3) For every $n \in \mathbf{N}$, $\hat{d}_H(\xi_n, \xi_{n+1}) \leqq 2^{-n}$.

We conclude that the sequence $(\xi_n)_n$ is Cauchy and that the formula

$$\beta(x) = \lim_{n\to\infty} \xi_n(x) = \bigcap_{n=1}^{\infty} \xi_n(x)$$

defines a continuous function from X to $\mathcal{K}(\mathbf{I})$. Also,

$$\lambda(\beta(x)) = \inf\{\lambda(\xi_n(x)): n \in \mathbf{N}\} = f(x)$$

for every $x \in X$. This completes the construction of β.

As announced, we now define $e: X \to \mathcal{K}([-1,1])$ by

$$e(x) = \alpha(x) \cup \beta(x) \qquad (x \in X).$$

Then e is clearly as required. $\square$

4. Typical bounded Baire 1 functions are not countably continuous

Before we explicitly formulate and prove the result indicated in the title of this section, we shall introduce some terminology which will allow us to apply the original idea of Sierpiński, Adjan and Novikov in the more general situation that we are dealing with.

Let $k \in \mathbf{N}$. $\Sigma(k)$ denotes the collection of all strings $\sigma = (i_1, \dots, i_p)$, where every i_j is a natural number $\leqq k$ and $p \leqq k$; the *length* of σ is p and the empty string which has length 0 is denoted by $\emptyset$. For convenience, put $\Sigma = \bigcup_{k=1}^{\infty} \Sigma(k)$. If $\sigma = (i_1, \dots, i_p) \in \Sigma$ and $i \in \mathbf{N}$ then $\sigma^\frown i$ denotes the string $(i_1, \dots, i_p, i)$.

Let X be a space. Given a compact set $C \subseteqq X$, we fix a countable basis $B_1(C), B_2(C), \dots$ for the open sets in C with $\lim_{i\to\infty} \operatorname{diam} B_i(C) = 0$.

Let $k \in \mathbf{N}$. A *k-system* $\mathcal{S}(k)$ in X consists of:

(1) a collection of Cantor subsets $\{C(\sigma): \sigma \in \Sigma(k)\}$ of X,

(2) a collection of Cantor subsets $\{D(\sigma): \sigma \in \Sigma(k)\}$ of X,

(3) a sequence $\{\varepsilon(\sigma): \sigma \in \Sigma(k)\}$ of positive numbers,

such that the following conditions are satisfied:

(i) $C(\emptyset) = D(\emptyset)$;

(ii) $\forall \sigma, \sigma^\frown i, \sigma^\frown j \in \Sigma(k)$:

(a) $C(\sigma^\frown i) \subseteqq D(\sigma^\frown i) \subseteqq B_i(C(\sigma))$;

(b) $C(\sigma^\frown i)$ has empty interior and $D(\sigma^\frown i)$ is clopen relative to $C(\sigma)$;

(c) if $i \neq j$ then $C(\sigma^\frown i) \cap C(\sigma^\frown j) = \emptyset$.

We say that a $(k+1)$-system $\mathcal{S}(k+1)$ *extends* a k-system $\mathcal{S}(k)$ if the objects in $\mathcal{S}(k+1)$ associated with the strings in $\Sigma(k)$ coincide with the corresponding objects in $\mathcal{S}(k)$.

We say that a function $f: X \to \mathbf{R}$ is *compatible* with a k-system $\mathcal{S}(k)$ in X if for any string $\sigma \in \Sigma(k)$,

$$(*) \qquad \sup\{f(x): x \in D(\sigma) \setminus C(\sigma)\} + \varepsilon(\sigma) < \inf\{f(x): x \in C(\sigma)\}.$$

For such an f we put

$$\eta(f) = \min\Big\{\inf\{f(x): x \in C(\sigma)\} - \varepsilon(\sigma) -$$

$$- \sup \{ f(x): x \in D(\sigma) \setminus C(\sigma) \} : \sigma \in \Sigma(k) \Big\}.$$

We call a function $f: X \to \mathbf{R}$ of *Sierpiński–Adjan–Novikov type*, if there exists a sequence of k-systems $\mathcal{S}(1), \mathcal{S}(2), \ldots, \mathcal{S}(k), \ldots$ such that for all $k \in \mathbf{N}$,

(1) $\mathcal{S}(k+1)$ extends $\mathcal{S}(k)$;

(2) f is compatible with $\mathcal{S}(k)$.

Observe that if $Y \subseteq X$ and $f: X \to \mathbf{R}$ has the property that $f \restriction Y: Y \to \mathbf{R}$ is of Sierpiński–Adjan–Novikov type then so is f.

The following lemma is implicit in Adjan and Novikov [1]. Since their paper is in Russian we include a proof for the convenience of the reader.

4.1. Lemma. *If $f: X \to \mathbf{R}$ is of Sierpiński–Adjan–Novikov type then it is not countably continuous.*

Proof. Let us fix a sequence $\mathcal{S}(1), \mathcal{S}(2), \ldots, \mathcal{S}(k), \ldots$ of k-systems compatible with f such that for every k the system $\mathcal{S}(k+1)$ extends $\mathcal{S}(k)$. Define

$$E = \bigcap_{p=1}^{\infty} \bigcup \{ C(\sigma): \sigma \text{ has length } p \}.$$

Write E as $E_1 \cup E_2 \cup \cdots$. We claim that for some $p \in \mathbf{N}$ and $\sigma \in \Sigma$ the set

$$(**) \qquad E_p \cap C(\sigma) \text{ is dense in } C(\sigma).$$

Otherwise (using (ii)(a)) we could choose inductively numbers $i_1, i_2, \ldots$ such that for every $p \in \mathbf{N}$, $E_p \cap C(i_1, \ldots, i_p) = \emptyset$. But then the non-empty set $\bigcap_{p=1}^{\infty} C(i_1, \ldots, i_p)$ is contained in $E \setminus \bigcup_{i=1}^{\infty} E_i$, which is a contradiction.

With p and σ as in $(**)$, choose any $x_0 \in E_p \cap C(\sigma)$. By the definition of E there exists $i \in \mathbf{N}$ with $x_0 \in C(\sigma^\frown i)$. By (ii)(b) and $(*)$ we can find a sequence $x_n \in \big(E_p \cap D(\sigma^\frown i) \big) \setminus C(\sigma^\frown i)$ converging to x_0. But then

$$f(x_0) > f(x_n) + \varepsilon(\sigma^\frown i)$$

for all n, demonstrating that $f \restriction E_p$ is not continuous at x_0. □

4.2. Remark. An inspection of the proof of Lemma 4.1 shows that condition $(*)$ above is much more than we need. It suffices for example if for every $\sigma \in \Sigma(k)$, $k \in \mathbf{N}$, there is a relatively open set $G(\sigma) \subseteq D(\sigma) \setminus C(\sigma)$ such that $C(\sigma) \subseteq \overline{G(\sigma)}$ while moreover

$$(*') \qquad \sup \{ f(x): x \in G(\sigma) \} + \varepsilon(\sigma) < \inf \{ f(x): x \in C(\sigma) \}.$$

By abuse of terminology we call functions satisfying such conditions also of Sierpiński–Adjan–Novikov type. The point is that the precise condition is not so important, as long as it is strong enough for the arguments in the proof of Lemma 4.1 to work. For the time being the definition of Sierpiński–Adjan–Novikov type is the one with the above condition $(*)$. We will warn the reader when it is time for a change.

A function $f\colon \mathbf{I} \to \mathbf{R}$ is of *first Baire class* if it is the pointwise limit of a sequence of continuous functions. The set $B_1(\mathbf{I})$ consists of all *bounded* functions of the first Baire class and is endowed with the supremum norm. It is well-known that with this norm, $B_1(\mathbf{I})$ is a (non-separable) Banach space.

4.3. THEOREM. *$B_1(\mathbf{I})$ contains a dense G_δ-subset consisting of functions of Sierpiński–Adjan–Novikov type.*

Consequently, by Lemma 4.1 we obtain the following corollary.

4.4. COROLLARY. *The set of all countably continuous functions in $B_1(\mathbf{I})$ is meager.*

Before presenting the proof of Theorem 4.3 we derive the following preliminary results.

4.5. LEMMA. *Let $\mathcal{S}(k)$ be a k-system. Then the set*

$$\{ f \in B_1(\mathbf{I}) \colon f \text{ is compatible with } \mathcal{S}(k) \}$$

is open in $B_1(\mathbf{I})$.

PROOF. Let $\mathcal{S}(k) = \langle C(\sigma), D(\sigma), \varepsilon(\sigma)\rangle_{\sigma\in\Sigma(k)}$. In addition, let f be compatible with $\mathcal{S}(k)$. It is easy to verify that if $g \in B_1(\mathbf{I})$ and $\|f-g\| < \eta(f)/3$ then g is compatible with $\mathcal{S}(k)$. □

4.6. LEMMA. *Let $K \subseteqq \mathbf{I}$ be a Cantor set, $u \in B_1(\mathbf{I})$ and $C_1 \subseteqq K$ a Cantor set with empty interior in K. Then if U is a nonempty open subset of K and $\delta > 0$ then there are a Cantor set $C \subseteqq U \setminus C_1$ having empty interior in K, a clopen neigborhood D of C in K and a nonempty open subset $W \subseteqq \subseteqq \{ v \in B_1(\mathbf{I}) \colon \|u-v\| < \delta \}$ such that for all $w \in W$:*

$$\sup\{ w(x) \colon x \in D \setminus C\} + \frac{\delta}{5} < \inf\{ w(x) \colon x \in C\}.$$

PROOF. Since u is of the first Baire class, there is a point $p \in V = U \setminus C_1$ at which $u \restriction K$ is continuous ([2, Theorem 8.3.1]). Let $D \subseteqq V$ be a clopen neighborhood of p in K such that

$$|u(x) - u(p)| < \frac{\delta}{5} \qquad (x \in D).$$

Let $C \subseteqq D$ be a Cantor set containing p and having empty interior in K. Define $v \in B_1(\mathbf{I})$ as follows:

$$v(x) = \begin{cases} u(x) & (x \notin C), \\ u(p) + \frac{3}{5}\delta & (x \in C). \end{cases}$$

Clearly $v \in B_1(\mathbf{I})$ and $\|u - v\| < \delta$. Also,

$$\inf\{v(x): x \in C\} - \sup\{v(x): x \in D \setminus C\} =$$

$$= u(p) + \frac{3}{5}\delta - \sup\{v(x): x \in D \setminus C\} \geqq \frac{2}{5}\delta.$$

So if W is a sufficiently small neighborhood of v then for every $w \in W$, $\inf\{w(x): x \in C\} - \sup\{w(x): x \in D \setminus C\} > \frac{\delta}{5}$. □

By a repeated application of Lemma 4.6 one obtains:

4.7. COROLLARY. *Let $f \in B_1(\mathbf{I})$ be compatible with the k-system $\mathcal{S}(k)$. Then for any $\alpha > 0$ one can extend $\mathcal{S}(k)$ to a $k+1$-system $\mathcal{S}(k+1)$ and one can find a function $g \in B_1(\mathbf{I})$ in the α-ball about f such that g is compatible with $\mathcal{S}(k+1)$.*

We are now in a position to present the proof of Theorem 4.3.

4.8. PROOF OF THEOREM 4.3. Let $\mathcal{U}_1$ be a family consisting of pairwise disjoint nonempty open subsets of $B_1(\mathbf{I})$ such that
(1) $\forall\, U \in \mathcal{U}_1$: $\operatorname{diam}(U) < 2^{-1}$,
(2) $\bigcup \mathcal{U}_1$ is dense in $B_1(\mathbf{I})$.
For every $U \in \mathcal{U}_1$ pick an arbitrary element $f_U^1 \in U$. Then every f_U^1 is compatible with the 0-system. So by applying Corollary 4.7 we find for every $U \in \mathcal{U}_1$ a 1-system $\mathcal{S}_U^1$ and a function $g_U^1 \in U$ compatible with $\mathcal{S}_U^1$. By Lemma 4.5, for every $U \in \mathcal{U}_1$ we may pick an open neighborhood $V_U \subseteqq U$ of g_U^1 such that every function in V_U is compatible with $\mathcal{S}_U^1$. Without loss of generality we may assume that every V_U has diameter less than 2^{-2}. For every $U \in \mathcal{U}$ enlarge $\{V_U\}$ to a pairwise disjoint family $\mathcal{V}_U$ consisting of nonempty open subsets of U of diameter less than 2^{-2} and dense union. Let $\mathcal{U}_2$ denote the collection $\bigcup_{U \in \mathcal{U}_1} \mathcal{V}_U$. Observe that there are two types of sets in $\mathcal{U}_2$. Now we repeat the same procedure. The sets in $\mathcal{U}_2$ that are "compatible" with a 1-system are being replaced by smaller sets that are "compatible" with a 2-system that extends the 1-system. Next, the sets that are "compatible" with the 0-system are being replaced by smaller sets that are "compatible" with a 1-system. Finally, we add sets that are compatible with the 0-system in order to get a family $\mathcal{U}_3$ with dense union. Then we again repeat the same procedure but now at three levels. At the end of the construction each function in the dense G_δ-set $\bigcap_{n=1}^{\infty} \bigcup \mathcal{U}_n$ is of Sierpiński–Adjan–Novikov type. □

5. Typical bounded derivatives are not countably continuous

The approach in this note provides also an answer to another question in Jackson and Mauldin [5].

5.1. THEOREM. *In the Banach space of bounded derivatives on* $\mathbf{I}$ *the countably continuous functions form a meager set.*

Let us indicate which modifications in the proof of Theorem 4.3 are necessary to obtain this result. Our terminology and facts from differentiation theory are all taken from Bruckner [3].

(A) We use here the definition of Sierpiński–Adjan–Novikov function with condition $(*)$ in §4 replaced by condition $(*')$ in Remark 4.2.

(B) We construct the Cantor sets $C(\sigma)$ in such a way that additionally each nonempty relatively open set in $C(\sigma)$ has positive Lebesgue measure.

(C) Because of (B), we can define the subsequent Cantor sets $C(\sigma)$ and the relatively open sets $G(\sigma)$ so that there exists an approximately continuous function $h\colon \mathbf{I} \to \mathbf{I}$ (hence a derivative by [3, Ch. II, Theorem 5.5(a)]) such that $h(x) \geqq \frac{3}{5}$ on $C(\sigma)$ and $h(x) = 0$ on $G(\sigma)$. The jump in condition $(*')$ can then be created by using the function $u + \delta \cdot h$ instead of v, where δ and v are as in Lemma 4.6.

Only (C) needs some additional justification. To this end, let C be a Cantor set in $\mathbf{I}$ such that nonempty relatively open sets in C have positive Lebesgue measure. Let $K \subsetneqq C$ be a Cantor set of positive Lebesgue measure such that $G = C \setminus K$ is dense in C, and let E be the set of Lebesgue density one points of K ([3, Ch. II, Theorem 5.1]). Removing a set of measure 0 if necessary, we can assume that E is as in [3, Ch. II, Theorem 6.5]; let $f\colon \mathbf{I} \to \mathbf{I}$ be the function described in that theorem. For every n, let $E_n = \{x \in E\colon f(x) \geqq \frac{1}{n}\}$ and pick n such that E_n has positive Lebesgue measure. There is a Cantor set $L \subsetneqq E_n$ having the property that all its nonempty relatively open subsets have positive measure. Then $\frac{1}{n} \leqq f(x) \leqq 1$ on L and $f(x) = 0$ on G. Finally, set $h = \ell \circ f$, where $\ell\colon I \to \mathbf{I}$ is a continuous function with $\ell(0) = 0$ and $\ell[\frac{1}{n}, 1] \subsetneqq [\frac{3}{5}, 1]$. Then h is approximately continuous by [3, Ch. II, Theorem 5.4].

6. Comments

6.1. *Explicit examples of functions that are not countably continuous.* We present here two explicit examples of first Baire class functions that are not countably continuous. Each, combined with Theorem 3.1, (re)proves the result of Jackson and Mauldin quoted in the introduction.

(A) Let $C \subsetneqq \mathbf{I}$ be the Cantor set. Since C is canonically homeomorphic to $\{0,1\}^\infty$ it follows that C is canonically homeomorphic to C^∞. The continuous function $\xi: C \times C \to [-1,1]$ defined by $\xi(x,y) = x - y$ is easily seen to be surjective. Consequently, there is an explicit map from C^2 onto $[-1,1]$. By taking the infinite product of this map, we conclude that there is an explicit map from C^∞ onto Q. Consequently, there is an explicit map from C onto Q, say f. (This is well-known of course.)

Define the functions $\ell, u: Q \to \mathbf{I}$ by

$$\ell(x) = \min f^{-1}(x) \qquad (x \in Q)$$

and

$$u(x) = \max f^{-1}(x) \qquad (x \in Q),$$

respectively.

6.1.1. THEOREM. *ℓ is lower semicontinuous and u is upper semicontinuous. Moreover, ℓ and u are not countably continuous.*

PROOF. We will prove that ℓ is lower semicontinuous. The proof that u is upper semicontinuous is similar and is left to the reader. To this end, let $r \in \mathbf{R}$ and $x \in \ell^{-1}(r,\infty)$. Then $\ell(x) > r$ and so $f^{-1}(x) \subseteqq (r,\infty)$. By compactness of $\mathbf{I}$ we have that the function f is closed. Consequently, there exists a neighborhood V of x in Q such that $f^{-1}[V] \subseteqq (r,\infty)$. Now for every $y \in V$ we have $\ell(y) > r$ which proves that $V \subseteqq \ell^{-1}(r,\infty)$. We conclude that $\ell^{-1}(r,\infty)$ is open.

We will next prove that ℓ is not countably continuous. The proof that u is not countably continuous is similar and is left to the reader. To this end, assume that $Q = E_1 \cup E_2 \cup \cdots$. Since Q is not the union of countably many zero-dimensional subspaces ([9, Corollary 4.8.5]) and every finite-dimensional separable metrizable space is the union of finitely many zero-dimensional subspaces ([9, Corollary 4.4.8]), it follows that for some i, $\dim E_i = \infty$. We claim that $\ell \restriction E_i$ is not continuous. Observe that the composition

$$f \circ \ell \restriction E_i$$

is the identity on E_i and that f is continuous. But then if $\ell \restriction E_i$ were continuous this would imply that $\ell \restriction E_i : E_i \to \ell[E_i]$ is a topological homeomorphism which is impossible because E_i is infinite-dimensional and every nonempty subspace of C is zero-dimensional. □

6.1.2. COROLLARY. $\sup: Q \to \mathbf{I}$ *is lower semicontinuous but not countably continuous. In addition,* $\inf: Q \to \mathbf{I}$ *is upper semicontinuous but not countably continuous.*

PROOF. The function $\frac{1}{2}\ell + \frac{1}{2}: Q \to [\frac{1}{2}, 1]$ is lower semicontinuous but not countably continuous (Theorem 6.1.1). The result for sup now easily follows from Theorem 2.2. The result for inf can be proved analogously. □

6.1.3. QUESTION. Is there a homeomorphism $\alpha: \mathcal{K}(\mathbf{I}) \to Q$ such that for every $A \in \mathcal{K}(\mathbf{I})$ we have

$$\lambda(A) = \inf\left(\alpha(A)\right),$$

i.e., are the pairs $\left(\mathcal{K}(\mathbf{I}), \lambda\right)$ and $(Q, \inf)$ topologically equivalent?

(B) The second example is a reformulation of the original construction of Adjan and Novikov. Again, let $C \subsetneqq \mathbf{I}$ be the Cantor set and let $D = \{d_1, d_2, \dots\}$ be a countable dense set in C. Define $\phi: C \to \mathbf{I}$ by the formula

$$\phi(x) = \begin{cases} 0 & (x \in C \setminus D), \\ \frac{1}{i} & (x = d_i) \end{cases}$$

and let $f: C \times C \times \cdots \to \mathbf{I}$ be defined by

$$f(x_1, x_2, \dots) = \sum_{i=1}^{\infty} 2^{-i} \phi(x_1) \cdots \phi(x_i).$$

The reasoning of Adjan and Novikov that was reproduced by us in the proof of Lemma 4.1 shows that f is not countably continuous. It is easily seen that f is upper semicontinuous.

