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# LOCAL UNIFORM CONVERGENCE OF THE EIGENFUNCTION EXPANSION ASSOCIATED WITH THE LAPLACE OPERATOR. I

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## 1. Introduction

The convergence properties of the expansions formed by a system of eigenfunctions of the Laplace operator are studied by many authors, see e.g. [1], [5], [6], [7], [8], [9], [10]. In this paper the following notions will be used. Consider a bounded domain

$$\Omega \subset \mathbf{R}, \qquad N > 1.$$

By an eigenfunction of the Laplace operator we mean a function $0 \not\equiv u \in$ $\in C^2(\Omega)$ satisfying

$$-\Delta u = \lambda u \quad \text{on} \quad \Omega;$$

$\lambda \in \mathbf{C}$ is called the eigenvalue of $u$. We consider a Riesz basis $(u_i) \subset L^2(\Omega)$ of the eigenfunctions of the Laplace operator and consider the biorthogonal system $(v_i) \subset L^2(\Omega)$:

$$(1) \qquad -\Delta u_i = \lambda_i u_i; \qquad \lambda_i \in \mathbf{C}; \qquad \langle u_i, v_j \rangle_{L^2(\Omega)} = \delta_{i,j}.$$

We do not assume that the $v_j$ are eigenfunctions. Introduce the notations

$$\mu_i := \sqrt{\lambda_i}, \qquad \rho_i := \operatorname{Re} \mu_i \geqq 0, \qquad \nu_i := \operatorname{Im} \mu_i.$$

Introduce further the Bessel–Macdonald kernel

$$(2) \qquad v_\alpha(r) := \frac{2^{\frac{2-\alpha}{2}}}{(2\pi)^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})} \frac{K_{\frac{N-\alpha}{2}}(r)}{r^{\frac{N-\alpha}{2}}}, \qquad 0 < \alpha$$

where $K_\nu(r)$ is the Macdonald function, see [1].

The Liouville classes $L_p^\alpha$, $1 \leqq p \leqq \infty$, $\alpha > 0$ are defined as follows ([4]). $L_p^\alpha(\mathbf{R}^N)$ consists of the functions $f : \mathbf{R}^N \to \mathbf{C}$ representable in the form

$$f(x) = \int_{R^N} v_\alpha\big(|x-y|\big)\, h(y)\, dy$$

with some $h \in L^p(\mathbf{R}^N)$ and we define the norm

$$\|f\|_{L_p^\alpha} := \|h\|_{L^p}.$$

This extends the notion of Soboleff spaces $W_p^\alpha$ for nonintegral $\alpha$.

The main results of this paper are the following statements published without proof in [12].

THEOREM 1. *Suppose* $\operatorname{supp} f \subset \Omega$ *is compact and*

$$f \in L_p^\alpha(\mathbf{R}^N), \qquad \alpha \geqq \frac{N-1}{2}, \qquad \alpha p > N, \qquad p \geqq 1.$$

*Then the partial sums*

$$S_\mu(f,x) := \sum_{\rho_i<\mu} f_i u_i(x), \qquad f_i := \int_\Omega f\overline{v}_i, \qquad \mu > 0 \tag{3}$$

*of the expansion of $f$ tend to $f$ locally uniformly in $\Omega$.*

THEOREM 2 (localization principle). *Let* $\operatorname{supp} f \subset \Omega$ *be compact and suppose that for some other domain $\Omega_0 \subset \Omega$ we have*

$$f \in L_2^\alpha(\mathbf{R}^N), \qquad \alpha \geqq \frac{N-1}{2}, \qquad f|_{\Omega_0} \equiv 0.$$

*Then the expansion of $f$ tends to zero locally uniformly in $\Omega_0$.*

THEOREM 3 (absolute convergence). *Suppose again that* $\operatorname{supp} f \subset \Omega$ *and*

$$f \in L_p^\alpha(\mathbf{R}^N), \qquad \alpha > \frac{N}{2}, \qquad \alpha p > N, \qquad p \geqq 1.$$

*Then the expansion of $f$ converges absolutely and locally uniformly in $\Omega$.*

REMARKS 1. The above theorems extend some results of E. C. Titchmarsh [1] and V. A. Il'in [5], [6]. They investigated the same problems for $\lambda_i \geqq 0$. For the case of arbitrary complex eigenvalues these theorems were obtained for $N = 3$ by I. Joó [8], [9], [10]. Earlier the case $N = 1$ with complex eigenvalues and for the Schrödinger operator was obtained by I.Joó and V. Komornik in [7].

2. Using the ideas of A. Bogmér [11] we can generalize the above theorems for the case of higher order eigenfunctions.

3. We can give examples of Riesz bases $(u_i)$ with complex eigenvalues $\lambda_i$ (even with $\sup |\nu_i| = \infty$) if $\Omega = (0, 2\pi)^N$ by the "direct product" of one-dimensional exponential bases, see e.g. [14].

## 2. Estimation of the square sums of eigenfunctions

In this section we consider a domain $\Omega \subset \mathbf{R}^N$ (not necessarily bounded) and a system of eigenfunctions $(u_i)_{i=1}^{\infty} \subset L^{(}\Omega)$,

$$-\Delta u_i = \lambda_i u_i, \qquad \lambda_i \in \mathbf{C}.$$

$(u_i)$ is called Bessel-system, if

$$\sum_{i=1}^{\infty} \left| \langle f, u_i \rangle \right|^2 \leqq c \|f\|_{L^2(\Omega)}^2, \qquad f \in L^2(\Omega) \tag{4}$$

holds with a constant $c > 0$ independent of $f$.

$(u_i)$ is a Hilbert-system if, conversely,

$$\sum_{i=1}^{\infty} \left| \langle f, u_i \rangle \right|^2 \geqq c \|f\|_{L^2(\Omega)}^2, \qquad f \in L^2(\Omega) \tag{5}$$

for some $c > 0$. In these formulas

$$\langle f, u_i \rangle := \int_{\Omega} f \overline{u}_i.$$

If $(u_i)$ is Bessel- and Hilbert-system, then

$$c_1 \|f\|_{L^2(\Omega)}^2 \leqq \sum_{i=1}^{\infty} \left| \langle f, u_i \rangle \right|^2 \leqq c_2 \|f\|_{L^2(\Omega)}^2.$$

In this case we shall use the notation

$$\sum_{i=1}^{\infty} \left| \langle f, u_i \rangle \right|^2 \asymp \|f\|_{L^2(\Omega)}^2. \tag{6}$$

We shall prove

LEMMA 1. *Let $(u_i) \subset L^2(\Omega)$ be a Bessel-system. Fix a compact set $K \subset \Omega$ and a number $R$,*

$$0 < R < \min \left\{ \frac{1}{2} \operatorname{dist}(K, \partial\Omega), \frac{\pi}{3} \right\}.$$

*Then there exists a constant*

$$c = c(R, K, N)$$

*independent of $x$ and $i$ such that*

$$\sum_{|\mu-\rho_i|\leqq 1}\left(|u_i(x)|e^{2|\nu_i|R}\right)^2 \leqq c\mu^{N-1}, \qquad x\in K, \quad \mu \geqq 1. \tag{7}$$

PROOF. We shall use the following mean-value formula:

$$\int_\Theta u_i(x+r\Theta)\,d\Theta = (2\pi)^{\frac{N}{2}} \frac{J_{\frac{N-2}{2}}(\mu_i r)}{(\mu_i r)^{\frac{N-2}{2}}}\cdot u_i(x). \tag{8}$$

Here $J_\nu(z)$ is the ordinary Bessel function [2], the integration is taken over the surface of the ball with centre $x$ and radius $r$, and $d\Theta$ is the normed Lebesgue measure. If $\mu_i=0$, then (8) is to be substituted by

$$\int_\Theta u_i(x+r\Theta)\,d\Theta = \frac{(2\pi)^{\frac{N}{2}}}{2^{\frac{N-2}{2}}\Gamma(\frac{N}{2})}\cdot u_i(x). \tag{8'}$$

Let now $0<R<\frac{1}{2}\operatorname{dist}(K,\partial\Omega)$ and define

$$d^1(r,\mu) := \begin{cases} \mu^{\frac{N}{2}}\cdot \dfrac{J_{\frac{N-2}{2}}(\mu r)}{r^{\frac{N-2}{2}}}, & R\leqq r\leqq 2R\\ 0, & \text{otherwise.}\end{cases}$$

Consider the coefficients $d_i^1$ of the expansion of $d^1\big(|x-y|,\mu\big)$ for fixed $x\in K$:

$$\overline{d_i^1} = \int_\Omega d^1\left(|x-y|,\mu\right)u_i(y)\,dy = \int_R^{2R} r^{N-1}d_1(r,\mu)\int_\Theta u_i(x+r\Theta)\,d\Theta\,dr =$$

$$= (2\pi)^{\frac{N}{2}}\int_R^{2R} r^{N-1}\mu^{\frac{N}{2}}\frac{J_{\frac{N-2}{2}}(\mu r)}{r^{\frac{N-2}{2}}}\frac{J_{\frac{N-2}{2}}(\mu_i r)}{(\mu_i r)^{\frac{N-2}{2}}}\,dr\cdot u_i(x)$$

i.e.

$$\overline{d_i^1} = (2\pi)^{\frac{N}{2}}\frac{\mu^{\frac{N}{2}}}{\mu_i^{\frac{N-2}{2}}}\int_R^{2R} rJ_{\frac{N-2}{2}}(\mu r)J_{\frac{N-2}{2}}(\mu_i r)\,dr\cdot u_i(x). \tag{9}$$

Here the integral can be explicity given by the formula ([3], 7.14.1)

$$\int rJ_\nu(\mu r)J_\nu(\mu_i r)\,dr = \tag{10}$$

$$= \begin{cases} \dfrac{r}{\mu_i^2 - \mu^2} \left[ \mu_i J_{\nu+1}(\mu_i r) J_\nu(\mu r) - \mu J_\nu(\mu_i r) J_{\nu+1}(\mu r) \right], & \mu \neq \mu_i \\ \dfrac{r^2}{4} \left[ 2\big(J_\nu(\mu r)\big)^2 - 2 J_{\nu+1}(\mu r) J_{\nu-1}(\mu r) \right], & \mu = \mu_i. \end{cases}$$

We shall use the asymptotical formula ([2])

(11)
$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \left(1 + O\left(\frac{1}{|z|^2}\right)\right) - \right.$$

$$\left. - \frac{4\nu^2 - 1}{8} \frac{\sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}{z} \left(1 + O\left(\frac{1}{|z|^2}\right)\right) \right\}, \quad \arg z < \pi.$$

In our case, when $z = \mu_i r$ the condition $|\arg z| \leqq \frac{\pi}{2}$ holds, so (11) can be applied. Using the Bessel property (4) we get that

(12)
$$\sum_{i=1}^{\infty} |d_i^1|^2 \leqq c \| d^1(|x-y|, \mu) \|_{L_y^2(\Omega)}^2 =$$

$$= c \int_R^{2R} r^{N-1} \big(d^1(r,\mu)\big)^2 \, dr = c\mu^N \int_R^{2R} r \left( J_{\frac{N-2}{2}}(\mu r) \right)^2 dr \leqq$$

$$\leqq c\mu^N \int_R^{2R} \frac{r}{\mu r} \, dr \leqq c\mu^{N-1}.$$

So our task is to estimate the integral in (9) from below. Consider separately the following cases:

A)
$$\rho_i \geqq B|\nu_i| \geqq B^2$$

for some large constant $B = B(K, R, N)$. Now if $|\mu - \rho_i| \leqq 1$ then $|\mu - \mu_i| \asymp \asymp |\mu_i|$, $|\mu + \mu_i| \asymp \rho_i \asymp \mu \asymp |\mu_i|$. Hence for $R \leqq r \leqq 2R$ we have

$$\mu_i J_{\frac{N}{2}}(\mu_i r) J_{\frac{N-2}{2}}(\mu r) - \mu J_{\frac{N}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r) =$$

$$= (\mu_i - \mu) J_{\frac{N}{2}}(\mu_i r) J_{\frac{N-2}{2}}(\mu r) +$$

$$+ \mu \left[ J_{\frac{N}{2}}(\mu_i r) J_{\frac{N-2}{2}}(\mu r) - J_{\frac{N}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r) \right] =$$

$$= O\left( |\nu_i| e^{|\nu_i| r} \big(|\mu_i| \mu\big)^{-\frac{1}{2}} \right) +$$

$$+\frac{\mu}{(\mu_i\mu)^{\frac{1}{2}}}\frac{2}{\pi r}\left[\cos\left(\mu_i r-\frac{N+1}{4}\pi\right)\cos\left(\mu r-\frac{N-1}{4}\pi\right)-\right.$$

$$\left.-\cos\left(\mu_i r-\frac{N-1}{4}\pi\right)\cos\left(\mu r-\frac{N+1}{4}\pi\right)+O\left(\frac{e^{|\nu_i|r}}{\mu}\right)\right]=$$

$$=\frac{2}{\pi r}\sqrt{\frac{\mu}{\mu_i}}\sin(\mu_i-\mu)r+O\left(\frac{|\nu_i|}{\mu}e^{|\nu_i|r}\right).$$

This means that for $B$ large enough we get

$$\left|\int rJ_{\frac{N-2}{2}}(\mu r)J_{\frac{N-2}{2}}(\mu_i r)\,dr\right|\asymp\frac{e^{|\nu_i|r}}{|\nu_i|\mu}$$

and then

$$\left|\int_R^{2R} rJ_{\frac{N-2}{2}}(\mu r)J_{\frac{N-2}{2}}(\mu_i r)\,dr\right|\geqq c\frac{e^{2|\nu_i|r}}{|\nu_i|\mu}.$$

Taking into account (9), (12) we obtain

$$\sum_{\substack{|\mu-\rho_i|\leqq 1\\ \rho_i\geqq B|\nu_i|\geqq B^2}}|u_i(x)|^2\frac{e^{4|\nu_i|R}}{|\nu_i|^2}\leqq\sum_{i=1}^{\infty}|d_i^1|^2\leqq c\mu^{N-1}. \tag{13}$$

B) $\qquad \mu\geqq\mu_0 \quad$ and $\quad |\nu_i|\leqq B$

where $\mu=\mu_0(B)$ is large enough. Then

$$|\mu_i|\asymp\mu,\qquad |\mu-\mu_i|\leqq 1+B,\qquad |\mu+\mu_i|\asymp\mu.$$

Define

$$f(z):=r\frac{zJ_{\frac{N}{2}}(rz)J_{\frac{N-2}{2}}(r\mu)-\mu J_{\frac{N}{2}}(r\mu)J_{\frac{N-2}{2}}(rz)}{z+\mu},$$

then

$$\int rJ_{\frac{N-2}{2}}(\mu r)J_{\frac{N-2}{2}}(\mu_i r)\,dr=\begin{cases}\dfrac{f(\mu_i)-f(\mu)}{\mu_i-\mu}, & \mu\neq\mu_i\\ f'(\mu), & \mu=\mu_i\end{cases}$$

and

$$f'(z)=\frac{r^2}{2}\left[\left(J_{\frac{N-2}{2}}(rz)\right)^2-J_{\frac{N}{2}}(rz)J_{\frac{N-4}{2}}(rz)\right].$$

From the equation

$$\frac{d}{dz}\left[z^{\nu} J_{\nu}(rz)\right] = rz^{\nu} J_{\nu-1}(rz)$$

and the estimate

$$|J_{\nu}(rz)| \leqq c|z|^{-\frac{1}{2}}$$

we get

$$|f''(z)| \leqq c|z|^{-1}, \qquad z \in [\mu, \mu_i].$$

Since $f$ is analytic on $[\mu, \mu_i]$, we get

$$\left|\frac{f(\mu_i) - f(\mu)}{\mu_i - \mu} - f'(\mu)\right| \leqq |\mu_i - \mu| \max_{z \in [\mu, \mu_i]} |f''(z)| \leqq c\left|1 - \frac{\mu_i}{\mu}\right|. \tag{14}$$

On the other hand,

$$f'(\mu) = \frac{r^2}{2}\left\{\frac{2}{\pi r \mu}\left[\cos^2\left(\mu r - \frac{N-1}{4}\pi\right) + O\left(\frac{1}{\mu}\right)\right] -\right.$$

$$\left. -\frac{2}{\pi r \mu}\left[\cos\left(\mu r - \frac{N+1}{4}\pi\right)\cos\left(\mu r - \frac{N-3}{4}\pi\right) + O\left(\frac{1}{\mu}\right)\right]\right\} =$$

$$= \frac{r}{\pi \mu}\left\{\cos^2\left(\mu r - \frac{N-1}{4}\pi\right) + \sin^2\left(\mu r - \frac{N-1}{4}\pi\right) + O\left(\frac{1}{\mu}\right)\right\} =$$

$$= \frac{r}{\pi \mu} + O\left(\frac{1}{\mu^2}\right).$$

Now if $|\mu - \mu_i| < c_0$ and $c_0$ is small enough with respect to the constant appearing in (14), then

$$\left|\int r J_{\frac{N-2}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r)\, dr\right| = \left|\frac{f(\mu_i) - f(\mu)}{\mu_i - \mu}\right| = \frac{r}{\pi \mu} + \eta, \qquad |\eta| < \frac{r}{8\pi \mu}$$

and hence

$$\left|\int_R^{2R} r J_{\frac{N-2}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r)\, dr\right| \geqq \frac{c}{\mu}. \tag{15}$$

If $c_0 \leqq |\mu_i - \mu|$ then, as we showed in A),

$$\int r J_{\frac{N-2}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r)\, dr =$$

$$= \frac{r}{\mu_i^2 - \mu^2} \left[ \frac{2}{\pi r} \sqrt{\frac{\mu}{\mu_i}} \sin(\mu_i - \mu) r + O\left(\frac{1}{\mu}\right) \right] =$$

$$= \frac{2}{\pi} \frac{\sin(\mu_i - \mu) r}{\mu_i^2 - \mu^2} + O\left(\frac{1}{\mu^2}\right).$$

Obviously, for $R < \frac{\pi}{3}$ we have $R|\mu - \rho_i| < \frac{\pi}{3}$, and then

$$\left| \left[ \frac{\sin(\mu_i - \mu) r}{\mu_i^2 - \mu^2} \right]_{r=R}^{2R} \right| \asymp \frac{1}{\mu} |\sin R(\mu_i - \mu)| \cdot$$

$$\cdot |2 \cos R(\mu_i - \mu) - 1| \geqq \frac{c}{\mu} |\sin R(\mu_i - \mu)| \geqq \frac{c}{\mu}$$

so (15) is proved also for $c_0 < |\mu - \mu_i|$. Consequently

$$\sum_{\substack{|\mu - \rho_i| \leqq 1 \\ |\nu_i| \leqq B}} |u_i(x)|^2 \leqq c \sum_{i=1}^{\infty} |d_i^1|^2 \leqq c\mu^{N-1} \quad \text{for} \quad \mu \geqq \mu_0. \tag{16}$$

C) $\mu \geqq \mu_0, \qquad \rho_i \leqq B|\nu_i|, \qquad B \leqq |\nu_i|.$

We expand in this case the function

$$d^2(r, \mu) := \begin{cases} r^{\frac{1-N}{2}}, & R \leqq r \leqq 2R \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\overline{d_i^2} = \int_{\Omega} d^2\big(|x - y|, \mu\big)\, u_i(y)\, dy = \tag{17}$$

$$= u_i(x) (2\pi)^{\frac{N}{2}} \int_R^{2R} r^{N-1} r^{\frac{1-N}{2}} \frac{J_{\frac{N-2}{2}}(\mu_i r)}{(\mu_i r)^{\frac{N-2}{2}}}\, dr =$$

$$= (2\pi)^{\frac{N}{2}} \mu_i^{\frac{2-N}{2}} \int_R^{2R} r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r)\, dr \cdot u_i(x).$$

We know that

$$\sum_{i=1}^{\infty} |d_i^2|^2 \leqq c \big\| d^2\big(|x - y|, \mu\big) \big\|_{L_y^2(\Omega)}^2 \leqq c. \tag{18}$$

Now

$$r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r) = \left(\frac{2}{\pi\mu_i}\right)^{\frac{1}{2}} \cos\left(\mu_i r - \frac{N-1}{4}\pi\right)\left(1 + O\left(\frac{1}{|\mu_i|}\right)\right)$$

and hence

$$\left|\int_R^{2R} r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r)\, dr\right| = \left|\sqrt{\frac{2}{\pi}} \frac{1}{\mu_i^{\frac{3}{2}}} \left[\sin\left(\mu_i r - \frac{N-1}{4}\pi\right)\right]_{r=R}^{2R} + \right.$$

$$\left. +O\left(\frac{1}{|\mu_i|^{\frac{3}{2}}} \int_R^{2R} e^{|\nu_i| r}\, dr\right)\right|.$$

Since $\operatorname{sh} |\operatorname{Im} z| \leqq |\sin z| \leqq \operatorname{ch} \operatorname{Im} z$ and

$$\int_R^{2R} e^{|\nu_i| r}\, dr \leqq c \frac{e^{2|\nu_i| R}}{|\nu_i|}$$

hence for large $B$

$$\left|\int_R^{2R} r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r)\, dr\right| \geqq c \frac{e^{2|\nu_i| R}}{|\mu_i|^{\frac{3}{2}}}$$

and so by (18)

$$\text{(19)} \qquad c \geqq \sum_{i=1}^{\infty} |d_i^2|^2 \geqq c \sum_{\substack{|\mu-\rho_i| \leqq 1 \\ \rho_i \leqq B|\nu_i| \\ B \leqq |\nu_i|}} |u_i(x)|^2 |\mu_i|^{2-N} \frac{e^{4|\nu_i| R}}{|\mu_i|^3} \geqq$$

$$\geqq c \sum_{\substack{|\mu-\rho_i| \leqq 1 \\ \rho_i \leqq B|\nu_i| \\ B \leqq |\nu_i|}} |u_i(x)|^2 \frac{e^{4|\nu_i| R}}{|\nu_i|^{N+1}} \quad \text{for} \quad \mu \geqq \mu_0.$$

D) $\qquad \mu \leqq \mu_0, \qquad |\nu_i| \geqq B_1$

where $B_1 = B_1(B, \mu_0)$ is large enough. Then

$$r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r) = \left(\frac{2}{\pi\mu_i}\right)^{\frac{1}{2}} \cos\left(\mu_i r - \frac{N-1}{4}\pi\right) + O\left(\frac{e^{|\nu_i| r}}{|\nu_i|^{\frac{3}{2}}}\right)$$

and hence

$$\left|\int_R^{2R} r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r)\,dr\right| \geqq c\frac{e^{2|\nu_i|R}}{|\nu_i|^{\frac{3}{2}}}.$$

Now (18) implies

$$c \geqq \sum_{i=1}^{\infty} |d_i^2|^2 \geqq c \sum_{\substack{\rho_i \leqq \mu_0+1 \\ |\nu_i| \geqq B_1}} |u_i(x)|^2 \frac{e^{4|\nu_i|R}}{|\nu_i|^{N+1}}. \tag{20}$$

E) $\mu \leqq \mu_0, \qquad |\nu_i| \leqq B_1.$

Here we take $0 < R_1 < R(B_1, \mu_0)$ small enough, and define the function $d^3(r,\mu)$, which is the same as $d^2$, only $R$ is changed in its definition by $R_1$. We have as above

$$\sum_{i=1}^{\infty} |d_i^3|^2 \leqq c,$$

$$\overline{d_i^3} = (2\pi)^{\frac{N}{2}} \mu_i^{\frac{2-N}{2}} \int_{R_1}^{2R_1} r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r)\,dr \cdot u_i(x).$$

Now

$$J_{\frac{N-2}{2}}(\mu_i r) = c(\mu_i r)^{\frac{N-2}{2}} + O\left(\left(|\mu_i| r\right)^{\frac{N+2}{2}}\right)$$

hence

$$\left|\mu_i^{\frac{2-N}{2}} \int_{R_1}^{2R_1} r^{\frac{1}{2}} J_{\frac{N-2}{2}}(\mu_i r)\,dr\right| = \left|cR_1^{\frac{N+1}{2}} + O\left(R_1^{\frac{N+5}{2}} |\mu_i|^2\right)\right| \geqq c(R_1) > 0$$

and then

$$\sum_{\substack{\rho_i \leqq \mu_0+1 \\ |\nu_i| \leqq B_1}} |u_i(x)|^2 \leqq c. \tag{21}$$

The estimates (13), (16), (19), (20), (21) show that

$$\sum_{|\mu-\rho_i| \leqq 1} |u_i(x)|^2 \frac{e^{4|\nu_i|R}}{(1+|\nu_i|)^{N+1}} \leqq c\mu^{N-1}, \qquad x \in K.$$

Take any $R < R' < \min\left\{\frac{1}{2}\operatorname{dist}(K,\partial\Omega), \frac{\pi}{3}\right\}$, then

$$e^{4|\nu_i|R} \leqq c\frac{e^{4|\nu_i|R'}}{\left(1+|\nu_i|\right)^{N+1}}$$

hence Lemma 1 is proved.

Next we consider the lower estimate.

LEMMA 2. *Let $(u_i) \subset L^2(\Omega)$ be a Bessel- and Hilbert-system. Then for any fixed compact set $K \subset \Omega$ and $0 < R$ there exist $M > 0$ and $c > 0$ satisfying*

$$\sum_{|\mu-\rho_i|\leqq M} |u_i(x)|^2 e^{4|\nu_i|R} \geqq c\mu^{N-1}, \qquad x \in K, \quad \mu \geqq 1. \tag{22}$$

PROOF. Obviously we can suppose that

$$R < \min\left\{\frac{1}{2}\operatorname{dist}(K,\partial\Omega), \frac{\pi}{3}\right\}$$

and then the upper estimate (7) holds. Consequently for any $\delta > 0$ we have

$$\sum_{i=1}^{\infty} \frac{u_i(x)|^2 e^{4|\nu_i|R}}{(1+\rho_i)^{N+\delta}} \leqq c(\delta) < \infty, \qquad x \in K. \tag{23}$$

Take the expansion of $d^1$ defined in Lemma 1, then

$$\left\| d^1(|x-y|,\mu)\right\|^2_{L^2_y(\Omega)} = c\mu^N \int_R^{2R} r\left(J_{\frac{N-2}{2}}(\mu r)\right)^2 dr =$$

$$= c\mu^N \int_R^{2R} r\left\{\frac{2}{\pi\mu r}\cos^2\left(\mu r - \frac{N-1}{4}\pi\right) + O\left(\frac{1}{\mu^2}\right)\right\} dr \geqq c\mu^{N-1}$$

for $\mu \geqq \mu_0$, $\mu_0 = \mu_0(R)$ large. Hence

$$c\mu^{N-1} \leqq \left\| d^1(|x-y|,\mu)\right\|^2_{L^2_y(\Omega)} \leqq c\sum_{i=1}^{\infty} |d_i^1|^2 = \tag{24}$$

$$= c\sum_{i=1}^{\infty} |u_i(x)|^2 \frac{\mu^N}{|\mu_i|^{N-2}} \left|\int_R^{2R} rJ_{\frac{N-2}{2}}(\mu r)J_{\frac{N-2}{2}}(\mu_i r)\, dr\right|^2.$$

We have to estimate the integrals

$$I_i = \int_R^{2R} rJ_{\frac{N-2}{2}}(\mu r)J_{\frac{N-2}{2}}(\mu_i r)\, dr.$$

Consider first the case $|\mu - \rho_i| \leqq \frac{\mu}{2}$. Then by the asymptotical expression of Bessel functions,

$$I_i = \frac{2}{\pi\sqrt{\mu\mu_i}}\left\{\int_R^{2R} \cos\left(\mu r - \frac{N-1}{4}\pi\right)\cos\left(\mu_i r - \frac{N-1}{4}\pi\right) dr + \right.$$

$$\left. +O\left(\frac{e^{2|\nu_i|R}}{\mu}\right)\right\}.$$

Consequently

$$(25) \qquad |I_i| \leqq \frac{c}{\mu}\frac{e^{2|\mu_i|R}}{1+|\mu-\rho_i|}, \qquad |\mu-\rho_i| \leqq \frac{\mu}{2}.$$

Take a large number

$$2^m = M > M(\mu_0, K, R, N)$$

to be specified later and suppose that

$$\mu > 2M.$$

Then (24) and (25) give

$$(26) \qquad \mu^{N-1} \leqq c \sum_{|\mu-\rho_i|\leqq M} |u_i(x)|^2 e^{4|\nu_i|R} + \sum_{|\mu-\rho_i|>M} |u_i(x)|^2 \frac{\mu^N}{|\mu_i|^{N-2}}|I_i|^2.$$

Here the constant $c$ is independent of $M$. Take the decomposition

$$\sum_{|\mu-\rho_i|>M} = \sum_{M<|\mu-\rho_i|<\frac{\mu}{2}} + \sum_{\rho_i\leqq 1} + \sum_{1<\rho_i\leqq\frac{\mu}{2}} + \sum_{\frac{3\mu}{2}\leqq\rho_i}.$$

Estimate the first sum. Suppose that

$$2^{p-1} < \frac{\mu}{2} \leqq 2^p.$$

Then

(27)
$$\sum_{M<|\mu-\rho_i|<\frac{\mu}{2}} |u_i(x)|^2 \frac{\mu^N}{|\mu_i|^{N-2}} |I_i|^2 \leqq$$

$$\leqq c \sum_{k=m+1}^{p} \sum_{2^{k-1} \leqq |\mu-\rho_i| \leqq 2^k} |u_i(x)|^2 \frac{e^{4|\nu_i|R}}{|\mu-\mu_i|^2} \leqq$$

$$\leqq c \sum_{k=m+1}^{p} \frac{1}{2^{2k}} \sum_{2^{k-1} \leqq |\mu-\rho_i| \leqq 2^k} |u_i(x)|^2 e^{4|\nu_i|R} \leqq$$

$$\leqq c \sum_{k=m+1}^{p} \frac{1}{2^{2k}} \mu^{N-1} \cdot 2^k \leqq c \frac{\mu^{N-1}}{M}$$

where $c$ is again independent of $M$.

In case $\rho_i \leqq 1$ suppose first that $|\mu_i| \leqq 1$. Then using the formulas

$$\int r^{\nu+1} J_\nu(\mu r)\, dr = r^{\nu+1} \frac{J_{\nu+1}(\mu r)}{\mu},$$

$$\frac{d}{dr}\left[r^{-\nu} J_\nu(\mu r)\right] = -\mu r^{-\nu} J_{\nu+1}(\mu r)$$

we get

(28)
$$I_i = \int_R^{2R} r^{\frac{N}{2}} J_{\frac{N-2}{2}}(\mu r) \cdot r^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(\mu_i r)\, dr =$$

$$= \left[\frac{r}{\mu} J_{\frac{N}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r)\right]_{r=R}^{2R} + \frac{\mu_i}{\mu} \int_R^{2R} r J_{\frac{N}{2}}(\mu r) J_{\frac{N}{2}}(\mu_i r)\, dr.$$

Using the estimates $|J_\nu(\mu_i r)| \leqq c|\mu_i|^\nu$, $|J_\nu(\mu r)| \leqq c\mu^{-\frac{1}{2}}$ we get

$$|I_i|^2 \leqq c \left\{ \frac{|\mu_i|^{N-2}}{\mu^3} + \frac{|\mu_i|^{N+2}}{\mu^3} \right\} \leqq c \frac{|\mu_i|^{N-2}}{\mu^3}.$$

If $\rho_i \leqq 1 \leqq |\mu_i|$ then we estimate by $|J_\nu(\mu_i r)| \leqq c \frac{e^{|\nu_i| r}}{|\mu_i|^{\frac{1}{2}}}$ to obtain

$$|I_i|^2 \leqq c \left\{ \frac{e^{2|\nu_i|R}}{\mu^{\frac{3}{2}} |\mu_i|^{\frac{1}{2}}} + \frac{|\mu_i|^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} \int_R^{2R} e^{|\nu_i| r}\, dr \right\}^2 \leqq c \frac{e^{4|\nu_i|R}}{\mu^3 |\mu_i|}$$

and then

$$\sum_{\rho_i \leqq 1} |u_i(x)|^2 \frac{\mu^N}{|\mu_i|^{N-2}} |I_i|^2 \leqq c\mu^{N-3} \sum_{\rho_i \leqq 1} |u_i(x)|^2 e^{4|\nu_i|R} \leqq c\mu^{N-3}. \tag{29}$$

In case $1 < \rho_i \leqq \frac{\mu}{2}$ we use again (28) and put the asymptotic expression of $J_\nu$ into the integral on the right to obtain

$$|I_i| \leqq c \left\{ \frac{e^{2|\nu_i|R}}{\mu^{\frac{3}{2}} |\mu_i|^{\frac{1}{2}}} + \frac{|\mu_i|}{\mu} \left[ \left(\mu|\mu_i|\right)^{-\frac{1}{2}} \frac{e^{2|\nu_i|R}}{|\mu - \mu_i|} + \frac{e^{2|\nu_i|R}}{\mu^{\frac{3}{2}} |\mu_i|^{\frac{1}{2}}} + \frac{e^{2|\nu_i|R}}{\mu^{\frac{1}{2}} |\mu_i|^{\frac{3}{2}}} \right] \right\},$$

$$\frac{\mu^N}{|\mu_i|^{N-2}} |I_i|^2 \leqq$$

$$\leqq c\mu^{N-\frac{3}{2}} e^{4|\nu_i|R} \left\{ \frac{1}{\mu^{\frac{3}{2}} |\mu_i|^{N-1}} + \frac{|\mu_i|}{\mu^{\frac{3}{2}} |\mu_i|^{N-1} |\mu - \mu_i|} + \frac{|\mu_i|^2}{\mu^{\frac{7}{2}} |\mu_i|^{N-1}} \right\} \leqq$$

$$\leqq c\mu^{N-\frac{3}{2}} e^{4|\nu_i|R} \left\{ \frac{1}{\rho_i^{N+\frac{1}{2}}} + \frac{1}{\rho_i^{N+\frac{1}{2}}} \frac{|\mu_i|}{|\mu - \mu_i|^2} + \frac{1}{\rho_i^{N+\frac{1}{2}}} \frac{|\mu_i|^2}{\mu^2} \right\} \leqq$$

$$\leqq c\mu^{N-\frac{3}{2}} e^{4|\nu_i|R} \frac{(1 + |\nu_i|)^2}{\rho_i^{N+\frac{1}{2}}} \leqq c\mu^{N-\frac{3}{2}} \frac{e^{4|\nu_i|R_1}}{\rho_i^{N+\frac{1}{2}}}$$

for some $R < R_1 < \min\left\{\frac{1}{2}\operatorname{dist}(K, \partial\Omega), \frac{\pi}{3}\right\}$. Consequently we have by (23)

$$\sum_{1 < \rho_i \leqq \frac{\mu}{2}} |u_i(x)|^2 \frac{\mu^N}{|\mu_i|^{N-2}} |I_i|^2 \leqq \tag{30}$$

$$\leqq c\mu^{N-\frac{3}{2}} \sum_{1 < \rho_i \leqq \frac{\mu}{2}} \frac{|u_i(x)|^2 e^{4|\nu_i|R_1}}{\rho_i^{N+\frac{1}{2}}} \leqq c\mu^{N-\frac{3}{2}}.$$

Finally if $\rho_i > \frac{3\mu}{2}$ then we take another integration by parts to get

$$I_i = \int_R^{2R} r^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(\mu r) \cdot r^{\frac{N}{2}} J_{\frac{N-2}{2}}(\mu_i r)\, dr =$$

$$= \left[ \frac{r}{\mu_i} J_{\frac{N-2}{2}}(\mu r) J_{\frac{N}{2}}(\mu_i r) \right]_{r=R}^{2R} + \frac{\mu}{\mu_i} \int_R^{2R} r J_{\frac{N}{2}}(\mu r) J_{\frac{N}{2}}(\mu_i r)\, dr.$$

Applying the asymptotics on the right, we obtain

$$|I_i| \leqq c\left\{\frac{e^{2|\nu_i|R}}{|\mu_i|^{\frac{3}{2}}\mu^{\frac{1}{2}}} + \frac{\mu}{|\mu_i|}\left[\frac{e^{2|\nu_i|R}}{\mu^{\frac{1}{2}}|\mu_i|^{\frac{1}{2}}|\mu-\mu_i|} + \frac{e^{2|\nu_i|R}}{\mu^{\frac{3}{2}}|\mu_i|^{\frac{1}{2}}} + \frac{e^{2|\nu_i|R}}{\mu^{\frac{1}{2}}|\mu_i|^{\frac{3}{2}}}\right]\right\} \leqq$$

$$\leqq c\frac{e^{2|\nu_i|R}}{|\mu_i|^{\frac{3}{2}}\mu^{\frac{1}{2}}}$$

and hence

$$\text{(31)} \qquad \sum_{\rho_i \geqq \frac{3\mu}{2}} |u_i(x)|^2 \frac{\mu^N}{|\mu_i|^{N-2}}|I_i|^2 \leqq c \sum_{\rho_i \geqq \frac{3\mu}{2}} \frac{|u_i(x)|^2 e^{4|\nu_i|R}}{|\mu_i|^{N+1}}\mu^{N-1} \leqq$$

$$\leqq c\mu^{N-\frac{3}{2}} \sum_{\rho_i \geqq \frac{3\mu}{2}} \frac{|u_i(x)|^2 e^{4|\nu_i|R}}{\rho_i^{N+\frac{1}{2}}} \leqq c\mu^{N-\frac{3}{2}}.$$

The estimates (26), (27), (29), (30), (31) show that

$$\mu^{N-1} \leqq c \sum_{|\mu-\rho_i| \leqq M} |u_i(x)|^2 e^{4|\nu_i|R} + c\mu^{N-1}\left(\frac{1}{M} + \frac{1}{\mu^{\frac{1}{2}}}\right)$$

where the constants $c$ are independent of $M$, $\mu$, $i$, $x$. If $M = M(\mu_0, K, N, R)$ is large enough, then we get the desired estimate for $\mu > 2M$. But we can substitute $M$ by its double and then (22) holds for all $\mu \geqq 1$. Lemma 2 is proved.

## 3. Estimates for the spectral function

Let $\Omega \subset \mathbf{R}^N$ be a (not necessarily bounded) domain and $(u_i)_{i=1}^{\infty} \subset L^2(\Omega)$ a Riesz basis in $L^2(\Omega)$ with arbitrary complex eigenvalues. The spectral function of this system is the function

$$\Theta(x, y, \mu) := \sum_{\rho_i < \mu} \overline{u_i(x)} v_i(y).$$

It may be an infinite sum, but the proof of Lemma 7 below will show that the sum converges. We shall show that the spectral function is bounded in norm when $\mu$ varies. First some technical lemmas are needed.

We know the general asymptotic expansion of $J_\nu(z)$, namely ([2]) fo $M \geqq 1$ and $|\arg z| < \pi$ we have

(32)

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \left[ \sum_{m=0}^{M-1} (-1)^m \frac{(\nu, 2m)}{(2z)^{2m}} + O\big(|z|^{-2M}\big) \right] - \right.$$

$$\left. - \sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \left[ \sum_{m=0}^{M-1} (-1)^m \frac{(\nu, 2m+1)}{(2z)^{2m+1}} + O\big(|z|^{-2M-1}\big) \right] \right\},$$

where the implicit constants contained in the $O$-term depend on $\nu$, $M$ and $\delta$ if $\pi - |\arg z| \geqq \delta > 0$. If we take the difference $z^{-\nu} J_\nu(z) - (\operatorname{Re} z)^{-\nu} J_\nu(\operatorname{Re} z)$ by (32), the remainder term does not give a good estimate using (32). So our first aim is to give an exact estimate for the remainder.

LEMMA 3. *Let $\nu \in \mathbf{R}$ be arbitrary fixed. Then the asymptotic expansion of*

$$\frac{J_\nu(z)}{z^\nu} - \frac{J_\nu(\operatorname{Re} z)}{(\operatorname{Re} z)^\nu}, \qquad \operatorname{Re} z \geqq 1$$

*whose main part is given corresponding to the main part of* (32), *has a remainder*

$$R_{M,\nu} = O\left( \frac{e^{|y|} - 1}{|z|^{\nu+2M+\frac{1}{2}}} + \frac{|y|}{|z|^{\nu+2M+\frac{3}{2}}} + \frac{|y|}{x^{\nu+2M+\frac{3}{2}}} \right) \tag{33}$$

*if* $x := \operatorname{Re} z \geqq 1$, $y := \operatorname{Im} z$.

PROOF. We know that ([2])

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2}.$$

We shall show separately that the corresponding remainder of the asymptotic expansions of $H_\nu^{(1)}(z)$, resp. $H_\nu^{(2)}(z)$ has an estimate of type (33). Remark that it is enough to prove (33) for large values of $M$ because for smaller $M$ the superfluous main terms satisfy the needed estimate, e.g.

$$\frac{\cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}{z^{\nu+\frac{1}{2}}} - \frac{\cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}{x^{\nu+\frac{1}{2}}} =$$

$$= \left( \frac{1}{z^{\nu+\frac{1}{2}}} - \frac{1}{x^{\nu+\frac{1}{2}}} \right) \cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) - i \sin\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \frac{\operatorname{sh} y}{z^{\nu+\frac{1}{2}}} +$$

$$+\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\frac{\operatorname{ch} y-1}{z^{\nu+\frac{1}{2}}}=$$

$$=O\left(\frac{|y|}{|z|^{\nu+\frac{3}{2}}}+\frac{|y|}{x^{\nu+\frac{3}{2}}}+\frac{e^{|y|}-1}{|z|^{\nu+\frac{1}{2}}}\right).$$

Suppose first that

$$\nu>-\frac{1}{2},$$

then ([2])

$$H_\nu^{(1)}(z)=\sqrt{\frac{2}{\pi z}}\frac{e^{i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(\nu+\frac{1}{2}\right)}\cdot\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{\nu-\frac{1}{2}}\left(1+\frac{iu}{2z}\right)^{\nu-\frac{1}{2}}du$$

if

$$|\beta|<\frac{\pi}{2},\qquad \arg z\neq\beta-\frac{\pi}{2}.$$

This last condition ensures that $1+\frac{iu}{2z}\neq 0$. Fix the value

$$\beta:=-\frac{\pi}{4}.$$

As it is known ([2]), for any integer $p>0$

$$\left(1+\frac{iu}{2z}\right)^{\nu-\frac{1}{2}}=\sum_{m=0}^{p-1}\binom{\frac{1}{2}-\nu}{m}\left(\frac{u}{2iz}\right)^m+ \tag{34}$$

$$+\frac{\left(\frac{1}{2}-\nu\right)_p}{(p-1)!}\left(\frac{u}{2iz}\right)^p\int_0^1(1-t)^{p-1}\left(1-\frac{ut}{2iz}\right)^{\nu-p-\frac{1}{2}}dt.$$

Let $p$ be large enough, namely

$$\nu-p-\frac{1}{2}<0,\qquad \nu+p-\frac{1}{2}>0.$$

Substituting (34) into the integral defining $H_\nu^{(1)}$ and applying the formula ([3], 1.1(6))

$$\Gamma(z)=\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{z-1}\,du,\qquad \operatorname{Re} z>0$$

the $m$-th mean term will be of the form

$$\sqrt{\frac{2}{\pi z}}\frac{e^{i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{\nu-\frac{1}{2}}\binom{\frac{1}{2}-\nu}{m}\left(\frac{u}{2iz}\right)^m du =$$

$$=\sqrt{\frac{2}{\pi z}}e^{i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}\frac{\binom{\frac{1}{2}-\nu}{m}}{\Gamma\left(\nu+\frac{1}{2}\right)}\frac{1}{(2iz)^m}\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{\nu-\frac{1}{2}+m}\,du =$$

$$=\sqrt{\frac{2}{\pi z}}e^{i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}\frac{\binom{\frac{1}{2}-\nu}{m}}{(2iz)^m}\frac{\Gamma\left(\nu+\frac{1}{2}+m\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} =$$

$$=\sqrt{\frac{2}{\pi z}}e^{i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}\frac{(-1)^m(\nu,m)}{(2iz)^m}.$$

We know that

$$\left|\arg\frac{iut}{2z}\right| = \left|\frac{\pi}{2}-\frac{\pi}{4}-\arg z\right| \leqq \frac{3\pi}{4}$$

hence $\left|1-\frac{ut}{2iz}\right| \geqq \frac{1}{\sqrt{2}}$ and then

$$\left|\int_0^1 (1-t)^{p-1}\left(1-\frac{ut}{2iz}\right)^{\nu-p-\frac{1}{2}} dt\right| \leqq c.$$

Consider the remainder for

$$\frac{H_\nu^{(1)}(z)}{z^\nu}-\frac{H_\nu^{(1)}(x)}{x^\nu};$$

apart from a constant factor, it is

$$\frac{e^{iz}}{z^{\nu+\frac{1}{2}}}\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{\nu-\frac{1}{2}}\left(\frac{u}{z}\right)^p\int_0^1 (1-t)^{p-1}\left(1+\frac{iut}{2z}\right)^{\nu-p-\frac{1}{2}} dt\,du-$$

$$-\frac{e^{ix}}{x^{\nu+\frac{1}{2}}}\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{\nu-\frac{1}{2}}\left(\frac{u}{x}\right)^p\int_0^1 (1-t)^{p-1}\left(1+\frac{iut}{2x}\right)^{\nu-p-\frac{1}{2}} dt\,du =$$

$$=\left[\frac{e^{iz}}{z^{\nu+p+\frac{1}{2}}}-\frac{e^{ix}}{x^{\nu+p+\frac{1}{2}}}\right]\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{\nu+p-\frac{1}{2}}\,\cdot$$

$$\cdot\int_0^1 (1-t)^{p-1}\left(1+\frac{iut}{2z}\right)^{\nu-p-\frac{1}{2}} dt\,du+$$

$$+\frac{e^{ix}}{x^{\nu+p+\frac{1}{2}}}\int_0^{\infty\cdot e^{i\beta}} e^{-u}u^{\nu+p-\frac{1}{2}}\int_0^1 (1-t)^{p-1}\left[\left(1+\frac{iut}{2z}\right)^{\nu-p-\frac{1}{2}}-\right.$$

$$\left.-\left(1+\frac{iut}{2x}\right)^{\nu-p-\frac{1}{2}}\right] dt\,du =: I_1+I_2.$$

For a regular function $f$ we have obviously

$$\left|f(z_2)-f(z_1)\right| \leqq |z_2-z_1| \max_{[z_1,z_2]} |f'|.$$

Applying this for $f(z)=z^{\nu+p+\frac{1}{2}}$ we get

$$\left|\frac{e^{iz}}{z^{\nu+p+\frac{1}{2}}}-\frac{e^{ix}}{x^{\nu+p+\frac{1}{2}}}\right| = \left|\frac{x^{\nu+p+\frac{1}{2}}e^{-y}-z^{\nu+p+\frac{1}{2}}}{(xz)^{\nu+p+\frac{1}{2}}}\right| \leqq$$

$$\leqq \frac{|e^{-y}-1|}{|z|^{\nu+p+\frac{1}{2}}}+\left|\frac{x^{\nu+p+\frac{1}{2}}-z^{\nu+p+\frac{1}{2}}}{(xz)^{\nu+p+\frac{1}{2}}}\right| \leqq$$

$$\leqq \frac{|e^{-y}-1|}{|z|^{\nu+p+\frac{1}{2}}}+c\left(\frac{|y|}{x^{\nu+p+\frac{3}{2}}}+\frac{|y|}{|z|^{\nu+p+\frac{3}{2}}}\right)$$

hence

$$|I_1| \leqq c\left(\frac{|e^{-y}-1|}{|z|^{\nu+p+\frac{1}{2}}}+\frac{|y|}{x^{\nu+p+\frac{3}{2}}}+\frac{|y|}{|z|^{\nu+p+\frac{3}{2}}}\right).$$

Analogously,

$$\left|\left(1+\frac{iut}{2z}\right)^{\nu-p-\frac{1}{2}}-\left(1+\frac{iut}{2x}\right)^{\nu-p-\frac{1}{2}}\right| \leqq$$

$$\leqq c|u|\left|\frac{1}{z}-\frac{1}{x}\right| = c\left|\frac{uy}{xz}\right|$$

and so

$$|I_2| \leqq c\frac{|y|}{x^{\nu+p+\frac{5}{2}}}.$$

We have proved that for large $p$, hence for all integers $p \geqq 0$,

$$\text{(35)}\qquad \frac{H_\nu^{(1)}(z)}{z^\nu}-\frac{H_\nu^{(1)}(x)}{x^\nu} = \sqrt{\frac{2}{\pi}}\frac{e^{i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}}{z^{\nu+\frac{1}{2}}}\sum_{m=0}^{p-1}\frac{(-1)^m(\nu,m)}{(2iz)^m}-$$

$$-\sqrt{\frac{2}{\pi}}\frac{e^{i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}}{x^{\nu+\frac{1}{2}}}\sum_{m=0}^{p-1}\frac{(-1)^m(\nu,m)}{(2ix)^m}+$$

$$+O\left(\frac{|e^{-y}-1|}{|z|^{\nu+p+\frac{1}{2}}}+\frac{|y|}{x^{\nu+p+\frac{3}{2}}}+\frac{|y|}{|z|^{\nu+p+\frac{3}{2}}}\right),\qquad x\geqq 1.$$

We know that

$$H_{-\nu}^{(1)}(z)=e^{\nu\pi i}H_\nu^{(1)}(z)$$

hence we get an integral representation also for $H_\nu^{(1)}$, $\nu\leqq -\frac{1}{2}$ and repeating the above proof we get that (35) holds for all real values $\nu$. The asymptotic expansion of $\frac{H_\nu^{(2)}(z)}{z^\nu}-\frac{H_\nu^{(2)}(x)}{x^\nu}$ can be dealt with similarly; using the representation

$$H_\nu^{(2)}(z)=\sqrt{\frac{2}{\pi z}}\frac{e^{-i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\infty\cdot e^{i\beta}}e^{-u}u^{\nu-\frac{1}{2}}\left(1-\frac{iu}{2z}\right)^{\nu-\frac{1}{2}}du,$$

$$|\beta|<\frac{\pi}{2},\qquad \nu>-\frac{1}{2},\qquad \arg z\neq\beta+\frac{\pi}{2}$$

with $\beta:=\frac{\pi}{4}$ we get for $\nu>-\frac{1}{2}$, and by the identity

$$H_{-\nu}^{(2)}(z)=e^{-\nu\pi i}H_\nu^{(2)}(z)$$

for all real $\nu$ the asymptotic expansion

$$\text{(36)}\qquad \frac{H_\nu^{(2)}(z)}{z^\nu}-\frac{H_\nu^{(2)}(x)}{x^\nu}=\sqrt{\frac{2}{\pi}}\frac{e^{-i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}}{z^{\nu+\frac{1}{2}}}\sum_{m=0}^{p-1}\frac{(\nu,m)}{(2iz)^m}-$$

$$-\sqrt{\frac{2}{\pi}}\frac{e^{-i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}}{x^{\nu+\frac{1}{2}}}\sum_{m=0}^{p-1}\frac{(\nu,m)}{(2ix)^m}+$$

$$+O\left(\frac{|e^{y}-1|}{|z|^{\nu+p+\frac{1}{2}}}+\frac{|y|}{x^{\nu+p+\frac{3}{2}}}+\frac{|y|}{|z|^{\nu+p+\frac{3}{2}}}\right),\qquad x\geqq 1.$$

Since $J_\nu=\frac{1}{2}\left(H_\nu^{(1)}+H_\nu^{(2)}\right)$, Lemma 3 is proved.

COROLLARY.

$$\left|\frac{J_\nu(\mu_i r)}{\mu_i^\nu}-\frac{J_\nu(\rho_i r)}{\rho_i^\nu}\right| \leqq \tag{37}$$

$$\leqq c\left(\frac{e^{|\nu_i|r}-1}{r^{\frac{1}{2}}|\mu_i|^{\nu+\frac{1}{2}}}+\frac{|\nu_i|}{r^{\frac{1}{2}}\rho_i^{\nu+\frac{3}{2}}}+\frac{|\nu_i|}{r^{\frac{1}{2}}|\mu_i|^{\nu+\frac{3}{2}}}\right), \qquad \rho_i r \geqq R,$$

$$\frac{J_\nu(\mu_i r)}{\mu_i^\nu}-\frac{J_\nu(\rho_i r)}{\rho_i^\nu}= \tag{38}$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{\cos\left(\mu_i r-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}{\mu_i^{\nu+\frac{1}{2}}r^{\frac{1}{2}}}-\frac{\cos\left(\rho_i r-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}{\rho_i^{\nu+\frac{1}{2}}r^{\frac{1}{2}}}\right]+$$

$$+\sqrt{\frac{2}{\pi}}\frac{4\nu^2-1}{8}\left[\frac{\cos\left(\mu_i r-\nu\frac{\pi}{2}+\frac{\pi}{4}\right)}{\mu_i^{\nu+\frac{3}{2}}r^{\frac{3}{2}}}-\frac{\cos\left(\rho_i r-\nu\frac{\pi}{2}+\frac{\pi}{4}\right)}{\rho_i^{\nu+\frac{3}{2}}r^{\frac{3}{2}}}\right]+$$

$$+O\left(\frac{e^{|\nu_i|r}-1}{|\mu_i|^{\nu+\frac{5}{2}}r^{\frac{5}{2}}}+\frac{|\nu_i|}{\rho_i^{\nu+\frac{7}{2}}r^{\frac{5}{2}}}+\frac{|\nu_i|}{|\mu_i|^{\nu+\frac{7}{2}}r^{\frac{5}{2}}}\right), \qquad \rho_i r \geqq R,$$

$$\left|\frac{J_\nu(\mu_i r)}{\mu_i^\nu}-\frac{J_\nu(\rho_i r)}{\rho_i^\nu}\right| \leqq \tag{39}$$

$$\leqq \begin{cases} c r^{\nu+1} e^{|\nu_i|R}, & \rho_i r \leqq R,\ \nu \geqq 0 \\[2ex] \dfrac{c}{\left(1+|\nu_i|\right)^{\frac{1}{2}}}\dfrac{e^{|\nu_i|R}}{\rho_i|\mu_i|^\nu}, & \rho_i r \leqq R,\ \nu \geqq 0,\ \rho_i \geqq 1, \end{cases}$$

$$\left|\mu_i J_1(\mu_i r)-\rho_i J_1(\rho_i r)\right| \leqq c e^{|\nu_i|R}, \qquad \rho_i r \leqq \frac{R}{2}. \tag{40}$$

PROOF. The estimates (37) and (38) are immediate consequences of Lemma 3. To show (39) and (40) we remark first that for $k=1,2,\dots$

$$|\mu_i^{2k}-\rho_i^{2k}| \leqq \sum_{l=1}^{2k}\binom{2k}{l}|\nu_i|^l\rho_i^{2k-l} \leqq$$

$$\leqq \rho_i^{2k-1} \sum_{l=0}^{2k} \binom{2k}{l} |\nu_i|^l = \rho_i^{2k-1} \big(1 + |\nu_i|\big)^{2k}$$

(here we use 1 instead of $\rho_i$ in case $\rho_i \leqq 1$). Consequently, using the power series of $J_\nu$ we get

$$\left| \frac{J_\nu(\mu_i r)}{\mu_i^\nu} - \frac{J_\nu(\rho_i r)}{\rho_i^\nu} \right| = \left| \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{r}{2}\right)^{\nu+2k} (\mu_i^{2k} - \rho_i^{2k}) \right| \leqq$$

$$\leqq c \sum_{k=1}^{\infty} \frac{r^{\nu+1}\big(1+|\nu_i|\big)}{k!\,\Gamma(k+\nu+1)} \left[\frac{r}{2}\rho_i\big(1+|\nu_i|\big)\right]^{2k-1} \leqq$$

$$\leqq c r^{\nu+1} \sum_{k=1}^{\infty} \frac{\big(1+|\nu_i|\big)}{k!\,\Gamma(k+\nu+1)} \left[\frac{R}{2}\big(1+|\nu_i|\big)\right]^{2k-1} \leqq$$

$$\leqq c r^{\nu+1} \sum_{k=1}^{\infty} \frac{\left[\frac{R}{2}\big(1+|\nu_i|\big)\right]^{2k}}{(k!)^2} \leqq c r^{\nu+1} \left( \sum_{k=1}^{\infty} \frac{\left[\frac{R}{2}\big(1+|\nu_i|\big)\right]^{k}}{k!} \right)^2 \leqq$$

$$\leqq c r^{\nu+1} e^{|\nu_i| R};$$

on the other hand

$$\left| \frac{J_\nu(\mu_i r)}{\mu_i^\nu} - \frac{J_\nu(\rho_i r)}{\rho_i^\nu} \right| \leqq \frac{c}{\rho_i} \frac{1}{\left[\rho_i\big(1+|\nu_i|\big)\right]^\nu} \sum_{k=1}^{\infty} \frac{\left[\frac{r}{2}\rho_i\big(1+|\nu_i|\big)\right]^{2k+\nu}}{k!\,\Gamma(k+\nu+1)} \leqq$$

$$\leqq \frac{c}{\rho_i |\mu_i|^\nu} \left( \sum_{k=1}^{\infty} \frac{\left[\frac{R}{2}\big(1+|\nu_i|\big)\right]^{k}}{k!} \right) \max_k \frac{\left[\frac{R}{2}\big(1+|\nu_i|\big)\right]^{k+\nu}}{\Gamma(k+\nu+1)} \leqq$$

$$\leqq \frac{c}{\rho_i |\mu_i|^\nu} e^{|\nu_i| \frac{R}{2}} \frac{e^{|\nu_i| \frac{R}{2}}}{\big(1+|\nu_i|\big)^{\frac{1}{2}}} = \frac{c}{\rho_i |\mu_i|^\nu} \frac{e^{|\nu_i| R}}{\big(1+|\nu_i|\big)^{\frac{1}{2}}}$$

so (39) is proved. The proof of (40) is similar:

$$\big|\mu_i J_1(\mu_i r) - \rho_i J_1(\rho_i r)\big| = \left| \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} \left(\frac{r}{2}\right)^{2k+1} (\mu_i^{2k+2} - \rho_i^{2k+2}) \right| \leqq$$

$$\leqq \sum_{k=0}^{\infty} \frac{1}{k!\,(k+1)!} \left[\frac{r}{2}\rho_i\big(1+|\nu_i|\big)\right]^{2k+1} \big(1+|\nu_i|\big) \leqq$$

$$\leqq c\big(1+|\nu_i|\big)\left(\sum_{k=0}^{\infty}\frac{\big[\frac{R}{4}\big(1+|\nu_i|\big)\big]^k}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{\big[\frac{R}{4}\big(1+|\nu_i|\big)\big]^{k+1}}{(k+1)!}\right)\leqq$$

$$\leqq c\big(1+|\nu_i|\big)\,e^{|\nu_i|\frac{R}{2}}\leqq ce^{|\nu_i|R}.$$

LEMMA 4. *Let $R>0$ and $\mu\geqq 1$, then*

$$\frac{e^{|\nu_i|r}-1}{r},\quad \frac{\operatorname{ch}\nu_i r-1}{r},\quad \frac{\operatorname{sh}|\nu_i|r}{r}\leqq ce^{|\nu_i|R},\qquad r\leqq R, \tag{41}$$

$$\int_{\frac{R}{\mu}}^{R}\frac{e^{|\nu_i|r}-1}{r^3}\,dr\leqq ce^{|\nu_i|R}\frac{\mu}{1+|\nu_i|}, \tag{42}$$

$$\left|\int_{\frac{R}{\mu}}^{t}\frac{\cos(\mu r+\alpha)\cos(\rho_i r+\beta)}{r}\,dr\right|\leqq c\frac{\mu}{1+|\mu-\rho_i|},\qquad \frac{R}{\mu}\leqq t\leqq R, \tag{43}$$

$$\begin{cases}\left|\displaystyle\int_{\frac{R}{\mu}}^{R}\frac{\cos(\mu r+\alpha)\cos(\rho_i r+\beta)}{r}\,\frac{\operatorname{ch}\nu_i r-1}{r}\,dr\right|\leqq c\frac{\mu e^{|\nu_i|R}}{1+|\mu-\rho_i|},\\[2ex] \left|\displaystyle\int_{\frac{R}{\mu}}^{R}\frac{\cos(\mu r+\alpha)\cos(\rho_i r+\beta)}{r}\,\frac{\operatorname{sh}\nu_i r}{r}\,dr\right|\leqq c\frac{\mu e^{|\nu_i|R}}{1+|\mu-\rho_i|}\end{cases} \tag{44}$$

*where the constants $c=c(R)$ are independent of the other variables.*

PROOF. (41) follows from the Taylor series of the corresponding functions. To show (42) we remark that

$$\int_{\frac{R}{2}}^{R}\frac{e^{|\nu_i|r}-1}{r^3}\,dr\leqq c\int_{\frac{R}{2}}^{R}e^{|\nu_i|r}\,dr\leqq c\frac{e^{|\nu_i|R}}{1+|\nu_i|},$$

$$\int_{\frac{R}{\mu}}^{\frac{R}{2}}\frac{e^{|\nu_i|r}-1}{r^3}\,dr\leqq ce^{|\nu_i|\frac{R}{2}}\int_{\frac{R}{\mu}}^{\frac{R}{2}}\frac{1}{r^2}\,dr\leqq c\frac{\mu e^{|\nu_i|R}}{1+|\nu_i|}.$$

In (43) we apply the addition formulas in case $|\mu-\rho_i|\geqq 1$:

$$\left|\int_{\frac{R}{\mu}}^{t}\frac{\cos\big((\mu+\rho_i)r+\gamma\big)+\cos\big((\mu-\rho_i)r+\delta\big)}{r}\,dr\right|\leqq$$

$$\leqq \left| \left[ \left( \frac{\sin\big((\mu+\rho_i)r+\gamma\big)}{\mu+\rho_i} + \frac{\sin\big((\mu-\rho_i)r+\delta\big)}{\mu-\rho_i} \right) \frac{1}{r} \right]_{r=\frac{R}{\mu}}^{t} \right| +$$

$$+ \left| \int_{\frac{R}{\mu}}^{t} \frac{1}{r^2} \left( \frac{\sin\big((\mu+\rho_i)r+\gamma\big)}{\mu+\rho_i} + \frac{\sin\big((\mu-\rho_i)r+\delta\big)}{\mu-\rho_i} \right) dr \right| \leqq$$

$$\leqq c \frac{\mu}{|\mu-\rho_i|} + \frac{c}{|\mu-\rho_i|} \int_{\frac{R}{\mu}}^{t} \frac{1}{r^2}\, dr \leqq c \frac{\mu}{1+|\mu-\rho_i|}$$

and in case $|\mu-\rho_i| \leqq 1$

$$\left| \int_{\frac{R}{\mu}}^{t} \frac{\cos\big((\mu+\rho_i)r+\gamma\big)+\cos\big((\mu-\rho_i)r+\delta\big)}{r}\, dr \right| \leqq$$

$$\leqq 2 \int_{\frac{R}{\mu}}^{t} \frac{1}{r}\, dr \leqq c \ln \mu \leqq c \frac{\mu}{1+|\mu-\rho_i|}$$

which proves (43). Finally (44) follows from (41), (43) if we use, as in [7], the inequalities

$$\left( \frac{\operatorname{ch} \nu_i r - 1}{r} \right)' > 0, \qquad \left( \frac{\operatorname{sh} |\nu_i| r}{r} \right)' > 0, \qquad r > 0$$

and integrate by parts in (44). Lemma 4 is proved.

We continue the proof in the next issue of this journal.

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# ARITHMETICS OF AGING DISTRIBUTIONS: MAXIMUM

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## 1. Introduction

**1.1. Objective and preliminaries.** The present paper is the counterpart of [6], where the convolution structure of the reliability semigroups **IFR**, **IFRA**, **NBU**, **NBUE**, **HNBUE** and **L** are discussed. It is proved there that in each semigroup every distribution can be decomposed into the convolution product of at most countably many irreducible distributions and a degenerate one; degenerate distributions are the only anti-irreducible or infinitely divisible elements; and the set of irreducible distributions is dense. Uniqueness of the decomposition cannot be expected, since at least in the four upper classes no distribution is prime. This time our aim is to investigate these classes when the semigroup operation is the pointwise multiplication of distribution functions (which corresponds to taking the maximum of independent random variables) instead of convolution. In order to make the present work easily readable without having read its counterpart [6] we repeat here all the necessary definitions.

Arithmetical properties of probability distributions were first studied by Khinchin and Lévy, as early as in the thirties. Since then lots of relevant papers have been published on the topic and interesting general theories have been developed. Most investigations have dealt with the convolution structure of probability distributions defined on more and more general structures, but there exist results concerning semigroups of distributions with other operations such as the multiplication of distribution function (maximum of random variables). A recent monograph of Ruzsa and Székely [8] provides an excellent synthesis of latest researches in the so called algebraic probability theory, which aims at proving arithmetical type results for a wide class of commutative semigroups. Their general theorems will be cited in this paper at every moment.

The arithmetic structure of the multiplicative semigroup of all real or nonnegative probability distributions is not so interesting, since every dis-

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[1] This paper was written while the author was visiting the Mathematical Institute of the Hungarian Academy of Sciences.

tribution is infinitely divisible, therefore most arithmetic problems become meaningless. The situation is quite different in higher dimensions, where the pointwise multiplication of distribution functions corresponds to the coordinatewise maximum of independent random vectors. Infinitely divisible elements in the multiplicative semigroup of multivariate distribution functions were characterized by Balkema and Resnick [1]; Zempléni found conditions for irreducibility and anti-irreducibility [9]–[11]. Another possibility to avoid triviality observed in one dimension is confining ourselves to certain subsemigroups, such as the aging classes of distributions. This is just what we wish to do.

Results to be communicated below were discovered in 1987. Although they were already reported in [5] and [8, Remark 6.3.9], no proofs have appeared so far. This may justify publishing the present paper.

**1.2. Aging distributions.** In order to introduce certain classes of aging distributions with finite expectation let us start with some definitions and notations.

As it is usual in reliability theory, we shall only deal with the set $D^+$ of nonnegative probability distributions. Distributions will be identified with their cumulative distribution functions defined right continuous. For an arbitrary distribution $F \in D^+$ let us introduce the corresponding *survival function* $\overline{F} = 1 - F$, *expectation* $\mathbf{E}(F)$, *variance* $\mathbf{Var}(F)$, *Laplace transform* $\varphi_F(t) = \int_0^\infty e^{-tx}\,dF(x)$, $t \geqq 0$, *starting point* $\alpha_F$ and *endpoint* $\omega_F$ defined as $\alpha_F = \inf\{t \in \mathbf{R}: F(t) > 0\}$ and $\omega_F = \sup\{t \in \mathbf{R}: F(t) < 1\}$, resp.

The exponential distribution with expectation $\mu > 0$ will be denoted by $\varepsilon_\mu$, that is, $\varepsilon_\mu(t) = 1 - \exp(-t/\mu)$, $t \geqq 0$. The corresponding Laplace transform is $\frac{1}{1+\mu t}$, $t \geqq 0$. In addition, we shall write $\delta_\mu$ for the degenerate distribution concentrated onto $\mu \geqq 0$. Clearly, $\delta_\mu(t) = 0$ or $1$ according as $t < \mu$ or $t \geqq \mu$, resp. The Laplace transform of $\delta_\mu$ is $\exp(-\mu t)$, $t \geqq 0$.

The most frequently used classes of aging distributions are as follows.

$F \in \mathbf{IFR}$ iff for every $s > 0$ the function $t \mapsto \overline{F}(t+s)/\overline{F}(t)$, $t \geqq 0$ is decreasing.

$F \in \mathbf{IFRA}$ iff the function $t \mapsto \overline{F}(t)^{1/t}$, $t > 0$ is decreasing, or, equivalently, $w(F,t) =: -\frac{1}{t}\log \overline{F}(t)$ is increasing in $t$, $0 < t < \omega_F$.

$F \in \mathbf{NBU}$ iff $\overline{F}(t+s) \leqq \overline{F}(t)\overline{F}(s)$ for every nonnegative $t$ and $s$.

$F \in \mathbf{NBUE}$ iff $\mathbf{E}(F) = \mu$ is finite and $\int_0^\infty \overline{F}(u)\,du \leqq \mu\overline{F}(t)$ for $t \geqq 0$.

$F \in \mathbf{HNBUE}$ iff $\mathbf{E}(F) = \mu$ is finite and $\int_0^\infty \overline{F}(u)\,du \leqq \mu\exp(-t/\mu)$ for $t \geqq 0$.

$F \in \mathbf{L}$ iff $\mathbf{E}(F) = \mu$ is finite and $\varphi_F(t) \leqq \frac{1}{1+\mu t}$, $t \geqq 0$.

These classes form an increasing sequence in the order of definition: **IFR** $\subset$ **IFRA** $\subset$ **NBU** $\subset$ **NBUE** $\subset$ **HNBUE** $\subset$ **L**. Properties of the first four classes are found in [2]. Classes **HNBUE** and **L** were first introduced and studied by Rolski [7] and Klefsjö [4], resp. Both exponential and degenerate distributions belong to the smallest class **IFR**. In addition, exponential distributions lie on the common boundary of the above aging classes, since they satisfy each inequality-type definition with equality and show constant rate where monotonicity is required.

All these classes are closed subsets of $D^+$ with respect to the usual topology of convergence in distribution (i.e., pointwise convergence at the continuity points of the limit distribution function).

**1.3. Arithmetical definitions.** For any pair of distributions $F, G \in \in D^+$ let $F \vee G$ denote their pointwise product. This corresponds to the maximum of independent random variables with distributions $F$ and $G$, resp. Operation $\vee$ is clearly commutative, associative and continuous with respect to the weak topology. This makes it possible to extend the operation to an infinite sequence of distributions as the weak limit of the finite sections. The main difference with respect to the case of convolution discussed in [6] is that the semigroup $(D^+, \vee)$ is not cancellative, which makes the basic arithmetical notions a little more complicated. Let us examine the above aging properties whether they are preserved under this operation.

As it is well-known, **IFR** is not closed in this sense; for example, $\varepsilon_\mu \vee \vee \varepsilon_\nu \notin$ **IFR** if $\mu \neq \nu$. Classes **IFRA** and **NBU** are closed under a more general (multivariate) operation: the life distribution of a coherent system with independent components all belonging to **IFRA** (**NBU**) is **IFRA** (**NBU**) again [2, Theorems 4.2.6 and 6.5.1]. Particularly, the case of parallel systems shows that both classes are subsemigroups of $D^+$, thus they can be subjects of our further investigations. Though **NBUE** is not preserved under the formation of coherent systems ([2] contains a counterexample consisting in a series system of two independent components), it can be shown that $\vee$ does not lead out from **NBUE**. Maybe this is well-known; for the sake of completeness we nevertheless give a short proof below.

LEMMA 1. *Let $F$ and $G$ belong to* **NBUE**, *then so does* $F \vee G$.

PROOF. Suppose $\mathbf{E}(F) \geqq \mathbf{E}(G)$. Since $\overline{F \vee G}(t) = \overline{F}(t) + F(t)\overline{G}(t)$, we have

$$\int_x^\infty \overline{F \vee G}(t)\,dt = \int_x^\infty \left(\overline{F}(t) + F(t)\overline{G}(t)\right)\,dt =$$

$$= \int_x^\infty \overline{F}(t)\,dt + \overline{F \vee G}(x) \int_x^\infty F(t)\overline{G}(t)\,dt + F(x)G(x) \int_x^\infty F(t)\overline{G}(t)\,dt.$$

Here the first two terms do not exceed $\mathbf{E}(F)\overline{F}(x)$ and $\big(\mathbf{E}(F \vee G) - - \mathbf{E}(F)\big)\,\overline{F \vee G}(x)$, resp. The third term can be estimated in the following way:

$$F(x)G(x)\int_x^\infty F(t)\overline{G}(t)\,dt \leqq F(x)\int_x^\infty \overline{G}(t)\,dt \leqq$$

$$\leqq F(x)\mathbf{E}(G)\overline{G}(x) \leqq \mathbf{E}(F)F(x)\overline{G}(x).$$

Hence

$$\int_0^\infty \overline{F\vee G}(t)\,dt \leqq$$

$$\leqq \mathbf{E}(F)\big(\overline{F}(x)+F(x)\overline{G}(x)\big)+\big(\mathbf{E}(F\vee G)-\mathbf{E}(F)\big)\,\overline{F\vee G}(x)=$$

$$=\mathbf{E}(F\vee G)\overline{F\vee G}(x).$$

REMARK 1. If $\alpha_F < x < \omega_F$ and $\alpha_G < x < \omega_G$, in the above line strict inequality holds.

**HNBUE** is not closed with respect to $\vee$, as it can be seen by considering the mixture $F = 0.98\,\delta_1 + 0.02\,\delta_5 \in$ **HNBUE**, $F \vee F = 0.9604\,\delta_1 + 0.0396\,\delta_5 \notin$ **HNBUE**. So far I have been unable to decide if **L** is closed or to find any reference on the subject.

In the arithmetic of $(D^+,\vee)$ the unity is $\delta_0$, the point mass at 0. Now, let $S$ be an arbitrary subsemigroup of $D^+$ containing $\delta_0$. For $F$ and $G$ belonging to $S$ let us introduce the following arithmetical notions.

$G$ is a *divisor* (or a *factor*) of $F$ if there exists an $H$ in $S$ such that $F = G \vee H$. We use the notation $G|F$. An equality of the form $F = G \vee H$ is called a *decomposition* of $F$. A pair of elements mutually dividing each other are called *associates*. Since $G|F$ implies $F \leqq G$ pointwise, it follows that $D^+$ is associate-free.

$G$ is an *effective divisor* of $F$ if $F = G \vee H$ with $H \in S$, $H \neq F$.

A *decomposition* $F = G \vee H$ is *effective*, if neither $G$, nor $H$ is equal to $F$.

$F$ is *irreducible*, if $F \neq \delta_0$ and it has no divisor but the unity and itself.

$F$ is *effectively irreducible*, if $F \neq \delta_0$ and it has no effective decomposition.

$F$ is *anti-irreducible*, if it has no irreducible effective divisor.

$F$ is *effectively anti-irreducible*, if it is not effectively divisible by any effectively irreducible element.

$F$ is *idempotent*, if $F^2 = F$. $F$ is *bald*, if it has no idempotent divisor but $\delta_0$.

$F$ is *infinitesimally divisible*, if it can be decomposed into a product of distributions all coming from an arbitrarily preassigned neighbourhood of $\delta_0$. $F$ is *infinitely divisible*, if for every positive integer $n$ there exists an $F_n \in S$ such that $F = F_n^n$ ($F_n$ is called the $n$th *root* of $F$).

$F$ is *prime*, if it is different from $\delta_0$ and $F|G \vee H$ implies $F|G$ or $F|H$.

## 2. Main results

**2.1. Decompositions.** In this section a Khinchin-type decomposition theorem will be proved in our three aging classes. This can be done by applying the general theory of Hun semigroups developed in [8, Chapter 2]. We start with some simple lemmas.

LEMMA 2. *$F \in D^+$ is idempotent iff $F = \delta_c$, $c > 0$, $F$ is bald iff $\alpha_F = 0$.*

PROOF. Obvious.

LEMMA 3. **IFRA**, **NBU** *and* **NBUE** *are stable normable Hun semigroups* (see Definitions 2.2.2, 2.10.6 and 2.15.2 in [8]).

PROOF. It is sufficient to show that $D^+$ itself possesses all these properties, since they are inheritable to closed subsemigroups. Since a distribution is always stochastically larger than its divisors, the set of divisors of a compact set is tight, hence relatively compact.

Thus $D^+$ is stable Hun. $D^+$ is normable, since for any non-degenerate $F$ and $\alpha_F < x < \omega_F$, $\Delta_F(G) = -\log G(x)$ defines an $F$-norm on the set of divisors of $F$ such that $\Delta_F(F) > 0$.

THEOREM 1. *In* **IFRA**, **NBU** *and* **NBUE** *every distribution can be decomposed*

(a) *into the product of at most countably many irreducible distributions and an anti-irreducible one,*

(b) *into the product of at most countably many effectively irreducible distributions and an effectively anti-irreducible one.*

PROOF. This is a simple corollary of our Lemma 3 and Theorem 2.23.3 of [8].

REMARK 2. The above decomposition can sometimes be simplified due to the results of the next section. Firstly, by Theorem 2.24.16 of [8], every effectively anti-irreducible element is infinitely divisible, hence we obtain that in **IFRA** every distribution can be decomposed into a (finite or countably infinite) product of effectively irreducible elements and a degenerate distribution. Secondly, the (effectively) anti-irreducible factor can be omitted for bald distributions in each of our semigroups, since by Theorem 2.8.9 of [8] every bald anti-irreducible element is infinitesimally divisible.

**2.2. Infinitely divisible distributions.** In this section we first deal with infinitesimally divisible elements.

LEMMA 4 [8, Remark 5.9.8]. *Expectation is a continuous operator on* **L** *(hence particularly on* **NBUE***).*

THEOREM 2. *There is no infinitesimally divisible element in* **IFRA**, **NBU** *and* **NBUE** *except the unity $\delta_0$.*

PROOF. Suppose $F$ is infinitesimally divisible in **NBUE** (it is clearly sufficient to deal with the largest class). Let us fix $x > 0$ and $\varepsilon > 0$ arbitrarily. Then by the previous lemma there exists a decomposition of the form $F = F_1 \vee F_2 \vee \ldots \vee F_n$, where $F_i(x) \geqq 1 - \varepsilon$ and $\mathbf{E}(F_i) \leqq \varepsilon$, $i = 1, \ldots, n$. Now let $y \geqq x$, then $F(y) > 0$ and

$$-\log F(y) = \sum_{i=1}^{n} \left(-\log F_i(y)\right) \leqq \sum_{i=1}^{n} \frac{\overline{F}_i(y)}{F_i(y)} \leqq \frac{1}{1-\varepsilon} \sum_{i=1}^{n} \overline{F}_i(y).$$

Integrating this and using the **NBUE** property we obtain that

$$\int_x^\infty \left(-\log F(y)\right) dy \leqq \frac{1}{1-\varepsilon} \sum_{i=1}^{n} \mathbf{E}(F_i)\overline{F}_i(x) \leqq \frac{\varepsilon}{1-\varepsilon} \sum_{i=1}^{n} \overline{F}_i(x) \leqq$$

$$\leqq \frac{\varepsilon}{1-\varepsilon} \sum_{i=1}^{n} \left(-\log F_i(x)\right) = \frac{\varepsilon}{1-\varepsilon} \left(-\log F(x)\right).$$

Since $\varepsilon$ can be arbitrarily small, it follows that $F(x) = 1$ for every $x > 0$.

THEOREM 3 (Characterization of infinitely divisible distributions).
(a) $F \in$ **IFRA** *is infinitely divisible iff* $F = \delta_c$, $c \geqq 0$.
(b) $F \in$ **NBU** *is infinitely divisible iff* $\omega_F \leqq 2\alpha_F$.
(c) $F \in$ **NBUE** *is infinitely divisible iff*

$$\int_x^\infty \left(-\log F(y)\right) dy \leqq \alpha_F \left(-\log F(x)\right), \qquad \forall x > \alpha_F. \tag{1}$$

PROOF. (a) Degenerate distributions are obviously infinitely divisible. On the other hand, consider a non-degenerate $F \in$ **IFRA** that is infinitely divisible. Then there exist positive numbers $x$ and $y$, $0 < x < y$, such that $0 < F(x) \leqq F(y) < 1$. Infinite divisibility means that $F^{1/n} \in$ **IFRA** for every $n = 1, 2, \ldots$, thus $w(F^{1/n}, x) \leqq w(F^{1/n}, y)$, from which it follows that

$$\frac{1}{x} - \frac{\log\left(n(1 - F^{1/n}(x))\right)}{x \log n} \leqq \frac{1}{y} - \frac{\log\left(n(1 - F^{1/n}(y))\right)}{y \log n}.$$

Letting $n \to \infty$ we obtain $\frac{1}{x} \leqq \frac{1}{y}$, a contradiction.

(b) Suppose $F$ is infinitely divisible in **NBU**. By the **NBU** property of $F^{1/n}$, for arbitrary $x > \alpha_F$, we have

$$1 - F^{1/n}(2x) \leqq \left(1 - F^{1/n}(x)\right)^2.$$

Since $n(1-c^{1/n}) \to -\log c$, multiplying the above inequality by $n^2$ then tending with $n$ to infinity we arrive at a contradiction unless $F(2x)=1$. Hence $\omega_F \leqq 2\alpha_F$.

The opposite implication easily follows from the simple observation that $\omega_F \leqq 2\alpha_F$ is sufficient for any $F \in D^+$ to have **NBU** property (the inequality in the definition holds automatically if $t$ or $s$ is less than $\alpha_F$ or $t+s \geqq \omega_F$).

(c) Suppose $F$ is infinitely divisible in **NBUE**. Since $F^{1/n} \in$ **NBUE**, for $x > \alpha_F$ we have

$$\int_x^\infty n\big(1-F^{1/n}(t)\big)\,dt \leqq n\mathbf{E}(F^{1/n})\big(1-F^{1/n}(x)\big). \tag{2}$$

Here $n\big(1-F^{1/n}(t)\big) \leqq -\log F(t)$ uniformly in $n$, and $-\log F$ is integrable over $(x,+\infty)$, since $-\log F(t) \sim \overline{F}(t)$ as $t \to \infty$. Let $n$ tend to infinity in (2). Then applying the monotone convergence theorem and using that $\mathbf{E}(F^{1/n}) \to \alpha_F$ we arrive at (1).

For the opposite it suffices to show that inequality (1) implies the **NBUE** property, because (1) is inheritable from $F$ to $F^{1/n}$.

Let $0 \leqq x < \alpha_F$, then clearly

$$\int_x^\infty \overline{F}(t)\,dt \leqq \mathbf{E}(F) = \mathbf{E}(F)\overline{F}(x).$$

If $\alpha_F \leqq x$ and $F(x)\mathbf{E}(F) \leqq \alpha_F$, we have

$$\int_x^\infty \overline{F}(t)\,dt = \mathbf{E}(F) - \int_0^x \overline{F}(t)\,dt \leqq \mathbf{E}(F) - \alpha_F \leqq \mathbf{E}(F)\overline{F}(x).$$

Finally, if $F(x)\mathbf{E}(F) > \alpha_F$, then

$$\int_x^\infty \overline{F}(t)\,dt \leqq \int_x^\infty -\log F(t)\,dt \leqq \alpha_F\big(-\log F(x)\big) \leqq$$

$$\leqq \alpha_F \frac{1-F(x)}{F(x)} \leqq \mathbf{E}(F)\overline{F}(x),$$

thus $F \in$ **NBUE**.

**2.3. Irreducible distributions.** Similarly to the case of convolution structure, we cannot give the full characterization of irreducible distributions, but there exists a simple sufficient condition for a distribution to be irreducible even in the largest class **NBUE**, which is still sufficiently general to imply that irreducible distributions are dense in all classes under consideration. The next lemma corresponds to Lemma 4 in [6]; in fact, formally it is the same assertion but with a different proof.

LEMMA 5. *Let $F \in$ **NBUE**, $F \neq \delta_0$, and suppose that*

$$\limsup_{x \downarrow 0} x^{-2} F(x) = +\infty.$$

*Then $F$ is irreducible.*

PROOF. By the **NBUE** property,

$$F(x) \leqq 1 - \frac{1}{\mathbf{E}(F)} \int_x^\infty \overline{F}(t)\, dt = \frac{1}{\mathbf{E}(F)} \int_0^x \overline{F}(t)\, dt \leqq x/\mathbf{E}(F). \tag{3}$$

Hence the lemma.

REMARK 3. Lemma 2.11.8 of [8] concerns the existence of an upper bound for the number of terms in certain decompositions of a bald element in a Hun semigroup. Inequality (3) above provides the following estimation in **NBUE**. Let $F = F_1 \vee F_2 \vee \ldots \vee F_n$, $\alpha_F = 0$, then

$$n \leqq \liminf_{x \downarrow 0} \frac{\log F(x)}{\log x}.$$

THEOREM 4. *The set of irreducible elements is dense in* **IFRA**, **NBU** *and* **NBUE**.

PROOF. By Lemma 5 it suffices to show that every distribution in each class can be approached by distributions with the property $F(x) \sim cx$, $x \downarrow 0$. This is just what has been done in Theorem 3 of [6].

REMARK 4. Effective irreducibility is weaker than irreducibility, for example, every irreducible distribution is necessarily bald, while there are lots of non-bald effective irreducible elements, as we shall show it in the next section (they all are anti-irreducible). Evidently, the set of effective irreducible elements is also dense in our semigroups. The set of irreducible elements, as well as the larger set of effective irreducible elements, are $G_\delta$ in each class, by Theorem 2.19.2 of [8].

**2.4. Anti-irreducible distributions.** In this section we are going to show that, unlike the case of convolution structure, the set of anti-irreducible elements is rich enough to be dense in each semigroup. Two simple steps of the proof may be worth separating as lemmas, being of independent interest.

LEMMA 6. *Let $F, G \in$ **IFRA**. Suppose that $w(F \vee G, x)$ is a positive constant on a nonnegative interval $(a, b)$. Then $\min\{\omega_f, \omega_G\} \leqq a$.*

PROOF. Suppose on the contrary that $F(a) < 1$, $G(a) < 1$. Let us denote $w(F, a) = 1/f$, $w(G, a) = 1/g$, $w(F \vee G, a) = 1/h$. By the **IFRA** property $F(x) \geqq \varepsilon_f(x)$ and $G(x) \geqq \varepsilon_g(x)$ for $x > a$. Hence in the interval

$(a,b)$ we have $\varepsilon_h(x) = F \vee G(x) \geqq \varepsilon_f \vee \varepsilon_g(x)$, and for $x = a$ equality holds. Here $\varepsilon_f \vee \varepsilon_g \in$ **IFRA**, consequently $\varepsilon_f \vee \varepsilon_g(x) \geqq \varepsilon_h(x)$, $x > a$. Thus

$$(1 - \exp(-fx))\,(1 - \exp(-gx)) = 1 - \exp(-hx) \tag{4}$$

for every $x \in (a,b)$. Both sides of this equality are analytic functions of $x$, hence (4) must hold for every real $x$. But this is impossible, since for negative $x$ the sides of (4) differ in sign.

LEMMA 7. *Let $F \in$ **NBU**. Suppose that $F$ is constant and less than* 1 *on a nonnegative interval $(a,b)$. Then $\alpha_F \geqq b - a$.*

PROOF. Let $0 < x < b - a$, then by the **NBU** property $\overline{F}(a) = \overline{F}(a + x) \leqq \overline{F}(a)\overline{F}(x)$, hence $F(x) = 0$.

THEOREM 5. *The set of anti-irreducible elements is dense in* **IFRA**, **NBU** *and* **NBUE**.

PROOF. The method of proof is common for **IFRA** and **NBUE**: we show that every distribution can be approximated by non-bald effective irreducible ones to an arbitrary extent. Since the only effective divisor of an effectively irreducible element is itself, and a non-bald distribution can never be irreducible, non-bald effective irreducible elements are anti-irreducible at the same time. In addition, in the proof of Theorem 4 we have seen that every distribution in each semigroup can be approximated by bald ones with arbitrary accuracy. Therefore we can be confined to approximating bald distributions.

*Case* **IFRA**. Let us define the approximating distributions $F_n$, $n = 1,2,\dots$ by

$$w(F_n, x) = \begin{cases} 0, & \text{if} \quad nx \leqq 1, \\ w\left(F, \dfrac{1}{kn}\right), & \text{if} \quad \dfrac{k+1}{k} \leqq nx \leqq \dfrac{k}{k-1},\ k = 2,3,\dots, \\ w\left(F, x - \dfrac{1}{n}\right), & \text{if} \quad 2 \leqq nx. \end{cases}$$

This function is increasing in $x$, that is, $F_n \in$ **IFRA**. Clearly, $\alpha_{F_n} \geqq 1/n$. Consider an arbitrary **IFRA**-decomposition of the form $F_n = G \vee H$. In virtue of Lemma 6 we can suppose $\omega_G \leqq 1/n$, hence $H = F_n$, thus the decomposition is not effective. Finally, $F_n \xrightarrow{w} F$, since $w(F_n, x) \to w(F_n, x)$ at every $F$-continuity point $x > 0$.

*Case* **NBUE**. Let $R$ be a fixed distribution, $\varepsilon$ and $\delta$ positive parameters which can be thought of as small. For any distribution $F \in$ **NBUE** let us define

$$F_\delta(x) = \begin{cases} \min\{\, R(x), F(\varepsilon)\}, & \text{if } x < \delta, \\ F(x - \delta + \varepsilon), & \text{if } x \geqq \delta. \end{cases}$$

Then $\mu(\delta) = \mathbf{E}(F_\delta)$ is continuous in $\delta$, since

$$\mu(\delta) = \int_\varepsilon^\infty \overline{F}(t)\,dt + \int_0^\delta \max\{\,\overline{R}(t), \overline{F}(\varepsilon)\}\,dt.$$

In addition, if $F(\varepsilon) < 1$, we have

$$\mu\big(\varepsilon\overline{F}(\varepsilon)\big) \leqq \int_\varepsilon^\infty \overline{F}(t)\,dt + \varepsilon\overline{F}(\varepsilon) \leqq \mathbf{E}(F).$$

$$\mu\big(\varepsilon/\overline{F}(\varepsilon)\big) \geqq \int_\varepsilon^\infty \overline{F}(t)\,dt + \overline{F}(\varepsilon)\big(\varepsilon/\overline{F}(\varepsilon)\big) \geqq \mathbf{E}(F).$$

Hence $\mu(\delta) = \mathbf{E}(F)$ for some $\delta = \delta(\varepsilon)$, $\varepsilon\overline{F}(\varepsilon) \leqq \delta \leqq \varepsilon/\overline{F}(\varepsilon)$. Let us choose $\delta$ in this way. We first show that if both $F$ and $R$ belong to **NBUE** and $\mathbf{E}(F) \leqq \mathbf{E}(R)$, then $F_\delta \in$ **NBUE**. Indeed, for $x \leqq \delta$ we can write

$$\int_0^x \overline{F}_\delta(t)\,dt \geqq \int_0^x \overline{R}(t)\,dt \geqq \mathbf{E}(R)R(x) \geqq \mathbf{E}(F_\delta)F_\delta(x),$$

while for $x > \delta$

$$\int_0^\infty \overline{F}_\delta(t)\,dt = \int_{x-\delta+\varepsilon}^\infty \overline{F}(t)\,dt \leqq \mathbf{E}(F)\overline{F}(x - \delta + \varepsilon) = \mathbf{E}(F_\delta)\overline{F}_\delta(x).$$

It is time now to specify $R$. Let $0 < c < \varepsilon\overline{F}(\varepsilon)$, $c < \mathbf{E}(F)F(\varepsilon)$, and

$$R(x) = \begin{cases} 0, & \text{if } x < c, \\ 1 - \left(1 - \dfrac{c}{\mathbf{E}(F)}\right)\exp\left(-\dfrac{x-c}{\mathbf{E}(F)}\right), & \text{if } x \geqq c. \end{cases}$$

Then $\mathbf{E}(R) = \mathbf{E}(F)$, and $\alpha_{F_\delta} = \alpha_R = c$. We want to show that no **NBUE**-decomposition of the form $F_\delta = G \vee H$ can be effective. Let $c < x < \delta$, then

$$\int_x^\infty \overline{F}_\delta(t)\,dt = \mathbf{E}(F) - c - \int_c^x \left(1 - \frac{c}{\mathbf{E}(F)}\right)\exp\left(-\frac{t-c}{\mathbf{E}(F)}\right)dt =$$

$$= \big(\mathbf{E}(F) - c\big)\exp\left(-\frac{x-c}{\mathbf{E}(F)}\right) = \mathbf{E}(F_\delta)\overline{F}_\delta(x).$$

By Remark 1, either $\omega_G \leqq c$ or $\omega_H \leqq c$; we can suppose the former. Consequently, $H(x) = F_\delta(x)$ for $x \geqq c$. If $\alpha_H$ were less than $c$, $H$ could not satisfy the **NBUE** inequality for $x \in (c, \delta)$, since

$$\int_x^\infty \overline{H}(t)\,dt = \int_x^\infty \overline{F}_\delta(t)\,dt = \mathbf{E}(F_\delta)\overline{F}_\delta(x) > \mathbf{E}(H)\overline{H}(x).$$

Thus $H = F_\delta$.

Finally, let $\varepsilon$ tend to 0, then the corresponding $\delta$ also tends to 0 ($\varepsilon \sim \delta$), and the distribution $F_\delta$ constructed above converges to $F$ weakly.

*Case* **NBU**. Now let $F_n(x) = F\big([nx]/n\big)$, then $F_n \in$ **NBU**, since

$$\overline{F}_n(t)\overline{F}_n(s) = \overline{F}\big([nt]/n\big)\,\overline{F}\big([ns]/n\big) \geqq \overline{F}\big(([nt]+[ns])/n\big) \geqq$$

$$\geqq \overline{F}\big([nt+ns]/n\big) = \overline{F}_n(t+s).$$

Clearly, $F_n \xrightarrow{w} F$ as $n \to \infty$. This time we prove the anti-irreducibility of $F_n$ directly. Suppose $F_n = G \vee H$ with some $G, H \in$ **NBU**. Then, together with $F_n$, both $G$ and $F$ are constant and positive on the intervals $\left(\frac{k}{n}, \frac{k+1}{n}\right)$, $k \geqq 1$. If $\omega_G > 1/n = \alpha_{F_n}$, then $\alpha_G \geqq 1/n$ by Lemma 7. thus $G$ is not irreducible. On the other hand, if $\omega_G \leqq 1/n$, then $H = F_n$, thus $G$ is not an effective divisor of $F_n$. Hence $F_n$ is anti-irreducible.

REMARK 5. In the above proof none of the constructed anti-irreducible distributions is bald. This is not a matter of chance: there are no bald anti-irreducible elements at all (see Remark 2). One can naturally ask if the (smaller) set of effectively anti-irreducible distributions is still dense. The answer is negative, not only in our semigroups, but also in every stable normable Hun semigroup, in which not every element is infinitely divisible. This follows from Theorem 2.24.16 and Statement 2.19.1 of [8]. The former claims that effective anti-irreducible elements are infinitely divisible; the latter, that the set of infinitely divisible elements is closed, thus it could be dense but in the trivial case when it exhausts the whole semigroup. Another question is whether effectively anti-irreducible distributions are dense among the infinitely divisible ones (this is obvious in **IFRA** but not decided in **NBU** and **NBUE**).

**2.5. Prime distributions.** There is no general result known about the existence or non-existence of primes in semigroups, should they be as special as stable, normable, metrizable Hun. For this reason it is always interesting to answer this question in specific semigroups. Ruzsa and Székely proved the non-existence of primes in the convolution semigroup of probability distributions on the Borel field of a locally compact Hausdorff group (to be more precise, they found a finite number of exceptions, all on small cyclic groups, see [8, Section 4.7]). Similar results are known for some other semigroups, such as the multiplicative structure of multivariate distribution functions (due to Zempléni, see [8, Corollary 6.3.7]), or the convolution structure of reliability classes **NBU**, **NBUE**, **HNBUE** and **L** [6].

Surprisingly, in out semigroups there are primes, however trivial.

THEOREM 6. *Degenerate distributions are the only primes in* **IFRA**, **NBU** *and* **NBUE**.

PROOF. The prime property of degenerate distributions is obvious, since $\delta_c | F$ iff $c \leqq \alpha_F$, and $\alpha_{G \vee H} = \max\{\alpha_G, \alpha_H\}$. In the opposite direction, the prime property of a given non-degenerate $F$ can be disproved, for example, by considering an appropriate decomposition of $F \vee \delta_c$ with $c < \omega_F$, in which none of the factors is divisible by $F$.

*Case* **IFRA**. Let $F$ be an arbitrary non-degenerate **IFRA** distribution. We distinguish two cases according as $\lim_{x \downarrow 0} w(F,x) = \lim_{x \downarrow 0} f(x)/x$ is positive or not.

Firstly, when this limit is equal to 0, let $G$ be defined as

$$w(G,x) = \begin{cases} w(F,c), & x < c, \\ w(F,x), & \text{if } x \geqq c, \end{cases}$$

where $\alpha_F < c < \omega_F$. Then $G \in \mathbf{IFRA}$, $G \neq F$, $F \vee \delta_c = G \vee \delta_c$, hence $F$ divides $G \vee \delta_c$, but does not divide either $G$ or $\delta_c$ (for $G$ is irreducible by Lemma 5 and $F(c) < 1$).

Secondly, when this limit is positive, then $\lim_{x \downarrow 0} w(F^2, x) = 0$; thus we can repeat the above argumentation with $F^2$ in place of $F$.

*Case* **NBU**. If $F \in \mathbf{NBU}$ is a non-degenerate infinitely divisible distribution, it cannot be prime, since $F$ divides $F = \sqrt{F} \vee \sqrt{F}$, but does not divide $\sqrt{F}$. By Theorem 3 we can suppose that $2\alpha_F < \omega_F$. Let $c$ be an $F$-continuity point, $2\alpha_F < c < \omega_F$, then by the **NBU** inequality $F(c) \geqq 1 - \big(1 - F(c/2)\big)^2 > F(c/2)$. Now define $G$ as

$$G(x) = \begin{cases} 0, & \text{if } x < c/2, \\ F(c/2), & \text{if } c/2 \leqq x < c, \\ F(x) & \text{if } c \leqq x. \end{cases}$$

Then $G \in \mathbf{NBU}$, because for $0 \leqq t \leqq s$, $t < c/2$ we have $\overline{G}(t)\overline{G}(s) = \overline{G}(s) \geqq \overline{G}(t+s)$, while for $c/2 \leqq t \leqq s$ we can write $\overline{G}(t)\overline{G}(s) \geqq \overline{F}(t)\overline{F}(s) \geqq \overline{F}(t+s) = \overline{G}(t+s)$. Again, $F \vee \delta_c = G \vee \delta_c$, thus $F | G \vee \delta_c$. Clearly, $\delta_c$ is not divisible by $F$; nor can $F$ divide $G$, since $\frac{G(c/2)}{F(c/2)} = 1$, but $\frac{G(c-)}{F(c-)} = \frac{F(c/2)}{F(c)} < 1$.

*Case* **NBUE.** If $\omega_F = c < \infty$ but $F \neq \delta_c$, then $\delta_{c/2} \vee F$ is non-degenerate and infinitely divisible (even in **NBU**). It is easy to see that $F$ does not divide $\sqrt{\delta_{c/2} \vee F}$ (because $F(x) < \sqrt{F(x)}$ if $\alpha_F < x < \omega_F$), thus $F$ is not prime.

Suppose now $\omega_F = \infty$. Since $-\log F \sim \overline{F}$ as $x \to \infty$, we have

$$\left(-\log F(x)\right)^{-1} \int_x^\infty \left(-\log F(t)\right) dt \sim \overline{F}(x)^{-1} \int_x^\infty \overline{F}(t)\, dt \leqq \mathbf{E}(F).$$

Hence there exists a positive constant $c$ for which

$$\int_x^\infty \left(-\log F(t)\right) dt \leqq c\left(-\log F(x)\right), \qquad \forall\, x \geqq c.$$

Consequently $F \vee \delta_c$ is infinitely divisible, in virtue of Theorem 3. Again, $F$ does not divide $\sqrt{F \vee \delta_c}$, thus $F$ is not prime.

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# $D$-COMPLETE EXTENSIONS OF QUASI-UNIFORM SPACES*

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**0. Introduction.** Let $(x,\mathcal{U})$ be a quasi-uniform space. A pair $(\mathbf{t},\mathbf{s})$ of filters in $X$ is said to be a *Cauchy filter pair* iff, for $U \in \mathcal{U}$, there are $T \in \mathbf{t}$ and $S \in \mathbf{s}$ such that $T \times S \subset U$. A filter $\mathbf{s}$ is said to be a *D-filter* (*Cauchy filter* in [8]–[12], *D-Cauchy-filter* in [13]) iff there exists a filter $\mathbf{t}$ such that $(\mathbf{t},\mathbf{s})$ is a Cauchy filter pair. The space $(X,\mathcal{U})$ (or the quasi-uniformity $\mathcal{U}$) is said to be *D-complete* (*complete* in [8]–[12]) iff every $D$-filter converges in $(X,\mathcal{U})$ (i.e. with respect to the topology $\mathcal{U}^{tp}$ induced by $\mathcal{U}$).

The papers [8]–[12] contain constructions of $D$-complete extensions for some classes of quasi-uniform spaces. More precisely, quiet spaces are concerned in [8]–[10]; $(X,\mathcal{U})$ and $\mathcal{U}$ are said to be *quiet* iff, for $U \in \mathcal{U}$, there exists $U_0 \in \mathcal{U}$ such that, if $x, y \in X$ and $U_0(x) \in \mathbf{s}$, $U_0^{-1}(y) \in \mathbf{t}$ for some Cauchy filter pair $(\mathbf{t},\mathbf{s})$, then $(x,y) \in U$. In [11]–[12], the construction is defined for stable spaces where $(X,\mathcal{U})$ or $\mathcal{U}$ is said to be *stable* iff every $D$-filter is stable, and the filter $\mathbf{s}$ is said to be *stable* (see e.g. [2], p.126) iff, for every $U \in \mathcal{U}$, $\bigcap\{U(S): S \in \mathbf{s}\} \in \mathbf{s}$.

The purpose of this paper is to present further constructions of $D$-complete extensions, permitting to establish the existence of such an extension for every quasi-uniform space (while special classes of spaces are considered in [7]–[12]).

**1. $D$-complete strict extensions.** Let $(X,\mathcal{T})$ be a topological space, $Y \supset X$, and let us be given, for $a \in Y$, a filter $\mathbf{s}(a)$ in $X$ that is open (i.e. it is generated by a filter base composed of open sets) in the topology $\mathcal{T}$; in particular, for $a \in X$, let $\mathbf{s}(a)$ denote the $\mathcal{T}$-neighbourhood filter of $a$. It is well-known (see e.g. [2], p.122) that there exist then topologies $\mathcal{T}'$ on $Y$ such that $\mathbf{s}(a)$ is the trace in $X$ of the $\mathcal{T}'$-neighbourhood filter $\mathbf{v}'(a)$ of $a \in Y$ (i.e. $\mathbf{v}'(a)|X = \mathbf{s}(a)$), consequently $\mathcal{T}'|X = \mathcal{T}$. Among these topologies, there is a coarsest one, called the *strict extension* of $\mathcal{T}$ for the *trace filter system* $\{\mathbf{s}(a): a \in Y\}$; for this topology, the sets $s(G) = \{a \in Y: G \in \mathbf{s}(a)\}$, where $G$ is $\mathcal{T}$-open, constitute a base. There is also a finest one, the *loose extension,* for which

$$\mathbf{v}'(a) = \{S \cup \{a\}: S \in \mathbf{s}(a)\} \qquad (a \in Y).$$

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Consider now a quasi-uniform space $(X,\mathcal{U})$ and define $\mathcal{T} = \mathcal{U}^{tp}$. If we look for a quasi-uniformity $\mathcal{U}'$ on $Y$ *compatible* with the trace filter system (i.e. such that $\mathbf{s}(a)$ is the trace in $X$ of the $\mathcal{U}'^{tp}$-neighbourhood filter of $a \in Y$) and with $\mathcal{U}$ (i.e. satisfying $\mathcal{U}'|X = \mathcal{U}$), then it is necessary that every $\mathbf{s}(a)$ should be $\mathcal{U}$-round (i.e. $S \in \mathbf{s}(a)$ has to imply the existence of $S_0 \in \mathbf{s}(a)$ and $U \in \mathcal{U}$ such that $U(S_0) \subset S$) (see [2], 1.1).

In the following, we understand by a *trace filter system in a quasi-uniform space* $(X,\mathcal{U})$ a family $\mathbf{S} = \{\mathbf{s}(a): a \in Y\}$ of $\mathcal{U}$-round filters in $X$ such that $Y \supset X$ and $\mathbf{s}(a)$ is the $\mathcal{U}^{tp}$-neighbourhood filter of $a$ if $a \in X$ (observe that a $\mathcal{U}^{tp}$-neighbourhood filter is always $\mathcal{U}$-round and a $\mathcal{U}$-round filter is necessarily $\mathcal{U}^{tp}$-open).

If $\mathcal{U}'$ is a quasi-uniformity on $Y$ such that $\mathcal{U}'|X = \mathcal{U}$, and $(\mathbf{t},\mathbf{s})$ is a filter pair in $X$, then, clearly, it is $\mathcal{U}$-Cauchy iff there is a $\mathcal{U}'$-Cauchy filter pair $(\mathbf{t}',\mathbf{s}')$ satisfying $\mathbf{t} = \mathbf{t}'|X$, $\mathbf{s} = \mathbf{s}'|X$. Therefore, if we look for a $D$-complete extension $\mathcal{U}'$, it is necessary that every $D$-filter in $(X,\mathcal{U})$ should be convergent for $\mathcal{U}'^{tp}$, i.e. it should be finer than some trace filter $\mathbf{s}(a)$. These considerations motivate the following terminology: a *trace filter system* $\{\mathbf{s}(a): a \in Y\}$ in $(X,\mathcal{U})$ is said to be *admissible* iff every $D$-filter in $(X,\mathcal{U})$ is finer than some $\mathbf{s}(a)$.

We would like to construct $D$-complete compatible extensions $\mathcal{U}'$ for a given admissible trace filter system $\mathbf{S}$. For this purpose, we need some lemmas.

LEMMA 1.1. *If $(Y,\mathcal{T}')$ is the strict extension of the topological space $(X,\mathcal{T})$ for the trace filter system $\{\mathbf{s}(a): a \in Y\}$, $\mathbf{s}'$ is a $\mathcal{T}'$-open filter and $\mathbf{s}'|X \to a \in Y$, then $\mathbf{s}' \to a$.*

PROOF. For a $\mathcal{T}$-open set $G$ such that $a \in s(G)$, there is a $\mathcal{T}'$-open set $G' \in \mathbf{s}'$ satisfying $G' \cap X \subset s(G)$. Then $G' \cap X \subset s(G) \cap X = G$, and $b \in G'$ implies $G' \cap X \in \mathbf{s}(b)$, $G \in \mathbf{s}(b)$, $b \in s(G)$, hence $G' \subset s(G)$. $\square$

For a filter $\mathbf{s}$ in the quasi-uniform space $(X,\mathcal{U})$, let us denote by $\mathcal{U}(\mathbf{s})$ the $\mathcal{U}$-*envelope* of $\mathbf{s}$, i.e. the filter $\{U(S): S \in \mathbf{s}, U \in \mathcal{U}\}$, coarser than s and $\mathcal{U}$-round (see [2], 4.6). By $\mathcal{U}^{-1}$ we denote the quasi-uniformity $\{U^{-1}: U \in \mathcal{U}\}$, by $\mathcal{U}^{-tp}$ its topology.

LEMMA 1.2. *If $(\mathbf{t},\mathbf{s})$ is a Cauchy filter pair in $(X,\mathcal{U})$, then $\big(\mathcal{U}^{-1}(\mathbf{t}), \mathcal{U}(\mathbf{s})\big)$ is a Cauchy filter pair as well. Consequently every $D$-filter is finer than a round $D$-filter.* $\square$

THEOREM 1.3. *Let $\mathbf{S} = \{\mathbf{s}(a): a \in Y\}$ be an admissible trace filter system for $(X,\mathcal{U})$. Suppose that $\mathcal{U}'$ is an extension of $\mathcal{U}$ compatible with $\mathbf{S}$ such that*

(a) *$\mathcal{U}'^{tp}$ is the strict extension of $\mathcal{U}^{tp}$,*

(b) *$X$ is dense both for $\mathcal{U}'^{tp}$ and $\mathcal{U}'^{-tp}$.*

*Then $\mathcal{U}'$ is $D$-complete.*

PROOF. By 1.2, it suffices to show that $\mathbf{s}'$ is convergent for $\mathcal{U}'^{tp}$ whenever $(\mathbf{t}',\mathbf{s}')$ is a $\mathcal{U}'$-Cauchy filter pair such that $\mathbf{s}'$ is $\mathcal{U}'$-round, $\mathbf{t}'$ is $\mathcal{U}'^{-1}$-round. Then by (b) $(\mathbf{t},\mathbf{s})$ is a $\mathcal{U}$-Cauchy filter pair for $\mathbf{t} = \mathbf{t}'|X$, $\mathbf{s} = \mathbf{s}'|X$. The $D$-filter $\mathbf{s}$ in $(X,\mathcal{U})$ is coarser than some trace filter $\mathbf{s}(a)$, hence it converges for $\mathcal{U}'^{tp}$, and the same is true for $\mathbf{s}'$ by 1.1. □

Let us say that an extension $\mathcal{U}'$ of $\mathcal{U}$, compatible with the trace filter system $\{\mathbf{s}(a): a \in Y\}$, is *uniformly strict* (*strict* in [2]) iff, for $U' \in \mathcal{U}'$, there is $U_0' \in \mathcal{U}'$ such that

$$s\big(U_0'(a) \cap X\big) \subset U'(a) \qquad (a \in Y).$$

Then clearly $\mathcal{U}'^{tp}$ is the strict extension of $\mathcal{U}^{tp}$. [2] 6.2 and 7.3 furnish necessary and sufficient conditions (in fact, rather complicated ones) for the existence of a uniformly strict extension for a given trace filter system; in particular, [2] 7.3 and (6.2.7) show that $X$ is $\mathcal{U}'^{-tp}$-dense whenever $\mathcal{U}'$ is uniformly strict and the trace filters $\mathbf{s}(a)$ are $D$-filters. Thus we can state:

COROLLARY 1.4. *If* $\mathbf{S} = \{\mathbf{s}(a): a \in Y\}$ *is an admissible trace filter system in* $(X,\mathcal{U})$ *such that every* $\mathbf{s}(a)$ *is a $D$-filter, and* $\mathcal{U}'$ *is a uniformly strict extension of* $\mathcal{U}$ *compatible with* $\mathbf{S}$, *then* $\mathcal{U}'$ *is $D$-complete.* □

COROLLARY 1.5 (J. Deák). *If* $\{\mathbf{s}(a): a \in Y\}$ *is an admissible trace filter system in* $(X,\mathcal{U})$ *such that every* $\mathbf{s}(a)$ *is a stable $D$-filter, then a $D$-complete, uniformly strict extension* $\mathcal{U}_s'(\mathbf{S})$, *compatible with* $\mathcal{U}$ *and this system, is generated by the entourages*

$$U' = \big\{(a,b): U(S) \in \mathbf{s}(b) \text{ for } S \in \mathbf{s}(a)\big\} \tag{1.5.1}$$

*where* $U \in \mathcal{U}$.

PROOF. By 1.4 and [2], 6.2 and 6.3, a $\mathcal{U}'$ satisfying these conditions is generated by the subbase composed of the entourages $W(U)$ $(U \in \mathcal{U})$ defined in the following manner. Suppose $x, y \in X$, $p, q \in Y - X$, and put

$$\sigma_U(p) = \bigcap\big\{U(S): S \in \mathbf{s}(p)\big\} \in \mathbf{s}(p), \tag{1.5.2}$$

$$(x,y) \in W(U) \quad \text{iff} \quad (x,y) \in U, \tag{1.5.3}$$

$$(p,x) \in W(U) \quad \text{iff} \quad x \in \sigma_U(p), \tag{1.5.4}$$

$$(x,p) \in W(U) \quad \text{iff} \quad U(x) \in \mathbf{s}(p), \tag{1.5.5}$$

$$(p,q) \in W(U) \quad \text{iff} \quad \sigma_U(p) \in \mathbf{s}(q). \tag{1.5.6}$$

We show that the entourages (1.5.1) generate $\mathcal{U}'_s(\mathbf{S})$ (they constitute a uniform base because $U_1 \subset U_2$ implies $U'_1 \subset U'_2$).

This is contained in

LEMMA 1.6 ([6], 1.7). *If* $\mathbf{S}$ *is a trace filter system in* $(X,\mathcal{U})$ *composed of stable filters, then the entourages* $U'$ *given by* (1.5.1) *and those* $W(U)$ *defined by* (1.5.2)–(1.5.6) *generate the same quasi-uniformity* $\mathcal{U}'_s(\mathbf{S})$.

PROOF. It is sufficient to show $U'_0 \subset W(U)$ and $W(U_0) \subset U'$ whenever $U, U_0 \in \mathcal{U}$, $U_0^2 \subset U$.

In fact, $(x,y) \in U'_0$ implies $U_0(U_0(x)) \in \mathbf{s}(y)$, $y \in U(x)$, and $(x,y) \in \in W(U_0)$ implies $y \in U_0(x)$, hence $y \in U_0(V)$ for $V \in \mathbf{s}(x)$, $U_0(y) \subset U(V)$, $U(V) \in \mathbf{s}(y)$, $(x,y) \in U'$.

Similarly $(p,x) \in U'_0$ implies $(p,x) \in U'$, so $U(S) \in \mathbf{s}(x)$ for $S \in \mathbf{s}(p)$, $x \in \sigma_U(p)$. Conversely, $(p,x) \in W(U_0)$ implies $x \in U_0(S)$ for $S \in \mathbf{s}(p)$, $U_0(x) \subset U(S) \in \mathbf{s}(x)$, $(p,x) \in U'$.

Further $(x,p) \in U'_0$ implies $U_0\big(U_0(x)\big) \in \mathbf{s}(p)$, $U(x) \in \mathbf{s}(p)$, $(x,p) \in \in W(U)$, and $(x,p) \in W(U_0)$ implies $U_0(x) \in \mathbf{s}(p)$, $U_0(S) \in \mathbf{s}(p)$ for $S \in \in \mathbf{s}(x)$, $(x,p) \in U'_0 \subset U'$.

Finally $(p,q) \in U'_0$ implies $U_0(S) \in \mathbf{s}(q)$ for $S \in \mathbf{s}(p)$, hence $U(S) \supset \supset U_0\big(U_0(S)\big)$, i.e. $U(S) \supset U_0(T)$ for some $T \in \mathbf{s}(q)$, consequently $\sigma_U(p) \supset \supset \sigma_{U_0}(q) \in \mathbf{s}(q)$, $(p,q) \in W(U)$. Conversely $(p,q) \in W(U_0)$ implies $\sigma_{U_0}(p) \in \in \mathbf{s}(q)$, hence $U_0(S) \in \mathbf{s}(q)$ for $S \in \mathbf{s}(p)$, $(p,q) \in U'_0 \subset U'$. □

Let us say that $(X,\mathcal{U})$ is a *D-space* iff every $D$-filter is finer than some stable $D$-filter. E.g. every $D$-complete space is a $D$-space because any neighbourhood filter is a stable $D$-filter:

$$\bigcap\{U(S): x \in \operatorname{int} S\} \supset U(x) \qquad (U \in \mathcal{U}),$$

and $T \times S \subset U$ if $T = \{x\}$, $S = U(x)$, so that $(\mathbf{t},\mathbf{s})$ is a Cauchy filter pair if $\mathbf{s}$ is the neighbourhood filter of $x$ and $\mathbf{t} = \dot{x}$, where $\dot{x} = \dot{A}$ for $A = \{x\}$ and $\dot{A} = \{S \subset X : S \supset A\}$.

Now we can state:

THEOREM 1.7 (J. Deák). *If* $(X,\mathcal{U})$ *is a D-space, then there is an admissible trace filter system* $\mathbf{S}$ *composed of stable D-filters, and then* $\mathcal{U}'_s(\mathbf{S})$ *is a D-complete extension of* $\mathcal{U}$*, compatible with* $\mathbf{S}$.

PROOF. 1.5 applies because, if $\mathbf{s}$ is a stable $D$-filter, then $\mathcal{U}(\mathbf{s})$ is round, stable ([2], 4.6), and a $D$-filter by 1.2; in particular, the $\mathcal{U}^{tp}$-neighbourhood filters are round, stable $D$-filters, so that it suffices to choose for $\{\mathbf{s}(p): p \in \in Y - X\}$ the collection of all non-$\mathcal{U}^{tp}$-convergent, round, stable $D$-filters in $(X,\mathcal{U})$. □

Instead of all these filters we may use a part of them provided every non-convergent $D$-filter is finer than one of the trace filters selected.

The hypotheses of 1.7 are fulfilled if $(X,\mathcal{U})$ is stable; then 1.7 furnishes the $D$-complete extension constructed in [11] (provided $\mathcal{U}^{tp}$ is $T_0$ and $\mathbf{S}$ is *reduced,* i.e. $x \in X$, $p,q \in Y - X$, $p \neq q$ imply $\mathbf{s}(x) \neq \mathbf{s}(p) \neq \mathbf{s}(q)$). However, a $D$-space need not be stable, even if it is $D$-complete:

EXAMPLE 1.8. For $x,y \in \mathbf{R}$, $\varepsilon > 0$, let $(x,y) \in U_\varepsilon$ hold iff $x = y$ or $x \leqq 0 \leqq y < x + \varepsilon$. Then clearly $U_\varepsilon^2 \subset U_{2\varepsilon}$, hence $\{U_\varepsilon : \varepsilon > 0\}$ is a base for a quasi-uniformity $\mathcal{U}$ on $\mathbf{R}$. $U_\varepsilon(x) = \{x\}$ if $x > 0$, $U_\varepsilon(x) = \{x\} \cup (0, x+\varepsilon)$ if $x \leqq 0$ (where $(0, x+\varepsilon) = \emptyset$ if $x + \varepsilon \leqq 0$), hence $\emptyset \neq T \times S \subset U_\varepsilon$ implies either $T = \{x\}$, $S \subset U_\varepsilon(x)$, or $T \cap (0,+\infty) = \{x\}$, $S = \{x\}$, $x \leqq \inf T + \varepsilon$, or $T \subset (-\infty, 0]$, $S \subset [0,+\infty)$, $\sup S \leqq \inf T + \varepsilon$. Consequently, for a Cauchy filter pair $(\mathbf{t},\mathbf{s})$, either $\mathbf{t} = \dot{x}$ and $\mathbf{s} \to x$, or $\mathbf{s} = \dot{x}$ (and then $\mathbf{s} \to x$ again), or $[0,+\infty) \in \mathbf{s}$ and $\mathbf{s} \to 0$: $(\mathbf{R},\mathcal{U})$ is $D$-complete. However, if

$$\mathbf{t} = \mathrm{fil}\{(-\varepsilon, 0) : \varepsilon > 0\}, \qquad \mathbf{s} = \mathrm{fil}\{(0,\varepsilon) : \varepsilon > 0\},$$

then $(\mathbf{t},\mathbf{s})$ is a Cauchy filter pair, but $\mathbf{s}$ is not stable. □

EXAMPLE 1.9 (cf. [4], 0.7). If $X = \mathbf{R} - \{0\}$ and we consider the subspace on $X$ of the space $(\mathbf{R},\mathcal{U})$ of the previous example, then any $D$-filter coarser than the $D$-filter $\mathbf{s}|X$ (using still the same notation) must coincide with it (because it is the trace of a filter that converges to 0 in $\mathcal{U}^{tp}$), and $\mathbf{s}|X$ is not stable: the subspace in question is not a $D$-space. □

A non-$D$-complete, non-stable $D$-space is presented by

EXAMPLE 1.10 (cf. [5], 7.12). For $X = \mathbf{R} - \{0\}$ again, let $(x,y) \in U_\varepsilon$ hold iff $x = y$ or $x < 0 < y$ and $-xy < \varepsilon$. Then $U_\varepsilon^2 = U_\varepsilon$, and $\{U_\varepsilon : \varepsilon > 0\}$ is a base for a quasi-uniformity $\mathcal{U}$.

For $c > 0$, $U_\varepsilon(c) = \{c\}$, $U_\varepsilon(-c) = \{-c\} \cup \left(0, \frac{\varepsilon}{c}\right)$. Hence, if $\emptyset \neq T \times S \subset U_\varepsilon$, then either $T = \{x\}$, $S \subset U_\varepsilon(x)$, or $T \cap (0,+\infty) = \{x\}$, $S = \{x\}$, or $T \subset (-a,0)$, $S \subset (0,b)$, $ab \leqq \varepsilon$. Thus any non-convergent $D$-filter $\mathbf{s}$ necessarily contains some bounded set $A \subset (0,+\infty)$, and then $\dot{A}$ is a stable $D$-filter coarser than $\mathbf{s}$. However, the Euclidean neighbourhood filter of $c > 0$ is a non-stable $D$-filter. □

THEOREM 1.11. *If $(X,\mathcal{U})$ is a uniform space, and every $\mathbf{S}(a)$ is a round, Cauchy filter in the trace filter system $\mathbf{S} = \{\mathbf{s}(a) : a \in Y\}$, then $\mathcal{U}'_s(\mathbf{S})$ is a uniformity.*

PROOF. Observe that, in a uniform space, Cauchy filters are stable and coincide with the $D$-filters, so that the hypotheses of 1.5 are fulfilled. For $U \in \mathcal{U}$, select a symmetric $U_1 \in \mathcal{U}$, $U_1^2 \subset U$. Then $U_1' \subset U'^{-1}$; in fact, $(a,b) \in U_1'$ implies $U_1(S) \in \mathbf{s}(b)$ for some $S \in \mathbf{s}(a)$ satisfying $S \times S \subset U_1$. For any $T \in \mathbf{s}(b)$, we have $U_1(S) \cap T \in \mathbf{s}(b)$, and $x \in S$, $y \in U_1(S) \cap T$ imply $(z,y) \in U_1$ for some $z \in S$, hence $x \in U_1(z) \subset U_1\big(U_1(y)\big) \subset U(y)$, $S \subset U\big(U_1(S) \cap T\big) \subset U(T) \in \mathbf{s}(a)$, consequently $(b,a) \in U'$. □

Thus $\mathcal{U}'_s(\mathbf{S})$ is the standard completion of a uniform space (cf. [1], p.256) provided the trace filter system $\mathbf{S}$ composed of all round Cauchy filters is reduced (cf. [8], Theorem 1, [12], Theorem 1).

**2. $D$-complete loose extensions.** Let $\mathbf{S}$ be a trace filter system in a quasi-uniform space $(X,\mathcal{U})$. It is known ([2], 4.8 and 6.1) that, in general, there is no compatible extension of $\mathcal{U}$ compatible with the strict extension of $\mathcal{U}^{tp}$ for $\mathbf{S}$. However, there is always an extension compatible with the loose extension ([2], 2.2). In order to construct it (see [2], 2.1), let us denote by $\Sigma$ the collection of all maps $\sigma : Y \to \exp X$ such that $\sigma(x) = \{x\}$ for $x \in X$, $\sigma(p) \in \mathbf{s}(p)$ for $p \in Y - X$, and let us keep the convention that $a, b, c$ denote points of $Y$, $x, y, z$ belong to $X$, and $p, q, r$ belong to $Y - X$. For $U \in \mathcal{U}$, $\sigma \in \Sigma$, let $W(U,\sigma)$ be the entourage on $Y$ defined by

$$(x,y) \in W(U,\sigma) \quad \text{iff} \quad (x,y) \in U, \tag{2.1}$$

$$(p,x) \in W(U,\sigma) \quad \text{iff} \quad x \in U\big(\sigma(p)\big), \tag{2.2}$$

$$(x,p) \notin W(U,\sigma), \tag{2.3}$$

$$(p,q) \in W(U,\sigma) \quad \text{iff} \quad p = q. \tag{2.4}$$

Then $\{W(U,\sigma) : U \in \mathcal{U}, \sigma \in \Sigma\}$ is a quasi-uniform base that generates a quasi-uniformity $\mathcal{U}'_\ell(\mathbf{S})$ compatible with $\mathcal{U}$ and with the loose extension of $\mathcal{U}^{tp}$ for the given trace filter system. Let us call it the *uniformly loose extension* of $\mathcal{U}$ for $\mathbf{S}$.

THEOREM 2.1. *If an admissible trace filter system is composed of $D$-filters, then the uniformly loose extension is $D$-complete.*

PROOF. Select, for every $a \in Y$, a filter $\mathbf{t}(a)$ in $X$ such that $\big(\mathbf{t}(a), \mathbf{s}(a)\big)$ is a Cauchy filter pair; in particular, let $\mathbf{t}(x) = \dot{x}$ for $x \in X$. Denote by $\Theta$ the collection of all maps $\tau : Y \to \exp X$ such that $\tau(a) \in \mathbf{t}(a)$, satisfying the condition $\tau(x) = \{x\}$ for $x \in X$. For $A \subset Y$, denote

$$\tau(A) = \bigcup\{\tau(a) : a \in A\}.$$

Let $(\mathbf{t}', \mathbf{s}')$ be a Cauchy filter pair in $\big(Y, \mathcal{U}'_\ell(\mathbf{S})\big)$. We have to show that $\mathbf{s}'$ is convergent. By 1.2 we can suppose that $\mathbf{s}'$ is $\mathcal{U}'_\ell(\mathbf{S})$-round so that $\mathbf{s} = \mathbf{s}'|X$ is a proper filter. The sets $\tau(T')$ ($T' \in t'$, $\tau \in \Theta$) clearly constitute a filter base in $X$ that generates a filter $\mathbf{t}$ in $X$. The pair $\mathbf{t}, \mathbf{s}$) is a $\mathcal{U}$-Cauchy filter pair.

In fact, given $U \in \mathcal{U}$, we can choose $U_1 \in \mathcal{U}$ such that $U_1^2 \subset U$, and then maps $\tau \in \Theta$, $\sigma \in \Sigma$ such that $\tau(a) \times \sigma(a) \subset U_1$ for each $a \in Y$. Choose $T' \in \mathbf{t}'$, $S' \in \mathbf{s}'$ such that $T' \times S' \subset W(U_1, \sigma)$. Then $\tau(T') \times (S' \cap X) \subset U$.

To see this, let $x \in \tau(T')$, $y \in S' \cap X$. Then $x \in \tau(a)$ for some $a \in T'$, hence $(a, y) \in W(U_1, \sigma)$, i.e. $(a, y) \in U_1$ if $a \in X$ or $y \in U_1(\sigma(a))$ if $a \in \in Y - X$. In the first case $x = a$ and $(x, y) \in U_1 \subset U$, in the second one $\tau(a) \times \sigma(a) \subset U_1$ implies $\sigma(a) \subset U_1(x)$, $y \in U_1(U_1(x)) \subset U(x)$.

The $D$-filter $\mathbf{s}$ converges with respect to $\mathcal{U}'_\ell(\mathbf{S})^{tp}$ since the trace filter system is admissible. Then $\mathbf{s}'$ converges as well. This is clear if $X \in \mathbf{s}'$, and this is the case if $\mathbf{t}'|X$ is a proper filter because then $T' \times S' \subset W(U_1, \sigma)$ implies $S' \subset X$ by $T' \cap X \neq \emptyset$ and (2.3). If $Y - X \in \mathbf{t}'$ and $X \notin \mathbf{s}'$ then the same inclusion implies $T' - X = \{p\} = S' - X$ by (2.4). Thus $\{p\} \in \mathbf{t}'$ and, for $U \in \mathcal{U}$, $\sigma \in \Sigma$, there is $S' \in \mathbf{s}'$ such that $\{p\} \times S' \subset W(U, \sigma)$, so that $\mathbf{s}' \to p$. □

COROLLARY 2.2. *Every quasi-uniform space has a $D$-complete uniformly loose extension.*

PROOF. By 1.2, there are admissible trace filter systems composed of $D$-filters. □

Instead of 2.1, we could use [3], Theorem 3.3. In fact, it is proved there that $\mathcal{U}'_\ell(\mathbf{S})$ is $SP$-complete if $\mathbf{S}$ is composed (of the neighbourhood filters and) of the non-convergent, round $SP$-filters. In $(X, \mathcal{U})$, a filter $\mathbf{s}$ is said to be an *$SP$-filter* iff, for $U \in \mathcal{U}$, there is $x \in X$ such that $U(x) \in \mathbf{s}$, and $(X, \mathcal{U})$ or $\mathcal{U}$ is *$SP$-complete* iff every $SP$-filter is convergent. Now a $D$-filter is obviously an $SP$-filter, so an $SP$-complete space is $D$-complete.

A further possible tool in proving 2.2 would be the observation that $\mathcal{U}'_\ell(\mathbf{S})$ is $SP$-complete (hence $D$-complete) if $\mathbf{S}$ is composed (of the neighbourhood filters and) of all non-convergent, round filters. In fact, the method of proof of [3], Theorem 3.3 furnishes the following statement: if $\mathbf{s}'$ is an $SP$-filter for $\mathcal{U}'_\ell(\mathbf{S})$ and $\mathbf{s}'|X$ is $\mathcal{U}'_\ell(\mathbf{S})^{tp}$-convergent then $\mathbf{s}'$, too, is convergent.

However, an extension involving a narrower class of new points is more valuable, hence 2.1 can be considered, in some sense, to be a better result than those yielding $SP$-complete extensions but using huge classes of trace filters.

As to 2.1, it is worth-while to observe that $\mathcal{U}'_\ell(\mathbf{S})$ is not necessarily $D$-complete if $\mathbf{S}$ is an admissible trace filter system:

EXAMPLE 2.3. Consider the upper half-plane of the Sorgenfrey plane, i.e. $X = \mathbf{R} \times (0, +\infty)$, and, for $x, y \in X$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\varepsilon > > 0$, $(x, y) \in U_\varepsilon$ iff $x_i \leqq y_i < x_1 + \varepsilon$ $(i = 1, 2)$, and let the quasi-uniformity $\mathcal{U}$ be generated by $\{U_\varepsilon : \varepsilon > 0\}$. Then $\mathcal{U}$ is $D$-complete so that any trace filter system is admissible. Let $Y - X = \mathbf{R} - \{0\}$ and the sets $S(p, \varepsilon) = = (p - \varepsilon, p + \varepsilon) \times (0, \varepsilon)$ $(\varepsilon > 0)$ constitute a base for $\mathbf{s}(p)$. Now

$$U_\varepsilon(S(p, \delta)) = (p - \delta, p + \delta + \varepsilon) \times (0, \delta + \varepsilon),$$

hence $U_\varepsilon\big(S(p,\varepsilon)\big) \subset S(p,2\varepsilon)$, and $\mathbf{s}(p)$ is round. For $\mathcal{U}' = \mathcal{U}'_\ell(\mathbf{S})$ (with $\mathbf{S}$ being composed of the given trace filters $\mathbf{s}(p)$ $(p \in Y - X)$ and the $\mathcal{U}^{tp}$-neighbourhood filters $\mathbf{s}(z)$ $(z \in X)$), $\mathbf{s}' = \operatorname{fil}_Y \mathbf{s}$ is a $D$-filter if $\mathbf{s}$ is generated by the base $\{Q(\varepsilon): \varepsilon > 0\}$, $Q(\varepsilon) = (0,\varepsilon) \times (0,\varepsilon)$; in fact, we can choose $\mathbf{t}' = \operatorname{fil}_Y\{T(\varepsilon): \varepsilon > 0\}$, $T(\varepsilon) = (-\varepsilon, 0) \subset Y - X$. In order to see this, choose $\varepsilon > 0$ and $\sigma \in \Sigma$, then $\delta_p > 0$ such that $S(p,\delta_p) \subset \sigma(p)$, and observe that $p \in T(\varepsilon)$ implies

$$U_{2\varepsilon}\big(\sigma(p)\big) \supset (p - \delta_p, p + \delta_p + 2\varepsilon) \times (0, \delta_p + 2\varepsilon) \supset$$

$$\supset (p, p + 2\varepsilon) \times (0, 2\varepsilon) \supset Q(\varepsilon),$$

i.e. $T(\varepsilon) \times Q(\varepsilon) \subset W(U_{2\varepsilon}, \sigma)$. However, $\mathbf{s}'$ does not $\mathcal{U}'^{tp}$-converge. □

The construction of the uniformly loose extension can be slightly generalized. For this purpose, let $\Sigma_0$ be a subset of $\Sigma$ such that

(2.5) for $a \in Y$, $S \in \mathbf{s}(a)$, there is a $\sigma \in \Sigma_0$ such that $\sigma(a) \subset S$

(automatically fulfilled for $a \in X$). Then it is easy to check (see [2], 2.1) that $W(U_1, \sigma)^2 \subset W(U, \sigma)$ if $U_1^2 \subset U$, so that

$$\big\{W(U,\sigma): U \in \mathcal{U}, \sigma \in \Sigma_0\big\} \tag{2.6}$$

is a subbase for a quasi-uniformity $\mathcal{U}'_\ell(\Sigma_0)$ (it is a base if, for $\sigma_1, \sigma_2 \in \Sigma_0$, there is $\sigma \in \Sigma_0$ such that $\sigma(p) \subset \sigma_1(p) \cap \sigma_2(p)$ for each $p \in Y - X$). By (2.5) $\mathcal{U}'_\ell(\Sigma_0)$ is compatible with the given trace filter system (and still induces the corresponding loose extension of $\mathcal{U}^{tp}$):

LEMMA 2.4 (cf. [5], 6.5 and [6], Problem 32). *If a trace filter system* $\mathbf{S}$ *is given in* $(X,\mathcal{U})$, *and* $\Sigma_0 \subset \Sigma$ *fulfils* (2.5), *then* (2.6) *is a subbase for a quasi-uniformity* $\mathcal{U}'_\ell(\Sigma_0)$ *compatible with* $\mathcal{U}$ *and with the loose extension of* $\mathcal{U}^{tp}$ *for* $\mathbf{S}$. □

If we look for a $D$-complete $\mathcal{U}'_\ell(\Sigma_0)$, we suppose that the trace filter system is admissible and it is composed of $D$-filters, and then we can repeat the reasoning in the proof of 2.1, provided the following condition is fulfilled:

(2.7) it is possible to select filters $\mathbf{t}(a)$ for $a \in Y$ such that $\big(\mathbf{t}(a), \mathbf{s}(a)\big)$ is a Cauchy filter pair, $\mathbf{t}(x) = \dot{x}$ for $x \in X$, and, for $U \in \mathcal{U}$, there are $\tau \in \Theta$ and $\sigma \in \Sigma_0$ such that $\tau(a) \times \sigma(a) \subset U$ for $a \in Y$.

COROLLARY 2.5. *If an admissible trace filter system composed of* $D$-*filters is given and* $\Sigma_0 \subset \Sigma$ *fulfils* (2.5) *and* (2.7), *then* $\mathcal{U}'_\ell(\Sigma_0)$ *is a* $D$-*complete extension.* □

$\mathcal{U}'_\ell(\Sigma_0)$ can be distinct from $\mathcal{U}'_\ell(\mathbf{S})$ (see 3.9).

If (2.5) holds without (2.7) being fulfilled, then $\mathcal{U}'_\ell(\Sigma_0)$ may or may not be $D$-complete:

EXAMPLE 2.6. Let $X = \mathbf{R} - \mathbf{Q}$, $\mathcal{U}$ be the restriction to $X$ of the Sorgenfrey quasi-uniformity, i.e. be generated by the collection of $U_\varepsilon$ ($\varepsilon > 0$) where $(x, y) \in U_\varepsilon$ iff $x, y \in X$, $x \leqq y < x + \varepsilon$. For $Y = \mathbf{R}$, $p \in Y - X = Q$, let $\mathbf{s}(p)$ be generated by $\{(p, p+\varepsilon) \cap X : \varepsilon > 0\}$. Then $\mathbf{s}(p)$ is a round $D$-filter and $\mathbf{S} = \{\mathbf{s}(a) : a \in Y\}$ is admissible. Choose a function $f: \mathbf{Q} \to \mathbf{R}$ such that $f(p) > 0$ for $p \in \mathbf{Q}$, and define

$$\sigma_n(p) = \left(p, p + \frac{1}{n} f(p)\right) \cap X, \qquad \Sigma_0 = \{\sigma_n : n \in \mathbf{N}\}.$$

Then $\Sigma_0$ fulfils (2.5), and (2.7) holds iff $f$ is bounded.

a) If $f\left(-\frac{1}{n}\right) = 2n$, $A = (0,1) \cap X$,

$$T_k = \left\{-\frac{1}{n} : n \in \mathbf{N}, n \geqq k\right\}, \qquad \mathbf{t} = \mathrm{fil}_Y \{T_k : k \in \mathbf{N}\},$$

then $(\mathbf{t}, \dot{A})$ is a $\mathcal{U}'_\ell(\Sigma_0)$-Cauchy pair because

$$W(U_\varepsilon, \sigma_k)(p) = U_\varepsilon\big(\sigma_k(p)\big) \supset (p, p + 2 + \varepsilon) \cap X \supset A$$

for $p \in T_n$, $\varepsilon > 0$, $n \geqq k$. However, $\dot{A}$ does not $\mathcal{U}'_\ell(\Sigma_0)^{tp}$-converge.

b) If $f(p) = p$ for $p > 0$, $f(p) = 1$ for $p \leqq 0$, then $\mathcal{U}'_\ell(\Sigma_0)$ is $D$-complete. In fact, consider the quasi-uniformity $\mathcal{U}'_\ell(\Sigma_1)$ obtained from the choice $f(p) = 1$ for $p \in \mathbf{Q}$ and denote by $\sigma'_n \in \Sigma_1$ the corresponding map. Now a round $\mathcal{U}'_\ell(\Sigma_0)$-$D$-filter $\mathbf{s}'$ clearly converges if there is a filter $\mathbf{t}'$ such that $(\mathbf{t}', \mathbf{s}')$ is a Cauchy pair and $Y - X \notin \mathbf{t}'$ or $Y - X \in \mathbf{s}'$. Suppose $\emptyset \neq T_0 \subset Y - X$, $\emptyset \neq S_0 \subset X$, $T_0 \in \mathbf{t}'$, $T_0 \times S_0 \subset W(U_{\varepsilon_0}, \sigma_{n_0})$ for some $\varepsilon_0 > 0$, $n_0 \in \mathbf{N}$. Then $S_0 \subset W((U_{\varepsilon_0}, \sigma_{n_0})(p)$ for a $p \in T_0$, hence $S_0$ is bounded, and $\sup T \leqq \inf S \leqq \sup S_0 = K$ whenever $T \in \mathbf{t}'$, $S \in \mathbf{s}'|X$, $T \subset Y - X$, $T \times S \subset W(U_\delta, \sigma_k)$ for some $\delta > 0$, $k \in \mathbf{N}$. Choose $\varepsilon > 0$, $n \in \mathbf{N}$, then $k \in \mathbf{N}$ such that $\frac{1}{k}K < \frac{1}{n}$, and $T \in \mathbf{t}'$, $S \in \mathbf{s}'$ satisfying $T \subset Y - X$, $T \times S \subset W(U_\varepsilon, \sigma_k)$. Then $p \in T$ implies

$$W(U_\varepsilon, \sigma_k)(p) \subset W(U_\varepsilon, \sigma'_n)(p)$$

i.e.

$$\left(p, p + \frac{1}{k} p + \varepsilon\right) \subset \left(p, p + \frac{1}{n} + \varepsilon\right),$$

so that $s'$ is a $D$-filter with respect to $\mathcal{U}'_\ell(\Sigma_1)$. The latter being $D$-complete by 2.5, $\mathbf{s}'$ converges for $\mathcal{U}'_\ell(\Sigma_0)^{tp} = \mathcal{U}'_\ell(\Sigma_1)^{tp}$. $\square$

**3. Extension of maps.** A reasonable concept of completeness involves an extension theorem of uniformly continuous maps into complete spaces to a complete extension of the space. A theorem of this kind corresponds to the extension $\mathcal{U}'_s(\mathbf{S})$ described in 1.5. In order to formulate it, we recall that a quasi-uniform space $(X,\mathcal{U})$ or a quasi-uniformity $\mathcal{U}$ is said to be *uniformly regular* (*regular* in [2]), iff, given $U \in \mathcal{U}$, there is $U_0 \in \mathcal{U}$ such that $\overline{U_0(x)} \subset U(x)$ for every $x \in X$ (and for the closure with respect to $\mathcal{U}^{tp}$).

