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STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF ORTHOGONAL RANDOM ELEMENTS IN BANACH SPACES

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Introduction

Several previous authors have investigated laws of large numbers for arrays of orthogonal Banach space-valued random elements. The general goal is to obtain conditions which yield the convergence

$$\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} \xrightarrow{\text{a.s.}} 0 \text{ as } \min(m,n)\to\infty \text{ or } \max(m,n)\to\infty$$

provided that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E\|X_{ij}\|^p}{i^{\alpha p}j^{\beta p}}\left[\log_2(i+1)\right]^p\left[\log_2(j+1)\right]^p<\infty,$$

where $\{X_{ij}\}$ is an array of orthogonal Banach space-valued random elements with zero means and

$$E\|X_{ij}\|^p<\infty,\quad 1\leqq p\leqq 2\quad\text{for all}\quad i,j\geqq 1.$$

Móricz [2] defined quasi-orthogonality for an array of random variables $\{X_{ik}\}$ as

$$\left|E(X_{ik}X_{jl}\right|\leqq\rho\left(|i-j|,|k-l|\right)\left(EX_{ik}^2\right)^{1/2}\left(EX_{jl}^2\right)^{1/2},$$

where $\rho(m,n)$ is a double sequence such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho(m,n)<\infty.$$

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For quasi-orthogonal real-valued random variables $\{X_{ij}\}$, Móricz [2] showed that the condition

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E(X_{ij}^2)}{i^2j^2}\left[\log_2(i+1)\right]^2\left[\log_2(j+1)\right]^2<\infty$$

implies

$$\lim_{\max\{m,n\}\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}=0 \quad \text{a.s.}$$

He also proved that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\sigma_{ij}^2}{i^2j^2}\left[\log_2(i+1)\right]^2\left[\log_2(j+1)\right]^2<\infty$$

is, in certain particular cases, the necessary condition for

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}=0 \quad \text{a.s.}$$

However, the sense of orthogonality in a Banach space must be quite different from that of the real numbers or even for Hilbert spaces. James type orthogonality for a Banach space is adopted in this paper since it is a generalized sense of orthogonality and will be described in detail in Section 2.

Howell and Warren [7] proposed the sufficient condition

$$\sum_{i=1}^{\infty}\frac{E\|X_i\|^{1+\alpha}}{i^{1+\alpha}}\log^{1+\alpha} i<\infty, \qquad 0<\alpha\leqq 1,$$

for the one-dimensional average $\frac{1}{n}\sum\limits_{i=1}^{n}X_i\xrightarrow{\text{a.s.}}0$ where $\{X_i\}$ is a sequence of B-valued random variables, B is a G_α-space, and $\{X_i\}$ is mutually James type orthogonal with

$$E\|X_i\|^{1+\alpha}<\infty \quad \text{for all} \quad i\geqq 1.$$

However, a G_α-space is a special type p space, and type p spaces will be addressed in this manuscript. Howell and Taylor [1] obtained the convergence in probability of $\sum\limits_{i=1}^{n}a_{ni}X_i$ for random elements in a separable Banach space satisfying various distributional conditions, including independence, conditional independence, and orthogonality, and weights $\{a_{ni}\}$ such that

$$\sum_{i=1}^{n} |a_{ni}|^p \leqq 1 \text{ for each } n \text{ and } \max_{1\leqq i\leqq n} |a_{ni}| \to 0 \text{ as } n \to \infty.$$

Móricz and Taylor [5] offered the sufficient condition

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha p}} \sum_{i=1}^{n} E\|X_{ni}\|^p < \infty$$

for the almost sure convergence of $\frac{1}{n^\alpha} \sum\limits_{i=1}^{n} X_{ni}$ for an array of rowwise orthogonal random variables in a Hilbert space or James type orthogonal random variables in a Banach space of type p for some $1 \leqq p \leqq 2$. In Section 3, it is shown that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|X_{ij}\|^p}{i^{\alpha p} j^{\beta p}} \left[\log_2(i+1)\right]^p \left[\log_2(j+1)\right]^p < \infty$$

is the sufficient condition for the strong convergence of

$$\frac{1}{m^\alpha n^\beta} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$$

for an array of James type orthogonal random elements in a separable Banach space of type p, $1 \leqq p \leqq 2$ and $\alpha, \beta > 0$, where $E\|X_{ij}\|^p < \infty$ for all $i, j \geqq 1$.

Preliminaries

The basic definitions and properties of Banach space-valued random variables (or random elements) are well established in the literature (cf: Chapter 2 of Taylor [6]). In these preliminaries, we will introduce only the concepts which are not easily found in the literature. Let B denote a separable Banach space. Let $\{X_{ik}: i, k \geqq 1\}$ be a double sequence of random elements in B with zero means (i.e., $E(X_{ik}) = 0$ for all i, k) and finite moments $E\|X_{ik}\|^p < \infty$, for all i, k, where $1 \leqq p < \infty$ and $\|\cdot\|$ denotes the norm of the separable Banach space B. When B is a Hilbert space, it is easy to relate orthogonality to the inner product. That is, the random elements $\{X_{ik}\}$ are said to be orthogonal if $E(X_{ik}, X_{jl}) = 0$ whenever $i \neq j$

or $k \neq l$ where $(\cdot,\cdot)$ denotes the inner product. However, it is not possible to create the same geometric sense of orthogonality in an arbitrary Banach space without the inner product. Consequently, James type orthogonality is adopted to circumvent this shortcoming.

For nonrandom elements x and y in a Banach space B, x is said to be *James orthogonal* to y (denoted $x \perp_J y$) if

$$\|x\| \leqq \|x+ty\| \quad \text{for all} \quad t \in \mathbf{R}.$$

If B is a Hilbert space, then James orthogonality agrees with the usual notion of orthogonality where the inner product is 0 since

$$\|x+ty\|^2 = (x+ty, x+ty) = \|x\|^2 + t^2\|y\|^2 + 2t(x,y) \geqq \|x\|^2$$

for all $t \in R$ if and only if $(x,y) = 0$. However, in a Banach space where the norm is not generated by an inner product, it is possible for $x \perp_J y$ but $y \not\perp_J x$ and for $x \perp_J y$ with $(x,y) \neq 0$. For example, let $\mathbf{R}^2 = \{(x_1,x_2): \|(x_1,x_2)\| = |x_1| + |x_2|,\ x_1,x_2 \in \mathbf{R}\}$ and let $x = (1,0)$ and $y = (1,1)$. It easily follows that the usual inner product $(x,y) = 1 \neq 0$. Also, $x \perp_J y$, since

$$\|x\| = 1 \quad \text{and} \quad \|x+ty\| = \big\|(1+t,t)\big\| = |1+t| + |t| \geqq 1 = \|x\|$$

for all $t \in \mathbf{R}$. However, $y \not\perp_J x$ since

$$\|y+tx\| = \|(1+t,1)\| = |1+t| + |1| = 3/2 < 2\|y\|$$

if we pick $t = -1/2$. Thus, the following definition (from Howell and Taylor [1]) is used for orthogonal random elements in a Banach space to achieve symmetry in the definition of orthogonality.

DEFINITION 2.1. An array of random elements $\{X_{ik}\}$ is *orthogonal* in $L^p(B)$, $1 \leqq p < \infty$, if

(i) $E\|X_{ik}\|^p < \infty$ for all i, k,

(ii)
$$E\left\| \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} a_{\pi_1(k),\pi_2(l)} X_{\pi_1(k),\pi_2(l)} \right\|^p \leqq$$
$$\leqq E\left\| \sum_{k=1}^{n_1+m_1} \sum_{l=1}^{n_2+m_2} a_{\pi_1(k),\pi_2(l)} X_{\pi_1(k),\pi_2(l)} \right\|^p,$$

for all arrays $\{a_{ik}\} \subseteqq \mathbf{R}$, for all n_1, n_2, m_1, and m_2, and for all permutations π_1, π_2 of the positive integers $\{1,2,\ldots,m_1+n_1\}$ and $\{1,2,\ldots,m_2+n_2\}$, respectively.

It is important to observe that Definition 2.1 is precisely a symmetric James orthogonal condition in $L^p(B)$ since (ii) implies

$$\left(E\|X\|^p\right)^{1/p} \leqq \left(E\|X+tY\|\right)^{1/p} \quad \text{and} \quad \left(E\|Y\|^p\right)^{1/p} \leqq \left(E\|Y+tX\|^p\right)^{1/p}$$

for all $t \in R$. The terminology "orthogonal in $L^p(B)$" used to indicate a dependence on the moment condition and for technical reasons later in addressing the geometry of the Banach space. The most recognizable case is when $p = 2$.

In order to obtain the desired results for arrays, it is necessary to consider results for the one-dimensional case. Let $\{X_i\}$ be a single sequence of orthogonal random elements in $L^p(B)$. A series of useful moment inequalities for later reference will be listed in the next four results.

PROPOSITION 2.2 (Howell and Taylor, 1981). *The following conditions are equivalent:*

(i) *B is of type p, $1 \leqq p \leqq 2$;*

(ii) *for each sequence $\{X_i\}$ of orthogonal random elements in $L^p(B)$, there exists a constant C such that, for all n,*

$$E\left\|\sum_{i=1}^{n} X_i\right\|^p \leqq C \sum_{i=1}^{n} E\|X_i\|^p.$$

The constant C in Proposition 2.2 depends on the particular orthogonal sequence $\{X_i\}$ and the Banach space B. On the other hand, in the case of independent random elements with zero means, the constant C depends only on the Banach space B (cf. Taylor [6], Theorem 4.4.6). Hence, independent random elements with p absolute moments and zero means in a type p space are orthogonal random elements in $L^p(B)$. Finite-dimensional spaces and separable Hilbert spaces are of type 2. Moreover, for $1 \leqq q < p \leqq 2$, type p implies type q, and every Banach space is of type 1. The following two theorems easily follow from theorems of Móricz [3] by replacing $|\cdot|$ by $\|\cdot\|$.

THEOREM 2.3. *Suppose that there exists a nonnegative function $g(F_{b,n})$ satisfying*

$$g(F_{b,k}) + g(F_{b+k,l}) \leqq g(F_{b,k+l})$$

for all $b \geqq 0$ and $1 \leqq k < k+l$, such that

$$E\left[\left\|\sum_{k=b+1}^{b+n} X_k\right\|^r\right] \leqq g(F_{b,n}) \quad \textit{for all} \quad b \geqq 0,\ n \geqq 1,$$

where $r > 0$, and the random variables X_k are Banach space-valued. Then

$$E\left[\left(\max_{1\leqq k\leqq n}\left\|\sum_{i=b+1}^{b+k} X_i\right\|\right)^r\right] \leqq (\log_2 2n)^r g(F_{b,n}).$$

The following corollary is obtained by setting $g(F_{b,n}) = \sum_{k=b+1}^{b+n} u_k$, where $\{u_k\}$ is a sequence of nonnegative numbers.

COROLLARY 2.4. *Suppose that there exist nonnegative numbers $\{u_k\}$ such that*

$$E\left\|\sum_{i=b+1}^{b+n} X_i\right\|^r \leqq \sum_{i=b+1}^{b+n} u_i \quad \textit{for all} \quad b \geqq 0,\ n \geqq 1.$$

Then

$$E\left[\left(\max_{1\leqq k\leqq n}\left\|\sum_{i=b+1}^{b+k} X_i\right\|\right)^r\right] \leqq (\log_2 2n)^r \left(\sum_{i=b+1}^{b+n} u_i\right).$$

LEMMA 2.5. *Let $\{X_n\}$ be orthogonal (in $L^p(B)$) random variables in a Banach space B of type p for some $1 \leqq p \leqq 2$. If*

$$\sum_{k=1}^{\infty} \frac{E\|X_k\|^p}{k^{\alpha p}} \left[\log_2(k+1)\right]^p < \infty$$

for some $\alpha > 0$, then

$$\lim_{n\to\infty} \left\|\frac{1}{n^\alpha}\sum_{k=1}^{n} X_k\right\| = 0 \quad a.s.$$

REMARK. The proof of Lemma 2.5 is similar to that in Theorem 3 of Móricz [4] and uses Proposition 2.2 and Corollary 2.4.

PROOF. Let

$$\xi_n = \frac{1}{n^\alpha}\sum_{k=1}^{n} X_k.$$

For any $\varepsilon > 0$

$$P\Big(\sup_{n>2^r} \|\xi_n\| > \varepsilon\Big) \leqq \sum_{q=r}^{\infty} P\Big(\max_{2^q<n\leqq 2^{q+1}} \|\xi_n\| > \varepsilon\Big). \tag{2.1}$$

Since

$$\|\xi_n\| = \left\| \frac{1}{n^\alpha} \sum_{k=1}^{n} X_k \right\| = \left\| \frac{1}{n^\alpha} \sum_{k=1}^{2^q} X_k + \frac{1}{n^\alpha} \sum_{k=2^q+1}^{n} X_k \right\| \leqq$$

$$\leqq \|\xi_{2^q}\| + \frac{1}{n^\alpha} \left\| \sum_{k=2^q+1}^{n} X_k \right\| \leqq \|\xi_{2^q}\| + \frac{1}{2^{\alpha q}} \left\| \sum_{k=2^q+1}^{n} X_k \right\|,$$

it follows that

$$\max_{2^q<n\leqq 2^{q+1}} \|\xi_n\| \leqq \|\xi_{2^q}\| + \frac{1}{2^{\alpha p}} \max_{2^q<n\leqq 2^{q+1}} \left\| \sum_{k=2^q+1}^{n} X_k \right\|. \tag{2.2}$$

Moreover,

$$E\|\xi_{2^q}\|^p = E \left\| \frac{1}{2^{q\alpha}} \sum_{k=1}^{2^q} X_k \right\|^p \leqq \frac{C_1}{2^{\alpha pq}} \sum_{k=1}^{2^q} E\|X_k\|^p, \quad \text{for some} \quad C_1 > 0,$$

since B is of type p.

Next,

$$E \left(\max_{2^q<n\leqq 2^{q+1}} \left\| \sum_{k=2^q+1}^{n} X_k \right\| \right)^p \leqq C_2 \left(\log_2 2^{q+1}\right)^p \sum_{k=2^q+1}^{2^{q+1}} E\|X_k\|^p$$

for some $C_2 > 0$ by Corollary 2.4. By Markov's inequality and (2.2),

$$P\Big(\max_{2^q<n\leqq 2^{q+1}} \|\xi_n\| > \varepsilon \Big) \leqq \tag{2.3}$$

$$\leqq P\big(\|\xi_{2^q}\| > \varepsilon/2\big) + P\left(\max_{2^q<n\leqq 2^{q+1}} \left\| \sum_{k=2^q+1}^{n} X_k \right\| > \varepsilon 2^{\alpha q - 1} \right) \leqq$$

$$\leqq \frac{2^p C}{\varepsilon^p} \left\{ \frac{1}{2^{\alpha pq}} \sum_{k=1}^{2^q} E\|X_k\|^p + \frac{1}{2^{\alpha pq}} \left(\log_2 2^{q+1}\right)^p \sum_{k=2^q+1}^{2^{q+1}} E\|X_k\|^p \right\},$$

where $C = \max\{C_1, C_2\}$. Consider

$$\sum_{q=r}^{\infty} \frac{1}{2^{\alpha pq}} \sum_{k=1}^{2^q} E\|X_k\|^p \leqq \tag{2.4}$$

$$\leqq \sum_{k=1}^{2^r} E\|X_k\|^p \sum_{q=r}^{\infty} \frac{1}{2^{\alpha pq}} + \sum_{k=2^r+1}^{\infty} E\|X_k\|^p \sum_{q:2^q>k} \frac{1}{2^{\alpha pq}} \leqq$$

$$\leqq \frac{2^{\alpha p}}{2^{\alpha p}-1}\left\{\frac{1}{2^{\alpha r p}} \sum_{k=1}^{2^r} E\|X_k\|^p + \sum_{k=2^r+1}^{\infty} \frac{E\|X_k\|^p}{k^{\alpha p}}\right\}$$

and

$$\sum_{q=r}^{\infty} \frac{1}{2^{\alpha pq}} \left(\log_2 2^{q+1}\right)^p \sum_{k=2^q+1}^{2^{q+1}} E\|X_k\|^p \leqq \tag{2.5}$$

$$\leqq 2^{(\alpha+1)p} \sum_{k=2^r+1}^{\infty} \left\{\frac{E\|X_k\|^p}{k^{\alpha p}} + \frac{E\|X_k\|^p}{k^{\alpha p}}\left[\log_2(k+1)\right]^p\right\}.$$

Combining (2.1), (2.3), (2.4), and (2.5), we can conclude that $\|\xi_n\| \to 0$ a.s. by letting $r = -1$ if we have a convention that $\sum_{k=1}^{2^r} = 0$ and $\sum_{k=2^r+1}^{\infty} = \sum_{k=1}^{\infty}$ for $r = -1$. □

Remark. Móricz and Taylor [5] showed that a sufficient condition for the strong law of large numbers for rowwise orthogonal random variables in a Banach space of type p is

$$\sum_{n=1}^{\infty} \frac{1}{n^{p\alpha}} \sum_{k=1}^{n} E\|X_{nk}\|^p < \infty.$$

Hence, to apply this result to a sequence and obtain the conclusion of Lemma 2.5, it requires that

$$E\|X_{n1}\|^p = \ldots = E\|X_{nn}\|^p = E\|X_n\|^p$$

for fixed n. However,

$$\sum_{n=1}^{\infty} \frac{1}{n^{p\alpha}} \sum_{k=1}^{n} E\|X_{nk}\|^p = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha p}} n E\|X_n\|^p < \infty$$

implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^{p\alpha}} E\|X_n\|^p \log_2^p(n+1) < \infty.$$

Major results

The following theorem for arrays of orthogonal random elements in Banach spaces is similar to Theorem 1 in Móricz [2] for arrays of orthogonal real-valued random variables in the special case $p = 2$. The proof of Theorem 3.1 depends very heavily on the geometric properties of the Banach space. However, Theorem 3.1 provides strong laws of large numbers with substantially lesser moment conditions even in the real-valued random variables case since almost all orthogonality results (including Menšov's SLLN and Theorem 1 in Móricz [2]) use 2nd moment conditions whereas Theorem 3.1 allows for p-th moments, $1 \leqq p \leqq 2$.

THEOREM 3.1. *Let $\{X_{ik}\}$ be an array of orthogonal (in $L^p(B)$) random elements in a Banach space B of type p for some $1 \leqq p \leqq 2$. If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\left[\log_2(i+1)\right]^p\left[\log_2(k+1)\right]^p<\infty \tag{3.1}$$

for some $\alpha, \beta > 0$, then

$$\lim_{\max\{m,n\}\to\infty}\left\|\frac{1}{m^\alpha n^\alpha}\sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}\right\|=0 \quad a.s.$$

Before starting the proof of Theorem 3.1, a supportive lemma will be established.

LEMMA 3.2. *If $\{X_{ik}\}$ is an array of orthogonal (in $L^p(B)$) random elements in a Banach space B of type p for some $1 \leqq p \leqq 2$, then*

$$E\left[\left(\max_{1\leqq j\leqq m}\left\|\sum_{i=a+1}^{a+j}\sum_{k=b+1}^{b+n}X_{ik}\right\|\right)^p\right]\leqq \tag{3.2}$$

$$\leqq C_1(\log_2 2m)^p\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}E\|X_{ik}\|^p \quad \textit{for some} \quad C_1>0$$

and

$$E\left[\max_{1\leqq j\leqq m}\max_{1\leqq l\leqq n}\left\|\sum_{i=a+1}^{a+j}\sum_{k=b+1}^{b+l}X_{ik}\right\|^p\right)\leqq \tag{3.3}$$

$$\leqq C_2(\log_2 2m)^p(\log_2 2n)^p\sum_{i=a+1}^{a+m}\sum_{k=a+1}^{b+n}E\|X_{ik}\|^p \quad \textit{for some} \quad C_2>0.$$

PROOF. Since $\left\{\sum_{k=1}^{n} X_{ik}: i = 1, 2, \dots\right\}$ is a sequence of orthogonal random elements in B and B is of type p for each $n \geqq 1$, there exists $A > 0$ such that

$$E\left\|\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n} X_{ik}\right\|^p = E\left\|\sum_{i=a+1}^{a+m} Y_i\right\|^p \leqq A\sum_{i=a+1}^{a+m} E\|Y_i\|^p,$$

where $Y_i = \sum_{k=b+1}^{b+n} X_{ik}$. Then, from Proposition 2.2 and Corollary 2.4, it follows that

$$E\left[\left(\max_{1\leqq j\leqq m}\left\|\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n} X_{ik}\right\|\right)^p\right] \leqq A(\log_2 2m)^p \sum_{i=a+1}^{a+m} E\left\|\sum_{k=b+1}^{b+n} X_{ik}\right\|^p \leqq$$

$$\leqq AC(\log_2 2m)^p \sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n} E\|X_{ik}\|^p \quad \text{for some} \quad C > 0,$$

which is (3.2) with $C_1 = AC$.

Similarly, we can obtain (3.3). □

PROOF OF THEOREM 3.1. Similar arguments to the proof of Theorem 1 in Móricz [2] and Lemma 3.2 will be used. For nonnegative integers u and v,

(3.4)
$$P\left[\sup_{m>2^u \text{ and } n>2^v} \|\xi_{mn}\| > \varepsilon\right] \leqq \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} P\left[\max_{2^r<m\leqq 2^{r+1}}\max_{2^s<n\leqq 2^{s+1}} \|\xi_{mn}\| > \varepsilon\right],$$

where

$$\xi_{mn} = \frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n} X_{ik}.$$

Let m and n be integers such that $2^r < m \leqq 2^{r+1}$ and $2^s < n \leqq 2^s + 1$. Then, like in the proof of Lemma 2.5, we have

$$\max_{2^r<m\leqq 2^{r+1}}\max_{2^s<n\leqq 2^{s+1}} \|\xi_{mn}\| \leqq \|\xi_{2^r,2^s}\| + \sum_{j=1}^{3} A_{rs}^{(j)}, \tag{3.5}$$

where

$$A_{rs}^{(1)} = \frac{1}{2^{\alpha r}2^{\beta s}} \max_{2^r<m\leqq 2^{r+1}} \left\| \sum_{i=2^r+1}^{m} \sum_{k=1}^{2^s} X_{ik} \right\|,$$

$$A_{rs}^{(2)} = \frac{1}{2^{\alpha r}2^{\beta s}} \max_{2^s<n\leqq 2^{s+1}} \left\| \sum_{i=1}^{2^r} \sum_{k=2^s+1}^{n} X_{ik} \right\|,$$

$$A_{rs}^{(3)} = \frac{1}{2^{\alpha r}2^{\beta s}} \max_{2^r<m\leqq 2^{r+1}} \max_{2^s<n\leqq 2^{s+1}} \left\| \sum_{i=2^r+1}^{m} \sum_{k=2^s+1}^{n} X_{ik} \right\|.$$

From (3.4),

(3.6)

$$P\left[\max_{2^r<m\leqq 2^{r+1}} \max_{2^s<n\leqq 2^{s+1}} \|\xi_{mn}\| > \varepsilon\right] \leqq P\left[\|\xi_{2^r,2^s}\| > \frac{\varepsilon}{4}\right] + \sum_{j=1}^{3} P\left[A_{rs}^{(j)} > \frac{\varepsilon}{4}\right].$$

First,

$$\text{(3.7)}\qquad \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} P\left[\|\xi_{2^r,2^s}\| > \frac{\varepsilon}{4}\right] \leqq \left(\frac{4}{\varepsilon}\right)^p \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} E\|\xi_{2^r,2^s}\|^p =$$

$$= \left(\frac{4}{\varepsilon}\right)^p \Gamma_1 \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} \frac{1}{2^{\alpha rp}2^{\beta sp}} \sum_{i=1}^{2^r}\sum_{k=1}^{2^s} E\|X_{ik}\|^p, \quad \text{some} \quad \Gamma_1 > 0$$

$$= \left(\frac{4}{\varepsilon}\right)^p \Gamma_1 \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} \frac{1}{2^{\alpha rp}2^{\beta sp}} \left\{ \sum_{i=1}^{2^u}\sum_{k=1}^{2^v} + \sum_{i=2^u+1}^{2^r}\sum_{k=1}^{2^s} + \right.$$

$$\left. + \sum_{i=1}^{2^u}\sum_{k=2^v+1}^{2^s} + \sum_{i=2^u+1}^{2^r}\sum_{k=2^v+1}^{2^s} \right\} E\|X_{ik}\|^p =$$

$$= \left(\frac{4}{\varepsilon}\right)^p \Gamma_1 \sum_{j=1}^{4} B_{uv}^{(j)}, \quad \text{say.}$$

Using the same technique as in the proof of Corollary 2.4, it follows that

$$\text{(3.8)}\qquad B_{uv}^{(1)} = \sum_{i=1}^{2^u}\sum_{k=1}^{2^v} E\|X_{ik}\|^p \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} \frac{1}{2^{\alpha rp}2^{\beta sp}} =$$

$$= \frac{2^{(\alpha+\beta)p}}{(2^{\alpha p}-1)(2^{\beta p}-1)} \frac{1}{2^{\alpha p u}} \frac{1}{2^{\beta p v}} \sum_{i=1}^{2^u} \sum_{k=1}^{2^v} E\|X_{ik}\|^p.$$

Next,

$$B_{uv}^{(2)} = \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} \frac{1}{2^{\alpha p r}} \frac{1}{2^{\beta p s}} \sum_{i=2^u+1}^{2^r} \sum_{k=1}^{2^s} E\|X_{ik}\|^p = \tag{3.9}$$

$$= \sum_{i=2^u+1}^{\infty} \sum_{r:2^r \geqq i} \frac{1}{2^{\alpha p r}} \left\{ \sum_{k=1}^{2^r} E\|X_{ik}\| \sum_{s=v}^{\infty} \frac{1}{2^{\beta p s}} + \sum_{k=2^v+1}^{\infty} E\|X_{ik}\|^p \sum_{s:2^s \geqq k} \frac{1}{2^{\beta p s}} \right\} \leqq$$

$$\leqq \frac{2^{(\alpha+\beta)p}}{(2^{\alpha p}-1)(2^{\beta p}-1)} \left\{ \sum_{i=2^u+1}^{\infty} \sum_{k=1}^{2^v} \frac{1}{2^{\beta p v}} \frac{E\|X_{ik}\|^p}{i^{\alpha p}} + \sum_{i=2^u+1}^{\infty} \sum_{k=2^v+1}^{\infty} \frac{E\|X_{ik}\|^p}{i^{\alpha p} k^{\beta p}} \right\}.$$

Similarly,

$$B_{uv}^{(3)} \leqq \frac{2^{(\alpha+\beta)p}}{(2^{\alpha p}-1)(2^{\beta p}-1)} \left\{ \sum_{k=2^v+1}^{\infty} \sum_{i=1}^{2^u} \frac{1}{2^{\alpha p u}} \frac{E\|X_{ik}\|^p}{k^{\beta p}} + \right. \tag{3.10}$$

$$\left. + \sum_{k=2^v+1}^{\infty} \sum_{i=2^u+1}^{\infty} \frac{E\|X_{ik}\|^p}{i^{\alpha p} k^{\beta p}} \right\}$$

and

$$B_{uv}^{(4)} = \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} \frac{1}{2^{\alpha r p} 2^{\beta s p}} \sum_{i=2^u+1}^{2^r} \sum_{k=2^v+1}^{2^s} E\|X_{ik}\|^p = \tag{3.11}$$

$$= \sum_{i=2^u+1}^{\infty} \sum_{k=2^v+1}^{\infty} E\|X_{ik}\|^p \sum_{r:2^r>i} \frac{1}{2^{\alpha r p}} \sum_{s:2^s>k} \frac{1}{2^{\beta s p}} \leqq$$

$$\leqq \sum_{i=2^u+1}^{\infty} \sum_{k=2^v+1}^{\infty} \frac{E\|X_{ik}\|^p}{i^{\alpha p} k^{\beta p}}.$$

Secondly,

$$\sum_{r=u}^{\infty} \sum_{s=v}^{\infty} P\left[A_{rs}^{(1)} > \frac{\varepsilon}{4}\right] \leqq \tag{3.12}$$

$$\leqq \sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\left(\frac{4}{\varepsilon}\right)^p \frac{1}{2^{\alpha pr}}\frac{1}{2^{\beta ps}} E\left(\max_{2^r<m\leqq 2^{r+1}}\left\|\sum_{i=2^r+1}^{m}\sum_{k=1}^{2^s} X_{ik}\right\|\right)^p \leqq$$

$$\leqq \sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\left(\frac{4}{\varepsilon}\right)^p \frac{1}{2^{\alpha pr}}\frac{1}{2^{\beta ps}}\Gamma_2(\log_2 2\cdot 2^r)^p \sum_{i=2^r+1}^{2^{r+1}}\sum_{k=1}^{2^s} E\|X_{ik}\|^p,$$

for some Γ_2, by Lemma 3.3,

$$\leqq \left(\frac{4}{\varepsilon}\right)^p \Gamma_2 \sum_{i=2^u+1}^{\infty}\sum_{s=v}^{\infty}\frac{1}{2^{\beta ps}}\sum_{k=1}^{2^s}\frac{E\|X_{ik}\|^p}{i^{\alpha p}}2^{(\alpha+1)p}\left\{1+\left[\log_2(i+1)\right]^p\right\} =$$

$$= \left(\frac{4}{\varepsilon}\right)^p \Gamma_2 \sum_{i=2^u+1}^{\infty}\sum_{s=v}^{\infty}\frac{2^{(\alpha+1)p}}{2^{\beta ps}}\left[\sum_{k=1}^{2^v}+\sum_{k=2^v+1}^{2^s}\right]\frac{E\|X_{ik}\|^p}{i^{\alpha p}}\left\{1+\log_2^p(i+1)\right\} =$$

$$= 2^{\alpha p}\left(\frac{8}{\varepsilon}\right)^p \Gamma_2\left\{\frac{1}{2^{\beta pv}}\frac{2^{\beta p}}{2^{\beta p}-1}\sum_{i=2^u+1}^{\infty}\sum_{k=1}^{2^v}\frac{E\|X_{ik}\|^p}{i^{\alpha p}}\left[1+\log_2^p(i+1)\right]+\right.$$

$$\left.+\sum_{i=2^u+1}^{\infty}\left(\sum_{i=2^v+1}^{\infty}\sum_{s:2^s>k}\frac{1}{2^{\beta ps}}\right)\frac{E\|X_{ik}\|^p}{i^{\alpha p}}\left[1+\log_2^p(i+1)\right]\right\} \leqq$$

$$\leqq 2^{\alpha p}\left(\frac{8}{\varepsilon}\right)^p \Gamma_2\left\{\frac{1}{2^{\beta pv}}\frac{2^{\beta p}}{2^{\beta p}-1}\sum_{i=2^u+1}^{\infty}\sum_{k=1}^{2^v}\frac{E\|X_{ik}\|^p}{i^{\alpha p}}\left[1+\log_2^p(i+1)\right]+\right.$$

$$\left.+\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\left[1+\log_2^p(i+1)\right]\right\}.$$

Next,

$$\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[A_{rs}^{(2)}>\frac{\varepsilon}{4}\right]\leqq \tag{3.13}$$

$$\leqq \sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\left(\frac{4}{\varepsilon}\right)^p\frac{1}{2^{\alpha pr}2^{\beta ps}}E\left(\max_{2^s<n\leqq 2^{s+1}}\left\|\sum_{i=1}^{2^r}\sum_{k=2^s+1}^{n}X_{ik}\right\|\right)^p\leqq$$

$$\leqq 2^{\beta p}\left(\frac{8}{\varepsilon}\right)^p \Gamma_3\left\{\frac{1}{2^{\alpha pu}}\frac{2^{\alpha p}}{2^{\alpha p}-1}\sum_{k=2^v+1}^{\infty}\sum_{i=1}^{2^u}\frac{E\|X_{ik}\|^p}{k^{\beta p}}\left[1+\log_2^p(k+1)\right]+\right.$$

$$+\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\left[1+\log_2^p(k+1)\right]\Bigg\},$$

for some $\Gamma_3 > 0$, and

$$\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[A_{rs}^{(3)} > \frac{\varepsilon}{4}\right] \leqq \tag{3.14}$$

$$\leqq \sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\left(\frac{4}{\varepsilon}\right)^p\frac{1}{2^{\alpha pr}2^{\beta ps}}E\left(\max_{2^r<m\leqq 2^{r+1}}\max_{2^s<n\leqq 2^{s+1}}\left\|\sum_{i=2^r+1}^{m}\sum_{k=2^s+1}^{n}X_{ik}\right\|\right)^p \leqq$$

$$\leqq \left(\frac{4}{\varepsilon}\right)^p\Gamma_4\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\frac{1}{2^{\alpha pr}2^{\beta ps}}\left(\log_2 2^{r+1}\right)^p\left(\log_2 2^{s+1}\right)^p\cdot$$

$$\cdot\sum_{i=2^r+1}^{2^{r+1}}\sum_{k=2^s+1}^{2^{s+1}}E\|X_{ik}\|^p \leqq 2^{(\alpha+\beta)p}\left(\frac{16}{\varepsilon}\right)^p\Gamma_4\cdot$$

$$\cdot\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\left[1+\log_2^p(i+1)\right]\left[1+\log_2^p(k+1)\right].$$

Finally, if

$$\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\left[\log_2(i+1)\right]^p\left[\log_2(k+1)\right]^p < \infty,$$

then by combining the results in (3.4), (3.6), (3.7), (3.12), (3.13), and (3.14), we have

$$P\Big[\sup_{m>2^u \text{ and } n>2^v}\|\xi_{mn}\| > \varepsilon\Big] < \infty.$$

The proof is completed by following the similar steps in the proof of Lemma 2.5 to obtain

$$P\Big[\sup_{m\geqq 1 \text{ and } n\geqq 1}\|\xi_{mn}\| > \varepsilon\Big] < \infty. \quad \square$$

For $p = 2$, Móricz [2] showed the necessity of condition (3.1). His result is stated in Theorem 3.3.

THEOREM 3.3. *If an array $\{\sigma_{ik} \geqq 0\}$ of real numbers is such that*

$$\frac{\sigma_{ik}}{i^\alpha k^\beta} \geqq \max\left\{\frac{\sigma_{i+1,k}}{(i+1)^\alpha k^\beta}, \frac{\sigma_{i,k+1}}{i^\alpha (k+1)^\beta}\right\} \quad \textit{for} \quad i,k \geqq 1$$

with some $\alpha, \beta > 0$, and if condition (3.1) *does not hold, that is,*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \frac{\sigma_{ik}^2}{i^{2\alpha}k^{2\beta}}\left[\log_2(i+1)\right]^2\left[\log_2(k+1)\right]^2 = \infty, \tag{3.15}$$

then there exists an array of orthogonal random variables $\{X_{ik}\}$ such that

$$E(X_{ik}) = 0 \quad \textit{and} \quad E|X_{ik}|^2 \leqq \sigma_{ik}^2, \quad \textit{for all} \quad i,k,$$

but

$$\lim_{\max\{m,n\}\to\infty} \left|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n} X_{ik}\right| = \infty \quad a.s.$$

If, in addition, for every $r \geqq 1$,

$$\sum_{i=r}^{\infty}\sum_{k=r}^{\infty} \frac{\sigma_{ik}^2}{i^{2\alpha}k^{2\beta}}\left[\log_2(i+1)\right]^2\left[\log_2(k+1)\right]^2 = \infty, \tag{3.16}$$

then we have

$$\lim_{\min\{m,n\}\to\infty} \left|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n} X_{ik}\right| = \infty \quad a.s.$$

It remains an open question as to whether Móricz's very long proof of the necessity of (3.1) for the strong law of arrays of orthogonal random variables can be appropriately modified to show that (3.1) is also necessary in the general Banach space case when $1 < p < 2$. Theorem 3.3 does provide the necessity of (3.1) for Theorem 3.1 when $p = 2$.

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ON COMPLETELY P-ADDITIVE FUNCTIONS WITH RESPECT TO INTERVAL-FILLING SEQUENCES OF TYPE P

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Notation. Let $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ denote the set of positive integers, reals and complex numbers, respectively. Let $N \in \mathbf{N}$, $-\infty < p_0 < p_1 < \ldots < p_N < < +\infty$ be fixed real numbers and $P = \{p_0, p_1, \ldots, p_N\}$. Denote by Λ the set of sequences $\lambda = (\lambda_n) \colon \mathbf{N} \to \mathbf{R}$ that satisfy

$$\text{(i)} \qquad |\lambda_n| > |\lambda_{n+1}| > 0 \quad \text{for every} \quad n \in \mathbf{N}$$

and

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} |\lambda_n| < \infty.$$

For $\lambda \in \Lambda$ and $k \in \mathbf{N} \cup \{0\}$ define

$$I_k(P,\lambda) = \left[p_0 \sum_{n=k+1}^{\infty} \lambda_n^+ - p_N \sum_{n=k+1}^{\infty} \lambda_n^-, \, -p_0 \sum_{n=k+1}^{\infty} \lambda_n^- + p_N \sum_{n=k+1}^{\infty} \lambda_n^+ \right]$$

where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. According to the definition in [1] a sequence $\lambda \in \Lambda$ is called interval-filling of type P, if for any $x \in I_0(P,\lambda)$ there exists a "coefficient sequence" $(\varepsilon_n) \colon \mathbf{N} \to P$ such that

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n.$$

We shall denote the set of interval-filling sequences of type P by $IF(P)$. Now we are ready to generalize the notion of completely additive functions.

DEFINITION. Let X be a (real or complex) Banach space, $\lambda = (\lambda_n) \in \in IF(P)$ and $f \colon I_0(P,\lambda) \to X$. We call f completely P-additive with respect to (λ_n), if there exists a sequence $(a_n) \colon \mathbf{N} \to X$ such that $\sum a_n$ is absolutely convergent and

$$\text{(1)} \qquad f\left(\sum_{n=1}^{\infty} \delta_n \lambda_n \right) = \sum_{n=1}^{\infty} \delta_n a_n$$

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holds for every coefficient sequence $(\delta_n)\colon \mathbf{N} \to P$.

Using the above notations set

$$CA(P,\lambda,X) = \{ f \in X^{I_0(P,\lambda)} | \\ f \text{ is completely } P\text{-additive with respect to } \lambda \}.$$

Naturally it would be much more convenient to write $f(\lambda_n)$ instead of a_n as it appears in [2], but here $\lambda_n \notin I_0(P,\lambda)$ may occur leaving $f(\lambda_n)$ undefined. Nevertheless the sequence (a_n) is uniquely determined by λ and f. (Let throughout this paper $\mathbf{K}$ denote either $\mathbf{R}$ or $\mathbf{C}$.)

LEMMA 1. *Let X be a Banach space over $\mathbf{K}$, $\lambda = (\lambda_n) \in IF(P)$ and $f \in CA(P,\lambda,X)$. Then the sequence (a_n) described in the Definition is uniquely given by*

$$a_n = \frac{1}{p_1 - p_0}\left(f\left(p_1\lambda_n + \sum_{k\in\mathbf{N}\setminus\{n\}} p_0\lambda_k \right) - f\left(\sum_{k=1}^{\infty} p_0\lambda_k \right)\right) \tag{2}$$

for every $n \in \mathbf{N}$.

PROOF. Applying (1) for coefficient sequences of the form $(p_0, p_0, \ldots)$ and $(p_0, \ldots, p_0, p_1, p_0, \ldots)$ respectively we get

$$f\left(\sum_{k=1}^{\infty} p_0\lambda_k \right) = \sum_{k=1}^{\infty} p_0 a_k$$

and

$$f\left(p_1\lambda_n + \sum_{k\in\mathbf{N}\setminus\{n\}} p_0\lambda_k \right) = p_1 a_n + \sum_{k\in\mathbf{N}\setminus\{n\}} p_0 a_k.$$

The difference of these two equations gives (2).

For a fixed $\lambda \in IF(P)$ define

$$\mathbf{c}_n(f) = \frac{1}{\lambda_n} a_n \quad (n \in \mathbf{N},\ f \in CA(P,\lambda,X))$$

where (a_n) is the sequence given by (2). When we wish to replace the sequence λ with another one (say μ), we will write $\mathbf{c}_n^{[\mu]}(f)$ instead of $\mathbf{c}_n(f)$. With this notation (1) can be written in the form

$$f\left(\sum_{n=1}^{\infty} \delta_n\lambda_n \right) = \sum_{n=1}^{\infty} \delta_n\lambda_n\mathbf{c}_n(f). \tag{3}$$

Conversely, if $f: I_0(P,\lambda) \to X$ and there exists a sequence $(\mathbf{c}_n(f)) : \mathbf{N} \to X$ such that $\sum |\lambda_n| \, \|\mathbf{c}_n(f)\|$ is convergent and (3) holds for every coefficient sequence $(\delta_n): \mathbf{N} \to X$, then $f \in CA(P,\lambda,X)$. For example, when $(\mathbf{c}_n(f))$ is a constant sequence, f is linear and completely P-additive with respect to (λ_n). It is worth to remark the following fact (whose proof is evident).

LEMMA 2. *If $\lambda \in IF(P)$ and X is a Banach space over $\mathbf{K}$, then $CA(P,\lambda,X)$ is a linear space over $\mathbf{K}$ and the mapping $\mathbf{c}_n: CA(P,\lambda,X) \to X$ is linear for every $n \in \mathbf{N}$.*

When $f: I_0(P,\lambda) \to X$ is linear i.e. there exists $\mathbf{c} \in X$such that $f(t) = t\mathbf{c}$ $(t \in I_0(P,\lambda))$, then f is obviously completely P-additive with respect to (λ_n), since $\mathbf{c}_n(f) = \mathbf{c}$ $(n \in \mathbf{N})$. Our main purpose is to prove the converse of this fact, but the way we do it (which is mostly a generalization of the method followed in [2]) requires a bit sharper hypothesis. For $m \in \mathbf{N} \cup \{0\}$ let $T^m\lambda$ denote the sequence whose nth element is λ_{m+n}. Put

$$IF(P)^\infty = \left\{ \lambda \in \Lambda | T^m\lambda \in IF(P) \quad \text{for every} \quad m \in \mathbf{N} \cup \{0\} \right\}.$$

The set $IF(P)^\infty$ is described by Theorem 4 in [1]. It also turns out, that in some cases (e.g. $P = \{0,1,\dots,N\}$) $IF(P)^\infty = IF(P)$, while in some other cases (e.g. $P = \{0,1,3,4\}$) $IF(P)^\infty \subsetneq IF(P)$.

THEOREM 1. *If $\lambda = (\lambda_n) \in IF(P)^\infty$ and $f \in CA(P,\lambda,\mathbf{R})$, then f is linear i.e. $f(x) = cx$ $(x \in I_0(P,\lambda))$ for some constant $c \in \mathbf{R}$.*

PROOF. For simplicity first suppose $p_0 = 0$. If $\mathbf{c}_n(f) = c$ for some constant c and for every $n \in \mathbf{N}$, then choosing arbitrarily $x \in I_0(P,\lambda)$ there exists $(\delta_n): \mathbf{N} \to P$ such that $x = \sum_{n=1}^\infty \delta_n\lambda_n$ and $f(x) = f\left(\sum_{n=1}^\infty \delta_n\lambda_n\right) = \sum_{n=1}^\infty \delta_n\lambda_n c = cx$ as it is stated. Throughout the rest of the proof (of the case $p_0 = 0$) we shall assume that $\mathbf{c}_n(f)$ is not a constant sequence. In this case there exist $m, r \in \mathbf{N}$ such that $\mathbf{c}_m(f) < \mathbf{c}_r(f)$. To simplify our notations set $c_n = \mathbf{c}_n(f)$ $(n \in \mathbf{N})$. We may (and will) assume that $c_1 > 0$, $c_n \neq 0$ for every $n \in \mathbf{N}$ and there exists $k \in \mathbf{N}$ for which $c_k < 0$. Indeed, the set $C_f = \{ c_n \mid n \in \mathbf{N} \}$ is countable thus we can choose a number $\gamma \in]c_m, c_r[\setminus C_f$ and define $f^\diamond: I_0(P,\lambda) \to \mathbf{R}$ by $f^\diamond(x) = f(x) - \gamma x$; then $f^\diamond$ and $-f^\diamond$ are completely P-additive with respect to (λ_n) by Lemma 2 and writing (the proper) one of them instead of f the above assumptions will hold and it clearly suffices to prove that the regarded function is linear.

We shall isolate two lemmas inside the proof.

LEMMA 3. *For every $n \in \mathbf{N}$ there exists $s(n) \in \{n, n+1, n+2, \dots, \infty\}$ such that*

$$p_1 c_n |\lambda_n| \leqq p_N \sum_{k=n+1}^{s(n)} c_k |\lambda_k|. \tag{4}$$

PROOF. Fix n and first assume $\lambda_n > 0$ adding it to our previous assumptions. Applying Theorem 4 in [1] we have

$$p_1\lambda_n \leqq p_N \sum_{k=n+1}^{\infty} (\lambda_k^+ + \lambda_k^-),$$

thus defining

$$x = p_1\lambda_n - \sum_{k=n+1}^{\infty} p_N\lambda_k^-$$

one can simply derive $x \in I_n(P,\lambda) = I_0(P,T^n\lambda)$. Hence there exist $\sigma \in P$ $(r = n+1, n+2, \ldots)$ satisfying $x = \sum_{r=n+1}^{\infty} \sigma_r\lambda_r$. An inductive construction of this coefficient sequence is described in [1] (in the proof of Theorem 1): let $s_n = 0$ and if s_{r-1} is already defined (for a considered $r > n$), put

$$A_r = \left\{ p \in P \mid x \in s_{r-1} + p\lambda_r + I_r(P,\lambda) \right\},$$

$\sigma_r \in A_r$ arbitrarily and $s_r = s_{r-1} + \sigma_r\lambda_r$. Now specify the coefficient sequence so that it should increase in the quickest possible way:

$$\sigma_r = \begin{cases} \max A_r & \text{if } \lambda_r > 0 \\ \min A_r & \text{if } \lambda_r < 0. \end{cases}$$

Following this representation of x we find that either

$$x = \sum_{k=n+1}^{\infty} p_N\lambda_k^+ \tag{i}$$

or there exists $m \in \mathbf{N}$ for which

$$x < \sum_{k=n+1}^{n+m} p_N\lambda_k^+ - \sum_{k=n+m+1}^{\infty} p_n\lambda_k^- \tag{ii-1}$$

and in the latter case there exist $\delta_k \in P$ $(k = n+m+1, n+m+2, \ldots)$ and $\alpha \in P$ such that $\alpha \neq p_N$ if $\lambda_{n+m} > 0$ while $\alpha \neq 0$ if $\lambda_{n+m} < 0$ and

$$x = \sum_{k=n+1}^{n+m-1} p_N\lambda_k^+ + \alpha\lambda_{n+m} + \sum_{k=n+m+1}^{\infty} \delta_k\lambda_k. \tag{ii-2}$$

In the first case the definition of x and formula (i) imply

$$p_1\lambda_n + \sum_{k=n+1}^{\infty} p_N(-\lambda_k^-) = \sum_{k=n+1}^{\infty} p_N\lambda_k^+.$$

These sums are taken from the set $S(P,\lambda)$ thus we can apply (3) to obtain two representations of $f(x)$ in the equation

$$p_1\lambda_n c_n + \sum_{k=n+1}^{\infty} p_N(-\lambda_k^-)c_k = \sum_{k=n+1}^{\infty} p_N\lambda_k^+ c_k,$$

from which noticing $\lambda_n^+ + \lambda_n^- = |\lambda_n|$ we get

$$p_1 c_n\lambda_n = p_N \sum_{k=n+1}^{\infty} c_k|\lambda_k|, \tag{5}$$

a special case of (4).

In the second case set

$$y = p_1\lambda_n - \sum_{k=n+1}^{n+m} p_N\lambda_k^-,$$

then due to inequality (ii-1) and the definition of x

$$y < \sum_{k=n+1}^{n+m} p_N\lambda_k^+$$

follows. Now put

$$z = y + \sum_{k=n+m+1}^{\infty} p_N\chi_S(k)\lambda_k,$$

where χ_S denotes the characteristic function of the set

$$S = \{r \in \mathbf{N} \mid c_r\lambda_r > 0\}$$

over $\mathbf{N}$. The above inequality for y then implies

$$z < \sum_{k=n+1}^{\infty} p_N\lambda_k^+.$$

On the other hand in view of (ii-2) we have

$$z \geqq y - \sum_{k=n+m+1}^{\infty} p_N \lambda_k^- = x \geqq \sum_{k=n+1}^{n+m-1} p_n \lambda_k^+ - \sum_{k=n+m}^{\infty} p_N \lambda_k^- .$$

Hence

$$z - \sum_{k=n+1}^{n+m-1} p_N \lambda_k^+ \in I_{n+m-1}(P,\lambda),$$

therefore there exist $\varepsilon_r \in P$ $(r = n+m, n+m+1, \ldots)$ such that

$$p_1 \lambda_n - \sum_{k=n+1}^{n+m} p_N \lambda_k^- + \sum_{k=n+m+1}^{\infty} p_n \chi_S(k) \lambda_k = z = \sum_{k=n+1}^{n+m-1} p_N \lambda_k^+ + \sum_{k=n+m}^{\infty} \varepsilon_k \lambda_k .$$

Now we can apply (3) to both expansion of $f(z)$ and an upper estimate to the second sum:

$$p_1 c_n \lambda_n + \sum_{k=n+1}^{n+m} p_N c_k (-\lambda_k^-) + \sum_{k=n+m+1}^{\infty} p_N \chi_S(k) c_k \lambda_k = f(z) =$$

$$= \sum_{k=n+1}^{n+m-1} p_N c_k \lambda_k^+ + \sum_{k=n+m}^{\infty} \varepsilon_k c_k \lambda_k \leqq \sum_{k=n+1}^{n+m-1} p_N c_k \lambda_k^+ + \sum_{k=n+m}^{\infty} p_N \chi_S(k) c_k \lambda_k ,$$

consequently

$$p_1 c_n \lambda_n \leqq \sum_{k=n+1}^{n+m-1} p_N c_k (\lambda_k^+ + \lambda_k^-) + \tag{6}$$

$$+ p_N c_{n+m} \lambda_{n+m}^- + p_N \chi_S(n+m) c_{n+m} \lambda_{n+m} = \sum_{k=n+1}^{s(n)} p_N c_k |\lambda_k| ,$$

where

$$s(n) = \begin{cases} n+m, & \text{if } c_{n+m} > 0, \\ n+m-1, & \text{if } c_{n+m} < 0. \end{cases}$$

So we have proved (4) for $\lambda_n > 0$. In case $\lambda_n < 0$ we may consider the sequence $-\lambda = (-\lambda_1, -\lambda_2, \ldots)$ which obviously also satisfies $-\lambda \in IF(P)^\infty$ with $I_0(P,-\lambda) = -I_0(P,\lambda)$, thus it is possible to apply our above result to the function $g \colon I_0(P,-\lambda) \to \mathbf{R}$ defined by $g(y) = -f(-y)$. But it gives (4) again, since $c_k^{[-\lambda]}(g) = c_k^{[\lambda]}(f)$ $(k \in \mathbf{N})$ as one can easily show.

LEMMA 4. *Set* $U = \{n \in \mathbf{N} \mid c_n > 0\}$. *The inequality*

$$p_1 c_n |\lambda_n| \leqq p_N \sum_{k=n+1}^{\infty} c_k |\lambda_k| \tag{7}$$

holds for every $n \in U$.

PROOF. First observe that in case $c_n > 0$ the left side of (4) (in Lemma 3) is positive, consequently the sum on the right side is non-void, i.e. $s(n) \geqq n+1$, furthermore it must have at least one positive summand, that is, there exists $m \in \mathbf{N}$ with $n+1 \leqq m \leqq s(n)$ and $c_m > 0$. If moreover m is the greatest integer with the above properties, (4) remains true by ignoring the negative summands of index k with $m < k \leqq s(n)$, i.e. putting $s(n) = m$. In other words, we may assume $s(n) > n$ and $c_{s(n)} > 0$ whenever $s(n) < \infty$. Now let $m_1 = s(n)$ and for $k \geqq 2$ define

$$m_k = \begin{cases} s(m_{k-1}), & \text{if } m_{k-1} \in \mathbf{N} \\ \infty, & \text{if } m_{k-1} = \infty \end{cases}$$

inductively. Then the sequence $(m_k)\colon \mathbf{N} \to \mathbf{N} \cup \{\infty\}$ is non-decreasing and in case $m_k < \infty$ we have $c_{m_k} > 0$ and $m_k < m_{k+1}$. Therefore $\sup\{ m_k \mid k \in \mathbf{N}\} = \infty$ and

$$p_1 c_n |\lambda_n| \leqq p_N \sum_{r=n+1}^{m_k} c_r |\lambda_r| \tag{8}$$

holds for every $k \in \mathbf{N}$, which can be proved by induction: for $k = 1$ it follows from Lemma 3; if $k \geqq 2$ and (8) holds for $k-1$, then in case $m_k = m_{k-1} = \infty$ it is the same for k while in case $m_{k-1} \in \mathbf{N}$ consider the inequalities

$$p_1 c_n |\lambda_n| \leqq p_1 c_n |\lambda_n| + p_1 c_{m_{k-1}} \left|\lambda_{m_{k-1}}\right| \leqq$$

$$\leqq p_N \sum_{r=n+1}^{m_{k-1}} c_r |\lambda_r| + p_N \sum_{r=m_{k-1}+1}^{s(m_{k-1})} c_r |\lambda_r| = p_N \sum_{r=n+1}^{m_k} c_r |\lambda_r|.$$

Since the sum in (8) is convergent, it immediately implies (7).

Now we can complete the proof of the theorem for the case $p_0 = 0$. Notice, that applying Lemma 4 for the function $-f$ we can write an inequality in case $c_n < 0$ as well, actually the negative of (7), since $\mathbf{c}_k(-f) = -\mathbf{c}_k(f)$. So we have

$$p_1 c_n |\lambda_n| \geqq p_N \sum_{k=n+1}^{\infty} c_k |\lambda_k| \tag{9}$$

for every $n \in \mathbf{N} \setminus U$. Due to our assumptions there exists $m \in \mathbf{N}$ such that $c_m > 0$ and $c_{m+1} < 0$. First apply (7) for $n = m$ and then apply (9) for $n = m + 1$ to obtain

$$0 < p_1 c_m |\lambda_m| \leqq p_N c_{m+1} |\lambda_{m+1}| + p_N \sum_{k=m+2}^{\infty} c_k |\lambda_k| \leqq$$
$$\leqq (p_N + p_1) c_{m+1} |\lambda_{m+1}| < 0,$$

which is a contradiction.

When $p_0 \neq 0$, introduce $p_j^\circ = p_j - p_0$ $(j = 0, 1, \dots, N)$ and $P^\circ = \{ p_j^\circ \,|\, j = 0, 1, \dots, N \}$. A simple argument (see [1]) shows, that $\lambda \in IF(P)^\infty$ implies $\lambda \in IF(P^\circ)^\infty$. Define a function $g: I_0(P^\circ, \lambda) \to \mathbf{R}$ by

$$g(x) = f\left(x + p_0 \sum_{n=1}^{\infty} \lambda_n\right) - \sum_{n=1}^{\infty} p_0 a_n$$

(where $(a_n): \mathbf{N} \to \mathbf{R}$ is the sequence in the Definition). For any sequence $(\varepsilon_n): \mathbf{N} \to P^\circ$ clearly $\varepsilon_n + p_0 \in P$ $(n \in \mathbf{N})$, thus

$$g\left(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) = f\left(\sum_{n=1}^{\infty} (\varepsilon_n + p_0) \lambda_n\right) - \sum_{n=1}^{\infty} p_0 a_n =$$
$$= \sum_{n=1}^{\infty} (\varepsilon_n + p_0) a_n - \sum_{n=1}^{\infty} p_0 a_n = \sum_{n=1}^{\infty} \varepsilon_n a_n,$$

i.e. g is completely P°-additive with the same sequence (a_n), thus applying the above result for P° and g it follows that $a_n = c\lambda_n$ with some constant $c \in \mathbf{R}$, consequently f is linear.

It is worth to mention the following consequences of Theorem 1.

THEOREM 2. *If $\lambda = (\lambda_n) \in IF(P)^\infty$ and $f \in CA(P, \lambda, \mathbf{C})$, then f is linear i.e. $f(x) = cx$ $\big(x \in I_0(P, \lambda)\big)$ for some constant $c \in \mathbf{C}$.*

PROOF. By hypothesis (1) holds for some sequence $(a_n): \mathbf{N} \to \mathbf{C}$ whose sum is absolutely convergent and for every coefficient sequence $(\delta_n): \mathbf{N} \to P$. Then the real valued functions $\operatorname{Re} f$ and $\operatorname{Im} f$ also satisfy (1) with the sequences $(\operatorname{Re} a_n)$ and $(\operatorname{Im} a_n)$, respectively, since Re and Im are continuous, additive and real homogeneous functions; furthermore the sums $\sum \operatorname{Re} a_n$ and $\sum \operatorname{Im} a_n$ are absolutely convergent as well. It means, that $\operatorname{Re} f$ and $\operatorname{Im} f$ are completely P-additive with respect to (λ_n), hence they are linear by Theorem 1. This proves that f is linear itself.

THEOREM 3. *If* $\lambda = (\lambda_n) \in IF(P)^\infty$, *$X$ is a Banach space over* $\mathbf{K}$ *and* $f\colon I_0(P,\lambda) \to X$ *is completely P-additive with respect to* (λ_n), *then f is linear, i.e. there exists* $c \in X$ *such that* $f(t) = tc$ $\left(t \in I_0(P,\lambda)\right)$.

PROOF. According to the hypothesis there exists a sequence $(a_n)\colon \mathbf{N} \to X$ with

$$\sum_{n=1}^{\infty} \|a_n\| < \infty$$

such that (1) holds for every coefficient sequence $(\delta_n)\colon \mathbf{N} \to P$. Now choose $\phi \in X^*$ (where X^* denotes the set of continuous linear functionals over X) arbitrarily and let $F = \phi \circ f$. Then

$$\sum_{n=1}^{\infty} |\phi(a_n)| \leqq \|\phi\| \sum_{n=1}^{\infty} \|a_n\| < \infty,$$

and

$$F\left(\sum_{n=1}^{\infty} \delta_n \lambda_n\right) = \phi\left(f\left(\sum_{n=1}^{\infty} \delta_n \lambda_n\right)\right) = \phi\left(\sum_{n=1}^{\infty} \delta_n a_n\right) = \sum_{n=1}^{\infty} \delta_n \phi(a_n)$$

holds for every coefficient sequence (δ_n). Hence $F \in CA(P,\lambda,\mathbf{K})$, therefore F is linear by Theorem 1 and Theorem 2. Thus there exists $c_\phi \in \mathbf{K}$ such that

$$\phi\big(f(t)\big) = F(t) = c_\phi t \quad \left(t \in I_0(P,\lambda)\right). \tag{10}$$

Setting $t_0 \in I_0(P,\lambda) \setminus \{0\}$ it implies

$$c_\phi = \frac{1}{t_0} c_\phi t_0 = \frac{1}{t_0}\phi\big(f(t_0)\big) = \phi\left(\frac{1}{t_0} f(t_0)\right),$$

from which with the notation $c = \frac{1}{t_0} f(t_0)$

$$\phi\big(f(t)\big) = tc_\phi = t\phi(c) = \phi(tc) \tag{11}$$

follows for every $t \in I_0(P,\lambda)$ and for every $\phi \in X^*$, since c does not depend on ϕ. If there exists $t \in I_0(P,\lambda)$ with $f(t) \neq tc$, then due to the Hahn–Banach theorem there exists $\phi_t \in X^*$ such that $\phi_t\big(f(t)\big) \neq \phi_t(tc)$ in contradiction with (11). This proves the theorem.

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LAGRANGE INTERPOLATION ON GENERALIZED JACOBI ZEROS WITH ADDITIONAL NODES[1]

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1. Introduction

Let $\mathcal{L}_m\left(v^{(\alpha,\beta)};f\right)$ be the Lagrange polynomial interpolating a given continuous function f at the zeros $\left\{x_{k,m}^{(\alpha,\beta)}\right\}_{k=1}^{m}$ of the mth Jacobi polynomial $p_m^{(\alpha,\beta)}$. It is well known that, if $\delta=\max(\alpha,\beta)>-\frac{1}{2}$, then the mth Lebesgue constant satisfies

$$\left\|\mathcal{L}_m\left(v^{(\alpha,\beta)}\right)\right\|_\infty=\sup_{\|f\|=1}\left\|\mathcal{L}_m\left(v^{(\alpha,\beta)};f\right)\right\|_\infty=\mathcal{O}\left(m^{\delta+\frac{1}{2}}\right).$$

Nevertheless, if in addition to the knots $\left\{x_{k,m}^{(\alpha,\beta)}\right\}_{k=1}^{m}$ we consider a suitable number of points near the endpoints ±1, then the new interpolating process $\overline{\mathcal{L}}_m\left(v^{(\alpha,\beta)};f\right)$ is optimal, in the sense that $\left\|\overline{\mathcal{L}}_m\left(v^{(\alpha,\beta)}\right)\right\|_\infty=\mathcal{O}(\log m)$. Similar result holds when the starting knots are the zeros of $p_m(w)$ with $w(x)=v^{(\alpha,\beta)}(x)|x-t|^\gamma$, $-1<t<1$, $-1<\gamma<0$ (see [9, Remark 3]).

The technique of adding nodes near the points ±1 was first introduced by Szabados [14]. Recently, this procedure has been extensively used by many authors in different contexts (see e.g. [2], [3], [5], [8], [9], [10], [11], [13]).

Nevertheless, if the interpolation knots are the zeros of $p_m(w)$ where w is the weight defined above but with the exponent $\gamma>0$, then the additional nodes near ±1 have no positive influence on the behaviour of the interpolating process (see Remark 1).

In this paper, generalizing the previous procedure we show that, if the interpolation knots are the zeros of $p_m(w)$ with $\gamma>0$, then adding a suitable number of nodes near ±1 and t we obtain an optimal interpolating process. Furthermore, we give estimates of simultaneous approximation. Finally, we consider the case of multiple additional nodes.

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2. Preliminaries

Spaces of functions. We will consider functions f with domain $[-1,1]$, and define the space of functions $C^{(q)}$ on the interval $[-1,1]$ in the usual way; thus $f \in C^{(q)}$ if and only if f is continuous with its derivatives $f^{(j)}$ $(j = 1, \dots, q)$ on $[-1,1]$.

Let

$$\|f\| = \max_{x\in[-1,1]} |f(x)|, \quad \|f\|_k = \max_{0\leqq i\leqq k} \left\| f^{(i)} \right\|, \quad \|f\|_0 = \|f\|.$$

We define the modulus of continuity $\omega(f;\ )$ of the function f by

$$\omega(f;\delta) = \sup_{h\leqq\delta} \|\Delta_h f\|_{[-1,1-h]},$$

where $\Delta_h f(x) = f(x+f) - f(x)$. Then we write $f \in \mathrm{Lip}_M \lambda$ if $\omega(f;\delta) \leqq \leqq M\delta^\lambda$. Finally, we set

$$E_m(f) = \min_{P_m\in\mathcal{P}_m} \|f - P_m\|,$$

where f is a given continuous function and $\mathcal{P}_m$ denotes the set of polynomials of degree at most m.

Special weights. Let

$$v^{(\alpha,\beta)}(x) = (1-x)^a(1+x)^b, \qquad a, b \in \mathbf{R},\ x \in [-1,1].$$

The *generalized Jacobi weight* $w \in \mathrm{GJ}$ is defined by

$$w(x) = g(x)v^{(\alpha,\beta)}(x)|t-x|^\gamma, \qquad x \in [-1,1], \tag{2.1}$$

where $\alpha > -1$, $\beta > -1$, $\gamma > -1$, and $-1 < t < 1$. Here g is a positive continuous function and its modulus of continuity ω satisfies $\int_0^1 \omega(g;t)t^{-1}\,dt < < \infty$. Now, let $\{p_m(w)\}_{m=0}^\infty$ be the system of orthonormal polynomials corresponding to the weight function $w \in \mathrm{GJ}$, that is, $p_m(w)$ is a polynomial of degree m with positive leading coefficient $\gamma_m(w)$ and

$$\int_{-1}^1 p_m(w;t)p_n(w;t)w(t)\,dt = \delta_{m,n}.$$

We denote by $\{x_{j,m}(w)\}_{j=1}^{m}$ the zeros of $p_m(w)$ indexed in increasing order, and by $\lambda_{j,m}(w) = \lambda_m\big(w; x_{j,m}(w)\big)$, $j = 1, 2, \dots, m$, the corresponding Cotes numbers, where

$$\lambda_m(w;x) = \left[\sum_{k=0}^{m-1} p_k^2(w;x)\right]^{-1}, \qquad x \in \mathbf{R},$$

is the mth Christoffel function.

3. Interpolation by generalized Jacobi zeros

Lagrange interpolation with simple additional nodes. We denote by $\mathcal{L}_m(w;f)$ the Lagrange polynomial interpolating f at the zeros $x_{j,m}(w)$ $(j = 1, \dots, m)$ of $p_m(w)$ with $w \in \mathrm{GJ}$ defined by (2.1). Then, given positive integers r and s, we introduce the points $y_j = y_{j,m}$ $(j = 1, 2, \dots, s)$ and $z_j = z_{j,m}$ $(j = 1, 2, \dots, r)$ defined by

$$y_j = -1 + (j-1)\frac{x_{1,m}(w) + 1}{s}, \quad j = 1, \dots, s, \tag{3.1}$$

$$z_j = 1 - (j-1)\frac{1 - x_{m,m}(w)}{r}, \quad j = 1, \dots, r. \tag{3.2}$$

Moreover, for any fixed $-1 < t < 1$ there exists an integer m_0 such that $x_{1,m}(w) \leqq t \leqq x_{m,m}(w)$ for $m \geqq m_0$. Then, given a positive integer ρ, we introduce also the points $\tau_j = \tau_{j,m}$ $(j = 1, 2, \dots, \rho)$ defined by

$$\tau_j = \begin{cases} t + j\frac{x_{i+1,m}(w) - t}{\rho+1} & \text{if } t - x_{i,m} \leqq x_{i+1,m} - t \\ x_{i,m} + j\frac{t - x_{i,m}(w)}{\rho+1} & \text{if } t - x_{i,m} > x_{i+1,m} - t \end{cases}, \quad j = 1, \dots, \rho, \tag{3.3}$$

where $x_{i,m}(w) \leqq t \leqq x_{i+1,m}(w)$.

So, we denote by $\mathcal{L}_{m,r,s,\rho}(w;f)$ the Lagrange polynomial interpolating the function f at the points $-1 = y_1 < \dots < y_s < x_{1,m}(w) < \dots < x_{i,m}(w) < \tau_1 < \dots < \tau_\rho < x_{i+1,m}(w) < \dots < x_{m,m}(w) < z_1 < \dots < z_r = 1$.

Previously we have assumed $r \geqq 1$, $s \geqq 1$ and $\rho \geqq 1$. We complete the definition by putting $\mathcal{L}_{m,0,0,0}(w;f) = \mathcal{L}_m(w;f)$.

Here, we have chosen the additional points y_j, z_j and τ_j equispaced. However, the results of this paper remain valid fixing the above points in several other ways. More precisely, the choices of the additional nodes satisfying $y_{j+1} - y_j \sim m^{-2}$, $j = 1, \ldots, s-1$,[1] $z_{j+1} - z_j \sim m^{-2}$, $j = 1, \ldots, r-1$, $x_{1,m}(w) - y_s \sim m^{-2} \sim z_1 - x_{m,m}(w)$, and $\tau_{j+1} - \tau_j \sim m^{-1}$, $j = 1, \ldots, \rho - 1$ with $\tau_1 - t \sim m^{-1} \sim x_{i+1,m}(w) - \tau_\rho$ or $\tau_1 - x_{i,m}(w) \sim m^{-1} \sim t - \tau_\rho$ according as $t - x_{i,m}(w) \leqq x_{i+1,m}(w) - t$ or $t - x_{i,m}(w) > x_{i+1,m}(w) - t$, are possible.

The following theorem determines the previous parameters r, s and ρ in order that $\mathcal{L}_{m,r,s,\rho}$ represent a good approximation for f and for its derivatives simultaneously.

THEOREM 3.1. *Let $w \in$ GJ be the weight function defined by (2.1). Let $f \in C^{(q)}$, $q \geqq 0$, and let $\ell \in \{0, 1, \ldots, q\}$. For any exponents $\alpha > -1$, $\beta > -1$ and $\gamma > -1$ of the weight w, there exist nonnegative integers r, s and ρ such that*

(3.4)

$$\left| f^{(j)}(x) - \mathcal{L}^{(j)}_{m,r,s,\rho}(w; f; x) \right| \leqq C \left[\frac{\sqrt{1-x^2}}{m} + \frac{1}{m^2} \right]^{\ell - j} E_{m-q}\big(f^{(q)}\big) \frac{\log m}{m^{q-\ell}},$$

$$x \in [-1, 1], \qquad j = 0, 1, \ldots, \ell,$$

for $m \geqq \max(4q + 4, m_0)$ and with some constant C independent of f, m and x, provided that the integers r, s and ρ are defined by

$$\frac{\alpha + \ell}{2} + \frac{1}{4} \leqq r < \frac{\alpha + \ell}{2} + \frac{5}{4}, \tag{3.5}$$

$$\frac{\beta + \ell}{2} + \frac{1}{4} \leqq s < \frac{\beta + \ell}{2} + \frac{5}{4}, \tag{3.6}$$

$$\frac{\gamma}{2} \leqq \rho < \frac{\gamma}{2} + 1. \tag{3.7}$$

Of course, one obtains the best estimate for $\ell = q$ in (3.4). Nevertheless, when only the first ℓ derivatives of f must be approximated and $\ell \ll q$, then (3.4) is useful in the applications; indeed, it holds by using a number of additional nodes depending on ℓ (cf. (3.5)–(3.6)).

From Theorem 2.1 the following corollary follows immediately.

COROLLARY 3.2. *Let $w \in$ GJ be the weight function defined by (2.1). Let $f \in C^{(q)}$, $q \geqq 0$, and $\ell \in \{0, 1, \ldots, q\}$. For any exponents $\alpha > -1$, $\beta > -1$ and $\gamma > -1$ of the weight w, there exist nonnegative integers r, s and ρ such that*

$$\big\| f - \mathcal{L}_{m,r,s,\rho}(w; f) \big\|_\ell \leqq C\, E_{m-q}\big(f^{(q)}\big) \frac{\log m}{m^{q-\ell}}, \tag{3.8}$$

[1] If A and B are two expressions depending on some variables then we write $A \sim B$ if $|A/B|^{\pm 1} \leqq C$ uniformly for the variables under consideration.

for $m \geqq \max(4q+4, m_0)$ *and with some constant* C *independent of* f *and* m*, provided that the integers* r*,* s *and* ρ *are defined by* (3.5)–(3.7).

REMARK 1. Assuming $\ell = 0$, Corollary 3.2 assures that if

$$\frac{\alpha}{2} + \frac{1}{4} \leqq r < \frac{\alpha}{2} + \frac{5}{4}, \tag{3.9}$$

$$\frac{\beta}{2} + \frac{1}{4} \leqq s < \frac{\beta}{2} + \frac{5}{4}, \tag{3.10}$$

$$\frac{\gamma}{2} \leqq \rho < \frac{\gamma}{2} + 1, \tag{3.11}$$

then the estimate

$$\| f - \mathcal{L}_{m,r,s,\rho}(w; f)\| \leqq C\, E_{m-q}\big(f^{(q)}\big)\frac{\log m}{m^q}, \tag{3.12}$$

holds. Therefore, choosing the zeros of the polynomial $p_m(w)$ with w defined by (2.1) as nodes of interpolation, by (3.9)–(3.11) we can always determine the numbers s, r and ρ of nodes that we must add near 1, -1 and t respectively, in order that (3.12) hold.

We further remark that, having fixed the zeros of the generalized Jacobi polynomial $p_m(w)$, in general it is necessary to use the additional nodes to obtain (3.12). Indeed, if $\alpha, \beta \leqq -1/2$ and $\gamma \leqq 0$ then (3.9)–(3.11) give $r = 0$, $s = 0$ and $\rho = 0$ and we find the estimate [12, p. 178, Theorem 12]

$$\| f - L_m(w; f)\| \leqq C\, E_{m-q}\big(f^{(q)}\big)\,\frac{\log m}{m^q}, \quad \alpha, \beta \leqq -1/2,\ \gamma \leqq 0.$$

However, in the case $\alpha, \beta > -1/2$, $\gamma \leqq 0$, if one does not use the additional nodes near ± 1, then [12]

$$\| f - L_m(w; f)\| \leqq C\, E_{m-q}\big(f^{(q)}\big)\,\frac{\log m}{m^{q-\delta-\frac{1}{2}}}, \quad \delta = \max(\alpha, \beta) > -1/2,\ \gamma \leqq 0, \tag{3.13}$$

which is worse than (3.12). On the other hand if $\alpha, \beta \leqq -1/2$ and $\gamma > 0$ and we do not add nodes near t, then we find

$$\| f - L_m(w; f)\| \leqq C\, E_{m-q}\big(f^{(q)}\big)\,\frac{\log m}{m^{q-\gamma/2}}, \quad \alpha, \beta \leqq -1/2,\ \gamma > 0, \tag{3.14}$$

which is also worse than (3.12).

Summarizing, in the case of generalized Jacobi zeros it is possible to change *bad* matrices (in the sense of (3.13) and (3.14)) into *good* ones (in the sense of (3.12)) by the simple technique of adding nodes near the zeros or singularities of the weight.

Furthermore, we remark that the number of nodes that we must add is independent of m; it depends only on the parameters α, β and γ of the weight (cf. (3.9)–(3.11)). On the other hand, for the simultaneous approximation the numbers r and s depend also on the order of derivatives ℓ that we would approximate, while ρ does not (cf. (3.5)–(3.7)).

REMARK 2. From the obvious inequality

$$\left\| \mathcal{L}_{m,r,s,\rho}(w;f)\right\|_{\ell} \leqq \|f\|_{\ell} + \left\| f - \mathcal{L}_{m,r,s,\rho}(w;f)\right\|_{\ell},$$

and from (3.8) with $q = \ell$ we deduce the useful estimate

$$\left\| \mathcal{L}_{m,r,s,\rho}(w)\right\|_{\ell} := \sup_{\|f\|_{\ell}=1} \left\| \mathcal{L}_{m,r,s,\rho}(w;f)\right\|_{\ell} \leqq C\log m,$$

which holds when $\ell \geqq 0$ and (3.5)–(3.7) are satisfied.

REMARK 3. The extension of the definition of $\mathcal{L}_{m,r,s,\rho}(w;f)$ when the weight w is defined by

$$w(x) = g(x)v^{(\alpha,\beta)}(x)\prod_{j=1}^{M} |t_j - x|^{\gamma_j} \in \mathrm{GJ},$$

with $-1 < t_1 < \dots < t_M < 1$, $\gamma_j > -1$, $j = 1,\dots,M$ and $M > 1$ instead of (2.1), is obvious. Further, all the previous results can be easily extended to this case and we omit the details for the sake of brevity.

Lagrange interpolation with multiple additional nodes. So far, starting from the zeros of generalized Jacobi polynomials, we have considered Lagrange interpolating polynomials on these zeros and on a suitable number of additional simple nodes in order to obtain an interpolating process by which f and its derivatives can be well approximated. Nevertheless, if the function f is sufficiently smooth we can do differently. We start again considering the points $x_{j,m}(w)$ with w defined by (2.1). Now, among the interpolation nodes we omit the point $x_{i^*,m}(w)$ closest to t and we add the point t with multiplicity ρ. Moreover, following a procedure used in [9] we also add the points -1 and 1 with multiplicities s and r, respectively. In this case we must assume $f \in C^{(q)}$ with $q \geqq r-1$, $q \geqq s-1$ and $q \geqq \rho - 1$. Thus, we define the interpolating polynomial $\widehat{\mathcal{L}}_{m,r,s,\rho}(w;f)$ at the points $x_{j,m}(w)$ $(j = 1,\dots,m$, $k \neq i^*)$ and at -1, 1 and t with multiplicities s, r and ρ respectively. Obviously, this is a mixed Lagrange–Hermite interpolating polynomial of degree $m + r + s + \rho - 2$ (Lagrange on the original points and Hermite on the nodes t, ± 1).

For this polynomial we can state the following theorem which gives an estimate of Telyakovskii–Gopengauz type.

THEOREM 3.3. *Let $w \in \mathrm{GJ}$ be the weight function defined by* (2.1) *with* $\alpha, \beta > -\frac{1}{2}$. *Let $f \in C^{(q)}$, $q \geqq 0$, with $q \geqq r-1,\ s-1,\ \rho-1$. If*

$$\frac{\alpha+q}{2}+\frac{1}{4} \leqq r < \frac{\alpha+q}{2}+\frac{5}{4}, \tag{3.15}$$

$$\frac{\beta+q}{2}+\frac{1}{4} \leqq s < \frac{\beta+q}{2}+\frac{5}{4}, \tag{3.16}$$

$$\frac{\gamma}{2}+1 \leqq \rho < \frac{\gamma}{2}+2, \tag{3.17}$$

then

(3.18)

$$\left| f^{(j)}(x) - \widehat{\mathcal{L}}^{(j)}_{m,r,s,\rho}(w;f;x)\right| \leqq C\, E_{m-q}\big(f^{(q)}\big) \left(\frac{\sqrt{1-x^2}}{m}\right)^{q-j} \log m,$$

$$x \in [-1,1], \qquad j = 0,1,\dots,\min(r-1, s-1),$$

for $m \geqq 4q+4$ and with some constant C independent of f, m and x.

4. Proofs of the main results

Given the weight $w \in \mathrm{GJ}$ defined by (2.1) and given the points $y_{j,m} = y_j$ $(j = 1,\dots,s)$, $z_{j,m} = z_j$ $(j = 1,\dots,r)$ and $\tau_{j,m} = \tau_j$ $(j = 1,\dots,\rho)$ defined by (3.1), (3.2) and (3.3) respectively, we define the matrices of points $Y = \{y_{j,m},\ j = 1,\dots,s,\ m \in \mathbf{N}\}$, $Z = \{z_{j,m},\ j = 1,\dots,r,\ m \in \mathbf{N}\}$ and $T = \{\tau_{j,m},\ j = 1,\dots,\rho,\ m \in \mathbf{N}\}$. Setting

$$A_0(x) \equiv 1, \qquad A_s(x) = \prod_{j=1}^{s}(x - y_{j,m}), \quad s > 0, \tag{4.1}$$

$$B_0(x) \equiv 1, \qquad B_r(x) = \prod_{j=1}^{r}(x - z_{j,m}), \quad r > 0, \tag{4.2}$$

$$C_0(x) \equiv 1, \qquad C_\rho(x) = \prod_{j=1}^{\rho}(x - \tau_{j,m}), \quad \rho > 0 \tag{4.3}$$

by the definition of the polynomial $\mathcal{L}_{m,r,s,\rho}(w;f)$ we can write

$$(4.4)\qquad \mathcal{L}_{m,r,s,\rho}(w;f;x)=A_s(x)B_r(x)C_\rho(x)\mathcal{L}_m\left(w;\frac{f}{A_sB_rC_\rho};x\right)+$$

$$+A_s(x)C_\rho(x)p_m(w;x)L_r\left(Z;\frac{f}{A_sC_\rho p_m(w)};x\right)+$$

$$+B_r(x)C_\rho(x)p_m(w;x)L_s\left(Y;\frac{f}{B_rC_\rho p_m(w)};x\right)+$$

$$+A_s(x)B_r(x)p_m(w;x)L_\rho\left(T;\frac{f}{A_sB_rp_m(w)};x\right),$$

where

$$(4.5)\qquad \mathcal{L}_m\left(w;\frac{f}{A_sB_rC_\rho};x\right)=$$

$$=\sum_{k=1}^{m}\ell_{k,m}(w;x)\frac{f\left(x_{k,m}(w)\right)}{A_s\left(x_{k,m}(w)\right)B_r\left(x_{k,m}(w)\right)C_\rho\left(x_{k,m}(w)\right)},$$

$\ell_{k,m}(w)$ being the kth fundamental Lagrange polynomial with respect to the weight w,

$$(4.6)\quad L_r\left(Z;\frac{f}{A_sC_\rho p_m(w)};x\right)=\sum_{j=1}^{r}\prod_{k=1,k\neq j}^{r}\frac{x-z_k}{z_j-z_k}\frac{f(z_j)}{A_s(z_j)C_\rho(z_j)p_m(w;z_j)},$$

$$(4.7)\quad L_s\left(Y;\frac{f}{B_rC_\rho p_m(w)};x\right)=\sum_{j=1}^{s}\prod_{k=1,k\neq j}^{s}\frac{x-y_k}{y_j-y_k}\frac{f(y_j)}{B_r(y_j)C_\rho(y_j)p_m(w;y_j)},$$

and

$$(4.8)\quad L_\rho\left(T;\frac{f}{A_sB_rp_m(w)};x\right)=\sum_{j=1}^{\rho}\prod_{k=1,k\neq j}^{\rho}\frac{x-\tau_k}{\tau_j-\tau_k}\frac{f(\tau_j)}{A_s(\tau_j)B_r(\tau_j)p_m(w;\tau_j)}.$$

In particular, if $r=0$ then we set $L_r\equiv 0$. Similarly, if $s=0$ or $\rho=0$ then $L_s\equiv 0$ and $L_\rho\equiv 0$.

Furthermore, setting

$$(4.9)\qquad C_\rho(x)=(x-t)^\rho,\qquad \rho\geqq 0,$$

and

$$\widehat{p}_m(w) = \frac{p_m(w)}{x - x_{i^*,m}(w)}, \tag{4.10}$$

where $x_{i^*,m}(w)$ is the zero of $p_m(w)$ closest to t, by the definition of $\widehat{L}_{m,r,s,\rho}(w;f)$ we can write

$$\widehat{\mathcal{L}}_{m,r,s,\rho}(w;f;x) = v^{(r,s)}(x)C_\rho(x)\widehat{\mathcal{L}}_m\left(w;\frac{f}{v^{(r,s)}C_\rho};x\right) + \tag{4.11}$$

$$+v^{(0,s)}(x)C_\rho(x)\widehat{p}_m(w;x)H_r\left(1;\frac{f}{v^{(0,s)}C_\rho\widehat{p}_m(w)};x\right) +$$

$$+v^{(r,0)}(x)C_\rho(x)\widehat{p}_m(w;x)H_s\left(-1;\frac{f}{v^{(r,0)}C_\rho\widehat{p}_m(w)};x\right) +$$

$$+v^{(r,s)}(x)\widehat{p}_m(w;x)H_\rho\left(t;\frac{f}{v^{(r,s)}\widehat{p}_m(w)};x\right),$$

where

$$\widehat{L}_m\left(w;\frac{f}{v^{(r,s)}C_\rho};x\right) = \tag{4.12}$$

$$= \sum_{\substack{k=1 \\ k\neq i^*}} \frac{x_{k,m}(w) - x_{i^*,m}(w)}{x - x_{i^*,m}(w)}\ell_{k,m}(w;x)\frac{f(x_{k,m}(w))}{v^{(r,s)}(x_{k,m}(w))\,C_\rho(x_{k,m}(w))},$$

$$H_r\left(1;\frac{f}{v^{(0,s)}C_\rho\widehat{p}_m(w)};x\right) = \tag{4.13}$$

$$= \sum_{j=0}^{r-1}\frac{(x-1)^j}{j!}\left[\frac{f(x)}{v^{(0,s)}(x)C_\rho(x)\widehat{p}_m(w;x)}\right]^{(j)}_{x=1},$$

$$H_s\left(-1;\frac{f}{v^{(r,0)}C_\rho\widehat{p}_m(w)};x\right) = \tag{4.14}$$

$$= \sum_{j=0}^{s-1}\frac{(x+1)^j}{j!}\left[\frac{f(x)}{v^{(r,0)}(x)C_\rho(x)\widehat{p}_m(w;x)}\right]^{(j)}_{x=-1},$$

$$H_\rho\left(t;\frac{f}{v^{(r,s)}\widehat{p}_m(w)};x\right) = \sum_{j=0}^{\rho-1}\frac{(x-t)^j}{j!}\left[\frac{f(x)}{v^{(r,s)}(x)\widehat{p}_m(w;x)}\right]^{(j)}_{x=t}. \tag{4.15}$$

In particular, if $r=0$ then we set $H_r \equiv 0$. Similarly, if $s=0$ or $\rho=0$ then $H_s \equiv 0$ and $H_\rho \equiv 0$ respectively.

In what follows, we assume that $w(x)=g(x)v^{(\alpha,\beta)}(x)|t-x|^\gamma \in \mathrm{GJ}$, and we denote the zeros of the mth orthonormal polynomial $p_m(w)$ corresponding to the weight w ordered increasingly by $\{x_{k,m}(w)\}_{k=1}^{m}$. For the convenience of the reader, we collect some properties of generalized Jacobi polynomials $p_m(w)$ which will be used in the proofs.

Let $x_{k,m}(w)=\cos\theta_{k,m}$ for $k=0,1,\dots,m+1$ where $x_{0,m}(w)=-1$, $x_{m+1,m}(w)=1$, and $0 \leqq \theta_{k,m} \leqq \pi$. Then

$$\theta_{k,m}-\theta_{k+1,m} \sim \frac{1}{m}, \tag{4.16}$$

uniformly for $m \in \mathbf{N}$ and $k=0,1,\dots,m$ (cf. [12, Theorem 9.22, p. 166]).

The Cotes numbers $\lambda_{k,m}(w)$ satisfy

(4.17)

$$\lambda_{k,m}(w) \sim m^{-1}\bigl(1-x_{k,m}(w)\bigr)^{\alpha+\frac{1}{2}}\bigl(|t-x_{k,m}(w)|+m^{-1}\bigr)^{\gamma}\bigl(1+x_{k,m}(w)\bigr)^{\beta+\frac{1}{2}},$$

uniformly for $m \in \mathbf{N}$ and $k=1,\dots,m$ (cf. [12, Theorem 6.3.28, p. 120]).

Furthermore,

$$\begin{aligned} &|p_m(w;x)| \leqq \\ &\leqq C\bigl(\sqrt{1-x}+m^{-1}\bigr)^{-\alpha-\frac{1}{2}}\bigl(|t-x|+m^{-1}\bigr)^{-\frac{\gamma}{2}}\bigl(\sqrt{1+x}+m^{-1}\bigr)^{-\beta-\frac{1}{2}}, \end{aligned} \tag{4.18}$$

uniformly for $-1 \leqq x \leqq 1$ and $m \in \mathbf{N}$ (cf. [1, Theorem 1.1, p. 226]).

In addition,

$$\begin{aligned} &\bigl|p_{m-1}\bigl(w;x_{k,m}(w)\bigr)\bigr| \sim \\ &\sim \bigl(1-x_{k,m}(w)\bigr)^{-\frac{\alpha}{2}+\frac{1}{4}}\bigl(|t-x_{k,m}(w)|+m^{-1}\bigr)^{-\frac{\gamma}{2}}\bigl(1+x_{k,m}(w)\bigr)^{-\frac{\beta}{2}+\frac{1}{4}}, \end{aligned} \tag{4.19}$$

uniformly for $m \in \mathbf{N}$ and $k=1,\dots,m$ (cf. [12, Lemma 9.30, p. 170]).

In particular, by (4.18) and (4.19) and taking into account that

$$\ell_{k,m}(w;x)=\frac{\gamma_{m-1}(w)}{\gamma_m(w)}\lambda_{k,m}(w)p_{m-1}\bigl(w;x_{k,m}(w)\bigr)\frac{p_m(w;x)}{x-x_{k,m}(w)},$$

where $\gamma_m(w)$ denotes the leading coefficient of $p_m(w)$, we deduce

$$\begin{aligned} &\bigl|\ell_{k,m}(w;x)\bigr| \leqq \frac{C}{m}\bigl(1-x_{k,m}(w)\bigr)^{\frac{\alpha}{2}+\frac{3}{4}} \cdot \\ &\cdot\bigl(|t-x_{k,m}(w)|+m^{-1}\bigr)^{\frac{\gamma}{2}}\bigl(1+x_{k,m}(w)\bigr)^{\frac{\beta}{2}+\frac{3}{4}}\frac{|p_m(w;x)|}{|x-x_{k,m}(w)|}. \end{aligned} \tag{4.20}$$

Moreover, denoting by d the index of the zero of $p_m(w)$ closest to $x \in \in [-1,1]$, we have

$$\left|\ell_{d,m}(w;x)\right| \sim 1, \tag{4.21}$$

(cf. [12, Theorem 33, p. 171]).

The following lemmas will be needed to prove the main results.

LEMMA 4.1. *Let $f \in C^{(q)}$. Then, there exists a sequence of polynomials $P_m \in \mathcal{P}_m$, $m \geqq 4q+4$, such that for $|x| \leqq 1$ and for $j = 0,1,\dots,q$*

$$\left|f^{(j)}(x) - P_m^{(j)}(x)\right| \leqq C\left(\frac{\sqrt{1-x^2}}{m}\right)^{q-j} E_{m-q}\big(f^{(q)}\big), \tag{4.22}$$

with some constant C independent of f, x and $m \geqq 4q+4$.

For the proof see Lemma 4.3 in [9].

LEMMA 4.2. *Let $Q_m \in \mathcal{P}_m$ be such that*

$$|Q_m(x)| \leqq C\left(\frac{\sqrt{1-x^2}}{m} + \frac{1}{m^2}\right)^a, \qquad |x| \leqq 1,\ a \in \mathbf{R}.$$

Then

$$\left|Q_m^{(j)}(x)\right| \leqq C\left(\frac{\sqrt{1-x^2}}{m} + \frac{1}{m^2}\right)^{a-j}, \qquad |x| \leqq 1,\ j < m.$$

Furthermore, if the polynomial Q_m has two zeros in ± 1 of multiplicity $[a]+1$, then

$$\left|Q_m^{(j)}(x)\right| \leqq C\left(\frac{\sqrt{1-x^2}}{m}\right)^{a-j}, \qquad |x| \leqq 1,\ j < m.$$

The first part of the lemma can be found in [6, Theorem 7.1.3], while the second one is in [7, p. 169].

LEMMA 4.3. *Let $f \in C^{(q)}$, $q \geqq 0$, and let $\ell \in \{0,1,\dots,q\}$. Let L_r and L_s be the polynomials defined by* (4.6) *and* (4.7) *respectively, such that* (3.5)–(3.7) *hold. Then, for $|x| \leqq 1$*

$$\left|A_s(x)C_\rho(x)p_m(w;x)L_r\left(Z;\frac{r_{m-1}}{A_sC_\rho p_m(w)};x\right)\right| \leqq \tag{4.23}$$

$$\leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^{\ell} E_{m-q}\big(f^{(q)}\big),$$

$$\left|B_r(x)C_\rho(x)p_m(w;x)L_s\left(Y;\frac{r_{m-1}}{B_rC_\rho p_m(w)};x\right)\right| \leqq \tag{4.24}$$

$$\leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^{\ell} E_{m-q}\big(f^{(q)}\big),$$

$$\left|A_s(x)B_r(x)p_m(w;x)L_\rho\left(T;\frac{r_{m-1}}{A_sB_r p_m(w)};x\right)\right| \leqq \tag{4.25}$$

$$\leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^{\ell} E_{m-q}\big(f^{(q)}\big),$$

where $r_{m-1}=f-P_{m-1}$, P_{m-1} being the polynomial defined by Lemma 4.1, *with some constant C independent of f, x and $m \geqq 4q+4$.*

PROOF. By the definition of L_r we have

$$\left|L_r\left(Z;\frac{r_{m-1}}{A_sC_\rho p_m(w)};x\right)\right| \leqq \sum_{j=1}^{r}\left|\prod_{k=1,k\neq j}^{r}\frac{x-z_k}{z_j-z_k}\right|\frac{|r_{m-1}(z_j)|}{|A_s(z_j)C_\rho(z_j)p_m(w;z_j)|},$$

where

$$|A_s(z_j)|^{-1}\leqq 1,\quad |C_\rho(z_j)|^{-1}\leqq C,\quad |p_m(w;z_j)| \sim m^{\alpha+\frac{1}{2}},$$

and by (4.22)

$$|r_{m-1}(z_j)| \leqq \frac{C}{m^{2q}}E_{m-q}\big(f^{(q)}\big).$$

Furthermore, by the choice of the points z_j we have

$$\left|\prod_{k=1,k\neq j}^{r}\frac{x-z_k}{z_j-z_k}\right| \leqq C\, m^{2r-2}\big(\sqrt{1-x}+m^{-1}\big)^{2r-2},$$

and consequently

$$\left|L_r\left(Z;\frac{r_{m-1}}{A_sC_\rho p_m(w)};x\right)\right| \leqq \big(\sqrt{1-x}+m^{-1}\big)^{2r-2}\frac{E_{m-q}\big(f^{(q)}\big)}{m^{2q+\alpha+\frac{1}{2}-2r+2}}.$$

If $x_{m,m}(w) \leqq x \leqq 1$ then $|A_s(x)| \leqq C$, $|C_\rho(x)| \leqq C$ and $|p_m(w;x)| \sim m^{\alpha+1/2}$. Thus, by (4.26) we deduce (4.23). On the other hand, if $-1 \leqq x \leqq x_{1,m}(w)$, then $|A_s(x)| \leqq C\, m^{-2s}$, $|C_\rho(x)| \leqq C$, $|p_m(w;x)| \sim m^{\beta+1/2}$ and we have

$$\left|A_s(x)C_\rho(x)p_m(w;x)L_r\left(Z;\frac{r_{m-1}}{A_sC_\rho p_m(w)};x\right)\right| \leqq$$

$$\leqq C\frac{E_{m-q}\big(f^{(q)}\big)}{m^{2q}}\,\frac{1}{m^{2s-\beta-\frac{1}{2}-\ell}}\,\frac{1}{m^{\alpha+\frac{5}{2}-2r+\ell}}.$$

Since (3.5) and (3.6) assure that $2s-\beta-\frac{1}{2}-\ell \geqq 0$ and $\alpha+\frac{5}{2}-2r+\ell > 0$, (4.23) follows again.

If $x_{1,m}(w) \leqq x \leqq x_{m,m}(w)$ then

$$|p_m(w;x)| \leqq C(1-x)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{-\frac{\beta}{2}-\frac{1}{4}}\big(|x-t|+m^{-1}\big)^{-\frac{\gamma}{2}},$$

$|A_s(x)| \sim (1+x)^s$ and $|C_\rho(x)| \leqq C\big(|x-t|+m^{-1}\big)^\rho$. Therefore, by (4.26) and $\rho-\frac{\gamma}{2} \geqq 0$

$$\left|A_s(x)C_\rho(x)p_m(w;x)L_r\left(Z;\frac{r_{m-1}}{A_sC_\rho p_m(w)};x\right)\right| \leqq$$

$$\leqq C\frac{E_{m-q}\big(f^{(q)}\big)}{m^{2q-\ell}}\,\frac{(1-x)^{r-1-\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{s-\frac{\beta}{2}-\frac{1}{4}}}{m^{\alpha+\frac{5}{2}-2r+\ell}} \leqq$$

$$\leqq C\frac{E_{m-q}\big(f^{(q)}\big)}{m^{2q-\ell}}(1-x)^{\frac{\ell}{2}}(1+x)^{s-\frac{\beta}{2}+1+\frac{\alpha+\ell}{2}-r},$$

being $m^{-2} \leqq C(1-x^2)$. Now, if $x \geqq 0$ then we use $(1+x)^{s-\frac{\beta}{2}+1+\frac{\alpha+\ell}{2}-r} \leqq$ $\leqq (1+x)^{\frac{\ell}{2}}$. On the other hand, if $x<0$ then we observe that from (3.5)–(3.6) $s-\frac{\beta}{2}+1+\frac{\alpha+\ell}{2}-r \geqq \frac{\ell}{2}$ follows. Thus, we have

$$\left|A_s(x)C_\rho(x)p_m(w;x)L_r\left(Z;\frac{r_{m-1}}{A_sC_\rho p_m(w)};x\right)\right| \leqq \left(\frac{\sqrt{1-x^2}}{m}\right)^\ell E_{m-q}\big(f^{(q)}\big),$$

and (4.23) follows also in this case. Similarly we can prove (4.24).

In order to prove (4.25) we recall that by the definition of the polynomial L_ρ we have

$$\left|L_\rho\left(T;\frac{r_{m-1}}{A_sB_rp_m(w)};x\right)\right| \leqq \sum_{j=1}^{\rho}\left|\prod_{k=1,k\neq j}^{\rho}\frac{x-\tau_k}{\tau_j-\tau_k}\right|\frac{|r_{m-1}(\tau_j)|}{\big|A_s(\tau_j)B_r(\tau_j)p_m(w;\tau_j)\big|},$$

where $\left|A_s(\tau_j)B_r(\tau_j)\right|^{-1} \leqq C$ and by (4.22), $|r_{m-1}(\tau_j)| \leqq \frac{C}{m^q} E_{m-q}\big(f^{(q)}\big)$. Let $x_{i,m}(w) \leqq t \leqq x_{i+1,m}(w)$; then for any $j \in \{1, \ldots, \rho\}$ the knot closest to τ_j is $x_{i,m}(w)$ or $x_{i+1,m}(w)$. Denoting this knot by $x_{\nu,m}(w)$ where $\nu \in \{i, i+1\}$, we have

$$\left|p_m(w;\tau_j)\right| \sim$$

$$\sim m^{\frac{1}{2}}\left|\tau_j - x_{\nu,m}(w)\right|\big(\sqrt{1-\tau_j} + m^{-1}\big)^{-1}\big(\sqrt{1+\tau_j} + m^{-1}\big)^{-1}\lambda_m^{-\frac{1}{2}}(w;\tau_j) \sim$$

$$\sim \big(\sqrt{1-\tau_j} + m^{-1}\big)^{-1}\big(\sqrt{1+\tau_j} + m^{-1}\big)^{-1}\big(|t-\tau_j| + m^{-1}\big)^{-\frac{\gamma}{2}},$$

(see [12, Theorem 33, p. 171]). Therefore,

$$\left|p_m(w;\tau_j)\right|^{-1} \leqq C\big(|t-\tau_j| + m^{-1}\big)^{\frac{\gamma}{2}} \leqq C\, m^{-\frac{\gamma}{2}}.$$

Furthermore, by the choice of the points τ_j we have

$$\left|\prod_{k=1,k\neq j}^{\rho} \frac{x-\tau_k}{\tau_j-\tau_k}\right| \leqq C\, m^{\rho-1}\big(|x-t| + m^{-1}\big)^{\rho-1},$$

and consequently

$$\left|L_\rho\left(T; \frac{r_{m-1}}{A_s B_r p_m(w)}; x\right)\right| \leqq C\,\big(|x-t| + m^{-1}\big)^{\rho-1} \frac{E_{m-q}\big(f^{(q)}\big)}{m^{q+\frac{\gamma}{2}-\rho+1}}. \tag{4.27}$$

If $-1 \leqq x \leqq x_{i,m}(w)$ or $x_{i+1,m}(w) \leqq x \leqq 1$ then proceeding very similarly as before we can deduce (4.25) from (4.27). On the other hand, if $x_{i,m}(w) \leqq x \leqq$ $\leqq x_{i+1,m}(w)$ then $\left|A_s(x)B_r(x)\right| \leqq C$ and $\left|p_m(w;x)\right| \leqq \big(|t-x| + m^{-1}\big)^{-\frac{\gamma}{2}}$. Thus, by the assumption $\rho - \frac{\gamma}{2} - 1 < 0$ in view of (4.27) we deduce (4.25) also in this case. □

Finally, we recall that if $w \in \mathrm{GJ}$ is the weight function defined by (2.1) and a, b and c are real numbers, then

$$\sum_{\substack{k=1\\k\neq d}}^{m} \frac{\big(1-x_{k,m}(w)\big)^{a}\big(1+x_{k,m}(w)\big)^{b}}{m|x-x_{k,m}(w)|} \leqq \tag{4.28}$$

$$\leqq C\big(\sqrt{1-x} + m^{-1}\big)^{2a-1}\big(\sqrt{1+x} + m^{-1}\big)^{2b-1}\log m,$$

$$\text{if} \quad -\frac{1}{2} \leqq a,b \leqq \frac{1}{2}, \qquad |x| \leqq 1,$$

$$(4.29)\qquad \sum_{\substack{k=1\\k\neq d}}^{m} \frac{\left(|t-x_{k,m}(w)|+m^{-1}\right)^{c}}{m|x-x_{k,m}(w)|}\left(1-x_{k,m}(w)^2\right)^{\frac{1}{2}} \leqq$$

$$\leqq C\begin{cases}\log m & \text{if } c=0\\ 1+m^{-c} & \text{if } c<0\end{cases}, \qquad |x|\leqq 1,$$

where d denotes the index corresponding to the knot closest to x and C is a constant independent of m and x.

For the proof of (4.28) and (4.29) see Lemma 4.1 in [9] and Lemma 5.9 in [4], respectively. Furthermore,

$$(4.30)\qquad \sum_{\substack{k=1\\k\neq d}}^{m} \frac{\left(1-x_{k,m}(w)\right)^{a}\left(1+x_{k,m}(w)\right)^{b}\left(|t-x_{k,m}(w)|+m^{-1}\right)^{c}}{m|x-x_{k,m}(w)|} \leqq$$

$$\leqq C\left(\sqrt{1-x}+m^{-1}\right)^{2a-1}\left(\sqrt{1+x}+m^{-1}\right)^{2b-1}\left(|x-t|+m^{-1}\right)^{c}\log m,$$

$$\text{if}\quad -\frac{1}{2}\leqq a,b\leqq\frac{1}{2},\quad -1<c\leqq 0,\quad |x|\leqq 1,$$

where d denotes the index corresponding to the knot closest to x and C is a constant independent of m and x.

Inequality (4.30) follows from (4.28) and (4.29) and by a routine and laborious but not enlightening computation. We omit the details.

PROOF OF THEOREM 3.1. Denoting $r_{m-1}=f-P_{m-1}$, where P_{m-1} is the polynomial defined by Lemma 4.1 and corresponding to the function f, we have

$$f^{(j)}(x)-\mathcal{L}^{(j)}_{m,r,s,\rho}(w;f;x)=r^{(j)}_{m-1}(x)-\mathcal{L}^{(j)}_{m,r,s,\rho}(w;r_{m-1};x)=$$

$$=r^{(j)}_{m-1}(x)-\left[A_s(x)C_\rho(x)p_m(w;x)L_r\left(Z;\frac{r_{m-1}}{A_sC_\rho p_m(w)};x\right)\right]^{(j)}-$$

$$-\left[B_r(x)C_\rho(x)p_m(w;x)L_s\left(Y;\frac{r_{m-1}}{B_rC_\rho p_m(w)};x\right)\right]^{(j)}-$$

$$-\left[A_s(x)B_r(x)p_m(w;x)L_\rho\left(T;\frac{r_{m-1}}{A_sB_r p_m(w)};x\right)\right]^{(j)}-$$

$$-\left[A_s(x)B_r(x)C_\rho(x)\mathcal{L}_m\left(w;\frac{r_{m-1}}{A_sB_rC_\rho};x\right)\right]^{(j)}=:$$

$$=:I_1(x)+I_2(x)+I_3(x)+I_4(x)+I_5(x),\quad j=0,1,\dots,\ell.$$

By Lemma 4.1

$$\left|I_1(x)\right| \leqq C\left(\frac{\sqrt{1-x^2}}{m}\right)^{q-j} E_{m-q}\left(f^{(q)}\right), \qquad j=0,1,\ldots,\ell.$$

In view of Lemmas 4.3 and 4.2 we also have

$$\left|I_2(x)\right| + \left|I_3(x)\right| + \left|I_4(x)\right| \leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^{\ell-j} E_{m-q}\left(f^{(q)}\right),$$

$$j=0,1,\ldots,\ell.$$

Therefore,

$$\left|f^{(j)}(x)-\mathcal{L}^{(j)}_{m,r,s,\rho}(w;f;x)\right| \leqq \tag{4.31}$$

$$\leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^{\ell-j} E_{m-q}\left(f^{(q)}\right)+$$

$$+\left|\left[A_s(x)B_r(x)C_\rho(x)\mathcal{L}_m\left(w;\frac{r_{m-1}}{A_sB_rC_\rho};x\right)\right]^{(j)}\right|.$$

To evaluate the second term on the right side of (4.31), we denote by $x_{d,m}$ the zero of $p_m(w)$ closest to $x \in [-1,1]$, and write

$$A_s(x)B_r(x)C_\rho(x)\mathcal{L}_m\left(w;\frac{r_{m-1}}{A_sB_rC_\rho};x\right) = \tag{4.32}$$

$$= A_s(x)B_r(x)C_\rho(x)\ell_{d,m}(w;x)\frac{r_{m-1}\left(x_{d,m}(w)\right)}{A_s\left(x_{d,m}(w)\right)B_r\left(x_{d,m}(w)\right)C_\rho\left(x_{d,m}(w)\right)}+$$

$$+A_s(x)B_r(x)C_\rho(x)\cdot$$

$$\cdot\sum_{\substack{k=1\\k\neq d}}^{m}\ell_{k,m}(w;x)\frac{r_{m-1}\left(x_{k,m}(w)\right)}{A_s\left(x_{k,m}(w)\right)B_r\left(x_{k,m}(w)\right)C_\rho\left(x_{k,m}(w)\right)} =:$$

$$=: J_1(x)+J_2(x).$$

Since $\left|A_s(x)B_r(x)C_\rho(x)\right| \sim \left|A_s\left(x_{d,m}(w)\right) B_r\left(x_{d,m}(w)\right) C_\rho\left(x_{d,m}(w)\right)\right|$ and $\left|\ell_{d,m}(w;x)\right| \sim 1$ (see (4.21)) and in view of Lemma 4.1 we can write

$$|J_1(x)| \leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^q E_{m-q}\left(f^{(q)}\right). \tag{4.33}$$

On the other hand, in view of the choice of the additional points we have

$$\begin{aligned}&\left|A_s\left(x_{k,m}(w)\right) B_r\left(x_{k,m}(w)\right) C_\rho\left(x_{k,m}(w)\right)\right| \geqq \\ &\geqq \left(1-x_{k,m}(w)\right)^r\left(1+x_{k,m}(w)\right)^s\left(\left|t-x_{k,m}(w)\right|+m^{-1}\right)^\rho.\end{aligned}$$

Thus, by Lemma 4.1 and (4.20)

$$|J_2(x)| \leqq C\left|A_s(x)B_r(x)C_\rho(x)p_m(w;x)\right| \frac{E_{m-q}\left(f^{(q)}\right)}{m^q} \times$$

$$\sum_{\substack{k=1\\k\neq d}}^m \frac{\left(1-x_{k,m}(w)\right)^{\frac{q}{2}-r+\frac{\alpha}{2}+\frac{3}{4}}\left(1+x_{k,m}(w)\right)^{\frac{q}{2}-s+\frac{\beta}{2}+\frac{3}{4}}\left(\left|t-x_{k,m}(w)\right|+m^{-1}\right)^{\frac{\gamma}{2}-\rho}}{m\left|x-x_{k,m}(w)\right|} \leqq$$

$$\leqq C\left|A_s(x)B_r(x)C_\rho(x)p_m(w;x)\right| \frac{E_{m-q}\left(f^{(q)}\right)}{m^q} \times$$

$$\times \sum_{\substack{k=1\\k\neq d}}^m \frac{\left(1-x_{k,m}(w)\right)^a\left(1+x_{k,m}(w)\right)^b\left(\left|t-x_{k,m}(w)\right|+m^{-1}\right)^c}{m\left|x-x_{k,m}(w)\right|},$$

with $a=\frac{\ell}{2}-r+\frac{\alpha}{2}+\frac{3}{4}$, $b=\frac{\ell}{2}-s+\frac{\beta}{2}+\frac{3}{4}$ and $c=\frac{\gamma}{2}-\rho$. By the assumptions (3.5)–(3.7) we have $-\frac{1}{2}<a,b<\frac{1}{2}$ and $-1<c\leqq 0$. So, by (4.30)

$$\sum_{\substack{k=1\\k\neq d}}^m \frac{\left(1-x_{k,m}(w)\right)^a\left(1+x_{k,m}(w)\right)^b\left(\left|t-x_{k,m}(w)\right|+m^{-1}\right)^c}{m\left|x-x_{k,m}(w)\right|} \leqq$$

$$\leqq C\left(\sqrt{1-x}+m^{-1}\right)^{\ell+\alpha+\frac{1}{2}-2r}\left(\sqrt{1+x}+m^{-1}\right)^{\ell+\beta+\frac{1}{2}-2r}\left(|t-x|+m^{-1}\right)^{\frac{\gamma}{2}-\rho}.$$

Thus,

$$|J_2(x)| \leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^\ell \frac{E_{m-q}\left(f^{(q)}\right)}{m^{q-\ell}}\log m. \tag{4.34}$$

Combining (4.33) and (4.34) with (4.32), we deduce

$$\left| A_s(x)B_r(x)C_\rho(x)\mathcal{L}_m\left(w; \frac{r_{m-1}}{A_sB_rC_\rho}; x\right)\right| \leqq$$

$$\leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^{\ell} \frac{E_{m-q}\left(f^{(q)}\right)}{m^{q-\ell}} \log m.$$

Finally, applying Lemma 4.2 with $0 \leqq j \leqq \ell$, we obtain

$$\left|\left[A_s(x)b_r(x)C_\rho(x)\mathcal{L}_m\left(w; \frac{r_{m-1}}{A_sB_rC_\rho}; x\right)\right]^{(j)}\right| \leqq$$

$$\leqq C\left(\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}\right)^{\ell-j} \frac{E_{m-q}\left(f^{(q)}\right)}{m^{q-\ell}} \log m.$$

The proof is completed by inserting the last inequality in (4.31). □

A further lemma will be needed to prove Theorem 3.3.

LEMMA 4.4. *Let $f \in C^{(q)}$, $q \geqq 0$. Let H_ρ be the polynomial defined by* (4.15) *such that* (3.15)–(3.17) *hold. Then, for $|x| \leqq 1$*

(4.35)

$$\left|v^{(r,s)}(x)\widehat{p}_m(w;x)H_\rho\left(t; \frac{r_{m-1}}{v^{(r,s)}\widehat{p}_m(w)}; x\right)\right| \leqq C\left(\frac{\sqrt{1-x^2}}{m}\right)^{q} E_{m-q}\left(f^{(q)}\right),$$

where $r_{m-1} = f - P_{m-1}$, P_{m-1} being the polynomial defined by Lemma 4.1, *and with some constant C independent of f, x and $m \geqq 4q+4$.*

PROOF. By the definition of H_ρ we can write

$$\left|H_\rho\left(t; \frac{r_{m-1}}{v^{(r,s)}\widehat{p}_m(w)}; x\right)\right| \leqq \sum_{j=0}^{\rho-1} \frac{|x-t|^j}{j!}\left|\left[\frac{r_{m-1}}{v^{(r,s)}\widehat{p}_m(w;x)}\right]^{(j)}_{x=t}\right|.$$

Now, since

$$\left[\frac{r_{m-1}(x)}{\widehat{p}_m(w;x)}\right]^{(j)}_{x=t} = \sum_{k=0}^{j}\binom{j}{k} r^{(j-k)}_{m-1}(t)\left(\frac{1}{\widehat{p}_m(w;x)}\right)^{(k)},$$

with

$$\left|\left(\frac{1}{\widehat{p}_m(w;t)}\right)^{(k)}\right| \leqq C\frac{m^{k-1}}{|\widehat{p}_m(w;t)|},$$

and observing that the polynomial $v^{(r,s)}$ and its derivatives are bounded in t we have

$$\left|v^{(r,s)}(x)\widehat{p}_m(w;x)H_\rho\left(t;\frac{r_{m-1}}{v^{(r,s)}\widehat{p}_m(w)};x\right)\right| \leqq$$

$$\leqq C(1+x)^s(1-x)^r\frac{\left|\widehat{p}_m(w;x)\right|}{\left|\widehat{p}_m(w;t)\right|}\sum_{j=0}^{\rho-1}\frac{|x-t|^j}{j!}m^j \leqq$$

$$\leqq C(1+x)^s(1-x)^r\frac{\left|\widehat{p}_m(w;x)\right|}{\left|\widehat{p}_m(w;t)\right|}\,\frac{E_{m-q}\left(f^{(q)}\right)}{m^{q-\rho+2}}\left(|x-t|+m^{-1}\right)^{\rho-1}.$$

Thus, taking into account that $\left|\widehat{p}_m(w;t)\right|^{-1} \leqq C\,m^{-\frac{\gamma}{2}-1}$ and $\left|\widehat{p}_m(w;x)\right| \leqq$ $\leqq C\,m\left|p_m(w;x)\right|$, we obtain

$$\left|v^{(r,s)}(x)\widehat{p}_m(w;x)H_\rho\left(t;\frac{r_{m-1}}{v^{(r,s)}\widehat{p}_m(w)};x\right)\right| \leqq$$

$$\leqq C\,(1+x)^s(1-x)^r\left|p_m(w;x)\right|\frac{E_{m-q}\left(f^{(q)}\right)}{m^{q-\rho+2+\frac{\gamma}{2}}}\left(|x-t|+m^{-1}\right)^{\rho-1}.$$

Comparing this last inequality with (4.27) we deduce that proceeding as in the proof of (4.25) we obtain (4.35). Indeed, the only difference is that the exponent $q-\rho+1+\frac{\gamma}{2}$ is replaced by $q-\rho+2+\frac{\gamma}{2}$, but now we have the assumption (3.17) replacing (3.7). □

Proof of Theorem 3.3. Let $r_{m-1}=f-P_{m-1}$ where P_{m-1} is the polynomial defined by Lemma 4.1 corresponding to f. Then $r^{(j)}_{m-1}(-1)=0$, $j=0,1,\dots,s-1$ and $r^{(j)}_{m-1}(1)=0$, $j=0,1,\dots,r-1$, $r,s\leqq q+1$. Thus, recalling (4.11) we can write

$$\tag{4.36} \left|f^{(j)}(x)-\mathcal{L}^{(j)}_{m,r,s,\rho}(w;f;x)\right| \leqq$$

$$\leqq \left|r^{(j)}_{m-1}(x)\right| + \left|\left[v^{(r,s)}(x)C_\rho(x)\widehat{C}_m\left(w;\frac{r_{m-1}}{v^{(r,s)}C_\rho};x\right)\right]^{(j)}\right| +$$

$$+\left|\left[v^{(r,s)}(x)\widehat{p}_m(w;x)H_\rho\left(t;\frac{r_{m-1}}{v^{(r,s)}\widehat{p}_m(w)};x\right)\right]^{(j)}\right| =:$$

$$=: I_1(x)+I_2(x)+I_3(x),\qquad j=0,1,\dots,\min(r-1,s-1).$$

By Lemmas 4.1 and 4.2 we have

(4.37)
$$I_1(x) \leqq C\left(\frac{\sqrt{1-x^2}}{m}\right)^{q-j} E_{m-q}\big(f^{(q)}\big), \quad j=0,1,\dots,\min(r-1,s-1).$$

Furthermore, in view of Lemmas 4.4 and 4.2 we also have

(4.38)
$$I_3(x) \leqq C\left(\frac{\sqrt{1-x^2}}{m}\right)^{q-j} E_{m-q}\big(f^{(q)}\big), \quad j=0,1,\dots,\min(r-1,s-1).$$

On the other hand, in view of the definition of $\widehat{\mathcal{L}}_m$ we can write

$$\left|v^{(r,s)}(x)C_\rho(x)\widehat{\mathcal{L}}_m\left(w;\frac{r_{m-1}}{v^{(r,s)}C_\rho};x\right)\right| \leqq$$
$$\leqq C(1+x)^s(1-x)^r|t-x|^\rho \times$$
$$\times \sum_{\substack{k=1\\k\neq i^*}}^m \frac{\left|x_{k,m}(w)-x_{i^*,m}(w)\right|}{\left|x-x_{i^*,m}(w)\right|}\left|\ell_{k,m}(w;x)\right| \times$$
$$\times \frac{\left|r_{m-1}\big(x_{k,m}(w)\big)\right|}{\big(1-x_{k,m}(w)\big)^r\big(1+x_{k,m}(x)\big)^s\left|t-x_{k,m}(w)\right|^\rho}.$$

Thus, denoting by d the index of the zero of $p_m(w)$ closest to x, by Lemma 4.1 and (4.20) we get

$$\left|v^{(r,s)}(x)C_\rho(x)\widehat{\mathcal{L}}_m\left(w;\frac{r_{m-1}}{v^{(r,s)}C_\rho};x\right)\right| \leqq$$
$$\leqq C\frac{E_{m-q}\big(f^{(q)}\big)}{m^q}\Bigg\{(1-x^2)^{\frac{q}{2}}+(1+x)^s|t-x|^{\rho-1}\left|p_m(w;x)\right| \times$$
$$\times \sum_{\substack{k=1\\k\neq d}}^m \frac{\big(1-x_{k,m}(w)\big)^a\big(1+x_{k,m}(x)\big)^b\big(\left|t-x_{k,m}(w)\right|+m^{-1}\big)^c}{m\left|x-x_{k,m}(w)\right|}\Bigg\},$$

where $a = \frac{q}{2} + \frac{\alpha}{2} + \frac{3}{4} - r$, $b = \frac{q}{2} + \frac{\beta}{2} + \frac{3}{4} - s$ and $c = \frac{\gamma}{2} - \rho + 1$. By the assumptions (3.15)–(3.17) we have $-\frac{1}{2} \leqq a, b \leqq \frac{1}{2}$ and $-1 < c \leqq 0$. Therefore, in view of (4.30) and since

$$(1+x)^s(1-x)^r|t-x|^{\rho-1}\left|p_m(w;x)\right| \leqq$$

$$\leqq C(1-x)^{r-\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{s-\frac{\beta}{2}-\frac{1}{4}}\left(|t-x|+m^{-1}\right)^{\rho-\frac{\gamma}{2}-1},$$

$$x_{1,m}(w) \leqq x \leqq x_{m,m}(w),$$

we deduce

(4.39)

$$\left|v^{(r,s)}(x)C_\rho(x)\widehat{\mathcal{L}}_m\left(w; \frac{r_{m-1}}{v^{(r,s)}C_\rho}; x\right)\right| \leqq C\left(\frac{\sqrt{1-x^2}}{m}\right)^q E_{m-q}\left(f^{(q)}\right)\log m,$$

$$x_{1,m}(w) \leqq x \leqq x_{m,m}(w).$$

Now we assume $x_{m,m}(w) < x \leqq 1$. Then, taking into account that

$$\left|p_m(w;x)\right| \leqq C\, m^{\alpha+\frac{1}{2}}\left(\sqrt{1+x}+m^{-1}\right)^{-\beta-\frac{1}{2}}\left(|x-t|+m^{-1}\right)^{-\frac{\gamma}{2}}$$

and proceeding as before we have

$$\left|v^{(r,s)}(x)C_\rho(x)\widehat{\mathcal{L}}_m\left(w; \frac{r_{m-1}}{v^{(r,s)}C_\rho}; x\right)\right| \leqq$$

$$\leqq C\frac{E_{m-q}\left(f^{(q)}\right)}{m^q}\Bigg\{(1-x^2)^{\frac{q}{2}}+$$

$$+(1+x)^s(1-x)^r\left(|t-x|+m^{-1}\right)^{\rho-\frac{\gamma}{2}-1}\left(\sqrt{1+x}+m^{-1}\right)^{-\beta-\frac{1}{2}}m^{\alpha+\frac{1}{2}}\times$$

$$\times\sum_{\substack{k=1\\k\neq d}}^{m}\frac{\left(1-x_{k,m}(w)\right)^a\left(1+x_{k,m}(x)\right)^b\left(|t-x_{k,m}(w)|+m^{-1}\right)^c}{m|x-x_{k,m}(w)|}\Bigg\} \leqq$$

$$\leqq C\frac{E_{m-q}\left(f^{(q)}\right)}{m^q}\Bigg\{(1-x^2)^{\frac{q}{2}}+$$

$$+(1+x)^s(1-x)^{\frac{q}{2}+\frac{\alpha}{2}+\frac{1}{4}}\left(|t-x|+m^{-1}\right)^{\rho-\frac{\gamma}{2}-1}\left(\sqrt{1+x}+m^{-1}\right)^{-\beta-\frac{1}{2}}m^{\alpha+\frac{1}{2}}\times$$

$$\times\sum_{\substack{k=1\\k\neq d}}^{m}\frac{\left(1-x_{k,m}(w)\right)^{\frac{1}{2}}\left(1+x_{k,m}(x)\right)^b\left(|t-x_{k,m}(w)|+m^{-1}\right)^c}{m|x-x_{k,m}(w)|}\Bigg\}.$$

Thus, by (4.30) and the assumption $\alpha > -\frac{1}{2}$, we can write

$$\left| v^{(r,s)}(x) C_\rho(x) \widehat{\mathcal{L}}_m \left(w; \frac{r_{m-1}}{v^{(r,s)} C_\rho}; x \right) \right| \leqq$$

$$\leqq C \left(\frac{\sqrt{1-x^2}}{m} \right)^q E_{m-q}\left(f^{(q)}\right) \left\{ 1 + (1-x)^{\frac{\alpha}{2}+\frac{1}{4}} m^{\alpha+\frac{1}{2}} \log m \right\} \leqq$$

$$\leqq C \left(\frac{\sqrt{1-x^2}}{m} \right)^q E_{m-q}\left(f^{(q)}\right) \log m, \qquad x_{m,m}(w) < x \leqq 1.$$

Then (4.39) still holds for $x_{m,m}(w) < x \leqq 1$. In the same way we can proceed in the case $-1 \leqq x < x_{1,m}(w)$.

Applying Lemma 4.2 to

$$\left| v^{(r,s)}(x) C_\rho(x) \widehat{\mathcal{L}}_m \left(w; \frac{r_{m-1}}{v^{(r,s)} C_\rho}; x \right) \right| \leqq$$

$$\leqq C \left(\frac{\sqrt{1-x^2}}{m} \right)^q E_{m-q}\left(f^{(q)}\right) \log m, \qquad |x| \leqq 1,$$

we get

$$I_2(x) \leqq C \left(\frac{\sqrt{1-x^2}}{m} \right)^{q-j} E_{m-q}\left(f^{(q)}\right) \log m, \tag{4.40}$$

$$j = 0, 1, \ldots, \min(r-1, s-1).$$

The proof is complete by combining (4.37), (4.38) and (4.40) with (4.36). □

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ALMOST COMPACT SUBSPACES OF HYPEREXTENSIONS

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0. Introduction

The paper [1] contains the construction, if E is a topological space and $\mathbf{S}$ is an open subbase in E satisfying $\emptyset$, $E \in \mathbf{S}$, of a space E^h, containing E as a subspace and having many interesting properties; in particular, E^h contains a series of subspaces with more or less nice behaviour, among them one which generalizes the concept of a Wallman-type compactification and another that generalizes the superextension introduced by J. de Groot [8]. Some further subspaces of E^h were examined in [2].

The purpose of the present paper is to study, under some restriction concerning the subbase $\mathbf{S}$, further subspaces of E^h that are almost compact. It will turn out, in particular, that we obtain in this manner a generalization of the theory of almost compact extensions due to J. Flachsmeyer [7] for Hausdorff spaces and generalized by K. Császár [5] to arbitrary topological spaces. Similar but weaker statements are contained in [6].

1. Terminology

We shall use the terminology and the notations of [1], so we do not recall here definitions that can be found there. On the other hand, we formulate here some definitions that are not generally used or known and are not contained in [1].

A topological space E is said to be *almost compact* if, in an arbitrary open cover of E, there is a finite number of members whose union is dense in E; this is equivalent to the condition that every open filter base (i.e. a filter base composed of open sets) has a cluster point in E where a point $x \in E$ is said to be a *cluster point* of a system $\mathbf{A}$ of subsets of E iff $x \in \overline{A}$ for every $A \in \mathbf{A}$. Another equivalent characterization of almost compact spaces is the property that every maximal open filter base is convergent in E (see e.g. [5], (1.5)).

Let $\mathbf{S}$ be a subbase (for the open sets) in E. The space E is said to be *almost supercompact relative to* $\mathbf{S}$ iff, in every open cover of E whose

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members belong to **S**, there are two members whose union is dense in E ([6], Definition 2.2, called super almost compact). If E is supercompact (see [1]) relative to **S** then it is obviously almost supercompact relative to **S**. On the other hand, if E is almost supercompact relative to a subbase **S**, then it is almost compact; in fact, we have the following analogue of the well-known Alexander Lemma (implying that [6], Example 2.1 is false):

LEMMA 1.1. *Let* **S** *be a subbase in* E. *If each cover of* E *whose members belong to* **S** *contains a finite number of members with union dense in* E *then* E *is almost compact.*

PROOF. Let $\mathbf{m}$ be a maximal open filter base in E and suppose that $\mathbf{m}$ does not converge. Then every point $x \in E$ would have an open neighbourhood $V_x \notin \mathbf{m}$; we can assume that V_x is a finite intersection of members of $\mathbf{S}$ and then at least one of them does not belong to $\mathbf{m}$, say $x \in S_x \in \mathbf{S}$, $S_x \notin$ $\notin \mathbf{m}$. Since $E = \bigcup_{x \in X} S_x$, there are $S_{x_1}, \dots, S_{x_n}$ such that $E = \bigcup_1^n \overline{S}_{x_i}$. By the maximality of $\mathbf{m}$, $S_{x_i} \notin \mathbf{m}$ implies the existence of an open $G_i \in \mathbf{m}$ such that $G_i \cap S_{x_i} = \emptyset$, hence $G_i \cap \overline{S}_{x_i} = \emptyset$. This would imply $\bigcap_1^n G_i = \emptyset$ which is impossible. □

[6], Theorem 2.1 contains a weaker statement, based on Definition 2.1 of [6] that introduces almost compactness relative to a subbase. According to 1.1, this concept coincides with almost compactness.

A subset $A \subset E$ will be said to be *ultradense* iff $E = \bigcup \{ \overline{\{x\}} : x \in A \}$. An ultradense subset is dense; in a T_1-space, there is no proper ultradense subset.

Now let X be a superspace (see [1]) of E. If $\mathbf{A}$ is a system of subsets of X, the trace of $\mathbf{A}$ in E will be denoted by $\mathbf{A}|E$.

The superspace X of E is said to be T_1-*reduced* iff $x \in X$, $y \in X - E$, $x \neq y$ implies that each of the points x and y has a neighbourhood not containing the other. X is said to be T_2-*reduced* iff $x \in X$, $y \in X - E$, $x \neq$ $\neq y$ implies that x and y have disjoint neighbourhoods. A superspace that is a T_i-space ($i = 1, 2$) is T_i-reduced. Under a slightly different terminology, these concepts have been investigated in [4].

Now let X be an extension of E (i.e. a superspace in which E is dense), and $\mathbf{A}$ a system of subsets of E. X is said to be *weakly* $\mathbf{A}$-*disjunctive* iff $A_1, A_2 \in \mathbf{A}$, $A_1 \cap A_2 = \emptyset$ implies $(\overline{A}_1 \cap \overline{A}_2) - E = \emptyset$ (cf. the concept of an $\mathbf{A}$-disjunctive extension in [1]). X is said to be $\mathbf{A}$-*hypercombinatorial* iff $A_1, A_2 \in \mathbf{A}$, $\operatorname{int}_E(A_1 \cap A_2) = \emptyset$ implies $\overline{A}_1 \cap \overline{A}_2 = A_1 \cap A_2$ for the closures in X (see for special systems $\mathbf{A}$ in [5], Definition (3.5), and in a still more special case in [9]).

Let X and Y be extensions of E. We shall say that X and Y are *weakly equivalent* iff there exists a bijection $h: X \to Y$ such that $h(x) = x$ for $x \in E$ (i.e. h *fixes* E) and, for $x \in X - E$, the trace in E of the neighbourhood filter of x (in X) is the same as the trace in E of the neighbourhood filter of $h(x)$ in Y. This is clearly an equivalence relation. If X and Y are equivalent extensions (see [1]), then they are obviously weakly equivalent.

2. The subspace E^a

As in [1], we consider an arbitrary topological space E and a subbase $\mathbf{S}$ in E satisfying $\emptyset, E \in \mathbf{S}$; we denote

$$\mathbf{T} = \{E - S : S \in \mathbf{S}\}, \quad \mathbf{V} = \mathbf{S} \cup \mathbf{T}. \tag{2.1}$$

In the sequel, we often assume the following standard hypothesis (cf. the concept of a C. C.-closed subbase in [6]):

$$S \in \mathbf{S} \quad \text{implies} \quad \overline{S} \cap E \in \mathbf{T}, \quad \text{i.e.} \quad E - \overline{S} \in \mathbf{S}. \tag{2.2}$$

If we understand the closure in E, then we can write simply $\overline{S} \in \mathbf{T}$; however, we prefer to denote by $\overline{A}$ the closure of A taken in the hyperextension E^h of E relative to $\mathbf{S}$ (see [1]). (2.2) is fulfilled e.g. if $\mathbf{S}$ coincides with the system of all open sets in E, or if $E = \mathbf{R}$ and $\mathbf{S}$ is composed of all intervals $(-\infty, c)$ and $(c, +\infty)$ $(-\infty \leqq c \leqq +\infty)$.

As a consequence of (2.2), let us observe:

THEOREM 2.3. *If the subbase* $\mathbf{S}$ *fulfils* (2.2), *the following are equivalent:*
(a) *E is almost supercompact relative to* $\mathbf{S}$.
(b) *Every linked system composed of members of* $\mathbf{S}$ *has a cluster point.*
(c) *Every* $\mathbf{S}$*-sieve has a cluster point.*
(d) *Every ultra-*$\mathbf{S}$*-sieve has a cluster point.*

PROOF. (a) $\Rightarrow$ (b): Let $\{S_i : i \in I\}$ be a linked system, $S_i \in \mathbf{S}$. If it did not have a cluster point, then $\bigcap_{i \in I} \overline{S}_i$ (with closures taken in E) would be empty so that the sets $E - \overline{S}_i = S_i' \in \mathbf{S}$ would cover E. For $i, j \in I$, $\overline{S}_i' \cup \overline{S}_j' = E$, $\overline{S}_i' = E - S_i$ implies $S_i \cap S_j = \emptyset$: a contradiction.

