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LAGRANGE INTERPOLATION FOR FUNCTIONS OF BOUNDED VARIATION

J. PRESTIN (Rostock)

1. Introduction. We investigate the Lagrange interpolation on Jacobi abscissas for functions of bounded variation on $[-1,1]$. Error estimates in weighted L^p-norms are established, which ensure convergence for a wide class of weights and Jacobi parameters α, β. The order of convergence is in most cases best possible. For some Jacobi nodes the results can be improved, if the function is of bounded variation and continuous. It turns out that the estimates depend essentially on the behaviour of the error near the endpoints of the interval.

2. Notations and preliminary results. Let

$$(1) \qquad -1 < x_n^{(\alpha,\beta)} < x_{n-1}^{(\alpha,\beta)} < \ldots < x_2^{(\alpha,\beta)} < x_1^{(\alpha,\beta)} < 1$$

be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ ($\alpha, \beta > -1$, $n = 1, 2, \ldots$; see e.g. G. Szegő [5]). In the sequel we will omit the superfluous notations α, β. Further we write $x_{n+1} = -1$, $x_0 = 1$ and $x_k = \cos \vartheta_k$, $x = \cos \vartheta$ with $0 \leqq \vartheta_k$, $\vartheta \leqq \pi$. For a function f defined in the interval $[-1,1]$ we denote, as usual,

$$L_n f(x) = \sum_{k=1}^{n} f(x_k) l_k(x)$$

the Lagrange interpolatory polynomials of degree $n-1$ based on the nodes (1). Here l_k are the k-th fundamental polynomials of the Lagrange interpolation

$$l_k(x) = \frac{P_n(x)}{P_n'(x_k)(x - x_k)}, \qquad k = 1, \ldots, n.$$

In [6] P. Vértesi proved for $f \in C \cap BV$ (f is continuous and of bounded variation in $[-1,1]$) and for $-1 < \alpha, \beta < 1/2$ that

$$(2) \qquad \max_{x \in [-1,1]} |f(x) - L_n f(x)| = o(1), \qquad n \to \infty.$$

Furthermore, it is proved in [7] that (2) does not hold in general for $\max(\alpha, \beta) \geqq 1/2$. However, for arbitrary $\alpha, \beta > -1$ and $f \in BV \cap C$, J. L. Geronimus [1] obtained pointwise convergence, too, but only in a compact subinterval of $(-1,1)$. Let us mention here that pointwise estimates for trigonometric interpolation of functions from $BV_{2\pi} \cap C_{2\pi}$ on equidistant nodes in $[0, 2\pi]$ are due to W. Quade [4]. An $L^2_{2\pi}$-error estimate is given by K. Zacharias in [8]. For the general $L^p_{2\pi}$-case see [2] and [3].

Now we define for $1 \leqq p < \infty$, $a, b > -1$ and $0 \leqq \varepsilon < 1$ the norm

$$\|f\|_{p,a,b,\varepsilon} = \left(\int_{-1+\varepsilon}^{1-\varepsilon} |f(x)|^p w(x)dx \right)^{1/p},$$

where $w(x) = (1-x)^a(1+x)^b$. Obviously, the weight w is only interesting for asymptotic results in the case $\varepsilon = 0$.

3. Results. The main result here is the following theorem.

THEOREM 1. *Let $f \in BV$. If $\varepsilon > 0$, then there exists a constant C, depending only on α, β, a, b, p and ε, such that for arbitrary $\alpha, \beta > -1$ and a, b*

$$\|f - L_n f\|_{p,a,b,\varepsilon} \leqq C \cdot V(f) \cdot n^{-1/p} \cdot \begin{cases} \ln n & \text{if } p = 1, \\ 1 & \text{if } 1 < p < \infty, \end{cases}$$

where $V(f)$ denotes the total variation of f on $[-1, 1]$. If $\varepsilon = 0$, then there exists a constant C, depending only on α, β, a, b and p such that

$$\|f - L_n f\|_{p,a,b,0} \leqq C \cdot V(f) \cdot (n^{d(p,a,\alpha)} D(p, a, \alpha, n) + n^{d(p,b,\beta)} D(p, b, \beta, n)),$$

where $d(p, a, \alpha) = \max\left(\alpha - \frac{1}{2} - \frac{2a+2}{p}, -\frac{2a+2}{p}, -\frac{1}{p}\right)$ and

$$D(p, a, \alpha, n) = \begin{cases} \ln^2 n & \text{if } p = 1,\ \alpha = 1/2,\ a = -1/2, \\ \ln n & \text{if } (p = 1,\ \alpha \neq 1/2,\ a \geqq -1/2,\ \alpha \leqq 2a + 3/2) \text{ or} \\ & \text{if } (p = 1,\ \alpha = 1/2,\ a \neq -1/2) \text{ or} \\ & \text{if } (p > 1,\ \alpha = 1/2,\ a \leqq -1/2), \\ 1 & \text{otherwise.} \end{cases}$$

For $\alpha \neq 1/2$ this means

$$D(p, a, \alpha, n) = \begin{cases} \ln n & \text{if } d(p, a, \alpha) = -1, \\ 1 & \text{if } d(p, a, \alpha) > -1. \end{cases}$$

We remark here that one can easily deduce the corresponding estimates for the one-sided L^p-norms on $[-1, 1-\varepsilon]$ and $[-1+\varepsilon, 1]$ with $\varepsilon > 0$. To illustrate the case $\varepsilon = 0$ we consider possible choices of the parameters.

COROLLARY 1. *Let $f \in BV$ be given. For every $\alpha, \beta > -1$ one can choose a, b, namely $a \geqq -1/2$ if $\alpha < 1/2$, $a > -1/2$ if $\alpha = 1/2$, $a \geqq$ $\geqq (\alpha - 1/2)p/2 - 1/2$ if $\alpha > 1/2$ and b respectively, such that*

$$\|f - L_n f\|_{p,a,b,0} \leqq C \cdot V(f) \cdot n^{-1/p} \cdot \begin{cases} \ln n & \text{if } p = 1, \\ 1 & \text{if } 1 < p < \infty. \end{cases}$$

Furthermore

$$\|f - L_n f\|_{1,0,0,0} = o(1) \quad \textit{if} \quad -1 < \alpha, \beta < 5/2,$$

$$\|f - L_n f\|_{2,0,0,0} = o(1) \quad \textit{if} \quad -1 < \alpha, \beta < 3/2,$$

and for all $1 \leqq p < \infty$

$$\|f - L_n f\|_{p,0,0,0} = o(1) \quad \textit{if} \quad -1 < \alpha, \beta \leqq 1/2$$

hold for the unweighted norm as $n \to \infty$.

Now it is interesting to investigate whether the order of convergence is best possible or not.

THEOREM 2. *Let* $a, b, \alpha, \beta > -1$. *Then there exists a function* $f_n \in BV$ *such that for* $0 \leqq \varepsilon < 1$

$$\|f_n - L_n f_n\|_{p,a,b,\varepsilon} \geqq C \cdot V(f_n) \cdot n^{-1/p} \cdot \begin{cases} \ln n & \textit{if} \ \ p = 1, \\ 1 & \textit{if} \ \ 1 < p < \infty, \end{cases} \tag{3}$$

and for $\varepsilon = 0$

$$\|f_n - L_n f_n\|_{p,a,b,0} \geqq C \cdot V(f_n) \cdot (n^{d(p,a,\alpha)} \tilde{D}(p,a,\alpha,n) + n^{d(p,b,\beta)} \tilde{D}(p,b,\beta,n))$$

with

$$\tilde{D}(p,a,\alpha,n) = \begin{cases} D(p,a,\alpha,n)/\ln n & \textit{if} \ \ (\alpha = 1/2, \ p = 1, \ a = -1/2) \ \textit{or} \\ & \textit{if} \ \ (\alpha = 1/2, \ p > 1, \ a \leqq -1/2), \\ D(p,a,\alpha,n) & \textit{otherwise}. \end{cases}$$

That means, the order of convergence in Theorem 1 is best possible, up to a $\log n$ term in the case $\alpha = 1/2$ and $a \leqq -1/2$. Nevertheless assuming continuity we can improve Theorem 1. Then we obtain the following estimate.

THEOREM 3. *Let* $f \in BV \cap C$ *and* $1 < p < \infty$. *If* $\varepsilon > 0$, *then*

$$\|f - L_n f\|_{p,a,b,\varepsilon} = o(n^{-1/p}), \quad n \to \infty.$$

Moreover, if $\varepsilon = 0$ *and* $-1 < \alpha, \beta < 1/2$, *then*

$$\|f - L_n f\|_{p,a,b,0} = o\left(n^{d(p,a,\alpha)} + n^{d(p,b,\beta)}\right), \qquad n \to \infty.$$

4. Proofs. 4.1. Let $\min_{1 \leqq k \leqq n} |x - x_k| = |x - x_j|$, further denote

$$A(x,m) = \sum_{k=1}^{m} l_k(x), \qquad m = 1, \dots, n$$

and

$$B(x,m) = 1 - A(x,m) = \sum_{k=m+1}^{n} l_k(x), \quad m = 1, \dots, n-1.$$

Now we prove the following estimate, which is a refinement of the corresponding Lemma 4.3 in [7], where a superfluous $\log n$ term is included.

LEMMA 1. *We have for $m = 1,2,\dots,n$ and $m \leqq j \leqq n$*

(4)
$$|A(x,m)| \leqq \frac{C}{j-m+1} + \frac{C}{n-j+1} \cdot \begin{cases} 1 & \text{if } \beta < 1/2, \\ \ln \frac{2n}{n-j+1} & \text{if } \beta = 1/2, \\ (n-j+1)^{-\beta+1/2} n^{\beta-1/2} & \text{if } \beta > 1/2, \end{cases}$$

and for $m = 1,\dots,n-1$, $1 \leqq j \leqq m$

$$|B(x,m)| \leqq \frac{C}{m-j+1} + \frac{C}{j} \cdot \begin{cases} 1 & \text{if } \alpha < 1/2, \\ \ln \frac{2n}{j} & \text{if } \alpha = 1/2, \\ j^{-\alpha+1/2} n^{\alpha-1/2} & \text{if } \alpha > 1/2. \end{cases} \tag{5}$$

Here as in the whole paper we are using the symbol C for positive constants, which depend only on the parameters a, b, α, β, p and ε, but are independent of f, n, j and m. Then C does not necessarily denote the same constant from term to term.

PROOF. I. We can write

$$|A(x,m)| \leqq \sum_{k=1}^{m-1} |l_k(x) + l_{k+1}(x)|.$$

To estimate now $|l_k(x) + l_{k+1}(x)|$ we use the ideas of [7]. Particularly, it is stated there (see [7], (4.16)) for $1 \leqq k \leqq n-1$ and $-1+\eta \leqq x \leqq 1$ that

(6)
$$|l_k(x) + l_{k+1}(x)| \leqq$$
$$\leqq C \cdot \left(\frac{k^{\alpha+1/2}}{j^{\alpha+1/2}(k+j)(|k-j|+1)} + \frac{k^{\alpha+5/2}}{j^{\alpha+1/2}(k+j)^2(|k-j|+1)^2} \right).$$

But (6) is valid only for $k < c_1 n$. (In the following $0 < c$, $c_1, c_2 < 1$ are constants independent of n, j, m and k.) To include the other cases j or $k > cn$ we write $\overline{P}_n = P_n^{(\beta,\alpha)}$ with the corresponding $\overline{x}_k$, $\overline{l}_k$. Then we can use the well-known fact that

$$P_n(x) = (-1)^n \overline{P}_n(-x),$$

which gives $x - x_k = x + \overline{x}_{n-k+1}$ and

$$l_k(x) = \overline{l}_{n-k+1}(-x). \tag{7}$$

To handle the term $l_k(x) + l_{k+1}(x)$ for $k > c_1 n$, $j < c_2 n$ we proceed in the same manner as in [7] and write

$$|l_k(x) + l_{k+1}(x)| = \left| \frac{P_n(x)}{P_n'(x_k)} \cdot \left(\frac{P_n'(x_k) + P_n'(x_{k+1})}{P_n'(x_{k+1})(x - x_{k+1})} + \frac{x_k - x_{k+1}}{(x-x_k)(x-x_{k+1})} \right) \right| =$$

$$= \left| \frac{P_n(x)}{\overline{P}'_n(\overline{x}_{n-k+1})} \cdot \left(\frac{\overline{P}'_n(\overline{x}_{n-k+1}) + \overline{P}'_n(\overline{x}_{n-k})}{\overline{P}'_n(\overline{x}_{n-k})(x - x_{k+1})} + \frac{x_k - x_{k+1}}{(x - x_k)(x - x_{k+1})} \right) \right|.$$

Now using for $k > c_1 n$, $n - k > C$ the inequality ([7], p. 422)

$$|(\overline{P}'_n(\overline{x}_{n-k+1}) + \overline{P}'_n(\overline{x}_{n-k}))/\overline{P}'_n(\overline{x}_{n-k})| \leqq C/(n-k+1)$$

and the well-known formulas ([5], [6], [7])

$$(8) \qquad |P_n(x)| \sim |x - x_j| \vartheta_j^{-\alpha-3/2} n^{1/2} < C \cdot j^{-\alpha-1/2} n^{\alpha} \qquad \text{for } j < cn,$$

$$(9) \qquad |\overline{P}'_n(x_{n-k+1})| \sim (n-k+1)^{-\beta-3/2} n^{\beta+2} \qquad \text{for } k > cn,$$

$$(10) \qquad |x - x_k| \sim |j-k|(2n+1-j-k)n^{-2} \qquad \text{for } k > cn, \ k \neq j,$$

$$x_k - x_{k+1} \sim (n-k+1)n^{-2} \qquad \text{for } k > cn,$$

we obtain for $k > c_1 n$ and $j < c_2 n$ that

$$|l_k(x) + l_{k+1}(x)| \leqq C \cdot n^{\alpha-\beta-1} j^{-\alpha-1/2} \frac{(n-k+1)^{\beta+1/2}}{|j-k|+1} \left(1 + \frac{(n-k+1)^2}{n(|j-k|+1)} \right).$$

Remark that the case $j = k$ is obviously seen from $c_1 n < k = j < c_2 n$ and $|l_j(x)| \leqq C$ for all $x_{j+1} \leqq x \leqq x_{j-1}$. Finally, if $j > c_1 n$, then by (7) and by the estimates (8)–(10) we can verify for $k < c_2 n$ that

$$(11) \qquad |l_k(x) + l_{k+1}(x)| \leqq$$

$$\leqq C \cdot n^{\beta-\alpha-1} (n-j+1)^{-\beta-1/2} \frac{k^{\alpha+1/2}}{|j-k|+1} \left(1 + \frac{k^2}{n(|j-k|+1)} \right).$$

Analogously we get for $j > c_1 n$ and $k > c_2 n$ that

$$|l_k(x) + l_{k+1}(x)| \leqq C \cdot \frac{(n-k+1)^{\beta+1/2}}{(n-j+1)^{\beta+1/2}(2n+2-j-k)(|k-j|+1)} \cdot$$
$$\cdot \left(1 + \frac{(n-k+1)^2}{(2n+2-k-j)(|k-j|+1)} \right).$$

II. To estimate $A(x, m)$ we distinguish two cases.

a) If $j > 2m$, then we can use formulas (6) and (11) for $k < cn$. If $j < c_1 n$ we conclude

$$A(x, m) \leqq C \cdot \sum_{k=1}^{m-1} \frac{k^{\alpha+1/2}}{j^{\alpha+1/2}(k+j)(|k-j|+1)} \left(1 + \frac{k^2}{(k+j)(|k-j|+1)} \right) \leqq$$

$$\leqq C \cdot \left(\frac{m^{\alpha+3/2}}{j^{\alpha+5/2}} + \frac{m^{\alpha+7/2}}{j^{\alpha+9/2}} \right) \leqq \frac{C}{j} \leqq \frac{C}{j-m}.$$

If $j > c_1 n$ one obtains

$$A(x,m) \leqq C \cdot \frac{n^{\beta-\alpha-1}}{(n-j+1)^{\beta+1/2}} \sum_{k=1}^{m-1} \frac{k^{\alpha+1/2}}{|j-k|+1} \cdot \left(1 + \frac{k^2}{n(|j-k|+1)}\right) \leqq$$

$$\leqq C \cdot \frac{n^{\beta-\alpha-2}}{(n-j+1)^{\beta+1/2}} \cdot \sum_{k=1}^{m-1} k^{\alpha+1/2} \leqq C \cdot \frac{n^{\beta-1/2}}{(n-j+1)^{\beta+1/2}}.$$

b) If $m < j \leqq 2m$, then we split up

$$\sum_{k=1}^{m-1} = \sum_{k=1}^{j/2} + \sum_{k=1+j/2}^{m-1}.$$

(Here and later $\sum\limits_{k=a}^{b}$ stands for $\sum\limits_{k=[a]}^{[b]}$ and let $\sum\limits_{k=a}^{b} = 0$ if $[a] > [b]$.) Now the case $j < c_1 n$ implies $k < c_1 n$ and

$$A(x,m) \leqq C \cdot \left(\frac{1}{j} + \frac{1}{j} \ln \frac{j}{j-m} + \frac{1}{j-m} \right) \leqq \frac{C}{j-m}.$$

If $j > c_2 n$ we can deduce (cf. (11))

$$\sum_{k=1}^{j/2} |l_k(x) + l_{k+1}(x)| \leqq C \cdot n^{\beta-\alpha-1}(n-j+1)^{-\beta-1/2} n^{\alpha+1/2} =$$

$$= C \cdot n^{\beta-1/2}(n-j+1)^{-\beta-1/2}$$

and

$$\sum_{k=1+j/2}^{m-1} |l_k(x) + l_{k+1}(x)| \leqq$$

$$\leqq C \cdot (n-j+1)^{-\beta-1/2} \sum_{k=1+j/2}^{m-1} \frac{(n-k+1)^{\beta+1/2}}{(2n+2-j-k)(j-k+1)} \cdot$$

$$\cdot \left(1 + \frac{(n-k+1)^2}{(2n+2-k-j)(j-k+1)}\right) \leqq$$

$$\leqq C \cdot (n-j+1)^{-\beta-1/2} \sum_{s=j-m+2}^{j/2} \frac{(s+n-j)^{\beta+1/2}}{s(s+2n-2j)} \cdot \left(1 + \frac{(s+n-j)^2}{s(s+2n-2j)}\right).$$

If $\beta \leqq -1/2$ we have $\left(\frac{s+n-j}{1+n-j}\right)^{\beta+1/2} \leqq 1$, which gives

$$\sum_{k=1+j/2}^{m-1} \leqq C \cdot \sum_{s=j-m+2}^{j/2} \left(\frac{1}{s^2} + \frac{(s+n-j)^2}{s^2(s+2n-2j)^2}\right) \leqq \frac{C}{j-m}.$$

On the other hand, if $\beta > -1/2$, we write

$$\sum_{k=1+j/2}^{m-1} \leqq C \cdot (n-j+1)^{-\beta-1/2}\Big(\sum_{s=j-m+2}^{n-j-1} + \sum_{s=t}^{j/2}\Big)$$

with $t = \max(n-j, j-m)$. Hence

$$C \cdot (n-j+1)^{-\beta-1/2} \sum_{s=j-m+2}^{n-j-1} \frac{(s+n-j)^{\beta+1/2}}{s(s+2n-2j)} \cdot \left(1 + \frac{(s+n-j)^2}{s(s+2n-2j)}\right) \leqq$$

$$\leqq C \cdot \sum_{s=j-m+2}^{n-j-1} \frac{1}{s^2} \leqq \frac{C}{j-m},$$

where we have used $\left(\frac{s+n-j}{1+n-j}\right)^{\beta+1/2} \leqq 2^{\beta+1/2}$.

Now we have only to deal with

$$C \cdot (n-j+1)^{-\beta-1/2} \sum_{s=t}^{j/2} \frac{(s+n-j)^{\beta+1/2}}{s(s+2n-2j)} \cdot \left(1 + \frac{(s+n-j)^2}{s(s+2n-2j)}\right) \leqq$$

$$\leqq C \cdot (n-j+1)^{-\beta-1/2} \sum_{s=1}^{j/2} (s+n-j)^{\beta-3/2} \leqq$$

$$\leqq C \cdot \begin{cases} (n-j+1)^{-\beta-1/2}(1+n-j)^{\beta-1/2} & \text{if } \beta < 1/2, \\ (n-j+1)^{-1} \ln \frac{2n}{1+n-j} & \text{if } \beta = 1/2, \\ (n-j+1)^{-\beta-1/2} n^{\beta-1/2} & \text{if } \beta > 1/2. \end{cases}$$

Here we used $s \geqq n-j$, i.e.,

$$\frac{1}{s(s+2n-2j)} \leqq 2(s+n-j)^{-2}.$$

Therefore the estimate (4) is proved.

III. The inequality (5) is a simple consequence of (4) and

$$B(x,m)=\sum_{k=m+1}^{n} l_k(x)=\sum_{k=m+1}^{n} \bar{l}_{n-k+1}(x)=\sum_{k=1}^{n-m} \bar{l}_k(-x). \qquad \square$$

4.2. Here we prove the theorems for some special step functions. Therefore let us consider any monotone increasing function g_m, $0 \leqq m \leqq n$, with

$$g_m(x)=\begin{cases} 0 & \text{if } \ -1 \leqq x \leqq x_{m+1}, \\ d_m & \text{if } \ x_m \leqq x \leqq 1. \end{cases} \tag{12}$$

Then we have the following result.

LEMMA 2. *The estimates of Theorem* 1 *are valid for*

$$\|g_m - L_n g_m\|_{p,a,b,\varepsilon}.$$

PROOF. I. We write

$$\|g_m - L_n g_m\|^p_{p,a,b,\varepsilon}=\sum_{j=1}^{n}\int_{I_j}\left|\sum_{k=1}^{n}(g_m(x)-g_m(x_k))l_k(x)\right|^p w(x)dx,$$

where

$$I_j=[-1+\varepsilon,1-\varepsilon]\cap\begin{cases}\left[\frac{x_{j+1}+x_j}{2},\frac{x_j+x_{j-1}}{2}\right] & \text{if } \ j=2,\dots,n-1,\\ \left[\frac{x_1+x_2}{2},1\right] & \text{if } \ j=1,\\ \left[-1,\frac{x_{n-1}+x_n}{2}\right] & \text{if } \ j=n.\end{cases}$$

Then we obtain by (12)

$$\|g_m - L_n g_m\|^p_{p,a,b,\varepsilon}=\sum_{j=1}^{m-1}\int_{I_j}\left|\sum_{k=m+1}^{n} d_m l_k(x)\right|^p w(x)dx+ \tag{13}$$

$$+\int_{I_m\cup I_{m+1}}\left|g_m(x)-\sum_{k=1}^{m} d_m l_k(x)\right|^p w(x)dx+\sum_{j=m+2}^{n}\int_{I_j}\left|\sum_{k=1}^{m} d_m l_k(x)\right|^p w(x)dx=$$

$$=d_m^p\left(\sum_{j=1}^{m-1}\int_{I_j}|B(x,m)|^p w(x)dx+\int_{I_m\cup I_{m+1}}\left|\frac{g_m(x)}{d_m}-A(x,m)\right|^p w(x)dx+\right.$$

$$\left.+\sum_{j=m+2}^{n}\int_{I_j}|A(x,m)|^p w(x)dx\right)=d_m^p\cdot(K_1+K_2+K_3).$$

From Lemma 1 and the inequality $(a+b)^p \leqq 2^{p-1}(a^p+b^p)$ it follows now for $\alpha, \beta > 1/2$ that

$$K_1 \leqq C \cdot \sum_{j=1}^{m-1} \left((m-j+1)^{-p} + n^{p\alpha-p/2} j^{-p\alpha-p/2}\right) \cdot \int_{I_j} w(x)dx, \tag{14}$$

$$K_2 \leqq C \cdot \int_{I_m \cup I_{m+1}} w(x)dx \cdot \left(1 + \frac{n^{p\beta-p/2}}{(n-m+1)^{p\beta+p/2}}\right)$$

and

$$K_3 \leqq C \cdot \sum_{j=m+2}^{n} \left((j-m+1)^{-p} + n^{p\beta-p/2}(n-j+1)^{-p\beta-p/2}\right) \cdot \int_{I_j} w(x)dx.$$

If $\alpha \leqq 1/2$ or $\beta \leqq 1/2$ we have to modify these estimates according to Lemma 1.

II. To estimate (13) we consider at first the easy case $\varepsilon > 0$. This means $|I_j| \neq 0$ only for $c_1 n \leqq j \leqq c_2 n$, which gives by (10)

$$\int_{I_j} w(x)dx \leqq \frac{C}{n}.$$

Then it follows for all $\alpha, \beta > -1$ that

$$K_1 \leqq \frac{C}{n} \sum_{j=2}^{m} j^{-p} = \begin{cases} \frac{C}{n} \ln n & \text{if } p = 1, \\ C/n & \text{if } p > 1, \end{cases}$$

$$K_2 \leqq C/n^{p+1}$$

and

$$K_3 \leqq \frac{C}{n} \sum_{j=3}^{n-m+1} j^{-p} \leqq \begin{cases} \frac{C}{n} \ln n & \text{if } p = 1, \\ C/n & \text{if } p > 1, \end{cases}$$

which immediately imply the assertion.

III. For $\varepsilon = 0$ we have to do more. Using the well-known relation $x_j - x_{j+1} \sim (j+1)n^{-2}$ if $j \leqq c_1 n$ and $x_j - x_{j+1} \sim (n-j+1)n^{-2}$ if $j \geqq c_2 n$ we get by a simple integration of the weight w

$$\int_{I_j} w(x)dx \leqq C \cdot \begin{cases} j^{2a+1} n^{-2a-2} & \text{if } 1 \leqq j \leqq c_1 n, \\ (n-j+1)^{2b+1} n^{-2b-2} & \text{if } c_2 n \leqq j \leqq n. \end{cases}$$

Hence we obtain from (14) for $\alpha > 1/2$ that

$$K_1 \leqq C \cdot \Big(\sum_{j=1}^{n/2} (m-j+1)^{-p} j^{2a+1} n^{-2a-2} + \sum_{j=n/2}^{m-1} (m-j+1)^{-p} (n-j+1)^{2b+1} \cdot$$

$$\cdot n^{-2b-2} + \sum_{j=1}^{n/2} n^{p\alpha - p/2 - 2a - 2} j^{2a+1-p\alpha - p/2} +$$

$$+ \sum_{j=n/2}^{m-1} n^{p\alpha - p/2 - 2b - 2} j^{-p\alpha - p/2} (n-j+1)^{2b+1} \Big) = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4 .$$

We estimate the four terms on the right hand side as follows: For $\sum_1$ we use in the case $2a+1 \geqq 0$ that $(j/n)^{2a+1} \leqq C$, which gives

$$\sum\nolimits_1 \leqq \frac{C}{n} \cdot \begin{cases} \ln n & \text{if} \quad p = 1, \\ 1 & \text{if} \quad p > 1. \end{cases} \tag{15}$$

In the opposite case $2a+1 < 0$ we obtain

$$\sum_{j=1}^{m-1} (\min(m-j+1), j))^{-p+2a+1} \leqq C.$$

Thus $\sum_1 \leqq C n^{-2a-2}$. Analogously we get the estimate for $\sum_2$ with b instead of a in the exponent. Furthermore, it is easy to see that

$$\sum\nolimits_3 \leqq C \cdot \begin{cases} n^{-p} & \text{if} \quad 2a+2 > \alpha p + p/2, \\ n^{-p} \ln n & \text{if} \quad 2a+2 = \alpha p + p/2, \\ n^{\alpha p - p/2 - 2a - 2} & \text{if} \quad 2a+2 < \alpha p + p/2, \end{cases}$$

and

$$\sum\nolimits_4 \leqq C \cdot \sum_{j=n/2}^{m-1} n^{-p-2b-2} (n-j+1)^{2b+1} \leqq C \cdot n^{-p}.$$

Now it remains to consider the case $\alpha \leqq 1/2$, where we have to modify only $\sum_3$ and $\sum_4$. Then we get for $\alpha = 1/2$

$$\sum\nolimits_3 \leqq C \cdot \sum_{j=1}^{n/2} n^{-2a-2} j^{2a+1-p} \ln^p (n/j) \leqq C \cdot \begin{cases} n^{-p} \ln^p n & \text{if} \quad 2a+2 > p, \\ n^{-p} \ln^{1+p} n & \text{if} \quad 2a+2 = p, \\ n^{-2a-2} \ln^p n & \text{if} \quad 2a+2 < p, \end{cases}$$

and

$$\sum\nolimits_4 \leqq C \cdot \sum_{j=n/2}^{m-1} n^{-p-2b-2} \cdot (n-j+1)^{2b+1} \ln^p \Big(\frac{n}{j} \Big) \leqq C \cdot n^{-p}.$$

For $\alpha < 1/2$ the inequalities are the same up to the $\log^p$-term. Summarizing these estimates we obtain for K_1

$$K_1 \leqq C \cdot n^{\max(\alpha p - p/2 - 2a - 2, -2a-2, -1)} D^p(p, a, \alpha, n) + C \cdot n^{-2b-2}$$

with $D(p, a, \alpha, n)$ defined as in Theorem 1. Analogously we have

$$K_2 \leqq C \cdot n^{\max(p\beta - p/2 - 2b - 2, -2a-2, -2b-2, -1)}$$

and

$$K_3 \leqq C \cdot n^{\max(p\beta - p/2 - 2b - 2, -2b-2, -1)} D^p(p, b, \beta, n) + C \cdot n^{-2a-2},$$

which proves the lemma. □

4.3. To obtain Theorem 1 we argue as follows. If $f \in BV$, then let $f = g - h$, where g and h are monotone increasing functions with $V(f) = V(g) + V(h)$. Further we assume w.l.o.g. $g(-1) = h(-1) = 0$. Now let us consider monotone functions g_m, $m = 0, \dots, n$, defined by (12) with $d_m = g(x_m) - g(x_{m+1})$ and

$$g_m(x) = g(x) - g(x_{m+1}) \quad \text{if} \quad x \in [x_{m+1}, x_m].$$

Thus we have

$$g(x) = \sum_{m=0}^{n} g_m(x) \quad \text{and} \quad V(g) = \sum_{m=0}^{n} V(g_m) = \sum_{m=0}^{n} d_m.$$

Using Lemma 2 it follows

$$\|g - L_n g\|_{p,a,b,e} \leqq \sum_{m=0}^{n} \|g_m - L_n g_m\|_{p,a,b,e} \leqq$$

$$\leqq C \cdot V(g) \left(n^{d(p,a,\alpha)} D(p, a, \alpha, n) + n^{d(p,b,\beta)} D(p, b, \beta, n) \right).$$

With

$$\|f - L_n f\|_{p,a,b,e} \leqq \|g - L_n g\|_{p,a,b,e} + \|h - L_n h\|_{p,a,b,e}$$

one obtains the assertion. □

4.4. To show Theorem 2 it is sufficient to consider such simple functions, which are zero except at one point, i.e.,

$$f_{n,k}(x) = \begin{cases} 1 & \text{if} \quad x = x_k, \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$\sup_{V(f_n)=1} \|f_n - L_n f_n\|_{p,a,b,e} \geqq \max_{1 \leqq k \leqq n} \|l_k\|_{p,a,b,e}.$$

With (8)–(10) it is easy to estimate

$$\|l_{2+[n/2]}\|^p_{p,a,b,\varepsilon} \geqq C\cdot\Big(\sum_{j=1}^{n/4}+\sum_{j=n/4}^{n/2}\Big)\int_{I_j}|l_{2+[n/2]}(x)|^p(1-x)^a dx = \sum{}_1+\sum{}_2.$$

For all $\varepsilon \geqq 0$ we have

$$\sum{}_2 \geqq \frac{C}{n}\cdot\sum_{j=n/4}^{n/2}\Big(1+\frac{n}{2}-j\Big)^{-p} \sim \begin{cases} n^{-1} & \text{if } p>1,\\ n^{-1}\ln n & \text{if } p=1.\end{cases}$$

This already proves (3). Furthermore, for $\varepsilon = 0$ and $\alpha-1/2-(2a+2)/p > -1$ we have by (8) and (9)

$$\sum{}_1 \geqq C\cdot\sum_{j=2}^{n/4}\int_{I_j}\left|\frac{P_n(x)}{P_n'(x_{2+[n/2]})}\right|^p(1-x)^a dx \geqq$$

$$\geqq C\cdot\sum_{j=2}^{n/4}\Big(\frac{j}{n}\Big)^{(-\alpha-3/2)p+2a}\int_{I_j}|x-x_j|^p dx \geqq$$

$$\geqq C\cdot n^{(\alpha-1/2)p-2a-2}\sum_{j=2}^{n/4} j^{(-\alpha-1/2)p+2a+1} \geqq C\cdot n^{(\alpha-1/2)p-2a-2}.$$

Finally we quote

$$\|l_3\|^p_{p,a,b,0} \geqq C\cdot\int_{I_1}|l_3(x)|^p(1-x)^a dx \geqq C\cdot\int_{I_1}(1-x)^a dx \geqq C\cdot n^{-2a-2}. \quad \square$$

4.5. To prove Theorem 3 let us write

$$\|f-L_n\|^p_{p,a,b,\varepsilon} \leqq \max_{|x|\leqq 1-\varepsilon}|f(x)-L_nf(x)|^s\cdot\int_{-1+\varepsilon}^{1-\varepsilon}|f(x)-L_nf(x)^{p-s}w(x)dx,$$

where $0 < s < p-1$. I.e.,

$$\|f-L_nf\|_{p,a,b,\varepsilon} \leqq \max_{|x|\leqq 1-\varepsilon}|f(x)-L_nf(x)|^{s/p}\cdot\|f-L_nf\|^{1-s/p}_{p-s,a,b,\varepsilon}.$$

Using Theorem 1 for $\alpha,\beta < 1/2$, $p > 1$ and $\varepsilon = 0$ we obtain immediately

$$\|f-L_nf\|^{1-s/p}_{p-s,a,b,\varepsilon} = O\left(n^{\max(-1,-2a-2,-2b-2)/p}\right), \qquad n\to\infty,$$

which gives with (2) the desired result. The case $\varepsilon > 0$ is obvious. $\square$

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ON BIRKHOFF QUADRATURE FORMULAS. II

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Dedicated to Professor O. Kis

1. Among the quadrature formulas (q.f.) (1.3), the most important are the Gauss formulas. As Gauss found in 1821, for each integer $n \geqq 1$ there exist points $-1 < x_{nn} < x_{n-1,n} < \ldots < x_{1,n}$ and constants $\lambda_{kn} > 0$, $k = 1, 2, \ldots, n$, so that the formula

$$(1.1) \qquad \int_{-1}^{1} f(x)dx = \lambda_{1n} f(x_{1n}) + \lambda_{2n} f(x_{2n}) + \ldots + \lambda_{nn} f(x_{nn})$$

is exact for all polynomials of degree $\leqq 2n-1$.

Next we consider q.f. of the type

$$(1.2) \qquad \int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} f(x_{kn})\lambda_{kn}^{(0)} + \sum_{k=1}^{n} f'(x_{kn})\lambda_{kn}^{(1)}$$

where $\lambda_{kn}^{(0)}, \lambda_{kn}^{(1)}$ are independent of f. Related to the q.f. (1.2), the following problem was raised and solved by P. Turán [4]. Does there exist a real system $(x_1, x_2, \ldots, x_n)$ for which the q.f. (1.2) is true for a greater variety of polynomials than those of degree $\leqq 2n-1$? He has shown that for no choice this formula (1.2) can be made precise even to all polynomials of degree $\leqq 2n$.

Now, P. Turán [4] went a step further. Consider the q.f. of the type

$$(1.3) \qquad \int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} f(x_{kn})\lambda_{kn}^{(0)} + \sum_{k=1}^{n} f'(x_{kn})\lambda_{kn}^{(1)} + \sum_{k=1}^{n} f''(x_{kn})\lambda_{kn}^{(2)}$$

where again $\lambda_{kn}^{(j)}$ are independent of f. It is easy to see that such a q.f. always exists uniquely and is precise for $f \in \pi_{3n-1}$ (for every choice of nodes $x_1, x_2, \ldots, x_n$). Turán asked the question to determine those systems $(x_1, x_2, \ldots, x_n)$ of n distinct points for which the q.f. (1.3) is valid for all polynomials $f \in \pi_{4n-1}$. The solution of this interesting question may be stated as follows:

THEOREM A (P. Turán [4]). *Among the q.f. (1.3) valid for all polynomials $f(x)$ of degree $\leqq 3n-1$ there is exactly one choice of $(x_1, x_2, \dots, x_n)$ such that the formula is valid for all polynomials of degree $\leqq 4n-1$. This system of nodes consists of the n real distinct zeros in the interior of $[-1,1]$ of that polynomial $\pi_{n,4}(x) = x^n + \dots$ which minimizes the integral*

$$I_4(\pi_n) = \int_{-1}^{1} |\pi_n(x)|^4 d.$$

In 1974 P. Turán [3] raised 89 problems on approximation theory. Some of them are on Birkhoff interpolation. (See the book of G. G. Lorentz et al [2] for detailed study of this subject.) Here we are concerned with the problem related to Birkhoff quadrature theory.

PROBLEM XXXII (P. Turán [3]). Determine the matrices A, if any, for which

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} f(x_{kn})\lambda_{kn}^{(0)} + \sum_{k=1}^{n} f''(x_{kn})\lambda_{kn}^{(1)} \tag{1.4}$$

is valid for all polynomials of degree $\leqq 2n$.

We may also ask the following analogous problem: Determine the matrices A, if any, for which

$$\int_{0}^{2\pi} f(\theta)d\theta = \sum_{k=0}^{n-1} f(\theta_{kn})\lambda_{kn}^{(0)} + \sum_{k=0}^{n-1} f^{(M)}(\theta_{kn})\lambda_{kn}^{(1)} \tag{1.5}$$

is valid for all trigonometric polynomials of highest possible degree.

The object of this paper is to provide the solution of this latter problem. We shall prove the following

THEOREM 1. *Let*

$$\theta_{kn} = \frac{2k\pi}{n}, \qquad k = 0, 1, \dots, n-1 \tag{1.6}$$

and M an even positive integer. Then

$$\int_{0}^{2\pi} f(\theta)d\theta = \frac{2\pi}{n}\sum_{k=0}^{n-1} f\left(\frac{2k\pi}{n}\right) + (-1)^{\frac{M-2}{2}} \frac{2\pi}{n^{M+1}} \sum_{k=0}^{n-1} f^{(M)}\left(\frac{2k\pi}{n}\right) \tag{1.7}$$

is a Gaussian type q.f. which is exact for all $f \in T_{2n}$ of the following form:

$$f(\theta) = a_0 + \sum_{k=1}^{2n-1} (a_k \cos k\theta + b_k \sin k\theta) + b_{2n} \sin 2n\theta.$$

Further (1.7) *is not valid if* $f(\theta)$ *is given by*

$$f(\theta) = a_0 + \sum_{k=1}^{2n} (a_k \cos k\theta + b_k \sin k\theta), \quad a_{2n} \neq 0. \tag{1.8a}$$

Next, we shall state the following

THEOREM 2. *Let* $f(\theta)$ *be any trigonometric polynomial of order* $3n$ *given by*

$$f(\theta) = a_0 + \sum_{k=1}^{3n-1} (a_k \cos k\theta + b_k \sin k\theta) + b_{3n} \sin 3n\theta \tag{1.9}$$

and let $M \neq P$ *be even integers. Then*

$$\int_0^{2\pi} f(\theta)d\theta = \frac{2\pi}{n} \sum_{k=0}^{n-1} f\left(\frac{2k\pi}{n}\right) + \frac{a\pi}{n^{M+1}} \sum_{k=0}^{n-1} f^{(M)}\left(\frac{2k\pi}{n}\right) + \frac{b\pi}{n^{P+1}} \sum_{k=0}^{n-1} f^{(P)}\left(\frac{2k\pi}{n}\right), \tag{1.10}$$

where

$$a = \frac{2(P-1)}{2^M - 2^P}(-1)^{\frac{M}{2}}, \quad b = \frac{2(-1)^{\frac{P}{2}}(2^M - 1)}{2^P - 2^M}. \tag{1.11}$$

REMARKS. 1. It is well known that the interpolation problem $(0, M, P)$ where M and P are even positive integers based on equidistant nodes does not exist uniquely, nevertheless the corresponding q.f. is indeed unique as demonstrated by Theorem 2 and even optimal in the sense of Gauss.

2. Following the ideas of Theorem 2, the q.f. $(0, M, N, P)$ where M, N, P are even positive integers can be obtained.

2. PROOF OF THEOREM 1. Here we need to show that if $f \in T_{2n}$ (of the form (1.8)) and M is an even positive integer then

$$\int_0^{2\pi} f(\theta)d\theta = \alpha \sum_{k=0}^{n-1} f(\theta_{kn}) + \beta \sum_{k=0}^{n-1} f^{(M)}(\theta_{kn}) \tag{2.1}$$

where

$$\alpha = \frac{2\pi}{n}, \qquad \beta = \frac{2\pi(-1)^{\frac{M-2}{2}}}{n^{M+1}}. \tag{2.2}$$

For this purpose, let $f(\theta) = 1$, then from (2.1) we have $2\pi = \alpha n$ as stated in the first part of (2.1). We next observe that

$$\sum_{k=0}^{n-1} \cos j\theta_{kn} = \sum_{k=0}^{n-1} \sin j\theta_{kn} = 0, \qquad j = 1, 2, \dots, n-1. \tag{2.3}$$

Next, we claim that no matter how we choose α and β, (2.1) is clearly valid for $f(\theta) = \cos j\theta$, or $f(\theta) = \sin j\theta$, $j = 1, 2, \dots, n-1$. Here we use (2.3). Next, in the case $f(\theta) = \cos(n+j)\theta$, $j = 1, 2, \dots, n-1$ we have

$$f(\theta_{kn}) = \cos j\theta_{kn}, \quad f^{(M)}(\theta_{kn}) = (-1)^{\frac{M}{2}}(n+j)^M \cos j\theta_{kn}. \tag{2.4}$$

Therefore, on using (2.3) and (2.4) we obtain (2.1). The same conclusion can be obtained if $f(\theta) = \sin(n+j)\theta$, $j = 0, 1, \dots, n$. Now β is uniquely determined as given by (2.2) by requiring that (2.1) is also valid for $f(\theta) = \cos n\theta$. This proves Theorem 1.

3. Proof of Theorem 2. The proof of this theorem is on the lines of the proof of Theorem 1. We only outline the ideas here. Clearly, if $f(\theta) = 1$, (1.10) is certainly valid. If $f(\theta) = \cos j\theta$, $j = 1, 2, \dots n-1, n+1, \dots,$ $2n-1, 2n+1, \dots, 3n-1$ or if $f(\theta) = \sin j\theta$, $j = 1, 2, \dots, 3n$ (1.10) is valid. Here one needs to use (2.8). Further, if $f(\theta) = \cos n\theta$, $f(\theta) = \cos 2n\theta$ then we determine a and b such that (1.7) is valid for these trigonometric polynomials. This will give us two equations in a and b which are uniquely solvable. (1.11) provides that solution.

4. Concerning the original Problem XXXII of P. Turán, the author strongly feels that such q.f. exist, for algebraic polynomials of degree even higher than $2n$. Here we only mention that in earlier works the author [5], [6] obtained the following q.f. based on the zeros of $\pi_n(x) = (1-x^2)P'_{n-1}(x)$ where $P_n(x)$ denotes the Legendre polynomial of degree n. It was proved that the q.f. $(n > 2)$

$$\int_{-1}^{1} f(x)dx = \frac{3}{n(2n-1)}(f(1) + f(-1)) + \frac{2(2n-3)}{n(n-2)(2n-1)} \sum_{k=2}^{n-1} \frac{f(x_{kn})}{P_{n-1}^2(x_{kn})} +$$

$$+ \frac{1}{n(n-1)(n-2)(2n-1)} \sum_{k=2}^{n-1} \frac{1-x_{kn}^2}{P_{n-1}^2(x_{kn})} f''(x_{kn})$$

is exact for all polynomials of degree $\leqq 2n-1$, though it only requires $2n-2$ information $(f(x_{kn}), k = 1, 2, \dots, n, f''(x_{kn}), k = 2, 3, \dots, n-1)$.

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A NOTE ON THE DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTIONS

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1. Introduction. Let $s_n(f;x)$ denote the n-th partial sum of the Fourier series at x of a 2π-periodic and L-integrable function over $[0,2\pi]$ and let $\omega(\delta)=\omega(\delta;f)$ denote the modulus of continuity of f (Zygmund [2], p. 42).

Let $A=(a_{n,k})$ $(k,n=0,1,\dots)$ be a lower-triangular infinite matrix of real numbers and let the A-transform of $(s_n(f;x))$ be given by

$$T_n(f;x)=\sum_{k=0}^{n} a_{n,k}s_k(f;x) \qquad (n=0,1,\dots).$$

Recently in 1988, we [1] have proved the following:

THEOREM A. *Let $(a_{n,k})$ satisfy the following:*

$$a_{n,k}\geqq 0 \ (n,k=0,1,2,\dots),\quad \sum_{k=0}^{n} a_{n,k}=1, \tag{1.1}$$

$$a_{n,k}\leqq a_{n,k+1} \quad (k=0,1,\dots,n-1,\ \ n=0,1,\dots). \tag{1.2}$$

Suppose $\omega(t)$ is such that

$$\int_u^{\pi} t^{-2}\omega(t)dt=O\{H(u)\} \quad (u\to 0+), \tag{1.3}$$

where $H\geqq 0$ and that

$$\int_0^{t} H(u)du=O\{tH(t)\} \quad (t\to 0+). \tag{1.4}$$

Then

$$\|T_n(f)-f\|=O\{a_{n,n}H(a_{n,n})\}, \tag{1.5}$$

provided that

$$tH(t)=o(1) \qquad (t\to 0+). \tag{1.6}$$

THEOREM B. *Suppose* (1.2) *is replaced by*

$$a_{n,k} \geqq a_{n,k+1} \qquad (k = 0, 1, \dots, n-1; \ n = 0, 1, \dots) \tag{1.7}$$

in Theorem A. *Then*

$$\|T_n(f) - f\| = O\{a_{n,0}H(a_{n,0})\}. \tag{1.8}$$

In this note, we show that estimates (1.5) and (1.8) may be obtained without requiring (1.6). Precisely we prove the following:

THEOREM. *Let* (1.1), (1.3) *and* (1.4) *hold. Then* (1.5) *and* (1.8) *hold provided* (1.2) *and* (1.7) *hold respectively.*

2. PROOF OF THE THEOREM. It has been observed that the hypothesis (1.6) has been used only to prove Lemma 1 of [1]:

$$\int_0^t u^{-1}\omega(u)du = O\{tH(t)\} \qquad (t \to 0+), \tag{2.1}$$

whenever (1.3), (1.4) and (1.6) hold. Therefore the proof of the theorem will follow from [1] if we show that (2.1) may be obtained whenever (1.3) and (1.4) hold.

We now prove (2.1) whenever (1.3) and (1.4) hold. Since $\omega(t)$ is non-decreasing, therefore we observe that

$$\omega(t)\Big(\frac{1}{t} - \frac{1}{\pi}\Big) \leqq \int_t^\pi u^{-2}\omega(u)du.$$

Hence

$$t^{-1}\omega(t) \leqq \int_t^\pi u^{-2}\omega(u)du + \frac{1}{\pi}\omega(t). \tag{2.2}$$

Also, for $0 < t < \pi/3$,

$$\int_t^\pi u^{-2}\omega(u)du \geqq \omega(t)\int_t^\pi u^{-2}du \geqq \frac{1}{\pi}\omega(t)\int_t^\pi u^{-1}du = \frac{1}{\pi}\omega(t)\log(\pi/t).$$

Hence, for $0 < t < \pi/3$,

$$\int_t^\pi u^{-2}\omega(u)du \geqq \frac{1}{\pi}\omega(t). \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$t^{-1}\omega(t) \leqq 2 \int_t^{\pi} u^{-2}\omega(u)du = O\{H(t)\},$$

by (1.3). Hence, as $t \to 0+$,

$$\int_0^t u^{-1}\omega(u)du = O(1) \int_0^t H(u)du = O\{tH(t)\},$$

by (1.4). This proves (2.1) under the hypotheses (1.3) and (1.4).

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SIMILARITY AND ISOMETRIC EQUIVALENCE OF L_p-NESTS

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Nests of subspaces and their associated operator algebras have occupied an important place in Hilbert space theory since their introduction by Ringrose [15, 16, 17]. A central problem in this area has been the determination of when two nests are similar or unitarily equivalent. There have been many important papers on this subject [3, 4, 5, 7, 8, 12, 13, 14]; for more detailed surveys of some of these results we refer the reader to [5] and [6]. The purpose of this paper is to consider these problems in the broader context of nests in the Lebesgue spaces $L_p = L_p[0,1]$, where $1 \leqq p < \infty$, with special emphasis on $p > 1, \neq 2$. In order to do this it is necessary to briefly recall some key results from the Hilbert space case. In [8] Kadison and Singer showed that continuous, multiplicity one nests are unitarily equivalent while in [12, 13] Larson showed that all continuous nests are similar and that the similarity could be induced by an arbitrarily small compact perturbation of a unitary operator. Using Larson's results and those of Lance [12], Anderson [4] showed that the similarity could be chosen to preserve the index. Finally, Davidson [9] gave a complete solution to the similarity classification problem and showed in the case of continuous nests that it was possible to choose a similarity which was an arbitrarily small compact perturbation of a unitary operator and which preserved the index. One might suppose that in the more general L_p setting that isometries would play the role that unitary operators do in the L_2 setting. But, in fact, due to the limited number of isometries on the L_p spaces when $p \neq 2$, results analogous to those obtained by Anderson and Davidson in the L_2 setting do not extend to the L_p setting. Specifically, we will consider here two particular types of continuous, multiplicity one nests in the L_p setting and characterize when there exists an index-preserving isometry carrying one nest onto another.

We will also show that, even when L_2-nests are unitarily equivalent, the equivalence may not be induced by an index-preserving, order bounded similarity with order bounded inverse. This last result uses a theorem of [1] where it was shown that the existence of an index-preserving similarity between certain L_1 nests implies the existence of an index-preserving isometry between the nests. The subject matter of [1] differs from that of the present paper in being confined to L_1; however, the nests considered there are of greater generality than those discussed here.

We now fix some terminology and notation. Throughout this paper p

will be in the range $[1,\infty)$ and we will use the symbol $L_p = L_p([0,1],m)$ to denote the Banach space of Lebesgue p-integrable functions defined on $[0,1]$. Lebesgue measure on $[0,1]$ will be denoted by m. For each measurable subset A of $[0,1]$, we let $P_A\colon L_p \to L_p$ denote the natural projection operator i.e. $P_A f = f\chi_A$ for all f in L_p. (Here χ_A is the characteristic function of the set A.) The term operator will always mean bounded linear operator. We will use $\mathcal{N}$ to denote the continuous L_p-nest consisting of the subspaces $N_t = \{f \in L_p \mid P_{[t,1]}f = 0\}$ for each $t \in [0,1]$. Throughout this paper, φ will denote any strictly increasing function from $[0,1]$ onto $[0,1]$ and we will use $\mathcal{M}(\varphi)$ to denote the continuous L_p-nest consisting of the subspaces $M_t(\varphi) = \{f \in L_p \mid P_{[\varphi(t),1]}f = 0\}$ for each t in $[0,1]$. If $\varphi =$ identity on $[0,1]$, then $\mathcal{M}(\varphi) = \mathcal{N}$. We will make frequent use of the obvious facts that φ is necessarily continuous, φ' exists and is $\geqq 0$ a.e. and that $\varphi[a,b] = [\varphi(a),\varphi(b)]$ for all a,b in $[0,1]$. The *diagonal* of $\mathcal{M}(\varphi)$ consists of those operators on L_p which leave invariant all of the subspaces in $\mathcal{M}(\varphi)$ and all of the complements of the subspaces in $\mathcal{M}(\varphi)$. It is easy to see that an operator T is in the diagonal of $\mathcal{M}(\varphi)$ if and only if the following condition is satisfied:

$$(*) \qquad \text{for every Borel set } E \text{ in } [0,1], \quad P_{[0,1]\setminus E}TP_E = 0.$$

Thus the diagonals of all the $\mathcal{M}(\varphi)$'s are identical. Now it is shown in [10, Theorem 4.2] that operators satisfying the condition $(*)$ are exactly those operators having the form $Tf = gf$ where g is in L_∞. Consequently, the diagonal of any $\mathcal{M}(\varphi)$ is a commutative algebra and the nest $\mathcal{M}(\varphi)$ is thus said to be of multiplicity one. Recall that in the Hilbert space case two nests are said to be similar (respectively, isometrically equivalent) if there exists an invertible operator (respectively, isometry) which takes each subspace of the first nest onto a subspace of the second nest. Obviously the nests $\mathcal{M}(\varphi)$ and $\mathcal{N}$ which we have defined above are isometrically equivalent for any φ via the identity. The purpose of this paper is to focus on the question of when there exists an *index-preserving* isometry between $\mathcal{M}(\varphi)$ and $\mathcal{N}$ i.e. when does there exist an isometry $T\colon L_p \to L_p$ so that $T(M_t(\varphi)) = N_t$ for all t in $[0,1]$. We note that if φ is absolutely continuous then $\varphi' > 0$ a.e. and the isometry T defined by $Tf = (\varphi')^{1/p}f(\varphi)$ is an index-preserving isometry between $\mathcal{M}(\varphi)$ and $\mathcal{N}$. The rest of this paper is concerned in one way or another with converses to this result. In the converses the following compatibility condition between T and φ plays a crucial role.

We shall say that an invertible operator T on L_p is φ-compatible if the function $\sigma\colon [0,1] \to [0,1]$ defined by

$$\sigma(t) = \sup\{\text{ess sup}(\varphi^{-1}\text{supp } T^{-1}f)) \colon f \text{ in } N_t\}$$

is absolutely continuous and $\sigma[0,t] = \cup\{\varphi^{-1}(\text{supp}\, T^{-1}f) \colon f \text{ in } N_t\}$ for all t in $[0,1]$. An easily distinguished class of φ-compatible operators are those invertible operators T which are index-preserving, i.e. $T(M_t(\varphi)) = N_t$ for all

t in $[0,1]$; in this case $\sigma(t)=t$. However, the notion is not limited to index-preserving operators. For example, the identity operator is φ-compatible if φ^{-1} is absolutely continuous.

Our first result contrasts with the results of Davidson [5], who showed that for any φ, the L_2-nests $\mathcal{M}(\varphi)$ and $\mathcal{N}$ are similar via an index-preserving operator. Moreover, that operator could also be chosen to be an arbitrarily small compact perturbation of a unitary operator. On the other hand, our first theorem shows that if $p \neq 2$ then the existence of a φ-compatible T on L_p whose condition number is sufficiently small is enough to force the absolute continuity of φ.

THEOREM 1. *Let $1 \leqq p < +\infty$, $p \neq 2$. There exists an $\varepsilon = \varepsilon(p) > 0$ so that if there exists a φ-compatible operator T on L_p with $\|T\|\|T^{-1}\| < 1+\varepsilon$, then φ is absolutely continuous. Hence there exists an index-preserving isometry from $\mathcal{M}(\varphi)$ to $\mathcal{N}$.*

PROOF. By a result of Alspach [2], there exists an $\varepsilon=\varepsilon(p)>0$ so that if $T\colon L_p \to L_p$ is an invertible operator with $\|T\|\|T^{-1}\| < 1+\varepsilon$ then there is an onto isometry $U\colon L_p \to L_p$ so that $\|T-U\|<1$. It it this ε which satisfies the conclusion of the theorem, as we shall see. First note that since T is φ-compatible, we have that $T(M_t(\varphi\circ\sigma)) = N_t$ for all t in $[0,1]$. Since σ is absolutely continuous, $(\varphi\circ\sigma)' = \varphi'(\sigma)\sigma'$ a.e. It follows that if $\varphi\circ\sigma$ were absolutely continuous, then φ would be absolutely continuous. Hence we may assume, without loss of generality, that T is an invertible operator satisfying the condition $T(M_t(\varphi)) = N_t$ for all t in $[0,1]$.

Now suppose $\|T\|\|T^{-1}\| < 1+\varepsilon$ and that U is an isometry so that $\|T-U\|<1$. By a result of Lamperti [11] there exists an essentially 1-1 Borel measurable function $\theta\colon [0,1]\to[0,1]$ and a.e. nonzero function h in L_p such that $Uf = h\cdot(f\circ\theta)$ for all f in L_p and $\int_E |h|^p dm = m(\theta(E))$ for all Borel sets E in $[0,1]$. We claim that, for each $t, \theta([0,t]) = \varphi([0,t])$ a.e., i.e. $m(\theta[0,t]\Delta\varphi[0,t]) = 0$. To see this, let $A = \theta([0,t])\setminus\varphi[0,t])$ and $B = \varphi([0,t])\setminus\theta[0,t])$. Suppose that $mB \neq 0$. Since $B \subset \varphi[0,t] = [0,\varphi(t)]$ we must have that supp $T(\chi_B) \subset [0,t]$. But (supp $U(\chi_B)) \cap [0,t]$ has Lebesgue measure zero since

$$U(\chi_B) = (\chi_B \circ \theta)h \text{ a.e. } = h\chi_{\theta^{-1}(B)} \text{ a.e.}$$

and $\theta^{-1}(B)\cap[0,t]$ has Lebesgue measure zero since θ is a.e. 1-1. Using the notation $\tilde{\chi}_B$ to denote the normalized function $\chi_B/mB^{1/p}$, we conclude $\|(T-U)\tilde{\chi}_B\|^p = \|T(\tilde{\chi}_B)\|^p + \|U(\tilde{\chi}_B)\|^p > 1$ since U is an isometry and T is invertible. But this contradicts the inequality $\|T-U\|<1$ and so $mB = 0$. Similarly, suppose $mA \neq 0$. Since $\theta^{-1}(A) \subset [0,t]$ a.e. we have that supp $U(\tilde{\chi}_A) \subset [0,t]$ a.e. Thus if $g = T^{-1}U(\tilde{\chi}_A)$ then supp $g \subset [0,\varphi(t)]$ a.e. But then $\|U(g-(\tilde{\chi}_A))\| = \|Ug - U(\tilde{\chi}_A)\| = \|Ug - Tg\| < \|g\|$ and yet

$\|g-(\tilde{\chi}_A)\|^p = \|g\|^p + \|\tilde{\chi}_A\|^p > \|g\|^p$, which contradicts the fact that U is an isometry. Hence $mA = 0$. We now have that

$$\int_{[0,t]} |h|^p dm = m(\theta[0,t]) = m(\varphi[0,t]) = m([0,\varphi(t)]) = \varphi(t) \text{ for all } t \text{ in } [0,1].$$

Hence $\varphi' = |h|^p$ a.e. and so φ is absolutely continuous. This completes the proof.

In [1], the question of when an order isomorphism between two continuous multiplicity one nests was implemented by a similar operator on L_1 was completely solved. Restricted to our present situation the theorem can be stated as follows:

THEOREM 2 [1]. *A necessary and sufficient condition that there exists an invertible operator T on L_1 so that $T(M_t(\varphi)) = N_t$, for all $t \in [0,1]$, is that φ is absolutely continuous.*

This theorem has, in fact, implications for $p > 1$. We first wish to derive a strengthening of Theorem 2. At first glance Theorem 2 is only about invertible operators which fix the index. But, in fact, as we saw at the start of the proof of Theorem 1, the existence of a φ-compatible T implies the existence of an absolutely continuous σ: $[0,1] \to [0,1]$ so that $T(M_{\sigma(t)}(\varphi)) = N_t$ for all t in $[0,1]$. This observation, coupled with Theorem 2, yields the following result.

THEOREM 3. *If there exists a φ-compatible operator on L_1, then φ is absolutely continuous and hence there exists an index preserving isometry between $\mathcal{M}(\varphi)$ and $\mathcal{N}$.*

PROOF. Let $T\colon L_1 \to L_1$ be a φ-compatible operator. As in the proof of Theorem 1, we may assume that $T(M_t(\varphi)) = N_t$ for all t in $[0,1]$. The result then follows by Theorem 2.

We can now apply Theorem 3 to L_p-nests with $p > 1$. In order to do this, we recall that, if $1 \leqq p < +\infty$, then an operator $T\colon L_p \to L_p$ is said to be *order bounded* if for every positive function g in L_p there exists a positive function h in L_p so that $|f| \leqq g$ implies $|Tf| \leqq h$. A *density* is a strictly positive function g in L_1 with $\int_{[0,1]} g\,dm = 1$. If λ_g is the measure given by $\lambda_g E = \int_E g\,dm$, then the spaces L_p and $L_p(\lambda_g)$ are isometric via the isometry $\varrho_p(f) = g^{-1/p} f$. Now if T is a φ-compatible operator on $L_1(\lambda_g)$ then $\varrho_p^{-1} T \varrho_p$ is a φ-compatible operator on L_1. Hence, by Theorem 3, φ is absolutely continuous. We will use this fact, together with the following lemma, to extend Theorem 3 to the $p > 1$ setting.

The following is a modification of a result of Weis [18, Theorem 2.1].

THEOREM 4. *Let $1 < p < +\infty$ and let T be an invertible operator on L_p such that T and T^{-1} are order bounded. Then there exists a density g so that the operators $\varrho_p T \varrho_p^{-1}$ and $\varrho_p T^{-1} \varrho_p^{-1}$ are bounded linear operators on $L_1(\lambda_g)$.*

PROOF. By Lemma 2.2 of [18] applied to the operator $|T| + |T^{-1}|$ there exists a constant c_p and density g so that $(|T| + |T^{-1}|)^*(g^{1/q}) \leqq c_p \||T| + |T^{-1}|\| g^{1/q}$ where $\frac{1}{p} + \frac{1}{q} = 1$. It follows that if $|f| \leqq 1$ then

$$|(\varrho_q T^* \varrho_q^{-1})(f)| \leqq (\varrho_q |T^*| \varrho_q^{-1})(1) \leqq (\varrho_q (|T| + |T^{-1}|)^* \varrho_q^{-1})(1) \leqq$$

$$\leqq g^{-1/q}(|T| + |T^{-1}|)^* g^{1/q} \leqq c_p \||T| + |T^{-1}|\|$$

and similarly $|(\varrho_q (T^*)^{-1} \varrho_q^{-1})(f)| \leqq c_p \||T| + |T^{-1}|\|$. Thus the operators $\varrho_q T^* \varrho_q^{-1}$ and $\varrho_q (T^*)^{-1} \varrho_q^{-1}$ are bounded linear operators on $L_\infty(\lambda_g)$. Hence it follows that their preadjoints are bounded linear operators on $L_1(\lambda_g)$. But since $\varrho_p^{-1} = \varrho_q^*$ and $\varrho_p = (\varrho_q^{-1})^*$ we have that $(\varrho_q T^* \varrho_q^{-1}) = (\varrho_q^* T \varrho_q^{-1})^*$ and $(\varrho_q (T^*)^{-1} \varrho_q^{-1})^* = \varrho_q T^{-1} \varrho_p^{-1}$ which yields the result.

THEOREM 5. *Let $1 \leqq p < +\infty$. If there exists an order bounded, φ-compatible operator on L_p whose inverse is also order bounded, then φ is absolutely continuous and hence there is an index-preserving isometry between $\mathcal{M}(\varphi)$ and $\mathcal{N}$.*

PROOF. Let $T \colon L_p \to L_p$ be an order bounded, φ-compatible operator with order bounded inverse. By Theorem 4, there exists a density g so that $\varrho_p T \varrho_p^{-1}$ is an invertible operator on $L_1(\lambda_g)$. Since the L_p nests $\mathcal{M}(\varphi)$ and $\mathcal{N}$ are L_1-dense in the L_1 nests $\mathcal{M}(\varphi)$ and $\mathcal{N}$, the operator $\varrho_p T \varrho_p^{-1}$ is a φ-compatible operator on $L_1(\lambda_g)$. Hence, by the remark after Theorem 3, φ is absolutely continuous.

We remark that in the case $p = 2$ the above result yields new information about unitary equivalence. To see this, recall that Kadison and Singer [8] have shown that, in the context of Hilbert space, all continuous multiplicity one nests are unitarily equivalent. Consequently, the L_2-nests $\mathcal{M}(\varphi)$ and $\mathcal{N}$ are unitarily equivalent for all φ. However, the unitary operator cannot in general be chosen to satisfy the additional conditions of Theorem 5 since this would imply that φ is absolutely continuous.

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WEIGHTED $(0,1,3)$ INTERPOLATION ON THE ZEROS OF HERMITE POLYNOMIALS

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1. J. Balázs [1] on the suggestion of P. Turán initiated the study of weighted $(0,2)$ interpolation, which means to determine a polynomial $R_n(x)$ of degree $\leqq 2n-1$ such that

$$(1.1)\qquad R_n(\xi_{\nu,n}) = \alpha_{\nu,n},\quad (wR_n)''(\xi_{\nu,n}) = \beta_{\nu,n},\quad \nu = 1,2,\dots,n$$

where $\alpha_{\nu,n}$, $\beta_{\nu,n}$ are arbitrary given numbers and $\xi_{\nu,n}$ are the zeros of the ultraspherical polynomial $P_n^{(\alpha)}(x)$, $\alpha > -1$ and the weight function $w(x) = (1-x^2)^{\frac{\alpha+1}{2}}$, $x \in [-1,1]$. He claimed that there exists no polynomial $R_n(x)$ of degree $\leqq 2n-1$ satisfying the conditions (1.1). However, taking n even, he proved the existence, uniqueness, explicit representation and a convergence theorem for polynomials $R_n(x)$ of degree $\leqq 2n$ satisfying conditions (1.1) together with a prescribed value of $R_n(0)$. If n is odd, uniqueness fails to hold. Recently L. Szili [10] carried over the same problem on the zeros of Hermite polynomial $H_n(x)$ and the weight function $w(x) = e^{-x^2/2}$. In another paper [2], the first author and A. Sharma considered the problem of existence, uniqueness and explicit representation of $(0,2)$ and $(0,1,3)$ interpolations on the abscissas as roots of the Hermite polynomial $H_n(x)$, but they could not settle the convergence problem in either case because of the complicated nature of the constants involved in them. For more literature in this direction one is referred to the interesting papers listed in the references.

The object of this paper is to extend the study of weighted $(0,2)$ interpolation due to J. Balázs [1] and L. Szili [10] to the case of weighted $(0,1,3)$ interpolation, which means to construct a polynomial $G_n(x)$ of degree $\leqq 3n-1$ such that

(1.2)
$$G_n(x_{\nu,n}) = a_{\nu,n},\ G_n'(x_{\nu,n}) = b_{\nu,n},\ (\varrho G_n)'''(x_{\nu,n}) = c_{\nu,n}\ \ (\nu = 1,\dots,n)$$

where $a_{\nu,n}$, $b_{\nu,n}$, $c_{\nu,n}$ are arbitrary given numbers and the weight function $\varrho(x) = e^{-x^2}$ $(x \in \mathbf{R})$ and $x_{\nu,n}$'s are the zeros of the Hermite polynomial $H_n(x)$, given by:

$$(1.3)\qquad -\infty < x_{n,n} < \dots < x_{1,n} < \infty \qquad (n \in \mathbf{N}).$$

2. We have the following theorems.

THEOREM 1. *There exists in general no polynomial* G_n *of degree* $\leqq 3n-1$, *satisfying the conditions* (1.2).

THEOREM 2. *If* n *is even, then there exists a unique polynomial* G_n *of degee* $\leqq 3n$ *satisfying the conditions:*

$$(2.1)\qquad \begin{cases} G_n(x_{\nu,n}) = a_{\nu,n},\ G_n'(x_{\nu,n}) = b_{\nu,n},\ (\varrho G_n)'''(x_{\nu,n}) = c_{\nu,n}, \\ \nu = 1,2,\dots,n \text{ and} \\ G_n(0) = \sum_{\nu=1}^{n} \left[(1+3x_{\nu,n}^2)a_{\nu,n} - x_{\nu,n}b_{\nu,n}\right] \ell_{\nu,n}^3(0). \end{cases}$$

THEOREM 3. *If* n *is odd, then there are infinitely many polynomials* G_n *of degree* $\leqq 3n$ *satisfying the conditions* (2.1) *of Theorem* 2.

THEOREM 4. *Under the conditions of Theorem* 2, *the polynomial* G_n *of degree* $\leqq 3n$ *is given by*

$$(2.2^*)\qquad G_n(x) = \sum_{\nu=1}^{n} a_{\nu,n}U_{\nu,n}(x) + \sum_{\nu=1}^{n} b_{\nu,n}V_{\nu,n}(x) + \sum_{\nu=1}^{n} c_{\nu,n}W_{\nu,n}(x)$$

such that

$$(2.3)\qquad G_n(0) = \sum_{\nu=1}^{n} \{(1+3x_\nu^2)a_\nu - x_\nu b_\nu\}\ell_\nu^3(0),$$

where $U_\nu(x)$, $V_\nu(x)$ *and* $W_\nu(x)$ $(1 \leqq \nu \leqq n)$ *are the fundamental polynomials of the first, second and third kind respectively of the weighted* $(0,1,3)$-*interpolation of degree* $\leqq 3n$, *which are uniquely determined by the following conditions:*

$$(2.4)\qquad U_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu \end{cases},\quad U_\nu'(x_j) = 0,\quad \left(e^{-x^2}U_\nu(x)\right)'''_{x=x_j} = 0,$$

$$(2.5)\qquad V_\nu(x_j) = 0,\quad V_\nu'(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu \end{cases},\quad \left(e^{-x^2}V_\nu(x)\right)'''_{x=x_j} = 0,$$

$$(2.6)\qquad W_\nu(x_j) = 0,\quad W_\nu'(x_j) = 0,\quad \left(e^{-x^2}W_\nu(x)\right)'''_{x=x_j} = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu \end{cases}.$$

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* From now onwards we shall drop the subscript n in $x_{\nu,n}$ etc.

Under the conditions (2.4), (2.5) and (2.6); $U_\nu(x)$, $V_\nu(x)$ and $W_\nu(x)$ are explicitly given by:

$$U_\nu(x) = \ell_\nu^3(x) - 3x_\nu V_\nu(x) + \tag{2.7}$$

$$+\frac{H_n^2(x)}{H_n'^2(x_\nu)} \int_0^x \frac{\{\lambda_\nu(t-x_\nu)^2 + \mu_\nu(t-x_\nu) + x_\nu\} l_\nu(t) - l_\nu'(t)}{(t-x_\nu)^2} dt$$

where

$$\lambda_\nu = x_\nu\Big(n - \frac{2}{3}x_\nu^2\Big) \quad \text{and} \quad \mu_\nu = \frac{1}{3}(x_\nu^2 - 2(n-1)). \tag{2.8}$$

$$V_\nu(x) = (x - x_\nu)\ell_\nu^3(x) + \frac{H_n^2(x)}{H_n'^2(x_\nu)} \int_0^x \frac{(s_\nu(t-x_\nu) + x_\nu)\ell_\nu(t) - \ell_\nu'(t)}{t - x_\nu} dt, \tag{2.9}$$

where

$$s_\nu = \frac{1}{3}(n + 2(1 - x_\nu^2)) \tag{2.10}$$

and

$$W_\nu(x) = \frac{e^{x_\nu^2} H_n^2(x)}{6H_n'^2(x_\nu)} \int_0^x \ell_\nu(t)dt. \tag{2.11}$$

THEOREM 5. *If the interpolated function $f\colon R \to R$ is twice continuously differentiable such that*

$$\begin{cases} \lim\limits_{|x|\to\infty} x^{2\nu}\varrho(x)f(x) = 0 \quad (\nu = 0,1,2,\dots) \\ \lim\limits_{|x|\to\infty} \varrho(x)f^{(r)}(x) = 0 \quad (r = 1,2); \end{cases} \tag{2.12}$$

then the weighted $(0,1,3)$-interpolation polynomials G_n $(n = 2,4,6,\dots)$ given by (2.2) together with

$$a_{\nu,n} = f(x_{\nu,n}), \quad b_{\nu,n} = f'(x_{\nu,n}); \tag{2.13}$$

$$c_{\nu,n} = O\left(e^{\beta x_{\nu,n}^2} n\omega\Big(f', \frac{1}{\sqrt{n}}\Big)\right), \quad 0 \leqq \beta \leqq \frac{1}{6}; \tag{2.14}$$

and

$$G_n(0) = \sum_{\nu=1}^{n} \left[(1 + 3x_{\nu,n}^2)f(x_{\nu,n}) - x_{\nu,n}f'(x_{\nu,n})\right] \ell_{\nu,n}^3(0) \tag{2.15}$$

satisfy the estimate

$$(2.16)\qquad e^{-\gamma x^2}|f(x)-G_n(x)| = O(\log n)\omega\Big(f', \frac{1}{\sqrt{n}}\Big) \quad \textit{for} \quad \gamma > \frac{3}{2}.$$

3. Preliminaries. In this section, we shall give well-known formulas, which we shall use in the sequel.

$$(3.1)\qquad H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

is the differential equation satisfied by $H_n(x)$. At $x = x_j$ $(1 \leqq j \leqq n)$

$$(3.2)\qquad \begin{cases} H_n''(x_j) = 2x_jH_n'(x_j),\ H_n'''(x_j) = 2[2x_j^2 - (n-1)]H_n'(x_j); \text{ and} \\ H_n^{(4)}(x_j) = 4x_j(3-2n+2x_j^2)H_n'(x_j). \end{cases}$$

Let $\ell_\nu(x)$ be the fundamental polynomial of Lagrange interpolation given by

$$(3.3)\qquad \ell_\nu(x) = \frac{H_n(x)}{H_n'(x_\nu)(x-x_\nu)}$$

from which one can obtain

$$(3.4)\qquad \ell_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu, \end{cases}$$

$$(3.5)\qquad \ell_\nu'(x_j) = \begin{cases} \frac{H_n'(x_j)}{H_n'(x_\nu)(x_j-x_\nu)} & \text{for } j \neq \nu \\ x_\nu & \text{for } j = \nu, \end{cases}$$

$$(3.6)\qquad \ell_\nu''(x_j) = \begin{cases} \frac{2H_n'(x_j)}{(x_j-x_\nu)H_n'(x_\nu)}\Big[x_j - \frac{1}{x_j-x_\nu}\Big] & \text{for } j \neq \nu \\ \frac{4x_\nu^2-2(n-1)}{3} & \text{for } j = \nu, \end{cases}$$

and

$$(3.7)\qquad l_\nu'''(x_j) = \begin{cases} \frac{1}{x_j-x_\nu}\Big[\frac{H_n'''(x_j)}{H_n'(x_\nu)} - 3\ell_\nu''(x_j)\Big] & \text{for } j \neq \nu \\ x_\nu(2x_\nu^2+3-2n) & \text{for } j = \nu. \end{cases}$$

For the roots of $H_n(x)$, we have

$$(3.8)\qquad x_\nu^2 \sim \frac{\nu^2}{n}\ (\nu = 1, 2\ldots, n),\ \ |1-x_\nu^2| \leqq e^{\beta x_\nu^2},\ \ 0 < \beta < \frac{1}{2}.$$

$$H_n(0) = n! \Big/ \left(\frac{n}{2}\right)!, \text{ when } n \text{ is even.} \tag{3.9}$$

$$H_n'(x) = 2nH_{n-1}(x). \tag{3.10}$$

$$H_n(x) = O\left(n^{-1/4}\sqrt{2^n n!}(1 + \sqrt[3]{|x|}\right) e^{x^2/2}, \quad x \in \mathbf{R}. \tag{3.11}$$

$$|H_n'(x_\nu)| \geqq c_1 2^{n+1}\left(\frac{n}{2}\right)! e^{(\delta/2)x_\nu^2}, \quad 0 < \delta < 1, \quad \nu = 1, 2, \dots, n. \tag{3.12}$$

$$|\ell_\nu(x)| = O(1) e^{r_1 \frac{x^2 + x_\nu^2}{2}} \frac{2^{n+1} n! \sqrt{n}}{H_n'^2(x_\nu)}, \quad r_1 > 1. \tag{3.13}$$

$$\left|\int_0^x \ell_\nu(t)dt\right| = O(1) \frac{2^{n+1} n! e^{r_1 \frac{x^2 + x_\nu^2}{2}}}{H_n'^2(x_\nu)} \log n, \quad r_1 > 1. \tag{3.14}$$

$$\sum_{\nu=1}^n e^{x_\nu^2} \ell_\nu^2(x) = O(e^{x^2}), \quad e^{x_\nu^2} \ell_\nu^2(x) = O(e^{x^2}), \quad \nu = 1, 2, \dots, n. \tag{3.15}$$

$$\sum_{\nu=1}^n e^{-\varepsilon x_\nu^2} = O(\sqrt{n}), \text{ where } \varepsilon > 0. \tag{3.16}$$

$$\sum_{\nu=1}^n e^{\delta_1 x_\nu^2} (H_n'(x_\nu))^{-2} = O\left((2^{n+1} n!)^{-1}\right), \quad 0 < \delta_1 < 1. \tag{3.17}$$

$$\sum_{\nu=1}^n \left(2 + \frac{1}{x_\nu^2}\right) \frac{1}{H_n'^2(x_\nu)} = O\left(\frac{1}{2^{n+1} n!}\right). \tag{3.18}$$

$$\frac{2^n \left[\left(\frac{n}{2}\right)!\right]^2}{(n+1)!} \sim n^{-1/2}, \qquad n = 1, 2, \dots. \tag{3.19}$$

LEMMA 1. *The fundamental polynomials of the first and second kind can be written in the following alternative forms:*

$$U_\nu(x) = \ell_\nu^3(x) - 3x_\nu V_\nu(x) + \frac{H_n^2(x)}{H_n'^2(x_\nu)} I, \tag{3.20}$$

where

$$I=\frac{2x_\nu}{3}(2n-x_\nu^2)\int_0^x \ell_\nu(t)dt-\frac{1}{6}(\ell_\nu''(x)-\ell_\nu''(0))+\frac{x_\nu}{2}(\ell_\nu'(x)-\ell_\nu'(0))-$$

$$-\frac{1}{3}(x_\nu^2+n+2)(\ell_\nu(x)-\ell_\nu(0))+\frac{1}{3H_n'(x_\nu)}(H_n'(x)-H_n'(0))-$$

$$-\frac{x_\nu}{3H_n'(x_\nu)}(H_n(x)-H_n(0))$$

and

$$\text{(3.21)}\quad V_\nu(x)=\frac{H_n(x)}{H_n'(x_\nu)}\left[\frac{1}{2}\ell_\nu^2(x)-\frac{2}{3}(2n+1-x_\nu^2)\frac{H_n(x)}{H_n'(x_\nu)}\int_0^x \ell_\nu(t)dt+\right.$$

$$\left.+\frac{1}{2}\frac{H_n(x)}{H_n'(x_\nu)}\ell_\nu'(x)-\frac{1}{2}\frac{H_n(x)}{H_n'(x_\nu)}\ell_\nu'(0)-x\frac{H_n(x)}{H_n'(x_\nu)}\ell_\nu(x)\right].$$

The lemma follows from expressions (2.7), (2.9) and Lemma 2 (given below), so we omit the details of the proof.

LEMMA 2.

$$\text{(3.22)}\qquad \int_0^x \frac{x_\nu\ell_\nu(t)-\ell_\nu'(t)}{t-x_\nu}dt=\frac{1}{2}(\ell_\nu'(x)-\ell_\nu'(0))-x\ell_\nu(x)+n\int_0^x \ell_\nu(t)dt.$$

If $\mu_\nu=\frac{1}{3}((x_\nu^2-2(n-1))$, *then*

$$\text{(3.23)}\qquad \int_0^x \frac{(\mu_\nu(t-x_\nu)+x_\nu)\ell_\nu(t)-\ell_\nu'(t)}{(t-x_\nu)^2}dt=\frac{nx_\nu}{3}\int_0^x \ell_\nu(t)dt-$$

$$-\frac{1}{6}(\ell_\nu''(x)-\ell_\nu''(0))+\frac{x_\nu}{2}(\ell_\nu'(x)-\ell_\nu'(0))-\frac{1}{3}(x_\nu^2+n+2)(\ell_\nu(x)-\ell_\nu(0))+$$

$$+\frac{1}{3H_n'(x_\nu)}(H_n'(x)-H_n'(0))-\frac{x_\nu}{3H_n'(x_\nu)}(H_n(x)-H_n(0)).$$

PROOF. From (3.1) and (3.3), we get

$$\text{(3.24)}\qquad H_n''(t)-2tH_n'(t)+2nH_n(t)=0,$$

$$\text{(3.25)}\qquad H_n'''(t)-2tH_n''(t)+2(n-1)H_n'(t)=0$$

and

$$(t-x_\nu)\ell_\nu(t)=\frac{H_n(t)}{H_n'(x_\nu)}. \tag{3.26}$$

Differentiating it, we get

$$(t-x_\nu)\ell_\nu'(t)+\ell_\nu(t)=\frac{H_n'(t)}{H_n'(x_\nu)}, \tag{3.27}$$

$$(t-x_\nu)\ell_\nu''(t)+2\ell_\nu'(t)=\frac{H_n''(t)}{H_n'(x_\nu)}, \tag{3.28}$$

and

$$(t-x_\nu)\ell_\nu'''(t)+3\ell_\nu''(t)=\frac{H_n'''(t)}{H_n'(x_\nu)} \tag{3.29}$$

From (3.24), (3.26), (3.27) and (3.28), we get

$$\frac{x_\nu\ell_\nu(t)-\ell_\nu'(t)}{t-x_\nu}=\frac{1}{2}[\ell_\nu''(t)-2t\ell_\nu'(t)+2(n-1)\ell_\nu(t)]$$

which on integrating under limits 0 to x, proves (3.22).

To prove (3.23), we see from (3.25), (3.27), (3.28) and (3.29) that

$$\begin{aligned}(t-x_\nu)[\ell_\nu'''(t)-2t\ell_\nu''(t)+2(n-1)\ell_\nu'(t)]+\\ +3\ell_\nu''(t)-4t\ell_\nu'(t)+2(n-1)\ell_\nu(t)=0.\end{aligned} \tag{3.30}$$

Again from (3.24), (3.26), (3.27) and (3.28), we get

$$(t-x_\nu)[\ell_\nu''(t)-2t\ell_\nu'(t)+2(n-1)\ell_\nu(t)]-2(x_\nu\ell_\nu(t)-\ell_\nu'(t))=0. \tag{3.31}$$

Multiplying (3.30) by $(t-x_\nu)$ and subtracting (3.31) after multiplying by 3, we get

$$\begin{aligned}\frac{1}{6}[\ell_\nu'''(t)-2t\ell_\nu''(t)+2n\ell_\nu'(t)]+\frac{1}{3}\frac{x_\nu\ell_\nu'(t)-2(n-1)\ell_\nu(t)}{t-x_\nu}+\\ +\frac{(x_\nu\ell_\nu(t)-\ell_\nu'(t))}{(t-x_\nu)^2}=0.\end{aligned}$$

Putting the values of $\ell_\nu'(t)$ from (3.27), we get

$$\begin{aligned}-\frac{x_\nu\ell_\nu(t)-\ell_\nu'(t)}{(t-x_\nu)^2}=\frac{1}{6}(\ell_\nu'''(t)-2t\ell_\nu''(t)+2n\ell_\nu'(t))+\frac{x_\nu^2-2(n-1)}{3(t-x_\nu)}\ell_\nu(t)-\\ -\frac{1}{6}x_\nu(\ell_\nu''(t)-2t\ell_\nu'(t)+2(n-1)\ell_\nu(t)),\end{aligned}$$

$$\frac{\left[\frac{1}{3}(x_\nu^2-2(n-1))x(t-x_\nu)+x_\nu\right]\ell_\nu(t)-\ell_\nu'(t)}{(t-x_\nu)^2}=$$

$$=-\frac{1}{6}(\ell_\nu'''(t)-2t\ell_\nu''(t)+2n\ell_\nu'(t))+\frac{1}{6}x_\nu(\ell_\nu''(t)-2t\ell_\nu'(t)+2(n-1)x\ell_\nu(t)),$$

which on integrating under limits 0 to x, yields (3.23).

4. Estimates relating to the fundamental polynomials $U_\nu(x)$, $V_\nu(x)$ and $W_\nu(x)$.

LEMMA 3. *Let n be even, $0 \leqq \beta \leqq \frac{1}{6}$ and $\gamma > \frac{3}{2}$, then*

$$\sum_{\nu=1}^{n} e^{\beta x_\nu^2} |W_\nu(x)| = O\left(e^{\gamma x^2} \frac{\log n}{n}\right), \tag{4.1}$$

$$\sum_{\nu=1}^{n} e^{x_\nu^2} |V_\nu(x)| = O(e^{\gamma x^2} \log n), \tag{4.2}$$

and

$$\sum_{\nu=1}^{n} e^{x_\nu^2} |U_\nu(x)| = O(e^{\gamma x^2} \sqrt{n} \log n). \tag{4.3}$$

PROOF. From (2.11), (3.11), (3.12), (3.14), (3.16) and (3.19), we get

$$\sum_{\nu=1}^{n} e^{\beta x_\nu^2} |W_\nu(x)| = \frac{1}{6} \sum_{\nu=1}^{n} e^{\beta x_\nu^2} e^{x_\nu^2} \frac{H_n^2(x)}{H_n'^2(x_\nu)} \left| \int_0^x \ell_\nu(t)dt \right| = \tag{4.4}$$

$$= O(1) \sum_{\nu=1}^{n} \frac{e^{(1+\beta+r_1/2)x_\nu^2} n^{-\frac{1}{2}} ((n+1)!)^2}{2^{2n+4} \left((\frac{n}{2})! \right)^4 e^{2\delta x_\nu^2}} e^{(1+\frac{r_1}{2})x^2} \log n =$$

$$= O\left(e^{(1+r_1/2)x^2} \frac{\log n}{n^{3/2}} \right) \sum_{\nu=1}^{n} \frac{1}{e^{(2\delta-\beta-\frac{r_1}{2}-1)x_\nu^2}} =$$

$$= O\left(e^{\gamma x^2} \frac{\log n}{n}\right), \quad 2\delta - \beta - \frac{r_1}{2} - 1 > 0, \quad \gamma > \frac{3}{2}.$$

To prove (4.2), due to (2.9), (3.1) and (3.3) after a little computation, we get

(4.5)

$$\sum_{\nu=1}^{n} e^{x_\nu^2} |V_\nu(x)| = \frac{1}{2} \sum_{\nu=1}^{n} e^{x_\nu^2} \ell_\nu^2(x) \left| \frac{H_n(x)}{H_n'(x_\nu)} \right| + \frac{4n}{3} \sum_{\nu=1}^{n} e^{x_\nu^2} \frac{H_n^2(x)}{H_n'^2(x_\nu)} \left| \int_0^x \ell_\nu(t)dt \right| +$$

$$+ \frac{2}{3} \sum_{\nu=1}^{n} (1-x_\nu^2) e^{x_\nu^2} \frac{H_n^2(x)}{H_n'^2(x_\nu)} \left| \int_0^x \ell_\nu(t)dt \right| + \frac{1}{2} \sum_{\nu=1}^{n} e^{x_\nu^2} \frac{|H_n(x)||H_n'(x)|}{H_n'^2(x_\nu)} |\ell_\nu(x)| +$$

$$+\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}|x\ell_\nu(x)|+\frac{1}{2}\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}|l_\nu'(0)|\equiv$$

$$\equiv I_1+I_2+I_3+I_4+I_5+I_6$$

(say). From (3.16) and (3.19), we have

$$I_1=O(1)\sum_{\nu=1}^{n}\frac{n^{-1/4}\sqrt{2^n n!}e^{x^2/2}}{2^{n+1}\left(\frac{n}{2}\right)!e^{(\delta/2)x_\nu^2}}e^{x^2}=O(e^{\gamma x^2}),\quad \gamma>\frac{3}{2},\tag{4.6}$$

$$I_2=O(e^{\gamma x^2}\log n),\quad \gamma>\frac{3}{2},\tag{4.7}$$

which follows on the same lines as given in (4.4). From (3.8) and (4.4), we get

$$I_3=O\Big(e^{\gamma x^2}\frac{\log n}{n}\Big),\quad \gamma>\frac{3}{2}.\tag{4.8}$$

From (3.10), (3.11), (3.12), (3.14), (3.16) and (3.19), we get

$$I_4=O(e^{\gamma x^2}),\quad \gamma>\frac{3}{2}\quad\text{and}\quad 2\delta-\frac{r_1}{2}-1>0.\tag{4.9}$$

Similarly,

$$I_5=O\Big(e^{\gamma x^2}\frac{1}{\sqrt{n}}\Big),\quad \gamma>\frac{3}{2}\quad\text{and}\quad 2\delta-\frac{r_1}{2}-1>0.\tag{4.10}$$

Further, using (3.3), we can easily get

$$\ell_\nu'(0)=-\frac{x_\nu H_n'(0)+H_n(0)}{x_\nu^2 H_n'(x_\nu)},$$

which gives

$$I_6=\frac{1}{2}\sum_{\nu=1}^{n}\frac{e^{x_\nu^2}H_n^2(x)}{|x_\nu|H_n'^2(x_\nu)|}\frac{1}{|H_n'(x_\nu)|}\left[|H_n'(0)|+\frac{|H_n(0)|}{|x_\nu|}\right]=\tag{4.11}$$

$$=O(1)\sum_{\nu=1}^{n}\frac{\sqrt{n}e^{(1-\delta/2)x_\nu^2}n^{-1/2}2^n n!}{H_n'^2(x_\nu)2^{n+1}\left(\frac{n}{2}\right)!}e^{x^2}\times$$

$$\times\left[2n(n-1)^{-1/4}\sqrt{2^{n-1}(n-1)!}+\frac{\sqrt{n}n!}{\nu\left(\frac{n}{2}\right)!}\right]=O(e^{\gamma x^2}\log n),$$

on using (3.17) and (3.19). Using (4.5)–(4.11), we get the proof of (4.2).