THEOREM 3.1. *If $(X,\mathcal{U})$ is a D-space, $\mathbf{S}$ is an admissible trace filter system composed of stable D-filters, $f: X \to Z$ is $(\mathcal{U},\mathcal{U}'')$-continuous for a D-complete, uniformly regular quasi-uniformity $\mathcal{U}''$ on $Z$, then $f$ admits a $\big(\mathcal{U}'_s(\mathbf{S}),\mathcal{U}''\big)$-continuous extension $g: Y \to Z$.*

PROOF. If $(\mathbf{t},\mathbf{s})$ is a $\mathcal{U}$-Cauchy filter pair in $X$, then $\big(\mathrm{fil}_Z\, f(\mathbf{t}),\mathrm{fil}_Z\, f(\mathbf{s})\big)$ is a $\mathcal{U}''$-Cauchy filter pair in $Z$. Hence the $\mathcal{U}''$-$D$-filters $\mathrm{fil}_Z\, f\big(\mathbf{s}(p)\big)$ (and the filter bases $f\big(\mathbf{s}(p)\big)$ are convergent in $\mathcal{U}''^{tp}$ $(p \in Y - X)$. Define $g: Y \to Z$ such that $g(x) = f(x)$ for $x \in X$, and $f\big(\mathbf{s}(p)\big) \to g(p)$ for $p \in Y - X$. Then, by the $(\mathcal{U}^{tp},\mathcal{U}''^{tp})$-continuity of $f$, $f\big(\mathbf{s}(a)\big) \to g(a)$ holds for every $a \in Y$.

Now, for a given $U'' \in \mathcal{U}''$, choose $U''_0 \in \mathcal{U}''$ such that $\overline{U''_0(z)} \subset U''(z)$ for $z \in Z$, then $U''_1 \in \mathcal{U}''$ such that $U''^2_1 \subset U''_0$, finally $U \in \mathcal{U}$ such that $(x,y) \in U$ implies $\big(f(x),f(y)\big) \in U''_1$ for $x,y \in X$. Define $U'$ be (1.5.1).

If $a,b \in Y$, $(a,b) \in U'$, then we can choose $S \in \mathbf{s}(a)$ such that $f(S) \subset$ $\subset U''_1\big(g(a)\big)$, and clearly

$$f\big(U(S)\big) \subset U''_1\big(U''_1(g(a))\big) \subset U''_0\big(g(a)\big).$$

As $U(S) \in \mathbf{s}(b)$ and $f\big(\mathbf{s}(b)\big) \to g(b)$, necessarily

$$g(b) \in \overline{f\big(U(S)\big)} \subset \overline{U''_0\big(g(a)\big)} \subset U''\big(g(a)\big). \quad \square$$

[11], Theorem 2 corresponds to 3.1 in the case of stable $T_0$-spaces and reduced trace filter systems.

An essentially better extension theorem can be proved for $D$-complete, uniformly loose extensions:

THEOREM 3.2. *If $\mathbf{S}$ is an admissible trace filter system composed of D-filters in a space $(X,\mathcal{U})$, $(Z,\mathcal{U}'')$ is D-complete and $f: X \to Z$ is $(\mathcal{U},\mathcal{U}'')$-continuous, then $f$ admits a $\big(\mathcal{U}'_\ell(\mathbf{S}),\mathcal{U}''\big)$-continuous extension $g: Y \to Z$.*

PROOF. We can again define $g$ satisfying $g|X = f$, $f\big(\mathbf{s}(a)\big) \to g(a)$ for $a \in Y$. Given $U'' \in \mathcal{U}''$, choose $U''_1 \in \mathcal{U}''$ such that $U''^2_1 \subset U''$, and $U \in \mathcal{U}$ such that $(x,y) \in U$ implies $\big(f(x),f(y)\big) \in U''_1$, finally define $\sigma: Y \to \exp X$

such that ($\sigma(x)=\{x\}$ for $x\in X$ and) $\sigma(p)\in \mathbf{s}(p)$ for $p\in Y-X$, satisfying $f\big(\sigma(p)\big)\subset U_1''\big(g(p)\big)$.

Now $(a,b)\in W(U,\sigma)$ implies $\big(g(a),g(b)\big)\in U''$. This is clear if $a,b\in\in X$, or $a\in X$, $b\in Y-X$, or $a,b\in Y-X$. If $a\in Y-X$, $b\in X$, then $b\in U\big(\sigma(a)\big)$, hence

$$f(b)\in U_1''\big(f(\sigma(a))\big)\subset U_1''\big(U_1''(g(a))\big)\subset U''\big(g(a)\big),$$

so that $g(b)=f(b)\in U''\big(g(a)\big)$. $\square$

It is easy to show that the condition of uniform regularity cannot be dropped in 3.1:

EXAMPLE 3.3. Let $Y=\mathbf{R}$, $X=\mathbf{Q}$, $\mathcal{U}$ be the restriction to $X$ of the Euclidean uniformity of $\mathbf{R}$, $\mathbf{s}(p)$ be generated by $\big\{(p-\varepsilon,p+\varepsilon)\cap X:\varepsilon>>0\big\}$ for $p\in Y-X$. Now $\mathcal{U}_s'(\mathbf{S})$ is the Euclidean uniformity of $\mathbf{R}$ by 1.11. For the same trace filter system $\mathbf{S}$, let $\mathcal{U}''=\mathcal{U}_\ell'(\mathbf{S})$, then $\mathcal{U}''$ is $D$-complete by 2.1. However, the $(\mathcal{U},\mathcal{U}'')$-continuous map $f=\mathrm{id}_X$ cannot be extended in a $\big(\mathcal{U}_2'(\mathbf{S})^{tp},\mathcal{U}''^{tp}\big)$-continuous manner, because such an extension would coincide with $\mathrm{id}_{\mathbf{R}}$ (since $\mathcal{U}''^{tp}$ is $T_2$), and the loose extension is strictly finer than the strict one. $\square$

In order to obtain a similar extension theorem for the extensions $\mathcal{U}_\ell'(\Sigma_0)$, let us introduce the following terminology: a quasi-uniform space $(X,\mathcal{U})$ or a quasi-uniformity $\mathcal{U}$ is *weakly quiet* iff, for $U\in\mathcal{U}$, there is $U_0\in\mathcal{U}$ such that if $(\mathbf{t},\mathbf{s})$ is a Cauchy filter pair, $\mathbf{s}\to x$ for $\mathcal{U}^{tp}$, $T\in\mathbf{t}$, $S\in\mathbf{s}$, and $T\times S\subset U_0$, then $S\subset U(x)$.

The terminology is justified by

LEMMA 3.4. *A quiet space is weakly quiet.*

PROOF. For $U\in\mathcal{U}$, choose $U_0\in\mathcal{U}$ such that if $(\mathbf{t},\mathbf{s})$ is a Cauchy filter pair, $U_0(x)\in\mathbf{s}$, $U_0^{-1}(y)\in\mathbf{t}$, then $(x,y)\in U$. Now if $(\mathbf{t},\mathbf{s})$ is a Cauchy filter pair, $\mathbf{s}\to x$, $T\in\mathbf{t}$, $S\in\mathbf{s}$, and $T\times S\subset U_0$, then $U_0(x)\in\mathbf{s}$, $U_0^{-1}(y)\supset T\in\mathbf{t}$ for $y\in S$, hence $S\subset U(x)$. $\square$

Conversely, a weakly quiet space need not be quiet:

EXAMPLE 3.5. Let $X$ be a regular topological space and $\mathcal{U}$ be its Pervin quasi-uniformity (see [14]), i.e. the entourages

$$U_G=(G\times G)\cup\big((X-G)\times X\big)\qquad (G\subset X\text{ open})$$

constitute a subbase for $\mathcal{U}$. It suffices to show that $U_0=U_G$ corresponds to $U=U_G$ in the sense of the weak quietness. In fact, let $(\mathbf{t},\mathbf{s})$ be a Cauchy filter pair, $\mathbf{s}\to x$, $T\in\mathbf{t}$, $S\in\mathbf{s}$, $T\times S\subset U_G$.

Now $T\subset X-G$ is impossible whenever $x\in G$. In fact, if $G_0$ is open and $x\in G_0\subset\overline{G_0}\subset G$, then $T\subset X-G$ would imply $T\subset X-\overline{G_0}=H$

and $T_0 \times S_0 \subset U_H$ for suitable $T_0 \in \mathbf{t}$, $S_0 \in \mathbf{s}$. But $y \in T \cap T_0$ implies $S_0 \subset U_H(y) = H$ in contradiction with $\mathbf{s} \to x$. Therefore $T \cap G \neq \emptyset$ and $z \in T \cap G$ implies $S \subset U_G(z) = G = U_G(x)$, while $S \subset U_G(x)$ is obvious if $x \in G$.

We know from [2], 8.2 that $\mathcal{U}$ is not uniformly regular in general (e.g. if $X = \mathbf{R}$), while a quiet space is necessarily uniformly regular ([13], Proposition 1.2). □

There are also weakly quiet, uniformly regular spaces that are not quiet:

EXAMPLE 3.6 (J. Deák). Let $X = (0,+\infty) \times \{-1,0,1\}$, and, for $\varepsilon > 0$, let $(z_1, z_2) \in U_\varepsilon$ iff $z_1 = z_2$ or $z_i = (x_i, y_i)$, $x_1 + x_2 < \varepsilon$, $y_1 < y_2$. Then $\{U_\varepsilon : \varepsilon > 0\}$ is a base for a quasi-uniformity $\mathcal{U}$. Clearly both $\mathcal{U}^{tp}$ and $\mathcal{U}^{-tp}$ are discrete, hence $\mathcal{U}$ is uniformly regular and weakly quiet (in fact, $\overline{U(z)} = U(z)$ for $z \in X$, and $(\mathbf{t}, \mathbf{s})$ can be Cauchy filter pair satisfying $\mathbf{s} \to z$ only if $\mathbf{t} = \mathbf{s} = \dot{z}$, so that $T \in \mathbf{t}$, $S \in \mathbf{s}$, $T \times S \subset U_\varepsilon$ imply $z \in T$, $S \subset U_\varepsilon(z)$). However, if $\mathcal{U}$ were quiet and $U_\delta$ $(\delta > 0)$ were suitable for $U_1$ in the sense of the quietness, then $z_i = (x_i, 0)$, $x_i < \delta$, $x_1 \neq x_2$ would imply $(z_1, z_2) \in U_1$ because $(\mathbf{t}, \mathbf{s})$ with

$$\mathbf{t} = \operatorname{fil}\big\{(0,\varepsilon) \times \{-1\} : \varepsilon > 0\big\},$$

$$\mathbf{s} = \operatorname{fil}\big\{(0,\varepsilon) \times \{1\} : \varepsilon > 0\big\}$$

is a Cauchy filter pair and $U_\delta(z_1) \in \mathbf{s}$, $U_\delta^{-1}(z_2) \in \mathbf{t}$, while $(z_1, z_2) \notin U_1$. □

On the other hand, there are uniformly regular spaces that are not weakly quiet:

EXAMPLE 3.7 (J. Deák). Let $X = \mathbf{R}$, and, for $\varepsilon > 0$, $U_\varepsilon(x) = \{x\} \cup [0,+\infty)$ if $-\varepsilon < x < 0$, $U_\varepsilon(0) = [0,\varepsilon)$, $U_\varepsilon(x) = \{x\}$ otherwise. Then $U_\varepsilon^2 \subset U_\varepsilon$, hence $\{U_\varepsilon : \varepsilon > 0\}$ is a base for a quasi-uniformity $\mathcal{U}$. For $\mathcal{U}^{tp}$, each $U_\varepsilon(x)$ is closed so that $\mathcal{U}$ is uniformly regular. However, any $U_\varepsilon$ is unsuitable for $U_1$ in the sense of the weak quietness, because $(\mathbf{t}, \mathbf{s})$ is a Cauchy filter pair if

$$\mathbf{t} = \operatorname{fil}\big\{(-\varepsilon,0) : \varepsilon > 0\big\}, \qquad \mathbf{s} = \operatorname{fil}\big\{[0,\varepsilon) : \varepsilon > 0\big\},$$

further $\mathbf{s} \to 0$, and $(-\varepsilon,0) \times [0,+\infty) \subset U_\varepsilon$, $(-\varepsilon,0) \in \mathbf{t}$, $[0,+\infty) \in \mathbf{s}$, but the latter set is not contained in $U_1(0)$. □

Now we can prove:

THEOREM 3.8. *Let* $\mathbf{S}$ *be an admissible trace filter system composed of D-filters in a quasi-uniform space* $(X,\mathcal{U})$, *and* $\Sigma_0 \subset \Sigma$ *satisfy* (2.5) *and* (2.7). *If* $(Z,\mathcal{U}'')$ *is D-complete and weakly quiet, and* $f: X \to Z$ *is* $(\mathcal{U},\mathcal{U}'')$-*continuous, then there is a* $\big(\mathcal{U}'_\ell(\Sigma_0),\mathcal{U}''\big)$-*continuous extension* $g: Y \to Z$ *of* $f$.

PROOF. We can define $g$ satisfying $g|X = f$ and $f\big(\mathbf{s}(a)\big) \to g(a)$ for $a \in Y$. Given $U'' \in \mathcal{U}''$, select $U_0'' \in \mathcal{U}''$ such that $U_0''^2 \subset U''$, then $U_1'' \in \mathcal{U}''$ such that $S'' \subset U_0''(z)$ whenever $(\mathbf{t}'', \mathbf{s}'')$ is a $\mathcal{U}''$-Cauchy filter pair, such that $S'' \subset U_0''(z)$ whenever $(\mathbf{t}'', \mathbf{s}'')$ is a $\mathcal{U}''$-Cauchy filter pair, $\mathbf{s}'' \to z$, $T'' \in \mathbf{t}''$, $S'' \in \mathbf{s}''$, $T'' \times S'' \subset U_1''$. We can suppose $U_1'' \subset U_0''$. For $U_1''$, choose first $U \in \mathcal{U}$ such that $(x,y) \in U$ implies $\big(f(x), f(y)\big) \in U_1''$, then $\mathbf{t}(a)$, $\tau \in \Theta$, $\sigma \in \Sigma_0$ according to (2.7).

We show that $(a,b) \in W(U,\sigma)$ implies $\big(g(a), g(b)\big) \in U''$. It suffices to consider the case $a \in Y - X$, $b \in X$. Then $b \in U\big(\sigma(a)\big)$, $\tau(a) \times \sigma(a) \subset U$, hence $f(b) \in U_1''\big(f(\sigma(a)\big)$, $f\big(\tau(a)\big) \times f\big(\sigma(a)\big) \subset U_1''$,

$$f\big(\tau(a)\big) \in \mathbf{t}'' = \mathrm{fil}_Z\, f\big(\mathbf{t}(a)\big), \qquad f\big(\sigma(a)\big) \in \mathbf{s}'' = \mathrm{fil}_Z\, f\big(\mathbf{s}(a)\big),$$

and $(\mathbf{t}'', \mathbf{s}'')$ is a $\mathcal{U}''$-Cauchy filter pair, $\mathbf{s}'' \to g(a)$. Thus

$$f\big(\sigma(a)\big) \subset U_0''\big(g(a)\big), \qquad U_1''\big(f(\sigma(a))\big) \subset U''\big(g(a)\big),$$

and $g(b) = f(b) \in U''\big(g(a)\big)$. □

The condition of weak quietness cannot be dropped in 3.8:

EXAMPLE 3.9 (cf. 1.9). Let $X = (\mathbf{R} - \{0\}) \times \mathbf{N}$, $(z_1, z_2) \in U_\varepsilon$ for $\varepsilon > 0$ iff either $z_1 = z_2$ or $z_i = (x_i, n_i)$, $n_1 = n_2$, $x_1 < 0 < x_2 < x_1 + \varepsilon$. Then $\{U_\varepsilon : \varepsilon > 0\}$ is a base for a quasi-uniformity $\mathcal{U}$. Define $Y = \mathbf{R} \times \mathbf{N}$, and let $\mathbf{s}(p_n)$ be generated, for $p_n = (0,n)$, by $\big\{(0,\delta) \times \{n\} : \delta > 0\big\}$, finally $\sigma_\delta(p_n) = (0,\delta) \times \{n\}$.

It is easy to see that the trace filter system $\mathbf{S} = \{\mathbf{s}(a) : a \in Y\}$ is composed of (round) $D$-filters: $U_\varepsilon\big(\sigma_\delta(p_n)\big) = \sigma_\delta(p_n)$, $\big(\mathbf{t}(p_n), \mathbf{s}(p_n)\big)$ is a Cauchy filter pair if $\mathbf{t}(p_n)$ is generated by $\big\{(-\delta, 0) \times \{n\} : \delta > 0\big\}$ because

$$\big((-\delta,0) \times \{n\}\big) \times \big((0,\delta) \times \{n\}\big) \subset U_{2\delta}.$$

Clearly (2.5) and (2.7) are fulfilled for $\Sigma_0 = \{\sigma_\delta : \delta > 0\}$. This trace filter system is admissible because, if $(\mathbf{t}, \mathbf{s})$ is a Cauchy filter pair distinct from those of the form $(\dot{z}, \dot{z})$, then necessarily $(-\infty, 0) \times \{n\} \in \mathbf{t}$, $(0, +\infty) \times \{n\} \in \mathbf{s}$ for some $n$, and $\mathbf{s}$ is finer than $\mathbf{s}(p_n)$.

Thus $\mathcal{U}_\ell'(\mathbf{S})$ and $\mathcal{U}_\ell'(\Sigma_0)$ both are $D$-complete extensions of $\mathcal{U}$, they induce the loose extension of $\mathcal{U}^{tp}$ (= the discrete topology of $X$).

The only $\big(\mathcal{U}_\ell'(\Sigma_0)^{tp}, \mathcal{U}_\ell'(\mathbf{S})^{tp}\big)$-continuous extension of $\mathrm{id}_X$ is $\mathrm{id}_Y$ (because the loose extension is $T_2$). However, $\mathrm{id}_Y$ is not $\big(\mathcal{U}_\ell'(\Sigma_0), \mathcal{U}_\ell'(\mathbf{S})\big)$-continuous. In fact, let $\sigma(p_n) = \big(0, \frac{1}{n}\big) \times \{n\}$, then $W(U_\varepsilon, \sigma_\delta) \subset W(U_1, \sigma)$ does not hold for any $\varepsilon > 0$, $\delta > 0$ because $\big(\frac{\delta}{2}, n\big) \in W(U_\varepsilon, \sigma_\delta)(p_n)$, while $\big(\frac{\delta}{2}, n\big) \notin W(U_1, \sigma)(p_n)$ if $\frac{1}{n} < \frac{\delta}{2}$. □

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# A NOTE ON CONGRUENCE DISTRIBUTIVE ALGEBRAS

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*Dedicated to my teacher, Prof. N. V. Subrahmanyam on his sixtieth birthday*

## Introduction

It is well known that the lattice Con$(A)$ of all congruence relations on a universal algebra $A$ is a complete and compactly generated lattice with $\triangle$ and $\bigtriangledown$, the smallest and the largest congruence relations, respectively. If $A$ is a congruence distributive algebra, i.e. Con$(A)$ is a distributive lattice, then it is well known that the infinite distributive law $\Theta \wedge \left( \bigvee_{i \in I} \alpha_i \right) = \bigvee_{i \in I} (\Theta \wedge \wedge \alpha_i)$ is satisfied in Con$(A)$ (where $\Theta$, $\alpha_i$ are in Con$(A)$). Hence it follows that Con$(A)$ is a complete lattice and is also pseudocomplemented. In their paper [4], T. Katriňák and S. El-Assar have stated that it is an open question whether the identity

$$\Big[\bigwedge_{t\in T}(\vee x_{ts} \mid s\in S)\Big]^{**} = \Big[\bigvee_{\phi\in S^T}\Big(\bigwedge_{t\in T} x_{t\phi(t)}\Big)\Big]^{**}$$

holds for all congruence distributive algebras. In this paper this question is answered in the negative by giving an example.

## Preliminaries

An element $a$ of a complete lattice $(L, \vee, \wedge)$ is said to be a compact element of $L$ if and only if $a \leqq \bigvee_{j \in I} x_j$ (where the elements $x_j$ are in $L$) implies $a \leqq \bigvee_{j \in F} x_j$ for some finite subset $F$ of $J$.

A lattice $L$ is said to be compactly generated if every element of $L$ is a join of compact elements of $L$. A pseudo complemented semilattice (=PCS) is an algebra $(S, \wedge, *, 0, 1)$ in which $(S, \wedge, 0, 1)$ is a bounded meet semilattice and for every element $a \in S$, the element $a^* \in S$ is the pseudo complement of $a$; that is, $x \leqq a^*$ if and only if $x \wedge a = 0$. If for any PCS $S$ we write $B(S)$ for $\{x \in S \mid x^{**} = x\}$ (the set of closed elements of $S$) and $D(S)$ for $\{x \in S \mid x^{**} = 1\}$ (the set of dense elements of $S$), then $\big(B(S), \sqcup, \wedge, *, 0, 1\big)$

is a Boolean algebra, where $a \sqcup b$ is defined to be $(a^* \wedge b^*)^*$ for $a, b \in B(S)$, and $D(S)$ is a filter in $S$.

An algebra $(L; \vee, \wedge, *, 0, 1)$ is called a $p$-algebra or a pseudo complemented lattice (=PCL) if $(L, \wedge, *, 0, 1)$ is a PCS and $(L, \vee, \wedge)$ is a lattice. If $(L, \vee, \wedge)$ is a complete lattice satisfying the identity

$$\text{(D)} \qquad x \wedge \Big(\bigvee_{i\in I} y_i\Big) = \bigvee_{i\in I}(x \wedge y_i),$$

then $L$ becomes a pseudocomplemented lattice in which for any $a \in L$ we have that $a^*$ is the join of all elements of $L$ that are disjoint from $a$, i.e. $a^* = \bigvee_{x\wedge a=0} x$. In such a lattice the set $B(L)$ of all closed elements of $L$ is closed under arbitrary meet and it is therefore a complete lattice, which is complemented. The join of a subset of $B(L)$ can be computed by taking the join in $L$ and then closing it (by applying $**$), i.e, if $A \subseteq B(L)$, then $\bigvee_{B(L)} A = \big(\bigvee_L A\big)^{**}$. These facts are straightforward.

## 3. Atomic congruence lattices

In their paper [4] the authors also consider the two identities

$$\text{(I)} \qquad \Big[\bigwedge_{t\in T}\Big(\bigvee_{s\in S} x_{ts}\Big)\Big]^{**} = \Big[\bigvee_{\phi\in S^T}\Big(\bigwedge_{t\in T} x_{t\phi(t)}\Big)\Big]^{**}$$

and

$$\text{(II)} \qquad \bigwedge_{t\in T}\Big(\bigvee_{s\in S} x_{ts}\Big)^{**} = \Big[\bigvee_{\phi\in S^T}\Big(\bigwedge_{t\in T} x_{t,\phi(t)}\Big)\Big]^{**}.$$

By omitting the asterisks, we obtain the complete distributive law from both identities. By a Theorem of Tarski (see [1] Theorem V. 17) every complete, completely distributive Boolean algebra is isomorphic to the Boolean algebra of all subsets of a set, that is, it is atomic. This result is applied in [4] (Theorem 2) to show that $L$ satisfies (II) if and only if $B(L)$ is atomic. It is asked in [4] whether (I) holds in Con($A$) for every congruence distributive algebra $A$. (It is easy to see that II implies (I).) In this paper an example is given to show that (I) need not hold in Con($A$) where $A$ is a congruence distributive algebra. Before going to the example the following two useful theorems may be observed. The proof of the following theorem is well known and hence its proof is omitted.

THEOREM 1. *If $B$ is a complete Boolean algebra, then $B$ is isomorphic to $B(\mathrm{Con}(B))$ as the closed ideals are exactly the principal ones.*

THEOREM 2. *If $B$ is a complete Boolean algebra and* $\mathrm{Con}(B)$ *satisfies* (I), *then $B \cong B(\mathrm{Con}\, B)$ is atomic.*

PROOF. If $B$ is a complete Boolean algebra, then by Theorem V.16 in [1] it follows that $B$ satisfies (D). Notice that $B(\mathrm{Con}(B))$ is a sublattice of $\mathrm{Con}(B)$, hence if $x$ and $x'$ are complements in $B$, then $(x] \vee (x'] = (1]$. Now let $C$ be the set of all complementary pairs $((x],(x'])$ with $x \in B$, $\Phi$ the set of all functions $\phi$ which select from each complementary pair one member and $(p_\phi]$ the intersection of the elements in the range of $\phi$. By (I) we have

$$(1] = \Big[\bigwedge_C \big((x] \wedge (x']\big)\Big]^{**} = \Big[\bigvee_\Phi (p_\phi]\Big]^{**} = \Big(\bigvee_\Phi p_\phi\Big],$$

(where the first join is understood in $\mathrm{Con}(B)$, the last one in $B$). Since from this point on, Tarski's proof does not use complete distributivity, only the identity (D) in $B$, our proof is complete.

Thus every complete Boolean algebra, which is not atomic, is a counterexample. Now the Boolean algebra $B/J$ where $B$ is the field of all Borel sets of real numbers and $J$ is the ideal of all sets of first category in in fact an uncountable complete atom free Boolean algebra (see e.g. Theorem XI.6 in [1]).

In conclusion I highly thank the referee for his valuable comments which helped in shaping the paper to the present form.

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# SOME NOTIONS OF NONSTATIONARY MULTISTEP ITERATION PROCESSES

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## I. Introduction

Iteration processes are extremely important in solving optimization problems, linear and nonlinear equations, and, in general, they are used in all fields of applied mathematics. A very important field of such applications can be found in solving optimization problems in economy and in solving nonlinear input-output systems. In recent years, the study of optimization problems has included a substantial effort to identify properties of iteration processes that will guarantee their convergence in some sense [1], [2], [3], [4], [5], [7]. Klessig, Polak and Tishyadhigama in [6] have made a very nice comparative study of several general convergence conditions for iteration processes modeled by stationary point-to-set maps.

In this paper the convergence conditions in [6] will be generalized and extended to nonstationary multistep iteration processes [1], [8] (to be precised later). We feel that iteration processes in this general form have real practical importance. Moreover, the extension of the convergence conditions for stationary point to set maps to nonstationary multistep maps is of extreme importance. Note first that one of the most popular solvers of nonlinear equations is the secant method, which is actually a two-step process. Many dynamic economic processes are based on the selection of optimal strategies by the participants at each time period. If the optimal solution is not unique, then the strategy for the next period can be selected from the set of optimal solutions. Hence the iteration is based on a set-valued mapping. In addition, if the participants' decisions are based on extrapolative expectations on the other's behaviour, then the process becomes multistep. Time dependency of the process follows from price changes, technological development, etc. For the description of such models in oligopoly theory see Okuguchi and Szidarovszky [8].

## II. Preliminaries

We will need the following definitions.

DEFINITION 1. Denote by $\Omega$ a Hausdorff topological space that satisfies the first axiom of countability and by $\Delta \subset \Omega$ the set of desirable points.

Note that the set $\Delta$ consists of points that we will accept as "solutions" to the problem being solved by the iteration process. For example, it may consist of all points satisfying a necessary condition of optimality. $\Omega$ is sometimes taken as the set of feasible points for a problem. Thus, $\Omega$ may be a subset of a larger topological space. If this is the case, the relative topology on $\Omega$ is used.

DEFINITION 2. Let $S$ be a set, and for $k \geqq 0$ the point-to-set mappings $f(k;\cdot)$ are defined on $\Omega = S^l = S \times S \times \dots S$, and for all

$$t^{(1)}, \dots, t^{(l)} \in S$$

and $k \geqq l-1$, $f\left(k; t^{(1)}, \dots, t^{(l)}\right)$ is nonempty in $S$. Define the iteration process as

$$x_{k+1} \in f\left(k; x_{k-l+1}, x_{k-l+2}, \dots, x_k\right), \tag{1}$$

where $k \geqq l-1$, $x_0, x_1, \dots, x_{l-1} \in S$, and arbitrary element from the set can be selected as the successor of $x_k$.

First we show that this multistep process is equivalent to a certain single-step method. To show this equivalence set

$$\begin{aligned} x_k^{(1)} &= x_k, \\ x_k^{(2)} &= x_{k+1}, \\ &\vdots \\ x_k^{(l-1)} &= x_{k+l-2} \end{aligned}$$

and

$$x_k^{(l)} = x_{k+l-1}.$$

Then let us define the vector $\overline{x}_k \in \Omega$ by

$$\overline{x}_k = \left(x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(l)}\right).$$

The vectors $\overline{x}_k$ $(k \geqq 0)$ obviously satisfy the following recursion:

$$x_{k+1}^{(1)} = x_k^{(2)},$$

$$x_{k+1}^{(2)} = x_k^{(3)},$$
$$\vdots$$
$$x_{k+1}^{(l-1)} = x_k^{(l)}$$

and

$$x_{k+1}^{(l)} \in f\left(k; x_k^{(1)}, \dots, x_k^{(l)}\right), \tag{2}$$

which is a single-step process. In the following part of this paper this process will be investigated.

We now state our concept of convergence.

DEFINITION 3. We say that the algorithm model described in Definition 2 is convergent if the accumulation points of the sequence $\left\{x_{k+1}^{(l)}\right\}$ are in $\Delta$.

DEFINITION 4. A map $V : \Omega \to R_+$ is called the Liapunov map of the iteration process (2), if for arbitrary $t^{(i)} \in S$, $i = 1, 2, \dots, l$, and $y \in \in f\left(k; t^{(1)}, \dots, t^{(l)}\right)$, $k \geqq l - 1$,

$$V\left(t^{(2)}, \dots, t^{(l)}, y\right) < V\left(t^{(1)}, t^{(2)}, \dots, t^{(l)}\right).$$

We can show that the chart on p. 173 of [6] where several sufficient conditions are compared with each other is correct for nonstationary multistep iteration processes. For brevity we will only prove two theorems that generalize Theorems (3.5) and (3.9) in [6], respectively. The rest of the theorems in [6] can be generalized similarly. The details are therefore omitted.

## III. Comparison of sufficient conditions

We will need the following definition:

DEFINITION 5. The Liapunov function $V$ is said to be locally bounded from below at $\left(t^{(1)}, t^{(2)}, \dots, t^{(l)}\right)$ if there exist a neighborhood $U$ of the point $\left(t^{(1)}, t^{(2)}, \dots, t^{(l)}\right)$ and $b \in R_+$ (possibly depending on $\left(t^{(1)}, t^{(2)}, \dots, t^{(l)}\right)$) such that

$$V\left(t^{(1)}, t_1^{(2)}, \dots, t_1^{(l)}\right) \geqq b$$

for all $\left(t_1^{(1)}, t_1^{(2)}, \dots, t_1^{(l)}\right) \in U$.

CONDITIONS 1. (i) The Liapunov function $V$ is locally bounded from below on $\Omega - \Delta$;

(ii) there exists an integer $k_1 \geqq l-1$ such that

$$V\left(t^{(2)},\ldots,t^{(l)},y\right) < V\left(t^{(1)},t^{(2)},\ldots,t^{(l)}\right)$$

for all

$$y \in f\left(k;t^{(1)},t^{(2)},\ldots,t^{(l)}\right), \quad \left(t^{(1)},t^{(2)},\ldots,t^{(l)}\right) \in \Omega$$

and $k \geqq k_1$.

(iii) For each $z \in \Omega - \Delta$, if $\left\{x_k^{(l)}\right\} \subset \Omega$ is such that $x_k^{(l)} \to z$ and $V\left(x_k^{(1)},x_k^{(2)},\ldots,x_k^{(l)}\right) \to c^*$, then there exists an integer $k_2 \geqq k_1$ such that

$$V\left(x_{k_1}^{(1)},x_{k_1}^{(2)},\ldots,y\right) < c^*$$

for all

$$y \in f\left(k_2;x_{k_2}^{(1)},x_{k_2}^{(2)},\ldots,x_{k_2}^{(l)}\right).$$

THEOREM 1. *If Condition* 1 *holds, then the iteration process* (2) *is convergent (in the sense of Definition* 3*).*

PROOF. Let $z^*$ be an accumulation point of the iteration sequence $\left\{x_k^{(l)}\right\}$, $k \geqq l-1$. We assume that $z^* \in \Omega - \Delta$ and establish a contradiction. Then there exists a subsequence $\left\{x_k^{(l)}\right\}_{k\in M'} \subset \{0,1,2,\ldots\}$ such that $x_k^{(l)} \to {}_M z^*$. Without loss of generality, we can also assume that $\left\{V\left(x_k^{(1)},x_k^{(2)},\ldots,x_k^{(l)}\right)\right\}_{k\in M}$ is monotonically decreasing because of (ii). Moreover, by (i) the sequence $\left\{V\left(x_k^{(1)},x_k^{(2)},\ldots,x_k^{(l)}\right)\right\}_{k\in M}$ is bounded from below. Therefore $V\left(x_k^{(1)},x_k^{(2)},\ldots,x_k^{(l)}\right) \to {}_M c^*$. But then,

$$V\left(x_k^{(1)},x_k^{(2)},\ldots,x_k^{(l)}\right) \to c^*$$

and

$$V\left(x_k^{(1)},x_k^{(2)},\ldots,x_k^{(l)}\right) \geqq c^* \quad \text{for all} \quad k \geqq k_1.$$

But from (iii) if $z^* \in \Omega - \Delta$, then

$$V\left(x_{k_3}^{(1)},x_{k_3}^{(2)},\ldots,x_{k_3}^{(l)}\right) < c^*, \quad k_3 = k_2+1$$

which contradicts the previous inequality. That is, $z^* \in \Delta$. Thus, the proof is complete. □

CONDITION 2. (i) The Liapunov function $V$ is locally bounded from below on $\Omega - \Delta$;

(ii) there exists an integer $k_1 \geqq l - 1$ such that

$$V\left(t^{(2)}, \ldots, t^{(l)}, y\right) < V\left(t^{(1)}, t^{(2)}, \ldots, t^{(l)}\right)$$

for all

$$y \in f\left(k; t^{(1)}, t^{(2)}, \ldots, t^{(l)}\right),$$

$$\left(t^{(1)}, t^{(2)}, \ldots, t^{(l)}\right) \in \Omega \quad \text{and} \quad k \geqq k_1.$$

(iii) For each $z \in \Omega - \Delta$, if $\left\{x_k^{(l)}\right\}, \left\{y_k^{(l)}\right\} \in \Omega$, $k \geqq l - 1$ are such that $x_k^{(l)} \to z$, $y_{k+1}^{(l)} \in f\left(k; x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(l)}\right)$, $V\left(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(l)}\right) \to c^*$ and $V\left(y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(l)}\right) \to \bar{c}$, then $\bar{c} < c^*$.

THEOREM 2. *Condition* 2 *implies Condition* 1.

PROOF. To show that Condition 2 implies Condition 1 we only need to show that (iii) of Condition 1 is satisfied. Let us assume that (iii) does not hold. Then there exists $y_{k+1}^{(l)} \in f\left(k; x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(l)}\right)$ such that

$$V\left(y_{k+1}^{(1)}, y_{k+1}^{(2)}, \ldots, y_{k+1}^{(l)}\right) \geqq c^*, \quad k \geqq l - 1.$$

But then we get

$$c^* \leqq V\left(y_{k+1}^{(1)}, y_{k+1}^{(2)}, \ldots, y_{k+1}^{(l)}\right) \leqq V\left(x_{k+1}^{(1)}, x_{k+1}^{(2)}, \ldots, x_{k+1}^{(l)}\right), k \geqq l - 1.$$

Hence, $c = c^*$ which contradicts (3.8)(iii).

That completes the proof of the theorem. □

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# ON CONVERGENCE COMPLETE STRONG QUASI-METRICS

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## 1. Introduction

Throughout this note all spaces are $T_1$, $\mathbf{N}$ will denote the set of positive integers and $\mathbf{R}$ the set of real numbers. Terms and concepts which are not defined are used as in [4].

A quasi-metric on a set $X$ is a non-negative real-valued function $d$ on $X \times X$ such that, for all $x, y, z \in X$: (i) $d(x,y) = 0$ if and only if $x = y$, and (ii) $d(x,y) \leqq d(x,z) + d(z,y)$.

Each quasi-metric $d$ on $X$ induces a topology $T(d)$ on $X$ which has as a base the family of $d$-balls $\{S_d(x,r) : x \in X,\ r > 0\}$ where $S_d(x,r) = \{y \in X : d(x,y) < r\}$.

If $d$ is a quasi-metric on $X$, let $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$. Then $d^{-1}$ is also a quasi-metric on $X$. A quasi-metric $d$ on $X$ is called strong if $T(d) \subseteqq T(d^{-1})$. A topological space $(X,T)$ is said to be (strongly) quasi-metrizable if there is a (strong) quasi-metric $d$ on $X$ such that $T = T(d)$. In this case we say that $d$ is compatible with $T$.

A quasi-metric $d$ on $X$ is called equinormal [4], [9], if $d(A,B) > 0$ whenever $A$ and $B$ are two disjoint non-empty $T(d)$-closed subsets of $X$ and $d$ is called a Lebesgue quasi-metric [9] if for each $T(d)$-open cover $\mathcal{C}$ of $X$ there exists an $n \in \mathbf{N}$ such that $\{S_d(x, 2^{-n}) : x \in X\}$ refines $\mathcal{C}$. It is well-known that every Lebesgue quasi-metric is equinormal and that every equinormal quasi-metric is strong.

Each quasi-metric $d$ on $X$ induces a quasi-uniformity $\mathcal{U}(d)$ on $X$ which is generated by the base $\{U_n : n \in \mathbf{N}\}$ where $U_n = \{(x,y) : d(x,y) < 2^{-n}\}$. Following [4, page 47], a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called a $\mathcal{U}$-Cauchy filter if for each $U \in \mathcal{U}$ there is an $x \in X$ such that $U(x) \in \mathcal{F}$. A quasi-uniformity $\mathcal{U}$ on a set $X$ is called convergence complete if each $\mathcal{U}$-Cauchy filter on $X$ converges in $X$ with respect to the topology $T(\mathcal{U})$ and it is said to be complete if each $\mathcal{U}$-Cauchy filter on $X$ has a $T(\mathcal{U})$-cluster point in $X$. Clearly, each convergence complete quasi-uniformity is complete. However, the converse is not true as it is shown in [8]. A quasi-

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metric $d$ on a set $X$ is called (convergence) complete if the quasi-uniformity $\mathcal{U}(d)$ is (convergence) complete.

In the light of the classical theorem [2, Theorem 4.3.26] that every metrizable Čech complete space is completely metrizable, one can conjecture that every quasi-metrizable Čech complete space has a compatible convergence complete quasi-metric. Unfortunately, this conjecture is false. In fact, H. P. Künzi [8, Example 7] has obtained a zero-dimensional space that has a compatible complete quasi-metric but does not have a compatible convergence complete quasi-metric. By [8, Proposition 4] this space is Čech complete. In Section 2 of this paper we observe that, nevertheless, the above conjecture is true in the class of strongly quasi-metrizable spaces. We will show that a Tychonoff space has a compatible convergence complete strong quasi-metric if and only if it is a Čech complete strongly quasi-metrizable space.

In Section 3 two interesting classes of complete strong quasi-metrics, namely equinormal quasi-metrics and Lebesgue quasi-metrics, are considered. In particular we will prove that if a space has a compatible Lebesgue quasi-metric then every compatible equinormal quasi-metric is convergence complete.

## 2. Spaces having a compatible convergence complete strong quasi-metric

Following [4, page 97] we say that a cover $\mathcal{C}$ of a subset $A$ of a quasi-uniform space $(X,\mathcal{U})$ is a quasi-uniform cover of $A$ if there is $U \in \mathcal{U}$ such that $\{U(x): x \in A\}$ refines $\mathcal{C}$. A quasi-uniformity $\mathcal{U}$ on a set $X$ is a Lebesgue quasi-uniformity provided that every $T(\mathcal{U})$-open cover of $X$ is a quasi-uniform cover. An open cover of a topological space $(X,T)$ is a quasi-normal cover provided it is a quasi-uniform cover of $(X,\mathcal{U})$ for some quasi-uniformity $\mathcal{U}$ compatible with $T$. Thus an open cover $\mathcal{C}$ of $(X,T)$ is a quasi-normal cover provided there is a normal neighbornet $U$ of $(X,T)$ such that $\{U(x): x \in X\}$ refines $\mathcal{C}$ (see [4, page 5] for the notion of a normal neighbornet). We say that a space $(X,T)$ is quasi-normal if every open cover of $X$ is a quasi-normal cover. So, the next result is obvious.

Proposition 1. *A space $(X,T)$ is quasi-normal if and only if the fine quasi-uniformity of $(X,T)$ is a Lebesgue quasi-uniformity.*

In [3] Fletcher and Lindgren introduced the notion of a strongly Čech complete space. Let $\alpha = \{\mathcal{G}_i : i \in I\}$ be a collection of open covers of a space $(X,T)$. An $\alpha$-Cauchy filter base on $X$ is a filter base $\beta$ on $X$ such that for each $\mathcal{G}_i \in \alpha$ there exists $G \in \mathcal{G}_i$ and $B \in \beta$ with $B \subseteq G$. The family $\alpha$ is called (weakly) complete provided that every open $\alpha$-Cauchy filter base (has a cluster point) is a convergent filter base. Frolík [5] showed that a

Tychonoff space is Čech complete if and only if it has a countable weakly complete collection of open covers. A space $(X,T)$ is said to be strongly Čech complete [3] if it has a countable complete collection of open covers.

Now we shall prove the following result (compare with [3, Theorem 4.4]).

PROPOSITION 2. *A quasi-normal space $(X,T)$ has a compatible convergence complete quasi-metric if and only if it is a strongly Čech complete space.*

PROOF. Suppose that $(X,T)$ is a quasi-normal strongly Čech complete space. Let $\mathcal{G} = \{\mathcal{G}_n : n \in \mathbf{N}\}$ be a countable complete collection of open covers of $(X,T)$. By Proposition 1 there exists a sequence $\langle U_n \rangle$ of neighbornets of $(X,T)$ such that, for each $n \in \mathbf{N}$, $U^3_{n+1} \subseteqq U_n$ and $\{U_n(x): x \in X\}$ refines $\mathcal{G}_n$. By Kelley's Lemma [6, page 185], there exists a quasi-pseudometric $d$ on $X$ such that, for each $n \in \mathbf{N}$, $U_{n+1} \subseteqq \{(x,y): d(x,y) < 2^{-n}\} \subseteqq U_n$. Thus, for each $x \in X$ and each $n \in \mathbf{N}$, we have $U_{n+1}(x) \subseteqq S_d(x, 2^{-n})$ and, hence, $T(d) \subseteqq T$. Now we will show that $T \subseteqq T(d)$. Given $x \in X$ let $\beta = \{S_d(x, 2^{-n}): n \in \mathbf{N}\}$. Clearly, $\beta$ is an open $\mathcal{G}$-Cauchy filter base and, hence, it converges to a point $y \in X$. Then, the filter base $\{\{x\}\}$ also converges to $y$. Consequently $x = y$ and, thus, $d$ is a quasi-metric on $X$ compatible with $T$. It remains to show that $d$ is convergence complete. Let $\mathcal{F}$ be a $\mathcal{U}(d)$-Cauchy filter on $X$. Then, there is a sequence $\langle x_n \rangle$ in $X$ such that $S_d(x_n, 2^{-n}) \in \mathcal{F}$ for all $n \in \mathbf{N}$. Let $\mathcal{H}$ be the filter generated by $\{S_d(x_n, 2^{-n}): n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$ there is $G_n \in \mathcal{G}_n$ such that $S_d(x_n, 2^{-n}) \subseteqq U_n(x_n) \subseteqq G_n$. So, $\mathcal{H}$ is an open $\mathcal{G}$-Cauchy filter. Therefore it converges to a point $x \in X$ and, consequently, $\mathcal{F}$ converges to $x$. We conclude that $d$ is convergence complete. The converse follows from the well-known fact (see [3, Theorem 4.4]) that every space that has a compatible convergence complete quasi-metric is a strongly Čech complete space.

COROLLARY 2.1. *Every countably metacompact strongly Čech complete quasi-metrizable space has a compatible convergence complete quasi-metric.*

PROOF. It is proved in [4, Corollary 7.22] that every countably metacompact quasi-metrizable space is a quasi-normal space. The result follows from Proposition 2.

PROPOSITION 3. *A space $(X,T)$ has a compatible convergence complete strong quasi-metric if and only if it is a strongly Čech complete space and a strongly quasi-metrizable space.*

PROOF. Suppose that $(X,T)$ is a strongly Čech complete space that has a compatible strong quasi-metric $d_1$. Since every strongly quasi-metrizable space is semi-stratifiable and every semi-stratifiable space is countably metacompact (see, for instance, [7, page 59] and [4, page 136]) it follows from Corollary 2.1 that $(X,T)$ has a compatible convergence complete quasi-metric $d_2$. Define $d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$ for all $x, y \in X$. Then, it is immediate to show that $d$ is a convergence complete strong quasi-metric on $X$ compatible with $T$. The converse is obvious.

REMARK. It is well-known [7, Theorem 1] that a quasi-metrizable space is strongly quasi-metrizable if and only if it is developable. By Proposition 2 we then have the following reformulation of Proposition 3: A space has a compatible convergence complete strong quasi-metric if and only if it is a quasi-normal developable strongly Čech complete space.

The next corollary should be compared with the result of Roberts [10] and Zippen [16] which says that a metrizable space is completely metrizable if and only if it is a complete Moore space . A related result may be found in [4, Theorem 7.40].

COROLLARY 3.1. *A regular space has a compatible convergence complete strong quasi-metric if and only if it is a complete Moore space and a quasi-metrizable space.*

PROOF. Let $(X,T)$ be a quasi-metrizable complete Moore space. By [3, Theorem 2.13] $(X,T)$ is strongly Čech complete. Since every developable quasi-metrizable space is strongly quasi-metrizable it follows from Proposition 3 that $(X,T)$ has a compatible convergence complete strong quasi-metric. The converse is obvious.

COROLLARY 3.2. *If a regular space has a compatible complete strong quasi-metric then it has a compatible convergence complete strong quasi-metric.*

PROOF. Let $(X,T)$ be a regular space having a compatible complete strong quasi-metric $d$. For each $x \in X$ and each $n \in \mathbf{N}$ there exists an open neighborhood $V_n(x)$ of $x$ such that $\overline{V_n(x)} \subseteqq S_d(x,2^{-n})$. Put, for each $n \in \mathbf{N}$, $\mathcal{G}_n = \{V_n(x): x \in X\}$. It is easy to see that $\{\mathcal{G}_n: n \in \mathbf{N}\}$ is a development for $(X,T)$. Now suppose that $\langle F_n\rangle$ is a decreasing sequence of non-empty closed sets such that $F_n \subseteqq \overline{V_n(x_n)}$ for all $n \in \mathbf{N}$. Let $\mathcal{F}$ be the filter generated by $\langle F_n\rangle$. Then, for each $n \in \mathbf{N}$, $S_d(x_n,2^{-n}) \in \mathcal{F}$ and, thus, $\mathcal{F}$ has a cluster point in $(X,T)$. Consequently, $\cap F_n \neq \emptyset$. We conclude that $(X,T)$ is a complete Moore space. By Corollary 3.1 $(X,T)$ has a compatible convergence complete strong quasi-metric.

COROLLARY 3.3. *A Tychonoff space has a compatible convergence complete strong quasi-metric if and only if it is a Čech complete strongly quasi-metrizable space.*

PROOF. Every Čech complete strongly quasi-metrizable Tychonoff space is strongly Čech complete [3, Corollary 2.13]. The result follows from Proposition 3. The converse is obvious.

EXAMPLE 1. It is well-known that the Niemytzki plane is a Čech complete space. Moreover it follows from [4, Example 5.17 and Corollary 7.24] that it is a strongly (non-archimedeanly) quasi-metrizable space. Therefore Corollary 3.3 shows that this space has a compatible convergence complete strong quasi-metric.

EXAMPLE 2. Let $(X,T)$ be the Dieudonné example [1] of a Tychonoff locally compact non-normal topological space. Then $(X,T)$ is a Čech complete space. On the other hand Stoltenberg [15] showed that this space is strongly quasi-metrizable. Therefore, Corollary 3.3 shows that it has a compatible convergence complete strong quasi-metric.

Certainly, the main problem in the area of strongly quasi-metrizable spaces is the question whether these spaces are orthocompact spaces (or equivalently, whether these spaces have a compatible strong non-archimedean quasi-metric). On the other hand we observe that the spaces of the above examples actually have a compatible convergence complete non-archimedean quasi-metric. It then seems that the following weaker question is also open: Let $(X,T)$ be a space admitting a compatible convergence complete strong quasi-metric. Is it an orthocompact space?

## 3. Equinormal and Lebesgue quasi-metrics

In [8, Proposition 6] Künzi proves that every equinormal quasi-metric is complete. He also gives [8, Example 5] an example of an equinormal quasi-metric $d$ on a set $X$ that is not convergence complete. However, it is not hard to see that the topological space $\big(X,T(d)\big)$ of Künzi's example is a strong Čech complete space.

A slight modification of the proof of Proposition 3 permits us to state the following result.

PROPOSITION 4. *Let $(X,T)$ be a space having a compatible equinormal quasi-metric. Then $(X,T)$ has a compatible convergence complete equinormal quasi-metric if and only if it is strongly Čech complete.*

The above observations suggest the following open question: Let $(X,T)$ be a space having a compatible equinormal quasi-metric. Is it a strongly Čech complete space? (A characterization of spaces which have a compatible equinormal quasi-metric is given in [11].)

It is noted in [8, Remark 2] that if $d$ is an equinormal quasi-metric on a set $X$ such that $\big(X,T(d)\big)$ is a Hausdorff space then $d$ is convergence complete. Since every Hausdorff space which has a compatible equinormal quasi-metric also has a compatible Lebesgue (quasi-)metric, our next result is a slight improvement of [8, Remark 2]. In the proof we use ideas of [8, Proposition 6] and the fact [12], [13], that a quasi-metrizable space $(X,T)$ has a compatible Lebesgue quasi-metric if and only if the set $X'$ of the nonisolated points of $X$ is compact.

PROPOSITION 5. *Let $(X,T)$ be a space having a compatible Lebesgue quasi-metric. Then every equinormal quasi-metric on $X$ compatible with $T$ is convergence complete.*

PROOF. Let $d$ be an equinormal quasi-metric on $X$ compatible with $T$ and let $\mathcal{F}$ be a $\mathcal{U}(d)$-Cauchy filter on $X$. Then there exists a sequence $\langle x_n \rangle$ of (distinct) points of $X$ such that $S_d(x_n, 2^{-n}) \in \mathcal{F}$ for all $n \in \mathbf{N}$. Suppose, firstly, that $\mathcal{F}$ has a finite element $F$. Then there exists an $x \in F$ and a subsequence $\langle x_{n(k)} \rangle$ of $\langle x_n \rangle$ such that $x \in S_d\left(x_{n(k)}, 2^{-n(k)}\right)$ for all $k \in \mathbf{N}$. Hence $\langle x_{n(k)} \rangle$ is $T(d^{-1})$-convergent to $x$ and, thus, $T(d)$-convergent to $x$. By the triangle inequality, $\mathcal{F}$ converges to $x$. Now suppose that each element in $\mathcal{F}$ is infinite. In this case, if $\langle x_n \rangle$ has a convergent subsequence, the triangle inequality permits us to show that the filter $\mathcal{F}$ converges. If the sequence $\langle x_n \rangle$ has no cluster point, we can suppose, without loss of generality, that each $x_n$ is an isolated point. Construct a sequence $\langle a_n \rangle$ in $X$ such that $a_1 \in S_d\left(x_1, 2^{-1}\right) \setminus \{x_1\}$ and $a_n \in S_d(x_n, 2^{-n}) \setminus \{x_1, \ldots, x_n, a_1, \ldots, a_{n-1}\}$ for all $n > 1$. Since each $x_n$ is isolated, it follows that $\overline{\{a_n : n \in \mathbf{N}\}} \cap \{x_n : n \in \mathbf{N}\} = \emptyset$. However, $d(x_n, a_n) < 2^{-n}$ for all $n \in \mathbf{N}$, a contradiction because $d$ is equinormal. Consequently, every $\mathcal{U}(d)$-Cauchy filter on $X$ is convergent.

The following example shows that the converse of the above Proposition is not true.

EXAMPLE 3. Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences of distinct points such that $A \cap B = \emptyset$ where $A = \{a_n : n \in \mathbf{N}\}$ and $B = \{b_n : n \in \mathbf{N}\}$. Let $X = A \cup B$ and define a quasi-metric $p$ on $X$ as follows

$$p(a_n, b_m) = 1/m \qquad \text{for all} \quad n, m \in \mathbf{N}$$

$$p(a_n, a_m) = 1 \quad \text{if} \quad n \neq m$$

$$p(b_n, x) = 1 \quad \text{if} \quad x \neq b_n$$

and

$$p(x, x) = 0 \quad \text{for all} \quad x \in X.$$

It is easy to see that $p$ is an equinormal quasi-metric on $X$. However the set $X'$ of the nonisolated points in $T(p)$ is not $T(p)$-compact. So $\left(X, T(p)\right)$ does not have a compatible Lebesgue quasi-metric. We want to show that if $d$ is an equinormal quasi-metric on $X$ compatible with $T(p)$ then it is convergence complete. Let $\mathcal{F}$ be a $\mathcal{U}(d)$-Cauchy filter on $X$. Then there is a sequence $\langle x_n \rangle$ of distinct points of $X$ such that $S_d(x_n, 2^{-n}) \in \mathcal{F}$ for all $n \in \mathbf{N}$. First assume that the sequence $\langle x_n \rangle$ has a cluster point $x$. Then, similarly to Proposition 5, the filter $\mathcal{F}$ converges to $x$. Second consider that $\langle x_n \rangle$ has no cluster point. Then we can suppose, without loss of generality, that $x_n \in A$ for all $n \in \mathbf{N}$. Put $F = \{x_n : n \in \mathbf{N}\}$. Note that if there exists a subsequence $\langle x_{n(k)} \rangle$ of $\langle x_n \rangle$ such that $d\left(x_{n(k)}, F \setminus \{x_{n(k)}\}\right) < 2^{-n(k)}$ for all

$k \in \mathbf{N}$, then we can obtain two subsequences $\langle x_{n(k_j)} \rangle$ and $\langle x'_{n(k_j)} \rangle$ of $\langle x_{n(k)} \rangle$ such that $\{ x_{n(k_j)}: j \in \mathbf{N} \} \cap \{ x'_{n(k_j)}: j \in \mathbf{N} \} = \emptyset$ and $d(x_{n(k_j)}, x'_{n(k_j)}) \to 0$, a contradiction. Hence we assume, without loss of generality, that $d(x_n, F \setminus \setminus \{x_n\}) \geqq 2^{-n}$ for all $n \in \mathbf{N}$. We now have two cases: (a) for every $k \in \in \mathbf{N}$ there is $y_k \in A$ such that $y_k \in S_d(x_k, 2^{-k}) \setminus \{x_k\}$; (b) there is $k \in \mathbf{N}$ such that $S_d(x_k, 2^{-k}) \cap A = \{x_k\}$. Assume (a). In this case, $x_m \neq y_k$ for all $m, k \in \mathbf{N}$ because $0 < d(x_k, x_m) - d(x_k, y_k) \leqq d(y_k, x_m)$. Thus, $F$ and $H = \{y_m : m \in \mathbf{N}\}$ are disjoint closed sets with $d(F, H) = 0$, a contradiction. Now assume (b). Then there is $k_0 \in \mathbf{N}$ such that $S_d(x_k, 2^{-k}) = \{x_k\} \cup B_k$ with $\{ b_n : n \geqq n(k) \} \subseteqq B_k \subseteqq B$. Since, for each $m \in \mathbf{N}$,

$$S_d\left(x_k, 2^{-k}\right) \bigcap \left[ \bigcap \{ S_d(x_{j+s}, 2^{-(j+s)}) : 1 \leqq s \leqq m \} \right] \in \mathcal{F}$$

it follows that there exists a strictly increasing sequence $\langle n(m) \rangle$ in $\mathbf{N}$ such that $\{ b_n : n \geqq n(m) \} \in \mathcal{F}$. Therefore $\mathcal{F}$ converges to every point in $A$. We conclude that $d$ is convergence complete.

Our next result extends the classical Niemytzki–Tychonoff theorem that a metrizable space is compact if and only if every compatible metric is complete to spaces having a compatible Lebesgue quasi-metric.

Let $(X, T)$ be a quasi-metrizable space and denote by $X'$ the set of the nonisolated points of $X$. Given a quasi-metric $d$ on $X$ compatible with $T$, define

$$d'(x, y) = \min\{ 1, d(x, y) \} \quad \text{if} \quad x \in X'$$

$$d'(x, y) = 1 \quad \text{if} \quad x \in X \setminus X' \quad \text{and} \quad x \neq y$$

and

$$d'(x, x) = 0 \quad \text{for all} \quad x \in X.$$

Clearly, $d'$ is a quasi-metric on $X$ compatible with $T$. We will say that $d'$ is the $X'$-associated quasi-metric to $d$.

It is well-known [4, Theorem 7.35] that a quasi-metrizable space $(X, T)$ is compact if and only if every quasi-metric on $X$ compatible with $T$ is (convergence) complete. We here prove the following extension.

PROPOSITION 6. *A quasi-metrizable space $(X, T)$ has a compatible Lebesgue quasi-metric if and only if for each quasi-metric on $X$ compatible with $T$, its $X'$-associated quasi-metric is (convergence) complete.*

PROOF. Suppose that $(X, T)$ has a compatible Lebesgue quasi-metric and let $d$ be a quasi-metric on $X$ compatible with $T$. If $\mathcal{F}$ is a $\mathcal{U}(d')$-Cauchy filter on $X$, then there exists a sequence $\langle x_n \rangle$ in $X$ such that $S_d(x_n, 2^{-n}) \in \in \mathcal{F}$ for all $n \in \mathbf{N}$. Assume that there is a subsequence $\langle x_{n(k)} \rangle$ of $\langle x_n \rangle$ such that $x_{n(k)} \in X \setminus X'$ for all $k \in \mathbf{N}$. Then all terms of $\langle x_{n(k)} \rangle$ coincide and, clearly, $\mathcal{F}$ converges. Otherwise there is $n_0 \in \mathbf{N}$ such that $x_n \in X'$ for all $n \geqq n_0$. Hence the sequence $\langle x_n \rangle$ has a cluster point $x \in X'$. It follows that $\mathcal{F}$ converges to $x$. Conversely, suppose that the set $X'$ is not compact. Then there exists a sequence $\langle x_n \rangle$ in $X'$ without cluster point. Let $p$ be a quasi-metric on $X$ compatible with $T$ such that $p \leqq 1$. Define, for each $n \in \mathbf{N}$, $A_n = \{x_m : m \geqq n\}$ and $d_n : X \times X \to \mathbf{R}$ by $d_n(x,y) = p(x,y)$ if $x, y \in X \setminus \overline{A}_n$, $d_n(x,y) = 1$ if $x \in X \setminus \overline{A}_n$ and $y \in \overline{A}_n$ and $d_n(x,y) = 0$ if $x \in \overline{A}_n$.

Now put $d(x,y) = \sup\{2^{-n} d_n(x,y) : n \in \mathbf{N}\}$ for all $x, y \in X$. Then it follows from [14, Lemma 2] that $d$ is a quasi-metric on $X$ compatible with $T$ such that the filter $\mathcal{F}$ generated by $\{A_n : n \in \mathbf{N}\}$ is a $\mathcal{U}(d)$-Cauchy filter. Consequently $\mathcal{F}$ is a $\mathcal{U}(d')$-Cauchy filter without cluster point. This contradiction concludes the proof.

Finally we observe that one can easily obtain an analogue of the preceding proposition for metrizable spaces. The necessary condition is an immediate consequence of it and the sufficiency should be derived from Hausdorff–Dugundji's extension theorem (see [2, Problem 4.5.20(c)]).

PROPOSITION 7. *A metrizable space $(X,T)$ has a compatible Lebesgue metric if and only if for each metric on $X$ compatible with $T$, its $X'$-associated quasi-metric is (convergence) complete.*

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# UNIQUENESS RESULTS FOR UNBOUNDED SOLUTIONS OF FIRST ORDER NON-LINEAR DIFFERENTIAL-FUNCTIONAL EQUATIONS

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## Introduction

Uniqueness results for first order differential equations are well known if boundedness of the domain or the solutions is assumed. Basic methods of uniqueness proofs were developed by Szarski [13], see also [10]. There are a lot of papers and books where the authors extend Szarski's results in a way. Let us mention only few of them, for instance [4] where uniqueness for unbounded solutions is shown, and [8] where Szarski's ideas are extended to the case of equations with functional variable. Particular results for equations with a special kind of functional dependence were established in [8]. Uniqueness and existence results for parabolic equations demand assumptions about a class of solutions, namely they assume that the solutions and their derivatives grow at most as $\exp\left(c\|x\|^2\right)$, cf. [9]. In this paper Besala's and Krzyżański's ideas of dealing with classes of unbounded functions are adapted to first order equations for which any restrictions were unnatural until there was no functional variable. This enables us to obtain uniqueness in a way which was done by Szarski. Let us notice that our results, written for one equation, can be easily proved for weakly coupled systems.

Now we formulate the differential-functional problem.

Let $E_0 = [-\tau_0, 0] \times \mathbf{R}^n$, $E = [0, a] \times \mathbf{R}^n$, $D = [-\tau_0, 0] \times [-\tau, \tau]$, where $a, \tau_0 \in \mathbf{R}_+$, $a > 0$, $\tau = (\tau_1, \ldots, \tau_n) \in \mathbf{R}^n_+$. Denote by $C(X, Y)$ the set of all continuous functions defined on $X$ taking values in $Y$. $X, Y$ are non-empty metric spaces. If $z \in C(E_0 \cup E, \mathbf{R})$ and $(t, x) \in E$ then $z_{(t,x)}: D \to \mathbf{R}$ is defined by $z_{(t,x)}(s, y) = z(t + s, x + y)$ for $(s, y) \in D$. Assume that $f: E \times \times \mathbf{R} \times C(D, \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}$ and $\varphi: E_0 \to \mathbf{R}$ are given functions.

We consider the problem

$$D_t z(t, x) = f\left(t, x, z(t, x), z_{(t,x)}, D_x z(t, x)\right), \tag{0.1}$$

$$z(t, x) = \varphi(t, x), \quad (t, x) \in E_0, \tag{0.2}$$

where $D_x z(t, x) = \left(D_{x_1} z(t, x), \ldots, D_{x_n} z(t, x)\right)$. We are concerned with classical solutions of problem (0.1)–(0.2), i.e. functions $u \in C(E_0 \cup E, R)$

having partial derivatives $D_t u$, $D_x u$ on $(0,a] \times \mathbf{R}^n$, and satisfying the differential equation on $(0,a] \times \mathbf{R}^n$ and the initial condition on $E_0$.

The differential inequalities methods seem to be basic tools in all investigations of solutions of initial-value problems for non-linear differential or differential-functional equations in partial derivatives of the first order. Uniqueness theorems require assumptions about an estimation of the right-side increase and about some regularity of the solutions. Uniqueness criteria are obtained as consequences of suitable comparison theorems for differential or differential-functional inequalities. In order to illustrate these methods consider two examples.

EXAMPLE 1. Let $X = \Big\{(t,x) \in \mathbf{R}^{1+n}: t \in (0,a),\ |x_i| \leqq c_i - M_i t,\ i = 1,\dots,n\Big\}$, $X_0 = [-\tau_0, 0] \times [-c,c]$, where $c = (c_1,\dots,c_n) \in \mathbf{R}^n_+$, $M_i > 0$, $c_i - M_i a \geqq 0$, $i = 1,\dots,n$. Any function $z: X_0 \cup X \to \mathbf{R}$ is said to be of class $\mathcal{D}$ iff $z \in C(X_0 \cup X, R)$, the derivatives $D_t z(t,x)$, $D_x z(t,x)$ exist for $(t,x) \in X$, and $z$ is differentiable on the set $\operatorname{Fr} X \cap \{(0,a) \times \mathbf{R}^n\}$. For $z \in C(X_0 \cup X, \mathbf{R})$ denote by $Tz: [-\tau_0, a) \to \mathbf{R}_+$ the function given by $(Tz)(t) = \max\Big\{|z(t,x)| : (t,x) \in X_0 \cup X\Big\}$ for $t \in [-\tau_0, a)$. Then $Tz \in C\big([-\tau_0,a), \mathbf{R}_+\big)$. Assume that the function $u: X_0 \cup X \to \mathbf{R}$ of class $\mathcal{D}$ satisfies the differential-functional inequality

$$(0.3)\qquad |D_t u(t,x)| \leqq \sigma\Big(t, |u(t,x)|, Tu\Big) + \sum_{i=1}^n M_i |D_{x_i} u(t,x)|,$$

for $(t,x) \in X$, where $\sigma: [0,a) \times \mathbf{R}_+ \times C\big([-\tau_0,a), \mathbf{R}_+\big) \to \mathbf{R}_+$. Suppose that for every $\eta \in C\big([-\tau_0, 0)\mathbf{R}_+\big)$ there exists an upper solution to the problem

$$(0.4)\qquad y'(t) = \sigma\big(t, y(t,y), y\big), \qquad y(t) = \eta(t) \quad \text{for} \quad t \in [-\tau_0, 0].$$

Let $\eta(t) = \max\Big\{|u(t,x)| : x \in [-c,c]\Big\}$, $t \in [-\tau_0, 0]$. Then under natural assumptions on $\sigma$ (see [7]) we have $|u(t,x)| \leqq \omega(t,\eta)$ for $(t,x) \in X_0 \cup X$, where $\omega(\cdot,\eta)$ is an upper solution of problem (0.4).

Consider the following Cauchy problem in the local version:

$$(0.5)\qquad D_t z(t,x) = F\big(t, x, z(t,x), z, D_x z(t,x)\big), \quad (t,x) \in X,$$

$$(0.6)\qquad z(t,x) = \psi(t,x), \quad (t,x) \in X_0,$$

where $F: X \times \mathbf{R} \times C(X_0 \cup X, \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}$ and $\psi: X_0 \to \mathbf{R}$. Assume that $F$ satisfies the following Volterra condition:

$$\begin{cases} \text{if } (t,x) \in X,\ (p,q) \in \mathbf{R}^{1+n},\ z, \bar z \in C(X_0 \cup X, R),\ z(s,y) = \bar z(s,y) \text{ for} \\ -\tau_0 \leqq s \leqq t,\ (s,y) \in X_0 \cup X, \text{ then } F(t,x,p,z,q) = F(t,x,p,\bar z,q). \end{cases}$$

If the estimation

$$\left| F(t,x,p,z,q) - F(t,x,\overline{p},\overline{z},q)\right| \leqq \sigma\big(t,|p-\overline{p}|,T(z-\overline{z})\big) + \sum_{i=1}^{n} M_i|q-\overline{q}_i|$$

is satisfied on $X \times \mathbf{R} \times C(X_0 \cup X, \mathbf{R}) \times \mathbf{R}^n$, and the upper solution to problem (0.4) for $\eta(t) = 0$, $t \in [-\tau_0, 0]$, is $y(t) = 0$, $t \in [-\tau_0, a)$, then problem (0.5)–(0.6) has at most one solution of class $\mathcal{D}$ on $X_0 \cup X$, see [7].

Thus we have obtained a uniqueness criterion of Perron type. Similarly, there is a uniqueness theorem with a comparison function of Kamke type, see [1].

It is easily seen that the above considerations do not cover our problem (0.1)–(0.2).

EXAMPLE 2. Suppose that $F: E \times \mathbf{R} \times C(E_0 \cup E, \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}$ and $\psi: E_0 \to \mathbf{R}$. Consider problem (0.5)–(0.6), which is now global with respect to $x$. The Volterra condition is modified there as follow:

($\mathcal{V}$)

$$\begin{cases} \text{if } z,\overline{z} \in C(E_0 \cup E, \mathbf{R}), \text{ and } z(s,y) = \overline{z}(s,y) \quad \text{for } \tau_0 \leqq s \leqq t, \\ \|y\| \leqq \|x\|, \text{ then } F(t,x,p,z,q) = F(t,x,p,\overline{z},q) \text{ for all } (p,q) \in \mathbf{R}^{1+n}. \end{cases}$$

Assume that the function $u: E_0 \cup E \to \mathbf{R}$ has partial derivatives on $E$ (we do not assume that $u$ is differentiable) and satisfies (instead of the comparison condition (0.3)) the inequality

$$\left| D_t u(t,x)\right| \leqq \sigma\Big(t, \left|u(t,x)\right|, \mathcal{T}_{(t,x)}u\Big) + \sum_{i=1}^{n} M_i \left| D_{x_i} u(t,x)\right|, \tag{0.7}$$

for $(t,x) \in E$, where $\mathcal{T}_{(t,x)}: [-\tau_0, a] \to \mathbf{R}_+$ is given by $(\mathcal{T}_{(t,x)}u)(s) = \max\Big\{ \left| u(s,y)\right| : \|y\| \leqq \|x\| \Big\}$ for $s \in [-\tau_0, a]$, and $\sigma: [0,a] \times \mathbf{R}_+ \times C\big([-\tau_0, a], \mathbf{R}_+\big) \to \mathbf{R}_+$ satisfies additionally the Lipschitz condition with respect to the last two variables. If $\left|u(t,x)\right| \leqq \eta(t)$ for $(t,x) \in E_0$, and the right-side solution $\omega(\cdot,\eta)$ of (0.4) exists on $[0,a]$, then $\left|u(t,x)\right| \leqq \omega(t,\eta)$ for $(t,x) \in E$.

As a consequence of the above consideration we get a uniqueness theorem ([8]) which is obtained under the assumption that $F$ satisfies condition ($\mathcal{V}$), and

$$\left| F(t,x,p,z,q) - F(t,x,\overline{p},\overline{z},\overline{q})\right| \leqq$$

$$\leqq K|p-\overline{p}| + M \max\Big\{ \big| z(s,y) - \overline{z}(s,y) \big| : -\tau_0 \leqq s \leqq t,\ \|y\| \leqq \|x\| \Big\} +$$

$$+\sum_{i=1}^{n} M_i \, |q_i - \overline{q}_i|$$

on $E \times \mathbf{R} \times C(E_0 \cup E, \mathbf{R}) \times \mathbf{R}^n$, where $K, M, M_i \in \mathbf{R}_+$, $i = 1, \dots, n$. This uniqueness result is proved as a natural generalization of [7], where partial differential equations were considered.

There exist simple natural examples of equations with do not come with the above uniqueness criterion.

Consider the equation (with an integral dependence)

$$D_t z(t,x) = G\big(t, x, z(t,x), \int_D z(t+s, x+y)\, ds\, dy, D_x z(t,x)\big), \tag{0.8}$$

where $G: E \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$. This equation does not satisfy a Volterra condition of type $(\mathcal{V})$, and it can be specified as a particular case of (0.1). It is enough to define

$$f(t,x,p,w,q) = G\left( t, x, p, \int_D w(s,y)\, ds\, dy, q \right)$$

$$\text{for} \quad (t,x,p,w,q) \in E \times \mathbf{R} \times C(E_0 \cup E, \mathbf{R}) \times \mathbf{R}^n.$$

Now we consider another equation (with a retarded variable):

$$D_t z(t,x) = G\Big(t, x, z(t,x), z\big(\alpha(t,x), \beta(t,x)\big), D_x z(t,x)\Big), \tag{0.9}$$

where $-\tau_0 \leqq \alpha(t,x) \leqq t$ and $\beta(t,x) \in \mathbf{R}^n$ for $(t,x) \in E$.

Studying equation (0.9) we come to the conclusion that condition $(\mathcal{V})$ implies $\big\|\beta(t,x)\big\| \leqq \|x\|$ for each $x \in \mathbf{R}^n$, whereas our equation (0.1) contains a relatively wide class of equations without such restrictions. It will be easier to check it if we define the function $f$ in (0.1) by

$$f(t,x,p,w,q) = G\Big(t, x, p, w\big(\alpha(t,x) - t, \beta(t,x) - x\big), q\Big)$$

for $(t,x,p,w,q) \in E \times \mathbf{R} \times C(E_0 \cup E, \mathbf{R}) \times \mathbf{R}^n$.

REMARK. There are numerous approaches to construct some appropriate models in many branches of science and technology which lead to differential equations. In order to describe the reality in a dependence in many circumstances as exactly as it is possible they must often permit non-linearity and appearance of an integral or retarded functional coefficients in deduced differential equations.

In [2] they reduce some problems of non-linear dispersive laser optics to quasilinear hyperbolic integral-differential systems, see also [3]. Let us list some further examples. Systems of differential equations containing an operator acting on an unknown density of populations in dependence on their age, size, DNA content and so on, are considered in [12]. An equation with a deviated variable ([5]) describes a density of households at time $t$, depending on their estates, in the theory of the distribution of wealth. Another system of integro-differential equations appears in modern research in biology in order to investigate an age-dependent epidemic of a disease with vertical transmission, ([6]). The authors of [11] consider also first order differential-functional equations motivated by applications.

## 1. Basic assumption and notations

In this short section we introduce some useful classes of functions and needed assumptions.

$H \in \mathcal{H}$ iff $H \in C\big(E_0 \cup E, (0,\infty)\big)$ and $H|_{E_0}$ and $H|_E$ are continuously differentiable, and

1° $H(\cdot, x)$ is non-decreasing for every $x = (x_1, \dots, x_n) \in \mathbf{R}^n$,

2° $x_i D_{x_i} H(t,x) \geqq 0$ for $i = 1, \dots, n$, $(t,x) \in E_0 \cup E$, $x = (x_1, \dots, x_n)$.

If $\mathcal{L} \in C(E, \mathbf{R}_+)$, $L_0, L_1, L_2 \in \mathbf{R}_+$ then $H \in \mathcal{H}_0(\mathcal{L}; L_0, L_1, L_2)$ iff $H \in \mathcal{H}$ and for $(t,x) \in E$ we have

$$(1.1)\quad D_t H(t,x) \geqq (L_0 + L_1) H(t,x) + \mathcal{L}(t,x) \big\| H_{(t,x)} \big\|_D + L_2 \big\| D_x H(t,x) \big\|,$$

where $D_x H(t,x) = \big(D_{x_1} H(t,x), \dots, D_{x_n} H(t,x)\big)$.

If $H \in \mathcal{H}$ then $z \in \mathcal{C}_H$ iff $z \in C(E_0 \cup E, \mathbf{R})$ and there is $\varepsilon \in C(E_0 \cup E, \mathbf{R}_+)$ such that $\big|z(t,x)\big| \leqq \varepsilon(t,x) H(t,x)$ for $t, x \in E_0 \cup E$, and $\varepsilon(t,x) \to 0$ as $\|x\| \to \infty$.

The class $\mathcal{C}_H$ is equipped with semi-norm $\|\cdot\|_{(t)}$ defined by

$$(1.2)\qquad \|z\|_{(t)} = \sup\Big\{ \big|z(s,y)\big| H^{-1}(s,y) \colon (s,y) \in E_0 \cup E,\ s \leqq t \Big\},$$

where $t \in [0,a]$.

We denote by $\|w\|_D$ the supremum norm of $w \in C(D, \mathbf{R})$. Denote $\Omega^{(0)} = E \times \mathbf{R} \times C(D, \mathbf{R}) \times \mathbf{R}^n$.

If $\mathcal{L} \in C(E, \mathbf{R}_+)$, $L_0, L_1, L_2 \in \mathbf{R}_+$ then $f \in \operatorname{Lip}(\Omega^{(0)}; \mathcal{L}, L_1, L_2)$ iff $f \in C(\Omega^{(0)}, \mathbf{R})$ and we have

$$
\begin{aligned}
(1.3) \qquad & \left| f(t,x,p,w,q) - f(t,x,\overline{p},\overline{w},\overline{q}) \right| \leqq \\
& \leqq L_1 |p - \overline{p}| + \mathcal{L}(t,x) \|w - \overline{w}\|_D + L_2 \|q - \overline{q}\|
\end{aligned}
$$

for all $(t,x,p,w,q), (t,x,\overline{p},\overline{w},\overline{q}) \in \Omega^{(0)}$.

Let $f \in \operatorname{Lip}(\Omega^{(0)}; \mathcal{L}, L_1, L_2)$ and $\varphi \in C(E_0, \mathbf{R})$. We will assume that the Cauchy problem (0.1)–(0.2) considered in our paper has a solution defined on $E_0 \cup E$.

## 2. The maximum principle and uniqueness result

Now we will prove the maximum principle for problem (0.1).

THEOREM 2.1. *Assume that*

1) $H \in \mathcal{H}_0(\mathcal{L}; L_0, L_1, L_2)$ *and* $f \in \operatorname{Lip}(\Omega^{(0)}; \mathcal{L}, L_1, L_2)$, *where* $L_0 > 0$, $L_1, L_2 \in \mathbf{R}_+$ *and* $\mathcal{L} \in C(E, \mathbf{R}_+)$,

2) *the functions* $u, \overline{u} \in \mathcal{C}_H$ *are solutions to problem* (0.1).

*Then we have*

$$
(2.1) \qquad \|u - \overline{u}\|_{(t)} \leqq \|u - \overline{u}\|_{(0)}, \quad t \in [0, a].
$$

PROOF. Denote $v(t,x) = u(t,x) - \overline{u}(t,x)$ for $(t,x) \in E_0 \cup E$, and $\omega(t,x) = v(t,x)\big(H(t,x)\big)^{-1}$. Of course, $v \in \mathcal{C}_H$. Thus $\left|\omega(t,x)\right| \leqq \varepsilon(t,x) \to 0$ as $\|x\| \to \infty$, where $\varepsilon \in C(E_0 \cup E, \mathbf{R}_+)$. Then there exists $(t^*, x^*) \in E_0 \cup E$ such that $\left|\omega(t,x)\right| \leqq \left|\omega(t^*,x^*)\right|$ for $(t,x) \in E_0 \cup E$. Either $\omega$ or $-\omega$ takes its maximum at the point $(t^*, x^*)$.

Assume that $(t^*, x^*) \in E \setminus E_0$, then

$$
(2.2) \qquad \omega(t^*,x^*) D_t \omega(t^*,x^*) \geqq 0,
$$

and

$$
(2.3) \qquad D_{x_i} \omega(t^*,x^*) = 0, \quad i = 1, \ldots, n.
$$

It is easy to get

$$
(2.4) \quad D_{x_i}\omega(t^*,x^*) = \big(H(t^*,x^*)\big)^{-1} \left[ D_{x_i} v(t^*,x^*) - \omega(t^*,x^*) D_{x_i} H(t^*,x^*) \right], \quad i = 1, \ldots, n.
$$

From (2.3) and (2.4) we obtain

$$D_{x_i}v(t^*,x^*) = \omega(t^*,x^*)D_{x_i}H(t^*,x^*), \quad i=1,\dots,n. \tag{2.5}$$

From (2.2) and $\omega(t,x)H(t,x) = v(t,x)$ we have

$$\omega(t^*,x^*)D_t v(t^*,x^*) \geqq \big(\omega(T^*,x^*)\big)^2 D_t H(t^*,x^*). \tag{2.6}$$

As $u$ and $\overline{u}$ are solutions of (0.1) we have

$$\begin{aligned}|D_t v(t^*,x^*)| = \Big| &f\big(t^*,x^*,u(t^*,x^*),u_{(t^*,x^*)},D_x u(t^*,x^*)\big) - \\ &- f\big(t^*,x^*,\overline{u}(t^*,x^*),\overline{u}_{(t^*,x^*)},D_x\overline{u}(t^*,x^*)\big)\Big|.\end{aligned}$$

By (1.3) this implies the inequality

$$\begin{aligned}&|D_t v(t^*,x^*)| \leqq \\ &\leqq L_1|v(t^*,x^*)| + \mathcal{L}(t^*,x^*)\|v_{(t^*,x^*)}\|_D + L_2\|D_x v(t^*,x^*)\|.\end{aligned} \tag{2.7}$$

From (2.5), (2.6), (2.7) we have

$$\begin{aligned}&\big(\omega(t^*,x^*)\big)^2 D_t H(t^*,x^*) \leqq \\ &\leqq \big(\omega(t^*,x^*)\big)^2\Big[L_1, H(t^*,x^*) + \mathcal{L}(t^*,x^*)\|H_{(t^*,x^*)}\|_D + \\ &\qquad + L_2\|D_x H(t^*,x^*)\|\Big],\end{aligned} \tag{2.8}$$

because

$$\|v_{(t^*,x^*)}\|_D = \|(\omega H)_{(t^*,x^*)}\|_D \leqq \|\omega_{(t^*,x^*)}\|_D\|H_{(t^*,x^*)}\|_D \tag{2.9}$$

and

$$\|\omega_{(t^*,x^*)}\|_D = |\omega(t^*,x^*)|. \tag{2.10}$$

Conditions (1.1) and (2.8) imply

$$\begin{aligned}&\big(\omega(t^*,x^*)\big)^2 D_t H(t^*,x^*) \leqq \\ &\leqq \big(\omega(t^*,x^*)\big)^2\big[D_t H(t^*,x^*) - L_0 H(t^*,x^*)\big].\end{aligned} \tag{2.11}$$

It follows from (2.11) that $\omega(t^*,x^*) = 0$, and then the maximum is taken on $E_0$, and condition (2.1) holds true. This finishes the proof.

Let us observe that continuous dependence on initial conditions and uniqueness are easily obtained from Theorem 2.1,

COROLLARY 2.1. *Suppose that*
1) *the assumptions of Theorem* 2.1 *are satisfied,*
2) $\left|u(t,x)-\overline{u}(t,x)\right| \leqq \mu H(t,x)$ *for* $(t,x) \in E_0$.
*Then* $\left|u(t,x)-\overline{u}(t,x)\right| \leqq \mu H(t,x)$ *for* $(t,x) \in E_0 \cup E$.

COROLLARY 2.2. *Suppose that*
1) *assumption* 1) *of Theorem* 2.1 *is satisfied,*
2) $u, \overline{u} \in \mathcal{C}_H$ *are solutions to problem* (0.1)–(0.2).
*Then* $u(t,x) = \overline{u}(t,x)$ *for* $(t,x) \in E_0 \cup E$.

## 3. Some modification of the theorem on uniqueness

One can consider another kind of functional dependence dealing with the Volterra function $f$ and without any trouble we obtain the maximum principle and uniqueness result assuming existence.

If $L, L_0, L_1, L_2 \in \mathbf{R}_+$ then $H \in \mathcal{H}_1(L, L_0, L_1, L_2)$ iff $H \in \mathcal{H}$ and for $(t,x) \in E$ we have

$$D_t H(t,x) \geqq (L + L_0 + L_1) H(t,x) + L_2 \left\| D_x H(t,x) \right\|. \tag{3.1}$$

Let $\Omega_H^{(1)} = E \times \mathbf{R} \times \mathcal{C}_H \times \mathbf{R}^n$. If $L, L_0, L_1, L_2 \in \mathbf{R}_n$ then $f \in \operatorname{Lip}\left(\Omega_H^{(1)}; L, L_1, L_2\right)$ iff $f \in C\left(\Omega_H^{(1)}, \mathbf{R}\right)$ and we have

$$\begin{aligned} &\left| f(t,x,p,z,q) - f(t,x,\overline{p},\overline{z},\overline{q}) \right| \leqq \\ &\leqq L_1 |p - \overline{p}| + L H(t,x) \|z - \overline{z}\|_{(t)} + L_2 \|q - \overline{q}\| \end{aligned} \tag{3.2}$$

for all $(t,x,p,w,q), (t,x,\overline{p},\overline{w},\overline{q}) \in \Omega_H^{(1)}$.

For $f \in \operatorname{Lip}\left(\Omega_H^{(1)}; L, L_1, L_2\right)$ and $\varphi \in C(E_0, \mathbf{R})$ we will consider the following Cauchy problem:

$$D_t z(t,x) = f\big(t, x, z(t,x), z, D_x z(t,x)\big), \tag{3.3}$$

$$z(t,x) = \varphi(t,x), \quad (t,x) \in E_0. \tag{3.4}$$

In a similar way as in Theorem 2.1 we can prove the maximum principle for problem (3.3).

THEOREM 3.1. *Assume that*
1) $H \in \mathcal{H}_1(L, L_0, L_1, L_2)$, $f \in \operatorname{Lip}\left(\Omega_H^{(1)}; L, L_1, L_2\right)$, *where* $L, L_0, L_1,$

$L_2 \in \mathbf{R}_+$, $L_0 > 0$,

2) *the functions* $u, \overline{u} \in C_H$ *are solutions to problem* (3.3).

*Then we have*

$$\|u - \overline{u}\|_{(t)} \leqq \|u - \overline{u}\|_{(0)}, \quad t \in [0, a]. \tag{3.5}$$

We omit the proof.

As a consequence of Theorem 3.1 we obtain continuous dependence on initial conditions and uniqueness for problem (3.3) and (3.4).

## 4. Examples of functions $H, \mathcal{L}$

Uniqueness criteria for differential-functional equations are based either on comparison theorems ([4], [7], [8], [13], see also Examples 1 and 2) or on the maximum principle. In comparison theorems the existence of solutions to some initial value problems for ordinary differential-functional equations (cf. (0.4)) is assumed. The maximum principle we have proved here works under the assumption on the existence of solutions to the partial differential-functional inequality (1.1) or (3.1). This is the main difference between the results from [7], [8] and our Theorems 2.1 and 3.1. To study more about them we give some examples of functions $H$ and $\mathcal{L}$ which satisfy conditions (1.1) or (3.1).

Let $\Gamma: \mathbf{R}_+ \to \mathbf{R}_+$ and $\kappa, \psi: [-\tau_0, a] \to (0, \infty)$ be continuously differentiable functions such that $\kappa'(t) \geqq 0$, $\psi'(t) \geqq 0$ for $t \in [0, a]$; $\kappa'(t) = 0$, $\psi'(t) = 0$ for $t \in [-\tau_0, 0]$; $\Gamma'(t) \geqq 0$ for $t \in \mathbf{R}_+$.

We define

$$H(t, x) = \Gamma\left(\psi(t)\sqrt{1 + \|x\|^2}\right) \tag{4.1}$$

for $(t, x) \in E_0 \cup E$, and

$$\mathcal{L}(t, x) = \overline{p}/\Gamma\left(\kappa(t)\sqrt{1 + \|x\|^2}\right) \tag{4.2}$$

for $(t, x) \in E$, where $\overline{p} \in \mathbf{R}_+$.

The first-order partial derivatives of $H$ are

$$\begin{cases} D_t H(t, x) = \Gamma'\left(\psi(t)\sqrt{1 + \|x\|^2}\right) \psi'(t)\sqrt{1 + \|x\|^2}, \\ D_{x_i} H(t, x) = \Gamma'\left(\psi(t)\sqrt{1 + \|x\|^2}\right) \psi(t)\dfrac{x_1}{\sqrt{1 + \|x\|^2}}, \ i = 1, \dots, n, \end{cases} \tag{4.3}$$

for $(t, x) \in E$.

Consider the inequalities

$$D_tH(t,x) \geqq K_0H(t,x) + \mathcal{L}(t,x)\|H_{(t,x)}\|_D + K_1\|D_xH(t,x)\| \tag{4.4}$$

for $(t,x) \in E$, where $K_0, K_1 \in \mathbf{R}_+$ and $\mathcal{L} \in C(E, \mathbf{R}_+)$, and

$$D_tH(t,x) \geqq K_0H(t,x) + KH(t,x)\|H\|_{(t)} + K_1\|D_xH(t,x)\| \tag{4.5}$$

for $(t,x) \in E$, where $K_0, K, K_1 \in \mathbf{R}_+$. These correspond to (1.1) and (3.1) respectively. We need not give any examples of functions $H$ satisfying inequality (4.5) because $\|H\|_{(t)} = 1$ (compare definition (2.1)) and it is in fact easy to solve it as a differential inequality without functional variables. We are concerned with the differential-functional inequality (4.4).

LEMMA 4.1. *Suppose that*
1) $\psi: [-\tau_0, a] \to (0,\infty)$ *satisfies*

$$\begin{aligned} &k\psi'(t)/\psi(t) \geqq \\ &\geqq K_0 + \overline{p}\big(\kappa(t)\big)^{-\beta}\left[1 + \frac{1}{2}\|\tau\|\left(\|\tau\| + \sqrt{4+\|\tau\|^2}\right)\right]^{k/2} + \frac{1}{2}K_1k \end{aligned} \tag{4.6}$$

*for* $t \in [0,a]$, *where* $\beta, k \in \mathbf{R}_+$, *and* $\kappa(t) > 0$ *for* $t \in [0,a]$,
2) $\Gamma(t) = t^k$ *for* $t \in \mathbf{R}_+$, *where* $k > 0$ *is fixed,*
3) $H$ *is defined by* (4.1) *and* $\mathcal{L}(t,x) = \overline{p}\left(\kappa(t)\sqrt{1+\|x\|^2}\right)^{-\beta}$ *for* $(t,x) \in E$.
*Then* $H \in \mathcal{H}$ *and inequality* (4.4) *is satisfied.*

PROOF. From (4.3) we obtain

$$\begin{cases} D_tH(t,x) = \psi'(t)k\big(\psi(t)\big)^{k-1}\big(1+\|x\|^2\big)^{k/2}, \\ D_{x_i}H(t,x) = k\big(\psi(t)\big)^k x_i\big(1+\|x\|^2\big)^{k/2-1}, \quad i = 1,\dots,n. \end{cases} \tag{4.7}$$

Moreover, for $(t,x) \in E$ we have

$$\begin{aligned} &\mathcal{L}(t,x)\|H_{(t,x)}\|_D \leqq \\ &\leqq \overline{p}\big(\psi(t)\big)^k\big(\kappa(t)\big)^{-\beta}\Big(1 + \big(\|x\| + \|\tau\|\big)^2\Big)^{k/2}\big(1+\|x\|^2\big)^{-\beta/2}. \end{aligned} \tag{4.8}$$

Of course we have

$$
(4.9)\qquad \begin{cases} \|x\|/\left(1+\|x\|^2\right) \leqq \dfrac{1}{2}, \\ \left(1+\left(\|x\|+\|\tau\|\right)^2\right)^{k/2}\left(1+\|x\|^2\right)^{-(k+\beta)/2} \leqq \\ \leqq \left[1+\dfrac{1}{2}\|\tau\|\left(\|\tau\|+\sqrt{4+\|\tau\|^2}\right)\right]^{k/2} \end{cases}
$$

for $x \in \mathbf{R}_+$. Then (4.4) follows from (4.6) and from (4.7)–(4.9). This finishes the proof.

LEMMA 4.2. *Suppose that $H$ is defined by* (4.1) *and we have*

$$
(4.10)\qquad \begin{cases} \psi'(t) \geqq (\beta_0 + K_1/2)\psi(t), & t \in [0,a], \\ \Gamma'(t) \geqq \dfrac{K_0+K}{t\beta_0}\Gamma(t), & t \in [\psi(0),\infty), \end{cases}
$$

*where $\beta_0 > 0$.*

*Then $H \in \mathcal{H}$ and inequality* (4.5) *is satisfied.*

PROOF. Using (4.3) we have that (4.5) is equivalent to

$$
(4.11)\qquad \Gamma'\left(\psi(t)\sqrt{1+\|x\|^2}\right)\left[\psi'(t)\sqrt{1+\|x\|^2} - K_1\psi(t)\frac{\|x\|}{\sqrt{1+\|x\|^2}}\right] \geqq
$$
$$
\geqq \Gamma\left(\psi(t)\sqrt{1+\|x\|^2}\right)(K_0+K).
$$

Condition (4.11) can be rewritten as

$$
(4.12)\qquad \Gamma'\left(\psi(t)\sqrt{1+\|x\|^2}\right)\psi(t)\sqrt{1+\|x\|^2}\times
$$
$$
\times\left\{\left[\psi'(t)-\psi(t)\Big(\beta_0+K_1\|x\|/\left(1+\|x\|^2\right)\Big)\right]+\psi(t)\beta_0\right\} \geqq
$$
$$
\geqq \Gamma\left(\psi(t)\sqrt{1+\|x\|^2}\right)(K_0+K)\psi(t)
$$

for $(t,x) \in E$. Condition (4.12) is a simple consequence of (4.10).

LEMMA 4.3. *Suppose that*

1) *the function* $H$ *is defined by* (4.1), *where* $\Gamma(t) = e^{t^k}$, $t \in \mathbf{R}_+$, *with fixed* $k > 0$,

2) $\mathcal{L}(t,x) \leqq \overline{p}\exp\left(-\left(\kappa(t)\sqrt{1+\|x\|^2}\right)^{k-1}\right)\left(\sqrt{1+\|x\|^2}\right)^k$ *for* $(t,x) \in$ $\in E$, *where* $\overline{p} \in \mathbf{R}_+$.

*Then* $H \in \mathcal{H}$ *and* $H$ *satisfies* (4.4) *if any of the conditions* (a), (b) *or* (c) *listed below holds true.*

(a) *Either* $0 < k \leqq 1$ *and* $\left(\kappa(t)\right)^{k-1} \geqq k\|\tau\|\left(\psi(t)\right)^k, t \in [0,a]$, *or* $1 < k$ *and*

$$\left(\kappa(t)\right)^{k-1} \geqq k\left(\psi(t)\right)^k\|\tau\|\left[1+\frac{1}{2}\|\tau\|\left(\|\tau\|+\sqrt{4+\|\tau\|^2}\right)\right]^{(k-1)/2},$$
$$t \in [0,a];$$

*and*

$$k\psi'(t)\left(\psi(t)\right)^{k-1} \geqq K_0 + \frac{1}{2}kK_1 + \overline{p}, \quad t \in [0,a]. \tag{4.13}$$

b) $1 < k \leqq 3$ *and*

$$\left(\kappa(t)\right)^{k-1} > k\psi(t)\|\tau\|\left(\sqrt{1+\|\tau\|^2}\right)^{k-3}\max\{1,(k-1)^{-1}\}$$
$$t \in [0,a],$$

*and*

$$k\psi'(t)\left(\psi(t)\right)^{k-1} \geqq K_0 + \frac{1}{2}kK_1\left(\psi(t)\right)^k + \tag{4.14}$$
$$+\overline{p}\exp\left\{-\left(\kappa(t)\right)^{k-1} + k\left(\psi(t)\right)^k\|\tau\|\left(1+\left(\|\tau\|+r_0(t)\right)^2\right)^{(k-1)/2}\right\},$$

*for* $t \in [0,a]$, *where* $r_0(t)$ *is given by*

$$r_0(t) = \tag{4.14'}$$
$$= \frac{k\left(\psi(t)\right)^k\left(\|\tau\|^2+\|\tau\|\right)\left(1+\|\tau\|^2\right)^{(k-3)/2}\max\{1,(k-1)^{-1}\}}{\left(\kappa(t)\right)^{k-1} - k\left(\psi(t)\right)^k\|\tau\|\left(1+\|\tau\|^2\right)^{(k-3)/2}\max\{1,(k-1)^{-1}\}}.$$

(c) $3 < k$ *and*

$$\big(\kappa(t)\big)^{k-1} > k\big(\psi(t)\big)^{k}\|\tau\|\left[\frac{1}{2}\left(\|\tau\| + \sqrt{4+\|\tau\|^2}\right)\right]^{k-3}\max\{1,(k-1)^{-1}\}$$

*for* $t \in [0,a]$, *and*

$$k\psi'(t)\big(\psi(t)\big)^{k-1} \geqq K_0 + \frac{1}{2}kK_1\big(\psi(t)\big)^{k} + \tag{4.15}$$
$$+\bar{p}\exp\left\{-\big(\kappa(t)\big)^{k-1} + k\big(\psi(t)\big)^{k}\|\tau\|\Big(1+\big(r_0(t)+\|\tau\|\big)^2\Big)^{(k-1)/2}\right\},$$

*for* $t \in [0,a]$, *where* $r_0(t)$ *is given by*

$$r_0(t) = \frac{k\big(\psi(t)\big)^{k}\big(\|\tau\|^2+\|\tau\|\big)\big(\varepsilon_0(t)\big)^{k-3}\max\{1,(k-1)^{-1}\}}{\big(\kappa(t)\big)^{k-1} - k\big(\psi(t)\big)^{k}\|\tau\|\big(\varepsilon_0(t)\big)^{k-3}\max\{1,(k-1)^{-1}\}}, \tag{4.16}$$
$$\varepsilon_0(t) = \frac{1}{2}\left(\|\tau\| + \sqrt{4+\|\tau\|^2}\right).$$

PROOF. Condition (4.4) follows from (4.3) and assumption 2) when the following inequality holds:

$$k\psi'(t)\big(\psi(t)\big)^{k-1} \geqq \tag{4.17}$$
$$\geqq K_0\big(1+\|x\|^2\big)^{-k/2} + kK_1\big(\psi(t)\big)^{k}\|x\|/\big(1+\|x\|^2\big) +$$
$$+\bar{p}\exp\left\{-\left(\kappa(t)\sqrt{1+\|x\|^2}\right)^{k-1} + \right.$$
$$\left. +\big(\psi(t)\big)^{k}\left[\Big(1+\big(\|x\|+\|\tau\|\big)^2\Big)^{k/2} - \big(1+\|x\|^2\big)^{k/2}\right]\right\}$$

where $(t,x) \in E$.

If we assume (a) then (4.17) follows from (4.13) and from

$$-\big(\kappa(t)\big)^{k-1}\big(1+\|x\|^2\big)^{(k-1)/2} + \big(\psi(t)\big)^{k}\Big[\Big(1+\big(\|x\|+\|\tau\|\big)^2\Big)^{k/2} - \tag{4.18}$$
$$-\big(1+\|x\|^2\big)^{k/2}\Big] \leqq 0, \quad (t,x) \in E.$$

Indeed, from the mean value theorem we have

$$(4.19)\qquad \Big(1+\big(\|x\|+\|\tau\|\big)^2\Big)^{k/2}-\big(1+\|x\|^2\big)^{k/2}=$$
$$=k\|\tau\|\big(\|x\|+\Theta\|\tau\|\big)\Big(1+\big(\|x\|+\Theta\|\tau\|\big)^2\Big)^{k/2-1},$$

where $\Theta\in(0,1)$. If $0<k\leqq 1$ then from (4.19) we have

$$(4.20)\quad \Big(1+\big(\|x\|+\|\tau\|\big)^2\Big)^{k/2}-\big(1+\|x\|^2\big)^{k/2}\leqq k\|\tau\|\big(1+\|x\|^2\big)^{(k-1)/2},$$

and inequality (4.18) follows immediately from $\kappa^{k-1}\geqq\|\tau\|\psi^k$.

If $1<k$ then (4.18) is a simple consequence of the condition

$$\big(\kappa(t)\big)^{k-1}\geqq k\big(\psi(t)\big)^k\|\tau\|\Big[1+\frac{1}{2}\|\tau\|\Big(\|\tau\|+\sqrt{4+\|\tau\|^2}\Big)\Big]^{(k-1)/2},$$

because (4.9) holds.

If we assume (b) then (4.17) follows from (4.14) and from

$$(4.21)\quad -\big(\kappa(t)\big)^{k-1}\big(1+\|x\|^2\big)^{(k-1)/2}+\big(\psi(t)\big)^k\Big[\Big(1+\big(\|x\|+\|\tau\|\big)^2\Big)^{k/2}-$$
$$-\big(1+\|x\|^2\big)^{k/2}\Big]\leqq-\big(\kappa(t)\big)^{k-1}+k\big(\psi(t)\big)^k\|\tau\|\Big(1+\big(\|\tau\|+r_0(t)\big)^2\Big)^{(k-1)/2}$$

for $t\in[0,a]$, where $r_0(t)$ for $t\in[0,a]$ is defined by (4.14′).

Inequality (4.21) holds for $\|x\|\leqq r_0(t)$, and the left-hand side of the inequality (4.21) is non-increasing with respect to $\|x\|$ for $\|x\|\geqq r_0(t)$, where $r_0(t)$ is defined by (4.14′).

If we assume (c) then inequality (4.21) with $r_0(t)$ defined by (4.16) holds for the same reason as in case (b). This finishes the proof.

Define $e_0(t)=t$, $e_{i+1}(t)=\exp\big(e_i(t)\big)$, $i=0,1,\dots$, $t\in\mathbf{R}$, $E_i(t)=e_i'(t)$ for $i=0,1,\dots$, $t\in\mathbf{R}$.

LEMMA 4.4. *Suppose that*

1) $H,\mathcal{L}$ *are defined by* (4.1), (4.2), *where*

$$(4.22)\qquad \Gamma(t)=e_m(t),\quad t\in\mathbf{R}_+,\quad \textit{with fixed}\quad m\in\{2,3,\dots,\},$$

2) *for* $t\in[0,a]$ *we have* $\kappa(t)>\psi(t)$ *and*

$$(4.23)\qquad \psi'(t)\geqq K_0(E_{m-1}\big(\psi(t)\big)^{-1}+\frac{1}{2}K_1\psi(t)+$$

$$+\overline{p}\exp\left\{e_{m-1}\left(\psi(t)\sqrt{1+\left(r_0(t)+\|\tau\|\right)^2}\right)-\right.$$

$$\left.-e_{m-1}\left(\kappa(t)\right)-\sum_{j=0}^{m-1}e_j\left(\psi(t)\right)\right\},$$

*where*

(4.24)

$$r_0(t)=\max\left\{\left(\|\tau\|+\sqrt{\|\tau\|^2\left(c_0(t)\right)^2-\left(\left(c_0(t)\right)^2-1\right)^2}\right)\left(\left(c_0(t)\right)^2-1\right)^{-1},\right.$$

$$\left.\left(\sqrt{\|\tau\|^2+4}-\|\tau\|\right)/2\right\},\ \textit{when}\ \|\tau\|>\left(\left(c_0(t)\right)^2-1\right)/c_0(t),$$

$$r_0(t)=\left(\sqrt{1+\|\tau\|^2}-\|\tau\|\right)/2,\ \textit{when}\ \|\tau\|\leqq\left(\left(c_0(t)\right)^2-1\right)/c_0(t),$$

$$c_0(t)=\kappa(t)/\psi(t).$$

*Then $H\in\mathcal{H}$ and* (4.4) *holds true.*

PROOF. From (4.22) and (4.1), (4.2) condition (4.4) is a consequence of the following:

$$E_m\left(\psi(t)\sqrt{1+\|x\|^2}\right)\psi'(t)\sqrt{1+\|x\|^2}\geqq \tag{4.25}$$

$$\geqq K_0e_m\left(\psi(t)\sqrt{1+\|x\|^2}\right)+K_1E_m\left(\psi(t)\sqrt{1+\|x\|^2}\right)\psi(t)\frac{\|x\|}{\sqrt{1+\|x\|^2}}+$$

$$+\overline{p}\exp\left\{-e_{m-1}\left(\kappa(t)\sqrt{1+\|x\|^2}\right)+e_{m-1}\left(\psi(t)\sqrt{1+\left(\|x\|+\|\tau\|\right)^2}\right)\right\}.$$

Condition (4.25) will be established by (4.23) if we prove that for $r\geqq 0$ we have

$$\text{(4.26)}\quad \phi(r)=-e_{m-1}\left(\kappa(t)\sqrt{1+r^2}\right)+e_{m-1}\left(\psi(t)\sqrt{1+\left(r+\|\tau\|\right)^2}\right)-$$

$$-\sum_{j=0}^{m-1}e_j\left(\psi(t)\sqrt{1+r^2}\right)\leqq$$

$$\leqq e_{m-1}\left(\psi(t)\sqrt{1+\left(r_0(t)\|\tau\|\right)^2}\right)-e_{m-1}\left(\kappa(t)\right)-\sum_{j=0}^{m-1}e_j\left(\psi(t)\right).$$

To do this let us observe that

$$\kappa(t)\sqrt{1+r^2}\geqq\psi(t)\sqrt{1+\left(r+\|\tau\|\right)^2}$$

for $r\geqq r_0(t)$, $r_0(t)$ defined by (4.24), moreover $\phi'(r)\leqq 0$ for $r\geqq r_0(t)$, because in this case the following inequalities hold:

$$\phi'(r)=-E_{m-1}\left(\kappa(t)\sqrt{1+r^2}\right)\frac{\kappa(t)r}{\sqrt{1+r^2}}+ \tag{4.27}$$

$$+E_{m-1}\left(\psi(t)\sqrt{1+\left(r+\|x\|\right)^2}\right)\frac{\psi(t)\left(r+\|\tau\|\right)}{\sqrt{1+\left(r+\|\tau\|\right)^2}}-$$

$$-\sum_{j=0}^{m-1}E_j\left(\psi(t)\sqrt{1+r^2}\right)\frac{\psi(t)r}{\sqrt{1+r^2}}\leqq$$

$$\leqq-\left\{E_{m-1}\left(\kappa(t)\sqrt{1+r^2}\right)\kappa(t)\sqrt{1+r^2}-\right.$$

$$\left.-E_{m-1}\left(\psi(t)\sqrt{1+\left(r+\|x\|\right)^2}\right)\psi(t)\sqrt{1+\left(r+\|x\|\right)^2}\right\}\frac{r}{1+r^2}-$$

$$-\psi(t)\sqrt{1+\left(r+\|x\|\right)^2}E_{m-1}\left(\psi(t)\sqrt{1+\left(r+\|x\|\right)^2}\right)\times$$

$$\times\left\{\frac{r}{1+r^2}-\frac{r+\|\tau\|}{1+\left(r+\|\tau\|\right)^2}\right\},$$

and

$$\frac{r}{1+r^2}-\frac{r+\|\tau\|}{1+\left(r+\|\tau\|\right)^2}\geqq 0 \tag{4.28}$$

for $r\geqq r_0(t)$. Thus, the maximum of $\phi$ must be taken at a point $r\in[0,r_0(t)]$, and estimation (4.26) holds true. This finishes the proof.

EXAMPLE 4.1. There are functions $\Gamma$ which grow faster than $e_m$. Let $\widetilde{\Gamma}(m)=e_m(m)$ for $m=0,1,\dots$, and $\widetilde{\Gamma}(r)=(r-m)\widetilde{\Gamma}(m+1)+(m+1-r)\widetilde{\Gamma}(n)$ for $m\leqq r\leqq m+1$, $m=0,1,\dots$. We define $\Gamma(r)$ for $r\in\mathbf{R}_+$ by

$$\Gamma(r) = \exp\left(\int_0^r \widetilde{\Gamma}(t+1)\,dt\right). \tag{4.29}$$

Let us observe that $\Gamma$ is increasing and $\Gamma(m) \geqq e_m(m)$ for $m = 1, 2, \ldots$. Thus, for every $m = 1, 2, \ldots$ we have $e_m(r)/\Gamma(r) \to 0$ as $r \to 0$. If we define $H, \mathcal{L}$ by (4.1),(4.2) with $\Gamma$ given by (4.29) then condition (4.4) will follow from the inequality

$$\begin{gathered}\psi'(t) \geqq K_0 + K_1 \frac{\psi(t)\|x\|}{1+\|x\|^2} + \\ +\overline{p}\exp\left(\int_{\kappa(t)\sqrt{1+\|x\|^2}}^{\psi(t)\sqrt{1+(\|x\|+\|\tau\|)^2}} \widetilde{\Gamma}(s+1)\,ds - \int_0^{\psi(t)\sqrt{1+\|x\|^2}} \widetilde{\Gamma}(s+1)\,ds\right).\end{gathered} \tag{4.30}$$

Let us observe that

$$\begin{gathered}-\widetilde{\Gamma}\left(\kappa(t)\sqrt{1+r^2}+1\right)\frac{\kappa(t)r}{\sqrt{1+r^2}} + \\ +\widetilde{\Gamma}\left(\psi(t)\sqrt{1+\left(r+\|\tau\|\right)^2}+1\right)\frac{\psi(t)\left(r+\|\tau\|\right)}{\sqrt{1+\left(r+\|\tau\|\right)^2}} \leqq 0\end{gathered} \tag{4.31}$$

for $r \geqq r_0(t)$, where $r_0(t)$ is defined by (4.24) with $c_0(t) = \kappa(t)/\psi(t) > 1$, $t \in [0, a]$. From (4.31) we obtain

$$\begin{gathered}\int_{\kappa(t)\sqrt{1+\|x\|^2}}^{\psi(t)\sqrt{1+(\|x\|+\|\tau\|)^2}} \widetilde{\Gamma}(s+1)\,ds - \int_0^{\psi(t)\sqrt{1+\|x\|^2}} \widetilde{\Gamma}(s+1)\,ds \leqq \\ \leqq \int_{\kappa(t)}^{\psi(t)\sqrt{1+(\|\tau\|+r_0(t))^2}} \widetilde{\Gamma}(s+1)\,ds - \int_0^{\psi(t)} \widetilde{\Gamma}(s+1)\,ds.\end{gathered} \tag{4.32}$$

and in order to obtain (4.30) from (4.31) and (4.32) we have to assume that

$$\begin{gathered}\psi'(t) \geqq K_0 + \frac{1}{2}K_1\psi(t) + \\ +\overline{p}\exp\left\{\int_{\kappa(t)}^{\psi(t)\sqrt{1+\left(\|\tau\|+r_0(t)\right)^2}} \widetilde{\Gamma}(s+1)\,ds - \int_0^{\psi(t)} \widetilde{\Gamma}(s+1)\,ds\right\}\end{gathered} \tag{4.33}$$

for $t \in [0, a]$.

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# A CONSISTENCY RESULT CONCERNING SET MAPPINGS

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## 0. Introduction

A *set mapping* is a function $f\colon [\kappa]^n \to [\kappa]^\lambda$ or $f\colon [\kappa]^n \to [\kappa]^{<\lambda}$ for some cardinals $\kappa$, $\lambda$, and natural number $n$. A subset $X \subseteqq \kappa$ is *free* (or $f$-*free*), if for different $x_1, \ldots, x_n, y \in X$, $y \notin f(x_1, \ldots, x_n)$ always holds. The most important problem in the theory of set mappings is how large free sets can be guaranteed, depending on $\kappa$, $\lambda$, and $n$. Several results have been proved by Sierpiński, Erdős, Hajnal, Máté, and others; see the excellent exposition [3].

Here we focus on the problem when $n = 2$. Erdős and Hajnal observed that if $\kappa > 2^\omega$, $f\colon [\kappa]^2 \to [\kappa]^{<\omega}$ is a set mapping, then there is an uncountable free set. Hajnal and Máté proved that it is consistent that $2^\omega = \omega_2$ and there is no uncountable free set for some $f\colon [\omega_2]^2 \to [\omega_2]^{<\omega}$. They asked if a similar statement can be proved for $\omega_3$. This is what we prove.

An easy argument (see Section 4) gives that it suffices to give a set mapping $f\colon [\omega_3]^2 \to [\omega_3]^{\aleph_0}$ with no uncountable free sets. We try to force this by countable approximations but the $\omega_2$-c.c. poses problems. To overcome the difficulty we use a (variant of a) forcing technique of Baumgartner–Shelah. Sections 2, 3 are devoted to the description of this technique (up to Lemma 3.3).

NOTATION. We identify cardinals with initial ordinals. If $S$ is a set, $\kappa$ a cardinal, $[S]^\kappa = \{X \subseteqq S\colon |X| = \kappa\}$, $[S]^{<\kappa} = \{X \subseteqq S\colon |X| < \kappa\}$. If $(S,<)$ is an ordered set, for $A$, $B \subseteqq S$, $A < B$ means that $x < y$ holds whenever $x \in A$, $y \in B$. We write $A < x$ for $A < \{x\}$, etc.

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## 1. The main forcing construction

In this section we show that if CH holds and a function $T\colon [\omega_3]^2 \to [\omega_3]^{\aleph_1}$ with some properties exists, then, in a certain forcing extension, there is a set mapping $H\colon [\omega_3]^2 \to [\omega_3]^{\aleph_0}$ with no uncountable free sets.

DEFINITION 1.1. Given $T\colon [\omega_3]^2 \to [\omega_3]^{\aleph_1}$, a forcing condition $q \in Q = Q(T)$ is a triplet $q = (s, f, \mathcal{F})$ such that $s$ is a countable subset of $\omega_3$, $f\colon [s]^2 \to [s]^{\aleph_0}$, $f(\alpha,\beta) \subseteqq T(\alpha,\beta)$ $(\alpha,\beta \in s)$, $\mathcal{F}$ is a countable set, every element of $\mathcal{F}$ is either

a subset $F \subseteqq s$, of limit type, $f$-free, for no $x > F$, $F \cup \{x\}$ is $f$-free,

or else

a pair $(F^0, F^1)$ with $F^0, F^1$ non-empty, $F^0 < F^1$, $F^0$ of limit type, $F^0 \cup F^1$ $f$-free, but for no $x \in s$, with $F^0 < x < F^1$ remains $F^0 \cup \{x\} \cup F^1$ $f$-free.

We order $Q$ as follows. $(s', f', \mathcal{F}') \leqq (s, f, \mathcal{F})$ iff $s' \supseteqq s$, $f' \supseteqq f$, $\mathcal{F}' \supseteqq \mathcal{F}$.

LEMMA 1.1. *$(Q, \leqq)$ is $\omega_1$-closed.*

PROOF. Straightforward. $\square$

The proof of the following statement will be given in Section 3.

LEMMA 1.2. *It is consistent that $2^\omega = \omega_1$ and there is a function $T\colon [\omega_3]^2 \to [\omega_3]^{\aleph_1}$ such that $(Q(T), \leqq)$ is $\omega_2$-c.c.*

When $G \subseteqq Q$ is generic, put

$$X = \bigcup\{s\colon (s, f, \mathcal{F}) \in G\}, \qquad H = \bigcup\{f\colon (s, f, \mathcal{F}) \in G\}.$$

LEMMA 1.3. $|X| = \omega_3$.

PROOF. As $Q$ is $\omega_3$-c.c. , there is an $\alpha < \omega_3$, such that if $q = (s, f, \mathcal{F}) \in Q$ with $s \cap \alpha = \emptyset$, then for every $\beta > \alpha$, there is a $q' = (s', f', \mathcal{F}')$ compatible with $q$, $s' \cap \beta = \emptyset$. If we remove the part up to $\alpha$ from $Q$, we get a poset satisfying the claim. $\square$

LEMMA 1.4. *$H\colon [X]^2 \to [X]^{\aleph_0}$ has no free set of size $\aleph_1$.*

PROOF. Assume that $q$ forces that $Y \subseteqq X$ is an uncountable free set. We can assume that $q$ forces that $Y$ is maximal and determines if $Y$ is of order type $\omega_1$. If $Y$ is of type $\omega_1$, by $\omega_1$-closure, we can assume that $q = (s, f, \mathcal{F})$ determines $s \cap Y$, which is the set of the first $\delta$ elements of $Y$, for some limit $\delta < \omega_1$. Let $F$ be this set, $s \cap Y$. Then $q' = (s, f, \mathcal{F} \cup \{F\})$ is a condition, extending $q$, forcing $Y \subseteqq \sup(F)$, a contradiction.

If $q$ forces that $Y$ is of type $> \omega_1$, we may assume that $q = (s, f, \mathcal{F})$ determines $\gamma$, the limit of the first $\omega_1$ elements of $Y$, determines $s \cap Y$, and forces that $s \cap \gamma \cap Y$ is the set of the first $\delta$ elements of $Y$, for some limit $\delta < \omega_1$. Put $F^0 = s \cap Y \cap \gamma$, $F^1 = (s \cap Y) - \gamma$. Then $(s, f, \mathcal{F} \cup \{(F^0, F^1)\}) \leqq q$ forces that $\sup(Y \cap \gamma) = \sup(F^0)$, a contradiction. □

## 2. The Baumgartner–Shelah forcing construction

In this section we describe a forcing notion which is an inessential variation of the historic forcing presented in [1], Section 9.

We are going to build a certain poset $P$. Every element $p$ of $P$ will be a set of size $\leqq \omega_1$ of functions $h$ such that $\mathrm{Dom}\,(h) = [a]^2$ for some $a \in [\omega_3]^{\leqq \aleph_1}$. In this case we write $a = \mathrm{supp}\,(h)$. We require that for $\alpha < \beta$, in the domain of $h$, $\alpha \notin h(\alpha, \beta) \subseteqq \beta$, $|h(\alpha, \beta)| \leqq \aleph_1$. The elements of $p$ must be compatible as functions, and one of them must extend all; $\bigcup p \in p$. This function is denoted as $\mathrm{base}\,(p)$, and let $\mathrm{supp}\,(p)$ be $\mathrm{supp}\,(\mathrm{base}\,(p))$.

Not all conditions of the above type will be in $P$, we put $P = \bigcup\{P_\alpha : \alpha < \omega_2\}$ where $P_\alpha$ is constructed by induction on $\alpha$.

$\{h\} \in P_0$ iff $\mathrm{Dom}\,(h) = \{\xi, \zeta\}$ for some $\xi < \zeta < \omega_3$, $h(\xi, \zeta) = \emptyset$.

$p \in P_{\alpha+1}$ iff there are $q$, $r \in P_\alpha$, $\mathrm{supp}\,(q) = a \cup b$, $\mathrm{supp}\,(r) = a \cup c$, $a < b < c$, there is an isomorphism between $(a \cup b, <, q)$ and $(a \cup c, <, r)$, and $p = q \cup r \cup \{f\}$ where $f \supseteqq \bigcup q$, $\bigcup r$ is such that $f(\xi, \zeta) = \big((a \cup b \cup c) \cap \zeta\big) - \{\xi\}$ for $\xi \in b$, $\zeta \in c$. In this case we say that $p$ is obtained by amalgamating $q$ and $r$.

If $\alpha < \omega_2$ is limit, $p \in P_\alpha$ iff for some limit $\eta < \omega_2$, $p_\xi \in P_{\alpha_\xi}$ ($\xi < \eta$, $\alpha_\xi < \alpha$) $p_\xi$ decreasing, $p$ contains the functions in the union of the $p_\xi$-s together with their union. Formally:

$$p = \bigcup\{p_\xi : \xi < \eta\} \cup \Big\{\bigcup\bigcup\{p_\xi : \xi < \eta\}\Big\}.$$

We put $p \leqq q$ iff $\mathrm{base}\,(q) \in p$ and $q = \{h \in p : h \subseteqq \mathrm{base}\,(q)\}$.

LEMMA 2.1. *If $p \in P_{\alpha+1}$ and $p$ is obtained by amalgamating $q$ and $r \in P_\alpha$, then $p \leqq q$, $r$.*

PROOF. Obvious. □

LEMMA 2.2. *If $2^{\omega_1} = \omega_2$, then $(P, \leqq)$ has the $\omega_3$-c.c.*

PROOF. By Lemma 2.1 and a $\Delta$-system argument. □

LEMMA 2.3. *If $p$, $q$, $r \in P$, $p \leqq q, r$ and* $\mathrm{base}\,(r) \in q$, *then $q \leqq r$.*

PROOF. Easy. □

LEMMA 2.4. *If $p$, $q$, $r \in P$, $p \leqq q$, $p \neq q$, $p \in P_\alpha$, $q \in P_\beta$, then $\beta < \alpha$.*

PROOF. By induction on $\alpha$. For $\alpha = 0$, the statement is vacuously true.

If $\alpha = \beta + 1$, and $p$ is obtained by amalgamating $p_1$ and $p_2$ then by Lemma 2.3 either $p_1 \leqq q$ or $p_2 \leqq q$ and we are done by induction.

If $\alpha$ is limit, and $p$ is obtained from $p_\xi$ $(\xi < \eta)$ for some $\eta < \omega_2$, then base$(q) \in p_\xi$ for some $\xi < \eta$ and by Lemma 2.3, $p_\xi \leqq q$ and we are done, again. $\square$

LEMMA 2.5. *$(P, \leqq)$ is $\omega_2$-closed.*

PROOF. From Lemma 2.4 and the definition of $(P, \leqq)$. $\square$

It follows from Lemmas 2.2 and 2.5 that if $2^{\omega_1} = \omega_2$, then forcing with $(P, \leqq)$ preserves cardinals and cofinalities. Let $G \subseteqq P$ be generic, put $A = \bigcup\{\operatorname{supp}(p): p \in G\}$, $T = \bigcup G: [A]^2 \to [A]^{\omega_1}$.

LEMMA 2.6. $|A| = \omega_3$.

PROOF. If $p \in P$, $\alpha < \omega_3$, there is a $q \in P$, isomorphic to $p$, such that $q$ lies entirely above $p$ and $\alpha$, and, therefore, $p$ and $q$ can be amalgamated. $\square$

For simplicity, from now on we pretend that $A = \omega_3$ and that $T$ is defined on $[\omega_3]^2$ (in fact, an isomorphism is needed).

## 3. Histories

LEMMA 3.1. *Suppose that $p \in P$, $h \in p$. Then $q = \{g \in p: g \subseteqq h\} \in P$ and $p \leqq q$.*

PROOF. By induction on $\alpha$, for $p \in P_\alpha$. $\square$

DEFINITION 3.1. Assume that $p \in P$. A *path through* $p$ is a sequence $\{h_\xi: \xi \leqq \zeta\}$ such that

(3.1) $\{h_0\} \in P_0$ ;

(3.2) $h_{\xi+1}$ is an immediate $\subseteqq$-successor of $h_\xi$ ;

(3.3) if $\xi$ is limit, then $h_\xi = \bigcup\{h_\eta: \eta < \xi\}$ ;

(3.4) $h_\zeta = $ base$(p)$.

LEMMA 3.2. *Suppose $h \in p \in P$. Then there is a path through $p$ which contains $h$.*

PROOF. By induction on $\alpha$, for $p \in P_\alpha$. $\square$

Now suppose that $p \in P$ and $s = \{p_\xi: \xi \leqq \zeta\}$ is a path through $p$. If $\alpha \in \operatorname{supp}(p)$, let $t(\alpha)$ be the least $\xi$ with $h_\xi$ defined on $\alpha$. $t(\alpha)$ is 0 or a successor ordinal. If $t(\alpha) = \xi + 1$, $\alpha$ is in $\operatorname{supp}(h_{\xi+1})$, where $h_{\xi+1}$ is obtained by amalgamating $h_\xi$ with some other function, say $g$. If $\pi$ is the isomorphism between $\operatorname{supp}(g)$ and $\operatorname{supp}(h_\xi)$, let $a(\alpha)$, the *ancestor of* $\alpha$, be $\pi(\alpha)$. If $t(\alpha) = 0$, then $a(\alpha)$ is undefined. Otherwise, $t\big(a(\alpha)\big) < t(\alpha)$, so the set $t^*(\alpha) = \big\{t(a^n(\alpha)): n < \omega\big\}$ is finite (the history of $\alpha$).

LEMMA 3.3. *Suppose* $p \in P$, $h = \mathrm{base}(p)$, $s = \{h_\xi : \xi \leqq \zeta\}$ *is a path through* $p$. *Suppose* $\alpha, \beta \in \mathrm{supp}(p)$, $t(\alpha) < t(\beta)$, $t(\alpha) \notin t^*(\beta)$. *Then* $h(\alpha,\beta) \supseteqq \min(\alpha,\beta) \cap \mathrm{supp}(h_{t(\alpha)})$.

PROOF. By induction on $t(\beta)$. If $t(\beta) = \xi + 1$, $h_{\xi+1}$ is obtained by amalgamating $h_\xi$ and $g$, on some supports $a \cup b$ and $a \cup c$. $\beta \in c$ and as $t(\alpha) < t(\beta)$, $\alpha \in a \cup b$.

If $\alpha \in b$, then $h(\alpha,\beta) = h_{\xi+1}(\alpha,\beta) = \big(\max(\alpha,\beta) \cap \mathrm{supp}(h_{\xi+1})\big) - \{\min(\alpha,\beta)\} \supseteqq \min(\alpha,\beta) \cap \mathrm{supp}(h_{t(\alpha)})$.

If $\alpha \in a$, $h_{\xi+1}(\alpha,\beta) = h_{\xi+1}\big(\alpha, a(\beta)\big)$, and $\alpha < a(\beta)$ as $a < b$.

If $t(\alpha) < t\big(a(\beta)\big)$, we are done by induction.

$t(\alpha) \neq t\big(a(\beta)\big)$ holds by hypothesis.

If $t\big(a(\beta)\big) < t(\alpha)$, at stage $t(\alpha)$, if the supports are $a \cup b$ and $a \cup c$, with $a < b < c$, $\alpha$ and $a(\beta)$ must be in different tails, and $\alpha < a(\beta)$, so $\alpha \in b$, $a(\beta) \in c$. Then, $h_{t(\alpha)}\big(\alpha, a(\beta)\big) \supseteqq \min(\alpha,\beta) \cap \mathrm{supp}(h_{t(\alpha)})$. □

LEMMA 3.4. = LEMMA 1.2. *In* $V^P$, $Q(T)$ *is* $\omega_2$*-c.c.*

PROOF. Assume that $p$ forces that $\{q_\xi : \xi < \omega_2\}$ is an antichain. We may assume that $q_\xi = (s_\xi, f_\xi, \mathcal{F}_\xi)$ are isomorphic, $s_\xi = \Delta \cup a_\xi$ with the sets $\{\Delta, a_\xi : \xi < \omega_2\}$ pairwise disjoint. Choose $p_0 \leqq p$ determining $\Delta$ and the isomorphism type of $q_\xi$. Then select a decreasing, continuous chain $\{p_\xi : \xi < \omega_2\}$ such that $p_{\xi+1}$ determines $s_\xi$. Let $\{h_\xi : \xi < \omega_2\}$ be a 'path' through $\bigcup\{p_\xi : \xi < \omega_2\}$ obtained by taking a path through $p_0$ and concatenating it with paths from $\mathrm{base}(p_\xi)$ to $\mathrm{base}(p_{(\xi+1)})$ for $\xi < \omega_2$. For every $p_\xi$ some initial segment of the path is a path through $p_\xi$.

For $\alpha \in \bigcup\{\mathrm{supp}(p_\xi) : \xi < \omega_2\}$ we can define $t(\alpha)$, $t^*(\alpha)$ as above. As $t^*(\alpha)$ is finite we can find $\xi < \eta < \omega_2$ such that

$$\text{if } \tau \in \Delta,\ \alpha \in a_\xi \cup a_\eta,\ \text{then } t(\tau) < t(\alpha); \tag{3.5}$$

$$\text{if } \alpha \in a_\xi,\ \beta \in a_\eta,\ \text{then } t(\alpha) < t(\beta),\ t(\alpha) \notin t^*(\beta). \tag{3.6}$$

We now prove that $p_{\eta+1}$ forces that $q_\xi$ and $q_\eta$ are compatible. As they are isomorphic structures the only thing we must prove is that $h_{\eta+1}$ allows us to extend $f_\xi \cup f_\eta$ to keep the elements of $\mathcal{F}_\xi \cup \mathcal{F}_\eta$ non-extendible, as required.

Assume first that $F \in \mathcal{F}_\xi$ is extended by $x \in a_\eta$. In $F$ there are $z < y$ such that $t(z) \leqq t(y)$ and $t(z) \notin t^*(x)$ (as $F$ is infinite and $t^*(x)$ is finite). By Lemma 3.3, $z \in h_{\eta+1}(y,x)$.

Assume next that $F \in \mathcal{F}_\eta$ is extended by $x \in a_\xi$, $F < x$. As $F$ is non-extendible in $q_\eta$, there is a $y \in F \cap a_\eta$. When $y$ is born, $x$ already exists, and as $y < x$, they are in different tails, so $h_{\eta+1}(y,x)$ covers $\{z \in F : z \neq y, t(z) \leqq t(y)\}$. This latter set is non-empty, if $F \cap \Delta \neq \emptyset$, in the other case, it is empty only for at most one $y \in F$.

Assume that $(F^0, F^1)$ suffers a forbidden extension by $x \in a_\eta$. As $x < < F^1$, when $x$ is born, $x$ and the elements of $F^1$ are in different tails, so for $y \in F^1$, $T(x,y)$ covers $F^0$.

Suppose, finally, that $(F^0, F^1) \in \mathcal{F}_\eta$ is extended by $x \in a_\xi$. If $F^0 \nsubseteq \Delta$, choose $z < y$ from $F^0$, $y \in F^0 - \Delta$, $t(z) \leqq t(y)$. When $y$ is born, $x$ already lives, so they are in different tails, and so $z \in T(x,y)$. If $F^0 \subseteq \Delta$, there is, as $(F^0, F^1)$ is non-extendible in $q_\eta$, a $y \in F^1 - \Delta$. Then, $x < y$, $t(x) < t(y)$, $t(x) \notin t^*(y)$, so we can apply Lemma 3.3 and get that $T(x,y)$ covers $F^0$. $\square$

## 4. End of the proof

THEOREM 4.1. *It is consistent that there is a set mapping $f: [\omega_3]^2 \to \omega_3$ with no uncountable free sets.*

PROOF. Assume that $H: [\omega_3]^2 \to [\omega_3]^{\aleph_0}$ is a set mapping with no uncountable free sets (see Lemma 1.4). Let $p \in P$ if $p = (s, g)$ where $s \in \in [\omega_3]^{<\omega}$, $g: [s]^2 \to s$ is a set mapping, with $g(\alpha, \beta) \in H(\alpha, \beta)$. $(s', g') \leqq \leqq (s, g)$ iff $s' \supseteq s$, $g' \supseteq g$.

LEMMA 4.2. *For $\alpha < \omega_3$, $\{(s,g): \alpha \in s\}$ is dense.*

PROOF. $(s \cup \{\alpha\}, g) \leqq (s, g)$. $\square$

LEMMA 4.3. *$(P, \leqq)$ is ccc.*

PROOF. Assume that $p_\xi \in P$ $(\xi < \omega_1)$. We may assume that $p_\xi = = (s \cup s_\xi, g_\xi)$. As $g_\xi(\alpha, \beta) \in H(\alpha, \beta)$, a countable set, we can assume that $g_\xi | s = g_\eta | s$ for $\xi < \eta$. Then, $(s \cup s_\xi \cup s_\eta, g_\xi \cup g_\eta)$ extends $p_\xi$ and $p_\eta$. $\square$

Therefore, forcing with $(P, \leqq)$ preserves cardinals and cofinality. Put $f = \bigcup\{g: (s, g) \in G\}$ for a generic $G \subseteq P$.

LEMMA 4.4. *In $V[G]$, $f$ has no uncountable free sets.*

PROOF. Assume that $Y$ is an uncountable free set. There are, for $\xi < < \omega_1$, conditions $p(\xi)$ and different ordinals $\alpha(\xi)$ that $p(\xi)$ forces that $\alpha(\xi)$ is in $Y$. We may assume that the $p(\xi)$-s are compatible, $p(\xi) = (s \cup s_\xi, g_\xi)$, $\alpha(\xi) \in s_\xi$ with $\{s, s_\xi: \xi < \omega_1\}$ pairwise disjoint. As $H$ has no uncountable free sets, there are $\xi_0$, $\xi_1$, $\xi_2$ such that $\alpha(\xi_2) \in H\big(\alpha(\xi_0), \alpha(\xi_1)\big)$. Now

$$q = \left( s \cup s_{\xi_0} \cup s_{\xi_1} \cup s_{\xi_2}, g_{\xi_0} \cup g_{\xi_1} \cup g_{\xi_2} \cup \Big\langle \big\{ \alpha(\xi_0), \alpha(\xi_1) \big\}, \alpha(\xi_2) \Big\rangle \right)$$

forces a contradiction. $\square$

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# LOCAL UNIFORM CONVERGENCE OF THE EIGENFUNCTION EXPANSION ASSOCIATED WITH THE LAPLACE OPERATOR. II

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In this paper we continue the proof of the two convergence theorems in [15]. We use the notations and symbols introduced in [15] and the numbering of formulas and statements follows the one used in [15] as well.