(b) $\Rightarrow$ (a): Let $E = \bigcup_{i \in I} S_i$, $S_i \in \mathbf{S}$. If $E = \overline{S}_i \cup \overline{S}_j$ were not true for any $i, j \in I$, then the sets $E - \overline{S}_i = S_i' \in \mathbf{S}$ would constitute a linked system, and $\bigcap_{i \in I} \overline{S}_i' \neq \emptyset$ would imply $\bigcap_{i \in I} (E - S_i) \neq \emptyset$: a contradiction.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b) is obvious. □

Let us denote by E^a the subspace of E^h composed of E and all points $x \in E^h - E$ such that $\mathbf{v}(x)$ is an $\mathbf{S}$-sieve; by [1], (3.6) the sieves $\mathbf{v}(x)$ $(x \in E^a - E)$ are precisely all nontrivial ultra-$\mathbf{S}$-sieves.

An important property of E^a is contained in

LEMMA 2.4. *If $y \in E^h$, $S_0 \in \mathbf{S}$ and $y \in \overline{v(S_0)}$, then there exists a point $x \in E^a \cap v(S_0)$ such that $y \in \overline{\{x\}}$.*

PROOF. For every set $S \in \mathbf{v}(y) \cap \mathbf{S}$, $v(S)$ is a neighbourhood of y in E^h, hence $v(S) \cap v(S_0) \neq \emptyset$. By [1], (4.5) this implies $S \cap S_0 \neq \emptyset$. Hence $\{S_0\} \cup \cup \left(\mathbf{v}(y) \cap \mathbf{S}\right)$ is a linked system contained in $\mathbf{S}$; by [1], (3.5) it is contained in an ultra-$\mathbf{S}$-sieve $\mathbf{s}$. If $\mathbf{s}$ is trivial, there is a point $x \in E$ satisfying $x \in S_0 \cap S$ for every $S \in \mathbf{v}(y) \cap \mathbf{S}$; if $\mathbf{s}$ is non-trivial, there is a point $x \in E^a - E$ such that $\mathbf{s} = \mathbf{v}(x)$. In both cases $x \in v(S_0) \cap \bigcap_1^n v(S_i)$ for any finite subsystem $\{S_1, \dots, S_n\}$ of $\mathbf{v}(y) \cap \mathbf{S}$. Hence $x \in E^a \cap v(S_0)$ is contained in every member $\bigcap_1^n v(S_i)$ of a neighbourhood base of y. □

COROLLARY 2.5. *E^a is ultradense in E^h.*

PROOF. For $S_0 = E$, $v(S_0) = E^h$ we obtain that every $y \in E^h$ is contained in some $\overline{\{x\}}$ with $x \in E^a$. □

LEMMA 2.6. *Let $S_0, S_1 \in \mathbf{S}$, $E \subset \overline{S}_0 \cup \overline{S}_1$. Then*
(a) $E^a - E \subset v(S_0) \cup v(S_1)$,
(b) $E^h = \overline{v(S_0)} \cup \overline{v(S_1)}$.

PROOF. (a) Assume $x \in E^a - E$, $x \notin v(S_0) \cup v(S_1)$. Then by [1], (3.3) $E - S_i \in \mathbf{v}(x)$ for $i = 0, 1$ and, since $\mathbf{v}(x)$ is an $\mathbf{S}$-sieve, there are $S_i' \in \mathbf{S}$ such that $S_i' \in \mathbf{v}(x)$, $S_i' \subset E - S_i$. The sets S_i' being open in E, we have also $S_i' \subset E - \overline{S}_i$, thus $E - \overline{S}_i \in \mathbf{v}(x)$. But this is impossible because $\left(E - \overline{S}_0\right) \cap \cap \left(E - \overline{S}_1\right) = \emptyset$ by hypothesis.

(b) The hypothesis and (a) yield

$$E^a \subset \overline{v(S_0)} \cup \overline{v(S_1)},$$

and by 2.5 E^a is dense in E^h. □

THEOREM 2.7. *If (2.2) is fulfilled and $E^a \subset X \subset E^h$ then X is almost supercompact relative to the subbase $\mathbf{S}^h | X$.*

PROOF. Consider a cover of X whose members belong to this subbase, i.e. assume

$$X \subset \bigcup_{i \in I} v(S_i) \tag{2.8}$$

where $S_i \in \mathbf{S}$. We claim that there are two sets S_i whose union is dense in E. By assuming the contrary, the system of the sets $E - \overline{S}_i$ would be linked and, by (2.2), contained in $\mathbf{S}$. Hence by [1], (3.5) there would be an ultra-$\mathbf{S}$-sieve $\mathbf{s}$ containing every set $E - \overline{S}_i$. From (2.8) we have $E = \bigcup_{i \in I} S_i$ so that

$\mathbf{s}$ cannot be trivial, hence $\mathbf{s} = \mathbf{v}(x)$ for some $x \in E^a - E$. Now if $x \in v(S_i)$ then $S_i \in \mathbf{v}(x)$ and $E - \overline{S}_i \in \mathbf{v}(x)$ cannot hold simultaneously.

Therefore there exist $i, j \in I$ such that

$$E \subset \overline{S}_i \cup \overline{S}_j.$$

Then by 2.6

$$X \subset \overline{v(S_i)} \cup \overline{v(S_j)}.$$

Now by 2.4 $y \in \overline{v(S_i)}$ implies

$$y \in \overline{v(S_i) \cap E^a} \subset \overline{v(S_i) \cap X}$$

and similarly $y \in \overline{v(S_j)}$ implies $y \in \overline{v(S_j) \cap X}$. Hence

$$X \subset \overline{v(S_i) \cap X} \cup \overline{v(S_j) \cap X}. \quad \square$$

Let us mention the following consequence of 2.6:

COROLLARY 2.9. *If* (2.2) *is fulfilled, then, for* $S \in \mathbf{S}$, *the set*

$$v(S) \cap (E^a - E) \tag{2.10}$$

is open and closed in $E^a - E$.

PROOF. For $S' = E - \overline{S} \in \mathbf{S}$ we have $E \subset \overline{S} \cup \overline{S'}$, hence by 2.6

$$E^a - E \subset v(S) \cup v(S').$$

On the other hand $S \cap S' = \emptyset$ implies $v(S) \cap v(S') = \emptyset$ so that the complement in $E^a - E$ of (2.10) coincides with $v(S') \cap (E^a - E)$. $\square$

We can add to 2.5:

THEOREM 2.11. *A set* $E \subset X \subset E^h$ *is ultradense in* E^h *iff* $E^a \subset X \subset$ $\subset E^h$.

PROOF. By 2.5 every set $E^a \subset X \subset E^h$ is ultradense in E^h. Conversely if $x \in E^a - E$, $y \in E^h$, $x \neq y$, then $\mathbf{v}(x) \neq \mathbf{v}(y)$ so that, by [1], (3.4), there exist $V_1, V_2 \in \mathbf{V}$ such that $V_1 \in \mathbf{v}(x)$, $V_2 \in \mathbf{v}(y)$, $V_1 \cap V_2 = \emptyset$. Since $\mathbf{v}(x)$ is an $\mathbf{S}$-sieve, we can assume $V_1 = S \in \mathbf{S}$. Hence $x \in v(S)$, $y \notin v(S)$ and $v(S)$ being a neighbourhood of x, we have $x \notin \overline{\{y\}}$. Therefore if $E \subset X \subset E^h$ and X is ultradense in E^h, it must contain every $x \in E^a$. $\square$

We can characterize the superspaces equivalent to a subspace of E^a:

THEOREM 2.12. *A superspace $Y \supset E$ is equivalent to a space X such that $E \subset X \subset E^a$ iff*

(a) *there is in Y a subbase $\mathbf{S}'$ such that $\mathbf{S}'|E = \mathbf{S}$, Y is $\mathbf{V}'$-exact* (see [1]) *for $\mathbf{V}' = \mathbf{S}' \cup \mathbf{T}'$, $\mathbf{T}' = \{X - S' : S' \in \mathbf{S}'\}$, and $y \in T' \in \mathbf{T}'$ implies the existence of $S' \in \mathbf{S}'$ with $y \in S' \subset T'$,*
and

(b) *Y is a reduced superspace.*

PROOF. If $E \subset X \subset E^a$ then $\mathbf{S}' = \mathbf{S}^h|X$ fulfils these conditions ([1], (4.8) and (4.2)), and the properties in question remain valid for an equivalent superspace.

Conversely, if Y fulfils (a) and (b), then, by [1], (4.8), there is a homeomorphism $h: Y \to X$ such that $E \subset X \subset E^h$, $h|E = \mathrm{id}_E$, and $h(V') = v(V' \cap E) \cap X$ for $V' \in \mathbf{V}'$. If $x \in X - E$, $x = h(y)$, and $T \in \mathbf{v}(x) \cap \mathbf{T}$, then $v(T) \cap X = h(T')$ for some $T' \in \mathbf{T}'$, $y \in T'$, $T = T' \cap E$, so that there exists $S' \in \mathbf{S}'$ such that $y \in \mathbf{S}' \subset \mathbf{T}'$. Thus $S = S' \cap E \in \mathbf{S}$ satisfies $S \subset T$, $x = h(y) \in h(S') = v(S) \cap X$, and $S \in \mathbf{v}(x)$. Therefore $\mathbf{v}(x)$ is an $\mathbf{S}$-sieve, $x \in E^a$, and $E \subset X \subset E^a$. $\square$

3. The subspace E^b

Let us now define the subspace $E^b \subset E^h$ by setting $x \in E^b$ iff either $x \in E$ or $x \in E^h - E$ and $\mathbf{v}(x)$ is an $\mathbf{S}$-sieve without cluster points in E. Clearly

$$E \subset E^b \subset E^a \subset E^h. \tag{3.1}$$

LEMMA 3.2. *If $y \in E^a - E^b$ then there exists a point $x \in E$ such that $x \in \overline{\{y\}}$.*

PROOF. For $y \in E^a - E^b$, $\mathbf{v}(y)$ has a cluster point $x \in E$. If $x \in S \in \mathbf{S}$, then $S \cap V \neq \emptyset$ for every $V \in \mathbf{v}(y)$, hence by [1], (3.3) $S \in \mathbf{v}(y)$, $y \in v(S)$. Therefore each neighbourhood of X of the form $\bigcap_1^n v(S_i)$, $S_i \in \mathbf{S}$, contains y. $\square$

From now on we always assume that the standard hypothesis (2.2) is fulfilled. First we prove the following analogue of 2.7:

THEOREM 3.3 (cf. [6], Theorem 3.1). *If $E^b \subset X \subset E^a$ then X is almost supercompact relative to $\mathbf{S}^h|X$.*

PROOF. Let $X \subset \bigcup_{i \in I} v(S_i)$, $S_i \in \mathbf{S}$. For a point $y \in E^a - E^b$, select $x \in E$ such that $x \in \overline{\{y\}}$. For an index i, we have $x \in v(S_i)$, hence $y \in v(S_i)$.

Therefore

$$E^a \subset \bigcup_{i \in I} v(S_i)$$

and, like in the proof of 2.7, there exist i and j such that $E \subset \overline{S}_i \cup \overline{S}_j$. Then by 2.6(a)

$$X \subset \overline{v(S_i) \cap X} \cup \overline{v(S_j) \cap X}. \quad \square$$

Another important property of E^b will follow from

LEMMA 3.4. *If $x \in E^a$, $y \in E^b - E$, $x \neq y$, then there are $S_0, S_1 \in \mathbf{S}$ such that $x \in v(S_0)$, $y \in v(S_1)$, and*

$$v(S_0) \cap v(S_1) = \emptyset.$$

PROOF. Suppose first $x \in E$. Then x is not a cluster point of $\mathbf{v}(y)$, hence there is an $S_1 \in \mathbf{S}$ in the $\mathbf{S}$-sieve $\mathbf{v}(y)$ such that $x \notin \overline{S}_1$. Then $x \in \in S_0 = E - \overline{S}_1 \in \mathbf{S}$ by (2.2), hence $S_0 \cap S_1 = \emptyset$ and, by [1], (4.2)(b),

$$x \in v(S_0), \quad y \in v(S_1), \quad v(S_0) \cap v(S_1) = \emptyset. \tag{3.5}$$

If $x \in E^a - E$, then $\mathbf{v}(x) \neq \mathbf{v}(y)$ both are $\mathbf{S}$-sieves, hence by [1], (3.4) there are $S_0, S_1 \in \mathbf{S}$ such that $S_0 \in \mathbf{v}(x)$, $S_1 \in \mathbf{v}(y)$, $S_0 \cap S_1 = \emptyset$. This implies (3.5) again. $\square$

COROLLARY 3.6 (see [6], Theorem 3.2). *E^b is a T_2-reduced superspace of E. More generally, if $E \subset X \subset E^a$ then $E^b \cup X$ is a T_2-reduced superspace of X.* $\square$

We shall need another important

LEMMA 3.7. *If $S_0 \in \mathbf{S}$, $x \in E \cap \overline{v(S_0)}$ then $x \in \overline{S}_0$.*

PROOF. Assuming $x \in E - \overline{S}_0$, by (2.2) $E - \overline{S}_0 = S \in \mathbf{S}$, and $S_0 \cap S = \emptyset$ implies $v(S_0) \cap v(S) = \emptyset$ so that the neighbourhood $v(S)$ of x does not intersect $v(S_0)$. $\square$

There is also a converse of 3.3:

THEOREM 3.8. *If $E \subset X \subset E^a$ and X is almost supercompact relative to $\mathbf{S}^h | X$, then $E^b \subset X \subset E^a$.*

PROOF. Suppose $E \subset X \subset E^a$ and $z \in E^b - X$. Let $\{S_i : i \in I\}$ be the system of those sets $S_i \in \mathbf{S}$ for which $E - \overline{S}_i \in \mathbf{v}(z)$. We have

$$X \subset \bigcup_{i \in I} v(S_i).$$

In fact, $x \in X$ implies by 3.4 the existence of $S, S' \in \mathbf{S}$ such that $x \in \in v(S)$, $z \in v(S')$, $S \cap S' = \emptyset$. Hence $\overline{S} \cap S' = \emptyset$, $S' \subset E - \overline{S}$, $S' \in \mathbf{v}(z)$, consequently $E - \overline{S} \in \mathbf{v}(z)$ so that $S = S_i$, $x \in v(S_i)$ for some $i \in I$.

Now the inclusion

$$X \subset \overline{v(S_i) \cap X} \cup \overline{v(S_j) \cap X}$$

cannot hold for any two indices $i, j \in I$. In fact, this would imply $E \subset \overline{v(S_i)} \cup \cup \overline{v(S_j)}$, hence $E \subset \overline{S}_i \cup \overline{S}_j$ by 3.7. However, this is impossible since $E - \overline{S}_i$, $E - \overline{S}_j \in \mathbf{v}(z)$ implies $(E - \overline{S}_i) \cap (E - \overline{S}_j) \neq \emptyset$.

Therefore X, satisfying $E \subset X \subset E^a$, cannot be almost supercompact relative to $\mathbf{S}^h | X$ unless it contains every $z \in E^b$. □

A converse of 3.6 will result from the following

LEMMA 3.9. *If $E \subset X \subset E^h$ and X is a strongly reduced superspace of E, then $X \cap (E^a - E^b) = \emptyset$.*

PROOF. 3.2. □

COROLLARY 3.10. *For a space X such that $E \subset X \dot{\subset} E^a$, the following statements are equivalent:*

(a) $E \subset X \subset E^b$,

(b) *X is a T_2-reduced superspace of E,*

(c) *X is a T_1-reduced superspace of E,*

(d) *X is a strongly reduced superspace of E.*

PROOF. (a) $\Rightarrow$ (b): 3.6.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d): Obvious.

(d) $\Rightarrow$ (a): 3.9. □

COROLLARY 3.11. *If $E \subset X \subset E^a$, X is strongly reduced and almost supercompact relative to $\mathbf{S}^h | X$, then $X = E^b$.*

PROOF. 3.10 and 3.8. □

It is essential in 3.11 to consider only subspaces X lying between E and E^a; without this restriction E^h can contain other subspaces that are almost supercompact relative to the trace of $\mathbf{S}^h$ and are even T_2-reduced.

E.g. let $E = \mathbf{R}$ with the usual topology, $\mathbf{S}$ be the system of all open subsets. Now $\mathbf{S}$ is a regular and normal subbase and E is T_2 so that E^s is supercompact relative to $\mathbf{S}^h | E^s$ and at the same time T_2 ([1], (5.2) and (5.10)). We have $E^s - E^a \neq \emptyset$, $E^b - E^s \neq \emptyset$; the first difference contains e.g. a point x such that $\mathbf{v}(x)$ is an ultra-$\mathbf{T}$-sieve containing the sets $\mathbf{N}$ and $[c, +\infty)$ for $c > 0$ ($\mathbf{v}(x)$ cannot be an $\mathbf{S}$-sieve since $\operatorname{int} \mathbf{N} = \emptyset$), and $y \in E^b - E^s$ if $\mathbf{v}(y)$ is an ultra-$\mathbf{S}$-sieve containing the sets

$$(0, +\infty), \quad (-\infty, \varepsilon) \quad \text{for} \quad \varepsilon > 0, \quad (-\infty, -c) \cup (c, +\infty) \quad \text{for} \quad c > 0$$

($\mathbf{v}(y)$ cannot be a $\mathbf{T}$-sieve because a closed subset of $(0,+\infty)$ is disjoint from a set $(-\infty,\varepsilon)$ if $\varepsilon > 0$ is small enough).

Similarly to the argument yielding 2.12 we can state (with the notation of 2.12):

THEOREM 3.12. *A space $Y \supset E$ is equivalent to a space X such that $E^b \subset X \subset E^a$ iff it fulfils* 2.12 (a) *and* (b) *and it is almost supercompact relative to* $\mathbf{S}'$. □

THEOREM 3.13. *A space $Y \supset E$ is equivalent to a space X such that $E \subset X \subset E^b$ iff it fulfils* 2.12 (a) *and it is strongly reduced.* □

THEOREM 3.14. *A space $Y \supset E$ is equivalent to E^b iff it fulfils* 2.12 (a), *it is almost supercompact relative to* $\mathbf{S}'$ *and strongly reduced.* □

4. The subspaces E^p and E^f

Let us now consider the closures of E in the subspaces E^a and E^b, i.e. the subspaces $E^a \cap E^c$, $E^b \cap E^c \subset E^h$. They will be denoted by E^p and E^f, respectively.

$x \in E^a$ belongs to E^c iff either $x \in E$ or $\mathbf{v}(x)$ is a centred (i.e. centrated in the terminology of [1]) ultra-$\mathbf{S}$-sieve (because $x \in E^c$ means that $\mathbf{v}(x) \cap \mathbf{S}$ is centred and now $\mathbf{v}(x)$ is an $\mathbf{S}$-sieve).

Our next purpose is to prove that E^p and E^f are almost compact. To this aim we need a series of lemmas.

LEMMA 4.1. *Every centred system contained in* $\mathbf{S}$ *is contained in a maximal centred system contained in* $\mathbf{S}$.

PROOF. An easy application of the Kuratowski–Zorn lemma. □

LEMMA 4.2. *If* $\mathbf{s}$ *is a maximal centred system contained in* $\mathbf{S}$, $S \in \mathbf{S}$, *and* $S \cap \bigcap_1^n S_i \neq \emptyset$ *whenever* $S_i \in \mathbf{s}$ $(i = 1,\dots,n)$, *then* $S \in \mathbf{s}$.

PROOF. By hypothesis $\mathbf{s} \cup \{S\}$ is a centred system contained in $\mathbf{S}$. □

LEMMA 4.3. *Let* $\mathbf{s}$ *be a maximal centred system contained in* $\mathbf{S}$ *and* $\mathbf{v}$ *the sieve generated by the linked system* $\mathbf{s}$. *Then* $\mathbf{v}$ *is an ultra-*$\mathbf{S}$*-sieve.*

PROOF. Consider a set $S \in \mathbf{S}$. If $S \cap \bigcap_1^n S_i \neq \emptyset$ whenever $S_i \in \mathbf{s}$ $(i = 1,\dots,n)$, then by 4.2 $S \in \mathbf{s} \subset \mathbf{v}$. If there are $S_1,\dots,S_n \in \mathbf{s}$ such that $S \cap \bigcap_1^n S_i = \emptyset$, then $\bigcap_1^n S_i \subset E - \overline{S}$ (since the intersection is open in E), hence the set $E - \overline{S} \in \mathbf{S}$ (by (2.2)) has a non-empty intersection with any finite number of members of $\mathbf{s}$ because such an intersection meets $\bigcap_1^n S_i$. By 4.2 we have $E - \overline{S} \in \mathbf{s}$ and $E - \overline{S} \subset E - S \in \mathbf{v}$. By [1], (3.3) the $\mathbf{S}$-sieve $\mathbf{v}$ is an ultra-$\mathbf{S}$-sieve. □

THEOREM 4.4 (cf. [6], Theorem 4.2). *If $E^f \subset X \subset E^c$, then X is almost compact.*

PROOF. By 1.1 it suffices to show that if

$$X \subset \bigcup_{i \in I} v(S_i) \quad (S_i \in \mathbf{S}) \tag{4.5}$$

then there are finitely many sets S_i such that the union of the corresponding sets $v(S_i) \cap X$ is dense in X.

Suppose (4.5) is valid. We claim that there are finitely many sets S_i such that their union is dense in E. Assume the contrary. Then the system of all sets $E - \overline{S}_i$ is a centred system contained in $\mathbf{S}$; let $\mathbf{s}$ be a maximal centred system contained in $\mathbf{S}$ such that $E - \overline{S}_i \in \mathbf{s}$ for every $i \in I$, and let $\mathbf{v}$ be the sieve generated by $\mathbf{s}$. By 4.3 $\mathbf{v}$ is an ultra-$\mathbf{S}$-sieve. If $x \in E$ then $x \in S_i$ for some i and then S_i does not meet the set $E - \overline{S}_i \in \mathbf{v}$. Hence $\mathbf{v}$ has no cluster point in E and $\mathbf{v} = \mathbf{v}(y)$ for some $y \in E^f$ (because $\mathbf{v}$ is clearly centred.) Now $y \in v(S_i)$, $S_i \in \mathbf{v}(y)$ for some i, in contradiction with $E - \overline{S}_i \in \mathbf{v}(y)$.

Therefore (4.5) implies

$$E \subset \bigcup_1^n \overline{S}_{i_j}$$

for suitable indices $i_j \in I$. Since E is dense in X, we also have

$$X \subset \bigcup_1^n \overline{S}_{i_j},$$

and clearly

$$X \subset \bigcup_1^n \overline{v\left(S_{i_j}\right) \cap X}. \quad \square$$

4.4 says more than the analogues of 2.7 and 3.3; it corresponds to a statement "$E^b \subset X \subset E^h$ implies that X is almost supercompact relative to $\mathbf{S}^h | X$". The author does not know whether this is true or not (for subbases fulfilling (2.2)).

The following converse precisely corresponds to 3.8:

THEOREM 4.6. *If $E \subset X \subset E^a$ and X is almost compact then $E^f \subset X$.*

PROOF. By 3.6 $X \cup E^f$ is a T_2-reduced, almost compact extension of X, hence [5], Theorem (1.5) (according to which an almost compact space has no proper T_2-reduced extension) implies $X \cup E^f = X$. □

LEMMA 4.7 (cf. [1], (7.1)). FOR $V \in \mathbf{V}$, $x \in E^p - E$, WE HAVE $x \in v(V)$ IFF $x \in \overline{V}$.

PROOF. $x \in v(V)$ implies $V \cap \bigcap_1^n S_i \neq \emptyset$ whenever $S_i \in \mathbf{v}(x) \cap \mathbf{S}$ ($i = 1, \dots, n$), i.e. every member of a neighbourhood base of x meets V and $x \in \overline{V}$. Conversely if $x \in \overline{V}$ then $V \cap v(S) \neq \emptyset$ for every $S \in \mathbf{v}(x) \cap \mathbf{S}$, hence $V \cap S \neq \emptyset$ for the S in question, whence $V \in \mathbf{v}(x)$, $x \in v(V)$ by [1], (3.3). □

THEOREM 4.8. *The space E^p is a weakly $\mathbf{V}$-disjunctive extension of E.*

PROOF. If $V_1, V_2 \in \mathbf{V}$, $V_1 \cap V_2 = \emptyset$, then $v(V_1) \cap v(V_2) = \emptyset$, so $\overline{V}_1 \cap \overline{V}_2 \cap (E^p - E) = \emptyset$ by 4.7. □

It is not difficult to show that the property of weak $\mathbf{V}$-disjunctivity coincides with that of being $\mathbf{T}$-hypercombinatorial:

LEMMA 4.9. *Let X be an arbitrary extension of E. Then X is weakly $\mathbf{V}$-disjunctive iff it is $\mathbf{T}$-hypercombinatorial.*

PROOF. Let X be weakly $\mathbf{V}$-disjunctive and suppose

$$T_1, T_2 \in \mathbf{T}, \quad \mathrm{int}_E(T_1 \cap T_2) = \emptyset. \tag{4.10}$$

Then, by introducing $S_i = E - T_i \in \mathbf{S}$ for $i = 1, 2$, by

$$\mathrm{int}_E(T_1 \cap T_2) = \mathrm{int}_E T_1 \cap \mathrm{int}_E T_2 = \left(E - \overline{S}_1\right) \cap \left(E - \overline{S}_2\right) = \emptyset$$

we have $E \subset \overline{S}_1 \cup \overline{S}_2$ (for the closures taken this time in X), hence $X = \overline{S}_1 \cup \overline{S}_2$. By hypothesis $\overline{S}_i \cap \overline{T}_i \cap (X - E) = \emptyset$ so that $\overline{T}_i - E \subset X - \overline{S}_i$ implies $\overline{T}_1 \cap \overline{T}_2 \subset E$ and

$$\overline{T}_1 \cap \overline{T}_2 = T_1 \cap T_2. \tag{4.11}$$

Conversely suppose that (4.10) implies (4.11) and consider $S \in \mathbf{S}$, $T = E - S \in \mathbf{T}$. By (2.2) $\overline{S} \cap E \in \mathbf{T}$ and $\overline{S} \cap E \cap T$ is the boundary of S in E, hence

$$\mathrm{int}_E \left(\overline{S} \cap E \cap T\right) = \emptyset.$$

Therefore, by hypothesis,

$$\overline{\overline{S} \cap E} \cap \overline{T} = \overline{S} \cap \overline{T} \subset E$$

and $\overline{S} \cap \overline{T} \cap (X - E) = \emptyset$. In the case $S_i \in \mathbf{S}$, $S_1 \cap S_2 = \emptyset$ we can consider S_1 and $T = E - S_1 \in \mathbf{T}$ to obtain $\overline{S}_1 \cap \overline{S}_2 \cap (X - E) = \emptyset$; if $T_i \in \mathbf{T}$, $T_1 \cap T_2 = \emptyset$, consider T_1 and $E - T_1 \in \mathbf{S}$. □

We can give now a characterization for the extensions equivalent to spaces lying between E and E^p:

THEOREM 4.12. *An extension Y of E is equivalent to an extension X satisfying $E \subset X \subset E^p$ iff it is a reduced, $\mathbf{T}$-strict, weakly $\mathbf{V}$-disjunctive (or T-hypercombinatorial) extension of E.*

PROOF. If $E \subset X \subset E^p$ then it is reduced together with E^h ([1], (4.4)), and $\mathbf{T}$-strict (see [1]) because $T \in \mathbf{T}$ implies $\overline{T} \cap X = v(T) \cap X$ by 4.7, so

$$X - \overline{T} = X - v(T) = v(S) \cap X$$

for $S = E - T \in \mathbf{S}$. The fact that E^p is weakly $\mathbf{V}$-disjunctive by 4.8 clearly implies the same property of X. If Y is equivalent to X, it possesses these properties as well.

Conversely let Y be an extension having the properties in question. Since weak $\mathbf{V}$-disjunctivity clearly implies $\mathbf{T}$-disjunctivity, Y is equivalent, by [1], (7.4), to an extension X such that $E \subset X \subset E^h$. More precisely, it is shown in the proof of [1], (7.4) that, with $\mathbf{T}' = \{\overline{T}: T \in \mathbf{T}\}$ (closure in Y), $\mathbf{S}' = \{Y - \overline{T}: T \in \mathbf{T}\}$ and $\mathbf{V}' = \mathbf{S}' \cup \mathbf{T}'$, [1], (4.8) (b) is satisfied ($E' = Y$); now the proof of [1], (4.8) yields that there is an extension X (denoted there by E'') between E and E^h, and a homeomorphism $h: Y \to X$ such that $h | E = \mathrm{id}_E$ and, for $y \in Y$, $\mathbf{v}(h(y))$ (which will be denoted by $\mathbf{v}'(y)$) is the sieve in E generated by $\{V' \cap E: y \in V' \in \mathbf{V}'\}$.

Now $E \subset X = h(Y) \subset E^p$ because $X \subset E^c$ (since E is dense in Y) and $X \subset E^a$. In fact, $\mathbf{v}'(y)$ is an $\mathbf{S}$-sieve for $y \in Y - E$ since $A \in \mathbf{v}'(y)$ implies the existence of $S \in \mathbf{v}'(y) \cap \mathbf{S}$ with $S \subset A$. To see this, consider $y \in V' \in \mathbf{V}'$, $V' \cap E \subset A$; if $V' \in \mathbf{S}'$ we are done. If $V' \in \mathbf{T}'$, $V' = \overline{T}$, $T \in \mathbf{T}$, then $T \subset A$, $S = E - \overline{S}_0 \in \mathbf{S}$ (see (2.2)) for $S_0 = E - T \in \mathbf{S}$. By the weak $\mathbf{V}$-disjunctivity of Y, we have

$$(\overline{S}_0 \cap \overline{T}) - E = \emptyset, \tag{4.13}$$

clearly $\overline{S}_0 = \overline{T}_0$ for $T_0 = \overline{S}_0 \cap E \in T$ (see (2.2)), so

$$Y - \overline{T}_0 = S' \in \mathbf{S}', \tag{4.14}$$

$$S' \cap E = E - \overline{T}_0 = E - \overline{S}_0 = S \subset E - S_0 = T \subset A, \tag{4.15}$$

and, by (4.13) and $y \in \overline{T}$, $y \in Y - \overline{S}_0 = S'$, $S = S' \cap E \in \mathbf{v}'(y)$. □

COROLLARY 4.16. *An extension Y of E is equivalent to an extension X satisfying $E \subset X \subset E^f$ iff it is strongly reduced, $\mathbf{T}$-strict, and weakly $\mathbf{V}$-disjunctive (or $\mathbf{T}$-hypercombinatorial).*

PROOF. If $E \subset X \subset E^f$ then X is strongly reduced by 3.10, and the other properties follow from 4.12. Conversely, if Y is an extension with the above properties, then it is equivalent to some X satisfying $E \subset X \subset E^p$. The existence of a point $y \in X - E^b \subset E^a - E^b$ would contradict by 3.2 the property of X of being strongly reduced. Hence $X \subset E^b \cap E^p \subset E^b \cap E^c = E^f$. □

COROLLARY 4.17. *An extension Y of E is equivalent to E^f iff it is strongly reduced, $\mathbf{T}$-strict, weakly $\mathbf{V}$-disjunctive (or $\mathbf{T}$-hypercombinatorial) and almost compact.*

PROOF. The necessity of these conditions follows from 4.16 and 4.4. Conversely if they are fulfilled, then Y is equivalent, by 4.16, to a space X such that $E \subset X \subset E^f$. By 4.6, $X = E^f$. □

5. Flachsmeyer-type extensions

Extensions with properties similar to E^f have been investigated in [7] in the case when E is a Hausdorff space and $\mathbf{S}$ is a base in E satisfying $\emptyset, E \in \mathbf{S}$, $E - \overline{S} \in \mathbf{S}$ for $S \in \mathbf{S}$ and $S_1 \cap S_2 \in \mathbf{S}$ for $S_1, S_2 \in \mathbf{S}$. In [5] the same conditions are assumed for $\mathbf{S}$, but E can be an arbitrary topological space; the present Theorem 4.17 shows that, under the above hypotheses concerning $\mathbf{S}$, the extension E^f is equivalent to the extension $\big(E', \sigma(\mathbf{S})\big)$ in the notation of [5] (see [5], Theorem (3.7)). Our Theorem 4.17 gives a generalization of [5], (3.7) by omitting the condition of being a $\cap$-semi-lattice for $\mathbf{S}$, and also by replacing the condition of being T_2-reduced by the weaker condition of being strongly reduced.

The following theorems show that further extensions studied in [5] (and in [7] for Hausdorff spaces) admit similar generalizations as well.

LEMMA 5.1. *Let $E \subset X \subset E^c$ and Y be an extension of E that is weakly equivalent to X. Then each point $y \in Y - E$ has a neighbourhood subbase $\mathbf{B}$ such that $\mathbf{B}| E \subset \mathbf{S}$.*

PROOF. Let $h: Y \to X$ be a bijection such that $h(y) = y$ for $y \in E$ and the traces in E of the neighbourhood filters of y and $h(y)$ coincide.

If $y \in Y - E$ and V is an arbitrary neighbourhood of y, then $V \cap E$ is the trace of a neighbourhood of $h(y) \in X$, hence

$$V \cap E \supset \bigcap_1^n v(S_i) \cap E = \bigcap_1^n S_i$$

for suitable sets $S_i \in \mathbf{S}$ such that $h(y) \in v(S_i)$, and the intersection $\bigcap_1^n S_i$ still belongs to the trace of the neighbourhood filter of $h(y)$, i.e. of y. In other words,

$$\bigcap_1^n S_i = W \cap E$$

for a suitable neighbourhood W of y. Now clearly

$$\bigcap_1^n S_i = W \cap V \cap E,$$

each of the sets $S_i \cup (W \cap V)$ is a neighbourhood of y,

$$\bigcap_1^n (S_i \cup (W \cap V)) \subset \left(\bigcap_1^n S_i\right) \cup (W \cap V) \subset V$$

and

$$(S_i \cup (W \cap V)) \cap E = S_i \in \mathbf{S}.$$

If we denote by $\mathbf{B}$ the system of all neighbourhoods of y whose trace in E belongs to $\mathbf{S}$, we obtain the statement. □

LEMMA 5.2. *If Y is an extension of E weakly equivalent to a space X such that $E \subset X \subset E^p$, then the traces $\mathbf{t}(y_1)$ and $\mathbf{t}(y_2)$ in E of the neighbourhood filters of two points $y_1 \neq y_2$ of $Y - E$ are distinct filters and $\mathbf{t}(y)$ is free for $y \in Y - E$.*

PROOF. It suffices to show this for $Y = X$. Now $y_1, y_2 \in E^p - E$, $y_1 \neq y_2$ imply $\mathbf{v}(y_1) \neq \mathbf{v}(y_2)$, hence there are $S_i \in \mathbf{v}(y_i) \cap \mathbf{S}$ such that $S_1 \cap S_2 = \emptyset$, and $y_i \in v(S_i)$ implies $S_i \in \mathbf{t}(y_i)$ so that $\mathbf{t}(y_1) \neq \mathbf{t}(y_2)$. If $x \in E$, $y \in E^p - E$, then $\mathbf{v}(x) \neq \mathbf{v}(y)$ and there are $V \in \mathbf{v}(x)$, $S \in \mathbf{v}(y) \cap \mathbf{S}$ such that $V \cap S = \emptyset$. Now $x \in V$, $S \in \mathbf{t}(y)$ show that $x \notin \cap\, \mathbf{t}(y)$. □

THEOREM 5.3. *An extension Y of E is weakly equivalent to a space X such that $E \subset X \subset E^p$ iff it is weakly $\mathbf{V}$-disjunctive, the points of $Y - E$ have for traces of the neighbourhood filters free filters having subbases composed of elements of $\mathbf{S}$, and distinct points of $Y - E$ have distinct trace filters.*

PROOF. If $E \subset X \subset E^p$ then X is weakly $\mathbf{V}$-disjunctive by 4.8. X is also a $\mathbf{T}$-strict extension which implies that it is a strict extension and therefore the topology of X is coarser than any other topology on X which induces on E the given topology of E and yields for the points $x \in X - E$ the same traces of neighbourhood filters (see e.g. [3], (6.1.10) and (6.1.8)). Therefore X, equipped with a topology of this kind, remains weakly $\mathbf{V}$-disjunctive, and the same holds for an extension weakly equivalent to X. The statement concerning the traces of the neighbourhood filters follows from 5.1 and 5.2.

Suppose now that Y fulfils the conditions in the theorem. Consider first the case when Y is a strict extension of E; then it is $\mathbf{T}$-strict. In fact, let $F \subset$ $\subset Y$ be closed. Then $F = \bigcap_{i \in I} \overline{A_i}$ for suitable sets $A_i \subset E$ (and closures taken in Y). If, for an $i \in I$, $y \in Y - \overline{A_i}$, then there are $S_1, \dots, S_n \in \mathbf{S}$ belonging to the trace of the neighbourhood filter of y such that $A_i \cap \bigcap_1^n S_k = \emptyset$. So $T_k = E - S_k \in \mathbf{T}$ implies $y \notin \overline{T}_k$, $A_i \subset \bigcup_1^n T_k$, and the sets $\overline{T}$ $(t \in \mathbf{T})$ constitute a subbase for the closed sets in Y. Now 4.12 implies that Y is equivalent to a space X such that $E \subset X \subset E^p$ (since the hypotheses imply that Y is a reduced extension).

In the general case we consider the topology on Y obtained as a strict extension with the same trace filters. It is weakly $\mathbf{V}$-disjunctive because this property depends on the trace filters only. Hence the homeomorphism $h: Y \to X$ corresponding to this topology of Y establishes a weak equivalence for the given topology. □

COROLLARY 5.4. *An extension Y of E is weakly equivalent to a space X such that $E \subset X \subset E^f$ iff it is T_2-reduced, weakly $\mathbf{V}$-disjunctive and each point $y \in Y - E$ has a neighbourhood subbase whose trace in E is contained in* $\mathbf{S}$.

PROOF. By 3.6 $E \subset X \subset E^f$ is T_2-reduced, and the same is true, as in the proof of 5.3, for any other topology on X inducing the given topology of E with the same traces of neighbourhood filters. The necessity of the remaining conditions follows from 5.3.

The converse can be deduced from 4.16 by considering a strict extension on Y (because the property of being a T_2-reduced extension depends on the trace filters only). □

In order to characterize the extensions weakly equivalent to E^f itself, we need two lemmas.

LEMMA 5.5. *Let E be an arbitrary topological space. An extension X of E is almost compact iff every filter base composed of open subsets of E has a cluster point in X.*

PROOF. Suppose X is almost compact and let $\mathbf{g}$ be a filter base such that every $G \in \mathbf{g}$ is open in E. Denote by $\mathbf{h}$ the system of all open subsets H of X such that $H \cap E \in \mathbf{g}$. Then $\mathbf{h}$ is an open filter base in X; in fact, $H_i \in \mathbf{h}$, $G_i = H_i \cap E \in \mathbf{g}$ $(i = 1, 2)$ implies the existence of $G_3 \in \mathbf{g}$ such that $G_3 \subset G_1 \cap G_2$, and if H_3 is open in X and $G_3 = H_3 \cap E$, then $G_3 = H_1 \cap H_2 \cap H_3 \cap E$, hence $H_1 \cap H_2 \cap H_3 \in \mathbf{h}$.

Let $x \in X$ be a cluster point of $\mathbf{h}$. Since H is open and E is dense in X, $x \in \overline{H} = \overline{H \cap E}$ for each $H \in \mathbf{h}$, i.e. x is a cluster point of $\mathbf{g}$.

Conversely suppose that each filter base composed of open subsets of E has a cluster point in X. If $\mathbf{h}$ is an arbitrary open filter base in X, let $\mathbf{g}$ denote its trace in E. Then $\mathbf{g}$ is a filter base of open subsets of E, hence by hypothesis it has a cluster point $x \in X$. Since $\mathbf{g}$ is finer than $\mathbf{h}$, x is a cluster point of $\mathbf{h}$ so that X is almost compact. □

LEMMA 5.6. *Let X and Y be two weakly equivalent extensions of an arbitrary topological space E. If X is almost compact then so is Y.*

PROOF. For $A \subset E$, the closure in the extension is determined by the trace filters. Thus 5.5 can be applied. □

THEOREM 5.7. *An extension Y of E is weakly equivalent to E^f iff Y is T_2-reduced, weakly $\mathbf{V}$-disjunctive, almost compact, and each point of $Y - E$ has a neighbourhood subbase whose trace in E is contained in $\mathbf{S}$.*

PROOF. The necessity follows from 5.4 and 5.6. Conversely if Y fulfils these conditions, then it is weakly equivalent by 5.4 to a space X such that $E \subset X \subset E^f$. Since X is almost compact by 5.6, $X = E^f$ by 4.6. □

Theorems 5.4 and 5.7 yield generalizations of [5], Lemma (4.2) and Theorem (4.3), respectively.

6. Weak equivalence and ϑ-equivalence

Lemma 5.6 says that almost compactness is invariant with respect to weak equivalence of extensions. This fact can be formulated in a more general manner with the help of the concept of ϑ-equivalent superspaces.

Let us recall that a map $h: X \to Y$ of topological spaces is said to be *ϑ-continuous* if, for $x \in X$ and an open neighbourhood W of $h(x)$, there is an open neighbourhood V of x such that $h\left(\overline{V}\right) \subset \overline{W}$. If X and Y are superspaces of the space E, they are said to be *ϑ-equivalent* whenever there exists a bijection $h: X \to Y$ such that $h|\,E = \mathrm{id}_E$ and both h and h^{-1} are ϑ-continuous. Taking into account the fact that a ϑ-continuous image of an almost compact space is almost compact, it is clear that almost compactness is invariant with respect to ϑ-equivalence of superspaces.

We show that 5.6 can be deduced from this remark. In fact:

LEMMA 6.1. *If X_1 and X_2 are extensions of a space E, $h: X_1 \to X_2$ satisfies $h|\,E = \mathrm{id}_E$ and, for $x \in X_1$, the trace in E of the neighbourhood filter of $h(x)$ is coarser than the trace of the neighbourhood filter of x, then h is ϑ-continuous.*

PROOF. Let us denote by $\mathbf{t}_i(x)$ the trace of the neighbourhood filter of $x \in X_i$ and by cl_i the closure with respect to X_i $(i = 1,2)$. Choose $x_0 \in X_1$ and let W be an open neighbourhood of $h(x_0)$ in X_2. For $G = W \cap E$, consider the open subset

$$V = \left\{x \in X_1 \colon G \in \mathbf{t}_1(x)\right\}$$

of X_1. Clearly $x_0 \in V$, since W is a neighbourhood of $h(x_0)$ so that G belongs to $\mathbf{t}_2\big(h(x_0)\big) \subset \mathbf{t}_1(x_0)$.

Now $x \in \mathrm{cl}_1 V$ implies that $x \in \mathrm{cl}_1(G)$ (because E is dense in X_1 and $G = V \cap E$), hence every element of $\mathbf{t}_1(x)$ meets G, and the same is, *a fortiori*, true for the elements of $\mathbf{t}_2\big(h(x)\big) \subset \mathbf{t}_1(x)$. Thus $h(x) \in \mathrm{cl}_2 G \subset \mathrm{cl}_2 W$. □

COROLLARY 6.2. *Two weakly equivalent extensions of a topological space are always ϑ-equivalent.* □

Let us notice that the converse is not true. For $E = \mathbf{R}$ with the usual topology, let $p \notin \mathbf{R}$ and X_i denote the (unique) extension of E on $\mathbf{R} \cup \{p\}$ corresponding to the trace filter $\mathbf{t}_i(p)$ $(i = 1,2)$, where $\mathbf{t}_1(p)$ is generated by the sets $(c, +\infty)$ $(c \in \mathbf{R})$, and $\mathbf{t}_2(p)$ by those $(c, +\infty) - \mathbf{N}$ $(c \in \mathbf{R})$. Now $h = \mathrm{id}_{X_1}$ is the only bijection $h\colon X_1 \to X_2$ satisfying $h|\,E = \mathrm{id}_E$, and $\mathbf{t}_1(p) \neq \mathbf{t}_2(p)$ shows that X_1 and X_2 fail to be weakly equivalent extensions.
However, the above h and its inverse h^{-1} are both ϑ-continuous by

$$\mathrm{cl}_1(a,b) = \mathrm{cl}_2(a,b) = [a,b] \quad (a < b),$$

$$\mathrm{cl}_1(c,+\infty) = \mathrm{cl}_2\big((c,+\infty) - \mathbf{N}\big) = [c,+\infty) \cup \{p\} \quad (c \in \mathbf{R}).$$

We can deduce from 6.1 (and the following Lemma 6.3) Theorem 6.4 that is a slight strengthening of [6], Theorem 4.4.

LEMMA 6.3. *If X is a T_2-reduced extension of E, $i\colon X \to X$ is ϑ-continuous, and $i|\,E = \mathrm{id}_E$, then $i = \mathrm{id}_X$.*

PROOF. Assume $x \in X - E$, $i(x) = y \neq x$. Then there are open neighbourhoods G and H of x and y, respectively, such that $G \cap H = \emptyset$. We can suppose $i\left(\overline{G}\right) \subset \overline{H}$, and then the existence of $z \in G \cap E$ leads to the contradiction $i(z) = z \in \overline{H}$. □

THEOREM 6.4. *Let Y be an almost compact extension of E that contains a subbase $\mathbf{S}'$ such that $\mathbf{S}'|\,E \subset \mathbf{S}$. Then*

(a) *for any such Y, there is a ϑ-continuous map $h\colon E^f \to Y$ with $h|\,E = \mathrm{id}_E$,*

(b) *the map h in* (a) *can be subject to the condition $h(E^f - E) \subset Y - E$,*

(c) *if Y is T_2-reduced then h in* (a) *is necessarily surjective,*

(d) *if E' is a T_2-reduced almost compact extension of E, containing a subbase $\mathbf{S}'$ such that $\mathbf{S}'|\,E \subset \mathbf{S}$, and for any almost compact extension $Y \supset E$ with subbase having a trace in E contained in $\mathbf{S}$, there is a ϑ-continuous map $h\colon E' \to Y$ such that $h|\,E = \mathrm{id}_E$, then E^f and E' are ϑ-equivalent.*

PROOF. (a) and (b): For $x \in E^f - E$, consider the filter base $\mathbf{g}$ composed of the sets $\bigcap\limits_1^n S_i$ such that $S_i \in \mathbf{v}(x) \cap \mathbf{S}$. Then, by 5.5, $\mathbf{g}$ has a cluster point $y \in Y$; clearly $y \notin E$. Select a y of this kind and define $y = h(x)$.
Then h satisfies the hypotheses of 6.1 provided $h(x) = x$ for $x \in E$. In fact, if $S' \in \mathbf{S}'$ is a neighbourhood of $h(x)$, then $S' \cap E = S \in \mathbf{S}$ meets every

element of $\mathbf{v}(x)$ so that $S \in \mathbf{v}(x)$. Since the finite intersections of the sets S of this kind generate the trace in E of the neighbourhood filter of $h(x)$, this trace is contained in the trace of the neighbourhood filter of x. By 6.1, h is ϑ-continuous.

(c): E^f being almost compact, the same holds for $h(E^f) \supset E$, and Y is a T_2-reduced extension of $h(E^f)$. Thus $Y = h(E^f)$ by [5], Theorem (1.5).

(d): Suppose E' satisfies the hypotheses. Then there are ϑ-continuous maps $h: E^f \to E'$ and $k: E' \to E^f$ such that $h| E = k| E = \mathrm{id}_E$. Consider the ϑ-continuous map $i = k \circ h: E^f \to E^f$; by 6.3 $i = \mathrm{id}_{E^f}$. Therefore h is injective and, by (c), it is surjective, too. Hence $k = h^{-1}$. □

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ON PSEUDOMANIFOLDS WITH BOUNDARY. III

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The first part of this paper [8] concerned nonorientable pseudomanifolds. The second one [9] discussed orientable pseudomanifolds without homologically singular interior points. This third part deals with locally orientable pseudomanifolds with boundary and without homologically singular interior points.

Without going in details the main result of this part can be sketched as follows:

Let (X, A) be a locally orientable n-pseudomanifold with boundary and without homologically singular interior points lying in R^{n+1}. Let K be a continuous closed path in $X \setminus A$. Under the circumstances K preserves its banks if and only if it preserves the orientation.

We shall use the definitions and notations of [8] and [9] without any comment.

First we deal with figures called (n,p)-manifolds where p is a prime.

Throughout this paper let p be a prime and n a positive integer. Let Z_p be the cyclic group of integers mod p and H the Čech homology theory defined on the category of compact pairs over the coefficient group Z_p. Let $\mathbf{N}$ be the set of positive integers and $\mathbf{I} = \{x; 0 \leqq x \leqq 1\}$ the unit interval with the usual topology.

1. (n,p)-manifolds

1.1. Definition. The compact pair (X, A) (i.e., X is a compact T_2-space and A is a closed subspace of X) is called an *(n,p)-manifold* if the following conditions are satisfied:

(a) $x \setminus A$ is a nonvoid connected space with countable base.

(b) There is a base σ of $X \setminus A$ such that for each $U \in \sigma$, $(X, X \setminus U)$ is an (n,p)-cell (cf. [8] 1.2).

(c) For each open subset U of $X \setminus A$ and for every $q > n$, $H_q(X, X \setminus U) =$ $= 0$ holds.

1.2. Observe that for each (n,p)-manifold (X, A) the subspace $X \setminus A$ of X is locally connected.

Namely by 1.1(b) and [8] 1.2(a) the base σ of $X \setminus A$ consists of connected sets.

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1.3. REMARK. Let (X, A) be an (n,p)-cell without c-singularity (cf. [9] 1.1 and [9] 1.6) satisfying the condition 1.1(c). Then by [9] 1.3 (X, A) is clearly an (n,p)-manifold.

1.4. To prepare the next section we make a preliminary remark.

Let X be a compact T_2-space and for $k \in \mathbf{N}$ let U_k be an open subset of X. Let $U = \bigcup_{k=1}^{\infty} U_k$ and for $k \in \mathbf{N}$ let $U^k = U_1 \cup \ldots \cup U_k$. Let q be an arbitrary integer. For $k \in \mathbf{N}$ let $j_{k*}: H_q(X, X \setminus U) \to H_q(X, X \setminus U^k)$ be the homomorphism induced by the inclusion $j_k: (X, X \setminus U) \subset (X, X \setminus U^k)$. Let b be an element of $H_q(X, X \setminus U)$ for which

$$j_{k*}(b) = 0 \quad \text{for all} \quad k \in \mathbf{N}. \tag{1}$$

Then $b = 0$.

Indeed $\mathcal{X} = \{(X, X \setminus U^k);\ k = 1, 2, \ldots\}$ is a nested system with the intersection $(X, X \setminus U)$ (see [10] 5.2). By (1) b determines the 0-thread of $\mathcal{X}$ (see [10] 5.4) and thus by the continuity of H (see [10] 5.5) we have $b = 0$ (see [10] 5.5) as required.

2. Quasiregular domains

Let (X, A) be an (n,p)-manifold.

Considering the domains in $X \setminus A$ we shall find that for such a domain U we have either $H_n(X, X \setminus U) = 0$ or $H_n(X, X \setminus U) \approx Z_p$ and in the latter case $(X, X \setminus U)$ is an (n,p)-cell.

First we introduce the notation of quasiregular domains.

2.1. DEFINITION. Let U be a domain in $X \setminus A$. We say that U is a *quasiregular domain* of the (n,p)-manifold (X, A) if it satisfies the following two conditions.

(a) For the inclusion $i: (X, \emptyset) \subset (X, X \setminus U)$ the induced homomorphism $i_*: H_n(X) \to H_n(X, X \setminus U)$ is trivial, i.e. $i_*(H_n(X)) = 0$.

(b) For each nonempty open subset U' of U and for the inclusion $j: (X, X \setminus U) \subset (X, X \setminus U')$ the induced homomorphism $j_*: H_n(X, X \setminus U) \to H_n(X, X \setminus U')$ is a monomorphism.

We are going to prove that each domain in $X \setminus A$ is a quasiregular domain of (X, A).

2.2. REMARK. Let U_1 and U_2 be open subsets of $X \setminus A$. Consider the segment

$$\longleftarrow H_{n+1}(X, X \setminus (U_1 \cap U_2))$$

$$H_n(X, X \setminus U_1) \oplus H_n(X, X \setminus U_2) \xleftarrow{\psi} H_n(X, X \setminus (U_1 \cup U_2)) \longleftarrow$$

of the relative Mayer–Vietoris sequence of the triad $(X; X \setminus U_1, X \setminus U_2)$, where ψ is defined by the formula

$$\psi(b) = \big(h_{1*}(b), -h_{2*}(b)\big) \tag{2}$$

and for $m = 1,2$ $h_{m*}: H_n\big(X, X \setminus (U_1 \cup U_2)\big) \to H_n(X, X \setminus U_m)$ is the homomorphism induced by the inclusion $h_m: \big(X, X \setminus (U_1 \cup U_2)\big) \subset (X, X \setminus U_m)$ (cf. [11] p. 42). According to 1.1(c) we have $H_{n+1}\big(X, X \setminus (U_1 \cap U_2)\big) = 0$ and thus by the exactness of the sequence in question, ψ is a monomorphism.

2.3. Let U_1 and U_2 be quasiregular domains of (X, A) intersecting each other. Then $U_1 \cup U_2$ is a quasiregular domain as well.

Indeed for $m = 1,2$ let h_m and h_{m*} be the same as in 2.2. We first show that h_{1*} and h_{2*} are monomorphisms.

To this end consider the diagram

$$\begin{tikzcd}
H_n(X, X \setminus U_1) \arrow[d, "j_{1*}"] & H_n\big(X, X \setminus (U_1 \cup U_2)\big) \arrow[l, "h_{1*}"'] \arrow[d, "h_{2*}"] \\
H_n\big(X, X \setminus (U_1 \cap U_2)\big) & H_n(X, X \setminus U_2), \arrow[l, "j_{2*}"']
\end{tikzcd}$$

where the homomorphisms j_{1*} and j_{2*} are induced by inclusions j_1 and j_2 respectively. This diagram is commutative. Since U_1 and U_2 are quasiregular domains it follows by 2.1(b) that j_{1*} and j_{2*} are monomorphisms. Let $b \in H_n\big(X, X \setminus (U_1 \cup U_2)\big)$ and suppose that $h_{1*}(b) = 0$. Then $j_{2*}h_{2*}(b) = j_{1*}h_{1*}(b) = 0$. Since j_{2*} is a monomorphism we get $h_{2*}(b) = 0$. Hence

$$\psi(b) = \big(h_{1*}(b), -h_{2*}(b)\big) = (0,0)$$

(cf. 2.2(2)). However by 2.2 ψ is a monomorphism and thus we obtain $b = 0$. h_{1*} is a monomorphism indeed. Likewise h_{2*} is a monomorphism as well.

Next we show that $U_1 \cup U_2$ satisfies condition 2.1(a).

Let

$$i_*: H_n(X) \to H_n\big(X, X \setminus (U_1 \cup U_2)\big) \text{ and } i_{1*}: H_n(X) \to H_n(X, X \setminus U_1)$$

be homomorphisms induced by the inclusions i and i_1 respectively. We then clearly have $i_{1*} = h_{1*}i_*$. Since U_1 is a quasiregular domain according to 2.1(a) for each $c \in H_n(X)$ we get $0 = i_{1*}(c) = h_{1*}i_*(c)$ and thus since h_{1*} is a monomorphism it follows $i_*(c) = 0$. i_* is a trivial homomorphism as required.

Finally we prove that $U_1 \cup U_2$ satisfies condition 2.1(b).

Indeed, let U' be a nonempty open subset of $U_1 \cup U_2$. Then U' intersects either U_1 or U_2. Without loss of generality we can suppose $U_1 \cap U' \neq \emptyset$. Consider the diagram

$$\begin{array}{ccc} H_n(X, X \setminus U_1) & \xleftarrow{h_{1*}} & H_n(X, X \setminus (U_1 \cup U_2)) \\ \Big\downarrow j''_* & & \Big\downarrow j_* \\ H_n(X, X \setminus (U_1 \cap U')) & \xleftarrow{r_*} & H_n(X, X \setminus U') \end{array}$$

where the homomorphisms j_*, j''_*, r_* are induced by the inclusions j, j'' and r respectively. This diagram is clearly commutative. However U_1 is a quasiregular domain and thus by 2.1(b) j''_* is a monomorphism. As we have seen above h_{1*} is a monomorphism as well. Consequently j_* is a monomorphism, too.

$U_1 \cup U_2$ satisfies condition 2.1(b) as required. $U_1 \cup U_2$ is a quasiregular domain indeed.

2.4. Theorem. *Every domain in $X \setminus A$ is quasiregular.*

Proof. First observe that if for a domain U in $X \setminus A$ the compact pair $(X, X \setminus U)$ is an (n,p)-cell then U is a quasiregular domain of (X, A) (see 2.1 and [8] 1.2). Thus according to 1.1(b) there is a base σ of $x \setminus A$ consisting of quasiregular domains of (X, A).

Now let U be an arbitrary domain in $X \setminus A$. Then U is the union of some domains of the base σ or even more, since $X \setminus A$ has a countable base it follows that U is the union of a countable subset of σ. However U is connected and so we can put this countable subset of σ in a sequence $U_1, \dots, U_m, \dots$ so that for $k \geqq 2$ U_k meets the set $\bigcup_{m=1}^{k-1} U_m$ and obviously $U = \bigcup_{k=1}^{\infty} U_k$. Denoting for $k \in \mathbf{N}$ $\bigcup_{m=1}^{k} U_m$ by U^k according to 2.3 we can state that each U^k is a quasiregular domain of (X, A).

Now for $k, s, t \in \mathbf{N}$ with $t \geqq s$ let $j^t_{s*}\colon H_n(X, X \setminus U^t) \longrightarrow H_n(X, X \setminus U^s)$ and $j_{k*}\colon H_n(X, X \setminus U) \longrightarrow H_n(X, X \setminus U^k)$ be the homomorphisms induced by the inclusions j^t_s and j_k respectively. Observe that

$$j_{s*} = j^t_{s*} j_{t*}. \tag{3}$$

Moreover since U^t is quasiregular and $\emptyset \neq U^s \subset U^t$ by 2.1(b) it follows that j^t_{s*} is a monomorphism.

We now show that j_{k*} is a monomorphism.

Indeed choose $b \in H_n(X, X \setminus U)$ so that $j_{k*}(b) = 0$. Then for $r \leqq k$ by (3) we have $j_{r*}(b) = j^k_{r*} j_{k*}(b) = 0$. On the other hand for $m \geqq k$ $j^m_{k*} j_{m*}(b) =$

$= j_{k*}(b) = 0$ and since j^m_{k*} is a monomorphism it follows that $j_{m*}(b) = 0$. Thus according to 1.4 we get $b = 0$. j_{k*} is a monomorphism as required.

Next we show that U satisfies condition 2.1(a).

Let $i_*: H_n(X) \to H_n(X, X \setminus U)$ and $i^1{}_*: H_n(X) \to H_n(X, X \setminus U^1)$ be the homomorphisms induced by the inclusions $i: (X, \emptyset) \subset (X, X \setminus U)$ and $i^1: (X, \emptyset) \subset (X, X \setminus U^1)$ respectively. We then clearly have

$$i^1{}_* = j_{1*} i_*. \tag{4}$$

Let $c \in H_n(X)$. Since $U^1 = U_1$ is a quasiregular domain of (X, A) by 2.1(a) and (4) we get $0 = i^1{}_*(c) = j_{1*} i_*(c)$ and since j_{1*} is a monomorphism it follows $i_*(c) = 0$. i_* is a trivial homomorphism as required.

Finally we prove that U satisfies 2.1(b).

Let U' be a nonempty open subset of U. Then there is a U^k meeting U'. Consider the diagram

$$\begin{tikzcd}
H_n(X, X \setminus U) \arrow[r, "j_{k*}"] \arrow[d, "j_*"] & H_n(X, X \setminus (U^k)) \arrow[d, "j'_{k*}"] \\
H_n(X, X \setminus (U) \arrow[r, "j'_*"] & H_n(X, X \setminus U^k \cap U'))
\end{tikzcd}$$

where the homomorphisms j_*, j'_*, j'_{k*} are induced by inclusions as well. This diagram is clearly commutative. U^k is a quasiregular domain, consequently j'_{k*} is a monomorphism. As we have seen above j_{k*} is a monomorphism as well. Hence $j'_* j_* = j'_{k*} j_{k*}$ is a monomorphism and so j_* is a monomorphism as required.

U satisfies condition 2.1(b) as well.

The proof of the Theorem is complete.

2.5. Corollary. *Let U be an arbitrary domain in $X \setminus A$. Since σ is a base of $X \setminus A$* (see 1.1(b)) *there is a member U' of σ contained in U. $(X, X \setminus U')$ is an (n,p)-cell and thus*

$$H_n(X, X \setminus U') \approx Z_p \tag{5}$$

(see [8] 1.2(b)). *However U is a quasiregular domain and thus for the inclusion $j: (X, X \setminus U) \subset (X, X \setminus U')$ the induced*

$$j_*: H_n(X, X \setminus U) \to (H_n X, X \setminus U')$$

is a monomorphism, consequently $H_n(X, X \setminus U)$ is isomorphic to a subgroup of Z_p. Hence we have either $H_n(X, X \setminus U) = 0$ or $H_n(X, X \setminus U) \approx Z_p$.

3. s-regular domains

Let (X, A) be an (n, p)-manifold.

3.1. Definition. A domain U of $X \setminus A$ is said to be an *s-regular domain* of (X, A) if $H_n(X, X \setminus U) \neq 0$.

3.2. Remark. 2.5 shows that for each s-regular domain U we have $H_n(X, X \setminus U) \approx Z_p$.

3.3. Proposition. *Let U be an s-regular domain of (X, A). Then $(X, X \setminus U)$ is an (n, p)-cell without c-singularity* (see [9] 1.6).

Proof. According to 3.2, 2.4, 1.1(a), 1.2 and [8] 1.2 $(X, X \setminus U)$ is an (n, p)-cell. Consider the base σ in 1.2(b) and let $\sigma' = \{U' \in \sigma ; U' \subset U\}$. Then σ' is a base of $U = X \setminus (X \setminus U)$ and for each $U' \in \sigma'$, $(X, X \setminus U')$ is an (n, p)-cell and thus U' is a c-regular domain of $(X, X \setminus U)$ (see [9] 1.3). Consequently $(X, X \setminus U)$ is without c-singularity (see [9] 1.7) as required.

3.4. Remark. Let U be an s-regular domain of (X, A). Let U' be a domain in U. Then U' is s-regular as well. Moreover U' is a c-regular domain in the (n, p)-cell $(X, X \setminus U)$.

Indeed for the inclusion $j: (X, X \setminus U) \subset (X, X \setminus U')$ the induced $j_*: H_n(X, X \setminus U) \to H_n(X, X \setminus U')$ is a monomorphism (see 2.1(b) and 2.4). However U is s-regular and thus $H_n(X, X \setminus U) \neq 0$. Consequently $H_n(X, X \setminus U') \neq 0$. U' is s-regular as well. Hence by 3.2 we have $H_n(X, X \setminus U') \approx Z_p$ and thus taking also 3.3 into account U' is a c-regular domain of the (n, p)-cell $(X, X \setminus U)$.

3.5. Let τ be the set of the s-regular domains of (X, A). According to 1.1(b), 3.1 and [8] 1.2(b) we have $\sigma \subset \tau$ and thus τ is a base of $X \setminus A$.

For each $U \in \tau$ let $H(U)$ denote the group $H_n(X, X \setminus U)$.

Now let $U, U' \in \tau$ and suppose that $U' \subset U$. Then the homomorphism $j_*: H(U) \to H(U')$ induced by the inclusion $j: (X, X \setminus U) \subset (X, X \setminus U')$ is an isomorphism.

Indeed by 2.1(b) and 2.4, j_* is a monomorphism. However by 3.2 $H(U) \approx \approx H(U') \approx Z_p$. Thus j_* is an isomorphism indeed.

We shall denote this isomorphism by $(U', U)_*$.

Observe that in case $U' = U$

$$(U', U)_* = \mathrm{id}_{H(U)}, \tag{6}$$

i.e., $(U', U)_*$ is the identity isomorphism of $H_n(X, X \setminus U)$, and if $U'' \subset U' \subset \subset U$ $(U, U', U'' \in \tau)$ then

$$(U'', U)_* = (U'', U')_*(U', U)_*. \tag{7}$$

3.6. For $U, U' \in \tau$ we say that U and U' are *compatible* (with each other) if either $U' \subset U$ or $U \subset U'$.

Now let U and U' be compatible members of τ. We define the isomorphism

$$[U', U]_*: H(U) \to H(U')$$

by letting

$$(8) \qquad [U', U]_* = \begin{cases} (U', U)_* & \text{if } U' \subset U, \\ ((U, U')_*)^{-1} & \text{if } U \subset U' \end{cases}.$$

According to 3.5(6) this isomorphism is well defined and we have

$$(9) \qquad [U, U]_* = \mathrm{id}_{H(U)}$$

for each $U \in \tau$. Moreover

$$(10) \qquad [U, U']_* = [U', U]_*^{-1}$$

holds for any two compatible members U, U' of τ. Observe also that in case $U'' \subset U' \subset U$ $(U, U', U'' \in \tau)$ we clearly have

$$(11) \qquad [U'', U]_* = [U'', U']_* [U', U]_*.$$

3.7. A *τ-chain* is defined as a sequence $C = (U_m, \ldots, U_1)$ of members of τ such that for $j = 2, \ldots, m$, U_j and U_{j-1} are compatible. The τ-chain is closed if $U_m = U_1$.

For any τ-chain $C = (U_m, \ldots, U_1)$ we define the isomorphism

$$C^*: H(U_1) \to H(U_m)$$

by letting

$$C^* = \mathrm{id}_{H(U_1)} \quad \text{if} \quad m = 1$$

and

$$C^* = [U_m, U_{m-1}]_* \cdots [U_2, U_1]_* \quad \text{if} \quad m > 1.$$

Observe that if C is a closed chain then C^* is an automorphism of $H(U_1)$. This automorphism is either the identity mapping of $H(U_1)$ or it differs from the identity. In the first case we say that C is an *orientation preserving closed chain* and in the second that C is an *orientation changing closed chain.*

Observe that in case $p = 2$ each closed τ-chain is clearly an orientation preserving chain.

3.8. Observe that if $U'' \subset U' \subset U$ $(U, U', U'' \in \tau)$ then by 3.6(10) and 3.6(11), $C = (U'', U, U', U'')$ is an orientation preserving closed τ-chain.

4. Orientation preserving closed paths

Let (X, A) be an (n,p)-manifold.

We now turn to the problem of defining the fact that a continuous closed path in $X \setminus A$ preserves the orientation. To this end we recall the notion and some properties of the i-category.

4.1. An *i-category* is a category $\mathcal{C}$ together with a contravariant functor $\mathcal{F}: \mathcal{C} \to \mathcal{C}$ such that $\mathcal{F} \cdot \mathcal{F} = \mathrm{id}_{\mathcal{C}}$ and $\mathcal{F}(B) = B$ for each object B of $\mathcal{C}$ (cf. [7]).

Now we shall use the symbols $\mathcal{C}^\bullet$, $\mathcal{D}^\bullet$ etc. for i-categories. If $\mathcal{C}^\bullet = (\mathcal{C}, \mathcal{F})$ is an i-category then the class of objects (morphisms) of $\mathcal{C}^\bullet$ will be denoted by $\mathrm{Ob}\,\mathcal{C}^\bullet$ ($\mathrm{Mor}\,\mathcal{C}^\bullet$). For any $\alpha \in \mathrm{Mor}\,\mathcal{C}^\bullet$ the symbol $\mathcal{F}(\alpha)$ will be sometimes replaced by $\alpha^\bullet$. $\alpha^\bullet$ is the *involutoric conjugate* of α.

A morphism $\alpha: A \to A'$ of an i-category $\mathcal{C}^\bullet$ is said to be *closed* if its domain is the same as its range (i.e. $A = A'$).

We now turn to the definition of the invariant subcategory.

An *invariant subcategory* of an i-category $\mathcal{C}^\bullet$ is a subcategory $\mathcal{Q}^\bullet$ of $\mathcal{C}^\bullet$ consisting of all objects and of some morphisms of $\mathcal{C}^\bullet$ such that

(a) for any $\alpha \in \mathrm{Mor}\,\mathcal{C}^\bullet$ we have $\alpha\alpha^\bullet \in \mathcal{Q}^\bullet$,

(b) if for any $\alpha_1, \alpha_2 \in \mathrm{Mor}\,\mathcal{C}^\bullet$ the morphism $\alpha_1\alpha_2$ is defined and it belongs to $\mathcal{Q}^\bullet$ then $\alpha_2\alpha_1$ is also defined and $\alpha_2\alpha_1 \in \mathcal{Q}^\bullet$,

(c) if α_1, α_2, α_3 are morphisms of $\mathcal{C}^\bullet$ such that $\alpha_1\alpha_2 = \alpha_3$ and $\alpha_2 \in \mathcal{Q}^\bullet$, $\alpha_3 \in \mathcal{Q}^\bullet$ then $\alpha_1 \in \mathcal{Q}^\bullet$.

If a morphism belongs to an invariant subcategory then it is clearly a closed morphism.

Suppose now that the i-category $\mathcal{C}^\bullet$ is small, i.e., the class of objects and morphisms of $\mathcal{C}^\bullet$ is a set. Let $\mathcal{Y}$ be an arbitrary subset of closed morphisms of $\mathcal{C}^\bullet$. Then there is a least invariant subcategory $\mathcal{Q}^\bullet(\mathcal{Y})$ of $\mathcal{C}^\bullet$ containing $\mathcal{Y}$. That means $\mathcal{Q}^\bullet(\mathcal{Y})$ is contained in each invariant subcategory $\mathcal{Q}^\bullet$ of $\mathcal{C}^\bullet$ containing $\mathcal{Y}$. $\mathcal{Q}^\bullet(\mathcal{Y})$ is called the *invariant hull* of $\mathcal{Y}$.

4.2. Let M be a partially ordered set, i.e., M is equipped with a binary relation $\leqq$ such that

(i) $a \leqq a$ for each $a \in M$,

(ii) $a \leqq b$ and $b \leqq a$ imply $a = b$,

(iii) $a \leqq b$ and $b \leqq c$ imply $a \leqq c$.

A sequence $\alpha = (x_k, \dots, x_1): x_1 \to x_k$ in M is called an *M-chain* if for $i = 2, \dots, k$ x_i and x_{i-1} are compatible elements of M, i.e., either $x_i \leqq x_{i-1}$ or $x_{i-1} \leqq x_i$. For the M-chains $\alpha = (x_k, \dots, x_1)$ and $\beta = (x_m, \dots, x_k)$ let $\beta\alpha = (x_m, \dots, x_{k+1}, x_k, \dots, x_1)$ and $\alpha^\bullet = (x_1, \dots, x_k)$. Thus $(M, \leqq)$ becomes an i-category $M^\bullet$ where $\mathrm{Ob}\,M^\bullet = M$ and $\mathrm{Mor}\,M^\bullet$ is the set of the M-chains.

$M^\bullet$ is a small i-category.

Let $\mathcal{Y}$ be the set of all M-chains of the type (b, d, c, b) where $b \leqq c \leqq d$. Then the invariant hull of $\mathcal{Y}$ is called the medial invariant subcategory of $M^\bullet$ (cf. [4] 3.13). We shall denote it by $M^\bullet_{\mathrm{med}}$.

4.3. Let R' be a T_2-space and R a subspace of R'. A system ω of open sets of R' is said to be an *external base* of R in R' if for each $q \in R$ the subsystem

$$\omega^q = \{U \in \omega : q \in U\}$$

of ω constitutes a base of neighbourhoods of the point q in the space R'.

Setting $U \leqq U'$ if $U \subset U'$ $(U, U' \in \omega)$ the external base ω can be considered as a partially ordered set. Hence the invariant subcategory $\omega^\bullet_{\text{med}}$ of the small i-category $\omega^\bullet$ is well defined.

4.4. Let $R' = R = X \setminus A$ and $\omega = \tau$, where τ is the same as in 3.5, i.e. it is the set of the s-regular domains of (X, A) (cf. 3.1). τ is obviously an external base of $X \setminus A$ in $X \setminus A$ (cf. 3.5). Moreover the members of τ together with the orientation preserving closed τ-chains (see 3.7 and 4.2) form clearly an invariant subcategory $\mathcal{Q}^\bullet$ of $\tau^\bullet$ and by 3.8 $\mathcal{Q}^\bullet$ contains the invariant subcategory $\tau^\bullet_{\text{med}}$ of $\tau^\bullet$.

4.5. Let R' and R be the same as in 4.3. Let ω be an external base of R in R' and let $K: q \to z$ be a continuous path of R (see [8] 2.4). As in [9] 2.8 we can define the ω-chains associated to K.

Indeed let $\alpha = (U_1, \ldots, U_{m+1})$ be an ω-chain. α is said to be *associated to* K if there exists a subdivision of K into factors $K = K_1, \ldots, K_m$, where $K_i = K_i: q_{i+1} \to q_i$ for $i = 1, \ldots, m$, $q_1 = z$, $q_{m+1} = q$ such that $q_i \in U_i$ and $\widetilde{K}_i \subset (U_i \cup U_{i+1})$ for $i = 1, \ldots, m$ and $q_{m+1} \in U_{m+1}$ (cf. [8] 2.4 and [8] 1.6).

Observe that for each continuous path $K: q \to z$ in R and for each $U, U' \in \in \omega$ with $q \in U$ and $z \in U'$ there is an ω-chain $\alpha = (U_1, \ldots, U_{m+1})$ associated to K such that $U_1 = U'$ and $U_{m+1} = U'$ (see [5] 6.8).

Consequently to each closed path K of R there is a closed ω-chain $\alpha = = (U_1, \ldots, U_{m+1} = U_1)$ associated to K.

Observe that if $\mathcal{Q}^\bullet$ is an invariant subcategory of $\omega^\bullet$ containing $\omega^\bullet_{\text{med}}$ then for each closed path K of R and for any closed ω-chains α and α' associated to K either both of the chains α and α' belong to $\mathcal{Q}^\bullet$ or neither of them belongs to $\mathcal{Q}^\bullet$ (see [5] 7.1).

4.6. DEFINITION. We say that a continuous closed path K in $X \setminus A$ *preserves the orientation* if each closed τ-chain associated to K is an orientation preserving chain.

According to 4.4 and 4.5 K preserves the orientation if and only if there exists an orientation preserving closed τ-chain associated to K.

We say that the closed continuous path K *changes the orientation* if K fails to preserve it, i.e., if each closed τ-chain associated to K is an orientation changing closed chain.

5. (n,p)-manifolds in R^{n+1}

We first show that the definition of the (n,p)-manifold may be simplified if it lies in R^{n+1}.

5.1. Let (X, A) be a compact pair in the Euclidean $(n+1)$-space R^{n+1}, i.e., the compact T_2-space X is a subspace of R^{n+1} and A is a closed subspace of X. Under the circumstances (X, A) is an (n,p)-manifold if the following conditions are satisfied:

(a) $X \setminus A$ is a nonvoid connected space.

(b) There is a base σ of $X \setminus A$ such that for each $U \in \sigma$ $(X, X \setminus U)$ is an (n,p)-cell.

Indeed, since R^{n+1} possesses a countable base so does $X \setminus A$. Thus condition 1.1(a) is satisfied. Condition 1.1(b) is the same as (b). We only need to show that condition 1.1(c) is also satisfied.

First observe that by [8] 2.16 for each $U \in \sigma$, $U = X \setminus (X \setminus U)$ is nowhere dense in R^{n+1}. Consequently by (b) $X \setminus A$ itself and so each open subset of $X \setminus A$ is nowhere dense in R^{n+1}. Thus according to [10] 6.23 and [10] 6.24 for each open subset U of $X \setminus A$ and for every $q > n$ we have $H_q(X, X \setminus U) = 0$ and so 1.1(c) is satisfied for the pair (X, A). (X, A) is an (n,p)-manifold as required.

5.2. COROLLARY. *Let* (X, A) *be an* (n,p)*-cell in* R^{n+1} *without c-singularity* (see [9] 1.6). *Then by* 5.1, [8] 1.2(a), [9] 1.3 *and* [9] 1.7 (X, A) *is an* (n,p)*-manifold in* R^{n+1}.

5.3. THEOREM. *Let* (X, A) *be an* (n,p)*-manifold in* R^{n+1}. *Then* (X, A) *is a k-manifold in* R^{n+1} (cf. [9] 2.6).

PROOF. By 5.1(a) condition [9] 2.6(a) is clearly satisfied. Moreover by 3.3 and [9] 2.7 for each $U \in \sigma$, $(X, X \setminus U)$ is a k-manifold in R^{n+1}. However for $U \in \sigma$ each $\mathrm{mod}(X, X \setminus U)$ k-regular domain of R^{n+1} is clearly k-regular $\mathrm{mod}(X, A)$. Thus for the compact pair (X, A) condition [9] 2.6(b) is satisfied as well. (X, A) is a k-manifold in R^{n+1} indeed.

In the remainder of this chapter let (X, A) be an (n,p)-manifold in R^{n+1}.

5.4. Let Ω be the set of the $\mathrm{mod}(X, A)$ k-regular domains in R^{n+1}. According to 5.3, Ω is an external base of $X \setminus A$ in R^{n+1}.

As in [9] 2.11 we can define the chains of banks associated to an Ω-chain (see also 4.2 and 4.3).

Indeed let $V \in \Omega$. Then by the *banks* of V we mean the components of $V \setminus X$ and we denote them by $P^1(V)$ and $P^2(V)$. Let $\alpha = (V_1, \dots, V_m)$ be an Ω-chain and let $P^1(V_1)$ and $P^2(V_1)$ be the banks of V_1. Then there is a numeration P_i^1 and P_i^2 of the banks of V_i such that

(a) $P_1^1 = P^1(V_1)$ and $P_1^2 = P^2(V_1)$,

(b) $P_i^1 \cap P_{i+1}^1 \neq \emptyset$ and $P_i^2 \cap P_{i+1}^2 \neq \emptyset$ for $i = 1, \dots, m-1$,

and this numeration is unique (see [9] 2.11). Hence two sequences $\alpha_P(1) =$

$= (P_1^1, \dots, P_m^1)$ and $\alpha_P(2) = (P_1^2, \dots, P_m^2)$ of the banks belong to the Ω-chain α. Moreover for $j = 1, 2$ we have $P_i^j \subset P_{i+1}^j$ in the case $V_i \subset V_{i+1}$ and $P_{i+1}^j \subset \subset P_i^j$ in the case $V_{i+1} \subset V_i$. The sequences $\alpha_P(1)$ and $\alpha_P(2)$ are called the *chains of banks associated to the Ω-chain α.*

If α is closed, i.e., if $V_m = V_1$ then two cases are possible:

(i) $P_m^1 = P_1^1$ and $P_m^2 = P_1^2$,

(ii) $P_m^1 = P_1^2$ and $P_m^2 = P_1^1$.

In the first case we say that α *preserves its banks* and in the second that α *changes its banks.*

Observe that the members of Ω together with the banks preserving closed Ω-chains form clearly an invariant subcategory of the i-category $\Omega^\bullet$. Denote this invariant subcategory by $\Omega^\bullet_{\text{pr}}$.

It is to be noted that if $V_1 \subset V_2 \subset V_3$ ($V_1, V_2, V_3 \in \Omega$) then the closed Ω-chain (V_1, V_3, V_2, V_1) clearly preserves its banks. Hence

$$\Omega^\bullet_{\text{med}} \subset \Omega^\bullet_{\text{pr}}.$$

5.5. Definition. Let K be a continuous closed path in $X \setminus A$. We say that K preserves *its banks* if each closed Ω-chain associated to K does (cf. 4.5). According to 4.5 and 5.4 K preserves its banks if and only if there exists a closed Ω-chain associated to K which preserves its banks.

Now we can formulate our first fundamental theorem.

5.6. Theorem. *Let (X, A) be an (n,p)-manifold in R^{n+1} with $p \geqq 3$. Let K be a continuous closed path in $X \setminus A$. Then K preserves its banks if and only if it preserves the orientation.*

We prepare the proof of this theorem.

5.7. Proposition. *Let V be a* $\operatorname{mod}(X, A)$ *k-regular domain in R^{n+1}. Then $V \cap X$ is an s-regular domain of (X, A).*

Proof. We argue by way of contradiction.

Suppose that $U = V \cap X$ is not s-regular, i.e., $H_n(X, X \setminus U) = 0$. V is a connected set in R^{n+1} containing $U = V \cap X$ and disjoint from $X \setminus U$. Hence by [10] 6.22, $V \setminus X$ is a connected set. However $V \setminus X$ consists of two components (see [9] 2.5) and this is a contradiction.

$V \cap X$ is an s-regular domain of (X, A) as required.

5.8. Proposition. *Suppose that $p \geqq 3$ and let U be an s-regular domain of (X, A). Then there is a k-regular domain V* $\operatorname{mod}(X, A)$ *such that $V \cap X = U$.*

Proof. By 3.3 $(X, X \setminus U)$ is an (n,p)-cell without c-singularity. Thus by [9] 2.14 $(X, X \setminus U)$ is a nonlinked (n,p)-cell. Hence U is an i-regular domain of $(X, X \setminus U)$ (cf. [8] 2.7, [8] 2.6 and [8] 1.11). Now let V be an e-regular domain of $(X, X \setminus U)$ such that $V \cap X = U$ (see [8] 2.14 and [8] 2.7). V is clearly a regularly intersecting domain of (X, A) (see [9] 2.4) and by [8] 2.15 $V \setminus X$ is nonconnected. Hence by [6] 3.4 V is a mod(X, A) k-regular domain in R^{n+1} as required.

5.9. Now let $\alpha = (V_1, \dots, V_k)$ be an Ω-chain and for $i = 1, \dots, k$ let $U_i = V_i \cap X$. Denote by C_α the chain $C_\alpha = (U_1, \dots, U_k)$. By 5.7 C_α is a τ-chain. Moreover if α is a closed chain then so is C_α.

Let K be a continuous path in $X \setminus A$ and let α be an Ω-chain associated to K. Then C_α is clearly a τ-chain associated to K. Hence in order to prove Theorem 5.6 it is necessary and sufficient to prove the following lemma.

5.10. LEMMA. *Let $\alpha = (V_1, \dots, V_k = V_1)$ be a closed Ω-chain* (cf. 5.4) *and let $C_\alpha = (U_1, \dots, U_k = U_1)$ where $U_i = V_i \cap X$ for $i = 1, \dots, k$. Then $C^*_\alpha = \mathrm{id}_{H(U_1)} = \mathrm{id}_{H_n(X, X \setminus U_1)}$* (cf. 3.7) *if α preserves its banks and $C^*_\alpha = -\mathrm{id}_{H(U_1)}$ (i.e., $C^*_\alpha(y) = -y$ for $y \in H(U_1)$) if α changes its banks.*

We now prepare the proof of the lemma.

5.11. A positive dilatation of R^{n+1} is a direction preserving transformation of R^{n+1}, i.e., $\varphi\colon R^{n+1} \to R^{n+1}$ is a positive dilatation if it is a bijective map and for any two distinct points a, b of R^{n+1} the vectors $\overrightarrow{ab}$ and $\overrightarrow{\varphi(a)\varphi(b)}$ have the same direction. The positive dilatations form a subgroup of the similarity group of R^{n+1}.

For any subset M of a topological space let $\overline{M}$ denote the closure of M.

Observe that for any two open balls $S_1 = S(q_1, \rho_1)$ and $S_2 = S(q_2, \rho_2)$ in R^{n+1} (cf. [8] 2.14) there is a unique positive dilatation φ_{S_1,S_2} mapping S_1 onto S_2. This similarity φ_{S_1,S_2} takes the closed ball $\overline{S_1}$ into $\overline{S_2}$. We shall denote the restriction $\varphi_{S_1,S_2}|_{\overline{S}_1}\colon (\overline{S}_1, \overline{S}_1 \setminus S_1) \to (\overline{S}_2, \overline{S}_2 \setminus S_2)$ by $\overline{\varphi}_{S_1,S_2}$.

Let G be a bounded open subset of R^{n+1}. Then let $H'(G)$ denote the group $H_{n+1}(\overline{G}, \overline{G} \setminus G)$. Moreover let ∂_G be the boundary homomorphism

$$\partial_G\colon H'(G) = H_{n+1}(\overline{G}, \overline{G} \setminus G) \to H_n(\overline{G} \setminus G) \tag{12}$$

of the compact pair $(\overline{G}, \overline{G} \setminus G)$.

Observe that for each open ball S in R^{n+1} we have

$$H'(S) \approx Z_p \quad \text{(cf. [11] I. 16.4 p. 45).} \tag{13}$$

Let S_1 and S_2 be open balls in R^{n+1}. Then there is a uniquely defined homomorphism

$$\overline{\varphi}_{S_1,S_2*}\colon H'(S_1) \to H'(S_2)$$

which is clearly an isomorphism. Moreover for each open ball S in R^{n+1} we clearly have

$$\overline{\varphi}_{S,S*} = \mathrm{id}_{H'(S)} \tag{14}$$

and for any three open balls S_1, S_2, S_3 in R^{n+1} the equality

$$\overline{\varphi}_{S_1,S_3*} = \overline{\varphi}_{S_2,S_3*}\overline{\varphi}_{S_1,S_2*} \tag{15}$$

holds. Consequently

$$\overline{\varphi}_{S_2,S_1*} = \left(\overline{\varphi}_{S_1,S_2*}\right)^{-1}. \tag{16}$$

Now let us fix an open ball S_0 in R^{n+1} and a nonzero element e_0 of $H'(S_0)$. (13) shows the existence of such an e_0.

For each open ball S in R^{n+1} let

$$e_S = \overline{\varphi}_{S_0,S*}(e_0). \tag{17}$$

Since $\overline{\varphi}_{S_0,S*}$ is an isomorphism we have

$$e_S \neq 0. \tag{18}$$

Moreover by (14) and (17) we clearly have

$$e_{S_0} = e_0. \tag{19}$$

On the other hand (15) and (17) show that

$$\overline{\varphi}_{S_1,S_2*}(e_{S_1}) = e_{S_2} \tag{20}$$

holds for any two open balls S_1 and S_2 in R^{n+1}.

5.12. Let G_1 and G_2 be bounded open subsets of R^{n+1} such that $G_2 \subset \subset G_1$. Let us denote by

$$k_{G_1,G_2*}\colon H'(G_1) \to H_{n+1}\left(\overline{G}_1, \overline{G}_1 \setminus G_1\right)$$

and

$$m_{G_1,G_2*}\colon H'(G_2) \to H_{n+1}\left(\overline{G}_1, \overline{G}_1 \setminus G_2\right)$$

the homomorphisms induced by the inclusions

$$k_{G_1,G_2}\colon \left(\overline{G}_1, \overline{G}_1 \setminus G_1\right) \subset \left(\overline{G}_1, \overline{G}_1 \setminus G_2\right)$$

and

$$m_{G_1,G_2}\colon \left(\overline{G}_2, \overline{G}_2 \setminus G_2\right) \subset \left(\overline{G}_1, \overline{G}_1 \setminus G_2\right)$$

respectively. Observe that since m_{G_1,G_2} is a relative homeomorphisms (cf. [10] 5.6) and the Čech homology theory H is invariant under relative homeomorphism (see [10] 5.6 and [10] 5.5) it follows that m_{G_1,G_2*} is an isomorphism.

Now let

$$r_{G_1,G_2*} = (m_{G_1,G_2*})^{-1} k_{G_1,G_2*}\colon H'(G_1) \to H'(G_2). \tag{21}$$

For each bounded open subset G of R^{n+1} we then clearly have

$$k_{G,G*} = m_{G,G*} = r_{G,G*} = \mathrm{id}_{H'(G)}\,. \tag{22}$$

Now let G_1, G_2, G_3 be bounded open subsets of R^{n+1} such that $G_3 \subset \subset G_2 \subset G_1$. Then

$$r_{G_2,G_3*} r_{G_1,G_2*} = r_{G_1,G_3*}. \tag{23}$$

Indeed let

$$i_{G_1,G_2,G_3*}\colon H_{n+1}\left(\overline{G}_2, \overline{G}_2 \setminus G_3\right) \to H_{n+1}\left(\overline{G}_1, \overline{G}_1 \setminus G_3\right)$$

and

$$j_{G_1,G_2,G_3*}\colon H_{n+1}\left(\overline{G}_1, \overline{G}_1 \setminus G_2\right) \to H_{n+1}\left(\overline{G}_1, \overline{G}_1 \setminus G_3\right)$$

be the homomorphisms induced by the inclusions

$$i_{G_1,G_2,G_3}\colon \left(\overline{G}_2, \overline{G}_2 \setminus G_3\right) \subset \left(\overline{G}_1, \overline{G}_1 \setminus G_3\right)$$

and

$$j_{G_1,G_2,G_3}\colon \left(\overline{G}_1, \overline{G}_1 \setminus G_2\right) \subset \left(\overline{G}_1, \overline{G}_1 \setminus G_3\right)$$

respectively. We then evidently have

$$j_{G_1,G_2,G_3} k_{G_1,G_2} = k_{G_1,G_3},$$

$$i_{G_1,G_2,G_3} m_{G_2,G_3} = m_{G_1,G_3}$$

and

$$i_{G_1,G_2,G_3} k_{G_2,G_3} = j_{G_1,G_2,G_3} m_{G_1,G_2}.$$

Consequently

$$j_{G_1,G_2,G_3*} k_{G_1,G_2*} = k_{G_1,G_3*},$$

$$i_{G_1,G_2,G_3*} m_{G_2,G_3*} = m_{G_1,G_3*}$$

and

$$i_{G_1,G_2,G_3*} k_{G_2,G_3*} = j_{G_1,G_2,G_3*} m_{G_1,G_2*}.$$

These yield

$$(m_{G_1,G_3*})^{-1} i_{G_1,G_2,G_3*} = (m_{G_2,G_3*})^{-1}$$

and thus

$$r_{G_1,G_3*} = (m_{G_1,G_3*})^{-1} k_{G_1,G_3*} = (m_{G_1,G_3*})^{-1} j_{G_1,G_2,G_3*} k_{G_1,G_2*} =$$

$$= (m_{G_1,G_3*})^{-1} j_{G_1,G_2,G_3*} m_{G_1,G_2*} (m_{G_1,G_2*})^{-1} k_{G_1,G_2*} =$$

$$= (m_{G_1,G_3*})^{-1} i_{G_1,G_2,G_3*} k_{G_2,G_3*} r_{G_1,G_2*} =$$

$$= (m_{G_2,G_3*})^{-1} k_{G_2,G_3*} r_{G_1,G_2*} = r_{G_2,G_3*} r_{G_1,G_2*}$$

as required.

5.13. Let S_1 and S_2 be open balls in R^{n+1} such that $S_2 \subset S_1$. Then

$$r_{S_1,S_2*} = \overline{\varphi}_{S_1,S_2*}. \tag{24}$$

Indeed consider the continuous map $h: \overline{S}_1 \times \mathbf{I} \to \overline{S}_1$ defined by $h(y,t) = (1-t)y + t\varphi_{S_1,S_2}(y)$. We then clearly have $h(y,0) = y = k_{S_1,S_2}(y)$, $h(y,1) = \varphi_{S_1,S_2}(y) = m_{S_1,S_2}\overline{\varphi}_{S_1,S_2}(y)$ and $h\Big(\big(\overline{S}_1 \setminus S_1\big) \times \mathbf{I}\Big) \subset \overline{S}_1 \setminus S_2$. Consequently

$$k_{S_1,S_2}: (\overline{S}_1, \overline{S}_1 \setminus S_1) \subset (\overline{S}_1, \overline{S}_1 \setminus S_2)$$

and

$$m_{S_1,S_2}\overline{\varphi}_{S_1,S_2}: (\overline{S}_1, \overline{S}_1 \setminus S_1) \to (\overline{S}_1, \overline{S}_1 \setminus S_2)$$

are homotopic maps and thus

$$k_{S_1,S_2*} = \big(m_{S_1,S_2}\overline{\varphi}_{S_1,S_2}\big)_* = m_{S_1,S_2*}\overline{\varphi}_{S_1,S_2*}.$$

Hence

$$\overline{\varphi}_{S_1,S_2*} = (m_{S_1,S_2*})^{-1} k_{S_1,S_2*} = r_{S_1,S_2*}$$

as required.

We mention here that by (24) and 5.11 (20)

$$r_{S_1,S_2*}(e_{S_1}) = e_{S_2}. \tag{25}$$

5.14. Let G be a bounded open subset of R^{n+1}. We define the element e'_G of $H'(G) = H_{n+1}\big(\overline{G}, \overline{G} \setminus G\big)$ by setting

$$e'_G = r_{S,G*}(e_S), \tag{26}$$

where S is an open ball containing G. This element e'_G is well defined.

Indeed, let S_1 and S_2 be open balls in R^{n+1} containing G and let S_3 be an open ball in R^{n+1} containing both S_1 and S_2. Then by 5.13 (25) and 5.12 (23) for $j = 1, 2$ we have

$$r_{S_j,G*}\left(e_{S_j}\right) = r_{S_j,G*} r_{S_3,S_j*}\left(e_{S_3}\right) = r_{S_3,G*}\left(e_{S_3}\right)$$

and thus

$$r_{S_1,G*}\left(e_{S_1}\right) = r_{S_2,G*}\left(e_{S_2}\right)$$

as required.

Observe that for any open ball S in R^{n+1} by (26) and 5.12(22) we clearly have

$$e'_S = r_{S,S*}(e_S) = e_S. \tag{27}$$

Moreover if G_1 and G_2 are bounded open subsets of R^{n+1} with $G_2 \subset G_1$ then

$$r_{G_1,G_2*}\left(e'_{G_1}\right) = e'_{G_2}. \tag{28}$$

Indeed let S be an open ball in R^{n+1} containing G_1. Then by (26) and 5.12 (23) we have

$$r_{G_1,G_2*}\left(e'_{G_1}\right) = r_{G_1,G_2*}\left(r_{S,G_1*}(e_S)\right) = r_{S,G_2*}(e_S) = e'_{G_2}.$$

5.15. Let V be a bounded mod(X, A) k-regular domain in R^{n+1} and let $U = V \cap X$. Let P be a bank of V, i.e., a component of $V \setminus X$. P is a bounded open set as well and thus $\overline{P}$ is compact. We shall define the homomorphism

$$\Delta_{V,P}: H'(P) = H_{n+1}\left(\overline{P}, \overline{P} \setminus P\right) \to H(U) = H_n(X, X \setminus U)$$

as follows.

First observe that by [9] 2.5(c) we have $U \subset \overline{P}$ and thus $\overline{U} \subset \overline{P} \setminus P$, $\overline{U} \setminus U = \overline{U} \setminus V \cap X = \overline{U} \setminus V \subset \overline{P} \setminus V = \overline{P} \setminus (P \cup U)$. Consequently $\left(\overline{P} \setminus P\right) \setminus \setminus \left(\overline{P} \setminus V\right) = U$.

Next, let

$$t_{V,P*}: H_n\left(\overline{P} \setminus P\right) \to H_n\left(\overline{P} \setminus P, \overline{P} \setminus V\right),$$

$$y_{V,P*}: H_n\left(\overline{U}, \overline{U} \setminus U\right) \to H_n\left(\overline{P} \setminus P, \overline{P} \setminus V\right)$$

and

$$z_{U*}: H_n\left(\overline{U}, \overline{U} \setminus U\right) \to H(U) = H_n(X, X \setminus U)$$

be homomorphisms induced by the inclusions

$$t_{V,P}: (\overline{P} \setminus P, \emptyset) \subset (\overline{P} \setminus P, \overline{P} \setminus V), \quad y_{V,P}: (\overline{U}, \overline{U} \setminus U) \subset (\overline{P} \setminus P, \overline{P} \setminus V)$$

and $z_U: (\overline{U}, \overline{U} \setminus U) \subset (X, X \setminus U)$ respectively. Since $y_{V,P}$ and z_U are relative homeomorphisms (cf. [10] 5.6) and H is invariant under relative homeomorphisms (see [10] 5.6 and [10] 5.5) it follows that $y_{V,P*}$ and z_{U*} are isomorphisms.

Now let

$$\Delta_{V,P} = z_{U*}(y_{V,P*})^{-1} t_{V,P*} \partial_P \tag{29}$$

(cf. 5.11 (12)). $\Delta_{V,P}: H'(P) \to H(U)$ is clearly a homomorphism.

5.16. PROPOSITION. *Let V, P and U be the same as in* 5.15. *Then*

$$\Delta_{V,P}(e'_P) \neq 0.$$

PROOF. Observe that since $y_{V,P*}$ and z_{U*} are isomorphisms we need only to prove $t_{V,P*}\partial_P(e'_P) \neq 0$, i.e., that

$$\partial_P(e'_P) \notin \ker t_{V,P*}.$$

Consider now the segment

$$H_n\left(\overline{P} \setminus P, \overline{P} \setminus V\right) \xleftarrow{t_{V,P*}} H_n(\overline{P} \setminus P) \xleftarrow{i*} H_n(\overline{P} \setminus V)$$

of the homology sequence of the compact pair $(\overline{P} \setminus P, \overline{P} \setminus V)$ where i_* is the homomorphism induced by the inclusion $i: (\overline{P} \setminus V) \subset (\overline{P} \setminus P)$. This sequence is exact and thus we have only to prove that

$$\partial_P(e'_P) \notin \operatorname{im} i_*. \tag{30}$$

To this end consider a proper linking theory $\mathfrak{V}$ of the type $\mathfrak{V} = \mathfrak{V}_{p,n,0}$ of compacts in R^{n+1}. $\mathfrak{V}$ is a mapping which makes correspond to each ordered pair (M, M') of disjoint compact subspaces of R^{n+1} a bihomomorphism (cf. [8] 1.7) $\mathfrak{v}_{M,M'}: H_n(M) \times \widetilde{H}_0(M) \to Z_p$ (cf. [8] 1.4) such that for any compact subspaces M, M', N, N' of R^{n+1} satisfying $M \subset N$, $M' \subset N'$ and of course $N \cap N' = \emptyset$ the condition $\mathfrak{v}_{M,M'}(u, u') = \mathfrak{v}_{N,N'}\big(j_*(u), j'_*(u')\big)$ is satisfied for every $u \in H_n(M)$ and $u' \in \widetilde{H}_0(M')$ where $j: M \subset N$ and $j': M' \subset N'$ are inclusion maps. Moreover for at least one ordered pair (M, M') of disjoint compact subspaces of R^{n+1}, $\mathfrak{v}_{M,M'}$ is a nontrivial bihomomorphism (see [8] 1.7). According to [10] 5.13 there exists a proper linking theory of the given type.

Notice that if M, N, M' are compacts in R^{n+1} such that $M \subset N \subset$ $\subset R^{n+1} \setminus M'$ then

$$\mathfrak{v}_{M,M'}(u,u') = \mathfrak{v}_{N,M'}\big(j_*(u),u'\big) \tag{31}$$

evidently holds for every $u \in H_n(M)$ and $u' \in \widetilde{H}_0(M')$ where $j\colon M \subset N$ is the inclusion map.

Now let Φ be an arbitrary compact subset of R^{n+1}, $u \in H_n(\Phi)$ and $Y =$ $= \{q,q'\}$ a 0-sphere in $R^{n+1} \setminus \Phi$. We say that Y *is linked by* u if there exists a $u' \in \widetilde{H}_0(Y)$ such that

$$\mathfrak{v}_{\Phi,Y}(u,u') \neq 0.$$

We now construct a 0-sphere Y in $V \setminus X$ linked by $\partial_P(e'_P)$.

Let S be an open ball in P and let P' be the bank of V distinct from P. Thus $P \cap P' = \emptyset$, $P \cup P' = V \setminus X$.

According to 5.14 (28), 5.14 (27) and 5.11 (18) we have

$$r_{P,S*}(e'_P) = e'_S = e_S \neq 0 \tag{32}$$

and thus

$$k_{P,S*}(e'_P) = m_{P,S*}(e_S). \tag{33}$$

Since $\overline{S}$ is homologically trivial (see [11] I. 16.1 p. 45) and $\overline{S} \setminus S$ is nonempty it follows that $\partial_S\colon H'(S) \to H_n\left(\overline{S} \setminus S\right)$ is an isomorphism (see [11] I. 9.4 p. 23) and thus by (32) we have

$$\partial_S(e_S) \neq 0. \tag{34}$$

Let

$$j^1_{S,P*}\colon H_n\left(\overline{P} \setminus P\right) \to H_n\left(\overline{P} \setminus S\right) \quad \text{and} \quad j^2_{S,P*}\colon H_n\left(\overline{S} \setminus S\right) \to H_n\left(\overline{P} \setminus S\right)$$

be homomorphisms induced by the inclusions $j^1_{S,P}\colon \overline{P} \setminus P \subset \overline{P} \setminus S$ and $j^2_{S,P}\colon$ $\colon \overline{S} \setminus S \subset \overline{P} \setminus S$ respectively. Moreover let $\partial_{P,S}\colon H_{n+1}\left(\overline{P},\overline{P} \setminus S\right) \to$ $\to H_n\left(\overline{P} \setminus S\right)$ be the boundary homomorphism of the compact pair $\left(\overline{P},\overline{P} \setminus S\right)$. We then have evidently $j^2_{S,P*}\partial_S = \partial_{P,S}m_{P,S*}$ and $j^1_{S,P*}\partial_P =$ $= \partial_{P,S}k_{P,S*}$. Hence by (33) we get

$$j^2_{S,P*}\partial_S(e_S) = j^1_{S,P*}\partial_P(e'_P). \tag{35}$$

Now let $Y' = \{q_1,q'_2\}$ be a 0-sphere in $R^{n+1} \setminus \left(\overline{S} \setminus S\right)$ linked by $\partial_S(e_S)$. (34) and [10] 6.13 show the existence of such a 0-sphere Y'. By [10] 6.8 q_1 and

q_2' lie in different components of $R^{n+1} \setminus (\overline{S} \setminus S)$, where the components of $R^{n+1} \setminus (\overline{S} \setminus S)$ are S and $R^{n+1} \setminus \overline{S}$. Hence we can suppose that $q_1 \in S \subset P$ and $q_2' \in R^{n+1} \setminus \overline{S}$. Let q_2 be a point in P' distinct from q_2'. There clearly exists such a point q_2. Let $Y = \{q_1, q_2\}$. Since $P' \cap \overline{P} = \emptyset$ and thus $P' \cap \overline{S} = \emptyset$ it follows that q_2' and q_2 belong to the same component of $R^{n+1} \setminus (\overline{S} \setminus S)$ and thus by [10] 6.12, Y is linked by $\partial_S(e_S)$ as well.

Let u' be an element of $\widetilde{H}_0(Y)$ such that

$$\mathfrak{v}_{\overline{S}\setminus S,Y}\left(\partial_S(e_S), u'\right) \neq 0. \tag{36}$$

Since Y is linked by $\partial_S(e_S)$ the existence of such $u' \in \widetilde{H}_0(Y)$ follows. However Y is disjoint from $\overline{P} \setminus S$ and thus by (36), (31) and (35) we get

$$0 \neq \mathfrak{v}_{\overline{S}\setminus S,Y}\left(\partial_S(e_S), u'\right) = \mathfrak{v}_{\overline{P}\setminus S,Y}\left(j^2_{S,P*}\partial_S(e_S), u'\right) =$$

$$= \mathfrak{v}_{\overline{P}\setminus S,Y}\left(j^1_{S,P*}\partial_P(e_P'), u'\right) = \mathfrak{v}_{\overline{P}\setminus P,Y}\left(\partial_P(e_P'), u'\right).$$

Y is linked by $\partial_P(e_P')$ indeed, where $Y \subset P \cup P' = V \setminus X$.

We have constructed the 0-sphere Y with the required properties.

Now let u be an arbitrary element of $H_n\left(\overline{P} \setminus V\right)$ and $u'' \in \widetilde{H}_0(Y)$. Since both points of $Y = \{q_1, q_2\}$ belong to V and thus q_1 and q_2 belong to the same component of $R^{n+1} \setminus \left(\overline{P} \setminus V\right)$ it follows that $\mathfrak{v}_{\overline{P}\setminus V,Y}(u, u'') = 0$ (see [10] 6.6). Consequently by (31) we get

$$\mathfrak{v}_{\overline{P}\setminus P,Y}\left(i_*(u), u''\right) = 0.$$

Thus Y fails to be linked by any element of im i_*. On the other hand Y is linked by $\partial_P(e_P')$. Consequently

$$\partial_P(e_P') \notin \operatorname{im} i_*$$

as required (see (30)). This yields $\Delta_{V,P}(e_P') \neq 0$.

The proof of the proposition is complete.

5.17. Proposition. *Let V be a bounded* mod(X, A) *k-regular domain in R^{n+1}. Let P^1 and P^2 be the banks of V. Then*

$$\Delta_{V,P^2}(e_{P^2}') = -\Delta_{V,P^1}(e_{P^1}').$$

Proof. Let $U = V \cap X$. We have to verify the equality

$$z_{U*}\left(y_{V,P^2*}\right)^{-1} t_{V,P^2*}\partial_{P^2}(e_{P^2}') =$$

$$= -z_{U*}\big(y_{V,P^1*}\big)^{-1} t_{V,P^1*}\partial_{P^1}(e'_{P_1}).$$

Hence we need only to show that

$$(37) \qquad (y_{V,P^2*})^{-1} t_{V,P^2*}\partial_{P^2}(e'_{P^2}) = -(y_{V,P^1*})^{-1} t_{V,P^1*}\partial_{P^1}(e'_{P^1}).$$

To this end for $j = 1,2$ consider the diagram

$$\begin{tikzcd}
H_n\left(\overline{P}^j \setminus P^j\right) \arrow[rr, "t_{V,P^j*}"] \arrow[d, "i^j_{1*}"] & & H_n\left(\overline{P}^j \setminus P^j, \overline{P}^j \setminus V\right) \arrow[dll, "i^j_{2*}"'] \\
H_n\big(\overline{V} \setminus (P^1 \cup P^2), \overline{V} \setminus V\big) & & H_n\left(\overline{U}, \overline{U} \setminus U\right) \arrow[ll, "i_{1*}"'] \arrow[u, "y_{V,P^j*}"']
\end{tikzcd}$$

where i_{1*}, i^j_{1*} and i^j_{2*} are homomorphisms induced by the respective inclusion maps. The diagram is clearly commutative and the inclusion $i_1\colon \left(\overline{U}, \overline{U} \setminus U\right) \subset \subset \left(\overline{V} \setminus (P^1 \cup P^2), \overline{V} \setminus V\right)$ is a relative homeomorphism. Hence i_{1*} is an isomorphism. Consequently we need to prove the equality

$$(38) \qquad i^2_{1*}\partial_{P^2}(e'_{P^2}) = -i^1_{1*}\partial_{P^1}(e'_{P^1}).$$

However by 5.14 (28) for $j = 1,2$ we have

$$e'_{P^j} = r_{V,P^j*}(e'_V) = \big(m_{V,P^j*}\big)^{-1} k_{V,P^j*}(e'_V)$$

(see also 5.12 (21)) and thus by (38) we have only to prove the equality

$$(39) \qquad i^2_{1*}\partial_{P^2}\big(m_{V,P^2*}\big)^{-1} k_{V,P^2*}(e'_V) = -i^1_{1*}\partial_{P^1}\big(m_{V,P^1*}\big)^{-1} k_{V,P^1*}(e'_V).$$

Observe that

$$P^1 \cup P^2 = V \setminus X = V \setminus (V \cap X) = V \setminus U.$$

Moreover since $\overline{P}^1 \supset U = V \cap X$ (see [9] 2.5(c)) we have

$$\overline{V \setminus X} = \overline{P^1} \cup \overline{P^2} = \overline{P^1 \cup U} \cup \overline{P^2} = \overline{P^1 \cup P^2 \cup U} = \overline{V}.$$

Consider now the diagram

$$\begin{tikzcd}
 & & H'(V) \arrow[dll, "k_{V,P^1*}"'] \arrow[drr, "k_{V,P^2*}"] \arrow[dd, "i_{2*}"] & & \\
H_{n+1}\left(\overline{V},\overline{V}\setminus P^1\right) & & & & H_{n+1}\left(\overline{V},\overline{V}\setminus P^2\right) \\
 & & H'(V\setminus X) \arrow[ull, "i^1_{3*}"'] \arrow[urr, "i^2_{3*}"] \arrow[dd, "\partial'_V"] & & \\
H'(P^1) \arrow[u, "m_{V,P^1*}"] \arrow[urr, "i^1_{4*}"'] \arrow[drr, "i^1_{1*}\partial_{P^1}"'] & & & & H'(P^2) \arrow[u, "m_{V,P^2*}"'] \arrow[ull, "i^2_{4*}"] \arrow[dll, "i^2_{1*}\partial_{P^2}"] \\
 & & H_n\left(\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V\right) & &
\end{tikzcd}$$

where the homomorphisms i^1_{3*}, i^2_{3*}, i^1_{4*}, i^2_{4*} and i_{2*} are induced by inclusions and

$$\partial'_V\colon H'(V\setminus X)=H_{n+1}\left(\overline{V},\overline{V}\setminus(P^1\cup P^2)\right)\to H_n\left(\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V\right)$$

is the boundary homomorphism of the triple $\left(\overline{V},\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V\right)$ i.e., $\partial'_V=i_{3*}\partial''_V$, where $\partial''_V\colon H_{n+1}\left(\overline{V},\overline{V}\setminus(P^1\cup P^2)\right)\to H_n\left(\overline{V}\setminus(P^1\cup P^2)\right)$ is the boundary homomorphism of the compact pair $\left(\overline{V},\overline{V}\setminus(P^1\cup P^2)\right)=$ $=\left(\overline{V\setminus X},\overline{V\setminus X}\setminus(V\setminus X)\right)$ and the homomorphism $i_{3*}\colon H_n\left(\overline{V}\setminus(P^1\cup P^2)\right)\to$ $\to H_n\left(\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V\right)$ is induced by inclusion. Hence to prove (39) we need only to show that the following conditions are satisfied:

(i) m_{V,P^1*} and m_{V,P^2*} are isomorphisms onto,

(ii) $\partial'_V i_{2*}=0$,

(iii) $\operatorname{im} i^2_{4*}=\ker i^1_{3*}$ and $\operatorname{im} i^1_{4*}=\ker i^2_{3*}$,

(iv) commutativity holds in each triangle of the diagram (see [11] I. Lemma 15.1 p. 38).

The homomorphisms m_{V,P^1*} and m_{V,P^2*} are isomorphisms indeed (see 5.12). Thus condition (i) is satisfied.

Considering the segment

$$H_n(\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V) \xleftarrow{\partial'_V} H_{n+1}(\overline{V},\overline{V}\setminus(P^1\cup P^2)) \xleftarrow{i_{2*}} H_{n+1}(\overline{V},\overline{V}\setminus V)$$

of the exact homology sequence of the triple $(\overline{V},\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V)$, where $H_{n+1}(\overline{V},\overline{V}\setminus(P^1\cup P^2)) = H'(V\setminus X)$ and $H_{n+1}(\overline{V},\overline{V}\setminus V) = H'(V)$, we obtain $\partial'_V i_{2*} = 0$. Thus condition (ii) is satisfied as well.

Now consider the commutative diagram

$$\begin{tikzcd}
H'(P^1) = H_{n+1}\left(\overline{P^1},\overline{P^1}\setminus P^1\right) \arrow[r, "i^1_{6*}"] \arrow[d, "i^1_{4*}"] & H_{n+1}(\overline{V}\setminus P^2,\overline{V}\setminus(P^1\cup P^2)) \arrow[dl, "i^1_{5*}"] \\
H'(V\setminus X) = H_{n+1}(\overline{V},\overline{V}\setminus(P^1\cup P^2)) &
\end{tikzcd}$$

where the homomorphisms i^1_{5*} and i^1_{6*} are induced by inclusions. Thus

$$\operatorname{im} i^1_{4*} = i^1_{5*}(\operatorname{im} i^1_{6*}).$$

However since $i^1_6\colon(\overline{P^1},\overline{P^1}\setminus P^1)\subset(\overline{V}\setminus P^2,\overline{V}\setminus(P^1\cup P^2))$ is a relative homeomorphism it follows that i^1_{6*} is an onto isomorphism and thus

$$\operatorname{im} i^1_{4*} = \operatorname{im} i^1_{5*}. \tag{40}$$

Now consider the segment

$$\xleftarrow{i^1_{5*}} H_{n+1}(\overline{V}\setminus P^2,\overline{V}\setminus(P^1\cup P^2))$$

$$H_{n+1}(\overline{V},\overline{V}\setminus P^2) \xleftarrow{i^2_{3*}} H_{n+1}(\overline{V},\overline{V}\setminus(P^1\cup P^2)) \xleftarrow{i^1_{5*}}$$

of the homology sequence of the compact triple $(\overline{V},\overline{V}\setminus P^2,\overline{V}\setminus(P^1\cup P^2))$, where $H_{n+1}(\overline{V},\overline{V}\setminus(P^1\cup P^2)) = H'(V\setminus X)$. By the exactness of this sequence (see [11] I. 10.2 p. 25) we get $\operatorname{im} i^1_{5*} = \ker i^2_{3*}$ and thus by (40) we obtain the required equality

$$\operatorname{im} i^1_{4*} = \ker i^2_{3*}.$$

Likewise we have

$$\operatorname{im} i^2_{4*} = \ker i^1_{3*}.$$

Thus condition (iii) is satisfied as well.

Observe that for $j=1,2$ the diagram

$$\begin{tikzcd}
H_{n+1}\left(\overline{V},\overline{V}\setminus P^j\right) & & H'(V) \arrow[ll, "k_{V,P^j*}"'] \arrow[dd, "i_{2*}"] \\
& & \\
H'(P^j) \arrow[uu, "m_{V,P^j*}"] \arrow[rr, "i^j_{4*}"'] & & H'(V\setminus X) \arrow[uull, "i^j_{3*}"']
\end{tikzcd}$$

is clearly commutative. Thus to prove (iv) we need only to show that for $j=1,2$ the diagram

$$\begin{tikzcd}
H'(V\setminus X) \arrow[drr, "\partial_{V'}"] & & \\
H'(P^j) \arrow[u, "i^j_{4*}"] \arrow[rr, "i^j_{1*}\partial_{P^j}"'] & & H_n\left(\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V\right)
\end{tikzcd}$$

is commutative as well, i.e., that

$$\partial'_V i^j_{4*} = i^j_{1*}\partial_{P^j}. \tag{41}$$

To this end consider the diagram

$$\begin{tikzcd}
H'(P^j)=H_{n+1}\left(\overline{P_j},\overline{P^j}\setminus P_j\right) \arrow[rr, "i^j_{4*}"] \arrow[dd, "\partial_{P^j}"'] & & H'(V\setminus X)=H_{n+1}\left(\overline{V},\overline{V}\setminus(P^1\cup P^2)\right) \arrow[dl, "\partial''_V"'] \arrow[dd, "\partial'_V"] \\
& H_n\left(\overline{V}\setminus(P^1\cup P^2)\right) \arrow[dr, "i_{3*}"] & \\
H_n\left(\overline{P^j}\setminus P^j\right) \arrow[ur, "i^j_{7*}"] \arrow[rr, "i^j_{1*}"'] & & H_n\left(\overline{V}\setminus(P^1\cup P^2),\overline{V}\setminus V\right)
\end{tikzcd}$$

where the homomorphism i^j_{7*} is induced by the respective inclusion. This diagram is clearly commutative and this proves the required equality (41).

Hence all the conditions (i), (ii), (iii) and (iv) are fulfilled and this proves equality (39) and also the original assertion.

The proof of the proposition is complete.

5.18. PROPOSITION. *Let V and V' be bounded compatible* $\operatorname{mod}(X,A)$ *k-regular domains in R^{n+1}* (i.e., either $V\subset V'$ or $V'\subset V$). *Let $U=V\cap X$ and $U'=V'\cap X$. Let P be a bank of V and P' a bank of V' such that $P\cap P'\neq\emptyset$* (cf. 5.4). *Then*

$$[U',U]_*\Delta_{V,P}(e'_P)=\Delta_{V',P'}(e'_{P'})$$

(see also 3.6 and 5.7).

PROOF. *By 3.6(10) we only need to consider the case $V' \subset V$ and in this case we have $U' \subset U$ and $P' \subset P$.*

Consider the following commutative diagram

$$\begin{tikzcd}
H'(P) \arrow[r, "k_{P,P'*}"] \arrow[d, "\partial_P"'] & H_{n+1}\left(\overline{P}, \overline{P} \setminus P'\right) \arrow[r, "m_{P,P'*}"] \arrow[d, "\partial_{P,P'}"] & H'(P') \arrow[d, "\partial_{P'}"] \\
H_n\left(\overline{P} \setminus P\right) \arrow[r, "j_{1*}"] \arrow[d, "t_{V,P*}"'] & H_n\left(\overline{P} \setminus P'\right) \arrow[r, "j_{2*}"] \arrow[d, "j_{7*}"] & H_n\left(\overline{P'} \setminus P'\right) \arrow[d, "t_{V',P'*}"] \\
H_n\left(\overline{P} \setminus P, \overline{P} \setminus V\right) \arrow[r, "j_{3*}"'] & H_n\left(\overline{P} \setminus P', \overline{P} \setminus V'\right) \arrow[r, "j_{4*}"'] & H_n\left(\overline{P'} \setminus P', \overline{P'} \setminus V'\right) \\
H_n\left(\overline{U}, \overline{U} \setminus U\right) \arrow[r, "j_{5*}"] \arrow[u, "y_{V,P*}"] \arrow[d, "z_{U*}"'] & H_n\left(\overline{U}, \overline{U} \setminus U'\right) \arrow[r, "j_{6*}"] \arrow[u, "j_{8*}"'] \arrow[dr, "j_{9*}"] & H_n\left(\overline{U'}, \overline{U'} \setminus U'\right) \arrow[u, "y_{V',P'*}"'] \arrow[d, "z_{U'*}"] \\
H_n(X, X \setminus U) \arrow[rr, "{[U',U]_*}"'] & & H_n(X, X \setminus U')
\end{tikzcd}$$

where $\partial_{P,P'}: H_{n+1}\left(\overline{P}, \overline{P} \setminus P'\right) \to H_n\left(\overline{P} \setminus P'\right)$ is the boundary homomorphism of the compact pair $\left(\overline{P}, \overline{P} \setminus P'\right)$ and for $s = 1, \dots, 9$ the homomorphism j_{s*} is induced by the respective inclusion. Further $U' \subset U$ and 3.6(8) show that $[U', U]_*$ is induced by the respective inclusion as well.

Observe that $j_8: \left(\overline{U}, \overline{U} \setminus U'\right) \subset \left(\overline{P} \setminus P', \overline{P} \setminus V'\right)$ is a relative homeomorphism and thus j_{8*} is an isomorphism. Moreover $e'_{P'} = r_{P,P'*}(e'_P)$ (see 5.14 (28)) and thus $e'_{P'} = \left(m_{P,P'*}\right)^{-1} k_{P,P'*}(e'_P)$ (see 5.12 (21)). Consequently

$$k_{P,P'*}(e'_P) = m_{P,P'*}(e'_{P'}) \tag{42}$$

and thus by the commutativity of the preceding diagram we get

$$[U', U]_* \Delta_{V,P}(e'_P) = [U', U]_* z_{U*} (y_{V,P*})^{-1} t_{V,P*} \partial_P(e'_P) =$$

$$= j_{9*}(j_{8*})^{-1} j_{7*} \partial_{P,P'} k_{P,P'*}(e'_P) = j_{9*}(j_{8*})^{-1} j_{7*} \partial_{P,P'} m_{P,P'*}(e'_{P'}) =$$

$$= z_{U'*}(y_{V',P'*})^{-1} t_{V'P'*} \partial'_P(e'_{P'}) = \Delta_{V',P'}(e'_{P'})$$

as required.

5.19. DEFINITION. Let $\alpha = (V_1, \dots, V_m)$ be an Ω-chain (see 5.4) and $C = (U_1, \dots, U_{m'})$ a τ-chain (see 3.7). We say that α and C are *associated to each other* or that α is *associated to* C or that C is *associated to* α if $m = m'$ and for $i = 1, \dots, m$, $U_i = V_i \cap X$.

5.7 shows that to each Ω-chain α there is a unique τ-chain C associated to α. Observe that if α is closed then so is C.

5.20. PROPOSITION. *Let* $\alpha = (V_1, \dots, V_m)$ *be a closed* Ω*-chain* (see 5.4) (i.e., $V_m = V_1$), *where each* V_i *is a bounded* k*-regular domain* mod(X, A). *Let* $C = (U_1, \dots, U_m)$ *be the closed* τ*-chain associated to* α (see 5.19). *Then for each* $w \in H(U_1)$ (cf. 3.5) *we have*

$$C^*(w) = \begin{cases} w & \textit{if } \alpha \textit{ preserves its banks} \\ -w & \textit{if } \alpha \textit{ changes its banks} \end{cases}$$

(cf. 3.7).

PROOF. If $m = 1$ then $C^* = \mathrm{id}_{H(U_1)}$ (see 3.7) and $\alpha = (V_1)$ clearly preserves its banks. Hence in this case the assertion is obviously true.

Now suppose that $m \geqq 2$ and let $\alpha_P(1) = (P_1^1, \dots, P_m^1)$ be a chain of banks associated to α (see 5.4). Then by 5.18 we clearly have

$$C^* \Delta_{V_m, P_m^1} \left(e'_{P_m^1} \right) = \Delta_{V_1, P_1^1} \left(e'_{P_1^1} \right).$$

Hence if α preserves its banks, i.e., if $V_m = V_1$ and $P_m^1 = P_1^1$ then

$$C^* \Delta_{V_m, P_m^1} \left(e'_{P_m^1} \right) = \Delta_{V_m, P_m^1} \left(e'_{P_m^1} \right), \tag{43}$$

and if α changes its banks, i.e., if $V_m = V_1$ and $P_m^1 \neq P_1^1$ then by 5.17

$$C^* \Delta_{V_m, P_m^1} \left(e'_{P_m^1} \right) = -\Delta_{V_m, P_m^1} \left(e'_{P_m^1} \right). \tag{44}$$

However by 5.16 $\Delta_{V_m, P_m^1} \left(e'_{P_m^1} \right) \neq 0$ and $H(U_m) = H(U_1) = H_n(X, X \setminus U_1) \approx Z_p$ (see 3.2 and 3.5). Thus $\Delta_{V_m, P_m^1} \left(e'_{P_m^1} \right)$ is a generator of the group $H(U_1)$. Consequently equalities (43) and (44) prove the assertion.

We are going now to prove Lemma 5.10.

5.21. Let $\alpha = (V_1, \dots, V_m = V_1)$ be a closed Ω-chain. Let $C_\alpha = (U_1, \dots, U_m = U_1)$ be the closed τ-chain associated to α. Then for $i = 1, \dots, m$ we have $U_i = V_i \cap X$.

Let S be an open ball in R^{n+1} containing X. For $i = 1, \dots, m$ let V_i' be the component of $V_i \cap S$ containing U_i. V_i' is clearly a bounded domain in R^{n+1} regularly intersecting the compact pair (X, A) in U_i. Thus by

[6] 3.5 and $V_i' \subset V_i \in \Omega$, V_i' is k-regular $\mathrm{mod}(X, A)$. Consequently $\alpha' = (V_1', \ldots, V_m' = V_1')$ is clearly a closed Ω-chain, where each V_i' is bounded and for $i = 1, \ldots, m$ we have $V_i' \cap X = U_i$. Moreover α' clearly preserves its banks if and only if α does. Hence Lemma 5.10 is an immediate corollary of Proposition 5.20.

The proof of Lemma 5.10 and also the proof of Theorem 5.6 is complete.

We now formulate some further theorems related to (n,p)-manifolds in R^{n+1}.

5.22. THEOREM. *Let (X, A) be an (n,p)-manifold in R^{n+1}. Let $C = (U_1, \ldots, U_m = U_1)$ be a closed orientation changing τ-chain* (cf. 3.7). *Then for each $w \in H(U_1)$ we have*

$$C^*(w) = -w.$$

PROOF. In case $p = 2$ each closed τ-chain is orientation preserving (see 3.7). Thus we may suppose $p \geqq 3$. According to Lemma 5.10 we have to prove the existence of a closed α-chain associated to C.

To this end we need only to show that for each finite subset $\{U_1', \ldots, U_r'\}$ of pairwise distinct members of τ there are members $V_1', \ldots, V_r'$ of Ω such that

$$U_i' = V_i' \cap X \quad \text{for} \quad i = 1, \ldots, r$$

and so that

$$U_i' \subset U_{i'}' \quad \text{implies} \quad V_i' \subset V_{i'}'.$$

To show this simple fact we proceed by induction. In case $r = 0$ there is nothing to prove. Suppose now that $r \geqq 1$ and the assertion is true if we replace r by $r - 1$.

Let $\{U_1', \ldots, U_r'\}$ be a finite subsystem of pairwise distinct members of τ and let U_j' be a minimal member of this system, i.e., $U_i' \subset U_j'$ implies $U_i' = U_j'$ and thus $i = j$. Without loss of generality we may suppose that $j = r$. By the induction hypothesis there are members $V_1', \ldots, V_{r-1}'$ of Ω such that $V_i' \cap X = U_i'$ for $i = 1, \ldots, r-1$ and $U_i' \subset U_{i'}'$ implies $V_i' \subset V_{i'}'$ for $i, i' \in \{1, \ldots, r-1\}$. Choose $V \in \Omega$ such that $V \cap X = U_r'$. By 5.8 and 5.4 this is possible. Let $A = \{i \in \{1, \ldots, r-1\}; U_r' \subset U_i'\}$ and let

$$V' = \begin{cases} V \cap \bigcap_{i \in A} V_i' & \text{if } A \neq \emptyset \\ V & \text{if } A = \emptyset. \end{cases}$$

Let V_r' be the component of V' containing U_r'. Then by [6] 3.5 V_r' is a $\mathrm{mod}(X, A)$ k-regular domain, i.e., $V_r' \in \Omega$. Moreover $V_r' \cap X = U_r'$ and the condition $U_i' \subset U_{i'}' \Longrightarrow V_i' \subset V_{i'}'$ holds for all $i, i' \in \{1, \ldots, r\}$.

The proof of the theorem is complete.

Now we are going to formulate the converse of Theorem 5.3.

First we recall a definition from [6].

5.23. DEFINITION. Let R be a T_2-space and (Y, B) a compact pair in R. A k-regular domain $V \operatorname{mod}(Y, B)$ is said to be a *subdividing domain* of (Y, B) if the two components of $V \setminus Y$ are contained in the same component of $R \setminus Y$.

In connection with this definition we recall a theorem.

5.24. THEOREM. *Let R and (Y, B) be the same as in* 5.23. *If at least one* $\operatorname{mod}(Y, B)$ *k-regular domain is a subdividing domain of (Y, B) then each* $\operatorname{mod}(Y, B)$ *k-regular domain has this property* (see [6] Theorem 4.1).

Next we give the definition of the bounded and closed k-manifold.

5.25. DEFINITION. Let R and (Y, B) be the same as in 5.23. (Y, B) is said to be a *bounded* (respectively *closed*) *k-manifold* if its k-regular domains are subdividing (non subdividing) domains of (Y, B).

Also, we recall the definition of the uniform decomposition of R^{n+1} (see [10] 6.25).

5.26. DEFINITION. We say that a compact pair (Z, C) in R^{n+1} *decomposes uniformly the space* R^{n+1} if for any two distinct points b and d of $R^{n+1} \setminus Z$ belonging to the same component of $R^{n+1} \setminus C$, b and d belong to the same component of $R^{n+1} \setminus Z$.

Now we can state the following complement of Theorem 5.3.

5.27. THEOREM. *Let (X, A) be an (n,p)-manifold in R^{n+1}. Then (X, A) is a bounded k-manifold in R^{n+1}.*

PROOF. Let V be a $\operatorname{mod}(X, A)$ k-regular domain in R^{n+1} and let q_1 and q_2 be points of $V \setminus X$ belonging to distinct components of $V \setminus X$. Since $V \cap \cap A = \emptyset$ it follows that q_1 and q_2 belong to the same component of $R^{n+1} \setminus A$.

According to Theorem 2.4 $X \setminus A$ is a quasiregular domain of (X, A) (cf. 2.1) and thus for the inclusion $i \colon (X, \emptyset) \subset \big(X, X \setminus (X \setminus A)\big) = (X, A)$ the induced $i_* \colon H_n(X) \to H_n(X, A)$ is trivial, i.e., $i_*\big(H_n(X)\big) = 0$. Consequently by [10] Theorem 6.28 (X, A) decomposes uniformly the space R^{n+1}. Hence q_1 and q_2 belong to the same component of $R^{n+1} \setminus X$ and thus the two components of $V \setminus X$ are contained in the same component of $R^{n+1} \setminus X$. V is a subdividing domain of (X, A) and according to 5.25 (X, A) is a bounded k-manifold in R^{n+1} indeed.

The converse of 5.27 is true as well.

5.28. THEOREM. *Let (X, A) be a bounded k-manifold in R^{n+1}. Then (X, A) is an (n,p)-manifold.*

We prepare the proof by two remarks and a lemma.

5.29. REMARK. Let S and S' be open balls in R^{n+1} such that $S' \subset S$. Then $\widetilde{H}_{n-1}\left(\overline{S} \setminus S'\right) = 0$.

Indeed, the inclusion $i: \overline{S} \setminus S \subset \overline{S} \setminus S'$ is clearly a homotopy equivalence and thus the induced $i_*: H_{n-1}(\overline{S} \setminus S) \to H_{n-1}(\overline{S} \setminus S')$ maps $\widetilde{H}_{n-1}(\overline{S} \setminus S)$ isomorphically onto $\widetilde{H}_{n-1}(\overline{S} \setminus S')$ (see [11] Theorem I. 11.3 p. 29). However for the n-sphere $\overline{S} \setminus S'$ we have $\widetilde{H}_{n-1}(\overline{S} \setminus S) = 0$ (see [11] Theorem I.16.6 p. 46) and thus $\widetilde{H}_{n-1}(\overline{S} \setminus S') = 0$ as required.

5.30. REMARK. Let (X, A) be a k-manifold in R^{n+1}. Then by [9] 2.4(b), [9] 2.5 and [9] 2.6, $X \setminus A$ is clearly a locally connected subset of X.

5.31. LEMMA. *Let (X, A) be a bounded k-manifold in R^{n+1}. Let S be an open ball in R^{n+1} containing X and let $q \in X \setminus A$. Let V be a* mod(X, A) *k-regular domain in R^{n+1} such that $q \in V \subset S$ and let S' be an open ball in R^{n+1} with the property $q \in S' \subset V$. Let U be the component of $S' \cap X$ containing q. U is a domain in $X \setminus A$* (see 5.30). *Let V' be a domain in S' regularly intersecting the pair (X, A) in U. There clearly exists such a domain V'. Let $B = \overline{S} \setminus V'$ and $Y = B \cup X$. Then*

$$H_n(Y) \approx Z_p \oplus Z_p$$

(where $Z_p \oplus Z_p$ is the external direct sum of Z_p and Z_p),

$$H_n(X, X \setminus U) \approx H_n(Y, B) \approx Z_p$$

and the homomorphism $k_*: H_n(Y) \to H_n(Y, B)$ induced by the inclusion $k: (Y, \emptyset) \subset (Y, B)$ is an epimorphism.