To prove (4.3), using Lemma 1 and (2.7), we get

$$(4.12)\qquad \sum_{\nu=1}^{n} e^{x_\nu^2}|U_\nu(x)| \leqq \sum_{\nu=1}^{n} e^{x_\nu^2}|\ell^3(x)| + 3\sum_{\nu=1}^{n}|x_\nu|e^{x_\nu^2}|V_\nu(x)|+$$

$$+\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}\left[\frac{2}{3}|x_\nu||2n-x_\nu^2|\left|\int_0^x \ell_\nu(t)dt\right| + \frac{1}{6}|\ell_\nu''(x)-\ell_\nu''(0)| + \frac{|x_\nu|}{2}|\ell_\nu'(x)|+\right.$$

$$\left.+\Big(\frac{n+2+x_\nu^2}{3}|\ell_\nu(x)|+\frac{2n+1}{6}|\ell_\nu(0)|\Big)+\frac{|2H_n'(x)+H_n'(0)|}{6|H_n'(x_\nu)|}+\frac{|x_\nu|}{3}\frac{|H_n(x)|}{|H_n'(x_\nu)|}\right] \equiv$$

$$\equiv J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8.$$

(4.13)

$$J_1=\sum_{\nu=1}^{n} e^{x_\nu^2}\ell_\nu^2(x)|l_\nu(x)| = O(e^{x^2})\sum_{\nu=1}^{n}\frac{e^{r_1(x^2+x_\nu^2)/2}2^{n+1}n!\sqrt{n}}{H_n'^2(x_\nu)} = O(e^{rx^2}\sqrt{n}),$$

$$r_1>1, r>\frac{3}{2},$$

owing to (3.13) and (3.17).

$$(4.14)\qquad J_2 = 3\sum_{\nu=1}^{n}|x_\nu|e^{x_\nu^2}|V_\nu(x)| = O(e^{rx^2}\sqrt{n}\log n),\quad r>\frac{3}{2},$$

using $|x_\nu| = O(\sqrt{n})$ and (4.2).

$$(4.15)\qquad J_3=\frac{2}{3}\sum_{\nu=1}^{n}|x_\nu||2n-x_\nu^2|e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}\left|\int_0^x \ell_\nu(t)dt\right| =$$

$$=O(n^{3/2})\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}\left|\int_0^x \ell_\nu(t)dt\right| = O(n^{3/2})O\Big(e^{rx^2}\frac{\log n}{n}\Big) =$$

$$=O(\sqrt{n}e^{rx^2}\log n).$$

$$J_4 \leqq \frac{1}{6}\sum_{\nu=1}^{n} e^{x_\nu^2}[|\ell_\nu''(x)|+|\ell_\nu''(0)|]\frac{H_n^2(x)}{H_n'^2(x_\nu)}.$$

From (3.3), we get

(4.16)

$$J_4 \leqq \frac{1}{3}\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}\left[\left|\frac{(x(x-x_\nu)-1)H_n'(x)}{(x-x_\nu)^2H_n'(x_\nu)}+\frac{\ell_\nu(x)}{(x-x_\nu)^2}-n\ell_\nu(x)\right|+\right.$$

$$+\left|-n\ell_\nu(0)-\frac{H_n'(0)}{x_\nu^2H_n'(x_\nu)}+\frac{\ell_\nu(0)}{x_\nu^2}\right|\Big]\leqq$$

$$\leqq\frac{1}{3}\sum_{\nu=1}^{n}e^{x_\nu^2}\left[|x(x-x_\nu)-1|\left|\frac{H_n'(x)}{H_n'(x_\nu)}\right|\ell_\nu^2(x)+(1+n(x-x_\nu)^2)|\ell_\nu^3(x)|\right]+$$

$$+\frac{1}{3}\sum_{\nu=1}^{n}e^{x_\nu^2}\left[\left(n+\frac{1}{x_\nu^2}\right)\frac{|H_n(0)|H_n^2(x)}{|x_\nu||H_n'(x_\nu)|H_n'^2(x_\nu)}+\frac{|H_n'(0)|H_n^2(x)}{|H_n'(x_\nu)|H_n^2(0)}\ell_\nu^2(0)\right]=$$

$$=O(e^{x^2})\sum_{\nu=1}^{n}(1+|x(x-x_\nu)|)\frac{|H_n'(x)|}{|H_n'(x_\nu)|}+\sum_{\nu=1}^{n}e^{x_\nu^2}(1+n(x-x_\nu)^2)|\ell_\nu^3(x)|+$$

$$+O(1)\sum_{\nu=1}^{n}\left(n+\frac{n}{\nu^2}\right)\frac{\sqrt{n}n!n^{-1/2}2^n n!e^{x^2}e^{(1-\delta/2)x_\nu^2}}{\nu(\frac{n}{2})!2^{n+1}(\frac{n}{2})!H_n'^2(x_\nu)}=$$

$$=O(e^{x^2})\sum_{\nu=1}^{n}\frac{(n-1)^{-1/4}\sqrt{(n+1)!}}{2^{n/2}(\frac{n}{2})!e^{\delta x_\nu^2/2}}+O(e^{x^2})\sum_{\nu=1}^{n}e^{r_1(x^2+x_\nu^2)/2}.$$

$$\cdot\frac{2^{n+1}n!\sqrt{n}}{H_n'^2(x_\nu)}+O(n)\sum_{\nu=1}^{n}\frac{e^{x_\nu^2(1+\frac{r_1}{2})}e^{r_1x^2/2}2^{n+1}n!n!e^{x^2}}{H_n'^2(x_\nu)2^{n+2}((\frac{n}{2})!)^2e^{\delta x_\nu^2}}+$$

$$+O(1)\sum_{\nu=1}^{n}\left(n+\frac{n}{\nu^2}\right)\frac{\sqrt{n}n!n^{-1/2}2^n n!e^{x^2}e^{(1-\delta/2)x_\nu^2}}{\nu(\frac{n}{2})!2^{n+1}(\frac{n}{2})!H_n'^2(x_\nu)}+$$

$$+O(1)\sum_{\nu=1}^{n}\frac{2n(n-1)^{-1/4}\sqrt{2^{n-1}(n-1)!}n^{-1/2}2^ne^{x^2}((\frac{n}{2})!)^2}{2^{n+1}(\frac{n}{2})!e^{\delta x_\nu^2/2}n!}=$$

$$=O(\sqrt{n}e^{x^2})+O\left(e^{(1+\frac{r_1}{2})x^2}\sqrt{n}\right)+O(\sqrt{n}e^{x^2})=O(\sqrt{n}e^{\gamma x^2}),\quad \gamma>\frac{3}{2}.$$

(4.17) $$J_5=\frac{1}{2}\sum_{\nu=1}^{n}e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}|x_\nu|\left|\frac{H_n'(x)}{(x-x_\nu)H_n'(x_\nu)}-\frac{\ell_\nu(x)}{x-x_\nu}\right|\leqq$$

$$\leqq\frac{1}{2}\sum_{\nu=1}^{n}|x_\nu|\frac{e^{x_\nu^2}|H_n(x)||H_n'(x)|}{H_n'^2(x_\nu)}|\ell_\nu(x)|+\frac{1}{2}\sum_{\nu=1}^{n}|x_\nu|e^{x_\nu^2}\frac{|H_n(x)|}{|H_n'(x_\nu)|}\ell_\nu^2(x)=$$

$$=O(\sqrt{n}e^{\gamma x^2}),\quad \gamma>\frac{3}{2}$$

as we have done in (4.6), (4.9) and using (3.9)

(4.18) $$J_6\leqq\frac{1}{3}\sum_{\nu=1}^{n}e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}[(n+2+x_\nu^2)|\ell_\nu(x)|+(n+1)|\ell_\nu(0)|]=$$

$$= O(n)\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}|\ell_\nu(x)| = O(\sqrt{n}e^{\gamma x^2}), \quad \gamma > \frac{3}{2}$$

owing to (4.10).

$$J_7 = \frac{1}{6}\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}\left[\frac{2|H_n'(x)| + |H_n'(0)|}{|H_n'(x_\nu)|}\right] = \tag{4.19}$$

$$= O(n)\sum_{\nu=1}^{n} e^{x_\nu^2}\frac{H_n^2(x)}{H_n'^2(x_\nu)}\frac{|H_{n-1}(x)|}{|H_n'(x_\nu)|} =$$

$$= O(n)\sum_{\nu=1}^{n}\frac{e^{x_\nu^2}n^{-1/2}2^n n! e^{(3/2)x^2}(n-1)^{-1/4}\sqrt{2^{n-1}(n-1)!}}{2^{n+1}(\frac{n}{2})! e^{(\delta/2)x_\nu^2}H_n'^2(x_\nu)} =$$

$$= O\left(ne^{\frac{3}{2}x^2}\right)\sum_{\nu=1}^{n}\frac{e^{(1-\delta/2)x_\nu^2}2^{n+1}n!n^{-3/4}((n+1)!)^{1/2}}{H_n'^2(x_\nu)\sqrt{n(n+1)}2^{n/2}(\frac{n}{2})!} =$$

$$= O\left(ne^{\frac{3}{2}x^2}\right)\frac{n^{-3/4}n^{1/4}}{n} = O\left(\frac{e^{\gamma x^2}}{\sqrt{n}}\right), \quad \gamma > \frac{3}{2}.$$

$$J_8 = \frac{1}{3}\sum_{\nu=1}^{n}|x_\nu|e^{x_\nu^2}\left|\frac{H_n^3(x)}{H_n'^3(x_\nu)}\right| = \tag{4.20}$$

$$= O(1)\sum_{\nu=1}^{n}\frac{\sqrt{n}e^{x_\nu^2(1-\delta/2)}n^{-3/4}(2^n n!)^{3/2}e^{3x^2/2}}{H_n'^2(x_\nu)2^{n+1}(\frac{n}{2})!} =$$

$$= O(e^{3x^2/2})\frac{\sqrt{n}n^{-3/4}((n+1)!)^{1/2}}{\sqrt{n+1}2^{\frac{n+1}{2}}(\frac{n}{2})!} = O(e^{\gamma x^2}/\sqrt{n}), \quad \gamma > \frac{3}{2}.$$

Using (4.12)–(4.20), we get the proof of (4.3).

5. Weighted polynomial approximation on the real line. G. Freud ([6], [8]) proved the following:

THEOREM A. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuously differentiable and $\varrho(x)$ be a weight function such that*

$$\lim_{|x|\to\infty} x^{2\nu}f(x)\varrho(x) = 0, \quad \nu = 0,1,2,\dots, \tag{5.1}$$

and

$$\lim_{|x|\to\infty} f'(x)\varrho(x) = 0.$$

Then there exists a polynomial $q_n(x)$ *of degree* $\leqq n$ *such that*

$$(5.2)\qquad \begin{cases} \varrho(x)|f(x)-q_n(x)| = O(1/\sqrt{n})\omega(f',1/\sqrt{n}),\\ \varrho(x)|f'(x)-q_n'(x)| = O(1)\omega(f',1/\sqrt{n}),\end{cases}$$

where ω *is the modulus of continuity of* f *defined by*

$$\omega(f,\delta) = \sup_{0\leqq t\leqq\delta} \|\varrho(x+t)f(x+t)-\varrho(x)f(x)\| + \|\tau(\delta x)\varrho(x)f(x)\|$$

where

$$\tau(x) = \begin{cases} |x| & \text{if } |x|\leqq 1\\ 1 & \text{if } |x|>1,\end{cases}$$

and $\|\cdot\|$ *denotes the sup norm in* $C(\mathbf{R})$. *If* $f\in C(\mathbf{R})$ *and* $\lim_{|x|\to\infty}\varrho(x)f(x)=0$, *then* $\lim_{\delta\to 0}\omega(f,\delta)=0$.

L. Szili [10] proved the following:

THEOREM B. *Let* $q_n(x)$ *be a polynomial of degree* $\leqq n$ *satisfying the condition* (5.2). *Then*

$$(5.3)\qquad \begin{cases} \varrho(x)|q_n(x)| = O(1), & x\in\mathbf{R}\\ \varrho(x)|q_n'(x)| = O(1), & x\in\mathbf{R}\\ \varrho(x)|q_n''(x)| = O(\sqrt{n})\omega(f',1/\sqrt{n}), & |x|<\sqrt{2n+1}.\end{cases}$$

To prove Theorem 5 we need

LEMMA 4. *Under the conditions* (5.2) *and* (5.3),

$$(5.4)\qquad \varrho(x)|q_n'''(x)| = O(n)\omega(f',1/\sqrt{n}),\quad |x|<\sqrt{2n+1}.$$

PROOF. To prove this lemma, we follow the methods of G. Freud ([7], [8]) and L. Szili [10]. The polynomial

$$p_{m+1}(x) = \sum_{k=1}^{m}\frac{x^k}{k!}$$

satisfies the inequalities:

$$(5.5)\qquad c_1e^x \leqq p_{m+1}(x)\leqq c_2e^x\quad(-m/4\leqq x\leqq 0)$$

and

$$(5.6)\qquad c_1e^x \leqq p_{m+1}'(x)\leqq c_2e^x\quad(-m/4\leqq x\leqq 0)$$

(see [8], Lemma 3). From (5.3) and (5.5), we get

$$(5.7)\quad p_m(-x^2)|q_n''(x)| = O(\sqrt{n})\omega(f',1/\sqrt{n})\quad(|x|<\sqrt{m/2},\ m<4n+2).$$

If Q_n is a polynomial of degree $\leqq n$, then

$$(5.8)\qquad |Q_n'(x)| = \frac{O(n)}{\sqrt{\frac{m}{2}-x^2}}\omega\Big(Q_n, \frac{\sqrt{m}}{n}\Big) \quad \text{for} \quad |x| < \sqrt{m/2}$$

holds, where ω is the modulus of continuity over the interval $[-m/2, m/2]$. Using this inequality on the interval $\left[-\frac{1}{2}\sqrt{\frac{m}{2}}, \frac{1}{2}\sqrt{\frac{m}{2}}\right]$ for the polynomial (5.7), we obtain

$$\left|\frac{d}{dx}[p_m(-x^2)q_n''(x)]\right| = |-2xp_m'(-x^2)q_n''(x) + p_m(-x^2)q_n'''(x)| \geqq$$
$$\geqq p_m(-x^2)|q_n'''(x)| - 2|xp_m'(-x^2)q_n''(x)|$$

which gives rise to

$$(5.9)\qquad \varrho(x)|q_n'''(x)| \leqq |p_m(-x^2)q_n'''(x)| \leqq 2|xp_m'(-x^2)q_n''(x)|+$$
$$+c_1\sqrt{n}\omega(p_m q_n'', 1/\sqrt{n}) \quad \text{for} \quad |x| < \sqrt{2n+1}.$$

By the definition of ω, we have

$$(5.10)\ \omega(p_m q_n''; 1/\sqrt{n}) = \sup_{0\leqq t\leqq 1/\sqrt{n}} |p_m[-(x+t)^2]q_n''(x+t) - p_m(-x^2)q_n''(x)| \leqq$$
$$\leqq 2c \sup_{|x|<\sqrt{m/2}} |p_m(-x^2)q_n''(x)| = O(\sqrt{n})\omega(f', 1/\sqrt{n})$$

on account of (5.7).

$$(5.11)\qquad |xp_m'(-x^2)q_n''(x)| = |x||p_m'(-x^2)||q_n''(x)| \leqq$$
$$\leqq c_2\sqrt{n}\varrho(x)|q_n''(x)| = O(n)\omega(f', 1/\sqrt{n}),$$

using (5.3) for $|x| < \sqrt{2n+1}$. Now (5.9), (5.10) and (5.11) prove the lemma.

6. PROOF OF THEOREM 5. Let n be even. According to Theorem 4, every polynomial P_n of degree $\leqq 3n$ is written as

(6.1)
$$P_n(x) = \sum_{\nu=1}^{n} P_n(x_\nu)U_\nu(x) + \sum_{\nu=1}^{n} P_n'(x_\nu)V_\nu(x) + \sum_{\nu=1}^{n}(\varrho P_n)'''(x_\nu)W_\nu(x) + d_n H_n^2(x)$$

where

$$(6.2)\ d_n = \frac{1}{H_n^2(0)}\left[P_n(0) - \sum_{\nu=1}^{n} P_n(x_\nu)(1+3x_\nu^2)\ell_\nu^3(0) + \sum_{\nu=1}^{n} x_\nu P_n'(x_\nu)\ell_\nu^3(0)\right].$$

Let q_n be a polynomial of degree $\leqq 3n$ satisfying the inequalities (5.2) and (5.3). Then we have

$$(6.3)\qquad e^{-\gamma x^2}|f(x)-G_n(x)| = O(1)\Bigg[\varrho(x)|f(x)-q_n(x)|+$$

$$+e^{-\gamma x^2}\left|\sum_{\nu=1}^{n}\varrho(x_\nu)\left(f(x_\nu)-q_n(x_\nu)\right)\frac{U_\nu(x)}{\varrho(x_\nu)}\right|+$$

$$+e^{-\gamma x^2}\left|\sum_{\nu=1}^{n}\varrho(x_\nu)\left(f'(x_\nu)-q_n'(x_\nu)\right)\frac{V_\nu(x)}{\varrho(x_\nu)}\right|+e^{-\gamma x^2}\left|\sum_{\nu=1}^{n}(c_\nu-(\varrho q_n)'''(x_\nu))W_\nu(x)\right|+$$

$$+|d_n|e^{-\gamma x^2}H_n^2(x)\Bigg] \equiv O(1)[s_1+s_2+s_3+s_4+s_5],$$

where

$$(6.4)\qquad s_1 = O(1/\sqrt{n})\omega(f',1/\sqrt{n}),$$

$$(6.5)\quad s_2 \leqq e^{-\gamma x^2}\sum_{\nu=1}^{n}\varrho(x_\nu)|f(x_\nu)-q_n(x_\nu)|\left|\frac{U_\nu(x)}{\varrho(x_\nu)}\right| = O(1)\log n\omega(f',1/\sqrt{n}),$$

owing to (5.2) and (4.3),

$$(6.6)\quad s_3 \leqq e^{-\gamma x^2}\sum_{\nu=1}^{n}\varrho(x_\nu)|f'(x_\nu)-q_n'(x_\nu)|\frac{|V_\nu(x)|}{\varrho(x_\nu)} = O(\log n)\omega(f',1/\sqrt{n}),$$

owing to (4.2) and (5.2).

$$s_4 \leqq e^{-\gamma x^2}\sum_{\nu=1}^{n}|c_\nu||W_\nu(x)| + e^{-\gamma x^2}\sum_{\nu=1}^{n}|(\varrho q_n)'''(x_\nu)||W_\nu(x)| =$$

$$= O(n)\omega(f',1/\sqrt{n})e^{-\gamma x^2}\left[\sum_{\nu=1}^{n}|W_\nu(x)|e^{\beta x_\nu^2}+\sum_{\nu=1}^{n}|W_\nu(x)|\right] =$$

$$= O(\log n)\omega(f',1/\sqrt{n}),$$

on account of Lemma 3, (2.14) and (4.1) of Lemma 2.

We now estimate the last term of (6.3). Replacing P_n by q_n in (6.2), we have

$$(6.7)\qquad s_5 = |d_n|e^{-\gamma x^2}H_n^2(x) =$$

$$= e^{-\gamma x^2}\frac{H_n^2(x)}{H_n^2(0)}\Big|q_n(0)-\sum_{\nu=1}^{n}q_n(x_\nu)(1+3x_\nu^2)\ell_\nu^3(0)+\sum_{\nu=1}^{n}x_\nu q_n'(x_\nu)\ell_\nu^3(0)\Big| =$$

$$= e^{-\gamma x^2}\frac{H_n^2(x)}{H_n^2(0)}\Big|\sum_{\nu=1}^{n}q_n(0)(1+3x_\nu^2)\ell_\nu^3(0)-\sum_{\nu=1}^{n}q_n(x_\nu)(1+3x_\nu^2)\ell_\nu^3(0)+$$

$$+\sum_{\nu=1}^{n}x_\nu q_n'(x_\nu)\ell_\nu^3(0)\Big|,$$

because

$$1=\sum_{\nu=1}^{n}(1+3x_\nu^2)\ell_\nu^3(0)$$

which can be obtained at $x = 0$ from Theorem 4, taking

$$a_\nu = G_n(x_\nu) = 1, \quad b_\nu = G_n'(x_\nu) = 0 \quad \text{and} \quad G_n(x) = 1.$$

Regarding the sums in (6.7), we get

$$\text{(6.8)} \qquad s_5 \leqq e^{-\gamma x^2}\frac{H_n^2(x)}{H_n^2(0)}\Big[\sum_{\nu=1}^{n}|q_n(0)-q_n(x_\nu)|(1+3x_\nu^2)|\ell_\nu^3(0)|+$$

$$+\sum_{\nu=1}^{n}|x_\nu||q_n'(x_\nu)||\ell_\nu^3(0)|\Big] = K(t_1+t_2),$$

where

$$\text{(6.9)} \qquad K = e^{-\gamma x^2}\frac{H_n^2(x)}{H_n^2(0)} = O(1),$$

owing to (3.9) and (3.11).

$$\text{(6.10)} \qquad t_1 = \sum_{\nu=1}^{n}|q_n(0)-q_n(x_\nu)|(1+3x_\nu^2)|\ell_\nu^3(0)| =$$

$$= \sum_{\nu=1}^{n}|x_\nu||q_n'(x_\nu)|(1+3x_\nu^2)|\ell^3(0)| = \sum_{\nu=1}^{n}|q_n'(\xi_\nu)|\frac{1+3x_\nu^2}{x_\nu^2}\Big|\frac{H_n^3(0)}{H_n'^3(x_\nu)}\Big| =$$

$$= O(1)\sum_{\nu=1}^{n}\frac{e^{2x_\nu^2}}{x_\nu^2}\left(\frac{n!}{(\frac{n}{2})!}\right)^3\frac{1}{2^{3n+3}((\frac{n}{2})!)^3e^{(3\delta/2)x_\nu^2}} =$$

$$= O(1/n^{3/2})\sum_{\nu=1}^{n}n/\nu^2 = O(1/\sqrt{n}) = O(1)\omega(f',1/\sqrt{n}), \quad 0<\xi_\nu<x_\nu,$$

for $3\delta/2 > 2$, owing to (3.19) and (3.8), because $1/\sqrt{n} \leqq \omega(f', 1/\sqrt{n})$, and

$$t_2 = \sum_{\nu=1}^{n} |x_\nu||q_n'(x_\nu)||\ell_\nu^3(0)| = O(1)\omega(f', 1/\sqrt{n}), \tag{6.11}$$

which follows mutatis mutandis to that of (6.10).

Thus (6.8)–(6.11) complete the estimation

$$s_5 = O(1)\omega(f', 1/\sqrt{n}). \tag{6.12}$$

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A NOTE ON CERTAIN NEXT-TO-INTERPOLATORY RATIONAL FUNCTIONS

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Dedicated to Prof. R. S. Varga on his 60th birthday

1. Introduction

This note owes its motivation to an extension of a simple and elegant theorem of J. L. Walsh ([6], p. 153). Let $f \in A_\varrho$ (the set of functions analytic in $|z| < \varrho$ but not in $|z| \leqq \varrho$, $\varrho > 1$). Let Π_s denote the class of all polynomials of degree $\leqq s$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then we put

$$S_{n-2}(z, f) = \sum_{j=0}^{n-2} a_j z^j. \tag{1.1}$$

Let $p^*_{n-2}(z, f)$ denote the polynomial of degree $\leqq n-2$ which minimizes

$$\max_{0 \leqq k \leqq n-1} |p_{n-2}(\lambda^k) - f(\lambda^k)|, \quad \lambda = \exp(2\pi i/n),$$

over all polynomials $p_{n-2} \in \Pi_{n-2}$. Then Cavaretta et al established:

THEOREM A ([2], Theorem 7). *Let $f \in A_\varrho$, $1 < \varrho < \infty$. Then*

$$\lim_{n\to\infty} \{S_{n-2}(z, f) - p^*_{n-2}(z, f)\} = 0, \quad |z| < \varrho^2, \tag{1.2}$$

the convergence being uniform and geometric for all $|z| \leqq \tau < \varrho^2$. Moreover, the result is sharp in the sense that (1.2) *is not valid at each point of $|z| = \varrho^2$ for all $f \in A_\varrho$.*

It may be noted that none of the sequences $\{S_{n-2}(z, f)\}_{n=1}^{\infty}$ and $\{p^*_{n-2}(z, f)\}_{n=1}^{\infty}$ in Theorem A converge beyond the region $|z| < \varrho$ whereas their difference converges in $|z| < \varrho^2$. This phenomenon introduced by Walsh ([6], p. 153) is known as overconvergence or equiconvergence in the literature. For further details on this topic we refer the interested reader to [1] and [5].

Recently, E. B. Saff and A. Sharma [4] discussed the equiconvergence of certain sequences of rational interpolants. The classic theorem of Walsh ([6], p. 153) is a special case of their result ([4], Theorem 2.3). Our object in the present paper is to generalize Theorem A in the spirit of the Saff–Sharma result. Building on results of Motzkin and Sharma [3], we solve a min-max problem in §3 for which the solution is the so-called next-to-interpolatory rational function. We prove our main result, i.e., Theorem 2.1, in §4.

2. Preliminaries and statement of main result

Let $m \geqq -n+1$ be a fixed integer and let $r_{n+m-1}(z,f)$ be a rational function of the form

$$(2.1) \qquad p(z)/(z^n-\sigma^n), \quad p(z)\in \Pi_{n+m-1}, \quad \sigma>1,$$

which minimizes (see [4])

$$\int_{|z|=1} |f(z)-r(z)|^2|dz|$$

over all rational functions $r(z)$ of the form (2.1). If $f\in A_\varrho$, then it is known [4] that the minimizing function $r_{n+m-1,n}(z,f)$ is given by

(2.2)

$$r_{n+m-1,n}(z,f)=\begin{cases}\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t-z)}\cdot\frac{t^m(t^n-\sigma^{-n})-z^m(z^n-\sigma^{-n})}{t^m(t^n-\sigma^{-n})}dt, \text{when } & m\geqq 0,\\ \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t-z)}\cdot\frac{t^{n-m}-z^{n-m}}{t^m(t^n-\sigma^{-n})}dt, \text{when} & m<0,\end{cases}$$

where Γ is a circle $|t|=\varrho'$, $1<\varrho'<\varrho$, and $z\in \mathbf{C}$ with $|z|\neq\sigma$.

Next, consider the following problem:

(PI) *Let m be a fixed integer with $m\geqq -n+1$, and let $\omega=\exp(2\pi i/(n+m+1))$. For $f\in A_\varrho$, minimize*

$$\max_{0\leqq k\leqq n+m} |f(\omega^k)-R(\omega^k,f)|$$

over all rational functions $R(\cdot,f)$ of the form (2.1).

The existence and uniqueness of the solution for (PI) is based on the results of Motzkin and Sharma ([3], Theorems 1 and 2). If the solution (see §3) is denoted by

$$(2.3) \quad R^*_{n+m-1,n}(z,f):=P^*_{n+m-1}(z,f)/(z^n-\sigma^n), \quad P^*_{n+m-1}\in\Pi_{n+m-1},$$

then we formulate our main result as follows:

THEOREM 2.1. *Let m be a fixed integer and let $\sigma>1$. If $f\in A_\varrho$, $1<\varrho<\infty$, then*

$$(2.4) \quad \lim_{n\to\infty}\{R^*_{n+m-1,n}(z,f)-r_{n+m-1,n}(z,f)\}=0\begin{cases}\text{if } |z|<\varrho^2 \ (\sigma\geqq\varrho^2),\\ \text{if } |z|\neq\sigma \ (\sigma<\varrho^2),\end{cases}$$

the convergence being uniform and geometric in any compact subset of the regions described above. Moreover, the result is sharp in the sense of Theorem A.

3. Solution of the minimization problem (PI)

Indeed, we want to solve the problem

$$\min_{p\in\Pi_{n+m-1}} \max_{0\leqq k\leqq n+m} \left| f(\omega^k) - \frac{p(\omega^k)}{\omega^{kn}-\sigma^n} \right|$$

(ω being as in (PI)), which is equivalent to

$$\min_{p\in\Pi_{n+m-1}} \max_{0\leqq k\leqq n+m} b_k |F_n(\omega^k) - p(\omega^k)| \tag{3.1}$$

where

$$b_k = |\omega^{kn}-\sigma^n|^{-1} \quad \text{and} \quad F_n(z) = (z^n - \sigma^n) f(z).$$

Based on a result of Motzkin and Sharma ([3], Theorem 2), it can be verified that the solution $P^*_{n+m-1}(z, f)$ of the problem (3.1) is given by

$$P^*_{n+m-1}(z,f) = \sum_{k=1}^{n+m+1} b_k^{-1} g_k(z) \Big/ \sum_{k=1}^{n+m+1} b_k^{-1} \tag{3.2}$$

where

$$g_k(z) := \sum_{\substack{j=1\\ j\neq k}}^{n+m+1} \frac{W_k(z)}{(z-\omega^j)W_k'(\omega^j)} F_n(\omega^j) \tag{3.3}$$

with

$$W_k(z) := (z^{n+m+1}-1)/(z-\omega^k).$$

In order to prove our main result, we shall require an integral representation of $P^*_{n+m-1}(z,f)$. For this we prove

LEMMA 3.1. *The polynomial $g_k(z)$, $1 \leqq k \leqq n+m+1$, given by (3.3) has the following representation:*

$$g_k(z) = \beta_{n+m,n}(z,f) - C_n(z^{n+m+1}-1)/(z-\omega^k) \tag{3.4}$$

where

$$\beta_{n+m,n}(z,f) := \frac{1}{2\pi i}\int_\Gamma \frac{F_n(t)}{t-z} \cdot \frac{t^{n+m+1}-z^{n+m+1}}{t^{n+m+1}-1} dt, \tag{3.5}$$

and

$$C_n := \frac{1}{2\pi i}\int_\Gamma \frac{F_n(t)}{t^{n+m+1}-1} dt. \tag{3.6}$$

Here Γ is a circle $|t| = \varrho'$, $1 < \varrho' < \varrho$.

PROOF. It is easy to see that

$$g_k(z) = \sum_{\substack{j=1\\ j\neq k}}^{n+m+1} \frac{(z^{n+m+1}-1)(\omega^j-\omega^k)\omega^j}{(n+m+1)(z-\omega^k)(z-\omega^j)} F_n(\omega^j).$$

Since

$$\frac{\omega^j-\omega^k}{(z-\omega^k)(z-\omega^j)} = \frac{1}{z-\omega^j} - \frac{1}{z-\omega^k},$$

we have

$$g_k(z) = \sum_{j=1}^{n+m+1} \frac{z^{n+m+1}-1}{z-\omega^j}\cdot\frac{\omega^j}{n+m+1}F_n(\omega^j) -$$
$$-\frac{z^{n+m+1}-1}{z-\omega^k}\sum_{j=1}^{n+m+1}\frac{\omega^j}{n+m+1}F_n(\omega^j).$$

The first summation on the right side of the above equation represents the Lagrange interpolation polynomial of degree $n+m$ to $F_n(z)$ on the $n+m+1$ roots of unity and therefore equals $\beta_{n+m,n}(z,f)$. The second summation upon using the Cauchy integral formula leads to C_n in (3.6).

REMARK 3.1. From (3.2) and (3.4) we have

(3.7)
$$P^*_{n+m-1}(z,f) = \beta_{n+m,n}(z,f) - C_n \sum_{k=1}^{n+m+1} b_k^{-1}\frac{z^{n+m+1}-1}{z-\omega^k} \Big/ \sum_{k=1}^{n+m+1} b_k^{-1}.$$

An alternate representation of $P^*_{n+m-1}(z,f)$ is therefore given by

$$P^*_{n+m-1}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^n-\sigma^n)f(t)}{t^{n+m+1}-1}\cdot\frac{t(t^{n+m}-z^{n+m})}{t-z}dt - \gamma_n(z,f) \tag{3.8}$$

where

$$\gamma_n(z,f) := C_n \sum_{k=1}^{n+m+1} b_k^{-1}\omega^k \frac{z^{n+m}-\omega^{k(n+m)}}{z-\omega^k} \Big/ \sum_{k=1}^{n+m+1} b_k^{-1}. \tag{3.9}$$

This can easily be seen upon writing the integral representation of $\beta_{n+m,n}(z,f)$ (cf. (3.5)) and considering the relation:

$$\frac{z^{n+m+1}-1}{z-\omega^k} = z^{n+m} + \omega^k\cdot\frac{z^{n+m}-\omega^{k(n+m)}}{z-\omega^k}$$

in (3.7).

4. Proof of the main result

When $|z| \leqq 1$, the direct estimates from (2.2), (3.2) and (3.3) show that

$$\lim_{n\to\infty} \{R^*_{n+m-1,n}(z,f) - r_{n+m-1}(z,f)\} = 0.$$

However, the proof of (2.4) for $|z| > 1$ requires a detailed analysis of $\gamma_n(z,f)$. For this, first we note that

$$|b_k^{-1} - \sigma^n| \leqq 1. \tag{4.1}$$

Also

$$\sum_{k=1}^{n+m+1} \frac{\omega^k(z^{n+m} - \omega^{k(n+m)})}{z - \omega^k} = 0, \quad \text{for all } z. \tag{4.2}$$

This follows from the properties of the roots of unity. From (4.1) and (4.2) we observe that

$$\left|\sum_{k=1}^{n+m+1} b_k^{-1} \frac{\omega^k(z^{n+m} - \omega^{k(n+m)})}{z - \omega^k}\right| \leqq |z|^{n+m}(n+m+1)^2 \frac{|z|^{n+m}+1}{|z^{n+m+1}-1|} \tag{4.3}$$

when $|z| > 1$. Since

$$\sum_{k=1}^{n+m+1} b_k^{-1} \geqq (n+m+1)(\sigma^n - 1) \tag{4.5}$$

and

$$C_n := \frac{1}{2\pi i} \int_\Gamma \frac{(t^n - \sigma^n)f(t)}{t^{n+m+1}-1}\,dt = O\left(\frac{\varrho^n + \sigma^n}{\varrho^n}\right), \tag{4.6}$$

we obtain the following inequality from (3.9) and (4.3)–(4.6):

$$|\gamma_n(z,f)| \leqq D_1(n+m+1)|z^{n+m}| \cdot \frac{|z|^{n+m}+1}{|z^{n+m+1}-1|} \cdot \frac{\varrho^n + \sigma^n}{\varrho^n \sigma^n}, \quad |z| > 1,$$

where D_1 is a constant independent of n. This shows that

$$\lim_{n\to\infty} \frac{\gamma_n(z,f)}{z^n - \sigma^n} = 0, \quad \text{for all } |z| > 1 \text{ and } |z| \neq \sigma. \tag{4.7}$$

Proof of Theorem 2.1. We shall prove this result for $m \geqq 0$. The proof for the case $m < 0$ would be similar. From (2.2), (2.3) and (3.8) we have

(4.8)

$$R^*_{n+m-1,n}(z,f)-r_{n+m-1,n}(z,f)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t-z)}K_n(t,z)dt-\frac{\gamma_n(z,f)}{z^n-\sigma^n}$$

where

$$K_n(t,z):=\frac{(t^{n+m}-z^{n+m})(1-t^{m+1}\sigma^{-n})}{(t^{n+m+1}-1)(t^n-\sigma^{-n})t^m}-\frac{(z^m-t^m)\sigma^{-n}}{t^m(t^n-\sigma^{-n})}. \tag{4.9}$$

Notice that Γ is a circle $|t|=\varrho'$, $1<\varrho'<\varrho$. Since $\sigma>1$, for all sufficiently large n we obtain

$$|R^*_{n+m-1}(z,f)-r_{n+m-1,n}(z,f)|\leqq D_2\frac{\varrho^n+\sigma^n}{|z^n-\sigma^n|}\left\{\frac{\varrho^n+|z|^n}{\varrho^{2n}}+\frac{\sigma^{-n}}{\varrho^n}\right\}+\frac{|\gamma_n(z,f)|}{|z^n-\sigma^n|}$$

where D_2 is a constant independent of n. Taking into account (4.7) and considering different cases for $\sigma\geqq\varrho^2$, and $\sigma<\varrho^2$, a straightforward analysis now yields (2.4).

Next, we show that the result is sharp in the sense that the first region of convergence in (2.4) cannot be improved. For this we follow the usual technique (cf. [4]) and select the function $\hat{f}(z)=(\varrho-z)^{-1}$ in A_ϱ along with the point $z=\varrho^2$ on the boundary of the region $|z|<\varrho^2$. It is easy to see that

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{(t^n-\sigma^n)(t^{n+m}-z^{n+m})\hat{f}(t)dt}{(z^n-\sigma^n)(t-z)(t^{n+m+1}-1)(t^n-\sigma^{-n})t^m}=$$

$$=\frac{(\varrho^n-\sigma^n)(\varrho^{n+m}-z^{n+m})}{(z^n-\sigma^n)(\varrho-z)(\varrho^{n+m+1}-1)(\varrho^n-\sigma^{-n})\varrho^m},$$

whereas for $z=\varrho^2$ and $\sigma>\varrho^2$ we have

$$\lim_{n\to\infty}\frac{\gamma_n(z,\hat{f})}{z^n-\sigma^n}=0$$

and

$$\lim_{n\to\infty}\int\limits_{\Gamma}\frac{(t^n-\sigma^n)\sigma^{-n}f(t)}{(z^n-\sigma^n)(t-z)(t^n-\sigma^{-n})t^m}\left\{\frac{(t^{n+m}-z^{n+m})t}{t^{n+m+1}-1}-(z^m-t^m)\right\}dt=0.$$

This along with (4.8) and (4.9) gives us

(4.10)

$$\lim_{n\to\infty}\left\{R^*_{n+m-1,n}(\varrho^2,\hat{f})-r_{n+m-1,n}(\varrho^2,\hat{f})\right\}=\frac{\varrho^{2m}}{(\varrho^2-\varrho)\varrho^{m+1}\varrho^m}\quad(\sigma>\varrho^2).$$

When $z=\varrho^2$ and $\sigma=\varrho^2$, the limit on the left side of (2.4) does not exist because of the factor $(z^n-\sigma^n)$ in the denominator. This completes the proof.

REMARK 4.3. If we choose $m=-1$ and let $\sigma\to\infty$ in (2.4), we obtain Theorem A.

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ON $(n+1)$-SUBWEBS OF AN $(n+1)$-WEB AND LOCAL ALGEBRAS ASSOCIATED WITH THEM

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0. Introduction

Let $W = W(n+1, n, r)$ be an $(n+1)$-web given on an (nr)-dimensional differentiable manifold X^{nr} by $n+1$ foliations λ_ξ, $\xi = 1, \dots, n+1$, of codimension r. For $n > 2$, subwebs of W defined by any $k+1$ foliations, $k < n$, out of the $n+1$ foliations λ_ξ were studied (see [9], [10], or [16]). It is natural to call such subwebs reduct $(k+1)$-subwebs.

On the other hand, one can consider (ns)-dimensional $(n+1)$-subwebs $\widetilde{W} = W(n+1, n, s) \subset X^{ns} \subset X^{nr}$ of the web W whose leaves are intersections of X^{ns} with the leaves of the web W. It is natural to call such subwebs transversal $(n+1)$-subwebs. The transversal $(n+1)$-subwebs were studied for $n = 2$, $s = 1$ in [1] and for $n > 2$, $s = 1$ in [9] (see also [10] or [16]). It was proved in these papers that n-dimensional $(n+1)$-subwebs, $n \geqq 2$, are induced on transversal geodesic surfaces of the $(n+1)$-web W. In the paper [6] transversal 3-subwebs of a 3-web $W(3,2,r)$ were studied for $s = 1, \dots, r-1$, the relationship of the Akivis algebras defined by the coordinate loops of the 3-web and its transversal 3-subwebs (see [3], [6] or [17]) was established, and isoclinic and Grassmann 3-webs were characterized in terms of these algebras.

In Section 1 of the present paper we study the transversal $(n+1)$-subwebs $\widetilde{W}$ of the web W for $n > 2$, establish their relation with reduct $(k+1)$-subwebs, some properties of their web spaces X^{ns} and the connections between the fundamental tensors of $\widetilde{W}$ and W. In addition, we prove that transversal subwebs $\widetilde{W}$ of the webs W of known special types are of the same types and give a characterization of Grassmannizable $(n+1)$-webs: they possess a maximal set of transversal subwebs.

In Section 2 we study algebra structures associated with coordinate n-loops of the web W and its transversal subwebs $\widetilde{W}$. For W and $\widetilde{W}$ such structures, which are called the (AC)-algebra and $\widetilde{(AC)}$-algebra respectively, are obtained as the sets of $\binom{n}{2}$ (Akivis) A-algebras associated with reduct 3-subwebs of W and $\widetilde{W}$ and $\binom{n}{3}$ (comtrans) C-algebras associated with reduct 4-subwebs of W and $\widetilde{W}$ (see [19] or [16]). The operations in both coordinate

n-loops (of W and $\widetilde{W}$) produce a series of h-ary operations, $h = 2, 3, 4, \ldots$, in their tangent spaces at the identity, and these tangent spaces with these h-ary operations give rise to the local $(AC)_h$-algebra and $(\widetilde{AC})_h$-algebra.

We prove that the $(\widetilde{AC})_h$-algebras are subalgebras of the corresponding $(AC)_h$-algebras and use these algebras to characterize isoclinic and Grassmannizable $(n+1)$-webs W.

1. $(n+1)$-subwebs of an $(n+1)$-web

1. Let $W(n+1, n, r)$ be an $(n+1)$-web defined on a differentiable manifold X^{nr} of dimension nr by $n+1$ foliations λ_ξ, $\xi = 1, \ldots, n+1$. Each foliation λ_ξ can be defined by a completely integrable system of Pfaffian equations (see [9], [10] or [16]):

$$\underset{\xi}{\omega}^i = 0, \quad \xi = 1, \ldots, n+1; \quad i = 1, \ldots, r. \tag{1.1}$$

The forms $\{\underset{\alpha}{\omega}^i\}$, $\alpha = 1, \ldots, n$, define a co-frame in the tangent bundle $T(X^{nr})$ and satisfy the following structure equations:

$$d\underset{\alpha}{\omega}^i - \underset{\alpha}{\omega}^j \wedge \omega_j^i = \sum_{\beta \neq \alpha} \underset{\alpha\beta}{a}{}^i_{jk} \underset{\alpha}{\omega}^j \wedge \underset{\beta}{\omega}^k, \tag{1.2}$$

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = \sum_{\alpha,\beta=1}^{n} \underset{\alpha\beta}{b}{}^i_{jkl} \underset{\alpha}{\omega}^k \wedge \underset{\beta}{\omega}^l, \tag{1.3}$$

where the quantities $\underset{\alpha\beta}{a}{}^i_{jk}$ and $\underset{\alpha\beta}{b}{}^i_{jkl}$ are connected by certain relations (see [9], [10], [7] or [16]). We indicate some of these relations:

$$\underset{\alpha\alpha}{a}{}^i_{jk} = 0, \tag{1.4}$$

$$\nabla \underset{\alpha\beta}{a}{}^i_{jk} = \sum_{\gamma=1}^{n} \Big(\underset{\alpha\beta\gamma}{a}{}^i_{jkl} + \underset{\alpha\beta}{a}{}^i_{mk} \underset{\gamma\alpha}{a}{}^m_{lj} + \underset{\alpha\beta}{a}{}^i_{jm} \underset{\beta\gamma}{a}{}^m_{kl} \Big) \underset{\gamma}{\omega}^l, \quad \alpha \neq \beta, \tag{1.5}$$

$$\underset{\alpha\beta}{a}{}^i_{jk} = \underset{\beta\alpha}{a}{}^i_{kj}, \tag{1.6}$$

$$\sum_{\alpha,\beta=1}^{n} \underset{\alpha\beta}{a}{}^i_{jk} = 0, \tag{1.7}$$

$$(1.8)\qquad \underset{\alpha\beta}{b}{}^{i}_{jkl} = \frac{1}{2}\Big(\underset{\gamma\alpha\beta}{a}{}^{i}_{jkl} - \underset{\beta\gamma\alpha}{a}{}^{i}_{ljk}\Big), \quad \gamma \neq \alpha, \beta,$$

$$(1.9)\qquad \underset{\alpha\beta}{b}{}^{i}_{[jkl]} = 0,$$

$$(1.10)\qquad \underset{\alpha\alpha}{b}{}^{i}_{ljk} + 2\,\underset{\alpha\beta}{b}{}^{i}_{[jk]l} = 0,$$

where $\nabla \underset{\alpha\beta}{a}{}^{i}_{jk} = d\,\underset{\alpha\beta}{a}{}^{i}_{jk} - \underset{\alpha\beta}{a}{}^{i}_{lk}\omega^l_j - \underset{\alpha\beta}{a}{}^{i}_{jl}\omega^l_k + \underset{\alpha\beta}{a}{}^{l}_{jk}\omega^i_l$. The quantities $\underset{\alpha\beta}{a}{}^{i}_{jk}$ and $\underset{\alpha\beta}{b}{}^{i}_{jkl}$ form the tensor fields in the tangent bundle $T(X^{nr})$ which are called respectively the *torsion* and *curvature tensors* of the web $W(n+1,n,r)$.

Equations (1.8) and (1.5) show that if $n > 2$, the curvature tensor of $W(n+1,n,r)$ is expressed in terms of the Pfaffian derivatives of its torsion tensor and the torsion tensor itself. This is not the case for $n = 2$ because in formulas (1.8) $\gamma \neq \alpha, \beta$.

Equations (1.2) and (1.3) define an affine connection γ_{n+1} on the manifold X^{nr} (see [9], [10] or [16]). This connection is induced on X^{nr} by the web $W(n+1,n,r)$. The subindex $n+1$ in the notation γ_{n+1} indicates that the foliation λ_{n+1} has been distinguished when we take the forms $\{\underset{\alpha}{\omega}{}^i\}$ as a co-frame and the equations (1.2) and (1.3) as the structure equations.

The geodesic lines of the connection γ_{n+1} are determined by the equations

$$(1.11)\qquad d\underset{\alpha}{\omega}{}^i + \underset{\alpha}{\omega}{}^j\omega^i_j = \varphi\underset{\alpha}{\omega}{}^i, \quad \alpha = 1, \dots, n;\ i = 1, \dots, r,$$

where φ is a 1-form. Note that in (1.11) d is the symbol of ordinary (not exterior) differentiation with respect to the parameter of a geodesic line.

It follows from the equations (1.11) that the leaves of all the foliations λ_ξ of the web $W(n+1,n,r)$ are totally geodesic submanifolds in the connection γ_{n+1}.

2. An r-surface $U_\alpha = \bigcap_{\hat\alpha\neq\alpha} F_{\hat\alpha}$, $\alpha = 1, \dots, n$, is the intersection of $n-1$ leaves $F_{\hat\alpha}$ passing through a point $p \in X^{nr}$. In a neighborhood of p the leaves of λ_{n+1} intersect each U_α at the point and give rise to a point correspondence among the surfaces U_α in which corresponding lines of U_α are tangent to vectors with equal coordinates. We denote by $\varphi_{\alpha\beta n+1}$ the projection map of the surface U_α onto the surface U_β defined in the neighborhood of p by means of the foliation λ_{n+1}. The local diffeomorphism $\varphi_{\alpha\beta n+1}$ maps geodesic lines onto geodesic lines. Let vectors $\{e^\alpha_i\}$ form a vectorial frame dual to the co-frame $\{\underset{\alpha}{\omega}{}^i\}$. If ξ is a tangent vector to X^{nr} at p, then

$$(1.12)\qquad \xi = \sum_{\alpha=1}^{n} \underset{\alpha}{\omega}{}^i e^\alpha_i.$$

Since the foliation λ_α, α fixed, is defined by $\underset{\alpha}{\omega}^i = 0$, equation (1.12) shows that the vectors $e_i^{\hat\alpha}$, $\hat\alpha \neq \alpha$, are tangent at p to the leaf F_α of λ_α passing through the point p and form a basis in $T_p(F_\alpha)$. The foliation λ_{n+1} is defined by $\underset{n+1}{\omega}{}^i = -\sum_\alpha \underset{\alpha}{\omega}^i = 0$. This and (1.12) imply that the vectors $e_i^{\hat\alpha} - e_i^\alpha$, $\hat\alpha \neq \alpha$, α fixed, form a basis in $T_p(F_{n+1})$. In addition, it is easy to see that

$$d\varphi_{\alpha\beta n+1}|_p(e_i^\alpha) = e_i^\beta. \tag{1.13}$$

In [9] (see also [10] or [16]) we introduced the notion of a transversally geodesic n-surface. We will generalize this notion, introducing transversally geodesic (ns)-surfaces. Let ns vectors

$$\xi_a^\alpha = \xi_a^i e_i^\alpha, \quad a, b, c = 1, \dots, s; \quad \alpha = 1, \dots, n, \tag{1.14}$$

be given in the tangent space $T_p(X^{nr})$. It follows from (1.13) and (1.14) that

$$d\varphi_{\alpha\beta n+1}|_p(\xi_a^\alpha) = \xi_a^\beta. \tag{1.15}$$

The equation (1.15) means that the subspace which is defined by the (ns)-vector

$$t = \xi_1^{\alpha_1} \wedge \dots \wedge \xi_s^{\alpha_1} \wedge \dots \wedge \xi_1^{\alpha_n} \wedge \dots \wedge \xi_s^{\alpha_n}, \quad a_1, \dots, \alpha_n = 1, \dots, n, \tag{1.16}$$

is invariant with respect to any of the operators $d\varphi_{\alpha\beta n+1}|_p$, $\alpha, \beta = 1, \dots, n$. The subspace in $T_p(X^{nr})$ defined by the (ns)-vector (1.16) is called a *transversal (ns)-subspace* of the web $W(n+1, n, r)$.

DEFINITION 1.1. A submanifold $\widetilde{M} = X^{ns}$ of dimension ns of the manifold $M = X^{nr}$ is said to be a *transversally geodesic submanifold* if its tangent spaces are transversal (ns)-subspaces of the web $W = W(n+1, n, r)$.

DEFINITION 1.2. Suppose that a web $\widetilde{W} = W(n+1, n, s)$ is given in a submanifold $\widetilde{M}$ of M. This $(n+1)$-web $\widetilde{W}$ is called a *subweb* of the web W if its leaves are intersections of $\widetilde{M}$ with the leaves of the web W.

In [9] (see also [10] or [16]) $(k+1)$-subwebs, $k < n$, of the web W were introduced in the following manner:

DEFINITION 1.3. Consider the intersection of $n-k$ leaves F_u, $u = k+1, \dots, n$, $1 < k < n$, of the web W. This intersection is defined by the system

$$\underset{\xi_{k+1}}{\omega}{}^i = 0, \dots, \underset{\xi_n}{\omega}{}^i = 0 \tag{1.17}$$

and has dimension kr. The leaves F_t, $t = 1, \dots, k, n+1$, of the other $k+1$ foliations cut on this intersection a $(k+1)$-web $W(k+1, k, r)$ of codimension r which is called a *$(k+1)$-subweb* of the web W.

We will denote such a subweb by $[\xi_{n+1},\xi_1,\dots,\xi_k]$. A web W has $\binom{n+1}{k+1}$ $(k+1)$-subwebs.

We can say that the $(n+1)$-subwebs of Definition 1.2 are *transversal* while $(k+1)$-subwebs of Definition 1.3 are *reduct* subwebs.

There are two kinds of reduct $(k+1)$-subwebs: a subweb $[n+1, \alpha_{n-k+1}, \dots, \alpha_n]$, which is defined by the system

$$\omega^i_{\alpha_1} = 0, \dots, \omega^i_{\alpha_{n-k}} = 0, \tag{1.18}$$

and a subweb $[\alpha_{n-k}, \dots, \alpha_n]$, which is defined by the system

$$\omega^i_{n+1} = 0, \quad \omega^i_{\alpha_1} = 0, \dots, \omega^i_{\alpha_{n-k-1}} = 0. \tag{1.19}$$

In the case (1.18), we can see from the structure equation (1.2) that the torsion tensor of the $(k+1)$-subweb $[n+1, \alpha_{n-k+1}, \dots, \alpha_n]$ is a subtensor of the torsion tensor of the web W.

In the case (1.19), we can write the structure equations of $[\alpha_{n-k}, \dots, \alpha_n]$ in the form

$$d\underset{\hat\alpha}{\omega^i} - \underset{\hat\alpha}{\omega^j} \wedge \underset{\hat\beta}{\omega^i_j} = \sum_{\gamma\neq\hat\beta,\hat\alpha} \Big(\underset{\hat\alpha\gamma}{a}{}^i_{jk} - \underset{\hat\beta\gamma}{a}{}^i_{jk} - \underset{\hat\alpha\hat\beta}{a}{}^i_{jk}\Big) \underset{\hat\alpha}{\omega^j} \wedge \underset{\gamma}{\omega^k}, \tag{1.20}$$

where

$$\underset{\hat\beta}{\omega^i_j} = \omega^i_j + \sum_{\hat\gamma\neq\hat\beta} \underset{\hat\beta\hat\gamma}{a}{}^i_{jk} \underset{\hat\gamma}{\omega^k},$$

and $\hat\alpha, \hat\beta, \hat\gamma$ take those values from $1, \dots, n$, which are distinct from $\alpha_1, \dots, \alpha_{n-k-1}$; $\hat\beta$ is fixed and $\hat\alpha \neq \hat\beta$.

Equations (1.20) show that the torsion tensor of the $(k+1)$-subweb defined by (1.19) is $\underset{\hat\alpha\gamma}{a}{}^i_{jk} - \underset{\hat\beta\gamma}{a}{}^i_{jk} - \underset{\hat\alpha\hat\beta}{a}{}^i_{jk}$.

If $k = 2$, we have $\binom{n}{2}$ 3-subwebs $\underset{\alpha\beta}{W} = [n+1, \alpha, \beta] \subset X^{2r} \subset X^{nr}$ of W which are defined by the foliations $\lambda_{n+1}, \lambda_\alpha, \lambda_\beta$, $\alpha, \beta = 1, \dots, n$.

The torsion and curvature tensors of the canonical affine connection $\overline{\gamma}_{\alpha\beta}$ (see [1]) of the 3-subweb $\underset{\alpha\beta}{W}$ are (see equation (28) in [10] or equations (1.3.23) and (1.3.25) in [16]):

$$\underset{\alpha\beta}{\overline{a}}{}^i_{jk} = \underset{\alpha\beta}{a}{}^i_{[jk]}, \tag{1.21}$$

$$\underset{\alpha\beta}{\overline{b}}{}^i_{jkl} = 2\,\underset{\alpha\beta}{b}{}^i_{jkl} - \underset{\beta\alpha\beta}{a}{}^i_{jkl} + \underset{\alpha\beta\alpha}{a}{}^i_{jlk} - \underset{\beta\alpha}{a}{}^m_{jk}\underset{\alpha\beta}{a}{}^i_{ml} + \underset{\alpha\beta}{a}{}^m_{jl}\underset{\beta\alpha}{a}{}^i_{mk}, \tag{1.22}$$

and the torsion and curvature tensors of the canonical affine connection of the 3-subweb $[\alpha, \beta, \gamma]$ are (see equation (2.48) and (2.49) in [8] or equations (1.3.69) and (1.3.71) in [16]):

$$\underset{\alpha\beta\gamma}{\overline{a}}{}^{i}_{jk} = \underset{\alpha\beta}{a}{}^{i}_{[jk]} + \underset{\beta\gamma}{a}{}^{i}_{[jk]} + \underset{\gamma\alpha}{a}{}^{i}_{[jk]}, \tag{1.23}$$

(1.24)

$$\begin{aligned} \underset{\alpha\beta\gamma}{\overline{b}}{}^{i}_{jkl} = &- \underset{\beta\alpha\beta}{a}{}^{i}_{jkl} + \underset{\alpha\beta\alpha}{a}{}^{i}_{jlk} - \underset{\beta\gamma\gamma}{a}{}^{i}_{jkl} + \underset{\beta\gamma\beta}{a}{}^{i}_{jkl} - \underset{\alpha\gamma\alpha}{a}{}^{i}_{jlk} + \underset{\alpha\gamma\gamma}{a}{}^{i}_{jlk} - \underset{\alpha\beta\gamma}{a}{}^{i}_{jlk} + \underset{\beta\alpha\gamma}{a}{}^{i}_{jkl} + \\ &+ \Big(\underset{\beta\alpha}{a}{}^{m}_{lj} + \underset{\gamma\beta}{a}{}^{m}_{lj} - \underset{\gamma\alpha}{a}{}^{m}_{lj}\Big)\Big(\underset{\beta\alpha}{a}{}^{i}_{mk} - \underset{\beta\gamma}{a}{}^{i}_{mk}\Big) + \Big(\underset{\beta\alpha}{a}{}^{m}_{jk} - \underset{\beta\gamma}{a}{}^{m}_{jk} + \underset{\alpha\gamma}{a}{}^{m}_{jk}\Big)\Big(\underset{\alpha\gamma}{a}{}^{i}_{ml} - \underset{\alpha\beta}{a}{}^{i}_{ml}\Big) + \\ &+ \Big(- \underset{\alpha\beta}{a}{}^{m}_{kl} + \underset{\gamma\beta}{a}{}^{m}_{kl} + \underset{\alpha\gamma}{a}{}^{m}_{kl}\Big)\Big(\underset{\alpha\beta}{a}{}^{i}_{mj} - \underset{\gamma\alpha}{a}{}^{i}_{mj}\Big). \end{aligned}$$

If $k = 3$, we have $\binom{n}{3}$ 4-subwebs $\underset{\alpha\beta\gamma}{W} = [n+1, \alpha, \beta, \gamma] \subset X^{3r} \subset X^{nr}$ of W which are defined by the foliations $\lambda_{n+1}, \lambda_\alpha, \lambda_\beta$ and λ_γ; $\alpha, \beta, \gamma = 1, \dots, n$. As we noted above, the torsion and curvature tensors of any of these 4-subwebs are subtensors of the corresponding tensors of the web W.

The following proposition immediately follows from Definitions 1.2 and 1.3:

PROPOSITION 1.4. *Every (ns)-dimensional $(n+1)$-subweb $\widetilde{W}$ of the (nr)-dimensional web W induces a (ks)-dimensional $(k+1)$-subweb on a (kr)-dimensional $(k+1)$-subweb $[\xi_{n+1}, \xi_1, \dots, \xi_k]$.* □

In other words: a transversal subweb of W induces a transversal subweb on a reduct subweb.

THEOREM 1.5. *If a web $\widetilde{W}$ is a subweb of the web W, then its web space $\widetilde{M}$ is a transversally geodesic submanifold of the web space M of the web W.*

PROOF. Let us denote by $\widetilde{U}_\alpha$ the s-surfaces associated with the subweb $\widetilde{W}$ which are similar to the r-surfaces U_α associated with the web W, i.e. $\widetilde{U}_\alpha = \bigcap_{\hat\alpha \neq \alpha} \widetilde{F}_{\hat\alpha}$, $\alpha = 1, \dots, n$, where $\widetilde{F}_\alpha$ are leaves of the subweb $\widetilde{W}$. Let $\widetilde{\varphi}_{\alpha\beta n+1}$ be the projection map of the surface $\widetilde{U}_\alpha$ onto the surface $\widetilde{U}_\beta$ defined in a neighborhood of p by means of the foliation $\tilde\lambda_{n+1}$ of the subweb $\widetilde{W}$. It is obvious that $\widetilde{F}_\alpha \subset F_\alpha$ and consequently $\widetilde{U}_\alpha \subset U_\alpha$. This implies that the maps $\widetilde{\varphi}_{\alpha\beta n+1}$ are restrictions of the maps $\varphi_{\alpha\beta n+1}$ on $\widetilde{U}_\alpha$:

$$\widetilde{\varphi}_{\alpha\beta n+1} = \varphi_{\alpha\beta n+1}|_{\widetilde{U}_\alpha}.$$

The tangent maps $d\widetilde{\varphi}_{\alpha\beta n+1}|_p \colon T(\widetilde{U}_\alpha) \to T(\widetilde{U}_\beta)$ are restrictions of the tangent maps $d\varphi_{\alpha\beta n+1}|_p \colon T(U_\alpha) \to T(U_\beta)$. This implies that the (ns)-vectors

$T(\widetilde{U}_{\alpha_1})\wedge\ldots\wedge T(\widetilde{U}_{\alpha_n})$ are invariant under the maps $d\varphi_{\alpha\beta n+1}|_p$. But the (ns)-vector $T(\widetilde{U}_1)\wedge\ldots\wedge T(\widetilde{U}_n)$ uniquely determines the tangent space $T_p(\widetilde{M})$ to the submanifold $\widetilde{M}$ at the point p. This means that this (ns)-vector is invariant with respect to any of the operators $d\varphi_{\alpha,\beta,n+1}|_p$. □

3. Let $f\colon \widetilde{M}\to M$ be an embedding of the subweb space $\widetilde{M}$ of $\widetilde{W}$ into the web space M of W. If the forms $\{\underset{\alpha}{\theta}^a\}$, $\alpha=1,\ldots,n$; $a,b,c=1,\ldots,s$, form an adapted co-frame for the subweb $\widetilde{W}$, then the differential equations of the embedding f can be written in the form

$$\underset{\alpha}{\omega}^i=\xi^i_a\underset{\alpha}{\theta}^a, \tag{1.25}$$

where ξ^i_a are coordinates of the tangent vectors of $T_p(M)$ defined by equation (1.14). The leaves $\widetilde{F}_\alpha\subset\tilde{\lambda}_\alpha$ and $\widetilde{F}_{n+1}\subset\tilde{\lambda}_{n+1}$ of the subweb $\widetilde{W}$ are defined respectively by the equations

$$\underset{\alpha}{\theta}^a=0,\quad \alpha=1,\ldots,n;\quad \sum_{\alpha=1}^{n}\underset{\alpha}{\theta}^a=0. \tag{1.26}$$

The forms $\underset{\alpha}{\theta}^a$ satisfy the structure equations of the web $\widetilde{W}=W(n+1,n,s)$:

$$d\underset{\alpha}{\theta}^a-\underset{\alpha}{\theta}^b\wedge\theta^a_b=\sum_{\beta\neq\alpha}\underset{\alpha\beta}{\tilde{a}}{}^a_{bc}\underset{\alpha}{\theta}^b\wedge\underset{\beta}{\theta}^c, \tag{1.27}$$

$$d\theta^a_b-\theta^c_b\wedge\theta^a_c=\sum_{\alpha,\beta=1}^{n}\underset{\alpha\beta}{\tilde{b}}{}^a_{bcd}\underset{\alpha}{\theta}^c\wedge\underset{\beta}{\theta}^d, \tag{1.28}$$

which are similar to the structure equations (1.2) and (1.3) of the web $W=$ $=W(n+1,n,r)$, and the quantities $\tilde{a}^a_{bc}$ and $\tilde{b}^a_{bcd}$ form the torsion and curvature tensors of $\widetilde{W}$ related by conditions similar to the conditions (1.4)–(1.10).

Other relations are obtained after taking exterior derivatives of the equations (1.25), which by means of (1.2) and (1.28) lead to the following exterior quadratic equations:

$$\left(\nabla\xi^i_a-\xi^i_b\theta^b_a\right)\wedge\underset{\alpha}{\theta}^a+\sum_{\beta\neq\alpha}\left(\xi^i_a\underset{\alpha\beta}{\tilde{a}}{}^a_{bc}-\underset{\alpha\beta}{a}{}^i_{jk}\xi^j_a\xi^k_b\right)\underset{\alpha}{\theta}^a\wedge\underset{\beta}{\theta}^b=0, \tag{1.29}$$

where $\nabla\xi^i_a=d\xi^i_a+\xi^j_a\omega^i_j$. Since the forms $\underset{\alpha}{\theta}^a$ are linearly independent, the equations (1.29) imply that

$$\nabla\xi^i_a=\xi^i_b\theta^b_a \tag{1.30}$$

and

$$a_{\alpha\beta}{}^{i}_{jk}\xi^j_b\xi^k_c = \xi^i_f \tilde{a}_{\alpha\beta}{}^{f}_{bc}. \tag{1.31}$$

Taking exterior derivatives of (1.31) and using (1.3) and (1.28), we will have the following equations connecting the curvature tensors of the webs W and $\widetilde{W}$ which are similar to the equations (1.31) connecting the torsion tensors of these two webs:

$$b_{\alpha\beta}{}^{i}_{jkl}\xi^j_a\xi^k_b\xi^l_c = \xi^i_f \tilde{b}_{\alpha\beta}{}^{f}_{abc}. \tag{1.32}$$

In addition, differentiating (1.31) and using (1.4) and (1.30), one can easily obtain similar relations between the Pfaffian derivatives of the torsion tensors of the webs W and $\widetilde{W}$:

$$a_{\alpha\beta\gamma}{}^{i}_{jkl}\xi^j_a\xi^k_b\xi^l_c = \xi^i_f \tilde{a}_{\alpha\beta\gamma}{}^{f}_{abc}. \tag{1.33}$$

It is easy to see that further differentiations of (1.32) and (1.33) give similar relations between the higher order Pfaffian derivatives of the torsion and curvature tensors of the webs W and $\widetilde{W}$.

Note that if $n > 2$, then the equations (1.32) follow from the equations (1.8) and (1.33). In this case only the equations (1.31) are essential.

In the case $k > 2$, it follows from the structure of the torsion tensors of reduct $(k+1)$-subwebs $[\xi_{n+1}, \xi_1, \dots, \xi_k]$ that the relations similar to (1.31) are a part of the equations (1.31) or are implied by (1.31). In the case $k = 2$, i.e. for 3-subwebs $\underset{\alpha\beta}{W}$ and $[\alpha, \beta, \gamma]$, we have respectively:

$$\bar{a}_{\alpha\beta}{}^{i}_{jk}\xi^j_a\xi^k_b = \xi^i_f \bar{\tilde{a}}_{\alpha\beta}{}^{f}_{ab}, \tag{1.34}$$

$$\bar{b}_{\alpha\beta}{}^{i}_{jkl}\xi^j_a\xi^k_b\xi^l_c = \xi^i_f \bar{\tilde{b}}_{\alpha\beta}{}^{f}_{abc} \tag{1.35}$$

and

$$\bar{a}_{\alpha\beta\gamma}{}^{i}_{jk}\xi^j_a\xi^k_b = \xi^i_f \bar{\tilde{a}}_{\alpha\beta\gamma}{}^{f}_{ab}, \tag{1.36}$$

$$\bar{b}_{\alpha\beta\gamma}{}^{i}_{jkl}\xi^j_a\xi^k_b\xi^l_c = \xi^i_f \bar{\tilde{b}}_{\alpha\beta}{}^{f}_{abc}, \tag{1.37}$$

where $\bar{a}_{\alpha\beta}{}^{i}_{jk}$, $\bar{b}_{\alpha\beta}{}^{i}_{jkl}$, $\bar{a}_{\alpha\beta\gamma}{}^{i}_{jk}$ and $\bar{b}_{\alpha\beta}{}^{i}_{jkl}$ are defined by (1.21)–(1.24), and this matches Theorem 2 of [6] proved for transversal 3-subwebs of a multidimensional 3-web.

Note also that if $s = 1$, the equations (1.31), (1.32) and (1.33) give

$$a_{\alpha\beta}{}^{i}_{jk}\xi^j\xi^k = \xi^i \underset{\alpha\beta}{\tilde{a}}, \tag{1.38}$$

$$a_{\alpha\beta\gamma}{}^{i}_{jkl}\xi^j\xi^k\xi^l = \xi^i \underset{\alpha\beta\gamma}{\tilde{a}} \tag{1.39}$$

and

$$b_{\alpha\beta}{}^{i}_{jkl}\xi^j\xi^k\xi^l = \xi^i \underset{\alpha\beta}{\tilde{b}}, \tag{1.40}$$

which were considered earlier (cf. equation (12) in [9] or equations (74) and (77) in [10] or equations (1.9.17), (1.9.21) and (1.9.22) in [16]).

We proved the following result:

THEOREM 1.6. *The functions* ξ^i_a *which define the embedding* $f: \widetilde{M} \to M$ *satisfy the equations* (1.31), (1.32), (1.33) *and similar equations connecting the torsion and curvature tensors of the webs* W *and* $\widetilde{W}$ *and their Pfaffian derivatives.* □

In the case $s = 1$, this theorem was proved in [9] (see also Theorem 15 in [10] or Theorem 1.9.5 in [16]).

THEOREM 1.7. *A transversally geodesic submanifold* $\widetilde{M}$ *of the manifold* M *is totally geodesic in the connection* γ_{n+1}.

PROOF. In addition to the affine connection γ_{n+1} defined on the manifold M by the forms ω^I_J, $I, J = 1, \dots, nr$, where the matrix (ω^I_J) has a block diagonal form with the matrix (ω^i_j) along its "diagonal", we have an induced affine connection $\tilde{\gamma}_{n+1}$ defined on the manifold $\widetilde{M}$ by the forms θ^A_B, $A, B = 1, \dots, ns$, where the matrix (θ^A_B) has a block diagonal form with the matrix (θ^a_b) along its "diagonal". The connection γ_{n+1} is associated with the web W, and the connection $\tilde{\gamma}_{n+1}$ is associated with its subweb $\widetilde{W}$.

Let $\tilde{\eta}^a_\alpha$ be a tangent vector in $T_p(\widetilde{M})$. The vector field η with coordinates $\tilde{\eta}^a_\alpha$ in $T_p(\widetilde{M})$ is parallel in the connection $\tilde{\gamma}_{n+1}$ if and only if its coordinates satisfy the equations

$$d\tilde{\eta}^a_\alpha + \tilde{\eta}^b_\alpha\theta^a_b = 0, \quad a = 1, \dots, n; \quad a, b = 1, \dots, s. \tag{1.41}$$

With respect to the frame $\{e^\alpha_i\}$ in $T_p(M)$, the vector field η has the following coordinates:

$$\eta^i_\alpha = \xi^i_a\tilde{\eta}^a_\alpha. \tag{1.42}$$

Differentiating (1.42) and using equations (1.30) and (1.41), we get

$$d\eta^i_\alpha + \eta^j_\alpha\theta^i_j = 0, \quad \alpha = 1, \dots, n; \quad i, j = 1, \dots, r. \tag{1.43}$$

Equation (1.43) proves that the vector field η is parallel on M in the connection γ_{n+1}. This means that a transversally geodesic submanifold $\widetilde{M}$ of the manifold M is totally geodesic in the connection γ_{n+1}. □

Note that Theorem 1.5 generalizes a theorem on transversally geodesic n-surfaces proved in [9] (see also Theorem 14 in [10] or Theorem 1.9.2 in [16]).

4. There are many special classes of $(n+1)$-webs W, $n \geqq 2$, which are characterized both geometrically and analytically (see [12], [15], [14], [5] or [16] and [1], [2], [18]).

We will now prove that if the web W is of a certain special type, then any of its transversal subwebs $\widetilde{W}$ is of the same kind.

First let us describe special classes of webs W by analytic characterization since these analytic conditions will be used in our proof.

We will consider here the following classes of $(n+1)$-webs W, $n > 2$:

1. Reducible $(n+1)$-webs W of different kinds, for example the reducible webs which are defined by the condition: $\underset{\alpha\beta}{a}{}^{i}_{jk} = \underset{\alpha\gamma}{a}{}^{i}_{jk}$.

2. Transversally geodesic $(n+1)$-webs W: $\underset{\alpha\beta}{a}{}^{i}_{(jk)} = \delta^i_{(j}\underset{\alpha\beta}{a}{}_{k)}$.

3. $(2n+2)$-hedral $(n+1)$-webs W: $\underset{\alpha\beta}{a}{}^{i}_{(jk)} = 0$.

4. Group $(n+1)$-webs W of different kinds, for example of the kind for which $\underset{\alpha\beta}{a}{}^{i}_{jk} = a^i_{jk}$, $a^i_{(jk)} = 0$, $\alpha < \beta$.

5. Parallelizable $(n+1)$-webs W: $\underset{\alpha\beta}{a}{}^{i}_{jk} = 0$.

6. Isoclinic $(n+1)$-webs W: $\underset{\alpha\beta}{a}{}^{i}_{[jk]} = \delta^i_{[k}\underset{\alpha\beta}{b}{}_{j]}$.

7. $(n+1)$-webs W with paratactical 3-subwebs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{a}{}^{i}_{[jk]} = 0$.

8. Hexagonal $(n+1)$-webs W: $\underset{\alpha\beta}{h}{}^{i}_{jkl} = 0$, where

$$\underset{\alpha\beta}{h}{}^{i}_{jkl} = 2\,\underset{\alpha\beta}{b}{}^{i}_{(jkl)} + \underset{\alpha\beta\alpha}{a}{}^{i}_{(jkl)} - \underset{\beta\alpha\beta}{a}{}^{i}_{(jkl)} - \underset{\alpha\beta}{a}{}^{m}_{(jk}\underset{\alpha\beta}{a}{}^{i}_{|m|l)} + \underset{\alpha\beta}{a}{}^{m}_{(jk}\underset{\beta\alpha}{a}{}^{i}_{|m|l)}.$$

9. Curvature-free $(n+1)$-webs W: $\underset{\alpha\beta}{b}{}^{i}_{jkl} = 0$.

10. Grassmannizable $(n+1)$-webs W: $\underset{\alpha\beta}{a}{}^{i}_{jk} = \delta^i_k\underset{\alpha\beta}{\lambda}{}_j + \delta^i_j\underset{\beta\alpha}{\lambda}{}_k$.

11. Algebraizable $(n+1)$-webs W: $\underset{\alpha\beta}{a}{}^{i}_{[jk]} = \delta^i_{[k}\underset{\alpha\beta}{b}{}_{j]}$, $\underset{\alpha\beta}{h}{}^{i}_{jkl} = 0$.

In the case $n = 2$, i.e. for 3-webs $W = W(3, 2, r)$, the following special classes are known (we will indicate the analytic characterization for 3-webs $\underset{\alpha\beta}{W}$):

1. Paratactical 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\overline{a}}{}^{i}_{jk} = 0$.

2. Bol 3-webs $\underset{\alpha\beta}{W}$ of three different kinds: $\underset{\alpha\beta}{\bar{b}}{}^{i}_{(jk)l} = 0$, $\underset{\alpha\beta}{\bar{b}}{}^{i}_{(j|k|l)} = 0$, or $\underset{\alpha\beta}{\bar{b}}{}^{i}_{j(kl)} = 0$.

3. Moufang 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{b}}{}^{i}_{jkl} = \underset{\alpha\beta}{\bar{b}}{}^{i}_{[jkl]}$.

4. Transversally geodesic 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{b}}{}^{i}_{(jkl)} = \delta^{i}_{(j}\underset{\alpha\beta}{\bar{b}}{}_{kl)}$.

5. Hexagonal 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{b}}{}^{i}_{(jkl)} = 0$.

6. Group 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{b}}{}^{i}_{jkl} = 0$.

7. Isoclinic 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{a}}{}^{i}_{[jk]} = \delta^{i}_{[j}\underset{\alpha\beta}{\bar{a}}{}_{k]}$.

8. 3-webs $\underset{\alpha\beta}{W}$ of three different kinds with a partially symmetric curvature tensor:

$$\underset{\alpha\beta}{\bar{b}}{}^{i}_{[jk]l} = 0, \quad \underset{\alpha\beta}{\bar{b}}{}^{i}_{[j|k|l]} = 0, \quad \text{or} \quad \underset{\alpha\beta}{\bar{b}}{}^{i}_{j[kl]} = 0.$$

9. 3-webs $\underset{\alpha\beta}{W}$ with a completely symmetric curvature tensor: $\underset{\alpha\beta}{\bar{b}}{}^{i}_{jkl} = \underset{\alpha\beta}{\bar{b}}{}^{i}_{(jkl)}$.

10. Grassmannizable 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{a}}{}^{i}_{jk} = \delta^{i}_{[j}\underset{\alpha\beta}{\tilde{\bar{a}}}{}_{k]}$, $\underset{\alpha\beta}{\bar{b}}{}^{i}_{jkl} = \underset{\alpha\beta}{f}{}_{jk}\delta^{i}_{l} + \underset{\alpha\beta}{g}{}_{lj}\delta^{i}_{k} + \underset{\alpha\beta}{h}{}_{kl}\delta^{i}_{j}$, where the tensors $\underset{\alpha\beta}{f}{}_{jk}$, $\underset{\alpha\beta}{g}{}_{jk}$ and $\underset{\alpha\beta}{h}{}_{jk}$ are symmetric.

11. Algebraizable 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{a}}{}^{i}_{jk} = \delta^{i}_{[j}\underset{\alpha\beta}{\bar{a}}{}_{k]}$, $\underset{\alpha\beta}{\bar{b}}{}^{i}_{(jkl)} = 0$.

12. Parallelizable 3-webs $\underset{\alpha\beta}{W}$: $\underset{\alpha\beta}{\bar{a}}{}^{i}_{jk} = 0$, $\underset{\alpha\beta}{\bar{b}}{}^{i}_{jkl} = 0$.

DEFINITION 1.8. We will say that a class of webs W is *tensorially defined* if its analytic characterization is of a tensorial nature.

It is easy to see that all classes of webs W, which were listed above, are tensorially defined classes.

THEOREM 1.9. *If an $(n+1)$-web W, $n \geqq 2$, is in a tensorially defined class, then each of its transversal $(n+1)$-subwebs $\widetilde{W}$ is in that class.*

PROOF. The proof is based on the fact (which can be also deduced from a so-called indirect test for tensor character (see [4]) that if some tensor $\underset{\alpha\beta}{Z}{}^{i}_{j_1\ldots j_p}$, $p = 2, 3$, satisfies the equation similar to the equations (1.31)–(1.33):

$$\underset{\alpha\beta}{Z}{}^{i}_{j_1\ldots j_p}\xi^{j_1}_{a_1}\ldots\xi^{j_p}_{a_p} = \xi^{i}_{b}\underset{\alpha\beta}{\widetilde{Z}}{}^{b}_{a_1\ldots a_p}, \tag{1.44}$$

and the tensor $\underset{\alpha\beta}{Z}{}^{i}_{j_1\ldots j_p} = 0$, then

$$\xi^{i}_{b}\underset{\alpha\beta}{\widetilde{Z}}{}^{b}_{a_1\ldots a_p} = 0. \tag{1.45}$$

Since the quantities ξ^i_b are the coordinates of an arbitrary tangent vector (1.14) satisfying the differential equations (1.30), it follows from the equation (1.45) that

$$\underset{\alpha\beta}{\widetilde{Z}}{}^b_{a_1\dots a_p} = 0. \tag{1.46}$$

For example, in the case of a transversally geodesic $(n+1)$-web W, we have

$$\Big(\underset{\alpha\beta}{a}{}^i_{(jk)} - \delta^i_{(j}\underset{\alpha\beta}{a}{}_{k)}\Big)\xi^j_a\xi^k_b = \xi^i_c\Big(\underset{\alpha\beta}{\tilde a}{}^c_{(ab)} - \delta^c_{(a}\underset{\alpha\beta}{\tilde a}{}_{b)}\Big), \tag{1.47}$$

where $\underset{\alpha\beta}{\tilde a}{}_b = \xi^k_b \underset{\alpha\beta}{a}{}_k$. Applying (1.45), we get $\xi^i_c\Big(\underset{\alpha\beta}{\tilde a}{}^c_{(ab)} - \delta^c_{(a}\underset{\alpha\beta}{\tilde a}{}_{b)}\Big) = 0$ and, as we indicated in our remark (see the equations (1.44)–(1.46)), it follows that $\underset{\alpha\beta}{\tilde a}{}^c_{(ab)} = \delta^c_{(a}\underset{\alpha\beta}{\tilde a}{}_{b)}$, which means that a transversal $(n+1)$-subweb $\widetilde{W}$ is transversally geodesic.

In some other cases, certain identities may be useful during the proof. For example, if $n > 2$, in the cases 4, 8 and 10, the following identities may be applied:

$$a^i_{jk}\xi^j_a\xi^k_b = \xi^i_c a^c_{ab},$$

$$\underset{\alpha\beta}{h}{}^f_{abc} = 2\underset{\alpha\beta}{b}{}^f_{(abc)} + \underset{\alpha\beta\alpha}{a}{}^f_{(abc)} - \underset{\beta\alpha\beta}{a}{}^f_{(abc)} - \underset{\alpha\beta}{a}{}^g_{(ab}\underset{\alpha\beta}{a}{}^f_{|g|c)} + \underset{\alpha\beta}{a}{}^g_{(ab}\underset{\beta\alpha}{a}{}^f_{|g|c)},$$

$$\underset{\alpha\beta}{\widetilde{\lambda}}{}_a = \underset{\alpha\beta}{\lambda}{}_j\xi^j_a. \qquad \square$$

Note also that a similar theorem is valid for other special classes of $(n+1)$-webs W which are intersections of the classes indicated above, for example, for reducible Grassmannizable $(n+1)$-webs W, reducible algebraic $(n+1)$-webs W, etc., since it is obvious that the intersection of two tensorially defined classes of webs W is again a tensorially defined class.

5. We will now describe the $(n+1)$-webs $W = W(n+1, n, r)$ given on an (nr)-dimensional manifold $M = X^{nr}$ which have the following property (property P_s): for every transversal tangent (ns)-dimensional subspace (s is fixed, $1 \leqq s \leqq r$) of the tangent space $T_p(M)$ passing through any point $p \in M$, there exists a transversally geodesic submanifold tangent to the given subspace.

Note that in [1] transversally geodesic 3-webs and in [9] (see also [10] or [16]) transversally geodesic $(n+1)$-webs W were defined as webs satisfying property P_1.

Note also that the Grassmann $(n+1)$-webs which were studied in [2] for $n = 2$, in [5] for $n = 3$ and in [13] for any $n \geqq 3$, satisfy property P_s for all $s = 1, \dots, r$. In fact, the Grassmann webs $W(n+1, n, r)$ are represented on the Grassmannian $G(n-1, r+n-1)$ of $(n-1)$-planes of a projective space

P^{r+n-1}, and every $(s+n-1)$-dimensional projective subspace of P^{r+n-1} determines a subweb $\widetilde{W} = W(n+1, n, s)$.

The following theorem shows that in the case $r > 2$, for an $(n+1)$-web W to be a Grassmannizable $(n+1)$-web (i.e. equivalent to a Grassmann $(n+1)$-web), property P_2 is sufficient.

THEOREM 1.10. *An $(n+1)$-web $W = W(n+1, n, r)$, $r > 2$, satisfying property P_2 is a Grassmannizable $(n+1)$-web.*

PROOF. First let us show that property P_2 implies property P_1. Let π^n be a transversal n-plane in the tangent space $T_p(M)$ and let π^{2n} and ϱ^{2n} be transversal $(2n)$-spaces such that $\pi^n = \pi^{2n} \cap \varrho^{2n}$. By property P_2, there exist transversally geodesic $(2n)$-submanifolds M_1 and M_2 tangent to π^{2n} and ϱ^{2n}, respectively. Consequently their intersection $M_1 \cap M_2$ is a transversally geodesic n-submanifold tangent to π^n. This gives property P_1. As we indicated earlier, the latter means that our $(n+1)$-web W is transversally geodesic.

If the web W has property P_2, then equations (1.31) are satisfied identically with respect to the coordinates ξ^i_a. Now let us alternate equation (1.31) with respect to the indices b and c. We obtain

$$\underset{\alpha\beta}{a}{}^i_{jk}\xi^j_{[b}\xi^k_{c]} = \xi^i_f \underset{\alpha\beta}{\tilde{a}}{}^f_{[bc]}. \tag{1.48}$$

One can easily see that this equation and equation (1.34) are the same.

The rest of the proof follows the lines of the proof of Theorem 3 in [6]. Since equation (1.48) is also satisfied identically with respect to the coordinates ξ^i_a, the tensor $\underset{\alpha\beta}{\tilde{a}}{}^a_{[bc]}$ is a linear form of these coordinates. Since this tensor is skew-symmetric in the indices b and c, we have

$$\underset{\alpha\beta}{\tilde{a}}{}^a_{[bc]} = \underset{\alpha\beta}{\lambda}{}^a_{bk}\xi^k_c - \underset{\alpha\beta}{\lambda}{}^a_{ck}\xi^k_b. \tag{1.49}$$

Substituting (1.49) into (1.31), we obtain

$$\underset{\alpha\beta}{a}{}^i_{jk}\xi^j_{[b}\xi^k_{c]} = \xi^i_a\left(\underset{\alpha\beta}{\lambda}{}^a_{bk}\xi^k_c - \underset{\alpha\beta}{\lambda}{}^a_{ck}\xi^k_b\right). \tag{1.50}$$

Since equation (1.50) is satisfied identically with respect to the coordinates ξ^i_a, it implies

$$\underset{\alpha\beta}{a}{}^i_{[jk]}\delta^{b'}_b\delta^{c'}_c = \underset{\alpha\beta}{\lambda}{}^{b'}_{bk}\delta^i_j\delta^{b'}_b - \underset{\alpha\beta}{\lambda}{}^{c'}_{cj}\delta^i_k\delta^{b'}_b. \tag{1.51}$$

Contracting equation (1.51) with respect to the pairs of indices b, b' and c, c', we obtain

$$\underset{\alpha\beta}{a}{}^i_{[jk]} = \underset{\alpha\beta}{b}{}_{[j}\delta^i_{k]}, \tag{1.52}$$

where $\underset{\alpha\beta}{b}_j = -\frac{2}{s}\underset{\alpha\beta}{\lambda}{}^b_{bj}$. As was proved in [11] (see also Theorem 1.11.4 in [16]), if $r > 2$, equation (1.52) holds if and only if the $(n+1)$-web W is isoclinic.

Since the $(n+1)$-web W is transversally geodesic and isoclinic, according to [11] (see also [12], [13] or Theorem 2.6.4 and Theorem 5.2.1 in [16]), this web is Grassmannizable. □

The following corollaries immediately follow from Theorem 1.10:

COROLLARY 1.11. *Property P_2 implies properties P_s, $s = 3, \dots, r$.* □

COROLLARY 1.12. *If $r > 2$, then a web W is a Grassmann $(n+1)$-web if and only if it satisfies properties P_s, $s = 1, \dots, r$.* □

In the proof of Theorem 1.10 we deduce the transversal geodesicity of the web W from property P_2. However, if $n > 2$, we could deduce this from conditions (1.31) using the same method which we used to prove the isoclinity of W. We will perform this in the following proposition:

PROPOSITION 1.13. *If $n > 2$, condition (1.31) implies that the web W is transversally geodesic.*

PROOF. Equations (1.31) are satisfied identically with respect to the coordinates ξ^i_a. Now let us symmetrize both members of equation (1.31) in indices b and c. We obtain

$$\underset{\alpha\beta}{a}{}^i_{jk}\xi^j_{(b}\xi^k_{c)} = \xi^i_f\, \underset{\alpha\beta}{\tilde{a}}{}^f_{(bc)}. \tag{1.53}$$

Since equation (1.53) is also satisifed identically with respect to the coordinates ξ^i_a, the tensor $\underset{\alpha\beta}{\tilde{a}}{}^a_{(bc)}$ is a linear form of these coordinates. Since this tensor is symmetric in the indices b and c, we have

$$\underset{\alpha\beta}{\tilde{a}}{}^a_{(bc)} = \underset{\alpha\beta}{\lambda}{}^a_{bk}\xi^k_c + \underset{\alpha\beta}{\lambda}{}^a_{ck}\xi^k_b. \tag{1.54}$$

Substituting (1.54) into (1.53), we obtain

$$\underset{\alpha\beta}{a}{}^i_{jk}\xi^j_{(b}\xi^k_{c)} = \xi^i_a(\underset{\alpha\beta}{\lambda}{}^a_{bk}\xi^k_c + \underset{\alpha\beta}{\lambda}{}^a_{ck}\xi^k_b). \tag{1.55}$$

Since equation (1.55) is satisfied identically with respect to the coordinates ξ^i_a, it implies

$$\underset{\alpha\beta}{a}{}^i_{(jk)}\delta^{b'}_b\delta^{c'}_c = \underset{\alpha\beta}{\lambda}{}^{b'}_{bk}\delta^i_j\delta^{c'}_c + \underset{\alpha\beta}{\lambda}{}^{c'}_{cj}\delta^i_k\delta^{b'}_b. \tag{1.56}$$

Contracting equation (1.56) with respect to the pairs of indices b, b' and c, c', we obtain

$$\underset{\alpha\beta}{a}{}^i_{(jk)} = \underset{\alpha\beta}{c}{}_{(j}\delta^i_{k)}, \tag{1.57}$$

where $\underset{\alpha\beta}{c}_j = \frac{2}{s}\underset{\alpha\beta}{\lambda}{}^b_{bj}$. As was proved in [9] (see also Theorem 16 in [10] or Theorem 1.9.7 in [16]), if $n > 2$, equation (1.57) holds if and only if the $(n+1)$-web W is transversally geodesic. □

2. Local algebras associated with an $(n+1)$-web and its $(n+1)$-subwebs

In this section we give an algebraic interpretation of relations (1.31)–(1.33).

1. It is known that an algebra structure is associated with any web $W(n+1,n,r)$ (see [19] or [16]). This structure consists of $\binom{n}{2}$ A-algebras (in [17] they were called Akivis algebras and originally in [3] they were called W-algebras) and $\binom{n}{3}$ C-algebras (in [19] and [16] they are called comtrans algebras).

We will describe these A-algebras and C-algebras in terms of our $(n+1)$-web W. For each 3-subweb $\underset{\alpha\beta}{W}$ and corresponding canonical affine connection $\overline{\gamma}_{n+1}$, in the tangent $(2r)$-space $\underset{\alpha\beta}{T}(e)$ of the coordinate binary loop $\underset{\alpha\beta}{F}(p)$ of the 3-subweb at the identity e, there exists an $\underset{\alpha\beta}{A}$-algebra $(\mathbf{R}^r, [\ ,\ ]_{\alpha\beta}, (\ ,\ ,\ )_{\alpha\beta})$ with two operations: the *binary commutator* defined as

$$(2.1)\qquad [\xi,\eta]^i_{\alpha\beta} = \underset{\alpha\beta}{\overline{a}}{}^i_{jk}(p)\xi^j\eta^k$$

and the *associator* defined as

$$(2.2)\qquad (\xi,\eta,\zeta)^i_{\alpha\beta} = \underset{\alpha\beta}{\overline{b}}{}^i_{jkl}(p)\xi^j\eta^k\zeta^k,$$

where $\underset{\alpha\beta}{\overline{a}}{}^i_{jk}$ and $\underset{\alpha\beta}{\overline{b}}{}^i_{jkl}$ are defined by (1.21) and (1.22), $\xi,\eta,\zeta \in \underset{\alpha\beta}{T}(e)$ are tangent vectors, and these two operations are connected by the Akivis identity (the generalized Jacobi identity):

$$(2.3)\qquad [[\xi,\eta]_{\alpha\beta},\zeta]_{\alpha\beta} + [[\eta,\zeta]_{\alpha\beta},\xi]_{\alpha\beta} + [[\zeta,\xi]_{\alpha\beta},\eta]_{\alpha\beta} =$$

$$= (\xi,\eta,\zeta)_{\alpha\beta} + (\eta,\zeta,\xi)_{\alpha\beta} + (\zeta,\xi,\eta)_{\alpha\beta} - (\eta,\xi,\zeta)_{\alpha\beta} - (\zeta,\eta,\xi)_{\alpha\beta} - (\xi,\zeta,\eta)_{\alpha\beta}$$

(see [3], [6], [17], [19] or [16]).

For each 4-subweb $\underset{\alpha\beta\gamma}{W}$, in the tangent $(3r)$-space $\underset{\alpha\beta\gamma}{T}(e)$ of the coordinate ternary loop $\underset{\alpha\beta\gamma}{F}(p)$ of the 4-subweb at the identity e, there exists a $\underset{\alpha\beta\gamma}{C}$-algebra $(\mathbf{R}^r, [\ ,\ ,\ ]_{\alpha\beta\gamma}, \langle\ ,\ ,\ \rangle_{\alpha\beta\gamma})$ with two operations: the *ternary commutator* defined as

$$(2.4)\qquad [\xi,\eta,\zeta]^i_{\alpha\beta\gamma} = (\lambda^i_{kjl} - \lambda^i_{jkl})(p)\xi^j\eta^k\zeta^k$$

and the *translator* defined as

$$(2.5)\qquad \langle\xi,\eta,\zeta\rangle^i_{\alpha\beta\gamma} = (\lambda^i_{klj} - \lambda^i_{jkl})(p)\xi^j\eta^k\zeta^k,$$

where $\xi,\eta,\zeta \in \underset{\alpha\beta\gamma}{T}(e)$ are tangent vectors and $\lambda^i_{jkl} = \underset{\alpha\beta\gamma}{\lambda}{}^i_{jkl}$ are the coefficients from the cubic chunk $\underset{\alpha\beta\gamma}{F}{}^i_3$ of the power series expansion of the functions $\underset{\alpha\beta\gamma}{F}{}^i$ defining the ternary loop $\underset{\alpha\beta\gamma}{F}(p)$. This cubic chunk is (see [19] or [16]):

(2.6)

$$\underset{\alpha\beta\gamma}{F}{}^i_3 = u^i_\alpha + u^i_\beta + u^i_\gamma + \sum_{(\varepsilon,\tau)} \underset{\varepsilon\tau}{\lambda}{}^i_{jk} u^j_\varepsilon u^k_\tau + \frac{1}{2}\sum_{(\varepsilon,\tau,\tau)} \underset{\varepsilon\tau\tau}{\lambda}{}^i_{jkl} u^j_\varepsilon u^k_\tau u^l_\tau + \underset{\alpha\beta\gamma}{\lambda}{}^i_{jkl} u^j_\alpha u^k_\beta u^l_\gamma,$$

where $\varepsilon,\tau = \alpha,\beta,\gamma$ and the notations (ε,τ) and (ε,τ,τ) mean that the summations are performed with respect to all combinations of (ε,τ) and (ε,τ,τ).

Note that the formulas (2.4) and (2.5) are obtained if one calculates the ternary commutator $[\xi,\eta,\zeta]_{\alpha\beta\gamma} = \xi\eta\zeta - \eta\xi\zeta$ and the translator $\langle\xi,\eta,\zeta\rangle_{\alpha\beta\gamma} = \zeta\eta\xi - \xi\eta\zeta$ of the cubic chunk M_3 of the masked version

$$\underset{\alpha\beta\gamma}{M} = \underset{\alpha\beta\gamma}{F}\{\alpha\} + \underset{\alpha\beta\gamma}{F}\{\beta\} + \underset{\alpha\beta\gamma}{F}\{\gamma\} - \underset{\alpha\beta\gamma}{F}$$

of the coordinate loop $\underset{\alpha\beta\gamma}{F}(p)$, where for example $\underset{\alpha\beta\gamma}{F}\{\gamma\}$ is the coordinate loop of the 3-subweb $\underset{\alpha\beta}{W}$, i.e. $\underset{\alpha\beta\gamma}{F}\{\gamma\} = \underset{\alpha\beta}{F}$ (see [19] or [16]). The coordinates $\underset{\alpha\beta\gamma}{M}{}^i_3$ of this cubic chunk $\underset{\alpha\beta\gamma}{M}{}_3$ have the following form:

$$\underset{\alpha\beta\gamma}{M}{}^i_3(u^j_\alpha, u^k_\beta, u^l_\gamma) = u^i_\alpha + u^i_\beta + u^i_\gamma - \lambda^i_{jkl} u^j_\alpha u^k_\beta u^l_\gamma. \tag{2.7}$$

2. We will study now the 4-web (2.7).

DEFINITION 2.1. The 4-web defined by the equations (2.7) is said to be the *comtrans 4-web associated with the 4-web* $\underset{\alpha\beta\gamma}{W}$.

Let us denote this 4-web by $C_{\alpha\beta\gamma}$. For this 4-web $C_{\alpha\beta\gamma}$ we will denote by $\underset{\varrho\mu}{A}{}^i_{jk}$, $\underset{\varrho\mu\nu}{A}{}^i_{jkl}$ and $\underset{\varrho\mu}{B}{}^i_{jkl}$, $\varrho,\mu,\nu = \alpha,\beta,\gamma$, the torsion tensor, its Pfaffian derivatives and the curvature tensor.