LEMMA 5. *We have*

$$(45)\qquad \mu\left|\int_0^R J_1(\mu r)\big[J_0(\mu_i r)-J_0(\rho_i r)\big]\,dr\right| \leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac12},\quad \mu\geqq 1.$$

PROOF. *Case* A: $\mu\leqq 2\rho_i$. Then in (45) we take the decomposition

$$\int_0^R=\int_0^{\frac{R}{2\rho_i}}+\int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}}+\int_{\frac{R}{\mu}}^{R}.$$

Using $|J_1(\mu r)|\leqq c\mu r$ we get by (39)

$$\mu\left|\int_0^{\frac{R}{2\rho_i}}\right|\leqq c\int_0^{\frac{R}{2\rho_i}}\mu^2 r\cdot re^{|\nu_i|R}\,dr\leqq ce^{|\nu_i|R}\frac{\mu^2}{\rho_i^3}\leqq$$

$$\leqq ce^{|\nu_i|R}\frac{\mu^{\frac12}}{\rho_i^{\frac32}}\leqq ce^{|\nu_i|R}\frac{1}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac12}.$$

In estimating $\mu\left|\int\limits_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}}\right|$ we use the formulas

$$J_1(\mu r)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)!}\left(\frac{\mu r}{2}\right)^{2k+1}, \tag{46}$$

$$J_0(\mu_i r)-J_0(\rho_i r)=c\left[\frac{\cos\left(\mu_i r-\frac{\pi}{4}\right)}{(\mu_i r)^{\frac12}}-\right. \tag{47}$$

$$\left.-\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{(\rho_i r)^{\frac12}}\right]+O\left(\frac{|\nu_i|}{r^{\frac32}\rho_i^{\frac52}}+\frac{e^{|\nu_i|r}-1}{(|\mu_i|r)^{\frac32}}\right).$$

Consider first the remainder term of (47) with $|(J_1(\mu r)|\leqq c\mu r$:

$$\mu\left|\int\limits_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}}\right|\leqq c\int\limits_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}}\mu^2 r\left(\frac{|\nu_i|}{r^{\frac32}\rho_i^{\frac52}}+\frac{e^{|\nu_i|r}-1}{(|\mu_i|r)^{\frac32}}\right)\,dr\leqq$$

$$\leqq c\mu^2\frac{|\nu_i|}{\rho_i^{\frac52}\mu^{\frac12}}+c\frac{\mu^2}{|\mu_i|^{\frac32}}\int\limits_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}}r^{\frac12}\frac{e^{|\nu_i|r}-1}{r}\,dr\leqq$$

$$\leqq c\frac{|\nu_i|}{\rho_i}\left(\frac{\mu}{\rho_i}\right)^{\frac12}+ce^{|\nu_i|R}\frac{\mu^2}{|\mu_i|^{\frac32}}\int\limits_0^{\frac{R}{\mu}}r^{\frac12}\,dr\leqq$$

$$\leqq c\frac{e^{|\mu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac12}.$$

The main term of (47) has a decomposition

$$\frac{\cos\left(\mu_i r-\frac{\pi}{4}\right)}{(\mu_i r)^{\frac12}}-\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{(\rho_i r)^{\frac12}}=\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r^{\frac12}}\left(\frac{1}{\mu_i^{\frac12}}-\frac{1}{\rho_i^{\frac12}}\right)+ \tag{48}$$

$$+\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{(\mu_i r)^{\frac{1}{2}}}(\operatorname{ch}\nu_i r-1)-j\frac{\sin\left(\rho_i r-\frac{\pi}{4}\right)}{(\rho_i r)^{\frac{1}{2}}}\operatorname{sh}\nu_i r$$

where $j^2=-1$. Estimating term by term we get from (48)

$$\mu\left|\int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} J_1(\mu r)\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r^{\frac{1}{2}}}\left(\frac{1}{\mu_i^{\frac{1}{2}}}-\frac{1}{\rho_i^{\frac{1}{2}}}\right)\,dr\right| \leqq$$

$$\leqq c\mu\frac{|\nu_i|}{\rho_i^{\frac{3}{2}}}\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(\frac{\mu}{2}\right)^{2k+1}\left|\int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} r^{2k+1}\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r^{\frac{1}{2}}}\,dr\right| \leqq$$

$$\leqq c\frac{\mu|\nu_i|}{\rho_i^{\frac{3}{2}}}\sum_{k=0}^{\infty}\frac{\left(\frac{\mu}{2}\right)^{2k+1}}{k!(k+1)!}\frac{1}{\mu^{2k+\frac{1}{2}}\rho_i}\leqq c\left(\frac{\mu}{\rho_i}\right)^{\frac{3}{2}}\frac{|\nu_i|}{\rho_i}\leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}},$$

$$\mu\left|\int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} J_1(\mu r)\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{(\rho_i r)^{\frac{1}{2}}}(\operatorname{ch}\nu_i r-1)\,dr\right|\leqq$$

$$\leqq c\frac{\mu}{|\mu_i|^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{\left(\frac{\mu}{2}\right)^{2k+1}}{k!(k+1)!}\left|\int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} r^{2k+\frac{3}{2}}\cos\left(\rho_i r-\frac{\pi}{4}\right)\frac{\operatorname{ch}\nu_i r-1}{r}\,dr\right|.$$

Since

$$\left|\int_{\frac{R}{2\rho_i}}^{t} r^{2k+\frac{3}{2}}\cos\left(\rho_i r-\frac{\pi}{4}\right)\,dr\right|\leqq c\frac{t^{2k+\frac{3}{2}}}{\rho_i}$$

and

$$\left|\int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} r^{2k+\frac{3}{2}}\cos\left(\rho_i r-\frac{\pi}{4}\right)\frac{\operatorname{ch}\nu_i r-1}{r}\,dr\right|\leqq$$

$$\leqq \frac{\operatorname{ch} \nu_i \frac{R}{\mu} - 1}{\frac{R}{\mu}} \left| \int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} r^{2k+\frac{3}{2}} \cos\left(\rho_i r - \frac{\pi}{4}\right) dr \right| +$$

$$+ \int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} \left| \int_{\frac{R}{2\rho_i}}^{t} r^{2k+\frac{3}{2}} \cos\left(\rho_i r - \frac{\pi}{4}\right) dr \right| \left( \frac{\operatorname{ch} \nu_i t - 1}{t} \right)' dt \leqq$$

$$\leqq \frac{c}{\rho_i \mu^{2k+\frac{3}{2}}} e^{|\nu_i| R}$$

hence

$$\mu \left| \int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} J_1(\mu r) \frac{\cos\left(\rho_i r - \frac{\pi}{4}\right)}{(\mu_i r)^{\frac{1}{2}}} (\operatorname{ch} \nu_i r - 1)\, dr \right| \leqq$$

$$\leqq c \frac{\mu}{|\mu_i|^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu}{2}\right)^{2k+1}}{k!(k+1)!} \frac{e^{|\nu_i| R}}{\rho_i \mu^{2k+\frac{3}{2}}} \leqq c \frac{\mu^{\frac{1}{2}}}{\rho_i^{\frac{3}{2}}} e^{|\nu_i| R} \leqq$$

$$\leqq c \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|} \left( \frac{\mu}{1 + \rho_i} \right)^{\frac{1}{2}}$$

and analogously

$$\mu \left| \int_{\frac{R}{2\rho_i}}^{\frac{R}{\mu}} J_1(\mu r) \frac{\sin\left(\rho_i r - \frac{\pi}{4}\right)}{(\rho_i r)^{\frac{1}{2}}} \operatorname{sh} \nu_i r \, dr \right| \leqq c \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|} \left( \frac{\mu}{1 + \rho_i} \right)^{\frac{1}{2}}.$$

So it is remains to estimate $\int_{\frac{R}{\mu}}^{R}$ in (45). In this case we use the following integration by parts:

$$\mu \int_{\frac{R}{\mu}}^{R} J_1(\mu r) \left[ J_0(\mu_i r) - J_0(\rho_i r) \right] dr =$$

$$= \mu \left[ J_1(\mu r) \left( \frac{J_1(\mu_i r)}{\mu_i} - \frac{J_1(\rho_i r)}{\rho_i} \right) \right]_{r=\frac{R}{\mu}}^{R} +$$

$$+\mu^2 \int_{\frac{R}{\mu}}^{R} J_2(\mu r) \left[ \frac{J_1(\mu_i r)}{\mu_i} - \frac{J_1(\rho_i r)}{\rho_i} \right] dr =: I_1 + I_2.$$

By Lemma 3,

$$\mu \left| J_1(\mu r) \left[ \frac{J_1(\mu_i r)}{\mu_i} - \frac{J_1(\rho_i r)}{\rho_i} \right] \right| \leqq c \frac{\mu^{\frac{1}{2}}}{r^{\frac{1}{2}}} \left( \frac{|\nu_i|}{r^{\frac{1}{2}} \rho_i^{\frac{5}{2}}} + \frac{e^{|\nu_i| r} - 1}{r^{\frac{1}{2}} |\mu_i|^{\frac{3}{2}}} \right),$$

whence

$$|I_1| \leqq c \frac{\mu^{\frac{3}{2}} |\nu_i|}{\rho_i^{\frac{5}{2}}} + c \frac{\mu^{\frac{1}{2}}}{|\mu_i|^{\frac{3}{2}}} e^{|\nu_i| R} \leqq c \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|} \left( \frac{\mu}{1 + \rho_i} \right)^{\frac{1}{2}}.$$

In $I_2$ we use the estimates

$$J_2(\mu r) = \sqrt{\frac{2}{\pi}} \frac{\cos\left(\mu r - \frac{5\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}} + \tag{49}$$

$$+\sqrt{\frac{2}{\pi}} \frac{15}{8} \frac{\cos\left(\mu r - \frac{3\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}} + O\left( \frac{1}{(\mu r)^{\frac{5}{2}}} \right),$$

$$\frac{J_1(\mu_i r)}{\mu_i} - \frac{J_1(\rho_i r)}{\rho_i} = \tag{50}$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{\cos\left(\mu_i r - \frac{3\pi}{4}\right)}{\mu_i^{\frac{3}{2}} r^{\frac{1}{2}}} - \frac{\cos\left(\rho_i r - \frac{3\pi}{4}\right)}{\rho_i^{\frac{3}{2}} r^{\frac{1}{2}}} \right] +$$

$$+\sqrt{\frac{2}{\pi}} \frac{3}{8} \left[ \frac{\cos\left(\mu_i r - \frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}} r^{\frac{3}{2}}} - \frac{\cos\left(\rho_i r - \frac{\pi}{4}\right)}{\rho_i^{\frac{5}{2}} r^{\frac{3}{2}}} \right] +$$

$$+O\left( \frac{|\nu_i|}{\rho_i^{\frac{9}{2}} r^{\frac{5}{2}}} + \frac{e^{|\nu_i| r} - 1}{|\mu_i|^{\frac{7}{2}} r^{\frac{5}{2}}} \right).$$

In $I_2$ the remainder terms give the following quantities:

$$\mu^2 \int_{\frac{R}{\mu}}^{R} \frac{1}{(\mu r)^{\frac{5}{2}}} \left( \frac{|\nu_i|}{r^{\frac{1}{2}} \rho_i^{\frac{5}{2}}} + \frac{e^{|\nu_i| r} - 1}{r^{\frac{1}{2}} |\mu_i|^{\frac{3}{2}}} \right) dt \leqq$$

$$\leqq \frac{c}{\mu^{\frac{1}{2}}} \frac{|\nu_i| \mu^2}{\rho_i^{\frac{5}{2}}} + \frac{c}{\mu^{\frac{1}{2}} |\mu_i|^{\frac{3}{2}}} \int_{\frac{R}{\mu}}^{R} \frac{e^{|\nu_i| r} - 1}{r^3} dr \leqq$$

$$\leqq c \left( \frac{\mu}{\rho_i} \right)^{\frac{1}{2}} \frac{|\nu_i|}{\rho_i} + c \frac{\mu^{\frac{1}{2}}}{|\mu_i|^{\frac{3}{2}}} e^{|\nu_i| R} \leqq c \frac{e^{|\nu_i| R}}{1 + |\nu - \rho_i|} \left( \frac{\mu}{1 + \rho_i} \right)^{\frac{1}{2}},$$

$$\mu^2 \int_{\frac{R}{\mu}}^{R} \frac{1}{(\mu r)^{\frac{1}{2}}} \left( \frac{|\nu_i|}{r^{\frac{5}{2}} \rho_i^{\frac{9}{2}}} + \frac{e^{|\nu_i| r - 1}}{r^{\frac{5}{2}} |\mu_i|^{\frac{7}{2}}} \right) dr \leqq$$

$$\leqq c \frac{\mu^{\frac{3}{2}} |\nu_i|}{\rho_i^{\frac{7}{2}}} \frac{\mu^2}{\rho_i} + c \frac{\mu^{\frac{3}{2}}}{|\mu_i|^{\frac{7}{2}}} \int_{\frac{R}{\mu}}^{R} \frac{e^{|\nu_i| r} - 1}{r^3} dr \leqq$$

$$\leqq c \frac{\mu^{\frac{1}{2}} |\nu_i|}{\rho_i^{\frac{3}{2}}} + c \frac{\mu^{\frac{5}{2}}}{|\mu_i|^{\frac{7}{2}}} e^{|\nu_i| R} \leqq c \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|} \left( \frac{\mu}{1 + \rho_i} \right)^{\frac{1}{2}}.$$

In the main terms of (50) we take a decomposition of type (48); e.g. the second term can be written in the form

$$\frac{\cos\left(\mu_i r - \frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}} r^{\frac{3}{2}}} - \frac{\cos\left(\rho_i r - \frac{\pi}{4}\right)}{\rho_i^{\frac{5}{2}} r^{\frac{3}{2}}} =$$

$$= \frac{\cos\left(\rho_i r - \frac{\pi}{4}\right)}{r^{\frac{3}{2}}} \left( \frac{1}{\mu_i^{\frac{5}{2}}} - \frac{1}{\rho_i^{\frac{5}{2}}} \right) +$$

$$+ \frac{\cos\left(\rho_i r - \frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}} r^{\frac{3}{2}}} (\operatorname{ch} \nu_i r - 1) - j \frac{\sin\left(\rho_i r - \frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}} r^{\frac{3}{2}}} \operatorname{sh} \nu_i r.$$

Now

$$\mu^2\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{5\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r^{\frac{3}{2}}}\left(\frac{1}{\mu_i^{\frac{5}{2}}}-\frac{1}{\rho_i^{\frac{5}{2}}}\right)\,dr\right|\leqq$$

$$\leqq c\mu^{\frac{3}{2}}\frac{|\nu_i|}{\rho_i^{\frac{7}{2}}}\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{5\pi}{4}\right)\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r}\cdot\frac{1}{r}\,dr\right|\leqq$$

$$\leqq c\frac{\mu^{\frac{3}{2}}|\nu_i|}{\rho_i^{\frac{7}{2}}}\frac{\mu}{1+|\mu-\rho_i|}\leqq\frac{c}{\rho_i}\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}},$$

$$\mu^2\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{5\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}}r^{\frac{3}{2}}}(\operatorname{ch}\nu_i r-1)\,dr\right|\leqq$$

$$\leqq c\frac{\mu^{\frac{3}{2}}}{|\mu_i|^{\frac{5}{2}}}\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{5\pi}{4}\right)\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r}\cdot\frac{\operatorname{ch}\nu_i r-1}{r}\,dr\right|\leqq$$

$$\leqq c\frac{\mu^{\frac{5}{2}}}{|\mu_i|^{\frac{5}{2}}}\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}$$

and similarly

$$\mu^2\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{5\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\frac{\sin\left(\rho_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}}r^{\frac{3}{2}}}\operatorname{sh}\nu_i r\,dr\right|\leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}$$

Using the estimates (41)–(44) we analogously get

$$\mu^2\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{3\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r^{\frac{3}{2}}}\left(\frac{1}{\mu_i^{\frac{5}{2}}}-\frac{1}{\rho_i^{\frac{5}{2}}}\right)\,dr\right|\leqq$$

$$\leqq c\frac{\mu^{\frac{1}{2}}|\nu_i|}{\rho_i^{\frac{7}{2}}}\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{3\pi}{4}\right)\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r}\frac{1}{r^2}\,dr\right|\leqq$$

$$\leqq c\frac{\mu^{\frac{1}{2}}}{\rho_i^{\frac{5}{2}}}\frac{|\nu_i|}{\rho_i}\frac{\mu^2}{1+|\mu-\rho_i|}\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}},$$

$$\mu^2\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{3\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}}r^{\frac{3}{2}}}(\operatorname{ch}\nu_i r-1)\,dr\right|\leqq$$

$$\leqq\frac{\mu^{\frac{1}{2}}}{|\mu_i|^{\frac{5}{2}}}\left|\int\limits_{\frac{R}{\mu}}^{R}\cos\left(\mu r-\frac{3\pi}{4}\right)\cos\left(\rho_i r-\frac{\pi}{4}\right)\frac{\operatorname{ch}\nu_i r-1}{r^3}\,dr\right|\leqq$$

$$\leqq\frac{\mu^{\frac{1}{2}}}{|\mu_i|^{\frac{5}{2}}}\left\{\left|\left[\frac{\operatorname{ch}\nu_i t-1}{t^3}\int\limits_{\frac{R}{\mu}}^{t}\cos\left(\mu r-\frac{3\pi}{4}\right)\cos\left(\rho_i r-\frac{\pi}{4}\right)\,dr\right]_{\frac{R}{\mu}}^{R}\right|+\right.$$

$$+\int\limits_{\frac{R}{\mu}}^{R}\left|\int\limits_{\frac{R}{\mu}}^{t}\cos\left(\mu r-\frac{3\pi}{4}\right)\cos\left(\rho_i r-\frac{\pi}{4}\right)\,dr\right|\left(\frac{1}{t^2}\left(\frac{\operatorname{ch}\nu_i t-1}{t}\right)'+\right.$$

$$\left.\left.+\frac{1}{t^3}\frac{\operatorname{ch}\nu_i t-1}{t}\right)\,dt\right\}\leqq c\left(\frac{\mu}{|\mu_i|}\right)^{\frac{5}{2}}\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}$$

and similarly

$$\mu^2\left|\int\limits_{\frac{R}{\mu}}^{R}\frac{\cos\left(\mu r-\frac{3\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\frac{\sin\left(\rho_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{5}{2}}r^{\frac{3}{2}}}\operatorname{sh}\nu_i r\,dr\right|\leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}.$$

Using the decomposition

$$\frac{\cos\left(\mu_i r - \frac{3\pi}{4}\right)}{\mu_i^{\frac{3}{2}} r^{\frac{1}{2}}} - \frac{\cos\left(\rho_i r - \frac{3\pi}{4}\right)}{\rho_i^{\frac{3}{2}} r^{\frac{1}{2}}} = \cos\left(\rho_i r - \frac{3\pi}{4}\right)\left(\frac{1}{\mu_i^{\frac{3}{2}}} - \frac{1}{\rho_i^{\frac{3}{2}}}\right) +$$

$$+ \frac{\cos\left(\rho_i r - \frac{3\pi}{4}\right)}{\mu_i^{\frac{3}{2}} r^{\frac{1}{2}}}(\operatorname{ch}\mu_i r - 1) - j\frac{\sin\left(\rho_i r - \frac{3\pi}{4}\right)}{\rho_i^{\frac{3}{2}} r^{\frac{1}{2}}}\operatorname{sh}\nu_i r$$

we see as above that

$$\mu^2\left|\int_{\frac{R}{\mu}}^{R} \frac{\cos\left(\mu r - \frac{5\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\left[\frac{\cos\left(\mu_i r - \frac{3\pi}{4}\right)}{\mu_i^{\frac{3}{2}} r^{\frac{1}{2}}} - \frac{\cos\left(\rho_i r - \frac{3\pi}{4}\right)}{\rho_i^{\frac{3}{2}} r^{\frac{1}{2}}}\right] dr\right| \leqq$$

$$\leqq c e^{|\nu_i| R} \frac{1}{1 + |\mu - \rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}},$$

$$\mu^2\left|\int_{\frac{R}{\mu}}^{R} \frac{\cos\left(\mu r - \frac{3\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\left[\frac{\cos\left(\mu_i r - \frac{3\pi}{4}\right)}{\mu_i^{\frac{3}{2}} r^{\frac{1}{2}}} - \frac{\cos\left(\rho_i r - \frac{3\pi}{4}\right)}{\rho_i^{\frac{3}{2}} r^{\frac{1}{2}}}\right] dr\right| \leqq$$

$$\leqq c\frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}.$$

Hence we obtain

$$|I_2| \leqq c\frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}$$

and so (45) is proved in case A).

*Case* B: $1 \leqq \rho_i \leqq \frac{\mu}{2}$. In this case we have to prove that

$$\mu\left|\int_0^R J_1(\mu r)\left[J_0(\mu_i r) - J_0(\rho_i r)\right]\, dr\right| \leqq c\frac{e^{|\nu_i| R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

Using the rule $J_0' = -J_1$ we obtain

$$\mu\int_0^R J_1(\mu r)\left[J_0(\mu_i r) - J_0(\rho_i r)\right]\, dr =$$

$$= -\Big[J_0(\mu r)\big(J_0(\mu_i r) - J_0(\rho_i r)\big)\Big]_{r=0}^{R} -$$

$$-\int_0^R J_0(\mu r)\big[\mu_i J_1(\mu_i r) - \rho_i J_1(\rho_i r)\big]\; dr =: I_1 + I_2.$$

Obviously

$$|I_1| = \Big|J_0(\mu R)\big[J_0(\mu_i R) - J_0(\rho_i R)\big]\Big| \leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

We have to obtain the same estimate for $I_2$. Consider the decompositi

$$\int_0^R = \int_0^{\frac{R}{\mu}} + \int_{\frac{R}{\mu}}^{\frac{R}{2\rho_i}} + \int_{\frac{R}{2\rho_i}}^{R}.$$

By (40) we have

$$\left|\int_0^{\frac{R}{\mu}} J_0(\mu r)\big[\mu_i J_1(\mu_i r) - \rho_i J_1(\rho_i r)\big]\; dr\right| \leqq$$

$$\leqq c e^{|\nu_i|R}\int_0^{\frac{R}{\mu}} dr \leqq c\frac{e^{|\nu_i|R}}{\mu} \leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}},$$

$$\left|\int_{\frac{R}{\mu}}^{\frac{R}{2\rho_i}} J_0(\mu r)\big[\mu_i J_1(\mu_i r) - \rho_i J_1(\rho_i r)\big]\; dr\right| \leqq$$

$$\leqq e^{|\nu_i|R}\int_0^{\frac{R}{\mu}} \frac{1}{(\mu r)^{\frac{1}{2}}}\; dr \leqq c\frac{e^{|\nu_i|R}}{\mu} \leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

In order to estimate $\int\limits_{\frac{R}{2\rho_i}}^{R}$, we take the expansions

$$J_0(\mu r) = \sqrt{\frac{2}{\pi}}\frac{\cos\left(\mu r - \frac{\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}} - \sqrt{\frac{2}{\pi}}\frac{1}{8}\frac{\cos\left(\mu r + \frac{\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}} + O\left(\frac{1}{(\mu r)^{\frac{5}{2}}}\right),$$

$$\mu_i J_1(\mu_i r) - \rho_i J_1(\rho_i r) =$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\cos\left(\mu_i r - \frac{3\pi}{4}\right)}{r^{\frac{1}{2}}}\mu_i^{\frac{1}{2}} - \frac{\cos\left(\rho_i r - \frac{3\pi}{4}\right)}{r^{\frac{1}{2}}}\rho_i^{\frac{1}{2}}\right] +$$

$$+\sqrt{\frac{2}{\pi}}\frac{3}{8}\left[\frac{\cos\left(\mu_i r - \frac{\pi}{4}\right)}{\mu_i^{\frac{1}{2}} r^{\frac{3}{2}}} - \frac{\cos\left(\rho_i r - \frac{\pi}{4}\right)}{\rho_i^{\frac{1}{2}} r^{\frac{1}{2}}}\right] + O\left(\frac{|\nu_i|}{(\rho_i r)^{\frac{5}{2}}} + \frac{e^{|\nu_i| r} - 1}{|\mu_i|^{\frac{3}{2}} r^{\frac{5}{2}}}\right).$$

The contribution of the remainder terms in $I_2$ can be estimated as follows:

$$\int\limits_{\frac{R}{2\rho_i}}^{R} \frac{1}{(\mu r)^{\frac{1}{2}}}\left(\frac{|\nu_i|}{(\rho_i r)^{\frac{5}{2}}} + \frac{e^{|\nu_i| r} - 1}{|\mu_i|^{\frac{3}{2}} r^{\frac{5}{2}}}\right) dr \leqq$$

$$\leqq c\frac{|\nu_i|\rho_i^2}{\mu^{\frac{1}{2}}\rho_i^{\frac{5}{2}}} + \frac{1}{\mu^{\frac{1}{2}}|\mu_i|^{\frac{3}{2}}}\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{e^{|\nu_i| r} - 1}{r^3}\,dr \leqq c\frac{e^{|\nu_i| R}}{(\mu\rho_i)^{\frac{1}{2}}},$$

$$\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{1}{(\mu r)^{\frac{5}{2}}}\left(\frac{|\nu_i|}{\rho_i}\left(\frac{\rho_i}{r}\right)^{\frac{1}{2}} + |\mu_i|^{\frac{1}{2}}\frac{e^{|\nu_i| r} - 1}{r^{\frac{1}{2}}}\right) dr \leqq$$

$$\leqq c\frac{|\nu_i|\rho_i^{\frac{1}{2}}}{\mu^{\frac{5}{2}}\rho_i}\rho_i^2 + \frac{|\mu_i|^{\frac{1}{2}}}{\mu^{\frac{5}{2}}}\frac{\rho_i}{1 + |\nu_i|} \leqq c\frac{e^{|\nu_i| R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

Consider the decomposition

$$\sqrt{\frac{\mu_i}{r}}\cos\left(\mu_i r - \frac{3\pi}{4}\right) - \sqrt{\frac{\rho_i}{r}}\cos\left(\rho_i r - \frac{3\pi}{4}\right) = \tag{51}$$

$$= \frac{\sqrt{\mu_i} - \sqrt{\rho_i}}{\sqrt{r}}\cos\left(\rho_i r - \frac{3\pi}{4}\right) + \sqrt{\frac{\mu_i}{r}}\cos\left(\rho_i r - \frac{3\pi}{4}\right)(\operatorname{ch}\nu_i r - 1) -$$

$$-j\sqrt{\frac{\mu_i}{r}}\sin\left(\rho_i r-\frac{3\pi}{4}\right)\operatorname{sh}\nu_i r.$$

We have

$$\left|\int_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r-\frac{\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\left(\sqrt{\mu_i}-\sqrt{\rho_i}\right)\frac{\cos\left(\rho_i r-\frac{3\pi}{4}\right)}{r^{\frac{1}{2}}}\,dr\right|\leqq$$

$$\leqq c\frac{|\nu_i|}{\mu^{\frac{1}{2}}\rho_i^{\frac{1}{2}}}\left|\int_{\frac{R}{\rho_i}}^{R}\frac{\cos\left(\mu r-\frac{\pi}{4}\right)\cos\left(\rho_i r-\frac{3\pi}{4}\right)}{r}\,dr\right|\leqq$$

$$\leqq c\frac{|\nu_i|}{(\mu\rho_i)^{\frac{1}{2}}}\frac{\rho_i}{1+|\mu-\rho_i|}\leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}},$$

$$\left|\int_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r-\frac{\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\frac{\mu_i^{\frac{1}{2}}-\rho_i^{\frac{1}{2}}}{r^{\frac{1}{2}}}\cos\left(\rho_i r-\frac{3\pi}{4}\right)(\operatorname{ch}\nu_i r-1)\,dr\right|\leqq$$

$$\leqq\frac{c}{\mu^{\frac{1}{2}}}\frac{|\nu_i|}{\rho_i^{\frac{1}{2}}}\int_{\frac{R}{2\rho_i}}^{R}\frac{\operatorname{ch}\nu_i r-1}{r}\,dr\leqq c\frac{|\nu_i|}{(\mu\rho_i)^{\frac{1}{2}}}\frac{e^{|\nu_i|R}}{1+|\nu_i|}\leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}},$$

$$\left|\int_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r-\frac{\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\left(\frac{\rho_i}{r}\right)^{\frac{1}{2}}\cos\left(\rho_i r-\frac{3\pi}{4}\right)(\operatorname{ch}\nu_i r-1)\,dr\right|\leqq$$

$$\leqq c\left(\frac{\rho_i}{\mu}\right)^{\frac{1}{2}}\left|\int_{\frac{R}{2\rho_i}}^{R}\cos\left(\mu r-\frac{\pi}{4}\right)\cos\left(\rho_i r-\frac{3\pi}{4}\right)\frac{\operatorname{ch}\nu_i r-1}{r}\,dr\right|\leqq$$

$$\leqq c\frac{\rho_i^{\frac{1}{2}}}{\mu^{\frac{3}{2}}}e^{|\nu_i|R}\leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}$$

and similarly for the third term of (51). So we proved that

$$\left|\int_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r-\frac{\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\left[\left(\frac{\mu_i}{r}\right)^{\frac{1}{2}}\cos\left(\mu_i r-\frac{3\pi}{4}\right)-\left(\frac{\rho_i}{r}\right)^{\frac{1}{2}}\cos\left(\rho_i r-\frac{3\pi}{4}\right)\right]dr\right|\leqq$$

$$\leqq ce^{|\nu_i|R}(\mu\rho_i)^{-\frac{1}{2}}.$$

Analogously we can see that

$$\left|\int\limits_{\frac{R}{2\rho_i}}^{R} \frac{\cos\left(\mu r+\frac{\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\left[\left(\frac{\mu_i}{r}\right)^{\frac{1}{2}}\cos\left(\mu_i r-\frac{3\pi}{4}\right)-\right.\right.$$

$$\left.\left.-\left(\frac{\rho_i}{r}\right)^{\frac{1}{2}}\cos\left(\rho_i r-\frac{3\pi}{4}\right)\right]dr\right| \leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

Indeed,

$$\left|\int\limits_{\frac{R}{2\rho_i}}^{R} \frac{\cos\left(\mu r+\frac{\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\left({\mu_i}^{\frac{1}{2}}-{\rho_i}^{\frac{1}{2}}\right)\frac{\cos\left(\rho_i r-\frac{3\pi}{4}\right)}{r^{\frac{1}{2}}}\,dr\right| \leqq$$

$$\leqq c\frac{|\nu_i|}{{\rho_i}^{\frac{1}{2}}\mu^{\frac{3}{2}}}\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{1}{r^2}\,dr \leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}},$$

$$\left|\int\limits_{\frac{R}{2\rho_i}}^{R} \frac{\cos\left(\mu r+\frac{\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\frac{\sqrt{\mu_i}-\sqrt{\rho_i}}{\sqrt{r}}\cos\left(\rho_i r-\frac{3\pi}{4}\right)(\operatorname{ch}\nu_i r-1)\,dr\right| \leqq$$

$$\leqq c\frac{|\nu_i|}{\rho^{\frac{1}{2}}\mu^{\frac{3}{2}}}\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{\operatorname{ch}\nu_i r-1}{r^2}\,dr \leqq c\frac{{\rho_i}^{\frac{1}{2}}|\nu_i|}{\mu^{\frac{3}{2}}}\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{\operatorname{ch}\nu_i r-1}{r}\,dr \leqq$$

$$\leqq c\frac{{\rho_i}^{\frac{1}{2}}}{\mu^{\frac{3}{2}}}e^{|\nu_i|R} \leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}},$$

$$\left|\int\limits_{\frac{R}{2\rho_i}}^{R} \frac{\cos\left(\mu r+\frac{\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\left(\frac{\rho_i}{r}\right)^{\frac{1}{2}}\cos\left(\rho_i r-\frac{3\pi}{4}\right)(\operatorname{ch}\nu_i r-1)\,dr\right| \leqq$$

$$\leqq c\frac{\rho_i^{\frac{1}{2}}}{\mu^{\frac{3}{2}}}\left|\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r+\frac{\pi}{4}\right)\cos\left(\rho_i r-\frac{3\pi}{4}\right)}{r}\frac{\operatorname{ch}\nu_i r-1}{r}\,dr\right|\leqq$$

$$\leqq c\frac{\rho_i^{\frac{1}{2}}}{\mu^{\frac{3}{2}}}\frac{\rho_i}{1+|\mu-\rho_i|}e^{|\nu_i|R}\leqq c\frac{e^{|\nu_i|R}}{\mu}\leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}$$

and similarly

$$\left|\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r+\frac{\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\left(\frac{\mu_i}{r}\right)^{\frac{1}{2}}\sin\left(\rho_i r-\frac{3\pi}{4}\right)\operatorname{sh}\nu_i r\,dr\right|\leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

Using the decomposition

$$\frac{\cos\left(\mu_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{1}{2}}r^{\frac{3}{2}}}-\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{\rho_i^{\frac{1}{2}}r^{\frac{3}{2}}}=\left(\frac{1}{\mu_i^{\frac{1}{2}}}-\frac{1}{\rho_i^{\frac{1}{2}}}\right)\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{r^{\frac{3}{2}}}+$$

$$+\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{1}{2}}r^{\frac{3}{2}}}(\operatorname{ch}\nu_i r-1)-j\frac{\sin\left(\rho_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{1}{2}}r^{\frac{3}{2}}}\operatorname{sh}\nu_i r$$

we get that

$$\left|\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r-\frac{\pi}{4}\right)}{(\mu r)^{\frac{1}{2}}}\left[\frac{\cos\left(\mu_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{1}{2}}r^{\frac{3}{2}}}-\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{\rho_i^{\frac{1}{2}}r^{\frac{3}{2}}}\right]dr\right|\leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

Finally we can put the absolute value into the integral to obtain

$$\left|\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{\cos\left(\mu r+\frac{\pi}{4}\right)}{(\mu r)^{\frac{3}{2}}}\left[\frac{\cos\left(\mu_i r-\frac{\pi}{4}\right)}{\mu_i^{\frac{1}{2}}r^{\frac{3}{2}}}-\frac{\cos\left(\rho_i r-\frac{\pi}{4}\right)}{\rho_i^{\frac{1}{2}}r^{\frac{3}{2}}}\right]dr\right|\leqq$$

$$\leqq c\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{1}{(\mu_i r)^{\frac{3}{2}}}\left(\frac{|\nu_i|}{(\rho_i r)^{\frac{3}{2}}}+\frac{e^{|\nu_i|r}-1}{|\mu_i|^{\frac{1}{2}}r^{\frac{3}{2}}}\right)dr\leqq c\frac{|\nu_i|}{\mu^{\frac{3}{2}}\rho_i^{\frac{3}{2}}}\rho_i^{2}+$$

$$+\frac{c}{\mu^{\frac{3}{2}}|\mu_i|^{\frac{1}{2}}}\int\limits_{\frac{R}{2\rho_i}}^{R}\frac{e^{|\nu_i|r}-1}{r^3}\,dr \leqq c\frac{e^{|\nu_i|R}}{(\mu\rho_i)^{\frac{1}{2}}}.$$

Consequently

$$|I_2| \leqq c\frac{e^{|\nu_i|r}}{(\mu\rho_i)^{\frac{1}{2}}}$$

and in case B) (45) is proved.

*Case* C: $\rho_i \leqq 1$. Then take the same transformation

$$\mu\int_0^R J_1(\mu r)\big[J_0(\mu_i r)-J_0(\rho_i r)\big]\;dr = I_1 + I_2$$

as in Case B; we have to prove that

$$|I_1|,\;|I_2| \leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{1}{2}}}.$$

Using (44) we obtain immediately that

$$|I_1| = \Big|J_0(\mu R)\big[J_0(\mu_i R)-J_0(\rho_i R)\big]\Big| \leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{1}{2}}}.$$

Now consider

$$I_2 = \int\limits_0^R J_0(\mu r)\big[\mu_i J_1(\mu_i r)-\rho_i J_1(\rho_i r)\big]\;dr;$$

here

$$\Bigg|\int\limits_0^{\frac{R}{2\mu}}\Bigg| \leqq c\int\limits_0^{\frac{R}{2\mu}} e^{|\nu_i|R}\,dr \leqq c\frac{e^{|\nu_i|R}}{\mu},$$

$$\Bigg|\int\limits_{\frac{R}{2\mu}}^{\min\left(\frac{R}{2\rho_i},R\right)}\Bigg| \leqq c\int\limits_{\frac{R}{2\mu}}^{\min\left(\frac{R}{2\rho_i},R\right)}\frac{1}{(\mu r)^{\frac{1}{2}}}e^{|\nu_i|R}\,dr \leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{1}{2}}}$$

and in case $\rho_i > \frac{1}{2}$,

$$\left|\int_{\frac{R}{2\rho_i}}^{R}\right| \leqq c\int_{\frac{R}{2\rho_i}}^{R} \frac{1}{(\mu r)^{\frac{1}{2}}}\left(\frac{|\nu_i|}{r^{\frac{1}{2}}} + |\mu_i|^{\frac{1}{2}}\frac{e^{|\nu_i| r}-1}{r^{\frac{1}{2}}}\right) dr \leqq$$

$$\leqq c\frac{|\nu_i|}{\mu^{\frac{1}{2}}} + \frac{c}{\mu^{\frac{1}{2}}}\frac{e^{|\nu_i|R}|\mu_i|^{\frac{1}{2}}}{1+|\nu_i|} \leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{1}{2}}}.$$

Lemma 5 is completely proved.

LEMMA 6. *For any dimension $N \geqq 1$ we have*

$$\left|\frac{\mu^{\frac{N}{2}}}{\mu_i^{\frac{N-2}{2}}}\int_0^R J_{\frac{N}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r)\, dr - \delta(\mu,\rho_i)\right| \leqq \tag{52}$$

$$\leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{N-1}{2}}, \quad \mu \geqq 1$$

*where*

$$\delta(\mu,\rho_i) := \begin{cases} 1 & \text{if} \quad \rho_i < \mu \\ \frac{1}{2} & \text{if} \quad \rho_i = \mu \\ 0 & \text{if} \quad \rho_i > \mu. \end{cases}$$

PROOF. The case $N = 1$ is proved in Joó, Komornik [7]. To prove the case $N = 2$, consider the decomposition

$$\mu\int_0^R J_1(\mu r) J_0(\mu_i r)\, dr =$$

$$= \mu\int_0^\infty J_1(\mu r) J_0(\rho_i r)\, dr - \mu\int_R^\infty J_1(\mu r) J_0(\rho_i r)\, dr +$$

$$+\mu\int_0^R J_1(\mu r)\left[J_0(\mu_i r) - J_0(\rho_i r)\right] dr =$$

$$=: I_1 + I_2 + I_3.$$

It is known [2] that

$$I_1 = \delta(\mu, \rho_i).$$

By Lemma 5

$$|I_3| \leqq c\frac{e^{|\nu_i|R}}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}.$$

We shall prove the same estimate for $I_2$.

*Case* A: $\rho_i \geqq \frac{\mu}{2}$. Then by integration by parts

$$|I_2| = \left|\mu\int_R^\infty J_1(\mu r)J_0(\rho_i r)\,dr\right| \leqq \frac{\mu}{\rho_i}\left|J_1(\mu R)J_1(\rho_i R)\right| +$$

$$+\frac{\mu^2}{\rho_i}\left|\int_R^\infty J_2(\mu r)J_1(\rho_i r)\,dr\right| \leqq c\frac{\mu^{\frac{1}{2}}}{\rho_i^{\frac{3}{2}}}+$$

$$+c\frac{\mu^{\frac{3}{2}}}{\rho_i^{\frac{3}{2}}}\left|\int_R^\infty\left[\frac{\cos\left(\mu r-\frac{5\pi}{4}\right)}{r^{\frac{1}{2}}}+O\left(\frac{1}{\mu^{\frac{3}{2}}}\right)\right]\cdot\right.$$

$$\left.\cdot\left[\frac{\cos\left(\rho_i r-\frac{3\pi}{4}\right)}{r^{\frac{1}{2}}}+O\left(\frac{1}{\rho_i r^{\frac{3}{2}}}\right)\right]dr\right| \leqq$$

$$\leqq c\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}\frac{1}{1+|\mu-\rho_i|}+c\left(\frac{\mu}{\rho_i}\right)^{\frac{3}{2}}\left|\int_R^\infty\frac{\cos\left(\mu r-\frac{5\pi}{4}\right)}{r}\cdot\right.$$

$$\left.\cdot\cos\left(\rho_i r-\frac{3\pi}{4}\right)dr\right| \leqq c\left(\frac{\mu}{1+\rho_i}\right)^{\frac{1}{2}}\frac{1}{1+|\mu-\rho_i|}.$$

*Case* B: $1 \leqq \rho_i \leqq \frac{\mu}{2}$. Then we make another integration by parts:

$$|I_2| \leqq \left|J_0(\mu R)J_0(\rho_i R)\right| + \rho_i\left|\int_R^\infty J_0(\mu r)J_1(\rho_i r)\,dr\right| \leqq$$

$$\leqq \frac{c}{(\mu\rho_i)^{\frac{1}{2}}}+c\left(\frac{\rho_i}{\mu}\right)^{\frac{1}{2}}\left|\int_R^\infty\left[\frac{\cos\left(\mu r-\frac{\pi}{4}\right)}{r^{\frac{1}{2}}}+O\left(\frac{1}{\mu r^{\frac{3}{2}}}\right)\right]\cdot\right.$$

$$\cdot\left[\frac{\cos\left(\rho_i r-\frac{3\pi}{4}\right)}{r^{\frac12}}+O\left(\frac{1}{\rho_i r^{\frac32}}\right)\right]dr\Bigg|\leqq$$

$$\leqq\frac{c}{(\mu\rho_i)^{\frac12}}+c\frac{\rho_i^{\frac12}}{\mu^{\frac32}}\leqq\frac{c}{1+|\mu-\rho_i|}\left(\frac{\mu}{1+\rho_i}\right)^{\frac12}.$$

*Case* C: $\rho_i\leqq 1$. Then in the decomposition given in Case B we make another partial integration:

$$\rho_i\left|\int_R^\infty J_0(\mu r)J_1(\rho_i r)\,dr\right|\leqq\frac{\rho_i}{\mu}|\,J_1(\mu R)J_1(\rho_i R)|\,+\,\frac{\rho_i^2}{\mu}\left|\int_R^\infty J_1(\mu r)J_2(\rho_i r)\,dr\right|.$$

Here the case $\rho_i=0$ is trivial, and if $\rho_i\neq 0$, then

$$\frac{\rho_i}{\mu}|\,J_1(\mu R)J_1(\rho_i R)|\leqq c\frac{\rho_i^{\frac12}}{\mu^{\frac32}}\leqq\frac{c}{\mu^{\frac12}},$$

$$\frac{\rho_i^2}{\mu_i}\left|\int_R^\infty J_1(\mu r)J_2(\rho_i r)\,dr\right|=$$

$$=\frac{\rho_i^2}{\mu_i}\left|\int_R^\infty\left[\frac{\cos\left(\mu r-\frac{3\pi}{4}\right)}{(\mu r)^{\frac12}}+O\left(\frac{1}{(\mu r)^{\frac32}}\right)\right]\cdot\right.$$

$$\left.\cdot\left[\frac{\cos\left(\rho_i r-\frac{5\pi}{4}\right)}{(\rho_i r)^{\frac12}}+O\left(\frac{1}{(\rho_i r)^{\frac32}}\right)\right]dr\right|\leqq$$

$$\leqq c\frac{\rho_i^{\frac12}}{\mu^{\frac32}}+c\frac{\rho_i^{\frac32}}{\mu^{\frac52}}\leqq\frac{c}{\mu^{\frac12}}.$$

So the statement of Lemma 6 is proved for $N=2$, too. Now consider the case $N\geqq 3$.

*Case* $A_1$: $\rho_i\leqq 2\mu$. In this case we can prove (52) by a simple induction on $N$. Using the rules

$$\int r^{-\alpha}J_{\alpha+1}(\mu r)\,dr=-\frac{1}{\mu}r^{-\alpha}J_\alpha(\mu r),$$

$$\left[r^{\alpha+1}J_{\alpha+1}(\mu r)\right]'=\mu r^{\alpha+1}J_\alpha(\mu r)$$

we get

$$\frac{\mu^{\frac{N}{2}}}{\mu_i^{\frac{N-2}{2}}} \int_0^R J_{\frac{N}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r)\, dr =$$

$$= -\left(\frac{\mu}{\mu_i}\right)^{\frac{N-2}{2}} J_{\frac{N-2}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i R) + \frac{\mu^{\frac{N-2}{2}}}{\mu_i^{\frac{N-4}{2}}} \int_0^R J_{\frac{N-2}{2}}(\mu r) J_{\frac{N-4}{2}}(\mu_i r)\, dr.$$

By the induction hypothesis

$$\left| \frac{\mu^{\frac{N-2}{2}}}{\mu_i^{\frac{N-4}{2}}} \int_0^R J_{\frac{N-2}{2}}(\mu r) J_{\frac{N-4}{2}}(\mu_i r)\, dr - \delta(\mu, \rho_i) \right| \leqq$$

$$\leqq c \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|} \left(\frac{\mu}{1+\rho_i}\right)^{\frac{N-3}{2}} \leqq c \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|} \left(\frac{\mu}{1+\rho_i}\right)^{\frac{N-1}{2}}.$$

In case $|\mu_i| \leqq 1$

$$\left| \left(\frac{\mu}{\mu_i}\right)^{\frac{N-2}{2}} J_{\frac{N-2}{2}}(\mu R) J_{\frac{N-2}{2}}(\mu_i R) \right| \leqq c \mu^{\frac{N-3}{2}} \leqq c \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|} \left(\frac{\mu}{1+\rho_i}\right)^{\frac{N-1}{2}}$$

and in case $|\mu_i| \geqq 1$

$$\left| \left(\frac{\mu}{\mu_i}\right)^{\frac{N-2}{2}} J_{\frac{N-2}{2}}(\mu R) J_{\frac{N-2}{2}}(\mu_i R) \right| \leqq c \frac{e^{|\nu_i| R}}{(\mu \mu_i)^{\frac{1}{2}}} \left(\frac{\mu}{|\mu_i|}\right)^{\frac{N-2}{2}} \leqq$$

$$\leqq c \left(\frac{\mu}{1+\rho_i}\right)^{\frac{N-1}{2}} \frac{e^{|\nu_i| R}}{1 + |\mu - \rho_i|}.$$

So in case $\rho_i \leqq 2\mu$ the estimate (52) is proved.

*Case* B$_1$: $\rho_i \geqq 2\mu$. We have to prove that

$$\left| \int_0^R J_{\frac{N}{2}}(\mu r) \frac{J_{\frac{N-2}{2}}(\mu_i r)}{\mu_i^{\frac{N-2}{2}}}\, dr \right| \leqq c \frac{e^{|\nu_i| R}}{\mu^{\frac{1}{2}} \rho_i^{\frac{N+1}{2}}}.$$

Consider the identity

$$\int_0^R J_{\frac{N}{2}}(\mu r)J_{\frac{N-2}{2}}(\mu_i r)\,dr = \frac{1}{\mu_i}J_{\frac{N}{2}}(\mu R)J_{\frac{N}{2}}(\mu_i R) + \frac{\mu}{\mu_i}\int_0^R J_{\frac{N+2}{2}}(\mu r)J_{\frac{N}{2}}(\mu_i r)\,dr.$$

Obviously

$$\left|\frac{1}{\mu_i}J_{\frac{N}{2}}(\mu R)J_{\frac{N}{2}}(\mu_i R)\right| \leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{1}{2}}|\mu_i|^{\frac{3}{2}}}$$

hence we have to show that

$$\left|\int_0^R J_{\frac{N+2}{2}}(\mu r)\frac{J_{\frac{N}{2}}(\mu_i r)}{\mu_i^{\frac{N}{2}}}\,dr\right| \leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{3}{2}}\rho_i^{\frac{N+1}{2}}}. \tag{53}$$

We substitute here $\mu_i$ by $\rho_i$:

$$\int_0^R J_{\frac{N+2}{2}}(\mu r)\frac{J_{\frac{N}{2}}(\mu_i r)}{\mu_i^{\frac{N}{2}}}\,dr =$$

$$= \frac{1}{\rho_i^{\frac{N}{2}}}\int_0^\infty J_{\frac{N+2}{2}}(\mu r)J_{\frac{N}{2}}(\rho_i r)\,dr - \frac{1}{\rho_i^{\frac{N}{2}}}\int_R^\infty J_{\frac{N+2}{2}}(\mu r)J_{\frac{N}{2}}(\rho_i r)\,dr +$$

$$+\int_0^R J_{\frac{N+2}{2}}(\mu r)\left[\frac{J_{\frac{N}{2}}(\mu_i r)}{\mu_i^{\frac{N}{2}}} - \frac{J_{\frac{N}{2}}(\rho_i r)}{\rho_i^{\frac{N}{2}}}\right]\,dr =: I_1 + I_2 + I_3.$$

From $\rho_i > \mu$ it follows that

$$I_1 = 0$$

see [2]. Further we have

$$|I_2| \leqq \frac{c}{\rho_i^{\frac{N}{2}}}\left|\int_R^\infty \left[\frac{\cos\left(\mu r - \frac{N+3}{4}\pi\right)}{(\mu r)^{\frac{1}{2}}} + O\left(\frac{1}{(\mu r)^{\frac{3}{2}}}\right)\right]\cdot\right.$$

$$\left.\cdot\left[\frac{\cos\left(\rho_i r - \frac{N+1}{4}\pi\right)}{(\rho_i r)^{\frac{1}{2}}} + O\left(\frac{1}{(\rho_i r)^{\frac{3}{2}}}\right)\right]\,dr\right| \leqq$$

$$\leqq \frac{c}{\mu^{\frac{3}{2}}\rho_i^{\frac{N+1}{2}}} + \frac{c}{\mu^{\frac{1}{2}}\rho_i^{\frac{N+1}{2}}}\left|\int_R^\infty \frac{\cos\left(\mu r - \frac{N+3}{4}\pi\right)\cos\left(\rho_i r - \frac{N+1}{4}\pi\right)}{r}\,dr\right| \leqq$$

$$\leqq \frac{c}{\mu^{\frac{3}{2}}\rho_i^{\frac{N+1}{2}}}.$$

Now consider $I_3$. We take the decomposition

$$\int_0^R = \int_0^{\frac{R}{\rho_i}} + \int_{\frac{R}{\rho_i}}^{\frac{R}{2\mu}} + \int_{\frac{R}{2\mu}}^R .$$

By (39) and (37)

$$\left|\int_0^{\frac{R}{\rho_i}} J_{\frac{N+2}{2}}(\mu r)\left[\frac{J_{\frac{N}{2}}(\mu_i r)}{\mu_i^{\frac{N}{2}}} - \frac{J_{\frac{N}{2}}(\rho_i r)}{\rho_i^{\frac{N}{2}}}\right] dr\right| \leqq$$

$$\leqq c\int_0^{\frac{R}{\rho_i}} (\mu r)^{\frac{N+2}{2}} r^{\frac{N+2}{2}} e^{|\nu_i|R}\,dr \leqq c e^{|\nu_i|R}\mu^{\frac{N+2}{2}}\rho_i^{-N-3} \leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{\rho_i^{\frac{N}{2}+2}} \leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{3}{2}}\rho_i^{\frac{N+1}{2}}},$$

$$\left|\int_{\frac{R}{\rho_i}}^{\frac{R}{2\mu}} J_{\frac{N+2}{2}}(\mu r)\left[\frac{J_{\frac{N}{2}}(\mu_i r)}{\mu^{\frac{N}{2}}} - \frac{J_{\frac{N}{2}}(\rho_i r)}{\rho_i^{\frac{N}{2}}}\right] dr\right| \leqq$$

$$\leqq c\int_{\frac{R}{\rho_i}}^{\frac{R}{2\mu}} (\mu r)^{\frac{N+2}{2}}\left(\frac{|\nu_i|}{r^{\frac{1}{2}}\rho_i^{\frac{N+3}{2}}} + \frac{e^{|\nu_i|R}-1}{|\mu_i|^{\frac{N+1}{2}} r^{\frac{1}{2}}}\right) dr \leqq$$

$$\leqq c\frac{|\nu_i|}{\mu^{\frac{1}{2}}\rho_i^{\frac{N+3}{2}}} + c\frac{\mu^{\frac{N+2}{2}}}{|\mu_i|^{\frac{N+2}{2}}}\int_{\frac{R}{\rho_i}}^{\frac{R}{2\mu}} r^{\frac{N+3}{2}}\frac{e^{|\nu_i|r}-1}{r}\,dr \leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{1}{2}}\rho_i^{\frac{N+3}{2}}}+c\frac{e^{|\nu_i|R}}{\mu^{\frac{3}{2}}|\mu_i|^{\frac{N+1}{2}}}\leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{3}{2}}\rho_i^{\frac{N+1}{2}}}.$$

For $\frac{R}{2\mu}\leqq r\leqq R$ we take the two term asymptotical expansion of the Bessel functions figuring in $I_3$:

$$J_{\frac{N+2}{2}}(\mu r)=c_1\frac{\cos\left(\mu r-\frac{N+3}{4}\pi\right)}{(\mu r)^{\frac{1}{2}}}+c_2\frac{\cos\left(\mu r-\frac{N+1}{4}\pi\right)}{(\mu r)^{\frac{3}{2}}}+O\left(\frac{1}{(\mu r)^{\frac{5}{2}}}\right),$$

$$\frac{J_{\frac{N}{2}}(\mu_i r)}{\mu_i^{\frac{N}{2}}}-\frac{J_{\frac{N}{2}}(\rho_i r)}{\rho_i^{\frac{N}{2}}}=c_3\left[\frac{\cos\left(\mu_i r-\frac{N+1}{4}\pi\right)}{r^{\frac{1}{2}}\mu_i^{\frac{N+1}{2}}}-\frac{\cos\left(\rho_i r-\frac{N+1}{4}\pi\right)}{r^{\frac{1}{2}}\rho_i^{\frac{N+1}{2}}}\right]+$$

$$+c_4\left[\frac{\cos\left(\mu_i r-\frac{N-1}{4}\pi\right)}{r^{\frac{3}{2}}\mu_i^{\frac{N+3}{2}}}-\frac{\cos\left(\rho_i r-\frac{N-1}{4}\pi\right)}{r^{\frac{3}{2}}\rho_i^{\frac{N+3}{2}}}\right]+$$

$$+O\left(\frac{|\nu_i|}{r^{\frac{5}{2}}\rho_i^{\frac{N+7}{2}}}+\frac{e^{|\nu_i|r}-1}{r^{\frac{5}{2}}|\mu_i|^{\frac{N+5}{2}}}\right).$$

We see from this expansion that the estimate

$$\left|\int\limits_{\frac{R}{2\mu}}^{R}J_{\frac{N+2}{2}}(\mu r)\left[\frac{J_{\frac{N}{2}}(\mu_i r)}{\mu_i^{\frac{N}{2}}}-\frac{J_{\frac{N}{2}}(\rho_i r)}{\rho_i^{\frac{N}{2}}}\right]dr\right|\leqq$$

$$\leqq c\frac{e^{|\nu_i|R}}{\mu^{\frac{3}{2}}\rho_i^{\frac{N+1}{2}}}$$

can be proved by the exact repetition of the steps of the proof of Lemma 5, Case A. Lemma 6 is proved.

Now we are able to estimate the spectral function. Introduce the function

$$v_R(r,\mu):=\begin{cases}(2\pi)^{-\frac{N}{2}}\left(\frac{\mu}{r}\right)^{\frac{N}{2}}J_{\frac{N}{2}}(\mu r) & \text{if}\quad 0\leqq r\leqq R\\ 0 & \text{if}\quad r>R.\end{cases}$$

LEMMA 2. *Let* $(u)_{i=1}^{\infty}$ *be a Riesz basis in* $L^2(\Omega)$, *and let* $K\subset\Omega$ *be compact,*

$$0<R<\min\left\{\operatorname{dist}(K,\partial\Omega),\frac{2\pi}{3}\right\}.$$

*Then*

$$\tag{54} \begin{cases} \Theta(x,y,\mu) = v_R\big(|x-y|,\mu\big) + \widehat{\Theta}(x,y,\mu), \\ \big\| \widehat{\Theta}(x,y,\mu)\big\|_{L^2_y(\Omega)} \leqq c\mu^{\frac{N-1}{2}}, \quad x \in K \end{cases}$$

*with $c$ independent of $x$ and $\mu$. If $\Omega$ is bounded and $1 \leqq q < \frac{2N}{N+1}$ then*

$$\tag{55} \big\| \Theta(x,\cdot,\mu)\big\|_{L^q(\Omega)} \leqq c\mu^{\frac{N-1}{2}}, \quad x \in K.$$

PROOF. Calculate the coefficients $\widehat{v}_i$ of $v_R$:

$$\tag{56} \overline{\widehat{v}_i} = \int_\Omega v_R\big(|x-y|,\mu\big)\, u_i(y)\, dy =$$

$$= \int_0^R r^{N-1} v_R(r,\mu) \int_\Theta u_i(x+r\Theta)\, d\Theta\, dr =$$

$$= \int_0^R r^{N-1}(2\pi)^{-\frac{N}{2}}\left(\frac{\mu}{r}\right)^{\frac{N}{2}} J_{\frac{N}{2}}(\mu r)\cdot(2\pi)^{\frac{N}{2}} \frac{J_{\frac{N-2}{2}}(\mu_i r)}{(\mu_i r)^{\frac{N-2}{2}}}\, dr \cdot u_i(x) =$$

$$= \frac{\mu^{\frac{N}{2}}}{\mu_i^{\frac{N-2}{2}}} u_i(x) \int_0^R J_{\frac{N}{2}}(\mu r) J_{\frac{N-2}{2}}(\mu_i r)\, dr.$$

Introduce the function

$$\tag{57} \widehat{\Theta}(x,y,\mu) := \sum_{\rho_i<\mu} \big(\overline{u_i(x)} - \widehat{v}_i\big)\, v_i(y) - \sum_{\rho_i \geqq \mu} \widehat{v}_i v_i(y).$$

Then

$$\big\| \widehat{\Theta}(x,\cdot,\mu)\big\|^2_{L^2(\Omega)} \asymp \sum_{\rho_i<\mu} \big| u_i(x) - \overline{\widehat{v}_i}\big|^2 + \sum_{\rho_i=\mu} |\widehat{v}_i|^2 + \sum_{\rho_i>\mu} |\widehat{v}_i|^2 =:$$

$$=: \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 .$$

From Lemma 6 we know that

$$\sum\nolimits_1 \leqq c \sum_{\rho_i<\mu} \frac{\big|u_i(x)\big|^2 e^{2|\nu_i|R}}{1+|\mu-\rho_i|^2} \left(\frac{\mu}{1+\rho_i}\right)^{N-1} \leqq$$

$$\leqq c \sum_{0 \leqq k \leqq [\mu]} \sum_{k \leqq \rho_i \leqq k+1} \frac{\left|u_i(x)\right|^2 e^{2|\nu_i|R}}{(k+1)^{N-1}\left(1+|\mu-k|\right)^2} \mu^{N-1} \leqq$$

$$\leqq c\mu^{N-1} \sum_{k=0}^{[\mu]} \frac{(k+1)^{N-1}}{(k+1)^{N-1}\left(1+|\mu-k|^2\right)} \leqq c\mu^{N-1},$$

$$\sum\nolimits_2 \leqq c \sum_{\rho_i=\mu} \left|u_i(x)\right|^2 \left(1+\frac{e^{2|\nu_i|R}}{1+|\mu-\rho_i|^2}\left(\frac{\mu}{\rho_i}\right)^{N-1}\right) \leqq$$

$$\leqq c \sum_{\rho_i=\mu} \left|u_i(x)\right|^2 e^{2|\nu_i|R} \leqq c\mu^{N-1},$$

$$\sum\nolimits_3 \leqq c \sum_{\rho_i>\mu} \frac{\left|u_i(x)\right|^2 e^{2|\nu_i|R}}{1+|\mu-\rho_i|^2}\left(\frac{\mu}{\rho_i}\right)^{N-1} \leqq$$

$$\leqq c\mu^{N-1} \sum_{k=[\mu]}^{\infty} \sum_{k \leqq \rho_i \leqq k+1} \frac{\left|u_i(x)\right|^2 e^{2|\nu_i|R}}{\left(1+|k-\mu|)^2\right) k^{N-1}} \leqq$$

$$\leqq c\mu^{N-1} \sum_{k=[\mu]}^{\infty} \frac{1}{1+|\mu-k|^2} \leqq c\mu^{N-1}$$

which proves (54). If $\Omega$ is bounded, then (54) obviously implies for $q<2$ that

$$\left\|\widehat{\Theta}(x,\cdot,\mu)\right\|_{L^q(\Omega)} \leqq c\left\|\widehat{\Theta}(x,\cdot,\mu)\right\|_{L^2(\Omega)} \leqq c\mu^{N-1}.$$

The main term of $\Theta$ can be estimated by

$$\left\|\mu^{\frac{N}{2}} v_R\left(|x-y|,\mu\right)\right\|_{L^q(\Omega)} =$$

$$= \mu^{\frac{N}{2}} \left(\int_0^R r^{N-1}\left(\frac{J_{\frac{N}{2}}(\mu r)}{r^{\frac{N}{2}}}\right)^q dr\right)^{\frac{1}{q}} \leqq$$

$$\leqq c\mu^{\frac{N-1}{2}} \left(\int_0^R r^{N-1}\left(\frac{1}{r^{\frac{N+1}{2}}}\right)^q dr\right)^{\frac{1}{q}} \leqq c\mu^{\frac{N-1}{2}}$$

if $N-1-\frac{N+1}{2}q>-1$ i.e. $q<\frac{2N}{N+1}$, which completes the proof of Lemma 7.

## 4. The kernel of fractional order; proof of Theorems 1, 2 and 3

Let $N > 1$, $\Omega \subset \mathbf{R}^N$ a domain and $(u_i) \subset L^2(\Omega)$ a Riesz basis. Let $0 < < \alpha < 2N$. As in [5–6], we introduce the notion of the kernel of fractional order $\alpha$; this is a kernel $T_\alpha(x,y)$, $x, y \in \Omega$, whose coefficients for all fixed $x$ are

$$\int_\Omega T_\alpha(x,y)\overline{u_i(y)}\,dy = \frac{\overline{u_i(x)}}{(1+\rho_i{}^2)^{\frac{\alpha}{2}}}, \qquad i = 1, 2, \ldots . \tag{58}$$

We can use the abbreviated form

$$T_\alpha(x,y) \sim \sum_{i=1}^{\infty} \frac{\overline{u_i(x)}v_i(y)}{(1+\rho_i{}^2)^{\frac{\alpha}{2}}}.$$

The question arises whether such a kernel exists. Our answer is positive for $\alpha > \frac{N-2}{2}$ (see the proof of Theorem 3 below). The proof which we will give here further develops the ideas of [5–6]. Recall the Bessel–Macdonald kernel

$$v_\alpha(r) = \frac{2^{\frac{2-\alpha}{2}}}{(2\pi)^{\frac{N}{2}}\Gamma\left(\frac{\alpha}{2}\right)} \frac{K_{\frac{N-\alpha}{2}}(r)}{r^{\frac{N-\alpha}{2}}}$$

defined in Section 1. For some fixed $0 < R$ and $n \in \{1, 2, \ldots\}$ define the polynomial

$$w_\alpha(r) := \sum_{k=0}^{n} a_k r^{2k}$$

by

$$v_\alpha(R) = w_\alpha(R), \quad v'_\alpha(R) = w'_\alpha(R), \ldots, v_\alpha^{(n)}(R) = w_\alpha^{(n)}(R); \tag{59}$$

we know that (59) determines uniquely the polynomial $w_\alpha$. Let further

$$v_\alpha^R(r) := \begin{cases} v_\alpha(r) & \text{if} \quad r \leqq R \\ 0 & \text{if} \quad r > R \end{cases}, \quad w_\alpha^R(r) := \begin{cases} w_\alpha(r) & \text{if} \quad r \leqq R \\ 0 & \text{if} \quad r > R \end{cases}.$$

For the sake of simplicity assume that $n > \alpha - \frac{N-1}{2}$. Define $\varphi_i = \varphi_i(x)$ as the $i$-th coefficient of the function $v_\alpha^R(|x-y|) - w_\alpha^R(|x-y|)$, then in case $0 < R < \operatorname{dist}(x, \partial\Omega)$ we have

$$\overline{\varphi}_i = \int_\Omega \Big[ v_\alpha^R(|x-y|) - w_\alpha^R(|x-y|) \Big] u_i(y)\,dy =$$

$$= \int_0^R r^{N-1}\left[v_\alpha(r) - w_\alpha(r)\right] \int_\Theta u_i(x + r\Theta)\, d\Theta\, dr =$$

$$= (2\pi)^{\frac{N}{2}} u_i(x) \int_0^R r^{N-1}\left[v_\alpha(r) - w_\alpha(r)\right] \frac{J_{\frac{N-2}{2}}(\mu_i r)}{(\mu_i r)^{\frac{N-2}{2}}}\, dr$$

We shall prove

LEMMA 8. *We have*

$$\left|\overline{\varphi}_i - \frac{u_i(x)}{(1+\rho_i{}^2)^{\frac{\alpha}{2}}}\right| \leqq c\frac{e^{|\nu_i|R}}{(1+\rho_i)^{\alpha+1}} \cdot \left|u_i(x)\right|. \tag{60}$$

PROOF. Introduce the operator

$$Df(r) := \frac{1}{r} f'(r).$$

Then (59) means that

$$0 = (v_\alpha - w_\alpha)(R) = D(v_\alpha - w_\alpha)(R) = \ldots = D^n(v_\alpha - w_\alpha)(R).$$

Using the rules ([2])

$$\int r^{\nu+1} J_\nu(\mu r)\, dr = \frac{1}{\mu} r^{\nu+1} J_{\nu+1}(\mu r),$$

$$D\big(r^{-\nu} K_\nu(r)\big) = -r^{-\nu-1} K_{\nu+1}(r)$$

we see that we can integrate by parts $n+1$ times such that the free terms of integrations vanish, namely

$$\overline{\varphi}_i = (2\pi)^{\frac{N}{2}} \frac{u_i(x)}{\mu_i{}^{\frac{N-2}{2}}} \int_0^R \left[v_\alpha(r) - w_\alpha(r)\right] r^{\frac{N}{2}} J_{\frac{N-2}{2}}(\mu_i r)\, dr =$$

$$= -(2\pi)^{\frac{N}{2}} \frac{u_i(x)}{\mu_i{}^{\frac{N}{2}}} \int_0^R D\left[v_\alpha(r) - w_\alpha(r)\right] r^{\frac{N+2}{2}} J_{\frac{N}{2}}(\mu_i r)\, dr = \ldots =$$

$$= (-1)^{n+1} (2\pi)^{\frac{N}{2}} u_i(x) \int_0^R D^{n+1}\left[v_\alpha(r) - w_\alpha(r)\right] r^{\frac{N}{2}+n+1} \frac{J_{\frac{N}{2}+n}(\mu_i r)}{\mu_i{}^{\frac{N}{2}+n}}\, dr.$$

We know that

$$D^{n+1} w_\alpha \equiv 0,$$

$$D^{n+1} v_\alpha(r) = \frac{2^{\frac{2-\alpha}{2}}}{(2\pi)^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right)} (-1)^{n+1} r^{\frac{\alpha-N}{2}-n-1} K_{\frac{N-\alpha}{2}+n+1}(r)$$

hence

$$\overline{\varphi}_i = \frac{2^{\frac{2-\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} u_i(x) \int_0^R r^{\frac{\alpha}{2}} K_{\frac{N-\alpha}{2}+n+1}(r) \frac{J_{\frac{N}{2}+n}(\mu_i r)}{\mu_i^{\frac{N}{2}+n}}\, dr =$$

$$= \frac{2^{\frac{2-\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} u_i(x) \Bigg\{ \int_0^\infty r^{\frac{\alpha}{2}} K_{\frac{N-\alpha}{2}+n+1}(r) \frac{J_{\frac{N}{2}+n}(\rho_i r)}{\rho_i^{\frac{N}{2}+n}}\, dr -$$

$$- \int_R^\infty r^{\frac{\alpha}{2}} K_{\frac{N-\alpha}{2}+n+1}(r) \frac{J_{\frac{N}{2}+n}(\rho_i r)}{\rho_i^{\frac{N}{2}+n}}\, dr +$$

$$+ \int_0^R r^{\frac{\alpha}{2}} K_{\frac{N-\alpha}{2}+n+1}(r) \left[ \frac{J_{\frac{N}{2}+n}(\mu_i)}{\mu_i^{\frac{N}{2}+n}} - \frac{J_{\frac{N}{2}+n}(\rho_i r)}{\rho_i^{\frac{N}{2}+n}} \right] dr \Bigg\} =$$

$$=: \frac{2^{\frac{2-\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} u_i(x) \{ I_1 + I_2 + I_3 \}.$$

Using the identity ([2])

$$\int_0^\infty \frac{K_\mu(at) J_\nu(bt)}{t^\lambda}\, dt = \frac{b^\nu \Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right)}{2^{\lambda+1} a^{\nu-\lambda+1} \Gamma(\nu+1)} \cdot$$

$$\cdot \Gamma\left(\frac{\nu-\lambda-\mu+1}{2}\right) {}_2F_1\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1, -\frac{b^2}{a^2}\right),$$

$$\operatorname{Re}(\nu+1) > |\operatorname{Re}\mu|, \quad \operatorname{Re} a > |\operatorname{Im} b|$$

we get

$$I_1 = \frac{\Gamma\left(\frac{\alpha}{2}\right)}{2^{\frac{2-\alpha}{2}}} \frac{1}{(1+\rho_i{}^2)^{\frac{\alpha}{2}}}.$$

Using the estimates

$$|K_\nu(r)| \leqq c\frac{e^{-r}}{r^{\frac{1}{2}}}, \quad |J_\nu(r)| \leqq \frac{c}{r^{\frac{1}{2}}}, \quad r \geqq R > 0$$

we get

$$|I_2| \leqq c\int_R^\infty e^{-r} r^{\frac{\alpha-1}{2}} \frac{1}{r^{\frac{1}{2}}\rho_i^{\frac{N+1}{2}+n}}\,dr \leqq \frac{c}{\rho_i^{\frac{N+1}{2}+n}}.$$

We estimate $I_3$ by (39) and (37). We know ([2]) that

$$|K_\nu(r)| \leqq \frac{c}{r^{|\nu|}}, \quad r \leqq R;$$

consequently we estimate in case $\rho_i \leqq 1$

$$|I_3| \leqq c\int_0^R r^{\frac{\alpha}{2}} r^{\frac{\alpha-N}{2}-n-1} r^{\frac{N}{2}+n+1} e^{|\nu_i|R}\,dr \leqq ce^{|\nu_i|R}$$

and in case $\rho_i \geqq 1$

$$\left|\int_0^{\frac{R}{\rho_i}} r^{\frac{\alpha}{2}} K_{\frac{N-\alpha}{2}+n+1}(r)\left[\frac{J_{\frac{N}{2}+n}(\mu_i r)}{\mu_i^{\frac{N}{2}+n}} - \frac{J_{\frac{N}{2}+n}(\rho_i r)}{\rho_i^{\frac{N}{2}+n}}\right]dr\right| \leqq$$

$$\leqq c\int_0^{\frac{R}{\rho_i}} r^{\frac{\alpha}{2}} r^{\frac{\alpha-N}{2}-n-1} r^{\frac{N}{2}+n+1} e^{|\nu_i|R}\,dr \leqq c\frac{e^{|\nu_i|R}}{(1+\rho_i)^{\alpha+1}},$$

$$\left|\int_{\frac{R}{\rho_i}}^{R} r^{\frac{\alpha}{2}} K_{\frac{N-\alpha}{2}+n+1}(r)\left[\frac{J_{\frac{N}{2}+n}(\mu_i r)}{\mu_i^{\frac{N}{2}+n}} - \frac{J_{\frac{N}{2}+n}(\rho_i r)}{\rho_i^{\frac{N}{2}+n}}\right]dr\right| \leqq$$

$$\leqq c\int_{\frac{R}{\rho_i}}^{R} r^{\frac{\alpha}{2}} r^{\frac{\alpha-N}{2}-n-1}\left(\frac{|\nu_i|}{r^{\frac{1}{2}}\rho_i^{\frac{N+3}{2}+n}} + \frac{e^{|\nu_i|r}-1}{r^{\frac{1}{2}}|\mu_i|^{\frac{N+1}{2}+n}}\right)dr \leqq$$

$$\leqq c\frac{|\nu_i|}{\rho_i^{\frac{N+3}{2}+n}}\rho_i^{-\alpha+\frac{N+1}{2}+n}+\frac{c}{\rho_i^{\frac{N+1}{2}+n}}\int\limits_{\frac{R}{\rho_i}}^{R} r^{\alpha-\frac{N+1}{2}-n}\frac{e^{|\nu_i|r}-1}{r}\,dr\leqq$$

$$\leqq c\frac{|\nu_i|}{\rho_i^{\alpha+1}}+c\frac{e^{|\nu_i|R}}{\rho_i^{\frac{N+1}{2}+n}}\int\limits_{\frac{R}{\rho_i}}^{R} r^{\alpha-\frac{N+1}{2}-n}\,dr\leqq c\frac{e^{|\nu_i|R}}{(1+\rho_i)^{\alpha+1}}.$$

Lemma 8 is proved.

Define the partial sums of the kernel $T_\alpha$:

$$E_\mu T_\alpha(x,y):=\sum_{\rho_i<\mu}\frac{\overline{u_i(x)}v_i(y)}{(1+\rho_i^2)^{\frac{\alpha}{2}}}.$$

LEMMA 9. *Let $N>1$, $\Omega\subset\mathbf{R}^N$ bounded and $K\subset\Omega$ a compact subset. Then*

$$\left\|E_\mu T_{\frac{N-1}{2}}(x,\cdot)\right\|_{L^q(\Omega)}\leqq c,\quad 1<q<\frac{2N}{N+1},\quad x\in K. \tag{61}$$

*Further if $K\subset\Omega_1\subset\Omega$ is another domain, then*

$$\left\|E_\mu T_{\frac{N-1}{2}}(x,\cdot)\right\|_{L^2(\Omega\setminus\Omega_1)}\leqq c,\quad c\in K. \tag{62}$$

PROOF. It is based on the partial integration

$$E_\mu T_{\frac{N-1}{2}}(x,y)=\int\limits_0^\mu\frac{1}{(1+t^2)^{\frac{N-1}{4}}}\,d_t\Theta(x,y,t)=$$

$$=\frac{\Theta(x,y,\mu)}{(1+\mu^2)^{\frac{N-1}{4}}}+\frac{N-1}{4}\int\limits_0^\mu\Theta(x,y,t)\frac{2t}{(1+t^2)^{1+\frac{N-1}{4}}}\,dt=:I_1+I_2.$$

Prove first (61). By (51) we have

$$\|I_1\|_{L^q_y(\Omega)}\leqq c.$$

In $I_2$ it is enough to restrict ourselves to $\int\limits_1^{\mu}$. Let $R < \min\left(\operatorname{dist}(K, \partial\Omega), \frac{2\pi}{3}\right)$. Take first the remainder $\widehat{\Theta}$. By (52) and (57) it has the form

$$\widehat{\Theta}(x,y,t) = \sum_{i=1}^{\infty} \overline{u_i(x)} v_i(y) \frac{e^{|\nu_i|R}}{1+|t-\rho_i|} \left(\frac{t}{1+\rho_i}\right)^{\frac{N-1}{2}} R_i(t)$$

where

$$|R_i| \leqq c$$

and $c$ does not depend on $i, t, x, y$. Consequently

$$\int\limits_1^{\mu} \widehat{\Theta}(x,y,t) \frac{t}{(1+t^2)^{\frac{N+3}{2}}}\, dt =$$

$$= \sum_{i=1}^{\infty} \overline{u_i(x)} v_i(y) \frac{e^{|\nu_i|R}}{(1+\rho_i{}^2)^{\frac{N-1}{2}}} \int\limits_1^{\mu} R_i(t) \frac{t^{\frac{N-1}{2}}}{1+|t-\rho_i|} \frac{t}{(1+t^2)^{\frac{N+3}{4}}}\, dt.$$

Now

$$\int\limits_1^{\mu} \frac{t^{\frac{N-1}{2}}}{1+|t-\rho_i|} \frac{t}{(1+t^2)^{\frac{N+3}{4}}}\, dt \leqq \int\limits_1^{\mu} \frac{1}{1+|t-\rho_i|} \frac{t}{1+t^2}\, dt \leqq$$

$$\leqq c \frac{\ln(1+\rho_i)}{1+\rho_i} \leqq \frac{c}{(1+\rho_i)^{\frac{2}{3}}}$$

hence

$$\left\| \int\limits_1^{\mu} \widehat{\Theta}(x,\cdot,t) \frac{t}{(1+t^2)^{\frac{N+3}{4}}}\, dt \right\|^2_{L^q(\Omega)} \leqq$$

$$\leqq c \left\| \sum_{i=1}^{\infty} |u_i(x)|\, |v_i(y)| \frac{e^{|\nu_i|R}}{(1+\rho_i)^{\frac{N-1}{2}+\frac{2}{3}}} \right\|^2_{L^2_y(\Omega)} \leqq$$

$$= c \sum_{i=1}^{\infty} \frac{|u_i(x)|^2 e^{2|\nu_i|R}}{(1+\rho_i)^{N+\frac{1}{3}}} \leqq c.$$

Consider the main term. Denote $r := |x-y|$, then in case $r \leqq \frac{1}{\mu}$ we have

$$\left|\int_1^\mu v_R(r,t)\frac{t}{(1+t^2)^{\frac{N+3}{4}}}\,dt\right| = c\left|\int_1^\mu \left(\frac{t}{r}\right)^{\frac{N}{2}} J_{\frac{N}{2}}(tr)\frac{t}{(1+t^2)^{\frac{N+3}{4}}}dt\right| \leqq$$

$$\leqq c\int_1^\mu \left(\frac{t}{r}\right)^{\frac{N}{2}}(tr)^{\frac{N}{2}}\frac{t}{(1+t^2)^{\frac{N+3}{4}}}\,dt \leqq \int_1^\mu t^{\frac{N-1}{2}}\,dt \leqq c\mu^{\frac{N+1}{2}}$$

and

$$\left\|\mu^{\frac{N+1}{2}}\right\|^q_{L^q\left(|x-y|<\frac{1}{\mu}\right)} = \int\limits_{|x-y|<\frac{1}{\mu}} \mu^{\frac{N+1}{2}q}\,dy =$$

$$= \mu^{\frac{N+1}{2}q}\int_0^{\frac{1}{\mu}} r^{N-1}\,dr \leqq c\mu^{\frac{N+1}{2}q-N} \leqq c.$$

In case $r \geqq \frac{1}{\mu}$ we have analogously

$$\left|\int_1^{\frac{1}{r}} v_R(r,t)\frac{t}{(1+t^2)^{\frac{N+3}{4}}}\,dt\right| \leqq c\int_1^{\frac{1}{r}} t^{\frac{N-1}{2}}\,dt \leqq \frac{c}{r^{\frac{N+1}{2}}}.$$

The integral $\int_{\frac{1}{r}}^{\mu}$ will be estimated by the asymptotical expansion

$$J_{\frac{N}{2}}(tr) = c\frac{\cos(tr+\beta)}{(tr)^{\frac{1}{2}}} + O\left(\frac{1}{(tr)^{\frac{3}{2}}}\right).$$

The remainder term gives

$$\int_{\frac{1}{r}}^{\mu}\left(\frac{t}{r}\right)^{\frac{N}{2}}\frac{1}{(tr)^{\frac{3}{2}}}\frac{t}{(1+t^2)^{\frac{N+3}{4}}}\,dt \leqq \frac{c}{r^{\frac{N+3}{2}}}\int_{\frac{1}{r}}^{\mu}\frac{t^{\frac{N-1}{2}}}{(1+t^2)^{\frac{N+3}{4}}}\,dt \leqq \frac{c}{r^{\frac{N+1}{2}}}$$

and the main term

$$\left|\int_{\frac{1}{r}}^{\mu}\left(\frac{t}{r}\right)^{\frac{N}{2}}\frac{\cos(tr+\beta)}{(tr)^{\frac{1}{2}}}\frac{t}{(1+t^2)^{\frac{N+3}{4}}}\,dt\right| =$$

$$= \frac{1}{r^{\frac{N+1}{2}}} \left| \int_{\frac{1}{r}}^{\mu} \frac{\cos(tr+\beta)}{t} \cdot \frac{t^{\frac{N+3}{2}}}{(1+t^2)^{\frac{N+3}{4}}} \, dt \right| \leqq$$

$$\leqq \frac{1}{r^{\frac{N+1}{2}}} \left\{ \left( \frac{\mu^2}{1+\mu^2} \right)^{\frac{N+3}{4}} \left| \int_{\frac{1}{r}}^{\mu} \frac{\cos(tr+\beta)}{t} \, dt \right| + \right.$$

$$\left. + \int_{\frac{1}{r}}^{\mu} \left| \int_{\frac{1}{r}}^{u} \frac{\cos(tr+\beta)}{t} \, dt \right| \left[ \left( \frac{u^2}{1+u^2} \right)^{\frac{N+3}{4}} \right]' du \right\} \leqq \frac{c}{r^{\frac{N+1}{2}}}$$

because

$$\left( \frac{u^2}{1+u^2} \right)' > 0, \qquad \left| \int_{\frac{1}{r}}^{u} \frac{\cos(tr+\beta)}{t} \, dt \right| \leqq c.$$

Consequently

$$\left\| \int_{1}^{\mu} v_R(r,t) \frac{t}{(1+t^2)^{\frac{N+3}{2}}} \, dt \right\|_{L^q(|x-y| \geqq \frac{1}{\mu})}^{q} \leqq c \int_{\frac{1}{\mu}}^{R} r^{N-1-q\frac{N+1}{2}} \, dr \leqq c.$$

Thus (61) is proved. To prove (62), let

$$0 < R < \min \left\{ \operatorname{dist}(K, \partial\Omega_1), \quad \frac{2\pi}{3} \right\}.$$

Then the main term $v_R(r) = 0$, since $r > R$, so

$$\Theta(x,y,t) = \widehat{\Theta}(x,y,t).$$

Consequently

$$\|I_1\|_{L^2(\Omega \setminus \Omega_1)}^2 \leqq \frac{c}{\mu^{N-1}} \left\| \widehat{\Theta}(x,\cdot,\mu) \right\|_{L^2(\Omega)}^2 \leqq \frac{c}{\mu^{N-1}} \left( \sum_{\rho_i = \mu} |u_i(x)|^2 + \right.$$

$$\left. + \sum_{i=1}^{\infty} \frac{|u_i(x)|^2 e^{2|\nu_i|R}}{1+|\mu-\rho_i|^2} \left( \frac{\mu}{1+\rho_i} \right)^{N-1} \right) \leqq c \sum_{k=1}^{\infty} \frac{1}{1+|\mu-k|^2} \leqq c,$$

$$\|I_2\|^2{}_{L^2(\Omega\setminus\Omega_1)} \leqq c + c\left\| \sum_{i=1}^{\infty} |u_i(x)|\, |v_i(y)| \frac{e^{|\nu_i|R}}{(1+\rho_i)^{\frac{N}{2}+\frac{1}{6}}} \right\|^2_{L^2(\Omega)} \leqq$$

$$\leqq c + c\sum_{i=1}^{\infty} \frac{|u_i(x)|^2 e^{2|\nu_i|R}}{(1+\rho_i)^{N+\frac{1}{3}}} \leqq c.$$

Lemma 9 is proved.

We finally need the following known result:

LEMMA 10 ([13]). *Let* $1 \leqq p \leqq \infty$, $\alpha \geqq 0$. *Then for every* $f \in L_p^\alpha(\mathbf{R}^N)$, $\operatorname{supp} f \subset \Omega$ *there exists a (unique)* $h \in L^p(\Omega)$ *such that* $\operatorname{supp} h \subset \Omega$,

$$f(x) = \int\limits_{\Omega} \left(v_\alpha^R - w_\alpha^R\right)\left(|x-y|\right) h(y)\, dy, \quad x \in \Omega$$

*further*

$$\|f\|_{L_p^\alpha} \asymp \|h\|_{L^p(\Omega)}$$

*and here the implicit constants may depend only on* $K_1$, $\operatorname{supp} f \subset K_1$, *and on* $R, \alpha, p, N$.

PROOF OF THEOREM 3. Remark first that there exists $\alpha \geqq \alpha' > \frac{N}{2}$ with

$$f \in L_2^{\alpha'}(\mathbf{R}^N).$$

Indeed, if $p \geqq 2$ then we can take $\alpha' = \alpha$ by the boundedness of $\Omega$ and if $1 \leqq p < 2$ then we apply the imbedding [4]

$$L_p^\alpha(\mathbf{R}^N) \subset L_2^{\alpha'}(\mathbf{R}^N), \quad \alpha' = \alpha - N\left(\frac{1}{p} - \frac{1}{2}\right) > 0.$$

In fact we have $\alpha' = \alpha - \frac{N}{p} + \frac{N}{2} > \frac{N}{2}$. So take $f \in L_2^\alpha$, $\alpha > \frac{N}{2}$ $\operatorname{supp} f \subset \Omega$ and let $h \in L^2(\Omega)$ be the function from Lemma 10. Define the function

$$\psi_\alpha(x,y) := \sum \gamma_i \overline{u_i(x)} v_i(y), \quad \gamma_i := -\frac{\varphi_i}{u_i(x)} + \frac{1}{(1+{\rho_i}^2)^{\frac{\alpha}{2}}}.$$

This series converges in $L_y^2(\Omega)$ for every fixed $x$ by Lemma 8 (even for $\alpha >$ $> \frac{N}{2} - 1$). Consequently the function

$$T_\alpha(x,y) := v_\alpha^R\left(|x-y|\right) - w_\alpha^R\left(|x-y|\right) + \psi_\alpha(x,y)$$

is the kernel of fractional order $\alpha$. It is enough to show that the expansions of the functions

$$g(x) := \int_\Omega \overline{\psi_\alpha(x,y)} h(y)\, dy,$$

$$F(x) := \int_\Omega \overline{T_\alpha(x,y)} h(y)\, dy$$

converge absolutely and locally uniformly. We know that

$$g(x) = \sum \overline{\gamma_i} u_i(x) \langle h, v_i \rangle$$

and

$$\sum_{i=k}^\infty \left| \overline{\gamma_i} u_i(x) \langle h, v_i \rangle \right| \leqq \left( \sum_{i=k}^\infty \left| \langle h, v_i \rangle \right|^2 \right)^{\frac{1}{2}} \cdot$$

$$\cdot \left( \sum_{i=1}^\infty \frac{\left| u_i(x) \right|^2 e^{2|\nu_i|R}}{(1+\rho_i)^{2\alpha+2}} \right)^{\frac{1}{2}} \leqq c(K) \cdot o_k(1).$$

Since for $\alpha > \frac{N}{2}$ the series of $T_\alpha(x,y)$ converges in $L^2_y(\Omega)$ for every fixed $x$, we get analogously

$$F(x) = \sum \frac{u_i(x)}{(1+\rho_i{}^2)^{\frac{\alpha}{2}}} \langle h, v_i \rangle$$

and then

$$\sum_{i=k}^\infty \frac{\left| u_i(x) \right|}{(1+\rho_i{}^2)^{\frac{\alpha}{2}}} \left| \langle h, v_i \rangle \right| \leqq \left( \sum_{i=k}^\infty \left| \langle h, v_i \rangle \right|^2 \right)^{\frac{1}{2}} \cdot$$

$$\cdot \left( \sum_{i=1}^\infty \frac{\left| u_i(x) \right|^2}{(1+\rho_i{}^2)^{\alpha}} \right)^{\frac{1}{2}} \leqq c(K) o_k(1)$$

which proves Theorem 3.

Proof of Theorem 1. We shall show first that

$$f \in L_q^{N-1}(\mathbf{R}^N) \quad \text{for some} \quad q > \frac{2N}{N-1}. \tag{63}$$

The tool of the proof is the following imbedding theorem [4]:

$$L_p^\alpha(\mathbf{R}^N) \subset L_q^{\alpha-N\left(\frac{1}{p}-\frac{1}{q}\right)}(\mathbf{R}^N)$$

if $1 < p < q < \infty$ and $\alpha > N\left(\frac{1}{p} - \frac{1}{q}\right)$.

If $\alpha = \frac{N-1}{2}$, then $q = p > \frac{N}{\alpha} = \frac{2N}{N-1}$ will be appropriate. If $\alpha > \frac{N-1}{2}$ then we want to apply the mentioned imbedding theorem, i.e. let $\frac{N-1}{2} = \alpha - N\left(\frac{1}{p} - \frac{1}{q}\right)$, $q = \frac{pN}{N-\alpha p+p\frac{N-1}{2}}$. Now if $N - \alpha p + p\frac{N-1}{2} > 0$ then $p < q$ is satisfied, because $\alpha > \frac{N-1}{2}$, which proves (63). If $N - \alpha p + p\frac{N-1}{2} \leqq 0$ then we can take some $\alpha > \beta > \frac{N-1}{2}$ satisfying $\beta p > N$ and $N - \beta p + p\frac{N-1}{2} > > 0$; in this case $f \in L_p^\beta(\mathbf{R}^N)$ and $L_p^\beta \subset L_q^{\frac{N-1}{2}}$. So (63) is proved and then it is enough to prove the following statement: Let $f \in L_p^{\frac{N-1}{2}}(\mathbf{R}^N)$ for some $p > \frac{2N}{N-1}$ and $\operatorname{supp} f \subset \Omega$. Then the expansion of $f$ converges locally uniformly in $\Omega$.

Let $0 < R < \min\left\{\operatorname{dist}(K, \partial\Omega), \frac{2\pi}{3}\right\}$, then $\psi_{\frac{N-1}{2}}(x, \cdot) \in L^2(\Omega)$ and

$$\left\| E_\mu \overline{\psi_{\frac{N-1}{2}}(x,\cdot)} \right\|_{L^q(\Omega)} \leqq c \left\| E_\mu \overline{\psi_{\frac{N-1}{2}}(x,\cdot)} \right\|_{L^2(\Omega)} \leqq$$

$$\leqq c \left( \sum_{\rho_i<\mu} \frac{\left|u_i(x)\right|^2 e^{2|\nu_i|R}}{(1+\rho_i)^{N+1}} \right)^{\frac{1}{2}} \leqq c.$$

Denote $f_i$ the coefficients of $f$, then

$$\sum_{\rho_i<\mu} f_i u_i(x) = \sum_{\rho_i<\mu} u_i(x) \int_\Omega \overline{v_i(z)} \int_\Omega \left(v_\alpha^R - w_\alpha^R\right)\left(|z-y|\right) h(y)\, dy\, dz =$$

$$= \int_\Omega h(y) \sum_{\rho_i<\mu} u_i(x) \int_\Omega \left(v_\alpha^R - w_\alpha^R\right)\left(|z-y|\right) \overline{v_i(z)}\, dz\, dy =$$

$$= \int_\Omega h(y) E_\mu \Big[ \left(v_\alpha^R - w_\alpha^R\right)\left(|x-y|\right) \Big]\, dy.$$

By (64) and (61) we obtain

$$\text{(65)} \qquad \left| \sum_{\rho_i<\mu} f_i u_i(x) \right| \leqq \|h\|_{L^p(\Omega)} \left\| E_\mu \Big[ \left(v_\alpha^R - w_\alpha^R\right)\left(|x-y|\right) \Big] \right\|_{L_y^q(\Omega)} \leqq$$

$$\leqq c\|h\|_{L^p(\Omega)}.$$

We know that $C_o^\infty(\mathbf{R}^N)$ is dense in $L_p^\alpha(\mathbf{R}^N)$ ([4]), hence we can take

$$f^{(k)} \in C_0^\infty(\Omega) \quad f^{(k)} \to f \quad \text{in} \quad L_p^\alpha.$$

Consider the functions $h^{(k)}$ representing $f^{(k)}$, then by Lemma 10

$$h^{(k)} \to h \quad \text{in} \quad L^p(\Omega).$$

Now we have

$$\left|f(x) - \sum_{\rho_i<\mu} f_i u_i(x)\right| \leqq \left|f^{(k)}(x) - \sum_{\rho_i<\mu} f_i^{(k)} u_i(x)\right| +$$

$$+\left|f^{(k)}(x) - f(x)\right| + \left|\sum_{\rho_i<\mu}\left(f_i - f_i^{(k)}\right)u_i(x)\right| =: I_1 + I_2 + I_3.$$

Since $L_p^{\frac{N-1}{2}}(\Omega) \subset L^\infty(\Omega)$ [4], for large $k$,

$$I_2 < \frac{\varepsilon}{3}.$$

If we apply (65) with $f - f^{(k)}$ instead of $f$, then for large $k$ we obtain

$$I_3 < \frac{\varepsilon}{3}.$$

Finally for fixed $k$ and for large $\mu > \mu(k)$ we obtain from Lemma 11 that

$$I_1 < \frac{\varepsilon}{3}.$$

Theorem 1 is proved.

PROOF OF THEOREM 2. Let $K \subset \Omega_0$ be compact, let

$$R_0 := \frac{1}{2}\operatorname{dist}(K, \partial\Omega_0),$$

$$K_{R_0} := \left\{x \in \Omega: \operatorname{dist}(x, K) \leqq R_0\right\}.$$

We can repeat the proof of Theorem 1 with $\Omega \setminus K_{R_0}$ instead of $\Omega$. This means that for the function $h$ representing $f$ we have

$$h|_{K_{R_0}} \equiv 0.$$

Hence (65) is substituted by the estimate

$$\left|\sum_{\rho_i<\mu} f_i u_i(x)\right| = \left|\int_\Omega h(y)E_\mu\Big[\left(v_\alpha^R - w_\alpha^R\right)\left(|x-y|\right)\Big]\,dy\right| \leqq$$

$$\leqq \|h\|_{L^2(\Omega)}\left\|E_\mu\Big[\left(v_\alpha^R - w_\alpha^R\right)\left(|x-y|\right)\Big]\right\|_{L^2(\Omega\setminus K_{R_0})} \leqq c\|h\|_{L^2(\Omega)}$$

(we used (64) and (62)). The remaining part of the proof is the same as in Theorem 1.

## References


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# AN ELEMENTARY PROOF FOR A RESULT ON SIMULATED FACTORING

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**1. Introduction.** Let $G$ be a finite abelian group written multiplicatively with identity element $e$. Let $A_1, \dots, A_n$ be subsets of $G$. If each element $g$ of $G$ is uniquely expressible in the form

$$g = a_1 \cdots a_n, \qquad a_1 \in A_1, \dots, a_n \in A_n,$$

then we say that the product $A_1 \cdots A_n$ is a *factorization* of $G$. In the most commonly used factorizations the factors are subgroups.

In [1] the case when the factors are close to being subgroup were considered. The subset $A_i$ of $G$ is said to be *simulated* if there is a subgroup $H_i$ of $G$ such that $|A_i| = |H_i| \geqq 3$ and $|A_i \cap H_i| + 1 \geqq |A_i|$. It was proved that if $A_1 \cdots A_n$ is a factorization of $G$, then one of the factors is a subgroup of $G$, that is, there is an $i$, $1 \leqq i \leqq n$ such that $A_i = H_i$. At a point the proof essentially made use of group characters. The purpose of this paper is to give a new proof which is elementary in the sense that does not use characters.

**2. The result.** The subset $A$ is defined to be *periodic* if there exists an element $g$ of $G \setminus \{e\}$ with $gA = A$. We refer to such elements $g$ as *periods* of $A$.

The periodicity of a subset can be tested in terms of its translates.

LEMMA 1. *If $A$ is a nonempty subset of a finite abelian group, then the nonidentity elements of the subgroup*

$$H = \bigcap_{a \in A} a^{-1} A$$

*are all the periods of $A$.*

PROOF. Let $A = \{a_1, \dots, a_n\}$ and suppose that $g \in H \setminus \{e\}$. There are elements $b_1, \dots, b_n \in A$ such that $g = b_1 a_1^{-1} = \cdots = b_n a_n^{-1}$. Since $b_1, \dots, b_n$ are different elements they are all the elements of $A$. Consequently,

$$gA = \{ga_1, \dots, ga_n\} = \{b_1 a_1^{-1} a_1, \dots, b_n a_n^{-1} a_n\} = \{b_1, \dots, b_n\} = A.$$

The converse, that each period $g$ of $A$ lies in $H$, is immediate. This completes the proof.

Let $A$ be a simulated subset of $G$ and let $H$ be the corresponding subgroup of $G$. There are elements $a \in A$ and $h \in H$ such that $a \notin H$ and $h \notin A$. Further there is an element $d \in G$ for which $a = hd$. Clearly, the subset $A$ uniquely determines the subgroup $H$ and the elements $h$ and $d$. Conversely, the subgroup $H$ and the elements $h$ and $d$ determine the subset $A$.

LEMMA 2. *Let $A$ be a simulated subset of $G$ and let $H$ be the corresponding subgroup of $G$. If $A$ is periodic, then $A = H$.*

PROOF. Let $g$ be a period of $A$ and suppose that $|g| = r$, where $r$ is prime. The permutation defined by

$$a \to ag, \quad a \in A$$

consists of cycles of length $r$. Consider the cycle which contains $hd$. If $r \geqq 3$, then there must be at least two further elements $a$ and $b$ in this cycle contained by $H$. So $g^t a = b$ for some $t$, $0 \leqq t \leqq r-1$. Therefore $g \in H$. Similarly, $g^s a = hd$ for some $s$, $0 \leqq s \leqq r-1$ and so $d \in H$.

If $r = 2$, then since $|A| \geqq 3$ the permutation must contain at least one additional cycle. As before, using this second cycle we can see that $g \in H$ and using the first cycle we have that $d \in H$. This completes the proof.

LEMMA 3. *Let $G = AB$ be a factorization of the finite abelian group $G$. Suppose that $A$ is a simulated subset of $G$ and $H$ is the corresponding subgroup of $G$. Then $G = HB$ is also a factorization of $G$.*

PROOF. Let $H = \{h_1, \dots, h_s\}$ and $A = \{h_1, \dots, h_{s-1}, h_s d\}$ and suppose that $h_1 = e$. The fact that $G = AB$ is a factorization of $G$ can be expressed such that $h_1 B, \dots, h_{s-1}B, h_s dB$ are disjoint subsets of $G$. If $G = HB$ is not a factorization of $G$, then the subsets $h_1 B, \dots, h_{s-1}B, h_s B$ are not disjoint. Thus $h_i B \cap h_s B$ is not empty for some $i$, $1 \leqq i \leqq s-1$.

If $i \neq 1$, then we have the contradiction $h_i h_i^{-1} B \cap h_s h_i^{-1} B = h_1 B \cap h_j B$ is not empty and $1 \leqq j \leqq s-1$.

If $i = 1$, then since $|H| = s \geqq 3$ there is a $h_k \in H$ with $2 \leqq k \leqq s-1$. Now we have the contradiction that $h_k h_1 B \cap h_k h_s B = h_k B \cap h_j B$ is not empty and $1 \leqq j, k \leqq s-1$. This completes the proof.

THEOREM 1. *Let*

$$G = A_1 \cdots A_n \tag{1}$$

*be a factorization of the finite abelian group $G$, where for each $i$, $1 \leqq i \leqq n$ there is a subgroup $H_i$ of $G$ such that $|A_i| = |H_i| \geqq 3$ and $|A_i \cap H_i| \geqq |A_i| - 1$. Then $A_i = H_i$ for some $i$, $1 \leqq i \leqq n$.*

PROOF. If $A_i = H_i$ for some $i$, $1 \leqq i \leqq n$, then the result holds. So we may suppose that $|A_i \cap H_i| = |A_i| - 1$, that is, $A_i$ is a simulated subset of $G$

for each $i$, $1 \leqq i \leqq n$. If $n = 1$, then the result is trivial so we may proceed by induction on $n$. By Lemma 3 the factor $A_1$ can be replaced by $H_1$ to give the factorization

$$G = H_1 A_2 \cdots A_n. \tag{2}$$

From this we have the factorization of the factorgroup $G/H_1$

$$G/H_1 = (H_1 A_2)/H_1 \cdots (H_1 A_n)/H_1.$$

By the inductive assumption some factor $(H_1 A_i)/H_1$ is a subgroup of $G/H_1$. We may assume that $i = 2$ because this is only a matter of indexing the factors. We may consider a suitable factorgroup again to get a new factorization. Continuing in this way we conclude that there is a subgroup $M$ of $G$ which contains all but one of the subsets $H, A_2, \ldots, A_n$. We suppose that $A_n \not\subset M$ since this is only a matter of indexing the factors.

Let $b \in A_n$. From factorization (2) multiplying by $b^{-1}$ we have that

$$G = H_1 A_2 \cdots A_{n-1}(b^{-1}A_n) \tag{3}$$

is also a normed factorization of $G$. Since $H_1, A_2, \ldots, A_{n-1} \subset M$ restricting the factorization (3) to $M$ we have the factorization $M = H_1 A_2 \cdots A_{n-1}$. We distinguish two cases depending on whether $A_1 \subset M$ or $A_1 \not\subset M$.

First consider the case when $A_1 \subset M$. Now $M = A_1 A_2 \cdots A_{n-1}$ is a factorization of $M$ as well. Since $M$ is a proper subgroup of $G$ by the inductive assumption $A_i = H_i$ for some $i$, $1 \leqq i \leqq n$.

Turn to the second case when $A_1 \not\subset M$. Note that in this case $G = M(b^{-1}A_n)$ is a factorization of $G$. Since $b^{-1}A_n$ is a complete set of representatives modulo $M$ there exists an element $c_b$ of $b^{-1}A_n$ such that the coset $c_b M$ contains the element $h_1^{-1}d_1^{-1}$, that is for which $h_1 d_1 c_b \in M$. Let

$$C_b = \{h_1 d_1 c_b\} \cup \left(H \setminus \{h_1 d_1\}\right) = \{h_1 d_1 c_b\} \cup \left(A_1 \setminus \{a_1\}\right).$$

Note that $M = C_b A_2 \cdots A_{n-1}$ is a factorization of $M$ as well. Indeed, products coming from $C_b A_2 \cdots A_{n-1}$ occur among the products coming from $A_1 A_2 \cdots A_{n-1}(b^{-1}A_n)$ and these latter are distinct since $A_1 \cdots A_{n-1}(b^{-1}A_n)$ is a factorization of $G$.

Using the fact that $M$ is a proper subgroup of $G$ the inductive assumption gives that $C_b$ is a subgroup of $M$. Clearly, $C_b$ is a periodic subset of $M$ and so by Lemma 2 $C_b = H_1$. Thus $h_1 d_1 c_b = h_1$, that is, $c_b = d_1^{-1}$ and so $d_1^{-1} \in b^{-1}A_n$. This gives

$$d_1^{-1} \in \bigcap_{b \in A_n} b^{-1}A_n.$$

Thus, by Lemma 1, $A_n$ is periodic.
This completes the proof.

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# ON THE MATRIX TRANSFORMATIONS OF ABSOLUTE SUMMABILITY FIELDS OF REVERSIBLE MATRICES

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## § 1. Introduction

In this paper, except otherwise stated, let $A = [\alpha_{nk}]$ be an arbitrary *reversible* matrix over $\mathbf{C}$, i.e. the infinite system of equations

$$z_n = \sum_{k=0}^{\infty} \alpha_{nk} x_k \tag{1.1}$$

has a unique solution for each convergent sequence $(z_n)$. Here and in what follows, free indices have the values $0, 1, \ldots$ and terms with negative indices the value 0. Moreover, let $B = [\beta_{nk}]$ be an arbitrary triangular matrix over $\mathbf{C}$ and $M = [m_{nk}]$ an infinite matrix over $\mathbf{C}$. We wish to determine necessary and sufficient conditions for $M$ in order that each of the series

$$y_n = \sum_{k=0}^{\infty} m_{nk} x_k \tag{1.2}$$

is convergent and the sequence $y = (y_n)$ is $B$-summable for each absolutely $A$-summable sequence $x = (x_k)$.

To characterize this problem in more details, we write (1.1) in the form $z = Ax$ or $z = (A_n x)$ where $A_n x = z_n$ as usual and introduce the notations

$$c = \{x = (x_k) \mid \text{the finite limit } \lim x_k \text{ exists}\},$$

$$c^0 = \{x = (x_k) \mid \lim x_k = 0\},$$

$$bv = \left\{x = (x_k) \mid |x_0| + \sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty\right\},$$

$$bv^0 = \{x = (x_k) \mid x \in bv \text{ and } \lim x_k = 0\},$$

$$c_A = \{x = (x_k) \mid \text{the series } A_n x \text{ are convergent} \\ \text{and } (A_n x) \in c\},$$

$$bv_A = \{x = (x_k) \mid \text{the series } A_n x \text{ are convergent} \\ \text{and } (A_n x) \in bv\},$$

$$(bv_A, c_B) = \{ M = [m_{nk}] \mid m_{nk} \in \mathbf{C}, \text{ the series } y_n = M_n x \text{ are convergent} \\ \text{and } (y_n) \in c_B \text{ for each } x \in bv_A\}.$$

Similarly to $(bv_A, c_B)$ we define the sets $(bv_A, bv_B)$ and $(c_A, c_B)$. The sets $c_A$ and $bv_A$ are called a *summability field* of the matrix $A$ and an *absolute summability field* of the matrix $A$, respectively. We are looking for necessary and sufficient conditions for $M$ to belong to $(bv_A, c_B)$ or $(bv_A, bv_B)$. We notice that in the special case when $M$ is a diagonal matrix, our problem reduces to the widely investigated problem of absolute summability factors. In several works [2–8,12] conditions for $M$ to belong to $(c_A, c_B)$ have been investigated also in the case of a non-diagonal matrix $M$. First results of this kind have been proved by L. Alpár [6–8] in the case if $A$ and $B$ are the methods of Cesàro of real order. In [12] B. Thorpe generalized L. Alpár's results considering the case when $B$ is an arbitrary normal matrix. Moreover, the case when $A$ is an arbitrary reversible matrix and $B$ is an arbitrary matrix has been considered in [3–5]. For non-reversible matrix $A$ this problem has been studied in [2]. Conditions for $M \in (bv_A, c_B)$ and $M \in (bv_A, bv_B)$ in the general case have been given in [1] without proofs. In the present paper we shall prove these results.

In §2 we are going to present some notations about reversible matrices, in §3 we shall give the solution of the stated problem for an arbitrary reversible matrix $A$ and an arbitrary triangular matrix $B$ and in §4 we shall apply the results, obtained in §3, in the case when $A$ is a Riesz method.

## § 2. Some notations concerning reversible matrices

**2.1.** It is known ([13, p.82) that the summability field $c_A$ of any reversible matrix $A$ is a *BK-space* (i.e. $c_A$ is a Banach space in which coordinatewise convergence is valid). Then each coordinate $x_k$ of the sequence $x = (x_k) \in c_A$ is a continuous linear functional on $c_A$. Hence there exist numbers $\alpha_k$ and absolutely convergent series $\sum_{l=0}^{\infty} \eta_{kl}$ such that

$$x_k = \alpha_k \mu + \sum_{l=0}^{\infty} \eta_{kl} z_l, \tag{2.1}$$

where $z_l = A_l x$ and $\mu = \lim z_l$ ([13], p.82). Taking now $z_l = \delta_{lr}$ (where $\delta_{lr} = 1$ if $l = r$ and $\delta_{lr} = 0$ if $l \neq r$) in (2.1) for fixed $r$ we see that $x_k = \eta_{kr}$ because $\mu = \lim_l \delta_{lr} = 0$. If $z_l = \delta_{ll}$ in (2.1) then

$$x_k = \alpha_k + \sum_{l=0}^{\infty} \eta_{kl}$$

since in this case $\mu = \lim \delta_{ll} = 1$. Consequently, $(\eta_{kr})$ for fixed $r$ and $(\eta_k)$ where

$$\eta_k = \alpha_k + \sum_{l=0}^{\infty} \eta_{kl}$$

are solutions of the system of equations $z = Ax$ for $z_l = \delta_{lr}$ and $z_l = \delta_{ll}$, respectively. Thus we may represent (2.1) in the form

$$x_k = \eta_k \mu + \sum_{l=0}^{\infty} \eta_{kl}(z_l - \mu).$$

As $bv_A \subset c_A$, (2.2) holds also for each $x \in bv_A$.

**2.2.** It is easy to see that for the convergence of the series (1.2) for each $x \in bv_A$ it is necessary and sufficient that the numbers $m_{nk}$ for fixed $n$ are absolute convergence factors for $A$. Therefore we have the following result (cf. also [10], Theorems 2 and 6):

LEMMA 1. *Let $A = [\alpha_{nk}]$ be a reversible matrix and $M = [m_{nk}]$ an infinite matrix. The necessary and sufficient conditions for the series $M_n x$ to be convergent whenever $x \in bv_A$, are the following:*

(I) *the series* $\sum_{l=0}^{\infty} m_{nl}\eta_l$ *and* $\sum_{l=0}^{\infty} m_{nl}\eta_{lk}$ *are convergent,*

(II) $\sum_{k=0}^{r} \sum_{l=0}^{p} m_{nl}\eta_{lk} = \mathcal{O}_n(1)$.

## § 3. The main results

We are going to prove two theorems. The first theorem characterizes the matrix transformations from $bv_A$ into $c_B$ and the second one from $bv_A$ into $bv_B$. First we introduce the matrix $G = [g_{nk}]$ where

$$g_{nk} = \sum_{s=0}^{n} \beta_{ns} m_{sk}$$

and the numbers

$$\gamma_{sk}^n = \sum_{l=0}^{s} g_{nl}\eta_{lk}.$$

THEOREM 1. *Let $A = [\alpha_{nk}]$ be a reversible matrix, $B = [\beta_{nk}]$ a triangular matrix and $M = [m_{nk}]$ an infinite matrix. The necessary and sufficient conditions for $M \in (bv_A, c_B)$ are:* (I), (II), *further*

(III) *the finite limit* $\lim_n \sum_{k=0}^{\infty} g_{nk}\eta_k$ *exists,*

(IV) *the finite limits* $\lim_n \gamma_{nk}$ *exist,*

(V) $\sum_{k=0}^{l} \gamma_{nk} = \mathcal{O}(1)$,

*where*

$$\gamma_{nk} = \lim_s \gamma_{sk}^n.$$

PROOF. *Necessity.* We suppose that $M \in (bv_A, c_B)$. Then the conditions (I) and (II) are fulfilled by Lemma 1 and

$$B_n y = G_n x \tag{3.1}$$

for each $x = (x_k) \in bv_A$ where $y = (y_k) = (M_k x)$. Therefore $bv_A \subset c_G$. Moreover, as $A_n\eta = \delta_{nn}$ where $\eta = (\eta_k)$ and

$$\delta_{00} + \sum_{n=1}^{\infty} |\delta_{nn} - \delta_{n-1,n-1}| = 1,$$

then $\eta \in bv_A$. Consequently $\eta \in c_G$, i.e. condition (III) is fulfilled.

As each coordinate $x_k$ of a sequence $x = (x_k) \in bv_A$ may be represented in the form (2.2), where $z_l = A_l x$, $\mu = \lim z_l$ and

$$\sum_{l=0}^{\infty} |\eta_{kl}| < \infty,$$

we have

$$\sum_{l=0}^{s} g_{nl}x_l = \mu \sum_{l=0}^{s} g_{nl}\eta_l + \sum_{k=0}^{\infty} \gamma_{sk}^n (z_k - \mu) \tag{3.2}$$

for each $x = (x_k) \in bv_A$, since

$$\sum_{k=0}^{\infty} |\eta_{lk}|\,|z_k - \mu| < \infty.$$

It follows from (3.2) that the finite limits

$$\lim_s \sum_{k=0}^{\infty} \gamma_{sk}^n (z_k - \mu)$$

exist for $(z_k - \mu) \in bv^0$, because the series $G_n x$ are convergent for each $x \in bv_A$ by (3.1) and $\eta \in c_G$. On the other hand, by the reversibility of the matrix $A$, for each $z = (z_l) \in bv^0$ there exists a sequence $x = (x_k) \in$ $\in bv_A$ so that $z_l = A_l x$ and $\mu = 0$. Hence the matrix $[\gamma_{sk}^n]$ (for each fixed $n$) transforms $bv^0$ into $c$. Consequently, it follows from (3.2) by Theorem 3.3 of [9] (cf. also [9], Remark in p.34) that

$$\sum_{l=0}^{\infty} g_{nl} x_l = \mu \sum_{l=0}^{\infty} g_{nl} \eta_l + \sum_{k=0}^{\infty} \gamma_{nk} (z_k - \mu) \tag{3.3}$$

for each $x \in bv_A$. Here we see that $(z_k - \mu) \in c_\Gamma$ where $\Gamma = [\gamma_{nk}]$. Moreover, $\Gamma$ transforms $bv^0$ into $c$ by the reversibility of the matrix $A$. Therefore the conditions (IV) and (V) are fulfilled (cf. [9], p.30–34).

*Sufficiency.* We suppose that the conditions (I)–(V) are fulfilled and show that in this case $M \in (bv_A, c_B)$.

It is easy to see that the series $M_n x$ are convergent for each $x \in bv_A$ by Lemma 1. Therefore (3.1) with $y = (y_k) = (M_k x)$ holds for each $x = (x_k) \in$ $\in bv_A$ and the series $G_n x$ are convergent for each $x \in bv_A$. Consequently (cf. the proof of necessity), (3.2) and (3.3) with $z_k = A_k x$ and $\mu = \lim z_k$ hold for each $x = (x_k) \in bv_A$. Moreover, (IV) and (V) imply that $(z_k - \mu) \in c_\Gamma$ (cf. [9], p.30–34). Hence $bv_A \subset c_G$ by (III). So $M \in (bv_A, c_B)$ by (3.1).

THEOREM 2. *Let $A = [\alpha_{nk}]$ be a reversible matrix, $B = [\beta_{nk}]$ a triangular matrix and $M = [m_{nk}]$ an infinite matrix. The necessary and sufficient conditions for $M \in (bv_A, bv_B)$ are:* (I), (II), *further*

(VI) $(\eta_k) \in bv_G$,

(VII) $\sum_{n=0}^{\infty} \left| \sum_{l=0}^{k} (\gamma_{kl} - \gamma_{n-1,l}) \right| = \mathcal{O}(1)$.

PROOF. *Necessity.* We suppose that $M \in (bv_A, bv_B)$. As $bv_B \subset c_B$, the conditions (I) and (II) are fulfilled and (3.1) holds for each $x \in bv_A$ by Theorem 1. Hence $bv_A \subset bv_G$. This implies that the condition (VI) is fulfilled because $(\eta_k) \in bv_A$ (cf. the proof of Theorem 1). Moreover, we have (3.3) for each $x = (x_k) \in bv_A$ where $z_k = A_k x$, $\mu = \lim z_k$ and the sequence $(z_k - \mu) \in bv^0$ (cf. the proof of Theorem 1). Therefore the condition (VII) is fulfilled by Proposition 100 of [11] since $\Gamma = [\gamma_{nk}]$ transforms $bv^0$ into $bv$ by the reversibility of the matrix $A$.

*Sufficiency* is proved by Theorem 3.3 of [9] and Proposition 100 of [11] similarly as it is done by Theorem 3.3 of [9] in the proof of Theorem 1.

## § 4. The applications of main results in the case of Riesz methods

Now we shall give some results for the case when $A$ is a Riesz method. Let $(\rho_n)$ be a sequence non-zero complex numbers, $P_n = \rho_0 + \ldots + \rho_n \neq 0$, $P_{-1} = 0$ and $P = [\alpha_{nk}]$ be the series-to-sequence Riesz method generated by $(\rho_n)$, i.e.

$$\alpha_{nk} = \begin{cases} 1 - P_{k-1}/P_n & \text{if } k \leqq n, \\ 0 & \text{if } k > n. \end{cases}$$

It is well-known that $P$ is a normal method. Therefore $P$ has the inverse matrix $P^{-1} = [\eta_{nk}]$ where

$$\eta_{nk} = \begin{cases} P_k/\rho_k & \text{if } n = k, \\ -P_k(1/\rho_k + 1/\rho_{k+1}) & \text{if } n = k+1, \\ P_k/\rho_{k+1} & \text{if } n = k+2, \\ 0 & \text{if } n < k \text{ or } n > k+2 \end{cases} \tag{4.1}$$

([9], p.116).

For the convergence of the series $M_n x$ for each $x \in bv_P$ it is necessary and sufficient that the numbers $m_{nk}$ are absolute convergence factors for $P$ in the case of fixed $n$. Therefore the following lemma holds by Theorems 17.2 and 22.4 of [9]:

LEMMA 2. *Let $P$ be an absolute convergence preserving Riesz method and $M = [m_{ns}]$ an infinite matrix. The necessary and sufficient conditions for the series $M_n x$ to be convergent whenever $x \in bv_P$, are the following:*
(VIII) $P_s m_{ns} = \mathcal{O}_n(\rho_s)$,
(IX) $P_s \Delta_s m_{ns} = \mathcal{O}_n(\rho_s)$
*where*

$$\Delta_s m_{ns} = m_{ns} - m_{n,s+1}.$$

Now we shall prove two theorems.

THEOREM 3. *Let $P$ be an absolute convergence preserving Riesz method, $B = [\beta_{nk}]$ a triangular matrix and $M = [m_{nk}]$ an infinite matrix. The necessary and sufficient conditions for $M \in (bv_P, c_B)$ are:* (VIII), (IX), *further*
(X) *the finite limits* $\lim_n g_{ns}$ *exist,*
(XI) $g_{ns} = \mathcal{O}(1)$,
(XII) $P_s \Delta_s g_{ns} = \mathcal{O}(\rho_s)$.

PROOF. *Necessity.* Let $M \in (bv_P, c_B)$. We shall show that in this case conditions (VIII)–(XII) are fulfilled.

It is easy to see that conditions (VIII) and (IX) are fulfilled by Lemma 2 and $bv_P \subset c_G$ (cf. the proof of Theorem 1). Therefore conditions (X) and

(XI) are fulfilled by Theorem 3.1 of [9] since the method $P$ preserves absolute convergence.

It remains to show that the condition (XII) is also fulfilled. We see by (4.1) that

$$\gamma_{nk} = P_k \Delta_k \frac{\Delta_k g_{nk}}{\rho_k}. \tag{4.2}$$

Therefore

$$\sum_{k=0}^{s-1} \gamma_{nk} = \sum_{l=0}^{s-1} \rho_l \left( \sum_{k=l}^{s-1} \Delta_k \frac{\Delta_k g_{nk}}{\rho_k} \right) = \sum_{l=0}^{s-1} \Delta_l g_{nl} - \frac{\Delta_s g_{ns}}{\rho_s} \sum_{l=0}^{s-1} \rho_l =$$

$$= g_{n0} - g_{n,s+1} - \frac{P_s}{\rho_s} \Delta_s g_{ns}$$

whence it follows that

$$\frac{P_s}{\rho_s} \Delta_s g_{ns} = g_{n0} - g_{n,s+1} - \sum_{k=0}^{s-1} \gamma_{nk}. \tag{4.3}$$

This implies that condition (XII) is fulfilled by (XI) and Theorem 1.

*Sufficiency.* We start with the hypothesis that the conditions (VIII)–(XII) hold and show that then the conditions of Theorem 1 are satisfied.

It is easy to see that the series $M_n x$ for each $x \in bv_P$ are convergent by Lemma 2. Therefore conditions (I) and (II) are fulfilled by Lemma 1.

As $P$ is a normal method and $\alpha_{n0} = 1$, we have $\eta_n = \delta_{n0}$ ([9], p.58). Hence condition (III) is fulfilled by (X) and condition (IV) by (X) and (4.2). Now we have by (4.3), (XI) and (XII) that condition (V) is also fulfilled. Thus $M \in (bv_P, c_B)$ by Theorem 1.

THEOREM 4. *Let $P$ be an absolute convergence preserving Riesz method, $B = [\beta_{nk}]$ a triangular matrix and $M = [m_{nk}]$ an infinite matrix. The necessary and sufficient conditions for $M \in (bv_P, bv_B)$ are:* (VIII), (IX), *further*

(XIII) $\sum\limits_{n=0}^{\infty} |g_{nk} - g_{n-1,k}| = \mathcal{O}(1).$

(XIV) $P_k \sum\limits_{n=0}^{\infty} \big| \Delta_k (g_{nk} - g_{n-1,k}) \big| = \mathcal{O}(\rho_k).$

PROOF. *Necessity.* We suppose that $M \in (bv_P, bv_B)$ and show that then conditions (VIII)–(IX) and (XIII)–(XIV) are fulfilled.

First we note that conditions (VIII) and (IX) are fulfilled by Lemma 2 and $bv_P \subset bv_G$ (cf. the proof of Theorem 2, taking there $A = P$). Hence condition (XIII) is fulfilled by Theorem 4.2 of [9] since the method $P$ transforms all absolutely convergent series into $bv$. Therefore condition (XIV) is fulfilled by (4.3) and Theorem 2.

*Sufficiency.* We shall show that conditions (VIII)–(IX) and (XIII)–(XIV) imply $M \in (bv_P, bv_B)$. It is sufficient to prove for this that the conditions of Theorem 2 are fulfilled.

First we notice that the series $M_n x$ are convergent for each $x \in bv_P$ by Lemma 2. Therefore conditions (I) and (II) are fulfilled by Lemma 1. Condition (VI) is fulfilled by (XIII) (since in the present case $\eta_n = \delta_{n0}$) and condition (VII) is fulfilled by (XIII) and (XIV). Thus, $M \in (bv_P, bv_B)$ by Theorem 2.

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# ON $h$-RECURRENT WAGNER SPACES OF $W$-SCALAR CURVATURE

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## Introduction

S. Numata [3] and T. Varga [5] have independently proved that if an $n$ ($\geqq 3$) dimensional Finsler space is a Berwald space of scalar curvature $K$, then it is a Riemannian space of constant curvature $K$, or a locally Minkowski space according as $K \neq 0$ or $K = 0$. Afterwards M. Hashiguchi and T. Varga [1] showed a similar result on Wagner spaces of $W$-scalar curvature. The purpose of the present paper is to examine $h$-recurrent Wagner spaces of $W$-scalar curvature and to prove a similar theorem.

Throughout the present paper we shall use the terminology and notations of Matsumoto's monograph [2].

## 1. $h$-recurrent Wagner connections and Wagner spaces

$h$-recurrent Wagner connections and Wagner spaces are defined as follows ([4]).

DEFINITION 1. The $h$-recurrent Wagner connection $W\Gamma$ of the Finsler space $F$ is a Finsler connection $\left(F^i_{jk}, N^i_k, C^i_{jk}\right)$ uniquely determined by the following five axioms:

(C1) $W\Gamma$ is $v$ metrical, i.e., $g_{ij|k} = 0$.

(C2) The $(v)v$-torsion $S'$ of $W\Gamma$ vanishes, i.e., $C^i_{jk} = C^i_{kj}$.

(C3) $W\Gamma$ is $h$-recurrent with respect to the vector field $a_k(x)$, i.e., $g_{ij|k} = a_k g_{ij}$.

(C4) The $(h)h$-torsion tensor $T$, of $W\Gamma$, is given by

$$T^i_{jk} \equiv F^i_{jk} - F^i_{kj} = \delta^i_j s_k - \delta^i_k s_j,$$

where $s_k(x)$ is a vector field.

(C5) The deflection tensor $D$ of $W\Gamma$ vanishes, i.e.

$$D^i_k \equiv y^j F^i_{jk} - N^i_k = 0.$$

(C1) and (C2) show that $C_{ijk}$ are the same as in the Cartan's connection $C\Gamma$.

DEFINITION 2. A Finsler space equipped with the $h$-recurrent Wagner connection $W\Gamma$ is called an $h$-recurrent Wagner space if the coefficients $F^i_{jk}$ depend on position alone.

For any Finsler connection $\left(F^i_{jk}, N^i_k, C^i_{jk}\right)$ the $(v)h$-torsion tensor $R^i_{jk}$ and $h$-curvature tensor $R^i_{hjk}$ are given by ([2])

$$R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k \tag{1.1}$$

and

$$R^i_{hjk} = \delta_k F^i_{hj} - \delta_j F^i_{hk} + F^m_{hj} F^i_{mk} - F^m_{hk} F^i_{mj} + C^i_{hm} R^m_{jk} \tag{1.2}$$

where $\delta_k = \partial_k - N^m_k \dot{\partial}_m$, $\partial_k = \partial/\partial x^k$, $\dot{\partial}_m = \partial/\partial y^m$.

THEOREM 1.1. *In an $h$-recurrent Wagner connection $W\Gamma$ with respect to a gradient recurrence vector $a_k = \delta_k a$ the $h$-curvature tensor defined by $R_{hijk} = g_{il} R^l_{hjk}$ is skew-symmetric in the first two indices.*

PROOF. Applying the Ricci identity ([2], (10.8′)) for the metric tensor $g_{ij}$ we get

$$g_{ij|k|l} - g_{ij|l|k} = -g_{ir} R^r_{jkl} - g_{rj} R^r_{ikl} - g_{ij}|_r T^r_{kl} - g_{ij|r} R^r_{kl}$$

which in view of $g_{ij|k} = a_k g_{ij}$, (C4) and $g_{ij}|_k = 0$ gives

$$R_{jikl} + R_{ijkl} = g_{ij}(\partial_k a_1 - \partial_1 a_k).$$

Therefore, since $a_k$ is a gradient vector, i.e., $a_k = \partial_k a$, the above equation reduces to

$$R_{ijkl} = -R_{jikl}.$$

## 2. $h$-recurrent Wagner spaces of $W$-scalar curvature

We consider one of the Bianchi identities ([2], (11.1′)) of a $W\Gamma$:

$$A_{(ijk)}\left\{T^h_{ir}T^r_{jk}+T^h_{ij|k}+C^h_{ir}R^r_{jk}-R^h_{ijk}\right\}=0, \tag{2.1}$$

where $A_{(ijk)}$ means the cyclic permutation of indices and summation for the expression in the brackets behind it.

First we derive the equation (2.5) which holds for some special $h$-recurrent Wagner connection of $W$-scalar curvature $K$, and will be needed in our subsequent proof. From (C4) we obtain

$$T^h_{ir}T^r_{jk}=-s_iT^h_{jk},\quad s_hT^h_{jk}=0,\quad A_{(ijk)}\left\{T^h_{ir}T^r_{jk}\right\}=0,$$

and hence

$$A_{(ijk)}\left\{T^h_{ij|k}\right\}=A_{(ijk)}\left\{\delta^h_j(s_{k|i}-s_{i|k}\right\},$$

$$s_{i|j}-s_{j|i}=\partial_js_i-\partial_is_j.$$

Therefore, if $s_i$ is a gradient vector, i.e., $s_i=\partial_is(x)$, then (2.1) becomes

$$A_{(ijk)}\left\{C^h_{ir}R^r_{jk}-R^h_{ijk}\right\}=0. \tag{2.2}$$

If $W\Gamma$ is $h$-recurrent with respect to the recurrence gradient vector $a_k=$ $=\partial_ka(x)$ then by Theorem 1.1 we have $R_{ihjk}=-R_{hijk}$. Therefore equation (2.2) becomes

$$A_{(ijk)}\left\{C_{ihr}R^r_{jk}+R_{hijk}\right\}=0 \tag{2.3}$$

where $C_{ihr}=g_{hl}C^l_{ir}$. Contracting equation (2.3) with $y^h$, we get

$$A_{(ijk)}\left\{R_{oijk}\right\}=0.$$

Then on account of [2] Theorem 13.3 we obtain $A_{(ijk)}\left\{R_{ijk}\right\}=0$, which implies $R_{ioj}=R_{joi}$.

Now we consider the equation

$$R_{iok}X^iX^k=KL^2h_{ik}X^iX^k \tag{2.4}$$

where $X=(X^i)$ is a tangent vector of a Finsler space at $x$, where $R_{iok}$ is formed from the coefficients of the $h$-recurrent Wagner connection, $L$ is the metric function, $h_{ik}=g_{ik}-l_il_k$ is the angular metric tensor, and $K$, called the $W$-sectional curvature, is a function of $x, y$ and of the vector field $X$ ((2.4) has the same form as (26.1) in [2] for the Cartan's connection $C\Gamma$.)

DEFINITION 3. If the $W$-sectional curvature $K(x, y, X)$ in (2.4) is a scalar field which does not depend on $X^i$, then the $h$-recurrent Wagner connection is called of $W$-scalar curvature $K$.

Consequently, if we consider $h$-recurrent Wagner connection $W\Gamma$ with gradient vector field $s_j(x) = \partial_j s(x)$, gradient recurrence vector $a_j(x) = \partial_j a$ and of $W$-scalar curvature $K$, then the symmetry property of $R_{ioj}$ leads us to the equation

$$R_{iok} = KL^2 h_{ik} \tag{2.5}$$

which is the same as the well-known equation in a Finsler space of scalar curvature with the Cartan's connection.

## 3. Berwald spaces conformal to $h$-recurrent Wagner spaces

Let a Finsler space $F^n$ be $h$-recurrent Wagner space with respect to the gradient vector field $s_j(x) = \partial_j s(x)$ and gradient recurrence vector $a_j(x) = \partial_j a(x)$. The metric function of $F^n$ is $L$, the metric tensor is $g_{ij}$, and the $h$-recurrent Wagner connection is given by $\left(F^i_{jk}, N^i_k, C^i_{jk}\right)$. Let us consider the Finsler space $^*F^n$ with the metric function $^*L = e^{-b(x)}L$ where $b = s + \frac{1}{2}a$. We define

$$^*F^i_{jk} = F^i_{jk} - \delta^i_j s_k, \tag{3.1}$$

$$^*N^i_k = N^i_k - y^i s_k, \tag{3.2}$$

$$^*C^i_{jk} = C^i_{jk}. \tag{3.3}$$

These $\left({}^*F^i_{jk}, {}^*N^i_k, {}^*C^i_{jk}\right)$ form the Cartan's connection on $^*F^n$. Namely, denoting quantities in $^*F^n$ by an asterisk we get

$$^*g_{ij} = e^{-2b} g_{ij} \tag{3.4}$$

and then we can easily check the validity of the relations $^*g_{ij|k} = 0$ with respect to $^*F^i_{jk}$ and $^*g_{ij|k} = 0$ with respect to $^*C^i_{jk}$, $^*D^i_k = 0$, and moreover the symmetry of $^*F^i_{jk}$ and $^*C^i_{jk}$ in $j$ and $k$. These together mean that $\left({}^*F^i_{jk}, {}^*N^i_k, {}^*C^i_{jk}\right)$ is the Cartan's connection on $^*F^n$ (see [2] Definition 17.2).

Since the $F^i_{jk}$ of the $h$-recurrent Wagner connection are functions of $x^i$ alone, so are the $^*F^i_{jk}$. Hence $^*F^n$ is a Berwald space (see also Theorem 3.6 of B. N. Prasad et al [4]).

We show that the Berwald space $^*F^n$ is of scalar curvature if the $h$-recurrent Wagner space $F^n$ is of $W$ scalar curvature. By virtue of (3.4) we have

$$^*R_{iok} = {^*g_{ir}}{^*R^r_{ok}} = e^{-2b} g_{ir} {^*R^r_{ok}}.$$

Since $s_i$ is gradient and $N^i_k$ is (1)$p$-homogeneous, from (1.1) and (3.2) we get $^*R^r_{jk} = R^r_{jk}$ and thus $^*R_{iok} = e^{-2b} R_{iok}$.

Assuming that $F^n$ is of $W$-scalar curvature, (2.4) yields (2.5), and we obtain

$$^*R_{iok} = e^{-2b} K L^2 h_{ik} = K\ {^*L^2} h_{ik}.$$

Furthermore

$$^*h_{ik} = {^*g_{ik}} - \dot\partial_i\ {^*L}\dot\partial_j\ {^*L} = e^{-2b}(g_{ik} - l_i l_k) = e^{-2b} h_{ik}.$$

Thus we have $^*R_{iok} = e^{2b} K\ {^*L^2}\ {^*h_{ik}}$, namely, the Berwald space $^*F^n$ is of scalar curvature $^*K = e^{2b} K$.

Thus, by the theorem of S. Numata ([3] Theorem 2) and T. Varga ([5] Theorem 11), this Berwald space $^*F^n$ for $n \geqq 3$ is a Riemannian space of constant curvature $^*K$, or a locally Minkowski space, according as $K \neq 0$ or $K = 0$. Furthermore the $h$-recurrent Wagner space $F^n$ is conformal to this $^*F^n$, so we have the following.

THEOREM 3.1. *If an $n$ ($\geqq 3$) dimensional Finsler space is an $h$-recurrent Wagner space with respect to the gradient vector $s_j(x) = \partial_j s(x)$ gradient recurrence vector $a_j(x) = \partial_j a$ and of $W$-scalar curvature $K$, then the space is conformal to a Riemannian space of constant curvature, or conformal to a locally Minkowski space, according as $K \neq 0$ or $K = 0$.*

It should be noted that the above scalar $K \neq 0$ does not depend on the supporting element $y^i$, but it is not necessarily constant.

As a special case, we get the following shown in the paper [1].

COROLLARY (Hashiguchi and Varga). *If an $n$ ($\geqq 3$) dimensional Wagner space with respect to a gradient vector field $s_i(x) = \partial_j s(x)$ is of non-zero $W$-scalar curvature $K$, then, the space is Riemannian and the scalar $K$ is written as $K = Ce^{-2s(x)}$, $C$ =constant.*

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# ON A NEW LAW OF THE ITERATED LOGARITHM OF ERDŐS AND RÉVÉSZ

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## 1. Introduction

Let $\{W(t),\, t \geqq 0\}$ be a standard Wiener process and consider the processes

$$\xi(t) = \sup\left\{ s\colon 0 \leqq s \leqq t,\, W(s) \geqq (2s \log\log s)^{1/2} \right\},\ t \geqq 0,$$

$$\xi_\varepsilon(t) = \sup\left\{ s\colon 0 \leqq s \leqq t,\, W(s) \geqq \big(2(1-\varepsilon)s \log\log s\big)^{1/2} \right\},\ t \geqq 0,\ 0 \leqq \varepsilon < 1,$$

$$\xi_\delta^{(p)}(t) = \sup\left\{ s\colon 0 \leqq s \leqq t,\, W(s) \geqq s^{1/2}\alpha(\delta, p, s) \right\},\ t \geqq 0,$$

where

$$(1.1)\quad \alpha(\delta,p,s) = \left(2\left(\log_2 s + \frac{3}{2}\log_3 s + \sum_{j=4}^{p} \log_j s - \delta \log_p s\right)\right)^{1/2},\ \delta \geqq 0,$$

$p = 3, 4, \ldots,\ \log_i t = \log_1(\log_{i-1} t),\ i = 2, 3, \ldots,\ \log_1 x = \ln x$ for $x > 0$ and $\log_1 x = 1$ if $x \leqq 0$. It is clear that

$$\lim_{t\to\infty} \xi(t) = \lim_{t\to\infty} \xi_\varepsilon(t) = \lim_{t\to\infty} \xi_\delta^{(p)}(t) = \infty \text{ a.s.}$$

and

$$\limsup_{t\to\infty} \frac{\xi(t)}{t} = \limsup_{t\to\infty} \frac{\xi_\varepsilon(t)}{t} = \limsup_{t\to\infty} \frac{\xi_\delta^{(p)}(t)}{t} = 1 \text{ a.s.}$$

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for each $0 \leqq \varepsilon < 1$, $\delta \geqq 0$, $p = 3, 4, \ldots$, by the law of the iterated logarithm. Erdős and Révész [2] considered the lower bound of $\xi(t)$ and obtained a new law of the iterated logarithm, which states that

$$\liminf_{t\to\infty} \frac{(\log_2 t)^{1/2}}{\log_3 t \cdot \log t} \cdot \log \frac{\xi(t)}{t} = -C_0 \text{ a.s.}$$

for some constant $C_0$ with $\frac{1}{4} \leqq C_0 \leqq 2^{14}$. At the end of their paper, Erdős and Révész proposed the challenging question of finding the lower bound for the general processes $\xi_\varepsilon(t)$ and $\xi_\delta^{(p)}(t)$.

The aim of the present note is to find the exact value of $C_0$ and the exact lower bound of $\xi_\varepsilon(t)$ and that of $\xi_\delta^{(p)}(t)$ as well.

THEOREM 1. *We have*

$$\liminf_{t\to\infty} \frac{(\log_2)^{1/2}}{\log_3 t \cdot \log t} \cdot \log \frac{\xi(t)}{t} = -3\sqrt{\pi} \ \textit{a.s.}, \tag{1.2}$$

$$\liminf_{t\to\infty} (\log t)^{\delta-1} (\log_2 t)^{-1/2} \cdot \log \frac{\xi_\delta(t)}{t} = -2\delta\sqrt{\pi/(1-\delta)} \ \textit{a.s.} \tag{1.3}$$

*for each* $0 < \delta \leqq 1/2$.

THEOREM 2. *We have*

$$\liminf_{t\to\infty} \frac{\log_p \xi_0^{(p)}(t) - \log_p t}{\log_{p+1} t} = -2\sqrt{\pi} \ \textit{a.s.}, \tag{1.4}$$

$$\liminf_{t\to\infty} \frac{\log_{p-1} \xi_\delta^{(p)}(t) - \log_{p-1} t}{(\log_{p-1} t)^{1-\delta} \log_p t} = -2\delta\sqrt{\pi} \ \textit{a.s. for } 0 < \delta < 1, \tag{1.5}$$

$$\liminf_{t\to\infty} \frac{\log_{p-2} \xi_\delta^{(p)}(t) - \log_{p-2} t}{\log_{p-2} t \cdot (\log_{p-1} t)^{1-\delta} \log_p t} = -2\delta\sqrt{\pi} \ \textit{a.s. for } \delta > 1 \tag{1.6}$$

*for* $p = 3, 4, \ldots$ .

Our Theorem 1 says that for any $t$ big enough, between

$$t^{1-3\sqrt{\pi}\log_3 t \cdot (\log_2 t)^{-1/2}} \quad \text{and} \quad t$$

there exists an $s$ such that $W(s) \geqq (2s \log_2 s)^{-1/2}$. The meaning of Theorem 2 can be interpreted in the same way.

## 2. Proofs

Throughout this section we will use the following notations: $\sum_{i=n}^{m} = 0$ and $\prod_{i=n}^{m} = 1$ if $m < n$; $[x]$ denotes the integer part of $x$; $\sum_{i=x}^{y}$ denotes $\sum_{i=[x]}^{[y]}$. Define $\alpha(\delta, p, s)$ as in (1.1). That is

$$\alpha(\delta,3,s) = \left(2\left(\log_2 s + \left(\frac{3}{2} - \delta\right)\log_3 s\right)\right)^{1/2}, \quad \delta \geqq 0,$$

$$\alpha(\delta,p,s) = \left(2\left(\log_2 s + \frac{3}{2}\log_3 s + \sum_{j=4}^{p}\log_j s - \delta\log_p s\right)\right)^{1/2}, \quad \delta \geqq 0,$$

$p = 4, 5, \ldots$ .

We also let

$$\alpha(\delta,2,s) = \left(2(1-\delta)\log_2 s\right)^{1/2}, \quad 0 \leqq \delta < 1.$$

It is easy to see that

$$\alpha(0,2,s) = \alpha(3/2,3,s), \tag{2.1}$$

$$\alpha(0,p,s) = \alpha(1,p+1,s), \quad p = 3,4,\ldots, \tag{2.2}$$

$$\lim_{s\to\infty} \alpha(\delta,p,s)/(2\log_2 s)^{1/2} = 1 \quad \text{for each} \delta \geqq 0,\ p \geqq 3, \tag{2.3}$$

$$\alpha(\delta,p,s) \quad \text{is non-decreasing for } s \text{ sufficiently large.} \tag{2.4}$$

Let $\{W(t),\ t \geqq 0\}$ be a standard Wiener process and $\{X(t),\ t \geqq 0\} = \{W(e^t)/e^{t/2},\ t \geqq 0\}$. Then $\{X(t),\ t \geqq 0\}$ is an Ornstein–Uhlenbeck process with $EX(t) = 0$, $EX^2(t) = 1$ and

$$\rho(t,s) = EX(t)X(s) = e^{-|t-s|/2}. \tag{2.5}$$

Clearly, we have

$$\rho(t,s) = 1 - \frac{1}{2}|t-s| + o(t-s) \quad \text{as} \quad t - s \to 0. \tag{2.6}$$

Hence, form Lemmas 2.5 and 2.9 of Picklands [4] it follows that (cf. [3], p.232)

$$\lim_{x\to\infty} \frac{P\left(\sup_{0\leqq s\leqq 1} X(s) > x\right)}{\psi(x)} = 1, \tag{2.7}$$

$$\lim_{x\to\infty} \frac{P\Big(\max\limits_{0\leqq j\leqq 2x^2/\theta} X\big(j\theta/(2x^2)\big) > x\Big)}{\psi(x)} = \frac{H(\theta)}{\theta}, \tag{2.8}$$

$$\lim_{\theta\to 0} \frac{H(\theta)}{\theta} = 1. \tag{2.9}$$

Here and in the sequel,

$$\psi(x) = \frac{1}{2\sqrt{2\pi}} x e^{-x^2/2}. \tag{2.10}$$

To prove our theorems, we need the following lemmas.

LEMMA 2.1. *Let $\{Y_t,\ t \in T\}$ and $\{Z_t,\ t \in T\}$ be centered Gaussian processes on a parameter set $T$ with $EY_t^2 = EZ_t^2$ and $EY_tY_s \leqq EZ_tZ_s$ for all $s,t \in T$. Then for all real $x$*

$$P\Big(\sup_{t\in T} Y_t \leqq x\Big) \leqq P\Big(\sup_{t\in T} Z_t \leqq x\Big).$$

This is the well-known Slepian lemma [5].

LEMMA 2.2. *For each $\delta > 0$, $p \geqq 2$, $0 < \eta < 1$, there exists a constant $N = N(\delta, p, \eta)$ such that*

$$P\bigg(\bigcap_{e^a\leqq s\leqq e^b} \bigg\{\frac{W(s)}{\sqrt{s}} \leqq \alpha(\delta,p,s) - \frac{\eta}{\alpha(\delta,p,s)}\bigg\}\bigg) \geqq \tag{2.11}$$

$$\geqq \begin{cases} \frac{1}{8}\exp\left(-\frac{1+8\eta}{2\delta}\sqrt{\frac{1-\delta}{\pi}}(\log b)^{1/2}(b^\delta - a^\delta)\right), & \text{if } 0<\delta<1,\ p=2, \\ \frac{1}{8}\exp\left(-\frac{1+8\eta}{2\delta\sqrt{\pi}}\left((\log_{p-2} b)^\delta - (\log_{p-2} a)^\delta\right)\right), & \text{if } \delta>0,\ p\geqq 3 \end{cases}$$

*for each $b \geqq a+2 \geqq N$.*

PROOF. Put

$$\tilde{\alpha}(\delta,p,s) = \alpha(\delta,p,s) - \frac{\eta}{\alpha(\delta,p,s)}.$$

Then

$$P\bigg(\bigcap_{e^a\leqq s\leqq e^b} \bigg\{\frac{W(s)}{\sqrt{s}} \leqq \alpha(\delta,p,s) - \frac{\eta}{\alpha(\delta,p,s)}\bigg\}\bigg) = \tag{2.12}$$

$$= P\Big( \sup_{e^a \leqq s \leqq e^b} \frac{W(s)}{\sqrt{s}\tilde{\alpha}(\delta, p, s)} \leqq 1 \Big) = P\Big( \sup_{a \leqq s \leqq b} \frac{X(s)}{\tilde{\alpha}(\delta, p, e^s)} \leqq 1 \Big) \geqq$$

$$\geqq P\Big( \max_{0 \leqq i \leqq b-a} \sup_{i \leqq s < i+1} \frac{X(a+s)}{\tilde{\alpha}(\delta, p, e^{a+s})} \leqq 1 \Big) \geqq$$

$$\geqq P\Big( \max_{0 \leqq i \leqq b-a} \sup_{i \leqq s < i+1} \frac{X(a+s)}{\tilde{\alpha}(\delta, p, e^{a+i})} \leqq 1 \Big),$$

provided that $a$ is large enough, by (2.4).

We now prove

$$P\Big( \max_{0 \leqq i \leqq b-a} \sup_{i \leqq s < i+1} \frac{X(a+s)}{\tilde{\alpha}(\delta, p, e^{a+i})} \leqq 1 \Big) \geqq \tag{2.13}$$

$$\geqq \prod_{i=0}^{b-a} P\Big( \sup_{i \leqq s < i+1} \frac{X(a+s)}{\tilde{\alpha}(\delta, p, e^{a+i})} \leqq 1 \Big) = \prod_{i=0}^{b-a} P\Big( \sup_{0 \leqq s < 1} X(s) \leqq \tilde{\alpha}(\delta, p, e^{a+i}) \Big).$$

Define

$$Z_t = \frac{X(a+t)}{\tilde{\alpha}(\delta, p, e^{a+i})} \quad \text{if} \quad i \leqq t < i+1, \quad i = 0, 1, \ldots, [b-a].$$

Let $\{Y_t,\ i \leqq t < i+1\}_{i=0}^{[b-a]}$ be independent stochastic processes and let $\{Y_t,\ k \leqq t < k+1\}$ have the same distribution as $\{Z_t,\ k \leqq t < k+1\}$ for each $0 \leqq k \leqq [b-a]$. Then, $\{Z_t,\ t \in [0, b-a+1)\}$ and $\{Y_t,\ t \in [0, b-a+1)\}$ are centered Gaussian processes with $EY_t^2 = EZ_t^2 = 1/\tilde{\alpha}^2(\delta, p, e^{a+i})$ if $i \leqq t < i+1$, $0 \leqq i \leqq [b-a]$. It is easy to see that

$$EY_tY_s \leqq EZ_tZ_s \quad \text{for all} \quad s, t \in [0, b-a+1).$$

Therefore, using the Slepian lemma (i.e., Lemma 2.1), we have

$$P\Big( \max_{0 \leqq i \leqq b-a} \sup_{i \leqq s < i+1} \frac{X(a+t)}{\tilde{\alpha}(\delta, p, e^{a+i})} \leqq 1 \Big) =$$

$$= P\Big( \sup_{t \in [0, b-a+1)} Z_t \leqq 1 \Big) \geqq P\Big( \sup_{t \in [0, b-a+1)} Y_t \leqq 1 \Big) =$$

$$= \prod_{i=0}^{b-a} P\Big( \sup_{t \in [i, i+1)} Y_t \leqq 1 \Big) = \prod_{i=0}^{b-a} P\Big( \sup_{t \in [i, i+1)} Z_t \leqq 1 \Big) =$$

$$= \prod_{i=0}^{b-a} P\Big( \sup_{t\in[i,i+1)} \frac{X(a+t)}{\tilde\alpha(\delta,p,e^{a+i})} \leqq 1 \Big) = \prod_{i=0}^{b-a} P\Big( \sup_{0\leqq s<1} X(s) \leqq \tilde\alpha(\delta,p,e^{a+i}) \Big)$$

as desired. By (2.7), there exists a positive constant $x_0$ such that for each $x \geqq x_0$

$$P\Big( \sup_{0\leqq s\leqq 1} X(s) \geqq x \Big) \leqq \left(1+\frac{\eta}{3}\right)\psi(x)$$

and

$$1-\left(1+\frac{\eta}{3}\right)\psi(x) \geqq \exp\left(-\left(1+\frac{\eta}{2}\right)\psi(x)\right).$$

Hence

$$\prod_{i=0}^{b-a} P\Big( \sup_{0\leqq s<1} X(s) \leqq \tilde\alpha(\delta,p,e^{a+i}) \Big) \geqq \tag{2.14}$$

$$\geqq \exp\left( -\left(1+\frac{\eta}{2}\right)\sum_{i=0}^{b-a} \psi\left(\tilde\alpha(\delta,p,e^{a+i})\right)\right),$$

provided $a$ is large enough. Write

$$I(a,b,\delta,p) = \sum_{i=0}^{b-a} \psi\left(\tilde\alpha(\delta,p,e^{a+i})\right).$$

Note that

$$\psi\left(\tilde\alpha(\delta,p,e^{a+i})\right) \leqq \frac{\alpha(\delta,p,e^{a+i})}{2\sqrt{2\pi}} \exp\left(\eta - \frac{1}{2}\alpha^2(\delta,p,e^{a+i})\right).$$

Therefore

(2.15)

$$\psi\left(\tilde\alpha(\delta,2,e^{a+i})\right) \leqq \frac{e^\eta}{2}\sqrt{\frac{1-\delta}{\pi}}(a+i)^{\delta-1}\log^{1/2}(a+i), \quad \text{for} \quad 0<\delta<1,$$

and

$$\psi\left(\tilde\alpha(\delta,p,e^{a+i})\right) \leqq \tag{2.16}$$

$$\leqq \frac{(1+\eta/2)e^\eta \log^{1/2}(a+i)}{2\sqrt{\pi}(a+i)\log^{3/2}(a+i)\cdot\left(\log_{p-2}(a+i)\right)^{-\delta}\cdot\prod_{j=2}^{p-2}\log_j(a+i)}$$

for $\delta > 0$, $p = 3, 4, \dots$, provided $a$ is sufficiently large, by (2.3).