PROOF. Since the inclusion $i: (X, X \setminus U) \subset (Y, B)$ is a relative homeomorphism it follows that

$$H_n(X, X \setminus U) \approx H_n(Y, B). \tag{45}$$

V' is contained in the mod(X, A) k-regular domain V and thus by [6] 3.5, V' is a mod(X, A) k-regular domain as well. Hence $V' \setminus X$ consists of two components, moreover $V' \cup X \subset \overline{S}$ and thus by

$$R^{n+1} \setminus Y = R^{n+1} \setminus (B \cup X) = \big(R^{n+1} \setminus (\overline{S} \setminus V')\big) \cap (R^{n+1} \setminus X) =$$

$$\big((R^{n+1} \setminus \overline{S}) \cup V'\big) \cap (R^{n+1} \setminus X) = (R^{n+1} \setminus \overline{S}) \cup (V' \setminus X)$$

$R^{n+1} \setminus Y$ has three components: the two components of $V' \setminus X$ and $R^{n+1} \setminus \overline{S}$. Now by the Decomposition Theorem (see [10] 6.5) we have $r_p\big(H_n(Y)\big) = 2$, where $r_p\big(H_n(Y)\big)$ is the p-rank of $H_n(Y)$ (cf. [10] 6.4 and [10] 5.17) and thus one has

$$H_n(Y) \approx Z_p \oplus Z_p. \tag{46}$$

Since $V' \subset \overline{S}$ it follows that $R^{n+1} \setminus B$ has two components namely $R^{n+1} \setminus \overline{S}$ and V'. Thus by the Decomposition Theorem we have $r_p\big(H_n(B)\big) = 1$ consequently

$$H_n(B) \approx Z_p. \tag{47}$$

Let $B_1 = \overline{S} \setminus S'$ and $Y_1 = B_1 \cup U$. B_1 is clearly compact. We show that Y_1 is compact as well.

Indeed U is a component of $S' \cap X$ and thus it is closed in $S' \cap X$. Hence $\overline{U} \cap (S' \cap X) = U$. On the other hand $\overline{U} \subset X \subset \overline{S}$ and thus $\overline{U} \cap S' = U$. Consequently $\overline{U} \subset (X \setminus S') \cup U \subset \big(\overline{S} \setminus S'\big) \cup U = B_1 \cup U = Y_1$ and this yields $Y_1 = \overline{U} \cup B_1$. Thus the union Y_1 of the compact sets $\overline{U}$ and B_1 is compact as required.

Now consider the segment

$$\widetilde{H}_{n-1}(B_1) \longleftarrow H_n(Y_1, B_1) \xleftarrow{k_{1*}} H_n(Y_1)$$

of the reduced homology sequence of the compact pair (Y_1, B_1), where the homomorphism k_{1*} is induced by the inclusion $k_1: (Y_1, \emptyset) \subset (Y_1, B_1)$. By 5.29 we have $\widetilde{H}_{n-1}(B_1) = \widetilde{H}_{n-1}\big(\overline{S} \setminus S'\big) = 0$ and thus by the exactness of the sequence in question we can conclude that k_{1*} is an epimorphism.

Next consider the commutative diagram

$$\begin{array}{ccc} H_n(Y_1) & \xrightarrow{\;i_{1*}\;} & H_n(Y) \\ {\scriptstyle k_{1*}}\big\downarrow & & \big\downarrow{\scriptstyle k_*} \\ H_n(Y_1, B_1) & \xrightarrow[\;i_{2*}\;]{} & H_n(Y, B) \end{array}$$

where the homomorphisms i_{1*} and i_{2*} are induced by the inclusions $i_1: Y_1 \subset \subset Y$ and $i_2: (Y_1, B_1) \subset (Y, B)$ respectively. However by $Y_1 \setminus B_1 = Y \setminus B = = U$ the inclusion i_2 is a relative homeomorphism and thus i_{2*} is an isomorphism. Consequently $k_* i_{1*} = i_{2*} k_{1*}$ is an epimorphism and thus k_* is an epimorphism as well.

Consider now the segment

$$H_n(Y, B) \xleftarrow{k_*} H_n(Y) \xleftarrow{j_*} H_n(B) \longleftarrow H_{n+1}(Y, B)$$

of the exact homology sequence of the compact pair (Y, B), where the homomorphism j_* is induced by the inclusion $j: B \subset Y$. $X \setminus A$ is nowhere dense in R^{n+1} (see [6] 1.8) and thus by $U \subset X \setminus A$, $U = Y \setminus B$ is nowhere dense in R^{n+1} as well. Consequently $H_{n+1}(Y, B) = 0$ (see [10] 6.23) and thus by the exactness of the sequence in question j_* is a monomorphism and im $j_* = \ker k_*$. Hence taking also (46) and (47) into account we get

$$H_n(Y)/\ker k_* \approx Z_p$$

and since k_* is an epimorphism we can conclude

$$H_n(X, X \setminus U) \approx H_n(Y, B) \approx Z_p.$$

The proof of the Lemma is complete.

5.32. Now we are going to prove Theorem 5.28.

For the compact pair (X, A) according to [9] 2.6(a) condition 5.1(a) is satisfied.

We now prove 5.1(b).

Let $q \in X \setminus A$ and let U' be a neighbourhood of q in $X \setminus A$. Let S be an open ball in R^{n+1} containing X and let V be a mod(X, A) k-regular domain such that $q \in V \subset S$. By [9] 2.6 there exists such a domain V. Let S' be an open ball in V such that $q \in S' \cap X \subset U'$, and let U be the component of $S' \cap X$ containing q. By 5.30, U is a domain in $X \setminus A$.

We only need to prove that $(X, X \setminus U)$ is an (n, p)-cell, i.e., that $(X, X \setminus U)$ satisfies conditions [8] 1.2(a), (b), (c) and (d).

U is nonempty and connected. It is an open subset of the locally connected space $X \setminus A$. Hence U is locally connected as well. U is a subspace of R^{n+1}. So it has a countable base. Thus $(X, X \setminus U)$ satisfies condition [8] 1.2(a).

By Lemma 5.31 $(X, X \setminus U)$ satisfies condition [8] 1.2(b).

Let V' be a domain in S' regularly intersecting the pair (X, A) in U. There clearly exists such a domain V'. By $V' \subset S' \subset V$ and [6] 3.5, V' is a mod(X, A) k-regular domain and since (X, A) is a bounded k-manifold in R^{n+1} it follows that the components of $V' \setminus X$ belong to the same component of $R^{n+1} \setminus X$. Hence by [10] 6.26 the compact pair $(X, X \setminus U)$ decomposes uniformly the space R^{n+1} and thus by [10] Theorem 6.28 the homomorphism $i_*\colon H_n(X) \to H_n(X, X \setminus U)$ induced by the inclusion $i\colon (X, \emptyset) \subset (X, X \setminus U)$ is a 0-homomorphism, i.e., $i_*\big(H_n(X)\big) = 0$. Consequently the compact pair $(X, X \setminus U)$ satisfies condition [8] 1.2(c) too.

Now we are going to prove that $(X, X \setminus U)$ satisfies condition [8] 1.2(d).

Let U_1 be a domain in U and $q_1 \in U_1$. Let S_1 be an open ball in V' such that $q_1 \in S_1 \cap X \subset U_1$. Let U_2 be the component of $S_1 \cap X$ containing q_1. By 5.30 U_2 is a domain in $X \setminus A$. Thus we have only to prove that for the inclusion $j_2\colon (X, X \setminus U) \subset (X, X \setminus U_2)$ the induced $j_{2*}\colon H_n(X, X \setminus U) \to H_n(X, X \setminus U_2)$ is a monomorphism.

Let V_2 be a domain in S_1 regularly intersecting the compact pair (X, A) in U_2. By $U_2 \subset S_1$ there exists such a V_2 and by $V_2 \subset S_1 \subset V'$, V_2 is a mod(X, A) k-regular domain (see [6] 3.5). Let $B = \overline{S} \setminus V'$, $Y = B \cup X$, $B_2 = \overline{S} \setminus V_2$ and $Y_2 = B_2 \cup X$. Then

$$(48) \qquad Y \setminus B = U \quad \text{and} \quad Y_2 \setminus B_2 = U_2.$$

First we show that the compact pair (Y_2, Y) decomposes uniformly the space R^{n+1}.

Indeed, let z_1 and z_2 be points in $R^{n+1} \setminus Y_2$ belonging to the same component of $R^{n+1} \setminus Y$. Then by $R^{n+1} \setminus Y = (R^{n+1} \setminus \overline{S}) \cup (V' \setminus X)$ and $R^{n+1} \setminus Y_2 = (R^{n+1} \setminus \overline{S}) \cup (V_2 \setminus X)$ either both of the points z_1 and z_2 belong to $R^{n+1} \setminus \overline{S}$ or both of these points lie in the same component of $V' \setminus X$, i.e., in the same bank of V' (see 5.4). In this latter case $z_1, z_2 \in V_2$ is satisfied as well. In the first case z_1 and z_2 lie in $R^{n+1} \setminus \overline{S}$ and thus in the same component of $R^{n+1} \setminus Y_2$. Since the banks of V_2 are the intersections of V_2 and the banks of V' (see [6] 2.3) it follows that in the second case z_1 and z_2 belong to the same bank of V_2 i.e., to the same component of $V_2 \setminus X$ and thus to the same component of $R^{n+1} \setminus Y_2$.

The compact pair (Y_2, Y) decomposes uniformly the space R^{n+1} as required. Consequently by [10] Theorem 6.28 the homomorphism m_{2*}: $: H_n(Y_2) \to H_n(Y_2, Y)$ induced by the inclusion $m_2: (Y_2, \emptyset) \subset (Y_2, Y)$ is a 0-homomorphism, i.e.,

$$(49) \qquad m_{2*}\big(H_n(Y_2)\big) = 0.$$

Consider now the segment

$$H_n(Y_2, Y) \xleftarrow{m_{2*}} H_n(Y_2) \xleftarrow{i_{2*}} H_n(Y)$$

of the homology sequence of the compact pair (Y_2, Y), where $i_2: Y \subset Y_2$ is the inclusion map. By the exactness of this sequence and by (49) we have $\operatorname{im} i_{2*} = \ker m_{2*} = H_n(Y_2)$ and thus i_{2*} is an epimorphism. However the diagram

$$\begin{tikzcd}
H_n(Y) \arrow[r, "i_{2*}"] \arrow[d, "k_*"] & H_n(Y_2) \arrow[d, "k_{2*}"] \\
H_n(Y, B) \arrow[r, "r_*"] & H_n(Y_2, B_2)
\end{tikzcd}$$

where $k: (Y, \emptyset) \subset (Y, B)$, $k_2: (Y_2, \emptyset) \subset (Y_2, B_2)$ and $r: (Y, B) \subset (Y_2, B_2)$ are inclusions — is commutative and by Lemma 5.31 k_{2*} is an epimorphism as well. Consequently r_* is an epimorphism, too. However by Lemma 5.31 we have $H_n(Y, B) \approx H_n(Y_2, B_2) \approx Z_p$ and thus the epimorphic r_* is an isomorphism.

Consider now the diagram

$$\begin{array}{ccc} H_n(X, X\setminus U) & \xrightarrow{j_{2*}} & H_n(X, X\setminus U_2) \\ \downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle t_{2*}} \\ H_n(Y, B) & \xrightarrow{r_*} & H_n(Y_2, B_2) \end{array}$$

where j_{2*} and r_* are the same as before and $t\colon (X, X\setminus U) \subset (Y, B)$, $t_2\colon (X, X\setminus U_2) \subset (Y_2, B_2)$ are inclusions. t and t_2 are relative homeomorphisms and thus t_* and t_{2*} are isomorphisms. Consequently by the commutativity of this diagram j_{2*} is an isomorphism as well and thus it is a monomorphism as required.

$(X, X\setminus U)$ satisfies condition [8] 1.2(d), $(X, X\setminus U)$ is an (n,p)-cell and (X, A) is an (n,p)-manifold indeed.

The proof of Theorem 5.32 is complete.

6. Locally orientable n-pseudomanifolds with boundary

6.1. Let N_0 be the set of nonnegative integers, i.e., $N_0 = \mathbf{N} \cup \{0\}$. For $k \in \mathbf{N}$ let Z_k be the cyclic group of integers mod k and let $Z_0 = Z$, where Z is the group of integers.

Moreover for $r \in N_0$ let H^r be the Čech homology theory defined on the category of compact pairs with the coefficient group Z_r (see [9] 3.4).

6.2. PROPOSITION. *Let $n \in \mathbf{N}$. Let K be a triangulation* (cf. [1] p. 118) *of dimension n. Let $Y = \|K\|$, where $\|K\|$ is the body of K* (see [1] p. 136). *Let F be a closed subset of Y. Let $r \in N_0$ and $q > n$. Then*

$$H_q^r(Y, F) = 0.$$

The proposition is an immediate corollary of [11] Lemma XI. 6.2 (p. 311) and of the remark "Hence all statements and proofs through 6.7 hold with cohomology replaced by homology" (see [11] p. 320).

6.3. Let K be a triangulation situated in some Euclidean space R^s and let L be a closed subcomplex of K (see [1] p. 126). Let $q \in \|K\| \setminus \|L\|$ and let $O_K(q)$ be the set of all simplexes $T \in K$ with $q \in \overline{T}$, where $\overline{T}$ is the closure of T. $O_K(q)$ is clearly an open subcomplex of $K \setminus L$.

Suppose that K is an n-dimensional combinatorial pseudomanifold with boundary L and $L \neq \emptyset$ (see [2] pp. 72, 74). Then clearly, for each $q \in \|K\| \setminus \|L\|$ the open subcomplex $O_K(q)$ of $K \setminus L$ can be uniquely represented in the form

$$O_K(q) = E_{q,1} \cup \ldots \cup E_{q,t(q)},$$

where for $j = 1, \dots, t(q)$ $E_{q,j}$ is a closed subcomplex of $O_K(q)$, it is an n-pseudomanifold and for $j \neq j'$ $\big(j, j' \in \{1, \dots, t(q)\}\big)$ the dimension of the subcomplex $E_{q,j} \cap E_{q,j'}$ of K is less than $n-1$. Moreover an easy computation shows that for $r \in N_0$ we have

$$\Delta_r^n\big(O_K(q)\big) \approx \Delta_r^n(E_{q,1}) \oplus \dots \oplus \Delta_r^n\big(E_{q,t(q)}\big) \tag{50}$$

(cf. [2] p. 50). In particular

$$\Delta_2^n\big(O_k(q)\big) \approx \Delta_2^n(E_{q,1}) \oplus \dots \oplus \Delta_2^n\big(E_{q,t(q)}\big) \approx \underbrace{Z_2}_{1} \oplus \dots \oplus \underbrace{Z_2}_{t(q)} . \tag{51}$$

Consider now the group $H_n^r\big(\|K\|, \|L\|, q\big)$ (see [9] 3.5), i.e., the n-dimensional local Betti group of the compact pair $\big(\|K\|, \|L\|\big)$ at the point q with respect to the coefficient group Z_r. According to [9] 3.6 we find that

$$H_n^r\big(\|K\|, \|L\|, q\big) \approx \Delta_r^n\big(O_k(q)\big)$$

and thus $M = H_n^2\big(\|K\|, \|L\|, q\big)$ is a finite elementary 2-group (see [10] 5.10) and for its 2-rank $r_2(M)$ (cf. [10] 5.17) we have

$$r_2(M) = r_2\big(H_n^2(\|K\|, \|L\|, q)\big) = t(q).$$

6.4. Definition. Let (Y, B) be a topological (nonclosed) n-pseudomanifold with boundary (see [8] 3.1). Then by 6.3 for each $q \in Y \setminus B$ the group $H_n^2(Y, B, q)$ is clearly a finite elementary 2-group. We say that (Y, B) is *locally orientable* if for each $q \in Y \setminus B$

$$r_2\big(H_n^2(Y, B, q)\big) = r\big(H_n^0(Y, B, q)\big)$$

where $r\big(H_n^0(Y, B, q)\big)$ is the rank of the Z-module $H_n^0(Y, B, q)$.

It is easy to see that in case $n \leqq 2$ each (Y, B) is locally orientable.

6.5. Let K be a triangulation in some Euclidean space R^s and let L be a closed subcomplex of K. Suppose that K is an n-dimensional (combinatorial) pseudomanifold with boundary L and $L \neq \emptyset$. Let $q \in \|K\| \setminus \|L\|$ and let

$$O_k(q) = E_{q,1} \cup \dots \cup E_{q,t(q)}$$

be the same as in 6.3. Then by 6.3 (50) the condition

$$r_2\Big(\Delta_2^n\big(O_k(q)\big)\Big) = r\Big(\Delta_0^n\big(O_k(q)\big)\Big),$$

i.e.,

$$r_2\Big(H_n^2(\|K\|,\|L\|,q)\Big) = r\Big(H_n^0(\|K\|,\|L\|,q)\Big)$$

(cf. [9] 3.6) is clearly satisfied if and only if the pseudomanifolds $E_{q,j}$ are all orientable ones. Moreover in this case we have

$$H_n^0(\|K\|,\|L\|,q) \approx \Delta_0^n(O_K(q)) \approx \underbrace{Z}_{1} + \ldots + \underbrace{Z}_{t(q)} . \tag{52}$$

6.6. Let (Y, B) be a topological (nonclosed) n-pseudomanifold with boundary. Then by 6.5 and 6.4 we can state that if (Y, B) is orientable then it is locally orientable as well (see also [8] 3.2 and [9] 3.8).

Observe that in case $n = 1$, Y is a simple arc and B is the couple of its endpoints. In this case (Y, B) is clearly orientable. However, clearly there exist locally orientable and nonorientable topological nonclosed 2-pseudomanifolds with boundary.

6.7. DEFINITION. Let (Y, B) be a topological nonclosed n-pseudomanifold with boundary. We say that (Y, B) *is without* 2-*singular interior points* if for each $q \in Y \setminus B$

$$r_2\big(H_n^2(Y, B, q)\big) = 1$$

holds, i.e.,

$$H_n^2(Y, B, q) \approx Z_2.$$

6.8. Let K and L be the same as in 6.5. Then by 6.3 (51) and [9] 3.6 $(\|K\|, \|L\|)$ is clearly without 2-singular interior points if and only if for each $q \in \|K\| \setminus \|L\|$, $t(q) = 1$, i.e., $O_k(q) = E_{q,1}$ is an n-pseudomanifold.

6.9. DEFINITION. Let (Y, B) be a topological (nonclosed) locally orientable n-pseudomanifold with boundary. We say that (Y, B) *is without homologically singular interior points* if for each $q \in Y \setminus B$ the local homology group $H_n^0(Y, B, q)$ with respect to the coefficient group Z is a cyclic group (cf. [9] 3.8). According to 6.4 and 6.5 (52) for each $q \in Y \setminus B$ we have $H_n^0(Y, B, q) \approx Z$ in this case.

Observe that in case $n = 1$ each (Y, B) is without homologically singular interior points, but if $n = 2$ this is not true (see e.g., [3] I:2.8 Fig. a)).

6.10. 6.4, 6.5 (52), 6.7 and 6.9 show that any locally orientable topological nonclosed n-pseudomanifold with boundary (Y, B) is without homologically singular interior points if and only if (Y, B) is without 2-singular interior points.

However if (Y,B) is not locally orientable then it may happen that for each $q \in Y \setminus B$, $H_n^0(Y,B,q)$ is a cyclic group and in spite of that for some $q \in Y \setminus B$ we have $r_2\big(H_n^2(Y,B,q)\big) \neq 1$.

We now prepare the second fundamental theorem by a lemma and by a remark.

6.11. LEMMA. *Let p be a prime and $n \in \mathbf{N}$. Let (X,A) and (X',A') be compact pairs such that X' is a closed subspace of X and $X \setminus A = X' \setminus A'$. Suppose that (X',A') is an (n,p)-cell and $H_n^p(X) = 0$. Then (X,A) is an (n,p)-cell as well.*

PROOF. [8] 1.2(a) is satisfied for (X',A') and thus by $X \setminus A = X' \setminus A'$ it is satisfied for (X,A) as well.

By $X \setminus A = X' \setminus A'$ the inclusion $k\colon (X',A') \subset (X,A)$ is a relative homeomorphism and thus the induced $k_*\colon H_n^p(X',A') \to H_n^p(X,A)$ is an isomorphism (see [11] p. 266). Hence by $H_n^p(X',A') \approx Z_p$ (see 1.2(b)) we have

$$H_n^p(X,A) \approx H_n^p(X',A') \approx Z_p.$$

1.2(b) is satisfied for (X,A) as well.

Since by assumption $H_n^p(X) = 0$ it follows that 1.2(c) is satisfied for the compact pair (X,A), too.

Now let V be a domain in $X \setminus A = X' \setminus A'$ and let U be a nonempty open subset of V such that for the inclusion $j'\colon (X',A') \subset (X',X' \setminus U)$ the induced homomorphism $j'_*\colon H_n^p(X',A') \to H_n^p(X',X' \setminus U)$ is a monomorphism. Let $k_1\colon (X',X' \setminus U') \subset (X,X \setminus U)$ and $j\colon (X,A) \subset (X,X \setminus U)$ be inclusions. Then the diagram

$$\begin{tikzcd}
H_n^p(X',A') \arrow[r, "j'_*"] \arrow[d, "k_*"] & H_n^p(X',X' \setminus U) \arrow[d, "k_{1*}"] \\
H_n^p(X,A) \arrow[r, "j_*"'] & H_n^p(X,X \setminus U)
\end{tikzcd}$$

is clearly commutative, where j_* and k_{1*} are homomorphisms induced by the inclusions j and k_1 respectively. k_1 is a relative homeomorphism as well and thus k_{1*} is an isomorphism. Consequently $k_{1*}j'_* = j_*k_*$ is a monomorphism and since k_* is an isomorphism too it follows that j_* is a monomorphism.

(X,A) satisfies condition [8] 1.2(d) as well.

(X,A) is an (n,p)-cell as required.

6.12. REMARK. If p is a prime, $n \in \mathbf{N}$, (X,A) and (X',A') are homeomorphic compact pairs, i.e., there is a homeomorphism $\varphi\colon X \to X'$ such that $\varphi(A) = A'$ and (X,A) is an (n,p)-cell then so is clearly (X',A').

The second fundamental theorem can be formulated as follows

6.13. THEOREM. *Let $n \in \mathbf{N}$. Let (Y,B) be a topological nonclosed n-pseudomanifold with boundary and without 2-singular interior points. Then*

(Y,B) is an $(n,2)$-manifold. Moreover if (Y,B) is also locally orientable then for each prime p, (Y,B) is an (n,p)-manifold.

PROOF. By assumption $Y=\|K\|$, $B=\|L\|$, where K is a triangulation situated in some Euclidean space R^s and L is a closed subcomplex of K, moreover K is an n-dimensional combinatorial pseudomanifold with boundary L and $L\neq\emptyset$ (see [8] 3.1).

By [8] Theorem 3.4 (Y,B) is an $(n,2)$-cell. Thus by [8] 1.2(a) condition 1.1(a) of the present paper is satisfied. By 6.2 condition 1.1(c) is satisfied as well for the compact pair (Y,B). Moreover by [8] 3.2(13) for each prime p we have

$$H_n^p(\|K\|)=H_n^p(Y)=0. \tag{53}$$

Now for $q\in\|K\|\setminus\|L\|$ let $O_k(q)$ be the same as in 6.3. Let $O'_K(q)$ be the subcomplex of K consisting of all simplexes of $O_k(q)$ and of all faces of such simplexes. Let $B_K(q)=O'_K(q)\setminus O_K(q)$. $O'_K(q)$ and $B_K(q)$ are closed subcomplexes of K. Since $(\|K\|,\|L\|)$ is without 2-singular interior points it follows by 6.8 that $O_K(q)=E_{q,1}$ is an n-pseudomanifold and thus $O'_K(q)$ is a combinatorial n-pseudomanifold with boundary $B_K(q)$ and $B_K(q)\neq\emptyset$. Moreover if $(\|K\|,\|L\|)$ is locally orientable then by 6.8, 6.4 and 6.5 for each $q\in\|K\|\setminus\|L\|$, $O_K(q)$ is an orientable n-pseudomanifold and thus the combinatorial n-pseudomanifold $O'_K(q)$ is orientable as well. Consequently by [8] Theorem 3.4 for each $q\in\|K\|\setminus\|L\|$ $(\|O'_K(q)\|,\|B_K(q)\|)$ is an $(n,2)$-cell and if $(\|K\|,\|L\|)$ is locally orientable then for each $q\in\|K\|\setminus\|L\|$ and for each prime p, $(\|O'_K(q)\|,\|B_K(q)\|)$ is an (n,p)-cell.

Now for $m\in\mathbf{N}$ and $q\in\|K\|\setminus\|L\|$ let $\psi_{m,q}$ be the positive dilatation of R^s with the invariant point q and with ratio $\frac{1}{m}$, i.e., $\overrightarrow{q\psi_{m,q}(q')}=\frac{1}{m}\overrightarrow{qq'}$ holds for each $q'\in R^s$ and let $U_{m,q}=\psi_{m,q}(\|O'_K(q)\|\setminus\|B_K(q)\|)$. $U_{m,q}$ is clearly an open subset of $\|K\|\setminus\|L\|$ and $\{U_{m,q}\,;\,q\in\|K\|\setminus\|L\|,\,m\in\mathbf{N}\}$ is a base of $\|K\|\setminus\|L\|=Y\setminus B$.

Now since $(\overline{U_{m,q}},\overline{U_{m,q}}\setminus U_{m,q})$ is homeomorphic to the compact pair $(\|O'_K(q)\|,\|B_K(q)\|)$ it follows by 6.12, (53) and 6.11 that for each $q\in$ $\in\|K\|\setminus\|L\|$ and $m\in\mathbf{N}$, $(\|K\|,\|K\|\setminus U_{m,q})$ is an $(n,2)$-cell. Moreover if $(\|K\|,\|L\|)$ is locally orientable then for each prime p, $(\|K\|,\|K\|\setminus U_{m,q})$ is an (n,p)-cell. Hence if $p=2$ then condition 1.1(b) is also satisfied for the compact pair $(\|K\|,\|L\|)$ and if $(\|K\|,\|L\|)$ is locally orientable then 1.1(b) is satisfied for each prime p.

Consequently $(\|K\|,\|L\|)=(Y,B)$ is an $(n,2)$-manifold and if (Y,B) is locally orientable then (Y,B) is an (n,p)-manifold for each prime p.

The Theorem is proved.

6.14. REMARK. Let p be a prime and let $n \in \mathbf{N}$. Let (X, A) and (Y, B) be homeomorphic compact pairs, where (Y, B) is an (n, p)-manifold. Then (X, A) is clearly an (n, p)-manifold as well.

6.15. Now by 6.10, 6.14 and by Theorems 6.13 and 5.6 we can state the following theorem.

THEOREM. *Let $n \in \mathbf{N}$. Let (X, A) be a compact pair lying in R^{n+1} and homeomorphic to a locally orientable nonclosed topological n-pseudomanifold with boundary and without homologically singular interior points. Let K be a continuous closed path in $X \setminus A$ and let p be a prime with $p \geqq 3$. Under the circumstances K preserves its banks if and only if it preserves the orientation in the (n, p)-manifold (X, A).*

Our program is finished.

Now we make an additional remark.

We can raise a problem converse in a certain sense to the statement of 6.13. We also formulate a theorem related to this question without proof.

6.16. THEOREM. *Let K be a triangulation in some Euclidean space and let L be a closed subcomplex of K. Suppose that $\left(\|K\|, \|L\|\right)$ is an (n, p)-manifold for some prime p. Then $K \setminus L$ is an n-dimensional pseudomanifold and for each $q \in \|K\| \setminus \|L\|$, $O_k(q)$ is an n-pseudomanifold as well. If $p \neq 2$ then in addition the $O_K(q)$-s are orientable n-pseudomanifolds.*

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PRODUCT SETS IN THE PLANE, SETS OF THE FORM $A+B$ ON THE REAL LINE AND HAUSDORFF MEASURES

Z. BUCZOLICH* (Budapest)

Introduction. From Theorem 2 in [1] it follows that every $E \subset [0,1] \times \times [0,1] = I^2$ of positive two-dimensional Lebesgue measure contains a set of the form $A \times B$ such that $\lambda_1(A) > 0$, and B is non-empty perfect. (We denote by $\lambda_m(A)$ the m-dimensional outer Lebesgue measure of the set A.) M. Laczkovich asked whether the set B can be of positive Hausdorff dimension. We show that the answer is negative. Moreover, in Theorem 1 we prove that for any Hausdorff measure κ^ϕ there exists a set $E \subset I^2$ of full measure such that if $A \times B \subset E$, $\lambda_1(A) > 0$, and the sets A, B are measurable then B is of zero κ^ϕ measure. (For the definition of the κ^ϕ measure see the Preliminaries.)

Sets of the form $A + B = \{a + b : a \in A,\ b \in B\}$ can be regarded as projections of $A \times B$ onto the line $y = x$. G. Petruska asked the following question. Assume that $\lambda_1(B) > 0$ and the Hausdorff dimension of $A \cap I$ equals $d \in [0,1]$ for any interval $I \neq \emptyset$. Is it true that the Hausdorff dimension of the complement of $A + B$ cannot be bigger than $1 - d$? In Theorem 2 we give a negative answer to this question. In fact we show that there exist B of full λ_1-measure, and a set A which satisfies the above conditions with $d = 1$ but the Hausdorff dimension of the complement of $A + B$ also equals 1.

Preliminaries. Assume that $\phi : [0,+\infty) \to [0,+\infty)$ is monotone increasing, $\phi(t) > 0$ for $t > 0$, $\phi(0) = 0$, and ϕ is continuous from the right for all $t \geqq 0$. If $E \subset \bigcup_i U_i$ and $\operatorname{diam}(U_i) \leqq \delta$ $(i = 1, 2, \dots)$ then we say that the system $\{U_i\}$ is a δ-cover of E. For an $E \subset \mathbf{R}$ put

$$\kappa_\delta^\phi(E) = \inf \sum_{i=1}^{\infty} \phi(\operatorname{diam} U_i)$$

where the inf is taken for all δ-covers of E. Put $\kappa^\phi(E) = \sup_{\delta>0} \kappa_\delta^\phi(E)$. It is well-known [2, Theorem 27, p. 50] that the Hausdorff measure κ^ϕ is a regular metric measure. Furthermore all Borel sets are κ^ϕ-measurable, and each κ^ϕ-measurable set of finite κ^ϕ-measure contains an F_σ-set of the same measure.

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When $\phi(t) = t^s$ then we obtain the s-dimensional Hausdorff measure. In this case we shall use the notation κ^s instead of κ^{t^s}, or κ^{ϕ}.

We say [cf. 2, Section 8.3, p.121] that the open set condition (OSC) holds for the contractions $\{\psi_j\}_1^m$, if there exists a non-empty bounded open set V such that

$$\bigcup_{j=1}^{m} \psi_j(V) \subset V$$

with this union disjoint. We also need [2, Theorem 8.6, p.122]

THEOREM A. *Suppose that $m > 1$ and the OSC holds for the similitudes ψ_j with ratios r_j, $1 \leqq j \leqq m$. Then the associated compact invariant set E is an s-set where s is determined by $\sum_1^m r_j^s = 1$; that is $0 < \kappa^s(E) < \infty$.*

We refer to [2], especially to Section 8.3 of [2] for the terminology used in the formulation of Theorem A.

Main results. THEOREM 1. *Let κ^{ϕ} denote any Hausdorff measure.*

(i) *For every $\varepsilon > 0$ there exists a measurable set $E \subset I^2$ such that $\lambda_2(I^2 \setminus \setminus E) < \varepsilon$ and whenever $A \times B \subset E$ then either $\lambda_1(A) = 0$ or $\kappa^{\phi}(B) = 0$.*

(ii) *There exists a measurable set $H \subset I^2$ such that $\lambda_2(I^2 \setminus H) = 0$ and whenever $A \times B \subset H$ with Lebesgue measurable A and Borel measurable B then either $\lambda_1(A) = 0$ or $\kappa^{\phi}(B) = 0$.*

REMARK. In statement (ii) the assumption about the measurability of A and B cannot be dropped. An unpublished result of R. O. Davies implies that assuming the Continuum Hypothesis, from $H \subset I^2$, $\lambda_2(I^2 \setminus H) = 0$ it follows that there exists $A \times B \subset H$ with $\lambda_1(A) = \lambda_1(B) = 1$. The proof of this fact is not difficult and can be obtained by transfinite induction.

PROOF. For $\rho \in (0,1)$ we define measurable sets $H(\rho) \subset I^2$ such that $\lambda_2(H(\rho)) > \rho$ and if $A \times B \subset H(\rho)$, $\lambda_1(A) > 0$ then $\kappa^{\phi}(B) = 0$. This proves (i). Then by using the sets $H(\rho)$ we construct a set E of full λ_2-measure in I^2 which satisfies (ii).

If k and M are given positive integers we define the sets $H_k(M) \subset \subset [0,1]$ by splitting $[0,1]$ into M^{k-1} many subintervals of length $1/M^{k-1}$ and deleting from each of these subintervals their last open sub-subinterval of length $1/M^k$, that is,

$$H_k(M) = \bigcup_{m=0}^{M^{k-1}-1} \bigcup_{\ell=0}^{M-2} \left[\frac{m}{M^{k-1}} + \frac{\ell}{M^k}, \frac{m}{M^{k-1}} + \frac{\ell+1}{M^k}\right].$$

Obviously the sets $H_k(M)$ are closed and $\lambda_1(H_k(M)) = 1 - \frac{1}{M}$. Furthermore it is easy to check that the sets $H_k(M)$ also satisfy the following:

Independence-property.

$$\lambda_1\Big(\bigcap_{j=1}^{r} H_{k_j}(M)\Big) = \Big(1 - \frac{1}{M}\Big)^r$$

for $k_1 < k_2 < \ldots < k_r$.

Assume that a positive integer N is also given. We define the closed set $H(N,M) \subset I^2$ so that its horizontal section at the height $y \in \left(\frac{k-1}{N}, \frac{k}{N}\right)$, $k = 1, \ldots, N$ equals $H_k(M)$, that is,

$$H(N,M) = \bigcup_{k=1}^{N} \left(H_k(M) \times \left[\frac{k-1}{N}, \frac{k}{N}\right]\right).$$

It is obvious that

$$\lambda_2\big(H(N,M)\big) = 1 - \frac{1}{M}. \tag{1}$$

Assume that $\rho \in (0,1)$ is given. Choose a sequence $M_1, M_2, \ldots, M_m, \ldots$ such that

$$1 - \sum_{m=1}^{\infty} \frac{1}{M_m} > \rho. \tag{2}$$

For $m = 1, 2, \ldots$ choose an integer L_m such that

$$\frac{1}{m} > \left(1 - \frac{1}{M_m}\right)^{L_m+1}. \tag{3}$$

Since $\phi(0) = 0$ and ϕ is continuous from the right we can also find integers N_m for $m = 1, 2, \ldots$ such that

$$L_m \cdot \phi\left(\frac{1}{N_m}\right) < \frac{1}{m}. \tag{4}$$

Put $H^m = H(N_m, M_m)$. Since sets of the form $H(N,M)$ are closed the sets H^m, $m = 1, 2, \ldots$ are also closed. By (1) we have

$$\lambda_2(H^m) = 1 - \frac{1}{M_m}.$$

Let $H(\rho) = \bigcap_{m=1}^{\infty} H^m$. As the intersection of closed sets $H(\rho)$ is obviously closed,

$$\lambda_2\big(H(\rho)\big) = \lambda_2\big(I^2 \setminus (I^2 \setminus H(\rho))\big) \geqq$$

$$\geqq \lambda_2(I^2) - \sum_{m=1}^{\infty} \lambda_2(I^2 \setminus H^m) = 1 - \sum_{m=1}^{\infty} \frac{1}{M_m} > \rho.$$

In this paragraph we show that if $A \times B \subset H(\rho)$, $\lambda_1(A) > 0$ then $\kappa^{\phi}(B) = 0$. Assume that the integer m is fixed with $1/m < \lambda_1(A)$. From $A \times B \subset H(\rho)$ it follows that $A \times B \subset H^m$ for $m = 1, 2, \ldots$. Choose the numbers $k_1 < k_2 < \ldots < k_{r_m}$ such that $\left(\frac{k_j-1}{N_m}, \frac{k_j}{N_m}\right) \cap B \neq \emptyset$, $j = 1, 2, \ldots, r_m$. From the definition of the set H^m it follows that its horizontal section at any height $y \in \left(\frac{k_j-1}{n}, \frac{k_j}{n}\right)$ equals $H_{k_j}(M_m)$. Since $A \times B \subset H^m$ we obtain that $A \subset \bigcap_{j=1}^{r_m} H_{k_j}(M_m)$. The Independence Property of the sets H_{k_j} and (3) imply that

$$\left(1 - \frac{1}{M_m}\right)^{r_m} \geqq \lambda_1(A) > \frac{1}{m} > \left(1 - \frac{1}{M_m}\right)^{L_m+1}.$$

Thus $r_m \leqq L_m$ and hence

$$B \subset \bigcup_{j=1}^{r_m} \left[\frac{k_j - 1}{N_m}, \frac{k_j}{N_m}\right] \cup \bigcup_{k=0}^{N_m} \left\{\frac{k}{N_m}\right\}.$$

Using the intervals $\left[\frac{k_j-1}{N_m}, \frac{k_j}{N_m}\right]$ and the points $\frac{k}{N_m}$, we obtain a δ-cover $U_1, U_2, \ldots$ of B with $\delta = \frac{1}{N_m}$ such that

$$\sum_i \phi(\operatorname{diam} U_i) < r_m \phi(1/N_m) + (N_m + 1) \cdot 0 \leqq L_m \phi(1/N_m) < 1/m$$

where at the last step we used (4). Since the above estimates are valid for any m large enough, we proved that $\kappa^{\phi}(B) = 0$.

For $n = 2, 3, \ldots$ put $E_n = H\left(1 - \frac{1}{n}\right)$ and $E = \bigcup_{n=2}^{\infty} E_n$. It is clear that E is of full λ_2-measure in I^2. Assume that $A \times B \subset E$, $\lambda_1(A) > 0$, A is Lebesgue and B is Borel measurable. In fact we can also make the auxiliary assumption that A and B are closed since if $\lambda_1(A) > 0$ then one can choose a closed subset of A of positive λ_1-measure and the same is true about κ^{ϕ} [3, Theorem 27, p.50]. Choose a sequence of intervals I_n, $n = 1, 2, \ldots$, which consists of all open intervals with rational endpoints. Denote by G the union of those intervals I_n for which $\kappa^{\phi}(I_n \cap B) = 0$. Obviously $\kappa^{\phi}(G \cap B) = 0$. If $G = \mathbf{R}$ then $\kappa^{\phi}(B) = 0$ and that is what we want to verify. Assume for a contradiction that $G \neq \mathbf{R}$. Put $B' = \mathbf{R} \setminus G$. Obviously B' is closed. Assume that $x \in B'$, $a < x < b$ and choose an I_n such that $x \in I_n \subset (a, b)$. Then $0 < \kappa^{\phi}(I_n \cap B) \leqq \kappa^{\phi}\big((a, b) \cap B\big)$. This implies that any neighborhood of any $x \in B'$ contains points of B. Since B is closed we obtain that $B' \subset B$. Furthermore $\kappa^{\phi}\big((a,b)\cap B'\big) \geqq \kappa^{\phi}\big((a,b)\cap B\big) - \kappa^{\phi}\big((a,b)\cap B\cap \cap G\big) = \kappa^{\phi}\big((a,b)\cap B\big)$ thus we obtain that for any $x\in(a,b)\cap B'$ we have $\kappa^{\phi}\big((a,b)\cap B'\big) > 0$. This also implies that B' is perfect. By using λ_1 instead of κ^{ϕ} and a process similar to the previous one we can find a closed $A'\subset A$ such that $\lambda_1(A') > 0$, A' is perfect and if $(a,b)\cap A' \neq \emptyset$ then $\lambda_1\big((a,b)\cap \cap A'\big) > 0$.

Therefore $A'\times B' \subset A\times B\subset E$ and $A'\times B'$ is perfect. Put $C_n = E_n\cap \cap (A'\times B')$. Since $A'\times B'\subset E = \bigcup_{n=2}^{\infty} E_n$ we have $\bigcup_{n=2}^{\infty} C_n = A'\times B'$. By Baire's Category Theorem there exists an n and an open set U such that $U\cap(A'\times B')\neq\emptyset$ and C_n is dense in $U\cap(A'\times B')$. Recall that $E_n = H\big(1 - -\frac{1}{n}\big)$ is a closed set. Thus $C_n\subset A'\times B'$ is also closed and it is dense in $U\cap(A'\times B')$. Then $U\cap C_n = U\cap(A'\times B')$. Choose (a,b) and (c,d) such that $A'' = (a,b)\cap A'\neq\emptyset$, $(c,d)\cap B'\neq\emptyset$, $[a,b]\times[c,d]\subset U$. Put $B'' = [c,d]\cap \cap B'$. Then $A''\times B''\subset U\cap(A'\times B') = U\cap C_n\subset C_n\subset E_n = H\big(1-\frac{1}{n}\big)$, $\lambda_1(A'') > 0$, B'' is closed and $\kappa^{\phi}(B'') > 0$ a contradiction proving that E satisfies the conclusion of Theorem 1 in I^2.

THEOREM 2. *There exist $A, B\subset \mathbf{R}$ such that the Hausdorff dimension of $A\cap I$ equals 1 for any non-empty interval I, $\lambda_1(\mathbf{R}\setminus B) = 0$ and the Hausdorff dimension of $\mathbf{R}\setminus(A+B)$ also equals 1.*

We shall show that there exists $P\subset[0,1]$ such that the Hausdorff dimension of P equals 1 and $\lambda_1(P-P) = 0$. (We remark, in contrast, that the Cantor triadic set has Hausdorff dimension $\log 2/\log 3$ and $\lambda_1(C-C) > 0$.) First assuming the existence of P we prove our theorem. If the sequence $\{q_n\}$ contains all the rationals put $A = \bigcup_{n=1}^{\infty}(P+q_n)$. Then the Hausdorff dimension of A in any interval equals 1. Put $B' = P - A$. Then $B' = P - -\bigcup_{n=1}^{\infty}(P+q_n) = \bigcup_{n=1}^{\infty}\big((P-P)-q_n\big)$. Since $\lambda_1(P-P) = 0$ we have $\lambda_1(B') = 0$. Put $B = \mathbf{R}\setminus B'$. If $x\in(A+B)\cap P$ then there exists $a\in A$, $b\in B$ such that $x = a+b$. Since $x\in P$, we have $b = x-a\in P-A = B'$ contradicting $b\in\mathbf{R}\setminus B'$. Therefore $P\subset\mathbf{R}\setminus(A+B)$ and hence the Hausdorff dimension of the complement of $A+B$ equals 1.

We now turn to the definition of the set P. Put $P = \{x\in[0,1]$: the decimal expansion of $x = 0.a_1a_2a_3\ldots$ and $a_{2^n} = 1$ for $n = 1,2,3,\ldots\}$.

To compute the Hausdorff dimension of P we need the auxiliary sets $P_n = \{x\in[0,1]$: the decimal expansion of $x = 0.a_1a_2a_3\ldots$, $a_k = 0$ if $k = 1,2,\ldots,2^n-1$, and $a_{\ell\cdot 2^n} = 1$ for $\ell = 1,2,3,\ldots\}$. Put $V = (10^{-2^n}, 2\cdot \cdot 10^{-2^n})$. Define the linear mappings ψ_j so that

$$\psi_j(V) = \left(10^{-2^n} + \left(j+\frac{1}{10}\right)\cdot 10^{-2^{n+1}+1}, 10^{-2^n} + \left(j+\frac{2}{10}\right)10^{-2^{n+1}+1}\right),$$

that is

$$\psi_j(x) = 10^{-2^n} + (x - 10^{-2^n}) \cdot 10^{-2^n} + \left(j + \frac{1}{10}\right) 10^{-2^{n+1}+1}$$

for $j = 0, 1, \ldots, 10^{2^n-1} - 1$. Then the sets $\psi_j(V) \subset V$ are disjoint for $j \neq \neq j'$ and this implies that the system ψ_j satisfies the OSC. It is also easy to check that P_n is the associated compact invariant set for the system ψ_j for $j = 0, 1, \ldots, 10^{2^n-1} - 1$. The contraction ratio $r_j = 10^{-2^n}$ for $j = = 0, 1, \ldots, 10^{2^n-1} - 1$. By Theorem A the Hausdorff dimension, s, of the set P_n can be computed from

$$1 = \sum_{j=0}^{10^{2^n-1}-1} \left(10^{-2^n}\right)^s = 10^{2^n-1}\left(10^{-2^n}\right)^s,$$

that is $s = \frac{2^n-1}{2^n}$.

Define

$$v_n = 0.w_1w_2\ldots.$$

such that $w_{2^k} = 1$ if $k = 1, \ldots, n-1$ and $w_j = 0$ otherwise. Put $P_n' = P_n + + v_n$. Then it is easy to check that $P_n' \subset P$. Thus the Hausdorff dimension of P is at least the Hausdorff dimension of P_n' which is $\frac{2^n-1}{2^n}$ for $n = 1, 2, \ldots$. Since $P \subset [0,1]$ its Hausdorff dimension cannot exceed 1. Thus the Hausdorff dimension of P equals 1.

If $y \in P - P$ and the decimal expansion of y equals $b_0.b_1b_2b_3\ldots$ then the definition of P implies that $b_{2^n} = 0$, or $b_{2^n} = 9$ holds for $n = 1, 2, \ldots$.

It is easy to verify that if $H = \{x \in \mathbf{R} : x = a_0.a_1a_2a_3\ldots$, and there exists at least one integer n such that $a_{2^n} \notin \{0, 9\}\}$ then the set H is of full λ_1-measure. Since $P - P \subset \mathbf{R} \setminus H$ this implies that $\lambda_1(P - P) = 0$. This concludes the proof.

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TURÁN TYPE PROBLEMS ON MEAN CONVERGENCE. I (LAGRANGE TYPE INTERPOLATIONS)

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1. Introduction. Preliminary results

1.1. Throughout this paper $X = \{x_{k_n} = \cos\vartheta_{k_n}\}$ denotes an infinite triangular *interpolatory matrix in* $[-1,1]$, that means

(1.1)

$$x_{n+1,n} \equiv -1 \leqq x_{n,n} < x_{n-1,n} < \ldots < x_{2n} < x_{1n} \leqq x_{0n} \equiv 1, \quad n = 1,2,\ldots.$$

For $M \geqq 1$, fixed integer, we consider the unique interpolatory polynomials (cf. (1.4) and (1.6))

$$(1.2) \qquad I_{nM}(f,X,x) := \sum_{k=1}^{n} f(x_{k_n}) h_{0knM}(X,x), \quad n = 1,2,\ldots$$

for a continuous $f(x)$ in $[-1,1]$ ($f \in C$, shortly) and the unique *Hermite* interpolatory polynomials (cf. (1.4) and (1.6)) defined by

$$(1.3) \qquad \mathcal{I}_{nM}(f,X,x) := \sum_{t=0}^{M-1} \sum_{k=1}^{n} f^{(t)}(x_{k_n}) h_{tknM}(X,x)$$

($f^{(M-1)} \in C$), where $h_{tknM} \in \mathcal{P}_{Mn-1}$ (the set of polynomials of degree at most $Mn-1$; actually, $h_{tknM} \in \mathcal{P}_{Mn-1} \setminus \mathcal{P}_{Mn-2}$), satisfying

$$(1.4) \quad h^{(i)}_{tknM}(X, x_{\ell n}) = \delta_{ti}\delta_{k\ell}, \quad t,i = 0,1,\ldots,M-1, \quad k,\ell = 1,2,\ldots,n.$$

By definition (using here and later some obvious short notations), $I_{nM}(f,x)$, $\mathcal{I}_{nM}(f,x) \in \mathcal{P}_{Mn-1}$ and

$$\mathcal{I}_{nM}(P,x) \equiv P(x) \quad \text{for any} \quad P \in \mathcal{P}_{Mn-1}.$$

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The so called *truncated Hermite* interpolatory polynomials are defined by

$$(1.5)\qquad I_{nMr}(f,X,x):=\sum_{t=0}^{r}\sum_{k=1}^{n}f^{(t)}(x_{kn})h_{tknM}(X,x),\quad n=1,2,\dots,$$

where $0\leqq r\leqq M-1$, fixed, $f^{(r)}\in C$. Obviously $I_{nM0}=I_{nM}$ and $I_{n,M,M-1}=\mathcal{I}_{nM}$, i.e. $I_{nMr}\in\mathcal{P}_{Mn-1}$ generalizes both I_{nM} and $\mathcal{I}_{nM}$. They satisfy the interpolatory properties

$$(1.6)\qquad I_{nMr}^{(i)}(f,X,x_{kn})=f^{(i)}(x_{kn}),\quad 1\leqq k\leqq n,\quad 0\leqq i\leqq r,$$

(cf. (1.4)).

A special case of a recent result in J. Szabados [1, Theorem 1] states: With $\|g\|:=\max_{-1\leqq x\leqq 1}|g(x)|$, we have the following. *If*

$$\Lambda_{0nM}(X):=\|\lambda_{0nM}(X,x)\|:=\left\|\sum_{k=1}^{n}h_{0knM}(X,x)\right\|,$$

then for any fixed interpolatory X

$$(1.7)\qquad \Lambda_{0nm}(X)\geqq c\log n,\quad m\quad \textit{is odd.}$$

From this Faber-type result using the Banach–Steinhaus theorem, we obtain that

$$(1.8)\qquad \overline{\lim_{n\to\infty}}\|I_{nm}(f,X,x)\|=\infty,\quad f\in C\quad \textit{is properly chosen}$$

(m is odd, X is arbitrary, fixed). ($I_{n1}\equiv L_n$ is the classical Lagrange interpolation).

1.2. However for even values of M we can find "good" matrices. Namely, if $X^{(\alpha,\beta)}$, $\alpha,\beta>-1$, denotes the interpolatory matrix whose n-th row consists of the roots of the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ *then if* $s=$ $=2,4,6,\dots$, *fixed,*

$$(1.9)\qquad \lim_{n\to\infty}\|I_{ns}(f,X^{(\alpha,\beta)},x)-f(x)\|=0\quad\text{for all } f\in C$$

whenever

$$(1.10)\qquad A_s:=-\frac{1}{2}-\frac{2}{s}\leqq\alpha,\beta<-\frac{1}{2}+\frac{1}{s}\quad\text{and}\quad|\alpha-\beta|\leqq\frac{2}{s}$$

(cf. the works of R. Sakai and P. Vértesi [2/I, Theorem 2.1] and [3/IV, Part 5.3]; $I_{n2} \equiv H_n$ is the classical Hermite–Fejér interpolation).

The previous considerations motivate the name *"Lagrange type interpolation"* for I_{nm} and the notation L_{nm} (instead of I_{nm}) whenever m is *odd*. Our present paper deals with this Lagrange type cases (i.e. with L_{nm}, m odd). The *even* values of M (called *Hermite–Fejér type interpolations*) will be considered in the second part of this paper. So from now on m is a *fixed odd positive integer*.

1.3. Throughout this paper $d\alpha$ denotes a *measure* generated by the non-decreasing bounded function $\alpha(x)$ supported in $[-1,1]$ such that $\operatorname{supp}(d\alpha)$ (= the set of points of increase of $\alpha(x)$) is an infinite set. We suppose that $0 < \int_{-1}^{1} d\alpha < \infty$. $p_n(d\alpha, x)$ denotes the corresponding orthonormal polynomial of degree exactly n. Its roots are $\{x_{kn}(d\alpha)\}$, $k = 1, 2, \ldots, n$. If $X = \{x_{kn}(d\alpha)\}$, we use the notations $X(d\alpha)$, $H_{nm}(f, d\alpha, x)$, $L_n(f, d\alpha, x)$, etc.

If α is *absolutely continuous* then $\alpha'(x) = w(x)$ a.e. where $w(x)$ is called a *weight(function).* In this case we write $X(w)$, $p_n(w,x)$, etc. Generally, $u(x)$ is a weight (on $[-1,1]$) iff $u(x) \geqq 0$ and $0 < \int_{-1}^{1} u(x)dx < \infty$; the above defined w obviously satisfies these conditions.

Let u be a weight. We define

$$\|f(x)\|_{p,u} = \|f\|_{p,u} := \begin{cases} \left(\int_{-1}^{1} |f(x)|^p u(x) dx \right)^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{-1 \leqq x \leqq 1} |f(x)|, & p = \infty. \end{cases}$$

Note that $\|\cdot\|_{p,u}$ is not a norm if $0 < p < 1$. With the above notations let

$$L^p_u := \left\{ f;\ \|f\|_{p,u} < \infty \right\}, \quad 0 < p \leqq \infty.$$

If $u(x) \equiv 1$, we write $\|f\|_p$ and L^p. Finally, if $f \in C$, $\|f\| := \|f\|_\infty$ ($= \|f\|_{\infty,u}$ for any u).

In 1937, P. Erdős and P. Turán [4] proved, in contrast to the Faber theorem (cf. (1.7) and (1.8) for $m = 1$), as follows.

THEOREM 1.1. *Let w be a fixed weight. Then*

$$\lim_{n\to\infty} \int_{-1}^{1} |L_n(f,w,x) - f(x)|^2 w(x)dx = 0 \quad \textit{for all } f \in C. \tag{1.11}$$

A natural question arises (cf. P. Turán [5, Problem IX]):
Do there exist a weight w and an $f \in C$ such that for every $p > 2$ the relation

$$\overline{\lim_{n\to\infty}} \int_{-1}^{1} |L_n(f,w,x) - f(x)|^p w(x)dx = \infty \tag{1.12}$$

holds?

In 1985, P. Nevai [8] improving his former result (cf. [6, Theorem 15, p. 180]) proved as follows.

THEOREM 1.2. *Let $\alpha \in S$ (= Szegő class, i.e. $\log \alpha'(x)/\sqrt{1-x^2} \in L^1$), $1 \leqq p_0 < \infty$ and $u(\geqq 0) \in L^1$. Suppose that*

$$I(\alpha', p, u) := \left\| 1/\sqrt{\alpha'\sqrt{1-x^2}} \right\|_{p,u} = \infty \quad \textit{for every} \quad p > p_0. \tag{1.13}$$

Then there exists an $f \in C$ such that

$$\overline{\lim_{n\to\infty}} \|L_n(f, d\alpha)\|_{p,u} = \infty \quad \textit{if} \quad p > p_0. \tag{1.14}$$

In 1991, combining some general properties of orthogonal polynomials with the investigation of the sum $\sum |x - x_{kn}(w)|\,|\ell_{kn}(w,x)|$ ($\ell_{kn} \equiv h_{0kn1}$ are the fundamental polynomials of Lagrange interpolation) Y. G. Shi [7, Theorem 4] proved the following general statement.

THEOREM 1.3. *Let u and w be two weight functions and $2 \leqq p_0 < \infty$. If*

$$\left\| 1/\sqrt{w\sqrt{1-x^2}} \right\|_{p,u} = \infty \quad \textit{for every} \quad p > p_0, \tag{1.15}$$

then there exists an $f \in C$ such that

$$\overline{\lim_{n\to\infty}} \|L_n(f, w)\|_{p,u} = \infty \quad \textit{if} \quad p > p_0. \tag{1.16}$$

(Notice that we do not suppose that $w \in S$.) The above theorems may serve as solutions for the Turán problem (cf. 2.2.1–2.2.4, especially 2.2.3).

2. New results

2.1. In our paper G. Mastroianni, P. Vértesi [10] we generalized Theorem 1.3. The aim of this work is to consider the analogous results for the process L_{nm}. (Again, m is odd.)

From now on $w \in \mathcal{J}$ (or $w \in \mathcal{J}(\alpha,\beta)$) means that $w(x) = (1-x)^{\alpha} \cdot \cdot (1+x)^{\beta}$, $\alpha,\beta > -1$. $w \in \mathcal{J}_M$ (or $w \in \mathcal{J}_M(\alpha,\beta)$ if, moreover, $\alpha,\beta > C_M := := -1/2 - 1/M$ (cf. (1.10)). These weights w will generally be denoted by v.

First we quote a result corresponding to the Erdős–Turán theorem (cf. Theorem 1.1). By a rather special case of P. Vértesi, Y. Xu [9, Theorem 2.1, (ii)] namely taking $r = \lambda = 0$ we get as follows.

Let $v \in \mathcal{J}_m$ be fixed. Then, if $p = 2/m$, we have

$$\lim_{n\to\infty} \int_{-1}^{1} |L_{nm}(f,v,x) - f(x)|^p \, v(x)dx = 0 \quad \text{for all } f \in C. \tag{2.1}$$

(As we mentioned, $m = 1,3,5,\dots$, fixed.)

The previous theorem has been obtained by verifying the *sufficient* condition of (2.1), namely the relation $(1-x^2)^{-\frac{pm}{4}} v(x)^{1-\frac{pm}{2}} \in L^1$ which, if $p = = 2/m$, turns into $1/\sqrt{1-x^2} \in L^1$. However, if $p = \frac{2}{m}(1+2\varepsilon)$, $\varepsilon > 0$, fixed, the condition becomes $(1-x^2)^{-1/2-\varepsilon} v(x)^{-2\varepsilon} \in L^1$, which certainly does not hold if $v \in \mathcal{J}_m(\gamma,\gamma)$ and $\gamma \geqq 1/(2\varepsilon)$, say.

The above argument suggests that for the process $L_{nm}(f, d\alpha)$ the critical exponent is $2/m$. Combining the previous methods with new ideas we can further strengthen this hint (cf. 2.2.3 and 2.2.5).

THEOREM 2.1. *Let* $\operatorname{supp}(d\alpha) = [-1,1]$, $\alpha'(x) > 0$ *a.e. in* $[-1,1]$, $0 < < p_0 \leqq \infty$ *and u be a weight. If*

$$I_m(\alpha',p,u) := \left\| 1/\left(\alpha'\sqrt{1-x^2}\right)^{\frac{m}{2}} \right\|_{p,u} = \infty \quad \text{for every} \quad p_0 < p \leqq \infty \tag{2.2}$$

then there exists an $f \in C$ such that

$$\overline{\lim_{n\to\infty}} \, \|L_{nm}(f,d\alpha)\|_{p,u} = \infty \quad \text{whenever} \quad p_0 < p \leqq \infty. \tag{2.3}$$

Now let $u = \alpha'$. Then, by definition $I_m(\alpha', 2/m, \alpha') = \int v^{(-1/2,-1/2)} < < \infty$. On the other hand, by Theorem 2.1 we obtain

COROLLARY 2.2. *Let* $\operatorname{supp}(d\alpha) = [-1, 1]$, $\alpha'(x) > 0$ *a.e. in* $[-1, 1]$. *If*

$$I_m(\alpha', p, \alpha') = \infty \quad \textit{for every} \quad 2/m < p \leqq \infty \tag{2.4}$$

then there exists a function $f \in C$ *such that*

$$\varlimsup_{n \to \infty} \|L_{nm}(f, d\alpha)\|_{p,\alpha'} = \infty \quad \textit{for every} \quad 2/m < p \leqq \infty.$$

2.2. REMARKS. 1. When $m = 1$, Theorem 2.1 was proved in G. Mastroianni, P. Vértesi [10].

2. It is easy to see that no Jacobi weight v satisfies $I_m(v, p, v) = \infty$ if p is "close" to $2/m$. Indeed, if $p = \frac{2}{m}(1 + 2\varepsilon)$, then $I_m^p(v, p, v) = \int_{-1}^{1} (1 - x^2)^{-1/2-\varepsilon} v(x)^{-2\varepsilon} dx < \infty$ if $\varepsilon > 0$ is small enough. On the other hand, let $w_\delta(x) = \exp\left(-(1 - x^2)^{-\delta}\right)$, $\delta > 0$. Then simple calculation shows that

$$I_m\left(w_\delta, \frac{2}{m}(1 + 2\varepsilon), w_\delta\right) = \int_{-1}^{1} (1 - x^2)^{-1/2-\varepsilon} \exp\left(\frac{1}{1 - x^2}\right)^{2\varepsilon\delta} dx = \infty.$$

Note that $w_\delta \in S$ if $\delta < 1/2$ (cf. [6, Definition 17, p. 181]).

3. By Corollary 2.2 and the above considerations we get the following Turán-type theorems.

THEOREM 2.3. *Let* $\delta > 0$ *be fixed. Then there is an* $f \in C$ *such that for any* p, $2/m < p \leqq \infty$,

$$\varlimsup_{n \to \infty} \int_{-1}^{1} |L_{nm}(f, w_\delta, x) - f(x)|^p w_\delta(x) dx = \infty$$

(cf (1.12) if $m = 1$; for a positive result, see (2.1)).

4. For arbitrary fixed $v \in \mathcal{J}_m$ with $\alpha, \beta \leqq -1/2$ as it comes from [9, Theorem 2.1, (ii)], (2.1) holds true for *arbitrary* $0 < p < \infty$ (By the way, now $I_m(v, p, v) \leqq \int v dx < \infty$.)

5. Applying [9, Theorem 2.1, (ii)] with $r = \lambda = 0$ and Remark 3.2.5.3, we have

STATEMENT 2.4. *Let $v \in \mathcal{J}_m$, $u \in \mathcal{J}$ and $0 < p < \infty$, fixed. Then*

$$\lim_{n\to\infty} \int_{-1}^{1} |L_{nm}(f, v, x) - f(x)|^p u(x)dx = 0 \quad \textit{for all } f \in C$$

iff

$$\frac{u(x)}{\left(v(x)\sqrt{1-x^2}\right)^{\frac{pm}{2}}} \in L.$$

When $m = 1$, cf. P. Nevai [18, Theorem 6, p. 695].

6. The following *problem* is rather natural.

Prove relation (2.1) *for arbitrary* (or at least "many") *weight(s)* w (cf. (1.11) when $m = 1$).
This Erdős–Turán-type theorem would supplement Remark 2.2.3.

2.3. To get our statements we prove the fairly general Theorem 2.5. First we give a

DEFINITION. The interpolatory matrix X is regular with respect to the weight w (X is w-regular, shortly) iff for any fixed interval $I \subset [-1, 1]$ with $\int_I w > 0$, there exists a subinterval $J = J(I) = [a, b] \subset I$ satisfying $\int_a^b w > 0$, further each of the intervals $[a, a + \varepsilon]$ and $[b - \varepsilon, b]$ contains at least one root of

$$(\omega(x) = \omega_n(x) =)\, \omega_n(X, x) := c_n \prod_{k=1}^{n} (x - x_{k_n}), \quad c_n > 0 \tag{2.5}$$

if $n \geqq n_0(\varepsilon)$. (Here $\varepsilon > 0$ is arbitrary fixed.)

REMARK. If $X = \{x_{k_n}(d\alpha)\}$ and $\int_{-1}^{1} \alpha' > 0$ (so α' is a weight) then X is α'-regular (cf. G. Szegő [11, Theorem 6.1.1, p. 111] and [10, Parts 2.1–2.2]).

Now let

$$\|L_{nm}(X)\|_{p,u} := \sup_{\|f\| \leqq 1} \|L_{nm}(f, X, x)\|_{p,u}, \quad n \geqq 1. \tag{2.6}$$

If $\chi_S(x)$ denotes the characteristic function of a Lebesgue measurable set S ($S \in \mathcal{M}$ shortly), and CS stands for $[-1, 1] \setminus S$, our statement is as follows (cf. [10, Theorem 2.2]).

THEOREM 2.5. *Let u and w be two weights, X be a w-regular interpolatory matrix and $q_0>0$ be fixed. Then there exists an $\varepsilon>0$ such that if $R_n\in\mathcal{M}$, $|R_n|\leqq\varepsilon$, otherwise arbitrary, we have for every p with $q_0<p\leqq\infty$ the relation*

$$(2.7)\qquad \|\omega_n^m(X,x)\|_{p,u}\leqq c\|\chi_{CR_n}(x)\omega_n^m(X,x)\|_{1,w}\|L_{nm}(X)\|_{p,u},\quad n\geqq 1,$$

with a proper $c>0$ not depending on p.

2.4. Let $\omega_N(x)=(1-x)^r(1+x)^s p_n(d\alpha,x)$, where $0\leqq r,\ s\leqq 1$, fixed, $N=n+r+s$. Let $L_{nmrs}(f,d\alpha,x)$ stand for the Lagrange type interpolation based on the roots of $\omega_N(x)$. As an application of Theorem 2.5, we state a generalization of Theorem 2.1 (cf. [10, Theorem 2.4]).

THEOREM 2.6. *Let* $\operatorname{supp}(d\alpha)=[-1,1]$, $\alpha'>0$ *a.e. in* $[-1,1]$, $0<p_0\leqq\leqq\infty$ *and u be a weight. If*

$$(2.8)\qquad \int_{-1}^{1}\left|\frac{(1-x)^r(1+x)^s}{\left(\alpha'(x)\sqrt{1-x^2}\right)^{1/2}}\right|^{mp}u(x)dx=\infty\quad\textit{for every}\quad p_0<p\leqq\infty$$

then there exists an $f\in C$ such that

$$(2.9)\qquad \varlimsup_{n\to\infty}\|L_{nmrs}(f,d\alpha)\|_{p,u}=\infty\quad\textit{whenever}\quad p_0<p\leqq\infty.$$

2.5. For completeness we formulate an inverse of Theorem 2.5.

THEOREM 2.7. *There exist weight functions u and w and a w-regular interpolatory matrix X such that for every p, $0<p<\infty$,*

$$(2.10)\qquad \|\omega_n^m(X,x)\|_{1,w}\,\|L_{nm}(X)\|_{p,u}\leqq c_p\|\omega_n^m(X,x)\|_{p,u},\quad n\geqq 1.$$

Similar inverse theorems can be stated considering Theorems 2.1 and 2.2. Note that Statement 2.4 actually contains an inverse of Theorem 2.1. When $m=1$, (2.10) comes from Nevai [18, Theorem 6] (cf. Part 3.4). We omit the further details.

3. Proofs

3.1. PROOF OF THEOREM 2.5. By J. Szabados [1, (7) and (12)]

$$(3.1)\qquad h_{0k}(x)=\ell_k^m(x)B_k(x),\quad k=1,2,\dots,$$

where $\ell_k(x) = \omega(x)\{\omega'(x_k)(x - x_k)\}^{-1} \in \mathcal{P}_{n-1} \setminus \mathcal{P}_{n-2}$ are the fundamental polynomials of Lagrange interpolation, $B_k \in \mathcal{P}_{m-1}$, further — which is fundamental —

$$(3.2) \qquad B_k(x) \geqq c_0 \left(\frac{x - x_k}{x_k - x_{k\pm1}} \right)^{m-1} \geqq 0, \quad x \in \mathbf{R}, \quad 1 \leqq k \leqq n,$$

with one of the signs in $x_{k\pm1}$; $\mathbf{R} := (-\infty, \infty)$. (Relation (3.2) made possible to prove (1.7) and (1.8). Further, again by (3.2), one can prove the inequality

$$\lambda_{0nm}(x) \geqq c \log n, \quad x \notin H_n$$

where $|H_n| \leqq \varepsilon$, $\varepsilon > 0$ is arbitrary fixed (cf. P. Vértesi [13, Theorem 2.1]).)

Let $I_n := [x_{j(n),n}, x_{i(n),n}]$, $0 \leqq i < j \leqq n+1$, $n = 1, 2, \ldots$.

If

$$(3.3) \qquad E_{knm}(x) = E_k(x) := \frac{|\ell_k^m(x)|\,|x - x_k|^m}{|x_k - x_{k\pm1}|^{m-1}} =$$

$$= \left| \frac{\omega(x)}{\omega'(x_k)} \right|^m \frac{1}{|x_k - x_{k\pm1}|^{m-1}}, \quad 1 \leqq k \leqq n,$$

we prove

3.1.1. LEMMA 3.1. *Let* X *and* $\varepsilon > 0$ *be fixed. Then there exist sets* H_n, $H_n \subset I_n$, $|H_n| \leqq \varepsilon$, *such that for any* $n \geqq 1$

$$(3.4) \qquad s_{nm}(I_n, x) := \sum_{x_k \in I_n} E_k(x) \geqq \eta(\varepsilon) \quad \text{if} \quad x \in I_n \setminus H_n,$$

where $\eta(\varepsilon) = c\varepsilon^{2m}$, $c > 0$ *does not depend on* n *or* I_n.

REMARK. The investigation of $\sum |x - x_k||\ell_k(x)|$ $(m = 1)$ was initiated by Y. G. Shi [12] and [7]. When $m \geqq 1$, by (3.1), (3.2) and Lemma 3.1

$$(3.5) \qquad S_{nm}(I_n, x) := \sum_{x_k \in I_n} |x - x_k||h_{0k}(x)| \geqq c_0 s_{nm}(I_n, x) \geqq$$

$$\geqq c_0 \eta(\varepsilon) \quad \text{if} \quad x \in I_n \setminus H_n.$$

PROOF OF LEMMA 3.1. The proof is based on ideas developed and used by P. Erdős, P. Vértesi and later Y. G. Shi (cf. J. Szabados, P. Vértesi [14, Sections III/2, III/6.1] further Y. G. Shi [12], [7] and P. Vértesi [13]).

First we recall some notations. Let $J_k = J_{kn} = [x_{k+1,n}, x_{kn}]$, $k = 0, 1, \dots, n$, $n = 1, 2, \dots$. With $0 < q_k = q_k(J_k) \leqq \frac{1}{2}$ let

$$J_k(q_k) = [x_{k+1} + q_k|J_k|, \quad x_k - q_k|J_k|], \tag{3.6}$$

$$\overline{J}_k = \overline{J}_k(q_k) = J_k \setminus J_k(q_k). \tag{3.7}$$

Let $z_k = z_k(q_k)$ be defined by

$$(0 <)\, |\omega(z_k)| = \min_{x_k \in J_k(q_k)} |\omega(z)|, \quad 0 \leqq k \leqq n, \tag{3.8}$$

further let

$$|J_i, J_k| = \max\left(|x_{i+1} - x_k|, |x_{k+1} - x_i|\right), \quad 0 \leqq i, k \leqq n. \tag{3.9}$$

We construct the set H_n as a sum of subsets $G_k = G_{kn}$, $1 \leqq k \leqq n$.

(1) If $|J_0| \leqq \varepsilon$ then let $G_0 = J_0$. If this is not the case, then using (3.1), (3.2) and the estimation $\ell_1(x) \geqq 1$ for $x \geqq x_1$ we get

$$|\ell_1(x)(x - x_1)|^m |x_1 - x_{1\pm 1}|^{1-m} \geqq (\varepsilon/2)^m 2^{1-m}, \quad x \geqq x_1 + \varepsilon/2 \tag{3.10}$$

whence we get (3.4) on J_0 apart from a set G_0 of measure $\leqq \varepsilon/2$. In both cases $|G_0| \leqq \varepsilon$.

(2) A similar argument for J_n results G_n with $|G_n| \leqq \varepsilon$.

(3) For the remaining intervals we define $G_k = J_k$ if $|J_k| \leqq \varepsilon/n$. For these J_k, $\sum |G_k| < \varepsilon$.

(4) Let $T_k = [x_{k+2}, x_{k-1}]$. If $|J_k|/|T_k| < \varepsilon$, again let $G_k = J_k$. The total measure of these intervals G_k, by $\varepsilon^{-1} \sum |G_k| < \sum |T_k| \leqq 6$, is less than 6ε.

By (1)–(4) for the remaining intervals J_k

$$|J_k| \geqq \varepsilon \max(|J_{k+1}|, |J_{k-1}|), \quad |J_k| > \varepsilon/n. \tag{3.11}$$

Denote by Δ_n the corresponding sets of indices (i.e. $k \in \Delta_n$ iff (3.11) holds). We prove (cf. [13, Lemma 3.2]):

If $n \geqq 2$, $k \in \Delta_n$ and $1 \leqq r \leqq n - 1$, we have

$$E_k(x) + E_{k+1}(x) \geqq c(m)\varepsilon^{m-1} q_k^m |J_k| \left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m \quad \textit{if} \quad x \in J_r(q_r). \tag{3.12}$$

Indeed, by (3.3) and (3.8), when $x \in J_r(q_r)$,

$$E_i(x) = \left|\frac{\omega(x)}{\omega(z_r)}\right|^m E_i(z_r) \geqq E_i(z_r), \quad i = k, k+1,$$

whence, using (3.3), (3.11), (3.6) and (3.7)

$$E_k(x) + E_{k+1}(x) \geqq E_k(z_r) + E_{k+1}(z_r) \geqq$$

$$\geqq \left(\frac{\varepsilon}{|J_k|}\right)^{m-1} \left\{ |\ell_k(z_r)(z_r - x_k)|^m + |\ell_{k+1}(z_r)(z_r - x_{k+1})|^m \right\} =$$

$$= \left(\frac{\varepsilon}{|J_k|}\right)^{m-1} \left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m \left\{ |\ell_k(z_k)(z_k - x_k)|^m + |\ell_{k+1}(z_k)(z_k - x_{k+1})|^m \right\} \geqq$$

$$\geqq \left(\frac{\varepsilon}{|J_k|}\right)^{m-1} \left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m (q_k|J_k|)^m \{\ell_k^m(z_k) + \ell_{k+1}^m(z_k)\},$$

which by $\ell_k^m(z_k) + \ell_{k+1}^m(z_k) \geqq 2^{1-m}$ (cf. [14, Lemma 3.6, p. 76]) gives (3.12).

(5) We now continue the proof of Lemma 3.1. Let $q_k = q = \varepsilon$ $(k \in \Delta_n)$. The point x, the intervals J_k and $J_k(q)$, the index k will be called *exceptional iff* $(s_{nm}(x) =)s_n(x) = s_n(I_n, x) < \eta$ *for* $x \in J_k(q)$, $k \in \Delta_n$ (n is fixed). *We state:*

$$\sum_{k\in\delta_n} |G_k| := \sum_{k\in\delta_n} |J_k| := \mu_n \leqq \varepsilon \quad \textit{if} \quad n \geqq n_0 = n_0(\varepsilon), \tag{3.13}$$

where $\delta_n(\subset \Delta_n)$ collects the exceptional indices k of Δ_n. *To prove* (3.13) let $\eta = c_1\varepsilon^{2m}$ (c_1 will be determined later).

Let $u_{kn} \in J_k(q)$ $(k \in \delta_n)$ be an exceptional point of $J_k(q)$. If for a fixed $n \geqq n_0$ there exists an index $t(n) \in \delta_n$ with

$$s_n(u_{tn}) \geqq \varepsilon^{-1}\eta\mu_n, \tag{3.14}$$

by $\eta > s_n(u_{tn})$ we get (3.13) for this n. We prove (3.14) for arbitrary $n \geqq n_0$.

Indeed, let us suppose that for a certain $N \geqq n_0$ (3.14) does not hold for any t. Then by $s_N(u_{rN}) < \varepsilon^{-1}\eta\mu_N$, $r \in \delta_N$ we get

$$\sum_{r\in\delta_N} |J_r| s_N(u_r) < \varepsilon^{-1}\eta\mu_N^2 = c_1\varepsilon^{2m-1}\mu_N^2\,. \tag{3.15}$$

On the other hand, by (3.12) for *arbitrary* $n \geqq n_0$

$$|J_r| \sum_{x_k\in I_n} E_k(u_r) \geqq \frac{1}{2}|J_r| \sum_{k\in\delta_n} \{E_k(u_r) + E_{k+1}(u_r)\} \geqq$$

$$\geqq \frac{c(m)}{2}\varepsilon^{2m-1} \sum_{k\in\delta_n} |J_r|\,|J_k| \left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m, \quad r \in \delta_n$$

whence by $a+\frac{1}{a}\geqq 2$, (3.13) and $c_1:=c(m)/8$,

$$\sum_{r\in\delta_n}|J_r|s_n(u_r)\geqq\frac{1}{2}\sum_{r\in\delta_n}|J_r|\sum_{k\in\delta_n}\{E_k(u_r)+E_{k+1}(u_r)\}\geqq$$

$$\geqq\frac{c(m)}{2}\varepsilon^{2m-1}\sum_{r\in\delta_n}\sum_{k\in\delta_n}|J_r|\,|J_k|\left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m>$$

$$>\frac{c(m)}{4}\varepsilon^{2m}\sum_{r\in\delta_n}\sum_{\substack{k\geqq r\\ k\in\delta_n}}|J_r|\,|J_k|\left(\left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m+\left|\frac{\omega(z_k)}{\omega(z_r)}\right|^m\right)\geqq$$

$$\geqq\frac{c(m)}{2}\varepsilon^{2m-1}\sum_{r\in\delta_n}\sum_{\substack{k\geqq r\\ k\in\delta_n}}|J_r|\,|J_k|>\frac{c(m)}{4}\varepsilon^{2m-1}\sum_{r\in\delta_n}\sum_{k\in\delta_n}|J_r|\,|J_k|=$$

$$=\frac{c(m)}{4}\varepsilon^{2m-1}\mu_n^2=2c_1\varepsilon^{2m-1}\mu_n^2$$

which contradicts (3.15). That means, (3.13) must hold.

(6) Finally, if $k\in\Delta_n\setminus\delta_n$, by the definition of δ_n, $s_n(x)\geqq\eta$ whenever $x\in J_k(q)$. For these values of k, let $G_k=\overline{J_k}$. Here

$$\sum|G_k|<2q\sum_{k=0}^{n}|J_k|=2q=2\varepsilon.$$

Summarizing, if H_n is the sum of sets G_k defined in (1)–(6), then $|H_n|\leqq\leqq 11\varepsilon$ which essentially gives Lemma 3.1 at least when $n\geqq n_0$. If $n\leqq n_0$, we can argue as in [14, Section 2.6.6, p. 87]. We omit the details. □

REMARK. One can investigate, instead of $S_{nm}(I_n,x)$, the expressions

$$S_{tnM}(I_n,x):=\sum_{x_k\in I_n}|x-x_k||h_{tknM}(x)|,\quad M-t\quad \textit{is odd}, \tag{3.16}$$

for arbitrary fixed $M=1,2,\ldots$, $t=0,1,\ldots,M-1$ (cf. (1.3) and (1.4)).

It is made possible by the formulas

$$\begin{cases} h_{tknM}(x)=\ell_{kn}^M(x)(x-x_k)^tB_{tk}(x), & \text{where}\\ B_{tk}(x)\geqq c_t\left(\dfrac{x-x_k}{x_k-x_{k\pm1}}\right)^{M-t-1}\geqq 0 & \text{if } M-t \text{ is odd, } x\in\mathbf{R},\end{cases}\tag{3.17}$$

where $c_t > 0$ $(1 \leqq k \leqq n)$ (cf. [1, (12) and (7)]). Now let $E_{tknM}(x) := |x_k - x_{k\pm 1}|^t E_{knM}(x)$ (cf. (3.3)). By the relation

$$s_{tnM}(I_n, x) := \sum_{x_k \in I_n} E_{tk}(x) \geqq \frac{\eta_t}{n^t}, \tag{3.18}$$

$$x \in I_n \setminus H_n, \quad M = 1, 2, \dots, \quad t = 0, 1, \dots,$$

(where $\eta_t = c\varepsilon^{2M+t}$, $H_n \subset I_n$ and $|H_n| \leqq \varepsilon$) (3.17) and (3.18) estimate S_{tnM}, whenever $M - t$ is odd.

The proof of (3.18) is a word for word repetition of the previous one if as a first step we define $G_k = J_k$ whenever $|J_k| \leqq \varepsilon/n$ (the total measure of these G_k is $\leqq \varepsilon$). Then for the remaining J_k, $|E_{tk}(x)| \geqq (\varepsilon/n)^t E_k(x)$ (actually for any fixed real $t \geqq 0$). Further details are left to the reader.

Estimations of Λ_{tnm} and $\lambda_{tnm}(x)$ are in [1] and [13], respectively.

3.1.2. In his paper [7, Lemma 7], Y. G. Shi obtained as follows.

Let $J \subset [-1, 1]$ denote an arbitrary interval with $\int_J w(x)dx > 0$. For $\delta > 0$ suppose $|J| > \delta > |Z(J)|$. (If $S \in \mathcal{M}$, $Z(S) = Z_w(S) := \{x; x \in S$ and $w(x) = 0\}$). Then we have

$$\rho(J, \delta) := \inf_{\substack{\Omega \subset J \\ |\Omega| = \delta}} \int_\Omega w(x)dx > 0. \tag{3.19}$$

Using Lemma 3.1, (3.19) (and the notation of (3.2)) we get (cf. [10, Statement 2.1]).

LEMMA 3.2. *Let X be w-regular. Then for every fixed interval $I \subset [-1, 1]$ with $\int_I w > 0$ there exists an ε such that if $R_n \in \mathcal{M}$, $|R_n| \leqq \varepsilon$, otherwise arbitrary, we have*

$$D_{nm}(I) := \sum_{x_k \in I} \frac{1}{|\omega_n'(x_k)|^m |x_k - x_{k\pm 1}|^{m-1}} \geqq \frac{c}{\int_{J(I)\setminus R_n} |\omega_n(x)|^m w(x)dx}$$

for $n \geqq n_0(\varepsilon)$. Here $c = c(I, \varepsilon) > 0$.

The proof of this lemma is similar to the one in [10, Statement 2.1]. Let $J = J(I) = [a, b]$. By $\int_J w > 0$, $|J| - |Z(J)| := 5\varepsilon > 0$.

Let $x_{j(n)} \in [a, a + \varepsilon]$ and $x_{i(n)} \in [b - \varepsilon, b]$ (X is w-regular so these nodes exist if $n \geqq n_0(\varepsilon)$). Then if $I_n := [x_j, x_i] \subset J$ we have $|I_n| - |Z(J)| \geqq 3\varepsilon$. Applying Lemma 3.1 for I_n and ε defined above, then using (3.19) with $J(I)$

and $\delta = |Z(J)| + \varepsilon$, we get, considering that $I \supset J \supset J \setminus R_n \supset U_n := I_n \setminus \setminus H_n \setminus R_n$ and $|U_n| \geqq |Z(J)| + \varepsilon$, as follows.

$$D_{nm}(I) \geqq D_{nm}(I_n) = \frac{\int\limits_{J\setminus R_n} s_n(I_n, x)w(x)dx}{\int\limits_{J\setminus R_n} |\omega_n(x)|^m w(x)dx} \geqq \frac{\int\limits_{U_n} \dots}{\dots} \geqq$$

$$\geqq \frac{\eta(\varepsilon) \int\limits_{U_n} w(x)dx}{\dots} \geqq \frac{\eta(\varepsilon)\rho(J, |Z(J)| + \varepsilon)}{\int\limits_{J\setminus R_n} |\omega_n(x)|^m w(x)dx}$$

(where by $|J| - |Z(J)| - \varepsilon \geqq 4\varepsilon$ the numerator is greater than zero). $\square$

REMARK. Using (3.18), the previous argument yields that the estimation of Lemma 3.2 can be replaced by

$$(3.20) \qquad D_{tnM}(I) := \sum_{x_k \in I} \frac{1}{|\omega_n'(x_k)|^M |x_k - x_{k\pm 1}|^{M-t-1}} \geqq$$

$$\geqq \frac{c}{n^t \int\limits_{J(I)\setminus R_n} |\omega_n(x)|^M w(x)dx}, \quad n \geqq n_0(\varepsilon)$$

for $M = 1, 2, 3, \dots$ and $t = 0, 1, 2, \dots$. Here $c = c(I, \varepsilon)$.

3.1.3. Now we can complete the proof of Theorem 2.5 (cf. [6, Theorem 10.15], [7, Part 2.6] and [10, Theorem 2.2]). Let $Z = Z_w([-1, 1])$. Denote $3\delta := 2 - |Z|$. By $\int\limits_{-1}^{1} w > 0$, clearly $\delta > 0$. Fix any interval $\tau \subset [-1, 1]$ with $|\tau| = \delta$. Then we can define two intervals in $[-1, 1]$, η_1 and η_2, so that $|\eta_1 \cap \cap \eta_2| = 0$, $|\Omega| = |Z| + \delta$, where $\Omega = \eta_1 \cup \eta_2$, finally dist$(\tau, \Omega) \geqq \delta/2$. Relation $|\Omega| > |Z|$ involves $\int\limits_{\Omega} w > 0$ whence $\int\limits_{\eta_1} w > 0$, say.

Define $f_n \in C$ as a function with $\|f_n\| \leqq 1$ further satisfying

$$(3.21) \qquad f_n(x_k) = \chi_\Omega(x_{k_n}) \operatorname{sign}\{\omega'(x_k)(C - x_k)\}.$$

where C is the center of τ.

Now let $x \in \tau$. Using that $B_k(x) \geqq 0$ (cf. (3.2)), by (3.21), using relations $|x - x_k| \leqq 2$, formula (3.2) and Lemma 3.2, we get

$$(3.22) \qquad \left| \sum_{k=1}^{n} f_n(x_k) \frac{B_k(x)}{\{\omega_n'(x_k)(x - x_k)\}^m} \right| = \sum_{x_k \in \Omega} \frac{B_k(x)}{|\omega_n'(x_k)(x - x_k)|^m} \geqq$$

$$\geqq \frac{1}{2} \sum_{x_k \in \eta_1} \frac{B_k(x)}{|\omega_n'(x_k)|^m |x - x_k|^{m-1}} \geqq \frac{d}{\|\chi_{CR_n} \omega_n^m\|_{1,w}}, \quad x \in \tau,$$

where $d = d(\eta_1, \varepsilon) = c_0 c(\eta_1, \varepsilon)/2$. By (3.22) and (3.1)

$$\frac{d\chi_\tau(x)|\omega_n^m(x)|}{\|\chi_{CR_n} \omega_n^m\|_{1,w}} \leqq \chi_\tau(x) \left| \sum_{k=1}^n f_n(x_k) \left\{ \frac{\omega_n(x)}{(\omega_n'(x_k)(x - x_k))} \right\}^m B_k(x) \right| \leqq$$

$$\leqq \left| \sum_{k=1}^n f_n(x_k) h_{0k}(x) \right| \quad \text{for } \textit{arbitrary} \quad x \in [-1, 1],$$

whence

$$d \frac{\|\chi_\tau \omega_n^m\|_{p,u}}{\|\chi_{CR_n} \omega_n^m\|_{1,w}} \leqq \|L_{nm}(X)\|_{p,u}. \tag{3.23}$$

Divide the interval $[-1, 1]$ into t subintervals τ_i, $i = 1, 2, \ldots, t$, of measure δ according to Figure 1.

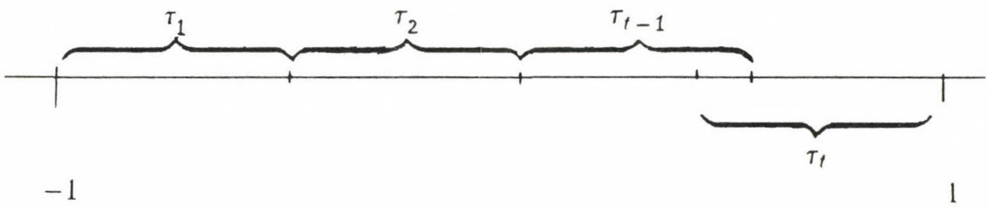

Fig. 1

Obviously $[2/\delta] \leqq t \leqq [2/\delta] + 1$. Choose $\varepsilon := \min_{1 \leqq i \leqq t} \varepsilon_i$ (cf. Lemma 3.2). Then (3.23) holds true for every τ_i with d_i and the same ε and R_n. So if $n \geqq n_0(\varepsilon)$, $0 < p < \infty$, by (3.23)

$$\int_{-1}^{1} |\omega_n|^{mp} u \leqq \sum_{i=1}^{t} \int_{-1}^{1} |\chi_{\tau_i} \omega_n|^{mp} u \leqq$$

$$\leqq t \left\{ \left(\max_i \frac{1}{d_i} \right) \|\chi_{CR_n} \omega_n^m\|_{1,w} \|L_{nm}(X)\|_{p,u} \right\}^p, \quad 0 < p < \infty,$$

whence taking p-th root (for $p \geqq q_0$) we get (2.7). If $p = \infty$, again by (3.23)

$$\|\omega_n^m\| = \max_{1 \leqq i \leqq t} \|\chi_{T_i} \omega_n^m\| \leqq \left(\max_i \frac{1}{d_i}\right) \|\chi_{CR_n} \omega_n^m\|_{1,w} \|L_{nm}(X)\|_{\infty,u}.$$

If $1 \leqq n \leqq n_0$, the statement is obvious whenever c is large enough. □

3.2. Proof of Theorem 2.1. We need the following general result of A. Máté, P. Nevai and V. Totik [15, Theorem 2, p. 317].

Let $0 < q \leqq \infty$. *Then there is a constant* $d > 0$ *with the property that for every measure with* $\operatorname{supp}(d\alpha) = [-1,1]$ *and* $\alpha'(x) > 0$ *a.e. in* $[-1,1]$, *the inequality*

$$\left\| g / \left(\alpha' \sqrt{1-x^2}\right)^{1/2} \right\|_q \leqq d \varliminf_{n \to \infty} \|g p_n(d\alpha)\|_q \tag{3.24}$$

holds for every Lebesgue measurable g.

Let $\omega_n(x) = p_n(d\alpha, x)$. Relation (3.24) yields (with $g^q = u$ and $q = mp$)

$$I_m(\alpha', p, u) \leqq d \varliminf_{n \to \infty} \|p_n^m(d\alpha)\|_{p,u}, \quad 0 < p \leqq \infty. \tag{3.25}$$

To estimate $\|\chi_{CR_n} p_n^m(d\alpha)\|_{1,w}$ we need another statement of [15].

Let $\operatorname{supp}(d\alpha) = [-1,1]$ *and* $\alpha' > 0$ *a.e. in* $[-1,1]$. *For a given real* $r > 0$ *and* $n \geqq 0$ *define the set* $B_{rn} = B_{rn}(d\alpha)$ *by*

$$B_{rn} := \{x; |p_n(d\alpha, x)| \Delta(x) \geqq r\} \tag{3.26}$$

where $\Delta(x) = \left(\alpha'(x)\sqrt{1-x^2}\right)^{1/2}$. *Then for every* $r > (2/\pi)^{1/2}$

$$\lim_{n \to \infty} |B_{rn}| = 0.$$

(see [15, Lemma]).

Now let in $[-1,1]$

$$\delta(x) = \begin{cases} \Delta(x) & \text{if} \quad \Delta(x) \leqq 1 \\ 1 & \text{if} \quad 1 < \Delta(x). \end{cases} \tag{3.27}$$

From $\alpha' > 0$ a.e. we conclude that $0 < \delta(x) \leqq 1$ a.e. whence δ^m is a weight. Further by definition $\delta(x) \leqq \Delta(x)$. Again by $\alpha' > 0$ a.e. we get that $X(d\alpha)$ is w-regular for an arbitrary weight w (cf. [11, Theorem 6.1.1, p. 111]), especially for δ^m. So by (3.26)

$$0 < \|\chi_{CB_{1n}} p_n^m(d\alpha)\|_{1,\delta^m} \leqq \|\chi_{CB_{1n}} \left(p_n(d\alpha)\Delta\right)^m\|_1 < 2, \quad n \geqq n_0. \tag{3.28}$$

By (2.7), (3.25), (3.26) and (3.28) we conclude the following important relation ·

$$\text{(3.29)} \qquad \left\| 1/\left(\alpha'\sqrt{1-x^2}\right)^{\frac{m}{2}} \right\|_{p,u} \leqq c \varliminf_{n\to\infty} \left\| L_{nm}(X(d\alpha)) \right\|_{p,u}, \quad q_0 \leqq p \leqq \infty.$$

3.2.1. Relations (3.29) and (2.2) yield that $\varlimsup_{n\to\infty} \left\| L_{nm}(X(d\alpha)) \right\|_{p,u} = \infty$ whence by the resonance principle applied for the Banach space C (with the usual norm) and the operator norms $\left\{ \|L_{nm}\|_{p,u} \right\}$ one would get (2.3) at least for any fixed $p \geqq 1$ (cf. K. Yosida [17, II. 1, Corollary 1, p. 69]). However if $0 < p < 1$, $\|\cdot\|_{p,u}$ is not a norm anymore. So to prove our statement in general, we define *prenorms* and *prenormed spaces* as follows.

DEFINITION. Let F be a real linear space.* If for every $f \in F$ we can define a real number $N(f) = N_F(f)$, the *prenorm* of f, such that with a fixed real $0 < A \leqq 1$

(i) $0 \leqq N(f) < \infty$ and $N(f) = 0$ iff $f = \mathbf{0}$ (zero-vector),
(ii) $N(f+g) \leqq N(f) + N(g)$ $f, g \in F$,
(iii) $N(cf) = |c|^A N(f)$, c real,

then F is a *prenormed* space. By (iii), $N(-f) = N(f)$ which yields that F is a *metric* space, too (with the metric $d(x,y) := N(x-y)$).

EXAMPLES. Every quasi-norm (cf. [17, I.2, p. 31]) so every norm, especially $\|\cdot\|_{p,u}$ $(p \geqq 1)$ is a prenorm with $A = 1$. Further, if $0 < p < 1$, $\|\cdot\|_{p,u}^p$ is a prenorm in L_u^p if $A = p$ ((i) and (iii) are obvious; for (ii), see [16, Ch. I, (9.13), p. 19]).

Now let B be a Banach space and F be a prenormed space endowed with $\|\cdot\|$ and $N(\cdot)$, respectively. If $\mathcal{M}$ is a linear continuous operator with $\mathcal{M} : B \to F$ (i.e. $\mathcal{M}(b) \in F$ if $b \in B$) we define $N(\mathcal{M}) = N_{F,B}(\mathcal{M})$ by

$$\text{(3.30)} \qquad N(\mathcal{M}) := \sup_{\substack{b\in B \\ \|b\|\leqq 1}} N(\mathcal{M}(b)).$$

By the usual argument one can see that

$$\text{(3.31)} \qquad N(\mathcal{M}) < \infty.$$

(Indeed, supposing the contrary, we can choose a sequence $\{b_n\} \subset B$, $\|b_n\| \leqq 1$, with $\lim_{n\to\infty} N(\mathcal{M}(b_n)) = \infty$. Then denoting $N(\mathcal{M}(b_n))$ by a_n, $b_n/a_n := \alpha_n \to \mathbf{0}$, whence, using the continuity of $\mathcal{M}$, $\mathcal{M}(\alpha_n) \to \mathbf{0}$, too. But (in

* We use the terminology of [17].

the metric space F) $\mathcal{M}(\alpha_n) \to \mathbf{0}$ iff $N(\mathcal{M}(\alpha_n)) \to 0$. On the other hand, by (iii) $N(\mathcal{M}(\alpha_n)) = N(a_n^{-1}\mathcal{M}(b_n)) = a_n^{1-A} \geqq 1$, a contradiction.)

Let $\{\mathcal{M}_n\}$ be a sequence of linear continuous operators with $\mathcal{M}_n : B \to F$ $(n = 1, 2, \dots)$. We claim

STATEMENT 3.3. *Let B, F and $\{\mathcal{M}_n : B \to F\}$ be defined as above. If*

$$\varlimsup_{n\to\infty} N(\mathcal{M}_n) = \infty \tag{3.32}$$

then with a proper $b \in B$

$$\varlimsup_{n\to\infty} N(\mathcal{M}_n(b)) = \infty\,. \tag{3.33}$$

The proof of this Banach–Steinhaus-type theorem goes along the original path (cf. [16, l.c.] for the classical version or [6, Part 10, Theorem 19, p. 182] for this prenorm form). For practical purpose we choose a proof based on the argument in [8, Lemma].

By (3.30) and (3.32) there exist $g_j \in B$ with $\|g_j\| \leqq 1$ and the subsequence $\{s_j\} \subset \mathbf{N}$ such that

$$\lim_{j\to\infty} N(\mathcal{M}_{s_j}(g_j)) = \infty. \tag{3.34}$$

Fix ℓ. If

$$\varlimsup_{n\to\infty} N(\mathcal{M}_n(g_\ell)) = \infty$$

then g_ℓ satisfy (3.32). If this is not the case we can suppose

$$\sup_n N(\mathcal{M}_n(g_\ell)) := a(g_\ell) < \infty, \quad \ell = 1, 2, \dots. \tag{3.35}$$

Now we can inductively define three sequences, $\{\varepsilon_k\}$, $\{b_k\} \subset \{g_j\}$ and $\{n_k\} \subset \{s_j\}$ such that $\varepsilon_1 = 1/2$, further for $k \geqq 1$

$$0 < \varepsilon_{k+1}^A \leqq \varepsilon_k^A/2, \tag{3.36}$$

$$\varepsilon_k^A N(\mathcal{M}_{n_k}(b_k)) \geqq k + 1 + \sum_{\ell=1}^{k-1} \varepsilon_\ell^A a(b_\ell), \tag{3.37}$$

$$2\varepsilon_{k+1}^A N(\mathcal{L}_{n_k}) \leqq 1. \tag{3.38}$$

Let

$$b := \sum_{k=1}^{\infty} \varepsilon_k b_k.$$

Then $b \in B$ and $\|b\| \leqq 1$. Further, by (ii) we can write

$$N(\mathcal{M}_{n_k}(b)) \geqq N(\varepsilon_k \mathcal{M}_{n_k}(b_k)) - N\left(\sum_{\ell=1}^{k-1} \varepsilon_\ell \mathcal{M}_{n_k}(b_\ell)\right) -$$

$$-N\left(\sum_{\ell=k+1}^{\infty} \varepsilon_\ell \mathcal{M}_{n_k}(b_\ell)\right) := S_1 - S_2 - S_3.$$

Here, by (iii), (3.35), $\|b_\ell\| \leqq 1$ and (3.36)

$$S_1 = \varepsilon_k^A N(\mathcal{M}_{n_k}(b_k)),$$

$$S_2 \leqq \sum_{\ell=1}^{k-1} \varepsilon_\ell^A N(\mathcal{M}_{n_k}(b_\ell)) \leqq \sum_{\ell=1}^{k-1} \varepsilon_\ell^A a(b_\ell),$$

$$S_3 \leqq \sum_{\ell=k+1}^{\infty} \varepsilon_\ell^A N(\mathcal{M}_{n_k}(b_\ell)) \leqq N(\mathcal{M}_{n_k}) \sum_{\ell=k+1}^{\infty} \varepsilon_\ell^A \leqq 2\varepsilon_{k+1}^A N(\mathcal{M}_{n_k}),$$

whence using (3.37) and (3.38) we get

$$N(\mathcal{M}_{n_k}(b)) \geqq k. \qquad \square$$

3.2.2. REMARKS. 1. Statement 3.3 remains true if we replace condition (iii) by

(iv) $N(cf) = \alpha(c)N(f)$, c real, $f \in F$

where $\alpha(x)$ is a continuous real function, $\alpha(x) > 0$ if $x \neq 0$, $\alpha(x)$ is even, $\alpha(x)$ is strictly increasing if $x \geqq 0$, $\alpha(0) = 0$, $\alpha(1) = 1$ and $\lim_{x\to\infty} \alpha(x) = \infty$. ($F$ remains linear!)

2. A *complete* prenormed space can be called pre-Banach space. Any Frechet space (cf. [17, p. 52]) especially any Banach space is a pre-Banach space. On the other hand, the spaces L_u^p with the prenorm $N(\cdot) = \|\cdot\|_{p,u}^p$, $0 < p < 1$, are pre-Banach spaces (cf. [16, Ch. I, (11.1), p. 26]), but not Banach spaces. One can check that Statement 3.3 remains true if B is replaced by a pre-Banach space. Again, (iii) can be replaced by (iv).

3. By Statements 3.3 and (3.29) one gets a "weak version" of Theorem 2.1. Namely

If for a fixed p, $0 < p \leqq \infty$, we have (2.2), *then for a proper $f \in C$*, $\overline{\lim\limits_{n\to\infty}} \|L_{nm}(f, d\alpha)\|_{p,u} = \infty$.

To verify this, we apply the cast $B = C$, $F = L_u^p$, $\mathcal{M}_n = L_{nm}$ and $N(\cdot\,\cdot) = \|\cdot\|_{p,u}^{A_p}$ where $A_p = \min(p, 1)$.

3.2.3. To complete proof of Theorem 2.1 we need a theorem which is a proper modification of the Lemma in [8] originally stated for Banach spaces.

STATEMENT 3.4. *Let F_p be prenormed spaces with prenorm N_p, $0 < p_0 < p \leqq \infty$, respectively. Suppose $F_p \subset F_r$ for $p > r(> p_0)$ moreover suppose, with $0 < \gamma < \infty$,*

$$N_r(f) \leqq \gamma N_p(f) \quad \text{if} \quad N_r(f) \geqq \gamma, \quad r < p \quad \text{and} \quad f \in F_p. \tag{3.39}$$

Let B be a Banach space with norm $\|\cdot\|$ and let $\{\mathcal{M}_n\}$ be a sequence of linear continuous operators with $\mathcal{M}_n : B \to F_\infty$ $(\subset F_p,\ p > p_0)$ such that

$$\overline{\lim_{n\to\infty}}\, N_p(\mathcal{M}_n) = \infty \quad \text{whenever} \quad p_0 < p \leqq \infty. \tag{3.40}$$

Then there exists a $b \in B$ such that

$$\overline{\lim_{n\to\infty}}\, N_p(\mathcal{M}_n(b)) = \infty \quad \text{for every} \quad p_0 < p \leqq \infty. \tag{3.41}$$

Before proving Statement 3.4 (cf. 3.2.4), we first use it to get Theorem 2.1. Let us define the prenormed spaces F_p by $F_p = L_u^p$ and $N_p(\cdot) = \|\cdot\|_{p,u}^{A_p}$ (as above, $A_p = \min(1,p)$). First we verify $L_u^p \subset L_u^r$, $0 < r < p \leqq \infty$ (cf. (3.43)), and relation (3.39), $0 < r < p \leqq \infty$ (see Lemma 3.5). Then, with the cast $B = C$ $(\subset L_u^\infty)$ and $\mathcal{M}_n = L_{nm}$, using relations (2.2) and (3.29), we find that formula (3.40) holds true whence by (3.41) we obtain (2.3).

Relation $L_u^p \subset L_u^r$ comes from the Hölder inequality. Indeed,* let $F = |f|^r$, $s = p/r$ and $t = s/(s-1)$. Then

$$\int |f|^r u = \int |F| u^{1/s+1/t} \leqq \left(\int |F|^s u\right)^{1/s} \left(\int u\right)^{1/t} = \tag{3.42}$$

$$= \left(\int |f|^p u\right)^{r/p} \left(\int u\right)^{1-\frac{r}{p}}$$

* Tacitly, we suppose that $p < \infty$. The case $p = \infty$ is left to the reader.

or

$$\|f\|_{r,u} \leqq \|f\|_{p,u} \, \|u\|_1^{\frac{1}{r}-\frac{1}{p}}, \quad 0 < r < p \leqq \infty . \tag{3.43}$$

By relations (3.42) and (3.43) it is easy to get

LEMMA 3.5. *For the prenormed spaces* L_u^p *with prenorms* $\|\cdot\|_{p,u}^{A_p}$, $0 < p \leqq \infty$, *we have relation* (3.39) *if* $\gamma = \max\left(1, \int_{-1}^{1} u(x)dx\right)$.

In the proof of (3.39) we distinguish three cases.

(1) Let $0 < r < p \leqq 1$. As in (3.42),

$$\gamma \leqq N_r(f) = \int |f|^r u \leqq \left(\int |f|^p u\right)^{r/p} \left(\int u\right)^{1-r/p} \leqq$$

$$\leqq (\ldots)^{r/p} \gamma^{1-r/p} \leqq (\ldots)^{r/p} \gamma := I .$$

Dividing by γ, we get $1 \leqq \int |f|^p u$ whence by $r/p < 1$ we can write $I = \left(\int |f|^p u\right)^{r/p} \gamma \leqq \gamma \int |f|^p u = \gamma N_p(f)$, as it was stated.

(2) $0 < r < 1 < p \leqq \infty$. By (3.42)

$$\gamma \leqq N_r(f) = \int |f|^r u \leqq \|f\|_{p,u}^r \gamma^{1-\frac{1}{p}} \leqq \|f\|_{p,u}^r \gamma := I$$

whence, as above, $1 \leqq \|f\|_{p,u}$, so by $r < 1$,

$$I \leqq \gamma \|f\|_{p,u} = \gamma N_p(f).$$

(3) $1 \leqq r < p \leqq \infty$. By (3.42)

$$N_r(f) = \|f\|_{r,u} \leqq \|f\|_{p,u} \gamma^{1/r-1/p} \leqq \|f\|_{p,u} \gamma^{1/r} \leqq \|f\|_{p,u} \gamma = \gamma N_p(f),$$

without using the restriction $\gamma \leqq N_r(f)$. □

3.2.4. *The proof of Statement* 3.4 is very similar to the one in [8].
Let $p_1 = p_0 + 1$. By (3.40) and Statement 3.3, there is a $b_1 \in B$ with $\|b_1\| \leqq 1$ such that

$$\overline{\lim_{n\to\infty}} \, N_{p_1}(\mathcal{M}_n(b_1)) = \infty . \tag{3.44}$$

Using (3.39) and (3.44) we obviously have

$$\overline{\lim_{n\to\infty}} N_{p_1+\delta}(\mathcal{M}_n(b_1)) = \infty \quad \text{if} \quad \delta \geqq 0. \tag{3.45}$$

So if $\overline{\lim\limits_{n\to\infty}} N_p(\mathcal{M}_n(b_1)) = \infty$ were true for every $p \in (p_0, p_1)$ we would be ready. If this is not the case then there must exist a $p_2 \in (p_0, p_0 + 1/2)$ such that $\overline{\lim\limits_{n\to\infty}} N_{p_2}(\mathcal{M}_n(b_1)) < \infty$. (Otherwise, if $\overline{\lim\limits_{n\to\infty}} N_{p_2}(\mathcal{M}_n(b_1)) = \infty$ were true for every $p_2 \geqq p_0 + 1/2$, by $\overline{\lim\limits_{n\to\infty}} N_{p_2+\delta}(\mathcal{M}_n(b_1)) = \infty$ (see the argument getting (3.45)) we would be ready.)

As above, there is a $b_2 \in B$ with $\|b_2\| \leqq 1$ such that $\overline{\lim\limits_{n\to\infty}} N_{p_2}(\mathcal{M}_n(b_2)) = \infty$ and there must exist a $p_3 \in (p_0, p_0 + 1/3)$ such that $\overline{\lim\limits_{n\to\infty}} N_{p_3}(\mathcal{M}_n(b_2)) < \infty$ (Otherwise, we would be ready.)

Continuing this process, either we find a $b \in B$ satisfying (3.41) or (and this case will be settled from now on) we define two infinite sequences $\{p_k\}$ and $\{b_k\}$ such that $b_k \in B$, $\|b_k\| \leqq 1$ and

$$p_0 < \ldots < p_3 < p_2 < p_1, \quad p_0 < p_k \leqq p_0 + 1/k,$$

$(k = 1, 2, \ldots)$, further they satisfy the relations

$$\overline{\lim_{n\to\infty}} N_{p_j}(\mathcal{M}_n(b_k)) = \infty \quad \text{if} \quad 1 \leqq j \leqq k \tag{3.46}$$

(cf. formula (3.45)) and

$$\overline{\lim_{n\to\infty}} N_{p_j}(\mathcal{M}_n(b_k)) < \infty \quad \text{if} \quad k < j. \tag{3.47}$$

(Indeed, by construction

$$\overline{\lim_{n\to\infty}} N_{p_{k+1}}(\mathcal{M}_n(b_k)) < \infty.$$

If $\overline{\lim\limits_{n\to\infty}} N_{p_{k+i_0}}(\mathcal{M}_n(b_2)) = \infty$ were true with a certain $i_0 \geqq 1$, then applying (3.45) with $\delta = p_{k+1} - p_{k+i_0}$, one would get $\overline{\lim\limits_{n\to\infty}} N_{p_{k+1}}(\mathcal{M}_n(b_2)) = \infty$, a contradiction.) Let

$$\sup N_{p_j}(\mathcal{M}_n(b_k)) := a_{k_j}, \quad 1 \leqq k < j.$$

By (3.47), every a_{k_j} is finite. Now, as in the proof of Statement 3.3, we can inductively define the sequences $\{\varepsilon_k\}$ and $\{n_k\} \subset \mathbf{N}$ such that $\varepsilon_1 = 1/2$, $0 < \varepsilon_{k+1} \leqq \varepsilon_k/2$, further for $k \geqq 1$

$$0 < \varepsilon_{k+1}^{\alpha_{k+1}} \leqq \varepsilon_k^{\alpha_k}/2,$$

$$\varepsilon_k^{\alpha_k} N_{p_k}(\mathcal{M}_{n_k}(b_k)) \geqq k+1+\sum_{\ell=1}^{k-1} \varepsilon_\ell^{\alpha_\ell} a_{\ell k},$$

$$2\varepsilon_{k+1}^{\alpha_{k+1}} N_{p_k}(\mathcal{M}_{n_k}) \leqq 1,$$

where α_k is the exponent A corresponding to the prenorm $N_{p_k}(\cdot)$ (cf. property (iii) in 3.2.1).

Let $b=\sum_{k=1}^{\infty} \varepsilon_k b_k$. Obviously $b \in B$ and $\|b\| \leqq 1$, further, as above, we get $N_{p_k}(\mathcal{M}_{n_k}(b)) \geqq k$ $(k \geqq 1)$. Now if $p>p_0$ is fixed (by $p_0<p_k \leqq p_0+\frac{1}{k}$) $p_k<p$ if $k \geqq k_0(p)$. Then by (3.39)

$$k \leqq N_{p_k}(\mathcal{M}_{n_k}(b))<N_p(\mathcal{M}_{n_k}(b)) \quad \text{whenever} \quad k \geqq k_0(p) \geqq \gamma. \qquad \square$$

3.2.5. REMARKS. 1. Although, as we mentioned, the proof is similar to the one in [8], we have verified Statement 3.4 in its general form, first of all because it is not a "l'art pour l'art" generalization but the one what we used (and will probably use).

2. Generalizations analogous to parts 3.2.2.1 and 3.2.2.2 can be considered. Details are left to the reader.

3. Let $\omega_n(x)=p_n(v,x)$ $(v \in \mathcal{J})$. By (3.25), (3.28), (2.7) and Statement 3.3, for a proper $f \in C$

$$\varlimsup_{n \to \infty} \|L_{nm}(f,v)\|_{p,u}=\infty$$

whenever $\left(v\sqrt{1-x^2}\right)^{-\frac{mp}{2}} u \notin L^1$. This fact has been used in Remark 2.2.3.

3.3. PROOF OF THEOREM 2.6. If we can prove the relation (cf. (3.29))

$$I:=\left(\int_{-1}^{1}\left|\frac{(1-x)^r(1+x)^s}{\left(\alpha'(x)\sqrt{1-x^2}\right)^{1/2}}\right|^{mp} u(x)dx\right)^{1/p} \leqq c \varliminf_{n \to \infty} \|L_{nmrs}(X(d\alpha))\|_{p,u} \tag{3.48}$$

$0<p \leqq \infty$, $n \geqq 1$ (where $\|\ldots\|_{p,u}$ corresponds to (2.6)), by Statement 3.4 we can complete the proof.

To get (3.48), first we remark that $|\omega_N(x)| \leqq 2|p_n(d\alpha,x)|$ whence by (3.28)

$$\|\chi_{CB_{1n}}\omega_N^m\|_{1,w}^2 \leqq 2^{2m+1}. \tag{3.49}$$

Further, by relation (3.24) (with $g(x) = \{(1-x)^r(1+x)^s\}^{mp}u(x)$ and $q = mp$)

$$I \leqq d \varliminf_{n\to\infty} \|\omega_N^m\|_{p,u}. \tag{3.50}$$

Now inequality (3.48) comes from (2.7), (3.49) and (3.50). □

3.4. PROOF OF THEOREM 2.7. Here is a good triple u, w, X. Let $u \in \in \mathcal{J}$ and $w = \left(v\sqrt{1-x^2}\right)^{\frac{m}{2}}$ where $v \in \mathcal{J}_m$. Then $w \in \mathcal{J}$, i.e. w is a weight further $X = X(v)$ is w-regular.

Let $\omega_n(X,x) = p_n(v,x)$. By [6, Theorem 33, p. 171]

$$|p_n(v,x)| \sim \frac{n|\vartheta - \vartheta_j|}{\left(v(x_j)\sqrt{1-x_j^2}\right)^{1/2}} \quad \text{uniformly in } n \text{ and } \quad x \in [-1,1], \tag{3.51}$$

where $x_j = x_{j(n),n}(v)$ is the (a) nearest root of $p_n(v)$ to x. By (3.51), $\|p_n^m(v)\|_{1,w} \leqq c$, further by [9, Theorem 2.1, (ii)] with $r = \lambda = 0$ and by Statement 3.3 we get $\varlimsup_{n\to\infty} \|L_{nm}(v)\|_{p,u} < \infty$ if $\left(v\sqrt{1-x^2}\right)^{-m/2} \in L_u^p$. However, by (3.51) the last condition is equivalent to $\varlimsup_{n\to\infty} \|p_n^m(v)\|_{p,u} < \infty$. □

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A GENERALIZATION OF J. ACZÉL'S INEQUALITY IN INNER PRODUCT SPACES

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1. Introduction

In 1956, J. Aczél has proved the following interesting inequality (see e.g. [13, p. 57]):

THEOREM A. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of real numbers such that*

$$a_1^2 - a_2^2 - \ldots - a_n^2 > 0 \quad or \quad b_1^2 - b_2^2 - \ldots - b_n^2 > 0.$$

Then

$$(1) \qquad (a_1^2 - a_2^2 - \ldots - a_n^2)(b_1^2 - b_2^2 - \ldots - b_n^2) \leqq \leqq (a_1b_1 - a_2b_2 - \ldots - a_nb_n)^2$$

with equality if and only if the sequences a and b are proportional.

Aczél's inequality was generalized by T. Popoviciu [15] (see also [13]):

$$(2) \qquad (a_1^p - a_2^p - \ldots - a_n^p)(b_1^p - b_2^p - \ldots - b_n^p) \leqq \leqq (a_1b_1 - a_2b_2 - \ldots - a_nb_n)^p.$$

The conditions

$$a_1^p - a_2^p - \ldots - a_n^p > 0 \quad \text{or} \quad b_1^p - b_2^p - \ldots b_n^p > 0 \quad \text{and} \quad p \geqq 1$$

given in [13] are not sufficient. This was pointed out by M. Bjelica [3] who also proved the following theorem:

THEOREM B. *If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are sequences of nonnegative real numbers such that*

$$(3) \qquad a_1^p - a_2^p - \ldots - a_n^p \geqq 0 \quad and \quad b_1^p - b_2^p - \ldots - b_n^p \geqq 0$$

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then for $0 < p \leqq 2$ *one has the inequality*

$$(a_1^p - a_2^p - \ldots - a_n^p)^{1/p}(b_1^p - b_2^p - \ldots - b_n^p)^{1/p} \leqq \leqq a_1b_1 - a_2b_2 - \ldots - a_nb_n \tag{4}$$

and conversely for $p < 0$.

For $p < 2$ equality holds in (4) iff

$$a = (a_1, 0, \ldots, 0) \quad \text{and} \quad b = (b_1, 0, \ldots, 0).$$

For $p = 2$ equality holds in (4) iff a and b are proportional.

Another result connected with Aczél's inequality was proved by R. Bellman in [2]. In this paper (see also [13]) the premise is sharper:

$$a_1^p - a_2^p - \ldots - a_n^p > 0 \quad \text{and} \quad b_1^p - b_2^p - \ldots - b_n^p > 0$$

which is weakened in the next theorem proved by M. Bjelica [3].

THEOREM C. *If* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are sequences of nonnegative real numbers which satisfy*

$$a_1^p - a_2^p - \ldots - a_n^p \geqq 0 \quad \textit{and} \quad b_1^p - b_2^p - \ldots - b_n^p \geqq 0,$$

then for $p > 1$ *one has the inequality*

$$(a_1^p - a_2^p - \ldots - a_n^p)^{1/p} + (b_1^p - b_2^p - \ldots - b_n^p)^{1/p} \leqq \leqq \left[(a_1 + b_1)^p - (a_2 + b_2)^p - \ldots - (a_n + b_n)^p\right]^{1/p}. \tag{5}$$

Equality holds in (5) *iff* a *and* b *are proportional.*

The main aim of this paper is to extend Aczél's inequality in inner product spaces. Some applications are also given.

2. The main results

We will start with the following theorem.

THEOREM 1. *Let* $(H; \langle , \rangle)$ *be an inner product space over the real or complex number field* K *and* a, b, c *real numbers satisfying the following condition*

$$a, c > 0 \quad \textit{and} \quad b^2 \geqq ac.$$

Then, for all $x, y \in H$ *so that*

$$a \geqq \|x\|^2 \quad \textit{or} \quad c \geqq \|y\|^2,$$

we have the inequality

$$(a - \|x\|^2)(c - \|y\|^2) \leqq \tag{6}$$
$$\leqq \min\Big\{(b \pm \operatorname{Re}\langle x,y\rangle)^2, \big(b \pm |\operatorname{Re}\langle x,y\rangle|\big)^2, (b \pm \operatorname{Im}\langle x,y\rangle)^2,$$
$$\big(b \pm |\operatorname{Im}\langle x,y\rangle|\big)^2, \big(b \pm |\langle x,y\rangle|\big)^2\Big\}.$$

PROOF. Suppose $a > \|x\|^2$ and consider the polynomial

$$P(t) := at^2 - 2bt + c, \quad t \in \mathbf{R}.$$

Since $a > 0$ and $b^2 \geqq ac$ it follows that there exists a $t_0 \in \mathbf{R}$ so that $P(t_0) = 0$.
Now, put

$$Q_1(t) := P(t) - \big(\|x\|^2 t^2 \mp 2\operatorname{Re}\langle x,y\rangle t + \|y\|^2\big), \quad t \in \mathbf{R}$$

and

$$\overline{Q}_1(t) := P(t) - \big(\|x\|^2 t^2 \mp 2|\operatorname{Re}\langle x,y\rangle| t + \|y\|^2\big), \quad t \in \mathbf{R}.$$

A simple calculation gives us

$$Q_1(t) = \big(a - \|x\|^2\big)t^2 - 2\big(b \pm \operatorname{Re}\langle x,y\rangle\big)t + \big(c - \|y\|^2\big), \quad t \in \mathbf{R}$$

and

$$\overline{Q}_1(t) = \big(a - \|x\|^2\big)t^2 - 2\big(b \pm |\operatorname{Re}\langle x,y\rangle|\big)t + \big(c - \|y\|^2\big), \quad t \in \mathbf{R}.$$

Now

$$Q_1(t_0) = -\big(\|x\|^2 t_0^2 \mp 2\operatorname{Re}\langle x,y\rangle t_0 + \|y\|^2\big) \leqq 0$$

because, by Schwarz's inequality in $(H; \langle , \rangle)$, one has

$$|\operatorname{Re}\langle x,y\rangle|^2 \leqq \|x\|^2\, \|y\|^2.$$

Thus

$$\|x\|^2 t^2 \mp 2\operatorname{Re}\langle x,y\rangle t + \|y\|^2 \geqq 0 \quad \text{for all} \quad t \in \mathbf{R},$$

and we conclude that Q_1 has at least one solution in $\mathbf{R}$, i.e.,

$$0 \leqq \frac{1}{4}\Delta_1 = \big(b \pm \operatorname{Re}\langle x,y\rangle\big)^2 - \big(a - \|x\|^2\big)\big(c - \|y\|^2\big).$$

Similarly, $\overline{Q}_1$ has at least one solution in $\mathbf{R}$ which is equivalent to

$$0 \leqq \frac{1}{4}\overline{\Delta}_1 = \big(b \pm |\operatorname{Re}\langle x,y\rangle|\big)^2 - \big(a - \|x\|^2\big)\big(c - \|y\|^2\big),$$

and the first part of (6) is proved.

The last part goes likewise, considering the polynomials

$$Q_2(t) := P(t) - \left(\|x\|^2 t^2 \mp 2\mathrm{Im}\langle x, y\rangle t + \|y\|^2 \right), \quad t \in \mathbf{R},$$
$$\overline{Q}_2(t) := P(t) - \left(\|x\|^2 t^2 \mp 2\mathrm{Im}|\langle x, y\rangle| t + \|y\|^2 \right), \quad t \in \mathbf{R}$$

and

$$Q_3(t) := P(t) - \left(\|x\|^2 t^2 \mp 2|\langle x, y\rangle| t + \|y\|^2 \right), \quad t \in \mathbf{R}.$$

The proof is thus finished.

REMARK 1. Let $(H; \langle , \rangle)$ be an inner product space over the real or complex number field and $M_1, M_2 \in \mathbf{R}$. Then for all $x, y \in H$ with

$$\|x\| \leqq |M_1| \quad \text{or} \quad \|y\| \leqq |M_2|$$

one has the inequality

$$\left(M_1^2 - \|x\|^2 \right) \left(M_2^2 - \|y\|^2 \right) \leqq \left(M_1 M_2 - \mathrm{Re}\langle x, y\rangle \right)^2. \tag{7}$$

This will be the corresponding Aczél inequality in inner product spaces. If $H = \mathbf{R}^{n-1}$ $(n \geqq 2)$ endowed with the usual inner product, we recapture from (7) the inequality (1).

Using the above theorem, we can give the following inverse of Schwarz's inequality in inner product spaces:

COROLLARY 1.1. *Suppose that a, b, c, x, y are as in Theorem 1. Then we have*

$$\begin{aligned} &0 \leqq \|x\|^2 \|y\|^2 - [\mathrm{Re}\langle x, y\rangle]^2 \leqq \\ &\leqq b^2 - ac + a\|y\|^2 + c\|x\|^2 + \min\left\{ \pm 2\mathrm{Re}\langle x, y\rangle b, \ \pm 2\mathrm{Re}|\langle x, y\rangle| b \right\}, \end{aligned} \tag{8}$$

$$\begin{aligned} &0 \leqq \|x\|^2 \|y\|^2 - [\mathrm{Im}\langle x, y\rangle]^2 \leqq \\ &\leqq b^2 - ac + a\|y\|^2 + c\|x\|^2 + \min\left\{ \pm 2\mathrm{Im}\langle x, y\rangle b, \ \pm 2|\mathrm{Im}\langle x, y\rangle| b \right\} \end{aligned}$$

and

$$0 \leqq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leqq b^2 - ac + a\|y\|^2 + c\|x\|^2 \pm 2|\langle x, y\rangle| b.$$

The proof follows from (6) by a simple computation.

COROLLARY 1.2. *Let H be as above and $M > 0$. Then for all $x, y \in H$ with*

$$\|x\| \leqq M \quad or \quad \|y\| \leqq M$$

one has the inequality

$$0 \leqq \|x\|^2\|y\|^2 - [\mathrm{Re}\langle x, y\rangle]^2 \leqq M^2 \min\{\, \|x-y\|^2,\ \|x+y\|^2\}.$$

The following theorem also holds.

THEOREM 2. *Let $(H; \langle,\rangle)$ be an inner product space and α, β, γ real numbers with*

$$\alpha, \gamma > 0 \quad and \quad \beta^2 \geqq \alpha\gamma.$$

Then for all $x, y \in H$ so that

$$\|x\| \leqq \alpha \quad or \quad \|y\| \leqq \gamma,$$

we have the inequality

$$\begin{aligned}(9) \qquad (\alpha - \|x\|)(\gamma - \|y\|) \leqq \min\Big\{ &\big(\beta \pm |\mathrm{Re}\langle x, y\rangle|^{1/2}\big)^2, \\ &\big(\beta \pm |\mathrm{Im}\langle x, y\rangle|^{1/2}\big)^2,\ \big(\beta \pm |\langle x, y\rangle|^{1/2}\big)^2\Big\}.\end{aligned}$$

PROOF. The argument is similar to that in the proof of the above theorem choosing the polynomials

$$\widetilde{Q}_1(t) := \widetilde{P}(t) - \big(\|x\|t^2 \mp 2|\mathrm{Re}\langle x, y\rangle|^{1/2}t + \|y\|\big), \quad t \in \mathbf{R},$$
$$\widetilde{Q}_2(t) := \widetilde{P}(t) - \big(\|x\|t^2 \mp 2|\mathrm{Im}\langle x, y\rangle|^{1/2}t + \|y\|\big), \quad t \in \mathbf{R}$$

and

$$\widetilde{Q}_3(t) := \widetilde{P}(t) - \big(\|x\|t^2 \mp 2|\langle x, y\rangle|^{1/2}t + \|y\|\big), \quad t \in \mathbf{R},$$

where

$$\widetilde{P}(t) = \alpha t^2 - 2\beta t + \gamma, \quad t \in \mathbf{R}.$$

We omit the details.

REMARK 2. Suppose that $\|x\| \leqq |M_1|$, $\|y\| \leqq |M_2|$ $(M_1, M_2 \in \mathbf{R})$. Then one has the inequality

$$(10) \qquad \big(|M_1| - \|x\|\big)^{1/2}\big(|M_2| - \|y\|\big)^{1/2} \leqq |M_1M_2|^{1/2} - |\langle x, y\rangle|^{1/2}.$$

If $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbf{R}^n$ are such that

$$a_1^2 - a_2^2 - \ldots - a_n^2 \geqq 0 \quad \text{and} \quad b_1^2 - b_2^2 - \ldots - b_n^2 \geqq 0$$

then

$$\left[|a_1| - (a_2^2 + \ldots + a_n^2)^{1/2}\right]^{1/2} \left[|b_1| - (b_2^2 + \ldots + b_n^2)^{1/2}\right]^{1/2} \leqq$$
$$\leqq |a_1 b_1|^{1/2} - |a_2 b_2 + \ldots + a_n b_n|^{1/2}.$$

This is a new inequality of Aczél type for real numbers (it is obvious by (10)).

COROLLARY 2.1. *let H, α, β, γ, x, y be as in Theorem 2. Then we have the following inverse of Schwarz's inequality:*

$$0 \leqq \|x\|\,\|y\| - |\mathrm{Re}\langle x, y\rangle| \leqq \beta^2 - \alpha\gamma + \alpha\|y\| + \gamma\|x\| \pm 2|\mathrm{Re}\langle x, y\rangle|^{1/2},$$
$$0 \leqq \|x\|\,\|y\| - |\mathrm{Im}\langle x, y\rangle| \leqq \beta^2 - \alpha\gamma + \alpha\|y\| + \gamma\|x\| \pm 2|\mathrm{Im}\langle x, y\rangle|^{1/2}$$

and

$$(11) \qquad 0 \leqq \|x\|\,\|y\| - |\langle x, y\rangle| \leqq \beta^2 - \alpha\gamma + \alpha\|y\| + \gamma\|x\| \pm 2|\langle x, y\rangle|^{1/2}.$$

The following corollary also holds.

COROLLARY 2.2. *Let H be as above and $M > 0$. Suppose $\|x\| \leqq M$ or $\|y\| \leqq M$. Then we have the inequality*

$$0 \leqq \|x\|\,\|y\| - |\langle x, y\rangle| \leqq M\left(\|x\| + \|y\| - 2|\langle x, y\rangle|^{1/2}\right).$$

The proof is obvious by inequality (11) for $\alpha = \beta = \gamma = M$.

For other inequalities in inner product spaces we refer to [4–14] where further references are given.

3. Applications

1. Let $x_i, y_i \in \mathbf{C}$ $(i = 1, \ldots, n)$ with $|x_i| \leqq M$ or $|y_i| \leqq M$ for all $i \in \{1, \ldots, n\}$. Then we have the following converse of Cauchy–Buniakowski–Schwarz inequality:

$$0 \leqq \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2 - \left|\sum_{i=1}^{n} \mathrm{Re}(x_i y_i)\right|^2 \leqq$$
$$\leqq n^2 M^2 \min\left\{\sum_{i=1}^{n} |x_i - y_i|^2,\ \sum_{i=1}^{n} |x_i + y_i|^2\right\}.$$

2. In the above assumptions for x_i, y_i we also have

$$0 \leqq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2} - \left|\sum_{i=1}^{n} x_i y_i\right| \leqq$$

$$\leqq nM \left[\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2} - 2\left|\sum_{i=1}^{n} x_i y_i\right|^{1/2}\right].$$

The proof is obvious by Corollaries 1.2 and 2.2. We omit the details.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbf{R} \cup \{\infty\}$. Denote $L^2(\Omega)$ the Hilbert space of all complex valued functions x defined on Ω and 2-integrable on Ω, i.e., $\int_\Omega |x(s)|^2 \, d\mu(s) < \infty$.

Suppose that $x, y \in L^2(\Omega)$ with

$$\int_\Omega |x(s)|^2 d\mu(s) < M^2 \quad \text{or} \quad \int_\Omega |y(s)|^2 d\mu(s) \leqq M^2.$$

Then we have the following converse of Cauchy–Buniakowski–Schwarz inequality for integrals:

$$0 \leqq \int_\Omega |x(s)|^2 \, d\mu(s) \int_\Omega |y(s)|^2 \, d\mu(s) - \left|\operatorname{Re} \int_\Omega x(s) y(s) \, d\mu(s)\right|^2 \leqq$$

$$\leqq M^2 \min \left\{ \int_\Omega |x(s) - y(s)|^2 \, d\mu(s), \int_\Omega |x(s) + y(s)|^2 \, d\mu(s) \right\}.$$

4. In the above assumptions for x, y in $L^2(\Omega)$ we also have

$$0 \leqq \left(\int_\Omega |x(s)|^2 \, d\mu(s)\right)^{1/2} \left(\int_\Omega |y(s)|^2 \, d\mu(s)\right)^{1/2} - \left|\int_\Omega x(s) y(s) \, d\mu(s)\right| \leqq$$

$$\leqq |M| \left[\left(\int_\Omega |x(s)|^2 \, d\mu(s)\right)^{1/2} + \left(\int_\Omega |y(s)|^2 \, d\mu(s)\right)^{1/2} - \right.$$

$$\left. -2\left|\int_\Omega x(s) y(s) \, d\mu(s)\right|^{1/2}\right].$$

The proofs are obvious by Corollaries 1.2 and 2.2; we omit the details.