Notice that one can identify $C \times C \times C \times \cdots$ with C in $\mathbf{I}$ which, as can easily be seen, provides a corresponding example defined on $\mathbf{I}$.

6.2. *Zero-dimensional spaces.* In the special case of zero-dimensional spaces it is possible to derive Theorem 6.1.1 from well-known selection theorems. To see this, let X be a compact zero-dimensional space and let $f: X \to \mathbf{I}$ be upper semicontinuous. Put

$$G = \left\{(x, A) \in X \times \mathcal{K}(\mathbf{I}) : f(x) = \lambda(A)\right\}.$$

Then G is a G_δ-subset of $X \times \mathcal{K}(\mathbf{I})$, and hence is completely metrizable. From the upper semicontinuity of the function f one readily concludes that the multifunction F which assigns to each $x \in X$ the vertical section of G at x is lower semicontinuous. There exists a continuous selection β for F by a selection theorem of Kuratowski and Ryll-Nardzewski [7] or Michael [8]. This function is what was needed in Claim 2 of the proof of Theorem 3.1.

Let us finally notice that the second function considered in §6.1 is defined on a zero-dimensional compact space.

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ON A CONVERGENT PÁL-TYPE (0,2) INTERPOLATION PROCESS

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1. Introduction

Let

$$(1)\qquad (-1 \leqq)x_{nn} < x_{n-1,n} < \ldots < x_{1n}(\leqq 1) \qquad (n = 2, 3, \ldots)$$

be an arbitrary triangular matrix of interpolation (shortly $x_k := x_{kn}$), and let

$$(2)\qquad (-1 <)y_{n-1,n} < y_{n-2,n} < \ldots < y_{1n}(< 1) \qquad (n = 2, 3, \ldots)$$

be the zeros of the derivative of the polynomial $\omega_n(x) = \prod_{k=1}^{n}(x - x_{kn})$ (shortly $y_k := y_{kn}$). Modifying the notion of the well-known Hermite–Fejér interpolation, L. G. Pál [5] introduced the polynomials $H_n(x) \in \Pi_{2n-1}$ (=the set of polynomials of degree at most $2n - 1$) satisfying the conditions

$$(3)\quad H_n(x_k) = z_{kn} \quad (k = 1, \ldots, n), \qquad H_n'(y_j) = z_{jn}' \quad (j = 1, \ldots, n-1)$$

where $z_k := z_{kn}$, $z_j' = z_{jn}'$ are arbitrary real numbers. It turned out that these polynomials are never uniquely determined, and in order to make them unique, one has to impose an additional condition. Recently, M. R. Akhlaghi [1] generalized this problem for successive higher order derivatives on the roots of successive higher order derivatives of $\omega_n(x)$, again imposing an additional condition on the interpolating polynomial. He, and earlier S. A. Eneduanya [4] proved convergence theorems for these polynomials on the roots of the polynomial (4) defined below. However, in doing so they assumed some higher order smoothness on the function to be approximated, and the order of convergence was far from the Jackson order.

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Motivated by the quoted work of Pál, in this paper we investigate the following related problem. Let (1) be the roots of the polynomial

$$\pi_n(x) = (1-x^2)P'_{n-1}(x) = -n(n-1)\int_{-1}^{x} P_{n-1}(t)\,dt \tag{4}$$

where $P_{n-1}(x) \in \Pi_{n-1}$ is the Legendre polynomial (normalized such that $P_{n-1}(1) = 1$). Then evidently, (2) are the roots of $P_{n-1}(x)$. Now, instead of (3), we are looking for polynomials $R_n(x) \in \Pi_{2n}$ satisfying

$$R_n(x_k) = z_k \quad (k = 1, \dots, n), \qquad R'_n(\pm 1) = z_{\pm}{}', \tag{5}$$
$$R''_n(y_j) = z''_j \quad (j = 1, \dots, n-1)$$

where z_k, $z_{\pm}{}'$, z''_j are arbitrary real numbers. It will turn out that these polynomials are uniquely determined, they have a relatively simple form, and the operators determined by them approximate in Telyakowski–Gopengauz order for continuous functions, and close to Telyakowski–Gopengauz order for continuously differentiable functions. These features (which, until now, were unknown for any Birkhoff type interpolation) prove that our R_n's are better than previously investigated Pál-type interpolating polynomials. Also, if we interpret Paul Turán's question about the existence of a convergent (0,2) interpolation process for all continuous functions in a broader sense, namely permitting Pál-type interpolation, then our Theorem 3 below is an answer to this question in the affirmative.

We also note that in a recent paper M. R. Akhlaghi and A. Sharma [2] considered basically the above Pál type problem (existence, uniqueness and fundamental functions) in a slightly different context, namely they did not prescribe first derivatives at the endpoints. As it will turn out, prescribing these data will enable us to prove Gopengauz–Telyakowski type error estimates. (The problem of convergence is not considered in [2].)

2. Existence and representation

Let

$$l_k(x) := l_{kn}(x) := \frac{\pi_n(x)}{\pi'_n(x_k)(x - x_k)} \qquad (k = 1, \dots, n)$$

be the fundamental polynomials of Lagrange interpolation based on (1) (i.e. the roots of (4)), let

(6)

$$\lambda_j := \lambda_{j,n-1} := \frac{2}{P_{n-1}'(y_j)^2(1-y_j^2)} = O\left(\frac{\sqrt{1-y_j^2}}{n}\right) \qquad (j=1,\dots,n-1)$$

be the Cotes numbers associated with the Legendre polynomial $P_{n-1}(x)$ (see G. Szegö [7], (15.3.2)), and let

$$\mu_i := \mu_{in} := \frac{1}{n(n-1)+i(i-1)}.$$

THEOREM 1. *If x_k, y_j are determined by (4) and z_k, $z_\pm'$, z_j'' are arbitrary real numbers, then there exists a polynomial $R_n(x) \in \Pi_{2n}$ satisfying (5) which can be written in the form*

$$R_n(x) = \sum_{k=1}^{n} z_k r_k(x) + z_+'\sigma_+(x) + z_-'\sigma_-(x) + \sum_{j=1}^{n-1} z_j''\varrho_j(x) \tag{7}$$

where r_k, $\sigma_\pm$, $\varrho_j \in \Pi_{2n}$ can be represented as

$$r_1(x) = r_n(-x) = \frac{(1+x)^2P_{n-1}'(x)P_{n-1}(x)}{2n(n-1)} - \frac{3\pi_n(x)(1+x)P_{n-1}'(x)}{4n(n-1)} + \tag{8a}$$

$$+\frac{(3n^2-3n+1)\pi_n(x)}{4n(n-1)}\left[P_{n-2}(x)+P_{n-1}(x)+2\sum_{i=2}^{n-1}\mu_i(2i-1)\pi_i(x)\right],$$

$$r_k(x) = l_k^2(x) + \frac{\pi_n(x)(1-x^2)}{\pi_n'(x_k)^2P_{n-1}(x_k)}\cdot \tag{8b}$$

$$\cdot\sum_{i=2}^{n-1}\mu_i(2i-1)P_{i-1}'(x)\left[P_{i-1}''(x_k)+\frac{n(n-1)}{1-x_k^2}P_{i-1}(x_k)\right]$$

$$(k=2,\dots,n-1),$$

$$\sigma_+(x) = -\sigma_-(-x) = -\frac{\pi_n(x)(1+x)P_{n-1}(x)}{2n(n-1)} - \tag{9}$$

$$-\frac{\pi_n(x)}{2n(n-1)}\sum_{i=2}^{n-1}\mu_i(2i-1)\pi_i(x) - \frac{\pi_n(x)^2}{4n(n-1)^2},$$

and

$$(10)\qquad \varrho_j(x) = -\frac{\lambda_j \pi_n(x)}{2P'_{n-1}(y_j)} \sum_{i=2}^{n-1} \mu_i \frac{2i-1}{i(i-1)} \pi_i(x) P'_{i-1}(y_j) - \frac{\lambda_j \pi_n(x)^2}{4n(n-1)^2}$$

$$(j = 1, \dots, n-1).$$

REMARK. Note that once this so-called modified (0,2) interpolation problem is solved, it is easy to obtain the fundamental polynomials of the original problem where the first derivative conditions at ± 1 are omitted. Namely, we can look for these polynomials in the forms

$$r_k(x) + \alpha_+ \sigma_+(x) + \alpha_- \sigma_-(x), \quad \varrho_k(x) + \beta_+ \sigma_+(x) + \beta_- \sigma_-(x),$$

where the constants α_+, α_-, β_+, β_- are determined such that these polynomials will be of degree $2n-2$.

PROOF. In order to prove the theorem we have to show that the polynomials of degree at most $2n$ defined in (8)–(10) satisfy the following conditions:

$$(11)\qquad r_k(x_p) = \delta_{kp}, \quad r'_k(\pm 1) = 0, \quad r''_k$$

$$(y_q) = 0 \quad (k, p = 1, \dots, n, \ q = 1, \dots, n-1),$$

$$(12)\qquad \sigma_+(x_p) = 0, \quad \sigma'_+(1) = 1, \ \sigma'_+(-1) = 0, \quad \sigma''_+(y_q) = 0$$

$$(p = 1, \dots, n, \ q = 1, \dots, n-1),$$

and

$$(13)\qquad \varrho_j(x_p) = 0, \quad \varrho'_j(\pm 1) = 0, \quad \varrho''_j(y_q) = \delta_{jq}$$

$$(p = 1 \dots, n; \ j, q = 1, \dots, n-1).$$

Some of these relations are trivial, and the others are easily proved by exact routine calculations. Therefore we omit the detailed proofs, and only indicate those identities which should be used in the course of the verifications:

$$(14)\qquad P_{n-1}(1) = 1, \quad P'_{n-1}(1) = \binom{n}{2}, \quad P''_{n-1}(1) = 3\binom{n+1}{4},$$

$$(15)\qquad \pi'_n(1) = -n(n-1),$$

$$(16)\qquad P''_{n-1}(y_q) = \frac{2y_q}{1-y_q^2} P'_{n-1}(y_q),$$

$$P'''_{n-1}(y_q)=\left[\frac{8y_q^2}{(1-y_q^2)^2}-\frac{(n+1)(n-2)}{1-y_q^2}\right]P'_{n-1}(y_q)$$

$$(q=1,\dots,n-1),$$

$$\left[\pi_n(x)\pi_i(x)\right]''_{x=y_q}=-\frac{1}{\mu_i}\pi_n(y_q)P'_{i-1}(y_q)\qquad(i=1,\dots,n,\ q=1,\dots,n-1),$$

(17) $$P_{n-2}(x_p)=x_pP_{n-1}(x_p),\quad P'_{n-2}(x_p)=-(n-1)P_{n-1}(x_p),$$

(18) $$P''_{n-2}(x_p)=-\frac{n(n-1)}{1-x_p^2}x_pP_{n-1}(x_p)\qquad(p=2,\dots,n-1),$$

$$P_{n-2}(y_q)=\frac{1-y_q^2}{n-1}P'_{n-1}(y_q),\quad P'_{n-2}(y_q)=y_qP'_{n-1}(y_q),$$

(19) $$P''_{n-2}(y_q)=\left(\frac{2}{1-y_q^2}-n\right)P'_{n-1}(y_q)\qquad(q=1,\dots,n-1),$$

(20) $$\sum_{j=2}^{i-1}(2j-1)\pi_j(x)=-i\pi_i(x)+2(1+x)P'_{i-1}(x)-$$

$$-i(i-1)(1+x)P_{i-1}(x)=$$

$$=(1+x)(i-1)\left[P'_{i-2}(x)-P'_{i-1}(x)+\frac{P_{i-2}(x)-P_{i-1}(x)}{1-x}\right]\qquad(i=3,4,\dots),$$

(21) $$\sum_{i=1}^{n-1}(2i-1)P_{i-1}(x)P_{i-1}(y)=$$

$$=(n-1)\frac{P_{n-1}(x)P_{n-2}(y)-P_{n-2}(x)P_{n-1}(y)}{x-y}\qquad(x\neq y),$$

(22) $$\sum_{i=2}^{n-1}\frac{2i-1}{i(i-1)}P'_{i-1}(x)P'_{i-1}(y)=$$

$$=\begin{cases}\frac{P'_{n-1}(x)P'_{n-2}(y)-P'_{n-2}(x)P'_{n-1}(y)}{(n-1)(x-y)} & \text{if } x\neq y,\\ \frac{P''_{n-1}(x)P'_{n-2}(x)-P'_{n-1}(x)P''_{n-2}(x)}{n-1} & \text{if } x=y.\end{cases}$$

All of these identities can be found in, or deduced from, relations in Chapter 4 of Szegö [7]. □

3. Uniqueness

THEOREM 2. *The polynomial $R_n(x)$ defined by the conditions* (5) *is uniquely determined.*

PROOF. The proof is basically the same as that of the corresponding result in [2]; we give it for the sake of completeness. Clearly,
it suffices to show that if $R_n(x) \in \Pi_{2n}$ satisfies

$$R_n(x_p) = 0 \quad (p = 1, \dots, n), \tag{23}$$

$$R'_n(\pm 1) = 0, \quad R''_n(y_q) = 0 \quad (q = 1, \dots, n-1)$$

then $R_n(x) \equiv 0$. By the first set of conditions in (23) we have $R_n(x) = \pi_n(x)q_n(x)$, where $q_n(x) \in \Pi_n$. By the last set of conditions in (23) we conclude to the relations

$$-n(n-1)P'_{n-1}(y_q)q_n(y_q) + (1-y_q^2)P'_{n-1}(y_q)q''_n(y_q) = 0 \qquad (q = 1, \dots, n-1),$$

i.e.

$$(1-y_q^2)q''_n(y_q) = n(n-1)q_n(y_q) \qquad (q = 1, \dots, n-1). \tag{24}$$

By the first derivative conditions in (23) we have $q_n(\pm 1) = 0$, i.e. (24) is also valid for $y_0 := -1$ and $y_n := 1$. But since $q_n(x) \in \Pi_n$, this implies

$$(1-x^2)q''_n(x) = n(n-1)q_n(x). \tag{25}$$

If $q_n(x) = c_m x^m +$lower degree terms $(m \leqq n)$, then this yields $-m(m-1)c_m = n(n-1)c_m$, whence $c_m = 0$, i.e. $q_n(x) \equiv 0$. □

4. Convergence

For an arbitrary $f(x) \in C[-1,1]$ (= the set of continuous functions in $[-1,1]$) we define the polynomial $R_n(f,x) \in \Pi_{2n}$ by the conditions

$$R_n(f, x_p) = f(x_p) \quad (p = 1, \dots, n),$$

$$R'_n(f, \pm 1) = 0, \quad R''_n(f, y_q) = 0 \quad (q = 1, \dots, n-1).$$

Also, if $f'(x) \in C[-1,1]$, then let $\bar{R}_n(f,x) \in \Pi_{2n}$ be defined by the conditions

$$\bar{R}_n(f, x_p) = f(x_p) \quad (p = 1, \dots, n), \qquad \bar{R}'_n(f, \pm 1) = f'(\pm 1),$$

$$\bar{R}''_n(f, y_q) = 0 \qquad (q = 1, \dots, n-1).$$

By Theorem 2, these polynomials are uniquely determined, and by Theorem 1 they can be written in the form

$$R_n(f,x)=\sum_{k=1}^{n} f(x_k)r_k(x)$$

and

$$\bar{R}_n(f,x)=\sum_{k=1}^{n} f(x_k)r_k(x)+f'(1)\sigma_+(x)+f'(-1)\sigma_-(x).$$

Let $\omega(f,h)$ be the modulus of continuity of $f(x)\in C[-1,1]$.

THEOREM 3. *We have*

$$|f(x)-R_n(f,x)|=O\left(\omega\left(f,\frac{\sqrt{1-x^2}}{n}\right)\right)\qquad \bigl(|x|\leqq 1,\ f\in C[-1,1]\bigr),$$

and

$$|f(x)-\bar{R}_n(f,x)|=O\left(\frac{\sqrt{1-x^2}}{n^2}\right)\left[\log n+\sum_{k=1}^{n}\omega\left(f',\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right]$$

$$\bigl(|x|\leqq 1,\ f'\in C[-1,1]\bigr).$$

In particular, if $f'\in \operatorname{Lip}\alpha$, then

$$|f(x)-\bar{R}_n(f,x)|=\begin{cases} O\left(\left(\frac{\sqrt{1-x^2}}{n}\right)^{1+\alpha}+\frac{\sqrt{1-x^2}}{n^{1+2\alpha}}\right) & \text{if } 0<\alpha<\frac{1}{2},\\ O\left(\left(\frac{\sqrt{1-x^2}}{n}\right)^{1+\alpha}+\frac{\sqrt{1-x^2}\log n}{n^2}\right) & \text{if } \frac{1}{2}\leqq\alpha<1,\\ O\left(\frac{\sqrt{1-x^2}\log n}{n^2}\right) & \text{if } \alpha=1\end{cases}$$

for $|x|\leqq 1$.

The proof will be given in Section 6. As a preparation, we need estimates of the fundamental functions (8) and (9).

5. Estimates of $r_k(x)$ and $\sigma_\pm(x)$

LEMMA 1. *Let $\omega(h)$ be an arbitrary modulus of continuity, or let $\omega(h) \equiv \equiv 1$. Then we have*

$$(26)\qquad \begin{cases} |r_1(x)| = O(1), \\ (1-x)\omega(1-x)|r_1(x)| = O\left(\frac{\sqrt{1-x^2}}{n}\right)\omega\left(\frac{\sqrt{1-x^2}}{n}\right) \end{cases} \qquad (|x| \leqq 1).$$

PROOF. Using Abel transform in (8a) with the factors μ_i, the relations

$$(27)\qquad P_{n-1}(x) = xP_{n-2}(x) - \frac{\pi_{n-1}(x)}{n-1}, \quad P_{n-2}(x) = \frac{\pi_n(x)}{n-1} + xP_{n-1}(x)$$

(cf. Szegö [6], (4.7.27)) and applying (20) we obtain

$$r_1(x) = \frac{(1+x)^2 P'_{n-1}(x)P_{n-1}(x)}{2n(n-1)} - \frac{(3n^2-3n+1)\pi_n(x)(1+x)P_{n-1}(x)}{4n(n-1)^2} +$$
$$+\frac{[(3n^2-3n+1)x+3n-2]\pi_n(x)P'_{n-1}(x)(1+x)}{4n(n-1)^3} +$$
$$+\frac{(3n^2-3n+1)\pi_n(x)}{2n(n-1)}\sum_{i=3}^{n-1}(\mu_{i-1}-\mu_i)\sum_{j=2}^{i-1}(2j-1)\pi_j(x).$$

Here and in the sequel, we shall use the well-known estimates

$$(28)\qquad |P_{n-1}(x)| = O\left(\frac{1}{(1-x^2)^{\mu/2}n^\mu}\right) \qquad \left(|x| \leqq 1,\ 0 \leqq \mu \leqq \frac{1}{2}\right),$$

$$(29)\qquad |P'_{n-1}(x)| = O\left(\frac{n^\nu}{(1-x^2)^{1-\nu/2}}\right) \qquad \left(|x| \leqq 1,\ \frac{1}{2} \leqq \nu \leqq 2\right)$$

and

$$(30)\qquad |\pi_n(x)| = O\left((1-x^2)^{\delta/2}n^\delta\right) \qquad \left(|x| \leqq 1,\ \frac{1}{2} \leqq \delta \leqq 2\right)$$

(cf. Szegö [7], Ch. VII). Hence we have

(31) $$\frac{(1+x)^2|P'_{n-1}(x)P_{n-1}(x)|}{2n(n-1)} = O\left(\frac{(1+x)(1-x)^{-\lambda/2}}{n^\lambda}\right)$$

$$\left(|x| \leqq 1,\, 0 \leqq \lambda \leqq 2\right).$$

Thus using (20) again we obtain

$$r_1(x) = -\frac{(3n^2-3n+1)\pi_n(x)(1+x)P_{n-1}(x)}{4n(n-1)^2} +$$

$$+\frac{[(3n^2-3n+1)x+3n-2]\pi_n(x)P'_{n-1}(x)(1+x)}{4n(n-1)^3} +$$

$$+\frac{(3n^2-3n+1)\pi_n(x)(1+x)}{2n(n-1)} \cdot$$

$$\cdot \sum_{i=3}^{n-1} \nu_i \left[P'_{i-2}(x) - P'_{i-1}(x) + \frac{P_{i-2}(x)-P_{i-1}(x)}{1-x}\right] +$$

$$+O\left(\frac{(1+x)(1-x)^{-\lambda/2}}{n^\lambda}\right) \qquad \left(|x| \leqq 1,\, 0 \leqq \lambda \leqq 2\right),$$

where

(32) $$\nu_i = (\mu_{i-1} - \mu_i)(i-1) = O\left(\frac{i^2}{n^4}\right) \qquad (i = 3, 4, \ldots).$$