From (2.16) it follows that, for $\delta > 0$, $p \geqq 3$

$$I(a, b, \delta\, p) \leqq \tag{2.17}$$

$$\leqq \frac{(1+\eta/2)e^{\eta}}{2\sqrt{\pi}} \sum_{i=0}^{b-a} \frac{1}{(a+i)\log(a+i)\left(\log_{p-2}(a+i)\right)^{-\delta} \cdot \prod_{j=2}^{p-2} \log_j(a+i)} \leqq$$

$$\leqq 1 + \frac{(1+\eta/2)e^{\eta}}{2\sqrt{\pi}} \cdot$$

$$\cdot \int_0^{[b-a]} \frac{dy}{(a+y)\log(a+y)\left(\log_{p-2}(a+y)\right)^{-\delta} \cdot \prod_{j=2}^{p-2} \log_j(a+y)} \leqq$$

$$\leqq 1 + \frac{(1+\eta/2)e^{\eta}}{2\sqrt{\pi}} \int_a^b \frac{dy}{y \log y (\log_{p-2} y)^{-\delta} \cdot \prod_{j=2}^{p-2} \log_j y} =$$

$$= 1 + \frac{(1+\eta/2)e^{\eta}}{2\delta\sqrt{\pi}} \left( (\log_{p-2} b)^{\delta} - (\log_{p-2} a)^{\delta} \right).$$

Similarly, one can get from (2.15) that for $0 < \delta < 1$

$$I(a, b, \delta, 2) \leqq 1 + \frac{e^{\eta}}{2\delta} \sqrt{\frac{1-\delta}{\pi}} \log^{1/2} b \cdot (b^{\delta} - a^{\delta}). \tag{2.18}$$

An elementary inequality says for $0 < \eta < 1$

$$\left(1 + \frac{\eta}{2}\right)^2 e^{\eta} \leqq 1 + 8\eta.$$

Now (2.11) follows from (2.12), (2.13), (2.14), (2.17) and (2.18).

LEMMA 2.3. *Let $\xi_1, \dots, \xi_n$ be standard normal variables with covariance matrix $\Lambda^1 = (\Lambda^1_{ij})$, and $\eta_1, \dots, \eta_n$ be standard normal variables with covariance matrix $\Lambda^0 = (\Lambda^0_{ij})$. Assume $n \geqq 3$,*

$$\Lambda^1_{ij} \geqq \Lambda^0_{ij} \quad \textit{for all} \quad 1 \leqq i, j \leqq n, \tag{2.19}$$

$$\Lambda^l_{ij} \geqq \Lambda^l_{ik} \Lambda^l_{kj} \geqq 0 \quad \textit{for} \quad l = 0, 1 \quad \textit{and for all} \quad 1 \leqq i, j, k \leqq n. \tag{2.20}$$

*Put $\rho_{ij} = \max\left(|\Lambda^1_{ij}|, |\Lambda^0_{ij}|\right)$. Then*

$$P\Big( \bigcap_{j=1}^{n} \{\xi_j \leqq u_j\} \Big) - P\Big( \bigcap_{j=1}^{n} \{\eta_j \leqq u_j\} \Big) \leqq \tag{2.21}$$

$$\leqq \frac{1}{2\pi} \sum_{1 \leqq i < j \leqq n} (\Lambda^1_{ij} - \Lambda^0_{ij})(1 - \rho^2_{ij})^{-1/2} \cdot$$

$$\cdot \exp\left(-\frac{u_i^2 + u_j^2}{2(1+\rho_{ij})}\right) P\left(\bigcap_{j=1}^{n}\{\xi_j \leqq u_j\}\right) +$$

$$+\frac{1}{2\pi} \sum_{1 \leqq i < j \leqq n} (\Lambda^1_{ij} - \Lambda^0_{ij})(1 - \rho^2_{ij})^{-1/2} \cdot$$

$$\cdot \exp\left(-\frac{u_i^2 + u_j^2}{2(1+\rho_{ij})}\right) \left(e^{-u_i^2/2} + e^{-u_j^2/2}\right)$$

*for all* $u_j \geqq 0$, $j = 1, \ldots, n$.

PROOF. The proof is along the lines of that of Theorem 4.2.1 of [3]. Write

$$\Lambda_h = h\Lambda^1 + (1-h)\Lambda^0 = (\Lambda^h_{ij}) \quad \text{for} \quad 0 \leqq h \leqq 1.$$

Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be normal random variables with covariance matrix $\Lambda_h$, let $f_h$ be the density function of $\zeta$ and

$$F(h, \mathbf{u}) = \int_{-\infty}^{\mathbf{u}'} \cdots \int f_h(y_1, \ldots, y_n)\, d\mathbf{y},$$

where $\mathbf{u} = (u_1, \ldots, u_n)$. Then

(2.22)
$$P\left(\bigcap_{j=1}^{n}\{\xi_j \leqq u_j\}\right) - P\left(\bigcap_{j=1}^{n}\{\eta_j \leqq u_j\}\right) = F(1,\mathbf{u}) - F(0,\mathbf{u}) = \int_0^1 F'(h,\mathbf{u})\, dh.$$

It is shown that (cf. [3], p.82)

$$(2.23) \quad F'(h, bu) = \sum_{1 \leqq i < j \leqq n} (\Lambda^1_{ij} - \Lambda^0_{ij}) \int_{-\infty}^{\mathbf{u}'} \cdots \int f_h(y_i = u_i,\, y_j = u_j)\, d\mathbf{y}',$$

where $f_h(y_i = u_i,\, y_j = u_j)$ denotes the function of $n-2$ variables formed by putting $y_i = u_i$, $y_j = u_j$, the integration being over the remaining variables.

Let $\phi(x, y; \gamma)$ be the standard bivariate normal density with correlation coefficient $\gamma$. Then we can write

$$\int \overset{\mathbf{u}'}{\underset{-\infty}{\cdots}} \int f_h(y_i = u_i,\ y_j = u_j)\, d\mathbf{y}' = \tag{2.24}$$

$$= \phi(u_i, u_j; \Lambda_{ij}^h) P\big(\zeta' \leqq \mathbf{u}' \mid \zeta_i = u_i,\ \zeta_j = u_j\big).$$

For the sake of simplicity, we work only with $i = 1$, $j = 2$. Since

$$\left\{\zeta_i - \frac{\Lambda_{i1}^h - \Lambda_{12}^h \Lambda_{i2}^h}{1 - {\Lambda_{12}^h}^2}\zeta_1 - \frac{\Lambda_{i2}^h - \Lambda_{12}^h \Lambda_{i1}^h}{1 - {\Lambda_{12}^h}^2}\zeta_2,\ i = 3, \ldots, n\right\} \quad \text{and} \quad \{\zeta_1, \zeta_2\}$$

are independent, we have

$$P\Big(\bigcap_{j=3}^{n}\{\zeta_j \leqq u_j\} \mid \zeta_1 = u_1,\ \zeta_2 = u_2\Big) =$$

$$= P\bigg(\bigcap_{j=3}^{n}\bigg\{\zeta_j - \frac{\Lambda_{j1}^h - \Lambda_{12}^h \Lambda_{j2}^h}{1 - {\Lambda_{12}^h}^2}\zeta_1 - \frac{\Lambda_{j2}^h - \Lambda_{12}^h \Lambda_{j1}^h}{1 - {\Lambda_{12}^h}^2}\zeta_2 \leqq$$

$$\leqq u_j - \frac{\Lambda_{j1}^h - \Lambda_{12}^h \Lambda_{j2}^h}{1 - {\Lambda_{12}^h}^2}u_1 - \frac{\Lambda_{j2}^h - \Lambda_{12}^h \Lambda_{j1}^h}{1 - {\Lambda_{12}^h}^2}u_2\bigg\}\bigg) \leqq$$

$$\leqq P(\zeta_1 \geqq u_1) + P(\zeta_2 \geqq u_2) +$$

$$+ P\bigg(\zeta_1 \leqq u_1,\ \zeta_2 \leqq u_2,\ \zeta_j - \frac{\Lambda_{j1}^h - \Lambda_{12}^h \Lambda_{j2}^h}{1 - {\Lambda_{12}^h}^2}\zeta_1 - \frac{\Lambda_{j2}^h - \Lambda_{12}^h \Lambda_{j1}^h}{1 - {\Lambda_{12}^h}^2}\zeta_2 \leqq$$

$$\leqq u_j - \frac{\Lambda_{j1}^h - \Lambda_{12}^h \Lambda_{j2}^h}{1 - {\Lambda_{12}^h}^2}u_1 - \frac{\Lambda_{j2}^h - \Lambda_{12}^h \Lambda_{j1}^h}{1 - {\Lambda_{12}^h}^2}u_2,\ j = 3, \ldots, n\bigg) \leqq$$

$$\leqq P(\zeta_1 \geqq u_1) + P(\zeta_2 \geqq u_2) + P\Big(\bigcap_{j=1}^{n}\{\zeta_j \leqq u_j\}\Big),$$

since (2.19) and (2.20) imply $\Lambda_{j1}^h \geqq \Lambda_{12}^h \Lambda_{j2}^h$ and $\Lambda_{j2}^h \geqq \Lambda_{12}^h \Lambda_{j1}^h$ for each $3 \leqq$ $\leqq j \leqq n$ and $0 \leqq h \leqq 1$.

Noting that $\Lambda_{ij}^h \leqq \Lambda_{ij}^1$ for all $1 \leqq i,\ j \leqq n$, $0 \leqq h \leqq 1$ and using the Slepian lemma, we obtain

$$P\Big(\bigcap_{j=1}^{n}\{\zeta_j \leqq u_j\}\Big) \leqq P\Big(\bigcap_{j=1}^{n}\{\xi_j \leqq u_j\}\Big).$$

Hence, we have

$$P\Big(\bigcap_{j=3}^{n}\{\zeta_j \leqq u_j\} \mid \zeta_1 = u_1,\, \zeta_2 = u_2\Big) \leqq$$

$$\leqq P(\zeta_1 \geqq u_1) + P(\zeta_2 \geqq u_2) + P\Big(\bigcap_{j=1}^{n}\{\xi_j \leqq u_j\}\Big) \leqq$$

$$\leqq e^{-u_1^2/2} + e^{-u_2^2/2} + P\Big(\bigcap_{j=1}^{n}\{\xi_j \leqq u_j\}\Big).$$

Generally, we have

$$(2.25)\quad P(\zeta' \leqq \mathbf{u}' \mid \zeta_i = u_i,\, \zeta_j = u_j) \leqq e^{-u_i^2/2} + e^{-u_j^2/2} + P\Big(\bigcap_{j=1}^{n}\{\xi_j \leqq u_j\}\Big).$$

On the other hand, it is easy to see that (cf. [3], p.83)

$$(2.26)\qquad \phi\big(u_i, u_j; \Lambda_{ij}^h\big) \leqq \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\right).$$

This proves (2.21) by (2.22)–(2.26).

Noting that for the Ornstein–Uhlenbeck process $\{X(t),\, t \geqq 0\}$

$$\rho(t,s) \geqq \rho(t,v)\rho(v,s), \quad \text{for all} \quad t,s,v \geqq 0,$$

by (2.5), as an immediate consequence of Lemma 2.3, we have the following.

LEMMA 2.4. *Let $\{X(t),\, t \geqq 0\}$ be an Ornstein–Uhlenbeck process satisfying* (2.5). *Let $E_n = \{X(t_{n,v}) \leqq x_{n,v}\colon v = 0, \dots, m_n\}$ with all $t_{n,v}$ distinct and $x_{n,v} \geqq 0$. Then*

$$P\Big(\bigcap_{k=1}^{n} E_k\Big) - \prod_{k=1}^{n} P(E_k) \leqq$$

$$\leqq \sum_{1\leqq i<j\leqq n} \sum_{u=0}^{m_j}\sum_{v=0}^{m_i} \rho(t_{i,v}, t_{j,u})\big(1-\rho^2(t_{i,v}, t_{j,u})\big)^{-1/2} P\Big(\bigcap_{k=1}^{n} E_k\Big)\,\cdot$$

$$\cdot \exp\left(-\frac{x_{i,v}^2 + x_{j,u}^2}{2\big(1+\rho(t_{i,v}, t_{j,u})\big)}\right) +$$

$$+\sum_{1\leqq i<j\leqq n}\sum_{u=0}^{m_j}\sum_{v=0}^{m_i}\rho(t_{i,v},t_{j,u})\big(1-\rho^2(t_{i,v},t_{j,u})\big)^{-1/2}\cdot$$

$$\cdot\exp\left(-\frac{x_{i,v}^2+x_{j,u}^2}{2\big(1+\rho(t_{i,v},t_{j,u})\big)}\right)\left(e^{-\frac{1}{2}x_{i,v}^2}+e^{-\frac{1}{2}x_{j,u}^2}\right).$$

Using Lemma 2.4, we can now establish another technical lemma.

LEMMA 2.5. *For each $\delta>0$, $p\geqq 2$, $0<\eta<\frac{1}{2}$, there exists a constant $N=N(\delta,p,\eta)$ such that*

$$P\left(\bigcap_{e^a\leqq s\leqq e^b}\left\{\frac{W(s)}{\sqrt{s}\alpha(\delta,p,s)}\leqq 1\right\}\right)\leqq \tag{2.27}$$

$$\leqq\begin{cases} 6\exp\left(-\frac{1-2\eta}{2\delta}\sqrt{\frac{1-\delta}{\pi}}(\log a)^{1/2}(b^\delta-a^\delta)\right)+ \\ \quad +N(b-a)a^{-1(1-\delta)(2+\rho/2)}\cdot\log^2 b, \quad \text{if } 0<\delta<1,\ p=2, \\ 6\exp\left(-\frac{1-2\eta}{2\delta\sqrt{\pi}}\left(\left(\log_{p-2}b\right)^\delta-\left(\log_{p-2}a\right)^\delta\right)\right)+ \\ \quad +Na^{-(1+\rho/2)}, \quad \text{if } \delta>0,\ p\geqq 3\end{cases}$$

*for each $b\geqq a+2\geqq N$ if $p\geqq 3$ and $N\leqq a+2\leqq b+a^{1-\delta}$ if $0<\delta<1$, $p=2$, where $\rho=1-e^{-\eta/2}$.*

PROOF. Note that

$$P\left(\bigcap_{e^a\leqq s\leqq e^b}\left\{\frac{W(s)}{\sqrt{s}\alpha(\delta,p,s)}\leqq 1\right\}\right)=P\left(\bigcap_{a\leqq s\leqq b}\left\{\frac{X(s)}{\alpha(\delta,p,e^s)}\leqq 1\right\}\right)\leqq$$

$$\leqq P\left(\max_{0\leqq i\leqq\frac{b-a-1}{1+\eta}}\ \sup_{a+i(1+\eta)\leqq s\leqq a+1+i(1+\eta)}\frac{X(s)}{\alpha(\delta,p,e^s)}\leqq 1\right)\leqq$$

$$\leqq P\left(\max_{0\leqq i\leqq\frac{b-a-1}{1+\eta}}\ \sup_{a+i(1+\eta)\leqq s\leqq a+1+i(1+\eta)}\frac{X(s)}{\alpha(\delta,p,e^{a+1+i(1+\eta)})}\leqq 1\right).$$

By (2.8) and (2.9), there exist $\theta>0$ and $x_0$ such that

$$P\left(\max_{0\leqq j\leqq 2x^2/\theta}X\left(j\frac{\theta}{2x^2}\right)>x\right)\geqq\left(1-\frac{\eta}{2}\right)\psi(x) \tag{2.28}$$

for each $x \geqq x_0$. Put

$$x_i := x_i(a,b) = \alpha\big(\delta, p, e^{a+1+i(1+\eta)}\big), \quad i = 0, 1, \dots .$$

Then, using Lemma 2.4, we obtain

$$P\left(\max_{0\leqq i\leqq \frac{b-a-1}{1+\eta}} \sup_{a+i(1+\eta)\leqq s\leqq a+1+i(1+\eta)} \frac{X(s)}{\alpha\big(\delta, p, e^{a+1+i(1+\eta)}\big)} \leqq 1\right) \leqq$$

$$\leqq P\left(\max_{0\leqq i\leqq \frac{b-a-1}{1+\eta}} \max_{0\leqq j\leqq \frac{2x_i^2}{\theta}} \frac{X\left(a+i(1+\eta)+j\frac{\theta}{2x_i^2}\right)}{x_i} \leqq 1\right) \leqq$$

$$\leqq \prod_{i=0}^{\frac{b-a-1}{1+\eta}} P\left(\max_{0\leqq j\leqq \frac{2x_i^2}{\theta}} \frac{X\left(a+i(1+\eta)+j\frac{\theta}{2x_i^2}\right)}{x_i} \leqq 1\right) +$$

$$+ \sum_{0\leqq i<j\leqq \frac{b-a-1}{1+\eta}} \sum_{u=0}^{\frac{2x_j^2}{\theta}} \sum_{v=0}^{\frac{2x_i^2}{\theta}} \rho_{i,j,u,v}\left(1-\rho_{i,j,u,v}^2\right)^{-1/2} \exp\left(-\frac{x_i^2+x_j^2}{2(1+\rho_{i,j,u,v})}\right) \cdot$$

$$\cdot P\left(\max_{0\leqq i\leqq \frac{b-a-1}{1+\eta}} \max_{0\leqq j\leqq \frac{2x_i^2}{\theta}} \frac{X\left(a+i(1+\eta)+j\frac{\theta}{2x_i^2}\right)}{x_i} \leqq 1\right) +$$

$$+ \sum_{0\leqq i<j\leqq \frac{b-a-1}{1+\eta}} \sum_{u=0}^{\frac{2x_j^2}{\theta}} \sum_{v=0}^{\frac{2x_i^2}{\theta}} \rho_{i,j,u,v}\left(1-\rho_{i,j,u,v}^2\right)^{-1/2} \cdot$$

$$\cdot \exp\left(-\frac{x_i^2+x_j^2}{2(1+\rho_{i,j,u,v})}\right)\left(e^{-\frac{1}{2}x_i^2}+e^{-\frac{1}{2}x_j^2}\right) =$$

$$= I_1(a,b) + I_2(a,b) +$$

$$+ I_3(a,b) P\left(\max_{0\leqq i\leqq \frac{b-a-1}{1+\eta}} \max_{0\leqq j\leqq \frac{2x_i^2}{\theta}} \frac{X\left(a+i(1+\eta)+j\frac{\theta}{2x_i^2}\right)}{x_i} \leqq 1\right),$$

where $\rho_{i,j,u,v} = \exp\left(-\frac{1}{2}\left((j-i)(1+\eta)+\frac{u\theta}{2x_j^2}-\frac{v\theta}{2x_i^2}\right)\right)$,

$$I_1(a,b)=\prod_{i=0}^{\frac{b-a-1}{1+\eta}} P\left(\max_{0\leqq j\leqq \frac{2x_i^2}{\theta}} \frac{X\left(j\frac{\theta}{2x_i^2}\right)}{x_i}\leqq 1\right),$$

$$I_2(a,b)=\sum_{0\leqq i<j\leqq \frac{b-a-1}{1+\eta}}\sum_{u=0}^{\frac{2x_j^2}{\theta}}\sum_{v=0}^{\frac{2x_i^2}{\theta}}\rho_{i,j,u,v}(1-\rho_{i,j,u,v}^2)^{-1/2}\cdot$$

$$\cdot\exp\left(-\frac{x_i^2+x_j^2}{2(1+\rho_{i,j,u,v})}\right)\left(e^{-\frac{1}{2}x_i^2}+e^{-\frac{1}{2}x_j^2}\right),$$

$$I_3(a,b)=\sum_{0\leqq i<j\leqq \frac{b-a-1}{1+\eta}}\sum_{u=0}^{\frac{2x_j^2}{\theta}}\sum_{v=0}^{\frac{2x_i^2}{\theta}}\rho_{i,j,u,v}(1-\rho_{i,j,u,v}^2)^{-1/2}\cdot$$

$$\cdot\exp\left(-\frac{x_i^2+x_j^2}{2(1+\rho_{i,j,u,v})}\right).$$

From (2.28) it follows that

$$I_1(a,b)\leqq\prod_{i=0}^{\frac{b-a-1}{1+\eta}}\left(1-\left(1-\frac{\eta}{2}\right)\psi(x_i)\right)\leqq \tag{2.29}$$

$$\leqq\exp\left(-\left(1-\frac{\eta}{2}\right)\sum_{i=0}^{\frac{b-a-1}{1+\eta}}\psi(x_i)\right)=$$

$$=\exp\left(-\left(1-\frac{\eta}{2}\right)\sum_{i=0}^{\frac{b-a-1}{1+\eta}}\psi\left(\alpha\left(\delta,p,e^{a+1+i(1+\eta)}\right)\right)\right).$$

Similarly to (2.17) and (2.18), one can see that

$$\sum_{i=0}^{\frac{b-a-1}{1+\eta}}\psi\left(\alpha\left(\delta,p,e^{a+1+i(1+\eta)}\right)\right)\geqq$$

$$\geqq \begin{cases} \frac{1}{2\delta(1+\eta)}\sqrt{\frac{1-\delta}{\pi}}(\log a)^{1/2}(b^\delta - a^\delta) - 1, & \text{if } 0<\delta<1,\, p=2, \\ \frac{1}{2\delta\sqrt{\pi}(1+\eta)}\left((\log_{p-2} b)^\delta - (\log_{p-2} a)^\delta\right) - 1, & \text{if } \delta>0,\, p\geqq 3. \end{cases}$$

Therefore

$$I_1(a,b) \leqq \begin{cases} 3\exp\left(-\frac{1-2\eta}{2\delta}\sqrt{\frac{1-\delta}{\pi}}(\log a)^{1/2}(b^\delta - a^\delta)\right), \\ \qquad \text{if } 0<\delta<1,\, p=2, \\ 3\exp\left(-\frac{1-2\eta}{2\delta\sqrt{\pi}}\left((\log_{p-2} b)^\delta - (\log_{p-2} a)^\delta\right)\right), \\ \qquad \text{if } \delta>0,\, p\geqq 3. \end{cases} \tag{2.30}$$

Noting that for $i<j$, $0 \leqq u \leqq 2x_j^2/\theta$, $0 \leqq v \leqq 2x_i^2/\theta$

$$\rho_{i,j,u,v} \leqq \exp\left(-\frac{1}{2}((j-i)(1+\eta)-1)\right) \leqq$$

$$\leqq \exp\left(-\frac{1}{2}(j-i)\eta\right) \leqq \exp\left(-\frac{\eta}{2}\right),$$

we have, with $\rho = 1-\exp\left(-\frac{\eta}{2}\right)$,

$$I_2(a,b) \leqq \tag{2.31}$$

$$\leqq \sum_{0\leqq i<j\leqq \frac{b-a-1}{1+\eta}} \sum_{u=0}^{\frac{2x_j^2}{\theta}} \sum_{v=0}^{\frac{2x_i^2}{\theta}} e^{-\frac{1}{2}(j-i)\eta}\rho^{-1/2}\,.$$

$$\cdot \exp\left(-\frac{x_i^2+x_j^2}{2(2-\rho)}\right)\left(e^{-\frac{1}{2}x_i^2} + e^{-\frac{1}{2}x_j^2}\right) \leqq$$

$$\leqq 16 \sum_{0\leqq i<j\leqq \frac{b-a-1}{1+\eta}} \frac{x_j^2 x_i^2}{\theta^2\rho^{1/2}} \exp\left(-\frac{x_i^2+x_j^2}{2(2-\rho)}\right)\left(e^{-\frac{1}{2}x_i^2} + e^{-\frac{1}{2}x_j^2}\right) e^{-\frac{1}{2}(j-i)\eta} \leqq$$

$$\leqq \frac{2^{10}}{\eta\theta^2\rho^{1/2}} \sum_{0\leqq i\leqq \frac{b-a-1}{1+\eta}} \exp\left(-\frac{x_i^2}{2-\rho} - \frac{x_i^2}{2}\right)\cdot \log^2\bigl(a+i(1+\eta)+1\bigr) \leqq$$

$$\leqq \begin{cases} C(\eta,\delta)(b-a)a^{-(1-\delta)(2+\rho/(2-\rho))}\log^2 b, & \text{if } 0<\delta<1,\, p=2, \\ C(\eta,\delta,p)a^{-(1+\rho/2)}, & \text{if } \delta>0,\, p\geqq 3, \end{cases}$$

by (2.3), where $C(\eta,\delta)$ and $C(\eta,\delta,p)$ are positive constants depending only on $\eta$, $\delta$, $p$. Similarly, we have

$$(2.32)\qquad I_3(a,b) \leqq \begin{cases} C(\eta,\delta)(b-a)a^{-(1-\delta)(1+\rho/(2-\rho))}, & \text{if } 0<\delta<1,\ p=2,\\ C(\eta,\delta,p)a^{-\rho/2}, & \text{if } \delta>0,\ p\geqq 3,\end{cases}$$

$$\leqq \begin{cases} \frac{1}{2}, & \text{if } 0<\delta<1,\ p=2,\ b-a\leqq a^{1-\delta},\\ \frac{1}{2}, & \text{if } \delta>0,\ p\geqq 3,\end{cases}$$

provided $a$ is sufficiently large. Combining the above inequalities, we get (2.27), as desired.

Let

$$a(t,0,2) = 3\sqrt{\pi}\frac{\log_3 t\cdot\log t}{(\log_2 t)^{1/2}},$$

$$a(t,\delta,2) = 2\delta\sqrt{\frac{\pi}{1-\delta}}(\log t)^{1-\delta}\cdot(\log_2 t)^{1/2},\quad 0<\delta<1,$$

$$a(t,0,p) = 2\sqrt{\pi}\log_{p+1} t,\quad p\geqq 3,$$

$$a(t,\delta,p) = 2\delta\sqrt{\pi}(\log_{p-1} t)^{1-\delta}\cdot\log_p t,\quad 0<\delta<1,\quad p\geqq 3,$$

$$a(t,\delta,p) = 2\delta\sqrt{\pi}\log_{p-2} t\cdot(\log_{p-1} t)^{1-\delta}\cdot\log_p t,\quad \delta>1,\quad p\geqq 3.$$

PROOF OF THEOREM 1. It is clear that

$$(2.33)\qquad \{\xi_\delta(t)\leqq a\} = \left\{\sup_{a\leqq s\leqq t}\frac{W(s)}{\sqrt{2(1-\delta)s\log\log s}}<1\right\}$$

for each $0<a\leqq t$. We first prove for each $0<\varepsilon<1/2$ that

$$(2.34)\qquad \liminf_{t\to\infty}\frac{(\log_2 t)^{1/2}}{\log_3 t\cdot\log t}\log\frac{\xi_0(t)}{t}\geqq -(1+2\varepsilon)3\sqrt{\pi}\ \text{a.s.}$$

Let

$$t_k = \exp\left(\exp(k^{2/3})\right),\qquad k=1,2,\dots\ .$$

Noting that

$$\frac{a(t,\delta,2)}{\log t - a(t,\delta,2)}\to 0\quad\text{as}\quad t\to\infty$$

for $0 \leqq \delta \leqq 1/2$, and using (2.33), (2.1) and (2.27), we have

$$(2.35) \qquad P\left(\log\frac{\xi_0(t_k)}{t_k} \leqq -(1+\varepsilon)a(t_k,0,2)\right) =$$

$$= P\left(\sup_{t_k\exp(-(1+\varepsilon)a(t_k,0,2))\leqq s\leqq t_k}\frac{W(s)}{\alpha(0,2,s)\sqrt{s}} < 1\right) =$$

$$= P\left(\sup_{t_k\exp(-(1+\varepsilon)a(t_k,0,2))\leqq s\leqq t_k}\frac{W(s)}{\alpha(3/2,3,s)\sqrt{s}} < 1\right) \leqq$$

$$\leqq 6\exp\left(-\frac{1-2\eta}{3\sqrt{\pi}}\left((\log_2 t_k)^{3/2} - \left(\log(\log t_k - (1+\varepsilon)a(t_k,0,2))\right)^{3/2}\right)\right) +$$

$$+N\left(\log t_k - (1+\varepsilon)a(t_k,0,2)\right)^{-1} \leqq$$

$$\leqq 6\exp\left(-\frac{1-2\eta}{3\sqrt{\pi}}\left((\log_2 t_k)^{3/2} -\right.\right.$$

$$\left.\left.-\left(\log_2 t_k + \log\left(1-\frac{(1+\varepsilon)a(t_k,0,2)}{\log t_k}\right)\right)^{3/2}\right)\right) + 2N(\log t_k)^{-1} \leqq$$

$$\leqq 6\exp\left(-\frac{1-2\eta}{3\sqrt{\pi}}\left((\log_2 t_k)^{3/2} -\right.\right.$$

$$\left.\left.-\left(\log_2 t_k - \frac{(1+\varepsilon/2)a(t_k,0,2)}{\log t_k}\right)^{3/2}\right)\right) + 2N(\log t_k)^{-1} \leqq$$

$$\leqq 6\exp\left(-\frac{1-2\eta}{3\sqrt{\pi}}\cdot\frac{3}{2}\cdot\frac{(1+\varepsilon/4)a(t_k,0,2)}{\log t_k}\cdot(\log_2 t_k)^{1/2}\right) +$$

$$+2N\exp(-k^{2/3}) \leqq$$

$$\leqq 6\exp\left(-\frac{3}{2}(1-2\eta)(1+\varepsilon/4)\log_3 t_k\right) + 2N\exp(-k^{2/3}) \leqq$$

$$\leqq 6k^{-(1+\varepsilon/8)} + 2N\exp(-k^{2/3})$$

for every $k$ sufficiently large, provided $1 > 1-2\eta \geqq \frac{1+\varepsilon/8}{1+\varepsilon/4}$. Hence

$$(2.36) \qquad P\left(\sup_{t_k\exp(-(1+\varepsilon)a(t_k,0,2))\leqq s\leqq t_k}\frac{W(s)}{\alpha(0,2,s)\sqrt{s}} \leqq 1, \text{ i.o.}\right) = 0,$$

by the Borel–Cantelli lemma. On the other hand, noting that

$$\log(t_{k+1}/t_k) = \exp\left((k+1)^{2/3}\right) - \exp(k^{2/3}) =$$

$$= \exp\left((k+1)^{2/3}\right)\left(1 - \exp\left(-(k+1)^{2/3} + k^{2/3}\right)\right) \sim$$

$$\sim \frac{2}{3}k^{-1/3}\exp\left((k+1)^{2/3}\right)$$

and

$$a(t_{k+1},0,2) \sim 3\sqrt{\pi}k^{-1/3}\exp\left((k+1)^{2/3}\right)\log(k+1)^{2/3},$$

we have

$$\text{(2.37)} \qquad \log(t_{k+1}/t_k) = o\big(a(t_{k+1},0,2)\big) \quad \text{as} \quad k \to \infty.$$

Therefore, for every $k$ big enough and for any $t_k \leqq t \leqq t_{k+1}$

$$t\exp\big(-(1+2\varepsilon)a(t,0,2)\big) \leqq t_{k+1}\exp\big(-(1+\varepsilon)a(t_{k+1},0,2)\big) =$$

$$= (t_{k+1}/t_k)\cdot t_k \cdot \exp\big(-(1+\varepsilon)a(t_{k+1},0,2)\big) \leqq$$

$$\leqq t_k\exp\big(-(1+\varepsilon)a(t_k,0,2)\big) \leqq t_k \leqq t.$$

Consequently, for any $t$ big enough, there exists a $t' = t'(t,\omega)$ between $t\exp\big(-(1+2\varepsilon)a(t,0,2)\big)$ and $t$ such that

$$W(t') \geqq \alpha(0,2,t')\sqrt{t'},$$

by (2.36). This proves (2.34).

We next show that for each $0 < \varepsilon < 1/5$

$$\text{(2.38)} \qquad \liminf_{t\to\infty} \frac{(\log_2 t)^{1/2}}{\log_3 t \cdot \log t}\log\frac{\xi_0(t)}{t} \leqq -(1-5\varepsilon)3\sqrt{\pi} \text{ a.s.}$$

Let

$$s_k := s_k(\varepsilon) = \exp\big(\exp(k^{\varepsilon+2/3})\big), \qquad k = 1,2,\dots .$$

Then

$$\log(s_{k+1}/s_k) \sim \left(\frac{2}{3}+\varepsilon\right)k^{\varepsilon-1/3}\exp\left((k+1)^{\varepsilon+2/3}\right),$$

$$a(s_{k+1},0,2) \sim 3\sqrt{\pi}\left(\frac{2}{3}+\varepsilon\right)k^{-1/3-\varepsilon/2}\exp\left((k+1)^{\varepsilon+2/3}\right)\log k$$

and hence

$$a(s_{k+1},0,2)=o\big(\log(s_{k+1}/s_k)\big).$$

Set $b_k=s_{k+1}\exp\big(-(1-5\varepsilon)a(s_{k+1},0,2)\big)$. Noting that

$$\left\{\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)-W(s_k)}{\alpha(0,2,s)\sqrt{s}}\leqq 1-\frac{(4s_k\log_2 s_k)^{1/2}}{b_k^{1/2}}\right\}\subset$$

$$\subset\left\{\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)}{\alpha(0,2,s)\sqrt{s}}\leqq 1\right\}\cup\left\{W(s_k)\geqq(4s_k\log_2 s_k)^{1/2}\right\}$$

and

$$P\left(W(s_k)\geqq(4s_k\log_2 s_k)^{1/2},\ \text{i.o.}\right)=0,$$

we have

$$P\left(\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)}{\alpha(0,2,s)\sqrt{s}}\leqq 1\ \text{i.o.}\right)\geqq \tag{2.39}$$

$$\geqq P\left(\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)-W(s_k)}{\alpha(0,2,s)\sqrt{s}}\leqq 1-\frac{(4s_k\log_2 s_k)^{1/2}}{b_k^{1/2}},\ \text{i.o.}\right).$$

To prove (2.38), it suffices to show that

$$\sum_{k=1}^{\infty}P\left(\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)-W(s_k)}{\alpha(0,2,s)\sqrt{s}}\leqq 1-\frac{(4s_k\log_2 s_k)^{1/2}}{b_k^{1/2}}\right)=\infty, \tag{2.40}$$

by the Borel–Cantelli lemma, since

$$\left\{\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)-W(s_k)}{\alpha(0,2,s)\sqrt{s}},\quad k\geqq 1\right\}$$

are independent.

It is easy to see that for each $0<\eta<1/3$

$$\left\{\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)}{\sqrt{s}}\leqq\alpha(0,2,s)-\frac{\eta}{\alpha(0,2,s)}\right\}\subset$$

$$\subset\left\{\sup_{b_k\leqq s\leqq s_{k+1}}\frac{W(s)-W(s_k)}{\alpha(0,2,s)\sqrt{s}}\leqq 1-\frac{(4s_k\log_2 s_k)^{1/2}}{b_k^{1/2}}\right\}\cup$$

$$\cup\left\{-W(s_k)\geqq(4s_k\log_2 s_k)^{1/2}\right\},$$

provided $k$ is big enough, since $\alpha^2(0,2,s_{k+1}) = o\left(\left(b_k/(s_k \log_2 s_k)\right)^{1/2}\right)$. Therefore, by (2.11) and (2.1)

$$(2.41) \qquad P\left(\sup_{b_k \leqq s \leqq s_{k+1}} \frac{W(s)-W(s_k)}{\alpha(0,2,s)\sqrt{s}} \leqq 1 - \frac{(4s_k \log_2 s_k)^{1/2}}{b_k^{1/2}}\right) \geqq$$

$$\geqq P\left(\sup_{b_k \leqq s \leqq s_{k+1}} \frac{W(s)}{\sqrt{s}} \leqq \alpha(0,2,s) - \frac{\eta}{\alpha(0,2,s)}\right) -$$

$$-P\left(-W(s_k) \geqq (4s_k \log_2 s_k)^{1/2}\right) \geqq$$

$$\geqq \frac{1}{8}\exp\left(-\frac{1+8\eta}{3\sqrt{\pi}}\left((\log_2 s_{k+1})^{3/2} - (\log_s b_k)^{3/2}\right)\right) - 2\exp(-2\log_2 s_k) \geqq$$

$$\geqq \frac{1}{8}\exp\left(-\frac{1+8\eta}{3\sqrt{\pi}}\left((\log_2 s_{k+1})^{3/2} - \right.\right.$$

$$\left.\left. - \left(\log_2 s_{k+1} - \frac{(1-4\varepsilon)a(s_{k+1},0,2)}{\log s_{k+1}}\right)^{3/2}\right)\right) - k^{-2} \geqq$$

$$\geqq \frac{1}{8}\exp\left(-\frac{1+8\eta}{3\sqrt{\pi}}(1-4\varepsilon)\frac{3}{2}\,\frac{a(s_{k+1},0,2)(\log_2 s_{k+1})^{1/2}}{\log s_{k+1}}\right) - k^{-2} \geqq$$

$$\geqq \frac{1}{8}\exp\big(-(1+8\eta)(1-2\varepsilon)\log k\big) - k^{-2} \geqq \frac{1}{16}k^{-(1-\varepsilon)},$$

provided $k$ is sufficiently large and $1 < 1+8\eta < (1-\varepsilon)/(1-2\varepsilon)$.

This proves (2.40). Now (2.38) is proved. This completes the proof of (1.2) by (2.34) and (2.38) and by the arbitrariness of $\varepsilon$.

We now turn to the proof of (1.3). We again formulate the proof in two steps.

*Step 1.* For each $0 < \varepsilon < 1/2$

$$(2.42) \quad \liminf_{t\to\infty} (\log t)^{\delta-1}(\log_2 t)^{-1/2} \cdot \log \frac{\xi_\delta(t)}{t} \geqq -(1+2\varepsilon)2\delta\sqrt{\frac{\pi}{1-\delta}} \quad \text{a.s.}$$

Put

$$t_k = \exp(k^{1/\delta}), \quad k = 1,2,\ldots, \quad 0 < \delta \leqq 1/2.$$

Similarly to (2.35), using (2.33) and (2.27), we have

$$P\left(\log\frac{\xi_\delta(t_k)}{t_k} \leqq -(1+\varepsilon)a(t_k,\delta,2)\right) =$$

$$= P\Bigg(\sup_{t_k\exp(-(1+\varepsilon)a(t_k,\delta,2))\leqq s\leqq t_k} \frac{W(s)}{\alpha(\delta,2,s)\sqrt{s}} \leqq 1\Bigg) \leqq$$

$$\leqq 6\exp\Bigg(-\frac{1-2\eta}{2\delta}\sqrt{\frac{1-\delta}{\pi}}\big(\log(\log t_k-(1+\varepsilon)a(t_k,\delta,2))\big)^{1/2}\cdot$$

$$\cdot\Big((\log t_k)^\delta - \big(\log t_k-(1+\varepsilon)a(t_k,\delta,2)\big)^\delta\Big)\Bigg)+$$

$$+Na(t_k,\delta,2)(\log_2 t_k)^2\cdot\big(\log t_k-(1+\varepsilon)a(t_k,\delta,2)\big)^{-(1-\delta)(2+\rho/2)} \leqq$$

$$\leqq 6\exp\left(-\frac{1-2\eta}{2\delta}\sqrt{\frac{1-\delta}{\pi}}(\log_2 t_k)^{1/2}\cdot\delta(1+\varepsilon/2)a(t_k,\delta,2)(\log t_k)^{\delta-1}\right)+$$

$$+2Na(t_k,\delta,2)(\log_2 t_k)^2\cdot(\log t_k)^{-(1-\delta)(2+\rho/2)} \leqq$$

$$\leqq 6\exp\big(-(1-2\eta)\delta(1+\varepsilon/2)\log_2 t_k\big)+$$

$$+N(\delta)(\log t_k)^{1-\delta}(\log_2 t_k)^3(\log t_k)^{-(1-\delta)(2+\rho/2)} \leqq$$

$$\leqq 6k^{-(1-2\eta)(1+\varepsilon/2)} + N(\delta)k^{-(1-\delta)(1+\rho/2)/\delta}(\log k)^4 \leqq$$

$$\leqq 6k^{-(1+\varepsilon/4)} + N(\delta)k^{-1-\frac{1-2\delta+(1-\delta)\rho/2}{\delta}}(\log k)^4$$

for every $k$ sufficiently large, provided $1 > 1-2\eta > (1+\varepsilon/4)/(1+\varepsilon/2)$. Hence

$$P\Bigg(\sup_{t_k\exp(-(1+\varepsilon)a(t_k,\delta,2))\leqq s\leqq t_k} \frac{W(s)}{\alpha(\delta,2,s)\sqrt{s}} \leqq 1,\ \text{i.o.}\Bigg) = 0, \tag{2.43}$$

by the Borel–Cantelli lemma and the assumption $0<\delta\leqq 1/2$. It is clear that

$$\log(t_{k+1}/t_k) = o\big(a(t_k,\delta,2)\big) \quad \text{as} \quad k\to\infty.$$

Now a repetition of the proof of (2.34) yields (2.42), by (2.43).

*Step 2.* For each $0 < \varepsilon < 1/5$, $0 < \delta < 1$

$$\liminf_{t\to\infty} (\log t)^{\delta-1}(\log_2 t)^{-1/2} \cdot \log \frac{\xi_\delta(t)}{t} \leqq -(1-5\varepsilon)2\delta\sqrt{\frac{\pi}{1-\delta}} \quad \text{a.s.} \tag{2.44}$$

Let

$$s_k := s_k(\varepsilon) = \exp(k^{\varepsilon+1/\delta}), \qquad k = 1, 2, \dots .$$

Then

$$a(s_{k+1}, \delta, 2) = o\big(\log(s_{k+1}/s_k)\big) \quad \text{as} \quad k \to \infty$$

and

$$\alpha^2(\delta, 2, s_{k+1}) = o\Big(\big(s_{k+1}\exp(-(1-5\varepsilon)a(s_{k+1}, 0, 2))/(s_k \log_2 s_k)\big)^{1/2}\Big)$$

as $k \to \infty$. Along the same lines of the proof of (2.38), one can arrive at (2.44). This proves (1.3) by (2.42) and (2.44).

PROOF OF THEOREM 2. The idea of the proof is completely the same as that of Theorem 1 and hence only a sketch will be given here. Write

$$e_1(x) = \exp x, \quad e_p(x) = \exp\big(e_{p-1}(x)\big), \quad p = 2, 3, \dots .$$

We first prove (1.5). It suffices to show that

$$\liminf_{t\to\infty} \frac{\log_{p-1}\xi_\delta^{(p)}(t) - \log_{p-1} t}{(\log_{p-1} t)^{1-\delta} \cdot \log_p t} \geqq -(1+2\varepsilon)2\delta\sqrt{\pi} \quad \text{a.s.} \tag{2.45}$$

and

$$\liminf_{t\to\infty} \frac{\log_{p-1}\xi_\delta^{(p)}(t) - \log_{p-1} t}{(\log_{p-1} t)^{1-\delta} \cdot \log_p t} \leqq -(1-5\varepsilon)2\delta\sqrt{\pi} \quad \text{a.s.} \tag{2.46}$$

for every $0 < \varepsilon < 1/5$, $0 < \delta < 1$, $p = 3, 4, \dots$ . Let

$$t_k = e_{p-1}(k^{1/\delta}), \qquad k = 1, 2, \dots .$$

Then

$$a(t, \delta, p) = o(\log_{p-1} t) \quad \text{as} \quad t \to \infty,\ 0 < \delta < 1,\ p \geqq 3. \tag{2.47}$$

Using (2.27), we get

$$P\left(\frac{\log_{p-1}\xi_\delta^{(p)}(t_k)-\log_{p-1}t_k}{(\log_{p-1}t_k)^{1-\delta}\log_p t_k}\leqq -(1+\varepsilon)2\delta\sqrt{\pi}\right)=$$

$$=P\Bigg(\sup_{e_{p-1}(\log_{p-1}t_k-(1+\varepsilon)a(t_k,\delta p))\leqq s\leqq t_k}\frac{W(s)}{\alpha(\delta,p,s)\sqrt{s}}\leqq 1\Bigg)\leqq$$

$$\leqq 6\exp\Bigg(-\frac{1-2\eta}{2\delta\sqrt{\pi}}\Big((\log_{p-1}t_k)^\delta-$$

$$-\left(\log_{p-2}\left(e_{p-2}(\log_{p-1}t_k-(1+\varepsilon)a(t_k,\delta,p))\right)\right)^\delta\Big)\Bigg)+$$

$$+N\left(e_{p-2}(\log_{p-1}t_k-(1+\varepsilon)a(t_k,\delta,p))\right)^{-1}\leqq$$

$$\leqq 6\exp\left(-\frac{1-2\eta}{2\delta\sqrt{\pi}}\left((\log_{p-1}t_k)^\delta-\left(\log_{p-1}t_k-(1+\varepsilon)a(t_k,\delta,p)\right)^\delta\right)\right)+$$

$$+N\exp\left(-\frac{1}{2}k^{1/\delta}\right)\leqq$$

$$\leqq 6\exp\left(-\frac{1-2\eta}{2\delta\sqrt{\pi}}(1+\varepsilon/2)\delta a(t_k,\delta,p)(\log_{p-1}t_k)^{1-\delta}\right)+$$

$$+N\exp\left(-\frac{1}{2}k^{1/\delta}\right)\leqq$$

$$\leqq 6k^{-(1-2\eta)(1+\varepsilon/2)}+N\exp\left(-\frac{1}{2}k^{1/\delta}\right)\leqq 6k^{-(1+\varepsilon/4)}+N\exp\left(-\frac{1}{2}k^{1/\delta}\right)$$

provided $1>1-2\eta>(1+\varepsilon/4)/(1+\varepsilon/2)$. Therefore

$$P\Bigg(\sup_{e_{p-1}(\log_{p-1}t_k-(1+\varepsilon)a(t_k,\delta p))\leqq s\leqq t_k}\frac{W(s)}{\alpha(\delta,p,s)\sqrt{s}}\leqq 1,\ \text{i.o.}\Bigg)=0. \tag{2.48}$$

It is easy to find that

$$\log_{p-1}t_{k+1}-\log_{p-1}t_k=o\left(a(t_k,\delta,p)\right)\quad\text{as}\quad k\to\infty.$$

Hence, (2.48) implies (2.45).

To prove (2.46), we let

$$s_k:=s_k(\varepsilon)=e_{p-1}(k^{\varepsilon+1/\delta}),\qquad k=1,2,\dots\ .$$

Then

$$a(s_k,\delta,p)=o(\log_{p-1}s_{k+1}-\log_{p-1}s_k)$$

and

$$\alpha^2(\delta,p,s_{k+1})=o\left(\left(s_{k+1}\exp(-(1-5\varepsilon)a(s_{k+1},\delta,p))/(s_k\log_2 s_k)\right)^{1/2}\right).$$

Now along the lines of the proof of (2.38), we get that (2.46) holds true.

Let again $t_k=e_{p-1}(k^{1/\delta})$ and $s_k=e_{p-1}(k^{\varepsilon+1/\delta})$. Then we can also obtain that

$$\liminf_{t\to\infty}\frac{\log_{p-2}\xi_\delta^{(p)}(t)-\log_{p-2}t}{\log_{p-2}t\cdot(\log_{p-1}t)^{1-\delta}\cdot\log_p t}\geqq-(1+2\varepsilon)2\delta\sqrt{\pi}\ \text{a.s.},\tag{2.49}$$

$$\liminf_{t\to\infty}\frac{\log_{p-2}\xi_\delta^{(p)}(t)-\log_{p-2}t}{\log_{p-2}t\cdot(\log_{p-1}t)^{1-\delta}\cdot\log_p t}\leqq-(1-5\varepsilon)2\delta\sqrt{\pi}\ \text{a.s.}\tag{2.50}$$

for every $0<\varepsilon<1/5$, $\delta>1$, $p\geqq 3$.

Taking $t_k=e_p(k)$ and $s_k=e_p(k^{1+\varepsilon})$, we can prove that

$$\liminf_{t\to\infty}\frac{\log_p\xi_0^{(p)}(t)-\log_p t}{\log_{p+1}t}\geqq-(1+2\varepsilon)2\sqrt{\pi}\ \text{a.s.},\tag{2.51}$$

$$\liminf_{t\to\infty}\frac{\log_p\xi_0^{(p)}(t)-\log_p t}{\log_{p+1}t}\leqq-(1-5\varepsilon)2\sqrt{\pi}\ \text{a.s.}\tag{2.52}$$

for every $p=3,4,\dots$ .

The proof of Theorem 2 is now complete.

REMARK 2.1. We conjecture that (1.3) holds true for every $0<\delta<1$.

REMARK 2.2. Let $X_1,X_2,\dots$ be a sequence of i.i.d.r.v.'s with $EX_i=0$, $EX_i^2=1$ and $E|X_i|^{2+\varepsilon}<\infty$ for some $\varepsilon>0$. Put $S_0=0$, $S_n=X_1+\dots+X_n$. Consider

$$\overline{\xi}(n)=\max\left\{k\colon 0\leqq k\leqq n,\ S_k\geqq(2k\log\log k)^{1/2}\right\},$$

$$\overline{\xi}_\delta(n)=\max\left\{k\colon 0\leqq k\leqq n,\ S_k\geqq\left(2(1-\delta)k\log\log k\right)^{1/2}\right\},\ 0\leqq\delta<1,$$

$$\overline{\xi}_\delta^{(p)}(n)=\max\left\{k\colon 0\leqq k\leqq n,\ S_k\geqq k^{1/2}\alpha(\delta,p,k)\right\},\ \delta\geqq 0,\ p\geqq 3.$$

Note that

$$\left\{\bar{\xi}_{\delta}^{(p)}(n) \leqq a(n)\right\} = \left\{ \max_{a(n) \leqq k \leqq n} \frac{S_k}{k^{1/2}\alpha(\delta,p,k)} \leqq 1 \right\}$$

and

$$S_n - W(n) = o\left(n^{1/(2+\varepsilon)}\right) \quad \text{a.s.}$$

in the sense of Strassen [6] (cf. [1]). Hence, we have, for every $0 < \eta < 1$

$$\left\{ \sup_{a(n) \leqq s \leqq n} \frac{W(s)}{s^{1/2}\alpha(\delta,p,s)} \leqq 1 - \frac{\eta}{\alpha^2(\delta,p,s)} \right\} \subset$$

$$\subset \left\{ \max_{a(n) \leqq k \leqq n} \frac{S_k}{k^{1/2}\alpha(\delta,p,k)} \leqq 1 \right\} \subset$$

$$\subset \left\{ \sup_{a(n) \leqq s \leqq n} \frac{W(s)}{s^{1/2}\alpha(\delta,p,s)} \leqq 1 + \frac{\eta}{\alpha^2(\delta,p,s)} \right\},$$

provided $n$ is sufficiently large, for any sequence $\{a(n),\ n \geqq 1\}$ with $a(n) \to \to \infty$ as $n \to \infty$. Now using the above relation and proceeding along the lines of the proof of Theorems 1 and 2, we can obtain the following results:

$$\liminf_{n\to\infty} \frac{(\log_2 n)^{1/2}}{\log_3 n \cdot \log n} \cdot \log \frac{\bar{\xi}(n)}{n} = -3\sqrt{\pi} \ \text{ a.s.},$$

$$\liminf_{n\to\infty} (\log n)^{\delta-1} (\log_2 n)^{-1/2} \cdot \log \frac{\bar{\xi}_{\delta}(n)}{n} = -2\delta\sqrt{\pi/(1-\delta)} \ \text{a.s.}, 0 < \delta \leqq 1/2,$$

$$\liminf_{n\to\infty} \frac{\log_p \bar{\xi}_0^{(p)}(n) - \log_p n}{\log_{p+1} n} = -2\sqrt{\pi} \ \text{ a.s.}, \ p \geqq 3,$$

$$\liminf_{n\to\infty} \frac{\log_{p-1} \bar{\xi}_{\delta}^{(p)}(n) - \log_{p-1} n}{(\log_{p-1} n)^{1-\delta} \log_p n} = -2\delta\sqrt{\pi} \ \text{ a.s.}, \ 0 < \delta < 1, \ p \geqq 3,$$

$$\liminf_{n\to\infty} \frac{\log_{p-2} \bar{\xi}_{\delta}^{(p)}(n) - \log_{p-2} n}{\log_{p-2} n \cdot (\log_{p-1} n)^{1-\delta} \log_p n} = -2\delta\sqrt{\pi} \ \text{ a.s.}, \ \delta > 1, \ p \geqq 3.$$

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# A NOTE ON KÖNIG'S MINIMAX THEOREM

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Recently G. Kassay [1] published an elementary proof of König's minimax theorem [2]. His method seems to be an interesting mixture of both of the so-called methods of level sets and cones, respectively. Formally, König's theorem is an extension of Ky Fan's classical minimax theorem [3] by restricting convexity to diadic rational convexity. It is well-known [4] that Ky Fan's theorem can be deduced from the Brézis–Nirenberg–Stampacchia level set minimax theorem by a function lifting. It is an old open question whether there is a short direct connection between König's and Ky Fan's minimax theorems.

The aim of this note is to show that the mentioned function lifting in [4] transforms a König-type saddle function into a Ky Fan-type saddle function with the same minimax values. A careful analysis of the proof of this fact leads also to new generalizations of König's theorem, which seem not be provable with a simple adaptation of Kassay's method.

Finally we remark that the question of König-type generalizations of M. Sion's minimax theorem [5] is still open.

## 1. On the continuity of convex functions

Throughout this section let $V$ denote an arbitrary vector space and let $\tau$ be the finest locally convex topology on $V$. It is immediate that the absorbing convex subsets of $V$ form a neighbourhood basis of 0 for $\tau$. We say that a subset $S \subset V$ is a *simplex* in $V$ if $S$ is the convex hull of a set $B \subset V$ such that the system $\{b - b_0 : b \in B, b \neq b_0\}$ is linearly independent for all $b_0 \in B$.

1.1. Lemma. *Assume $S$ is a simplex in $V$, $K$ is a convex subset of $S$ and $x \in K$. Then the following statements are equivalent:*

(i) *$K$ is a neighbourhood of $x$ in the relative topology of $\tau$ on $S$,*

(ii) $\{v \in V : \exists\varepsilon > 0,\ x + \varepsilon v \in K\} = \{v \in V : \exists\varepsilon > 0,\ x + \varepsilon v \in S\}$.

Proof. By shifting a suitable vertex of $S$ into the origin and restricting ourselves to the subspace spanned by $S$, we may assume without loss of generality that $S = \operatorname{co}\big(B \cup \{0\}\big)$ the convex hull of some Hamel basis $B$ of $V$ with the origin and $x = \sum_{i=1}^{n} \beta_i b_i$ for some $b_1, \dots, b_n \in B$ and

$\beta_1, \ldots, \beta_n > 0$ with $\sum_{i=1}^n \beta_i = 1$. Let us write

$$C_0 := \{b_i - x : i = 1, \ldots, n-1\}, \quad C_1 := \big\{b - x : b \in B \setminus \{b_1, \ldots, b_n\}\big\}.$$

Then $C := \{-x\} \cup C_0 \cup C_1$ is again a Hamel basis of $V$ and

$$\text{(1.2)} \qquad \{v \in V : \exists \varepsilon > 0,\ x + \varepsilon v \in S\} = \operatorname{co}\big((\mathbf{R}_+ C) \cup (-\mathbf{R}_+ C_0)\big).$$

Therefore for each $c \in C$ there exists $\varepsilon(c) > 0$ with $x + [0, \varepsilon(c)]\, c \subset K$ for $c \in C \setminus C_0$ and $x + [-\varepsilon(c), \varepsilon(c)]\, c \subset K$ for $c \in C_0$. Define

$$U := \operatorname{co}\Big(\bigcup_{c \in C} [-\varepsilon(c), \varepsilon(c)]\, c + x\Big).$$

Since $C$ is a Hamel basis of $V$, $U$ is a convex $\tau$-neighbourhood of $x$ and

$$U = \Big\{x + \sum_{c \in C} \lambda_c c : (c \mapsto \lambda_c) \in \Lambda,\ \sum_{c \in C} |\lambda_c| / \varepsilon(c) \leqq 1\Big\}$$

where $\Lambda := \{\text{functions } C \to \mathbf{R} \text{ with finite support}\}$. By (1.2) we obtain

$$U \cap S = \Big\{x + \sum_{c \in C} \lambda_c c : (c \mapsto \lambda_c) \in \Lambda,\ \lambda_c \geqq 0\ (c \in C),\ \sum_{c \in C} \lambda_c / \varepsilon(c) \leqq 1\Big\} =$$

$$= \operatorname{co}\Big(\bigcup_{c \in C} [0, \varepsilon(c)]\, c + x\Big).$$

Since $x, \varepsilon(c)c + x \in K$ $(c \in C)$, we have $U \cap S \subset K$ which completes the proof.

1.3. Corollary. *If $\{g_i : i \in \mathcal{I}\}$ is a family of affine functions on $V$ such that the function $f := \sup_{i \in \mathcal{I}} g_i$ is finite on the simplex $S$ then $f$ is continuous on $S$ with respect to the relative topology of $\tau$.*

Proof. First of all remark that convex functions of one real variable are always upper semicontinuous. Hence, for any $x \in S$, $\eta > 0$ and $u \in \{v \in V : \exists \varepsilon > 0,\ x + \varepsilon v \in S\}$ there exists $\varepsilon > 0$ such that $f(x + \xi u) < \eta + f(x)$ for all $\xi \in [0, \varepsilon]$. Thus the convex level sets $K_\gamma := \{x \in S : f(x) < \gamma\}$ are all open in the relative topology of $\tau$ on $S$ by 1.1. That is, the function $f$ is upper semicontinuous. On the other hand, affine functions are all $\tau$-continuous on $V$. Hence $f$ as the supremum of a family of continuous functions is lower semicontinuous on $S$ in the relative topology of $\tau$.

## 2. König-convex and Ky Fan-convex mappings

2.1. Definition. Let $E$ be an ordered vector space and $Z$ be any set. We say that a mapping $\Phi : Z \to E$ is *König-convex* if for every $z_1, z_2 \in Z$ there exists $z \in Z$ with $\Phi(z) \leqq (1/2)\Phi(z_1) + (1/2)\Phi(z_2)$. The mapping $\Phi$ is said to be *Ky Fan-convex* if for every $z_1, z_2 \in Z$ and $t \in [0,1]$ there exists $z \in Z$ with $\Phi(z) \leqq (1-t)\Phi(z_1) + t\Phi(z_2)$. If $-\Phi$ is König-convex (resp. Ky Fan-convex) then we say that $\Phi$ is *König-concave* (resp. *Ky Fan-concave*).

Throughout the whole work we write $\mathcal{D}$ for the field of diadic rationals.

2.2. Lemma. *If $\Phi : Z \to E$ is König-convex then for every finite seqence $z_1, \dots, z_n \in Z$ and $0 \leqq \delta_1, \dots, \delta_n \in \mathcal{D}$ with $\sum_{i=1}^n t_i = 1$ there exists $z \in Z$ with $\Phi(z) \leqq \sum_{i=1}^n \delta_i \Phi(z_i)$.*

Proof. Define $\overline{Z} := \{\text{functions } Z \to \mathbf{R} \text{ with finite support}\}$ and

$$\overline{\Phi}(\overline{z}) := \sum_{z \in Z} \overline{z}(z)\Phi(z) \qquad (\overline{z} \in \overline{Z}).$$

Let $\overline{T} := \left\{\overline{z} \in \overline{Z} : \exists z \in Z,\ \Phi(z) \leqq \overline{\Phi}(\overline{z})\right\}$. We have to prove that

$$\overline{T} \supset \left\{\overline{z} \in \overline{Z} : \operatorname{range}(\overline{Z}) \subset \mathcal{D},\ \overline{z} \geqq 0,\ \sum_{z \in Z} \overline{z}(z) = 1\right\}. \tag{2.3}$$

By writing $1_z$ for the characteristic function of the set $\{z\}$, we have $1_z \in \overline{T}$ because $\overline{\Phi}(1_z) = \Phi(z)$ $(z \in Z)$. Furthermore, if $\overline{z}_1, \overline{z}_2 \in \overline{Z}$ then for some $z_1, z_2 \in Z$ we have $\Phi(z_i) \leqq \sum_{z \in Z} \overline{z}_i(z)\Phi(z)$ $(i = 1, 2)$. Since $\Phi$ is König-convex, hence there exists $z_3 \in Z$ with

$$\Phi(z_3) \leqq \frac{1}{2}\Phi(z_1) + \frac{1}{2}\Phi(z_2) \leqq \sum_{z \in Z} \left(\frac{1}{2}\overline{z}_1(z) + \frac{1}{2}\overline{z}_2(z)\right)\Phi(z).$$

Thus $\overline{T} \supset (1/2)\overline{T} + (1/2)\overline{T}$ and $1_z \in \overline{T}$ $(z \in Z)$ whence 2.3 is immediate.

2.4. Lemma. *Let $E$ be a function space (with its natural ordering) and let $Z$ be a compact topological space. Assume $\Phi : Z \to E$ is a lower semicontinuous*[1] *König-convex mapping. Then $\Phi$ is necessarily Ky Fan-convex.*

Proof. Fix any $z_0, z_1 \in Z$. By 2.2, for every $\delta \in \mathcal{D} \cap [0,1]$ there exists $z_\delta \in Z$ such that $\Phi(z_\delta) \leqq (1-\delta)\Phi(z_0) + \delta\Phi(z_1)$. Given any $t \in [0,1]$,

[1] I.e. if $E \subset \{\text{functions } \Omega \to \mathbf{R}\}$ then for each fixed $\omega \in \Omega$ the function $z \mapsto \Phi(z)(\omega)$ is lower semicontinuous on $Z$.

choose a sequence $\delta_1, \delta_2, \ldots \in \mathcal{D} \cap [0,1]$ such that $t = \lim_{n\to\infty} \delta_n$. By the compactness of the space $Z$, there exists an index net $(n_i : i \in \mathcal{I})$ with $\lim_{i\in\mathcal{I}} z_{\delta_{n_i}} = z^*$ for some $z^* \in Z$. Then

$$\Phi(z^*) \leqq \liminf_{i\in\mathcal{I}} \Phi(z_{\delta_{n_i}}) \leqq \liminf_{i\in\mathcal{I}} \left[(1-\delta_{n_i})\Phi(z_0) + \delta_{n_i}\Phi(z_1)\right] \leqq$$

$$\leqq (1-t)\Phi(z_0) + t\Phi(z_1).$$

## 3. König's theorem via Ky Fan's minimax theorem

3.1. Definition. Henceforth let $X$ denote a compact topological space, $Y$ a non-empty set and let $F$ be a function $X \times Y \to \mathbf{R}$. We write $E_X$ (resp. $E_Y$) for the space of all real functions on $X$ (resp. $Y$). We denote by $\tau_X$ the finest locally convex topology on the subspace $\overline{X} := \{\overline{x} : \operatorname{supp}(\overline{x})$ finite$\}$ and we embed the set $X$ into $\overline{X}$ by identifying each point $x \in X$ with its characteristic function $1_x$. In accordance with this embedding, we denote the simplex $\left\{\overline{x} \in \overline{X} : \overline{x} \geqq 0,\ \sum_{x\in X} \overline{x}(x) = 1\right\}$ by $\operatorname{co}(X)$. The objects $\overline{Y}, \tau_Y, \operatorname{co}(Y)$ are defined analogously.

We say that the function $f$ is of *König-type* if the mapping $x \mapsto f(x,\cdot)$ is König-concave and upper semicontinuous from $X$ into the function space $E_Y$ (see footnote [1]) and $y \mapsto f(\cdot, y)$ is König-convex from $Y$ into $E_X$.

Similarly we speak of fuctions of *Ky Fan-type* when replacing König-convexity (concavity) in the above definition by Ky Fan-convexity (concavity).

Finally we shall write shortly $\inf\sup f$ (resp. $\sup\inf f$) instead of $\inf_{y\in Y}\sup_{x\in X} f(x,y)$ (resp. $\sup_{x\in X}\inf_{y\in Y} f(x,y)$).

3.2. Proposition. *Let $f : X \times Y \to \mathbf{R}$ be a function of König-type. Then the lifted function $\overline{f} : X \times \operatorname{co}(Y) \to \mathbf{R}$ defined by*

$$\overline{f}(x,\overline{y}) := := \sum_{y\in Y} \overline{y}(y) f(x,y) \ (x \in X,\ \overline{y} \in \operatorname{co}(Y))$$

*is of Ky Fan-type and it satisfies*

$$\inf\sup \overline{f} = \inf\sup f, \qquad \sup\inf \overline{f} = \sup\inf f.$$

Proof. For any $x \in X$, clearly $\inf_{\overline{y}\in\operatorname{co}(Y)} \overline{f}(x,\overline{y}) = \inf_{y\in Y} f(x,y)$. Hence $\sup\inf \overline{f} = \sup\inf f$.

We have also $\inf\sup f = \inf_{y\in Y}\sup_{x\in X} \overline{f}(x, 1_y) \geqq \inf\sup \overline{f}$.

To prove the converse inequality, notice that $\operatorname{co}(Y)$ is a simplex in $\overline{Y}$ and for any $x \in X$, the function $g_x : \overline{y} \mapsto \sum_{y\in Y} \overline{y}(y) f(x,y)$ is affine on $\overline{Y}$. Moreover, if $\overline{y} \in \operatorname{co}(Y)$ and $\operatorname{supp}(\overline{y}) = \{y_1, \ldots, y_n\}$ then

$$g_x(\overline{y}) = \sum_{i=1}^{n} \overline{y}(y_i) f(x, y_i) \leqq \sum_{i=1}^{n} \max_{x' \in X} f(x', y_i) < \infty \qquad (x \in X)$$

since, for any fixed $y \in Y$, the function $x' \mapsto f(x', y)$ is upper semicontinuous on the compact space $X$. Thus we may apply 1.3 to conclude that the function $\overline{y} \mapsto \sup_{x \in X} \overline{f}(x, \overline{y})$ is convex and continuous when restricted to any finite dimensional affine section of $\mathrm{co}(Y)$. Fix again an arbitrary $\overline{y} \in \in \mathrm{co}(Y)$ and let $\mathrm{supp}(\overline{y}) = \{y_1, \ldots, y_n\}$. Given any $\varepsilon > 0$, we can choose diadic rationals $\delta_1, \ldots, \delta_n \geqq 0$ with $\sum_{i=1}^{n} \delta_i = 1$ such that

$$\sup_{x \in X} \overline{f}\Big(x, \sum_{i=1}^{n} \delta_i 1_{y_i}\Big) \leqq \sup_{x \in X} \overline{f}(x, \overline{y}) + \varepsilon.$$

Since the mapping $y \mapsto f(\cdot, y)$ is supposed to be König-convex, by 2.2 there exists $y^* \in Y$ with

$$f(\cdot, y^*) \leqq \sum_{i=1}^{n} \delta_i f(\cdot, y_i) = \overline{f}\Big(\,\cdot\,, \sum_{i=1}^{n} \delta_i 1_{y_i}\Big).$$

Therefore

$$\inf \sup f = \sup_{x \in X} f(x, y^*) \leqq \sup_{x \in X} \overline{f}(x, \overline{y}) + \varepsilon.$$

By the arbitrariness of $\overline{y} \in \mathrm{co}(Y)$ and $\varepsilon > 0$, hence $\inf \sup f \leqq \sup \inf \overline{f}$.

By 2.4, the mapping $x \mapsto f(x, \cdot)$ is also Ky Fan-concave. Hence, given any $t \in [0,1]$, $x_0, x_1 \in X$, there exists $x_t \in X$ with

$$f(x_t, y) \geqq (1-t) f(x_0, y) + t f(x_1, y) \qquad (y \in Y).$$

If $\overline{y} \in \mathrm{co}(Y)$ then

$$\overline{f}(x_t, \overline{y}) = \sum_{y \in Y} \overline{y}(y) f(x_t, y) \geqq$$

$$\geqq \sum_{y \in Y} \overline{y}(y) \big[(1-t) f(x_0, y) + t f(x_1, y)\big] = (1-t)\overline{f}(x_0, y) + t\overline{f}(x_1, y).$$

Thus the mapping $x \mapsto \overline{f}(x, \cdot)$ is Ky Fan-concave $X \to E_{\mathrm{co}(Y)}$.

For any fixed $\overline{y} \in \mathrm{co}(Y)$ the function $\overline{f}(\cdot, \overline{y}) = \sum_{y \in Y} \overline{y}(y) f(\cdot . y)$ is a finite convex combination of upper semicontinuous functions on $X$. Thus the mapping $x \mapsto \overline{f}(x, \cdot)$ is upper semicontinuous $X \to E_{\mathrm{co}(Y)}$.

Finally the mapping $\overline{y} \mapsto \overline{f}(\cdot, \overline{y})$ is affine $\mathrm{co}(Y) \to E_X$, whence it is in particular also Ky Fan-convex.

3.3. Corollary (König's theorem [2]). *If $f : X \times Y \to \mathbf{R}$ is a function of König-type then* $\inf \sup f = \sup \inf f$.

Proof. We may apply Ky Fan's minimax theorem to the fuction $\overline{f}$ in 3.2. Hence $\inf\sup\overline{f} = \sup\inf\overline{f}$.

## 4. Generalizations

Throughout this section let $X, Y$ denote two non-void sets, let $f$ be a function $X \times Y \to \mathbf{R}$. We shall keep the notations $\overline{X}, \tau_X, \operatorname{co}(X)$ resp. $\overline{Y}, \tau_Y, \operatorname{co}(Y)$ established in 3.1. We denote by $\overline{f}$ the affine lifting

$$\overline{f}(x,\overline{y}) := \sum_{y\in Y} \overline{y}(y) f(x,y) \qquad (x \in X,\ \overline{y} \in \operatorname{co}(Y))$$

of the function $f$ in the second variable to $X \times \operatorname{co}(Y)$.

4.1. Proposition. *Assume the function* $f : X \times Y \to \mathbf{R}$ *has the following properties:*

(i) $\sup_{x\in X} f(x,y) < \infty \quad (y \in Y)$,

(ii) *the set* $\{\overline{y} \in \operatorname{co}(Y) : \exists\, y \in Y\ f(\cdot,y) \leqq \overline{f}(\cdot,\overline{y})\}$ *is dense in* $\operatorname{co}(Y)$ *with respect to* $\tau_Y$.

*Then we have* $\inf\sup\overline{f} = \inf\sup f$ *and* $\sup\inf\overline{f} = \sup\inf f$.

Proof. The simple arguments at the beginning of the proof of 3.2 show that $\sup\inf\overline{f} = \sup\inf f$ and $\inf\sup f \geqq \inf\sup\overline{f}$.

Since $\operatorname{co}(Y)$ is a simplex in $\overline{Y}$ and since the family $\{\overline{f}(x,\cdot) : x \in X\}$ of affine functions on $X$ is bounded from above (by assumption (i)) for each $\overline{y} \in \operatorname{co}(Y)$, it follows from 1.3 that the function $\operatorname{co}(Y) \ni \overline{y} \mapsto \sup_{x\in X}\overline{f}(x,\overline{y})$ is continuous with respect to the topology $\tau_Y$. Then, given any $\varepsilon > 0$ and $\overline{y} \in \operatorname{co}(Y)$, by assumption (ii) there exist $y^* \in Y$ and $\overline{y}^* \in \operatorname{co}(Y)$ with

$$\sup_{x\in X}\overline{f}(x,\overline{y}^*) \leqq \sup_{x\in X}\overline{f}(x,\overline{y}) + \varepsilon \quad \text{and} \quad f(\cdot\,,y^*) \leqq \overline{f}(\cdot\,,\overline{y}^*).$$

Thus $\inf\sup f \leqq \sup_{x\in X} f(\cdot\,,y^*) \leqq \sup_{x\in X}\overline{f}(x,\overline{y}^*) \leqq \sup_{x\in X}\overline{f}(x,\overline{y}) + \varepsilon$ for every $\overline{y} \in \operatorname{co}(Y)$ and $\varepsilon > 0$. This implies $\inf\sup f \leqq \inf\sup\overline{f}$.

4.2. Theorem. *Let $X$ be a compact topological space, $Y$ an abstract set and* $f : X \times Y \to \mathbf{R}$ *be a function satisfying* 4.1(ii) *and such that the mapping* $x \mapsto f(x,\cdot)$ *is Ky Fan-concave and upper semicontinuous (cf. footnote* [1]*). Then* $\inf\sup f = \sup\inf f$.

Proof. The lifted function $\overline{f} : X \times \operatorname{co}(Y) \to \mathbf{R}$ is of Ky Fan-type (for definition see 3.1). Hence, by Ky Fan's minimax theorem $\inf\sup\overline{f} = \sup\inf\overline{f}$. Since for every fixed $y \in Y$, the function $x \mapsto f(x,y)$ is upper semicontinuous on the compact space $X$, also 4.1(i) holds. Thus, by 4.1, also $\inf\sup f = \inf\sup\overline{f} = \sup\inf\overline{f} = \sup\inf f$.

4.3. REMARK. Several equivalent but seemingly weaker formulations can be given for the conditions of 4.2.

(i) The Ky Fan-concavity of $x \mapsto f(x,\cdot)$ can be replaced by König-concavity in view of 2.4.

(ii) Observe that, by writing $\mathcal{M}_Y := \{\text{functions } Y \to (0,\infty)\}$, the family of all figures

$$U_\mu := \Big\{\overline{y} \in \overline{Y} : \sum_{y\in Y} |\overline{y}(y)|\mu(x) < 1\Big\} \qquad (\mu \in \mathcal{M}_Y)$$

forms a neighbourhood basis of 0 for the topology $\tau_Y$ on the space $\overline{Y}$. Therefore condition 4.1(ii) can be formulated elementarily as follows:

For every finite family $\{y_1,\ldots,y_n\} \subset Y$ and $t_1,\ldots,t_n \geqq 0$ with $\sum_{i=1}^n t_i = 1$ and for every $\mu \in \mathcal{M}_Y$ there exist $y^* \in Y$ and $\{(y_i', t_i') : i = 1,\ldots,n'\} \subset Y \times \mathbf{R}_+$ such that $n' \geqq n$, $y_i' = y_i$ $(i = 1,\ldots,n)$, $\sum_{i=1}^{n'} t_i' = 1$,

$$f(\cdot,y^*) \leqq \sum_{i=1}^{n'} t_i' f(\cdot,y_i') \quad \text{and} \quad \sum_{i\leqq n} |t_i - t_i'|\mu(y_i) + \sum_{i>n} t_i'\mu(y_i') < 1.$$

4.4. COROLLARY. *If $X$ is a compact space, $Y$ is a set and $f : X \times Y \to \mathbf{R}$ is a function such that*

$\{\overline{y} \in \operatorname{co}(Y) : \exists y^* \in Y,\ f(\cdot,y^*) \leqq \sum_{y\in Y} \overline{y}(y) f(\cdot,y)\}$ *is dense in* $\operatorname{co}(Y)$ *with respect to the topology* $\tau_Y$,

$\{\overline{x} \in \operatorname{co}(X) : \exists x^* \in X,\ f(\cdot,x^*) \geqq \sum_{x\in X} \overline{x}(x) f(x,\cdot)\}$ *is dense in* $\operatorname{co}(X)$ *with respect to the topology* $\tau_X$ *and the mapping* $x \mapsto f(x,\cdot)$ *is continuous (cf. footnote* [1]*) then* $\inf\sup f = \sup\inf f$.

PROOF. In view of 4.3(i) we need only to verify the König concavity of $x \mapsto f(x,\cdot)$.

Let $x_1, x_2 \in X$ be arbitrarily fixed. We have to find $x^* \in X$ such that $f(x^*,y) \geqq \left(f(x_1,y) + f(x_2,y)\right)/2$ for all $y \in Y$.

Given any $\varepsilon > 0$ and finite subset $F \subset Y$, define

$$\mu_{\varepsilon,F}(x) := \sum_{y\in F} \max |f(x,y)|/\varepsilon \qquad (x \in X).$$

By the continuity of the mapping $x \mapsto f(x,\cdot)$, the function $\mu_{\varepsilon,F}$ belongs to $\mathcal{M}_X$ (for definition see 4.3(ii)). By assumption, we can choose $x_{\varepsilon,F} \in X$ and $\overline{x}_{\varepsilon,F} \in \operatorname{co}(X)$ such that

$$f(\overline{x}_{\varepsilon,F},\cdot) \geqq \sum_{x\in X} \overline{x}_{\varepsilon,F}(x) f(x,\cdot) \quad \text{and} \quad \overline{x}_{\varepsilon,F} - \overline{x}^* \in U_{\mu_{\varepsilon,f}}$$

where $\overline{x}^* := (1/2)1_{x_1} + (1/2)1_{x_1}$ and $U_{\mu_\varepsilon,f}$ denotes the $\tau_X$-neighbourhood of 0 defined in 4.3(ii). It follows from the definition of $U_{\mu_\varepsilon,f}$ that

$$\left|\sum_{x\in X}\overline{x}_{\varepsilon,F}(x)f(x,y) - \sum_{x\in X}\overline{x}^*(x)f(x,y)\right| \leqq$$
$$\leqq \sum_{x\in X}\left|\overline{x}_{\varepsilon,F}(x) - \overline{x}^*(x)\right| \max\left|f(\cdot,y)\right| \leqq \varepsilon \quad (y\in F).$$

In particular

$$f(x_{\varepsilon,F},y) \geqq \sum_{x\in X}\overline{x}_{\varepsilon,F}(x)f(x,y) \geqq \sum_{x\in X}\overline{x}^*(x)f(x,y) - \varepsilon =$$
$$= \frac{1}{2}f(x_1,y) + \frac{1}{2}f(x_2,y) - \varepsilon \quad (y\in F).$$

If $x^*$ is an accumulation point (with respect to the topology of $X$) of the net $(x_{\varepsilon,F} : \varepsilon > 0, F$ finite $\subset Y)$ then, by the continuity of the functions $x \mapsto$ $\mapsto f(x,y)$ $(y \in Y)$ on the space $X$, we have $f(x^*,y) \geqq \big(f(x_1,y) + f(x_2,y)\big)/2$ for all $y\in Y$.

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# COBORDISM GROUPS OF IMMERSIONS OF ORIENTED MANIFOLDS

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## 1. Introduction

In this paper we compute the cobordism groups of immersions of oriented manifolds by projecting the images of the immersions into a hyperplane. Such a "project in a hyperplane" method has been used by T. Banchoff [1] and U. Koschorke [5]. Banchoff considers the singular set as an invariant. Koschorke takes into account also naturally arising bundles and bundle maps over the singularity set and forms some rather complicated bordism groups from these data and relates these groups by exact sequences to the cobordism groups of immersions.

The essential new feature in our approach is that we consider the whole singular map and *construct a classifying space* (i.e. an analogue of the Thom space) *for the cobordism of these singular maps.*

This classifying space by its construction has a filtration with simple quotient spaces. Koschorke's exact sequences arise now simply as homotopy exact sequences of pairs of spaces. Moreover we can consider the spectral sequence arising from this filtration and this allows us to get results in a much wider range than just by using the exact sequences.

By this method we have investigated the cobordism groups of immersions and embeddings of *unoriented* manifolds in [16]. There the computation was trivial since the spectral sequence had a single nonzero column and so it degenerated.

In the present case of *oriented* manifolds the spectral sequence is non trivial and the main work is devoted to its computation. (I tried to make this paper independent of [16].)

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## 2. Formulation of the theorems

NOTATIONS. 1. The cobordism group of immersions of *oriented* $n$ dimensional smooth manifolds into the euclidean space $R^{n+k}$ will be denoted by $\mathrm{Imm}^{SO}(n,k)$. (A cobordism joining two such immersions is an immersion into $R^{n+k}\times I$, see for example [7].)

2. Let $\mathcal{C}(2,3)$ denote the class of finite Abelian groups (in the sense of Serre) having only 2 and 3 primary torsions. Let $\mathcal{C}(2)$ be the class of finite 2-primary groups and $\mathcal{C}(3)$ that of finite 3-primary groups.

The aim of this paper is to prove the following

THEOREM 1. *For $n<3k$ the following sequences of groups are exact modulo $\mathcal{C}(2,3)$:*

(a) *If $k$ is even then*

$$0\to\Omega_{n-k}\to\mathrm{Imm}^{SO}(n,k)\xrightarrow{\varphi_{n,k}}\mathrm{Imm}^{SO}(n,k+1)\to 0.$$

(b) *If $k$ is odd then*

$$0\to\mathrm{Imm}^{SO}(n,k)\xrightarrow{\varphi_{n,k}}\mathrm{Imm}^{SO}(n,k+1)\xrightarrow{\sigma_1\oplus\sigma_{11}}\Omega_{n-k-1}\oplus\Omega_{n-2k-2}\to 0$$

*where the groups and maps are defined as follows:*

(1) *$\Omega_i$ is the group of cobordism classes of oriented manifolds.*

(2) *$\varphi_{n,k}:\mathrm{Imm}^{SO}(n,k)\to\mathrm{Imm}^{SO}(n,k+1)$ sends $[f]\in\mathrm{Imm}^{SO}(n,k)$ to $[i\circ f]\in\mathrm{Imm}^{SO}(n,k+1)$ where $i$ denotes the inclusion $R^{n+k}=R^{n+k}\times 0\subset\subset R^{n+k}\times R=R^{n+k+1}$.*

(3) *The maps $\sigma_1:\mathrm{Imm}^{SO}(n,k+1)\to\Omega_{n-k-1}$ and $\sigma_{1,1}:\mathrm{Imm}^{SO}(n,k+1)\to\Omega_{n-2k-2}$ are defined as follows. Let $p:R^{n+k+1}\to R^{n+k}$ be the standard projection and let $[f]$ be the cobordism class of an immersion $f:M^n\to R^{n+k+1}$. For a generic immersion $f$ the sets $\Sigma^1(p\circ f)$ and $\Sigma^{1,1}(p\circ f)$ are oriented manifolds of dimensions $n-k-1$ and $n-2k-2$ respectively. Let $[\Sigma^1(p\circ f)]$ and $[\Sigma^{1,1}(p\circ f)]$ denote their cobordism classes in $\Omega_{n-k-1}$ and $\Omega_{n-2k-2}$ respectively. They depend only on the class of $f$.*

*Let us put*

$$\sigma_1([f])=[\Sigma^1(p\circ f)]\quad\textit{and}\quad\sigma_{1,1}([f])=[\Sigma^{1,1}(p\circ f)].$$

*Now the map $\mathrm{Imm}^{SO}(n,k+1)\to\Omega_{n-k-1}\oplus\Omega_{n-2k-2}$ in the second sequence (for $k$ odd) is $\sigma_1\oplus\sigma_{1,1}$.*

*For $k$ even the maps $\sigma_1$ and $\sigma_{1,1}$ are zero modulo $\mathcal{C}(2,3)$ (and then actually $2\cdot\sigma_1=0$ and $2\cdot\sigma_{1,1}=0$ since $\Omega_*$ has only second order torsion elements).*

(4) *The map* $\Omega_{n-k} \to \mathrm{Imm}^{SO}(n,k)$ *of the first sequence can be described as follows. Let*

$$[f] \in \operatorname{Ker}\left(\varphi_{n,k} : \mathrm{Imm}^{SO}(n,k) \to \mathrm{Imm}^{SO}(n,k+1)\right).$$

*Then there exists an immersion* $F$ *into* $R^{n+k+1} \times I$ *such that* $\partial F = f$. *Let us project the image of* $F$ *into* $R^{n+k} \times I$. *For a generic* $F$ *we obtain an* $n-k$ *dimensional manifold of singular points of this projection. Let* $[\Sigma] \in \Omega_{n-k}$ *be the cobordism class of this manifold. It turns out that the correspondence* $[f] \to [\Sigma]$ *defines a* $\mathcal{C}(2,3)$ *isomorphism* $\operatorname{Ker}\varphi_{n,k} \xrightarrow{\approx} \Omega_{n-k}$. *Its inverse composed with the inclusion* $\operatorname{Ker}\varphi_{n,k} \subset \mathrm{Imm}^{SO}(n,k)$ *gives the map* $\Omega_{n-k} \to$ $\to \mathrm{Imm}^{SO}(n,k)$. *(For the correctness of this definition see below in Remark 1.)*

*Addenda to the Theorem:*

ADDENDUM 1. The first sequence (where $k$ is even) has a splitting map

$$\Delta : \mathrm{Imm}^{SO}(n,k) \to \Omega_{n-k}$$

which associates with the cobordism class $[f]$ of an immersion the cobordism class of the manifold of the double points of $f$.

REMARK 1. The double points of the projection of $F$ to $R^{n+k} \times I$ form an oriented cobordism between the singularity manifold of this projection and the double point manifold of $f$. (The latter is orientable since $k$ is even.) This shows that

a) the class $[\Sigma]$ depends only on $[f]$ and not on $[F]$.

b) the map $\Delta$ is a splitting map.

ADDENDUM 2. (a) The first sequence (for $k$ even) is actually exact modulo the class of finite 2-primary groups.

(b) The homology groups of the second sequence (where $k$ is odd) do have 3-torsion (for some $n < 3k$). Nevertheless it becomes exact modulo the class of finite 2-primary groups if we replace in it the group $\Omega_{n-2k-2}$ by the cobordism group $\Omega^{3\gamma}_{n-2k-2}$ of those oriented $n-2k-2$ dimensional manifolds whose stable normal bundles are split into the direct sum of three isomorphic bundles. The groups $\Omega_i$ and $\Omega_i^{3\gamma}$ are isomorphic modulo their 3-primary torsion but $\Omega_i^{3\gamma}$ does have 3-torsion (for some $i$) while $\Omega_i$ does not.

COROLLARY. *The group* $\mathrm{Imm}^{SO}(n,k)$ *has no p-torsion if* $p > 3$ *and* $n < 3k$.

PROOF. For $k$ big enough $\mathrm{Imm}^{SO}(n,k) \approx \Omega_n$. By Theorem 1

$$\operatorname{Ker}\left(\varphi_{n,k} : \mathrm{Imm}^{SO}(n,k) \to \mathrm{Imm}^{SO}(n,k+1)\right)$$

has no $p$ torsion. $\square$

REMARK 2. (a) The ranks of the groups $\mathrm{Imm}^{SO}(n,k)$ are well-known (see e.g. [3]). They can be computed as follows:

$$\mathrm{Imm}^{SO}(n,k)\otimes Q\approx\pi^s_{n+k}\big(MSO(k)\otimes Q\big)\approx H_{n+k}\big(MSO(k);Q\big)\approx$$
$$\approx H_{n+k}\big(MSO(k);Q\big)\approx H_n\big(BSO(k);Q\big).$$

(b) The groups $\mathrm{Imm}^{SO}(n,k)$ have no odd torsion if $n<2k$.

Actually Theorem 1 allows us to compute the groups $\mathrm{Imm}^{SO}(n,k)$ completely modulo 2 and 3 torsion.

MAIN COROLLARY. *Let $P_{n,k}$ be the subgroup of the cobordism group $\Omega_n$ consisting of those cobordism classes $[M]$ for which each Pontryagin number corresponding to a monomial divisible by a normal Pontryagin class $\bar{p}_i$ for $2i>k$ are zero. Then for $n<3k$ the following isomorphism holds modulo $\mathcal{C}(2,3)$:*

$$\mathrm{Imm}^{SO}(n,k)\approx\begin{cases}P_{n,k} & \text{if } k \text{ is odd}\\ P_{n,k}\oplus\Omega_{n-k} & \text{if } k \text{ is even.}\end{cases}$$

When $k$ is even then the second factor is the cobordism class of the manifold of double points.

NOTATIONS. 1. Let $\mathrm{Emb}(n,k)$ denote the cobordism group of embeddings of oriented $n$-manifolds into $R^{n+k}$. (Hence $\mathrm{Emb}(n,k)\approx$ $\approx\pi_{n+k}\big(MSO(k)\big)$.)

2. Let $\mathrm{Emb}(n,k\oplus 1)$ denote the cobordism group of embeddings of oriented $n$-manifolds into $R^{n+k+1}$ with nonzero normal vectorfield. (Hence $\mathrm{Emb}(n,k\oplus 1)\approx\pi_{n+k+1}\big(SMSO(k)\big)$.)

THEOREM 2. *For $n<3k$ the following sequences of groups are exact modulo $\mathcal{C}(2,3)$:*

(a) *If $k$ is even then*

$$0\to\Omega_{n-k}\to\mathrm{Emb}(n,k\oplus 1)\to\mathrm{Emb}(n,k+1)\to 0.$$

(b) *If $k$ is odd then*

$$0\to\Omega_{n-k}\oplus\Omega_{n-2k-1}\to\mathrm{Emb}(n,k\oplus 1)\to\mathrm{Emb}(n,k+1)$$

*where the maps are defined as follows:*

*The map*

$$\mathrm{Emb}(n,k\oplus 1)\to\mathrm{Emb}(n,k+1)$$

*is given by forgetting the normal vector field.*

*The map* $\Omega_{n-k} \to \mathrm{Emb}(n, k \oplus 1)$ *can be defined as above for immersions (see part* (3) *in Theorem* 1*).*

*The map* $\Omega_{n-2k-1} \to \mathrm{Emb}(n, k \oplus 1)$ *can be defined analogously but taking only the* $\Sigma^{1,1}$ *singular set of the projection (instead of the whole singular set).*

More precisely the definitions of the last two maps are the following. Let $f$ be a map representing an element of the kernel of the map $\mathrm{Emb}(n, k \oplus \oplus 1) \to \mathrm{Emb}(n, k+1)$. Then $f$ is an embedding with normal field, and there is an embedding $F : W^{n+1} \to R^{n+k+1} \times I$ of a manifold $W^{n+1}$ with boundary, such that the restriction of $F$ to the boundary of $W^{n+1}$ coincides with $f$.

By Hirsch's theorem $f$ is regularly homotopic to an immersion into $R^{n+k}$. Therefore — after a possible regular homotopy — the embedding $F$ projected to $R^{n+k} \times I$ will have no singularities at the boundary. Let $\pi$ denote the projection $R^{n+k+1} \times I \to R^{n+k} \times I$. Then the map $\pi \circ F$ has only $\Sigma^1$ singularities and by dimensional reasons these are only $\Sigma^{1,0}$ and $\Sigma^{1,1}$ singular points. The manifolds $\Sigma^1(\pi \circ F)$ and $\Sigma^{1,1}(\pi \circ F)$ are oriented manifolds of dimensions $n-k$ and $n-2k-1$ respectively with cobordism classes $\left[\Sigma^1(\pi \circ F)\right] \in \Omega_{n-k}$ and $\left[\Sigma^{1,1}(\pi \circ F)\right] \in \Omega_{n-2k-1}$. The correspondence $[f] \to \left(\left[\Sigma^1(\pi \circ F)\right], \left[\Sigma^{1,1}(\pi \circ F)\right]\right)$ defines a $\mathcal{C}(2,3)$ isomorphism of the Ker $\left(\mathrm{Emb}(n, k \oplus 1) \to \mathrm{Emb}(n, k+1)\right)$ to $\Omega_{n-k} \oplus \Omega_{n-2k-1}$. Its inverse composed with the inclusion into $\mathrm{Emb}(n \oplus 1)$ gives the map

$$\Omega_{n-k} \oplus \Omega_{n-2k-1} \to \mathrm{Emb}(n, k \oplus 1)$$

from the exact sequence.

*Addenda to Theorem 2.*

ADDENDUM 1. An embedding with normal field into $R^{n+k+1}$ is regularly homotopic to an immersion into $R^{n+k}$. Taking the double points set of this immersion we obtain a splitting map $\Delta$ analogous to the one in Theorem 1.

ADDENDUM 2. The analogue of Addendum 2 to Theorem 1 holds, i.e. the sequence for $k$ even is exact modulo the finite 2-primary groups, while the homologies of the sequence for $k$ odd have 3-torsion.

TERMINOLOGY. From now on we shall say that a certain sequence is exact or a homomorphism is an isomorphism if it is such modulo $\mathcal{C}(2,3)$.

## 3. The scheme of the proof of the theorems

For Theorem 1: We project the immersions $M^n \to R^{n+k+1}$ into $R^{n+k}$ and get singular maps $M^n \to R^{n+k}$ of certain type ($\Sigma^{1,1}$-prim maps in the terminology of [16] i.e.: these maps may have only $\Sigma^{1,0}$ and $\Sigma^{1,1}$ singular points and their kernel bundles are trivial over the sets of singular points). The cobordism group of these singular maps $M^n \to N^{n+k}$ can be identified with the cobordism group of immersions into $R^{n+k+1}$ i.e. with $\mathrm{Imm}^{SO}(n, k+1)$. One can construct a classifying space $X(k)$ for the cobordisms of singular maps of this type. (See [16] and Section 4 below.) Now our task will be to compute the homotopy groups of this space $X(k)$. The construction of $X(k)$ provides a filtration $\Gamma(k) \subset Z(k) \subset Y(k) \subset X(k)$. Investigating the spectral sequence associated with this filtration in the stable homotopy groups and in the homotopy groups we shall obtain Theorem 1.

Theorem 2 can be proved mainly in the same way; we only have to replace the space $X(k)$ by another space $\bar{X}(k)$ which classifies the projections of *embeddings,* see [14]. It also has an analogous filtration:

$$\bar{X}(k) \supset \bar{Y}(k) \supset \bar{Z}(k) \supset \bar{\Gamma}(k).$$

The computation of the spectral sequence of this filtration will prove Theorem 2.

## 4. The construction of the spaces $X(k) \supset Y(k) \supset Z(k) \supset \Gamma(k)$

**4.1. The space** $\Gamma(k) \overset{\text{def}}{=} \Omega^\infty S^\infty MSO(k)$. Here $\Omega$ is the loop-space functor and $S$ is the suspension.

REMARK $\Gamma$. This is the classifying space of codimension $k$ immersions having oriented normal bundles in the following sense. The $n+k$-th homotopy group of this space is isomorphic to the cobordism group of immersions of oriented $n$ manifolds in $R^{n+k}$, i.e.:

$$\pi_{n+k}\big(\Gamma(k)\big) \approx \mathrm{Imm}^{SO}(n,k).$$

Moreover any codimension $k$ immersion with oriented normal bundle defines a unique homotopy class of maps of its target manifold into $\Gamma(k)$.

**4.2. The space** $Z(k)$. (Compare with the space $X(k)$ in [11] and [13].) Before constructing this space we need some definitions.

DEFINITION. (1) A smooth map $f : M \to N$ is a *prim* map if it is the composition of an immersion $M \to N \times R$ with the projection $N \times R \to \to N$. (*prim* = *pro*jected *im*mersion.) An equivalent definition is that the differential of the map at each point has at most one dimensional kernel and the line bundle formed by these kernels over the set of singular points is a trivial line bundle.)

(2) A prim map $f : M^n \to N^{n+k}$ will be called $\Sigma^{1_r}$ *prim* map ($\Sigma^{1_r} = \Sigma^{1,\dots,1}$, the number of digits 1 is $r$) if it has no $\Sigma^{1_{r+1}}$ singular points and

the $\Sigma^{1_r}$ singular points are not multiple points. For such a map we denote by $\Sigma(f)$ the set of $\Sigma^{1_r}$ singular points and by $\tilde{\Sigma}(f)$ its image.

The space $Z(k)$ which we are going to construct now will be the classifying space of $\Sigma^{1,0}$ prim maps. ($\Sigma^{1,1}$ prim maps will be called also $Z$-maps.) Let $f : M^n \to N^{n+k}$ be a $\Sigma^{1,0}$ prim map. Let us denote by $T$ and $\tilde{T}$ the tubular neighbourhoods of $\Sigma(f)$ and $\tilde{\Sigma}(f)$ in the manifolds $M^n$ and $N^{n+k}$ respectively. Then the following commutative diagram arises:

$$(*)\qquad \begin{array}{ccc} T & \xrightarrow{f|T} & \tilde{T} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ \Sigma(f) & \xrightarrow{f|\Sigma(f)} & \tilde{\Sigma}(f) \end{array}$$

We recall from [17] that for these commutative diagrams there exists a universal one. Namely there exists a universal $\Sigma^{1,0}$ prim map

$$(**)\qquad \begin{array}{ccc} \xi & \xrightarrow{\Phi} & \tilde{\xi} \\ {\scriptstyle R^{k+1}}\downarrow & & \downarrow{\scriptstyle R^{2k+1}} \\ B & \xrightarrow{\text{identity}} & B \end{array}$$

such that for any $\Sigma^{1,0}$ prim map $f$ the diagram $(*)$ can be induced from this diagram, i.e. there exist vector bundle maps $j : T \to \xi$ and $\tilde{j} : \tilde{T} \to \tilde{\xi}$ which are isomorphisms on each fibre and

$$\Phi \circ j = \tilde{\xi} \circ (f \mid T).$$

The bundles $\xi$ and $\tilde{\xi}$ are the $k+1$ and $2k+1$ dimensional universal vectorbundles over the space $BSO(k)$ associated with the representations $\alpha : SO(k) \to SO(k+1)$, $\alpha(A) = \operatorname{diag}\langle 1, A\rangle$ and $\beta : SO(k) \to SO(2k+1)$, $\beta(A) = \operatorname{diag}\langle 1, A, A\rangle$.

(Here $\operatorname{diag}\langle A_1, A_2, \ldots\rangle$ denotes the matrix with the blocks $A_1, A_2, \ldots$ on its diagonal and with zero elements otherwise.) We shall not repeat here the description of the map $\Phi$ in details. (See [17].) We only recall that $\Phi$ is described locally by the local normal form of $\Sigma^{1,0}$ maps. ($\Phi$ is a fiberwise map but it is not linear. Its restriction to any fiber $R_x^{k+1}$ of $\xi$, $x \in B = BSO(k)$, gives a map to the corresponding fiber $R_x^{2k+1}$ of $\tilde{\xi}$ and the image of this map is the $k+1$ dimensional Whitney umbrella in $R^{2k+1} = R_x^{2k+1}$.)

The image of $\Phi$ intersected with the sphere bundle $S(\tilde{\xi})$ defines an (infinite dimensional) immersed submanifold of codimension $k$. Therefore by Remark $\Gamma$ a unique homotopy class of maps $\rho_Z : S(\tilde{\xi}) \to \Gamma(k)$ arises.

The definition of the space $Z(k)$ is

$$Z(k) \stackrel{\text{def}}{=} \Gamma(k) \cup_{\rho_Z} D(\tilde{\xi}).$$

Corollary. $Z(k)/\Gamma(k) = S(T2\gamma_k)$.

Here $S$ denotes the suspension, $T$ is the Thom space functor, $\gamma_k$ is the universal $k$ dimensional oriented vectorbundle and $2\gamma_k$ denotes the sum $\gamma_k \oplus \gamma_k$.

Remark Z. The constructed space $Z(k)$ is the classifying space for the $\Sigma^{1,0}$ prim maps of codimension $k$. This means that any $\Sigma^{1,0}$ prim map defines a unique homotopy class of maps of its target manifold to $Z(k)$.

**4.3. The space $Y(k)$.**

Definition. A point $P$ in the source of a map $f$ is called *stationary* point if

1) it is a $\Sigma^{1,0}$ singular point, and

2) $f^{-1}(f(P))$ consists of two points: the point $P$ and a nonsingular point.

Let us denote by $\mathrm{St}(f)$ the set of stationary points of a map $f : M^n \to \to N^{n+k}$ and by $\tilde{\mathrm{St}}(f)$ its image, $\tilde{\mathrm{St}}(f) = f(\mathrm{St}(f))$.

The analogue of the universal diagram $(**)$ exists for the stationary points of prim maps too, see [17], Example 2.

The universal bundle $\tilde{\xi}$ in this case will be $(2\gamma_k \oplus 1) \times \gamma_k$ over the space $BSO(k) \times BSO(k)$. The space $Y(k)$ will be the following:

$$Y(k) \stackrel{\text{def}}{=} Z(k) \cup_{\rho_Y} D\big((2\gamma_k \oplus 1) \times \gamma_k\big)$$

where $\rho_Y : \partial D\big((2\gamma_k \oplus 1) \times \gamma_k\big) \to Z(k)$ is an attaching map analogous to $\rho_Z$ (existing by Remark Z).

Corollary. $Y(k)/Z(k) = ST2\gamma_k \wedge T\gamma_k$.

Remark Y. The space $Y(k)$ is the classifying space for those codimension $k$ maps, which have (besides the regular points) only simple $\Sigma^{1,0}$ points and stationary points. Such a map will be called $Y$-map. Any codimension $k$ $Y$-map with oriented source and target manifolds induces a unique homotopy class of maps of its target manifold in $Y(k)$.

**4.4. The space $X(k)$.** The analogue of the universal diagram $(**)$ holds also if we substitute for $\Sigma(f)$ the set $\Sigma^{1,1}(f)$ of $\Sigma^{1,1}$ points of $f$, where $f$ is a codimension $k$ $\Sigma^{1,1}$-prim map. (See [17], Example 3.)

The target bundle $\tilde{\xi}$ in this case is $3\gamma_k \oplus 2\epsilon^1$, where $\epsilon^1$ is the trivial line bundle.

The definition of the space $X(k)$ is the following:

$$X(k) \overset{\text{def}}{=} Y(k) \cup_{\rho_X} D(3\gamma_k \oplus 2\epsilon^1)$$

where $\rho_X : \partial D(3\gamma_k \oplus 2\epsilon^1) \to Y(k)$ is an attaching map defined according to Remark Y by the image of the map $\Phi : \xi \to \tilde{\xi}$ in $S(\tilde{\xi})$.

REMARK X. The space $X(k)$ is the classifying space of $\Sigma^{1,1}$ prim maps with oriented source and target manifolds. This means that any such map induces a unique homotopy class of maps of its target manifold in the space $X(k)$. ($\Sigma^{1,1}$ prim maps will be called also $X$-maps.)

COROLLARY. $X(k)/Y(k) = S^2T3\gamma_k$.

**Summary.** So far we have constructed the spaces $X(k) \supset Y(k) \supset \supset Z(k) \supset \Gamma(k)$.

The space $X(k)$ classifies the prim $\Sigma^{1,1}$-maps of codimension $k$ of oriented manifolds. The dimension of the set of $\Sigma^{1,1}$ singular points of a generic map $f : M^n \to N^{n+k}$ is $n - 2(k+1)$. If $n + 1 < 2(k+1) + k = 3k + 2$, then the $\Sigma^{1,1}$ points are not multiple points either for the maps or the cobordisms joining them. Since the more complicated singularities have even bigger codimension we have:

$$\pi_{n+k}\big(X(k)\big) \approx \operatorname{Imm}^{SO}(n, k+1).$$

Now we use the spectral sequence arising from this filtration to compute the homotopy groups of the space $X(k)$.

## 5. Generalities about spectral sequences

Our reference for spectral sequences will be Chapter 15 of Switzer's book [10] with a slight modification. Switzer treats spectral sequences for arbitrary (extraordinary) homology theories. We are going to apply them for homotopy groups, which of course do not form a homology theory. But the arguments in [10] use only the exact sequences arising from pairs and triples of spaces. So the only point where we can not follow those arguments is that we must not replace the groups of pairs by the groups of the corresponding quotient spaces (unless we are in the range of dimensions where the homotopy excision theorem holds.)

The groups and differentials of the spectral sequence associated with the filtration $X(k) \supset Y(k) \supset Z(k) \supset \Gamma(k)$ can be seen on Fig. 1.

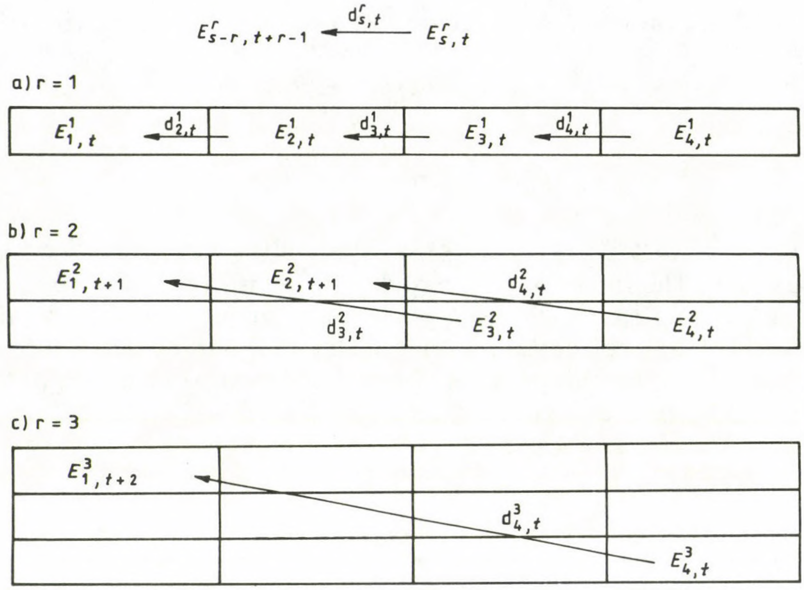


*Fig. 1*

$$E^1_{1,t} \approx \pi_{t+1}\bigl(\Gamma(k)\bigr)\,,$$

$$E^1_{2,t} \approx \pi_{t+2}\bigl(Z(k),\Gamma(k)\bigr)\,,$$

$$E^1_{3,t} \approx \pi_{t+3}\bigl(Y(k),Z(k)\bigr)\,,$$

$$E^1_{4,t} \approx \pi_{t+4}\bigl(X(k),Y(k)\bigr)\,.$$

The final term of this spectral sequence is $E^\infty \approx E^4$. This final term is associated with a filtration of the group $\pi_*\bigl(X(k)\bigr)$, i.e. $\pi_{t+1}(X)$ has a filtration $\pi_{t+1}\bigl(X(k)\bigr) = F_{t,1} \supset F_{t-1,2} \supset F_{t-2,3} \supset F_{t-3,4} \supset F_{t-4,5} = 0$ such that $F_{p,q}/F_{p-1,q+1} \approx E^\infty_{p,q}$.

Similar spectral sequence holds for the stable homotopy groups too. In this case the relative homotopy groups of pairs can be replaced by the (absolute) homotopy groups of the quotient spaces and this makes easier to handle this spectral sequence. In the next section we shall see how one can gain information from the spectral sequence in the stable homotopies for that in the non stable homotopies.

## 6. The relationship between the stable and non stable homotopy groups of the classifying spaces

LEMMA 1. *Let $A$ and $B$ be two spaces, $B \subset A$ and let both of them be one of the spaces $X, Y, \Gamma, \emptyset$.** *Then there exist maps*

$$\pi_i(A,B) \xrightarrow{s} \pi_i^s(A/B) \xrightarrow{r} \pi_i(A,B)$$

*such that $r \circ s =$ identity, and these maps commute with the homomorphisms of the exact sequences of the pair $A, B$. (If $B = \emptyset$ then $\pi_i(A,\emptyset) \stackrel{\text{def}}{=} \pi_i(A)$, $A/\emptyset = A \cup *$ and $s$ is the composition of the map induced by the inclusion $A \subset A \cup *$ with the iterated suspension.)*

PROOF. For simplicity we prove this lemma for $A = X$, $B = \emptyset$. For any other case it can be proved modifying this proof in an obvious way.

The following *geometric interpretation of the groups* $\pi^s_{n+k}(X(k) \cup *)$ will be useful for the proof. The stable homotopy group $\pi^s_{n+k}(X(k) \cup *)$ is isomorphic to the cobordism group of triples consisting of

(1) an $n+k$-dimensional stably parallelizable manifold $\Pi$,

(2) a trivialization of its stable normal bundle, and

(3) an $X$-map of a closed smooth oriented $n$ dimensional manifold into $\Pi$.

To show this consider the natural fibrewise map $\theta : S^N(X \cup *) \to S^N$ such that $\theta(X)$ is one point. Let us denote this point by $P$. The composition of $\theta$ with a map $F : S^{N+n+k} \to S^N(X \cup *)$ gives a map $g = \theta \circ \circ F : S^{N+n+k} \to S^N$. We can suppose that $g$ is transversal to $P$ and so $g^{-1}(P)$ is a smooth $(n+k)$ dimensional stably parallelizable manifold $\Pi^{n+k}$. The map $g$ defines a trivialization of the stable normal bundle of $\Pi$. We shall denote this trivialization also by $\theta$. The restriction of the map $F$ to $\Pi$ i.e. $F|_\Pi : \Pi \to X$ defines an $X$-map (up to cobordism) of codimension $k$ into $\Pi$ which we denote by $f : M^n \to \Pi^{n+k}$. Now it is easy to see that the correspondence between the homotopy classes $[F]$ and the cobordism classes of the triples $(\Pi, \theta, f)$ is one to one.

Similar geometric interpretation can be given for the spaces $Y(k)$, $\Gamma(k)$ and for the pairs $(X,Y)$, $(X,\Gamma)$, $(Y,\Gamma)$ as well.

Now the map $s : \pi_i(X) \to \pi_i^s(X \cup *)$ is the obvious map induced by the iterated suspension. It sends an $X$-cobordism class $[f]$ represented by the $X$-map $f : M^n \to S^{n+k}$ into the triple $(S^{n+k}, \theta, f)$ where $\theta$ is the standard trivialization of the stable normal bundle of $S^{n+k}$. The map $r : \pi_i^s(A,B) \to \to \pi_i(A,B)$ can be defined as follows. Let $(\Pi, \theta, f : M^n \to \Pi)$ be a triple representing an element $\alpha$ of $\pi^s_{n+k}(X \cup *)$. We can consider $\Pi^{n+k}$ as a

* $k$ will be fixed all the time and it will be often omitted. So instead of $X(k)$, $Y(k)$, $Z(k)$, $\Gamma(k)$ we shall write just $X$, $Y$, $Z$, $\Gamma$ respectively.

submanifold in the Euclidean space $R^{N+n+k}$ $(N \gg 1)$. Remove a point $Q$ from $\Pi$ which does not belong to $f(M)$. The manifold $\Pi' = \Pi \setminus Q$ is an open submanifold of $R^{N+n+k}$ and has trivialized normal bundle. By the Smale–Hirsh theory there exists a regular homotopy joining the embedding $\Pi' \subset R^{N+n+k}$ with an immersion $g : \Pi' \to R^{n+k}(\subset R^{N+n+k})$. Now the composition $g \circ f$ is an $X$-map $M^n \to R^{n+k}$ and so it represents an element $\beta \in \pi_{n+k}\big(X(k)\big)$. It can be shown that this element depends only on $\alpha$.

We put $r(\alpha) = \beta$. Obviously $r \circ s =$ identity. The lemma is proved. □

REMARK 3. (a) Note that the space $Z$ was excluded in the lemma above. (Recall that a $Z$-map is a map which has only $\Sigma^{1,0}$ singular points, its kernel bundle is trivial *and the singular points are non-multiple.*) If $f$ had been a $Z$-map in the proof above then its composition with the immersion $g : \Pi' \to$ $\to R^{n+k}$ might have not been a $Z$-map (the singular points might have become double points). But if $f$ is an $X$-, $Y$- or $\Gamma$-map then its composition with an immersion is also a map of the same type.

(b) For the space $\Gamma$ the lemma follows from Proposition 3.6 of [2] saying that for any pointed space $A$ there exists a commutative diagram

$$\begin{tikzcd} \Gamma^+ A \arrow[r] \arrow[dr, "1"'] & \Gamma^+\Gamma^+ A \arrow[d, "h_A"] \\ & \Gamma^+ A \end{tikzcd}$$

Indeed $\pi_*^s(\Gamma^+ A) = \pi_*(\Gamma^+\Gamma^+ A)$ and for $A = MSO(k)$ the space $\Gamma^+ A$ is $\Gamma(k)$ (see [12]).

## 7. The spectral sequence in the stable homotopy groups

Let us consider the spectral sequence $\{E^r_{**}\}$ associated with the filtration $X \cup * \supset Y \cup * \supset Z \cup * \supset \Gamma \cup *$ in the generalized homology theory formed by the stable homotopy groups.

**7.1. The groups $E^1_{**}$.** From Section 5 by the corollaries in Section 4 we have that the groups $E^1_{**}$ are the following:

$$E^1_{1,t} = \pi^s_{t+1}(\Gamma \cup *),$$

$$E^1_{2,t} = \pi^s_{t+2}(Z/\Gamma) = \pi^s_{t+2}(ST2\gamma_k) = \pi^s_{t+1}(T2\gamma_k),$$

$$E^1_{3,t} = \pi^s_{t+3}(Y/Z) = \pi^s_{t+3}(ST2\gamma_k \wedge T\gamma_k) = \pi^s_{t+2}(T2\gamma_k \wedge T\gamma_k),$$

$$E^1_{4,t} = \pi^s_{t+4}(X/Y) = \pi^s_{t+4}(S^2T3\gamma_k) = \pi^s_{t+2}(T3\gamma_k).$$

In the next lemma we compute these groups expressing them by the cobordism groups of manifolds $\Omega_*$.

LEMMA 2. *If* $i < 4k$ *then*

(a) 1. $\pi_i^s(T2\gamma_k) \approx \Omega_{i-2k}$ *modulo* $\mathcal{C}(2)$ *if* $k$ *is odd,*

2. $\pi_i^s(T2\gamma_k) \approx \Omega_{i-2k} \oplus \Omega_{i-3k}^{3\gamma}$ *modulo* $\mathcal{C}(2)$ *if* $k$ *is even,*

(b) 1. $\pi_i^s(T3\gamma_k) \approx \Omega_{i-3k}$ *modulo* $\mathcal{C}(3)$,

2. $\pi_i^s(T3\gamma_k)$ *has* 3-*torsion for* $i = 3k + 4$,

(c) $\pi_i^s(T2\gamma_k \wedge T\gamma_k) \approx$ *modulo* $\mathcal{C}(2)$.

PROOF. (a) 1. For $k$ odd the inclusion $T2\gamma_k \subset T\gamma_{2k}$ induces isomorphisms of the cohomology groups with coefficients $Z_p$, $p \neq 2$ in dimensions less than $4k$ and $\pi_i^s(T\gamma_{2k}) \approx \Omega_{i-2k}$ for $i < 4k$.

(a) 2. For $k$ even the natural inclusion $T2\gamma_k \subset T\gamma_{2k} \times T3\gamma_k$ (defined by the inclusions $2\gamma_k \subset \gamma_{2k}$ and $2\gamma_k \subset 3\gamma_k$) induces $\mathcal{C}(2)$ isomorphisms of the cohomology groups in dimensions less than $4k$. Indeed the Thom class of the bundle $3\gamma_k$ is mapped onto $U \cdot \chi$ where $\chi$ is the Euler class of $\gamma_k$ and $U$ is the Thom class of $2\gamma_k$. The ring $H^*(T3\gamma_k)$ (in dimensions less than $4k$) is mapped isomorphically onto the ideal generated by $U \cdot \chi \in H^*(T2\gamma_k)$, while $H^*(T\gamma_{2k})$ is mapped isomorphically onto the factorring $H^*(T2\gamma_k)/(U \cdot \cdot \chi)$ (which is generated by the Pontrjagin classes). $\pi_i(T\gamma_{2k}) \to \pi_i^s(T\gamma_{2k})$ is an isomorphism if $i < 4k$, since $T\gamma_{2k}$ is $(2k-1)$-connected. Idem for $\pi_i(T\gamma_{3k}) \to \pi_i^s(T\gamma_{3k})$. Therefore $\pi_i^s(T\gamma_{2k}) = \pi_i^s(T\gamma_{2k}) \oplus \pi_i^s(T\gamma_{3k})$ $i < 4k$, and the isomorphism (a) 2 follows again since $i < 4k$.

(b) The inclusion $T3\gamma_k \subset T\gamma_{3k}$ induces isomorphisms of the cohomology groups with coefficients $Z_p$, $p \neq 3$ in dimensions less than $4k$. Indeed it is enough to show that the corresponding map of the base spaces

$$j : BSO(k) \to BSO(3k)$$

defines isomorphisms in dimensions less than $k$. Notice that $j$ is *not* the standard inclusion, it is defined by the equation $j^*\gamma_{3k} = 3 \cdot \gamma_k$. Hence the induced homomorphism $j^*$ in dimensions less than $4k$ is defined by the formula

$$j^*\big(p(\gamma_{3k})\big) = p(\gamma_k)^3$$

where $p(\ \ )$ denotes the total Pontryagin class $\Sigma p_i$. Hence

$$j^* p_r = 3p_r + f(p_1, \dots, p_{r-1}) \text{ for } r = 1, 2, \dots, [k/2].$$

So if 3 is invertible in the coefficient ring, then $j^*$ is an isomorphism and this proves (b) 1, but $j^*$ is not an isomorphism for coefficients $Z_3$ and this implies (b) 2.

(c) Trivial computation shows that the groups on the left and the right sides have the same ranks. Hence it is enough to show the following proposition:

PROPOSITION 1. *The groups $\pi_j(T2\gamma_k \wedge T\gamma_k)$ have no odd torsion for $j < 4k$.*

PROOF. We shall use the following lemma due to Milnor (see [9] page 113).

LEMMA (Milnor). *Let $X$ be a stable spectrum such that the group $\tilde{H}^*(X; Z)$ has no p-torsion and $\tilde{H}^*(X; Z_p)$ is a free $A_p/(Q_0)$ module. Then the homotopy groups of the spectrum have no p-torsion.* □

We shall use also the following fact (see [9] Chapter IX, section "Odd primary results"):

PROPOSITION 2. *The spectrum $TBSO$ satisfies the conditions of Milnor's lemma for any odd prime p, i.e. $\tilde{H}^*(TBSO; Z)$ has no p-torsion and $\tilde{H}^*(TBSO; Z_p)$ is a free $A_p/Q_0$ module.* □

Let us consider the spectrum $TBSO \wedge TBSO$. The cohomology ring of this spectrum is the tensor product $\tilde{H}^*(TBSO \wedge TBSO) \approx \tilde{H}^*(TBSO) \otimes \otimes \tilde{H}^*(TBSO)$. Therefore the spectrum $TBSO \wedge TBSO$ also satisfies the conditions of Milnor's lemma and so its homotopy groups $\pi_*(TBSO \wedge \wedge TBSO)$ do not have odd torsion. It remained to show the following lemma.

LEMMA. *$\pi_j(T2\gamma_k \wedge T\gamma_k)$ is isomorphic to the $(j-3k)$-th homotopy group of the spectrum $TBSO \wedge TBSO$ for $j < 4k$.*

PROOF. By part (a) of Lemma 2 the inclusion $T2\gamma_k \subset T\gamma_{2k}$ induces isomorphism of the homologies in dimensions less than $3k$. The space $T\gamma_k$ is $k-1$ connected. Therefore in dimensions less than $4k$ the inclusion $T2\gamma_k \wedge \wedge T\gamma_k \subset T\gamma_{2k} \wedge T\gamma_k$ induces isomorphisms.

Finally notice that the inclusion $ST\gamma_N \wedge T\gamma_M \subset T\gamma_{N+1} \wedge T\gamma_M$ induces isomorphisms in dimensions less than $2N + M - 1$ for any $N$ and $M$. Therefore the homotopy groups $\pi_j(T2\gamma_k \wedge T\gamma_k)$ for $j < 4k$ coincide with the homotopy groups of the spectrum $TBSO \wedge TBSO$. □

COROLLARY. 1. *The groups $E^1_{s,t}$ have no odd torsion for $s = 2$ or $3$ and $t < 4k$.*

2. *The groups $E^1_{4,t}$ have no $p$ torsion for $p > 3$ and $t < 4k$.*

**Summary of 7.1.** The groups $E^1$ of the spectral sequence in the stable homotopy groups are the following:

$$E^1_{1,t} \approx \pi^s_{t+1}(\Gamma \cup *),$$

$$E^1_{2,t} \approx \pi^s_{t+2}(Z/\Gamma) \approx \Omega_{t-2k+1} \text{ for } k \text{ odd},$$

$$E^1_{2,t} \approx \pi^s_{t+2}(Z/\Gamma) \approx \Omega_{t-2k+1} \oplus \Omega^{3\gamma}_{t-3k+1} \text{ for } k \text{ even},$$

$$E^1_{3,t} \approx \pi^s_{t+3}(Y/Z) \approx \pi^s_{t+2}(T2\gamma_k \wedge T\gamma_k) \approx$$

$$\approx \oplus \{\Omega_a \otimes \Omega_b \mid a+b = t+2-3k\},$$

$$E^1_{4,t} \approx \pi^s_{t+4}(X/Y) \approx \pi^s_{t+2}(T3\gamma_k) \approx \Omega_{t+2-3k} \mod \mathcal{C}(3).$$

**7.2. The differentials of the spectral sequence in $\pi^s_*$.**

CLAIM 1A. *If $k$ is odd then the differentials $d^1_{2,t}$, $d^2_{3,t}$ going into the first column are zero modulo $\mathcal{C}(2)$ (i.e their images are contained in the 2-torsion parts.)*

PROOF. Let us consider the spaces $\hat{Y}(k)$ and $\hat{Z}(k)$ which classify the same maps as $Y(k)$ and $Z(k)$ but without the condition that the kernel bundles are trivial i.e. $\pi_{n+k}(\hat{Y}(k))$ = the cobordism group of the maps of oriented $n$ dimensional manifolds into $R^{n+k}$ having only simple (= non-multiple) $\Sigma^{1,0}$ singular points and having no quadruple points and $\pi_{n+k}(\hat{Z}(k))$ = the cobordism group of the maps which have only $\Sigma^{1,0}$ singular points and the preimage of a point may consist of

1) at most 3 nonsingular points, or
2) a $\Sigma^{1,0}$ singular point and a nonsingular point, or
3) a $\Sigma^{1,0}$ singular point alone.

These spaces $\hat{Y}(k)$ and $\hat{Z}(k)$ have been constructed in [14]. Notice that $\hat{Y}(k) \supset \hat{Z}(k) \supset \Gamma(k)$ and there exists a natural map $(Y, Z, \Gamma) \to (\hat{Y}, \hat{Z}, \Gamma)$ which is the identity on $\Gamma$. Therefore in order to show Claim 1a it is enough to show that the corresponding differentials $\hat{d}^1_{2,t}$ and $\hat{d}^2_{3,t}$ of the *second* triple are zero modulo $\mathcal{C}(2)$. This follows from the following

LEMMA 3. *The factorspaces $\hat{Y}/\hat{Z}$ and $\hat{Z}/\Gamma$ have finite 2-primary cohomology groups.*

Hence the stable homotopy groups of these spaces are finite 2-primary groups as well. The source groups of the differentials $\hat{d}^1_{2,t}$ and $\hat{d}^2_{3,t}$ are these stable homotopy groups and so Claim 1a will follow from Lemma 3.

PROOF OF LEMMA 3. It follows from the construction of the spaces $\hat{Y}$ and $\hat{Z}$ that there exists a bundle $\xi$ such that $\hat{Z}/\Gamma = ST\xi$ and $\hat{Y}/\hat{Z} = ST\xi \wedge$

$\wedge T\gamma_k$. Here $T\xi$ is the Thom space of the bundle $\xi$ and $S$ is the suspension. It is enough to show that $H^*(T\xi; Z_p) = 0$ for $p$ odd. The Thom space $T\xi$ can be obtained from $T2\gamma_k \times S^\infty$ factorising out by an involution which changes the factors in $2\gamma_k = \gamma_k \oplus \gamma_k$ and which is the multiplication by $-1$ on $S^\infty$. Hence $H^*(T\xi; Z_p)$ is isomorphic to the invariant part of $H^*(T2\gamma_k; Z_p)$ under this involution. But this involution changes the sign of the Thom class (because $k$ is odd) and keeps unchanged the cohomology classes of the base space $BSO(k)$. Hence $H^*(T2\gamma_k; Z_p) = 0$. $\square$

CLAIM 1B. *For $k$ even the restriction of the differential*

$$d^1_{2,t-1} : \Omega_{t-2k} \oplus \Omega^{3\gamma}_{t-3k} \to \pi^s_t(\Gamma(k) \cup *)$$

(a) *to the first factor is a monomorphism* mod $\mathcal{C}(2)$.
(b) *to the second factor is zero* mod $\mathcal{C}(2)$.

PROOF. Part (a) follows by mapping the spectral sequence into the analogous spectral sequence of the space $X'(k)$ which classifies the same maps as $X(k)$ without the conditions of orientability of the source manifolds. This space has a filtration analogous to the one of $X(k)$ and so a spectral sequence arises. (See [16].) In this spectral sequence the map

$$d^1_{2,t-1} : \Omega_{t-2k} \to \pi^s_t\big(QMO(k) \cup *\big)$$

is the only nonzero (modulo $\mathcal{C}(2)$) differential and it is an isomorphism. Hence $d^1_{2,t-1}$ in our spectral sequence restricted to $\Omega_{t-2k}$ is a monomorphism.

Part (b) will follow from the fact, which we prove later, that for $k$ even the differential $d^2_{4,t-2}$ is a $\mathcal{C}(2)$-epimorphism onto the second factor of the source of $d^1_{2,t-1}$. $\square$

CLAIM 2. *The differential $d^1_{3,t} : E^1_{3,t} \to E^1_{2,t}$ is zero.*

PROOF. We have

$$E^1_{3,t} = \pi^s_{t+2}(T2\gamma_k \wedge T\gamma_k), \quad E^1_{2,t} = \pi^s_{t+1}(T2\gamma_k).$$

Let us put $T_1 = T\gamma_k$ and $T_2 = T2\gamma_k$. Let us consider the following part of the exact sequence of the pair $\big((T_1 \times T_2)/T_1, T_2\big)$:

$$\to \pi^s_j\big((T_1 \times T_2)/T_1, T_2\big) \xrightarrow{\partial} \pi^s_{j-1}(T_2) \xrightarrow{i_*} \pi^s_{j-1}\big((T_1 \times T_2)/T_1\big) \to$$

The map $\partial$ is the differential $d^1_{3,t}$.

There exists a commutative diagram:

$$\begin{tikzcd} & (T_1 \times T_2)/T_1 \arrow[dr, "p"] & \\ T_2 \arrow[ur, "i"] \arrow[rr, "1"] & & T_2 \end{tikzcd}$$

where $p$ is the natural projection and $i$ is the inclusion. Hence $i_*$ is a monomorphism and so $\partial = 0$. □

CLAIM 3. *If $k$ is even then $E^{\infty}_{s,t} \otimes Q = 0$ for $s > 1$ and $t < 4k$.*

PROOF. It is enough to show that the inclusion

$$\Gamma(k) \cup * \subset X(k) \cup *$$

induces epimorphisms of the rational stable homotopy groups in dimensions less than $4k$. Since the inclusion $i : SMSO(k) \subset MSO(k+1)$ induces a monomorphism of the rational cohomology rings

$$H^*\big(MSO(k+1); Q\big) \to H^*\big(SMSO(k); Q\big)$$

so does the inclusion $i_Q : QSMSO(k) \subset QMSO(k+1)$, where $Q = \Omega^\infty S^\infty$. This follows for example from the stable decomposition

$$QA = \Gamma^+ A \cong_{st} \bigvee_{i=1}^{\infty} D_i A$$

which holds for any space $A$, see [2]. Since the spaces $QSMSO(k)$ and $QMSO(k+1)$ are $H$-spaces they can be replaced from rational point of view by some products of Eilenberg–Maclane spaces and the fact that $i_Q^*$ is a monomorphism in rational cohomology implies that the map

$$i_* : \pi_*\big(QSMSO(k)\big) \otimes Q \to \pi_*\big(QMSO(k+1)\big) \otimes Q$$

is onto. Now it is obvious that the map $\Omega i : \Omega QSMSO(k) \subset \Omega QMSO(k+ +1)$ induces an epimorphism in rational homotopy. But $\Omega QSMSO(k) = = QMSO(k)$ and this space is $(4k-1)$-equivalent to $\Gamma(k)$. Moreover the space $\Omega QMSO(k+1)$ is $4k-1$ equivalent to $X(k)$. Hence the inclusion $\Gamma(k) \cup * \subset X(k) \cup *$ induces epimorphisms of the rational stable homotopy groups

$$\pi_j^s(\Gamma(k) \cup *) \otimes Q \twoheadrightarrow \pi_j^s(X(k) \cup *) \otimes Q \quad \text{for} \quad j < 4k. \quad \square$$

CLAIM 4. *$d^1_{4,t} : E^1_{4,t} \to E^1_{3,t}$ is zero.*

PROOF. The map $d^1_{4,t}$ can be visualized as it is shown on Fig. 2.

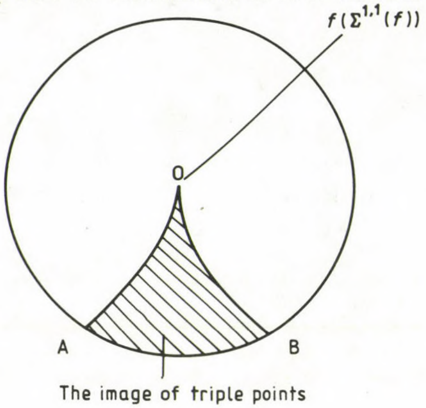


*Fig. 2*

This figure shows a disc normal to the submanifold formed by the images of $\Sigma^{1,1}$ singular points in the image of the set of triple points. Such a normal disc can be defined although the image of the triple points set is not a smooth manifold (it has a sharp edge at the image of $\Sigma^{1,1}$ points). Now as the point $O$ runs over the manifold $f\left(\Sigma^{1,1}(f)\right)$ the points $A$ and $B$ run over some manifolds $\hat{A}$ and $\hat{B}$. The differential $d^1_{4,t}$ sends the cobordism class of $f\left(\Sigma^{1,1}(f)\right)$ (with the appropriate normal structure) into the cobordism class of $\hat{A} \cup \hat{B}$ (with the appropriate normal structure). The manifolds $\hat{A}$ and $\hat{B}$ are diffeomorphic but they are provided with opposite orientations and so $\hat{A} \cup \hat{B}$ is null-cobordant. (The fact that for an arbitrary $\Sigma^{1,1}$ prim map $f$ the manifold $\hat{A} \cup \hat{B}$ is null-cobordant can be seen as follows. The manifold $\hat{A} \cup \hat{B}$ is the intersection of the set of stationary (= multiple singular) points $\mathrm{St}(f)$ with the boundary of a tubular neighbourhood $\tilde{T}_{\Sigma^{1,1}}$ of the manifold $\tilde{\Sigma}^{1,1} = f(\Sigma^{1,1})$. Now $\hat{A} \cup \hat{B}$ is the boundary of the submanifold formed by those stationary points which lie outside the tubular neighbourhood $\tilde{T}_{\Sigma^{1,1}}$ i.e $\hat{A} \cup \hat{B} = \partial\{\mathrm{St}(f) \setminus \tilde{T}_{\Sigma^{1,1}}\}$.) □

COROLLARY. *There exists a retraction $r : X/Z \to Y/Z$.*

PROOF. By definition $X/Z = Y/Z \cup_\rho D(3\gamma_k \oplus 2)$ where $D(3\gamma_k \oplus 2)$ is the disc bundle $3\gamma_k \oplus 2$. This bundle is the sum of 3 copies of the universal oriented $k$ dimensional bundle $\gamma_k$ with a 2 dimensional trivial bundle. $\rho$ is an attaching map $\rho : \partial D(3\gamma_k \oplus 2) \to Y/Z$, which is defined by $\hat{A} \cup \hat{B}$ (and its normal structure). Claim 4 shows that $\rho$ is null-homotopic hence it can be extended to a map

$$\rho^* : D(3\gamma_k \oplus 2) \to Y/Z.$$

Now $\rho^*$ defines a retraction $r : X/Z \to Y/Z$. $\square$

We shall use this Corollary later in the computation of $d^3_{4,t}$.

**Summary of 7.2.** The $E^2$ terms of the spectral sequence in the stable homotopy groups are the following:

For $k$ odd $E^2 \approx E^1$.

For $k$ even the last two columns are the same as before i.e.

$$E^2_{3,t} \approx E^1_{3,t} \text{ and } E^2_{4,t} \approx E^1_{4,t}.$$

The first two columns change as follows:

$$E^2_{2,t} \approx \Omega_{t-3k} \text{ and } E^2_{1,t} \approx E^1_{1,t}/\Omega_{t-3k} \approx \pi^s_{t+1}(\Gamma \cup *)/\Omega_{t-3k}.$$

**7.3 The differentials $d^2$ and the terms $E^3$ of the spectral sequence in the stable homotopy groups.**

REMARK 4. For $k$ odd the differential $d^2_{3,t}$ is zero by Claim 1a.

CLAIM 5. *If $k$ is even then the differential $d^2_{3,t}$ is monomorphic modulo* $\mathcal{C}(2)$.

PROOF.

$$\begin{tikzcd} E^2_{3,t} \arrow[r, "d^2_{3,t}"] \arrow[d, "\approx"] & E^2_{1,t+1} \arrow[d, "\approx"] \\ \pi^s_{t+3}(Y/Z) \arrow[r] & \pi^s_{t-2}(\Gamma \cup *) \end{tikzcd}$$

$$\pi^s_{t+3}(Y/Z) = \pi^s_{t+3}(ST2\gamma_k \wedge T\gamma_k) \approx \oplus\{\Omega_a \otimes \Omega_b | a+b = t+2-3k\} \bmod \mathcal{C}(2).$$

Since $E^\infty_{3,t} \otimes Q = 0$ (see Claim 3) and there are no nonzero differentials going into $E^r_{3,t}$ the differential $d^2_{3,t}$ must be a rational monomorphism (i.e. it must have finite kernel). But $\oplus\{\Omega_a \otimes \Omega_b\}$ has no odd torsion so $d^2_{3,t}$ is a $\mathcal{C}(2)$-monomorphism. $\square$

CLAIM 6. *If $k$ is odd then* $d^2_{4,t} = 0 \bmod \mathcal{C}(2)$.

PROOF. The source of the map $d^2_{4,t} : E^2_{4,t} \to E^2_{2,t+1}$ is $\Omega_{t-3k+2}$ modulo $\mathcal{C}(3)$, its target is $\Omega_{t-2k+2}$ modulo $\mathcal{C}(2)$. At least one of these groups is finite (since the difference of the dimensions of the manifolds occurring in these cobordism groups is $k$ which is odd) and their torsion groups are always 2-primary. Hence $d^2_{4,t}$ is zero modulo $\mathcal{C}(2)$ if $k$ is odd. $\square$

In order to compute the differential $d^2_{4,t}$ for $k$ even we shall need the following remark.

REMARK 5. In Lemma 2 we have shown that the following $\mathcal{C}(2)$ isomorphism holds:

$$\pi^s_{i+1}(Z,\Gamma) \approx \Omega_{i-2k} \oplus \Omega^{3\gamma}_{i-3k}.$$

Here we give a geometric description of this $\mathcal{C}(2)$ isomorphism.

The group $\pi^s_{i+1}(Z,\Gamma)$ can be identified with the cobordism group of embeddings of $i-2k$ dimensional manifolds in $R^{i+1}$ with normal bundle isomorphic to the sum of two isomorphic bundles plus the trivial line bundle. (Such a manifold will be called a $2\gamma$ manifold.) Now let $j : N \subset R^{i+1}$ be an embedding such that its normal bundle $\nu_j$ is isomorphic to $1 \oplus 2 \cdot \xi_k$ where $\xi_k$ is a $k$ dimensional bundle over $N$. Then the first component of the element corresponding to the class of $j$ is the cobordism class $[N] \in \Omega_*$.

We shall call the bundle $\xi_k$ *the half normal bundle of the* $2\gamma$ *manifold* $N$.

In order to obtain the second component — which we shall call the $3\gamma$ component — take a generic (i.e. transversal to the zero-section) section $\tau$ of the half normal bundle $\xi^k \to N$ and let $\tau^{-1}(0)$ be the set where $\tau$ vanishes. Then the second component is represented by the manifold $\tau^{-1}(0)$.

Notice that the normal bundle of $\tau^{-1}(0)$ in $N$ is isomorphic to $\xi_k|\tau^{-1}(0)$ and so the normal bundle of $\tau^{-1}(0)$ in $R^{i+1}$ is isomorphic to $3\cdot\xi_k|\tau^{-1}(0)\oplus 1$. Hence $\tau^{-1}(0)$ represents an element of $\Omega^{3\gamma}_*$.

The $3\gamma$-cobordism class of the manifold $\tau^{-1}(0)$ will be called *the Euler cobordism class of the half normal bundle of the* $2\gamma$ *cobordism class of the manifold* $N$.

Let $k$ be even and let us consider the differential

$$d^2_{4,t-2} : \pi^s_{t+2}(X,Y) \to \operatorname{Ker}\left[\partial : \pi^s_{t+1}(Z,\Gamma) \to \pi^s_t(\Gamma \cup *)\right].$$

The source of this differential is isomorphic to $\Omega^{3\gamma}_{t-3k}$ and its target is modulo $\mathcal{C}(2)$ a subgroup of $\Omega^{3\gamma}_{t-3k}$. Hence we can consider this differential (modulo $\mathcal{C}(2)$) as a homomorphism:

$$(*) \qquad d^2_{4,t-2} : \Omega^{3\gamma}_{t-3k} \to \Omega^{3\gamma}_{t-3k}.$$

CLAIM 7. *The map* $(*)$ *is the multiplication by* 2 *(for* $t < 4k$ *and* $k$ *even).*

PROOF. This differential can be described as follows. Put $n = t+2-k$. Pick up an element $\alpha \in \pi^s_{t+2}(X,Y)$. It can be represented by an $X$-map (i.e. a $\Sigma^{1,1}$ prim map) $f : (M^n, \partial M^n) \to (D^{n+k}, S^{n+k-1})$ such that its restriction to the boundary $f|_{\partial M} : \partial M \to S^{n+k-1}$ is a $Y$-map (i.e. has no $\Sigma^{1,1}$ point). Take a small tubular neighbourhood $T$ of the set $f\big(\Sigma^{1,1}(f)\big)$ and let us denote by $Z(f)$ the intersection of $\partial T$ with $f\big(\Sigma^{1,0}(f)\big)$. $Z(f)$ is a submanifold of $\partial T$ of codimension $2k+1$. The normal bundle $\nu(Z(f) \subset \subset \partial T)$ of $Z(f)$ in $\partial T$ is the direct sum of two isomorphic bundles and a trivial line bundle

$$\nu\big(Z(f) \subset \partial T\big) = 2 \cdot \xi^k \oplus 1.$$

The manifold $Z(f)$ with this normal structure represents the image of $\alpha$ at the map $\partial$:

$$\pi^s_{n+k}(X/Y) \xrightarrow{\partial} \pi^s_{n+k-1}(Z/\Gamma) \approx \Omega_{n-k-2} \oplus \Omega^{3\gamma}_{n-2k-2}.$$

The last "isomorphism" is an isomorphism modulo the class $\mathcal{C}(2)$. The element $d^2_{4,*}(\alpha)$ is the second component of $\partial(\alpha)$, i.e

$$\begin{tikzcd} \pi^s_i(X,Y) \arrow[r, "\partial"] \arrow[rrr, bend right, "d^2_{4,*}"'] & \pi^s_{i-1}(Z/\Gamma) \arrow[r, phantom, "\approx"] & \Omega_{i-2k-2} \oplus \Omega^{3\gamma}_{i-3k-2} \arrow[r, "p"] & \Omega^{3\gamma}_{i-3k-2} \end{tikzcd}$$

$d^2_{4,*} = p \circ \partial$, where $p$ is the projection onto the second factor.