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DISTAL COMPACT RIGHT TOPOLOGICAL GROUPS

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Let G be a compact group for which the left multiplications $t \mapsto st$ are all continuous; we call G a compact *left* topological group, and have the left translation flow (λ_G, G). W. Ruppert has studied the case where the compact left topological G is *equicontinuous*, i.e., (λ_G, G) is equicontinuous; one of his conclusions about equicontinuous groups is that the topological centre

$$\{\eta \in G \mid \nu \mapsto \nu\eta,\ G \to G, \text{ is continuous}\}$$

is closed in G. This implies that none of the non-trivial compact left topological groups coming from distal flows are equicontinuous (the "trivial" ones being the topological ones). In this paper, we study a class of compact left topological groups broader than that of the equicontinuous ones, a class that includes some of the compact left topological groups coming from distal flows. The class we consider consists of the *distal* compact left topological groups, the ones for which the flow (λ_G, G) is distal. In our analysis of this class, we present, among other things, conditions that are at the same time equivalent to the distality of G and analogous to conditions of Ruppert that are equivalent to the equicontinuity of G; we also deal with a significant aspect of the resulting theory of distal compact left topological groups: a process, which effectively terminates after one step for equicontinuous G, can be meaningfully repeated for distal (non-equicontinuous) G. We discuss some examples of distal G, and illustrate with one of them how the process just mentioned not only does not terminate after one step, but can be repeated indefinitely. We also present some non-distal G.

I. Preliminaries

A *flow* (S,X) consists of a compact Hausdorff space X and a group S with identity e acting on it on the left (as in [8]): each $s \in S$ determines a homeomorphism $x \mapsto sx$ of X and the conditions $ex = x$ and $s\big(t(x)\big) = (st)x$ for all $s, t \in S$ and $x \in X$ are satisfied. So, S determines a subgroup (denoted here also by S) of the semigroup X^X of all transformations of X. The closure S^- of S in X^X is a subsemigroup of X^X called the *enveloping semigroup* of the flow. With the relative topology from X^X, S^- is a compact

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right topological semigroup, i.e., for all $\eta \in S^-$, right multiplication by η, $\nu \mapsto \nu\eta$, $S^- \to S^-$, is continuous. The set

$$\Lambda(S^-) := \{\eta \in S^- \mid \nu \mapsto \eta\nu,\ \ S^- \to S^- \text{ is continuous}\}$$

is called the *topological centre* of S^-; here $S \subset \Lambda(S^-)$, so $\Lambda(S^-)$ is dense in S^-. The flow is *distal* if $s_\alpha x_1 \to x_0$ and $s_\alpha x_2 \to x_0$ for a net $\{s_\alpha\} \subset S$ and $x_0, x_1, x_2 \in X$ always implies $x_1 = x_2$. We quote a famous theorem of Ellis [3, or 4].

1. THEOREM. *A flow (S, X) is distal if and only if its enveloping semigroup S^- is a group (i.e., a subgroup of X^X).*

For a distal flow (S, X), S^- is called the *Ellis group* of the flow. There is a powerful structure theorem for compact right topological groups that come from topological dynamics like this [6,7]. A consequence of it is the existence of Haar measure μ for such groups; μ is a probability measure on the group and is invariant under all right translations and all continuous left translations. We do not need these results here.

Thus far we have considered flows (S, X) with S acting on the left ($x \mapsto sx$), $S^- \subset X^X$ right topological, etc. We shall need the "other-sided" notions as well (for example, the statement of Theorem 3 needs compact right topological groups and also compact left topological groups). To be specific, we may also consider flows (X, S) with S acting on the right, $x \mapsto xs$ (as in [4]); then $(xt)s = x(ts)$ and $S^- \subset X^X$ is a compact *left topological* semigroup, i.e., $\nu \mapsto \eta\nu$ is continuous for all $\eta \in S^-$; also S is contained in the topological centre

$$\mathfrak{R}(S^-) := \{\eta \in S^- \mid \nu \mapsto \nu\eta,\ \ S^- \to S^-, \text{ is continuous}\},$$

which is therefore dense in S^-. A flow (X, S) is distal if and only if $S^- \subset X^X$ is a compact left topological group.

We need to establish notation for Schreier's analysis of group extensions. Suppose that G_1 and G_2 are groups, the identity of each of them being denoted by e. Suppose that there is a mapping of G_2 into the automorphism group of G_1, that is, for every $t \in G_2$, there is an automorphism $s \mapsto t(s)$ of G_1 (acting on the left). Suppose also that there is a function $(t', t) \mapsto [t', t]$ from $G_2 \times G_2$ into G_1, so that all of the following conditions are satisfied:

$$e(s) = s \quad \text{and} \quad [t, e] = [e, t] = e \quad \text{for} \quad s \in G_1,\ t \in G_2, \text{ and also}$$

$$[t, t']tt'(s'') = t\big(t'(s'')\big)\,[t, t'] \quad \text{and} \quad [t, t'][tt', t''] = t([t', t''])[t, t't'']$$

$$\text{for} \quad t,\ t',\ t'' \in G_2 \quad \text{and} \quad s,\ s',\ s'' \in G_1.$$

Note that the function sending $t \in G_2$ to the automorphism $s \mapsto t(s)$ of G_1 is not necessarily a homomorphism. However, the hypotheses do ensure that the formula

$$(1) \qquad (s', t')(s, t) = \big(s't'(s)[t', t], t't\big)$$

defines a group operation on the set $G = G_1 \times G_2$, that $G_1 \times \{e\}$ is a normal subgroup isomorphic to G_1, and that $G/G_1 \cong G_2$. We say that G is an *extension of* G_1 *by* G_2. Further, if G is a group with a normal subgroup G_1 and if $G_2 := G/G_1$, then one can find functions satisfying the conditions above, so that G is canonically isomorphic (algebraically) to $G_1 \times G_2$ with group operation (1).

Because of the asymmetry of continuity in the definition of a compact right topological group, we also need the notation for the analogous situation, where $G = G_1 \times G_2$ is an *extension of* G_2 *by* G_1, i.e., G_2 is a normal subgroup of G and $G_1 \cong G/G_2$. In this case, we have automorphisms $t \mapsto (t)s$ of G_2 (acting on the *right*) and the multiplication formula is

$$(s', t')(s, t) = (s's, [s', s](t')st). \tag{2}$$

A situation in which left and right notations can be used at the same time is that of Zappa products [11, or 2], where a group G has subgroups G_1 and G_2 with

$$G = G_1G_2 = \{st \mid s \in G_1,\ \ t \in G_2\} \quad \text{and} \quad G_1 \cap G_2 = \{e\}.$$

Then G is (algebraically) isomorphic to $G_1 \times G_2$ with operation

$$(s', t')(s, t) = \big(s't'(s), (t')st\big);$$

here the functions $s \mapsto \mathcal{L}_{t'}(s) := t'(s)$ and $t' \mapsto (t')_s\mathcal{R} := (t')s$ are not necessarily automorphisms or even endomorphisms of G_1 and G_2, respectively; however, the maps

$$t' \mapsto \mathcal{L}_{t'},\ G_2 \to G_1^{G_1} \text{ and } s \mapsto {}_s\mathcal{R},\ G_1 \to G_2^{G_2}$$

are homomorphisms (the semigroup operation in $G_1^{G_1}$ and $G_2^{G_2}$ being composition of functions). The conditions that make the Zappa operation associative are

$$\mathcal{L}_{t'}(s's) = \mathcal{L}_{t'}(s')\Big[\mathcal{L}_{((t')_{s'}\mathcal{R})}(s)\Big] \quad \text{and} \quad (t't)_s\mathcal{R} = \Big[(t')_{(\mathcal{L}_t(s))}\mathcal{R}\Big](t)_s\mathcal{R}.$$

We mention that $G = G_1 \times G_2$ is an extension and a Zappa product precisely when it is a semidirect product, i.e., when both of $G_1 \times \{e\}$ and $\{e\} \times G_2$ are subgroups of G in the extension format, and at least one of them is a normal subgroup in the Zappa format. The group Γ of Example 6(e) below has a feature that seems peculiar. It is a Zappa product $\Gamma = M \times H$ of two of its subgroups, neither of which is normal, so it is not an extension of one of these groups by the other; nonetheless, the functions ${}_s\mathcal{R}$ are automorphisms of H.

II. Distal compact left topological groups

We now consider a situation where we start with a compact group with one-sided continuity; we want to embed it in another compact group with continuity on the other side. By the symmetry of the situation, we shall then want to repeat the process and embed the second group in a third with continuity on the original side; and so forth. Clearly, the difference made by starting with one side, rather than the other, is in the notation. Accordingly, we start in the setting where the notation feels most familiar at the first stage, with a compact left topological group.

Let G be a compact left topological group. G acts on itself by left translation, each $s \in G$ determines a homeomorphism λ_s of G, $t \mapsto \lambda_s(t) := st$ for all $t \in G$. Setting $\lambda_G := \{\lambda_s \mid s \in G\}$, we call G *distal* if (λ_G, G) is a distal flow. When G is a compact topological group, G is distal; in fact, (λ_G, G) is equicontinuous and the Ellis group λ_G^- is a compact topological group that is topologically isomorphic to G. At first glance, it may seem unexpected that there exist non-topological G for which (λ_G, G) is equicontinuous, and so λ_G^- is a compact topological group. Ruppert [10] studied compact left topological groups with this equicontinuity property. Indeed, one thrust of our work here is to present appropriate extensions to our more general setting of results of Ruppert; also, Example 6(e) below is taken from [10]. The term "distal group" was used by Rosenblatt [9], who showed that distality and polynomial growth are equivalent concepts for almost connected, locally compact, topological groups; thus, our work here is in quite a different direction. It is obvious that a direct product of distal groups is distal, as is the homomorphic image of a distal group (since the homomorphic image of a distal flow is distal [4; Corollary 5.7]). Noting that the semidirect product of distal (even topological) groups need not be distal (Example 6(b)), we state the following lemma for ease of reference.

2. LEMMA. *Suppose that G_1 and G_2 are compact left topological groups and that $G = G_1 \times G_2$ is the product space.*

(i) *Let G have Schreier operation* (1). *Then G is a left topological group if the function*

$$(s,t) \mapsto t'(s)[t',t], \quad G \to G_1$$

is continuous for all $t' \in G_2$. If, as well, G_2 is distal and the automorphisms $s \mapsto t'(s)$ of G_1 are all trivial, then G is distal.

(ii) *Let G have Schreier operation* (2). *Then G is a left topological group if G_2 is a topological group and the function*

$$s \mapsto [s',s](t')s, \quad G_1 \to G_2$$

is continuous for all $(s',t') \in G$. If, as well, G_1 is distal, then G is distal.

The next theorem gives conditions that are equivalent to distality of G; they are analogous to the conditions of Ruppert [10; p. 160] that are equivalent to equicontinuity of G.

3. THEOREM. *For a compact left topological group G, the following assertions are equivalent.*

(i) *G is a distal group, i.e., (λ_G, G) is a distal flow.*

(ii) *$\lambda_G^- \subset G^G$ is a compact right topological group.*

(iii) *If $s \in G$ and $\{s_\alpha\} \subset G$ with $\lambda_{s_\alpha}(e) = s_\alpha \to e$ and $\lambda_{s_\alpha}(s) = s_\alpha s \to e$, then $s = e$ (i.e., (λ_G, G) is a point distal flow with e as distal point).*

(iv) *There exist a compact right topological group Γ and an algebraic isomorphism ψ of G onto a dense subgroup M of Γ. Also, there is a continuous map $\delta : \Gamma \to G$ with $\delta(\psi(s)) = s$ for all $s \in G$; the kernel $H = \{T \in \Gamma \mid \delta(T) = e\}$ is a compact subgroup of Γ, $M \cap H = \{e\}$, $\Gamma = MH$, and δ induces a homeomorphism between the quotient space Γ/H and G.*

PROOF. (i) and (ii) are equivalent by Theorem 1, and (i) obviously implies (iii). If (iii) holds and $\lambda_{r_\alpha} t_i \to t_0$, $i = 1, 2$, put $s_\alpha := t_0^{-1} r_\alpha t_1$ and $s := t_1^{-1} t_2$. Then $s_\alpha \to e$ and $s_\alpha s \to e$, so $s = e$ and $t_1 = t_2$, and (i) holds.

(ii) implies (iv). If (ii) holds, set $\Gamma = \lambda_G^-$, so that

$$\psi : s \mapsto \lambda_s,\ G \to G^G \text{ with } M = \lambda_G,$$

and

$$\delta : T \mapsto T(e),\ \Gamma \to G \text{ with } H = \{T \in \Gamma \mid T(e) = e\}.$$

Then Γ, ψ, M, δ and H have the desired properties. We mention that, if $T = \lim_\alpha \lambda_{s_\alpha} \in \Gamma$, then $\lambda_{s_\alpha}(e) = s_\alpha \to s_1 := T(e)$ in G and

$$T = \lim_\alpha \lambda_{s_\alpha} = \lim_\alpha \lambda_{s_1} \lambda_{s_1^{-1}} \lambda_{s_\alpha} = \lambda_{s_1} h \in MH,$$

where $h := \lambda_{s_1^{-1}} T \in H$. (Recall that $\lambda_G \subset \Lambda(\Gamma)$.)

(iv) implies (iii). Suppose that (iv) holds and that $s_\alpha \to e$ and $s_\alpha s \to e$. To show that $s = e$, we may assume that $\{\psi(s_\alpha)\}$ converges in Γ (since we can take a subnet of $\{\psi(s_\alpha)\}$, if necessary). Then the limit $h := \lim_\alpha \psi(s_\alpha)$ is in H, since

$$\delta(h) = \lim_\alpha \delta\big(\psi(s_\alpha)\big) = \lim_\alpha s_\alpha = e;$$

also

$$\lim_\alpha \psi(s_\alpha s) = \lim_\alpha \psi(s_\alpha)\psi(s) = h\psi(s),$$

since Γ is right topological, and $h\psi(s) \in H$. Thus $\psi(s) \in M \cap H = \{e\}$, so $s = e$. □

4. Remarks, the first and third of which are analogous to remarks of Ruppert [10] about equicontinuous groups.

(a) The map δ (in Theorem 3(iv)) is a homomorphism if and only if H is a normal subgroup of Γ; in this case $G \cong \Gamma/H$ is also a right topological group, and hence is a topological group (so $G \cong \Gamma$ and $H = \{e\}$). Indeed, when $\Gamma = \lambda_G^{-} \subset G^G$ as in the proof that (ii) implies (iv), $\{e\}$ is the only closed subgroup of H that is normal in Γ. For, if $N \subset H$ is closed and is a normal subgroup of Γ, we want to get a compact, Hausdorff, right topological group that is a homomorphic image of Γ. The problem is that Γ/N is Hausdorff if and only if N is also closed in the weaker topology σ of Γ [8, or 2]. To get around this, we take the σ-closure N_1 of N, which is a normal subgroup of Γ and is contained in H, which is σ-closed as Γ/H is Hausdorff. Then $\Gamma_1 := \Gamma/N_1$ is a compact, Hausdorff, right topological group with closed subgroup $H_1 := H/N_1$, so that Γ_1/H_1 is homeomorphic to G (and to Γ/H). The flow $(G, \Gamma_1/H_1)$, $(s, (TN_1)H_1) = (s, TH_1) \mapsto sTH_1$, is isomorphic to (λ_G, G), but its Ellis group is a homomorphic image of Γ_1, while that of (λ_G, G) is Γ. Thus $\Gamma_1 \cong \Gamma$ and $N_1 = N = \{e\}$.

In the general setting of (iv), the left action of Γ on Γ/H gives a continuous homomorphism θ of Γ onto the enveloping semigroup of (λ_G, G); the kernel of θ is

$$\bigcap_{\gamma\in\Gamma} \gamma H \gamma^{-1},$$

the largest normal subgroup of Γ that is contained in H. $\ker(\theta)$ can properly contain $\{e\}$ even in the "trivial" situation. Let G be the circle group $\mathbf{T}$ (or any infinite compact abelian topological group), and let ψ be the natural homomorphism of $\mathbf{T}$ into $\mathbf{T}_2 := \mathbf{T}_d\hat{}_d\hat{} \cong \mathbf{T} \times \widehat{\mathbf{T}}^{\perp}$. (See Lemma 5 below; $\mathbf{T}_2$ is isomorphic to the almost periodic compactification $\mathbf{T}_d{}^{\mathcal{AP}}$ of $\mathbf{T}$ with the discrete topology.) Here $\ker(\theta) = \{e\} \times \widehat{\mathbf{T}}^{\perp}$.

(b) Since $\Gamma = MH$ for subgroups satisfying $M \cap H = \{e\}$, Γ is a Zappa product of M and H. It is impossible for Γ to be an extension of H by M, since H is not a normal subgroup of Γ. However, it sometimes happens that M is a normal subgroup of Γ (e.g., Example 6(a)). Then Γ is a semidirect product of H and M, which seems very strange, since the members of H correspond to automorphisms of G, while being pointwise limits of left translations $\lambda_s \colon G \to G$, which are never automorphisms (except in the trivial situation $s = e$).

(c) Let $N(H)$ be the normalizer of H in Γ,

$$N(H) := \{T \in \Gamma \mid TH = HT\}.$$

Then $\delta(N(H)) = \mathfrak{N}(G)$. To see this, give Γ/H the multiplication of G, $(T_1H)(TH) = \psi(s_1)TH$, where $\psi(s_1)$ is the unique element of $T_1H \cap M$.

Note that $(\psi(s)H)(TH) = \psi(s)TH$ for all $s \in G, T \in \Gamma$. So, for $T \in N(H)$, $\delta(T) = TH$, and it follows from the fact that Γ is right topological that

$$\psi(s)H \mapsto (\psi(s)H)\,\delta(T) = (\psi(s)H)(TH) = \psi(s)TH$$

is a homeomorphism of $\Gamma/H \cong G$, i.e., $\delta(T) \in \mathfrak{R}(G)$. Conversely, if $\delta(T) \in \mathfrak{R}(G)$, then from $(\psi(s)H)(TH) = \psi(s)TH$ for all $s \in G$ we get $(T_1H)(TH) = T_1TH$ for all $T_1 \in \Gamma$, using continuity in both $\Gamma/H \cong G$ and Γ. Setting $T_1 = T^{-1}$ shows that $T \in N(H)$.

(d) As pointed out by Ruppert [10], when λ_G acts equicontinuously on G and Γ is a topological group, then $N(H)$ is closed in Γ; so $\mathfrak{R}(G) = \delta(N(H))$ must be closed in G and cannot be dense in G unless $\mathfrak{R}(G) = G$, in which case G is a topological group. Similarly, when $\mathfrak{R}(G)$ is not closed in Γ, e.g., if $\mathfrak{R}(G)$ is dense in G and not equal to G, then $N(H)$ is not closed in Γ, Γ is not a topological group, and λ_G does not act equicontinuously on G.

The next step. Starting with a distal left topological group G, we now have a compact right topological group Γ. So, Γ acts on itself by right translation. We use right notation, $(s')_s\rho = s's$, and can ask if the flow $(\Gamma,{}_\Gamma\rho)$ is distal, i.e., if Γ is a distal right topological group. If (λ_G, G) is equicontinuous, instead of merely distal, then Γ is a compact topological group (and therefore distal), and the Ellis group ${}_\Gamma\rho^- \subset \Gamma^\Gamma$ is isomorphic to Γ. Here are two

QUESTIONS. Suppose G is a distal compact left topological group. Can the resulting right topological group Γ fail to be distal? Can $(\Gamma,{}_\Gamma\rho)$ be not only a distal flow, but an equicontinuous one? We do not know the answer to the first question. The answer to the second question is: if and only if Γ is a topological group, i.e., (λ_G, G) is equicontinuous. This follows from the density of $\Lambda(G)$ in Γ and the last sentence of Remark 4(d) (in its other-sided form).

III. Examples

We apply the methods outlined above to some examples that appear in [2] (and elsewhere). A consequence of Pontrjagin duality is useful in the presentation of the examples; we give a proof of it for completeness. First we need some definitions. For a locally compact abelian group $\mathfrak{G}$, $\mathfrak{G}^\wedge = \widehat{\mathfrak{G}}$ denotes the dual group (consisting of the continuous characters of $\mathfrak{G}$, i.e., the continuous homomorphisms from $\mathfrak{G}$ into the circle group $\mathbf{T}$) and $\mathfrak{G}_d$ denotes the group $\mathfrak{G}$ with the discrete topology; so $\mathfrak{G}_1 := \mathfrak{G}_d^\wedge$ is the dual of $\mathfrak{G}_d$ and $\mathfrak{G}_1 \supset \widehat{\mathfrak{G}}$. We need also $\mathfrak{G}_2 := (\mathfrak{G}_1)_d^\wedge$, and identify $\mathfrak{G}$ with its canonical image in $\mathfrak{G}_2$ ($s(h) := h(s)$ for $s \in \mathfrak{G}$, $h \in \mathfrak{G}_1$), recalling that this image is dense in $\mathfrak{G}_2$. If $\mathfrak{K}$ is a subgroup of $\mathfrak{G}$, then $\mathfrak{K}^\perp := \{h \in \widehat{\mathfrak{G}} \mid h(\mathfrak{K}) = \{1\}\}$. One reference for all this is [5; §24].

5. LEMMA. *Let $\mathfrak{G}$ be a compact abelian topological group, and let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be as above. For a neighbourhood V of $e \in \mathfrak{G}$, let V^- denote the closure of (the image of) V in $\mathfrak{G}_2$. Then*

$$\bigcap V^- = \widehat{\mathfrak{G}}^\perp := \{\varphi \in \mathfrak{G}_2 \mid \varphi(\widehat{\mathfrak{G}}) = \{1\}\} = (\mathfrak{G}_1/\widehat{\mathfrak{G}})_d\widehat{\ },$$

and $\mathfrak{G}_2$ is the direct product of $\mathfrak{G}$ and $\widehat{\mathfrak{G}}^\perp$.

PROOF. Clearly, if $s_\alpha \to e$ in $\mathfrak{G}$ and $s_\alpha \to \varphi \in \mathfrak{G}_2$ (i.e., $h(s_\alpha) \to \varphi(h)$ for all $h \in \mathfrak{G}_1$), then $\varphi(h) = 1$ for all $h \in \widehat{\mathfrak{G}}$, i.e., $\varphi \in \widehat{\mathfrak{G}}^\perp$. Conversely, let V be a neighbourhood of e in $\mathfrak{G}$, and let W be a neighbourhood in $\mathfrak{G}_2$ of some $\varphi \in \widehat{\mathfrak{G}}^\perp \subset \mathfrak{G}_2$. To show that $\widehat{\mathfrak{G}}^\perp \subset V^- \subset \mathfrak{G}_2$, we must demonstrate that $V \cap W \neq \emptyset$. Now, we may assume that V contains a neighbourhood of e in G of the form

$$V_1 := \{s \in G \mid |h(s) - 1| < \varepsilon \quad \text{for all} \quad h \in F\},$$

where $F \subset \widehat{\mathfrak{G}}$ is finite, and also that W contains a neighbourhood of φ in $\mathfrak{G}_2$ of the form

$$W_1 := \{\varphi' \mid |\varphi'(h) - \varphi(h)| < \varepsilon \quad \text{for all} \quad h \in F_1\},$$

where $F_1 \subset \mathfrak{G}_1$ is finite. But $\varphi(F) = \{1\}$, since $\varphi \in \widehat{\mathfrak{G}}^\perp$, so the neighbourhood W_2 of φ in $\mathfrak{G}_2$,

$$W_2 := \{\varphi' \in \mathfrak{G}_2 \mid |\varphi'(h) - \varphi(h)| < \varepsilon \quad \text{for all} \quad h \in F \cup F_1\},$$

is contained in W and contains a member $s_1 \in \mathfrak{G}$, since $\mathfrak{G}$ is dense in $\mathfrak{G}_2$. By the definition of W_2, $s_1 \in V$ as required.

The last equality of the display in the statement of the lemma is part of [5; 24.10-11]. The proof that $\mathfrak{G}_2 = \mathfrak{G} \times \widehat{\mathfrak{G}}^\perp$ is now easy: if $\varphi \in \mathfrak{G}_2$ and net $\{s_\alpha\} \subset \mathfrak{G} \subset \mathfrak{G}_2$ converges to φ, then $\{s_\alpha\}$ converges to some $s \in \mathfrak{G}$ and

$$\varphi = \lim_\alpha s(s^{-1}s_\alpha) = s(s^{-1}\varphi),$$

where $\varphi_1 := s^{-1}\varphi \in \widehat{\mathfrak{G}}^\perp$. □

6. EXAMPLES. (a) Let $\mathbf{T}$ be the circle group, and let $E := \mathbf{T}_1 = \mathbf{T}_d\widehat{\ }$ be the set of all endomorphisms of $\mathbf{T}$. ($E \cong \mathbf{Z}^{\mathcal{AP}}$, the almost periodic compactification of the integers $\mathbf{Z}$, a compact topological abelian group [1, or 2].) Then $G = \mathbf{T} \times E$ with multiplication in Schreier formulation (1)

$$(w,h)(w',h') = (ww'h' \circ h(e^{2i}), hh') = \lambda_{(w,h)}(w',h')$$

is a compact left topological group. Here $[h,h'] = h' \circ h(e^{2i})$ and the automorphisms of $G_1 = \mathbf{T}$ are all trivial, so G is distal by Lemma 2(i). We write $E_1 := E_d\widehat{\ } = \mathbf{T}_2$, and can then identify $\lambda_{(w,h)}$ with $\big(w, h, h(e^{2i})\big) \in \mathbf{T} \times E \times \times E_1$, where $\mathbf{T}$ is regarded as a subset of E_1, $w(h) := h(w)$, as above, and $\mathbf{T} \times E \times E_1$ acts on G,

$$(w,h,\varphi)\colon (w',h') \mapsto (ww'\varphi(h'), hh').$$

To ensure that the map from λ_G onto

$$M := \big\{ \big(w,h,h(e^{2i})\big) \mid (w,h) \in G \big\} \subset \mathbf{T} \times E \times E_1$$

is an isomorphism, we give $\mathbf{T} \times E \times E_1 = (\mathbf{T} \times E) \times E_1$ the multiplication in Schreier formulation (1)

$$(w_1,h_1,\varphi_1)(w,h,\varphi) = (w_1\varphi_1(h)w, h_1h, \varphi_1\varphi),$$

which makes it a compact right topological group. (Here $\varphi_1(w,h) = = (\varphi_1(h)w, h)$ and the function $[\ ,\ ]\colon E_1 \times E_1 \to \mathbf{T} \times E$ is trivial.) One checks readily that the map is also a homeomorphism. Now, we claim that

$$\Gamma\ (\cong M^-) = A := \big\{ \big(w,h,h(e^{2i})\varphi_1\big) \mid \varphi_1 \in \widehat{\mathbf{T}}^\perp = (E_d/\mathbf{Z})\widehat{\ } \big\} \subset \mathbf{T} \times E \times E_1$$

(where $\widehat{\mathbf{T}}^\perp = \mathbf{Z}^\perp \subset E_1$, the image of $n \in \mathbf{Z}$ in $E = \mathbf{T}_d\widehat{\ } = \widehat{\mathbf{Z}}_d\widehat{\ }$ being the character $w \mapsto w^n$). For, if

$$\lambda_{s_\alpha} := \lambda_{(w_\alpha,h_\alpha)} \to T \in \Gamma,$$

we refer to the last sentence of the proof that (ii) implies (iv) in Theorem 1:

$$\lambda_{s_\alpha}(e) \to s_1 := T(e) = (w,h),$$

say, in G, and $h_\alpha(e^{2i}) \to h(e^{2i})$, so $\lambda_{s_\alpha} = \lambda_{s_1}\lambda_{s_1^{-1}s_\alpha} \to T$. Now

$$s_1^{-1}s_\alpha = (w^{-1}h \circ h(e^{2i}), h^{-1})(w_\alpha,h_\alpha) =$$

$$= (w^{-1}[(h^{-1}h_\alpha) \circ h^{-1}](e^{2i})w_\alpha, h^{-1}h_\alpha) \to e$$

in G, and one sees that the last coordinate of $\lambda_{s_1^{-1}s_\alpha}$, namely $[h^{-1}h_\alpha](e^{2i}) \in \mathbf{T} \subset E_1$, converges to 1 in $\mathbf{T}$, and so $\lambda_{s_1^{-1}s_\alpha} \to (1, e, \varphi_1)$ in $\mathbf{T} \times E \times E_1$ for some $\varphi_1 \in \mathbf{Z}^\perp$ by Lemma 5. Thus

$$T = \lim_\alpha \lambda_{s_\alpha} = \big(w, h, h(e^{2i})\big)(1, e, \varphi_1) = (w, h, h(e^{2i})\varphi_1),$$

which shows that $\Gamma \subset A$; it follows also from Lemma 5 that $\Gamma = A$. Note that, although $\Gamma \subsetneqq \mathbf{T} \times E \times E_1$, the projection $(w, h, \varphi) \mapsto \varphi$ maps Γ onto E_1. The map $\delta : \Gamma \to G$ is given by $\delta(w, h, h(e^{2i})\varphi_1) \mapsto (w, h)$; its kernel $H = \big\{(1, e, \varphi_1) \mid \varphi_1 \in \widehat{\mathbf{T}}^\perp\big\} \cong \widehat{\mathbf{T}}^\perp$. M is a normal subgroup of Γ, so Γ is a semidirect product of $\widehat{\mathbf{T}}^\perp$ and M. The action of $\widehat{\mathbf{T}}^\perp$ on M is given by

$$\varphi_1 \colon \big(w, h, h(e^{2i})\big) \mapsto \big(\varphi_1(h)w, h, h(e^{2i})\big).$$

We now start with the compact right topological group

$$\Gamma = \big\{(w, h, h(e^{2i})\varphi') \mid (w, h) \in G,\ \ \varphi' \in \widehat{\mathbf{T}}^\perp\big\},$$

whose multiplication is given by

$$(w_1, h_1, \varphi_1)(w, h, \varphi) = (w_1\varphi_1(h)w, h_1h, \varphi_1\varphi) = (w_1, h_1, \varphi_1)_{(w,h,\varphi)}\rho.$$

Lemma 2 cannot be used directly to determine the distality of Γ, because Γ is not an extension of closed subgroups. However, Lemma 2 (in its other-sided form) does show that $\mathbf{T} \times E \times E_1$ is distal. Then, referring to the remarks preceding Lemma 2 (or checking directly), one verifies that Γ is distal, $(\Gamma, {}_\Gamma\rho)$ is a distal flow. So, setting $E_2 := E_{1d}\widehat{\ }$, we identify E with a subset of E_2 $\big(h(\varphi') = \varphi'(h)\big)$ and ${}_{(w,h,\varphi)}\rho \in \Gamma^\Gamma$ with

$$(w, h, \varphi, h) \in \mathbf{T} \times E \times E_1 \times E_2.$$

The identification of the character $\varphi_1 \mapsto \varphi_1(h)$ with a member of E_2 in the fourth coordinate is correct, because the projection $(w_1, h_1, \varphi_1) \mapsto \varphi_1$ maps Γ onto E_1. Writing out $\big((w_1, h_1, \varphi_1)_{(w,h,\varphi)}\rho\big)_{(w',h',\varphi')}\rho$, we see that the appropriate multiplication in $\mathbf{T} \times E \times E_1 \times E_2$ is

$$(w, h, \varphi, \psi)(w', h', \varphi', \psi') = \big(ww'\psi'(\varphi), hh', \varphi\varphi', \psi\psi'\big),$$

which makes $\mathbf{T} \times E \times E_1 \times E_2$ a compact left topological group. As above, if

$${}_{(w_\alpha,h_\alpha,\varphi_\alpha)}\rho \to (w, h, \varphi, \psi) \in \mathbf{T} \times E \times E_1 \times E_2,$$

then $w_\alpha \to w$ in $\mathbf{T}$, $h_\alpha \to h$ in E, $\varphi_\alpha \to \varphi$ in E_1, and also $h_\alpha h^{-1} \to e$ in E, so $\psi = \psi_1 h$, where

$$\psi_1 = \lim_\alpha h_\alpha h^{-1} \in \mathbf{T}^\perp \subset E_2$$

by Lemma 5, and $[\psi_1 h](\varphi') = \psi_1(\varphi')\varphi'(h)$ for $\varphi' \in E_1$. The identification of ${}_\Gamma\rho^- \subset \Gamma^\Gamma$ that we end up with is

$${}_\Gamma\rho^- = G^1 :=$$

$$:= \left\{ (w, h, h(e^{2i})\varphi_1, h\psi_1) \in \mathbf{T} \times E \times E_1 \times E_2 \mid \varphi_1 \in \mathbf{Z}^\perp,\ \ \psi_1 \in \mathbf{T}^\perp \right\}.$$

We remark that, although $G^1 \subsetneqq \mathbf{T} \times E \times E_1 \times E_2$, the projections onto the third and fourth coordinates map G^1 onto E_1 and E_2, respectively, and the function

$$(w, h, h(e^{2i})\varphi_1, h\psi_1) \mapsto (\varphi_1, \psi_1)$$

maps G^1 onto $\mathbf{Z}^\perp \times \mathbf{T}^\perp$. The map $\delta^1 : G^1 \to \Gamma$ (from the other-sided form of Theorem 3(iv)) just removes the last coordinate; its kernel H_1 equals

$$\{1\} \times \{e\} \times \{e\} \times \mathbf{T}^\perp.$$

The map $\kappa\colon (w,h) \mapsto (w,h,h(e^{2i}),h)$, $G \to G^1$ (a composition of discontinuous isomorphisms), is a discontinuous isomorphism of G into G^1. $\kappa(G)$ is a normal subgroup of G^1; the density of $\kappa(G)$ in G^1 would follow if each $(\varphi_1, \psi_1) \in \mathbf{Z}^\perp \times \mathbf{T}^\perp \subset E_1 \times E_2$ could be approximated by members $(h(e^{2i}), h) \in \mathbf{T} \times E$ with h's close to the identity in E. We doubt this can be done, and therefore think that $\kappa(G)$ is not dense in G^1.

We discuss briefly the next step, where we start with the compact left topological group

$$G^1 = \left\{ (w, h, h(e^{2i})\varphi_1, h\psi_1) \mid (w,h) \in G,\ \ \varphi_1 \in \mathbf{Z}^\perp,\ \ \psi_1 \in \mathbf{T}^\perp \right\} \subset$$

$$\subset \mathbf{T} \times E \times E_1 \times E_2,$$

whose multiplication is given by

$$(w,h,\varphi,\psi)(w',h',\varphi',\psi') = (ww'\psi'(\varphi), hh', \varphi\varphi', \psi\psi') =$$

$$= \lambda_{(w,h,\varphi,\psi)}(w',h',\varphi',\psi').$$

G^1 is distal, and we identify $\lambda_{(w,h,\varphi,\psi)}$ with

$$(w,h,\varphi,\psi,\varphi) \in \mathbf{T} \times E \times E_1 \times E_2 \times E_3$$

(where $E_3 := E_{2d}\widehat{\ }$), for which the appropriate multiplication is

$$(w_1, h_1, \varphi_1, \psi_1, \theta_1)(w, h, \varphi, \psi, \theta) = (w_1 w \theta_1(\psi), h_1 h, \varphi_1 \varphi, \psi_1 \psi, \theta_1 \theta);$$

$\mathbf{T} \times E \times E_1 \times E_2 \times E_3$ is a compact right topological group. The Ellis group $\Gamma^1 := \lambda_{G^1}^{-} \subset (G^1)^{G^1}$ is (isomorphic to)

$$\{ (w, h, h(e^{2i})\varphi_1, h\psi_1, h(e^{2i})\varphi_1\theta_1) \mid$$

$$(w,h) \in G,\ \ \varphi_1 \in \mathbf{Z}^{\perp},\ \ \psi_1 \in \mathbf{T}^{\perp},\ \ \theta_1 \in E^{\perp} \}.$$

The (discontinuous) isomorphism of G into Γ^1 is

$$(w, h) \mapsto (w, h, h(e^{2i}), h, h(e^{2i}))\,;$$

the image of G is clearly not dense in Γ^1, since any non-trivial $\theta_1 \in E^{\perp} \subset E_3$ cannot be approximated by members $h(e^{2i}) \in \mathbf{T} \subset E_1 \subset E_3$.

By this point the reader will be able to guess the form of subsequent groups. As a concluding remark on this example, we make the observation in connection with the structure theorem for compact groups with one-sided continuity (mentioned after Theorem 1) that, for each of G, Γ, G^1 and Γ^1, (the subgroup isomorphic to) $\mathbf{T}$ is a compact normal topological subgroup yielding a compact abelian topological quotient group.

(b) Let $G = \mathbf{T}^{\mathbf{T}} \times \mathbf{T}$ with multiplication in Schreier formulation (1)

$$(h', w')(h, w) = (h' L_{w'} h, w' w);$$

here $L_{w'}h$ is the left translate of the function $h \in \mathbf{T}^{\mathbf{T}}$ by $w' \in \mathbf{T}$, $L_{w'}h(v) = h(w'v)$. G is a compact left topological group, and is not distal. For, define $h \in \mathbf{T}^{\mathbf{T}}$ by $h(e^{2i\pi p/q}) = -1$ if $p \in \mathbf{Z}$ and $q \in \mathbf{N}$, $h(w) = 1$ otherwise, and let $\{w_n\} \subset \mathbf{T}$ be linearly independent [5] and such that $w_n \to 1$. Then

$$\lambda_{(1,w_n)}(1,1) = (1, w_n) \to (1,1), \quad \text{and} \quad \lambda_{(1,w_n)}(h,1) = (L_{w_n}h, w_n) \to (1,1)$$

as well, since for a given $v \in \mathbf{T}$, $L_{w_n}h(v) = h(w_n v)$ can be equal to -1 for at most one value of n. This is because of the linear independence; if $w_n v = e^{2i\pi p_1/q_1}$ and $w_m v = e^{2i\pi p_2/q_2}$, then

$$w_n^{q_1 q_2} w_m^{-q_1 q_2} = \left(w_n v (w_m v)^{-1} \right)^{q_1 q_2} = 1,$$

which is a contradiction (unless $n = m$).

(c) This example is much like the first, so we discuss it only briefly. Let $G = E \times \mathbf{T} \times \mathbf{T}$ with multiplication (representable in Schreier formulation (2) in two different ways)

$$(h, v, w)(h', v', w') = (hh', vv', h'(v)ww') = \lambda_{(h,v,w)}(h', v', w').$$

G is a distal left topological group, by Lemma 2(ii). We identify $\lambda_{(h,v,w)}$ with

$$(v, h, v, w) \in E_1 \times E \times \mathbf{T} \times \mathbf{T},$$

which gets multiplication

$$(\varphi_1, h_1, v_1, w_1)(\varphi, h, v, w) = (\varphi_1\varphi, h_1h, v_1v, \varphi_1(h)w_1w);$$

$\Gamma = \lambda_G^- \subset G^G$ is (isomorphic to)

$$\{(\varphi' v, h, v, w) \in E_1 \times E \times \mathbf{T} \times \mathbf{T} \mid (h, v, w) \in G,\ \varphi' \in \mathbf{Z}^\perp\}.$$

The last two examples are quite different from all the other groups that have appeared in this paper so far, in that the topological centre $\mathfrak{R}(G)$ for these last two is not dense in G.

(d) Let G be the semidirect product $\mathbf{T} \times \{\pm 1\}$ with multiplication in Schreier formulation (1)

$$(u, \varepsilon)(v, \delta) = (uv^\varepsilon, \varepsilon\delta) = \lambda_{(u,\varepsilon)}(v, \delta).$$

Give G the topology for which a typical basic neighbourhood of $(e^{ia}, 1)$ or $(e^{ib}, -1)$, where $a < b$, is

$$A := \{(e^{ia}, 1), (e^{ib}, -1)\} \cup \{(e^{i\theta}, \varepsilon) \mid \varepsilon = \pm 1,\ a < \theta < b\};$$

these basic neighbourhoods are open and closed. Then G is a compact left topological group, and $\mathfrak{R}(G) = \{(1,1)\}$; ${}_{(1,-1)}\rho$ is at least measurable, but all the other right translations are not even measurable. (Nonetheless, this group does admit Haar measure [6].) Furthermore, G is not distal (and not point distal), since

$$\lambda_{(e^{i/n},1)}(1,1) = (e^{i/n}, 1) \to (1,1), \text{ and } \lambda_{(e^{i/n},1)}(1,-1) = (e^{i/n}, -1) \to (1,1)$$

as well ($(1,1)$ being the identity of G).

We mention that (λ_G, G) is a minimal flow, and yields an interesting "six-circle" enveloping semigroup, which we believe has useful applications.

(e) [10; pp. 164-5] For this example, we need a discontinuous automorphism φ of $\mathbf{T}$ satisfying $\varphi^2 = 1$. [Let $H' := \{2\pi\} \cup H$ be a Hamel basis for $\mathbf{R}$ over $\mathbf{Q}$: each $x \in \mathbf{R}$ has a unique representation $x = \Sigma_{a \in H'} a x_a$, where the x_a's are in $\mathbf{Q}$ and only a finite number of them are not equal to 0. Then each $w \in \mathbf{T}$ has the corresponding unique representation $w = w_0 \Pi_{a \in H} e^{iax_a}$, where $w_0^k = 1$ for some $k \in \mathbf{N}$. Taking a_1 and a_2 in H, we can then define a suitable φ by

$$\varphi(w) := w_0 e^{ia_1 x_{a_2}} e^{ia_2 x_{a_1}} \Pi_{a \in H \setminus \{a_1, a_2\}} e^{iax_a}.]$$

Let G be the semidirect product $\{\varphi, 1\} \times \mathbf{T}$ with multiplication (essentially) in Schreier formulation (2)

$$(\varepsilon, u)(1, v) = (\varepsilon, uv), \quad (\varepsilon, u)(\varphi, v) = (\varepsilon\varphi, \varphi(u)v)$$

(and the product topology). G is a compact left topological group, and $\mathfrak{R}(G) = \{1\} \times \mathbf{T}$, which is closed and not dense in G. Also, G is distal by Lemma 2(ii), but in fact is equicontinuous. The Ellis group $\Gamma = \lambda_G^- \subset G^G$ may be identified with

$$\{\varphi, 1\} \times \mathbf{T} \times \mathbf{T} = \{\varphi, 1\} \times (\mathbf{T} \times \mathbf{T}),$$

a topological group with multiplication in Schreier formulation (2)

$$(\varepsilon, u_1, u_2)(1, v_1, v_2) = (\varepsilon, u_1 v_1, u_2 v_2),$$

$$(\varepsilon, u_1, u_2)(\varphi, v_1, v_2) = (\varepsilon\varphi, u_2 v_1, u_1 v_2).$$

$\lambda_{(\varepsilon,u)}$ corresponds to $(\varepsilon, u, \varphi(u)) \in \{\varphi, 1\} \times \mathbf{T} \times \mathbf{T}$, and the density of the image M of λ_G in $\{\varphi, 1\} \times \mathbf{T} \times \mathbf{T}$ follows from Kronecker's theorem [5]. M is not a normal subgroup of Γ; for example,

$$(1, v, 1)(\varphi, 1, 1)(1, v, 1)^{-1} = (\varphi, v^{-1}, v) \notin M$$

(unless $v = 1$). So, although Γ is a semidirect product of $\{\varphi, 1\}$ and $\mathbf{T} \times \times \mathbf{T}$, Γ is a Zappa product, and not a semidirect product, of M and H, $H = \{1\} \times \{1\} \times \mathbf{T}$ being the kernel of the continuous map $\delta : \Gamma \to G$. (δ just drops the last coordinate.) We obtain the Zappa operators $\mathcal{L}$ and $\mathcal{R}$ for $\{\varphi, 1\} \times \mathbf{T} \times \mathbf{T} = MH$ by taking $(\varphi, v, \varphi(v)) \in M$ and $(1, 1, u) \in H$ and rewriting

$$(1, 1, u)(\varphi, v, \varphi(v)) = (\varphi, uv, \varphi(v)) =$$

$$= (\varphi, uv, \varphi(u)\varphi(v))\,(1, 1, \varphi(u)^{-1}) \in MH;$$

we see that

$$\mathcal{L}_{(1,1,u)}\big(\varphi, v, \varphi(v)\big) = \big(\varphi, uv, \varphi(u)\varphi(v)\big)$$

and

$$(1,1,u)_{(\varphi,v,\varphi(v))}\mathcal{R} = (1,1,\varphi(u)^{-1}).$$

Also,

$$\mathcal{L}_{(1,1,u)}\big(1, v, \varphi(v)\big) = \big(1, v, \varphi(v)\big) \text{ and } (1,1,u)_{(1,v,\varphi(v))}\mathcal{R} = (1,1,u),$$

since $(1,1,u)\big(1,v,\varphi(v)\big) = \big(1,v,\varphi(v)\big)(1,1,u)$. The maps

$$(1,1,u) \mapsto \mathcal{L}_{(1,1,u)},\ H \to M^M \quad \text{and} \quad \big(\varepsilon, v, \varphi(v)\big) \mapsto {}_{(\varepsilon,v,\varphi(v))}\mathcal{R},\ M \to H^H$$

are homomorphisms (as they should be). It happens that the ${}_{(\varepsilon,v,\varphi(v))}\mathcal{R}$'s are automorphisms of H; however, $\mathcal{L}_{(1,1,u)}$ is a homomorphism of M only if $u = 1$. The isomorphism of $\Gamma = MH$ onto the Zappa product $M \times H$ is given by

$$(\varepsilon, v_1, v_2) = \big(\varepsilon, v_1, \varphi(v_1)\big)\big(1,1,\varphi(v_1)^{-1}v_2\big) \mapsto$$

$$\mapsto \Big(\big(\varepsilon, v_1, \varphi(v_1)\big), \big(1,1,\varphi(v_1)^{-1}v_2\big)\Big).$$

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DIFFERENCE SETS WITHOUT κ-TH POWERS

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1. Introduction

For any positive integers N, κ let $\sigma(N, \kappa)$ denote the maximal number of integers $1 \leqq a_1 < \dots < a_\sigma \leqq N$ such that the difference set of $\mathcal{A} = \{a_1, \dots, a_\sigma\}$ contains no perfect κ-th powers, i.e. $a_i - a_j = m^\kappa$, $i > j$ has no solution.

It follows from the work of Kamae and Mendés-France [3] that $\sigma(N, \kappa) = o(N)$ for any fixed $\kappa \geqq 2$ as N tends to infinity. Quantitative results are known only in the case of $\kappa = 2$. Pintz, Steiger and Szemerédi [5] proved that

$$\sigma(N, 2) \ll \frac{N}{(\log N)^{\frac{1}{12} \log\log\log\log N}}. \tag{1}$$

Our aim with this paper is to extend (1) for any $\kappa \geqq 2$.

THEOREM. *For any $\kappa \geqq 2$ there are positive constants C_0 and N_0 such that*

$$\sigma(N, \kappa) \leqq C_0 \frac{N}{(\log N)^{\frac{1}{4} \log\log\log\log N}} \tag{2}$$

for any $N \geqq N_0$.

The proof is based on the method developed in [5] so on one hand we can be brief in some technical details, on the other hand we can attempt to give a cleaner explanation of the argument.

Note that optimizing the parameters in the finest way would lead to a constant $\frac{1}{\log 3}$ in place of $\frac{1}{4}$.

Our calculations are effective everywhere and it is not hard to get the final result uniform in κ, i.e. to get the dependence of C_0 and N_0 on κ.

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2. Outline of the proof

We choose C_0 and N_0 large enough to provide that

$$C_0 \geqq (\log N)^{\frac{1}{4}\log\log\log\log N}$$

whenever $N_0 \leqq N \leqq N_0^2$ and let $\kappa \geqq 2$ be fixed. Thus (2) is certainly true for this range. Our argument is indirect. Suppose that (2) is not true and let $N > N_0^2$ be the smallest integer such that

$$\sigma_0 = \sigma(N, \kappa) > C_0 \frac{N}{(\log N)^{\frac{1}{4}\log\log\log\log N}}, \tag{3}$$

and let $\mathcal{A}_0 = \{1 \leqq a_1 < \cdots < a_{\sigma_0} \leqq N\}$ be an extremal set of σ_0 integers such that the difference set of $\mathcal{A}_0$ contains no perfect κ-th powers. C_0, N_0, N, κ and $\mathcal{A}_0$ are now fixed.

For shorter reference we denote the right hand side of (2) and (3) by $\gamma(N)$ and observe that $\gamma(x)$ is an increasing function of the real variable $x \geqq N_0$, if N_0 was chosen large enough.

The simple assumption that $\mathcal{A}_0$ is an extremal set will imply that $\mathcal{A}_0$ is, in fact, very well distributed in a certain sense. For example, let us define

$$\mathcal{A}_1 = \mathcal{A}_0 \cap \left[1, \frac{N}{2}\right], \qquad \mathcal{A}_2 = \mathcal{A}_0 \cap \left(\frac{N}{2}, N\right], \tag{4}$$

and $\sigma_1 = |\mathcal{A}_1|$, $\sigma_2 = |\mathcal{A}_2|$. Clearly $\sigma_1 + \sigma_2 = \sigma_0 > \gamma(N)$, but as the difference set of $\mathcal{A}_0$ and thus $\mathcal{A}_1$ and $\mathcal{A}_2$ contain no perfect κ-th powers, we have $\sigma_1 \leqq \gamma\left(\frac{N}{2}\right)$ and $\sigma_2 \leqq \gamma\left(\frac{N+1}{2}\right)$. Standard computation gives that

$$\begin{cases} \sigma_i = \dfrac{\sigma_0}{2}\left(1 + O\left(\dfrac{\log\log\log\log N}{\log N}\right)\right), & i = 1, 2; \\[2ex] \sigma_0 = \gamma(N)\left(1 + O\left(\dfrac{\log\log\log\log N}{\log N}\right)\right). & \end{cases} \tag{5}$$

The implied constants here and later depend at most on κ, C_0 and N_0 and are effectively calculable.

Let us define the generating functions

$$F_i(\alpha) = \sum_{a \in \mathcal{A}_i} e(\alpha a), \quad i = 0, 1, \qquad S(\alpha) = \sum_{m \leqq M} w_m e(\alpha m^\kappa),$$

where $e(x) = e^{2\pi i x}$ as usual, $M = \left[\left(\frac{N}{2}\right)^{1/\kappa}\right]$, and the non-negative weights w_m will be defined later. $\mathcal{A}_2$ contains no additional information to us thus $\mathcal{A}_2$ will not appear in the proof later on. From Parseval identity we have

$$\sum_{t=0}^{N-1} \left| F_i\left(\frac{t}{N}\right)\right|^2 = \sigma_i N, \qquad i = 0, 1. \tag{6}$$

On the other hand as $\mathcal{A}_o - \mathcal{A}_1$ is free of perfect κ-th powers we have

$$\sum_{t=0}^{N-1} F_0\left(-\frac{t}{N}\right) F_1\left(\frac{t}{N}\right) S\left(\frac{t}{N}\right) = N \sum_{\substack{a-a' \equiv m^\kappa (N) \\ a\in\mathcal{A}_0,\, a'\in\mathcal{A}_1 \\ m^\kappa \leqq \frac{N}{2}}} w_m = 0, \tag{7}$$

which imply

$$\sum_{t=1}^{N-1} \left| F_0\left(\frac{t}{N}\right) F_1\left(\frac{t}{N}\right) S\left(\frac{t}{N}\right)\right| \geqq \left|F_0(0)F_1(0)S(0)\right| = \sigma_0\sigma_1 \sum_{m\leqq M} w_m. \tag{8}$$

The first key-point of the proof is that according to (8) $F_1\left(\frac{t}{N}\right)$ must take large values frequently. If $\left|F_1\left(\frac{t}{N}\right)\right| > \varepsilon\sigma_1$ for a set $\mathcal{P}$ of points $\frac{t}{N}$ then from (6) we have

$$|\mathcal{P}|(\varepsilon\sigma_1)^2 \leqq \sum_{t=0}^{N-1} \left|F_1\left(\frac{t}{N}\right)\right|^2 = \sigma_1 N,$$

that is

$$\sigma_1 \leqq \frac{N}{\varepsilon^2|\mathcal{P}|} \tag{9}$$

contradicting (3) and (5) if ε could be chosen properly. This argument alone would give a weaker result than the stated one.

The second key-point of the proof is to try to find the best ε such that $\varepsilon^2|\mathcal{P}|$ is biggest possible. As in (7) we have

$$\sum_{t=0}^{N-1} F_0\left(-\frac{t}{N}\right) F_1\left(\alpha + \frac{t}{N}\right) S\left(\frac{t}{N}\right) = 0$$

and so

$$\sum_{t=1}^{N-1} \left| F_0\left(\frac{t}{N}\right) F_1\left(\alpha+\frac{t}{N}\right) S\left(\frac{t}{N}\right) \right| \geqq$$

$$\geqq \left| F_0(0) F_1(\alpha) S(0) \right| \geqq \varepsilon\sigma_0\sigma_1 \sum_{m \leqq M} w_m,$$

if $\alpha \in \mathcal{P}$, a point where $F_1(\alpha)$ is large. Similarly to the above, $F_1\left(\alpha+\frac{t}{N}\right)$ must take large values frequently. An interesting property of the rational numbers will provide that these points $\alpha+\frac{t}{N}$, where α runs through $\mathcal{P}$, are basically different. This would increase $\varepsilon^2|\mathcal{P}|$ unless $\varepsilon^2|\mathcal{P}|$ is already large.

3. Notations and preliminaries

We have already fixed the value of C_0, N_0, N, κ, σ_0, σ_1, M, and have defined $\mathcal{A}_0$, $\mathcal{A}_1$, $S(\alpha)$, $F_0(\alpha)$, $F_1(\alpha)$, $\gamma(x)$. Now we introduce some more parameters.

$$Q = e^{(\log\log N)^4}, \; z = e^{(\log\log N)^2},$$

$$P = \prod_{p<z} p, \qquad W = \prod_{p<z}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{(\log\log N)^2},$$

$$W_q = \prod_{\substack{p<z\\ p\nmid q}}\left(1-\frac{1}{p}\right) \quad \text{for any} \quad q \geqq 1,$$

$$w_m = \begin{cases} \kappa\left(\dfrac{m}{M}\right)^{\kappa-1}, & \text{if} \quad (m,P)=1,\, m \leqq M \\ 0, & \text{otherwise;} \end{cases}$$

$$T(q,a) = \sum_{\substack{m=1\\ ((m,q),P)=1}}^{q} e\left(\frac{am^\kappa}{q}\right),$$

$\mu(n)$, $\tau_k(n)$ and $\nu(n)$ denote the Möbius-, the divisor- and the counting function of different prime factors, resp.

To fully understand the behaviour of $S(\alpha)$ we need the next sieve result.

LEMMA 1. *If* $(a, q) = 1$, $1 \leqq y \leqq x$ *then*

$$\sum_{\substack{x-y<n\leqq x \\ n\equiv a(q) \\ (n,P)=1}} 1 = \frac{y}{q} W_q\Big(1 + O\Big(e^{-\frac{1}{2}\frac{\log \frac{y}{q}}{\log z}}\Big)\Big).$$

PROOF. This is Theorem 7.2 in Halberstam-Richert [1].

We also need some bounds for $T(q, a)$.

LEMMA 2. *If* $(a, q) = 1$ *then*

$$T(q, a) \leqq \begin{cases} \kappa^{\nu(q)} q^{1/2} & \text{if} \quad q < z, \\ \kappa^{\nu(q)} q^{1-\frac{1}{\kappa}} & \text{generally.} \end{cases}$$

PROOF. This is basically Theorem 4.2 in Vaughan [6]. Like Lemma 2.10 in that book we have for $(a, q) = 1$

$$T(q, a) = \prod_{p^\alpha || q} T(p^\alpha, a_p),$$

where $a_p \pmod{p^\alpha}$ is determined by a, q, p^α and $p \nmid a_p$. Then Lemma 4.3 of the book says that

$$|T(p, a)| \leqq \kappa p^{1/2} \leqq \kappa p^{1-\frac{1}{\kappa}}. \tag{10}$$

For higher prime powers let $\tau \geqq 0$ be an integer such that $\kappa = p^\tau \kappa'$, $p \nmid \kappa'$, and $\gamma = \tau + 2$ if $p = 2$, $\tau > 0$, $\gamma = \tau + 1$ in all other cases. A reduced residue class $\bmod p^\alpha$ is a κ-th power residue if and only if it is a κ-th power residue $\bmod p^\gamma$. For $\alpha > \gamma$ we have

$$\sum_{\substack{m=1 \\ p\nmid m}}^{p^\alpha} e\left(\frac{am^\kappa}{p^\alpha}\right) = \sum_{\substack{m=1 \\ p\nmid m}}^{p^\gamma} \sum_{h=0}^{p^{\alpha-\gamma}} e\left(\frac{a(hp^\gamma + m^\kappa)}{p^\alpha}\right) = 0.$$

This means (see Lemma 4.4 of [6])

$$T(p^\alpha, a) = \begin{cases} 0 & \text{if} \quad p < z,\, \alpha > \gamma \\ p^{\alpha-1} & \text{if} \quad p \geqq z,\, \gamma < \alpha \leqq \kappa \\ p^{\kappa-1} T(p^{\alpha-\kappa}, a) & \text{if} \quad p \geqq z,\, \kappa < \alpha,\, \gamma < \alpha. \end{cases}$$

This with the trivial bound for the remaining cases together with (10) imply

$$\left|T(p^\alpha, a)\right| \leqq \begin{cases} \kappa p^{\alpha/2} \leqq \kappa p^{\alpha(1-\frac{1}{\kappa})} & \text{if} \quad p < z \\ \kappa p^{\alpha(1-\frac{1}{k})} & \text{if } p \geqq z. \end{cases}$$

The next ingredient, we need for studying $S(\alpha)$, is Weyl's inequality.

LEMMA 3. *If* $(a, q) = 1$ *then*

$$\left|\sum_{m\leqq y} e\left(\frac{a}{q} m^\kappa\right)\right| \leqq \left(y^{1-\frac{1}{2\kappa}} + \frac{y}{q^{1/2^\kappa}} + y\left(\frac{q}{y^\kappa}\right)^{\frac{1}{2^\kappa}}\right)(1 + \log yq)$$

with an absolute implied constant.

PROOF. This is basically Lemma 2.4 in [6]. The unnecessary y^ε can be removed by an additional use of the Cauchy–Schwarz inequality along with a bound like

$$\sum_{m\leqq Y} \tau^2_{\kappa-1}(m) \leqq Y \sum_{m\leqq Y} \frac{\tau^2_{\kappa-1}(m)}{m} \leqq$$

$$\leqq Y\left(\sum_{m\leqq Y} \frac{1}{m}\right)^{(\kappa-1)^2} \leqq Y(1+\log Y)^{(\kappa-1)^2}.$$

Finally we also need Hua's lemma.

LEMMA 4. *For* $s > 2^\kappa$ *we have*

$$\int_0^1 \left|\sum_{m\leqq M} e(\alpha m^\kappa)\right|^s d\alpha \ll M^{s-\kappa}.$$

PROOF. This is well known from the theory of Waring's problem, see [6] and follows from Theorem 4, Lemma 3.6 and Lemma 7.12 of [2].

4. The generating function $S(\alpha)$

In this section we study the behaviour of $S(\alpha)$. We first state all the lemmas and we give the very technical proofs later on.

LEMMA S1. *For $q < z$, $(a,q) = 1$ we have*

$$\left|S\left(\frac{a}{q}+\eta\right)\right| \ll \frac{\kappa^{\nu(q)}}{q^{1/2}}\frac{W_q}{W}|S(\eta)| + \left(1+|\eta|N\right)Me^{-\sqrt{\log N}}.$$

LEMMA S2. *For $z \leqq q \leqq \frac{N}{Q}$, $(a,q)=1$ and $\left|\alpha - \frac{a}{q}\right| < \frac{Q}{qN}$ we have*

$$|S(\alpha)| \ll Me^{-\frac{1}{4\kappa}(\log\log N)^2}.$$

LEMMA S3. *For $\frac{1}{10} < h \leqq M^{1/4}$ we have*

$$\left|S\left(\frac{h}{N}\right)\right| \ll Me^{-\sqrt{\log N}} + \frac{M}{h}$$

(but the second term appears only when h is not an even integer).

LEMMA S4.

$$\sum_{t=0}^{N-1}\left|S\left(\frac{t}{N}\right)\right|^{2^\kappa+2} \ll M^{2^\kappa+2}.$$

PROOF OF S1. Let us write $g(x) = e(\eta x^\kappa)\kappa\left(\frac{x}{M}\right)^{\kappa-1}$ with any fixed real η. By partial summation

(11)

$$S\left(\frac{a}{q}+\eta\right) = \sum_{\substack{m\leqq M\\(m,P)=1}} e\left(\frac{am^\kappa}{q}\right)g(M) - \int_0^M g'(x)\sum_{\substack{m\leqq x\\(m,P)=1}} e\left(\frac{am^\kappa}{q}\right)dx \ll$$

$$\ll \left(1+|\eta|N\right)\max_{1\leqq x\leqq M}\left|\sum_{\substack{m\leqq x\\(m,P)=1}} e\left(\frac{am^\kappa}{q}\right)\right|.$$

We use Lemma 1 to study this sum. We have in case of $x \geqq qz$

$$\sum_{\substack{m\leqq x\\(m,P)=1}} e\left(\frac{am^\kappa}{q}\right) = \sum_{b=1}^{q} e\left(\frac{ab^\kappa}{q}\right)\sum_{\substack{m\leqq x\\ m\equiv b(q)\\(m,P)=1}} 1 = \tag{12}$$

$$= \sum_{\substack{b=1\\((b,q),P)=1}}^{q} e\left(\frac{ab^\kappa}{q}\right)\frac{x}{q}\prod_{\substack{p<z\\ p\nmid\frac{q}{(b,q)}}}\left(1-\frac{1}{p}\right)\left(1+O\left(e^{-\frac{\log\frac{x}{q}}{2\log z}}\right)\right).$$

Note that $\big((b, q), P\big) = 1$ implies $p < z$, $p \nmid \frac{q}{(b,q)}$ is equivalent to $p < z$, $p \nmid q$. Note also that writing an error term $O(qz)$ makes (12) true for any x. Writing back everything to the first line of (11) we arrive at

$$S\left(\frac{a}{q} + \eta\right) = \frac{M}{q} W_q T(q, a) g(M) - \tag{13}$$

$$- \int_0^M \frac{x}{q} W_q T(q, a) g'(x) dx + O\Big(qz + Me^{-\frac{\log \frac{M}{q}}{2\log z}}\Big) +$$

$$+ O\left(\int_0^M \left(qz + xe^{-\frac{\log \frac{x}{q}}{2\log z}}\right) \big|g'(x)\big|\, dx\right) =$$

$$= \frac{W_q}{qW} T(q, a)\left(MWg(M) - \int_0^M xWg'(x)dx\right) +$$

$$+ O\left((1 + |\eta| N)\Big(qz + Me^{-\frac{\log \frac{M}{q}}{4\log z}}\Big)\right) =$$

$$= \frac{W_q}{qW} T(q, a) S(\nu) + O\left((1 + |\eta| N) M e^{-\frac{\log M}{4\log z}}\right)$$

supposing $q \leqq \sqrt{M}$. Lemma S1 now follows from the bound for $T(q, a)$ given in Lemma 2.

Note that from the first line of (13) we also have

$$S(0) = MW\left(1 + O\left(e^{-\sqrt{\log N}}\right)\right). \tag{14}$$

Proof of S2. We can use the previous analysis when $z < q \leqq \sqrt{M}$. The trivial bounds $|S(\eta)| \leqq \kappa M$, $W_q \leqq 1$, and Lemma 2 for $T(q, a)$ give

$$|S(\alpha)| \ll \log z \kappa^{\nu(q)} q^{-\frac{1}{\kappa}} M + QMe^{-\frac{\log M}{4\log z}} \leqq$$

$$\ll Me^{-\frac{\log z}{2\kappa}} \ll Me^{-\frac{1}{2\kappa}(\log\log N)^2}.$$

In the remaining range $\sqrt{M} < q \leqq \frac{N}{Q}$ we use a different approach based on Weyl's inequality (Lemma 3). We have from (11) that

$$|S(\alpha)| \ll \max_{1 \leqq x \leqq M} \Bigg| \sum_{\substack{m \leqq x \\ (m, P) = 1}} e\left(\frac{am^\kappa}{q}\right) \Bigg|. \tag{15}$$

This time we detect $(m, P) = 1$ by the elementary sieve of Erathostenes.

$$\sum_{\substack{m\leqq x\\(m,P)=1}} e\left(\frac{am^\kappa}{q}\right) = \sum_{d|P}\mu(d)\sum_{m\leqq\frac{x}{d}} e\left(\frac{ad^\kappa m^\kappa}{q}\right) \ll \tag{16}$$

$$\ll \sum_{d\leqq Q^{\frac{1}{2\kappa}}}\left|\sum_{m\leqq\frac{x}{d}} e\left(\frac{ad^\kappa m^\kappa}{q}\right)\right| + \sum_{\substack{d|P\\d>Q^{\frac{1}{2\kappa}}}}\frac{x}{d}.$$

We use Lemma 3 for the inner sum together with the trivial bound

$$q \geqq \frac{q}{(q,\, d^\kappa)} \geqq \frac{q}{d^\kappa} \geqq \frac{q}{\sqrt{Q}}.$$

We have

$$\sum_{d\leqq Q^{\frac{1}{2\kappa}}}\left|\sum_{m\leqq\frac{x}{d}} e\left(\frac{ad^\kappa m^\kappa}{q}\right)\right| \ll \tag{17}$$

$$\ll \log N\left(x^{1-\frac{1}{2^\kappa}}Q^{\frac{1}{\kappa 2^{\kappa+1}}} + (\log Q)xM^{-\frac{1}{2^{\kappa+2}}} + x\left(\frac{N}{\sqrt{Q}x^\kappa}\right)^{\frac{1}{2^\kappa}}\right) \ll$$

$$\ll (\log N)MQ^{-\frac{1}{2^{\kappa+1}}}.$$

For the second sum on the right hand side of (16) we note that $d|P$, $d \geqq Q^{\frac{1}{2\kappa}}$ imply $Q^{\frac{1}{2\kappa}} \leqq d \leqq z^{\nu(d)}$ and thus

$$\nu(d) \geqq \frac{\log d}{\log z} \geqq \frac{\log Q}{2\kappa\log z}; \qquad 1 \leqq 2^{\nu(d)-\frac{\log Q}{2\kappa\log z}}$$

and

$$\sum_{\substack{d|P\\d>Q^{\frac{1}{2\kappa}}}}\frac{x}{d} \leqq x2^{-\frac{\log Q}{2\kappa\log z}}\sum_{d|P}\frac{2^{\nu(d)}}{d} = \tag{18}$$

$$\leqq x2^{-\frac{\log Q}{2\kappa\log z}}\prod_{p<z}\left(1+\frac{2}{p}\right) \ll (\log^2 z)M2^{-\frac{\log Q}{2\kappa\log z}}.$$

Now (15), (16), (17) and (18) prove Lemma S2 in the range $\sqrt{M} < q \leqq \frac{N}{Q}$.

Proof of S3. We are going to split the interval $0 < m \leqq M$ into subintervals where $e\left(\frac{hm^\kappa}{N}\right)$ is close to constant. w_m will be well distributed in these subintervals explaining the choice $\kappa\left(\frac{m}{M}\right)^{\kappa-1}$ in the definition of $S(\alpha)$. Note that $\frac{1}{10} \leqq h \leqq M^{1/4}$ is now any real number, not necessarily integer. Let $J = [M^{1/4}]$ and $I(n, j)$ denote the interval

$$\frac{N}{h}\left(n + \frac{j}{J}\right) < m^\kappa \leqq \frac{N}{h}\left(n + \frac{j+1}{J}\right).$$

These intervals cover $0 < m^\kappa \leqq \frac{[\frac{h}{2}]}{h}N$ when $n = 0, \ldots, [\frac{h}{2}] - 1$; $j = 0, \ldots, J-1$. When $h < 2$ we do not split anything at all and when h is an even integer we cover exactly the interval $0 < m \leqq M$. We have

$$S\left(\frac{h}{N}\right) = \sum_{n=0}^{[\frac{h}{2}]-1} \sum_{j=0}^{J-1} \sum_{m \in I(n,j)} e\left(\frac{hm^\kappa}{N}\right) w_m + \sum_{\frac{[\frac{h}{2}]}{h} N < m^\kappa \leqq M^\kappa} e\left(\frac{hm^\kappa}{N}\right) w_m.$$

The last sum contains at most $\ll \frac{M}{h}$ terms and appears only when h is not an even integer. This is responsible for the second term in Lemma S3. If $m \in I(n, j)$ then

$$e\left(\frac{hm^\kappa}{N}\right) = e\left(m + \frac{j}{J} + O\left(\frac{1}{J}\right)\right) = e\left(\frac{j}{J}\right) + O\left(\frac{1}{J}\right).$$

We arrive at

$$S\left(\frac{h}{N}\right) = \sum_{n=0}^{[h/2]-1} \sum_{j=0}^{J-1} e\left(\frac{j}{J}\right) \sum_{m \in I(n,j)} w_m + O\left(\frac{M}{J} + \frac{M}{h}\right).$$

We use the sieve (Lemma 1) with $a = q = 1$ to show that the inner sum is independent of j within a reasonable error term. The summation over j then kills the main terms. In fact for $X - Y < m^\kappa \leqq X$ we have

$$m^{\kappa-1} = X^{1-\frac{1}{\kappa}} + O\left(YX^{-\frac{1}{\kappa}}\right)$$

while

$$X^{1/\kappa} - (X-Y)^{1/\kappa} = \frac{1}{\kappa} Y X^{\frac{1}{\kappa}-1} + O\left(Y^2 X^{\frac{1}{\kappa}-2}\right) = O\left(YX^{\frac{1}{\kappa}-1}\right)$$

so we get from Lemma 1 that

$$\sum_{\substack{X-Y<m^\kappa\leqq X\\(m,P)=1}} \kappa m^{\kappa-1} = YW\left(1+O\left(e^{-\frac{u}{2}}\right)\right)+O\left(\frac{Y^2}{X}\right)$$

where $u=\frac{\log\left(\frac{1}{\kappa}YX^{\frac{1}{\kappa}-1}\right)}{\log z}$. We need this with $Y=\frac{N}{hJ}$, $X=\frac{N}{h}\left(n+\frac{j+1}{J}\right)$ and the main term is really independent of j (and n as well). The contribution of the error terms after adding over j and n is

$$\ll \frac{M\log N}{hJ}+Me^{-\frac{v}{2}}$$

where v is the smallest among the u's, which is at $X=N$ and

$$v\geqq \frac{\log N}{2\kappa\log z}\geqq 2\sqrt{\log N}.$$

The choice of J provides that the first error term is much smaller. This proves Lemma S3.

Proof of S4. From Sobolev's inequality (Lemma 1.2 [4]) we have

$$\sum_{t=1}^{N-1}\left|S\left(\frac{t}{N}\right)\right|^s \leqq N\int_0^1|S(\alpha)|^s\,d\alpha+\frac{s}{2}\int_0^1\left|S(\alpha)^{s-1}S'(\alpha)\right|d\alpha\leqq$$

$$\leqq N\int_0^1|S(\alpha)|^s\,d\alpha+\frac{s}{2}\left(\int_0^1|S(\alpha)|^s\,d\alpha\right)^{\frac{s-1}{s}}\left(\int_0^1|S'(\alpha)|^s\,d\alpha\right)^{\frac{1}{s}}.$$

When s is even both of these integrals can be expressed as a sum over the solutions of an equation weighted by w_m or $2\pi i m^\kappa w_m$ respectively. Thus we get an upper bound if we write κ or $2\pi\kappa M^\kappa$ in place of the weights. We arrive at

$$\sum_{t=1}^{N-1}\left|S\left(\frac{t}{N}\right)\right|^s\leqq(\kappa^sN+s\pi\kappa^sM^\kappa)\int_0^1\left|\sum_{m\leqq M}e(\alpha m)\right|^s d\alpha\ll$$

$$\ll(N+M^\kappa)M^{s-\kappa}\ll M^s$$

by Hua's lemma (Lemma 4) with $s=2^\kappa+2$.

5. The generating function $F_0(\alpha)$

In this section we state and prove some properties of the generating function $F_0(\alpha)$. We have to emphasize that these properties hold because $\mathcal{A}_0$ is an extremal set, which means $\mathcal{A}_0 \subset \{1,\ldots,N\}$, $\mathcal{A}_0 - \mathcal{A}_0$ contains no perfect κ-th power, $|\mathcal{A}_0| = \sigma_0$ satisfies (3) but any set $\mathcal{A}'$ for which $\mathcal{A}' \subset \{1,\ldots,N'\}$, $\mathcal{A}' - \mathcal{A}'$ contains no perfect κ-th powers will satisfy (2) whenever $N' < N$. The first consequence of this situation has been derived in (5), which says that $\mathcal{A}_0$ is equally distributed in the two halves of the interval $(1,N]$. We extend this result.

LEMMA F. *For* $1 \leqq q \leqq Q$ *we have*

$$\sum_{a=1}^{q} \sum_{\substack{t=1 \\ \left|\frac{t}{N}-\frac{a}{q}\right| < \frac{Q}{N}}}^{N-1} \left|F_0\left(\frac{t}{N}\right)\right|^2 \ll \frac{(\log\log N)^5}{\log N}\sigma_0^2.$$

PROOF. First we note that, if $N/Q^{3\kappa} \leqq N' \leqq N$, $\mathcal{A}' \subset \{1,\ldots,N'\}$ and $\mathcal{A}' - \mathcal{A}'$ contains no perfect κ-th powers then

$$|\mathcal{A}'| \leqq \left(1 + O\left(\frac{(\log\log N)^5}{\log N}\right)\right)\sigma_0\frac{N'}{N}.$$

This follows immediately from (5) and $|\mathcal{A}'| \leqq \gamma(N')$ (the extremality of N) by

$$\frac{\gamma(N')}{N'} = \frac{\gamma(N)}{N}\left(1 + O\left(\frac{(\log\log N)^5}{\log N}\right)\right).$$

Set $H = \left[\frac{N}{Q^{\kappa+3}}\right]$ and

$$H(\alpha) = \frac{1}{H}\sum_{|h|\leqq H}\left(1 - \frac{|h|}{H}\right)e\left(hq^\kappa\alpha\right) =$$

$$= \left(\frac{\sin \pi H q^\kappa \alpha}{H \sin \pi q^\kappa \alpha}\right)^2 = 1 + O\left(H^2\|q^\kappa\alpha\|^2\right).$$

where $\|x\|$ denotes the distance of x from its nearest integer. On one hand we have

$$\sum_{t=0}^{N-1}\left|F_0\left(\frac{t}{N}\right)\right|^2 H\left(\frac{t}{N}\right) \geqq \sum_{a=1}^{q} \sum_{\left|\frac{t}{N}-\frac{a}{q}\right| < \frac{Q^2}{N}} \left|F_0\left(\frac{t}{N}\right)\right|^2 \left(1 + O\left(\frac{1}{Q^2}\right)\right) =$$

$$= \sigma_0^2 + \sum_{a=1}^{q} \sum_{\substack{t=1 \\ \left|\frac{t}{N} - \frac{a}{q}\right| < \frac{Q^2}{N}}}^{N-1} \left|F_0\left(\frac{t}{N}\right)\right|^2 \left(1 + O\left(\frac{1}{Q^2}\right)\right) + O\left(\frac{\sigma_0^2}{Q^2}\right).$$

On the other hand

$$\sum_{t=0}^{N-1} \left|F\left(\frac{t}{N}\right)\right|^2 H\left(\frac{t}{N}\right) = \frac{N}{H} \sum_{\substack{a-a'+q^\kappa h \equiv 0(N) \\ |h| \leqq H}} \left(1 - \frac{|h|}{H}\right) =$$

$$= \frac{2N}{H} \sum_{\substack{a-a'+q^\kappa h \equiv 0(N) \\ |h| \leqq H,\, a>a'}} \left(1 - \frac{|h|}{H}\right) + O\left(\frac{\sigma_0 N}{H}\right).$$

Here and later a, a' represent elements of $\mathcal{A}_0$. As $1 - q^\kappa H \leqq a - a' + q^\kappa h \leqq \leqq N + q^\kappa H - 1$ we either have $a - a' + q^\kappa h = 0$ or $a - a' + q^\kappa h = N$. In the latter case $a = N + O(q^\kappa H)$, $a' = O(q^\kappa H)$, $a' \equiv a - N\ (q^\kappa)$. The number of choices for a is $O(q^\kappa H)$, for a' is $O(H)$, and we arrive at

$$\text{(19)} \qquad \sigma_0^2 + \sum_{\substack{a=1 \\ \left|\frac{t}{N} - \frac{a}{q}\right| < \frac{Q^2}{N}}}^{q} \left|F\left(\frac{t}{N}\right)\right|^2 \left(1 + O\left(\frac{1}{Q^2}\right)\right) \leqq$$

$$\leqq \frac{2N}{H} \sum_{\substack{a-a' \equiv 0(q^\kappa) \\ a' < a \leqq a' + q^\kappa H}} \left(1 - \frac{a-a'}{q^\kappa H}\right) + O\left(\frac{\sigma_0^2}{Q^2}\right).$$

Set $J = [\log N]$ and split the interval $(0,\, H]$ into J equal subintervals of type $\left(\frac{jH}{J}, \frac{(j+1)H}{J}\right]$, $j = 0, \ldots, J-1$. Let us fix a'. For every j we can construct a set $\mathcal{A}'$ by

$$\mathcal{A}' = \left\{\frac{a-a'}{q^\kappa} : a' + \frac{jq^\kappa H}{J} < a \leqq a' + \frac{(j+1)q^\kappa H}{J}, a \equiv a'(q^\kappa)\right\}.$$

Here $\mathcal{A}' - \mathcal{A}'$ contains no perfect κ-th power as $\frac{a_1-a'}{q^\kappa} - \frac{a_2-a'}{q^\kappa} = m^\kappa$ implies $a_1 - a_2 = (qm)^\kappa$. As we noted in the beginning

$$|\mathcal{A}'| \leqq \left(1 + O\left(\frac{(\log\log N)^5}{\log N}\right)\right) \sigma_0 \frac{H}{JN}$$

and we get

$$\sum_{\substack{a-a'\equiv 0(q^\kappa)\\ a'<a<a'+q^\kappa H}} \left(1-\frac{a-a'}{q^\kappa H}\right) \leqq \tag{20}$$

$$\leqq \sum_{a'\in\mathcal{A}_0}\sum_{j=0}^{J-1}\left(1-\frac{j}{J}\right)\left(1+O\left(\frac{(\log\log N)^5}{\log N}\right)\right)\sigma_0\frac{H}{JN} \leqq$$

$$\leqq \frac{\sigma_0^2 H}{2N}\left(1+O\left(\frac{(\log\log N)^5}{\log N}\right)\right).$$

(19) and (20) prove Lemma F.

6. Combinatorics of rational numbers

Let $K \geqq 1$ and $L \geqq 1$ be given integers, moreover let $\mathcal{K}$ be a given set of rational numbers with denominators at most K, i.e.

$$\mathcal{K} \subset \left\{\frac{a}{k}; 1 \leqq a \leqq k \leqq K, (a, k) = 1\right\}.$$

For every $\frac{a}{k} \in \mathcal{K}$ we have another given set $\mathcal{L}_{a/k}$ of rational numbers with denominators at most L, i.e.

$$\mathcal{L}_{a/k} \subset \left\{\frac{b}{l}; 1 \leqq b \leqq l \leqq L, (b,l) = 1\right\}.$$

We want to conclude that the set

$$\mathcal{Q} = \left\{\frac{a}{k}+\frac{b}{l}; \frac{a}{k}\in\mathcal{K}, \frac{b}{l}\in\mathcal{L}_{a/k}\right\}$$

is big. Of course, without any additional condition we can say nothing. However, if we know that for every fixed $l \leqq L$ there are only a few possible numerators in the union of all the sets $\mathcal{L}_{a/k}$ then we get our result.

LEMMA CR. *Let τ be the maximal value of the divisor function up to KL, and let $G \geqq 1$, $H \geqq 1$, $B \geqq 1$ be integers such that*

$$|\mathcal{K}| = G; |\mathcal{L}_{a/k}| \geqq H \text{ for all } a/k \in \mathcal{K};$$

$$\left|\left\{b;\ \frac{b}{l} \in \bigcup \mathcal{L}_{a/k}\right\}\right| \leqq B \ for\ all\ l \leqq L.$$

Then

$$|\mathcal{Q}| \geqq GH\left(\frac{H}{LB\tau^8(1+\log K)}\right).$$

PROOF. Let us fix an $\frac{a}{k} \in \mathcal{K}$ first. For any $\frac{b}{l} \in \mathcal{L}_{a/k}$ we associate a pair of integers (d, f) such that $d = (k, l)$, $k = dk'$, $l = dl'$, $(k', l') = 1$, $f = (al' + bk', d)$. Note that $f|d$, $d|k$, $(f, k') = (f, l') = 1$ and

$$\frac{a}{k} + \frac{b}{l} = \frac{ad + bk}{kl} = \frac{\frac{al'+bk'}{f}}{\frac{l'k'd}{f}}$$

where this last fraction can not be simplified any more. The number of possible pairs (d, f) is at most $\tau_3(k) \leqq \tau^2$ so there is a pair associated to more than $|\mathcal{L}_{a/k}|/\tau^2$ rational numbers $\frac{b}{l} \in \mathcal{L}_{a/k}$. We associate this popular pair (d, f) to $\frac{a}{k}$ and set

$$\mathcal{L}^*_{a/k} = \left\{\frac{b}{l} \in \mathcal{L}_{a/k};\ (k, l) = d,\ (al' + bk', d) = f\right\}.$$

We have $|\mathcal{L}^*_{a/k}| \geqq \frac{1}{\tau^2}|\mathcal{L}_{a/k}| \geqq \frac{H}{\tau^2}$.

Next we fix $k \leqq K$ and set $A(k) = \{a;\ \frac{a}{k} \in \mathcal{K}\}$. We have

$$\sum_{k \leqq K} |A(k)| = |\mathcal{K}| = G.$$

Again there is a pair (d, f) and a set $A^*(k) \subset A(k)$ such that $|A^*(k)| \geqq \frac{1}{\tau^2}|A(k)|$ and (d, f) is associated to all $\frac{a}{k}$, $a \in A^*(k)$. We finally set

$$\mathcal{K}^* = \left\{\frac{a}{k};\ k \leqq K,\ a \in A^*(k)\right\},$$

$$r(k) = \left|\{a \bmod f;\ a \in A^*(k)\}\right|,$$

and we note that

$$|\mathcal{K}^*| = \sum_{k \leqq K} |A^*(k)| \geqq \frac{G}{\tau^2}.$$

If $k \leqq K$ is fixed and (d, f) is the associated pair then on the one hand

$$\left|\bigcup_a \mathcal{L}^*_{a/k}\right| \leqq \sum_{\substack{l \leqq L \\ d|l}} \left|\left\{b;\ \frac{b}{l} \in \bigcup \mathcal{L}_{a/k}\right\}\right| \leqq \frac{LB}{d}, \tag{21}$$

but on the other hand $\frac{b}{l} \in \mathcal{L}^*_{a_1/k} \cap \mathcal{L}^*_{a_2/k}$ implies $(a_1 l' + bk', d) = (a_2 l' + bk', d) = f$ and so $a_1 l' \equiv a_2 l' \pmod f$. Hence $a_1 \equiv a_2 \pmod f$ in view of $(f, l') = 1$. This means $a_1 \not\equiv a_2 \pmod f$ implies $\mathcal{L}^*_{a_1/k} \cap \mathcal{L}^*_{a_2/k} = \emptyset$ and so

$$\left|\bigcup_a \mathcal{L}^*_{a/k}\right| \geqq r(k)\frac{H}{\tau^2}. \tag{22}$$

(21) and (22) together give

$$r(k)d \leqq \frac{LB\tau^2}{H}. \tag{23}$$

Now we fix a fraction $\frac{c}{q}$ and check how many solutions the equation

$$\frac{c}{q} = \frac{a}{k} + \frac{b}{l}; \quad \frac{a}{k} \in \mathcal{K}^*, \quad \frac{b}{l} \in \mathcal{L}^*_{a/k} \tag{24}$$

has. If we write $q = k'l'e$, $(k', l') = 1$ (we can do this at most in $\tau_3(q) \leqq \tau^2$ different ways) then for every $f \leqq \min\left(\frac{K}{ek'}, \frac{L}{el'}\right)$ we have $d = ef$, $k = k'd$, $l = l'd$ are determined. Also $a \pmod{k'}$ is determined by $al' + bk' = cf$. For $a \pmod f$ there are $r(k)$ choices and as $(f, k') = 1$ there are $r(k)\frac{k}{k'f} = r(k)\frac{d}{f}$ choices of a. Finally, a determines b. Thus the number of solutions of (24) is by (23) at most

$$\sum_{q = l'k'e} \sum_f r(efk')\frac{d}{f} \leqq \sum_{q = l'k'e} \sum_{f \leqq K} \frac{LB\tau^2}{fH} \leqq \frac{LB\tau^4(1 + \log K)}{H}.$$

This means

$$|\mathcal{Q}| \geqq \frac{H}{LB\tau^4(1 + \log K)} \sum_{\frac{a}{k} \in \mathcal{K}^*} |\mathcal{L}^*_{a/k}| \geqq \frac{GH^2}{LB\tau^8(1 + \log K)}$$

which completes the proof.

7. Proof of the theorem

We are going to detect solutions of the form $a - a' = m^\kappa$ where $a \in \mathcal{A}_0$, $a' \in \mathcal{A}_1$, $(m, P) = 1$. At first sight this looks curious as $2 \nmid m$ but $\mathcal{A}_0$ could be very big without any $2 \nmid a - a'$. We stress again that this can not happen with an extremal $\mathcal{A}_0$ which must be well distributed in residue classes to small moduli. This fact is implicit in Lemma F.

For any $1 \leqq \lambda$, $1 \leqq K$, $1 \leqq U$ we define

$$P_\lambda(K, U) = \left| \left\{ \frac{a}{k}; 1 \leqq a \leqq k \leqq K,\ (a, k) = 1,\ \max_{\left|\frac{t}{N} - \frac{a}{k}\right| < \frac{\lambda Q}{N}} \left| F_1\left(\frac{t}{N}\right)\right| \geqq \frac{\sigma_1}{U} \right\} \right|;$$

$$Q_\lambda = Q_{\lambda-1}^4 Q_1 = Q_1^{\frac{4^\lambda - 1}{3}}, \quad Q_1 > 1; \quad \mu_\lambda = \max_{\substack{1 \leqq K \leqq Q_\lambda \\ 1 \leqq U}} P_\lambda(K, U)/U^2.$$

Q_1 is our most important parameter which will determine the exponent in the Theorem. We choose Q_1 later optimally. We want that for $K \leqq Q_\lambda$ the intervals in the definition of $P_\lambda(K, U)$ should be disjoint, and also $Q_\lambda < z^{\frac{1}{8\kappa}} = e^{\frac{1}{8\kappa}(\log\log N)^2}$. Both follow if we can assume

$$4^\lambda \log Q_1 \leqq \frac{3}{8\kappa}(\log\log N)^2. \tag{25}$$

K_λ and U_λ will denote that pair where μ_λ takes its maximum. If this happens for different K_λ, let K_λ be minimal such. As $K = U = 1$ is considered in the definition of μ_λ we have

$$1 \leqq \mu_\lambda \leqq \frac{K_\lambda^2}{U_\lambda^2}. \tag{26}$$

Like in (9), Section 2 we have

$$P_\lambda(K, U)\frac{\sigma_1^2}{U^2} \leqq \sum_{t=0}^{N-1} \left| F_1\left(\frac{t}{N}\right)\right|^2 = \sigma_1 N,$$

especially

$$\sigma_1 \leqq \frac{N}{\mu_\lambda} \tag{27}$$

for any λ, Q_1 satisfying (25).

Clearly $\mu_\lambda \leqq \mu_{\lambda+1}$ and either there is a $1 \leqq \lambda \leqq \Lambda$ such that

$$\mu_\lambda \leqq \mu_{\lambda+1} \leqq \mu_\lambda (\log N)^{\frac{1}{2}} \tag{28}$$

or

$$\mu_\Lambda \geqq (\log N)^{\frac{\Lambda}{2}}. \tag{29}$$

We will show that (28) does not happen when $\lambda \leqq \frac{1}{2} \log\log\log\log N$ and (27), (29) will prove our Theorem.

Let $1 \leqq \lambda$ be fixed and let K_λ, U_λ provide the corresponding maximum. By (26), (25) and the definitions

$$1 \leqq U_\lambda \leqq K_\lambda \leqq Q_\lambda \leqq e^{\frac{1}{8\kappa}(\log\log N)^2}. \tag{30}$$

We can select a set $\mathcal{P} = \left\{\frac{t}{N}\right\}$ such that $\left|F_1\left(\frac{t}{N}\right)\right| \geqq \frac{\sigma_1}{U_\lambda}$ is the maximal value in $\left|\frac{t}{N} - \frac{a}{k}\right| < \frac{\lambda Q}{N}$, $1 \leqq a \leqq k \leqq K_\lambda$, $(a, k) = 1$ and $|\mathcal{P}| = P_\lambda(K_\lambda, U_\lambda)$. For any $\alpha \in \mathcal{P}$ we have

$$\sum_{t=0}^{N-1} F_0\left(-\frac{t}{N}\right) F_1\left(\alpha + \frac{t}{N}\right) S\left(\frac{t}{N}\right) = \sum_{-a+a'+m^\kappa \equiv 0(N)} e(\alpha a') w_m = 0$$

since $-N+1 \leqq -a + a' + m^\kappa \leqq N-1$. Thus by (14), Section 3 we have

$$\sum_{t=1}^{N-1} \left| F_0\left(\frac{t}{N}\right) F_1\left(\alpha + \frac{t}{N}\right) S\left(\frac{t}{N}\right)\right| \geqq \frac{\sigma_0\sigma_1}{U_\lambda} S(0) \geqq \frac{\sigma_0\sigma_1 M}{2U_\lambda(\log\log N)^2}.$$

The contribution of those t for which either $\left|F_0\left(\frac{t}{N}\right)\right| \leqq \frac{\sigma_0}{Q}$ or $\left|F_1\left(\alpha + \frac{t}{N}\right)\right| \leqq$ $\leqq \frac{\sigma_1}{Q}$ is negligible by Hölder inequality, Parseval identity (6), Lemma S4, and (5), (25), (30). Indeed

$$\sum \left| F_0\left(\frac{t}{N}\right) F_1\left(\alpha + \frac{t}{N}\right) S\left(\frac{t}{N}\right)\right| \leqq$$

$$\leqq \max |F_i|^{\frac{1}{2^{\kappa-1}+1}} \left(\sum |F_i|^2\right)^{\frac{1}{2} - \frac{1}{2^\kappa+2}} \times \left(\sum |F_{1-i}|^2\right)^{\frac{1}{2}} \left(\sum |S|^{2^\kappa+2}\right)^{\frac{1}{2^\kappa+2}} \ll$$

$$\ll \left(\frac{\sigma_i}{Q}\right)^{\frac{1}{2^{\kappa-1}+1}} (\sigma_i N)^{\frac{1}{2} - \frac{1}{2^\kappa+2}} (\sigma_{1-i} N)^{\frac{1}{2}} M \ll \frac{\sigma_0 M N}{Q^{\frac{1}{2^{\kappa-1}+1}}} < \frac{\sigma_0 \sigma_1 M}{U_\lambda (\log\log N)^3}.$$

Similarly the contribution of those t for which

$$\left|S\left(\frac{t}{N}\right)\right| \leqq \frac{\sigma_1 M}{Q_\lambda N(\log\log N)^3} \tag{32}$$

is negligible by Cauchy–Schwarz inequality and Parseval identity (6)

$$\sum\left|F_0\left(\frac{t}{N}\right)F_1\left(\alpha+\frac{t}{N}\right)S\left(\frac{t}{N}\right)\right| \leqq$$

$$\leqq \max|S|\left(\sum|F_0|^2\right)^{1/2}\left(\sum|F_1|^2\right)^{1/2} \leqq$$

$$\leqq \frac{\sigma_1 M}{Q_\lambda N(\log\log N)^3}(\sigma_0 N)^{1/2}(\sigma_1 N)^{1/2} \leqq \frac{\sigma_0\sigma_1 M}{U_\lambda N(\log\log N)^3}.$$

By Dirichlet approximation theorem we can find a $\frac{b}{l}$, $1 \leqq b \leqq l \leqq \frac{N}{Q}$, $(b, l) = 1$ to each $\frac{t}{N}$ such that

$$\left|\frac{t}{N} - \frac{b}{l}\right| < \frac{Q}{lN}.$$

If $z \leqq l \leqq \frac{N}{Q}$ then Lemma S2 says

$$\left|S\left(\frac{t}{N}\right)\right| \ll Me^{-\frac{1}{4\kappa}(\log\log N)^2} < \frac{\sigma_1 M}{Q_\lambda N(\log\log N)^4}$$

thus (32) is really the case. If $Q_{\lambda+1}/Q_\lambda < l < z$ then by Lemma S1

$$\left|S\left(\frac{t}{N}\right)\right| \ll \frac{M}{l^{1/3}} \leqq \frac{M}{Q_\lambda Q_1^{1/3}} < \frac{\sigma_1 M}{Q_\lambda N(\log\log N)^4}$$

provided

$$Q_1 \geqq \left(\frac{N}{\sigma_1}\right)^3 (\log\log N)^{12}. \tag{33}$$

Let $\tau(l, b, \alpha)$ be the set of $\frac{t}{N} \neq 0$ such that $\left|\frac{t}{N} - \frac{l}{b}\right| < \frac{Q}{N}$, $\left|F_0\left(\frac{t}{N}\right)\right| \geqq \frac{\sigma_0}{Q}$, $\left|F_1\left(\alpha+\frac{t}{N}\right)\right| \geqq \frac{\sigma_1}{Q}$. We get from (31) that

$$\frac{\sigma_0\sigma_1 M}{U_\lambda(\log\log N)^2} \ll$$

$$\ll \sum_{l \leqq \frac{Q_{\lambda+1}}{Q_\lambda}} \sum_{(b,l)=1} \sum_{\frac{t}{N} \in \tau(l,b,\alpha)} \left| F_0\left(\frac{t}{N}\right) F_1\left(\alpha + \frac{t}{N}\right) S\left(\frac{t}{N}\right) \right| \ll$$

$$\ll \sum_{l \leqq \frac{Q_{\lambda+1}}{Q_\lambda}} \sum_{(b,l)=1} \max_{\tau(l,b,\alpha)} \left| F_0\left(\frac{t}{N}\right) \right| \max_{\tau(l,b,\alpha)} \left| F_1\left(\alpha + \frac{t}{N}\right) \right| \sum_{\frac{t}{N} \in \tau(l,b,\alpha)} \left| S\left(\frac{t}{N}\right) \right|.$$

Lemma S1 and Lemma S3 provide

$$\sum_{\frac{t}{N} \in \tau(l,\, b,\, \alpha)} \left| S\left(\frac{t}{N}\right) \right| \ll \frac{\kappa^{\nu(l)}}{l^{1/2}} (\log\log N)^2 \sum_{\frac{t}{N} \in \tau(l,\, b,\, \alpha)} \left| S\left(\frac{t}{N} - \frac{b}{l}\right) \right| +$$

$$+ Q^2 M e^{-\sqrt{\log N}} \ll \frac{\kappa^{\nu(l)}}{l^{1/2}} (\log\log N)^6 M.$$

This means that there are integers $1 \leqq V_\alpha \leqq Q$, $1 \leqq W_\alpha \leqq Q$, $1 \leqq L_\alpha \leqq \frac{Q_{\lambda+1}}{Q_\lambda}$ such that the set $\mathcal{L}(\alpha)$ defined by

$$\mathcal{L}(\alpha) = \left\{ \frac{b}{l};\ \frac{L_\alpha}{2} < l \leqq L_\alpha,\ (b,\, l) = 1,\quad \frac{\sigma_0}{V_\alpha} \leqq \max_{\tau(l,\, b,\, \alpha)} \left| F_0\left(\frac{t}{N}\right) \right| < \frac{2\sigma_0}{V_\alpha}, \right.$$

$$\left. \frac{\sigma_1}{W_\alpha} \leqq \max_{\tau(l,\, b,\, \alpha)} \left| F_1\left(\alpha + \frac{t}{N}\right) \right| < \frac{2\sigma_1}{W_\alpha} \right\}$$

satisfies

$$|\mathcal{L}(\alpha)| \gg \frac{V_\alpha W_\alpha L_\alpha^{1/2}}{U_\lambda R (\log\log N)^{18}}$$

where $R = \max\{\kappa^{\nu(l)};\ l \leqq Q_{\lambda+1}\}$. Here we chose V_α, W_α, L_α as diadic integers so the possible number of choices was

$$\ll \log Q \log Q \log z \leqq (\log\log N)^{10}.$$

But this also means that for at least

$$P_\lambda(K_\lambda,\, U_\lambda)/(\log\log N)^{10}$$

different $\alpha \in \mathcal{P}$ we chose the same triplet V, W, L. With this triplet there is a set $\mathcal{P}^* \subset \mathcal{P}$ such that

$$(34) \qquad |\mathcal{P}^*| \geqq \frac{P_\lambda(K_\lambda,\, U_\lambda)}{(\log\log N)^{10}}, \qquad \left| F_1\left(\frac{t}{N}\right) \right| \geqq \frac{\sigma_1}{U_\lambda}, \text{ for } \frac{t}{N} \in \mathcal{P}^*.$$

Further, for any $\frac{t}{N} \in \mathcal{P}^*$ there is an $\frac{a}{k}$ with $\left|\frac{t}{N} - \frac{a}{k}\right| < \frac{\lambda Q}{N}$, $1 \leqq a \leqq k \leqq K_\lambda$, $(a, k) = 1$ and also there is a set $\mathcal{L}\left(\frac{t}{N}\right)$ of rational numbers $\frac{b}{l}$, $\frac{1}{2}L < l \leqq \leqq L$, $(b, l) = 1$, with corresponding numbers $\frac{v}{N} \neq 0$, $\frac{w}{N} \neq 0$, $\left|\frac{v}{N} - \frac{b}{l}\right| < \frac{Q}{N}$, $\left|\frac{w}{N} - \frac{b}{l}\right| < \frac{Q}{N}$ such that

$$\text{(35)} \qquad \begin{cases} \left|\mathcal{L}\left(\dfrac{t}{N}\right)\right| \gg \dfrac{VWL^{1/2}}{U_\lambda R\,(\log\log N)^{18}}, \\ \dfrac{\sigma_0}{V} \leqq \left|F_0\left(\dfrac{v}{N}\right)\right| < \dfrac{2\sigma_0}{V}, \quad \dfrac{\sigma_1}{W} \leqq \left|F_1\left(\dfrac{t}{N} + \dfrac{w}{N}\right)\right| < \dfrac{2\sigma_1}{W}. \end{cases}$$

We are going to use Lemma CR to show that the number of different $\frac{t+w}{N}$ is large. Because of the small size of k and l this follows if the number of different $\frac{a}{k} + \frac{b}{l}$ is large. $\mathcal{K}$ is the set of $\frac{a}{k}$, so G satisfies (34), $\mathcal{L}_{a/k}$ is the set of the corresponding $\frac{b}{l}$, so H satisfies (35). Finally Lemma F says that for any fixed $\frac{1}{2}L < l \leqq L$

$$\left|\left\{b;\ \frac{b}{l} \in \cup\,\mathcal{L}_{a/k}\right\}\right| \left(\frac{\sigma_0}{V}\right)^2 \leqq \sum_{b=1}^{l} \sum_{\substack{v=1 \\ \left|\frac{v}{N} - \frac{b}{l}\right| < \frac{Q}{N}}}^{N-1} \left|F_0\left(\frac{v}{N}\right)\right|^2 \ll \frac{(\log\log N)^5}{\log N}\sigma_0^2,$$

i.e.

$$B \ll \frac{V^2(\log\log N)^5}{\log N}.$$

Lemma CR says that the number of different $\frac{a}{k} + \frac{b}{l}$ is at least

$$\gg \frac{P_\lambda(K_\lambda, U^\lambda)}{(\log\log N)^{10}} \frac{V^2W^2L}{U_\lambda^2 R^2(\log\log N)^{36}} \frac{\log N}{LV^2(\log\log N)^7\tau^8} \geqq$$

$$\geqq \frac{W^2\log N}{R^2\tau^8(\log\log N)^{53}} \frac{P_\lambda(K_\lambda, U_\lambda)}{U_\lambda^2}.$$

As the denominator of $\frac{a}{k} + \frac{b}{l}$ is at most $kl \leqq Q_{\lambda+1}$ and $\left|\frac{t+w}{N} - \frac{a}{k} - \frac{b}{l}\right| < < \frac{(\lambda+1)Q}{N}$ we get that

$$P_{\lambda+1}(Q_{\lambda+1}, W) \gg \frac{\mu_\lambda W^2 \log N}{R^2\tau^8(\log\log N)^{53}},$$

i.e.

$$\mu_{\lambda+1} \geqq \mu_\lambda \frac{\log N}{R^2\tau^8(\log\log N)^{54}}. \tag{36}$$

As is well-known

$$\log\tau,\ \log R \ll \frac{\log Q_{\lambda+1}}{\log\log Q_{\lambda+1}} \ll \frac{4^\lambda \log Q_1}{\lambda + \log\log Q_1}.$$

If

$$\frac{4^\lambda \log Q_1}{\lambda + \log\log Q_1} \leqq C_1 \log\log N \tag{37}$$

with a suitably chosen $C_1 > 0$ then (25) is satisfied trivially and (28) is not true for λ. We have to balance this with (33). The close to optimal choice of Q_1 is

$$Q_1 = (\log N)^{(\log\log\log N)^{1/4}}$$

and then (37) says

$$4^\lambda \ll (\log\log\log N)^{3/4}$$

which follows if

$$\lambda \leqq \frac{1}{2}\log\log\log\log N.$$

(27) and (29) prove our Theorem.

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MARKOV AND BERNSTEIN TYPE INEQUALITIES ON SUBSETS OF $[-1,1]$ AND $[-\pi,\pi]$

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The primary purpose of this note is to extend Markov's and Bernsteins's inequalities to arbitrary subsets of $[-1,1]$ and $[-\pi,\pi]$, respectively.

We denote by $\mathcal{P}_n$ the set of all real algebraic polynomials of degree at most n, and let $m(\cdot)$ denote the Lebesgue measure of a subset of $\mathbf{R}$. We were led to the results of this paper by the following problem. Can one give polynomials $p_n \in \mathcal{P}_n$ and numbers $a_n \in (0,1)$, $n = 1,2,\dots$, such that

(i) $m(\{x \in [0,1]: |p_n(x)| \leqq 1\}) \geqq 1 - a_n$,

(ii) $\max\limits_{0 \leqq x \leqq a_n} |p_n(x)| \leqq 1$

and

(iii) $\lim\limits_{n\to\infty} n^{-2}|p_n'(0)| = \infty$

are satisfied? This question was asked by Vilmos Totik, and a positive answer would have been used in proving a conjecture in the theory of orthogonal polynomials. However, Theorem 2 of this note shows that the answer to the above question is negative, in fact, it gives slightly more. In addition, our Theorem 1 answers the corresponding question for trigonometric polynomials. Though our results cannot be used for Totik's original purpose, our proofs illustrate well, how Remez-type inequalities can be used in proving various other polynomial inequalities.

In this note we prove the following pair of theorems.

THEOREM 1. *Let $0 < a \leqq 2\pi$, $0 < L \leqq 1$, let A be a closed subset of $[0,2\pi]$ with Lebesgue measure $m(A) \geqq 2\pi - a$. There is an absolute constant $c_1 > 0$ such that*

$$(1) \qquad \max_{t\in I} |p'(t)| \leqq c_1 L^{-1}(n + n^2 a) \max_{t \in A} |p(t)|$$

for every real trigonometric polynomial p of degree at most n, and for every subinterval I of A with length at least La.

THEOREM 2. *Let $0 < a \leqq 1$, $0 < M \leqq 1$, let A be a closed subset of $[0,1]$ with Lebesgue measure $m(A) \geqq 1 - a$. There is an absolute constant $c_2 > 0$*

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such that

$$\max_{x \in I} |p'(x)| \leqq c_2 M^{-1} n^2 \max_{x \in A} |p(x)| \tag{2}$$

for every real algebraic polynomial p of degree at most n, and for every subinterval I of A with length at least Ma.

Up to the constant c_1, Theorem 1 is an extension of both Bernstein's [5, pp. 39–41] and Videnskii's [6] inequalities, while up to the constant c_2, Theorem 2 contains Markov's inequality [5, pp. 39–41] as a special case.

The key to the proof of Theorem 1 is a Remez-type inequality [2] proved recently for trigonometric polynomials, while the proof of Theorem 2 relies on Theorem 1.

PROOF OF THEOREM 1. Denote by $\mathcal{T}_n$ the set of all real trigonometric polynomials of degree at most n. If $\pi/2 \leqq a \leqq 2\pi$, then the theorem follows from an extension [1, Theorem 5] of an inequality of Videnskii [6]. Therefore, in the sequel we assume that $0 < a < \pi/2$. Let I be a subinterval of A such that $m(I) \geqq La$ and $\pi \in I$. It is sufficient to prove that there is an absolute constant $c_1 > 0$ such that

$$|p'(\pi)| \leqq c_1 L^{-1}(n + n^2 a) \max_{t \in A} |p(t)| \tag{3}$$

for every $p \in \mathcal{T}_n$. Let T_n be the Chebyshev polynomial of degree n given by

$$T_n(x) = \cos(n \arccos x), \qquad -1 \leqq x \leqq 1, \tag{4}$$

and let

$$Q_{n,La}(t) := T_{2n}\Big(\sin(t/2)\big(\cos(La/4)\big)^{-1}\Big)\Big(T_{2n}\big((\cos(La/4))^{-1}\big)\Big)^{-1}. \tag{5}$$

A simple calculation shows that $Q_{n,La} \in \mathcal{T}_n$,

$$Q_{n,La}(\pi) = 1, \quad Q'_{n,La}(\pi) = 0, \quad \max_{t \in \mathbf{R}} |Q_{n,La}(t)| = 1, \tag{6}$$

and there is an absolute constant $c_3 > 0$ such that

$$|Q_{n,La}(t)| \leqq \exp(-c_3 n La), \qquad t \in [0, \pi - La/2] \cup [\pi + La/2, 2\pi]. \tag{7}$$

Let $p \in \mathcal{T}_n$ be such that

$$\max_{t \in A} |p(t)| = 1. \tag{8}$$

The Remez-type inequality for trigonometric polynomials [2, Theorem 2], $m(A) \geqq 2\pi - a$, $0 < a \leqq \pi/2$, and (8) yield that there is an absolute constant $c_4 > 0$ such that

$$\max_{0 \leqq t \leqq 2\pi} |p(t)| \leqq \exp(c_4 n a). \tag{9}$$

Denote the endpoints of the interval I by $\alpha < \beta$. Since $\beta - \alpha = m(I) \geqq La$ and $\pi \in I$, we have either $\alpha \leqq \pi - La/2$ or $\beta \geqq \pi + La/2$. We may assume that

$$\beta \geqq \pi + La/2, \tag{10}$$

otherwise we consider the trigonometric polynomial $\tilde{p} \in \mathcal{T}_n$ defined by $\tilde{p}(t) := := p(\pi - t)$. Now let

$$m := [c_4 c_3^{-1} L^{-1} n] + 1 \quad \text{and} \quad Q := Q_{m,La}. \tag{11}$$

Observe that (6)–(11) imply

$$|(pQ)(t)| \leqq 1, \qquad t \in E, \tag{12}$$

where

$$E := [0, \pi - La/2] \cup [\pi, 2\pi]. \tag{13}$$

Note that E is an interval of the period with length $2\pi - La/2$, and $\pi \in E$. Therefore an extension [1, Theorem 5] of an inequality of Videnskii [6], $0 < < L \leqq 1$ and (8) yield that there are absolute constants $c_5 > 0$ and $c_1 > 0$ such that

(14)
$$|(pQ)'(\pi)| \leqq \left((n+m) + c_5(n+m)^2 La/2\right) \leqq c_1 L^{-1}(n + n^2 a) \max_{t \in A} |p(t)|.$$

Recalling (6), we have

$$p'(\pi) = (pQ)'(\pi), \tag{15}$$

which, together with (14) gives the theorem. □

PROOF OF THEOREM 2. If $1/4 \leqq a \leqq 1$, then the theorem follows from the Markov inequality [5, pp. 39–41]. Therefore, in what follows we may assume that $0 < a \leqq 1/4$. Without loss of generality we may also assume that $I=[0,b]$, where $Ma \leqq b \leqq 1$, the general case can be deduced from this easily by a linear transformation. Let $p \in \mathcal{P}_n$,

$$y(t) := 1/2 + (1/2 + a)\cos t, \tag{16}$$

$$\tilde{p}(t) := p\big(y(t)\big) \in \mathcal{T}_n, \tag{17}$$

$$\tilde{A} := \{t \in [0, 2\pi] : \ y(t) \in A\}, \tag{18}$$

$$\tilde{I} := \{t \in [0, \pi] : \ y(t) \in I\}, \tag{19}$$

and

$$\tilde{a} := 2\pi - m(\tilde{A}), \quad \text{i.e.} \quad m(\tilde{A}) = 2\pi - \tilde{a}. \tag{20}$$

It is easy to see that $0 < a \leqq 1/4$, $A \subset [0,1]$, $m(A) \geqq 1-a$, $m(I) \geqq Ma$, (16), (18), (19), and (20) imply that

$$\tilde{a} \leqq c_6\sqrt{a} \tag{21}$$

and

$$m(\tilde{I}) \geqq c_7 M\sqrt{a} \geqq c_7 c_6^{-1} M\tilde{a} \tag{22}$$

with suitable absolute constants $c_6 > 0$ and $c_7 > 0$. If $L := c_7c_6^{-1}M \leqq 1$ and $a \geqq n^{-2}$, then Theorem 1, (20), (21), and (22) yield

(23)
$$\max_{t\in\tilde{I}} \left|\tilde{p}'(t)\right| \leqq c_1 c_7^{-1} c_6 M^{-1}(n + n^2\tilde{a}) \max_{t\in\tilde{A}} \left|\tilde{p}(t)\right| \leqq c_8 M^{-1} n^2\sqrt{a} \max_{x\in A} \left|p(x)\right|$$

with a suitable absolute constant $c_8 > 0$. Also, (16)–(19) and $I \subset [0,1]$ imply that

$$\left|\tilde{p}'(t)\right| = \left|p'\big(y(t)\big)\, y'(t)\right| = \left|p'\big(y(t)\big)\right| (1/2 + a)\sin t \geqq c_9 \left|p'\big(y(t)\big)\right| \sqrt{a}$$

for every $t \in \tilde{I}$ with a suitable absolute constant $c_9 > 0$. Since every $x \in I$ is of the form $x = y(t)$ with some $t \in \tilde{I}$, (23) and (24) imply that

$$\max_{x\in I} \left|p'(x)\right| \leqq c_8 c_9^{-1} M^{-1} n^2 \max_{x\in A} \left|p(x)\right|, \tag{25}$$

whenever $c_7c_6^{-1}M \leqq 1$ and $a \geqq n^{-2}$. If $c_7c_6^{-1}M \geqq 1$, that is $M \geqq c_6c_7^{-1}$, and $a \geqq n^{-2}$, then I can be divided into subintervals of length $k^{-1}m(I)$, where $k := [c_6c_7^{-1}] + 1$, and the already proved part gives the theorem. If

$0 < a < n^{-2}$, $A \subset [0,1]$ and $m(A) \geqq 1 - a$, then the Remez inequality [4, pp. 119–121] or [3] yields that

$$\max_{0 \leqq x \leqq 1} |p(x)| \leqq c_{10} \max_{x \in A} |p(x)| \tag{26}$$

for every $p \in \mathcal{P}_n$, where $c_{10} > 0$ is a suitable absolute constant. Combining this with the Markov inequality [5, pp. 39–41], we obtain

$$\max_{x \in I} |p'(x)| \leqq \max_{0 \leqq x \leqq 1} |p'(x)| \leqq 2n^2 \max_{0 \leqq x \leqq 1} |p(x)| \leqq \tag{27}$$

$$\leqq 2c_{10} n^2 \max_{x \in A} |p(x)|,$$

and the theorem is completely proved. □

It may be interesting to compare Theorem 2 with the following

EXAMPLE 3. Let $0 < a \leqq 1/2$, $A = [0, 1-a] \cup \{1\}$ and

$$P_n(x) = (x-1) T_n\big(2(1-a)^{-1} x - 1\big), \qquad n = 1, 2, \ldots,$$

where T_n is the Chebyshev polynomial of degree n defined by $T_n(x) = \cos(n \arccos x)$, $-1 \leqq x \leqq 1$. Then

$$\max_{x \in A} |P_n'(x)| \geqq |P_n'(1)| = T_n\big(2(1-a)^{-1} - 1\big) \geqq T_n(1 + 2a) \geqq$$

$$\geqq 2^{-1}\big(1 + 2\sqrt{a}\big)^n \geqq 2^{-1}\big(1 + 2\sqrt{a}\big)^n \max_{x \in A} |P_n(x)|.$$

A similar example can be given in the trigonometric case.

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SOME LATTICE HORN SENTENCES FOR SUBMODULES OF PRIME POWER CHARACTERISTIC

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For a ring R with unit the class of lattices embeddable in the submodule lattices of R modules is known to be a quasivariety (cf. Makkai and McNulty [6]). This quasivariety will be denoted by

$$\mathcal{L}(R) = \{\,\mathrm{Su}({}_RM) : {}_RM \text{ is an } R\text{-module}\,\}.$$

We will consider rings with prime power characteristic p^k where $k > 1$. All the rings in the sequel, unless otherwise stated, will be assumed to be of characteristic p^k. Let $\mathbf{W}(p^k)$ denote the class $\{\,\mathcal{L}(R) : \operatorname{char} R = p^k\,\}$. While the *variety* $\mathbf{H}\,\mathcal{L}(R)$ depends only on p^k, the characteristic of R (cf. [5]), and $\mathbf{W}(p)$ is a singleton (cf. [3, p. 88]), $\mathbf{W}(p^k)$ consists of continuously many quasivarieties $\mathcal{L}(R)$, cf. [2]. This result was proved by the following powerful tool. Let τ denote the similarity type consisting of operation symbols $\vee, \wedge, \cdot, \uparrow, \downarrow, \mathbf{0}, \mathbf{1}$ with respective arities 2,2,2,1,1,0,0. The set $\mathcal{I}(R)$ of two-sided ideals of R becomes a τ-algebra in a natural way: $\vee, \wedge$ are the lattice operations, $\mathbf{0} = \{0\}$, $\mathbf{1} = R$, $\cdot$ is the usual product of ideals, $\downarrow X = \{\,px : x \in \in X\,\}$, and $\uparrow X = \{\,x : px \in X\,\}$. Let $K(R)$ denote the set of all nullary τ-terms σ such that $\sigma = \mathbf{1}$ $(= R)$ holds in $\mathcal{I}(R)$, and let $\Sigma(R)$ denote the set of (universal) lattice Horn sentences satisfied in $\mathcal{L}(R)$.

THEOREM A (Hutchinson [2]). *If* $\mathcal{L}(R_1) \subseteqq \mathcal{L}(R_2)$ *then* $K(R_1) \supseteqq K(R_2)$.

The proof of this theorem is based on the following

THEOREM B (Hutchinson [3] and [4]). $\mathcal{L}(R_1) \subseteqq \mathcal{L}(R_2)$ *is equivalent to the existence of an exact embedding functor* R_1-**Mod** $\to$ R_2-**Mod**.

Note that $\Sigma(R_1) \supseteqq \Sigma(R_2)$ is also equivalent to $\mathcal{L}(R_1) \subseteqq \mathcal{L}(R_2)$. Therefore our present investigation based on Horn sentences might be interesting from abelian category theoretical point of view, too.

Our goal is to deal with the following two open problems, the first of which is related to the converse of Theorem A.

PROBLEM C. Does $K(R_1) \supseteqq K(R_2)$ imply $\mathcal{L}(R_1) \subseteqq \mathcal{L}(R_2)$?

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PROBLEM D. Is $\mathbf{W}(p^k)$ closed with respect to arbitrary joins (taken in the lattice of all lattice quasivarieties)?

Note that $\mathbf{W}(p^k)$ is closed with respect to finite joins. It is shown in [2] that $(\mathbf{W}(p^k); \subseteqq)$ contains large chains and antichains and it has a nontrivial automorphism, namely $\mathcal{L}(R) \mapsto \mathcal{L}(R^{\mathrm{op}})$, but we do not know if it is a lattice. An affirmative answer to Problem D or (much less trivially!) to Problem C would imply that $\mathbf{W}(p^k)$ is a lattice. The analogous problems for the *set* of lattice varieties $\mathbf{H}\mathcal{L}(S)$, where the S are rings of any characteristic, have positive solutions (cf.[5]).

MAIN THEOREM. *At least one of Problems C and D has a negative answer.*

The *proof* of the Main Theorem is based on certain lattice Horn sentences $\chi(m,p)$, which might be of separate interest. Note that $\chi(2,2)$ appeared in [1] but without any application that time. Our proof is divided into several lemmas.

First we define appropriate rings. The ring of integers modulo p^k will be denoted by $\mathbf{Z}_{p^k}$. For a given n let F_n denote the polynomial ring

$$\mathbf{Z}_{p^k}[\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n].$$

Let I_n be the ideal generated by

$$\{\,\xi_i\eta_i - p^{k-1}\xi_{i-1} : 1 \leqq i \leqq n\,\} \cup \{\,p\eta_i : 1 \leqq i \leqq n\,\} \cup \{\,p^{k-1}\xi_n\,\} \cup$$

$$\cup \{\,\xi_i\xi_j : 1 \leqq i \leqq n,\ 1 \leqq j \leqq n\,\} \cup \{\,\eta_i\eta_j : 1 \leqq i \leqq n,\ 1 \leqq j \leqq n\,\} \cup$$

$$\cup \{\,\xi_i\eta_j : 1 \leqq i \leqq n,\ 1 \leqq j \leqq n,\ i \neq j\,\},$$

where $\xi_0 = 1$. Put $R_n = F_n/I_n$, $x_i = \xi_i + I_n$, $y_i = \eta_i + I_n$. Note that $x_0 = 1$. By the definition of R_n we have

$$x_iy_i = p^{k-1}x_{i-1},\quad y_iy_j = 0,\quad x_ix_j = 0,\quad x_iy_l = 0,\quad p^kx_i = 0, \tag{1}$$

$$p^{k-1}x_n = 0,\quad p\,y_i = 0 \qquad \text{for } i,j,l \in \{\,1,2,\dots n\,\},\ i \neq l.$$

LEMMA 1. *The elements x_i $(i = 0,1,\dots,n-1)$, x_n and y_i $(i = 1,2,\dots,n)$ are of respective additive order p^k, p^{k-1} and p. Further, the additive group of R_n is the direct sum of the additive cyclic subgroups generated by these elements. In other words, each element of R_n has a unique canonical form*

$$\sum_{i=0}^{n-1}\alpha_ix_i + \beta x_n + \sum_{i=1}^{n}\gamma_iy_i \tag{2}$$

where $\alpha_i \in \{0,1,\dots,p^k-1\}$, $\beta \in \{0,1,\dots,p^{k-1}-1\}$ *and* $\gamma_i \in \{0,1,\dots, p-1\}$. *The rules of computation in* R_n *are* (1) *together with the axioms of unital commutative rings of characteristic* p^k.

PROOF. It suffices to show the uniqueness of (2); the rest is clear. Assume that $0 \in R_n$ is of the form (2). Then, by the definition of I_n, we have

$$(3)\qquad \sum_{i=0}^{n-1}\alpha_i\xi_i + \beta\xi_n + \sum_{i=1}^{n}\gamma_i\eta_i = \sum_{i=1}^{n} f_i\cdot(\xi_i\eta_i - p^{k-1}\xi_{i-1}) + \sum_{i=1}^{n} g_i\cdot p\eta_i +$$

$$+g_0p^{k-1}\xi_n + \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij}\cdot\xi_i\xi_j + \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\cdot\eta_i\eta_j + \sum_{i=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n} s_{ij}\cdot\xi_i\eta_l$$

where f_i, g_i, h_{ij}, r_{ij}, $s_{ij} \in F_n$. We treat the elements of F_n as polynomials in the usual canonical form. Hence these polynomials are sums of uniquely determined summands and each summand consists of uniquely determined factors (i.e. powers of indeterminants) and a unique coefficient (from $\mathbf{Z}_{p^k}$). Suppose we have performed the operations on the right hand side of (3). Then each summand on the right hand side in which η_i is the only indeterminant has a coefficient divisible by p. Therefore $\gamma_i = 0$ for all i. We obtain $\beta = 0$ similarly.

Suppose $\alpha_i \neq 0$ for some i. The only source of ξ_i on the right is $f_{i+1}\cdot \cdot(\xi_{i+1}\eta_{i+1} - p^{k-1}\xi_i)$. Since p^k does not divide α_i, the constant δ in f_{i+1} is not divisible by p. But then $\delta\xi_{i+1}\eta_{i+1}$ cannot be cancelled by other summands. This contradiction completes the proof.

Before describing $K(\mathbf{Z}_{p^k})$ we make the set $\{0,1,2,\dots k\}$ into an algebra of type τ via putting $x \vee y = \max\{x,y\}$, $x \wedge y = \min\{x,y\}$, $\uparrow x = \min\{x+ +1,k\}$, $\downarrow x = \max\{x-1,0\}$, $\mathbf{0} = 0$, $\mathbf{1} = k$ and $x\cdot y = \max\{x+y-k,0\}$. (To avoid confusion, the ordinary product of x and y will be denoted by the concatenation xy.) Denoting the set of nullary τ-terms by $\mathcal{P}_0$, let h be the map associating with any element of $\mathcal{P}_0$ its value in the above-defined algebra $\{0,1,2,\dots k\}$.

LEMMA 2. $K(\mathbf{Z}_{p^k}) = \{\sigma \in \mathcal{P}_0 : h(\sigma) = k\}$.

PROOF. An easy induction on the length of σ yields that the value of σ in $\mathcal{I}(\mathbf{Z}_{p^k})$ is $p^{k-h(\sigma)}\mathbf{Z}_{p^k} = \downarrow^{k-h(\sigma)}\mathbf{Z}_{p^k}$, whence the lemma follows.

LEMMA 3. $\bigcap_{n=1}^{\infty} K(R_n) = K(\mathbf{Z}_{p^k})$.

Proof. For $0 \leqq t \leqq n-1$ and $0 \leqq j \leqq k$ we consider the following subsets of R_n:

$$A_{j,t}^{(n)} = \{\, p^i x_l : 1 \leqq l < n-t,\ i \geqq k-j,\ i \geqq 0 \},$$

$$B_{j,t}^{(n)} = \{\, p^i x_l : n-t \leqq l \leqq n-1,\ i+l \geqq n-t+k-j-1,\ i \geqq 0 \},$$

$$C_{j,t}^{(n)} = \{\, p^i x_n : i \geqq k-j-1,\ i \geqq 0 \},$$

$$D_{j,t}^{(n)} = \{ y_l : 1 \leqq l \leqq n,\ j > 0 \} \quad \text{and}$$

$$E_{j,t}^{(n)} = \{\, p^i : i \geqq k-j \} \cup A_{j,t}^{(n)} \cup B_{j,t}^{(n)} \cup C_{j,t}^{(n)} \cup D_{j,t}^{(n)}\,.$$

Note that $D_{j,t}^{(n)} = \{ y_1, \dots, y_n \}$ for $j > 0$ and $D_{0,t}^{(n)} = \emptyset$. Let $I_{j,t}^{(n)}$ be the additive subgroup of R_n generated by $E_{j,t}^{(n)}$. With the help of Lemma 1 it is not hard to see that the $I_{j,t}^{(n)}$ are ideals of R_n, $I_{k,t}^{(n)} = R_n$, $0 \leqq t_1 \leqq t_2 \leqq n-1$ implies $I_{j,t_1}^{(n)} \subseteqq I_{j,t_2}^{(n)}$, and $0 \leqq j_1 \leqq j_2 \leqq k$ implies $I_{j_1,t}^{(n)} \subseteqq I_{j_2,t}^{(n)}$. Further, $\downarrow I_{j,t}^{(n)} \subseteqq I_{\downarrow j,t}^{(n)}$, and $\uparrow I_{j,t}^{(n)} \subseteqq I_{\uparrow j,t}^{(n)}$. Now we claim that $I_{j,t}^{(n)} \cdot I_{s,t}^{(n)} \subseteqq I_{j\cdot s,t+1}^{(n)}$. Suppose $a \in E_{j,t}^{(n)}$ and $b \in E_{s,t}^{(n)}$. It suffices to check $ab \in E_{j\cdot s,t+1}^{(n)}$. We omit the straightforward but long details and consider only the case $a \in B_{j,t}^{(n)}$ and $b \in D_{s,t}^{(n)}$. Then $a = p^i x_l$, $n-t \leqq l \leqq n-1$, $i+l \geqq n-t+k-j-1$ and $s > 0$. We may assume that $b = y_l$ as otherwise $ab = 0$. We conclude $ab = p^{i+k-1}x_{l-1}$, $n-(t+1) \leqq l-1 \leqq n-1$ and $(i+k-1)+(l-1) = i+l+k-2 \geqq n-t+k-j-1+k-2 = n-(t+1)+k-(j+1-k)-1 \geqq n-(t+1)+k-(j+s-k)-1 \geqq n-(t+1)+k-j\cdot s-1$, yielding $ab \in B_{j\cdot s,t+1}^{(n)} \subseteqq E_{j\cdot s,t+1}^{(n)}$.

For a τ-term $\sigma \in \mathcal{P}_0$ let σ_{R_n} denote the value of σ in $\mathcal{I}(R_n)$. The *length* $|\sigma|$ of σ is defined via induction: $|\mathbf{0}| = |\mathbf{1}| = 1$, $|\uparrow \sigma| = |\downarrow \sigma| = |\sigma| + 1$, $|\sigma_1 \vee \sigma_2| = |\sigma_1 \wedge \sigma_2| = |\sigma_1 \cdot \sigma_2| = |\sigma_1| + |\sigma_2| + 1$. The inclusions among the $I_{j,t}^{(n)}$ we have already established yield

$$\sigma_{R_n} \subseteqq I_{h(\sigma),|\sigma|}^{(n)}, \quad \text{provided} \quad |\sigma| < n\,, \tag{4}$$

via an easy induction on $|\sigma|$.

Now the proof of Lemma 3 will be completed easily. Suppose that $\sigma \notin K(\mathbf{Z}_{p^k})$. Then $h(\sigma) \leqq k-1$ by Lemma 2. Choose an n with $n > |\sigma| + 2$. Then, by (4) and Lemma 1,

$$\sigma_{R_n} \subseteqq I_{h(\sigma),|\sigma|}^{(n)} \subseteqq I_{k-1,|\sigma|}^{(n)} \subseteqq I_{k-1,n-2}^{(n)} \not\ni 1,$$

whence $\sigma \notin K(R_n)$. Therefore $\bigcap_{l=1}^{\infty} K(R_l) \not\supset K(\mathbf{Z}_{p^k})$.

Conversely, an easy induction on $|\sigma|$ yields $\sigma_{R_n} \supseteqq \downarrow^{k-h(\sigma)} R_n$. In particular, if $h(\sigma) = k$ then $\sigma_{R_n} = R_n$. Hence Lemma 2 yields $\bigcap_{l=1}^{\infty} K(R_l) \supseteqq \supseteqq K(\mathbf{Z}_{p^k})$, proving Lemma 3.

Now let $m = p^{k-1}$. On the set of variables $\{x, y, z, t\}$ we define the following lattice terms:

$$r = (x \vee y) \wedge (z \vee t), \quad h_0 = g_0 = t, \quad h_i' = (h_i \vee y) \wedge (x \vee z)$$

$$h_{i+1} = (h_i' \vee r) \wedge (x \vee t), \quad g_i' = (g_i \vee x) \wedge (y \vee z), \quad g_{i+1} = (g_i' \vee r) \wedge (y \vee t),$$

$$r_0 = (h_{m-1} \vee z) \wedge y, \quad q_0 = x \vee z \vee g_{p-1}, \quad q = r_0 \vee x.$$

Let $\chi(m,p)$ denote the lattice Horn sentence

$$r_0 \leqq q_0 \Longrightarrow r \leqq q.$$

LEMMA 4. *$\chi(m,p)$ does not hold in $\mathcal{L}(\mathbf{Z}_{p^k})$.*

PROOF. Let M be the $\mathbf{Z}_{p^k}$-module freely generated by $\{f_1, f_2, f_3\}$. Consider the submodules $x = [f_2]$, $y = [f_1 - f_2]$, $z = [f_3]$, $t = [f_1 - f_3]$. An easy calculation gives $r = [f_1]$. (We do not make a notational distinction between lattice terms and the submodules obtained from them by substituting the *submodules* x, y, z, t for their variables.) It is not hard to check, via induction on i, that $h_i' = [(i+1)f_2 - f_3]$, $h_i = [f_1 + if_2 - f_3]$, $g_i' = [(i+ +1)f_1 - (i+1)f_2 - f_3]$, $g_i = [(i+1)f_1 - if_2 - f_3]$. These equations yield $r_0 = \{\alpha(f_1 - f_2) : m\alpha = 0\} = [p(f_1 - f_2)]$, $q_0 = [pf_1, f_2, f_3]$, $q = [pf_1, f_2]$. Therefore $\chi(m,p)$ does not hold in $\mathrm{Su}(M)$.

LEMMA 5. *$\chi(m,p)$ holds in $\mathcal{L}(R_n)$ for every $n \geqq 1$.*

PROOF. Assume that x, y, z, t are submodules of an R_n-module M such that $r_0 \subseteqq q_0$, and let $f_1 \in M$ be an arbitrary element of r. Our aim is to show $f_1 \in q$. Since $f_1 \in r = (x+y) \cap (z+t)$, we can choose $f_2, f_3 \in M$ such that $f_2 \in x$, $f_1 - f_2 \in y$, $f_3 \in z$, $f_1 - f_3 \in t$. An easy calculation, essentially the same as in the previous lemma, gives $(i+1)f_2 - f_3 \in h_i'$, $f_1 + if_2 - f_3 \in \in h_i$, and $\{\alpha(f_1 - f_2) : m\alpha = 0\} \subseteqq r_0$. In particular, $x_n(f_1 - f_2) \in r_0$.