PROPOSITION 2.2. *The torsion tensor, its Pfaffian derivatives and the curvature tensor of the comtrans 4-web* $C_{\alpha\beta\gamma}$ *are expressed in terms of the coefficients of the expansions* (2.7) *and variables* u^i_α *by the following expressions:*

$$\begin{cases} \underset{\alpha\beta}{A}{}^i_{jk} = \frac{1}{6}(5\lambda^i_{jkq} - \lambda^i_{kjq})u^q_\gamma - \frac{1}{3}\lambda^i_{q(jk)}u^q_\alpha - \frac{1}{3}\lambda^i_{(j|q|k)}u^q_\beta + o(\varrho^2), \\ \underset{\beta\gamma}{A}{}^i_{jk} = \frac{1}{6}(5\lambda^i_{qjk} - \lambda^i_{qkj})u^q_\alpha - \frac{1}{3}\lambda^i_{(j|q|k)}u^q_\beta - \frac{1}{3}\lambda^i_{(jk)q}u^q_\gamma + o(\varrho^2), \\ \underset{\gamma\alpha}{A}{}^i_{jk} = \frac{1}{6}(5\lambda^i_{jqk} - \lambda^i_{kqj})u^q_\beta - \frac{1}{3}\lambda^i_{q(jk)}u^q_\alpha - \frac{1}{3}\lambda^i_{(jk)q}u^q_\gamma + o(\varrho^2), \end{cases} \tag{2.8}$$

$$(2.9)\qquad \begin{cases} \underset{\alpha\beta\gamma}{A}{}^{i}_{jkl} = \frac{1}{6}(5\lambda^{i}_{jkl} + \lambda^{i}_{kjl}) + o(\varrho), \\ \underset{\alpha\beta\beta}{A}{}^{i}_{jkl} = \underset{\gamma\beta\beta}{A}{}^{i}_{jkl} = -\frac{1}{6}(\lambda^{i}_{jlk} + \lambda^{i}_{klj}) + o(\varrho), \\ \underset{\beta\alpha\alpha}{A}{}^{i}_{jkl} = \underset{\gamma\alpha\alpha}{A}{}^{i}_{jkl} = -\frac{1}{6}(\lambda^{i}_{ljk} + \lambda^{i}_{lkj}) + o(\varrho), \\ \underset{\beta\gamma\gamma}{A}{}^{i}_{jkl} = \underset{\alpha\gamma\gamma}{A}{}^{i}_{jkl} = -\frac{1}{6}(\lambda^{i}_{jkl} + \lambda^{i}_{kjl}) + o(\varrho), \end{cases}$$

$$(2.10)\qquad \begin{cases} \underset{\alpha\beta}{B}{}^{i}_{jkl} = \frac{1}{12}(\lambda^{i}_{kjl} - \lambda^{i}_{jlk}) + o(\varrho), \\ \underset{\beta\gamma}{B}{}^{i}_{jkl} = \frac{1}{12}(\lambda^{i}_{lkj} - \lambda^{i}_{kjl}) + o(\varrho), \\ \underset{\gamma\alpha}{B}{}^{i}_{jkl} = \frac{1}{12}(\lambda^{i}_{jlk} - \lambda^{i}_{lkj}) + o(\varrho), \end{cases}$$

where $\varrho = \max|u^{i}_{\alpha}|$, $\frac{o(t)}{t} \to 0$ *if* $t \to 0$.

PROOF. To prove the formulae (2.8)–(2.10), one should apply the corresponding formulae of [12] and [7] (see all of them in Section 3.3 of [16]) and equation (1.5). □

COROLLARY 2.3. *The torsion tensor, its Pfaffian derivatives and the curvature tensor of the comtrans 4-web* $C_{\alpha\beta\gamma}$ *at the identity e of the coordinate ternary loop of the 4-web* $\underset{\alpha\beta\gamma}{W}$ *are expressed in terms of the coefficients of the expansions* (2.7) *as follows:*

$$(2.11)\qquad \underset{\alpha\beta}{A}{}^{i}_{jk} = \underset{\beta\gamma}{A}{}^{i}_{jk} = \underset{\gamma\alpha}{A}{}^{i}_{jk} = 0,$$

$$(2.12)\qquad \begin{cases} \underset{\alpha\beta\gamma}{A}{}^{i}_{jkl} = \frac{1}{6}(5\lambda^{i}_{jkl} + \lambda^{i}_{kjl}), \\ \underset{\alpha\beta\beta}{A}{}^{i}_{jkl} = \underset{\gamma\beta\beta}{A}{}^{i}_{jkl} = -\frac{1}{6}(\lambda^{i}_{jlk} + \lambda^{i}_{klj}), \\ \underset{\beta\alpha\alpha}{A}{}^{i}_{jkl} = \underset{\gamma\alpha\alpha}{A}{}^{i}_{jkl} = -\frac{1}{6}(\lambda^{i}_{ljk} + \lambda^{i}_{lkj}), \\ \underset{\beta\gamma\gamma}{A}{}^{i}_{jkl} = \underset{\alpha\gamma\gamma}{A}{}^{i}_{jkl} = -\frac{1}{6}(\lambda^{i}_{jkl} + \lambda^{i}_{kjl}), \end{cases}$$

$$(2.13)\qquad \begin{cases} \underset{\alpha\beta}{B}{}^{i}_{jkl} = \frac{1}{12}(\lambda^{i}_{kjl} - \lambda^{i}_{jlk}), \\ \underset{\beta\gamma}{B}{}^{i}_{jkl} = \frac{1}{12}(\lambda^{i}_{lkj} - \lambda^{i}_{kjl}), \\ \underset{\gamma\alpha}{B}{}^{i}_{jkl} = \frac{1}{12}(\lambda^{i}_{jlk} - \lambda^{i}_{lkj}). \end{cases}$$

PROOF. The proof immediately follows from the formulae (2.8)–(2.10) where $u^{i}_{\mu} = 0$, $\mu = \alpha, \beta, \gamma$. □

COROLLARY 2.4. *Every 3-subweb $\underset{\varrho\mu}{W}$, $\varrho, \mu = \alpha, \beta, \gamma$, of the comtrans 4-web $C_{\alpha\beta\gamma}$ is parallelizable.*

PROOF. This follows from equations (2.7): if for example $u^i_\gamma = 0$, then

$$\underset{\alpha\beta\gamma}{M}{}^i_3(u^j_\alpha, u^k_\beta, 0) = u^i_\alpha + u^i_\beta, \tag{2.14}$$

which means that the 3-subweb $\underset{\alpha\beta}{W}$ is parallelizable. Of course, the same result can be obtained if, using the formulae similar to the formulae (1.21) and (1.22), one calculates the torsion and curvature tensors of this 3-subweb and observes that both of them vanish. □

3. We will now find the expressions of the ternary operations associated with the 4-web $\underset{\alpha\beta\gamma}{W}$.

PROPOSITION 2.5. *The ternary commutator and the translator of the 4-web $\underset{\alpha\beta\gamma}{W}$ at the identity e of a ternary coordinate loop of this 4-web have the following expressions in terms of the torsion and curvature tensors of the comtrans 4-web $C_{\alpha\beta\gamma}$:*

$$[\xi, \eta, \zeta]^i_{\alpha\beta\gamma} = (\underset{\alpha\beta\gamma}{A}{}^i_{kjl} - \underset{\alpha\beta\gamma}{A}{}^i_{jkl})(p)\xi^j\eta^k\zeta^l, \tag{2.15}$$

$$\begin{aligned}\langle \xi, \eta, \zeta\rangle^i_{\alpha\beta\gamma} &= 6(\underset{\beta\gamma\gamma}{A}{}^i_{jkl} - \underset{\gamma\alpha\alpha}{A}{}^i_{ljk})(p)\xi^j\eta^k\zeta^l = -12\underset{\alpha\beta}{B}{}^i_{kjl}(p)\xi^j\eta^k\zeta^l = \\ &= -12\underset{\beta\gamma}{B}{}^i_{lkj}(p)\xi^j\eta^k\zeta^l = -12\underset{\gamma\alpha}{B}{}^i_{jlk}(p)\xi^j\eta^k\zeta^l.\end{aligned} \tag{2.16}$$

PROOF. The formulae (2.15) and (2.16) follow from the formulae (2.4), (2.5) and the formulae (2.12) and (2.13) of Corollary 2.3. □

DEFINITION 2.6. We will call the ensemble consisting of the $\underset{\alpha\beta\gamma}{C}$-algebra of the 4-subweb $\underset{\alpha\beta\gamma}{W}$ and the three $\underset{\varrho\mu}{A}$-algebras of its 3-subwebs $\underset{\varrho\mu}{W}$, $\varrho, \mu = \alpha, \beta, \gamma$, the $\underset{\alpha\beta\gamma}{(AC)}$*-algebra associated with the* 4*-web* $\underset{\alpha\beta\gamma}{W}$. The full set of $\binom{n}{3}$ such algebras will be called the (AC)*-algebra associated with the* $(n+1)$*-web* W.

Note that, given the indicated (AC)-algebra, i.e. $\binom{n}{2}$ $\underset{\alpha\beta}{A}$-algebras and $\binom{n}{3}$ $\underset{\alpha\beta\gamma}{C}$-algebras, one can construct (but not uniquely) a local analytic n-ary loop such that the constructed n-loop in turn determines these algebras as its A- and C-algebras (see [19] or [16]).

4. In addition to the binary and ternary operations in the $\underset{\alpha\beta}{A}$-algebra of each 3-subweb $\underset{\alpha\beta}{W}$, and to the ternary operations in the $\underset{\alpha\beta\gamma}{C}$-algebra of each 4-subweb $\underset{\alpha\beta\gamma}{W}$, further operations of arity $h = 4, 5, \ldots$ can be introduced in the vector spaces tangent to the coordinate loops $\underset{\alpha\beta}{T}(e)$ and $\underset{\alpha\beta\gamma}{T}(e)$ of these 3- and 4-subwebs at the identity. They can be defined by means of the Pfaffian derivatives of the fundamental tensors $\underset{\alpha\beta}{\overline{a}}{}^i_{jk}$ and $\underset{\alpha\beta}{\overline{b}}{}^i_{jkl}$ of the 3-subweb and of the fundamental tensor $\underset{\alpha\beta}{A}{}^i_{jkl}$ of the comtrans 4-web of the 4-subweb: the tensors $\underset{\alpha\beta}{t}{}_2 = \{\underset{\alpha\beta}{\overline{a}}{}^i_{jk}\}$, $\underset{\alpha\beta}{t}{}_3 = \{\underset{\alpha\beta}{\overline{b}}{}^i_{jkl}\}$, $\underset{\alpha\beta}{t'}{}_3 = \{\underset{\alpha\beta\mu}{\overline{a}}{}^i_{jkl}\}$, $\underset{\alpha\beta}{t}{}_4 = \{\underset{\alpha\beta\mu}{\overline{c}}{}^i_{jklm}\}$, $\mu = \alpha, \beta$, etc. and the tensors

$$\underset{\alpha\beta\gamma}{r}{}_3 = \{\underset{\alpha\beta\gamma}{A}{}^i_{kjl} - \underset{\alpha\beta\gamma}{A}{}^i_{jkl}\},$$

$$\underset{\alpha\beta\gamma}{s}{}_3 = \{6(\underset{\beta\gamma\gamma}{A}{}^i_{jkl} - \underset{\gamma\alpha\alpha}{A}{}^i_{ljk})\} = \{-12\underset{\alpha\beta}{B}{}^i_{kjl}\} = \{-12\underset{\beta\gamma}{B}{}^i_{lkj}\} = \{-12\underset{\gamma\alpha}{B}{}^i_{jlk}\},$$

$$\underset{\alpha\beta\gamma}{r}{}_4 = \{\underset{\alpha\beta\gamma\varrho}{A}{}^i_{kjlm} - \underset{\alpha\beta\gamma\varrho}{A}{}^i_{jklm}\},$$

$$\underset{\alpha\beta\gamma}{s}{}_4 = \{6(\underset{\beta\gamma\gamma}{A}{}^i_{jklm} - \underset{\gamma\alpha\alpha}{A}{}^i_{ljkm})\} = \{-12\underset{\alpha\beta\varrho}{B}{}^i_{kjlm}\} =$$
$$= \{-12\underset{\beta\gamma\varrho}{B}{}^i_{lkjm}\} = \{-12\underset{\gamma\alpha\varrho}{B}{}^i_{jlkm}\}$$

$(\varrho = \alpha, \beta, \gamma)$ of type $\binom{1}{h}$ define h-ary operations $(h = 2, 3, 4, \ldots)$:

$$(2.17)\quad \begin{cases} \underset{\alpha\beta}{t}{}_2 : (\underset{\alpha\beta}{T})^2(e) \to \underset{\alpha\beta}{T}(e), \\ \underset{\alpha\beta}{t}{}_3 : (\underset{\alpha\beta}{T})^3(e) \to \underset{\alpha\beta}{T}(e), \\ \underset{\alpha\beta}{t'}{}_3 : (\underset{\alpha\beta}{T})^3(e) \to \underset{\alpha\beta}{T}(e), \\ \underset{\alpha\beta}{t}{}_4 : (\underset{\alpha\beta}{T})^4(e) \to \underset{\alpha\beta}{T}(e), \\ \underset{\alpha\beta\gamma}{r}{}_3 : (\underset{\alpha\beta\gamma}{T})^3(e) \to \underset{\alpha\beta\gamma}{T}(e), \\ \underset{\alpha\beta\gamma}{r}{}_4 : (\underset{\alpha\beta\gamma}{T})^4(e) \to \underset{\alpha\beta\gamma}{T}(e), \\ \underset{\alpha\beta\gamma}{s}{}_3 : (\underset{\alpha\beta\gamma}{T})^3(e) \to \underset{\alpha\beta\gamma}{T}(e), \\ \underset{\alpha\beta\gamma}{s}{}_4 : (\underset{\alpha\beta\gamma}{T})^4(e) \to \underset{\alpha\beta\gamma}{T}(e). \end{cases}$$

DEFINITION 2.7. The vector space $T(e)$ with the operations (2.17) is called a *local* $(AC)_h$*-algebra* of the web W at the point p.

It is easy to see that the (AC)-algebra introduced in Definition 2.6 is the $(AC)_3$-algebra.

In a similar manner the operations

$$
(2.18)\qquad \begin{cases}
\underset{\alpha\beta}{\tilde{t}}{}_h : (\underset{\alpha\beta}{\widetilde{T}})^h(e) \to \underset{\alpha\beta}{\widetilde{T}}(e), \\
\underset{\alpha\beta}{\tilde{t}'}{}_h : (\underset{\alpha\beta}{\widetilde{T}})^h(e) \to \underset{\alpha\beta}{\widetilde{T}}(e), \\
\underset{\alpha\beta\gamma}{\tilde{r}}{}_h : (\underset{\alpha\beta\gamma}{\widetilde{T}})^h(e) \to \underset{\alpha\beta\gamma}{\widetilde{T}}(e), \\
\underset{\alpha\beta\gamma}{\tilde{s}}{}_h : (\underset{\alpha\beta\gamma}{\widetilde{T}})^h \to \underset{\alpha\beta\gamma}{\widetilde{T}}(e),
\end{cases}
$$

where $h = 2, 3, 4, \dots$, and the corresponding $(\underset{\alpha\beta\gamma}{\widetilde{AC}})_h$-algebras are defined in the tangent $(2r)$-space $\underset{\alpha\beta}{\widetilde{T}}(e)$ of the coordinate binary loop $\underset{\alpha\beta}{\widetilde{F}}(p)$ of each 3-subweb $\underset{\alpha\beta}{W}$ and in the tangent $(3r)$-space $\underset{\alpha\beta\gamma}{\widetilde{T}}(e)$ of the coordinate ternary loop $\underset{\alpha\beta\gamma}{\widetilde{F}}(p)$ of any 4-subweb $\underset{\alpha\beta\gamma}{W}$, where $\tilde{t}_h$, $\tilde{t}'_h$, $\tilde{r}_h$ and $\tilde{s}_h$ map $(\widetilde{T})^h(e)$ into $\widetilde{T}(e)$.

THEOREM 2.8. *The local $(\widetilde{AC})_h$-algebra of a subweb $\widetilde{W}$ of the web W is a subalgebra of the local $(AC)_h$-algebra of the web W at every point $p \in \widetilde{M}$.*

PROOF. First, as was proved in Theorem 4 of [6], any $\underset{\alpha\beta}{\widetilde{A}}{}_h$-algebra associated with a 3-subweb $\underset{\alpha\beta}{\widetilde{W}}$ of the 3-web $\underset{\alpha\beta}{W}$ is a subalgebra of the $\underset{\alpha\beta}{A}{}_h$-algebra of the 3-web $\underset{\alpha\beta}{W}$.

Next, the comtrans web associated with the 4-subweb $\underset{\alpha\beta}{\widetilde{W}}$ is a 4-subweb of the comtrans web associated with the 4-web $\underset{\alpha\beta\gamma}{W}$, since for any (ns)-dimensional submanifold $\widetilde{M}$, its leaves are intersections of $\widetilde{M}$ with the leaves $\underset{\alpha\beta\gamma}{W}$. This implies that for the embedding $i_{\alpha\beta\gamma} \colon \underset{\alpha\beta\gamma}{\widetilde{T}}(e) \to \underset{\alpha\beta\gamma}{T}(e)$ which is defined by equations (1.14) or by

$$
(2.19)\qquad u^i_\mu = \xi^i_a \tilde{u}^a_\mu,
$$

the torsion and curvature tensors of these two comtrans webs and their Pfaffian derivatives are connected by the relations similar to (1.31)–(1.33). In other words, for example for the operations r_h and $\tilde{r}_h$ we have the following

(1.31). On the other hand, if we have (1.31) identically satisfied for all ξ^i_a, we can deduce from it the isoclinity of W as was done in the proof of Theorem 1.9. Moreover, if $n > 2$, in Proposition 1.12 we proved that this condition also implies the transversal geodesicity of W.

We will define now almost trivial $(AC)_h$-algebras. This notion will help us to find a characterization of Grassmannizable $(n+1)$-webs W.

DEFINITION 2.11. The $(AC)_h$-algebra of the coordinate loop $F(p)$ of an $(n+1)$-web W is said to be *almost trivial* if each of its linear subspaces is an $(\widetilde{AC})_h$-subalgebra.

Now we are able to prove a theorem giving a necessary and sufficient condition for an $(n+1)$-web W, $n > 2$, to be a Grassmannizable $(n+1)$-web W in terms of almost triviality of its $(AC)_2$-algebra.

THEOREM 2.12. *An $(n+1)$-web W, $n > 2$, is a Grassmannizable $(n+1)$-web if and only if its $(AC)_2$-algebra is almost trivial.*

PROOF. Since a Grassmannizable $(n+1)$-web satisfies property P_s for any $s = 1, 2, \dots, r$, it follows from Definition 2.11 that its $(AC)_2$-algebra is almost trivial. Conversely, suppose that the $(AC)_2$-algebra of an $(n+1)$-web W, $n > 2$, is almost trivial. Then equations (1.31) are identically satisfied for all ξ^i_a. As we mentioned above, this implies that the web W is both isoclinic and transversally geodesic. According to [11] (see also [12], [13] or Theorem 2.6.4 and Theorem 5.2.1 in [16]), this web is Grassmannizable. □

Note that a similar theorem for 3-webs $W = W(3, 2, r)$ which has been proved in [6] (Theorem 6) requires almost triviality of their A_3-algebras. This is explained by the fact that the transversal geodesicity condition for a 3-web $W(3, 2, r)$ is expressed by the equation $\underset{\alpha\beta}{\bar{b}}{}^i_{(jkl)} = \delta^i_{(j}\,\underset{\alpha\beta}{\bar{b}}{}_{kl)}$ containing the curvature tensor, while the transversal geodesicity condition for an $(n+1)$-web $W(n+1, n, r)$, $n > 2$, is expressed by the equation $\underset{\alpha\beta}{a}{}^i_{(jk)} = \delta^i_{(j}\,\underset{\alpha\beta}{a}{}_{k)}$ containing the torsion tensor.

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commutative diagram:

$$\begin{array}{ccc} (\underset{\alpha\beta\gamma}{\widetilde{T}})^h(e) & \xrightarrow{\tilde{r}_h} & (\underset{\alpha\beta\gamma}{\widetilde{T}})(e) \\ {\scriptstyle i_h}\big\downarrow & & \big\downarrow \\ (\underset{\alpha\beta\gamma}{T})^h(e) & \xrightarrow[r_h]{} & (\underset{\alpha\beta\gamma}{T})(e) \end{array}$$

Similar diagram is valid for the operations s_h and $\tilde{s}_h$. This means that the $\underset{\alpha\beta\gamma}{\widetilde{C}}{}_h$-algebra of a 4-subweb $\underset{\alpha\beta\gamma}{\widetilde{W}}$ is a subalgebra of the $\underset{\alpha\beta\gamma}{C}{}_h$-algebra of a 4-web $\underset{\alpha\beta\gamma}{W}$. □

5. In [6] the notion of an almost trivial A_2-algebra of the coordinate binary loop was introduced. In our notation and terminology this definition can be formulated as follows:

DEFINITION 2.9. The $\underset{\alpha\beta}{A}{}_h$-algebra of the coordinate loop $\underset{\alpha\beta}{F}$ of a 3-web $\underset{\alpha\beta}{W}$ is said to be *almost trivial* if each of its linear subspaces is an $\underset{\alpha\beta}{\tilde{A}}{}_h$-subalgebra.

Definition 2.9 describes property $\underset{\alpha\beta}{P}{}_s$ for a 3-subweb $\underset{\alpha\beta}{W}$ in terms of its $\underset{\alpha\beta}{A}{}_h$-algebras. It was proved in [6] (Theorem 5) that if $r > 2$, the local $\underset{\alpha\beta}{A}{}_2$-algebras of a 3-web $\underset{\alpha\beta}{W}$ are almost trivial if and only if $\underset{\alpha\beta}{W}$ is isoclinic. This implies the following result:

THEOREM 2.10. *If $r > 2$, the local $\underset{\alpha\beta}{A}{}_2$-algebras of all $\binom{n}{2}$ reduct 3-subwebs $\underset{\alpha\beta}{W}$ of an $(n+1)$-web W are almost trivial if and only if W is isoclinic.*

PROOF. In fact, if $r > 2$ and W is isoclinic, then all its 3-subwebs $\underset{\alpha\beta}{W}$ are isoclinic (see Theorem 4 in [11] or Corollary 1.11.7 in [16]). Analytically this means that we have equations (1.64) or equivalent equations (1.52) for any α, β; $\alpha \neq \beta$. By Theorem 5 of [6], it follows that the local $\underset{\alpha\beta}{A}{}_2$-algebras of all $\binom{n}{2}$ reduct 3-subwebs $\underset{\alpha\beta}{W}$ of an $(n+1)$-web W are almost trivial.

Conversely, if all these $\underset{\alpha\beta}{A}{}_2$-algebras are almost trivial, then, by Theorem 5 of [6], all the 3-subwebs $\underset{\alpha\beta}{W}$ are isoclinic. This proves the isoclinity of the whole web W (see again Theorem 4 in [11] or Corollary 1.11.7 in [16]). □

6. The almost triviality of all $\underset{\alpha\beta}{A}{}_h$-algebras implies condition (1.48) (or the equivalent condition (1.34)). However, this does not imply condition

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MULTIPLICATIVE FUNCTIONS SATISFYING A CONGRUENCE PROPERTY. V

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An arithmetic function $f(n) \not\equiv 0$ is said to be multiplicative if $(n,m)=1$ implies

$$f(nm) = f(n)f(m).$$

If this equation holds for all pairs of positive integers n and m, then we say that $f(n)$ is completely multiplicative. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the set of integer-valued multiplicative and completely multiplicative functions, respectively. M. V. Subbarao [5] proved that if $f \in \mathcal{M}$ satisfies the relation

$$(1) \qquad f(n+m) \equiv f(m) \pmod{n}$$

for every positive integer n and m, then

$$(2) \qquad f(n) = n^a \qquad (n = 1, 2, \ldots),$$

where a is a non-negative integer. In [2], A. Iványi extended this result proving that if $f \in \mathcal{M}^*$ and (1) holds for a fixed m and for every positive integer n, then $f(n)$ also has the form (2). For some generalizations of these results we refer to [3], [4]. For example, from [4] it follows that the result of A. Iványi mentioned above is true for a multiplicative function $f(n)$.

Our purpose in this paper is to give a complete characterization of those functions $f \in \mathcal{M}$ which satisfy

$$f(An+B) \equiv C \pmod{n}$$

for every integer $n \geqq N$. Here $A>0$, $B>0$, $C \neq 0$, $N>0$ are arbitrarily fixed integers with the condition $(A,B)=1$. I. Joó examined the special case $A=p$ prime and $C=f(B)$ in [6].

We shall prove the following

THEOREM. *Let $A>0$, $B>0$, $C \neq 0$ and $N>0$ be integers with the condition $(A,B)=1$. If $f \in \mathcal{M}$ satisfies the relation*

$$(3) \qquad f(An+B) \equiv C \pmod{n}$$

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for every integer $n \geqq N$, then there are a non-negative integer a and a real-valued Dirichlet character χ (mod A) such that

$$f(n) = \chi(n)n^a \tag{4}$$

for all positive integers n which are prime to A.

COROLLARY. *Let A and N be positive integers. If $f \in \mathcal{M}$ and $g \in \mathcal{M}$ satisfy the relation*

$$f(An+m) \equiv g(m) \qquad (\mathrm{mod}\ n)$$

for every integer $n \geqq N$, $m \geqq 1$, then there are a positive integer a and a real-valued Dirichlet character χ (mod A) such that

$$f(n) = g(n) = \chi(n)n^a$$

for all positive integers n which are prime to A.

We shall use three lemmas in the proof of our theorem.

LEMMA 1. *Assume that A, B, C, N and f satisfy the conditions of the theorem and* (3) *holds for every integer $n \geqq N$. Then*

$$f(B) = C \tag{5}$$

and

$$f(Q^sB)f(Q) = Cf(Q^{s+1}) \qquad (s = 1, 2, \ldots) \tag{6}$$

holds for each positive integer Q coprime to A.

PROOF. Let Q, s be positive integers for which $(Q, A) = 1$. First we prove that (6) holds.

Let p be a prime with $(p, QB) = 1$. Let e_0 be a positive integer for which $p^{e_0} > \max(C, N)$. Then for each positive integer $e \geqq e_0$ there are positive integers $x = x(e)$, $y = y(e)$ such that

$$Q^s x = 1 + Ap^e y \quad \text{and} \quad (x, QB) = 1. \tag{7}$$

Since $p^e \geqq N$, by using (3) and (7) we have

$$f(Q^sB)f(x) = f(Q^sxB) = f(B + Ap^eyB) \equiv C \qquad (\mathrm{mod}\ p^e), \tag{8}$$

and so

$$f(Q^sB)f(Q)f(x) \equiv Cf(Q) \qquad (\mathrm{mod}\ p^e). \tag{9}$$

Since $(Q, Axp) = 1$ we can choose positive integers $u = u(e)$ and $v = v(e)$ to satisfy

$$Qu = B + Axp^ev \quad \text{and} \quad (u, Q) = 1. \tag{10}$$

Let

(11) $$X = xu \quad \text{and} \quad Y = By + x^2Q^s v.$$

By using (7), (10) and (11) we have

$$Q^{s+1}X = (Q^s x)(Qu) = Q^s x(B + Axp^e v) = (1 + Ap^e y)B + Ax^2p^eQ^s v = \\ = B + Ap^e(By + x^2Q^s v) = B + AYp^e.$$

Thus, applying (3) we obtain the congruence

$$f(Q^{s+1})f(X) \equiv C \pmod{p^e},$$

consequently

(12) $$f(Q^{s+1})f(QX) \equiv Cf(Q) \pmod{p^e}.$$

On the other hand, from (3), (10) and (11) we get

(13) $$f(QX) = f(x(B + Axp^e v)) = f(x)f(B + Axp^e v) \equiv Cf(x) \pmod{p^e}.$$

Thus, by (9), (12) and (13) we have

(14) $$f(Q^s B)f(Q)f(x) \equiv Cf(Q^{s+1})f(x) \pmod{p^e}.$$

It can be easily seen by (8) that

$$f(x) \not\equiv 0 \pmod{p^{e_0}},$$

since $p^{e_0} > C$. This with (14) implies

$$f(Q^s B)f(Q) \equiv Cf(Q^{s+1}) \pmod{p^{e-e_0+1}}$$

for every integer $e \geqq e_0$, which leads to

$$f(Q^s B)f(Q) = Cf(Q^{s+1}).$$

Thus (6) is true. It is obvious that (5) follows from (6) with $Q = 1$. Lemma 1 is proved.

LEMMA 2. *Assume that A, B, C, N and f satisfy the conditions of the theorem and* (3) *holds for every integer $n \geqq N$. Then there is a non-negative integer a such that*

(15) $$|f(nB)| = n^a |f(B)|$$

for all positive integers n which are prime to A.

PROOF. Let Q be a positive integer for which $(Q, AB) = 1$. By using Lemma 1 we have

(16) $$f(Q^s) = f(Q)^s \qquad (s = 1, 2, \dots).$$

We show that if q is prime, then

$$q \mid f(Q) \quad \text{implies} \quad q \mid Q. \tag{17}$$

Assume indirectly that $q \mid f(Q)$ and $q \nmid Q$. We choose a positive integer s_0 to satisfy $q^{s_0} > \max(C, N)$. Then there are positive integers z and t such that

$$Q^{s_0} z = B + Aq^{s_0} t \quad \text{and} \quad (z, Q) = 1. \tag{18}$$

By using (3), (16) and (18) we get

$$0 \equiv f(Q)^{s_0} f(z) = f(Q^{s_0} z) = f(B + Aq^{s_0} t) \equiv C \pmod{q^{s_0}},$$

which is a contradiction, since $q^{s_0} > C$. Thus (17) is proved.

By (17) it follows that for each prime p for which $(p, AB) = 1$, we have

$$f(p) = \pm p^{a(p)}, \tag{19}$$

where $a(p) \geqq 0$ is an integer. Now we prove that for distinct primes p, q we have

$$a(p) = a(q). \tag{20}$$

Let p, q be distinct primes for which $(pq, AB) = 1$ and let $a(p) \geqq a(q)$. Since $(pq, AB) = 1$, from Euler Theorem we get that

$$\left(pq^k\right)^{2\varphi(A)} \equiv 1 \pmod{A}$$

holds for every positive integer k, where φ denotes Euler's totient function. Let

$$\left(pq^k\right)^{2\varphi(A)} = 1 + AT(k). \tag{21}$$

It is obvious that there is a positive integer k_0 such that $T(k) \geqq N$ for every integer $k \geqq k_0$. By using (3), (19), (21) and Lemma 1 we see that

$$f\left[\left(pq^k\right)^{2\varphi(A)} B\right] \equiv C = f(B) \pmod{T(k)}$$

and

$$f\left[\left(pq^k\right)^{2\varphi(A)} B\right] = f(p)^{2\varphi(A)} f(q)^{2k\varphi(A)} f(B) =$$

$$= p^{2(a(p)-a(q))\varphi(A)} \left(pq^k\right)^{2a(q)\varphi(A)} f(B) \equiv p^{2(a(p)-a(q))\varphi(A)} f(B) \pmod{T(k)}$$

holds for every integer $k \geqq k_0$. These imply $a(p) = a(q)$, since $T(k) \to \infty$ as $k \to \infty$. So (20) is proved. By (19) and (20) there is a non-negative integer a such that

$$|f(n)| = n^a \quad \text{if} \quad (n, AB) = 1. \tag{22}$$

Now we prove (15).

Let n be a positive integer for which $(n,A)=1$. Then there is a positive integer $Q=Q(n)$ such that $nQ\equiv 1 \pmod A$ and $(Q,AB)=1$. For each positive integer t let us define $h(t)$, $H(t)$ by the relations

$$h(t):=1+2\varphi(A)t \quad \text{and} \quad nQ^{h(t)}=1+AH(t). \tag{23}$$

Let t_0 be a positive integer for which $H(t)\geqq N$ if $t\geqq t_0$. Then, by using (3), (22), (23) and Lemma 1 we get

$$\begin{aligned} f(B)=C\equiv f(nQ^{h(t)}B)=f(Q^{h(t)})f(nB)= \\ =f(Q)^{h(t)}f(nB)=\pm Q^{ah(t)}f(nB) \pmod{H(t)} \end{aligned}$$

holds for every $t\geqq t_0$, where a is a non-negative integer which is given in (22). Using (23) we have

$$n^a f(B)\equiv \pm\left(nQ^{h(t)}\right)^a f(nB)\equiv \pm f(nB) \qquad \pmod{H(t)},$$

which implies $|f(nB)|=n^a|f(B)|$, since $H(t)\to\infty$ as $t\to\infty$. This completes the proof of Lemma 2.

LEMMA 3. *Let $g(n)$ be a complex-valued multiplicative function with $g(An+B)=c\neq 0$ for fixed integers $A>0$, B and all sufficiently large integers n. Then there is a Dirichlet character $\chi(\mathrm{mod}\ A)$ so that $g(n)=$ $=\chi(n)$ for all positive integers n which are prime to A.*

PROOF. This lemma is identical to Lemma 19.3 in [1].

PROOF OF THE THEOREM. Assume that A,B,C,N and f satisfy the conditions of the theorem and (3) holds for every integer $n\geqq N$.

Now consider the function

$$g(n)=\frac{f(n)}{n^a} \qquad (n=1,2,\dots),$$

where a is a non-negative integer determined in Lemma 2. It is obvious that $g(n)$ is a multiplicative function.

In the following we shall prove that for all sufficiently large integers n

$$g(An+B)=g(B) \quad \text{and } g(B)\neq 0, \tag{24}$$

which, by using Lemma 3, proves the theorem.

First we note that from Lemma 2 we have

$$|f(n)|=n^a \quad \text{if} \quad (n,AB)=1. \tag{25}$$

We shall prove that

$$|f(n)|=n^a \quad \text{if} \quad (n,A)=1. \tag{26}$$

Let d be a positive divisor of B. Then there are infinitely many positive integers m which satisfy

$$(27)\qquad m \geqq N \quad \text{and} \quad \left(Am + \frac{B}{d}, dAB\right) = 1.$$

By using (3), (25) and (27) we have

$$f(B) \equiv f(Amd + B) = f(d) f\left(Am + \frac{B}{d}\right) =$$
$$= \pm f(d)\left(Am + \frac{B}{d}\right)^a \equiv \pm f(d)\left(\frac{B}{d}\right)^a \pmod{m},$$

which gives

$$(28)\qquad \frac{|f(d)|}{d^a} = \frac{|f(B)|}{B^a}.$$

Applying (28) in the case $d = 1$, we get that $|f(B)| = B^a$, and so

$$(29)\qquad |f(d)| = d^a \quad \text{for all} \quad d \mid B.$$

Now let p be a prime for which $p \mid B$, $p^{k_0} \parallel B$. From (29)

$$(30)\qquad |f(p^k)| = p^{ka}$$

for all positive integers $k \leqq k_0$. If $k \geqq k_0$, then (30) and Lemma 2 give

$$|f(p^k)| = (p^{k-k_0})^a \left|f(p^{k_0})\right| = (p^{k-k_0})^a p^{k_0 a} = p^{ka}.$$

In both cases we have

$$|f(p^k)| = p^{ka} \qquad (k = 1, 2, \dots),$$

which with (25) proves (26).

Since $(An + B, A) = (A, B) = 1$ for every positive integer n, we get from (3) and (26) that $g(B) = \pm 1$, $g(An + B) = \pm 1$ and

$$g(An + B)(An + B)^a = f(An + B) \equiv f(B) = g(B)B^a \pmod{n}$$

holds for every integer $n \geqq N$. These imply

$$g(An + B) = g(B) \quad \text{for all} \quad n > \max(N, 2B^a).$$

This proves (24) and so the proof of the theorem is finished.

Proof of the Corollary. Assume that $f, g \in \mathcal{M}$ and

$$f(An + m) \equiv g(m) \pmod{n}$$

for every integer $n \geqq N$, $m \geqq 1$, where A and N are given positive integers. Applying the above congruence in the case $m = 1$, from our theorem we get that $f(n) = \chi(n)n^a$, for all n coprime to A, where $a \geqq 0$ is an integer, χ is a Dirichlet character (mod A). Thus we have

$$g(m) \equiv f(An + m) = \chi(An + m)(An + m)^a \equiv \chi(m)m^a \pmod{n}$$

for every positive integer $n \geqq N$ and for every positive integer m for which $(m, A) = 1$. This congruence shows that $g(m) = \chi(m)m^a$ for all m coprime to A.

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PALEY SETS AND TERM BY TERM DYADIC DIFFERENTIATION OF WALSH SERIES

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1. Dyadic differentiation

Let $\mathbf{G}$ be the *dyadic group*, $\mathbf{m}$ be its Haar measure, and $\psi_0, \psi_1, \ldots$ represent its characters. Thus $\mathbf{G} = \{(x_0, x_1, \ldots) : x_i = 0 \text{ or } 1\}$, the sum of two elements in $\mathbf{G}$ is defined by

$$(x_0, x_1, \ldots) + (y_0, y_1, \ldots) = (|x_0 - y_0|, |x_1 - y_1|, \ldots)$$

and $\mathbf{m}(E + x) = \mathbf{m}(E)$ for all $x \in \mathbf{G}$ and Borel subsets E of $\mathbf{G}$. Also, if $k = \sum_{j=0}^{\infty} k_j 2^j$ for a given integer $k \geqq 0$ where each $k_j = 0$ or 1, then

$$\psi_k(x) = \prod_{j=0}^{\infty} (-1)^{k_j x_j}$$

for $x = (x_0, x_1, \ldots) \in \mathbf{G}$ and $\psi_k(x + y) = \psi_k(x)\psi_k(y)$ for all $x, y \in \mathbf{G}$ and $k = 0, 1, \ldots$. Moreover, the ψ_k's are essentially the Walsh functions. (See Fine [3] for details.)

For each integer $j \geqq 0$ let $h^{(j)} = (h_0^{(j)}, h_1^{(j)}, \ldots) \in \mathbf{G}$ be defined by

$$h_i^{(j)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Butzer and Wagner [2] defined pointwise dyadic differentiability in the following way. Let $x \in \mathbf{G}$ and suppose f is a function on $\mathbf{G}$ defined at the points x and $x + h^{(j)}$, $j = 0, 1, \ldots$. For each integer $n \geqq 1$ set

$$d_n(f, x) = \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x + h^{(j)})).$$

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Then f is said to be *dyadically differentiable* at x if

$$df(x) := \lim_{n\to\infty} d_n(f,x)$$

exists and is finite. The function $df(x)$ is called the dyadic derivative of f at x.

Given integers $n, k \geqq 0$ let $\langle k \rangle_n$ represent the integer congruent to k modulo 2^n. Thus $\langle k \rangle_n := p$ where $0 \leqq p < 2^n$ and $k = l2^n + p$ for some integer $l \geqq 0$. Butzer and Wagner proved

$$d_n\psi_k = \langle k \rangle_n \psi_k \quad \text{for } n, k = 0, 1, \dots . \tag{1}$$

Thus each Walsh function ψ_k is everywhere dyadically differentiable and $d\psi_k = k\psi_k$ for $k = 1, 2, \dots$.

The Rademacher system is given by $\phi_n := \psi_{2^n}$, $n = 0, 1, \dots$. Onneweer [4] characterized dyadic differentiability of absolutely convergent Rademacher series on Vilenkin groups (a class of compact abelian groups which contains the dyadic group). Specializing to the dyadic group, his results can be stated as follows. If

$$f := \sum_{k=0}^{\infty} a_k \phi_k$$

converges absolutely on $\mathbf{G}$ then f is dyadically differentiable at a point $x \in \mathbf{G}$ if and only if the derived series $\sum 2^k a_k \phi_k(x)$ converges in which case

$$df(x) = \sum_{k=0}^{\infty} 2^k a_k \phi_k(x).$$

In [8] it was conjectured that this result holds for all lacunary Walsh series, not just the Rademacher ones. We shall verify this conjecture. In fact, we shall show that term by term dyadic differentiation holds for a large class of gap Walsh series whether they converge globally or not. Our techniques extend easily to any group of integers of a p-series field, but it is not clear how to proceed for Vilenkin groups of unbounded type.

2. Walsh series with Paley sets for indices

For each finite set E we shall represent the number of elements in E by $\#E$.

Let $0 < \lambda_1 < \lambda_2 < \dots$ be integers. The sequence $\boldsymbol{\lambda} = \{\lambda_j\}$ is called *lacunary* if there is a number q such that

$$\frac{\lambda_{j+1}}{\lambda_j} \geqq q > 1 \quad \text{for } j = 1, 2, \dots .$$

It is called a *Paley set* if

$$\#\{j: 2^N \leqq \lambda_j < 2^{N+1}\} = O(1) \quad \text{as } N \to \infty$$

(see Rudin [6]). It is easy to check that every lacunary sequence is a Paley set, with

$$\#\{j: 2^N \leqq \lambda_j < 2^{N+1}\} \leqq \log 2/\log q.$$

It is clear that there exist Paley sets which are not lacunary.

LEMMA 1. *A sequence of positive integers* $\boldsymbol{\lambda} = \{\lambda_j\}$ *is a Paley set if and only if*

$$\sum_{k=N}^{\infty} \frac{1}{\lambda_k} = O\left(\frac{1}{\lambda_N}\right) \qquad \textit{as} \quad N \to \infty. \tag{2}$$

PROOF. Clearly, every Paley set satisfies (2).

Conversely, if $\boldsymbol{\lambda}$ is not a Paley set then for every integer M there is an integer N such that

$$\#\{j: 2^N \leqq \lambda_j < 2^{N+1}\} > M.$$

Fix N and let j_0 be the smallest index j which satisfies $2^N \leqq \lambda_j < 2^{N+1}$. Then

$$\sum_{j=j_0}^{\infty} \frac{1}{\lambda_j} \geqq \sum_{2^N \leqq \lambda_j < 2^{N+1}} \frac{1}{\lambda_j} > \frac{m}{2^{N+1}} \geqq \frac{M}{2\lambda_{j0}}.$$

In particular, $\boldsymbol{\lambda}$ cannot satisfy (2). This completes the proof of Lemma 1.

Some authors call $\boldsymbol{\lambda}$ weakly lacunary if $\lambda_{j+1}/\lambda_j \downarrow 1$ as $j \to \infty$. Notice that if $\boldsymbol{\lambda}$ is weakly lacunary then

$$\sup_{N \geqq 0} \lambda_N \sum_{j=N}^{\infty} \frac{1}{\lambda_j} = \infty.$$

In particular, no Paley set is weakly lacunary.

Let $f \in L^1(\mathbf{G})$. Recall that *Walsh–Fourier coefficients* are defined by

$$\hat{f}(k) := \int_{\mathbf{G}} f \psi_k d\mathrm{m},$$

Walsh–Fourier series by

$$S[f] := \sum_{k=0}^{\infty} \hat{f}(k)\psi_k,$$

partial sums of $S[f]$ by

$$S_n[f] := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

and *Cesaro means* of $S[f]$ by

$$\sigma_n[f] := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \hat{f}(k)\psi_k$$

for $n = 1, 2, \ldots$.

We shall say that a Walsh series S is of *Paley type* if

$$S = \sum_{j=1}^{\infty} a_{\lambda_j} \psi_{\lambda_j}$$

for some real coefficients a_{λ_j} and some Paley set $\{\lambda_j\}$. It is known (see Fine [3]) that a lacunary Walsh–Fourier series converges if and only if its Cesaro means converge. The following result shows that this property extends to Walsh series of Paley type.

THEOREM 1. *Let $S = \sum a_k\psi_k$ be a Walsh series whose coefficients satisfy $a_k \to 0$ as $k \to \infty$. Set*

$$S_n = \sum_{k=0}^{n-1} a_k\psi_k \quad \text{and} \quad \sigma_n = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k\psi_k,$$

for $n = 1, 2, \ldots$. If S is of Paley type then $\lim_{n\to\infty} (S_n - \sigma_n) = 0$ uniformly on $\mathbf{G}$.

PROOF. Fix an integer $n > 1$ and recall that each Walsh function takes on only the values $+1$ or -1. Thus we have by hypothesis that

$$|S_n - \sigma_n| \leqq \frac{1}{n} \sum_{\lambda_j < n} \lambda_j |a_{\lambda_j}|, \tag{3}$$

for some Paley set $\{\lambda_j\}$.

Let $\varepsilon > 0$. Choose N_0 so large that $|a_{\lambda_j}| < \varepsilon$ for $j \geqq N_0$. We may suppose that $n > \lambda_{N_0}$ and choose N so large that $\lambda_N < n \leqq \lambda_{N+1}$. By (3) we have

$$|S_n - \sigma_n| \leqq \frac{1}{n} \sum_{j=1}^{N_0-1} \lambda_j |a_{\lambda_j}| + \frac{\varepsilon}{\lambda_N} \sum_{j=N_0}^{N} \lambda_j \equiv A_N + \varepsilon B_N.$$

Since N_0 is fixed, it is clear that $A_N \to 0$ as $N \to \infty$. In particular, it suffices to show there is an absolute constant C such that

$$\frac{1}{\lambda_N}\sum_{j=1}^{N}\lambda_j \leqq C \tag{4}$$

for $N > N_0$.

Fix $N \geqq 1$. Apply (2) to choose $M > 1$ such that

$$\lambda_k \leqq M\left(\sum_{j=k}^{\infty}\frac{1}{\lambda_j}\right)^{-1} \tag{5}$$

for $k = 1, 2, \ldots$. Thus

$$\sum_{j=N}^{\infty}\frac{1}{\lambda_j} = \sum_{j=N-1}^{\infty}\frac{1}{\lambda_j} - \frac{1}{\lambda_{N-1}} \leqq \left(1 - \frac{1}{M}\right)\sum_{N-1}^{\infty}\frac{1}{\lambda_j}.$$

In fact, an easy induction argument establishes

$$\sum_{j=N}^{\infty}\frac{1}{\lambda_j} \leqq r^{N-k}\sum_{j=k}^{\infty}\frac{1}{\lambda_j} \tag{6}$$

for $k = 1, 2, \ldots, N$ and $r = 1 - 1/M$. Combining (5) and (6) we obtain

$$\frac{1}{\lambda_N}\sum_{k=1}^{N}\lambda_k \leqq \sum_{j=N}^{\infty}\frac{1}{\lambda_j}\left(\sum_{k=1}^{N}\lambda_k\right) \leqq \sum_{k=1}^{N}\sum_{j=N}^{\infty}\frac{M}{\lambda_j}\left(\sum_{j=k}^{\infty}\frac{1}{\lambda_j}\right)^{-1} \leqq M\sum_{k=1}^{N}r^{N-k} \leqq M\sum_{\ell=0}^{\infty}r^{\ell}.$$

Since $0 < r < 1$ we conclude that (4) holds for $C = M/(1-r)$. This completes the proof of Theorem 1.

We notice that Theorem 1 holds for any bounded orthonormal system in place of the Walsh system. In particular, in spite of the fact that I cannot find it in Zygmund [9], Theorem 1 holds for trigonometric Fourier series.

Let $C(\mathbf{G})$ represent the space of functions continuous on the group $\mathbf{G}$.

COROLLARY 1. *If $S[f]$ is of Paley type then $S[f]$ converges a.e. when $f \in L^1(\mathbf{G})$ and converges uniformly when $f \in C(\mathbf{G})$.*

Recall that $\mathrm{Lip}(1, \mathbf{G})$ is the collection of functions $f \in C(\mathbf{G})$ such that

$$\sup_{|h|\leqq\delta}\sup_{x\in\mathbf{G}}|f(x+h) - f(x)| = O(\delta)$$

as $\delta \to 0$. Fine [3] proved that $\hat{f}(k) = O(1/k)$ as $k \to \infty$ for all $f \in$ $\in \mathrm{Lip}(1, \mathbf{G})$. The following result shows that when $S[f]$ is of Paley type, this growth condition characterizes $\mathrm{Lip}(1, \mathbf{G})$.

THEOREM 2. *Suppose $f \in C(\mathbf{G})$ has a Walsh–Fourier series of Paley type. Then $f \in \mathrm{Lip}(1, \mathbf{G})$ if and only if $\hat{f}(k) = O(1/k)$ as $k \to \infty$.*

PROOF. Choose $M > 0$ such that

$$|\hat{f}(k)| \leqq \frac{M}{k} \tag{7}$$

for $k = 0, 1, \ldots$. Let $\delta > 0$ and choose an integer n so large that $2^{1-n} \leqq \delta < < 2^{-n}$. Since Walsh functions are locally constant, it is clear that $k < 2^n$ and $|h| \leqq \delta$ imply $\psi_k(x + h) = \psi_k(x)$. Thus we can use Corollary 1 to write

$$f(x+h) - f(x) = \sum_{\lambda_j \geqq 2^n} \hat{f}(\lambda_j)(\psi_{\lambda_j}(x+h) - \psi_{\lambda_j}(x)) \tag{8}$$

for all $|h| \leqq \delta$, $x, h \in \mathbf{G}$, and some Paley set $\boldsymbol{\lambda} = \{\lambda_j\}$.

Choose N so that $\lambda_{N-1} < 2^n \leqq \lambda_N$. By (7) and (8) we have

$$|f(x+h) - f(x)| \leqq 2M \sum_{j=N}^{\infty} \frac{1}{\lambda_j}$$

for all $|h| \leqq \delta$, $x, h \in \mathbf{G}$. It follows from (2) and the choice of n that

$$\sup_{|h| \leqq \delta} |f(x+h) - f(x)| \leqq \frac{M'}{\lambda_N} \leqq M' 2^{-n} \leqq \frac{M'}{2}\delta$$

for some absolute constant M'. We conclude that $f \in \mathrm{Lip}(1, \mathbf{G})$.

(Note: This same proof establishes the trigonometric analogue of Theorem 2.)

Since the coefficients of a term by term dyadically differentiable Walsh series satisfy $a_k = o(1/k)$ as $k \to \infty$ we see that the condition of term by term differentiability of a Walsh series of Paley type is much stronger than a Lipschitz condition.

3. Term by term dyadic differentiation

The following result was proved in [5].

LEMMA 2. *Let $b_k^{(n)}$, b_k, and x_k be real numbers for $n, k = 0, 1, \ldots$, and suppose $\sum_{k=0}^{\infty} x_k$ converges to a finite real number.*

(i) *If $\sum_{k=0}^{\infty} |b_k - b_{k+1}| < \infty$ then $\sum_{k=0}^{\infty} b_k x_k$ converges to a finite real number.*

(ii) *If*

$$\sum_{k=0}^{\infty} |b_k^{(n)} - b_{k+1}^{(n)}| \leqq M < \infty$$

for $n = 0, 1, \ldots$ *and* $b_k^{(n)} \to b_k$ *as* $n \to \infty$ *then*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} b_k^{(n)} x_k = \sum_{k=0}^{\infty} b_k x_k$$

exist and are finite.

We shall use this result to obtain a local criterion for term by term dyadic differentiation of Walsh series of Paley type.

THEOREM 3. *Suppose* $\boldsymbol{\lambda} = \{\lambda_j\}$ *is a Paley set and* $a_0, a_1, \ldots$ *is a sequence of real numbers such that* $a_k \neq 0$ *if and only if* $k = \lambda_j$ *for some* $j \geqq 1$. *Let* $x \in \mathbf{G}$ *and suppose* $\sum_{k=0}^{\infty} k a_k \psi_k(x)$ *converges to a finite real number. Then*

$$f := \sum_{k=0}^{\infty} a_k \psi_k$$

exists and is finite at x *and* $x + h^{(n)}$ *for* $n = 0, 1, \ldots,$ *is dyadically differentiable at* x, *and*

$$df(x) = \sum_{k=0}^{\infty} k a_k \psi_k(x).$$

PROOF. Fix $n \geqq 0$ and define a sequence $b_0, b_1, \ldots$ by

$$b_k = \begin{cases} \psi_{\lambda_j}(h^{(n)})/\lambda_j & \text{if } k = \lambda_j \text{ for some } j \\ 0 & \text{if } k \neq \lambda_j \text{ for all } j. \end{cases}$$

Since $\psi_k(x + h^{(n)}) = \psi_k(x)\psi_k(h^{(n)})$ it is clear that

$$\sum_{k=0}^{\infty} a_k \psi_k(x + h^{(n)}) = \sum_{k=0}^{\infty} k a_k b_k \psi_k(x).$$

Moreover, we have by (2) that

$$\sum_{k=0}^{\infty} |b_k| \leqq \sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty.$$

Hence the sequence $\{b_k\}$ is of bounded variation and it follows from Lemma 2 i) that $f(x + h^{(n)})$ exists and is finite. A similar argument shows $f(x)$ is also defined. Hence $d_n(f, x)$ makes sense.

Use (1) to write

$$d_n(f, x) = \sum_{k=0}^{\infty} \langle k \rangle_n a_k \psi_k(x) = \sum_{k=0}^{\infty} \frac{\langle k \rangle_n}{k} k a_k \psi_k(x).$$

Since $\langle k\rangle_n \to k$ as $n \to \infty$ it suffices by Lemma 2 ii) to find an absolute constant M such that

$$A_n := \sum_{j=1}^{\infty} \left| \frac{\langle \lambda_j \rangle_n}{\lambda_j} - \frac{\langle \lambda_{j+1} \rangle_n}{\lambda_{j+1}} \right| \leqq M$$

for $n = 1, 2, \ldots$.

Suppose n is large enough so that $\lambda_{N-1} < 2^n \leqq \lambda_N$ for some $N \geqq 2$. Since $\langle k\rangle_n = k$ for $k < 2^n$ we have

$$\left| \frac{\langle \lambda_j \rangle_n}{\lambda_j} - \frac{\langle \lambda_{j+1} \rangle_n}{\lambda_{j+1}} \right| \equiv 0$$

for $j < N$. Consequently,

$$A_n = \sum_{j=N}^{\infty} \left| \frac{\langle \lambda_j \rangle_n}{\lambda_j} - \frac{\langle \lambda_{j+1} \rangle_n}{\lambda_{j+1}} \right| \leqq \frac{\langle \lambda_N \rangle_n}{\lambda_N} + 2 \sum_{j=N+1}^{\infty} \frac{\langle \lambda_j \rangle_n}{\lambda_j} \leqq$$

$$\leqq 1 + 2(2^n) \sum_{j=N+1}^{\infty} \frac{1}{\lambda_j} \leqq 1 + 2\lambda_N \sum_{j=N+1}^{\infty} \frac{1}{\lambda_j}.$$

In particular, (2) implies A_n is uniformly bounded in n and the proof of Theorem 3 is complete.

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CONVOLUTION PROCESSES OF FEJÉR TYPE AND THE DIVERGENCE ALMOST EVERYWHERE OF A POINTWISE COMPARISON

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1. Introduction

The present paper is concerned with negative results to the pointwise comparison within a class of convolution processes, previously employed by Shapiro (see [8, Chapter 5]).

To this end let $C_{2\pi} = C_{2\pi}(\mathbf{R}^N)$ be the Banach space of functions f, defined and continuous on the N-dimensional Euclidean space $\mathbf{R}^N$, 2π-periodic in each variable, endowed with the usual norm $\|f\| := \max\{|f(x)| : x \in \mathbf{R}^N\}$. Let $\mathcal{M}_0 = \mathcal{M}_0(\mathbf{R}^N)$ be the class of real-valued, bounded measures μ on $\mathbf{R}^N$, satisfying $\left(\langle v, u\rangle := \sum_{i=1}^{N} v_i u_i\right)$

$$(1.1)\qquad \mu^\wedge(0) = 0, \qquad \mu^\wedge(v) := (2\pi)^{-N/2} \int_{\mathbf{R}^N} e^{-i\langle v,u\rangle} d\mu(u).$$

For $\mu \in \mathcal{M}_0$ consider the convolution process of Fejér type, for $n \in \mathbf{N}$ (set of natural numbers) given by

(1.2)

$$T_n^\mu f(x) := (2\pi)^{-N/2} \int_{\mathbf{R}^n} f(x - u/n) d\mu(u) = (2\pi)^{-N/2} \int_{\mathbf{R}^N} f(x-u) d\mu_n(u),$$

where $\mu_n \in \mathcal{M}_0$ is generated by μ via its Fourier transform $\mu^\wedge{}_n(v) := \mu^\wedge \wedge(v/n)$. Note that the normalization (1.1) takes care of the fact that in the applications the sequence $\{\mu_n\}$ will represent the remainders of processes, approximating the identity.

Concerning the uniform comparison of two processes Shapiro (see [8, p. 78], also [1, p. 494]) has given a sufficient condition in terms of the global divisibility of the Fourier transforms of the generating measures $\mu, \nu \in \mathcal{M}_0$.

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Indeed, if $\nu^\wedge$ divides $\mu^\wedge$ globally, i.e., there exists a bounded measure λ such that $\mu^\wedge(v) = \lambda^\wedge(v)\nu^\wedge(v)$ for all $v \in \mathbf{R}^N$, then for $f \in C_{2\pi}$

$$\|T_n^\mu f\| \leqq M \cdot \|T_n^\nu f\| \tag{1.3}$$

with constant M, independent of f and $n \in \mathbf{N}$. Shapiro further remarks (see [8, p. 120]): 'Another interesting area for study is how far *pointwise* (rather than norm) approximation theorems can be inferred from the Fourier transform of the kernel.' In this connection a pointwise interpretation of (1.3) in the sense that

$$|T_n^\mu f(x)| \leqq M \cdot |T_n^\nu f(x)| \tag{1.4}$$

for many points $x \in \mathbf{R}^N$ will certainly be possible for smooth functions, for example, for polynomials (cf. (1.6)). But for functions which are less smooth, the situation is quite different. In fact, Theorem 3.3 establishes the existence of counterexamples $f_0 \in C_{2\pi}$, even belonging to some Lipschitz class, such that

$$\limsup_{n\to\infty} \frac{|T_n^\mu f_0(x)|}{|T_n^\nu f_0(x)|} = \infty \tag{1.5}$$

holds true almost everywhere (for counterexamples, satisfying (1.5) at one, prescribed point, see [4]). Apart from the linear independence, the condition upon the Fourier transforms of the generating measures will be that they have the same asymptotic behaviour at the origin in the sense that for some $r \in \mathbf{N}$

$$\mu^\wedge(k/n) = O_k(n^{-r}), \qquad \nu^\wedge(k/n) = O_k(n^{-r}) \qquad (n \to \infty) \tag{1.6}$$

for each $k \in \mathbf{Z}^N$, the set of integral lattice points. This then extends the result, obtained in [3] for the special case of one-dimensional Fejér and Abel–Poisson means.

In Section 2 some ingredients are prepared, needed for the proof of divergence assertions of type (1.5). These include a lemma of A. P. Calderón (Lemma 2.1) and a quantitative resonance principle (Lemma 2.3), dealing with the comparison of two families of sublinear functionals, depending upon rather arbitrary index sets. Section 3 starts with the aforementioned result for periodic functions (cf. Section 3.1), but the problem will also be discussed in the space $C_0 = C_0(\mathbf{R}^N)$ of functions f, continuous on $\mathbf{R}^N$, vanishing at infinity, i.e., $\lim_{|x|\to\infty} f(x) = 0$ (cf. Section 3.2). These general results are then applied in Section 4 to some specific examples, establishing, e.g., the divergence almost everywhere on $\mathbf{R}^N$ of a pointwise comparison of the convolution processes of Bochner–Riesz and Gauss–Weierstrass.

2. Preliminaries

A basic tool will be the following lemma of A. P. Calderón. For a proof see [9; 11, p. 165].

LEMMA 2.1. *Let* $\{H_k\}$ *be a sequence of measurable subsets of* $\mathbf{R}^N$, 2π-*periodic in each component such that with* N*-dimensional Lebesgue measure* λ_N

$$\sum_{k=1}^{\infty} \lambda_N(H_k \cap [-\pi,\pi]^N) = \infty. \tag{2.1}$$

Then there exist points $\{x_k\} \subset \mathbf{R}^N$ *such that*

$$H := \limsup_{k\to\infty}(H_k - x_k) := \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x - x_k \colon x \in H_k\}$$

is a set of full measure, i.e., $\lambda_N(\mathbf{R}^N \setminus H) = 0$. *Hence almost every point of* $\mathbf{R}^N$ *belongs to infinitely many sets* $H_k - x_k$.

To verify (2.1), the next lemma outlines a situation where this can be reduced to a one-dimensional problem.

LEMMA 2.2. *Let* f, g *be real-valued functions,* 2π*-periodic and continuous on* $\mathbf{R}$. *If*

$$\lambda_1(\{s \in [-\pi,\pi] \colon f(s) \leqq a_1, g(s) \geqq a_2\}) = a_3 \tag{2.2}$$

for some constants $a_1, a_2, a_3 \in \mathbf{R}$, *then*

$$L := \lambda_N\left(\{\tilde{x} \in [-\pi,\pi]^N \colon f(\langle \tilde{t},\tilde{x}\rangle) \leqq a_1, g(\langle \tilde{t},\tilde{x}\rangle) \geqq a_2\}\right) = (2\pi)^{N-1}\cdot a_3$$

for each $\tilde{t} = (\tilde{t}_1,\dots,\tilde{t}_N) \in \mathbf{Z}^N \setminus \{0\}$.

PROOF. Given $\tilde{t} \in \mathbf{Z}^n \setminus \{0\}$, let $t = (t_1,\dots,t_j) = (\tilde{t}_{k_1},\dots,\tilde{t}_{k_j}) \in \mathbf{Z}^j$ be the vector, determined by the nonzero components $\tilde{t}_{k_i}$ of $\tilde{t}$. Correspondingly for $\tilde{x} \in \mathbf{R}^N$ let $x = (x_1,\dots,x_j) = (\tilde{x}_{k_1},\dots,\tilde{x}_{k_j})$. Then $\langle t,x\rangle = \langle \tilde{t},\tilde{x}\rangle$ for every $\tilde{x} \in \mathbf{R}^N$. It follows that

$$L = (2\pi)^{N-j}\int_A d(x_1,\dots,x_j),\quad A := \{x \in [-\pi,\pi]^j \colon f(\langle t,x\rangle) \leqq a_1, g(\langle t,x\rangle) \geqq a_2\}$$

Substitute $x = h(z) = h(z_1,\dots,z_j) = ((z_1 - z_2 - \ldots - z_j)/t_1, z_2/t_2,\dots,z_j/t_j)$, thus $\det(h'(z)) = \prod_{i=1}^{j} t_i^{-1} \neq 0$, and consider the sets

$$A_1 := \{z_1 \in [-t_1\pi + (z_2 + \ldots + z_j), t_1\pi + (z_2 + \ldots + z_j)] \colon f(z_1) \leqq a_1, g(z_1) \geqq a_2\},$$
$$A_2 := \{(z_2/t_2,\dots,z_j/t_j) \in [-\pi,\pi]^{j-1}\}.$$

Since $t_1 \in \mathbf{Z} \setminus \{0\}$ and since the functions f, g are 2π-periodic, one then has by Fubini's theorem and (2.2) that

$$L = (2\pi)^{N-j} \left| \prod_{i=1}^{j} t_i \right|^{-1} \int_{A_2} \int_{A_1} dz_1 d(z_2, \dots, z_j) =$$

$$= (2\pi)^{N-j} \left| \prod_{i=1}^{j} t_i \right|^{-1} t_1 a_3 \int_{A_2} d(z_2, \dots, z_j) = (2\pi)^{N-1} \cdot a_3. \qquad \square$$

The divergence assertions mentioned result from an application of the following quantitative resonance principle. For a Banach space X (with norm $\|\cdot\|$) let X^* be the set of nonnegative, sublinear, and bounded functionals T on X, i.e., T maps X into $[0, \infty)$ such that for all $f, g \in X$ and scalars a

$$T(f+g) \leqq Tf + Tg, \quad T(af) = |a| Tf, \quad \|T\|_{X^*} := \sup\{Tf \colon \|f\| \leqq 1\} < \infty.$$

Let ω be an abstract modulus of continuity, i.e. a function, continuous on $[0, \infty)$ with

$$(2.3) \qquad 0 = \omega(0) < \omega(s) \leqq \omega(s+t) \leqq \omega(s) + \omega(t) \qquad (s, t > 0),$$

additionally satisfying

$$(2.4) \qquad \lim_{t \to 0+} \omega(t)/t = \infty.$$

Note that (2.3) implies

$$(2.5) \qquad \omega(s)/s \leqq 2\omega(t)/t \qquad (0 < t \leqq s).$$

Let $\sigma(t)$ be a function, (strictly) positive on $(0, \infty)$, and $\{\varphi_n\}$ be a sequence, (strictly) decreasing with $\lim_{n \to \infty} \varphi_n = 0$.

LEMMA 2.3. *Let A, B be arbitrary index sets. Suppose that for families of functionals $\{U_t \colon t \in (0, \infty)\}$, $\{V_{n,\alpha} \colon n \in \mathbf{N}, \alpha \in A\}$, $\{W_{n,\alpha} \colon n \in \mathbf{N}, \alpha \in A\} \subset X^*$ with*

$$(2.6) \qquad \|V_{n,\alpha}\|_{X^*} + \|W_{n,\alpha}\|_{X^*} \leqq C_1 \qquad (n \in \mathbf{N}, \alpha \in A)$$

there exist test elements $\{g_{n,\beta} \colon n \in \mathbf{N}, \beta \in B\} \subset X$ such that

$$(2.7) \qquad \|g_{n,\beta}\| \leqq C_2 \qquad (n \in \mathbf{N}, \beta \in B),$$

$$(2.8) \qquad U_t g_{n,\beta} \leqq C_3 \min\{1, \sigma(t)/\varphi_n\} \qquad (t \in (0, \infty), n \in \mathbf{N}, \beta \in B),$$

$$V_{n,\alpha}g_{j,\beta} + W_{n,\alpha}g_{j,\beta} = O_j(\varphi_n) \qquad (\alpha \in A, \beta \in B, n \to \infty). \tag{2.9}$$

Moreover, for each subsequence $\{n_j\} \subset \mathrm{N}$ *let there exist a sequence* $\{M_k\}$ *of subsets of* A *(more exactly* $M_{\{n_j\},k}$*), a sequence of points* $\{\beta_k\} \subset B$*, a family of sequences* $\{\{\varepsilon_{\alpha,k}\} : \alpha \in A\}$ *with* $\lim_{k\to\infty} \varepsilon_{\alpha,k} = 0$*, and a constant* $C_4 > 0$ *such that for* $\alpha \in M_k$

$$V_{n_k,\alpha}g_{n_k,\beta_k} \geqq C_4 - \varepsilon_{\alpha,k}, \tag{2.10}$$

$$W_{n_k,\alpha}g_{n_k,\beta_k} \leqq \varepsilon_{\alpha,k}. \tag{2.11}$$

Then for each modulus ω *satisfying* (2.4) *there exist a subsequence* $\{n_j\}$ *and a counterexample* $f_\omega \in X$ *with*

$$U_t f_\omega = O(\omega(\sigma(t))) \qquad (t \to 0+), \tag{2.12}$$

$$V_{n,\alpha} f_\omega \neq o(\omega(\varphi_n)), \tag{2.13}$$

$$V_{n,\alpha} f_\omega \neq O(W_{n,\alpha} f_\omega) \qquad (n \to \infty), \tag{2.14}$$

simultaneously for each $\alpha \in M_{\{n_j\}} := \limsup_{k\to\infty} M_{\{n_j\},k}$.

Thus the negative result (2.14) on the comparison of the processes $\{V_{n,\alpha}\}$, $\{W_{n,\alpha}\}$ is given in quantitative terms inasmuch as (2.12) assures a certain smoothness of the counterexample f_ω, whereas (2.13) may be interpreted as a precision of its nonsmoothness.

PROOF. Let us proceed via a suitable quantitative gliding hump method (cf. [2; 5; 6] and the literature cited there). Starting with $n_1 = 1$, in view of the properties of ω and $\{\varphi_n\}$ one may successively select a strictly increasing subsequence $\{n_k\} \subset \mathrm{N}$ such that for $k \geqq 2$

$$\omega(\varphi_{n_k}) \leqq \omega(\varphi_{n_{k-1}})/k^2, \tag{2.15}$$

$$\sum_{j=1}^{k-1} \omega(\varphi_{n_j}) \max\{1/\varphi_{n_j}, C_{5,n_j}\} \leqq \omega(\varphi_{n_k})/k^2 \varphi_{n_k}, \tag{2.16}$$

where the constants C_{5,n_j} result from (2.9) (and (2.6,7)). For this subsequence $\{n_k\} \subset \mathrm{N}$ there exist, by assumption, sets $M_k \subset A$ and points $\beta_k \in B$ such that (2.10,11) hold true. Consider the candidate

$$f_\omega := \sum_{j=1}^{\infty} \omega(\varphi_{n_j}) g_{n_j,\beta_j}.$$

By (2.7,15) it follows that

$$\sum_{j=1}^{\infty}\|\omega(\varphi_{n_j})g_{n_j,\beta_j}\| \leqq C_2\sum_{j=1}^{\infty}\omega(\varphi_{n_j}) \leqq C_2\omega(\varphi_1)\sum_{j=1}^{\infty}2^{-j+1} < \infty.$$

Since X is complete, f_ω is well-defined as an element of X. Moreover, f_ω satisfies (2.12). Indeed, suppose that $t \in (0,\infty)$ is such that $\sigma(t) \leqq \varphi_1$. Then there exists $k \in \mathbf{N}$ with $\varphi_{n_{k+1}} < \sigma(t) \leqq \varphi_{n_k}$, and it follows by (2.5,8,15,16) that

$$U_t f_\omega \leqq \left(\sum_{j=1}^{k} + \sum_{j=k+1}^{\infty}\right)\omega(\varphi_{n_j})U_t g_{n_j,\beta_j} \leqq$$

$$\leqq C_3\sigma(t)\sum_{j=1}^{k}\omega(\varphi_{n_j})/\varphi_{n_j} + C_3\sum_{j=k+1}^{\infty}\omega(\varphi_{n_j}) \leqq$$

$$\leqq 2C_3\sigma(t)\omega(\varphi_{n_k})/\varphi_{n_k} + 2C_3\omega(\varphi_{n_{k+1}}) \leqq 4C_3\omega(\sigma(t)) + 2C_3\omega(\sigma(t)) = 6C_3\omega(\sigma(t)).$$

If $t \in (0,\infty)$ is such that $\sigma(t) > \varphi_1$, then by (2.8,15)

$$U_t f_\omega \leqq C_3\sum_{j=1}^{\infty}\omega(\varphi_{n_j}) \leqq 2C_3\omega(\varphi_1) \leqq 2C_3\omega(\sigma(t)),$$

thus in any case (2.12). Now let $\alpha \in M_{\{n_j\}}$, i.e., $\alpha \in M_k$ for infinitely many $k \in \mathbf{N}$. For these k it follows by (2.6,7,9,10,15,16) that

$$(2.17)\quad V_{n_k,\alpha}f_\omega \geqq \omega(\varphi_{n_k})V_{n_k,\alpha}g_{n_k,\beta_k} - \left(\sum_{j=1}^{k-1} + \sum_{j=k+1}^{\infty}\right)\omega(\varphi_{n_j})V_{n_k,\alpha}g_{n_j,\beta_j} \geqq$$

$$\geqq \omega(\varphi_{n_k})(C_4 - \varepsilon_{\alpha,k}) - \sum_{j=1}^{k-1}\omega(\varphi_{n_j})C_{5,n_j}\varphi_{n_k} - \sum_{j=k+1}^{\infty}\omega(\varphi_{n_j})C_1C_2 \geqq$$

$$\geqq \omega(\varphi_{n_k})[C_4 - o_\alpha(1) - (1 + 2C_1C_2)/k^2] = \omega(\varphi_{n_k})[C_4 - o_\alpha(1)].$$

Analogously one obtains, now by (2.11) instead of (2.10), that

$$(2.18)\qquad W_{n_k,\alpha}f_\omega = o_\alpha(\omega(\varphi_{n_k})).$$

Obviously, (2.17,18) yield the assertions (2.13,14). □

3. Divergence almost everywhere of a pointwise comparison

Let $\mu \in \mathcal{M}_0(\mathbf{R}^N)$ and $n \in \mathbf{N}$. If X denotes one of the spaces $C_{2\pi}$ or C_0, then T_n^μ (cf. (1.2)) is a bounded linear operator of X into itself with norm $\|T_n^\mu\| = \|\mu\|_{\mathcal{M}}$ (total variation of μ) and

$$(3.1)\qquad T_n^\mu(e^{i\langle v,\cdot\rangle})(x) = \mu^\wedge(v/n)e^{i\langle v,x\rangle} \qquad (v,x \in \mathbf{R}^N).$$

For an abstract modulus of continuity ω and $r \in \mathbf{N}$ let r-th (radial) Lipschitz classes be defined by

$$\mathrm{Lip}_r(\omega, X) := \{f \in X : \omega_r(f,t) = O(\omega(t^r)), t \to 0+\},$$
$$\omega_r(f,t) := \sup\left\{\left\|\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f(x+kh)\right\| : h \in \mathbf{R}^N, |h| \leqq t\right\}.$$

Since Lemma 2.3 yields counterexamples for which (2.14) holds true simultaneously on the limsup of certain sets M_k, the problem is to find, for each choice of a subsequence $\{n_k\} \subset \mathbf{N}$, sets M_k and points β_k such that the limsup is of full measure. Using Lemma 2.1 the following constructs appropriate sets M_k and points β_k for testelements $g_{n,\beta}$ to be chosen later.

LEMMA 3.1. *Let $a, b, c \in \mathbf{Z}^N$ with $a \neq b$, $c \neq 0$, and let $\{n_k\} \subset \mathbf{N}$ be an arbitrary subsequence. There exist points $\{x_k\} \subset \mathbf{R}^N$ such that*

(a) $H_i := \limsup_{k\to\infty}(H_{i,k} + x_k)$ *for each $i = 1, 2$ is a set of full measure, where*

$$H_{1,k} := \left\{x \in \mathbf{R}^N : |\sin n_k\langle c, x\rangle| \geqq \frac{1}{2}\right\}, \quad H_{2,k} := \left\{x \in \mathbf{R}^N : |\sin n_k\langle c, x\rangle| \leqq \frac{1}{k}\right\};$$

(b) $H^s := \limsup_{k\to\infty}(H_k^s + x_k)$ *or* $H^c := \limsup_{k\to\infty}(H_k^c + x_k)$ *is a set of full measure, where*

$$H_k^s := \left\{x \in \mathbf{R}^N : \left|\sin \frac{1}{2} n_k\langle a-b, x\rangle\right| \leqq \frac{1}{k}, |\sin n_k\langle b, x\rangle| \geqq \frac{1}{2}\right\},$$
$$H_k^c := \left\{x \in \mathbf{R}^N : \left|\sin \frac{1}{2} n_k\langle a-b, x\rangle\right| \leqq \frac{1}{k}, |\cos n_k\langle b, x\rangle| \geqq \frac{1}{2}\right\}.$$

PROOF. (a) Consider the one-dimensional sets

$$H_k^1 := \left\{s \in \mathbf{R} : |\sin n_k s| \geqq \frac{1}{2}\right\}, \quad H_k^2 := \left\{s \in \mathbf{R} : |\sin n_k s| \leqq \frac{1}{k}\right\}.$$

For $k \in \mathbf{N}$ one has $\lambda_1(H_k^1 \cap [-\pi,\pi]) = 4\pi/3$, $\lambda_1(H_k^2 \cap [-\pi,\pi]) \geqq 4/k$. Setting $a_3 \geqq 4/k$, $\tilde{t} = c$ as well as $a_1 = 0$, $a_2 = 1/2$, $f(s) = 0$, $g(s) = |\sin n_k s|$ or $a_1 = 1/k$, $a_2 = 0$, $f(s) = |\sin n_k s|$, $g(s) = 0$, respectively, by Lemma 2.2 one obtains that for $i = 1, 2$

$$\lambda_N(H_{i,k} \cap [-\pi,\pi]^N) \geqq (2\pi)^{N-1}\frac{4}{k}. \tag{3.2}$$

Thus Lemma 2.1 yields the assertion.

(b) For $c = a - b \neq 0$, $\tilde{n}_k = n_k/2$ one has $H_{2,k} = H_k^s \cup H_k^c$. By (3.2) one obtains (note that the sets are nevertheless 2π-periodic)

$$(2\pi)^{N-1}\frac{4}{k} \leqq \lambda_N(H_k^s \cap [-\pi,\pi]^N) + \lambda_N(H_k^c \cap [-\pi,\pi]^N)$$

so that the assumptions of Lemma 2.1 are fulfilled for at least one of the families $\{H_k^s\}$ or $\{H_k^c\}$. □

Turning to the pointwise comparison of two convolution processes of Fejér type, it is clear that, if the generating measures μ and ν are linearly dependent, i.e., $\mu = A \cdot \nu$ for some $A \in \mathbf{C}$, there cannot exist functions f_0 such that assertions of type (1.5) hold true. So one has to exclude this case which is characterized by (with $\mathbf{Q} \subset \mathbf{R}$, the set of rational numbers)

LEMMA 3.2. *Let $\mu, \nu \in \mathcal{M}_0(\mathbf{R}^N)$ with $\nu^\wedge \neq 0$ and*

$$\mu^\wedge(p)\nu^\wedge(q) = \mu^\wedge(q)\nu^\wedge(p) \qquad (p, q \in \mathbf{Q}^N). \tag{3.3}$$

Then there exists a constant $A \in \mathbf{C}$ such that $\mu^\wedge = A \cdot \nu^\wedge$.

PROOF. Since the continuous function $\nu^\wedge$ does not vanish identically, there exists $p_0 \in \mathbf{Q}^N$ such that $\nu^\wedge(p_0) \neq 0$. Then by (3.3)

$$\mu^\wedge(q) = [\mu^\wedge(p_0)/\nu^\wedge(p_0)]\nu^\wedge(q) =: A \cdot \nu^\wedge(q) \qquad (q \in \mathbf{Q}^N). \qquad \square$$

3.1. Periodic counterexamples.

THEOREM 3.3. *Let $\mu, \nu \in \mathcal{M}_0(\mathbf{R}^N)$ be linearly independent such that* (1.6) *holds true for some $r \in \mathbf{N}$. Then for each modulus ω satisfying* (2.4) *there exists a (real-valued) counterexample $f_\omega \in \mathrm{Lip}_r(\omega, C_{2\pi}(\mathbf{R}^N))$ such that*

$$|T_n^\mu f_\omega(x)| \neq o(\omega(n^{-r})), \tag{3.4}$$

$$|T_n^\mu f_\omega(x)| \neq O(|T_n^\nu f_\omega(x)|) \qquad (n \to \infty) \tag{3.5}$$

for almost every $x \in \mathbf{R}^N$.

PROOF. If necessary, we will represent the complex values $\mu^\wedge(a), \nu^\wedge(a) \in \in \mathbf{C} \setminus \{0\}$ in polar coordinates via $\mu^\wedge(a) = |\mu^\wedge(a)|e^{i\vartheta_a}$, $\nu^\wedge(a) = |\nu^\wedge(a)|e^{i\Theta_a}$ with $\vartheta_a, \Theta_a \in [0, 2\pi)$, respectively. Let $0 \neq a = a_1/a_2 \in \mathbf{Q}^N$ ($a_1 \in \mathbf{Z}^N$, $a_2 \in \mathbf{N}$) with $\mu^\wedge(a) \neq 0$. Since $\mu^\wedge(v) = \overline{\mu^\wedge(-v)}$ ($\overline{z}$ = complex conjugate of $z \in \mathbf{C}$), in view of (3.1) one obtains for arbitrary $\vartheta \in \mathbf{R}$

$$T_{a_2}^\mu(\sin(\langle a_1, \cdot\rangle - \vartheta))(x) = |\mu^\wedge(a)| \sin(\langle a_1, x\rangle - \vartheta + \vartheta_a). \tag{3.6}$$

Obviously, the same relation holds true for ν with Θ_a instead of ϑ_a.

Let us first look at the case that μ, ν are such that for some $p = p_1/p_2 \in \in \mathbf{Q}^N$ ($0 \neq p_1 \in \mathbf{Z}^N$, $p_2 \in \mathbf{N}$) one has $\mu^\wedge(p) \neq 0$ but $\nu^\wedge(p) = 0$. To apply Lemma 2.3 set

$$X = C_{2\pi}(\mathbf{R}^N), \quad A = B = \mathbf{R}^N, \quad \varphi_n = n^{-r}, \quad \sigma(t) = t^r, \quad U_t f = \omega_r(f, t),$$
$$V_{n,x} f = |T_{np_2}^\mu f(x)|, \quad W_{n,x} f = |T_{np_2}^\nu f(x)|, \quad g_{n,y}(x) = \sin(n\langle p_1, x - y\rangle - \vartheta_p).$$

Obviously, (2.6,7) are fulfilled with $C_1 = \|\mu\|_{\mathcal{M}} + \|\nu\|_{\mathcal{M}}$, $C_2 = 1$. In view of (1.6,3.6) one obtains (2.9), whereas (2.8) is valid since

$$\omega_r(g_{n,y}, t) \leqq \min\Big\{2^r \|g_{n,y}\|, t^r \sum_{|\tau|=r} \|D^\tau g_{n,y}\|\Big\}. \tag{3.7}$$

Now let $\{n_k\} \subset \mathbf{N}$ be an arbitrary subsequence. By Lemma 3.1 (a) for $c = p_1$ there exist points $\{x_k\} \subset \mathbf{R}^N$ such that $H_1 := \limsup_{k\to\infty}(H_{1,k} + x_k)$ is a set of full measure. In view of (3.6) and the definition of $H_{1,k}$ it follows for $x \in H_{1,k} + x_k$ that

$$V_{n_k,x} g_{n_k,x_k} = |\mu^\wedge(p)|\,|\sin n_k\langle p_1, x - x_k\rangle| \geqq \frac{1}{2}|\mu^\wedge(p)|, \quad W_{n_k,x} g_{n_k,x_k} \leqq |\nu^\wedge(p)| = 0.$$

Thus assumptions (2.10,11) are fulfilled, too, and Lemma 2.3 yields the assertion in this special case.

It remains the case of measures μ, ν for which $\nu^\wedge(a) = 0$ for $a \in \mathbf{Q}^N$ necessarily implies $\mu^\wedge(a) = 0$. Note that then $\nu^\wedge \neq 0$ since μ, ν are linearly independent. We first exclude the situation that for each $a \in \mathbf{Q}^N$ where $\mu^\wedge(a) \neq 0$ (hence $\nu^\wedge(a) \neq 0$) there exists $\varepsilon_a \in \{0, 1\}$ with

$$\Theta_a = \vartheta_a + \varepsilon_a \pi \qquad (\mathrm{mod}\ 2\pi). \tag{3.8}$$

Thus, since $\mu^\wedge$ does not vanish identically, there exists $p = p_1/p_2 \in \mathbf{Q}^N$ such that $\mu^\wedge(p) \neq 0$ and for $\varepsilon_p \in \{0, 1\}$

$$\Theta_p \neq \vartheta_p + \varepsilon_p \pi \qquad (\mathrm{mod}\ 2\pi). \tag{3.9}$$

Setting as above but with $g_{n,y}(x) = \sin(n\langle p_1, x - y\rangle - \Theta_p)$, for any subsequence $\{n_k\} \subset \mathbf{N}$ one obtains the existence of points $\{x_k\} \subset \mathbf{R}^N$ such that H_2 is a set of full measure (cf. Lemma 3.1 (a)). Now, if $x \in H_{2,k} + x_k$, then

$$\begin{aligned} V_{n_k,x} g_{n_k,x_k} &= |\mu^\wedge(p)|\,|\sin(n_k\langle p_1, x - x_k\rangle - \Theta_p + \vartheta_p)| \geqq \\ &\geqq |\mu^\wedge(p)|\,[|\sin(\vartheta_p - \Theta_p)|\,|\cos n_k\langle p_1, x - x_k\rangle| - |\sin n_k\langle p_1, x - x_k\rangle|] \geqq \\ &\geqq |\mu^\wedge(p)| \left[|\sin(\vartheta_p - \Theta_p)|\sqrt{1 - k^{-2}} - \frac{1}{k}\right] \geqq C_4 - o(1), \end{aligned}$$

$$W_{n_k,x} g_{n_k,x_k} = |\nu^\wedge(p)|\,|\sin n_k\langle p_1, x - x_k\rangle| \leqq |\nu^\wedge(p)|/k = o(1)$$

with $C_4 > 0$ in view of (3.9) and $|\mu^\wedge(p)| \neq 0$.

Finally, let (3.8) be valid for each $a \in \mathbf{Q}^N$ with $\mu^\wedge(a) \neq 0$ (and thus $\nu^\wedge$ $\wedge(a) \neq 0$). Since μ, ν are linearly independent and $\nu^\wedge \neq 0$, in view of Lemma 3.2 there are rationals $p = p_1/p_2$, $q = q_1/q_2 \in \mathbf{Q}^N$, $p \neq q$, such that

$$\mu^\wedge(p)\nu^\wedge(q) \neq \mu^\wedge(q)\nu^\wedge(p).$$

This can be rewritten as (without loss of generality assume $\mu^\wedge(p)\nu^\wedge(q) \neq 0$, and if $\mu^\wedge(q) = 0$, set $\vartheta_q := \Theta_q$, $\varepsilon_q := 0$)

$$|\mu^\wedge(p)\nu^\wedge(q)|e^{i(\vartheta_p+\Theta_q)} \neq |\mu^\wedge(q)\nu^\wedge(p)|e^{i(\vartheta_q+\Theta_p)},$$

hence by (3.8) with $\varepsilon := \varepsilon_p - \varepsilon_q$

$$|\mu^\wedge(p)\nu^\wedge(q)| \neq |\mu^\wedge(q)\nu^\wedge(p)|(-1)^\varepsilon.$$

Set $X, A, \varphi_n, \sigma(t), U_t$ as above, but $B = \mathbf{R}^n \times \{s, c\}$,

$$V_{n,x}f = |T^\mu_{np_2q_2}f(x)|, \qquad W_{n,x}f = |T^\nu_{np_2q_2}f(x)|,$$

$$g_{n,(y,s)}(x) = |\nu^\wedge(q)|\sin(n\langle a, x-y\rangle - \vartheta_p) - (-1)^\varepsilon|\nu^\wedge(p)|\sin(n\langle b, x-y\rangle - \vartheta_q),$$

$$g_{n,(y,c)}(x) = |\nu^\wedge(q)|\cos(n\langle a, x-y\rangle - \vartheta_p) - (-1)^\varepsilon|\nu^\wedge(p)|\cos(n\langle b, x-y\rangle - \vartheta_q)$$

with $a := q_2p_1$, $b := p_2q_1 \in \mathbf{Z}^N$. Again (2.6–9) are valid. Now let $\{n_k\} \subset \mathbf{N}$ be an arbitrary subsequence. Since $p \neq q$, and therefore $a \neq b$, Lemma 3.1 (b) delivers the existence of points $\{x_k\} \subset \mathbf{R}^N$ such that H^s or H^c is a set of full measure. Suppose that H^s is this set (the other case can be treated analogously). Set $\beta_k = (x_k, s)$. Then (2.10,11) hold true on $H^s_k + x_k$. Indeed, if $x \in H^s_k + x_k$, then ($d := (-1)^\varepsilon|\mu^\wedge(q)\nu^\wedge(p)|/|\mu^\wedge(p)\nu^\wedge(q)|$)

$$V_{n_k,x}g_{n_k,(x_k,s)} =$$
$$= \Big||\mu^\wedge(p)\nu^\wedge(q)|\sin n_k\langle a, x-x_k\rangle - (-1)^\varepsilon|\mu^\wedge(q)\nu^\wedge(p)|\sin n_k\langle b, x-x_k\rangle\Big| =$$
$$= |\mu^\wedge(p)\nu^\wedge(q)|\Big|[\sin n_k\langle a, x-x_k\rangle - \sin n_k\langle b, x-x_k\rangle] + (1-d)\sin n_k\langle b, x-x_k\rangle\Big| \geqq$$
$$\geqq |\mu^\wedge(p)\nu^\wedge(q)|\Big[|1-d|\,|\sin n_k\langle b, x-x_k\rangle| - 2\Big|\sin\frac{1}{2}n_k\langle a-b, x-x_k\rangle\Big|\Big] \geqq$$
$$\geqq |\mu^\wedge(p)\nu^\wedge(q)|\Big[\frac{|1-d|}{2} - \frac{2}{k}\Big],$$

$$W_{n_k,x}g_{n_k,(x_k,s)} =$$
$$= |\nu^\wedge(q)\nu^\wedge(p)||\sin(n_k\langle a, x-x_k\rangle - \vartheta_p + \Theta_p) - (-1)^\varepsilon\sin(n_k\langle b, x-x_k\rangle - \vartheta_q + \Theta_q)| =$$
$$= |\nu^\wedge(q)\nu^\wedge(p)|\,|\sin n_k\langle a, x-x_k\rangle - \sin n_k\langle b, x-x_k\rangle| \leqq$$
$$\leqq 2\|\nu^\wedge\|^2\Big|\sin\frac{1}{2}n_k\langle a-b, x-x_k\rangle\Big| \leqq 2\|\nu^\wedge\|^2/k,$$

which completes the proof of Theorem 3.3. □

3.2. Counterexamples vanishing at infinity. Since $C_{2\pi} \subset \mathrm{UCB}$, the space of functions, uniformly continuous and bounded on $\mathbf{R}^N$, the assertions of Theorem 3.3 hold true in UCB, too. In C_0, however, one cannot immediately proceed as for Theorem 3.3, since the trigonometric test elements do not belong to C_0. Indeed, instead of (1.6) we now make use of the rate of convergence (3.10) on the subset $C^\infty_{00}(\mathbf{R}^N) \subset C_0$ of those functions, which have compact support and continuous partial derivatives of all orders.

THEOREM 3.4. *Let $\mu, \nu \in \mathcal{M}_0(\mathbf{R}^N)$ be linearly independent such that*

$$(3.10) \qquad \|T_n^\mu g\| = O_g(n^{-r}), \qquad \|T_n^\nu g\| = O_g(n^{-r}) \qquad (g \in C_{00}^\infty(\mathbf{R}^N))$$

for some $r \in \mathbf{N}$. Then for each modulus ω satisfying (2.4) there exists a (real-valued) counterexample $f_\omega \in \mathrm{Lip}_r(\omega, C_0(\mathbf{R}^N))$ such that (3.4,5) hold true for almost every $x \in \mathbf{R}^N$.

PROOF. Choose a function $K \in C_{00}^\infty$ such that $0 \leqq K(x) \leqq 1$ for $x \in \mathbf{R}^N$, $K(x) = 1$ for $|x| \leqq 1$. Setting as for the three different parts in the proof of Theorem 3.3, but now with testfunctions $h_{n,y}(x) := K(x/n) g_{n,y}(x)$, it follows that $h_{n,y} \in C_{00}^\infty$ with (2.7) and (2.8) (cf. (3.7)). By (3.10) one also has (2.9). For an arbitrary subsequence $\{n_k\} \subset \mathbf{N}$ and for $\sigma \in \mathcal{M}_0$, $a \in \mathbf{N}$

$$|T_{an_k}^\sigma h_{n_k,y}(x)| \overset{(\leqq)}{\geqq}$$

$$\overset{(\leqq)}{\geqq} |T_{an_k}^\sigma g_{n_k,y}(x)| \overset{(+)}{-} (2\pi)^{-N/2} \|g_{n_k,y}\| \int_{\mathbf{R}^N} (1 - K(x/n_k - u/an_k^2)) |d\sigma(u)| =$$

$$= |T_{an_k}^\sigma g_{n_k,y}(x)| \overset{(+)}{-} o_x(1),$$

the latter by dominated convergence. Therefore (2.10,11) hold true for points $x \in M_k$, the sets being given as in the proof of Theorem 3.3. Hence Lemma 2.3 yields the assertion. □

4. Examples

As a first explicit example consider the N-dimensional (product) integrals of Fejér and Abel–Poisson, defined by

$$\sigma_n f(x) := \left(\frac{2}{\pi n}\right)^N \int_{\mathbf{R}^N} f(x-u) \prod_{j=1}^N \left[\frac{\sin n u_j/2}{u_j}\right]^2 du,$$

$$A_n f(x) := \left(\frac{n}{\pi}\right)^N \int_{\mathbf{R}^N} f(x-u) \prod_{j=1}^N \frac{1}{1+n^2 u_j^2} du,$$

respectively. The remainders $f - \sigma_n f$, $f - A_n f$ may be rewritten as convolutions of type (1.2) with measures $\mu, \nu \in \mathcal{M}_0(\mathbf{R}^N)$, given by

$$\mu^\wedge(v) = 1 - \prod_{j=1}^N \max\{1 - |v_j|, 0\}, \qquad \nu^\wedge(v) = 1 - \prod_{j=1}^N e^{-|v_j|},$$

respectively. Obviously (1.6) is fulfilled for $r = 1$, thus Theorem 3.3 delivers (for $N = 1$ see [3])

COROLLARY 4.1. *For each modulus ω satisfying* (2.4) *there exists a counterexample $f_\omega \in \mathrm{Lip}_1(\omega, C_{2\pi}(\mathbf{R}^N))$ such that*

$$|f_\omega(x) - \sigma_n f_\omega(x)| \neq o(\omega(1/n)),$$

$$|f_\omega(x) - \sigma_n f_\omega(x)| \neq O(|f_\omega(x) - A_n f_\omega(x)|) \ (n \to \infty)$$

for almost every $x \in \mathbf{R}^N$.