Here we use another Abel transform with the factors ν_i, and then apply (27) as well as

(33) $$P'_{n-2}(x) = xP'_{n-1}(x) - (n-1)P_{n-1}(x),$$

$$P'_{n-1}(x) = xP'_{n-2}(x) + (n-1)P_{n-2}(x)$$

(cf. Szegö [7], (4.7.28)) to get

$$r_1(x) = -\frac{(3n^2-3n+1)\pi_n(x)(1+x)P_{n-1}(x)}{4n(n-1)^2} +$$

$$+\frac{[(3n^2-3n+1)x+3n-2]\pi_n(x)P'_{n-1}(x)(1+x)}{4n(n-1)^3} -$$

$$-\frac{(3n^2-3n+1)\pi_n(x)(1+x)}{2n(n-1)}\Bigg\{\nu_{n-1}\left[P'_{n-2}(x)+\frac{P_{n-2}(x)-1}{1-x}\right]+$$

$$+\sum_{i=4}^{n-1}(\nu_{i-1}-\nu_i)\left[P'_{i-2}(x)+\frac{P_{i-2}(x)-1}{1-x}\right]\Bigg\}+O\left(\frac{(1+x)(1-x)^{-\lambda/2}}{n^\lambda}\right)=$$

$$=-\frac{(3n^2-3n+1)\pi_n(x)(1+x)P_{n-1}(x)}{4n(n-1)^2}+$$

$$+\frac{[(3n^2-3n+1)x+3n-2]\pi_n(x)P'_{n-1}(x)(1+x)}{4n(n-1)^3}-$$

$$-\frac{(3n^2-3n+1)\pi_n(x)(1+x)}{2n(n-1)}\Bigg\{\frac{(n-2)^2}{2(n-1)^2(n^2-3n+3)}\left[\frac{nx+1}{n-1}P'_{n-1}(x)+\right.$$

$$\left.+\left(\frac{1}{1-x}-n\right)P_{n-1}(x)\right]-\frac{\nu_3}{1-x}+\sum_{i=4}^{n-1}(\nu_{i-1}-\nu_i)\left[P'_{i-2}(x)+\frac{P_{i-2}(x)}{1-x}\right]\Bigg\}+$$

$$+O\left(\frac{(1+x)(1-x)^{-\lambda/2}}{n^\lambda}\right)=$$

$$=-\frac{(3n^2-3n+1)\pi_n(x)(1+x)}{2n(n-1)}\sum_{i=4}^{n-1}(\nu_{i-1}-\nu_i)P'_{i-2}(x)+$$

$$+O\Bigg(\frac{(1+x)\pi_n(x)}{(1-x)n^4}\sum_{i=4}^{n-1}i|P_{i-2}(x)|+\frac{1+x}{n^4}\big|\pi_n(x)P'_{n-1}(x)\big|+$$

$$+\frac{(1+x)\big|\pi_n(x)P_{n-1}(x)\big|}{(1-x)n^2}+\frac{(1+x)(1-x)^{-\lambda/2}}{n^\lambda}\Bigg)\qquad\big(|x|\leqq 1,\ 0\leqq\lambda\leqq 2\big).$$

Here by (28)–(30) again,

$$\frac{(1+x)\pi_n(x)}{(1-x)n^4}\sum_{i=4}^{n-1}i\big|P_{i-2}(x)\big|=O\left(\frac{(1+x)(1-x)^{-\lambda/2}}{n^\lambda}\right)\tag{34}$$

$$\big(|x|\leqq 1,\ 0\leqq\lambda\leqq 2\big),$$

$$\pi_n(x)P'_{n-1}(x)=O\left(\frac{n^\lambda}{(1-x^2)^{1-\lambda/2}}\right)\qquad\big(|x|\leqq 1,\ 1\leqq\lambda\leqq 4\big)\tag{35}$$

and

(36)
$$\frac{(1+x)\pi_n(x)P_{n-1}(x)}{(1-x)n^2}=O\left(\frac{(1+x)(1-x)^{-\lambda/2}}{n^\lambda}\right)\qquad \left(|x|\leqq 1,\ 0\leqq\lambda\leqq 2\right).$$

With the notation
$$\kappa_i=\frac{\nu_{i-1}-\nu_i}{2i-3}=O(n^{-4}),$$
an easy calculation shows that
$$\kappa_{i-1}-\kappa_i=O\left(\frac{i}{n^6}\right),$$

whence we obtain by (20) and (28)–(30), (34)–(36), performing one more Abel transform

$$\frac{(3n^2-3n+1)\pi_n(x)(1+x)}{2n(n-1)}\sum_{i=4}^{n-1}\kappa_i(2i-3)P'_{i-2}(x)=$$
$$=O\left(\frac{\pi_n(x)}{n^4(1-x)}\right)\left\{\frac{1}{n^4}\left|\sum_{i=4}^{n-1}(2i-3)P'_{i-2}(x)\right|+\right.$$
$$\left.+\sum_{i=5}^{n-1}|\kappa_{i-1}-\kappa_i|\sum_{j=4}^{i-1}\left[j|\pi_j(x)|+(1+x)|P'_{j-1}(x)|+(1+x)j^2|P_{j-1}(x)|\right]\right\}=$$
$$=O\left(\frac{(1+x)^{\frac{3-\lambda}{2}}(1-x)^{\frac{1-\lambda}{2}}}{n^\lambda}+\frac{(1+x)^{2-\lambda/2}(1-x)^{-\lambda/2}}{n^\lambda}\right)=$$
$$=O\left(\frac{\sqrt{1+x}(1-x)^{-\lambda/2}}{n^\lambda}\right)$$
$$(0\leqq\lambda\leqq 2,\ |x|\leqq 1).$$

Collecting all these estimates we obtain

(37)
$$|r_1(x)|=O\left(\frac{\sqrt{1+x}(1-x)^{-\lambda/2}}{n^\lambda}\right)\qquad\left(|x|\leqq 1,\ 0\leqq\lambda\leqq 2\right).$$

Now the proof of the lemma can be finished. The first relation in (26) is obtained from (37) with $\lambda = 0$. As for the second relation, if $n\sqrt{1-x} \leqq \leqq \sqrt{1+x}$, then applying (37) with $\lambda = 1$ and using the monotonicity of ω we get (26). If $n\sqrt{1-x} \geqq \sqrt{1+x}$ then we put $\lambda = 2$ in (37), and use the well-known inequality

$$\omega(1-x) \leqq \frac{2n(1-x)}{\sqrt{1-x^2}}\omega\left(\frac{\sqrt{1-x^2}}{n}\right)$$

to get (26) in this case. □

In what follows we shall use the notations $x = \cos t$, $x_k = \cos t_k$ $(k = = 2, \dots, n-1)$.

LEMMA 2. *We have*

$$\left|r_k(x)\right| = O\left(\frac{\sin t}{n^3 \sin^3 \frac{|t-t_k|}{2} \sin t_k}\right)$$

$$\left(|x| \leqq 1,\ |t-t_k| > c/n,\ k = 2, \dots, n-1\right),$$

where $c > 0$ is an arbitrary constant.

PROOF. Using the differential equation

$$(1-x^2)P''_{i-1}(x) - 2xP'_{i-1}(x) + i(i-1)P_{i-1}(x) = 0,$$

we obtain from (8b)

$$r_k(x) = l_k^2(x) + \frac{\pi_n(x)(1-x^2)}{\pi'_n(x_k)^2 P_{n-1}(x_k)(1-x_k^2)} \cdot \tag{38}$$

$$\left[2x_k \sum_{i=2}^{n} a_i \frac{2i-1}{i(i-1)} P'_{i-1}(x) P'_{i-1}(x_k) + \right.$$

$$\left. + \sum_{i=2}^{n} \alpha_i (2i-1) P'_{i-1}(x) P_{i-1}(x_k)\right] \qquad (k = 2, \dots, n-1),$$

where

(39)

$$a_i = \frac{i(i-1)}{n(n-1)+i(i-1)}, \quad \alpha_i = \frac{n(n-1)-i(i-1)}{n(n-1)+i(i-1)} \qquad (i = 1, \dots, n).$$

At first we estimate the second sum. Using (21), the notation

$$(40)\qquad \beta_1 = 0, \quad \beta_i = (\alpha_{i-1} - \alpha_i)(i-1) \qquad (i = 2, \dots, n)$$

and applying Abel transform we get

$$(41)\qquad B := \sum_{i=2}^{n} \alpha_i (2i-1) P'_{i-1}(x) P_{i-1}(x_k) =$$

$$= \frac{1}{x - x_k} \sum_{i=2}^{n} \beta_i [P'_{i-1}(x) P_{i-2}(x_k) - P'_{i-2}(x) P_{i-1}(x_k)] -$$

$$- \frac{1}{(x - x_k)^2} \sum_{i=2}^{n} \beta_i \big[P_{i-1}(x) P_{i-2}(x_k) - P_{i-2}(x) P_{i-1}(x_k) \big] := B_1 - B_2.$$

Here, using the identities (33),

$$P'_{i-1}(x) P_{i-2}(x_k) - P'_{i-2}(x) P_{i-1}(x_k) =$$

$$= x \big[P'_{i-2}(x) P_{i-2}(x_k) - P'_{i-1}(x) P_{i-1}(x_k) \big] +$$

$$+ (i-1) \big[P_{i-2}(x) P_{i-2}(x_k) + P_{i-1}(x) P_{i-1}(x_k) \big],$$

whence and from (41), with the notations

$$(42)\qquad \gamma_i = \frac{\beta_i - \beta_{i+1}}{2i-1}, \quad \delta_i = \frac{(i-1)\beta_i + i\beta_{i+1}}{2i-1} \qquad (i = 1, \dots, n-1)$$

we obtain, after applying another Abel transform, that

$$(43)\qquad B_1 = \frac{x}{x - x_k} \sum_{i=2}^{n} \beta_i \big[P'_{i-2}(x) P_{i-2}(x_k) - P'_{i-1}(x) P_{i-1}(x_k) \big] +$$

$$+ \frac{1}{x - x_k} \sum_{i=2}^{n} (i-1) \beta_i \big[P_{i-2}(x) P_{i-2}(x_k) + P_{i-1}(x) P_{i-1}(x_k) \big] =$$

$$= \frac{x}{x - x_k} \left[- P'_{n-1}(x) P_{n-1}(x_k) - \sum_{i=2}^{n-1} \gamma_i (2i-1) P'_{i-1}(x) P_{i-1}(x_k) \right] +$$

$$+ \frac{1}{x - x_k} \left[\sum_{i=1}^{n-1} \delta_i (2i-1) P_{i-1}(x) P_{i-1}(x_k) + + (n-1) P_{n-1}(x) P_{n-1}(x_k) \right]$$

(since by (39)–(40) $\beta_n = 1$). An easy calculation from (39),(40) and (42) shows that

$$\gamma_{i-1} - \gamma_i = O\left(\frac{i}{n^4}\right) \qquad (i = 2, \dots, n-1), \tag{44}$$

and using the estimate

$$\sum_{j=2}^{i} (2j-1) P'_{j-1}(x) P_{j-1}(x_k) =$$

$$= O\left(\frac{i}{\sin\frac{|t-t_k|}{2}\sin^{3/2} t \sin^{1/2} t_k} + \frac{1}{|x-x_k|\sin\frac{|t-t_k|}{2}\sin^{1/2} t\sin^{1/2} t_k}\right)$$
$$(i = 2, 3, \dots)$$

(this follows from [6], Lemma 2, with a slight modification of the proof therein), we obtain by using another Abel transform, that

$$\sum_{i=2}^{n-1} \gamma_i (2i-1) P'_{i-1}(x) P_{i-1}(x_k) = \tag{45}$$

$$= \gamma_{n-1} \sum_{i=2}^{n-1} (2i-1) P'_{i-1}(x) P_{i-1}(x_k) +$$

$$+ \sum_{i=3}^{n-1} (\gamma_{i-1} - \gamma_i) \sum_{j=2}^{i-1} (2j-1) P'_{j-1}(x) P_{j-1}(x_k) =$$

$$= O\left(\sum_{i=1}^{n} \frac{i}{n^4}\left[\frac{i}{\sin\frac{|t-t_k|}{2}\sin^{3/2} t\sin^{1/2} t_k} + \right.\right.$$

$$\left.\left. + \frac{1}{|x-x_k|\sin\frac{|t-t_k|}{2}\sin^{1/2} t\sin^{1/2} t_k}\right]\right) =$$

$$= O\left(\frac{1}{n\sin\frac{|t-t_k|}{2}\sin^{3/2} t\sin^{1/2} t_k}\right) \qquad \big(|t-t_k| > c/n,\ k = 2, \dots, n-1\big)$$

(since by (39), (40) and (42) $\gamma_{n-1} = O(n^{-3})$, and

$$\max(\sin t, \sin t_k) = O\big(n|x-x_k|\big)$$

from the condition $|t-t_k|>c/n$; the latter estimate will be frequently used in the sequel). Hence (43) yields

$$(46)\qquad B_1=-\frac{xP'_{n-1}(x)P_{n-1}(x_k)}{x-x_k}+\frac{(n-1)P_{n-1}(x)P_{n-1}(x_k)}{x-x_k}+$$

$$+\frac{1}{x-x_k}\sum_{i=1}^{n-1}\delta_i(2i-1)P_{i-1}(x)P_{i-1}(x_k)+O\left(\frac{1}{n\sin\frac{|t-t_k|}{2}\sin^{3/2}t\sin^{1/2}t_k}\right)$$

$$\big(|t-t_k|>c/n,\ k=2,\dots,n-1\big).$$

We still have to estimate the sum here on the right hand side. With the notation

$$(47)\qquad \varepsilon_1=0,\quad \varepsilon_i=(\delta_{i-1}-\delta_i)(i-1)\qquad (i=2,\dots,n-1),$$

and using Abel transform as before we get

$$(48)\qquad C:=\sum_{i=1}^{n-1}\delta_i(2i-1)P_{i-1}(x)P_{i-1}(x_k)=$$

$$=\delta_{n-1}\sum_{i=1}^{n-1}(2i-1)P_{i-1}(x)P_{i-1}(x_k)+$$

$$+\frac{1}{x-x_k}\sum_{i=2}^{n-1}\varepsilon_i\big[P_{i-1}(x)P_{i-2}(x_k)-P_{i-2}(x)P_{i-1}(x_k)\big].$$

Here by (39), (40) and (42)

$$\delta_{n-1}=1+O(n^{-1}),$$

by (21), (18) and (27)

$$\sum_{i=1}^{n-1}(2i-1)P_{i-1}(x)P_{i-1}(x_k)=-(n-1)P_{n-1}(x)P_{n-1}(x_k)-\frac{\pi_n(x)P_{n-1}(x_k)}{x-x_k},$$

and by (27)

$$P_{i-1}(x)P_{i-2}(x_k)-P_{i-2}(x)P_{i-1}(x_k)=$$

$$=x\big[P_{i-2}(x)P_{i-2}(x_k)-P_{i-1}(x)P_{i-1}(x_k)\big]-$$

$$-\frac{1-x^2}{i-1}\big[P'_{i-2}(x)P_{i-2}(x_k)+P'_{i-1}(x)P_{i-1}(x_k)\big].$$

Thus, after applying another Abel transform and using the obvious estimates

$$\varepsilon_i = O\left(\frac{i^2}{n^2}\right), \quad \varepsilon_i - \varepsilon_{i+1} = O\left(\frac{i}{n^2}\right)$$

we get

$$C = -(n-1)P_{n-1}(x)P_{n-1}(x_k) - \frac{\pi_n(x)P_{n-1}(x_k)}{x - x_k} + \tag{49}$$

$$+\frac{x}{x-x_k}\sum_{i=2}^{n-1}\varepsilon_i\left[P_{i-2}(x)P_{i-2}(x_k) - P_{i-1}(x)P_{i-1}(x_k)\right] -$$

$$-\frac{1-x^2}{x-x_k}\sum_{i=2}^{n-1}(\delta_{i-1}-\delta_i)\left[P'_{i-2}(x)P_{i-2}(x_k) + P'_{i-1}(x)P_{i-1}(x_k)\right] +$$

$$+O\left(\frac{1}{n\sin\frac{|t-t_k|}{2}\sin^{1/2}t\sin^{1/2}t_k}\right) =$$

$$= -(n-1)P_{n-1}(x)P_{n-1}(x_k) - \frac{\pi_n(x)P_{n-1}(x_k)}{x-x_k} +$$

$$+\frac{x}{x-x_k}\Bigg\{\varepsilon_{n-1}\left[1 - P_{n-2}(x)P_{n-2}(x_k)\right] +$$

$$+\sum_{i=3}^{n-1}(\varepsilon_{i-1}-\varepsilon_i)\left[1 - P_{i-2}(x)P_{i-2}(x_k)\right]\Bigg\} +$$

$$-\frac{1-x^2}{x-x_k}\Bigg[\sum_{i=2}^{n-2}(\delta_{i-1}-\delta_{i+1})P'_{i-1}(x)P_{i-1}(x_k) +$$

$$+(\delta_{n-2}-\delta_{n-1})P'_{n-2}(x)P_{n-2}(x_k)\Bigg] + O\left(\frac{1}{n\sin\frac{|t-t_k|}{2}\sin^{1/2}t\sin^{1/2}t_k}\right) =$$

$$= -(n-1)P_{n-1}(x)P_{n-1}(x_k) - \frac{\pi_n(x)P_{n-1}(x_k)}{x-x_k} -$$

$$-\frac{1-x^2}{x-x_k}\sum_{i=2}^{n-2}(\delta_{i-1}-\delta_{i+1})P'_{i-1}(x)P_{i-1}(x_k) + O\left(\frac{1}{n|x-x_k|\sin^{1/2}t\sin^{1/2}t_k}\right).$$

To finish this estimate, we consider the sum on the right hand side. This can be estimated the same way as we did in (45); namely by (42) and (44) we have

$$\frac{\delta_{i-1}-\delta_{i+1}}{2i-1}-\frac{\delta_i-\delta_{i+2}}{2i+1}=$$

$$=\frac{i-2}{2i-1}(\gamma_{i-1}-\gamma_i)+\frac{4i^2-4}{4i^2-1}(\gamma_i-\gamma_{i+1})+\frac{i+2}{2i+1}(\gamma_{i+1}-\gamma_{i+2})=O\left(\frac{i}{n^4}\right).$$

Thus we get

$$\sum_{i=2}^{n-2}(\delta_{i-1}-\delta_{i+1})P'_{i-1}(x)P_{i-1}(x_k)=O\left(\frac{1}{n\sin\frac{|t-t_k|}{2}\sin^{3/2}t\sin^{1/2}t_k}\right)$$

$$\left(|t-t_k|>c/n,\ k=2,\dots,n-1\right).$$

Substituting this into (49) we obtain

$$C=-(n-1)P_{n-1}(x)P_{n-1}(x_k)-\frac{\pi_n(x)P_{n-1}(x_k)}{x-x_k}+$$

$$+O\left(\frac{1}{n|x-x_k|\sin^{1/2}t\sin^{1/2}t_k}+\frac{\sin^{1/2}t}{n|x-x_k|\sin\frac{|t-t_k|}{2}\sin^{1/2}t_k}\right)$$

$$\left(|t-t_k|>c/n,\ k=2,\dots,n-1\right).$$

Finally, substituting this into (46) we get

$$(50)\quad B_1=\frac{(xx_k-1)P'_{n-1}(x)P_{n-1}(x_k)}{(x-x_k)^2}+O\left(\frac{1}{n\sin\frac{|t-t_k|}{2}\sin^{3/2}t\sin^{1/2}t_k}+\right.$$

$$\left.+\frac{1}{n(x-x_k)^2\sin^{1/2}t\sin^{1/2}t_k}+\frac{\sin^{1/2}t(1}{n(x-x_k)^3\sin\frac{|t-t_k|}{2}\sin^{1/2}t_k}\right)$$

$$\left(|t-t_k|>c/n,\ k=2,\dots,n-1\right).$$

In order to estimate B_2 in (41) we note that, in the order of magnitude, this sum contributes the same as the second sum for C in (48), since the β_i's behave similarly to the ε_i's (the fact that the sum extends to n instead of $n-1$ is indifferent). Thus the estimate for B is the same as that of B_1 in (50).