Now let us suppose that $x_j(f_1 - f_2) \in r_0$ for some $j > 0$. We intend to show $x_{j-1}(f_1 - f_2) \in r_0$; then $f_1 - f_2 = x_0(f_1 - f_2) \in r_0$ follows by (downward) induction on j. From $r_0 \subseteqq q_0$ we infer $x_j(f_1 - f_2) \in q_0 = x + + z + g_{p-1}$. Hence there exist elements e_0 and e_1 in M such that $e_0 \in x$, $e_1 - e_0 \in z$ and $x_j(f_1 - f_2) - e_1 \in g_{p-1} = (g_{p-2}' + r) \cap (y + t)$. This implies the existence of two elements, say e_2^{p-1} and $e_4^{p-1} \in M$ such that $e_1 - e_4^{p-1} \in \in y$, $x_j(f_1 - f_2) - e_4^{p-1} \in t$, $e_1 - e_2^{p-1} \in g_{p-2}'$, and $x_j(f_1 - f_2) - e_2^{p-1} \in r$. Continuing this parsing and denoting $x_j(f_1 - f_2)$ by e_1^p we obtain that there exist elements $e_l^i \in M$ for $i = 1, 2, \ldots, p-1$ and $l = 1, 2, \ldots, 6$ such that for $i \in \{1, 2, \ldots, p-1\}$

$$e_1 - e_3^i \in y, \quad e_1 - e_4^i \in y, \quad e_2^i - e_3^i \in z, \quad e_4^i - e_1^{i+1} \in t, \quad e_1^i - e_2^i \in x,$$

$$e_2^i - e_5^i \in x, \quad e_1^{i+1} - e_5^i \in y, \quad e_2^i - e_6^i \in z, \quad e_1^{i+1} - e_6^i \in t, \quad e_1 - e_1^1 \in t.$$

Clearly, $e_1^p = x_j(f_1 - f_2) \in y$. Let us observe that x contains $u_0 = x_j f_2 + e_0 + \sum_{i=1}^{p-1}(e_2^i - e_1^i)$. But

$$u_0 = \sum_{i=1}^{p-2}(e_2^i - e_6^i) + \sum_{i=1}^{p-2}(e_6^i - e_1^{i+1}) - (x_j(f_1 - f_2) - e_4^{p-1}) + x_j(f_1 - f_3) +$$

$$+ x_j f_3 + (e_0 - e_1) + (e_1 - e_1^1) + (e_2^{p-1} - e_6^{p-1}) + (e_6^{p-1} - e_1^p) + (e_1^p - e_4^{p-1}),$$

whence $u_0 \in r$. Now $u_0 \in x$ and $u_0 \in r$ imply $u_0 \in h_i$ for all $i > 0$. In particular, $u_0 \in h_{m-1}$. Let $u_i = e_0 - e_1 - e_2^i + e_3^i$ for $1 \leqq i \leqq p-1$. We have, for $i > 0$,

$$u_i = e_0 - (e_1 - e_3^i) - e_1^p + \sum_{l=i}^{p-1}(e_1^{l+1} - e_5^l) +$$

$$+ \sum_{l=i}^{p-1}(e_5^l - e_2^l) + \sum_{l=i+1}^{p-1}(e_2^l - e_1^l) \in x + y$$

and $u_i = (e_0 - e_1) - (e_2^i - e_3^i) \in z$, whence $u_i \in r$. Let $v_i = e_1 + e_1^i - e_3^i$. Since $e_1^1 - e_3^1 = (e_1^1 - e_1) + (e_1 - e_3^1) \in y + t$ and, for $i > 1$, $e_1^i - e_3^i = (e_1^i - e_4^{i-1}) - (e_1 - e_4^{i-1}) + (e_1 - e_3^i) \in y + t$, we have $v_i = (e_1 - e_4^{p-1}) + (e_4^{p-1} - e_1^p) + e_1^p + (e_1^i - e_3^i) \in y + t$. But $v_i = e_0 - (e_0 - e_1) + (e_1^i - e_2^i) + (e_2^i - e_3^i) \in x + z$, whence $v_i \in h_0'$ $(i = 1, 2, \ldots, p-1)$. For $1 \leqq i \leqq p-1$ let $w_i = e_0 + e_1^i - e_2^i$. From $w_i = v_i + u_i \in h_0' + r$ and $w_i = e_0 + (e_1^i - e_2^i) \in x$ we infer that $w_i \in h_1$. This together with $w_i \in x$ yield $w_i \in h_{m-1}$.

Now $x_{j-1}(f_1 - f_2) \in y$ and, by $y_j x_j = m x_{j-1}$ and $p y_j = 0$, $x_{j-1}(f_1 - f_2) = x_{j-1}(f_1 + (m-1)f_2 - f_3) - y_j u_0 - \sum_{i=1}^{p-1} y_j w_i + x_{j-1} f_3 \in h_{m-1} + z$. Thus $x_{j-1}(f_1 - f_2) \in r_0$, as intended.

Finally, $f_1 = (f_1 - f_2) + f_2 \in r_0 + x = q$ completes the proof of Lemma 5.

PROOF OF THE MAIN THEOREM. Let us assume that Problem C has an affirmative answer. We claim that

$$\bigvee_{n=1}^{\infty} \mathcal{L}(R_n) = \mathcal{L}(\mathbf{Z}_{p^k}) \tag{5}$$

where the join is formed in $(\mathbf{W}(p^k); \subseteqq)$. Since $K(R_n) \supseteqq K(\mathbf{Z}_{p^k})$ by Lemma 3, we obtain $\mathcal{L}(R_n) \subseteqq \mathcal{L}(\mathbf{Z}_{p^k})$, for every n, by the assumption. (Note that $\mathcal{L}(R_n) \subseteqq \mathcal{L}(\mathbf{Z}_{p^k})$ also follows from Theorem B.) On the other hand, suppose $\mathcal{L}(S) \in \mathbf{W}(p^k)$ and, for all n, $\mathcal{L}(R_n) \subseteqq \mathcal{L}(S)$. Theorem A yields $K(R_n) \supseteqq K(S)$. From Lemma 3 we conclude $K(\mathbf{Z}_{p^k}) = \bigcap_{n=1}^{\infty} K(R_n) \supseteqq K(S)$, and the assumption on Problem C gives $\mathcal{L}(\mathbf{Z}_{p^k}) \subseteqq \mathcal{L}(S)$. This proves (5).

Now if Problem D had an affirmative answer then (5) would be true even in the lattice of all quasivarieties of lattices. But this would contradict Lemmas 4 and 5.

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SOME REMARKS ON S-CLOSED SPACES

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0. Introduction

S-closed spaces have been introduced in [5], and some of their properties were investigated there. In particular, it was shown that they constitute a rather peculiar class of topological spaces (see below). The purpose of the present paper is to establish further statements on S-closed spaces together with the result that, no matter how strange these spaces may be, every topological space has S-closed extensions.

In a topological space X, a set G is said to be *r-open* (regular(ly) open) iff $G = \operatorname{int}\overline{G}$, i.e. iff G is the interior of a closed set. F is said to be *r-closed* (regular(ly) closed) iff $X - F$ is r-open, i.e. iff $F = \overline{\operatorname{int} F}$, i.e. iff F is the closure of an open set.

A set S is said to be *semi-open* iff $G \subset S \subset \overline{G}$ for some open set G. Open sets and r-closed sets are semi-open. T is said to be *semi-closed* iff $X - T$ is semi-open, i.e. iff there is a closed set F such that $\operatorname{int} F \subset T \subset F$.

According to [5], X is said to be *S-closed* iff every cover of X composed of semi-open sets contains a finite number of members whose closures cover X. Clearly an S-closed space X is almost compact (i.e. every open cover of X contains a finite number of members whose closures cover X). It is shown that S-closed, first countable, regular spaces are finite ([5], Theorem 3), and a regular compact space is S-closed iff it is extremally disconnected (briefly EDC, i.e. the closure of any open set is open) ([5], Corollary of Theorem 7). By this, there exist compact spaces that are not S-closed; the existence of noncompact S-closed spaces was proved in [4], see also Corollary 10.

A *filter* in X will be called *open* (*r-open*) iff it is generated by a filter base composed of open (r-open) sets; a maximal open (r-open) filter is said to be *ultra-open* (*ultra-r-open*). Since the intersection of two open (r-open) sets is open (r-open), [2] (6.1.29) shows that every open (r-open) filter is contained in an ultraopen (ultra-r-open) one, [2] (6.1.26) implies that an open (r-open) filter $\mathbf{s}$ is ultraopen (ultra-r-open) iff either $G \in \mathbf{s}$ or $X - G \in$

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$\in \mathbf{s}$ for every open (r-open) set G, and [2] (6.1.28) says that two distinct ultraopen (ultra-r-open) filters $\mathbf{s}_1$ and $\mathbf{s}_2$ contain members $G_i \in \mathbf{s}_i$ such that $G_1 \cap G_2 = \emptyset$.

A topological space is said to be S_1 ([2], p. 93) iff x has a neighbourhood not containing y whenever y has a neighbourhood not containing x. The space is S_2 ([2], p. 95) iff two points having distinct neighbourhood filters have disjoint neighbourhoods. Every Hausdorff space and every regular space is S_2.

Let X be a topological space and $Y \supset X$. A *trace filter system* on X is a map $\mathbf{s}$ that assigns to each $a \in Y$ a filter $\mathbf{s}(a)$ in X such that $\mathbf{s}(a)$ is an open filter and, in particular, $\mathbf{s}(x)$ is the neighbourhood filter of x if $x \in X$. A topology on Y is an *extension compatible* with the trace filter system iff $\mathbf{s}(a)$ is the trace in X of the neighbourhood filter of $a \in Y$. Then, in particular, the restriction on X of this topology coincides with the given topology of X.

It is well-known that, for every trace filter system, there exist compatible extensions on Y. Among them, there are a coarsest one, called *strict extension* relative to the trace filter system, and determined by the base composed of the sets

$$s(G) = \{a \in Y : G \in \mathbf{s}(a)\}$$

where $G \subset X$ is open in X ([2] (6.1.2)), and a finest one, called *loose extension* relative to the trace filter system, and determined by the bases $\{S \cup \{a\} : S \in \in \mathbf{s}(a)\}$ for the neighbourhood filters of the points $a \in Y$.

An extension on Y is *reduced* iff $a, b \in Y$, $a \neq b$ have distinct neighbourhood filters except when $a, b \in X$ ([2], p. 218). A loose extension is always reduced; a strict one is reduced iff distinct points of $Y - X$ have distinct trace filters, and $\mathbf{s}(p)$ does not coincide with any neighbourhood filter in X if $p \in Y - X$.

A filter $\mathbf{s}$ is said to be *fixed* or *free* according as $\bigcap \mathbf{s} \neq \emptyset$ or $\bigcap \mathbf{s} = \emptyset$.

1. Characterization of S-closed spaces

THEOREM 1. *The following are equivalent for a topological space X:*
(a) *X is S-closed.*
(b) *In X, every cover composed of r-closed sets contains a finite subcover.*
(c) *In X, every r-open filter is fixed.*
(d) *In X, every ultra-r-open filter is fixed.*

PROOF. (a) $\Rightarrow$ (b). An r-closed set is semi-open and closed.

(b) $\Rightarrow$ (c). If $\mathbf{s}$ were a free r-open filter, then the r-closed complements $X - G_i$ of the r-open elements $G_i \in \mathbf{s}$ would cover X. A finite subcover $\{X - G_i: i \in I\}$ (I is finite) would imply $\bigcap_{i \in I} G_i = \emptyset$: a contradiction.

(c) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (a). Assume $\{S_i: i \in I\}$ is a cover of X composed of semi-open sets such that $\bigcup\{\overline{S}_i: i \in I'\} \neq X$ for every finite subset I' of I. Then $\{\bigcap\{X - \overline{S}_i: i \in I'\}\}$ is a filter base composed of r-open sets (when I' runs over all finite subsets of I). Let $\mathbf{s}$ be an ultra-r-open filter containing it. For $x \in \bigcap \mathbf{s}$ there is an i such that $x \in S_i$: a contradiction since $X - \overline{S}_i \in \mathbf{s}$. □

For (a) $\Leftrightarrow$ (b), see [1]; Theorem 2 and [4], Theorem 3.2.

2. S-closed S_2-spaces

The following lemma is contained in [5], Theorem 2:

LEMMA 2. *If* $\mathbf{r}$ *is a filter base in an S-closed space X, then there is a point $x \in X$ such that each r-closed set containing x intersects each member of* $\mathbf{r}$.

PROOF. Assume the contrary. Then, for $x \in X$, there would be an r-closed set F_x and a set $R_x \in \mathbf{r}$ such that $x \in F_x$, $F_x \cap R_x = \emptyset$. The cover $\{F_x: x \in X\}$ would contain by Theorem 1 a finite subcover $\{F_x: x \in F\}$ (F finite), implying $\bigcap\{R_x: x \in F\} = \emptyset$: a contradiction. □

The following theorem generalizes [5], Theorem 6 and Theorem 7:

THEOREM 3. *An S-closed S_2-space is extremally disconnected.*

PROOF. Assume X is S-closed and S_2, but not EDC. Then there is an open set G such that $\overline{G}$ is not open. We can suppose that G is r-open by substituting $\operatorname{int}\overline{G}$ for G. For a point $y \in \overline{G} - G$, let $\mathbf{r}$ denote the collection of the intersections of the open neighbourhoods of y with G. By Lemma 2 there is $x \in X$ such that each r-closed set containing x intersects each member of $\mathbf{r}$. Now $x \in X - G$ is impossible since $X - G$ is r-closed and it does not intersect the members of $\mathbf{r}$. Hence $x \in G$, $x \neq y \notin G$, and X being S_2, there are open sets U and V satisfying $x \in U$, $y \in V$, $U \cap V = \emptyset$. The r-closed set $\overline{U}$ would intersect each member of $\mathbf{r}$ in contradiction with $V \cap G \in \mathbf{r}$, $\overline{U} \cap V = \emptyset$. □

The following theorem generalizes [5], Theorem 5:

THEOREM 4 ([1], Corollary 1 and [4], Theorem 3.4). *An almost compact, extremally disconnected space is S-closed.*

PROOF. The members of an r-closed cover of an EDC space are clopen, and if the space is almost compact, this cover contains a finite subcover so that Theorem 1 can be applied. □

3. S-closed extensions

Consider an extension Y of the space X compatible with the trace filter system $\mathbf{s}$. The following lemma is probably known:

LEMMA 5. *If $G' \subset Y$ is r-open then $G = G' \cap X$ is r-open in X and $G' = s(G)$.*

PROOF. Let F' be closed in Y, $G' = \operatorname{int}_Y F'$. We show $G = \operatorname{int}_X F$ for the set $F = F' \cap X$ closed in X. The inclusion $G \subset \operatorname{int}_X F$ holds since $G \subset \subset F$ and G is open in X. Conversely, $x \in \operatorname{int}_X F$ implies the existence of an H, open in X, such that $x \in H \subset F$. Let H' be open in Y, $H = H' \cap X$. Then $H' - F' \neq \emptyset$ would imply $(H' - F') \cap X \neq \emptyset$, $y \in (H' - F') \cap X$ for some y, yielding the contradiction $y \in H' \cap X = H$, $y \notin F' \cap X = F$. Thus $H' \subset F'$, $H' \subset G' = \operatorname{int}_Y F'$, $H \subset G$, $x \in G$, $\operatorname{int}_X F \subset G$.

$p \in G'$ implies $G \in \mathbf{s}(p)$, $p \in s(G)$. Conversely if $p \in s(G)$, say $p \in H'$, $H = H' \cap X \subset G$ for some H' open in Y, then $H' \subset F'$ again since $y \in (H' - - F') \cap X$ would furnish $y \in H \subset G$, $y \notin F' \supset G'$. Thus $H' \subset \operatorname{int}_Y F' = G'$, $p \in G'$. Consequently $G' = s(G)$. □

LEMMA 6. *If every free r-open filter in X is coarser than some trace filter $\mathbf{s}(p)$ $(p \in Y)$ then Y is S-closed.*

PROOF. By Theorem 1 we have to show that each r-open filter $\mathbf{s}'$ in Y is fixed. Now $\mathbf{s}'|X = \mathbf{s}$ is an r-open filter in X by Lemma 5. If $\mathbf{s}$ is fixed then so is $\mathbf{s}'$. If $\mathbf{s}$ is free and $\mathbf{s} \subset \mathbf{s}(p)$ for some $p \in Y$, then $G = G' \cap X \in \mathbf{s} \subset \mathbf{s}(p)$ for any r-open $G' \in \mathbf{s}'$, hence $G' = s(G)$ by Lemma 5 and $p \in s(G) = G'$, $p \in \bigcap \mathbf{s}'$. □

COROLLARY 7. *Every topological space possesses S-closed extensions.*

PROOF. Take an extension compatible with the trace filter system $\mathbf{s}$ such that $\{\mathbf{s}(p) : p \in Y - X\}$ is the collection of all free ultra-r-open filters in X. □

The extensions figuring in Corollary 7 are reduced if $\mathbf{s}|Y - X$ is injective. Hence they are T_0 if X is T_0. Moreover, two distinct points of $Y - X$ have in this case disjoint neighbourhoods (because their trace filters contain disjoint members.) However, by Theorem 3, X cannot have an S-closed S_2 extension unless it is (S_2 and) EDC (since a dense subspace of an EDC space is EDC).

On the other hand, for EDC spaces there exist S-closed extensions with better properties:

THEOREM 8. *Let X be an* EDC *space, and $Y \supset X$ an extension compatible with a trace filter system $\mathbf{s}$ such that $\{\mathbf{s}(p) : p \in Y - X\}$ is the collection of all non-convergent ultraopen filters in X. Then Y is S-closed.*

PROOF. It is easy to see that each open filter in X has a cluster point in Y. Hence Y is almost compact by [3], 5.5.

Y is EDC. In fact, if $G' \subset Y$ is open and $G = G' \cap X$, then clearly

$$\mathrm{cl}_Y G' = \mathrm{cl}_Y G = \mathrm{cl}_Y H$$

for $H = \mathrm{cl}_X G$ that is clopen in X. Now $x \in \mathrm{cl}_Y H \cap X$ iff $x \in H$, and $p \in \in \mathrm{cl}_Y H - X$ iff $p \in Y - X$ and each member of $\mathbf{s}(p)$ intersects H, i.e. iff $H \in \mathbf{s}(p)$ (since $\mathbf{s}(p)$ is ultraopen). Thus $\mathrm{cl}_Y H = s(H)$ is open in Y.

By Theorem 4 Y is S-closed. □

COROLLARY 9. *An* EDC *space that is $S_2(T_2)$ possesses an S-closed extension with the same property.*

PROOF. In Theorem 8, consider a reduced, loose extension. It is $S_2(T_2)$ because a non-convergent ultraopen filter in X does not admit any cluster point in X. □

COROLLARY 10. ([4], Example 3.17.) *There exists an S-closed T_2-space that is not compact.*

PROOF. Apply Corollary 9 for an infinite discrete space. Then Y is not compact because it contains an infinite, closed, discrete subspace. □

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ON A PROBLEM OF TURÁN CONCERNING SUMS OF POWERS OF COMPLEX NUMBERS

A. BIRÓ (Budapest)

Let $z_1, z_2, \dots, z_n$ be complex numbers and write $S_j = \sum_{t=1}^{n} z_t^j$ ($j = 1, 2, \dots$) for their power sums. Paul Turán started the investigation of the sequence

$$R_n = \min_{z_1, z_2, \dots, z_n} \max_{1 \leqq j \leqq n} |S_j|$$

under the condition

$$(*) \qquad \max_{1 \leqq t \leqq n} |z_t| = 1.$$

This minimum exists by Weierstrass' theorem, and one can easily see that condition $(*)$ can be replaced by $z_1 = 1$.

Turán proved in 1942 that $R_n > \frac{1}{n}$, this was improved to $R_n > \frac{0.5}{\sum_{k=1}^{n} \frac{1}{k}}$ by Paul Erdős, then to $R_n > \frac{\log 2}{\sum_{k=1}^{n} \frac{1}{k}}$ by Turán (see [1]). The relation $R_n > c$ for some positive constant c independent of n has already been conjectured by Turán in 1942. In the special case when $z_1 = 1$ and the system $z_1, z_2, \dots, z_n$ is symmetric to the real axis, $\max_{1 \leqq j \leqq n} |S_j| \geqq 1$ holds, as it was shown by M. Schweitzer (see [1]). This is obviously the best possible result in this direction in view of the example $z_1 = 1$, $z_2 = z_3 = \dots = z_n = 0$. The next result concerning the general case was N. G. de Bruijn's one: $R_n > \frac{c \log \log n}{\log n}$ for some $c > 0$ and for sufficiently large n. It was shown subsequently by S. Uchiyama that c may be chosen to be $1 - \varepsilon$ with arbitrary $\varepsilon > 0$ (see [2]).

The conjecture of Turán was proved by F. V. Atkinson in 1961; using complex analysis he showed that $R_n > \frac{1}{6}$ (see [3]). Atkinson improved his estimate in two further papers, firstly he proved that $R_n > \frac{1}{3}$, then in [4] he obtained the following results: $R_n > \frac{\pi}{8}$ for $n < 1600$, and, for sufficiently large n, $R_n > s_0$, where $0 < s_0 < \frac{1}{2}$ and s_0 satisfies the equation

$$\frac{s_0^2}{2\pi} \int_0^\infty \Big(\exp\big(2 s_0 T(x)\big) - 1\Big)^2 x^{-2}\, dx = 1,$$

where $T(x) = \int\limits_0^x y^{-1}|\sin y|\,dy$. (Atkinson did not give the exact value of s_0 in [4], only that $s_0 < \frac{\pi}{8}$.) These last results were the best lower estimates known.

In his book ([5], Problem 12) Turán posed the problem of finding the best possible constant c for which $R_n > c$. This problem is still unsolved.

The best upper estimate known is due to J. Komlós, A. Sárközy and E. Szemerédi [6]: $R_n < 1 - \frac{1}{250n}$ for $n > n_0$, and $R_n < 1 - \frac{1}{3}\frac{\log n}{n}$ for infinitely many n.

In this paper I improve Atkinson's result showing that $R_n > \frac{1}{2}$ (Theorem 1). Theorem 2 is on the one hand a more precise form of Theorem 1, on the other hand it deals also with the case when there are more 1's among the numbers $z_1, z_2, \dots, z_n$ explaining why it is not worth seeking "near" extremal systems with more 1's.

THEOREM 1. *If $z_1, z_2, \dots, z_n$ are complex numbers and $z_1 = 1$, then* $\max\limits_{1 \leqq j \leqq n} |S_j| > \frac{1}{2}$. *So $R_n > \frac{1}{2}$ for every n.*

PROOF. Let $(z - z_2)(z - z_3)\dots(z - z_n) = z^{n-1} + b_1 z^{n-2} + \dots + b_{n-1}$. From the Newton–Girard formulas for this polynomial $\left(\text{let } T_j = \sum_{t=2}^{n} z_t^j\right)$ $T_k + b_1 T_{k-1} + \dots + b_{k-1}T_1 + kb_k = 0$ $(k = 1, 2, \dots, n-1)$ and $T_n + b_1 T_{n-1} + \dots + b_{n-1}T_1 = 0$. Taking into account that $T_j = S_j - 1$ we get

$$S_k + b_1 S_{k-1} + \dots + b_{k-1}S_1 = 1 + b_1 + \dots + b_{k-1} - kb_k \qquad (k = 1, 2, \dots, n-1), \tag{1}$$

$$S_n + b_1 S_{n-1} + \dots + b_{n-1}S_1 = 1 + b_1 + \dots + b_{n-1}. \tag{2}$$

LEMMA 1. *Let $0 < \alpha < \frac{\pi}{2}$. If $1 \leqq k \leqq n-1$, then one of the following two inequalities holds:*

$$|1 + b_1 + \dots + b_{k-1} - kb_k| \geqq \sin\alpha |1 + b_1 + \dots + b_{k-1}|, \tag{3}$$

$$|1 + b_1 + \dots + b_{k-1} + b_k| \geqq |1 + b_1 + \dots + b_{k-1}| + \cos\alpha |b_k|. \tag{4}$$

If (4) is valid for $k = 1, 2, \dots, s$ $(s \leqq n-1)$, then

$$|1 + b_1 + \dots + b_s| > \cos\alpha\bigl(1 + |b_1| + \dots + |b_s|\bigr).$$

PROOF. If $1 + b_1 + \dots + b_{k-1} = 0$ or $b_k = 0$ then both (3) and (4) are true. Hence assume that $1 + b_1 + \dots + b_{k-1} \neq 0$ and $b_k \neq 0$, and consider these complex numbers as vectors of the plane. Elementary geometric consideration shows that if the angle of these two vectors $\bigl((1 + b_1 + \dots + b_{k-1})$ and $b_k\bigr)$ is not greater than α then (4) is true, and if this angle is greater than α, then (3) is. (If one wants to avoid geometric arguments, it is possible to apply Lemma 2 with $A = k$, $z = \frac{b_k}{1+b_1+\ldots+b_{k-1}}$.) Finally, if (4) is valid for $k = 1, 2, \ldots, s$, then obviously

$$|1 + b_1 + \ldots + b_s| \geqq 1 + \cos\alpha\big(|b_1| + \ldots + |b_s|\big) >$$

$$> \cos\alpha\big(1 + |b_1| + \ldots + |b_s|\big). \quad \square$$

Continuing the proof of the theorem we distinguish between two cases.

a) (4) holds for $k = 1, 2, \ldots, n-1$. Then by Lemma 1 and (2) we have

$$\left(\max_{1\leqq j\leqq n} |S_j|\right)\left(1 + |b_1| + \ldots + |b_{n-1}|\right) \geqq |S_n + b_1 S_{n-1} + \ldots + b_{n-1}S_1| =$$

$$= |1 + b_1 + \ldots + b_{n-1}| > \cos\alpha\big(1 + |b_1| + \ldots + |b_{n-1}|\big),$$

hence $\max\limits_{1\leqq j\leqq n} |S_j| > \cos\alpha$.

b) Case a) is not satisfied. Let $1 \leqq k_0 \leqq n-1$ be the least positive integer for which (4) is not valid. Then by Lemma 1

$$\big|1 + b_1 + \ldots + b_{k_0-1}\big| > \cos\alpha\Big(1 + |b_1| + \ldots + \big|b_{k_0-1}\big|\Big).$$

(The inequality holds for $k_0 = 1$, too.) Applying this, (1), and the fact that (3) is true for k_0 (because (4) is not valid) we obtain that

$$\left(\max_{1\leqq j\leqq k_0} |S_j|\right)\Big(1 + \big|b_1\big| + \ldots + \big|b_{k_0-1}\big|\Big) \geqq \big|S_{k_0} + b_1 S_{k_0-1} + \ldots + b_{k_0-1}S_1\big| =$$

$$= \big|1 + b_1 + \ldots + b_{k_0-1} - k_0 b_{k_0}\big| \geqq \sin\alpha\big|1 + b_1 + \ldots + b_{k_0-1}\big| >$$

$$> \sin\alpha\cos\alpha\Big(1 + \big|b_1\big| + \ldots + \big|b_{k_0-1}\big|\Big).$$

From this

$$\max_{1\leqq j\leqq n} |S_j| \geqq \max_{1\leqq j\leqq k_0} |S_j| > \sin\alpha\cos\alpha.$$

So in both cases we have the estimate

$$\max_{1\leqq j\leqq n} |S_j| > \sin\alpha\cos\alpha = \frac{\sin 2\alpha}{2}.$$

With the choice $\alpha = \frac{\pi}{4}$ we get the assertion of the theorem. $\square$

LEMMA 2. *Let $z \neq 0$ be a complex number, $0 < \alpha < \frac{\pi}{2}$, and $A > 0$. Then* (5) *or* (6) *is satisfied:*

$$|1 - Az|^2 \geqq \sin^2\alpha\left(1 + \frac{\cos^2\alpha}{A + \sin^2\alpha}\right), \tag{5}$$

$$|1 + z| \geqq 1 + \cos\alpha |z|. \tag{6}$$

PROOF. Let $z = r(\cos\phi + i\sin\phi)$ be the trigonometric form of z. Assume that (6) is not true, then

$$|1+z|^2 = |1 + r\cos\phi + ir\sin\phi|^2 = 1 + 2r\cos\phi + r^2 <$$
$$< 1 + 2r\cos\alpha + r^2\cos^2\alpha,$$

hence $2\cos\phi < 2\cos\alpha - r\sin^2\alpha$. Applying this

$$|1 - Az|^2 = |1 - Ar\cos\phi - iAr\sin\phi|^2 = 1 - 2Ar\cos\phi +$$
$$+ A^2r^2 > 1 - Ar(2\cos\alpha - r\sin^2\alpha) + A^2r^2 =$$
$$= \left(1 + \frac{1}{A}\sin^2\alpha\right)\left(Ar - \frac{\cos\alpha}{1 + \frac{1}{A}\sin^2\alpha}\right)^2 - \frac{\cos^2\alpha}{1 + \frac{1}{A}\sin^2\alpha} + 1 \geqq$$
$$\geqq 1 - \frac{\cos^2\alpha}{1 + \frac{1}{A}\sin^2\alpha} = \sin^2\alpha\left(1 + \frac{\cos^2\alpha}{A + \sin^2\alpha}\right),$$

so in this case (5) is valid, which proves the lemma. □

THEOREM 2. *Let m be a positive integer and assume that*

$$z_1 = z_2 = \ldots = z_m = 1. \tag{7}$$

For arbitrary $n > m$ and every system $z_1, z_2, \ldots, z_n$ satisfying (7) *we have*

$$\max_{1 \leqq j \leqq n-m+1} |S_j| > m\left(\frac{1}{2} + \frac{1}{8}\frac{m}{n} + \frac{3}{64}\left(\frac{m}{n}\right)^2\right).$$

PROOF. Now we apply the Newton–Girard formulas for the polynomial

$$(z - z_{m+1})(z - z_{m+2})\ldots(z - z_n) = z^{n-m} + b_1 z^{n-m-1} + \ldots + b_{n-m},$$

and obtain

$$T_k + b_1 T_{k-1} + \ldots + b_{k-1} T_1 + k b_k = 0 \quad (k = 1, 2, \ldots, n-m),$$
$$T_{n-m+1} + b_1 T_{n-m} + \ldots + b_{n-m} T_1 = 0$$

where $T_j = \sum_{t=m+1}^{n} z_t^j$. Here $T_j = S_j - m$, hence

$$S_k + b_1 S_{k-1} + \ldots + b_{k-1} S_1 = m(1 + b_1 + \ldots + b_{k-1}) - k b_k \qquad (k = 1, 2, \ldots, n-m), \tag{8}$$

$$S_{n-m+1} + b_1 S_{n-m} + \ldots + b_{n-m} S_1 = m(1 + b_1 + \ldots + b_{n-m}). \tag{9}$$

LEMMA 3. *Let $0 < \alpha < \frac{\pi}{2}$. If $1 \leqq k \leqq n-m$, then one of the following two inequalities holds:*

$$\left| m(1 + b_1 + \ldots + b_{k-1}) - k b_k \right|^2 \geqq \geqq m^2 \sin^2 \alpha \left(1 + \frac{\cos^2}{\frac{n}{m} - \cos^2 \alpha} \right) |1 + b_1 + \ldots + b_{k-1}|^2, \tag{10}$$

$$|1 + b_1 + \ldots + b_{k-1} + b_k| \geqq |1 + b_1 + \ldots + b_{k-1}| + \cos \alpha |b_k|. \tag{11}$$

If (11) *is valid for $k = 1, 2, \ldots, s$ ($s \leqq n-m$), then*

$$|1 + b_1 + \ldots + b_s| > \cos \alpha \big(1 + |b_1| + \ldots + |b_s| \big).$$

PROOF OF LEMMA 3. If $1 + b_1 + \ldots + b_{k-1} = 0$ or $b_k = 0$, then (11) is satisfied. Otherwise we apply Lemma 2 with the choice $z = \frac{b_k}{1+b_1+\ldots+b_{k-1}}$ and $A = \frac{k}{m}$. From this we get that (12) or (13) is true:

$$\left| 1 - \frac{k}{m} \frac{b_k}{1 + b_1 + \ldots + b_{k-1}} \right|^2 \geqq \sin^2 \alpha \left(1 + \frac{\cos^2 \alpha}{\frac{k}{m} + \sin^2 \alpha} \right), \tag{12}$$

$$\left| 1 + \frac{b_k}{1 + b_1 + \ldots + b_{k-1}} \right| \geqq 1 + \cos \alpha \frac{|b_k|}{|1 + b_1 + \ldots + b_{k-1}|}. \tag{13}$$

Hence we have proved that (10) or (11) is valid, because (13) implies (11), and (12) implies (10) (taking into account that $k \leqq n-m$). The last assertion of the lemma does not differ from the last assertion of Lemma 1. □

Now there are two possible cases.

a) (11) holds for $k = 1, 2, \ldots, n-m$. Then by (9) and Lemma 3 we have

$$\left(\max_{1 \leqq j \leqq n-m+1} |S_j| \right) \left(1 + |b_1| + \ldots + |b_{n-m}| \right) \geqq$$

$$\geqq |S_{n-m+1} + b_1 S_{n-m} + \ldots + b_{n-m} S_1| = m |1 + b_1 + \ldots + b_{n-m}| >$$

$$> m \cos \alpha \big(1 + |b_1| + \ldots + |b_{n-m}| \big),$$

hence

$$\max_{1 \leqq j \leqq n-m+1} |S_j| > m \cos \alpha. \tag{14}$$

b) Case a) is not satisfied. Let $1 \leqq k_0 \leqq n-m$ be the least positive integer for which (11) is not valid. Then by Lemma 3 (and for $k_0 = 1$ obviously)

$$|1 + b_1 + \ldots + b_{k_0-1}| > \cos \alpha \left(1 + |b_1| + \ldots + |b_{k_0-1}|\right).$$

Applying this, (8) and (10), which is true for k_0, we obtain

$$\left(\max_{1 \leqq j \leqq k_0} |S_j|\right)\left(1 + |b_1| + \ldots + |b_{k_0-1}|\right) \geqq$$

$$\geqq \left|S_{k_0} + b_1 S_{k_0-1} + \ldots + b_{k_0-1} S_1\right| = \left|m\left(1 + b_1 + \ldots + b_{k_0-1}\right) - k_0 b_{k_0}\right| \geqq$$

$$\geqq m \sin \alpha \left|1 + b_1 + \ldots + b_{k_0-1}\right| \sqrt{1 + \frac{\cos^2 \alpha}{\frac{n}{m} - \cos^2 \alpha}} >$$

$$> m \sin \alpha \cos \alpha \left(1 + |b_1| + \ldots + |b_{k_0-1}|\right) \sqrt{1 + \frac{\cos^2 \alpha}{\frac{n}{m} - \cos^2 \alpha}}.$$

From this

$$\max_{1 \leqq j \leqq n-m} |S_j| \geqq \max_{1 \leqq j \leqq k_0} |S_j| > \tag{15}$$

$$> m \sin \alpha \cos \alpha \sqrt{1 + \frac{\cos^2 \alpha}{\frac{n}{m} - \cos^2 \alpha}}.$$

Let $\alpha = \frac{\pi}{4}$, then from (14) and (15) we know that

$$\max_{1 \leqq j \leqq n-m+1} |S_j| > \min\left(\frac{m}{\sqrt{2}}, \frac{m}{2}\sqrt{1 + \frac{m}{2n - m}}\right).$$

Now

$$\sqrt{1 + \frac{m}{2n - m}} = \left(1 - \frac{m}{2n}\right)^{-\frac{1}{2}} =$$

$$= \sum_{t=0}^{\infty} \frac{\binom{2t}{t}}{2^{2t}} \left(\frac{m}{2n}\right)^t > 1 + \frac{1}{2}\frac{m}{2n} + \frac{3}{8}\left(\frac{m}{2n}\right)^2.$$

On the other hand

$$\frac{1}{\sqrt{2}} > \frac{1}{2}\sqrt{1 + \frac{m}{2n - m}}$$

as $m < n$, and this proves the theorem. □

REMARK 1. The function $\sin\alpha\cos\alpha\sqrt{1 + \frac{\cos^2\alpha}{\frac{n}{m} - \cos^2\alpha}}$ is maximal not with $\alpha = \frac{\pi}{4}$, but with $\cos^2\alpha = \frac{1}{1+\sqrt{1-\frac{m}{n}}}$. This choice of α improves the coefficient of $\left(\frac{m}{n}\right)^2$ in Theorem 2 to $\frac{1}{16}$.

REMARK 2. From the proof it can be seen too (see inequalities (14) and (15)) that for arbitrary $n > m$, if $\max\limits_{1 \leqq j \leqq n-m} |S_j| \leqq m\frac{\sin 2\alpha}{2}$, then $|S_{n-m+1}| > m\cos\alpha$. For example, if $\max\limits_{1 \leqq j \leqq n-m} |S_j| \leqq \frac{m}{2}$, then $|S_{n-m+1}| > \frac{m}{\sqrt{2}}$ (with $\alpha = \frac{\pi}{4}$). The assertion is also interesting for $\alpha < \frac{\pi}{4}$, because then the condition for $\max\limits_{1 \leqq j \leqq n-m} |S_j|$ is stronger, but we obtain a better estimate for $|S_{n-m+1}|$.

REMARK 3. A possibility to improve Theorem 2 in the case $m = 1$ is the following. Let $\alpha = \frac{\pi}{4}$, $d < 1$ (d is a constant). In case a) from (14) $\max\limits_{1 \leqq j \leqq n} |S_j| > \frac{1}{\sqrt{2}}$, so it suffices to consider Case b). If $k_0 < dn$, then we use this inequality instead of $k_0 \leqq n - 1$ in Lemma 3, and this improves the estimate. But if d is sufficiently close to 1 (this means more precisely that $2^{-\frac{1}{8}} < d < 1$), and n is sufficiently large, then $k_0 \geqq dn$ implies $\max\limits_{1 \leqq j \leqq n} |S_j| > \frac{1}{2} + h$, where $h > 0$ is a constant. Indeed, for $k_0 \leqq k \leqq n - 1$ either $|kb_k| < 2|1 + b_1 + \ldots + b_{k-1}|$ or $|kb_k| \geqq 2|1 + b_1 + \ldots + b_{k-1}|$. If k_1 is the first k for which the latter inequality is true (if there is no such k, let $k_1 = n$), then in view of (8) (or (9))

$$\max_{1 \leqq j \leqq n} |S_j| \geqq \frac{\left|1 + b_1 + \ldots + b_{k_1-1}\right|}{1 + \left|b_1\right| + \ldots + \left|b_{k_1-1}\right|}.$$

As

$$\frac{\left|1 + b_1 + \ldots + b_{k_0-1}\right|}{1 + |b_1| + \ldots + |b_{k_0-1}|} \geqq \frac{1}{\sqrt{2}}$$

(this settles the case $k_0 = k_1$), and

$$|b_k| < 2\frac{|1 + b_1 + \ldots + b_{k-1}|}{k} \quad \text{for} \quad k_0 \leqq k < k_1,$$

so it is easy to see (since $k_0 \geqq dn$) that

$$\frac{|1+b_1+\ldots+b_{k_1-1}|}{1+|b_1|+\ldots+|b_{k_1-1}|}>\frac{1}{2}+h \quad \text{for some} \quad h,$$

if d is sufficiently close to 1. Combining this with the above geometric arguments we get that $R_n > \frac{1}{2}+\frac{0.159}{n}$ for sufficiently large n.

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PURE SUBGROUPS OF A-PROJECTIVE GROUPS

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1. Introduction

The perhaps most significant difference between torsion-free groups and p-groups is that there are only few results that guarantee the splitting of an exact sequence of torsion-free abelian groups. One of the best known is Baer's result that a pure subgroup of a homogeneous completely decomposable group of finite rank is a direct summand [9]. While possible generalizations of other results of [9] have been discussed by various authors ([2], [8], [7], and others), relatively little attention has been given to the previously mentioned result. Before we can give a summary of what has been done, it is necessary to introduce some notation:

Consider abelian groups A and G. The group G is *A-generated* if $G = S_A(G) = \sum \{ \phi(A) \mid \phi \in \operatorname{Hom}(A, G) \}$. An A-generated group G is *A-projective of finite A-rank* if it is isomorphic to a direct summand of $\bigoplus_n A$ for some $n < \omega$. In [3], it was shown that A-generated subgroups of A-projective groups of finite A-rank are quasi-summands if A has a semi-simple Artinian quasi-endomorphism ring. Moreover, if A is a faithfully flat as an $E(A)$-module, then this condition on $QE(A)$ is necessary too.

Unfortunately, this result is of limited use if we are to decide whether a pure subgroup U of an A-projective group of finite A-rank is a direct summand. In [5], we addressed this question in the case that U itself is A-projective; but the problem remained open for arbitrary A-generated groups. It is the purpose of this paper to address some of the questions that have been left unanswered in [5]. We say that A *splits pure A-socles* if a pure A-generated subgroup of an A-projective group of finite A-rank is a direct summand. Necessary and sufficient conditions for a torsion-free group A to have this property are given in Theorem 2.1. Several corollaries improve on the conditions in Theorem 2.1 if A is flat, respectively faithfully flat as an $E(A)$-module. In the last case, we obtain that A splits pure A-socles iff it has a semi-hereditary endomorphism ring and a semi-simple Artinian quasi-endomorphism ring.

In Section 3, we consider a dual version of the splitting problem for pure subgroups U of an A-projective group P of finite A-rank. We say that A has *the radical splitting property* if every subgroup U of an A-projective group P of finite A-rank with $R_A(P/U) = 0$ is a direct summand. Here the *A-radical of an abelian group* G is $R_A(G) = \bigcap\{\ker\phi \mid \phi \in \mathrm{Hom}(G, A)\}$. Theorem 3.3 characterizes the abelian groups which have the radical splitting property. The section concludes with the surprising result that, for a torsion-free group A of finite rank, the two splitting properties of this paper are equivalent. Moreover, we show that an abelian group A which is faithfully flat as an $E(A)$-module and has the splitting property for pure A-socles also has the radical splitting property. We give an example that the converse may fail if A has infinite rank.

2. Pure A-socles

Consider abelian groups A and G. The group $H_A(G) = \mathrm{Hom}(A, G)$ carries a natural right $E(A)$-module structure which is induced by the composition of maps. Since A is a left $E(A)$-module,

$$T_A(M) = M \otimes_{E(A)} A$$

defines a functor from the category of right $E(A)$-modules to the category $\mathcal{A}b$ of abelian groups which is an adjoint of H_A. We obtain induced homomorphisms $\theta_G \colon T_A H_A(G) \to G$ and $\phi_M \colon M \to H_A T_A(M)$ for all abelian groups G and right $E(A)$-modules M which are defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(m)](a) = m \otimes a$ for all $a \in A$, $\alpha \in H_A(G)$ and $m \in M$. If G is A-projective of finite A-rank, then θ_G is an isomorphism, while ϕ_M is an isomorphism for all finitely generated projective $E(A)$-modules M [8].

THEOREM 2.1. *The following conditions are equivalent for a torsion-free abelian group A:*

a) i) *A has the splitting property for pure A-socles.*

ii) $\mathrm{Tor}^1(E(A)/I, A) = 0$ *for all right ideals I of $E(A)$ such that $(E(A)/I)^+$ is torsion.*

b) *If M is a finitely generated right $E(A)$-module, then $M = U \oplus P$, where P is projective, and $T_A(U)$ is torsion.*

PROOF. a) $\Rightarrow$ b). The torsion-subgroup tM of M is a submodule of M. We consider an exact sequence $0 \to K \xrightarrow{\alpha} \bigoplus_n E(A) \xrightarrow{\beta} M/tM \to 0$, and choose a submodule V of $\bigoplus_n E(A)$ containing K such that $V/K \cong tM$. Assume that it has been shown that $T_A(N)$ is torsion-free for all $E(A)$-modules N whose additive group is torsion-free. Then, the kernel of the induced epimorphism $T_A(\beta)$ is a pure A-generated subgroup of $T_A(E(A)^n)$. The map $T_A(\beta)$ splits by a); and the top-row of the commutative diagram

$$\begin{array}{ccccc}
H_AT_A\big(A(A)^n\big) & \xrightarrow{H_AT_A(\beta)} & H_AT_A(M/tM) & \longrightarrow & 0 \\
\uparrow{\scriptstyle \phi_{E(A)^n}} & & \uparrow{\scriptstyle \phi_{M/tM}} & & \\
\bigoplus_n E(A) & \xrightarrow{\beta} & M/tM & \longrightarrow & 0
\end{array}$$

is split-exact. This shows that $\phi_{M/tM}$ is an epimorphism. Since $H_AT_A(M/tM)$ is projective, we obtain $M/tM = \ker\phi_{M/tM} \oplus Q$ for some projective submodule Q of M/tM. Write $Q = [tM \oplus P]/tM$ and $\ker\phi_{M/tM} = U/tM$. Therefore, $M = U \oplus P$, and it remains to show that $T_A(U)$ is torsion.

Since the inclusion $\varepsilon: U/tM \to M/tM$ splits, the induced map $H_AT_A(\varepsilon)$ is a monomorphism. Consider $x \in U/tM$, and observe that $H_AT_A(\varepsilon)\phi_{U/tM}(x) = \varepsilon\phi_{M/tM}(x) = 0$ implies $0 = \left[\phi_{U/tM}(x)\right](a) = x \otimes a \in \in T_A(U/tM)$ for all $a \in A$. Thus, $T_A(U/tM) = 0$, and $T_A(U)$ is torsion as an epimorphic image of $T_A(tM)$.

Now consider a right $E(A)$-module V whose additive group is torsion. To show that $\mathrm{Tor}^1(V,A) = 0$, it is enough to consider the case that V is finitely generated, say by r elements $v_1,\dots,v_r$. By part i) of a), we may assume $r > 1$. Let W be the submodule of V which is generated by $v_1,\dots,v_{r-1}$. Since both, $\mathrm{Tor}^1(V/W,A)$ and $\mathrm{Tor}^1(W,A)$ vanish, the same holds for $\mathrm{Tor}^1(V,A)$. If N is an $E(A)$-module whose additive group is torsion-free, then multiplication by a non-zero integer m induces a monomorphism $\mu: N \to N$. The induced sequence $0 = \mathrm{Tor}^1(N/mN,A) \to T_A(N) \overset{T_a(\mu)}{\to} T_A(N)$ is exact. Since $T_A(\mu)$ is multiplication by m, $T_A(N)$ is torsion-free.

b) $\Rightarrow$ a). To show that the second part of a) holds, we first establish that $E(A)$ is a right semi-hereditary ring: If U is a finitely generated submodule of a finitely generated module F, then we have a decomposition $U = P \oplus V$ where P is projective and $T_A(V)$ is torsion. The inclusion $V \subseteqq F$ is denoted by ι, and satisfies $\left[H_AT_A(\iota)\phi_V(x)\right](a) = T_A(\iota)\left[\phi_V(x)\right](a) \in T_A(\iota)\big(T_A(V)\big)$ for all $x \in V$ and $a \in A$. Since $T_A(F)$ is torsion-free, this is only possible if $0 = H_AT_A(\iota)\phi_V = \phi_F\iota$. Since ϕ_F is an isomorphism, $V = 0$. Hence, $E(A)$ is right semi-hereditary.

Let M be a right $E(A)$-module whose additive group is torsion. There exists an exact sequence $0 \to U \to \bigoplus_I E(A) \to M \to 0$. We observe that $\mathrm{Tor}^1(M,A)$ is torsion and isomorphic to a subgroup of $T_A(U)$. Since $E(A)$ is right semi-hereditary, U is a flat $E(A)$-module. But this is only possible if $T_A(U)$ is torsion-free. Therefore, $\mathrm{Tor}^1(M,A) = 0$.

To verify part i) of a), we consider the pure-exact sequence $0 \to U \overset{\alpha}{\to} \overset{\alpha}{\to} A^n \overset{\beta}{\to} G \to 0$ of abelian groups in which U is A-generated. It induces the exact sequence $0 \to H_A(U) \overset{H_A(\alpha)}{\to} H_A(A^n) \overset{H_A(\beta)}{\to} M \to 0$ of right $E(A)$-

modules in which $M = \operatorname{im} H_A(\beta)$ is a submodule of $H_A(G)$. We write $M = V \oplus P$ with P projective and $T_A(V)$ torsion, and consider the induced diagram

$$\begin{array}{ccccccccc}
 & & T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(A^n) & \xrightarrow{T_AH_A(\beta)} & T_A(M) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \theta_U} & & \Big\downarrow{\scriptstyle \theta_{A^n}} & & \Big\downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & U & \xrightarrow{\alpha} & A^n & \xrightarrow{\beta} & G & \longrightarrow & 0
\end{array}$$

in which θ is the evaluation map. By the Snake-Lemma, θ is an isomorphism. In particular, $T_A(V) = 0$ since G is torsion-free, and V is a direct summand of M. On the other hand, if $V \neq 0$, then there are $\sigma \in V$ and $a \in A$ with $0 \neq \sigma(a) = \theta(\sigma \otimes a) = 0$. The resulting contradiction shows that M is projective. Thus, the top-row of the last diagram splits; and the same holds for the bottom row.

We want to remind the reader that a ring R is *right strongly non-singular* if the finitely generated non-singular right $E(A)$-modules precisely are the finitely generated submodules of free modules.

COROLLARY 2.2. *The following are equivalent for a torsion-free abelian group A which is flat as an $E(A)$-module:*

a) *A has the splitting property for pure A-socles.*

b) i) *A/IA is torsion for all essential right ideals I of $E(A)$.*

ii) *$E(A)$ is a right strongly non-singular, semi-hereditary ring.*

PROOF. a) $\Rightarrow$ b). Let I be an essential right ideal of $E(A)$, and denote its Z-purification by I_*. The induced sequence $0 \to T_A(I_*) \to T_A\big(E(A)\big) \to T_A\big(E(A)/I_*\big) \to 0$ is split-exact by a). Thus, $A = I_*A \oplus C$ for some subgroup C of A. If C were non-zero, then $H_A(C)$ would be a non-zero right ideal of $E(A)$ with $I \cap H_A(C) \neq 0$, which is not possible. Therefore, $A = I_*A$, and $A/IA \cong T_A(I_*/I)$ is torsion.

Consider a finitely generated, non-singular right $E(A)$-module M. By Theorem 2.1, we can write $M = U \oplus P$ with P projective and $T_A(U)$ torsion. Since A is flat, the group $T_A(U)$ is torsion-free. Therefore, $T_A(U) = 0$. If U were non-zero, then we could find a non-zero right ideal I of $E(A)$, which is isomorphic to a submodule of U. Then, the non-zero group IA would be isomorphic to a subgroup of $T_A(U)$ since A is flat. The resulting contradiction shows $U = 0$.

b) $\Rightarrow$ a). If M is a finitely generated right $E(A)$-module, then $M = Z(M) \oplus P$ where $Z(M)$ is the singular submodule of M, and P is projective. Since $Z(M)$ is finitely generated, say by $\{x_1, \ldots, x_n\}$, we can find an essential right ideal I of $E(A)$ with $x_iI = 0$ for all i. Therefore, $Z(M)$ is an epimorphic image of $\bigoplus_n E(A)/I$, and $T_A\big(Z(M)\big)$ is torsion as an image of the torsion group $\bigoplus_n T_A(A/IA)$. Now apply Theorem 2.1.

COROLLARY 2.3. *The following conditions are equivalent for a torsion-free abelian group A which is faithful as an $E(A)$-module:*

a) i) *A has the splitting property for pure A-socles.*

ii) *A is flat as an $E(A)$-module.*

b) *$E(A)$ is a right semi-hereditary ring such that $QE(A)$ is semi-simple Artinian.*

PROOF. It remains to show that a) implies that $QE(A)$ is semi-simple Artinian. Let I be an essential right ideal of $E(A)$ whose Z-purification, I_*, in $E(A)$ is not $E(A)$. By Theorem 2.1, we have $E(A)/I_* = U \oplus P$ where P is projective, and $T_A(U)$ is torsion. Since I_* is essential, this is only possible if $P = 0$ and $T_A(U) = 0$. By the faithfulness of A, we have $U = 0$.

Clearly, b) implies a) without the faithfulness assumption on A. However, a later example shows that the converse fails in general.

COROLLARY 2.4. *Let A be a torsion-free abelian group which is faithfully flat as an $E(A)$-module. If A has the splitting property for pure A-socles, then it has the quasi-splitting property for A-socles.*

PROOF. Let U be an A-generated subgroup of A^n. Since A is flat, we obtain that the Z-purification, U_*, of U in A^n is of the form $T_A(V)$ where V is the Z-purification of $H_A(U)$ in $H_A(A^n)$. Since $E(A)$ is right semi-hereditary by Corollary 2.3 and $QE(A)$ is semi-simple Artinian, we obtain that $H_A(A^n)/V$ is projective. Thus, V is finitely generated; and $V/H_A(U)$ is bounded. Consequently, U is quasi-equal to U_*. Since A has the splitting property for pure A-socles, U_* is a direct summand of A^n.

The last result fails if A is not faithfully flat as an $E(A)$-module as the following example shows:

EXAMPLE 2.5. *Let $A = \oplus_P Z_p$ where P is an infinite set of primes. Then, A is flat as an $E(A)$-module, has the splitting property for pure A-socles, but not the quasi-splitting property for A-socles.*

PROOF. By [7], A is flat as an $E(A)$-module. Moreover, [1] shows that $E(A) = \prod_P Z_p$ is strongly non-singular and semi-hereditary. If I is an essential ideal of $E(A)$, then $IZ_p \neq 0$ for all $p \in P$, since otherwise I would annihilate all maps $\alpha_p : A \to Z_p$. Thus, $H_A(Z_p) \subseteqq Z\big(E(A)\big) = 0$, which is not possible. Therefore, Z_p/IZ_p is torsion. Since $\bigoplus_P IZ_p \subseteqq IA$, we obtain that A/IA is torsion. Thus, A has the splitting property for pure A-socles by Corollary 2.2. On the other hand, $\bigoplus_P Z_p p$ is an A-generated subgroup of A which is not a quasi-summand.

The last example shows that implication a) $\Rightarrow$ b) of Corollary 2.3 may fail if A is not faithful as an $E(A)$-module.

The final result of this section shows that the converse of Corollary 2.4 fails in general:

EXAMPLE 2.6. Let A be a torsion-free abelian group which is faithfully flat as an $E(A)$-module, and has a semi-simple Artinian quasi-endomorphism ring. If $E(A)$ is not semi-hereditary, then A has the quasi-splitting property for A-socles, but not the pure splitting property.

PROOF. By [3], A has the quasi-splitting property for A-socles. Apply Corollary 2.3.

3. Radicals and splitting

Let A and G be abelian groups. The abelian group $G^* = \mathrm{Hom}(G, A)$ carries a natural left $E(A)$-module structure. Similarly, we set $M^* = \mathrm{Hom}_{E(A)}(G, A)$ for all left $E(A)$-modules M. The natural map $G \to G^{**}$ is denoted by ψ_G. Its kernel is $R_A(G) = \bigcap\{\ker f \mid f \in G^*\}$. A similar notation is used for left $E(A)$-modules.

LEMMA 3.1. *The following conditions are equivalent for a torsion-free abelian group A:*

a) *A has the radical splitting property.*

b) *For every index-set I, finitely generated submodules of A^I are projective.*

PROOF. a) $\Rightarrow$ b). Let M be a finitely generated submodule of A^I for some index-set I. Choose a projective resolution $0 \to U \xrightarrow{\alpha} E(A)^m \xrightarrow{\beta} M \to 0$ of M. It induces the exact sequence $0 \to M^* \xrightarrow{\beta^*} [E(A)^m]^* \xrightarrow{\alpha^*} K \to 0$ of abelian groups where $K \subseteq U^*$. Thus, $R_A(K) = 0$; and the last sequence splits. We obtain the commutative diagram

$$\begin{array}{ccccc}
[E(A)^m]^{**} & \xrightarrow{\beta^{**}} & M^{**} & \longrightarrow & 0 \\
\uparrow{\scriptstyle \psi_{E(A)^m}} & & \uparrow{\scriptstyle \psi_M} & & \\
E(A)^m & \xrightarrow{\beta} & M & \longrightarrow & 0
\end{array}$$

whose top-row splits. The map ψ_M is one-to-one since M is a submodule of A^I. Consequently, the vertical maps in the diagram are isomorphisms; and the bottom row splits too.

b) $\Rightarrow$ a). We consider an exact sequence $0 \to U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \to 0$ with $R_A(G) = 0$. It induces the exact sequence $0 \to G^* \xrightarrow{\beta^*} [A^n]^* \xrightarrow{\alpha^*} M \to 0$ of left $E(A)$-modules where M is a finitely generated submodule of U^*. There is an index-set I such that $U^* \subseteq A^I$ as a left $E(A)$-module. By b), we obtain that M is projective. An argument similar to the one used to prove the previous implication yields the splitting of the original sequence.

In particular, we obtain that a group A which has the radical splitting property has a left semi-hereditary endomorphism ring and is flat as a left $E(A)$-module by [7].

LEMMA 3.2. *The following conditions are equivalent for a torsion-free abelian group A whose endomorphism ring is left strongly non-singular:*

a) *A has the radical splitting property.*

b) *$E(A)$ is a left semi-hereditary ring, and A is non-singular as an $E(A)$-module.*

PROOF. It remains to show that b) implies a). If M is a finitely generated submodule of A^I, then M is non-singular. Consequently, M is isomorphic to a submodule of a finitely generated free module since $E(A)$ is strongly non-singular. By b), M is projective. Apply Lemma 3.1.

THEOREM 3.3. *The following conditions are equivalent for a torsion-free abelian group A:*

a) *$E(A)$ is a right (and left) semi-hereditary ring, and $QE(A)$ is semi-simple Artinian.*

b) i) *A has the splitting property for pure A-socles.*

ii) *A is a flat $E(A)$-module, and $A \neq IA$ for all pure proper right ideals I of A^n.*

c) i) *A has the radical splitting property.*

ii) *If I is a pure, proper left ideal of $E(A)$, then* $\operatorname{ann}(I) \neq 0$.

PROOF. b) $\Rightarrow$ a). It remains to show that $QE(A)$ is semi-simple Artinian. If I is an essential right ideal of $E(A)$, then we denote its Z-purification in $E(A)$ by I_*. Since A is flat, I_*A is a pure A-generated subgroup of A. Corollary 2.2, on the other hand, yields that A/I_*A is a torsion group. Thus, $A = I_*A$. By b), we obtain $E(A) = I_*$. Hence, $QE(A)$ is semi-simple Artinian.

a) $\Rightarrow$ b). Observe that non-singular modules over the ring in a) are flat. If I is a proper, pure right ideal of $E(A)$, then $E(A)/I$ and A are flat $E(A)$-modules. In particular, $E(A)/I$ is projective, and $E(A) = I \oplus J$ for some non-zero right ideal J of $E(A)$. Then, $A = IA \oplus JA$, and $A \neq IA$.

c) $\Rightarrow$ a). Let I_* be the Z-purification of the essential left ideal I of $E(A)$. The exact sequence $0 \to I_* \overset{\alpha}{\to} E(A) \overset{\beta}{\to} E(A)/I_* \to 0$ induces the sequence $0 \to [E(A)/I_*]^* \overset{\beta^*}{\to} E(A)^* \overset{\alpha^*}{\to} (I_*)^*$ which splits by b). We obtain the commutative diagram

$$\begin{array}{ccccc} E(A)^{**} & \xrightarrow{\beta^{**}} & \left[E(A)/I_*\right]^{**} & \longrightarrow & 0 \\ \uparrow{\scriptstyle \psi_{E(A)}} & & \uparrow{\scriptstyle \psi_{E(A)/I_*}} & & \\ E(A) & \xrightarrow{\beta} & E(A)/I_* & \longrightarrow & 0 \end{array}$$

whose top-row splits. Consider a splitting map τ for β^{**}. For $x \in I_* \cap \cap \operatorname{im}(\psi_{E(A)}^{-1}\tau)$, there is $y \in \left[E(A)/I_*\right]^{**}$ with $\psi_{E(A)}(x) = \tau(y)$. We obtain $y = \beta^{**}\tau(y) = \beta^{**}\psi_{E(A)}(x) = \psi_{E(A)/I_*}\beta(x) = 0$. Therefore, $\operatorname{im}(\psi_{E(A)}^{-1}\tau) = = 0$ since I_* is essential in $E(A)$. Because $\psi_{E(A)}^{-1}\tau$ is one-to-one, we obtain $\left[E(A)/I_*\right]^{**} = 0$. Consequently, $\left[E(A)/I_*\right]^{*} = 0$ since the latter is isomorphic to a direct summand of A. If $I_* \neq 0$, then there is a non-zero $a \in A$ with $I_*a \neq 0$. The assignment $1 \to a$ induces a non-zero $E(A)$-homomorphism $\sigma: E(A) \to A$ with $\sigma(I_*) = 0$, which is not possible. Thus, $I_* = E(A)$.

a) $\Rightarrow$ c). By Lemma 3.2, it enough to show that part ii) of b) holds. If I is a pure, proper left ideal of $E(A)$, then $E(A) = I \oplus J$ for some non-zero left ideal J of $E(A)$. We choose an idempotent $e \in E(A)$ with $I = E(A)e$, and $a \in A$ with $(1-e)(a) \neq 0$. Clearly, $(1-e)(a) \in \operatorname{ann}(I)$.

COROLLARY 3.4. *The following conditions are equivalent for a torsion-free abelian group A whose quasi-endomorphism ring is a finite dimensional Q-algebra:*

a) *A is a generalized rank* 1 *group.*

b) *A has the splitting property for pure A-socles.*

c) *A has the radical splitting property.*

PROOF. Conditions b) and c) yield that $E(A)$ is a right, respectively left, semi-hereditary ring. By [12], A is a generalized rank 1-group. The converse follows from Theorems 2.1 and 3.3.

In [3], we showed that the ring R of algebraic integers is semi-hereditary. If A is an abelian group with $E(A) = R$, then the quasi-endomorphism ring of A is a field; and A satisfies conditions b) and c) of the previous corollary by Theorem 3.3, although A is not a generalized rank 1 group.

COROLLARY 3.5. *Let A be a torsion-free abelian group which is faithfully flat as an $E(A)$-module. If A splits pure A-socles, then A has the radical splitting property.*

PROOF. By Corollary 2.3, $E(A)$ is right semi-hereditary, and $QE(A)$ is semi-simple Artinian. Consider a map $\phi: A^n \to A^I$ for some $n < \omega$ and some index-set I. Suppose that there is no finite subset J of I such that $\ker \pi_J\phi = \ker\phi$, where π_J is the projection of A^I onto A^J whose kernel is

A^{I-J}. Inductively, we obtain a descending chain $\{U_i \mid i < \omega\}$ of subgroups of A^n such that A^n/U_i is isomorphic to a subgroup of an A-projective group of finite A-rank and $U_i \neq U_{i+1}$ for all $i < \omega$. By [1], each U_i is an A-generated subgroup of A^n. Therefore, $\{H_A(U_i) \mid i < \omega\}$ is an infinite strictly descending chain of submodules of $H_A(A^n)$. Since A is torsion-free, each of these submodules is pure as an abelian group. Thus, $\{Q \otimes_Z H_A(U_i) \mid i < < \omega\}$ is an infinite descending chain of $QE(A)$-submodules of the finitely generated $QE(A)$-module $Q \otimes_Z H_A(A^n)$. Since $QE(A)$ is Artinian, this is not possible.

Therefore, $A^n/\ker\phi \cong \bigoplus_J A$ for some finite subset J of I. In particular, $A^n/\ker\phi$ is A-solvable. Since A is faithfully flat as an $E(A)$-module, [2] yields that $\ker\phi$ is A-solvable. Since A splits pure A-socles, $\ker\phi$ is a direct summand of A^n.

We now give an example that the converse of the last result fails in general:

EXAMPLE 3.6. Let A be an abelian group with $E(A) = Z^\omega$ which is faithfully flat as an $E(A)$-module. Such an A exists by [2] and [10]. By Corollary 2.3, A does not split pure A-socles. Since R is strongly non-singular and semi-hereditary, in view of Lemma 3.2 it is enough to show that A is non-singular as a left $E(A)$-module to ensure that A has the radical splitting property. But this is guaranteed by the fact that A is flat as an $E(A)$-module.

4. Faithfully flat S-groups

Consider a torsion-free abelian group G. If A is a subgroup of Q of type τ, then $S_A(G) = G(\tau)$ where $G(\tau) = \{x \in G \mid \text{type}(x) \geqq \text{type}(\tau)\}$ is a subgroup of G. While the A-socle of G, for groups A which are faithfully flat as an $E(A)$-module, resembles $G(\tau)$, there are significant differences if $r_0(A) > 1$. One of these is that $S_A(G)$ is not necessarily a pure subgroup of G. In this section, we investigate which conditions have to be satisfied by an abelian group A to ensure that A-socles of torsion-free groups are pure. As in [11], a torsion-free abelian group A is an *S-group* if $S_A(B) = B$ for all subgroups B of A of finite index.

THEOREM 4.1. *Let A be an abelian group which is faithful as an $E(A)$-module. Then, A is an S-group if and only if $S_A(B)$ is a pure subgroup of B for all torsion-free abelian groups B.*

PROOF. Suppose that A is a faithful S-group. By [11, Theorem III.1 and Corollary III.2], we obtain that $\mathrm{Ext}(A, A)$ is torsion-free, and that A has finite p-rank for all primes p of Z. Set $C = S_A(B)$. We claim that $\mathrm{Ext}(A, C)$ is torsion-free.

To show this, we observe that [11, Theorem III.1] guarantees that A is projective with respect to the sequence $0 \to pA \to A \to A/pA \to 0$. Let $\pi: C \to C/pC$ be the natural projection, and consider $f \in H_A(C/pC)$. Since A has finite p-rank, there are $x_1, \dots, x_n \in C$ such that $f(A) \subseteqq \subseteqq \langle x_1, \dots, x_n, pC\rangle/pC$. For each $i \in \{1, \dots, n\}$, there are $m_i < \omega$ and elements $g_{i1}, \dots, g_{im_i} \in H_A(C)$ and $a_{i1}, \dots, a_{im_i} \in A$ with $x_i = \sum_{j=1}^{m_i} g_{ij}(a_{ij})$. Set $m = m_1 + \dots + m_n$, and define a map $\phi: A^m = a^{m_1} \oplus \dots \oplus A^{m_n} \to C$ by $\phi[(b_1^1, \dots, b_{m_1}^1), \dots, (b_{m_1}^n, \dots, b_{m_n}^n)] = \sum_{i,j} g_{ij}(b_j^i)$, and denote its kernel by K. The group $G = \operatorname{im}\phi$ contains $x_1, \dots, x_n$. We obtain the exact sequence $\operatorname{Ext}(A, K) \to \operatorname{Ext}(A, A^m) \to \operatorname{Ext}(A, G) \to 0$. Since $\operatorname{Ext}(A, K)$ is divisible, $\operatorname{Ext}(A, G)$ is isomorphic to a direct summand of $\operatorname{Ext}(A, A^m)$. In particular, $\operatorname{Ext}(A, G)$ is torsion-free. By [11, Theorem II.2], the diagram

$$\begin{tikzcd} & A \arrow[d, "f"] & \\ G \arrow[r, "\pi"] & (G + pC)/pC \arrow[r] & 0 \end{tikzcd}$$

can be completed by a map $g \in H_A(C)$. Thus, $\operatorname{Ext}(A, C)$ is torsion-free by [11, Theorem II.2].

Suppose that C_1 is a subgroup of B containing C such that $pC_1 \subseteqq C$ for some prime p. Since B is torsion-free, and C is A-generated, $A \neq pA$. Consider the exact sequence $0 \to H_A(C) \xrightarrow{\alpha} H_A(C_1) \xrightarrow{\beta} H_A(C_1/C) \to \to \operatorname{Ext}(A, C)$ which is induced by the inclusion $C \subseteqq C_1$. Since $S_A(C_1) = C$, the map α is an isomorphism. Thus, β is a monomorphism, and $\operatorname{Ext}(A, C)$ is not torsion-free unless $C = C_1$. This shows that $S_A(B)$ is pure in B.

The converse is obvious.

Combining the last result with those of the previous sections yields

COROLLARY 4.2. *Let A be a faithful S-group such that $QE(A)$ is Artinian:*

a) *If U is a pure subgroup of A^n for some $n < \omega$, then $S_A(U)$ is a direct summand of U.*

b) *If U is a quasi-summand of A^n for some $n < \omega$, then U is A-projective.*

c) *$R_A(B/S_A(B)) = 0$ if B is a subgroup of A^n for some $n < \omega$.*

PROOF. a) By [11], $E(A)$ is a right hereditary. Corollary 2.2 yields that A splits pure A-socles. The previous result guarantees that $S_A(U)$ is a pure subgroup of A^n. Thus, $S_A(U)$ is a direct summand of U.

b) If U is a quasi-summand of A^n, then $U \oplus V$ is quasi-equal to A^n. Since $S_A(U \oplus V)$ is a pure subgroup of $U \oplus V$ by the last result and quasi-equal to A^n, we obtain that $U \oplus V$ is A-generated. Since $E(A)$ is hereditary, U is A-projective.

c) Let U be the Z-purification of $S_A(B)$ in A^n. By [1], U is an A-generated subgroup of A^n. Thus, we obtain a decomposition $A^n = U \oplus V$. By Theorem 4.1, $S_A(B)$ is a pure subgroup of B, and hence $B \cap U = S_A(B)$. Furthermore, $B/S_A(B) \cong [B+U]/U \subseteqq V$ shows that $R_A\big(B/S_A(B)\big) = 0$.

In conclusion, we want to remark that, in general, quasi-summands of A-projective groups of finite A-rank need not be A-projective even if $QE(A)$ is semi-simple Artinian [4].

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ON PROBLEMS OF APPROXIMATION IN L_2 SPACES

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1. Introduction

Let L_2^* be the class of all 2π-periodic, real valued functions $f(x)$ square integrable in the interval $[0,2\pi]$. Goyaliya [2] proved the following

THEOREM A. *If $f^{(s)} \in L_2^*$, then for $n \in \mathbf{N}$, $s \in \{0\} \cup \mathbf{N}$*

(1) $$E_n\big(f^{(s)}\big)_{L_2^*} \leqq 2^{-1/2}\omega\big(f^{(s)}, \pi/(n+1)\big)_{L_2^*},$$

(2) $$E_n(f)_{L_2^*} \leqq Mn^{-s}E_n\big(f^{(s)}\big)_{L_2^*},$$

where M is a constant and $E_n(f)_{L_2^*} = \|f - S_n(f)\|_{L_2^*}$, $S_n(f,x)$ is the partial sum of the Fourier series of $f(x)$.

In his paper Goyaliya asked the following two questions:

(1) Can Theorem A be extended to $(C,1)$ summability or matrix summability?

(2) Can the result be extended to some other series, viz. Legendre series, ultraspherical series, Bessel series, etc.?

The purpose of this paper is to answer these questions. We shall prove that (1) can be extended to $(C,1)$ summability, but (2) can not.

Let $L_2 = L_2[-1,1]$ be the class of all real functions $f(x)$ such that

$$\|f\|_{L_2} := \left\{1/2\int_{-1}^{1} f^2(x)\,dx\right\}^{1/2} < \infty.$$

Suppose

(3) $$f(x) \sim \textstyle\sum_{k=0}^{\infty} a_k P_k(x)$$

is its Fourier–Legendre series, where

$$a_k = (k + 1/2) \int_{-1}^{1} f(x) P_k(x)\, dx. \tag{4}$$

Define

$$E_n(f)_{L_2} = \left\| f - S_n(f) \right\|_{L_2} = \| f(x) - \textstyle\sum_{k=0}^{n} a_k P_k(x) \|_{L_2}.$$

For Problem 2 we shall establish an estimate similar to (1) and prove that (2) can be extended to Legendre series.

Throughout the paper, c always denotes a constant independent of f, n, and k, but not the same at each appearance.

2. Main results

Our main results are the following

THEOREM 2.1. *Let $f \in L_2^*$ and let $\sigma_n(f,x)$ be the $(C,1)$-means of its Fourier series. Then there exists a constant $c > 0$ such that*

$$F_n(f)_{L_2^*} := \left\| f(x) - \sigma_n(f,x) \right\|_{L_2^*} \leqq c\omega\left(f, \pi/(n+1)\right)_{L_2^*}. \tag{5}$$

If $f' \in L_2^*$ and $\|f'\| > 0$, then

$$F_n(f)_{L_2^*} \sim n^{-1} \|f'\|_{L_2^*}. \tag{6}$$

REMARK 1. (6) shows that (2) can not be extended to $(C,1)$ summability.

We say that the function $\omega(t) \in \mathbf{N}^\alpha$ $(\alpha > 0)$ if

(i) $\omega(t)$ defined on $[0,2]$ is nondecreasing and $\omega(t) \to 0$ as $t \to 0$.

(ii) For $0 < \delta < \eta \leqq 2$, there exists a constant $K = K(\alpha)$ such that

$$\eta^{-\alpha}\omega(\eta) \leqq K\delta^{-\alpha}\omega(\delta).$$

Define the Legendre transformation of f by

$$f_h(x) := \pi^{-1} \int_0^\pi f\left(x \cos h + \sqrt{1-x^2}\sin h \cos\theta\right) d\theta.$$

Butzer, Stens and Wehrens [1] introduced the modulus of continuity of $f \in \mathbf{C}$ by this transformation. Here we define the integral modulus of continuity of $f^{(s)} \in L_2$:

$$W^L\left(f^{(s)}, t\right)_{L_2} := \sup_{|h| \leqq t} \left\| \left(f_h^{(s)}(x) - f^{(s)}(x)\right)\left(1-x^2\right)^{s/2} \right\|_{L_2}.$$

It is easy to see that

$$W_L\left(f^{(s)},t\right)_{L_2} \to 0 \quad (t \to +0).$$

THEOREM 2.2. *Let $f^{(s)} \in L_2$ and let $\omega(t)$ be a given modulus of continuity. Then*

(a) *for $n \geqq 2$*

$$E_n(f_2)_{L_2} \leqq M\omega(1/n)$$

if and only if

$$W^L(f,t)_{L_2} \leqq cM\omega(t);$$

(b) *if $s \in \mathbf{N}$ and $\omega(t) \in \mathbf{N}^\alpha$ $(0 < \alpha \leqq 1)$, then for $n \geqq s$*

$$E_n(f)_{L_2} \leqq Mn^{-s}\omega(1/n) \tag{7}$$

if and only if

$$W^L\left(f^{(s)},t\right)_{L_2} \leqq cM\omega(t). \tag{8}$$

THEOREM 2.3. *If $f^{(s)} \in L_2$, then*

$$E_n(f)_{L_2} \leqq cn^{-s}W^L\left(f^{(s)},1/n\right)_{L_2}, \tag{9}$$

$$E_n(f)_{L_2} \leqq cn^{-s}E_n\left(f^{(s)}\right)_{L_2}. \tag{10}$$

REMARK 2. Theorem 2.2 is an improvement of a result of Zidkov [3].

3. Proofs

PROOF OF THEOREM 2.1. Zygmund [5] proved that if $p \geqq 1$ then

$$\left\| f(x) - \sigma_n(f,x)\right\|_{L_p^*} \leqq c\omega\left(\widetilde{f},\pi/(n+1)\right)_{L_p^*}, \tag{11}$$

where $\widetilde{f}$ is the conjugate function to $f(x)$. Then, (5) is a consequence of (11) and Riesz' Theorem (see [6, Ch. VII]).

Now we turn to prove (6). We have

$$\| f(x) - \sigma_n(f,x)\|^2_{L_2^*} = (n+1)^{-2}\sum_{k=1}^{\infty} k^2(a_k^2+b_k^2)+ \tag{12}$$

$$+\sum_{k=n+1}^{\infty}(a_k^2+b_k^2) - (n+1)^{-2}\sum_{k=n+1}^{\infty} k^2(a_k^2+b_k^2).$$

From [2]

$$\sum_{k=n+1}^{\infty}(a_k^2+b_k^2) \leqq cn^{-2}E_n^2(f')_{L_2^*} = o(n^{-2}) \quad (n\to\infty). \tag{13}$$

It is clear that

$$(n+1)^{-2}\sum_{k=n+1}^{\infty} k^2(a_k^2+b_k^2) = \tag{14}$$

$$= (n+1)^{-2}\| f' - S_n(f')\|^2_{L_2^*} = o(n^{-2}) \quad (n\to\infty).$$

Hence from (12)–(14) it follows that

$$\| f(x) - \sigma_n(f,x)\|_{L_2^*} \sim n^{-1}\{\sum_{k=1}^{\infty} k^2(a_k^2+b_k^2)\}^{1/2} = n^{-1}\|f'\|_{L_2^*}.$$

Q.E.D.

Proof of Theorem 2.2. Set $x=\cos\beta$. Using (3) and the addition formula of the Legendre polynomials (cf. [4])

$$P_k(\cos\beta\cos h+\sin\beta\sin h\cos\theta) = P_k(\cos\beta)P_k(\cos h)+$$

$$+2\sum_{m=1}^{k}(k-m)!((k+m))^{-1}P_k^m(\cos\beta)P_k^m(\cos h)\cos m\theta,$$

where

$$P_k^m(x) = (1-x^2)^{m/2}d^mP_k(x)/dx^m \quad (m=1,2,\dots,k),$$

we get

$$f_h(x) = \pi^{-1}\int_0^{\pi} f(\cos\beta\cos h+\sin\beta\sin h\cos\theta)\,d\theta =$$

$$= \sum_{k=0}^{\infty} a_kP_k(x)P_k(\cos h),$$

where a_k is denoted by (4). Observing the properties of the function $P_k^m(x)$:

$$\int_{-1}^{1} P_k^m(x)P_l^m(x)\,dx = 0, \quad (k\neq l),\ [4] \tag{15}$$

and

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = 2(n+m)!/\left[(2n+1)(n-m)!\right], \ [4] \tag{16}$$

we obtain (set $b_k^2 = 2a_k^2/(2k+1)$)

$$\left\|\left(f_h^{(s)}(x) - f^{(s)}(x)\right)(1-x^2)^{s/2}\right\|_{L_2}^2 = \tag{17}$$

$$= \sum_{k=s}^{\infty}\left(1 - P_k(\cos h)\right)^2 2a_k^2(k+s)!/\left[(2k+1)(k-s)!\right] =$$

$$= \sum_{k=s}^{\infty}\left(1 - P_k(\cos h)\right)^2 b_k^2(k+s)!/(k-s)!.$$

First assume that for $s = 0, 1, 2, \ldots$ (8) is satisfied. Since

$$(k+s)!/(k-s)! \sim k^{2s},$$

from (17), we have for $|h| \leqq t$

$$\sum_{k=s}^{\infty}\left(1 - P_k(\cos h)\right)^2 k^{2s} b_k^2 \leqq \tag{18}$$

$$\leqq c\sum_{k=s}^{\infty}\left(1 - P_k(\cos h)\right)^2 b_k^2(k+s)!/(k-s)! =$$

$$= c\left\|\left(f_h^{(s)}(x) - f^{(s)}(x)\right)(1-x^2)^{s/2}\right\|_{L_2}^2 \leqq c\left\{W^L(f^{(s)}, t)_{L_2}\right\}^2 \leqq cM^2\omega^2(t).$$

If $h = 2/n$ and $k \geqq n \geqq 2$, using the known estimate of $P_k(\cos h)$,

$$\left|P_k(\cos h)\right| \leqq \left(2/(\pi k \sin h)\right)^{1/2} \leqq 2^{-1/2}.$$

Hence

$$1 - P_k(\cos h) \geqq 1/4 \quad (k \geqq n,\ h = 2/n,\ n \geqq 2).$$

Then using (15), (16) and (18), we get

$$E_n^2(f)_{L_2} = \left\|f - S_n(f)\right\|_{L_2}^2 = \sum_{k=n+1}^{\infty} b_k^2 \leqq \tag{19}$$

$$\leqq 16n^{-2s}\sum_{k=n+1}^{\infty}\left(1 - P_k(\cos 2/n)\right)^2 k^{2s} b_k^2 \leqq$$

$$\leqq 16n^{-2s}\sum_{k=s}^{\infty}\left(1 - P_k(\cos 2/n)\right)^2 k^{2s} b_k^2 \leqq \tag{20}$$

$$\leqq cM^2 n^{-2s}\omega^2(1/n). \tag{21}$$

That is (7) holds for $s = 0, 1, 2, \ldots$.

Now assume (7) holds for $s = 0, 1, 2, \ldots$. Set $n = [h^{-1}]$. Observing (17), we write

$$\left\| \left(f_h^{(s)}(x) - f^{(s)}(x) \right) (1 - x^2)^{s/2} \right\|_{L_2}^2 = \tag{22}$$

$$= \left(\sum_{k=s}^{n-1} + \sum_{k=n}^{\infty} \right) \left(1 - P_k(\cos h) \right)^2 (k+s)! b_k^2 / (k-s)! := \sum_1 + \sum_2.$$

If $s \in \mathbf{N}, \omega(t) \in \mathbf{N}^\alpha$ $(0 < \alpha \leqq 1)$, then observing (19), by Abel transformation we get

$$\gamma_n := \sum_{k=n}^{\infty} k^{2s} b_k^2 = n^{2s} \sum_{k=n}^{\infty} b_k^2 + \tag{23}$$

$$+ \sum_{k=n+1}^{\infty} \left(k^{2s} - (k-1)^{2s} \right) \sum_{i=k}^{\infty} b_i^2 \leqq$$

$$\leqq n^{2s} E_{n-1}^2(f)_{L_2} + c \sum_{k=n+1}^{\infty} k^{2s-1} E_{k-1}^2(f)_{L_2} \leqq$$

$$\leqq cM^2 \omega^2(1/n) + cM^2 \sum_{k=n+1}^{\infty} k^{-1} \omega^2(1/k) \leqq$$

$$\leqq cM^2 \omega^2(1/n) + cM^2 n^{2\alpha} \omega^2(1/n) \sum_{k=n+1}^{\infty} k^{-1-2\alpha} \leqq cM^2 \omega^2(1/n).$$

If $s = 0$, then by assumption

$$\gamma_n = \sum_{k=n}^{\infty} b_k^2 = E_{n-1}^2(f) \leqq cM^2 \omega^2(1/n). \tag{24}$$

From (23)–(24) it follows that for $s = 0, 1, 2, \ldots$

$$\sum_2 \leqq c\gamma_n = c \sum_{k=n}^{\infty} k^{2s} b_k^2 \leqq cM^2 \omega^2(1/n). \tag{25}$$

Using Abel transformation again, from (23)–(24) and the estimate (cf. [3])

$$\left| 1 - P_k(\cos h) \right| \leqq k^2 h^2 / 2,$$

we have

$$\sum_1 \leqq c \sum_{k=1}^{n} h^4 k^{4+2s} b_k^2 = \tag{26}$$

$$= ch^4 \left(\sum_{k=1}^{n} k^4 \gamma_k - \sum_{k=2}^{n+1} (k-1)^4 \gamma_k \right) \leqq ch^4 \sum_{k=1}^{n} k^3 \gamma_k \leqq$$

$$\leqq cM^2 h^4 \sum_{k=1}^{n} k^3 \omega^2(1/k) \leqq$$

$$\leqq cM^2 h^4 \sum_{k=1}^{n} k^3 (1 + n/k)^2 \omega^2(1/n) \leqq cM^2 \omega^2(1/n).$$

Combining (22), (25) and (26) yields

$$\left\|\left(f_h^{(s)}(x)-f^{(s)}(x)\right)\left(1-x^2\right)^{s/2}\right\|_{L_2} \leqq cM\omega(1/n) \leqq cM\omega(h).$$

From the above it follows that for $s=0,1,2,\dots$

$$W^L(f^{(s)},t)_{L_2} \leqq cM\omega(t).$$

Q.E.D.

PROOF OF THEOREM 2.3. (9) follows from (20) and (17) immediately. Now turn to prove (10). Obviously, it is sufficient to prove (10) for the case $s=1$. By induction we can easily obtain that (10) holds for all $s \geqq 1$. Using the method of proof of Theorem 2.2, we can prove that if $s \geqq 1$ and

$$\left\|\left(f_h^{(s)}(x)-f^{(s)}(x)\right)\left(1-x^2\right)^{s/2}\right\|_{L_2} \leqq M,$$

then

$$E_n(f)_{L_2} \leqq cMn^{-s}.$$

In fact, if $n \geqq 2/h$, then from (20) and (17)

$$E_n^2(f)_{L_2} \leqq cn^{-2s}\textstyle\sum_{k=s}^{\infty}\left(1-P_k(\cos 2/n)\right)^2 k^{2s}b_k^2 \leqq$$

$$\leqq cn^{-2s}\textstyle\sum_{k=s}^{\infty}\left(1-P_k(\cos 2/n)\right)^2(k+s)!b_k^2/(k-s)! =$$

$$= cn^{-2s}\left\|\left(f_h^{(s)}(x)-f^{(s)}(x)\right)\left(1-x^2\right)^{s/2}\right\|_{L_2}^2 \leqq cn^{-2s}M^2.$$

Let $s=1$ and $f' \in L_2$. Then, using Parseval equality and (17), we get

$$2\|f'\|_{L_2}^2 = \int_{-1}^{1}\left(f'(x)\right)^2 dx \geqq \int_{-1}^{1}\left(f'(x)\right)^2(1-x^2)\,dx =$$

$$= \int_{-1}^{1}\left(\textstyle\sum_{k=1}^{\infty}a_kP_k'(x)\right)^2(1-x^2)\,dx =$$

$$= \textstyle\sum_{k=1}^{\infty}a_k^2 2(k+1)!/\left[(2k+1)(k-1)!\right] \geqq$$

$$\geqq \textstyle\sum_{k=1}^{\infty}(1/4)b_k^2\left[1-P_k(\cos h)\right]^2(k+1)!/(k-1)! =$$

$$= (1/4)\left\|\left(f_h'(x)-f'(x)\right)\left(1-x^2\right)^{1/2}\right\|_{L_2}^2.$$

Hence

$$\left\| \left(f_h'(x) - f'(x) \right) \left(1 - x^2\right)^{1/2} \right\|_{L_2} \leqq c \|f'\|_{L_2}.$$

Using the above proved result with $s = 1$ and $M = \|f'\|_{L_2}$, we get

$$E_n(f)_{L_2} \leqq cn^{-1} \|f'\|_{L_2}.$$

Now let P_n' be the best approximating polynomial of f', then

$$E_n(f)_{L_2} = E_n(f - P_n)_{L_2} \leqq c\|f' - P_n'\|_{L_2}/n \leqq cn^{-1} E_n(f')_{L_2}.$$

Q.E.D.

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TURÁN TYPE PROBLEMS ON MEAN CONVERGENCE. II (HERMITE–FEJÉR TYPE INTERPOLATIONS)

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This paper is the second part of [19]. That means many notations, definitions and theorems used here are detailed in the three chapters of [19]. Here we refer to them without any further explanation. Moreover, for example, (1.2) means [19, (1.2)], etc., while the references [1]–[18] are detailed in [19].

4. Introduction. Notations

While in [19] theorems were proved for *odd* values of M (Lagrange type interpolation) throughout this paper we mainly deal with the process I_{nM} for *even* values of M (Hermite–Fejér type interpolations; cf (1.1) and Part 1.1). From now on we denote them by H_{ns} where $s = 2, 4, 6, \ldots$, is fixed. If $s = 2$, we often write H_n (the classical Hermite–Fejér interpolation).

5. Results on Hermite–Fejér type interpolations

Contrary to the Lagrange type cases, for many matrices we have uniform convergence results for every $f \in C$ taking H_{ns} (cf. (1.8)–(1.10)). However the complete analogoue of the Erdős–Turán result is still missing (cf. (1.11)). Instead, we have

THEOREM 5.1. *Let w be a fixed weight. Then*

$$(5.1)\quad \lim_{n\to\infty} \int_{-1}^{1} \left| H_n(R, w, x) - R(x) \right| w(x)dx = 0 \quad \textit{for every polynomial} \quad R.$$

The relatively simple proof is in P. Nevai, P. Vértesi [20, p. 46].

If $s = 2, 4, 6, \ldots$, by P. Vértesi [21, Theorem 2.1, p. 371] (using definitions in Part 2.1), we have

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THEOREM 5.2. *Let $v \in \mathcal{J}_s$ and $u \in \mathcal{J}$. Then for every p, $0 < p < \infty$,*

$$\lim_{n\to\infty} \| H_{ns}(f,v) - f\|_{p,u} = 0 \qquad \forall f \in C \tag{5.2}$$

iff

$$\frac{\sqrt{1-x^2}}{\left(v(x)\sqrt{1-x^2}\right)^{s/2}} \in L_u^p \tag{5.3}$$

(Compare Remark 2.2.5.) A simple consequence of Theorem 5.2 is

COROLLARY 5.3. *Let $v \in \mathcal{J}_s$. Then if $p = 2/s$,*

$$\lim_{n\to\infty} \int_{-1}^{1} | H_{ns}(f,v,x) - f(x)|^p v(x)dx = 0 \qquad \forall f \in C. \tag{5.4}$$

(Cf. (2.1).) Results (5.1) and (5.4) suggest that the critical exponent is (again) $2/s$. The aim of this paper is to further strengthen this hint by verifying a Turán type result in Theorem 6.4. The scheme and some arguments are similar to those in [19].

6. New results

6.1. First we prove the analogue of Theorem 2.5. Let $f_1(x) = x$ and $\Delta_n = [-1, x_{n,n}] \cup [x_{1n}, 1]$.

THEOREM 6.1. *Let u and w be two weights, X be a w-regular interpolatory matrix. Then there exists an $\varepsilon > 0$ such that if $R_n \in \mathcal{M}$, $|R_n| \leqq \varepsilon$, otherwise arbitrary, we have for every p, $0 < p \leqq \infty$, the relation*

$$\begin{aligned} \| f_1(x) - H_{ns}(f_1, X, x)\|_{p,u} \cdot \| \chi_{CR_n}(x)\omega_n^s(X,x)\|_{1,w} \geqq \\ \geqq \frac{c}{n} \| \chi_{\Delta_n}(x)\omega_n^s(X,x)\|_{p,u}, \quad n \geqq 1 \end{aligned} \tag{6.1}$$

with a proper $c > 0$ not depending on p.

Considering the factor $1/n$ and that $|\Delta_n|$ is generally much smaller than 2, one may think that (6.1) is not sharp (cf. (2.7)). However, it turns out that the estimation generally gives the best result.

THEOREM 6.2. *There exist weights u and w and a w-regular interpolatory matrix X such that for every p, $0 < p < \infty$,*

$$\overline{\lim_{n\to\infty}} \| f(x) - H_{ns}(f, X, x) \|_{p,u} \| \omega_n^s(X, x) \|_{1,w} \leqq \tag{6.2}$$

$$\leqq \lim_{n\to\infty} \frac{\varphi_n}{n} \| \chi_{\Delta_n} \omega_n^s(X, x) \|_{p,u} \qquad \forall f \in C.$$

Here $\{\varphi_n\}$ is an arbitrary sequence with $0 < \varphi_1 < \varphi_2 < \ldots$, $\lim_{n\to\infty} \varphi_n = \infty$.

6.2. An important consequence of Theorem 6.1 is as follows (cf. Theorem 2.1).

THEOREM 6.3. *Let* $\operatorname{supp}(d\alpha) = [-1, 1]$, $\alpha'(x) > 0$ *a.e. in* $[-1, 1]$ *and* u *be a weight. Then for any fixed* $0 < p \leqq \infty$ *we have*

$$\| f_1(x) - H_{ns}(f_1, d\alpha, x) \|_{p,u} \geqq \frac{c}{n} \| \chi_{\Delta_n}(x) p_n^s(d\alpha, x) \|_{p,u}. \tag{6.3}$$

However, for the right hand side of (6.3) we cannot apply relation (3.24) (which has previously led to the Turán type Theorem 2.3).

Nevertheless, using straightforward calculations Theorem 6.3 yields as follows (cf. Theorem 2.3).

Let

$$W(x) = \frac{\exp\left((\vartheta - \frac{\pi}{2}) \cot \vartheta\right)}{\cosh(\frac{\pi}{2} \cot \vartheta)}, \quad x = \cos \vartheta, \quad 0 \leqq \vartheta \leqq \pi \tag{6.4}$$

be the Pollaczek-weight $w(\cos \vartheta; 1, 0)$ (cf. [11, Appendix, (1.9), p. 392]).

THEOREM 6.4. *For any fixed p with $2/s < p \leqq \infty$ we have*

$$\lim_{n\to\infty} \| f_1(x) - H_{ns}(f_1, W, x) \|_{p,W} = \infty. \tag{6.5}$$

6.5. Simple considerations show that

$$W(x) = g(x) \exp\left(-\frac{1}{\sqrt{1 - x^2}}\right), \quad -1 \leqq x \leqq 1,$$

where $g(x)$ is positive and continuous on $[-1, 1]$ (cf. [6, Example 14, p. 82]). So combining Corollary 2.2, Statement 2.3 (with $\delta = 1/2$) and Theorem 6.4, we immediately obtain the following Turán type theorem valid for an *arbitrary* process I_{nM} (cf. Part 1.1 (in [19]) for the definition of I_{nM}).

STATEMENT 6.5. *Let $M = 1, 2, 3, \ldots$ be fixed. Then there is an $f \in C$ such that for any p, $2/M < p \leqq \infty$,*

$$\overline{\lim_{n\to\infty}} \int_{-1}^{1} \left| I_{nM}(f, W, x) - f(x) \right|^p W(x)dx = \infty. \tag{6.6}$$

If $p = 2/M$, the corresponding convergence results were stated in Theorem 1.1 ($M = 1$), formula (2.1) ($M = 1, 3, 5, \ldots$), Theorem 5.1 ($M = 2$) and Corollary 5.3 ($M = 2, 4, 6, \ldots$).

Finally, let us remark that similar other weights can be found to get statements similar to Theorems 6.4 and 6.5. Details are left to the reader (cf. [6, Example 14, p. 82]).

6.6. Some natural problems arise.

1. Prove relation (5.1) for arbitrary $f \in C$.
2. Prove Corollary 5.3 for other weights.
3. Prove Theorems 6.1 and 6.4 for other polynomials or continuous functions.

7. Proofs

7.1. PROOF OF THEOREM 6.1. By definition and using (3.17), relation $x - x_k \leqq 2$ and (3.20), we get for $x_1 \leqq x \leqq 1$ (whence $x - x_k \geqq 0$, $1 \leqq k \leqq n$)

$$\begin{aligned} f_1(x) - H_{ns}(f_1, x) &= \sum_{k=1}^{n} h_{1k}(x) = \sum_{k=1}^{n} \ell_k^s(x)(x - x_k)B_{1k}(x) \geqq \\ &\geqq c_1\omega_n^s(x) \sum_{k=1}^{n} \frac{(x - x_k)^{s-1}}{\left\{\omega_n'(x_k)(x - x_k)\right\}^s (x_k - x_{k\pm1})^{s-2}} \geqq \\ &\geqq \frac{c_1}{2}\omega_n^s(x) \sum_{k=1}^{n} \frac{1}{\left(\omega_n'(x_k)\right)^s (x_k - x_{k\pm1})^{s-2}} = \frac{c_t}{2} D_{1ns}\left([-1, 1]\right)\omega_n^s(x) \geqq \\ &\geqq \frac{c}{n}\frac{\omega_n^s(x)}{\|\chi_{CR_n}\omega_n^s\|_{1,w}}, \quad x_1 \leqq x \leqq 1. \end{aligned} \tag{7.1}$$

If $-1 \leqq x \leqq x_n$, then $x_k - x \geqq 0$, which yields the same estimation for $H_{ns}(f_1, x) - f_1(x)$. By these relation (6.1) is obvious. $\square$

7.2. Proof of Theorem 6.2 (cf. Section 3.4). Let $u \in \mathcal{J}(c,d)$, $w = \left(v\sqrt{1-x^2}\right)^{s/2}$ with $v \in \mathcal{J}_s(a,b)$, $X = X(v)$ and $\omega_n(X,x) = p_n(v,x)$. Then by (3.51), $\left\| p_n^s(v)\right\|_{1,w} \sim 1$, further, using (3.51) again, one gets

$$(7.2)\qquad \frac{1}{n}\left\| \chi_{\Delta_n} p_n^s(v)\right\|_{p,u} \sim n^{s(a+1/2)-\frac{2}{p}(c+1)-1} + n^{s(b+1/2)-\frac{2}{p}(d+1)-1} := r_n.$$

Now, if (5.3) does not hold, by (5.3) $\Longleftrightarrow$ (5.2), the left hand side of (6.2) is greater than zero for a proper $f \in C$. On the other hand,

$$\sqrt{1-x^2}\left(v\sqrt{1-x^2}\right)^{-s/2} \notin L_u^p$$

means that the exponent of $1-x^2$ is less than or equal to -1, whence

$$\frac{p}{2}+c-\frac{sp}{2}(a+1/2) \leqq -1 \quad \text{or} \quad \frac{p}{2}+d-\frac{sp}{2}(b+1/2) \leqq -1$$

whence $\lim\limits_{n\to\infty} r_n \geqq 1$, i.e. $\lim\limits_{n\to\infty} \varphi_n r_n = \infty$.

So we obtained (6.2) whenever (5.3) does not hold.

When relation (5.3) is true, by (5.3) $\Longleftrightarrow$ (5.2), (6.2) is obvious. □

7.3. Proof of Theorem 6.3. It goes like the first part of the proof of Theorem 2.1. We apply Theorem 6.1. Let $X = X(d\alpha)$ and $\omega_n(X) = p_n(d\alpha)$. Then if $w = \delta^s$ ($X(d\alpha)$ is δ^s-regular) and $R_n = B_{1n}$, we obtain

$$(7.3)\qquad 0 < \left\| \chi_{CB_{1n}} p_n^s(d\alpha)\right\|_{1,\delta^s} < 2, \quad n \geqq n_0$$

(cf. (3.26)–(3.28)), whence by relation (6.1) we obtain (6.3). □

7.4. Proof of Theorem 6.4. Using formulae (6.4), [11; Appendix (1.8), (5.3) and (5.5)] we obtain

$$(7.4)\, W(\cos\vartheta) = 2\left(\exp\left(1-\frac{\pi}{\vartheta}\right)\right)(1+O(\vartheta)) \quad \text{if} \quad \vartheta > 0 \quad \text{is small enough,}$$

$$(7.5)\ p_n\left(W, \cos\frac{t}{\sqrt{n}}\right) = \frac{1}{2\varepsilon\sqrt{e\pi}\, n^{1/4}} \exp\left\{\sqrt{n}\left(\frac{\pi}{2t}+d\varepsilon^6\right)\right\}\left(1+O\left(\frac{1}{n}\right)\right),$$

where $|d| \leqq 2$ if $0 < t = \sqrt{1-\varepsilon^4}$, $0 < \varepsilon \leqq \frac{1}{2}$, fixed, $n \geqq 1$,

$$(7.6)\qquad \lim_{n\to\infty} \sqrt{n}\vartheta_{1n}(W) = 1,$$

respectively. Now using (6.3), (7.4)–(7.6), the fact that $p_n(x)$ is strictly increasing when $x \geqq x_{1n}$ and writing the condition $p > 2/s$ as $p = \frac{2+4\delta}{s}$, we get with a certain fixed $\rho > 0$, as follows (supposing that $p < \infty$).

$$\|f_1(x) - H_{ns}(f_1, W, x)\|_{p,W}^p \geqq \left(\frac{c}{n}\right)^p \int_{x_{1n}}^{1} |p_n|^{sp} W \geqq \tag{7.7}$$

$$\geqq \left(\frac{c}{n}\right)^p \int_{\cos\frac{t}{\sqrt{n}}}^{\cos\frac{t}{(1+\rho)\sqrt{n}}} |p_n|^{2+4\delta} W \geqq$$

$$\geqq c\left(\frac{1}{n}\right)^p \cdot \frac{1}{n} n^{\frac{1}{(2+4\delta)/4}} \exp\left\{\frac{\sqrt{n}}{t}\pi\left(1 + 2\delta - 1 - \rho + \frac{2+4\delta}{\pi} t d\varepsilon^6\right)\right\}, n \geqq n_0$$

(the length of the interval is cn^{-1}). Here $(\dots) \geqq \delta$ whenever ε and ρ are small enough. This gives the theorem when $p < \infty$. The case $p = \infty$ comes from (6.3), (7.5), and (7.6). □

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ASYMPTOTIC STABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

We consider a system of functional differential equations with finite delay written as

$$(1) \qquad x'(t) = f(t, x_t), \qquad ' = d/dt,$$

where $f\colon [0,\infty) \times C_H \to R^m$ is continuous and takes bounded sets into bounded sets and $f(t,0) = 0$. Here, $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi\colon [-h, 0] \to R^m$ with the supremum norm, h is a non-negative constant, C_H is the open H-ball in C, and $x_t(s) = x(t+s)$ for $-h \leqq s \leqq 0$. Standard existence theory shows that if $\phi \in C_H$ and $t_0 \geqq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ on $[t_0, t_0 + \alpha)$ satisfying (1) for $t > t_0$, $x_t(t_0, \phi) = \phi$ and α some positive constant; if there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Also, $|\cdot|$ will denote the norm in R^m with $|x| = \max_{1 \leqq i \leqq m} |x_i|$.

We are concerned here with asymptotic stability in the context of Liapunov's direct method. Thus, we are concerned with continuous, strictly increasing functions $W_i\colon [0,\infty) \to [0,\infty)$ with $W_i(0) = 0$, called wedges, and with Liapunov functionals V.

DEFINITION. A continuous functional $V\colon [0,\infty) \times C_H \to [0,\infty)$ which is locally Lipschitz in ϕ is called a Liapunov functional for (1) if there is a wedge W with

(i) $W(|\phi(0)|) \leqq V(t,\phi)$, $V(t,0) = 0$, and

(ii) $V'_{(1)}(t, x_t) = \limsup\limits_{\delta \to 0} \frac{1}{\delta} \left\{ V\left(t+\delta, x_{t+\delta}(t_0,\phi)\right) - V\left(t, x_t(t_0,\phi)\right) \right\} \leqq 0.$

REMARK. A standard result states that if there is a Liapunov functional for (1), then $x = 0$ is stable. Definitions will be given in the next section.

The classical result on asymptotic stability may be traced back to Marachkov [9] through Krasovskii [7, pp. 151–154]. It may be stated as follows.

THEOREM MK. *Suppose there are a constant M, wedges W_i, and a Liapunov functional V (so $W_1(|\phi(0)|) \leqq V(t,\phi)$ and $V(t,0)=0$) with*

(i) $V'_{(1)}(t,x_t) \leqq -W_2(|x(t)|)$ *and*
(ii) $|f(t,\phi)| \leqq M$ *if* $t \geqq 0$ *and* $\|\phi\| < H$.

Then $x=0$ is asymptotically stable.

Condition (ii) is troublesome, since it excludes many examples of considerable interest. And there are several results which reduce or eliminate (ii). For example, we showed [2] that if

(iii) $V(t,\phi) \leqq W_3(|x| + |x_t|_2)$,

where $|\cdot|_2$ is the L^2-norm, then uniform asymptotic stability would result. Other alternatives may be found in [3, 4, 5, 6], for example.

We reduce (ii) in a variety of ways and obtain results on asymptotic stability, partial stability, and uniform asymptotic stability. Our work was motivated in part by the fact that the zero solution of

$$x'' + tx' + x = 0 \tag{2}$$

is asymptotically stable [1, 5, 10, 11], so that a substantial weakening of (ii) is indicated.

The following is a simplified corollary to our results and is stated here to focus the paper.

THEOREM A. *Suppose there is a Liapunov functional V, wedges W_i, positive constants K and J, a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \leqq K$ such that*

(i) $V(t_n,\phi) \leqq W_2(\|\phi\|)$,
(ii) $V'_{(1)}(t,x_t) \leqq -W_3(|x(t)|)$ *if* $t_n - h \leqq t \leqq t_n$, *and*
(iii) $|f(t,\phi)| \leqq J(t+1)\ln(t+2)$ *for* $t \geqq 0$ *and* $\|\phi\| < H$.

Then $x=0$ is AS.

2. Statement of results and an example

We now define the terminology to be used here.

DEFINITION. The solution $x=0$ of (1) is:

(a) *stable* if for each $\varepsilon > 0$ and $t_0 \geqq 0$ there is a $\delta > 0$ such that $[\|\phi\| < \delta, t \geqq t_0]$ imply that $|x(t,t_0,\phi)| < \varepsilon$;

(b) *uniformly stable (US)* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t_0 \geqq 0, \|\phi\| < \delta, t \geqq t_0]$ imply that $|x(t,t_0,\phi)| < \varepsilon$;

(c) *asymptotically stable (AS)* if it is stable and if for each $t_0 \geqq 0$ there is a $\gamma > 0$ such that $\|\phi\| < \gamma$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$;

(d) *uniformly asymptotically stable (UAS)* if it is US and if there is a $\gamma > 0$ and for each $\mu > 0$ there is a $T > 0$ such that $\left[t_0 \geqq 0,\ \|\phi\| < \gamma,\ t \geqq t_0 + T\right]$ imply that $\left|x(t, t_0, \phi)\right| < \mu$.

In preparation for our main result we remind the reader that if V is a Liapunov functional, then $W_1\left(|\phi(0)|\right) \leqq V(t, \phi)$, $V(t, 0) = 0$, and $V'_{(1)}(t, x_t) \leqq 0$. So that our result applies also to ODE's we introduce a positive number k which will replace h found in (1).

THEOREM 1. *Let $k > 0$, $k \geqq h$, let V be a Liapunov functional for* (1) *(so that $W_1\left(|\phi(0)|\right) \leqq V(t, \phi)$, $V(t, 0) = 0$, and $V'_{(1)}(t, x_t) \leqq 0$) and $x = (x_1, \dots, x_m)$. Consider the following conditions for a given i ($1 \leqq i \leqq m$) and a given sequence $\{t_n\}$ with $t_n \uparrow \infty$:*

(i) *there are wedges W_i, U_i, Q_i,*

(ii) *there are locally integrable functions $M_i, P_i \colon [0, \infty) \to [0, \infty)$,*

(iii) *there is a sequence $\left\{\lambda_n^{(i)}\right\}$ with $\lambda_n^{(i)} \geqq \lambda > 0$ (λ is constant) such that if $a, b \in [t_n - k, t_n]$ with $a < b$, then $\int_a^b M_i(t)\, dt \leqq \lambda_n^{(i)}(b - a)$,*

(iv) *for each $D > 0$ with $D/\lambda_n^{(i)} \leqq k$ there is a sequence $\left\{c_n^{(i)}\right\}$, $c_n^{(i)} > 0$, such that $\int_{s_n}^{s_n + D/\lambda_n^{(i)}} P_i(s)\, ds \geqq c_n^{(i)}$ for all $s_n \in \left[t_n - k, t_n - D/\lambda_n^{(i)}\right]$,*

(v) *$V'_{(1)}(t, x_t) \leqq -P_i(t) U_i\left(|x_i|\right)$ for $\|x_t\| < H$ and $t \in [t_n - k, t_n]$, and*

(vi) *$V'_{(1)}(t, x_t) \leqq -Q_i\left(|x'_i|\right) + M_i(t)$ for $\|x_t\| < H$ and $t \in [t_n - k, t_n]$ with Q_i convex downward.*

We then have the following conclusions:

(I) *If* (i)–(vi) *hold for all i satisfying $1 \leqq i \leqq m$ and for some $\{t_n\} \uparrow \infty$ with $c_n^{(i)} \geqq c_0 > 0$ for all n and all i, if $t_n - t_{n-1}$ is bounded, and if $V(t, \phi) \leqq W\left(\|\phi\|\right)$, then $x = 0$ is UAS.*

(II) *If* (i)–(vi) *hold for an arbitrary sequence $\{t_n\} \uparrow \infty$ and for some i satisfying $1 \leqq i \leqq m$, if $c_n^{(i)} \geqq c_0 > 0$ for all n then any solution $x(t)$ which remains in C_H satisfies $x_i(t) \to 0$ as $t \to \infty$.*

(III) *If* (i)–(vi) *hold for all i satisfying $1 \leqq i \leqq m$ and for some sequence $\{t_n\} \uparrow \infty$, if $V(t_n, \phi) \leqq W\left(\|\phi\|\right)$, if $c_n^{(i)} \geqq c_n$ for $1 \leqq i \leqq m$ and some c_n with $\sum_{n=1}^{\infty} c_n = \infty$, then $x = 0$ is AS.*

REMARK. Theorem 1 is long because it is stated in terms of separate components of x. However, to grasp the significance we will now state some useful corollaries.

COROLLARY 1. *Suppose there is a Liapunov functional V, a locally integrable function $M \colon [0, \infty) \to [0, \infty)$ and a monotone increasing function $\lambda \colon [0, \infty) \to (1, \infty)$ such that if $0 < b - a < h$ then*

(i) $\int_a^b M(t)\,dt \leqq \lambda(b)(b-a)$ *and* $\int_1^\infty \frac{dt}{\lambda(t)} = \infty$.

Suppose also that there are wedges, a constant $K > 0$, *and a sequence* $\{t_n\} \uparrow \infty$ *with* $t_n - t_{n-1} \leqq K$ *such that*

(ii) $V(t_n, \phi) \leqq W(\|\phi\|)$

and if $t_n - h \leqq t \leqq t_n$ *then*

(iii) $V'_{(1)}(t, x_t) \leqq -W_2(|x(t)|)$ *and*
(iv) $V'_{(1)}(t, x_t) \leqq -W_3(|x'(t)|) + M(t)$, W_3 *is convex downward.*

Then $x = 0$ *is AS.*

COROLLARY 2. *Suppose there is a Liapunov functional* V, *wedges* W_i, *positive constants* K *and* J, *a sequence* $\{t_n\} \uparrow \infty$ *with* $t_n - t_{n-1} \leqq K$ *such that*

(i) $V(t_n, \phi) \leqq W_2(\|\phi\|)$,
(ii) $V'_{(1)}(t, x_t) \leqq -W_3(|x(t)|)$ *if* $t_n - h \leqq t \leqq t_n$, *and*
(iii) $|f(t, \phi)| \leqq J(t+1)\ln(t+2)$ *for* $t \geqq 0$ *and* $\|\phi\| < H$.

Then $x = 0$ *is AS.*

COROLLARY 3. *Suppose there are a Liapunov functional* V *and a wedge* W_2 *with*

(i) $V(t, \phi) \leqq W_2(\|\phi\|)$.

In addition, suppose there are locally integrable functions $M, P\colon [0, \infty) \to [0, \infty)$, *a positive constant* K, *sequences* $\{t_n\} \uparrow \infty$ *and* $\{\lambda_n\}$ *with* $t_n - t_{n-1} \leqq K$, *such that if* $0 < b - a < h$ *and if* $t_n - h \leqq t \leqq t_n$ *with* $b \leqq t_n$, *then for each* $D > 0$ *there is a* $c > 0$ *with*

(ii) $\int_a^b M(s)\,ds \leqq \lambda_n(b-a)$ *and* $\int_t^{t+D/\lambda_n} P(s)\,ds \geqq c$,
(iii) $V'_{(1)}(t, x_t) \leqq -P(t)W_3(|x(t)|)$ *for* $t_n - h \leqq t \leqq t_n$, *and*
(iv) $V'_{(1)}(t, x_t) \leqq -W_4(|x'(t)|) + M(t)$, W_4 *is convex downward.*

Then $x = 0$ *if UAS.*

COROLLARY 4 (Marachkov–Krasovskii). *If there is a Liapunov functional* V, *wedges* W_i, *and a constant* M *such that*

(i) $V(t, \phi) \leqq W_2(\|\phi\|)$,
(ii) $V'_{(1)}(t, x_t) \leqq -W_3(|x(t)|)$,
(iii) $|f(t, \phi)| \leqq M$ *if* $t \geqq 0$ *and* $\|\phi\| < H$,

then $x = 0$ *is UAS.*

We now give an example of Corollary 2.

EXAMPLE. Let $a, b\colon [0,\infty) \to R$ be continuous and suppose there are constants $c_1, c_2, c_3 > 0$ with

(a) $a(t) - \big|b(t+1)\big| =: \alpha(t) \geqq c_1$,

(b) there is a sequence $\{t_n\} \uparrow \infty$ and $K > 0$ with $t_{n+1} - t_n \leqq K$ and $\int_{t_n-1}^{t_n} \big|b(s+1)\big|\,ds \leqq c_2$.

(c) $a(t) + |b(t)| \leqq c_3(t+1)\ln(t+2)$.

Then the zero solution of

$$x'(t) = -a(t)x + b(t)x(t-1) \tag{3}$$

is AS.

PROOF. Define

$$V(t, x_t) = |x(t)| + \int_{t-1}^{t} \big|b(s+1)\big|\,|x(s)|\,ds$$

so that

$$\begin{aligned} V'_{(3)}(t, x_t) \leqq & -a(t)|x| + |b(t)|\,\big|x(t-1)\big| + \big|b(t+1)\big|\,|x| - |b(t)|\,\big|x(t-1)\big| \leqq \\ & \leqq -\big[a(t) - \big|b(t+1)\big|\big]\,|x| \leqq -\alpha(t)|x|. \end{aligned}$$

Take $H = 1$ and $W(r) = r$. Then for $\|\phi\| < H$ we have

$$|\phi(0)| \leqq V(t,\phi), \qquad V(t,0) = 0, \qquad V(t_n,\phi) \leqq |\phi(0)| + c_2\|\phi\|$$

and

$$V'(t, x_t) \leqq -c_1|x(t)|.$$

The conditions of Corollary 2 are satisfied.

Examples of $a(t)$ and $b(t)$ are easily constructed so that this equation is not uniformly stable. Let $u(t) = -[t]\sin 2\pi t$, $w(t) = -[t](\cos 2\pi t - 1)/2\pi$ and $z(t) = |\sin \pi t| - \sin \pi t$, where $[\cdot]$ stands for the greatest integer function. Consider the scalar equation

$$x' = \big(u(t) - 1 - e^2\ln(t+1)\big)\,x(t) + \frac{1}{2}z(t)(\ln t)x(t-1)$$

for $t \geqq 1$. Note that

$$\int_n^{n+1} [t]\sin 2\pi t\,dt = n\int_n^{n+1} \sin 2\pi t\,dt = -\frac{n}{2\pi}\big(\cos 2\pi(n+1) - \cos 2\pi n\big) = 0$$

so that if $n \leqq t < n+1$ then

$$w(t) = \int_0^t -[2] \sin 2\pi s \, ds = \frac{n}{2\pi}(\cos 2\pi t - 1) = \frac{[t]}{2\pi}(\cos 2\pi t - 1).$$

Let

$$V(t) = V(t, x_t) = e^{2w(t)} x^2 + \frac{1}{2} \int_{t-1}^t e^{2w(s+1)} \ln(s+1) z(s+1) x^2(s) \, ds$$

so that

$$V'(t) \leqq$$

$$\leqq \left(-2u(t) + 2u(t) - 2 - 2e^2 \ln(t+1)\right) e^{2w(t)} x^2 + z(t)(\ln t) x(t) x(t-1) e^{2w(t)} +$$

$$+\frac{1}{2} e^{2w(t+1)} \ln(t+1) z(t+1) x^2 - \frac{1}{2} e^{2w(t)} (\ln t) z(t) x^2(t-1) \leqq$$

$$\leqq -\left(2 + 2e^2 \ln(t+1)\right) e^{2w(t)} x^2 + (\ln t) e^{2w(t)} x^2 + \frac{z(t)}{2} (\ln t) e^{2w(t)} x^2(t-1) +$$

$$+ e^{2w(t+1)} \ln(t+1) x^2(t) - \frac{1}{2} e^{2w(t)} (\ln t) z(t) x^2(t-1).$$

Now

$$e^{2w(t+1)} = e^{-2([t]+1)(\cos 2\pi t - 1)/2\pi} \leqq e^2 e^{2w(t)}$$

so $V'(t) \leqq -2x^2(t)$. Also, $V(t) \geqq x^2(t)$. Finally, when n is even

$$V(n) =$$

$$= x^2 + \frac{1}{2} \int_{n-1}^n e^{2w(s+1)} \ln(s+1)\left(|\sin \pi(s+1)| - \sin \pi(s+1)\right) x^2(s) \, ds = x^2.$$

Hence, the conditions of Corollary 2 are satisfied and $x = 0$ is AS.

REMARK. This result will not follow from the work of Busenberg and Cooke [6] because they require that for each $\eta > 0$ there exists $\tau > 0$ such that $\int_t^{t+\eta} a(s) \, ds \leqq \tau$. It will not follow from Burton [2] because that result requires that $V(t, \phi) \leqq W_2\left(|\phi(0)|\right) + W_3\left(|\phi|_2\right)$, where $|\cdot|_2$ is the L^2-norm. It will not follow from Burton–Hatvani [5] for the same reason. It will not follow from Makay [8] because he requires $V(t, \phi) \leqq W\left(\|\phi\|\right)$. It will not follow from Wang [12] because he requires uniform stability.

3. Proof of Theorem 1

We prove (I) first. Since V is a Liapunov functional we have $W_1(|\phi(0)|) \leqq V(t,\phi)$ and $V'_{(1)}(t,x_t) \leqq 0$. The additional assumption that $V(t,\phi) \leqq W(\|\phi\|)$ yields US. For $\varepsilon_1 = H$ find δ_1 of US and take $\gamma = \delta_1$ in the definition of UAS. Let $\mu > 0$ be given and find the δ_2 of US so that $[\|\phi\| < \delta_2,\ t_0 \geqq 0,\ t \geqq t_0]$ imply that $|x(t,t_0,\phi)| < \mu$.

We will find $T > 0$ such that if $\phi \in C_\gamma$ and $t_0 \geqq 0$, then $|x(t,t_0,\phi)| < \mu$ if $t \geqq t_0 + T$. Let $x(t) = x(t,t_0,\phi)$ and $V(t) = V(t, x_t(t_0,\phi))$.

Consider the intervals $S_n = [t_n - k, t_n]$, where we may suppose, by renumbering, that $t_n - k \geqq t_{n-1}$. For a given n, suppose $\|x_{t_n}\| \geqq \delta_2$. Then there is an $r_n \in S_n$ with $|x_i(r_n)| \geqq \delta_2$ for some i. Let $-\alpha_n = V(t_n) - V(t_n - k)$.

(a) If $|x_i(t)| \geqq \delta_2/2$ for $t \in S_n$, then by (v) we have $V'(t) \leqq$ $\leqq -P_i(t)U_i(\delta_2/2)$ on S_n. Let $D = k\lambda$ in (iv), so that

$$-\alpha_n = V(t_n) - V(t_n - k) \leqq -U_i(\delta_2/2)\int_{t_n-k}^{t_n} P_i(s)\,ds \leqq -c_n^{(i)}U_i(\delta_2/2).$$

(b) If (a) fails, then there are $p_n < q_n$ with $[p_n, q_n] \subset S_n$ and with $|x_i(t)|$ between $\delta_2/2$ and δ_2 on $[p_n; q_n]$; to be definite, say $|x_i(p_n)| = \delta_2/2$ and $|x_i(q_n)| = \delta_2$. To simplify arithmetic in Jensen's inequality, let $k \leqq 1$. Then we integrate (vi), use Jensen's inequality, and have

$$-\alpha_n \leqq V(q_n) - V(p_n) \leqq -Q_i\left(\int_{p_n}^{q_n} |x_i'(s)|\,ds\right) + \int_{p_n}^{q_n} M_i(s)\,ds \leqq$$
$$\leqq -Q_i(\delta_2/2) + (q_n - p_n)\lambda_n^{(i)}.$$

(bi) If $\alpha_n \geqq Q_i(\delta_2/2)/2$, this will suffice for our proof.

(bii) If $\alpha_n < Q_i(\delta_2/2)/2$, then $D := Q_i(\delta_2/2)/2 \leqq (q_n - p_n)\lambda_n^{(i)}$. We then integrate (v) and have

$$-\alpha_n \leqq V(q_n) - V(p_n) \leqq -U_i(\delta_2/2)\int_{p_n}^{q_n} P_i(s)\,ds \leqq$$
$$\leqq -U_i(\delta_2/2)\int_{p_n}^{p_n + D/\lambda_n^{(i)}} P_i(s)\,ds \leqq -c_n^{(i)}U_i(\delta_2/2).$$

From (a), (b), (bi) and (bii) we find

$$\alpha_n \geqq \min_i\left[c_n^{(i)}U_i(\delta_2/2), Q_i(\delta_2/2)/2\right] \geqq \min_i\left[c_0U_i(\delta_2/2), Q_i(\delta_2/2)/2\right] =: \alpha.$$

If $t > t_n$, then

$$0 \leqq V(t) \leqq V(t_0) - n\alpha \leqq W(\delta_1) - n\alpha,$$

a contradiction if $n > W(\delta_1)/\alpha$. Now there is a $K > 0$ with $t_n - t_{n-1} \leqq K$ so we may select $N > W(\delta_1)/\alpha$ and then $T = NK$. This completes the proof of (I).

The other proofs are parallel. We must only change t_n for (II), while in (III) we need to change t_n and $c_n^{(i)}$.

To prove (II) we first note that it is not vacuous. The zero solution is stable so there are solutions remaining in C_H. Suppose that $x(t)$ remains in C_H and $x_i(t) \nrightarrow 0$ as $t \to \infty$. Then there is an $\varepsilon > 0$ and a sequence $\{t_n\} \uparrow \infty$ with $t_{n+1} \geqq t_n + k$ and $|x_i(t_n)| \geqq \varepsilon$. Let $S_n = [t_n - k, t_n]$ and $-\alpha_n = V(t_n) - V(t_n - k)$ where $V(t) = V(t, x_t)$. Using the same proof as in (I) we have

$$\alpha_n \geqq \min\left[c_0 U_i(\varepsilon/2), Q_i(\varepsilon/2)/2\right] =: \alpha.$$

If $t > t_n$, then $0 \leqq V(t) \leqq V(t_0) - n\alpha$, a contradiction for large n. This proves (II).

To prove (III), we note again that it is not vacuous, as in (II), and we consider a solution $x(t)$ remaining in C_H on an interval $[t_0, \infty)$. Suppose that $x(t) \nrightarrow 0$ and note that $V'(t, x_t) \leqq 0$ so that if $t \geqq t_n$ then $W_1(|x(t)|) \leqq$ $\leqq V(t, x_t) \leqq V(t_n, xt_n) \leqq W(\|x_{t_n}\|)$; thus there is an $\varepsilon > 0$ with $\|x_{t_n}\| \geqq \varepsilon$ and so there is an i for each n with $|x_i(r_n)| \geqq \varepsilon$, where $r_n \in [t_n - h, t_n]$. Let $S_n = [t_n - k, t_n]$. Once again the same proof gives

$$(*) \qquad \alpha_n \geqq \min_i\left[c_n^{(i)} U_i(\varepsilon/2), Q_i(\varepsilon/2)/2\right] \geqq \min_i\left[c_n U_i(\varepsilon/2), Q_i(\varepsilon/2)/2\right].$$

Since $t > t_n$ yields

$$(**) \qquad 0 \leqq V(t, x_t) \leqq V(t_1, x_{t_1}) - \sum_{i=2}^{n} \alpha_i \leqq W(\|x_{t_1}\|) - \sum_{i=2}^{n} \alpha_i,$$

the second choice in $(*)$ can hold only for finitely many n. Since $\sum_{n=0}^{\infty} c_n =$ $= \infty$, a contradiction results in $(**)$ for large n. This completes the proof.

4. Proofs of the corollaries

First, note that Corollary 1 is just a statement of Theorem 1 (III) without a separate statement for each component. Also, $\lambda_n = \lambda(t_n)$ will suffice, since

$P(t) = 1$ and so

$$\int_{s_n}^{s_n+D/\lambda_n} 1\,dt = \int_{s_n}^{s_n+D/\lambda(t_n)} dt = \frac{D}{\lambda(t_n)} =: c_n$$

and $\sum c_n$ diverges since $\int_1^\infty \frac{dt}{\lambda(t)}$ diverges and λ is increasing.

Corollary 2 follows from Corollary 1 when we note that (iv) of Corollary 1 is satisfied, because for $\|\phi\| < 1$ we have

$$V'(t,x_t) \leqq -W_2\big(|x(t)|\big) \leqq -\big|f(t,x_t)\big| + J(t+1)\ln(t+2)$$

and $M(t) = J(t+1)\ln(t+2)$ satisfies condition (iv) of Corollary 1.

Corollary 3 plays the role for Theorem 1 (I) that Corollary 1 plays for Theorem 1 (III). It merely avoids the component conditions.

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COMPLEMENTARY RADICALS REVISITED

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1. Introduction

Initially the idea with the notion of a radical was solely to have it fill the bad part in an algebraic structure, which was to be factored out for the purpose of obtaining structure theorems on the quotient (cf. [10]). This idea was supplemented by Andrunakievich in 1958 in [3] where he highlighted the fact that "bad" and "good" radicals go hand in hand to provide a fertile source of algebraic knowledge.

In this expanded radical scene Andrunakievich has given the concept of *complementary radicals* a fair amount of prominence, with good effect. His main result in this respect is his Theorem 10 in which he exhibited (in the case of *associative* rings) the existence of two mutually complementary dual Kurosh–Amitsur radicals associated with each class of subdirectly irreducible rings with idempotent hearts of some definite kind; and also emphasized that all dual supernilpotent radicals and all dual subidempotent radicals are obtained in this manner.

The present paper deals with the problem of complementary radicals in more general settings. We study ideal-mappings more general than Kurosh–Amitsur radicals, these mappings moreover being taken on universal classes of *not necessarily* associative rings (Section 2); and then investigate complementarity in *Andrunakievich s-varities* (Section 3); and for *abstract affine near-rings* (Section 4).

Complementary radicals have been revisited recently also by Gardner [7] from a similar point of view; he too tends for generality; there is, however, no overlapping in the results.

We shall use the standard terminology and basic facts of radical theory (see [9] and [11]).

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2. Results on generalizations of Kurosh–Amitsur radicals

Let $\mathcal{V}$ be a universal class of not necessarily associative rings, or algebras over a commutative ring with unity. For brevity these objects will be referred to as *rings*. A mapping ρ which assigns to every $A \in \mathcal{V}$ an ideal ρA of A is called an *ideal-mapping* on $\mathcal{V}$. An ideal-mapping ρ on $\mathcal{V}$ is said to be

hereditary if $I \cap \rho A \subseteqq \rho I$ for all $I \triangleleft A \in \mathcal{V}$;

idempotent if $\rho\rho A = \rho A$ for all $A \in \mathcal{V}$;

complete if ($I \triangleleft A \in \mathcal{V}$ and $\rho I = I$) implies $I \subseteqq \rho A$;

a *preradical* if $f(\rho A) \subseteqq \rho f(A)$ for every homomorphism $f : A \to f(A)$ with $A \in \mathcal{V}$;

a *Plotkin radical* if it is a complete and idempotent preradical;

a *Kurosh–Amitsur radical* if it is a Plotkin radical with the property $\rho(A/\rho A) = 0$ for all $A \in \mathcal{V}$.

We now prove various results within this framework.

PROPOSITION 2.1. *If ρ is a hereditary ideal-mapping then its semisimple class $\mathbf{S}_\rho := \{A | \rho A = 0\}$ has the inductive property, that is, if $I_1 \subseteqq \cdots \subseteqq I_\alpha \subseteqq \subseteqq \cdots$ is an ascending chain of ideals of a ring A such that $I_\alpha \in \mathbf{S}_\rho$ for each index α, then also $\cup I_\alpha \in \mathbf{S}_\rho$.*

PROOF. Suppose that $I := \cup I_\alpha \notin \mathbf{S}_\rho$, i.e., $\rho I \neq 0$. Then there is an index α such that $I_\alpha \cap \rho I \neq 0$. By $I_\alpha \in \mathbf{S}_\rho$ and the heredity of ρ we get $I_\alpha \cap \rho I \subseteqq \rho I_\alpha = 0$, a contradiction.

Let us recall that an ideal I of a ring A is said to be *essential* in A, if $I \cap K \neq 0$ for every ideal $K \neq 0$ of A. This fact will be denoted by $I \triangleleft \cdot A$. A class $\mathbf{C}$ of rings is said to be *closed under essential extensions*, if $I \triangleleft \cdot A$ *and* $I \in \mathbf{C}$ imply $A \in \mathbf{C}$.

PROPOSITION 2.2. *If ρ is a hereditary ideal-mapping then its semisimple class $\mathbf{S}_\rho$ is closed under essential extensions, that is, if $I \triangleleft \cdot A$ and $I \in \mathbf{S}_\rho$ then also $A \in \mathbf{S}_\rho$.*

PROOF. Let $I \in \mathbf{S}_\rho$ be an essential ideal of a ring $A \neq 0$. Then $I \cap \rho A \subseteqq \subseteqq \rho I = 0$. Since I is essential in A it follows that $\rho A = 0$, i.e., $A \in \mathbf{S}_\rho$.

PROPOSITION 2.3. *If ρ is a hereditary preradical then its semisimple class $\mathbf{S}_\rho$ is closed under extensions.*

PROOF. Let $B, A/B \in \mathbf{S}_\rho$ and consider an ideal C of A which is maximal with respect to $B \cap C = 0$. From $B \cong (B + C)/C$ and the well-known fact that $(B + C)/C \triangleleft \cdot A/C$ we infer (using Proposition 2.2) that $A/C \in \mathbf{S}_\rho$. Since ρ is a preradical, $\mathbf{S}_\rho$ is closed under subdirect sums (cf. [4], Proposition 1.1). Hence A, being a subdirect sum of the rings A/B and A/C, is in $\mathbf{S}_\rho$.

Given two ideal-mappings γ and δ, we say that *δ is greater than γ* and write $\gamma \leqq \delta$, if $\gamma A \subseteqq \delta A$ for all $A \in \mathcal{V}$.

Now let ρ be a preradical and consider all Plotkin radicals δ such that $\rho A \cap \delta A = 0$ for all $A \in \mathcal{V}$. If there exists a largest Plotkin radical $\langle\rho\rangle$ among these δ's, then $\langle\rho\rangle$ is called the *complementary (Plotkin) radical of* ρ. In the case where ρ is hereditary, this complement exists and it is in fact a Kurosh–Amitsur radical. This is

THEOREM 2.4. *If ρ is a hereditary preradical, then the largest homomorphically closed subclass*

$$\mathbf{H}_\rho := \{A \in \mathbf{S}_\rho \mid \text{every homomorphic image of } A \text{ is in } \mathbf{S}_\rho\}$$

in the semisimple class $\mathbf{S}_\rho$, is a Kurosh–Amitsur radical class, and the radical $\langle\rho\rangle$ defined by $\mathbf{H}_\rho$ is the complementary radical of ρ. Moreover, $\mathbf{H}_\rho = \mathbf{R}_{\langle\rho\rangle} := \{A \mid \langle\rho\rangle A = A\}$ is the largest preradical class inside $\mathbf{S}_\rho$.