Our next example is concerned with the singular integrals of Bochner–Riesz and Gauss–Weierstrass

$$B_{n,\lambda} f(x) := \left(\frac{n}{2\pi}\right)^{N/2-\lambda} \int_{\mathbf{R}^N} f(x-u)\pi^{-\lambda}\Gamma(\lambda+1)\mathcal{J}_{N/2+\lambda}(n|u|)|u|^{-N/2-\lambda}du,$$

$$W_n f(x) := \left(\frac{n}{2\sqrt{\pi}}\right)^N \int_{\mathbf{R}^N} f(x-u)e^{-n^2|u|^2/4}du,$$

respectively, where $\lambda > \lambda_0$ is assumed to be beyond the critical index $\lambda_0 = (N-1)/2$ and $\mathcal{J}_\alpha(s)$ denotes Bessel's function. The remainders $f - B_{n,\lambda}f$, $f - W_n f$ are convolutions of type (1.2) with $\mu_\lambda, \nu \in \mathcal{M}_0(\mathbf{R}^N)$, given by (cf. [10, p. 171])

$$\mu^\wedge_\lambda(v) := \begin{cases} 1-(1-|v|^2)^\lambda, & |v| \leqq 1 \\ 1, & |v| > 1 \end{cases}, \qquad \nu^\wedge(v) := 1 - e^{-|v|^2},$$

respectively. Since $\nu^\wedge$ divides $\mu^\wedge_\lambda$ globally, the uniform estimate (1.3) holds true, nevertheless, since (1.6) is valid for $r = 2$, a corresponding pointwise estimate is not possible. Indeed,

COROLLARY 4.2. *Let $\lambda > (N-1)/2$. For each modulus ω satisfying* (2.4) *there exists a counterexample $f_\omega \in \mathrm{Lip}_2(\omega, C_{2\pi}(\mathbf{R}^N))$ such that*

$$|f_\omega(x) - B_{n,\lambda} f_\omega(x)| \neq O(|f_\omega(x) - W_n f_\omega(x)|) \qquad (n \to \infty)$$

for almost every $x \in \mathbf{R}^N$.

Moreover, in view of (1.3) and the well-known (uniform) direct estimate

$$\|f - W_n f\| \leqq M \cdot \omega_2(f, 1/n) \qquad (f \in C_0(\mathbf{R}^N)), \tag{4.1}$$

condition (3.10) is fulfilled for $r = 2$, too, so that the assertion of Corollary 4.2 is also valid for the space C_0.

Finally, consider the pointwise comparison of the N-dimensional (product) Fejér integral σ_n and the (radial) singular integral of Cauchy–Poisson

$$P_y f(x) := \frac{\Gamma((N+1)/2)}{\pi^{(N+1)/2}} \int\limits_{\mathbf{R}^N} f(x-u) \frac{y}{(u^2+y^2)^{(N+1)/2}} du$$

which solves Dirichlet problem for the upper half space $\mathbf{R}^N \times (0,\infty)$, satisfying the boundary condition $\lim\limits_{y\to 0+} \|f - P_y f\| = 0$. Again the remainder $f - P_{1/n} f$ is a convolution (1.2) with measure $\nu \in \mathcal{M}_0(\mathbf{R}^N)$, given by $\nu^\wedge$ $\wedge(v) = 1 - e^{-|v|}$. In view of the weak-type inequality (cf. [7, Corollary 4.2], together with (4.1))

$$\text{(4.2)} \qquad \|f - P_{1/n} f\| \leqq M \cdot \frac{1}{n} \int\limits_{1/n}^{\infty} \frac{\omega_2(f,u)}{u^2} du \qquad (f \in C_0(\mathbf{R}^N))$$

one has (3.10) for $r = 1$ (cf. (3.7)), which also holds true for the (product) Fejér means σ_n, since the analogue of (4.2) is valid for the one-dimensional integral (cf. [1, p. 146]). Hence Theorem 3.4 delivers

COROLLARY 4.3. *For each modulus ω satisfying* (2.4) *there exists a counterexample $f_\omega \in \mathrm{Lip}_1(\omega, C_0(\mathbf{R}^N))$ such that*

$$|f_\omega(x) - \sigma_n f_\omega(x)| \neq O(|f_\omega(x) - P_{1/n} f_\omega(x)|) \qquad (n \to \infty)$$

for almost every $x \in \mathbf{R}^N$.

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ON THE PRODUCT OF TWO b_f-SPACES

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1. Introduction. The topological spaces used here will always be completely regular Hausdorff spaces. Let α be a cover of a space X. A function g from a space X into a space Y is *α-continuous* if the restriction of g to each member of α is continuous, and g is *α_f-continuous* if the restriction of g to each member of α can be extended to a continuous function on X. A space such that every real-valued α-continuous (resp. α_f-continuous) function is continuous is called an *α_R-space* (resp. an *α_f-space*). Clearly every α_R-space is an α_f-space. A subset B of X is said to be *bounded* if every continuous real-valued function on X is bounded on B. Write b for the family of all bounded subsets of a space.

The b_R-spaces and the b_f-spaces arise in the study of z-closed projections ([10]) and also in the problem of the distribution of the functor of the topological completion ([4], [13]). This class of spaces also appears studying compactness of function spaces in the topology of pointwise convergence ([1], §2).

In this paper we are concerned with determining conditions under which b_f-continuous functions on a product space $X \times Y$ will be continuous. We apply our results to characterize the class of all spaces X such that the product $X \times Y$ is a b_f-space for every b_f-space Y.

Notations and preliminaries. Throughout this paper we adopt the notation and terminology of [7]. We write $C(X)$ for the ring of all continuous real-valued functions on the space X and $C^*(X)$ for the subring of all bounded functions in $C(X)$. N is the discrete space of positive integers.

Write k for the family of all compact subsets. We shall say that a subspace S of X is *C-embedded* in X if every continuous real-valued function on S can be extended to a continuous function on X. Since every compact set in a completely regular Hausdorff space is C-embedded ([7], 3.11(c)) then every k_f-space is a k_R-space. It is an open question if every b_f-space is a b_R-space. First countable spaces and locally compact spaces are k_R-spaces and therefore b_f-spaces.

A *Frolík sequence* in a space X is a sequence $\{U_n\}_{n\in N}$ of open subsets of X such that for each filter G of infinite subsets of N,

$$\bigcap_{F\in G} \mathrm{cl}_X\Big(\bigcup_{n\in F} U_n\Big) \neq \emptyset.$$

A subset B of a space X is *strongly bounded in* X ([14]) if each infinite family of mutually disjoint open subsets of X meeting B contains an infinite subfamily $\{U_n\}_{n\in N}$ which is a Frolík sequence. According to [11], Proposition 2.3 *a subset* B *of* X *is bounded if and only if for each locally finite family* U *of mutually disjoint, non-empty open sets in* X *only finitely many members of* U *meet* B. Therefore each strongly bounded in X is bounded. As we will see in the remark preceding Corollary 6, the converse does not hold.

A space X is said to be *pseudocompact* if $C(X) = C^*(X)$. It is well-known that X is pseudocompact if and only if every sequence of non-empty open sets has a cluster point. This result suggests us the following question: Is every closed strongly bounded subset of a space X contained in a pseudocompact subspace of X? We will answer negatively this question with Example 7 (ii).

The results. The following proposition was stated by Noble in [11], Theorem 2.6 and he asserts that the implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(a) follow by the obvious adaptation of the proof of Theorem 3.6 in [6]. But the proof of necessity in Frolík's Theorem is not correct. A correct proof is given in [2], Theorem 1. For the sake of completeness we include a sketch of the same one, in order to prove the implication (c)$\Rightarrow$(a) in the following proposition.

PROPOSITION 1. *Let* S *be a subset of a space* X. *The following conditions are equivalent:*

(a) S *is strongly bounded in* X.

(b) *For each space* Y *and each bounded subset* B *of* Y, $S \times B$ *is bounded in* $X \times Y$.

(c) *For each pseudocompact space* Y, $S \times Y$ *is bounded in* $X \times Y$.

PROOF. The implication (b)$\Rightarrow$(c) is trivial and the implication (a)$\Rightarrow$ $\Rightarrow$(b) follows by the adaptation of the proof of sufficiency in [6], Theorem 3.6.

(c)$\Rightarrow$(a). Suppose that S is not strongly bounded in X. Then there exists a sequence $\{U_n\}_{n\in N}$ of mutually disjoint non-empty open subsets of X, meeting S, such that for each infinite subset M of N,

$$\bigcap_{F\in G(M)} \operatorname{cl}_X\Big(\bigcup_{n\in F} U_n\Big) = \emptyset$$

for some filter $G(M)$ of infinite subsets of M.

For each infinite subset M of N, we choose an ultrafilter U_M on N containing $G(M)$ and consider the following subspace of βN:

$$Y = N \cup \{p(M) \in \beta N - N \mid U_M \text{ converges to } p(M), M \subset N, M \text{ infinite}\}.$$

Then, the space Y is pseudocompact and the family $\{U_n \times \{n\}\}_{n\in N}$ is locally finite in $X \times Y$. Therefore $S \times Y$ is not bounded in $X \times Y$. $\square$

We say that a space X *has property* (b) if for each space Y, the product $A \times B$ of each pair of bounded subsets $A \subset X$ and $B \subset Y$ is bounded in $X \times Y$.

From Proposition 1 we have the following result.

COROLLARY 2. *X has property* (b) *if and only if every bounded subset of X is strongly bounded.*

A space is said to be a μ-space if every closed bounded subset is compact. Realcompact spaces (closed subspaces of a product of real lines) and P-spaces (spaces in which every G_δ subset is open) are μ-spaces ([7], 8E.1, 4K.3). Clearly every compact subset of a space is strongly bounded, hence each μ-space has property (b).

The family of all strongly bounded subsets of a space is denoted by sb.

THEOREM 3. *Let X be a b_f-space. Then X has the property* (b) *if and only if it is an sb_f-space.*

PROOF. The necessity follows from the fact that X is a b_f-space and each bounded subset of X is strongly bounded (by Corollary 2).

To prove sufficiency, let B be a bounded subset of X and we are going to show that B is strongly bounded.

Let $\{U_n\}_{n \in N}$ be a sequence of mutually disjoint non-empty open sets in X each of which meets B and suppose that each strongly bounded subset of X intersects finitely many U_n.

For each $n \in N$, let $x_n \in B \cap U_n$ and let f_n be a function in $C(X)$ such that $f_n(x_n) = 1$ and $f_n(X - U_n) = \{0\}$. Since each strongly bounded subset of X intersects finitely many U_n, the function

$$f = \sum_{n \geqq 1} f_n$$

is sb_f-continuous and as X is an sb_f-space, f is continuous.

On the other hand, if

$$V_n = \{x \in X \mid |f_n(x)| > 1/2\}, \qquad n \in N$$

then V_n is a subset of U_n and $\{V_n\}_{n \in N}$ is a sequence of non-empty open sets each of which meets B. Since B is bounded there exists a point $z \in X$ such that each neighborhood of z intersects infinitely many V_n ([11], Proposition 2.3). Therefore $z \notin \bigcup\{U_n \mid n \in N\}$ and f is not continuous in z, which is a contradiction.

Thus, there exists a strongly bounded set which intersects infinitely many U_n and consequently there is a subsequence of the sequence $\{U_n\}_{n \in N}$ which is a Frolík sequence. Then B is strongly bounded. From Corollary 2, X has property (b). □

Since every compact subset of a space is strongly bounded, we have the following

COROLLARY 4. *Every k_R-space has the property* (b).

Later we will give an example of a space X with property (b) which is neither a b_f-space nor a μ-space (Example 7 (i)).

LEMMA 5. *Let W be a non-empty regular closed subset of a space X. If the set $W \times B$ is bounded in $X \times Y$, then it is bounded in $W \times Y$.*

PROOF. We shall prove that $W \times B$ is bounded in $W \times Y$, by showing that if U_n and V_n are open sets in X and Y, respectively, such that

$$(U_n \times V_n) \cap (W \times B) \neq \emptyset, \qquad n \in N$$

then the family $\{U_n \times V_n\}_{n \in N}$ has a cluster point in $W \times Y$.

Write $H_n = U_n \cap \mathrm{int}_X W$. Since W is regular closed, we have that H_n is an open subset of X such that $(H_n \times V_n) \cap (W \times B) \neq \emptyset$ for each $n \in N$. By our hypothesis the set $W \times B$ is bounded in $X \times Y$, hence the family $\{H_n \times V_n\}$ has a cluster point $(x, y) \in X \times Y$. Since $H_n \subset W = \mathrm{cl}_X W$ it follows that $(x, y) \in W \times Y$. Then $W \times B$ is bounded in $W \times Y$. □

Let $\mathcal{B}$ be the class of spaces X such that for every pseudocompact space Y the product $X \times Y$ is pseudocompact. According to Frolík's Theorem ([6], Theorem 3.6) a space X belongs to $\mathcal{B}$ if and only if it is strongly bounded in itself. Consequently, if X is a pseudocompact space which is not in $\mathcal{B}$ ([7], Example 9.15), then X is a bounded set that is not strongly bounded.

COROLLARY 6. *Let X be a locally bounded space. Then X has the property* (b) *if and only if each point of X has a neighborhood in $\mathcal{B}$.*

PROOF. *Necessity.* Let $x \in X$ and let V be a bounded neighborhood of x. The set $W = \mathrm{cl}_X\ (\mathrm{int}_X V)$ is a bounded neighborhood of x and by our hypothesis, the product $W \times Y$ is bounded in $X \times Y$ for each pseudocompact space Y. From Lemma 5 the product $W \times Y$ is pseudocompact for each pseudocompact space Y. Hence we have that W belongs to $\mathcal{B}$.

Sufficiency. From Frolík's Theorem ([6], Theorem 3.6) every subset of X which belongs to $\mathcal{B}$ is strongly bounded in X. Then by hypothesis, each point of X has a strongly bounded neighborhood and consequently X is an sb_f-space. From Theorem 3, X has the property (b). □

EXAMPLE 7. We write βX (resp. vX) for the Stone–Čech compactification (resp. Hewitt realcompactification) of a space X. The smallest subspace of βX that contains X and is a μ-space is denoted by μX. Since every realcompact space is a μ-space, μX is a subspace of vX, and therefore X is C-embedded in μX ([7], Theorem 8.7, (II)).

If α is an ordinal, we write $\alpha + 1$ for the ordinal which follows it and ω_0 (resp. ω_1) for the first infinite (resp. uncountable) ordinal. $W(\alpha)$ is the space of all ordinals less than α endowed with the order topology.

Let Q be the product space $W(\omega_1 + 1) \times W(\omega_0 + 1)$ and let us consider the following subspaces:

$R = Q - \{(\alpha, \omega_0) \mid \alpha \text{ is a countable limit ordinal}\}$,
$U = R - \{\{\omega_1\} \times W(\omega_0)\}$,
$T = U - \{(\omega_1, \omega_0)\}$.

In ([8], p. 102]), the following facts are proved:

(1) $\mu T = vT = R$, (2) *A base of bounded sets of T is the family $\mathcal{J}$ of all subsets of T which are of the form*

$$B = \{W(\omega_1) \times \{n_1, \dots n_k\}\} \cup \{\{\alpha_1, \dots, \alpha_k\} \times W(\omega_0 + 1)\}$$

where $n_1, \dots, n_k$ are elements of $W(\omega_0)$ and $\alpha_1, \dots, \alpha_k$ are isolated ordinals of $W(\omega_1)$.

We now prove

(i) *The subspace U has the property* (b) *but it is neither a b_f-space nor a μ-space.*

From $T \subset U \subset R$ and (1), we have $\mu U = vU = R$ and hence U is not a μ-space. By (2), if F is a bounded set in U such that (ω_1, ω_0) is not in F, then (ω_1, ω_0) is not in $\mathrm{cl}_U F$. Consequently the characteristic function of the point (ω_1, ω_0) is b_f-continuous but not continuous. Therefore U is not a b_f-space.

According to [12], Theorem 2.2 every locally compact pseudocompact space belongs to $\mathcal{B}$. Hence the members of the base $\mathcal{J}$ considered in (2) belong to $\mathcal{B}$. Therefore each bounded set in U is strongly bounded and by Corollary 2, U has the property (b).

(ii) *In the locally compact space $X = R - \{(\omega_1, \omega_0)\}$, the set $V = \{\omega_1\} \times \times W(\omega_0)$ is a closed copy of N strongly bounded in X, which is not included in any pseudocompact subset of X (i.e. V does not have bounded neighborhoods).*

Let $\mathcal{F}$ be an infinite family of open subsets of X meeting V. Then there exist two sequences $\{n_k\}_{k \in N}$ and $\{U_k\}_{k \in N}$ in $W(\omega_0)$ and $\mathcal{F}$, respectively, such that $n_k < n_{k+1}$ and $(\omega_1, n_k) \in U_k$ for each $k \in N$.

Since U_k is open for each $k \in N$, there exists $\alpha_k < \omega_1$ such that $\{(\beta, n_k) \mid \mid \alpha_k \leqq \beta \leqq \omega_1\} \subset U_k$.

If $\mu = \sup\{a_k \mid k \in N\}$, then $\mu < \omega_1$ and for each $k \in N$, the set $\{(\beta, n_k) \mid \mu \leqq \beta \leqq \omega_1\}$ is contained in U_k.

Then $\{\beta + 1\} \times W(\omega_0 + 1)$ is a compact subset of X meeting each U_k and therefore $\{U_k\}_{k \in N}$ is a Frolík sequence. So V is strongly bounded in X.

Now let us consider a subset Z of X containing V. Suppose that Z is pseudocompact. Since V is closed in Z, there exists a positive integer n such that for every $j \geqq n$ the point (ω_1, j) is not isolated in Z. Inductively, choose points (β_j, j) in Z such that $\beta_j < \beta_{j+1}$, $j \geqq n$. Since the set $F = \{(\beta_j, j) \mid \mid j \geqq n\}$ is not included in any member of the base $\mathcal{J}$, F is not bounded in T. From (1) T is C-embedded in R, therefore F is not bounded in X. This is a contradiction since $F \subset Z$ and Z is pseudocompact. □

The following result is a consequence of 3.10(b) in [7].

LEMMA 8. *If X is a b_f-sapce, then each b_f-continuous function g from X into a (completely regular) space Y is continuous.*

In the proof of the following theorem we use the technique utilized in [3], Theorem 5).

THEOREM 9. *Let X be a locally bounded space and let Y be a b_f-space. If the product $A \times B$ of each pair of bounded subsets $A \subset X$ and $B \subset Y$ is bounded in $X \times Y$, then $X \times Y$ is a b_f-space.*

PROOF. Firstly, suppose that X is pseudocompact and let f be a real-valued b_f-continuous function in $X \times Y$. If A is a bounded set in Y, then $X \times A$ is bounded in $X \times Y$ by hypothesis. Thus, there exists a function $h \in C^*(X \times Y)$ which agrees with f in $X \times A$.
Let $e(h)$ be the function from Y into $C^*(X)$ defined by

$$e(h)(y) = h(\cdot, y), \qquad y \in Y.$$

According to [4], Theorem 5.3, $e(h)$ is a continuous function from Y into the Banach space $C^*(X)$ with the supremum-norm. Define $e(f)$ analogously. Since $e(f)_{|A} = e(h)_{|A}$ it follows that $e(f)$ is b_f-continuous on Y and by Lemma 8, $e(f)$ is continuous on Y. It is routine to check that f is continuous on $X \times Y$.
We now prove the result when X is locally bounded. Let $x \in X$ and let V be a bounded neighborhood of x. Then $W = \mathrm{cl}_X\,(\mathrm{int}_X V)$ is a pseudocompact neighborhood of x and by Lemma 5 we have that the product $W \times A$ is bounded in $W \times Y$ for each bounded subset A of Y. From the preceding case, $W \times Y$ is a b_f-space and hence each point of $X \times Y$ has a neighborhood which is a b_f-space. Thus $X \times Y$ is a b_f-space. □

An open question is if the above result holds for b_R-spaces. From Corollaries 4 and 6 we obtain the following result.

COROLLARY 10. *The product $X \times Y$ is a b_f-space in the following cases:*
(a) *Each point of X has a neighborhood in $\mathcal{B}$ and Y is a b_f-space.*
(b) *X is locally bounded and Y is a k_R-space.*

In the following examples we shall see that Theorem 9 fails to hold if some of the conditions are omitted.

EXAMPLE 11. In [3], Theorem 1 Blasco proved the following result: *Let V be a non-empty regular closed subset of a space X. If V is pseudocompact and does not belong to the class $\mathcal{B}$, then there is a pseudocompact subspace Z of βN and a b_R-continuous function f on $X \times Z$ which is not continuous.* It is easy to verify that the function f is b_f-continuous on $X \times Z$. Therefore, if X is a pseudocompact space which is not in $\mathcal{B}$, there exists a pseudocompact space Z such that $X \times Z$ is not a b_f-space. □

EXAMPLE 12. Hušek ([9], Theorem 1) proved the following result: *If X is not locally bounded, there are a paracompact k_R-space $H(X)$ and a k_R-continuous function f on $X \times H(X)$ which is not continuous.* Actually, the function f is b_f-continuous on $X \times H(X)$ as an easy check shows. If Q is the space of all rational numbers with the usual topology, then Q is a k_R-space (i.e. has property (b) by Corollary 4) and $Q \times H(Q)$ is not a b_f-space. □

THEOREM 13. *Let $\mathcal{S}$ be the class of all spaces X such that $X \times Y$ is a b_f-space whenever Y is. Then X belongs to $\mathcal{S}$ if and only if each point of X has a neighborhood in $\mathcal{B}$.*

PROOF. The sufficiency follows from part (a) of Corollary 10.

Necessity. If X is a space in $\mathcal{S}$, from Hušek's result quoted in Example 12, X is locally bounded and therefore locally pseudocompact ([5], Proposition 4.2). According to Blasco's result stated in Example 11, if V is a pseudocompact regular closed neighborhood of a point of X, then V belongs to $\mathcal{B}$. □

We denote by p the family of all pseudocompact subsets of a space. From [3], Theorem 6 we have the following.

COROLLARY 14. *The class $\mathcal{S}$ coincides with the class of all spaces X such that $X \times Y$ is a p_R-space whenever Y is.*

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SOME FURTHER TYPICAL RESULTS ON BOUNDED BAIRE ONE FUNCTIONS

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The goal of the present paper is to answer most of the problems raised in [3], Section 4. Therefore, we will widely use the notations introduced in that paper. Hence $\mathrm{b}\mathcal{A}$, $\mathrm{b}\Delta$, $\mathrm{b}\mathcal{DB}^1$, $\mathrm{b}\mathcal{B}^1$, $\mathrm{b}\mathcal{M}_k$, $k = 1, \dots, 5$, denotes the set of bounded approximately continuous functions, bounded derivatives, bounded Darboux Baire one functions, bounded Baire one functions, and bounded functions belonging to the kth class of Zahorski, resp., all defined on $[0,1]$. Further, for any function f we denote by $\mathcal{C}_f$, $\mathcal{A}_f$, and $\mathcal{R}_f$ the set of continuity points, the set of approximate continuity points, and the range of f, resp.

Since Problem 1 from [3] is closely related to the main part of Problem 4, we will start with Problem 2 and return to Problem 1 later.

PROBLEM 2. *What are the typical properties of* $\lambda(\mathrm{cl}\, f^{-1}(y))$ *and* $\mu(\mathrm{cl}\, f^{-1}(y))$ *in the three subclasses considered?*

For $\lambda(\mathrm{cl}\, f^{-1}(y))$ this question was answered in [10], our treatment of the case of a general continuous[1] measure μ is heavily based on Rinne's result.

THEOREM 1. *Let* μ *be a continuous, nonzero (and finite) measure on* $[0,1]$ *and let* X *be one of the spaces* $\mathrm{b}\Delta$, *and* $\mathrm{b}\mathcal{M}_k$, $k = 1, \dots, 5$. *Then for a typical* $f \in X$ *there is a nonvoid open interval* $I \subset \mathbf{R}$ *such that*

$$\inf\{\mu(\mathrm{cl}\, f^{-1}(y)); y \in I\} > 0.$$

PROOF. Let M be the interior of the set of all $f \in X$ such that there exist $a < b$ with $\inf\{\mu(\mathrm{cl}\, f^{-1}(y)); a < y < b\} > 0$. Obviously, it suffices to show that M is *dense* in X. For this purpose we fix an arbitrary $f \in X$. We have to distinguish two cases. If $\mu([0,1] \setminus \mathcal{C}_f)$ is positive, then we find rationals $p < q$ such that the Borel set

$$S = \mathrm{cl}\,\{x \in [0,1]; f(x) < p\} \cap \mathrm{cl}\,\{x \in [0,1]; f(x) > q\}$$

has positive μ-measure. Since any $g \in X$ is a *Darboux* function, we get $S \subset \mathrm{cl}\, g^{-1}(y)$ whenever $\|f - g\| < \frac{q-p}{4}$ and $y \in \left(\frac{3p+q}{4}, \frac{p+3q}{4}\right)$. This shows that $f \in M$. In the other case we can select $x_1 \in (0,1) \cap \mathrm{spt}\, \mu \cap \mathcal{C}_f$. Then

[1] If μ is not continuous then the answer is obvious.

for any $\varepsilon > 0$ we find $x_0 \in (0, x_1)$ and $x_2 \in (x_1, 1)$ with $\sup\{f(x); x \in \in [x_0, x_2]\} < \inf\{f(x); x \in [x_0, x_2]\} + (\varepsilon/2)$. Because $x_1 \in \operatorname{spt}\mu$, we have $\mu([x_0, x_2]) > 0$ and we can apply Rinne's construction (see [10], Theorem 6) to obtain a function $g \in X$, a nonempty open interval $I \subseteqq \mathbf{R}$, a (nowhere dense) compact set $K \subseteqq [x_0, x_2]$ with $\mu(K) > 0$, and a positive δ such that $\|f - g\| < \varepsilon/2$ and that $\operatorname{cl} h^{-1}(y) \supseteqq K$ whenever $y \in I$ and $h \in X$ fulfils $\|h - g\| < \delta$. We conclude that $g \in M$ and $f \in \operatorname{cl} M$.

PROBLEM 3. *Is it true that there is a "large" set $Y \subseteqq \mathcal{R}_f$ such that $f^{-1}(y)$ is countable (finite or singleton) for a typical $f \in \mathcal{F}$; $\mathcal{F} = \mathrm{b}\mathcal{A}$, $\mathrm{b}\Delta$, $\mathrm{b}\mathcal{DB}^1$, $\mathrm{b}\mathcal{B}^1$ and $y \in Y$?*

Our answer given below shows that the space $\mathrm{b}\mathcal{B}^1$ from this point of view behaves completely differently from the other classes considered here.

PROPOSITION 1. *A typical $f \in \mathrm{b}\mathcal{B}^1$ is an injective function.*

PROOF. For $n \geqq 1$ we denote $\mathcal{G}_n$ to be the set of all $f \in \mathrm{b}\mathcal{B}^1$ such that there exist sets $M_1, \ldots, M_k$ satisfying $\bigcup_{i=1}^{k} M_i = [0,1]$, $\operatorname{diam}(M_i) < 1/n$, and $\operatorname{dist}(f(M_i), f(M_j)) > 0$ whenever $1 \leqq i \neq j \leqq k$. Obviously, any set $\mathcal{G}_n$ is open in $\mathrm{b}\mathcal{B}^1$ and it is easy to see that $\bigcap_{n=1}^{\infty} \mathcal{G}_n$ contains only injective functions. Hence it only remains to show that each $\mathcal{G}_n$ is dense in $\mathrm{b}\mathcal{B}^1$. But this can be easily done using "simple" functions, see Theorems 1.7 and 1.8 in [3] for more details.

PROPOSITION 2. *For a typical f in any of $\mathrm{b}\Delta$, $\mathrm{b}\mathcal{M}_k$, $k = 1, \ldots, 5$, the following holds:*

$$f^{-1}(y) \text{ is of power of continuum whenever } \inf f < y < \sup f.$$

PROOF. According to Theorems 3 and 4 in [9] and the remark at the end of [10], for a typical f any real number is a derived number at any point $x \in (0,1)$. This implies that for a typical f each point $x \in [0,1]$ is either a cluster point of $f^{-1}(f(x))$ or a strong local extremum; for $x \in \{0,1\}$ this alternative is always true. Furthermore, it is easy to show that for a typical f and two distinct strong local extrema x and y of this function $f(x) \neq f(y)$ holds. Consequently, in typical case $f^{-1}(y)$ has at most one isolated point. But if for the strong local extremum $x \in [0,1]$ $\inf f < f(x) < \sup f$ holds then obviously $f^{-1}(f(x)) \cap ([0,1] \setminus \{x\}) \neq \emptyset$. Summarizing we conclude that for a typical f and $y \in (\inf f, \sup f)$ the level set $f^{-1}(y)$ contains at least two different points and at most one isolated point. Since moreover

$$f^{-1}(y) = f^{-1}((-\infty, y]) \cap f^{-1}([y, \infty))$$

is a $\mathbf{G}_\delta$-set, it is of the power of continuum.

REMARK. Using some additional results from [5] we could prove also a more precise result: For a typical f in any of the classes considered above we have

(a) There exist exactly two points $y_+ = \max f$ and $y_- = \min f$ such that the corresponding level sets are not of power of continuum, moreover both $f^{-1}(y_-)$ and $f^{-1}(y_+)$ are singletons.

(b) There exists a countable infinite set $M \subset \mathcal{R}_f$ such that for any $y \in M$ $f^{-1}(y)$ is a set of power of continuum having exactly one isolated point.

(c) For all other $y \in \mathcal{R}_f$ the set $f^{-1}(y)$ is a set of power of continuum without isolated points.

PROBLEM 4. *How large is the Hausdorff measure of $f^{-1}(y)$, $\mathcal{C}_f$, $f(\mathcal{C}_f)$ for a typical f?*

To formulate our results precisely we have to introduce some further notations.

A function $\varphi\colon [0,\infty) \to [0,\infty)$ will be said to be a *Hausdorff function* if φ is nondecreasing and $\lim_{x\to 0_+} \varphi(x) = 0$. The *Hausdorff measure $\mathcal{H}^\varphi$ generated by the Hausdorff function φ* is defined as follows. For $M \subseteqq \mathbf{R}$ and an integer $n \geqq 1$ we put

$$\mathcal{H}^\varphi_n(M) = \inf\left\{\sum_{k=1}^{\infty} \varphi(\operatorname{diam} E_k);\ \bigcup_{k=1}^{\infty} E_k \supseteqq M \ \text{ and } \operatorname{diam} E_k < \frac{1}{n} \text{ if } k \geqq 1\right\}.$$

Clearly $\mathcal{H}^\varphi_{n+1}(M) \geqq \mathcal{H}^\varphi_n(M)$ and hence it is possible to define the measure of the set M by

$$\mathcal{H}^\varphi(M) = \lim_{n\to\infty} \mathcal{H}^\varphi_n(M).$$

At first we will answer the questions concerning $\mathcal{H}^\varphi(\mathcal{C}_f)$ and $\mathcal{H}^\varphi(f(\mathcal{C}_f))$ since our results are quite satisfactory in these cases.

LEMMA 1. *Let $f \in \mathrm{b}\mathcal{B}^1([0,1])$, let U be an open, dense subset of $[0,1]$, and let ε be positive. Then there exists an approximately continuous map $g\colon [0,1] \to [-\varepsilon,\varepsilon]$ such that*

$$\{x \in [0,1]; \operatorname{osc}(f+g,x) < \varepsilon\} \subseteqq U.$$

PROOF. Since $\mathcal{C}_f$ is residual in $[0,1]$, we can select a countable set $M = \{x_i; i \geqq 1\} \subset U \cap \mathcal{C}_f$ such that $x \in [0,1]$ is a cluster point of M if and only if $x \notin U$. By the Lusin–Menchoff Theorem, see Theorem 6.9.(g) in [7], we find an open set $G \supset M$ such that any $x \in [0,1]\setminus U$ is a dispersion point of G. Because M has no cluster point in U, we can choose a sequence $\{r_i\}_{i=1}^{\infty}$ of positive numbers such that the balls $B(x_i,r_i)$, $i \geqq 1$, are mutually disjoint, contained in G, and fulfil

$$\sup f(B(x_i,r_i)) < \inf f(B(x_i,r_i)) + \varepsilon \ \text{ for any } \ i \geqq 1.$$

Now one easily verifies that the map g: $[0,1] \to [-\varepsilon, \varepsilon]$ defined by $g(x) = \sum_{i=1}^{\infty} \varepsilon g_i(x)$, where

$$g_i(x) = \begin{cases} 0 & \text{if } x \notin B(x_i, r_i) \\ \sin\left(2\pi \frac{|x-x_i|}{r_i}\right) & \text{if } x \in B(x_i, r_i) \end{cases}$$

has all properties as required, $g|_G$ is even continuous.

THEOREM 2. *Let X be one of the spaces $\mathrm{b}\mathcal{M}_k$, $k = 1,4,5$, $\mathrm{b}\mathcal{B}^1$ and $\mathrm{b}\Delta$, and let φ be a Hausdorff function. Then for a typical $f \in X$ both $\mathcal{H}^\varphi(\mathcal{C}_f) = 0$ and $\mathcal{H}^\varphi(\operatorname{cl} f(\mathcal{C}_f)) = 0$ hold.*

PROOF. For $n \geqq 1$ let $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ be the set of all $f \in X$ such that there is an $\varepsilon > 0$ with

$$\mathcal{H}_n^\varphi(\{x; \operatorname{osc}(f,x) < \varepsilon\}) < \frac{1}{n}, \tag{1}$$

and

$$\mathcal{H}_n^\varphi(B(\{f(x); \operatorname{osc}(f,x) < \varepsilon\}, \varepsilon)) < \frac{1}{n}, \tag{2}$$

resp. It is easy to see that any of the sets $\mathcal{G}_n^1$, $\mathcal{G}_n^2$ is open in X and that $\mathcal{H}^\varphi(\mathcal{C}_f) = \mathcal{H}^\varphi(\operatorname{cl} f(\mathcal{C}_f)) = 0$ whenever $f \in \bigcap_{n=1}^{\infty} \mathcal{G}_n^1 \cap \mathcal{G}_n^2$. It remains to show that for every $n \geqq 1$ both sets $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ are dense in X.

At first we consider $\mathcal{G}_n^1$. Since the approximately continuous functions form an additive class for $\mathrm{b}\mathcal{M}_4$, $\mathrm{b}\mathcal{M}_5$, $\mathrm{b}\mathcal{B}^1$ and $\mathrm{b}\Delta$, accordingly to Lemma 1 $\mathcal{G}_n^1$ is dense in X for $X = \mathrm{b}\mathcal{M}_4$, $\mathrm{b}\mathcal{M}_5$, $\mathrm{b}\mathcal{B}^1$ or $\mathrm{b}\Delta$. In case $X = \mathrm{b}\mathcal{DB}^1$ we use Maximoff's Theorem, see page 36 in [1]. We fix $f \in \mathrm{b}\mathcal{DB}^1$ and $\varepsilon > 0$ arbitrarily. Then there is a homeomorphism h: $[0,1] \to [0,1]$ such that $f \circ h \in \mathrm{b}\mathcal{M}_5([0,1])$. Further, we choose an open dense set $U \subset [0,1]$ with $\mathcal{H}_n^\varphi(U) < 1/n$ and an approximately continuous map g: $[0,1] \to [-\varepsilon, \varepsilon]$ such that

$$\operatorname{osc}(g + f \circ h, x) > \varepsilon \quad \text{whenever} \quad x \in [0,1] \setminus h(U).$$

Obviously, $(g + f \circ h) \circ h^{-1} \in \mathrm{b}\mathcal{DB}^1$, $\|f - [(g + f \circ h) \circ h^{-1}]\| \leqq \varepsilon$, and $\{x; \operatorname{osc}((g + f \circ h) \circ h^{-1}, x) < \varepsilon\} \subseteqq U$.

Finally we will show that also the sets $\mathcal{G}_n^2$ are dense in X. Let $\mathcal{F}$ be the set of all $f \in X$ such that $f(\mathcal{C}_f)$ is finite. As shown in Theorems 4, 9 and 10 from [4], in any case the set $\mathcal{F}$ is dense in X. Hence in order to finish the proof it suffices to show that $\mathcal{F} \subseteqq \mathcal{G}_n^2$. But as remarked in the proof of Theorem 5 in [4], we have $\operatorname{dist}(f(x), f(\mathcal{C}_f)) \leqq \operatorname{osc}(f,x)$ for any $x \in [0,1]$. (Indeed, this easily follows from the fact that $\mathcal{C}_f$ is dense in $[0,1]$.)

Consequently, for $f \in \mathcal{F}$ and $\varepsilon \in (0, 1/4n)$ with $\operatorname{card}(f(\mathcal{C}_f)) \cdot \varphi(4\varepsilon) < 1/n$ we have

$$M_\varepsilon = B(\{f(x); \operatorname{osc}(f,x) < \varepsilon\}, \varepsilon) \subseteqq \\ \subseteqq B(\{f(x); \operatorname{dist}(f(x), f(\mathcal{C}_f)) < \varepsilon\}, \varepsilon) \subseteqq B(f(\mathcal{C}_f), 2\varepsilon)$$

and this means $\mathcal{H}^\varphi_n(M_\varepsilon) < 1/n$, as required.

REMARK. It is easy to see that using Theorems 4 and 5 from [4], our proof can be modified to show that even $\mathcal{H}^\varphi(\operatorname{cl} f(\mathcal{A}_f)) = 0$ for a typical $f \in \mathrm{b}\mathcal{DB}^1$.

Now we turn to the problem concerning the Hausdorff measure of the level sets of typical functions. Simultaneously we also solve

PROBLEM 1. *What are the typical properties of $\mu(f^{-1}(y))$ in $\mathrm{b}\mathcal{A}$, $\mathrm{b}\Delta$, and $\mathrm{b}\mathcal{DB}^1$ for finite, continuous and Borel regular measure μ?*

In [8] the result that $\lambda(f^{-1}(y)) = 0$ for a typical $f \in b\mathcal{M}_k$, $k = 1, \dots, 5$ (which is not covered by [3]) was announced. Here we prove that $\mu(f^{-1}(y)) = 0$ for a typical $f \in \mathrm{b}\mathcal{M}_k$, $k = 1, \dots, 5$, and all $y \in \mathbf{R}$. The same question for Hausdorff measure seems to be more difficult. One basic contrast between $\mathcal{H}^\varphi$ and a finite measure μ is that the set

$$B_{\mu,\delta} = \{f \in b\mathcal{B}^1; \text{ there is } y \text{ with } \mathcal{H}^\varphi(f^{-1}(y)) \geqq \delta\}$$

need not be closed in $\mathrm{b}\mathcal{B}^1$; compare with Lemma 1.2 in [3]. This can be easily seen using a sequence f_n of continuous maps on $[0,1]$ such that the f_n's uniformly converge to the identity and that every f_n is constant on a nondegenerate interval.

So we have to choose a new approach and will show that in many cases a typical f is injective on a large set. Consequently, any level set is contained in the union of a singleton and of the (small) complement of this large set. This moreover demonstrates that the level sets are somehow "uniformly" small. Because this point of view seems to be new, we will also deal with the question of injectivity on residual sets. Here our result shows a difference between continuous and $\mathcal{B}^1$-functions.

The following table demonstrates our present knowledge whether for a typical $f \in X$ (see the first row) there is a set M small in the sense indicated in the first column such that $f|_{([0,1]\setminus M)}$ is injective.

X	$\mathrm{b}\mathcal{DB}^1$	$\mathrm{b}\mathcal{M}_2, \mathrm{b}\mathcal{M}_3$	$\mathrm{b}\Delta$	$\mathrm{b}\mathcal{M}_4$	$\mathrm{b}\mathcal{A}$	$\mathcal{C}$
M first category	YES	?	YES	YES	YES	NO
$\mathcal{H}^\varphi(M) = 0$	YES	?	?	?	?	YES
$\mu(M) = 0$	YES	YES	YES	YES	YES	YES

Before proving our statements given above, we make two remarks. First, since any σ-porous set is a first category one, it follows that strong porosity features like those derived in [2] can not be proved by our method. The question[2] concerning σ-porosity remains open for the classes $b\Delta$ and $b\mathcal{M}_k$, $k = 1, \dots, 5$. Second, even more unpleasant is the fact that I am not able to answer the questions marked by a bold question mark. Of course, in the light of an "interpolation" between $b\mathcal{DB}^1$ and the space $\mathcal{C}$ of continuous functions, affirmative answers seem highly probable.

THEOREM 3. a) *A typical* $f \in b\mathcal{DB}^1$ *is injective on* $\mathcal{A}_f$.

b) *A typical* $f \in b\Delta$, $b\mathcal{M}_4$, $b\mathcal{A}$ *is injective on* $\mathcal{C}_f$.

c) *For a typical continuous* f *defined on* $[0,1]$ *there is no residual subset of* $[0,1]$ *on which* f *is injective.*

PROOF. a) For $n \geqq 1$ we define $\mathcal{G}_n$ to be the set of all $f \in b\mathcal{DB}^1$ such that there are sets $M_1, \dots, M_k$ of diameter less than $1/n$ and an $\varepsilon > 0$ fulfilling $\operatorname{dist}(f(M_i), f(M_j)) > 0$ for $1 \leqq i \neq j \leqq k$ and

$$\bigcup_{i=1}^{k} M_i \supseteqq \{x \in [0,1]; \operatorname{osc}_d(f,x) < \varepsilon\}. \text{ [3]}$$

Again it is easy to see that each $\mathcal{G}_n$ is open and that any $f \in \bigcap_{n=1}^{\infty} \mathcal{G}_n$ is injective on $\mathcal{A}_f$. Finally, the fact that every set $\mathcal{G}_n$ is dense in $b\mathcal{DB}^1$ can be easily demonstrated applying Theorem 3 from [4] to the simple functions appearing in the proof of Proposition 1.

b) Using a slight modification of the proof of Theorem 9 in [4] we obtain the following statement. *Let* $f \in b\mathcal{B}^1$ *and* $\delta > 0$ *be given. Then there exists* $g \in b\mathcal{A}$ *with* $\|g\| < \delta$ *such that there are sets* $S_1, \dots, S_m$ *of diameter less than* δ *fulfilling* $\bigcup_{i=1}^{m} S_i \supseteqq \{x \in [0,1];\ \operatorname{osc}(f+g,x) < \varepsilon\}$ *and* $\operatorname{dist}((f+g)(S_i), (f+g)(S_j)) > 0$ *for* $1 \leqq i \neq j \leqq m$. (Indeed, using the notations of [4] it suffices to ensure that $\operatorname{diam} A_k < \delta$ and to put $\varphi(x) = v_k$ for $x \in A_k$, where the v_k's are mutually distinct with $|v_k - f(m_k)| < \delta/2$.)[4] Now the proof of b) closely follows that of a).

c) It suffices to prove that a continuous map f is not injective on any residual subset M of $[0,1]$ provided f is not constant on any nondegenerate interval and there are $0 \leqq a < b < c < d \leqq 1$ with $f([a,b]) = f([c,d])$. (In fact, these properties are typical.) For this purpose we only need to prove that any dense $\mathbf{G}_\delta$-subset G of $[a,b]$ is mapped on a set residual in $f([a,b])$.

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[2]See the Remark after Theorem 3.

[3]Here $\operatorname{osc}_d(f,x) = \inf\{\sup f(A) - \inf f(A);\ x \in A \text{ is a dispersion point of } [0,1]\backslash A\}$.

[4]Compare with the proof mentioned above.

Therefore, let $S = f([a,b])\setminus f(G)$ be a second category set. Since $f(G)$ is an analytic set, according to [6] §35.II, Corollary 1 the set S has the Baire property. Hence we find $p \leqq q$ with $p, q \in f([a,b])$ and a $\mathbf{G}_\delta$-set $S_1 \subseteqq S$ which is dense in (p,q). Then $T = f^{-1}(S_1) \cap [a,b]$ is a $\mathbf{G}_\delta$-set and $T \cap G = \emptyset$. Consequently, T is a nowhere dense set and we can choose $s < t$ such that $(s,t) \subset f^{-1}((p,q))\setminus T$. Then $f((s,t)) \subseteqq (p,q)\setminus S_1$ and simultaneously $\operatorname{int} f((s,t)) \neq \emptyset$, a contradiction finishing the proof.

REMARK. It is well known (and easy to see) that the typical function in any of the spaces under consideration is nowhere monotone, i.e. (Darboux property!) nowhere injective. Therefore a typical f can not be injective on a nonempty *open* set.

THEOREM 4. *Let φ be a Hausdorff function. Then the following property is typical in* $\mathrm{b}\mathcal{DB}^1$: *There is a set $M \subset [0,1]$ with $\mathcal{H}^\varphi(M) = 0$ such that $f|_{([0,1]\setminus M)}$ is injective.*

PROOF. Let $\mathcal{G}_n$, $n \geqq 1$, be the set of all $f \in \mathrm{b}\mathcal{DB}^1$ such that there exist sets $M_i = M_i^n(f)$, $i = 0, 1, \dots, k$, satisfying:

$$\bigcup_{i=0}^{k} M_i = [0,1], \tag{1}$$

$$\operatorname{diam}(M_i) < \frac{1}{n} \quad \text{for} \quad i \geqq 1, \tag{2}$$

$$\operatorname{dist}(f(M_i), f(M_j)) > 0 \quad \text{for} \quad 1 \leqq i \neq j \leqq k, \quad \text{and} \tag{3}$$

$$\mathcal{H}_n^\varphi(M_0) < 2^{-n}. \tag{4}$$

Obviously, any $\mathcal{G}_n$ is open in $\mathrm{b}\mathcal{DB}^1$. Next, assume $f \in \bigcap_{n=1}^{\infty} \mathcal{G}_n$ and denote $M = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} M_0^k(f)$. Then $\mathcal{H}^\varphi(M) = 0$ and it is easy to see that $f|_{([0,1]\setminus M_0)}$ is injective. Hence it remains to prove that each $\mathcal{G}_n$ is dense in $\mathrm{b}\mathcal{DB}^1$.

For this purpose, fix arbitrary $n \geqq 1$, $\varepsilon > 0$ and $f \in \mathrm{b}\mathcal{DB}^1$. As already remarked in the proof of Proposition 1 we can choose a function $g \in \mathrm{b}\mathcal{B}^1$ with finite range $\mathcal{R}_g = \{z_1, \dots, z_m\}$ such that $\|f - g\| < \varepsilon$ and that the sets $S_i = g^{-1}(\{z_i\})$ fulfil $\operatorname{diam}(S_i) < 1/n$ whenever $1 \leqq i \leqq m$. Now we define for any $x \in \mathbf{R}$

$$M^+(x) = \{y \in g^{-1}(x) \cap [0,1);\ \text{there is } \delta > 0 \text{ with } (y, y+\delta) \cap g^{-1}(x) = \emptyset\}$$

and

$$M^-(x) = \{y \in g^{-1}(x) \cap [0,1);\ \text{there is } \delta > 0 \text{ with } (y-\delta, y) \cap g^{-1}(x) = \emptyset\}.$$

Standard arguments now show that any of the sets $M^+(x)$ and $M^-(x)$ is countable, consequently also

$$M = \bigcup\{M^+(x) \cup M^-(x); x \in \mathcal{R}_g\}$$

is a countable set. We order the members of M in a sequence $\{x_k\}_{k=1}^\infty$ with $x_i \neq x_j$ for $i \neq j$. Next, we construct by induction a sequence $\{r_k\}_{k=1}^\infty$ of real numbers. We put $r_i = 0$ in case $x_i \in \bigcup\limits_{j<i}(x_j - r_j, x_j + r_j)$ and otherwise we select $r_i \in \frac{1}{2}\cdot\varphi^{-1}((0, 2^{-in})) \cap \left(0, \frac{1}{2n}\right)$ such that

$$x_i - r_i, x_i + r_i \in \mathbf{C}_g \cup (-\infty, 0) \cup (1, \infty) \tag{5}$$

(since $\mathbf{C}_g$ is residual in $[0,1]$), and

$$(x_i - r_i, x_i + r_i) \cap \bigcup_{j<i}(x_j - r_j, x_j + r_j) = \emptyset \tag{6}$$

(since $|x_i - x_j| = r_j$ for some $j < i$ would imply that $x_i \in \mathbf{C}_g$, but $\mathbf{C}_g \cap M = \emptyset$ by definition of the sets $M^+(x)$ and $M^-(x)$).

Next, we set $G = [0,1] \cap \bigcup\limits_{i=1}^{\infty}(x_i - r_i, x_i + r_i)$ and define the required map $\tilde{g} \in \mathcal{G}_n$ as follows. Let J be a (nonvoid) component of G.[5] According to Theorem 3 from [4] there is a map $\hat{g} \in \mathrm{b}\mathcal{DB}^1(\mathrm{cl}\,J)$ satisfying $\|g - \hat{g}\| \leqq 4\|g - f\| < 4\varepsilon$ and $g = \hat{g}$ a.e. on $\mathrm{cl}\,J$. Further, we choose $y_1 < \ldots < y_l$ from J and a continuous map $h\colon \mathrm{cl}\,J \to [-4\varepsilon, 4\varepsilon]$ such that $g(J) = \{g(y_1), \ldots, g(y_l)\}$, $h(y_i) = g(y_i) - \hat{g}(y_i)$ for $1 \leqq i \leqq l$, and $h(x) = 0$ if $x \in \partial J \cap \mathbf{C}_g$. Finally, we put

$$\tilde{g}(x) = \begin{cases} h(x) + \hat{g}(x) & \text{if } x \in J \text{ and } J \text{ is a component of } G \\ g(x) & \text{if } x \in [0,1]\setminus G. \end{cases}$$

Obviously, $\|\tilde{g} - f\| < 9\varepsilon$ and, if we denote $M_0 = G$, $M_i = S_i \setminus M_0$, $i = 1, \ldots, m$, then the sets $M_i^n(\tilde{g}) = M_i$, $i = 0, \ldots, m$, satisfy the conditions (1)–(4) stated above. Since $\varepsilon > 0$ and $n \geqq 1$ have been chosen arbitrarily, in order to conclude that each $\mathcal{G}_n$ is dense in $\mathrm{b}\mathcal{DB}^1$ we need only to show that $\tilde{g} \in \mathrm{b}\mathcal{DB}^1$ indeed. But it is easy to see that $\tilde{g}$ is a Baire one function. According to Theorem 1.1 in Chapter II of [1] it suffices to prove that

$$\tilde{g}(x) \in \mathrm{Cl}^+(x, \tilde{g}) = \bigcap_{t>0} \mathrm{cl}\,\tilde{g}((x, x+t)) \text{ if } x \in [0,1)$$

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[5] I.e. $J = (x_i - r_i, x_i + r_i)$ for some i with $r_i > 0$.

and

$$\tilde{g}(x) \in \mathrm{Cl}^-(x,\tilde{g}) = \bigcap_{t>0} \mathrm{cl}\,\tilde{g}((x-t,x)) \text{ if } x \in (0,1].$$

Since $\tilde{g}\mid_G$ and $\tilde{g}\mid_{([0,1]\setminus \mathrm{cl}\,G)}$ are Darboux functions, we may assume that $x \in \in \partial G$ and for brevity we consider only $\mathrm{Cl}^+(x,\tilde{g})$ where $0 \leqq x < 1$, the second consideration being similar. If $(x, x+\delta) \subseteqq G$ for some positive δ then $\tilde{g}(x) = = g(x) \in \mathrm{Cl}^+(x,\tilde{g})$ because $x = x_i - r_i$ for some $i \geqq 1$, and this implies that $x \in \mathbf{C}_g$, $\lim_{t\to x+} h(t) = 0$, and $g = \hat{g}$ a.e. on $(x, x+\delta)$. In case $(x, x+\delta) \cap \cap G = \emptyset$ for some $\delta > 0$ we have $\tilde{g}\mid_{[x,x+\delta)} = g\mid_{x,x+\delta)}$ and $\tilde{g}(x) \in \mathrm{Cl}^+(x,\tilde{g})$ since $x \notin M^+(g(x)) \subseteqq G$. Finally, we easily verify that in the remaining case there must be a sequence $t_k \searrow x$ with $g((x, x+t_k)) \subseteqq \tilde{g}((x, x+t_k))$. Indeed, we have $\tilde{g}(J) \supseteqq g(J)$ for any component J of G. But then $\tilde{g}(x) = g(x) \in \in \mathrm{Cl}^+(x,\tilde{g}) \supseteqq \mathrm{Cl}^+(x,g)$ again follows from $x \notin M^+(g(x))$. This finishes the proof.

THEOREM 5. *Let φ be a Hausdorff function. Then for a typical continuous f on $[0,1]$ there is a set $M \subset [0,1]$ with $\mathcal{H}^\varphi(M) = 0$ such that $f\mid_{([0,1]\setminus M)}$ is injective.*

PROOF. We define the sets $\mathcal{G}_n$ analogous to those in the preceding proof. Using uniform continuity it is quite easy to show that $\mathrm{cl}\,\mathcal{G}_n = \mathcal{C}([0,1])$ for any $n \geqq 1$. Details are left to the reader.

THEOREM 6. *Let X be a set of bounded Borel functions on $[0,1]$ which is complete with respect to the supremum metric and closed under the addition of continuous functions. Then for any finite and Borel regular measure μ on $[0,1]$ a typical $f \in X$ is injective on a set of full μ-measure.*

PROOF. We define the $\mathcal{G}_n$'s like in the proof of Theorem 4, but instead of condition (4) we require that $\mu(M_0) < 2^{-n}$. The following lemma now finishes the proof by showing that each $\mathcal{G}_n$ is dense in X.

LEMMA 2. *Let $f\colon [0,1] \to \mathbf{R}$ be μ-measurable, where μ fulfils the conditions stated in Theorem 6. Then for an arbitrary $\varepsilon > 0$ there exist a continuous function $g\colon [0,1] \to [-\varepsilon, \varepsilon]$ and sets $M_0, \dots, M_k$ such that*

$$\bigcup_{i=0}^{k} M_i = [0,1], \tag{1}$$

$$\mathrm{diam}(M_i) < \varepsilon \quad \textit{for} \quad i \geqq 1, \tag{2}$$

$$\mathrm{dist}((f+g)(M_i), (f+g)(M_j)) > 0 \quad \textit{for} \quad 1 \leqq i \neq j \leqq k, \quad \textit{and} \tag{3}$$

$$\mu(M_0) < \varepsilon. \tag{4}$$

PROOF. According to Lusin's theorem we can find a compact set $K \subseteqq [0,1]$ with $\mu([0,1]\setminus K) < \varepsilon/3$ such that $f\mid_K$ is continuous. Since both sets $\{x \in [0,1]; \mu(\{x\}) > 0\}$ and $\{y \in \mathcal{R}_f; \mu(f^{-1}(y)) > 0\}$ are countable, we can choose mutually disjoint closed sets $K_i \subset [0,1]$, $i = 1, \dots, m$, with $\operatorname{diam} K_i < \varepsilon$ and $\mu([0,1]\setminus \bigcup_{i=1}^{m} K_i) < \varepsilon/3$ and mutually disjoint closed sets $K^j \subset \mathbf{R}$, $j = 1, \dots, n$ with $\operatorname{diam} K^j < \varepsilon/2$ and

$$\mu\Big\{x \in [0,1]; f(x) \notin \bigcup_{j=1}^{n} f^{-1}(K^j)\Big\} < \frac{\varepsilon}{3}.$$

We set $k = m \cdot n$,

$$M_0 = [0,1]\setminus\Big[K \cap \bigcup_{i=1}^{m} K_i \cap \bigcup_{j=1}^{n} f^{-1}(K^j)\Big],$$

and define the mutually disjoint closed sets $M_{i \cdot n+j} = K_{i+1} \cap f^{-1}(K^j) \cap K$ for $0 \leqq i \leqq m-1$ and $1 \leqq j \leqq n$. Finally, we select mutually distinct numbers $t_1, \dots, t_k$ such that $\operatorname{dist}(t_{i \cdot n+j}, K^j) < \varepsilon/2$ if $0 \leqq i \leqq m-1$ and $1 \leqq j \leqq n$. Tietze's extension theorem now guarantees the existence of a continuous function $g\colon [0,1] \to [-\varepsilon, \varepsilon]$ such that $g(x) = t_l - f(x)$ whenever $x \in M_l$, $l = 1, \dots, k$. It is easy to verify that g and M_l, $l = 0, 1, \dots, k$, exhibit all properties required.

REMARK. Since for a fixed *continuous* measure μ the value $\mu(M)$ is controlled by the diameter of M, the following refinement of "*Preiss's Lemma*" (see Lemma 1.9 in [3]), which can be used to answer Problem 1 in the manner of Theorem 1.10 from [3], is obviously new.

LEMMA 3. *Let μ be a continuous (finite Borel regular) measure on* $[0,1]$ *and let the function $f\colon [0,1] \to \mathbf{R}$ be μ-measurable. Then for any $\varepsilon > 0$ there exists a continuous map $g\colon [0,1] \to [0,\varepsilon]$ such that $\mu((f+g)^{-1}(y)) < \varepsilon$ for any $y \in \mathbf{R}$.*

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DIRICHLET SETS IN VILENKIN GROUPS

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For the purposes of this paper, G will denote a compact, totally disconnected, abelian, metric group, i.e. a Vilenkin group, and Γ will denote the Pontryagin dual of G. We set $A(G)$ to be the algebra of absolutely convergent Fourier series on G and $PM(G)$ the Banach space dual of $A(G)$. Elements of $PM(G)$ are called pseudomeasures. If S is a pseudomeasure, write $\hat{S}(\gamma) = \langle S, \gamma \rangle$ where $\langle\ ,\ \rangle$ denotes the dual pairing of $PM(G)$ and $A(G)$. We call S a pseudofunction if $\hat{S}(\gamma) \to 0$ as $\gamma \to \infty$ in Γ. The collection of all pseudofunctions is called $PF(G)$.

A pseudomeasure is said to be supported on a closed set E if it is supported as a distribution with test functions from $A(G)$. A closed set is called a U-set if it supports no nontrivial pseudofunction. The set E is called a strong U-set if

$$\limsup_{\gamma \to \infty} |\hat{S}(\gamma)| = \sup_{\gamma \in \Gamma} |\hat{S}(\gamma)|$$

for all pseudomeasures supported on E.

A set E is called a Dirichlet set if the constant function 1 can be approximated uniformly on E by elements of Γ which go to infinity.

Our first result will be to show that certain sets are U-sets.

THEOREM 1. *Let E be a subset of the Vilenkin group G such that there is a sequence γ_n in Γ with $\operatorname{ord}(\gamma_n) \to \infty$ and an open subset U of $\mathbf{T}$ (the circle group), such that $\gamma_n(E) \cap U = \emptyset$. (Recall that γ_n is a map from G into $\mathbf{T}$). Then E is a U-set in G.*

PROOF. We use the Rajchman Theorem in compact, 0-dimensional, metric groups, see Theorem 3 of [1]. To do this, let

$$f(e^{it}) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

be a function in $A(\mathbf{T})$ with support in U. Then the functions $f \circ \gamma_n$ are in $A(G)$ and are 0 in a neighborhood of E. Also,

$$f \circ \gamma_n(x) = \sum_{m=-\infty}^{\infty} c_m \gamma_n^m = \sum_{m=1}^{\operatorname{ord}(\gamma_n)} \sum_{j=-\infty}^{\infty} c_{m+j\cdot\operatorname{ord}(\gamma_n)} \gamma_n^m. \tag{1}$$

Since $\text{ord}(\gamma_n) \to \infty$, and since $\sum |c_n| < \infty$, we see that $\widehat{f \circ \gamma_n}(\chi) \to 0$ if $\chi \neq 1$ and $\to c_0$ if $\chi = 1$. Since $\|f \circ \gamma_n\|_{A(G)} \leqq \|f\|_{A(\mathbf{T})}$, an application of Theorem 3 of [1] gives the result. □

In particular, we see that Dirichlet sets are U-sets by taking U above to be any open set separated from 1. However, even more can be seen from the next result.

THEOREM 2. *A Dirichlet set is a strong U-set.*

PROOF. Our proof is essentially that of [2]. See Corollary 1 of [4] and Corollary 2.4 of [5] for the case of the dyadic group.

Let E be a Dirichlet set and S a pseudomeasure supported on E. By considering γS in place of S, it is enough to show that

$$\limsup_{\gamma \to \infty} |\hat{S}(\gamma)| \geqq |\hat{S}(1)|. \tag{2}$$

Since the singleton containing 1 is a set of synthesis on $\mathbf{T}$, for every $\varepsilon > 0$, there is an $f_\varepsilon \in A(\mathbf{T})$ such that for $|z-1| < \varepsilon$ we have $f_\varepsilon(z) = z - 1$ and $\|f_\varepsilon\|_{A(\mathbf{T})} \leqq C\varepsilon$ where C is some constant independent of ε.

Let $\varepsilon > 0$ and pick γ in Γ with $\|\gamma - 1\|_{C(E)} < \varepsilon$. Then $f_\varepsilon \circ \gamma \in A(G)$ with $\|f_\varepsilon \circ \gamma\|_{A(G)} \leqq C\varepsilon$. Furthermore $f_\varepsilon \circ \gamma(x) = \gamma(x) - 1$ for all x in a neighborhood of E. Since S is supported on E, we have

$$|\hat{S}(\gamma) - \hat{S}(1)| = |\langle S, \gamma - 1 \rangle| = |\langle S, f_\varepsilon \circ \gamma \rangle| \leqq \|S\| \cdot C\varepsilon. \tag{3}$$

Since we may take $\gamma \to \infty$ as $\varepsilon \to 0$, this shows that (2) holds. □

In [4], K. Yoneda showed that a closed subgroup of measure 0 is a strong U-set in the group 2^ω. With the last result, we can show this same result for general Vilenkin groups.

COROLLARY 3. *A closed subgroup of measure 0 is a Dirichlet set. Conversely, in a group of bounded order, a Dirichlet set is a subset of a closed subgroup of measure 0.*

PROOF. If H is a closed subgroup of measure 0, the group G/H is of infinite order so $H^\perp$ is infinite. Therefore, there is a sequence $\gamma_n \to \infty$ in $H^\perp$. This shows H to be a Dirichlet set.

For the converse, let E be a Dirichlet set in G and assume that $x^p = 1$ for all $x \in G$. Then $\gamma^p = 1$ for all $\gamma \in \Gamma$. This shows that if

$$\sup_{x \in E} |\gamma(x) - 1| < 2 \sin \frac{\pi}{p} \tag{4}$$

then γ is identically 1 on E. Since E is a Dirichlet set, it is possible to find a sequence $\gamma_n \to \infty$ such that (4) holds. Then

$$E \subseteqq \bigcap_{n=1}^{\infty} \operatorname{Ker} \gamma_n. \tag{5}$$

But the right hand side of this equation is a subgroup of infinite index, so of measure 0. □

COROLLARY 4. *Translates of subgroups of measure* 0 *are strong* U*-sets.*
□

For comparison, see Theorem 6 of [1] and Corollary 2.4 of [6].

The next result is an analogue of a classical result of Salem. See [2] for a statement and [3] for a proof of the classical theorem. For the dyadic case, Yoneda has a similar result in [6]. However, Yoneda's result restricts the sequence of subgroups K_n to a particular case and the group has to be the dyadic group. Here, even groups of unbounded order are allowed.

THEOREM 5. *Let* G *be a Vilenkin group and* E *a subset of* G. *Assume that* $(K_n)_{n=0}^{\infty}$ *is a sequence of open subgroups in* G *and set* $A_n = \operatorname{card}\{xK_n : E \cap K_n$ *is non-void*$\}$. *If*

$$\liminf_{n\to\infty} \frac{A_n}{\log[G : K_n]} = 0, \tag{6}$$

then E *is a Dirichlet set.*

PROOF. Since G is a Vilenkin group, there is an increasing sequence of finite subgroups H_n of Γ with $\bigcup_{n=0}^{\infty} H_n = \Gamma$.

Fix a positive integer p. Our assumption shows that

$$\liminf_{n\to\infty} \frac{p^{A_n}+1}{[G : K_n]} = 0.$$

Fix an integer n with

$$\frac{p^{A_n}+1}{[G : K_n]} < \frac{1}{2\operatorname{card} H_p}.$$

Then

$$p^{A_n}+1 < \frac{1}{2}\frac{\operatorname{card} K_n^{\perp}}{\operatorname{card} H_p}. \tag{7}$$

This shows that $K_n^{\perp}$ intersects at least $p^{A_n}+1$ cosets of H_p. Let $\chi_1, \chi_2, \cdots, \chi_{p^{A_n}+1} \in K_n^{\perp}$ be representatives of these different cosets of H^p. By definition of A_n, there are points $x_1, x_2, \cdots, x_{A_n}$ in E with

$$E \subseteqq \bigcup_{m=1}^{A_n} x_m K_n. \tag{8}$$

Consider the $p^{A_n}+1$ points of $\mathbf{T}^{A_n}$ defined by $\chi_j(x_1), \cdots, \chi_j(x_{A_n})$ for $1 \leqq j \leqq p^{A_n}+1$. By dividing $\mathbf{T}^{A_n}$ into p^{A_n} equal rectangles, we find that there exist j and k with $1 \leqq j \neq k \leqq p^{A_n}+1$ with

$$|\chi_j(x_m) - \chi_k(x_m)| < 2\sin\frac{\pi}{p} \tag{9}$$

for $1 \leqq m \leqq A_n$.

Setting $\gamma_p = \chi_j \chi_k^{-1}$, we find

$$|\gamma_p(x_m) - 1| < 2 \sin \frac{\pi}{p} \tag{10}$$

for $1 \leqq m \leqq A_n$.

Since $E \subseteqq \bigcup_{m=1}^{A_n} x_m K_n$ and $\gamma_p \in K_n^{\perp}$ is constant on cosets of K_n,

$$|\gamma_p(x) - 1| < 2 \sin \frac{\pi}{p} \tag{11}$$

for all $x \in E$. Since χ_j and χ_k are in different cosets of H_p, $\gamma_p \notin H_p$. Thus $\gamma_p \to \infty$ in Γ as $p \to \infty$. But

$$\sup\{|\gamma_p(x) - 1| : x \in E\} < 2 \sin \frac{\pi}{p} \to 0$$

as $p \to \infty$, showing that E is a Dirichlet set. □

EXAMPLES 1. The group of p-adic integers, Δ_p, has very few subgroups. In fact, the only non-trivial closed subgroups are of finite index. Therefore, the results in [1] are of little value in producing U-sets. However, if we set

$$E = \{x = x_0 + x_1 p + x_2 p^2 + \cdots \mid x_n \neq 1 \text{ for all } n \geqq 0\}, \tag{12}$$

and let γ_n be the character on Δ_p such that

$$\gamma_n(x) = e^{2\pi i (x_0 + x_1 p + \cdots + x_{n-1} p^{n-1})/p^n}$$

we see that E is a U-set. The much smaller set obtained by letting $x_n = 0$ or 1 was shown to be a U-set by Y. Meyer in [7].

2. It is possible for a set to be a Dirichlet set even thoguh the condition of the last theorem does not hold. For example, if we let G be the group $\prod_{n=1}^{\infty} \mathbf{Z}(p_n)$, and set E to be the set $\prod_{n=1}^{\infty} [0, a_n]$, then E is a Dirichlet set whenever $\lim a_n/p_n = 0$.

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THE ABSTRACT PRIME NUMBER THEOREM FOR FUNCTION FIELDS

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I. Introduction

In a monograph [7] Knopfmacher, motivated by earlier work of Fogels [3] on polynomial rings and algebraic function fields, developed the concept of an arithmetical semigroup satisfying Axiom $A^{\#}$. He showed how further concrete motivation for introducing Axiom $A^{\#}$ is provided by various asymptotic enumeration theorems regarding several arithmetical categories, some of which are not usually viewed in a number theoretical way, and investigated a variety of more basic (number theoretical) consequences of the axiom.

One of the cores of his investigations is an abstract prime number theorem for additive arithmetical semigroups satisfying Axiom $A^{\#}$. This paper will give a report on recent results about such a prime number theorem ([1], [4]) and concerns itself with the intrinsic connection between the prime number theorem and a (new) Axiom $\overline{A^{\#}}$.

In order to formulate the basic Axiom and the results in question, first recall that an *additive arithmetical semigroup* G is by definition a free commutative semigroup with identity element 1, generated by a countable set $\mathcal{P}$ of primes and admitting an integer valued degree mapping $\delta\colon G \to \mathbf{N} \cup \{0\}$ which satisfies

(i) $\delta(1) = 0$ and $\delta(p) > 0$ for all $p \in \mathcal{P}$,

(ii) $\delta(ab) = \delta(a) + \delta(b)$ for all $a, b \in G$,

and

(iii) the total number $N_G(x)$ of elements $a \in G$ of degree $\delta(a) \leqq x$ is finite for each real $x > 0$.

The investigations described in [7] are particularly concerned with arithmetical consequences of assumptions about the total number $\overline{P}(n)$ of primes of degree n in G or about the total number $\overline{G}(n)$ of elements of degree n in G, known as Axiom $A^{\#}$.

Axiom $A^{\#}$. *There exist constants $A > 0$, $q > 1$ and ν with $0 \leqq \nu < 1$ (all depending on G), such that*

$$\overline{G}(n) = Aq^n + O(q^{\nu n}) \quad as \quad n \to \infty. \tag{1}$$

The *abstract prime number* theorem for additive arithmetical semigroups satisfying Axiom $A^{\#}$ is an estimate of $\overline{P}(n)$. The proof of such an estimate depends on the knowledge about the distribution of the zeros of the *generating function*

$$Z_G^{\#}(y) = \sum_{n=1}^{\infty} \overline{G}(n) y^n = \prod_{p \in \mathcal{P}} (1 - y^{\delta(p)})^{-1}, \tag{2}$$

which is, by (1), a meromorphic function in $|y| < q^{-\nu}$ with a simple pole at $y = q^{-1}$ with residue A. Lemma 8.5 in [7] states that $Z_G^{\#}(y) \neq 0$ for $|y| = q^{-1}$, but the proof works only for $y \neq -q^{-1}$. There are examples (see Indlekofer–Manstavicius–Warlimont [4] and Remark 1) of additive arithmetical semigroups G satisfying (1) with $\nu = 1/2$ and $Z_G^{\#}(-q^{-1}) = 0$. On the other hand Theorem 1 (see [4]) shows that, if G satisfies (1) with $\nu < 1/2$, then $Z_G^{\#}(-q^{-1}) \neq 0$.

Now, if $Z_G^{\#}(y) \neq 0$ for $|y| = q^{-1}$ the following abstract prime number theorem, due to Knopfmacher [7], holds: *For any* $\alpha > 1$,

$$\overline{P}(n) = \frac{q^n}{n} + O\left(\frac{q^n}{n^{\alpha}}\right) \quad \textit{as} \quad n \to \infty. \tag{3}$$

This result does not cover the case where $Z_G^{\#}(-q^{-1}) = 0$ and leaves open some problems (cf. Knopfmacher [7], p. 77–78), such as that of deriving sharper estimates or of proving the same theorem under weaker hypotheses than Axiom $A^{\#}$ or, conversely, of deducing Axiom $A^{\#}$ or weaker forms of it from asymptotic assumptions about $\overline{P}(n)$.

A first contribution towards the complete solution of these problems is a recent result by Cohen [1], who gave a sharper estimate than (3), namely

$$\overline{P}(n) = \frac{q^n}{n} + O(q^{n\theta}) \quad \text{as} \quad n \to \infty \tag{4}$$

with some constant θ satisfying $\max(1/2, \nu) < \theta < 1$, but this result is only valid in the case $Z_G^{\#}(-q^{-1}) \neq 0$.

In [4] Indlekofer, Manstavicius and Warlimont gave (in a more general setting) much sharper results valid also in the case $Z_G(-q^{-1}) = 0$. The key for these results is the fact that, if $Z_G(-q^{-1}) = 0$, then $Z_G(y) \neq 0$ for all $|y| < q^{-\nu}$, $y \neq -q^{-1}$ (Theorem 1). The results in [4] improve (4) by giving an asymptotic formula for $\overline{P}(n)$ with a remainder term $O(q^{n(\nu+\varepsilon)})$, where $\varepsilon > 0$ is arbitrarily small (Theorem 2) which holds also in the case $Z_G^{\#}(-q^{-1}) = 0$. Conversely, such an asymptotic formula for $\overline{P}(n)$ with a remainder term $O(q^{n(\nu+\delta)})$ for each $\delta > 0$ yields a weaker form of (1), namely (Corollary 1)

$$\overline{G}(n) = Aq^n + O(q^{n(\nu+\varepsilon)}) \quad \text{for each} \quad \varepsilon > 0;$$

in fact, it shows that the generating function $Z_G^{\#}(y)$ of G is a meromorphic function in the disc $|y| < q^{-\nu}$ with a simple pole at $y = q^{-1}$.

Assuming some mild conditions on the boundary behaviour of $Z_G^{\#}(y)$, which are fulfilled by all the examples treated in [7], we obtain an exact formula for $\overline{P}(n)$ (Theorem 5 together with (8)) and the equivalence of the asymptotic formulas of $\overline{G}(n)$ and $\overline{P}(n)$ (Corollary 3).

This suggests that we consider arithmetical semigroups satisfying a modified version of Axiom A$^{\#}$ which we call Axiom $\overline{\text{A}^{\#}}$, the abstract theory of which is especially convenient for the purpose of deriving asymptotic conclusions about the specific systems which are considered in [7].

Many conclusions in one axiom system are parallel to conclusions in another. However, the consequences of Axiom $\overline{\text{A}^{\#}}$ tend to give more precise information and have simpler proofs.

We shall not pursue the search for consequences of Axiom $\overline{\text{A}^{\#}}$ further here but we intend to come back to this topic in a different place.

We will end this paper by applying our considerations to some special cases of semigroups satisfying Axiom A$^{\#}$ and Axiom $\overline{\text{A}^{\#}}$.

2. Associated power series

Any complex-valued function on G is called an arithmetical function on G. For a given arithmetical function f we put

$$\overline{f}(n) = \sum_{\delta(a)=n} f(a), \quad n \in \mathbf{N}, \qquad f^{\#}(y) = \sum_{n=0}^{\infty} \overline{f}(n) y^n.$$

$f^{\#}(y)$ is called the *associated power series* of the arithmetical function f.

We recall the definition (2) of the *generating function of G* and put

$$Z^{\#}(y) := Z_G^{\#}(y) = \sum_{n=0}^{\infty} \overline{G}(n) y^n.$$

The *von Mangoldt function* Λ on G is defined by

$$\Lambda(a) = \begin{cases} \delta(p) & \text{if } a \text{ is a prime-power } p^r \neq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then the *associated power series of* Λ is (see [7], p. 77)

$$\Lambda^{\#}(y) = y \frac{Z^{\#\prime}(y)}{Z^{\#}(y)}. \tag{6}$$

Furthermore, by definition,

$$\overline{\Lambda}(n) = \sum_{\substack{p\in\mathcal{P},r\geqq 1\\ \delta(p^r)=n}} \delta(p) = \sum_{r|n} \frac{n}{r}\overline{p}\left(\frac{n}{r}\right) \tag{7}$$

and, by the Möbius inversion formula,

$$n\overline{P}(n) = \sum_{r|n} \overline{\Lambda}(r)\mu\left(\frac{n}{r}\right), \tag{8}$$

where μ denotes the Möbius function (on $\mathbf{N}$).

This shows that one may study $\overline{P}(n)$ instead of investigating $\overline{\Lambda}(n)$.

3. von Mangoldt's function and the Nevanlinna class N

By Axiom $A^{\#}$ we can write

$$Z^{\#}(y) = A\sum_{n=0}^{\infty} q^n y^n + \sum_{n=0}^{\infty} r_n y^n = \frac{A}{1-qy} + \sum_{n=0}^{\infty} r_n y^n, \tag{9}$$

where $r_n = \overline{G}(n) - Aq^n = O(q^{\nu n})$. Therefore one sees that the last series represents a holomorphic function of y in the open disc $|y| < q^{-\nu}$.

We define a function $Z(y)$ holomorphic in the open disc $|y| < q^{-\nu}$ by means of

$$Z(y) = (1-qy)Z^{\#}(y). \tag{10}$$

Then

$$\Lambda^{\#}(y) = \frac{qy}{1-qy} + y\frac{Z'(y)}{Z(y)}. \tag{11}$$

If $Z(y) \neq 0$ for $|y| \leqq q^{-1}$ then $Z(y) \neq 0$ for $|y| < q^{-\theta}$ with some $\nu < \theta < 1$ because the set of zeros of $Z(y)$ has no limit point in $|y| < q^{-\nu}$. This shows that, under the above assumptions about the zeros of $Z(y)$, $\Lambda^{\#}(y) - qy/(1-qy)$ is holomorphic for $|y| < q^{-\theta}$. This implies immediately that

$$\overline{\Lambda}(n) = q^N + O(q^{\theta N}) \tag{12}$$

and, by (8),

$$P(N) = \frac{q^N}{N} + \frac{1}{N}\sum_{\substack{r|N\\ r<N}} q^r\mu(N/r) + O\left(\frac{q^{N\theta}}{N}\right). \tag{13}$$

This sharpens (3) and (4), if $Z(y) \neq 0$ for $|y| \leqq q^{-1}$, but we shall see that as a consequence of the next theorems a much stronger result is true.

Knopfmacher [7], Lemma 8.5, stated that $Z^{\#}(y) \neq 0$ for $|y| = q^{-1}$ but his proof shows only the following

LEMMA. *$Z^{\#}(y)$ has no zeros for $|y| \leqq q^{-1}$ except possibly a simple zero at $y = -q^{-1}$.*

REMARK 1. R. Warlimont and (as the author was told) Wen-Bin Zhang independently noticed that Lemma 8.5 of Knopfmacher's work [7] is not correct. Both of them gave an example of a generating function $Z^{\#}(y) = (1-qy)^{-1}Z(y)$ such that $Z(y)$ is holomorphic in the disc $|y| < q^{-1/2}$ and $Z(-q^{-1}) = 0$.

Concerning the zeros of the generating function of an additive arithmetical semigroup we have

THEOREM 1 (see [4], Theorems 1 and 2). *Let $Z(y) = (1-qy)Z^{\#}(y)$ be holomorphic in the open disc $|y| < q^{-\nu}$ $(0 \leqq \nu < 1)$ and $Z(q^{-1}) > 0$. Then the following two assertions hold.*

(i) *If $Z(-q^{-1}) = 0$ then $Z(y) \neq 0$ for all $|y| < q^{-\nu}$, $y \neq -q^{-1}$.*

(ii) *Let $\nu = 1/2$. If*

$$Z(y)Z(-y) = o((1-q^{1/2}y)^{-1}) \quad \text{as} \quad y \to q^{-1/2},\ 0 < y < q^{-1/2}, \tag{14}$$

holds then $Z(-q^{-1}) \neq 0$.

REMARK 2. In [4] we gave an example of a generating function $Z^{\#}(y) = (1-qy)^{-1}Z(y)$ such that $Z(y)$ is holomorphic for $|y| < q^{-1/2}$, $Z(-q^{-1}) = 0$ and

$$Z(y)Z(-y)(1-q^{1/2}y) \to c \neq 0 \quad \text{as} \quad y \to q^{-1/2},\ 0 < y < q^{-1/2}.$$

More exactly we showed

$$Z(y) = (1+qy)(1+qy^2)^{1/2}(1-qy^2)^{-1/2}H(y)$$

where $H(y)$ is holomorphic in the closed disc $|y| \leqq q^{-1/2}$. Axiom $A^{\#}$ is fulfilled with $\nu = 1/2$.

REMARK 3. Observe that the hypothesis

$$\overline{G}(n) = Aq^n + O(q^{n/2}/n^{\gamma}), \qquad \gamma > 1/2, \tag{15}$$

implies (14). The author has been informed that the implication "(15)⇒ $Z(-q^{-1}) \neq 0$" has been shown by Wen-Bin Zhang, too.

THEOREM 2 (see [4], Theorem 1). *Let G satisfy Axiom $A^{\#}$, and let $0 < \varepsilon < 1-\nu$. Then the following assertions are true.*

(i) *Assume that $Z(-q^{-1}) \neq 0$. Then there exist $l = l(\varepsilon) \in \mathbf{N}_0$, $0 < \varepsilon' < \varepsilon$ and complex numbers β_i, $i = 1, \dots, l$, such that the following holds:*

(a) $\Lambda^{\#}(y) = \frac{y}{q^{-1}-y} - y\sum_{i=1}^{l}\frac{1}{\beta_i - y} + yR(y)$

where

(16) $$q^{-1} < \min_{i=1,\dots,l} |\beta_i| \leqq \max_{i=1,\dots,l} |\beta_i| \leqq q^{-\nu-\varepsilon}$$

and $R(y)$ *is holomorphic for* $|y| \leqq q^{-\nu-\varepsilon'}$; *furthermore*

(17) $$\overline{\Lambda}(N) = q^N - \sum_{i=1}^{l} \beta_i^{-N} + O(q^{N(\nu+\varepsilon')}) \quad as \quad N \to \infty.$$

(b) *The numbers* β_i $(i = 1, \dots, l)$ *are the zeros of* $Z(y)$ *in the disc* $|y| \leqq q^{-\nu-\varepsilon}$.

(ii) *Assume that* $Z(-q^{-1}) = 0$. *Then*

$$\Lambda^{\#}(y) = \frac{y}{q^{-1} - y} + \frac{y}{q^{-1} + y} + yR(y),$$

where $R(y)$ *is holomorphic for* $|y| < q^{-\nu}$; *furthermore,*

$$\overline{\Lambda}(N) = q^N(1 - (-1)^N) + O(q^{N(\nu+\varepsilon)})$$

as $N \to \infty$.

REMARK 4. Of course, the zeros in (i), (b) are counted according to their multiplicities. ($l = 0$ means that $Z(y)$ has no zeros in the disc $|y| \leqq q^{-\nu-\varepsilon}$.) The assertion contains the fact that the set of zeros of a function $f \neq 0$, which is holomorphic in a region Ω, has no limit points in Ω.

THEOREM 3 (see [4], Remark 2). (i) *If for every* $\varepsilon > 0$ *the assertion* (i), (a), *of Theorem* 2 *is true with given* $0 < \varepsilon' < \varepsilon$, $l = l(\varepsilon) \in \mathbf{N}_0$ *and complex numbers* $\beta_1, \dots, \beta_l$ *satisfying* (16) *then a solution* $Z^{\#}(y)$ *of* (6) *has the form* $Z^{\#}(y) = Z(y)/(1 - qy)$, *where* $Z(y)$ *is holomorphic in the disc* $|y| < q^{-\nu}$ *and* $\beta_1, \dots, \beta_l$ *are the zeros of* $Z(y)$ *in the disc* $|y| < q^{-\nu-\varepsilon}$.

(ii) *If the assertion* (ii) *of Theorem* 2 *is true then a solution* $Z^{\#}(y)$ *of* (6) *has the form* $Z^{\#}(y) = Z(y)/(1 - qy)$ *where* $Z(y)$ *is holomorphic in the disc* $|y| < q^{-\nu}$ *and* $-q^{-1}$ *is the only zero of* $Z(y)$ *in this disc.*

COROLLARY 1. *Under the assumptions of Theorem* 3

$$\overline{G}(N) = Z(q^{-1})q^N + O(q^{N(\nu+\delta)}) \quad as \quad N \to \infty$$

holds for each $\delta > 0$.

Motivated by examples of semigroups satisfying Axiom $A^{\#}$ (see [7]) we impose some (mild) conditions on $Z(y)$ and obtain sharper results. In order to formulate the next theorem let us define some spaces of holomorphic functions.

If f is holomorphic in the unit disc then for $r \in [0,1)$ put

$$M_0(f;r) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta\right),$$

$$M_p(f;r) := \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\right\}^{1/p} \qquad (0 < p < \infty),$$

$$M_\infty(f;r) := \sup |f(re^{i\theta})|.$$

Here, $\log^+ x = \max(\log x, 0)$ for $x > 0$.

It is well-known that M_o, M_p and M_∞ are monotonically increasing functions of r in $[0,1)$ (see Rudin [8], Theorem 17.6). This suggests the following

DEFINITION. If f is holomorphic in the unit disc then we put, for $0 \leqq p \leqq \infty$,

$$\|f\|_p := \lim_{r\to 1} M_p(f;r).$$

For $0 < p \leqq \infty$ and $p = 0$ respectively the class H^p (for Hardy) and N (for Nevanlinna) are defined to consist of all f, holomorphic in the open unit disc, for which $\|f\|_p < \infty$. It is clear that

$$H^\infty \subset H^p \subset H^s \subset N \quad \text{if} \quad 0 < s < p < \infty.$$

In the following we assume that the radius of convergence for $Z(y) = (1-qy)Z^\#(y)$ is equal to $q^{-\nu}$, $0 \leqq \nu < 1$. Then, by well-known results about functions from N, we obtain the following (see Rudin [8], pp. 370).

PROPOSITION. *Let $\overline{Z}(z) := Z(q^{-\nu}z)$. Suppose Z is not identically zero and $\beta_1, \beta_2, \ldots$ are the zeros of Z in the disc $|y| < q^{-\nu}$, listed according to their multiplicities. Then, if $\overline{Z} \in N$, the following holds:*

(i)

$$\sum_{n=1}^{\infty}(1 - |\alpha_n|) < \infty, \tag{18}$$

where α_n is defined by $\alpha_n = q^\nu \beta_n$ $(n = 1, 2, \ldots)$.

(ii)

$$Z(y) = c\prod_{n=1}^{\infty}\frac{q^{-\nu}\alpha_n - y}{q^{-\nu} - \overline{\alpha}_n y}\cdot\frac{|\alpha_n|}{\alpha_n}\exp\left(\int_{-\pi}^{\pi}\frac{e^{it} + q^\nu y}{e^{it} - q^\nu y}d\lambda(t)\right), \tag{19}$$

where c is a constant, $|c| = 1$, and λ is a real measure.

(iii) *The radial limit* Z^* *of* Z, *i.e.*

$$\lim_{r\to q^{-\nu}} Z(re^{it}) = Z^*(q^{-\nu}e^{it})$$

exists almost everywhere, $Z^* \neq 0$ *a.e. and* $\log|Z^*| \in L^1([-\pi,\pi])$.

REMARK 5. a) By Lebesgue's decomposition theorem we may assume that

$$d\lambda(t) = \frac{1}{2\pi}\log|Z^*(q^{-\nu}e^{it})|dt + d\lambda_s(t)$$

where λ_s and the Lebesgue measure are mutually singular.

b) Suppose (18) holds for a sequence $\{\alpha_n\}$, $|\alpha_n| < 1$, and λ is a real measure, then (19) defines a function $\overline{Z} \in N$ by $\overline{Z}(z) = Z(q^{-\nu}z)$.

As a consequence of the Proposition we state the following

THEOREM 4. *Suppose* Z *is as in the Proposition. Then*

(20)

$$\frac{\Lambda^{\#}(y)}{y} = \frac{q}{(1-qy)} + \sum_{n=1}^{\infty}\left(\frac{\overline{\alpha}_n q^{\nu}}{1-\overline{\alpha}_n q^{\nu}y} - \frac{q^{\nu}}{\alpha_n - q^{\nu}y}\right) + \frac{d}{dy}\left(\int_{-\pi}^{\pi}\frac{e^{it}+q^{\nu}y}{e^{it}-q^{\nu}y}d\lambda(t)\right).$$

REMARK 6. If $\Lambda^{\#}(y)/y$ has the form (20) with a real measure λ and complex numbers $\alpha_1, \alpha_2, \dots, |\alpha_n| < 1$ $(n = 1, 2, \dots)$ satisfying (15), then a solution Z of (11) is given by (19) up to a constant factor.

An easy computation gives

THEOREM 5. *If* $\Lambda^{\#}(y)/y$ *has the form* (20) *then*

$$\overline{\Lambda}(N) = q^N - \sum_{n=1}^{\infty}(1-|\alpha_n|^N)(1+|\alpha_n|^N)\beta_n^{-N} + 2Na_Nq^{\nu N}, \tag{21}$$

where

$$a_N = \int_{-\pi}^{\pi} e^{-iNt}d\lambda(t).$$

The preceding results and the examples by Fogels [3] and Knopfmacher [7] motivate:

AXIOM $\overline{\text{A}^{\#}}$. *There exist constants* $q > 1$ *and* ν *with* $0 \leqq \nu < 1$ *(all depending on* G*), such that*

(i) *the function* $Z(y) := (1-qy)Z_G^{\#}(y)$ *is holomorphic in the open disc* $|y| < q^{-\nu}$, *and* $Z(q^{-1}) > 0$,

(ii) *the function* $\overline{Z}(z) := Z(q^{-\nu}z)$ *is an element of the Nevanlinna class* N.

REMARK 7. From Axiom $A^{\#}$ one can only deduce that the function $Z(y)$ is holomorphic in the open disc $|y| < q^{-\nu}$. On the other hand, Axiom $\overline{A^{\#}}$ does not imply that, if $Z(y) = \sum_{n=1}^{\infty} a_n y^n$, the estimate $a_n = O(q^{\nu n})$ holds. Furthermore, we note that all the examples treated in Knopfmacher [7] actually satisfy Axiom $\overline{A^{\#}}$ where in each case the function $Z(y)$ has the form

$$Z(y) = \Big(\frac{a}{1-q^{\nu}y} + \frac{b}{1+q^{\nu}y}\Big)H(y) \quad \text{with some} \quad q>1, \quad \text{and} \quad a,b \in \mathbf{C}$$

and a function H which is holomorphic in $|y| \leqq q^{-\nu}$. This implies that if $Z(y) = \sum_{n=1}^{\infty} a_n y^n$, then for some $\varepsilon > 0$,

$$a_n = \big(aH(q^{-\nu}) + b(-1)^n H(-q^{-\nu})\big)\, q^{\nu n} + \int_{|y|=q^{-\nu+\varepsilon}} Z(y)\frac{dy}{y^{n+1}} =$$
$$= \big(aH(q^{-\nu}) + b(-1)^n H(-q^{-\nu})\big)\, q^{\nu n} + O(q^{(\nu-\varepsilon)n}) \quad \text{as} \quad n \to \infty.$$

We observe that, by formula (8), Theorems 2 and 5 give prime number theorems for additive arithmetical semigroups satisfying Axiom $A^{\#}$ and Axiom $\overline{A^{\#}}$, respectively. Now, if $Z(-q^{-1}) = 0$ then Theorem 2, (ii) gives for even N only the estimate

$$\overline{\Lambda}(2m) = O\Big(\frac{q^{2m(\nu+\varepsilon)}}{2m}\Big).$$

Therefore it is worthwhile to formulate

COROLLARY 2. *Assume that the additive arithmetical semigroup G satisfies Axiom $\overline{A^{\#}}$. Let $Z(-q^{-1}) = 0$. Then, with the notation of the Proposition,*

$$\overline{\Lambda}(N) = q^N(1-(-1)^N) + 2Na_N q^{\nu N},$$

where

$$a_N + \int_{-\pi}^{\pi} e^{-iNt}\, d\lambda(t).$$

There is a converse result which follows from Theorems 4 and 5 and from Remark 5 made after the statement of Theorem 4. We have

COROLLARY 3. *Axiom $\overline{A^{\#}}$ is equivalent to the asymptotic formula* (21) *for $\overline{\Lambda}(N)$ and also to the corresponding formula for $\overline{P}(N)$.*

4. Examples of arithmetical semigroups satisfying Axiom $\overline{\mathrm{A}}^{\#}$

We consider some examples of arithmetical semigroups given in Knopfmacher [7] and we will freely use the properties described there (see [7], §1 and §6).

EXAMPLE 1: *Galois polynomial rings.* A simple but nevertheless quite interesting example is provided by the semigroup $G = G[q,t]$ of all monic polynomials in a Galois polynomial ring $GF[q,t]$, i.e. the polynomial ring $F[t]$ in an indeterminate t over the finite Galois field $F = GF(q)$ with q elements.

G forms a semigroup under multiplication and, together with the usual degree mapping on polynomials, an additive arithmetical semigroup with generating function

$$Z_G^{\#}(y) = \frac{1}{1-qy}.$$

Thus G trivially satisfies Axiom $\overline{\mathrm{A}}^{\#}$.

EXAMPLE 2: *Finite modules on* $GF[q,t]$. Let F_q denote the category of all finitely generated torsion modules over the above ring $GF[q,t]$. F_q is an additive arithmetical category and its associated additive arithmetical semigroup, i.e. the set of all isomorphism classes of modules in F_q, has the generating function

$$Z_{F_q}^{\#}(y) = \prod_{r=1}^{\infty}(1-qy^r)^{-1}.$$

Then

$$Z(y) = Z_{F_q}(y) = \frac{1}{1-qy^2}\prod_{r=3}^{\infty}(1-qy^r)^{-1} = \frac{1}{1-qy^2}H(y),$$

and we conclude that $\nu = 1/2$ and H is holomorphic and different from zero in the disc $|y| < q^{-1/3}$.