So we have to show that the Euler cobordism class of the half normal bundle of the class $\partial(\alpha)$ is equal to $2\alpha$. This will follow from part (c) of the following lemma.

LEMMA 4. *Let $Z(f)$, $\Sigma^{1,1}(f)$, and $\xi^k$ denote the same as above. Then there exists a bundle $\eta^k \to \Sigma^{1,1}(f)$ such that*

(a) *$Z(f)$ is diffeomorphic to the total space of the spheric bundle of the vector bundle $\eta^k \oplus 1$. Let us denote this spheric bundle by*

$$\pi_S : S(\eta^k \oplus 1) \to \Sigma^{1,1}(f).$$

(b) *The half normal bundle $\xi^k \to Z(f)$ of the $2\gamma$-manifold $Z(k)$ can be identified with the bundle $\pi^*_S \eta^k$.*

(c) *The generic section $\tau$ of the half normal bundle of the $2\gamma$ manifold $Z(f)$ can be chosen in such a way that the zeroset $\tau^{-1}(0)$ intersects each fibre of the bundle $\pi_S : S(\eta^k \oplus 1) \to \Sigma^{1,1}(f)$ at two points and the projection $\pi_S|\tau^{-1}(0) : \tau^{-1}(0) \to \Sigma^{1,1}(f)$ give an orientation preserving double covering map.*

PROOF OF LEMMA 4. First of all we show the bundle $\eta^k \to \Sigma^{1,1}(f)$. In a neighbourhood of the set $\Sigma^{1,1}(f)$

1) the stationary points (those which are both double and singular) form a set $\mathrm{St}\,(f)$ diffeomorphic to $\Sigma^{1,1}(f) \times I$ (where $I = [-1,1]$), and

2) the singular points form a $k$ dimensional disc bundle over the set $\mathrm{St}\,(f)$. We denote this bundle by $\zeta^k$.

The restriction of the bundle $\zeta^k$ to $\Sigma^{1,1}(f)$ is $\eta^k$.

(a) The set $Z(f)$ was defined as the intersection of $f\big(\Sigma^1(f)\big)$ with the boundary of a neighbourhood of $f\big(\Sigma^{1,1}(f)\big)$ in the target manifold. Since $f$ restricted to the set of singular points is a 1 to 1 map, $Z(f)$ is diffeomorphic to the intersection of the set of singular points with the boundary of a neighbourhood of $\Sigma^{1,1}(f)$ in the source manifold $M^n$, i.e.

$$Z(f) = \pi_\zeta^{-1}\big(\Sigma^{1,1}(f) \times (-1)\big) \cup \pi_\zeta^{-1}\big(\Sigma^{1,1}(f) \times (1)\big) \cup S(\zeta^k)$$

where $\pi_\zeta$ is the projection of the disc bundle $\zeta^k$. Now it is clear that the set $Z(f)$ can be identified with the set $S(\eta^k \oplus 1)$. Part (a) is proved.

(b) In order to show that the half normal bundle $\xi^k$ of the $2\gamma$-manifold $Z(f)$ is isomorphic to the bundle $\pi_S^*\eta^k$ it is enough to find a bundle over $D(\eta^k \oplus 1)$ such that its restrictions to $S(\eta^k \oplus 1)$ and $\Sigma^{1,1}(f)$ are isomorphic to $\xi^k$ and $\eta^k$ respectively. The normal bundle of the closure of the set of double points in the source manifold restricted to $D(\eta^k \oplus 1)$ is such a bundle.

All the statements made here during the proofs of parts (a) and (b) follow directly from the local normal form of a $\Sigma^{1,1}$ singular map given by Morin [6]. This normal form is the following:

*Morin's formulae:*

$$R^n \ni (t_1, \dots, t_{n-1}, x) \xrightarrow{f} (y_1, \dots, y_{n-1}, z_1, \dots, z_k, z_{k+1}) \in R^{n+k},$$

$$\begin{aligned} y_i &= t_i, \qquad i = 1, 2, \dots, n-1, \\ z_1 &= t_1 \cdot x + t_2 \cdot x^2, \\ &\vdots \\ z_k &= t_{2k-1} \cdot x + t_{2k} \cdot x^2, \\ z_{k+1} &= t_{2k+1} \cdot x + x^3, \end{aligned}$$

The equations defining the set $\Sigma^{1,1}(f)$ of $\Sigma^{1,1}$ singular points are

$$t_1 = t_2 = \ldots = t_{2k+1} = 0 \quad \text{and} \quad x = 0.$$

The set $\mathrm{St}\,(f)$ of stationary points (i.e those $\Sigma^{1,0}$ singular points which have the same image as another — nonsingular — point) is defined by the equations

$$t_1 = \ldots = t_{2k} = 0 \quad \text{and} \quad t_{2k+1} = -3x^2.$$

In the local coordinates $(t_1, \dots, t_{n-1}, x)$ the equations of the set of $\Sigma^1$ points are the following:

$$t_1 + 2x \cdot t_2 = 0,$$
$$t_3 + 2x \cdot t_4 = 0,$$
$$\vdots$$
$$t_{2k-1} + 2x \cdot t_{2k} = 0,$$
$$t_{2k+1} + 3x^2 = 0.$$

Hence over each point of the manifold $\mathrm{St}(f)$ there is a $k$ dimensional linear space formed by the $\Sigma^1$ singular points (those which have the same coordinates $t_j$ for $j > 2k+1$). Above we denoted this bundle by $\zeta^k \to \mathrm{St}(f)$.

PROOF OF PART (c). We have to define a section $\tau$ of the bundle $\xi^k = \pi_S^* \eta^k$ which is transversal to the zero section and intersects with the zero section in the union of the "poles", i.e. in the spheric bundle of the trivial factor:

$$\tau\big(S(\eta^k \oplus 1)\big) \cap S(\eta^k \oplus 1) = S(1).$$

Let $b \in \Sigma^{1,1}(f)$. Then the fibre of $S(\eta^k \oplus 1)$ over $b$ is

$$S(\eta^k \oplus 1)_b = \left\{(x,y) | x \in \eta_b^k,\ y \in 1 \text{ and } |x|^2 + |y|^2 = 1\right\}.$$

The fibre of $\pi_S^* \eta^k$ over $(x,y) \in S(\eta^k \oplus 1)_b$ can be identified with $\eta_b^k$. Let us define the section $\tau : S(\eta^k \oplus 1) \to \pi_S^* \eta^k$ as follows:

$$\tau(x,y) = x \in \eta_b^k.$$

Obviously this section is zero if and only if $x = 0$ and then $(x,y) = (0,1) \in$ $\in S(1)$. (Hence the intersection $\tau^{-1}(0) = \tau\big(S(\eta^k \oplus 1)\big) \cap S(\eta^k \oplus 1)$ is $S(1)$.) In order to show that $\tau$ is transversal to the zero section of $\xi^k$ consider its composition $c$ with the projection of the nearby fibers onto the fibre over a point of $S(1)$. The tangent space to the fibre of $S(\eta^k \oplus 1)$ at this pole can be identified with the corresponding fibre of $\eta^k$. After this identification the differential of $c$ restricted to the fibre of $S(\eta^k \oplus 1)$ will be the identity map (up to sign). Hence $\tau$ is transversal to the zero section.

**The orientation.** The fibers of $S(1)$ consist of two points: $+1$ and $-1$. The union of $(+1)$ points and the union of $(-1)$ points have opposite orientations since they form the oriented boundary of $D(1)$. On the other hand the bundle $\pi_S^* \eta^k$ restricted to a fibre $S^k$ of the sphere bundle $S(\eta^k \oplus 1)$ is trivial. Hence the signs of intersection of $\tau(S^k)$ with $S^k$ at the two poles are opposite (since the Euler class of the trivial bundle over the fibre $S^k$ is zero). Eventually the union of $(+1)$ points (North poles) and the union of $(-1)$ points (South poles) have to be taken with the same orientation when we compute $p \circ \partial(\alpha) = d^2_{4,*}(\alpha)$. Lemma 4 is proved and so Claim 7 is proved as well. (Notice that for $k$ odd the orientations of the components — namely the union of the North poles and union of the South poles would get opposite orientations — the Euler cobordism class of an odd dimensional bundle has order 2, and so the differential $d^2_{4,t}$ is zero for $k$ odd, according to Claim 6.) $\square$

**Summary of 7.3.** The $E^3$ terms of the spectral sequence in the stable homotopy groups are the following:

For $k$ odd $\quad E^3 \approx E^2$.

For $k$ even $\quad E^3_{4,t} \approx 0,$

$$E^3_{3,t} \approx 0,$$

$$E^3_{2,t} \approx E^2_{2,t} \approx \oplus \{\Omega_a \otimes \Omega_b \mid a + b = i - 3k\},$$

$$E^3_{1,t} \approx E^2_{1,t}/\operatorname{im} d^2_{3,t}.$$

### 7.4 The differentials $d^3$ and the $E^4$ terms of the spectral sequence in the stable homotopy groups.

REMARK 6. For $k$ even the differential $d^3_{4,t}$ is automatically zero because the group $E^3_{4,t}$ is zero.

CLAIM 8. *If $k$ is odd then* $d^3_{4,t} = 0 \mod \mathcal{C}(2)$.

PROOF. We shall use the standard notation for the groups of cycles and boundaries associated with the filtration $X \supset Y \supset Z \supset \Gamma$. (See for example [10]. Here $X^4 = X$; $X^3 = Y$; $X^2 = Z$; $X^1 = \Gamma$.)

The differential

$$d^3_{4,t} : E^3_{4,t} \to E^3_{1,t+2}$$

is the following composition:

$$E^3_{4,t} = Z^3_{4,t}/B^3_{4,t} \to Z^3_{4,t}/Z^4_{4,t} \approx B^4_{1,t+2}/B^3_{1,t+2} \to Z^3_{1,t+2}/B^3_{1,t+2},$$

where the first map is epimorphic and the second one is monomorphic.

In our case this is the following:

$$\frac{\operatorname{im}\left[\pi^s_{t+4}(X,\Gamma) \to \pi^s_{t+4}(X,Y)\right]}{0} \to \frac{\operatorname{im}\left[\pi^s_{t+4}(X,\Gamma) \to \pi^s_{t+4}(X,Y)\right]}{\operatorname{im}\left[\pi^s_{t+4}(X \cup *) \to \pi^s_{t+4}(X,Y)\right]} \approx$$

$$\approx \frac{\operatorname{im}\left[\partial_X : \pi^s_{t+4}(X,\Gamma) \to \pi^s_{t+3}(\Gamma \cup *)\right]}{\operatorname{im}\left[\partial_Y : \pi^s_{t+4}(Y,\Gamma) \to \pi^s_{t+3}(\Gamma \cup *)\right]} \to$$

$$\to \frac{\pi^s_{t+3}(\Gamma)}{\operatorname{im}\left[\partial_Y : \pi^s_{t+4}(Y,\Gamma) \to \pi^s_{t+3}(\Gamma \cup *)\right]}$$

It is enough to show that the images of the maps

$$\partial_X : \pi^s_*(X,\Gamma) \to \pi^s_{*-1}(\Gamma \cup *) \quad \text{and} \quad \partial_Y : \pi^s_*(Y,\Gamma) \to \pi^s_{*-1}(\Gamma \cup *)$$

coincide modulo $\mathcal{C}(2)$ for $* < 4k$.

Obviously $\operatorname{im}\partial_X \supset \operatorname{im}\partial_Y$. Now we prove the inclusion in the other direction. By the Corollary of Claim 4 there exists a retraction $r : X/Z \to$ $\to Y/Z$. Hence the following commutative diagram arises:

$$\begin{tikzcd}
\pi^s_*(Y,Z) \arrow[r, "\partial_2"] \arrow[d] & \pi^s_{*-1}(Z \cup *) \arrow[d, "\text{identity}"] \\
\pi^s_*(X,Z) \arrow[r, "\partial_1"] \arrow[d, "r_*"'] & \pi^s_{*-1}(Z \cup *) \arrow[d, "\text{identity}"] \\
\pi^s_*(Y,Z) \arrow[r, "\partial_2"] & \pi^s_{*-1}(Z \cup *)
\end{tikzcd}$$

It follows that

$$\operatorname{im}\left[\partial_1 : \pi^s_*(X,Z) \to \pi^s_{*-1}(Z \cup *)\right] = \operatorname{im}\left[\partial_2 : \pi^s_*(Y,Z) \to \pi^s_{*-1}(Z \cup *)\right].$$

Since $d^1_{2,t} : \pi^s_*(Z,\Gamma) \to \pi^s_{*-1}(\Gamma \cup *)$ is zero modulo $\mathcal{C}(2)$ (by Claim 1a) the map $i : \pi^s_*(\Gamma \cup *) \to \pi^s_*(Z \cup *)$ is monomorphic modulo $\mathcal{C}(2)$. Now the following diagram implies that $\operatorname{im}\partial_X \subset \operatorname{im}\partial_Y$.

$$\begin{tikzcd}
\pi^s_*(Y,\Gamma) \arrow[rrr, "\partial_Y"] \arrow[dr] \arrow[ddd] & & & \pi^s_{*-1}(\Gamma \cup *) \arrow[dl, "i"'] \arrow[ddd, "1"] \\
& \pi^s_*(Y,Z) \arrow[r, "\partial_2"] \arrow[d] & \pi^s_{*-1}(Z \cup *) \arrow[d, "1"] & \\
& \pi^s_*(X,Z) \arrow[r, "\partial_1"] & \pi^s_{*-1}(Z \cup *) & \\
\pi^s_*(X,\Gamma) \arrow[rrr, "\partial_X"'] \arrow[ur] & & & \pi^s_{*-1}(\Gamma \cup *) \arrow[ul, "i"]
\end{tikzcd}$$

Indeed let $\alpha \in \pi^s_{*-1}(\Gamma \cup *)$ belong to $\operatorname{im} \partial_X$. Then $i(\alpha) \in \operatorname{im} \partial_1$ modulo $\mathcal{C}(2)$ and the latter group is isomorphic to $\operatorname{im} \partial_2$. Hence $i(\alpha) \in (\operatorname{im} i) \cap \cap \operatorname{im} \partial_2 \bmod \mathcal{C}(2)$. Then $\alpha \in \operatorname{im} \partial_Y \bmod \mathcal{C}(2)$. □

**Summary of 7.4.** The $E^4$ terms are the same as the $E^3$ terms modulo $\mathcal{C}(2,3)$.

**Summary of Section 7.** Having computed all the differentials, we can write down the $E^\infty$ members of the spectral sequence associated with the filtration

$$\Gamma(k) \cup * \subset Z(k) \cup * \subset Y(k) \cup * \subset X(k) \cup *$$

in the generalized homology theory formed by the *stable* homotopy groups:

For $k$ *even* the groups $E^\infty_{s,t}$ modulo $\mathcal{C}(2)$ are the following:

$$E^\infty_{1,t} \approx \left\{ \pi^s_t(\Gamma(k) \cup *)/\Omega_{t-2k} \right\} / \oplus \left\{ \Omega_a \otimes \Omega_b \mid a+b=t+1-3k \right\}$$

and

$$E^\infty_{s,t} = 0 \quad \text{for } s = 2,3,4\dots .$$

For $k$ *odd* these groups modulo $\mathcal{C}(2)$ are:

$$E^\infty_{1,t} \approx \pi^s_t(\Gamma(k) \cup *),$$

$$E^\infty_{2,t} \approx \Omega_{t-2k},$$

$$E^\infty_{3,t} \approx \oplus \{\Omega_a \otimes \Omega_b \mid a+b=t+1-3k\},$$

$$E^\infty_{4,t} \approx \Omega^{3\gamma}_{t+1-3k}.$$

## 8. The spectral sequence of the filtration $\Gamma \subset Z \subset Y \subset X$ in the non stable homotopy groups

From now on the differentials and groups of the spectral sequence that we considered in Section 3 (i.e the one for the stable homotopy groups) will be denoted with an additional upper index $s$. For example instead of $E^2_{4,t}$ we shall write ${}^sE^2_{4,t}$. The notation without this upper index $s$ will refer to the objects of the spectral sequence we consider in the present section, i.e. to the one in the *non stable* homotopy groups.

**8.1. The groups $E^1$.** The $E^1$ groups are the following:

$$E^1_{1,t} \approx \pi_{t+1}(\Gamma),$$

$$E^1_{2,t} \approx \pi_{t+2}(Z,\Gamma),$$

$$E^1_{3,t} \approx \oplus \{\Omega_a \otimes \Omega_b \mid a+b = t+2-3k\},$$

$$E^1_{4,t} \approx \Omega^{3\gamma}_{t+2-3k}.$$

So the third and fourth columns are the same as in the previous spectral sequence while in the second column we have $\pi_*(Z,\Gamma)$ instead of $\pi^s_*(Z/\Gamma) \approx \approx \pi_*(Z/\Gamma)$ (this last isomorphism holds for $* < 4k$) and in the first $\pi_{t+1}(\Gamma)$ instead of $\pi^s_{t+1}(\Gamma \cup *)$.

LEMMA 5. *The natural map $\pi_t(Z,\Gamma) \to \pi_t(Z/\Gamma)$ is an epimorphism if $t < 4k$.*

PROOF. Let $\alpha$ be an element of $\pi_t(Z/\Gamma)$ and let $f : V^{t-2k-1} \to R^t$ be a map representing $\alpha$. Then $f$ is an embedding and the normal bundle $\nu$ of $f(V)$ in $R^t$ is split into the direct sum of a trivial line bundle and two isomorphic $k$ dimensional bundles $\xi^k_1 \approx \xi^k_2$ i.e.

$$\nu \approx 1 \oplus \xi^k_1 \oplus \xi^k_2.$$

Let $D^t$ be a ball in $R^t$ containing $f(V)$ and let $S^{t-1}$ be its boundary. Since $t < 4k$ there exists an embedding $j$ of the cylinder $V \times [0,1]$ into $D^t$ such that

(1) $j|_{V\times 0} = f$,

(2) $j(V \times 1) \subset S^{t-1}$,

(3) $j(x \times [0,1])$ is tangent to the line bundle at $f(x) = j(x \times 0)$ for $x \in \in V$, and

(4) $j(x \times [0,1])$ is transversal to $S^{t-1}$.

Obviously the normal bundle of the cylinder $j(V \times [0,1])$ in $D^t$ is the direct sum $\eta = \zeta^k_1 \oplus \zeta^k_2$ where $\zeta^k_i = \pi^*\xi^k_i$, $i = 1,2$ and $\pi : V \times [0,1] \to V$ is the projection. Let $w = (v,t)$ be a point of the cylinder $V \times [0,1]$. Let us denote the normal fibre to the cylinder $j(V \times [0,1])$ over $w$ by $\eta_w$. Let us join the points which correspond to each other at the isomorphism $\xi^k_1 \approx \xi^k_2$ and lie on the $\varepsilon$-spheres of $\eta_w$ for some fixed, small enough $\varepsilon$ by a segment in $\eta_w$. The union of these segments in $\eta_w$ together with the $\varepsilon$-discs of the fibers $(\xi^k_1)_w$ and $(\xi^k_2)_w$ form the image of a topological immersion of a $k$-sphere. It can be smoothed and we get a (differentiable) immersion $i_w : S^k \to R^{2k} = \eta_w$ with a single double point at the origin. Let us denote by $H$ the union of images of the immersions $i_w$ for $w = (v,t)$, $v \in V$ and $\delta \leqq t \leqq 1$ where $\delta$ is a small number. $H$ is the image of an immersion with double points at $j(V \times [\delta, t])$.

Its boundary $\partial H$ consists of two parts $\partial_1 H$ and $\partial_2 H$. The first one lies in $S^{t-1}$ and the second one in the boundary of a tubular neighbourhood $T$ of $f(V)$. Now in each fibre of the $2k+1$ dimensional disc bundle $T \to f(V)$ we can put a $(k+1)$-dimensional Whitney umbrella in such a way that the union of these umbrellas form the image of a smooth map of a manifold $W : (M^{t-k}, \partial M) \to (T, \partial T)$ such that

(1) $W$ has only simple $\Sigma^{1,0}$ singular points and the image of its singular set is $f(V)$.

(2) $W(\partial M) = \partial_2 H$. Then $W(M) \cup H$ forms the image of a map which represents an element $\beta$ of $\pi_t(Z, \Gamma)$. The image of $\beta$ is $\alpha$.

NOTATION. Let $G_t$ denote the kernel of the epimorphism

$$\pi_t(Z, \Gamma) \twoheadrightarrow \pi_t(Z/\Gamma).$$

REMARK 7. Actually we have shown that

$$\pi_t(Z, \Gamma) \approx \pi_t(Z/\Gamma) \oplus G_t \quad \text{for} \quad t < 4k.$$

**8.2 The differentials of the spectral sequence in the non-stable homotopy groups $\pi_*$.**

CLAIM 9. $d^1_{4,t} = 0$.

PROOF. $d^1_{4,t} = {}^s d^1_{4,t} = 0$. □

CLAIM 10. $d^1_{2,t}(G_{t+2}) = 0$.

NOTATION. Let & denote the natural map (induced by the iterated suspension) from the spectral sequence $\{E^r_{**}\}$ of the filtration $X \supset Y \supset \supset Z \supset \Gamma$ (in the groups $\pi_*$) into the spectral sequence $\{{}^s E^r_{**}\}$ of the same filtration in the stable groups $\pi^s_*$).

PROOF OF CLAIM 10. We have

$$\& d^1_{2,t}(G_{t+2}) = {}^s d^1_{2,t}(\& G_{t+2}) = {}^s d^1_{2,t}(0) = 0.$$

The map & on the left is the map $s : \pi_*(\Gamma) \to \pi^s_*(\Gamma \cup *)$ which is monomorphic by Section 6. □

CLAIM 11. *The restriction of the differential $d^1_{2,t}$ to the group $\pi^s_{t+2}(Z/\Gamma)$ is the same as before.* □

CLAIM 12.

$$\operatorname{im} d^1_{3,t} = G_{t+2} \quad \textit{modulo} \quad \mathcal{C}(2,3).$$

PROOF. For any $x \in \pi_{t+3}(Y/Z)$ $(= E^1_{3,t})$ we have

$$\& \circ d^1_{3,t}(x) = {}^s d^1_{3,t} \circ \ \& \ (x) = 0$$

(because ${}^s d^1_{3,t} = 0$). Hence im $d^1_{3,t} \subset G_{t+2}$. Now

$$G_{t+2}/\operatorname{im} d^1_{3,t} \subset \operatorname{Coker}\big(\pi_{t+2}(\Gamma) \to \pi_{t+2}(X)\big) \subset$$
$$\subset \operatorname{Coker}\big(\pi^s_{t+2}(\Gamma \cup *) \to \pi^s_{t+2}(X \cup *)\big)$$

and the last group is zero modulo $\mathcal{C}(2)$ if $k$ is even (since ${}^s E^\infty_{s,t} \in \mathcal{C}(2)$ for $s > 1$). Thus for $k$ even the Claim is proved.

For $k$ odd the group Coker $\big(\pi^s_*(\Gamma \cup *) \to \pi^s_*(X)\big)$ will have free part but its torsion part is the sum of a 2-primary group and a 3-primary group. Hence the Claim will follow if we show that $G_{t+2}/\operatorname{im} d^1_{3,t}$ is finite.

LEMMA 6. *$G_{t+3}/\operatorname{im} d^1_{3,t}$ is finite if $k$ is odd.*

PROOF. We shall show later that $d^2_{4,t} = 0$, $d^3_{4,t} = 0$, $d^2_{3,t} = 0 \mod \mathcal{C}(2)$ and so the $E^\infty_{p,q}$ members on a line where $p + q = t$ will be the following modulo $\mathcal{C}(2)$:

$$E^\infty_{1,t-1} = \pi_t(\Gamma),$$

$$E^\infty_{2,t-2} = \Omega_{t-2k} \oplus (G_t/\operatorname{im}\ d^1_{3,t-2}),$$

$$E^\infty_{3,t-3} = 0,$$

$$E^\infty_{4,t-4} = \Omega^{3\gamma}_{t-3k-3}.$$

On the other hand the rank of Coker $\big(\pi_t(\Gamma) \to \pi_t(X)\big)$ can be computed easily (because $\pi_t(\Gamma) \approx \operatorname{Imm}(t-k,k)$ and $\pi_t(X) \approx \operatorname{Imm}(t-k,k+1)$.) This computation implies that $G_{t+3}/\operatorname{im}\ d^1_{3,t}$ is finite. □

Now we show that actually the previous Claim is true modulo $\mathcal{C}(2)$ too.

CLAIM 12′.

$$\operatorname{im} d^1_{3,t} = G_{t+2} \quad \textit{modulo} \quad \mathcal{C}(2).$$

PROOF. For $k$ even we have shown this claim above.

For $k$ odd:

$$G_{t+2}/\operatorname{im}\ d^1_{3,t} \subset \operatorname{coker}\big(\pi_{t+2}(\Gamma) \to \pi_{t+2}(Y)\big) \subset$$
$$\subset \operatorname{coker}\big(\pi^s_{t+2}(\Gamma \cup *) \to \pi^s_{t+2}(Y \cup *)\big)$$

and the last group has only 2-primary torsion. The last inclusion follows from the following

LEMMA 7. *Let us denote by $\{F^*_{**}\}$ and $\{{}^sF^*_{**}\}$ the spectral sequences of the triples $X \supset Y \supset \Gamma$ and $X \cup * \supset Y \cup * \supset \Gamma \cup *$ in the non stable and in the stable homotopy groups, respectively. Then there exist maps*

$$\{F^*_{**}\} \xrightarrow{\&} \{{}^sF^*_{**}\} \xrightarrow{\mathcal{R}} \{F^*_{**}\}$$

*such that $\mathcal{R} \cdot \& =$ identity.*

PROOF. This follows from Lemma 1. □

CLAIM 13.
$$d^2_{3,t} = 0 \quad modulo \quad \mathcal{C}(2).$$

PROOF. For $k$ odd this can be proved in the same way as in the stable groups namely by mapping into the spectral sequence of the filtration $\hat{Y} \supset \supset \hat{Z} \supset \Gamma$ and noticing that the groups $\pi_*(\hat{Y}, \hat{Z})$ are 2-primary for $* < 4k$.

For $k$ even we show that the group $E^2_{3,t}$ (which is the domain of the differential $d^2_{3,t}$) is zero modulo $\mathcal{C}(2)$. We know that for $k$ even the natural map $\varphi_t : \pi_t(\Gamma) \otimes Q \to \pi_t(X) \otimes Q$ is epimorphic. Hence $E^\infty_{s,t} \otimes Q = 0$ for $s = 2, 3, 4$. Especially rank $E^\infty_{3,t} = 0$. The rank of the kernel of the map $\varphi_t$ can also be computed easily, since this is the rank of the kernel of the map $H_*\big(MSO(k)\big) \to H_*\big(MSO(k+1)\big)$. On the other hand the kernel of the map $\varphi_t$ equals the sum of the images of all the differentials going into the groups $E^r_{1,t-1}$; $r = 1, 2, \dots$, i.e.

$$\text{rank } \ker \varphi_t = \text{rank } \operatorname{im} d^1_{2,t-1} + \text{rank } \operatorname{im} d^2_{3,t-2} + \text{rank } \operatorname{im} d^3_{4,t-3}$$

We have computed the differential $d^1_{2,t-1}$ and the computation gives that

$$\text{rank } \operatorname{im} d^1_{2,t-1} = \text{rank } \ker \varphi_t.$$

Hence all the other differentials mapping into the first column have finite images. Especially the image of $d^2_{3,*}$ is finite. Hence

$$\text{rank } \ker d^2_{3,*} = \text{rank } E^2_{3,*}$$

and the latter equals rank ker $d^1_{3,*}$. On the other hand

$$\text{rank } \ker d^2_{3,*} = \text{rank } E^\infty_{3,*}$$

(because all the differentials $d^r$ for $r > 2$ mapping into the groups $E^r_{3,*}$ are zero). But rank $E^\infty_{3,*}$ is zero. So ker $d^1_{3,*}$ is finite. But ker $d^1_{3,*} \subset \pi_*(Y/Z)$ and $\pi_*(Y/Z)$ has only 2-primary torsion. So ker $d^1_{3,*} \in \mathcal{C}(2)$. Since $E^2_{3,*}$ is a quotient-group of ker $d^1_{3,*}$ it also belongs to $\mathcal{C}(2)$. The group $E^2_{3,*}$ is the domain of the differential $d^2_{3,*}$ and so this differential is zero modulo $\mathcal{C}(2)$ (if $k$ is even). $\square$

CLAIM 14. *If $k$ is odd then $d^2_{4,t} = 0$ mod $\mathcal{C}(2)$ for $t < 4k$.*

LEMMA 8. *Let $\partial_1 : \pi_t(X,Y) \to \pi_{t-1}(Y,\Gamma)$ denote the boundary map of the triple $X,Y,\Gamma$. Then for $k$ odd*

$$\ker\left[\partial_1 : \pi_t(X,Y) \to \pi_{t-1}(Y,\Gamma)\right] = \ker\left[d^2_{4,*-4} : \pi_t(X,Y) \to \pi_{t-1}(Z,\Gamma)\right].$$

PROOF. By definition the map $d^2_{4,t}$ is the following composition:

$$\pi_t(X,Y) \xrightarrow{d'} \frac{\pi_t(X,Y)}{\operatorname{im}\left[\pi_t(X,\Gamma) \to \pi_t(X,Y)\right]} \overset{i}{\approx}$$

$$\overset{i}{\approx} \frac{\operatorname{im}\left[\partial : \pi_t(X,Z) \to \pi_{t-1}(Z,\Gamma)\right]}{\operatorname{im}\left[\partial : \pi_t(Y,Z) \to \pi_{t-1}(Z,\Gamma)\right]} \xrightarrow{d''}$$

$$\xrightarrow{d''} \frac{\operatorname{im}\left[\pi_{t-1}(Z) \to \pi_{t-1}(Z,\Gamma)\right]}{\operatorname{im}\left[\partial : \pi_t(Y,Z) \to \pi_{t-1}(Z,\Gamma)\right]},$$

$$d^2_{4,t} = d'' \circ i \circ d'$$

Since $d''$ and $i$ are monomorphic maps, $d^2_{4,t}(x) = 0$ if and only if $d'(x) = 0$. Now from the exact sequence

$$\pi_t(X,\Gamma) \to \pi_t(X,Y) \xrightarrow{\partial_1} \pi_{t-1}(Y,\Gamma)$$

we have ker $\partial_1 = \operatorname{im}\left[\pi_t(X,\Gamma) \to \pi_t(X,Y)\right] = \ker d' = \ker d^2_{4,t}$. $\square$

REMARK 8. The analogue of this lemma holds in the stable homotopy groups as well i.e

$$\operatorname{Ker}\left({}^s\partial_1 : \pi^s_t(X,Y) \to \pi^s_{t-1}(Y,\Gamma)\right) = \operatorname{Ker}^s d^2_{4,t-4}.$$

The proof is the same.

LEMMA 9. *$\partial_1 : \pi_t(X,Y) \to \pi_{t-1}(Y,\Gamma)$ is zero* mod $\mathcal{C}(2)$ *for $k$ odd and $t < 4k$.*

PROOF. Since ${}^s d^2_{4,t} = 0$ the map ${}^s\partial_1 : \pi^s_t(X,Y) \to \pi^s_{t-1}(Y,\Gamma)$ is zero mod $\mathcal{C}(2)$. The maps $\partial_1$ and ${}^s\partial_1$ are differentials of the spectral sequences $\{F^*_{**}\}$ and $\{{}^sF^*_{**}\}$ respectively arising from the filtration $X \supset Y \supset \Gamma$ in the non stable and in the stable homotopy groups respectively. Using the maps $\mathcal{R}$

and & between these spectral sequences we can write $\& \circ \partial_1 = {}^s\partial_1 \circ \& = 0$ and so $\mathcal{R} \circ \& \circ \partial_1 = 0$ but $\mathcal{R} \circ \& =$ identity. Eventually we have $\partial_1 = 0$. □

Lemmas 8 and 9 imply Claim 14. □

CLAIM 15.

$$d^3_{4,t} = 0$$

PROOF. This can be deduced from the fact that

$$^s d^3_{4,t} = 0$$

using the maps $\mathcal{R}$ and & as above. □

*The* $E^\infty$ *members modulo* $\mathcal{C}(2)$ in the $t$-th row are the following.

For $k$ even

$$\frac{\pi_t(\Gamma)}{\Omega_{t-2k}} \qquad 0 \qquad 0 \qquad 0,$$

for $k$ odd

$$\pi_t(\Gamma) \qquad \Omega_{t-2k} \qquad 0 \qquad \Omega^{3\gamma}_{t+1-3k}.$$

## 9. Proof of Theorem 1

Now we show that having computed the $E^\infty$ terms of this spectral sequence, Theorem 1 follows immediately. The easiest way to see this is to use part (3) of Leray's theorem (see [4], page 133) which we recall here.

THEOREM (Leray).

$$E^{p,q}_\infty \approx \frac{\operatorname{Im}\left\{ H_{p+q}(X_p) \to H_{p+q}(X) \right\}}{\operatorname{Im}\left\{ H_{p+q}(X_{p-1}) \to H_{p+q}(X) \right\}}$$

*where*

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_{k-1} \subset X_k = X$$

*is a filtration.*

(Repeating what we have said about the spectral sequence for homotopy groups in Section 5 we replace here the homology groups $H$ by the homotopy groups $\pi$.)

Our spectral sequence converges to the group $\pi_t\big(X(k)\big) \approx \operatorname{Imm}^{SO}(t - - k, k+1)$.

Let $n$ and $k$ be fixed as in the formulation of Theorem 1. *For $k$ even* on the skew line $p + q = n + k$ we have the group $\pi_{n+k}\big(\Gamma(k)\big) / \Omega_{n-k}$ standing in the first coloumn as the only non zero group. By Leray's theorem this is the image of the map $\pi_{n+k}\big(\Gamma(k)\big) \to \pi_{n+k}\big(X(k)\big)$ i.e. the image of the map $\mathrm{Imm}^{SO}(n,k) \to \mathrm{Imm}^{SO}(n,k+1)$.

Since all the other columns contain only zero groups this map is epimorphic and we get the exact sequence for $k$ even:

$$0 \to \Omega_{n-k} \to \mathrm{Imm}^{SO}(n,k) \xrightarrow{\varphi_{n,k}} \mathrm{Imm}^{SO}(n,k+1) \to 0$$

Retracking the computation we see that the kernel group $\Omega_{n-k}$ arises as the homotopy group of the quotient space $Z/\Gamma$ which is the Thom space of the normal bundle of the submanifold formed by the $\Sigma^{1,0}$ points. This implies the geometric interpretation of this kernel (part (4) of Theorem 1).

*For $k$ odd* we have on the skew line $p+q=n+k$ the groups

$$\pi_{n+k}(\Gamma) \qquad \Omega_{n-k-1} \qquad 0 \qquad \Omega^{3\gamma}_{n-2k-2}.$$

By the theorem of Leray mentioned above we conclude that the map $\pi_{n+k}(\Gamma) \to \pi_{n+k}(X)$ is monomorphic, i.e. the map $\mathrm{Imm}^{SO}(n,k) \to \mathrm{Imm}^{SO}(n,k+1)$ is monomorphic and its cokernel is $\Omega_{n-k-1} \oplus \Omega_{n-2k-2}$. Now the exact sequence

$$0 \to \mathrm{Imm}^{SO}(n,k) \xrightarrow{\varphi_{n,k}} \mathrm{Imm}^{SO}(n,k+1) \xrightarrow{\sigma_1 \oplus \sigma_{11}} \Omega_{n-k-1} \oplus \Omega_{n-2k-2} \to 0$$

follows.

Retracking the computation we see that the groups $\Omega_{n-k-1}$ and $\Omega_{n-2k-2}$ arise from the homotopy groups of qoutient spaces $Z/\Gamma$ and $X/Y$ which are the Thom spaces of the normal bundles of submanifolds formed by the $\Sigma^{1,0}$ and $\Sigma^{1,1}$ singular points respectively. This implies the geometric interpretations of the maps $\sigma_1$ and $\sigma_{1,1}$.

Theorem 1 is proved. $\square$.

PROOF OF THE MAIN COROLLARY. First we prove

LEMMA 10. *If $k$ is odd then any self-tranverse immersion $f : M^n \to R^{n+k}$ has a 0-cobordant double point set.*

PROOF. Let us consider the map $f \times f : M \times M \to R^{n+k} \times R^{n+k}$ restricted to $M \times M \setminus U$ where $U$ is a small neighbourhood of the diagonal in $M \times M$. The map $f \times f \mid M \times M \setminus U$ is transversal to the diagonal of $R^{n+k} \times R^{n+k}$ and its preimage consists of the pairs of double points:

$$\tilde{\Delta}(f) = \big\{ (x,y) \mid f(x) = f(y) \ \text{ and } \ x \neq y \big\}.$$

This set is an oriented $n-k$ dimensional oriented manifold invariant under the involution $M \times M \to M \times M$ induced by the change of orders $(x,y) \to (y,x)$. This involution changes the orientation on $\tilde{\Delta}(f)$ (because $k$ is odd). An oriented manifold with an orientation reversing involution is 0-cobordant. □

LEMMA 11. *Let $f : M^n \to R^{n+k}$ be a generic immersion where $k$ is even. Then the double point set $\tilde{\Delta}(f)$ of $f$ is cobordant (in oriented sense) to the singularity manifold $\Sigma^1(\pi \circ f)$, where $\pi$ is a projection into a hyperplane, i.e.*

$$\tilde{\Delta}(f) \sim \Sigma^1(\pi \circ f).$$

PROOF. See Remark 1. □

NOTATION. Let $J(n,k)$ denote the subgroup of $\mathrm{Imm}^{SO}(n,k)$ formed by the cobordism classes of those (self-transverse) immersions, which have 0-cobordant double-point sets. (If a (self-transverse) immersion has 0-cobordant double-point set then any other (self-transverse) immersion in its cobordism class also has 0-cobordant double-point set.)

By Theorem 1 and Lemma 10 we have the following isomorphism mod $C(2,3)$:

$$\mathrm{Imm}^{SO}(n,k) \approx \begin{cases} J(n,k), & \text{for } k \text{ odd} \\ J(n,k) \oplus \Omega_{n-k}, & \text{for } k \text{ even.} \end{cases}$$

Let $k$ be any even natural number. Then $J(n,k) \approx J(n,k+1)$ and

$$0 \to J(n,k-1) \to J(n,k) \xrightarrow{\sigma_{1,1}} \Omega_{n-2k} \to 0.$$

Therefore for any even $k$ greater than 2

$$0 \to J(n,k-2) \to J(n,k) \xrightarrow{\sigma_{1,1}} \Omega_{n-2k} \to 0.$$

LEMMA 12. a) *Given an immersion $f : M^n \to R^{n+k}$ with $k$ even and denoting by $\pi$ the projection into a hyperplane, the homology class realised by the submanifold $\Sigma^{1,1}(\pi \circ f)$ is dual to the top normal Pontrjagin class $\bar{p}_{k/2}$.*

b) *There is an automorphism $\theta$ of the linear space formed by all polynomials of total degree $n-k$ in the variables $p_1, p_2, \dots$, where $\deg p_i = 4i$, such that for any polynomial $\omega(p_1, \dots)$ of degree $n-k$ we have the equality*

$$\left\langle \omega\left(\bar{p}(\Sigma^{1,1})\right), [\Sigma^{1,1}] \right\rangle = \left\langle \bar{p}_{k/2}(M) \cup \theta\left(\omega(\bar{p}(M))\right), [M] \right\rangle.$$

*Here $\Sigma^{1,1}$ denotes the manifold $\Sigma^{1,1}(\pi \circ f)$ and $\omega\left(\bar{p}(\Sigma^{1,1})\right)$ denotes the rational cohomology class in $H^*(\Sigma^{1,1}; Q)$ obtained from $\omega(p_1, \dots)$ by substituting $\bar{p}_i(\Sigma^{1,1})$ for $p_i$. Similarly we define $\omega\left(\bar{p}(M)\right)$. In other words the normal Pontrjagin numbers of $\Sigma^{1,1}$ coincide with those of $M$ corresponding to the polynomials divisible by $\bar{p}_{k/2}$.*

PROOF. We prove the following more general statement:

PROPOSITION. *For any natural r*

$$\mathcal{D}\left[\Sigma^{1_r}(\pi \circ f)\right] = \bar{\chi}^r$$

*where $\bar{\chi}$ is the normal Euler class of the immersion $f$ and $\mathcal{D}$ is the Poincare duality operator.*

PROOF. *Induction on r.* First for $r = 1$. Let us call the chosen hyperplane (onto which the projection $\pi$ maps) horizontal and the orthogonal direction vertical. Let us choose for each $x \in M$ a vertical upward directed short vector starting at the point $f(x)$ and let us project it orthogonally into the normal fibre at $f(x)$. This projection gives a section of the normal bundle vanishing precisely at the points of the singular set of $\pi \circ f$. For generic immersions this section will be transversal to the zero section, therefore

$$\mathcal{D}\left[\Sigma^{1}(\pi \circ f)\right] = \bar{\chi}.$$

*The induction step.* Suppose that for $r$ we know already the claim. Let us choose a direction different from the vertical. Let $\pi'$ be the projection in this direction.

Let us consider the normal bundle of $f$ pulled back to $\Sigma^{1_r}$. The projection of a small vector field parallel to the chosen non vertical direction into the pulled back normal bundle gives a section transversal to the zero section and its zero set is $\Sigma^{1_r}(\pi \circ f) \cap \Sigma^1(\pi' \circ f)$. The homology class in $M^n$ realized by this submanifold is

$$\left[\Sigma^{1_r}(\pi \circ f) \cap \Sigma^1(\pi' \circ f)\right] = \mathcal{D}(\bar{\chi}^r \cup \bar{\chi}) = \mathcal{D}\bar{\chi}^{r+1}.$$

We get a cobordant submanifold of $\Sigma^{1_r}$ (cobordant inside $\Sigma^{1_r}$) if we choose any other generic direction instead of $\pi'$. Especially we get the same homology class realized by the zero set if we choose the vertical direction for $\pi'$ too. (For generic immersions the vertical direction will be generic.) But the zero set in this case will be precisely the set $\Sigma^{1_{r+1}}$. Part a) of Lemma 12 is proven.

PROOF OF PART b). Let us denote the $\Sigma^{1,1}$-prim map $\pi \circ f$ by $g$. It follows from the description of $\Sigma^{1,1}$-prim maps that the stable normal bundle of $g(\Sigma^{1,1}(g))$ is stably isomorphic to the triple of the normal bundle of $f$ restricted to $\Sigma^{1,1}$. (Here we identified $\Sigma^{1,1}$ with its image $g(\Sigma^{1,1})$.) Therefore if $i$ is the inclusion of $\Sigma^{1,1}$ in $M^n$ then for the total normal Pontrjagin classes the following holds: $\bar{p}(\Sigma^{1,1}) = \left(i^*\bar{p}(M)\right)^3$. Let us denote by $q_s$ the class $i^*\left(\bar{p}_s(M)\right)$. Then $\bar{p}_s(\Sigma^{1,1}) = 3q_s +$ polynomial of $q_1, q_2, \dots, q_{s-1}$. Therefore any monomial $\omega$ in the variables $\bar{p}_1(\Sigma^{1,1}), \bar{p}_2(\Sigma^{1,1}), \dots$ of the degree $(n - - k)/4 = \dim \Sigma^{1,1}$ can be expressed in a unique way as a homogeneous polynomial $\theta(q_1, q_2, \dots)$. Then

$$\left\langle \omega\left(\bar{p}_1(\Sigma^{1,1}), \bar{p}_2(\Sigma^{1,1}), \dots\right), [\Sigma^{1,1}]\right\rangle = \left\langle \theta(q_1, q_2, \dots), [\Sigma^{1,1}]\right\rangle =$$

$$= \langle i^*\theta(\bar{p}_1(M), \bar{p}_2(M), \dots), [\Sigma^{1,1}] \rangle = \langle \theta(\bar{p}_1(M), \bar{p}_2(M), \dots), i_*[\Sigma^{1,1}] \rangle =$$

$$= \langle \theta(\bar{p}_1(M), \bar{p}_2(M), \dots), \mathcal{D}\bar{p}_{k/2}(M) \rangle =$$

$$= \langle \bar{p}_{k/2}(M) \cup \theta(\bar{p}_1(M), \bar{p}_2(M), \dots), [M] \rangle$$

Part b) is proven. □

Obviously if $k \geqq n + 2$ then $J(n,k) \approx \Omega_n$ and so the Main Corollary holds for big enough $k$. Now we deduce the Main Corollary for arbitrary $k$ by an induction going downward using the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \longrightarrow J(n,k-2) & \longrightarrow & J(n,k) & \xrightarrow{\sigma_{1,1}} & \Omega_{n-2k} \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow P(n,k-2) & \longrightarrow & P(n,k) & \longrightarrow & Z^{\pi((n-2k)/4)}
\end{array}$$

Here $\pi(x)$ denotes the number of partitions of $x$. ($\pi(x) = 0$ if $x$ is not a natural number.)

The right side vertical arrow is the mod $\mathcal{C}(2)$ monomorphism given by the normal Pontrjagin numbers, the midle arrow is an isomorphism modulo $\mathcal{C}(2,3)$ by the hypothesis of the induction. The map $P(n,k) \to Z^{\pi(n-2k)/4)}$ is given by normal Pontrjagin numbers divisible by $\bar{p}_{k/2}$.

The arguments above show that the natural map $J(n,k) \to P_{n,k}$ that associates with the cobordism class of an immersion that of its domain manifold maps the subgroup $J(n,k-2)$ in $P_{n,k-2}$, therefore the left side arrow arises and the left hand square of the diagram commutes. The commutativity of the right hand square follows from part b) of Lemma 12. By the commutativity of this diagram the map $J(n,k-2) \to P(n,k-2)$ is an isomorphism (modulo $\mathcal{C}(2,3)$.) The proof of the Main Corollary is finished. □.

## 10. Proof of Theorem 2

Let us define the spaces $\bar{\Gamma}(k) \subset \bar{Z}(k) \subset \bar{Y}(k) \subset \bar{X}(k)$ as follows:

(1) $\bar{\Gamma}(k) = \Omega SMSO(k)$.

(2) $\bar{Z}(k)$ can be obtained from $\bar{\Gamma}(k)$ in the same way as $Z(k)$ was obtained from $\Gamma(k)$. More precisely: Let us recall that the space $Z(k)$ was obtained by attaching a disc bundle $D(\xi)$ to $\Gamma(k)$ by a gluing map $\rho : \partial D(\xi) \to \Gamma(k)$. Notice that $\bar{\Gamma}(k) \subset \Gamma(k)$ and the map $\rho$ can be decomposed as follows:

$$\rho : \partial D(\xi) \xrightarrow{\bar{\rho}} \bar{\Gamma}(k) \subset \Gamma(k).$$

Now use the map $\bar{\rho}$ as a gluing map to attach $D(\xi)$ to $\bar{\Gamma}(k)$. The space obtained in this way will be $\bar{Z}(k)$.

(3) Analogously $\bar{Y}(k)$ and $\bar{X}(k)$ can be obtained from $\bar{Z}(k)$ and $\bar{Y}(k)$ in the same way as $Y(k)$ and $X(k)$ were obtained from $Z(k)$ and $Y(k)$, respectively.

REMARK 9. It follows from this description of the spaces $\bar{X}(k)$, $\bar{Y}(k)$, $\bar{Z}(k)$, $\bar{\Gamma}(k)$ that there exists a map

$$\vartheta : \big(\bar{X}(k), \bar{Y}(k), \bar{Z}(k), \bar{\Gamma}(k)\big) \to \big(X(k), Y(k), Z(k), \Gamma(k)\big)$$

such that

(a) $\vartheta|_{\bar{\Gamma}(k)}$ is the natural inclusion of $\bar{\Gamma}(k) = \Omega SMSO(k)$ into $\Gamma(k) = \Omega^\infty S^\infty MSO(k)$.

(b) The spaces $\bar{X} \setminus \bar{Y}, \bar{Y} \setminus \bar{Z}$, $\bar{Z} \setminus \Gamma$ are mapped by $\vartheta$ in a 1-1 way onto the spaces $X \setminus Y$, $Y \setminus Z$, $Z \setminus \Gamma$ respectively.

PROPOSITION. *If $n < 3k$ then*

$$\pi_{n+k}\big(\bar{X}(k)\big) \approx \pi_{n+k+1}\big(MSO(k+1)\big) = \mathrm{Emb}(n, k+1)$$

*and*

$$\pi_{n+k}\big(\bar{\Gamma}(k)\big) \approx \pi_{n+k+1}\big(SMSO(k)\big) = \mathrm{Emb}(n, k \oplus 1).$$

PROOF. See the analogous propositions on $X(k)$ and $\Gamma(k)$ in [15]. □

To prove Theorem 2 we have to compute the homomorphism $\pi_*\big(\bar{\Gamma}(k)\big) \to \pi_*\big(\bar{X}(k)\big)$ induced by the inclusion $\bar{\Gamma}(k) \subset \bar{X}(k)$. For this purpose we should consider the spectral sequences of the filtration $\bar{\Gamma}(k) \subset \bar{Z}(k) \subset \bar{Y}(k) \subset \bar{X}(k)$ first in the stable and then in the non stable homotopy groups. In the stable homotopy groups the computation is quite similar to the case of the immersions so we do not repeat it. This computation gives the following result.

CLAIM 16. *The group* $\mathrm{Coker}(\pi^s_*\big(SMSO(k)\big) \to \pi^s_*\big(MSO(k+1)\big)$ *is a finite* 2*-primary group if $k$ is even and it has only* 2*- and* 3*-primary torsions if $k$ is odd.*

Now we sketch the computation of the spectral sequence in the non stable homotopy groups.

(A) Let $k$ be even. *Then modulo $\mathcal{C}(2)$ the $E^1$ groups are the following:*

$$E^1_{1,t-1} \approx \pi_t\big(\bar{\Gamma}(k)\big) \approx \pi_t\big(\Omega SMSO(k)\big),$$

$$E^1_{2,t-1} \approx \pi_{t+1}(\bar{Z}, \bar{\Gamma}) \approx \Omega_{t-2k} \oplus \Omega^{3\gamma}_{t-3k} \oplus \bar{G}_{t+1}$$

Here $\bar{G}_{t+1}$ is the kernel of the natural epimorphic map

$$\pi_{t+1}(\bar{Z}, \bar{\Gamma}) \to \pi_{t+1}(\bar{Z}/\bar{\Gamma}).$$

The groups $E^1_{3,*}$ and $E^1_{4,*}$ are the same as for the filtration $X \supset Y \supset Z \supset \Gamma$:

$$E^1_{3,t-1} \approx \pi_{t+2}(\bar{Y}/\bar{Z}) = \pi_{t+2}(Y/Z) \approx \oplus \{\Omega_a \otimes \Omega_b | a+b = t+1-3k\}$$

$$E^1_{4,t-1} \approx \pi_{t+3}(\bar{X}/\bar{Y}) = \pi_{t+3}(X/Y) \approx \Omega_{t+1-3k}.$$

*The differentials for k even.* In computing the differetials of this spectral sequence, the main tool will be the map $\vartheta : \bar{X}(k) \to X(k)$ which respects the filtrations of these spaces and so it induces a map $\vartheta_*$ of the corresponding spectral sequences. Using $\vartheta_*$ we obtain

(1) $d^1_{4,t} = 0$. $d^1_{3,t}$ is monomorphic and its image is contained in the group $\bar{G}_{t+3}$. $d^1_{2,t-1}$ restricted to $\Omega_{t-2k}$ is monomorphic.

(2) In the same way as in the case of immersions it can be proved that $d^2_{4,t-2}$ is an isomorphism modulo $\mathcal{C}(2)$ onto $\Omega^{3\gamma}_{t-3k}$. This implies that $d^1_{2,t-1}$ is zero modulo $\mathcal{C}(2)$ on $\Omega^{3\gamma}_{t-3k}$.

(3) Let us consider the spectral sequence of the filtration $\bar{X}(k) \supset \bar{Y}(k) \supset \supset \bar{Z}(k) \supset \bar{\Gamma}(k)$ in the groups $\pi^s_*$ and the maps & and $\mathcal{R}$ analogous to those defined for immersions. We obtain that

$$d^1_{2,t}(\bar{G}_{t+2}) = 0.$$

(4) By a theorem of Burlet [3] the map

$$\pi_*(SMSO(k)) \to \pi_*(MSO(k+1))$$

is a rational epimorphism. Hence the factorgroup $\bar{G}_*/\operatorname{im} d^1_{3,*}$ is finite.

$$\text{(5)} \qquad \begin{aligned} &\operatorname{Coker} \pi_*(SMSO(k)) \to \pi_*(MSO(k+1)) \subset \\ &\subset \operatorname{Coker} \pi^s_*(SMSO(k)) \to \pi^s_*(MSO(k+1)) \end{aligned}$$

and the last group belongs to $\mathcal{C}(2,3)$ by Claim 16. Then $E^\infty_{2,*} = \bar{G}_*/\operatorname{im} d^1_{3,*} \in \in \mathcal{C}(2,3)$ because this group is a factor in a subgroup chain of the previous group.

*So we get the following $E^\infty$ groups for k even modulo $\mathcal{C}(2,3)$:*

$$\frac{\pi_t(\Omega SMSO(k))}{\Omega_{t-2k}} \quad | \quad 0 \quad | \quad 0 \quad | \quad 0 \quad |$$

This implies Theorem 2 for $k$ even.

(B) Let $k$ be odd. *Then the $E^1$ members are the following:*

$$\pi_t(\Omega SMSO(k)) \quad | \; \Omega_{t-2k} \oplus \bar{G}_{t+1} \; | \; \oplus \{\Omega_a \otimes \Omega_b\} \; | \; \Omega^{3\gamma}_{t+1-3k}.$$

*The differentials for $k$ odd:*
$d^1_{4,t} = 0$
$d^1_{2,t}$ restricted to $\Omega_{t-2k}$ is monomorphic.
$d^1_{2,t}(\bar{G}_{t+2}) = 0$ as before.
$d^1_{3,t}$ is isomorphic onto $\bar{G}_{t+2}$ modulo $\mathcal{C}(2,3)$.
$d^2_{4,t} = 0$.
$d^3_{4,t}$ is monomorphic modulo $\mathcal{C}(2,3)$.

(*Proof*: Because of the theorem of Burlet mentioned above, $\ker d^3_{4,t}$ is finite. On the other hand the torsion group of the domain of this differential belongs to $\mathcal{C}(2,3)$.)

*Hence the $E^\infty$ members are the following:*
$E^\infty_{1,t-1} = \pi_t(\Omega SMSO(k)) / \Omega_{t-2k} \oplus \Omega_{t-3k-2}$
$E^\infty_{s,*} = 0$ for $s > 1$.

This implies Theorem 2 for $k$ odd too. □

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# RELATIONS BETWEEN NEW TOPOLOGIES OBTAINED FROM OLD ONES

T. HATICE YALVAÇ (Ankara)

So far, topologies $\tau_s$, $\tau_\theta$, $\tau^\alpha$, $\tau_{PO(X)}$ were defined on $X$ by using the given topology $\tau$ on $X$.

We construct new topologies on $X$ by using supratopologies on $X$. We obtain the topology $\tau_{SPO(X)}$ in this way and investigate some relations between $\tau$, $\tau_s$, $\tau_\theta$, $\tau^\alpha$, $\tau_{PO(X)}$, and $\tau_{SPO(X)}$.

$\dot{A}$, $\bar{A}$ will stand for the interior and the closure of the subset $A$ of $X$ respectively in the topological space $(X,\tau)$ only.

In a topological space $(X,\tau)$, the following families of subsets of $X$ are well known:

(1) $\tau^\alpha = \{A \subset X : A \subset \dot{\bar{\dot{A}}}\}$. Sets in $\tau^\alpha$ were introduced by Njåstad [8] and called $\alpha$-sets.

(2) $SO(X) = \{A \subset X : A \subset \bar{\dot{A}}\}$. Sets in $SO(X)$ were introduced by Levine [4] and called semi-open sets.

(3) $PO(X) = \{A \subset X : A \subset \dot{\bar{A}}\}$. Sets in $PO(X)$ were introduced by Mashhour et al. [6] and called pre-open sets.

(4) $SPO(X) = \{A \subset X : A \subset \bar{\dot{\bar{A}}}\}$. Sets in $SPO(X)$ were introduced by Andrijević [1] and called semi-preopen sets.

(5) $A$ is called regular open if $A = \dot{\bar{A}}$ [2, Problem 22, p. 92], $RO(X) = \{A \subset X : A = \dot{\bar{A}}\}$.

The topology $\tau_s$ on $X$ which has as its base $RO(X)$ is called semiregularization topology of $(X,\tau)$.

(6) $\tau_\theta = \{A \subset X : A \text{ is } \theta\text{-open}\} = \{A \subset X : A = \theta - \operatorname{int} A = \{x \in A :$ there exists an open set $U$ such that $x \in U \subset \bar{U} \subset A\}\}$.

$\theta$-open sets were introduced by Long and Herrington [5].

A set $A$ is called $\alpha$-closed (resp. semi-closed, pre-closed, semi-preclosed, regular closed) if $X - A$ is an $\alpha$-set (resp. semi-open, pre-open, semi-preopen, regular open set).

$\alpha$-closure (resp. semi-closure, pre-closure, semi-preclosure) of a set $A$ is the intersection of all $\alpha$-closed (resp. semi-closed, preclosed, semi-preclosed) sets containing $A$, and we will denote these sets respectively by $\alpha\mathrm{cl}\, A$, $\mathrm{scl}\, A$, $\mathrm{pcl}\, A$, $\mathrm{spcl}\, A$.

It is well known that $\tau^\alpha$ is a topology [8] and $\tau^\alpha = PO(X) \cap SO(X)$ [9], $PO(X) \cup SO(X) \subset SPO(X)$ [1].

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$X$ is extremely disconnected, shortly e.d. (means every regular open set is closed), iff $\tau^\alpha = SO(X)$ [8]. $X$ is e.d. iff $SO(X) \subset PO(X)$ [3].

$\tau_\theta$ is a topology and at the same time $(X, \tau)$ is almost regular iff $\tau_s = \tau_\theta$ [5]. $(X, \tau)$ is semi-regular iff $\tau_s = \tau$, $X$ is regular iff it is semi-regular and almost regular.

DEFINITION [7]. Let $\mathcal{A} \subset P(X)$. If $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$ and $\mathcal{A}$ is closed under arbitrary union, then $\mathcal{A}$ is called a supratopology.

It is known that $SO(X)$, $PO(X)$, $SPO(X)$ are supratopologies [8], [6], [1].

THEOREM 1. *If $\mathcal{A}$ is supratopology on $X$, then $\tau_\mathcal{A} = \{T \subset X : A \in \mathcal{A} \Rightarrow \Rightarrow T \cap A \in \mathcal{A}\}$ is a topology and $\tau_\mathcal{A} \subset \mathcal{A}$.*

PROOF. $A \in \mathcal{A} \Rightarrow \emptyset \cap A = \emptyset \in \mathcal{A}$ and $X \cap A = A \in \mathcal{A}$. Hence $\emptyset \in \tau_\mathcal{A}$ and $X \in \tau_\mathcal{A}$.

Let $\{T_i\} \subset \tau_\mathcal{A}$ and $A \in \mathcal{A}$. $(\bigcup T_i) \cap A = \bigcup (T_i \cap A) \in \mathcal{A}$. We have $\bigcup T_i \in \tau_\mathcal{A}$.

Let $T_1 \in \tau_\mathcal{A}$, $T_2 \in \tau_\mathcal{A}$ and $A \in \mathcal{A}$. Then $(T_1 \cap T_2) \cap A = T_1 \cap (T_2 \cap A)$. Since $T_2 \cap A \in \mathcal{A}$ and $T_1 \in \tau_\mathcal{A}$, we have $T_1 \cap (T_2 \cap A) \in \mathcal{A}$. Thus $\tau_\mathcal{A}$ is a topology.

If $T \in \tau_\mathcal{A}$, then, since $X \in \mathcal{A}$, $T \cap X = T \in \mathcal{A}$.

COROLLARY 1. *$\tau_{SPO(X)} = \{T \subset X : A \in SPO(X) \Rightarrow T \cap A \in \in SPO(X)\}$ is a topology and $\tau_{SPO(X)} \subset SPO(X)$.*

If we take $\mathcal{A}$ as $SO(X)$ $(PO(X))$ then we get the topology $\tau^\alpha$ [8, Proposition 1] ($\tau_\gamma$ defined by Andrijević [1]).

We know that $\tau \subset \tau^\alpha \subset \tau_{PO(X)} \subset PO(X) \subset SPO(X)$ [1]. Clearly $\tau^\alpha = \tau_{PO(X)} \cap SO(X)$.

THEOREM 2. *If a supratopology $\mathcal{A}$ is a topology then $\tau_\mathcal{A} = \mathcal{A}$.*

COROLLARY 2. (1) (Njåstad [8]). *$\tau^\alpha = SO(X)$ iff $SO(X)$ is a topology.*
(2) *$\tau_{PO(X)} = PO(X)$ iff $PO(X)$ is a topology.*
(3) *$\tau_{SPO(X)} = SPO(X)$ iff $SPO(X)$ is a topology.*

COROLLARY 3. *If $PO(X) \subset SO(X)$ then*

$$\tau_{PO(X)} = \tau_{SO(X)} = \tau^\alpha = PO(X).$$

PROOF. If $PO(X) \subset SO(X)$ then $\tau^\alpha = PO(X)$. It is clear now since $\tau^\alpha = \tau_{SO(X)}$ and from Corollary 2.

But as we can see from the following example, we can have $\tau_{PO(X)} = = \tau_{SO(X)}$ without $PO(X) \subset SO(X)$.

EXAMPLE 1. Let $X = \mathbf{N}$, $\tau$ the cofinite topology on it.

$SO(X) = \tau = \tau_{SO(X)}$, $PO(X) = \{A \subset X : A \text{ is infinite or void}\}$.

If $A \in PO(X)$ and $A \notin \tau$ then $A \notin \tau_{PO(X)}$, because $X \setminus A$ is infinite and if we choose $B = (X \setminus A) \cup C$ (where $C$ is a finite subset of $A$), $B \in PO(X)$ but $A \cap B \notin PO(X)$. Hence we have $\tau_{PO(X)} = \tau$.

LEMMA 1. *If $A \in \tau^\alpha$ and $B \in SPO(X)$ then $A \cap B \in SPO(X)$.*

PROOF. Let $A \in \tau^\alpha$ and $B \in SPO(X)$.

$$A \cap B \subset \dot{\bar{\dot{A}}} \cap \bar{\dot{\bar{B}}} \subset \left(\dot{\bar{\dot{A}}} \cap \dot{\bar{B}}\right)^{-} = \left(\bar{\dot{A}} \cap \dot{\bar{B}}\right)^{\bar{\cdot}} \subset \left(\dot{A} \cap \dot{\bar{B}}\right)^{\bar{\dot{-}}} = \left(\dot{A} \cap \bar{B}\right)^{\dot{\bar{\dot{-}}}} =$$

$$= \left(\dot{A} \cap \bar{B}\right)^{\bar{\cdot}} \subset \left(\dot{A} \cap B\right)^{\bar{\dot{-}}} \subset \left(A \cap B\right)^{\bar{\dot{-}}}.$$

COROLLARY 4. $\tau^\alpha \subset \tau_{SPO(X)} \subset SPO(X)$.

THEOREM 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be supratopologies on $X$.*
(1) *If $\mathcal{A} = \mathcal{B}$ then $\tau_{\mathcal{A}} = \tau_{\mathcal{B}}$.*
(2) *$\mathcal{A} \subset \mathcal{B}$ does not imply $\tau_{\mathcal{A}} \subset \tau_{\mathcal{B}}$ or $\tau_{\mathcal{B}} \subset \tau_{\mathcal{A}}$.*
(3) *It can be $\tau_{\mathcal{A}} = \tau_{\mathcal{B}}$ without $\mathcal{A} = \mathcal{B}$.*

In Example 1, $\tau_{PO(X)} = \tau_{SO(X)}$ but $PO(X) \neq SO(X)$.

EXAMPLE 2. Let

$$X = \{a,b,c\} \qquad \tau = \big\{\emptyset, X, \{b\}, \{c\}, \{b,c\}\big\},$$

$$SO(X) = \big\{\emptyset, X, \{b\}, \{c\}, \{b,c\}, \{a,b\}, \{a,c\}\big\},$$

$$PO(X) = \tau. \qquad \tau^\alpha = SO(X) \cap PO(X) = \tau = \tau_{SO(X)}.$$

Let $\tau^* = \big\{\emptyset, X, \{b\}, \{c\}, \{b,c\}, \{a,b\}\big\}$. $\tau^* \subset SO(X)$. Since $\tau^*$ is a topology on $X$, $\tau_{\tau^*} = \tau^*$. But $\tau_{\tau^*} \not\subset \tau_{SO(X)}$.

EXAMPLE 3. Let

$$X = \{a,b,c\}, \qquad \tau = \big\{\emptyset, X, \{a,b\}\big\},$$

$$\tau = SO(X) = \tau_{SO(X)}, \qquad PO(X) = \big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\big\},$$

$$\tau_{PO(X)} = \tau \cup \big\{\{a\}, \{b\}\big\},\ SO(X) \subset PO(X), \text{ but } \tau_{PO(X)} \not\subset \tau_{SO(X)}.$$

LEMMA 2. *If $U$ is open and closed then $U \in \tau_\theta$.*

PROOF. $x \in U \Rightarrow x \in U \subset \bar{U} \subset U$. Hence $U$ is $\theta$-open.

LEMMA 3. *$X$ is e.d. iff $SPO(X) = PO(X)$.*

PROOF. Let $X$ be e.d. $A \in SPO(X) \Rightarrow A \subset \bar{\dot{\bar{A}}} = \dot{\bar{A}}$. Hence $A \in PO(X)$. Since $PO(X) \subset SPO(X)$, we have $SPO(X) = PO(X)$. Conversely let $SPO(X) = PO(X)$. Since $SO(X) \subset SPO(X) \subset PO(X)$, $X$ is e.d.

COROLLARY 5. *$SO(X) \subset PO(X)$ iff $SPO(X) = PO(X)$.*

COROLLARY 6. *If $X$ is e.d. then*
(1) $\tau_s = \tau_\theta$,
(2) $\tau^\alpha = \tau_{SO(X)} = SO(X)$ [8],
(3) $\tau_{SPO(X)} = \tau_{PO(X)}$.

PROOF. (1) Since every regular open set is closed, $\tau_\theta$ contains the base of $\tau_s$. We have $\tau_s \subset \tau_\theta$. Hence $\tau_s = \tau_\theta$.
(3) $SPO(X) = PO(X)$ gives us $\tau_{SPO(X)} = \tau_{PO(X)}$.

COROLLARY 7. *If $SO(X) = PO(X)$ then $\tau_s = \tau_\theta$, $\tau^\alpha = \tau_{PO(X)} = = \tau_{SPO(X)} = PO(X) = SO(X) = SPO(X)$.*

THEOREM 4. *The followings are equivalent:*
(1) *each subset of $X$ is pre-open,*
(2) *for each $x \in X$, $\{x\} \subset \dot{\overline{\{x\}}}$,*
(3) *each subset of $X$ is pre-closed,*
(4) *each open set is pre-closed,*
(5) *each open set is closed (equivalently each closed set is open),*
(6) *each pre-open set is pre-closed,*
(7) *each pre-closed set is pre-open.*

PROOF. Since $PO(X)$ is a supratopology and a set $A$ is pre-closed iff $X - A$ is pre-open, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ is clear.

($4 \Rightarrow 5$) Let $U$ be an open set. Since $U$ is pre-closed, $\bar{U} = \bar{\dot{U}} \subset U$. So $U$ is closed.

($5 \Rightarrow 6$) Let $U$ be a pre-open set. $\bar{\dot{U}} = \dot{U} \subset U$. So $U$ is pre-closed.

($6 \Rightarrow 7$) is clear.

($6 \Rightarrow 5$) Let $U$ be an open set. $U$ is pre-open, semi-open and pre-closed. Since semi-open and pre-closed is regularly closed [3], $U$ is closed.

($5 \Rightarrow 1$) Let $U \subset X$. $\bar{U}$ is closed, $\dot{\bar{U}} = \bar{U} \supset U$. So $U$ is pre-open.

COROLLARY 8. *If every open set is closed in $(X, \tau)$ then*
(1) $\tau = \tau_s = \tau_\theta = \tau^\alpha = SO(X)$,
(2) $PO(X) = \tau_{SPO(X)} = \tau_{PO(X)} =$ *discrete topology on $X$.*

PROOF. (1) Since every open set is closed, every open set is $\theta$-open from Lemma 2. We know that $\tau_\theta \subset \tau_s \subset \tau$. Hence $\tau = \tau_s = \tau_\theta$. $A \in \tau^\alpha \Rightarrow \Rightarrow A \subset \dot{\bar{\dot{A}}} = \dot{A} \Rightarrow A$ is open, and since X is e.d. $\tau = \tau^\alpha = SO(X)$. Thus $\tau = \tau_s = \tau_\theta = \tau^\alpha = SO(X) = \tau_{SO(X)}$.

This corollary is interesting for an indiscrete space $(X, \tau)$.

EXAMPLE 4. *Let $X$ be any set and $\tau = \{\emptyset, X\}$. $\tau = \tau_s = \tau^\alpha = \tau_\theta = = SO(X) = \tau_{SO(X)}$, $PO(X) = SPO(X) = \tau_{SPO(X)} = \tau_{PO(X)} = P(X)$.*

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# ON A PROBLEM IN ADDITIVE NUMBER THEORY

A. SÁRKÖZY and E. SZEMERÉDI (Budapest), corresponding member of the Academy

**1.** Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sequences of non-negative integers. Denote the number of solutions of

$$(1)\qquad a+b=n, \qquad a\in\mathcal{A}, \qquad b\in\mathcal{B}$$

by $f(n)$, and denote the counting function of the sequences $\mathcal{A}$ and $\mathcal{B}$ by $A(x)$ and $B(x)$, respectively:

$$A(x)=\sum_{\substack{a\leqq x\\ a\in\mathcal{A}}} 1 \qquad \text{and} \qquad B(x)=\sum_{\substack{b\leqq x\\ b\in\mathcal{B}}} 1.$$

Assume that there exists an integer $n_0$ with

$$(2)\qquad f(n)\geqq 1 \quad \text{for} \quad n>n_0,$$

i.e., every large integer $n$ can be represented in the form (1). This implies that for all $x$ we have

$$(3)\qquad A(x)B(x)=\sum_{\substack{a\leqq x\\ a\in\mathcal{A}}}\ \sum_{\substack{b\leqq x\\ b\in\mathcal{B}}} 1 \geqq \sum_{\substack{a+b\leqq x\\ a\in\mathcal{A},b\in\mathcal{B}}} 1 \geqq$$

$$\geqq \sum_{n_0<n\leqq x}\ \sum_{\substack{a+b=n\\ a\in\mathcal{A},b\in\mathcal{B}}} 1 \geqq \sum_{n_0<n\leqq x} 1 = [x]-n_0$$

and hence

$$(4)\qquad \liminf_{x\to+\infty}\frac{A(x)B(x)}{x}\geqq 1.$$

Starting out from a problem of Hanani and Erdős [2], [3], Danzer [1] conjectured that if also

$$(5)\qquad \limsup_{x\to+\infty}\frac{A(x)B(x)}{x}\leqq 1$$

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holds and hence, in view of (4),

$$\lim_{x\to+\infty} \frac{A(x)B(x)}{x} = 1, \tag{6}$$

then we have

$$\liminf_{x\to+\infty} \big( A(x)B(x) - x \big) = +\infty. \tag{7}$$

(See also [4], p. 10, [5], p. 75 and [6].) The goal of this paper is to prove this conjecture.

THEOREM. *If $\mathcal{A}$ and $\mathcal{B}$ are infinite sequences satisfying* (2) *and* (5), *then* (7) *must hold.*

**2.** PROOF OF THE THEOREM. We start out from the indirect assumption that $\mathcal{A}$ and $\mathcal{B}$ satisfy (2) and (5), however, (7) does not hold. In view of (3) this implies that the limit on the left hand side of (7) is finite:

$$-\infty < \liminf_{x\to+\infty} \big( A(x)B(x) - x \big) < +\infty. \tag{8}$$

By (2), it follows that

$$+\infty > \liminf_{x\to+\infty} \big( A(x)B(x) - x \big) = \liminf_{x\to+\infty} \Bigg( \Big( \sum_{\substack{a\leqq x\\ a\in\mathcal{A}}} 1 \Big) \Big( \sum_{\substack{b\leqq x\\ b\in\mathcal{B}}} 1 \Big) - x \Bigg) \geqq$$

$$\geqq \liminf_{x\to+\infty} \Bigg( \Big( \sum_{\substack{a+b\leqq x\\ a\in\mathcal{A}, b\in\mathcal{B}}} 1 \Big) - x \Bigg) = \liminf_{x\to\infty} \Big( \sum_{n=0}^{x} f(n) - x \Big) \geqq$$

$$\geqq \liminf_{x\to+\infty} \Big( \sum_{n=n_0+1}^{x} f(n) - x \Big) \geqq \liminf_{x\to+\infty} \Bigg( [x] - n_0 + \sum_{\substack{n_0<n\leqq x\\ f(n)>1}} 1 - x \Bigg) \geqq$$

$$\geqq \liminf_{x\to+\infty} \Bigg( \sum_{\substack{n_0<n\leqq x\\ f(n)>1}} 1 \Bigg) - (n_0+1)$$

and thus by (2), there exists an integer $n_1$ with

$$f(n) = 1 \qquad \text{for} \qquad n \geqq n_1. \tag{9}$$

Furthermore, since $\mathcal{A}$ and $\mathcal{B}$ are infinite, it follows from (5) that

$$A(x) = o(x) \qquad \text{and} \qquad B(x) = o(x). \tag{10}$$

Let us write

$$\liminf_{n\to+\infty}\big(A(n)B(n) - n\big) = L$$

($L$ is finite by (8)), and let $\mathcal{X} = \{x_1, x_2, \ldots\}$ (where $x_1 < x_2 < \ldots$) be a sequence of positive integers with

$$A(x_k)\,B(x_k) - x_k = L \qquad (k = 1, 2, \ldots). \tag{11}$$

To every $k$ we assign certain numbers $y_k, b^{(k)}$, etc., and we will study these numbers and their functions as $k \to +\infty$. Correspondingly, we write $\varphi \overset{k}{\to} +\infty$, $\varphi = o_k(\psi)$, $\varphi = O_k(\psi)$ (where $\varphi$ and $\psi$ depend on $k$) if $\lim\limits_{k\to+\infty} \varphi = +\infty$, $\lim\limits_{k\to+\infty} \frac{\varphi}{\psi} = 0$ and $\limsup\limits_{k\to+\infty} |\frac{\varphi}{\psi}| < +\infty$, respectively.

We may assume without loss of generality that the greatest term of $(\mathcal{A} \cup \mathcal{B}) \cap [0, x_k]$ belongs to $\mathcal{B}$, and let us denote this term by $b^{(k)}$. We put $y_k = x_k - b^{(k)}$. Then in view of (2) and (11) we have

$$\begin{aligned}
L = A(x_k)(x_k) - x_k &= \Bigg(\sum_{\substack{a \leqq x_k \\ a \in \mathcal{A}}} 1\Bigg)\Bigg(\sum_{\substack{b \leqq x_k \\ b \in \mathcal{B}}} 1\Bigg) - x_k \geqq \\
&\geqq \sum_{\substack{a+b \leqq x_k \\ a \in \mathcal{A}, b \in \mathcal{B}}} 1 + \Bigg(\sum_{\substack{y_k < a \leqq x_k \\ a \in \mathcal{A}}} 1\Bigg)\Bigg(\sum_{\substack{x_k - y_k \leqq b \leqq x_k \\ b \in \mathcal{B}}} 1\Bigg) - x_k = \\
&= \sum_{n=0}^{x_k} f(n) + \big(A(x_k) - A(y_k)\big) - x_k \geqq \\
&\geqq (x_k - n_0) + \big(A(x_k) - A(y_k)\big) - x_k = \big(A(x_k) - A(y_k)\big) - n_0.
\end{aligned} \tag{12}$$

Since $\mathcal{A}$ is infinite, we have $A(x_k) \overset{k}{\to} +\infty$. Thus (12) implies that

$$A(y_k) \overset{k}{\to} +\infty \tag{13}$$

and hence

$$y_k \overset{k}{\to} +\infty. \tag{14}$$

On the other hand, by (6) we have

$$\big(1+o_k(1)\big)x_k = A(x_k)B(x_k) = A\big(b^{(k)}\big)\,B\big(b^{(k)}\big) =$$

$$= \big(1+o_k(1)\big)\,b^{(k)} = \big(1+o_k(1)\big)\,(x_k-y_k)$$

and hence

$$y_k = o_k(x_k). \tag{15}$$

Furthermore, (12) implies that

$$A(x_k)-A(y_k) \leqq L+n_0. \tag{16}$$

Now we are going to show that

$$\max_{0\leqq z\leqq x_k}\big(B(z+y_k)-B(z)\big) = o_k(y_k). \tag{17}$$

(Note that $y_k \overset{k}{\to} +\infty$ by (14).) In fact, assume that contrary to (17), for some $\varepsilon$ and infinitely many $k$ there exists a $z_k$ with $0\leqq z_k < x$ and

$$B\,(z_k+y_k)-B\,(z_k) > \varepsilon y_k. \tag{18}$$

If $0\leqq z\leqq 2y_k$, then in view of (10) we have

$$B\,(z+y_k)-B(z)\leqq B\,(3y_k) = o\,(y_k)$$

so that for large $k$ (18) can not hold (with $z$ in place of $z_k$).

Assume now that (18) holds with some $z_k > 2y_k$. Let us form all the sums $a+b$ with $0\leqq a\leqq y_k$, $z_k < b\leqq z_k+y_k$. In view of (18), the number of these sums is

$$A(y_k)\big(B(z_k+y_k)-B(z_k)\big) > \varepsilon A\,(y_k)\,y_k,$$

and since $\mathcal{A}$ is infinite, for large $k$ this is greater than $2y_k$. On the other hand, all these sums belong to the interval $(z_k, z_k+2y_k]$ which contains $2y_k$ integers. Thus by the pigeon hole principle, there exist two equal sums:

$$a+b=a'+b'>z_k \qquad (a\in\mathcal{A}, a'\in\mathcal{A}, a\neq a', b\in\mathcal{B}, b'\in\mathcal{B}, b\neq b').$$

For large $k$ this contradicts (9), and this contradiction completes the proof of (17).

Next we will show that

$$B(x_k - y_k) - B(x_k - 2y_k) \xrightarrow{k} +\infty. \tag{19}$$

In fact, by (9), (15), (16) and (17), for $k \to +\infty$ we have

$$y_k = \sum_{n=x_k-y_k+1}^{x_k} f(n) = \sum_{\substack{x_k - y_k < a+b \leqq x_k \\ a\in\mathcal{A}, b\in\mathcal{B}}} 1 \leqq$$

$$\leqq \sum_{\substack{y_k < a \leqq x_k \\ a\in\mathcal{A}}} \Bigg( \sum_{\substack{x_k - a - y_k < b \leqq x_k - a \\ b\in\mathcal{B}}} 1 \Bigg) + \Bigg( \sum_{\substack{a \leqq y_k \\ a\in\mathcal{A}}} 1 \Bigg) \Bigg( \sum_{\substack{x_k - 2y_k < b \leqq x_k \\ b\in\mathcal{B}}} 1 \Bigg) =$$

$$= \sum_{\substack{y_k < a \leqq x_k \\ a\in\mathcal{A}}} \big( B(x_k - a) - B(x_k - a - y_k) \big) + A(y_k)\big( B(x_k - y_k) - B(x_k - 2y_k) \big) =$$

$$= \Bigg( \sum_{\substack{y_k < a \leqq x_k \\ a\in\mathcal{A}}} 1 \Bigg) o_k(y_k) + A(y_k)\big( B(x_k - y_k) - B(x_k - 2y_k) \big) =$$

$$= O_k(1) o_k(y_k) + A(y_k)\big( B(x_k - y_k) - B(x_k - 2y_k) \big) =$$

$$= o_k(y_k) + A(y_k)\big( B(x_k - y_k) - B(x_k - 2y_k) \big).$$

By (10) this implies (19).

By (2) and (11) we have

$$L = A(x_k)B(x_k) - x_k = \Bigg( \sum_{\substack{a \leqq x_k \\ a\in\mathcal{A}}} 1 \Bigg) \Bigg( \sum_{\substack{b \leqq x_k \\ b\in\mathcal{B}}} 1 \Bigg) - x_k \geqq$$

$$\geqq \sum_{\substack{a+b \leqq x_k \\ a\in\mathcal{A}, b\in\mathcal{B}}} 1 + \Bigg( \sum_{\substack{2y_k < a \leqq x_k \\ a\in\mathcal{A}}} 1 \Bigg) \Bigg( \sum_{\substack{x_k - 2y_k < b \leqq x_k - y_k \\ b\in\mathcal{B}}} 1 \Bigg) - x_k =$$

$$= \sum_{n=0}^{x_k} f(n) + \big( A(x_k) - A(2y_k) \big) \big( B(x_k - y_k) - B(x_k - 2y_k) \big) - x_k \geqq$$

$$\geqq \sum_{n=n_0+1}^{x_k} 1 + \big(A(x_k) - A(2y_k)\big)\big(B(x_k - y_k) - B(x_k - 2y_k)\big) - x_k =$$

$$= \big(A(x_k) - A(2y_k)\big)\big(B(x_k - y_k) - B(x_k - 2y_k)\big) - n_0.$$

By (19), this implies for large $k$ that

$$A(x_k) - A(2y_k) = 0,$$

or in equivalent form,

$$A \cap (2y_k, x_k] = \emptyset \qquad \text{for} \qquad k > k_0. \tag{20}$$

Finally, let us write

$$\mathcal{D} = \big\{(b,a): b \in \mathcal{B},\, a \in \mathcal{A},\, b \leqq x_k - y_k,\, a \leqq x_k - y_k,\, b - a > y_k\big\}.$$

It suffices to show that for large $k$,

$$|\mathcal{D}| > x_k - 2y_k. \tag{21}$$

Namely, if $(b,a) \in \mathcal{D}$, then

$$y_k < b - a \leqq x_k - y_k \qquad (\text{for } (b,a) \in \mathcal{D}). \tag{22}$$

By the pigeon hole principle, (21) and (22) imply that there exist $(b,a) \in \mathcal{D}$, $(b',a') \in \mathcal{D}$ with $b \neq b', a \neq a'$,

$$b - a = b' - a'$$

hence

$$a + b' = a' + b.$$

Then writing $n = a + b' = a' + b$, we have

$$f(n) \geqq 2$$

and

$$n \geqq b \geqq b - a > y_k$$

which, for large $k$, contradicts (9). This proves (7) and thus it completes the proof of our theorem.

In order to prove (21), first we write

$$\mathcal{D}_1 = \big\{(b,a): b \in \mathcal{B},\, a \in \mathcal{A},\, 2y_k < b \leqq x_k - y_k,\, a < b < y_k\big\}$$

and

$$\mathcal{D}_2 = \left\{(b,a): b \in \mathcal{B},\ a \in \mathcal{A},\ \frac{3}{2}y_k < b \leqq 2y_k,\ 0 \leqq a \leqq \frac{1}{2}y_k\right\}.$$

Then, in view of (15) and (20), we have $\mathcal{D}_1 \subset \mathcal{D}$, $D_2 \subset \mathcal{D}$ and clearly $\mathcal{D}_1 \cap \cap \mathcal{D}_2 \neq 0$, hence

$$|\mathcal{D}| > |\mathcal{D}_1| + |\mathcal{D}_2|. \tag{23}$$

In view of (20), $b \leqq x_k - y_k$, $a < b - y_k (< x_k)$ imply that $a \leqq 2y_k$. Thus writing

$$\mathcal{D}_1^+ = \{(b,a): b \in \mathcal{B},\ a \in \mathcal{A},\ 2y_k < b \leqq x_k - y_k,\ a \leqq 2y_k\}$$

and

$$\mathcal{D}_2^- = \{(b,a): b \in \mathcal{B},\ a \in \mathcal{A},\ 2y_k < b \leqq x_k - y_k,\ a \leqq 2y_k,\ a \geqq b - y_k\},$$

we have $\mathcal{D}_1^+ = \mathcal{D}_1 \cup \mathcal{D}_1^-$, $\mathcal{D}_1 \cap \mathcal{D}_1^- = \emptyset$, hence

$$|\mathcal{D}_1| = |\mathcal{D}_1^+| - |\mathcal{D}_1^-|. \tag{24}$$

By (6), (11), (14) and (20),

$$\begin{aligned} |\mathcal{D}_1^+| &= \big(B(x_k - y_k) - B(2y_k)\big)\, A(2y_k) = \\ &= B(x_k - y_k)A(2y_k) - B(2y_k)A(2y_k) = \\ &= B(x_k)A(x_k) - B(2y_k)A(2y_k) = \big(x_k O_k(1)\big) - \big(1 + o_k(1)\big) \cdot 2y_k = \\ &= (x_k - 2y_k) + o_k(y_k). \end{aligned} \tag{25}$$

Furthermore, $2y_k < b$, $a \leqq 2y_k$, $a \geqq b - y_k$ imply that $2y_k \geqq a \geqq \geqq b - y_k > 2y_k - y_k = y_k$ and $2y_k < b \leqq a + y_k \leqq 3y_k$, hence, by (16) and (17),

$$|\mathcal{D}_1^-| \leqq \big(A(2y_k) - A(2y_k)\big)\big(B(3y_k) - B(2y_k)\big) = O_k(1)o_k(y_k) = o_k(y_k). \tag{26}$$

It follows from(24), (25) and (26) that

$$|\mathcal{D}_1| = (x_k - 2y_k) + o_k(y_k). \tag{27}$$

Finally, we are going to estimate $|\mathcal{D}_2|$. It follows from the definition of $\mathcal{D}_2$ that

$$|\mathcal{D}_2| = A\left(\frac{1}{2}y_k\right)\left(B(2y_k) - B\left(\frac{3}{2}y_k\right)\right). \tag{28}$$

In view of (6) and (14), here we have

$$A\left(\frac{1}{2}y_k\right) \geqq A\left(\frac{1}{2}y_k\right)\frac{B\left(\frac{1}{2}y_k\right)}{B\left(y_k\right)} = \frac{A\left(\frac{1}{2}y_k\right)B\left(\frac{1}{2}y_k\right)}{A\left(y_k\right)B\left(y_k\right)}A\left(y_k\right) = \tag{29}$$

$$= \left(1 + o_k(1)\right)\frac{\frac{1}{2}y_k}{y_k}A(y_k) = \left(\frac{1}{2} + o_k(1)\right)A(y_k),$$

and, in view of (6), (10), (14) and (16),

$$B(2y_k) - B\left(\frac{3}{2}y_k\right) = \frac{B\left(2y_k\right)A(y_k) - B\left(\frac{3}{2}y_k\right)A\left(y_k\right)}{A(y_k)} = \tag{30}$$

$$= \frac{B(2y_k)\big(A(2y_k) + O_k(1)\big) - B\left(\frac{3}{2}y_k\right)\left(A\left(\frac{3}{2}y_k\right) + O_k(1)\right)}{A(y_k)} =$$

$$= \frac{A(2y_k)B(2y_k) + O_k\big(B(2y_k)\big) - A\left(\frac{3}{2}y_k\right)B\left(\frac{3}{2}y_k\right) + O_k\left(B\left(\frac{3}{2}y_k\right)\right)}{A(y_k)} =$$

$$= \frac{\left(2 + o_k(1)\right)y_k + o_k(y_k) - \left(\frac{3}{2} + o_k(1)\right)y_k + o_k(y_k)}{A(y_k)} =$$

$$= \left(\frac{1}{2} + o_k(1)\right)\frac{y_k}{A(y_k)}.$$

It follows from (28), (29) and (30) that

$$|\mathcal{D}_2| \geqq \left(\frac{1}{4} + o_k(1)\right)y_k. \tag{31}$$

If $k$ is large enough then (23), (27) and (31) imply (21) and this completes the proof of the theorem.

**3.** By using the same method, it could be shown that if (2) and (7) hold then

$$A(x)B(x) - x = o\big(\min(A(x), B(x))\big)$$

is impossible. On the other hand, we guess that

$$A(x)B(x) - x = O\big( \min(A(x), B(x))\big) \tag{32}$$

is possible and, in fact, we guess that for every function $f(x)$ with $\lim_{x\to+\infty} f(x) = +\infty$, there exist infinite sequences satisfying $A(x) = O\big(f(x)\big)$, (2), (5) and (32).

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# APPROXIMATE MEAN CONTINUOUS INTEGRAL

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## 1. Introduction

By extending the concept of ordinary limits Burkill introduced the approximate continuous Perron integral or the AP-integral [1] and the Cesaro–Perron integral or the CP-integral [2]. These integrals have been extended further in various ways [4, 6, 3]. The extension of CP-integral to $\mathrm{GM}_1$-integral by Ellis [4], based on the descriptive definition of the general Denjoy integral, uses approximate derivative and mean continuous ACG function. Replacing the mean continuity in the definition of $\mathrm{GM}_1$-integral, by the approximate mean continuity — called $\mathrm{D}_1$-continuity — we introduce, in the present paper, an integral which is called $\mathrm{D}_1$-integral. This integral which is an approximate extension of $\mathrm{GM}_1$-integral, is shown to possess various properties of Denjoy integrals including integration by parts and the Cauchy and the Harnack properties. The special Denjoy integral or the $\mathrm{D}^*$-integral and the general Denjoy integral or the D-integral, used in this paper, are in [7].

## 2. Preliminaries

DEFINITION 2.1. A function $f: E \to \mathbf{R}$, where $\mathbf{R}$ is the set of reals and $E \subset \mathbf{R}$, is said to be generalized absolutely continuous on $E$ if $E$ can be expressed as a countable union of closed sets on each of which $f$ is absolutely continuous and is written $f \in \mathrm{ACG}(E)$.

Note that this definition of ACG differs from that in [7, p. 223] in that we are not using continuity of $f$. Since a continuous function $f$ is absolutely continuous on the closure of a set on which $f$ is absolutely continuous, it follows that if $f$ is ACG in the sense of [7], then $f$ is also ACG in our sense. The converse is not true. It is clear that if $f \in \mathrm{ACG}(E)$ then $f$ is VBG on $E$ in the sense of [7, p. 221] and is measurable and so by [7, Theorem 4.3, p. 222] the approximate derivative $f'_{\mathrm{ap}}$ exists almost everywhere on $E$. It can be verified that if $f, g \in \mathrm{ACG}(E)$ then $\alpha f + \beta g \in \mathrm{ACG}(E)$ and $fg \in \mathrm{ACG}(E)$, where $\alpha$ and $\beta$ are constants.

LEMMA 2.2. *If $F \in \mathrm{ACG}(E)$ then every closed subset of $E$ contains a portion on which $F$ is absolutely continuous.*

PROOF. Let $E = \bigcup_k E_k$ where each $E_k$ is closed and $F$ is absolutely continuous on $E_k$. Let $Q$ be any closed subset of $E$. By Baire's theorem there is a portion $P$ of $Q$ which is contained in some $E_k$ and hence $F$ is absolutely continuous on $P$.

LEMMA 2.3. *If $F \in \mathrm{ACG}(E)$ then $F$ fulfils the Lusin condition* (N) *on $E$.*

PROOF. The proof given in [7, p. 225, Theorem 6.1] will suffice.

THEOREM 2.4. *Let $F \in \mathrm{ACG}[a,b]$ and let $F$ have Darboux property in $[a,b]$. If*

$$\limsup_{h\to 0} \frac{F(x+h)-F(x)}{h} \geqq 0 \tag{2.1}$$

*for almost all $x \in [a,b]$ then $F$ is continuous and nondecreasing in $[a,b]$.*

PROOF. Let $G$ be the set of all points $x$ in $[a,b]$ such that there is a neighbourhood of $x$ in which $F$ is nondecreasing (for the endpoints $a$ and $b$ we consider one sided neighbourhoods). Then $G$ is open. Let $H = [a,b] \sim \sim G$. Then $H$ is closed. If possible let $H$ be nonvoid. If $(c,d)$ is a contiguous interval of $H$ then $F$ is nondecreasing in $(c,d)$ and so by the Darboux property $F$ is continuous and nondecreasing in $[c,d]$. Hence $H$ cannot have isolated points. So $H$ is perfect. By Lemma 2.2, there is a portion $(p,q)\cap H$ of $H$ on which $F$ is absolutely continuous. Let $\alpha$ and $\beta$ be such that $p \leqq \alpha < < \beta \leqq q$ and $(\alpha,\beta)\cap H \neq \emptyset$. Then $F$ is absolutely continuous in $[\alpha,\beta]\cap H$. Since $F$ is continuous and nondecreasing in the closure of the complementary intervals of $[\alpha,\beta]\cap H$, $F$ is continuous and of bounded variation in $[\alpha,\beta]$. Since by Lemma 2.3, $F$ fulfils the Lusin condition (N), $F$ is absolutely continuous on $[\alpha,\beta]$. The condition (2.1) almost everywhere then ensures that $F$ is nondecreasing in $[\alpha,\beta]$. But this is a contradiction, since $(\alpha,\beta)\cap \cap H \neq \emptyset$. Thus $H$ is void. Hence $F$ is nondecreasing in $[a,b]$.

COROLLARY 2.5. *If $F$ has Darboux property on $[a,b]$ and $F \in \in \mathrm{ACG}([a,b])$ and $F'_{\mathrm{ap}} = 0$ almost everywhere in $[a,b]$, then $F$ is constant.*

## 3. The $\mathrm{D}_1$-integral

DEFINITION 3.1. Let $f\colon [a,b] \to \mathbf{R}$ and let $x \in [a,b]$. Let $f$ be D-integrable in some neighbourhood of $x$ and let $F$ be its indefinite D-integral.

If $F$ is approximately differentiable at $x$ then the approximate derivative $F'_{\mathrm{ap}}(x)$ is called the $\mathrm{D}_1$-limit of $f$ at $x$ and we write

$$\mathrm{D}_1\text{-}\lim_{t\to x} f(t) = F'_{\mathrm{ap}}(x).$$

The function $f$ is said to be $\mathrm{D}_1$-continuous at $x$ if

$$\mathrm{D}_1\text{-}\lim_{t\to x} f(t) = f(x).$$

In other words, $f$ is $\mathrm{D}_1$-continuous at $x \in [a,b]$ if $f$ is D-integrable in some neighbourhood of $x$ and $f(x)$ is the approximate derivative at $x$ of its indefinite D-integral; $f$ is said to be $\mathrm{D}_1$-continuous on $[a,b]$ if it is $\mathrm{D}_1$-continuous at every point of $[a,b]$. (If $x = a$ or $x = b$ then appropriate one sided neighbourhood and one sided limit are to be considered in the above definition).

Clearly if $f$ is continuous in $[a,b]$ then $f$ is the derivative of its indefinite integral and so $f$ is $\mathrm{D}_1$-continuous in $[a,b]$. The converse is not true. In fact, there exists a function $f$ and there is a set $E_0$ of positive measure in its domain such that $f$ is $\mathrm{D}_1$-continuous at each point of $E_0$ but nowhere continuous on $E_0$. Let $F$ be an ACG function on an interval which is not differentiable at the points of a set $E$ of positive measure (cf. [7, p. 224]). The approximate derivative $F'_{\mathrm{ap}}$ exists almost everywhere (cf. [7, p. 222, Theorem 4.3]). Let $f = F'_{\mathrm{ap}}$ where $F'_{\mathrm{ap}}$ exists and $f = 0$ otherwise. Clearly $f$ is $\mathrm{D}_1$-continuous almost everywhere on $E$ but $f$ is not continuous on $E$.

It may be recalled that a function $f$ is said to be $\mathrm{C}_1$-continuous at $x$ if $f$ is $\mathrm{D}^*$-integrable in some neighbourhood of $x$ and if $F'(x) = f(x)$ where $F$ is an indefinite $\mathrm{D}^*$-integral of $f$ (see [2]). Replacing $\mathrm{D}^*$-integral by D-integral, Ellis [4] introduced the concept of $\mathrm{M}_1$-continuity. Clearly $\mathrm{C}_1$-continuity implies $\mathrm{M}_1$-continuity and $\mathrm{M}_1$-continuity implies $\mathrm{D}_1$-continuity.

Definition 3.2. A function $f\colon [a,b] \to \mathbf{R}$ is said to be $\mathrm{D}_1$-integrable on $[a,b]$ if there is a $\mathrm{D}_1$-continuous, ACG function $\Phi\colon [a,b] \to \mathbf{R}$ such that $\Phi'_{\mathrm{ap}} = f$ almost everywhere in $[a,b]$. Then the function $\Phi$ is said to be an indefinite $\mathrm{D}_1$-integral of $f$ and $\Phi(b) - \Phi(a)$ is the definite integral of $f$ on $[a,b]$. Since a $\mathrm{D}_1$-continuous function is an approximate derivative, it has Darboux property and so by Corollary 2.5, $\Phi$ is unique up to an additive constant whence the definite integral is unique. The definite integral is denoted by

$$(\mathrm{D}_1)\int_a^b f(t)\,dt \quad \text{or simply} \quad (\mathrm{D}_1)\int_a^b f.$$

Recall that a function $f\colon [a,b] \to \mathbf{R}$ is $\mathrm{GM}_1$-integrable on $[a,b]$ if there is an $\mathrm{M}_1$-continuous, ACG function $\Phi\colon [a,b] \to \mathbf{R}$ such that $\Phi'_{\mathrm{ap}} = f$ almost everywhere in $[a,b]$. Since $M_1$-continuity implies $D_1$-continuity, it follows that if $f$ is $GM_1$-integrable then it is $D_1$-integrable and the integrals are equal. In Example 6.1 we shall show that the $D_1$-integral is strictly more general than the $GM_1$-integral [4]. Since the $GM_1$-integral includes the CP-integral [2] the $D_1$-integral is more general than the CP-integral and hence more general than the D- and D*-integrals. In Examples 6.2, 6.3 and 6.4 we shall show that the $D_1$-integral and the AP-integral [1] (and also the AD-integral [6]) are not comparable and even not compatible. It may be noted that AP-integral and the AD-integral have the disadvantages that the indefinite integrals (which are expected to have properties nicer than the integrand) may not be integrable (cf. [3] and Example 6.3 below).

The function $f$ is said to be $D_1$-integrable on a measurable subset $E$ of $[a,b]$ if $f_E$ is $D_1$-integrable on $[a,b]$ where $f_E$ is defined by

$$f_E(x) = \begin{cases} f(x), & x \in E \\ 0, & x \notin E \end{cases}$$

and we write $f \in D_1(E)$. We shall take

$$(D_1)\int_E f = (D_1)\int_a^b f_E.$$

THEOREM 3.3. *If $f$ is $D_1$-integrable in $[a,b]$, then $F(x) = (D_1)\int_a^x f$ is D-integrable and $D_1$-continuous on $[a,b]$.*

THEOREM 3.4. *If $f$ and $g$ are $D_1$-integrable in $[a,b]$ and $\alpha, \beta$ are constants then $\alpha f + \beta g$ is $D_1$-integrable in $[a,b]$ and*

$$(D_1)\int_a^b (\alpha f + \beta g) = \alpha (D_1)\int_a^b f + \beta (D_1)\int_a^b g.$$

The proofs of Theorems 3.3 and 3.4 follow from the definition of the $D_1$-integral.

THEOREM 3.5. *If $f$ is $D_1$-integrable in $[a,b]$ and in $[b,c]$ then it is $D_1$-integrable in $[a,c]$ and conversely if $f$ is $D_1$-integrable in $[a,c]$ and $a < b < c$ then it is so in $[a,b]$ and $[b,c]$.*

*In either case*

$$(D_1)\int_a^c f = (D_1)\int_a^b f + (D_1)\int_b^c f. \tag{3.1}$$

PROOF. Let $F_1$ and $F_2$ be the indefinite $\mathrm{D}_1$-integrals of $f$ in $[a,b]$ and in $[b,c]$ respectively. We may suppose that $F_2(b)=0$. Let

$$F(x)=\begin{cases} F_1(x) & \text{for } x\in[a,b] \\ F_1(b)+F_2(x) & \text{for } x\in[b,c]. \end{cases}$$

Then $F$ is ACG in $[a,c]$ and $F'_{\mathrm{ap}}=f$ almost everywhere in $[a,c]$. So we are to show that $F$ is $\mathrm{D}_1$-continuous in $[a,c]$. Since $F_1$ and $F_2$ are $\mathrm{D}_1$-continuous in $[a,b]$ and in $[b,c]$ respectively, we are only to consider the point $x=b$. Since $F_1$ and $F_2$ are D-integrable in $[a,b]$ and $[b,c]$ respectively, $F$ is D-integrable in $[a,c]$. Let $\Phi$ be an indefinite D-integral of $F$. Since $F_1$ is $\mathrm{D}_1$-continuous at $b$,

$$\lim_{h\to 0+}\operatorname{ap}\frac{\Phi(b-h)-\Phi(b)}{-h}=\lim_{h\to 0+}\operatorname{ap}\frac{1}{h}(\mathrm{D})\int_{b-h}^{b}F(t)\,dt=$$

$$=\lim_{h\to 0+}\operatorname{ap}\frac{1}{h}(\mathrm{D})\int_{b-h}^{b}F_1(t)\,dt=F_1(b).$$

Also, since $F_2$ is $\mathrm{D}_1$-continuous at $b$,

$$\lim_{h\to 0+}\operatorname{ap}\frac{\Phi(b+h)-\Phi(b)}{h}=\lim_{h\to 0+}\operatorname{ap}\frac{1}{h}(\mathrm{D})\int_{b}^{b+h}F(t)\,dt=$$

$$=\lim_{h\to 0+}\operatorname{ap}\frac{1}{h}(\mathrm{D})\int_{b}^{b+h}\left[F_1(b)+F_2(t)\right]\,dt=F_1(b)+F_2(b)=F_1(b).$$

Hence $F$ is $\mathrm{D}_1$-continuous at $b$. Thus $F$ is indefinite $\mathrm{D}_1$-integral of $f$. Since $F(b)-F(a)+F(c)-F(b)=F(c)-F(a)$, the relation (3.1) is clear. The converse is easy.

THEOREM 3.6. *If $f$ is $\mathrm{D}_1$-integrable and $f\geqq 0$ almost everywhere in $[a,b]$, then $f$ is Lebesgue integrable in $[a,b]$ and the integrals are equal.*

PROOF. Let $F(x)=(\mathrm{D}_1)\int_a^x f$. Then $F$ is $\mathrm{D}_1$-continuous and hence is an approximate derivative. So $F$ has Darboux property. Also $F$ is in $\mathrm{ACG}([a,b])$ and $F'_{\mathrm{ap}}=f\geqq 0$ almost everywhere in $[a,b]$. Hence by Theorem 2.4, $F$ is nondecreasing in $[a,b]$. So $F'$ exists almost everywhere and is Lebesgue integrable in $[a,b]$. Since $F'_{\mathrm{ap}}=f$ almost everywhere in $[a,b]$, $f$ is Lebesgue integrable in $[a,b]$. The rest is clear.

THEOREM 3.7. *If both $f$ and $g$ are* $\mathrm{D}_1$*-integrable on $[a,b]$ and if $f \geqq g$ almost everywhere then*

$$(\mathrm{D}_1)\int_a^b f \geqq (\mathrm{D}_1)\int_a^b g.$$

PROOF. Let $F$ and $G$ be indefinite integrals of $f$ and $g$ respectively. Then both $F$ and $G$ are $\mathrm{D}_1$-continuous and ACG on $[a,b]$ and $F'_{\mathrm{ap}} = f$, $G'_{\mathrm{ap}} = g$ almost everywhere on $[a,b]$. Hence $F'_{\mathrm{ap}} \geqq G'_{\mathrm{ap}}$ almost everywhere on $[a,b]$. If $\Phi = F - G$, then $\Phi$ is $\mathrm{D}_1$-continuous and ACG on $[a,b]$ and $\Phi'_{\mathrm{ap}} \geqq 0$ almost everywhere on $[a,b]$ and hence by Theorem 2.4, we have

$$\Phi(b) - \Phi(a) \geqq 0.$$

So

$$(\mathrm{D}_1)\int_a^b f \geqq (\mathrm{D}_1)\int_a^b g.$$

THEOREM 3.8. *If $f$ is* $\mathrm{D}_1$*-integrable then $f$ is measurable and finite almost everywhere.*

PROOF. Let $F$ be an indefinite $\mathrm{D}_1$-integral of $f$ in $[a,b]$. Then, since $F$ is ACG in $[a,b]$, $[a,b] = \bigcup_n E_n$, $E_n$ closed and $F$ is absolutely continuous on each $E_n$. For each $n$, let $F_n = F$ on $E_n$ and $F_n$ is linear in the closure of each contiguous interval. Then $F_n$ is of bounded variation in $[a,b]$. Since $f = F'_{\mathrm{ap}} = F'_n$ almost everywhere on $E_n$ and since the derivative of a function of bounded variation is finite almost everywhere and measurable, the result follows.

THEOREM 3.9 (Dominated convergence theorem). *If*

(i) *for each $n$, $g \leqq f_n \leqq h$ almost everywhere in $[a,b]$ where $g$, $f_n$, $h$ are* $\mathrm{D}_1$*-integrable and*

(ii) $\lim_{n\to\infty} f_n(x) = f(x)$ *almost everywhere on $[a,b]$,*

*then $f$ is* $\mathrm{D}_1$*-integrable and*

$$\lim_{n\to\infty}(\mathrm{D}_1)\int_a^b f_n = (\mathrm{D}_1)\int_a^b f.$$

PROOF. Write $\Phi_n = f_n - g$, $\Phi = f - g$, $\Psi = h - g$. Then $\Phi_n$ and $\Psi$ are non-negative almost everywhere and $\mathrm{D}_1$-integrable in $[a,b]$. By Theorem 3.6, they are Lebesgue integrable in $[a,b]$. Since $0 \leqq \Phi_n \leqq \Psi$, the Lebesgue theory of limits under the integral sign shows that $\Phi$ is Lebesgue integrable and

$$\lim_{n\to\infty}(\mathrm{L})\int_a^b \Phi_n = (\mathrm{L})\int_a^b \Phi.$$

That is

$$\lim_{n\to\infty}\left[(\mathrm{D}_1)\int_a^b f_n - (\mathrm{D}_1)\int_a^b g\right] = (\mathrm{D}_1)\int_a^b f - (\mathrm{D}_1)\int_a^b g.$$

Hence the result.

THEOREM 3.10 (Monotone convergence theorem). *If $\{f_n\}$ is a non-decreasing sequence of* $\mathrm{D}_1$*-integrable functions on $[a,b]$ and if the sequence $\{(\mathrm{D}_1)\int_a^b f_n\}$ is bounded above then the function $f(x) = \lim_{n\to\infty} f_n(x)$ is* $\mathrm{D}_1$*-integrable on $[a,b]$ and*

$$\lim_{n\to\infty}(\mathrm{D}_1)\int_a^b f_n = (\mathrm{D}_1)\int_a^b f. \tag{3.2}$$

PROOF. Since $f_n - f_1$ is $\mathrm{D}_1$-integrable and nonnegative, by Theorem 3.6 it is Lebesgue integrable. Since $f_n - f_1 \to f - f_1$, by the Lebesgue theory

$$\lim_{n\to\infty}(\mathrm{L})\int_a^b (f_n - f_1) = (\mathrm{L})\int_a^b (f - f_1). \tag{3.3}$$

Since the sequence of integrals $\{(\mathrm{D}_1)\int_a^b f_n\}$ is bounded above, so is the sequence $\{(\mathrm{L})\int_a^b (f_n - f_1)\}$ and therefore

$$0 \leqq (\mathrm{L})\int_a^b (f - f_1) < \infty.$$

Hence $f - f_1$ is Lebesgue integrable and a fortiori, is $\mathrm{D}_1$-integrable and $f_1$ being $\mathrm{D}_1$-integrable, $f$ is also so. Hence (3.2) follows from (3.3).

THEOREM 3.11. *If $f$ is $\mathrm{D}_1$-integrable on $[a,b]$ then for every closed set $E \subset [a,b]$ there is a closed interval $J \subset [a,b]$ containing points of $E$ in its interior such that*

(i) *$f$ is Lebesgue integrable on $J \cap E$,*

(ii) *if $\{I_k\}$ is the sequence of contiguous closed intervals of $J \cap E$ then*

$$\sum_k \left| (\mathrm{D}_1) \int_{I_k} f \right| < \infty.$$

PROOF. Let $F(x) = (\mathrm{D}_1) \int_a^x f$. Then since $F$ is ACG on $[a,b]$ by Lemma 2.2 there is a closed interval $J \subset [a,b]$ containing points of $E$ in its interior such that $F$ is absolutely continuous on $J \cap E$. Let $G$ be the function on $J$ which coincides with $F$ on $J \cap E$ and is linear on each contiguous closed interval of $J \cap E$. Then $G$ is continuous and of bounded variation on $J$ and $F$ satisfies Lusin condition on $J$. So $G$ is absolutely continuous on $J$ and hence $G'$ is Lebesgue integrable on $J$. Since $G' = F'_{\mathrm{ap}} = f$ almost everywhere on $J \cap E$, $f$ is Lebesgue integrable on $J \cap E$, proving (i).

To prove (ii), note that since $F$ is absolutely continuous on $J \cap E$, $F$ is also of bounded variation on $J \cap E$ and hence

$$\sum_k \left| (\mathrm{D}_1) \int_{I_k} f \right| = \sum_k \left| F(b_k) - F(a_k) \right| < \infty$$

where $I_k = [a_k, b_k]$. This proves (ii).

## 4. Integration by parts

LEMMA 4.1. *Let $\Phi$ be $\mathrm{D}_1$-continuous at $x_0 \in [a,b]$ and $F$ be an indefinite $L$-integral of a function $f$ of bounded variation in $[a,b]$. Then $\Phi F$ is $\mathrm{D}_1$-continuous at $x_0$.*

PROOF. Let $a \leqq x_0 < b$. Since $\Phi$ is $\mathrm{D}_1$-continuous at $x_0$, it is D-integrable in some neighbourhood of $x_0$. Let $\Phi_1 = (\mathrm{D}) \int_{x_0}^x \Phi$. Then $\Phi_1$ is continuous in that neighbourhood of $x_0$. Hence for $\varepsilon > 0$ there is $\delta > 0$ such that

$$(4.1) \qquad \left| \Phi_1(t) \right| < \varepsilon \quad \text{whenever} \quad |t - x_0| < \delta.$$

Since $f$ is bounded in $[a,b]$, there is $M > 0$ such that

$$(4.2) \qquad \left| f(t) \right| \leqq M, \quad \text{for all} \quad t \in [a,b].$$

Since $\Phi$ is D-integrable in a neighbourhood of $x_0$ and $F$ is absolutely continuous in $[a,b]$, by [7, p. 246, Theorem 2.5] $\Phi F$ is D-integrable in that neighbourhood of $x_0$ and

$$(\mathrm{D})\int_{x_0}^{x_0+h}\Phi F=\Phi_1(x_0+h)F(x_0+h)-(\mathrm{S})\int_{x_0}^{x_0+h}\Phi_1\,dF.$$

Let

$$H(x)=(\mathrm{D})\int_{x_0}^{x}\Phi F.$$

Let $f=g_1-g_2$ where $g_1$ and $g_2$ are nonnegative nondecreasing functions and let $F=F_1-F_2$ where $F_1$ and $F_2$ are indefinite $L$-integrals of $g_1$ and $g_2$ respectively.

Then, if $0<|h|<\delta$

$$\frac{H(x_0+h)-H(x_0)}{h}-F(x_0)\Phi(x_0)= \tag{4.3}$$

$$=\frac{1}{h}(\mathrm{D})\int_{x_0}^{x_0+h}\Phi F-F(x_0)\Phi(x_0)=$$

$$=\frac{1}{h}F(x_0+h)\Phi_1(x_0+h)-\frac{1}{h}(\mathrm{S})\int_{x_0}^{x_0+h}\Phi_1\,dF-F(x_0)\Phi(x_0).$$

If $S(x)=(\mathrm{S})\int_{x_0}^{x}\Phi_1\,dF_1$, then since $F_1$ is continuous and nondecreasing, $S(x)$ is continuous. Also by [7, p. 244, Theorem 2.1(ii)], $S^{(1)}(x)=\Phi_1(x)g_1(x)$ except on an enumerable set and hence by [7, p. 235, Theorem 10.5], $S$ is ACG*. Since $\Phi_1 g_1$ is Riemann integrable

$$(\mathrm{S})\int_{x_0}^{x}\Phi_1\,dF_1=S(x)=(\mathrm{R})\int_{x_0}^{x}\Phi_1 g_1.$$

Considering similarly for $g_2$ and taking the difference, we have

$$(\mathrm{S})\int_{x_0}^{x}\Phi_1\,dF=(\mathrm{R})\int_{x_0}^{x}\Phi_1 f.$$

Hence from (4.3), (4.1) and (4.2)

$$\left|\frac{H(x_0+h)-H(x_0)}{h} - F(x_0)\Phi(x_0)\right| =$$

$$= \left|\frac{1}{h}F(x_0+h)\Phi_1(x_0+h) - F(x_0)\Phi(x_0) - \frac{1}{h}(\mathrm{R})\int_{x_0}^{x_0+h}\Phi_1 f\right| \leqq$$

$$\leqq \left|F(x_0+h)\cdot\frac{\Phi_1(x_0+h)}{h} - F(x_0+h)\Phi(x_0)\right| +$$

$$+\left|F(x_0+h)\Phi(x_0) - F(x_0)\Phi(x_0)\right| + \frac{1}{h}(\mathrm{R})\int_{x_0}^{x_0+h}|\Phi_1|\cdot|f| \leqq$$

$$\leqq |F(x_0+h|\cdot\left|\frac{\Phi_1(x_0+h)}{h} - \Phi(x_0)\right| +$$

$$+|\Phi(x_0)|\cdot|F(x_0+h)-F(x_0)| + \frac{M}{h}\cdot\varepsilon\cdot h.$$

Therefore, since $\varepsilon$ is arbitrary,

$$\lim_{h\to 0+}\mathrm{ap}\left[\frac{H(x_0+h)-H(x_0)}{h} - F(x_0)\Phi(x_0)\right] = 0.$$

Similarly if $a < x_0 \leqq b$ then

$$\lim_{h\to 0+}\mathrm{ap}\left[\frac{H(x_0)-H(x_0-h)}{h} - F(x_0)\Phi(x_0)\right] = 0.$$

Hence the result.

LEMMA 4.2. *Let $\varphi$ be $\mathrm{D}_1$-integrable and $f$ be of bounded variation in $[a,b]$. Let*

$$\Phi(x) = (\mathrm{D}_1)\int_a^x \varphi, \quad F(x) = (\mathrm{R})\int_a^x f, \quad a \leqq x \leqq b.$$

*Then the function $\Psi$ defined by*

$$\Psi(x) = F(x)\Phi(x) - (\mathrm{D})\int_a^x \Phi f, \quad a \leqq x \leqq b,$$

*is $\mathrm{D}_1$-continuous,* ACG *on $[a,b]$.*

PROOF. *By Theorem 3.3, $\Phi$ is* D*-integrable and so by [7, p. 246, Theorem 2.5], $\Phi f$ is* D*-integrable and so $\Psi$ is well defined. Since $\Phi$ is* $\mathrm{D}_1$*-continuous, by Lemma 4.1, $F\Phi$ is* $\mathrm{D}_1$*-continuous. Also since $\Phi$ is ACG and $F$ is absolutely continuous, $F\Phi$ is ACG in $[a,b]$. Since a continuous function is* $\mathrm{D}_1$*-continuous,* $(\mathrm{D})\int_a^x \Phi f$ *is* $\mathrm{D}_1$*-continuous and ACG. So, $\Psi$ is* $\mathrm{D}_1$*-continuous and ACG in $[a,b]$.*

THEOREM 4.3 (Integration by parts). *If $\varphi$ is* $\mathrm{D}_1$*-integrable and $f$ is of bounded variation in $[a,b]$ and if*

$$\Phi(x)=(\mathrm{D}_1)\int_a^x \varphi,\quad F(x)=(\mathrm{R})\int_a^x f,\quad a\leqq x\leqq b$$

*then $\varphi F$ is* $\mathrm{D}_1$*-integrable in $[a,b]$ and*

$$(\mathrm{D}_1)\int_a^b \varphi F=[\Phi F]_a^b-(\mathrm{D})\int_a^b \Phi f.$$

PROOF. By Lemma 4.2, the function $\Psi$ defined by

$$\Psi(x)=F(x)\Phi(x)-(\mathrm{D})\int_a^x \Phi f,\quad a\leqq x\leqq b.$$

is $\mathrm{D}_1$-continuous and ACG in $[a,b]$. Also almost everywhere in $[a,b]$,

$$F'=f,\quad \Phi'_{\mathrm{ap}}=\varphi\quad\text{and}\quad \left((\mathrm{D})\int_a^x \Phi f\right)'_{\mathrm{ap}}=\Phi f.$$

Hence almost everywhere in $[a,b]$,

$$\Psi'_{\mathrm{ap}}=f\Phi+F\varphi-\Phi f=F\varphi.$$

So $F\varphi$ is $\mathrm{D}_1$-integrable and $\Psi$ is an indefinite $\mathrm{D}_1$-integral of $F\varphi$. Hence

$$(\mathrm{D}_1)\int_a^b F\varphi=\Psi(b)-\Psi(a)=\left[F(x)\Phi(x)\right]_a^b-(\mathrm{D})\int_a^b \Phi f.$$

## 5. Cauchy and Harnack property

THEOREM 5.1 (Cauchy property). *If $f$ is* $\mathrm{D}_1$*-integrable in* $[a,\beta]$ *for every* $\beta$, $a<\beta<b$, *and if*

$$\mathrm{D}_1\text{-}\lim_{\beta\to b-}(\mathrm{D}_1)\int_a^\beta f = L$$

*then $f$ is* $\mathrm{D}_1$*-integrable in* $[a,b]$ *and*

$$(\mathrm{D}_1)\int_a^b f = L.$$

PROOF. Let $b_1, b_2, \dots, b_n, \dots$ be an increasing sequence which converges to $b$ with $b_1 = a$. Then $f$ is $\mathrm{D}_1$-integrable on each $I_n = [b_n, b_{n+1}]$ and so there is a function $F_n$ which is $\mathrm{D}_1$-continuous and ACG on $I_n$ and $(F_n)'_{\mathrm{ap}} = f$ almost everywhere on $I_n$. We may suppose $F_n(b_n) = 0$ for all $n$. Let

$$F(x) = \begin{cases} F_1(x), & x \in I_1 \\ F_n(x) + \sum_{k=1}^{n-1} F_k(b_{k+1}), & x \in I_n, \quad n \geqq 2 \\ L, & x = b. \end{cases}$$

Then since $F$ is ACG on each $I_n$, $F$ is ACG on $[a,b]$. Also $F'_{\mathrm{ap}} = f$ almost everywhere in $[a,b]$. Since

$$F(x) = (\mathrm{D}_1)\int_a^x f, \quad a \leqq x < b,$$

we have from the given condition $\mathrm{D}_1\text{-}\lim\limits_{\beta\to b-} F(\beta) = L$ and so $F$ is D-integrable in some neighbourhood of $b$ and hence $F$ is D-integrable in $[a,b]$. Let

$$\Phi(x) = (\mathrm{D})\int_a^x F, \quad a \leqq x \leqq b.$$

Then

$$\Phi'_{\mathrm{ap}}(b) = \lim_{x\to 0+}\mathrm{ap}\,\frac{1}{x}(\mathrm{D})\int_{b-x}^b F = \mathrm{D}_1\text{-}\lim_{\beta\to b-} F(\beta) = L = F(b).$$

Thus $F$ is $\mathrm{D}_1$-continuous at $x = b$. Also $F$ is $\mathrm{D}_1$-continuous on each $I_n$. So $f$ is $\mathrm{D}_1$-integrable in $[a,b]$ and $F$ is an indefinite $\mathrm{D}_1$-integral in $[a,b]$. Thus

$$(\mathrm{D}_1)\int_a^b f = F(b) - F(a) = F(b) = L.$$

THEOREM 5.2 (Harnack property). *Let $E \subset [a,b]$ be a closed set with complementary intervals $I_k = (a_k,b_k)$, $k = 1,2,\dots$. Let $f \in \mathrm{D}_1(E)$ and $f \in \mathrm{D}_1\big([a_k,b_k]\big)$ for each $k$ with*

$$F_k(x) = (\mathrm{D}_1)\int_{a_k}^{x} f, \quad a_k \leqq x \leqq b_k.$$

*Let (if there are infinite number of intervals $I_k$)*

(i) $\displaystyle\sum_{k=1}^{\infty}\left|(\mathrm{D}_1)\int_{a_k}^{b_k} f\right| < \infty,$

(ii) $\displaystyle\lim_{k\to\infty}\ \sup_{x\in(a_k,b_k]}\left|\frac{1}{x-a_k}(\mathrm{D})\int_{a_k}^{x} F_k(t)\,dt\right| = 0.$

*Then $f$ is $\mathrm{D}_1$-integrable in $[a,b]$ and*

$$(\mathrm{D}_1)\int_a^b f = (\mathrm{D}_1)\int_E f + \sum_k (\mathrm{D}_1)\int_{a_k}^{b_k} f.$$

PROOF. This result is known for the D-integral [7. p. 257, Theorem 5.1] with the condition (ii) replaced by

(ii)′ $$\lim_{k\to\infty} O(F_k; a_k, b_k) = 0$$

where $O(F_k; a_k, b_k)$ denotes oscillation of $F_k$ in $[a_k,b_k]$. Note that for the D-integral the condition (ii)′ implies (ii). In fact, for the D-integral $F_k$ is continuous and

$$\left|\frac{1}{x-a_k}(\mathrm{R})\int_{a_k}^{x} F_k(t)\,dt\right| \leqq \frac{1}{x-a_k}(\mathrm{R})\int_{a_k}^{x}\left|F_k(t)\right|\,dt \leqq O(F_k; a_k, b_k)$$

and so if (ii)′ holds then (ii) holds.

Let

$$\Psi_k(x) = \begin{cases} F_k(x), & a_k \leqq x \leqq b_k, \\ F_k(b_k), & x > b_k, \\ O, & x < a_k \end{cases}$$

and

$$F(x) = \sum_{k=1}^{\infty} \Psi_k(x).$$

We shall show that $F$ is $\mathrm{D}_1$-continuous in $[a, b]$.

If $x$ is an interior point of some $I_k$, say $I_m$, then since $F(t) = \sum' \Psi_k(b_k) + F_m(t)$, for $a_m < t \leqq b_m$, where $\sum'$ is taken for those $k$ for which $I_k \subset [a, a_m)$ and since $F_m$ is $\mathrm{D}_1$-continuous, $F$ is $\mathrm{D}_1$-continuous at $x$. If $x$ is an isolated point of $E$ then $x$ is the common endpoint of two intervals $I_k$, say $I_p$ and $I_m$ where $b_p = x = a_m$. Then for small $h > 0$, $F(x+h) - F(x) = F_m(x+h)$ and $F(x) - F(x-h) = F_p(x) - F_p(x-h)$. Since $F_m$ and $F_p$ are $\mathrm{D}_1$-continuous in $[a_m, b_m]$ and in $[a_p, b_p]$ respectively,

$$\mathrm{D}_1\text{-}\lim_{h \to 0+} \left[ F(x+h) - F(x) \right] = F_m(a_m) = 0$$

and

$$\mathrm{D}_1\text{-}\lim_{h \to 0+} \left[ F(x) - F(x-h) \right] = F_p(x) - F_p(b_p) = 0.$$

So $F$ is $\mathrm{D}_1$-continuous at $x$. If $E$ has component intervals then clearly these intervals are closed and $F$ is constant in these intervals and hence is $\mathrm{D}_1$-continuous there with one sided $\mathrm{D}_1$-continuity at the endpoints. Also the endpoints of the component intervals are the endpoints of suitable contiguous intervals $(a_k, b_k)$ and so, applying the second case above, $F$ is both sided $\mathrm{D}_1$-continuous at the endpoints of the component intervals of $E$. The only remaining case we are to consider is that $x$ is a limit point of endpoints of the contiguous intervals. Let $x$ be such a limit point from the right. Let $\varepsilon > 0$ be arbitrary. Then from the conditions (i) and (ii) there is $k_0$ such that

$$\sum_{k > k_0} \left| F_k(b_k) \right| < \varepsilon, \tag{5.1}$$

$$\sup_{x \in (a_k, b_k]} \left| \frac{1}{x - a_k} (\mathrm{D}) \int_{a_k}^{x} F_k \right| < \varepsilon, \quad \text{for all} \quad k \geqq k_0. \tag{5.2}$$

Let $\delta > 0$ be such that $(x, x+\delta)$ does not contain the intervals $I_k$, $1 \leqq k \leqq \leqq k_0$. Let $t \in (x, x+\delta) \cap E$. Then denoting by $\sum_1$ the summation taken over those $k$ for which $I_k \subset [x, t]$, we have from (5.1)

$$(5.3)\quad |F(t) - F(x)| = \left|\sum\nolimits_1 F_k(b_k)\right| \leqq \sum\nolimits_1 |F_k(b_k)| \leqq \sum_{k>k_0} |F_k(b_k)| < \varepsilon.$$

Since the functions $F_k$ are D-integrable in $[a_k, b_k]$, $\Psi_k$ is measurable in $[a, b]$ for each $k$ by [7, p. 243, Theorem 1.3] and so $F$ is measurable. So by (5.3), $F(t) - F(x)$ is Lebesgue integrable in $(x, x+\delta) \cap E$ and

$$(5.4)\quad (\mathrm{L}) \int_{(x,x+h)\cap E} |F(t) - F(x)|\, dt \leqq \varepsilon \cdot \mu\big((x, x+h) \cap E\big) \quad \text{for} \quad 0 < h < \delta$$

where $\mu$ is the Lebesgue measure.

Let $t \in (x, x+\delta) \sim E$. Then $t \in I_k$ for some $k$. Let $t \in I_m$. Denoting by $\sum_2$ the summation taken over those $k$ for which $I_k \subset [x, a_m]$, we have

$$F(t) - F(x) = \sum\nolimits_2 F_k(b_k) + F_m(t).$$

That is

$$(5.5)\qquad |F(t) - F(x) - F_m(t)| \leqq \sum\nolimits_2 |F_k(b_k)| \leqq \sum_{k>k_0} |F_k(b_k)| < \varepsilon.$$

Hence as above $F(t) - F(x) - F_m(t)$ is Lebesgue integrable and so it is D-integrable in $I_m$. The function $F_m$ which is an indefinite $\mathrm{D}_1$-integral is D-integrable and hence $F(t) - F(x)$ is D-integrable in $I_m$. Taking D-integral in (5.5),

$$\left|\frac{1}{t - a_m} (\mathrm{D}) \int_{a_m}^{t} [F(\xi) - F(x)]\, d\xi\right| < \varepsilon + \left|\frac{1}{t - a_m} (\mathrm{D}) \int_{a_m}^{t} F_m(\xi)\, d\xi\right|,$$

for $a_m \leqq t \leqq b_m$. Hence by (5.2), since $m > k_0$

(5.6)

$$-2\varepsilon(t - a_m) < (\mathrm{D}) \int_{a_m}^{t} [F(\xi) - F(x)]\, d\xi < 2\varepsilon(t - a_m), \quad \text{for} \quad a_m \leqq t \leqq b_m.$$

Hence

$$\left|(\mathrm{D})\int_{a_m}^{b_m} F(\xi)\,d\xi\right| < \Bigl(2\varepsilon + |F(x)|\Bigr)\cdot(b_m - a_m). \tag{5.7}$$

Since $I_m$ is any interval $I_k \subset (x,\, x+\delta) \sim E$, (5.7) is true for all such intervals and so adding for these intervals

$$\sum\nolimits_3 \left|(\mathrm{D})\int_{a_k}^{b_k} F(\xi)\,d\xi\right| < \Bigl(2\varepsilon + |F(x)|\Bigr)\delta, \tag{5.8}$$

where $\sum_3$ denotes summation over those $k$ for which $I_k \subset (x, x+\delta) \sim E$.

Also if

$$H_k(t) = (\mathrm{D})\int_{a_k}^{t} F(\xi)\,d\xi, \quad a_k \leqq t \leqq b_k,$$

then

$$H_k(t) = \sum\nolimits_4 F_p(b_p)\cdot(t - a_k) + G_k(t) \tag{5.9}$$

where $\sum_4$ denotes summation over those $p$ for which $I_p \subset (a, a_k)$, and $G_k$ is defined by

$$G_k(t) = (\mathrm{D})\int_{a_k}^{t} F_k(\xi)\,d\xi, \quad a_k \leqq t \leqq b_k.$$

Then for $u, v \in [a_k, b_k]$

$$\begin{aligned} |G_k(u) - G_k(v)| &\leqq |G_k(u) - G_k(a_k)| + |G_k(v) - G_k(a_k)| \leqq \\ &\leqq \sup_{z\in(a_k,b_k]} \left|\frac{1}{z-a_k}(\mathrm{D})\int_{a_k}^{z} F_k(\xi)\,d\xi\right| \{(u-a_k)+(v-a_k)\} \leqq \\ &\leqq 2(b_k - a_k) \sup_{z\in(a_k,b_k)} \left|\frac{1}{z-a_k}(\mathrm{D})\int_{a_k}^{z} F_k(\xi)\,d\xi\right| \end{aligned}$$

and hence

$$\lim_{k\to\infty} O(G_k; a_k, b_k) = 0.$$

Therefore from (5.9)

$$\lim_{k\to\infty} O(H_k; a_k, b_k) = 0. \tag{5.10}$$

The relations (5.8) and (5.10) show that $F$ satisfies the hypothesis of the corresponding theorem for D-integral [7, p. 257, Theorem 5.1]. Hence by [7, p. 257, Theorem 5.1], $F$ is D-integrable in $[x, x+h]$ and

$$(\mathrm{D})\int_x^{x+h} F = (\mathrm{D})\int_{E\cap[x,x+h]} F + {\sum}_3 (\mathrm{D})\int_{a_k}^{b_k} F, \tag{5.11}$$

where $\sum_3$ is the summation over all $k$ for which $I_k \subset [x, x+h] \sim E$. The relation (5.6) being true for all intervals $I_k \subset [x, x+\delta] \sim E$, by adding all the relations in (5.6) with (5.4), we get from (5.11)

$$\left|(\mathrm{D})\int_x^{x+h} \big[F(t) - F(x)\big]\, dt\right| \leqq 2\varepsilon h \quad \text{for all} \quad h, \quad 0 < h < \delta.$$

Dividing by $h$ and letting $h \to 0+$, since $\varepsilon$ is arbitrary,

$$\lim_{h\to 0+} \frac{1}{h}(\mathrm{D})\int_x^{x+h} F(t)\, dt = F(x).$$

If $x$ is a limit point of endpoints of the contiguous intervals from the left then we get similarly

$$\lim_{h\to 0+} \frac{1}{h}(\mathrm{D})\int_{x-h}^{x} F(t)\, dt = F(x).$$

In fact, in this case the left side of (5.5) will be $\big|F(t) - F(x) + F_m(t)\big|$ and the summation $\sum_2$ there will be taken over those $k$ for which $I_k \subset [a_m, x]$.

Hence $F$ is $\mathrm{D}_1$-continuous at $x$. Thus $F$ is $\mathrm{D}_1$-continuous in $[a, b]$. Next we shall show that $F$ is ACG on $[a, b]$. Clearly $F$ is ACG on each interval $[a_k, b_k]$. So we are to show that $F$ is absolutely continuous on $E$. Let

$$g(x) = \begin{cases} 0, & x \in E \\ \frac{1}{b_k - a_k}(\mathrm{D}_1)\int_{a_k}^{b_k} f, & x \in [a_k, b_k]. \end{cases}$$

By the condition (i), $g$ is Lebesgue integrable in $[a,b]$. Put

$$G(x) = (\mathrm{L}) \int_a^x g, \quad a \leqq x \leqq b.$$

If $x \in E$, then $G(x) = F(x)$. Since $G$ is absolutely continuous on $[a,b]$, $F$ is absolutely continuous on $E$. Thus $F$ is ACG on $[a,b]$.

Finally, since $G = F$ on $E$, we have $F'_{\mathrm{ap}} = G' = g = 0$ almost everywhere on $E$. Also in $I_k$, $F$ and $F_k$ differ by a constant and hence $F'_{\mathrm{ap}} = (F_k)'_{\mathrm{ap}} = f$ almost everywhere in $I_k$. Hence $F'_{\mathrm{ap}} = f$ almost everywhere in $[a,b] \sim E$. So it follows that $F$ is an indefinite $\mathrm{D}_1$-integral of $\Phi$ where $\Phi(x) = f(x)$ if $x \in [a,b] \sim E$ and $\Phi(x) = 0$ if $x \in E$.

On the other hand if $\Psi(x) = f(x)$ if $x \in E$ and $\Psi(x) = 0$ if $x \in [a,b] \sim \sim E$, then by hypothesis $\Psi$ is $\mathrm{D}_1$-integrable in $[a,b]$. Hence $\Phi + \Psi = f$ is $\mathrm{D}_1$-integrable in $[a,b]$ and

$$(\mathrm{D}_1) \int_a^b f = (\mathrm{D}_1) \int_a^b \Phi + (\mathrm{D}_1) \int_a^b \Psi = F(b) - F(a) + (\mathrm{D}_1) \int_a^b \Psi =$$

$$= \sum_k \Psi_k(b) - \sum_k \Psi_k(a) + (\mathrm{D}_1) \int_E f =$$

$$= \sum_k F_k(b_k) + (\mathrm{D}_1) \int_E f = \sum_k (\mathrm{D}_1) \int_{a_k}^{b_k} f + (\mathrm{D}_1) \int_E f.$$

Hence the result.

## 6. Examples

For the definitions of $\mathrm{GM}_1$-integral, AP-integral and AD-integral, considered below, we refer to [4], [1] and [6] respectively. Note that AD-integral includes AP-integral.

EXAMPLE 6.1. There is a function which is $\mathrm{D}_1$-integrable but not $\mathrm{GM}_1$-integrable.

Let $\{I_n = (a_n, b_n)\}$ be a sequence of intervals such that

(i) $I_n \subset (0,1)$, for all $n$,

(ii) $b_1 > a_1 > b_2 > a_2 > \ldots > b_n > a_n > \ldots$ and $\lim\limits_{n\to\infty} b_n = 0$,

(iii) 0 is a point of dispersion of the set $\bigcup_{n=1}^{\infty} I_n$,

(iv) $\dfrac{a_n + b_n}{b_n - a_n} \to \infty$ as $n \to \infty$.

(One may take $a_n = \frac{1}{n} + \frac{1}{(n+1)^2}$, $b_n = \frac{1}{n} + \frac{1}{n^2}$).

Let

$$F(x) = \begin{cases} \frac{a_n+b_n}{2} \sin^2 \frac{\pi(x-a_n)}{b_n-a_n} & \text{for } x \in I_n, \\ 0 & \text{for } x \notin \bigcup_{n=1}^{\infty} I_n. \end{cases}$$

Then $F(x)$ is continuous in $[0,1]$. Also $F'$ exists in $(0,1]$ but $F'$ does not exist at 0 while $F'_{\mathrm{ap}}(0)$ exists and is 0. Since $F$ is continuous and is ACG in $[0,1]$ and since $F'_{\mathrm{ap}}$ exists everywhere in $[0,1]$, $F'_{\mathrm{ap}}$ is $\mathrm{D}_1$-continuous in $[0,1]$. Since

$$F'_{\mathrm{ap}}(x) = \begin{cases} \frac{a_n+b_n}{b_n-a_n} \cdot \pi \sin \frac{\pi(x-a_n)}{b_n-a_n} \cos \frac{\pi(x-a_n)}{b_n-a_n}, & \text{for } x \in I_n, \\ 0, & \text{for } x \notin \bigcup_{n=1}^{\infty} I_n, \end{cases}$$

$F'_{\mathrm{ap}}$ is ACG in $[0,1]$. Also $(F'_{\mathrm{ap}})'_{\mathrm{ap}}$ exists almost everywhere on $[0,1]$. Hence the function $f$, where

$$f(x) = \begin{cases} (F'_{\mathrm{ap}})'_{\mathrm{ap}}(x) & \text{if } (F'_{\mathrm{ap}})'_{\mathrm{ap}}(x) \text{ exists}, \\ 0 & \text{otherwise} \end{cases}$$

is $\mathrm{D}_1$-integrable and $F'_{\mathrm{ap}}$ is its indefinite $\mathrm{D}_1$-integral. But $f$ is not $\mathrm{GM}_1$-integrable. For, if it is so then there exists an $\mathrm{M}_1$-continuous ACG function $\Phi$ such that $\Phi'_{\mathrm{ap}} = f$ almost everywhere on $[0,1]$. Since $\mathrm{M}_1$-continuity implies $\mathrm{D}_1$-continuity $\Psi = \phi - F'_{\mathrm{ap}}$ is $\mathrm{D}_1$-continuous, ACG in $[0,1]$ and $\Psi'_{\mathrm{ap}} = 0$ almost everywhere on $[0,1]$. Hence by Corollary 2.5, $\Phi$ and $F'_{\mathrm{ap}}$ differ by a constant. Since $\Phi$ is $\mathrm{M}_1$-continuous at 0, $F'_{\mathrm{ap}}$ is also so at 0. Since $F$ is indefinite D-integral of $F'_{\mathrm{ap}}$, the derivative $F'(0)$ exists. But this is a contradiction, since $F'(0)$ does not exist.

EXAMPLE 6.2. There is a function which is $\mathrm{D}_1$-integrable but not AD-integrable.

Let

$$f(x) = \begin{cases} -\frac{1}{x^2} \cos \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $f$ is $\mathrm{D}_1$-integrable in $[0,1]$ and

$$F(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is its indefinite $D_1$-integral. In fact $F$ is the derivative of

$$G(x) = \begin{cases} x^2 \cos \frac{1}{x} - 2(\mathrm{R}) \int\limits_0^x x \cos \frac{1}{x}\, dx, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

and so $F$ is $D_1$-continuous in $[0,1]$. Moreover $F \in \mathrm{ACG}([0,1])$ and $F' = f$ almost everywhere. Hence $f$ is $D_1$-integrable in $[0,1]$ (in fact, $f$ is CP-integrable in [0,1]). But $f$ is not AD-integrable in [0,1]. For, if possible let $\varphi$ be an indefinite AD-integral of $f$. Let $0 < \alpha < 1$. Then since $\varphi$ is approximately continuous and ACG in [0,1], $F - \varphi$ is approximately continuous and ACG in $[\alpha, 1]$. Also since $\varphi'_{\mathrm{ap}} = f$ almost everywhere, $(F - \varphi)'_{\mathrm{ap}} = 0$ almost everywhere in $[\alpha, 1]$. Since approximately continuous functions possess Darboux property, by Corollary 2.5 there is a constant $K$ such that $\varphi(x) = \sin \frac{1}{x} + K$ for $x \in [\alpha, 1]$. The constant $K$ cannot be different for different $\alpha$ and hence $\varphi(x) = \sin \frac{1}{x} + K$ for $x \in (0,1]$. Hence $\lim\limits_{x \to 0} \operatorname{ap} \varphi(x)$ does not exist which is a contradiction since $\varphi$ is approximately continuous at $x = 0$.

Example 6.3. There is a function which is AP-integrable (and hence AD-integrable) but not $D_1$-integrable.