PROOF. By definition $\mathbf{H}_\rho$ is homomorphically closed. To prove that $\mathbf{H}_\rho$ is a Kurosh–Amitsur radical class we have to show that $\mathbf{H}_\rho$ has the inductive property and is closed under extensions.

Let $I_1 \subseteqq \cdots \subseteqq I_\alpha \subseteqq \cdots$ be an ascending chain of ideals of a ring A such that $I_\alpha \in \mathbf{H}_\rho$ for all indices α and let $I := \cup I_\alpha$. We prove that $I \in \mathbf{H}_\rho$ by showing that $I/K \in \mathbf{S}_\rho$ for all ideals K of I. So let K be any ideal of I and consider the chain

$$(I_1 + K)/K \subseteqq \cdots \subseteqq (I_\alpha + K)/K \subseteqq \cdots.$$

Since each I_α is in $\mathbf{H}_\rho \subseteqq \mathbf{S}_\rho$ and since $\mathbf{H}_\rho$ is homomorphically closed, we have $(I_\alpha + K)/K \cong I_\alpha/(I_\alpha \cap K) \in \mathbf{S}_\rho$ for all α. By Proposition 2.1 $\mathbf{S}_\rho$ has the inductive property, so we get $I/K = \cup (I_\alpha + K)/K \in \mathbf{S}_\rho$.

Next we prove that $\mathbf{H}_\rho$ is closed under extensions. Suppose that I, $A/I \in \mathbf{H}_\rho$ and consider an arbitrary ideal K of A. Since $\mathbf{H}_\rho$ is homomorphically closed, $A/I \in \mathbf{H}_\rho$ implies that $A/(I + K) \in \mathbf{H}_\rho$. Using $I \in \mathbf{H}_\rho$ we get $(I + K)/K \cong I/(I \cap K) \in \mathbf{H}_\rho \subseteqq \mathbf{S}_\rho$. Since also

$$\frac{A/K}{(I + K)/K} \cong A/(I + K) \in \mathbf{H}_\rho \subseteqq \mathbf{S}_\rho$$

it follows by Proposition 2.3 that $A/K \in \mathbf{S}_\rho$.

We now show that $\langle\rho\rangle$ is the complementary radical of ρ. By the heredity of ρ we have that $\langle\rho\rangle A \cap \rho A \subseteqq \rho(\langle\rho\rangle A)$ for any A. Since $\langle\rho\rangle$ is a Kurosh–Amitsur radical and hence idempotent, it follows that $\langle\rho\rangle A \in \mathbf{R}_{\langle\rho\rangle} = \mathbf{H}_\rho \subseteqq \mathbf{S}_\rho$, i.e., $\rho(\langle\rho\rangle A) = 0$. Thus we have that $\langle\rho\rangle A \cap \rho A = 0$ for all A. And we now claim that $\delta A \subseteqq \langle\rho\rangle A$ for all A whenever δ is a Plotkin radical such that $\rho A \cap \delta A = 0$ for all A. For let $B \in \mathbf{R}_\delta$. Then $B = \delta B$, and so we have that $\rho B = \rho B \cap B = \rho B \cap \delta B = 0$, so that $B \in \mathbf{S}_\rho$. Thus we have that the homomorphically closed class $\mathbf{R}_\delta$ is contained in $\mathbf{S}_\rho$; and consequently $\mathbf{R}_\delta \subseteqq \mathbf{H}_\rho = \mathbf{R}_{\langle\rho\rangle}$. Now for any ring A, $\delta A = \sum(I \triangleleft A \mid I \in \mathbf{R}_\delta)$; and from $\mathbf{R}_\delta \subseteqq \mathbf{R}_{\langle\rho\rangle}$ we get $\delta A \subseteqq \sum(I \triangleleft A \mid I \in \mathbf{R}_{\langle\rho\rangle}) = \langle\rho\rangle A$. Thus $\langle\rho\rangle$ is the complementary radical of ρ.

The final assertion in the theorem is obvious in view of the fact that the radical class of any preradical is homomorphically closed (cf.[4], Proposition 1.1).

From [4] Proposition 4.1 we know that a hereditary preradical is idempotent. In proving our next theorem we shall need also that the hereditary preradical be complete. We shall therefore consider a hereditary *Plotkin* radical. We note that a Plotkin radical ρ is hereditary if an only if its radical class $\mathbf{R}_\rho$ is a hereditary class.

We shall denote by $H(A)$ the *heart* of a subdirectly irreducible ring A, i.e., $H(A) = \cap\, (I \triangleleft A \mid I \neq 0)$.

THEOREM 2.5. *Let ρ be a hereditary Plotkin radical and let*

$$s(\rho) := \{\textit{all subdirectly irreducible rings } A \mid H(A) \in \mathbf{R}_\rho\}.$$

The complementary radical class $\mathbf{R}_{\langle\rho\rangle}$ coincides with the class $\mathcal{U}s(\rho) :=$ $:= \{A \mid A/I \neq 0 \Rightarrow A/I \notin s(\rho)\}$, and hence $\langle\rho\rangle$ is the upper radical of the class $s(\rho)$.

PROOF. For proving $\mathbf{R}_{\langle\rho\rangle} \subseteqq \mathcal{U}s(\rho)$ let us consider an arbitrary $A \in \mathbf{R}_{\langle\rho\rangle}$ and any subdirectly irreducible homomorphic image B of A. Since $A \in \mathbf{R}_{\langle\rho\rangle}$ we have by Theorem 2.4 that $B \in \mathbf{S}_\rho$. Hence $H(B) \notin \mathbf{R}_\rho$, otherwise we would have $0 \neq H(B) = \rho H(B) \subseteqq \rho B$ by the completeness of ρ. Thus it follows that $A \in \mathcal{U}s(\rho)$; and $\mathbf{R}_{\langle\rho\rangle} \subseteqq \mathcal{U}s(\rho)$.

Conversely, suppose that $A \notin \mathbf{R}_{\langle\rho\rangle}$. Then by Theorem 2.4 A has a nonzero homomorphic image B which is not in $\mathbf{S}_\rho$. Let $\{B_\alpha = B/K_\alpha \mid \alpha \in$ $\in \Lambda\}$ be the set of all subdirectly irreducible homomorphic images of B and consider the subdirect sum representation $B = \sum_{\text{s.d.}}(B_\alpha \mid \alpha \in \Lambda)$. Since $\mathbf{S}_\rho$ is closed under subdirect sums and $B \notin \mathbf{S}_\rho$, at least one subdirectly irreducible component B_α is not in $\mathbf{S}_\rho$, i.e., $\rho B_\alpha \neq 0$. Hence by the heredity of ρ we have $H(B_\alpha) = H(B_\alpha) \cap \rho B_\alpha \subseteqq \rho H(B_\alpha)$, showing that $H(B_\alpha) \in \mathbf{R}_\rho$. Since B_α is a homomorphic image also of A, we conclude that $A \notin \mathcal{U}s(\rho)$; and $\mathcal{U}s(\rho) \subseteqq \mathbf{R}_\rho$.

3. Complementary radicals in Andrunakievich s-varieties

In this section the universal class $\mathcal{V}$ under consideration is an Andrunakievich s-variety. We briefly recall the definition: For $A \in \mathcal{V}$ we define inductively:

$$A^{(0)} := A;\; A^{(n)} := A^{(n-1)} \cdot A^{(n-1)} \quad \text{for integers } n > 0$$

and

$$A^1 := A;\ A^s := \sum_{i=1}^{s-1} A^i \cdot A^{i-1} \quad \text{for integers } s > 1.$$

If $A^{(n)} = 0$ for some n ($A^s = 0$ for some s) then A is said to be *solvable* (*nilpotent*). (Nilpotency implies solvability, but not conversely. In associative rings these two concepts coincide.) The universal class $\mathcal{V}$ is called an *Andrunakievich s-variety* if the following two conditions hold:

(A) If $C \triangleleft B \triangleleft A \in \mathcal{V}$ then $\bar{C}/C$ is solvable, where $\bar{C}$ denotes the ideal of A generated by its subring C.

(s) There exists an integer $s > 1$ such that whenever $B \triangleleft A \in \mathcal{V}$ then $B^s \triangleleft A$. Examples of Andrunakievich s-varieties are given in [1].

We shall study the behaviour of the complementary radicals of Plotkin radicals in $\mathcal{V}$, and prove an Andrunakievich s-variety version of Andrunakievich's fundamental Theorem 10 in [3]. We shall need the following statement which is well-known for associative rings:

PROPOSITION 3.1. *The heart of a subdirectly irreducible ring in $\mathcal{V}$ is either simple and idempotent, or nilpotent. Every ideal I of a subdirectly irreducible ring A with idempotent heart $H(A)$ is itself subdirectly irreducible with heart $H(I) = H(A)$.*

PROOF. The first statement is Proposition 4 in [6].

Let $0 \neq K \triangleleft I \triangleleft A$ and $\bar{K}$ the ideal of A generated by K. $H(A) \nsubseteq K$ implies that $0 \neq \bar{K}/K$ is solvable by condition (A). Since $\bar{K}/K$ contains the idempotent ring $\big(H(A) + K\big)/K \cong H(A)/\big(H(A) \cap K\big)$, the latter must be 0 and $H(A) \subseteq K$, a contradiction.

A Plotkin radical ρ is said to be *supersolvable* if its radical class $\mathbf{R}_\rho$ is hereditary and contains all solvable rings. The supersolvability of ρ has a favorable influence on its complementary radical. This is

THEOREM 3.2. *Let ρ be a supersolvable Plotkin radical. Then the radical class $\mathbf{R}_{\langle\rho\rangle}$ of its complementary radical $\langle\rho\rangle$ is hereditary.*

PROOF. We first show that $K \triangleleft I \triangleleft A \in \mathbf{R}_{\langle\rho\rangle}$ implies that $K \triangleleft A$. Since $A \in \mathbf{R}_{\langle\rho\rangle}$, $A/\bar{K}^s \in \mathbf{S}_\rho$, $\bar{K}$ being the ideal of A generated by K. Moreover, $\bar{K}/\bar{K}^s$ is nilpotent and hence solvable, so $\bar{K}/\bar{K}^s \in \mathbf{R}_\rho$; and consequently $\bar{K}/\bar{K}^s \subseteq \rho(A/\bar{K}^s) = 0$ as $A/\bar{K}^s \in \mathbf{S}_\rho$. Hence $\bar{K} = \bar{K}^s$, showing that $\bar{K}$ is idempotent. Consequently, since $\bar{K}/K$ is solvable, $\bar{K}/K = 0$ so that $K = \bar{K} \triangleleft A$.

We now verify the heredity of the class $\mathbf{R}_{\langle\rho\rangle}$. Let I/K be a subdirectly irreducible homomorphic image of an ideal I of a ring A in $\mathbf{R}_{\langle\rho\rangle}$. Now $K \triangleleft A$, and by Zorn's lemma we may choose an ideal L of A such that $I \cap L = K$ and L is maximal with respect to this property. We now have that $I/K \cong (I + L)/L \triangleleft A/L$. Let H/L be the heart of the subdirectly irreducible ring $(I+L)/L$. Using the fact that $A \in \mathbf{R}_{\langle\rho\rangle}$ we get from $H/L \triangleleft (I+L)/L \triangleleft \triangleleft A/L \in \mathbf{R}_{\langle\rho\rangle}$ that $H/L \triangleleft A/L \in \mathbf{R}_{\langle\rho\rangle} \subseteqq \mathbf{S}_\rho$. This implies that $H/L \notin \mathbf{R}_\rho$, and so H/L is not solvable. Thus, by Proposition 3.1, H/L is a simple idempotent ring and so, again by $H/L \notin \mathbf{R}_\rho$, it follows that $\rho(H/L) = 0$, i.e., $H/L \in \mathbf{S}_\rho$. From the isomorphism $I/K \cong (I+L)/L$ we get that the heart of I/K is isomorphic to H/L, and so it is in $\mathbf{S}_\rho$. Since $\mathbf{S}_\rho$ is closed under essential extensions (Proposition 2.2) we conclude that $I/K \in \mathbf{S}_\rho$. Since I/K was an arbitrary subdirectly irreducible homomorphic image of I, we have that $I \in \mathcal{U}s(\rho) = \mathbf{R}_{\langle\rho\rangle}$ by Theorem 2.5. Thus $\mathbf{R}_{\langle\rho\rangle}$ is a hereditary class.

Following Kurosh–Amitsur radical theory we may call a Plotkin radical γ *subidempotent* if $\mathbf{R}_\gamma$ is hereditary and consists of idempotent rings. The radical class $\mathbf{R}_{\langle\rho\rangle}$ in Theorem 3.2 does consist of idempotent rings: let $A \in \mathbf{R}_{\langle\rho\rangle}$. Since A/A^2 is a solvable ring, $A/A^2 \in \mathbf{R}_\rho$. Since $\mathbf{R}_{\langle\rho\rangle}$ is homomorphically closed also $A/A^2 \in \mathbf{R}_{\langle\rho\rangle}$. Hence $A/A^2 \in \mathbf{R}_\rho \cap \mathbf{S}_\rho = 0$, showing that $A^2 = A$. In view of this and Theorem 2.4 we may now reformulate Theorem 3.2 as:

COROLLARY 3.3. *If ρ is a supersolvable Plotkin radical in $\mathcal{V}$, then its complementary radical $\langle\rho\rangle$ is a subidempotent Kurosh–Amitsur radical.*

Since $\langle\rho\rangle$ is a hereditary Kurosh–Amitsur radical, by Theorem 2.4 its complementary, radical $\langle\langle\rho\rangle\rangle$ exists, and by Theorem 2.5 we have $\mathbf{R}_{\langle\langle\rho\rangle\rangle} = \mathcal{U}s(\langle\rho\rangle)$ where $s(\langle\rho\rangle)$ is the class

$$s(\langle\rho\rangle) = \{\text{all subdirectly irreducible rings with heart in } \mathbf{R}_{\langle\rho\rangle}\}.$$

Clearly if ρ is supersolvable, then the heart of any subdirectly irreducible ring in $s(\langle\rho\rangle)$ is simple and idempotent in view of Proposition 3.1. Hence the class $s(\langle\rho\rangle)$ is hereditary and also closed under essential extensions.

Let us consider $C \triangleleft B \triangleleft A$ and the ideal $\bar{C}$ of A generated by C. Assume that $B/C \in s(\langle\rho\rangle)$. Being in an Andrunakievich variety, $\bar{C}/C$ is a solvable ideal of B/C. Since $B/C \in s(\langle\rho\rangle)$, the heart $H(B/C)$ is idempotent, and therefore $\bar{C}/C$ has to be 0, that is $C = \bar{C} \triangleleft A$. Thus the class $s(\langle\rho\rangle)$ satisfies also condition

(F) *if $C \triangleleft B \triangleleft A$ and $B/C \in s(\langle\rho\rangle)$, then $C \triangleleft A$.*

Now [2] Theorem 1 and its Corollary 1 is applicable to the class $s(\langle\rho\rangle)$ yielding the following

THEOREM 3.4. *Let ρ be a supersolvable Plotkin radical with complementary radical $\langle\rho\rangle$, $\mathbf{R}_{\langle\rho\rangle} = \mathcal{U}s(\rho)$. The complementary radical $\langle\langle\rho\rangle\rangle$ of $\langle\rho\rangle$ is the upper radical of the class*

$$s(\langle\rho\rangle) = \{\textit{all subdirectly irreducible rings with heart in } \mathbf{R}_{\langle\rho\rangle}\},$$

that is, $\mathbf{R}_{\langle\langle\rho\rangle\rangle} = \mathcal{U}s(\langle\rho\rangle)$. *Moreover, the class* $\mathbf{R}_{\langle\langle\rho\rangle\rangle}$ *is hereditary and so is the semisimple class* $\mathbf{S}_{\langle\langle\rho\rangle\rangle}$. *If* A *is any ring of* $\mathbf{S}_{\langle\langle\rho\rangle\rangle}$, *then* A *is a subdirect sum*

$$A = \sum_{\text{s.d.}} (A_\alpha \mid A_\alpha \in s(\langle\rho\rangle))$$

of subdirectly irreducible rings with heart in $\mathbf{R}_{\langle\rho\rangle}$.

In order to prove a counterpart of Theorem 3.2, we need the following

PROPOSITION 3.5 ([6] Lemma 2). *If* $K \triangleleft I \triangleleft A$ *and* I/K *has no nonzero solvable ideals and* M/K *is an idempotent minimal ideal in* I/K, *then there exists an ideal* L *of* A *such that* A/L *is subdirectly irreducible with heart* $H(A/L) = (M+L)/L \cong M/K$.

THEOREM 3.6. *If* ρ *is a subidempotent Plotkin radical, then the radical class* $\mathbf{R}_{\langle\rho\rangle}$ *of its complementary radical* $\langle\rho\rangle$ *is hereditary.*

PROOF. Let us consider an ideal I of a ring $A \in \mathbf{R}_{\langle\rho\rangle}$, and let I/K be any subdirectly irreducible factor ring with heart $H(I/K) = M/K$. Let us suppose that $M/K \in \mathbf{R}_\rho$. Since ρ is subidempotent, M/K has to be a simple idempotent ring in view of Proposition 3.1. Now Proposition 3.5 is applicable, yielding the existence of an ideal L of A such that A/L is subdirectly irreducible with heart $H(A/L) = (M+L)/L \cong M/K \in \mathbf{R}_\rho$. Hence $A/L \notin \mathbf{S}_\rho$ contradicting $A \in \mathbf{R}_{\langle\rho\rangle}$. Thus necessarily $H(I/K) \in \mathbf{S}_\rho$, regardless whether $H(I/K)$ is simple idempotent or solvable. Since by Proposition 2.2 the semisimple class $\mathcal{S}_\rho$ is closed under essential extensions, it follows that $I/K \in \mathbf{S}_\rho$. Thus $I \in \mathcal{U}s(\rho) = \mathbf{R}_{\langle\rho\rangle}$ holds in view of Theorem 2.5, proving the heredity of the class $\mathbf{R}_{\langle\rho\rangle}$.

Theorem 2.4 and Theorem 3.6 give immediately

COROLLARY 3.7. *If* ρ *is a subidempotent Plotkin radical in* $\mathcal{V}$, *then its complementary radical* $\langle\rho\rangle$ *is a supersolvable Kurosh–Amitsur radical.*

A direct consequence of Theorems 2.4 and 2.5 and Corollaries 3.3 and 3.7 is the following

COROLLARY 3.8. *If* ρ *is a supersolvable or subidempotent Plotkin radical, then* $\rho \leqq \langle\langle\rho\rangle\rangle$ *where* $\langle\langle\rho\rangle\rangle$ *is the complementary radical of the complementary radical* $\langle\rho\rangle$ *of* ρ.

As in the case of associative rings, a hereditary radical ρ is called a *dual radical* of $\langle\rho\rangle$, if $\rho = \langle\langle\rho\rangle\rangle$. A dual radical is always Kurosh–Amitsur radical in view of Theorem 2.4, and in general the same can be said on dual radicals in Andrunakievich s-varieties as on those of associative rings ([3], [7], [9]). The next theorem extends [3] Theorem 10 to Andrunakievich s-varieties and adds some new aspects to it.

THEOREM 3.9. *Let* $\mathbf{Q}$ *be any class of simple idempotent rings in* $\mathcal{V}$, *and define classes* $s(\mathbf{Q})$ *and* $t(\mathbf{Q})$ *by*

$$s(\mathbf{Q}) = \{\textit{all subdirectly irreducible rings in } \mathcal{V} \textit{ with heart in } \mathbf{Q}\}$$

and

$$t(\mathbf{Q}) = \{\textit{all subdirectly irreducible rings in } \mathcal{V} \textit{ with heart not in } \mathbf{Q}\}.$$

The upper radical class $\mathcal{U}s(\mathbf{Q})$ *is a supersolvable dual radical, and it is the unique largest universal subclass* $\mathbf{U}$ *of* $\mathcal{V}$ *such that* $\mathbf{U} \cap \mathbf{Q} = 0$. *The upper radical class* $\mathcal{U}t(\mathbf{Q})$ *is a subidempotent dual radical, and it is the largest universal subclass* $\mathbf{V}$ *of* $\mathcal{V}$ *such that the simple rings of* $\mathbf{V}$ *are in* $\mathbf{Q}$. $\mathcal{U}s(\mathbf{Q})$ *and* $\mathcal{U}t(\mathbf{Q})$ *are mutually dual radicals.*

PROOF. The class $s(\mathbf{Q})$ is hereditary by Proposition 3.1, whence $\mathbf{R}_\gamma = \mathcal{U}s(\mathbf{Q})$ is a Kurosh–Amitsur radical class. Next, we prove that γ is hereditary. To this end, let $I \triangleleft A \in \mathbf{R}_\gamma$, and suppose that $I \notin \mathbf{R}_\gamma$. Then I has a homomorphic image I/K in $s(\mathbf{Q})$. Let $\bar{K}$ denote the ideal of A generated by K. In the case $K \neq \bar{K}$ the ring $\bar{K}/K$ is solvable by condition (A). Further, $\bar{K}/K$ contains the heart of I/K which is an idempotent ring. This contradiction proves that only $K = \bar{K}$ is possible, that is, $K \triangleleft A$. Next, put $J = I/K$ and $B = A/K$, and consider an ideal L of B which is maximal with respect to $J \cap L = 0$. It is well-known that $J \cong (J + L)/L$ is an essential ideal in B/L, and so $J \in s(\mathbf{Q})$ implies $B/L \in s(\mathbf{Q})$, contradicting $A \in \mathbf{R}_\gamma$ and $B = A/L \in \mathbf{R}_\gamma$. Thus γ is hereditary. Since $\mathbf{R}_\gamma$ contains clearly all solvable rings, γ is supersolvable.

An application of Theorem 2.5 yields that the complementary radical of γ is $\mathbf{R}_\delta = \mathcal{U}t(\mathbf{Q})$, the latter being a subidempotent Kurosh–Amitsur radical by Corollary 3.3. By the same token, the complementary radical of δ is $\mathbf{R}_\gamma = \mathcal{U}s(\mathbf{Q})$. Hence by Corollary 3.8 γ and δ are mutually dual supersolvable, resp. subidempotent radicals, and consequently both $\mathbf{R}_\gamma = \mathcal{U}s(\mathbf{Q})$ and $\mathbf{R}_\delta = \mathcal{U}t(\mathbf{Q})$ are universal subclasses of $\mathcal{V}$.

We have to prove that $\mathbf{R}_\gamma$ and $\mathbf{R}_\delta$ are the largest universal subclasses with the additional properties demanded in the theorem. Let B be any subdirectly irreducible homomorphic image of a ring $A \in \mathcal{V}$.

Assume that $A \in \mathbf{U}$. Then $B \in \mathbf{U}$ and also the heart $H(B)$ is in $\mathbf{U}$. Hence $H(B) \notin \mathbf{Q}$, and so $B \notin s(\mathbf{Q})$, implying $A \in \mathcal{U}s(\mathbf{Q})$.

Suppose that $A \in \mathbf{V}$. Then also $B \in \mathbf{V}$ and $H(B) \in \mathbf{V}$. Let us assume that $H = H(B)$ is solvable. Then $H^2 \neq H$ and $H^2 \triangleleft H$ holds. Since $\mathbf{V}$ is a universal class and $H \in \mathbf{V}$, we have also $H/H^2 \in \mathbf{V}$ which is a ring with trivial multiplication. So any cyclic subgroup of H/H^2 is an ideal of H/H^2, and it has a simple cyclic homomorphic image $C \in \mathbf{V}$. But the simple rings of $\mathbf{V}$ are in $\mathbf{Q}$ by the assumption, and $\mathbf{Q}$ consists of simple idempotent rings. This contradiction shows that H cannot be solvable. Hence by Proposition

3.1 H is a simple idempotent ring in $\mathbf{V}$, and so by the assumption $H \in \mathbf{Q}$. This proves that $A \in \mathcal{U}t(\mathbf{Q})$.

For instance, Theorem 3.9 yields the following special cases:

i) *the Brown-McCoy radical* (the upper radical of all simple rings with unity) *is the largest universal subclass not containing simple rings with unity*;

ii) *Behrens radical* (the upper radical of all subdirectly irreducible rings having a nonzero idempotent element in the heart) *is the largest universal subclass not containing simple rings with nonzero idempotents*;

iii) *the antisimple radical* (the upper radical of all subdirectly irreducible prime rings) *is the largest universal subclass not containing simple idempotent rings.*

iv) *the class of hereditarily idempotent rings* (the dual radical of the antisimple radical which is the upper radical of all subdirectly irreducible rings having solvable heart) *is the largest universal subclass such that the simple rings are idempotent.*

4. Complementary radicals of abstract affine near-rings

A (right) near-ring N is called an *abstract affine near-ring*, if N is abelian (i.e. the addition is commutative) and the 0-symmetric part N_0 of N coincides with the set of distributive elements of N. As is well-known the constant part N_c is an ideal in the abstract affine near-ring N and every ideal $I = I_0 + I_c$ of N is given by $I_0 \triangleleft N_0$ and $I_c \triangleleft N_c$ such that $I_0 I_c \subseteqq I_c$ and $I_0 N_c \subseteqq I_c$. For details we refer the reader to [8].

In this section we prove that analogous results are valid in the variety of abstract affine near-rings to those of Section 3. Our main objective will be, therefore, to prove corresponding statements to Theorems 3.2 and 3.6. For that purpose we need some specific results on abstract affine near-rings. The following assertion states that abstract affine near-rings form a 2-variety.

PROPOSITION 4.1. *If I and K are ideals of an abstract affine near-ring N, then also IK is an ideal of N. In particular, $I^2 \triangleleft N$ for every $I \triangleleft N$.*

The proof is a straightforward verification.

PROPOSITION 4.2. *If $K \triangleleft I \triangleleft N$ and G is the ideal of N generated by K, then $(G/K)^3$ is a constant near-ring. Hence abstract affine near-rings do not form an Andrunakievich variety.*

PROOF. By $G_0 G_c \subseteqq G_c$ we have

$$G^2 = (G_0 + G_c)G = G_0 G + G_c = G_0^2 + G_0 G_c + G_c = G_0^2 + G_c,$$

and therefore

$$G^3 = (G_0^2 + G_c)G = G_0^2 G + G_c = G_0^3 + G_0^2 G_c + G_c = G_0^3 + G_c.$$

Since $K_0 \triangleleft G_0 \triangleleft N_0$, N_0 is a ring and G_0 is just the ideal of N_0 generated by K_0, the Andrunakievich Lemma is applicable yielding $G_0^3 \subseteqq K_0 \subseteqq K$. Thus we get

$$(G/K)^3 = (G^3 + K)/K = (G_0^3 + G_c + K)/K = (G_c + K)/K \cong G_c/(G_c \cap K),$$

proving the assertion.

PROPOSITION 4.3. *If $K \triangleleft I \triangleleft N$ and I/K is a subdirectly irreducible ring with idempotent heart $H(I/K) = M/K$, then there exists an ideal L of N such that N/L is subdirectly irreducible with heart $H(N/L) = (M + L)/L \cong M/K$.*

PROOF. First, we prove that $K \triangleleft N$. Let us consider the ideal G of N generated by K. Now we have $G/K \triangleleft I/K$. Since I/K is a subdirectly irreducible ring, either $M/K \subseteqq G/K$ or $G/K = 0$. In the first case Proposition 4.2 ensures that $(G/K)^3$ is a constant near-ring. By $G/K \subseteqq I/K$, however, G/K is also a ring, and therefore $(G/K)^3 = 0$. Since M/K is idempotent we get $M/K = (M/K)^3 \subseteqq (G/K)^3 = 0$, a contradiction. Hence $G/K = 0$ is valid, that is, $K = G \triangleleft N$.

Next, we show that $M \triangleleft N$. Since $K \triangleleft N$, we may consider $M_K = M/K$ $I_K = I/K$ and $N_K = N/K$. Now, for the ideal J_K of N_K generated by M_K an application of Proposition 4.2 yields that $(J_K/M_K)^3$ is a constant near-ring, and by $J_K/M_K \subseteqq I_K/M_K$ it is a ring too. Hence we conclude that $(J_K/M_K)^3 = 0$, i.e., $J_K^3 \subseteqq M_K$. Taking into account that M_K is idempotent, we get $M_K = M_K^3 \subseteqq J_K^3$, and so by Proposition 4.1 we have $M_K = J_K^3 \triangleleft N_K$. Thus also M is an ideal of N.

Finally, let us consider an ideal L of N which is maximal relative to $M \cap L = K$. Since $K \triangleleft N$, by Zorn's lemma such an L does exist. $(M + L)/L$ is a minimal ideal of N/L, because of $M/K \cong (M + L)/L$ and of Proposition 3.1. For any nonzero ideal J/L of N/L the maximality of L yields $M \cap J \neq K$, and so by $0 \neq (M \cap J)/K \subseteqq M/K$ the simplicity of M/K implies $M \cap J = M$, i.e., $M \subseteqq J$. Hence $(M + L)/L$ is the unique minimal ideal of N/L, proving that N/L is subdirectly irreducible with heart $(M + L)/L$.

A Plotkin radical ρ of abstract affine near-rings will be said to be *supernilpotent and superconstant,* if its radical class $\mathbf{R}_\rho$ is hereditary and contains all nilpotent rings as well as all constant near-rings. By definition the semisimple class $\mathbf{S}_\rho$ of a supernilpotent and superconstant Plotkin radical ρ consists of rings. Let us mention that the semisimple class $\mathbf{S}_\rho$ of a Kurosh–Amitsur radical ρ of abstract affine near-rings is hereditary if and only if the radical class $\mathbf{R}_\rho$ contains all constant near-rings ([5] Theorem 3.4). Considering, however, a supernilpotent and superconstant Plotkin radical, we do not know whether its semisimple class is hereditary.

THEOREM 4.4. *Let ρ be a supernilpotent and superconstant Plotkin radical of abstract affine near-rings. Then the radical class $\mathbf{R}_{\langle\rho\rangle}$ of its complementary radical $\langle\rho\rangle$ is hereditary.*

PROOF. Notice that the results of Section 2 are valid also for abstract affine near-rings, in particular by Theorem 2.4 the complementary radical $\langle\rho\rangle$ exists.

Since the semisimple class $\mathbf{S}_\rho$ and so also the radical class $\mathbf{R}_{\langle\rho\rangle}$ consists of rings, the same proof as that of Theorem 3.2 yields the assertion.

A Plotkin radical ρ of abstract affine near-rings is said to be *subidempotent,* if its radical class $\mathbf{R}_\rho$ is hereditary and consists of idempotent rings. By the above quoted [5] Theorem 3.4 we know that the corresponding semisimple class $\mathbf{S}_\rho$ is in general not hereditary. Thus proving that the complementary radical class of a subidempotent Plotkin radical is hereditary, needs more effort inasmuch as Proposition 4.3 is used.

THEOREM 4.5. *If ρ is a subidempotent Plotkin radical, then the radical class $\mathbf{R}_{\langle\rho\rangle}$ of its complementary radical ρ is hereditary.*

PROOF. The proof of Theorem 3.6 can be followed and then one uses Proposition 4.3 instead of Proposition 3.5.

In the possession of the key statements of Theorems 4.4 and 4.5, which correspond to those of Theorems 3.2 and 3.6, the theory of complementary and dual radicals of abstract affine near-rings can be developed in the same way as in Section 3.

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ON THE QUOTIENTS OF COUNTABLE DIRECT PRODUCTS OF MODULES MODULO DIRECT SUMS

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§1. **Introduction.** A remarkable result in abelian group theory is Hulanicki's theorem (see [6] or [2, p.176]) stating that, for any countable family $A_1, \dots, A_n, \dots$ of abelian groups, the quotient $\prod A_n / \oplus A_n$ of the direct product modulo the direct sum is an algebraically compact group; thus it is a direct sum of a divisible group and a $\mathbf{Z}$-complete group. Though most theorems on abelian groups easily extend to modules over P.I.D.'s, and moreover, over Dedekind domains, this result fails for P.I.D.'s unless the number of primes is countable.

In view of the most recent developments on R-completions over domains R whose field Q of quotients satisfies p.d. $Q = 1$ (see [3]), it seems reasonable to have a new look at the quotients $M^* = \prod M_n / \oplus M_n$ for countable sets $\{M_1, \dots, M_n, \dots\}$ of R-modules. We start from the fundamental observation due to Mycielski [9] which states that M^* is *algebraically* (or *equationally*) $\aleph_1$-*compact* in the sense that any countable system of equations with unknowns x_j,

$$\sum_j r_{ij} x_j = a_i \in M^* \qquad (r_{ij} \in R; i, j = 1, 2, \dots, n, \dots)$$

(for a fixed i, almost all $r_{ij} = 0$) has a solution in M^* provided that every finite subsystem is solvable in M^*. We will prove that the first Ulm submodule M^{*1} of M^* is a divisible module, and though the 0-th Ulm factor M^*/M^{*1} is in general not R-complete, it is always *torsion-complete*, i.e. it is Hausdorff in the R-topology and its RD-extensions by torsion divisible R-modules split. (Recall that a submodule N of M is an *RD-submodule* if $rN = N \cap rM$ for all $r \in R$; see [4, p.39]. Those extensions of M by a module A in which M is an *RD-submodule* form a subgroup $\mathrm{RDext}^1(A, M)$ of $\mathrm{Ext}^1(A, M)$.) As torsion-free modules are always torsion-complete, the result is more meaningful for torsion modules M_n.

Our main result (Theorem 6) shows that for torsion modules M_n, the structure of M^*/M^{*1} can be given more explicitly with the aid of R-complete modules.

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§2. **Preliminaries.** We assume that R is a commutative domain with 1, and $R \neq Q$ for the field Q of quotients of R. Let S be a subsemigroup of $R\backslash 0$. An R-module M is said to be *S-divisible* if $sM = M$ for all $s \in S$. If S is countable, then the localization R_S of R at S satisfies for S-divisible M

$$\operatorname{Ext}^1(R_S, M) = 0 \quad \text{and} \quad \operatorname{Ext}^1(R_S/R, M) = 0.$$

In fact, by the S-divisibility of $M, R_S, R_S/R$ and the countability of S it is easy to show that every extension of M by R_S and R_S/R is splitting.

If the projective dimension p.d. $Q = 1$, then $K = Q/R$ decomposes into the direct sum of countably generated divisible modules, $K = \oplus A_i/R$; see Lee [7]. By Matlis [8, p.401], each A_i is a flat overring of R. Hamsher [5] shows that A_i is contained in a localization R_{S_i} of R at a suitable countable subsemigroup S_i of $R\backslash 0$ such that R_{S_i}/R is a summand of K.

The *A_i-component* M_{A_i} of a reduced torsion R-module M (reduced means that $\operatorname{Hom}_R(Q, M) = 0$) is defined via

$$M_{A_i} = \operatorname{Tor}_1^R(A_i/R, M) = B_i \otimes_R M = \operatorname{Hom}_R(B_i, M)$$

where $B_i = \sum_{j \neq i} A_j$. The A_i-component M_{A_i} is *A_i-torsion* ($A_i \otimes_R M_{A_i} = 0$), *$A_i$-reduced* ($\operatorname{Hom}_R(A_i, M_{A_i}) = 0$), *$B_i$-torsion-free* ($\operatorname{Tor}_1^R(B_i/R, M_{A_i}) = 0$) and *$B_i$-divisible* ($\operatorname{Ext}_R^1(B_i/R, M_{A_i}) = 0$). Furthermore, we have

$$M = \bigoplus_i M_{A_i}. \tag{1}$$

For a subsemigroup S of $R\backslash 0$, we set $M^S = \bigcap_{s \in S} sM$. The *first Ulm submodule* of M is defined as $M^1 = \bigcap_{0 \neq r \in R} rM$, and M/M^1 is called the *0-th Ulm factor* of M. A module M is *S-complete* (*R-complete*) if $M^S = 0$ ($M^1 = 0$) and it is complete in the S-topology (R-topology) where $\{sM\}_{s \in S}$ ($\{rM\}_{0 \neq r \in R}$) is a subbase of neighborhoods of 0. In [3] it is shown that — under the hypothesis p.d. $Q = 1$ — an R-module M with $M^1 = 0$ is R-complete exactly if $\operatorname{Ext}_R^1(Q, M) = 0$, in which case $M \cong \operatorname{Ext}_R^1(K, M)$, while Pretorius–Schoeman [10] show that M with $M^1 = 0$ is torsion-complete if and only if $\operatorname{RDext}_R^1(K, M) = 0$ (for the definition of the group RDext^1 of RD-extensions, see e.g. [4, p.59]).

§3. **Algebraically $\aleph_1$-compact modules.** We wish to establish a couple of properties of algebraically $\aleph_1$-compact modules, in particular, in case p.d. $Q = 1$.

LEMMA 1. *Let R_S be the localization of the domain R at a countable semigroup S. For an algebraically $\aleph_1$-compact R-module M, we have*

(i) M^S *is* S*-divisible and algebraically* $\aleph_1$*-compact;*

(ii) M/M^S *is* S*-complete.*

If M *is reduced, then moreover*

(iii) $\mathrm{Hom}_R(R_S, M) \cong M^S$;

(iv) $\mathrm{Ext}^1_R(R_S, M) = 0$ *and* $\mathrm{Ext}^1_R(R_S/R, M) \cong M/M^S$.

PROOF. For (i), we first show that M^S is contained in the union $d_S M$ of all S-divisible submodules of M. Setting $S = \{s_1, \ldots, s_n, \ldots\}$, observe that $a \in M^S$ if and only if the finite subsystems of the countable system of equations

$$s_1 x_1 = a, \quad s_n x_n = x_{n-1} \qquad (n = 2, 3, \ldots) \tag{2}$$

with unknowns $x_1, \ldots, x_n, \ldots$ are solvable in M. The algebraic $\aleph_1$-compactness of M ensures the existence of a global solution of (2), say $x_n = b_n \in M$. Thus the submodule $N = \langle a, b_1, \ldots, b_n, \ldots \rangle$ of M is an epic image of R_S under the map $1 \mapsto a, s_1^{-1} \ldots s_n^{-1} \mapsto b_n (n \geqq 1)$, so $a \in d_S M$.

Let $\sum_j r_{ij} x_j = a_i \in M^S$ $(i, j = 1, 2, \ldots)$ be a countable system of equations which is finitely solvable in M^S. Adding to this system countably many equations

$$s_1 s_2 \cdots s_n x_{jn} - x_j = 0 \qquad (j, n = 1, 2, \ldots),$$

the arising new system will still be finitely solvable. By hypothesis it has a solution $x_j = b_j, x_{jn} = b_{jn}$ in M. Evidently, $b_j \in M^S$ for each j, thus M^S is algebraically $\aleph_1$-compact.

To verify (ii), notice that M/M^S is evidently Hausdorff in the S-topology. Let $\{a_n\} \subset M$ be a Cauchy sequence in the S-topology; without loss of generality, we may assume that for each $n, a_n - a_m \in s_1 \ldots s_n M$ whenever $m \geqq n$. This sequence has a limit in M if and only if the countable system

$$x - s_1 \ldots s_n x_n = a_n \qquad (n = 1, 2, \ldots) \tag{3}$$

with unknowns $x, x_1, \ldots, x_n, \ldots$ is solvable in M. The fact that $\{a_n\}$ is Cauchy ensures the finite solvability of the system (3), thus by algebraic $\aleph_1$-compactness (3) has a global solution in M. Consequently, (ii) holds.

For a reduced M, from the exactness of $0 \to R \to R_S \to R_S/R \to 0$ and from the divisibility of R_S/R we deduce the exact sequence

$$0 \to \mathrm{Hom}(R_S, M^S) \to \mathrm{Hom}(R, M^S) \to \mathrm{Ext}^1(R_S/R, M^S) = 0;$$

thus $\mathrm{Hom}(R_S, M^S) \cong M^S$. But R_S is S-divisible, so $\mathrm{Hom}(R_S, M) = \mathrm{Hom}(R_S, M^S)$.

(i) implies $\mathrm{Ext}^1(R_S, M^S) = 0$ and (ii) implies $\mathrm{Ext}^1(R_S, M/M^S) = 0$. As M is an extension of M^S by $M/M^S, \mathrm{Ext}^1(R_S, M) = 0$ follows.

From $0 \to R \to R_S \to R_S/R \to 0$ we also obtain the exact sequence $0 \to$ $\to \operatorname{Hom}(R_S, M) \cong M^S \to \operatorname{Hom}(R, M) \cong M \to \operatorname{Ext}^1(R_S/R, M) \to$ $\to \operatorname{Ext}^1(R_S, M) = 0$. Hence the stated isomorphism in (iv) follows. □

We can improve on the second part of the preceding lemma.

LEMMA 2. *Let M be a reduced algebraically $\aleph_1$-compact R-module and A/R a summand of R_S/R where S is a countable semigroup. Then setting $M^{(A)} = \operatorname{Hom}_R(A, M)$, we have*

$$\operatorname{Ext}^1_R(A, M) = 0 \quad \textit{and} \quad \operatorname{Ext}^1_R(A/R, M) \cong M/M^{(A)}. \tag{4}$$

PROOF. If $C \leqq R_S$ is such that $R_S/R = A/R \oplus C/R$, then from the exact sequence $0 \to R \to A \oplus C \to R_S \to 0$ we derive, in view of (iv), that $\operatorname{Ext}^1_R(A, M) = 0$. The exact sequence $0 \to R \to A \to A/R \to 0$ yields the exactness of $0 \to M^{(A)} \to M \to \operatorname{Ext}^1(A/R, M) \to \operatorname{Ext}^1(A, M) = 0$ whence the stated isomorphism follows. □

Turning to the global case, we have the following result.

PROPOSITION 3. *Over a domain R with* p.d. $Q = 1$, *an algebraically $\aleph_1$-compact module M satisfies*

(i) *M^1 is divisible, and*

(ii) *M/M^1 is torsion-complete.*

PROOF. Let A/R be a countably generated summand of K, $K = A/R \oplus$ $\oplus B/R$, and S a countable semigroup, $A \leqq R_S$ and R_S/R a summand of K. The exact sequence $0 \to M^S/M^1 \to M/M^1 \to M/M^S \to 0$ induces the exact sequence

$$\operatorname{Ext}^1(R_S/R, M^S/M^1) \to \operatorname{Ext}^1(R_S/R, M/M^1) \to \operatorname{Ext}^1(R_S/R, M/M^S) \to 0.$$

If M is algebraically $\aleph_1$-compact, then by Lemma 1(i), the first Ext vanishes, while by (ii) the third Ext is $\cong M/M^S$. Hence there is a natural isomorphism

$$\operatorname{Ext}^1(R_S/R, M/M^1) \cong M/M^S. \tag{5}$$

As in Lemma 2, we obtain $\operatorname{Ext}^1(A/R, M/M^1) \cong M/M^{(A)}$ where, evidently, $M/M^{(A)}$ is a summand of M/M^S. If we set $K = \bigoplus A_i/R$ with A_i/R countably generated, then we are led to the natural isomorphism

$$\operatorname{Ext}^1(K, M/M^1) \cong \prod_i \operatorname{Ext}^1(A_i/R, M/M^1) \cong \prod_i M/M^{(A_i)}. \tag{6}$$

Manifestly, $\left(M/M^{(A)}\right)^1 \leqq \left(M/M^S\right)^S = 0$. We infer that the first Ulm submodule $\mathrm{RDext}^1(K, M/M^1)$ of $\mathrm{Ext}^1(K, M/M^1)$ (cf. [4, p.105]) vanishes, thus M/M^1 is torsion-complete.

From the exactness of $0 \to M^1 \to M^S \to M^S/M^1 \to 0$ we deduce that the sequence

$$\mathrm{Hom}(R_S/R, M^S/M^1) \to \mathrm{Ext}^1(R_S/R, M^1) \to \mathrm{Ext}^1(R_S/R, M^S) = 0$$

is exact; the last term vanishes because of Lemma 1 (i). Observe that R_S/R is divisible, while M^S/M^1 is reduced (otherwise M^1 would be larger). Hence the Hom vanishes, and $\mathrm{Ext}^1(R_S/R, M^1) = 0$. Therefore, $\mathrm{Ext}^1(K, M^1) = 0$, which amounts to the divisibility of M^1. □

From the proof it is clear that M/M^S and $M/M^{(A)}$ are R-complete. Moreover, from (6) we obtain immediately:

COROLLARY 4. *Under the hypotheses of Proposition* 3, *the R-completion of M/M^1 (and hence that of M) is $\prod M/M^{(A_i)}$.* □

Later on (Corollary 7) we will see that in Proposition 3 (ii), 'torsion-completeness' can not be replaced by 'R-completeness'.

§4. The structure of $\prod M_n/\oplus M_n$. By Mycielski [9], this quotient is algebraically $\aleph_1$-compact, so in case p.d.$Q = 1$ the last Proposition applies. In order to improve on (ii), it is natural to concentrate on torsion modules M_n, since all torsion-free modules are torsion-complete.

LEMMA 5. *Suppose R is a domain with* p.d.$Q = 1$, *$K = A/R \oplus B/R$ where A/R is countably generated. If $\{M_n\}$ is a countable set of A-torsion R-modules, then*

$$M^* = \prod M_n/\oplus M_n$$

satisfies:

(a) *M^{*1} is divisible, and*

(b) *M^*/M^{*1} is R-complete.*

PROOF. By Mycielski [9], M^* is an algebraically $\aleph_1$-compact R-module. Proposition 3 implies (a).

As we have noticed earlier, an A-torsion module is B-divisible. This property is inherited by direct products and quotients, so M^*/M^{*1} is B-divisible, i.e. $\mathrm{Ext}^1(B/R, M^*/M^{*1}) = 0$. If S is a countably generated semigroup with $A \leqq R_S$ and R_S/R a summand of K, then the vanishing of the last Ext implies $\mathrm{Ext}^1(A/R, M^*/M^{*1}) = \mathrm{Ext}^1(R_S/R, M^*/M^{*1})$. This is, in view of (5), naturally isomorphic to M^*/M^{*S}. Hence $\mathrm{Ext}^1(K, M^*/M^{*1}) \cong$ $\cong M^*/M^{*S}$. Here the Ext contains M^*/M^{*1} as a submodule and M^*/M^{*S}

is a summand of M^*/M^{*1}; this can happen only if $M^*/M^{*1} \cong M^*/M^{*S}$. (b) follows at once. □

For a set $\{N_i (i \in I)\}$ of R-modules, let $\prod^{\aleph_1} N_i$ denote the submodule of the product $\prod N_i$ consisting of all vectors with countable support.

We are now ready to prove:

THEOREM 6. *Let R be a domain such that* p.d. $Q = 1$, *and M_n ($n = 1, 2, \ldots$) a countable set of reduced torsion modules. Then $M^* = \prod M_n / \bigoplus M_n$ satisfies:*

(a) *M^{*1} is a divisible R-module; and*

(b) *M^*/M^{*1} is isomorphic to a submodule of elements of countable support in a product of R-complete modules.*

PROOF. On account of Mycielski [9], statement (a) follows at once from Proposition 3(i).

Write $K = \bigoplus A_i/R$ where each A_i/R is countably generated, and let M_{ni} denote the A_i-component of M_n.

Hypothesis implies that $M_n = \oplus M_{ni}$ for each n. We view $\prod_n M_n$ as a submodule of $\prod_n \prod_i M_{ni} = \prod_{(n,i)} M_{ni}$ and write $x \in \prod_n M_n$ in the form $x = (x_1, \ldots, x_n, \ldots)$ with $x_n \in M_n$. Since each x_n can have but finitely many nonzero coordinates $x_{ni} \in M_{ni}$, it is evident that $\prod_n M_n \leqq \prod^{\aleph_1}_{(n,i)} M_{ni}$. Factoring out $\oplus M_n = \oplus M_{ni}$, we obtain the inclusion

$$M^* \leqq \prod_i{}^{\aleph_1} M_i^*.$$

where $M_i^* = \prod_n M_{ni} / \bigoplus M_{ni}$.

For each index i, M_{ni} is a summand of M_n whence it is easy to conclude that M_i^* is a summand of M^*. Therefore M^* is an RD-submodule in $\prod M_i^*$, thus

$$M^*/M^{*1} \leqq \prod_i{}^{\aleph_1} M_i^*/M_i^{*1} \tag{7}$$

where, for each i, M_i^*/M_i^{*1} is a summand of M^*/M^{*1} and is R-complete by Lemma 5.

For a countable subset J of the index set I, we can form the $A_J = \sum_{i \in J} A_i$-components M_{nJ} of M_n and argue that M_J^*/M_J^{*1} is a summand of M^*/M^{*1} where $M_J^* = \prod M_{nJ} / \oplus M_{nJ}$. Because of Lemma 5, M_J^*/M_J^{*1} is R-complete, and therefore by Corollary 4 it must coincide with $\prod_{i \in J} M_i^*/M_i^{*1}$. Consequently, the inclusion in (7) is not proper. □

A comparison of Theorem 6(b) and Corollary 4 leads us to the following

COROLLARY 7. *Under the hypotheses of Theorem* 6, M^*/M^{*1} *is* R-*complete for every choice of torsion* R-*modules* M_n *if and only if* K *is a countably generated* R-*module.* □

It is easy to extend our results to quotients $\prod_{j\in J}^{\aleph_1} M_j / \oplus M_j$ with an arbitrarily large index set J.

If the modules M_n in Theorem 6 are torsion-free, then M^* is likewise torsion-free. In this case, M^{*1} is injective, and from the proof of Proposition 3 we can conclude that M^*/M^{*1} is a subdirect sum of R-complete modules $M/M^{(A_i)}$.

The case $M_n = R$ was considered (for arbitrary domain R) by Dimitrić [1]. Then $M^{*1} = 0$ whenever Q is an uncountably generated R-module.

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CONVERGENCE OF INTERPOLANTS BASED ON THE ROOTS OF FABER POLYNOMIALS

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§1. Introduction

Let D be a Jordan domain in the complex plane $\mathbf{C}$ bounded by Γ, let U be the unit disc $\{w: |w| < 1\}$. We denote by $E^p(D)$ the space of all functions $f(z)$ that are analytic in D and satisfy

$$\sup_{0<r<1} \int_0^{2\pi} \left| f \circ \psi(re^{i\theta})\right|^p \left|\psi'(re^{i\theta})\right| d\theta < \infty$$

where $\psi(w)$ is a conformal map of U onto D.

It is well-known that if Γ is rectifiable, for any $f(z) \in E^p(D)$, $f(z)$ has nontangential boundary values almost everywhere on Γ and $E^p(D)$ can be equipped with the norm

$$\|f\|_p = \left\{ \int_\Gamma \left| f(z)\right|^p |dz| \right\}^{\frac{1}{p}}.$$

Let $z = \Psi(w)$ be the conformal map of $\{w: |w| > 1\}$ onto the complement of $D \cup \Gamma$ such that $\Psi(\infty) = \infty$, $\Psi'(\infty) > 0$, and let $w = \Phi(z)$ be the inverse map of Ψ. When $|z|$ is sufficiently large, Ψ has the Laurent expansion

$$\Psi(z) = dz + d_0 + \frac{d_1}{z} + \dots$$

and

$$\left[\Psi(z)\right]^n = d^n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k + \sum_{k<0} d_{n,k} z^k.$$

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The polynomial

$$F_n(z) = d^n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k$$

is called the n-th Faber polynomial with respect to the domain D. We know that it is an effective tool to construct approximation polynomials by means of Faber expansion. Comparing with the Faber expansion, we can see that interpolation polynomials can be constructed more directly. Early works dealing with interpolation polynomials in the complex plane often assume that the function to be interpolated can extend continuously, even analytically on $\bar{D}$ (see [3], [4], [5]). In 1989, X. C. Shen and L. Zhong [6] constructed a series of interpolation nodes in D under the assumption $\Gamma \in C(2, \alpha)$, and showed that the interpolation polynomials have the same order of convergence as the best approximation polynomials in $E^p(D)$ for $1 < p < \infty$. Recently, L. Y. Zhu [7] obtained similar result under the assumption $\Gamma \in C(1, \alpha)$ by choosing the zeros of Faber polynomials of D as the interpolation nodes. In the above works Γ does not admit corners. Since many typical domains in the complex plane have corners (for example, the rectangle), to study interpolation in such a domain is of interest. In this paper, we shall show that the interpolation polynomials based on the zeros of Faber polynomials converge in $E^p(D)$ for $1 < p < \infty$, under the condition that Γ is piecewise VR smooth.

Before stating the theorem, we introduce some concepts and notations. Let γ be an oriented rectifiable curve. For $z \in \gamma$, $\delta > 0$ we denote by $s_+(z, \delta)$ (respectively $s_-(z, \delta)$) the subarc of γ in the positive (respectively negative) orientation of γ with z the starting point, and arclength from z to each point is not more than δ. We say that the smooth curve γ is of vanishing rotation smoothness (shortly VR), if

$$\lim_{\delta \to 0} \int_{s_-(z,\delta)} \left| d_\zeta \arg(\zeta - z) \right| + \int_{s_+(z,\delta)} \left| d_\zeta \arg(\zeta - z) \right| = 0 \tag{1.1}$$

uniformly for $z \in \gamma$. The VR condition is slightly stronger than smoothness. If the angle of inclination $\theta(s)$ of tangent to γ as a function of the arclength s along γ satisfies the Al'per condition [8], that means

$$\int_0^\delta \frac{\omega(t)}{t} \, dt < +\infty \tag{1.2}$$

where $\omega(t)$ is the modulus of continuity of $\theta(s)$, then γ is VR (see the appendix).

When all the zeros of the n-th Faber polynomial $F_n(z)$ are in D, we denote by $L_n(f, z)$ the $(n-1)$-th interpolation polynomial to $f(z) \in E^p(D)$

based on the zeros of $F_n(z)$, and denote

$$E_n(f)_p = \inf_{\deg P_n \leqq n} \|f - P_n\|_p.$$

The main result of this paper is following

THEOREM. *If* Γ *consists of finitely many VR curves, and none of its exterior angles equals* 0 *or* 2π, *then for sufficiently large* n, *the zeros of* $F_n(z)$ *are in* D. *Furthermore, for any* $f(z) \in E^p(D)$, $1 < p < \infty$, *we have*

$$\| f(z) - L_n(f, z)\|_p \leqq cE_{n-1}(f)_p \tag{1.3}$$

where the constant c *depends only on* D *and* p.

§2. Some preliminaries

In this section we shall always assume that Γ satisfies the condition of the theorem. For $z \in \Gamma$, by [2],

$$F_n(z) = \frac{1}{\pi} \int_\Gamma [\Phi(\zeta)]^n d_\zeta \arg(\zeta - z)$$

where the jump of $\arg(\zeta - z)$ at $\zeta = z$ equals the exterior angle $\alpha_z \pi$. Therefore

$$F_n(z) - [\Phi(z)]^n = \frac{1}{\pi} \int_{\Gamma\setminus\{z\}} [\Phi(\zeta)]^n d_\zeta \arg(\zeta - z) + (\alpha_z - 1)[\Phi(z)]^n. \tag{2.1}$$

Setting

$$\beta = \max_{z\in\Gamma} |\alpha_z - 1| \tag{2.2}$$

in view of the fact that none of the exterior angles is 0 or 2π, then $0 \leqq \beta < 1$.

LEMMA 1. *For an arbitrary* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\int_{s_-(z,\delta)} |d_\zeta \arg(\zeta - z)| + \int_{s_+(z,\delta)} |d_\zeta \arg(\zeta - z)| \leqq \beta\pi + \varepsilon \tag{2.3}$$

for any $z \in \Gamma$. *Furthermore, if* z *is a joint between two VR curves, then*

$$\int_{s_-(z,\delta)} |d_\zeta \arg(\zeta - z)| + \int_{s_+(z,\delta)} |d_\zeta \arg(\zeta - z)| < \varepsilon. \tag{2.4}$$

The proof of this lemma uses elementary calculus and is tedious a little, we leave it at the end of the paper.

For any $\delta > 0$, $\theta \in [0, 2\pi]$, we denote by $I_{\theta,\delta}$ the image of

$$s_-\big(\Psi(e^{i\theta}), \delta\big) \cup s_+\big(\Psi(e^{i\theta}), \delta\big)$$

under the map Φ. Let

$$\nu(t, \theta; \delta) = \begin{cases} \frac{e^{it}\Psi'(e^{it})}{\Psi(e^{it}) - \Psi(e^{i\theta})}, & \text{if} \quad e^{it} \notin I_{\theta,\delta} \\ 0 & \text{if} \quad e^{it} \in I_{\theta,\delta}. \end{cases}$$

LEMMA 2. *For arbitrary $\varepsilon > 0$, $\delta > 0$, there exists an integer $N > 0$ such that for $\theta \in [0, 2\pi]$, there is a trigonometric polynomial $T_\theta(t)$ of t with degree at most N satisfying*

$$\int_0^{2\pi} \big|\nu(t, \theta; \delta) - T_\theta(t)\big|\, dt < \varepsilon. \tag{2.5}$$

PROOF. For $\delta > 0$, $e^{it} \notin I_{\theta,\delta}$, there is a constant c_δ such that

$$\frac{1}{\big|\Psi(e^{it}) - \Psi(e^{i\theta})\big|} \leqq c_\delta.$$

Thus

$$\big|\nu(t, \theta; \delta)\big| \leqq c_\delta \big|\Psi'(e^{it})\big|, \qquad e^{it} \notin I_{\theta,\delta}. \tag{2.6}$$

For an arbitrary sequence $\{\theta_n\} \subset [0, 2\pi]$, there exists a convergent subsequence $\{\theta_{n_k}\}$. Writing the limits θ_0, we have

$$\nu(t, \theta_{n_k}; \delta) \to \nu(t, \theta_0; \delta)$$

for almost all t. By the dominated convergence theorem and (2.6) we have

$$\int_0^{2\pi} \big|\nu(t; \theta_{n_k}; \delta) - \nu(t, \theta_0; \delta)\big| \to 0.$$

This implies that the set of functions $\big\{\nu(t, \theta; \delta) : \theta \in [0, 2\pi]\big\}$ is sequentially compact in L^1. Consequently, given any $\varepsilon > 0$, there exists a finite $\frac{\varepsilon}{2}$-net $\big\{\nu(t, \theta_j; \delta),\ j = 1, 2, \ldots, M\big\}$ such that

$$\min_{1 \leqq j \leqq M} \int_0^{2\pi} \big|\nu(t, \theta; \delta) - \nu(t, \theta_j; \delta)\big|\, dt < \frac{\varepsilon}{2}, \qquad \theta \in [0, 2\pi].$$

Therefore there are a finite number of trigonometric polynomials $\{T^{(j)}(t)\}$ satisfying

$$\int_0^{2\pi} \left|\nu(t;\theta_j;\delta) - T^{(j)}(t)\right| dt < \frac{\varepsilon}{2}, \qquad j = 1,2,\dots,M.$$

Hence

$$\min_{1 \leqq j \leqq M} \int_0^{2\pi} \left|\nu(t;\theta_j;\delta) - T^{(j)}(t)\right| dt < \varepsilon, \qquad \theta \in [0,2\pi].$$

Let N be the largest of the degrees of $\{T^{(j)}(t),\, j = 1,2,\dots,M\}$. This completes the proof of Lemma 2. □

From Lemmas 1 and 2 we have

LEMMA 3. *For an arbitrary $\varepsilon > 0$, there exists an integer N, such that*

$$\left|F_n(z) - \left[\Psi(z)\right]^n\right| < \beta + \varepsilon, \qquad z \in \Gamma \tag{2.7}$$

holds for $n > N$.

PROOF. By Lemma 1, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that (2.2) and (2.3) are valid. For the chosen ε and δ, by Lemma 2, there exists an integer N such that (2.5) is valid. For sake of simplicity, we write

$$s(z) = s_-(z,\delta) \cup s_+(z,\delta) \setminus \{z\}, \qquad z \in \Gamma.$$

Therefore, by (2.1) for $z = \Psi(e^{i\theta})$ we have

$$F_n(z) - \left[\Phi(z)\right]^n = \frac{1}{\pi}\int_{s(z)} \left[\Phi(\zeta)\right]^n d_\zeta \arg(\zeta - z) +$$

$$+\frac{1}{\pi}\int_{\Gamma\setminus s(z)} \left[\Phi(\zeta)\right]^n d_\zeta \arg(\zeta - z) + (\alpha_z - 1)e^{in\theta} =$$

$$= \frac{1}{\pi}\int_{s(z)} \left[\Phi(\zeta)\right]^n d_\zeta \arg(\zeta - z) +$$

$$+\frac{1}{\pi}\int_{e^{it} \notin I_{(\theta,\delta)}} e^{int} d_t \arg\left(\Psi(e^{it}) - \Psi(e^{i\theta})\right) + (\alpha_z - 1)e^{in\theta} =$$

$$= \frac{1}{\pi}\int_{s(z)} \left[\Phi(\zeta)\right]^n d_\zeta \arg(\zeta - z) +$$

$$+\frac{1}{\pi}\int_0^{2\pi} e^{int}\, \mathrm{Im}\left[i\nu(t,\theta;\delta)\right] dt + (\alpha_z - 1)e^{in\theta} =$$

$$= \frac{1}{\pi} \int_{s(z)} \left[\Phi(\zeta) \right]^n d_\zeta \arg(\zeta - z) +$$

$$+ \frac{1}{\pi} \int_0^{2\pi} e^{int} \operatorname{Im} \left[i\nu(t, \theta; \delta) - T_\theta(t) \right] dt + (\alpha_z - 1) e^{in\theta};$$

the last equality is from the fact that e^{int} is orthogonal to $T_\theta(t)$ as $n > N$.

If z is not a joint of two VR curves, then $\alpha_z = 1$, by (2.3) and (2.5) we have (2.6). If $\alpha_z \neq 1$, then z must be a joint of two VR curves. By (2.2), (2.4) and (2.5) we also have (2.7). This completes the proof of Lemma 3. □

For any $g \in L^p(\Gamma)$, $1 < p < \infty$, we define the Cauchy integral operator $\mathcal{H}$ by

$$\mathcal{H}g(z) = \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta, \qquad z \in D.$$

Then $\mathcal{H}: L^p(\Gamma) \to E^p(D)$ is bounded, that means

$$\|\mathcal{H}g\|_p \leqq c_1 \|g\|_p$$

where the constant c_1 only depends on p and D [9].

§3. Proof of the theorem

First of all, we claim that all the zeros of $F_n(z)$ are in D when n is sufficiently large. Setting $\varepsilon = \frac{1+\beta}{2}$ in Lemma 3, for n sufficiently large we have

$$\left| F_n(z) - \left[\Phi(z) \right]^n \right| < \frac{1+\beta}{2}, \qquad z \in \Gamma.$$

Since $F_n(z) - \left[\Phi(z) \right]^n$ is analytic on the exterior of $\bar{D}$, by the maximum principle we have

$$\left| F_n(z) - \left[\Phi(z) \right]^n \right| < \frac{1+\beta}{2}, \qquad z \notin \bar{D}.$$

Therefore

$$|F_n(z)| \geqq |\Phi(z)|^n - \frac{1+\beta}{2} \geqq \frac{1-\beta}{2} > 0, \qquad z \notin \bar{D}.$$

This implies that the zeros of $F_n(z)$ are all in D.

Since the $n-1$-th interpolation polynomial operator $L_n(f, z)$ is linear and reproduces polynomials of degree at most $n-1$, we only need to show that $L_n(f,z)$ is bounded uniformly in $E^p(D)$ as $n \to \infty$. In fact, denoting by $P_{n-1}(z)$ the $n-1$-th best approximation polynomial to $f(z)$ in $E^p(D)$, we have

$$\left\| f(z) - L_n(f,z) \right\|_p = \left\| f(z) - P_{n-1}(z) - L_n(f - P_{n-1}, z) \right\|_p \leqq$$
$$\leqq \left(1 + \|L_n\|\right) \left\| f(z) - P_{n-1}(z) \right\|_p .$$

Noticing that the interpolation nodes of $L_n(f,z)$ are the zeros of $F_n(z)$, for $f(z) \in E^p(D)$, we have

$$f(z) - L_n(f,z) = \frac{F_n(z)}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{F_n(\zeta)(\zeta - z)} = F_n(z) \mathcal{H}\left(\frac{f}{F_n}\right)(z) \quad z \in D.$$

It follows that

$$\left\| f(z) - L_n(f,z) \right\|_p \leqq \max_{z \in \Gamma} |F_n(z)| \left\| \mathcal{H}\left(\frac{f}{F_n}\right) \right\|_p \leqq$$
$$\leqq c_1 \max_{\zeta, z \in \Gamma} \left| \frac{F_n(z)}{F_n(\zeta)} \right| \|f\|_p \leqq c_1 \frac{3+\beta}{1-\beta} \|f\|_p .$$

Hence the operators $L_n(f,z)$ are uniformly bounded in $E^p(D)$. This completes the proof of the theorem.

§4. Appendix

This section includes

i) An example of a smooth curve which is not VR smooth curve.
ii) Showing that a curve is VR smooth if it satisfies (1.2).
iii) Proof of Lemma 1.

i) Let

$$h(t) = \begin{cases} \frac{t^2 \sin(1/t)}{\log(1/t)}, & t \in (0, 1/2], \\ 0, & t \in [-1/2, 0]. \end{cases}$$

Set

$$\gamma = \left\{ t + ih(t) : t \in [-1/2, 1/2] \right\}.$$

Then γ is smooth. Evidently

$$\int_{s_+(0,\delta)} \left| d_\zeta \arg(\zeta) \right| = \int_{s_+(0,\delta)} \left| d \arctan \frac{h(t)}{t} \right| \geqq$$

$$\geqq \int_0^{\frac{\delta}{20}} \left| d \arctan \frac{h(t)}{t} \right| = +\infty.$$

Consequently, γ is not a VR smooth curve.

ii) Suppose γ satisfies (1.2). Let γ have the representation $\zeta = \zeta(s)$, where s is the arclength parameter. For $z = \zeta(s_0)$, $s > s_0$,

$$d_s \arg\big(\zeta(s) - z\big) = d_s \operatorname{Im}\big[\ln(\zeta(s) - z)\big] =$$

$$= \operatorname{Im} \frac{\zeta'(s)}{\zeta(s) - z}\, ds.$$

Noticing $|\zeta'(s)| = 1$, we have

$$\left| \operatorname{Im} \frac{\zeta'(s)}{\zeta(s) - z} \right| = \frac{\big|\sin\big[\arg \zeta'(s) - \arg(\zeta(s) - z)\big]\big|}{\big|\zeta(s) - z\big|} \leqq$$

$$\leqq \frac{\big|\arg \zeta'(s) - \arg(\zeta(s) - z)\big|}{\big|\zeta(s) - z\big|}.$$

Since γ is smooth, there exist a constant c_2 and an $\tilde{s} \in (s_0, s)$ such that

$$\frac{1}{\big|\zeta(s) - z\big|} \leqq \frac{c_2}{s - s_0}$$

and

$$\arg\big(\zeta(s) - z\big) = \arg \zeta'(\tilde{s}).$$

Therefore

$$\big|d_s \arg(\zeta(s) - z)\big| \leqq c_2 \frac{\big|\arg \zeta'(s) - \arg \zeta'(\tilde{s})\big|}{|s - s_0|} \leqq c_2 \frac{\omega\big(|s - s_0|\big)}{|s - s_0|}.$$

Similarly, the above inequality is valid as $s < s_0$. Consequently,

$$\int_{s_-(z,\delta)} \big| d_\zeta \arg(\zeta - z)\big| + \int_{s_+(z,\delta)} \big| d_\zeta \arg(\zeta - z)\big| \leqq 2c_2 \int_0^{\delta} \frac{\omega(t)}{t}\, dt.$$

It follows from (1.2) that the inequality (1.1) is valid uniformly. This means that γ is a VR smooth curve.

iii) Since Γ consists of finite VR smooth curves, we can take $\delta > 0$ so small that $s_-(z,\delta) \cup s_+(z,\delta)$ contains at most one corner. So Lemma 1 is a consequence of the following assertion:

LEMMA 1$'$. *Let γ_1, γ_2 be two VR smooth curves with the same starting point z_0, at which γ_1, γ_2 have the angle $\alpha\pi$ $(0 < \alpha < 2)$. Set $\gamma = \gamma_1^- \cup \gamma_2$, where γ_1^- is the same curve γ_1 with opposite orientation. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$(4.1) \qquad \int_{s_-(z,\delta)} \left| d_\zeta \arg(\zeta - z) \right| + \int_{s_+(z,\delta)} \left| d_\zeta \arg(\zeta - z) \right| \leqq \alpha\pi + \varepsilon, \quad z \in \gamma$$

and

$$(4.2) \qquad \int_{s_-(z_0,\delta)} \left| d_\zeta \arg(\zeta - z_0) \right| + \int_{s_+(z_0,\delta)} \left| d_\zeta \arg(\zeta - z_0) \right| < \varepsilon.$$

PROOF. Noticing that $s_+(z_0,\delta)$ and $s_-(z_0,\delta)$ are $s_+(z_0,\delta)$ on γ_2 and $s_-(z_0,\delta)$ on γ_1^- respectively, (4.2) follows from the definition of VR smoothness.

Next we consider (4.1). Without loss of generality we may assume $z_0 = 0$, and that the tangent of γ_2 at $z_0 = 0$ coincides with the real axis. Therefore γ_2 has the representation in polar coordinates near $z_0 = 0$

$$\theta = \theta(r), \qquad re^{i\theta(r)} \in \gamma_2$$

which satisfies

$$\theta(0) = \lim_{r \to 0} \theta(r) = 0.$$

Since the angle $\alpha\pi$ is not 0 or 2π, γ has the same order of arclength and chord length locally. Thus for any $z \in \gamma_1$, $\zeta = re^{i\theta(r)}\gamma_2$, we have

$$|z| + r \leqq |\,\widehat{zz_0}\,| + |\,\widehat{z_0\zeta}\,| = |\,\widehat{z\zeta}\,| \leqq c_3 \left| z - re^{i\theta(r)} \right|.$$

Then

$$\left| d_\zeta \arg(\zeta - z) \right| \leqq$$

$$\leqq \left| \frac{e^{i\theta(r)}}{re^{i\theta(r)} - z} - \frac{1}{r - z} \right| dr + \left| d_r \arg(r - z) \right| + \frac{r|\theta'(r)|}{\left| re^{i\theta(r)} - z \right|}\, dr.$$

It follows from (4.3), (4.4) and (4.7) that

$$\left| \frac{e^{i\theta(r)}}{re^{i\theta(r)} - z} - \frac{1}{r - z} \right| = \frac{\left| e^{i\theta(r)} - 1 \right| |z|}{\left| re^{i\theta(r)} - z \right| |r - z|} \leqq \frac{|z|\varepsilon}{8(r + |z|)^2}$$

and

$$\frac{r\left|\theta'(r)\right|}{\left|re^{i\theta(r)}-z\right|} \leqq c_3\left|\theta'(r)\right|.$$

Denote by $[0,\eta]$ the set $\left\{r\colon re^{i\theta(r)} \in s_+(0,\delta)\right\}$. Since $s_+(z,\delta)\cap\gamma_2 \subset$ $\subset s_+(0,\delta)$, we have

$$\int_{s_+(z,\delta)\cap\gamma_2}\left|d_\zeta \arg(\zeta-z)\right| \leqq \int_{s_+(0,\delta)}\left|d_\zeta \arg(\zeta-z)\right| \leqq$$

$$\leqq \frac{\varepsilon}{8}\int_0^\eta \frac{|z|}{\left(r+|z|\right)^2} + \int_0^\eta \left|d_r \arg(r-z)\right| + c_3\int_0^\eta \left|\theta'(r)\right|\,dr \leqq$$

$$\leqq \frac{\varepsilon}{8}\int_0^\infty \frac{|z|}{\left(r+|z|\right)^2} + \int_0^\infty \left|d_r \arg(r-z)\right| + c_3\int_0^\eta \left|\theta'(r)\right|\,dr.$$

Noticing that the first integral equals 1 and that $\arg(r-z)$ is a monotonic function of r, it follows from (4.6) that

$$\int_0^\infty \left|d_r \arg(r-z)\right| = \left|\arg(r-z)\right|_{r=0}^{r=\infty} = \left|\arg(-z)\right| \leqq |\alpha-1|\pi + \frac{\varepsilon}{8}.$$

Therefore by (4.8) we have

$$\int_{s_+(z,\delta)\cap\gamma_2}\left|d_\zeta \arg(\zeta-z)\right| \leqq |\alpha-1|\pi + \frac{\varepsilon}{2}.$$

Together with (4.6) we have proved (4.1) for $z\in\gamma_1$. For $z\in\gamma_2$, (4.1) can be proved in the same way.

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NOWHERE DIFFERENTIABLE FUNCTIONS CONSTRUCTED FROM PROBABILISTIC POINT OF VIEW

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§1. Berman's principle

Since Weierstrass gave an example of nowhere differentiable functions, many people have investigated various types of nowhere differentiable functions.

In this paper, we will construct a class of nowhere differentiable functions based on i.i.d. (independent identically distributed) random variables and we shall prove a variety of irregularity of the functions including nowhere differentability by making use of a notion of local times following Berman's idea. The notion of local times originated by P. Lévy (cf. Ito–McKean [11]) plays a very important role especially in the theory of Markov processes, but it is S. M. Berman [1] who first applied local times to investigate sample path properties of a wide class of stochastic processes. The definition of local time itself is a purely real analytic one, namely, a Borel measurable real function f defined on $I = [0,1]$ maps the Lebesgue measure on I to $\Re$ by

$$\mu_f(I,E) = \big|\{t \in I,\ f(t) \in E\}\big|, \quad E \in \mathcal{B},$$

where $|A|$ means the linear Lebesgue measure of a Borel set A and $\mathcal{B}$ is the Borel field of $\Re$. Here we assume that $\mu_f(I,E)$ is absolutely continuous with respect to the Lebesgue measure, i.e.

$$\mu_f(I,E) = \int_E \alpha_f(x)dx.$$

Now we restrict f to $A \in \mathcal{B}(I)$, the Borel field on I. Then the induced measure $\mu_f(A,E)$ is also absolutely continuous with respect to the Lebesgue measure, i.e.

$$\mu_f(A,E) = \int_E \alpha_f(x,A)dx$$

and in general we have

$$0 \leqq \alpha_f(x,A) \leqq \alpha_f(x) \quad \text{a.e.} \quad x \in \Re,$$

$$\alpha_f(x,A)+\alpha_f(x,B)=\alpha_f(x,A\cup B) \quad \text{a.e.} \quad x\in\Re \text{ if } A\cap B=\emptyset.$$

We shall write $\alpha_f(x,t)$ when $A=[0,t]$.

Since $\alpha_f(x,t)$ is a density function, we can choose a nice version satisfying the following conditions:

LEMMA 1 ([3], [9]). (i) *$\alpha_f(x,t)$ is right continuous and non-decreasing in t,*

(ii) *$\alpha_f(x,t)$ is $\mathcal{B}\times\mathcal{B}(I)$-measurable,*

(iii) *almost every x, the support of the measure $\alpha_f(x,dt)$ is carried by $\{t; f(t)=x\}$,*

(iv) *for almost every t, for any $\varepsilon>0$,*

$$\alpha_f\big(f(t),[t,t+\varepsilon)\big)>0 \quad \textit{and} \quad \alpha_f(f(t),[t-\varepsilon,t])>0,$$

(v) *for all $J=$ subinterval with rational end points, $\alpha_f(x,J)=0$ if $x\notin$ the closure of $\{$the range of $f(t)$; $t\in J\}$.*

DEFINITION 1. The above function $\alpha_f(x,t)$ is called local time at x.

Berman [4] first pointed out the relation between the original function f and the local times. In short the irregularity (regularity) of the original function reflects regularity (irregulariy) of the local times. After S. M. Berman we shall call it "Berman's principle". For example if f is a C^1-function such that $\{t\in I;\ f'(t)=0\}$ consists of isolated points, then

$$\alpha_f(x,t)=\sum_{s\leqq t,f(s)=x}\frac{1}{|f'(s)|}.$$

In this case, for fixed x, $\alpha_f(x,t)$ is a step function in t and at the local extremal points, $\alpha_f(x,t)$ is divergent in x. So, it is very difficult to imagine a real continuous function such that the function $\alpha_f(x,t)$ is also continuous in (x,t). The sample functions of a Brownian motion are such a case with probability one. An example of a deterministic continuous function (not a sample function of a stochastic process) was first "discovered" by Kôno [12]. Let $P(t)=\big(x(t),y(t)\big)$ be the famous Peano's surface filling function [15], then $f(t)=x(t)-y(t)$ is an example of a continuous function whose $\alpha_f(x,t)$ is also continuous in (x,t). We will generalize this example to obtain a class of nondifferentiable continuous functions and analyze them through Berman's principle.

§2. Construction of nowhere differentiable functions

It is well known that coordinate functions $\big(x(t), y(t)\big)$ of the famous Peano curve are stochastically independent as random variables on a probability space $\big(I, \mathcal{B}(I), dt\big)$ having uniform distribution and 1/2-Hölder continuous, moreover [14] the sequence defined by

$$y_1(t) = x(t),\ y_2(t) = x\big(y(t)\big), \quad \cdots, \quad y_n(t) = x\big(y^{(n-1)}(t)\big),$$

is i.i.d. having uniform distribution. From this fact we can easily observe that $P_n(t) = \big(y_1(t), y_2(t), \cdots, y_n(t)\big)$; $t \in I$ maps I continuously onto the n-dimensional cube $[0,1]^n$ ([17], [18]).

Since the above $\{y_n\}$ are not mutually orthogonal, we let

$$z_n(t) = y_n(t) - \frac{1}{2},$$

then $\{z_n\}$ is i.i.d. having mean 0 and variance 1/12 with uniform distribution on $[-1/2, 1/2]$. Therefore, for $\{a_n\} \in \ell^2$,

$$(*) \qquad f(t) = \sum_{n=1}^{\infty} a_n z_n(t)$$

converges not only in $L^2(I, dt)$ but also converges almost surely with respect to the Lebesgue measure. Clearly, if $\{a_n\} \in \ell^1$, the series $(*)$ converges unformly and f is a continuous function and if $\tau = \max\{n;\ a_n \neq 0\} < +\infty$, then f is $2^{-\tau n}$-Hölder continuous. If $\{a_n\} \in \ell^2$ but $\notin \ell^1$, then by taking account of Theorem 1, the image of f is $[-\infty, +\infty]$, the extended real line, and the cardinal number of the level set $L_x = \{t \in I;\ f(t) = x\}$ is continuum for all x.

To avoid triviality, we always assume that $\{a_n\} \in \ell^2$ and $r =$ the number of $\{n;\ a_n \neq 0\}$ is positive. Now we claim that f has local times satisfying some regularities.

THEOREM. (i) *f has local times $\alpha_f(x,t)$ and*

$$\big|\alpha_f(x,t) - \alpha_f(x,s)\big| \leqq 6D|t-s|^{1/2}, \quad \forall x,\ \forall t,\ \forall s$$

holds, where $D = |a_\sigma|^{-1}$, $\sigma = \min\{n;\ a_n \neq 0\}$.

(ii) *If $r \geqq 2$, then $\alpha_f(x,t)$ is continuous in (x,t), and $\alpha_f(x) \in C^{r-2}$ (including the case $r = \infty$).*

(iii) *For every subinterval J, $\alpha_f(x, J) > 0$ at every interior point x of the set $\big\{f(t);\ t \in J\big\}$.*

(iv) *If $\{a_n\} \in \ell^2$ and $\notin \ell^1$, then $\alpha_f(x) > 0$ for $\forall x \in \Re$.*

(v) *If there exists $c>0$ such that* $\liminf_{n\to+\infty} n|a_n| \geqq c$, *then* $\alpha_f(x)$ *has analytic extension in the domain* $|\Im z| < ce^{-1}/2$.

PROOF. Let $\widehat{\mu_f}(\theta)$ be the characteristic function of f, i.e.

$$\widehat{\mu_f}(\theta) = \int_0^1 e^{i\theta f(t)}dt.$$

Then we have

$$\widehat{\mu_f}(\theta) = \int_0^1 \prod_{n=1}^{\infty} e^{i\theta a_n z_n(t)}dt = \prod_{n=1}^{\infty} \int_0^1 e^{i\theta a_n z_n(t)}dt =$$

$$= \prod_{n=1}^{\infty} \int_{-1/2}^{1/2} e^{i\theta a_n x}dx = \prod_{n=1}^{\infty} \frac{2}{\theta a_n} \sin\frac{\theta a_n}{2}.$$

(by independence). (Here $\frac{1}{0}\sin 0 = 1$.) Now let

$$N(\theta) = \{n;\, |\theta a_n| \geqq 2\}.$$

(a) If $\lim_{|\theta|\to\infty} \sharp N(\theta) = 1$, then the only one $a_n \neq 0$, (say a), so f has the uniform distribution on $[-a/2, a/2]$ and

$$\alpha_f(x) = \begin{cases} 1/a, & \text{on } [-a/2, a/2], \\ 0, & \text{otherwise}. \end{cases}$$

(b) If $2 \leqq r < +\infty$, then $\exists\theta_0 > 0$, and for all $|\theta| \geqq \theta_0$ we have $N(\theta) \geqq r$. Therefore it follows that

$$|\widehat{\mu_f}(\theta)| \leqq \prod_{n\in N(\theta)} \frac{2}{|\theta a_n|} \leqq \prod_{n\in N(\theta_0)} \frac{|\theta_0|}{|\theta|}\frac{2}{|\theta_0 a_n|} \leqq \left(\frac{|\theta_0|}{|\theta|}\right)^r.$$

From this estimation we have $\widehat{\mu_f}(\theta) \in L^1(\Re, d\theta)$, so there exists a continuous density $\alpha_f(x)$ such that

$$\widehat{\mu_f}(\theta) = \int_{-\infty}^{\infty} e^{i\theta x}\alpha_f(x)dx.$$

More precisely we have $|\theta|^{r-2}\widehat{\mu_f}(\theta) \in L^1(\Re, d\theta)$ and $\alpha_f(x)$ belongs to the class C^{r-2}. If $r = +\infty$, a simple modification tells us $\alpha_f(x)$ belongs to the class C^∞.

Now we shall investigate $\alpha_f(x,t)$. Set

$$g(\theta)=\frac{\sin\theta/2}{\theta/2}$$

and

$$\widehat{\mu_f}(\theta,t)=\int_0^t e^{i\theta f(s)}ds.$$

Then for $t_{N,k}=k9^{-N}$ we have

$$\widehat{\mu_f}(\theta,t_{N,k})=\sum_{j=0}^{k-1}\int_{j9^{-N}}^{(j+1)9^{-N}}\exp\Big(i\theta a_1\big(x(s)-1/2\big)+$$

$$+i\theta\sum_{n=2}^{\infty}a_n\big(x(y^{(n-1)}(s))-1/2\big)\Big)\,ds.$$

Since $x(t)$ and $y(t)$ are self-affine functions (cf. N. Kôno [12]), for $j9^{-N}\leqq s\leqq(j+1)9^{-N}$ they are expressed by

$$x(s)=x_j+T^x_{N,j}3^{-N}x(9^Nh_j),$$

$$y(s)=y_j+T^y_{N,j}3^{-N}y(9^Nh_j),$$

where $x_j=x(j9^{-N})$, $y_j=y(j9^{-N})$, $T^x_{N,j}$ and $T^y_{N,j}=+1$, or -1 and $h_j=s-j9^{-N}$.

Therefore we have

$$\widehat{\mu_f}(\theta,t_{N,k})=\sum_{j=0}^{k-1}\int_{j9^{-N}}^{(j+1)9^{-N}}\exp\big(i\theta a_1\big(x_j-1/2+T^x_{N,j}3^{-N}x(9^Nh_j)\big)\big)\times$$

$$\times\exp\Big(i\theta\sum_{n=2}^{\infty}a_n\Big(x\Big(y^{(n-2)}\big(y_j+T^y_{N,j}3^{-N}y(9^Nh_j)\big)\Big)-1/2\Big)\Big)\,dh_j=$$

$$=9^{-N}\sum_{j=0}^{k-1}\int_0^1\exp\big(i\theta a_1(x_j-1/2+T^x_{N,j}3^{-N}s)\big)\,ds\times$$

$$\times\int_0^1\exp\Big(i\theta\sum_{n=2}^{\infty}a_n\Big(x\big(y^{(n-2)}(y_j+T^y_{N,j}3^{-N}s)\big)-1/2\Big)\Big)\,ds=$$

$$= 9^{-N}\sum_{j=0}^{k-1}\exp\big(i\theta a_1(x_j-1/2)\big)\exp(T^x_{N,j}i\theta a_1 3^{-N}/2)g(\theta a_1 3^{-N})\times$$

$$\times\int_0^1\exp\Bigg(i\theta a_2\big(x(y_j+T^y_{N,j}3^{-N}s)-1/2\big)+$$

$$+i\theta\sum_{n=3}^{\infty}a_n\big(x\big(y^{(n-2)}(y_j+T^y_{N,j}3^{-N}s)\big)-1/2\big)\Bigg)\,ds=$$

$$= 9^{-N}\sum_{j=0}^{k-1}\exp\big(i\theta a_1(x_j-1/2)\big)\exp(T^x_{N,j}i\theta a_1 3^{-N}/2)g(\theta a_1 3^{-N})\times$$

$$\times\Bigg(\sum_{j_1=p_1}^{q_1-1}3^N\int_{j_1 9^{-N}}^{(j_1+1)9^{-N}}\exp\Bigg(i\theta a_2\big(x(s)-1/2\big)+$$

$$+i\theta\sum_{n=3}^{\infty}a_n\Big(x\big(y^{(n-2)}(s)\big)-1/2\Big)\Bigg)\,ds\Bigg),$$

where

$$p_1 9^{-N}=y_j+3^{-N}(T^y_{N,j}-1)/2,\ q_1 9^{-N}=y_j+3^{-N}(T^y_{N,j}+1)/2,\ q_1-p_1=3^N.$$

By the same procedure we have

$$\widehat{\mu_f}(\theta,t_{N,k})=9^{-N}\sum_{j=0}^{k-1}\exp\big(i\theta a_1(x_j-1/2+T^x_{N,j}3^{-N}/2)\big)\,g(\theta a_1 3^{-N})\times$$

$$\times\prod_{m=1}^{\infty}\Bigg(3^{-N}\sum_{j_m=p_m}^{q_m-1}\exp\big(i\theta a_{m+1}(x_{j_m}-1/2+T^x_{N,j_m}3^{-N}/2)\big)\,g(\theta a_{m+1}3^{-N})\Bigg).$$

Denoting by χ the indicator function of the interval $I'=[-1/2,1/2]$, Fourier inversion formula tells us that if $a_1\neq 0$, then

$$\alpha_f\big(x,(t_{N,k-1},t_{N,k}]\big)=\frac{3^{-N}}{a_1}\chi\left(\frac{x-a_1(x_{k-1}-1/2+T^x_{N,k-1}3^{-N})/2}{3^{-N}a_1}\right)\star$$

$$\star\prod_{m=1}^{\infty}\star\left(\sum_{j_m=p_m}^{q_m-1}\frac{1}{a_{m+1}}\chi\left(\frac{x-a_{m+1}(x_{j_m}-1/2+T^x_{N,j_m}3^{-N}/2)}{3^{-N}a_{m+1}}\right)\right).$$

Therefore

$$\alpha_f\big(x,(t_{N,k-1},t_{N,k}]\big) \leqq \frac{3^{-N}}{|a_1|}\prod_{m=1}^{\infty}\Bigg(\sum_{j_m=p_m}^{q_m-1} 3^{-N}\Bigg) = \frac{3^{-N}}{|a_1|}.$$

In general, we have

$$\alpha_f\big(x,(t_{N,k-1},t_{N,k}]\big) \leqq \frac{3^{-N}}{|a_\sigma|}.$$

Since $\alpha_f(x,t)$ is non-decreasing in t, for $9^{-N-1} \leqq t-s \leqq 9^{-N}$ there exists k such that $t_{N,k-1} \leqq s < t \leqq t_{N,k+1}$ and we have

$$\alpha_f(x,t)-\alpha_f(x,s) \leqq \alpha_f(x,t_{N,k+1})-\alpha_f(x,t_{N,k-1}) =$$

$$= \alpha_f\big(x,(t_{N,k-1},t_{N,k+1}]\big) \leqq$$

$$\leqq 2\times 3^{-N}|a_\sigma|^{-1} \leqq 6|t-s|^{1/2}|a_\sigma|^{-1} \quad (\forall x).$$

The direct expression for $\alpha_f(x)$ or $\alpha_f(x,t_{N,k})$ yields the proof of (iii) and (iv).

Now let us prove (v). Setting $N^\star(\theta)=\big\{n;\ |\theta a_n| \geqq 2e\big\}$, we have

$$|\widehat{\mu_f}(\theta)| = \prod_{n=1}^{\infty}\frac{2}{\theta a_n}\sin\frac{\theta a_n}{2} \leqq \prod_{n\in N^\star(\theta)}\frac{2}{|\theta a_n|} \leqq e^{-\#N^\star(\theta)}.$$

Since for $\forall\varepsilon>0$, $\exists n_\varepsilon$, $\forall n \geqq n_\varepsilon$ we have $|a_n| \geqq (c-\varepsilon)/n$, it follows for $n_0 = = \big[(c-\varepsilon)|\theta|/(2e)\big] > n > n_\varepsilon$ that

$$|a_n\theta| \geqq (c-\varepsilon)|\theta|/n = (n_0/n)(c-\varepsilon)|\theta|/n_0 \geqq 2e.$$

This means $\#N^\star(\theta) \geqq n_0-n_\varepsilon$. Hence for $|\theta| \geqq 2en_\varepsilon(c-\varepsilon)^{-1}$ we have

$$|\widehat{\mu_f}(\theta)| \leqq e^{-(c-\varepsilon)|\theta|/(2e)+n_\varepsilon+1}.$$

So finally for $0< \ b<(c-\varepsilon)/(2e)$, we have

$$\int_0^{+\infty} e^{b\theta}|\widehat{\mu_f}(\theta)|d\theta \ < \ +\infty.$$

This yields the proof of (v) by Berman ([2], Lemma 8.1).

§3. Relation between local times and the original function

In this section we shall sumarize the known facts about the relation between regularity of local times and irregularity of the original function.

THEOREM A. (a) *If the local time* $\alpha_f(x,t)$ *is continuous in* t *for almost every* x, *then*

(i) *([8], Theorem* A-(a)*)*

$$\text{ap-}\lim_{s\to t} \frac{|f(t)-f(s)|}{|t-s|} = +\infty \quad a.e. \quad t \in [0,1],$$

where "ap-lim" *stands for approximate limit, for the definition see* [16], *p.*220.

(ii) *([8], Theorem* A-(b)*) the level set*

$$L_{f(t)} = \{0 \leqq s \leqq 1;\ f(s) = f(t)\}$$

is uncountable for almost every t, *and*

(iii) *([6], Theorem* 1*) for almost every* t, f *is not locally increasing or decreasing at* t.

(iv) *([7]) Let* f *be a continuous function, then on every subinterval* $J \subset \subset [0,1]$, f *has multiple image of order* m, *all* $m \geqq 2$, *i.e.* $J \supset \exists I_1, \cdots, I_m$ *disjoint invervals such that*

$$\left|\bigcap_{k=1}^{m} f(I_k)\right| > 0.$$

(b) *If the local time* $\alpha_f(x,t)$ *is jointly continuous, then*

(i) *([3], Lemma* 3.1, *in the original statement the approximate limit is taken as a bilateral limit, but the proof actually gives one sided limits.)*

For all t,

$$\text{ap-}\lim_{s\downarrow t} \frac{|f(t)-f(s)|}{|t-s|} = +\infty,$$

$$\text{ap-}\lim_{s\uparrow t} \frac{|f(t)-f(s)|}{|t-s|} = +\infty$$

hold.

(ii) *([3], Lemma* 3.2*) Let* f *be continuous, then* $\{x;\ \sharp\{t;\ f(t)=x\}$ *is countable*$\}$ *is nowhere dense in the range of* f.

(c) *([6]) Let* f *be continuous and* $\alpha_f(x,t)$ *be also jointly continuous. Moreover, if* $\alpha_f(x,I)$ *is positive on the interior of the image of* I *by* f, *then*

f is nowhere locally incereasing or decreasing in I. Combining this and (b)-(i), *we obtain that f is nowhere differentiable in the sense of Weierstrass (it does not allow $f'(t) = +\infty$ or $f'(t) = -\infty$).*

(d) *If $\alpha_f(x,t)$ is jointly continuous and Hölder-continuous in t, i.e. there exist $0 < \beta < 1$, $D > 0$, such that*

$$\left|\alpha_f(x,t) - \alpha_f(x,s)\right| \leqq D|t-s|^{\beta}, \quad \textit{for all} \quad x,t,s$$

holds, then

(i) *([5], Theorem* (10.1)*) for all t*

$$\text{ap-}\lim_{s\to t} \frac{\left|f(t)-f(s)\right|}{|t-s|^{\gamma}} = \infty, \quad \forall\gamma > 1-\beta,$$

holds,

(ii) *([5], Lemma* 5.1*) let f be continuous, then*

$$\max\{t \in J;\ f(t)\} - \min\{t \in J;\ f(t)\} \geqq |J|^{1-\beta}/D$$

holds for all sub-intervals $J \subseteqq I$, that is, at each point t, the graph of f is not contained in any domain $\{(u,v);\ u \geqq t,\ |v-f(t)| \leqq a(u-t)^{\gamma}\}$ nor $\{(u,v);\ u \leqq t,\ |v-f(t)| \leqq a(t-u)^{\gamma}\}$ (for all $a > 0$, $\gamma > 1-\beta$) with the vertex $(t,f(t))$ and

(iii) *([5], Lemma* 6.2.*) let f be continuous, then*

$$\text{Hausdorff-dim}\, L_x \geqq \beta \quad \textit{for } x \in \{y;\ \alpha_f(y) \neq 0\}$$

($\alpha_f(x,t)$ is not necessarily continuous in x).

We remark that if f is α-Hölder continuous, that is $\left|f(t)-f(s)\right| \leqq \leqq D|t-s|^{\alpha}$ and $\alpha_f(x,t)$ is bounded at the neighberhood of x, then for all x,

$$\text{Hausdorff-dim}\, L_x \leqq 1-\alpha$$

holds ([5], Lemma 7.2).

As for the connection between the Fourier transform of α and the Hölder continuity and the variation of f, see [1], Lemmas 4.1 and 4.3.

§4. Conclusion

From the Theorem and Theorem A(c), (d–iii), if $r \geqq 2$ and $\{a_n\} \in \in \ell^1$, then f is a continuous nowhere differentiable function in the sense of Weierstrass and

$$\text{Hausdorff-dim}\, L_x \geqq \frac{1}{2}, \quad \forall |x| < \sum_{n=1}^{\infty} |a_n|/2,$$

more precisely, for every subinterval J, the Hausdorff-dimension of $L_x = \{t \in J;\ f(t) = x\} \geqq 1/2$ at the interior point x of the set $\{f(t);\ t \in J\}$.

Example. For the Peano curve $\big(x(t), y(t)\big)$ let

$$f = a\left(x(t) - \frac{1}{2}\right) + b\left(x\big(y(t)\big) - \frac{1}{2}\right) \quad (a, b \neq 0)$$

then f is nowhere differentiable and

$$\text{Hausdorff-dim}\, L_x = \frac{1}{2}, \quad \forall\, |x| < \big(|a| + |b|\big)/2.$$

Since $\big(x(t), x(y(t))\big)$ and $\big(x(t), y(t)\big)$ have the same probability laws, the above $x\big(y(t)\big)$ can be replaced by $y(t)$.

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RANGES OF POLYNOMIALS WITH CURVED MAJORANTS

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Let $\mathcal{P}_n^r$ denote the set of real polynomials of degree at most n. Suppose $w(x) \geqq 0$ on $[-1,1]$. Define

$$C_n := \left\{ p_n \in \mathcal{P}_n^r \mid |w(x)p_n(x)| \leqq 1, \quad \text{for} \quad -1 \leqq x \leqq 1 \right\}.$$

According to Rahman [5], a polynomial $p_n \in C_n(w)$ is called a polynomial with curved majorant $1/w(x)$. (We find it more convenient to use $|w(x)p_n(x)| \leqq 1$ than $|p_n(x)| \leqq \phi(x)$ to define the polynomials with curved majorant in this paper.) This paper is concerned with the ranges of polynomials in $C_n(w)$, i.e., we want to describe the set

$$\mathcal{R}_n(w) := \left\{ p_n(x) \mid p_n \in C_n(w) \quad \text{and} \quad x \in \mathbf{R} \right\}$$

in terms of w. Since $p_n \in C_n(w)$ and $r \in [-1,1]$ imply $rp_n \in C_n(w)$, we need only to determine the boundary of $\mathcal{R}_n(w)$ which is given by

$$B_w(x) := \sup_{p \in C_n(w)} |p(x)|, \quad x \in \mathbf{R}$$

and $-B_w(x)$.

In the case when $w(x) = (1-x^2)^{1/2}$, Newman and Rivlin obtained the following result which can be stated in our notation as follows.

THEOREM 1 ([1]). *If* $w(x) = (1-x^2)^{1/2}$, *then*

$$B_w(x) = \begin{cases} (1-x^2)^{-1/2}, & x \in \left[-\cos\frac{\pi}{2(n+1)}, \cos\frac{\pi}{2(n+1)} \right], \\ |U_n(x)|, & x \notin \left[-\cos\frac{\pi}{2(n+1)}, \cos\frac{\pi}{2(n+1)} \right], \end{cases}$$

where $U_n(x)$ *is the* n*-th Chebyshev polynomial of the second kind.*

REMARK. The following observations will be helpful in formulating the results for more general weight functions.

(i) The end points of the interval $\left[-\cos\frac{\pi}{2(n+1)}, \cos\frac{\pi}{2(n+1)} \right]$ are the first and the last extremal points of $(1-x^2)^{1/2}U_n(x)$ over the interval $[-1,1]$.

(ii) The polynomial

$$\frac{1}{2^{n+1}(n+1)}U_n(x)$$

is the n-th weighted Chebyshev polynomial with respect to the weight $(1-x^2)^{1/2}$ over the interval $[-1,1]$.

Our goal is to find $B_w(x)$ for more general weights $w(x)$. In [3], we considered the case when $d^2/dx^2\big(1/w(x)\big)$ is continuous in $(-1,1)$. To restate the result in [3] which will be used in this paper, we need to introduce the weighted Chebyshev polynomials and related concepts.

For a continuous weight function $w(x)$ on $[-1,1]$ with $w(x)>0$ for $x\in(-1,1)$, we know that (cf., e.g., [3, §2]) there exists $\big\{T_n(x;w)\big\}_{n=0}^{\infty}$, $T_n(x;w)=x^n+\ldots\in\mathcal{P}_n^r$, $n=0,1,2,\ldots$, satisfying

$$\max_{x\in[-1,1]}\big|T_n(x;w)w(x)\big| = \inf_{p(x)=x^n+\ldots\in\mathcal{P}_n^r}\ \max_{x\in[-1,1]}\big|p(x)w(x)\big|.$$

The polynomial $T_n(x;w)$ is called the n-th weighted Chebyshev polynomial with respect to w. By Chebyshev's maximum equioscillation theorem, there are ξ_k, $k=0,1,\ldots,n$, such that

$$-1\leqq\xi_n<\xi_{n-1}<\ldots<\xi_0\leqq 1$$

and

$$T_n(\xi_k;w)=(-1)^k\max_{x\in[-1,1]}\big|T_n(x;w)w(x)\big|,\quad k=0,1,\ldots,n.$$

We call $\{\xi_k\}_{k=0}^n$ a set of points of equioscillation of $T_n(x,w)w(x)$. Generally, such a set is not unique. Denote

$$\hat{\xi}_n(n)=\sup\xi_n\quad\text{and}\quad\tilde{\xi}_0(n)=\inf\xi_0$$

among all sets of points of equioscillation. Define

$$\hat{T}_n(x;w)=T_n(x;w)//\|T_nw\|.$$

THEOREM 2 ([3, Theorem 3]). *Let $w\colon[-1,1]\to[0,\infty)$ be continuous, $w(x)>0$ for $x\in(-1,1)$ and $d^2/dx^2\big(1/w(x)\big)$ continuous in $(-1,1)$. Given $r\in(0,1)$, there exists $N=N(r,w)>0$ such that, for $n\geqq N$,*

$$\hat{\xi}_n(n)<-r,\quad\tilde{\xi}_0(n)>r$$

and

$$B_w(x)=\begin{cases}1/w(x), & \text{if}\quad x\in(-r,r),\\ \big|\hat{T}_n(x,w)\big|, & \text{if}\quad x\in\big(-\infty,\hat{\xi}_n(n)\big]\cup\big[\tilde{\xi}_0(n),\infty\big).\end{cases}$$

As suggested by the Question at the end of Section 2 in [3], however, we do not know what happens over $\left(\hat{\xi}(n), -r\right] \cup \left[r, \tilde{\xi}_0(n)\right)$. The result of this paper will fill this gap for certain classes of weight functions.

Now, let us define the weight functions we will consider.

Let ρ be a real polynomial of degree m with $\rho(x) > 0$ on $[-1,1]$. Following Freund [2], set

$$s_0(x) = 1, \quad s_{1/2}(x) = \sqrt{x+1}, \quad s_1(x) = i\sqrt{1-x^2}$$

and

$$w_j(x) = \left|s_j(x)\right| / \sqrt{\rho(x)}, \quad j = 0, 1/2, 1,$$

on $[-1,1]$.

Let $\{a_k\}_{k=1}^m$ be the zeros of ρ. If we write

$$x = \frac{1}{2}\left(v + \frac{1}{v}\right) \qquad (|v| \leqq 1)$$

and

$$a_k = \frac{1}{2}\left(\alpha_k + \frac{1}{\alpha_k}\right) \quad (|\alpha_k| < 1), \quad k = 1, 2, \ldots, m,$$

and set

$$h(v) := \prod_{k=1}^{m}(v - \alpha_k),$$

then h is a real polynomial of degree m and

$$\rho(x) = \rho_0 h(v) h\left(\frac{1}{v}\right)$$

with $\rho_0 > 0$. Furthermore, if we define, for $j = 0, 1/2, 1$,

$$\hat{T}_{n,j}(x) = \frac{\sqrt{\rho_0}}{2s_j(x)}\left\{v^{n+j-m}h(v) + \frac{(-1)^{[j]}}{v^{n+j-m}}h\left(\frac{1}{v}\right)\right\},$$

then we have the following proposition.

PROPOSITION 3 (cf., e.g., [1, 2]). *For each j $(j = 0, 1/2, 1)$, $\hat{T}_{n,j}$ is a real polynomial in x and the degree of $\hat{T}_{n,j}$ is given by*

$$\operatorname{degree} \hat{T}_{n,j} = \begin{cases} n, & \text{if} \quad n \geqq n_j, \\ m - n - 2j, & \text{if} \quad 0 \leqq n < n_j, \end{cases}$$

where

$$n_j = \begin{cases} 0, & \textit{if} \quad (j,m) = (1,0), \\ [m/2 + 1/2 - j], & \textit{otherwise.} \end{cases}$$

When $v = e^{i\varphi}$ ($\varphi \in [0,\pi]$), one can write

$$w_j(x)\hat{T}_{n,j}(x) = \begin{cases} \cos\big((n+j-m/2)\varphi + \gamma(\varphi)\big), & \text{if} \quad j = 0, 1/2, \\ \sin\big((n+j-m/2)\varphi + \gamma(\varphi)\big), & \text{if} \quad j = 1, \end{cases}$$

where the function $\gamma\colon [0,\pi] \to \mathbf{R}$ is defined continuously by

$$\frac{\sqrt{h(v)}}{v^{m/2}\sqrt{h(1/v)}} = \sqrt{\prod_{k=1}^{m} \frac{v - \alpha_k}{1 - \bar{\alpha}_k v}} = e^{i\gamma(\varphi)}$$

and $\gamma(0) = 0$. Then $\gamma(\pi) = m\pi/2$, and $\gamma \equiv 0$ if $m = 0$. We have

PROPOSITION 4. *The function γ is differentiable on $(0,\pi)$. Furthermore, if $m \neq 0$ then*

$$\gamma'(\varphi) > 0, \qquad \varphi \in (0,\pi).$$

PROOF. For $v = e^{i\varphi}$ ($\varphi \in [0,\pi]$), write

$$\frac{v - \alpha_k}{1 - \bar{\alpha}_k v} = e^{i\gamma_k(\varphi)} \quad (|\gamma_k(0) - \gamma_k(\varphi)| < 2\pi), \quad k = 1, 2, \dots, m,$$

then

$$\gamma(\varphi) = \frac{1}{2}\sum_{k=1}^{m} \gamma_k(\varphi) - \frac{1}{2}\sum_{k=1}^{m} \gamma_k(0).$$

Now

$$\frac{d\gamma_k(\varphi)}{d\varphi} = \frac{1 - |\alpha_k|^2}{|1 - \bar{\alpha}_k e^{i\varphi}|^2},$$

so $\gamma'(\varphi)$ exists on $(0,\pi)$ and is positive unless $m = 0$ (in this case $\gamma \equiv 0$). $\square$

By Chebyshev's maximum equioscillation theorem and the above representations, it can be verified that (cf. [1]) if $a_{n,j}$ denotes the leading coefficient of $\hat{T}_{n,j}$ then $\hat{T}_n(x; w_j) = \hat{T}_{n,j}(x)/a_{n,j}$.

Define $\xi_{k,n}$ ($k = 0, 1, \dots, n$; $n = 0, 1, 2, \dots$) by means of

$$(n + 1 - m/2)\xi_{k,n} + \gamma(\xi_{k,n}) = k\pi + \pi/2$$

and $\xi_{k,n} \in (0,\pi]$. Then $\{\cos \xi_{k,n}\}_{k=0}^{n}$ is the only set of the equioscillation of $\hat{T}_{n,1}(x)w_1(x)$ over $[-1,1]$. Our main result is the following theorem.

THEOREM 5. *For $w_1(x)$ as defined above, there exists $N = N(w_1) > 0$ such that, for $n > N$, we have*

$$B_{w_1}(x) = \begin{cases} 1/w_1(x), & \cos\xi_{n,n} \leqq x \leqq \cos\xi_{0,n}. \\ \left|\hat{T}_{m,1}(x)\right|, & \textit{otherwise}. \end{cases}$$

REMARK. Sharp estimates for the growth along the imaginary axis of polynomials with curved majorants $1/w_j(x)$ $(j = 0, 1/2, 1)$ are given by Freund in [2, Corollary 1].

In light of Theorem 2, we need only to prove Theorem 5 for those points $x \in (\cos\xi_{n,n}, \cos\xi_{m',m'}] \cup [\cos\xi_{0,m'}, \cos\xi_{0,n})$ with $m' = [m/2] + 1$ (taking $r = \max(\cos\xi_{m',m'}, \cos\xi_{0,m'})$ in Theorem 2). The idea of our proof is essentially a refinement and generalization of that of Newman and Rivlin in [4]. We need the following lemmas.

LEMMA 6. *The following assertions hold.*
(i) $\xi_{k,n} < \xi_{k,n-1} < \pi$, $k = 0, 1, \dots, n-2$.
(ii) $\xi_{k,n-1} < \xi_{k+1,n}$, $k = 0, 1, \dots, n-2$.
(iii) *If $n \geqq m/2$, then $\xi_{0,n} < \pi/2$.*

PROOF. (i) If $\xi_{k,n} \geqq \xi_{k,n-1}$, then $\gamma(\xi_{k,n}) \geqq \gamma(\xi_{k,n-1})$ by Proposition 3. Now

$$(n + 1 - m/2)\xi_{k,n} > (n - m/2)\xi_{k,n} \geqq (n - m/2)\xi_{k,n-1},$$

so

$$k\pi + \pi/2 = (n + 1 - m/2)\xi_{k,n} + \gamma(\xi_{k,n}) >$$
$$> (n - m/2)\xi_{k,n-1} + \gamma(\xi_{k,n-1}) = k\pi + \pi/2,$$

which is a contradiction. Thus $\xi_{k,n} < \xi_{k,n-1}$.

Similarly, note that $\xi_{k,n-1} \geqq \pi$ would imply

$$k\pi + \pi/2 = (n - m/2)\xi_{k,n-1} + \gamma(\xi_{k,n-1}) \geqq (n - m/2)\pi + (m/2)\pi,$$

which is impossible. Hence we have established (i).

(ii) Assume $\xi_{k+1,n} \leqq \xi_{k,n-1}$, then by the definition of $\xi_{k,n-1}, \xi_{k+1,n}$ and Proposition 3,

$$k\pi + \pi/2 = (n - m/2)\xi_{k,n-1} + \gamma(\xi_{k,n-1}) \geqq (n - m/2)\xi_{k+1,n} + \gamma(\xi_{k+1,n}) =$$
$$= (k+1)\pi + (\pi/2) - \xi_{k+1,n}.$$

So $\xi_{k+1,n} \geqq \pi$, which contradicts (i).

(iii) In fact, by the definition of $\xi_{0,n}$,

$$(n + 1 - m/2)\xi_{0,n} + \gamma(\xi_0, n) = \pi/2,$$

so we have

$$(n - m/2)\xi_{0,n} + \gamma(\xi_{0,n}) = \pi/2 - \xi_{0,n}.$$

The left side of the equality is positive if $n \geqq m/2$. $\square$

For $\lambda > 0$, define

$$S_n(\varphi) := S_n(\varphi; \lambda) = w_1(\cos\varphi)\left[\hat{T}_{n,1}(\cos\varphi) + \lambda\hat{T}_{n-1,1}(\cos\varphi)\right].$$

We have

LEMMA 7. *When $n \geqq m/2$, the relative extrema of $|S_n(\varphi)|$ on $[0,\pi]$ is strictly decreasing.*

PROOF. Define

$$C_n(\varphi) := \cos\big((n+1-m/2)\varphi + \gamma(\varphi)\big) + \lambda\cos\big((n-m/2)\varphi + \gamma(\varphi)\big)$$

and

$$f(\varphi) = \left|C_n(\varphi) + iS_n(\varphi)\right|.$$

Then

$$f(\varphi) = \left|e^{[(n+1-m/2)\varphi+\gamma(\varphi)]} + \lambda e^{i[(n-m/2)\varphi+\gamma(\varphi)]}\right| =$$

$$= |e^{i\varphi} + \lambda| = (1 + \lambda^2 + 2\lambda\cos\varphi)^{1/2}$$

and

$$|S_n(\varphi)| \leqq \left(|C_n(\varphi)|^2 + |S_n(\varphi)|^2\right)^{1/2} = f(\varphi).$$

Now we claim that

(1) $$\operatorname{sgn} S_n'(\xi_{k,n}) = -\operatorname{sgn} S_n'(\xi_{k,n-1}) = (-1)^k,$$

for $k = 0, 1, \ldots, n-1$. In fact,

$$S_n'(\xi_{k,n}) = (-1)^k\lambda\left[(n - m/2) + \gamma'(\xi_{k,n})\right]\sin\xi_{k,n}$$

and

$$S_n'(\xi_{k,n-1}) = (-1)^{k+1}\left[(n+1-m/2) + \gamma'(\xi_{k,n-1})\right]\sin\xi_{k,n-1}.$$

Also, $\xi_{k,n} \in (0,\pi)$, so (1) holds for $k = 0, 1, \ldots, n-1$.

Similarly, one can check that

(2) $$\operatorname{sgn} C_n(\xi_{k,n}) = -\operatorname{sgn} C_n(\xi_{k,n-1}) = (-1)^k$$

for $k = 0, 1, \ldots, n-1$. Therefore, from (1) and (2), it follows that for each k ($k = 0, 1, \ldots, n-1$), there exist $\zeta_{k,n}, \eta_{k,n} \in (\xi_{k,n}, \xi_{k,n-1})$ such that $S'_n(\zeta_{k,n}) = 0$ and $C_n(\eta_{k,n}) = 0$. Each $\zeta_{k,n}$ is a relative maximum point of $|S_n(\varphi)|$. Furthermore, since $|S_n(\varphi)|$ has exactly n relative maxima over $[0, \pi]$, the set $\{\zeta_{k,n}\}_{k=0}^{n-1}$ consists of all the relative maximum points of $|S_n(\varphi)|$ over $[0, \pi]$.

Now at $\eta_{k,n}$ ($k = 0, 1, \ldots, n-1$) there holds $|S_n(\eta_{k,n})| = f(\eta_{k,n})$. But f is strictly decreasing over $[0, \pi]$, while

$$f(\zeta_{k,n}) > |S_n(\zeta_{k,n})| \geqq |S_n(\eta_{k,n})| = f(\eta_{k,n}).$$

So we must have $\eta_{k,n} > \zeta_{k,n}$. Hence

$$|S_n(\zeta_{k,n})| \geqq |S_n(\eta_{k,n})| = f(\eta_{k,n}) > f(\zeta_{k+1,n}) \geqq |S_n(\zeta_{k+1,n})|,$$

for $k = 0, 1, \ldots, n-1$. $\square$

Now we can give the proof of Theorem 5.

Proof of Theorem 5. As remarked after the statement of Theorem 5, we need only to consider the case when $x \in (\cos \xi_{n,n}, \cos \xi_{m',m'}] \cup [\cos \xi_{0,m'}, \cos xi_{0,n}) =: I_- \cup I_+$. Now, assume $x_0 \in I_+$, then

$$x_0 = \cos \theta \quad \text{with} \quad \theta \in [\xi_{0,n}, \xi_{0,m'}).$$

We need to construct $p_\theta \in C_n(w)$ such that

$$\big|p_\theta(\cos \theta)\big| = 1/w(\cos \theta).$$

From Lemma 6,

$$\xi_{0,n} < \xi_{0,n-1} < \ldots < \xi_{0,m'},$$

so there exists j_θ with $m' \leqq j_\theta \leqq n$ such that

$$\theta \in (\xi_{0,j_\theta}, \xi_{0,j_{\theta-1}}].$$

Define

$$\lambda_\theta := \frac{\cos\big[(n+1-m/2)\theta + \gamma(\theta)\big]\,\big(n+1-m/2+\gamma'(\theta)\big)}{\cos\big[(n-m/2)\theta + \gamma(\theta)\big]\,\big(n-m/2+\gamma'(\theta)\big)}.$$

and set

$$p_\theta(x) := \frac{\hat{T}_{j_\theta,1}(x) + \lambda_\theta \hat{T}_{j_{\theta-1},1}(x)}{\hat{T}_{j_\theta,1}(\cos\theta) + \lambda_\theta \hat{T}_{j_{\theta-1},1}(\cos\theta)}.$$

Then, with $x = \cos\varphi$,

$$w_1(x) p_\theta(x) = \frac{S_{j_\theta}(\varphi; \lambda_\theta)}{S_{j_\theta}(\theta; \lambda_\theta)}.$$

By Lemma 7 and the definition of λ_θ,

$$\max_{x\in[-1,1]} |w_1(x)p_\theta(x)| = \left| w_1(\cos\theta)p_\theta(\cos\theta)\right|.$$

So

$$p_\theta \in C_{j_\theta}(w_1) \subsetneqq C_n(w_1)$$

and

$$p_\theta(\cos\theta) = 1/w_1(\cos\theta).$$

By considering $w_1(-x)$, we can obtain the result for the case when $x \in I_-$. This completes the proof of Theorem 5. $\square$

Similar results can be proved for the weight functions w_0 and $w_{1/2}$. The details are omitted here.

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SEMI-NORMAL SPACES AND SOME FUNCTIONS

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

1. Introduction

Arya and Bhamini [1] and Dorsett [7] have introduced the notion of semi-normal spaces by using semi-open sets due to Levine [9]. Recently, in [2], the concept of semi-generalized open sets has been introduced as a generalization of semi-open sets. In the present paper, we obtain further characterizations of semi-normal spaces by using semi-generalized open sets. Moreover, in order to obtain preservation theorems of semi-normal spaces, we introduce the concepts of pre sg-continuous functions and pre sg-closed functions.

2. Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let X be a space and A a subset of X. We denote the closure of A and the interior of A by $\mathrm{Cl}\,(A)$ and $\mathrm{Int}\,(A)$, respectively. A subset A is said to be *semi-open* [9] if there exists an open set U of X such that $U \subset A \subset \mathrm{Cl}\,(U)$. The complement of a semi-open set is said to be *semi-closed*. The family of all semi-open (resp. semi-closed) sets of X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{SC}(X)$). The intersection of all semi-closed sets containing A is called the *semi-closure* of A [3] and is denoted by $\mathrm{sCl}(A)$. The *semi-interior* of A, denoted by $\mathrm{sInt}(A)$, is defined to be the union of all semi-open sets contained in A.

DEFINITION 1. A subset A of a space X is said to be *semi-generalized closed* (briefly sg-closed) [2] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathrm{SO}(X)$.

Every semi-closed set is sg-closed but the converse is false [2, Example 3]. The complement of a sg-closed set is said to be *semi-generalized open* (briefly sg-open) [2]. A subset A is sg-open if and only if $F \subset \mathrm{sInt}(A)$ whenever $F \in \mathrm{SC}(X)$ and $F \subset A$ [2, Theorem 6].

DEFINITION 2. A function $: X \to Y$ is said to be *semi-continuous* [9] (resp. *irresolute* [4]) if $f^{-1}(V) \in \mathrm{SO}(X)$ for every open set V of Y (resp. $V \in \mathrm{SO}(Y)$).

It is obvious that semi-continuity is implied by both continuity and irresoluteness.

DEFINITION 3. A function $f: X \to Y$ is said to be *semi-closed* [10] (resp. *presemiclosed* [11]) if $f(F) \in \mathrm{SC}(Y)$ for every closed set F of X (resp. $F \in \in \mathrm{SC}(X)$).

DEFINITION 4. A function $f: X \to Y$ is said to be *sg-continuous* [12] (resp. *sg-irresolute* [12]) if $f^{-1}(F)$ is sg-closed in X for every closed (resp. sg-closed) set F of Y.

It was shown that semi-continuity implies sg-continuity but the converse is false [12, Example 3.4].

DEFINITION 5. A space X is said to be *semi-normal* [7] if for each pair of disjoint semi-closed sets A and B, there exist disjoint $U, V \in \mathrm{SO}(X)$ such that $A \subset U$ and $B \subset V$.

In [1], Arya and Bhamini called semi-normal spaces *s-normal*. However, in this paper, we shall use the term "*semi-normal*" in the sequel.

DEFINITION 6. A space X is said to be *semi-*$T_{\frac{1}{2}}$ [2] if every sg-closed set of X is semi-closed in X.

3. Semi-normal spaces

We shall obtain the further characterizations of semi-normal spaces by using sg-open sets and sg-closed sets.

THEOREM 1. *The following properties are equivalent for a space X:*

(a) *X is semi-normal;*

(b) *for each pair of disjoint $A, B \in \mathrm{SC}(X)$, there exists disjoint* sg-*open sets U and V such that $A \subset U$ and $B \subset V$;*

(c) *for each $A \in \mathrm{SC}(X)$ and each $U \in \mathrm{SO}(X)$ containing A, there exists a* sg-*open set G such that $A \subset G \subset \mathrm{sCl}(G) \subset U$;*

(d) *for each $A \in \mathrm{SC}(X)$ and each sg-open set U containing A, there exists $G \in \mathrm{SO}(X)$ such that $A \subset G \subset \mathrm{sCl}(G) \subset \mathrm{sInt}(U)$;*

(e) *for each* sg-*closed set A and each $U \in \mathrm{SO}(X)$ containing A, there exists $G \in \mathrm{SO}(X)$ such that $A \subset \mathrm{sCl}(A) \subset G \subset \mathrm{sCl}(G) \subset U$;*

(f) *for each $A \in \mathrm{SC}(X)$ and each $U \in \mathrm{SO}(X)$ containing A, there exists $G \in \mathrm{SO}(X) \cap \mathrm{SC}(X)$ such that $A \subset G \subset U$.*

PROOF. (a) $\Rightarrow$ (b). This is obvious since every semi-open set is sg-open.

(b) $\Rightarrow$ (c). Let $A \in \mathrm{SC}(X)$ and $U \in \mathrm{SO}(X)$ containing A. Then $A \cap \cap (X - U) = \emptyset$ and $X - U \in \mathrm{SC}(X)$. There exist sg-open sets G and V such that $A \subset G$, $X - U \subset V$, and $G \cap V = \emptyset$. Therefore, we have $A \subset G \subset X - - V \subset U$ and hence $\mathrm{sCl}(G) \subset \mathrm{sCl}(X - V) \subset U$ since $X - V$ is sg-closed and $U \in \mathrm{SO}(X)$. Consequently, we obtain $A \subset G \subset \mathrm{sCl}(G) \subset U$.

(c) $\Rightarrow$ (d). Let $A \in \mathrm{SC}(X)$ and U be a sg-open set containing A. We have $A \subset \mathrm{sInt}(U)$ [2, Theorem 6] and $\mathrm{sInt}(U) \in \mathrm{SO}(X)$. There exists a sg-open set V such that $A \subset V \subset \mathrm{sCl}(V) \subset \mathrm{sInt}(U)$. Put $G = \mathrm{sInt}(V)$, then we obtain $G \in \mathrm{SO}(X)$ and $A \subset G \subset \mathrm{sCl}(G) \subset \mathrm{sInt}(U)$.

(d) $\Rightarrow$ (e). Let A be any sg-closed set and $U \in \mathrm{SO}(X)$ containing A. Then, we have $\mathrm{sCl}(A) \subset U$ and $\mathrm{sCl}(A) \in \mathrm{SC}(X)$. Since every semi-open set is sg-open, there exists $G \in \mathrm{SO}(X)$ such that $A \subset \mathrm{sCl}(A) \subset G \subset \mathrm{sCl}(G) \subset \subset U$.

(e) $\Rightarrow$ (f). Let $A \in \mathrm{SC}(X)$ and $U \in \mathrm{SO}(X)$ containing A. There exists $V \in \mathrm{SO}(X)$ such that $A \subset V \subset \mathrm{sCl}(V) \subset U$. Put $G = \mathrm{sCl}(V)$, then G is semi-open and semi-closed [6, Proposition 2.2] and $A \subset G \subset U$.

(f) $\Rightarrow$ (a). Let A and B be any pair of disjoint semi-closed sets. Then, we have $A \subset X - B \in \mathrm{SO}(X)$ and there exists $U \in \mathrm{SO}(X) \cap \mathrm{SC}(X)$ such that $A \subset U \subset X - B$. Now, put $V = X - U$, then we obtain $A \subset U$, $B \subset \subset V \in \mathrm{SO}(X)$, and $U \cap V = \emptyset$. This shows that X is semi-normal.

4. Pre sg-continuous functions

In this section we introduce a new class of functions called pre sg-continuous functions.

DEFINITION 7. A function $f\colon X \to Y$ is said to be *pre sg-continuous* if $f^{-1}(F)$ is *sg-closed* in X for every $F \in \mathrm{SC}(Y)$.

It is obvious that $f\colon X \to Y$ is pre sg-continuous if and only if $f^{-1}(V)$ is sg-open in X for every $V \in \mathrm{SO}(Y)$. From Definitions 2, 4 and 7, for the properties of a function we obtain the following relations.

$$\begin{array}{ccccc} & & & & sg\text{-irresoluteness} \\ & & & & \Downarrow \\ & & \text{irresoluteness} & \Longrightarrow & \text{pre } sg\text{-continuity} \\ & & \Downarrow & & \Downarrow \\ \text{continuity} & \Longrightarrow & \text{semi-continuity} & \Longrightarrow & sg\text{-continuity} \end{array}$$

Diagram I

REMARK 1. By the three examples stated below we obtain the following properties:

(a) none of the implications in Diagram I are reversible;

(b) sg-irresoluteness, irresoluteness, and continuity are pairwise independent.

(c) pre sg-continuity and continuity are independent of each other;

(d) pre sg-continuity and semi-continuity are independent of each other.

EXAMPLE 1. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $f\colon$ $\colon (X,\tau) \to (X,\tau)$ be a function defined as follows: $f(a) = f(b) = a$ and

$f(c) = c$. Then f is continuous but it is not pre sg-continuous since $\{a\} \in \mathrm{SC}(X,\tau)$ and $f^{-1}(\{a\}) = \{a,b\}$ is not sg-closed in (X,τ).

EXAMPLE 2. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b,c\}\}$. Let $f\colon (X,\tau) \to (X,\sigma)$ be the identity function. Then f is irresolute but it is neither sg-irresolute nor continuous. There exists a sg-closed set $\{a,b\}$ in (X,σ) such that $f^{-1}(\{a,b\})$ is not sg-closed in (X,τ).

EXAMPLE 3. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$, and $\sigma = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$. Let $f\colon (X,\tau) \to (X,\sigma)$ be the identity function. Then f is sg-irresolute but it is not semi-continuous since $f^{-1}(\{a,c\}) \notin \mathrm{SO}(X,\tau)$.

THEOREM 2. *If a function $f\colon X \to Y$ is pre* sg*-continuous and presemiclosed, then f is* sg*-irresolute.*

PROOF. Let K be any sg-closed set of Y and $U \in \mathrm{SO}(X)$ containing $f^{-1}(K)$. Since f is presemiclosed, it follows from [8, Theorem 3.5] that there exists $V \in \mathrm{SO}(Y)$ such that $K \subset V$ and $f^{-1}(V) \subset U$. Since K is sg-closed in Y, we have $\mathrm{sCl}(K) \subset V$ and hence $f^{-1}(\mathrm{sCl}(V)) \subset f^{-1}(V) \subset U$. Since f is pre sg-continuous, $f^{-1}(\mathrm{sCl}(V))$ is sg-closed in X and hence $\mathrm{sCl}(f^{-1}(K)) \subset \mathrm{sCl}\big(f^{-1}(\mathrm{sCl}(V))\big) \subset U$. This shows that $f^{-1}(K)$ is sg-closed in X. Therefore, f is sg-irresolute.

The following two corollaries are immediate consequences of Theorem 2.

COROLLARY 1 (Sundaram et al. [12]). *Every irresolute presemiclosed function is* sg*-irresolute.*

COROLLARY 2 (Sundaram et al. [12]). *Semi-$T_{\frac{1}{2}}$ spaces are preserved under irresolute presemiclosed surjections.*

PROPOSITION 1. *Let X be a semi-$T_{\frac{1}{2}}$ space. A function $f\colon X \to Y$ is pre* sg*-continuous if and only is f is irresolute.*

PROOF. Suppose that f is pre sg-continuous. Let K be any semi-closed set of Y. Then $f^{-1}(K)$ is sg-closed in X and hence $f^{-1}(K) \in \mathrm{SC}(X)$ since X is semi-$T_{\frac{1}{2}}$. Therefore, it follows from [4, Theorem 1.4] that f is irresolute. The converse is obvious.

COROLLARY 3 (Sundaram et al. [12]). *If $f\colon X \to Y$ is* sg*-irresolute and X is semi-$T_{\frac{1}{2}}$, then f is irresolute.*

THEOREM 3. *If $f\colon X \to Y$ is a pre* sg*-continuous presemiclosed injection and Y is a semi-normal space, then X is semi-normal.*

PROOF. Let A and B be any disjoint semi-closed sets of X. Since f is a presemiclosed injection, $f(A)$ and $f(B)$ are disjoint semi-closed sets of Y. By the semi-normality of Y, there exist disjoint $U, V \in \mathrm{SO}(Y)$ such that

$f(A) \subset U$ and $f(B) \subset V$. Since f is pre sg-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint sg-open sets containing A and B, respectively. It follows from Theorem 1 that X is semi-normal.

COROLLARY 4 (Arya and Bhamini [1]). *The inverse image of a semi-normal space under an irresolute presemiclosed injection is semi-normal.*

5. Pre sg-closed functions

In this section we introduce a new class of functions called pre sg-closed functions

DEFINITION 8. A function $f: X \to Y$ is said to be *pre* sg-*closed* (resp. *sg-closed* [5]) if $f(F)$ is sg-closed in Y for every semi-closed (resp. closed) set F of X.

By definition 3 and 8, we easily obtain the following diagram:

$$\begin{array}{ccccc} & & \text{presemiclosed} & \Longrightarrow & \text{pre } sg\text{-closed} \\ & & \Downarrow & & \Downarrow \\ \text{closed} & \Longrightarrow & \text{semi-closed} & \Longrightarrow & sg\text{-closed} \end{array}$$

Diagram II

REMARK 2. By the two examples stated below, we obtain the following properties:

(a) none of the implications in Diagram II are reversible;
(b) a continuous closed open surjection need not be pre sg-closed;
(c) closedness and pre sg-closedness are independent of each other;
(d) semi-closedness and pre sg-closedness are independent of each other.

EXAMPLE 4. Let $f: (X, \tau) \to (X, \sigma)$ be the same function as in Example 2. Then f is pre sg-closed but it is not semi-closed. Moreover, f^{-1} is presemiclosed but it is not closed.

EXAMPLE 5. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}$, $Y = \{a, b, c\}$, and $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = f(d) = a$, $f(b) = b$, and $f(c) = c$. Then f is a continuous closed open surjection. However, f is not pre sg-closed since $\{a\} \in \mathrm{SC}(X, \tau)$ and $f(\{a\})$ is not sg-closed in (Y, σ).

PROPOSITION 2. *If $f: X \to Y$ is an irresolute pre* sg-*closed function and A is a* sg-*closed set of X, then $f(A)$ is* sg-*closed in Y.*

PROOF. Let A be a sg-closed set of X and $V \in \mathrm{SO}(Y)$ containing $f(A)$. Since f is irresolute, we have $A \subset f^{-1}(V) \in \mathrm{SO}(X)$ and hence $\mathrm{sCl}(A) \subset f^{-1}(V)$. Since f is pre sg-closed and $\mathrm{sCl}(A) \in \mathrm{SC}(X)$, $f(\mathrm{sCl}(A))$ is

sg-closed in Y and $f(\mathrm{sCl}(A)) \subset V$. Therefore, we obtain $\mathrm{sCl}(f(A)) \subset \subset \mathrm{sCl}\Big(f(\mathrm{sCl}(A))\Big) \subset V$. This shows that $f(A)$ is sg-closed in Y.

PROPOSITION 3. *A surjective function $f\colon X \to Y$ is pre* sg*-closed if and only if for each subset B of Y and each $U \in \mathrm{SO}(X)$ containing $f^{-1}(B)$, there exists a* sg*-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

PROOF. *Necessity*. Suppose that f is pre sg-closed. Let B be any subset of Y and $U \in \mathrm{SO}(X)$ containing $f^{-1}(B)$. Put $V = Y - f(X - U)$. Then, V is sg-open in $Y, B \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any semi-closed set of X. Put $B = Y - f(F)$, then we have $f^{-1}(B) \subset X - F \in \mathrm{SO}(X)$. There exists a sg-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is sg-closed in Y. This shows that f is pre sg-closed.