It is easy to show that if $0 < p < 1$, the estimate

$$\int_{-\pi}^{\pi}\frac{dt}{|1-qr^2e^{2it}|^p} \ll \int_{-\pi}^{\pi}\frac{dt}{|1-q^{1/2}re^{it}|^p} + \int_{-\pi}^{\pi}\frac{dt}{|1+q^{1/2}re^{it}|^p} \ll 1$$

holds uniformly for all $r \in [0, q^{-1/2})$. This yields $\overline{Z} \in H^p \subset N$, and Axiom $\overline{\mathrm{A}}^{\#}$ is satisfied.

REMARK 8. We observe that Z has no zeros in the disc $|y| < q^{-1/2}$, and it is not difficult to show (see Rudin [8], Theorem 17.17) that the measure λ in the representation (19) of Z is determined by

$$d\lambda(t) = \frac{1}{2\pi}\log\left|Z^*(q^{-1/2}e^{it})\right| dt.$$

EXAMPLE 3: *Integral divisors on algebraic function fields.* Let K denote a field of algebraic functions in one variable over an exact finite constant field $GF(q)$ with q elements, i.e. let K be an extension field of finite degree over the field of fractions $GF(q,t)$ of the polynomial ring $GF[q,t]$.

We consider here the multiplicative semigroup G_K of all integral divisors of K. Then G_K, together with the degree function $\delta(j) = \log_q N(j)$ where $N(j)$ denotes the "absolute norm" of j, forms an additive arithmetical semigroup, and the generating function is given by

$$Z_K^{\#}(y) = \frac{L(y)}{(1-qy)(1-y)},$$

where $L(y)$ is a polynomial with rational integer coefficients whose degree is twice the "genus" of K. By a theorem of A. Weil, every zero of $L(y)$ lies on the circle $|y| = q^{-1/2}$ (see Eichler [2], Chapter V, §5). Thus

$$Z(y) = Z_{G_K}(y) = \frac{L(y)}{1-y}.$$

As in the previous example we conclude $\overline{Z} \in H^p \subset N$ $(0 < p < 1)$. Here $\nu = 0$ and the zeros of $Z(y)$ are exactly the zeros of $L(y)$. Therefore G_K satisfies Axiom $\overline{\mathrm{A}^{\#}}$ and

$$\overline{\Lambda}(N) = \overline{\Lambda}_{G_K}(N) = q^N - \sum_{i=1}^{2q} \beta_i^{-N} + 1,$$

where $\beta_1, \ldots, \beta_{2g}$ are the zeros of $L(y)$ $(|\beta_i| = q^{-1/2}, i = 1, \ldots, 2g)$.

EXAMPLE 4: *Ideals in the principal orders of an algebraic function field.* Let D denote the ring of all integral functions in the algebraic function field K discussed in the previous example. The set G_D of all non-zero ideals of the ring D may be identified with a subsemiring of G_K, namely with the set of those integral divisors of K that are not divisible by prime divisors of K induced by the denominator divisor of t in $GF(q,t)$.

The generating function of G_D is given by

$$Z_{G_D}^{\#}(y) = \frac{Q(y)}{1-qy}$$

where $Q(y)$ is a polynomial with rational integer coefficients whose zeros lie on the circles $|y| = q^{-1/2}$ and $|y| = 1$. G_D obviously satisfies Axiom $\overline{\mathrm{A}^{\#}}$.

EXAMPLE 5: *Finite modules over a ring of integral functions..* Let $F = F_D$ denote the category of all finitely generated torsion modules over the ring D considered above, and denote by G_F the associated additive arithmetical semigroup of all isomorphism classes of modules in F. Then the generating function of G_F is

$$Z^{\#}_{G_F}(y) = \prod_{r=1}^{\infty} \frac{Q(y^r)}{1 - qy^r}$$

with the polynomial $Q(y)$ that occurs in Example 4. This yields

$$Z(y) = Z_{G_F}(y) = \frac{Q(y)Q(y^2)}{1 - qy^2} \prod_{r=3}^{\infty} \frac{Q(y^r)}{1 - qy^r}.$$

In the same way as above we deduce $\nu = 1/2$ and $\overline{Z} \in H^p \subset N$ $(0 < p < 1)$. Furthermore, $Z(y) \neq 0$ for $|y| < q^{-1/2}$ (Z has only zeros of modulus $\geqq q^{-1/2}$). Thus G_F satisfies Axiom $\overline{\mathrm{A}^{\#}}$.

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ON THE DENSITY OF PRIME VECTORS IN LATTICES

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Let n be a given natural number. Lattices of $\mathbf{Z}^n$ are the cosets of any nonzero subgroup of the additive group $\mathbf{Z}^n$. Each lattice $\Lambda \subsetneqq \mathbf{Z}^n$ has the form $\mathbf{Z}^k A + b$, where $1 \leqq k \leqq n$ is the dimension of Λ, A is an integral $k \times n$ matrix of rank k, and b is a vector of $\mathbf{Z}^n$. An n dimensional lattice in $\mathbf{Z}^n$ is called complete.

A vector of $\mathbf{Z}^n$ is called prime if all coordinates are primes. $\pi(N, \Lambda)$ will denote the number of prime vectors of the lattice Λ all of whose coordinates are positive primes not exceeding N.

Hua Loo Keng [1] asked, under which conditions there are infinitely many prime vectors in a lattice of dimension 2. A partial answer was given for complete lattices by A. I. Vinogradov [2]. The first named author [3] indicated necessary and sufficient conditions, under which there are infinitely many prime vectors in a lattice of any dimension. On the basis of the multi-dimensional circle method, Wu Fang [4] proved an asymptotic formula for the number of solutions of certain systems of linear equations in prime variables. This enabled us to obtain results to $\pi(N, \Lambda)$ for a more extensive class of lattices in [5]. Finally a more precise result was discussed in [6] with an extended class of lattices.

However, all of these results refer to lattices with dimension $k > n/2$ in $\mathbf{Z}^n$. In the present work we drop this condition on the dimension but we derive only upper estimates.

We introduce some notations. $\lambda_j \colon \mathbf{Z}^n \to \mathbf{Z}$ $(j = 1, \dots, n)$ is the j-th projection, i.e. $\lambda_j(x)$ is the j-th coordinate of the vector x. $|x$ is the maximum norm. For the lattice $\Lambda = Z^k A + b$ we denote by A' the matrix obtained by attaching the vector b to the matrix A as an additional row.

The lattices Λ_1 and Λ_2 are called equivalent if $\pi(N, \Lambda_1) = \pi(N, \Lambda_2)$ for all sufficiently large N. We denote by P the class of lattices $\Lambda = \mathbf{Z}^k A + b$ whose matrix A has non-zero columns, each two columns of A' are linearly independent, and elements of any column of A' are coprime. Any integral lattice containing more than one prime vector is equivalent to some lattice of the class P.

THEOREM 1. *Let $\Lambda = \mathbf{Z}^k A + b$ be a lattice of the class P in $\mathbf{Z}^n$. Let $d_j(A)$ be the greatest common divisor of all elements of the j-th column of the matrix A, let $d_{ij}(A)$ be the greatest common divisor of all second order* *minors formed by the i-th and j-th columns of A',*

$$g=\prod_{j=1}^{n} d_j(A) \prod_{1\leqq i<j\leqq n} d_{ij}(A'), \tag{1}$$

and let $\omega(p)$ be the number of solutions of the congruence

$$\prod_{j=1}^{n} \lambda_j(xA+b)\equiv 0 \pmod p \qquad (x\in \mathbf{Z}^k), \tag{2}$$

and

$$\pi(N,x_0,A,b)=\operatorname{card}\left\{x\in \mathbf{Z}^k, |x-x_0|\leqq \frac{N}{2},\ xA+b \text{ is a prime vector}\right\}.$$

Then for $x_0\in \mathbf{R}^k$, $N\geqq 2$ we have

$$\pi(N,x_0,A,b)\leqq c(n)\frac{N^k}{\log^n N}\exp\sum_{p|g}\frac{np^{k-1}-\omega(p)}{p^k}, \tag{3}$$

where $c(n)$ is a positive constant, depending only on n.

THEOREM 2. *If with the notations of Theorem 1*

$$m=\inf_{\substack{x\in\mathbf{R}^k\\ x\neq 0}}\frac{|xA|}{|x|},$$

then for $N\geqq 2m$ the inequality

$$\pi(N,\Lambda)\leqq \frac{c(n)}{\log^n \frac{N}{m}}\left(\frac{N}{m}\right)^k \exp\sum_{p|g}\frac{np^{k-1}-\omega(p)}{p^k} \tag{4}$$

is valid.

The lattice $\mathbf{Z}^kA+b$ is called unimodular if the k-th order minors of the matrix A are coprime. Any lattice in $\mathbf{Z}^n$ is the intersection of a unimodular and a complete lattice of the form $d\mathbf{Z}^n+b$, where $d\neq 0$ is an integer (see [3]). (A more precise notation is $\mathbf{Z}^nD+b$ where D is the identity matrix multiplied by d.)

We can improve Theorem 2 by making use of this structure of the lattice. The next theorem shows some similarity to the well-known Brun–Titchmarsh inequality. The lattice $d\mathbf{Z}^n+b$ is called primitive if all coordinates of b are coprime to d.

THEOREM 3. *If under the conditions of Theorem* 1 Λ *is a unimodular lattice,* $d\mathbf{Z}^n + b$ *is a primitive lattice,* $d > 0$, z_0 *is a vector in* $\mathbf{R}^n$, *and*

$$\pi(N, z_0, \Lambda_1) = \operatorname{card}\left\{z_0 \in \Lambda_1, |z - z_0| \leqq \frac{N}{2}, \ z \text{ is a prime vector}\right\}$$

then for $N \geqq 2md$ *we have*

$$\pi(N, z_0, \Lambda \cap (d\mathbf{Z}^n + b)) \leqq \tag{5}$$

$$\leqq \frac{c(n)d^{n-k}}{\varphi^n(d)\log^n \frac{N}{md}}\left(\frac{N}{m}\right)^k \exp \sum_{\substack{p|g \\ p\nmid d}} \frac{np^{k-1} - \omega(p)}{p^k}.$$

The main ingredient of the proof of Theorems 1, 2, 3 is Selberg's sieve in the following form (see [7], Theorem 4.1):

Let $\mathcal{A}$ be a finite set of integers (possibly with repetitions), $\mathcal{P}$ a set of primes, $S(\mathcal{A}, \mathcal{P}, z)$ the number of elements of $\mathcal{A}$ not divisible by any prime $p < z$, $p \in \mathcal{P}$ (the product of these primes is abbreviated by $\mathcal{D}$), while S_d is the number of elements of $\mathcal{A}$ divisible by d. We suppose that

$$S_d = \frac{\Omega(d)}{d} X + R(A, d) \tag{6}$$

for all $d|\mathcal{D}$, where $X > 0$ is a good approximation of the size of $\mathcal{A}$, $\Omega(d)$ is a non-negative multiplicative function satisfying $\Omega(p) < \min(p, C)$ for all $p|\mathcal{D}$ (or equivalently $p < z$, $p \in \mathcal{P}$), with a positive constant C. Then we have

$$S(A, \mathcal{D}, z) \leqq BX \prod_{\substack{p|\mathcal{D} \\ p \leqq z}} \left(1 - \frac{\Omega(p)}{p}\right) + \sum_{\substack{d \leqq z^2 \\ d|\mathcal{D}}} |R(A, d)|, \tag{7}$$

where the constant $B \geqq 0$ depends on C only.[1]

The most important step of the proof is the next lemma. (Note that $\omega(p)$ is defined as the number of solutions of (2).)

LEMMA. *Under the conditions of Theorem* 1 *we have*

$$\omega(p) \leqq np^{k-1} \tag{8}$$

for all primes p, and in addition we have

$$\omega(p) \geqq np^{k-1} - \frac{n(n-1)}{2} p^{k-2} \tag{9}$$

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[1] The first variant of this work was based on the multidimensional analogue of the large sieve. The authors express their thanks to Professor Antal Balog, who drew their attention to Selberg's sieve, which made a remarkable simplification possible.

for primes p not dividing g.

PROOF. Since Λ is a lattice of the class P we have $g \neq 0$. (Here we used the convention $\gcd(0,0) = 0$.) Let α_{ij} be the entries of the matrix $A \in \mathbf{Z}^{k\times n}$ and

$$\xi_j = \lambda_i(x), \quad \beta_j = \lambda_j(b) \quad (i = 1,2,\dots,k, \quad j = 1,2,\dots,n).$$

If x satisfies the congruence (2) then it also satisfies at least one of the congruences

$$\sum_{i=1}^{k} \alpha_{ij}\xi_i + \beta_j = 0 \pmod p, \qquad j = 1,2,\dots,n. \tag{10}$$

However, if $p|d_j(A)$ then p does not divide β_j and (10) has no solution for that j. If $p \nmid d_j(A)$ then $p \nmid \alpha_{ij}$ for some $i = i_0$. We can fix ξ_i for all $i \neq i_0$ in an arbitrary way and then ξ_{i_0} is uniquely determined modulo p. Therefore (10) has exactly p^{k-1} solutions for a fixed j if $p \nmid d_j(A)$. Since we have n congruences in (10) this proves (8).

Now let $p \nmid g$ be fixed. Then for each j the congruence (10) has exactly p^{k-1} solutions. We are going to estimate the number of common solutions of any two congruences of (10), i.e. for $j_1 \neq j_2$

$$\begin{cases} \sum_{i=1}^{k} \alpha_{ij_1}\xi_i + \beta_{j_1} \equiv 0 \pmod p \\ \sum_{i=1}^{k} \alpha_{ij_2}\xi_i + \beta_{j_2} \equiv 0 \pmod p. \end{cases} \tag{11}$$

We can consider (11) as a system of equations over $\mathbf{Z}_p$, the field of residue classes modulo p. Since $p \nmid g$ the rank of the extended matrix of the coefficients (including the row β) is 2. If $k = 1$ then there is no solution of this system for any pair $j_1 \neq j_2$ and $\omega(p) = n$. If $k \geqq 2$ and the rank of the restricted matrix of the coefficients (excluding the row β) $\leqq 1$ then the system still has no solution. In the remaining case there is a non-zero minor of the form

$$\begin{vmatrix} \alpha_{i_1 j_1} & \alpha_{i_1 j_2} \\ \alpha_{i_2 j_1} & \alpha_{i_2 j_2} \end{vmatrix}.$$

We can fix ξ_1 for all $i \neq i_1, i_2$ in an arbitrary way and then ξ_1 and ξ_2 are uniquely determined in $\mathbf{Z}_p$. Therefore (11) has exactly p^{k-2} solutions. Since the total number of systems (11) is $\frac{n(n-1)}{2}$, this proves (9).

Notice that both the number g and the function $\omega(p)$ are determined by the lattice Λ.

PROOF OF THEOREM 1. We define

$$f(x) = \prod_{j=1}^{n} \lambda_j(xA + b).$$

If $Q \geqq 2$, $x \in \mathbf{Z}^k$ and $xA + b$ is a prime vector all of whose components are larger than Q then $f(x) \not\equiv 0 \pmod p$ for all primes $p \leqq Q$. Therefore

(12)
$$\pi(N, x_0, A, b) \leqq \sum_{j=1}^{n} \operatorname{card}\left\{x \in \mathbf{Z}^k, |x - x_0| \leqq \frac{N}{2},\ 2 \leqq \lambda_j(xA + b) \leqq Q\right\} +$$
$$+\operatorname{card}\left\{x \in \mathbf{Z}^k, |x - x_0| \leqq \frac{N}{2}, (f(x), \mathcal{D}) = 1\right\}$$

where $\mathcal{D} = \prod_{p \leqq Q} p$.

We keep the notation from the proof of the Lemma, i.e. the entries of the matrix A are denoted by α_{ij} and the components of the vector b are denoted by β_j. For each given $j = 1, 2, \ldots, n$ there exists at least one i such that $\alpha_{i_1 j} \neq 0$. Since $|x - x_0| \leqq N/2$, all the components ξ_i of the vector x are in certain intervals of length N. When all but ξ_{i_1} are fixed, ξ_{i_1} itself is restricted in an interval of length at most $Q - 2$ by

$$2 \leqq \sum_{i=1}^{k} \alpha_{ij}\xi_i + \beta_j \leqq Q.$$

Consequently we have for all $j = 1, 2, \ldots, n$ that

$$\text{(13)} \quad \operatorname{card}\left\{x \in \mathbf{Z}^k, |x - x_0| \leqq \frac{N}{2},\ 2 \leqq \lambda_j(xA + b) \leqq Q\right\} \leqq Q(N + 1)^{k-1}.$$

We are going to apply Selberg's sieve to the situation

$$\mathcal{A} = \left\{f(x) | x \in \mathbf{Z}^k, |x - x_0| \leqq \frac{N}{2}\right\}, \quad z = Q, \quad \mathcal{D} = \prod_{p \leqq Q} p.$$

We have

$$\sum_{\substack{x \in \mathbf{Z}^k \\ |x - x_0| \leqq \frac{N}{2} \\ f(x) \equiv 0 \pmod d}} = \sum_{\substack{\ell \bmod d \\ \ell \in \mathbf{Z}^k \\ f(\ell) \equiv 0 \pmod d}} \sum_{\substack{x \in \mathbf{Z}^k \\ |x - x_0| \leqq \frac{N}{2} \\ x \equiv \ell \pmod d}} 1,$$

and as the inner sum is

$$\left(\frac{N}{d}\right)^k + \theta_\ell k \cdot 2^{k-1}\left(\frac{N}{d}\right)^{k-1}, \qquad |\theta_\ell| \leqq 1,$$

we get (6) with

$$\Omega(d) = \frac{\omega(d)}{d^{k-1}}, \quad X = N^k, \quad |R(A, d)| \leqq k \cdot 2^{k-1} \cdot N^{k-1}\frac{\omega(d)}{d^{k-1}}.$$

Our lemma provides that $\Omega(p) \leqq n$ for all primes p. Also $\omega(p) \leqq p^k$, that is $\Omega(p) \leqq p$. In case of $\Omega(p) = p$ for some prime $p \leqq Q$ we have $S(\mathcal{A}, \mathcal{P}, Q) = 0$. (7) implies that

$$\text{(14)} \qquad \operatorname{card}\left\{x \in \mathbf{Z}^k, |x - x_0| \leqq \frac{N}{2}, (f(x), \mathcal{D}) = 1\right\} =$$

$$= S(A, \mathcal{D}, Q) \leqq c_1(n)\left(N^k \prod_{p \leqq Q}\left(1 - \frac{\omega(p)}{p^k}\right) + N^{k-1} \sum_{\substack{d \leqq Q^2 \\ d | \mathcal{D}}} \frac{\omega(d)}{d^{k-1}}\right\}.$$

Since

$$\prod_{p \leqq Q}\left(1 - \frac{\omega(p)}{p^k}\right) \leqq \exp\left(-\sum_{p \leqq Q} \frac{\omega(p)}{p^k}\right),$$

$$\sum_{\substack{d \leqq Q^2 \\ d | \mathcal{D}}} \frac{\omega(d)}{d^{k-1}} \leqq Q^2 \sum_{d | \mathcal{D}} \frac{\omega(d)}{d^k} = Q^2 \prod_{p \leqq Q}\left(1 + \frac{\omega(p)}{p^k}\right) \leqq Q^2 \exp\left(\sum_{p \leqq Q} \frac{\omega(p)}{p^k}\right),$$

it follows from (13) and (14) that

$$\text{(15)} \quad \pi(N, x_0, A, b) \leqq c_1(n) N^k \exp\left(-\sum_{p \leqq Q} \frac{\omega(p)}{p^k}\right)\left(1 + \frac{Q^2}{N} \exp 2 \sum_{p \leqq Q} \frac{\omega(p)}{p^k}\right).$$

Applying our Lemma again

$$\sum_{p \leqq Q} \frac{\omega(p)}{p^k} \leqq n \sum_{p \leqq Q} \frac{1}{p} \leqq n \log\log Q + c_2(n),$$

$$\sum_{p \leqq Q} \frac{\omega(p)}{p^k} \geqq n \sum_{\substack{p \leqq Q \\ p \nmid g}} \frac{1}{p} - \frac{n(n-1)}{2} \sum_{\substack{p \leqq Q \\ p \nmid g}} \frac{1}{p^2} + \sum_{\substack{p \leqq Q \\ p | g}} \frac{\omega(p)}{p^k} \geqq$$

$$\geqq n \log\log Q + \sum_{p | g} \frac{\omega(p) - np^{k-1}}{p^k} - c_3(n).$$

Collecting all these estimates and choosing $Q = N^{1/3}$ Theorem 1 follows.

PROOF OF THEOREM 2. Theorem 2 is a simple consequence of Theorem 1.

Let $z = xA + b \in \Lambda$, $x \in \mathbf{Z}^k$, $|z - z_0| \leqq \frac{N}{2}$. There is a vector $x_0 \in \mathbf{R}^k$ such that

$$|x - x_0| \leqq \frac{N}{2m}.$$

According to our Lemma for $N > 2m$

$$\pi(N, z_0, \Lambda) \leqq \pi\Big(\frac{N}{m}, x_0, A, b\Big) \leqq \frac{c(n)}{\log^n \frac{N}{m}} \Big(\frac{N}{m}\Big)^k \exp \sum_{p|g} \frac{np^{k-1} - \omega(p)}{p^k}.$$

Supposing that $z_0 = \frac{N}{2}e$ where $e \in \mathbf{Z}^n$ is a vector with identity components we obtain inequality (4).

PROOF OF THEOREM 3. Let $\Lambda_1 = \Lambda \cap (d\mathbf{Z}^n + b)$. By virtue of the unimodularity of the lattice

$$\Lambda = \mathbf{Z}^k A + b, \ \ \mathbf{Z}^k A \cap d\mathbf{Z}^n = d\mathbf{Z}^k A = \mathbf{Z}^k dA,$$

so that $\Lambda_1 = \mathbf{Z}^k dA + b$. If Λ is a lattice of the class P then so is Λ_1. We apply inequality (4) to the lattice Λ_1. We denote by m_d, g_d and $\omega_d(p)$ those characteristics of the lattice Λ which correspond to m, g and $\omega(p)$ of Λ. Notice that $m_d = md$, the prime factors of g_d are the same as the prime factors of gd, and as $d\mathbf{Z}^n + b$ is primitive we have

$$\omega_d(p) = \begin{cases} \omega(p), & \text{if } (p, d) = 1 \\ 0, & \text{if } p|d. \end{cases}$$

Therefore we get

$$\sum_{p|gd} \frac{np^{k-1} - \omega_d(p)}{p^k} = n \sum_{p|d} \frac{1}{p} + \sum_{\substack{p|g \\ (p,d)=1}} \frac{np^{k-1} - \omega(p)}{p^k}.$$

Finally using the inequality

$$\exp \sum_{p|d} \frac{1}{p} \leqq \prod_{p|d} \Big(1 - \frac{1}{p}\Big)^{-1} = \frac{d}{\varphi(d)},$$

we get (5) by applying (4) to the lattice Λ.

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A GELFAND–NEUMARK THEOREM FOR COMMUTATIVE SEMIFINITE RANK J^*-ALGEBRAS

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Introduction

L. A. Harris in [5, p. 360] has asked: Give a definition of commutative J^*-algebra which involves only the J^*-structure and show that such J^*-algebra is J^*-isomorphic to the space $C_0(X)$ of continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space X. It is shown in [6, Proposition 1.2.1] that every singly generated J^*-algebra is J^*-isomorphic to $C_0(X)$. In this note we shall define a commutative J^*-algebra of semifinite rank $\mathcal{A}$ and we shall prove that $\mathcal{A}$ is J^*-isomorphic to $C_0(X)$.

1. Definitions and basic results

Suppose H and K are complex Hilbert spaces. Let $B(H,K)$ denote the Banach space of all bounded transformations from H to K with the operator norm. For each element $A \in B(H,K)$ there is a uniquely determined element $A^* \in B(K,H)$ such that

$$(Ax, y) = (x, A^*y) \quad \text{for all} \quad x \in H \quad \text{and} \quad y \in K.$$

A^* is said to be the adjoint of A.

A closed subspace $\mathcal{A}$ of $B(H,K)$ is called a J^*-algebra if $AA^*A \in \mathcal{A}$, whenever $A \in \mathcal{A}$. For example, every C^*-algebra or JC^*-algebra is a J^*-algebra. Cartan factors of Type I–IV are J^*-algebras ([3], [5]).

Suppose $\mathcal{A}$ is a J^*-algebra. $\mathcal{A}$ is said to have *semifinite rank* if $\operatorname{sp}(A^*A)$ has no non-zero limit point for each $A \in \mathcal{A}$.

A J^*-algebra $\mathcal{A}$ has *finite rank* if there exists a number n such that $\operatorname{sp}(A^*A)$ has at most n non-zero elements for each $A \in \mathcal{A}$.

For example, the set of compact operators in a Cartan factor of any one of the types I–III is a J^*-algebra having semifinite rank. A C^*-algebra has semifinite rank if and only if it is isomorphic to a C^*-algebra of compact operators [2, 4.7.20] and has finite rank if and only if it is finite dimensional [2, p. 184].

We recall that an operator V is a *partial isometry* if and only if $VV^*V = V$.

Suppose $\mathcal{A}$ is a J^*-algebra and $A \in \mathcal{A}$. A partial isometry V in $\mathcal{A}$

(1) *covers* A if $VV^*A = AV^*V = A$,

(2) *commutes* with A if $VV^*A = AV^*V$,

(3) is a *unitary* element of $\mathcal{A}$ if V covers each $A \in \mathcal{A}$,

(4) is a *central element* of $\mathcal{A}$ if V commutes with each $A \in \mathcal{A}$,

(5) is a *minimal element* of $\mathcal{A}$ if for each $A \in \mathcal{A}$ there is an $\alpha \in \mathbf{C}$ with $VA^*V = \alpha V$,

(6) is *orthogonal to* A if $AV^* = 0$ and $V^*A = 0$.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are J^*-algebras. A map $\phi\colon \mathcal{A} \to \mathcal{B}$ is called a *J^*-isomorphism* if ϕ is a bounded linear bijection of $\mathcal{A}$ onto $\mathcal{B}$ satisfying

$$\phi(AA^*A) = \phi(A)\phi(A)^*\phi(A)$$

for all $A \in \mathcal{A}$. Note that the above properties 1–6 are preserved by J^*-isomorphism [5, Proposition 2.1].

A J^*-algebra $\mathcal{A}$ need not have any non-trivial partial isometries. For example, let $\mathcal{A} = C(X)$, where X is a compact connected space. Nevertheless many J^*-algebras are rich in partial isometries.

1.1. THEOREM [5, Theorem 3.3]. *Suppose $\mathcal{A}$ has semifinite rank, and let $A \in \mathcal{A}$ with $A \neq 0$. Then there exist a set $\{V_n\}$ of mutually orthogonal non-zero minimal partial isometries in $\mathcal{A}$ and a sequence (a_n) of positive numbers such that $A = \sum_n a_n V_n$.*

Suppose $\mathcal{A}$ is a J^*-algebra. A *J^*-ideal* in $\mathcal{A}$ is a closed subspace $\mathcal{J}$ of $\mathcal{A}$ such that $A, B, C \in \mathcal{A}$, then $AB^*C + CB^*A \in \mathcal{J}$ whenever $B \in \mathcal{J}$ or $C \in \mathcal{J}$. A J^*-algebra $\mathcal{A}$ is simple if the only J^*-ideals in $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$. It is obvious that the kernel of any J^*-isomorphism is a J^*-ideal.

Suppose $\mathcal{A}$ is a C^*-algebra of semifinite rank. Then $\mathcal{A}$ is isomorphic to the restricted product of a family of the C^*-algebras of compact operators on Hilbert spaces [2, 4.7.20]. In the case of a J^*-algebra of semifinite rank the following is true.

1.2. THEOREM [5, Theorem 5.9]. *Suppose $\mathcal{A}$ is a J^*-algebra and M is the set of all non-zero minimal partial isometries in $\mathcal{A}$. Let $\mathcal{A}$ be the J^*-ideal generated by M. For each $V \in M$, let $\mathcal{J}_V$ denote the intersection of all J^*-ideals in $\mathcal{A}$ containing V, and let $\mathcal{J}$ be the family of all nonzero simple J^*-ideals in $\mathcal{A}$. Then*

(a) $\mathcal{J} = \{\mathcal{J}_V\colon V \in M\}$,

(b) *$\mathcal{A}$ is J^*-isomorphic to the restricted product of $\mathcal{J}$.*

2. Commutative J^*-algebra of semifinite rank

Suppose $\mathcal{A}$ is a J^*-algebra of semifinite rank. Then $\mathcal{A}$ is said to be *commutative* if each minimal partial isometry V in $\mathcal{A}$ is central, that is

(1) $$VV^*A = AV^*V \quad \text{for all} \quad A \in \mathcal{A}.$$

Note that (1) is equivalent to

$$VV^*A + AV^*V = 2VV^*AV^*V$$

which is J^*-invariant.

2.1. LEMMA. *Suppose $\mathcal{A}$ is a commutative J^*-algebra of semifinite rank. Then $AB^*C = CB^*A$ for all $A, B, C \in \mathcal{A}$.*

PROOF. Suppose A and B are non-zero elements in $\mathcal{A}$. Since $\mathcal{A}$ has semifinite rank, it follows from Theorem 1.1 that there exist a set $\{V_n\}$ of mutually orthogonal non-zero partial isometries in $\mathcal{A}$ and a sequence (b_n) of positive numbers such that $B = \sum_n b_n V_n$. Note that the adjoint operation is continuous and

$$V_i V_j^* = V_j^* V_i = 0 \qquad (i \neq j), \quad i, j \in N.$$

So

$$BB^* = \sum_n b_n^2 V_n V_n^*.$$

However, each minimal partial isometry in $\mathcal{A}$ is central and so

$$AV_i^* V_i = V_i V_i^* A, \qquad i \in N.$$

Consequently $AB^*B = BB^*A$ for all $A, B \in \mathcal{A}$.

Suppose A, B and C are in $\mathcal{A}$. Then by the first part

$$A(B+C)^*(B+C) = (B+C)(B+C)^*A,$$

or

(1) $$ABC^* + AC^*B = BC^*A + CB^*A$$

for all $A, B, C \in \mathcal{A}$. Replace C by iC in (1). Then

(2) $$AB^*C - AC^*B = -BC^*A + CB^*A$$

for all $A, B, C \in \mathcal{A}$. Add (1) and (2). Then $AB^*C = CB^*A$ for all $A, B, C \in \in \mathcal{A}$.

2.2. THEOREM. *Suppose $\mathcal{A}$ is a commutative J^*-algebra of semifinite rank. Then $\mathcal{A}$ is J^*-isomorphic to $C_0(X)$, the space of all complex-valued continuous functions vanishing at infinity on a locally compact Hausdorff space X.*

PROOF. Since $\mathcal{A}$ is commutative, by Lemma 2.1,

$$AB^*C = CB^*A \quad \text{for all} \quad A, B, C \in \mathcal{A}.$$

However, by [3, Proposition 1], $AB^*C + CB^*A \in \mathcal{A}$ and so

$$AB^*C \in \mathcal{A}, \quad \text{whenever} \quad A, B, C \in \mathcal{A}.$$

Let M be the set of all non-zero minimal partial isometries in $\mathcal{A}$. For each $V \in M$, let $\mathcal{J}_V$ denote the intersection of all J^*-ideals in $\mathcal{A}$ containing V. Then by Theorem 1.2 $\mathcal{J}_V$ is a simple J^*-ideal in $\mathcal{A}$. Define the map $\phi_V : \mathcal{J}_V \to V^*\mathcal{J}_V$ by

$$\phi_V(A) = V^*A \quad \text{for all} \quad A \in \mathcal{J}_V.$$

Suppose $\mathcal{B}_V = V^*\mathcal{J}_V$. We claim that $\mathcal{B}_V$ is a commutative C^*-algebra and ϕ_V is a J^*-isomorphism of $\mathcal{J}_V$ onto $\mathcal{B}_V$ for each $V \in M$. Indeed, let S and T be in $\mathcal{B}_V$ and $S = V^*A$, $T = V^*B$ for some $A, B \in \mathcal{J}_V$. Then

$$S^* = (V^*A)^* = A^*V = A^*VV^*V = V^*(VA^*V).$$

But VA^*V is an element of $\mathcal{J}_V$ and so $S^* \in \mathcal{B}_V$. Therefore $\mathcal{B}_V$ is self-adjoint. Similarly,

$$ST = (V^*A)(V^*B) = V^*(AV^*B),$$

and so $ST \in \mathcal{B}_V$. Since S and T are arbitrary non-zero elements of $\mathcal{B}_V$, it follows that $\mathcal{B}_V$ is a C^*-algebra in $B(H)$. Also,

$$S^*S = (V^*A)^*(V^*A) = A^*VV^*A = V^*VA^*A = V^*AA^*V = SS^*.$$

So each element in $\mathcal{B}_V$ is normal and therefore $\mathcal{B}_V$ is commutative.

Suppose $A \in \mathcal{J}_V$, then $AA^*A \in \mathcal{J}_V$. So

$$\begin{aligned}\phi_V(AA^*A) &= V^*AA^*A = V^*VV^*AA^*A = V^*AV^*VA^*A = \\ &= V^*AA^*VV^*A = \phi_V(A)\phi_V(A)^*\phi_V(A).\end{aligned}$$

Therefore ϕ_V is a J^*-homomorphism of $\mathcal{J}_V$ onto $\mathcal{B}_V$. Suppose

$$\operatorname{Ker} \phi_V = \{A \in \mathcal{J}_V : \phi_V(A) = 0\};$$

then $\operatorname{Ker} \phi_V$ is a closed J^*-ideal in $\mathcal{J}_V$. But $\mathcal{J}_V$ is simple and so either $\operatorname{Ker} \phi_V = \{0\}$ or $\operatorname{Ker} \phi_V = \mathcal{J}_V$. Since V is non-zero, it follows that V does not belong to $\operatorname{Ker} \phi_V$ and so $\operatorname{Ker} \phi_V = \{0\}$. Therefore ϕ_V is a J^*-isomorphism of $\mathcal{J}_V$ onto $\mathcal{B}_V$ for each V in M.

Suppose $\mathcal{J} = \{\mathcal{J}_V, V \in M\}$. Then by Theorem 1.2, $\mathcal{A}$ is J^*-isomorphic to the restricted product of $\mathcal{J}$. However for each $V \in M$, $\mathcal{J}_V$ is J^*-isomorphic to a commutative C^*-algebra which in turn can be identified with $C(X_V)$ for some compact Hausdorff space X_V. Since the restricted product of C^*-algebras is a C^*-algebra [2, 1.9.14], it follows that $\mathcal{A}$ is J^*-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

REMARK. M. C. V. Berglund [1, Theorem 5.5] proves that a commutative C^*-algebra $\mathcal{B}$ has semifinite rank if and only if it is isomorphic to $C_0(X)$, where X is discrete.

Note that the semifinite rank property is J^*-invariant and so by the above observation and Theorem 2.2 we have the following result.

2.3. COROLLARY. *Suppose $\mathcal{A}$ is a commutative J^*-algebra of semifinite rank. Then $\mathcal{A}$ is J^*-isomorphic to $C_0(X)$, where X is discrete.*

Next we show that the proof of Theorem 2.2 can be simplified in the case of commutative J^*-algebra of finite rank. First we need a lemma.

2.4. LEMMA. *Suppose $\mathcal{A}$ is a commutative J^*-algebra of finite rank. Then $\mathcal{A}$ contains a unitary element.*

PROOF. Suppose $S = \{V_1, V_2, \dots, V_k\}$ is a maximal set of mutually orthogonal non-zero partial isometries in $\mathcal{A}$. Since $\mathcal{A}$ has finite rank, it follows from Theorem 1.1, that S is non-empty. Let V be the sum of the elements of S. Then

$$V_i^* = V^*VV_i^*, \qquad i = 1, 2, \dots, k.$$

Let $\mathcal{B} = \mathcal{A}(I - V^*V)$. Then $\mathcal{B}$ is contained in $\mathcal{A}$. Suppose $\mathcal{B} \neq \{0\}$, and note that $\mathcal{B}$ has finite rank. So there is a non-zero partial isometry W in $\mathcal{B}$, by Theorem 1.1. Therefore,

$$WV_i^* = WV^*VV_i^* = (WV^*V)V_i^* = 0 \quad \text{for} \quad i = 1, 2, \dots, k.$$

Similarly $V_i^*W = 0$ for $i = 1, 2, \dots, k$. But S is maximal and so $W = 0$. Therefore by Theorem 1.1, $\mathcal{B} = \{0\}$ and consequently $A = AV^*V = VV^*A$ for all $A \in \mathcal{A}$ and V is unitary.

2.5. PROPOSITION. *Suppose $\mathcal{A}$ is a commutative J^*-algebra of finite rank. Then $\mathcal{A}$ is J^*-isomorphic to $C(X)$ for some compact Hausdorff space X.*

PROOF. By Lemma 2.4, $\mathcal{A}$ contains a unitary element V. Define a map $\phi\colon \mathcal{A} \to V^*\mathcal{A}$, by

$$\phi(A) = V^*A \quad \text{for all} \quad A \in \mathcal{A}.$$

Then, as in the proof of Theorem 2.2, $\mathcal{B} = V^*\mathcal{A}$ is a commutative C^*-algebra with the identity V and ϕ is a J^*-homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Since V is a unitary element of $\mathcal{A}$ and Ker $\phi = \{0\}$, ϕ is a J^*-isomorphism of $\mathcal{A}$ onto $C(X)$ for some compact Hausdorff space X.

REMARK. We recall that a C^*-algebra has finite rank if and only if it is finite dimensional. So a commutative C^*-algebra $\mathcal{B}$ has finite rank if and only if it is isomorphic to $C(X)$, where X is finite.

2.6. COROLLARY. *Suppose $\mathcal{A}$ is a commutative J^*-algebra of finite rank. Then it is J^*-isomorphic to $C(X)$, where X is finite.*

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NOTE ON MULTIPLICATIVE FUNCTIONS SATISFYING A CONGRUENCE PROPERTY. II

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1. Introduction

An arithmetical function $f(n) \not\equiv 0$ is said to be multiplicative if $(n, m) = 1$ implies

$$f(nm) = f(n)f(m)$$

and it is called completely multiplicative if the above equation holds for all pairs of positive integers n and m. In the following let $\mathcal{M}$ and $\mathcal{M}^*$ denote the set of integer-valued multiplicative and completely multiplicative functions, respectively.

The problem concerning the characterization of an integer-valued power function as an integer-valued multiplicative function satisfying a congruence property was studied by several authors. The first such characterization is apparently that of M. V. Subbarao [7]. He proved that if $f \in \mathcal{M}$ and satisfies the relation

$$f(n+m) \equiv f(m) \pmod n \tag{1}$$

for every positive integer n and m, then $f(n)$ is a power of n with non-negative integer exponent. A. Iványi [2] extended this result proving that if $f \in \mathcal{M}^*$ and (1) holds for a fixed positive integer m and for every positive integer n, then $f(n)$ has also the same form. Recently, B. M. Phong and J. Fehér [6] improved the results of Subbarao and Iványi mentioned above, proving that if $f \in \mathcal{M}$ and (1) holds for a fixed m with $f(m) \neq 0$ and for every positive integer n, then there is a non-negative integer a such that $f(n) = n^a$ $(n = 1, 2, \dots)$.

For a positive integer k let N_k be the arithmetical function defined by $N_k(n) = m$ if $n = m^k h$, where h is k-free. It is obvious that $N_k \in \mathcal{M}$ for every positive k and

$$N_1(n) = n \qquad (n = 1, 2, \dots). \tag{2}$$

In this paper we would like to determine all $f \in \mathcal{M}$ which satisfy the relation

$$f(n+M) \equiv f(M) \qquad (\mathrm{mod}\ N_k(n)) \tag{3}$$

for every positive integer n, where k, M are fixed positive integers. It is obvious that $f(n) = n^a$ ($a \geqq 0$ is an integer) is a solution of (3).

We shall prove the following results.

THEOREM 1. *Let k and M be fixed positive integers. Assume that $f \in \in \mathcal{M}$ with $f(M) \neq 0$ and (3) holds for every positive integer n. Then there is a non-negative integer a such that*

$$f(n) = n^a \qquad (n = 1, 2, \dots).$$

COROLLARY. *Let k be a fixed integer. Assume that $f, g \in \mathcal{M}$ satisfy the relation*

$$f(n+m) \equiv g(m) \qquad (\mathrm{mod}\ N_k(n))$$

for every positive integer n, m. Then

$$f(n) = g(n) = n^a \qquad (n = 1, 2, \dots),$$

where a is a non-negative integer.

REMARKS. 1. The theorem of B. M. Phong and J. Fehér in [6] is a special case $k = 1$ of our Theorem 1. Indeed, by using (2) the congruence (1) is equivalent to (3) in the case $k = 1$.

2. Let k, A, M be positive integers for which $(A, M) = 1$. Let $f \in \mathcal{M}$ with the condition $f(M) \neq 0$ and consider the congruence

$$f(An + M) \equiv f(M) \qquad (\mathrm{mod}\ N_k(n)) \tag{4}$$

for every positive integer n. Combining the method of the present paper and that of [5] one can prove that if $f \in \mathcal{M}$ and (4) holds for every positive integer n, then there are a positive integer a and a real-valued Dirichlet character χ (mod A) such that $f(n) = \chi(n)n^a$ for all integers n which are prime to A. This statement was proved by the author in [3] for $A = p$ prime and $k = 1$, further it was proved for any non-negative integer A and $k = 1$ by B. M. Phong in [5] generalizing the ideas of the paper [3].

The proof of Theorem 1 of the present paper is based on the ideas of the papers [3], [5] and [6]. The new idea in the proof is contained in Lemma 3, which gives some new information on the prime power divisors of special second order linear sequences.

The following theorem is an easy consequence of Theorem 1 and an improvement of the above result of A. Iványi.

THEOREM 2. *Let k be a fixed positive integer. Assume that f satisfies the relation*

$$f(n^k + m) \equiv f(n^k) + f(m) \qquad (\mathrm{mod}\ n) \tag{5}$$

for every positive integer n, m. Then there is a positive integer a such that $f(n) = n^a$ $(n = 1, 2, \dots)$.

2. Auxiliary results

The simple idea to prove Theorem 1 is the following:

(i) The congruence (3) implies that $f \in \mathcal{M}^*$ and $f(Q) = Q^{a(Q)}$ holds for every prime Q, where $a(Q) \geqq 0$ is an integer.

(ii) For every prime Q we have $a(Q) = a(2)$.

In this section we prove (i).

LEMMA 1. *Assume that k, M, f satisfy the conditions of Theorem* 1 *and* (3) *holds for every positive integer n. Then $f \in \mathcal{M}^*$ and*

$$f(Q) = Q^{a(Q)} \tag{6}$$

holds for every prime Q, where $a(Q) \geqq 0$ is an integer.

PROOF. We first prove that

$$f(ab)f(M) = f(aM)f(b) \tag{7}$$

for all positive integers a and b.

Let a, b positive integers and let m be a positive integer for which

$$(m, abMf(M)) = 1. \tag{8}$$

Then there are positive integers x, y, u and v such that

$$ax = 1 + m^k y, \qquad (x, abM) = 1 \tag{9}$$

and

$$bu = M + m^k v, \qquad (u, abx) = 1. \tag{10}$$

From (9) and (10) we have

$$abxu = M + m^k T, \tag{11}$$

where $T = My + v + m^k yv$. By using (3), (9), (10) and (11) we have

$$f(aM)f(x) = f(aMx) = f(M + m^k My) \equiv f(M) \pmod m, \tag{12}$$

$$f(b)f(u) = f(bu) = f(M + m^k v) \equiv f(M) \pmod m \tag{13}$$

and

$$f(ab)f(x)f(u) = f(abxu) = f(M + m^k T) \equiv f(M) \pmod m \tag{14}$$

since $N_k(m^k My)$, $N_k(m^k v)$, $N_k(m^k T)$ are divisible by m. From (12) it follows that

$$(f(x), m) = 1, \tag{15}$$

since $(m, f(M)) = 1$. From (12), (14) and (15) we get that

$$f(aM) \equiv f(ab)f(u) \pmod{m},$$

which together with (13) implies

$$f(aM)f(b) \equiv f(ab)f(b)f(u) \equiv f(ab)f(M) \pmod{m}.$$

This congruence shows that (7) holds, since there are infinitely many positive integers m satisfying (8). So (7) is proved. Let Q be a prime. In order to show $f \in \mathcal{M}^*$ it is enough to prove that

$$f(Q^s) = (f(Q))^s \tag{16}$$

for every positive integer s. It is obvious that (16) is true for $s = 1$. Assume that (16) holds for s.

If $(Q, M) = 1$, then applying (7) with a, b given by Q^s and Q respectively, we have

$$f(Q^{s+1})f(M) = f(Q^sM)f(Q) = f(Q^s)f(Q)f(M)$$

which implies

$$f(Q^{s+1}) = f(Q^s)f(Q) = (f(Q))^{s+1}. \tag{17}$$

If $Q^\gamma \| M$, where $\gamma \geqq 1$ is an integer, then applying (7) with a, b given by Q and 1 respectively, we get

$$f(Q^{\gamma+1}) = f(Q)f(Q^\gamma). \tag{18}$$

By using (18), we apply (7) in the case $a = Q$ and $b = Q^s$ to get

$$f(Q^{s+1})f(Q^\gamma)f(M/Q^\gamma) = f(Q^{\gamma+1})f(Q^s)f(M/Q^\gamma) =$$
$$= f(Q)f(Q^\gamma)f(Q^s)f(M/Q^\gamma),$$

which implies

$$f(Q^{s+1}) = f(Q)f(Q^s) = (f(Q))^{s+1}. \tag{19}$$

From (17) and (19) it follows that (16) holds for $s + 1$, and so (16) is true for every positive integer s. By (16) $f \in \mathcal{M}^*$ follows.

Now we prove (6). We shall prove that for a prime p and a positive integer c

$$p \mid f(c) \quad \text{implies} \quad p \mid c. \tag{20}$$

Assume indirectly that for a prime p, $p \mid f(c)$ and $(p, c) = 1$. Let $p^{k_0} \nmid f(M)$. Thus there exist positive integers H, L such that

$$Hc^{k_0} = M + p^{k_0 k}L. \tag{21}$$

By using (21) and applying (3) with $n = p^{k_0 k} L$ we have

$$(22) \qquad f(H)f(c^{k_0}) = f(Hc^{k_0}) = f\left(M + (p^{k_0})^k L\right) \equiv f(M) \pmod{p^{k_0}}.$$

On the other hand, it follows from $f \in \mathcal{M}^*$ and $p \mid f(c)$ that

$$f(c^{k_0}) = f(c)^{k_0} \equiv 0 \pmod{p^{k_0}}$$

which together with (22) implies that $f(M) \equiv 0 \pmod{p^{k_0}}$. This is a contradiction, since $p^{k_0} \nmid f(M)$. So (20) is proved.

From (20) it follows that for a prime Q we have

$$(23) \qquad f(Q) = \pm Q^{a(Q)}$$

where $a(Q) \geqq 0$ is an integer. In order to prove (6) it is enough to show that

$$(24) \qquad f(Q) > 0$$

for every prime Q.

Let $t > 1$ be an odd integer. It is obvious that there exists an odd prime p such that

$$(25) \qquad Q^t \equiv 1 \pmod{p}.$$

Let j be a positive integer for which

$$(26) \qquad p^j \nmid 2f(M).$$

It can be easily seen by (25) that

$$(27) \qquad Q^{t \cdot p^{kj-1}} \equiv 1 \pmod{p^{kj}},$$

i.e.

$$(28) \qquad Q^{t \cdot p^{kj-1}} = 1 + p^{kj} K,$$

where K is a positive integer. By using (3) and (28) we have

$$(29) \qquad f\left(Q^{t \cdot p^{kj-1}} M\right) = f(M + p^{kj} MK) \equiv f(M) \pmod{p^j}.$$

If $f(Q) < 0$, then by (23) $f(Q) = -Q^{a(Q)}$ follows. Since $f \in \mathcal{M}^*$, from (27) we have

$$(30) \qquad f(Q^{t \cdot p^{kj-1}} M) = (f(Q))^{t \cdot p^{kj-1}} f(M) =$$
$$= -Q^{t \cdot a(Q) p^{kj-1}} f(M) \equiv -f(M) \pmod{p^j}$$

which together with (29) implies $2f(M) \equiv 0 \pmod{p^j}$. This is a contradiction, since from (26) we have $p^j \nmid 2f(M)$. Thus $f(Q) > 0$ and so (24) is proved.

One can deduce (24) also from the result of [1].

From (23) and (24) the proof of (6) is finished. Lemma 1 is proved.

3. Prime power divisors of the number $Q \cdot 2^n - 1$

In order to prove (ii) we need some further information on prime power divisors of special second order linear recurrences.

Let Q be a positive integer. Let $G(Q) = \{G_n\}_{n=0}^{\infty}$ be a second order linear recurrence defined by the relation

$$G_n = Q2^n - 1 \qquad (n = 0, 1, 2, \dots).$$

In the case of $Q = 1$ the sequence $G(1)$ will be denoted by R and its terms by R_n ($R_n = 2^n - 1$). R_n is called the n-th Mersenne number.

If for a positive integer m there are terms in G divisible by m, then $g(m)$ denotes the least positive integer g, for which $m \mid G_g$. The number $g(m)$ is called the rank of apparation of m in the sequence G. For a prime p, which divides some terms of the sequence G, we denote by $e(p)$ the greatest exponent e for which p^e divides $G_{g(p)}$, i.e. $p^{e(p)} \| G_{g(p)}$. In the sequence R we denote the rank of apparation of a positive integer m by $r(m)$.

P. Kiss and B. M. Phong [4] have given necessary and sufficient conditions for the existence of the rank of apparation of a prime power in the general second order linear recurrence. From their theorems one can deduce immediately the following

LEMMA 2. *Assume that for a prime p there is the rank of apparation of p in the sequence $G(Q)$. Then there is $g(p^{e(p)+n})$ for every positive integer n if and only if*

$$r(p^{e(p)}) \neq r(p^{e(p)+1}), \tag{31}$$

PROOF. We prove Lemma 2 directly. Assume that for a prime p there is $g(p^k)$ for every positive integer k. Let $e = e(p)$ and $h = g(p^{e+1})$. It is obvious that there is a positive integer x such that $h = g + xr$, where $g = g(p^e)$, $r = r(p^e)$. We have

$$Q2^h - 1 = (Q2^g - 1)2^{xr} + (2^{xr} - 1) \equiv 0 \qquad (\text{mod } p^{e+1}),$$

and so

$$\frac{Q2^g - 1}{p^e} 2^{xr} + \frac{2^{xr} - 1}{p^e} \equiv 0 \qquad (\text{mod } p). \tag{32}$$

Since $2^r \equiv 1 \pmod p$, we have $2^{rx} \equiv 1 \pmod p$ and

$$\frac{2^{xr} - 1}{p^e} = \frac{2^r - 1}{p^e}(2^{r(x-1)} + \ldots + 1) \equiv \frac{2^r - 1}{p^e} x \qquad (\text{mod } p).$$

Using these congruences we get from (32) that

$$\frac{Q2^g - 1}{p^e} + \frac{2^r - 1}{p^e} x \equiv 0 \qquad (\text{mod } p). \tag{33}$$

This shows that $\frac{2^r-1}{p^e} \not\equiv 0 \pmod p$, i.e. (31) holds.

Assume that (31) holds, that is $r(p^e) \neq r(p^{e+1})$. It is well-known that in this case we have

$$r(p^{e+n}) = p^n r(p^e) \qquad (n = 1, 2, \dots). \tag{34}$$

Assume that there is $g(p^{e+n})$ for some integer $n \geqq 0$ and $r(p^{e+n}) \neq r(p^{e+n+1})$. We shall show that $g(p^{e+n+1})$ also exists.

Let $g^* := g(p^{e+n})$ and $r^* := r(p^{e+n})$. As we have seen in the proof above, we have

$$Q2^{g^*+xr^*} - 1 \equiv 0 \qquad (\mathrm{mod}\ p^{e+n+1})$$

if and only if

$$\frac{Q2^{g^*} - 1}{p^{e+n}} + \frac{2^{r^*} - 1}{p^{e+n}} x \equiv 0 \qquad (\mathrm{mod}\ p). \tag{35}$$

By using (34) we have $r^* = r(p^{e+n}) \neq r(p^{e+n+1})$ for every integer $n \geqq 0$, and so the congruence (35) has solutions. This shows that there is $g(p^{e+n+1})$. This completes the proof of Lemma 2.

We shall deduce from Lemma 2 the following result.

LEMMA 3. *For each fixed prime Q there exists a prime p such that there is $g(p^h)$ for every positive h in the sequence $G(Q)$.*

PROOF. By using Lemma 2, in order to prove Lemma 3 it is enough to show that there is a prime p for which there is $g(p)$ and $r(p) \neq r(p^2)$.

If $Q > 3$, then $Q \equiv 1 \pmod 3$ or $Q \equiv -1 \pmod 3$, since Q is a prime. We have $r(3) = 2 \neq r(3^2)$.

If $Q \equiv 1 \pmod 3$, then $G_2(Q) = Q2^2 - 1 \equiv 0 \pmod 3$ and so $g(3) = 2$. If $Q \equiv -1 \pmod 3$, then $G_1(Q) = Q2 - 1 \equiv 0 \pmod 3$, i.e. $g(3) = 1$. Thus, in both cases there is $g(3)$, which together with the condition $r(3) \neq r(3^2)$ implies that there is $g(3^h)$ for every positive integer h.

If $Q = 3$, then we choose $p = 5$. In our case $r(5) = 4 \neq r(5^2)$ and $G_3(3) = 3 \cdot 3^3 - 1 \equiv 0 \pmod 5$, which also implies that there is $g(5^h)$ for every positive integer h. The proof of Lemma 3 is complete.

4. Proof of Theorem 1

By using Lemma 1, in order to prove Theorem 1 it is enough to show that (ii) holds, that is $a(Q) = a(2)$ for every odd prime Q. Let Q be an odd prime. Let $G_n(Q) = Q2^n - 1$. By using Lemma 3 we have $\limsup_{n\to\infty} N_k(Q2^n - 1) = \infty$.

Applying (3) and using Lemma 1 we get that

$$(36) \qquad f(Q)f(2)^n f(M) = f(Q2^n M) =$$

$$= f[(Q2^n - 1)M + M] \equiv f(M) \pmod{N_k(Q2^n - 1)}$$

holds for every positive integer n.

On the other hand we have

$$(37) \qquad f(Q)f(2)^n f(M) = Q^{a(Q)} 2^{a(2)n} f(M).$$

From (36) and (37) it follows that

$$Q^{a(Q)} 2^{a(2)n} f(M) \equiv f(M) \pmod{N_k(Q2^n - 1)}$$

and so

$$f(M)Q^{a(2)} \equiv Q^{a(Q)}(Q2^n)^{a(2)} f(M) \equiv Q^{a(Q)} f(M) \pmod{N_k(Q2^n - 1)}$$

for every positive integer n. This shows that $a(Q) = a(2)$ since $\limsup_{n\to\infty} N_k(Q2^n - 1) = \infty$. This completes the proof of Theorem 1.

2. Proof of the Corollary

Assume that $f, g \in \mathcal{M}$ and satisfy

$$(38) \qquad f(n+m) \equiv g(m) \pmod{N_k(n)}$$

for every positive integer n, m. Applying (38) with $m = 1$ we have

$$(39) \qquad f(n+1) \equiv g(1) \equiv f(1) \pmod{N_k(n)}$$

for every positive integer n. Applying Theorem 1 with $M = 1$ we get that $f(n) = n^a$ $(n = 1, 2, \dots)$, where $a \geqq 0$ is a non-negative integer.

Let m be a positive integer. Then from (38) we have

$$g(m) \equiv f(n+m) = (n+m)^a \equiv m^a \pmod{N_k(n)}$$

for every positive integer n, which implies $g(m) = m^a$, since $\limsup_{n\to\infty} N_k(n) = \infty$. This completes the proof of the Corollary.

6. Proof of Theorem 2

Assume that $f \in \mathcal{M}$ and (5) holds, i.e.

$$f(n^k + m) \equiv f(n^k) + f(m) \pmod n \tag{5}$$

for every positive integer n and m. It is easy to see that from (5) we have

$$f(mn^k + 1) \equiv mf(n^k) + 1 \pmod n \tag{40}$$

for every positive integer n, m. Indeed, (40) holds for $m = 1$ and from (5) we get

$$f[(m+1)n^k + 1] = f[n^k + (mn^k + 1)] \equiv f(n^k) + f(mn^k + 1) \pmod n$$

which by using induction on m proves (40).

Applying (40) with m replaced by mn we get

$$f(mn^{k+1} + 1) \equiv mnf(n^k) + 1 \equiv 1 \pmod n \tag{41}$$

holds for every integer n, m. From (41), it follows that

$$f(n+1) \equiv 1 \pmod{N_{k+1}(n)}$$

for every positive integer n, which by using Theorem 1 implies $f(n) = n^a$ $(n = 1, 2, \dots)$, where $a \geqq 0$ is an integer. But in the case $a = 0$ (5) is not true. Thus Theorem 2 is proved.

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WEAK ASYMPTOTIC FORMULAS FOR PARTITIONS FREE OF SMALL SUMMANDS. II

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1. Introduction

Denote by $p(n)$ the number of (unrestricted) partitions of the positive integer n and by $q(n)$ the number of partitions of n into distinct parts. G. H. Hardy and S. Ramanujan [8] showed[2] that for $n \to \infty$

$$p(n) = \{1 + O(n^{-1/2})\}(4\sqrt{3}n)^{-1} \exp\{\sqrt{2/3}\pi\sqrt{n}\} \tag{1}$$

and

$$q(n) = \{1 + O(n^{-1/2})\}(4 \cdot 3^{1/4} \cdot n^{3/4})^{-1} \exp\{\sqrt{1/3}\pi\sqrt{n}\}. \tag{2}$$

In [4] P. Erdős and J. Lehner investigated the distribution of the parts n_i in the partitions

$$n = n_1 + \ldots + n_r \qquad (1 \leqq n_r \leqq n_1,\ n_i \in \mathbf{Z}) \tag{3}$$

of n. They proved that almost all (i.e. with exception of at most $o(p(n))$) partitions of n consist of

$$r = (1 + o(1)) \left(\sqrt{2/3}\pi\right)^{-1} \sqrt{n} \log n \tag{4}$$

parts and that, again for almost all partitions of n, the largest summand n_1 in (3) is of the same order:

$$n_1 = (1 + o(1)) \left(\sqrt{2/3}\pi\right)^{-1} \sqrt{n} \log n. \tag{5}$$

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[2] Actually, [8] contains much sharper results. The method of Hardy and Ramanujan was subsequently refined by H. Rademacher [11], who obtained a representation of $p(n)$ as the sum of a convergent series involving nothing but elementary functions such as the exponential function and certain roots of unity. L. K. Hua [10] employed Rademacher's technique to derive a similar series expansion for $q(n)$.

Analogous results hold for $q(n)$, and in [7] P. Erdős and P. Turán considered similar problems for more general partition functions.

It is interesting to study the influence of the "small" parts of the partitions of n. To this end one might investigate the behavior of $p_y(n)$, which is the number of partitions of n with smallest part $n_r \geqq y$. In view of (4) and (5) it can be expected that $p_y(n)$ and $p(n)$ — or at least their logarithms — are not "too far" apart as long as y is not "too large", e.g. $y \leqq n^{1/2-\varepsilon}$. On the other hand, it would not come as a big surprise if $p_y(n)$ is much smaller than $p(n)$ (even on the logarithmic scale), once y is "very large", e.g. $y \geqq n^{1/2+\varepsilon}$. In fact, results of J. Dixmier and J.-L. Nicolas [2], [3] and the author [9] show that[3]

$$(6)\quad \log p_y(n) = \sqrt{2/3}\pi n^{1/2} - y\log(\sqrt{n}\cdot y^{-1}) - cy + O\left(y^2/\sqrt{n} + n^{1/4}\sqrt{\log n}\right)$$

as $n \to \infty$, uniformly in $y = o(\sqrt{n})$.

For applications it is important that asymptotic formulas for $p_y(n)$ hold uniformly with respect to y in "large" y intervals depending on n (cf. [2], [5]).

The purpose of the present paper is to provide asymptotic formulas for the logarithms of *general* partition functions related to partitions into "very large" parts (resp. distinct parts) which are uniform with respect to the relevant parameter. As in [9], these results are deduced from a Tauberian theorem which is similar to a theorem of W. Schwarz [12], but more involved due to the presence of an additional parameter.

In the case of the "ordinary" partition function $p_y(n)$ considered above we shall see that $\log p_y(n)$ decreases rapidly if the parameter y increases in $[n^{1/2+\varepsilon}, n^{1-\varepsilon}]$, in accordance with our heuristic considerations based upon (4) and (5).

2. Basic definitions and results

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ be an unbounded sequence of real numbers with counting function

$$(7)\qquad N(u) = \sum_{\lambda_\nu < u} 1.$$

For[4] $\ell \in \mathrm{Lin}_{\mathbf{N}_0}\{\lambda_1, \lambda_2, \dots\}$ we consider the Diophantine equation

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[3] Dixmier and Nicolas [2] prove that $p_m(n) \sim p(n)\ (\pi/\sqrt{6n})^{m-1}(m-1)!$ as $n \to \infty$, uniformly in $m = o(n^{1/4})$, $1 \leqq m \in \mathbf{Z}$, and extend the m-range to $m \leqq n^{1/3-\varepsilon}$ in [3]. In [9] the author investigates more general partition functions.

[4] $\mathbf{N}_0 = \{0, 1, 2, \dots\}$.

(8) $$\ell = \sum_{\lambda_\nu \geqq y} r_\nu \lambda_\nu \qquad (r_\nu \in \mathbf{N}_0).$$

The number of solutions of (8) in nonnegative integers r_ν is denoted by $p_y(\ell)$, and $q_y(\ell)$ denotes the number of solutions in $r_\nu \in \{0,1\}$.

These partition functions are generated by

(9) $$f_y(s) = \sum_\ell p_y(\ell)e^{-\ell s} = \prod_{\lambda_\nu \geqq y} \{1 - \exp(-\lambda_\nu s)\}^{-1}$$

and

(10) $$\tilde{f}_y(s) = \sum_\ell q_y(\ell)e^{-\ell s} = \prod_{\lambda_\nu \geqq y} \{1 + \exp(-\lambda_\nu s)\}$$

respectively, the series and products being convergent in $\operatorname{Re} s > 0$ if $N(u)$ does not grow too rapidly,

(11) $$N(u) \ll u^B \qquad (u > \lambda_1)$$

with a positive constant B, say.[5]

Both generating functions have holomorphic logarithms in $\operatorname{Re} s > 0$ given by

(12) $$\varphi_y(s) = \log f_y(s) = -\sum_{\lambda_\nu \geqq y} \log\{1 - \exp(-\lambda_\nu s)\}$$

and

(13) $$\tilde{\varphi}_y(s) = \log \tilde{f}_y(s) = \sum_{\lambda_\nu \geqq y} \log\{1 + \exp(-\lambda_\nu s)\}.$$

Here log denotes the principal branch of the logarithm.

In the theorems below we deal with the logarithms of the summatory functions

(14) $$P_y(u) = \sum_{\ell < u} p_y(\ell)$$

and

(15) $$Q_y(u) = \sum_{\ell < u} q_y(\ell).$$

---

[5] $\log N(u) = o(u)$ as $u \to \infty$ is sufficient.

THEOREM 1. *Let c, C and α be positive real constants satisfying*

$$0 < C \cdot 2^{-\alpha} < c \leqq C \tag{16}$$

and assume

$$c \cdot u^{\alpha} L(u) \leqq N(u) \leqq C \cdot u^{\alpha} L(u) \qquad (u > \lambda_1) \tag{17}$$

for some slowly varying function L.[6] *Define $\sigma_u(y)$ for sufficiently large u by*[7]

$$-\varphi_y'(\sigma_u(y)) = u. \tag{18}$$

Then for every small positive number ε the asymptotic formula

$$\log P_y(u) = \varphi_y(\sigma_u(y)) + u \cdot \sigma_u(y) + O\left(\sqrt{u/y} \cdot (\log u)^{3/2}\right) \tag{19}$$

holds uniformly in $u^{\frac{1}{\alpha+1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$ as $u \to \infty$.

REMARKS. (A) We shall show that $\frac{u}{y} \log y \ll u\sigma_u(y) \ll \frac{u}{y} \log y$.

(B) If $N(u) = c_1 u^{\alpha} + O(u^{\beta})$, $0 \leqq \beta < \alpha$, $c_1 > 0$, then Theorem 1 implies that

$$\log P_y(u) = \frac{u}{y} \log(y^{\alpha+1}/u) \cdot \{1 + O(\operatorname{loglog} u / \log u)\} \quad (u \to \infty) \tag{20}$$

uniformly in $u^{\frac{1}{\alpha+1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$.

In the case of the ordinary partition function (i.e. $\lambda_\nu = \nu$, $\nu = 1, 2, 3, \dots$) we have $N(u) = u + O(1)$, hence Theorem 1 yields

$$\log P_y(y) = \frac{u}{y} \log(y^2/u) - \frac{u}{y} \operatorname{loglog}(y^2/u) + \frac{u}{y} + O_\varepsilon\left(\frac{u \operatorname{loglog} u}{y \log u}\right) \tag{21}$$

uniformly in $y \in [u^{1/2+\varepsilon}, u^{1-\varepsilon}]$ as $u \to \infty$. Since $n \mapsto p_y(n)$ is increasing, (21) remains true if $P_y(u)$ and u are replaced by $p_y(n)$ and n, respectively.

The corresponding result for partitions into distinct parts is also true.

THEOREM 2. *Let N(u) be as in the preceding theorem, and define $\tilde{\sigma}_u(y)$ by*

$$-\tilde{\varphi}_y'(\tilde{\sigma}_u(y)) = u \qquad (u \text{ sufficiently large}). \tag{22}$$

Then the asymptotic relation

$$\log Q_y(u) = \tilde{\varphi}_y(\tilde{\sigma}_u(y)) + u \cdot \tilde{\sigma}_u(y) + O\left(\sqrt{u/y} \cdot (\log u)^{3/2}\right) \tag{23}$$

---

[6] I.e. L is a positive Lebesgue-measurable function satisfying $L(\mu x)/L(x) \to 1$ ($x \to \infty$) for every $\mu > 0$.

[7] φ_y is given by (12). The proof will show that $\sigma_u(y)$ is well defined.

holds uniformly in $u^{\frac{1}{\alpha+1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$ *as* $u \to \infty$ *for each small* $\varepsilon > 0$.

REMARK. If we consider the special case $N(u) = c_1 u^\alpha + O(u^\beta)$, $0 \leqq \beta < \alpha$, $c_1 > 0$ again, it turns out that uniformly in $u^{\frac{1}{\alpha+1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$

$$\log Q_y(u) = \frac{u}{y}\log(y^{\alpha+1}/u)\{1 + O(\operatorname{loglog} u/\log u)\} \quad (u \to \infty). \tag{24}$$

Comparing this with (20) we observe that $\log Q_y(u) \sim \log P_y(u)$ uniformly in y for this special type of counting function.

The following theorem generalizes the observation.

THEOREM 3. *Let* $N(u)$ *satisfy the hypotheses of Theorem* 1. *Then as* $u \to \infty$

$$\log P_y(u) = (1 + O(1/\log u)) \cdot \log Q_y(y) \tag{25}$$

uniformly in $u^{\frac{1}{\alpha+1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$.

Both Theorems 1 and 2 are consequences of the following Tauberian theorem for Laplace-transforms depending on a parameter, which is closely related to Theorem 1 in [9]. However, the present theorem allows one to deal with much larger values of the parameter. Both of these results have their roots in a Tauberian theorem of W. Schwarz ([12], Satz 1).[8]

THEOREM 4. *Let* $\{A_y : [0,\infty[\to [0,\infty[; y \geqq 0\}$ *be a family of nondecreasing functions such that each* A_y *is dominated by a (fixed) nondecreasing function* $A\colon [0,\infty[\to [0,\infty[$ *whose Laplace-transform*

$$\int_0^\infty A(u)e^{-u\sigma}\,du \tag{26}$$

converges in $\sigma > 0$. *Therefore, for every* $y \geqq 0$, *the function*

$$f_y(\sigma) = \sigma \cdot \int_0^\infty A_y(u)e^{-u\sigma}\,du \tag{27}$$

is well defined in $\sigma > 0$.

We assume the existence of constants $C_1 \geqq 0$, $\mu > 0$ *and nonnegative functions* $\varphi_y \in \mathcal{C}^2]0,\mu[$ *with*

$$|\log f_y(\sigma) - \varphi_y(\sigma)| \leqq C_1 \qquad (y \geqq 0,\ 0 < \sigma < \mu) \tag{28}$$

---

[8]Note that there is a typographical error in the hypotheses of Satz 1 in [12]: Formula (3.6) should read $\sigma(\varphi''(\sigma))^{\varrho+1}|\varphi'(\sigma)|^{-2\varrho-1} \leqq C$.

and, for each fixed $y \geqq 0$,

$$(29) \qquad -\varphi_y'(\sigma) \to \infty \quad \textit{as} \quad \sigma \to 0+,$$

$$(30) \qquad 0 < \varphi_y''(\sigma) \quad \textit{increases as } \sigma \textit{ decreases to zero,}$$

$$(31) \qquad -\varphi_y'(\sigma) \leqq \psi(\sigma) \qquad (0 < \sigma < \mu)$$

with a fixed nonnegative function $\psi \in \mathcal{C}^1]0,\mu[$, $\psi' < 0$. *Furthermore we suppose that there are real constants* $a_2 \geqq a_1 > 1$, $0 < b_2 \leqq b_1$ *and* C_2, C_3, M, $y_0 > 0$ *such that the following bounds for* $-\varphi_y'(\sigma)$, $y \geqq y_0$, *hold:*

$$(32) \qquad -\varphi_y'(\sigma) \geqq C_2 \cdot y^{a_1} \cdot \exp(-b_1 y \sigma) \qquad (0 < \sigma < \mu),$$

$$(33) \qquad -\varphi_y'(\sigma) \leqq C_3 \cdot y^{a_2} \cdot \exp(-b_2 y \sigma) \quad \textit{if} \quad 0 < \sigma < \mu \textit{ and } y\sigma \geqq M.$$

Finally we require the following relation between the first and second derivatives of φ_y: *Given real constants* $0 < A_1 \leqq B_1$ *there exist constants* $0 <$ $< A_2 \leqq B_2$, *which may depend on* A_1, B_1 *but are independent of* σ *and* y, *such that, for sufficiently large* y,

$$(34) \quad A_1 \log y \leqq y\sigma \leqq B_1 \log y \quad \textit{implies} \quad A_2 y \leqq \varphi_y''(\sigma) \cdot |\varphi_y'(\sigma)|^{-1} \leqq B_2 y.$$

Now define the quantity $\sigma_u(y)$ *for sufficiently large* u *by*

$$(35) \qquad -\varphi_y'(\sigma_u(y)) = u.$$

Then for each positive real number $\varepsilon < \frac{a_1 - 1}{2a_1}$ *the asymptotic formula*

$$(36) \qquad \log A_y(u) = \varphi_y(\sigma_u(y)) + u \cdot \sigma_u(y) + O_\varepsilon(\sqrt{u/y}(\log u)^{3/2})$$

holds uniformly in $u^{\frac{1}{a_1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$ *as* $u \to \infty$. *The O constant in* (36) *may depend on* ε, *but not on* u *or* y.

REMARKS. (A) As we shall show in a moment, the order of the main term in (36) is at least $\frac{u}{y} \log u$.

(B) In our applications the functions A_y and $-\varphi_y'$ will be decreasing with respect to y (i.e. $\eta > y$ implies $A_\eta(u) \leqq A_y(u)$ and $-\varphi_\eta'(\sigma) \leqq -\varphi_y'(\sigma)$). In this case we obviously may set $A = A_0$ and $\psi = -\varphi_0'$.

The bounds A and ψ are useful: Since

$$\sigma \int_0^w A_y(u) e^{-u\sigma} du \leqq \sigma \int_0^w A(u) e^{-u\sigma} du \leqq A(w),$$

we have

$$f_y(\sigma) = \sigma \int_w^\infty A_y(u)e^{-u\sigma}\,du + O(1) \tag{37}$$

uniformly in y for every fixed positive number w.

From relations (29)–(31) it follows that $\sigma_u(y)$ is well defined by equation (35) for all $y \geqq 0$ and $u \geqq u_0$, where u_0 is independent of y: Let u_0 be such that the equation $\psi(\sigma_u) = u$ has a unique solution for all $u \geqq u_0$.

2. Some preliminary estimates

Before we actually start proving the theorems, we recall some simple but useful consequences of the hypotheses of Theorem 4.

For the rest of this section we assume that

$$0 < \varepsilon < \frac{a_1 - 1}{2a_1}, \tag{38}$$

$$u^{\frac{1}{a_1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon} \tag{39}$$

and that $u_0(\varepsilon), c_1(\varepsilon), c_2(\varepsilon), \dots$ are sufficiently large (resp. small) positive real numbers which may depend upon ε but not on u or y.

From (35) and (32) it follows that

$$u = -\varphi_y'(\sigma_u(y)) \geqq C_2 y^{a_1} \exp\{-b_1 y \sigma_u(y)\}$$

hence

$$y\sigma_u(y) \geqq \frac{1}{b_1}\{a_1 \log y - \log u + \log C_2\} \geqq \frac{1}{b_1}\left\{\frac{a_1 \cdot \varepsilon}{1-\varepsilon}\log y + \log C_2\right\}$$

by (39). This implies

$$y\sigma_u(y) \geqq c_1(\varepsilon)\log y \qquad (u \geqq u_0(\varepsilon)). \tag{40}$$

If $u_0(\varepsilon)$ is large enough, (39) and (40) show that $y\sigma_u(y) \geqq M$. Therefore (33) may be applied with $\sigma = \sigma_u(y)$:

$$u = -\varphi_y'(\sigma_u(y)) \leqq C_3 y^{a_2} \exp\{-b_2 y \sigma_u(y)\}$$

which yields

$$y\sigma_u(y) \leqq \frac{a_2}{b_2}\log y \qquad (u \geqq u_0(\varepsilon)). \tag{41}$$

In view of (40) and (41), relation (34) guarantees the existence of positive constants $A_2(\varepsilon) \leqq B_2(\varepsilon)$ such that

$$(42) \qquad A_2(\varepsilon)\cdot y \leqq \frac{\varphi_y''(\sigma_u(y))}{u} \leqq B_2(\varepsilon)\cdot y \qquad (u \geqq u_0(\varepsilon)).$$

Using the right hand inequality in (42) together with (39) we deduce

$$(43) \qquad u^2/\varphi_y''(\sigma_u(y)) \geqq u\{B_2(\varepsilon)\cdot y\}^{-1} \geqq c_2(\varepsilon)\cdot u^{\varepsilon} \quad (u \geqq u_0(\varepsilon)).$$

Inequalities (40), (41) and (42) yield

$$(44) \qquad c_3(\varepsilon)\cdot \log y \leqq \frac{\sigma_u(y)\cdot\varphi_y''(\sigma_u(y))}{u} \leqq c_4(\varepsilon)\cdot\log y \quad (u \geqq u_0(\varepsilon))$$

which shows that

$$(45) \qquad u/\varphi_y''(\sigma_u(y)) \leqq \varepsilon\cdot\sigma_u(y) \quad (u \geqq u_0(\varepsilon)).$$

If $u \leqq v \leqq 2u$ and $u \geqq u_0(\varepsilon)$ then

$$(46) \qquad 1 \leqq \sigma_u(y)/\sigma_v(y) \leqq c_5(\varepsilon).$$

The left-hand inequality is an immediate consequence of the monotonicity of $u \mapsto \sigma_u(y)$, while the upper bound in (46) follows from (40) and (41).

3. Proofs of the results on partitions

Proof of Theorem 1. We want to apply the Tauberian theorem with $A_y = P_y$, $A = P_0$ and $\varphi_y(\sigma) = \log f_y(\sigma) = -\sum_{\lambda_\nu \geqq y} \log\{1-\exp(-\lambda_\nu\cdot\sigma)\}$. To this end we have to verify that the hypotheses of Theorem 4 are satisfied.

Since we define $\varphi_y(\sigma) = \log f_y(\sigma)$, (28) holds trivially with $C_1 = 0$ and $\mu = 1$, say. The properties (29) and (30) follow easily from

$$(47) \qquad -\varphi_y'(\sigma) = \sum_{\lambda_\nu \geqq y} \frac{\lambda_\nu}{\exp(\lambda_\nu\sigma)-1} = -\int_y^\infty (N(u)-N(y))\frac{d}{du}\frac{u}{e^{u\sigma}-1}\,du$$

and

$$(48) \qquad \varphi_y''(\sigma) = \sum_{\lambda_\nu \geqq y} \lambda_\nu^2 e^{\lambda_\nu\sigma}(\exp(\lambda_\nu\sigma)-1)^{-2}$$

since $N(u)-N(y) \to \infty$ as $u \to \infty$ for fixed y. We may choose $\psi = -\varphi_0'$ according to our remark following Theorem 4.

In the sequel we shall need some basic properties of slowly varying (abbreviated s.v.) functions, namely

$$\text{(49)}\qquad \begin{cases} \text{If } L \text{ is s.v., then } \lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \text{ holds uniformly} \\ \text{with respect to } t \in [a,b] \text{ for any finite interval } [a,b] \subset]0,\infty[. \end{cases}$$

$$\text{(50)}\qquad \begin{cases} \text{Let } L \text{ be s.v. Then } x^{\delta}L(x) \to \infty \text{ and } x^{-\delta}L(x) \to 0 \\ \text{for any positive real number } \delta \text{ as } x \to \infty. \end{cases}$$

Proofs of (49) and (50) may be found in the books of E. Seneta [13] and N. H. Bingham, C. M. Goldie and J. L. Teugels [1].

Relation (17) in Theorem 1 implies that

$$N(2u) - N(u) \geqq u^{\alpha}L(u)\{c2^{\alpha}L(2u)/L(u) - C\}.$$

Since $2^{\alpha}c > C$ by (16) and $L(2u)/L(u) \to 1$ as $u \to \infty$ there exist positive constants β and u_0 such that for $u \geqq u_0$

$$\text{(51)}\qquad N(2u) - N(u) \geqq \beta u^{\alpha}L(u) \geqq \beta^{*}N(u) \qquad (\beta^{*} = \beta C^{-1})$$

whence

$$-\varphi_y'(\sigma) \geqq \sum_{y \leqq \lambda_\nu < 2y} \lambda_\nu \cdot e^{-\lambda_\nu \sigma} \geqq y \cdot e^{-2y\sigma}\{N(2y) - N(y)\} \geqq$$
$$\geqq \beta y^{\alpha+1}L(y)e^{-2y\sigma}.$$

Therefore, using (50), we obtain for any small positive number δ

$$\text{(52)}\qquad -\varphi_y'(\sigma) \geqq \frac{\beta}{2} y^{\alpha+1-\delta} e^{-2y\sigma}$$

provided y is large enough. Hence we may take $a_1 = \alpha + 1 - \delta$ and $b_1 = 2$ in (32), where δ is any real number in $]0, \alpha[$.

In order to verify (33) we define $M = 2(\alpha + 2)$ and observe that by (17) and (50)

$$\text{(53)}\qquad N(u) \leqq u^{\alpha+1}$$

for all sufficiently large u.

Using (53) and the estimate

$$0 < -\frac{d}{dw}\{w \cdot (e^{w} - 1)^{-1}\} < 2we^{-w} \qquad (w \geqq 4)$$

we infer that for $y\sigma \geqq M$

$$-\varphi_y'(\sigma) = -\sigma^{-1} \int_{y\sigma}^{\infty} N\left(\frac{w}{\sigma}\right) \frac{d}{dw} \frac{w}{e^w - 1} dw - N(y) \cdot y \cdot \{e^{y\sigma} - 1\}^{-1} \leqq$$

$$\leqq 2 \cdot \sigma^{-\alpha-2} \cdot \int_{y\sigma}^{\infty} w^{\alpha+2} e^{-w} dw.$$

Now it is easy to see that for $0 < r < x$

$$\int_{x}^{\infty} w^r e^{-w} dw \leqq \left\{1 - \frac{r}{x}\right\}^{-1} x^r e^{-x}. \tag{54}$$

Applying this estimate in the inequality above yields

$$-\varphi_y'(\sigma) \leqq 2\left\{1 - \frac{\alpha+2}{M}\right\}^{-1} \sigma^{-\alpha-2} (y\sigma)^{\alpha+2} e^{-y\sigma} \leqq 4y^{\alpha+2} e^{-y\sigma} \tag{55}$$

if $y\sigma \geqq M = 2(\alpha + 2)$ and y is large enough. Hence we may set $a_2 = \alpha + 2$ and $b_2 = 1$ in (33).

It remains to show that relation (34) holds. For all $\sigma > 0$ and $y > 0$ we have

$$\varphi_y''(\sigma) = \sum_{\lambda_\nu \geqq y} \lambda_\nu^2 e^{\lambda_\nu \sigma} \{\exp(\lambda_\nu \sigma) - 1\}^{-2} > y \cdot \sum_{\lambda_\nu \geqq y} \frac{\lambda_\nu}{\exp(\lambda_\nu \sigma) - 1} = y \cdot |\varphi_y'(\sigma)|.$$

Thus we may choose $A_2 = 1$ in (34) independently of the magnitude of $y\sigma$.

The verification of the corresponding upper bound is somewhat more involved. Here the key step is to establish the existence of a constant $C^* = C^*(A_1, B_1)$ such that, for sufficiently large y,

$$A_1 \log y \leqq y\sigma \leqq B_1 \log y \text{ implies } \varphi_y''(\sigma) \leqq C^* \sum_{y \leqq \lambda_\nu \leqq 2y} \frac{\lambda_\nu^2 e^{\lambda_\nu \sigma}}{\{\exp(\lambda_\nu \sigma) - 1\}^2}. \tag{56}$$

From (56) the desired estimate is easily deduced: If $y \geqq y_1$ we obtain

$$\varphi_y''(\sigma) \leqq C^* \cdot 2y \cdot e^{y\sigma} \{e^{y\sigma} - 1\}^{-1} \cdot \sum_{y \leqq \lambda_\nu \leqq 2y} \frac{\lambda_\nu}{\exp(\lambda_\nu \sigma) - 1} \leqq$$

$$\leqq C^{**} \cdot y(-\varphi_y'(\sigma))$$

where $C^{**} = 2C^* \cdot y_1^{A_1} \{y_1^{A_1} - 1\}^{-1}$.

As for the proof of (56) we assume that $y \geqq y_1$ is sufficiently large and start with

(57)
$$\sum_{y \leqq \lambda_\nu \leqq 2y} \frac{\lambda_\nu^2 e^{\lambda_\nu \sigma}}{\{\exp(\lambda_\nu \sigma) - 1\}^2} \geqq y^2 e^{-2y\sigma} \{N(2y) - N(y)\} \geqq \beta y^{\alpha+2} L(y) e^{-2y\sigma}$$

where we have used (51). Furthermore, since $y\sigma \geqq A_1 \log y$, we have

$$\sum_{\lambda_\nu \geqq 2y} \frac{\lambda_\nu^2 e^{\lambda_\nu \sigma}}{\{\exp(\lambda_\nu \sigma) - 1\}^2} \leqq \{1 - y_1^{-2A_1}\}^{-2} \cdot \sum_{\lambda_\nu \geqq 2y} \lambda_\nu^2 e^{-\lambda_\nu \sigma}. \tag{58}$$

Partial summation and some simple estimates yield

(59)
$$\sum_{\lambda_\nu \geqq 2y} \lambda_\nu^2 e^{-\lambda_\nu \sigma} = -\int_{2y}^{\infty} \{N(u) - N(2y)\} \frac{d}{du} \frac{u^2}{e^{u\sigma}} du \leqq \sigma^{-2} \int_{2y\sigma}^{\infty} N\Big(\frac{w}{\sigma}\Big) \frac{w^2}{e^w} dw.$$

Let $T = 2 + 2/A_1$; from (49) and (50) we know that

$$y^{-1} \leqq L(y) \leqq y \qquad (y \geqq y_1) \tag{60}$$

and

$$L(v) \leqq 2 \cdot L(y) \qquad (y \geqq y_1 \text{ and } y \leqq v \leqq Ty) \tag{61}$$

provided y_1 is large enough.

Now split the right hand integral in (59) at $Ty\sigma$ and use (17) as well as (61) resp. (60) to estimate $N(w/\sigma)$ in $[2y\sigma, Ty\sigma]$ resp. in $[Ty\sigma, \infty[$, thus obtaining

$$\sum_{\lambda_\nu \geqq 2y} \lambda_\nu^2 e^{-\lambda_\nu \sigma} \leqq$$
$$\leqq 2C\sigma^{-\alpha-2} L(y) \int_{2y\sigma}^{Ty\sigma} w^{\alpha+2} e^{-w} dw + C\sigma^{-\alpha-3} \int_{Ty\sigma}^{\infty} w^{\alpha+3} e^{-w} dw.$$

Utilizing inequality (54) in both integrals above we infer[9] that for sufficiently large $y \geqq y_1$

$$\sum_{\lambda_\nu \geqq 2y} \lambda_\nu^2 e^{-\lambda_\nu \sigma} \leqq 9C 2^\alpha y^{\alpha+2} L(y) e^{-2y\sigma} + C 2 T^{\alpha+3} y^{\alpha+3} \cdot \exp\Big\{-2y\sigma\Big(1 + \frac{1}{A_1}\Big)\Big\}.$$

[9] Recall that $T = 2 + 2/A_1$.

If we use (60) and the estimate $e^{-2y\sigma/A_1} \leqq y^{-2}$, which follows from $y\sigma \geqq \geqq A_1 \log y$, we obtain

$$(62)\qquad \sum_{\lambda_\nu \geqq 2y} \lambda_\nu^2 e^{-\lambda_\nu \sigma} \leqq \{9C2^\alpha + C2T^{\alpha+3}\} \cdot y^{\alpha+2} L(y) e^{-2y\sigma}.$$

The estimates (57), (58) and (62) yield (56). Now Theorem 1 follows from Theorem 4.

A remark to the proof of Theorem 2. Of course, Theorem 2 may be proved in a similar way, but it is easier to make use of the inequalities given in the proof above. We illustrate this by showing that

$$\tilde\varphi_y''(\sigma) \ll y \cdot (-\tilde\varphi_y'(\sigma)) \quad \text{if} \quad A_1 \log y \leqq y\sigma \leqq B_1 \log y, \quad y \geqq y_1.$$

Here we have

$$\tilde\varphi_y''(\sigma) = \sum_{\lambda_\nu \geqq y} \lambda_\nu^2 e^{\lambda_\nu \sigma} \{\exp(\lambda_\nu \sigma) + 1\}^{-2} \leqq \sum_{\lambda_\nu \geqq y} \lambda_\nu^2 e^{\lambda_\nu \sigma} \{\exp(\lambda_\nu \sigma) - 1\}^{-2} =$$

$$= \varphi_y''(\sigma) \ll y \cdot (-\varphi_y'(\sigma)) = y \cdot \sum_{\lambda_\nu \geqq y} \lambda_\nu \{\exp(\lambda_\nu \sigma) - 1\}^{-1} \leqq$$

$$\leqq \left(1 + 2/(y_1^{A_1} - 1)\right) \cdot y \cdot \sum_{\lambda_\nu \geqq y} \lambda_\nu \{\exp(\lambda_\nu \sigma) + 1\}^{-1} = \tilde C \cdot y \cdot \left(-\tilde\varphi_y'(\sigma)\right).$$

PROOF OF THEOREM 3. In view of Theorems 1 and 2 it suffices to show that uniformly in $u^{\frac{1}{\alpha+1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$

$$(63)\qquad \varphi_y(\sigma_u(y)) + u \cdot \sigma_u(y) = \{1 + O(1/\log u)\} \cdot (\tilde\varphi_y(\tilde\sigma_u(y)) + u \cdot \tilde\sigma_u(y))$$

as $u \to \infty$.

In all subsequent estimates the $\ll$ constants are independent of u and y. For $y\sigma \gg \log y$ it is not difficult to see that

$$-\varphi_y'(\sigma) + \tilde\varphi_y'(\sigma) = -\sigma^{-1} \int_{y\sigma}^{\infty} \left\{N\left(\frac{w}{\sigma}\right) - N(y)\right\} \cdot \frac{d}{dw}\left\{\frac{w}{e^w - 1} - \frac{w}{e^w + 1}\right\} dw \ll$$

$$\ll N(y) \cdot y \cdot e^{-2y\sigma}.$$

Hence by the mean value theorem there exists a $\sigma^* \in [\tilde\sigma_u(y), \sigma_u(y)]$ such that

$$0 = -\varphi_y'(\sigma_u(y)) + \varphi_y'(\tilde\sigma_u(y)) + \tilde\varphi_y'(\tilde\sigma_u(y)) - \varphi_y'(\tilde\sigma_u(y)) =$$

$$= (\tilde\sigma_u(y) - \sigma_u(y)) \cdot \varphi_y''(\sigma^*) + O\left(N(y) \cdot y \cdot \exp(-2y\tilde\sigma_u(y))\right).$$

In view of (30) and (42) this implies uniformly in $u^{\frac{1}{\alpha+1}+\varepsilon} \leqq y \leqq u^{1-\varepsilon}$

$$0 < \sigma_u(y) - \tilde{\sigma}_u(y) \ll \frac{N(y) \cdot y \cdot \exp(-2y\tilde{\sigma}_u(y))}{y \cdot u} \ll y^{-1}. \tag{64}$$

Here the second $\ll$ estimate requires an application of the inequality

$$-\tilde{\varphi}_y'(\sigma) \gg y \cdot N(y) \cdot \exp(-2y\sigma)$$

with $\sigma = \tilde{\sigma}_u(y)$; in case of $\varphi_y'(\sigma)$ this inequality has been verified in the proof of Theorem 1, cf. the lines following relation (51).

Since, for $y\sigma \gg \log y$,

$$\varphi_y(\sigma) - \tilde{\varphi}_y(\sigma) = \int_{y\sigma}^{\infty} \left\{ N\left(\frac{w}{\sigma}\right) - N(y) \right\} \cdot \left\{ \frac{1}{e^w - 1} - \frac{1}{e^w + 1} \right\} dw \ll N(y) \cdot e^{-2y\sigma}$$

the estimate

$$\varphi_y(\sigma_u(y)) - \tilde{\varphi}_y(\sigma_u(y)) \ll u \cdot y^{-1} \tag{65}$$

may be proved in almost the same way as (64). Now (63) follows from (64), (65) and (40), and our proof of Theorem 3 is complete.

4. Proof of the Tauberian theorem

PROOF OF THEOREM 4. Since our proof is closely related to the proof of Theorem 1 in [9] it suffices to sketch the main ideas and indicate the relevant modifications. See [9] for full details.

The monotonicity of $u \mapsto A_y(u)$ shows that for any $T > 0$ and all $\sigma \in]0, \mu[$, $y \geqq 0$

$$f_y(\sigma) \geqq \sigma \int_T^{\infty} A_y(y) \cdot e^{-u\sigma} du \geqq A_y(T) \cdot e^{-T\sigma}$$

whence by (28)

$$A_y(T) \leqq e^{C_1} \cdot \exp\{\varphi_y(\sigma) + T \cdot \sigma\}.$$

For sufficiently large T_0 and all[10] $T \geqq T_0$ the right-hand side above attains its minimum at $\sigma = \sigma_T(y)$, and therefore we use the upper bound

$$A_y(T) \leqq e^{C_1} \cdot \exp\{\varphi_y(\sigma_T(y)) + T\sigma_T(y)\} \qquad (y \geqq y_0,\ T \geqq T_0). \tag{66}$$

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[10] Cf. remark (B) following Theorem 4.

For the rest of the proof we assume that

$$T^{\frac{1}{a_1}+\delta} \leqq y \leqq T^{1-\delta} \tag{67}$$

where $\delta > 0$ is a small real number. All $\ll$ constants appearing subsequently may depend on δ but are independent of T and y.