In [3] a nonnegative function $\varphi$ has been constructed such that $\varphi'$ exists finitely everywhere on (0,1] and $\varphi'_{\mathrm{ap}}(0)$ exists finitely but $\varphi$ is not Lebesgue integrable in [0,1]. Clearly $\varphi'_{\mathrm{ap}}$ is AP-integrable and $\varphi$ is its indefinite AP-integral. Note that $\varphi$ is not even AD-integrable in [0,1]. For, if $\varphi$ is so then since $\varphi \geqq 0$, $\varphi$ would be Lebesgue integrable (as in Theorem 3.6). But $\varphi'_{\mathrm{ap}}$ is not $D_1$-integrable in [0,1]. For, if possible, let $F$ be its indefinite $D_1$-integral. Then for $0 < \alpha < 1$, $F - \varphi \in \mathrm{ACG}([\alpha, 1])$ and $F - \varphi$ is $D_1$-continuous in $[\alpha, 1]$ and so as in Example 6.2, $F(x) = \varphi(x) + K$ for $x \in (0,1]$ where $K$ is a constant. Since $F$ is D-integrable in [0,1], $\varphi$ is also D-integrable in [0,1]. Since $\varphi \geqq 0$, $\varphi$ is Lebesgue integrable in [0,1] which is a contradiction.

Example 6.4. The $D_1$-integral and the AP-integral (and hence the AD-integrals) are not compatible.

Ellis [5] constructed a function $f$ such that $f$ is AP-integrable and is CP-integrable but the values of the integrals are different. Since the $D_1$-integral includes CP-integral and the AD-integral includes AP-integral the result follows.

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# ON EXTENSIONS OF SOME THEOREMS OF FLETT. I

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**1. Introduction.** In [2] T. M. Flett defined a very useful extension of absolute Cesàro summability. According to his definition we shall say that a series $\sum a_n$ is summable $|C,\alpha,\gamma|_k$, where $k \geqq 1$, $\alpha > -1$, $\gamma \geqq 0$, if the series $\sum n^{k\gamma+k-1}|\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k$ is convergent, $\sigma_n^\alpha$ being the $n$th Cesàro mean of order $\alpha$ of the series $\sum_0^\infty a_n$.

Among others, he proved the following result.

THEOREM A. *Let $r \geqq k > 1$, $\gamma \geqq 0$, $\alpha > \gamma - 1$, $\beta \geqq \alpha + 1/k - 1/r$. Then if $\sum_0^\infty a_n$ is summable $|C,\alpha,\gamma|_k$, it is summable $|C,\beta,\gamma|_r$ and with $\tau_n^\alpha := n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$*

$$\left\{\sum n^{r\gamma-1}|\tau_n^\beta|^r\right\}^{1/r} \leqq B\left\{\sum n^{k\gamma-1}|\tau_n^\alpha|^k\right\}^{1/k}. \tag{1.1}$$

*If $k = 1$, result* (1.1) *holds when $r \geqq 1$, $\gamma \geqq 0$, $\alpha > \gamma - 1$, $\beta > \alpha + 1/k - 1/r$.*

This theorem is a very important result, also in itself; moreover it has turned out that inequality (1.1) is crucial in the proofs of theorems concerning strong approximation of orthogonal series having approximation order $o_x(1/n^\gamma)$ (see e.g. G. Sunouchi [8], and [4], [5], [6]). Recently we intended to generalize these results replacing the factor $1/n^\gamma$ by a more general factor $1/\gamma(n)$. We had to recognize that this can be done, in our view and experience, only if previously we can generalize Theorem A by the same way, that is, if in Theorem A we can replace the factor $n^\gamma$ by a suitable factor $\gamma(n)$.

This is our motivation for generalizing this important result of Flett. Naturally, having found the method for such an extension, we shall use it for generalizing some further interesting theorems of Flett [2] relevant to our present interest; maybe in some subsequent papers, too.

We shall not cite the theorems of Flett to be generalized here, because the theorems to be proved in our present paper in the special case $\gamma(t) = t^\gamma$ will reduce to the appropriate results of Flett, except Theorem 4.

**2. Notations and definitions.** We shall use the notions and notations of the paper [2] and introduce some new ones. Let $\alpha$ be any real number,

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and let

$$E_n^\alpha = \frac{(\alpha+1)\dots(\alpha+n)}{n!} \quad (n>0), \quad E_0^\alpha = 1.$$

For any given series $\sum_0^\infty a_n$ and any $n \geqq 0$ we write $\sigma_n^\alpha$ and $\tau_n^\alpha$ for the $n$th Cesàro means of order $\alpha$ of the series $\sum_0^\infty a_n$ and the sequence $na_n$, respectively. We have then the following identities, valid for all $\alpha$ and $\delta$ :

$$\tau_n^{\alpha+\delta} = \frac{1}{E_n^{\alpha+\delta}} \sum_{\nu=1}^{n} E_{n-\nu}^{\delta-1} E_\nu^\alpha \tau_\nu^\alpha, \tag{2.1}$$

$$\sigma_n^{\alpha+\delta} = \frac{1}{E_n^{\alpha+\delta}} \sum_{\nu=0}^{n} E_{n-\nu}^{\delta-1} E_\nu^\alpha \sigma_\nu^\alpha \tag{2.2}$$

and

$$\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha) \quad \left( = \alpha\{\sigma_n^{\alpha-1} - \sigma_n^\alpha\} \quad \text{if} \quad \alpha > 0 \right). \tag{2.3}$$

In view of the identity (2.3) we may restate the definition of summability $|C,\alpha,\gamma|_k$ in terms of the series

$$\sum_1^\infty n^{k\gamma-1} |\tau_n^\alpha|^k, \tag{2.4}$$

the series $\sum_0^\infty a_n$ being summable $|C,\alpha,\gamma|_k$ if series (2.4) is convergent.

Let $\gamma(t)$ be a positive non-decreasing function defined for $1 \leqq t < \infty$. We shall say that the series $\sum_0^\infty a_n$ is summable $|C,\alpha,\gamma(t)|_k$ if the series

$$\sum_1^\infty \gamma(n)^k n^{-1} |\tau_n^\alpha|^k \tag{2.5}$$

is convergent.

We extend also the definition of the summability $|A,\gamma|_k$ introduced by Flett [2] as follows: We shall say that the series $\sum_0^\infty a_n$ is summable $|A,\gamma(t)|_k$, where $k \geqq 1$, if the series $\sum a_n x^n$ is convergent for any $0 \leqq x < 1$ and its sum-function $\phi(x)$ satisfies the condition

$$\int_0^1 (1-x)^{k-1} \gamma\left( (1-x)^{-1} \right)^k |\phi'(x)|^k dx < \infty. \tag{2.6}$$

It is clear that if $\gamma(t) = t^\gamma$ this definition coincides with that of Flett.

A summation sign $\sum$ in which the limits of summation are omitted will denote summation from 1 to $\infty$.

We use $B$ to denote a positive constant depending on the parameters $c, d, \dots$ concerning the particular problem in which it appears. If we wish to express the dependence explicitly, we write $B$ in the form $B(c, d, \dots)$. The constants are not necessarily the same at any two occurrences.

Inequalities of the form

$$L \leqq BR$$

are to be interpreted as meaning "if the expression $R$ is finite, then the expression $L$ is finite and satisfies the inequality."

**3. Theorems.** First we extend Theorem A.

THEOREM 1. *Let $r \geqq k > 1$, $\alpha > -1$, $\beta \geqq \alpha + 1/k - 1/r$, and $\gamma(t)$ a non-decreasing positive function defined for $1 \leqq t < \infty$ so that with some $C > 1$*

$$\limsup \frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1}. \tag{3.1}$$

*Then if the series*

$$\sum_{n=0}^{\infty} a_n \tag{3.2}$$

*is summable $|C, \alpha, \gamma(t)|_k$, it is summable $|C, \beta, \gamma(t)|_k$ and*

$$\left\{\sum \gamma(n)^r n^{-1} |\tau_n^\beta|^r\right\}^{1/r} \leqq B \left\{\sum \gamma(n)^k n^{-1} |\tau_n^\alpha|^k\right\}^{1/r}. \tag{3.3}$$

*If $k = 1$, the result holds when $r \geqq 1$, $\beta > \alpha + 1 - 1/r$ and* (3.1) *is satisfied.*

We mention that using the reasoning of the proof of Theorem 1 to be given the following result follows:

THEOREM 1A. *Under the assumptions of Theorem* 1, *we have for any $s$*

$$\left\{\sum_{n=0}^{\infty} \gamma(n+1)^r (n+1)^{-1} |\sigma_n^\beta - s|^r\right\}^{1/r} \leqq$$

$$\leqq B \left\{\sum_{n=0}^{\infty} \gamma(n+1)^k (n+1)^{-1} |\sigma_n^\alpha - s|^k\right\}^{1/k}.$$

If we keep $r = k$, then the factor $\gamma(n)$ on the left-hand side of (3.3) can be replaced by another factor $\mu(n)$ as follows.

THEOREM 2. *Let $k \geqq 1$, $\alpha > -1$, $\delta > 0$, $\beta \geqq \alpha - \delta$, and $\beta > -1$, furthermore let $\mu(t)$ be a positive monotone function, and $\gamma(t)$ a non-decreasing positive function defined for $1 \leqq t < \infty$, so that*

$$C^{\delta} \limsup \frac{\mu(Ct)}{\mu(t)} < \liminf \frac{\gamma(Ct)}{\gamma(t)} \leqq \limsup \frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1} \tag{3.4}$$

*with some $C > 1$. Then if series* (3.2) *is summable $|C, \alpha, \gamma(t)|_k$, it is summable $|C, \beta, \mu(t)|_k$ and*

$$\sum \mu(n)^k n^{-1} |\tau_n^{\beta}|^k \leqq B \sum \gamma(n)^k n^{-1} |\tau_n^{\alpha}|^k. \tag{3.5}$$

In case of the strict inequality $\beta > \alpha - \delta$ we can prove a consistency result for $r < k$, too.

THEOREM 3. *Let $k > r \geqq 1$, $\alpha > -1$, $\delta > 0$, and $\beta > \max(\alpha - \delta, -1)$. If $\mu(t)$ is a positive monotone function, and $\gamma(t)$ is a positive non-decreasing function defined for $1 \leqq t < \infty$, furthermore they satisfy* (3.4) *and series* (3.2) *is summable $|C, \alpha, \gamma(t)|_k$, then* (3.2) *is also summable $|C, \beta, \mu(t)|_k$ and*

$$\left\{\sum \mu(n)^r n^{-1} |\tau_n^{\beta}|^r\right\}^{1/r} \leqq B \left\{\sum \gamma(n)^k n^{-1} |\tau_n^{\alpha}|^k\right\}^{1/k}. \tag{3.6}$$

Finally we present two theorems in connection with summability $|A, \gamma(t)|_k$.

THEOREM 4. *Let $k \geqq 1$, $\alpha > -1$ and $\gamma(t)$ be a non-decreasing positive function defined for $1 \leqq t < \infty$ so that*

$$\limsup \frac{\gamma(Ct)}{\gamma(t)} < C^{(\alpha+1)/k} \tag{3.7}$$

*with some $C > 1$. Then if series* (3.2) *is summable $|C, \alpha, \gamma(t)|_k$, it is summable $|A, \gamma(t)|_k$. Moreover, if*

$$\phi(x) := \sum_{n=0}^{\infty} a_n x^n, \tag{3.8}$$

*then*

$$\int_0^1 (1-x)^{k-1} \gamma\left((1-x)^{-1}\right)^k |\phi'(x)|^k dx \leqq B \sum \gamma(n)^k n^{-1} |\tau_n^{\alpha}|^k. \tag{3.9}$$

THEOREM 5. *Let $k > r \geqq 1$ and $\delta > 0$, furthermore let $\mu(t)$ be a positive monotone function, and $\gamma(t)$ a positive non-decreasing function defined for*

$1 \leqq t < \infty$ *with properties* (3.4). *Then if series* (3.2) *is summable* $|A, \gamma(t)|_k$, *it is summable* $|A, \mu(t)|_r$. *Moreover, if* $\phi(x)$ *is given by* (3.8), *then*

$$\left\{\int_0^1 (1-x)^{r-1}\mu\left((1-x)^{-1}\right)^r |\phi'(x)|^r dx\right\}^{1/r} \leqq \tag{3.10}$$

$$\leqq B\left\{\int_0^1 (1-x)^{k-1}\gamma\left((1-x)^{-1}\right)^k |\phi'(x)|^k dx\right\}^{1/k}.$$

For further cases of the summability $|C, \alpha, \gamma|_k$ and $|A, \gamma|_k$ useful hints can be found in the fundamental paper of T.M. Flett [2], p. 361.

**4. Lemmas.** We require the following lemmas.

LEMMA 1. *If* $\beta > 0$, $\rho \geqq 1$, *then*

$$B(\beta)\rho^{-\beta} \leqq \int_0^1 (1-x)^{\beta-1}x^\rho dx \leqq \Gamma(\beta)\rho^{-\beta}. \tag{4.1}$$

This lemma is due to H. P. Mulholland [7].
Using Lemma 1 we can prove the following result.

LEMMA 2. *If* $\alpha > -1$, $\rho \geqq 1$, $k \geqq 1$ *and* $\gamma(t)$ *is a non-decreasing positive function defined for* $1 \leqq t < \infty$ *satisfying the condition*

$$\limsup \frac{\gamma(Ct)}{\gamma(t)} < C^{(\alpha+1)/k} \tag{4.2}$$

*with some* $C > 1$, *then*

$$I := \int_0^1 (1-x)^\alpha \gamma\left((1-x)^{-1}\right)^k x^\rho dx \leqq B\gamma(\rho)^k \rho^{-\alpha-1}. \tag{4.3}$$

PROOF. Since

$$I = \left(\int_0^{1-1/\rho} + \int_{1-1/\rho}^1\right)(1-x)^\alpha \gamma\left((1-x)^{-1}\right)^k x^\rho dx =: I_1 + I_2,$$

and by (4.1)

$$I_1 \leqq \gamma(\rho)^k \int_0^{1-1/\rho} (1-x)^\alpha x^\rho dx \leqq B(\alpha)\gamma(\rho)^k \rho^{-\alpha-1}$$

holds for any $\rho > 1$; so it is enough to verify that

$$I_2 \leqq B\gamma(\rho)^k \rho^{-\alpha-1} \tag{4.4}$$

also holds.

An elementary transformation gives that

(4.5)

$$I_2 = \int_\rho^\infty y^{-\alpha-2}\gamma(y)^k \left(1-\frac{1}{y}\right)^\rho dy \leqq B\sum_{n=0}^{\infty}\sum_{m=\rho N^n}^{\rho N^{n+1}} m^{-\alpha-2}\gamma(m)^k =: I_3.$$

From (4.2) we can easily derive that there exist an integer $N(\geqq 1)$ and a positive $t_0$ so that

$$\frac{\gamma(Nt)}{(Nt)^{(\alpha+1)/k}} < \frac{1}{4}\frac{\gamma(t)}{t^{(\alpha+1)/k}}$$

holds for any $t \geqq t_0$. Hence we get for any $\rho \geqq t_0$ $(\geqq 2)$ that

$$I_3 \leqq B(N)\sum_{n=0}^{\infty}(\rho N^n)^{-\alpha-1}\gamma(\rho N^n)^k \leqq B(N)\gamma(\rho)^k\rho^{-\alpha-1}.$$

This inequality and (4.5) prove (4.4), and this completes the proof of Lemma 2.

LEMMA 3. *Let* $r > k > 1$, $\delta = 1/k - 1/r$, $c_n \geqq 0$,

$$C_n := \sum_{\nu=1}^{n-1}(n-\nu)^{\delta-1}c_\nu.$$

*Then*

$$\left\{\sum C_n^r\right\}^{1/r} \leqq B(k,r)\left\{\sum c_n^k\right\}^{1/k}. \tag{4.6}$$

This lemma was proved by G. H. Hardy, J. E. Littlewood and G. Pólya [3].

**5. Proofs of the theorems.** It seems to be the most convenient to follow the treatments of the proofs of Flett with the required changes, so we do this helping the comparisons as well.

PROOF OF THEOREM 1. First we consider the case $r \geqq k > 1$. Let $S$ denote the expression on the right hand side of (3.3), and let $\delta > 1/k - 1/r$

$(\geqq 0)$, $\eta := \frac{k}{k+r(k-1)}$ and $\lambda := 1 - 1/k - \varepsilon$ where $\varepsilon > 0$ and will be chosen later. Setting $k' := k/(k-1)$, then we have that

$$k'(\delta-1)(1-\eta) > -1. \tag{5.1}$$

Now using (2.1) and applying the triple form of Hölder's inequality with indices $r$, $k'$ and $kr/(r-k)$, we get

$$|\tau_n^{\alpha+\delta}| \leqq Bn^{-n-\delta}\sum_{\nu=1}^{n}(n+1-\nu)^{\delta-1}\nu^{\alpha}|\tau_n^{\alpha}| = \tag{5.2}$$

$$= Bn^{-\alpha-\delta}\sum_{\nu=1}^{n}\left\{(n+1-\nu)^{(\delta-1)\eta}\nu^{\alpha+\lambda+(r-k)/rk}\gamma(\nu)^{k/r-1}|\tau_\nu^{\alpha}|^{k/r}\right\}\times$$

$$\times\left\{(n+1-\nu)^{(\delta-1)(1-\eta)}\nu^{-\lambda}\right\}\left\{\gamma(\nu)^{k}\nu^{-1}|\tau_\nu^{\alpha}|^{k}\right\}^{(r-k)/kr} \leqq$$

$$\leqq Bn^{-\alpha-\delta}\left\{\sum_{\nu=1}^{n}(n+1-\nu)^{(\delta-1)\eta r}\nu^{r(\alpha+\lambda)+\frac{r}{k}-1}\gamma(\nu)^{k-r}|\tau_\nu^{\alpha}|^{k}\right\}^{1/r}\times$$

$$\times\left\{\sum_{\nu=1}^{n}(n+1-\nu)^{(\delta-1)(1-\eta)k'}\nu^{-\lambda k'}\right\}^{1/k'}\times\left\{\sum_{\nu=1}^{n}\gamma(\nu)^{k}\nu^{-1}|\tau_\nu^{\alpha}|^{k}\right\}^{(r-k)/kr}.$$

Using the definition of $S$ and $\lambda$, by (5.1) and (5.2) we get

$$|\tau_n^{\alpha+\delta}| \leqq Bn^{-\alpha-\delta}n^{(\delta-1)(1-\eta)-\lambda+1/k'}S^{1-k/r}\times \tag{5.3}$$

$$\times\left\{\sum_{\nu=1}^{n}(n+1-\nu)^{(\delta-1)\eta r}\nu^{r(\alpha+\lambda)+r/k-1}\gamma(\nu)^{k-r}|\tau_\nu^{\alpha}|^{k}\right\}^{1/r}.$$

Setting $\mu := (\delta-1)\eta r$ and $\omega := r(\alpha+\lambda)+r/k$ we get from (5.3)

$$\sum_{n=1}^{m}\gamma(n)^{r}n^{-1}|\tau_n^{\alpha+\delta}|^{r} \leqq \tag{5.4}$$

$$\leqq B\ S^{r-k}\sum_{n=1}^{m}\gamma(n)^{r}n^{-1}n^{-\omega-\mu}\sum_{\nu=1}^{n}(n+1-\nu)^{\mu}\nu^{\omega-1}\gamma(\nu)^{k-r}|\tau_\nu^{\alpha}|^{k} \leqq$$

$$\leqq B\ S^{r-k}\sum_{\nu=1}^{m}\nu^{\omega-1}\gamma(\nu)^{k-r}|\tau_\nu^{\alpha}|^{k}\sum_{n=\nu}^{m}(n+1-\nu)^{\mu}n^{-\omega-\mu-1}\gamma(n)^{r}.$$

If we can show that

$$(5.5)\qquad \sum_{n=\nu}^{m}(n+1-\nu)^{\mu}n^{-\omega-\mu-1}\gamma(n)^{r} \leqq B\gamma(\nu)^{r}\nu^{-\omega}$$

holds for any $m$ and $\nu$ large enough ($\nu < m$), then (5.4) verifies (3.3) when $r \geqq k > 1$ and $\beta := \alpha + \delta > \alpha + 1/k - 1/r$.

Now we start the proof of (5.5). For this purpose we first show that in view of (3.1) there exist an integer $N$ and positive numbers $\varepsilon$ and $t_0$ so that

$$(5.6)\qquad \frac{(tN)^{\varepsilon}\gamma(Nt)}{(Nt)^{\alpha+1}} < \frac{1}{2}\frac{t^{\varepsilon}\gamma(t)}{t^{\alpha+1}}$$

holds for any $t \geqq t_0$. A straightforward calculation gives that (3.1) implies, with some $t_0$ and $N$, that

$$(5.7)\qquad \frac{\gamma(Nt)}{(Nt)^{\alpha+1}} < \frac{1}{4}\frac{\gamma(t)}{t^{\alpha+1}}$$

for any $t \geqq t_0$. Hence, by choosing $\varepsilon := (\log^2 N)^{-1} (< 1/k')$, we get (5.6).

Now we turn back to the proof of (5.5). In view of $\mu > -1$ and $\gamma(2\nu) \leqq \leqq B\gamma(\nu)$

$$(5.8)\qquad \sum_{n=\nu}^{2\nu}(n+1-\nu)^{\mu}n^{-\omega-\mu-1}\gamma(n)^{r} \leqq B\nu^{-\omega-\mu-1}\gamma(\nu)^{r}\sum_{i=1}^{\nu} i^{\mu} \leqq$$
$$\leqq B\gamma(\nu)^{r}\nu^{-\omega}.$$

On the other hand, if $\nu \geqq t_0$, then by (5.6) and $\omega = r(\alpha + 1 - \varepsilon)$, we have

$$(5.8^{*})\qquad \sum_{n=2\nu}^{\infty}(n+1-\nu)^{\mu}n^{-\omega-\mu-1}\gamma(n)^{r} \leqq B\sum_{n=\nu}^{\infty}n^{-\omega-1}\gamma(n)^{r} \leqq$$
$$\leqq B\sum_{m=0}^{\infty}\sum_{n=\nu N^{m}}^{\nu N^{m+1}} n^{-\omega-1}\gamma(n)^{r} \leqq B(N)\sum_{m=0}^{\infty}(\nu N^{m})^{-\omega}\gamma(\nu N^{m})^{r} =$$
$$= B(N)\sum_{m=0}^{\infty}\left[(\nu N^{m})^{-\alpha-1}(\nu N^{m})^{\varepsilon}\gamma(\nu N^{m})\right]^{r} \leqq B(N)\nu^{-\omega}\gamma(\nu)^{r}.$$

Thus (5.5) is proved, consequently Theorem 1 holds if $r \geqq k > 1$ and $\beta = \alpha + \delta > \alpha + 1/k - 1/r$.

Now we consider the case $r \geqq k > 1$ and $\delta := 1/k - 1/r$.

If $r = k$ then $\delta = 0$, consequently $\beta = \alpha$ thus (3.3) is trivial.

If $r > k$ then $\delta = 1/k - 1/r$ is positive. The proof of this case needs a longer calculation. By (2.1)

$$|\tau_n^{\alpha+\delta}| \leqq \frac{1}{E_n^{\alpha+\delta}} \sum_{\nu=1}^{n} E_{n-\nu}^{\delta-1} E_\nu^\alpha |\tau_\nu^\alpha| \leqq \tag{5.9}$$

$$\leqq \frac{1}{E_n^{\alpha+\delta}} \left( \sum_{\nu=1}^{n/2} + \sum_{\nu \geqq n/2}^{n} \right) E_{n-\nu}^{\delta-1} E_\nu^\alpha |\tau_\nu^\alpha| =: T_1 + T_2.$$

Here

$$T_1 \leqq \frac{B}{E_n^{\alpha+1}} \sum_{\nu=1}^{n/2} E_\nu^\alpha |\tau_\nu^\alpha| \leqq \frac{B}{E_n^{\alpha+1}} \sum_{\nu=1}^{n} E_\nu^\alpha |\tau_\nu^\alpha|.$$

Repeating the procedure used above with 1 in place of $\delta$ we can conclude that

$$\left\{ \sum \gamma(n)^r n^{-1} |T_1|^r \right\}^{1/r} \leqq B \left\{ \sum \gamma(n)^k n^{-1} |\tau_n^\alpha|^k \right\}^{1/k} \tag{5.10}$$

holds.

To estimate the expression with $T_2$ we use Lemma 3. Writing $c_n := := \gamma(n) n^{-1/k} |\tau_n^\alpha|$, we have

$$\gamma(n) n^{-1/r} T_2 \leqq B \sum_{\nu \geqq n/2}^{n} E_{n-\nu}^{\delta-1} \gamma(\nu) \nu^{-1/k} |\tau_n^\alpha| \leqq$$

$$\leqq B \sum_{\nu=1}^{n-1} (n-\nu)^{\delta-1} \gamma(\nu) \nu^{-1/k} |\tau_\nu^\alpha| + B\gamma(n) n^{-1/k} |\tau_n^\alpha| \leqq$$

$$\leqq BC_n + B\gamma(n) n^{-1/k} |\tau_n^\alpha|.$$

Hence, by (4.6),

$$\left\{ \sum \gamma(n)^r n^{-1} T_2^r \right\}^{1/r} \leqq \tag{5.11}$$

$$\leqq B \left\{ \sum C_n^r \right\}^{1/r} + \left\{ \sum \gamma(n)^r n^{-r/k} |\tau_n^\alpha|^r \right\}^{1/r} \leqq$$

$$\leqq B \left\{ \sum c_n^k \right\}^{1/k} + B \left\{ \sum \gamma(n)^k n^{-1} |\tau_n^\alpha|^k \right\}^{1/k} =$$

$$= 2B\left\{\sum \gamma(n)^k n^{-1}|\tau_n^\alpha|^k\right\}^{1/k}.$$

The required inequality (3.3) now follows from inequalities (5.9), (5.10) and (5.11).

Finally we deal with the case $k = 1$. Then $\delta > 1 - 1/r \geqq 0$. In this special case we define $\lambda := 0$ and $\eta := 1$. With these parameters we can repeat the reasoning made in (5.2) and (5.3) which gives that

$$(5.12)\qquad |\tau_n^{\alpha+\delta}| \leqq Bn^{-\alpha-\delta}S^{1-1/r} \times$$
$$\times\left\{\sum_{\nu=1}^{n}(n+1-\nu)^{(\delta-1)r}\nu^{r\alpha+r-1}\gamma(\nu)^{1-r}|\tau_\nu^\alpha|\right\}^{1/r}.$$

Now setting $\mu^* := (\delta-1)r$ and $\omega^* := r(\alpha+1)$, then by (5.12), we get

$$(5.13)\qquad \sum_{n=1}^{m}\gamma(n)^r n^{-1}|\tau_n^{\alpha+\delta}|^r \leqq B\ S^{r-1}\sum_{n=1}^{m}\gamma(n)^r n^{-1}n^{-\mu^*-\omega^*}\times$$
$$\times\sum_{\nu=1}^{n}(n+1-\nu)^{\mu^*}\nu^{\omega^*-1}\gamma(\nu)^{1-r}|\tau_\nu^\alpha| \leqq$$
$$\leqq BS^{r-1}\sum_{\nu=1}^{m}\nu^{\omega^*-1}\gamma(\nu)^{1-r}|\tau_\nu^\alpha|\sum_{n=\nu}^{m}(n+1-\nu)^{\mu^*}n^{-\mu^*-\omega^*-1}\gamma(n)^r.$$

Here we have the same question as at (5.5) with $\mu^*$ and $\omega^*$ in place of $\mu$ and $\omega$, respectively. Since $\mu^* > -1$ and $\omega^* > 0$ we can carry out all of the considerations made in (5.8) and (5.8*), and so we conclude that

$$\sum_{n=\nu}^{m}(n+1-\nu)^{\mu^*}n^{-\mu^*-\omega^*-1}\gamma(n)^r \leqq B\gamma(\nu)^r\nu^{-\omega^*}$$

holds if $\nu$ and $m$ are large enough. Regarding this, (5.13) proves that (3.3) holds when $k = 1$, $r \geqq 1$ and $\beta > \alpha + 1 - 1/r$.

Herewith Theorem 1 is completely proved.

PROOF OF THEOREM 1A. Using the identity (2.2) instead of (2.1) and following the arguments used in the proof of Theorem 1, we get the statement of Theorem 1A. To give the details it seems to be a good practice, so we omit it.

PROOF OF THEOREM 2. First of all we show that it is enough to prove the special case in which $\beta = \alpha - \delta > -1$. For if $\beta > \alpha - \delta$ and $\beta > -1$, then

$\beta = \alpha_1 + \delta$, where $\alpha_1 > \alpha$. Series (3.2) is summable $|C, \alpha_1, \gamma(t)|_k$ whenever it is summable $|C, \alpha, \gamma(t)|_k$ (see Theorem 1), and the result will therefore follow immediately from the special case mentioned above with $\alpha$ replaced by $\alpha_1$.

Therefore we can suppose that $\beta = \alpha - \delta > -1$. Then we have to prove that

$$\sum \mu(n)^k n^{-1} |\tau_n^{\alpha-\delta}|^k \leqq B \sum \gamma(n)^k n^{-1} |\tau_n^{\alpha}|^k.$$

Consider first the case $k > 1$. By (2.1) we have

$$|\tau_n^{\alpha-\delta}| \leqq B n^{-\alpha+\delta} \sum_{\nu=1}^{n} |E_{n-\nu}^{-\delta-1}| \nu^{\alpha} |\tau_\nu^{\alpha}|. \tag{5.14}$$

Let $0 < \varepsilon < \min\left(\delta, (k+k')^{-1}\right)$ and $\lambda := 1/k' - \varepsilon \quad (k' = k/(k-1))$. Then, by Hölder's inequality,

$$|\tau_n^{\alpha-\delta}|^k \leqq B n^{k(\delta-\alpha)} \left\{ \sum_{\nu=1}^{n} |E_{n-\nu}^{-\delta-1}| \nu^{k(\alpha+\lambda)} |\tau_\nu^{\alpha}|^k \right\} \times \tag{5.15}$$

$$\times \left\{ \sum_{\nu=1}^{n} |E_{n-\nu}^{-\delta-1}| \nu^{-k'\lambda} \right\}^{k/k'} \leqq B n^{k(\delta-\alpha-\lambda)} \left\{ \sum_{\nu=1}^{n} |E_{n-\nu}^{-\delta-1}| \nu^{k(\alpha+\lambda)} |\tau_\nu^{\alpha}|^k \right\}.$$

Furthermore it is easy to see that in view of (3.4) there exists a number $t_0$ so that

$$\frac{\mu(Ct)C^{\delta}}{\mu(t)} < \frac{\gamma(Ct)}{\gamma(t)} < C^{\alpha+1}$$

holds for any $t \geqq t_0$. Hence, considering the monotonicity of $\mu(t)$ and $\gamma(t)$, we get

$$\mu(y) y^{\delta} \leqq B(t_0, \mu, \gamma, C, \alpha) \gamma(y)$$

for any $y \geqq t_0$, whence, obviously, it follows

$$\mu(y) y^{\delta} \leqq B \gamma(y) \tag{5.16}$$

for any $y \geqq 1$ with an appropriately chosen constant $B$.

Thus, by (5.15) and (5.16), we have for any $m \geqq 1$

$$\sum_{n=1}^{m} \mu(n)^k n^{-1} |\tau_n^{\alpha-\delta}|^k \leqq B \sum_{n=1}^{m} \gamma(n)^k n^{-k(\alpha+\lambda)-1} \times \tag{5.17}$$

$$\times \sum_{\nu=1}^{n} |E_{n-\nu}^{-\delta-1}| \nu^{k(\alpha+\lambda)} |\tau_\nu^\alpha|^k \leqq$$

$$\leqq B \sum_{\nu=1}^{m} \nu^{k(\alpha+\lambda)} |\tau_\nu^\alpha|^k \sum_{n=\nu}^{m} |E_{n-\nu}^{-\delta-1}| n^{-k(\alpha+\lambda)-1} \gamma(n)^k.$$

Next we show that

$$\sum_{\nu}^{m} := \sum_{n=\nu}^{m} |E_{n-\nu}^{-\delta-1}| n^{-k(\alpha+\lambda)-1} \gamma(n)^k \leqq B\gamma(\nu)^k \nu^{-k(\alpha+\lambda)-1}. \tag{5.18}$$

A reasoning similar to that given in the proof of Theorem 1 shows that

$$\sum_{n=\nu}^{2\nu} \leqq B\nu^{-k(\alpha+\lambda)-1} \gamma(\nu)^k \sum_{n=1}^{\nu} n^{-\delta-1} \leqq B\nu^{-k(\alpha+\lambda)-1} \gamma(\nu)^k \tag{5.19}$$

and, by $k\varepsilon < \delta$,

$$\sum_{n=2\nu}^{\infty} \leqq B \sum_{n=\nu}^{\infty} n^{-\delta-2-k(\alpha+\lambda)} \gamma(n)^k \leqq B\nu^{k\varepsilon-\delta} \sum_{n=\nu}^{\infty} n^{-k(\alpha+1)-1} \gamma(n)^k. \tag{5.20}$$

The second inequality in (3.4) is the same as (3.1) in Theorem 1. Therefore (3.4) also implies (5.7) as we have seen above, whence

$$\sum_{n=\nu}^{\infty} n^{-k(\alpha+1)-1} \gamma(n)^k \leqq \sum_{i=0}^{\infty} \sum_{n=\nu N^i}^{\nu N^{i+1}} n^{-k(\alpha+1)-1} \gamma(n)^k \leqq$$

$$\leqq B \sum_{i=0}^{\infty} (\nu N^i)^{-k(\alpha+1)} \gamma(\nu N^i)^k \sum_{n=\nu N^i}^{\nu N^{i+1}} n^{-1} \leqq B(N) \nu^{-k(\alpha+1)} \gamma(\nu)^k$$

follows. Hence, in view of (5.20), we get

$$\sum_{n=2\nu}^{\infty} \leqq B\nu^{k\varepsilon-\delta} \nu^{-k(\alpha+1)} \gamma(\nu)^k = B\nu^{-\delta-1} \nu^{-k(\alpha+\lambda)} \gamma(\nu)^k. \tag{5.21}$$

This and (5.19) verify (5.18), whence, by (5.17), (3.5) follows for $k > 1$.

If $k = 1$ then we cannot apply Hölder's inequality in (5.15). But then we can use inequality (5.14) instead of (5.15). If we set $\lambda = \varepsilon = 0$, it is easy to check that all of the estimations (5.17)—(5.21) will be fulfilled with these parameters. Collecting these estimations we again get the required statement (3.5).

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. As an additional hypothesis, we may suppose that $\beta < \alpha$. For if $\beta \geqq \alpha$, then $\beta = \alpha_0 - \delta$, where $\alpha_0 > \alpha$ (since $\delta > 0$). Then the hypotheses continue to hold with $\alpha$ replaced by any $\alpha_1$ between $\alpha$ and $\alpha_0$ and thus (since $\alpha_0 > \alpha$), there exists some $\alpha_1$ so that $\alpha_1 > \beta$. From $\alpha_1$ to $\alpha$ we can get over by using Theorem 1 with $\beta = \alpha_1$.

In the case $\beta < \alpha$ we have $\delta > \alpha - \beta > 0$, further, by Hölder's inequality with indices $k/r$ and $k/(k-r)$,

$$(5.22) \qquad \sum \mu(n)^r n^{-1} |\tau_n^\beta|^r \leqq \left\{ \sum \mu(n)^k n^{k(\alpha-\beta)-1} |\tau_n^\beta|^k \right\}^{r/k} \times$$

$$\times \left\{ \sum n^{-1+rk(\beta-\alpha)/(k-r)} \right\}^{1-r/k} \leqq B \left\{ \sum \mu(n)^k n^{k(\alpha-\beta)-1} |\tau_n^\beta|^k \right\}^{r/k}.$$

Now we apply Theorem 2 with $\mu(t)t^{\alpha-\beta}$ and $\delta - (\alpha - \beta)$ in place of $\mu(t)$ and $\delta$, respectively; the application is legitimate in view of $\delta > \alpha - \beta > 0$ and (3.4). Then we get that

$$\sum \mu(n)^k n^{k(\alpha-\beta)-1} |\tau_n^\beta|^k \leqq B \sum \gamma(n)^k n^{-1} |\tau_n^\alpha|^k.$$

This and (5.22) prove (3.6), which completes the proof.

PROOF OF THEOREM 4. The convergence of series (3.8) follows from the obvious implication $|C, \alpha, \gamma(t)|_k \Rightarrow |C, \alpha, 0|_k \equiv |C, \alpha|_k$ and from a result of T.M. Flett [1] (Theorem 2) stating that if $k \geqq 1$ and $\alpha > -1$ then summability $|C, \alpha|_k$ implies summability $|A|_k \equiv |A, 0|_k$.

Thus it is enough to verify that (3.9) holds. Using the identity

$$x\phi'(x) = (1-x)^\alpha \sum E_n^\alpha \tau_n^\alpha x^n \quad (|x| < 1),$$

where $\phi(x)$ is the sum-function of the convergent series (3.8), and applying Hölder's inequality with indices $k$ and $k'$, we have for $0 \leqq x < 1$

$$|\phi'(x)|^k \leqq (1-x)^{k\alpha} \left( \sum E_n^\alpha |\tau_n^\alpha| x^{n-1} \right)^k \leqq$$

$$\leqq (1-x)^{k\alpha} \left( \sum E_n^\alpha |\tau_n^\alpha|^k x^{n-1} \right) \left( \sum E_n^\alpha x^{n-1} \right)^{k-1} \leqq$$

$$\leqq B(k, \alpha)(1-x)^{\alpha-k+1} \sum E_n^\alpha |\tau_n^\alpha|^k x^{n-1}.$$

Hence we get

$$I := \int_0^1 (1-x)^{k-1}\gamma\left((1-x)^{-1}\right)^k |\phi'(x)|^k dx \leqq$$

$$\leqq B(k,\alpha)\sum E_n^\alpha |\tau_n^\alpha|^k \int_0^1 (1-x)^\alpha \gamma\left((1-x)^{-1}\right)^k x^{n-1}dx.$$

Regarding (4.3), the previous inequality yields

$$I \leqq B\sum |\tau_n^\alpha|^k \gamma(n)^k n^{-1},$$

which proves (3.9), and hereby the proof is complete.

PROOF OF THEOREM 5. As we have seen at the proof of Theorem 2, (3.4) implies (5.16), whence

$$\mu(y)\gamma(y)^{-1} \leqq By^{-\delta}$$

follows for any $y \geqq 1$, or equivalently, for any $0 \leqq x < 1$,

$$\mu((1-x)^{-1})\gamma\left((1-x)^{-1}\right)^{-1} \leqq B(1-x)^\delta.$$

Regarding this estimation, and applying Hölder's inequality, we get

$$\int_0^1 (1-x)^{r-1}\mu\left((1-x)^{-1}\right)^r |\phi'(x)|^r dx \leqq$$

$$\leqq \left\{\int_0^1 (1-x)^{k-1}\gamma\left((1-x)^{-1}\right)^k |\phi'(x)|^k dx\right\}^{r/k} \times$$

$$\times\left\{\int_0^1 (1-x)^{-1}\left(\mu\left((1-x)^{-1}\right)/\gamma\left((1-x)^{-1}\right)\right)^{rk/(k-r)} dx\right\}^{1-r/k} \leqq$$

$$\leqq B\left\{\int_0^1 (1-x)^{k-1}\gamma\left((1-x)^{-1}\right)^k |\phi'(x)|^k\right\}^{r/k} \cdot$$

$$\cdot\left\{\int_0^1 (1-x)^{-1+\delta rk/(k-r)}dx\right\}^{1-r/k}.$$

Hence (3.10) clearly follows as desired.

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# A MATRIX OPERATIONAL CALCULUS

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The vectorial-matrix model and the matrix model of the Bittner operational calculus are introduced in this article. The primary properties of those models are also described.

## 1. Preliminaries

The Bittner operational calculus [2] is the system

$$CO(L^0, L^1, S, T_q, s_q, q, Q),$$

where $L^0$ and $L^1$ are linear spaces (over the same field $\Gamma$ of scalars) such that $L^1 \subset L^0$; the linear operation $S: L^1 \to L^0$ (denoted as $S \in L(L^1, L^0)$), called the (abstract) derivative, is a surjection. Moreover, a nonempty set $Q$ is the set of indices $q$ for the operations $T_q \in L(L^0, L^1)$ such that $ST_q f = f$, $f \in L^0$, called integrals and for the operations $s_q \in L(L^1, L^1)$ such that $s_q x = x - T_q S x$, $x \in L^1$, called limit conditions. The kernel of $S$, i.e. the set $\operatorname{Ker} S := \{c \in L^1: Sc = 0\}$, is called the space of constants for the derivative $S$.

THEOREM 1 [2]. *The abstract differential equation*

$$Sx = f, \quad f \in L^0, \quad x \in L^1$$

*with the limit condition*

$$s_q x = x_0, \quad q \in Q, \quad x_0 \in \operatorname{Ker} S$$

*has a unique solution*

$$x = x_0 + T_q f.$$

## 2. Basic properties

Let $\widetilde{Q} \subset Q$ be a set, which has more than one element. Consider an operational calculus

$$CO(L^0, L^1, S, T_q, s_q, q, Q),$$

in which

— $L^0$ is a commutative algebra with a unity $e \in L^1$, and $L^1$ is its subalgebra;

— the derivative $S$ satisfies the Leibniz condition

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy, \quad x, y \in L^1 \tag{1}$$

— the limit conditions $s_q, q \in \widetilde{Q}$ are multiplicative, i.e.

$$s_q(x \cdot y) = s_q x \cdot s_q y, \quad q \in \widetilde{Q}, \quad x, y \in L^1. \tag{2}$$

The mapping $I_{q_1}^{q_2} \in L(L^0, \operatorname{Ker} S)$ described by the formula

$$I_{q_1}^{q_2} f := (T_{q_1} - T_{q_2}) f = s_{q_2} T_{q_1} f, \quad q_1, q_2 \in Q, \quad f \in L^0$$

is called the operation of definite integration.

It is easy to verify that [13]

$$I_{q_1}^{q_2} Sx = R_{q_1}^{q_2} x, \quad q_1, q_2 \in Q, \quad x \in L^1 \tag{3}$$

and

$$I_{q_1}^{q_2}(x \cdot Sy) = R_{q_1}^{q_2}(x \cdot y) - I_{q_1}^{q_2}(Sx \cdot y), \quad q_1, q_2 \in Q, \quad x, y \in L^1, \tag{4}$$

where the operation $R_{q_1}^{q_2} \in L(L^1, \operatorname{Ker} S)$ is defined by the formula

$$R_{q_1}^{q_2} x := (s_{q_2} - s_{q_1}) x, \quad q_1, q_2 \in Q, \quad x \in L^1.$$

(3), (4) are called the Leibniz–Newton formula and the integration by parts formula, respectively.

We also have $e \in \operatorname{Ker} S$ and

$$(c, d \in \operatorname{Ker} S) \Rightarrow (cd \in \operatorname{Ker} S)$$

$$(c \in \operatorname{Ker} S \text{ is invertible in } L^1) \Rightarrow (c^{-1} \in \operatorname{Ker} S).$$

The other properties of the derivative $S$ satisfying the Leibniz condition (1) are discussed in the works [10]–[13].

Let $\mathrm{Mat}_{m\times n}(Z)$, $m, n \in N$ denote the set of all matrices with $m$ rows and $n$ columns, with elements belonging to the set $Z$. In the sets $\mathrm{Mat}_{m\times n}(L^k)$, $k = 0,1$ we define the usual operations of addition of matrices and multiplication of a matrix by a scalar.

COROLLARY 1. *The sets* $\mathrm{Mat}_{m\times n}(L^k)$, $k = 0,1$, $\mathrm{Mat}_{m\times n}(\mathrm{Ker}\, S)$ *are linear spaces (over the field* $\Gamma$*) such that*

$$\mathrm{Mat}_{m\times n}(\mathrm{Ker}\, S) \subset \mathrm{Mat}_{m\times n}(L^1) \subset \mathrm{Mat}_{m\times n}(L^0).$$

For the elements $\widehat{X} \in \mathrm{Mat}_{m\times r}(L^k)$, $\widehat{Y} \in \mathrm{Mat}_{r\times n}(L^k)$, $k = 0,1$ the product $\widehat{X} \cdot \widehat{Y}$ is defined as the usual matrix multiplication.

COROLLARY 2. *The sets* $\mathrm{Mat}_{n\times n}(L^k)$, $k = 0,1$ *are algebras (over the field* $\Gamma$*) with unity* $\widehat{E} := [\delta_{ij}e]_{n\times n}$, *where* $\delta_{ij}$ *denotes the Kronecker symbol.*

Let

$$S\widehat{X} := [Sx_{ij}]_{m\times n}, \quad T_q\widehat{F} := [T_q f_{ij}]_{m\times n}, \quad s_q\widehat{X} := [s_q x_{ij}]_{m\times n},$$

where $\widehat{F} := [f_{ij}] \in \mathrm{Mat}_{m\times n}(L^0)$, $\widehat{X} := [x_{ij}] \in \mathrm{Mat}_{m\times n}(L^1)$, $q \in Q$. Using the formulas (1)–(4) and the definitions of matrix operations, it is easy to show the following relations:

(5) $$S(\alpha\widehat{X} + \beta\widehat{Y}) = \alpha S\widehat{X} + \beta S\widehat{Y},\ \alpha,\beta \in \Gamma,\ \widehat{X},\widehat{Y} \in \mathrm{Mat}_{m\times n}(L^1),$$

(6) $$s_q(\alpha\widehat{X} + \beta\widehat{Y}) = \alpha s_q\widehat{X} + \beta s_q\widehat{Y},\ q \in Q,\ \alpha,\beta \in \Gamma,\ \widehat{X},\widehat{Y} \in \mathrm{Mat}_{m\times n}(L^1),$$

(7) $$T_q(\alpha\widehat{F} + \beta\widehat{G}) = \alpha T_q\widehat{F} + \beta T_q\widehat{G},\ q \in Q,\ \alpha,\beta \in \Gamma,\ \widehat{F},\widehat{G} \in \mathrm{Mat}_{m\times n}(L^0),$$

(8) $$ST_q\widehat{F} = \widehat{F},\ s_q\widehat{X} = \widehat{X} - T_qS\widehat{X},$$

$$q \in Q,\ \widehat{F} \in \mathrm{Mat}_{m\times n}(L^0),\ \widehat{X} \in \mathrm{Mat}_{m\times n}(L^1),$$

(9) $$S\widehat{C} = \widehat{0}, \quad \widehat{C} \in \mathrm{Mat}_{m\times n}(\mathrm{Ker}\, S),$$

(10) $$S(\widehat{X} \cdot \widehat{Y}) = S\widehat{X} \cdot \widehat{Y} + \widehat{X} \cdot S\widehat{Y},\ \widehat{X} \in \mathrm{Mat}_{m\times r}(L^1),\ \widehat{Y} \in \mathrm{Mat}_{r\times n}(L^1)$$

(the Leibniz formula),

(11) $$S(\widehat{X}^k) = \sum_{i=0}^{k-1} \widehat{X}^i \cdot S\widehat{X} \cdot \widehat{X}^{k-i-1}, \quad \widehat{X} \in \mathrm{Mat}_{n\times n}(L^1), \quad k \in N,$$

(12) $\quad$ if $\widehat{X}\cdot S\widehat{X}=S\widehat{X}\cdot\widehat{X}$, then $S(\widehat{X}^k)=k\widehat{X}^{k-1}S\widehat{X}=kS\widehat{X}\cdot\widehat{X}^{k-1}$,

$$\widehat{X}\in \mathrm{Mat}_{n\times n}(L^1),\quad k\in N,$$

(13) $\quad S(\widehat{C}\cdot\widehat{X})=\widehat{C}\cdot S\widehat{X},\quad S(\widehat{X}\cdot\widehat{D})=S\widehat{X}\cdot\widehat{D},\quad \widehat{X}\in \mathrm{Mat}_{n\times p}(L^1),$

$$\widehat{C}\in \mathrm{Mat}_{m\times n}(\mathrm{Ker}\,S),\ \widehat{D}\in \mathrm{Mat}_{p\times r}(\mathrm{Ker}\,S),$$

(14) $\quad s_q\widehat{C}=\widehat{C},\quad q\in Q,\quad \widehat{C}\in \mathrm{Mat}_{m\times n}(\mathrm{Ker}\,S),$

(15) $s_q(\widehat{X}\cdot\widehat{Y})=s_q\widehat{X}\cdot s_q\widehat{Y},\quad q\in\widetilde{Q},\quad \widehat{X}\in \mathrm{Mat}_{m\times r}(L^1),\widehat{Y}\in \mathrm{Mat}_{r\times n}(L^1)$

(the multiplication condition),

(16) $s_q(\widehat{C}\cdot\widehat{X})=\widehat{C}\cdot s_q\widehat{X},\ s_q(\widehat{X}\cdot\widehat{D})=s_q\widehat{X}\cdot\widehat{D},\ q\in\widetilde{Q},\ \widehat{X}\in \mathrm{Mat}_{n\times p}(L^1),$

$$\widehat{C}\in \mathrm{Mat}_{m\times n}(\mathrm{Ker}\,S),\ \widehat{D}\in \mathrm{Mat}_{p\times r}(\mathrm{Ker}\,S),$$

(17) $T_q(\widehat{C}\cdot\widehat{F})=\widehat{C}\cdot T_q\widehat{F},\ T_q(\widehat{F}\cdot\widehat{D})=T_q\widehat{F}\cdot\widehat{D},\ q\in\widetilde{Q},\ \widehat{F}\in \mathrm{Mat}_{n\times p}(L^0),$

$$\widehat{C}\in \mathrm{Mat}_{m\times n}(\mathrm{Ker}\,S),\ \widehat{D}\in \mathrm{Mat}_{p\times r}(\mathrm{Ker}\,S),$$

(18) $\quad I_{q_1}^{q_2}S\widehat{X}=R_{q_1}^{q_2}\widehat{X},\quad q_1,q_2\in Q,\quad \widehat{X}\in \mathrm{Mat}_{m\times n}(L^1)$

(the Leibniz–Newton formula),

(19) $I_{q_1}^{q_2}(\widehat{X}\cdot S\widehat{Y})=R_{q_1}^{q_2}(\widehat{X}\cdot\widehat{Y})-I_{q_1}^{q_2}(S\widehat{X}\cdot\widehat{Y}),\ q_1,q_2\in Q,\ \widehat{X}\in \mathrm{Mat}_{m\times r}(L^1),$

$$\widehat{Y}\in \mathrm{Mat}_{r\times n}(L^1)$$

(the integration by parts formula).

COROLLARY 3. $\mathrm{Mat}_{n\times n}(\mathrm{Ker}\,S)$ *is the subalgebra of the algebras* $\mathrm{Mat}_{n\times n}(L^k)$, $k=0,1$. *Moreover,* $\widehat{E}\in \mathrm{Mat}_{n\times n}(\mathrm{Ker}\,S)$.

The determinant of a matrix $\widehat{X}=[x_{ij}]\in \mathrm{Mat}_{n\times n}(L^k)$, $k=0,1$ is defined similarly to the numerical determinant. Namely, it is an element of the algebra $L^k$, $k=0,1$ defined by the formula

$$\det\widehat{X}=\begin{vmatrix} x_{11} & \cdots & x_{1n}\\ x_{21} & \cdots & x_{2n}\\ \cdots & \cdots & \cdots\\ x_{n1} & \cdots & x_{nn}\end{vmatrix}:=\sum_p(-1)^{I_p}x_{1j_1}x_{2j_2}\cdots x_{nj_n},$$

where the summation is extended to all permutations $p = (j_1, j_2, \dots, j_n)$ of numbers $1, 2, \dots, n$, whereas $I_p$ denotes the number of inversions in the permutation $p$. The rules of computing $\det \widehat{X}$ are the same as for a numerical determinant.

COROLLARY 4. *If* $\widehat{C} \in \mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S)$, *then* $\det \widehat{C} \in \mathrm{Ker}\, S$.

Let $\mathrm{Inv}(Z)$ denote the set of invertible elements in an algebra $Z$.

COROLLARY 5. $\widehat{X} \in \mathrm{Inv}(\mathrm{Mat}_{n\times n}(Z))$ *iff* $\det \widehat{X} \in \mathrm{Inv}(Z)$, *where* $Z$ *is the algebra* $L^k$, $k = 0, 1$ *or* $\mathrm{Ker}\, S$. *If* $\widehat{X} \in \mathrm{Inv}(\mathrm{Mat}_{n\times n}(L^1))$, *then* $s_q\widehat{X} \in \mathrm{Inv}(\mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S))$, $q \in \widetilde{Q}$. *Moreover,*

$$s_q(\widehat{X}^{-1}) = (s_q\widehat{X})^{-1}, \quad q \in \widetilde{Q}, \tag{20}$$

$$s_q(\det \widehat{X}) = \det(s_q\widehat{X}), \quad q \in \widetilde{Q}, \tag{21}$$

$$S(\widehat{X}^{-1}) = -\widehat{X}^{-1} \cdot S\widehat{X} \cdot \widehat{X}^{-1}. \tag{22}$$

THEOREM 2. *If* $\widehat{X} = [x_{ij}] \in \mathrm{Mat}_{n\times n}(L^1)$, *then*

$$S(\det \widehat{X}) = \sum_{i=1}^{n} \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Sx_{i1} & \cdots & Sx_{ij} & \cdots & Sx_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix}. \tag{23}$$

PROOF. (23) follows from the determinant definition and from the Leibniz formula (1).

## 3. The Cauchy matrix

Assume that there exists a solution $\widehat{X} \in \mathrm{Inv}(\mathrm{Mat}_{n\times n}(L^1))$ of the abstract matrix differential equation

$$S\widehat{X} = \widehat{A}\widehat{X}, \tag{24}$$

where $\widehat{A} \in \mathrm{Mat}_{n\times n}(L^0)$ is the given matrix. From (13) it follows that for an arbitrary matrix $\widehat{D} \in \mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S)$,

$$\widehat{Y} = \widehat{X}\widehat{D} \tag{25}$$

is also the solution of the equation (24). Let

$$FM(\widehat{A}) := \left\{\widehat{X} \in \operatorname{Inv}\left(\operatorname{Mat}_{n\times n}(L^1)\right) : S\widehat{X} = \widehat{A}\widehat{X}\right\}.$$

REMARK. For any pair $\widehat{X}, \widehat{Y} \in FM(\widehat{A})$, $\widehat{Y}$ has the form (25), where $\widehat{D} \in \operatorname{Inv}\left(\operatorname{Mat}_{n\times n}(\operatorname{Ker} S)\right)$. Indeed, if $\widehat{X}, \widehat{Y} \in FM(\widehat{A})$, then $\widehat{X}^{-1} \cdot \widehat{Y} \in \operatorname{Inv}\left(\operatorname{Mat}_{n\times n}(L^1)\right)$. Therefore, on the basis of (10) and (22), we obtain

$$S(\widehat{X}^{-1}\widehat{Y}) = -\widehat{X}^{-1}\widehat{A}\widehat{Y} + \widehat{X}^{-1}\widehat{A}\widehat{Y} = \widehat{0},$$

i.e. $\widehat{X}^{-1}\widehat{Y} = \widehat{D} \in \operatorname{Inv}\left(\operatorname{Mat}_{n\times n}(\operatorname{Ker} S)\right)$.

THEOREM 3. *The differential equation* (24) *with the limit condition*

$$s_{q_0}\widehat{X} = \widehat{X}_0, \quad q_0 \in \widetilde{Q}, \quad \widehat{X}_0 \in \operatorname{Inv}\left(\operatorname{Mat}_{n\times n}(\operatorname{Ker} S)\right) \tag{26}$$

*has exactly one solution in the set* $FM(\widehat{A})$.

PROOF. Assume that the problem (24), (26) has two solutions $\widehat{X}, \widehat{Y} \in FM(\widehat{A})$. Then by the Remark $\widehat{Y}$ is of the form (25) where $D$ is invertible and $s_{q_0}\widehat{Y} = \widehat{X}_0$. Hence and from (16) it follows that

$$\widehat{X}_0 = s_{q_0}\widehat{Y} = s_{q_0}(\widehat{X}\widehat{D}) = s_{q_0}\widehat{X} \cdot \widehat{D} = \widehat{X}_0 \cdot \widehat{D}. \tag{27}$$

Thus from (27) we obtain $\widehat{D} = \widehat{E}$. So $\widehat{X} = \widehat{Y}$.

In the case, when $n = 1$ the set $FM(a)$, where $a \in L^0$, is the set of all invertible solutions of the equation

$$Sx = ax, \quad x \in L^1. \tag{28}$$

So, if there exists a solution $x \in \operatorname{Inv}(L^1)$ of the equation (28) satisfying the limit condition

$$s_{q_0}x = x_0 \in \operatorname{Ker} S, \quad q_0 \in \widetilde{Q}, \tag{29}$$

then it is unique. That is the statement of Theorem 1 in [7] (see also [5]).

DEFINITION 1. Each element $\widehat{Y} \in FM(\widehat{A})$ will be called a fundamental matrix of the equation (24).

DEFINITION 2. The element

$$\widehat{\Phi}(q, q_0)(\widehat{A}) := (s_q\widehat{X}) \cdot \left(s_{q_0}\widehat{X}\right)^{-1}, \quad q, q_0 \in \widetilde{Q}, \tag{30}$$

where $\widehat{X} \in FM(\widehat{A})$, will be called the Cauchy matrix (belonging to $q, q_0$) of the equation (24).

THEOREM 4. 1. *The Cauchy matrix does not depend on choosing the fundamental matrix.*

2. $\widehat{\Phi}(q,q_0)(\widehat{A}) \in \operatorname{Inv}\bigl(\operatorname{Mat}_{n\times n}(\operatorname{Ker} S)\bigr)$.
3. $\det \widehat{\Phi}(q,q_0)(\widehat{A}) \in \operatorname{Inv}(\operatorname{Ker} S)$.
4. $\widehat{\Phi}(q_0,q_0)(\widehat{A}) = \widehat{E}$.
5. $\widehat{\Phi}^{-1}(q,q_0)(\widehat{A}) := \bigl[\widehat{\Phi}(q,q_0)(\widehat{A})\bigr]^{-1} = \widehat{\Phi}(q_0,q)(\widehat{A})$.
6. $\widehat{\Phi}(q_2,q_1)(\widehat{A}) \cdot \widehat{\Phi}(q_1,q_0)(\widehat{A}) = \widehat{\Phi}(q_2,q_0)(\widehat{A})$, $q, q_0, q_1, q_2 \in \widetilde{Q}$.

PROOF. 1. If $\widehat{Y}, X$ are the fundamental matrices of (24), then $\widehat{Y} = \widehat{X}\widehat{D}$, where $\widehat{D} \in \operatorname{Inv}\bigl(\operatorname{Mat}_{n\times n}(\operatorname{Ker} S)\bigr)$. Therefore

$$\widehat{\Phi}_{\widehat{Y}}(q,q_0)(\widehat{A}) = (s_q\widehat{Y}) \cdot (s_{q_0}\widehat{Y})^{-1} = \bigl[s_q(\widehat{X}\widehat{D})\bigr] \cdot \bigl[s_{q_0}(\widehat{X}\widehat{D})\bigr]^{-1} =$$

$$= (s_q\widehat{X}) \cdot \widehat{D} \cdot \widehat{D}^{-1} \cdot (s_{q_0}\widehat{X})^{-1} = \widehat{\Phi}_{\widehat{X}}(q,q_0)(\widehat{A}).$$

Properties 2 and 3 follow from Corollary 5.

4. $\widehat{\Phi}(q_0,q_0)(\widehat{A}) = (s_{q_0}\widehat{X}) \cdot (s_{q_0}\widehat{X})^{-1} = \widehat{E}$.

5. $\widehat{\Phi}^{-1}(q,q_0)(\widehat{A}) = \bigl[(s_q\widehat{X}) \cdot (s_{q_0}\widehat{X})^{-1}\bigr]^{-1} = (s_{q_0}\widehat{X}) \cdot (s_q\widehat{X})^{-1} = \widehat{\Phi}(q_0,q)(\widehat{A})$.

6. $\widehat{\Phi}(q_2,q_1)(\widehat{A}) \cdot \widehat{\Phi}(q_1,q_0)(\widehat{A}) = (s_{q_2}\widehat{X}) \cdot (s_{q_1}\widehat{X})^{-1} \cdot (s_{q_1}\widehat{X}) \cdot (s_{q_0}\widehat{X})^{-1} = (s_{q_2}\widehat{X}) \cdot (s_{q_0}\widehat{X})^{-1} = \widehat{\Phi}(q_2,q_0)(\widehat{A})$.

DEFINITION 3. An element $\widehat{X} \in FM(\widehat{A})$ being the solution of the problem

$$S\widehat{X} = \widehat{A}\widehat{X}, \quad s_{q_0}\widehat{X} = \widehat{E}, \quad q_0 \in \widetilde{Q}$$

will be called a normalized fundamental matrix (belonging to $q_0$) and denoted as $\widehat{\Phi}_{q_0}(\widehat{A})$.

COROLLARY 6. *The solution of the problem* (24), (26) *has the form*

$$\widehat{X} = \widehat{\Phi}_{q_0}(\widehat{A}) \cdot \widehat{X}_0. \tag{31}$$

DEFINITION 4 (cf. [11]). The mapping defined by the formula

$$\widehat{X}_0 \mapsto \widehat{\Phi}_{q_0}(\widehat{A}) \cdot \widehat{X}_0$$

will be called the resolvent of the equation (24) (belonging to $q_0$).

It follows from (31) and from the definition of the Cauchy matrix that

$$s_q\widehat{X} = s_q\widehat{\Phi}_{q_0}(\widehat{A}) \cdot \widehat{X}_0 = \widehat{\Phi}(q, q_0)(\widehat{A}) \cdot \widehat{X}_0.$$

COROLLARY 7 (cf. [9]). $s_q\widehat{\Phi}_{q_0}(\widehat{A}) = \widehat{\Phi}(q, q_0)(\widehat{A})$.

This is a direct consequence of Definitions 1 and 3.

## 4. The Ostrogradski–Liouville–Jacobi formula

DEFINITION 5. The element

$$w = W(\widehat{X}) := \det \widehat{X}$$

will be called the Wroński determinant (briefly: the Wrońskian) of the fundamental matrix $\widehat{X} \in FM(\widehat{A})$.

COROLLARY 8. $W(\widehat{X}) \in \mathrm{Inv}(L^1)$ *for all* $\widehat{X} \in FM(\widehat{A})$.

Let $\widehat{X} = [x_{ij}]_{n\times n}$ be a fixed fundamental matrix of (24). Using Theorem 2 we obtain for the derivative of the Wrońskian

$$Sw = \sum_{i=1}^{n} w_i,$$

where

$$w_i := \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Sx_{i1} & \cdots & Sx_{ij} & \cdots & Sx_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix}, \qquad i = 1, 2, \ldots, n. \tag{32}$$

Equation (24) is equivalent to the system of the following $n^2$ "scalar" abstract differential equations:

$$Sx_{ij} = \sum_{k=1}^{n} a_{ik} \cdot x_{kj}, \qquad i, j = 1, 2, \ldots, n.$$

Substituting these into (32), and using the known determinant properties, we obtain

$$w_i = a_{ii} \cdot w, \qquad i = 1, 2, \ldots, n.$$

Finally,

$$Sw = \operatorname{tr}\widehat{A}\cdot w,$$

where $\operatorname{tr}\widehat{A} \in L^0$ is the trace of the matrix $\widehat{A} \in \mathrm{Mat}_{n\times n}(L^0)$. Since $w \in \operatorname{Inv}(L^1)$, therefore

$$w = \Phi_{q_0}(\operatorname{tr}\widehat{A})\cdot w_0, \tag{33}$$

where $\Phi_{q_0}(\operatorname{tr}\widehat{A}) \in FM(\operatorname{tr}\widehat{A})$ is the normalized fundamental element and $w_0 = s_{q_0}w$, $q_0 \in \widetilde{Q}$ (cf. the problem (28), (29)).

(33) will be called the Ostrogradski–Liouville–Jacobi formula.

## 5. The linear transformation

Let

$$\widehat{Y} = \widehat{P}\widehat{X}, \tag{34}$$

where $\widehat{P} \in \operatorname{Inv}\bigl(\mathrm{Mat}_{n\times n}(L^1)\bigr)$, $\widehat{X} \in FM(\widehat{A})$. Then

$$S\widehat{Y} = S\widehat{P}\cdot\widehat{X} + \widehat{P}\cdot S\widehat{X} = (S\widehat{P} + \widehat{P}\cdot\widehat{A})\widehat{X} = (S\widehat{P}\cdot\widehat{P}^{-1} + \widehat{P}\cdot\widehat{A}\cdot\widehat{P}^{-1})\widehat{Y} \tag{35}$$

and

$$\widehat{Y}_0 = S_{q_0}\widehat{Y} = \widehat{P}_0\cdot\widehat{X}_0, \qquad q_0 \in \widetilde{Q}, \tag{36}$$

where $\widehat{P}_0 = s_{q_0}\widehat{P}$, $\widehat{X}_0 = s_{q_0}\widehat{X}$. Hence $\widehat{P}FM(\widehat{A},n) = FM(\widehat{B},n)$, where $\widehat{B} = S\widehat{P}\cdot\widehat{P}^{-1} + \widehat{P}\cdot\widehat{A}\cdot\widehat{P}^{-1}$. From (35) and (36) we obtain

$$\widehat{Y} = \widehat{\Phi}_{q_0}(\widehat{B})\widehat{Y}_0 = \widehat{\Phi}_{q_0}(S\widehat{P}\cdot\widehat{P}^{-1} + \widehat{P}\cdot\widehat{A}\cdot\widehat{P}^{-1})\cdot\widehat{P}_0\widehat{X}_0.$$

Therefore

$$\widehat{\Phi}_{q_0}(\widehat{A})\cdot\widehat{X}_0 = \widehat{X} = \widehat{P}^{-1}\widehat{Y} = \widehat{P}^{-1}\cdot\widehat{\Phi}_{q_0}(S\widehat{P}\cdot\widehat{P}^{-1} + \widehat{P}\cdot\widehat{A}\cdot\widehat{P}^{-1})\cdot\widehat{P}_0\widehat{X}_0,$$

whence

$$\widehat{\Phi}_{q_0}(\widehat{A}) = \widehat{P}^{-1}\widehat{\Phi}_{q_0}(S\widehat{P}\cdot\widehat{P}^{-1} + \widehat{P}\widehat{A}\widehat{P}^{-1})\cdot s_{q_0}\widehat{P} = \widehat{P}^{-1}\widehat{\Phi}_{q_0}(\widehat{B})s_{q_0}\widehat{P}. \tag{37}$$

This shows the connection between the normalized fundamental matrices $\Phi_{q_0}(\widehat{A}) \in FM(\widehat{A},n)$ and $\widehat{\Phi}_{q_0}(\widehat{B}) \in FM(\widehat{B},n) = \widehat{P}FM(\widehat{A},n)$.

## 6. The adjoint equations

If $\Gamma$ is the field of reals or complexes and $\widehat{A}^+ \in \mathrm{Mat}_{n\times n}(L^0)$ denotes the adjoint matrix with the matrix $\widehat{A} \in \mathrm{Mat}_{n\times n}(L^0)$, i.e. $\widehat{A}^+ := (\widehat{A}^*)^t$ (the symbols "$*$", "$t$" denote the complex conjugation and the transposition, respectively), then we can introduce the following

DEFINITION 6. The abstract matrix differential equation

$$S\widehat{Y} = -\widehat{A}^+ \cdot \widehat{Y}, \tag{38}$$

where $\widehat{Y} \in \mathrm{Mat}_{n\times n}(L^1)$, will be called the adjoint equation to (24).

Between the elements $\widehat{X} \in FM(\widehat{A})$ and $\widehat{Y} \in FM(-\widehat{A}^+)$ the relation

$$\widehat{Y}^+ \cdot \widehat{X} = \widehat{C}, \qquad \widehat{C} \in \mathrm{Inv}\big(\mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S)\big) \tag{39}$$

holds.

Indeed, applying $S\widehat{Y}^+ = (S\widehat{Y})^+$ (cf. [8]), we have

$$S(\widehat{Y}^+ \cdot \widehat{X}) = (S\widehat{Y})^+ \cdot \widehat{X} + \widehat{Y}^+ \cdot S\widehat{X} = -\widehat{Y}^+\widehat{A}\widehat{X} + \widehat{Y}^+\widehat{A}\widehat{X} = \widehat{0}$$

which means that $\widehat{Y}^+ \cdot \widehat{X} \in \mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S)$. The invertibility of $\widehat{Y}^+ \cdot \widehat{X}$ is evident. From (39) we obtain

$$\widehat{Y} = (\widehat{X}^{-1})^+ \cdot \widehat{C}^+, \quad s_q\widehat{Y} = (s_q\widehat{X}^{-1})^+ \cdot \widehat{C}^+, \quad q \in \widetilde{Q},$$

because $s_q\widehat{Y}^+ = (s_q\widehat{Y})^+$, $q \in Q$ (cf. [8]). Hence, in particular, if $\widehat{C} = \widehat{E}$, we obtain the relation between the normalized fundamental matrices of equations (24) and (38):

$$\widehat{\Phi}_{q_0}(-\widehat{A}^+) = \big[\widehat{\Phi}_{q_0}^{-1}(\widehat{A})\big]^+.$$

COROLLARY 9. *If the equation* (24) *is self-adjoint, i.e.* $\widehat{A} = \widehat{A}^+$, *then*

$$\widehat{\Phi}_{q_0}(\widehat{A}) = \big[\widehat{\Phi}_{q_0}^{-1}(-\widehat{A})\big]^+.$$

## 7. The reducible equations

DEFINITION 7. If $\widehat{A} \in \mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S)$, then the abstract matrix differential equation (24) will be called the stationary equation. In the opposite case (24) is nonstationary.

DEFINITION 8. Equation (24) will be called reducible if there exists an element $\widehat{P} \in \mathrm{Inv}(\mathrm{Mat}_{n\times n}(L^1))$ such that applying the linear transformation (34) with this $\widehat{P}$, (35) will turn to a stationary equation $S\widehat{Y} = \widehat{B}\widehat{Y}$. In this case $\widehat{P}$ and the transformation (34) will be called the Liapunov matrix and the Liapunov transformation, respectively.

THEOREM 5. *The differential equation* (24) *is reducible iff its normalized fundamental matrix takes the form*

$$\widehat{\Phi}_{q_0}(\widehat{A}) = \widehat{P}^{-1} \cdot \widehat{\Phi}_{q_0}(\widehat{B}) \cdot s_{q_0}\widehat{P} \quad (\textit{the Yerugin formula}), \tag{40}$$

*where* $\widehat{P} \in \mathrm{Inv}(\mathrm{Mat}_{n\times n}(L^1))$, $\widehat{\Phi}_{q_0}(\widehat{B}) \in FM(\widehat{B})$, $\widehat{B} \in \mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S)$.

PROOF. *Necessity.* Assume that (24) is reducible. Then some Liapunov transformation $\widehat{Y} = \widehat{P}\widehat{X}$ transforms it into the stationary equation

$$S\widehat{Y} = \widehat{B}\widehat{Y} \tag{41}$$

for which $\widehat{\Phi}_{q_0}(\widehat{B})$ is the normalized fundamental matrix. Equation (41) with the limit condition

$$s_{q_0}\widehat{Y} = s_{q_0}\widehat{P} \cdot \widehat{X}_0, \qquad q_0 \in \widetilde{Q},$$

where $\widehat{X}_0 = s_{q_0}\widehat{X}$, has the solution

$$\widehat{Y} = \widehat{\Phi}_{q_0}(\widehat{B}) \cdot s_{q_0}\widehat{P} \cdot \widehat{X}_0.$$

Hence

$$\widehat{X} = \widehat{\Phi}_{q_0}(\widehat{A}) \cdot \widehat{X}_0 = \widehat{P}^{-1} \cdot \widehat{Y} = \widehat{P}^{-1} \cdot \widehat{\Phi}_{q_0}(\widehat{B}) \cdot s_{q_0}\widehat{P} \cdot \widehat{X}_0,$$

whence the Yerugin formula (40) follows.

*Sufficiency.* Assume that the matrix $\widehat{P} \in \mathrm{Inv}(\mathrm{Mat}_{n\times n}(L^1))$ satisfies (40) with the matrices $\widehat{A} \in \mathrm{Mat}_{n\times n}(L^0)$ and $\widehat{B} \in \mathrm{Mat}_{n\times n}(\mathrm{Ker}\, S)$. Then

$$\widehat{P} = \widehat{\Phi}_{q_0}(\widehat{B}) \cdot s_{q_0}\widehat{P} \cdot \widehat{\Phi}_{q_0}^{-1}(\widehat{A}).$$

Since $\widehat{\Phi}_{q0}(\widehat{A}) \in FM(\widehat{A})$, we have for the derivative of the transformed matrix $\widehat{Y} = \widehat{P}\widehat{\Phi}_{q0}(\widehat{A})$

$$S\widehat{Y} = S\big(\widehat{P}\widehat{\Phi}_{q0}(\widehat{A})\big) = (S\widehat{P} + \widehat{P}\widehat{A})\widehat{\Phi}_{q0}(\widehat{A}). \tag{42}$$

Taking into account that $S\widehat{\Phi}_{q0}^{-1}(\widehat{A}) = -\widehat{\Phi}_{q0}^{-1}(\widehat{A}) \cdot \widehat{A}$ and therefore

$$S\widehat{P} = \widehat{B}\widehat{P} - \widehat{P}\widehat{A},$$

after substituting (42) we have

$$S\widehat{Y} = (S\widehat{P} + \widehat{P}\widehat{A})\widehat{\Phi}_{q0}(\widehat{A}) = (\widehat{B}\widehat{P} - \widehat{P}\widehat{A} + \widehat{P}\widehat{A})\widehat{\Phi}_{q0}(\widehat{A}) = \widehat{B}\widehat{P}\widehat{\Phi}_{q0}(\widehat{A}) = \widehat{B}\widehat{Y},$$

where by assumption $\widehat{B} \in \mathrm{Mat}_{n\times n}(\operatorname{Ker} S)$. On the other hand by (35)

$$\widehat{B} = S\widehat{P} \cdot \widehat{P}^{-1} + \widehat{P}\widehat{A}\widehat{P}^{-1}.$$

Hence (24) is reducible and $\widehat{P}$ is the Liapunov matrix.

## 8. The vectorial-matrix Cauchy model of the operational calculus

THEOREM 6. *If for a fixed matrix $\widehat{A} \in \mathrm{Mat}_{n\times n}(L^0)$ and for all $q \in \widetilde{Q}$ the elements $\widehat{\Phi}_q(\widehat{A})$ exist, then the system*

$$(L_n^0, L_n^1, \overline{S}, \overline{T}_q, \overline{s}_q, q, \widetilde{Q}),$$

*where*

$$L_n^0 := \mathrm{Mat}_{n\times 1}(L^0), \qquad L_n^1 := \mathrm{Mat}_{n\times 1}(L^1)$$

*and*

$$\overline{S}\overline{x} := S\overline{x} - \widehat{A} \cdot \overline{x}, \quad \overline{x} \in L_n^1,$$

$$\overline{T}_q\overline{f} := \widehat{\Phi}_q(\widehat{A}) \cdot T_q\big[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{f}\big], \quad q \in \widetilde{Q}, \quad \overline{f} \in L_n^0,$$

$$\overline{s}_q\overline{x} := \widehat{\Phi}_q(\widehat{A}) \cdot s_q\overline{x}, \quad q \in \widetilde{Q}, \quad \overline{x} \in L_n^1$$

*forms an operational calculus.*

PROOF. It is easy to notice that $\overline{S}$, $\overline{T}_q$, $\overline{s}_q$ are linear operations. Utilizing the axioms of the operational calculus $CO(L^0, L^1, S, T_q, s_q, q, Q)$ and the known properties of the derivative, the integrals and the limit conditions, we obtain

$$\overline{ST_q}\overline{f} = \overline{S}\{\widehat{\Phi}_q(\widehat{A}) \cdot T_q[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{f}]\} = S\{\widehat{\Phi}_q(\widehat{A}) \cdot T_q[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{f}]\} -$$

$$-\widehat{A} \cdot \widehat{\Phi}_q(\widehat{A}) \cdot T_q\big[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{f}\big] =$$

$$= \widehat{A} \cdot \widehat{\Phi}_q(\widehat{A}) \cdot T_q\big[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{f}\big] + \widehat{\Phi}_q(\widehat{A}) \cdot \widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{f} -$$

$$-\widehat{A} \cdot \widehat{\Phi}_q(\widehat{A}) \cdot T_q\big[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{f}\big] = \overline{f}$$

or

$$\overline{ST_q}\overline{f} = \overline{f}, \quad q \in \widetilde{Q}, \quad , \overline{f} \in L_n^0.$$

We also have

$$\overline{T}_q\overline{S}\overline{x} = \overline{T}_q(S\overline{x} - \widehat{A} \cdot \overline{x}) = \widehat{\Phi}_q(\widehat{A}) \cdot T_q\big[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot S\overline{x} - \widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \widehat{A} \cdot \overline{x}\big] =$$

$$= \widehat{\Phi}_q(\widehat{A}) \cdot T_q S\big[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{x}\big] = \widehat{\Phi}_q(\widehat{A})\big[\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot \overline{x} - s_q\widehat{\Phi}_q^{-1}(\widehat{A}) \cdot s_q\overline{x}\big] =$$

$$= \overline{x} - \widehat{\Phi}_q(\widehat{A}) \cdot s_q\overline{x}$$

or

$$\overline{T}_q\overline{S}\overline{x} = \overline{x} - \overline{s}_q\overline{x}, \quad q \in \widetilde{Q}, \quad \overline{x} \in L_n^1.$$

Therefore $\overline{S}$ is the derivative, $\overline{T}_q$, $q \in \widetilde{Q}$ are integrals and $\overline{s}_q$, $q \in \widetilde{Q}$ are limit conditions (cf. Theorem 2 [7]).

DEFINITION 9. The system

$$CO(L_n^0, L_n^1, \overline{S}, \overline{T}_q, \overline{s}_q, q, \widetilde{Q}) \tag{43}$$

will be called the vectorial-matrix Cauchy model of the Bittner operational calculus.

COROLLARY 10. *If $\widehat{A} \in \mathrm{Mat}_{n\times n}(L^0)$, $\widehat{B} \in \mathrm{Mat}_{n\times m}(L^0)$, $\overline{u} \in L_m^0$ are given and for a certain $q_0 \in \widetilde{Q}$ the element $\widehat{\Phi}_{q_0}(\widehat{A})$ exists, then the abstract vectorial-matrix differential equation*

$$S\overline{x} = \widehat{A}\overline{x} + \widehat{B}\overline{u}, \qquad \overline{x} \in L_n^1 \tag{44}$$

*with the limit condition*

$$s_{q_0}\overline{x} = \overline{x}_0, \qquad \overline{x}_0 \in \mathrm{Mat}_{n\times 1}(\mathrm{Ker}\, S) \tag{45}$$

*has a unique solution defined by the Cauchy formula*

$$\overline{x} = \widehat{\Phi}_{q_0}(\widehat{A}) \cdot \overline{x}_0 + \widehat{\Phi}_{q_0}(\widehat{A}) \cdot T_{q_0}\left[\widehat{\Phi}_{q_0}^{-1}(\widehat{A}) \cdot \widehat{B}\overline{u}\right]. \tag{46}$$

PROOF. In the Cauchy model (43) of the operational calculus, the problem (44), (45) takes the form

$$\overline{S}\overline{x} = \widehat{B}\overline{u}, \qquad \overline{s}_{q_0}\overline{x} = \widehat{\Phi}_{q_0}(\widehat{A}) \cdot \overline{x}_0.$$

Hence and from Theorem 1 the Cauchy formula (46) follows.

## 9. The matrix Cauchy model of the operational calculus

THEOREM 7. *If for fixed matrices $\widehat{A}_1, \widehat{A}_2 \in \mathrm{Mat}_{n\times n}(L^0)$ and for all $q \in \widetilde{Q}$ the elements $\widehat{\Phi}_q(\widehat{A}_1)$ and $\widehat{\Phi}_q(\widehat{A}_2)$ exist, then the system*

$$\left(\mathrm{Mat}_{n\times n}(L^0), \mathrm{Mat}_{n\times n}(L^1), \widehat{S}, \widehat{T}_q, \widehat{s}_q, q, \widetilde{Q}\right),$$

*where*

$$\widehat{S}\widehat{X} := S\widehat{X} - \widehat{A}_1 \cdot \widehat{X} - \widehat{X} \cdot \widehat{A}_2, \quad \widehat{X} \in \mathrm{Mat}_{n\times n}(L^1),$$

$$\widehat{T}_q\widehat{F} := \widehat{\Phi}_q(\widehat{A}_1) \cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1) \cdot \widehat{F} \cdot \left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\} \cdot \widehat{\Phi}_q^t(\widehat{A}_2^t),$$

$$q \in \widetilde{Q}, \quad \widehat{F} \in \mathrm{Mat}_{n\times n}(L^0),$$

$$\widehat{s}_q\widehat{X} := \widehat{\Phi}_q(\widehat{A}_1) \cdot s_q\widehat{X} \cdot \widehat{\Phi}_q^t(\widehat{A}_2^t), \quad q \in \widetilde{Q}, \quad \widehat{X} \in \mathrm{Mat}_{n\times n}(L^1)$$

*forms an operational calculus.*

PROOF. It is easy to verify that $\widehat{S}$, $\widehat{T}_q$, $\widehat{s}_q$ are linear operations. We also have

$$\widehat{S}\widehat{T}_q\widehat{F} = S\left\{\widehat{\Phi}_q(\widehat{A}_1) \cdot T_q\left[\widehat{\Phi}_q^{-1}(\widehat{A}_1) \cdot \widehat{F} \cdot \left(\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right)^t\right] \cdot \widehat{\Phi}_q^t(\widehat{A}_2^t)\right\} -$$

$$-\widehat{A}_1 \cdot \widehat{\Phi}_q(\widehat{A}_1) \cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1) \cdot \widehat{F} \cdot \left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\} \cdot \widehat{\Phi}_q^t(\widehat{A}_2^t) -$$

$$-\widehat{\Phi}_q(\widehat{A}_1) \cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1) \cdot \widehat{F} \cdot \left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\} \cdot \widehat{\Phi}_q^t(\widehat{A}_2^t) \cdot \widehat{A}_2 =$$

$$= \widehat{A}_1 \cdot \widehat{\Phi}_q(\widehat{A}_1) \cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1) \cdot \widehat{F} \cdot \left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\} \cdot \widehat{\Phi}_q^t(\widehat{A}_2^t) +$$

$$+\widehat{\Phi}_q(\widehat{A}_1)\cdot\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot\widehat{F}\cdot\left[\widehat{\Phi}_q(\widehat{A}_2^t)\cdot\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t+$$

$$+\widehat{\Phi}_q(\widehat{A}_1)\cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot\widehat{F}\cdot\left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\}\cdot\widehat{\Phi}_q^t(\widehat{A}_2^t)\cdot\widehat{A}_2-$$

$$-\widehat{A}_1\cdot\widehat{\Phi}_q(\widehat{A}_1)\cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot\widehat{F}\cdot\left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\}\cdot\widehat{\Phi}_q^t(\widehat{A}_2^t)-$$

$$-\widehat{\Phi}_q(\widehat{A}_1)\cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot\widehat{F}\cdot\left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\}\cdot\widehat{\Phi}_q^t(\widehat{A}_2^t)\cdot\widehat{A}_2=\widehat{F}$$

or

$$\widehat{S}\widehat{T}_q\widehat{F}=\widehat{F},\quad q\in\widetilde{Q},\quad \widehat{F}\in\operatorname{Mat}_{n\times n}(L^0).$$

Moreover,

$$\widehat{T}_q\widehat{S}\widehat{X}=$$

$$=\widehat{\Phi}_q(\widehat{A}_1)\cdot T_q\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot[S\widehat{X}-\widehat{A}_1\cdot\widehat{X}-\widehat{X}\cdot\widehat{A}_2]\cdot\left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\}\cdot\widehat{\Phi}_q^t(\widehat{A}_2^t)=$$

$$=\widehat{\Phi}_q(\widehat{A}_1)\cdot T_qS\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot\widehat{X}\cdot\left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\}\cdot\widehat{\Phi}_q^t(\widehat{A}_2^t)=$$

$$=\widehat{\Phi}_q(\widehat{A}_1)\cdot\left\{\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot\widehat{X}\cdot\left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t-\right.$$

$$\left.-s_q\widehat{\Phi}_q^{-1}(\widehat{A}_1)\cdot s_q\widehat{X}\cdot s_q\left[\widehat{\Phi}_q^{-1}(\widehat{A}_2^t)\right]^t\right\}\cdot\widehat{\Phi}_q^t(\widehat{A}_2^t)=$$

$$=\widehat{X}-\widehat{\Phi}_q(\widehat{A}_1)\cdot s_q\widehat{X}\cdot\widehat{\Phi}_q^t(\widehat{A}_2^t)$$

or

$$\widehat{T}_q\widehat{S}\widehat{X}=\widehat{X}-\widehat{s}_q\widehat{X},\quad q\in\widetilde{Q},\quad \widehat{X}\in\operatorname{Mat}_{n\times n}(L^1).$$

Therefore $\widehat{S}$ is the derivative, $\widehat{T}_q$, $q\in\widetilde{Q}$ are integrals and $\widehat{s}_q$, $q\in\widetilde{Q}$ are limits conditions.

DEFINITION 10. The system

$$CO\big(\operatorname{Mat}_{n\times n}(L^0),\operatorname{Mat}_{n\times n}(L^1),\widehat{S},\widehat{T}_q,\widehat{s}_q,q,\widetilde{Q}\big)\tag{47}$$

will be called the matrix Cauchy model of the Bittner operational calculus.

COROLLARY 11. *If $\widehat{A}_1,\widehat{A}_2,\widehat{F}\in\operatorname{Mat}_{n\times n}(L^0)$ are given and for a certain $q_0\in\widetilde{Q}$ the elements $\widehat{\Phi}_{q_0}(\widehat{A}_1)$, $\widehat{\Phi}_{q_0}(\widehat{A}_2)$ exist, then the abstract matrix differential equation*

$$S\widehat{X}=\widehat{A}_1\widehat{X}+\widehat{X}\widehat{A}_2+\widehat{F},\quad \widehat{X}\in\operatorname{Mat}_{n\times n}(L^1)\tag{48}$$

*with the limit condition*

$$s_{q_0}\widehat{X} = \widehat{X}_0, \qquad \widehat{X}_0 \in \mathrm{Mat}_{n\times n}(\operatorname{Ker} S) \tag{49}$$

*has a unique solution defined by the Cauchy formula*

$$\widehat{X} = \widehat{\Phi}_{q_0}(\widehat{A}_1)\cdot \widehat{X}_0 \cdot \widehat{\Phi}^t_{q_0}(\widehat{A}^t_2) + \tag{50}$$

$$+\widehat{\Phi}_{q_0}(\widehat{A}_1)\cdot T_{q_0}\left\{\widehat{\Phi}^{-1}_{q_0}(\widehat{A}_1)\cdot \widehat{F}\cdot \left[\widehat{\Phi}^{-1}_{q_0}(\widehat{A}^t_2)\right]^t\right\}\cdot \widehat{\Phi}^t_{q_0}(\widehat{A}^t_2).$$

PROOF. In the Cauchy model (47) of the operational calculus, the problem (48), (49) takes the form

$$\widehat{S}\widehat{X} = \widehat{F}, \quad \widehat{s}_{q_0}\widehat{X} = \widehat{\Phi}_{q_0}(\widehat{A}_1)\cdot \widehat{X}_0 \cdot \widehat{\Phi}^t_{q_0}(\widehat{A}^t_2).$$

Hence and from Theorem 1 the Cauchy formula (50) follows.

## 10. Examples

Let us consider the abstract differential equation

$$S^2 x = u \tag{51}$$

with the limit conditions

$$s_{q_0}x = c_0, \qquad s_{q_0}Sx = c_1, \tag{52}$$

where $x \in L^2 := \{x \in L^1 \colon Sx \in L^1\}$, $u \in L^0$, $c_0, c_1 \in \operatorname{Ker} S$, $q_0 \in Q$. Putting $x_1 := x$, $x_2 := Sx_1$ we can represent the problem (51), (52) in the vectorial-matrix form

$$S\overline{x} = \widehat{A}\overline{x} + \widehat{B}u, \qquad s_{q_0}\overline{x} = \overline{x}_0, \tag{53}$$

where

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} 0 \\ e \end{bmatrix}, \quad \overline{x}_0 = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

and $e$ is the unity in the algebra $L^0$. If the derivative $S$ satisfies the Leibniz condition and the limits condition $s_{q_0}$ is multiplicative, then the solution of the problem (53) can be obtained on the basis of the Cauchy formula (46).

A. The non-homogeneous Euler differential equation of the second order

$$t^2\ddot{x} + t\dot{x} = u \tag{54}$$

with the initial conditions

$$x(t_0) = d_0, \qquad \dot{x}(t_0) = d_1 \tag{55}$$

can be reduced into the form (53) if we consider the operational calculus in which

$$L^0 := C^0(Q, R^1), \qquad L^1 := C^1(Q, R^1)$$

and

$$Sx := \left\{ t\frac{dx}{dt} \right\}, \quad T_{t_0} f := \left\{ \int_{t_0}^{t} \frac{f(\tau)}{\tau}\, d\tau \right\}, \quad s_{t_0} x := \left\{ x(t_0) \right\},$$

where $f = \{ f(t) \} \in L^0$, $x = \{ x(t) \} \in L^1$, $q_0 = t_0 \in Q := (0, +\infty)$. The initial conditions (55) determine the limit conditions (52). Namely, $c_0 = \{d_0\}$, $c_1 = \{t_0 d_1\}$.

With the usual multiplication of functions, the spaces $L^0$, $L^1$ are commutative algebras with unity $e = \{1\}$, the derivative $S$ satisfies the Leibniz condition and the operations $s_{t_0}$ are multiplicative.

It is easy to verify that in the considered model of the operational calculus the matrix

$$\widehat{\Phi}_{t_0}(\widehat{A}) = \begin{bmatrix} 1 & \ln \frac{t}{t_0} \\ 0 & 1 \end{bmatrix}$$

is the normalized fundamental matrix corresponding to the matrix $\widehat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore from the Cauchy formula (46) we obtain

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & \ln \frac{t}{t_0} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ t_0 d_1 \end{bmatrix} + \begin{bmatrix} 1 & \ln \frac{t}{t_0} \\ 0 & 1 \end{bmatrix} \int_{t_0}^{t} \begin{bmatrix} 1 & \ln \frac{t_0}{\tau} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{u(\tau)}{\tau}\, d\tau,$$

whence it follows that the function

$$x = \left\{ d_0 + t_0 d_1 \ln \frac{t}{t_0} + \int_{t_0}^{t} \frac{u(\tau)}{\tau} \ln \frac{t}{\tau}\, d\tau \right\}$$

is the solution of the problem (54), (55).

B. In the operational calculus [3, 7] with the derivative

$$S_x := \left\{ \frac{\partial x}{\partial t} + \frac{\partial x}{\partial z} \right\},$$

the integrals

$$T_{t_0} f := \left\{ \int_{t_0}^{t} f(\tau, z - t + \tau)\, d\tau \right\}$$

and the limits conditions

$$s_{t_0}x := \{ x(t_0, z - t + t_0) \},$$

where

$$f = \{ f(t,z) \} \in L^0 := C^1(R^1 \times R^1, R^1),$$

$$x = \{ x(t,z) \} \in L^1 := \{ x \in L^0 \colon Sx \in L^0 \}, \quad q_0 = t_0 \in Q := R^1,$$

the differential equation (51) takes the form of the partial equation of the second order of the parabolic type

$$\frac{\partial^2 x}{\partial t^2} + 2\frac{\partial^2 x}{\partial t \partial z} + \frac{\partial^2 x}{\partial z^2} = u. \tag{56}$$

The limit conditions (52) are determined by the Cauchy conditions

$$x(t_0, z) = \varphi(z), \qquad x'_t(t_0, z) = \psi(z), \tag{57}$$

where $\varphi \in C^3(R^1, R^1)$, $\psi \in C^2(R^1, R^1)$. Then

$$c_0 = \{ \varphi(z - t + t_0) \}, \quad c_1 = \{ \varphi'(z - t + t_0) + \psi(z - t + t_0) \}.$$

With the usual multiplication of functions of two variables, the spaces $L^0$, $L^1$ are commutative algebras with unity $e = \{1\}$, whereas the derivative $S$ satisfies the Leibniz condition and $s_{t_0}$ are multiplicative operations.

It is easy to verify that

$$\widehat{\Phi}_{t_0}(\widehat{A}) = \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix}$$

is the normalized fundamental matrix corresponding to the matrix $\widehat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. On the basis of the Cauchy formula (46) we obtain

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi(z - t + t_0) \\ \varphi'(z - t + t_0) + \psi(z - t + t_0) \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix} \int_{t_0}^{t} \begin{bmatrix} 1 & t_0 - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau, z - t + \tau)\, d\tau,$$

whence it follows that the function

$$x = \Big\{\varphi(z-t+t_0) + (t-t_0)\big[\varphi'(z-t+t_0) + \psi(z-t+t_0)\big] +$$

$$+ \int_{t_0}^{t} (t-\tau)u(\tau, z-t+\tau)\,d\tau\Big\}$$

is the solution of the problem (56), (57).

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# HAJÓS' THEOREM FOR MULTIPLE FACTORIZATIONS

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**1. Introduction.** Let $G$ be a finite abelian group written multiplicatively with identity element $e$. Let $A_1, \ldots, A_n$ be subsets of $G$. If each element $g$ of $G$ is expressible precisely $k$ ways in the form

$$g = a_1 \cdots a_n, \qquad a_1 \in A_1, \ldots, a_n \in A_n,$$

then we say that the product $A_1 \cdots A_n$ is a *k-factorization* of $G$. When the product $A_1 \cdots A_n$ is direct then it is a 1-factorization of $G$ and will be called simply a *factorization* of $G$. The subset $A$ of $G$ is defined to be *cyclic* if it is of form

$$\{e, a, a^2, \ldots, a^{r-1}\}.$$

In 1942 G. Hajós [1] solving a famous geometrical conjecture of H. Minkowski proved that in every factorization of a finite abelian group by cyclic subsets at least one subgroup must occur among the factors.

Hajós' theorem does not extend to multiple factorizations as it is shown by examples. Let $G$ be the direct product of two cyclic groups of order four with basis $x, y$ and let

$$\begin{array}{lll} A_1 = \{e, y\}, & A_2 = \{e, x^2 y\}, & A_3 = \{e, x\}, \\ A_4 = \{e, xy\}, & A_5 = \{e, xy^2\}, & A_6 = \{e, xy^3\}. \end{array}$$

The product $A_1A_2A_3A_4A_5A_6$ is a 4-factorization of $G$ and none of the factors is a subgroup of $G$.

However, we will show that there must occur a subgroup among the factors in every multiple factorization of a finite abelian group, provided the multiplicity of the factorization is relative prime to the order of the group.

**2. The result.** The group ring $Z(G)$ provides an adequate tool to deal with $k$-factorizations. We identify the subset $A$ of $G$ with the element

$$\overline{A} = \sum_{a \in A} a$$

of $Z(G)$. The product $A_1 \cdots A_n$ is a $k$-factorization of $G$ if and only if

$$k\overline{G} = \overline{A}_1 \cdots \overline{A}_n.$$

Rédei [2] developed a method using characters of $G$ to study factorizations of $G$. Characters of $Z(G)$ which are linear extensions of characters of $G$ can be used to study multiple factorizations.

Let $\chi_i$ be the $i$th character of $G$ and let $g_j$ be the $j$th element of $G$. Let $A, B \in Z(G)$,

$$A = \sum_{j=1}^{|G|} a_j g_j, \qquad B = \sum_{j=1}^{|G|} b_j g_j.$$

If $\chi_i(A) = \chi_i(B)$ for each $i$, $1 \leqq i \leqq |G|$, then

$$\sum_{j=1}^{|G|} (a_j - b_j)\chi_i(g_j) = 0.$$

By the standard orthogonality relations the matrix $\chi_i(g_j)$ is orthogonal and so its determinant is nonzero. Hence it follows that $a_j - b_j = 0$ for each $j$, $1 \leqq j \leqq |G|$. Therefore $A = B$. Thus $k\overline{G} = \overline{A}_1 \cdots \overline{A}_n$ if and only if $\chi(k\overline{G}) = \chi(\overline{A}_1 \cdots \overline{A}_n)$ for each character $\chi$ of $G$. For the principal character this reduces to $k|G| = |A_1| \cdots |A_n|$. For nonprincipal characters we have $0 = \chi(\overline{A}_1) \cdots \chi(\overline{A}_n)$. Therefore, the product $A_1 \cdots A_n$ is a $k$-factorization of $G$ if and only if $k|G| = |A_1| \cdots |A_n|$ and for each nonprincipal character $\chi$ of $G$ there is an $i$, $1 \leqq i \leqq n$ such that $\chi(\overline{A}_i) = 0$.

THEOREM. *In a multiple factorization of a finite abelian group by cyclic subsets always occurs a subgroup among the factors if the multiplicity is relatively prime to the order of the group.*

PROOF. Let $A_1, \ldots, A_n$ be cyclic subsets of the finite abelian group $G$ and suppose that the product $A_1 \cdots A_n$ is a $k$-factorization of $G$, where $k$ is relatively prime to $|G|$. In other words, suppose that the equation

$$k\overline{G} = \overline{A}_1 \cdots \overline{A}_n$$

holds in $Z(G)$.

First we show that we may focus our attention on factorizations in which the orders of the factors are primes. To show this suppose that

$$A = \{e, a, a^2, \ldots, a^{rs-1}\}$$

is one of the factors of the factorization and let

$$B = \{e, a, a^2, \ldots, a^{r-1}\}, \qquad C = \{e, a^r, a^{2r}, \ldots, a^{(s-1)r}\}.$$

Note that $\overline{A} = \overline{B}\,\overline{C}$. If $C$ is a subgroup of $G$, then $a^{rs} = e$ and so $A$ is also a subgroup of $G$. If $B$ is a subgroup of $G$, then $a^r = e$. But this is impossible since the elements $e, a, a^2, \ldots, a^{rs-1}$ are the elements of $A$ and so they are different. Thus cyclic factors of composite order can be further factored and if a subgroup occurs in the resulting factorization then a subgroup must occur in the original factorization.

Let $B_1 \cdots B_m$ be the resulting $k$-factorization of $G$, where $|B_i|$ is a prime for each $i$, $1 \leqq i \leqq m$. Clearly,

$$k\overline{G} = \overline{B}_1 \cdots \overline{B}_m.$$

Let $\chi$ be a nonprincipal character of $G$. There must be a factor, say $B$, for which $\chi(\overline{B}) = 0$. Let

$$B = \{e, b, b^2, \ldots, b^{r-1}\}.$$

Now

$$0 = \chi(\overline{B}) = \sum_{i=0}^{r-1} \chi(b^i) = \sum_{i=0}^{r-1} \big(\chi(b)\big)^i.$$

Then $\chi(b) \neq 1$ and so

$$0 = \chi(\overline{B}) = \frac{1 - \big(\chi(b)\big)^r}{1 - \chi(b)}.$$

Consequently, $\big(\chi(b)\big)^r = 1$. Since $r$ is prime and since $\chi(b)$ is a $|G|$-th root of unity it follows that $r$ is a divisor of $|G|$. Thus for each nonprincipal character $\chi$ of $G$ there is a factor $B_i$ such that $\chi(\overline{B}_i) = 0$ and $|B_i|$ is a divisor of $|G|$. This means that the product of factors $B_i$ whose order divide $|G|$ form a multiple factorization of $G$ as well.

Using the facts that the multiplicity $k$ is relatively prime to $|G|$ and each $|B_i|$ is a prime and $k|G| = |B_1| \cdots |B_m|$ we have that $|G|$ is equal to the product of all the $|B_i|$'s for which $|B_i|$ divides $|G|$. Thus certain factors form a 1-factorization of $G$. So by Hajós' theorem one of the factors $B_1, \ldots, B_m$ is a subgroup of $G$ and finally one of the original factors $A_1, \ldots, A_n$ must be a subgroup of $G$.

This completes the proof.

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# DECOMPOSITIONS OF CONTINUITY

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Tong [10], [11] and Ganster–Reilly [5] gave some decompositions of continuity. In this paper we will give a decomposition of continuity which improves the decompositions given in [10] and [5], and some properties between special families of subsets of a topological space $(X, \tau)$.

$\dot{A}$, $\bar{A}$ will stand for the interior and the closure of subset $A$ of $X$ in the topological space $(X, \tau)$ only, resp.

A set $S$ is called regular-open if $S = \dot{\bar{S}}$, regular closed if $X - S$ is regular open (equivalently $S = \bar{\dot{S}}$).

Tong [11] defined a set $S$ $t$-set if $\dot{S} = \dot{\bar{S}}$.

Definition 1. A subset $S$ of $(X, \tau)$ is called

(i) an $\alpha$-set (or $\alpha$-open) if $S \subset \dot{\bar{\dot{S}}}$,

(ii) a semi-open set if $S \subset \bar{\dot{S}}$,

(iii) a pre-open set if $S \subset \dot{\bar{S}}$,

(iv) an $\mathcal{A}$ set if $S = U \cap F$, where $U$ is open and $F$ is regular closed,

(v) locally closed if $S = U \cap F$ where $U$ is open and $F$ is closed,

(vi) a $\mathcal{B}$ set if $S = U \cap A$ where $U$ is open and $A$ is a $t$-set,

(vii) a semi-preopen set if $S \subset \bar{\dot{\bar{S}}}$.

The notions in Definition 1 were introduced by Njåstad [7], Levine [6], Mashhour et al. [8], Tong [10], Bourbaki [4], Tong [11] and Andrijević [1].

$S$ is called $\alpha$-closed (resp. semi-closed, pre-closed, semi-preclosed) set if $X - S$ is $\alpha$-open (resp. semi-open, pre-open, semi-preopen).

It is known that $S$ is $\alpha$-closed (resp. semi-closed, pre-closed, semi-preclosed) iff $\bar{\dot{\bar{S}}} \subset S$ (resp. $\dot{\bar{S}} \subset S$, $\bar{\dot{S}} \subset S$, $\dot{\bar{\dot{S}}} \subset S$).

$\alpha$-closure (resp. semi-closure, pre-closure, semi-preclosure) of a set $S$ is the intersection of all $\alpha$-closed (semi-closed, pre-closed, semi-preclosed) sets containing $S$, and we will denote these sets by $\alpha\mathrm{cl}\, S$, $\mathrm{scl}\, S$, $\mathrm{pcl}\, S$, $\mathrm{spcl}\, S$ respectively.

It is well known that a set $S$ is semi-closed iff $\dot{S} = \dot{\bar{S}}$. Hence $t$-sets and semi-closed sets are the same. Clearly $S$ is a $\mathcal{B}$ set iff there exists an open set $U$ and a semi-closed set $A$ such that $S = U \cap A$. We note that a subset $S$ is locally closed iff $S = U \cap \bar{S}$ for some open set $U$ ([4], I. 3.3, Proposition 5).

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We will denote family of $\alpha$-sets (resp. pre-open sets, semi-open sets, locally closed sets, $\mathcal{A}$-sets, $\mathcal{B}$-sets) shortly by $\tau^\alpha$, $PO(X)$, $SO(X)$, $LC(X)$, $\mathcal{A}(X)$, $\mathcal{B}(X)$.

In [5] the following relations were given:

(i) $\mathcal{A}(X) = SO(X) \cap LC(X)$,

(ii) $\tau = \tau^\alpha \cap LC(X)$,

(iii) $\tau = PO(X) \cap LC(X)$,

(iv) $\tau = PO(X) \cap \mathcal{A}(X)$.

From (ii), (iii) and (iv), three decompositions of continuity were obtained [5] and all of them improve the decomposition of continuity given by Tong [10].

In the topological space $(X,\tau)$ we will define the following families:

$$\mathcal{A}_1 = \tau^\alpha = \{U \cap N : U \in \tau,\ \bar{\dot{\bar{N}}} = X\} \qquad [7],$$

$$\mathcal{A}_2 = PO(X) = \{U \cap N : U \in \tau,\ \bar{N} = X\} \qquad [3],$$

$$\mathcal{A}_3 = \{U \cap N : U \in \tau,\ \dot{\bar{\dot{N}}} \subset N\},$$

$$\mathcal{A}_4 = \mathcal{A}(X) = \{U \cap N : U \in \tau,\ N = \bar{\dot{N}}\},$$

$$\mathcal{A}_5 = \mathcal{B}(X) = \{U \cap N : U \in \tau,\ \dot{\bar{N}} \subset N\},$$

$$\mathcal{A}_6 = LC(X) = \{U \cap N : U \in \tau,\ \bar{N} = N\},$$

$$\mathcal{A}_7 = \{U \cap N : U \in \tau,\ \bar{\dot{N}} \subset N\}.$$

If we take $X$ instead of $N$ in every $\mathcal{A}_i$ we can easily see that $\tau$ is a subfamily of all these families:

$$\mathcal{A}_1 \subset \mathcal{A}_2 \quad [8], \qquad \mathcal{A}_4 \subset \mathcal{A}_6 \quad [5], \qquad \mathcal{A}_4 \subset \mathcal{A}_5 \quad [11].$$

It is easy to see that $\mathcal{A}_5 \subset \mathcal{A}_3$, $\mathcal{A}_6 \subset \mathcal{A}_7 \subset \mathcal{A}_3$, $\mathcal{A}_4 \subset \mathcal{A}_7$, $\mathcal{A}_6 \subset \mathcal{A}_5$.

Let $f$ be a function from $(X,\tau)$ to any topological space. If the inverse image of each open set is in $\mathcal{A}_i$, then we call $f$ $\mathcal{A}_i$-continuous.

An $\mathcal{A}_1$-continuous (resp. $\mathcal{A}_2$-, $\mathcal{A}_4$-, $\mathcal{A}_5$-, $\mathcal{A}_6$-continuous) function was called $\alpha$-continuous (resp. pre-continuous, $\mathcal{A}$-, $\mathcal{B}$-, $LC$-continuous) before.

THEOREM 1. *Let $S$ be a subset of $(X,\tau)$.*

(1) *$S \in \mathcal{A}_3$ iff there exists an open set $U$ such that $S = U \cap \operatorname{spcl} S$.*

(2) *$S \in \mathcal{A}_4$ iff there exists an open set $U$ such that $S = U \cap \bar{\dot{S}}$.*

(3) *$S \in \mathcal{A}_5$ iff there exists an open set $U$ such that $S = U \cap \operatorname{scl} S$.*

(4) *$S \in \mathcal{A}_7$ iff there exists an open set $U$ such that $S = U \cap \operatorname{pcl} S$.*

PROOF. (1) Let $S \in \mathcal{A}_3$. Then there exists an open set $U$ and a semi-preclosed set $N$ such that $S = U \cap N$.

We have $S \subset U$, $S \subset N$, $S \subset \operatorname{spc} N$, $S \subset U \cap \operatorname{spcl} S \subset U \cap N = S$.

Hence $S = U \cap \operatorname{spcl} S$.

The converse is obvious since $\operatorname{spcl} S$ is semi-preclosed.

Proofs of (3) and (4) are similar.

(2) If $S = U \cap \bar{\dot{S}}$ (where $U \in \tau$), then, since $\left(\bar{\dot{S}}\right)^{\dot{-}} = \bar{\dot{S}}$, $S \in \mathcal{A}_4$.

Let $S \in \mathcal{A}_4$. Then there exists an open set $U$ and a regular closed set $N$ such that $S = U \cap N$.

$$S \subset N \Rightarrow \bar{\dot{S}} \subset \bar{\dot{N}} = N.$$

We get $U \cap \bar{\dot{S}} \subset U \cap N$.

$$U \cap N = U \cap \bar{\dot{N}} \subset \left(U \cap \dot{N}\right)^{-} = (U \cap N)^{\dot{-}} = \bar{\dot{S}}.$$

Hence $U \cap N \subset U \cap \bar{\dot{S}}$.

So $U \cap \bar{\dot{S}} = U \cap N = S$.

THEOREM 2. (1) $\mathcal{A}_4 = SO(X) \cap \mathcal{A}_7$,

(2) $SO(X) \cap \mathcal{A}_3 \subset \mathcal{A}_5$.

PROOF. (1) Let $S \in \mathcal{A}_4$. Since $\mathcal{A}_4 \subset SO(X)$ ([10], Theorem 3.1) and $\mathcal{A}_4 \subset \mathcal{A}_7$, $\mathcal{A}_4 \subset SO(X) \cap \mathcal{A}_7$.

Conversely let $S$ be semi-open and in $\mathcal{A}_7$. Then from Theorem 1, there exists an open set $U$ such that

$$S = U \cap \operatorname{pcl} S = U \cap \left(S \cup \bar{\dot{S}}\right) = U \cap \bar{\dot{S}}.$$

(We use here that, for any set $S$, $\operatorname{pcl} S = S \cup \bar{\dot{S}}$ [2].)

So $S$ is an $\mathcal{A}$ set, i.e. $S \in \mathcal{A}_4$.

(2) let $S \in SO(X) \cap \mathcal{A}_3$. Then $S \subset \bar{\dot{S}}$ and there exists an open set $U$ such that

$$S = U \cap \operatorname{spcl} S = U \cap \left(S \cup \dot{\bar{\dot{S}}}\right) =$$

$$= U \cap \left(S \cup \dot{\bar{S}}\right) \qquad \left(S \subset \bar{\dot{S}} \Rightarrow \bar{S} = \bar{\dot{S}} \Rightarrow \dot{\bar{S}} = \dot{\bar{\dot{S}}}\right)$$

$$= U \cap \operatorname{scl} S \qquad (\text{for any set } S,\ \operatorname{scl} S = S \cup \dot{\bar{S}},\ [2]).$$

Hence $S \in \mathcal{A}_5$.

COROLLARY 1. (1) *$f$ is $\mathcal{A}_4$-continuous iff $f$ is semi-continuous and $\mathcal{A}_7$-continuous.*

(2) *([5], Theorem 5) $\mathcal{A}_4 = SO(X) \cap \mathcal{A}_6$.*

(3) *([5], Theorem 4(i)) $f$ is $\mathcal{A}_4$-continuous iff $f$ is semi-continuous and $\mathcal{A}_6$-continuous.*

(4) *$\mathcal{A}_4 \subset SO(X) \cap \mathcal{A}_3$.*

(5) *If $f$ is $\mathcal{A}_4$-continuous, then it is semi-continuous and $\mathcal{A}_3$-continuous.*

THEOREM 3. *$\tau = \mathcal{A}_1 \cap \mathcal{A}_3$.*

PROOF. It is clear that $\tau \subset \mathcal{A}_1 \cap \mathcal{A}_3$.

Conversely $\mathcal{A}_1 \cap \mathcal{A}_3 \subset PO(X) \cap SO(X) \cap \mathcal{A}_3 \subset PO(X) \cap \mathcal{A}_5 = \tau$. (from Theorem 2, and from Proposition 9 in [11])

COROLLARY 2. (1) *$f$ is continuous iff $f$ is $\mathcal{A}_1$-continuous and $\mathcal{A}_3$-continuous.*

(2) *$\tau = \mathcal{A}_1 \cap \mathcal{A}_7$.*

(3) *$f$ is continuous iff $f$ is $\mathcal{A}_1$-continuous and $\mathcal{A}_7$-continuous.*

(4) *(Given in [5]) $\tau = \mathcal{A}_1 \cap \mathcal{A}_6$.*

(5) *([5], Theorem 4(ii)). $f$ is continuous iff $f$ is $\mathcal{A}_1$-continuous and $\mathcal{A}_6$-continuous.*

(6) *(Given in [5])*

$$\tau = \mathcal{A}_1 \cap \mathcal{A}_6 = PO(X) \cap SO(X) \cap \mathcal{A}_6 = PO(X) \cap \mathcal{A}_4.$$

(7) *([5], Theorem 4(v)) $f$ is continuous iff $f$ is precontinuous and $\mathcal{A}_4$-continuous.*

The following theorem was given by Tong [11]. But we will give a different proof.

THEOREM 4 (Proposition 9 in [11]). *$\tau = PO(X) \cap \mathcal{A}_5$.*

PROOF. Clearly $\tau \subset PO(X) \cap \mathcal{A}_5$. If $S \in PO(X) \cap \mathcal{A}_5$ then $S \subset \dot{\bar{S}}$ and there exists an open set $U$ such that

$$S = U \cap \operatorname{scl} S = U \cap \left(S \cup \dot{\bar{S}}\right) = U \cap \dot{\bar{S}} \in \tau.$$

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# INTEGRAL OPERATORS ACTING ON CONTINUOUS FUNCTIONS

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## Introduction

Let $\Omega$ be a locally compact Hausdorff space, and let $C(\Omega)$ be the Banach space of all bounded complex valued continuous functions on $\Omega$. The complete norm on $C(\Omega)$ is, as usual,

$$\|f\|_\Omega = \sup_{x\in\Omega}\left| f(x)\right|.$$

Fix $\mu$ a $\sigma$-finite positive regular Borel measure on $\Omega$ which is strictly positive in the sense that

$$\mu(U) > 0 \quad \text{for all nonempty open subsets} \quad U \subseteqq \Omega.$$

In this paper we study the class of all linear integral operators on $C(\Omega)$ (relative to the fixed measure $\mu$). Such an operator is determined by a kernel (measurable function) $K(x,t)$ on $\Omega^2 = \Omega \times \Omega$ according to the formula

$$(I) \qquad K_c(f)(x) \equiv \int_\Omega K(x,t)f(t)\,d\mu(t) \quad \bigl(f \in C(\Omega)\bigr).$$

For convenience of notation, $\mu$ being fixed, we often write $dt$ in place of $d\mu(t)$ in integral expressions. The class of integral operators considered in this paper is defined as follows.

DEFINITION 1. A kernel $K$ on $\Omega^2$ is in $\mathcal{A}_c = \mathcal{A}_c(\Omega,\mu)$ if

(i) $\tau_c(K) = \sup\limits_{x\in\Omega} \int_\Omega \left| K(x,t)\right| \, dt < \infty$; and

(ii) $K_c(f) \in C(\Omega)$ whenever $f \in C(\Omega)$.

We derive the basic properties of $\mathcal{A}_c$, including the fact that $(\mathcal{A}_c, \tau_c)$ is a Banach algebra in §1. For $K \in \mathcal{A}_c$, formula (I) also defines an operator $K_\infty(f)$ for $f \in L^\infty = L^\infty(\Omega,\mu)$. In §2 we consider compactness properties of the operators $K_c$ and $K_\infty$. The main results are in §3 where the spectral and Fredholm theory of the Banach algebra $\mathcal{A}_c$ is developed. These results are all expressed in terms of the spectral and Fredholm properties of the operators $K_c$ and $K_\infty$.

This paper is motivated by the beautiful and useful theory of integral operators on $C(\Omega)$ as developed by K. Jörgens in his book [5].

## The algebra $\mathcal{A}_c$

In this section we derive the basic properties of the algebra of kernels, $\mathcal{A}_c$. Throughout this paper the space $\Omega$ and the measure $\mu$ are as described in the Introduction. For $K, J \in \mathcal{A}_c$

$$\sup_{x\in\Omega}\left\{\int_\Omega |K(x,z)|\left(\int_\Omega |J(z,t)|\,dt\right)dz\right\} \leqq \tau_c(K)\tau_c(J).$$

Therefore the convolution of the kernels $K$ and $J$,

$$(K*J)(x,t) = \int_\Omega K(x,z)J(z,t)\,dz, \quad x,t\in\Omega,$$

is again a kernel in $\mathcal{A}_c$, and clearly, $(K*J)_c = K_cJ_c$. We make the understanding that two kernels $K, J \in \mathcal{A}_c$ are considered equal if for each $x\in\Omega$, $J(x,t) = K(x,t)$ a.e. on $\Omega$. Equivalently, $K$ and $J$ are considered equal if $\tau_c(K-J) = 0$. Thus $(\mathcal{A}_c, \tau_c)$ is a normed algebra of equivalence classes of kernels.

PROPOSITION 2. *For $K \in \mathcal{A}_c$, $\tau_c(K) = \|K_c\|$ (the operator norm $K_c$).*

PROOF. For each $x \in \Omega$, define the functional $\alpha_x$ on $C = C(\Omega)$ by

$$\alpha_x(g) = \int_\Omega K(x,t)g(t)\,dt \quad (g\in C).$$

Then the norm of $\alpha_x$ on $C$ is $\|\alpha_x\| = \int_\Omega |K(x,t)|\,dt$ [7, Theorem 6.13 and Theorem 6.19]. Suppose for some $x_o \in \Omega$,

$$\int_\Omega |K(x_o,t)|\,dt > \|K_c\|.$$

Then there exists $g \in C$, $\|g\|_\Omega = 1$, with $\left|\int\limits_\Omega K(x_o,t)g(t)\,dt\right| = |\alpha_{x_o}(g)| >$ $> \|K_c\|$. But this implies $\|K_c(g)\|_\Omega > \|K_c\|$, a contradiction. Therefore

$$\tau_c(K) \leqq \|K_c\|.$$

The reverse inequality is immediately obvious.

THEOREM 3. *$(\mathcal{A}_c, \tau_c)$ is a Banach algebra. Also, if $\{K_n\}$ is a $\tau_c$-Cauchy sequence, then some subsequence of $\{K_n\}$ converges a.e. on $\Omega^2$.*

PROOF. We prove completeness as this requires a technical argument which is not completely obvious. Let $\{K_n\} \subseteqq \mathcal{A}_c$ be $\tau_c$-Cauchy. There exists a subsequence $\{K_{n_j}\}_{j\geqq 1}$ such that $\tau_c\left(K_{n_{j+1}} - K_{n_j}\right) < 2^{-j}$ for $j \geqq 1$. It suffices to show that this subsequence converges. For convenience we use the relabeling $K_j = K_{n_j}, j \geqq 1$. So we have

$$\tau_c(K_{j+1} - K_j) < 2^{-j}, \qquad j \geqq 1.$$

Set $J(x,t) = \sum\limits_{j=1}^{\infty} |K_{j+1}(x,t) - K_j(x,t)|$. For all $x \in \Omega$, integrating term by term we have

$$\left\| J(x,t)\right\|_1 \leqq \sum_{j=1}^{\infty} \tau_c(K_{j+1} - K_j) \leqq 1.$$

Let

$$\Gamma = \left\{(x,t) \in \Omega^2 : J(x,t) = +\infty\right\}.$$

Also, for each fixed $x, \Gamma_x = \left\{t \in \Omega : (x,t) \in \Gamma\right\}$ is a set of $\mu$-measure zero since $\left\| J(x,t)\right\|_1 < \infty$. Therefore by definition [7, Definition 8.7] $\mu_2(\Gamma) =$ $= \int\limits_\Omega \mu(\Gamma_x)\,d\mu(x) = 0$ where $\mu_2$ denotes the usual product measure $\mu \times \mu$. When $\Delta$ is a set, we use the notation $\Delta^c$ for the complement of $\Delta$. Fix $x$. If $(x,t) \in \Gamma^c$, then let

$$K(x,t) = \sum_{j=1}^{\infty}\left(K_{j+1})(x,t) - K_j(x,t) + K_1(x,t)\right) = \lim_{n\to\infty} K_n(x,t)$$

(note that the series converges absolutely). If $(x,t) \in \Gamma$, then set $K(x,t) =$ $= 0$. Since $\Gamma$ is a set of $\mu_2$-measure zero, $K$ is a measurable function on $\Omega^2$.

Now for each $x \in \Omega$

$$|K_{n+1}(x,t)| \leqq \sum_{j=1}^{n} |K_{j+1}(x,t) - K_j(x,t)| + |K_1(x,t)| \leqq$$

$$\leqq J(x,t) + |K_1(x,t)| \in L^1(\Omega),$$

so by Dominated Convergence Theorem

$$(\alpha) \qquad \|K(x,t) - K_n(x,t)\|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\varepsilon > 0$ be arbitrary. Fix $N$ such that

$$n, m \geqq N \Rightarrow \tau_c(K_n - K_m) < \varepsilon/2.$$

Fix $x$ and assume $n \geqq N$. Using $(\alpha)$, choose $m \geqq N$ such that

$$\int_\Omega |K(x,t) - K_m(x,t)| \, dt < \varepsilon/2.$$

Then

$$\int_\Omega |K(x,t) - K_n(x,t)| \, dt \leqq \int_\Omega |K(x,t) - K_m(x,t)| \, dt + \tau_c(K_m - K_n) <$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore if $n \geqq N$, $\tau_c(K - K_n) < \varepsilon$. Thus, $K$ is the $\tau_c$-limit of the sequence $\{K_n\}$.

Assume $K(x,t)$ is a measurable function on $\Omega^2$, the minimal requirements for $K$ to determine an integral operator on $C(\Omega)$ are:
(i) for all $x \in \Omega$, $K(x,t) \in L^1$; and
(ii) for all $g \in C$, $\int_\Omega K(x,t)g(t) \, dt \in C$.

In fact if $K$ satisfies (i) and (ii), then $K \in \mathcal{A}_c$, as the following argument shows. It suffices to show $\tau_c(K) < \infty$. For each $x \in \Omega$, let $\alpha_x$ be the continuous linear functional on $C$ given by

$$\alpha_x(g) = \int_\Omega K(x,t)g(t) \, dt \quad (g \in C).$$

Now $\|\alpha_x\| = \int\limits_\Omega |K(x,t)|\, dt$. The collection $\{\alpha_x : x \in \Omega\}$ is pointwise bounded on $C$ since $\alpha_x(g) = \int\limits_\Omega K(x,t)g(t)\, dt \in C$. Therefore by the Uniform Boundedness Theorem,

$$\tau_c(K) = \sup_{x\in\Omega} \int\limits_\Omega |K(x,t)|\, dt = \sup_{x\in\Omega} \|\alpha_x\| < \infty.$$

EXAMPLE 4. In [5], K. Jörgens introduces the algebra of all kernels $K$ on $\Omega^2$ having the property that

$$x \to K(x,\cdot) \quad \text{is a bounded continuous function from } \Omega \text{ into } L^1(\Omega,\mu).$$

Let $\ell = \ell(\Omega,\mu)$ be the collection of all such kernels. Then $\ell$ is a closed subalgebra of $\mathcal{A}_c$ [5, Theorem 12.2]. In fact, $\ell$ is a closed right ideal in $\mathcal{A}_c$ as the following argument proves:

For $K \in \ell$, $J \in \mathcal{A}_c$, $x, y \in \Omega$;

$$\begin{aligned}
&\|(K*J)(x\,\cdot) - (K*J)(y,\cdot)\|_1 \leqq \\
&\leqq \int\limits_\Omega \left( \int\limits_\Omega |K(x,z) - K(y,z)|\,|J(z,t)|\, dz \right) dt = \\
&= \int\limits_\Omega |K(x,z) - K(y,z)| \left( \int\limits_\Omega (J(z,t)|\, dt \right) dz \leqq \\
&\leqq \|K(x\,\cdot) - K(y,\cdot)\|_1 \tau_c(J).
\end{aligned}$$

When $X$ is a Banach space, let $\mathcal{K}(X)$ denote the space of all compact operators on $X$. Now $\ell$ may be a proper ideal of $\mathcal{A}_c$, for when $\Omega$ is compact, then $\ell \subseteqq \mathcal{K}(C)$ by [5, Theorem 12.1]. But it is not difficult to find examples where $\Omega$ is compact, $K \in \mathcal{A}_c$, but $K \notin \mathcal{K}(C)$; see Example 6.

The algebra $\ell$ is a very interesting Banach algebra of operators. Now we give two examples with the aim of showing that there are important operators in $\mathcal{A}_c$ which are not in $\ell$. This is one justification for studying the larger algebra, $\mathcal{A}_c$.

EXAMPLES 5. Let $\Omega = [0,1]$, let $m$ be the Lebesgue measure, and let $\delta_0$ be the point mass at the origin. Set $\mu = m + \delta_0$. Let $\chi_E$ denote the characteristic function of a set $E$. Fix $\alpha > 0$, and define $K$ by

$$K(x,t) = x^{-\alpha} t^{\alpha-1} \chi_{(0,x]}(t), \quad x > 0;$$

$$K(0,t) = \alpha^{-1}\chi_{\{0\}}(t).$$

For $f \in L^\infty(\mu)$

$$K_\infty(f)(x) = x^{-\alpha}\int_0^x t^{\alpha-1}f(t)\,dm(t), \quad x > 0;$$

$$K_\infty(f)(0) = \alpha^{-1}f(0).$$

If $f \in C[0,1]$, $f$ real valued, then clearly $K_\infty(f)(x)$ is continuous on $(0,1]$. If $\varepsilon > 0$ is given, choose $\delta > 0$ such that $0 \leqq t < \delta \Rightarrow f(0) - \varepsilon \leqq \alpha^{-1}f(t) \leqq \leqq f(0) + \varepsilon$. If $0 < x < \delta$, then

$$f(0) - \varepsilon \leqq K_\infty(f)(x) = x^{-\alpha}\int_0^x t^{\alpha-1}f(t)\,dt \leqq f(0) + \varepsilon.$$

Therefore $K_\infty(f)(x)$ is continuous at 0. Thus, $K \in \mathcal{A}_c$.

Note that for $g = \chi_{\{0\}} \in L^\infty(\mu)$, $K_\infty(g) = g$, so $K_\infty(L^\infty) \nsubseteq C$. It is not difficult to see that $K \notin \ell(\Omega,\mu)$. Operators of this type are used as examples in Jörgens book [5]. The case $\alpha = 1$ is the familiar Cesaro operator.

EXAMPLE 6. Let $c$ be the usual Banach space of sequences which converge. For $a = \{a_k\}_{k \geqq 1} \in c$, let $a_0 = \lim\limits_{k\to\infty} a_k$. Let $\mathbf{N} = \{1,2,3,\dots\}$ be the natural numbers with the discrete topology and let $\Omega = \mathbf{N} \cup \{\infty\}$ be the one-point compactification of $\mathbf{N}$, formed by adding the point $\infty$. A set $U \subseteqq \Omega$ is an open neighborhood of $\infty$ if $U^c$ is a finite subset of $\mathbf{N}$. Each sequence $a = \{a_k\} \in c$ is naturally associated with a continuous function on $\Omega$ by setting

$$\widetilde{a}(k) = a_k, \quad k \geqq 1, \quad \widetilde{a}(\infty) = a_0.$$

Then $a \to \widetilde{a}$ is a linear isometry of $c$ onto $C(\Omega)$. Let $\mu$ be counting measure on $\Omega$.

The following proposition is a restatement of the well-known characterization of when an infinite matrix determines an operator in $\mathcal{B}(c)$; see [6, Theorem 6.4].

PROPOSITION 7. *Let $\{K(j,k)\}_{k \geqq 0\, j \geqq 0}$ be an infinite matrix. Then $K \in \in \mathcal{A}_c(\Omega,\mu)$ if and only if*

(i) $\operatorname*{lub}\limits_{1 \leqq j < \infty} \sum\limits_{k=0}^{\infty} |K(j,k)| = \alpha < \infty$*; and*

(ii) $K(0,k) = \lim\limits_{j\to\infty} K(j,k)$ *for* $k \geqq 1$*; and*

(iii) $\lim\limits_{j\to\infty} \sum\limits_{k=0}^{\infty} K(j,k) = \beta$ *exists,*

*and* $K(0,0) = \beta - \sum\limits_{k=1}^{\infty} K(0,k)$.

When conditions (i)–(iii) hold, then for $a = \{a_k\}_{k \geqq 1} \in c$,

$$K_c(\widetilde{a})(j) = \sum_{k=0}^{\infty} K(j,k)a_k \quad (j \geqq 0),$$

and $\tau_c(K) = \alpha$.

In this case $\mathcal{A}_c(\Omega,\mu)$ is completely identified with $\mathcal{B}(c)$ via the map $K \to$ $\to K_c$. There are many interesting operators in $\mathcal{B}(c)$ which are not in the ideal $\ell(\Omega,\mu)$, for example the Hausdorff matrix $H \notin \ell$:

$$H(j,k) = j^{-1}, \quad 1 \leqq k \leqq j;$$

$$H(j,k) = 0, \quad k > j \geqq 1;$$

$$H(0,k) = 0, \quad k \geqq 1;$$

$$H(j,0) = 0, \quad j \geqq 1; \quad H(0,0) = 1.$$

## 2. Compactness properties

PROPOSITION 8. *For $K \in \mathcal{A}_c$, $K_c$ is compact $\Leftrightarrow$ $K_\infty$ is compact.*

PROOF. Assume that $K_\infty$ is compact, and let $\{f_n\} \subseteqq C$ with $\|f_n\|_\Omega \leqq 1$. Since $K_\infty$ is compact, there is a subsequence $\{f_{n_k}\}$ such that $\{K_\infty(f_{n_k})\}$ is a Cauchy sequence in $L^\infty$. Because the measure $\mu$ is strictly positive, $\|g\|_\Omega = \|g\|_\infty$ for all $g \in C$. Therefore

$$\left\|K_c(f_{n_k}) - K_c(f_{n_j})\right\|_\Omega = \left\|K_\infty(f_{n_k}) - K_\infty(f_{n_j})\right\|_\infty \to 0 \text{ as } k,j \to \infty.$$

This proves $K_c \in \mathcal{K}(C)$.

Now suppose $K_c$ is compact. Note that $L^1(\Omega,\mu)$ can be identified as a closed subspace of $C(\Omega)'$, the dual space of $C(\Omega)$, as follows: For $f \in L^1$, the functional

$$\alpha_f(g) = \int_\Omega fg\,d\mu \quad (g \in C)$$

has the property $\|\alpha_f\| = \|f\|_1$. Since $K_c$ is compact, we have $K_c'$, the adjoint of $K_c$, is compact [6, Theorem 7.3]. The operator $K_c'$ acts on $g \in L^1$ (considered as a subspace of $C'$) according to the formula

$$K_c'(g)(t) = \int_\Omega K(x,t)g(x)\,dx.$$

Therefore

$$\|K_c'(g)\|_1 \leqq \int_\Omega \left( \int_\Omega |K(x,t)|\,|g(x)|\,dx \right) dt =$$

$$= \int_\Omega \left( \int_\Omega |K(x,t)|\,dt \right) |g(x)|\,dx \leqq \tau_c(K)\|g\|_1.$$

This proves that $K_c'(L^1) \subseteqq L^1$, so the restriction of $K_c'$ to $L^1$ is in $\mathcal{K}(L^1)$. Let $J$ denote this restriction. It is straighforward to verify that $J'$, the adjoint of $J$ on $L^\infty$, is the operator $K_\infty$. Finally, since $J$ is compact, we have $K_\infty = J'$ is compact.

In the next two results we assume $\Omega$ is $\sigma$-compact. Then there exists $\{\Omega_n\}_{n\geqq 1}$, a sequence of compact subsets of $\Omega$ with

$$\Omega_n \subseteqq \operatorname{int}(\Omega_{n+1}) \subseteqq \Omega_{n+1} \quad (n \geqq 1),$$

and $\Omega = \bigcup_{n=1}^\infty \Omega_n$. We fix this sequence for the remainder of this section. It is easy to see that if $\Delta$ is any compact subset of $\Omega$, then $\Delta \subseteqq \Omega_n$ for some $n$. We need to define a condition on kernels.

II. Condition on a kernel $K \in \mathcal{A}_c$:

$$\lim_{n\to\infty} \left\| \left( \int_{\Omega_n^c} |K(x,t)|\,dt \right) \right\|_\Omega = 0.$$

We use the notation $C_c(\Omega)$ for the subspace of $f \in C(\Omega)$ such that $f$ has compact support.

THEOREM 9. *Let $\Omega$ be $\sigma$-compact. Assume $K \in \mathcal{A}_c$ with $K$ compact on $C$. Then $K$ satisfies the condition in* II.

PROOF. Suppose not. Then there exists $\delta > 0$ and a sequence $\{y_n\} \subseteqq \Omega$ with

$$\int_{\Omega_n^c} |K(y_n,t)|\,dt > \delta, \quad n \geqq 1.$$

Define the functional $\alpha_n$ on $C_c(\Omega_n^c)$ by

$$\alpha_n(g) = \int\limits_{\Omega_n^c} K(y_n,t)g(t)\,dt.$$

Now $\|\alpha_n\| = \int\limits_{\Omega_n^c} \big|K(y_n,t)\big|\,dt > \delta$, $n \geqq 1$. Choose $g_n \in C_c(\Omega_n^c)$, $\|g_n\|_{\Omega_n^c} = 1$, with

$$\left|\int\limits_{\Omega_n^c} K(y_n,t)g_n(t)\,dt\right| = \big|\alpha_n(g_n)\big| \geqq \delta.$$

Extend $g_n$ to be 0 on $\Omega_n$, and note that this extension, which we also denote by $g_n$, is in $C_c(\Omega)$. For each fixed $x$, $K(x,t)g_n(t) \to 0$ everywhere on $\Omega$. Therefore

$$K(g_n)(x) = \int\limits_{\Omega} K(x,t)g_n(t)\,dt \to 0$$

by the Dominated Convergence Theorem. Since $K$ is compact, there exists a subsequence $\{g_{n'}\}$ of $\{g_n\}$ and $f \in C$ such that

$$K(g_{n'}) \to f \quad \text{uniformly on} \quad \Omega.$$

By the argument above, $f \equiv 0$. Therefore $\big\|K(g_{n'})\big\|_\Omega \to 0$. But $\big|K(g_{n'})(y_{n'})\big| = \left|\int\limits_{\Omega_n^c} K(y_{n'},t)g_{n'}(t)\,dt\right| \geqq \delta$, a contradiction!

We say a sequence $\{f_k\} \subseteqq C(\Omega)$ converges locally to $f \in C(\Omega)$, if $\{\|f_k\|_\Omega\}_{k \geqq 1}$ is bounded and $f_k \to f$ a.e. on $\Omega$.

Corollary 10. *Let $\Omega$ be $\sigma$-compact. If $K_\infty$ or $K_c$ is compact, then $K_\infty(L^\infty) \subseteqq C$.*

Proof. By Proposition 8 we may assume that $K_c$ is compact. Let $f \in L^\infty$, $\|f\|_\infty = 1$. Let $\varepsilon > 0$ be arbitrary. By Theorem 9 there exists $n$ such that $\left\|\left(\int\limits_{\Omega_n^c} \big|K(x,t)\big|\,dt\right)\right\|_\Omega < \varepsilon$. Fix $w \in C$ with $w \equiv 1$ on $\Omega_n$, $w \equiv 0$ on $\Omega_{n+1}^c$, $0 \leqq w \leqq 1$. By [7, Corollary, p. 56] there exists $\{f_k\} \subseteqq C$ with $f_k \to \chi_{\Omega_{n+1}} f$ locally on $\Omega_{n+1}$. Therefore $wf_k \to wf$ locally on $\Omega$. Since $K_c$

is compact, there exists a subsequence $\{f_{n_k}\}$ such that $K_c((wf_{n_k})) \to g$ for some $g \in C$. Now $K_c(wf_{n_k}) \to K_\infty(wf) = g$. Finally,

$$\|g - K_\infty(f)\|_\Omega = \left\|K_\infty((w-1)f)\right\|_\Omega \leqq \left\| \int_{\Omega_n^c} |K(x,t)|\,|f(t)|\,dt \right\|_\Omega \leqq \varepsilon.$$

Since $\varepsilon$ was arbitrary, $K_\infty(f)$ must be continuous.

## 3. Spectral theory in $\mathcal{A}_c$

Let $X$ be a Banach space. Denote the collection of all $T \in \mathcal{B}(X)$ which are Fredholm operators by $\Phi(X)$ [5,§ 5.3]. For $T \in \Phi(X)$, let $\operatorname{nul}(T)$, $\operatorname{def}(T)$, and $\operatorname{ind}(T)$, denote the nullity, defect, and index of $T$, respectively. For $T \in \mathcal{B}(X)$, let $\sigma(T)$ be the usual spectrum of $T$ as an operator in $\mathcal{B}(X)$, and let $\omega(T)$ be the usual Fredholm essential spectrum of $T$:

$$\omega(T) = \{\lambda : \lambda - T \notin \Phi(X)\}.$$

In this section we describe the spectral and Fredholm properties of a kernel $K \in \mathcal{A}_c$ relative to the algebra $\mathcal{A}_c$ in terms of the spectral and Fredholm properties of the operators $K_c$ and $K_\infty$.

Another algebra of operators plays a role here. We describe this algebra now. Let $\Omega$ and $\mu$ be as in the Introduction. The spaces $C = C(\Omega)$ and $L^1 = L^1(\Omega, \mu)$ form a dual system where the nondegenerate bounded bilinear form on $C \times L^1$ is given by

$$\langle f, g\rangle = \int_\Omega fg\,d\mu \quad (f \in C,\ g \in L^1).$$

Dual systems are discussed in [5, pp. 43–44]. Let $\mathcal{A}(C, L^1)$ be the algebra of all operators $T \in \mathcal{B}(C)$ for which there exists (a necessarily unique) operator $T^\dagger \in \mathcal{B}(L^1)$ with

$$\langle Tf, g\rangle = \langle f, T^\dagger g\rangle \quad (f \in C,\ g \in L^1).$$

The algebra $\mathcal{B} = \mathcal{A}(C, L^1)$ is a Banach algebra, and the spectral and Fredholm theory in $A$ is well established; see [5] and [2].

Let $K \in \mathcal{A}_c$, and set

$$K^\dagger(x,t) = K(t,x) \quad (x, t \in \Omega).$$

For $f \in C$, $g \in L^1$, using Fubini's Theorem we have

$$\langle K_c(f)g\rangle = \int\limits_{\Omega} \left( \int\limits_{\Omega} K(x,t)f(t)\,dt \right) g(x)\,dx =$$

$$= \int\limits_{\Omega} f(t) \left( \int\limits_{\Omega} K(x,t)g(x)\,dx \right) dt = \langle f, K^{\dagger}(g)\rangle .$$

Therefore $\mathcal{A}_c$ is a subalgebra of $\mathcal{B}$. We prove that the spectral and Fredholm properties of $K \in \mathcal{A}_c$ are the same as those of $K$ relative to the larger algebra $\mathcal{B}$. It is clear what this means in the case of spectral theory: For $K \in \mathcal{A}_c$, $\sigma_{\mathcal{A}}(K) = \sigma_{\mathcal{B}}(K)$ (the spectrum of $K$ relative to the algebra $\mathcal{A}$ is equal to the spectrum of $K$ relative to the algebra $\mathcal{B}$). In order to understand the Fredholm theory in $\mathcal{B}$ and $\mathcal{A}_c$ we need some preliminary definitions.

First note that the set

$$\mathcal{F}_c = \operatorname{span}\{ \varphi(x)\psi(t) : \varphi \in C,\ \psi \in L^1\}$$

is an ideal of both $\mathcal{A}_c$ and $\mathcal{B} = \mathcal{A}(C, L^1)$. The set

$$\Phi_{\mathcal{B}} = \{T \in \mathcal{B} : T \text{ is invertible in } \mathcal{B} \text{ modulo } \mathcal{F}_c\}$$

is called the collection of Fredholm elements of $\mathcal{B}$ (relative to $\mathcal{F}_c$); see [2,§ 2] concerning properties of this set. The algebra $\mathcal{A}_c$ in general does not contain the identity operator. For $K \in \mathcal{A}_c$ and $\lambda \in \mathbf{C}$, $\lambda \neq 0$, we write $\lambda - K \in \Phi_{\mathcal{A}}$ when there exists $J \in \mathcal{A}_c$ such that $J * K = K * J$ and $K + \lambda J - J * K \in \in \mathcal{F}_c$. We show in Theorem 12 that when $\lambda \neq 0$, $K \in \mathcal{A}_c$, then $\lambda - K \in \Phi_{\mathcal{A}}$ if and only if $\lambda - K \in \Phi_{\mathcal{B}}$.