Now we turn to estimating

$$A := \sum_{i=2}^{n} a_i \frac{2i-1}{i(i-1)} P'_{i-1}(x) P'_{i-1}(x_k)$$

in (38), by using the same method as above. The first Abel transform yields

$$A = a_n \sum_{i=2}^{n-1} \frac{2i-1}{i(i-1)} P'_{i-1}(x) P'_{i-1}(x_k) +$$

$$+ \frac{1}{x-x_k} \sum_{i=2}^{n} b_i \left[P'_{i-1}(x) P'_{i-2}(x_k) - P'_{i-2}(x) P'_{i-1}(x_k) \right],$$

where

(51) $$b_i = \frac{a_{i-1} - a_i}{i-1} \qquad (i = 2, \ldots, n).$$

Here by (39) $a_n = \frac{1}{2}$, and using (22), (17) and (33),

$$a_n \sum_{i=2}^{n-1} \frac{2i-1}{i(i-1)} P'_{i-1}(x) P'_{i-1}(x_k) = -\frac{P'_{n-1}(x) P_{n-1}(x_k)}{2(x-x_k)}$$

and

$$P'_{i-1}(x) P'_{i-2}(x_k) - P'_{i-2}(x) P'_{i-1}(x_k) =$$

$$= x_k \left[P'_{i-1}(x) P'_{i-1}(x_k) - P'_{i-2}(x) P'_{i-2}(x_k) \right] -$$

$$-(i-1) \left[P'_{i-1}(x) P_{i-1}(x_k) + P'_{i-2}(x) P_{i-2}(x_k) \right].$$

Hence applying another Abel transform we get

(52) $$A = -\frac{P'_{n-1}(x) P_{n-1}(x_k)}{2(x-x_k)} + \frac{x_k}{x-x_k} \sum_{i=2}^{n-1} (b_i - b_{i+1}) P'_{i-1}(x) P'_{i-1}(x_k) +$$

$$+ \frac{1}{x-x_k} \left[\sum_{i=2}^{n-1} (a_{i+1} - a_{i-1}) P'_{i-1}(x) P_{i-1}(x_k) + (a_n - a_{n-1}) P'_{n-1}(x) P_{n-1}(x_k) \right]$$

$$\left(|t - t_k| > c/n, \ k = 2, \ldots, n-1 \right).$$

An easy calculation shows that, using (39),

$$\frac{a_{i+2}-a_i}{2i+1}-\frac{a_{i+1}-a_{i-1}}{2i-1}=O\left(\frac{i}{n^4}\right)\qquad (i=2,\dots,n-2), \tag{53}$$

thus for estimating the sum

$$D:=\sum_{i=2}^{n-1}(a_{i+1}-a_{i-1})P'_{i-1}(x)P_{i-1}(x_k)$$

we can use the same method as in (45). Hence we obtain

$$D=\frac{a_n-a_{n-2}}{2n-1}\sum_{i=2}^{n-1}(2i-1)P'_{i-1}(x)P_{i-1}(x_k)+$$

$$+O\left(\sum_{i=1}^{n}\frac{i}{n^4}\left[\frac{i}{\sin\frac{|t-t_k|}{2}\sin^{3/2}t\sin^{1/2}t_k}+\right.\right.$$

$$\left.\left.+\frac{1}{|x-x_k|\sin\frac{|t-t_k|}{2}\sin^{1/2}t\sin^{1/2}t_k}\right]\right)=O\left(\frac{1}{n\sin\frac{|t-t_k|}{2}\sin^{3/2}t\sin^{1/2}t_k}\right)$$

$$\bigl(|t-t_k|>c/n,\ k=2,\dots,n-1\bigr).$$

Substituting this into (52) we get

$$A=-\frac{P'_{n-1}(x)P_{n-1}(x_k)}{2(x-x_k)}+\frac{x_k}{x-x_k}\sum_{i=2}^{n-1}(b_i-b_{i+1})P'_{i-1}(x)P'_{i-1}(x_k)+ \tag{54}$$

$$+O\left(\frac{1}{n|x-x_k|\sin\frac{|t-t_k|}{2}\sin^{3/2}t\sin^{1/2}t_k}\right)$$

$$\bigl(|t-t_k|>c/n,\ k=2,\dots,n-1\bigr).$$

Here we have to estimate

$$E:=\sum_{i=2}^{n-1}(b_i-b_{i+1})P'_{i-1}(x)P'_{i-1}(x_k). \tag{55}$$

In doing so, we apply the same method as in (52), but now with

$$a'_i:=(b_i-b_{i+1})\frac{i(i-1)}{2i-1}=O\left(\frac{i^2}{n^4}\right)\qquad (i=1,\dots,n).$$

Then it is easily seen from (38) and (51) that

$$a'_{i+1} - a'_{i-1} = O\left(\frac{i}{n^4}\right) \qquad (i = 2, \dots, n-1)$$

and with the notation $b'_i = \frac{a'_{i-1} - a'_i}{i-1}$ $(i = 2, \dots, n)$,

$$b'_{i-1} - b'_i = O\left(\frac{i}{n^6}\right) \qquad (i = 2, \dots, n).$$

Thus we obtain from (55), just like in (52), but now using term-by-term estimates in the sums, that

$$E = -a'_{n-1}\frac{P'_{n-1}(x)P_{n-1}(x_k)}{x - x_k} + \frac{x_k}{x - x_k}\sum_{i=2}^{n-1}(b'_i - b'_{i+1})P'_{i-1}(x)P'_{i-1}(x_k) +$$

$$+\frac{1}{x - x_k}\left[\sum_{i=2}^{n-1}(a'_{i+1} - a'_{i-1})P'_{i-1}(x)P_{i-1}(x_k) + \right.$$

$$\left. +(a'_n - a'_{n-1})P'_{n-1}(x)P_{n-1}(x_k)\right] =$$

$$= O\left(\frac{1}{n^2|x - x_k|\sin^{3/2} t \sin^{1/2} t_k} + \frac{1}{n^3|x - x_k|\sin^{3/2} t \sin^{3/2} t_k} + \right.$$

$$\left. + \frac{1}{n^2|x - x_k|\sin^{3/2} t \sin^{1/2} t_k} + \frac{1}{n^3|x - x_k|\sin^{3/2} t \sin^{1/2} t_k}\right) =$$

$$= O\left(\frac{1}{n^2|x - x_k|\sin^{3/2} t \sin^{1/2} t_k}\right) \qquad \left(|t - t_k| > c/n,\ k = 2, \dots, n-1\right).$$

Substituting this into (54) we obtain

$$(56)\quad A = -\frac{P'_{n-1}(x)P_{n-1}(x_k)}{2(x - x_k)} + O\left(\frac{1}{n|x - x_k|\sin\frac{|t - t_k|}{2}\sin^{3/2} t \sin^{1/2} t_k}\right)$$

$$\left(|t - t_k| > c/n,\ k = 2, \dots, n-1\right).$$

Now we are in the position to finish the proof of the lemma. Substituting (56) and (50) into (38) we get

$$r_k(x) = l_k^2(x) + \frac{\pi_n(x)(1 - x^2)}{\pi'_n(x_k)^2 P_{n-1}(x_k)(1 - x_k^2)}(2x_k A + B_1 - B_2) =$$

$$= l_k^2(x) + \frac{\pi_n(x)(1-x^2)}{\pi_n'(x_k)^2(1-x_k^2)}\left[-\frac{x_k P_{n-1}'(x)}{x-x_k} + \frac{(xx_k-1)P_{n-1}'(x)}{(x-x_k)^2}\right]+$$

$$+O\left(\frac{\sin t}{n^3|x-x_k|\sin\frac{|t-t_k|}{2}\sin t_k} + \frac{\sin^3 t}{n^3(x-x_k)^2\sin\frac{|t-t_k|}{2}\sin t_k}\right) =$$

$$= O\left(\frac{\sin t}{n^3\sin^3\frac{|t-t_k|}{2}\sin t_k}\right) \qquad \big(|t-t_k| > c/n,\ k=2,\dots,n-1\big)$$

which is exactly the statement of the lemma. □

LEMMA 3. *If ω is an arbitrary modulus of continuity, or $\omega \equiv 1$, then*

$$(57)\qquad \begin{cases} \sum_{k=1}^{n} \big|r_k(x)\big| = O(1), \\ \sum_{k=1}^{n} |x-x_k|\omega\big(|x-x_k|\big)\,\big|r_k(x)\big| = \\ = O\left(\frac{\sqrt{1-x^2}}{n^2}\right)\left[\log n + \sum_{k=1}^{n}\omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right] \end{cases}$$

$$\big(|x| \leqq 1\big).$$

PROOF. First note that by Lemma 1 and by symmetry, it is sufficient to consider

$$(58)\qquad \sum_{k=2}^{n-1} = \sum_{|t-t_k|\leqq c/n} + \sum_{|t-t_k|>c/n}$$

on the left hand sides of (57), and we may assume that $0 \leqq t \leqq \pi/2$. Here the first sums are easily settled if we use term-by-term estimate in (8) and obtain $\big|r_k(x)\big| = O(1)$. Namely, in estimating the first sums we may assume that $c_1/n < t \leqq \pi/2$ with some $c_1 > 0$ (otherwise, by choosing $0 < c < c_1$, the first sums in (58) would be empty), and then by $|x-x_k| = O(\frac{\sin t}{n})$ the contribution of this sum will be $O(\frac{\sin t}{n})\omega(\frac{\sin t}{n})$, which is less than the right hand sides of (57).

In estimating the second sum in (58) we use Lemma 2 the inequality

$$\sin\frac{t+t_k}{2} \leqq \sin\frac{|t-t_k|}{2} + \sin t$$

to obtain

$$\sum_{|t-t_k|>c/n} |r_k(x)| = O\left[\sum_{|t-t_k|>c/n}\left(\frac{1}{n^3\sin^3\frac{|t-t_k|}{2}}+\frac{1}{n^3\sin^2\frac{t-t_k}{2}\sin t_k}\right)\right]$$

and

(59) $$\sum_{|t-t_k|>c/n} |x-x_k|\omega(|x-x_k|)\,|r_k(x)| =$$

$$= O\left(\frac{\sin t}{n^2}\right)\sum_{k=2}^{n-1}\left[\frac{\omega\left(\sin\frac{|t-t_k|}{2}\sin t+\sin^2\frac{t-t_k}{2}\right)}{n\sin^2\frac{t-t_k}{2}}+\right.$$

$$\left.+\frac{\omega\left(\sin\frac{|t-t_k|}{2}\sin t+\sin^2\frac{t-t_k}{2}\right)}{n\sin\frac{|t-t_k|}{2}\sin t_k}\right] = O\left(\frac{\sin t}{n^2}\right)(A+B).$$

Here by the regular distribution of the roots x_k the first sum is estimated by $\sum_{k=1}^{n} k^{-3} = O(1)$. As for the second sum, if $\sin\frac{|t-t_k|}{2} < \sin t_k$, then it is equivalent to the first sum, while in the opposite case it is $\sum_{k=1}^{n}(n\sin t_k)^{-3} = O(1)$ again.

Similarly, in (59) the first sum, A, is easily seen to be equivalent to

$$n\sum_{k=2}^{n-1}\frac{1}{k^2}\omega\left(\frac{k}{n}\sin t+\frac{k^2}{n^2}\right) = O\left(\int_{1/n}^{1}\frac{\omega(u\sin t+u^2)}{u^2}\,du\right) =$$

$$= O\left(\int_1^n \omega\left(\frac{\sin t}{u}+\frac{1}{u^2}\right)du\right) = O\left(\sum_{k=1}^{n}\omega\left(\frac{\sin t}{k}+\frac{1}{k^2}\right)\right).$$

In estimating the last sum, B, if $\sin t_k > \sin\frac{|t-t_k|}{2}$ then it is a part of A. If $\sin t_k \leqq \sin\frac{|t-t_k|}{2}$ then using a property of the modulus of continuity we obtain

$$B = \sum_{k=2}^{n-1}\frac{\omega\left(\sin\frac{|t-t_k|}{2}\sin t+\sin^2\frac{t-t_k}{2}\right)}{n\sin\frac{|t-t_k|}{2}\sin t_k} \leqq 2\sum_{k=2}^{n-1}\frac{\omega(\sin t_k\sin t)}{n\sin^2 t_k}+$$

$$+\left(\sum_{t_k\geqq\frac{2\pi+t}{3}}+\sum_{t_k\leqq\frac{t}{3}}\right)\frac{\omega\left(\sin^2\frac{t-t_k}{2}\right)}{n\sin\frac{|t-t_k|}{2}\sin t_k},$$

and here the first sum is again estimated like A. The second sum, since here $t_k - t \geqq 2(\pi - t)/3 \geqq \pi/3$, easily seen to be

$$O\left(\frac{1}{n}\sum_{k=2}^{n-1}\frac{1}{\sin t_k}\right) = O(\log n).$$

Finally, in the third sum we use $\sin\frac{|t-t_k|}{2} \leqq 2\sin t$, and then it becomes

$$2\sum_{k=2}^{n-1}\frac{\omega\left(\sin\frac{|t-t_k|}{2}\sin t\right)}{n\sin\frac{|t-t_k|}{2}\sin t_k},$$

which has been already estimated (see the first sum in B above). Collecting all of these estimates, the proof of Lemma 3 is complete. □

We now estimate the fundamental functions $\sigma_\pm(x)$.

LEMMA 4. *We have*

$$\left|\sigma_\pm(x)\right| = O\left(\frac{(1\pm x)\sin^\alpha t}{n^{2-\alpha}} + \frac{\sin t}{n^2}\right) \qquad \left(|x| \leqq 1\right)$$

where $\alpha \geqq 0$ is an arbitrary constant.

PROOF. By symmetry, it suffices to prove the statement for $\sigma_+(x)$. Using (22) with $y = 1$, as well as (33), (14), the differential equation of the Legendre polynomials and (28)–(29) with $\mu = \nu = 1/2$ we get

$$\sum_{j=2}^{i-1}(2j-1)P'_{j-1}(x) = 2\frac{P'_{i-1}(x)P'_{i-2}(1) - P'_{i-2}(x)P'_{i-1}(1)}{(i-1)(x-1)} =$$

$$= (2-i)P'_{i-1}(x) + (1+x)P''_{i-1}(x) = O\left(\frac{i^{3/2}}{\sin^{3/2}t} + \frac{(1+x)i^{3/2}}{\sin^{5/2}t}\right)$$

$$\left(|x| < 1,\ i = 2,\dots\right).$$

Now applying Abel transform in (9) and using (32) and (30) with $\delta = 1/2$ and $\delta = 1/2 + \alpha$ we obtain

$$\sigma_+(x) = O\left(\frac{(1+x)\sin^\alpha t}{n^{2-\alpha}} + \frac{\sin t}{n^2}\right) + \frac{\mu_{n-1}\pi_n(x)\sin^2 t}{2n(n-1)}\sum_{i=2}^{n-1}(2i-1)P'_{i-1}(x)+$$

$$+O\left(\frac{|\pi_n(x)|\sin^2 t}{n^6}\right)\sum_{i=3}^{n-1} i\left|\sum_{j=2}^{i-1}(2j-1)P'_{j-1}(x)\right| =$$

$$= O\left(\frac{(1+x)\sin^\alpha t}{n^{2-\alpha}}+\frac{\sin t}{n^2}\right)+$$

$$+O\left(\frac{|\pi_n(x)|\sin^2 t}{n^6}\right)\sum_{i=3}^{n-1}\left(\frac{i^{5/2}}{\sin^{3/2} t}+\frac{(1+x)i^{5/2}}{\sin^{5/2} t}\right) =$$

$$= O\left(\frac{(1+x)\sin^\alpha t}{n^{2-\alpha}}+\frac{\sin t}{n^2}\right) \qquad \big(|x|<1\big). \quad \square$$

6. Proof of Theorem 3

Let first $f'(x) \in C[-1,1]$. By uniqueness of the polynomials $\bar{R}_n(f,x)$ we obtain the identities

$$\sum_{k=1}^n r_k(x) = 1, \qquad x = \sum_{k=1}^n x_k r_k(x) + \sigma_+(x) + \sigma_-(x),$$

whence

$$\sum_{k=1}^n (x-x_k) r_k(x) = \sigma_+(x) + \sigma_-(x).$$

Thus using the relation

$$f(x) - f(x_k) = f'(x)(x-x_k) + O\big(|x-x_k|\big)\omega\big(f',|x-x_k|\big)$$

we obtain by the second relation in Lemma 3

$$f(x) - \bar{R}_n(f,x) = \sum_{k=1}^n \big[f(x)-f(x_k)\big]\, r_k(x) - f'(-1)\sigma_-(x) - f'(1)\sigma_+(x) =$$

$$= f'(x)\sum_{k=1}^n (x-x_k) r_k(x) + O\left(\sum_{k=1}^n |x-x_k|\omega\big(f',|x-x_k|\big)\,|r_k(x)|\right) -$$

$$-f'(-1)\sigma_-(x) - f'(1)\sigma_+(x) = \big[f'(x)-f'(-1)\big]\,\sigma_-(x)+$$

$$+[f'(x)-f'(1)]\,\sigma_+(x)+O\left(\frac{\sin t}{n^2}\right)\left[\log n+\sum_{k=1}^{n}\omega\left(f',\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right]=$$

$$=O\big(\omega(f',1-x)|\sigma_+(x)|\big)+O\left(\frac{\sin t}{n^2}\right)\left[\log n+\sum_{k=1}^{n}\omega\left(f',\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right],$$

by symmetry. If $-1\leqq x\leqq 0$ then using Lemma 4 with $\alpha=0$, we have $\left|\sigma_+(x)\right|=O\left(\frac{\sin t}{n^2}\right)$ and we are done. If $0\leqq x\leqq 1$, then by

$$\omega(f',1-x)=\omega\left(f',2\sin^2\frac{t}{2}\right)\leqq\left(1+2n\tan\frac{t}{2}\right)\omega\left(f',\frac{\sin t}{n}\right)$$

we obtain, on using Lemma 4 again with $\alpha=1$ and $\alpha=0$,

$$\omega(f',1-x)\left|\sigma_+(x)\right|=O\left(\omega\left(f',\frac{\sin t}{n}\right)\right)\left(\frac{\sin t}{n}+\frac{1+\cos t+\sin t}{n}\tan\frac{t}{2}\right)=$$

$$=O\left(\frac{\sin t}{n}\omega\left(f',\frac{\sin t}{n}\right)\right),$$

and the second statement in Theorem 3 is completely proved.

In order to prove the statement concerning the operator $R_n(f,x)$, first assume that $\left|f'(x)\right|$ is bounded. Then we obtain from the second relation in Lemma 3 applied with $\omega\equiv 1$

$$\left|f(x)-R_n(f,x)\right|\leqq\sum_{k=1}^{n}\left|f(x)-f(x_k)\right|\left|r_k(x)\right|=$$

$$=O\big(\|f'\|\big)\sum_{k=1}^{n}|x-x_k|\left|r_k(x)\right|=O\big(\|f'\|\big)\frac{\sqrt{1-x^2}}{n}.$$

Thus by the first relation in Lemma 3 and by Theorem 2.3 of R. DeVore [3], the first statement of Theorem 3 for an arbitrary $f(x)\in C[-1,1]$ follows. □

7. The optimal order of convergence

Theorem 3 does not give $O(n^{-2})$ as the order of convergence for the operator $\bar{R}_n$. However, for $f(x) = x^2$ we have

$$x^2 - \bar{R}_n(x^2, x) = 2\sum_{k=1}^{n-1} \varrho_k(x),$$

and this is shown to be of order $O(n^{-2})$. (We do not go into details; see Section I.6 of [6].)