In order to obtain a good lower bound for $\log A_y(T)$ we define[11]

$$\varepsilon = \varepsilon(T,y) = K \cdot \left\{ \frac{\varphi_y''(\sigma_T(y))}{T^2} \cdot \log \frac{T^2}{\varphi_y''(\sigma_T(y))} \right\}^{1/2}, \tag{68}$$

$$R = R(T,y) = (1+\varepsilon)\cdot T \tag{69}$$

where the constant K is specified below, and consider

$$f_y(\sigma_T(y)) = \sigma_T(y) \left\{ \int_0^{T_0} + \int_{T_0}^{(1-\varepsilon)T} + \int_{(1-\varepsilon)T}^{R} + \int_R^{\infty} \right\} A_y(u)\cdot \exp(-u\cdot\sigma_T(y))du =$$
$$= l_0 + l_1 + l_2 + l_3.$$

The estimate

$$l_0 + l_1 + l_3 \ll_\delta \exp\{\varphi_y(\sigma_T(y))\}\cdot e^U \tag{70}$$

with

$$U = U(T,y) = -\frac{1}{4}\frac{\varepsilon^2 R^2}{\varphi_y''(\sigma_R(y))} - \log\varepsilon + \log\frac{\sigma_T(y)\varphi_y''(\sigma_R(y))}{R} \tag{71}$$

depends on monotonicity properties of $-\varphi_y'$ and φ_y'' and may be proved in almost the same way as the corresponding estimate in [9] (proof of Theorem 1). Only the approximation of $\sigma_R(y) - \sigma_T(y)$ has to be modified slightly, since the inequality $\varphi_y''(\sigma)\sigma \geqq |\varphi_y'(\sigma)|$ is not necessarily true in the present situation:

The mean value theorem implies the existence of $\sigma^* \in [\sigma_R(y), \sigma_T(y)]$ such that

$$\varepsilon T = -\varphi_y'(\sigma_R(y)) + \varphi_y'(\sigma_T(y)) = (\sigma_T(y) - \sigma_R(y))\cdot\varphi_y''(\sigma^*),$$

and by the monotonicity of φ_y'', (40) and (42) it follows that

$$\frac{1}{2}\frac{\varepsilon T}{\varphi_y''(\sigma_R(y))} \leqq \sigma_T(y) - \sigma_R(y) \ll_\delta \varepsilon y^{-1} \leqq \varepsilon\sigma_T(y). \tag{72}$$

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[11] Note that $\varepsilon(T,y) \to 0$ as $T \to \infty$ by (43).

Next we show that $U \to -\infty$ ad $T \to \infty$, uniformly in y. From our definition of ε and relation (42) we deduce that there is a constant $C_\delta > 0$ which is independent of T and y such that

$$\frac{\varepsilon^2 R^2}{\varphi_y''(\sigma_R(y))} \geqq K^2 C_\delta \log \frac{T^2}{\varphi_y''(\sigma_T(y))}$$

and

$$-\log\varepsilon \leqq \frac{1}{2}\log\frac{T^2}{\varphi_y''(\sigma_T(y))}.$$

In view of (44) and (46) the third summand on the right-hand side of (71) is $O_\delta(\log\log T)$. Hence, if we choose the constant K in (68) such that $K^2 \cdot C_\delta \geqq 6$, we obtain

$$U \leqq -\log\frac{T^2}{\varphi_y''(\sigma_T(y))} + O_\delta(\log\log T), \tag{73}$$

whence $U \ll_\delta -\log T \to -\infty$ uniformly in y as $T \to \infty$ by (43). Therefore it follows that, uniformly in y,

$$l_0 + l_1 + l_3 = o(\exp\{\varphi_y(\sigma_T(y))\}) \qquad (T \to \infty)$$

implying that

$$l_2 = f_y(\sigma_T(y)) - \{l_0 + l_1 + l_3\} \gg \exp\{\varphi_y(\sigma_T(y))\}$$

for sufficiently large T.

On the other hand, from the monotonicity of $u \mapsto A_y(u)$ we deduce that

$$I_2 = \sigma_T(y)\int\limits_{(1-\varepsilon)T}^{R} A_y(u)\cdot\exp(-u\cdot\sigma_T(y))du \leqq A_y(R)\cdot\exp\{-\sigma_T(y)\cdot(1-\varepsilon)T\}$$

and consequently

$$\log A_y(R) \geqq \varphi_y(\sigma_T(y)) + T\cdot\sigma_T(y) - \varepsilon\cdot T\cdot\sigma_T(y) + O(1). \tag{74}$$

It is not difficult to replace $\varphi_y(\sigma_T(y))$ resp. $T\cdot\sigma_T(y)$ by $\varphi_y(\sigma_R(y))$ resp. $R\cdot\sigma_R(y)$ in the inequality above: If we apply the mean value theorem and (72) we see that

$$\varphi_y(\sigma_R(y)) - \varphi_y(\sigma_T(y)) \leqq (\sigma_T(y) - \sigma_R(y))\cdot R \ll \varepsilon T\sigma_T(y),$$

and another application of (72) yields

$$|R\cdot\sigma_R(y) - T\cdot\sigma_T(y)| \leqq T\cdot(\sigma_T(y) - \sigma_R(y)) + \varepsilon T\sigma_R(y) \ll \varepsilon T\sigma_T(y).$$

Therefore we have

$$\log A_y(R) \geqq \varphi_y(\sigma_R(y)) + R \cdot \sigma_R(y) + O(\varepsilon \cdot T \cdot \sigma_T(y)) \tag{75}$$

where the O-constant depends at most on δ, but not on T or y. Our definition of ε and the relations (41), (42) and (43) imply that the error term in (74) does not exceed

$$K^* \cdot \sqrt{T/y} \cdot \log T^{3/2}$$

where K^* is a constant that is independent of T and y for all y satisfying (67); of course, K^* may depend upon δ. This finishes our proof of Theorem 4.

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ON CARDINAL INVARIANTS OF CONTINUOUS IMAGES OF TOPOLOGICAL GROUPS

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Introduction. In 1949, A. S. Esenin-Vol'pin [8] proved the coincidence of the character and the weight of every dyadic compact space. Later this result was generalized in different directions. Thus, in the paper of A. V. Arhangel'skiĭ and V. I. Ponomarev [4] it was shown that the weight of every dyadic compact space is equal to its tightness. The deep result of Efimov, Gerlits and Hagler is the following statement: if a dyadic compact space X of an uncountable weight τ is not a union of countably many closed subsets of smaller weight, then there exists a continuous mapping of X onto the Tychonoff cube I^τ (see [7, 9, 12]). In particular, the conclusion holds if $\operatorname{cf}\tau > \aleph_0$.

We shall call a space quasidyadic if it is a continuous image of a dense subset of a generalized Cantor cube. The problem of investigation of quasidyadic compact spaces seems to be more complicated than that of compact dyadic spaces. One of the earliest results in this area is Theorem 6 in [10], which claims that every compact continuous image of a Σ-product of doubletons is metrizable. In [20] this result is extended to all compact spaces which are continuous images of dense subsets of Σ-products of doubletons (of separable metrizable spaces).

The next step in the development of quasidyadic compact spaces theory was made by L. V. Shirokov [17]. In particular, Theorems 1 and 2 of [17] imply coincidence of the weight of a quasidyadic compact space X with the tightness of X, and of the character of X at any point $p \in X$ with the hereditary π-character of X at the point p.

For every cardinal τ the generalized Cantor cube D^τ has the intrinsic structure of a compact topological group. Taking this into account, A. V. Arhangel'skiĭ [3] suggested to extend the sphere of investigations by considering "weakly dyadic" compact spaces, that is, compact continuous images of dense subsets of σ-compact topological groups. For example, does the weight of a weakly dyadic compact space coincide with its tightness? This problem of Arhangel'skiĭ was partially solved by V. V. Uspenskiĭ. He proved in [22] that the equality $w = \chi$ of Esenin-Vol'pin holds for all weakly dyadic compact spaces. The complete solution of the above problem is given in [21], whose Corollary 2.23 states that $w(X) = t(X) = \operatorname{id}(X)$ for every weakly dyadic compact space X (here w, t, id denote the weight, tightness and dyadicity index resp., see [3]). However it remained open the question of

Uspenskiĭ if a "local" version of Corollary 2.23 in [21] is true, i.e., whether the character and the tightness coincide at every point of a weakly dyadic compact space.

Here we answer the question of Uspenskiĭ in the affirmative. Our main result, Theorem 1, claims that if S is dense in a product of σ-compact topological groups and a compact space X is a continuous image of S, then $\chi(p,X) = t(p,X)$ for every point $p \in X$. The same conclusion remains valid in a more general case when S is dense in a product of Lindelöf Σ-groups (the corresponding definition is given below).

Recall that a space X is said to be perfectly k-normal [19, 5] if the closure of each open subset of X is a G_δ-set in X. It is well-known that D^τ is perfectly k-normal for each cardinal τ. Nevertheless a dyadic compact space need not be perfectly k-normal: to get an example it suffices to identify two distinct points of D^τ with $\tau > \aleph_0$. Thus, weakly dyadic compact spaces need not be perfectly k-normal. However, if a continuous mapping $f\colon H \to X$ of a Lindelöf Σ-group H onto a compact space X is open, then X is perfectly k-normal; moreover, the closure of each $G_{\delta,\Sigma}$-subset of X is a G_δ-set in X (Theorem 2). Note that the above mapping f is not assumed to be closed.

Terminology and notations. All topological groups are assumed to be Hausdorff, and topological spaces to be Tychonoff. Following [16, 1] we call X a Lindelöf Σ-space if there exists a countable family $\mathcal{F}$ of closed subsets of the Čech–Stone compactification βX with the following property: for every $x \in X$ and $y \in \beta X \setminus X$ there exists $F \in \mathcal{F}$ such that $x \in F \subseteqq \beta X \setminus \{y\}$. If there exists a family $\mathcal{F}'$ of closed subsets of βX with the same property and $|\mathcal{F}'| \leqq \tau$, then we write $\mathrm{Nag}(X) \leqq \tau$. So X is a Lindelöf Σ-space iff $\mathrm{Nag}(X) \leqq \aleph_0$. One can easily verify that if $X = \cup\mathcal{P}$ and $\mathrm{Nag}(P) \leqq \tau$ for each $P \in \mathcal{P}$, then $\mathrm{Nag}(X) \leqq |\mathcal{P}|\cdot\tau$. This implies that if a topological group G is generated by its subspace X, then $\mathrm{Nag}(G) \leqq \mathrm{Nag}(X)\cdot\aleph_0$ [22]. Obviously, $\mathrm{Nag}(X) \leqq \aleph_0$ for every σ-compact space X. The same inequality holds for a space X with a countable network [16].

We call a union of arbitrary many G_δ-subsets of X a $G_{\delta,\Sigma}$-set in X. By $w(X)$, $\mathrm{nw}(X)$, $t(X)$, $\pi\chi(X)$ we denote the weight, the net weight, the tightness and the π-character of a space X, resp. If $p \in X$, the symbols $\chi(p,X)$, $\pi\chi(p,X)$ and $t(p,X)$ stand for the character, the π-character and the tightness of X at the point p, resp. (see [13]).

The main results. The aim of the article is the proof of the following theorem.

THEOREM 1. *Let S be a dense subset in a product of σ-compact topological groups, and let the compact space X be a continuous image of S. Then $\chi(p,X) = t(p,X)$ for every point $p \in X$.*

REMARK. In case S coincides with a product of spaces with countable networks, the conclusion of Theorem 1 follows from [10, Theorem 9]. Generalizations of Theorem 1 (see Corollaries 1 and 2 below) cover this case with a stock.

Now we turn to the other result.

THEOREM 2. *Let f be a continuous open mapping of a Lindelöf Σ-group H onto a space X of pointwise countable type. Then $\mathrm{cl}_X Q$ is a G_δ-set in X for every $G_{\delta,\Sigma}$-subset Q of X. In particular, X is perfectly k-normal.*

PROOF. Let $\mathcal{P}$ be a family of G_δ-sets in X, and let $Q = \cup\mathcal{P}$. Put $B = \mathrm{cl}_X Q$ and $T = \mathrm{cl}_H f^{-1}(Q)$. Then $T = f^{-1}(B)$ because f is open. Evidently $f^{-1}(Q)$ is a $G_{\delta,\Sigma}$-set in H, so Lemma 2 of [22] implies that T is of type G_δ in H. Hence there exists a countable family γ of open subsets of H such that $T = \cap\gamma$.

Extend f to a continuous mapping $g\colon \beta H \to \beta X$. Since H is a Lindelöf Σ-group, there exists a countable family $\mathcal{F}$ of closed subsets of βH which separates points of H from the points of $\beta H \setminus H$. One can assume that $\mathcal{F}$ is closed under finite intersections. For every $U \in \gamma$ put $U^* = \beta H \setminus \mathrm{cl}_{\beta H}(H \setminus U)$. Then the following assertion holds:

(*) for each $g \in H$ and $U \in \gamma$ there exists $F \in \mathcal{F}$ such that $g \in F$ and $g^{-1}(B) \cap (F \setminus U^*) = \emptyset$.

Assume that (*) fails. Then there exist $g \in H$ and $U \in \gamma$ which contradict (*). By virtue of compactness of βH we have $K = (g^{-1}(B) \cap \cap \bigcap \mathcal{F}(g)) \setminus U^* \neq \emptyset$, where $\mathcal{F}(g) = \{F \in \mathcal{F} : g \in F\}$. But the definition of $\mathcal{F}$ implies $\bigcap \mathcal{F}(g) \subseteqq H$, therefore $g^{-1}(B) \cap \bigcap \mathcal{F}(g) \subseteqq g^{-1}(B) \cap H = T$. This inclusion and the fact that $K \neq \emptyset$ imply that $T \setminus U^* \neq \emptyset$. This contradicts the obvious inclusions $T \subseteqq U \subseteqq U^*$.

Put $\mathcal{K} = \{g(F \setminus U^*) : F \in \mathcal{F}, U \in \gamma\}$. Clearly, $\mathcal{K}$ is a countable family of closed subsets of βX. By (*), for every $x \in X \setminus B$ one can find $K \in \mathcal{K}$ so that $x \in K \subseteqq \beta X \setminus B$, and hence B is of type G_δ in X.

Let us turn to Theorem 1. We shall prove a more general result (Theorem 3) that implies Theorem 1 as a special case. To start with, we formulate a notion due to L. V. Shirokov [18].

DEFINITION 1. Suppose $S \subseteqq \Pi$ and $f\colon S \to Y$ is a continuous mapping. Then f is said to be regular with respect to Π provided there exists an operator e_f assigning to each open set $0 \subseteqq Y$ an open subset $e_f(0)$ of Π such that

(R1) $e_f(0) \cap S = f^{-1}(0)$;

(R2) if U, V are open in Y and disjoint, then $e_f(U)$ and $e_f(V)$ are disjoint.

Clearly, if S is dense in Π, then every continuous mapping $f\colon S \to Y$ is regular with respect to Π. With the help of the notion of a regular mapping Shirokov [18] gave a characterization of k-metrizable compact spaces in terms of their embeddings into Tychonoff cubes. From technical point of view the notion of a regular mapping is convenient because of the following reason. Assume that S is dense in Π, f is a continuous mapping of S onto Y, T is dense in Y and g is a continuous mapping of T onto Z. Then the set

$f^{-1}(T)$ need not be dense in Π, i.e., the space Z need not be an image of a dense subset of Π under the mapping $g \circ f$. However the mapping $g \circ f \colon f^{-1}(T) \to Z$ is regular (with respect to Π) by the following lemma (see [21]):

LEMMA 1. *Let $S \subseteqq \Pi$ and suppose that f is a regular (with respect to Π) mapping of S onto Y, $T \subseteqq Y$, and g is a regular (with respect to Y) mapping of T onto Z. Then the mapping $h = g \circ f\big|_{f^{-1}(T)}$ of $f^{-1}(T)$ onto Z is regular with respect to Π.* □

To prove Theorem 1 (and its generalizations) we need several lemmas. All of them, except Lemma 5, are given in [21]. For the reader's convenience we formulate them here.

LEMMA 2 (see [21, Lemma 2.16]). *Let ξ be a closed countable cover of a space Z of point-countable type. Then, for every non-empty G_δ-set Φ in Z there exists a non-empty closed G_δ-set Φ^* in Z such that $\Phi^* \subseteqq \Phi$ and for each $B \in \xi$ either $\Phi^* \subseteqq B$ or $\Phi \cap B = \emptyset$.* □

In what follows we use some notations. If $\Pi = \prod_{\alpha \in A} X_\alpha$, then for each $B \subseteqq A$ put $\Pi_B = \prod_{\alpha \in B} X_\alpha$ and denote by π_B the projection of Π onto Π_B.

LEMMA 3 (see [21, Lemma 1.7 and Corollary 1.8]). *Suppose that $\Pi = \prod_{\alpha \in A} X_\alpha$, where for each $\alpha \in A$, X_α is a $G_{\delta,\Sigma}$-set in a Lindelöf Σ-group. If T is of type $G_{\delta,\Sigma}$ in Π, then $F = \mathrm{cl}_\Pi T$ is a G_δ-set in Π. Moreover, there exist a countable set $B \subseteqq A$ and a closed G_δ-set F_B of Π_B such that $F = \pi_B^{-1}(F_B)$.* □

The lemma above and Lemma 2.15, Theorem 1.5 in [21] together imply the following.

LEMMA 4. *Let Π be as in Lemma 3, and assume that λ is a countable family of open subsets of Π. Then for every filter $\mathcal{R}$ on the set λ the upper limit $\overline{\lim}\,\mathcal{R} = \bigcap_{R \in \mathcal{R}} \mathrm{cl}_\Pi(\cup\{U \in \lambda : U \in R\})$ is of type G_δ in Π.* □

The proof of the lemma is based on the fact that a Lindelöf Σ-group has a large stock of open homomorphisms onto groups with a countable network [22]. Lemma 4 does not infer that $\overline{\lim}\,\mathcal{R} \neq \emptyset$.

LEMMA 5. *Suppose that R is of type $G_{\delta,\Sigma}$ in a space Y of point-countable type, and $y \in \mathrm{cl}_Y R = B$. If $\pi\chi(y, B) \leqq \aleph_0$, then there exists a countable π-base μ of Y at the point y such that $V \cap B \neq \emptyset$ for each $V \in \mu$.*

PROOF. Since $\pi\chi(y, B) \leqq \aleph_0$, there exists a countable π-base λ of B at the point y. Clearly, $U \cap R \neq \emptyset$ for each $U \in \lambda$. Taking into account that Y is of point-countable type, for every $U \in \lambda$ one can find a non-empty compact set $K_U \subseteqq U \cap R$ so that $\chi(K_U, Y) \leqq \aleph_0$. Let γ_U be a countable base of K_U in Y. We put $\mu = \cup\{\gamma_U : U \in \lambda\}$. It is easy to see that the π-base μ of Y at y is as required. □

Now we formulate the main result of the paper in a sufficiently general form.

THEOREM 3. *For every $\alpha \in A$ let X_α be a $G_{\delta,\Sigma}$-set in a Lindelöf Σ-group H_α, and assume that $\Pi = \prod_{\alpha\in A} X_\alpha$, $S \subseteqq \Pi$ and f is a regular (with respect to Π) mapping of S onto a space Y of point-countable type. Then $\chi(p,Y) = t(p,Y)$ for each point $p \in Y$.*

PROOF. It is sufficient to show that $\chi(p,Y) \leqq t(p,Y)$. Assume that there exists a point $p \in Y$ with $t(p,Y) < \chi(p,Y)$. Then the point p is not isolated in Y. Let Φ be a closed subset of Y and $p \in \Phi$. One can find a compact set Φ_0 of a countable character in Φ so that $p \in \Phi_0 \subseteqq \Phi$. By a recent theorem of I. Juhász and S. Shelah [14] we have $\pi\chi(p, \Phi_0) \leqq$ $\leqq t(p, \Phi_0)$ (if the cardinal $\pi\chi(p, \Phi_0)$ is regular, this inequality follows from [13, Theorem 3.14.b]). Clearly, $t(p, \Phi_0) \leqq t(p,Y)$. Lemma 1 in [2] implies that $\pi\chi(p, \Phi) \leqq \pi\chi(p, \Phi_0) \cdot \chi(\Phi_0, \Phi)$, and hence $\pi\chi(p, \Phi) \leqq t(p,Y)$. Consequently a hereditary π-character of Y at the point p does not exceed $t(p,Y)$.

For the sake of simplicity we assume that $t(p,Y) = \aleph_0$; this special case is completely analogous to the general one. For every $\alpha \in A$ define $Y_\alpha = \mathrm{cl}_{H_\alpha} X_\alpha$. By Theorem 2 in [22] Y_α is a G_δ-set in the group H_α (this follows also from our Lemma 3), and since Y_α is closed in H_α, we have $\mathrm{Nag}(Y_\alpha) \leqq \mathrm{Nag}(H_\alpha) \leqq \aleph_0$. The set Π is dense in $\hat{\Pi} = \prod_{\alpha\in A} Y_\alpha$, so the mapping f is regular with respect to $\hat{\Pi}$. Let e_f be a lifting operator corresponding to f (see Definition 1), where $e_f(0)$ is open in $\hat{\Pi}$ for each open set $0 \subseteqq Y$. By $\mathcal{T}(p)$ denote the family of all open subsets of Y containing p, and put $F_p^* = \cap\{\mathrm{cl}_{\Pi^*} e_f(0) : 0 \in \mathcal{T}(p)\}$, $F_p = F_p^* \cap \hat{\Pi}$, where $\Pi^* = \prod_{\alpha\in A} \beta Y_\alpha$ is the product of Čech–Stone compactifications βY_α.

For a subset $B \subseteqq A$, denote π_B the projection of $\hat{\Pi}$ onto $\hat{\Pi}_B = \prod_{\alpha\in B} Y_\alpha$. By p_B denote the projection of Π^* onto $\Pi_B^* = \prod_{\alpha\in B} \beta Y_\alpha$. Let T be the union of all G_δ-sets of $\hat{\Pi}$ contained in F_p. Lemma 3 implies that the set $\mathrm{cl}_{\hat{\Pi}} T \subseteqq F_p$ is of type G_δ in $\hat{\Pi}$, and therefore $T = \mathrm{cl}_{\hat{\Pi}} T$ is the maximal G_δ-set of $\hat{\Pi}$ lying in F_p.

For a subset $\Phi \subseteqq Y$ let $\widetilde{\Phi}$ be the union of all G_δ-sets in Y, lying in Φ. If $\mathcal{F}$ is a family of subsets of Π^* and $z \in \Pi^*$, then put $\mathcal{F}(z) = \{F \in \mathcal{F} : z \in F\}$.

We claim the following.

ASSERTION 1. *For every countable family $\mathcal{F}$ of closed subsets of Π^* satisfying the condition $\hat{\Pi} \setminus F_p \subseteqq \cup\mathcal{F}$, there exists a point $z \in S \setminus F_p$ such that $p \in \mathrm{cl}_Y \widetilde{\Phi}_F$ for each $F \in \mathcal{F}(z)$, where $\Phi_F = \mathrm{cl}_Y f(F \cap S)$.*

Assume the contrary. Then there exists a family $\mathcal{F}$ which contradicts the assertion above. Therefore for every point $z \in S \setminus F_p$ one can find $F(z) \in \in \mathcal{F}(z)$ so that $p \notin \mathrm{cl}_Y \widetilde{\Phi}_{F(z)}$. For every $z \in S \setminus F_p$ put $P_z = \Phi_{F(z)}$ and consider the family $\xi = \{P_z : z \in S \setminus F_p\}$. Obviously $|\xi| \leqq |\mathcal{F}| \leqq \aleph_0$, and all elements of ξ are closed in Y. Let us verify the inclusion $Y \setminus \{p\} \subseteqq \cup\xi$. First, note that $F_p \cap S = f^{-1}(p)$. Indeed, suppose that $y \in Y \setminus \{p\}$. Choose disjoint open neighborhoods 0 and U in Y of the points p and y resp. Then $e_f(0) \cap \cap e_f(U) = \emptyset$ and $f^{-1}(y) \subseteqq f^{-1}(U) \subseteqq e_f(U) \cap S$, so $F_p \subseteqq \mathrm{cl}_{\hat{\Pi}} e_f(0) \subseteqq \hat{\Pi} \setminus f^{-1}(y)$. Consequently $F_p \cap S = f^{-1}(p)$. The inclusion $Y \setminus \{p\} \subseteqq \cup\xi$ follows now from the obvious facts that $Y \setminus \{p\} \subseteqq f(S \setminus F_p)$ and $f(z) \in P_z$ for each $z \in \in S \setminus F_p$. If $p \notin \cup\xi$, then, since elements of ξ are closed in Y, we have $\chi(p, Y) = \psi(p, Y) \leqq |\xi| \leqq \aleph_0$, a contradiction. Thus the equality $Y = \cup\xi$ is valid.

For every $P \in \xi$ choose an open subset V_P of Y so that $p \in V_P$ and $V_P \cap \tilde{P} = \emptyset$. Denote $\Phi = \cap\{V_P : P \in \xi\}$. Then Φ is a non-empty G_δ-set in Y and by Lemma 2 there exists a nonempty G_δ-set Φ' in Y such that $\Phi' \subseteqq \Phi$ and for each $P \in \xi$ either $\Phi' \cap P = \emptyset$ or $\Phi' \subseteqq P$. Since ξ is a cover of Y, some element $P^* \in \xi$ meets Φ'. Then the definition of Φ' implies that $\Phi' \subseteqq P^*$, and hence $\Phi' \subseteqq \widetilde{P^*}$. This contradicts the facts that $\Phi' \subseteqq \Phi \subseteqq V_{P^*}$ and $V_{P^*} \cap P^* = \emptyset$. Thus the assertion is proved.

Since T is a closed G_δ-set in $\hat{\Pi}$, one can find a countable subset $C \subseteqq A$ and a closed G_δ-set T_C in $\hat{\Pi}_C$ so that $T = \pi_C^{-1}(T_C)$ (Lemma 3). Fix a countable family θ_C of open subsets of Π_C^* so that $T_C = \hat{\Pi}_C \cap \bigcap \theta_C$, and put $\theta = p_C^{-1}(\theta_C) = \{p_C^{-1}(U) : U \in \theta_C\}$.

Now we will carry out an inductive construction. Put $\mathcal{F}_0 = \mu_0 = \emptyset$ and $B(1) = C$. Let $n \in \mathrm{N}^+$ and assume that a subset $B(n) \subseteqq A$ and families $\mathcal{F}_{n-1}$, μ_{n-1} have already been defined so that $|B(n)| \cdot |\mathcal{F}_{n-1}| \cdot |\mu_{n-1}| \leqq \aleph_0$. Since $\mathrm{Nag}(Y_\alpha) \leqq \aleph_0$ for each $\alpha \in B(n)$, we have $\mathrm{Nag}\left(\hat{\Pi}_{B(n)}\right) \leqq \aleph_0$ (see [1]). Hence there exists a countable family $\mathcal{F}_n^*$ of closed sets in $\Pi_{B(n)}^*$, which separates points of $\hat{\Pi}_{B(n)}$ from points of $\Pi_{B(n)}^* \setminus \hat{\Pi}_{B(n)}$ (Definition 1 of a Lindelöf Σ-space X does not depend on the choice of the Čech–Stone compactification βX of X; one can substitute βX by any other compactification [1]). Denote by $\mathcal{F}_n$ the smallest family of closed sets in Π^* which is closed under finite intersections and contains the families $\mathcal{F}_{n-1}$, $\{p_{B(n)}^{-1}(K) : K \in \mathcal{F}_n^*\}$ and $\{\Pi^* \setminus U : U \in \theta\}$. It is clear that $\hat{\Pi} \subseteqq \cup\mathcal{F}_n$. Put $\xi_n = \{\mathrm{cl}_Y f(F \cap S) : F \in \mathcal{F}_n\}$. Then $|\xi_n| \leqq |\mathcal{F}_n| \leqq \aleph_0$. Let $\widetilde{\xi_n}$ be the family of all $P \in \xi_n$ satisfying $p \in \mathrm{cl}\, \widetilde{P}$. By Assertion 1, the family $\widetilde{\xi_n}$ is not empty. With the help of Lemma 5, for every $P \in \widetilde{\xi_n}$ one can find a countable π-base μ_P of Y at the point p so that $V \cap P \neq \emptyset$ for each $V \in \mu_P$. Put $\mu_n = \mu_{n-1} \cup \bigcup\{\mu_P : P \in \widetilde{\xi_n}\}$. Clearly, $|\mu_n| \leqq \aleph_0$. For every $V \in \mu_n$ there exists a countable subset $B(V) \subseteqq A$ such that $\mathrm{cl}_{\hat{\Pi}} e_f(V) = \pi_{B(V)}^{-1} \pi_{B(V)} \mathrm{cl}_{\hat{\Pi}} e_f(V)$ (apply Lemma 3). Since the projection $p_{B(V)}$ is open, we have $K = p_{B(V)}^{-1} p_{B(V)}(K)$, where $K = \mathrm{cl}_{\Pi^*} e_f(V)$. Then define $B(n+1) = B(n) \cup \bigcup\{B(V) : V \in \mu_n\}$. Obviously $|B(n+1)| \leqq \aleph_0$.

To end our construction, put $B = \cup\{B(n) : n \in \mathbf{N}^+\}$, $\mu = \cup\{\mu_n : n \in \mathbf{N}\}$ and $\mathcal{F} = \cup\{\mathcal{F}_n : n \in \mathbf{N}\}$. It is clear that $|B| \cdot |\mu| \cdot |\mathcal{F}| \leqq \aleph_0$. It follows from the construction that $F = p_B^{-1} p_B(F)$ for each $F \in \mathcal{F}$, and the same equality holds if one substitutes F by $\mathrm{cl}_{\Pi^*} e_f(V)$, $V \in \mu$. Furthermore, the family $\mathcal{F}_B = \{p_B(F) : F \in \mathcal{F}\}$ separates points of $\hat{\Pi}_B$ from points of $\Pi_B^* \setminus \hat{\Pi}_B$, and the family $\mathcal{F}$ is closed under finite intersections. The inclusion $\hat{\Pi} \subseteqq \cup\mathcal{F}$ is obvious.

Applying Assertion 1 to the family $\mathcal{F}$ choose a point $z \in S \setminus F_p$. The above construction implies that μ is a π-base of Y at the point p, and for each $P \in \widetilde{\xi}$ the subfamily $\mu_P = \{V \in \mu : V \cap P \neq \emptyset\}$ of μ is a π-base of Y at p as well, where $\widetilde{\xi} = \{\mathrm{cl}_Y f(F \cap S) : z \in F \in \mathcal{F}\}$.

For every $0 \in \mathcal{T}(p)$ put $V_0 = \cup\{e_f(U) : U \in \mu,\ U \subseteqq 0\}$. Put also $R^* = \cap \cap\{\mathrm{cl}_{\Pi^*} V_0 : 0 \in \mathcal{T}(p)\}$ and $R = R^* \cap \hat{\Pi}$. An easy verification shows that $R^* = p_B^{-1} p_B(R^*)$ and $R = \pi_B^{-1} \pi_B(R)$.

ASSERTION 2. *The set R is of type G_δ in $\hat{\Pi}$, $R \subseteqq F_p$, and $R \setminus T \neq \emptyset$.*

Indeed, the inclusion $R \subseteqq F_p$ follows from the definitions of R and F_p. Lemma 4 implies that R is a G_δ-set in $\hat{\Pi}$. It remains to show that $R \setminus T \neq \emptyset$. Consider the set $K = \cap\mathcal{F}(z)$. Clearly $K = p_B^{-1} p_B(K)$. Since the family $\mathcal{F}_B$ separates points of $\hat{\Pi}_B$ from points of $\Pi_B^* \setminus \hat{\Pi}_B$, we have $p_B(K) \subseteqq \hat{\Pi}_B$. The fact that μ_P is a π-base of Y at the point p for each $P \in \widetilde{\xi}$ implies that $V_0 \cap F \neq \emptyset$ whenever $0 \in \mathcal{T}(p)$ and $F \in \mathcal{F}(z)$. The family $\{V_0 : 0 \in \mathcal{T}(p)\}$ is directed by inclusion and the family $\mathcal{F}(z)$ consists of compact sets, so we have $R^* \cap K \neq \emptyset$. Since $R^* = p_B^{-1} p_B(R^*)$, $K = p_B^{-1} p_B(K)$ and $p_B(K) \subseteqq \hat{\Pi}_B$, the set $R^* \cap \hat{\Pi} \cap K$ is not empty, i.e., $R \cap K \neq \emptyset$. However, $K \cap T = \emptyset$, because $z \notin T$ and $K = \cap\mathcal{F}(z) \subseteqq \Pi^* \setminus \cap\theta \subseteqq \Pi^* \setminus T$. Consequently $R \setminus T \neq \emptyset$, and Assertion 2 is proved.

It remains to note that the existence of the set R with the above properties contradicts the definition of the set T. This contradiction means that $\chi(p, X) \leqq \aleph_0$. □

The following corollary of Theorem 3 generalizes Theorem 1.

COROLLARY 1. *Let S be a dense subset of a product of Lindelöf Σ-groups and assume that a compact space Y is a continuous image of S. Then $\chi(p, Y) = t(p, Y)$ for each point $p \in Y$.* □

Another corollary of Theorem 3 has a purely topological character.

COROLLARY 2. *Let a compact space Y be a continuous image of a dense subspace of a product of cosmic* spaces. Then $\chi(p, Y) = t(p, Y)$ for each point $p \in Y$.*

PROOF. Assume that S is dense in the product $\Pi = \prod_{\alpha \in A} X_\alpha$ of cosmic spaces X_α and Y is a continuous image of S. Let $H_\alpha = F(X_\alpha)$ be the free topological group over X_α, $\alpha \in A$ (see [11, 15]). Since H_α is generated by its subspace X_α (in an algebraic sense), we have $\mathrm{nw}(H_\alpha) \leqq \mathrm{nw}(X_\alpha) \leqq \leqq \aleph_0$. In particular, H_α is a Lindelöf Σ-group of a countable pseudocharacter. Consequently X_α is a $G_{\delta,\Sigma}$-subset of H_α, $\alpha \in A$. An application of Theorem 3 completes the proof. □

Theorem 3 and Lemma 1 yield together the following.

COROLLARY 3. *Suppose that S_0 is dense in a product of Lindelöf Σ-groups, Y_0 is a continuous image of S_0, S_1 is dense in Y_0, Y_1 is a continuous image of S_1, and so on. If Y_n is a compact space for some $n \in \mathbf{N}$, then $\chi(p, Y_n) = t(p, Y_n)$ for each point $p \in Y_n$.* □

Finally, we formulate three questions connected with the above considerations. The first and the second questions provide a way of generalizing Theorem 2.

QUESTION 1. Suppose that a compact space Y is an image of a dense subspace of a Lindelöf Σ-group under an open continuous mapping. Must Y be perfectly k-normal?

QUESTION 2. Let a compact space Y be an image of a product of Lindelöf Σ-groups under a continuous open mapping. Is it true then that Y is perfectly k-normal?

QUESTION 3. Suppose there exists a continuous mapping of a dense supbspace of a σ-compact topological group H onto a Tychonoff cube I^τ, $\tau > \aleph_0$. Is there a continuous mapping of H onto I^τ?

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*A space X is said to be cosmic provided $\mathrm{nw}(X) \leqq \aleph_0$.

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CONTENTS

Prestin, J., Lagrange interpolation for functions of bounded variation ... 1
Varma, A. K., On Birkhoff quadrature formulas. II ... 15
Chandra, P., A note on the degree of approximation of continuous functions ... 21
Allen, G. D., *Andrews, K. T.* and *Ward, J. D.*, Similarity and isometric equivalence of L_p-nests ... 25
Mathur, K. K. and *Saxena, R. B.*, Weighted $(0,1,3)$ interpolation on the zeros of Hermite polynomials ... 31
Bokhari, M. A., A note on certain next-to-interpolatory rational functions 49
Goldberg, V. V., On $(n+1)$-subwebs of an $(n+1)$-web and local algebras associated with them ... 57
Phong, B. M., Multiplicative functions satisfying a congruence property 81
Powell, C. H. and *Wade, W. R.*, Paley sets and term by term dyadic differentiation of Walsh series ... 89
Kirchhoff, N. and *Nessel, R. J.*, Convolution processes of Fejér type and the divergence almost everywhere of a pointwise comparison ... 97
Blasco, J. L. and *Sanchis, M.*, On the product of two b_f-spaces ... 111
Kirchheim, B., Some further typical results on bounded Baire one functions ... 119
Grubb, D. J., Dirichlet sets in Vilenkin groups ... 131
Indlekofer, K.-H., The abstract prime number theorem for function fields 137
Tulyaganova, M. I. and *Fainleib, A. S.*, On the density of prime vectors in lattices ... 149
Zahedani, H. Z., A Gelfand–Neumark theorem for commutative semifinite rank ... 157
Joó, I., Note on multiplicative functions satisfying a congruence property. II ... 163
Herzog, J., Weak asymptotic formulas for partitions free of small summands. II ... 173
Tkačenko, M. G., On cardinal invariants of continuous images of topological groups ... 189

Index: 26.659

Acta Mathematica Hungarica

VOLUME 62, NUMBERS 3–4, 1993

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ON SOME GENERALIZATIONS OF THE KAKUTANI–STONE AND STONE–WEIERSTRASS THEOREMS

M. I. GARRIDO and F. MONTALVO (Badajoz)

1. Introduction

For a completely regular Hausdorff space $X, C^*(X)$ denotes the algebra of all bounded real-valued continuous functions over X. We consider the topology of uniform convergence over $C^*(X)$.

When K is a compact space, the Stone–Weierstrass and Kakutani–Stone theorems provide necessary and sufficient conditions under which a function $f \in C^*(K)$ can be uniformly approximated by members of an algebra, lattice or vector lattice of $C^*(K)$. In this way, the uniform closure and in particular the uniform density of algebras and lattices of $C^*(K)$, can be characterized. Other authors, like Császár, Nöbeling, Bauer and others have studied the uniform density of lattices of $C^*(K)$ with some additional properties (subtractive lattices, affine lattices, semi-affine lattices, etc.).

When X is a noncompact space, different versions and generalizations of these theorems have been given. Thus, Hewitt [6] in 1947 gives a uniform density theorem for algebras of $C^*(X)$ containing all the real constant functions. This is done through the identification between the rings $C^*(X)$ and $C(\beta X)$ (βX is the Stone–Čech compactification of X), and it is a generalization of the Stone–Weierstrass theorem. In a paper devoted to the study of the comparison of certain compactifications of X [1], Blasco establishes the relationship existing between an extension problem of bounded continuous functions and a problem of uniform approximation. In this context he characterizes the uniform closure for certain lattices of $C^*(X)$.

In this paper we make a systematic study about uniform approximation for algebras and lattices of $C^*(X)$. If $\mathfrak{F}$ is an algebra or lattice (vector lattice, affine lattice, etc.) we shall characterize its uniform closure and we shall give necessary and sufficient conditions for uniform density in $C^*(X)$. For our purposes we shall also identify the rings $C^*(X)$ and $C(\beta X)$. In this way, we generalize the classical results in the compact case. Likewise we also obtain, in particular, the mentioned generalizations by Hewitt and Blasco.

2. Notation

As we have already mentioned, for a completely regular space X, which has more than one point, $C^*(X)$ denotes the set of all bounded real-valued continuous functions endowed with the uniform convergence topology. $C^*(X)$ can also be provided with an algebraic structure and an order structure, defining pointwise the suitable operations. Thus, if $\mathfrak{F}$ is a subset of $C^*(X)$ we shall say that $\mathfrak{F}$ is:

1. *a ring*, if $f, g \in \mathfrak{F}$ then $f - g \in \mathfrak{F}$ and $fg \in \mathfrak{F}$.
2. *a linear space*, if $f, g \in \mathfrak{F}$ and $\lambda, \mu \in \mathbf{R}$ then $\lambda f + \mu g \in \mathfrak{F}$.
3. *an algebra*, if $\mathfrak{F}$ is a ring and it is a linear space.
4. *a lattice*, if $f, g \in \mathfrak{F}$ then $f \vee g \in \mathfrak{F}$ and $f \wedge g \in \mathfrak{F}$, where $f \vee g = \sup\{f, g\}$ and $(f \wedge g) = \inf\{f, g\}$.
5. *a vector lattice or linear lattice*, if $\mathfrak{F}$ is a lattice and it is a linear space.
6. *a subtractive lattice*, if $\mathfrak{F}$ is a lattice and if $f, g \in \mathfrak{F}$ then $f - g \in \mathfrak{F}$.
7. *an affine lattice*, if $\mathfrak{F}$ is a lattice and if $f \in \mathfrak{F}$ and $a \in \mathbf{R}$ then $f + a \in \mathfrak{F}$ and $af \in \mathfrak{F}$.
8. *a semi-affine lattice*, if $\mathfrak{F}$ is a lattice and if $f \in \mathfrak{F}$ then $f + a \in \mathfrak{F}$ and $\mu f \in \mathfrak{F}$ for every $a \in \mathbf{R}$ and $\mu \in \Gamma$ where Γ is a set of real numbers containing 0 and unbounded both from above and from below.

Given $f \in C^*(X)$ and a real number a, we let $L_a(f) = \{x \in X : f(x) \leqq a\}$, $L^a(f) = \{x \in X : f(x) \geqq a\}$ and we refer to $L_a(f)$ and to $L^a(f)$ as the Lebesgue sets of f. We let $Z(f)$ denote the set $\{x \in X : f(x) = 0\}$ and call it zero-set of f.

Suppose that $\mathfrak{F} \subset C^*(X)$ and that A and B are subsets of X. We say that $\mathfrak{F}$:

1. *S_1-separates or completely separates* A and B when there is $g \in \mathfrak{F}$ such that $0 \leqq g \leqq 1$, $g(x) = 0$ if $x \in A$ and $g(x) = 1$ if $x \in B$.
2. *S_2-separates or separates* A and B when there is $g \in \mathfrak{F}$ such that $\overline{g(A)} \cap \overline{g(B)} = \emptyset$.
3. *S_i-separates* $(i = 1, 2)$ *the Lebesgue sets of a function* f when for every $a < b$, $\mathfrak{F}$ S_i-separates $L_a(f)$ and $L^b(f)$.

We also say that $\mathfrak{F}$ *separates* (resp. *strongly separates*) points of X when for every $x_1 \neq x_2$ in X there is $g \in \mathfrak{F}$ such that $g(x_1) \neq g(x_2)$ (resp. when for every $x_1 \neq x_2$ in X and for every pair of real numbers a, b, there is $g \in \mathfrak{F}$ such that $g(x_1) = a$ and $g(x_2) = b$).

Finally, βX denotes the Stone–Čech compactification of X. This space can be characterized by many forms. For instance βX is the unique (up to homeomorphism that leaves X pointwise fixed) compactification which has the following property:

"Every function in $C^*(X)$ can be continuously extended to βX".

Thus if $f \in C^*(X)$ and f^β is its (unique) extension to βX, the map $f \longrightarrow f^\beta$ is an isomorphism of rings from $C^*(X)$ onto $C(\beta X)$. In this way we can identify the rings $C^*(X)$ and $C(\beta X)$. If $\mathfrak{F} \subset C^*(X)$ and $\mathfrak{F}^\beta$ is the set of all continuous extensions to βX of functions in $\mathfrak{F}$, then $\mathfrak{F}$ and $\mathfrak{F}^\beta$ have the same algebraic properties and it is easy to check that $\overline{\mathfrak{F}^\beta} = \overline{\mathfrak{F}}^\beta$ (being $^-$ the uniform closure).

The above exposition is contained essentially in Gillman–Jerison's book [5].

3. The classical results

In this section we state (without proof) the classical results for the compact case. Although they are well-known we prefer to recall them because they will be generalized one by one.

THEOREM 1 (Kakutani–Stone, [7], [11], [12]). *Let K be a compact space, $\mathfrak{F} \subset C^*(K)$ a lattice and $f \in C^*(K)$. Then $f \in \overline{\mathfrak{F}}$ if and only if for any $x_1 \neq x_2 \in K$ and $\varepsilon > 0$ there is $g \in \mathfrak{F}$ such that $|g(x_1) - f(x_1)| < \varepsilon$ and $|g(x_2) - f(x_2)| < \varepsilon$.*

Consequently, $\mathfrak{F}$ is uniformly dense in $C^*(K)$ if and only if for any $x_1 \neq x_2 \in K$, for any pair of real numbers $a, b \in \mathbf{R}$ and $\varepsilon > 0$ there is $g \in \mathfrak{F}$ such that $|g(x_1) - a| < \varepsilon$ and $|g(x_2) - b| < \varepsilon$.

THEOREM 2 (Kakutani–Stone, [7], [11], [12]). *Let K be a compact space, $\mathfrak{F} \subset C^*(K)$ a vector lattice and $f \in C^*(K)$. Then $f \in \overline{\mathfrak{F}}$ if and only if for any $x_1 \neq x_2$ in K there is $g \in \mathfrak{F}$ such that $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2)$.*

Consequently, $\mathfrak{F}$ is uniformly dense in $C^*(K)$ if and only if $\mathfrak{F}$ strongly separates points of K.

The following theorem gives some sufficient conditions under which certain lattices containing all real constant functions are uniformly dense in $C^*(K)$. It can be obtained as a consequence of Theorems 1 and 2, though they have been studied in a different framework, exactly, in the theory of uniform spaces.

THEOREM 3. *Let K be a compact space and $\mathfrak{F}$ a subset of $C^*(K)$ containing all the real constant functions and separating points of K. Then $\mathfrak{F}$ is uniformly dense in $C^*(K)$ if any of the following conditions is satisfied:*

(i) *$\mathfrak{F}$ is a vector lattice* (Nöbeling–Bauer [10]).

(ii) *$\mathfrak{F}$ is a subtractive lattice* (Császár–Czipszer [3]).

(iii) *$\mathfrak{F}$ is an affine lattice* (Császár–Czipszer [3]).

(iv) *$\mathfrak{F}$ is a semi-affine lattice* (Császár [2]).

THEOREM 4 (Stone–Weierstrass, [11], [12]). *Let K be a compact space, $\mathfrak{F} \subset C^*(K)$ a subalgebra and $f \in C^*(K)$. Then $f \in \overline{\mathfrak{F}}$ if and only if the following conditions are satisfied:*

(i) *$\mathfrak{F}$ separates values of f (i.e., if for every $x_1, x_2 \in K$ with $f(x_1) \neq f(x_2)$ there is $g \in \mathfrak{F}$ with $g(x_1) \neq g(x_2)$).*

(ii) $\mathfrak{F}$ *is non-vanishing on the points* $x \in K$ *such that* $f(x) \neq 0$ *(i.e., if* $f(x) \neq 0$ *there is* $g \in \mathfrak{F}$ *with* $g(x) \neq 0$*).*

Consequently, $\mathfrak{F}$ *is uniformly dense in* $C^*(K)$ *if and only if* $\mathfrak{F}$ *separates points of* K *and* $\mathfrak{F}$ *is non-vanishing on* K.

The proofs of Theorems 1, 2 and 4 are contained essentially in Nachbin's book [9].

4. Uniform approximation for sublattices of $C^*(X)$

In this section we generalize the theorems by Kakutani, Stone, Császár, and others (i.e., Theorems 1, 2 and 3) for sublattices of $C^*(X)$. For this, we identify $C^*(X)$ and $C(\beta X)$. This identification allows us to translate conditions related to points in βX into conditions related to Lebesgue sets or zero-sets in X.

THEOREM 5. *Let* $\mathfrak{F}$ *be a sublattice of* $C^*(X)$ *and let* $f \in C^*(X)$. *Suppose that for every* $a < b$ *and* $\varepsilon > 0$ *there exists* $g \in \mathfrak{F}$ *such that*

$$|g(x) - a| < \varepsilon \quad \text{if} \quad x \in L_a(f),$$
$$|g(x) - b| < \varepsilon \quad \text{if} \quad x \in L^b(f).$$

Then $f \in \overline{\mathfrak{F}}$.

PROOF. We are going to apply Theorem 1 to prove that $f^\beta \in \overline{\mathfrak{F}^\beta}$. Let $p \neq q$ be two points in βX and let $\varepsilon > 0$. Firstly, suppose that $f^\beta(p) \neq f^\beta(q)$ (for instance, $f^\beta(p) < f^\beta(q)$). Let $0 < \delta < \frac{1}{2}(f^\beta(q) - f^\beta(p))$, then

$$p \in \{y \in \beta X : f^\beta(y) < f^\beta(p) + \delta\} \subset \mathrm{Cl}_{\beta X}\{y \in \beta X : f^\beta(y) < f^\beta(p) + \delta\} =$$
$$= \mathrm{Cl}_{\beta X}\{y \in X : f(y) < f^\beta(p) + \delta\} \subset \mathrm{Cl}_{\beta X} L_{f^\beta(p)+\delta}(f).$$

Similarly $q \in \mathrm{Cl}_{\beta X} L^{f^\beta(q)-\delta}(f)$. By hypothesis there exists $g \in \mathfrak{F}$ with

$$|g(x) - (f^\beta(p) + \delta)| < \delta \quad \text{if} \quad x \in L_{f^\beta(p)+\delta}(f),$$
$$|g(x) - (f^\beta(q) - \delta)| < \delta \quad \text{if} \quad x \in L^{f^\beta(q)-\delta}(f).$$

Therefore $|g^\beta(p) - f^\beta(p)| \leqq 2\delta$ and $|g^\beta(q) - f^\beta(q)| \leqq 2\delta$.

When $f^\beta(p) = f^\beta(q)$, the points p and q belong to $\mathrm{Cl}_{\beta X} L_{f^\beta(p)+\delta}(f)$ for any $\delta > 0$. Taking $g \in \mathfrak{F}$ such that $|g(x) - (f^\beta(p) + \delta)| < \delta$ if $x \in L_{f^\beta(p)+\delta}(f)$ and $|g(x) - b| < \delta$ if $x \in L^b(f)$, being b any real number with $b > f^\beta(p) + \delta$, we obtain that $|g^\beta(p) - f^\beta(p)| \leqq 2\delta$ and $|g^\beta(q) - f^\beta(q)| \leqq 2\delta$.

The proof is concluded when we take $\delta < \varepsilon/2$.

THEOREM 6. *Let $\mathfrak{F}$ be a sublattice of $C^*(X)$. The following conditions are equivalent:*

(a) *$\mathfrak{F}$ is uniformly dense in $C^*(X)$.*

(b) *For every pair of disjoint zero-sets in X, Z_1 and Z_2, for every pair of real numbers a, b and $\varepsilon > 0$ there exists $g \in \mathfrak{F}$ such that $|g(x) - a| < \varepsilon$ if $x \in Z_1$ and $|g(x) - b| < \varepsilon$ if $x \in Z_2$.*

PROOF. From Theorem 5 it follows that (b) implies (a). To prove that (a) implies (b) it is enough to note that if $Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$ are disjoint zero sets in X, then the function $g = a + (b-a)f_1^2/(f_1^2 + f_2^2)$ belongs to $\overline{\mathfrak{F}}$ and it takes the values a and b on Z_1 and Z_2 respectively.

REMARK. The condition in Theorem 5 is not necessary even when $\mathfrak{F}$ is a vector sublattice of $C^*(X)$. For instance if $\mathfrak{F}$ is the set of all functions on $[0,1]$ of the form $f(x) = ax$ ($a \in \mathbf{R}$), then $\mathfrak{F}$ is a (vector) sublattice which does not satisfy that condition for its functions. Let us note that $\mathfrak{F}$ does not contain the constant functions.

For sublattices containing all the real constant functions the next theorem can be established.

THEOREM 7. *Let $\mathfrak{F}$ be a sublattice of $C^*(X)$ containing all the real constant functions and let $f \in C^*(X)$. Then $f \in \overline{\mathfrak{F}}$ if and only if for every $a < b$ and $\varepsilon > 0$ there exists $g \in \mathfrak{F}$ such that*

$$|g(x) - a| < \varepsilon \quad \textit{if} \quad x \in L_a(f),$$
$$|g(x) - b| < \varepsilon \quad \textit{if} \quad x \in L^b(f).$$

PROOF. It is enough to prove that this condition is necessary. Suppose $f \in \overline{\mathfrak{F}}$, $a < b$ and $\varepsilon > 0$. Take $0 < \delta < \frac{b-a}{3}$ and $h \in \mathfrak{F}$ with $|h - f| < \delta$. Thus, we have

$$a \leqq (h(x) \vee a) \wedge b \leqq a + \delta \quad \text{if} \quad x \in L_a(f),$$
$$b - \delta \leqq (h(x) \vee a) \wedge b \leqq b \quad \text{if} \quad x \in L^b(f).$$

Now, if $\delta < \varepsilon$ and $g = (h \vee a) \wedge b$, then g belongs to $\mathfrak{F}$ and hence g is the required function.

Now we shall see that if the lattice has some additional properties then the separation condition in the above theorems admits several equivalent formulations. To this end, we need the next result.

LEMMA 1. *Let $\mathfrak{F}$ be a semi-affine sublattice of $C^*(X)$. If $\mathfrak{F}$ separates points in X then $\mathfrak{F}$ strongly separates points in X.*

PROOF. Let $x \neq y$ be two points in X, let $a < b$ and $g \in \mathfrak{F}$ such that $g(x) \neq g(y)$. We can suppose that $g(x) < g(y)$, because in the other case taking $\mu < 0$ such that $\mu g \in \mathfrak{F}$, it is obvious that $\mu g(x) < \mu g(y)$. Let $\lambda >$ $> \frac{b-a}{g(y)-g(x)}$ with $\lambda g \in \mathfrak{F}$. Then the function $h = (a \vee (\lambda g + a - \lambda g(x))) \wedge b$ is in $\mathfrak{F}$ and it satisfies $h(x) = a$ and $h(y) = b$.

THEOREM 8. *Let $\mathfrak{F}$ be a semi-affine sublattice of $C^*(X)$ and let $f \in C^*(X)$. The following conditions are equivalent:*

(a) $f \in \overline{\mathfrak{F}}$.

(b) *For every $a < b$ and $\varepsilon > 0$ there exists $g \in \mathfrak{F}$ such that*

$$|g(x) - a| < \varepsilon \quad \text{if} \quad x \in L_a(f) \quad \text{and} \quad |g(x) - b| < \varepsilon \quad \text{if} \quad x \in L^b(f).$$

(c) *For every $a < b$ there exists $g \in \mathfrak{F}$ such that*

$$g(x) = a \quad \text{if} \quad x \in L_a(f) \quad \text{and} \quad g(x) = b \quad \text{if} \quad x \in L^b(f).$$

(d) *$\mathfrak{F}$ S_1-separates the Lebesgue sets of f.*

(e) *For every $a < b$ there exists $g \in \mathfrak{F}$ such that*

$$\sup\{g(x) : x \in L_a(f)\} < \inf\{g(x) : x \in L^b(f)\}$$

(whenever $L_a(f)$ and $L^b(f)$ are not empty).

(f) *$\mathfrak{F}$ separates the Lebesgue sets of f.*

PROOF. Since every semi-affine sublattice contains all the real constant functions, then (a) and (b) are equivalent (Theorem 7).

(b) *implies* (c). Let $a < b$, $0 < \varepsilon < \frac{b-a}{2}$ and $g \in \mathfrak{F}$ such that $|g(x) - a| < \varepsilon$ if $x \in L_a(f)$ and $|g(x) - b| < \varepsilon$ if $x \in L^b(f)$. The function $h = ((a+\varepsilon) \vee g) \wedge \wedge (b - \varepsilon)$ belongs to $\mathfrak{F}$ and satisfies $h(x) = a + \varepsilon$ if $x \in L_a(f)$ and $h(x) = b - \varepsilon$ if $x \in L^b(f)$.

If $\mathfrak{L}$ denotes the set of functions $\varphi \in C^*(\mathbf{R})$ such that $\varphi \circ h \in \mathfrak{F}$, then it is easy to check that $\mathfrak{L}$ is a semi-affine sublattice separating points of $\mathbf{R}$, and so $\mathfrak{L}$ strongly separates points of $\mathbf{R}$ (Lemma 1). Thus, there exists $\varphi \in \mathfrak{L}$ with $\varphi(a + \varepsilon) = a$ and $\varphi(b - \varepsilon) = b$ and hence the function $\varphi \circ h$ is in $\mathfrak{F}$ and it satisfies $(\varphi \circ h)(x) = a$ if $x \in L_a(f)$ and $(\varphi \circ h)(x) = b$ if $x \in L^b(f)$.

Trivially (c) implies (e) and (e) implies (f).

(f) *implies* (b). Let $a < b$ and $\varepsilon > 0$. By hypothesis there is $g \in \mathfrak{F}$ which separates $L_a(f)$ and $L^b(f)$. If $K = \overline{g(X)}$ and if $\mathfrak{L}$ again denotes the set of the functions $\varphi \in C^*(K)$ such that $\varphi \circ g \in \mathfrak{F}$, then $\mathfrak{L}$ is a semi-affine sublattice separating points of the compact space K, and so $\mathfrak{L}$ is uniformly dense in $C^*(K)$ (Theorem 3).

Since $\overline{g(L_a(f))}$ and $\overline{g(L^b(f))}$ are disjoint closed sets in K, there exists $\psi \in C^*(K)$ such that $\psi(x) = a$ if $x \in \overline{g(L_a(f))}$ and $\psi(x) = b$ if $x \in \overline{g(L^b(f))}$. If now we choose $\varphi \in \mathfrak{L}$ with $|\varphi - \psi| < \varepsilon$, then the function $\varphi \circ g \in \mathfrak{F}$ and hence the condition (b) holds.

We have proved the equivalence among the conditions (a), (b), (c), (e) and (f). Obviously (d) implies (f). If now we repeat the proof of "(b) implies (c)" but taking $\varphi \in \mathfrak{L}$ such that $\varphi(a + \varepsilon) = 0$ and $\varphi(b - \varepsilon) = 1$ then the function $((\varphi \circ h) \vee 0) \wedge 1$ is in $\mathfrak{F}$ and completely separates $L_a(f)$ and $L^b(f)$. Hence (b) implies (d).

REMARK. In [1], Blasco pointed out the equivalence between the conditions (a) and (e).

Next, we show by means of some examples that in the above theorem we cannot allow $\mathfrak{F}$ to be an arbitrary sublattice containing all the constant functions. Namely, in order that a function is in the uniform closure of a sublattice containing all the real constant functions, conditions (e) and (f) are only necessary (Example 1), condition (c) is only sufficient (Example 2) and condition (d) is neither necessary nor sufficient (Examples 1 and 2).

EXAMPLES. (1) *Conditions* (d), (e) *and* (f) *are not sufficient.* Let $\mathfrak{F}$ be the set of the functions in $C^*(\mathbf{R})$ which are constant or they take values in $[0,1]$. $\mathfrak{F}$ is a uniformly closed sublattice containing all the real constant functions which separates the Lebesgue sets of every $f \in C^*(\mathbf{R})$ as in (d), (e) and (f), but it is obvious that not every function in $C^*(\mathbf{R})$ is in $\overline{\mathfrak{F}}$.

(2) *Conditions* (c) *and* (d) *are not necessary.* Let $\mathfrak{F}$ be the set of polygonals on $[0,1]$ whose straight lines have slope strictly less than 1. Then $\mathfrak{F}$ is a sublattice containing all the real constant functions and it is easy to see that the identity function f belongs to $\overline{\mathfrak{F}}$ (the straight line joining $(0,\varepsilon)$ with $(1,1-\varepsilon)$ is in $\mathfrak{F}$ and at a distance ε from the identity function). Nevertheless any function $g \in \mathfrak{F}$ does not satisfy that for $0 \leqq a < b \leqq 1$, $g([0,a]) = \{a\}$ and $g([b,1]) = \{b\}$. In particular, this implies that $\mathfrak{F}$ cannot completely separate $L_0(f)$ and $L^1(f)$.

THEOREM 9. *Let $\mathfrak{F}$ be a semi-affine sublattice of $C^*(X)$. The following conditions are equivalent:*

(a) *$\mathfrak{F}$ is uniformly dense in $C^*(X)$.*

(b) *For every pair of disjoint zero-sets in X, Z_1 and Z_2, for every pair of real numbers a, b and $\varepsilon > 0$ there exists $g \in \mathfrak{F}$ such that*

$$|g(x) - a| < \varepsilon \quad \textit{if} \quad x \in Z_1 \quad \textit{and} \quad |g(x) - b| < \varepsilon \quad \textit{if} \quad x \in Z_2.$$

(c) *For every pair of disjoint zero-sets Z_1 and Z_2 of X and for every pair of real numbers a, b there exists $g \in \mathfrak{F}$ such that*

$$g(x) = a \quad \textit{if} \quad x \in Z_1 \quad \textit{and} \quad g(x) = b \quad \textit{if} \quad x \in Z_2.$$

(d) *$\mathfrak{F}$ S_1-separates every pair of disjoint zero-sets in X.*

(e) *For every pair of non-empty disjoint zero-sets in X, Z_1 and Z_2 there exists $g \in \mathfrak{F}$ such that*

$$\sup\{g(x) \colon x \in Z_i\} < \inf\{g(x) \colon x \in Z_j\}$$

where $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$.

(f) *$\mathfrak{F}$ separates every pair of disjoint zero-sets in X.*

PROOF. It is straightforward after Theorem 8.

REMARK. Let us note that since every affine, subtractive or vector sublattice which contains all the real constant functions is a semi-affine lattice, the above two theorems remain true for them.

5. Uniform approximation for subalgebras of $C^*(X)$

As before, we shall now make a similar study for subalgebras of $C^*(X)$. In this way we shall generalize the Stone–Weierstrass theorem and we shall obtain, as a consequence, the well-known result by Hewitt.

THEOREM 10. *Let $\mathfrak{F}$ be a subalgebra of $C^*(X)$ and let $f \in C^*(X)$. Then $f \in \overline{\mathfrak{F}}$ if and only if the following conditions are satisfied:*

(i) *$\mathfrak{F}$ separates the Lebesgue sets of f.*

(ii) *For every $\varepsilon > 0$ there are $\delta > 0$ and $g \in \mathfrak{F}$ such that $L^\varepsilon(|f|) \subset L^\delta(g)$.*

PROOF. *Necessity.* Suppose $f \in \overline{\mathfrak{F}}$. Let $a < b$ and take $0 < \varepsilon < \frac{b-a}{3}$ and $g \in \mathfrak{F}$ with $|g - f| < \varepsilon$. Then $g(L_a(f)) \subset (-\infty, a + \varepsilon]$ and $g(L^b(f)) \subset [b - \varepsilon, \infty)$. Hence $\overline{g(L_a(f))} \cap \overline{g(L^b(f))} = \emptyset$ and so (i) holds.

To prove (ii), we take $\varepsilon > 0$, $\delta = \varepsilon/2$ and $g \in \mathfrak{F}$ with $|g - |f|| < \delta$. (It is well-known that the uniform closure of a subalgebra of bounded functions is also a lattice, and so $|f| \in \overline{\mathfrak{F}}$.) Obviously, $g(x) \geqq \delta$ if $|f(x)| \geqq \varepsilon$, hence $L^\varepsilon(|f|) \subset L^\delta(g)$.

Sufficiency. Since the set $\mathfrak{F}^\beta$ of all continuous extensions to βX of functions in $\mathfrak{F}$, is also an algebra and $\overline{\mathfrak{F}^\beta} = \overline{\mathfrak{F}}^\beta$, it is sufficient to prove that $f^\beta \in \overline{\mathfrak{F}^\beta}$. For that we shall apply Theorem 4.

Let p, q be two points in βX such that $f^\beta(p) \neq f^\beta(q)$, for instance $f^\beta(p) < f^\beta(q)$, and let $0 < \varepsilon < \frac{1}{2}(f^\beta(q) - f^\beta(p))$. Then, from an argument already used in Theorem 5 it follows that $p \in \mathrm{Cl}_{\beta X} L_{f^\beta(p)+\varepsilon}(f)$ and $q \in \mathrm{Cl}_{\beta X} L^{f^\beta(q)-\varepsilon}(f)$. Now, let $g \in \mathfrak{F}$ be separating $L_{f^\beta(p)+\varepsilon}(f)$ and $L^{f^\beta(q)-\varepsilon}(f)$. Clearly $g^\beta(p) \neq g^\beta(q)$, hence $\mathfrak{F}^\beta$ separates values of f^β.

Moreover $\mathfrak{F}^\beta$ is non-vanishing on the points where f^β is non-vanishing. Indeed, let $p \in \beta X$ be with $f^\beta(p) \neq 0$, and let $\varepsilon = |f^\beta(p)|$. By (ii) there exist $\delta > 0$ and $g \in \mathfrak{F}$ such that $L^{\varepsilon/2}(|f|) \subset L^\delta(g)$. As before, we obtain that $p \in \mathrm{Cl}_{\beta X} L^{\varepsilon/2}(|f|)$ because $p \in \{y \in \beta X : |f^\beta(y)| > \varepsilon/2\}$. It follows that $g^\beta(p) \geqq \delta > 0$ because $\mathrm{Cl}_{\beta X} L^{\varepsilon/2}(|f|) \subset \mathrm{Cl}_{\beta X} L^\delta(g)$.

We can conclude that $f^\beta \in \overline{\mathfrak{F}^\beta}$ and so $f \in \overline{\mathfrak{F}}$.

THEOREM 11. *Let $\mathfrak{F}$ be a subalgebra of $C^*(X)$. Then $\mathfrak{F}$ is uniformly dense in $C^*(X)$ if and only if the following conditions are satisfied:*

(i) *$\mathfrak{F}$ separates every pair of disjoint zero-sets in X.*

(ii) *$\mathfrak{F}$ contains a unity of $C^*(X)$ (i.e., there is $f \in \mathfrak{F}$ with $f \geqq \varepsilon > 0$).*

PROOF. The sufficiency follows from Theorem 10. Conversely, if $\mathfrak{F}$ is uniformly dense in $C^*(X)$ then, trivially (ii) is true. To prove (i) it is enough to note that if $Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$ are disjoint zero-sets, then $g = f_1^2/(f_1^2 + f_2^2)$ is in $\overline{\mathfrak{F}}$ and it is equal to 0 on Z_1 and to 1 on Z_2.

REMARK. In [6], Hewitt shows that any subalgebra of $C^*(X)$ which separates every pair of disjoints zero-sets in X and contains all the real constant functions is uniformly dense in $C^*(X)$. But condition (ii) in the above theorem is not equivalent to the condition on $\mathfrak{F}$ to contain the real constant functions. In fact, (ii) is equivalent for these functions to belong to $\overline{\mathfrak{F}}$.

COROLLARY (Hewitt [6]). *Let $\mathcal{H}$ be a subset of $C^*(X)$ such that for every pair of non-empty disjoint zero-sets Z_1 and Z_2 of X, there is $g \in \mathcal{H}$ with $\sup\{|g(x)|: x \in Z_j\} < \inf\{g(x): x \in Z_j\}$ where $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$. Under these conditions the subring $\mathfrak{F}$ generated by $\mathcal{H}$ and all the real constant functions is uniformly dense in $C^*(X)$.*

PROOF. It follows at once from Theorem 11.

In [6] Hewitt also shows that his results and, of course, Theorems 10 and 11, are true generalizations to the non-compact case of the Stone–Weierstrass theorem. Namely, the condition over $\mathfrak{F}$ of separation of zero-sets cannot be weakened to the requirement that $\mathfrak{F}$ separates points in X.

All the preceding results lie in the framework of approximation theory, although they can also be used in other topics, for instance, in the study of compactifications of a completely regular space, algebraic-topological characterizations of $C^*(X)$, extension problems of continuous functions, etc. (See, for instance, Mrówka [8], Blasco [1], Garrido [4] and others.)

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A NOTE ON SUMS OF POWERS OF COMPLEX NUMBERS

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Let n be a positive integer, z_j complex numbers such that $1 \geqq |z_1| \geqq |z_2| \geqq \dots \geqq |z_n|$ and $g_0(v) = \sum_{j=1}^{n} z_j^v$, where v is an integer. Using probabilistic methods Erdős and Rényi [1] have shown that one can find complex numbers z_j, $|z_j| = 1$ for $j = 1, 2, \dots, n$ such that for $B > 0$

$$\max_{v=1,2,\dots,n^B} |g_0(v)| \leqq C(B)(n \log n)^{1/2}. \tag{1}$$

Under the same assumptions Leenman and Tijdeman [2] proved that

$$\max_{v=1,2,\dots,2_n} |g_0(v)| \geqq \frac{1}{2} n^{1/2}$$

and gave an explicit construction of unimodular complex numbers z_j, $j = 1, \dots, n$ satisfying (1) with $C(B) = 9B$. On the other hand H. Montgomery (see [4], p. 83) found unimodular complex numbers z_j such that for every $\varepsilon > 0$ there exist infinitely many integers n for which

$$\max_{1 \leqq v \leqq n^{2-\varepsilon}} |g_0(v)| < 2n^{1/2}. \tag{2}$$

Related to (2) P. Turán raised the problem: what is $\inf_{z_j} \max_{1 \leqq v \leqq n^2} |g_0(v)|$ if $\min_j |z_j| = 1$.

The purpose of this note is to throw some light on Turán's question and simultaneously improve (2). In fact we shall show that there exist infinitely many integers n and unimodular complex numbers z_j, $j = 1, \dots, n$ such that

$$\max_{1 \leqq v \leqq n^2+n} |g_0(v)| = O(n^{1/2}). \tag{3}$$

Let p be a prime number and $q = p^2 + p + 1$. By Singer Theorem [3] there exist integers $n_0 = 0$, $n_1 = 1$, $n_2, \dots, n_p$ between 0 and q such that all $(p+1)p$ differences $n_j - n_k$ ($j \neq k$, $j, k = 0, 1, \dots, p$) are distinct modulo q. Putting

$z_j = \exp\left(\frac{2\pi i}{q} n_j\right)$, $j = 0, 1, \dots, p$ we have:

$$|g_0(v)|^2 = \left|\sum_{j=1}^{p} \exp\Big(\frac{2\pi i}{q} v n_j\Big)\right|^2 = \left|\sum_{j=0}^{p} \exp\Big(\frac{2\pi i}{q} v n_j\Big) - 1\right|^2 \leqq$$

$$\leqq \sum_{j,k=0,1,\dots,p} \exp\Big(\frac{2\pi i}{q} v(n_j - n_k)\Big) + 2\left|\sum_{j=0}^{p} \exp\Big(\frac{2\pi i}{q} v n_j\Big)\right| + 1 \leqq$$

$$\leqq p+1+ \sum_{\substack{j\neq k\\ j,k=0,1,\dots,p}} \exp \frac{2\pi i}{2} v(n_j - n_k) + 2(p+1)+1 = p+2(p+1)+1 = 3(p+1)$$

for all $v = 1, 2, \dots, q-1$ so (3) follows.

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DIRECT SUM DECOMPOSITIONS OF CONFORMAL RINGS

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1. Introduction. In [4] Chatters and Jordan defined a ring R to be a *noetherian unique factorisation ring (noetherian UFR)* if it satisfies the following conditions:

(a) R is prime and

(b) R is (left-right) noetherian;

(c) every non-minimal prime $P \in \operatorname{spec}(R)$ contains a non-minimal prime of form $Rp = pR$ for some $p \in R$.

More recently [2] Chatters, Gilchrist and Wilson have extended many of the results of [4] to the case of any ring satisfying only conditions (a) and (c), which they call *unique factorisation rings (UFRs)* (and which we here call prime UFRs).

There are, however, two main results of [4] which have so far not been extended to the most general context discussed in [2]. The first is Theorem 2.4, which states that a prime noetherian UFR R is a maximal order in its (left and right) artinian ring of quotients Q; we have extended this result (in [19], [20]) in the following form: a prime UFR R has a *unique* minimal simple ring of left and right quotients, say $S(R)$, which coincides with its symmetric ring of Martindale quotients and which embeds (over R) in any simple ring of left or right (classical) quotients of R, and R is a maximal order in $S(R)$.

The other ungeneralised result from [4] (Theorem 2.5) asserts that, if we drop the requirement (a) that R be prime, then R is a finite direct sum of prime (noetherian) UFRs and of noetherian rings in which every prime is maximal. This implies that, if R is semiprime and satisfies conditions (b) and (c), then R is a finite direct sum of prime noetherian UFRs and of noetherian simple rings (which are themselves 'trivial' prime UFRs).

The aim of this note is to explore generalisations (and the limits to generalisations) of this direct sum decomposition. There are two potential lines of attack on the problem: first, simply to consider arbitrary rings satisfying the condition (c) on prime ideals; secondly, to exploit the fact that in a prime UFR R every non-zero ideal I contains a non-zero, hence regular, normal element (i.e., an element a such that $Ra = aR$: see Remark 3.1, or Proposition 3.1 of [2]).

Prime (noetherian) rings with this latter property were termed *conformal* in [7], where the fact that such a ring has a simple ring of fractions with respect to the left-right Ore set of regular normal elements was used in an analysis of *noetherian UFN rings*, that is noetherian prime rings in which there is a well-defined notion ([7], [8]) of unique factorisation of normal elements. We used the concept of a prime conformal ring (under a less satisfactory name) in [15], employing the simple rings of fractions of certain (not-necessarily noetherian) prime rings in analysing selected bimodules over *bi-noetherian polynormal (BPN) rings* (see Section 2 for definition).

It turns out that the second line of attack is more fruitful, both because it yields non-trivial results about a broader class of rings than the UFRs and because there is a severe technical difficulty (see Remark 5.2), which we have failed to overcome, in deciding whether a UFR must necessarily have finitely many minimal primes; this problem does not arise if we follow Jordan's approach of requiring the presence of 'nice' normal elements in sufficient supply.

We begin, therefore, by extending the concept of a conformal ring from the prime rings to arbitrary rings in two different but natural ways. For convenience, we say that a ring with finitely many minimal primes is *Goldie finite*. With this terminology, one definition — *T-conformal* rings — turns out to be very restrictive, and we show that a T-conformal ring must necessarily be semiprime Goldie finite, and conformal in the other definition. Using that more general definition of an (arbitrary) *conformal* ring, such a ring must be Goldie finite, but need not be semiprime. (Section 7 discusses the relative merits of the two alternative generalisations in the light of preceding sections.)

In general, a conformal ring in our definition need not have a good direct sum decomposition (even if it is one-sided noetherian *and* a UFR — see Example 5.7). But our two main results show that mild *two-sided* restrictions do force conformal rings to have 'good' direct sum decompositions. It is known [16] that BPN rings have properties at least as pleasing as those of left-right noetherian rings, and our first main result (Theorem 4.4) shows that a left-right noetherian or BPN ring is conformal if and only if it is a direct sum of a *semiprime* conformal ring with no maximal minimal primes and a certain type of ring with d.c.c. on two-sided ideals; this extends the Chatters–Jordan direct sum decomposition of UFRs in less specific form.

Our second main result (Theorem 5.4) shows that, provided a UFR is semiprime Goldie finite — a very different, but also a two-sided, condition — then the full Chatters–Jordan decomposition can still be obtained: such a ring is a finite direct sum of prime UFRs. Combining these two results at Theorem 5.5 yields an extension (to BPN rings) — and offers a new proof for left-right noetherian rings — of the original Chatters–Jordan decomposition.

As noted, these results highlight yet another marked contrast between rings in which two-sided ideals are both left and right finitely generated and rings which are noetherian only on one side. It is also worth mentioning that, in the uniform proofs which we present for left-right noetherian and for BPN rings, crucial roles are played by those prime ideals Q of a ring R such that

R/Q is subdirectly irreducible (i.e., has non-zero, hence simple idempotent, heart) and by partial quotient rings which have bimodule composition series, but need not be right artinian.

Section 2 of the paper covers terminology and notation, Section 3 discusses preliminaries, and Section 6 discusses examples. Most of the work presented here was inspired by [4] and [7], and grateful thanks are due to Drs Chatters and Jordan for their comments and encouragement.

2. Terminology and notation. Terminology and notation are standard or self-explanatory with a handful of exceptions. The term *ring* always means an associative ring with unity, and all *(bi)modules* are unital. The term *ideal* (unqualified by 'left' or 'right') always means two-sided ideal, and $\mathbf{I}(R)$ denotes the ideal lattice of a ring R. As in [12], we say that a bimodule or ring *is bi-noetherian (bi-artinian)* or *satisfies the bi-a.c.c. (bi-d.c.c.)* (or some similar expression) if it satisfies the ascending (descending) chain condition on subbimodules (ideals, respectively).

If S, R are rings then an element u of an $S-R$ bimodule B is *normalising* if $Su = uR$. In an S–R bimodule B over rings S, R, a sequence $u_1, u_2, \ldots$ is *subnormalising* if u_1 is normalising in M, $u_2 + M/(u_1R)$ is normalising in $M/(u_1R)$, and so on. (The reasons for this terminology are explained in [13], though other names are widely used.)

We then call a ring or bimodule *polynormal (PN)* if each *principal* (two-sided) ideal or subbimodule is generated by a *finite* subnormalising sequence. Thus a *bi-noetherian polynormal (BPN)* ring or bimodule is one in which every ideal or subbimodule is generated by a finite subnormalising sequence. It is known that, if ${}_SB_R$ is a left (right) faithful BPN bimodule then $S(R)$ is a BPN ring ([16], Theorem 4.8). Extensive classes of non-noetherian BPN rings are identified in [16] and [14]; note that the terminology used here follows that of [16], not [15] or [14].

If $B = {}_SB_R$ is a bimodule, we say a subbimodule M of B is *N-cyclic* if $M = Sv = vR$ for some $v \in B$. In this situation, every subbimodule $L \subseteqq M$ is of form $L = Yv = vX$ for some $Y \in \mathbf{I}(S)$, $X \in \mathbf{I}(R)$, and when using the equation *we always assume* that Y, X are the unique ideals maximal with respect to satisfying it, e.g. that $X = \{x \in R : vx \in L\}$. A ring is *noetherian (artinian)* if it is left and right noetherian (artinian), and (see [5], [6]) a bimodule ${}_SB_R$ is *noetherian* if ${}_SB$, B_R are left, right noetherian respectively. It is standard that this implies $S/L\,\mathrm{Ann}(B)$, $R/R\,\mathrm{Ann}(B)$ are left, right noetherian rings respectively. If ${}_SB_R$ is a bimodule, we say it is *fully noetherian (fully BPN)* if each of S, R, B is noetherian (respectively, BPN).

A *prime* ideal Q of a ring R satisfies $Q \neq R$. The symbols spec(R), max(R), min(R) denote the sets of prime, maximal and minimal prime ideals of a ring R, and $P = P(R)$ denotes its prime radical. A ring is *polysimple* if it is a finite direct sum of simple rings. If $Q \in \mathbf{I}(R)$ then Q is *quasi-primitive** if R/Q is prime with simple (idempotent) heart, e.g. if $Q \in \max(R)$; the set of such ideals is denoted by $\text{quas}(R)$; R itself is *quasi-primitive* if $0 \in \text{quas}(R)$. A ring R is *Goldie finite* if $\min(R)$ is finite: it is well known that this is equivalent to each of: (i) $R/P(R)$ satisfies the a.c.c. on annihilator ideals; (ii) $R/P(R)$ satisfies the d.c.c. on annihilator ideals; (iii) $R/P(R)$ has finite bimodule uniform dimension; it is also equivalent to: (iv) $P(R)$ contains a product of prime ideals. Every bi-noetherian ring is Goldie finite.

If $I \in \mathbf{I}(R)$, then $C(I)$ (or $C_R(I)$) denotes the set of elements of R which are regular mod I, and I is *essential* (or occasionally, for emphasis, *bi-essential*) if it is an essential subbimodule of ${}_RR_R$. The symbol $\subset$ denotes strict inclusion.

3. Preliminaries. For any ring Γ, let $N(\Gamma)$ denote the set of non-zero normal elements, and $U(\Gamma)$ the group of units. Then $U(\Gamma) \subseteqq N(\Gamma)$, and when Γ is prime we have (i) $N(\Gamma) \subseteqq C_\Gamma(0) = C_\Gamma(P(\Gamma))$; (ii) $0 = P(\Gamma) \subseteqq \Gamma u = u\Gamma$ for all $u \in N(\Gamma)$. By definition [7], a prime (noetherian) ring Λ is *conformal* if $N(\Lambda) \cap I \neq \emptyset$ for all $0 \neq I \in \mathbf{I}(\Lambda)$. In any ring R, let $N_T(R)$, $N_P(R)$ denote respectively $N(R) \cap C_R(0)$ and $\{n \in N(R) \cap C_R(P(R)) \colon P(R) \subseteqq Rn = nR\}$. It is easy to check that $n \in N_P(R)$ implies $P(R) = nP(R) = P(R)n$, and hence that $N_P(R)$: (a) is multiplicatively closed, and (b) does not contain 0. (Clearly $N_T(R)$ has the same two properties.) If Q is an ideal of a ring R maximal with respect to $Q \cap N_P(R) = \emptyset$ then $Q \in \text{spec}(R)$ and R/Q is prime conformal.

Let R be a ring. Then $J \in \mathbf{I}(R)$ is *P-essential* if $P(R) \subseteqq J$ and $J/P(R)$ is essential in $R/P(R)$. A subset M of R is said to be *ubiquitous* if $M \cap I \neq \emptyset$ for each (bi-)essential $I \in \mathbf{I}(R)$, and *P-ubiquitous* if $M \cap J \neq \emptyset$ for each P-essential $J \in I(R)$. We shall say that an arbitrary ring R is *T-conformal* if $N_T(R)$ is ubiquitous, and that it is *conformal* if $N_P(R)$ is P-ubiquitous. Clearly a prime ring R is conformal as defined (for noetherian rings) in [7] if and only if it is conformal as defined here if and only if it is T-conformal, since for R prime $N(R) = N_T(R) = N_P(R)$. A semiprime ring R is conformal if and only if it is T-conformal, since $P(R) = 0$.

REMARK 3.1. If R is a prime ring then R is conformal if and only if $Q \cap N(R) \neq \emptyset$ for each $0 \neq Q \in \text{spec}(R)$ (by Zorn's Lemma).

The next result is probably well-known (and is true for 'associative rings not necessarily with unity'), but we give it for completeness.

PROPOSITION 3.1. *Let R be a ring such that the set of right regular elements is ubiquitous. Then R is semiprime Goldie finite.*

PROOF. Let $A \in \mathbf{I}(R)$ and suppose $A^2 = 0$. If B is a complement ideal for A, then $A \oplus B$ is essential, so contains a right regular element $t = a + b$,

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*The term *quasi-primitive* was used in a different sense by Szász, but it turned out that every quasi-primitive ideal was primitive (see [10], p. 120).

where $a \in A$, $b \in B$. But then $tA = (a+b)A = bA \subseteq BA = 0$, so $A = 0$ since t is right regular. Hence R is semiprime. Now a semiprime ring which is not Goldie finite contains an infinite direct sum of non-zero ideals, which by adding a single complement if necessary may be assumed to be essential. No such infinite direct sum can contain a right regular element, so R must be Goldie finite. □

COROLLARY 3.2. *A conformal ring is Goldie finite and a T-conformal ring is semiprime conformal. Suppose R is a semiprime Goldie-finite ring, and $u \in R$ is normal. Then u is regular if and only if $u \notin Q$ for all $Q \in \min(R)$ if and only if Ru is an essential ideal.*

PROOF. Straightforward and/or well-known. □

COROLLARY 3.3. *Suppose R is a conformal ring and $Q \in \min(R)$. Then R/Q is a conformal ring.*

PROOF. If $Q \in \max(R)$ this is trivial. So suppose $Q \notin \max(R)$. If $Q \subset X \in \operatorname{spec}(R)$ then $X \cap N_P(R) \neq \emptyset$, so $(X/Q) \cap N(R/Q) \neq \emptyset$. Since R/Q is prime the result follows from Remark 3.1. □

REMARKS. 3.2. In an arbitrary ring, the elements of $N_P(R)$ need not be regular. For example, let p be a rational prime, $C = C_p\infty$, and R be the ring $R = \mathbf{Z} \oplus C$ with multiplication defined (necessarily) by $C^2 = 0$. Then $N_P(R) = R \backslash C$, $P(R) = C$, and of course $nP(R) = P(R)n = P(R)$ for every $n \in N_P(R)$. But $n \in N_P(R)$ is regular if and only if $n \notin pR$.

3.3. In any ring R, $N_P(R) \cap N_T(R)$ is multiplicatively closed: Proposition 3.6 shows that this observation and Remark 3.2 can both be improved on for binoetherian rings.

EXAMPLE 3.4. Let $R = k[x,y]$ be the commutative noetherian ring generated over the field k by x, y subject to the relation $x^2y^2 = 0$. Then $x + y \in N_T(R) \backslash N_P(R)$. In fact $N_P(R) = U(R)$, so R is not conformal. The element $z = x + y$ is regular, but $P(R) \nsubseteq zR$.

In an arbitrary ring it is not possible to 'lift' normal elements from $R/P(R)$ to R. For example, if $R = UT_2(\mathbf{Z})$ is the ring of 2×2 upper triangular matrices over $\mathbf{Z}$, let x be the matrix $\operatorname{diag}(n, 1)$ for some $1 < n \in \mathbf{N}$. Then $x + P(R)$ is regular and normal in $R/P(R)$, but x is not normal in R. In a conformal ring, however, things behave better. We denote the subgroup $\{1 + x : x \in P(R)\}$ of $U(R)$ by $1 + P(R)$.

PROPOSITION 3.5. *Let R be a ring. Then:*

(i) *$v \in N_P(R)$ if and only if $v + \xi \in N_P(R)$ for all $\xi \in P(R)$.*

Suppose further that R is conformal. Then:

(ii) *$u \in C(P(R))$ is normal modulo $P(R)$ if and only if $u \in N_P(R)$;*

(iii) *if $u, v, w \in N_P(R)$, $x, y \in R$ and $v = xu$ (or $v = wy$), then $x(y) \in N_P(R)$;*

(iv) *if $A \in \mathbf{I}(R)$, $A \subseteq P(R)$, then $N_P(R/A) = \{x + A : x \in N_P(R)\}$, and R/A is a conformal ring;*

(v) *if* $p, x, y \in N_P(R)$ *and* $px = py$ *then* $x = \lambda y$ *for some* $\lambda \in 1 + P(R)$.

PROOF. Throughout, let $P = P(R)$.

(i) One implication is trivial, so suppose $v \in N_P(R)$, $\xi \in P$. Then $P = Pv$, so $\xi = \mu v$ for some $\mu \in P$. Hence $v + \xi = v + \mu v = (1+\mu)v$. Since $1 + \mu \in U(R)$, it follows that $(1+\mu)v \in N_P(R)$ as required.

(ii) Again, one way round is trivial, so suppose $u \in C(P)$ and $Ru + P = uR + P$. Then $Ru + P$ is P-essential, so $v = xu + \xi = uy + \mu$ for some $v \in N_P(R)$, $x, y \in R$ and $\xi, \mu \in P$. From (i), $xu, uy \in N_P(R)$. Let $r \in R$. Then $ru = us + \tau$ for some $s \in R$, $\tau \in P$. But $P = uyP$ since $uy \in N_P(R)$. Hence $\tau = uy\lambda$ for some $\lambda \in P$. It follows that $ru = u(s + y\lambda)$, so $Ru \subseteqq uR$, and by symmetry $Ru = uR$. Finally, $P = Pv = vP = Pxu = uyP$, so $P = Pu = uP$.

(iii) It is easy to check that in the situations described $x + P(R)$, $y + P(R) \in N(R/P(R)) \cap C_{R/P(R)}(0)$, and the result then follows from (ii).

(iv) Suppose $w \in R$ and $w + A \in N_P(R/A)$. Then $w \in C(P)$, and $P + A \subseteqq Rw + A = wR + A$. Since $A \subseteqq P$, $Rw + P = wR + P$, so $w \in N_P(R)$ by (ii). It follows that $N_P(R/A) = \{x + A : x \in N_P(R)\}$ and that R/A is conformal.

(v) It is easy to 'lift' this result from the trivial semiprime case. □

PROPOSITION 3.6. *Let* R *be a ring.*

(i) *If* R *satisfies the a.c.c. on left and on right annihilator ideals then* $N_P(R) \subseteqq N_T(R)$;

(ii) *if* R *is Goldie finite and* $P(R)$ *is nilpotent then* $N_T(R) \cap Q = \emptyset$ *for all* $Q \in \min(R)$;

(iii) *if* R *is conformal and* $P(R)$ *is nilpotent then* $N_T(R) \subseteqq N_P(R)$;

(iv) *if* R *is conformal and bi-noetherian then* $N_P(R) = N_T(R)$.

PROOF. (i) This follows by an easy adaptation of the proof of [12], Proposition 3.6.

(ii) Suppose that $Q_1, \dots, Q_r$ are the distinct elements of $\min(R)$ and (without loss of generality) that $t \in N_T(R) \cap Q_1$. Since $P(R)$ is nilpotent 0 is a product $0 = X_1 \dots X_s$ of minimal primes, with every Q_j cropping up among the X_i (so that $r \leqq s$). We may suppose that s is minimal, and if $s = 1$ the result is trivial (since R is then prime). So suppose $1 < s$; the minimality of s and the regularity of t imply that $t \notin X_1, X_s$. But $t \in Q_1 = X_v$ for some $1 < v < s$, so we have an equation $X_1 \dots X_{v-1} t X_{v+1} \dots X_s = 0$. Since $t \in N_T(R)$ the map $\varrho \to \varrho'$ defined by $\varrho t = t\varrho'$ for all $\varrho \in R$ is an automorphism; hence there exist $W_w \in \min(R)$, $1 \leqq w < v$, such that $X_w t = tW_w$ for each $1 \leqq w < v$. Hence $tW_1 \dots W_{v-1} X_{v+1} \dots X_s = 0$. But, since $t \in C_R(0)$, this contradicts the minimality of s.

(iii), (iv) Since R is conformal it is Goldie finite, and it follows from (ii) and Corollary 3.2 that $N_T(R) \subseteqq C_R(P(R))$. Now use Proposition 3.5 (ii) to obtain (iii); (iv) then follows using (i). □

The next result is needed to obtain a suitable direct sum decomposition for noetherian conformal rings. In its statement, the symbol ϱ stands for the reduced rank (see [3], Chapter 2) of a one-sided noetherian module over a similarly noetherian ring.

PROPOSITION 3.7. *Let $B = {}_SB_R$ be a noetherian bimodule, and suppose that S, R are simple rings. Then B has a bimodule composition series of length $r \leqq \min(\varrho({}_SB), \varrho(B_R))$.*

PROOF. By [5], Lemma 1.1 B is left-right torsion-free, as is any subbimodule or subfactor bimodule of B. Since the functions ϱ are additive (on each side), any strictly descending chain of subbimodules must terminate. □

COROLLARY 3.8. *Suppose R is a noetherian ring and that* $\operatorname{spec}(R) = \max(R)$*; then R is bi-artinian.* □

REMARKS. 3.4. If e, f are coprime integers then there exist finite fields K, P, F, where $K, P \subseteqq F$, such that $\varrho({}_KF) = e$, $\varrho(F_P) = f$ and ${}_KF_P$, ${}_KF_F$ are simple bimodules.

3.5. The converse of Corollary 3.8 is well known to be false: in [10] Robson exhibits a noetherian domain D with exactly three ideals, whose (idempotent) heart H is a height 1 prime of D.

3.6. *Neither the previous proposition nor its corollary is true for right noetherian objects.* For example, let S be a simple, right noetherian, non-artinian ring, and F be its centre. Then the matrix ring $R = \begin{pmatrix} F & S \\ 0 & S \end{pmatrix}$ is right noetherian and every prime ideal of R is maximal, but R is not bi-artinian.

3.7. We shall say that a ring is *G-bi-artinian* if it is bi-artinian and every prime ideal is maximal. It is known ([16], Section 6) that — like a commutative ring — a BPN ring R is bi-artinian if and only if $\operatorname{spec}(R) = \max(R)$ (and hence if and only if it is G-bi-artinian). There is an analogue of Proposition 3.7 for BPN rings and bimodules, which shows that a BPN bimodule B over simple rings has a bimodule composition series of length the least number of terms in a finite subnormalising sequence generating B.

3.8. Let ${}_SB_R$ be fully noetherian or fully BPN, and suppose S, R are G-bi-artinian. Then it is easy to use Proposition 3.7 and its corollary, or ([16], Proposition 4.1), to show that B has a bimodule composition series.

PROPOSITION 3.9. *Let R be a bi-noetherian conformal ring, and $x \in N_P(R)$. If $A \in \mathbf{I}(R)$, $A \subsetneqq P(R)$ then $A = Ax = xA$.*

PROOF. Since $A \subseteqq P(R) \subsetneqq xR$, $A = xA'$ for some $A' \in \mathbf{I}(R)$. Since $x \in C(P(R))$, $A' \subsetneqq P(R)$, and evidently $A \subseteqq A'$. But by Proposition 3.6, $x + A$ is regular in R/A, so $A' = A$, and the final statement follows by symmetry. □

PROPOSITION 3.10. (1) *Suppose R is a ring and $\emptyset \neq D \subseteqq N_T(R)$ is multiplicatively closed. Then D is left-right Ore and R has a ring W of left and right fractions with respect to D. Moreover:*

(i) *If $J \in \mathbf{I}(W)$ then $J = WJ' = J'W$ for $J' = R \cap W \in \mathbf{I}(R)$;*

(ii) *If $Q \in \operatorname{spec}(R)$ then $WQ = QW = WQW$ and exactly one of the following occurs:* (a) *$D \cap Q \neq \emptyset$ and $WQW = W$;* (b) *$D \cap Q = \emptyset$, $WQW \in \in \operatorname{spec}(W)$ and $Q = R \cap WQW$;*

(iii) *If $M \in \max(W)$ then $M' = M \cap R \in \operatorname{spec}(R)$, $M = WM' = M'W$, and M' is maximal with respect to $I \in \mathbf{I}(R)$ and $I \cap D = \emptyset$;*

(iv) *If $K \in \mathbf{I}(R)$ is maximal with respect to $K \cap D = \emptyset$ then $K \in \operatorname{spec}(R)$ and $WK = KW \in \max(W)$;*

(2) *Suppose further that R is conformal, and noetherian or BPN. Then R has a bi-artinian quotient ring W with respect to $N_P(R)$, and W is also noetherian or BPN respectively.*

PROOF. Straightforward using standard facts about left-right Ore sets of regular elements plus the observation that $u \in N(R)$, $u \notin Q \in \operatorname{spec}(R)$ implies $u \in C_R(Q)$. □

Finally, we need a technical result about quasi-primitive rings. A more complicated proof of effectively the same result is to be found at [16], Proposition 7.9.