We need to prove a preliminary result. Let $\mathcal{A}_{\infty\infty}$ be the set of all kernels $K$ on $\Omega^2$ such that $\int\limits_{\Omega} |K(x,t)|\,dt \in L^{\infty}$.

For $K \in \mathcal{A}_{\infty\infty}$, let $\tau_{\infty}(K) = \left\| \left( \int\limits_{\Omega} |K(x,t)|\,dt \right) \right\|_{\infty}$. Call $g \in L^{\infty}$ essentially continuous if there exists $f \in C$ with $g = f$ a.e. on $\Omega$.

Proposition 11. *Assume $K \in \mathcal{A}_c$, and $J \in \mathcal{A}_{\infty\infty}$ has the property that for all $f \in C$, $J(f)$ is essentially continuous. Then $K * J \in \mathcal{A}_c$.*

Proof. For every $x \in \Omega$,

$$\int\limits_{\Omega} |(K * J)(x,t)|\,dt \leqq \int\limits_{\Omega} |K(x,z)| \left( \int\limits_{\Omega} |J(z,t)|\,dt \right) dz \leqq$$

$$\leqq \int_\Omega |K(x,z)|\, dz\tau_\infty(J).$$

Therefore $\tau_c(K * J) \leqq \tau_c(K)\tau_\infty(J)$.

Also, if $f \in C$, then $J(f)$ is essentially continuous, and so $K\big(J(f)\big) \in C$. This proves $K * J \in \mathcal{A}_c$.

For $T \in \mathcal{B} = \mathcal{A}(C, L^1)$, let

$$\omega_\mathcal{B}(T) = \{\lambda \in \mathbf{C} \colon \lambda - T \notin \Phi_\mathcal{B}\}.$$

If $K \in \mathcal{A}_c$, then there is a similar definition of $\omega_\mathcal{A}(K)$. If the algebra $\mathcal{A}_c$ does not have an identity, then let

$$\omega_\mathcal{A}(K) = \{\lambda \in \mathbf{C},\ \lambda \neq 0 \colon \lambda - K \notin \Phi_\mathcal{A}\} \cup \{0\}.$$

If $\mathcal{A}_c$ has an identity, let

$$\omega_\mathcal{A}(K) = \{\lambda \in \mathbf{C} \colon \lambda - K \text{ is not invertible in } \mathcal{A}_c \text{ modulo } \mathcal{F}_c\}.$$

An element $T$ of an algebra $\mathcal{B}$ has a quasi-inverse $S \in \mathcal{B}$ when $S + T = ST = TS$. When $\mathcal{B}$ has a unit $I$, then $T \in \mathcal{B}$ has a quasi-inverse in $\mathcal{B}$ exactly when $I - T$ has an inverse in $\mathcal{B}$.

THEOREM 12. *The algebra $\mathcal{A}_c$ is a subalgebra of $\mathcal{B} = \mathcal{A}(C, L^1)$. For $K \in \mathcal{A}_c$:*

(1) *$\sigma_\mathcal{A}(K) = \sigma_\mathcal{B}(K) = \sigma(K_c) \cup \sigma(K_\infty)$;*

(2) *For $\lambda \neq 0$, $\lambda - K \in \Phi_\mathcal{A}$ if and only if $\lambda - K \in \Phi_\mathcal{B}$;*

(3) *$\omega_\mathcal{A}(K) = \omega(K_c) \cup \omega(K_\infty) \cup \omega_0$, where $\omega_0 = \big\{\lambda : \lambda - K_c \in \Phi(C)$ and $\lambda - K_\infty \in \Phi(L^\infty)$, but $\operatorname{ind}(\lambda - K_c) \neq \operatorname{ind}(\lambda - K_\infty)\big\}$.*

PROOF. Assume $K \in \mathcal{A}_c$ has a quasi-inverse $T \in \mathcal{A}(C, L^1)$, so $K + T = KT = TK$. Since $T \in \mathcal{A}(C, L^1)$, we have that $(T^\dagger)'$ is an extension of $T$, and it is easy to see that $(T^\dagger)' \in \mathcal{A}(L^\infty, L^1)$. By [5, Proof of Theorem 11.11, p. 293] there exists $M \in \mathcal{A}_{\infty\infty}$ such that

$$M(f) = K\Big(\big(T^\dagger\big)'(f)\Big) \quad \text{a.e.} \quad (f \in L^\infty).$$

For $f \in C$, $M(f) = K\big(T(f)\big)$ a.e., so $M(f)$ is essentially continuous. Therefore $T = -K + M \in \mathcal{A}_{\infty\infty}$ and has the property that $T(f)$ is essentially continuous for all $f \in C$. Thus by Proposition 11

$$KT = K(-K + M) \in \mathcal{A}_c.$$

Therefore $T \in \mathcal{A}_c$. It follows that $\sigma_{\mathcal{A}}(K) = \sigma_{\mathcal{B}}(K)$. Now by [2, Theorem 2.7 (1)], $\sigma_{\mathcal{B}}(K) = \sigma(K_c) \cup \sigma\left(K_c^\dagger\right)$. Also, $\left(K_c^\dagger\right)' = K_\infty$, so $\sigma_{\mathcal{B}}(K) = \sigma(K_c) \cup \cup\, \sigma(K_\infty)$. This proves (1).

The proof of (2) is similar to the proof of (1). Note that $\mathcal{F}$ as defined in [2, p. 2] is exactly $\mathcal{F}_c$. Suppose $K \in \mathcal{A}_c$ and there exists $T \in \mathcal{A}(C, L^1)$ with

$$K + T - KT \in \mathcal{F} = \mathcal{F}_c.$$

Following the same steps as in the proof of (1), we have $T \in \mathcal{A}_c$. This suffices to prove (2).

To prove (3), we have by (2) that when $K \in \mathcal{A}_c$, then $\omega_{\mathcal{A}}(K) = \omega_{\mathcal{B}}(K)$. Now by [2, Theorem 2.7]

$$\omega_{\mathcal{B}}(K) = \omega(K_c) \cup \omega\left(K_c^\dagger\right) \cup \omega_0$$

where $\omega_0 = \{\lambda : \lambda - K_c \in \Phi(C)\}$, $\lambda - K_c^\dagger \in \Phi(L^1)$, but $\operatorname{ind}(\lambda - K_c) \neq \neq -\operatorname{ind}\left(\lambda - K_c^\dagger\right)$. As we have noted previously, $\left(K_c^\dagger\right)' = K_\infty$. From the Fredholm theory of operators [5, Corollary 3, p. 91] $\omega\left(K_c^\dagger\right) = \omega\left(\left(K_c^\dagger\right)'\right) = = \omega(K_\infty)$, and when $\lambda - K_c^\dagger \in \Phi(L^1)$, $\operatorname{ind}\left(\lambda - K_c^\dagger\right) = -\operatorname{ind}\left(\lambda - \left(K_c^\dagger\right)'\right) = = -\operatorname{ind}(\lambda - K_\infty)$. Combining these various equalities, we have the statement in (3).

For an operator $T$, we let $\mathcal{N}(T)$ denote the null space of $T$ and $\mathcal{R}(T)$ denote the range of $T$. For $E \subseteqq C$ and $F \subseteqq L^1$, let

$$E^\perp = \{f \in L^1 : \langle g, f\rangle = 0 \quad \text{for all} \quad g \in E\},$$

$$^\perp F = \{g \in C : \langle g, f\rangle = 0 \quad \text{for all} \quad f \in F\}.$$

Corollary 13. *If $K \in \mathcal{A}_c$, $\lambda \neq 0$, and $\lambda - K \in \Phi_{\mathcal{A}}$, then:*

(a) $\mathcal{N}(\lambda - K_c) = {}^\perp\mathcal{R}\left(\lambda - K_c^\dagger\right)$;

(b) $\mathcal{N}(\lambda - K_c)^\perp = \mathcal{R}\left(\lambda - K_c^\dagger\right)$;

(c) $\mathcal{N}\left(\lambda - K_c^\dagger\right) = \mathcal{R}(\lambda - K_c)^\perp$;

(d) $^\perp\mathcal{N}\left(\lambda - K_c^\dagger\right) = \mathcal{R}(\lambda - K_c)$;

(e) $\operatorname{nul}(\lambda - K_c) = \operatorname{def}\left(\lambda - K_c^\dagger\right)$;

(f) $\operatorname{nul}\left(\lambda - K_c^\dagger\right) = \operatorname{def}(\lambda - K_c)$.

Proof. Since $\lambda - K \in \Phi_{\mathcal{B}}$ where $\mathcal{B} = \mathcal{A}(C, L^1)$, that (a)–(f) hold follows from the general Fredholm theory in algebras such as $\mathcal{B}$ as developed in [5]

PROOF. Since $\lambda - K \in \Phi_{\mathcal{B}}$ where $\mathcal{B} = \mathcal{A}(C, L^1)$, that (a)–(f) hold follows from the general Fredholm theory in algebras such as $\mathcal{B}$ as developed in [5] and [2]. We give some details. By [2, Theorem 2.5 (3)] $\operatorname{ind}(\lambda - K_c) = - \operatorname{ind}\left(\lambda - K_c^{\dagger}\right)$ follows immediately that (a), (c), (e), and (f) hold by applying [5, Theorem 5.16, p. 111].

Now we prove (b). Let $\{f_1, \ldots, f_n\}$ be a basis for $\mathcal{N}(\lambda - K_c)$. By [5, Exercise 3.17, p. 45] there exists $\{g_1, \ldots, g_n\} \subseteq L^1$ with $\langle f_k, g_j \rangle = \delta_{k,j}$, $1 \leqq k, j \leqq n$. If $\lambda_k \in \mathbf{C}$ and $\lambda_1 g_1 + \ldots + \lambda_n g_n \in \mathcal{N}(\lambda - K_c)^{\perp}$, then $\lambda_j = 0$ for all $j$. Using (a) this implies,

$$\operatorname{codim}\left({}^{\perp}\mathcal{R}(\lambda - K_c^{\dagger})\right)^{\perp} = \operatorname{codim}\left(\mathcal{N}(\lambda - K_c)^{\perp}\right) \geqq n.$$

By (e) $\operatorname{codim}\left(\mathcal{R}(\lambda - K_c^{\dagger})\right) = n$. Since $\mathcal{R}\left(\lambda - K_c^{\dagger}\right) \subseteq \left({}^{\perp}\mathcal{R}(\lambda - K_c)\right)^{\perp}$, it follows using (a) that $\mathcal{R}\left(\lambda - K_c^{\dagger}\right) = \left({}^{\perp}\mathcal{R}(\lambda - K_c^{\dagger})\right)^{\perp} = \mathcal{N}(\lambda - K_c)^{\perp}$.

A similar argument proves (d).

Corollary 13 has application to equations involving $K_c$ and $K_c^{\dagger}$. For example, assume $K \in \mathcal{A}_c$, $\lambda_0 \neq 0$, and $\lambda_0 - K \in \Phi_{\mathcal{A}}$. Let $\{g_1, \ldots, g_n\}$ be a basis for $\mathcal{N}\left(\lambda_0 - K_c^{\dagger}\right)$. Then by Corollary 13, for $h \in C(\Omega)$ the equation

$$\lambda_0 f(x) - \int_{\Omega} K(x,t) f(t)\, dt = h(x)$$

has a solution $f \in C(\Omega)$ exactly when

$$\int_{\Omega} g_k(t) h(t)\, dt = 0 \quad \text{for} \quad 1 \leqq k \leqq n.$$

A general Fredholm theory relative to a Banach algebra is developed in [1]. This is the setting for the Fredholm theory in the algebras studied in [2]. One consequence of Theorem 12 is that spectral and Fredholm properties of $K \in \mathcal{A}_c$ relative to the Banach algebra $\mathcal{A}_c$ is identical to the spectral and Fredholm properties of $K$ as a member of $\mathcal{A}(C, L^1)$.

It is often the case that a kernel $K \in \mathcal{A}_c$ has the property $K_{\infty}(L^{\infty}) \subseteq C$. For example whenever $K \in \ell(\Omega, \mu)$ then $K_{\infty}(L^{\infty}) \subseteq C$ [5, Exercise 12.4 (a), p. 306]. Let $\mathcal{J} = \left\{K \in \mathcal{A}_c : K_{\infty}(L^{\infty}) \subseteq C\right\}$. It is easy to verify that $\mathcal{J}$ is a closed ideal in $\mathcal{A}_c$. When $K \in \mathcal{J}$, then the spectral and Fredholm theory of $K_c$ in $\mathcal{B}(C)$, $K_{\infty}$ in $\mathcal{B}(L^{\infty})$, and $K$ in $\mathcal{A}_c$ are essentially the same. We prove this result now.

If $\Delta$ is a subset of $\mathbf{C}$, then we use the notation $\Delta' = \Delta \setminus \{0\}$.

THEOREM 14. *Assume* $K \in \mathcal{A}_c$ *and* $K_\infty(L^\infty) \subseteqq C$.

(1) $\sigma'_{\mathcal{A}}(K) = \sigma'(K_c) = \sigma'(K_\infty)$;

(2) $\omega'_{\mathcal{A}}(K) = \omega'(K_c) = \omega'(K_\infty)$;

(3) *For* $\lambda \neq 0$,

$$\lambda - K \in \Phi_{\mathcal{A}} \Leftrightarrow \lambda - K_c \in \Phi(C) \Leftrightarrow \lambda - K_\infty \in \Phi(L^\infty).$$

*Furthermore, when* $\lambda - K_c \in \Phi(C)$, *then* $\operatorname{ind}(\lambda - K_c) = \operatorname{ind}(\lambda - K_\infty)$.

(4) *If* $\lambda \neq 0$ *and* $\lambda - K_c \in \Phi^0(C)$, *then there exists* $F \in \mathcal{F}_c$ *with* $\lambda - K - F$ *invertible in the algebra* $\mathcal{A}_c$ *with the identity adjoined.*

PROOF. Parts (1), (2) and (3) follow from applying Theorem 12 in combination with [2, Theorem 2.7] and [3, Theorem 4]. For example, letting $\mathcal{B}$ be as before, we have by Theorem 12 that $\sigma_{\mathcal{A}}(K) = \sigma_{\mathcal{B}}(K)$ for $K \in \mathcal{A}_c$. Also, by [2, Theorem 2.7],

$$\sigma_{\mathcal{B}}(K) = \sigma(K_c) \cup \sigma\left(K_c^\dagger\right).$$

Now $K_\infty = \left(K_c^\dagger\right)'$, and therefore, $\sigma(K_\infty) = \sigma\left(K_c^\dagger\right)$. By assumption $K_\infty(L^\infty) \subseteqq C$, so applying [3, Theorem 4] we have $\sigma'(K_c) = \sigma'(K_\infty)$. Combining all these equalities has the result:

$$\sigma'_{\mathcal{A}}(K) = \sigma'_{\mathcal{B}}(K) = \sigma'(K_c) = \sigma'(K_\infty).$$

This proves (1), and the proofs of (2) and (3) are not much different.

To prove (4), assume that $\lambda - K_c \in \Phi^0(C)$. Then by (3) $\lambda - K \in \Phi^0_{\mathcal{A}} = \Phi^0_{\mathcal{B}}$. By [2, Theorem 2.5 (2) and Corollary 2.6] there exists $F \in \mathcal{F}$ with $\lambda - K - \mathcal{F}$ invertible in $\mathcal{B}$. As noted before $\mathcal{F} = \mathcal{F}_c$, so $\lambda - K - \mathcal{F}$ is invertible in $\mathcal{A}_c$ with identity adjoined.

THEOREM 15. *Assume* $K \in \mathcal{A}_c$.

(1) $\partial\sigma_{\mathcal{A}}(K) \subseteqq \sigma(K_\infty)$ *and* $\partial\sigma_{\mathcal{A}}(K) \subseteqq \sigma(K_c)$;

(2) $\partial\sigma(K_c) \subseteqq \sigma(K_\infty)$;

(3) *If* $\sigma(K_\infty)$ *or* $\sigma(K_c)$ *is totally disconnected (abbreviation: t.d.), then*

$$\sigma_{\mathcal{A}}(K) = \sigma(K_\infty) = \sigma(K_c);$$

(4) *If* $\partial\sigma(K_c) = \sigma(K_c)$ *and* $\partial\sigma(K_\infty) = \sigma(K_\infty)$, *then*

$$\sigma_{\mathcal{A}}(K) = \sigma(K_\infty) = \sigma(K_c).$$

PROOF. The algebra isometry $K \to K_c$ maps $\mathcal{A}_c$ onto a closed subalgebra of $\mathcal{B}(C)$. Therefore by standard Banach algebra theory

$$\partial\sigma_{\mathcal{A}}(K) \subseteqq \sigma(K_c).$$

Essentially the same argument proves

$$\partial\sigma_{\mathcal{A}}(K) \subseteqq \sigma(K_\infty).$$

This proves (1).

(2) follows from the obvious fact that every approximate eigenvalue of $K_c$ is an approximate eigenvalue of $K_\infty$.

If $\sigma(K_\infty)$ or $\sigma(K_c)$ is t.d., then by (1), $\partial\sigma_{\mathcal{A}}(K)$ is t.d. Therefore $\sigma_{\mathcal{A}}(K) = \partial\sigma_{\mathcal{A}}(K)$. Then the conclusion of (3) follows from (1).

Now assume $\partial\sigma(K_c) = \sigma(K_c)$ and $\partial\sigma(K_\infty) = \sigma(K_\infty)$. Since $\sigma_{\mathcal{A}}(K) = \sigma(K_\infty) \cup \sigma(K_c)$, it follows that $\partial\sigma_{\mathcal{A}}(K) = \sigma_{\mathcal{A}}(K)$. Therefore by (1)

$$\sigma_{\mathcal{A}}(K) = \sigma(K_\infty) = \sigma(K_c).$$

QUESTION. Is it true for all $K \in \mathcal{A}_c$ that

$$\sigma_{\mathcal{A}}(K) = \sigma(K_c) = \sigma(K_\infty)?$$

It seems very unlikely that this question would have an affirmative answer. However, despite some effort, we have been unable to find a kernel $K \in \mathcal{A}_c$ for which $\sigma(K_c) \neq \sigma(K_\infty)$.

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# ON ENTROPY FUNCTIONALS OF STATES OF OPERATOR ALGEBRAS

D. PETZ (Budapest)

For two finite probability distributions $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ the quantity

$$\sum_{k=1}^{n} p_k(\log p_k - \log q_k) \tag{1}$$

was introduced in 1951 by Kulback and Leibler. They called it information for discrimination [12, 13]. Some years later Rényi suggested the name information gain [23]. As a natural analogue of (1) Umegaki defined the relative entropy of two density matrices in 1962 [27] by the formula

$$\operatorname{Tr} \varrho(\log \varrho - \log \varphi) \tag{2}$$

and this notion was extended by Araki [2] to states of C*-algebras as follows.

Let the von Neumann algebra $\mathcal{M}$ act on a Hilbert space $\mathcal{H}$ and let the normal state $\omega$ be given by a vector $\Omega \in \mathcal{H}$. Let $\varphi$ be another normal state. Then there exists a positive selfadjoint operator $\Delta(\varphi, \omega)$ such that

(i) $\| \Delta(\varphi,\omega)^{1/2} a\Omega \|^2 = \varphi(apa^*)$ for every $a \in \mathcal{M}$ and for the support projection $p$ of $\omega$,

(ii) the support of $\Delta(\varphi,\omega)$ is in the closure of $\mathcal{M}\Omega$,

(iii) $\mathcal{M}\Omega$ is a core for the restriction of $\Delta(\varphi,\omega)^{1/2}$ to the closure of $\mathcal{M}\Omega$.

For normal states Araki defined the relative entropy as

$$S(\omega, \varphi) = -\langle \log \Delta(\varphi, \omega)\Omega, \Omega \rangle \tag{3}$$

which turns out to be independent of the representation. For positive functionals of an arbitrary C*-algebra $\mathcal{A}$ the relative entropy may be determined through the GNS-construction. Let $(\mathcal{H}, \Phi, \pi)$ stand for the GNS-triplet for the unital C*-algebra $\mathcal{A}$ and the positive functional $\phi$ of $\mathcal{A}$. Let $\psi$ be another positive functional on $\mathcal{A}$. We write $\bar{\psi}$ for the normal state of $\pi(\mathcal{A})''$ such that

$$\bar{\psi}(\pi(a)) = \psi(a) \qquad (a \in \mathcal{A})$$

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if a normal functional with this property exists. Let

$$S(\psi,\phi)=\begin{cases} S(\bar\psi,\bar\phi) & \text{if } \bar\psi \text{ exists,}\\ +\infty & \text{otherwise.}\end{cases}$$

Properties of the relative entropy functional were established in many papers and the highlight of this development was Lieb's convexity theorem [15]. The notion received much attention in quantum mechanics [16]. Concerning the details we refer to the survey papers [3] and [19].

The aim of the present paper is to characterize the relative entropy functional through its well-known properties and to prove some results related to a net of mappings approximating the identity. As a frame we consider nuclear C*-algebras [10, p. 858] and injective von Neumann algebras [24, p. 143]. Such algebras are well-aproximated by finite dimensional ones and we shall benefit from the characterization of the relative entropy functional on matrix algebras [22].

Our crucial postulate for the relative entropy includes the notion of conditional expectation. Let us recall that in the setting of operator algebras conditional expectation (or projection of norm one) is defined as a positive unital idempotent linear mapping onto a subalgebra [25, p. 131].

Now we list properties of the relative entropy functional needed in the characterization. Let us recall that a separating state gives rise to a separating cyclic vector in the GNS Hilbert space for the generated von Neumann algebra.

(i) Conditional expectation property: Assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and there exists a projection of norm one $E$ of $\mathcal{B}$ onto $\mathcal{A}$ which leaves invariant the separating state $\varphi$. Then for every state $\omega$ of $\mathcal{B}$ the equality $S(\omega,\varphi)=S(\omega|\mathcal{A},\varphi|\mathcal{A},)+S(\omega,\omega\circ E,)$ holds.

(ii) Monotonicity property: For every completely unital positive mapping $\alpha$ of a C*-algebra $\mathcal{A}$ into $\mathcal{B}$ we have $S(\omega,\varphi)\geqq S(\omega\circ\alpha,\varphi\circ\alpha)$.

(iii) Direct sum property: Assume that $\mathcal{B}=\mathcal{B}_1\oplus\mathcal{B}_2$ and $\varphi_{12}(a\oplus b)=$ $=\lambda\varphi_1(a)+(1-\lambda)\varphi_2(b)$ and $\omega_{12}(a\oplus b)=\lambda\omega_1(a)+(1-\lambda)\omega_2(b)$ for every $a\in\mathcal{B}_1, b\in\mathcal{B}_2$ and some $0<\lambda<1$. Then $S(\omega_{12},\varphi_{12})=\lambda S(\omega_1,\varphi_1)+(1-$ $-\lambda)S(\omega_2,\varphi_2)$.

(iv) Nilpotence property: $S(\varphi,\varphi)=0$.

(v) Lower semicontinuity: The function $(\omega,\varphi)\mapsto S(\omega,\varphi)$ is weak* lower semicontinuous on the state space of a nuclear C*-algebra $\mathcal{B}$ (when $\varphi$ is assumed to be separating).

The properties (i)–(v) are well-known for the relative entropy functional. Among them the conditional expectation property is the most crucial (it was obtained in [21] in full generality, cf. [20]). The monotonicity has been proven by Uhlmann [26] and weak* lower semicontinuity is a consequence of Kosaki's formula [11] stated here for further use:

$$(4) \qquad S(\omega,\varphi) =$$
$$= \sup \sup \left\{ \omega(I)\log n - \int_{1/n}^{\infty} \omega\big(y(t)^* y(t)\big) + t^{-1}\varphi\big(x(t)x(t)^*\big)\frac{dt}{t} \right\}$$

where the first sup is taken over all natural numbers $n$, the second one is over all step functions $x : (1/n,\infty) \to \mathcal{A}$ with finite range and $y(t) = I - x(t)$.

THEOREM 1 [22]. *If a real valued functional $S'(\varphi,\omega)$ defined for separating states $\varphi$ and arbitrary states $\omega$ of finite dimensional $C^*$-algebras posesses the properties* (i)–(v) *then there exists a constant $C \in \mathbf{R}$ such that*

$$S'(\varphi,\omega) = C \operatorname{Tr} D_\omega(\log D_\omega - \log D_\varphi).$$

The proof consists of several steps. It is shown that for larger and larger class of states

$$S'(\varphi,\omega) = C\, S(\varphi,\omega)$$

must hold.

A C*-algebra $\mathcal{A}$ is said to be nuclear if, for every C*-algebra $\mathcal{B}$, there is only one C*-norm on $\mathcal{A} \odot \mathcal{B}$. Finite dimensional and abelian C*-algebras are nuclear. A C*-algebra $\mathcal{A}$ is called AF-algebra if it contains an increasing sequence of finite dimensional subalgebras such that their union is norm dense in $\mathcal{A}$. It can be proved that the inductive limit of nuclear C*-algebras is nuclear itself. In particular, every AF-algebra is nuclear. Let $(\alpha_i : \mathcal{A}_i \to \mathcal{A})_i$ be a net of unital completely positive mapping defined on finite dimensional algebras. We shall call $(\alpha_i)_i$ a norm approximating net if for each $i$ there exists a unital completely positive mapping $\beta_i : \mathcal{A} \to \mathcal{A}_i$ such that

$$\|\alpha_i \circ \beta_i(a) - a\| \to 0 \quad (a \in \mathcal{A}).$$

(The net $\alpha_i \circ \beta_i$ approximates the identity of $\mathcal{A}$ in the topology of pointwise norm convergence.) The class of nuclear C*-algebras is characterized by the existence of a norm approximating net. A C*-algebra admitting the existence of a norm approximating net is often called semidiscrete. The equivalence of nuclearity and semidiscretness was proved in [4]. The works [8] and [14] review this subject. Most physically important C*-algebras are nuclear. For example, the algebra of the canonical commutation relation is a nonseparable nuclear C*-algebra.

THEOREM 2. *Let $\mathcal{A}$ be a nuclear $C^*$-algebra with states $\varphi$ and $\omega$. If $(\alpha_i : \mathcal{A}_i \to \mathcal{A})_i$ is a norm approximating net then*

$$S(\omega,\varphi) = \lim_i S(\omega \circ \alpha_i, \varphi \circ \alpha_i).$$

*Consequently, $S(\omega,\varphi)$ is the lowest upper bound of the quantities $S(\omega\circ\alpha,\varphi\circ\alpha)$ where $\alpha$ ranges all completely positive unital mappings from a finite dimentional algebra into $\mathcal{A}$.*

PROOF. Since $S(\omega\circ\alpha,\varphi\circ\alpha)\leqq S(\omega,\varphi)$ holds for any completely positive mapping $\alpha$, we show that given a generating net $(\alpha_i:\mathcal{A}_i\to\mathcal{A})_i$ and numbers $0\leqq u<S(\omega,\varphi)$, $0<\varepsilon$, for large enough $i$

$$S(\omega\circ\alpha_i,\varphi\circ\alpha_i)\geqq u-\varepsilon \tag{5}$$

holds.

The main ingredient of the proof will be Kosaki's formula (4). There exists an $n\in\mathbf{N}$ and a step function $x:[1/n,\infty)\to\mathcal{A}$ such that it has finite range, $x(t)=I$ for large $t$ and

$$\log n-\int_{1/n}^{\infty} t^{-1}\omega\big(y(t)^*y(t)\big)+t^{-2}\varphi\big(x(t)x(t)^*\big)\,dt\geqq u.$$

For large $i$ we have

$$\log n-\int_{1/n}^{\infty} t^{-1}\omega(\alpha_i\circ\beta_i\big(y(t)^*y(t)\big)+t^{-2}\varphi(\alpha_i\circ\beta_i\big(x(t)x(t)^*\big)\,dt\geqq u-\varepsilon$$

where $\|\alpha_i\circ\beta_i(a)-a\|\to 0$ for every $a\in\mathcal{A}$. So writing $x_i(t)$ and $y_i(t)$ for $\beta_i\big(x_i(t)\big)$ and $\beta_i\big(y_i(t)\big)$, respectively, we obtain from the Schwarz inequality

$$\log n-\int_{1/n}^{\infty} t^{-1}(\omega\circ\alpha_i)\big(y_i(t)^*y_i(t)\big)+t^{-2}(\varphi\circ\alpha_i)\big(x_i(t)x_i(t)^*\big)\,dt\geqq u-\varepsilon$$

and Kosaki's formula yields (5) for large enough $i$. □

THEOREM 3. *If a real valued functional $S'(\varphi,\omega)$ defined for separating states $\varphi$ and arbitrary states $\omega$ of nuclear $C^*$-algebras posesses the properties* (i)–(v) *then there exists a constant $C\in\mathbf{R}$ such that*

$$S'(\varphi,\omega)=C\,S(\varphi,\omega). \tag{6}$$

PROOF. Theorem 1 tells us that $S'$ must be a constant multiple on finite dimensional algebras. The rest is in Theorem 2. For an arbitrary nuclear C*-algebra let $(\alpha_i:\mathcal{A}_i\to\mathcal{A})_i$ be a norm approximating net and let $\beta_i:\mathcal{A}\to\mathcal{A}_i$ be the corresponding completely positive mappings from the definition of such a net. From the monotonicity

$$S'(\omega,\varphi)\geqq\limsup_i S'(\omega\circ\alpha_i,\varphi\circ\alpha_i)\geqq\limsup_i S'(\omega\circ\alpha_i\circ\beta_i,\varphi\circ\alpha_i\circ\beta_i).$$

According to the weak* lower semicontinuity we have

$$S'(\omega,\varphi) \leqq \liminf_i S'(\omega\circ\alpha_i\circ\beta_i,\varphi\circ\alpha_i\circ\beta_i)\,.$$

Therefore

$$S'(\omega,\varphi) = \lim_i S'(\omega\circ\alpha_i,\varphi\circ\alpha_i)$$

and (6) must holds. □

Let $\mathcal{M}$ be a von Neumann algebra and let $(\alpha_i : \mathcal{A}_i \to \mathcal{M})$ be a net of unital completely positive mappings with finite dimensional algebras $(\mathcal{A}_i)_i$. We shall call $(\alpha_i)_i$ a weak* approximating net if for each $i$ there exists a normal unital completely positive mapping $\beta_i : \mathcal{M} \to \mathcal{A}_i$ such that

$$\lim_i \psi(\alpha_i\circ\beta_i a) = \psi(a)$$

for every $a \in \mathcal{M}$ and for every $\psi \in \mathcal{M}_*$.

Assume that a von Neumann algebra $\mathcal{M}$ contains an ascending net $(\mathcal{N}_i)_i$ of finite dimensional subalgebras so that $\cup_i\mathcal{N}_i$ is strongly dense in $\mathcal{M}$. Let $\kappa_i$ be the embedding of $\mathcal{N}_i$ into $\mathcal{M}$. Then $(\kappa_i)_i$ is a strong approximating net. Indeed, for the generalized conditional expectation $E_i : \mathcal{M} \to \mathcal{M}_i$ we have $\kappa_i \circ E_i \to \mathrm{id}$ strongly, due to the martingale convergence theorem [9] and [18].

Injective von Neumann algebras admit the existence of a weak* approximating net (for the identity, see [8]) and completely similarly to the previous proofs one obtains the following.

THEOREM 4. *Let $\mathcal{M}$ be an injective von Neumann algebra with normal states $\varphi$ and $\omega$. Then $S(\omega,\varphi)$ is the supremum of all the quantities $S(\omega\circ\alpha,\varphi\circ\alpha)$ where $\alpha$ runs over all completely positive unital mappings from a finite dimensional algebra into $\mathcal{M}$.*

THEOREM 5. *If a real valued functional $S'(\varphi,\omega)$ defined for separating states $\varphi$ and arbitrary states $\omega$ of injective von Neumann algebras posesses the properties* (i)–(v) *then the functional $S'$ is a constant multiple of Araki's relative entropy.*

While this characterization of the relative entropy on injective algebras is based on finite dimensional approximation, we note that another characterization was given in [6] which benefited from the fact that an injective von Neumann algebra is the range of a conditional expectation of some $B(\mathcal{H})$.

Let $\mathcal{M} \subset B(\mathcal{H})$ be an injective von Neumann algebra and let $\varphi,\omega$ be normal states on $\mathcal{M}$. Then $S(\omega,\varphi) = \mathrm{ent}_{\mathcal{M}}(\omega,\varphi)$ where $\mathrm{ent}_{\mathcal{M}}$ is defined in the following way:

(i) $\mathrm{ent}_{B(\mathcal{H})}(\sigma,\rho) = S(\sigma,\rho)$ when $\sigma$ and $\rho$ are normal.

(ii) $\mathrm{ent}_{B(\mathcal{H})}(\sigma,\rho) = \sup\big\{F(\sigma,\rho) : F$ is $w^*$ lower semicontinuous, convex, and coincides with $S(\sigma,\rho)$ when $\sigma$ and $\rho$ are normal$\}$.

(iii) $\mathrm{ent}_{\mathcal{M}}(\sigma,\rho) = \inf\{\, \mathrm{ent}_{B(\mathcal{H})}(\sigma',\rho') : \sigma'|\mathcal{M} = \sigma \text{ and } \rho'|\mathcal{M} = \rho\}$.

The following definition for the entropy of states of arbitrary C*-algebras was proposed in [17]:

$$S(\varphi) = \sup\{\, \textstyle\sum_i \lambda_i S(\varphi_i,\varphi) : \sum_i \lambda_i\varphi_i = \varphi\}\,. \tag{7}$$

Here the supremum is over all decompositions of $\varphi$ into finite (or equivalently countable) convex combinations of other states. This definition was generalized in [5]. Let $\alpha : \mathcal{B} \to \mathcal{A}$ be a completely positive unital map and $\varphi$ a state of $\mathcal{A}$. The quantity

$$H_\varphi(\alpha) = \sup\{\, \textstyle\sum_i \lambda_i S(\varphi_i\circ\alpha,\varphi\circ\alpha) : \sum_i \lambda_i\varphi_i = \varphi\}\,. \tag{8}$$

can be called the entropy of the mapping $\alpha$.

THEOREM 6. *Let $\mathcal{A}$ be a nuclear $C^*$-algebra with an approximating net $(\alpha_i : \mathcal{A}_i \to \mathcal{A})_i$. Then for every state $\varphi$ of $\mathcal{A}$*

$$S(\varphi) = \lim_i H_\varphi(\alpha_i)$$

*holds.*

PROOF. By the definition of the entropy we can find a finite convex decomposition $\sum_{k=1}^n \lambda_k\varphi_k$ of $\varphi$ for an $\varepsilon > 0$ so that

$$S(\varphi) \leqq \sum_{k=1}^n \lambda_k S(\varphi_k,\varphi) + \varepsilon.$$

For $i$ big enough we have

$$S(\varphi_k,\varphi) \leqq S(\varphi_k\circ\alpha_i,\varphi\circ\alpha_i) + \varepsilon \quad (k = 1,2,\dots,n)$$

due to Theorem 2. Hence

$$S(\varphi) \leqq \sum_{k=1}^n \lambda_k S(\varphi_k\circ\alpha_i,\varphi\circ\alpha_i) + 2\varepsilon \leqq H_\varphi(\alpha_i) + 2\varepsilon$$

for large $i$. Since $H_\varphi(\alpha_i) \leqq S(\varphi)$, the proof is complete. □

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# LEBESGUE FUNCTION TYPE SUMS OF HERMITE INTERPOLATIONS

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## 1. Introduction. Preliminary results

Let $X = \{x_{kn} = \cos\vartheta_{kn}\} \subset [-1,1]$,

$$-1 \leqq x_{nn} < x_{n-1,n} < \ldots < x_{1n} \leqq 1, \quad n = 1,2,\ldots, \tag{1.1}$$

be an infinite triangular interpolatory matrix. For $m \geqq 1$ we consider the unique Hermite interpolatory polynomials

$$I_{nm}(\mathcal{F}, X, x) := \sum_{t=0}^{m-1} \sum_{k=1}^{n} f_{tkn} h_{tknm}(X, x) \tag{1.2}$$

of degree $\leqq mn-1$ where $\mathcal{F} = \{f_{tkn}\}$ ($f_{tkn}$ real), and the polynomials $h_{tknm}(X,x) \in \mathcal{P}_{mn-1}$ satisfy

$$h_{tknm}^{(p)}(X, x_{qn}) = \delta_{tp}\delta_{kq}, \qquad t,p = 0,\ldots,m-1, \quad k,q = 1,\ldots,n \tag{1.3}$$

($\delta$ is the Kronecker delta). When $f \in C$ or $f^{(m-1)} \in C$,

$$\begin{cases} H_{nm}(f,X,x) := \displaystyle\sum_{k=1}^{n} f(x_{kn}) h_{0knm}(X,x), \\ \mathcal{H}_{nm}(f,X,x) := \displaystyle\sum_{t=0}^{m-1}\sum_{k=1}^{n} f^{(t)}(x_{kn}) h_{tknm}(X,x) \end{cases} \tag{1.4}$$

are two important special cases of $I_{nm}$. By (1.2) and (1.3) (using obvious short notations) $H_{nm}^{(t)}(f, x_k) = \delta_{0t} f(x_k)$ ($1 \leqq k \leqq n$, $0 \leqq t \leqq m-1$) —

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this is why $H_{nm}$ is called Hermite–Fejér interpolatory polynomials of higher order. (If $m = 1$, we get the Lagrange interpolation; $H_{n2}$ is the classical HF interpolation.) For $X = X^{(\alpha,\beta)}$ (Jacobi nodes) using fairly precise asymptotic formulae for the corresponding $h_{0k}^{(\alpha,\beta)}$, it turned out that for arbitrary *odd* $m$, $H_{nm}^{(\alpha,\beta)}(f,x)$ is of "*Lagrange type*", i.e. for a proper $f_1 \in C$

$$\lim_{n\to\infty} \| H_{nm}^{(\alpha,\beta)}(f,x)\| = \infty \tag{1.5}$$

($\|g\| = \max_{x\in[-1,1]} |g(x)|$). On the other hand, if $m$ is *even*, for proper $\alpha$ and $\beta$

$$\lim_{n\to\infty} \| H_{nm}^{(\alpha,\beta)}(f,x) - f(x)\| = 0 \quad \text{for all } f \in C$$

i.e., $H_{nm}^{(\alpha,\beta)}(f,x)$ is of "*HF-type*" for *even* values of $m$ (cf. R. Sakai, P. Vértesi [1, Theorem 3.3], [2, Part 5.3]).

Actually, (1.5) follows from the estimations

(1.6)

$$\left\| \sum_{k=1}^{n} \left| h_{0knm}^{(\alpha,\beta)}(x)\right| \right\| \geqq c \max\left(\log n, n^{m(\alpha+1/2)}, n^{m(\beta+1/2)}\right), \quad m = 1,3,\dots,$$

which makes the conjecture

$$\left\| \sum_{k=1}^{n} | h_{0knm}(X,x)| \right\| \geqq c \log n \quad \text{for any} \quad X \subset [-1,1] \tag{1.7}$$

reasonable. (1.7) was proved by J. Szabados [3]. More exactly, using a nice idea of G. Halász, he proved the following very general theorem.

*With*

(1.8)

$$\Lambda_{tnm}(X) := \|\lambda_{tnm}(X,x)\| := \left\| \sum_{k=1}^{n} |h_{tknm}(X,x)| \right\|, \quad t = 0,\dots,m-1,$$

*we have for an arbitrary system* $X$

$$\Lambda_{tnm}(X) \geqq \begin{cases} c_1 \dfrac{\log n}{n^t}, & m-t \quad \textit{odd}, \\[2mm] \dfrac{c_2}{n^t}, & m-t \quad \textit{even}, \end{cases} \qquad t = 0,1,\dots,m-1. \tag{1.9}$$

(If $m = 1$, $t = 0$, we get the well-known Faber theorem; $\Lambda_{0n3}(X) \geqq$ $\geqq c \log n$ was proved in J. Szabados, A. K. Varma [4].)

## 2. The result

**2.1.** The aim of this paper is to prove some estimations for the Lebesgue function type sums $\lambda_{tnm}(X,x)$, $m-t$ odd. We have

THEOREM 2.1. *For arbitrary fixed $X$, $m$ and $t$, $0 \leqq t \leqq m-1$, $m-t$ odd, there exists a constant $c = c(m,t) > 0$ such that if $\varepsilon > 0$ is any fixed positive number then there exist sets $H_n = H_n(\varepsilon, X, m, t)$ with $|H_n| \leqq \varepsilon$ such that with $\eta = \eta(\varepsilon, m, t) > 0$*

$$\lambda_{tnm}(X,x) > \eta \frac{\log n}{n^t} \quad \text{if} \quad x \in (-\infty, \infty) \setminus H_n, \quad n = 1, 2, \ldots. \tag{2.1}$$

**2.2.** REMARKS AND PROBLEMS. 1. For $\lambda_{m-1,nm}(x)$ $(m = 1, 2, \ldots)$ and $\lambda_{0n3}(x)$ see J. Szabados, P. Vértesi [5; Theorem 3.5, p. 75] and its references, further P. Vértesi [6].

2. Using the Chebyshev matrix $T = \left\{ \cos \frac{2k-1}{2n}\pi \right\}_{k=1}^{n}$ one can prove that the order of estimation (2.1) is optimal. However our present proof gives $\eta = c\varepsilon^{2m}$. We think $\eta = c\varepsilon^m$ can be obtained (cf. the case $X = T$).

3. Estimate (2.1) gives

$$\int_{-1}^{1} \lambda_{tnm}(X,x)\,dx \geqq c\eta \frac{\log n}{n^t}, \quad n \geqq 1.$$

4. Analogous results certainly hold when $m-t$ is even, but we can not prove them at present.

5. It would be interesting to get estimates analogous to those in (1.9) and (2.1) considering trigonometric and complex cases (cf. [5, Ch. III. §3]).

6. Our theorem is important proving divergence of $|H_{nm}(f,x)|$ almost everywhere ($m$ is odd) and in the investigation of the mean convergence. We consider them in other papers.

## 3. Proof

**3.1.** First we recall that by the fundamental [3, Lemma 3] *we have with an absolute constant $a > 0$*

$$|h_{tk}(x)| \geqq a \frac{|x - x_k|^{m-1} |\ell_k(x)|^m}{(x_k - x_{k\pm 1})^{m-t-1}}, \quad m-t \text{ odd}, \quad 0 \leqq t \leqq m-1 \tag{3.1}$$

$(-\infty < x < \infty)$ *with one of the signs in* $x_{k\pm1}$ (cf. [3, Corollary]).

Here $\ell_k(x) = \ell_{kn}(X, x)$ are the fundamental polynomials of Lagrange interpolation, i.e. with $\omega_n(x) = c_n \prod_{k=1}^{n}(x - x_k)$, $c_n \neq 0$

$$\ell_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)}, \quad 1 \leqq k \leqq n.$$

**3.2.** In what follows we use many ideas of some previous works of P. Erdős and P. Vértesi (cf. [5; References [E4], [EV2], [V14], [V15]]). First we recall some notations.

Let $J_k = J_{kn} = [x_{k+1,n}, x_{kn}]$, $k = 0, 1, \ldots, n$, $x_{01} \equiv 1$, $x_{n+1,n} \equiv -1$, $n = 1, 2, \ldots$. With $0 < q_k = q_{kn} \leqq \frac{1}{2}$ let

$$\begin{cases} J_k(q_k) = [x_{k+1} + q_k|J_k|, x_k - q_k|J_k|], \\ \overline{J}_k = \overline{J}_k(q_k) = J_k \setminus J_k(q_k). \end{cases}$$

Let $z_k = z_k(q_k)$ be defined by

$$(3.2) \qquad (0<)\bigl|\omega_n(z_k)\bigr| = \min_{x \in J_k(q_k)} \bigl|\omega_n(x)\bigr|, \quad 0 \leqq k \leqq n,$$

further let

$$(3.3) \qquad |J_i, J_k| = \max\bigl(|x_{i+1} - x_k|, |x_{k+1} - x_i|\bigr), \quad 0 \leqq i,\ k \leqq n,$$

$$(3.4) \qquad \rho(J_i, J_k) = \min\bigl(|x_{i+1} - x_k|, |x_{k+1} - x_i|\bigr), \quad 0 \leqq i,\ k \leqq n.$$

A simple consequence of [5, Lemma 3.9 or [V14, Lemma 3.1] from the references therein] is

LEMMA 3.1. *Let $|J_{kn}| > \delta_n := n^{-1/6}$ ($k$ is fixed, $0 < k < n$). Then for any $\{q_{kn}\}$ with $(\log n)^{-2} \leqq q_{kn} \leqq 1/4$, we can define the index $u = u(k, n)$ and the set $r_{kn}$, $\overline{J}_{kn} \subset r_{kn} \subset J_{kn}$, so that $|r_{kn}| \leqq 4q_{kn}|J_{kn}|$, moreover*

$$\bigl|\ell_{un}(x)\bigr| \geqq 3^{\sqrt{n}} \quad \text{if} \quad x \in J_{kn} \setminus r_{kn} \quad \text{and} \quad n \geqq n_1$$

*($n_1 > 0$ is an absolute constant, large enough).*

By Lemma 3.1, if $q_{kn} = s_n = (\log n)^{-2}$, say, for a "long" interval ($|J_k| > \delta_n$) we have if $x \in J_k \setminus r_k$, using (3.1) and Lemma 3.1,

$$\lambda_{tnm}(x) \geqq \bigl|h_{tk}(x)\bigr| \geqq c\frac{1}{s_n^{m-1}}\frac{\left(3^{\sqrt{n}}\right)^m}{2^{m-t-1}} > \log^2 n \quad (n \geqq n_0(\varepsilon)),$$

say. Further $\sum |r_k| \leqq 4s_n \sum |J_k| < \varepsilon$, $n \geqq n_0(\varepsilon)$, whence (2.1) holds true for the long intervals excluding the set $H_{1n} := \bigcup\limits_{|J_k|>\delta_n} J_k$ of measure $\leqq \varepsilon$. (Here and later we exclude certain sets where (2.1) may not hold. We have to prove that their total measure is "small" and the validity of (2.1) for the remaining part of $(-\infty,\infty)$.)

**3.3.** We now consider the "short" intervals, i.e. when $|J_k| \leqq \delta_n$ $(= n^{-1/6})$, $k = 0, 1, \dots, n$. If $J_k$, $k = 0, n$, is short we exclude it (them). The measure of the excluded set is $\leqq 2n^{-1/6} < \varepsilon$ $(n \geqq n_0)$.

**3.4.** From the remaining short intervals we omit those for which $|J_k|/|T_k| < \varepsilon$, where $T_k = [x_{k+2}, x_{k-1}]$. The total measure of these excluded short intervals $J_k$ is less than or equal to $\sum |J_k| < \varepsilon \sum |T_k| \leqq 6\varepsilon$. (By the definition of $T_k$, the relation $\sum\limits_{k=1}^{n-1} |T_k| < 6$ is clear.)

Remark that for the remaining short intervals we have

$$|J_k| \geqq \varepsilon \max\left(|J_{k+1}|, |J_{k-1}|\right), \quad k \in E_n \tag{3.5}$$

where $E_n$ denotes the corresponding set of indices. If $k \in E_n$, then (cf. [6, Lemma 3.2])

LEMMA 3.2. *Let $m - t$ be odd. If $k \in E_n$ and $1 \leqq r \leqq n-1$, then*

$$\left|h_{tk}(x)\right| + \left|h_{t,k+1}(x)\right| \geqq c(m)\varepsilon^{m-t-1} q_k^m \left|\frac{\omega_n(z_r)}{\omega_n(z_k)}\right|^m \frac{|J_k|^{t+1}}{|J_r, J_k|}, \quad n \geqq 6, \tag{3.6}$$

*whenever $x \in J_r(q_r)$, $\rho(J_r, J_k) \geqq 2\delta_n$ and $|J_r| \leqq \delta_n$.*

PROOF. By [6, (3.7), (3.8)]

$$\frac{2}{3} \leqq \left|\frac{z_r - x_s}{x - x_s}\right| \leqq 2, \quad s = k, k+1, \quad x \in J_r(q_r) \tag{3.7}$$

and

$$\left|\ell_s(x)\right| \geqq \frac{2}{3}\left|\ell_s(z_r)\right|, \quad s = k, k+1, \quad x \in J_r(q_r). \tag{3.8}$$

By (3.1), (3.8), (3.5), (3.7) and again by (3.7)

$$\left|h_{tk}(x)\right| + \left|h_{t,k+1}(x)\right| \geqq a\left(\frac{2}{3}\right)^m \left(\frac{\varepsilon}{x_k - x_{k+1}}\right)^{m-t-1} \cdot$$

$$\cdot \left\{|x - x_k|^{m-1}\left|\ell_k(z_r)\right|^m + |x - x_{k+1}|^{m-1}\left|\ell_{k+1}(z_r)\right|^m\right\} =$$

$$= a\left(\frac{2}{3}\right)^m \left(\frac{\varepsilon}{x_k - x_{k+1}}\right)^{m-t-1} \left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m \cdot$$

$$\cdot \left\{ \frac{\ell_k^m(z_k)}{|x_k - z_r|}|x_k - z_k|^m \left|\frac{x - x_k}{x_k - z_r}\right|^{m-1} + \right.$$

$$\left. + \frac{\ell_{k+1}^m(z_k)}{|x_{k+1} - z_r|}|x_{k+1} - z_k|^m \left|\frac{x - x_{k+1}}{x_{k+1} - z_r}\right|^{m-1} \right\} \geqq$$

$$\geqq a\left(\frac{2}{3}\right)^m \frac{\varepsilon^{m-t-1}}{2^{m-1}} \left|\frac{\omega(z_r)}{\omega(z_k)}\right|^m \frac{q_k^m |J_k|^m}{|J_k|^{m-t-1}} \left\{ \frac{\ell_k^m(z_k)}{|x_k - z_r|} + \frac{\ell_{k+1}^m(z_k)}{|x_{k+1} - z_r|} \right\}$$

$$(x \in J_r(q_r)),$$

whence by $|x_s - z_r| < |J_s, J_r|$ $(s = k, k+1)$ and $\ell_k^m(z_k) + \ell_{k+1}^m(z_k) \geqq 2^{1-m}$ (cf. [5, Lemma 3.6, p. 76]) we get (3.6). □

**3.5.** The following argument is a combination of Erdős–Vértesi [5, [EV2]] and [5, [V15]].

Let $q_k = q = \varepsilon$ $(1 \leqq k \leqq n-1)$. The point $x$, the intervals $J_k$ and $J_k(q)$, the index $k$ will be called *exceptional* iff $\lambda_{tnm}(x) \leqq \eta \dfrac{\log n}{n^t}$ for $x \in J_k(q)$ $(k \in E_n$, $n$ is fixed). We state

$$\text{(3.9)} \qquad H_{2n} := \sum_{k \in e_n} |J_{kn}| := 2\mu_n \leqq 2\varepsilon \quad \text{if} \quad n \geqq n_0 = n_0(\varepsilon),$$

where $e_n = \{k\colon k \in E_n \text{ and } J_{kn} \text{ is exceptional}\}$. To prove (3.9), first we quote a slight modification of [5, Lemma 3.8, p. 78].

LEMMA 3.2. *Let $I_k = [a_k, b_k]$, $1 \leqq k \leqq t$, $t \geqq 2$, be any $t$ intervals in $[-1,1]$ with $|I_k \cap I_j| = 0$ $(k \neq j)$, $|I_k| \leqq \delta$ $(1 \leqq k \leqq t)$ and $\sum_{k=1}^{t} |T_k| \geqq \mu$, where $T_k \subseteqq I_k$, are arbitrary measurable sets. Let $\xi \geqq \delta$ be fixed. If for a certain integer $R \geqq 2$ we have the relation $\mu \geqq 2^R \xi$, then with a proper $s$, $1 \leqq s \leqq t$, we have*

$$\text{(3.10)} \qquad F = F_s := \sum_{\substack{k=1 \\ \rho(I_s, I_k) \geqq \xi}}^{t} \frac{|T_k|}{|I_s, I_k|} \geqq \frac{R}{8}\mu - \frac{3}{2}.$$

*$I_s$ will be called (an) accumulation interval of $\{I_k\}_{k=1}^t$.*

(The proof is a word-for-word repetition of the original lemma.)

To prove (3.9), it is enough to take those $\{n_i\}_{i=1}^{\infty} := N$ for which $\mu_{n_i} \geqq \varepsilon/20$, say. Omitting those exceptional intervals for which $|J_k| \leqq \frac{\mu_n}{n}$, let $k \in f_n$ iff $k \in e_n$ and $|J_k| > \frac{\mu_n}{n}$. Obviously

$$2\mu_n \geqq \sum_{k \in f_n} |J_k| > \mu_n, \quad n \geqq n_0. \tag{3.11}$$

Apply Lemma 3.2 with $\underbrace{\{I_k\}}_{2} = \{J_k,\ k \in f_n\} := S_n$, $T_k \equiv I_k$, $\mu = \mu_n$, $\xi = 2\delta_n$, $\delta = \delta_n$, $R = [\log n^{1/7}] + 1$ if $n \in N$, $n \geqq n_0(\varepsilon)$ (shortly $n \in N_1$).

Denote by $M_1 = M_{1n}$ an accumulation interval. Dropping $M_1$ we apply Lemma 3.2 for the remaining intervals with $\mu = \mu_n - |M_1| > \frac{\mu_n}{2}$ with the same $\xi, \delta$ and $R$ ($n \in N_1$). An accumulation interval now is $M_2$. At the $i$-th step ($2 \leqq i \leqq \psi_n$) we drop $M_1, M_2, \dots, M_{i-1}$ and apply Lemma 3.2 for the remaining intervals of $S_n$ with $\mu = \mu_n - \sum_{j=1}^{i-1} |M_i|$ using the above $\xi, \delta$ and $R$, where $\psi_n$ is the first index with

$$\sum_{i=1}^{\psi_n - 1} |M_i| \leqq \frac{\mu_n}{2} \quad \text{but} \quad \sum_{i=1}^{\psi_n} |M_i| > \frac{\mu_n}{2}, \quad n \in N_1. \tag{3.12}$$

Denoting by $M_i$, $i = \psi_n + 1, \psi_n + 2, \dots, \varphi_n$ (where $\varphi_n = |f_n|$), the remaining (i.e. not accumulation) intervals of $S_n$, using (3.10) and (3.12) we have (by $20\mu_n \geqq \varepsilon$)

$$\sum_{k=r}^{\varphi_n}{}' \frac{|M_k|}{|M_r, M_k|} \geqq \frac{\mu_n \log n}{2 \cdot 7 \cdot 8} - \frac{3}{2} \geqq \frac{\mu_n \log n}{113}, \quad 1 \leqq r \leqq \psi_n, \quad n \in N_1. \tag{3.13}$$

Here and later the dash indicates that we omit $k$ whenever $\rho(M_r, M_k) < 2\delta_n$.

**3.6.** Now let $\eta = c_1 \varepsilon^{T+1}$ (where $c_1 > 0$ and $T > 0$ will be given later), and let $u_{in} \in M_{in}(q)$ ($1 \leqq i \leqq \varphi_n$, $n \in N_1$) be exceptional points.

If for a fixed $n \in N_1$ there exists a $v = v(n)$, $1 \leqq v \leqq \varphi_n$, such that

$$\lambda_{tnm}(u_{vn}) \geqq c_1 \varepsilon^T \frac{\mu_n \log n}{n^t}, \tag{3.14}$$

by $\eta \frac{\log n}{n^t} \geqq \lambda_{tnm}(u_{vn})$, we obtain (3.9) for this $n$. We prove the existence of $v(n)$ for *arbitrary* $n \in N_1$. Indeed, let us suppose that for a certain $p \in N_1$

$$\lambda_{tnm}(u_{rp}) < c_1 \varepsilon^T \frac{\mu_n \log n}{n^t} \quad \text{for each} \quad u_{rn} \in M_{rn}(q), \quad 1 \leqq r \leqq \varphi_n. \tag{3.15}$$

By (3.15) and (3.9)

$$\sum_{r=1}^{\varphi_p} |M_{rp}|\lambda_{tpm}(u_{rp}) < 2c_1\,\varepsilon^T \frac{\mu_n^2 \log n}{n^t}, \quad p \in N_1. \tag{3.16}$$

On the other hand, by (3.6) for arbitrary $n \in N_1$ we have (with $\overline{z}_k$ corresponding to (3.2))

$$|M_r| \sum_{k=1}^{n} |h_{tk}(u_r)| \geqq \frac{1}{2}|M_r| \sum_{k\in f_n}{}' \left(|h_{tk}(u_r)| + |h_{t,k+1}(u_r)|\right) \geqq$$

$$\geqq \frac{c(m)}{2}\varepsilon^{m-t-1}q^m \sum_{k=1}^{\varphi_n}{}' \left|\frac{\omega(\overline{z}_r)}{\omega(\overline{z}_k)}\right|^m \frac{|M_k|^{t+1}|M_r|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{c(m)}{2}\varepsilon^{m-t-1}\frac{q^m \mu_n^t}{n^t} \sum_{k=1}^{\varphi_n}{}' \left|\frac{\omega(\overline{z}_r)}{\omega(\overline{z}_k)}\right|^m \frac{|M_r||M_k|}{|M_r, M_k|}, \quad 1 \leqq r \leqq \varphi_n$$

(if $k \in f_n$, then $|J_k| > \mu_n/n$). So by $|y| + \frac{1}{|y|} \geqq 2$, (3.12), (3.13), $\mu_n \geqq \varepsilon/20$ and $q = \varepsilon$, we have

(3.17)

$$\sum_{r=1}^{\varphi_n} |M_r|\lambda_{tn}(u_r) \geqq \frac{c(m)}{2}\varepsilon^{m-t-1}\frac{q^m \mu_n^t}{n^t} \sum_{r=1}^{\varphi_n}\sum_{k=1}^{\varphi_n}{}' \left|\frac{\omega(\overline{z}_r)}{\omega(\overline{z}_k)}\right|^m \frac{|M_r||M_k|}{|M_r, M_k|} =$$

$$= \frac{c(m)}{2}\varepsilon^{m-t-1}\frac{q^m \mu_n^t}{n^t} \sum_{r=1}^{\varphi_n}\sum_{k=r}^{\varphi_n}{}' \left(\left|\frac{\omega(\overline{z}_r)}{\omega(\overline{z}_k)}\right|^m + \left|\frac{\omega(\overline{z}_k)}{\omega(\overline{z}_r)}\right|^m\right) \frac{|M_r||M_k|}{|M_r, M_k|} \geqq$$

$$\geqq c(m)\varepsilon^{m-t-1}\frac{q^m \mu_n^t}{n^t} \sum_{r=1}^{\varphi_n} |M_r| \sum_{k=r}^{\varphi_n}{}' \frac{|M_k|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{c(m)}{2\cdot 113}\,\frac{\varepsilon^{m-t-1}q^m \mu_n^{t+2}\log n}{n^t} \geqq \frac{c(m)\varepsilon^{2m+1}\mu_n^2 \log n}{2\cdot 113\cdot 12^m\cdot 20^t\cdot n^t},$$

which contradicts (3.16) if $c_1 = c(m)(4\cdot 113\cdot 12^m\cdot 20^t)^{-1}$ and $T = 2m+1$. That means (3.9) holds true, i.e. (2.1) is valid if $x \in J_k(q)$, $k \in E_n \setminus e_n$. The total measure of the set $H_{3n}$ omitted in 2.3–2.6 on which (2.1) may not hold is less than or equal to

$$\varepsilon + 6\varepsilon + |H_{2n}| + \sum_{k\in E_n\setminus e_n} 2q|J_k| \leqq 9\varepsilon + 4q = 13\varepsilon \quad (n \geqq n_0).$$

**3.7.** If $|x| \geqq 1+\varepsilon$, e.g. $x \geqq 1+\varepsilon$, say, then

$$(1+\varepsilon)^{n-1} \leqq x^{n-1} = \sum_{k=1}^{n} x_k^{n-1} \ell_k(x) \leqq \sum_{k=1}^{n} |\ell_k(x)|$$

whence by (3.1) it is easy to get (2.1) for $|x| \geqq 1+\varepsilon$.

Summarizing, we obtained (2.1) apart from a set of total measure $|H_{1n}| + |H_{3n}| + 2\varepsilon < 16\varepsilon$, which essentially gives our theorem whenever $n \geqq n_0$.

**3.8.** Finally, let $n \leqq n_0(\varepsilon)$. For $q_{kn} = q_n := \frac{\varepsilon}{n}$ we get by (3.1), $\sum\limits_{k=1}^{n} |\ell_k(x)| \geqq 1$ and $n \leqq n_0$ with a proper $\eta$

$$\lambda_{tnm}(x) \geqq c \frac{\varepsilon^{m-1}}{n_0^{m-1} 2^{m-t-1}} > \eta \max_{1 \leqq i \leqq n} \left\{ \frac{\log i}{i^t} \right\} \quad (t = 0, \dots, m-1). \quad \square$$

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# ON THE ZEROS OF JACOBI POLYNOMIALS+

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## 1. Introduction

For $\alpha > -1$, $\beta > -1$ we denote by $x_{ni} = x_{ni}(\alpha, \beta)$ the $i$th zero, in decreasing order, of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$:

$$1 > x_{n1} > x_{n2} > \ldots > x_{nn} > -1.$$

In the literature there are many inequalities for $x_{ni}$. We mention, for example, the well-known Buell inequalities [3, p.125]

$$\frac{i + (\alpha+\beta-1)/2}{n + (\alpha+\beta+1)/2}\pi < \vartheta_{ni} < \frac{i}{n + (\alpha+\beta+1)/2}\pi, \quad i = 1, 2, \ldots, n$$

where $\vartheta_{ni} = \arccos x_{ni}$ and $-1/2 \leqq \alpha \leqq 1/2$, $-1/2 \leqq \beta \leqq 1/2$ excluding the case $\alpha^2 = \beta^2 = 1/4$.

These inequalities are stringent for fixed $\alpha$ and $\beta$ and large values of $n$. In this paper we present a procedure based on the Sturm comparison theorem, to obtain inequalities for $x_{ni}$. The results obtained in this way are valid for $\alpha > -1/2$ and $\beta > -1/2$. The method employed here has been already used in the case of ultraspherical polynomials $P_n^{(\lambda)}(x)$ (see [1]). In that case, however, the situation was simpler because of the symmetry relation $P_n^{(\lambda)}(-x) = (-1)^n P_n^{(\lambda)}(x)$ and the point $x = 0$ played a key role in the application of Sturm comparison theorem.

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## 2. Preliminaries

The function

$$u(x) = (1-x)^{(\alpha+1)/2}(1+x)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x) \tag{2.1}$$

is a solution of the differential equation [3, p.67]

$$u'' + q(x)u = 0 \tag{2.2}$$

where

$$q(x) = \frac{1}{4}\frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4}\frac{1-\beta^2}{(1+x)^2} + \frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{1-x^2}$$

and, as usual, $\alpha > -1, \beta > -1$.

Introducing the notations

$$\tilde{\alpha} = \alpha + \frac{1}{2}, \qquad \tilde{\beta} = \beta + \frac{1}{2}$$

and

$$A = (2n+\tilde{\alpha}+\tilde{\beta})^2, \quad B = 4n^2 + 4n(\tilde{\alpha}+\tilde{\beta}) - (\tilde{\alpha}-\tilde{\beta})^2, \quad C = 2(\tilde{\alpha}^2 - \tilde{\beta}^2)$$

the function $q(x)$ assumes the form

$$q(x) = \frac{B - Cx - Ax^2}{4(1-x^2)^2} + \frac{\frac{3}{4}+\tilde{\alpha}}{4(1-x)^2} + \frac{\frac{3}{4}+\tilde{\beta}}{4(1+x)^2} + \frac{(\tilde{\alpha}+\tilde{\beta})/4 + \frac{1}{8}}{1-x^2}. \tag{2.3}$$

The quadratic polynomial $B - Cx - Ax^2$ can be written as

$$B - Cx - Ax^2 = A(a-x)(x-b)$$

where

$$a, b = \frac{\tilde{\beta}^2 - \tilde{\alpha}^2 \pm \sqrt{16n(n+\tilde{\alpha})(n+\tilde{\beta})(n+\tilde{\alpha}+\tilde{\beta})}}{(2n+\tilde{\alpha}+\tilde{\beta})^2}, \quad a > b. \tag{2.4}$$

It is clear that $-1 \leqq b < a \leqq 1$ and, from (2.4)

$$\sqrt{(1-a)(1-b)} = \frac{2\tilde{\alpha}}{2n+\tilde{\alpha}+\tilde{\beta}}, \quad \sqrt{(1+a)(1+b)} = \frac{2\tilde{\beta}}{2n+\tilde{\alpha}+\tilde{\beta}}. \tag{2.5}$$

Let us introduce the function $\varphi(x)$ by

$$\varphi(x) = \int_x^a \sqrt{\frac{B - Cs - As^2}{4(1-s^2)^2}}\, ds = \tag{2.6}$$

$$= \frac{2n + \tilde{\alpha} + \tilde{\beta}}{2} \int_x^a \frac{\sqrt{(a-s)(s-b)}}{1-s^2}\, ds, \quad b \leqq x \leqq a.$$

$\varphi(x)$ is clearly decreasing with respect to $x$.

Making use of the substitution

$$t^2 = \frac{a-x}{x-b}, \qquad 0 \leqq t < \infty$$

in (2.6) we find

$$\varphi(x) = (2n + \tilde{\alpha} + \tilde{\beta}) \arctan t - \tilde{\alpha} \arctan\left(\sqrt{\frac{1-b}{1-a}}\, t\right) - \tilde{\beta} \arctan\left(\sqrt{\frac{1+b}{1+a}}\, t\right).$$

This formula can be checked directly by differentiation with respect to $x$, taking into account formula (2.5).

Moreover we have

$$\varphi(a) = 0, \qquad \varphi(b) = n\pi.$$

Now we can prove the following result.

LEMMA. *Let $\rho(x)$ be defined by*

$$\rho(x) = (1 - x^2)^{1/2} \left[(a-x)(x-b)\right]^{-1/4}. \tag{2.7}$$

*Then the functions*

$$v_1(x) = \rho(x) \sin \varphi(x), \qquad v_2(x) = \rho(x) \cos \varphi(x) \tag{2.8}$$

*are linearly independent solutions of the differential equation*

$$v'' + \left(\varphi'^2 - \frac{\rho''}{\rho}\right) v = 0, \qquad b < x < a. \tag{2.9}$$

PROOF. The result follows immediately by direct substitution of $v_1''$ and $v_2''$ in (2.9).

By (2.8) and (2.7) we get

$$\lim_{x\to a-0} v_1(x) = \lim_{x\to b+0} v_1(x) = 0.$$

Therefore the function $v_1(x)$ has zeros at $\xi_i$, where

$$\varphi(\xi_i) = i\pi, \qquad i = 0,1,2,\dots,n \tag{2.10}$$

with the special values

$$\xi_0 = a, \qquad \xi_n = b.$$

## 3. The main result

Now we are in the position to prove the main result.

THEOREM 3.1. *For $i = 1,2,\dots,n$ let $x_{ni} = x_{ni}(\alpha,\beta)$ be the $i^{th}$ zero, in decreasing order, of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Then for $\alpha > -1/2$, $\beta > -1/2$ the following inequalities*

$$\xi_i < x_{ni} < \xi_{i-1}, \qquad i = 1,2,\dots,n$$

*hold, where $\xi_0,\xi_1,\dots,\xi_n$ are defined by* (2.10). *In particular $\xi_0 = a$ and $\xi_n = b$ are given by* (2.4).

PROOF. We shall apply the Sturm comparison theorem [3, p.19] to the differential equations (2.2) and (2.9). Actually we show that differential equation (2.2) is a Sturmian majorant of (2.9).

By (2.3) and (2.6) we have to show the inequality

$$\frac{\frac{3}{4}+\tilde\alpha}{4(1-x)^2} + \frac{\frac{3}{4}+\tilde\beta}{4(1+x)^2} + \frac{(\tilde\alpha+\tilde\beta)/4+\frac{1}{8}}{1-x^2} + \frac{\rho''}{\rho} > 0 \quad \text{for} \quad b < x < a, \tag{3.1}$$

where $\tilde\alpha > 0$, $\tilde\beta > 0$ and $\rho$ is defined by (2.7).

By (2.7) we have

$$\frac{\rho'}{\rho} = \frac{1}{4}\,\frac{1}{a-x} - \frac{1}{4}\,\frac{1}{x-b} - \frac{x}{1-x^2}$$

and

$$\frac{\rho''}{\rho} = \left(\frac{\rho'}{\rho}\right)^2 + \left(\frac{\rho'}{\rho}\right)'.$$

Using these relations in (3.1) we have to prove the inequality

$$2\left[(\tilde{\alpha}-\tilde{\beta})x+\tilde{\alpha}+\tilde{\beta}\right](x-a)^2(x-b)^2+P(x)>0$$

where

$$P(x)=(x^2-2)(x-a)^2(x-b)^2-2x(1-x^2)(2x-a-b)(a-x)(x-b)+$$
$$+(1-x^2)^2\left[3x^2-3(a+b)x+\frac{5a^2+2ab+5b^2}{4}\right].$$

Since

$$(\tilde{\alpha}-\tilde{\beta})x+\tilde{\alpha}+\tilde{\beta}>0, \qquad -1\leqq x\leqq 1$$

we need only to show that $P(x)\geqq 0$ for $b<x<a$.

With the notations

$$\frac{a+b}{2}=F, \qquad ab=G,$$

the polynomial $P(x)$ can be written in the form

$$P(x)=P(x;F,G)=(x^2-2)(x^2-2Fx+G)^2+4x(1-x^2)(x-F)\cdot$$
$$\cdot(x^2-2Fx+G)+(x^2-1)(3x^2-6Fx+5F^2-2G).$$

Since

$$P(x;F,G)=P(-x;-F,G),$$

it is sufficient to prove the inequality $P(x)\geqq 0$ only for $F\geqq 0$. We observe that

$$2F-1\leqq G\leqq 2xF-x^2, \qquad b\leqq x\leqq a.$$

Indeed, the first inequality is equivalent to $(1-a)(1-b)\geqq 0$ which is clearly true. The second one is equivalent to $(a-x)(x-b)\geqq 0$ which is also true for $b\leqq x\leqq a$.

Now $P(x;F,G)$ is a quadratic polynomial in $G$ and the coefficient of $G^2$ is $x^2-2$ which is negative in our cases. Therefore, in order to prove that $P(x)\geqq 0$, we have only to check the inequalities at the endpoints $G=2F-1$, $G=2xF-x^2$, i.e.

$$P(x;F,2F-1)\geqq 0, \qquad P(x;F,2xF-x^2)\geqq 0.$$

In the first case we get

$$P(x;F,2F-1)=(x-1)^3F\left[(x+3)F+2(x+1)(x-2)\right].$$

Clearly $(x-1)^3 < 0$ and $F \geqq 0$, thus we have only to show that the expression in the brackets is negative. Since

$$F = \frac{a+b}{2} \leqq \frac{1+x}{2}$$

we need to prove that $(x+3)\frac{1+x}{2} + 2(x+1)(x-2) < 0$. But this is true because it is equivalent to $\frac{5}{2}(1+x)(x-1) < 0$ which clearly holds. This proves that $P(x; F, 2F-1) \geqq 0$. The inequality $P(x; F, 2xF - x^2) \geqq 0$ follows immediately observing that it is equivalent to

$$5(x-1)^2(F-x)^2 \geqq 0.$$

Thus we can conclude that $P(x) > 0$ on $b < x < a$ or, equivalently, that the equation (2.2) is a Sturmian majorant of (2.9). Therefore we are in the position to apply the Sturm comparison theorem to equations (2.2) and (2.9) obtaining that between two consecutive zeros $\xi_i$, $\xi_{i-1}$ of $v_1(x)$ occurs at least one zero of $P_n^{(\alpha,\beta)}(x)$. But we know that on the interval $(-1,1)$ there are exactly $n$ zeros of $P_n^{(\alpha,\beta)}(x)$, hence we can conclude that in each interval $(\xi_i, \xi_{i-1})$ it occurs exactly the zero $x_{ni}(\alpha,\beta)$ of $P_n^{(\alpha,\beta)}(x)$ and the conclusion of Theorem 3.1 follows. □

A consequence of the above theorem is the following result.

COROLLARY 3.1. *Under the conditions of Theorem* 3.1 *we get*

$$\varphi(x_{n,i+1}) - \varphi(x_{ni}) < \pi, \qquad i = 1, 2, \ldots, n-1.$$

PROOF. The functions $u(x)$ defined by (2.1) and the function

$$v(x) = \rho(x)\sin\left[\varphi(x) - \varphi(x_{ni})\right] = \cos\varphi(x_{ni})v_1(x) - \sin\varphi(x_{ni})v_2(x)$$

with $v_1(x)$ and $v_2(x)$ defined by (2.8) have a common zero at $x = x_{ni}$. The largest zero $\hat{x}$ of $v(x)$ on the left of $x_{ni}$ satisfies the relation

$$\varphi(\hat{x}) - \varphi(x_{ni}) = \pi.$$

The existence of $\hat{x}$ is ensured by the facts that the function $\varphi(x)$ is decreasing and that by Theorem 3.1 we have

$$\varphi(b) - \varphi(x_{ni}) > \varphi(b) - \varphi(\xi_{n-1}) = \pi.$$

Again by the Sturm comparison theorem we conclude that the zero of $v(x)$ next to $\hat{x}$, occurs before $x_{n,i+1}$, the next zero of $u(x)$, i.e.

$$\hat{x} < x_{n,i+1}.$$

Using again the decreasing character of $\varphi(x)$ we obtain

$$\pi = \varphi(\hat{x}) - \varphi(x_{ni}) > \varphi(x_{n,i+1}) - \varphi(x_{ni})$$

which proves Corollary 3.1.

The proof of the following result is based also on the Sturm comparison theorem.

COROLLARY 3.2. *Suppose that all the conditions of Theorem* 3.1 *are satisfied and let $x'' < x'$ be two values on $[b,a]$ such that*

$$\varphi(x'') - \varphi(x') \geqq \pi.$$

*Then the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ has a zero on $(x'', x')$.*

PROOF. The function

$$v(x) = \rho(x)\sin\left[\varphi(x) - \varphi(x')\right]$$

has a zero at $x = x'$. The largest zero $\bar{x}$ of $v(x)$ on the left of $x'$ satisfies the relation

$$\varphi(\bar{x}) - \varphi(x') = \pi.$$

The existence of $\bar{x}$ is ensured by the fact that

$$\varphi(b) - \varphi(x') \geqq \pi.$$

The Sturm comparison theorem gives that between $\bar{x}$ and $x'$ a zero of $P_n^{(\alpha,\beta)}(x)$ occurs, i.e.

$$\bar{x} < x_{ni} < x'$$

for some value of $i$. Since

$$\varphi(x'') - \varphi(x') \geqq \pi,$$

using the decreasing character of $\varphi(x)$ we get $x'' \leqq \bar{x}$ and this proves the desired result.

COROLLARY 3.3. *Let $\gamma$ and $\delta$ be nonnegative real numbers such that $\gamma + \delta > 0$. Let $\hat{\alpha}_n$, $\hat{\beta}_n$ be defined by $\hat{\alpha}_n = -1/2 + n\gamma$, $\hat{\beta}_n = -1/2 + n\delta$. Then for the zeros $x_{n1}^{(\hat{\alpha}_n,\hat{\beta}_n)}$, $x_{n,n}^{(\hat{\alpha}_n,\hat{\beta}_n)}$ of the Jacobi polynomial $P_n^{(\hat{\alpha}_n,\hat{\beta}_n)}(x)$ we have*

$$a > x_{n1}^{(\hat{\alpha}_n,\hat{\beta}_n)} > \ldots > x_{n,n}^{(\hat{\alpha}_n,\hat{\beta}_n)} > b$$

*and*

$$\lim_{n\to\infty} x_{n1}^{(\hat{\alpha}_n,\hat{\beta}_n)} = a, \qquad \lim_{n\to\infty} x_{n,n}^{(\hat{\alpha}_n,\hat{\beta}_n)} = b$$

*where*

$$a,b=\frac{\delta^2-\gamma^2\pm\sqrt{16(1+\gamma)(1+\delta)(1+\gamma+\delta)}}{(2+\gamma+\delta)^2}. \tag{3.2}$$

PROOF. Let $\varepsilon>0$, $\varepsilon<a-b$. In order to prove the first limit relation we need only to show that there exists a zero of $P_n^{(\alpha,\beta)}(x)$ on the interval $(a-\varepsilon,a)$. We observe that now $a$ and $b$ are independent of $n$ and that the function $\varphi(x)$ in (2.6) can be written in the form $\varphi(x)=n\hat{\varphi}(x)$ where $\hat{\varphi}(x)$ is independent of $n$. Thus we find

$$\lim_{n\to\infty}\left[\varphi(a-\varepsilon)-\varphi(a)\right]=\lim_{n\to\infty}n\hat{\varphi}(a-\varepsilon)=\infty.$$

Therefore by Corollary 3.2 we can conclude that there is at least one zero of $P_n^{(\alpha,\beta)}(x)$ on the interval $[a-\varepsilon,a]$ if $n$ is sufficiently large.

The proof of the second limit relation is similar. We get

$$\lim_{n\to\infty}\left[\varphi(b)-\varphi(b-\varepsilon)\right]=\lim_{n\to\infty}n\left[\pi-\hat{\varphi}(b-\varepsilon)\right]=\infty.$$

The proof is now complete.

We observe that Corollary 3.3 shows that our result in Theorem 3.1 cannot be improved and that in some sense it is optimal. By Theorem 3.1 we have that the possible values of $\varphi(x_{ni})$ $(i=1,2,\dots,n)$ belong to the interval $(0,n\pi)$. Supported by numerical calculations we guess that the interval $(0,n\pi)$ could be replaced by $\left(\frac{\pi}{2},\left(n-\frac{1}{2}\right)\pi\right)$.

Finally we observe that by the continuous dependence of $a,b$ on $\gamma,\delta$ in (3.2) we can generalize Corollary 3.3 to the cases

$$\gamma=\lim_{n\to\infty}\frac{\hat{\alpha}_n}{n},\qquad \delta=\lim_{n\to\infty}\frac{\hat{\beta}_n}{n}$$

which is equivalent to the results proved by Moak, Saff and Varga [2].

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# LACUNARY INTERPOLATION BY COSINE POLYNOMIALS

C. R. SELVARAJ (Sharon)

**1. Introduction.** For a given positive integer $n$, we shall consider the nodes $\{x_k\}_{k=0}^{n-1}$, $x_k = \frac{k\pi}{n}$, $k = 0, 1, \cdots, n-1$ and the space $\mathcal{T}_m$ of even trigonometric polynomials spanned by $\{\cos jx\}_{j=0}^{m-1}$. Since the polynomials in $\mathcal{T}_m$ are even, it seems reasonable to raise the problem of regularity of interpolation from $\mathcal{T}_m$ on the $n$ nodes $x_k$.

The regularity of $(0, M)$ interpolation is established in Section 2. In the remark at the end of Section 2 we indicate why the regularity is not true when $M$ is odd. The fundamental polynomials are given explicitly in Section 3. In Section 4 we discuss the convergence of certain sequences of interpolating polynomials to $f$. This requires a lemma by Sharma and Varma [2, p. 350] in which the authors use the inequalities due to O. Kiš [1, p. 268].

**2. The regularity theorem.** The result on regularity can be stated as follows:

THEOREM 2.1. *Given a positive even integer $M$ and the equidistant nodes $\{x_k\}_{k=0}^{n-1}, n \geq 1$ in $[0, \pi]$, there exists a unique trigonometric polynomial $T(x) \in \mathcal{T}_N$ where*

$$\mathcal{T}_N = \operatorname{span}\left\{1, \cos x, \cos 2x, \cdots, \cos(N-1)x\right\}$$

*depending on $n$ such that*

$$T(x_k) = \alpha_k; \quad T^{(M)}(x_k) = \beta_k \quad (k = 0, 1, \ldots, n-1) \tag{2.1}$$

*for any given $2n (= N)$ complex numbers $\{\alpha_k\}$ and $\{\beta_k\}$.*

PROOF. It suffices to consider the homogeneous $(0, M)$ interpolation problem and show that if the trigonometric polynomial $T(x)$ from $\mathcal{T}_N$ satisfies

$$T(x_k) = 0; \quad T^{(M)}(x_k) = 0 \quad (k = 0, 1, \ldots, n-1), \tag{2.2}$$

then $T(x) \equiv 0$.

Since the total number of interpolation conditions in (2.1) is $N = 2n$, the required unique trigonometric interpolant $T(x) \in \mathcal{T}_N$ must be of the form

$$T_{2n}(x) = \sum_{j=0}^{2n-1} a_j \cos jx.$$

Equivalently we may write

$$T_{2n}(x) = a_0 + \sum_{j=1}^{n-1} a_j \cos jx + \sum_{j=0}^{n-1} a_{n+j} \cos(n+j)x. \tag{2.3}$$

Applying the conditions (2.2) to $T_{2n}(x)$ given by (2.3) we get (for $k = 0, 1, \ldots, n-1$),

(2.4)

$$\begin{cases} a_0 + (-1)^k a_n + \sum_{j=1}^{n-1} a_j \cos jx_k + \sum_{j=1}^{n-1} a_{n+j} \cos(n+j)x_k = 0, \\ n^M(-1)^k a_n + \sum_{j=1}^{n-1} j^M a_j \cos jx_k + \sum_{j=1}^{n-1} (n+j)^M a_{n+j} \cos(n+j)x_k = 0. \end{cases}$$

It is easy to verify that the system (2.4) is equivalent to

(2.5)

$$\begin{cases} a_0 + (-1)^k a_n + \sum_{j=1}^{n-1} \left(a_j \cos jx_k + a_{n+j} \cos(n-j)x_k\right) = 0, \\ n^M(-1)^k a_n + \sum_{j=1}^{n-1} \left(j^M a_j \cos jx_k + (n+j)^M a_{n+j} \cos(n-j)x_k\right) = 0. \end{cases}$$

In (2.5), we write $\sum_{1}^{n-1} = s_1 + s_2$ and replace $j$ by $n - j$ in $s_2$. Thus we have

$$
(2.6) \qquad \begin{cases} a_0 + (-1)^k a_n + \sum_{j=1}^{n-1} (a_j + a_{2n-j}) \cos jx_k = 0, \\ \\ n^M (-1)^k a_n + \sum_{j=1}^{n-1} (j^M a_j + (2n-j)^M a_{2n-j}) \cos jx_k = 0 \end{cases}
$$

$$
(k = 0, 1, \ldots, n-1).
$$

Now, we will show that $a_j = 0$ for $j = 0, 1, \cdots, 2n-1$. In order to establish this, we will first prove that $a_0 = a_n = 0$ by taking the cases when $n$ is odd and $n$ is even in (2.6).

Suppose that $n$ is odd. Then by considering only those equations with even nodes (i.e., $k = 2\mu, \quad \mu = 0, 1, 2, \cdots, \frac{n-1}{2}$) we get

$$
(2.7) \qquad \begin{cases} a_0 + a_n + \sum_{j=1}^{n-1} (a_j + a_{2n-j}) \cos jx_{2\mu} = 0, \\ \\ n^M a_n + \sum_{j=1}^{n-1} \left( j^M a_j + (2n-j)^M a_{2n-j} \right) \cos jx_{2\mu} = 0. \end{cases}
$$

Since $\cos jx_{2\mu} = \cos(n-j)x_{2\mu}$, we can combine the $j$-th term and the $(n-j)$-th term ($j = 1, 2, \cdots, \frac{n-1}{2}$) in each of the summations above to get

$$
(2.8) \qquad \begin{cases} a_0 + a_n + \sum_{j=1}^{\frac{n-1}{2}} (a_j + a_{n-j} + a_{n+j} + a_{2n-j}) \cos jx_{2\mu} = 0, \\ \\ n^M a_n + \sum_{j=1}^{\frac{n-1}{2}} \Big( j^M a_j + (n-j)^M a_{n-j} + (n+j)^M a_{n+j} + \\ \qquad\qquad + (2n-j)^M a_{2n-j} \Big) \cos jx_{2\mu} = 0. \end{cases}
$$

The equations (2.8) imply that two cosine polynomials of the form

$$
p(x) = \sum_{j=0}^{\frac{n-1}{2}} \lambda_j \cos jx
$$

each of degree $\frac{n-1}{2}$ vanish at $\frac{n+1}{2}$ distinct points in $[0,\pi]$. Therefore $p(x) \equiv \equiv 0$, whence $\lambda_j = 0$ for all $j = 0, 1, \cdots, \frac{n}{2}$. In particular $\lambda_0 = 0$ which implies that $a_n = 0$ and $a_0 = 0$.

Now suppose that $n$ is even in (2.6). Then by considering only those equations with odd nodes (i.e., for $k = 2\mu - 1$, $\mu = 1, 2, \cdots, \frac{n}{2}$) we get

$$
\text{(2.9)}\qquad \begin{cases} a_0 - a_n + \sum_{j=1}^{n-1} (a_j + a_{2n-j}) \cos x_{2\mu-1} = 0, \\[2ex] -n^M a_n + \sum_{j=1}^{n-1} (j^M a_j + (2n-j)^M a_{2n-j}) \cos jx_{2\mu-1} = 0. \end{cases}
$$

Rewriting (2.9) in a way similar to the case for $k$ even and noting that $\cos \frac{n}{2} x_{2\mu-1} = 0$ for $\mu = 1, 2, \cdots, \frac{n}{2}$ we obtain that

$$
\text{(2.10)}\qquad \begin{cases} a_0 - a_n + \sum_{j=1}^{\frac{n}{2}-1} (a_j - a_{n-j} - a_{n+j} + a_{2n-j}) \cos jx_{2\mu-1} = 0, \\[2ex] -n^M a_n + \sum_{j=1}^{\frac{n}{2}-1} \Big(j^M a_j - (n-j)^M a_{n-j} - (n+j)^M a_{n+j} + \\[1ex] \qquad +(2n-j)^M a_{2n-j}\Big) \cos jx_{2\mu-1} = 0. \end{cases}
$$

The equation (2.10) consists of two cosine polynomials each of degree $\frac{n}{2} - - 1$ vanishing at $\frac{n}{2}$ distinct points. Hence all coefficients in (2.10) vanish implying that $a_n = 0$ and $a_0 = 0$. Hence we have shown that $a_n = a_0 = 0$ in (2.6) for all $n$.

Therefore the system (2.6) becomes

$$
\text{(2.11)}\qquad \begin{cases} \sum_{j=1}^{n-1} (a_j + a_{2n-j}) \cos jx_k = 0, \\[2ex] \sum_{j=1}^{n-1} (j^M a_j + (2n-j)^M a_{2n-j}) \cos jx_k = 0, \end{cases}
$$

where $k = 0, 1, \cdots, n-1$. The system of equations in (2.11) shows that the cosine polynomials of degree $n - 1$ have $n$ zeros in $[0, \pi]$. Therefore, the polynomials are identically zero. Hence, their coefficients must be zero. Thus we have

$$\tag{2.12} \begin{cases} a_j + a_{2n-j} = 0, \\ j^M a_j + (2n-j)^M a_{2n-j} = 0 \end{cases} \qquad (j = 1, 2, \cdots, n-1).$$

Since $(2n-j)^M - j^M > 0$ for $j = 1, 2, \cdots, n-1$ we conclude that

$$a_j = 0 \quad \text{for} \quad j = 1, 2, \cdots, 2n-1.$$

Hence $T_{2n}(x) = 0$.

We would like to remark that the $(0, M)$ interpolation is not regular if $M$ is odd. For, if $M$ is odd, then the conditions given in (2.2) yield a homogeneous system of $2n-1$ equations in $2n$ unknowns. This fact can be easily verified by a simple case of $M = 3$.

**3. Fundamental polynomials.** Given any function $f$, the trigonometric polynomial which coincides with $f$ and $f^{(M)}$ at the points $\{x_j\}_{j=0}^{n-1}$ is given by

$$I_n(f, x) = \sum_{\nu=0}^{n-1} f(x_\nu) A_\nu(x) + \sum_{\nu=0}^{n-1} f^{(M)}(x_\nu) B_\nu(x).$$

The fundamental polynomials $A_\nu(x)$ and $B_\nu(x)$ satisfy the following conditions:

$$\tag{3.1} A_\nu(x_k) = \delta_{\nu k}, \quad A_\nu^{(M)}(x_k) = 0,$$

$$\tag{3.2} B_\nu(x_k) = 0, \quad B_\nu^{(M)}(x_k) = \delta_{\nu k}$$

where $x_k = \frac{k\pi}{n}$, for $k = 0, 1, \cdots, n-1$ and $A_\nu(x)$ and $B_\nu(x)$ are each of order $2n-1$.

Consider the cosine polynomial

$$\tag{3.3} F(x) = \frac{1}{2n}\left[1 + 2\sum_{\substack{j=1 \\ j\neq n}}^{2n-1} \frac{(2n-j)^M \cos jx}{(2n-j)^M - j^M}\right].$$

The $M$-th derivative of this polynomial vanishes at each of the points $x_k$, because

$$F^{(M)}(x_k) = \frac{(-1)^{\frac{M}{2}}}{n}\left[\sum_{\substack{j=1 \\ j\neq 1}}^{2n-1} \frac{(2n-j)^M j^M}{(2n-j)^M - j^M} \cos jx_k\right] =$$

$$= \frac{(-1)^{\frac{M}{2}}}{n}\left[\sum_{j=1}^{n-1}\frac{(2n-j)^M j^M}{(2n-j)^M - j^M}\cos jx_k + \right.$$

$$\left. + \sum_{j=1}^{n-1}\frac{(2n-j)^M j^M}{j^M-(2n-j)^M}\cos(2n-j)x_k\right]$$

and $\cos jx_k = \cos(2n-j)x_k$. Now to get the values of $F(x_k)$ for $k=0,1,\dots,$ first we notice that $F(x)$ is $2\pi$-periodic. Then we consider

$$F(x_k) = \frac{1}{2n}\left[1 + 2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{(2n-j)^M\cos jx_k}{(2n-j)^M-j^M}\right] = \frac{1}{2n}\left[1+2\sum_{j=1}^{n-1}\cos jx_k\right].$$

Therefore,

$$F(0) = \frac{2n-1}{2n}, \qquad F(\pi) = \frac{(-1)^{n+1}}{2n}$$

and

$$F(x_k) = \frac{(-1)^{k+1}}{2n} \quad \text{for} \quad k = 1,2,\dots,n-1,n+1,\dots,2n-1,$$

using the fact that $\displaystyle\sum_{j=1}^{n-1} \cos jx_k = \frac{(-1)^{k+1}-1}{2}$.

Now we define the fundamental polynomial $A_\nu(x)$ as

(3.4)

$$A_\nu(x) := \begin{cases} F(x-x_\nu)+F(x+x_\nu)+2(-1)^{\nu+n+1}F(x-\pi), & \nu\neq 0,\\[2ex] F(x)+(-1)^{n+1}F(x-\pi), \quad \nu = 0. \end{cases}$$

Then $A_\nu(x)$ is a trigonometric cosine polynomial of degree $\leqq 2n-1$. Next we will prove that this $A_\nu(x)$ satisfies the conditions given in (3.1). It is clear that $A_\nu^{(M)}(x_k)=0$. Also, $A_0(0)=1$ and for $k\neq 0$, $A_0(x_k)=0$. Moreover, when $\nu\neq 0$,

$$A_\nu(x_\nu) = F(0) + F\left(\frac{2\nu\pi}{n}\right) + 2(-1)^{\nu+n+1}F\left(\frac{\nu\pi}{n}-\pi\right) =$$

$$= \frac{2n-1}{2n} + \frac{(-1)^{2\nu+1}}{2n} + (-1)^{\nu+n+1}\frac{(-1)^{\nu-n+1}}{n} = 1$$

and for $k \neq \nu$

$$A_\nu(x_k) = F\left(\frac{k-\nu}{n}\pi\right) + F\left(\frac{k+\nu}{n}\pi\right) + 2(-1)^{\nu+n+1}F\left(\frac{k-n}{n}\pi\right) =$$

$$= \frac{(-1)^{\nu+k+1}}{2n} + \frac{(-1)^{\nu+k+1}}{2n} + \frac{(-1)^{\nu+k}}{n} = 0.$$

Now let us consider the cosine polynomials $G(x)$ and $H(x)$ given by

$$G(x) = \frac{(-1)^{\frac{M}{2}-1}}{n} \sum_{\substack{j=1 \\ j\neq n}}^{2n-1} \frac{\cos jx}{(2n-j)^M - j^M}$$

and

$$H(x) = (-1)^{\frac{M}{2}-1} \left[\frac{1-\cos nx}{2 \quad n^{M+1}} + \frac{1}{n} \sum_{\substack{j=1 \\ j\neq n}}^{2n-1} \frac{\cos jx}{(2n-j)^M - j^M}\right]$$

which are also $2\pi$-periodic. We can easily verify that $G(x_k) = 0$ for $k = 0, 1, \ldots$, and to get the values of the $M$-th derivative of this polynomial at the points $x_k$, $k = 0, 1, \ldots$, we consider that

$$G^{(M)}(x_k) = \frac{(-1)^{M-1}}{n} \sum_{\substack{j=1 \\ j\neq n}}^{2n-1} \frac{j^M \cos jx_k}{(2n-j)^M - j^M} = \frac{1}{n} \sum_{j=1}^{n-1} \cos jx_k.$$

Therefore,

$$G^{(M)}(0) = \frac{n-1}{n} \quad \text{and} \quad G^{(M)}(x_k) = \frac{(-1)^{k+1}-1}{2n},$$

for $k = 1, 2, \ldots, 2n-1$, using the fact that $\sum_{j=1}^{n-1} \cos jx_k = \dfrac{(-1)^{k+1}-1}{2}$.

Also, it is easy to see that $H(0) = 0$ and

$$H(x_k) = \frac{(-1)^{\frac{M}{2}-1}}{2n^{M+1}} \left[1-(-1)^k\right], \quad \text{for } k = 1, 2, \ldots, n-1.$$

Next, we consider that

$$H^{(M)}(x_k) = (-1)^{\frac{M}{2}-1}\left[\frac{(-1)^{\frac{M}{2}-1}\cos nx_k}{2n} + \right.$$

$$\left. + \frac{(-1)^{\frac{M}{2}}}{n}\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{j^M\cos jx_k}{(2n-j)^M - j^M}\right] =$$

$$= (-1)^{\frac{M}{2}-1}\left[\frac{(-1)^{\frac{M}{2}+k+1}}{2n} + \frac{(-1)^{\frac{M}{2}-1}}{n}\sum_{j=1}^{n-1}\cos jx_k\right].$$

Therefore,

$$H^{(M)}(0) = 1 - \frac{1}{2n}, \quad \text{and} \quad H^{(M)}(x_k) = -\frac{1}{2n} \quad \text{for} \quad k = 1, 2, \ldots, 2n-1.$$

Set

$$(3.5)\qquad \begin{cases} B_0(x) = \begin{cases} G(x) - G(x-\pi), & n \text{ even} \\ H(x) - H(x-\pi) - \\ \quad - \dfrac{(-1)^{\frac{M}{2}}}{n^{M+1}}\displaystyle\sum_{i=0}^{n-1}(-1)^i A_i(x), & n \text{ odd} \end{cases} \\ \text{and for } \nu \neq 0 \text{ set} \\ B_\nu(x) = \begin{cases} G(x-x_\nu) + G(x+x_\nu) + (-1)^{\nu+n+1}2G(x-\pi), \\ \qquad \text{if } \nu \text{ and } n \text{ have same parity,} \\ H(x+x_\nu) + H(x-x_\nu) - 2H(x-\pi) + \\ \qquad + \dfrac{2(-1)^{\frac{M}{2}+n}}{n^{M+1}}\displaystyle\sum_{i=0}^{n-1}(-1)^i A_i(x), \\ \qquad \text{if } \nu \text{ and } n \text{ have opposite parity.} \end{cases} \end{cases}$$

Clearly $B_\nu(x)$ is a trigonometric cosine polynomial of degree $\leqq 2n-1$. Also, it satisfies the conditions given in (3.2) which can be verified by similar arguments used in the case of the polynomial $A_\nu(x)$.

**4. Convergence.** The convergence problem requires the estimates on the sums $\sum_{\nu=0}^{n-1}|A_\nu(x)|$ and $\sum_{\nu=0}^{n-1}|B_\nu(x)|$ where $A_\nu(x)$ and $B_\nu(x)$ are the fundamental polynomials given in (3.4) and (3.5) respectively. First we will obtain the bounds on $F(x)$, $H(x)$ and $G(x)$.

From (3.3) we have

$$F(x)=\frac{1}{2n}\left[1+2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{\cos jx}{1-\left(\frac{j}{2n-j}\right)^M}\right].$$

Now using the identity given in [2, p. 349]

$$\frac{1}{1-a^M}=\frac{1}{M}\frac{1}{1-a}+h(a)$$

where $h(a)=\frac{1}{M}\frac{(M-1)+(M-2)a+\dots+a^{M-1}}{1+a+a^2+\dots a^{M-1}}$ we obtain

$$F(x)=\frac{1}{2n}\left[1+\frac{1}{M}\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{2n-j}{n-j}\cos jx+2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h\left(\frac{j}{2n-j}\right)\cos jx\right]. \tag{4.1}$$

Here $h(a)=h(\frac{j}{2n-j})$ is a decreasing function and hence $h(a)<1$. This implies that

$$\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\left|h\left(\frac{j}{2n-j}\right)\cos jx\right|=O(n). \tag{4.2}$$

Also, consider the absolute value sum

$$\left|\frac{1}{M}\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{2n-j}{n-j}\cos jx\right|\leqq\frac{1}{M}\sum_{\substack{j=1\\ j\neq n}}^{2n-1}|\cos jx|+\frac{n}{M}\left|\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{\cos jx}{n-j}\right|\leqq$$

$$\leqq\frac{2(n-1)}{M}+\frac{n}{M}\left|\sum_{j=1}^{n-1}\frac{\cos jx}{n-j}+\sum_{j=n+1}^{2n-1}\frac{\cos jx}{n-j}\right|.$$

Replacing $j$ by $2n-j$ in the second summation on the right side of the above inequality we obtain,

$$\left|\frac{1}{M}\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{2n-j}{n-j}\cos jx\right| \leqq \frac{2(n-1)}{M}+\frac{n}{M}\left|\sum_{j=1}^{n-1}\frac{\cos jx-\cos(2n-j)x}{n-j}\right| \leqq$$

$$=\frac{2(n-1)}{M}+\frac{2n}{M}\left|\sum_{j=1}^{n-1}\frac{\sin nx\sin(n-j)x}{n-j}\right|$$

by the identity $\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$. Thus we have

(4.3)
$$\left|\frac{1}{M}\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\frac{2n-j}{n-j}\cos jx\right| \leqq \frac{2n}{M}\left(1+\left|\sum_{j=1}^{n-1}\frac{\sin(n-j)x}{n-j}\right|\right), \quad 0\leqq x\leqq\pi.$$

$$\leqq \frac{2n}{M}(1+3\sqrt{\pi})=O(n).$$

Using (4.2) and(4.3) in (4.1) yields

$$|F(x)|=O(1). \tag{4.4}$$

A similar method of obtaining the bound for $F(x)$ can be found in [3, p. 247].

We observe that the cosine polynomials $H(x)$ and $G(x)$ defined in Section 3 to form the fundamental polynomial $B_\nu(x)$ is similar to the cosine polynomial given in the equation (6) by Sharma and Varma in [2, p. 342]. Following the method used in the proof of Lemma 3 in [2, p. 350] for the case $M$ even, we get

$$|H(x)|=O\left(\frac{1}{n^M}\right) \quad \text{and} \quad |G(x)|=O\left(\frac{1}{n^M}\right). \tag{4.5}$$

Substituting the values of $|F(x)|$, $|H(x)|$, and $|G(x)|$ in the expression for the fundamental polynomials $A_\nu(x)$ and $B_\nu(x)$ respectively we obtain the following estimates:

$$\sum_{\nu=0}^{n-1}|A_\nu(x)|=O(n), \tag{4.6}$$

and

$$\sum_{\nu=0}^{n-1} |B_\nu(x)| = O\left(\frac{1}{n^{M-1}}\right). \tag{4.7}$$

Now we state the convergence theorem.

THEOREM 4.1. *If $f(x)$ is an even $2\pi$-periodic function and continuous on $[0,\pi]$ satisfying the Zygmund condition*

$$f(x+h) - 2f(x) + f(x-h) = o(h)$$

*and if we set*

$$R_n(x) := \sum_{\nu=0}^{n-1} f(x_\nu)A_\nu(x) + \sum_{\nu=0}^{n-1} \beta_\nu B_\nu(x)$$

*where $\beta_\nu = o(n^{M-1})$, $\nu = 0,1,\dots,n-1$, then $R_n(x)$ converges uniformly to $f(x)$ on every finite interval on the $x$ axis.*

PROOF. We know that [2, p. 356] if $f(x)$ is continuous and $2\pi$-periodic and satisfies the Zygmund condition then there exists a trigonometric polynomial $T_n(x)$ of order $n-1$ such that

$$f(x) - T_n(x) = o(\frac{1}{n}) \quad \text{and} \quad T_n^{(M)}(x) = o(n^{M-1}). \tag{4.8}$$

Since $f(x)$ is even, $T_n(x)$ must be a cosine polynomial. Then

$$f(x) - R_n(x) = f(x) - T_n(x) + T_n(x) - R_n(x) =$$

$$= f(x) - T_n(x) + \sum_{\nu=0}^{n-1} T_n(x_\nu)A_\nu(x) + \sum_{\nu=0}^{n-1} T_n^{(M)}(x_\nu)B_\nu(x) -$$

$$-\sum_{\nu=0}^{n-1} f(x_\nu)A_\nu(x) - \sum_{\nu=0}^{n-1} \beta_\nu B_\nu(x)$$

so that by (4.6), (4.7) and (4.8) we get

$$|f(x) - R_n(x)| \leqq |f(x) - T_n(x)| + \sum_{\nu=0}^{n-1} |T_n(x_\nu) - f(x_\nu)|\,|A_\nu(x)| +$$

$$+\sum_{\nu=0}^{n-1} |T_n^{(M)}(x_\nu)|\,|B_\nu(x)| + (\max_\nu |\beta_\nu|) \sum_{\nu=0}^{n-1} |B_\nu(x)| =$$

$$= o(\frac{1}{n}) + o\left(\frac{1}{n}\right)\sum_{\nu=0}^{n-1} |A_\nu(x)| + o(n^{M-1})\sum_{\nu=o}^{n-1} |B_\nu(x)| = o(1).$$

which proves the theorem.

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# EXTREMAL PROPERTIES OF DERIVATIVE OF ALGEBRAIC POLYNOMIALS

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## Introduction

The following problem was raised by P. Turán at a conference held in Varna, Bulgaria (1970). Let $\varphi(x) \geqq 0$ for $-1 \leqq x \leqq 1$ and consider the class $p_{n,\varphi}$ of all polynomials $p_n(x) = \sum\limits_{k=0}^{n} a_k x^k$ of degree at most $n$ such that $|p_n(x)| \leqq \varphi(x)$ for $-1 \leqq x \leqq 1$. How large can $\max\limits_{-1 \leqq x \leqq 1} \left|p_n^{(k)}(x)\right|$ become if $p_n(x)$ is an arbitrary polynomial belonging to $p_{n,\varphi}$? Important contributions to the problem of Turán have been made by Prof. Rahman and his associates. In the case of circular majorants $\left(\varphi(x) = (1-x^2)^{\frac{1}{2}}\right)$, Rahman [4] proved the following result:

Theorem A (Q. I. Rahman). *If $p_n(x)$ is an algebraic polynomial of degree $n$ such that $|p_n(x)| \leqq (1-x^2)^{\frac{1}{2}}$, for $-1 \leqq x \leqq 1$, then $\max\limits_{-1 \leqq x \leqq 1} |p_n'(x)| \leqq$ $\leqq 2(n-1)$. Equality iff $p_n(x) = (1-x^2)u_{n-2}(x)$, $u_{n-2}(x) = \frac{\sin(n-1)\theta}{\sin\theta}$, $x =$ $= \cos\theta$.*

For other interesting results concerning Turáns problem, we refer to the works of Rahman and Pierre [2], [3] and Rahman and Schmeisser [5].

Recently, Varma [7], [9] and Varma, Mills, and Smith [10] obtained an analogue of Theorem A in the $L_2$ norm. We state some of these results as follows:

Theorem B (A. K. Varma). *Let $p_{n+1}(x)$ be any real algebraic polynomial of degree at most $n+1$ such that*

$$|p_{n+1}(x)| \leqq (1-x^2)^{\frac{1}{2}}, \quad \textit{for} \quad -1 \leqq x \leqq 1.$$

*Then*

$$\int_{-1}^{1} \left[p_{n+1}^{(j)}(x)\right]^2 (1-x^2)^{\frac{1}{2}}\, dx \leqq \int_{-1}^{1} \left[f_0^{(j)}(x)\right]^2 (1-x^2)^{\frac{1}{2}}\, dx \textit{ for } j = 1,2,3$$

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*where* $f_0(x) = (1-x^2)u_{n-1}$, $u_{n-1}(x) = \frac{\sin n\theta}{\sin\theta}$, $x = \cos\theta$. *Equality iff* $p_{n+1}(x) = \pm f_0(x)$.

THEOREM C (Varma, Mills, and Smith). *Let* $p_{n+2}(x)$ *be any real algebraic polynomial of degree at most* $n+2$ *such that*

$$|p_{n+2}(x)| \leqq (1-x^2), \quad \text{for} \quad -1 \leqq x \leqq 1.$$

*Then,* $\int_{-1}^{1} \left[p''_{n+2}(x)\right]^2 dx \leqq \int_{-1}^{1} \left[f''_1(x)\right]^2 dx$ *where* $f_1(x) = (1-x^2)T_n(x)$; $T_n(x) = \cos n\theta$, $x = \cos\theta$. *Equality iff* $p_{n+2}(x) = \pm f_1(x)$.

*In addition, if all* $n+2$ *zeros of* $p_{n+2}(x)$ *are real and lie inside* $[-1,1]$, *then* $\int_{-1}^{1} \left[p'_{n+2}(x)\right]^2 dx \leqq \int_{-1}^{1} \left[f'_1(x)\right]^2 dx$. *Equality iff* $p_{n+2}(x) = \pm f_1(x)$.

In this paper, we shall prove the following analogue of Theorem B and Theorem C.

THEOREM 1. *Let* $p_{n+1}(x)$ *be a real algebraic polynomial of degree* $n+1$ *such that* $|p_{n+1}(x)| \leqq (1-x^2)^{\frac{1}{2}}$, *for* $-1 \leqq x \leqq 1$. *Then, for* $k = 2,3,\dots$ *we have*

$$\int_{-1}^{1} \left[p_{n+1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx \leqq \int_{-1}^{1} \left[f_0^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx \tag{1.1}$$

*where* $f_0(x) = (1-x^2)u_{n-1}(x)$. *Equality iff* $p_{n+1}(x) = \pm f_0(x)$.

REMARK 1. The case $k = 1$, under the further assumption that the polynomial $p_{n+1}(x)$ has all real zeros that lie inside $[-1,1]$, is also treated in [7].

Next we shall prove

THEOREM 2. *Let* $p_{n+2}(x)$ *be a real algebraic polynomial of degree* $n+2$ *such that* $|p_{n+2}(x)| \leqq 1-x^2$, *for* $-1 \leqq x \leqq 1$. *Then for* $k = 3,4,\dots$ *we have*

$$\int_{-1}^{1} \left[p_{n+2}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx \leqq \int_{-1}^{1} \left(f_1^{(k)}(x)\right)^2 (1-x^2)^{\frac{2k-5}{2}} dx \tag{1.2}$$

*where* $f_1(x) = (1-x^2)T_n(x)$. *Equality iff* $p_{n+2}(x) = \pm f_1(x)$.

REMARK 2. In the case $k = 2$, we were not able to resolve the inequality (1.2).

THEOREM 3. *Let* $P_{n+2}(x)$ *be any real polynomial of degree* $n+2$ *such that*

$$|P_{n+2}(x)| \leqq 1-x^2 \quad \text{for} \quad -1 \leqq x \leqq 1. \tag{1}$$

*Then we have*

$$(2)\qquad \int_{-1}^{1} \left| P'_{n+2}(x)\right|^{2p} dx \leqq \frac{2}{2p+1} n^{2p} + c_p n^{2p-1}$$

*where $p$ is any fixed positive integer and $c_p$ is a constant that depends on $p$ $(c_p \geqq 0)$. The above inequality is best possible in the following sense:*

$$(3)\qquad \lim_{n\to\infty} \frac{1}{n^{2p}} \int_{-1}^{1} \left| P'_{n+2}(x)\right|^{p} dx = \frac{2}{2p+1},$$

*where $P_{n+2}(x) = (1-x^2)T_n(x) = \cos n\theta$, $\cos\theta = x$.*

## 2. Lemmas

Here we state and prove some lemmas which are needed in the proofs of our theorems.

LEMMA 2.1. *Let $q_{n-1}(x)$ be any algebraic polynomial of degree at most $n-1$ with real coefficients. Further let*

$$(2.1)\qquad \left|q_{n-1}(x)\right| \leqq (1-x^2)^{-\frac{1}{2}}, \quad \textit{for} \quad -1 < x < 1.$$

*Then we have*

$$(2.2)\qquad \int_{-1}^{1} \left[q'_{n-1}(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx \leqq \frac{\pi}{2}(n^2-1).$$

*Equality iff $q_{n-1}(x) = \pm\frac{\sin n\theta}{\sin\theta}$, $x = \cos\theta$.*

Proof of this lemma is given in [8].

LEMMA 2.2. *Let $q_{n-1}(x)$ be any algebraic polynomial of degree $n-1$ with real coefficients such that $\left|q_{n-1}(x)\right| \leqq (1-x^2)^{-\frac{1}{2}}$, for $-1 < x < 1$. Then we have for $k = 1, 2, \dots$*

$$(2.3)\qquad \int_{-1}^{1} \left[q^{(k)}_{n-1}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx \leqq \int_{-1}^{1} \left[u^{(k)}_{n-1}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx.$$

*Equality iff $q_{n-1}(x) = \pm u_{n-1}(x)$; $u_{n-1}(x) = \frac{\sin n\theta}{\sin\theta}$, $x = \cos\theta$.*

PROOF. We begin by setting $q_{n-1}(x) = \sum_{j=0}^{n-1} \beta_j u_j(x)$. Now using the orthogonal properties of $\left\{u_j^{(k)}(x)\right\}$ and $\left\{u_j'(x)\right\}$, we obtain

(2.4)
$$\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx = \sum_{j=k}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx$$

and

$$\int_{-1}^{1} \left[q_{n-1}'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx = \sum_{j=1}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx. \tag{2.5}$$

Next, we note that $y = u_j(x)$ satisfies the differential equation

$$(1-x^2)y'' - 3xy' + j(j+2)y = 0. \tag{2.6}$$

From (2.6) it follows that

(2.7)
$$(1-x^2)u_j^{(k)}(x) - (2k-1)xu_j^{(k-1)}(x) + \left((j+1)^2 - (k-1)^2\right) u_j^{(k-2)}(x) = 0.$$

Now on using integration by parts and (2.7), we have

$$\int_{-1}^{1} \left[u_j^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx = \tag{2.8}$$
$$= -\int_{-1}^{1} u_j^{(k-1)}(x)(1-x^2)^{\frac{2k-1}{2}} \left[(1-x^2)u_j^{(k+1)}(x) - (2k+1)xu_j^{(k)}(x)\right] dx =$$
$$= \left(\int_{-1}^{1} \left[u_j^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\, dx\right) \left[(j+1)^2 - k^2\right].$$

Through repeated application of (2.8) we have

$$\int_{-1}^{1} \left[u_j^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx = \tag{2.9}$$
$$= \left[\prod_{i=2}^{k} \left((j+1)^2 - i^2\right)\right] \int_{-1}^{1} \left[u_j'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx$$

for $k = 1, 2, \ldots$, where we define for $k = 1$, $\prod_{i=2}^{k}\left((j+1)^2 - i^2\right) \equiv 1$.
Hence

$$\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}} =$$

$$= \sum_{j=k}^{n-1} \left[\prod_{i=2}^{k}\left((j+1)^2 - i^2\right)\right] \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx \leqq$$

$$\leqq \left[\prod_{i=2}^{k}(n^2 - i^2)\right] \sum_{j=k}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx \leqq$$

$$\leqq \left[\prod_{i=2}^{k}(n^2 - i^2)\right] \sum_{j=1}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx =$$

$$= \left[\prod_{i=2}^{k}(n^2 - i^2)\right] \int_{-1}^{1} \left[q_{n-1}'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx \leqq$$

$$\leqq \left[\prod_{i=2}^{k}(n^2 - i^2)\right] \int_{-1}^{1} \left[u_{n-1}'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\, dx =$$

$$= \int_{-1}^{1} \left[u_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx.$$

Equality, iff $q_{n-1}(x) = \pm u_{n-1}(x)$. This completes the proof of Lemma 2.2.

LEMMA 2.3. *Let $q_{n-1}(x)$ be any real algebraic polynomial of degree $n-1$ such that $\left|q_{n-1}(x)\right| \leqq (1-x^2)^{-\frac{1}{2}}$ for $-1 < x < 1$. Then, we have for $k = 0, 1, \ldots$*

$$\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\, dx \leqq \int_{-1}^{1} \left[u_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\, dx. \tag{2.10}$$

*Equality holds iff $q_{n-1}(x) = \pm u_{n-1}(x)$.*

PROOF. Let $v_1, v_2, \ldots, v_{n-k}$ be the zeros of $T_n^{(k)}(x)$. Then

$$\left|q_{n-1}^{(k)}(v_i)\right| \leqq \left|u_{n-1}^{(k)}(v_i)\right| \quad \text{for} \quad i = 1, 2, \ldots, n-k \tag{2.11}$$

(c.f. [8]). Now, using Gaussian quadrature formula, based on $v_1, v_2, \ldots, v_{n-k}$, we obtain

$$\int_{-1}^{1}\left[q_{n-1}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx=\sum_{i=1}^{n-k}\left[q_{n-1}^{(k)}(v_i)\right]^2 M_i$$

where

$$M_i=\int_{-1}^{1}\frac{\left[T_n^{(k)}(x)\right]^2}{(x-v_i)^2\left[T_n^{(k+1)}(v_i)\right]^2}(1-x^2)^{\frac{2k-1}{2}}\,dx\geqq$$
$$\geqq 0 \quad \text{for} \quad i=1,2,\ldots,n-k.$$

In view of (2.11), we have

$$\int_{-1}^{1}\left[q_{n-1}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx\leqq\sum_{i=1}^{n-k}\left[u_{n-1}^{(k)}(v_i)\right]^2 M_i=$$
$$=\int_{-1}^{1}\left[u_{n-1}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx$$

This completes the proof of Lemma 2.3.

The previous three lemmas are needed for the proof of Theorem 1, while the subsequent three lemmas are used in the proof of Theorem 2.

LEMMA 2.4. *Let $q_{n-1}(x)$ be any real algebraic polynomial of degree at most $n-1$ such that $|q_{n-1}(x)|\leqq 1$ for $-1\leqq x\leqq 1$. Then we have*

$$\int_{-1}^{1}[q'_{n-1}(x)]^2(1-x^2)^{\frac{1}{2}}\,dx\leqq\frac{\pi}{2}(n-1)^2. \tag{2.12}$$

*Equality iff $q_{n-1}(x)=\pm\cos(n-1)\theta$, $x=\cos\theta$.*

The proof of this lemma follows from a known result of Calderon and Klein [9].

LEMMA 2.5. *Let $q_n(x)$ be any real algebraic polynomial of degree $n$ such that $|q_n(x)|\leqq 1$, for $-1\leqq x\leqq 1$. Then we have for $k=1,2,\ldots$*

$$\int_{-1}^{1}\left[q_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx\leqq\int_{-1}^{1}\left[T_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx. \tag{2.13}$$

*Equality iff* $q_n(x) = \pm T_n(x)$; $T_n(x) = \cos n\theta$, $x = \cos\theta$.

PROOF. Let $q_n(x) = \sum_{j=0}^{n} \alpha_j T_j(x)$. From the orthogonal properties of $\{T_j^{(k)}(x)\}$ and $\{T_j'(x)\}$, we obtain

(2.14)
$$\int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\,dx = \sum_{j=k}^{n} \alpha_j^2 \int_{-1}^{1} \left[T_j^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\,dx$$

and

$$\int_{-1}^{1} \left[q_n'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx = \sum_{j=1}^{n} \alpha_j^2 \int_{-1}^{1} \left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx. \tag{2.15}$$

Following as in (2.8), we obtain

(2.16)
$$\int_{-1}^{1} \left[T_j^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\,dx = \left[\prod_{i=1}^{k-1}(j^2-i^2)\right] \int_{-1}^{1} \left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx$$

where for $k=1$, we define $\prod_{i=1}^{k-1}(j^2-i^2) \equiv 1$. Hence,

$$\begin{aligned}
&\int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\,dx = \\
&= \sum_{j=k}^{n} \left[\prod_{i=1}^{k-1}(j^2-i^2)\right] \alpha_j^2 \int_{-1}^{1} \left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx \leqq \\
&\leqq \prod_{i=1}^{k-1}(n^2-i^2) \sum_{j=k}^{n} \alpha_j^2 \int_{-1}^{1} \left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx \leqq \\
&\leqq \prod_{i=1}^{k-1}(n^2-i^2) \sum_{j=1}^{n} \alpha_j^2 \int_{-1}^{1} \left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx = \\
&= \prod_{i=1}^{k-1}(n^2-i^2) \int_{-1}^{1} \left[q_n'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx \leqq
\end{aligned}$$

$$\leqq \prod_{i=1}^{k-1}(n^2-i^2)\int_{-1}^{1}\left[T_n'(x)\right]^2(1-x^2)^{\frac{1}{2}}\,dx =$$

$$= \int_{-1}^{1}\left[T_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx.$$

This completes the proof of Lemma 2.5.

LEMMA 2.6. *Let $q_n(x)$ be any real algebraic polynomial of degree $n$ such that $|q_n(x)| \leqq 1$ for $-1 \leqq x \leqq 1$. Then we have for $k = 1, 2, \ldots$*

$$\int_{-1}^{1}\left[q_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx \leqq \int_{-1}^{1}\left[T_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx. \tag{2.17}$$

*Equality holds iff $q_n(x) = \pm T_n(x)$.*

PROOF. Let $u_1, u_2, \ldots, u_{n-k+1}$ be the zeros of $T_n^{(k-1)}(x)$. Then

$$\left|q_n^{(k)}(u_i)\right| \leqq \left|T_n^{(k)}(u_i)\right|, \quad \text{for} \quad i = 1, 2, \ldots, n-k+1. \tag{2.18}$$

Equality possible for any $i$ if $q_n(x) = \pm T_n(x)$.

For the proof of the above statement we refer to [8], page 104, formula (2.7.1). Now, using Gaussian quadrature formula, based on $u_1, u_2, \ldots, u_{n-k+1}$, we obtain

$$\int_{-1}^{1}\left[q_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx = \sum_{i=1}^{n-k+1}\left[q_n^{(k)}(u_i)\right]^2 H_i$$

where $H_i = \int_{-1}^{1}\left[\frac{T_n^{(k-1)}(x)}{(x-u_i)T_n^{(k)}(u_i)}\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx \geqq 0$.

Now, using (2.18), we have

$$\int_{-1}^{1}\left[q_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx \leqq \sum_{i=1}^{n-k+1}\left[T_n^{(k)}(u_i)\right]^2 H_i =$$

$$= \int_{-1}^{1}\left[T_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx.$$

From this Lemma 2.6 follows.

The proof of Theorem 3 is based on the following lemma.

LEMMA 2.7. *Let $t_n(\theta)$ be a trigonometric polynomial of order $n$. For $p$ and $k$ fixed positive integers, we have the following inequality*

(2.19)
$$\int_0^\pi \left[t_n'(\theta)\right]^{2p} \sin^k \theta \, d\theta \leqq \frac{2p-1}{2p} n^2 \int_0^\pi \left[t_n'(\theta)\right]^{2p-2} \sin^k \theta \, d\theta + \frac{k\pi}{2p} n^{2p-1}.$$

*The above inequality is best possible in the following sense:*

$$\lim_{n\to\infty} \frac{1}{n^{2p}} \int_0^\pi \left[t_n'(\theta)\right]^{2p} \sin^k \theta \, d\theta =$$
$$= \lim_{n\to\infty} \frac{1}{n^{2p-2}} \left(\frac{2p-1}{2p}\right) \int_0^\pi \left[t_n'(\theta)\right]^{2p-2} \sin^k \theta \, d\theta$$

*when $t_n(\theta) = \cos n\theta$.*

PROOF. Define

$$I_1 = \int_0^\pi \left[t_n'(\theta)\right]^{2p} \sin^k \theta \, d\theta. \tag{2.20}$$

Clearly, we may write

$$I_1 = I_2 + I_3,$$

where

$$I_2 = \int_0^\pi \left[t_n'(\theta)\right]^{2p-2} \Big[\big(t_n'(\theta)\big)^2 - t_n(\theta) t_n''(\theta)\Big] \sin^k \theta \, d\theta \tag{2.21}$$

and

$$I_3 = \int_0^\pi t_n''(\theta) t_n(\theta) \left[t_n'(\theta)\right]^{2p-2} \sin^k \theta \, d\theta. \tag{2.22}$$

From integration by parts,

$$I_3 = -\int_0^\pi t_n'(\theta) \Big\{ \left[t_n'(\theta)\right]^{2p-1} \sin^k \theta + \tag{2.23}$$
$$+ (2p-2) t_n''(\theta) t_n(\theta) \left[t_n'(\theta)\right]^{2p-3} \sin^k \theta + k t_n(\theta) \left[t_n'(\theta)\right]^{2p-2} \sin^{k-1} \theta \cos \theta \Big\} d\theta.$$

From (2.23), we have

$$I_3 = -I_1 - (2p-2) I_3 + E, \tag{2.24}$$

where

$$E = -k \int_0^\pi t_n(\theta) \left[t_n'(\theta)\right]^{2p-1} \sin^{k-1} \theta \cos \theta \, d\theta.$$

Hence,

$$I_3 = \frac{-1}{2p-1} I_1 + \frac{1}{2p-1} E. \tag{2.25}$$

From (2.21) and (2.25), we have

$$\frac{2p}{2p-1} I_1 = I_2 + \frac{1}{2p-1} E. \tag{2.26}$$

Further

$$\int_0^{\pi} \left[t_n'(\theta)\right]^{2p} \sin^k \theta \, d\theta = \tag{2.27}$$

$$= \frac{2p-1}{2p} \int_0^{\pi} \left[t_n'(\theta)\right]^{2p-2} \left[t_n'(\theta)^2 - t_n(\theta) t_n''(\theta)\right] \sin^k \theta \, d\theta -$$

$$- \frac{k}{2p} \int_0^{\pi} t_n(\theta) \left[t_n'(\theta)\right]^{2p-1} \sin^{k-1} \theta \cos \theta \, d\theta.$$

We now state the following inequality (see Varma [8])

$$\left| \left[t_n'(\theta)\right]^2 - t_n''(\theta) t_n(\theta) \right| \leqq n^2 \quad \text{for} \quad \left|t_n(\theta)\right| \leqq 1 \tag{2.28}$$

valid for any real trigonometric polynomial $t_n(\theta)$ of order $n$. Here equality holds for $t_n(\theta) = \cos n\theta$. We now apply (2.28) to the first and Berstein's inequality to the second term on the right hand side of (2.27). We obtain

(2.29)
$$\int_0^{\pi} \left[t_n'(\theta)\right]^{2p} \sin^k \theta \, d\theta \leqq \frac{2p-1}{2p} n^2 \int_0^{\pi} \left[t_n'(\theta)\right]^{2p-2} \sin^k \theta \, d\theta + \frac{k\pi}{2p} n^{2p-1}.$$

The proof of Lemma 2.7 is now complete.

Proof of Theorem 1. We let $p_{n+1}(x)$ be any real algebraic polynomial of degree $n+1$ such that $\left|p_{n+1}(x)\right| \leqq (1-x^2)^{\frac{1}{2}}$ for $-1 \leqq x \leqq 1$.

Now we write

$$p_{n+1}(x) = (1-x^2) q_{n-1}(x) \tag{3.1}$$

where $q_{n-1}(x)$ is a real algebraic polynomial of degree $n-1$. Further we have,

$$\left|q_{n-1}(x)\right| \leqq (1-x^2)^{-\frac{1}{2}}, \quad \text{for} \quad -1 < x < 1. \tag{3.2}$$

Through repeated differentiation of (3.1), we obtain

$$p_{n+1}^{(k)}(x) = (1-x^2)q_{n-1}^{(k)}(x) - 2kxq_{n-1}^{(k-1)}(x) - (k-1)kq_{n-1}^{(k-2)}(x) \quad \text{for} \quad k = 0,1,2,\dots \tag{3.3}$$

From (3.3) we have

$$\int_{-1}^{1} \left[p_{n+1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}}\, dx = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \tag{3.4}$$

where

$$\begin{cases} I_1 = \displaystyle\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\, dx, \\ I_2 = 4k^2 \displaystyle\int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 x^2(1-x^2)^{\frac{2k-3}{2}}\, dx, \\ I_3 = (k-1)^2k^2 \displaystyle\int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}}\, dx, \\ I_4 = -4k \displaystyle\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)q_{n-1}^{(k-1)}(x)\right] x(1-x^2)^{\frac{2k-1}{2}}\, dx. \end{cases} \tag{3.5}$$

Upon integration by parts

$$\begin{cases} I_4 = 2k \displaystyle\int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} (1-2kx^2)\, dx, \\ I_5 = 4(k-1)k^2 \displaystyle\int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)q_{n-1}^{(k-2)}(x)\right] x(1-x^2)^{\frac{2k-3}{2}}\, dx. \end{cases} \tag{3.6}$$

Similarly, we obtain

$$I_5 = -2(k-1)k^2 \int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} \left[1-2(k-1)x^2\right]\, dx. \tag{3.7}$$

Next,

$$I_6 = -2(k-1)k \int_{-1}^{1} \left[q_{n-1}^{(k)}(x)q_{n-1}^{(k-2)}(x)\right] (1-x^2)^{\frac{2k-1}{2}}\, dx =$$

$$= 2(k-1)k \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\, dx -$$

$$-2(2k-1)(k-1)k\int_{-1}^{1}\left[q_{n-1}^{(k-1)}(x)q_{n-1}^{(k-2)}(x)\right]x(1-x^2)^{\frac{2k-3}{2}}\,dx.$$

Note that

$$\int_{-1}^{1}\left[q_{n-1}^{(k-1)}(x)q_{n-1}^{(k-2)}(x)\right]x(1-x^2)^{\frac{2k-3}{2}}\,dx=$$

$$=-\frac{1}{2}\int_{-1}^{1}\left[q_{n-1}^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\left[1-2(k-1)x^2\right]\,dx.$$

Hence,

$$I_6=2(k-1)k\int_{-1}^{1}\left[q_{n-1}^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx+ \tag{3.8}$$

$$+(2k-1)(k-1)k\int_{-1}^{1}\left[q_{n-1}^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\left[1-2(k-1)x^2\right]\,dx.$$

From (3.4)–(3.8), we obtain

$$\int_{-1}^{1}\left[p_{n+1}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx=\int_{-1}^{1}\left[q_{n-1}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k+1}{2}}\,dx+ \tag{3.9}$$

$$+2(k-1)k\int_{-1}^{1}\left[q_{n-1}^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx+$$

$$+2k\int_{-1}^{1}\left[q_{n-1}^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx+$$

$$+(k-2)(k-1)^2k\int_{-1}^{1}\left[q_{n-1}^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx+$$

$$+(2k-3)(k-1)k\int_{-1}^{1}\left[q_{n-1}^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx.$$

Finally, we use Lemmas 2.2 and 2.3 along with (3.2) and (3.9) to get

$$\int_{-1}^{1}\left[p_{n+1}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx\leqq\int_{-1}^{1}\left[u_{n-1}^{(k)}(x)\right](1-x^2)^{\frac{2k+1}{2}}\,dx+$$

$$+2(k-1)k\int_{-1}^{1}\left[u_{n-1}^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx+$$

$$+2k\int_{-1}^{1}\left[u_{n-1}^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx+$$

$$+(k-2)(k-1)^2k\int_{-1}^{1}\left[u_{n-1}^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx+$$

$$+(2k-3)(k-1)k\int_{-1}^{1}\left[u_{n-1}^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx=$$

$$=\int_{-1}^{1}\left[f_0^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx \quad \text{where} \quad f_0(x)=(1-x^2)u_{n-1}(x).$$

Equality iff $p_{n+1}(x)=\pm f_0(x)$. The proof of Theorem 1 is now complete.

PROOF OF THEOREM 2. We let $p_{n+2}(x)$ be any real algebraic polynomial of degree $n+2$ such that $|p_{n+2}(x)|\leqq 1-x^2$, for $-1\leqq x\leqq 1$. Now we write

$$p_{n+2}(x)=(1-x^2)q_n(x) \tag{3.10}$$

where $q_n(x)$ is a real algebraic polynomial of degree $n$. Further, we have

$$|q_n(x)|\leqq 1, \quad \text{for} \quad -1\leqq x\leqq 1. \tag{3.11}$$

Through repeated differentition of (3.10), we get

$$p_{n+2}^{(k)}(x)=(1-x^2)q_n^{(k)}(x)-2kxq_n^{(k-i)}(x)-(k-1)kq_n^{(k-2)}(x) \quad \text{for} \quad k=0,1,2,\dots \tag{3.12}$$

From (3.12) we have

$$\int_{-1}^{1}\left[p_{n+2}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx=I_1+I_2+I_3+I_4+I_5+I_6 \tag{3.13}$$

where

$$\begin{cases} I_1=\int_{-1}^{1}\left[q_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx,\\ I_2=4k^2\int_{-1}^{1}\left[q_n^{(k-1)}(x)\right]^2x^2(1-x^2)^{\frac{2k-5}{2}}\,dx,\\ I_3=(k-1)^2k^2\int_{-1}^{1}\left[q_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx,\\ I_4=-4k\int_{-1}^{1}\left[q_n^{(k)}(x)q_n^{(k-1)}(x)\right]x(1-x^2)^{\frac{2k-3}{2}}\,dx. \end{cases} \tag{3.14}$$

Using integration by parts,

$$(3.15)\qquad \begin{cases} I_4 = 2k\displaystyle\int_{-1}^{1}\left[q_n^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\left[1-2(k-1)x^2\right]\,dx, \\ I_5 = 4(k-1)k^2\displaystyle\int_{-1}^{1}\left[q_n^{(k-2)}(x)q_n^{(k-1)}(x)\right]x(1-x^2)^{\frac{2k-5}{2}}\,dx. \end{cases}$$

Similarly, we obtain

$$(3.16)\quad I_5 = -2(k-1)k^2\int_{-1}^{1}\left[q_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-7}{2}}\left[1-2(k-2)x^2\right]\,dx.$$

Next,

$$I_6 = -2(k-1)k\int_{-1}^{1}\left[q_n^{(k-2)}(x)q_n^{(k)}(x)\right](1-x^2)^{\frac{2k-3}{2}}\,dx =$$

$$= 2(k-1)k\int_{-1}^{1}\left[q_n^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx-$$

$$-2(2k-3)(k-1)k\int_{-1}^{1}\left[q_n^{(k-1)}(x)q_n^{(k-2)}(x)\right]x(1-x^2)^{\frac{2k-5}{2}}\,dx.$$

Observe that

$$\int_{-1}^{1}\left[q_n^{(k-1)}(x)q_n^{(k-2)}(x)\right]x(1-x^2)^{\frac{2k-5}{2}}\,dx =$$

$$= -\frac{1}{2}\int_{-1}^{1}\left[q_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-7}{2}}\left[1-2(k-2)x^2\right]\,dx.$$

Hence,

$$(3.17)\qquad I_6 = 2(k-1)k\int_{-1}^{1}\left[q_n^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx+$$

$$+(2k-3)(k-1)k\int_{-1}^{1}\left[q_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-7}{2}}\left[1-2(k-2)x^2\right]\,dx.$$

From 3.13–3.17, we can write

$$(3.18)\quad \int_{-1}^{1}\left[p_{n+2}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx = \int_{-1}^{1}\left[q_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx+$$

$$+2(k-3)k\int_{-1}^{1}\left[q_n^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx+$$

$$+6k\int_{-1}^{1}\left[q_n^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx+$$

$$+(k-4)(k-3)(k-1)k\int_{-1}^{1}\left[q_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx+$$

$$+3(2k-5)(k-1)k\int_{-1}^{1}\left[q_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-7}{2}}\,dx.$$

Finally, we use Lemmas 2.5 and 2.6 along with (3.11) and (3.18) to obtain

$$\int_{-1}^{1}\left[p_{n+2}^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx \leqq \int_{-1}^{1}\left[T_n^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-1}{2}}\,dx+$$

$$+2(k-3)k\int_{-1}^{1}\left[T_n^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-3}{2}}\,dx+$$

$$+6k\int_{-1}^{1}\left[T_n^{(k-1)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx+$$

$$+(k-4)(k-3)(k-1)k\int_{-1}^{1}\left[T_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx+$$

$$+3(2k-5)(k-1)k\int_{-1}^{1}\left[T_n^{(k-2)}(x)\right]^2(1-x^2)^{\frac{2k-7}{2}}\,dx=$$

$$=\int_{-1}^{1}\left[f_1^{(k)}(x)\right]^2(1-x^2)^{\frac{2k-5}{2}}\,dx$$

where $f_1(x)=(1-x^2)T_n(x)$. Equality iff $p_{n+2}(x)=\pm f_1(x)$. This conclude the proofs of Theorems 1 and 2.

PROOF OF THEOREM 3. Define $f_0(x)$ such that $P_{n+2}(x)=(1-$ $-x^2)f_0(x)$. Then $f_0(x)$ is a real algebraic polynomial of degree $n$ such that $|f_0(x)|\leqq 1$ for $-1\leqq x\leqq 1$. We now define

$$t_n(\theta)=f_0(\cos\theta)\quad\text{for}\quad 0\leqq\theta\leqq\pi. \tag{3.19}$$

Then $t_n(\theta)$ is a trigonometric polynomial of order $n$ such that

$$|t_n(\theta)|\leqq 1\quad\text{for}\quad 0\leqq\theta\leqq\pi. \tag{3.20}$$

From the binomial theorem and Bernstein's inequality,

$$(3.21)\quad \int_{-1}^{1} |P'_{n+2}(x)|^{2p}\,dx = \int_0^\pi \left[\sin\theta t'_n(\theta) + 2\cos\theta t_n(\theta)\right]^{2p} \sin\theta d\theta =$$

$$= \binom{2p}{0} \int_0^\pi \left[\sin\theta t'_n(\theta)\right]^{2p} \sin\theta\, d\theta +$$

$$+\binom{2p}{1} \int_0^\pi \left[\sin\theta t'_n(\theta)\right]^{2p-1} \left[2\cos\theta t_n(\theta)\right]^{1} \sin\theta\, d\theta + \ldots +$$

$$+\ldots+\binom{2p}{2p} \int_0^\pi \left[2\cos\theta t_n(\theta)\right]^{2p} \sin\theta d\theta \leqq \int_0^\pi \left[t'_n(\theta)\right]^{2p} \sin^{2p+1}\theta\, d\theta +$$

$$+\binom{2p}{1}(2)^1\pi n^{2p-1} + \binom{2p}{2} 2^2\pi n^{2p-2} + \ldots + \binom{2p}{2p}(2)^{2p}\pi n^0.$$

After repeated applications of Lemma 2.7, we have

$$\int_0^\pi \left[t'_n(\theta)\right]^{2p} \sin^{2p+1}\theta\, d\theta \leqq$$

$$\leqq \left[\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{2p-1}{2p}\right)\int_0^\pi \sin^{2p+1}\theta d\theta\right] +$$

$$+\frac{(2p+1)\pi}{2}\left[\frac{n^{2p-1}}{p} + \frac{n^{2p-3}}{p-1} + \ldots + \frac{n^3}{2} + \frac{n}{1}\right].$$

We conclude that

$$(3.23)\quad \int_{-1}^{1} |\,P'_{n+2}(x)|^{2p}\,dx \leqq$$

$$+\left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{2p-1}{2p}\right)\int_0^\pi \sin^{2p+1}\theta d\theta\right] n^{2p} +$$

$$+\frac{(2p+1)\pi}{2}\left[\frac{n^{2p-1}}{p} + \frac{n^{2p-3}}{p-1} + \ldots + \frac{n^3}{2} + \frac{n}{1}\right] +$$

$$+\binom{2p}{1} 2\pi n^{2p-1} + \binom{2p}{2} 2^2\pi n^{2p-2} + \ldots + \binom{2p}{2p} 2^{2p}\pi n^0.$$

The theorem now follows.

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# ON EXTENSIONS OF SOME THEOREMS OF FLETT. II

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**1.** In [4] we generalized some interesting theorems of T. M. Flett [2] concerning the extended absolute Cesàro and Abel summability of numerical series. Applying our procedure of proof introduced in [4] we continue the generalizations of Flett's theorems and improve some of his results concerning the generalized absolute summability of power series.

We do not cite the theorems of Flett to be generalized here, because our results in the special case $\gamma(t) = t^\gamma$ will reduce to his appropriate theorems proved in [2].

We use throughout this paper the following notations, and agreements: Let $\alpha$ be any real number, and let

$$E_n^\alpha := \frac{(\alpha+1)\cdot\ldots\cdot(\alpha+n)}{n!} \quad (n>0), \quad E_0^\alpha := 1.$$

Let $\phi(z) := \sum_{n=1}^\infty c_n z^n$ be regular in $|z|<1$, and let $\tau_n^\alpha := \tau_n^\alpha(Q)$ denote the $n$th Cesàro mean of order $\alpha$ of the sequence $nc_ne^{niQ}$.

A summation sign $\sum$ in which the limits of summation are omitted will denote summation from 1 to $\infty$.

We use $B$ to denote a positive constant depending only on the parameters concerned in the particular problem in which it appears. If we wish to express the dependence explicitly, we write $B(c,d,\ldots)$, say. The constants are not necessarily the same at any two occurrences.

Inequalities of the form $L \leqq BR$ mean: "if $R$ is finite, then $L$ is also finite and the inequality holds".

Let $\gamma(t)$ be a positive non-decreasing function defined for $0<t<\infty$ such that

$$\gamma(t)\gamma(1/t) = 1 \tag{1.1}$$

for any $t$.