On the other hand, $O(n^{-2})$ cannot be further improved:

THEOREM 4. *We have*

$$\|f(x) - \bar{R}_n(f, x)\| = o(n^{-2})$$

if and only if $f(x)$ is a linear function.

The proof is an exact analogue of that of Theorem 2 from [6].

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SASAKIAN MANIFOLDS WITH VANISHING C-BOCHNER CURVATURE TENSOR

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§**1. Introduction.** As a complex analogue to the Weyl conformal curvature tensor, Bochner and Yano [1], [15] (see also, Tachibana [13]) introduced a Bochner curvature tensor in a Kählerian manifold. Many subjects for vanishing Bochner curvature tensors with constant scalar curvature have been studied by Ki and Kim [6], Kubo [8], Matsumoto [9], Matsumoto and Tanno [11], Yano and Ishihara [16] and so on. One of those, done by Ki and Kim, asserts the following theorem:

THEOREM A ([6]). *Let M be a Kählerian manifold with vanishing Bochner curvature tensor. Then the scalar curvature is constant if and only if* Tr $\mathrm{Ric}^{(m)}$ *is constant for a positive integer* $m(\geqq 2)$.

In a Sasakian manifold, a C-Bochner curvature tensor is constructed from the Bochner curvature tensor in a Kählerian manifold by the fibering of Boothby–Wang. Recently, the Sasakian manifold with vanishing C-Bochner curvature tensor and the constant scalar curvature is studied, and in [12], the following theorem was proved:

THEOREM B. *Let M^n $(n \geqq 5)$ be a Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes. If the Ricci tensor is positive semi-definite, then M is a space of constant ϕ-holomorphic sectional curvature.*

Also, when M is compact, the following theorems were proved:

THEOREM C ([4]). *Let M^n $(n \geqq 5)$ be a compact Sasakian manifold with vanishing C-Bochner curvature tensor. If the length of the Ricci tensor is constant and the length of the η-Einstein tensor is less than $\frac{\sqrt{2}(R-n+1)}{\sqrt{(n-1)(n-3)}}$, then M is a space of constant ϕ-holomorphic sectional curvature.*

THEOREM D ([10]). *Let M^n $(n \geqq 5)$ be a compact Sasakian manifold with vanishing C-Bochner curvature tensor and constant scalar curvature. If the smallest Ricci curvature is greater than -2, then M is a space of constant ϕ-holomorphic sectional curvature.*

We shall prove Theorem A as a Sasakian analogue in §3. Moreover in §4 we shall discuss when the smallest Ricci curvature is greater than or equal to

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-2 in a Sasakian manifold with vanishing C-Bochner curvature tensor and Tr $\mathrm{Ric}^{(m)}$ is constant for a positive integer m.

§2. Preliminaries. Let M be an n-dimensional Riemannian manifold. Throughout this paper, we assume that manifolds are connected and of class C^∞. Denote by g_{ji}, $R_{kji}{}^h$, $R_{ji} = R_{rji}{}^r$ and R the metric tensor, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively, in terms of local coordinates $\{x^h\}$, where Latin indices run over the range $\{1, 2, \ldots, n\}$.

An $n(= 2l + 1)$-dimensional Riemannian manifold is called a Sasakian manifold if there exists a unit Killing vector field ξ^h satisfying

$$\tag{2.1} \begin{cases} \eta_i = g_{ir}\xi^r, \quad \phi_{ji} = \nabla_j\eta_i, \quad \phi_{ji} + \phi_{ij} = 0, \quad \phi_r{}^h\xi^r = 0, \quad \phi_j{}^r\eta_r = 0, \\ \phi_i{}^r\phi_r{}^h = -\delta_i{}^h + \eta_i\xi^h, \qquad \nabla_k\phi_{ji} = -g_{kj}\eta_i + g_{ki}\eta_j, \end{cases}$$

where ∇ denotes the operator of the Riemannian covariant derivative.

It is well known that in a Sasakian manifold the following equations hold:

$$\tag{2.2} R_{jr}\xi^r = (n-1)\eta_j,$$

$$\tag{2.3} H_{ji} + H_{ij} = 0,$$

$$\tag{2.4} R_{ji} = R_{rs}\phi_j{}^r\phi_i{}^s + (n-1)\eta_j\eta_i,$$

$$\tag{2.5} \nabla_k R_{ji} - \nabla_j R_{ki} = (\nabla_t R_{kr})\phi_j{}^r\phi_i{}^t - \\ -\eta_j\{H_{ki} - (n-1)\phi_{ki}\} - 2\eta_i\{H_{kj} - (n-1)\phi_{kj}\},$$

$$\tag{2.6} \nabla_k R_{ji} - (\nabla_k R_{rs})\phi_j{}^r\phi_i{}^s = \\ = -\eta_i\{H_{kj} - (n-1)\phi_{kj}\} - \eta_j\{H_{ki} - (n-1)\phi_{ki}\},$$

$$\tag{2.7} \xi^r\nabla_r R_{kji}{}^h = 0,$$

where we put $H_{ji} = \phi_j{}^r R_{ri}$.

We denote a tensor field $\mathrm{Ric}^{(m)}$ with components $R_{ji}{}^{(m)}$ and a function $R_{(m)}$ as follows:

$$R_{ji}{}^{(m)} = R_{ji_1}R_{i_2}{}^{i_1}\cdots R_i^{i_{m-1}}, \qquad R_{(m)} = \text{ Tr Ric}^{(m)} = g^{ji}R_{ji}{}^{(m)}.$$

Then, from (2.2) and (2.3), we get

$$\tag{2.8} R_{jr}{}^{(m)}\xi^r = (n-1)^m\eta_j,$$

$$\tag{2.9} R_{jr}{}^{(m)}\phi_i{}^r + R_{ir}{}^{(m)}\phi_j{}^r = 0.$$

Also, we define the η-Einstein tensor T_{ji} by

$$(2.10)\qquad T_{ji} = R_{ji} - \left(\frac{R}{n-1} - 1\right) g_{ji} + \left(\frac{R}{n-1} - n\right) \eta_j \eta_i.$$

If the η-Einstein tensor vanishes, then M is called an η-Einstein manifold. From (2.2) and (2.3), we have

$$(2.11)\qquad \operatorname{Tr} T = 0,$$

$$(2.12)\qquad T_{jr}\xi^r = 0,$$

$$(2.13)\qquad T_{jr}\phi_i{}^r + T_{ir}\phi_j{}^r = 0.$$

A Sasakian manifold M is called a space of constant ϕ-holomorphic sectional curvature c if the curvature tensor of M has the form

$$R_{kji}{}^h = \frac{c+3}{4}(g_{ji}\delta_k{}^h - g_{ki}\delta_j{}^h) +$$

$$+\frac{c-1}{4}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k{}^h - \phi_{ki}\phi_j{}^h + \phi_{ji}\phi_k{}^h - 2\phi_{kj}\phi_i{}^h).$$

Matsumoto and Chūman [10] introduced the C-Bochner curvature tensor $B_{kji}{}^h$ defined by

$$(2.14)\; B_{kji}{}^h = R_{kji}{}^h + \frac{1}{n+3}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + g_{ki}R_j{}^h - g_{ji}R_k{}^h + H_{ki}\phi_j{}^h -$$

$$-H_{ji}\phi_k{}^h + \phi_{ki}H_j{}^h - \phi_{ji}H_k{}^h + 2H_{kj}\phi_i{}^h + 2\phi_{kj}H_i{}^h -$$

$$-R_{ki}\eta_j\xi^h + R_{ji}\eta_k\xi^h - \eta_k\eta_iR_j{}^h + \eta_j\eta_iR_k{}^h) -$$

$$-\frac{k+n-1}{n+3}(\phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h) - \frac{k-4}{n+3}(g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h) +$$

$$+\frac{k}{n+3}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k{}^h),$$

where $k = \frac{R+n-1}{n+1}$. It is well-known that if a Sasakian manifold with vanishing C-Bochner curvature tensor is an η-Einstein manifold, then it is a space of constant ϕ-holomorphic sectional curvature.

§3. A Sasakian manifold with vanishing C-Bochner curvature tensor. Let M^n ($n \geqq 5$) be a Sasakain manifold with vanishing C-Bochner curvature tensor. By a straightforward computation, we can prove

$$(3.1)\qquad \frac{n+3}{n-1}\nabla_r B_{kji}{}^r = \nabla_k R_{ji} - \nabla_j R_{ki} - \eta_k\{H_{ji} - (n-1)\phi_{ji}\} +$$

$$+\eta_j\{H_{ki}-(n-1)\phi_{ki}\}+2\eta_i\{H_{kj}-(n-1)\phi_{kj}\}+$$

$$+\frac{1}{2(n+1)}\{(g_{ki}-\eta_k\eta_i)\delta_j{}^r-(g_{ji}-\eta_j\eta_i)\delta_k{}^r+$$

$$+\phi_{ki}\phi_j{}^r-\phi_{ji}\phi_k{}^r+2\phi_{kj}\phi_i{}^r\}R_r,$$

where we put $R_j=\nabla_j R$.

By virtue of (2.1), (2.2), (2.5)–(2.7) and (3.1), we obtain

$$\nabla_k R_{ji}=\{R_{kr}-(n-1)g_{kr}\}(\phi_j{}^r\eta_i+\phi_i{}^r\eta_j)+ \tag{3.2}$$

$$+\frac{1}{2(n+1)}\{2R_k(g_{ji}-\eta_j\eta_i)+R_j(g_{ki}-\eta_k\eta_i)+$$

$$+R_i(g_{kj}-\eta_k\eta_j)-\phi_{kj}\phi_i{}^rR_r-\phi_{ki}\phi_j{}^rR_r\}$$

and consequently from (2.7), we find

$$(n+1)(\nabla_k R_{ji})R^jR^i=2\lambda^2R_k, \tag{3.3}$$

where we put $\lambda^2=R_rR^r$.

The following lemma is needed for later use.

LEMMA 3.1. *Let M^n ($n\geqq 5$) be a Sasakian manifold with vanishing C-Bochner curvature tensor. Then $R_{jr}{}^{(m)}R^r=0$ holds for a positive integer m if and only if the scalar curvature R is constant.*

PROOF. If $R_{jr}{}^{(m)}R^r=0$ holds, then we get $R_{jr}{}^{(2m-2)}R^r=0$ which implies that $\left|R_{jr}{}^{(m-1)}R^r\right|^2=0$. Accordingly, we obtain $R_{jr}{}^{(m-1)}R^r=0$. By the inductive method, we get $R_{jr}R^r=0$. Operating ∇_k to this, we find $(\nabla_k R_{jr})R^jR^r=0$. By means of (3.3), we see that the scalar curvature R is constant. The converse is trivial.

For the sake of brevity, we shall define a function $\alpha(m)$ as follows:

$$\alpha(m)=R_{ji}{}^{(m)}R^jR^i.$$

Then, it is clear from (3.2) that

$$2(n+1)(\nabla_k R_{ji})R^j\left(R^{ir(m)}R_r\right)=\lambda^2R_{kr}{}^{(m)}R^r+3\alpha(m)R_k, \tag{3.4}$$

$$2(n+1)(\nabla_k R_{ji})\left(R^{jr(\ell)}R_r\right)\left(R^{is(m)}R_s\right)= \tag{3.5}$$

$$=\alpha(\ell)R_{kr}{}^{(m)}R^r+\alpha(m)R_{kr}{}^{(\ell)}R^r+2\alpha(\ell+m)R_k,$$

where we have used (2.7), (2.8) and (2.9).

Operating $R^{ji(m)}$ to (3.2) and owing to (2.1), (2.7), (2.8) and (2.9), we find

$$(3.6)\quad (n+1)\nabla_k R_{(m+1)} = (m+1)\left[2R_{kr}{}^{(m)}R^r + \left\{R_{(m)} - (n-1)^m\right\}R_k\right].$$

Therefore, if the scalar curvature R is constant, then $R_{(m)}$ is constant for any integer $m(\geqq 2)$.

Now, we shall prove that the scalar curvature R is constant if $R_{(m)}$ is constant for any fixed integer $m(\geqq 2)$.

At first, suppose that $R_{(2\ell+3)}$ $(\ell = 0,1,2,\ldots)$ is constant. Then, from (3.6), we can get

$$2R_{kr}{}^{(2\ell+2)}R^r + \left\{R_{(2\ell+2)} - (n-1)^{2\ell+2}\right\}R_k = 0,$$

which yields that $2\alpha(2\ell+2) + \lambda^2\{R_{(2\ell+2)} - (n-1)^{2\ell+2}\} = 0$, that is,

$$2\left|R_{jr}{}^{(\ell+1)}R^r\right|^2 + \lambda^2\left|R_{ji}{}^{(\ell+1)} - (n-1)^{\ell+1}\eta_j\eta_i\right|^2 = 0.$$

Thus, from Lemma 3.1, the scalar curvature R is constant.

In the next place, we shall consider when $R_{(2\ell+2)}$ $(\ell = 0,1,2,\ldots)$ is constant. From (3.6), we have

$$(3.7)\quad 2R_{jr}{}^{(2\ell+1)}R^r + \left\{R_{(2\ell+1)} - (n-1)^{2\ell+1}\right\}R_j = 0.$$

Operating ∇_k to this and owing to (3.7), we get

$$(3.8)\quad 2\left(\nabla_k R_{jr}{}^{(2\ell+1)}\right)R^jR^r + \lambda^2\nabla_k R_{(2\ell+1)} = 0.$$

From (3.3) and (3.8), we find the scalar curvature R is constant if $\ell = 0$. Because of (3.4), (3.5) and (3.6), equation (3.8) is rewritten as follows:

$$(3.9)\quad 4(\ell+1)\lambda^2R_{kr}{}^{(2\ell)}R^r + 2\sum_{i=1}^{2\ell-1}\alpha(i)R_{kr}{}^{(2\ell-i)}R^r +$$

$$+4(\ell+1)\alpha(2\ell)R_k + (2\ell+1)\lambda^2\left|R_{ji}{}^{(\ell)} - (n-1)^\ell\eta_j\eta_i\right|^2R_k = 0.$$

By virtue of (3.9) and Lemma 3.1, it is clear that the scalar curvature R is constant if $\ell = 1$.

On the other hand, we have

$$(3.10)\qquad \lambda^6\alpha(2\ell)+2\lambda^4\alpha(s)\alpha(2\ell-s)+\lambda^4\alpha(2s)\alpha(2\ell-2s)=$$

$$=\lambda^2\big|\lambda^2 R_{jr}{}^{(\ell)}R^r+\alpha(s)R_{jr}{}^{(\ell-s)}R^r\big|^2+\alpha(2\ell-2s)\big|\lambda^2 R_{jr}{}^{(s)}R^r-\alpha(s)R_j\big|^2.$$

Because of (3.9) and (3.10), it is easy to see that the following equations hold: if $\ell=2,6,10,\ldots$,

$$(7\ell+8)\lambda^6\alpha(2\ell)+(2\ell+1)\lambda^8\big|R_{ji}{}^{(\ell)}-(n-1)^\ell\eta_j\eta_i\big|^2+$$

$$+4\lambda^4\sum_{i=1}^{(\ell-2)/4}\alpha(4i)\alpha(2\ell-4i)+$$

$$+2\lambda^2\sum_{i=1}^{\ell/2}\big|\lambda^2 R_{js}{}^{(\ell)}R^s+\alpha(2i-1)R_{js}{}^{(\ell-2i+1)}R^s\big|^2+$$

$$+2\sum_{i=1}^{\ell/2}\alpha(2\ell-4i+2)\big|\lambda^2 R_{js}{}^{(2i-1)}R^s-\alpha(2i-1)R_j\big|^2=0,$$

if $\ell=4,8,12,\ldots$,

$$(7\ell+8)\lambda^6\alpha(2\ell)+(2\ell+1)\lambda^8\big|R_{ji}{}^{(\ell)}-(n-1)^\ell\eta_j\eta_i\big|^2+$$

$$+4\lambda^4\sum_{i=1}^{(\ell-4)/4}\alpha(4i)\alpha(2\ell-4i)+2\lambda^4\alpha(\ell)^2+$$

$$+2\lambda^2\sum_{i=1}^{\ell/2}\big|\lambda^2 R_{js}{}^{(\ell)}R^s+\alpha(2i-1)R_{js}{}^{(\ell-2i+1)}R^s\big|^2+$$

$$+2\sum_{i=1}^{\ell/2}\alpha(2\ell-4i+2)\big|\lambda^2 R_{js}{}^{(2i-1)}R^s-\alpha(2i-1)R_j\big|^2=0$$

and if $\ell=3,5,7,\ldots$,

$$(7\ell+9)\lambda^6\alpha(2\ell)+(2\ell+1)\lambda^8\big|R_{ji}{}^{(\ell)}-(n-1)^\ell\eta_j\eta_i\big|^2+$$

$$+2\lambda^4\sum_{i=1}^{(\ell-1)/2}\alpha(2i)\alpha(2\ell-2i)+2\lambda^4\alpha(\ell)^2+$$

$$+2\lambda^2 \sum_{i=1}^{(\ell-1)/2} \left| \lambda^2 R_{js}{}^{(\ell)} R^s + \alpha(2i-1) R_{js}{}^{(\ell-2i+1)} R^s \right|^2 +$$

$$+2 \sum_{i=1}^{(\ell-1)/2} \alpha(2\ell-4i+2) \left| \lambda^2 R_{js}{}^{(2i-1)} R^s - \alpha(2i-1) R_j \right|^2 = 0.$$

Thus we find from Lemma 3.1 that the scalar curvature R is constant if $R_{(2\ell+2)}$ ($\ell = 2, 3, 4, \ldots$) is constant. Hence, we have

THEOREM 3.2. *Let M^n $(n \geqq 5)$ be a Sasakian manifold with vanishing C-Bochner curvature tensor. Then the scalar curvature R is constant if and only if* Tr $\mathrm{Ric}^{(m)}$ *is constant for an integer $m (\geqq 2)$.*

REMARK. In the proof of Theorem 3.2, we use only equation (3.1). Thus Theorem 3.2 is valid for the parallel C-Bochner curvature tensor.

Also, we have from Theorems B and 3.2

THEOREM 3.3. *Let M^n $(n \geqq 5)$ be a Sasakian manifold whose C-Bochner curvature tensor vanishes. If the Ricci tensor is positive semi-definite and* Tr $\mathrm{Ric}^{(m)}$ *is constant for a positive integer m, then M is a space of constant ϕ-holomorphic sectional curvature.*

Furthermore, it is easy to see from the proof of Theorem C and Theorem 3.2 that the following theorem holds:

THEOREM 3.4. *Let M^n $(n \geqq 5)$ be a Sasakian manifold with vanishing C-Bochner curvature tensor. If* Tr $\mathrm{Ric}^{(m)}$ *is constant for a positive integer m and the length of the η-Einstein tensor is less than $\frac{\sqrt{2}(R-n+1)}{\sqrt{(n-1)(n-3)}}$, then M is a space of constant ϕ-holomorphic sectional curvature.*

§4. The smallest Ricci curvature. Let M be an $n(\geqq 5)$-dimensional Sasakian manifold with vanishing C-Bochner curvature tensor. Suppose that $R_{(m)}$ is constant for any positive integer m. By Theorem 3.2, equation (3.2) is reduced to

$$\nabla_k R_{ji} = \left\{ R_{kr} - (n-1) g_{kr} \right\} (\phi_j{}^r \eta_i + \phi_i{}^r \eta_j), \tag{4.1}$$

which implies $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$, namely, the Ricci tensor is cyclic parallel. Therefore, using the Ricci formula, we find

$$\nabla^k \nabla_k R_{ji} = 2\left(R_{rjis} R^{rs} - R_{ji}{}^{(2)} \right).$$

Applying ∇^k to (4.1) and owing to (2.1) and (2.2), we get

$$\nabla^k \nabla_k R_{ji} = -2 \left[R_{ji} - (n-1) g_{ji} - \left\{ R - n(n-1) \right\} \eta_j \eta_i \right].$$

On the other hand, by virtue of (2.1)–(2.4) and (2.14), it is clear that the following equation holds:

$$(n+3)R_{rjis}R^{rs} =$$

$$= 4R_{ji}{}^{(2)} - (4n - R + 2k)R_{ji} + \left\{ R_{(2)} - (k-4)R + (n-1)k \right\} g_{ji} -$$

$$-\left\{ R_{(2)} + (n-1)^2 - (n-1)k - kR \right\} \eta_j \eta_i.$$

From the last three equations, we have

$$R_{ji}{}^{(2)} = \beta R_{ji} + \gamma g_{ji} + \left\{ (n-1)^2 - (n-1)\beta - \gamma \right\} \eta_j \eta_i, \tag{4.2}$$

where the constants β and γ are given by

$$(n+1)\beta = R - 3n - 5, \tag{4.3}$$

$$(n-1)\gamma = R_{(2)} - \frac{1}{n+1}R^2 + 4R - \frac{n-1}{n+1}(n^2+3n+4). \tag{4.4}$$

Thus, equation (4.2) tells us that M has at most three constant Ricci curvatures $n-1$, x_1 and x_2, where we have put

$$x_1 = \frac{1}{2}\left(\beta - \sqrt{D}\right), \qquad x_2 = \frac{1}{2}\left(\beta + \sqrt{D}\right), \qquad D = \beta^2 + 4\gamma (\geqq 0), \tag{4.5}$$

moreover, denote by s and $n-1-s$ the multiplicities of x_1 and x_2, respectively. Therefore we have (cf. [7])

LEMMA 4.1. *Let M^n $(n \geqq 5)$ be a Sasakian manifold with vanishing C-Bochner curvature tensor such that* Tr $\mathrm{Ric}^{(m)}$ *is constant for a positive integer m. Then M has at most three constant Ricci curvatures.*

Now, we shall prove the following theorem.