PROPOSITION 3.11. *A quasi-primitive conformal ring R is simple.*

PROOF. Let H be the heart of R, and $u \in H \cap N(R)$. Then $H = Ru = = Ru^2$, so $u = xu^2$ for some $x \in R$. Since u is regular, $Ru = R$, so $u \in U(R)$. □

4. Noetherian and BPN conformal rings. If R is a ring, let $m-\min(R) = = \min(R) \cap \max(R)$ and $g\text{-}\min(R) = \min(R) \backslash m\text{-}\min(R)$. Throughout this section we confine attention to Goldie finite rings, and fix the following notation: $K = \bigcap_{Q \in g-\min(R)} Q$, $L = \bigcap_{M \in m-\min(R)} M$, and $P = K \cap L$. Note that by elementary arguments $K + L = R$.

LEMMA 4.1. *Let R be a conformal ring. Then* $\min(R) \cap \operatorname{quas}(R) = = m\text{-}\min(R)$.

PROOF. If $Q \in \min(R)$ then R/Q is conformal (Corollary 3.3), so if also $Q \in \operatorname{quas}(R)$ then $Q \in \max(R)$ by Proposition 3.11. □

LEMMA 4.2. *Let $B = {}_SB_R$ be a simple bimodule, and suppose B is either fully noetherian or fully BPN, and is left (right) faithful. Then S (respectively, R) is quasi-primitive.*

PROOF. This is an easy application of [5], Lemma 5.2 in the noetherian case and of [16], Proposition 4.1 in the BPN case. □

The following is the main technical result of this section.

PROPOSITION 4.3. *Let R be a noetherian or BPN conformal ring, and let $n \in \mathbf{N}$ be the least integer such that $P^n = 0$. Then $K \cap L^n = 0$, and for each $r \in \mathbf{N}$, $P^r = K \cap L^r = KL^r = L^rK$. Moreover, $R \simeq (R/K) \oplus (R/L^n)$, so that K, L^n are generated by central idempotents.*

PROOF. From Propositions 3.6 (iii) and 3.10 the quotient ring W of R with respect to $N_P(R)$ exists and is bi-artinian and noetherian or BPN as appropriate, so has a bimodule composition series. Suppose $A \in \mathbf{I}(R)$, and define $\delta(A)$ to be the bimodule composition length of WAW.

The case $n = 0$ is trivial, so suppose that $n > 0$. We can find $X \in \in \mathbf{I}(R)$, $X \subseteqq P$, such that $X = WXW \cap R$ and $\delta(X) = 1$ (Proposition 3.10 (1) (i)), and then $WXW = WX = XW$. Now $F = L\,\mathrm{Ann}_R(XW) = = L\,\mathrm{Ann}_W(XW) \cap R$, and $L\,\mathrm{Ann}_W(XW) \in \mathrm{spec}(W) = \max(W)$. It follows using Proposition 3.10 and the definition of a conformal ring that $F \in \in \min(R)$. There exists $Y \in \mathbf{I}(R)$ such that $Y \subseteqq X$ and X/Y is a simple $R-R$ bimodule; let $H = L\,\mathrm{Ann}_R(X/Y)$. Then $H \in \min(R)$ by considering the factor ring R/Y. However $F \subseteqq H$, so $F = H$. But by Lemma 4.2 $H \in \mathrm{quas}(R)$, so by Lemma 4.1 $F \in m\text{–}\min(R)$. Using symmetry, it follows easily that $LX = XL = 0$.

Now let $T \in \mathbf{I}(R)$, $T \subseteqq P$, and suppose $KT \subset T$. Pick $Z \in \mathbf{I}(R)$ such that $KT \subseteqq Z \subset T$ and T/Z is a simple R–R bimodule. Let $S = R/Z$, $V = T/Z$. Then S is conformal (Proposition 3.5 (iv)), and is noetherian or BPN according as R is, so has a bi-artinian quotient ring with respect to $N_P(S)$. Moreover, the map $\mathrm{spec}(R) \to \mathrm{spec}(S)$ is a poset isomorphism and the map $N_P(R) \to N_P(S)$ is surjective (Proposition 3.5 (iv)). Using Propositions 3.5 and 3.9 it follows successively that $uV = Vu = V$ for all $u \in N_P(S)$, that $\delta(V) = 1$, that $L\,\mathrm{Ann}_S(V) \in \min(S)$, and finally that $E = L\,\mathrm{Ann}_R(T/Z) \in m\text{–}\min(R)$ (as in the previous paragraph). But then $(K + E)T \subseteqq Z$, contradicting the evident fact that $K + E = R$. It follows, using symmetry, that $KT = TK$ for all $T \in \mathbf{I}(R)$, $T \subseteqq P$.

Now $P = K \cap L = KP \subseteqq KL \subseteqq K \cap L$, so (by symmetry) $K \cap L = = KL = LK$; let $r \in \mathbf{N}$. It follows that $(KL)^r = K^rL^r = KL^r$. But then $KL^r = K^rL^r = (KL)^r \subseteqq (K \cap L)^r \subseteqq K \cap L^r = K(K \cap L^r) \subseteqq KL^r$. Hence (by symmetry) $K \cap L^r = KL^r = L^rK = (K \cap L)^r$, as required.

The remaining assertions then follow at once. □

We say that a conformal ring R is a *proper* conformal ring if $\min(R) = = g - \min(R)$, and a *trivial* conformal ring if $\min(R) = m - \min(R)$ (see Remark 4.1 below). Let R be a trivial conformal ring: then R is itself its only P-essential ideal, so $N_P(R) = U(R) = N(R) \cap C(P(R))$. Conversely, the same argument shows that any Goldie finite ring Γ in which $\min(\Gamma) = = \max(\Gamma)$ is trivial conformal. In particular, a noetherian or a BPN ring is trivial conformal if and only if it is G-bi-artinian (Corollary 3.8 and Remark 3.7).

THEOREM 4.4. *Let R be a noetherian or BPN ring. Then R is conformal if and only if it is the direct sum of a proper semiprime conformal ring and* *a trivial conformal ring.*

PROOF. The direct sum of two conformal rings is clearly conformal. Conversely, in the decomposition of Proposition 4.3, R/K is a proper semiprime conformal ring (since if Q is a non-minimal prime of R/K then $Q\oplus \oplus(R/L^n)$ is a non-minimal prime of $(R/K)\oplus(R/L^n)$), and R/L^n is a trivial conformal ring. □

REMARKS. 4.1. At Example 5.7 we exhibit an idecomposable right noetherian conformal ring which is neither trivial nor proper.

4.2. We have already noted that, in a conformal ring R, if $A \in \mathbf{I}(R)$, $A \subsetneqq P(R)$ or if $Q \in \min(R)$ then R/A, R/Q are conformal. Let R be the commutative algebra $k[x, y]$ defined over a field k by the relation $xy = =0$. Then R is a proper, noetherian, BPN conformal ring, as is R/xR, but R/x^2R is indecomposable while neither semiprime nor G-bi-artinian, hence not conformal. Thus $x^2R = A \in \mathbf{I}(R)$, $A \subsetneqq Q = xR \in \min(R)$, but R/A is not conformal.

4.3. In Theorem 4.4, the ring R/L^n embeds in R as the unique largest bi-artinian ideal, so the isomorphism classes of R/K, R/L^n as rings are unique.

4.4. According to Theorem 4.1.9 of [9], a noetherian ring R is the direct sum of a semiprime ring and an artinian ring if and only if $P(R) = c(P(R)) = = P(R)c$ for all $c \in C(P(R))$. There is a clear parallel between this result and Theorem 4.4 above.

4.5. As noted in Section 6 and [14], BPN rings are 'almost always' not noetherian on either side, and of course there is a plentiful supply of noetherian rings which are not BPN. But both classes of rings have a common feature which they do not share with right noetherian rings: ideals are both left and right finitely generated. It is tempting to speculate that Theorem 4.4 can be extended to rings in which it is only assumed that ideals are finitely generated on both sides. We therefore formulate:

CONJECTURE 4.5. *Let R be a ring in which ideals are left and right finitely generated. Then R is conformal if and only if it is the direct sum of a proper semiprime conformal ring and a trivial conformal ring.*

Using the methods of [14] this hypothesis can be reduced to the case in which ideals are cyclic on both sides, but we have been unable to make any further progress in deciding if it is true or false.

5. Unique factorisation rings. Recall that an arbitrary ring is a *unique factorisation ring (UFR)* if every non-minimal prime contains an N-cyclic non-minimal prime. *Prime* UFRs are discussed in [2], but here we seek to deal with more general rings. Although we do not restrict the definition to the Goldie finite case, in practice almost all our results deal with Goldie finite UFRs: see the Introduction and Remark 5.2.

Examples of prime, non-noetherian UFRs are given in Section 6. The ring $R = \mathbf{Z} \oplus C$ mentioned in Remark 3.2 is a Goldie finite UFR in which the generator p of the height 1 maximal ideal $Rp = pR$ is not regular, but if R is bi-noetherian then *every* normal generator of a non-minimal prime is regular ([12], Proposition 3.6).

We say that an element p of a ring R is *prime* if $Rp = pR \in \operatorname{spec}(R)\setminus \min(R)$, and then $\Delta(R)$, $D(R)$ denote respectively the sets of prime elements and of finite products of prime elements and/or units of R. Hence (see Proposition 5.3 (i)) $U(R) \subseteqq D(R) \subseteqq C_R(P(R))$, $D(R)$ is left-right Ore, and $D(R) \subseteqq C_R(Q)$ for each $Q \in \min(R)$; thus $0 \notin D(R)$. *This notation is fixed for the rest of Sections* 5 *and* 6.

PROPOSITION 5.1. *Let R be a Goldie finite UFR; then R is conformal.*

PROOF. Suppose $D(R) \cap J = \emptyset$ for some P-essential $J \in \mathbf{I}(R)$; by Zorn's Lemma there is a P-essential ideal K maximal with respect to $K \cap D(R) = \emptyset$. Evidently K is prime, and since R is Goldie finite it follows that $K \notin \min(R)$; contradiction. □

PROPOSITION 5.2. *Let R be a conformal ring with prime radical $P = P(R)$, and suppose that $A \in \mathbf{I}(R)$, $A \subseteqq P$. Then R is a UFR if and only if R/A is a UFR.*

PROOF. Suppose R is a UFR. Then every $Q' \in \operatorname{spec}(R/A)\setminus \min(R/A)$ is of form Q/A for some $Q \in \operatorname{spec}(R)\setminus \min(R)$; if $p \in \Delta(R) \cap Q$ then since $A \subseteqq P \subseteqq Rp$, Q' contains the non-minimal N-cyclic prime Rp/A.

Conversely, suppose R/A is a UFR, and let $X \in \operatorname{spec}(R)\setminus \min(R)$. Then X/A contains a non-minimal N-cyclic prime; we can suppose $x \in R$ and $\overline{x} = x + A \in X/A$ is a prime element of R/A, so that $(R/A)\overline{x} = \overline{x}(R/A) \subseteqq X/A$. Then $\overline{x} \in N_P(R/A)$, so by Proposition 3.5 (iv) there exists $\hat{x} \in N_P(R)$ such that $\hat{x} + A = \overline{x}$. Since $A \subseteqq P \subseteqq R\hat{x} = \hat{x}R$, it follows that $R\hat{x} = \hat{x}R \in \operatorname{spec}(R)\setminus \min(R)$, and $R\hat{x} \subseteqq X$. □

The next result shows that, if a ring R is a Goldie finite UFR, that fact decisively affects the structure of $\operatorname{spec}(R)$.

PROPOSITION 5.3. *Let R be a UFR, $Q \in \min(R)$, and $p \in \Delta(R)$. Then:*

(i) *$p \in C(P(R))$ and $p \in C(Q)$;*

(ii) *$P(R) = pP(R) = P(R)p$ and $L\operatorname{Ann}(p^i) + R\operatorname{Ann}(p^j) \subseteqq P(R)$ for all $i, j \in \mathbf{N}$;*

(iii) *if $Q(p) = \bigcap_{n=1}^{\infty} Rp^n$ then $Q(p) \in \operatorname{spec}(R)$ and $Q(p) \subset Rp$;*

(iv) *Rp is a height* 1 *prime of R;*

(v) *$Q(p) \in \min(R)$ and $Q(p) = pQ(p) = Q(p)p$;*

(vi) *if $X \in \operatorname{spec}(R)$ and $X \subset Rp$ then $X = Q(p)$;*

(vii) *if $X \in \min(R)$ then $Xp = pX$, and $Xp = X$ if and only if $X = Q(p)$;*

Suppose further that R is Goldie finite. Then:

(viii) *$Q \in g - \min(R)$ if and only if $Q = Q(q)$ for some $Q \in \Delta(R)$;*

(ix) *if $X \in \operatorname{spec}(R)$ then X contains a unique $K \in \min(R)$, so distinct minimal primes of R are comaximal, and* $\operatorname{spec}(R)$ *is the disjoint union of* *the sets* m–min(R) *and*

$$\{X \in \operatorname{spec}(R) \colon X \supseteqq K \in g\text{-}\min(R)\}$$

as K *ranges over* g–min(R).

(x) *if* $X \in \operatorname{spec}(R)$, $q \in \Delta(R)$ *then* $q \in X$ *implies* $Q(q) \subseteqq X$, *and if* $Q(q) \nsubseteqq X$ *then* $X + Q(q) = X + Rq = R$.

PROOF. (i) We may suppose that R is semiprime, and then the non-minimal prime ideal $Rp = pR$ is (bi)-essential, so $L\operatorname{Ann}(pR) = L\operatorname{Ann}(p) = R\operatorname{Ann}(Rp) = R\operatorname{Ann}(p) = 0$. The other assertion is obvious.

(ii) Let $L \in \min(R)$, so that $p \in C(L)$. Now $P(R) \subset Rp$, so $P(R) = Ap = pB$ for some $A, B \in \mathbf{I}(R)$. Hence $Ap = A(pR) = P(R) \subseteqq L$, and $p \in C(L)$, so $A \subseteqq L$. It follows that $A \subseteqq P(R) \subseteqq A$, as required, and then $B = P(R)$ by symmetry. If $i, j \in \mathbf{N}$ then $L\operatorname{Ann}(p^i) + R\operatorname{Ann}(p^j) \subseteqq P(R)$ since $p^n \in C(P(R))$ for all $n \in \mathbf{N}$.

(iii) By [12], Lemma 3.4, Rp has no idempotent power, so $Q(p) \subset Rp$. Suppose $x, y \in R \backslash Q(p)$. Then there exist $X, Y \in R \backslash Rp$, $i, j \in \mathbf{Z}$, $i, j \geqq 0$ such that $x = Xp^i$, $y = Yp^j$. Let $Y' \in R$, $Y'p^i = p^iY$. If $Y' \in Rp$ then $Y' = \lambda p$ for some $\lambda \in R$, so $Y'p^i = \lambda p^{i+1} = p^{i+1}\mu = p^iY$ for some $\mu \in R$. Hence $p\mu - Y = \xi \in R\operatorname{Ann}(p^i) \subseteqq P(R)$. But $P(R) = pP(R)$ so $\xi = p\chi$ for some $\chi \in P(R)$. Hence $p(\mu - \chi) = Y$, i.e. $Y \in Rp = pR$; contradiction.

Hence $X, Y' \in R \backslash Rp$, so since Rp is prime there exists $z \in R$ such that $XzY' \in R \backslash Rp$. Let $z' \in R$, $zp^i = p^iz'$, and suppose $xz'y \in Q$. Then $xz'y = Xp^iz'Yp^j = XzY'p^{i+j} \in Q(p) \subseteqq Rp^{i+j+1}$, so $XzY'p^{i+j} = \tau p^{i+j+1}$ for some $\tau \in R$. But then $ZxY' - \tau p \in L\operatorname{Ann}(p^{i+j})$, and it follows as above that $XzY' \in Rp$; contradiction. Hence $Q(p)$ is prime.

(iv) and (v) From (iii), height $(Rp) \geqq 1$. Suppose height $(Rp) > 1$. Then there exists $q \in \Delta(R)$ such that $Q(q) \subset Rq \subset Rp$. Passing to $R/Q(q)$, we may assume q, p are regular. But such a situation is impossible by [12], Lemma 3.5. Hence height $(Rp) = 1$ and $Q(p) \in \min(R)$. That $Q(p) = pQ(p) = Q(p)p$ follows by adapting the proof of (ii).

(vi) and (vii) Straightforward.

Suppose now that R is Goldie finite. Then it is easy to use Proposition 5.2 to reduce (viii), (ix) and (x) to the case in which R is semiprime. But in that case each element of m–min(R) gives rise to a simple direct summand of R, which can easily be eliminated from the discussion. We can therefore assume that R is semiprime and that $\min(R) = g$–min(R).

(viii) Certainly, there exists $q \in \Delta(R)$ such that $Q(q) \in \min(R)$. Suppose there exists $Z \in g - \min(R)$, $Z \neq Q(q)$ for all $q \in \Delta(R)$. We may partition $\min(R)$ into two disjoint non-empty sets:

$$\mathbf{X} = \{X_1, \ldots, X_n\}, \qquad \mathbf{Y} = \{Y_1, \ldots, Y_m\},$$

where each X_i is of form $Q(q)$ for some $q \in \Delta(R)$ and no Y_j is of that form.

Let $X = \bigcap_{i=1}^{n} X_i$, $Y = \bigcap_{j=1}^{m} Y_j$, $J = \prod_{j=1}^{m} Y_j$. It is easy to adapt the proof of (ii) to show that $X = pX = Xp$ for each $p \in \Delta(R)$. Moreover, p is regular, so the map $s \to s'$ defined by $sp = ps'$ is an automorphism of R, and hence $Yp = pY$. If $Yp = Y$ then $J = \prod_{j=1}^{m} Y_j \subseteqq Rp \in \operatorname{spec}(R)$, so $Y_k \subseteqq Rp$ for some k, $1 \leqq k \leqq m$. But by (vi) this contradicts the definition of Y_k; hence $Yp = pY \subset Y$.

Let S be the partial ring of fractions of R with respect to $D(R)$. Since $D(R) \subseteqq C(0)$, the map $R \to S$ is injective. It follows that:

$$\operatorname{spec}(S) = \{SMS : M \in \min(R)\} = \max(S).$$

Moreover, from (vi) and (vii) and Proposition 5.1 it is easy to check (compare Proposition 3.10) that:

$$SX_iS = SX_i = X_iS, \quad 1 \leqq i \leqq n;$$
$$SY_jS = SY_j = Y_jS, \quad 1 \leqq j \leqq m;$$

and:

$$SXS = SX = XS = \bigcap_{i=1}^{n} SX_iS \quad \text{and} \quad SYS = SY = YS = \bigcap_{j=1}^{m} SY_jS.$$

Hence $S = YS + XS$, so by a common denominator argument $1 \doteq (y+a)s^{-1}$ for some $y \in Y$, $a \in X$, $s \in D(R)$. Hence $s = y + a$ in R. But $uX = Xu = X$ for every $u \in \Delta(R)$, and hence for every $s \in D(R)$. Hence $a = bs$ for some $b \in X$, so $(1-b)s = y \in Y$. But clearly $s \in C(Y)$, so $1 - b = v \in Y$. Hence $1 = b + v \in X + Y$. It follows that:

$$\forall 1 \leqq i \leqq n, \quad \forall 1 \leqq j \leqq m, \quad X_i + Y_j = R.$$

Fix j, $1 \leqq j \leqq m$. Then $Y_j \in g\text{–}\min(R)$, so there exists $T \in \operatorname{spec}(R)$ such that $Y_j \subset T$, and then

$$(*) \qquad \forall 1 \leqq i \leqq n, \quad X_i + T = R.$$

Now, by hypothesis, there exists $z \in T \cap \Delta(R)$, so $Q(z) \subset Rz \subseteqq T$. On the other hand, $Q(z) \in \mathbf{X}$, so $Q(z) + T = R$ from $(*)$. This contradiction implies that $\mathbf{Y} = \emptyset$ as required.

(ix) If $\min(R)$ is a singleton then R is prime (since we have reduced to the semiprime case), and (ix) is then trivial. So suppose that $1 < w$ and $V_1, \dots, V_w$ are the distinct elements of $\min(R) = g\text{–}\min(R)$. Let $V = V_1$, $L = V_2 \cap V_3 \cap \dots \cap V_w$, so that $V \cap L = 0 \neq L$. From (viii), for each $1 \leqq j \leqq w$ there exists $q_j \in \Delta(R)$ such that $V_j = Q(q_j)$, and it is easy to check that:

$$(**) \qquad \begin{cases} 1 < j & \text{implies} \quad q_jL = L = Lq_j \quad \text{but} \quad q_jV = Vq_j \neq V \\ 1 = j & \text{implies} \quad q_1L = Lq_1 \neq L \quad \text{but} \quad q_1V = V = Vq_1. \end{cases}$$

Let

$$\Delta(1) = \{q \in \Delta(R) : V \subseteqq qR\}, \qquad \Delta(w) = \{t \in \Delta(R) : L \subseteqq tR\};$$

then

$$(***) \qquad \Delta(R) = \Delta(1) \cup \Delta(w), \quad \Delta(1) \cap \Delta(w) = \emptyset.$$

Letting $D(1)$, $D(w)$ be the sets of finite products of elements of $\Delta(1)$, $\Delta(w)$ respectively, it follows that:

$$D(1) \cap D(w) = \{1\}, \quad D(R) = U(R)D(1)D(w) = D(w)D(1)U(R).$$

Now R has a polysimple partial ring of fractions W with respect to $D(R)$, and hence with respect to $D(1)D(w)$. It follows from $(**)$ and $(***)$ that $WV = VW = WVW$, $WL = LW = WLW$; it is then easy to check that $WVW \cap WLW = 0$, $WVW + WLW = W$.

Hence $1 = av^{-1} + bu^{-1}$ for some $a \in V$, $b \in L$ and $v, u \in D(R)$. Again using $(**)$ and $(***)$ and divisibility, we may without loss of generality assume that $v \in D(w)$, $u \in D(1)$. But then $v = a + bu^{-1}v = a + bvy^{-1}$ for some $y \in D(1)$, so $vy = ay + bv$. Hence $(v-a)y = bv \in L$. Since $y \in C(L)$, $v - a = c \in L$. Since $v \in D(w)$, $vL = Lv = L$, so $c = vd$ for some $d \in L$. Hence $v - a = vd$, so $a = v(1-d) \in V$. But $v \in C(V)$, so $1 - d = e \in V$. It follows that $1 = d + e \in V + L$, and hence that the elements of $\min(R)$ are pairwise comaximal.

(x) This is immediate from (ix). □

COROLLARY 5.4. *Let R be a Goldie finite ring.*

(i) *Suppose that R is a UFR and $A \in \mathbf{I}(R)$ has the property that each $Q \in \operatorname{spec}(R)$ minimal over A belongs to $\min(R)$; then R/A is a UFR;*

(ii) *Suppose that R is conformal; then R is a UFR if and only if R/A is a UFR for some $A \in \mathbf{I}(R)$, $A \subseteqq P(R)$, if and only if R/B is a UFR for every $B \in \mathbf{I}(R)$, $B \subseteqq P(R)$.*

PROOF. (i) follows by a straightforward adaptation of the argument sketched at the beginning of the proof of Proposition 5.3 (ix); for (ii), use Proposition 5.2. □

REMARKS. 5.1. As noted in the proof of Proposition 5.3 (viii), a semiprime Goldie finite UFR has a polysymple ring of fractions (and, of course, each simple direct summand is trivially a prime UFR).

5.2. We know no example of a non-trivial UFR which is not Goldie finite.

5.3. The proof of Proposition 5.3 (iii) in fact shows that, in any ring R, if $Rp = pR \in \operatorname{spec}(R) \setminus \min(R)$, then $Q(p) = \bigcap_{n=1}^{\infty} Rp^n \in \operatorname{spec}(R)$. But without some further hypothesis it does not follow that $Q(p) \in \min(R)$. Thus at [12], Example 5.5 we give for each $n \in \mathbf{N}$ a prime bi-noetherian ring R_n in which every maximal ideal is of height $n+1$ and is generated by a (regular) central element, but whose non-minimal primes of lesser height contain no non-zero normal (indeed, no regular) element.

5.4. There is no analogue of Proposition 5.3 if we merely assume that every non-minimal prime ideal contains a non-minimal prime $P = Ra = bR$ for some $a, b \in R$. In [14] we exhibit a prime BPN ring (which therefore satisfies the d.c.c. on prime ideals, from [15]) in which there are prime ideals of arbitrarily large height, but in which every ideal is left and right cyclic.

5.5. If R is a ring, and $R/P(R)$ is a UFR, it need not follow that R is a UFR without the assumption (as in Corollary 5.4) that R is conformal. For example, the ring R of 2×2 upper triangular matrices over $\mathbf{Z}$ is not a UFR but $R/P(R)$ is.

The main application of Proposition 5.3 is to the case of semiprime Goldie finite UFRs.

THEOREM 5.5. *Let R be a semiprime Goldie finite UFR. Then R is a finite direct sum of prime UFRs, $R \simeq \oplus(R/Q_i)$, where $Q_1, \dots, Q_n$ are the distinct elements of* $\min(R)$.

PROOF. If $Q_1, \dots, Q_n$ are the distinct elements of $\min(R)$ then $1 \leqq i$, $j \leqq n$ and $i \neq j$ implies $Q_i + Q_j = R$ (Proposition 5.3 (ix)). The result then follows from standard arguments and Corollary 5.4 (i). □

In the case of noetherian or BPN UFRs, the explicit assumption in Theorem 5.5 that the ring in question is semiprime can be dropped.

THEOREM 5.6. *Let R be a UFR, and suppose that R is noetherian or BPN. Then $R \simeq \Big(\bigoplus_{Q \in g-\min(R)} R/Q \Big) \oplus S$ for a G-bi-artinian ring S (which is the unique largest bi-artinian ideal of R).*

PROOF. Since R is bi-noetherian it is Goldie finite, and hence conformal (Proposition 5.1). By Theorem 4.4 $R \simeq (R/K) \oplus (R/L^n)$, where $K = \bigcap_{Q \in g-\min(R)} Q$ and $L = \bigcap_{M \in m-\min(R)} M$ and $n \in \mathbf{N}$. By Theorem 5.5 $R/K \simeq \bigoplus_{Q \in g-\min(R)} R/Q$. Identifying R with $(R/K) \oplus (R/L^n)$, we have $S = R/L^n$, as required. □

EXAMPLE 5.7. *An indecomposable, right noetherian (conformal) UFR which is neither semiprime nor G-bi-artinian, and neither proper nor trivial conformal.*

Let D be a commutative noetherian unique factorisation domain, and $K \neq D$ be its quotient field. Let S be a simple, right noetherian, non-artinian ring whose centre contains K. Then the matrix ring $R = \begin{pmatrix} D & S \\ 0 & S \end{pmatrix}$ has the required properties. The elements of $N_P(R)$ are regular, since R

is bi-noetherian, and the quotient ring of R with respect to $N_P(R)$ is $W = \begin{pmatrix} K & S \\ 0 & S \end{pmatrix}$, which is not bi-artinian.

REMARK 5.6. We use the name *unique factorisation ring* because it can be shown (see [19], Proposition 2.1) that, if R is a prime UFR and $0 \neq x \in R$, then there exist $n \geqq 0$, $p_i \in \Delta(R)$, $e(i) \in \mathbf{N}$, $1 \leqq i \leqq n$, and an element $c \in R \setminus \bigcup_{q \in \Delta} Rq$, such that $x = cp_1^{e(1)} \dots p_n^{e(n)}$, where the ideals Rp_i and the integers n, $e(i)$ are uniquely determined by x, and c is determined by x up to multiplication by a unit of R. This extends the factorisation properties of noetherian UFRs (see [4]) to the case of an arbitrary prime UFR; indeed *via* Theorem 5.5 they may be extended to semiprime Goldie finite UFRs.

5.7. If, however, a UFR R is Goldie finite but not semiprime, there is no satisfactory notion of unique factorisation of *all* non-zero elements; on the contrary, ${}_RP(R)_R$ is always infinitely divisible on either side by the prime elements of R.

6. Examples. Throughout this section, if R is a ring, $n \in \mathbf{N}$ and $t > 1$ is a cardinal, then $M_n(R)$, $G_t(R)$ denote respectively the full $n \times n$ matrix algebra over R and the algebra over R obtained by using the so-called *G-functor* $G_t(-)$ defined in [14], [18].

1. A finite direct sum of conformal rings (UFRs) is conformal (a UFR respectively). A direct summand of a conformal ring (a UFR) is conformal (a UFR respectively).

2. The classes of conformal rings and of Goldie finite UFRs are closed under suitable — but not all — epimorphic images (Proposition 3.5, Corollary 5.4 and Remark 4.2). In the other direction, for conformal rings the property of being a UFR lifts modulo ideals contained in the prime radical (Corollary 5.4).

3. Any right artinian ring is a Goldie finite, trivial conformal UFR (since it is right noetherian and every prime is maximal co-artinian).

4. Any Goldie finite semiprime commutative ring is conformal. More generally, so is any Goldie finite semiprime ring R in which every ideal is generated by a finite sequence $c_1, \dots, c_n$ such that c_1 is central, $c_2 + c_1R$ is central in R/c_1R, and so on. (This can be seen by observing that the regular *seminormalising* elements of such a ring — discussed in [15] — can then be chosen to be central.) In [16] we discuss other properties of these rings, under the name of *bi-noetherian polycentral (BPC)* rings.

5. Any prime BPN or BPC ring is conformal.

6. If R is semiprime, and is conformal (a Goldie finite UFR) and X is a central indeterminate then the polynomial and Laurent series rings $S = R[X]$, $W = R[X, X^{-1}]$ are conformal (Goldie finite UFRs respectively). To see the first assertion, note that in either case R has a polysimple ring of fractions Q with respect to the left-right Ore set $D(R) \subsetneqq C(0)$, and that S, W embed in the obvious ways in $Q[X]$, $Q[X, X^{-1}]$. But every bi-essential

ideal of $Q[X]$ is generated by a monic central polynomial (since the same is true when Q is simple), and it then follows — 'multiplying up' by elements of $D(R)$ — that S and W are conformal. To see that S, W are UFRs when R is, adapt the proof for the prime noetherian case in [4]: see Theorem 3.16 of [2].

7. A commutative domain is a UFR if and only if it is a unique factorisation domain by the classical definition.

8. A ring is a *quasi-commutative principal ideal ring (QCPIR)* if every ideal is generated by a normal element [12]. Theorem 4.4 of [12] shows that any QCPIR is a (BPN) UFR, and that for such a ring the decomposition given here at Theorem 5.6 can be slightly improved on (in respect of the bi-artinian direct summand).

9. Let k be a commutative field, and let K be the commutative field extension generated over k by two families $\{X_i : -\infty < i < \infty\}$ and $\{Y_j : -\infty < j < \infty\}$ of central ideterminates. Let α, β be the k-automorphisms of K defined by: $\alpha(X_i) = X_{i+1}$, $\alpha(Y_j) = Y_j$, $\beta(Y_j) = Y_{j+1}$, $\beta(X_i) = X_i$ for all $-\infty < i, j < \infty$. Then $\alpha\beta = \beta\alpha$, so we may construct the skew polynomial extension $R = K[T, V; \alpha, \beta]$ in which T, V are commuting indeterminates such that $\alpha(x)T = Tx$ and $\beta(x)V = Vx$ for all $x \in K$. Let S be the factor ring R/TVR, and $t = T + TVR$, $v = V + TVR$. Then S is a semiprime Goldie finite BPN ring, whose only non-minimal prime is $M = StS + SvS$. But S is not conformal, because $N_P(R) = U(R)$, although the rings S/vS, S/tS are conformal. This shows that a finite subdirect product of prime conformal rings, although semiprime, need not be conformal. Using the techniques in [15], it can be shown that the element $z = t + v$ is regular seminormalising, and that S has a polysimple ring of fractions with respect to the left-right Ore set $\{z^n : n \in \mathbf{Z}, n \geqq 0\}$.

10. If R is a conformal ring then so are each $M_n(R)$ and each $G_t(R)$. If R is a UFR then so are each $M_n(R)$ and each $G_t(R)$. Since the cardinal t may take arbitrarily large values, there are (from [14]) arbitrarily many pairwise non-isomorphic non-noetherian UFRs of form $G_t(R)$ for any UFR R (noetherian or otherwise). In particular, this is the case for any prime noetherian UFR R in one of the classes of noetherian UFRs identified in [4].

11. Let R be a non-trivial conformal ring, and suppose the elements of $D = N_P(R)$ are regular (e.g., R is bi-noetherian), and that S is the partial ring of fractions of R with respect to D. For any integer $n \in \mathbf{N}$, $n > 1$, let $T^*(R)$ be any ring of $n \times n$ (upper) triangular matrices $\{a_{ij}\}$ in which $A_{ij} \in S$ for all $1 \leqq i, j \leqq n$, but for at least one $1 \leqq k \leqq n$, every a_{kk} is in R. Then $T^*(R)$ is a non-trivial conformal ring, and if R is a UFR then so is $T^*(R)$ (since $T^*(R)/P(T^*(R))$ is). These examples are, of course, not noetherian or BPN.

12. If R is prime conformal or a prime UFR and $\{X_i\}$ is a collection of indeterminates such that each X_i 'skews' R by an automorphism α_i, where the collection $\{\alpha_i\}$ satisfies conditions identified in [17], then the iterated polynomial ring $T = R[X_i; \alpha_i]$ generated by all the indeterminates X_i is

prime conformal or a prime UFR respectively. In particular, if $\alpha_i\alpha_j = \alpha_j\alpha_i$ for each pair i, j and R is simple then T is a UFR [17].

13. In [1] Chatters and Clark identify large classes of group rings which are UFRs.

7. Polynomial and Laurent extensions. It was noted at §6 of Section 6 that, if R is semiprime and is conformal or a Goldie finite UFR then the rings $S = R[X]$ and $T = R[X, X^{-1}]$ have the corresponding property. But if R is conformal but not semiprime then it is easy to see that $X \in N_T(S)$ but $X \notin N_P(S)$, since $P(S) = SP(R) = P(R)S \not\supseteqq SX = XS$. Since the central indeterminate X is as 'nice' a normal element as one could wish for, this suggests that the generalisation of conformal rings should be restricted to semiprime rings, for which the concepts of *conformal* and *T-conformal* coincide.

Against this viewpoint, however, is the curious fact that, taking *skew* Laurent extensions, the conformal property of non-semiprime conformal rings *can* in certain circumstances be preserved. We recall from Chapter 10 of [9] that, if Γ is a ring and σ is an automorphism of Γ, then $I \in \mathbf{I}(R)$ is *σ-stable* if $\sigma(I) = I$, and is *σ-prime* if, for all σ-stable ideals J, K, $JK \subseteqq I$ implies $J \subseteqq I$ or $K \subseteqq I$, so that Γ is *σ-prime* if 0 is a σ-prime ideal. Moreover, it is noted in the same source that if Γ is σ-prime then there is a well-defined notion of a *right Martindale ring of quotients* $Q_r(\Gamma)$ of Γ with respect to the filter $\mathbf{F}$ of non-zero σ-stable ideals of Γ. Hence there is also a well-defined notion of an *X-inner* (or *generalised inner*) automorphism of a σ-prime ring Γ: it is an automorphism τ of Γ such that, for some unit u of $Q_r(\Gamma)$, $\tau(r)u = ur$ for all $r \in \Gamma$. With this background, we note the following result:

PROPOSITION 7.1. *Suppose that R is a proper conformal ring, that σ is an automorphism of R, and that the following conditions are satisfied:*

(a) *$P = P(R)$ is nilpotent;*

(b) *for each $Q \in \operatorname{spec}(R)\setminus \min(R)$, there exists $v \in N_P(R) \cap Q$ such that $\sigma(v) = \alpha v$ for some $\alpha \in U(R)$;*

(c) *σ does not become X-inner on any minimal σ-prime ideal of R.*

then the skew Laurent extension $T = R[X, X^{-1}; \sigma]$ is conformal.

PROOF. Since R is conformal, $\min(R)$ is finite, and then since P is nilpotent, the minimal σ-primes of R are the intersections of the orbits under σ of the elements of $\min(R)$, so are finite intersections of elements of $\min(R)$. Let

$$D = \{v \in N_P(R) : \exists \alpha \in U(R),\ \sigma(v) = \alpha v\}.$$

It is easy to check that $1 \in D$ and that D is multiplicatively closed; suppose that $L \in \mathbf{I}(R)$ is P-essential but $L \cap D = \emptyset$. Then L can be expanded by Zorn's Lemma to an ideal L' maximal with respect to $L' \cap D = \emptyset$, and evidently $L' \in \operatorname{spec}(R)\setminus \min(R)$, contradicting supposition (b). By Theorem 10.6.17 of [9], if $K \in \operatorname{spec}(T)\setminus \min(T)$ then $K \cap R$ is P-essential in R, so that

$D \cap K \neq \emptyset$. It follows easily that

$$(*) \qquad \forall u \in D,\ \forall n \in \mathbf{Z},\ RuX^n = X^n uR$$

and hence that $D \subseteqq N(T)$. The prime radical of T is $TP = PT$, and if $w \in D$ then $P \subseteqq Rw = wR$. It follows by $(*)$ that $TP = PT \subseteqq Tw = wT$ for all $w \in D$. □

REMARKS. 7.1. (a) It does not follow, in the circumstances of Proposition 7.1, that $T = R[X, X^{-1}; \sigma]$ is a *proper* conformal ring. For example, let K be a commutative field of characteristic 0, $R = K[Y]$, Y an indeterminate, and σ be the K-automorphism defined by $\sigma(Y) = Y + 1$. Then σ is of infinite order, and it follows that $T = R[X, X^{-1}]$ is simple, hence trivial conformal.

(b) The assumptions in Proposition 7.1 need not be satisfied by an arbitrary conformal ring R and automorphism σ, but neither are they particularly farfetched. Indeed, they may easily be satisfied by conformal rings which are also algebras over (suitable) fields, as the following example shows.

EXAMPLE 7.2. Let K be a commutative field, and X, Y, Z, W be central indeterminates; form the extension field $Q = K(X, Y, Z, W)$. Let σ, τ be the K-automorphisms of Q defined by: $\sigma(X) = X = \tau(X)$, $\sigma(Y) = Y = \tau(Y)$, $\sigma(W) = W$, $\tau(W) = YW$, $\sigma(Z) = XZ$, $\tau(Z) = Z$. Then σ, τ are of infinite order, and $\sigma\tau = \tau\sigma$. Hence σ has an extension to the skew polynomial ring $S = Q[T; \tau]$, uniquely determined by the property that $\sigma(T) = T$; we denote this also by σ. The non-trivial ideals of S are of form $ST^n = T^n S$, $n \in \mathbf{N}$, and the set $D = \{T^n : n \geqq 0\} \subseteqq C_S(0)$ is multiplicatively closed and left-right Ore. Let A be the ring of quotients of S with respect to D; then A is a simple ring.

Now let R be the matrix ring $\begin{pmatrix} S & A \\ 0 & S \end{pmatrix}$; it follows from Section 6 §11 that R is a UFR, and evidently σ extends *via* A to R; again we denote this extension by σ. Now $P = P(R) = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, so $P^2 = 0$. The minimal σ-primes of R are simply its minimal primes, and it is easy to check that every P-essential ideal I of R contains the diagonal matrix $v = \operatorname{diag}\{T^n, T^m\}$ for some integers $n, m \geqq 0$. But clearly $v \in N_P(R)$, and $\sigma(v) = v$. Thus conditions (a) and (b) of Proposition 7.1 are satisfied, and all we still need to show is that σ is not X-inner on S. Now $0 \neq I \in \mathbf{I}(S)$ implies $I \cap N(S) \neq \emptyset$, and $N(S) \subseteqq C_S(0)$, so standard techniques show that the group $G_X(S)$ of X-inner automorphisms of S is generated (as a group) by the automorphisms induced by the elements of $N(S)$. But then $G_X(S)$ must be generated by τ, so $\sigma \notin G_X(S)$, and hence condition (c) of Proposition 7.1 is satisfied. It follows that the skew Laurent extension ring $B = R[V, V^{-1}; \sigma]$ is also conformal.

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ON THE MATRIX VALUED EXPONENTIALLY CONVEX, TOTALLY POSITIVE FUNCTIONS AND SEQUENCES

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0. Introduction

The concept of exponentially convex functions was introduced by S. N. Bernstein and D. V. Widder independently. This concept is extended in this paper for the matrix-valued functions and sequences; moreover we deal with the Hankelian totally positive (non-negative, matrix-valued) functions and sequences.

The paper consists of six parts. In the first one we introduce the usual concepts and notations. The subject of the second part is the positive definite (semidefinite) hypermatrices of grade p. We prove among others an extension of the well-known Lagrange transformation. The third part contains an appropriate generalization of the Landsberg theorem, which has an importance in the theory of total positivity. Using the results of the third part we give a procedure to construct totally positive matrices, and hypermatrices of grade p, respectively, in the fourth part. After these preparatory parts we deal with matrix-valued exponentially convex functions, with matrix-valued Hankelian totally positive functions, and with matrix-valued absolutely monotone functions in the fifth part. At the same place we give characterizations for these matrix-valued functions, and also some relations among them. The last part contains a generalization of the Hamburger moment problem for matrix-valued sequences, as well as a new proof of the full Stieltjes moment problem.

The results of the paper can be applied in several fields of probability theory, for example in the theory of birth and death processes. We shall return to these questions in another paper.

1. Concepts, notations

A function $\alpha(x)$ of bounded variation defined on a finite or infinite interval $a \leqq x \leqq b$ is called a distribution function if it is non-decreasing. It is

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called of finite type if its range contains finitely many values; otherwise it is called of infinite type.

The distribution function $\alpha(x)$ defined on the whole real line is referred to as probability distribution function if it is right continuous and $\alpha(-\infty)=0$, $\alpha(\infty)=1$.

The $p\times p$ matrix-valued function $F(x)$ defined on the finite or infinite interval $a \leqq x \leqq b$ is said to be a matrix-valued distribution function if the following properties are satisfied:

(a) For each value of x, $F(x)$ is a symmetric matrix with entries of bounded variation.

(b) $F(y)-F(x)$ is a positive definite or semidefinite matrix for $a \leqq x < y \leqq b$, i.e.

$$\alpha(x;v)=v^*F(x)v \qquad (a \leqq x \leqq b) \tag{1.1}$$

is a distribution function for all $v\in R_p$, $v\neq 0$. $F(x)$ is of infinite type if $\alpha(x;v)$ in (1.1) is a distribution function of infinite type for all $z\in R_p$, $z\neq 0$.

The finite or infinite matrix A is said to be a hypermatrix of grade p if the entries of A are $p\times p$ matrices.

Let

$$A=(a_{jk})_{j,k=1}^{n},$$

and let

$$1\leqq j_1<\ldots<j_s\leqq n, \qquad 1\leqq k_1<\ldots<k_s\leqq n.$$

We shall use the following notation:

$$(a_{j_\alpha k_\beta})_{\alpha,\beta=1}^{s}=A\begin{pmatrix} j_1\ldots j_s\\ k_1\ldots k_s\end{pmatrix}.$$

If A is a hypermatrix of grade p then of course

$$A\begin{pmatrix} j_1\ldots j_s\\ k_1\ldots k_s\end{pmatrix}$$

is a hypermatrix of grade p as well.

The matrices

$$A\begin{pmatrix} j_1\ldots j_s\\ j_1\ldots j_s\end{pmatrix}, \quad 1\leqq j_1<\ldots j_s\leqq n, \quad s=1,2,\ldots,n$$

are said to be the principal minor matrices of A.

The infinite matrix A is called positive definite (semidefinite) if all principal minor matrices of A are positive definite (semidefinite).

The derivative (integral) of a matrix-valued function is, as usual, the matrix formed from the derivatives (integrals) of its elements.

2. Symmetric positive definite (semidefinite) matrices of order p

We need the following generalization of the symmetric positive definite (semidefinite) matrices.

DEFINITION 2.1. The symmetric hypermatrix $A = (a_{jk})_{j,k=1}^{n}$ of grade p with entries of real elements is positive definite (semidefinite) of order p if for all

$$v \in R_p, \quad v^*v > 0; \quad z = (z_j) \in R_n, \quad z^*z > 0 \tag{2.1}$$

the inequality

$$\sum_{j=1}^{n}\sum_{k=1}^{n}(v^* a_{jk} v) z_j z_k > 0 \qquad (\geqq 0) \tag{2.2}$$

holds.

THEOREM 2.1. *Let the symmetric hypermatrix $A = (a_{jk})_{j,k=1}^{n}$ of grade p be positive definite of order p. Then there exists a hypermatrix*

$$B = \begin{pmatrix} E & 0 & 0 & \dots & 0 \\ \beta_{12} & E & 0 & \dots & 0 \\ \beta_{13} & \beta_{23} & E & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \beta_{1n} & \beta_{2n} & \beta_{3n} & \dots & E \end{pmatrix}$$

of grade p such that

$$A = B \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & (0) & \\ & (0) & \ddots & \\ & & & \gamma_m \end{pmatrix} B^*, \tag{2.3}$$

where γ_i $(i = 1, \dots, n)$ are $p \times p$ symmetric positive definite matrices, and E is the $p \times p$ unit matrix.

In case of $p = 1$ the procedure expressed in Theorem 2.1 is due to Lagrange. Therefore this procedure is called as Lagrange transformation too.

PROOF OF THEOREM 2.1. Denote $H_n(v; z_1, \dots, z_n)$ the quadratic form (2.2) of the symmetric positive definite hypermatrix A of grade p and of order p. Since this quadratic form is positive under the conditions (2.1), we get that the $p \times p$ matrices a_{jj} $(j = 1, \dots, n)$ are symmetric positive definite.

Now let

$$\beta_{1j}^* = a_{1j} \cdot a_{11}^{-1} \qquad (j = 2, \dots, n), \tag{2.4}$$

and

$$\xi_1 = (z_1 + z_2\beta_{12} + \ldots + z_n\beta_{1n})v.$$

Since

$$\xi_1^* a_{11}\xi_1 = v^*(a_{11}z_1^2 + a_{12}z_1z_2 + \ldots + a_{1n}z_1z_n +$$
$$+ a_{21}z_2z_1 + \ldots + a_{n1}z_nz_1)v + \text{members which do not contain } z_1.$$

Using the notation $\gamma_1 = a_{11}$, we obtain

$$H(v; z_1, \ldots, z_n) = \xi_1^*\gamma_1\xi + H_{n-1}(v; z_2, \ldots, z_n), \tag{2.5}$$

where

$$H_{n-1}(v; z_2, \ldots, z_n) = H_n(v; z_1, z_2, \ldots, z_n) - \xi_1^*\gamma_1\xi_1$$

is a quadratic form with a symmetric positive definite hypermatrix of grade p and of order p. Continuing this procedure we get that

$$H_n(v; z_1, \ldots, z_n) = \sum_{k=1}^{n} \xi_k^*\gamma_k\xi_k,$$

where

$$(\xi_1, \ldots, \xi_n) = v(z_1, \ldots, z_n)B,$$

and γ_i $(i = 1, \ldots, n)$ are $p\times p$ symmetric positive definite matrices. Moreover

$$\operatorname{Det} A = \prod_{k=1}^{n} \operatorname{Det} \gamma_k > 0$$

by the representation (2.3).

COROLLARY 2.1. *If the symmetric hypermatrix A of grade p is positive definite of order p, then the determinants of all principal minor hypermatrices of grade p are positive.*

THEOREM 2.2. *The symmetric hypermatrix $A = (a_{jk})_{j,k=1}^{m}$ of grade p is positive definite of order p if and only if the determinants of all principal minor hypermatrices of grade p are positive.*

PROOF. The first part of the proof is contained in Corollary 1.2. We have to show that if the determinants of all principal minor hypermatrices of grade p are positive, then

$$H_n(v; z_1, \ldots, z_n) > 0$$

for all vectors defined by (2.1).

Evidently, the statement holds for $n = 1$. Suppose that the theorem is valid for the positive integers $k = 1, \ldots, n-1$. Since the determinants of the

$p \times p$ matrices a_{jj} $(j = 1, \ldots, n)$ are positive by Corollary 2.1, the transformation (2.4) is applicable. We get identity (2.5), where the determinants of all principal minor hypermatrices of grade p of the matrix of the quadratic form $H_{n-1}(v; z_2, \ldots, z_n)$ are positive. Thus this quadratic form is positive for all

$$v \in R_p, \quad z_j \in R \quad (j = 2, \ldots, n), \quad v^*v \sum_{j=2}^{n} z_j^2 > 0$$

by inductive assumption. Using (2.5) again, one gets that the quadratic form $H_n(v; z_1, \ldots, z_n)$ is positive for all (2.1).

3. On an integral identity

The following theorem is important in the verification of the theorems concerning total positivity (non-negativity) of hypermatrices of grade p. This theorem gives for $p = 1$ the Theorem of G. Landsberg ([7]), which has a fundamental role in the theory of total positivity ([6], 16–17).

The following concept and result is needed. Let p and n be positive integers. The Rados product of the matrices

$$A_i = (a_{k\ell}^{(i)})_{k,\ell=1}^{p} \qquad (i = 1, \ldots, n),$$

$$B_j = (b_{k\ell}^{(j)})_{k,\ell=1}^{n} \qquad (j = 1, \ldots, p)$$

with real or complex entries is defined by the matrix

$$N = (N_{jk})_{j,k=1}^{n}, \tag{3.1}$$

where

$$N_{jk} = \begin{pmatrix} a_{11}^{(j)} & \ldots & a_{1p}^{(j)} \\ \cdot & \ldots & \cdot \\ a_{p1}^{(j)} & \ldots & a_{pp}^{(j)} \end{pmatrix} \begin{pmatrix} b_{kj}^{(1)} & & (0) \\ & \ddots & \\ (0) & & b_{kj}^{(p)} \end{pmatrix} \qquad (j, k = 1, \ldots, n).$$

Obviously, the Rados product of matrices is a straightforward generalization of the direct product, or Kronecker product of matrices.

The following statement was proved by G. Rados ([9]):

$$\operatorname{Det} N = \prod_{i=1}^{n} \operatorname{Det} A_i \prod_{j=1}^{p} \operatorname{Det} B_j.$$

THEOREM 3.1. *Let the matrix-valued function*

$$f(t) = (f_{jk}(t))_{j,k=1}^{p} \qquad (a \leqq t \leqq b)$$

and the functions

$$g_j(t),\ \ h_j(t) \qquad (a \leqq t \leqq b,\ \ j = 1, \dots, n)$$

be given. Suppose that the matrix-valued functions

$$f(t)g_j(t)h_k(t) \qquad (j, k = 1, \dots, n) \tag{3.2}$$

are integrable (elementwise) with respect to the distribution function $\psi(t)$, $a \leqq t \leqq b$. Then the identity

$$D = \operatorname{Det} \int_a^b (f(x)g_j(x)h_k(x))_{j,k=1}^n d\psi(x) =$$

$$= \underbrace{\int \dots \int}_{\Omega} \prod_{i=1}^n \operatorname{Det} \begin{pmatrix} f_{11}(x_{i1}) & \dots & f_{1p}(x_{ip}) \\ \cdot & \dots & \cdot \\ f_{p1}(x_{i1}) & \dots & f_{pp}(x_{ip}) \end{pmatrix} \cdot$$

$$\cdot \prod_{j=1}^p \operatorname{Det} \begin{pmatrix} g_1 & \dots & g_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix} \operatorname{Det} \begin{pmatrix} h_1 & \dots & h_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix} \cdot$$

$$\cdot d\psi(x_{11}) \dots d\psi(x_{1p}) \dots d\psi(x_{n1}) \dots d\psi(x_{np})$$

holds, where

$$\Omega = \{a \leqq x_{1j} < \dots < x_{nj} \leqq b \ \ (j = 1, \dots, p)\},$$

and

$$\begin{pmatrix} g_1 & \dots & g_n \\ x_1 & \dots & x_m \end{pmatrix} = (g_j(x_k))_{j,k=1}^n . \tag{3.3}$$

PROOF. It is very simple to verify that

$$D = \underbrace{\int_a^b \dots \int_a^b}_{np} \operatorname{Det} N \, d\psi(x_{11}) \dots d\psi(x_{np}),$$

where N is the Rados product (3.1) of the matrices

$$A_i = \begin{pmatrix} f_{11}(x_{i1}) & \dots & f_{1p}(x_{ip}) \\ \cdot & \dots & \cdot \\ f_{p1}(x_{i1}) & \dots & f_{pp}(x_{ip}) \end{pmatrix} \qquad (i = 1, \dots, n),$$

and

$$B_j = \begin{pmatrix} g_1 & \dots & g_n \\ x_{ij} & \dots & x_{nj} \end{pmatrix}^* \begin{pmatrix} h_1(x_{1j}) & & (0) \\ & \ddots & \\ (0) & & h_n(x_{nj}) \end{pmatrix} \quad (j = 1, \dots, p)$$

in so far as

$$N_{jk} = \begin{pmatrix} f_{11}(x_{j1}) & \dots & f_{1p}(x_{jp}) \\ \cdot & \dots & \cdot \\ f_{p1}(x_{j1}) & \dots & f_{pp}(x_{jp}) \end{pmatrix} \begin{pmatrix} g_k(x_{j1})h_j(x_{n1}) & & (0) \\ & \ddots & \\ (0) & & g_k(x_{jp})h_j(x_{np}) \end{pmatrix}.$$

Applying the theorem of Rados, we get

$$D = \underbrace{\int_a^b \dots \int_a^b}_{np} \prod_{i=1}^{n} \operatorname{Det} \begin{pmatrix} f_{11}(x_{i1}) & \dots & f_{1p}(x_{ip}) \\ \cdot & \dots & \cdot \\ f_{p1}(x_{i1}) & \dots & f_{pp}(x_{ip}) \end{pmatrix} \cdot$$

$$\cdot \prod_{j=1}^{p} h_1(x_{1j}) \dots h_n(x_{nj}) \operatorname{Det} \begin{pmatrix} g_1 & \dots & g_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix} \cdot$$

$$\cdot d\psi(x_{11}) \dots d\psi(x_{1p}) \dots d\psi(x_{n1}) \dots d\psi(x_{np}).$$

Now let j be a fixed integer. Denote by $D_{i_1 \dots i_n}(j)$ the integral which can be obtained from D if

$$h_1(x_{1j}) \dots h_n(x_{nj})$$

is replaced by

$$(-1)^I h_1(x_{i_1 j}) \dots h_n(x_{i_n j})$$

where $i_1, \dots, i_n$ is an arbitrary permutation of the elements $1, \dots, n$, and I is the number of inversions of $i_1, \dots, i_n$ with respect to $1, \dots, n$. If $x_{1j}, \dots, x_{nj}$ is replaced by $x_{i_1 j}, \dots, x_{i_n j}$ in the integral $D_{i_1 \dots i_n}(j)$, then

$$\operatorname{Det} \begin{pmatrix} g_1 & \dots & g_n \\ x_{i,j} & \dots & x_{i_n j} \end{pmatrix} = (-1)^I \operatorname{Det} \begin{pmatrix} g_1 & \dots & g_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix},$$

thus

$$h_1(x_{i_1 j}) \dots h_n(x_{i_n j})(-1)^I \operatorname{Det} \begin{pmatrix} g_1 & \dots & g_n \\ x_{i_1 j} & \dots & x_{i_n j} \end{pmatrix} =$$

$$= h_1(x_{1j}) \dots h_n(x_{nj}) \operatorname{Det} \begin{pmatrix} g_1 & \dots & g_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix},$$

i.e. $D_{i_1\dots i_n}(j) = D$, since the transformation in question means only an exchange of the variables in the first factor of $D_{i_1\dots i_n}(j)$. Consequently

$$\sum_{(i_1,\dots,i_n)} D_{i_1\dots i_n}(j) = n!D \qquad (j=1,\dots,p),$$

i.e.

$$D = \left(\frac{1}{n!}\right)^p \underbrace{\int_a^b \cdots \int_a^b}_{np} \prod_{i=1}^n \operatorname{Det}\begin{pmatrix} f_{11}(x_{i1}) & \dots & f_{1p}(x_{ip}) \\ \cdot & \dots & \cdot \\ f_{p1}(x_{i1}) & \dots & f_{pp}(x_{ip}) \end{pmatrix} \cdot$$

$$\prod_{j=1}^n \operatorname{Det}\begin{pmatrix} g_1 & \dots & g_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix} \operatorname{Det}\begin{pmatrix} h_1 & \dots & h_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix} d\psi(x_{11})\dots d\psi(x_{np}).$$

Changing over to the domain Ω, we get the statement of Theorem 3.1.

Now let

$$f(t) = \begin{pmatrix} Q_1(t)R_1(t) & \dots & Q_1(t)R_p(t) \\ \cdot & \dots & \cdot \\ Q_p(t)R_1(t) & \dots & Q_p(t)R_p(t) \end{pmatrix} \qquad (a \leqq t \leqq b), \tag{3.4}$$

where

$$Q_j(t), \quad R_j(t) \quad (a \leqq t \leqq b, \quad j = 1,\dots,p)$$

are functions such that the matrix-valued functions (3.2) are integrable with respect to the distribution function $\psi(t)$. In this case

$$\operatorname{Det}\begin{pmatrix} f_{11}(x_{i1}) & \dots & f_{1p}(x_{ip}) \\ \cdot & \dots & \cdot \\ f_{p1}(x_{i1}) & \dots & f_{pp}(x_{ip}) \end{pmatrix} =$$

$$= \operatorname{Det}\begin{pmatrix} Q_1 & \dots & Q_p \\ x_{i_1} & \dots & x_{ip} \end{pmatrix} R_1(x_{i1})\dots R_p(x_{ip}).$$

If we apply again the method used in the proof of Theorem 3.1, we have the following result.

THEOREM 3.2. *Let*

$$Q_j(t), \quad R_j(t) \qquad (a \leqq t \leqq b,\ j = 1,\dots,p)$$

be functions such that the matrix-valued functions (3.2) *are integrable with respect to the distribution function* $\psi(t)$, *where the matrix-valued function*

$f(t)$ is defined by (3.4). *Then*

$$D = \underbrace{\int \cdots \int}_{\Omega} \prod_{i=1}^{n} \operatorname{Det}\begin{pmatrix} Q_1 & \dots & Q_p \\ x_{i_1} & \dots & x_{i_p} \end{pmatrix} \operatorname{Det}\begin{pmatrix} R_1 & \dots & R_p \\ x_{i_1} & \dots & x_{ip} \end{pmatrix} \cdot$$

$$\cdot \prod_{j=1}^{p} \operatorname{Det}\begin{pmatrix} g_1 & \dots & g_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix} \operatorname{Det}\begin{pmatrix} h_1 & \dots & h_n \\ x_{1j} & \dots & x_{nj} \end{pmatrix} \cdot$$

$$\cdot \psi(x_{11}) \dots d\psi(x_{1p}) \dots d\psi(x_{n1}) \dots d\psi(x_{np}),$$

where

$$\Omega = \{a \leqq x_{11} < \dots < x_{1p} < \dots < x_{n1} < \dots < x_{np} \leqq b\}.$$

4. On totally positive (non-negative) matrices

It seems that the concept of totally positive (non-negative) matrices was introduced by F. R. Gantmacher and M. G. Krein (see [2], [3], [5]).

DEFINITION 4.1. A finite or infinite matrix A is said to be totally positive (non-negative) if its subdeterminants of all order are positive (non-negative).

The fundamental properties of these matrices can be found in the monograph [4].

An extension of Definition 4.1 is the following.

DEFINITION 4.2. The finite or infinite hypermatrix A of grade p is said to be totally positive (non-negative) of order p if

$$\operatorname{Det} A \begin{pmatrix} j_1 & \dots & j_s \\ k_1 & \dots & k_s \end{pmatrix} > 0 \qquad (\geqq 0)$$

for all possible integers

$$1 \leqq j_1 < \dots < j_s, \quad 1 \leqq k_1 < \dots < k_s, \quad s = 1, 2, \dots .$$

In the following we shall show that there is a hypermatrix of grade p which is totally positive (non-negative) of order p for an arbitrary positive integer p.

The sequence

$$(4.1) \qquad f_j(x) \qquad (a \leqq x \leqq b, \quad j = 1, \dots, n)$$

of real continuous functions is said to be Tchebycheff system in this interval, if all linear combinations

$$\sum_{k=1}^{n} c_k f_k(x) \qquad \Big(c_k \in R\,(k = 1, \dots, n), \sum_{k=1}^{n} c_k^2 > 0\Big)$$

vanish at most $n-1$ times in the interval $a \leqq x \leqq b$ ([4], p. 157, Definition 2).

The following theorem of S. N. Bernstein is a characterization of the Tchebycheff systems ([4], p. 157, Hilfssatz 2).

The system of real continuous functions (4.1) defined in the interval $a \leqq x \leqq b$ is a Tchebycheff system if and only if

$$\text{Det}\begin{pmatrix} f_1 & \dots & f_n \\ x_1 & \dots & x_n \end{pmatrix} \tag{4.2}$$

is different from zero for all choices of $a \leqq x_1 < \dots < x_n \leqq b$ (thus these determinants have the same sign).

A Tchebycheff system (4.1) is called of positive type if the determinants (4.2) are positive. In the opposite case it is called of negative type.

LEMMA 4.1 ([8], II. p. 48, Example 75). *If $\alpha_1 < \dots < \alpha_n$ are arbitrary real numbers, then*

$$f_j(x) = x^{\alpha_j} \qquad (0 < x < \infty, \quad j = 1, \dots, n)$$

is a Tchebycheff system of positive type.

The finite or infinite sequence

$$f_j(x) \qquad (a \leqq x \leqq b, \quad j = 1, 2, \dots)$$

of functions is said to be a Markov sequence of positive or negative type if the sequence (4.1) is a Tchebycheff system in this interval for all possible integers n ([4], p. 207, Definition 2).

DEFINITION 4.3. The continuous (elementwise) matrix-valued function

$$f(t) = f_{jk}(t))_{j,k=1}^{p} \qquad (a \leqq t \leqq b) \tag{4.3}$$

is said to be of Tchebycheff type in this interval, if

$$\text{Det}\begin{pmatrix} f_{11}(x_1) & \dots & f_{1p}(x_p) \\ \cdot & \dots & \cdot \\ f_{p1}(x_1) & \dots & f_{pp}(x_p) \end{pmatrix} > 0$$

for all $a \leqq x_1 < \dots < x_n \leqq b$.

If e.g. the sequence of functions $f_j(t)$ $(j = 1, \dots, p)$ is a Tchebycheff system in the interval $a \leqq t \leqq b$, then the $p \times p$ matrix-valued function

$$f(t) = \begin{pmatrix} f_1(t) & \dots & f_p(t) \\ \cdot & \dots & \cdot \\ f_1(t) & \dots & f_p(t) \end{pmatrix}$$

is of Tchebycheff type in this interval.

We get the following theorem by Theorem 3.1.

THEOREM 4.1. *Suppose the matrix-valued function* (4.3) *is of Tcheby-cheff type in the interval* $a \leqq t \leqq b$, *moreover*

$$\{g_j(t)\}_1^\infty, \quad \{h_j(t)\}_1^\infty \qquad (a \leqq t \leqq b) \tag{4.4}$$

are Markov sequences of the same type. Suppose the matrix-valued functions

$$f(t)g_j(t)h_k(t) \qquad (j,k = 1,2,\dots) \tag{4.5}$$

are integrable in the interval $a \leqq t \leqq b$ *with respect to the distribution function* $\psi(t)$ *defined in the same interval. Then the hypermatrix*

$$\int_a^b (f(t)g_j(t)h_k(t))_{j,k=1}^\infty \, d\psi(t) \tag{4.6}$$

of grade p is a totally positive matrix of order p.

The following statement holds by Theorem 3.2.

THEOREM 4.2. *Let* (4.4) *be Markov sequences of the same type. Let*

$$Q_j(t) \qquad (a \leqq t \leqq b, \ j = 1,\dots,p)$$

be linearly independent functions for which the matrix-valued functions (4.5) *are integrable in the interval* $a \leqq t \leqq b$ *with respect to the distribution function* $\psi(t)$ *defined in the same interval, where*

$$f(t) = (Q_j(t)Q_k(t))_{j,k=1}^p \qquad (a \leqq t \leqq b).$$

Then the hypermatrix (4.6) *of grade p is a totally non-negative matrix of order p.*

The following statement can be obtained from Theorem 4.1 in case of $p = 1$.

THEOREM 4.3. *If* $f(t) > 0$ $(a \leqq t \leqq b)$, *and if the functions*

$$f(t)g_j(t)h_k(t) \qquad (j,k = 1,2,\dots)$$

are integrable in the interval $a \leqq t \leqq b$ *with respect to the distribution function* $\psi(t)$ *defined in the same interval, where the sequences* (4.4) *are Markov sequences of the same type, then*

$$\int_a^b (f(t)q_j(t)h_k(t))_{j,k=1}^\infty \, d\psi(t)$$

is a totally positive matrix.

5. On functions of special type

5.1. The following definitions play an important role in this section.

DEFINITION 5.1. The $p \times p$ matrix-valued function

$$f(t) \qquad (a < t < b, \ -\infty \leqq a < b \leqq \infty)$$

is said to be positive definite (semidefinite) exponentially convex of order p, if

(a) $f(t)$ is a real symmetric matrix for each value of t,
(b) $f(t)$ is measurable (elementwise),
(c) it is finite almost everywhere (elementwise), and
(d) the hypermatrix

$$(f(t_j + t_k))_{j,k=1}^{n}$$

of grade p is positive definite (semidefinite) of order p for an arbitrary positive integer n, and for all choices $a < t_1 < \dots < t_n < b$ satisfying $a < t_j + t_k < b$ $(j, k = 1, 2, \dots, n)$.

This concept was introduced by S. N. Bernstein for $p = 1$. In this case $f(t)$ is called a positive definite (semidefinite) exponentially convex function.

DEFINITION 5.2. The $p \times p$ matrix-valued function

$$f(t) \qquad (a < t < b, \ -\infty \leqq a < b \leqq \infty)$$

is said to be Hankelian totally positive (non-negative) of order p, if the following conditions are satisfied:

(a) $f(t)$ is measurable (elementwise),
(b) it is finite almost everywhere (elementwise), and
(c) the hypermatrix

$$(f(t_j + \tau_k))_{j,k=1}^{n}$$

of grade p is totally positive (non-negative) of order p for an arbitrary positive integer n, and for all choices

$$a < t_1 < \dots < t_n < b, \quad a < \tau_1 < \dots < \tau_n < b$$

satisfying

$$a < t_j + \tau_k < b \qquad (j, k = 1, \dots, n).$$

The following theorem contains the identity of the class of exponentially convex positive definite (semidefinite) functions with the class of the Hankelian totally positive (non-negative) functions of order $p = 1$.

THEOREM 5.1. *The function* $f(t)$ $(a < t < b)$ *is Hankelian totally positive (non-negative) of order* $p = 1$ *if and only if it is exponentially convex positive definite (semidefinite) in the same interval, i.e. if and only if the representation*

$$f(t) = \int_{-\infty}^{\infty} e^{tx} d\alpha(x) \qquad (a < t < b) \tag{5.1}$$

holds, where $\alpha(x)$ *is a distribution function of infinite (finite) type defined on* R.

PROOF. The sufficiency follows by Lemma 4.1 using Theorem 4.3. The necessity can be obtained by the fact that if $f(t)$ is Hankelian totally positive (non-negative) of order $p = 1$ in $a < t < b$, then it is exponentially convex in this interval ([1], pp. 211–212). Therefore the representation theory (5.1) of Bernstein and Widder ([1], Theorem 5.5.4) holds.

The following theorem is a partial extension of the above mentioned theorem of Bernstein and Widder.

THEOREM 5.2. *The* $p \times p$ *matrix-valued function* $f(t)$ $(a < t < b)$ *is exponentially convex positive definite of order* p *in this interval if and only if the representation*

$$f(t) = \int_{-\infty}^{\infty} e^{tx} dF(x) \qquad (a < t < b) \tag{5.2}$$

holds, where $F(x)$, $x \in R$ *is a* $p \times p$ *matrix-valued distribution function of infinite type.*

PROOF. *Sufficiency.* If the representation (5.2) holds, and if v and z are vectors defined by (2.1), then

$$\sum_{j=1}^{n}\sum_{k=1}^{n} (v^* f(x_j + x_k)v)\, z_j z_k = \int_{-\infty}^{\infty} \Big(\sum_{j=1}^{n} e^{tx_j} z_j\Big)^2 dv^* F(t) v > 0$$

for all positive integers n, and for all choices

$$a < x_1 < \dots < x_n < b, \quad a < x_j + x_k < b \quad (j, k = 1, \dots, n),$$

i.e. $f(t)$ $(a < t < b)$ is an exponentially convex positive definite $p \times p$ matrix-valued function of order p.

Necessity. If the $p \times p$ matrix-valued function

$$f(t) = (f_{jk}(t))_{j,k=1}^{p} \qquad (a < t < b)$$

is positive definite exponentially convex of order p, then

$$\sum_{j=1}^{n}\sum_{k=1}^{n}(v^{*}f(t_j+t_k)v)\,z_j z_k > 0$$

for given $v \in R_p$, $v^{*}v > 0$, for an arbitrary positive integer n and for arbitrary choices

$$a < t_1 < \dots < t_n < b, \quad a < t_j + t_k < b \quad (j,k = 1,\dots,n),$$

$$z = (z_j) \in R_n, \quad z^{*}z > 0,$$

i.e. $v^{*}f(t)v$ is a positive definite exponentially convex function in the interval $a < t < b$. Consequently the functions

$$f_{jj}(t), \quad f_{jj}(t) + f_{kk}(t) - 2f_{jk}(t) \quad (j \neq k,\ j,k = 1,\dots,n)$$

are positive definite exponentially convex functions in the same interval. Using the representation theory (5.1) of Bernstein, there are distribution functions

$$F_{jj}(x), \quad G_{jk}(x) \quad (x \in R,\ j,k = 1,\dots,n)$$

of infinite type, such that

$$f_{jj}(t) = \int_{-\infty}^{\infty} e^{tx}\,dF_{jj}(x),$$

$$f_{jj}(t) + f_{kk}(t) - 2f_{jk}(t) = \int_{-\infty}^{\infty} e^{tx}\,dG_{jk}(x) \quad (j \neq k,\ a < t < b).$$

From this

$$f_{jk}(t) = \int_{-\infty}^{\infty} e^{tx}\,dF_{jk}(x) \quad (j \neq k) \tag{5.3}$$

where

$$F_{jk}(x) = \frac{1}{2}\left[F_{jj}(x) + F_{kk}(x)\right] - G_{jk}(x) \quad (j \neq k) \tag{5.4}$$

is a function of bounded variation. Thus we get that the representation

$$f(t) = \int_{-\infty}^{\infty} e^{tx}\,dF(x), \quad F(x) = (F_{jk}(x))_{j,k=1}^{p}$$

holds. Since

$$v^* f(t) v = \int_{-\infty}^{\infty} e^{tx} d(v^* F(x) v) \quad (a < t < b)$$

is a positive definite exponentially convex function for $v \in R_p$, $v^* v > 0$, the function $F(x)$ $(x \in R)$ is a $p \times p$ matrix-valued distribution function of infinite type defined on the real line.

Thus Theorem 5.2 is proved.

A statement similar to Theorem 5.1 does not hold generally in the case of $p > 1$. Whereas all symmetric Hankelian totally positive functions of order p are positive definite exponentially convex functions of order p by Theorem 2.2, i.e. the representation (5.2) holds with a $p \times p$ matrix-valued distribution function $F(x)$. But generally the matrix-valued function (5.2) is not a Hankelian totally positive function of order $p > 1$ for all $p \times p$ distribution functions $F(x)$ of infinite type.

However, if the matrix-valued distribution function $F(x)$ of infinite type is absolutely continuous with respect to a distribution function $\psi(x)$, i.e. if

$$dF(x) = \omega(x) d\psi(x),$$

where $\omega(x)$ is a $p \times p$ matrix-valued function, then

$$f(t) = \int_{-\infty}^{\infty} e^{tx} \omega(x) d\psi(x) \qquad (a < t < b)$$

by (5.2). Since $\{e^{t_k x}\}_{k=1}^{n}$ is a Tchebycheff system of positive type for $a < t_1 < \cdots < t_n < b$ by Lemma 4.1, and if $\omega(x)$, $x \in R$ satisfies the conditions of Theorem 4.1 or 4.2, respectively, then $f(t)$ is a Hankelian totally positive (non-negative) matrix-valued function of order p.

5.2. Absolutely monotone functions. S. N. Bernstein calls the function $f(x)$ $(a < x < b)$ absolutely monotone, if it satisfies the inequalities

$$f(x) \geqq 0, \quad \Delta_h^n f = \sum_{k=0}^{m} (-1)^{n-k} \binom{n}{k} f(x + nh) \geqq 0$$

for all positive integers n, and $x, h > 0$, for which $a < x < b$, $a < x + nh < b$. The trivial case where $f(x) \equiv \infty$ is assumed to be excluded.

The following statement is due to Widder ([1], Theorem 5.5.2). The function $f(x)$ $(0 < x < \infty)$ can be represented in the form

$$f(x) = \int_{0}^{\infty} e^{-xt} d\alpha(t) \quad (0 < x < \infty)$$

with distribution function $\alpha(x)$ $(x \geqq 0)$ if and only if $f(x)$ is absolutely monotone in this interval.

Comparing this result with Theorem 5.1, we have the following result.

THEOREM 5.3. *The function $f(x)$ $(-\infty < x < 0)$ is Hankelian totally positive (non-negative) if and only if it is absolutely monotone in this interval.*

THEOREM 5.4. *The function $f(x)$ $(-\infty < x < 0)$ is Hankelian totally positive (non-negative) and exponentially convex if and only if it is absolutely monotone in this interval.*

DEFINITION 5.3. The real symmetric $p \times p$ matrix-valued function $f(x)$, $a < x < b$ is said to be absolutely monotone in this interval if $v^*f(x)v$, $a < x < b$ is an absolutely monotone function for all $v \in R_p$, $v^*v > 0$.

THEOREM 5.5. *The $p \times p$ matrix-valued function $f(x)$ $(0 < x < \infty)$ is absolutely monotone in this interval if and only if the representation*

$$f(t) = \int_0^\infty e^{-tx} dF(x) \qquad (0 < t < \infty) \tag{5.5}$$

holds, where $F(x)$, $(x \geqq 0)$ is a $p \times p$ matrix-valued distribution function.

PROOF. If the representation (5.5) is valid, then

$$v^*f(t)v = \int_0^\infty e^{-tx} d(v^*F(x)v) \qquad (0 < x < \infty)$$

is an absolutely monotone function for all $v \in R_p$, $v^*v > 0$, since $v^*F(x)v$ $(x \geqq 0)$ is a distribution function. If $f(x)$ $(0 < x < \infty)$ is a $p \times p$ matrix-valued absolutely monotone function, then $v^*f(t)v$ is an exponentially convex function in the same interval for $v \in R_p$, $v^*v > 0$ by Theorem 5.4. The necessity can be proved similarly like in the proof of necessity in Theorem 5.2. Thus we get finally that the representation (5.5) holds indeed.

6. Exponentially convex, and Hankelian totally positive sequences

In this section the following definition will be used.

DEFINITION 6.1. The sequence $\{M_k\}_0^\infty$ of the $p \times p$ real (Hermite) symmetric matrices is said to be positive definite (semidefinite) exponentially convex of order p, if the Hankelian hypermatrix

$$H = (M_{j+k})_{j,k=0}^\infty$$

of grade p is a positive definite (semidefinite) matrix of order p.

6.1. In this section our aim is to prove the following extension of the theorem of Hamburger.

THEOREM 6.1. *The sequence* $\{M_k\}_0^\infty$ *of the* $p\times p$ *real symmetric matrices is positive definite and exponentially convex of order* p *if and only if the representation*

$$M_k = \int_{-\infty}^{\infty} x^k dF(x) \quad (k = 0,1,2,\dots) \tag{6.1}$$

holds, where $F(x)$, $x \in R$ *is a* $p \times p$ *matrix-valued distribution function of infinite type.*

PROOF. If $p = 1$, the statement of the Theorem was proved by H. Hamburger ([1], Theorem 2.1.1), i.e. the following moment problem holds:

"The necessary and sufficient condition in order that a non-decreasing function $\alpha(x)$ $(x \in R)$ having an infinite number of points of increase and such that

$$\int_{-\infty}^{\infty} x^k d\alpha(x) = M_k \qquad (k = 0,1,2,\dots)$$

exist, is that the sequence $\{M_k\}_0^\infty$ should be positive definite and exponentially convex."

Returning to the proof of Theorem 6.1, suppose first that the representation (6.1) holds. Let n be an arbitrary non-negative integer, moreover let $v \in R_p$, $v^*v > 0$; $z = (z_j)_0^n \in R_{n+1}$, $z^*z > 0$, then

$$\sum_{j=0}^{n}\sum_{k=0}^{n}(v^*M_{j+k}v)z_jz_k = \int_{-\infty}^{\infty}\Big(\sum_{j=0}^{n} z_jx^j\Big)^2 dv^*F(x)v > 0,$$

since $\sum_{j=0}^n z_jx^j$ is a polynomial, and $v^*F(x)v$ is a distribution function of infinite type. I.e. the $p\times p$ matrix-valued sequence $\{M_k\}_0^\infty$ is positive definite and exponentially convex of order p by Theorem 2.2.

Now let $\{M_k\}_0^\infty$ be a positive definite exponentially convex sequence of order p. Let $v \in R_p$, $v^*v > 0$ be given. Then

$$\sum_{j=0}^{n}\sum_{k=0}^{n}(v^*M_{j+k}v)z_jz_k > 0$$

for an arbitrary non-negative integer n, and for arbitrary

$$z = (z_j)_0^n \in R_{n+1}, \qquad z^*z > 0,$$

i.e. $\{v^*M_jv\}_0^\infty$ is a positive definite exponentially convex sequence. By Hamburger's Theorem there is a distribution function $\alpha(x;v)$ $(x \in R)$ of infinite type such that

$$v^*M_kv = \int_{-\infty}^{\infty} x^k d\alpha(x;v) \qquad (k=0,1,2,\dots).$$

Let

$$M_k = (m_{ij}^{(k)})_{i,j=1}^{p}.$$

By the last formula we get that

$$m_{jj}^{(k)} = \int_{-\infty}^{\infty} x^k d\alpha_{jj}(x), \quad m_{ii}^{(k)} + m_{jj}^{(k)} - 2m_{ij}^{(k)} = \int_{-\infty}^{\infty} x^k d\beta_{ij}(x),$$

$$(i \neq j, \quad i,j = 1,\dots,p),$$

where

$$\alpha_{jj}(x); \;\; \beta_{ij}(x) \qquad (x \in R, \; i \neq j, \; i,j=1,\dots,p)$$

are distribution functions of infinite type. Thus

$$m_{ij}^{(k)} = \int_{-\infty}^{\infty} x^k d\alpha_{ij}(x) \qquad (i \neq j)$$

where

$$\alpha_{ij}(x) = \frac{1}{2}(\alpha_{ii}(x) + \alpha_{jj}(x)) - \beta_{ij}(x) \qquad (x \in R, \; i \neq j)$$

is a function of bounded variation. Using the notation

$$F(x) = (\alpha_{jk}(x))_{j,k=1}^{p} \qquad (x \in R),$$

we get that

$$v^*F(x)v = \alpha(x;v) \qquad (x \in R)$$

is a distribution function of infinite type for all $v \in R_n$, $v^*v > 0$. Thus the representation

$$M_k = \int_{-\infty}^{\infty} x^k dF(x) \qquad (k=0,1,2,\dots)$$

holds, where $F(x)$ $(x \in R)$ is a $p \times p$ matrix-valued distribution function of infinite type. This completes the proof.

6.2. In this section we give a characterization of the Hankelian totally positive sequences.

DEFINITION 6.2.. The sequence $\{M_k\}_0^{\infty}$ of real numbers is said to be Hankelian totally positive (non-negative) if the infinite matrix

$$H = (M_{j+k})_{j,k=0}^{\infty}$$

is totally positive (non-negative).

THEOREM 6.2. *The sequence* $\{M_k\}_0^\infty$ *is Hankelian totally positive if and only if a distribution function* $\alpha(x)$ $(x \in R)$ *of infinite type with* $\alpha(x) = 0$ $(x < 0)$ *exists such that*

$$M_k = \int_0^\infty x^k d\alpha(x) \qquad (k = 0, 1, 2, \dots). \tag{6.2}$$

PROOF. Suppose that the representation (6.2) holds with a distribution function $\alpha(x)$ $(x \geqq 0)$ of infinite type. By Lemma 4.1 $\{x^k\}_0^\infty$ is a Markov sequence of positive type for $x > 0$. Since $\alpha(x)$ is a distribution function of infinite type, we get that $\{M_k\}_0^\infty$ is a Hankelian totally positive sequence by Theorem 4.3.

Now let us suppose that $\{M_k\}_0^\infty$ is a Hankelian totally positive sequence. We show that the representation (6.2) holds in this case with a distribution function $\alpha(x)$ $(x \geqq 0)$ of infinite type. To prove this we use the procedure ([1], pp. 30–32) which can be applied in the proof of the Hamburger moment problem ([1], Theorem 2.1.1).

Starting from the polynomials

$$P_n(x) = \operatorname{Det} \begin{pmatrix} M_0 & M_1 & \dots & M_{n-1} & 1 \\ M_1 & M_2 & \dots & M_n & x \\ \cdot & \cdot & \dots & \cdot & \cdot \\ M_n & M_{n+1} & \dots & M_{2n-1} & x^n \end{pmatrix} \qquad (n = 1, 2, \dots)$$

it can be shown that if $\{M_k\}_0^\infty$ is a Hankelian totally positive sequence, then the roots of these polynomials are positive numbers with multiplicity one. Denote

$$0 < \xi_1^{(n)} < \dots < \xi_n^{(n)} \tag{6.3}$$

the roots of $P_n(x)$ in order of magnitude.

Let us introduce the functional Φ defined on the set of polynomials with real coefficients in the following way. If

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n \qquad (a_n \neq 0),$$

then

$$\Phi(P(x)) = a_0 M_0 + a_1 M_1 + \ldots + a_n M_n.$$

Let

$$A_k^{(m)} = \Phi\left(\frac{P_n(x)}{(x - \xi_k^{(n)}) P'(\xi_k^{(n)})}\right) \qquad (k = 1, \dots, n).$$

These quantities are positive and

$$\sum_{k=1}^n A_k^{(n)} = M_0. \tag{6.4}$$

We now introduce the distribution function $g_n(x)$ of finite type with jumps $A_k^{(n)}$ at $\xi_k^{(n)}$ $(k=1,\dots,n)$, which has correct moments of order $\leqq 2n-1$, i.e.

$$\int_0^\infty x^k dg_n(x) = M_k \qquad (k=0,1,\dots,2n-1).$$

We get that $g_n(x)=0$ $(x \leqq 0)$, $g_n(\infty)=M_0$ by (6.3) and (6.4), i.e. the functions $\{g_n(x)\}_1^\infty$ are uniformly bounded. Thus the elements of this sequence satisfy the conditions of the well-known Helly's theorem, i.e. there exists a non-decreasing function $g(x)$, and a sequence $\{g_{m_k}(x)\}_1^\infty$ such that

$$g(x) = \lim_{k\to\infty} g_{n_k}(x)$$

in all points of continuity of $g(x)$. Finally it can be proved that $\alpha(x)=g(x)$ in the expression (6.2). This completes the proof of Theorem 6.2.

THEOREM 6.3. *The sequence $\{M_k\}_0^\infty$ is Hankelian totally positive if and only if the matrices*

$$(6.5) \qquad (M_{j+k})_{j,k=0}^\infty, \qquad (M_{j+k+1})_{j,k=0}^\infty$$

are positive definite.

A similar result due to Gantmacher and Krein in the case if $\alpha(x)$ is a distribution function of finite type ([4]).

PROOF. The full Stieltjes moment problem ([1], p. 76) says the following. The system of equations

$$M_k = \int_0^\infty x^k d\alpha(x) \qquad (k=0,1,2,\dots)$$

has a distribution function of infinite type solution if and only if the matrices (6.5) are positive definite. Comparing this result of Stieltjes with Theorem 6.2, we get the statement of our Theorem.

THEOREM 6.4. *If $\{M_j\}_0^\infty$ and $\{N_j\}_0^\infty$ are Hankelian totally positive matrices, then so is $\{M_jN_j\}_0^\infty$.*

PROOF. By the assumption the matrices

$$(M_{j+k})_{j,k=0}^\infty, \qquad (M_{j+k+1})_{j,k=0}^\infty,$$

and

$$(N_{j+k})_{j,k=0}^\infty, \qquad (N_{j+k+1})_{j,k=0}^\infty$$

are positive definite by Theorem 6.3. Then the matrices

$$(M_{j+k}N_{j+k})_{j,k=0}^{\infty}, \qquad (M_{j+k+1}N_{j+k+1})_{j,k=0}^{\infty}$$

are also positive definite by the following theorem of I. J. Schur [10]: When the matrices

$$A = (a_{jk})_{j,k=1}^{n}, \qquad B = (b_{jk})_{j,k=1}^{n}$$

are positive definite, then so is their Hadamard product $(a_{jk}b_{jk})_{j,k=1}^{n}$. Using Theorem 6.3 again, we get the statement of our Theorem.

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ON A CLASS OF FIVE-POINT BOUNDARY VALUE PROBLEMS IN SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PARAMETER

S. STANĚK (Olomouc)

1. Introduction

Let $a, b, t_1, t_5 \in R$, $a < b$, $t_1 < t_5$ and let $J = \langle t_1, t_5 \rangle$, $I = \langle a, b \rangle$. Finally, let $X = C^0(J)$ be the Banach space with the norm $\|y\| = \max\{|y(t)|; t \in J\}$. Consider the functional differential equation

$$y''(t) - Q[y, y'](t)y(t) = F[y, y', \mu](t) \tag{1}$$

in which $Q\colon X^2 \to X$, $F\colon X^2 \times I \to X$ are continuous operators, $Q[y, z](t) > 0$ on X^2 for all $t \in J$, depending on the parameter μ.

Let $t_1 < t_2 < t_3 < t_4 < t_5$. The purpose of this paper is to use the Schauder linearization technique and to obtain sufficient conditions on Q, F such that for a suitable value of μ (1) admits a solution y satisfying some of the following boundary conditions:

$$y(t_1) - y(t_2) = 0, \quad y(t_3) = 0, \quad y(t_4) - y(t_5) = 0, \tag{2}$$

$$y'(t_1) = 0, \quad y(t_3) = 0, \quad y'(t_5) = 0, \tag{3}$$

$$y'(t_1) = 0, \quad y(t_3) = 0, \quad y(t_4) - y(t_5) = 0, \tag{4}$$

$$y(t_1) - y(t_2) = 0, \quad y(t_3) = 0, \quad y'(t_5) = 0. \tag{5}$$

A special case of (1) is the differential equation

$$y'' - q(t, y, y')y = f(t, y, y', \mu) \tag{6}$$

in which $q \in C^0(J \times R^2)$, $f \in C^0(J \times R^2 \times I)$, $q(t, y, z) > 0$ for all $(t, y, z) \in$ $\in J \times R^2$. For (6) the question of uniqueness of the boundary value problems is also discussed.

Various k-point boundary value problems for second-order differential equations with $k > 2$ have been studied for example in [1]–[13], and from them linear differential equations have been considered in [4], [6]–[9]. In [4, 6] a suitable implementation of a parameter into the homogeneous equation guarantees the existence of a solution y satisfying $y(t_1) = y(t_2) = y(t_3) = 0$. For functional differential equations depending on a parameter, this problem has been studied in [13]–[15].

2. Notations, lemmas

Let $\varphi \in C^1(J)$ and let u_φ, v_φ be the solutions of the differential equation

$$y'' = Q[\varphi, \varphi'](t)y, \tag{7}$$

$u_\varphi(t_1) = 0$, $u'_\varphi(t_1) = 1$, $v_\varphi(t_1) = 1$, $v'_\varphi(t_1) = 0$. For $(t, s) \in J^2$ and $\varphi \in C^1(J)$ define $r(t, s; \varphi)$, $r'(t, s; \varphi)$ by

$$r(t, s; \varphi) = u_\varphi(t)v_\varphi(s) - u_\varphi(s)v_\varphi(t) \qquad (= -r(s, t; \varphi)),$$

$$r'(t, s; \varphi) = u'_\varphi(t)v_\varphi(s) - u_\varphi(s)v'_\varphi(t) \qquad \left(= \frac{\partial r}{\partial t}(t, s, \varphi)\right)$$

and for $(s, z, t) \in J^3$ and $\varphi \in C^1(J)$ define $k(s, z, t; \varphi)$ by

$$k(s, z, t; \varphi) = r(s, t; \varphi) - r(z, t; \varphi).$$

Then $r(t, s; \varphi) > 0$ for $t_1 \leqq s < t \leqq t_5$, $r'(t, s; \varphi) > 1$ for all $(t, s) \in J^2$, $t \neq s$ and $r'(t, t; \varphi) = 1$ for all $t \in J$ (see [13]).

LEMMA 1. *Let $s, z \in J$, $s > z$ and let $\varphi \in C^1(J)$. Then*

$$k(s, z, t; \varphi) > 0 \quad \textit{for all} \quad t \in J.$$

PROOF. Setting $w(t) = k(s, z, t; \varphi)$ for all $t \in J$, w is a solution of (7), $w(s) = -r(z, s; \varphi) > 0$, $w(z) = r(s, z; \varphi) > 0$, $w'(s) = r'(s, z; \varphi) - 1 > 0$, $w'(z) = 1 - r'(z, s; \varphi) < 0$ and since $Q[\varphi, \varphi'](t) > 0$ on J, one sees $w(t) > 0$ for all $t \in J$.

LEMMA 2. *Assume $\varphi \in C^1(J)$, $h \in C^0(J \times I)$, $h(t, \cdot)$ is increasing on I for any fixed $t \in J$ and*

$$h(t, a) \cdot h(t, b) \leqq 0 \quad \textit{for all} \quad t \in J. \tag{8}$$

Then there is a unique $\mu_0 \in I$ such that the equation

$$y'' - Q[\varphi, \varphi'](t)y = h(t, \mu) \tag{9}$$

with $\mu = \mu_0$ admits a solution y satisfying (2). Moreover this solution y is unique.

PROOF. Setting

$$y(t, \mu) = \frac{r(t_3, t; \varphi)}{k(t_2, t_1, t_3; \varphi)} \left[\int_{t_1}^{t_3} r(t_1, s; \varphi)h(s, \mu)ds - \int_{t_2}^{t_3} r(t_2, s; \varphi)h(s, \mu)ds\right] + \\ + \int_{t_3}^{t} r(t, s; \varphi)h(s, \mu)ds$$

for all $(t,\mu) \in J \times I$, y is a solution of (9), $y(t_1,\mu) - y(t_2,\mu) = 0$, $y(t_3,\mu) = 0$. For the functions $A, B\colon I \to R$ defined by

$$A(\mu) = \int_{t_1}^{t_3} r(t_1,s;\varphi)h(s,\mu)ds - \int_{t_2}^{t_3} r(t_2,s;\varphi)h(s,\mu)ds,$$

$$B(\mu) = \int_{t_3}^{t_5} r(t_5,s;\varphi)h(s,\mu)ds - \int_{t_3}^{t_4} r(t_4,s;\varphi)h(s,\mu)ds,$$

we have

$$A(\mu) = \int_{t_1}^{t_2} r(t_1,s;\varphi)h(s,\mu)ds - \int_{t_2}^{t_3} k(t_2,t_1,s;\varphi)h(s,\mu)ds,$$

$$B(\mu) = \int_{t_3}^{t_4} k(t_5,t_4,s;\varphi)h(s,\mu)ds + \int_{t_4}^{t_5} r(t_5,s;\varphi)h(s,\mu)ds,$$

thus $A(\cdot)$ is decreasing on I and $B(\cdot)$ is increasing on I. Since

$$y(t_5,\mu) - y(t_4,\mu) = -\frac{k(t_5,t_4,t_3;\varphi)}{k(t_2,t_1,t_3;\varphi)}A(\mu) + B(\mu),$$

$y(t_5,\cdot) - y(t_4,\cdot)$ is an increasing function on I and (by (8)) $y(t_5,a) - y(t_4,a) \leqq 0$, $y(t_5,b) - y(t_4,b) \geqq 0$. Consequently there is a unique $\mu_0 \in I$ such that $y(t_5,\mu_0) - y(t_4,\mu_0) = 0$, and (9) admits a solution y satisfying (2) if and only if $\mu = \mu_0$. The uniqueness of y follows from the fact that the homogeneous equation $y'' = Q[\varphi,\varphi'](t)y$ admits only the trivial solution satisfying (2).