In Example 5, (X,τ) is semi-normal, (Y,σ) is not semi-normal, and $f\colon(X,\tau) \to (Y,\sigma)$ is a closed irresolute surjection. Therefore, semi-normality is not preserved under closed irresolute surjections.

THEOREM4. *If $f\colon X \to Y$ is a pre* sg*-closed irresolute surjection and X is a semi-normal space, then Y is semi-normal.*

PROOF. Let F and K be any pair of disjoint semi-closed sets of Y. Since f is irresolute, $f^{-1}(F)$ and $f^{-1}(K)$ are disjoint semi-closed sets of X. By the semi-normality of X, there exist $U, V \in \mathrm{SO}(X)$ such that $f^{-1}(F) \subset U$, $f^{-1}(K) \subset V$, and $U \cap V = \emptyset$. By Proposition 3, there exist sg-open sets G and H such that $F \subset G$, $K \subset H$, $f^{-1}(G) \subset U$, and $f^{-1}(H) \subset V$. Since f is surjective and $U \cap V = \emptyset$, we have $G \cap H = \emptyset$. It follows from Theorem 1 that Y is semi-normal.

COROLLARY 5 (Arya and Bhamini [1]). *Semi-normality is preserved under presemiclosed irresolute surjections.*

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NOTE ON THE LOWER ESTIMATE OF OPTIMAL LEBESGUE CONSTANTS

R. GÜNTTNER (Osnabrück)

1. Introduction. Let $X\colon -1 \leqq x_{n-1} < x_{n-2} < \ldots < x_0 \leqq 1$ denote an array of n arbitrary points in the interval $[-1,1]$. Given the values of some function f at these points, it is well known that there exists a unique polynomial $P_{n-1}[f](X,x)$ of degree at most $n-1$ such that $P_{n-1}[f](X,x_k) = f(x_k)$, $k=0,1,2,\ldots,n-1$. We may write this interpolating polynomial in the Lagrange form

$$P_{n-1}[f](X,x) = \sum_{k=0}^{n-1} f(x_k)\cdot l_k(X,x), \quad \text{where} \quad l_k(X,x) = \prod_{\substack{i=0\\ i\neq k}}^{n-1} \frac{x-x_i}{x_k-x_i}.$$

The Lebesgue function

$$L_{n-1}(X,x) := \sum_{k=0}^{n-1} \left| l_k(X,x)\right|$$

and the Lebesgue constant

$$\Lambda_{n-1}(X) := \max_{-1\leqq x\leqq 1} L_{n-1}(X,x)$$

are of central importance in the theory of interpolation.

It is known by Luttman and Rivlin [8] that the Lebesgue function $L_{n-1}(X,x)$ on each of the intervals $(x_0,1),(x_1,x_0),(x_2,x_1),\ldots$ is a polynomial, which we denote by $L_{n-1}^{(0)}(X,x), L_{n-1}^{(1)}(X,x), L_{n-1}^{(2)}(X,x),\ldots$, possessing there a single maximum $\Lambda_{n-1}^{(0)}(X),\Lambda_{n-1}^{(1)}(X),\Lambda_{n-1}^{(2)}(X),\ldots$ $(n>2)$, respectively. Let further

$$\underline{\Lambda}_{n-1}(X) := \min\left\{\Lambda_{n-1}^{(0)}(X),\Lambda_{n-1}^{(1)}(X),\ldots,\Lambda_{n-1}^{(n)}(X)\right\},$$

$$\overline{\Lambda}_{n-1}(X) := \max\left\{\Lambda_{n-1}^{(0)}(X),\Lambda_{n-1}^{(1)}(X),\ldots,\Lambda_{n-1}^{(n)}(X)\right\}.$$

Of course we have $\overline{\Lambda}_{n-1}(X) = \Lambda_{n-1}(X)$.

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It is an open question to get the exact value of the optimal Lebesgue constants

$$\Lambda^*_{n-1} := \min_X \Lambda_{n-1}(X).$$

By the Erdős conjecture, now verified by Kilgore [7], de Boor and Pinkus [1], we have

$$\Lambda^*_{n-1} \geqq \underline{\Lambda}_{n-1}(X) \tag{1}$$

for arbitrary X.

2. The statement. In [10] Vértesi proved the famous result

$$O\left(\frac{1}{(\log n)^{\frac{1}{3}}}\right) > \Lambda^*_{n-1} - \frac{2}{\pi}\log n - \chi > \begin{cases} \frac{\pi}{18n^2} + O\left(\frac{1}{n^4}\right), & n \text{ even}, \\ -\frac{2}{\pi n} + O\left(\frac{1}{n^2}\right), & n \text{ odd}, \end{cases} \tag{2}$$

where the constant χ is defined by

$$\chi = \frac{2}{\pi}\left(\gamma + \log\frac{4}{\pi}\right) = 0.5212\ldots, \tag{3}$$

($\gamma = 0.5772\ldots$ Euler's constant).

In [11] the upper estimate was improved to $O\big((\log\log n/\log n)^2\big)$. For a general survey on this topic see [12].

In this note we focus attention on the lower estimate in (2). As it was already obtained in [5, p.513] we have

$$\Lambda^*_{n-1} - \frac{2}{\pi}\log n - \chi > 0 \qquad (n = 1, 2, 3, \ldots).$$

In view of (1) we have $\Lambda^*_{n-1} \geqq \underline{\Lambda}_{n-1}(T)$. Brutman [3] proved that $\underline{\Lambda}_{n-1}(T) = \Lambda_{m-1}(T)$, $m = \frac{n}{2}$, which of course means that (n even)

$$\Lambda^*_{n-1} \geqq \Lambda_{m-1}(T), \qquad m = \frac{n}{2}. \tag{4}$$

From [6] (theorem 1) we know

(5)

$$\Lambda_{m-1}(T) > \frac{2}{\pi}\log m + \frac{2}{\pi}\left(\gamma + \log\frac{8}{\pi}\right) + \frac{\pi}{72m^2} - \frac{49\pi^3}{172800m^4}, \quad m = 1, 2, 3, \ldots$$

If n is even we easily get from (3)–(5) the result cited in the following theorem. The purpose of this note is to prove this result if n is odd, i.e. to verify the following

THEOREM.

$$\Lambda_{n-1}^{*} - \frac{2}{\pi}\log n - \chi > \frac{\pi}{18n^2} - \frac{49\pi^3}{10800n^4}.$$

3. Proof. Let T denote the zeros of the Chebysev polynomials, thus

$$T: \qquad x_k = \cos\frac{(2k+1)\pi}{2n}, \qquad k = 0, 1, \ldots, n-1.$$

In view of (1) and our preliminary remarks it is sufficient to prove that

$$\underline{\Lambda}_{n-1}(T) > \frac{2}{\pi}\log n + \chi + \frac{\pi}{18n^2} - \frac{49\pi^3}{10800n^4}, \quad n \text{ odd}. \tag{6}$$

From [5] (Section 4) we derive

(7)

$$\underline{\Lambda}_{n-1}(T) = \Lambda_{n-1}^{\lfloor\frac{n}{2}\rfloor}(T) \geqq \Lambda_{n-1}(T) - 2\cdot\left(\sum_{k=n-\lfloor\frac{n}{2}\rfloor}^{n-1} + \sum_{k=\lfloor\frac{n}{2}\rfloor}^{2\lfloor\frac{n}{2}\rfloor-1}\right)\frac{1}{2n}\cdot\cot\frac{(2k+1)\pi}{4n}.$$

If n is odd, then $\lfloor\frac{n}{2}\rfloor = \frac{n-1}{2}$; therefore by (7) we have to deal with

$$\frac{1}{n}\cdot\left(\sum_{k=\frac{n+1}{2}}^{n-1} + \sum_{k=\frac{n-1}{2}}^{n-2}\right)\cot\frac{(2k+1)\pi}{4n} = \tag{8}$$

$$= \frac{4}{\pi}\cdot\left\{\frac{\pi}{2n}\cdot\left[\frac{1}{2}\cot\frac{n\pi}{4n} + \cot\frac{(n+2)\pi}{4n} + \cot\frac{(n+4)\pi}{4n} + \ldots + \right.\right.$$
$$\left.\left. + \cot\frac{(2n-3)\pi}{4n} + \frac{1}{2}\cot\frac{(2n-1)\pi}{4n}\right]\right\}.$$

We make use of the well known trapezoidal rule

$$Q^{\mathrm{tr}}(f) = h\cdot\left[\frac{1}{2}f(a) + f(a+h) + f(a+2h) + \ldots + f(b-h) + \frac{1}{2}f(b)\right] \tag{9}$$

and the error estimate (cf. [2], p.176)

$$Q^{\mathrm{tr}}(f) = \int_a^b f(x)\,dx + \frac{1}{12}h^2\left[f'(b) - f'(a)\right] - \frac{b-a}{720}h^4\cdot f^{(4)}(\xi), \tag{10}$$

(assuming $f^{(4)}$ to be continuous on $[a,b]$). Considering $f(x) = \cot x$ on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ we know that $f \geqq 0$ and $f' = -(1+f^2) \leqq 0$; by further differentiation of f' it can be seen that $f^{(2)} \geqq 0$, $f^{(3)} \leqq 0$, $f^{(4)} \geqq 0$. From (7)–(10) this yields

$$\underline{\Lambda}_{n-1}(T) \geqq$$

$$\geqq \Lambda_{n-1}(T) - \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{\pi}{4n}} \cot x\, dx - \frac{\pi}{12n^2}\cdot\left[\cot'\left(\frac{\pi}{4}-\frac{\pi}{4n}\right) - \cot'\left(\frac{\pi}{4}\right)\right] =$$

$$= \Lambda_{n-1}(T) - \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx + \frac{4}{\pi}\int_{\frac{\pi}{2}-\frac{\pi}{4n}}^{\frac{\pi}{2}} \cot x\, dx -$$

$$-\frac{\pi}{12n^2}\cdot\left[\frac{-1}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{4n}\right)} + \frac{1}{\sin^2\frac{\pi}{4}}\right] =$$

$$= \Lambda_{n-1}(T) - \frac{2}{\pi}\log 2 + \frac{4}{\pi}\int_0^{\frac{\pi}{4n}} \tan x\, dx + \frac{\pi}{12n^2}\cdot\left[\frac{1}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{4n}\right)} - \frac{1}{\sin^2\frac{\pi}{4}}\right] >$$

$$> \Lambda_{n-1}(T) - \frac{2}{\pi}\log 2 + \frac{4}{\pi}\int_0^{\frac{\pi}{4n}} x\, dx + \frac{\pi}{12n^2}\cdot(1-2) =$$

$$= \Lambda_{n-1}(T) - \frac{2}{\pi}\log 2 + \frac{\pi}{8n^2} - \frac{\pi}{12n^2}.$$

We deduce from this

$$\underline{\Lambda}_{n-1}(T) > \Lambda_{n-1}(T) - \frac{2}{\pi}\log 2 + \frac{\pi}{24n^2}. \tag{11}$$

Now (11), (5) and (3) yield (6).

4. Remarks. The theorem was proved independently by W. Stolzmann [9] developing and using an asymptotic expansion of

$$S_a := \frac{1}{n}\sum_{k=0}^{a-1} \cot\frac{(2k+1)\pi}{4n} \qquad (a \in \mathbf{N},\ 1 \leqq a \leqq n).$$

The proof is somewhat lengthy and cannot be cited here.

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ON APPROXIMATE PEANO DERIVATIVES

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1. Introduction

In [1], the authors attempted to explain why many classes of functions behave like derivatives. They introduced a new class of functions which they denoted by $[\Delta']$ and they showed that $f \in [\Delta']$ if and only if there are differentiable functions g, h and k so that $f = g' + hk'$. The latter decomposition shows how far a function $f \in [\Delta']$ is from being a derivative. In the same paper, the authors showed that the class of all approximately continuous functions, the class of all approximate derivatives and the so-called B_1^* class are subclasses of $[\Delta']$. In [7] the present author showed that the class of all Peano derivatives is also a subclass of $[\Delta']$. The first goal of this paper is to show that approximate Peano derivatives are in $[\Delta']$. An immediate consequence is that approximate Peano derivatives are Baire 1 functions. This result was originally proved by M. Evans in [5]. His proof is very complicated as are the other early investigations of approximate Peano derivatives. (For example see [2].)

It is also shown that a k-th approximate Peano derivative is a composite derivative of the corresponding $k-1$-st approximate Peano derivative. As a consequence it is shown that a k-th approximate Peano derivative is a path derivative of the $k-1$-st approximate Peano derivative with a nonporous system of paths satisfying the I.C. condition as defined in [4]. This result is obtained without using any of the known properties of k-th approximate derivatives. In [4] it is shown that a path derivative for a nonporous system of paths satisfying the I.C. condition has all of the known properties of a k-th approximate Peano derivative. Consequently the results in this paper constitute a fresh, new approach to obtaining the basic properties of k-th approximate derivatives thus avoiding the complicated approach alluded to above. Finally it is shown that the system of paths mentioned above actually can be modified to have the I.I.C. condition. Again applying a result from [4] it follows that a k-th approximate derivative is a selective derivative of the $k-1$-st approximate Peano derivative.

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For definitions and properties of the class $[\Delta']$ and path derivatives, the reader is referred to [1] and [4] respectively. For the definition of composite derivatives the reader should consult [7].

2. Decomposition

Throughout the paper k is a fixed positive integer. In addition for $A \subset \mathbf{R}$ the closure of A is denoted by $\overline{A}$ and if A is Lebesgue measurable, then $m(A)$ denotes the Lebesgue measure of A. This section begins with the definition of k-th approximate Peano derivative. Then the major decomposition theorem, Theorem 2.6, is proved and several consequences are established.

DEFINITION 2.1. Let $f : \mathbf{R} \to \mathbf{R}$ and let $x \in \mathbf{R}$. Then f is k times approximately Peano differentiable at x means that there are numbers $f_1(x), \dots, f_k(x)$ and a set V_x of density 1 at x so that

$$f(x+t) = f(x) + tf_1(x) + \dots + \frac{t^k}{k!} f_k(x) + t^k \varepsilon_k(x,t)$$

where $\lim_{x+t \in V_x,\, t \to 0} \varepsilon_k(x,t) = 0$. The coefficient $f_k(x)$ is called the k-th approximate Peano derivative of f at x.

For $k = 1$ the above definition is just that of the classical approximate derivative. In [1] and [9] it is shown that approximate derivatives are in $[\Delta']$. For that reason throughout this section it is assumed that $k \geqq 2$. The next lemma is used to show that k-th approximate Peano derivatives are composite derivatives. The formula of the lemma is almost the same as the corresponding formula in Theorem 1 in [7]. As might be expected, the proof is also quite similar. Consequently the proof consists of a comment as to how to alter the proof of the theorem in [7] to fit the present circumstance.

LEMMA 2.2. *Let $f : \mathbf{R} \to \mathbf{R}$ and let $y, x \in \mathbf{R}$. Suppose f is k times approximately Peano differentiable at x and at y. Then for $t \neq 0$*

$$\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x) = \frac{t}{y-x} \frac{k-1}{2} \big(f_k(x) - f_k(y) \big) + \tag{1}$$

$$+ \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \frac{(y-x+jt)^k}{t^{k-1}(y-x)} \varepsilon_k(x, y-x+jt) -$$

$$- \frac{t}{y-x} \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(y, jt)$$

and

$$(2)\qquad f_k(y)-f_k(x)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(y-x+jt)^k}{t^k}\varepsilon_k(x,y-x+jt)-$$
$$-\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^k\varepsilon_k(y,jt).$$

PROOF. The proof of (1) is the same as the proof of Theorem 1 in [7]. The proof of (2) is similar to the same proof with some modification of (1). Instead of using Δ_{k-1} as is done in the proof of Theorem 1 in [7], one should use Δ_k where $\Delta_k=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(y+jt)$. □

To establish that a k-th approximate derivative is a composite derivative, a decomposition of $\mathbf{R}$ is needed. Next those sets are introduced.

DEFINITION 2.3. Let $f:\mathbf{R}\to\mathbf{R}$. Suppose f is k times approximately Peano differentiable at each $x\in\mathbf{R}$. For each $x\in\mathbf{R}$ set $A(x)=\{x+ +t:\sum_{j=0}^{k}|\varepsilon_k(x,jt)|\leqq 1\}$ and let

$$H_n=\Big\{x:|f_k(x)|\leqq n \text{ and } m(A(x)\cap I)>\frac{3}{4}m(I)$$
$$\forall \text{ interval } I \text{ containing } x \text{ with } m(I)<1/n\Big\}.$$

The statement of Theorem 2.6 is that under the assumptions of Definition 2.3 the derivative of f_{k-1} on $\overline{H_n}$, computed relative to $\overline{H_n}$ is f_k. According to the definition of composite derivatives this result means that f_k is the composite derivative of f_{k-1}.

The following convention and notation will be used in the remainder of this section. If f is k times approximately Peano differentiable at x, then for all t with $x+t\in V_x$ it is assumed that $|\varepsilon_k(x,t)|\leqq\frac{1}{k}$. Let γ be the characteristic function of the set V_x of Definition 2.1, and set $\Gamma(u)= =\int_0^u\gamma(x+t)\,dt$.

LEMMA 2.4. *Let $f:\mathbf{R}\to\mathbf{R}$ be k times approximately Peano differentiable at each $x\in\mathbf{R}$. Then $\bigcup_{n=1}^{\infty}H_n=\mathbf{R}$.*

PROOF. Let $x\in\mathbf{R}$. Let $I=[a,b]$ be an interval containing x. For $j= =1,2,\dots,k$ let $T_j=\{x+t\in[a,b]:|\varepsilon_k(x,jt)|\leqq\frac{1}{k}\}$. Then,

$$m(T_j)\geqq\int_{a-x}^{b-x}\gamma(x+jt)\,dt=\frac{1}{j}\big(\Gamma\big(j(b-x)\big)-\Gamma\big(j(a-x)\big)\big).$$

Let $n \in \mathbf{N}$ be such that $|f_k(x)| \leqq n$ and $|\Gamma(u)| > \left(1 - \frac{1}{4k}\right)|u|$ whenever $|u| < \frac{k}{n}$. If $m(I) < \frac{1}{n}$, then

$$m(T_j) \geqq \frac{1 - \frac{1}{4k}}{j}\big(j(b-x) + j(x-a)\big) = \left(1 - \frac{1}{4k}\right) m(I).$$

Setting $T = \bigcap_{j=1}^{k} T_j$ it follows that $m(T) \geqq \left(1 - k\frac{1}{4k}\right) m(I) = \frac{3}{4}m(I)$. Obviously $T \subset A(x) \cap I$. Thus $m\big(A(x) \cap I\big) > \frac{3}{4}m(I)$ whenever $m(I) < \frac{1}{n}$. Hence $x \in H_n$. $\square$

The next lemma and Lemma 2.4 to follow constitute the crux in the proof of Theorem 2.6.

LEMMA 2.5. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$, let $\varepsilon \in \left(0, \frac{1}{2k}\right)$ and let $x \in \mathbf{R}$. Then there is a $\delta > 0$ such that whenever $y \in H_n$, with $|y - x| < \delta$ there are t_1 and t_2 so that*

1) $y + jt_1 \in V_x$ for $j = 1, \dots, k$, $y + t_1 \in A(y)$ and $\varepsilon^2 < |t_1|/|y - x| < \varepsilon$,

2) $y + jt_2 \in V_x$ for $j = 1, \dots, k$, $y + t_2 \in A(y)$ and $1 > |t_2|/|y - x| > 1/4$.

PROOF. It is enough to consider the case when $y > x$. For $j = 1, 2, \dots, k$ let $T_j = \left\{t_1 \in \left[\varepsilon^2|y-x|, \varepsilon|y-x|\right] : y + jt_1 \in V_x\right\}$. Then

$$m(T_j) = \int_{\varepsilon^2|y-x|}^{\varepsilon|y-x|} \gamma(y + jt)\,dt = \int_{\varepsilon^2|y-x|}^{\varepsilon|y-x|} \gamma(x + y - x + jt)\,dt =$$

$$= \frac{1}{j}\int_{(1+j\varepsilon^2)|y-x|}^{(1+j\varepsilon)|y-x|} \gamma(x+u)\,du = \frac{1}{j}\left(\Gamma\big((1+j\varepsilon)|y-x|\big) - \Gamma\big((1+j\varepsilon^2)|y-x|\big)\right).$$

Now let $\lambda \in \left(0, \frac{\varepsilon - \varepsilon^2}{1 + k\varepsilon}\right)$ and let $\delta_1 \in \left(0, \frac{1}{n}\right)$ be such that $|\Gamma(u)| > (1 - \lambda\varepsilon)|u|$ whenever $|u| < 2\delta_1$. If $|y - x| < \delta_1$, then

$$m(T_j) > \frac{1}{j}\big((1-\lambda\varepsilon)(1+j\varepsilon)|y-x| - (1+j\varepsilon^2)|y-x|\big) =$$

$$= (\varepsilon - \varepsilon^2)|y-x| - \frac{\lambda\varepsilon(1+j\varepsilon)}{j}|y-x| >$$

$$> (\varepsilon - \varepsilon^2)|y-x| - \frac{\varepsilon - \varepsilon^2}{1 + j\varepsilon}\varepsilon(1+j\varepsilon)|y-x| = (1-\varepsilon)(\varepsilon - \varepsilon^2)|y-x|.$$

Therefore if $T = \bigcap_{j=1}^{k} T_j$, then

$$m(T) > (1 - k\varepsilon)(\varepsilon - \varepsilon^2)|y-x| > \frac{\varepsilon - \varepsilon^2}{2}|y-x|.$$

If $y \in H_n$, $|y-x| < \delta_1$, then

$$m\left(A(y) \cap \left[y + \varepsilon^2|y-x|, y + \varepsilon|y-x|\right]\right) >$$

$$> \frac{3}{4}\varepsilon|y-x| - \varepsilon^2|y-x| > \frac{\varepsilon - \varepsilon^2}{2}|y-x|.$$

Therefore $(y+T) \cap A(y) \neq \emptyset$. Choose any t_1 so that $y + t_1 \in (y+T) \cap A(y)$. Then t_1 satisfies condition 1) of the lemma.

Turning to the proof of 2) set $C_j = \{t_2 \in [\frac{1}{4}|y-x|, |y-x|] : y + jt_2 \in V_x\}$ for each $j = 1, 2, \ldots, k$. By an argument similar to the argument above

$$m(C_j) = \frac{1}{j}\left(\Gamma((1+j)|y-x|) - \Gamma\left(\left(1 + \frac{j}{4}\right)|y-x|\right)\right).$$

Let $\lambda \in \left(0, \frac{3}{4(1+k)}\right)$ and let $\delta_2 \in \left(\frac{1}{n}, 0\right)$ be such that $|\Gamma(u)| > (1 - \lambda\varepsilon)|u|$ whenever $|u| < (1+k)\delta_2$. If $|y-x| < \delta_2$, then

$$m(C_j) > \frac{1}{j}\left((1-\lambda\varepsilon)(1+j)|y-x| - \left(1 + \frac{j}{4}\right)|y-x|\right) =$$

$$= \frac{3}{4}|y-x| - \frac{\lambda\varepsilon(1+j)}{j}|y-x| >$$

$$> \frac{3}{4}|y-x| - \frac{3}{4(1+j)}\varepsilon(1+j)|y-x| = \frac{3}{4}(1-\varepsilon)|y-x|.$$

Therefore if $C = \bigcap_{j=1}^{k} C_j$, then

$$m(C) > \frac{3}{4}(1-k\varepsilon)|y-x| > \frac{1}{2}\frac{3}{4}|y-x|.$$

If $y \in H_n$, $|y-x| < \delta_2$, then

$$m\left(A(y) \cap \left[y + \frac{1}{4}|y-x|, y + |y-x|\right]\right) > \frac{3}{4}|y-x| - \frac{1}{4}|y-x| > \frac{1}{2}\frac{3}{4}|y-x|.$$

Therefore the set $(y+C) \cap A(y) \neq \emptyset$. Choose any t_2 so that $y + t_2 \in (y + {} + C) \cap A(y)$. Then t_2 satisfies condition 2) of the lemma. Finally set $\delta = {} = \min(\delta_1, \delta_2)$. $\square$

THEOREM 2.6. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then there is a sequence of closed sets $\{E_n\}$ so that*

1) $\bigcup_{n=1}^{\infty} E_n = \mathbf{R}$,

2) f_{k-1} *is differentiable on* E_n, *with respect to* E_n *with the expected equality; namely* $f_{k-1}|'_{E_n}(x) = f_k(x)$ *for each* $n \in \mathbf{N}$.

PROOF. Let $E_n = \overline{H}_n$. By Lemma 2.4, $\bigcup_n E_n = \mathbf{R}$ establishing condition 1). It will be shown that the sets E_n also satisfy condition 2). Let $x \in E_n$, let $\varepsilon \in \left(0, \frac{1}{2k}\right)$ and let δ, t_1 and t_2 be as in Lemma 2.5. If $y \in H_n$ with $|y-x| < \delta$, then $y + t_2 \in A(y)$ and $y + jt_2 \in V_x$ for $j = 1, 2, \dots, k$. Therefore $\left|\varepsilon_k(y, jt_2)\right| \leqq 1$, $\left|\varepsilon_k(x, y-x+jt_2)\right| \leqq \frac{1}{k}$ and there is a positive constant N depending only on x so that $\left|f_k(y) - f_k(x)\right| \leqq N$. Also by the choice of t_2 it follows that $\frac{|t_2|}{|y-x|} < 1$ and $\frac{|y-x+jt_2|^k}{|y-x|^k} \leqq (1+k)^k$ for $j = 1, 2, \dots, k$. Now from formula 2) of Lemma 2.2, there is a positive constant L depending only on x so that if $y \in H_n$ with $|y-x| < \delta$, then $\left|\varepsilon_k(x, y-x)\right| \leqq L$. Since $\frac{|t_2|}{|y-x|} > \frac{1}{4}$, formula 1) of Lemma 2.2 implies that there is a positive constant M depending only on x so that if $y \in H_n$ with $|y-x| < \delta$, then $\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right| \leqq M$. Solving (1) of Lemma 2.2 for $\varepsilon_k(x, y-x)$ it follows that

$$\varepsilon_k(x, y-x) = (-1)^{k+1} \frac{t^{k-1}}{(y-x)^{k-1}} \left(\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right) +$$

$$+(-1)^k \frac{t^k}{(y-x)^k} \frac{(k-1)\left(f_k(x) - f_k(y)\right)}{2} +$$

$$+(-1)^k \sum_{j=1}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \frac{(y-x+jt)^k}{(y-x)^k} \varepsilon_k(x, y-x+jt) -$$

$$-(-1)^k \frac{t^k}{(y-x)^k} \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(y, jt).$$

Choose $\delta_1 > 0$ so that if $x + t \in V_x$ with $|t| < 2\delta_1$, then $|\varepsilon_k(x,t)| < \varepsilon^{2k}$. Set $\delta_2 = \min(\delta, \delta_1)$. If $y \in H_n$ with $|y-x| < \delta_2$, then the formula above and the choice of t_1 gives

$$\left|\varepsilon_k(x, y-x)\right| <$$

$$< M\varepsilon^{k-1} + \frac{(k-1)N}{2}\varepsilon^k + \sum_{j=1}^{k-1} \binom{k-1}{j} (1+j\varepsilon)^k \varepsilon^{2k} + \sum_{j=1}^{k-1} \binom{k-1}{j} j^k \varepsilon^k =$$

$$= K\varepsilon^{k-1}.$$

In the above formula K is a constant that depends only on k and x. Since $k \geqq 2$, $\lim_{y\in H_n, y\to x} \varepsilon_k(x, y-x) = 0$. Choose $\delta_3 > 0$ so that if $y \in H_n$ with $|y-x| < \delta_3$, then $|\varepsilon_k(x, y-x)| < \varepsilon^{2k}$. Let $\delta_4 = \min(\delta_2, \delta_3)$. By formula (1) of Lemma 2.2, if $y \in H_n$ with $|y-x| < \delta_4$, then

$$\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right| \leqq \tag{3}$$

$$\leqq \varepsilon\frac{(k-1)N}{2} + \sum_{j=0}^{k-1}\binom{k-1}{j}(1+j\varepsilon)^k\frac{\varepsilon^{2k}}{\varepsilon^{2(k-1)}} + \sum_{j=0}^{k-1}\binom{k-1}{j} j^k\varepsilon = R\varepsilon.$$

Since the constant R in (3) depends only on k and x,

$$\lim_{y\in H_n,\, y\to x} \frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} = f_k(x). \tag{4}$$

To complete the proof let $x \in E_n$ and let $\{x_j\}$ be a sequence in H_n such that $x_j \to x$. By what was just proved, for each $j \in \mathbf{N}$ there is a $y_j \in H_n$ such that

$$\left|\frac{f_{k-1}(y_j) - f_{k-1}(x_j)}{y_j - x_j} - f_k(x_j)\right| \leqq 1. \tag{5}$$

In addition assume that $|y_j - x_j| \leqq \frac{1}{j}|x_j - x|$. Since $x_j \in H_n$, $|f_k(x_j)| \leqq n$ for every $j \in \mathbf{N}$. Consequently (5) implies that

$$\left|\frac{f_{k-1}(y_j) - f_{k-1}(x_j)}{y_j - x_j}\right| \leqq n+1. \tag{6}$$

Now

$$\frac{f_{k-1}(x_j) - f_{k-1}(x)}{x_j - x} - f_k(x) = \frac{f_{k-1}(x_j) - f_{k-1}(y_j)}{x_j - y_j}\,\frac{x_j - y_j}{x_j - x} +$$

$$+ \left\{\frac{f_{k-1}(y_j) - f_{k-1}(x)}{y_j - x} - f_k(x)\right\}\frac{y_j - x}{x_j - x} - f_k(x)\frac{x_j - y_j}{x_j - x}.$$

So by (6)

$$\left|\frac{f_{k-1}(x_j) - f_{k-1}(x)}{x_j - x} - f_k(x)\right| \leqq \tag{7}$$

$$\leqq (n+1)\frac{1}{j} + \left|\frac{f_{k-1}(y_j) - f_{k-1}(x)}{y_j - x} - f_k(x)\right|\left(1 + \frac{1}{j}\right) + n\frac{1}{j}.$$

Finally since $\lim_{j\to\infty} x_j = x$, $\lim_{j\to\infty} y_j = x$. Since $y_j \in H_n$ for each $j \in \mathbf{N}$, (4) implies

$$\lim_{j\to\infty} \frac{f_{k-1}(y_j) - f_{k-1}(x)}{y_j - x} = f_k(x). \tag{8}$$

Therefore by (7) and (8)

$$\lim_{j\to\infty} \frac{f_{k-1}(x_j) - f_{k-1}(x)}{x_j - x} = f_k(x). \quad \square$$

COROLLARY 2.7. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then f_k is a composite derivative of f_{k-1}.*

It is well known that if a function g, defined on a closed set A, is differentiable relative to A, then there is a differentiable function G on $\mathbf{R}$ with $G = g$ and $G' = g'$ on A. (See Mařík [12].) This result yields the following corollary to Theorem 2.6.

CORLOOARY 2.8. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then $f_k \in [\Delta']$.*

The next corollary follows from the fact that every function in $[\Delta']$ is a Baire 1 function.

COROLLARY 2.9. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then f_k is a Baire 1 function.*

Corollary 2.9 was first proved by M. Evans in [5] using a long and complicated proof. The proof presented here is much shorter, but does require the work done in [1].

3. Approximate Peano derivatives and path derivatives

This section presents a new development of the basic properties of approximate Peano derivatives. It has already been proved in Corollary 2.9 that they are Baire 1 functions. Here using the notion of path derivatives as developed in [4] and the results established in that paper, it is shown that approximate Peano derivatives have the Darboux property, the Denjoy property and, when bounded either above or below on an interval, are ordinary derivatives on that interval. In addition the so-called $-M, M$ property introduced in [10] is also established for approximate Peano derivative. The sets E_n from Theorem 2.6 are employed to construct the paths that are used.

LEMMA 3.1. *Let $l \in \mathbf{N}$ with $l \leqq k-1$. Assume that if I is an interval and if $g : I \to \mathbf{R}$ is l times approximately Peano differentiable on I, then g_l has* *the Darboux property. Moreover if* $g_l \geqq 0$ *on* I, *then* $g_l = g^{(l)}$ *on* I. *Suppose* $f : \mathbf{R} \to \mathbf{R}$ *is* k *times approximately Peano differentiable at each* $x \in \mathbf{R}$. *Fix* $x \in \mathbf{R}$ *and for each* $y \in \mathbf{R}$ *set* $P(y) = \sum_{i=0}^{k} (y-x)^i \frac{f_i(x)}{i!}$. *Let* $\varepsilon > 0$, $\eta > 0$. *Then there is a* $\delta > 0$ *such that if* I *is a subinterval of* $(x-\delta, x+\delta)$, j *an integer with* $0 < j < k$ *and* $|f_j(y) - P^{(j)}(y)| \geqq \varepsilon |y-x|^{k-j}$ *on* I, *then* $m(I) \leqq \eta \operatorname{dist}(x, I)$. *(Here* $\operatorname{dist}(x,I)$ *is the distance from* x *to* I.*)*

PROOF. The proof of this assertion parallels the proof of Theorem 3 in [11]. The difference is that the use of Lemma 4 in [11] is replaced here by the assumption concerning the general function g. □

We need Lemma 3.1 to state and prove the following result.

LEMMA 3.2. *Under the assumptions of Lemma* 3.1, *for each point* $x \in \mathbf{R}$ *there is a nonporous path* E_x *leading to* x *so that*

$$\lim_{y \in E_x,\, y \to x} \frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} = f_k(x).$$

PROOF. The assertion of Lemma 3.2 follows directly from Lemma 3.1 with $j = k-1$ and Lemma 3.6.1 in [4]. □

THEOREM 3.3. *Let* $l \in \mathbf{N}$ *with* $l \leqq k-1$. *Assume that if* I *is an interval and if* $g : I \to \mathbf{R}$ *is* l *times approximately Peano differentiable on* I, *then* g_l *has the Darboux property. Moreover if* $g_l \geqq 0$ *on* I, *then* $g_l = g^{(l)}$ *on* I. *Suppose* $f : \mathbf{R} \to \mathbf{R}$ *is* k *times approximately Peano differentiable at each* $x \in \mathbf{R}$. *Then there is a bilateral nonporous system of paths* $E = \{E_x : x \in \mathbf{R}\}$ *satisfying the I.C. intersection condition such that* f_k *is the* E*-derivative of* f_{k-1}.

PROOF. For each $x \in \mathbf{R}$ let E'_x be a path satisfying the conclusions of Lemma 3.2. For $x \in \mathbf{R}$ let $E_x = E'_x \cup E_n$ where $n \in \mathbf{N}$ is such that $x \in E_n$. That E is nonporous (therefore bilateral) follows directly from Lemma 3.2. Also Lemma 3.2 and Theorem 2.6 imply that f_{k-1} is E differentiable with $f_{k-1}|'_E(x) = f_k(x)$ for every $x \in \mathbf{R}$. It remains only to prove that E satisfies the I.C. intersection condition. In fact it is shown that $E_x \cap E_y \cap [x,y] \neq \emptyset$ for any two distinct points x and y; a condition stronger than the I.C. condition.

Let x and y be any two distinct points and let $n, m \in \mathbf{N}$ be such that $x \in E_n$ and $y \in E_m$. If $n \leqq m$, then $E_m \supset E_n$ and hence $y \in E_x$. If $n \geqq m$, then $E_n \supset E_m$ and hence $x \in E_y$. Therefore $E_x \cap E_y \cap [x,y] \neq \emptyset$. Thus E satisfies the I.C. condition. □

Next it is shown that the assumption concerning the general function g of Theorem 3.3 can be dropped. To accomplish this goal the results from [4] are used extensively. In particular those dealing with properties of path derivatives that are Baire 1 functions and with a nonporous system of paths satisfying the I.C. condition are employed.

THEOREM 3.4. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then there is a bilateral nonporous system of paths $E = \{E_x : x \in \mathbf{R}\}$ satisfying the I.C. intersection condition such that f_k is the E-derivative of f_{k-1}.*

PROOF. The proof is by induction on k. It is well known and easy to see that an approximate derivative is the path derivative of its primitive for the system of paths consisting of the sets of density one through which the approximate derivative is computed. Consequently the assertion is true for $k = 1$.

Suppose the assertion of the theorem is true for every $1 \leqq j \leqq k-1$, and every function h, defined on some closed interval J, which is j times approximately Peano differentiable on J. (Note that this seemingly stronger induction hypothesis is justified because any such h can always be extended to $\mathbf{R}$ so that h_j exists on $\mathbf{R}$. For example if $J = [a, b]$, then set

$$h(y) = \sum_{i=0}^{j} (y-x)^i \frac{f_i(a)}{i!} \quad \text{for} \quad y \in (-\infty, a)$$

and

$$h(y) = \sum_{i=0}^{j} (y-x)^i \frac{f_i(b)}{i!} \quad \text{for} \quad y \in (b, \infty).)$$

Let $1 \leqq l \leqq k-1$ and let a function g, defined on some closed interval I, have an l-th approximate Peano derivative on I. By Corollary 2.9, g_l is a Baire 1 function. By the induction hypothesis and Theorem 6.4 from [4], g_l has the Darboux property. Suppose that $g_l \geqq 0$ on I. Again by the induction hypothesis but now using Theorem 4.7.1 of [4], g_{l-1} is nondecreasing on I. By Theorem 4.4.3 from [4] $g'_{l-1} = g_l$ on I. Also there is an $\alpha \in \mathbf{R}$ such that $g_{l-1} - \alpha \geqq 0$ on I. Let $h(x) = g(x) - \alpha \frac{x^{l-1}}{(l-1)!}$. Then $h_{l-1} = g_{l-1} - \alpha$ and hence $h_{l-1} \geqq 0$ on I. Proceeding as before $h'_{l-2} = h_{l-1}$ on I. This implies $g'_{l-2} = g_{l-1}$ on I. Continuing in this fashion one can deduce that $g^{(l)}$ exists on I. Now apply Theorem 3.3. $\square$

COROLLARY 3.5. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then f_k has the Darboux property.*

PROOF. The assertion follows directly from Theorem 3.4 and Theorem 6.4 in [4]. $\square$

COROLLARY 3.6. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Let $[a, b]$ be an interval, and $\alpha \in \mathbf{R}$. If $f_k \geqq \alpha$ (or $f_k \leqq \alpha$) on $[a, b]$, then*

a) $f_{k-1}(x) - \alpha x$ $\left(\alpha x - f_{k-1}(x)\right)$ *is nondecreasing and continuous on*

$[a, b]$, *and*

b) $f^{(k)}$ *exists and* $f^{(k)} = f_k$ *on* $[a, b]$.

PROOF. The assertion follows directly from Theorem 3.4, and Theorems 6.6.1 and 4.4.3 of [4]. □

COROLLARY 3.7. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be* k *times approximately Peano differentiable at each* $x \in \mathbf{R}$. *Then* f_k *has the Denjoy property.*

PROOF. The assertion follows directly from Theorem 3.4, Theorems 6.6.1 and 6.7 from [4] and Corollary 3.5. □

An immediate consequence of Theorem 3.4 and Theorem 8.1 of [4] is the following corollary.

COROLLARY 3.8. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be* k *times approximately Peano differentiable at each* $x \in \mathbf{R}$ *and let* $M > 0$. *If* f_k *attains both* M *and* $-M$ *on an interval,* I_0, *then there is a subinterval* I *of* I_0 *on which* $f_k = f^{(k)}$ *and* $f^{(k)}$ *attains both* M *and* $-M$ *on* I.

4. Approximate Peano derivatives and selective derivatives

The goal of the final section is to prove that every approximate Peano derivative is a selective derivative. This result is obtained by constructing a different system of paths along which the derivative of f_{k-1} is f_k. This system of paths will be shown to have the I.I.C. intersection property. Then a result from [4] will achieve the desired result.

First the system of paths is defined. For that purpose it is assumed that $f : \mathbf{R} \to \mathbf{R}$ is k times approximately Peano differentiable at each $x \in \mathbf{R}$. The desired system is constructed from the sequence of sets $\{E_n\}$ of Theorem 2.6 augmented by the sets introduced next. For each $y \in \mathbf{R}$ let P_y be a set containing y, having y as a bilateral point of accumulation of P_y and so that

$$\lim_{z \in P_y,\, z \to y} \frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} = f_k(y)$$

and

$$\left| \frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} - f_k(y) \right| \leqq 1 \text{ for every } z \in P_y. \tag{9}$$

Theorem 3.4 assures the existence of such a set P_y.

NOTATION. For $x, y \in \mathbf{R}$ let $\delta(x, y) = \min\{1, \frac{|y-x|}{2}\}$. For $x \in \mathbf{R}$ and $n \in \mathbf{N}$ let

$R_{x,n} = \bigcup \{ P_y \cap [y, y + \delta^2(x, y)) : y \in E_n$ and y is right isolated from E_m for all $m \in \mathbf{N} \}$

and let

$$L_{x,n} = \bigcup \{ P_y \cap (y - \delta^2(x,y), y] : y \in E_n \text{ and } y \text{ is left isolated from } E_m \text{ for all } m \in \mathbf{N} \}.$$

Next a specific set from the sequence $\{E_n\}$ is selected for each $x \in \mathbf{R}$ and the set E_x is defined depending on this choice. If there is an $n \in \mathbf{N}$ such that x is a bilateral accumulation point of E_n, then select M_x to be the smallest such n and let

$$E_x = E_{M_x} \cup R_{x,M_x} \cup L_{x,M_x}.$$

Suppose no such n exists. Assume there is an n such that x is a left accumulation point of E_n. Let M_x be the smallest such n and note that in this case x is right isolated from E_n for each $n \in \mathbf{N}$. Similarly if there is an n such that x is a right accumulation point of E_n, then select M_x to be the smallest such n and note that in this case x is left isolated from E_n for each $n \in \mathbf{N}$. Finally if x is an isolated point of E_n for every $n \in \mathbf{N}$, then let $M_x = 1$. In each of these three cases let

$$E_x = E_{M_x} \cup P_x \cup R_{x,M_x} \cup L_{x,M_x}.$$

Let E be the system of paths, E_x.

THEOREM 4.1. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then E is bilateral and satisfies the I.I.C. intersection condition.*

PROOF. Clearly E is bilateral. A slightly stronger condition than I.I.C. will actually be established; namely that $E_x \cap E_y \cap (x,y) \neq \emptyset$ for *any* two points $x, y \in \mathbf{R}$. Let $x < y$ and suppose first that $M_x \leqq M_y$. Then $E_{M_x} \subset$ $\subset E_{M_y}$ and consequently $x \in E_{M_y}$. If x is a right accumulation point of E_{M_x}, then $E_x \cap E_y \cap (x,y) \neq \emptyset$. If x is right isolated from E_{M_x}, then by the choice of M_x, x is right isolated from E_n for every $n \in \mathbf{N}$. Thus

$$\emptyset \neq P_x \cap \left[x, x + \delta^2(x,y)\right) \cap (x,y) \subset E_x \cap E_y \cap (x,y).$$

Now suppose $M_x > M_y$. Then $E_{M_y} \subset E_{M_x}$. If y is a left accumulation point of E_{M_y}, then $E_x \cap E_y \cap (x,y) \neq \emptyset$. If y is left isolated from E_{M_y}, then by an argument similar to that above $E_x \cap E_y \cap (x,y) \neq \emptyset$. Therefore E satisfies the I.I.C. condition. $\square$

THEOREM 4.2. *Suppose $f : \mathbf{R} \to \mathbf{R}$ is k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then f_{k-1} is E differentiable with $f'_{(k-1)E}(x) =$ $= f_k(x)$.*

PROOF. Let $x \in \mathbf{R}$ and $\varepsilon > 0$. Then there is an $\eta \in (0, \varepsilon)$ such that

$$\left| \frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x) \right| < \varepsilon \tag{10}$$

whenever $|y-x| < \eta$ and $y \in E_{M_x} \cup P_x$. Let $z \in E_x$ be such that $|z-x| < \frac{\eta}{2}$. If $z \in E_{M_x} \cup P_x$, then (10) holds with y replaced by z. So assume $z \in R_{x,M_x} \cup \cup L_{x,M_x}$. Then $z \in P_y$ for some $y \in E_{M_x}$ and y is left or right isolated from E_n for every $n \in \mathbf{N}$. Then $\frac{\eta}{2} > |z-x| \geqq |x-y| - |y-z| \geqq 2\delta(x,y) - \delta^2(x,y) \geqq \geqq \delta(x,y)$. Therefore $|y-x| \leqq |y-z| + |x-z| < \delta(x,y) + \eta/2 < \eta$. Hence

$$\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right| < \varepsilon.$$

Thus

$$\left|\frac{f_{k-1}(z) - f_{k-1}(x)}{z-x} - f_k(x)\right| = \left|\left(\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right)\frac{y-x}{z-x} + \right.$$

$$\left. + \left(\frac{f_{k-1}(z) - f_{k-1}(y)}{z-y} - f_k(y)\right)\frac{z-y}{z-x} + \frac{z-y}{z-x}\big(f_k(y) - f_k(x)\big)\right| \leqq$$

$$\leqq \left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y-x} - f_k(x)\right|\left|1 - \frac{z-y}{z-x}\right| +$$

$$+ \left|\frac{f_{k-1}(z) - f_{k-1}(y)}{z-y} - f_k(y)\right|\left|\frac{z-y}{z-x}\right| + \left|\frac{z-y}{z-x}\right|\big(|f_k(x)| + |f_k(y)|\big).$$

By (10), (9) and the relationship among x, y and z, the last two lines in the above estimate are no more than

$$\varepsilon\left(1 + \frac{\delta^2(x,y)}{\delta(x,y)}\right) + 1 \cdot \frac{\delta^2(x,y)}{\delta(x,y)} + \frac{\delta^2(x,y)}{\delta(x,y)}\big(|f_k(x)| + M_x\big) \leqq$$

$$\leqq 2\varepsilon + \delta(x,y)\big(1 + |f_k(x)| + M_x\big) \leqq 2\varepsilon + \frac{\varepsilon}{2}\big(1 + |f_k(x)| + M_x\big)$$

and since ε was arbitrary,

$$\lim_{z \in E_x,\, z \to x} \frac{f_{k-1}(z) - f_{k-1}(x)}{z-x} = f_k(x). \quad \square$$

COROLLARY 4.3. *Let $f : \mathbf{R} \to \mathbf{R}$ be k times approximately Peano differentiable at each $x \in \mathbf{R}$. Then f_k is a selective derivative of f_{k-1}.*

PROOF. The assertion follows immediately from Theorems 4.1 and 4.2 and Theorem 3.4 of [4]. $\square$

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UNIFORM EMBEDDING OF ABELIAN TOPOLOGICAL GROUPS IN EUCLIDEAN SPACES

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Introduction

LCA-groups are in this paper locally compact Abelian T_0-topological groups with countable bases. For each nonnegative integer m let $\mathbf{R}^m$ denote the m-dimensional Euclidean space.

In the papers [1], [2], [3], [4] the question of topological embeddability of any LCA-group in $\mathbf{R}^m$ was investigated and solved. However each Abelian topological group determines also uniquely a uniform space, the underlying (uniform) space of the topological group. $\mathbf{R}^m$ itself is a uniform space as well. Thus we can raise the question of the uniform embeddability of an arbitrary Abelian topological group in $\mathbf{R}^m$. This is the aim of the present paper.

1. Locally compact groups

We shall prove the following theorems.

THEOREM A. *Each n-dimensional LCA-group can be uniformly embedded in $\mathbf{R}^{n+2}$.*

THEOREM B. *Each n-dimensional LCA-group with locally connected components can be uniformly embedded in $\mathbf{R}^{n+1}$.*

THEOREM C. *An n-dimensional LCA-group can be uniformly embedded in $\mathbf{R}^n$ if and only if it is isomorphic to the vector group $\mathbf{R}^n$.*

THEOREM D. *An n-dimensional LCA-group without locally connected components cannot be uniformly embedded in $\mathbf{R}^{n+1}$.*

First we are going to prove Theorems A and B.

Let Γ be an n-dimensional LCA-group. Then there exists a subgroup Γ_1 in Γ satisfying the following conditions:

(a) the factor group Γ/Γ_1 is discrete,

(b) Γ_1 is the direct sum of a compact subgroup Γ_2 and a vector subgroup V (see [7] Ch. V. §35 E) and Theorem 41, pp.160–161).

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Since the subgroup V is divisible, it is an injective group and thus Γ is the direct sum of a subgroup Γ_3 and V, where $\Gamma_3 \cap \Gamma_1 = \Gamma_2$. Consequently Γ_3/Γ_2 is isomorphic to Γ/Γ_1 thus it is discrete and countable.

Let Γ_4 be the component of zero of the group Γ_3. Then Γ_4 is a subgroup of Γ_2. Let Γ_5 be the component of zero of the group Γ. We then clearly have $\Gamma_5 = \Gamma_4 + V$.

Suppose that V is a k-dimensional group. Then Γ_3 and Γ_2 are $(n-k)$-dimensional groups (see [3] 1.2 and 1.3).

Now suppose that there is a topological embedding $\varphi\colon \Gamma_2 \to \mathbf{R}^m$ in some Euclidean space $\mathbf{R}^m$, where $m \geqq 1$. Γ_2 is a compact group and thus φ is also a uniform embedding. Let $C = \varphi(0)$. $\varphi(\Gamma_2)$ is then contained in an open sphere $S(C, \varepsilon)$ in $\mathbf{R}^m$. Choose the points $C_1, \dots, C_r, \dots$ in $\mathbf{R}^m$ so that $C_1 = C$ and $d(C_i, C_j) > 3\varepsilon$ for $i \neq j$, where $d(C_i, C_j)$ is the distance between the points C_i and C_j.

If Γ_3/Γ_2 is finite then let $H_1, \dots, H_s$ and if Γ_3/Γ_2 is infinite then let $H_1, \dots, H_r, \dots$ be the cosets of Γ_3 modulo Γ_2, where in both cases $H_1 = \Gamma_2$. Choose a representative g_r from each coset H_r so that $g_1 = 0$.

Now for each coset H_r let $\psi_r\colon H_r \to \mathbf{R}^m$ be defined by the formula

$$\psi_r(g) = C_r - C_1 + \varphi(g - g_r).$$

Then $\psi_1 = \varphi$ and the map $\psi\colon \Gamma_3 \to \mathbf{R}^m$ defined by $\psi|_{H_r} = \psi_r$ $(r = 1, \dots, s$ or $r = 1, 2, \dots)$ is clearly a uniform embedding of Γ_3 in $\mathbf{R}^m$. Further for $h = g + v$ $(g \in \Gamma_3, v \in V)$ let

$$\zeta(h) = \psi(g) + v \in \mathbf{R}^m + V = \mathbf{R}^{m+k},$$

where $\mathbf{R}^m + V$ is the direct sum of the vector groups $\mathbf{R}^m$ and V. Then ζ is clearly a uniform embedding of Γ in $\mathbf{R}^{m+k}$.

Now according to [3] Theorem D (see also [1] and [2]) Γ_2 can be topologically embedded in $\mathbf{R}^{n-k+2}$. Hence by the preceding considerations Γ can be uniformly embedded in $\mathbf{R}^{n-k+2+k} = \mathbf{R}^{n+2}$. The proof of Theorem A is complete.

Suppose now that Γ_5 is locally connected. Since $\Gamma_5 = \Gamma_4 + V$, it follows that Γ_4 is locally connected as well. Hence by [3] Theorem B Γ_2 can be topologically embedded in $\mathbf{R}^{n-k+1}$. According to the preceding considerations we obtain that Γ can be uniformly embedded in $\mathbf{R}^{n-k+1+k} = \mathbf{R}^{n+1}$. The proof of Theorem B is complete as well.

We now prove Theorem C.

If Γ is an n-dimensional vector group then clearly it can be uniformly embedded in $\mathbf{R}^n$.

Now suppose that Γ is an n-dimensional LCA-group and it can be uniformly embedded in $\mathbf{R}^n$. Let $\zeta\colon \Gamma \to \mathbf{R}^n$ be a uniform embedding. Then ζ is also a topological embedding. Γ has a subspace homeomorphic to $\mathbf{R}^n$

(see [3] Lemma 1.1). Thus each element of Γ is contained in a subspace of Γ homeomorphic to $\mathbf{R}^n$. Consequently, according to Brouwer's theorem on invariance of domain it follows that $\xi(\Gamma)$ is open in $\mathbf{R}^n$. On the other hand by the local compactness of Γ it follows that Γ is a complete uniform space (see [5] 11.3.21 p.466). Hence $\zeta(\Gamma)$ is complete as well and thus $\zeta(\Gamma)$ is a closed subspace of $\mathbf{R}^n$ (see [5] 5.1.16 p.185).

Thus by the connectedness of $\mathbf{R}^n$ it follows that $\zeta(\Gamma)=\mathbf{R}^n$. Hence Γ is connected and locally connected. Consequently Γ is the direct sum of a k-dimensional $(0 \leqq k \leqq n)$ toroidal subgroup Γ_1 and an $(n-k)$-dimensional vector subgroup V (see [3] 1.2 and [7] Theorem 43. p.170). Thus Γ and Γ_1 are of the same homotopy type and since $\mathbf{R}^n$ is contractible to a point over itself so is Γ and Γ_1. This yields $k=0$ and thus $\Gamma = V$. Γ is isomorphic to $\mathbf{R}^n$ as required. The proof of Theorem C is complete.

Finally we prove Theorem D. If Γ is an n-dimensional LCA-group without locally connected components then by [3] Theorem A and [3] 1.3 Γ cannot be topologically embedded in $\mathbf{R}^{n+1}$. Hence it cannot be uniformly embedded in R^{n+1} indeed.

2. Abelian topological groups

Let Γ be an Abelian topological group and $\mathbf{R}^m$ a Euclidean m-space, where m is a nonnegative integer. Suppose the existence of a uniform embedding $\zeta\colon\Gamma \to \mathbf{R}^m$ of Γ in $\mathbf{R}^m$. Γ is then clearly a T_0-group with a countable base. Let $\widetilde{\Gamma}$ be the completion of Γ (see [6] 8.5.15 p.571 and [5] 11.3.d. pp.463–466). The closure $\overline{\zeta(\Gamma)}$ of $\zeta(\Gamma)$ in $\mathbf{R}^m$ is a complete subspace of $\mathbf{R}^m$ and $\zeta(\Gamma)$ is dense in $\overline{\zeta(\Gamma)}$. Thus the uniform isomorphism $\zeta\colon\Gamma \to \zeta(\Gamma)$ is extendable to a uniform isomorphism $\widetilde{\zeta}\colon\widetilde{\Gamma} \to \overline{\zeta(\Gamma)}$ (see [6] 8.3.11 p.549). However $\overline{\zeta(\Gamma)}$ is a closed subspace of the locally compact space $\mathbf{R}^m$, consequently $\overline{\zeta(\Gamma)}$ and so $\widetilde{\Gamma}$ are locally compact spaces and thus the 0-element of Γ has a precompact neighbourhood (see [5] 11.3.24 p.466).

Hence we have the following theorem.

THEOREM E. *If an Abelian topological group Γ can be uniformly embedded in $\mathbf{R}^m$ then it is a T_0-group with a countable base, its 0-element has a precompact neighbourhood, its completion $\widetilde{\Gamma}$ is an LCA-group and $\widetilde{\Gamma}$ can be uniformly embedded in $\mathbf{R}^m$ as well.*

Now suppose that Γ is an Abelian T_0-topological group with a countable base and the zero element of Γ has a precompact neighbourhood. Then the topological space Γ is separable and Γ is also a T_0-uniform space with a countable uniform base. Hence the uniform space Γ is metrizable (see [8] II.2.8 Satz 2. p.119) and so Γ is a metric space with a countable base. Consider now the completion $\widetilde{\Gamma}$ of Γ. Then $\widetilde{\Gamma}$ is a T_0-group as well with a countable base (see [5] 6.3.29 p.256, [6] 8.3.12 p.549 and [6] 4.3.19 p.340) and $\widetilde{\Gamma}$ is locally compact (see [5] 11.3.24 p.466).

We now define the *uniform dimension* $\dim\Gamma$ of Γ as the topological dimension of $\widetilde{\Gamma}$. Γ is said to be *m-dimensional* if $\dim\Gamma = m$. If Γ is locally compact then it is complete (see [5] 11.3.21. p.466) and thus in this case we have $\widetilde{\Gamma} = \Gamma$ and so the topological dimension of Γ coincides with its uniform dimension. However if Γ is not locally compact then its topological dimension may differ from its uniform one.

Now according to Theorems A, B, C, D and E we have the following theorems.

THEOREM F. *An Abelian topological group Γ can be uniformly embedded in some Euclidean space if and only if the following conditions are satisfied.*

(a) *Γ is a T_0-group with a countable base.*

(b) *The zero element of Γ has a precompact neighbourhood.*

(c) *The uniform dimension of Γ is finite.*

In what follows we shall say that an Abelian topological group Γ is *simple* if it satisfies the conditions (a), (b) and (c) of Theorem F.

THEOREM G. *Each n-dimensional simple Abelian topological group can be uniformly embedded in $\mathbf{R}^{n+2}$.*

THEOREM H. *A simple n-dimensional Abelian topological group can be uniformly embedded in $\mathbf{R}^{n+1}$ whenever the components of its completion are locally connected.*

THEOREM I. *A simple n-dimensional Abelian topological group can be uniformly embedded in $\mathbf{R}^{n}$ if and only if it is isomorphic to a dense subgroup of $\mathbf{R}^{n}$.*

THEOREM J. *A simple n-dimensional Abelian topological group cannot be uniformly embedded in $\mathbf{R}^{n+1}$ if the components of its completion are not locally connected.*

THEOREM K. *If $m < n$ then no simple n-dimensional Abelian topological group can be uniformly embedded in $\mathbf{R}^{m}$.*

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ON QUASI $\varphi(L)^*$-A.E. CONVERGENCE OF FOURIER SERIES OF FUNCTIONS IN ORLICZ SPACES

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§1. Introduction

Let f be a real valued integrable function defined on $\mathbf{T} = [-\pi, \pi]$ and $S_n(f;x)$ be the n-th partial sum of the Fourier series of f. The majorant function $S^*(f)$ is defined by

$$S^*(f)(x) := \sup\left\{\left|S_n(f;x)\right| : n \geqq 0\right\} \quad \text{for} \quad x \in \mathbf{T}. \tag{1.1}$$

In this paper $\|\cdot\|_p$ means the usual norm on $L^p(\mathbf{T})$ and meas(E) means the Lebesgue measure of the set $E \subset \mathbf{T}$. Hunt [1] proved the following theorem.

THEOREM 1.1. *When $1 < p < +\infty$, we get*

$$\left\|S^*(f)\right\|_p \leqq C_p\|f\|_p \quad \textit{for all} \quad f \in L^p(\mathbf{T}), \tag{1.2}$$

where C_p is a constant depending only on p and satisfies $C_p = O(p)$ as $p \to +\infty$. When $p = +\infty$, we get

$$\operatorname{meas}\left\{x \in \mathbf{T}: S^*(f)(x) > t\right\} \leqq C_1 \exp\left(-C_2 t/\|f\|_\infty\right) \textit{ for all } t > 0, \tag{1.3}$$

where C_1 and C_2 are absolute positive constants.

The following are proved in [3], [4].

THEOREM 1.2 (see [3]). *Let $\varphi(t) = e^t - 1$. If f is a continuous 2π-periodic function, then we get*

$$\int_{-\pi}^{\pi} \varphi\left(\alpha S^*(f)(x)\right) dx < +\infty \quad \textit{for all} \quad \alpha > 0,$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \varphi\left(\alpha\left|S_n(f;x) - f(x)\right|\right) dx = 0 \quad \textit{for all} \quad \alpha > 0.$$

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THEOREM 1.3 (see [4]). *Let* $\varphi(t) = e^t - 1$. *Then there exists a bounded function* f *such that* $\|f\|_\infty = 1$,

$$\int_{-\pi}^{\pi} \varphi\big(\alpha_0 S^*(f)(x)\big)\, dx = +\infty \tag{1.4}$$

and

$$\overline{\lim_{n\to\infty}} \int_{-\pi}^{\pi} \varphi\Big(\alpha_0 \big| S_n(f;x) - f(x)\big|\Big)\, dx = +\infty$$

for some positive constant α_0.

The aim of this paper is to consider the case of the Fourier series of functions in Orlicz spaces L^*_φ.

§2. Notations and definitions

Let φ be a continuous function defined on $[0,\infty)$ satisfying the following properties:

$$\begin{cases} \varphi(0) = 0, \quad \varphi(t) > 0 & \text{if } t > 0; \\ \varphi(t) \uparrow +\infty & \text{as } t \to +\infty. \end{cases} \tag{2.1}$$

We denote by Φ the set of all functions φ satisfying (2.1).

DEFINITION 2.1. When $\varphi \in \Phi$ and $\alpha > 0$, $\varphi(\alpha L)$ is a set of real valued functions f such that $\int_{-\pi}^{\pi} \varphi(\alpha|f(x)|\, dx < +\infty$. The sets L^*_φ and $\varphi(L)^*$ are defined as follows:

$$L^*_\varphi := \bigcup_{\varepsilon>0} \varphi(\varepsilon L); \tag{2.2}$$

$$\varphi(L)^* := \bigcap_{\alpha>0} \varphi(\alpha L). \tag{2.3}$$

The space L^*_φ is termed an Orlicz space which is a generalization of the space $L^p(\mathbf{T})$.

DEFINITION 2.2. We say that a function $\varphi \in \Phi$ satisfies the Δ_2-condition if there exist positive constants $C_0 > 0$ and $t_0 > 0$ such that

$$\varphi(2t) \leqq C_0\varphi(t) \quad \text{for all} \quad t > t_0. \tag{2.4}$$

From (2.2) and (2.3) it is clear that $\varphi(L)^* \subset L_\varphi^*$. In general the equality $\varphi(L)^* = L_\varphi^*$ does not hold. However we get the following result ([6], [8]).

THEOREM 2.3. *When $\varphi \in \Phi$, the equality $\varphi(L)^* = L_\varphi^*$ holds if and only if $\varphi(t)$ satisfies the Δ_2-condition.*

We are interested in the case when φ does not satisfy the Δ_2-condition. First we give an interesting example which does not satisfy the Δ_2-condition and plays an important role later.

EXAMPLE 2.4. Let $0 < \gamma < +\infty$ and put

$$\varphi(t) := \sum_{k=2}^{\infty} \frac{(t^\gamma)^k}{k!} = \exp(t^\gamma) - t^\gamma - 1 \quad \text{for} \quad t \geqq 0; \tag{2.5}$$

$$\psi(t) := \sum_{k=2}^{\infty} \frac{t^k}{(k!)^{1/\gamma}} \quad \text{for} \quad t \geqq 0. \tag{2.6}$$

Since $\lim_{t\to\infty} \varphi(t)/t^n = +\infty$ and $\lim_{t\to\infty} \psi(t)/t^n = +\infty$ for every positive integer n, neither φ nor ψ satisfy the Δ_2-condition. Therefore, by Theorem 2.3 it follows that $\varphi(L)^* \subsetneqq L_\varphi^*$ and $\psi(L)^* \subsetneqq L_\psi^*$. The functions $\varphi(t)$ and $\psi(t)$ have the different expressions (2.5) and (2.6). However they define the same Orlicz space. Namely, we have

$$\varphi(L)^* = \psi(L)^* \quad \text{and} \quad L_\varphi^* = L_\psi^*. \tag{2.7}$$

We prove (2.7). In fact, when $1 < \gamma < +\infty$, it follows that

$$\varphi(t) = \sum_{k=2}^{\infty} \left(\frac{t^k}{(k!)^{1/\gamma}} \right)^\gamma \leqq \left(\sum_{k=2}^{\infty} \frac{t^k}{(k!)^{1/\gamma}} \right)^\gamma = (\psi(t))^\gamma.$$

Since $\varphi(t) \geqq (1/2)\exp(t^\gamma)$ for sufficiently large $t > 0$, $\varphi\left((1/\gamma)^{1/\gamma} t\right) \leqq 2\psi(t)$ holds for sufficiently large $t > 0$. By Theorem 3.1 in [6] $\psi(L)^* \subset \varphi(L)^*$ holds. On the other hand, Hölder's inequality gives

$$\psi(t/2) = \sum_{k=2}^{\infty} (1/2^k) \frac{t^k}{(k!)^{1/\gamma}} \leqq$$

$$\leqq \left(\sum_{k=2}^{\infty} (1/2^k)^{\gamma'} \right)^{1/\gamma'} \left(\sum_{k=2}^{\infty} \frac{(t^\gamma)^k}{k!} \right)^{1/\gamma} \leqq \left(\frac{1}{1-(1/2^{\gamma'})} \right)^{1/\gamma'} (\varphi(t))^{1/\gamma},$$

where $\gamma' = \gamma/(\gamma - 1)$ is the exponent conjugate to $\gamma > 1$. Therefore we get $\psi(t/2) \leqq 4\varphi(t)$ for sufficiently large t. Theorem 3.1 in [6] gives $\varphi(L)^* \subset \subset \psi(L)^*$, and so $\varphi(L)^* = \psi(L)^*$ holds.

When $0 < \gamma < 1$, arguing in the same way, we find that $\varphi(L)^* = \psi(L)^*$. By Theorem 2.5 in [7] $L^*_\varphi = L^*_\psi$ holds. □

DEFINITION 2.5. Let φ be a function defined by (2.5). Then $L^*(\exp t^\gamma)$ denotes the set L^*_φ given by (2.2) and $(\exp L^\gamma)^*$ denotes the set $\varphi(L)^*$ given by (2.3).

§3. Control functions of a.e. convergence

For the a.e. convergence of sequences of functions, Yoneda [11] proved the following result. When $\{f_n(x); n \geqq 1\}$ is a sequence of real valued functions defined on the closed interval $[a, b]$ and

$$(3.1) \qquad \lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e. on} \quad [a, b],$$

there exists a positive and a.e. finite valued function $\delta(x)$ such that for every $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ satisfying $|f_n(x) - f(x)| \leqq \varepsilon\delta(x)$ everywhere for all $n \geqq n(\varepsilon)$. The function $\delta(x)$ is termed a control function of the a.e. convergence (3.1).

Wagner and Wilczyński [10] showed that Yoneda's result is equivalent to the well known Egoroff's and Taylor's theorem [9]. We apply the control function to the a.e. convergences of Fourier series of functions in Orlicz spaces. We say that (3.1) has an L^*_φ-integrable (or $\varphi(L)^*$-integrable) control function, when (3.1) has a control function in the space L^*_φ (or $\varphi(L)^*$).

In [3] the following theorem was proved:

THEOREM 3.1. *If f is a continuous 2π-periodic function, then the a.e. convergence*

$$(3.2) \qquad \lim_{n\to\infty} S_n(f; x) = f(x) \quad \textit{a.e.} \quad x \in \mathbf{T}$$

has an $(\exp L)^*$-integrable control function.

When f is a bounded function, the following result has been given in [4].

THEOREM 3.2. *There exists a bounded function f such that $\|f\|_\infty = 1$ and the a.e. convergence (3.2) has no $\varphi(L)^*$-integrable control function.*

On the other hand $S^*(f) \in L^*(\exp t)$, whenever $f \in L^\infty(\mathbf{T})$ in virtue of (1.3). A problem arises whether a control function of the almost everywhere convergence of (3.2) can be chosen in the Orlicz space $L^*(\exp t)$, whenever $f \in L^\infty(\mathbf{T})$. The following lemma plays an important role.

LEMMA 3.3. *Let $f \in \varphi(L)^*$ and let $S^*(f)$ be a majorant function defined by* (1.1). *Then it follows that*

(1) *if $S^*(f) \in \varphi(L)^*$, then* (3.2) *has a $\varphi(L)^*$-integrable control function;*

(2) *if $S^*(f) \notin \varphi(L)^*$, then* (3.2) *has no L^*_φ-integrable control function.*

PROOF. The statement (1) was proved in [12]. We have to prove (2). Suppose (3.2) has an L^*_φ-integrable control function δ. Then there exists a positive number ε_0 such that

$$\int_{-\pi}^{\pi} \varphi(\varepsilon_0\delta(x))\,dx < +\infty. \tag{3.3}$$

Since δ is a control function of (3.2), for any $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ satisfying

$$\left|S_n(f;x) - f(x)\right| \leqq \varepsilon\delta(x) \quad \text{everywhere for all} \quad n \geqq n(\varepsilon).$$

Therefore it follows that

$$S^*(f)(x) \leqq \varepsilon\delta(x) + \left|f(x)\right| + M_\varepsilon(f) \quad \text{for} \quad x \in \mathbf{T}, \tag{3.4}$$

where $M_\varepsilon(f) := \sup\Big\{\left|S_n(f;x)\right| : 1 \leqq n \leqq n(\varepsilon), x \in \mathbf{T}\Big\}$.

Let α be any positive number and fix it. Put $\varepsilon = \varepsilon_0/2\alpha$, where ε_0 is a positive number which is given in (3.3). Then it follows from (3.4) that

$$\int_{-\pi}^{\pi} \varphi(\alpha S^*(f)(x))\,dx \leqq \int_{-\pi}^{\pi} \varphi(\alpha\varepsilon\delta(x) + \alpha\left|f(x)\right| + \alpha M_\varepsilon(f))\,dx \leqq$$

$$\leqq \int_{-\pi}^{\pi} \varphi(2\alpha\varepsilon\delta(x))\,dx + \int_{-\pi}^{\pi} \varphi\Big(2\alpha\left|f(x)\right| + 2\alpha M_\varepsilon(f)\Big)\,dx \leqq$$

$$\leqq \int_{-\pi}^{\pi} \varphi(\varepsilon_0\delta(x))\,dx + \int_{-\pi}^{\pi} \varphi\Big(4\alpha\left|f(x)\right|\Big)\,dx + 2\pi\varphi(4\alpha M_\varepsilon(f)) < +\infty.$$

Hence $S^*(f)$ is $\varphi(L)^*$-integrable, which contradicts our assumption. □

By Theorem 1.3 and Lemma 3.3 we have the following theorem.

THEOREM 3.4. *There exists a bounded function f such that $\|f\|_\infty = 1$ and the a.e. convergence* (3.2) *has no $L^*(\exp t)$-integrable control function.*

PROOF. We consider a function f mentioned in Theorem 1.3. It is easy to see that $S^*(f) \in L^*(\exp t)$ in virtue of (1.3). However, from (1.4) $S^*(f) \notin$ $\notin (\exp L)^*$ holds. Therefore by Lemma 3.3 the desired result follows. □

§4. $\varphi(L)^*$-a.e. convergence

We now turn to the discussion of almost everywhere convergence of a sequence of functions $\{f_n(x); n \geqq 1\}$.

DEFINITION 4.1. We say that a sequence of functions $\{f_n(x); n \geqq 1\}$, $x \in [-\pi, \pi]$ converges to f $\varphi(L)^*$-a.e., if (3.1) has a $\varphi(L)^*$-integrable control function.

DEFINITION 4.2. We denote by Φ_0 a subset of Φ such that each function φ in Φ_0 has the expansion

$$\varphi(t) = \sum_{n=2}^{\infty} \beta_n t^n \quad \text{for all} \quad t > 0, \tag{4.1}$$

where $\beta_n \geqq 0$ for all $n \geqq 2$ and $\lim\limits_{n\to\infty} \sqrt[n]{\beta_n} = 0$.

THEOREM 4.3. *Let $\varphi \in \Phi_0$ have an expansion* (4.1). *Put*

$$\psi(t) := \sum_{n=2}^{\infty} \frac{\beta_n}{n!} t^n \quad \textit{for all} \quad t > 0. \tag{4.2}$$

If $f \in \varphi(L)^*$, then the Fourier series of f converges to f $\psi(L)^*$-a.e.

PROOF. Since our theorem is clear by Theorem 1.1, if $\beta_n = 0$ for sufficiently large n, we consider the case when $\beta_n > 0$ for infinitely many positive integers n. Then we get $\varphi(L)^* \subset \bigcap\limits_{p>1} L^p$. From (1.2) it follows that

$$\|S^*(f)\|_p \leqq C \cdot p\|f\|_p \quad \text{for} \quad p \geqq 2, \tag{4.3}$$

where $C > 0$ is an absolute constant.

Let α be any positive number. In virtue of (4.2) and (4.3), it follows that

$$\int_{-\pi}^{\pi} \psi(\alpha S^*(f)(x))\, dx = \sum_{n=2}^{\infty} \frac{\beta_n \alpha^n}{n!} \|S^*(f)\|_n^n \leqq$$

$$\leqq \sum_{n=2}^{\infty} \beta_n (n^n/n!) \int_{-\pi}^{\pi} \left(\alpha C |f(x)|\right)^n dx.$$

Since $n^n/n! \leqq e^n$, we get

$$\int_{-\pi}^{\pi} \psi\big(\alpha S^*(f)(x)\big)\,dx \leqq \sum_{n=2}^{\infty} \beta_n \int_{-\pi}^{\pi} \Big(e\alpha C|\,f(x)|\Big)^n\,dx =$$

$$= \int_{-\pi}^{\pi} \varphi\Big(e\alpha C|\,f(x)|\Big)\,dx < +\infty.$$

Therefore $S^*(f) \in \psi(L)^*$ holds and by Lemma 3.3 our desired result follows. □

COROLLARY 4.4. *When $f \in (\exp L^\gamma)^*$ for $0 < \gamma < +\infty$, then the Fourier series of f converges to f $\big(\exp L^{\gamma/(\gamma+1)}\big)^*$-a.e.*

PROOF. Let $0 < \gamma < +\infty$ and put

$$\psi_\gamma(t) := \sum_{k=2}^{\infty} \frac{t^k}{(k!)^{1/\gamma}} \quad \text{for} \quad t \geqq 0.$$

From (2.7) $(\exp L^\gamma)^* = \psi_\gamma(L)^*$ holds. By Theorem 4.3 it follows that the Fourier series of f converges to f $\psi_{\gamma/(\gamma+1)}(L)^*$-a.e. Applying (2.7) again, the desired result follows. □

§5. Quasi $\varphi(L)^*$-a.e. convergence

In this section we consider the control function of a.e. convergence of functions in the Orlicz space $L^*(\exp t^\gamma)$. We obtain the following result.

THEOREM 5.1. *Let $\varphi \in \Phi_0$ be a function defined by* (4.1) *and let $\psi \in \Phi_0$ be a function defined by* (4.2). *If $f \in L^*_\varphi$, then $S^*(f) \in L^*_\psi$.*

PROOF. Arguing in the same way as in the proof of Theorem 4.3, the desired conclusion follows. □

DEFINITION 5.2. Let $\varphi_0 \in \Phi$. A set $\Phi(\varphi_0)$ consists of functions $\varphi \in \Phi$ such that for any $0 < \varepsilon \leqq 1$,

$$\varphi(t) = o\big(\varphi_0(\varepsilon t)\big) \quad \text{as} \quad t \to +\infty. \tag{5.1}$$

As an example, it is easy to see that if $0 < \gamma < 1$, $\widetilde{\varphi}_\gamma(t) := e^{t^\gamma} - 1$ and $\widetilde{\varphi}_0(t) := e^t - 1$ for $t \geqq 0$, then $\widetilde{\varphi}_\gamma \in \Phi\big(\widetilde{\varphi}_0\big)$ for $0 < \gamma < 1$.

DEFINITION 5.3. Let $\varphi_0 \in \Phi$. We say that a sequence of functions $\{f_n(x); n \geqq 1\}$, $x \in [-\pi, \pi]$ converges to f quasi $\varphi_0(L)^*$-a.e., if for any $\varphi \in \Phi(\varphi_0)$, (3.1) has a $\varphi(L)^*$-integrable control function δ_φ.

By Definitions 4.1 and 5.3 we have the following lemma.

LEMMA 5.4. *Let $\varphi_0 \in \Phi$. If a sequence of functions $\{f_n(x); n \geqq 1\}$, $x \in [-\pi, \pi]$ converges to f $\varphi_0(L)^*$-a.e., then it converges to f quasi $\varphi_0(L)^*$-a.e.*

PROOF. If a sequence of functions $\{f_n(x); n \geqq 1\}$ converges to f $\varphi_0(L)^*$-a.e., then there exists a $\varphi_0(L)^*$-integrable control function δ.

For any $\varphi \in \Phi(\varphi_0)$ it is easy to see that $\varphi_0(L)^* \subset \varphi(L)^*$ in virtue of (5.1). Therefore $\delta \in \varphi(L)^*$ holds. □

The concept of quasi $\varphi_0(L)^*$-a.e. convergence is a generalization of quasi uniform convergence (see [5], [12]). The following theorem plays an important role in deciding a control function of a.e. convergence of Fourier series of $f \in L^*(\exp t^\gamma)$.

THEOREM 5.5. *If $\varphi_0 \in \Phi$, then*

$$L^*_{\varphi_0} = \cap\{\varphi(L)^* : \varphi \in \Phi(\varphi_0)\}. \tag{5.2}$$

PROOF. First we prove that $L^*_{\varphi_0} \subset \cap\, \varphi(L)^*$. Let $f \in L^*_{\varphi_0}$. Then there exists a positive number ε_0 such that

$$\int_{-\pi}^{\pi} \varphi_0\Big(\varepsilon_0 |f(x)|\Big)\, dx < +\infty. \tag{5.3}$$

For any $\alpha \geqq 1$, choose a positive number ε such that $0 < \varepsilon\alpha < \varepsilon_0$. From (5.1) there exists a positive constant t_0 such that $\varphi(t) \leqq \varphi_0(\varepsilon t)$ for $t \geqq t_0$. Therefore we get

$$\varphi(\alpha t) \leqq \varphi_0(\varepsilon\alpha t) \leqq \varphi_0(\varepsilon_0 t) \quad \text{for all} \quad t \geqq t_0.$$

Thus it follows from (5.3) that

$$\int_{-\pi}^{\pi} \varphi\Big(\alpha |f(x)|\Big)\, dx \leqq 2\pi\varphi(\alpha t_0) + \int_{|f(x)| \geqq t_0} \varphi_0\Big(\varepsilon_0 |f(x)|\Big)\, dx < +\infty.$$

Therefore $f \in \varphi(L)^*$ holds.

Next we show that $L^*_{\varphi_0} \supset \bigcap_{\varphi\in\Phi(\varphi_0)} \varphi(L)^*$. Let $f \in \varphi(L)^*$ for all $\varphi \in \Phi(\varphi_0)$. Suppose for each $\eta > 0$

$$\int_{-\pi}^{\pi} \varphi_0\Big(\eta\big|f(x)\big|\Big)\,dx = +\infty. \tag{5.4}$$

It will be proved that there exists a function $\varphi \in \Phi(\varphi_0)$ such that $f \notin \varphi(L)^*$.

For each $0 \leqq t \leqq u \leqq +\infty$, put

$$F(x;u,t) := \begin{pmatrix} \big|f(x)\big| & \text{if } u > \big|f(x)\big| \geqq t; \\ 0 & \text{if } \big|f(x)\big| \geqq u \quad\text{or}\quad t > \big|f(x)\big|. \end{pmatrix} \tag{5.5}$$

In virtue of (5.4) it follows that for each $\eta > 0$

$$\lim_{u\to\infty} \int_{-\pi}^{\pi} \varphi_0\big(\eta F(x;u,t)\big)\,dx = +\infty. \tag{5.6}$$

Choose two sequences of numbers $\{u_n; n \geqq 1\}$ and $\{t_n; n \geqq 0\}$ satisfying the following properties:

$$0 = t_0 < u_1 < t_1 < u_2 < \ldots < u_n < t_n < \ldots \uparrow +\infty \text{ as } n \to +\infty; \tag{5.7}$$

$$\frac{1}{n}\varphi_o(u_n/n) = \frac{1}{n+1}\varphi_0\big(t_n/(n+1)\big) \quad \text{for} \quad n \geqq 1; \tag{5.8}$$

$$\frac{1}{n}\varphi_0(u_n/n) \geqq n \quad \text{for} \quad n \geqq 1; \tag{5.9}$$

$$\int_{-\pi}^{\pi} \frac{1}{n}\varphi_0\left(\frac{1}{n}F(x;u_n,t_{n-1})\right)dx \geqq 1 \quad \text{for} \quad n \geqq 1. \tag{5.10}$$

We show that it is indeed possible to choose such sequences satisfying (5.6)–(5.10). Put $t_0 = 0$. From (5.5) and (5.6) we can choose a positive number $u_1 > t_0$ such that $\varphi_0(u_1) \geqq 1$ and

$$\int_{-\pi}^{\pi} \varphi_0\big(F(x;u_1,t_0)\big)\,dx \geqq 1.$$

Since $\varphi_0(u_1) > \frac{1}{2}\varphi_0\left(\frac{1}{2}u_1\right)$, there exists a positive number $t_1 > u_1$ such that $\varphi_0(u_1) = \frac{1}{2}\varphi_0\left(\frac{1}{2}t_1\right)$.

Suppose $t_0, u_1, \dots, u_n, t_n$ have been chosen. From (5.6) we can choose a positive number u_{n+1} such that $u_{n+1} > t_n$,

$$\frac{1}{n+1}\varphi_0\left(\frac{1}{n+1}u_{n+1}\right) \geqq n+1$$

and

$$\int_{-\pi}^{\pi} \frac{1}{n+1}\varphi_0\left(\frac{1}{n+1}F(x; u_{n+1}, t_n)\right) dx \geqq 1.$$

Since

$$\frac{1}{n+1}\varphi_0\left(\frac{1}{n+1}u_{n+1}\right) \geqq \frac{1}{n+1}\varphi_0\left(\frac{1}{n+2}u_{n+1}\right) > \frac{1}{n+2}\varphi_0\left(\frac{1}{n+2}u_{n+1}\right),$$

there exists a positive number $t_{n+1} > u_{n+1}$ such that

$$\frac{1}{n+1}\varphi_0\left(\frac{1}{n+1}u_{n+1}\right) = \frac{1}{n+1}\varphi_0\left(\frac{1}{n+2}t_{n+1}\right).$$

This selection procedure produces a sequence of numbers satisfying the properties (5.7)–(5.10). We note that

$$u_n \uparrow +\infty \quad \text{as} \quad n \to +\infty. \tag{5.11}$$

In fact, suppose there exists a positive constant C such that $0 < u_n \leqq C < < +\infty$ for ɔ'l $n \geqq 1$. From (5.5) it is clear that

$$\varphi_0\left(\frac{1}{n}F(x; u_n, t_{n-1})\right) \leqq \varphi_0\left(\frac{1}{n}F(x; C, 0)\right) \leqq \varphi_0\left(\frac{1}{n}C\right).$$

Therefore if follows that

$$\int_{-\pi}^{\pi} \frac{1}{n}\varphi_0\left(\frac{1}{n}F(x; u_n, t_{n-1})\right) dx \leqq \frac{2\pi}{n}\varphi_0\left(\frac{1}{n}C\right) \to 0 \quad \text{as} \quad n \to +\infty.$$

We have arrived at a contradiction to (5.10). Thus (5.11) holds.

Define a function $\varphi(t)$ as follows:

$$\varphi(t) := \begin{cases} \frac{1}{n}\varphi_0\left(\frac{1}{n}t\right) & \text{if } t_{n-1} \leqq t < u_n \text{ for } n \geqq 1; \\ \frac{1}{n}\varphi_0\left(\frac{1}{n}u_n\right) & \text{if } u_n \leqq t < t_n \text{ for } n \geqq 1. \end{cases} \tag{5.12}$$

By the construction of $\varphi(t)$ it is clear that $\varphi(0)=0$ and $\varphi(t)$ is a monotone increasing function. Moreover, in virtue of (5.9) and (5.12) we get $\varphi(u_n) \geqq n$ for $n \geqq 1$. Therefore we get $\varphi(t) \uparrow +\infty$ as $t \to +\infty$.

It remains to prove that $\varphi \in \Phi(\varphi_0)$ and $f \notin \varphi(L)^*$. For any $\varepsilon > 0$, a positive integer $n(\varepsilon)$ can be chosen such that

$$0 < \frac{1}{n(\varepsilon)+1} < \varepsilon. \tag{5.13}$$

For any $t \geqq t_{n(\varepsilon)}$, there exists a positive integer $n \geqq n(\varepsilon)$ such that $t_n \leqq t < t_{n+1}$. Then combining (5.12) and (5.13) we can at once obtain that

$$\frac{\varphi(t)}{\varphi_0(\varepsilon t)} \leqq \frac{\frac{1}{n+1}\varphi_0\left(\frac{t}{n+1}\right)}{\varphi_0(\varepsilon t)} \leqq \frac{\frac{1}{n+1}\varphi_0\left(\frac{t}{n(\varepsilon)+1}\right)}{\varphi_0(\varepsilon t)} \leqq$$
$$\leqq \frac{\frac{1}{n+1}\varphi_0(\varepsilon t)}{\varphi_0(\varepsilon t)} = \frac{1}{n+1}.$$

Consequently, $\varphi(t) = o\big(\varphi_0(\varepsilon t)\big)$ as $t \to +\infty$ holds.

We show that $f \notin \varphi(L)^*$. Put

$$E_n := \left\{x \in [-\pi,\pi] : t_n \leqq \left|f(x)\right| < t_{n+1}\right\} \quad \text{for} \quad n \geqq 0.$$

Then it follows from (5.5) and (5.10) that

$$\int_{-\pi}^{\pi} \varphi\Big(\left|f(x)\right|\Big)\,dx = \sum_{n=0}^{\infty} \int_{E_n} \varphi\Big(\left|f(x)\right|\Big)\,dx =$$
$$= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \varphi\big(F(x;t_{n+1},t_n)\big)\,dx \geqq \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \varphi\big(F(x;u_{n+1},t_n)\big)\,dx =$$
$$= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \frac{1}{n+1}\varphi_0\left(\frac{1}{n+1}F(x;u_{n+1},t_n)\right)dx \geqq \sum_{n=0}^{\infty} 1 = +\infty.$$

The proof is complete. □

It was proved in [12] that if (3.1) holds and $f^*(x) = \sup\Big\{\left|f_n(x)\right| : n \geqq 1\Big\} \in \varphi(L)^*$, then there exists a $\varphi(L)^*$-integrable control function δ. Combining this with Theorem 5.5, the following theorem holds.

THEOREM 5.6. *Let* $\varphi_0 \in \Phi$. *If* (3.1) *holds and* $f^* \in L^*_{\varphi_0}$, *then* f_n *converges to* f *quasi* $\varphi_0(L)^*$*-a.e.*

COROLLARY 5.7. *Suppose* $f \in L^*(\exp t^\gamma)$ *for* $\gamma > 0$. *Then* $S^*(f) \in L^*\big(\exp t^{\gamma/(\gamma+1)}\big)$. *Consequently,* $S_n(f;x)$ *converges to* $f(x)$ *quasi* $\big(\exp L^{\gamma/(\gamma+1)}\big)^*$*-a.e.*

PROOF. Put

$$\psi_\gamma(t) := \sum_{k=2}^{\infty} \frac{t^k}{(k!)^{1/\gamma}} \quad \text{for} \quad t \geqq 0.$$

In virtue of (2.7), we get $L^*_{\psi_\gamma} = L^*(\exp t^\gamma)$. Therefore, by Theorem 5.1 it follows that if $f \in L^*(\exp t^\gamma)$, then $S^*(f) \in L^*_{\psi_{\gamma/(\gamma+1)}} = L^*\big(\exp t^{\gamma/(\gamma+1)}\big)$. By Theorem 5.6 our desired conclusion holds. □

COROLLARY 5.8. *If* $f \in \mathrm{BMO}(\mathbf{T})$, *then* $S^*(f) \in L^*\big(\exp t^{1/2}\big)$. *Consequently,* $S_n(f;x)$ *converges to* $f(x)$ *quasi* $\big(\exp L^{1/2}\big)^*$*-a.e.*

PROOF. John and Nirenberg [2] proved that if $f \in \mathrm{BMO}(\mathbf{T})$ and I is any interval in $\mathbf{T}$, then there are positive constants C_1 and C_2 independent of f and I such that

$$\text{(5.14)} \qquad \operatorname{meas}\Big\{x \in I: \big|f(x) - f_I\big| > t\Big\} \leqq C_1|I|\exp\big(-C_2 t/\|f\|_*\big)$$

for all $t > 0$, where $f_I = \frac{1}{|I|}\int_I f(x)\,dx$ and $\|f\|_* = \sup_I \Big\{\frac{1}{|I|}\int_I \big|f(x) - f_I\big|\,dx\Big\}$.

From (5.14) it is easy to see that if $f \in \mathrm{BMO}(\mathbf{T})$ then $f \in L^*(\exp t)$. By Corollary 5.7 the desired result follows. □

§6. Example

In the preceding section we proved that if $f \in L^*(\exp t^\gamma)$ for $\gamma > 0$, then $S^*(f) \in L^*\big(\exp t^{\gamma/(\gamma+1)}\big)$. A problem arises whether we can find a control function of (3.2) in $L^*\big(\exp t^{\gamma/(\gamma+1)}\big)$, whenever $f \in L^*(\exp t^\gamma)$. In this section a negative example will be given.

THEOREM 6.1. *For each* $\gamma > 0$ *there exists a function* $f \in L^*(\exp t^\gamma)$ *such that for some positive integers* $N_1 < N_2 < \ldots < N_k < \ldots$ *we get*

$$\text{(6.1)} \qquad S_{N_k}(f;x) \geqq C(\log N_k)^{(\gamma+1)/\gamma} \quad \textit{for all} \quad 0 \leqq x \leqq \pi/3N_k,$$

where $C > 0$ *is a constant depending only on* $\gamma > 0$.

THEOREM 6.2. *Let $\gamma > 0$. Then there exists a function $f \in L^*(\exp t^\gamma)$ such that $S^*(f) \notin \left(\exp L^{\gamma/(\gamma+1)}\right)^*$ and*

$$\overline{\lim_{n\to\infty}} \int_{-\pi}^{\pi} \varphi_{\gamma/(\gamma+1)}\Big(\alpha_0 \big| S_n(f;x) - f(x)\big|\Big)\, dx = +\infty$$

for sufficiently large $\alpha_0 > 0$, where $\varphi_{\gamma/(\gamma+1)}(t) = \exp\left(t^{\gamma/(\gamma+1)}\right) - t^{\gamma/(\gamma+1)} - 1$.

By Lemma 3.3 and Theorem 6.2 we get the following result.

COROLLARY 6.3. *For any $\gamma > 0$, there exists a function $f \in L^*(\exp t^\gamma)$ such that the a.e. convergence of (3.2) has no $L^*\left(\exp t^{\gamma/(\gamma+1)}\right)$-integrable control function.*

PROOF OF THEOREM 6.1. Let $\{n_k; k \geqq 1\}$ be an increasing sequence of positive integers which will be defined later. Set

$$N_0 = 1, \quad N_k = n_1 n_2 \dots n_k \quad \text{for} \quad k \geqq 1, \tag{6.2}$$

and define the closed intervals I_k and J_k as follows:

$$I_k = [\pi/N_k, \pi/N_{k-1}], \quad J_k = [2\pi/N_k, \pi/N_{k-1} - \pi/N_k] \quad \text{for} \quad k \geqq 1.$$

If we choose the sequence $\{N_k; k \geqq 1\}$ as

$$3N_{k-1} < N_k \quad \text{for} \quad k \geqq 1, \tag{6.3}$$

then we get $2\pi/N_k < \pi/N_{k-1} - \pi/N_k$ and so $J_k \subset I_k$.

Now we construct a function $f(x) \in L^*(\exp t^\gamma)$. Put

$$f(x) := \begin{cases} c_k \sin N_k x & \text{if } x \in J_k \quad \text{for} \quad k \geqq 1; \\ 0 & \text{if } x \in [0,\pi] \setminus \bigcup\limits_{k=1}^{\infty} J_k, \end{cases} \tag{6.4}$$

where $\{c_k; k \geqq 1\}$ is an increasing sequence of positive numbers which will be defined later. Put $f(-x) = f(x)$ for $0 < x \leqq \pi$ and extend it to a function with period 2π.

We consider the Fourier series of $f(x)$. As is known, we have

$$S_n(f;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin nt}{t}\, dt + o(1) \quad \text{as} \quad n \to +\infty.$$

where $o(1)$ is uniformly convergent. Therefore it follows that

$$S_{N_k}(f;x)=\frac{1}{\pi}\int\limits_0^\pi f(x+t)\frac{\sin N_k t}{t}\,dx+\frac{1}{\pi}\int\limits_0^\pi f(x-t)\frac{\sin N_k t}{t}\,dt+o(1):=$$

$$:=S^{(1)}_{N_k}(f;x)+S^{(2)}_{N_k}(f;x)+o(1).$$

In order to evaluate $S^{(1)}_{N_k}(f;x)$, we divide it into three terms:

$$S^{(1)}_{N_k}(f;x)=\frac{1}{\pi}\int\limits_0^{\pi/N_k} f(x+t)\frac{\sin N_k t}{t}\,dt+$$

$$+\frac{1}{\pi}\int\limits_{\pi/N_k}^{\pi/N_{k-1}} f(x+t)\frac{\sin N_k t}{t}\,dt+\frac{1}{\pi}\int\limits_{\pi/N_{k-1}}^{\pi} f(x+t)\frac{\sin N_k}{t}\,dt:=$$

$$:=T^{(1)}_k(x)+T^{(2)}_k(x)+T^{(3)}_k(x).$$

If $0\leqq x<\pi/N_k$ and $\pi/N_{k-1}\leqq t\leqq\pi$, then $\pi/N_{k-1}\leqq x+t\leqq\pi/N_k+\pi\leqq$ $\leqq\pi/N_1+\pi$. Therefore it follows from (6.4) that

$$\text{(6.5)}\qquad \left|T^{(3)}_k(x)\right|\leqq\frac{1}{\pi}\int\limits_{\pi/N_{k-1}}^{\pi}\left|f(x+t)\right|\frac{1}{t}\,dt\leqq\frac{c_{k-1}}{\pi}\log N_{k-1}.$$

For $\left|T^{(1)}_k(x)\right|$ it is easy to see that if $0\leqq x<\pi/N_k$, then it follows that

$$\left|T^{(1)}_k(x)\right|\leqq\frac{N_k}{\pi}\int\limits_0^{\pi/N_k}\left|f(x+t)\right|dt\leqq$$

$$\leqq\frac{N_k}{\pi}\int\limits_0^{2\pi/N_k}\left|f(s)\right|ds=\frac{N_k}{\pi}\int\limits_0^{\pi/N_k}\left|f(s)\right|ds=$$

$$=\frac{N_k}{\pi}\sum_{j=k+1}^{\infty}\int\limits_{\pi/N_j}^{\pi/N_{j-1}}\left|f(s)\right|ds\leqq N_k\cdot\sum_{j=k+1}^{\infty}\frac{c_j}{N_j-1}.$$

Thus we get

$$(6.6) \qquad \left|T_k^{(1)}(x)\right| \leqq N_k \cdot \sum_{j=k+1}^{\infty} \frac{c_j}{N_{j-1}} \quad \text{for} \quad 0 \leqq x < \pi/N_k \quad \text{and} \quad k \geqq 1.$$

It remains to estimate $T_k^{(2)}(x)$. Suppose $0 \leqq x \leqq \pi/3N_k$ and put $\alpha_k = 2\pi/N_k$ and $\beta_k = \pi/N_{k-1} - \pi/N_k$. Then it follows from (6.4) that

$$T_k^{(2)}(x) = \frac{1}{\pi} \int_{\alpha_k - x}^{\beta_k - x} c_k \sin N_k(x+t) \frac{\sin N_k t}{t} \, dt =$$

$$= \frac{c_k \sin N_k x}{\pi} \int_{\alpha_k - x}^{\beta_k - x} \frac{\cos N_k t \cdot \sin N_k t}{t} \, dt +$$

$$+ \frac{c_k \cos N_k x}{\pi} \int_{\alpha_k - x}^{\beta_k - x} \frac{(\sin N_k t)^2}{t} \, dt = c_k U_k^{(1)}(x) + c_k U_k^{(2)}(x).$$

According to the second mean value theorem, taking into account that $1/(2t)$ is positive and decreasing monotonically in the range of integration, we get

$$U_k^{(1)}(x) = \frac{\sin N_k x}{\pi} \int_{\alpha_k - x}^{\xi(x)} \frac{\sin 2N_k t}{2(\alpha_k - x)} \, dt,$$

where $\xi(x) \in [\alpha_k - x, \beta_k - x]$. Since $0 \leqq x \leqq \pi/3N_k < \pi/N_k$ and $\alpha_k = 2\pi/N_k$, it follows that

$$(6.7) \qquad \left|U_k^{(1)}(x)\right| \leqq \frac{1}{2\pi(\alpha_k - x)} \cdot \frac{1}{2N_k} \cdot 2 \leqq \frac{1}{2\pi^2} \quad \text{for} \quad 0 \leqq x \leqq \pi/3N_k.$$

We estimate $U_k^{(2)}(x)$. Since $0 \leqq x \leqq \pi/3N_k$, we get

$$U_k^{(2)}(x) \geqq \frac{1}{2\pi} \int_{\alpha_k - x}^{\beta_k - x} \frac{(\sin N_k t)^2}{t} \, dt = \frac{1}{4\pi} \int_{\alpha_k - x}^{\beta_k - x} \frac{1}{t} \, dt - \frac{1}{4\pi} \int_{\alpha_k - x}^{\beta_k - x} \frac{\cos 2N_k t}{t} \, dt.$$

Now we can assume that the sequence $\{n_k; k \geqq 1\}$ satisfies the following properties:

(6.8) $\qquad 4 < n_1 < n_2 < \ldots < n_k < \ldots$, that is, $4N_{k-1} < N_k$ for $k \geqq 1$.

From (6.8) we find $\alpha_k - x \leqq 2\pi/N_k < \pi/N_{k-1} - 2\pi/N_k$, therefore we have

$$\frac{1}{4\pi}\int_{\alpha_k - x}^{\beta_k - x} \frac{1}{t}\,dt \geqq \frac{1}{4\pi}\int_{2\pi/N_k}^{\pi/N_{k-1}-2\pi/N_k} \frac{1}{t}\,dt =$$

$$= \frac{1}{4\pi}\log\big((N_k/2N_{k-1}) - 1\big) = \frac{1}{4\pi}\log(n_k/2 - 1) \geqq$$

$$\geqq \frac{1}{4\pi}\log(n_k/4) = \frac{1}{4\pi}\log(N_k/4N_{k-1}).$$

We choose the sequence $\{n_k; k \geqq 1\}$ such that

(6.9) $$16N_{k-1}^2 < N_k \quad \text{for} \quad k \geqq 1.$$

Then we get

(6.10) $$\frac{1}{4\pi}\int_{\alpha_k - x}^{\beta_k - x} \frac{1}{t}\,dt \geqq \frac{1}{8\pi}\log N_k \quad \text{for} \quad k \geqq 1.$$

By the second mean value theorem, we get

(6.11) $$\left|\frac{1}{\pi}\int_{\alpha_k - x}^{\beta_k - x} \frac{\cos 2N_k t}{t}\,dt\right| \leqq \frac{1}{4\pi(\alpha_k - x)}\cdot\frac{1}{N_k} \leqq \frac{1}{4\pi^2}.$$

From (6.10) and (6.11) we get

(6.12) $$U_k^{(2)}(x) \geqq \frac{1}{8\pi}\log N_k - \frac{1}{4\pi^2} \quad \text{for} \quad k \geqq 1.$$

From (6.7) and (6.12) it follows that if $0 \leqq x \leqq \pi/3N_k$, then

(6.13) $$T_k^{(2)}(x) \geqq c_k\left\{\frac{1}{8\pi}\log N_k - \frac{3}{4\pi^2}\right\} \quad \text{for} \quad k \geqq 1.$$

Thus it follows from (6.5), (6.6) and (6.13) that if $0 \leqq x \leqq \pi/3N_k$, then we get

$$\frac{1}{\pi}\int_0^\pi f(x+t)\frac{\sin N_k t}{t}\,dt \geqq T_k^{(2)}(x) - \left|T_k^{(1)}(x)\right| - \left|T_k^{(3)}(x)\right| \geqq$$

$$\geqq c_k\left(\frac{1}{8\pi}\log N_k - \frac{1}{\pi}\log N_{k-1} - \frac{3}{4\pi^2}\right) - N_k \cdot \sum_{j=k+1}^{\infty}\frac{c_j}{N_{j-1}} \quad \text{for} \quad k \geqq 1.$$

Now we consider the sequence $N_k \cdot \sum\limits_{j=k+1}^{\infty}\frac{c_j}{N_{j-1}}$ for $k \geqq 1$. Put

(6.14) $$c_k = (\log N_{k-1})^{1/\gamma} \quad \text{for} \quad k \geqq 1.$$

Then from (6.2) we get

$$N_k \cdot \sum_{j=k+1}^{\infty}\frac{c_j}{N_{j-1}} = N_k \cdot \sum_{j=k+1}^{\infty}\frac{(\log N_{j-1})^{1/\gamma}}{N_{j-1}} =$$

$$= (\log N_k)^{1/\gamma} + \sum_{s=1}^{\infty}\frac{(\log N_{k+s})^{1/\gamma}}{n_{k+1}n_{k+2}\dots n_{k+s}}.$$

Since $k^{2/\gamma} \leqq 2^k < n_k$ for sufficiently large k in virtue of (6.9), it is easy to see that

$$\frac{(\log N_{k+s})^{1/\gamma}}{n_{k+s}} \leqq \frac{\left((k+s)\log n_{k+s}\right)^{1/\gamma}}{n_{k+s}} \leqq \frac{\sqrt{n_{k+s}}(\log n_{k+s})^{1/\gamma}}{n_{k+s}} =$$

$$= \frac{(\log n_{k+s})^{1/\gamma}}{\sqrt{n_{k+s}}} \to 0 \quad \text{as} \quad s \to +\infty.$$

Therefore there exists a positive constant $M(\gamma)$ depending only on $\gamma > 0$ such that

$$N_k \cdot \sum_{j=k+1}^{\infty}\frac{(\log N_{j-1})^{1/\gamma}}{N_{j-1}} \leqq$$

$$\leqq (\log N_k)^{1/\gamma} + M(\gamma)\left\{1 + \sum_{s=2}^{\infty}\frac{1}{n_{k+1}\dots n_{k+s-1}}\right\} \leqq$$

$$\leqq (\log N_k)^{1/\gamma} + M(\gamma)\left\{1 + \sum_{s=1}^{\infty} \frac{1}{2^s}\right\} = (\log N_k)^{1/\gamma} + 2M(\gamma).$$

Consequently we get the following inequality:

$$(6.15) \qquad N_k \cdot \sum_{j=k+1}^{\infty} \frac{(\log N_{j-1})^{1/\gamma}}{N_{j-1}} \leqq 2(\log N_k)^{1/\gamma}$$

for sufficiently large k.

Now we choose the sequence $\{N_k; k \geqq 1\}$ such that

$$(6.16) \qquad N_{k-1}^{16} < N_k < N_{k-1}^{17} \quad \text{for} \quad k \geqq 1.$$

Then it is clear from (6.14) that

$$(6.17) \qquad \log N_{k-1} < \frac{1}{16} \log N_k \quad \text{and} \quad (\log N_k)^{1/\gamma} < 17^{1/\gamma} c_k.$$

When $0 \leqq x \leqq \pi/3N_k$, from (6.15) and (6.17) the following inequality follows:

$$(6.18) \qquad \frac{1}{\pi} \int_0^{\pi} f(x+t) \frac{\sin N_k t}{t}\, dt \geqq C_1(\gamma)(\log N_k)^{(1+\gamma)/\gamma}$$

for sufficiently large k, where $C_1(\gamma) > 0$ is a constant depending only on $\gamma > 0$.

In order to estimate $S_{N_k}^{(2)}(f;x)$ we divide it into four terms:

$$S_{N_k}^{(2)}(f;x) = \frac{1}{\pi} \int_0^{\pi/N_k} f(x-t) \frac{\sin N_k t}{t}\, dt + \frac{1}{\pi} \int_{\pi/N_k}^{2\pi/N_k} f(x-t) \frac{\sin N_k t}{t}\, dt +$$

$$+ \frac{1}{\pi} \int_{2\pi/N_k}^{\pi/N_{k-1}} f(x-t) \frac{\sin N_k t}{t}\, dt + \frac{1}{\pi} \int_{\pi/N_{k-1}}^{\pi} f(x-t) \frac{\sin N_k t}{t}\, dt.$$

Arguing in the same way, we find that if $0 \leqq x \leqq \pi/N_k$,

$$(6.19) \qquad S_{N_k}^{(2)}(f;x) \geqq C_2(\gamma)(\log N_k)^{(1+\gamma)/\gamma} \quad \text{for sufficiently large} \quad k,$$

where $C_2(\gamma) > 0$ is a constant depending only on $\gamma > 0$.

If we choose the sequence $\{n_k; k \geqq 1\}$ satisfying the conditions (6.3), (6.8), (6.9) and (6.16), then from (6.18) and (6.19) we get (6.1).

Finally, it will be proved that the function f constructed above is in the Orlicz space $L^*(\exp t^\gamma)$. Put $\varphi(t) = \exp(t^\gamma) - t^\gamma - 1$ and $\varepsilon_0 = (1/2)^{1/\gamma}$. It follows from (6.14) that

$$\int_{-\pi}^{\pi} \varphi\Big(\varepsilon_0 \big| f(x) \big| \Big)\, dx = 2 \int_0^{\pi} \varphi\Big(\varepsilon_0 \big| f(x) \big| \Big)\, dx =$$

$$= 2 \sum_{k=1}^{\infty} \int_{\pi/N_k}^{\pi/N_{k-1} - \pi/N_k} \varphi\Big(\varepsilon_0 \big| f(x) \big| \Big)\, dx \leqq$$

$$\leqq 2 \sum_{k=1}^{\infty} \varphi(\varepsilon_0 c_k) \cdot \frac{\pi}{N_{k-1}} \leqq 2 \sum_{k=1}^{\infty} \exp\big((\varepsilon_0 c_k)^\gamma \big) \cdot \frac{\pi}{N_{k-1}} =$$

$$= 2 \sum_{k=1}^{\infty} \exp(c_k^\gamma / 2) \cdot \frac{\pi}{N_{k-1}} = 2 \sum_{k=1}^{\infty} \exp\left(\frac{1}{2} \log N_{k-1} \right) \cdot \frac{\pi}{N_{k-1}} =$$

$$= 2 \sum_{k=1}^{\infty} \frac{\pi}{\sqrt{N_{k-1}}} \leqq 2\pi \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} < +\infty.$$

Therefore $f \in L^*(\exp t^\gamma)$.

PROOF OF THEOREM 6.2. Let f be a function constructed in the proof of Theorem 6.1. We have to prove that $S^*(f) \notin \big(\exp L^{\gamma/(\gamma+1)}\big)^*$. Suppose $S^*(f) \in \big(\exp L^{\gamma/(\gamma+1)}\big)^*$. Put

$$\lambda_k := C(\log N_k)^{(1+\gamma)/\gamma} \quad \text{for} \quad k \geqq 1, \tag{6.20}$$

where $C > 0$ is a constant given in the inequality (6.1). Since $S^*(f) \in \in \big(\exp L^{\gamma/(\gamma+1)}\big)^*$ by assumption, for any positive number α we get the following inequality:

$$\operatorname{meas}\big\{ x \in \mathbf{T} \colon S^*(f)(x) \geqq \lambda_k \big\} \leqq \tag{6.21}$$

$$\leqq \exp\big(-(\alpha \lambda_k)^{\gamma/(\gamma+1)} \big) \int_{-\pi}^{\pi} \exp\big(\alpha S^*(f)(x) \big)^{\gamma/(\gamma+1)}\, dx \quad \text{for} \quad k \geqq 1.$$

On the other hand, by Theorem 6.1 it is easy to see that

$$\operatorname{meas}\big\{ x \in \mathbf{T} \colon S^*(f)(x) \geqq \lambda_k \big\} \geqq \operatorname{meas}\big\{ x \in \mathbf{T} \colon S_{N_k}(f; x) \geqq \lambda_k \big\} \geqq \tag{6.22}$$

$$\geqq \frac{\pi}{3N_k} \quad \text{for} \quad k \geqq 1.$$

From (6.21) and (6.22) we find that

$$(6.23) \qquad \frac{\pi}{3N_k} \cdot \exp\left((\alpha\lambda_k)^{\gamma/(\gamma+1)}\right) \leqq \int_{-\pi}^{\pi} \exp\left(\alpha S^*(f)(x)\right)^{\gamma/(\gamma+1)} dx < +\infty$$

for $k \geqq 1$. In virtue of (6.20), $N_k = \exp\left((\lambda_k/C)^{\gamma/(\gamma+1)}\right)$. Therefore it follows from (6.23) that

$$(6.24) \qquad \frac{\pi}{3}\exp\left(\left(\alpha^{\gamma/(\gamma+1)} - (1/C)^{\gamma/(\gamma+1)}\right)\lambda_k^{\gamma/(\gamma+1)}\right) \leqq$$

$$\leqq \int_{-\pi}^{\pi} \exp\left(\alpha S^*(f)(x)\right)^{\gamma/(\gamma+1)} dx < +\infty \quad \text{for} \quad k \geqq 1.$$

If we choose α such that $\alpha > 1/C$, then the left side of the inequality (6.24) diverges to infinity because of the fact $\lim\limits_{k\to\infty} \lambda_k = +\infty$. We arrive at a contradiction. We get $S^*(f) \notin \left(\exp L^{\gamma/(\gamma+1)}\right)^*$.

Finally, we show that if we choose a positive number α_0 such that

$$(6.25) \qquad \frac{1}{2}(\alpha_0 C/2)^{\gamma/(\gamma+1)} > 2,$$

where $C > 0$ is a constant given in (6.1), then we get

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} \varphi_{\gamma/(\gamma+1)}\left(\alpha_0 \left| S_{N_k}(f;x) - f(x)\right|\right) dx = +\infty.$$

When $\pi/N_{k+1} \leqq x \leqq \pi/N_k$, it follows from (6.4) that

$$\left| S_{n_k}(f;x) - f(x)\right| \geqq C(\log N_k)^{(1+\gamma)/\gamma} - c_{k+1} =$$

$$= C(\log N_k)^{(1+\gamma)/\gamma} - (\log N_k)^{1/\gamma} \geqq \frac{C}{2}(\log N_k)^{(1+\gamma)/\gamma}$$

for sufficiently large k. Therefore from (6.25) we get

$$\varphi_{\gamma/(\gamma+1)}\Big(\alpha_0\big|S_{N_k}(f;x)-f(x)\big|\Big) \geqq \varphi_{\gamma/(\gamma+1)}\left(\frac{\alpha_0 C}{2}(\log N_k)^{(1+\gamma)/\gamma}\right) \geqq$$

$$\geqq \exp\left\{\frac{1}{2}\left(\frac{\alpha_0 C}{2}(\log N_k)^{(1+\gamma)/\gamma}\right)^{\gamma/(\gamma+1)}\right\} =$$

$$= \exp\left\{\frac{1}{2}\left(\frac{\alpha_0 C}{2}\right)^{\gamma/(\gamma+1)}\log N_k\right\} \geqq \exp(2\log N_k) = N_k^2.$$

Thus we get

$$\int_{-\pi}^{\pi} \varphi_{\gamma/(\gamma+1)}\Big(\alpha_0\big|S_{N_k}(f;x)-f(x)\big|\Big)\,dx \geqq$$

$$\geqq \int_{\pi/N_{k+1}}^{\pi/3N_k} \varphi_{\gamma/(\gamma+1)}\Big(\alpha_0\big|S_{N_k}(f;x)-f(x)\big|\Big)\,dx \geqq$$

$$\geqq N_k^2\left(\frac{\pi}{3N_k}-\frac{\pi}{N_{k+1}}\right) = N_k\left(\frac{\pi}{3}-\frac{\pi}{n_{k+1}}\right) \to +\infty \quad \text{as} \quad k \to +\infty.$$

Our desired result follows. □

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ON A DIOPHANTINE PROBLEM CONCERNING STIRLING NUMBERS

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Denote by S_k^n the Stirling number of second kind with parameters (n,k), that is the number of partitions of the set $\{1,2,\dots,n\}$ into k non-empty subsets. The Stirling numbers play an important rôle in mathematics, especially in combinatorics, number theory and a little bit surprisingly in algebraic topology, too. For a survey on Stirling numbers and their applications we refer to [3].

The arithmetical structure of Stirling numbers have been studied by several authors (see [2], [5] and [10]). We proved in [1] that for fixed positive integers u, v, the equation

$$S_{x-u}^x = vy^z$$

in integers x,y,z with $x > u$, $|y| > 1$, $z \geqq 2$ has only finitely many solutions and gave an effective upper bound for these solutions. The proof of this result is based ultimately on Baker's method.

Let $b > a > 1$ be rational integers. In this note we consider the equation

$$S_a^x = S_b^y \quad \text{in integers } x, y \text{ with } x > a,\ y > b. \tag{1}$$

Using again the theory of linear forms in logarithms we obtain

THEOREM. *All the solutions of equation* (1) *satisfy*

$$\max(x,y) < C \cdot b \cdot (\log b)^3 \cdot \log\left(\frac{b!}{a!}\right) \cdot \log a,$$

where C is an effectively computable absolute constant.

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Preliminaries

To the proof of the Theorem we need two auxiliary results.

LEMMA 1. *Let n, k be rational integers with $1 \leqq k < n$. Then*

$$\frac{k^n}{k!} - \frac{(k-1)^n}{(k-1)!} \leqq S_k^n \leqq \frac{k^n}{k!} \tag{2}$$

and

$$\frac{1}{2}(k^2+k+2)k^{n-k-1} - 1 \leqq S_k^n \leqq \frac{1}{2}\binom{n}{k}k^{n-k}. \tag{3}$$

PROOF. For (2), see Satz 2.1 in [12] and the inequalities (3) are due to Dobson and Rennie [4].

Let $a_1, a_2, \dots, a_r$ be rational integers with $a_i \geqq 2$. The next lemma is a special case of a deep result of [6]. For a more explicit version, see the recent paper of Waldschmidt [11].

LEMMA 2. *Let $b_1, \dots, b_r$ be rational integers such that*

$$a_1^{b_1} \cdots a_r^{b_r} \neq 1,$$

and put $B = \max\{2, |b_1|, \dots, |b_2|\}$. Then

$$\left| a_1^{b_1} \cdots a_r^{b_r} - 1 \right| > \exp(-c_1 \cdot \log a_1 \cdots \log a_r \cdot \log B),$$

where c_1 is an effectively computable positive number depending only on r.

Proof of the Theorem

In the sequel c_2, c_3 will denote effectively computable positive constants. Let (x, y) be an arbitrary but fixed solution to (1). By (3) we obtain $a^{x-a} \leqq \leqq 2^{b-1}b^{y-b}$ and $b^{y-b} \leqq 2^{a-1}a^{x-a}$. These inequalities imply

$$x \leqq 2(\log b)y \quad \text{and} \quad y \leqq x + b, \quad \text{respectively.} \tag{4}$$

It is known (cf. [8] or [9]), that

$$S_k^n = \frac{1}{k!}\left(k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \ldots + (-1)^k 0^n\right). \tag{5}$$

Set

$$S(k,n)=\binom{k}{1}\left(\frac{k-1}{k}\right)^{n}-\binom{k}{2}\left(\frac{k-2}{k}\right)^{n}+\ldots+(-1)^{k-2}\binom{k}{k-1}\left(\frac{1}{k}\right)^{n}.$$

We may assume that $\min(x,y)>2b\log b$, for otherwise our Theorem is proved by (4). Using (1), (2), (5) and the assumption $x>2b\log b$ we get

$$\left|\frac{a^x b!}{a!b^y}-1\right|\leqq\frac{\max\big(S(a,x),S(b,y)\big)}{1-S(a,x)}\leqq 2\max\big(S(a,x),S(b,y)\big)\leqq \tag{6}$$

$$\leqq 2b\left(\frac{b-1}{b}\right)^{\min(x,y)}\leqq 2b\mathrm{e}^{-\frac{\min(x,y)}{b}}.$$

One can see that $\frac{a^x b!}{b^y a!}\neq 1$. Indeed, suppose the contrary

$$b!a^x=a!b^y \tag{7}$$

for some $x>a$ and $y>b>2$. Then (7) gives that $b-1$ divides b^{y-1}, which is a contradiction. Since $\min(x,y)>2b\log b$, we deduce from (4) that

$$\max(x,y)\leqq 2\log b\min(x,y), \tag{8}$$

and Lemma 2 yields

$$\left|\frac{a^x b!}{b^y a!}-1\right|>\exp\left\{-c_2\cdot\log\left(\frac{b!}{a!}\right)\cdot\log a\cdot\log b\cdot\log\max(x,y)\right\}.$$

Comparing now this inequality with (6) we infer that

$$\min(x,y)<c_3\cdot b\cdot\log\left(\frac{b!}{a!}\right)\cdot\log a\cdot(\log b)^2,$$

and (8) completes the proof of the Theorem.

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SIMULTANEOUS EXTENSIONS OF CAUCHY STRUCTURES

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0. Introduction. The papers [3] deal, among others, with the following problem: in a closure space (X, c), let a (possibly empty) family of subsets X_i $(i \in I)$ be given, and, for each $i \in I$, a merotopy $\mathbf{M}_i$ on X_i; look for an extension of $\{c; \mathbf{M}_i\}$, i.e. for a merotopy $\mathbf{M}$ on X such that $\mathbf{M}$ induces the closure c and its restriction to X_i coincides with $\mathbf{M}_i$. The author considered in [2] the same problem in the case when $\mathbf{M}_i$ and $\mathbf{M}$ are filter merotopies or (equivalently) $\mathbf{S}_i$ and $\mathbf{S}$ are screens on X_i and X, respectively (for the terminology, see Chapter 1 below). The present paper intends to investigate from this point of view a still more special kind of structures, namely Cauchy structures; in fact, our results will concern two classes of Cauchy structures only and questions related with Cauchy structures in general remain open until future publications.

1. Preliminaries. For a set X, let us denote by $\operatorname{Fil} X$ the collection of all filters in X (including the improper filter $\exp X$). For $\mathbf{a} \subset \exp X$, denote by $\operatorname{fil}_X \mathbf{a} = \operatorname{fil} \mathbf{a}$ the smallest filter containing $\mathbf{a}$, by $\sec_X \mathbf{a} = \sec \mathbf{a}$ the collection of all subsets of X that meet each element of $\mathbf{a}$. In particular, we write $\dot{A} = \operatorname{fil}_X\{A\}$ for $A \subset X$ and $\dot{x} = \dot{A}$ for $x \in X$, $A = \{x\}$.

If $\mathbf{a} \subset \exp X$ and $X_0 \subset X$, we denote by $\mathbf{a}|X_0$ the collection of all intersections $A \cap X_0$ where $A \in \mathbf{a}$. If $\mathbf{s}$ is a filter in X then $\mathbf{s}|X_0$ is a filter in X_0. Conversely, if $\mathbf{s}_0 \in \operatorname{Fil} X_0$, then $\mathbf{s}_0^X = \operatorname{fil}_X \mathbf{s}_0$ is the finest filter $\mathbf{s}$ in X such that $\mathbf{s}|X_0 = \mathbf{s}_0$.

For $\mathbf{a}, \mathbf{b} \subset \exp X$, let us introduce the notation $\mathbf{a}\Delta\mathbf{b}$ to denote the situation that each $A \in \mathbf{a}$ meets each $B \in \mathbf{b}$; we write $\mathbf{a}\overline{\Delta}\mathbf{b}$ in the opposite case.

A *screen* on X (see [2]) is a set $\emptyset \neq \mathbf{S} \subset \operatorname{Fil} X$ such that

(1.1) $\quad x \in X$ implies that there is $\mathbf{s} \in \mathbf{S}$ such that $x \in \bigcap \mathbf{s}$,

(1.2) $\quad \mathbf{s} \in \mathbf{S}$ and $\mathbf{s} \subset \mathbf{s}' \in \operatorname{Fil} X$ imply $\mathbf{s}' \in \mathbf{S}$.

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If $\emptyset \neq \mathbf{S} \subset \operatorname{Fil} X$ fulfils (1.1), we say that $\mathbf{S}$ is a *screen base* on X; it *generates* the screen $\mathbf{S}'$ composed of all filters in X finer than some element of $\mathbf{S}$. We also say that $\mathbf{S}$ is a *screen base for the screen* $\mathbf{S}'$.

If $\mathbf{S}$ is a screen on X and $X_0 \subset X$ then $\mathbf{S}|X_0 = \{\mathbf{s}|X_0 : \mathbf{s} \in \mathbf{S}\}$ is a screen on X_0.

The screen $\mathbf{S}$ on X induces a *closure* (see [3], 0.1) $c = c(\mathbf{S})$ on X defined by $x \in c(A)$ iff there is $\mathbf{s} \in \mathbf{S}$ such that $\{x\}, A \in \sec \mathbf{s}$. Clearly if $\mathbf{s} \in \mathbf{S}$ and $x \in \bigcap \mathbf{s}$ then $\mathbf{s} \to x$ for $c(\mathbf{S})$. Observe that the free filters in $\mathbf{S}$ do not have any influence on $c(\mathbf{S})$. If $X_0 \subset X$ and $\mathbf{S}$ is a screen on X, we have $c(\mathbf{S}|X_0) = c(\mathbf{S})|X_0$.

A screen $\mathbf{S}$ on X is said to be *Riesz* iff $\mathbf{v}_c(x) \in \mathbf{S}$ for $x \in X$, where $\mathbf{v}_c(x)$ denotes the c-neighbourhood filter of x for the closure $c = c(\mathbf{S})$; $\mathbf{S}$ is said to be *Lodato* iff $\mathbf{v}_c(\mathbf{s}) \in \mathbf{S}$ for each $\mathbf{s} \in \mathbf{S}$ where $c = c(\mathbf{S})$ again and, for an arbitrary $\mathbf{s} \in \operatorname{Fil} X$, $\mathbf{v}_c(\mathbf{s})$ is composed of all sets $V \subset X$ such that there is $S \in \mathbf{s}$ satisfying $V \in \mathbf{v}_c(x)$ for $x \in S$. If c is a topology then $\mathbf{v}_c(\mathbf{s})$ is the filter generated by the filter base composed of the c-open elements of $\mathbf{s}$. If $\mathbf{S}$ is a Lodato screen then $c(\mathbf{S})$ is a topology; if c is a topology and $c = c(\mathbf{S})$ then $\mathbf{S}$ is Lodato iff it is generated by a screen base composed of c-open filters.

If $\mathbf{S}$ is a Riesz or Lodato screen on X then so is $\mathbf{S}|X_0$ on $X_0 \subset X$.

A *Cauchy structure* $\mathbf{S}$ on X (see e.g. [4], p. 12) is a screen satisfying

$$\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{S}, \quad \mathbf{s}_1 \Delta \mathbf{s}_2 \quad \text{imply} \quad \mathbf{s}_1 \cap \mathbf{s}_2 \in \mathbf{S}. \tag{1.3}$$

We shall call *Cauchy screen* or briefly C-*screen* on X a Cauchy structure on X in the above sense. A CR-*screen* or CL-*screen* is a C-screen that is Riesz or Lodato, respectively.

If a screen base $\mathbf{S}$ fulfils (1.3) then the screen generated by $\mathbf{S}$ is a C-screen since $\mathbf{s}_1 \subset \mathbf{s}_1'$, $\mathbf{s}_2 \subset \mathbf{s}_2'$, $\mathbf{s}_1' \Delta \mathbf{s}_2'$ imply $\mathbf{s}_1 \Delta \mathbf{s}_2$. This is the case in particular when $\mathbf{s}_1 \overline{\Delta} \mathbf{s}_2$ for $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{S}$, $\mathbf{s}_1 \neq \mathbf{s}_2$.

If $\mathbf{S}$ is a C-screen on X and $X_0 \subset X$ then $\mathbf{S}|X_0$ is a C-screen on X_0 because $\mathbf{s}_1|X_0 \Delta \mathbf{s}_2|X_0$ implies $\mathbf{s}_1 \Delta \mathbf{s}_2$, further $\mathbf{s}_1 \cap \mathbf{s}_2|X_0 = (\mathbf{s}_1|X_0) \cap (\mathbf{s}_2|X_0)$.

Let us agree in saying that $(\mathbf{s}_0, \dots, \mathbf{s}_n)$ is a *Cauchy chain* on X iff $\mathbf{s}_i \in \operatorname{Fil} X$ for $i = 0, \dots, n$ and $\mathbf{s}_{i-1} \Delta \mathbf{s}_i$ for $i = 1, \dots, n$. An easy induction based on the observation

$$\begin{cases} \text{for} \quad \mathbf{s}_i, \mathbf{t}_j \in \operatorname{Fil} X, \ \bigcap\limits_0^n \mathbf{s}_i \Delta \bigcap\limits_0^m \mathbf{t}_j \quad \text{iff there are} \\ i \ \text{ and } \ j \ \text{ such that } \ \mathbf{s}_i \Delta \mathbf{t}_j \end{cases} \tag{1.4}$$

shows that $\bigcap\limits_0^n \mathbf{s}_i \in \mathbf{S}$ whenever $\mathbf{S}$ is a C-screen and $(\mathbf{s}_0, \dots, \mathbf{s}_n)$ is a Cauchy chain such that $\mathbf{s}_i \in \mathbf{S}$ for $i = 0, \dots, n$.

For screens $\mathbf{S}_1$ and $\mathbf{S}_2$ on X, $\mathbf{S}_1$ is said to be *coarser* than $\mathbf{S}_2$, $\mathbf{S}_2$ *finer* than $\mathbf{S}_1$, iff $\mathbf{S}_1 \supset \mathbf{S}_2$. If $\mathbf{S}_1$ is coarser than $\mathbf{S}_2$ then $c(\mathbf{S}_1)$ is coarser than $c(\mathbf{S}_2)$.

For a screen $\mathbf{S}$ on X, the finest C-screen $\mathbf{S}^{\mathrm{C}}$ coarser than $\mathbf{S}$ is generated by the screen base $\mathbf{S}'$ composed of all intersections $\bigcap_0^n \mathbf{s}_i$ where $(\mathbf{s}_0, \dots, \mathbf{s}_n)$ is a Cauchy chain such that $\mathbf{s}_i \in \mathbf{S}$ $(i = 0, \dots, n)$; $\mathbf{S}'$ fulfils (1.3) in consequence of (1.4). If $\mathbf{B}$ is a screen base for $\mathbf{S}$, it suffices to take intersections $\bigcap_0^n \mathbf{s}_i$ such that the elements of the Cauchy chain $(\mathbf{s}_0, \dots, \mathbf{s}_n)$ belong to $\mathbf{B}$.

We are now able to formulate more precisely the purpose of the present paper. Let (X, c) be a closure space, $X_i \subset X$ for $i \in I$ ($I = \emptyset$ can happen), and $\mathbf{S}_i$ a given screen on X_i. We look for CR- or CL-*extensions* of $\{c; \mathbf{S}_i\}$, i.e. for a CR-screen or CL-screen $\mathbf{S}$ on X such that

$$(1.5) \qquad c(\mathbf{S}) = c, \quad \mathbf{S}|X_i = \mathbf{S}_i \quad \text{for} \quad i \in I.$$

In Sections 2 and 3, we always assume the following standard hypotheses: (X, c) is a closure space, $X_i \subset X$, $\mathbf{S}_i$ is a screen on X_i for $i \in I$, and extension will mean a screen $\mathbf{S}$ on X fulfilling (1.5). We write $c_i = c(\mathbf{S}_i)$ and $X_{ij} = X_i \cap X_j$ for $i, j \in I$.

2. CR-extensions. A simple necessary condition can very easily be obtained:

LEMMA 2.1. *If* $\mathbf{S}$ *is a* CR-*screen then* $c = c(\mathbf{S})$ *fulfils the condition*

$$(2.1.1) \qquad \textit{for} \quad x, y \in X, \quad \mathbf{v}_c(x) \neq \mathbf{v}_c(y) \quad \textit{implies} \quad \mathbf{v}_c(x) \overline{\Delta} \mathbf{v}_c(y).$$

PROOF. If $\mathbf{v}_c(x) \Delta \mathbf{v}_c(y)$ then $\mathbf{s} = \mathbf{v}_c(x) \cap \mathbf{v}_c(y) \in \mathbf{S}$ since the CR-screen $\mathbf{S}$ contains both $\mathbf{v}_c(x)$ and $\mathbf{v}_c(y)$. Now $\mathbf{s} \in \mathbf{S}$, $x \in \cap\, \mathbf{s}$ imply $\mathbf{s} \to x$ for c and $\mathbf{v}_c(x) \subset \mathbf{s}$. Similarly $\mathbf{v}_c(y) \subset \mathbf{s}$ so that $\mathbf{s} = \mathbf{v}_c(x) = \mathbf{v}_c(y)$. □

For topological spaces, condition (2.1.1) is often called axiom (S_2) (see e.g. [1], p. 95). Therefore we shall say that the closure c (or the closure space (X, c)) is S_2 iff (2.1.1) holds. This is the case of course if c is *Hausdorff* (or T_2), i.e. if $x \neq y$ implies $\mathbf{v}_c(x) \overline{\Delta} \mathbf{v}_c(y)$. According to [5], Definition 3.1, the C-screen $\mathbf{S}$ is said to be Hausdorff iff $c(\mathbf{S})$ is T_1 (i.e. separated).

It is easy to prove the converse of 2.1 in the following form:

LEMMA 2.2. *If* c *is an* S_2 *closure then* $\mathbf{B} = \{\mathbf{v}_c(x) : x \in X\}$ *is a screen base for a* CR-*screen* $\mathbf{S}$ *such that* $c = c(\mathbf{S})$.

PROOF. By (S_2) the screen generated by $\mathbf{B}$ is Cauchy. If $x \in c(A)$, $\mathbf{v}_c(x) \in \mathbf{S}$ satisfies $x \in \cap\, \mathbf{v}_c(x)$, $A \in \sec \mathbf{v}_c(x)$. If $\mathbf{s} \in \mathbf{S}$, $x \in \cap\, \mathbf{s}$, $A \in \sec \mathbf{s}$, then $\mathbf{s} \supset \mathbf{v}_c(y)$ for some $y \in X$; by $x \in \cap\, \mathbf{v}_c(y)$ and (S_2), $\mathbf{v}_c(y) = \mathbf{v}_c(x)$, hence $A \in \sec \mathbf{v}_c(x)$ and $x \in c(A)$. Thus $c = c(\mathbf{S})$ and $\mathbf{S}$ is Riesz. □

Observe that a C-screen need not be a CR-screen although it induces a T_2 closure:

EXAMPLE 2.3. In a set X, let us say that $\mathbf{s} \in \operatorname{Fil} X$ is an *elementary filter* if $\mathbf{s} = \bigcap_{0}^{n} \mathbf{u}_i$ where $\mathbf{u}_i$ is an ultrafilter for $i = 0, \dots, n$. Then by (1.4) $\mathbf{u} \Delta \mathbf{s}$ for an ultrafilter $\mathbf{u}$ implies $\mathbf{u} = \mathbf{u}_i$ for some i; hence $\mathbf{s}$ is an elementary filter iff there are finitely many ultrafilters only finer than $\mathbf{s}$. Therefore a filter finer than an elementary filter is elementary itself.