THEOREM 4.2. *Let M^n $(n \geqq 5)$ be a Sasakian manifold with vanishing C-Bochner curvature tensor such that* Tr $\mathrm{Ric}^{(m)}$ *is constant for a positive integer m. If the smallest Ricci curvature is greater than or equal to -2, then M is a space of constant ϕ-holomorphic sectional curvature -3.*

PROOF. By means of (4.3), (4.5) and Lemma 4.1, we find

$$R + n - 1 = \frac{n+1}{n+3}(n-1-2s)\sqrt{D}. \tag{4.6}$$

Because of (4.3), (4.4) and (4.6), we have

$$\frac{n-1}{4}\left\{1-\left(\frac{n-1-2s}{n+3}\right)^2\right\}D =$$

$$= R_{(2)} - \frac{1}{n+1}\{R^2 - 2(n+3)R + (n-1)^2(n+2)\},$$

which yields that

$$(n+1)R_{(2)} \geqq R^2 - 2(n+3)R + (n-1)^2(n+2). \tag{4.7}$$

Let x_1 be the smallest Ricci curvature. Then, by virtue of (4.5), we obtain $\gamma \leqq 2\beta + 4$ which means from (4.4) that

$$(n+1)R_{(2)} \leqq R^2 - 2(n+3)R + (n-1)^2(n+2).$$

Combining this with (4.7), we get that D vanishes identically, which implies that equation (4.6) gives $R = -n+1$. We find $|R_{ji} + 2g_{ji} - (n+1)\eta_j\eta_i|^2 = 0$ which yields that M is an η-Einstein manifold. Thus, it is easy to see from (2.14) that M is of constant ϕ-holomorphic sectional curvature -3.

REMARK. In [10], this theorem was proved under the condition that M is compact.

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ON DERIVATIONS AND COMMUTATIVITY IN PRIME RINGS

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We have shown in [4] that if R is a semiprime ring admitting a derivation d, and if K is a two-sided ideal such that either $xy + d(xy) = yx + d(yx)$ for all $x, y \in K$, or $xy - d(xy) = yx - d(yx)$ for all $x, y \in K$, then K is a central ideal of R. More recently we have proved that if R is a semiprime ring admitting a derivation d such that $xy - d(x)d(y) = yx - d(y)d(x)$ for all x, y in some nonzero right ideal U, then U must be central [3]. Of course, in the event that R is prime, any of the conditions mentioned implies that R is commutative.

In this paper we study conditions which are in some sense related to all the conditions above. Suppose that R is a prime ring having a nonzero right ideal U. If d is a derivation on R such that $d(x)d(y) + d(xy) = d(y)d(x) + d(yx)$ for all $x, y \in U$, we say that d is a $U-^{*}$ derivation; and if $d(x)d(y) + d(yx) = d(y)d(x) + d(xy)$ for all $x, y \in U$, we call d a $U-^{**}$ derivation. We prove that if d is a nonzero $U-^{*}$ or $U-^{**}$ derivation, then either R is commutative or $d^2(U) = \{0\} = d(U)d(U)$. This result yields as a corollary an earlier result of Bell and Kappe [2]; and it facilitates the study of derivations d such that $d(xy) = d(yx)$ for all $x, y \in U$ — a study which constitutes the final section of the paper.

1. Some preliminaries

Throughout the paper, we make extensive use of the basic commutator identities $[x, yz] = y[x, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]y$. Moreover, we shall require the following known results.

(A) [1, Theorem 4] *Let R be a prime ring and U a nonzero right ideal. If R admits a nonzero derivation d such that $[x, d(x)]$ is central for all $x \in U$, then R is commutative.*

(B) (Cf. [6. Lemma 1]) *Let R be a prime ring and U a nonzero two-sided ideal. If d is a nonzero derivation on R, and if $a \in R$ is such that $d(U)a = \{0\}$ or $ad(U) = \{0\}$, then $a = 0$.*

(C) [1, Lemma 3] *Let U be a nonzero left ideal of a prime ring R. If d is*

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a nonzero derivation of R, then d is nonzero on U.

(D) [2, Lemma 2(a)] *Let U be a subring of a ring R, and let d be a derivation of R such that $d(xy) = d(x)d(y)$ for all $x, y \in U$. Then $d(x)x(y - d(y)) = 0$ for all $x, y \in U$.*

(E) [4, Lemma 1] *Let R be a semiprime ring and I a nonzero ideal of R. Let $[I, I] = \{[x, y] \mid x, y \in I\}$. If $z \in R$ and z centralizes $[I, I]$, then z centralizes I.*

(F) *If R is a prime ring, the centralizer of any one-sided ideal is equal to the center of R.*

2. Results on $U-^{*}$ and $U-^{**}$ derivations

THEOREM 1. *Let R be a prime ring and U a nonzero right ideal. If R admits a nonzero $U-^{*}$ derivation d, then either R is commutative or $d^2(U) = d(U)d(U) = \{0\}$.*

PROOF. Since d is a $U-^{*}$ derivation, we have

$$(1) \qquad [d(x), d(y)] = [d(y), x] + [y, d(x)] \quad \text{for all} \quad x, y \in U.$$

Substituting xy for y, we get

$$(2) \qquad d(x)[y, x] = [d(x), x]\, d(y) + d(x)[d(x), y] \quad \text{for all} \quad x, y \in U.$$

Replacing y by yx and using (2), we have

$$(3) \qquad [d(x), x]\, yd(x) + d(x)y[d(x), x] = 0 \quad \text{for all} \quad x, y \in U.$$

In (2) we substitute $yd(x)$ for y, since U is a right ideal, to get

$$(4) \qquad d(x)y[d(x), x] - [d(x), x]\, y\, d^2(x) = 0 \quad \text{for all} \quad x, y \in U.$$

From (3) and (4) we obtain

$$(5) \qquad [d(x), x]\, y\big(d(x) + d^2(x)\big) = 0 \quad \text{for all} \quad x, y \in U.$$

Thus, (5) yields

$$(6) \qquad [d(x), x]\, UR\big(d(x) + d^2(x)\big) = \{0\} \quad \text{for all} \quad x, y \in U.$$

But R is prime, hence for each $x \in U$, we have either $[d(x), x]\, U = \{0\}$ or $d(x) + d^2(x) = 0$. If $[d(x), x]\, U = \{0\}$, then (4) shows that $d(x)y[d(x), x] =$

$= 0$ for all $y \in U$, so that $d(x)UR\big[d(x),x\big] = \{0\}$. Therefore, either $d(x)U = \{0\}$ or $\big[d(x),x\big] = 0$.

On the other hand, suppose $d(x) + d^2(x) = 0$. In (1), put $y = yd(x)$ to get

$$(7)\quad y\big[d(x),d^2(x)\big] + \big[d(x),y\big]d^2(x) = d(y)\big[d(x),x\big] + y\big[d^2(x),x\big] + [y,x]d^2(x) \text{ for all } y \in U.$$

But $d(x) = -d^2(x)$, hence (7) implies

$$(8)\quad d(y)\big[d(x),x\big] - [y,x]d(x) + \big[d(x),y\big]d(x) = y\big[d(x),x\big] \text{ for all } y \in U.$$

If in (1) we put $y = yx$, we get

$$(9)\qquad [y,x]d(x) = \big[d(x),y\big]d(x) + d(y)\big[d(x),x\big] \quad \text{for all} \quad x,y \in U.$$

Thus substituting from (9) in (8), we get $y\big[d(x),x\big] = 0$ for all $y \in U$, that is

$$(10)\qquad U\big[d(x),x\big] = \{0\}.$$

But U is a right ideal, hence $\big[d(x),x\big] = 0$. Thus, in any event, for each $x \in U$, either $\big[d(x),x\big] = 0$ or $d(x)U = \{0\}$.

Suppose that $\big[d(x),x\big] = 0$. Then by (2), we have

$$(11)\qquad d(x)[y,x] = d(x)\big[d(x),y\big] \quad \text{for all} \quad y \in U.$$

Replacing y by yz in (11) and using (11), we get $d(x)y[z,x] = d(x)y\big[d(x),z\big]$ for all $y \in U$, $z \in R$; i.e., $d(x)y\big[z,x+d(x)\big] = 0$ for all $y \in U$, $z \in R$. Thus, $d(x)yR\big[z,x+d(x)\big] = \{0\}$ for all $y \in U$, $z \in R$; hence we have either $d(x)U = \{0\}$ or $x + d(x) \in Z$, when Z denotes the center of R. The sets of x for which these conditions hold are additive subgroups of U with union equal to U; hence either $d(U)U = \{0\}$ or $x + d(x) \in Z$ for all $x \in U$. In the latter case, R is commutative by (A); therefore we assume henceforth that $d(U)U = \{0\}$.

Under this assumption, the condition that $\big[d(x),d(yz)\big] = \big[d(yz),x\big] + \big[yz,d(x)\big]$ for all $x,y,z \in U$ becomes $\big[d(x),yd(z)\big] = \big[yd(z),x\big] + \big[yz,d(x)\big]$, or

$$y\big[d(x),d(z)\big] + \big[d(x),y\big]d(z) = y\big[d(z),x\big] + [y,x]d(z) + y\big[z,d(x)\big] + \big[y,d(x)\big]z.$$

Using (1) to eliminate the terms with first factor y, and noting that the last summand on the right is 0, we get

$$(12) \qquad yd(x)d(z) = [x, y]d(z) \quad \text{for all} \quad x, y, z \in U;$$

hence,

$$(13) \qquad yd(z)d(x) = [z, y]d(x) \quad \text{for all} \quad x, y, z \in U.$$

Thus (12) and (13) give $y\big[d(x), d(z)\big] = [x, y]d(z) - [z, y]d(x)$ for all $x, y, z \in$ $\in U$. Using (1), we reduce this to

$$(14) \qquad xy\,d(z) - zy\,d(x) = 0 \quad \text{for all} \quad x, y, z \in U.$$

Replacing x by xt in (14) and using (14) itself, we obtain

$$(15) \qquad [x, zy]\,d(t) = 0 \quad \text{for all} \quad x, y, z, t \in U.$$

From (12), we have $[x, zy]d(t) = zy\,d(x)\,d(t)$. Substituting in (15) we get

$$(16) \qquad zy\,d(x)d(t) = 0 \quad \text{for all} \quad x, y, z, t \in U.$$

Since $zyRd(x)\,d(t) = \{0\}$ for all $x, y, z, t \in U$ and since $U^2 \neq \{0\}$, we conclude that $d(x)d(t) = 0$ for all $x, t \in U$, which is the desired conclusion that $d(U)d(U) = \{0\}$. In particular,

$$(17) \qquad \big[d(x), d(t)\big] = 0 \quad \text{for all} \quad x, t \in U.$$

From (1), (17), and $d(U)U = \{0\}$, we now get

$$(18) \qquad yd(x) = xd(y) \quad \text{for all} \quad x, y \in U.$$

Replacing y by yr for arbitrary $r \in R$, we get $xyd(r) = yrd(x) - xd(y)r$; and substituting $yd(x)$ for $xd(y)$ now yields

$$(19) \qquad xyd(r) = y\big[r, d(x)\big] \quad \text{for all} \quad x, y \in U,\ r \in R.$$

For r we substitute $d(z)$, $z \in U$, obtaining

$$xyd^2(z) = y\big[d(z), d(x)\big] \quad \text{for all} \quad x, y, z \in U;$$

and using (17), we get

$$xyd^2(z) = 0 \quad \text{for all} \quad x, y, z \in U.$$

Since $U^2 \neq \{0\}$, we conclude that $d^2(U) = \{0\}$; and our theorem is proved.

Using similar arguments, we get

THEOREM 2. *Let R be a prime ring and U a nonzero right ideal. If R admits a nonzero $U-^{**}$ derivation d, then either R is commutative or $d^2(U) = d(U)d(U) = \{0\}$.*

From Theorems 1 and 2 we can get the following corollaries.

COROLLARY 1. *Let R be a prime ring and U a nonzero right ideal of R. If R admits a nonzero $U-^{*}$ or $U-^{**}$ derivation d with $d^2(U) \neq \{0\}$, then R is commutative.*

COROLLARY 2. *Let R be a prime ring and U a nonzero two-sided ideal. If R admits a nonzero $U-^{*}$ or $U-^{**}$ derivation d, then R is commutative.*

This corollary follows from our theorems and (B).

The next corollary is a result of Bell and Kappe, who say a derivation acts as a homomorphism on U (resp. an anti-homomorphism on U) if $d(xy) = d(x)d(y)$ for all $x, y \in U$ (resp. $d(xy) = d(y)d(x)$ for all $x, y \in U$).

COROLLARY 3. [2. Theorem 3]. *Let R be a prime ring and U a nonzero right ideal. If d is a derivation which acts as an anti-homomorphism or a homomorphism on U, then $d = 0$.*

PROOF. Whether we assume that d acts as a homomorphism or as an anti-homomorphism, the condition that $d(U)d(U) = \{0\}$ shows that $d(U^2) = \{0\}$; and by (C), we have $d = 0$. Thus, by Theorems 1 and 2 we may assume that R is commutative, hence is a domain, and that d acts as a homomorphism on U. If we assume $d \neq 0$, it follows from (D) that $d(y) = y$ for all $y \in U$; therefore, if $u \in U \setminus \{0\}$ and $r \in R$, we have $ur = ud(r) + d(u)r$ and hence $ud(r) = 0$. But this contradicts (B), so in fact $d = 0$.

We conclude this section with an example showing that the non-commutative case in Theorems 1 and 2 actually does occur.

EXAMPLE. Let R be the ring of 2×2 matrices over a field F; let $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in F \right\}$. Let d be the inner derivation given by $d(x) = x \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$ for all $x \in R$. It is readily verified that d is a $U-^{*}$ and $U-^{**}$ derivation.

3. Derivations with $d(xy) = d(yx)$

Long ago Herstein [5] proved that if R is a prime ring of characteristic not 2 which admits a nonzero derivation such that $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, then R is commutative. In view of this result, it seems appropriate to study derivations such that $d(xy) = d(yx)$ for all x, y in some distinguished subset of R. To our surprise, the results and methods of the previous section are applicable in such a study.

THEOREM 3. *Let R be a prime ring and U a nonzero two-sided ideal of R. If R admits a nonzero derivation d such that $d(xy) = d(yx)$ for all $x, y \in U$, then R is commutative.*

PROOF. Let $c \in U$ be a constant — i.e. an element such that $d(c) = 0$; and let z be an arbitrary element of U. The condition that $d(cz) = d(zc)$ yields $cd(z) = d(z)c$. Now for each $x, y \in U$, $[x, y]$ is a constant; hence

$$d(z)[x,y] = [x,y]d(z) \quad \text{for all} \quad x, y, z \in U. \tag{20}$$

By (E) and (F), $d(z)$ is central for all $z \in U$; hence d is a $U-^*$ derivation and R is therefore commutative by Corollary 2.

The example in the previous section shows that in Theorem 3, U cannot be replaced by a one-sided ideal. However, we do have the following extension of Theorem 3.

THEOREM 4. *Let R be a prime ring of characteristic different from 2, and let U be a nonzero right ideal. If d is a nonzero derivation such that $d(xy) = d(yx)$ for all $x, y \in U$, then either R is commutative, or $d^2(U) = \{0\} = d(U)d(U)$.*

PROOF. Writing $d(xy) = d(yx)$ in the form $[x, d(y)] = [y, d(x)]$ and replacing x by x^2, we get

$$[y,x]d(x) + d(x)[y,x] = 0 \quad \text{for all} \quad x, y \in U.$$

Recalling (20) and using the fact that char $R \neq 2$, we have

$$[y,x]d(x) = 0 \quad \text{and} \quad d(x)[y,x] = 0 \quad \text{for all} \quad x, y \in U. \tag{21}$$

In the first of these equalities replace y by yw, $w \in U$, thereby obtaining

$$[y,x]Ud(x) = \{0\} = [y,x]URd(x) \quad \text{for all} \quad x, y \in U.$$

Since $d \neq 0$, we can conclude from the usual additive-group argument that

$$[y,x]U = \{0\} \quad \text{for all} \quad x, y \in U. \tag{22}$$

On the other hand, the second equality of (21) yields $d(x)U[y,x] = \{0\} = d(x)UR[y,x]$ for all $x, y \in U$; thus,

$$\text{for each} \quad x \in U, \quad \text{either } x \text{ is central or} \quad d(x)U = \{0\}. \tag{23}$$

Assume that R is not commutative, and hence that U is not central. By (22) and (23) we have $[y,x]U = \{0\}$ for all $x, y \in U$ and $d(U)U = \{0\}$.

These conditions, together with the $d(xy)=d(yx)$ condition, yield

$$yd(x)=xd(y) \quad \text{for all} \quad x,y\in U.$$

But this is just (18); and as in the proof of Theorem 1, we have

$$xyd^2(z)=y\big[d(z),d(x)\big] \quad \text{for all} \quad x,y,z\in U. \tag{24}$$

Now by applying d to the condition $zd(x)=xd(z)$, we obtain $zd^2(x)+ + d(z)d(x)=xd^2(z)+d(x)d(z)$; hence $zd^2(x)+\big[d(z),d(x)\big]=xd^2(z)$ and

$$y\big[d(z),d(x)\big]=yxd^2(z)-yzd^2(x). \tag{25}$$

Substituting in (24) now yields

$$yzd^2(x)=[y,x]d^2(z) \quad \text{for all} \quad x,y,z\in U. \tag{26}$$

Since $[y,x]$ is constant, applying d to (22) shows that $[y,x]d(U)=\{0\}= =[y,x]d^2(U)$ for all $x,y\in U$; and (26) yields $U^2d^2(U)=\{0\}$. Since $U^2\neq\{0\}$ and R is prime, we conclude that $d^2(U)=\{0\}$. Finally, since $\operatorname{char} R\neq 2$, using the fact that $d^2(xy)=0$ for all $x,y\in U$ gives $d(U)d(U)=\{0\}$.

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ON MODULI OF CONTINUITY FOR A TWO-PARAMETER ORNSTEIN–UHLENBECK PROCESS*

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1. Introduction and conclusions

Given $\sigma > 0$ and an n-dimensional vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i > 0$, $i = 1, \ldots, n$, define the n-parameter Ornstein–Uhlenbeck process (OUP_n) $\{X(t), t \in R^n_+\}$ by

$$X(t) = e^{-\langle \alpha, t\rangle} \left\{ X_0 + \sigma \int_0^t e^{\langle \alpha, x\rangle} dW(x) \right\},$$

where W is an n-parameter Brownian motion, X_0 is a random variable independent of W, $\langle \cdot, \cdot \rangle$ stands for the inner product in R^n. This definition was introduced by Wang [1], who investigated some Markov properties of OUP_2 in his paper. Chen [2] studied sample path properties of OUP_2 by giving Hausdorff dimension of the graph and image sets of OUP_2. Xiao [3] generalized these results to the case of n-dimensional processes. In this paper, we give some direct depictions of sample path properties of OUP_2 by establishing its Lévy's exact moduli of continuity not only for one of two parameters but also for both parameters.