We define the following functions:

$$g_{k,\gamma(t)}(Q) := \left\{\int_0^1 \gamma(1-\rho)^{-k}(1-\rho)^{k-1}\left|\phi'(\rho e^{iQ})\right|^k d\rho\right\}^{1/k},$$

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$$h_{k,\alpha,\gamma(t)}(Q) := \left\{ \sum \gamma(n)^k n^{-1} \left| \tau_n^\alpha(Q) \right|^k \right\}^{1/k},$$

and

$$T_{k,\gamma(t),\sigma}(Q) := \left\{ \int_0^1 \int_{-\pi}^{\pi} \gamma(1-\rho)^{-k}(1-\rho)^{\sigma+k-2} \frac{\left|\phi'(\rho e^{iQ+it})\right|^k}{|1-\rho e^{it}|^\sigma}\, dt\, d\rho \right\}^{1/k}.$$

**2.** In terms of functions introduced above we can state our results.

THEOREM 1. *If* $k \geqq 1$, $\eta := \max(0, 1-2/k)$, $\alpha > \eta + (1/k) - 1$ *and* $\alpha_k := \alpha + \min(1/k, 1-1/k)$ $(> 0)$, *furthermore if there exists a constant* $C > 1$ *so that*

$$\limsup_{y\to\infty} \frac{\gamma(Cy)}{\gamma(y)} < C^{\alpha_k}, \tag{2.1}$$

*then*

$$h_{k,\alpha,\gamma(t)}(Q) \leqq B\ T_{k,\gamma(t),k\alpha-k\eta}(Q).$$

THEOREM 2. *If* $\lambda \geqq k \geqq 1$, *and* $\sigma > 1$, *then*

$$\int_{-\pi}^{\pi} T^\lambda_{k,\gamma(t),\sigma}(Q) dQ \leqq B \int_{-\pi}^{\pi} g^\lambda_{k,\gamma(t)}(Q)\, dQ. \tag{2.2}$$

These theorems imply the most interesting inequality formulated as

THEOREM 3. *If* $\lambda \geqq k \geqq 1$, $\alpha > \max(1/k, 1-1/k)$, $\alpha_k := \alpha + \min(1/k, 1-1/k)$, *and* (2.1) *holds with some* $C > 1$, *then*

$$\int_{-\pi}^{\pi} h^\lambda_{k,\alpha,\gamma(t)}(Q) dQ \leqq B \int_{-\pi}^{\pi} g^\lambda_{k,\gamma(t)}(Q)\, dQ.$$

**3.** To prove the theorems we require the following lemmas.

LEMMA 1. *Let* $\xi(Q)$ *be integrable in* $(-\pi,\pi)$ *and periodic with period* $2\pi$, *and let*

$$\xi^*(Q) := \sup_{0<|t|\leqq\pi} \left\{ \frac{1}{t} \int_0^t |\xi(Q+\tau)|\, d\tau \right\}.$$

*Then*

$$\|\xi^*\|_r \leqq \frac{r}{r-1} \|\xi\|_r \quad (r > 1),$$

*where* $\|\cdot\|_p$ *denotes the usual* $L^p$*-norm.*

This lemma is a part of a well-known theorem of Hardy and Littlewood [3].

LEMMA 2. *If $\xi(Q)$ and $\xi^*(Q)$ are the functions given in Lemma 1, and $\lambda > 1$, then*

$$\int_{-\pi}^{\pi} \frac{\xi(Q+\psi+t)}{|1-\rho e^{it}|^{\lambda}}\,dt \leqq B(\lambda)\frac{\xi^*(Q)|1-\rho e^{i\psi}|}{(1-\rho)^{\lambda}}. \tag{3.1}$$

This lemma is due to Flett [1] (Lemma 4).

The following lemma can be found in [1] (p. 368) implicitly. For details see, for example, Zygmund [5], Chapter XII, §3–5.

LEMMA 3. *If $k \geqq 1$, $\alpha > -1$ and $\eta := \max(0, 1-2/k)$, then*

$$\sum n^{k\alpha-k\eta+k-2}|\tau_n^{\alpha}|^k \rho^{kn} \leqq B \int_{-\pi}^{\pi} \frac{|\phi'(\rho e^{iQ+it})|^k}{|1-\rho e^{it}|^{k\alpha-k\eta}}\,dt. \tag{3.2}$$

LEMMA 4. *If $\beta > -1$, $\nu \geqq 1$, $\ell \geqq 1$ and $\rho(t)$ is a positive non-decreasing function defined for $1 \leqq t < \infty$ satisfying the condition*

$$\limsup_{t\to\infty} \frac{\rho(Ct)}{\rho(t)} < C^{(\beta+1)/\ell} \tag{3.3}$$

*with some $C > 1$, then*

$$\int_0^1 (1-x)^{\beta}\rho\big((1-x)^{-1}\big)^{\ell} x^{\nu}\,dx \leqq B\,\rho(\nu)^{\ell}\nu^{-\beta-1}.$$

This lemma was proved in [4].

4. PROOF OF THEOREM 1. Let us apply Lemma 3, furthermore multiply both sides of (3.2) by $(1-\rho)^{k\alpha-k\eta+k-2}\gamma(1-\rho)^{-k}$ and integrate with respect to $\rho$ from 0 to 1; then we get that

$$\sum n^{k\alpha-k\eta+k-2}|\tau_n^{\alpha}|^k \int_0^1 (1-\rho)^{k\alpha-k\eta+k-2}\gamma(1-\rho)^{-k}\rho^{kn}\,d\rho \leqq \tag{4.1}$$

$$\leqq B\int_0^1\int_{-\pi}^{\pi}(1-\rho)^{k\alpha-k\eta+k-2}\gamma(1-\rho)^{-k}\frac{|\phi'(\rho e^{iQ+it})|^k}{|1-\rho e^{it}|^{k\alpha-k\eta}}\,dt\,d\rho.$$

In view of (1.1) it is obvious that

$$I := \int_0^1 (1-\rho)^{k\alpha-k\eta+k-2}\gamma\big((1-\rho)^{-1}\big)^k\rho^{kn}\,d\rho \geqq \tag{4.2}$$

$$\geqq \int_{1-2/n}^{1-1/2n} \geqq B\gamma(n)^k n^{-k\alpha+k\eta-k+1};$$

furthermore, by Lemma 4, $I < \infty$ also holds. Indeed, the use of Lemma 4 with $\rho(t) = \gamma(t)$, $\beta = k\alpha - k\eta + k - 2$, $\nu = kn$ and $\ell = k$ is permissible, because then $\beta > -1$ and $(\beta + 1)/\ell = \alpha_k$, that is, (2.1) coincides with (3.3), thus every condition of Lemma 4 is satisfied.

Finally (4.1) and (4.2) plainly yield the statement of Theorem 1.

PROOF OF THEOREM 2. First we verify the case $\lambda = k$ of (2.2). Then

$$\int_{-\pi}^{\pi} T^k_{k,\gamma(t),\sigma}(Q)\, dQ =$$

$$= \int_{-\pi}^{\pi} \left\{ \int_0^1 \int_{-\pi}^{\pi} \gamma(1-\rho)^{-k}(1-\rho)^{\sigma+k-2} \frac{\left|\phi'(\rho e^{iQ+iy})\right|^k}{|1-\rho e^{iy}|^\sigma}\, dy\, d\rho \right\} dQ =$$

$$= \int_0^1 \gamma(1-\rho)^{-k}(1-\rho)^{\sigma+k-2} d\rho \,\cdot$$

$$\cdot \left\{ \int_{-\pi}^{\pi} \frac{dy}{|1-\rho e^{iy}|^\sigma} \right\} \left\{ \int_{-\pi}^{\pi} \left|\phi'(\rho e^{iQ+iy})\right|^k dQ \right\},$$

namely the innermost integral is actually independent of $y$, and is equal to

$$\int_{-\pi}^{\pi} \left|\phi'(\rho e^{iQ})\right|^k dQ.$$

Furthermore, by $\sigma > 1$,

$$\int_{-\pi}^{\pi} \frac{dy}{|1-\rho e^{iy}|^\sigma} \leqq B(\sigma)(1-\rho)^{1-\sigma}.$$

In view of these

$$\int_{-\pi}^{\pi} T^k_{k,\gamma(t),\sigma}(Q) dQ \leqq$$

$$\leqq B(\sigma) \int_0^1 \gamma(1-\rho)^{-k}(1-\rho)^{k-1} \left\{ \int_{-\pi}^{\pi} \left|\pi'(\rho e^{iQ})\right|^k dQ \right\} d\rho =$$

$$= B(\sigma) \int_{-\pi}^{\pi} g^k_{k,\gamma(t)}(Q)\, dQ$$

and this is (2.2) with $\lambda = k$.

If $\lambda > k$ then we set $\mu := \lambda/(\lambda - k)$. Then

$$(4.3)\qquad \left\{\int_{-\pi}^{\pi} T_{k,\gamma(t),\sigma}^{\lambda}\, dQ\right\}^{k/\lambda} = \left\{\int_{-\pi}^{\pi} (T_{k,\gamma(t),\sigma}^{k})^{\lambda/k}\, dQ\right\}^{k/\lambda} = $$
$$= \sup_{\xi} \int_{-\pi}^{\pi} T_{k,\gamma(t),\sigma}^{k} \xi\, dQ,$$

the supremum being taken over all non-negative functions $\xi$ such that $\|\xi\|_{\mu} \leqq 1$.

Since

$$(4.4)\qquad \int_{-\pi}^{\pi} T_{k,\gamma(t),\sigma}^{k} \xi\, dQ =$$
$$= \int_{-\pi}^{\pi} \left\{ \int_{0}^{1} \int_{-\pi}^{\pi} \gamma(1-\rho)^{-k} (1-\rho)^{\sigma+k-2} \frac{|\phi'(\rho e^{iy})|^{k}}{|1-\rho e^{iy-iQ}|^{\sigma}}\, dy\, d\rho \right\} \xi(Q)\, dQ =$$
$$= \int_{0}^{1} \int_{-\pi}^{\pi} \gamma(1-\rho)^{-k} (1-\rho)^{\sigma+k-2} |\phi'(\rho e^{iy})|^{k} \left\{ \int_{-\pi}^{\pi} \frac{\xi(Q) dQ}{|1-\rho e^{iy-iQ}|^{\sigma}} \right\} dy\, d\rho,$$

and, by Lemma 2, using the notations introduced in Lemma 1, the integral in the brackets does not exceed

$$(4.5)\qquad B(\sigma) \xi^{*}(y) (1-\rho)^{1-\sigma}.$$

Combining (4.4) and (4.5), and applying Hölder's inequality with indices $\lambda/k$ and $\mu$, we obtain that

$$(4.6)\qquad \int_{-\pi}^{\pi} T_{k,\gamma(t),\sigma}^{k} \xi dQ \leqq$$
$$\leqq B(\sigma) \int_{0}^{1} \int_{-\pi}^{\pi} \gamma(1-\rho)^{-k} (1-\rho)^{k-1} |\phi'(\rho e^{iy})|^{k} \xi^{*}(y)\, dy\, d\rho =$$
$$= B(\sigma) \int_{-\pi}^{\pi} g_{k,\gamma(t)}^{k}(y) \xi^{*}(y)\, dy \leqq B(\sigma) \|g_{k,\gamma(t)}\|_{\lambda}^{k} \|\xi^{*}\|_{\mu}.$$

Combining (4.3), (4.6) and the inequality

$$\|\xi^{*}\|_{\mu} \leqq \frac{\mu}{\mu-1} \|\xi\|_{\mu} \leqq \frac{\lambda}{k}$$

which comes from Lemma 1 and the definition of $\mu$, we get immediately (2.2); this completes the proof of Theorem 2.

Proof of Theorem 3. Theorem 3 is an immediate consequence of Theorems 1 and 2; we have only to observe that if the conditions of Theorem 3 are satisfied, then all of the conditions of Theorems 1 and 2 are also fulfilled. Indeed, $\alpha > \max(1/k, 1-1/k)$ implies both $\sigma = k\alpha - k\eta > 1$, and $\alpha > \eta + (1/k) - 1$, furthermore the rest of the conditions is obviously satisfied.

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# THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS USING MODIFIED LACUNARY SPLINE FUNCTIONS OF TYPE (0;2;3)

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## 1. Introduction

In this paper we are going to approximate the solution of the differential equation

$$
(1)\qquad \begin{cases} y^{(m)}(x) = f\left[x, y(x), y'(x), \dots, y^{(m-1)}(x)\right], & x \in [0;1], \\ y^{(j)}(0) = y_0^{(j)} & (j = 0, 1, \dots, m-1;\ 2 \leqq m \leqq 5) \end{cases}
$$

supposing

$$
(1.1)\qquad f\left[x, y(x), y'(x), \dots, y^{(m-1)}(x)\right] \in C^{(r)}\left([0;1]\right)
$$

where $r$ is a fixed integer,

$$
(1.2)\qquad \left| f^{(q)}\left[x, y_1^{(0)}, y_1^{(1)}, \dots, y_1^{(m-1)}\right] - f^{(q)}\left[x, y_2^{(0)}, y_2^{(1)}, \dots, y_2^{(m-1)}\right] \right| \leqq
$$

$$
\leqq L \sum_{i=0}^{m-1} \left| y_1^{(i)} - y_2^{(i)} \right| \quad (q = 0, 1, \dots, r)
$$

(Lipschitz condition), where

$$
f^{(0)} = f,
$$

$$
f^{(q+1)} = f_x^{(q)} + f_y^{(q)} \cdot y' + f_{y'}^{(q)} \cdot y'' + \dots + f_{y^{(m-2)}}^{(q)} \cdot y^{(m-1)} + f_{y^{(m-1)}}^{(q)} \cdot f,
$$
$$
(q = 0, 1, 2, \dots, r-1).
$$

The problem of approximating the solution of differential equations with Hermite type spline functions was discussed in [1], [2], [3] and [7]. In [4], [5], [6] lacunary spline functions are used. The main idea of our method is the following: we approximate the values of $y^{(q)}(x_k)$ using the values of

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$y_0^{(j)}$ $(j=0,1,\dots,m-1)$ and the function $f$ (like in [5], [7]). Denote these approximate values by $\overline{y}_k^{(q)}$ $(k=0,1,\dots,n;\ q=0,1,2,3)$. Then, using these approximate values we construct the modified lacunary spline function of type $(0;2;3)$, defined in [8]. In the approximation theorems we use the average moduli defined in [10].

## 2. The first approximation process

**2.1.** *Definition of the approximate values* $\overline{y}_k^{(q)}$. Let

$$x_k=\frac{k}{n};\quad h=\frac{1}{n};\quad x_{k+1/2}=x_k+\frac{h}{2}\quad (k=0,1,\dots,n);$$

$$\omega\left(f^{(r)},x,h\right)=\sup_{t_1,t_2\in\left[x-\frac{h}{2};x+\frac{h}{2}\right]\cap[0;1]}\left|f^{(r)}(t_1)-f^{(r)}(t_2)\right|,$$

$$\tau\left(f^{(r)},h\right)=\int_0^1\omega\left(f^{(r)},x,h\right)\,dx.$$

DEFINITION ([7]). Let

$$\overline{y}_0^{(j)}:=y_0^{(j)}\quad (j=0,1,\dots,m-1),$$

$$\overline{y}_0^{(m+q)}:=f^{(q)}\left[x_0,y_0^{(0)},y_0^{(1)},\dots,y_0^{(m-1)}\right]\quad (q=0,1,\dots,r),$$

$$\overline{y}_k^{(q)}:=G_k^{(q)}(x_k)\quad (k=1,2,\dots,n-1;\ q=0,1,\dots,m+r),$$

$$\overline{y}_n^{(q)}:=G_{n-1}^{(q)}(x_n)\quad (q=0,1,\dots,m+r),$$

where

$$G_0(x):=\sum_{j=0}^{m-1}\frac{y_0^{(j)}}{j!}(x-x_0)^j+ \tag{2.1.1}$$

$$+\sum_{q=0}^{r}\frac{f^{(q)}\left[x_0,y_0^{(0)},y_0^{(1)},\dots,y_0^{(m-1)}\right]}{(m+q)!}(x-x_0)^{m+q}$$

if $x_0\leqq x\leqq x_1$;

$$G_k(x) := \sum_{j=0}^{m-1} \frac{G_{k-1}^{(j)}(x_k)}{j!}(x-x_k)^j + \tag{2.1.2}$$

$$+\sum_{q=0}^{r} \frac{f^{(q)}\left[x_k, G_{k-1}(x_k), G'_{k-1}(x_k), \dots, G_{k-1}^{(m-1)}(x_k)\right]}{(m+q)!}(x-x_k)^{m+q}$$

if $x_k \leqq x \leqq x_{k+1}$ $(k = 1, 2, \dots, n-1)$.

**2.2.** *The convergence process.* In [7] a theorem was proved which showed how the approximating values $\overline{y}_k^{(j)}$ converge to the exact values of $y_k^{(j)} = y^{(j)}(x_k)$ $(k = 1, 2, \dots, n;\ j = 0, 1, \dots, m+r)$.

THEOREM 2.1 (see [7, Theorem 2.2.1]). *We have*

$$e_k^{(j)} := \left| y_k^{(j)} - \overline{y}_k^{(j)} \right| \leqq C_{j,k} h^r \tau\left(y^{(m+r)}, h\right)$$
$$(k = 1, 2, \dots, n; \quad j = 0, 1, \dots, m+r)$$

*where the constants $C_{j,k}$ are independent of $n$.*

## 3. The second approximation process

As we have seen before, we have the set of approximate values

$$\overline{y}_0^{(q)}, \overline{y}_1^{(q)}, \dots, \overline{y}_n^{(q)}, \overline{y}_0^{(1)}, \overline{y}_n^{(1)} \quad (q = 0, 2, 3)$$

which are the approximating values for the exact solution $y(x)$ of (1) and its derivatives at the points $x_0, x_1, \dots, x_n$.

Using these approximate values $\overline{y}_k^{(q)}$ $(q = 0, 2, 3;\ k = 0, 1, \dots, n)$ and $\overline{y}_0^{(1)}, \overline{y}_n^{(1)}$ on the basis of [8] we construct the modified lacunary spline function $\overline{S}_\Delta(x)$ of type (0;2;3). Then we prove some approximation theorems.

**3.1.** *The construction of the modified lacunary spline function.* We construct the lacunary spline function $\overline{S}_\Delta(x)$ of type (0;2;3) ($\overline{S}_\Delta(x) :=$ $:= \overline{S}_k(x)$ if $x_k \leqq x \leqq x_{k+1}$) similarly to [8].

THEOREM 3.1.1. *Let $\overline{y}_k^{(q)}$ $(q = 0, 2, 3;\ k = 0, 1, \dots, n)$, $\overline{y}_0^{(1)}$, $\overline{y}_n^{(1)}$ be the approximate values defined above. Then there exists a unique spline function $\overline{S}_\Delta(x)$ satisfying:*

$$\overline{S}_0^{(j)}(x_0) := \overline{y}_0^{(j)} \quad (j = 0, 1, 2, 3), \tag{3.1.1}$$

$$\overline{S}_k^{(q)}(x_k) := \overline{y}_k^{(q)} \quad (q = 0,2,3;\ k = 1,2,\dots,n-1), \tag{3.1.2}$$

$$\overline{S}_{n-1}^{(j)}(x_n) := \overline{y}_n^{(j)} \quad (j = 0,1,2,3), \tag{3.1.3}$$

$$\overline{S}_k^{(q)}(x_{k+1}) := \overline{S}_{k+1}^{(q)}(x_{k+1}) := \overline{y}_{k+1}^{(q)} \ (q = 0,2,3;\ k = 0,1,\dots,n-2), \tag{3.1.4}$$

$$\overline{S}_\Delta(x) \quad \textit{is a polynomial of minimal degree on} \quad [x_k; x_{k+1}]. \tag{3.1.5}$$

We quote the following relations (see [8] § 2):

$$\begin{aligned}\overline{S}_0(x) &= \overline{y}_0^{(0)} + \overline{y}_0^{(1)}(x-x_0) + \frac{\overline{y}_0^{(2)}}{2}(x-x_0)^2 + \frac{\overline{y}_0^{(3)}}{6}(x-x_0)^3 + \\ &+\overline{a}_{4,0}(x-x_0)^4 + \overline{a}_{5,0}(x-x_0)^5 + \overline{a}_{6,0}(x-x_0)^6, \text{ if } \quad x_0 \leqq x \leqq x_1,\end{aligned} \tag{3.1.6}$$

$$\begin{aligned}\overline{S}_k(x) &= \overline{y}_k^{(0)} + \overline{a}_{1,k}(x-x_k) + \frac{\overline{y}_k^{(2)}}{2}(x-x_k)^2 + \frac{\overline{y}_k^{(3)}}{6}(x-x_k)^3 + \\ &+\overline{a}_{4,k}(x-x_k)^4 + \overline{a}_{5,k}(x-x_k)^5, \quad \text{if} \quad x_k \leqq x \leqq x_{k+1} \\ &(k = 1,2,\dots,n-2),\end{aligned} \tag{3.1.7}$$

$$\begin{aligned}\overline{S}_{n-1}(x) &= \overline{y}_{n-1}^{(0)} + \overline{a}_{1,n-1}(x-x_{N-1}) + \frac{\overline{y}_{n-1}^{(2)}}{2}(x-x_{n-1})^2 + \\ &+\frac{\overline{y}_{n-1}^{(3)}}{6}(x-x_{n-1})^3 + \overline{a}_{4,n-1}(x-x_{n-1})^4 + \\ &+\overline{a}_{5,n-1}(x-x_{n-1})^5 + \overline{a}_{6,n-1}(x-x_{n-1})^6, \quad \text{if} \quad x_{n-1} \leqq x \leqq x_n,\end{aligned} \tag{3.1.8}$$

where

$$\overline{a}_{4,0} = \frac{5}{h^4}\left\{\overline{F}_0^{(0)} - \overline{y}_0^{(1)}h\right\} - \frac{1}{2h^2}\overline{F}_0^{(2)} + \frac{1}{12h}\overline{F}_0^{(3)}, \tag{3.1.9}$$

$$\overline{a}_{5,0} = \frac{-6}{h^5}\left\{\overline{F}_0^{(0)} - \overline{y}_0^{(1)}h\right\} + \frac{4}{5h^3}\overline{F}_0^{(2)} - \frac{3}{20h^2}\overline{F}_0^{(3)}, \tag{3.1.10}$$

$$\overline{a}_{6,0} = \frac{2}{h^6}\left\{\overline{F}_0^{(0)} - \overline{y}_0^{(1)}h\right\} - \frac{3}{10h^4}\overline{F}_0^{(2)} + \frac{1}{15h^3}\overline{F}_0^{(3)}, \tag{3.1.11}$$

$$\overline{a}_{1,k} = \frac{1}{h}\overline{F}_k^{(0)} - \frac{3}{20}h\overline{F}_k^{(2)} + \frac{1}{30}h^2\overline{F}_k^{(3)} \quad (k = 1, 2, \dots, n-2), \tag{3.1.12}$$

$$\overline{a}_{4,k} = \frac{1}{4h^2}\overline{F}_k^{(2)} - \frac{1}{12h}\overline{F}_k^{(3)} \quad (k = 1, 2, \dots, n-2), \tag{3.1.13}$$

$$\overline{a}_{5,k} = \frac{-1}{10h^3}\overline{F}_k^{(2)} + \frac{1}{20h^2}\overline{F}_k^{(3)} \quad (k = 1, 2, \dots, n-2), \tag{3.1.14}$$

$$\overline{a}_{1,n-1} = \frac{2}{h}\overline{F}_{n-1}^{(0)} - \overline{F}_{n-1}^{(1)} + \frac{1}{5}h\overline{F}_{n-1}^{(2)} - \frac{1}{60}h^2\overline{F}_{n-1}^{(3)}, \tag{3.1.15}$$

$$\overline{a}_{4,n-1} = \frac{-5}{h^4}\overline{F}_{n-1}^{(0)} + \frac{5}{h^3}\overline{F}_{n-1}^{(1)} - \frac{3}{2h^2}\overline{F}_{n-1}^{(2)} + \frac{1}{6h}\overline{F}_{n-1}^{(3)}, \tag{3.1.16}$$

$$\overline{a}_{5,n-1} = \frac{6}{h^5}\overline{F}_{n-1}^{(0)} - \frac{6}{h^4}\overline{F}_{n-1}^{(1)} + \frac{2}{h^3}\overline{F}_{n-1}^{(2)} - \frac{1}{4h^2}\overline{F}_{n-1}^{(3)}, \tag{3.1.17}$$

$$\overline{a}_{6,n-1} = \frac{-2}{h^6}\overline{F}_{n-1}^{(0)} + \frac{2}{h_5}\overline{F}_{n-1}^{(1)} - \frac{7}{10h^4}\overline{F}_{n-1}^{(2)} + \frac{1}{10h^3}\overline{F}_{n-1}^{(3)}, \tag{3.1.18}$$

$$\overline{F}_k^{(0)} = \overline{y}_{k+1}^{(0)} - \overline{y}_k^{(0)} - \frac{\overline{y}_k^{(2)}}{2}h^2 - \frac{\overline{y}_k^{(3)}}{6}h^3 \quad (k = 0, 1, \dots, n-1), \tag{3.1.19}$$

$$\overline{F}_k^{(2)} = \overline{y}_{k+1}^{(2)} - \overline{y}_k^{(2)} - \overline{y}_k^{(3)}h \quad (k = 0, 1, \dots, n-1), \tag{3.1.20}$$

$$\overline{F}_k^{(3)} = \overline{y}_{k+1}^{(3)} - \overline{y}_k^{(3)} \quad (k = 0, 1, \dots, n-1), \tag{3.1.21}$$

$$\overline{F}_{n-1}^{(1)} = \overline{y}_n^{(1)} - \overline{y}_{n-1}^{(2)}h - \frac{\overline{y}_{n-1}^{(3)}}{2}h^2. \tag{3.1.22}$$

**3.2.** *A general convergence process.* In this section we prove the essential theorems concerned with the convergence of modified lacunary spline functions. We also prove that this function satisfies the differential equation as $n \to \infty$, for $2 \leqq m$ and $m + r \leqq 5$.

THEOREM 3.2.1. *If* $y(x) \in C^{m+r}([0;1])$ *is the exact solution of* (1) *and* $\overline{S}_\Delta(x)$ *is the spline function constructed in Theorem* 3.1.1 *then the following inequalities hold:*

$$\left|y^{(q)}(x) - \overline{S}_0^{(q)}(x)\right| \leqq M_{0,q} h^{m+r-q} \tau\left(y^{(m+r)}, h\right)$$

$$(q = 0, 1, \ldots, m + r), \quad \textit{if} \quad x_0 \leqq x \leqq x_1,$$

$$\left|y^{(q)} - \overline{S}_k^{(q)}(x)\right| \leqq M_{k,q} h^{r-q} \tau\left(y^{(m+r)}, h\right)$$

$$(q = 0, 1), \quad \textit{if} \quad x_k \leqq x \leqq x_{k+1} \quad (k = 1, 2, \ldots, n-2),$$

$$\left|y^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq M_{k,q} h^{r+2-q} \tau\left(y^{(m+r)}, h\right)$$

$$(q = 2, 3, \ldots, m + r), \quad \textit{if} \quad x_k \leqq x \leqq x_{k+1} \quad (k = 1, 2, \ldots, n-2),$$

$$\left|y^{(q)}(x) - \overline{S}_{n-1}^{(q)}(x)\right| \leqq M_{n-1,q} h^{r-q} \tau\left(y^{(m+r)}, h\right)$$

$$(q = 0, 1, \ldots, m + r), \quad \textit{if} \quad x_{n-1} \leqq x \leqq x_n,$$

*where the constants* $M_{k,q}$ *are independent of* $n$.

We prove this theorem later.

Let

$$\overline{S}_\Delta^*(x) := \overline{S}_k^*(x) := f\left[x, \overline{S}_k(x), \overline{S}_k'(x), \ldots, \overline{S}_k^{(m-1)}(x)\right],$$
$$\text{if} \quad x_k \leqq x \leqq x_{k+1} \quad (k = 0, 1, \ldots, n-1, \quad 2 \leqq m \leqq 5).$$

THEOREM 3.2.2. *If the function* $f$ *in* (1) *satisfies the conditions* (1.1) *and* (1.2), *then the following inequalities hold:*

$$\left|y^{(m)}(x) - \overline{S}_0^*(x)\right| \leqq D_{0,m} h^{r+1} \tau\left(y^{(m+r)}, h\right), \quad \textit{if} \quad x_0 \leqq x \leqq x_1,$$

$$\left|y^{(m)}(x) - \overline{S}_k^*(x)\right| \leqq D_{k,m} h^{r-1} \tau\left(y^{(m+r)}, h\right), \quad \textit{if} \quad x_k \leqq x \leqq x_{k+1}$$
$$(k = 1, 2, \ldots, n-2),$$

$$\left|y^{(m)}(x) - \overline{S}_{n-1}^*(x)\right| \leqq D_{n-1,m} h^{r+1-m} \tau\left(y^{(m+r)}, h\right), \quad \textit{if} \quad x_{n-1} \leqq x \leqq x_n,$$

*where the constants* $D_{k,m}$ *are independent of* $n$.

THEOREM 3.2.3. *If the function* $f$ *in* (1) *satisfies the conditions* (1.1) *and* (1.2), *then the following inequalities hold:*

$$\left|\overline{S}_0^{(m)}(x) - \overline{S}_0^*(x)\right| \leqq E_{0,m} h^r \tau\left(y^{(m+r)}, h\right), \quad \textit{if} \quad x_0 \leqq x \leqq x_1,$$

$$\left|\overline{S}_k^{(m)}(x) - \overline{S}_k^*(x)\right| \leqq E_{k,m} h^s \tau\left(y^{(m+r)}, h\right), \quad \textit{if} \quad x_k \leqq x \leqq x_{k+1}$$

$$(k = 1, 2, \ldots, n-2) \quad \textit{where} \quad s = \min(r + 2 - m, r - 1),$$

$$\left|\overline{S}_{n-1}^{(m)}(x) - \overline{S}_{n-1}^*(x)\right| \leqq E_{n-1,m} h^{r-m} \tau\left(y^{(m+r)}, h\right), \quad \textit{if} \quad x_{n-1} \leqq x \leqq x_n,$$

*where the constants* $E_{k,m}$ *are independent of* $n$.

To prove these theorems we need some lemmas, which can be easily proved using formulae (3.1.6–3.1.22) and Theorem 2.1.

LEMMA 3.2.1. *Let* $F_k^{(j)}$ *and* $\overline{F}_k^{(j)}$ $(j = 0, 2, 3;\ k = 0, 1, \ldots, n-1)$ *denote the values defined in* (3.1.19–3.1.21) *with the exact values* $y^{(j)}(x_k) = y_k^j$ *and the approximate values* $\overline{y}_k^{(j)}$, *respectively. Then we have*

$$\left|F_0^{(j)} - \overline{F}_0^{(j)}\right| \leqq A_{0,j} h^{m+r-j} \tau\left(y^{(m+r)}, h\right) \quad (j = 0, 2, 3),$$

$$\left|F_k^{(j)} - \overline{F}_k^{(j)}\right| \leqq A_{k,j} h^r \tau\left(y^{(m+r)}, h\right) \quad (j = 0, 2, 3;\ k = 1, 2, \ldots, n-1),$$

*where the constants* $A_{k,j}$ *are independent of* $n$.

LEMMA 3.2.2. *Let* $a_{j,k}$ *and* $\overline{a}_{j,k}$ *denote the coefficients of the lacunary spline function* $S_\Delta(x)$ *and* $\overline{S}_\Delta(x)$ *of type* (0;2;3) *constructed in* (3.1.9–3.1.18) *with the exact values* $y^{(j)}(x_k) = y_k^{(j)}$ *and the approximate values* $\overline{y}_k^{(j)}$, *respectively.*

*Then we have*

$$\left|a_{j,0} - \overline{a}_{j,0}\right| \leqq B_{j,0} h^{m+r-j} \tau\left(y^{(m+r)}, h\right) \quad (j = 4, 5, 6),$$

$$\left|a_{1,k} - \overline{a}_{1,k}\right| \leqq B_{1,k} h^{r-1} \tau\left(y^{(m+r)}, h\right) \quad (k = 1, 2, \ldots, n-2),$$

$$\left|a_{j,k} - \overline{a}_{j,k}\right| \leqq B_{j,k} h^{r+2-j} \tau\left(y^{(m+r)}, h\right) \quad (j = 4, 5;\ k = 1, 2, \ldots, n-2),$$

$$\left|a_{j,n-1} - \overline{a}_{j,n-1}\right| \leqq B_{j,n-1} h^{r-j} \tau\left(y^{(m+r)}, h\right) \quad (j = 1, 4, 5, 6),$$

*where the constants* $B_{j,k}$ *are independent of* $n$.

LEMMA 3.2.3. *Let* $S_\Delta(x)$ *and* $\overline{S}_\Delta(x)$ *denote the lacunary spline function of type* (0;2;3) *constructed with the exact values* $y^{(j)}(x_k) = y_k^{(j)}$ *(Theorem 2.1 in* [8]*) and the approximate values* $\overline{y}_k^{(j)}$ *(see Theorem 3.1.1). Then we have* $(2 \leqq m \leqq 5)$

$$\left|S_0^{(q)}(x) - \overline{S}_0^{(q)}(x)\right| \leqq K_{0,q} h^{m+r-q} \tau\left(y^{(m+r)}, h\right), \quad \textit{if} \quad x_0 \leqq x \leqq x_1$$

$$(q = 0, 1, \ldots, 6),$$

$$\left|S_k^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq K_{k,q} h^{r-q} \tau\left(y^{(m+r)}, h\right), \quad \text{if} \quad x_k \leqq x \leqq x_{k+1}$$

$$(q = 0, 1;\ k = 1, 2, \ldots, n-2),$$

$$\left|S_k^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq K_{k,q} h^{r+2-q} \tau\left(y^{(m+r)}, h\right), \quad \text{if} \quad x_k \leqq x \leqq x_{k+1}$$

$$(q = 2, 3, 4, 5;\ k = 1, 2, \ldots, n-2),$$

$$\left|S_{n-1}^{(q)}(x) - \overline{S}_{n-1}^{(q)}(x)\right| \leqq K_{n-1,q} h^{r-q} \tau\left(y^{(m+r)}, h\right), \quad \text{if} \quad x_{n-1} \leqq x \leqq x_n$$

$$(q = 0, 1, \ldots, 6),$$

*where the constants $K_{k,q}$ are independent of $n$.*

PROOF OF THEOREM 3.2.1. Lemma 3.2.3, Theorem 3.1.2 of [9], Theorem 3.1 of [8] and the inequality

$$\left|y^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq \left|y^{(q)}(x) - S_k^{(q)}(x)\right| + \left|S_k^{(q)}(x) - \overline{S}_k^{(q)}(x)\right|$$

$$(q = 0, 1, \ldots, m+r)$$

imply Theorem 3.2.1.

PROOF OF THEOREM 3.2.2. Using condition (1), (1.1) and (1.2) we obtain

$$\left|y^{(m)}(x) - \overline{S}_\Delta^*(x)\right| = \left|f\left[x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right] - \right.$$

$$\left. - f\left[x, \overline{S}_\Delta(x), \overline{S}'_\Delta(x), \ldots, \overline{S}_\Delta^{(m-1)}(x)\right]\right| \leqq L \sum_{q=0}^{m-1} \left|y^{(q)}(x) - \overline{S}_\Delta^{(q)}(x)\right|$$

which implies Theorem 3.2.2 by the help of Theorem 3.2.1.

PROOF OF THEOREM 3.2.3. Theorems 3.2.1, 3.2.2 and the inequality

$$\left|\overline{S}_\Delta^{(m)}(x) - \overline{S}_\Delta^*(x)\right| \leqq \left|\overline{S}_\Delta^{(m)}(x) - y^{(m)}(x)\right| + \left|y^{(m)}(x) - \overline{S}_\Delta^*(x)\right|$$

imply Theorem 3.2.3.

REMARK 1. If $f^{(r)}$ has finite variation then $\tau\left(f^{(r)}, h\right) = O(h)$ (see [10]). Using this equation we get the following versions of Theorems 3.2.1 through 3.2.3:

THEOREM 3.2.1.a. *We have*

$$\left|y^{(q)}(x)-\overline{S}_0^{(q)}(x)\right| \leqq M_{0,q}^* h^{m+r+1-q}$$
$$(q=0,1,\ldots,m+r), \quad \textit{if} \quad x_0 \leqq x \leqq x_1,$$
$$\left|y^{(q)}(x)-\overline{S}_k^{(q)}(x)\right| \leqq M_{k,q}^* h^{r+1-q}$$
$$(q=0,1), \quad \textit{if} \quad x_k \leqq x \leqq x_{k+1} \quad (k=1,2,\ldots,n-2),$$
$$\left|y^{(q)}(x)-\overline{S}_k^{(q)}(x)\right| \leqq M_{k,q}^* h^{r+3-q}$$
$$(q=2,3,\ldots,m+r) \quad \textit{if} \quad x_k \leqq x \leqq x_{k+1} \quad (k=1,2,\ldots,n-2),$$
$$\left|y^{(q)}(x)-\overline{S}_{n-1}^{(q)}(x)\right| \leqq M_{n-1,q}^* h^{r+1-q}$$
$$(q=0,1,\ldots,m+r), \quad \textit{if} \quad x_{n-1} \leqq x \leqq x_n,$$

*where the constants* $M_{k,q}^*$ *are independent of* $n$.

THEOREM 3.2.2.a. *We have*

$$\left|y^{(m)}(x)-\overline{S}_0^*(x)\right| \leqq D_{0,m}^* h^{r+2}, \quad \textit{if} \quad x_0 \leqq x \leqq x_1,$$

$$\left|y^{(m)}(x)-\overline{S}_k^*(x)\right| \leqq D_{k,m}^* h^{r}, \quad \textit{if} \quad x_k \leqq x \leqq x_{k+1} \quad (k=1,2,\ldots,n-2),$$

$$\left|y^{(m)}(x)-\overline{S}_{n-1}^*(x)\right| \leqq D_{n-1,m}^* h^{r+2-m}, \quad \textit{if} \quad x_{n-1} \leqq x \leqq x_n,$$

*where the constants* $D_{k,m}^*$ *are independent of* $n$.

THEOREM 3.2.3.a. *We have*

$$\left|\overline{S}_0^{(m)}(x)-\overline{S}_0^*(x)\right| \leqq E_{0,m}^* h^{r+1}, \quad \textit{if} \quad \leqq x \leqq x_1,$$

$$\left|\overline{S}_k^{(m)}(x)-\overline{S}_k^*(x)\right| \leqq E_{k,m}^* h^{s+1}, \quad \textit{if } x_k \leqq x \leqq x_{k+1} \quad (k=1,2,\ldots,n-2),$$

*where* $s=\min(r+2-m,\ r-1)$,

$$\left|\overline{S}_{n-1}^{(m)}(x)-\overline{S}_{n-1}^*(x)\right| \leqq E_{n-1,m}^* h^{r+1-m}, \quad \textit{if} \quad x_{n-1} \leqq x \leqq x_n,$$

*where the constants* $E_{k,m}^*$ *are independent of* $n$.

REMARK 2. Theorems 3.2.1 through 3.2.3 show that modified lacunary spline functions of type (0;2;3) give better approximation than the Hermite spline function of type (0;1;2) discussed in [7].

**3.3.** *A numerical example.* In this section we have tested our method numerically by the example of [11] and [9]:

$$y'' = 4y + 4\,\mathrm{ch}1, \quad y(0) = 0; \quad y'(0) = -2\,\mathrm{sh}1.$$

The exact solution is

$$y(x) = \mathrm{ch}\,(2x-1) - \mathrm{ch}\,1$$

which is symmetric at $x = 0.5$.

We have made the calculation at $r = 3$, $n = 9$ (see in [11]) and $n = 16$ (see in [9]). We have given the result by $x = 0.1 \cdot k$ $(k = 0, 1, 2, \dots, 10)$.

$n = 9$

| $x$ | $y(x)$<br>$\overline{S}_\Delta(x)$ | $y^{(1)}(x)$<br>$\overline{S}_\Delta^{(1)}(x)$ | $y^{(2)}x$<br>$\overline{S}_\Delta^{(2)}(x)$ | $y^{(3)}x$<br>$\overline{S}_\Delta^{(3)}(x)$ | $\overline{S}_\Delta^{*}(x)$ |
|---|---|---|---|---|---|
| 0.0 | −0.00000000<br>+0.00000000 | −2.35040239<br>−2.35040239 | +6.17232254<br>+6.17232254 | −9.40160955<br>−9.40160955 | +6.17232254 |
| 0.1 | −0.20564569<br>−0.20564589 | −1.77621196<br>−1.77621646 | +5.34973979<br>+5.34972941 | −7.10484786<br>−7.10330507 | +5.34973898 |
| 0.2 | −0.35761542<br>−0.35761704 | −1.27330716<br>−1.27332379 | +4.74186087<br>+4.74184217 | −5.09322866<br>−5.09244521 | +4.74185439 |
| 0.3 | −0.46200826<br>−0.46201239 | −0.82150465<br>−0.82153350 | +4.32428949<br>+4.32425381 | −3.28601861<br>−3.28545893 | +4.32427297 |
| 0.4 | −0.52301388<br>−0.52302161 | −0.40267201<br>−0.40271301 | +4.08026702<br>+4.08021256 | −1.61068802<br>−1.61048730 | +4.08023609 |
| 0.5 | −0.54308063<br>−0.54309314 | +0.00000000<br>−0.00005406 | +4.00000000<br>+3.99992494 | +0.00000000<br>−0.00021020 | +3.99994997 |
| 0.6 | −0.52301388<br>−0.52303248 | +0.40267201<br>+0.40260302 | +4.08026702<br>+4.08016893 | +1.61068802<br>+1.61005813 | +4.08019261 |
| 0.7 | −0.46200826<br>−0.46203452 | +0.82150465<br>+0.82141784 | +4.32428949<br>+4.32416502 | +3.28601861<br>+3.28500274 | +4.32418446 |
| 0.8 | −0.35761542<br>−0.35765122 | +1.27330716<br>+1.27319848 | +4.74186087<br>+4.74170510 | +5.09322866<br>+5.09194165 | +4.74171767 |
| 0.9 | −0.20564569<br>−0.20569335 | +1.77621196<br>+1.77607699 | +5.34973979<br>+5.34954909 | +7.10484786<br>+7.10428859 | +5.34954914 |
| 1.0 | +0.00000000<br>−0.00006243 | +2.35040239<br>+2.35023330 | +6.17232254<br>+6.17152163 | +9.40160955<br>+9.38095072 | +6.17207282 |

$n = 16$

| $x$ | $y(x)$ <br> $\overline{S}_\Delta(x)$ | $y^{(1)}(x)$ <br> $\overline{S}^{(1)}_\Delta(x)$ | $y^{(2)}x$ <br> $\overline{S}^{(2)}_\Delta(x)$ | $y^{(3)}x$ <br> $\overline{S}^{(3)}_\Delta(x)$ | $\overline{S}^{*}_\Delta(x)$ |
|---|---|---|---|---|---|
| 0.0 | −0.00000000 <br> +0.00000000 | −2.35040239 <br> −2.35040239 | +6.17232254 <br> +6.17232254 | −9.40160955 <br> −9.40160955 | +6.17232254 |
| 0.1 | −0.20564569 <br> −0.20564573 | −1.77621196 <br> −1.77621288 | +5.34973979 <br> +5.34973649 | −7.10484786 <br> −7.10476629 | +5.34973962 |
| 0.2 | −0.35761542 <br> −0.35761561 | −1.27330716 <br> −1.27330939 | +4.74186087 <br> +4.74185889 | −5.09322866 <br> −5.09338271 | +4.74186010 |
| 0.3 | −0.46200826 <br> −0.46200872 | −0.82150465 <br> −0.82150769 | +4.32428949 <br> +4.32428654 | −3.28601861 <br> −3.28589435 | +4.32428767 |
| 0.4 | −0.52301388 <br> −0.52301471 | −0.40267201 <br> −0.40267634 | +4.08026702 <br> +4.08026133 | −1.61068802 <br> −1.61076838 | +4.08026371 |
| 0.5 | −0.54308063 <br> −0.54308195 | +0.00000000 <br> −0.00000576 | +4.00000000 <br> +3.99999473 | +0.00000000 <br> −0.00002288 | +3.99999473 |
| 0.6 | −0.52301388 <br> −0.52301582 | +0.40267201 <br> +0.40266503 | +4.08026702 <br> +4.08025687 | +1.61068802 <br> +1.61072373 | +4.08025924 |
| 0.7 | −0.46200826 <br> −0.46201099 | +0.82150465 <br> +0.82149575 | +4.32428949 <br> +4.32427743 | +3.28601861 <br> +3.28584653 | +4.32427856 |
| 0.8 | −0.35761542 <br> −0.35761913 | +1.27330716 <br> +1.27329635 | +4.74186087 <br> +4.74184482 | +5.09322866 <br> +5.09333049 | +4.74184602 |
| 0.9 | −0.20564569 <br> −0.20565063 | +1.77621196 <br> +1.77619828 | +5.34973979 <br> +5.34971688 | +7.10484786 <br> +7.10470868 | +5.34972003 |
| 1.0 | +0.00000000 <br> −0.00000646 | +2.35040239 <br> +2.35038513 | +6.17232254 <br> +6.17223840 | +9.40160955 <br> +9.39779356 | +6.17229669 |

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# THE CONGRUENCE LATTICE OF AN EXTENSION OF COMPLETELY 0-SIMPLE SEMIGROUPS

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## 1. Introduction and summary

Much has been said about congruences on a completely 0-simple semigroup $S$. If $S$ is represented, indeed identified, with a Rees matrix semigroup, proper congruences on $S$ can be themselves represented by admissible triples which involve all the ingredients of a Rees matrix semigroup. Manipulation of these triples makes it possible to obtain almost any information we may desire about the congruences on $S$ or the congruence lattice $\mathcal{C}(S)$ of $S$.

All of this changes radically if we consider congruences on an (ideal) extension $S$ of a completely 0-simple semigroup $S_0$ by a completely 0-simple semigroup $S_1$; this is especially true about the congruence lattice $\mathcal{C}(S)$. Proper congruences $\rho$ on $S$ are of two kinds: $S/\rho$ is either completely 0-simple or an (ideal) extension of completely 0-simple semigroups. In studying the structure of $\mathcal{C}(S)$, one may consider the partitions of $\mathcal{C}(S)$ induced by restrictions of congruences to $S_0$ or to $S_1^*$. Another approach consists in considering the kernel relation $K$ and the trace relation $T$ on $\mathcal{C}(S)$. In this context, it is of interest to find necessary and sufficient conditions on $S$ in order that $K$ be a congruence. If this is not feasible, we may consider certain restrictions on $S$ and then attempt to answer such a question. Our purpose here is to consider each one of these subjects.

In Section 2 we prepare the notation, terminology and some preliminary results which will be used extensively throughout the paper. Congruences on $S$ in the preceding paragraph are constructed in Section 3 in a rather transparent form; certain simple properties of these are also proved in this section to be used repeatedly. Meets and joins of all types of congruences on $S$ are described in Section 4. In Section 5, the relation on $\mathcal{C}(S)$ induced by the restrictions of congruences to $S_0$ and $S_1^*$, respectively, and their classes are described in some detail. The kernel and the trace relations, $K$ and $T$, are then studied in Section 6 with the particular attention to $K$ and the conditions ensuring that it be a congruence. The case when $S$ is a strict extension of $S_0$ by $S_1$ is considered in Section 7, in particular necessary and sufficient conditions on $S$ are found for $K$ to be a congruence when

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the extension is strict or $S_0$ has no contractions, as well as certain ends of intervals which make up the $K$- or $T$-classes. In the final Section 8, necessary and sufficient conditions for $K$ to be a congruence when $S_1$ has no zero divisors are found.

We are unable to find necessary and sufficient conditions on $S$ in order that $K$ be a congruence, but can do this in three special cases: with a condition on $S_0$ (no contractions), with a condition on $S_1$ (no zero divisors) and with a condition on the kind of extension (strict).

## 2. Preliminaries

We follow the standard notation and terminology which can be found in the books [1] and [5]. In addition, or for emphasis, we shall adhere to the following notation, nomenclature and convention.

The equality and the universal relations on any set $X$ are denoted by $\varepsilon$ and $\omega$, respectively. The restriction of a function or a relation $\theta$ to a set $X$ is denoted by $\theta|_X$. If $\theta$ is an equivalence relation on a set $X$ and $x \in X$, then $x\theta$ denotes the $\theta$-class containing $x$; $\theta$ is *proper* if $\theta \neq \omega$; if $A \subseteqq X$ is a union of $\theta$-classes, then $\theta$ *saturates* $A$. If $A$ and $B$ are sets, then $A \setminus B = \{a \in A | a \notin B\}$.

If $\alpha$ and $\beta$ are elements of a lattice $L$ such that $\alpha \leqq \beta$, then $[\alpha, \beta]$ denotes the interval in $L$ with lower end $\alpha$ and upper end $\beta$.

Let $S$ be a semigroup. Its set of idempotents is denoted by $E(S)$. If $S$ has an identity, then $S^1 = S$ otherwise $S^1$ stands for $S$ with an identity adjoined. If $S$ has a zero and $A \subseteqq S$, then $A^* = A \setminus \{0\}$. The congruence lattice of $S$ is denoted by $\mathcal{C}(S)$. If $S$ has a zero, $\mathcal{C}_0(S)$ denotes the set of all congruences on $S$ having $\{0\}$ as a class. For any relation $\theta$ on $S$, $\theta^*$ stands for the congruence on $S$ generated by $\theta$. If $\theta$ is an equivalence relation, $\theta^o$ denotes the greatest congruence on $S$ contained in $\theta$; it is given by

$$a\theta^o b \Leftrightarrow xay \; \theta \; xby \quad \text{for all} \quad x, y \in S^1.$$

Now let $S$ be a regular semigroup. The *natural partial order* on $S$ is given by

$$a \leqq b \Leftrightarrow a = eb = bf \quad \text{for some} \quad e, f \in E(S),$$

Restricted to $E(S)$, it has the form

$$e \leqq f \Leftrightarrow e = ef = fe.$$

For $\rho \in \mathcal{C}(S)$,

$$\ker \rho = \{a \in S | a\rho e \text{ for some } e \in E(S)\}, \quad \operatorname{tr} \rho = \rho|_{E(S)}$$

are the *kernel* and the *trace* of $\rho$, respectively. The relations $K$ and $T$ are defined on $\mathcal{C}(S)$ by

$$\lambda K \rho \Leftrightarrow \ker\lambda = \ker\rho, \quad \lambda T \rho \Leftrightarrow \operatorname{tr}\lambda = \operatorname{tr}\rho.$$

Then $K$ is a complete $\wedge$-congruence and $T$ is a complete congruence. Their classes are intervals, in the notation

$$\rho K = \left[\rho_K, \rho^K\right], \quad \rho T = \left[\rho_T, \rho^T\right].$$

Next let $S$ be a completely 0-simple semigroup. We may consider its Rees matrix representation and in fact set $S = \mathcal{M}^\circ(I, G, \Lambda; P)$. A proper congruence $\rho$ on $S$ is represented by an admissible triple $(r, N, \pi)$ so that

$$(i, g, \lambda)\rho(j, h, \mu) \Leftrightarrow irj, \quad p_{\theta i} g p_{\lambda k} N = p_{\theta j} h p_{\mu k} N, \quad \lambda \pi \mu$$

for some $\theta \in \Lambda$ and $k \in I$ such that $p_{\theta i} \neq 0$, $p_{\lambda k} \neq 0$. If $\rho \subseteqq \mathcal{H}$, this simplifies to

$$(i, g, \lambda)\rho(j, h, \mu) \Leftrightarrow i = j, \quad gh^{-1} \in N, \ \lambda = \mu.$$

For a complete discussion of this subject, we refer to ([1], III.4). In particular, define $r$ and $\pi$ by:

$$irj \Leftrightarrow \left(p_{\lambda i} \neq 0 \Leftrightarrow p_{\lambda j} \neq 0 \quad \text{for all} \quad \lambda \in \Lambda\right),$$

$$\lambda \pi \mu \Leftrightarrow \left(p_{\lambda i} \neq 0 \Leftrightarrow p_{\mu i} \neq 0 \quad \text{for all} \quad i \in I\right).$$

Then the triple $(r, G, \pi)$ is easily seen to be admissible and the corresponding congruence $\zeta$ to be the greatest proper congruence on $S$. In fact, $\zeta$ is the principal congruence on $S$ relative to the set $\{0\}$.

Throughout the whole paper we fix the following notation. *$S$ stands for an (ideal) extension of a completely 0-simple semigroup $S_0$ by a completely 0-simple semigroup $S_1$ such that $S_0 S_1 \neq \{0\}$.* The equality and the universal relations on $S_i$ are denoted by $\varepsilon_i$ and $\omega_i$, respectively, for $i = 0, 1$. This extension is *strict* (or retract) if for every $a \in S_1^*$ there exists $a' \in S_0$ such that $ax = a'x$ and $xa = xa'$ for all $x \in S_0$. For a complete discussion of this subject we refer to ([5], Chapter III).

The reason for limiting ourselves to the case $S_0 S_1 \neq \{0\}$ is that in the contrary case, the semigroup $S$ is an orthogonal sum of $S_0$ and $S_1$. The study of $\mathcal{C}(S)$ in this case is routine and is omitted. In particular $K$ is a congruence in this case ([7], Theorem 3.6).

NOTATION 2.1. Let $\kappa$ be the congruence on $S$ generated by the relation

$$\left\{(e, f) \in E(S) \times E(S) | e > f > 0\right\},$$

and let $\kappa_0 = \kappa|_{S_0}$. Also let $\zeta_1$ be the greatest proper congruence on $S_1$.

We shall see in Lemma 3.6(i) that either $\kappa = \omega$ or $\kappa$ is the least completely 0-simple congruence on $S$. We shall need the following auxiliary results.

LEMMA 2.2. *For every $a \in S_1^*$, there exists $b \in S_0^*$ such that $a > b$.*

PROOF. Let

$$I = \{a \in S_1^* | S_0 a S_0 = \{0\}\} \cup \{0\}.$$

Then $I$ is an ideal of $S_1$ so either $I = \{0\}$ or $I = S_1$. The second alternative would contradict the hypothesis that $S_0 S_1 \neq \{0\}$. Hence $I = \{0\}$ so for every $a \in S_1^*$, we have $S_0 a S_0 \neq \{0\}$, and thus $ua, av \neq 0$ for some $u, v \in S_0^*$. Let $s \in S_0$ be such that $avsua \neq 0$ and let $p$ be an inverse of $avsua$. For $e = avsuap$, $f = vsuapa$ and $b = avsuapa$, we obtain

$$\begin{aligned} e^2 &= avsua(pavsuap) = avsuap = e, \\ f^2 &= vsua(pavsuap)a = vsuap = f, \\ b &= ea = af \neq 0 \end{aligned}$$

so that $a > b$ with $b \in S_0^*$.

COROLLARY 2.3. *For every $e \in E(S_1^*)$, there exists $f \in E(S_0^*)$ such that $e > f$.*

PROOF. By Lemma 2.2, there exists $b \in S_0^*$ such that $e > b$. Hence $b = fe = eg$ for some $f, g \in E(S)$ so that

$$b^2 = (fe)(eg) = (fe)g = bg = b \in E(S_0^*).$$

LEMMA 2.4 ([5], Chapter III). *The extension $S$ of $S_0$ by $S_1$ is strict if and only if for any $e, f, g \in E(S)$, $e > f > 0$ and $e > g > 0$ implies $f = g$.*

LEMMA 2.5 ([7], Theorem 3.6). *The relation $K$ is a congruence for a completely 0-simple semigroup.*

## 3. Congruences

We extract here a description of congruences on $S$ from [3] in terms of congruences on $S_0$ and $S_1$ as follows.

NOTATION 3.1. Let $\rho_0 \in \mathcal{C}(S_0)$ be such that for every $a \in S_1^*$ there exists $a' \in S_0$ with the property that

(1) $$a x \rho_0 a' x, \ x a \rho_0 x a' \quad (x \in S_0).$$

In such a case we say that $a$ and $a'$ are $\rho_0$-*linked*. On $S$ define a relation $[\rho_0]$ by

$$a[\rho_0]b \Leftrightarrow \begin{cases} a\rho_0 b & \text{if } a,b \in S_0, \\ a\rho_0 b' & \text{if } a \in S_0,\ b \in S_1^* \\ a'\rho_0 b & \text{if } a \in S_1^*,\ b \in S_0, \\ a'\rho_0 b' & \text{if } a,b \in S_1^* \end{cases} \tag{2}$$

for some [all] $a'$ $\rho_0$-linked to $a$ and $b'$ $\rho_0$-linked to $b$.

Let $\rho_0 \in \mathcal{C}(S_0)$ and $\rho_1 \in \mathcal{C}_0(S_1)$ be such that

$$a,b \in S_1^*,\ a\rho_1 b,\ x\rho_0 y \Rightarrow ax\rho_0 by,\ xa\rho_0 yb.$$

On $S$ define a relation $[\rho_0,\rho_1]$ by

$$a[\rho_0,\rho_1]b \Leftrightarrow \begin{cases} a\rho_0 b & \text{if } a,b \in S_0, \\ a\rho_1 b & \text{if } a,b \in S_1^*. \end{cases}$$

The alternative "for all $a'$" to "for some $a'$" in Notation 3.1 is justified by noticing that if $a'$ and $a''$ are $\rho_0$-linked to $a$, then $a'x\rho_0 a''x$ and $xa'\rho_0 xa''$ for all $x \in S_0$ and thus $a'\rho_0 a''$ by weak reductivity of $S_0/\rho_0$.

All our discussion of the congruence lattice $\mathcal{C}(S)$ is based upon the following representation of congruences on $S$.

THEOREM 3.2. *The relations $[\rho_0]$ and $[\rho_0,\rho_1]$ are congruences on $S$. Conversely, every congruence on $S$ can be so represented.*

This representation is evidently unique. Clearly $\varepsilon = [\varepsilon_0,\varepsilon_1]$, $\omega = [\omega_0]$ and $[\omega_0,\varepsilon_1]$ is the Rees congruence relative to the ideal $S_0$. The congruences $[\rho_0,\rho_1]$ are precisely those which saturate $S_0$. For $\rho \in \mathcal{C}(S)$, we have that $S/\rho$ is completely 0-simple if and only if either $\rho = [\rho_0]$ for some $\rho_0 \in \mathcal{C}(S_0)$ such that $\rho_0 \neq \omega_0$ or $\rho = [\omega_0,\rho_1]$ for some $\rho_1 \in \mathcal{C}_0(S_1)$. For the remaining congruences $\rho = [\rho_0,\rho_1]$, we have that $S/\rho$ is an extension of $S_0/\rho_0$ by $S_1/\rho_1$, both of which are completely 0-simple. We now give the inclusion relation for these congruences.

LEMMA 3.3. (i) *Let $[\lambda_0],[\rho_0] \in \mathcal{C}(S)$. Then $[\lambda_0] \subseteqq [\rho_0]$ if and only if $\lambda_0 \subseteqq \rho_0$.*

(ii) *Let $[\lambda_0,\lambda_1],[\rho_0,\rho_1] \in \mathcal{C}(S)$. Then $[\lambda_0,\lambda_1] \subseteqq [\rho_0,\rho_1]$ if and only if $\lambda_0 \subseteqq \rho_0$ and $\lambda_1 \subseteqq \rho_1$.*

(iii) *Let $[\lambda_0,\lambda_1],[\rho_0] \in \mathcal{C}(S)$. Then $[\lambda_0,\lambda_1] \subset [\rho_0]$ if and only if $\lambda_0 \subseteqq \rho_0$.*

(iv) *Let $[\lambda_0] \in \mathcal{C}(S)$, $\rho_0 \in \mathcal{C}(S_0)$ and $\lambda_0 \subseteqq \rho_0$. Then $[\rho_0] \in \mathcal{C}(S)$.*

PROOF. (i) Note that $[\lambda_0]|_{S_0} = \lambda_0$. Hence if $[\lambda_0] \subseteqq [\rho_0]$, then $\lambda_0 \subseteqq \rho_0$. Conversely, assume that $\lambda_0 \subseteqq \rho_0$ and let $a[\lambda_0]b$. For $a,b \in S_1^*$, let $a$, $a'$ and

$b$, $b'$ be $\lambda_0$-linked. Then $a'\lambda_0 b'$. Since $\lambda_0 \subseteqq \rho_0$, we have that $a$, $a'$ and $b$, $b'$ are $\rho_0$-linked; in addition, $a'\rho_0 b'$. Consequently $a[\rho_0]b$. The other three cases require similar arguments. Therefore $[\lambda_0] \subseteqq [\rho_0]$.

(ii) This follows directly from $[\rho_0, \rho_1]|_{S_0} = \rho_0$ and $[\rho_0, \rho_1]|_{S_1^*} = \rho_1|_{S_1^*}$.

(iii) Note that $[\lambda_0, \lambda_1]|_{S_0} = \lambda_0$ and $[\rho_0]|_{S_0} = \rho_0$. Hence if $[\lambda_0, \lambda_1] \subset \subset [\rho_0]$, then $\lambda_0 \subseteqq \rho_0$. Conversely, assume that $\lambda_0 \subseteqq \rho_0$ and let $a[\lambda_0, \lambda_1]b$. By the remarks made, if $a, b \in S_0$, then $a[\rho_0]b$. Let $a, b \in S_1^*$ and $x \in S_0$. Then $ax\lambda_0 bx$ and $xa\lambda_0 xb$ so that $ax\rho_0 bx$ and $xa\rho_0 xb$. Let $a$, $a'$ and $b$, $b'$ be $\rho_0$-linked. Then $ax\rho_0 a'x$, $xa\rho_0 xa'$, $bx\rho_0 b'x$ and $xb\rho_0 xb'$ which implies that $a'x\rho_0 b'x$ and $xa'\rho_0 xb'$. Since this holds for all $x \in S_0$ and $S_0/\rho_0$ is weakly reductive, we conclude that $a'\rho_0 b'$. Therefore $a[\rho_0]b$ which completes the proof that $[\lambda_0, \lambda_1] \subseteqq [\rho_0]$. However $[\lambda_0, \lambda_1] \neq [\rho_0]$ since the former saturates $S_0$ and the latter does not.

(iv) Indeed, if $a'$ is $\lambda_0$-linked to $a$, then also $a'$ is $\rho_0$-linked to $a$.

The next lemma will come in very handy.

LEMMA 3.4. *Let $[\rho_0] \in \mathcal{C}(S)$ and $a > b > 0$ in $S$. Then $a[\rho_0]b$.*

PROOF. By hypothesis, $b = ea = af$ for some $e, f \in E(S)$. The conclusion holds trivially if $\rho_0 = \omega_0$, so we may assume that $\rho_0 \neq \omega_0$. Then $S/[\rho_0]$ is completely 0-simple and we may let $S/[\rho_0] = \mathcal{M}^\circ(I, G, \Lambda; P)$. Then

$$b[\rho_0] = (i, g, \lambda), \quad e[\rho_0] = (i, p_{\mu i}^{-1}, \mu),$$
$$f[\rho_0] = (j, p_{\lambda j}^{-1}, \lambda), \quad a[\rho_0] = (i, h, \lambda)$$

with

$$(i, g, \lambda) = (i, p_{\mu i}^{-1}, \mu)(i, h, \lambda) = (i, h, \lambda)(j, p_{\lambda j}^{-1}, \lambda)$$

whence $g = h$. Therefore $a[\rho_0] = b[\rho_0]$, that is $a[\rho_0]b$.

Based on this lemma, we can give alternative expressions to those in (1) and (2) for definability and the form of $[\rho_0]$ in terms of the natural partial order which will prove often more convenient than the original definition.

LEMMA 3.5. *Let $\rho_0 \in \mathcal{C}(S_0)$.*

(i) *$[\rho_0] \in \mathcal{C}(S)$ if and only if for every $a \in S_1^*$ and some [all] $b \in S_0^*$ such that $a > b$, we have $ax\rho_0 bx$ and $xa\rho_0 xb$ for all $x \in S_0$.*

(ii) *Assume that $[\rho_0] \in \mathcal{C}(S)$. If $a, b \in S_1^*$, then*

$$a[\rho_0]b \Leftrightarrow \bar{a}\rho_0\bar{b} \text{ for some [all] } \bar{a}, \bar{b} \text{ such that } a > \bar{a} > 0,\ b > \bar{b} > 0.$$

*If $a \in S_1^*$ and $b \in S_0$, then*

$$a[\rho_0]b \Leftrightarrow \bar{a}\rho_0 b \text{ for some [all] } \bar{a} \text{ such that } a > \bar{a} > 0.$$

PROOF. (i) Assume that $[\rho_0] \in \mathcal{C}(S)$ and let $a > b > 0$, $x \in S_0$. By Lemma 3.4, we have $a[\rho_0]b$ whence $ax[\rho_0]bx$ so that $ax\rho_0 bx$ and similarly $xa\rho_0 xb$. The converse is obvious.

(ii) Let $a, b \in S_1^*$. Assume that $a[\rho_0]b$ and let $a > \overline{a} > 0$ and $b > \overline{b} > > 0$. By Lemma 3.4, we have $a[\rho_0]\overline{a}$ and $b[\rho_0]\overline{b}$. Thus $\overline{a}[\rho_0]\overline{b}$ so that $\overline{a}\rho_0\overline{b}$. Conversely, assume that $\overline{a}\rho_0\overline{b}$ where $a > \overline{a} > 0$ and $b > \overline{b} > 0$. By Lemma 3.4, we have $a[\rho_0]\overline{a}$ and $b[\rho_0]\overline{b}$. Hence, for all $x \in S_0$, we get $ax[\rho_0]\overline{a}x$, $xa[\rho_0]x\overline{a}$, $bx[\rho_0]\overline{b}x$ and $xb[\rho_0]x\overline{b}$. But then $ax\rho_0\overline{a}x$, $xa\rho_o x\overline{a}$, $bx\rho_0\overline{b}x$ and $xb\rho_0 x\overline{b}$, that is $a, \overline{a}$ and $b, \overline{b}$ are $\rho_0$-linked. Therefore $a[\rho_0]b$.

The argument for the case $a \in S_1^*$ and $b \in S_0$ runs along the same lines and is omitted.

We collect some of the properties of $\kappa$ and $\kappa_0$, introduced in Notation 2.1, in the next simple result.

LEMMA 3.6. (i) *$\kappa$ is the least congruence on $S$ of the form $[\rho_0]$.*

(ii) *Let $\rho_0 \in \mathcal{C}(S_0)$. Then $[\rho_0] \in \mathcal{C}(S)$ if and only if $\kappa_0 \subseteqq \rho_0$.*

(iii) *Let $e, f, g \in E(S)$ be such that $e > f > 0$, $e > g > 0$ and $fg = 0$. Then $\kappa = \omega$.*

(iv) *Let $S_0$ be a Brandt semigroup with at least two nonzero idempotents. If $S$ is a strict extension of $S_0$, then $\kappa = [\varepsilon]$; otherwise $\kappa = \omega$.*

PROOF. (i) By Corollary 2.3, for every $e \in E(S_1^*)$, there exists $f \in \in E(S_0^*)$ such that $e > f$. Hence $\kappa$ does not saturate $S_0$ and must be of the form $[\rho_0]$, in fact $\kappa = [\kappa_0]$. Let $[\rho_0] \in \mathcal{C}(S)$ and $e, f \in E(S)$ be such that $e > f > 0$. Then by Lemma 3.4, we have $e[\rho_0]f$. This proves that $\kappa^* \subseteqq [\rho_0]$, as required.

(ii) If $[\rho_0] \in \mathcal{C}(S)$, then $\kappa \subseteqq [\rho_0]$ by part (i) and thus $\kappa_0 \subseteqq \rho_0$. Conversely, if $\kappa_0 \subseteqq \rho_0$, then $[\kappa_0] \in \mathcal{C}(S)$ together with Lemma 3.3(iv) implies that $[\rho_0] \in \in \mathcal{C}(S)$.

(iii) Lemma 3.4 implies that $e\kappa f$ and $e\kappa g$ so that $f\kappa g$. Multiplying this on the right by $g$ gives $0\kappa g$. Therefore $\kappa = \omega$.

(iv) If $S$ is a strict extension, then for every $a \in S_1^*$ there exists $a' \in S_0^*$ such that $ax = a'x$ and $xa = xa'$ for all $x \in S_0$ which shows that $[\varepsilon_0] \in \mathcal{C}(S)$ and thus $\kappa = [\varepsilon_0]$ by part (i). If $S$ is not a strict extension of $S_0$, then by Lemma 2.4 there exist $e, f, g \in E(S)$ such that $e > f > 0$, $e > g > 0$ and $f \neq g$ so that $fg = 0$ and part (iii) gives that $\kappa = \omega$.

## 4. Meets and joins

We shall represent meets and joins of the congruences on $S$ in our representation.

LEMMA 4.1. *Let* $[\lambda_0], [\rho_0] \in \mathcal{C}(S)$.
(i) $[\lambda_0] \wedge [\rho_0] = [\lambda_0 \wedge \rho_0]$.
(ii) $[\lambda_0] \vee [\rho_0] = [\lambda_0 \vee \rho_0]$.

PROOF. (i) By Lemma 3.6(ii), $[\lambda_0 \wedge \rho_0] \in \mathcal{C}(S)$. For $a > b > 0$ in $S$, by Lemma 3.4, we have $a[\lambda_0] \wedge [\rho_0]b$ and hence $[\lambda_0] \wedge [\rho_0]$ is of the form $[\theta_0]$. Clearly $\left([\lambda_0] \wedge [\rho_0]\right)\big|_{S_0} = \lambda_0 \wedge \rho_0$ whence, by the uniqueness of the representation, $[\lambda_0] \wedge [\rho_0] = [\lambda_0 \wedge \rho_0]$.

(ii) By Lemma 3.6(ii), $[\lambda_0 \vee \rho_0] \in \mathcal{C}(S)$. Since $\lambda_0, \rho_0 \subseteqq \lambda_0 \vee \rho_0$, by Lemma 3.3(i), we have $[\lambda_0], [\rho_0] \subseteqq [\lambda_0 \vee \rho_0]$ so that $[\lambda_0] \vee [\rho_0] \subseteqq [\lambda_0 \vee \rho_0]$. Also $[\lambda_0] \vee [\rho_0]$ does not saturate $S_0$ and hence is of the form $[\theta_0]$. Thus $[\theta_0] \subseteqq [\lambda_0 \vee \rho_0]$. For the opposite inclusion, in view of Lemma 3.3(i), it suffices to consider their restrictions to $S_0$. Indeed, let $a\lambda_0 \vee \rho_0 b$. There exists a sequence

$$a\lambda_0 c_1 \rho_0 c_2 \ldots c_n \rho_0 b$$

with $c_1, c_2, \ldots, c_n \in S_0$, so that

$$a[\lambda_0]c_1[\rho_0]c_2 \ldots c_n[\rho_0]b$$

and thus $a[\lambda_0] \vee [\rho_0]b$. Therefore $[\lambda_0 \vee \rho_0] \subseteqq [\lambda_0] \vee [\rho_0]$ and equality prevails.

LEMMA 4.2. *Let* $[\lambda_0, \lambda_1], [\rho_0, \rho_1] \in \mathcal{C}(S)$.
(i) $[\lambda_0, \lambda_1] \wedge [\rho_0, \rho_1] = [\lambda_0 \wedge \rho_0, \lambda_1 \wedge \rho_1]$.
(ii) $[\lambda_0, \lambda_1] \vee [\rho_0, \rho_1] = [\lambda_0 \vee \rho_0, \lambda_1 \vee \rho_1]$.

PROOF. (i) A simple argument shows that $[\lambda_0 \wedge \rho_0, \lambda_1 \wedge \rho_1] \in \mathcal{C}(S)$. The required equality follows easily from the definitions.

(ii) Let $a, b \in S_1^*$ and $x, y \in S_0$ be such that $a\lambda_1 \vee \rho_1 b$ and $x\lambda_0 \vee \rho_0 y$. There exist sequences

$$a\lambda_1 c_1 \rho_1 c_2 \ldots c_m \rho_1 b, \quad c_1, c_2, \ldots, c_m \in S_1^*, \tag{3}$$

$$x\lambda_0 z_1 \rho_0 z_2 \ldots z_n \rho_0 y, \quad z_1, z_2, \ldots, z_n \in S_0. \tag{4}$$

By possibly repeating parts of the above sequences, we may assume that $m = n$. It follows that

$$ax\lambda_1 c_1 z_1 \rho_0 c_2 z_2 \ldots c_n z_n \rho_0 by$$

and thus $ax\lambda_0 \vee \rho_0 by$ and similarly $xa\lambda_0 \vee \rho_0 yb$. Therefore

$$[\lambda_0 \vee \rho_0, \lambda_1 \vee \rho_1] \in \mathcal{C}(S).$$

The inclusion $[\lambda_0, \lambda_1] \vee [\rho_0, \rho_1] \subseteqq [\lambda_0 \vee \rho_0, \lambda_1 \vee \rho_1]$ follows directly from Lemma 3.3(ii). If $a, b \in S_1^*$ and $a[\lambda_0 \vee \rho_0, \lambda_1 \vee \rho_1]b$, then we have a sequence (3) so that

$$a[\lambda_0, \lambda_1]c_1[\rho_0, \rho_1]c_2 \ldots c_m[\rho_0, \rho_1]b$$

whence $a[\lambda_0, \lambda_1] \vee [\rho_0, \rho_1]b$. If $x, y \in S_0$ and $x[\lambda_0 \vee \rho_0, \lambda_1 \vee \rho_1]y$, then we have a sequence (4) so that

$$x[\lambda_0, \lambda_1]z_1[\rho_0, \rho_1]z_2 \dots z_n[\rho_0, \rho_1]y$$

whence $x[\lambda_0, \lambda_1] \vee [\rho_0, \rho_1]y$. Therefore

$$[\lambda_0 \vee \rho_0, \lambda_1 \vee \rho_1] \subseteqq [\lambda_0, \lambda_1] \vee [\rho_0, \rho_1]$$

and equality prevails.

For the meet of mixed types of congruences, we shall need some preparation.

NOTATION 4.3. For $[\rho_0] \in \mathcal{C}(S)$, define a relation $\rho_0'$ on $S_1$ by

$$a\rho_0'b \Leftrightarrow ax\rho_0by, xa\rho_0yb \quad \text{for all} \quad x\rho_0y$$

if $a, b \in S_1^*$ and $0\rho_0'0$.

Straightforward verification shows that $\rho_0'$ is an equivalence relation on $S_1$. Let $\overline{\rho}_0 = (\rho_0')^o$ be the greatest congruence on $S_1$ contained $\rho_0'$. Recall that $\zeta_1$ denotes the greatest proper congruence on $S_1$.

LEMMA 4.4. *Let $[\rho_0] \in \mathcal{C}(S)$. Then $[\rho_0, \overline{\rho}_0] \in \mathcal{C}(S)$ and is the greatest congruence $\rho$ on $S$ with the properties: $\rho$ saturates $S_0$ and $\rho|_{S_0} = \rho_0$. In addition, $[\rho_0, \overline{\rho}_0] = [\rho_0] \wedge [\omega_0, \zeta_1]$.*

PROOF. First note that $\overline{\rho}_0$ is a proper congruence on $S_1$. That $[\rho_0, \overline{\rho}_0] \in \mathcal{C}(S)$ follows directly from the definition of $\overline{\rho}_0$. Let $[\rho_0, \rho_1] \in \mathcal{C}(S)$ and $a, b \in S_1^*$ be such that $a\rho_1b$. Then $ax\rho_0by$ and $xa\rho_0yb$ for all $x, y \in S_0$ such that $x\rho_0y$ and thus $a\overline{\rho}_0b$. Therefore $\rho_1 \subseteqq \overline{\rho}_0$ so that $[\rho_0, \rho_1] \subseteqq [\rho_0, \overline{\rho}_0]$.

By Lemma 3.3(iii), we have $[\rho_0, \overline{\rho}_0] \subset [\rho_0]$. Maximality of $\zeta_1$ implies that $\overline{\rho}_0 \subseteqq \zeta_1$ which by Lemma 3.3(ii) yields $[\rho_0, \overline{\rho}_0] \subseteqq [\omega_0, \zeta_1]$. Therefore $[\rho_0, \overline{\rho}_0] \subseteqq [\rho_0] \wedge [\omega_0, \zeta_1]$.

Since $[\rho_0] \wedge [\omega_0, \zeta_1] \subseteqq [\omega_0, \zeta_1]$ we have that $[\rho_0] \wedge [\omega_0, \zeta_1]$ is of the form $[\theta_0, \theta_1]$. Let $a[\rho_0] \wedge [\omega_0, \zeta_1]b$. For $a, b \in S_0$, $a[\rho_0]b$ implies that $a\rho_0b$. Let $a, b \in S_1^*$, $u, v \in (S_1^*)^1$ and $x\rho_0y$. Then $a\rho_0b$ implies $(uav)x[\rho_0](ubv)y$ so that $(uav)x\rho_0(ubv)y$ and similarly $x(uav)\rho_0y(ubv)$. Also $uav[\omega_0, \zeta_1]ubv$ and hence $uav \in S_0$ if and only if $ubv \in S_0$. In $S_1$ this means that $uav = 0$ if and only if $ubv = 0$. It follows that $uav\rho_0'ubv$. Since this holds for all $u, v \in (S_1)^1$, we get $a\overline{\rho}_0b$. Therefore $[\rho_0] \wedge [\omega_0, \zeta_1] \subseteqq [\rho_0, \overline{\rho}_0]$ and equality prevails.

LEMMA 4.5. *Let $[\rho_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$.*

(i) $[\lambda_0] \wedge [\rho_0, \rho_1] = [\lambda_0 \wedge \rho_0, \overline{\lambda}_0 \wedge \rho_1]$.

(ii) $[\lambda_0] \vee [\rho_0, \rho_1] = [\lambda_0 \vee \rho_0]$.

PROOF. (i) Indeed,

$$\begin{aligned}[\lambda_0] \wedge [\rho_0, \rho_1] &= [\lambda_0] \wedge [\omega_0, \zeta_1] \wedge [\rho_0, \rho_1] && \text{by maximality of } [\omega_0, \zeta_1] \\ &= [\lambda_0, \overline{\lambda}_0] \wedge [\rho_0, \rho_1] && \text{by Lemma 4.4} \\ &= [\lambda_0 \wedge \rho_0, \overline{\lambda}_0 \wedge \rho_1] && \text{by Lemma 4.2(i).}\end{aligned}$$

(ii) By Lemma 3.3(i) (iii), we have $[\lambda_0] \subseteqq [\lambda_0 \vee \rho_0]$ and

$$[\rho_0, \rho_1] \subset [\lambda_0 \vee \rho_0]$$

so that $[\lambda_0] \vee [\rho_0, \rho_1] \subseteqq [\lambda_0 \vee \rho_0]$.

Since $[\lambda_0] \subseteqq [\lambda_0] \vee [\rho_0, \rho_1]$, we have that $[\lambda_0] \vee [\rho_0, \rho_1]$ is of the form $[\theta_0]$. We already have $\theta_0 \subseteqq \lambda_0 \vee \rho_0$. For the opposite inclusion, by Lemma 3.3(i), it suffices to show that $\lambda_0 \vee \rho_0 \subseteqq \theta_0$, that is

$$\lambda_0 \vee \rho_0 \subseteqq \left([\lambda_0] \vee [\rho_0, \rho_1]\right)|_{S_0}.$$

If $x \lambda_0 \vee \rho_0 y$, then there exists a sequence (4) whence

$$x[\lambda_0] z_1 [\rho_0, \rho_1] z_2 \dots z_n [\rho_0, \rho_1] y$$

and thus $x[\lambda_0] \vee [\rho_0, \rho_1] y$, as required.

## 5. Relations induced by restrictions

In order to study the structure of the congruence lattice of $S$, we consider here the relations $R_0$ and $R_1$ on $\mathcal{C}(S)$ induced by the restrictions of the congruences on $S$ to $S_0$ and $S_1^*$, respectively.

NOTATION 5.1. Define a relation $R_0$ by

$$\lambda R_0 \rho \Leftrightarrow \lambda|_{S_0} = \rho|_{S_0} \quad (\lambda, \rho \in \mathcal{C}(S)).$$

Considering various cases, straightforward checking shows that $R_0$ is a congruence on $\mathcal{C}(S)$. We now describe its classes. Recall that $[\alpha, \beta]$ denotes an interval in a lattice and $\overline{\rho}_0$ is defined in Notation 4.3.

PROPOSITION 5.2. *Let* $[\rho_0, \rho_1] \in \mathcal{C}(S)$ *and*

$$\Gamma_{\rho_0} = [[\rho_0, \varepsilon_1], [\rho_0, \overline{\rho}_0]].$$

*Then*

$$[\rho_0, \rho_1] R_0 = \begin{cases} \Gamma_{\rho_0} \cup \{[\rho_0]\} = [[\rho_0, \varepsilon_1]], [\rho_0]] & \text{if } [\rho_0] \in \mathcal{C}(S), \\ \Gamma_{\rho_0} & \text{otherwise.} \end{cases}$$

PROOF. By Lemma 4.4, $[\rho_0, \overline{\rho}_0]$ is the greatest congruence on $S$ of the form $[\rho_0, \theta_1]$ for some $\theta_1 \in \mathcal{C}_0(S_1)$. If $[\rho_0] \in \mathcal{C}(S)$, then by Lemma 3.3(iii), it is the greatest element of $[\rho_0, \rho_1]R_0$; otherwise its greatest element is $[\rho_0, \overline{\rho}_0]$. In either case, $[\rho_0, \varepsilon_1]$ is the least element of $[\rho_0, \rho_1]R_0$. Clearly $[\rho_0, \rho_1]R_0$ is convex, which then implies the assertion of the proposition.

The above result also takes care of $[\rho_0]R_0$ when $[\rho_0] \in \mathcal{C}(S)$. We shall need some more symbolism.

NOTATION 5.3. Define a relation $R_1$ by

$$\lambda R_1 \rho_1 \Leftrightarrow \lambda|_{S_1^*} = \rho|_{S_1^*} \quad (\lambda, \rho \in \mathcal{C}(S)).$$

Also let

$$A = \{b \in S_0^* | \text{ there exists } a \in S_1^* \text{ such that } a > b\}.$$

Clearly $R_1$ is an equivalence relation on $\mathcal{C}(S)$.

LEMMA 5.4. (i) *For* $[\lambda_0], [\rho_0] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0]R_1[\rho_0] \Leftrightarrow \lambda_0|_A = \rho_0|_A.$$

(ii) *For* $[\lambda_0, \lambda_1], [\rho_0, \rho_1] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0, \lambda_1]R_1[\rho_0, \rho_1] \Leftrightarrow \lambda_1 = \rho_1.$$

(iii) *For* $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0]R_1[\rho_0, \rho_1] \Leftrightarrow \text{ (for any } a > \overline{a} > 0,\ b > \overline{b} > 0\text{: } \overline{a}\lambda_0\overline{b} \Leftrightarrow a\rho_1 b).$$

PROOF. We shall use Lemma 3.5(ii) freely.

(i) Let $a > \overline{a} > 0$ and $b > \overline{b} > 0$. If $[\lambda_0]R_1[\rho_0]$, then

$$\overline{a}\lambda_0\overline{b} \Leftrightarrow a[\lambda_0]b \Leftrightarrow a[\rho_0]b \Leftrightarrow \overline{a}\rho_0\overline{b}$$

which shows that $\lambda_0|_A = \rho_0|_A$. Conversely, if $\lambda_0|_A = \rho_0|_A$, then

$$a[\lambda_0]b \Leftrightarrow \overline{a}\lambda_0\overline{b} \Leftrightarrow \overline{a}\rho_0\overline{b} \Leftrightarrow a[\rho_0]b$$

so that $[\lambda_0]R_1[\rho_0]$.

(ii) This is obvious.

(iii) Again let $a > \overline{a} > 0$ and $b > \overline{b} > 0$. If $[\lambda_0]R_1[\rho_0, \rho_1]$, then

$$\overline{a}\lambda_0\overline{b} \Leftrightarrow a[\lambda_0]b \Leftrightarrow a[\rho_0, \rho_1]b \Leftrightarrow a\rho_1 b.$$

Conversely, if the above condition holds, then

$$a[\lambda_0]b \Leftrightarrow \overline{a}\lambda_0\overline{b} \Leftrightarrow a\rho_1 b \Leftrightarrow a[\rho_0,\rho_1]b.$$

In order to describe the $R_1$-classes, we shall need some more notation.

NOTATION 5.5. For $\rho_1 \in \mathcal{C}_0(S_1)$, let

$$\overline{\rho}_1 = \cap\{\, \lambda_0 \in \mathcal{C}(S_0) | [\lambda_0, \rho_1] \in \mathcal{C}(S)\,\}.$$

For $\rho_0 \in \mathcal{C}(S)$, let

$$\widetilde{\rho}_0 = \cap\{\, \lambda_0 \in \mathcal{C}(S_0) | [\lambda_0] \in \mathcal{C}(S) \text{ and } \lambda_0|_A = \rho_0|_A\,\},$$

and define a relation $\widehat{\rho}_0$ on $S_0$ by

$$a\widehat{\rho}_0 b \Leftrightarrow \left(\text{if } x, y \in (S_0)^1 \text{ and } xay, xby \in A, \text{ then } xay\rho_0 xby\right).$$

We shall see below that the following notation for certain intervals is meaningful.

$$\Gamma_{\rho_1} = \left[[\overline{\rho}_1, \rho_1], [\omega_0, \rho_1]\right] \quad \text{if} \quad \rho_1 \in \mathcal{C}(S_1),$$

$$\Delta_{\rho_0} = \left[[\widetilde{\rho}_0], [\widehat{\rho}_0]\right] \quad \text{if} \quad [\rho_0] \in \mathcal{C}(S).$$

We call $[\rho_0] \in \mathcal{C}(S)$ *pure* if $[\rho_0]$ is not $R_1$-related to any $[\lambda_0, \lambda_1]$, and we call $[\rho_0, \rho_1] \in \mathcal{C}(S)$ *pure* if $[\rho_0, \rho_1]$ is not $R_1$-related to any $[\lambda_0]$.

We are now ready for the main result of this section.

THEOREM 5.6. $[\rho_0, \rho_1]R_1 = \Gamma_{\rho_1}$ *if* $[\rho_0, \rho_1] \in \mathcal{C}(S)$ *is pure,*

$$[\rho_0]R_1 = \Delta_{\rho_0} \quad \textit{if} \quad [\rho_0] \in \mathcal{C}(S) \quad \textit{is pure},$$

$$[\rho_0]R_1 = \Gamma_{\rho_1} \cup \Delta_{\lambda_0} \quad \textit{if} \quad [\lambda_0]R_1[\rho_0, \rho_1].$$

PROOF. Let $[\rho_0, \rho_1]$ be pure. It suffices to prove that $[\overline{\rho}_1, \rho_1]$ is the least element of $[\rho_0, \rho_1]R_1$. We verify first that $[\overline{\rho}_0, \rho_1] \in \mathcal{C}(S)$. Hence let $a, b \in \in S_1^*$ be such that $a\rho_1 b$ and let $x\overline{\rho}_1 y$. Then for every $[\lambda_0, \rho_1] \in \mathcal{C}(S)$, we have $x\lambda_0 y$ and hence $ax\lambda_0 by$ and $xa\lambda_0 yb$ which implies that $ax\overline{\rho}_1 by$ and $xa\overline{\rho}_1 yb$. Therefore $[\overline{\rho}_1, \rho_1] \in \mathcal{C}(S)$. The minimality of $[\overline{\rho}_1, \rho_1]$ is obvious.

Next let $[\rho_0]$ be pure. We show first that $[\widetilde{\rho}_0]$ is the least element of $[\rho_0]R_1$. Let

$$\mathcal{F} = \{\, \lambda_0 \in \mathcal{C}(S_0) | [\lambda_0] \in \mathcal{C}(S) \text{ and } \lambda_0|_A = \rho_0|_A\,\}.$$

Then $\rho_0 \in \mathcal{F}$ and thus $\mathcal{F} \neq \emptyset$. Let $a > b > 0$ in $S$ and $x \in S_0$. Then $ax\lambda_0 bx$ and $xa\lambda_0 xb$ for all $\lambda_0 \in \mathcal{F}$ by Lemma 3.4 and hence $ax\widetilde{\rho}_0 bx$ and $xa\widetilde{\rho}_0 xb$. Therefore $[\widetilde{\rho}_0] \in \mathcal{C}(S)$.

Since $\widetilde{\rho}_0 \subseteqq \rho_0$, we have $\widetilde{\rho}_0|_A \subseteqq \rho_0|_A$. Let $a\rho_0|_A b$. Then $a\lambda_0 b$ for all $\lambda_0 \in \mathcal{F}$ and thus $a\widetilde{\rho}_0 b$. Hence $\rho_0|_A \subseteqq \widetilde{\rho}_0|_A$ and equality prevails. By Lemma 5.4(i), we get $[\widetilde{\rho}_0]R_1[\rho_0]$. Let $[\lambda_0]R_1[\rho_0]$. Then by Lemma 5.4(i), we have $\lambda_0|_A = \rho_0|_A$ so that $\lambda_0 \in \mathcal{F}$ and thus $\widetilde{\rho}_0 \subseteqq \lambda_0$. But then Lemma 3.3(i) yields $[\widetilde{\rho}_0] \subseteqq [\lambda_0]$.

We now show that $[\widehat{\rho}_0]$ is the greatest element of $[\rho_0]R_1$. Clearly $\widehat{\rho}_0$ is an equivalence relation. Let $a\widehat{\rho}_0 b, c \in S_0$ and $x, y \in (S_0)^1$. If $x(ac)y$, $x(bc)y \in A$, then $xa(cy), xb(cy) \in A$ and the hypothesis implies that $xa(cy)\rho_0 xb(cy)$, that is $x(ac)y\rho_0 x(bc)y$. Hence $ac\widehat{\rho}_0 bc$ and similarly $ca\widehat{\rho}_0 cb$. Therefore $\widehat{\rho}_0 \in \mathcal{C}(S_0)$.

If $a\rho_0 b$, then $xay\rho_0 xby$ for all $x, y \in (S_0)^1$ so that $\rho_0 \subseteqq \widehat{\rho}_0$. Now Lemma 3.3(v) implies that $[\widehat{\rho}_0] \in \mathcal{C}(S)$. In addition, $\rho_0 \subseteqq [\widehat{\rho}_0]|_A$. If $a\widehat{\rho}_0|_A b$, then $a, b \in A$ so with $x = y = 1$, we get $a\rho_0 b$. Therefore $\widehat{\rho}_0|_A \subseteqq \rho_0|_A$ and equality prevails. By Lemma 5.4(i), we obtain that $[\widehat{\rho}_0]R_1[\rho_0]$. Let $[\lambda_0]R_1[\rho_0]$ and $a\lambda_0 b$. By Lemma 5.4(i), we have $\lambda_0|_A = \rho_0|_A$. If for $x, y \in (S_0)^1$ we have $xay, xby \in A$, then $xay\lambda_0 xby$ which then yields $xay\rho_0 xby$. Therefore $a\widehat{\rho}_0 b$ which proves that $\lambda_0 \subseteqq \widehat{\rho}_0$. Now Lemma 3.3(i) gives $[\lambda_0] \subseteqq [\widehat{\rho}_0]$.

Since $[\rho_0]R_1$ is obviously convex, we deduce that $[\rho_0]R_1 = \Gamma_{\rho_0}$.

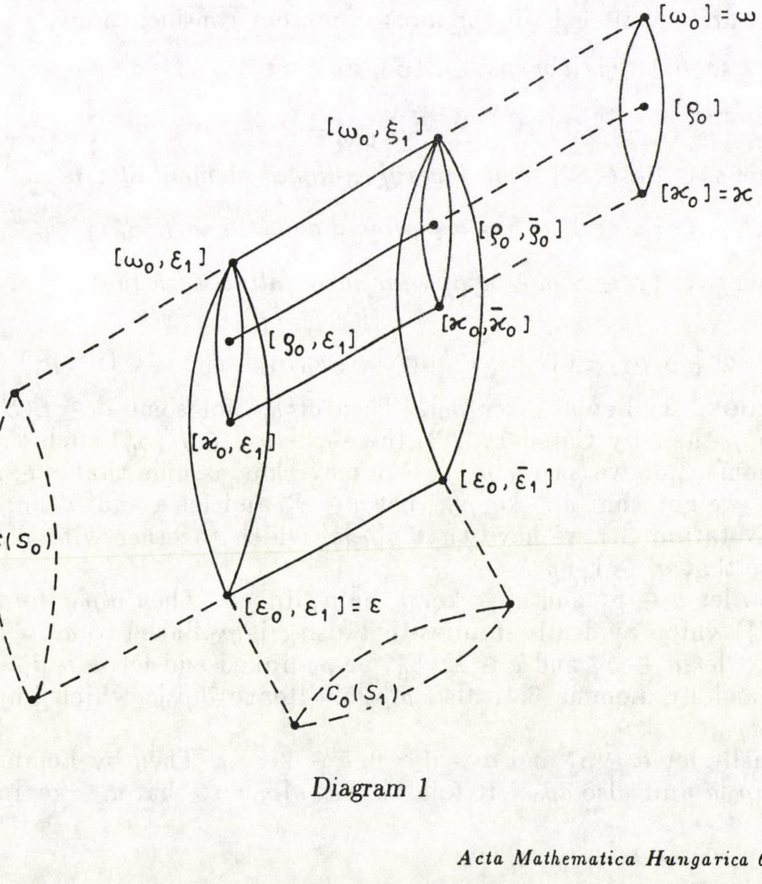


*Diagram 1*

Finally assume that $[\lambda_0]R_1[\rho_0,\rho_1]$. By the above $\Gamma_{\rho_1} \subseteqq [\rho_0,\rho_1]R_1$ and $\Delta_{\lambda_0} \subseteqq [\lambda_0]R_1$. If $[\theta_0]R_1[\lambda_0]$, then as above, we get $[\theta_0] \in \Delta_{\lambda_0}$, and if $[\theta_0,\theta_1]R_1[\rho_0,\rho_1]$ then again as above, we obtain $[\theta_0,\theta_1] \in \Gamma_{\rho_1}$. Consequently $[\lambda_0]R_1 = [\rho_0,\rho_1]R_1 = \Gamma_{\rho_1} \cup \Delta_{\lambda_0}$.

Diagram 1 represents the lattice $\mathcal{C}(S)$. In it are visible $R_0$-classes and the $\varepsilon R_1$-class. The intersection $R_0 \cap R_1$ is "almost" the equality relation; it is possible, however, that $[\rho_0]R_0 \cap R_1[\rho_0,\rho_1]$.

## 6. The kernel and the trace relations

We have recalled the definitions of the kernel and trace of a congruence and the kernel and trace relations $K$ and $T$ in Section 2. We start here with a study of the relation $K$ on our semigroup $S$ with a view of obtaining necessary and sufficient conditions on $S$ in order that $K$ be a congruence. This does not succeed in a satisfactory manner, so in the next two sections we consider special classes for which the answer is complete. We also consider briefly some of the ends of the intervals which make up the $K$- and $T$-classes on $\mathcal{C}(S)$.

Our first result is basic for most of present considerations.

LEMMA 6.1. (i) *For* $[\rho_0] \in \mathcal{C}(S)$, *we have*

$$\ker[\rho_0] =$$
$$= \ker\rho_0 \cup \{a \in S_1^* \mid \textit{some [every] } \rho_0\textit{-linked element of } a \textit{ is in } \ker\rho_0\}$$
$$= \ker\rho_0 \cup \{a \in S_1^* \mid a \ \textit{ has a } \rho_0\textit{-linked element in } E(S_0^*)\}$$
$$= \ker\rho_0 \cup \{a \in S_1^* \mid \overline{a} \in \ker\rho_0 \textit{ for some [all] } \overline{a} \textit{ such that } a > \overline{a} > 0\}.$$

(ii) *For* $[\rho_0,\rho_1] \in \mathcal{C}(S)$, *we have* $\ker[\rho_0,\rho_1] = \ker\rho_0 \cup (\ker\rho_1)^*$.

PROOF. (i) Let $a \in \ker[\rho_0]$. Then $a[\rho_0]e$ for some $e \in E(S)$. If $e \in E(S_1^*)$, then by Corollary 2.3, there exists $f \in E(S_0^*)$ such that $e > f$. By Lemma 3.4, we have $e[\rho_0]f$. We may thus assume that $e \in E(S_0)$. If $a \in S_0$, we get that $a \in \ker\rho_0$. Let $a \in S_1^*$ and let $a$ and $a'$ be $\rho_0$-linked. From Notation 3.1 we have that $a[\rho_0]a'$ which together with $a[\rho_0]e$ gives $a'\rho_0 e$ so that $a' \in \ker\rho_0$.

Now let $a \in S_1^*$ and $a' \in \ker\rho_0$ be $\rho_0$-linked. Then $a'\rho_0 e$ for some $e \in E(S_0^*)$ which evidently implies that also $e$ is $\rho_0$-linked to $a$.

Next let $a \in S_1^*$ and $e \in E(S_0^*)$ be $\rho_0$-linked and let $a > \overline{a} > 0$. Then $a[\rho_0]e$ and by Lemma 3.4, also $a[\rho_0]\overline{a}$. Hence $\overline{a}[\rho_0]e$ which implies that $\overline{a} \in \ker\rho_0$.

Finally let $a \in S_1^*$ and $a > \overline{a} > 0$, $\overline{a} \in \ker\rho_0$. Then by Lemma 3.4, we have $a[\rho_0]\overline{a}$ and also $\overline{a}\rho_0 e$. It follows that $a[\rho_0]e$ so that $a \in \ker[\rho_0]$.

Since obviously $\ker\rho_0 \subseteqq \ker[\rho_0]$, this proves all the equalities in part (i).

(ii) This is evident.

LEMMA 6.2. *Let* $[\rho_0, \rho_1] \in \mathcal{C}(S)$, $a \in (\ker\rho_1)^*$ *and* $a > b > 0$. *Then* $b \in \ker\rho_0$.

PROOF. By hypothesis, $a\rho_1 e$ for some $e \in E(S_1^*)$ and $b = fa = ag$ for some $f, g \in E(S)$. Hence $b = fa\rho_0 fe$ and $b = ag\rho_0 eg$ so that

$$b^2\rho_0(fe)(eg) = (fe)g\rho_0 bg = b$$

and $b \in \ker\rho_0$.

We can now characterize the relation $K$ as follows.

COROLLARY 6.3. (i) *For* $[\lambda_0], [\rho_0] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0]K[\rho_0] \Leftrightarrow \lambda_0 K\rho_0.$$

(ii) *For* $[\lambda_0, \lambda_1], [\rho_0, \rho_1] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0, \lambda_1]K[\rho_0, \rho_1] \Leftrightarrow \lambda_0 K\rho_0, \lambda_1 K\rho_1.$$

(iii) *For* $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0]K[\rho_0, \rho_1] \Leftrightarrow \lambda_0 K\rho_0 \text{ and } (a > b > 0, b \in \ker\rho_0 \Rightarrow a \in \ker\rho_1).$$

PROOF. (i) Indeed, by Lemma 6.1(i), we get

$$\begin{gathered}\ker[\lambda_0] = \ker[\rho_0] \Leftrightarrow \\ \Leftrightarrow \ker\lambda_0 = \ker\rho_0 \text{ and } (\text{for } a > b > 0 : b \in \ker\lambda_0 \Leftrightarrow b \in \ker\rho_0) \Leftrightarrow \\ \Leftrightarrow \ker\lambda_0 = \ker\rho_0.\end{gathered}$$

(ii) This is an obvious consequence of Lemma 6.1(ii).

(iii) Indeed, by Lemma 6.1, we have

$$\begin{gathered}\ker[\lambda_0] = \ker[\rho_0, \rho_1] \Leftrightarrow \\ \Leftrightarrow \ker\lambda_0 = \ker\rho_0 \text{ and } (\text{for } a > b > 0 : b \in \ker\lambda_0 \Leftrightarrow b \in \ker\rho_1)\end{gathered}$$

which in view of Lemma 6.2 gives the desired statement.

The manipulation with kernels and the relation $K$ would be more effective if $K$ were always a $\vee$-congruence; recall that $K$ is always a (complete)

$\wedge$-congruence. It is thus of some importance to know for which regular semigroups (belonging to some classes) is $K$ a congruence on their congruence lattices. The next theorem provides a general criterion on our $S$ which is necessary and sufficient for its kernel relation $K$ to be a congruence. In fact, representing the congruences on $S$ as either $[\rho_0]$ or $[\rho_0, \rho_1]$, we shall examine which combinations of these respect the $\vee$-congruence property and which, in general, do not. Even though this result expresses the congruence property of $K$ again in terms of congruences on $S$, and not directly in terms of the structure of $S$, it will turn out quite useful when we consider specializations of $S$ in the next two sections. Even in its full generality, it indicates which combinations of congruences $[\rho_0]$ and $[\rho_0, \rho_1]$ are crucial for the congruence property of $K$.

THEOREM 6.4. *The following conditions on $S$ are equivalent.*

(i) *$K$ is a congruence.*

(ii) *For $[\rho_0], [\rho_0, \rho_1], [\theta_0, \theta_1] \in \mathcal{C}(S)$, $\lambda_0 \neq \omega_0$,*

$$[\lambda_0]K[\rho_0, \rho_1] \Rightarrow [\lambda_0 \vee \theta_0]K[\rho_0 \vee \theta_0, \rho_1 \vee \theta_1].$$

(iii) *For $[\lambda_0], [\rho_0, \rho_1]; [\theta_0, \theta_1] \in \mathcal{C}(S)$, $\lambda_0 \neq \omega_0$,*

$$\ker \lambda_0 = \ker \rho_0, \ (x > y > 0, \ y \in \ker \rho_0 \Rightarrow x \in \ker \rho_1),$$

$$a > b > 0, \ b \in \ker(\rho_0 \vee \theta_0) \Rightarrow a \in \ker(\rho_1 \vee \theta_1).$$

PROOF. By Lemma 4.5(ii), (i) implies (ii). Also (ii) and (iii) are equivalent in view of Corollary 6.3(iii) and Lemma 2.5.

(ii) *implies* (i). We have to check that all cases, except the one in part (ii), are automatically satisfied. We shall use Lemmas 2.5, 4.1(ii), 4.2(ii), 4.5(ii) and Corollary 6.3 freely.

1. $[\lambda_0]K[\rho_0]$. Then $\lambda_0 K \rho_1$ and
$[\lambda_0] \vee [\theta_0] = [\lambda_0 \vee \theta_0]K[\rho_0 \vee \theta_0] = [\rho_0] \vee [\theta_0]$,
$[\lambda_0] \vee [\theta_0, \theta_1] = [\lambda_0 \vee \theta_0]K[\rho_0 \vee \theta_0] = [\rho_0] \vee [\theta_0, \theta_1]$.

2. $[\lambda_0, \lambda_1]K[\rho_0, \rho_1]$. Then $\lambda_0 K \rho_0$, $\lambda_1 K \rho_1$ and
$[\lambda_0, \lambda_1] \vee [\theta_0] = [\lambda_0 \vee \theta_0]K[\rho_0 \vee \theta_0] = [\rho_0, \rho_1] \vee [\theta_0]$,
$$[\lambda_0, \lambda_1] \vee [\theta_0, \theta_1] = [\lambda_0 \vee \theta_0, \lambda_1 \vee \theta_1]K[\rho_0 \vee \theta_0, \rho_1 \vee \theta_1] = [\rho_0, \rho_1] \vee [\theta_0, \theta_1].$$

3. $[\lambda_0]K[\rho_0, \rho_1]$. Then $\lambda_0 K \rho_0$ and
$[\lambda_0] \vee [\theta_0] = [\lambda_0 \vee \theta_0]K[\rho_0 \vee \theta_0] = [\rho_0] \vee [\theta_0]$.

Let$\lambda_0 = \omega_0$. By Lemma 6.1, $\ker \rho_0 = S_0$ and $\ker \rho_1 = S_1$ and thus

$$\ker[\omega_0 \vee \theta_0] = S = \ker[\rho_0 \vee \theta_0, \rho_1 \vee \theta_1]$$

so that $[\omega_0 \vee \theta_0]K[\rho_0 \vee \theta_0, \rho_1 \vee \theta_1]$.

Therefore $K$ may fail to be a congruence because of the single case in Theorem 6.4(ii). Sufficient conditions for the congruence property of $K$ can thus be easily provided.

COROLLARY 6.5. *If* $\ker[\lambda_0] \neq \ker[\rho_0, \rho_1]$ *for all choices of* $[\lambda_0]$, $[\rho_0, \rho_1] \in \mathcal{C}(S)$ *with* $\lambda_0 \neq \omega_0$ *or* $a > b > 0$ *implies* $b \in E(S)$ *or* $\kappa = \omega$, *then* $K$ *is a congruence.*

PROOF. The first antecedent uses Theorem 6.4(ii), the second Theorem 6.4(iii). Assume that $\kappa = \omega$. By Lemma 3.6(ii), $\omega = [\omega_0]$ is the only congruence on $S$ of the form $[\rho_0]$ and Theorem 6.4(ii) is satisfied.

A concrete sufficient condition for the first condition in Corollary 6.5 is the following.

LEMMA 6.6. *Suppose that there exists* $a \in S_1^*$ *such that* $a > e$ *for some* $e \in E(S_0^*)$ *and* $a^2 \in S_0$. *Then* $\ker[\lambda_0] \neq \ker[\rho_0, \rho_1]$ *for all* $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$ *and thus* $K$ *is a congruence.*

PROOF. By contrapositive, assume that $\ker[\lambda_0] = \ker[\rho_0, \rho_1]$ and that $a > e$ for some $e \in E(S_0^*)$. By Corollary 6.3(iii), we have $\ker \lambda_0 = \ker \rho_0$ and $x > y > 0$, $y \in \ker \rho_0$ imply $x \in \ker \rho_1$. In particular, $a > e > 0$ and $e \in \ker \rho_0$ imply that $a \in \ker \rho_1$. Hence $a^2 \rho a$ so that $a^2 \in S_1^*$.

We shall see in Lemma 7.4 that the converse of this lemma holds for strict extensions.

The $K$-classes are intervals in $\mathcal{C}(S)$, in the notation $\rho K = [\rho_K, \rho^K]$ for any $\rho \in \mathcal{C}(S)$. If we represent the congruences on $S$ as either $[\rho_0]$ or $[\rho_0, \rho_1]$, one may ask about the form of

$$(5) \qquad [\rho_0]_K,\ [\rho_0, \rho_1]_K,\ [\rho_0]^K, [\rho_0, \rho_1]^K$$

in terms of $\rho_0$ or $\rho_0$ and $\rho_1$, respectively. Toward this goal, we find only the following result.

PROPOSITION 6.7. *For* $[\rho_0] \in \mathcal{C}(S)$, *we have* $[\rho_0]^K = [(\rho_0)^K]$.

PROOF. Since $\rho_0 \subseteqq (\rho_0)^K$ and $[\rho_0] \in \mathcal{C}(S)$, Lemma 3.3(iv) implies that $[(\rho_0)^K] \in \mathcal{C}(S)$. By Lemma 6.1(i), we obtain

$$\begin{aligned}\ker\left[(\rho_0)^K\right] &= \ker(\rho_0)^K \cup \left\{a \in S_1^* \mid a > \overline{a} > 0 \Rightarrow \ker(\rho_0)^K\right\} = \\ &= \ker\rho_0 \cup \{a \in S_1^* \mid a > \overline{a} > 0 \Rightarrow a \in \ker\rho_0\} = \ker[\rho_0].\end{aligned}$$

If $[\lambda_0]K[\rho_0]$, then by Corollary 6.3(i), we have $\lambda_0 K \rho_0$ whence $\lambda_0 \subseteqq (\rho_0)^K$ so that $[\lambda_0] \subseteqq [(\rho_0)^K]$ by Lemma 3.3(i).

Let $[\lambda_0, \lambda_1]K[\rho_0]$ and $a[\lambda_0, \lambda_1]b$. By Corollary 6.3(iii), we have $\lambda_0 K \rho_0$. Hence $\lambda_0 \subseteqq (\rho_0)^K$ so that, if $a, b \in S_0$, then $a\lambda_0 b$ and thus $a\left[(\rho_0)^K\right]b$ whence $a\left[(\rho_0)^K\right]b$. Assume next that $a, b \in S_1^*$. Then $a\lambda_1 b$. Let $x \in S_0$. Then $ax\lambda_0 bx$ so that $ax(\rho_0)^K bx$ and thus $ax\left[(\rho_0)^K\right]bx$. Let $a > \overline{a} > 0$ and $b > \overline{b} > 0$. By Lemma 3.4, we have $a\left[(\rho_0)^K\right]\overline{a}$ and $b\left[(\rho_0)^K\right]\overline{b}$ and thus $\overline{a}x\left[(\rho_0)^K\right]\overline{b}x$ whence $\overline{a}x\left[(\rho_0)^K\right]\overline{b}x$. Symmetrically, we get $x\overline{a}(\rho_0)^K x\overline{b}$. Since this holds for all $x \in S_0$, weak reductivity of $S_0/(\rho_0)^K$ yields that $\overline{a}(\rho_0)^K\overline{b}$. Therefore $\overline{a}\left[(\rho_0)^K\right]\overline{b}$ and thus $a\left[(\rho_0)^K\right]b$. Consequently $[\lambda_0, \lambda_1] \subseteqq \subseteqq \left[(\rho_0)^K\right]$ completing the proof of the maximality of the latter. The assertion of the proposition follows.

We now cast a brief look at the relation $T$.

LEMMA 6.8. (i) *For* $[\lambda_0], [\rho_0] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0]T[\rho_0] \Leftrightarrow \lambda_0 T \rho_0.$$

(ii) *For* $[\lambda_0, \lambda_1], [\rho_0, \rho_1] \in \mathcal{C}(S)$, *we have*

$$[\lambda_0, \lambda_1]T[\rho_0, \rho_1] \Leftrightarrow \lambda_0 T \rho_0, \lambda_1 T \rho_1.$$

(iii) *For* $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$, *we have* $[\lambda_0] \not{T} [\rho_0, \rho_1]$.

PROOF. (i) Trivially $[\lambda_0]T[\rho_0]$ implies $\lambda_0 T \rho_0$. Conversely assume that $\lambda_0 T \rho_0$ and let $e, f \in E(S)$ be such that $e[\lambda_0]f$. If $e, f \in S_0$, then $e\lambda_0 f$ and hence $e\rho_0 f$ so that $e[\rho_0]f$. Let $e, f \in S_1^*$. By Corollary 2.3, there exist $\overline{e}, \overline{f} \in E(S_0^*)$ such that $e > \overline{e}$ and $f > \overline{f}$. Now Lemma 3.4 implies that $\overline{e}[\lambda_0]\overline{f}$ whence $\overline{e}\lambda_0\overline{f}$ so that $\overline{e}\rho_0\overline{f}$. But then $\overline{e}[\rho_0]\overline{f}$ and again by Lemma 3.4, we conclude that $e[\rho_0]f$. The case $e \in S_1^*$ and $f \in S_0$ is treated similarly whereas the case $e \in S_0^*$ and $f \in S_1^*$ is symmetric to it. Therefore $e[\rho_0]f$ in all cases so that $\operatorname{tr}[\lambda_0] \subseteqq \operatorname{tr}[\rho_0]$. By symmetry, equality prevails and gives $[\lambda_0]T[\rho_0]$.

(ii) This is obvious.

(iii) For any $e \in E(S_1^*)$, by Corollary 2.3, there exists $f \in E(S_0^*)$ such that $e > f$. By Lemma 3.4, we have $e[\lambda_0]f$, whereas $e[\rho_0, \rho_1]f$ is contrary to the definition of $[\rho_0, \rho_1]$.

We can also consider expressions in (5) with $T$ instead of $K$. The next result concerns $[\rho_0]$.

PROPOSITION 6.9. *For* $[\rho_0] \in \mathcal{C}(S)$, *we have* $[\rho_0]_T = \left[(\rho_0)_T\right]$ *and* $[\rho_0]^T = \left[(\rho_0)^T\right]$.

PROOF. We show first that $[(\rho_0)_T] \in \mathcal{C}(S)$. Recall that we have defined $\kappa$ in Notation 2.1 as the congruence on $S$ generated by the set

$$\{(e,f) \in E(S) \times E(S) | e > f > 0\}.$$

It then follows that $\kappa = (\operatorname{tr}\kappa)^*$ and by ([2], Theorem 3.2), $(\times\kappa)^* = \kappa_T$. Consequently $\kappa = \kappa_T$. It also follows that the operator $\operatorname{sub} T$ is monotone. Hence $\kappa = \kappa_T \subseteqq ((\rho_0)_T)_T = (\rho_0)_T$ so by Lemma 3.6(ii), we have $[(\rho_0)_T] \in \mathcal{C}(S)$. By Lemma 6.8(i), we easily get that $[\rho_0]_T = [(\rho_0)_T]$. By Lemma 3.3(iv), we have $[(\rho_0)^T] \in \mathcal{C}(S)$ so again by Lemma 6.8(i), we easily see that $[(\rho_0)]^T = [(\rho_0)^T]$.

## 7. Strict extensions

In the case of a strict extension $S$ of $S_0$ by $S_1$, we shall be able to obtain explicit answers to several queries left open in the general case, in particular, by providing converses of some results obtained in the preceding section.

Recall from Section 2 that $S$ is a strict extension (of $S_0$ by $S_1$) if for every $a \in S_1^*$, there exists $a' \in S_0$ such that $ax = a'x$ and $xa = xa'$ for all $x \in S_0$. In fact, the mapping $\varphi: a \to a'$ $(a \in S_1^*)$ is a partial homomorphism $\varphi: S_1^* \to S_0$ which determines the multiplication of $S$. Since we have excluded the case when $S_0S_1 = \{0\}$, that is when $S$ is an orthogonal sum of $S_0$ and $S_1$, our partial homomorphism maps $S_1^*$ into $S_0^*$. It will be convenient occasionally to give both $S_0$ and $S_1$ the Rees matrix representation. In such a case, we may set

$$S_0 = \mathcal{M}^\circ(I_0, G_0, \Lambda_0; P), \quad S_1 = \mathcal{M}^\circ(I_1, G_1, \Lambda_1; Q)$$

and our partial homomorphism $\varphi$ takes on the following form. Let

$$\xi: I_1 \to I_0, \quad u: I_1 \to G_0, \quad \omega: G_1 \to G_0, \quad v: \Lambda_1 \to G_0, \quad \eta: \Lambda_1 \to \Lambda_0$$

be functions with $u: i \to u_i$, $\omega$ a homomorphism, $v: \lambda \to v_\lambda$, such that

$$p_{\lambda i} \neq 0 \Rightarrow p_{\lambda i}\omega = v_\lambda q_{\lambda\eta, i\xi} u_i \tag{6}$$

Now define

$$\varphi: (i,g,\lambda) \to (i\xi, u_i(g\omega)v_\lambda, \lambda\eta) \quad ((i,g,\lambda) \in S_1^*).$$

Then $\varphi$ is a partial homomorphism of $S_1^*$ into $S_0^*$. Conversely, every such is of the above form.

Also note that the mapping

$$\psi\colon a \to \begin{cases} a\varphi & \text{if } a \in S_1^* \\ a & \text{if } a \in S_0 \end{cases}$$

is a retraction of $S$ onto $S_0$, so $S$ is also called a retract extension (of $S_0$ by $S_1$). For a full discussion of this subject, see ([5], Chapter III).

We shall abbreviate the designation of the above situation to "Let $S$ be a strict extension." In order to place the strict extension case into our general context, we prove the following simple result.

PROPOSITION 7.1. *The following conditions on $S$ are equivalent.*
(i) *$S$ is a strict extension.*
(ii) $[\varepsilon_0] \in \mathcal{C}(S)$.
(iii) $[\rho_0] \in \mathcal{C}(S)$ *for some* $\rho_0 \subseteqq \mathcal{H}_0$.
(iv) $[\mathcal{H}_0] \in \mathcal{C}(S)$.
(v) $[\rho_0]$ *is defined for all* $\rho_0 \in \mathcal{C}(S)$.
(vi) $\kappa_0 = \varepsilon_0$.

PROOF. Items (i) and (ii) are equivalent by the very definitions. Also items (v) and (vi) are equivalent by Lemma 3.6(ii). Lemma 3.3(iv) yields that (ii) implies (v) and (iii) implies (iv). Trivially (ii) implies (iii) and (v) implies (ii).

(iv) *implies* (i). Let $e, f, g \in E(S)$ be such that $e > f > 0$ and $e > g > 0$. By Lemma 3.4, we have $e[\mathcal{H}_0]f$ and $e[\mathcal{H}_0]g$ so that $f\mathcal{H}_0 g$ and hence $f = g$. By Lemma 2.4, we conclude that the extension is strict.

It is a useful consequence of this lemma that $[\rho_0] \in \mathcal{C}(S)$ for all $\rho_0 \in \in \mathcal{C}(S_0)$. In fact, for any $a \in S_1^*$ and $[\rho_0] \in \mathcal{C}(S)$, we have that $a\varphi$ is $\rho_0$-linked to $a$. Using the retraction $\psi$ mentioned above, the expression for $[\rho_0]$ simplifies to:

$$a[\rho_0]b \Leftrightarrow a\psi\rho_0 b\psi \quad (a, b \in S_0).$$

LEMMA 7.2. *Let $S$ be a strict extension. Then for $\rho_0 \in \mathcal{C}(S_0)$, we have*

$$\ker[\rho_0] = \ker \rho_0 \cup \left\{ a \in S_1^* \mid a\varphi \in \ker \rho_0 \right\}.$$

In this case, even the conditions for membership of $[\rho_0, \rho_1]$ in $\mathcal{C}(S)$ simplify, as we shall now see.

LEMMA 7.3. *Let $S$ be a strict extension and let $\rho_0 \in \mathcal{C}(S_0)$ and $\rho_1 \in \in \mathcal{C}_0(S_1)$. Then $[\rho_0, \rho_1] \in \mathcal{C}(S)$ if and only if for any $a, b \in S_1^*$, $a\rho_1 b$ implies $a\varphi\rho_0 b\varphi$.*

PROOF. This is a special case of ([3], Theorem 3(ii)).

We also have the converse of Lemma 6.6 for strict extensions.

LEMMA 7.4. *Let $S$ be a strict extension. If* $\ker[\lambda_0] \neq \ker[\rho_0, \rho_1]$ *for all* $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$, *then there exists $a \in S_1^*$ such that $a\varphi \in E(S_0^*)$ and $a^2 \in S_0$.*

PROOF. By contrapositive, assume that if $a\varphi \in E(S_0^*)$, then $a^2 \in S_1^*$. We shall construct $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$ such that $\ker[\lambda_0] = \ker[\rho_0, \rho_1]$. With the notation for $\varphi$ above, we get that $(\varepsilon, \ker\omega, \varepsilon)$ is an admissible triple for the Rees matrix semigroup $S_1$; let $\rho_1$ be the corresponding congruence. We show next that $[\varepsilon_0, \rho_1] \in \mathcal{C}(S)$.

Let $a = (i, g, \lambda)$ and $b = (j, h, \mu) \in S_1^*$ be such that $a\rho_1 b$. Then $i = j$, $gh^{-1} \in \ker\omega$ and $\lambda = \mu$ so that

$$a\varphi = (i, g, \lambda)\varphi = \big(i\xi, u_i(g\omega)v_\lambda, \lambda\eta\big) =$$

$$= \big(j\xi, u_j(h\omega)v_\mu, \mu\eta\big) = (j, h\mu)\varphi = b\varphi.$$

Hence $a\rho_1 b$ implies $a\varphi\varepsilon_0 b\varphi$ and Lemma 7.3 implies that $[\varepsilon_0, \rho_1] \in \mathcal{C}(S)$. By Lemma 7.1, we have $[\varepsilon_0] \in \mathcal{C}(S)$. It follows from Lemma 7.2 that

$$\ker[\varepsilon_0] = E(S_0) \cup \big\{a \in S_1^* \,|\, a\varphi \in E(S_0)\big\}$$

and from Lemma 6.1(ii) that $\ker[\varepsilon_0, \rho_1] = E(S_0) \cup (\ker\rho_1)^*$ with

$$(\ker\rho_1)^* = \big\{(i, g, \lambda) \in S_1^* \,|\, p_{\lambda i} \neq 0, gp_{\lambda i} \in \ker\omega\big\}.$$

Hence $\ker[\varepsilon_0] = \ker[\varepsilon_0, \rho_1]$ is equivalent to

$$(i, g, \lambda)\varphi \in E(S_0^*) \Leftrightarrow p_{\lambda i} \neq 0,\ gp_{\lambda i} \in \ker\omega. \tag{7}$$

Now

$$(i, g, \lambda)\varphi = \big(i\xi, u_i(g\omega)v_\lambda, \lambda\eta\big) \in E(S_0^*)$$

$$\Leftrightarrow q_{\lambda\eta, i\xi} \neq 0, \quad u_i(g\omega)v_\lambda = {q_{\lambda\eta, i\xi}}^{-1}$$

$$\Leftrightarrow q_{\lambda\eta, i\xi} \neq 0 \quad (g\omega)^{-1} = v_\lambda q_{\lambda\eta, i\xi} u_i$$

$$\Leftrightarrow q_{\lambda\eta, i\xi} \neq 0, \quad (g\omega)^{-1} = p_{\lambda i}\omega \quad \text{by (6)}$$

$$\Leftrightarrow p_{\lambda i} \neq 0, \quad gp_{\lambda i} \in \ker\omega.$$

Therefore $\ker[\varepsilon_0] = \ker[\varepsilon_0, \rho_1]$, as required.

COROLLARY 7.5. *Let $S$ be a strict extension. Then* $\ker[\lambda_0] \neq \neq \ker[\rho_0, \rho_1]$ *for all* $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$ *if and only if there exists* $a \in S_1^*$ *such that* $a\varphi \in E(S_0^*)$ *and* $a^2 \in S_0$.

PROOF. This follows directly from Lemmas 6.6 and 7.4.

We are now ready for a complete answer to the question: when is $K$ a congruence if the extension is strict. For the case when $S$ is also an inverse semigroup, a similar characterization can be found in ([6], Theorem 5.7).

THEOREM 7.6. *Let the multiplication in $S$ be determined by a partial homomorphism* $\varphi: S_1^* \to S_0^*$. *Then $K$ is a congruence if and only if either* $\varphi: S_1^* \to E(S_0^*)$ *or there exists* $a \in S_1^*$ *such that* $a\varphi \in E(S_0^*)$ *and* $a^2 \in S_0$.

PROOF. *Necessity.* Assume that the second alternative does not take place. By Corollary 7.5, there exist $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$ such that $\ker[\lambda_0] = = \ker[\rho_0, \rho_1]$. Then

$$\ker[\varepsilon_0] = \ker\big([\varepsilon_0] \wedge [\lambda_0]\big) = \ker[\varepsilon_0] \cap \ker[\lambda_0] =$$

$$= \ker[\varepsilon_0] \cap \ker[\rho_0, \rho_1] = \ker\big([\varepsilon_0] \wedge [\rho_0, \rho_1]\big)$$

and $[\varepsilon_1] \wedge [\rho_0, \rho_1] = [\theta_0, \theta_1]$ for some $\theta_0$ and $\theta_1$ so that $[\varepsilon_0]K[\theta_0, \theta_1]$. The hypothesis implies that

$$[\varepsilon_0] \vee [\omega_0, \varepsilon_1]K[\theta_0, \theta_1] \vee [\omega_0, \varepsilon_1]$$

whence $[\omega_0]K[\omega_0, \theta_1]$ by Lemmas 4.5(ii) and 4.2(ii). Hence $\ker[\omega_0, \theta_1] = S$ and thus $\ker\theta_1 = S_1$. Consequently

$$E(S_0) \cup \{a \in S_1^* \mid a\varphi \in E(S_0^*)\} = \ker[\varepsilon_0] \quad \text{by Lemma 7.2}$$

$$= \ker[\theta_0, \theta_1] = \ker\theta_0 \cup (\ker\theta_1)^* \quad \text{by Lemma 6.1(ii)}$$

$$= \ker\theta_0 \cup S_1^*$$

which implies that $S_1^*\varphi \subseteqq E(S_0^*)$. Therefore $\varphi: S_1^* \to E(S_0^*)$.

*Sufficiency.* Assume the first alternative and let $[\lambda_0]K[\rho_0, \rho_1]$. By Lemma 7.2 and the hypothesis, we obtain

$$\ker[\lambda_0] = \ker\lambda_0 \cup S_1^*. \tag{8}$$

Now $\ker[\lambda_0] = \ker[\rho_0, \rho_1]$ by Corollary 6.3(iii) implies that $\ker\lambda_0 = \ker\rho_0$ and $\ker\rho_1 = S_1$. Let $[\theta_0, \theta_1] \in \mathcal{C}(S)$. Then

$$\ker[\lambda_0 \vee \theta_0] = \ker(\lambda_0 \vee \theta_0) \cup S_1^* \quad \text{by Lemma 6.1 and (8)}, \tag{9}$$

(10)
$$\ker[\rho_0 \vee \theta_0, \rho_1 \vee \theta_1] = \ker(\rho_0 \vee \theta_0) \cup \big(\ker(\rho_1 \vee \theta_1)\big)^* \quad \text{by Lemma 6.1(ii).}$$

By Lemma 2.5, we have $\ker(\lambda_0 \vee \theta_0) = \ker(\rho_0 \vee \theta_0)$. Since $\ker\rho_1 = S_1$, we have $\ker(\rho_1 \vee \theta_1) = S_1$. Now (9) and (10) give

$$[\lambda_0 \vee \theta_0] K [\rho_0 \vee \theta_0, \rho_1 \vee \theta_1].$$

By Theorem 6.4, we conclude that $K$ is a congruence.

If the second alternative takes place, then by Corollary 7.5, we have $\ker[\lambda_0] \neq \ker[\rho_0, \rho_1]$ for all $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$ which by Theorem 6.4 yields that $K$ is a congruence.

COROLLARY 7.7. *Let $S$ be a strict extension and suppose that $\varphi$ is a homomorphism. Then $K$ is a congruence if and only if $\varphi\colon S_1^* \to E(S_0^*)$.*

PROOF. The second alternative in Theorem 7.6 can not occur since if $a \in S_1^*$, $a\varphi \in E(S_0^*)$, then $a^2\varphi = a\varphi$ so that $a^2 \in S_1^*$.

In the present case, we can find some more ends of the intervals making up the $K$- and $T$-classes in $\mathcal{C}(S)$.

THEOREM 7.8. *Let $S$ be a strict extension.*
(i) *For $[\rho_0, \rho_1] \in \mathcal{C}(S)$, we have*

$$[\rho_0, \rho_1]_K = \big[(\rho_0)_K, (\rho_1)_K\big], \quad [\rho_0, \rho_1]^T = \big[(\rho_0)^T, (\rho_1)^T\big].$$

(ii) *For $\rho_0 \in \mathcal{C}(S_0)$, we have*

$$[\rho_0]_K = \begin{cases} \big[(\rho_0)_K\big] & \textit{if } \ker[\rho_0] \neq \ker[\lambda_0, \lambda_1] \textit{ for all } [\lambda_0, \lambda_1] \in \mathcal{C}(S)), \\ \big[(\rho_0)_K, \theta_1\big] & \textit{for some } \theta_1 \in \mathcal{C}_0(S_1) \textit{ otherwise.} \end{cases}$$

PROOF. (i) We show first that $\big[(\rho_0)_K, (\rho_1)_K\big] \in \mathcal{C}(S)$. In the Rees matrix representation of $S_1$, let $a = (i, g, \lambda)$ and $b = (j, h, \mu)$ and assume that $a(\rho_1)_K b$. Let $\rho_1$ be represented by the admissible triple $(r_1, N_1, \pi_1)$. Clearly $(\varepsilon, N_1, \varepsilon)$ is admissible and it represents the congruence $(\rho_1)_K$. It follows that $i = j, gh^{-1} \in N_1$ and $\lambda = \mu$. Furthermore, $a\rho_1 b$ and hence $a\varphi\rho_0 b\varphi$ by Lemma 7.3.

Now assume that $\rho_0 \neq \omega_0$ and represent it by the admissible triple $(r_0, N_0, \pi_0)$. Hence

$$a\varphi = \big(i\xi, u_i(g\omega)v_\lambda, \lambda\eta\big)\, \rho_0 b\varphi = \big(i\xi, u_i(h\omega)v_\lambda, \lambda\eta\big)$$

implies that $(gh^{-1})\omega \in N_0$. Since $(\rho_0)_K$ is represented by the triple $(\varepsilon, N_0, \varepsilon)$, similarly as above for $\rho_1$, we conclude that $a\varphi(\rho_0)_K b\varphi$.

Consider the case $\rho_0 = \omega_0$. Note that $\omega_0 K$ consists precisely of band congruences on $S_0$. First assume that $S_0$ has zero divisors. Then there exists $a \in S_0^*$ such that $a^2 = 0$. If $\theta_0 K \omega_0$, then $a\theta_0 a^2 = 0$ and thus $\theta_0 = \omega_0$. In this case we have $(\omega_0)_K = \omega_0$. Suppose next that $S_0$ has no zero divisors. Then $\mathcal{H}_0$ is the least band congruence on $S_0$ so that $(\omega_0)_K = \mathcal{H}_0$. In both of these cases we have $a\varphi(\rho_0)_K b\varphi$.

Therefore $\left[(\rho_0)_K, (\rho_1)_K\right] \in \mathcal{C}(S)$. It now follows from Corollary 6.3(ii) that $[\rho_0, \rho_1]_K = \left[(\rho_0)_K, (\rho_1)_K\right]$.

For the second assertion, we prove first that $\left[(\rho_0)^T, (\rho_1)^T\right] \in \mathcal{C}(S)$. With the same notation as above, let $a(\rho_1)^T b$. Since $\rho_1$ is represented by the triple $(r_1, N_1, \pi_1)$, clearly $(\rho_1)^T$ is represented by $(r_1, G_1, \pi_1)$. Hence $ir_1 j$ and $\lambda \pi_1 \mu$. Let $p_{\theta i} \neq 0$ and $p_{\lambda k} \neq 0$. Then

$$p_{\theta i} 1 p_{\lambda k} = p_{\theta j}\left(p_{\theta j}{}^{-1} p_{\theta i} p_{\lambda k} p_{\mu k}{}^{-1}\right) p_{\mu k}$$

implies that

$$(i, 1, \lambda)\rho_1 \left(j, p_{\theta j}{}^{-1} p_{\theta i} p_{\lambda k} p_{\mu k}{}^{-1}, \mu\right)$$

whence

$$(i, 1, \lambda)\varphi\rho_0 \left(j, p_{\theta j}{}^{-1} p_{\theta i} p_{\lambda k} p_{\mu k}{}^{-1}, \mu\right)\varphi. \tag{11}$$

If $\rho_0 = \omega_0$, then trivially $a\varphi\rho_0 b\varphi$. Assume that $\rho_0 \neq \omega_0$. Then (11) implies that $i\xi r_0 j\xi$ and $\lambda\eta\pi_0\mu\eta$. Since $(\rho_0)^T$ is represented by the triple $(r_0, G_0, \pi_0)$, it follows that

$$a\varphi = \left(i\xi, u_i(g\omega)v_\lambda, \lambda\eta\right)(\rho_0)^T\left(j\xi, u_j(h\omega)v_\mu, \mu\eta\right) = b\varphi,$$

as required. Therefore $\left[(\rho_0)^T, (\rho_1)^T\right] \in \mathcal{C}(S)$. Now Lemma 6.8(ii) implies that $[\rho_0, \rho_1]^T = \left[(\rho_0)^T, (\rho_1)^T\right]$.

(ii) Assume first that $\ker[\rho_0] \neq \ker[\lambda_0, \lambda_1]$ for all $[\lambda_0, \lambda_1] \in \mathcal{C}(S)$. It follows that $[\rho_0]_K$ is of the form $[\theta_0]$. Now Lemma 6.3(i) implies that $[\rho_0]_K = \left[(\rho_0)_K\right]$.

Suppose next that $\ker[\rho_0] = \ker[\lambda_0, \lambda_1]$ for some $[\lambda_0, \lambda_1] \in \mathcal{C}(S)$. Then $[\rho_0]_K = [\lambda_0, \lambda_1]_K = \left[(\lambda_0)_K, (\lambda_1)_K\right]$ by part (i). By Corollary 6.3(iii), we have $\lambda_0 K \rho_0$ and thus $(\lambda_0)_K = (\rho_0)_K$. Therefore $[\rho_0]_K = \left[(\rho_0)_K, \theta_1\right]$ where $\theta_1 = (\lambda_1)_K$.

We can derive one more result from our considerations. As above, we let $S_0 = M^0(I_0, G_0, \Lambda_0; P)$. Following ([4], Definition 6.1), we say that $P$ has no contractions if

$$p_{\lambda i} \neq 0 \Leftrightarrow p_{\mu i} \neq 0 \quad \text{for all} \quad i \in I_0 \quad \text{implies} \quad \lambda = \mu,$$

$$p_{\lambda i} \neq 0 \Leftrightarrow p_{\lambda j} \neq 0 \quad \text{for all} \quad \lambda \in \Lambda_0 \quad \text{implies} \quad i = j.$$

Since for another Rees matrix representation of $S_0$ with sandwich matrix $Q$, we have that $P$ has no contractions if and only if $Q$ has no contractions, we may say that $S_0$ *has no contractions.*

THEOREM 7.9. *Let $S_0$ have no contractions. Then $K$ is a congruence if and only if*

*either the extension is not strict,*

*or the extension is determined by a partial homomorphism $\varphi\colon S_1^* \to \to E(S_0^*)$,*

*or the extension is determined by a partial homomorphism $\varphi\colon S_1^* \to \to S_0^*$ and there exists $a \in S_1^*$ such that $a\varphi \in E(S_0^*)$ and $a^2 \in S_0$.*

PROOF. *Necessity.* It suffices to consider a strict extension. This was taken care of in Theorem 7.6.

*Sufficiency.* Assume first that the extension is not strict. In view of Lemma 2.4, there exist $e, f, g \in E(S)$ such that $e > f > 0$, $e > g > 0$ and $f \neq g$. By Lemma 3.4, we have that $e\kappa f$ and $e\kappa g$ whence $f\kappa g$. Hence $\kappa$ is not idempotent separating. By ([4], Lemma 6.2), the hypothesis implies that all proper congruences on $S_0$ are idempotent separating. Therefore $\kappa_0 = \omega_0$ whence $\kappa = [\kappa_0] = [\omega_0] = \omega$. By Corollary 6.5, we have that $K$ is a congruence on $\mathcal{C}(S)$. If the extension is strict, the same conclusion follows directly from Theorem 7.6.

## 8. The case when $S_1$ has no zero divisors

We first characterize this case in terms of congruences on $S$ and then give necessary and sufficient conditions for $K$ to be a congruence.

PROPOSITION 8.1. *The following conditions on $S$ are equivalent.*

(i) *$S_1$ has no zero divisors.*

(ii) *If $[\rho_0] \in \mathcal{C}(S)$, then* $\ker[\rho_0] = \ker[\rho_0, \rho_1]$ *for some $\rho_1 \in \mathcal{C}_0(S_1)$.*

(iii) $\ker[\omega_0, \mathcal{H}_1] = S$.

PROOF. (i) *implies* (ii). Let $[\rho_0] \in \mathcal{C}(S)$. We define $\rho_1$ on $S_1$ as the proper congruence on $S_1$ with the property that $\rho_1|_{S_1^*} = [\rho_0]|_{S_1^*}$. It follows easily that $[\rho_0, \rho_1] \in \mathcal{C}(S)$. If $a \in S_1 \cap \ker[\rho_0]$, then $a[\rho_0]$ is an idempotent $[\rho_0]$-class and thus also of $\rho_1$ and $a \in (\ker\rho_1)^*$. Conversely, if $a \in (\ker\rho_1)^*$, then clearly $a \in S_1^* \cap [\rho_0]$. Therefore $\ker[\rho_0] = \ker[\rho_0, \rho_1]$.

(ii) *implies* (iii). Since $[\omega_0] \in \mathcal{C}(S)$, by hypothesis $\ker[\omega_0] = \ker[\omega_0, \rho_1]$ for some $\rho_1 \in \mathcal{C}_0(S_1)$. Hence $\ker[\omega_0, \rho_1] = S$ so that $\ker\rho_1 = S_1$. Let $a \in \in S_1$. If $a^2 = 0$, then $a\rho_1 a^2 = 0$ which is possible only for $a = 0$. Hence $S_1$ has no zero divisors which evidently implies that $\ker[\omega_0, \mathcal{H}_1] = S$.

(iii) *implies* (i). The hypothesis implies that $\ker\mathcal{H}_1 = S_1$ which clearly implies that $S_1$ has no zero divisors.

Recall Notations 2.1 and 5.3.

THEOREM 8.2. *Let $S_1$ have no zero divisors. Then $K$ is a congruence if and only if $A \subseteqq \ker \kappa$.*

PROOF. *Necessity.* Since $S_1$ has no zero divisors, we may define a proper congruence $\kappa_1$ on $S_1$ by the requirement that $\kappa_1|_{S_1^*} = \kappa|_{S_1^*}$. It is easily checked that $[\kappa_0, \kappa_1] \in \mathcal{C}(S)$. By Lemma 3.3(iii), we have $[\kappa_0, \kappa_1] \subset [\kappa_0]$ and thus $\ker[\kappa_0, \kappa_1] \subseteqq \ker[\kappa_0]$. Conversely, let $a \in S_1^* \cap \ker \kappa$. Then $a\kappa e$ for some $e \in E(S)$. If $e \in S_1^*$, then $a\kappa_1 e$ and hence $a \in \ker \kappa_1$. If $e \in \in S_0$, then $a\kappa \cap S_1^*$ is an idempotent $\kappa_1$-class and hence $a \in \ker \kappa_1$. Since $[\kappa_0, \kappa_1]|_{S_0} = \kappa|_{S_0} = \kappa_0$, it follows that $\ker[\kappa_0] \subseteqq \ker[\kappa_0, \kappa_1]$ and equality prevails. Therefore $[\kappa_0]K[\kappa_0, \kappa_1]$.

The hypothesis then implies that

$$[\kappa_o] \vee [\omega_0, \varepsilon_1] K [\kappa_0, \kappa_1] \vee [\omega_0, \varepsilon_1]$$

which by Lemmas 4.5(ii) and 4.2(ii) implies that $[\omega_0]K[\omega_0, \kappa_1]$ whence $\ker \kappa_1 = S_1$. Now let $a > b > 0$ in $S$. By Lemma 3.4, we have $a[\kappa_0]b$ which together with $a \in \ker \kappa_1 \subseteqq \ker[\kappa_0]$ implies that $b \in \ker[\kappa_0]$. Therefore $A \subseteqq \ker \kappa$.

*Sufficiency.* By Theorem 6.4, it suffices to consider the case: $\lambda_0 \neq \omega_0$, $\ker[\lambda_0] = \ker[\rho_0, \rho_1]$ and $[\theta_0, \theta_1] \in \mathcal{C}(S)$. Let $a \in S_1^*$. By Lemma 2.2, there exists $b \in S_0^*$ such that $a > b$. It follows that $a[\lambda_0]b$ by Lemma 3.4. Since $b \in A$, the hypothesis implies that $b \in \ker \kappa_0$. Since $[\lambda_0] \in \mathcal{C}(S)$, Lemma 3.6(ii) implies that $\kappa_0 \subseteqq \lambda_0$ which yields $b \in \ker \lambda_0$. Now Lemma 6.1(i) gives $a \in \ker[\lambda_0]$. We deduce that $\ker[\lambda_0] = \ker \lambda_0 \cup S_1^*$ again by Lemma 6.1(i). In addition, Lemma 6.1(ii) implies that $\ker[\rho_0, \rho_1] = \ker \rho_0 \cup (\ker \rho_1)^*$. Now the equality $\ker[\lambda_0] = \ker[\rho_0, \rho_1]$ gives $\ker \rho_1 = S_1$. Finally, by Corollary 6.3(iii), we have $\ker \lambda_0 = \ker \rho_0$ so that

$$\begin{aligned}\ker[\lambda_0 \vee \theta_0] &= \ker(\lambda_0 \vee \theta_0) \cup S_1^* \quad \text{by Lemma 6.1(i)}\\ &= \ker(\rho_0 \vee \theta_0) \cup S_1^* \quad \text{by Lemma 2.5}\\ &= \ker[\rho_0 \vee \theta_0, \rho_1 \vee \theta_0] \quad \text{by Lemma 6.1(ii)},\end{aligned}$$

as required.

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