Let S be a screen on X and denote by S^e the collection of $\exp X$ and of all elementary filters belonging to S. Then S^e is a screen since $\dot{x}$ is an elementary filter for $x \in X$. If S is a C-screen then the same holds for S^e. We have $c(\mathsf{S}^e) = c(\mathsf{S})$ because $\mathsf{S}^e \subset \mathsf{S}$ implies that $c(\mathsf{S}^e)$ is finer than $c(\mathsf{S})$, and if $\mathbf{s} \in \mathsf{S}$, $x \in \bigcap \mathbf{s}$, $A \in \sec \mathbf{s}$, then $\dot{x} \supset \mathbf{s}$, $(\mathbf{s}|A)^X \supset \mathbf{s}$, so that by taking an ultrafilter $\mathbf{u}$ finer than $(\mathbf{s}|A)^X$, we obtain an elementary filter $\mathbf{s}' = \dot{x} \cap \mathbf{u}$ finer than $\mathbf{s}$ and satisfying $\{x\}$, $A \in \sec \mathbf{s}'$. Hence $c(\mathsf{S})$ is finer than $c(\mathsf{S}^e)$.

Consider now a non-discrete T_2 topology c that is first countable and let S be a C-screen such that $c = c(\mathsf{S})$ (by 2.2, S may be chosen to be a CR-screen). Then $c = c(\mathsf{S}^e)$ and S^e is a C-screen; however, S^e cannot contain $\mathbf{v}_c(x)$ for a point x that is the limit of a sequence (x_n) such that $x \neq x_n \neq x_m$ for $n \neq m$. In fact, there are infinitely many subsequences of (x_n) each two of which correspond to disjoint sets of indices, and by choosing ultrafilters finer than the corresponding Fréchet filters, it turns out that there are infinitely many ultrafilters finer than $\mathbf{v}_c(x)$, and the latter cannot belong to S^e. □

LEMMA 2.4 (cf. [2], (2.7.1)). *If S is a Riesz extension then $\mathbf{v}_c(x)|X_i \in \mathsf{S}_i$ for $x \in X$, $i \in I$.* □

LEMMA 2.5. *If S is an extension and $\mathbf{s}_i \in \mathsf{S}_i$ then $\mathbf{s}_i^X \in \mathsf{S}$.*

PROOF. There is $\mathbf{s} \in \mathsf{S}$ such that $\mathbf{s}|X_i = \mathbf{s}_i$ and $\mathbf{s}_i^X$ is finer than $\mathbf{s}$. □

COROLLARY 2.6. *If S is a Cauchy extension and $(\mathbf{t}_0, \dots, \mathbf{t}_n)$ is a Cauchy chain such that $\mathbf{t}_j = \mathbf{s}_j^X$, $\mathbf{s}_j \in \mathsf{S}_{i_j}$, $i_j \in I$, then*

$$\left(\bigcap_{0}^{n} \mathbf{t}_j\right) |X_k \in \mathsf{S}_k \quad (k \in I). \quad \square \tag{2.6.1}$$

COROLLARY 2.7. *If S is a CR-extension, $\mathbf{s}_i \in \mathsf{S}_i$, and $x \in X$ is a cluster point for c of $\mathbf{s}_i^X$, then $\mathbf{s}_i^X \to x$ with respect to c.*

PROOF. $\mathbf{v}_c(x) \Delta \mathbf{s}_i^X$ by hypothesis and both filters belong to S, hence $\mathbf{s} = \mathbf{v}_c(x) \cap \mathbf{s}_i^X \in \mathsf{S}$ and $x \in \bigcap \mathbf{s}$, so $\mathbf{s} \to x$ for c, and $\mathbf{s}_i^X$ is finer than $\mathbf{s}$. □

Now we can prove:

THEOREM 2.8. *There is a CR-extension of $\{c; \mathsf{S}_i\}$ iff the following conditions hold:*

(a) *c is an S_2 closure,*

(b) *$\mathbf{v}_c(x)|X_i \in \mathsf{S}_i$ for $x \in X$, $i \in I$,*

(c) *if* $\mathbf{s}_i \in \mathbf{S}_i$ *and* $x \in X$ *is a cluster point for* c *of* $\mathbf{s}_i^X$ *then* $\mathbf{s}_i^X \to x$ *for* c,

(d) *if* $(\mathbf{t}_0, \dots, \mathbf{t}_n)$ *is a Cauchy chain such that* $\mathbf{t}_j = \mathbf{s}_j^X$, $\mathbf{s}_j \in \mathbf{S}_{i_j}$, $i_j \in I$, *then* $\left(\bigcap_0^n \mathbf{t}_j\right) | X_k \in \mathbf{S}_k$ *for* $k \in I$.

If these conditions are fulfilled then the filters $\mathbf{v}_c(x)$ $(x \in X)$ *and the filters*

$$(2.8.1) \qquad \bigcap_0^n t_j \qquad (\mathbf{t}_j \quad \textit{as in} \quad \text{(d)})$$

constitute a screen base for the finest CR-*extension* $\mathbf{S}_{\mathrm{CR}}^1 = \mathbf{S}_{\mathrm{CR}}^1(c; \mathbf{S}_i)$.

PROOF. *Necessity*: 2.1, 2.4, 2.7, 2.6.

Sufficiency: The collection $\mathbf{B}$ of the neighbourhood filters $\mathbf{v}_c(x)$ and the filters (2.8.1) is obviously a screen base. It generates a Cauchy screen because $\mathbf{s}', \mathbf{s}'' \in \mathbf{B}$, $\mathbf{s}' \Delta \mathbf{s}''$ imply $\mathbf{s}' \cap \mathbf{s}'' \in \mathbf{B}$. This is a consequence of (a) if both $\mathbf{s}'$ and $\mathbf{s}''$ are neighbourhood filters, and of (d) if both have the form (2.8.1) (cf. (1.4)). If $\mathbf{s}' = \mathbf{v}_c(x)$, $\mathbf{s}'' = \bigcap_0^n \mathbf{t}_j$ as in (2.8.1), then by (1.4) one of the filters $\mathbf{t}_j$ has x for cluster point (with respect to c) and then, by (c), $\mathbf{t}_j \to x$. Now $\mathbf{t}_{j-1} \Delta \mathbf{t}_j$ (except for $j = 0$) and $\mathbf{t}_j \Delta \mathbf{t}_{j+1}$ (except for $j = n$) imply, again by (c), $\mathbf{t}_{j-1} \to x$, $\mathbf{t}_{j+1} \to x$. After a finite number of steps we obtain $\mathbf{t}_j \to x$ for each j, hence $\mathbf{s}'' \to x$, and $\mathbf{s}' \cap \mathbf{s}'' = \mathbf{s}'$.

By 2.2 $c(\mathbf{S})$ is coarser than c for the screen $\mathbf{S}$ generated by $\mathbf{B}$. If $\mathbf{s} \in \mathbf{S}$, $\{x\}$, $A \in \sec \mathbf{s}$, then $x \in c(A)$ (and $c(\mathbf{S})$ is finer than c). In fact, we may suppose that $\mathbf{s} \in \mathbf{B}$, and the case $\mathbf{s} = \mathbf{v}_c(x)$ is settled by 2.2. If $\mathbf{s}$ is of the form (2.8.1) then $\{x\} \in \sec \mathbf{t}_j = \mathbf{s}_j^X$ for a j, hence by (c) $\mathbf{s}_j^X \to x$ for c, and a successive application of (c) as above furnishes $\mathbf{s} \to x$, $x \in c(A)$.

Therefore $\mathbf{S}$ is a Riesz screen. (b) and (d) show that $\mathbf{s}|X_i \in \mathbf{S}_i$ $(i \in I)$ if $\mathbf{s} \in \mathbf{B}$, and then for $\mathbf{s} \in \mathbf{S}$, too. On the other hand, $\mathbf{s}_i \in \mathbf{S}_i$ implies $\mathbf{s}_i^X \in \mathbf{B} \subset \mathbf{S}$ and $\mathbf{s}_i^X | X_i = \mathbf{s}_i$.

By this, $\mathbf{S} = \mathbf{S}_{\mathrm{CR}}^1$ is a CR-extension. If $\mathbf{S}'$ is another CR-extension then $\mathbf{B} \subset \mathbf{S}'$ by 2.5, so that $\mathbf{S} \subset \mathbf{S}'$. □

Observe that 2.8 (c) and (d) hold as soon as they are fulfilled for filters $\mathbf{s}_i$ or $\mathbf{s}_j$ taken from screen bases generating $\mathbf{S}_i$ or $\mathbf{S}_j$, respectively.

Further necessary conditions can be easily formulated for the existence of a CR-extension:

LEMMA 2.9. *If* $\mathbf{S}$ *is a* CR-*extension then*

(a) $c_i = c|X_i$ *for* $i \in I$,

(b) $\mathbf{S}_i | X_{ij} = \mathbf{S}_j | X_{ij}$ *for* $i, j \in I$,

(c) $\mathbf{S}_i$ *is a* CR-*screen for* $i \in I$.

PROOF. (a) and (b) hold for every extension ([2], (1.19) and (1.20)). □

We show that each of the conditions 2.8 (a) to (d) is independent of the others even if 2.9 (a) to (c) hold. For (a), this is shown by the case $I = \emptyset$.

EXAMPLE 2.10. Let $x = \mathbf{R}$, c be the Euclidean topology, $X_0 = (0, +\infty)$, $c_0 = c|X_0$, and let S_0 be generated by the screen base composed of all c_0-neighbourhood filters. By 2.2 S_0 is a CR-screen and $c(\mathrm{S}_0) = c_0$; 2.9 (b) is obvious since $|I| = 1$. 2.8 (c) holds because $\mathbf{v}_{c_0}(x)^X$ has the only c-cluster point $x \in X_0$. 2.8 (d) is always fulfilled if $I = \{0\}$ and S_0 is a C-screen by

$$\mathbf{s}_i^X \Delta \mathbf{s}_j^X \quad \text{iff} \quad \mathbf{s}_i \Delta \mathbf{s}_j \quad \text{for} \quad \mathbf{s}_i, \mathbf{s}_j \in \operatorname{Fil} X_0, \tag{2.10.1}$$

$$\bigcap_0^n \mathbf{s}_i^X = \left(\bigcap_0^n \mathbf{s}_i\right)^X \quad \text{for} \quad \mathbf{s}_i \in \operatorname{Fil} X_0. \tag{2.10.2}$$

However, $\mathbf{v}_c(0)|X_0 \notin \mathrm{S}_0$. □

EXAMPLE 2.11. Consider X, c, X_0, c_0 as in 2.10, and let S_0 be generated by the filters $\mathbf{v}_{c_0}(x)$ $(x \in X_0)$ and by

$$\mathbf{s}_0 = \left(\mathbf{v}_c(0)|X_0\right) \cap \operatorname{fil}_{X_0} \mathbf{r}$$

where $\mathbf{r} = \left\{(a, +\infty) : a \in X_0\right\}$. We have $c_0 = c(\mathrm{S}_0)$ as above because $\mathbf{s}_0$ is a free filter. Hence S_0 is a Riesz screen again and 2.9 (b) holds for the same reason as in 2.10. S_0 is Cauchy since $\mathbf{s}_0$ does not have any c_0-cluster point. Now

$$\mathbf{v}_c(x)|X_0 = \mathbf{v}_{c_0}(x) \in \mathrm{S}_0 \quad \text{for} \quad x \in X_0,$$

$$\mathbf{v}_c(x)|X_0 = \exp X_0 \in \mathrm{S}_0 \quad \text{for} \quad x < 0,$$

$$\mathbf{v}_c(0)|X_0 \supset \mathbf{s}_0,$$

so that 2.8 (b) holds. However, $\mathbf{s}_0^X$ has the c-cluster point 0 without converging to 0. □

EXAMPLE 2.12. Let $Y_i = \mathbf{R} \times \{i\}$ for $i = 0, 1, 2$, $X = \bigcup_0^2 Y_i$, c the Euclidean topology of $\mathbf{R}^2$ restricted to X, $X_i = X - Y_i$, $c_i = c|X_i$, $\mathbf{r}_i = \left\{(a, +\infty) \times \{i\} : a \in \mathbf{R}\right\}$. Let S_i be generated by the screen base composed of the filters $\mathbf{v}_c(x)|X_i$ $(x \in X_i)$ and, for $i = 0$, of

$$\mathbf{s}_{01} = \operatorname{fil}_{X_0} \mathbf{r}_1 \quad \text{and} \quad \mathbf{s}_{02} = \operatorname{fil}_{X_0} \mathbf{r}_2,$$

for $i = 1$ of

$$\mathbf{s}_1 = (\operatorname{fil}_{X_1} \mathbf{r}_0) \cap (\operatorname{fil}_{X_1} \mathbf{r}_2),$$

for $i = 2$ of

$$\mathbf{s}_2 = (\mathrm{fil}_{X_2}\, \mathbf{r}_0) \cap (\mathrm{fil}_{X_2}\, \mathbf{r}_1).$$

Now $c_i = c(\mathrm{S}_i)$ since $\mathbf{s}_{01}$, $\mathbf{s}_{02}$, $\mathbf{s}_1$, $\mathbf{s}_2$ are free filters, hence the screens S_i are Riesz. They are Cauchy because $\mathbf{s}_{01}$, $\mathbf{s}_{02}$, $\mathbf{s}_1$, $\mathbf{s}_2$ do not have any c-cluster points and $\mathbf{s}_{01}\overline{\Delta}\mathbf{s}_{02}$. For the same reason, 2.8 (c) holds, and 2.8 (b) is obviously valid. 2.9 (b) follows from the formulae

$$\mathbf{s}_{01}|X_{01} = \exp X_{01} = \exp X_1|X_{01},$$

$$\mathbf{s}_{01}|X_{02} = \mathrm{fil}_{Y_1}\, \mathbf{r}_1 = \mathbf{s}_2|X_{02},$$

$$\mathbf{s}_{02}|X_{01} = \mathrm{fil}_{Y_2}\, \mathbf{r}_2 = \mathbf{s}_1|X_{01},$$

$$\mathbf{s}_{02}|X_{02} = \exp X_{02} = \exp X_2|X_{02},$$

$$\mathbf{s}_1|X_{12} = \mathrm{fil}_{Y_0}\, \mathbf{r}_0 = \mathbf{s}_2|X_{12}$$

and the obvious ones concerning the neighbourhood filters.

However, $(\mathbf{s}_1^X, \mathbf{s}_2^X)$ is a Cauchy chain and

$$(\mathbf{s}_1^X \cap \mathbf{s}_2^X)\, |X_0 = (\mathrm{fil}_{X_0}\mathbf{r}_1) \cap (\mathrm{fil}_{X_0}\mathbf{r}_2) \notin \mathrm{S}_0. \quad \square$$

From a certain point of view, Example 2.12 is the best one; in fact, we can show:

LEMMA 2.13. *If* $|I| \leqq 2$, *each* S_i *is a Cauchy screen, and* 2.9 (b) *is fulfilled, then* 2.8 (d) *holds.*

PROOF. This is obvious for $I = \emptyset$ and we have shown it in 2.10 for $|I| = 1$. Let $I = \{0, 1\}$ and consider a Cauchy chain $(\mathbf{t}_0, \dots, \mathbf{t}_n)$ such that $\mathbf{t}_j = \mathbf{s}_j^X$, $\mathbf{s}_j \in \mathrm{S}_0$ or S_1 for each j.

Suppose $\mathbf{s}_j \in \mathrm{S}_0$ for $j_1 \leqq j \leqq j_2$. Then by (2.10.2)

$$\bigcap_{j=j_1}^{j_2} \mathbf{s}_j^X = \mathbf{s}^X \quad \text{for} \quad \mathbf{s} = \bigcap_{j=j_1}^{j_2} \mathbf{s}_j$$

and by (2.10.1) $(\mathbf{s}_{j_1}, \dots, \mathbf{s}_{j_2})$ is a Cauchy chain in X_0. Hence $\mathbf{s} \in \mathrm{S}_0$ and

$$\mathbf{s}_{j_1-1}^X \Delta \mathbf{s}_{j_1}^X \quad (\text{except for} \quad j_1 = 0),$$

$$\mathbf{s}_{j_2}^X \Delta \mathbf{s}_{j_2+1}^X \quad (\text{except for} \quad j_2 = n)$$

clearly imply

$$\mathbf{s}_{j_1-1}^X \Delta \mathbf{s}^X, \quad \mathbf{s}^X \Delta \mathbf{s}_{j_2+1}^X.$$

A similar statement is valid if $\mathbf{s}_j \in \mathbf{S}_1$ for $j_1 \leqq j \leqq j_2$. Therefore it suffices to consider Cauchy chains $(\mathbf{t}_0, \dots, \mathbf{t}_n)$ such that $\mathbf{t}_j = \mathbf{s}_j^X$ and, say, $\mathbf{s}_j \in \mathbf{S}_0$ for $j = 2k$, $\mathbf{s}_j \in \mathbf{S}_1$ for $j = 2k+1$.

Now we proceed by induction according to n. For $n = 0$, $\mathbf{s}_0 \in \mathbf{S}_0$ implies $\mathbf{s}_0^X | X_0 = \mathbf{s}_0 \in \mathbf{S}_0$ and $\mathbf{s}_0^X | X_1 \in \mathbf{S}_1$, too. In fact,

$$\mathbf{s}_0^X | X_1 = \left(\mathbf{s}_0 | X_{01}\right)^{X_1} \quad \text{for} \quad \mathbf{s}_0 \in \operatorname{Fil} X_0. \tag{2.13.1}$$

Namely, $\mathbf{s}_0$ is a base in X for $\mathbf{s}_0^X$, hence $\mathbf{s}_0 | X_1 = \mathbf{s}_0 | X_{01}$ is a base in X_1 for $\mathbf{s}_0^X | X_1$. On the other hand, $\mathbf{s}_0 | X_{01}$ is a base in X_1 for $\left(\mathbf{s}_0 | X_{01}\right)^{X_1}$, too.

From $\mathbf{s}_0 \in \mathbf{S}_0$ and (2.13.1) we obtain by 2.9 (b)

$$\mathbf{s}_0^X | X_1 = \left(\mathbf{s}_1 | X_{01}\right)^{X_1}$$

for some $\mathbf{s}_1 \in \mathbf{S}_1$, and $\left(\mathbf{s}_1 | X_{01}\right)^{X_1} \supset \mathbf{s}_1$ implies $\mathbf{s}_0^X | X_1 \in \mathbf{S}_1$ as stated.

Suppose the statement holds for some n and consider a Cauchy chain $(\mathbf{t}_0, \dots, \mathbf{t}_{n+1})$ such that $\mathbf{t}_j = \mathbf{s}_j^X$, $\mathbf{s}_j \in \mathbf{S}_0$ if $j = 2k$, $\mathbf{s}_j \in \mathbf{S}_1$ if $j = 2k+1$. Assume $n+1$ is even (the other case is established by interchanging the roles of $\mathbf{S}_0$ and $\mathbf{S}_1$). By the induction hypothesis, $\mathbf{s} = \bigcap_0^n \mathbf{t}_j$ satisfies $\mathbf{s} | X_0 \in \mathbf{S}_0$, $\mathbf{s} | X_1 \in \mathbf{S}_1$. Now $\mathbf{t}_{n+1} = \mathbf{s}_{n+1}^X$, $\mathbf{s}_{n+1} \in \mathbf{S}_0$ implies

$$\left(\bigcap_0^{n+1} \mathbf{t}_j\right) | X_0 = \left(\mathbf{s} \cap \mathbf{s}_{n+1}^X\right) | X_0 = (\mathbf{s} | X_0) \cap \mathbf{s}_{n+1}, \tag{2.13.2}$$

further $\mathbf{s} \subset \mathbf{t}_n$ and $\mathbf{t}_n \Delta \mathbf{s}_{n+1}^X$ imply $\mathbf{s} \Delta \mathbf{s}_{n+1}^X$ and then $\mathbf{s} | X_0 \Delta \mathbf{s}_{n+1}$ by $X_0 \in \in \mathbf{s}_{n+1}^X$ so that the right hand side of (2.13.2) belongs to $\mathbf{S}_0$. On the other hand,

$$\left(\bigcap_0^{n+1} \mathbf{t}_j\right) | X_1 = \left(\mathbf{s} \cap \mathbf{s}_{n+1}^X\right) | X_1 = \left(\mathbf{s} | X_1\right) \cap \left(\mathbf{s}_{n+1}^X | X_1\right) \tag{2.13.3}$$

where $\mathbf{s}_{n+1}^X | X_1 \in \mathbf{S}_1$ by the reasoning applied above for $n = 0$. Now $\mathbf{t}_n \Delta \mathbf{t}_{n+1}$ and $\mathbf{t}_n = \mathbf{s}_n^X$, $\mathbf{s}_n \in \mathbf{S}_1$, $X_1 \in \mathbf{s}_n^X$ imply $\mathbf{t}_n | X_1 \Delta \mathbf{t}_{n+1} | X_1$, hence, by $\mathbf{s} \subset \mathbf{t}_n$, $\mathbf{s} | X_1 \Delta \mathbf{s}_{n+1}^X | X_1$, so that the right hand side of (2.13.3) belongs to $\mathbf{S}_1$. □

COROLLARY 2.14. *If* (2.8) (a), (b), (c), (2.9) (b), (c) *are fulfilled and* $|I| \leqq 2$, *then* $\mathbf{S}_{\mathrm{CR}}^1$ *constructed as in* 2.8 *is the finest* CR*-extension.* □

Observe that 2.9 (c) can be weakened to: $\mathbf{S}_i$ is a Cauchy screen for each i.

It is shown in [2], 2.8 that if c is S_2 (or even satisfies a weaker hypothesis) and 2.8 (b), 2.9 (a) and (b) hold, then the finest Riesz extension $\mathbf{S}_R^1$ is generated by the screen base composed of the filters $\mathbf{v}_c(x)$ $(x \in X)$ and $\mathbf{s}_i^X$ $(\mathbf{s}_i \in \mathbf{S}_i,\ i \in I)$. From this, we easily deduce

THEOREM 2.15. *Under the hypotheses of* 2.8, *we have*

$$\mathbf{S}^1_{\mathrm{CR}} = \left(\mathbf{S}^1_{\mathrm{CR}}\right)^C.$$

PROOF. If $\mathbf{S}$ is a CR-extension then $\mathbf{S}^1_R \subset \mathbf{S}$, consequently $\left(\mathbf{S}^1_R\right)^C \subset \mathbf{S}$. From $\mathbf{S}^1_R \subset \left(\mathbf{S}^1_R\right)^C \subset \mathbf{S}$ we deduce that $\left(\mathbf{S}^1_R\right)^C$ is an extension, namely a CR one; therefore it coincides with the finest CR-extension $\mathbf{S}^1_{\mathrm{CR}}$ by 2.8. □

$\mathbf{S}^1_{\mathrm{CR}}$ can be distinct from $\mathbf{S}^1_R$:

EXAMPLE 2.16. Consider X, X_i, c, $\mathbf{S}_i$ as in 2.12 but for $i = 1, 2$ only. Then by 2.14 and 2.15 (whose hypotheses are fulfilled now by 2.13) there exists $\mathbf{S}^1_{CR} = \left(\mathbf{S}^1_R\right)^C$. Now clearly $\mathbf{s}_1^X, \mathbf{s}_2^X \in \mathbf{S}^1_R$ but $\mathbf{s}_1^X \cap \mathbf{s}_2^X \in \left(\mathbf{S}^1_R\right)^C$ does not belong to $\mathbf{S}^1_R$. □

In contrast to [2], 2.7, according to which there exists a coarsest Riesz extension (if there exist any), we can show that a coarsest CR-extension need not exist:

EXAMPLE 2.17. Let $X = \mathbf{R} \times \{0, 1\}$, let c be the restriction to X of the Euclidean topology of the plane,

$$\mathbf{r}_{0+} = \left\{(a, +\infty) \times \{0\} : a \in \mathbf{R}\right\},$$

$$\mathbf{r}_{0-} = \left\{(-\infty, a) \times \{0\} : a \in \mathbf{R}\right\},$$

$$\mathbf{r}_1 = \left\{(a, +\infty) \times \{1\} : a \in \mathbf{R}\right\},$$

and

$$\mathbf{s}_{0+} = \left(\operatorname{fil}_X \mathbf{r}_{0+}\right) \cap \left(\operatorname{fil}_X \mathbf{r}_1\right),$$

$$\mathbf{s}_{0-} = \left(\operatorname{fil}_X \mathbf{r}_{0-}\right) \cap \left(\operatorname{fil}_X \mathbf{r}_1\right).$$

For $X_0 = \mathbf{R} \times \{0\}$, let $\mathbf{S}_0$ be generated by the screen base composed of the filters $\mathbf{v}_c(x)|X_0$ $(x \in X_0)$ and $\mathbf{s}_{0+}|X_0$, $\mathbf{s}_{0-}|X_0$. On the set X, consider the screens $\mathbf{S}$ and $\mathbf{S}'$ generated by the screen bases composed of the filters $\mathbf{v}_c(x)$ $(x \in X)$ and, in the case of $\mathbf{S}$, of $\mathbf{s}_{0+}$ and $\operatorname{fil}_X \mathbf{r}_{0-}$, in the case of $\mathbf{S}'$, of $\mathbf{s}_{0-}$ and $\operatorname{fil}_X \mathbf{r}_{0+}$. As the elements of both screen bases are pairwise in relation $\overline{\Delta}$, and clearly $c(\mathbf{S}) = c(\mathbf{S}') = c$, $\mathbf{S}|X_0 = \mathbf{S}'|X_0 = \mathbf{S}_0$, both $\mathbf{S}$ and $\mathbf{S}'$ are CR-extensions of $\mathbf{S}_0$. However, any Cauchy screen coarser than both $\mathbf{S}$ and $\mathbf{S}'$ necessarily contains $\mathbf{s}_{0+} \cap \mathbf{s}_{0-}$ whose trace on X_0 does not belong to $\mathbf{S}_0$. □

Observe that this example shows the lack of a coarsest Cauchy extension in general, in contrast to [2], 2.6, according to which there is a coarsest extension of $\{c; \mathrm{S}_i\}$ whenever an extension exists at all.

3. CL-extensions. A great deal of questions arising on this field can be treated with methods applied for CR-extensions. If S is a Lodato screen and $x \in c\big(c(A)\big)$, $c = c(\mathrm{S})$, then there is $\mathbf{s} \in \mathrm{S}$ such that $\{x\}$, $c(A) \in \sec \mathbf{s}$ and clearly $A \in \sec \mathbf{v}_c(\mathbf{s})$, $\mathbf{v}_c(\mathbf{s}) \in \mathrm{S}$, so that $x \in c(A)$ and c is a topology. If $\mathrm{S}_0 = \mathrm{S}|X_0$, $\mathbf{s}_0 \in \mathrm{S}_0$, then $\mathbf{s}_0^X \in \mathrm{S}$ by 2.5 and $\mathbf{v}_c\big(\mathbf{s}_0^X\big) \in \mathrm{S}$. By this, taking into account that a Lodato screen is Riesz (since $\mathbf{v}_c(x) = \mathbf{v}_c(\dot{x})$), we immediately obtain, using 2.8, the necessity part of

THEOREM 3.1. *There exists a* CL*-extension iff the following conditions are satisfied:*

(a) *c is an S_2 topology,*

(b) *$\mathbf{v}_c(x)|X_i \in \mathrm{S}_i$ for $x \in X$, $i \in I$,*

(c) *if $\mathbf{s}_i \in \mathrm{S}_i$ and $x \in X$ is a cluster point for c of $\mathbf{v}_c\big(\mathbf{s}_i^X\big)$ then $\mathbf{v}_c\big(\mathbf{s}_i^X\big) \to x$ for c,*

(d) *if $(\mathbf{t}_0, \dots, \mathbf{t}_n)$ is a Cauchy chain such that $\mathbf{t}_j = \mathbf{v}_c\big(\mathbf{s}_j^X\big)$, $\mathbf{s}_j \in \mathrm{S}_{i_j}$, $i_j \in I$, then*

$$\left(\bigcap_0^n \mathbf{t}_j\right) |X_k \in \mathrm{S}_k \qquad (k \in I).$$

If these conditions are fulfilled then the screen base B *composed of all filters $\mathbf{v}_c(x)$ $(x \in X)$ and of all intersections*

$$\bigcap_0^n \mathbf{t}_j \qquad (\mathbf{t}_j \quad \textit{as in} \quad (\mathrm{d})) \tag{3.1.1}$$

generates the finest CL*-extension* $\mathrm{S}^1_{\mathrm{CL}} = \mathrm{S}^1_{\mathrm{CL}}(c; \mathrm{S}_i)$.

PROOF. We only have to check the sufficiency. Similarly to the proof of 2.8, (a) and (c) show that B is a screen base for a Cauchy screen S such that $c(\mathrm{S}) = c$. Thus S is a CL-screen because c is a topology and the elements of B are c-open filters. Now (b) and (d) show $\mathrm{S}|X_i \subset \mathrm{S}_i$ for $i \in I$ and $\mathrm{S}_i \subset \mathrm{S}|X_i$ follows from $\mathbf{s}_i^X \supset \mathbf{v}_c\big(\mathbf{s}_i^X\big)$. If S' is an arbitrary CL-extension then $\mathrm{B} \subset \mathrm{S}'$, hence $\mathrm{S} \subset \mathrm{S}'$. □

If c is a topology and X_0 is c-open, $\mathbf{s}_0$ is a $c_0 = c|X_0$-open filter in X_0, then clearly $\mathbf{v}_c\big(\mathbf{s}_0^X\big) = \mathbf{s}_0^X$. Hence, if each X_i is c-open and S_i is Lodato, then 2.8 (c) and (d) coincide with 3.1 (c) and (d), respectively.

For the existence of a CL-extension, further necessary conditions are, of course, 2.9 (a) and (b), and the one that the screens S_i have to be CL-screens. Examples 2.10, 2.11, 2.12 show that each of the conditions 3.1 (a) to (d) is independent of the others even if the above necessary conditions are satisfied; in fact, in 2.10, 2.11, 2.12, c is always a T_2 topology, the sets X_i are c-open and the filters in the screen bases generating $\mathbf{S}_i$ are c-open, too, so that all screens $\mathbf{S}_i$ are CL-screens and the conditions 3.1 (b), (c), (d) coincide with the respective conditions in 2.8.

Moreover, it can happen that all conditions in 2.8 and all but one conditions in 3.1 are fulfilled and the exceptional condition in 3.1 fails to be true.

EXAMPLE 3.2. Let $X = \mathbf{R}$, c^* be the Euclidean topology on X, c the Hausdorff topology for which $\mathbf{v}_c(x) = \mathbf{v}_{c^*}(x)$ if $x \neq 0$, and a base for $\mathbf{v}_c(0)$ is composed of the sets $V - N$ where V is a c^*-neighbourhood of 0 and

$$N = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\}.$$

Define $X_0 = N$ so that $c_0 = c|X_0$ is discrete, and let the screen base generating $\mathbf{S}_0$ be composed of the filters $\dot{x}|X_0$ ($x \in X_0$) and $\mathbf{s}_0 = \mathbf{v}_{c^*}(0)|X_0$. Then $c(\mathbf{S}_0) = c_0$ since $\mathbf{s}_0$ is free, so $\mathbf{S}_0$ is a CL-screen and 2.9 (a), (b) are fulfilled. The same holds for 3.1 (a), (b), (d) since $\mathbf{v}_c(\mathbf{s}_0^X)$ does not have any c-cluster point in X_0 and $\mathbf{v}_c(\mathbf{s}_0^X)|X_0 = \mathbf{s}_0$. 2.8 (c) is satisfied, too, since $\mathbf{s}_0^X$ does not have any c-cluster points. However, 3.1 (c) fails to hold since 0 is a c-cluster point of $\mathbf{v}_c(\mathbf{s}_0^X)$ but $\mathbf{s}_0^X \supset \mathbf{v}_c(\mathbf{s}_0^X)$ does not c-converge to 0. □

EXAMPLE 3.3. Let c^* denote the Euclidean topology on $\mathbf{R}^2$, $X = \mathbf{R} \times \times [0, +\infty)$, and c be the Niemytzki topology on X ($\mathbf{v}_c(p) = \mathbf{v}_{c^*}(p)|X$ for $p = (x, y)$, $y > 0$, and, for $p = (x, 0)$, a base for $\mathbf{v}_c(p)$ is composed of c^*-closed disks contained in X and containing p). Then c is T_2 and, on $X_0 = \mathbf{R} \times \{0\}$, c induces the discrete topology $c_0 = c|X_0$. Define $Q = \mathbf{Q} \times \times \{0\}$, $P = X_0 - Q$, and let $\mathbf{s}_P$ ($\mathbf{s}_Q$) be composed of those sets $S \subset X_0$ for which $P - S$ ($Q - S$) is finite. Let a screen base for $\mathbf{S}_0$ be composed $\dot{x}|X_0$ ($x \in X_0$) and of $\mathbf{s}_P$ and $\mathbf{s}_Q$. Then $c_0 = c(\mathbf{S}_0)$ ($\mathbf{s}_P$ and $\mathbf{s}_Q$ are free filters), and $\mathbf{S}_0$ is a CL-screen ($\mathbf{s}_P \overline{\Delta} \mathbf{s}_Q$ by $P \in \mathbf{s}_P$, $Q \in \mathbf{s}_Q$). Consequently 2.9 (a), (b), 3.1 (a), (b), 2.8 (d) are fulfilled (for the latter, consider $I = \{0\}$). 3.1 (c) holds since $\mathbf{v}_c(\mathbf{s}_P^X)$ clearly does not have any cluster points in $X - X_0$ and, if $p \in X_0$, choose $S \in \mathbf{s}_P$ such that $p \notin S$, a disk $D \subset X$ such that $p \in D$, and disks $D_s \subset X$ for $s \in S$ such that $s \in D_s$, $D_s \cap D = \emptyset$, so that $V = \bigcup_{s \in S} D_s \in \mathbf{v}_c(\mathbf{s}_P^X)$, $V \cap D = \emptyset$. A similar reasoning shows that $\mathbf{v}_c(\mathbf{s}_Q^X)$ does not have any cluster points at all for c.

However, $\mathbf{v}_c(\mathbf{s}_P^X) \Delta \mathbf{v}_c(\mathbf{s}_Q^X)$. In fact, for $S \in \mathbf{s}_P$ and a c-open $V \supset S$, let P_n denote the set of all $p \in P$ for which the disk $D \subset X$ containing p and of diameter $\frac{1}{n}$ is contained in V. Then P is the union of the sets P_n and of the finite set $P - S$. Since P is a G_δ subset of X_0 for the topology $c^*|X_0$, the Baire Category Theorem furnishes a P_n that is c^*-dense in an open interval $(a, b) \subset X_0$. Then $(a, b) \times \left(0, \frac{1}{n}\right) \subset V$, hence, for an arbitrary $S' \in \mathbf{s}_Q$ and a

c-open $V' \supset S'$, there is a $q \in S'$ whose first coordinate belongs to (a,b) and then necessarily $V \cap V' \neq \emptyset$.

Now $\mathbf{v}_c(\mathbf{s}_p^X) \cap \mathbf{v}_c(\mathbf{s}_Q^X)$ has a trace on X_0 that is composed of all sets $S \subset X_0$ for which $X_0 - S$ is finite; this filter does not belong to S_0, and 3.1 (d) fails to hold. □

This example shows that an analogue of 2.14 (by substituting 3.1 (a), (b), (c) to the respective conditions in 2.8) cannot be true for CL-extensions and $|I| = 1$. It also shows that an analogue of [2], 2.14 fails to be valid for $I = \{0\}$ and a c-closed X_0. On the other hand:

THEOREM 3.4. *The conditions* 3.1 (b) *to* (d) *follow from* 3.1 (a) *and* 2.8 (b) *to* (d) *provided X_i is c-open and S_i is a Lodato screen for $i \in I$. Therefore, under these hypotheses, there exists a* CL-*extension.*

PROOF. By 2.8 there exists an extension, so 2.9 (a) holds and then 3.1 (c) and (d) coincide with 2.8 (c) and (d), respectively. □

According to [2], 2.17, the finest Lodato extension S_L^1 is generated by the screen base composed of the filters $\mathbf{v}_c(x)$ $(x \in X)$ and $\mathbf{v}_c(\mathbf{s}_i^X)$ $(\mathbf{s}_i \in \mathrm{S}_i$, $i \in I)$ (whenever a Lodato extension exists at all). Hence 2.16 (in which c is a T_2 topology, the sets X_i are c-open, and the screen bases for S_i are composed of c-open filters) furnishes an example where $\mathrm{S}_R^1 = \mathrm{S}_L^1$, $\mathrm{S}_{\mathrm{CR}}^1 = \mathrm{S}_{\mathrm{CL}}^1$, consequently $\mathrm{S}_{\mathrm{CL}}^1 \neq \mathrm{S}_L^1$. In the following example $\mathrm{S}_R^1 = \mathrm{S}_{\mathrm{CR}}^1$, $\mathrm{S}_L^1 = \mathrm{S}_{\mathrm{CL}}^1$, but $\mathrm{S}_R^1 \neq \mathrm{S}_L^1$ and so $\mathrm{S}_{\mathrm{CR}}^1 \neq \mathrm{S}_{\mathrm{CL}}^1$:

EXAMPLE 3.5. Let $X = \mathbf{R}$, c be the Euclidean topology on X, $X_0 = \mathbf{N}$, $c_0 = c|X_0$, and S_0 be composed of $\exp X_0$ and of all ultrafilters in X_0. Then 3.1 (a) to (d) are fulfilled: $\mathbf{v}_c(\mathbf{s}_0^X)$ does not have any c-cluster points for a free ultrafilter $\mathbf{s}_0$ in X_0, $\mathbf{v}_c(\mathbf{s}_1^X)\,\overline{\Delta}\,\mathbf{v}_c(\mathbf{s}_2^X)$ for two distinct free ultrafilters $\mathbf{s}_1, \mathbf{s}_2 \in \mathrm{S}_0$, and $\mathbf{v}_c(\mathbf{s}_0^X)\,|X_0 = \mathbf{s}_0$. However, $\mathbf{s}_0^X$ is an ultrafilter in X, whereas $\mathbf{v}_c(\mathbf{s}_0^X)$ does not contain any of the complementary sets X_0 and $X - X_0$. □.

Example 2.17 (in which c is a T_2 topology and the screen base defining S and S' are composed of c-open filters) shows that CL-extensions may exist without existing a coarsest one among them (in contrast to the case of Lodato extensions, see [2], 2.13).

The following analogue of 2.15 can be proved in the same way:

THEOREM 3.6. *Under the conditions of* 3.1, *we have*

$$\mathrm{S}_{\mathrm{CL}}^1 = \left(\mathrm{S}_L^1\right)^C.$$

PROOF. $\left(\mathrm{S}_L^1\right)^C$ is Lodato since it admits a base composed of open filters. □

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GENERALIZED ARITHMETICAL PROGRESSIONS AND SUMSETS

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1. Introduction

A famous theorem of Freiman [3, 4, 5] describes the structure of sets whose sum-set is not much larger than the original set. In this paper we present a novel approach to this problem. We prove a result, which is essentially equivalent to Freiman's though expressed in different terms, and the proof goes along completely different lines. The connection between Freiman's and our formulation is discussed in the last section.

Like Freiman's, our method works equally for sets in finite dimensional Euclidean spaces, or abstract torsionfree groups, so for greater flexibility we present it in this form. Probably a generalization to every commutative group is possible, though this seems to present some difficulties.

Let $q_1, \ldots, q_d$ and a be elements of an arbitrary commutative group, $l_1, \ldots, l_d$ positive integers. By a *d-dimensional (generalized) arithmetical progression* we mean a set of the form

$$(1.1)\quad P(q_1, \ldots, q_d; l_1, \ldots, l_d; a) = \{n = a + x_1q_1 + \ldots + x_dq_d, 0 \leqq x_i \leqq l_i\}.$$

(More exactly, we think of it as a set together with a fixed representation in the form (1.1); this representation is in general not unique.) We call d the *dimension* of P, and by its *size* we mean the quantity

$$\|P\| = \prod_{i=1}^{d}(l_i + 1),$$

which is the same as the number of elements if all sums in the right side of (1.1) are distinct. In this case we say that P is *proper*.

1.1. THEOREM. *Let A, B be finite sets in a torsionfree commutative group satisfying $|A| = |B| = n$, $|A + B| \leqq \alpha n$. There are numbers d, C*

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depending on α only such that A is contained in a generalized arithmetical progression of dimension at most d and size at most Cn.

Two important cases, which we do not formulate separately, are $A = B$ and $A = -B$.

2. "Bohr neighbourhoods" in certain sumsets

For iterated sumsets we introduce the notation

$$kA = A + \ldots + A, \quad k \text{ summands.}$$

For distinction we denote

$$Ak = \{ka : a \in A\}.$$

If G is a commutative group, $\gamma_1, \ldots, \gamma_k$ are characters of G and $\varepsilon_j > 0$, we write

$$B(\gamma_1, \ldots, \gamma_k; \varepsilon_1, \ldots, \varepsilon_k) = \{g \in G : |\arg \gamma_j(g)| \leqq 2\pi\varepsilon_j \text{ for all } j = 1, \ldots, k\}$$

and call these sets *Bohr sets*. In particular, if $\varepsilon_1 = \ldots = \varepsilon_k = \varepsilon$, we shall speak of a *Bohr* (k, ε)*-set*. (We take the branch of arg that lies in $[-\pi, \pi)$.)

In locally compact groups these sets form a base for the Bohr topology; we shall work with finite groups, but we preserve the name that suggests certain ideas.

2.1. LEMMA. *Let G be a finite commutative group, $|G| = m$. Let A be a nonempty subset of G and write $|A| = n = \beta m$. The set $D = 2A - 2A$ (the second difference set of A) contains a Bohr k, ε-set with some integer $k < \beta^{-2}$ and $\varepsilon = 1/4$.*

This is essentially a result of Bogolyubov [1] which he used to study the Bohr topology on the integers. We include a proof for sake of completeness. With certain additional ideas, a similar result can be achieved for three summands , see Freiman–Halberstam–Ruzsa [6], but the situation for two is different, see Bourgain [2] and Ruzsa [7].

PROOF. Let Γ denote the group of characters. For $\gamma \in \Gamma$ put

$$f(\gamma) = \sum_{a \in A} \gamma(a).$$

We have

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = mn = \beta m^2$$

and $f(\gamma_0) = n$ for the principal character γ_0. We have $x \in D$ for those elements x for which

$$\sum_{\gamma \in \Gamma} \left| f(\gamma) \right|^4 \gamma(x) \neq 0. \tag{2.1}$$

To estimate (2.1), we split the characters $\gamma \neq \gamma_0$ into two groups. We put those for which $|f(\gamma)| \geqq \sqrt{\beta} n$ into Γ_1 and the rest into Γ_2. We claim that $x \in D$ whenever $\operatorname{Re} \gamma(x) \geqq 0$ is satisfied for all $\gamma \in \Gamma_1$. Indeed, we have

$$\left| \sum_{\gamma \in \Gamma_2} \left| f(\gamma) \right|^4 \gamma(x) \right| < \beta n^2 \sum_{\gamma \in \Gamma_2} \left| f(\gamma) \right|^2 < \beta^2 m^2 n^2 = n^4,$$

consequently

$$\operatorname{Re} \sum_{\gamma \in \Gamma} \left| f(\gamma) \right|^4 \gamma(x) \geqq n^4 + \operatorname{Re} \sum_{\gamma \in \Gamma_2} \left| f(\gamma) \right|^4 \gamma(x) \geqq n^4 - \left| \sum_{\gamma \in \Gamma_2} \left| f(\gamma) \right|^4 \gamma(x) \right| > 0.$$

The condition $\operatorname{Re} \gamma(x) \geqq 0$ is equivalent to $|\arg \gamma(g)| \leqq \pi/2$, thus we have a Bohr $(k, 1/4)$ set with $k = |\Gamma_1|$. We estimate k. We have

$$k\beta n^2 \leqq \sum_{\gamma \in \Gamma_1} \left| f(\gamma) \right|^2 < \sum_{\gamma \in \Gamma} \left| f(\gamma) \right|^2 = \beta m^2,$$

hence $k \leqq (m/n)^2 = \beta^{-2}$ as claimed. □

3. A generalized arithmetical progression in a Bohr neighbourhood

We show that Bohr sets contain large generalized arithmetical progressions. We are able to do this only for cyclic groups; this will be sufficient for our present aims, but it would be interesting to decide whether a similar result holds in general groups.

Let G be the group of residues modulo m. The characters of G are of the form

$$\gamma(x) = e^{2\pi i u x/m}$$

for some residue u. Consequently Bohr sets are sets of the form

$$B(u_1, \ldots, u_k; \varepsilon_1, \ldots, \varepsilon_k) = \left\{ x : \|u_j x/m\| \leqq \varepsilon_j \text{ for all } j = 1, \ldots, k \right\}. \tag{3.1}$$

Here $\|t\|$ denotes the distance of a real number t from the nearest integer. With a slight abuse of notation, we define $\|u/m\|$ for a residue u modulo m as the common value of $\|v/m\|$ for representants v of this residue class.

3.1. THEOREM. *Let m be a positive integer, $u_1, \dots, u_k$ residues modulo m such that $(u_1, u_2, \dots, u_k, m) = 1$, $\varepsilon_1, \dots, \varepsilon_k$ real numbers satisfying $0 < < \varepsilon_j < 1/2$. Write*

$$\delta = \frac{\varepsilon_1 \dots \varepsilon_k}{k^k}. \tag{3.2}$$

There are residues $q_1, \dots, q_k$ and nonnegative integers $l_1, \dots, l_k$ such that the set

$$P = \left\{ q_1 x_1 + \dots q_k x_k : |x_i| \leqq l_i \right\} \tag{3.3}$$

satisfies

$$P \subset B(u_1, \dots, u_k; \varepsilon_1, \dots, \varepsilon_k), \tag{3.4}$$

the sums in (3.3) are all distinct and

$$\|P\| = \prod (2l_j + 1) \geqq \prod (l_j + 1) > \delta m. \tag{3.5}$$

PROOF. Let L be the k dimensional lattice of integer vectors $(x_1, \dots, x_k)$ satisfying

$$x_1 \equiv x u_1, \dots, x_k \equiv x u_k \pmod m$$

with some integer x. This lattice is the union of m translations of the lattice $(\mathbf{Z}m)^k$ (here we need the coprimality condition, otherwise there may be coincidences), hence its determinant is m^{k-1}.

Let Q be the rectangle determined by $|x_j| \leqq \varepsilon_j, j = 1, \dots, k$ and let $\lambda_1, \dots, \lambda_k$ denote the successive minima of Q with respect to the lattice L. These are the smallest positive numbers such that there are linearly independent vectors $a_1, \dots, a_k \in L$, $a_i \in Q\lambda_i$. By Minkowski's inequality we have

$$\lambda_1 \dots \lambda_k \leqq 2^k \frac{\det L}{\operatorname{vol} Q} = \frac{m^{k-1}}{\varepsilon_1 \dots \varepsilon_k}. \tag{3.6}$$

Write

$$a_i = (a_{i1}, \dots, a_{ik}).$$

The condition $a_i \in Q\lambda_i$ means that $|a_{ij}| \leqq \lambda_i \varepsilon_j$. Since $a_i \in L$, there are residues q_i such that $a_{ij} \equiv q_i u_j \pmod m$. These are our q_j's and we put

$$l_i = \left[\frac{m}{k\lambda_i} \right].$$

First we show that $P \subset B$. Consider an $x \in P$, $x = x_1q_1 + \ldots + x_kq_k$. We have

$$xu_j = \sum x_iq_iu_j \equiv \sum x_ia_{ij} \pmod m,$$

consequently

$$\left\| \frac{xu_j}{m} \right\| = \left\| \sum \frac{x_ia_{ij}}{m} \right\| \leqq \sum \left| \frac{x_ia_{ij}}{m} \right| \leqq \sum \frac{l_i\lambda_i\varepsilon_j}{m} \leqq \sum \frac{\varepsilon_j}{k} = \varepsilon_j.$$

Next we show that these elements are all distinct. If $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ give the same sum, then with $z_j = x_j - y_j$ we have

$$\sum z_iq_i \equiv 0 \pmod m, \quad |z_i| \leqq 2l_i.$$

Multiplying this congruence by u_j we infer that

$$\sum z_ia_{ij} \equiv 0 \pmod m$$

for all j. Moreover a calculation like above yields

$$\left| \sum z_ia_{ij} \right| \leqq \sum l_i\lambda_i\varepsilon_j \leqq 2\varepsilon_jm < m.$$

Consequently $\sum z_ia_{ij} = 0$ for every j, which means that $\sum z_ia_i = 0$; by view of the linear independence of the vectors a_i, $z_i = 0$ for all i, qu.e.d.

Finally we prove (3.5). We have

$$l_i + 1 > \frac{m}{k\lambda_i},$$

hence

$$\prod(l_i + 1) > \frac{m^k}{k^k\lambda_1 \ldots \lambda_k} \geqq \frac{m}{k^k}\varepsilon_1 \ldots \varepsilon_k = \delta m$$

by (3.6). □

It is easy to see that the result need not hold if $(u_1, \ldots, u_k, m) = d > 1$; consider, for instance, the case $m = d^2$, $k = 1$, $u_1 = d$. It can be shown that a $k + 1$ dimensional arithmetical progression can always be found in B.

3.2. Lemma. *Let m be a prime, and let A be a nonempty set of residues modulo m with $|A| = \beta m$. There are residues $q_1, \ldots, q_k$ and nonnegative integers $l_1, \ldots, l_k$ such that the set*

$$P = \left\{ q_1x_1 + \ldots + q_kx_k : |x_i| \leqq l_i \right\} \tag{3.7}$$

satisfies $P \subset D = 2A - 2A$, *the sums in* (3.7) *are all distinct and*

$$\|P\| = \prod(2l_j + 1) \geqq \prod(l_j + 1) > \delta m. \tag{3.8}$$

where $k \leqq \beta^{-2}$ *and*

$$\delta = (4k)^{-k} \leqq (\beta^2/4)^{1/\beta^2} \tag{3.9}$$

PROOF. This follows from a combination of Lemma 2.1 and Theorem 3.1. The assumption that m is a prime guarantees the coprimality assumption required in Theorem 3.1. □

4. Freiman isomorphy

Let G_1, G_2 be commutative groups, $A_1 \subset G_1$, $A_2 \subset G_2$. We say that a mapping $\phi : A_1 \to A_2$ is a *homomorphism of order* r *in the sense of Freiman*, or an F_r*-homomorphism* for short, if for every $x_1, \dots, x_r, y_1, \dots, y_r \in A_1$ (not necessarily distinct) the equation

$$x_1 + x_2 + \dots + x_r = y_1 + y_2 + \dots + y_r \tag{4.1}$$

implies

$$\phi(x_1) + \phi(x_2) + \dots + \phi(x_r) = \phi(y_1) + \phi(y_2) + \dots + \phi(y_r). \tag{4.2}$$

We call ϕ an F_r*-isomorphism,* if it is (1-1) and its inverse is a homomorphism as well, that is, (4.2) holds if and only if (4.1) does. When we do not specify the order, we mean a homomorphism of order 2.

4.1. LEMMA. *Let* G, G' *be commutative groups. If a set* $P' \subset G'$ *is the homomorphic image of a generalized arithmetical progression* $P(q_1, \dots, q_d; l_1, \dots, l_d; a) \subset G$, *then there are elements* $q_1', \dots, q_d', a' \in G'$ *such that*

$$P' = P(q_1', \dots, q_d'; l_1, \dots, l_d; a') \tag{4.1}$$

and the homomorphism is given by

$$\phi(a + x_1 q_1 + \dots + x_d q_d) = a' + x_1 q_1' + \dots + x_d q_d'. \tag{4.2}$$

PROOF. Define a' and q_i' by

$$a' = \phi(a), \quad q_i' = \phi(a + q_i) - \phi(a).$$

We prove (4.2) by induction on $r = x_1 + \ldots + x_d$. For $r \leqq 1$ it is an immediate consequence of the definition. Assume that $r \geqq 2$ and the statement holds for every smaller value. Consider an element

$$x = a + x_1 q_1 + \ldots + x_d q_d, \quad x_1 + \ldots + x_d = r.$$

Since $r \geqq 2$, either there are subscripts $i \neq j$ such that $x_i \geqq 1$ and $x_j \geqq 1$, or there is a subscript for which $x_i \geqq 2$. In the second case write $j = i$. In both cases the sums

$$y = x - x_i, \ z = x - x_j, \ u = x - x_i - x_j$$

are in P, their sums of coefficients are at most $r - 1$ and they satisfy $x + u = y + z$. This implies $\phi(x) + \phi(u) = \phi(y) + \phi(z)$, that is, $\phi(x) = \phi(y) + \phi(z) - \phi(u)$. Substituting (4.2) for y, z and u into this equation we conclude that (4.2) holds for x as well, which completes the inductive step. □

4.2. LEMMA. *Let G, G' be commutative groups, and let $A \subset G$, $A' \subset G'$ be F_r isomorphic sets. Assume that $r = r'(k + l)$ with nonnegative integers r', k, l. The sets $kA - lA$ and $kA' - lA'$ are $F_{r'}$ isomorphic.*

PROOF. Let ϕ be the isomorphism between A and A'. For an

$$x \in kA - lA, \ x = a_1 + \ldots + a_k - b_1 - \ldots - b_l$$

we define naturally

$$\psi(x) = \phi(a_1) + \ldots + \phi(a_k) - \phi(b_1) - \ldots - \phi(b_l).$$

The facts that this depends only on x and not on the particular representation, and that ψ is an $F_{r'}$ isomorphism, follow immediately from the definition. □

5. Proof of the main theorem

We need the following results from Ruzsa [7].

5.1. LEMMA. *Let A be a set of integers, $|A| = n$, $r \geqq 2$ an integer and $D = rA - rA$. Write $|D| = N$. For every $m > 2r(N - 1)$ there exists a set $A' \subset A$, $|A'| \geqq n/r$ which is F_r-isomorphic to a set T of residues* mod m.

5.2. LEMMA. *Let A, B be subsets of an arbitrary Abelian group. Write $|B| = n$, $|B + A| = \alpha n$. For arbitrary positive integers k, l we have*

$$|kA - lA| \leqq \alpha^{k+l} n. \tag{5.1}$$

Proof of Theorem 1.1. The subgroup generated by A, like any finitely generated torsionfree group, is isomorphic to $\mathbf{Z}^v$ for some integer v. Let A_1 be the image of A; the group isomorphy implies the Freiman isomorphy of arbitrary order between A and A_1.

For arbitrary fixed r we can find a set $A_2 \subset \mathbf{Z}$ which is F_r-isomorphic to A_1. Indeed, consider the mapping

$$(a_1, \dots, a_v) \to a_1 + ta_2 + \dots + t^{v-1}a_v. \tag{5.2}$$

This is an F_r-homomoprhism for every r. If H denotes the maximal absolute value of coordinates of elements of A_1 and $t > 2rH$, then the coincidence of two r-fold sums of numbers of the form (5.2) implies the coincidence of the coordinates, thus this mapping is an isomorphism of order r. We shall use this with $r = 8$; so let A_2 be a set of integers which is F_8-isomorphic to A_1, hence to A.

Lemma 5.2 implies that

$$|2A_2 - 2A_2| = |2A - 2A| \leqq \alpha^4 n. \tag{5.3}$$

(The first equality follows from the F_8 isomorphy.) We apply Lemma 5.1 for $r = 8$ and a prime number $m > 2r|2A - 2A|$. By (5.3) and Chebyshev's theorem we can find such a prime with

$$m < 4r|2A - 2A| \leqq 32\alpha^4 n.$$

Lemma 5.1 gives us a set $A' \subset A_2$ F_8-isomorphic to a set T of residues modulo m, $|A'| \geqq n/r = n/8$.

Applying Lemma 3.2 we find a k dimensional proper arithmetical progression $P \subset 2T - 2T$ of size $\geqq \delta n$, where $k = k(\alpha)$ and $\delta = \delta(\alpha) > 0$ depend on α only.

Now T is F_8-isomorphic to a subset A^* of A. By Lemma 4.2, this F_8 isomorphism can be extended to an F_2 isomorphism between $2T - 2T$ and $2A^* - 2A^*$. The image P^* of P is a proper k-dimensional arithmetical progression by Lemma 4.1 and we have $P^* \subset 2A^* - 2A^* \subset 2A - 2A$.

Select a maximal collection of elements $a_1, \dots, a_s \in A$ such that the sets $P^* + a_i$ are pairwise disjoint. We estimate s. Since these sets are all subsets of $A + P^* \subset 3A - 2A$, we have

$$s \leqq \frac{|3A - 2A|}{\|P^*\|} \leqq \frac{\alpha^5 n}{\delta n} = \alpha^5/\delta(\alpha).$$

Here in the second inequality we used Lemma 5.2.

For every $a \in A$ there is an a_i such that

$$(a + P^*) \cap (a_i + P^*) \neq \emptyset.$$

Thus there are $p, p' \in P^*$ such that $a + p = a_i + p'$, that is, $a = a_i + p' - p$. This means that

$$A \subset \{a_1, \dots, a_s\} + P^* - P^*. \tag{5.4}$$

Since P^* is a k-dimensional arithmetical progression, so is $P^* - P^*$, and obviously

$$\|P^* - P^*\| \leqq 2^k \|P^*\| \leqq 2^k |2A - 2A| \leqq 2^k \alpha^4 n.$$

The set $\{a_1, \dots, a_s\}$ can be covered by the s-dimensional arithmetical progression

$$P(a_1, \dots, a_s; 1, \dots, 1; 0).$$

Hence the right side of (5.4) can be covered by an arithmetical progression of dimension $d = s + k$ and size $C = 2^{s+k}\alpha^4 n$. Since both s and k were bounded in terms of α, the proof is complete. □

6. Concluding remarks

One can imagine many results in the form "if $|2A| \leqq \alpha|A|$, then ...". Such a description is *adequate*, if the condition involved implies that $|2A| \leqq \alpha'|A|$ with some α' depending on α only. Among adequate descriptions one can distinguish on two grounds: first, the smaller the value of α', the better our result is; second, simplicity. For instance, the statement "if $|2A| \leqq \alpha|A|$, then $|A - A| \leqq \alpha^2|A|$", a particular case of Lemma 5.2, is an adequate description with $\alpha' = \alpha^4$, but one cannot say that it helps much to understand the structure of these sets. Our Theorem 1.1 uses a very simple structure. The value we get for α' is $2^d C$. We did not express d and C from α, but if we did so, we would get an exponential bound for d and a doubly exponential one for C, so a doubly exponential bound for α'. An improvement of these bounds would be interesting; I think the correct value of d is about α, and that of C is about $\exp \alpha$.

There are other possibilities for improvement. It would be desirable to have a proper d-dimensional arithmetical progression. Also, a d-dimensional progression is the image of a set of lattice points in $\mathbf{Z}^d$. The following two properties would be useful to have:

i) this map between $\mathbf{Z}^d$ and our set is an isomorphism

ii) the inverse image of A in $\mathbf{Z}^d$ is proper d-dimensional.

Since proper d-dimensional sets satisfy $|2A| \geqq (d+1)|A| - d(d+1)/2$ (Freiman [4]), these properties would automatically yield the (optimal) bound $d \leqq [\alpha - 1]$ for $n > n_0(\alpha)$.

Freiman [3, 4, 5] expresses the result in a rather different way. He asserts that the set A is isomorphic to a subset of the set of lattice points in a convex region of volume Cn in $\mathbf{R}^d$, where for d he gives the optimal bound $\alpha - 1$

(he does not specify the bound for C). I think my formulation and his are essentially equivalent, though this is not obvious. I plan to return to this and the problems mentioned above in another paper.

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OPTIMAL PACKINGS OF ELEVEN EQUAL CIRCLES IN AN EQUILATERAL TRIANGLE

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An interesting problem in the theory of packing is that of finding the densest packings of circles in a triangle. Malfatti [6], for instance, studied the question of how to cut three circular cylinders of largest total volume from a right triangular prism. He surmised that the solution would be obtained by finding three circles in the triangle that each touches both the other two circles as well as two of the sides of the triangle: '... cosicchè ciascun de circolo toccasse gli alti due ed insieme due lati del triangolo'. Unfortunately, Malfatti had been wrong here; Goldberg [3, 4] showed that these so-called Malfatti circles in fact never solve the densest packing problem. For a description of similar packing problems we refer to [1, 2, 9].

In this article we will restrict our attention to equilateral triangles and to packings with congruent circles. By stacking the circles on a regular triangular lattice, it is easy to find obvious candidates for the optimal packings of $n = k(k+1)/2$ circles in an equilateral triangle. The proof, however, is not so straightforward as the simplicity of the solution would suggest; it was given in 1961 by Oler [10]. Recently, optimal configurations and proofs for $n = 4, 5, 7, 8, 9$ and 12 were obtained by the author [7]. A different proof for the triangular numbers is also contained in that article. Conjectured configurations for $n = 17$ are given in [8].

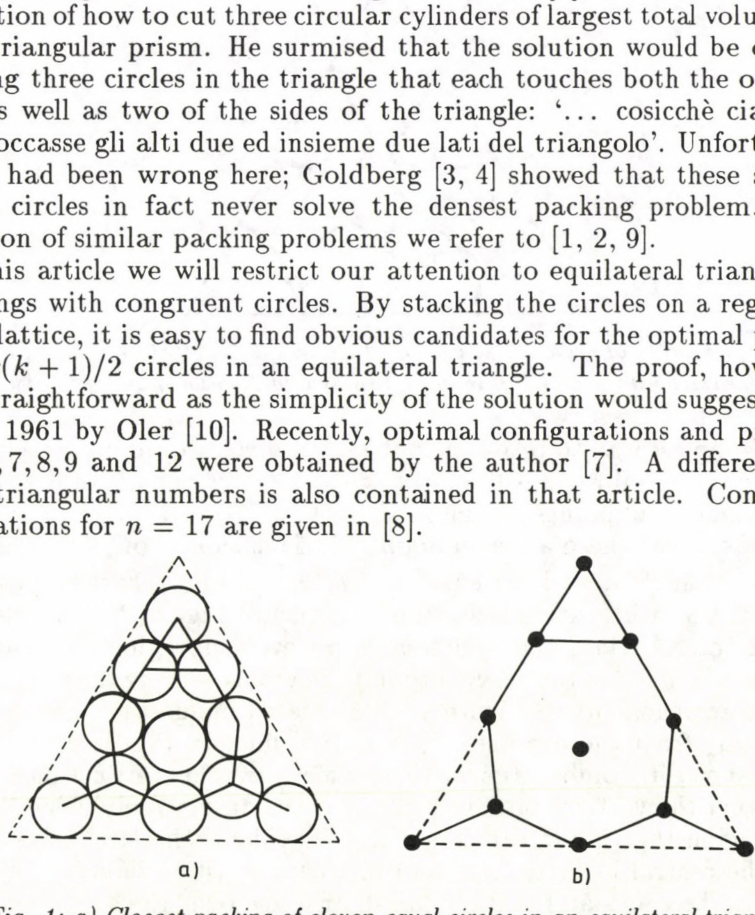

Fig. 1: a) Closest packing of eleven equal circles in an equilateral triangle.
b) Maximum least distance arrangement of eleven points in an equilateral triangle. The solid line segments between the points are of equal length.

We will now give a proof for the optimality of certain arrangements of eleven circles in an equilateral triangle. Since the inner parallel domains

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of a triangle are similar to the original triangle, the problem of finding the densest packing of n equal circles in a triangle is equivalent to placing n points inside that triangle such that the minimum distance between the points (the *separation distance*) is maximal. We shall use this last formulation.

The optimal arrangements of eleven points in a unilateral triangle (up to rotations) are shown in Figure 1b. The separation distance between the points follows from these configurations as $d_{11} = \left(3-\sqrt{6}\right)/2 = 0.275255\ldots$. The position of the central point is not unique, similar to the optimal arrangements of seven points in an equilateral triangle [7], or eight points in a circle [11].

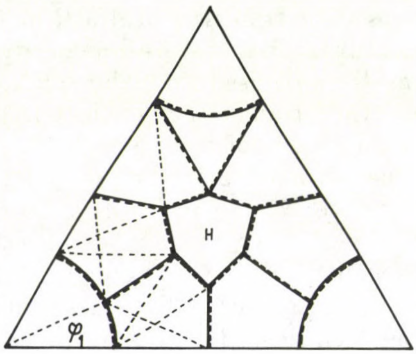

Fig. 2: Partition of the triangle used in the proof. The dotted lines are of length d_{11}. The dashed/solid lines and arcs indicate to which of the subregions each edge belongs.

The simplest way to prove the optimality of d_{11} would be to partition the triangle into ten subregions that all have a diameter of at most d_{11}. For some smaller numbers of points this method works well [7]. It is a result of Graham [5], however, that the maximum diameter of ten subregions whose union is a unilateral triangle is at least equal to $1/\left(2\sqrt{3}\right) > d_{11}$. For the proof of the case $n = 11$ we will, instead, use the partition in Figure 2. This subdivision of the unilateral triangular region T into eleven subregions is based on the three disc segments of radius d_{11} around the vertices of the triangle, on points from the arrangement in Figure 1b, its rotated images and some points at distance d_{11} from these points. The dotted lines in Figure 2 are of length d_{11} and show, in combination with symmetry in the vertical bisector, how to construct the vertices of the partition. Each dashed/solid edge belongs to the subregion that is indicated by the solid side of the line. All subregions except the central hexagon H are of diameter d_{11}; the diameter of H is less than d_{11}. The edges and vertices are distributed over the subregions in such a way that no single subregion can contain points that are a distance d_{11} apart.

Suppose that we have a configuration of eleven points in T for which the maximal separation distance is equal to $d \geqq d_{11}$, then there must be exactly one point in each of the eleven regions. The points in the three extreme circle segments may be assumed to lie on the vertices of the triangle. To see this let x_0 be a vertex of T and suppose that x_1 and x_2 are points in T such that $\|x_1 - x_2\| \geqq d$ and $\|x_2 - x_0\| \geqq \|x_1 - x_0\|$. If $\|x_1 - x_0\| \geqq d$, then $\|x_2 - x_0\| \geqq \|x_1 - x_0\| \geqq d$. The region of those points in T that have a distance to x_0 of at most d is a circular disc segment around x_0. This region has a diameter d, so if $\|x_1 - x_0\| < d$, then $\|x_2 - x_0\| \geqq d$. In both cases x_1 can be replaced by x_0 in the arrangement without decreasing the separation distance. We can therefore assume that the three vertices of T are part of the solution. As it turns out, this assumption will not restrict the number of solutions found.

Let U be the closure of the region that is obtained by removing the three disc segments of radius d around the vertices from the triangular region, and let Γ_1 be the boundary of U. Without restricting generality, it may also be assumed that the seven points in $U\backslash H$ lie on Γ_1. This can be seen as follows. Region H can be divided into six congruent triangles. By symmetry it can be assumed that the point in H is in one particular triangle S. The point in S restricts the seven other points in U to lie in an annular-shaped region $U\backslash\tilde{H}$, where $\tilde{H} = \{x \mid \operatorname{dist}(S,x) \leqq d_{11}\}$. The boundary $\Gamma_2 = \partial\tilde{H}$ of $\tilde{H}$ consists of three circular arcs. Now consider the polygon that is formed by connecting each of the seven points of the configuration in $U\backslash\tilde{H}$ with the two points from the neighbouring regions in $U\backslash\tilde{H}$. If the interior angle at a vertex is less than or equal to π, this vertex can be moved onto Γ_1 without decreasing the separation distance. In the case that the angle exceeds π, the point can first be moved onto Γ_2, and subsequently it can be reflected in the line through the two neighbouring vertices. Some tedious, but straightforward calculations show that the reflected vertex always remains inside $U\backslash\tilde{H}$ (only three subregions need to be considered). It can then be moved onto Γ_1. This means that it can now be assumed that all seven points lie on Γ_1.

There can only be seven points on Γ_1 at a mutual distance of at least d if the maximal separation distance d is equal to d_{11}, as the following detailed analysis of the location of these points will show.

First, suppose that d is equal to d_{11}. The points on the circle segments can be parametrised by the use of angles $\psi_1, \psi_2, \psi_3 \in [0, \pi/3[$ as shown in Figure 3. Now we will determine the tightest possible arrangement of points on Γ_1 by starting with the point on the first circular arc at angle ψ_1. There is a point on the horizontal line segment at distance d_{11} from the first point. The next point at distance d_{11} may lie on the same line segment, or on the next arc, depending on the value of ψ_1. For the angle ψ_2 on the next arc the following relations hold:

$$\psi_1 \in [0, \varphi_1[\Rightarrow \psi_2 = \frac{\pi}{3} - \arccos(\alpha - \cos\psi_1) \in [\varphi_1, \varphi_2[\ ,$$

$$\psi_1 \in [\varphi_1, \varphi_2[\Rightarrow \psi_2 = \frac{\pi}{3} - \arccos(\alpha - \cos\psi_1) \in \left[\varphi_2, \frac{\pi}{3}\right[\ ,$$

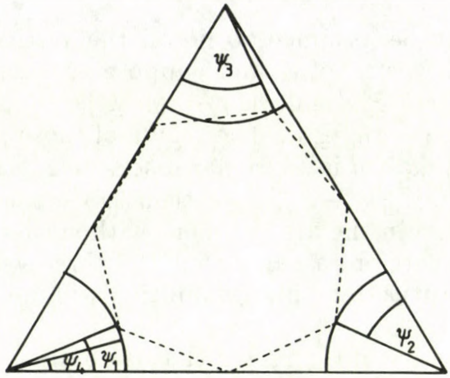

Fig. 3: Definition of the angles ψ_j required in the proof. The length of the dotted line segments is equal to d_{11}.

$$\psi_1 \in \left[\varphi_2, \frac{\pi}{3}\right[\Rightarrow \psi_2 = \frac{\pi}{3} - \arccos\left(\alpha - \frac{1}{2} - \cos\psi_1\right) \in [0, \varphi_1[\, .$$

Here $\varphi_1 = \arccos\left(1/2 + 1/\sqrt{6}\right)$, $\varphi_2 = \pi/3 - \varphi_1 = \arccos\left(\sqrt{6}/3\right)$ and $\alpha = 1/(2d_{11})$. In the first two cases there is one point of the solution on the interjacent line segment, whereas in the third case the line segment contains two points. After finding this point on the second arc, the same construction can be continued to obtain a point on the third arc at an angle of ψ_3, and another point on the first arc at angle ψ_4. The seven points can, of course, only be accommodated on Γ_1 if $\psi_4 \geqq \psi_1$ for some choice of ψ_1. For $\psi_1 \in \,]0, \varphi_1[$ we have

$$\frac{d\psi_2}{d\psi_1} = \frac{\sin\psi_1}{\sqrt{1 - (\alpha - \cos\psi_1)^2}} > 0,$$

and

$$\frac{d^2\psi_2}{d\psi_1^2} = \alpha\frac{(1 - \cos\psi_1)^2 + (2 - \alpha)\cos\psi_1}{\left(1 - (\alpha - \cos\psi_1)^2\right)^{3/2}} > 0,$$

so ψ_2 is increasing and strictly convex as a function of ψ_1. This results in the following inequality

$$\psi_2 < \varphi_1 + \frac{\varphi_2 - \varphi_1}{\varphi_1}\psi_1 \quad \text{for} \quad \psi_1 \in \,]0, \varphi_1[.$$

By similar arguments it follows that

$$\psi_3 < \varphi_2 + \frac{\frac{\pi}{3} - \varphi_2}{\varphi_2 - \varphi_1}(\psi_2 - \varphi_1) \quad \text{and} \quad \psi_4 < \frac{\varphi_1}{\frac{\pi}{3} - \varphi_2}(\psi_3 - \varphi_2).$$

Combination of these three inequalities shows that ψ_4 is always smaller than ψ_1 if $\psi_1 \in\ ]0, \varphi_1[$. The same is true for $\psi_1 \in\ ]\varphi_1, \varphi_2[$ and for $\psi_1 \in$ $\in\ ]\varphi_2, \pi/3[$. Only when ψ_1 is equal to 0, φ_1 or φ_2, we can have that $\psi_4 = \psi_2$, so it is only in these cases that the seven points can be fitted on Γ_1. The corresponding solutions are the optimal configurations depicted in Figure 1, and their rotated images. If $d > d_{11}$, the seven points obviously cannot be placed on the curve at a mutual distance of at least d. □

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NOTE ON ε-SADDLE POINT AND SADDLE POINT THEOREMS

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1. Introduction

Let X, Y be Hausdorff topological spaces and f be a real-valued function on the product space $X \times Y$. If $\varepsilon > 0$, a point $(x_\varepsilon^*, y_\varepsilon^*) \in X \times Y$ is said to be an ε-saddle point of f if for each $(x, y) \in X \times Y$, we have

$$f(x, y_\varepsilon^*) - \varepsilon < f(x_\varepsilon^*, y_\varepsilon^*) < f(x_\varepsilon^*, y) + \varepsilon.$$

Also a point $(x^*, y^*) \in X \times Y$ is said to be a saddle point of f if for each $(x, y) \in X \times Y$, we have

$$f(x, y^*) \leqq f(x^*, y^*) \leqq f(x^*, y).$$

We shall denote by $\mathbf{R}$ and $\mathbf{N}$ the set of all real numbers and the set of all natural numbers, respectively. If X is a convex subset of a vector space and $f : X \to \mathbf{R}$, then f is said to be quasi-concave if for each $t \in \mathbf{R}$, the set $\{x \in X : f(x) > t\}$ is convex. f is said to be quasi-convex if $-f$ is quasi-concave. If Y is a compact Hausdorff space, $C(Y)$ denotes the Banach space of all continuous real-valued functions on Y with supremum norm.

In this note, we shall obtain a new ε-saddle point theorem and two new saddle point theorems. Our results generalize the corresponding results of Komiya [5].

2. Main results

We begin with the following result:

THEOREM 1. *Let X be a non-empty convex subset of a Hausdorff topological vector space, Y be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that $f : X \times Y \to \mathbf{R}$ satisfies the following conditions:*

(i) *for each $(x, y) \in X \times Y$, $\inf_{v \in Y} f(x, v) > -\infty$ and $\sup_{u \in X} f(u, y) < < +\infty$;*

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(ii) *for each fixed* $y \in Y$, $x \mapsto f(x,y)$ *is quasi-concave and* $x \mapsto f(x,y) - \inf_{v\in Y} f(x,v)$ *is upper semicontinuous;*

(iii) *for each fixed* $x \in X$, $y \mapsto f(x,y)$ *is quasi-convex and* $y \mapsto f(x,y) - \sup_{u\in X} f(u,y)$ *is lower semicontinuous.*

Then f *has an* ε*-saddle point* $(x^*_\varepsilon, y^*_\varepsilon) \in X \times Y$ *for each* $\varepsilon > 0$.

PROOF. Let $\varepsilon > 0$ be given. For each $(x,y) \in X \times Y$, define

$$T(x) = \left\{ y \in Y : f(x,y) - \inf_{v\in Y} f(x,v) < \varepsilon \right\},$$

and

$$S(y) = \left\{ x \in X : f(x,y) - \sup_{u\in X} f(u,y) > -\varepsilon \right\},$$

then by (i), (ii) and (iii), $T(x)$ and $S(y)$ are non-empty and convex. For each $(x,y) \in X \times Y$, the sets

$$T^{-1}(y) = \left\{ x \in X : f(x,y) - \inf_{v\in Y} f(x,v) < \varepsilon \right\},$$

$$S^{-1}(x) = \left\{ y \in Y : f(x,y) - \sup_{u\in X} f(u,y) > -\varepsilon \right\}$$

are open by (ii) and (iii).

Thus, by Theorem 1 of [3] (which is equivalent to Theorem 1 of [4]; see also [7, Corollary 1.7]), there exists $(x^*_\varepsilon, y^*_\varepsilon) \in X \times Y$ such that $y^*_\varepsilon \in T(x^*_\varepsilon)$ and $x^*_\varepsilon \in S(y^*_\varepsilon)$, i.e.,

$$f(x, y^*_\varepsilon) - \varepsilon < f(x^*_\varepsilon, y^*_\varepsilon) < f(x^*_\varepsilon, y) + \varepsilon$$

for each $(x,y) \in X \times Y$. $\square$

THEOREM 2. *Let* X *be a non-empty convex subset of a Hausdorff topological vector space,* Y *be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that* $f : X \times Y \to \mathbf{R}$ *satisfies the following conditions:*

(i) *for each fixed* $y \in Y$, $\sup_{u\in X} f(u,y) < +\infty$;

(ii) *for each fixed* $y \in Y$, $x \mapsto f(x,y)$ *is quasi-concave and* $x \mapsto f(x,y) - \inf_{v\in Y} f(x,v)$ *is upper semicontinuous;*

(iii) *for each fixed* $x \in X$, $y \mapsto f(x,y)$ *is quasi-convex and lower semicontinuous and* $y \mapsto f(x,y) - \sup_{u\in X} f(u,y)$ *is lower semicontinuous;*

(iv) *for each sequence* $\{(x_k, y_k)\}_{k\in\mathbf{N}}$ *in* $X \times Y$ *where for each* $k \in \mathbf{N}$, (x_k, y_k) *is an* ε_k*-saddle point of* f *and* $\varepsilon_k \to 0^+$, *there exist a subsequence* $\{x_{n_k}\}_{k\in\mathbf{N}}$ *and* $x^* \in X$ *such that for each* $y \in Y$,

$$\overline{\lim_{k\to\infty}} f(x_{n_k}, y) \leqq f(x^*, y).$$

Then f has saddle point $(x^*, y^*) \in X \times Y$.

PROOF. For each fixed $x \in X$, since Y is compact and $y \mapsto f(x, y)$ is lower semicontinuous by (ii), we must have $\inf_{v \in Y} f(x, v) > -\infty$.

For each $k \in \mathbf{N}$, by Theorem 1, there exists $(x_k^*, y_k^*) \in X \times Y$ such that for each $(x, y) \in X \times Y$,

$$f(x, y_k^*) - \varepsilon_k < f(x_k^*, y_k^*) < f(x_k^*, y) + \varepsilon_k. \tag{1}$$

By (iv), there exist a subsequence $\{x_{n_k}^*\}_{k \in \mathbf{N}}$ of $\{x_k^*\}_{k \in \mathbf{N}}$ and $x^* \in X$ such that for each $y \in Y$,

$$\overline{\lim_{k \to \infty}} f(x_{n_k}^*, y) \leqq f(x^*, y).$$

Since Y is compact, there exist a subnet $\{y_\alpha\}_{\alpha \in \Gamma}$ of $\{y_{n_k}^*\}_{k \in \mathbf{N}}$ and $y^* \in Y$ such that $y_\alpha^* \to y^*$. We shall prove that (x^*, y^*) is a saddle point of f.

For each $y \in Y$ and each $\alpha \in \Gamma$, by (1),

$$\begin{aligned}
f(x^*, y^*) &= f(x^*, y^*) - f(x_\alpha^*, y_\alpha^*) + f(x_\alpha^*, y_\alpha^*) < \\
&< f(x^*, y^*) - f(x_\alpha^*, y_\alpha^*) + f(x_\alpha^*, y) + \varepsilon_\alpha = \\
&= \left[f(x^*, y^*) - f(x^*, y_\alpha^*)\right] + \left[f(x^*, y_\alpha^*) - f(x_\alpha^*, y_\alpha^*)\right] + f(x_\alpha^*, y) + \varepsilon_\alpha < \\
&< \left[f(x^*, y^*) - f(x^*, y_\alpha^*)\right] + f(x_\alpha^*, y) + 2\varepsilon_\alpha.
\end{aligned}$$

By (iii) and (iv), it follows that

$$\begin{aligned}
f(x^*, y^*) &\leqq \overline{\lim_{\alpha}}\left[f(x^*, y^*) - f(x^*, y_\alpha^*)\right] + \overline{\lim_{\alpha}} f(x_\alpha^*, y) \leqq \\
&\leqq f(x^*, y^*) - \underline{\lim_{\alpha}} f(x^*, y_\alpha^*) + f(x^*, y) \leqq f(x^*, y).
\end{aligned}$$

Next, for each $x \in X$ and each $\alpha \in \Gamma$, by (1),

$$\begin{aligned}
f(x^*, y^*) &= f(x^*, y^*) - f(x_\alpha^*, y_\alpha^*) + f(x_\alpha^*, y_\alpha^*) > \\
&> f(x^*, y^*) - f(x_\alpha^*, y_\alpha^*) + f(x, y_\alpha^*) - \varepsilon_\alpha = \\
&= \left[f(x^*, y^*) - f(x_\alpha^*, y^*)\right] + \left[f(x_\alpha^*, y^*) - f(x_\alpha^*, y_\alpha^*)\right] + f(x, y_\alpha^*) - \varepsilon_\alpha > \\
&> \left[f(x^*, y^*) - f(x_\alpha^*, y^*)\right] + f(x, y_\alpha^*) - 2\varepsilon_\alpha.
\end{aligned}$$

By (iii) and (iv), it follows that

$$\begin{aligned}
f(x^*, y^*) &\geqq \underline{\lim_{\alpha}}\left[f(x^*, y^*) - f(x_\alpha^*, y^*)\right] + \underline{\lim_{\alpha}} f(x, y_\alpha^*) \geqq \\
&\geqq f(x^*, y^*) - \overline{\lim_{\alpha}} f(x_\alpha^*, y^*) \geqq f(x, y^*).
\end{aligned}$$

Therefore $(x^*, y^*) \in X \times Y$ is a saddle point of f. □

LEMMA 3. *Let X be a non-empty convex subset of a Hausdorff topological vector space, Y be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that $f : X \times Y \to \mathbf{R}$ satisfies the following conditions:*

(i) *$f(x, y)$ is continuous on $X \times Y$;*

(ii) *for each fixed $y \in Y$, $x \mapsto f(x, y)$ is quasi-concave;*

(iii) *for each fixed $x \in X$, $y \mapsto f(x, y)$ is quasi-convex;*

(iv) $\sup_{x\in X} \min_{y\in Y} f(x, y) < \infty$;

(v) *the family $\{f(x, \cdot) : x \in X\}$ of real-valued functions on Y is equicontinuous and closed in the Banach space $C(Y)$;*

(vi) *there exists a sequence $\{(x_k, y_k)\}_{k\in \mathbf{N}}$ in $X \times Y$ where for each $k \in \mathbf{N}$, (x_k, y_k) is an ε_k-saddle point of f on $X \times Y$ and $\varepsilon_k \to 0^+$.*

Then $\sup\{ |f(x_k, y)| : k \in \mathbf{N}, y \in Y\} < +\infty$.

PROOF. Since Y is compact, by (i), (ii), (iii) and (iv), the beginning of proof in Theorem 3 of [5] shows that there exists a number M_1 such that $f(x, y) \leqq M_1$ for all $x \in X$ and all $y \in Y$. It follows that the map $y \mapsto \mapsto \sup_{x\in X} f(x, y)$ is real-valued and lower semicontinuous. By compactness of Y again, there exists a number M_2 such that $\sup_{x\in X} f(x, y) \geqq M_2$ for all $y \in Y$.

Since $\varepsilon_k \to 0^+$, we may suppose that $\varepsilon_k < 1$ for each $k \in \mathbf{N}$. For each $k \in \mathbf{N}$ and each $y \in Y$, by (vi),

$$M_1 \geqq f(x_k, y) \geqq f(x_k, y_k) - \varepsilon_k \geqq \sup_{x\in X} f(x, y_k) - 2\varepsilon_k \geqq M_2 - 2$$

so that

$$\sup\{ |f(x_k, y)| : k \in \mathbf{N}, y \in Y\} \leqq \max(|M_1|, |M_2 - 2|). \quad \square$$

As an application of Theorem 2, we have the following result which is Theorem 3 of [5].

COROLLARY 4. *Let X be a non-empty convex subset of a Hausdorff topological vector space, Y be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that $f : X \times Y \to \mathbf{R}$ satisfies the following conditions:*

(i) *$f(x, y)$ is continuous on $X \times Y$;*

(ii) *for each fixed $y \in Y$, $x \mapsto f(x, y)$ is quasi-concave;*

(iii) *for each fixed $x \in X$, $y \mapsto f(x, y)$ is quasi-convex;*

(iv) $\sup_{x\in X} \min_{y\in Y} f(x, y) < +\infty$;

(v) *the family $\{f(x, \cdot) : x \in X\}$ of real-valued functions on Y is equicontinuous and closed in the Banach space $C(Y)$.*

Then f has a saddle point $(x^*, y^*) \in X \times Y$.

PROOF. For each fixed $y \in Y$, by Corollary 2 of [1, p.53],

$$x \mapsto \sup_{v \in Y}[-f(x,v)] = -\inf_{v \in Y} f(x,v)$$

is upper semicontinuous so that by (i), $x \mapsto f(x,y) - \inf_{v\in Y} f(x,v)$ is upper semicontinuous.

For each fixed $y \in Y$ and for any $\eta > 0$, by (v), there exists a neighborhood V of y such that

$$f(u,y') - f(u,y) < \eta$$

for any $u \in X$ and $y' \in V$; thus

$$\sup_{u\in X} f(u,y') \leqq \sup_{u\in X} f(u,y) + \eta.$$

It follows that $y \mapsto \sup_{u\in X} f(u,y)$ is upper semicontinuous so that by (i), $y \mapsto f(x,y) - \sup_{u\in X} f(u,y)$ is also lower semicontinuous.

Now if $\{(x_k,y_k)\}_{k\in\mathbf{N}}$ is a sequence in $X \times Y$ where for each $k \in \mathbf{N}$, (x_k, y_k) is an ε_k-saddle point of f and $\varepsilon_k \to 0^+$, by Lemma 3 and Ascoli's Theorem (e. g., see [6, p.369]), there exist a subsequence $\{f(x_{n_k},\cdot)\}_{k\in\mathbf{N}}$ of $\{f(x_k,\cdot)\}_{k\in\mathbf{N}}$ and $x^* \in X$ such that $\{f(x_{n_k},\cdot)\}_{k\in\mathbf{N}}$ converges uniformly to $f(x^*,\cdot)$ on Y. The conclusion now follows from Theorem 2. □

In order to obtain another new saddle point theorem, we need the concept of an escaping sequence introduced in [2, p.34]: Let E be a Hausdorff topological vector space and Y be a subset of E such that $Y = \bigcup_{n=1}^{\infty} K_n$ where $\{K_n\}_{n\in\mathbf{N}}$ is an increasing sequence of non-empty compact sets, then a sequence $\{y_n\}_{n\in\mathbf{N}}$ in Y is said to be escaping from Y (relative to $\{K_n\}_{n\in\mathbf{N}}$) if for each $n \in \mathbf{N}$, there exists a positive integer M such that $y_k \notin K_n$ for all $k \geqq M$.

THEOREM 5. *Let X be a non-empty convex subset of a Hausdorff topological vector space, Y be a non-empty convex subset of a Hausdorff topological vector space such that $Y = \bigcup_{n=1}^{\infty} K_n$ where $\{K_n\}_{n\in\mathbf{N}}$ is an increasing sequence of non-empty compact convex subsets of Y. Suppose that $f : X \times \times Y \to \mathbf{R}$ satisfies the following conditions:*

(i) *$f(x,y)$ is continuous on $X \times Y$;*

(ii) *for each fixed $y \in Y$, $x \mapsto f(x,y)$ is quasi-concave;*

(iii) *for each fixed $x \in X$, $y \mapsto f(x,y)$ is quasi-convex;*

(iv) *for each $n \in \mathbf{N}$, $\sup_{x\in X} \min_{y\in K_n} f(x,y) < +\infty$;*

(v) *for each $n \in \mathbf{N}$, the family $\{f(x,\cdot) : x \in X\}$ of real-valued functions on K_n is equicontinuous and closed in the Banach space $C(K_n)$;*

(vi) *for each sequence $\{(x_n,y_n)\}_{n\in\mathbf{N}}$ in $X \times Y$ where (x_n,y_n) is a saddle point of f on $X \times K_n$ for each $n \in \mathbf{N}$ which is escaping from $X \times Y$ relative to $\{X \times K_n\}_{n\in\mathbf{N}}$, there exist $n_0 \in \mathbf{N}$ and $x'_{n_0} \in X$ (or $y'_{n_0} \in K_{n_0}$) such that*

$$f(x'_{n_0}, y_{n_0}) > f(x_{n_0}, y_{n_0}) \quad (\text{or } f(x_{n_0}, y_{n_0}) > f(x_{n_0}, y'_{n_0})).$$

Then f has a saddle point $(x^*, y^*) \in X \times Y$.

PROOF. By Corollary 4, for each $n \in \mathbf{N}$, there exists a saddle point $(x_n, y_n) \in X \times K_n$ of f on $X \times K_n$.

Suppose that the sequence $\{(x_n, y_n)\}_{n\in\mathbf{N}}$ in $X \times Y$ were escaping from $X \times Y$ relative to $\{X \times K_n\}_{n\in\mathbf{N}}$, then by (vi) there exist $n_0 \in \mathbf{N}$ and $x'_{n_0} \in X$ (or $y'_{n_0} \in K_{n_0}$) such that

$$f(x'_{n_0}, y_{n_0}) > f(x_{n_0}, y_{n_0}) \quad (\text{or } f(x_{n_0}, y_{n_0}) > f(x_{n_0}, y'_{n_0}))$$

which contradicts the fact that $(x_{n_0}, y_{n_0}) \in X \times K_{n_0}$ is a saddle point of f on $X \times K_{n_0}$. Therefore the sequence $\{(x_n, y_n)\} \in \{X \times K_n\}_{n\in\mathbf{N}}$ is not escaping from $X \times Y$ relative to $\{X \times K_n\}_{n\in\mathbf{N}}$; thus some subsequence $\{(x_{n_k}, y_{n_k})\}_{k\in\mathbf{N}}$ of $\{(x_n, y_n)\}_{n\in\mathbf{N}}$ must lie entirely in some $X \times K_s$, where $s \in \mathbf{N}$.

By (v), Lemma 3 and Ascoli's Theorem again, there exist a subsequence $\{f(x_{n_k}, \cdot)\}_{k\in\mathbf{N}}$ of $\{f(x_n, \cdot)\}_{n\in\mathbf{N}}$ and $x^* \in X$ such that

(2) $\{f(x_{n_k}, \cdot)\}_{k\in\mathbf{N}}$ converges uniformly to $f(x^*, \cdot)$ on K_s.

Since K_s is compact, there exist a subnet $\{y_\alpha\}_{\alpha\in\Gamma}$ of $\{y_{n_k}\}_{k\in\mathbf{N}}$ and $y^* \in K_s$ such that $y_\alpha \to y^*$.

For each $y \in Y$, there exists $s_1 > s$ such that $y \in K_{s_1}$. Let $\alpha_0 \in \Gamma$ be such that $\alpha_0 \geqq s_1$, then for any $\alpha \geqq \alpha_0$, $y \in K_\alpha$ and

$$f(x^*, y^*) = f(x^*, y^*) - f(x^*_\alpha, y^*_\alpha) + f(x^*_\alpha, y^*_\alpha) \leqq$$

$$\leqq f(x^*, y^*) - f(x^*_\alpha, y^*_\alpha) + f(x^*_\alpha, y) =$$

$$= \left[f(x^*, y^*) - f(x^*, y^*_\alpha)\right] + \left[f(x^*, y^*_\alpha) - f(x^*_\alpha, y^*_\alpha)\right] + f(x^*_\alpha, y)$$

so that by (i) and (2), we have $f(x^*, y^*) \leqq f(x^*, y)$.

For each $x \in X$,

$$f(x^*, y^*) = f(x^*, y^*) - f(x^*_\alpha, y^*_\alpha) + f(x^*_\alpha, y^*_\alpha) \geqq$$

$$\geqq f(x^*, y^*) - f(x^*_\alpha, y^*_\alpha) + f(x, y^*_\alpha) =$$

$$= \left[f(x^*, y^*) - f(x^*, y^*_\alpha)\right] + \left[f(x^*, y^*_\alpha) - f(x^*_\alpha, y^*_\alpha)\right] + f(x, y^*_\alpha)$$

so that by (i) and (2) again, we also have $f(x^*, y^*) \geqq f(x, y^*)$. Therefore (x^*, y^*) is a saddle point of f. $\square$

Finally, for application of escaping sequences in minimax inequalities, variational inequalities and equilibrium points, we refer to [8–10].

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THE STRONG SUMMABILITY OF FOURIER TRANSFORMS

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We prove that the Fourier transform of a nonnegative integrable function f and its conjugate Fourier transform are strongly summable with every exponent $q > 0$ at every Lebesgue point of f, at which the Hilbert transform $\tilde{f}$ exists. Hence it follows that both the Fourier transform of any integrable function f and its conjugate Fourier transform are strongly summable with every $q > 0$ almost everywhere. In the particular case where $f \in L^1 \cap L^p$ for some $p > 1$, we give an essentially shorter proof.

The proof of our main theorem can be modified for the case of the Fourier series of a periodic, integrable function to obtain an improvement of the strong summability theorem of Marcinkiewicz and Zygmund.

1. Introduction

We consider complex-valued functions f defined on the real line $\mathbf{R} := := (-\infty, \infty)$. Denote by L^p the class of measurable functions whose pth power is Lebesgue integrable on $\mathbf{R}$ in the case $1 \leqq p < \infty$, or essentially bounded on $\mathbf{R}$ in the case $p = \infty$. As is well known, L^p endowed with the norm

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p} \qquad \text{for} \qquad 1 \leqq p < \infty,$$

or

$$\|f\|_\infty := \operatorname{ess\,sup} \{ |f(t)| : -\infty < t < \infty \}$$

is a Banach space.

It is easy to check that

$$(1.1) \qquad L^1 \cap L^r \subseteqq L^1 \cap L^p \qquad \text{for} \qquad 1 < p < r \leqq \infty.$$

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In fact, in the case $r = \infty$ it is plain that

$$\int_{-\infty}^{\infty} |f(t)|^p \, dt \leqq \|f\|_\infty^{p-1} \|f\|_1,$$

whence (1.1) follows. In the case $1 < r < \infty$, we apply Hölder's inequality with the conjugate exponents

$$\lambda := \frac{r-1}{p-1} \qquad \text{and} \qquad \lambda' := \frac{r-1}{r-p}$$

to obtain

$$\int_{-\infty}^{\infty} |f(t)|^p \, dt = \int_{-\infty}^{\infty} |f(t)|^{r/\lambda} |f(t)|^{p-r/\lambda} \, dt \leqq$$

$$\leqq \left(\int_{-\infty}^{\infty} |f(t)|^r \, dt \right)^{1/\lambda} \left(\int_{-\infty}^{\infty} |f(t)|^{\lambda'(p-r/\lambda)} \, dt \right)^{1/\lambda'} = \|f\|_r^{r/\lambda} \|f\|_1^{1/\lambda'},$$

whence (1.1) follows again.

We remind the reader that the cosine Fourier transform $a(u)$ and sine Fourier transform $b(u)$ of a function $f \in L^1$ are defined by

$$a(u) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos ut \, dt, \tag{1.2}$$

$$b(u) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin ut \, dt \qquad \text{for} \qquad u \in \mathbf{R}; \tag{1.3}$$

the partial integral $s_\nu(f, x)$ and Cesàro mean $\sigma_\nu(f, x)$ of the Fourier transform of f are defined by

$$s_\nu(f, x) := \int_0^\nu \{a(u) \cos xu + b(u) \sin xu\} \, du,$$

$$\sigma_T(f, x) := \frac{1}{T} \int_0^T s_\nu(f, x) \, d\nu =$$

$$= \int_0^T \left(1 - \frac{u}{T}\right) \left\{ a(u) \cos xu + b(u) \sin xu \right\} du \quad \text{for} \quad T > 0 \quad \text{and} \quad x \in \mathbf{R};$$

the conjugate partial integral $\tilde{s}_\nu(f, x)$ and conjugate Cesàro mean $\tilde{\sigma}_\nu(f, x)$ of the Fourier transform of f are defined by

$$\tilde{s}_\nu(f, x) := \int_0^\nu \{a(u) \sin xu - b(u) \cos xu\} \, du,$$

$$\tilde{\sigma}_T(f,x) := \frac{1}{T}\int_0^T \tilde{s}_\nu(f,x)\,d\nu = \int_0^T \left(1-\frac{u}{T}\right)\left\{a(u)\sin xu - b(u)\cos xu\right\}du.$$

As is well known,

$$(1.4)\qquad s_\nu(f,x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\nu(t-x)}{t-x}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\frac{\sin\nu t}{t}\,dt.$$

Given a function g locally integrable on $\mathbf{R}$, in sign : $g\in L^1_{\mathrm{loc}}$, we agree to write

$$\int_{\to-\infty}^{\to\infty} g(t)\,dt := \lim_{N\to\infty}\int_{-N}^{N} g(t)\,dt,$$

provided this limit exists. Accordingly,

$$(1.5)\qquad \int_{\to-\infty}^{\to\infty}\frac{\sin\nu t}{t}\,dt = \pi \qquad \text{for} \qquad \nu>0.$$

Combining (1.4) and (1.5) gives

$$(1.6)\qquad s_\nu(f,x)-f(x) = \frac{1}{\pi}\int_{\to-\infty}^{\to\infty}\left\{f(x+t)-f(x)\right\}\frac{\sin\nu t}{t}\,dt.$$

Let $f\in L^p_{\mathrm{loc}}$ for some p, $1\leqq p<\infty$. We say that $x\in\mathbf{R}$ is a Lebesgue point of f of order p if

$$(1.7)\qquad g(t) := \int_0^t \left|f(x+u)-f(x)\right|^p du = o\left(|t|\right) \qquad \text{as} \qquad t\to 0.$$

We call the set of such points $x\in\mathbf{R}$ the Lebesgue set of f of order p and denote it by $E(f,p)$. Due to Hölder's inequality, we have

$$E(f,r)\subseteqq E(f,p) \qquad \text{for} \qquad 1\leqq p<r<\infty.$$

In the case $p=1$, we write $E(f):=E(f,1)$.

As is known (see, e.g., [10, Vol. 1, p.65]), almost every $x\in\mathbf{R}$ is a Lebesgue point of order p of every function $f\in L^p_{\mathrm{loc}}$ for every $1\leqq p<\infty$, which means that the complement of $E(f,p)$ with respect to $\mathbf{R}$ is of Lebesgue measure zero.

As is well known, if $f\in L^1$ and $x\in E(f)$, then

$$(1.8)\qquad \lim_{\nu\to\infty}\sigma_\nu(f,x) = f(x),$$

$$(1.9)\qquad \lim_{\nu\to\infty}\tilde{\sigma}_\nu(f,x) = \tilde{f}(x),$$

in the latter case provided that the Hilbert transform of f defined by

$$(1.10)\quad \tilde{f}(x) := \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{|t|\geqq\varepsilon} \frac{f(x+t)}{t}\, dt = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\varepsilon}^{\infty} \frac{f(x+t)-f(x-t)}{t}\, dt$$

exists. Since the existence of $\tilde{f}$ for any $f \in L^1$ is proven for almost every $x \in \mathbf{R}$, both (1.8) and (1.9) hold almost everywhere on $\mathbf{R}$ (in abbreviation: a.e.).

Let $q > 0$. We say that the Fourier transform of a function $f \in L^1$ is strongly summable with the exponent q, or briefly: summable H_q, at $x \in \mathbf{R}$ if

$$(1.11)\qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T \left| s_\nu(f,x) - f(x)\right|^q d\nu = 0.$$

Clearly, summability H_1 implies Cesàro summability, i.e., the fulfillment of (1.8). Hölder's inequality shows that if (1.11) is satisfied for some q, then it is true for any smaller $q_1 > 0$:

$$(1.12)\qquad \left(\frac{1}{T}\int_0^T \left| s_\nu(f,x) - f(x)\right|^{q_1} d\nu\right)^{1/q_1} \leqq$$

$$\leqq \left(\frac{1}{T}\int_0^T \left| s_\nu(f,x) - f(x)\right|^{q} d\nu\right)^{1/q} \qquad \text{for} \qquad 0 < q_1 < q < \infty.$$

Summability H_1 indicates that the mean value of $s_\nu(f,x) - f(x)$ tends to zero, not because of the cancellation of positive and negative terms, but because the indices ν for which $\left| s_\nu(f,x) - f(x)\right|$ is large form a set of "small" measure.

On the other hand, ordinary convergence of $s_\nu(f,x)$ to $f(x)$ as $\nu \to \infty$ implies summability H_q at x, for every $q > 0$. To sum up, strong summability lies between ordinary convergence and Cesàro summability.

Of course, we may speak of summability H_q of the conjugate Fourier transform of a function $f \in L^1$ at some $x \in \mathbf{R}$:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \tilde{s}_\nu(f,x) - \tilde{f}(x)\right|^q d\nu = 0,$$

provided the Hilbert transform $\tilde{f}$ exists at x.

Concerning the definitions and results in this section, we refer the reader to the monographs [7, Ch. 1] and [8, Ch. 1].

2. Main results

Among others, we will prove that the Fourier transform of every function $f \in L^1$ is a.e. summable H_q with every exponent $q > 0$.

The particular case where $f \in L^1 \cap L^p$ for some $p > 1$ can be proved in an easy way.

THEOREM 1. *If $f \in L^1 \cap L^p$ for some $p > 1$, $x \in E(f,p)$, and $q > 0$, then the Fourier transform of f is summable H_q at x.*

In the more general setting where we assume merely $f \in L^1$, it is no longer true that the Fourier transform of f is strongly summable at every Lebesgue point of f. However, if we assume that some $x \in \mathbf{R}$ is not only a Lebesgue point of a nonnegative f, but the Hilbert transform $\tilde{f}$ also exists at x, then we are able to conclude strong summability at x. Even the following finer conclusion is true, which is our main result.

THEOREM 2. *If $0 \leqq f \in L^1$, $x \in E(f)$, the Hilbert transform $\tilde{f}$ exists at x, and $q > 0$, then both the Fourier transform of f and its conjugate Fourier transform are summable H_q at x.*

REMARK 1. From Theorem 2 it follows immediately that both the Fourier transform of every function $f \in L^1$ and its conjugate Fourier transform are a.e. summable H_q with every $q > 0$.

REMARK 2. As is known (see [5] and also [1]), there exists a function $f \in L^1$ such that

$$\limsup_{\nu \to \infty} |s_\nu(f,x)| = \infty \qquad \text{a.e.}$$

This example shows that a.e. strong summability of the Fourier transform of an integrable function may take place when ordinary convergence fails a.e.

REMARK 3. The reader will have no difficulty in modifying the proof of Theorem 2 contained in Sections 4 and 5 to obtain the following result for the Fourier series of a nonnegative, periodic, integrable function f, in sign : $f \in L^1_{2\pi}$.

THEOREM 3. *If $0 \leqq f \in L^1_{2\pi}$, $x \in E(f)$, the conjugate series of f is Cesàro summable at x, and $q > 0$, then both the Fourier series of f and its conjugate series are summable H_q at x.*

We note that Marcinkiewicz [4] (in the case $q = 2$) and Zygmund [9] (in the case $q > 0$) proved that if $f \in L^1_{2\pi}$, then both the Fourier series of f and its conjugate series are a.e. summable H_q. (See also [10, Vol. 2, p.184].) Furthermore, according to [10, Vol. 1, p.92], the Cesàro summability of the conjugate series of f at a Lebesgue point x of f is equivalent to the existence of the conjugate function of f at x.

3. Proof of Theorem 1

We begin with the following

REMARK 4. It is enough to prove Theorem 1 in the special case where q is the conjugate exponent to p, i.e., $q := p/(p-1)$. Indeed, by (1.1) then $f \in L^1 \cap L^{p'}$ for every $1 < p' < p$. Hence we conclude summability $H_{q'}$ for $q' := p'/(p'-1)$. By (1.12), summability H_q follows for every $q < q'$. It remains to take into account that by taking $p' - 1$ sufficiently small we get q' arbitrarily large.

Let $0 < \nu < T$. By (1.6), we may write

$$\pi\{ s_\nu(f,x) - f(x)\} = \tag{3.1}$$

$$= \left\{ \int_{-1/T}^{1/T} + \left(\int_{\to -\infty}^{-1/T} + \int_{1/T}^{\to\infty} \right) \right\} \{ f(x+t) - f(x)\} \frac{\sin \nu t}{t}\, dt =$$

$$= I_1(T,\nu) + I_2(T,\nu),$$

say. An elementary estimate shows that

$$|I_1(T,\nu)| \leqq \nu \int_{-1/T}^{1/T} \left| f(x+t) - f(x) \right| dt,$$

whence, by Fubini's theorem,

$$\left\{ \frac{1}{T} \int_0^T \left| I_1(T,\nu) \right|^q d\nu \right\}^{1/q} \leqq \tag{3.2}$$

$$\leqq \frac{1}{(q+1)^{1/q}} T \int_{-1/T}^{1/T} \left| f(x+t) - f(x) \right| dt \to 0 \qquad \text{as} \qquad T \to \infty,$$

due to (1.7).

Next, we estimate $I_2(T,\nu)$. Without loss of generality, we may assume that $1 < p < 2$. Making use of the Hausdorff–Young inequality (see [8, p.96]), then exploiting (1.7) gives

$$\left\{ \frac{1}{T} \int_0^T \left| I_2(T,\nu) \right|^q d\nu \right\}^{1/q} \leqq \tag{3.3}$$

$$\leqq \frac{C_p}{T^{1/q}} \left\{ \int_{|t| \geqq 1/T} \left| \frac{f(x+t) - f(x)}{t} \right|^p dt \right\}^{1/p} = \frac{C_p}{T^{1/q}} \left\{ \int_{|t| \geqq 1/T} \frac{g'(t)}{|t|^p}\, dt \right\}^{1/p},$$

where C_p is a constant depending only on p. By integrating by parts, we get

$$\int_{1/T}^{\infty} \frac{g'(t)}{t^p}\,dt = \left[\frac{g(t)}{t^p}\right]_{1/T}^{\infty} + p\int_{1/T}^{\infty} \frac{g(t)}{t^{p+1}}\,dt = \tag{3.4}$$

$$= o\left(\int_{1/T}^{\infty} \frac{dt}{t^p}\right) = o(T^{p-1}) \quad \text{as} \quad T \to \infty.$$

Combining (3.3), (3.4), and the symmetric counterpart of (3.4) for $\int_{-1}^{-1/T}$ yields

$$\left\{\frac{1}{T}\int_0^T |\,I_2(T,\nu)|^q\,d\nu\right\}^{1/q} \to 0 \qquad \text{as} \qquad T \to \infty. \tag{3.5}$$

Clearly, (1.11) follows from (3.1), (3.2) and (3.5). The proof of Theorem 1 is complete.

REMARK 5. By [5] (see also [1]), the celebrated results of [2] and [3] can be extended to Fourier transforms, as well. Among others, the following is true: If $f \in L^1 \cap L^p$ for some $p > 1$, then

$$\lim_{\nu\to\infty} s_\nu(f,x) = f(x) \qquad \text{a.e.}$$

Hence a.e. summability H_q follows immediately for all $q > 0$. However, our proof of Theorem 1 gives slightly more: namely, strong summability takes place at every Lebesgue point of f.

4. Auxiliary notions and results

The proof of Theorem 2 is much more difficult than that of Theorem 1. We will still use the Hausdorff–Young inequality. However, since f need not belong to any L^p, $p > 1$, it will be necessary to deal not with f itself, but with its Poisson integral $U(f,x,y)$ as well as with its conjugate Poisson integral $\tilde{U}(f,x,y)$, and then to make y tend to 0.

First, we remind the reader of the definition of the Poisson integrals U and $\tilde{U}$. Given a function $f \in L^1$, set

$$U(f,x,y) := \int_{-\infty}^{\infty} f(t)P(x-t,y)\,dt, \tag{4.1}$$

$$\tilde{U}(f,x,y) := \int_{-\infty}^{\infty} f(t)\tilde{P}(x-t,y)\,dt, \tag{4.2}$$

where

$$P(x,y) := \frac{y}{\pi(x^2+y^2)} \tag{4.3}$$

is the Poisson kernel, while

$$\tilde{P}(x,y) := \frac{x}{\pi(x^2+y^2)}$$

is the conjugate Poisson kernel, where $(x,y) \in \mathbf{R}^2_+ := \{(x,y) : x \in \mathbf{R}$ and $y > 0\}$.

The following properties of the Poisson kernel P are of vital importance in the sequel:

$$\int_{-\infty}^{\infty} P(x,y)dx = 1, \tag{4.4}$$

$$\int_{-\infty}^{\infty} P(x-u,y)P(x-t,y)dx = P(u-t,2y). \tag{4.5}$$

For a complex number $z := x + iy$ with $y > 0$, define

$$\Phi(z) := \int_0^{\infty} \{a(u) - ib(u)\} e^{izu}\, du, \tag{4.6}$$

where $a(u)$ and $b(u)$ are defined in (1.2) and (1.3). By Fubini's theorem,

$$\begin{aligned} \Phi(z) &= \frac{1}{\pi}\int_0^{\infty} f(t)\, dt \int_0^{\infty} e^{iu(z-t)}\, du = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt = \\ &= \frac{1}{i\pi}\int_{-\infty}^{\infty} f(t)\frac{(t-x)+iy}{(t-x)^2+y^2}\, dt = U(f,x,y) - i\tilde{U}(f,x,y), \end{aligned} \tag{4.7}$$

where the Poisson integrals U and $\tilde{U}$ are defined in (4.1) and (4.2). It is plain that the function Φ of the complex variable $z := x + iy$ is analytic on the upper half-plane $\mathbf{R}^2_+$.

As is known (see, e.g. [6, p.62]), for every $f \in L^1$ we have

$$\lim_{y\to 0} U(f,x,y) = f(x) \quad \text{and} \quad \lim_{y\to 0} \tilde{U}(f,x,y) = \tilde{f}(x) \quad \text{a.e.},$$

where the Hilbert transform $\tilde{f}$ of f is defined in (1.10).

In Section 5, we will rely on the following two auxiliary results.

LEMMA 1. *If $\alpha > 1$, $f \in L^1$, and for some $k > 0$ we have*

$$\int_0^y |f(t)|\,dt, \quad \int_{-y}^0 |f(t)|\,dt \leqq ky \quad \textit{for all} \quad y > 0, \tag{4.8}$$

then

$$\int_{|t|\geqq y} \frac{|f(t)|}{|t|^\alpha}\,dt \leqq \frac{2\alpha}{\alpha-1} k y^{1-\alpha} \qquad \textit{for all} \qquad y > 0. \tag{4.9}$$

PROOF. Introduce the auxiliary function

$$\psi(t) := \int_0^t |f(u)|\,du. \tag{4.10}$$

As is well known,

$$\psi'(t) = |f(t)| \qquad \text{a.e.} \tag{4.11}$$

By integrating by parts and (4.8), we have

$$\int_y^\infty \frac{|f(t)|}{|t|^\alpha}\,dt = \int_y^\infty \frac{\psi'(t)}{t^\alpha}\,dt = \tag{4.12}$$

$$= \left[\frac{\psi(t)}{t^\alpha}\right]_y^\infty + \alpha \int_y^\infty \frac{\psi(t)}{t^{\alpha+1}}\,dt < \alpha \int_y^\infty \frac{k}{t^\alpha}\,dt = \frac{\alpha k}{\alpha-1} y^{1-\alpha} \quad \text{for all} \quad y > 0.$$

Analogously, we have

$$\int_{-\infty}^y \frac{|f(t)|}{|t|^\alpha}\,dt \leqq \frac{\alpha k}{\alpha-1} y^{1-\alpha} \qquad \text{for all} \qquad y < 0. \tag{4.13}$$

Collecting (4.12) and (4.13) yields (4.9).

LEMMA 2. *If $0 \leqq f \in L^1$ and condition (4.8) is satisfied, then*

$$0 \leqq U(f,x,y) \leqq k \qquad \textit{for all} \qquad (x,y) \in \mathbf{R}^2_+, \tag{4.14}$$

where the Poisson integral U is defined in (4.1).

PROOF. We use notation (4.10). By (4.3) and (4.11), while integrating by parts, we have

$$U(f,x,y) = \int_{-\infty}^\infty \psi'(t) P(x-t,y)\,dt = \tag{4.15}$$

$$= \left[\psi(t)P(x-t,y)\right]_{t=-\infty}^{\infty} + \int_{-\infty}^{\infty} \psi(t)P_t(x-t,y)\,dt =$$

$$= \left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} \psi(t)P_t(x-t,y)\,dt := I_1 + I_2,$$

say, where $P_t := (\partial/\partial t)P$. Another integration by parts and (4.8) gives

$$I_2 \leqq k\int_0^{\infty} tP_t(x-t,y)\,dt = \tag{4.16}$$

$$= k\left[\,-tP(x-t,y)\right]_{t=0}^{\infty} + k\int_0^{\infty} P(x-t,y)\,dt = k\int_0^{\infty} P(x-t,y)\,dt.$$

By symmetry,

$$I_1 \leqq k\int_{-\infty}^{0} P(x-t,y)\,dt. \tag{4.17}$$

Taking into account (4.4) and (4.15)-(4.17), the right inequality in (4.14) follows.

The left inequality in (4.14) is obvious, since the Poisson kernel P is nonnegative.

5. Proof of Theorem 2

Without loss of generality, we may assume that $x = 0$ and $f(0) = 0$ in Theorem 2. Accordingly, the assumptions in Theorem 2 can be formulated as follows:

$$g(t) := \int_0^t f(u)\,du = o(|t|) \qquad \text{as} \qquad t \to 0 \tag{5.1}$$

(cf. (1.7)), the Hilbert transform $\tilde{f}$ defined in (1.10) exists at $x = 0$, and

$$\lim_{\nu\to\infty}\left\{\sigma_\nu(f,0) + i\tilde{\sigma}_\nu(f,0)\right\} = i\tilde{f}(0) \tag{5.2}$$

(cf. (1.8) and (1.9)). In the sequel, we will adopt the abbreviations

$$s_\nu := s_\nu(f,0) + i\tilde{s}_\nu(f,0) \qquad \text{and} \qquad \sigma_\nu := \sigma_\nu(f,0) + i\tilde{\sigma}_\nu(f,0).$$

Since, by (5.2), the limit of σ_ν exists as $\nu \to \infty$, it is enough to show that for all $q > 2$

$$\int_0^T |s_\nu - \sigma_\nu|^q \, d\nu = o(T) \qquad \text{as} \qquad T \to \infty. \tag{5.3}$$

This in turn will follow if we show that

$$\int_0^\infty \nu^q |s_\nu - \sigma_\nu|^q e^{-\nu q y} \, d\nu = o(y^{-1-q}) \qquad \text{as} \qquad y \to 0. \tag{5.4}$$

In fact, take (5.4) valid for the moment and set $y := 1/T$. Clearly,

$$e^{-q} \int_0^T \nu^q |s_\nu - \sigma_\nu|^q \, d\nu \leqq \int_0^T \nu^q |s_\nu - \sigma_\nu|^q e^{-\nu q/T} \, d\nu.$$

Consequently, from (5.4) it follows that

$$\int_0^T \nu^q |s_\nu - \sigma_\nu|^q \, d\nu = o(T^{1+q}) \qquad \text{as} \qquad T \to \infty. \tag{5.5}$$

Now, introducing the auxiliary function

$$\psi(t) := \int_0^t \nu^q |s_\nu - \sigma_\nu|^q \, d\nu,$$

by l'Hopital's rule, we see that

$$\lim_{t \to 0} \frac{\psi(t)}{t^q} = \lim \frac{\psi'(t)}{q t^{q-1}} = \lim \frac{t|s_t - \sigma_t|^q}{q} = 0. \tag{5.6}$$

By integration by parts, while making use of (5.5) and (5.6), we conclude

$$\int_0^T |s_\nu - \sigma_\nu|^q \, d\nu = \int_0^T \frac{\psi'(\nu)}{\nu^q} \, d\nu =$$

$$= \left[\frac{\psi(\nu)}{\nu^q} \right]_0^T + q \int_0^T \frac{\psi(\nu)}{\nu^{q+1}} \, d\nu = \frac{\nu(T)}{T^q} + o(T) = o(T) \qquad \text{as} \qquad T \to \infty,$$

which is (5.3).

Returning to (5.4), we start with the representation

$$\nu(s_\nu - \sigma_\nu) = \int_0^\nu u\{a(u) - ib(u)\} \, du.$$

Hence, by Fubini's theorem,

$$\int_{-\infty}^{\infty} \nu(s_\nu - \sigma_\nu)e^{i\nu z}\,d\nu = \int_0^{\infty} u\{a(u) - ib(u)\}\,du \int_u^{\infty} e^{i\nu z}\,d\nu =$$

$$= \frac{1}{z}\int_0^{\infty} \{a(u) - ib(u)\}e^{izu} iu\,du = \frac{\Phi'(z)}{z},$$

where Φ is defined in (4.6). Let p be the conjugate exponent to $q : 1/p + 1/q = 1$. We remind the reader that $q > 2$, so $1 < p < 2$. By the Hausdorff–Young inequality,

$$\left\{\int_0^{\infty} \nu^q |s_\nu - \sigma_\nu|^q e^{-\nu q y}\,d\nu\right\}^{1/q} \leqq C_p \left\{\int_{-\infty}^{\infty} \left|\frac{\Phi'(z)}{z}\right|^p dx\right\}^{1/p}$$

(cf. (3.3)). Since $(-1 - q)p/q = 1 - 2p$, (5.4) will be established if we show that

$$\int_{-\infty}^{\infty} \left|\frac{\Phi'(z)}{z}\right|^p dx = o(y^{1-2p}) \qquad \text{as} \qquad y \to 0. \tag{5.7}$$

Finally, we recall that, by (4.7),

$$|\Phi'(z)| = \left|\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(t-z)^2}\,dt\right| = \left|\frac{1}{i\pi}\int_{-\infty}^{\infty} f(t)\frac{(t - x + iy)^2}{\left[(t-x)^2 + y^2\right]^2}\,dt\right| \leqq \tag{5.8}$$

$$\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{1}{(t-x)^2 + y^2}\,dt = \frac{1}{y}U(f, x, y) \quad \text{for} \quad z = x + iy \quad \text{and} \quad y > 0$$

(cf. (4.1) and (4.3)). Consequently, (5.7) will follow if we show that

$$I(f, y) := \int_{-\infty}^{\infty} \frac{U^p(f, x, y)}{(x^2 + y^2)^{p/2}}dx = o(y^{1-p}) \qquad \text{as} \qquad y \to 0. \tag{5.9}$$

The rest of this section is devoted to the proof of (5.9). Writing $U^p = U^{p-1}U$, by Fubini's theorem, we have

$$I(f, y) = \int_{-\infty}^{\infty} \frac{U^{p-1}(f, x, y)}{(x^2 + y^2)^{p/2}}dx \int_{-\infty}^{\infty} f(t)P(x - t, y)\,dt = \tag{5.10}$$

$$= \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \frac{U^{p-1}(f, x, y)}{(x^2 + y^2)^{p/2}} P(x - t, y)dx.$$

We will estimate $I(f, y)$ by splitting the integrals in (5.10) into a few parts, appealing in each case to appropriate hypotheses about the behavior of f at $x = 0$.

First, we split the range t of the outer integral in (5.10) into two parts: $|t| \leqq y$ and the remainder. Correspondingly,

$$I(f,y) =: I_1(f,y) + I_2(f,y). \tag{5.11}$$

It is not hard to see that

$$\begin{aligned} I_1(f,y) &:= \int_{-y}^{y} f(t)\,dt \int_{-\infty}^{\infty} \frac{U^{p-1}(f,x,y)}{(x^2+y^2)^{p/2}} P(x-t,y)dx \leqq \\ &\leqq y^{-p} \int_{-y}^{y} f(t)\,dt \int_{-\infty}^{\infty} U^{p-1}(f,x,y)P(x-t,y)dx = \\ &= y^{-p} \int_{-y}^{y} f(t)\,dt \int_{-\infty}^{\infty} U^{p-1}(f,x,y)P^{1/r}(x-t,y)P^{1/r'}(x-t,y)dx, \end{aligned} \tag{5.12}$$

where

$$r := \frac{1}{p-1} \quad \text{and} \quad \frac{1}{r} + \frac{1}{r'} = 1 \qquad (1 < p < 2). \tag{5.13}$$

Observe that, by (4.5),

$$\int_{-\infty}^{\infty} U(f,x,y)P(x-t,y)dx = U(f,t,2y). \tag{5.14}$$

Making use of Hölder's inequality, (4.4) and (5.14), from (5.12) it follows that

$$\begin{aligned} I_1(f,y) &\leqq y^{-p} \int_{-y}^{y} f(t)\,dt \left\{ \int_{-\infty}^{\infty} U(f,x,y)P(x-t,y)dx \right\}^{p-1} \times \\ &\times \left\{ \int_{-\infty}^{\infty} P(x-t,y)dx \right\}^{1/r'} = y^{p-1} \int_{-y}^{y} f(t)U^{p-1}(f,t,2y)\,dt. \end{aligned} \tag{5.15}$$

By (5.1), there exists a positive constant $k = k(f)$ such that

$$\int_0^y f(t)\,dt, \qquad \int_{-y}^0 f(t)\,dt \leqq ky \qquad \text{for all} \qquad y \geqq 0. \tag{5.16}$$

Consequently, from (4.14) and (5.15) it follows that

$$I_1(f,y) \leqq y^{-p}k^{p-1}\int_{-y}^{y} f(t)\,dt = o(k^p y^{1-p}) \qquad \text{as} \qquad y \to 0. \tag{5.17}$$

Next, we split the inner integral in $I_2(f,y)$ according as $|x| \leqq |t|/2$ or $|x| \geqq |t|/2$:

$$I_2(f,y) = \int_{|t|\geqq y} f(t)\,dt \int_{|x|\leqq |t|/2} \frac{U^{p-1}(f,x,y)}{(x^2+y^2)^{p/2}} P(x-t,y)dx+ \tag{5.18}$$

$$+\int_{|t|\geqq y} f(t)\,dt \int_{|x|\geqq |t|/2} \frac{U^{p-1}(f,x,y)}{(x^2+y^2)^{p/2}} P(x-t,y)dx =: I_{21}(f,y) + I_{22}(f,y),$$

say. The inner integral in $I_{21}(f,y)$ does not exceed

$$P(t/2,y)\int_{|x|\leqq |t|/2} \frac{U^{p-1}(f,x,y)}{(x^2+y^2)^{p/2}}dx \leqq$$

$$\leqq P(t/2,y)\left\{\int_{|x|\leqq |t|/2} U(f,x,y)dx\right\}^{p-1}\left\{\int_{|x|\leqq |t|/2}\frac{dx}{(x^2+y^2)^{r'p/2}}\right\}^{1/r'},$$

where r' is defined in (5.13), and so, by (4.14), does not exceed

$$P(t/2,y)\big(k|t|\big)^{p-1}\left\{\int_{-\infty}^{\infty}\frac{dx}{(x^2+y^2)^{pr'/2}}\right\}^{1/r'} =$$

$$= C^{1/r'}(pr'/2)P(t/2,y)\big(k|t|\big)^{p-1}y^{(1-pr')/r'} \leqq \frac{4}{\pi}C^{1/r'}(pr'/2)k^{p-1}|t|^{p-3}y^{3-2p},$$

where we used the obvious estimate

$$P(t/2,y) \leqq \frac{4}{\pi}yt^{-2} \qquad \text{for} \qquad t \in \mathbf{R} \qquad \text{and} \qquad y > 0;$$

and the notation

$$C(q) := \int_{-\infty}^{\infty}\frac{du}{(u^2+1)^q} \qquad \text{for} \qquad q > 1. \tag{5.19}$$

Thus, by (5.16) and Lemma 1, we have

$$(5.20)\qquad I_{21}(f,y) \leqq \frac{4}{\pi} C^{1/r'}(pr'/2)k^{p-1}y^{3-2p} \int_{|t|\geqq y} f(t)|t|^{p-3}\,dt \leqq$$

$$\leqq \frac{8(3-p)}{\pi(2-p)} C^{1/r'}(pr'/2)k^p y^{1-p} \qquad \text{for all} \qquad y>0.$$

Now, we deal with $I_{22}(f,y)$. Similarly to (5.12) and (5.15), we may proceed as follows:

$$I_{22}(f,y) \leqq 2^p \int_{|t|\geqq y} \frac{f(t)}{|t|^p}\,dt \int_{-\infty}^{\infty} U^{p-1}(f,x,y)P(x-t,y)dx \leqq$$

$$\leqq 2^p \int_{|t|\geqq y} \frac{f(t)}{|t|^p} U^{p-1}(f,t,2y)\,dt.$$

By (4.14), (5.16), and Lemma 1,

$$(5.21)\qquad I_{22}(f,y) \leqq 2^p k^{p-1} \int_{|t|\geqq y} \frac{f(t)}{|t|^p}\,dt \leqq$$

$$\leqq \frac{p2^{p+1}}{p-1} k^p y^{1-p} \qquad \text{for all} \qquad y>0.$$

To sum up, by (5.11), (5.17), (5.18), (5.20) and (5.21), we have

$$(5.22)\qquad I(f,y) = O(k^p y^{1-p}) \qquad \text{for all} \qquad y>0.$$

In order to arrive at the wanted estimate (5.9), we have to improve "O" to "o" in (5.22) as $y \to 0$. To this effect, we set $f = f_1 + f_2$, where $f_1(t) :=$ $:= f(t)$ for $|t| \leqq 2\eta$ and $f_1(t) = 0$ otherwise. It is clear that the value of k for f_1 may be as small as we wish, provided η is sufficiently small. Thus, for every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon)$ such that

$$I(f_1,y) \leqq \varepsilon y^{1-p} \qquad \text{for all} \qquad y>0$$

(cf. (5.1) and (5.16)).

Therefore, it remains to verify that

$$(5.23)\qquad I(f_2,y) = o(y^{1-p}) \qquad \text{as} \qquad y \to 0.$$

To achieve this goal, we consider

$$(5.24)\quad I(f_2,y)=\left\{\int_{|x|\leqq\eta}+\int_{|x|\geqq\eta}\right\}\frac{U^p(f_2,x,y)}{(x^2+y^2)^{p/2}}dx=:J_1(f_2,y)+J_2(f_2,y),$$

say (cf. (5.9)).

On the one hand,

$$\lim_{y\to 0}U(f_2,x,y)=0 \qquad \text{uniformly for all} \qquad |x|\leqq\eta$$

(see, e.g., [7, p.10] or [8, p.28]). In other words, given any $\varepsilon>0$ there exists $\delta_1=\delta_1(\varepsilon)$ such that

$$0\leqq U(f_2,x,y)\leqq\varepsilon^{1/p} \qquad \text{for all} \qquad |x|\leqq\eta \qquad \text{and} \qquad 0<y<\delta_1.$$

Hence it follows that

$$(5.25)\quad J_1(f_2,y)\leqq\varepsilon\int_{-\eta}^{\eta}\frac{dx}{(x^2+y^2)^{p/2}}\leqq C(p/2)\varepsilon y^{1-p} \quad \text{for all} \quad 0<y<\delta_1,$$

where $C(p/2)$ is defined in (5.19).

On the other hand, by (4.14) and (5.16) (observe that $k(f_2)\leqq k(f)=k$), we have

$$(5.26)\qquad J_2(f_2,y)\leqq\frac{1}{\eta^p}\{\max_{|x|\geqq\eta}U^{p-1}(f_2,x,y)\}\int_{-\infty}^{\infty}U(f_2,x,y)dx\leqq$$

$$\leqq\frac{k^{p-1}}{\eta^{p+1}}\|f_2\|_1\leqq\varepsilon y^{1-p} \qquad \text{for all} \qquad 0<y<\delta_2,$$

provided that δ_2 is small enough. (We recall that $p>1$.)

Combining (5.24)-(5.26) yields

$$I(f_2,y)\leqq\big(1+C(p/2)\big)\varepsilon y^{1-p} \qquad \text{for all} \qquad 0<y<\min(\delta_1,\delta_2).$$

This proves (5.23). The proof of Theorem 2 is complete.

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CORRECTION TO THE PAPER "ON THE RIEMANNIAN CURVATURE OF A TWISTOR SPACE"[1]

J. DAVIDOV and O. MUŠKAROV (Sofia)

The system at the bottom of p.330 is not written correctly. This system and the subsequent arguments up to line 9, p.331 should be changed as follows:

$$b_{ii} - (3t/4)\sum_{k=1}^{3} a_{ki}^2 + (3t/4)a_{ii}^2 = \mathcal{X},$$

$$b_{ii} + b_{jj} - (3t/4)\sum_{k=1}^{3}(a_{ki}^2 + a_{kj}^2) + (3t/4)(a_{ij} + a_{ji})^2 + (3t/2)a_{ii}a_{jj} = 2\mathcal{X},$$

$$b_{ij} + b_{ji} - (3t/2)\sum_{k=1}^{3} a_{ki}a_{kj} + (3t/2)a_{ii}(a_{ij} + a_{ji}) = 0$$

for $1 \leqq i \neq j \leqq 3$. These identities imply $a_{ii} = a_{jj}$ and $a_{ij} = -a_{ji}$ for $i \neq j$, i.e.

$$g\big(\mathcal{R}(s_i), X \wedge S_i X\big) = g\big(\mathcal{R}(s_j), X \wedge S_j X\big),$$

$$g\big(\mathcal{R}(s_i), X \wedge S_j X\big) = -g\big(\mathcal{R}(s_j), X \wedge S_i X\big), \quad i \neq j.$$

Now varying X over the unit sphere of T_pM gives

$$g\big(\mathcal{R}(s_i), s_j\big) = \delta_{ij} g\big(\mathcal{R}(s_1), s_1\big),$$

$$g\big(\mathcal{R}(s_i), \bar{s}_j\big) = 0, \qquad 1 \leqq i,\ j \leqq 3.$$

[1] *Acta Math. Hung.*, **58** (1991), 319–332.

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Hence M is Einstein and self-dual. Since $X \wedge SX \in \mathbf{R} \cdot \sigma \oplus \bigwedge_{+}^{2} T_pM$ for any $X \in T_pM$, it follows that $R(X \wedge SX)\sigma = 0$ and (5.1) shows that M is of constant sectional curvature $\mathcal{X}$. The rest of the proof is unchanged.

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