For simplicity, we assume that $\sigma = 1$, $EX_0 = 0$, $EX_0^2 = 1$, $E\exp(tX_0^2) < \infty$ for any $0 < t < \frac{1}{2}$. OUP_2 can be rewritten as

$$X(t, v) = e^{-\alpha t - \beta v} \left\{ X_0 + \int_0^t \int_0^v e^{\alpha x + \beta y} dW(x, y) \right\} \tag{1}$$

with $\alpha > 0$, $\beta > 0$. Then the increment

$$\begin{aligned} X(t+s, v) - X(t, v) &= e^{-\alpha(t+s) - \beta v}(1 - e^{\alpha s})X_0 + \\ &+ e^{-\alpha(t+s) - \beta v}(1 - e^{\alpha s}) \int_0^{t+s} \int_0^v e^{\alpha x + \beta y} dW(x, y) + \end{aligned} \tag{2}$$

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$$+e^{-\alpha t-\beta v}\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y)=:$$

$$=:\xi_1(t,s,v)+\xi_2(t,s,v)+\xi_3(t,s,v).$$

Hence

$$E\big(X(t+s,v)-X(t,v)\big)^2= \tag{3}$$

$$=e^{-2\alpha(t+s)-2\beta v}\big(1-e^{\alpha s}\big)^2\left\{1+\frac{1}{4\alpha\beta}\big(e^{2\alpha(t+s)}-1\big)\big(e^{2\beta v}-1\big)\right\}+$$

$$+\frac{1}{4\alpha\beta}\big(e^{2\alpha s}-1\big)\big(1-e^{-2\beta v}\big)+\frac{1}{2\alpha\beta}\big(e^{-\alpha s}-1\big)\big(e^{2\alpha s}-1\big)\big(1-e^{-2\beta v}\big)=$$

$$=\alpha^2s^2e^{-2\alpha(t+s)-2\beta v}+\frac{s}{2\beta}\big(1-e^{-2\beta v}\big)+O(s^2)\quad\text{as}\quad s\to 0$$

for any $v>0$. Put $\sigma^2(t,s,v)=\alpha^2s^2e^{-2\alpha(t+s)-2\beta v}$, $\sigma^2(s,v)=\frac{s}{2\beta}\big(1-e^{-2\beta v}\big)$.

REMARK 1. We take $\sigma^2(t,s,v)$ into consideration since

$$\sigma(s,v)=o\big(\sigma(t,s,v)\big)\quad\text{as}\quad v\to 0$$

for any fixed $t\geqq 0$ and $s>0$.

Consider the increment of $X(t,v)$ for both t and v. Put

$$X\big(R(t,s,v,u)\big):=X(t+s,v+u)-X(t+s,v)-X(t,v+u)+X(t,v).$$

Similarly to (2) we have

$$X\big(R(t,s,v,u)\big)=e^{-\alpha(t+s)-\beta(v+u)}\big(1-e^{\alpha s}\big)\big(1-e^{\beta u}\big)X_0+$$

$$+e^{-\alpha(t+s)-\beta(v+u)}\big(1-e^{\alpha s}\big)\int_0^{t+s}\int_v^{v+u}e^{\alpha x+\beta y}dW(x,y)+$$

$$+e^{-\alpha(t+s)-\beta(v+u)}\big(1-e^{\alpha s}\big)\big(1-e^{\beta u}\big)\int_0^{t+s}\int_0^{v}e^{\alpha x+\beta y}dW(x,y)+$$

$$+e^{\alpha t-\beta(v+u)}\int_t^{t+s}\int_v^{v+u}e^{\alpha x+\beta y}dW(x,y)+$$

$$+e^{-\alpha t-\beta(v+u)}\big(1-e^{\beta u}\big)\int_t^{t+s}\int_0^{v}e^{\alpha x+\beta y}dW(x,y)$$

and

$$(4)\qquad EX^2\big(R(t,s,v,u)\big) = e^{-2\alpha(t+s)-2\beta(v+u)}\big(1-e^{\alpha s}\big)^2\big(1-e^{\beta u}\big)^2$$

$$\cdot\left\{1+\frac{1}{4\alpha\beta}\big(e^{2\alpha(t+s)}-1\big)\big(e^{2\beta v}-1\big)\right\}+$$

$$+e^{-2\alpha(t+s)-2\beta(v+u)}\big(1-e^{\alpha s}\big)^2\frac{1}{4\alpha\beta}\big(e^{2\alpha(t+s)}-1\big)\big(e^{2\beta(v+u)}-e^{2\beta v}\big)+$$

$$+e^{-2\alpha t-2\beta(v+u)}\frac{1}{4\alpha\beta}\big(e^{2\alpha(t+s)}-e^{2\alpha t}\big)\big(e^{2\beta(v+u)}-e^{2\beta v}\big)+$$

$$+e^{-2\alpha t-2\beta(v+u)}\big(1-e^{\beta u}\big)^2\frac{1}{4\alpha\beta}\big(e^{2\alpha(t+s)}-e^{2\alpha t}\big)\big(e^{2\beta v}-1\big)+$$

$$+2e^{-\alpha(2t+s)-2\beta(v+u)}\big(1-e^{\alpha s}\big)\frac{1}{4\alpha\beta}\big(e^{2\alpha(t+s)}-e^{2\alpha t}\big)\big(e^{2\beta(v+u)}-e^{2\beta v}\big)+$$

$$+2e^{-\alpha(2t+s)-2\beta(v+u)}\big(1-e^{\alpha s}\big)\big(1-e^{\beta u}\big)^2\frac{1}{4\alpha\beta}\big(e^{2\alpha(t+s)}-e^{2\alpha t}\big)\big(e^{2\beta v}-1\big)=$$

$$=\frac{1}{4\alpha\beta}\big(e^{2\alpha s}-1\big)\big(1-e^{-2\beta u}\big)+o(su)\quad\text{as}\quad s\to 0,\ u\to 0.$$

Put $\sigma_1(t,s,v)=\sigma(t,s,v)+\sigma(s,v)$, $\sigma_2(t,s,v)=\sigma(t,s,v)\wedge\sigma(s,v)$. At first, we consider moduli of continuity for one of two parameters.

THEOREM 1. *Suppose that a_h is a function of h with $a_h=o(h^{-\delta})$ as $h\to 0$ for any $\delta>0$ and $\varliminf_{h\to 0} a_h>0$. Then we have*

(5)

$$\limsup_{h\to 0}\sup_{v>0}\sup_{0\leqq t\leqq a_h}\sup_{0\leqq s\leqq h}\frac{\big|X(t+s,v)-X(t,v)\big|}{\sigma_1(t,h,v)\Big\{2\big(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v)\big)\Big\}^{\frac{1}{2}}}=1\ a.s.$$

and

$$(6)\qquad \lim_{h\to 0}\sup_{0\leqq t\leqq a_h}\frac{\big|X(t+h,v)-X(t,v)\big|}{\sigma_1(t,h,v)\Big\{2\big(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v)\big)\Big\}^{\frac{1}{2}}}=1\ a.s.$$

for any fixed $v>0$.

REMARK 2. By symmetry of $X(t,v)$ in t and v, we can write alternatively

$$\lim_{h\to 0}\sup_{t>0}\sup_{0\leqq v\leqq a_h}\sup_{0<u\leqq h}\frac{|X(t,v+u)-X(t,v)|}{\nu(t,v,h)}=1 \quad \text{a.s.}$$

and

$$\lim_{h\to 0}\sup_{0\leqq v\leqq a_h}\frac{|X(t,v+h)-X(t,v)|}{\nu(t,v,h)}=1 \quad \text{a.s.}$$

where $\nu(t,v,h)$ is an analogue of the normalized factor in (5) and (6).

As to moduli of continuity of $X(t,v)$ for both parameters, we have

THEOREM 2. *Suppose that a_h and b_h are functions of h with $\underline{\lim}_{h\to 0} a_h b_h > 0$ and c_h is a continuous non-increasing function of h with $c_h \to 0$ and $a_h b_h = o\big((hc_h)^{-\delta}\big)$ as $h\to 0$ for any $\delta>0$. Then we have*

$$\lim_{h\to 0}\sup_{0\leqq t\leqq a_h}\sup_{0<s\leqq h}\sup_{0\leqq v\leqq b_h}\sup_{0<u\leqq c_h}\frac{\big|X\big(R(t,s,v,u)\big)\big|}{\big(2hc_h\log(hc_h)^{-1}\big)^{\frac12}}=1 \quad a.s. \tag{7}$$

and

$$\lim_{h\to 0}\sup_{0\leqq t\leqq a_h}\sup_{0\leqq v\leqq b_h}\frac{\big|X\big(R(t,h,v,c_h)\big)\big|}{\big(2hc_h\log(hc_h)^{-1}\big)^{\frac12}}=1 \quad a.s. \tag{8}$$

2. Proofs

In order to prove our theorems, we need some exponential inequalities.

LEMMA 1. *For any $0<\varepsilon<\frac12$, there exist $h=h(\varepsilon)>0$ and $C=C(\varepsilon)>0$ such that for any fixed $t\geqq 0$ and $0<s\leqq h$*

$$P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,s,v)\big(x^2+2\log\log\sigma_2^{-1}(t,s,v)\big)^{\frac12}}\geqq 1+2\varepsilon\right\}\leqq \tag{9}$$

$$\leqq C\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}.$$

PROOF. Let $0 < \vartheta < 1$, $\delta > 0$ be specified later on. Define v_k and v'_k by

$$\sigma^2(s, v_k) = \vartheta^k, \quad k = k_0, k_0 + 1, \dots,$$

where $k_0 = \left[\frac{\log(\delta s/2\beta)}{\log \vartheta}\right]$, and

$$\sigma^2(t, s, v'_k) = \vartheta^k, \quad k = k_1, k_1 + 1, \dots,$$

where $k_1 = \left[\frac{\log \sigma^2(t,s,v_{k_0})}{\log \vartheta}\right]$. By the definition, it is easy to see that

$$v_k \to 0 \quad \text{and} \quad v'_k \to \infty \quad \text{as} \quad k \to \infty, \tag{10}$$

$$v'_{k_1} \leqq v_{k_0}, \tag{11}$$

$$1 - e^{-2\beta v_{k_0+1}} \leqq \delta \leqq 1 - e^{-2\beta v_{k_0}} \tag{12}$$

and

$$\vartheta\left(1 - e^{-2\beta v_k}\right) = 1 - e^{-2\beta v_{k+1}}. \tag{13}$$

(12) and (13) imply that for $k \geqq k_0$

$$e^{-2\beta(v_k - v_{k+1})} = 1 - (1-\vartheta)\left(1 - e^{-2\beta v_k}\right) e^{2\beta v_{k+1}} \geqq 1 - \frac{1-\vartheta}{1-\delta} = \frac{\vartheta - \delta}{1-\delta}. \tag{14}$$

Moreover, obviously

$$e^{-2\beta(v'_{k+1} - v'_k)} = \vartheta \tag{14$'$}$$

and

$$1 - e^{-2\beta v'_k} = 1 - \frac{1}{\vartheta} e^{-2\beta v'_{k+1}} = \tag{13$'$}$$

$$= \frac{1}{\vartheta}\left(1 - e^{-2\beta v'_{k+1}}\right) - \left(\frac{1}{\vartheta} - 1\right) \geqq \frac{1}{\sqrt{\vartheta}}\left(1 - e^{-2\beta v'_{k+1}}\right)$$

for $k \geqq k_1$, provided that ϑ is close enough to 1 since

$$1 - e^{-2\beta v'_{k+1}} \geqq 1 - e^{-2\beta v'_{k_1+1}} \geqq 1 - e^{-2\beta v_{k_0}} \geqq \delta.$$

From (2) we have

$$(15)\qquad P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,s,v)\big(x^2+2\log\log\sigma_2^{-1}(t,s,v)\big)^{\frac12}}\geqq 1+2\varepsilon\right\}\leqq$$

$$\leqq\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leqq v_k}\frac{|\xi_1(t,s,v)|}{\sigma(t,s,v)}\geqq(1+2\varepsilon)(x^2+2\log\log\vartheta^{-k})^{\frac12}\right\}+$$

$$+\sum_{k=k_1}^{\infty}P\left\{\sup_{v'_k\leqq v<v'_{k+1}}\frac{|\xi_1(t,s,v)|}{\sigma(t,s,v)}\geqq(1+2\varepsilon)(x^2+2\log\log\vartheta^{-k})^{\frac12}\right\}+$$

$$+\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leqq v_k}\frac{|\xi_2(t,s,v)|}{\sigma(s,v)}\geqq\frac{\varepsilon}{2}(x^2+2\log\log\vartheta^{-k})^{\frac12}\right\}+$$

$$+\sum_{k=k_1}^{\infty}P\left\{\sup_{v'_k\leqq v<v'_{k+1}}\frac{|\xi_2(t,s,v)|}{\sigma(s,v)}\geqq\frac{\varepsilon}{2}(x^2+2\log\log\vartheta^{-k})^{\frac12}\right\}+$$

$$+\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leqq v_k}\frac{|\xi_3(t,s,v)|}{\sigma(s,v)}\geqq\left(1+\frac{3\varepsilon}{2}\right)(x^2+2\log\log\vartheta^{-k})^{\frac12}\right\}+$$

$$+\sum_{k=k_1}^{\infty}P\left\{\sup_{v'_k\leqq v<v'_{k+1}}\frac{|\xi_3(t,s,v)|}{\sigma(s,v)}\geqq\left(1+\frac{3\varepsilon}{2}\right)(x^2+2\log\log\vartheta^{-k})^{\frac12}\right\}=$$

$$=:\sum_{j=1}^{6}p_j.$$

Estimate p_1 at first. By the assumption on X_0, for s small enough we have

$$(16)\quad p_1=\sum_{k=k_0}^{\infty}P\Big\{(e^{\alpha s}-1)|X_0|\geqq(1+2\varepsilon)\alpha s(x^2+2\log\log\vartheta^{-k})^{\frac12}\Big\}\leqq$$

$$\leqq\sum_{k=k_0}^{\infty}P\Big\{|X_0|\geqq(1+\varepsilon)(x^2+2\log\log\vartheta^{-k})^{\frac12}\Big\}\leqq$$

$$\leqq\sum_{k=k_0}^{\infty}E\exp\left(\frac{1-\varepsilon/2}{2}X_0^2\right)\exp\left\{-\frac12(1+\varepsilon)(x^2+2\log\log\vartheta^{-k})\right\}\leqq$$

$$\leqq c \exp\left\{-\frac{1}{2}(1+\varepsilon)x^2\right\} \sum_{k=k_0}^{\infty} k^{-(1+\varepsilon)} \leqq c \exp\left\{-\frac{1}{2}(1+\varepsilon)x^2\right\};$$

here and in the sequel c stands for a positive constant, whose value is irrelevant. For p_2 we have a similar estimation.

Consider p_3. Let

$$Y(v) = \int_0^{t+s} \int_0^v e^{\alpha s+\beta y}\, dW(x,y),$$

which is a Gaussian process with independent increments and

$$EY^2(v) = \frac{1}{4\alpha\beta}\left(e^{2\alpha(t+s)} - 1\right)\left(e^{2\beta v} - 1\right).$$

Noting (13) and (14), we have

$$\text{(17)} \qquad p_3 \leqq \sum_{k=k_0}^{\infty} P\Bigg\{ \sup_{v_{k+1} < v \leqq v_k} \left|Y(v)\right| \geqq$$

$$\geqq \frac{\varepsilon}{2} e^{\alpha(t+s)+\beta v_{k+1}} (e^{\alpha s} - 1)^{-1} \sigma(s, v_{k+1})(x^2 + 2\log\log\vartheta^{-k})^{\frac{1}{2}}\Bigg\} \leqq$$

$$\leqq 2\sum_{k=k_0}^{\infty} P\Big\{\left|Y(v_k)\right| / \left(EY^2(v_k)\right)^{\frac{1}{2}} \geqq \frac{\varepsilon}{2}\left(EY^2(v_k)\right)^{-\frac{1}{2}} e^{\alpha(t+s)+\beta v_{k+1}} \cdot$$

$$\cdot (e^{\alpha s} - 1)^{-1} \sigma(s, v_{k+1})(x^2 + 2\log\log\vartheta^{-k})^{\frac{1}{2}}\Big\} \leqq$$

$$\leqq c \sum_{k=k_0}^{\infty} \exp\left\{-\frac{\varepsilon^2 \vartheta}{8\alpha s} e^{-2\beta(v_k - v_{k+1})}(x^2 + 2\log\log\vartheta^{-k})^{\frac{1}{2}}\right\} \leqq$$

$$\leqq c \sum_{k=k_0}^{\infty} \exp\left\{-\frac{\varepsilon^2 \vartheta(\vartheta - \delta)}{8\alpha s(1-\delta)}(x^2 + 2\log\log\vartheta^{-k})^{\frac{1}{2}}\right\} \leqq c\exp(-x^2)$$

provided that s is small enough. For p_4 we have a similar estimation by using (13)' and (14)' instead of (13) and (14).

We now turn to p_5. Let

$$Z(v) = \int_t^{t+s} \int_0^v e^{\alpha x+\beta y}\, dW(x,y),$$

which is also a Gaussian process with independent increments and

$$EZ^2(v)=\frac{1}{4\alpha\beta}e^{2\alpha t}\left(e^{2\alpha s}-1\right)\left(e^{2\beta v}-1\right).$$

Similarly to (17) we obtain

$$p_5\leqq\sum_{k=k_0}^{\infty}P\Big\{\sup_{v_{k+1}<v\leqq v_k}\left|Z(v)\right|\geqq \tag{18}$$

$$\geqq\left(1+\frac{3\varepsilon}{2}\right)e^{\alpha t+\beta v_{k+1}}\sigma(s,v_{k+1})(x^2+2\log\log\vartheta^{-k})^{\frac{1}{2}}\Big\}\leqq$$

$$\leqq c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{1}{2}\left(1+\frac{3\varepsilon}{2}\right)\vartheta e^{-2\beta(v_k-v_{k+1})}(x^2+2\log\log\vartheta^{-k})\right\}\leqq$$

$$\leqq c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{\vartheta}{2}\left(1+\frac{3\varepsilon}{2}\right)\frac{\vartheta-\delta}{1-\delta}(x^2+2\log\log\vartheta^{-k})\right\}\leqq$$

$$\leqq c\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}$$

provided that ϑ is close enough to 1 and δ is small enough.

For p_6 we have a similar estimation.

Inserting these inequalities into (15), we obtain (9). Lemma 1 is proved.

LEMMA 2. *Let $a>0$, $0<\varepsilon<\frac{1}{2}$. There exist $h=h(\varepsilon)>0$ and $C_1=C_1(\varepsilon)>0$ such that*

(19)

$$P\left\{\sup_{v>0}\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\frac{\left|X(t+s,v)-X(t,v)\right|}{\sigma_1(t,h,v)\left(x^2+2\log\log\sigma_2^{-1}(t,h,v)\right)^{\frac{1}{2}}}\geqq 1+4\varepsilon\right\}\leqq$$

$$\leqq\frac{C_1 a}{h}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}.$$

PROOF. Without loss of generality, we assume that $x^2\geqq 2$.

Let k be an integer specified later on and

$$t_j=(t2^j/h)h/2^j,\quad j=k,k+1,\dots,$$

for any $t \geqq 0$. It is easy to show that $X(t,v)$ is almost surely continuous in (t,v). Hence we can write

$$
(20)\qquad \left|X(t+s,v)-X(t,v)\right| \leqq \left|X\big((t+s)_k,v\big)-X(t_k,v)\right|+
$$
$$
+\sum_{j=0}^{\infty}\left|X\big((t+s)_{k+j+1},v\big)-X\big((t+s)_{k+j},v\big)\right|+
$$
$$
+\sum_{j=0}^{\infty}\left|X(t_{k+j+1},v)-X(t_{k+j},v)\right|.
$$

By definitions, for h small enough, k large enough and $0<s\leqq h$,

$$
\sigma^2\big(t_k,(t+s)_k-t_k,v\big)\leqq \alpha^2(1+2^{-k})^2h^2e^{-2\alpha(t-2^{-k}h)-2\beta v}\leqq
$$
$$
\leqq\left(1+\frac{1}{2}\varepsilon\right)\sigma^2(t,h,v),
$$
$$
\sigma^2\big((t+s)_k-t_k,v\big)\leqq(1-2^{-k})\frac{h}{2\beta}(1-e^{-2\beta v})\leqq\left(1+\frac{1}{2}\varepsilon\right)\sigma^2(h,v)
$$

and

$$
\sigma^2\big((t+s)_{k+j},h/2^{k+j+1},v\big)\leqq\alpha^2 2^{-2(k+j+1)}h^2e^{-2\alpha t-2\beta v}\leqq
$$
$$
\leqq 2^{-(k+j+1)}\sigma^2(t,h,v),
$$
$$
\sigma^2\big(h/2^{k+j+1},v\big)\leqq 2^{-(k+j+1)}\frac{h}{2\beta}\big(1-e^{-2\beta v}\big)\leqq 2^{-(k+j+1)}\sigma^2(h,v).
$$

From these inequalities and Lemma 1, we have

$$
P\left\{\sup_{v>0}\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\frac{\left|X\big((t+s)_k,v\big)-X(t_k,v)\right|}{\sigma_1(t,h,v)\big(x^2+2\log\log\sigma_2^{-1}(t,h,v)\big)^{\frac{1}{2}}}\geqq 1+3\varepsilon\right\}\leqq
$$
$$
\leqq c2^{2k}\frac{a}{h}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\},
$$

$$
P\left\{\sup_{v>0}\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\sum_{j=0}^{\infty}\frac{\left|X\big((t+s)_{k+j+1},v\big)-X\big((t+s)_{k+j},v\big)\right|}{\sigma_1(t,h,v)\big(x^2+2\log\log\sigma_2^{-1}(t,h,v)\big)^{\frac{1}{2}}}\geqq\frac{\varepsilon}{2}\right\}\leqq
$$

$$\leqq c\sum_{j=0}^{\infty} 2^{2(k+j+1)}\frac{a}{h}\exp\left\{-\frac{(1+\varepsilon)\varepsilon^2}{8(1+2\varepsilon)^2}2^{k+j+1}x^2\right\} \leqq$$

$$\leqq c\frac{a}{h}e^{-x^2}\sum_{j=0}^{\infty} 2^{2(k+j+1)}\exp(-\varepsilon^2 2^{k+j+4}) \leqq c\frac{a}{h}e^{-x^2}$$

provided that k is large enough, where we have used the inequalities $bd \geqq b+d$ for any $b \geqq 2$ and $d \geqq 2$. For the second series on the right hand side of (20), we have a similar estimation. Combining these inequalities with (20) yields (19).

LEMMA 3. *Let $a > 0$, $b > 0$, $0 < \varepsilon < \frac{1}{2}$. There exist $h = h(\varepsilon) > 0$, $d = d(\varepsilon) > 0$, $C_2 = C_2(\varepsilon) > 0$ such that*

$$(21)\ P\left\{\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\sup_{0\leqq v\leqq b}\sup_{0<u\leqq d}\left|X\left(R(t,s,v,u)\right)\right|/(su)^{\frac{1}{2}} \geqq (1+2\varepsilon)x\right\} \leqq$$

$$\leqq C_2\frac{ab}{hd}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}$$

for any $x > 0$.

PROOF. Without loss of generality, we assume that $x \geqq \sqrt{2}$.
Let k be an integer specified later on and

$$t_j = [t2^j/h]h/2^j, \quad v'_j = [v2^j/d]d/2^j, \quad j = k, k+1, \ldots,$$

for any $t \geqq 0$, $v \geqq 0$. Similarly to (20), we write

$$(22)\ \left|X\left(R(t,s,v,u)\right)\right| \leqq \left|X\left(R(t_k,(t+s)_k - t_k, v'_k, (v+u)'_k - v'_k)\right)\right| +$$

$$+\left|X\left(R((t+s)_k, (t+s)-(t+s)_k, v'_k, (v+u)'_k - v'_k)\right)\right| +$$

$$+\left|X\left(R(t_k, t-t_k\, v'_k, (v+u)'_k - v'_k)\right)\right| +$$

$$+\left|X\left(R(t,s,v'_k, v-v'_k)\right)\right| + \left|X\left(R(t,s,(v+u)'_k, (v+u)-(v+u)'_k)\right)\right| \leqq$$

$$\leqq \left|X\left(R(t_k,(t+s)_k - t_k, v'_k, (v+u)'_k - v'_k)\right)\right| +$$

$$+\sum_{j=0}^{\infty}\left|X\left(R((t+s))_{k+j}, (t+s)_{k+j+1} - (t+s)_{k+j}, v'_k, (v+u)'_k - v'_k)\right)\right| +$$

$$+\sum_{j=0}^{\infty}\Big|X\big(R(t_{k+j},t_{k+j+1}-t_{k+j},v'_k,(v+u)'_k-v'_k)\big)\Big|+$$

$$+\Big|X\big(R(t,s,v'_k,v-v'_k)\big)\Big|+\Big|X\big(R(t,s,(v+u)'_k,(v+u)-(v+u)'_k)\big)\Big|.$$

Furthermore, by recalling (4), as $s\to 0$ and $u\to 0$,

$$EX^2\Big(R\big(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\big)\Big)\leqq\left(1+2^{-k}\right)^2 su+o(su),$$

$$EX^2\Big(R\big((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j},v'_k,(v+u)'_k-v'_k)\big)=$$

$$=2^{-2(k+j+1)}su+o\big(2^{-2(k+j+1)}su\big),$$

and

$$EX^2\big(R(t,s,v'_k,v-v'_k)\big)=2^{-k}su+o(su).$$

Therefore, for large k, small s and u,

$$P\left\{\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\sup_{0\leqq v\leqq b}\sup_{0<u\leqq d}\Big|X\big(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\big)\Big|/\right.$$

$$\left./(su)^{\frac12}\geqq(1+\varepsilon)x\right\}\leqq 2^{4k}\frac{ab}{hd}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\},$$

$$P\left\{\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\sup_{0\leqq v\leqq b}\sup_{0<u\leqq d}\sum_{j=0}^{\infty}\Big|X\big(R(t+s)_{k+j},\right.$$

$$(t+s)_{k+j+1}-(t+s)_{k+j},v'_k,(v+u)'_k-v'_k)\big)\Big|$$

$$\left./\sum_{j=0}^{\infty}\left(2^{-(j+1)}su\right)^{\frac12}\geqq\frac{\sqrt2-1}{4}\varepsilon x\right\}\leqq$$

$$\leqq\sum_{j=0}^{\infty}2^{4(k+j+1)}\frac{ab}{hd}\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\sup_{0\leqq v\leqq b}\sup_{0<u\leqq d}P\left\{\Big|X\big(R((t+s)_{k+j},\right.$$

$$(t+s)_{k+j+1}-(t+s)_{k+j},v'_k,(v+u)'_k-v'_k)\big)\Big|\geqq$$

$$\geqq \frac{\sqrt{2}-1}{4}\varepsilon x 2^{k+(j+1)/2}\left(2^{-2(k+j+1)}su\right)^{\frac{1}{2}}\Bigg\} \leqq$$

$$\leqq \frac{ab}{hd}\sum_{j=0}^{\infty} 2^{4(k+j+1)}\exp\left\{-\frac{\varepsilon^2}{200}2^{2k+j+1}x^2\right\} \leqq c\frac{ab}{hd}e^{-x^2}.$$

For the second sum on the right hand side of (22), we have a similar estimation. As to $X\left(R(t,s,v_k',v-v_k')\right)$, we have

$$P\left\{\sup_{0\leqq t\leqq a}\sup_{0<s\leqq h}\sup_{0\leqq v\leqq b}\sup_{0<u\leqq d}\left|X\left(R(t,s,v_k',v-v_k')\right)\right|/(su)^{\frac{1}{2}} \geqq \frac{\varepsilon}{4}x\right\} \leqq$$

$$\leqq 2^{4k}\frac{ab}{hd}\exp\left\{-\frac{\varepsilon^2}{40}2^k x^2\right\} \leqq c\frac{ab}{hd}e^{-x^2}.$$

For the last term on the right hand side of (22), we have also a similar estimation. Combining these inequalities with (22) yields (21).

Proof of Theorem 1. First, we prove

(23)

$$\limsup_{h\to 0}\sup_{v>0}\sup_{0\leqq t\leqq a_h}\sup_{0<s\leqq h}\frac{\left|X(t+s,v)-X(t,v)\right|}{\sigma_1(t,h,v)\left\{2\left(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v)\right)\right\}^{\frac{1}{2}}} \leqq$$

$$\leqq 1 \quad \text{a.s.}$$

Without loss of generality, we assume that a_h is non-increasing for $0\leqq h\leqq 1$; otherwise we consider $a_h^* = \sup_{h\leqq s\leqq 1} a_s$ instead of a_h.

Let $0<\varepsilon<\frac{1}{2}$, $\vartheta = 1-\varepsilon$. Define $h_j = \vartheta^j$. For j large enough, using Lemma 2 we obtain

$$P\left\{\sup_{v>0}\sup_{0\leqq t\leqq a_{h_{j+1}}}\sup_{0<s\leqq h_j}\frac{\left|X(t+s,v)-X(t,v)\right|}{\sigma_1(t,h_j,v)\left\{2\left(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v)\right)\right\}^{\frac{1}{2}}} \geqq\right.$$

$$\left.\geqq 1+\varepsilon\right\} \leqq$$

$$\leqq C_1\frac{a_{h_{j+1}}}{h_j}\exp\left\{-\left(1+\frac{\varepsilon}{4}\right)\log h_j^{-1}\right\} \leqq C_1\frac{(h_{j+1})^{-\varepsilon/8}}{h_j}h_j^{1+\varepsilon/4} \leqq C_1\vartheta^{(j-1)\varepsilon/8},$$

which, in combination with the Borel–Cantelli lemma, implies

$$\limsup_{j\to\infty}\sup_{v>0}\sup_{0\leqq t\leqq a_{h_{j+1}}}\sup_{0<s\leqq h_j}$$

$$\frac{\left|X(t+s,v)-X(t,v)\right|}{\sigma_1(t,h_j,v)\left\{2\left(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v)\right)\right\}^{\frac{1}{2}}}\leqq 1+\varepsilon \quad \text{a.s.}$$

Furthermore

$$\limsup_{h\to 0}\sup_{v>0}\sup_{0\leqq t\leqq a_h}\sup_{0<s\leqq h}\frac{\left|X(t+s,v)-X(t,v)\right|}{\sigma_1(t,h,v)\left\{2\left(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v)\right)\right\}^{\frac{1}{2}}}\leqq$$

$$\leqq \limsup_{j\to\infty}\sup_{v>0}\sup_{0\leqq t\leqq a_{h_{j+1}}}\sup_{0<s\leqq h_j}\left|X(t+s,v)-X(t,v)\right|/$$

$$/\vartheta\sigma_1(t,h_j,v)\left\{2\left(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v)\right)\right\}^{\frac{1}{2}}\leqq$$

$$\leqq(1-\varepsilon)^{-1}(1+\varepsilon) \quad \text{a.s.}$$

This proves (23) by the arbitrariness of ε.

Next, we prove that for fixed $v>0$

$$\liminf_{h\to 0}\sup_{0\leqq t\leqq a_h}\frac{\left|X(t+h,v)-X(t,v)\right|}{\sigma_1(t,h,v)\left\{2\left(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v)\right)\right\}^{\frac{1}{2}}}\geqq 1 \quad \text{a.s.} \tag{24}$$

Noting the fact that for fixed $v>0$ and $t\geqq 0$,

$$\sigma(t,h,v)=o\left(\sigma(h,v)\right) \quad \text{as} \quad h\to 0$$

and recalling the proof of Lemma 1 we find that (24) is equivalent to

$$\liminf_{h\to 0}\sup_{0\leqq t\leqq a_h}\frac{\left|\xi_3(t,h,v)\right|}{\sigma(h,v)(2\log h^{-1})^{\frac{1}{2}}}\geqq 1 \quad \text{a.s.} \tag{25}$$

Put $t_i=ih$, $i=0,1,\ldots,i_h:=[a_h/h]$. Since $\xi_3(t_i,h,v)$, $i=0,1,\ldots,i_h$, are independent, we have for any $\varepsilon>0$

$$P\left\{\max_{0\leqq i\leqq i_h}\frac{\left|\xi_3(t_i,h,v)\right|}{\sigma(h,v)(2\log h^{-1})^{\frac{1}{2}}}\leqq 1-\varepsilon\right\}= \tag{26}$$

$$= \prod_{i=0}^{i_h} \left\{ 1 - P \left\{ \frac{|\xi_3(t_i, h, v)|}{\sigma(h, v)(2\log h^{-1})^{\frac{1}{2}}} > 1 - \varepsilon \right\} \right\} \leqq$$

$$\leqq \prod_{i=0}^{i_h} \left\{ 1 - \exp\{-(1-\varepsilon)\log h^{-1}\} \right\} \leqq \exp(-i_h h^{1-\varepsilon}) \leqq \exp(-h^{-\varepsilon/2}).$$

Let $h_k = k^{-1}$. (26) implies

$$\liminf_{h \to 0} \sup_{0 \leqq t \leqq a_h} \frac{|\xi_3(t, h, v)|}{\sigma(h, v)(2\log h^{-1})^{\frac{1}{2}}} \geqq$$

$$\geqq \liminf_{k \to \infty} \max_{0 \leqq i \leqq i_{h_k}} \frac{|\xi_3(t_i, h_k, v)|}{\sigma(h_k, v)(2\log h_k^{-1})^{\frac{1}{2}}} \geqq 1 - \varepsilon \quad \text{a.s.}$$

Hence (24) is proved. Combining (23) and (24) yields the conclusion of Theorem 1.

PROOF OF THEOREM 2. At first, we prove

$$\limsup_{h \to 0} \sup_{0 \leqq t \leqq a_h} \sup_{0 < s \leqq h} \sup_{0 \leqq v \leqq b_h} \sup_{0 < u \leqq c_h} \frac{\left| X\left(R(t, s, v, u)\right) \right|}{\left(2hc_h \log (hc_h)^{-1}\right)^{\frac{1}{2}}} \leqq 1 \text{ a.s.} \tag{27}$$

We also assume that a_h and b_h are non-increasing, otherwise we consider $a_h^* = \sup_{h \leqq s \leqq 1} a_s$ and $b_h^* = \sup_{h \leqq s \leqq 1} b_s$. Let $0 < \varepsilon < \frac{1}{2}$, $\vartheta = 1 - \varepsilon$. Define h_j by $h_j c_{h_j} = \vartheta^j$, $j = 0, 1, \ldots$. Then by Lemma 3

$$P\left\{ \sup_{0 \leqq t \leqq a_{h_{j+1}}} \sup_{0 < s \leqq h_j} \sup_{0 \leqq v \leqq b_{h_{j+1}}} \sup_{0 < u \leqq c_{h_j}} \frac{\left| X\left(R(t, s, v, u)\right) \right|}{\left(2h_j c_{h_j} \log (h_j c_{h_j})^{-1}\right)^{\frac{1}{2}}} \geqq 1 + 2\varepsilon \right\} \leqq$$

$$\leqq C_2 \frac{a_{h_{j+1}} b_{h_{j+1}}}{h_j c_{h_j}} \exp\left\{ -\frac{1+\varepsilon}{2} \log (h_j c_{h_j})^{-1} \right\} \leqq$$

$$\leqq C_2 \frac{\left(h_{j+1} c_{h_{j+1}}\right)^{-\varepsilon/2}}{h_j c_{h_j}} (h_j c_{h_j})^{1+\varepsilon} = C_2 \vartheta^{(j-1)\varepsilon/2}$$

which implies

$$\limsup_{j \to \infty} \sup_{0 \leqq t \leqq a_{h_{j+1}}} \sup_{0 < s \leqq h_j} \sup_{0 \leqq v \leqq b_{h_{j+1}}} \sup_{0 < u \leqq c_{h_j}} \frac{\left| X\left(R(t, s, v, u)\right) \right|}{\left(2h_j c_{h_j} \log (h_j c_{h_j})^{-1}\right)^{\frac{1}{2}}} \leqq$$

$$\leqq 1 + 2\varepsilon \quad \text{a.s.}$$

Furthermore

$$\limsup_{h\to 0} \sup_{0\leqq t\leqq a_h} \sup_{0<s\leqq h} \sup_{0\leqq v\leqq b_{h_{j+1}}} \sup_{0<u\leqq c_h} \frac{\left|X\left(R(t,s,v,u)\right)\right|}{\left(2hc_h \log\left(hc_h\right)^{-1}\right)^{\frac{1}{2}}} \leqq$$

$$\leqq \limsup_{j\to\infty} \sup_{0\leqq t\leqq a_{h_{j+1}}} \sup_{0<s\leqq h_j} \sup_{0<v\leqq b_{h_{j+1}}} \sup_{0<u\leqq c_{h_j}} \frac{\left|X\left(R(t,s,v,u)\right)\right|}{\vartheta^{\frac{1}{2}}\left(2h_j c_{h_j} \log\left(h_j c_{h_j}\right)^{-1}\right)^{\frac{1}{2}}} \leqq$$

$$\leqq (1-\varepsilon)^{-\frac{1}{2}}(1+2\varepsilon) \quad \text{a.s.}$$

This proves (27) by the arbitrariness of ε.

Next we prove

$$\liminf_{h\to 0} \sup_{0\leqq t\leqq a_h} \sup_{0\leqq v\leqq b_h} \frac{\left|X\left(R(t,s,v,c_h)\right)\right|}{\left(2hc_h \log\left(hc_h\right)^{-1}\right)^{\frac{1}{2}}} \geqq 1 \quad \text{a.s.} \tag{28}$$

Put $t_i = ih$, $i = 0, 1, \ldots, i_h := [a_h/h]$, $v_j = jc_h$, $j = 0, 1, \ldots, j_h := [b_h/c_h]$. Then for any given $\varepsilon > 0$,

$$P\left\{\max_{0\leqq i\leqq i_h} \max_{0\leqq j\leqq j_h} \frac{\left|X\left(R(t_i,h,v_j,c_h)\right)\right|}{\left(2hc_h \log\left(hc_h\right)^{-1}\right)^{\frac{1}{2}}} \leqq 1-\varepsilon\right\} \leqq \tag{29}$$

$$\leqq \prod_{i=0}^{i_h}\prod_{j=0}^{j_h}\left\{1 - P\left\{\frac{\left|X\left(R(t_i,h,v_j,c_h)\right)\right|}{\left(2hc_h \log\left(hc_h\right)^{-1}\right)^{\frac{1}{2}}} > 1-\varepsilon\right\}\right\} \leqq$$

$$\leqq \prod_{i=0}^{i_h}\prod_{j=0}^{j_h}\left\{1 - \exp\left(-(1-\varepsilon)\log\left(hc_h\right)^{-1}\right)\right\} \leqq$$

$$\leqq \exp\left\{-i_h j_h (hc_h)^{1-\varepsilon}\right\} \leqq \exp\left\{-\frac{1}{2}a_h b_h (hc_h)^{-\varepsilon}\right\} \leqq \exp\left\{-c(hc_h)^{-\varepsilon}\right\}$$

provided that h is small enough. Define h_k by $h_k c_{h_k} = k^{-1}$. Then (29) implies

$$\liminf_{h\to 0} \sup_{0\leqq t\leqq a_h} \sup_{0\leqq v\leqq b_h} \frac{\left|X\left(R(t,h,v,c_h)\right)\right|}{\left(2hc_h \log\left(hc_h\right)^{-1}\right)^{\frac{1}{2}}} \geqq$$

$$\geqq \liminf_{k\to\infty} \max_{0\leqq i\leqq i_{h_k}} \max_{0\leqq j\leqq j_{h_k}} \frac{\left|X\left(R(t,h_k,v,c_{h_k})\right)\right|}{\left(2h_k c_{h_k}\log\left(h_k c_{h_k}\right)^{-1}\right)^{\frac{1}{2}}} \geqq$$

$$\geqq 1-\varepsilon \quad \text{a.s.}$$

i.e. (28) holds true. (27) and (28) together yield the conclusion of Theorem 2.

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