Next we shall assume that Q, F and g, f satisfy for positive constants r_0, r_1 the following assumptions:

(i) $|F[y,z,\mu](t)| \leqq r_0 \cdot Q[y,z](t)$ for all $t \in J$ and $[y,z,\mu] \in D \times I$, where $D = \{[y,z]; y,z \in X,\ \|y\| \leqq r_0,\ \|z\| \leqq r_1\}$;

(ii) $F[y,z,\cdot](t)$ is an increasing function on I for any fixed $t \in J$ and $[y,z] \in D$;

(iii) $F[y,z,a](t) \cdot F[y,z,b](t) \leqq 0$ for all $t \in J$ and $[y,z] \in D$;

(iv) $\min\{(A + r_0B)\tau, 2\sqrt{r_0}\sqrt{A + r_0B}\} \leqq r_1$, where $A = \sup\{\|F[y,z,\mu]\|; [y,z,\mu] \in D \times I\}$, $B = \sup\{\|Q[y,z]\|; [y,z] \in D\}$ and $\tau = \max\{t_3 - t_1, t_5 - t_3\}$; and

(j) $|f(t,y,z,\mu)| \leqq r_0 \cdot q(t,y,z)$ for all $(t,y,z,\mu) \in H \times I$, where $H = J \times \langle -r_0, r_0\rangle \times \langle -r_1, r_1\rangle$;

(jj) $f(t,y,z,\cdot)$ is an increasing function on I for every fixed $(t,y,z) \in H$;

(jjj) $f(t,y,z,a) \cdot f(t,y,z,b) \leqq 0$ for all $(t,y,z) \in H$;

(jv) $\min\{(A_1 + r_0B_1)\tau, 2\sqrt{r_0}\sqrt{A_1 + r_0B_1}\} \leqq r_1$, where $A_1 = \sup\{|f(t,y,z,\mu)|; (t,y,z,\mu) \in H \times I\}$, $B_1 = \sup\{q(t,y,z); (t,y,z) \in H\}$ and $\tau = \max\{t_3 - t_1, t_5 - t_3\}$.

LEMMA 3. *Assume that assumptions* (i)–(iv) *are satisfied for positive constants* r_0, r_1. *Then to every* $\varphi \in C^1(J)$, $\|\varphi^{(i)}\| \leqq r_i$, $i = 0,1$, *there is a unique* $\mu_0 \in I$ *such that the equation*

$$y'' - Q[\varphi, \varphi'](t)y = F[\varphi, \varphi', \mu](t) \tag{10}$$

with $\mu = \mu_0$ *admits a solution* y *satisfying* (2). *Moreover this solution* y *is unique and*

$$\|y^{(i)}\| \leqq r_i, \qquad i = 0, 1. \tag{11}$$

PROOF. Let $\varphi \in C^1(J)$, $\|\varphi^{(i)}\| \leqq r_i$ for $i = 0,1$. Setting $h(t,\mu) = F[\varphi, \varphi', \mu](t)$ for all $(t,\mu) \in J \times I$, h fulfils the assumptions of Lemma 2 and consequently, there is a unique $\mu_0 \in I$ such that (10) with $\mu = \mu_0$ admits a (unique) solution y for which (2) holds.

Let $|y(\xi)| \geqq |y(t)|$ for all $t \in J$ and some $\xi \in J$. Since $y(t_1) = y(t_2)$, $y(t_4) = y(t_5)$ we may assume $\xi \in (t_1, t_5)$. If $y(\xi) > r_0$ ($y(\xi) < -r_0$) then (by (i)) $y''(\xi) > 0$ ($y''(\xi) < 0$) contradicts the fact that y has a local maximum (minimum) at the point $t = \xi$. This proves (11) for $i = 0$. As in [13] we may prove (11) for $i = 1$.

3. Existence theorems

THEOREM 1. *Assume that assumptions* (i)–(iv) *are fulfilled with positive constants* r_0, r_1. *Then there is* $\mu_0 \in I$ *such that* (1) *with* $\mu = \mu_0$ *admits a solution* y *satisfying* (2) *and* (11).

PROOF. Let $Y = C^1(J)$ be a Banach space with the norm $\|y\|_1 = \|y\| + \|y'\|$ and let $K = \{y; y \in Y,\ y \text{ satisfies (2) and (11)}\}$. K is a convex bounded closed subset of Y. Let $\varphi \in K$. By Lemma 3 there is a unique $\mu_0 \in I$ such that (10) with $\mu = \mu_0$ admits a (unique) solution y, $y \in K$. Setting $T(\varphi) = y$ we obtain an operator $T\colon K \to K$. To prove Theorem 1 it is sufficient to show that T has a fixed point.

Let $\{y_n\} \subset K$ be a convergent sequence, $\lim\limits_{n\to\infty} y_n = y$, and let $z_n = T(y_n)$, $z = T(y)$. Then there are sequences $\{\mu_n\} \subset I$ and $\mu_0 \in I$ such that we have (see the proof of Lemma 2)

$$z_n(t) = \frac{r(t_3, t; y_n)}{k(t_2, t_1, t_3; y_n)} \left[\int_{t_1}^{t_3} r(t_1, s; y_n) F[y_n, y_n', \mu_n](s)ds - \right.$$

$$\left. - \int_{t_2}^{t_3} r(t_2, s; y_n) F[y_n, y_n', \mu_n](s)ds \right] + \int_{t_3}^{t} r(t, s; y_n) F[y_n, y_n', \mu_n](s)ds,$$

$$z(t) = \frac{r(t_3,t;y)}{k(t_2,t_1,t_3;y)}\Bigg[\int_{t_1}^{t_3} r(t_1,s;y)F[y,y',\mu_0](s)ds-$$

$$-\int_{t_2}^{t_3} r(t_2,s;y)F[y,y',\mu_0](s)ds\Bigg] + \int_{t_3}^{t} r(t,s;y)F[y,y',\mu_0](s)ds$$

for all $t \in J$ and $n \in N$. If $\{\mu_n\}$ is not a convergent sequence, then there are convergent subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$, $\lim\limits_{n\to\infty} \mu_{k_n} = \lambda_1$, $\lim\limits_{n\to\infty} \mu_{r_n} = \lambda_2$, $\lambda_1 < \lambda_2$, and we have

$$\lim_{n\to\infty} z_{k_n}(t) = \frac{r(t_3,t;y)}{k(t_2,t_1,t_3;y)}\Bigg[\int_{t_1}^{t_3} r(t_1,s;y)F[y,y',\lambda_1](s)ds-$$

$$-\int_{t_2}^{t_3} r(t_2,s;y)F[y,y',\lambda_1](s)ds\Bigg] + \int_{t_3}^{t} r(t,s;y)F[y,y',\lambda_1](s)ds,$$

$$\lim_{n\to\infty} z_{r_n}(t) = \frac{r(t_3,t;y)}{k(t_2,t_1,t_3;y)}\Bigg[\int_{t_1}^{t_3} r(t_1,s;y)F[y,y',\lambda_2](s)ds-$$

$$-\int_{t_2}^{t_3} r(t_2,s;y)F[y,y',\lambda_2](s)ds\Bigg] + \int_{t_3}^{t} r(t,s;y)F[y,y',\lambda_2](s)ds$$

uniformly on J. Setting $h(t,\mu) = F[y,y',\mu](t)$ for all $(t,\mu) \in J \times I$ and using the inequality $h(t,\lambda_1) < h(t,\lambda_2)$, one can prove, like in the proof of Lemma 2, that

$$\lim_{n\to\infty} (z_{k_n}(t_5) - z_{r_n}(t_4)) < \lim_{n\to\infty} (z_{r_n}(t_5) - z_{r_n}(t_4))$$

which contradicts $z_n(t_5) - z_n(t_4) = 0$ for all $n \in N$. Consequently, $\{\mu_n\}$ is a convergent sequence, $\lim\limits_{n\to\infty} \mu_n = \mu^*$ and

$$(z^*(t) :=) \lim_{n\to\infty} z_n(t) = \frac{r(t_3,t;y)}{k(t_2,t_1,t_3;y)}\Bigg[\int_{t_1}^{t_3} r(t_1,s;y)F[y,y',\mu^*](s)ds-$$

$$-\int_{t_2}^{t_3} r(t_2,s;y)F[y,y',\mu^*](s)ds\Bigg] + \int_{t_3}^{t} r(t,s;y)F[y,y',\mu^*](s)ds$$

uniformly on J. Hence z^* is a solution of the equation

$$z'' - Q[y,y'](t)z = F[y,y',\mu^*](t),$$

$z^*(t_1)-z^*(t_2)=0$, $z^*(t_3)=0$, $z^*(t_4)-z^*(t_5)=0$ and therefore (by Lemma 3) $\mu^*=\mu_0$, $z^*=z$. Since $\lim_{n\to\infty} z_n'(t)=z'(t)$ uniformly on J, $\lim_{n\to\infty} T(y_n)=T(y)$ and T is a continuous operator.

Let $\varphi\in K$ and $T(\varphi)=y$. Then it follows from the equality $y''(t)=Q[\varphi,\varphi'](t)y(t)+F[\varphi,\varphi',\mu_0](t)$ holding on J for some $\mu_0\in I$ that $\|y''\|\leqq B\|y\|+A\leqq A+r_0B$, consequently, $K\subset\{y;\, y\in C^2(J)\cap K,\ \|y''\|\leqq A+r_0B\}$. Therefore K is a compact subset of Y and by the Schauder fixed point theorem there is a fixed point of T.

THEOREM 2. *Assume that assumptions* (i)–(iv) *are fulfilled with positive constants* r_0, r_1. *Then there are* $\mu_0,\mu_1,\mu_2\in I$ *such that* (1) *with* $\mu=\mu_0$ ($\mu=\mu_1$; $\mu=\mu_2$) *admits a solution* y *satisfying* (3) ((4); (5)) *and* (11).

PROOF. By Theorem 1 there is a sequence $\{\mu_n\}\subset I$ such that (1) with $\mu=\mu_n$ admits a solution y_n,

$$y_n(t_1)-y_n\Big(t_1+\frac{1}{n}\Big)=0,\quad y_n(t_3)=0,\quad y_n\Big(t_5-\frac{1}{n}\Big)-y_n(t_5)=0$$

and $\|y_n^{(i)}\|\leqq r_i$, $\|y_n''\|\leqq A+r_0B$ for all $n\in N$ and $i=0,1$. Using the Ascoli theorem without loss of generality we may assume $\{y_n^{(i)}(t)\}$ are uniformly convergent on J for $i=0,1$ and further $\{\mu_n\}$ is a convergent sequence. Let $\lim_{n\to\infty} y_n(t)=y(t)$ for all $t\in J$ and let $\lim_{n\to\infty}\mu_n=\mu_0$. Then y is a solution of (1) with $\mu=\mu_0$ satisfying (11) and $y(t_3)=0$. Let $y_n'(\xi_n)=y_n'(\eta_n)=0$ for all $n\in N$ with $t_1<\xi_n<t_1+\frac{1}{n}$, $t_5-\frac{1}{n}<\eta_n<t_5$. Then $\lim_{n\to\infty}\xi_n=t_1$, $\lim_{n\to\infty}\eta_n=t_5$ and since $\lim_{n\to\infty}\|y_n'-y'\|=0$ we see $y'(t_1)=y'(t_5)=0$. This proves y is a solution of (1) with $\mu=\mu_0$ satisfying (3) and (11).

Since the proofs of the two remaining cases are very similar to the above one, they are omitted.

EXAMPLE 1. Consider the equation

$$(12)\qquad y''(t)-k\Big(\int_0^1 |y'(s)|ds+\exp(t|y(h_0(t))|)\Big)y(t)=$$

$$=\frac{1}{2}(1+\|y\|^{\nu})\sin(2\pi t)\ln\Big(e+\Big|y'\Big(\int_0^t |y(h_1(s))|ds\Big)\Big|\Big)+\mu\cdot p(t)$$

where $h_0,h_1,p\in C^0(\langle 0,1\rangle)$, $h_0,h_1\colon\langle 0,1\rangle\to\langle 0,1\rangle$, $0<p_1\leqq p(t)\leqq 2p_1$, $\|y\|=\sup\{|y(t)|;\, t\in\langle 0,1\rangle\}$, k, p_1, ν are positive constants, $k\geqq 5e$. Let

$0 = t_1 < t_2 < t_3 < t_4 < t_5 = 1$. One can verify that the assumptions of Theorems 1 and 2 are fulfilled with $r_0 = 1$, $r_1 = 2\sqrt{k}\left(\sqrt{k} + \sqrt{k+1+e}\right)$, $\mu \in \in \left\langle -\frac{1}{p_1}\ln(e + r_1), \frac{1}{p_1}\ln(e + r_1)\right\rangle$. Hence there are $\mu_0, \mu_1, \mu_2, \mu_3 \in \in \left\langle -\frac{1}{p_1}\ln(e + r_1), \frac{1}{p_1}\ln(e + r_1)\right\rangle$ such that (12) with $\mu = \mu_0$ ($\mu = \mu_1$; $\mu = \mu_2$; $\mu = \mu_3$) admits a solution y satisfying (2) ((3); (4); (5)) and $\|y\| \leqq 1$, $\|y'\| \leqq 2\sqrt{k}\left(\sqrt{k} + \sqrt{k+1+e}\right)$.

COROLLARY 1. *Let the assumptions* (i)–(iii) *be fulfilled with positive constants* r_0, r_1 *and let* B *be defined as in* (iv). *If* $r_1 > 2r_0\sqrt{B}$ *then for some positive constant* $\delta > 0$ *we have:*

To every ε, $0 < \varepsilon \leqq \delta$ *there are* $\mu_{\varepsilon 0}, \mu_{\varepsilon 1}, \mu_{\varepsilon 2}, \mu_{\varepsilon 3} \in I$ *such that the equation*

$$y''(t) - Q[y, y'](t)y(t) = \varepsilon \cdot F[y, y', \mu](t)$$

with $\mu = \mu_{\varepsilon 0}$ ($\mu = \mu_{\varepsilon 1}$; $\mu = \mu_{\varepsilon 2}$; $\mu = \mu_{\varepsilon 3}$) *admits a solution* y *satisfying* (2) ((3); (4); (5)) *and* (11).

PROOF. Setting $\delta = \min\left\{\frac{r_0\overline{B}}{A}, \frac{1}{A}\left(\frac{r_1^2}{4r_0} - r_0 B\right)\right\}$, where A is defined as in (iv) and

$$\overline{B} = \inf\{\|Q[y, z]\|; y, z \in X, \|y\| \leqq r_0, \|z\| \leqq r_1\} > 0,$$

$\varepsilon \cdot F$ satisfies the same assumptions as F in Theorems 1 and 2 for all ε, $0 < \varepsilon \leqq \delta$ and therefore Corollary 1 follows immediately from Theorems 1 and 2.

For (6) the following corollary follows from Theorems 1 and 2.

COROLLARY 2. *Let the assumptions* (j)–(jv) *be fulfilled with positive constants* r_0, r_1. *Then there are* $\mu_0, \mu_1, \mu_2, \mu_3 \in I$ *such that* (6) *with* $\mu = \mu_0$ ($\mu = \mu_1$; $\mu = \mu_2$; $\mu = \mu_3$); *admits a solution* y *satisfying* (2) ((3); (4); (5)) *and* (11).

COROLLARY 3. *Let* $p(t, y, z)$, $\frac{\partial p}{\partial y}(t, y, z) \in C^0(J \times R^2)$ *and set*

$$q(t, y, z) = \int_0^1 \frac{\partial p}{\partial y}(t, \tau y, z)d\tau$$

for all $(t, y, z) \in J \times R^2$. *If the assumptions* (j)–(jv) *are fulfilled with positive constants* r_0, r_1 *then there are* $\mu_0, \mu_1, \mu_2, \mu_3 \in I$ *such that the equation*

$$y'' = p(t, y, y') - p(t, 0, y') + f(t, y, y', \mu)$$

with $\mu = \mu_0$ ($\mu = \mu_1$; $\mu = \mu_2$; $\mu = \mu_3$) *admits a solution* y *satisfying* (2) ((3); (4); (5)) *and* (11).

PROOF. Since

$$p(t,y,z) = p(t,0,z) + y\int_0^1 \frac{\partial p}{\partial y}(t,\tau y,z)d\tau = p(t,0,z) + q(t,y,z)y$$

for all $(t,y,z) \in J \times R^2$, Corollary 3 follows immediately from Corollary 2.

4. Uniqueness theorem for equation (6)

THEOREM 3. *Assume that conditions* (j)–(jv) *are satisfied with positive constants* r_0, r_1. *If* $\frac{\partial q}{\partial y}, \frac{\partial q}{\partial z} \in C^0(H)$, $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \in C^0(H \times I)$ *and*

$$q(t,y_1,z) + y_2\frac{\partial q}{\partial y}(t,\xi_1,z) + \frac{\partial f}{\partial y}(t,\xi_2,z,\mu) \geqq 0 \tag{13}$$

for all $\xi_i \in \langle \min\{y_1,y_2\}, \max\{y_1,y_2\}\rangle$, $i = 1,2$, *then there is a unique* $\mu_0 \in \in I$ *such that* (6) *with* $\mu = \mu_0$ *admits a solution* y *satisfying* (2) *and* (11). *Moreover this solution* y *is unique in the set* $S = \{y; y \in C^2(J), \|y^{(i)}\| \leqq r_{i+1}, i = 0,1\}$.

The theorem is true also when the boundary condition (2) is replaced by some of the boundary conditions (3), (4) and (5).

PROOF. By Corollary 2 there is $\mu_0 \in I$ such that (6) with $\mu = \mu_0$ admits a solution $y \in S$ satisfying (2). Suppose there is $\mu_1 \in I$, $\mu_0 \leqq \mu_1$, such that (6) with $\mu = \mu_1$ admits a solution $y_1 \in S$ satisfying (2) where in place of y we have y_1. Setting $w = y - y_1$ then $w(t_1) - w(t_2) = 0$, $w(t_3) = 0$, $w(t_4) - w(t_5) = 0$, consequently, $w'(\tau_1) = w'(\tau_2) = 0$ for $t_1 < \tau_1 < t_2$, $t_4 < \tau_2 < t_5$. Next we have

$$w''(t) = a(t)w(t) + b(t)w'(t) + c(t) \quad \text{for all} \quad t \in J,$$

where $a, b, c \in C^0(J)$, $a(t) \geqq 0$ on J (by (13)) and if $\mu_0 < \mu_1$ ($\mu_0 = \mu_1$) then $c(t) < 0$ ($c(t) = 0$) for all $t \in J$. Setting $p(t) = \exp\left(-\int_{t_3}^{t} b(s)ds\right)$, $k(t) = p(t)w(t)w'(t)$ for all $t \in J$ (see [16]) we see $k(t_3) = 0$, $k' = p(w'^2 + aw^2 + cw)$. Let $w \neq 0$.

If $c = 0$, $k' = p(w'^2 + aw^2) \geqq 0$ and therefore $k(t) > 0$ for all $t \in (t_3, t_5\rangle$ which contradicts $k(\tau_2) = 0$.

Let $c(t) < 0$ on J, that is $\mu_0 < \mu_1$. If $w'(t_3) = 0$ then $w''(t_3) = c(t_3) < 0$ and therefore $w(t) < 0$ in a set $(t_3 - \varepsilon, t_3) \cup (t_3, t_3 + \varepsilon) \subset J$ with a positive

$\varepsilon > 0$. Consequently, we may always suppose that either $w(t) < 0$, $w'(t) \neq 0$ for all $t \in (t_3, \xi)$ and $k(\xi) = 0$ or $w(t) < 0$, $w'(t) \neq 0$ for all $t \in (\eta, t_3)$ and $k(\eta) = 0$. In the first case we have $k'(t) > 0$ on $\langle t_3, \xi\rangle$ which contradicts $k(\xi) = 0$, in the second case $k'(t) > 0$ on $\langle \eta, t_3\rangle$ which contradicts $k(\eta) = 0$.

Hence $w = 0$ and the theorem is proved.

If the boundary condition (2) is replaced by some of the boundary conditions (3), (4), (5), the proof is very similar and therefore it is omitted.

EXAMPLE 2. Let k, p_1, g_1, g_2 be positive constants, $k \geqq (1+p_1)\left(1+\frac{g_2}{g_1}\right)$, $g_1 \leqq g_2$ and let n be a positive integer. Consider the equation

$$(14) \qquad y'' - ke^t\left(1 + (\operatorname{arctg} y')^2\right) y = ty^{2n+1}\cos^2(y') + p(t) + \mu \cdot g(t),$$

where $p, g \in C^0(\langle 0,1\rangle)$, $|p(t)| \leqq p_1$, $g_1 \leqq g(t) \leqq g_2$ for all $t \in \langle 0,1\rangle$. Let $0 = t_1 < t_2 < t_3 < t_4 < t_5 = 1$. The assumptions of Theorem 3 are fulfilled for $r_0 = 1$, $r_1 \geqq ke\left(1 + \left(\frac{\pi}{4}\right)^2\right) + (1+p_1)\left(1 + \frac{g_2}{g_1}\right)$ and $\mu \in \left\langle -\frac{1+p_1}{g_1}, \frac{1+p_1}{g_1}\right\rangle$, consequently, there are unique $\mu_0, \mu_1, \mu_2, \mu_3 \in \left\langle -\frac{1+p_1}{g_1}, \frac{1+p_1}{g_1}\right\rangle$ such that (14) with $\mu = \mu_0$ ($\mu = \mu_1$; $\mu = \mu_2$; $\mu = \mu_3$) admits a solution y satisfying (2) ((3); (4); (5)) and $\|y\| \leqq 1$, $\|y'\| \leqq ke\left(1 + \left(\frac{\pi}{2}\right)^2\right) + (1+p_1)\left(1+\frac{g_2}{g_1}\right)$. Moreover this solution y is unique even in the set $S = \{y; y \in C^2(\langle 0,1\rangle), \|y\| \leqq 1\}$.

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known to be poised for any given data. Special cases of this problem, namely

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

have earlier been considered in [7] and [4] respectively.

In Section 2 we shall find an explicit solution of the above problem. We shall study the convergence problem in the case when the knots are equi-spaced. Section 3 deals with estimates needed for the purpose and Section 4 treats the proofs of our convergence results.

2. Existence and uniqueness

(a) *The fundamental polynomials.* We shall first consider the polynomial interpolation problem with respect to E on the points $0, \frac{1}{2}, 1$. Let $A_{n,k}(x)$, $B_{n,k}(x)$, $k = 0, \dots, n-1$, $C_n(x)$ and $D_n(x)$ denote the fundamental polynomials of interpolation at the points $0, \frac{1}{2}, 1$ with respect to the matrix E. In other words, we have

$$(2.1) \quad \begin{cases} A_{n,k}^{(j)}(0) = \delta_{j,k}, \ A_{n,k}^{(j)}(1) = 0, \ A_{n,k}^{(n+1)}(0) = 0, \ A_{n,k}^{(n+1)}\left(\frac{1}{2}\right) = 0, \\ B_{n,k}^{(j)}(0) = 0, \ B_{n,k}^{(j)}(1) = \delta_{j,k}, \ B_{n,k}^{(n+1)}(0) = 0, \ B_{n,k}^{(n+1)}\left(\frac{1}{2}\right) = 0, \\ C_n^{(j)}(0) = 0, \ C_n^{(j)}(1) = 0, \ C_n^{(n+1)}(0) = 1, \ C_n^{(n+1)}\left(\frac{1}{2}\right) = 0, \\ D_n^{(j)}(0) = 0, \ D_n^{(j)}(1) = 0, \ D_n^{(n+1)}(0) = 0, \ D_n^{(n+1)}\left(\frac{1}{2}\right) = 1; \end{cases}$$

where $k, j = 0, \dots, n-1$.

These polynomials can be easily obtained from the fundamental polynomials of two point Hermite interpolation at 0 and 1:

$$\begin{matrix} 0 \\ 1 \end{matrix} \begin{pmatrix} 1 & 1 & \dots & 1 & \\ & & & & 0 \\ 1 & 1 & \dots & 1 & \end{pmatrix},$$

where each row consists of n ones. Denote these polynomials by $M_{n-1,k}(x)$, determined by

$$(2.2) \quad \begin{cases} M_{n-1,k}^{(j)}(0) = \delta_{k,j}, & j, k = 0, \dots, n-1 \\ M_{n-1,k}^{(j)}(1) = 0, & j, k = 0, \dots, n-1. \end{cases}$$

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ODD DEGREE SPLINES OF HIGHER ORDER

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1. Introduction

In this paper we consider a special problem of Birkhoff interpolation by odd degree splines. It includes as particular cases the earlier results of Guo Zhu-rui [7] and the second author [4].

Let

$$\Delta\colon 0 = x_0 < x_1 < \ldots < x_N = 1$$

be an arbitrary partition of the interval $I = [0,1]$ and let $h_i = x_{i+1} - x_i$, $i = 0, \ldots, N-1$ and $f(x) \in C^{n+1}(I)$. We consider a sort of interpolating spline $s(x)$ satisfying the following conditions:

(i) $s(x) \in C^n(I)$,

(ii) $s(x) \in \Pi_{2n+1}$ in each interval $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$,

(iii) $s^{(k)}(x_i) = f^{(k)}(x_i)$, $i = 0, \ldots, N$, $k = 0, \ldots, n-1$,

(iv) $s^{(n+1)}\left(\frac{x_i + x_{i+1}}{2}\right) = f^{(n+1)}\left(\frac{x_i + x_{i+1}}{2}\right)$, $i = 0, \ldots, N-1$,

(v) $s^{(n)}(0) = f^{(n)}(0)$ or (v′) $s^{(n+1)}(0) = f^{(n+1)}(0)$.

We are interested in the following

PROBLEM. Determine $s(x)$ satisfying the conditions (i)–(v) or (i)–(v′).

The problem is related to Birkhoff interpolation by algebraic polynomials concerning the interpolation matrix

$$(1.1) \qquad E = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 1 & \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 1 & 1 & \ldots & 1 & 0 & 0 & \end{pmatrix},$$

where the first and third rows have Hermite sequences of length n, and two ones, in row one and row two, occur in the $(n+2)$th column. This problem is

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It is easy to see that (cf. [2] Example 3, p. 37; also [1] formula (4.1.20), p. 118)

$$M_{n-1,k}(x) = (1-x)^n \frac{x^k}{k!} \sum_{j=0}^{n-k-1} \binom{n+j-1}{j} x^j. \tag{2.3}$$

We now set

$$\begin{cases} A_{n,k}(x) = (a_{n,k}x + b_{n,k})x^n(1-x)^n + M_{n-1,k}(x), \\ B_{n,k}(x) = (c_{n,k}x + d_{n,k})x^n(1-x)^n + M_{n-1,k}(1-x), \\ C_n(x) = (a_n x + b_n)x^n(1-x)^n, \\ D_n(x) = (c_n x + d_n)x^n(1-x)^n; \end{cases} \tag{2.4}$$

where $a_{n,k}$ and $b_{n,k}$ are determined by the conditions

$$A_{n,k}^{(n+1)}(0) = 0, \quad A_{n,k}^{(n+1)}\left(\frac{1}{2}\right) = 0,$$

and $c_{n,k}$ and $d_{n,k}$ are determined by the conditions

$$B_{n,k}^{(n+1)}(0) = 0, \quad B_{n,k}^{(n+1)}\left(\frac{1}{2}\right) = 0.$$

Also,

$$C_n(x) = \begin{cases} \dfrac{x^n(1-x)^n(2x-1)}{(n+2)!}, & \text{if } n \text{ is odd} \\ -\dfrac{1}{n(n+1)!}x^n(1-x)^n, & \text{if } n \text{ is even,} \end{cases}$$

and

$$D_n(x) = \begin{cases} \dfrac{2}{(n+2)D^{n+1}[x^n(1-x)^n]_{\frac{1}{2}}}x^n(1-x)^n(nx+1), & \text{if } n \text{ is odd} \\ \dfrac{1}{n(n+1)D^n[x^n(1-x)^n]_{\frac{1}{2}}}x^n(1-x)^n(nx+1), & \text{if } n \text{ is even.} \end{cases}$$

For later references we note that

$$C_n^{(n)}(0) = \begin{cases} -\frac{1}{n(n+1)} = C_n^{(n)}(1), & \text{if } n \text{ is even} \\ -\frac{1}{(n+1)(n+2)} = C_n^{(n)}(1), & \text{if } n \text{ is odd.} \end{cases}$$

In fact, $C_n^{(n)}(0) = C_n^{(n)}(1)$, either n is even or odd. Any polynomial $p(x)$ of degree at most $2n+1$ can be written as

$$p(x) = \sum_{k=0}^{n-1} [p^{(k)}(0)A_{n,k}(x) + p^{(k)}(1)B_{n,k}(x)] + \\ + p^{(n+1)}(0)C_n(x) + p^{(n+1)}\left(\frac{1}{2}\right)D_n(x). \tag{2.5}$$

(b) *Solution of the problem.* We turn to the solution of the above problem. Denote $s^{(n+1)}(x_i)$ by $s_i^{(n+1)}$ $(i = 0, \dots, N)$ and let $x_i \leqq x \leqq x_{i+1}$. Then by scaling the fundamental polynomials of (a) above, we have

$$(2.6)\qquad s(x) = \sum_{k=0}^{n-1} h_i^k \left[f_i^{(k)} A_{n,k}\left(\frac{x - x_i}{h_i}\right)\right) + f_{i+1}^{(k)} B_{n,k}\left(\frac{x - x_i}{h_i}\right)\Big] +$$
$$+h_i^{n+1}\left[s_i^{(n+1)} C_n\left(\frac{x - x_i}{h_i}\right) + f_{i+\frac{1}{2}}^{(n+1)} D_n\left(\frac{x - x_i}{h_i}\right)\right],$$

where we write

$$f_{i+\frac{1}{2}}^{(n+1)} = f^{(n+1)}\left(\frac{x_i + x_{i+1}}{2}\right).$$

Since $s(x) \in C^n[0, 1]$,

$$s^{(n)}(x_i+) = s^{(n)}(x_i-), \qquad i = 1, \dots, N - 1.$$

Hence from (2.6), after some simplifications, we get the following system of equations for $s_i^{(n+1)}$, $i = 1, \dots, N - 1$;

$$(2.7)\qquad h_i s_i^{(n+1)} C_n^{(n)}(0) - h_{i-1} s_{i-1}^{(n+1)} C_n^{(n)}(1) = \Delta_i,$$

where

$$\Delta_i = \sum_{k=0}^{n-1} \left[h_{i-1}^{k-n}(f_{i-1}^{(k)} A_{n,k}^{(n)}(1) + f_i^{(k)} B_{n,k}^{(n)}(1)) - h_i^{k-n}(f_i^{(k)} A_{n,k}^{(n)}(0) + f_{i+1}^{(k)} B_{n,k}^{(n)}(0)\right] +$$
$$+h_{i-1} f_{i-\frac{1}{2}}^{(n+1)} D_n^{(n)}(1) - h_i f_{i+\frac{1}{2}}^{(n+1)} D_n^{(n)}(0).$$

Writing $M_i = s^{(n+1)}(x_i)$, $i = 0, \dots, N-1$ and observing that $C_n^{(n)}(0) = C_n^{(n)}(1)$, we have

$$(2.8)\qquad h_i M_i - h_{i-1} M_{i-1} = \frac{1}{C_n^{(n)}(0)} \Delta_i, \qquad i = 1, \dots, N - 1.$$

From (2.6) and (v)

$$(2.9)\quad f^{(n)}(0) = \sum_{k=0}^{n-1} h_0^{k-n} \left[f_0^{(k)} A_{n,k}^{(n)}(0) + f_1^{(k)} B_{n,k}^{(n)}(0)\right] + h_0 C_n^{(n)}(0) M_0 +$$
$$+h_0 D_n^{(n)}(0) f^{(n+1)}\left(\frac{x_0 + x_1}{2}\right)$$

and from (v′)

$$(2.9')\qquad M_0 = f^{(n+1)}(0).$$

From (2.8), (2.9) and (2.8), (2.9′) we find that the interpolation problems (iii)–(v) and (iii)–(v′) have unique solutions. Thus we have proved

THEOREM 1. *Given a partition of the unit interval I with $h_i = x_{i+1} - x_i$, $i = 0, \dots, N-1$, there exists a unique spline $s(x)$ defined by* (i)–(ii) *which satisfies the conditions* (iii)–(v) *or* (iii)–(v′).

3. Error bounds (knots equispaced)

We shall now take the knots to be equispaced so that $h_i = \frac{1}{N} = h$. Let $f \in C^{(n+1)}(I)$. Set

$$f_i^{(n+1)} = f^{(n+1)}(x_i) \quad \text{and} \quad s_i^{(n+1)} = s^{(n+1)}(x_i).$$

We shall prove

THEOREM 2. *Let $s(x)$ be the spline of Theorem 1 when the knots are equispaced. Then we have*

(3.1)
$$\|s^{(n+1)} - f^{(n+1)}\| = O(h^{\ell-n-2}\omega_\ell(h)), \quad \text{if } f \in C^\ell(I),\ \ell = n+2, \dots, 2n+1,$$

(3.2)
$$\|s^{(n+1)} - f^{(n+1)}\| = Ch^{n+1}\|f^{(2n+2)}\| + O(h^{n+1}\omega_{2n+2}(h)), \quad \text{if } f \in C^{2n+2}(I).$$

Here, as usual $\omega_\ell(f, \delta)$ denotes the modulus of continuity of $f^{(\ell)}$ and $\|\cdot\|$ is the uniform norm.

REMARK. The O depends only on n and is independent of N.

By successive integration we have the following

COROLLARY. *Under the conditions of Theorem 2, we have for $r = 0, \dots, n+1$,*

$$\|s^{(r)} - f^{(r)}\| = O(h^{\ell-r-1}\omega_\ell(h)),$$

and

$$\|s^{(r)} - f^{(r)}\| = Ch^r\|f^{(2n+2)}\| + O(h^r\omega_{2n+2}(h)).$$

For the proof of the above theorem we need some auxiliary results which we obtain in the following. At the very outset we see from (2.5) that the following identities are true.

$$\frac{x^k}{k!} = A_{n,k}(x) + \sum_{j=0}^{k} \frac{B_{n,j}(x)}{(k-j)!}, \qquad k = 0, \dots, n-1, \tag{3.3}$$

$$\frac{x^n}{n!} = \sum_{j=0}^{n-1} \frac{B_{n,j}(x)}{(n-j)!}, \tag{3.4}$$

(3.5)
$$\frac{x^{n+1}}{(n+1)!}=\sum_{j=0}^{n-1}\frac{B_{n,j}(x)}{(n+1-j)!}+C_n(x)+D_n(x),$$

(3.6)
$$\frac{x^k}{k!}=\sum_{j=0}^{n-1}\frac{B_{n,j}(x)}{(k-j)!}+\frac{1}{(k-n-1)!}\frac{D_n(x)}{2^{k-n-1}},\quad k=n+2,\dots,2n+1;$$

(3.7)
$$\frac{(1-x)^k}{k!}=\sum_{j=0}^{k}\frac{(-1)^j}{(k-j)!}A_{n,j}(x)+(-1)^k B_{n,k}(x),\quad k=0,\dots,n-1,$$

(3.8)
$$\frac{(1-x)^n}{n!}=\sum_{j=0}^{n-1}\frac{(-1)^j}{(n-j)!}A_{n,j}(x),$$

(3.9)
$$\frac{(1-x)^{n+1}}{(n+1)!}=\sum_{j=0}^{n-1}\frac{(-1)^j}{(n+1-j)!}A_{n,j}(x)+(-1)^{n+1}[C_n(x)+D_n(x)],$$

(3.10)
$$\frac{(1-x)^k}{k!}=\sum_{j=0}^{n-1}\frac{(-1)^j}{(k-j)!}A_{n,j}(x)+\frac{(-1)^{n+1}}{(k-n-1)!}\left[C_n(x)+\frac{D_n(x)}{2^{k-n-1}}\right],\ k=n+2,\dots,2n+1.$$

From these equations, on differentiation, we have

(3.3′)
$$0=A_{n,k}^{(n)}(x)+\sum_{j=0}^{k}\frac{B_{n,j}^{(n)}(x)}{(k-j)!},\quad k=0,\dots,n-1,$$

(3.4′)
$$1=\sum_{j=0}^{n-1}\frac{B_{n,j}^{(n)}(x)}{(n-j)!},$$

(3.5′)
$$x=\sum_{j=0}^{n-1}\frac{B_{n,j}^{(n)}(x)}{(n+1-j)!}+C_n^{(n)}(x)+D_n^{(n)}(x),$$

(3.6′)
$$\frac{x^{k-n}}{(k-n)!}=\sum_{j=0}^{n-1}\frac{B_{n,j}^{(n)}(x)}{(k-j)!}+\frac{D_n^{(n)}(x)}{(k-n-1)!2^{k-n-1}},\quad k=n+2,\dots,2n+1;$$

$$(3.7') \qquad 0=\sum_{j=0}^{k}\frac{(-1)^j}{(k-j)!}A_{n,j}^{(n)}(x)+(-1)^k B_{n,k}^{(n)}(x), \quad k=0,\dots,n-1,$$

$$(3.8') \qquad (-1)^n=\sum_{j=0}^{n-1}\frac{(-1)^j}{(n-j)!}A_{n,j}^{(n)}(x),$$

$$(3.9') \quad (-1)^n(1-x)=\sum_{j=0}^{n-1}\frac{(-1)^j}{(n+1-j)!}A_{n,j}^{(n)}(x)+(-1)^{n+1}[C_n^{(n)}(x)+D_n^{(n)}(x)],$$

$$(3.10') \qquad (-1)^n\frac{(1-x)^{k-n}}{(k-n)!}=\sum_{j=0}^{n-1}\frac{(-1)^j}{(k-j)!}A_{n,j}^{(n)}(x)+$$

$$+\frac{(-1)^{n+1}}{(k-n-1)!}\left[C_n^{(n)}(x)+\frac{D_n^{(n)}(x)}{2^{k-n-1}}\right], \qquad k=n+2,\dots,2n+1.$$

Next we see that in case of equidistant knots the system (2.7) is reduced to

$$s_i^{(n+1)}C_n^{(n)}(0)-s_{i-1}^{(n+1)}C_n^{(n)}(1)=\sum_{k=0}^{n-1}h^{k-n-1}\left[(f_{i-1}^{(k)}A_{n,k}^{(n)}(1)+f_i^{(k)}B_{n,k}^{(n)}(1))-\right.$$

$$\left.-(f_i^{(k)}A_{n,k}^{(n)}(0)+f_{i+1}^{(k)}B_{n,k}^{(n)}(0))\right]+f_{i-\frac{1}{2}}^{(n+1)}D_n^{(n)}(1)-f_{i+\frac{1}{2}}^{(n+1)}D_n^{(n)}(0).$$

Putting $N_i=s_i^{(n+1)}-f_i^{(n+1)}$, we have

$$N_iC_n^{(n)}(0)-N_{i-1}C_n^{(n)}(1)=\bar{\Delta}_i,$$

where

$$\bar{\Delta}_i=\sum_{k=0}^{n-1}h^{k-n-1}\left[f_{i-1}^{(k)}A_{n,k}^{(n)}(1)+f_i^{(k)}B_{n,k}^{(n)}(1)\right]+f_{i-1}^{(n+1)}C_n^{(n)}(1)+$$

$$+f_{i-\frac{1}{2}}^{(n+1)}D_n^{(n)}(1)-\sum_{k=0}^{n-1}h^{k-n-1}\left[f_i^{(k)}A_{n,k}^{(n)}((0)+f_{i+1}^{(k)}B_{n,k}^{(n)}(0)\right]-$$

$$-f_i^{(n+1)}C_n^{(n)}(0)-f_{i+\frac{1}{2}}^{(n+1)}D_n^{(n)}(0).$$

We prove the following lemmas.

LEMMA 3.1. *If $f \in C^{\ell}(I)$, then for $i = 0, \dots, N-1$,*
(i) $\|N_i\| = O(h^{\ell-n-2}\omega_\ell(h)), \quad \ell = n+2, \dots, 2n+1.$
Also for $f \in C^{2n+2}(I)$,
(ii) $\|N_i\| = Ch^n \|f^{(2n+2)}\| + O(h^n \omega_{2n+2}(h)).$

PROOF OF (i). Let $i = 1, \dots, N-1$. Then on writing finite Taylor sums for $f_{i-1}^{(k)}$, $f_{i+1}^{(k)}$, $f_{i+\frac{1}{2}}^{(n+1)}$ and $f_{i-\frac{1}{2}}^{(n+1)}$ about x_i and simplifying, we have

$$\bar{\Delta}_i = \sum\nolimits_1 - \sum\nolimits_2,$$

where

$$\sum\nolimits_1 = \sum_{k=0}^{n-1} h^{k-n-1}\left[f_{i-1}^{(k)} A_{n,k}^{(n)}(1) + f_i^{(k)} B_{n,k}^{(n)}(1)\right] + f_{i-1}^{(n+1)} C_n^{(n)}(1) + f_{i-\frac{1}{2}}^{(n+1)} D_n^{(n)}(1) =$$

$$= \sum_{k=0}^{n-1}\left[h^{k-n-1} f_i^{(k)} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} A_{n,j}^{(n)}(1)\right] + \sum_{k=n}^{\ell}\left[h^{k-n-1} f_i^{(k)} \sum_{j=0}^{n-1} \frac{(-1)^{k-j}}{(k-j)!} A_{n,j}^{(n)}(1)\right] +$$

$$+ O(h^{\ell-n-1}\omega_\ell(h)) + \sum_{k=0}^{n-1} h^{k-n-1} f_i^{(k)} B_{n,k}^{(n)}(1) +$$

$$+ \sum_{k=n+1}^{\ell} \frac{(-h)^{k-n-1}}{(k-n-1)!} f_i^{(k)} \left[C_n^{(n)}(1) + \frac{D_n^{(n)}(1)}{2^{k-n-1}}\right] + O(h^{\ell-n-1}\omega_\ell(h)) =$$

$$= \sum_{k=0}^{n-1} h^{k-n-1} f_i^{(k)} \left[\sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} A_{n,j}^{(n)}(1) + B_{n,k}^{(n)}(1)\right] +$$

$$+ h^{-1} f_i^{(n)} \sum_{j=0}^{n-1} \frac{(-1)^{n-j}}{(n-j)!} A_{n,j}^{(n)}(1) +$$

$$+ f_i^{(n+1)} \left[\sum_{j=0}^{n-1} \frac{(-1)^{n+1-j}}{(n+1-j)!} C_n^{(n)}(1) + D_n^{(n)}(1)\right] + \sum_{k=n+2}^{\ell} h^{k-n-1} f_i^{(k)}.$$

$$\left[\sum_{j=0}^{n-1} \frac{(-1)^{k-j}}{(k-j)!} A_{n,j}^{(n)}(1) + \frac{(-1)^{k-n-1}}{(k-n-1)!} C_n^{(n)}(1) + \frac{(-1)^{k-n-1}}{(k-n-1)!} \frac{D_n^{(n)}(1)}{2^{k-n-1}}\right] +$$

$$+ O(h^{\ell-n-1}\omega_\ell(h)) =$$

$$= h^{-1} f_i^{(n)} + O(h^{\ell-n-1}\omega_\ell(h)), \qquad \ell = n+2, \dots, 2n+1;$$

and

$$\sum\nolimits_2 = \sum_{k=0}^{n-1} h^{k-n-1}\left[f_i^{(k)}A_{n,k}^{(n)}(0) + f_{i+1}^{(k)}B_{n,k}^{(n)}(0)\right] + f_i^{(n+1)}C_n^{(n)}(0) + f_{i+\frac{1}{2}}^{(n+1)}D_n^{(n)}(0) =$$

$$= \sum_{k=0}^{n-1} h^{k-n-1} f_i^{(k)} A_{n,k}^{(n)}(0) + \sum_{k=0}^{n-1}\left[h^{k-n-1} f_i^{(k)} \sum_{j=0}^{k}\frac{1}{(k-j)!}B_{n,j}^{(n)}(0)\right] +$$

$$+ \sum_{k=n}^{\ell}\left[h^{k-n-1} f_i^{(k)} \sum_{j=0}^{n-1}\frac{1}{(k-j)!}B_{n,j}^{(n)}(0)\right] + f_i^{(n+1)}[C_n^{(n)}(0) + D_n^{(n)}(0)] +$$

$$+ \sum_{k=n+2}^{\ell} \frac{h^{k-n-1}}{(k-n-1)!2^{k-n-1}} f_i^{(k)} D_n^{(n)}(0) + O(h^{\ell-n-1}\omega_\ell(h)) =$$

$$= \sum_{k=0}^{n-1} h^{k-n-1} f_i^{(k)}\left[A_{n,k}^{(n)}(0) + \sum_{j=0}^{k}\frac{1}{(k-j)!}B_{n,j}^{(n)}(0)\right] +$$

$$+ h^{-1} f_i^{(n)} \sum_{j=0}^{n-1}\frac{1}{(n-j)!}B_{n,j}^{(n)}(0) +$$

$$+ f_i^{(n+1)}\left[\sum_{j=0}^{n-1}\frac{1}{(n+1-j)!}B_{n,j}^{(n)}(0) + C_n^{(n)}(0) + D_n^{(n)}(0)\right] +$$

$$+ \sum_{k=n+2}^{\ell} h^{k-n-1} f_i^{(k)} \cdot \left[\sum_{j=0}^{n-1}\frac{B_{n,j}^{(n)}(0)}{(k-j)!} + \frac{1}{(k-n-1)!}\frac{D_n^{(n)}(0)}{2^{k-n-1}}\right] + O(h^{\ell-n-1}\omega_\ell(h)) =$$

$$= h^{-1} f_i^{(n)} + O(h^{\ell-n-1}\omega_\ell(h)).$$

Hence

$$\bar{\Delta}_i = O(h^{\ell-n-1}\omega_\ell(h)), \qquad \ell = n+2, \dots, 2n+1.$$

Also from (2.9)

$$hC_n^{(n)}(0)[M_0 - f_0^{(n+1)}] = f^{(n)}(0) - f^{(n+1)}(0)hC_n^{(n)}(0) -$$

$$-hD_n^{(n)}(0) f^{(n+1)}\left(\frac{x_0+x_1}{2}\right) - \sum_{k=0}^{n-1} h^{k-n}\left[f_0^{(k)}A_{n,k}^{(n)}(0) + f_1^{(k)}B_{n,k}^{(n)}(0)\right] = \bar{\Delta}_0,$$

where

$$\bar{\Delta}_0 = f_0^{(n)}\left[1 - \sum_{j=0}^{n-1}\frac{B_{n,j}^{(n)}(0)}{(n-j)!}\right] - hf_0^{(n+1)}\left[\sum_{j=0}^{n-1}\frac{B_{n,j}^{(n)}(0)}{(n+1-j)!} + C_n^{(n)}(0) + D_n^{(n)}(0)\right] -$$

$$-\sum_{k=0}^{n-1} h^{k-n} f_0^{(k)} \left[A_{n,k}^{(n)}(0) + \sum_{j=0}^{k} \frac{B_{n,j}^{(n)}(0)}{(k-j)!} \right] -$$

$$-\sum_{k=n+2}^{\ell} h^{k-n} f_0^{(k)} \left[\sum_{j=0}^{n-1} \frac{B_{n,j}^{(n)}(0)}{(k-j)!} + \frac{1}{(k-n-1)!} \frac{D_n^{(n)}(0)}{2^{k-n-1}} \right] + O(h^{\ell-n}\omega_\ell(h)) =$$

$$= O(h^{\ell-n}\omega_\ell(h)), \quad \ell = n+2, \dots, 2n+1.$$

Hence for $i = 1, \dots, N-1$

$$N_i - N_{i-1} = \frac{1}{C_n^{(n)}(0)} \bar{\Delta}_i = O(h^{\ell-n-1}\omega_\ell(h))$$

and

$$N_0 = \frac{1}{hC_n^{(n)}(0)} \bar{\Delta}_0 = O(h^{\ell-n-1}\omega_\ell(h)), \qquad \ell = n+2, \dots, 2n+1,$$

from which we obtain

$$N_j = O(h^{\ell-n-2}\omega_\ell(h)), \qquad j = 0, \dots, N-1, \qquad \ell = n+2, \dots, 2n+1.$$

PROOF OF (ii). Going through the argument of part (i) we have for $\ell = 2n+2$,

$$\bar{\Delta}_i = h^{n+1} f_i^{(2n+2)} \left[\sum_{j=0}^{n-1} \frac{(-1)^j}{(2n+2-j)!} A_{n,j}^{(n)}(1) + \frac{(-1)^{n+1}}{(n+1)!} \left(C_n^{(n)}(1) + \frac{D_n^{(n)}(1)}{2^{n+1}} \right) \right] -$$

$$-h^{n+1} f_i^{(2n+2)} \left[\sum_{j=0}^{n-1} \frac{1}{(2n+2-j)!} B_{n,j}^{(n)}(0) + \frac{D_n^{(n)}(0)}{2^{n+1}(n+1)!} \right] + O(h^{n+1}\omega_{2n+2}(h)) =$$

$$= h^{n+1} f_i^{(2n+2)} \left[\sum_{j=0}^{n-1} \frac{1}{(2n+2-j)!} ((-1)^j A_{n,j}^{(n)}(1) - B_{n,j}^{(n)}(0)) + \right.$$

$$\left. + \frac{1}{(n+1)!} ((-1)^{n+1} C_n^{(n)}(1) + \frac{1}{2^{n+1}} ((-1)^{n+1} D_n^{(n)}(1) - D_n^{(n)}(0)) \right] +$$

$$+ O(h^{n+1}\omega_{2n+2}(h))$$

and

$$\bar{\Delta}_0 = Ch^{n+2} \| f^{(2n+2)} \| + O(h^{n+2}\omega_{2n+2}(h)).$$

This gives

$$N_j = Ch^n \| f^{(2n+2)} \| + O(h^n \omega_{2n+2}(h)),$$

which proves Lemma 3.1.

LEMMA 3.2. *We have*

(i)
$$\sum_{k=0}^{n-1} h^{k-n-1}\left[f_i^{(k)} A_{n,k}^{(n+1)}(t) + f_{i+1}^{(k)} B_{n,k}^{(n+1)}(t)\right] + f_i^{(n+1)} C_n^{(n+1)}(t) + f_{i+\frac{1}{2}}^{(n+1)} D_n^{(n+1)}(t) =$$
$$= \sum_{k=n+1}^{\ell} \frac{(ht)^{k-n-1}}{(k-n-1)!} f_i^{(k)} + O(h^{\ell-n-1}\omega_\ell(h)), \quad \text{if } f \in C^\ell(I),\ \ell = n+2, \dots, 2n+1,$$

and

(ii)
$$\sum_{k=0}^{n-1} h^{k-n-1}\left[f_i^{(k)} A_{n,k}^{(n+1)}(t) + f_{i+1}^{(k)} B_{n,k}^{(n+1)}(t)\right] + f_i^{(n+1)} C_n^{(n+1)}(t) + f_{i+\frac{1}{2}}^{(n+1)} D_n^{(n+1)}(t) =$$
$$= \sum_{k=n+1}^{2n+1} \frac{(ht)^{k-n-1}}{(k-n-1)!} f_i^{(k)} + C h^{n+1} f_i^{(2n+2)} + O(h^{n+1}\omega_{2n+2}(h)), \quad \text{if } f \in C^{2n+2}(I).$$

PROOF. Writing Taylor's expansion for $f_{i+1}^{(k)}$ and $f_{i+\frac{1}{2}}^{(n+1)}$ in powers of h and collecting the coefficients of $f_i^{(k)}$ in the above summation on the left, we obtain for $f \in C^\ell(I)$, $\ell = n+2, \dots, 2n+1$,

$$\sum_{k=0}^{n-1} h^{k-n-1}\left[f_i^{(k)} A_{n,k}^{(n+1)}(t) + f_{i+1}^{(k)} B_{n,k}^{(n+1)}(t)\right] + f_i^{(n+1)} C_n^{(n+1)}(t) + f_{i+\frac{1}{2}}^{(n+1)} D_n^{(n+1)}(t) =$$
$$= \sum_{k=0}^{n-1} h^{k-n-1} f_i^{(k)}\left[A_{n,k}^{(n+1)}(t) + \sum_{j=0}^{k} \frac{B_{n,j}^{(n+1)}(t)}{(k-j)!}\right] + h^{-1} f_i^{(n)} \sum_{j=0}^{n-1} \frac{B_{n,j}^{(n+1)}(t)}{(n-j)!} +$$
$$+ f_i^{(n+1)}\left[\sum_{j=0}^{n-1} \frac{B_{n,j}^{(n+1)}(t)}{(n+1-j)!} + C_n^{(n+1)}(t) + D_n^{(n+1)}(t)\right] +$$
$$+ \sum_{k=n+2}^{\ell} h^{k-n-1} f_i^{(k)}\left[\sum_{j=0}^{n-1} \frac{B_{n,j}^{(n+1)}(t)}{(k-j)!} + \frac{1}{(k-n-1)!}\frac{D^{(n+1)}(t)}{2^{k-n-1}}\right] + O(h^{\ell-n-1}\omega_\ell(h)) =$$
$$= \sum_{k=n+1}^{\ell} \frac{(ht)^{k-n-1}}{(k-n-1)!} f_i^{(k)} + O(h^{\ell-n-1}\omega_\ell(h))$$

by the use of the identities (3.3)–(3.6).

Part (ii) of the lemma can be proved easily by the same arguments.

4. Proof of Theorem 2

Let $x \in [x_i, x_{i+1}]$, $i = 0, \dots, N-1$ and $x = x_i + ht$, $0 \leqq t \leqq 1$. Then from (2.6) for equidistant knots

$$s(x) = \sum_{k=0}^{n-1} h^k [f_i^{(k)} A_{n,k}(t) + f_{i+1}^{(k)} B_{n,k}(t)] + \tag{4.1}$$
$$+ h^{n+1} \left[s_i^{(n+1)} C_n(t) + f_{i+\frac{1}{2}}^{(n+1)} D_n(t) \right].$$

Differentiating (4.1) $n+1$ times with respect to x we have

$$s^{(n+1)}(x) = \sum_{k=0}^{n-1} h^{k-n-1} \left[f_i^{(k)} A_{n,k}^{(n+1)}(t) + f_{i+1}^{(k)} B_{n,k}^{(n+1)}(t) \right] +$$
$$+ [s_i^{(n+1)} C_n^{(n+1)}(t) + f_{i+\frac{1}{2}}^{(n+1)} D_n^{(n+1)}(t)],$$

which on using Lemma 3.2 gives

$$s^{(n+1)}(x) = \sum_{k=n+1}^{\ell} \frac{(ht)^{k-n-1}}{(k-n-1)!} f_i^{(k)} + O(h^{\ell-n-1}\omega_\ell(h)) + N_i C_n^{(n+1)}(t).$$

Since

$$\sum_{k=n+1}^{\ell} \frac{(ht)^{k-n-1}}{(k-n-1)!} f_i^{(k)} - f^{(n+1)}(x) = O(h^{\ell-n-1}\omega_\ell(h)),$$

we have owing to Lemma 3.1 (i)

$$\|s^{(n+1)} - f^{(n+1)}\| = O(h^{\ell-n-1}\omega_\ell(h)) + O(h^{\ell-n-2}\omega_\ell(h)) =$$
$$= O(h^{\ell-n-2}\omega_\ell(h)), \qquad \ell = n+2, \dots, 2n+1,$$

which proves (3.1).

To prove (3.2), we observe that for $\ell = 2n+2$

$$s^{(n+1)}(x) = \sum_{k=n+1}^{2n+1} \frac{(ht)^{k-n-1}}{(k-n-1)!} f_i^{(k)} +$$

$$+ h^{n+1} f_i^{(2n+2)} \left[\sum_{j=0}^{n-1} \frac{B_{n,j}^{(n+1)}(t)}{(2n+2-j)!} + \frac{1}{(n+1)!} \frac{D_n^{(n+1)}(t)}{2^{n+1}} \right] + O(h^{n+1}\omega_{2n+2}(h)).$$

Hence arguing as above we have

$$\|s^{(n+1)} - f^{(n+1)}\| = Ch^{n+1} \|f^{(2n+2)}\| + O(h^{n+1}\omega_{2n+2}(h)).$$

Thus the theorem is completely proved.

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ON INTEGRAL FORMULAS FOR CONVEX DOMAINS

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1. Generalized Holditch's theorem

Let C be a closed convex regular C^1-curve in the Euclidean plane, parametrized by arc length s. We denote by L the perimeter of C. Moreover, we denote by $\langle\ ,\ \rangle$ and $[\ ,\ ]$ the Euclidean scalar product and the determinant, repsectively.

Let $\nu\colon [0,+\infty)\to R$ be a function satisfying the condition

$$(1)\qquad \begin{cases} \nu \text{ is differentiable and } \nu'>0,\\ s<\nu(s)<s+L \text{ for all } s\geqq 0,\\ \nu(s+L)=\nu(s)+L \text{ for all } s\geqq 0.\end{cases}$$

With a curve C, $s\to z(s)=(x(s),y(s))$ for all $0\leqq s\leqq L$ and a function ν we associate the vector field q defined as follows:

$$(2)\qquad q(s)=z(s)-z(\nu(s)) \text{ for } 0\leqq s\leqq L.$$

We denote by $\alpha(s)$ the angle contained between the vectors $z_0=z(0)$ and $q(s)$. Differentiating the relation

$$\cos\alpha=\frac{\langle q,z_0\rangle}{|q||z_0|}$$

we obtain

$$-\alpha'\sin\alpha=\frac{1}{|q|^3|z_0|}\{\langle q',z_0\rangle|q|^2-\langle q,z_0\rangle\langle q,q'\rangle\}=$$
$$=\frac{1}{|q|^3|z_0|}[q,z_0][q,q']=$$
$$=\frac{1}{|q|^3|z_0|}(-|q||z_0|\sin\alpha)[q,q']=\frac{-\sin\alpha}{|q|^2}[q,q']$$

and

$$(3)\qquad \alpha'=\frac{[q,q']}{|q|^2}.$$

Hence we immediately get the following integral formula:

$$\oint \frac{[q,q']}{|q|^2} ds = 2\pi. \tag{4}$$

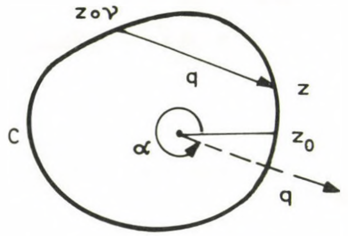

Fig. 1

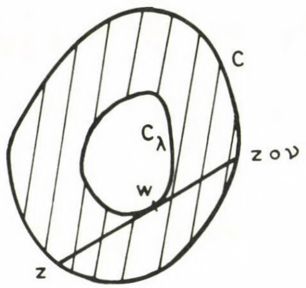

Fig. 2

Now, we give some application of the above formula.

Let us fix an arbitrary number $\lambda \in (0,1)$. We consider the curve C_λ, $s \to w(s) = (1-\lambda)z(s) + \lambda z(\nu(s))$ for $0 \leqq s \leqq L$. We note that C_λ is a closed curve. We find the area A of the region bounded by C and C_λ. By S and S_λ we denote the areas of the regions bounded by C and C_λ respectively. Using the Green formula we obtain

$$2A = 2S - 2S_\lambda = \oint([z,z'] - [w,w'])ds =$$

$$= \oint([z,z'] - (1-\lambda)^2)[z,z'] - \lambda(1-\lambda)[z, z'\circ\nu]\nu' -$$

$$-\lambda(1-\lambda)[z\circ\nu, z'] - \lambda^2[z\circ\nu, z'\circ\nu]\nu')ds =$$

$$= \oint((2\lambda-\lambda^2)[z,z'] - \lambda^2[z\circ\nu, z'\circ\nu]\nu')ds -$$

$$-\lambda(1-\lambda)\oint([z\circ\nu, z'] + [z, z'\circ\nu]\nu')ds =$$

$$= (2\lambda - 2\lambda^2)2S - \lambda(1-\lambda)\oint([z\circ\nu - z, z'] + [z,z'] -$$

$$-[z\circ\nu - z, z'\circ\nu]\nu' + [z\circ\nu, z'\circ\nu]\nu')ds =$$

$$= 4\lambda(1-\lambda)S - \lambda(1-\lambda)4S - \lambda(1-\lambda)\oint(-[q,z'] + [q, z'\circ\nu]\nu')ds =$$

$$= \lambda(1-\lambda)\oint[q, z\circ\nu\cdot\nu' - z']ds = \lambda(1-\lambda)\oint[q,q']ds.$$

Thus we have

$$2A = \lambda(1-\lambda)\oint [q, q']ds. \tag{5}$$

Making use of the integral formula (4) and the mean value theorem for integrals we get

$$2A = \lambda(1-\lambda)\oint |q|^2 \frac{[q, q']}{|q|^2} ds = \lambda(1-\lambda)|q(r)|^2 \oint \frac{[q, q']}{|q|^2} ds =$$

$$= 2\pi\lambda(1-\lambda)|q(r)|^2$$

for some r, $0 \leqq r \leqq L$.

THEOREM 1. *The area A of the region bounded by C and C_λ is given by the formula*

$$A = \pi\lambda(1-\lambda)|q(r)|^2 \tag{6}$$

for some r, $0 \leqq r \leqq L$.

As a simple consequence of the above theorem we obtain the Holditch theorem [4], [1]. Indeed, if $|q(s)| \equiv \text{const.} = a + b$ $(a, b > 0)$, for $0 \leqq s \leqq L$ and $\lambda = a/(a+b)$, $1 - \lambda = b/(a+b)$, then $A = \pi ab$.

2. A Crofton-type integral formula

In this section we give some Crofton-type formula for a convex domain. Crofton-type integral formulas for a convex domain can be found in [4] and for a star domain in [2].

Let C be a curve as in the previous section. We denote by $\operatorname{ext} C$ the exterior of the region bounded by C. Let $(u, v) \in \operatorname{ext} C$ and let (x, y) be a point of C such that a straight line passing through (x, y) and (u, v) is tangent to C (see Fig. 3). We denote by T_1 the tangent vector of C at (x, y). By (p, q) we denote the point of C realising the shortest distance between C and (u, v). Then by T_2 we denote the tangent vector of C at (p, q) and by r the distance between (p, q) and (u, v). Finally, let α, φ and ψ be as on Fig. 3.

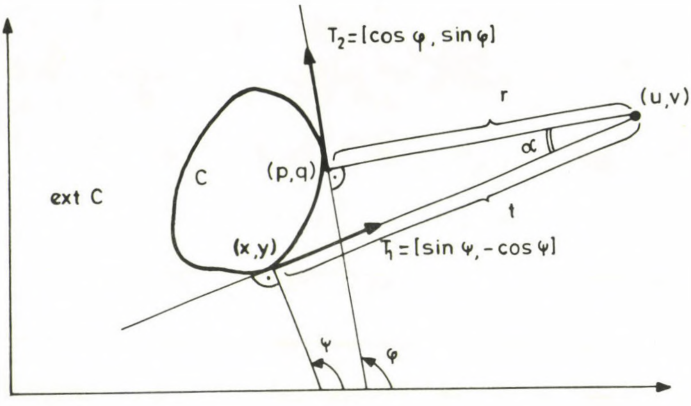

Fig. 3

We note that

$$\begin{cases} \alpha = \psi - \varphi \\ r = (u-p)\sin\varphi - (v-q)\cos\varphi \\ 0 = (u-p)\cos\varphi + (v-q)\sin\varphi \\ t = (u-x)\sin\psi - (v-y)\cos\psi \\ 0 = (u-x)\cos\psi + (v-y)\sin\psi. \end{cases}$$

Hence we get

$$dr = (du-dp)\sin\varphi + (u-p)\cos\varphi d\varphi - (dv-dq)\cos\varphi + (v-q)\sin\varphi d\varphi =$$
$$= \sin\varphi du - \cos\varphi dv - \sin\varphi dp + \cos\varphi dq + ((u-p)\cos\varphi + (v-q)\sin\varphi)d\varphi =$$
$$= \sin\varphi du - \cos\varphi dv - \sin\varphi dp + \cos\varphi dq,$$
$$0 = (du-dx)\cos\psi - (u-x)\sin\psi d\psi + (dv-dy)\sin\psi + (v-y)\cos\psi d\psi =$$
$$= \cos\psi du + \sin\psi dv - ((u-x)\sin\psi - (v-y)\cos\psi)d\psi - \cos\psi dx - \sin\psi dy =$$
$$= \cos\psi du + \sin\psi dv - td\psi - \cos\psi dx - \sin\psi dy.$$

Moreover, we have

$$dp = \cos\varphi ds, \qquad dq = \sin\varphi ds,$$
$$dx = \sin\psi ds, \qquad dy = -\cos\psi ds.$$

These imply

$$\sin\varphi dp - \cos\varphi dq = 0, \quad \cos\psi dx + \sin\psi dy = 0.$$

Thus we have

$$dr = \sin\varphi du - \cos\varphi dv, \quad td\psi = \cos\psi du + \sin\psi dv,$$

and

$$tdr \wedge d\psi = \cos\alpha du \wedge dv. \tag{8}$$

The relation (8) leads us immediately to the following result.

THEOREM 2. *If C is a closed, convex and regular C^1-curve then the following integral formula holds:*

$$\iint\limits_{\operatorname{ext} C} \exp(-\operatorname{dist}((u,v),C)) \frac{\cos\alpha(u,v)}{t(u,v)} dudv = 2\pi, \tag{9}$$

where $\operatorname{dist}((u,v),C)$ *denotes the distance between a point* $(u,v) \in \operatorname{ext} C$ *and the curve C.*

3. Integral formula for ovaloids

Let M be an ovaloid, i.e. a compact surface $M \subset R^3$ which has strictly positive Gauss curvature, [3]. Moreover, let a point x_0 lie in the interior of the region bounded by M (we will write $x_0 \in I(M)$).

By H, K, N we denote the mean curvature, Gauss curvature and outward normal vector field on M, respectively. Let S^2 denote the unit sphere. Let M be an ovaloid and let dM denote the volume form on M.

PROPOSITION 3. *If M is an ovaloid, then for an arbitrary point $x_0 \in$ $\in I(M)$ the following integral formula holds:*

$$\iint\limits_{M} \langle x - x_0, N\rangle \Big(4\pi H(x) - \frac{s(M)}{|x-x_0|^3}\Big) dM = 0, \tag{10}$$

where $s(M)$ denotes the surface area of M.

PROOF. We take an arbitrary atlas of S^2. Using the central projection from x_0 we obtain an atlas on M. Let (u^2, u^2) denote a local coordinate system at some point of M and

$$p(u^1,u^2) = \frac{x(u^1,u^2) - x_0}{|x(u^1,u^2) - x_0|}. \tag{11}$$

We denote by $d\omega$ the volume form of S^2. Moreover, let x_α, p_α denote the partial derivatives of x and p, respectively and let $\det G = |x_1 \wedge x_2|^2$ where the wedge denotes exterior multiplication. Then we have

$$d\omega = |p_1 \wedge p_2| du^1 du^2 = \langle p, p_1 \wedge p_2\rangle du^1 du^2 =$$

$$= \frac{\langle x - x_0, x_1 \wedge x_2 \rangle}{|x - x_0|^3} du^1 du^2 = \frac{\langle x - x_0, N \rangle}{|x - x_0|^3} \sqrt{\det G} du^1 du^2.$$

Hence we get

$$\iint_M \frac{\langle x - x_0, N \rangle}{|x - x_0|^3} dM = 4\pi. \tag{13}$$

Now, making use of Minkowski integral formula ([3], Lemma 6.2.9)

$$\iint_M H(x) \langle x - x_0, N \rangle dM = s(M) \tag{14}$$

and (13), we get (10).

Let us consider a pair (M, x_0), where M is an ovaloid and $x_0 \in I(M)$. We introduce a notion of x_0-mean curvature lines.

DEFINITION 1. A regular line C on M will be called x_0-mean curvature line if

$$4\pi |x - x_0|^3 H(x) = s(M) \quad \text{for all} \quad x \in C. \tag{15}$$

A line C on M will be called a mean curvature line for some point $x_0 \in I(M)$.

An x_0-mean curvature line is an isometric invariant of the pair (M, x_0). Now, we consider the problem of existence of such invariant.

1. Let $M = S^2$ and let x_0 coincide with the center of S^2. In this case condition (15) is an identity. Thus each regular line on S^2 is an x_0-mean curvature line.

2. Let $M = S^2$ and let $x_0 \in I(S^2)$ be different from the center of S^2. Then condition (15) reduces to $|x - x_0| = 1$. This means that the common circle for the spheres $|x| = 1$ and $|x - x_0| = 1$ is an x_0-mean curvature line.

We note that the integral formula (10) implies the existence of points on M satisfying (15). Now, we additionally assume that the mean curvature of an ovaloid M is a differentiable function and at some point $c = x(u_0) \in M$ satisfying condition (15), the expressions

$$3|c - x_0| \langle c - x_0, x_2(u_0) \rangle + \frac{s(M) H_2(u_0)}{4\pi H(u_0)^2},$$

$$3|c - x_0| \langle c - x_0, x_1(u_0) \rangle + \frac{s(M) H_1(u_0)}{4\pi H(u_0)^2},$$

where H_α denotes partial derivatives of H, do not vanish simultaneously. Let us consider the system of differential equations

$$\begin{cases} \dot{u}^1 = -3|x(u) - x_0| \langle x(u) - x_0, x_2(u) \rangle - \frac{s(M) H_2(u)}{4\pi H(u)^2}, \\ \dot{u}^2 = -3|x(u) - x_0| \langle x(u) - x_0, x_1(u) \rangle + \frac{s(M) H_1(u)}{4\pi H(u)^2} \end{cases} \tag{16}$$

with the initial condition $u(0) = (u^1(0), u^2(0)) = u_0$. We prove the existence of an x_0-mean curvature line on M passing through the point c. Indeed, let $t \to u(t)$ be a solution of the system (16). Then the curve $x(u(t))$ is a required x_0-mean curvature line because for the function

$$t \to g(t) = |x(u(t)) - x_0|^3 - \frac{s(M)}{4\pi H(u(t))}$$

$g(0) = 0$ and

$$\frac{dg}{dt} = \left(3|x - x_0|\langle x - x_0, x_\alpha\rangle + \frac{s(M)H_\alpha}{4\pi H^2}\right)\dot{u}^\alpha \equiv 0.$$

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NOTES TO MY PAPER "ON THE CONVERGENCE OF EIGENFUNCTION EXPANSIONS IN THE NORM OF SOBOLEFF SPACES"

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In this note we shall correct a proof of our paper [3]. For the convenience of the reader we repeat here the statement of the problem. Let $S_k \subset \mathbf{R}^n$ $(n \geqq 3,\ k = 1, \dots, \ell)$ be manifolds of dimension $\sim \dim S_k = m_k \leqq n-3$ such that

$$S_k = \{(\xi, y) \in \mathbf{R}^n : y = \varphi_k(\xi)\}, \qquad S := \bigcup_{k=1}^{\ell} S_k,$$

where

$$\varphi_k \in C^1(\mathbf{R}^{m_k} \to \mathbf{R}^{n-m_k}), \qquad |\nabla \varphi_k(\xi)| \leqq C_k < \infty.$$

Let $q \in C^\infty(\mathbf{R}^n \setminus S)$ be a real-valued function for which

(1) $$|D^\alpha q(x)| \leqq c[\operatorname{dist}(x, S)]^{-\tau-|\alpha|} \qquad (x \in \mathbf{R}^n,\ \ 0 \leqq |\alpha| \leqq 2)$$

holds for some $\tau > 0$. Consider the Schrödinger operator

$$L_0 = -\Delta + q(x), \qquad D(L_0) = C_0^\infty(\mathbf{R}^n).$$

We proved in [3] that L_0 has a selfadjoint extension L with $L \geqq -cI$ and such that $D(L) = H^2(\mathbf{R}^n)$. Let $L = \int_{-c}^{\infty} \lambda dE_\lambda$ be the spectral expansion of L. We can improve the main result of [3] as follows:

THEOREM. *Suppose* $\tau \in [0, 3/2)$ *and* $0 \leqq s < \frac{7}{2} - \tau$. *Then for* $f \in \in H^s(\mathbf{R}^n)$

(2) $$\|E_\lambda f - f\|_{H^s(\mathbf{R}^n)} \to 0 \quad \textit{as} \quad \lambda \to \infty.$$

We have to correct the proof of Lemma 1 and improve Lemma 9 of [3].

LEMMA 1. *Let* $m := \min_k (n - m_k) \geqq 2$, $1 < p$, $0 < s < m/p$. *Then*

(3) $$\|\varrho_k^s f\|_{L_p} \leqq c\|f\|_{L_p^s}, \qquad f \in L_p^s(\mathbf{R}^n),$$

$$\varrho_k(x) := |y - \varphi_k(\xi)|^{-1}, \qquad x = (\xi, y),$$

where $c = c(s, m, n, p)$ is independent of f.

PROOF. Let $g(\xi, z) \in L_p^1(\mathbf{R}^{m_k} \times \mathbf{R}^{n-m_k})$; then $|g(\xi, y)|$ and hence $|g(\xi, y)|^p$ is absolutely continuous. Consequently, taking the polar coordinates $z = (r, \theta)$ we obtain

$$|g(\xi, z)|^p = -p \int_r^\infty |g(t, \theta; \xi)|^{p-1} \frac{\partial |g(t, \theta; \xi)|}{\partial t} dt \quad (g \in L_p^1(\mathbf{R}^n)) \tag{4}$$

since the integral is convergent by the Hölder inequality. We know that for $s \geqq 1$

$$\|\varrho_k^s f\|_{L_p}^p = \int_{\mathbf{R}^n} \frac{|f(\xi, y)|^p}{|y - \varphi_k(\xi)|^{sp}} dy\, d\xi = $$

$$= \int_{\mathbf{R}^n} |z|^{-sp} |f(\xi, \varphi_k(\xi) + z)|^p \, dz\, d\xi.$$

The function $g(z, \xi) := f(\xi, \varphi_k(\xi) + z)$ belongs to $L_p^1(\mathbf{R}^n)$ because $|\nabla \varphi_k| \leqq c$. Hence

$$\|\varrho_k^s f\|_{L_p}^p = -p \int_{\mathbf{R}^n} |z|^{-sp} \int_r^\infty |g(t, \theta, \xi)|^{p-1} \frac{\partial |g(t, \theta; \xi)|}{\partial t} dt\, dz\, d\xi =$$

$$= -p \int_{\mathbf{R}^{m_k}} \int_\theta \int_0^\infty r^{n-m_k-1-sp} \int_r^\infty |g(t, \theta; \xi)|^{p-1} \frac{\partial |g(t, \theta; \xi)|}{\partial t} dt\, dr\, d\theta\, d\xi =$$

$$= -p \int_{\mathbf{R}^{m_k}} \int_\theta \int_0^\infty |g(t, \theta; \xi)|^{p-1} \frac{\partial |g(t, \theta; \xi)|}{\partial t} \int_0^t r^{n-m_k-1-sp} dr\, dt\, d\theta\, d\xi =$$

$$= \frac{-p}{n - m_k - sp} \int_{\mathbf{R}^{m_k}} \int_\theta \int_0^\infty t^{n-m_k-sp} |g(t, \theta; \xi)|^{p-1} \frac{\partial |g(t, \theta; \xi)|}{\partial t} dt\, d\theta\, d\xi =$$

$$= \frac{-p}{n - m_k - sp} \int_{\mathbf{R}^{n-m_k}} \int_{\mathbf{R}^{m_k}} |z|^{1-sp} |f(\xi, z + \varphi_k(\xi))|^{p-1} \frac{\partial |f(\xi, z + \varphi_k(\xi))|}{\partial t} d\xi\, dz \leqq$$

$$\leqq c \int_{\mathbf{R}^n} |z|^{1-sp} |f(\xi, z + \varphi_k(\xi))|^{p-1} |\nabla_z f(\xi, z + \varphi_k(\xi))| d\xi\, dz =$$

$$= c \int_{\mathbf{R}^{n-m_k}} \int_{\mathbf{R}^{m_k}} |z|^{1-s} |\nabla_z f(\xi, z + \varphi_k(\xi))| |z|^{s(1-p)} |f(\xi, z + \varphi_k(\xi))|^{p-1} d\xi\, dz.$$

By twofold application of Hölder inequality we obtain

$$\|\varrho_k^s f\|_{L_p}^p \leqq c \int\limits_{\mathbf{R}^{m_k}} \Bigg(\int\limits_{\mathbf{R}^{n-m_k}} |z|^{(1-s)p} |\nabla_z f(\xi, z+\varphi_k(\xi))|^p dz \Bigg)^{\frac{1}{p}} \cdot$$

$$\cdot \Bigg(\int\limits_{\mathbf{R}^{n-m_k}} |z|^{s(1-p)p^*} |f(\xi, z+\varphi_k(\xi))|^{(p-1)p^*} dz \Bigg)^{\frac{1}{p^*}} d\xi \leqq$$

$$\leqq c \Bigg(\int\limits_{\mathbf{R}^{n-m_k}} \int\limits_{\mathbf{R}^{m_k}} |z|^{(1-s)p} |\nabla_z f(\xi, z+\varphi_k(\xi))|^p d\xi\, dz \Bigg)^{\frac{1}{p}} \cdot$$

$$\cdot \Bigg(\int\limits_{\mathbf{R}^{n-m_k}} \int\limits_{\mathbf{R}^{m_k}} |z|^{-sp} |f(\xi, z+\varphi_k(\xi))|^p d\xi\, dz \Bigg)^{\frac{1}{p^*}}.$$

Multiplying by $\|\varrho_k^s f\|_{L_p}^{-p/p^*}$ we finally get

$$\|\varrho_k^s f\|_{L_p(\mathbf{R}^n)} \leqq c \|\varrho_k^{s-1} |\nabla_y f|\|_{L_p(\mathbf{R}^n)} \tag{5}$$

$$\left(f \in L_p^s(\mathbf{R}^n), \quad s \geqq 1, \quad c = \frac{p}{n - m_k - sp} \right).$$

Using the trivial estimate

$$\|\varrho_k^{s-1} |\nabla_y f|\|_{L_p} \leqq \sum_{j=1}^{n} \|\varrho_k^{s-1} D_{x_j} f\|_{L_p}$$

we can prove by induction the statement of Lemma 1 for entire values of s, $s < m/p$. Denote s_0 the greatest integer with $s_0 < m/p$. Then, as we have seen, the identity mapping

$$\text{Id:} \quad \begin{aligned} L_p^{s_0} &\to L_p(\varrho_k^{s_0 p}) \\ L_p &\to L_p \end{aligned}$$

is continuous; $L_p(\varrho_k^{s_0 p})$ means the L_p space weighted by $\varrho_k^{s_0 p}$. Using complex interpolation we get for $0 < \theta < 1$ that

$$(L_p, L_p^{s_0})_{[\theta]} = L_p^{\theta s_0}, \qquad (L_p, L_p(\varrho_k^{s_0 p}))_{[\theta]} = L_p(\varrho_k^{s_0 \theta p}),$$

see [2, 2.4.2 (11)] and [4, 5.5.3]. Consequently Id: $L_p^{\theta s_0} \to L_p(\varrho_k^{s_0 \theta p})$ is also continuous, i.e.

$$\|\varrho_k^{s_0 \theta} f\|_{L_p} \leqq c \|f\|_{L_p^{s_0 \theta}}, \quad f \in L_p^{s_0 \theta}(\mathbf{R}^n), \tag{6}$$

which proves Lemma 1 for $0 \leqq s \leqq s_0$. It remains to show it for $s_0 < s < m/p$.

Distinguish two cases.

(A) $s_0 = 0$. In this case choose $1 < p_0 < \infty$, $1 < p_1 < m$ arbitrarily. It follows from (5) that

$$\mathrm{Id}\colon \begin{array}{l} L^1_{p_1} \to L_{p_1}(\varrho_k^{p_1}) \\ L_{p_0} \to L_{p_0} \end{array}$$

is continuous. As above we get that

$$\mathrm{Id}\colon L^\theta_{p^*} \to L_{p^*}\left(\varrho_k^{\theta p^*}\right), \qquad \frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

is also continuous. Let here $\theta = s$. Then the value of $1/p^*$ can be varied arbitrarily between the bounds

$$1 > \frac{1}{p^*} > \frac{\theta}{m} = \frac{s}{m}$$

i.e. for appropriate p_0, p_1 we have $p = p^*$ and then

$$\|\varrho_k^\theta f\|_{L_p} = \|\varrho_k^s f\|_{L_p} \leqq c\|f\|_{L^s_p} \qquad (f \in L^s_p)$$

as we asserted.

(B) $s_0 > 0$. Let $s_0 < s < m/p$. Then by (5) and (6) we obtain

$$\|\varrho_k^s f\|_{L_p} \leqq c\sum_{j=1}^n \|\varrho_k^{s-1} D_{x_j} f\|_{L_p} \leqq c\sum_{j=1}^n \|D_{x_j} f\|_{L_p^{s-1}} =$$

$$= c\sum_{j=1}^n \left\| F^{-1}\left((1+|\xi|^2+|y|^2)^{\frac{s-1}{2}} F(D_{x_j} f)\right)\right\|_{L_p} =$$

$$= c\sum_{j=1}^n \left\| F^{-1}\left((1+|\xi|^2+|y|^2)^{s/2} \frac{x_j}{\sqrt{1+|\xi|^2+|y|^2}} Ff\right)\right\|_{L_p}.$$

Here F denotes the Fourier transform.

Now we need the following famous theorem of Marcinkiewicz:

THEOREM (Marcinkiewicz [1]). *Suppose the function $\lambda\colon \mathbf{R}^n \to \mathbf{R}$ satisfies the following property: let $1 < k_1 < k_2 < \ldots < k_r \leqq n$, $r \leqq n$ be an arbitrary index sequence, then the derivative*

$$D^{\mathbf{k}}\lambda := D_{x_{k_1}} \ldots D_{x_{k_r}} \lambda(x)$$

exists and is continuous at the points $x \in \mathbf{R}^n$, $x_{k_1} \neq 0, \dots, x_{k_r} \neq 0$; *further*

$$|x^{\mathbf{k}} D^{\mathbf{k}} \lambda| = |x_{k_1} \dots x_{k_r} D_{x_{k_1}} \dots D_{x_{k_r}} \lambda(x)| \leqq M, \quad x \in \mathbf{R}^n.$$

In this case for any $1 < p < \infty$ *there exists a constant* c_p, *independent of* f *and* M *such that*

$$\|F^{-1}(\lambda F f)\|_{L_p} \leqq c_p M \|f\|_{L_p}, \qquad f \in L_p(\mathbf{R}^n).$$

(Such function $\lambda(x)$ *is called multiplicator.)*

Since the functions

$$\lambda(\xi, y) := \frac{x_j}{\sqrt{1 + |\xi|^2 + |y|^2}}$$

are multiplicators (see [1], 1.5.5), we can apply Marcinkiewicz's just mentioned theorem to obtain

$$\|\varrho_k^s f\|_{L_p} \leqq c \left\| F^{-1}\left((1 + |\xi|^2 + |y|^2)^{s/2} F f \right) \right\|_{L_p} = \|f\|_{L_p^s}.$$

Lemma 1 is proved. □

Next we improve Lemma 9 of [3].

LEMMA 9. *Suppose* $0 \leqq \tau < 3/2$, $0 \leqq s < \frac{7}{2} - \tau$. *Then for any* $\mu \geqq \mu_0$ *and* $g \in H^s(\mathbf{R}^n)$ *we have*

$$\|g\|_{H^s(\mathbf{R}^n)} \leqq c \|L_\mu^{s/2} g\|_{L_2(\mathbf{R}^n)}.$$

PROOF. In case $0 \leqq s \leqq 2$ the proof is the same as in [3]. Now let $2 < s < \frac{7}{2} - \tau$. We have for $\delta := s - 2$

$$\|g\|_{H^s} = \|(I - \Delta) g\|_{H^\delta} \leqq c \|L_\mu^{\delta/2} (I - \Delta) g\|_{L_2} \leqq \tag{7}$$

$$\leqq c \left[\|L_\mu^{\delta/2} g\|_{L_2} + \|L_\mu^{\delta/2} L_\mu g\|_{L_2} + \|L_\mu^{\delta/2} (qg)\|_{L_2} \right] \leqq$$

$$\leqq c \left[\|L_\mu^{s/2} g\|_{L_2} + \|L_\mu^{\delta/2} (qg)\|_{L_2} \right] \leqq c \left[\|L_\mu^{s/2} g\|_{L_2} + \|qg\|_{H^\delta} \right].$$

We know further that

$$\|gq\|_{L_{p_0}} \leqq c \|g\|_{L_{p_0}^\tau} \qquad \text{for} \quad 1 < p_0 < 3/\tau,$$

$$\|gq\|_{L_{p_2}^2} \leqq c \|(I - \Delta)(gq)\|_{L_{p_2}} \leqq c \|g\|_{L_{p_2}^{2+\tau}} \qquad \left(1 < p_2 < \frac{3}{\tau + 2} \right).$$

Indeed, the first inequality follows from

$$|q(x)| \leqq c [\operatorname{dist}(x, S)]^{-\tau} \leqq c \sum |y_k - \varphi_k(\xi)|^{-\tau}$$

by Lemma 1. The second inequality has a similar proof, only in $(I-\Delta)(gq)$ the derivatives are decomposed by the Leibniz rule and taking into account (1) we can use Lemma 1 and

$$|D^\alpha q(x)| \leqq c\sum |y_k-\varphi_k(\xi)|^{-\tau-|\alpha|}, \quad |\alpha|=0,1,2.$$

By complex interpolation we get

$$\|gq\|_{L_p^{2\theta}} \leqq c\|g\|_{L_p^{(1-\theta)\tau+\theta(\tau+2)}} = c\|g\|_{L_p^{\tau+2\theta}}$$

$$\Big(0<\theta<1, \quad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_2}\Big).$$

We choose θ to satisfy $2\theta=\delta=s-2$; then we have

$$1>p^{-1}>\frac{\tau(1-\theta)}{3}+\frac{(\tau+2)\theta}{3}=\frac{\tau+2\theta}{3}=\frac{\tau+s-2}{3}.$$

Since $\frac{\tau+s-2}{3}<\frac{1}{2}$, we can choose p_0, p_1 to obtain $p=2$ and then

(8) $$\|gq\|_{H^\delta} \leqq c\|g\|_{H^{\delta+\tau}}.$$

Next we show that for any $\varepsilon>0$ there exists $c(\varepsilon)>0$ such that

(9) $$\|g\|^2_{H^{\delta+\tau}} \leqq \varepsilon\|g\|^2_{H^s}+c(\varepsilon)\|g\|^2_{L_2}.$$

Indeed, taking the Fourier transform this can be reformulated as

$$\int_{\mathbf{R}^n}(1+|x|^2)^{\delta+\tau}|Fg(x)|^2dx \leqq \varepsilon\int_{\mathbf{R}^n}(1+|x|^2)^s|Fg(x)|^2dx+$$

$$+c(\varepsilon)\int_{\mathbf{R}^n}|Fg(x)|^2dx$$

and this holds since $\delta+\tau<s$ implies

$$(1+|x|^2)^{\delta+\tau} \leqq \varepsilon(1+|x|^2)^s+c(\varepsilon), \qquad x\in\mathbf{R}^n$$

which proves (9). Finally we have

(10) $$\|g\|_{L_2} \leqq c\|L_\mu^{s/2}g\|_{L_2} \qquad (g\in H^s).$$

Indeed, Lemma 8 of [3] states that $g\in H^s$ implies $f=L_\mu^{s/2}g\in L_2$. Since $L_\mu^{-s/2}\colon L_2\to L_2$ is bounded for large μ (cf. the spectral theorem), hence

$$\|g\|_{L_2}=\|L_\mu^{-s/2}f\|_{L_2} \leqq c\|f\|_{L_2}=c\|L_\mu^{s/2}g\|_{L_2}$$

as we asserted. Now unify (7)–(9) to obtain

$$\|g\|_{H^s} \leqq c\left[\left\|L_\mu^{s/2} g\right\|_{L_2} + c\left(\varepsilon\|g\|_{H^s} + c(\varepsilon)\left\|L_\mu^{s/2} g\right\|_{L_2}\right)\right].$$

If $\varepsilon > 0$ is sufficiently small, this implies the statement of Lemma 9. □

The proof of the Theorem follows from Lemma 9 as in [3]:

$$\|f - E_\lambda f\|_{H^s} \leqq c\left\|L_\mu^{s/2}(I - E_\lambda)f\right\|_{L_2} = c\left\|(I - E_\lambda)L_\mu^{s/2} f\right\|_{L_2} \to 0$$

if $\lambda \to \infty$. □

Finally we state two problems.

PROBLEM 1. Does the Theorem hold for some $s \geqq \frac{7}{2} - \tau$?

PROBLEM 2. Does Lemma 1 hold in case $p = 1$, $0 \leqq s < m$?

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THE SIZE OF THE MIDDLE SUMMANDS IN PARTITIONS INTO DISTINCT s-TH POWERS*

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1. Introduction. The irreducible representations of the symmetric group S_n of order n are closely connected with the partitions of n by a theorem of Frobenius and Schur ([6], p. 123). According to this theorem, there is a one-to-one correspondence between the irreducible representations of S_n and the ordinary partitions of n in the following way. Let

$$\Pi : \begin{cases} \lambda_1 + \lambda_2 + \ldots + \lambda_m = n, \quad \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m (\geqq 1); \\ \lambda_j\text{'s integers} \end{cases} \tag{1.1}$$

be a generic partition of n. Then for the corresponding irreducible representation Γ_Π

$$\dim \Gamma_\Pi = n! \frac{\prod\limits_{1 \leqq \mu < \nu \leqq m} (\lambda_\nu - \lambda_\nu + \nu - \mu)}{\prod\limits_{\mu=1}^{m} (\lambda_\mu + m - \mu)!}. \tag{1.2}$$

Using the classical results of Erdős and Lehner (see [1] eqn. (1.4) and the paragraph thereafter), Szalay [7] showed that, for almost all Π's,

$$\dim \Gamma_\Pi = \exp\left\{\frac{1}{2} n \log n - O(n \log\log n)\right\}. \tag{1.3}$$

Erdős noted that this cannot be improved to

$$\exp\{g(n) + O(n^{1/2} \log^{-1} n)\}$$

for any function $g(n)$.

In order to estimate the dimensions given by (1.2), we need precise estimates of the λ_i's. This led Szalay and Turán to study the value distribution of the summands in (1.1). For example they obtained the following

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THEOREM 1 [8]. *For* $\log^6 n \leqq \mu \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 5\sqrt{n}\log\log n$ *the relation*

$$(1.4)\qquad \lambda_\mu = \left(1 + O(\log^{-1} n)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)}$$

holds uniformly for almost all ordinary partitions of n.

Using these results, they were able to improve the estimate (1.3) to

$$\exp\left\{\frac{1}{2}n\log n - \left(\frac{1}{2} + A'\right)n + O(n^{7/8}\log^4 n)\right\}$$

where A' is a computable constant.

Let Π' be a generic unequal partition of n into distinct positive integers, that is,

$$(1.5)\qquad \Pi' : \begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_m = n, & \alpha_1 > \alpha_2 > \cdots > \alpha_m (\geqq 1); \\ \alpha_j\text{'s integers.} \end{cases}$$

Recently, Erdős and Szalay studied the size of summands in unequal partitions. They pointed out that there exists a formula analogous to (1.4) for the α_μ's. In (1.5) instead of listing the summands of Π' in decreasing order, we can rearrange them in ascending order, i.e.,

$$\Pi' : \quad n = \alpha'_1 + \alpha'_2 + \cdots + \alpha'_m, \quad \alpha'_1 < \alpha'_2 < \cdots < \alpha'_m$$

where $\alpha'_\mu = \alpha_{m-\mu+1}$.

For convenience we call α'_μ *the* μ*-th smallest part of* Π'. In [2], Erdős and Szalay studied the size of the μ-th smallest part. Essentially they showed that for *most* of the unequal partitions Π' of n, α'_μ does not differ much from 2μ for *moderate* value of μ. The purpose of this paper is to extend their results to partitions into distinct s-th powers.

2. The middle summands. Let Π^* be a generic partition of n into distinct squares, i.e.,

$$(2.1)\qquad \Pi^* : \begin{cases} \alpha_1^2 + \alpha_2^2 + \cdots \alpha_m^2 = n, & \alpha_1 > \alpha_2 > \cdots > \alpha_m (\geqq 1); \\ \alpha_j\text{'s integers.} \end{cases}$$

As to the number $q_2(n)$ of restricted square partitions of n, we have (see [4])

$$(2.2)\qquad q_2(n) = (1 + o(1))(6\pi)^{-1/2}2^{-1/3}c_2^{1/3}n^{-5/6}\exp\{3\cdot 2^{-2/3}c_2^{2/3}n^{1/3}\},$$

where $c_2 = \Gamma(3/2)(1 - 1/\sqrt{2})\zeta(3/2)$.

Given any Π^* of n, let $(\alpha'_\mu)^2 = (\alpha_{m+1-\mu})^2$ be the μ-th smallest part of Π^*. In this section we consider the asymptotic behaviour of α'_μ. We show that for most of the restricted partitions of n into squares, if μ is *not too small*, α'_μ is asymptotic to 2μ. More precisely, we have

THEOREM 2. *For $\omega_1(n) \to \infty$, $\omega_2(n) \to \infty$ and $\omega_1(n)\log n \leqq \mu \leqq \leqq n^{1/3}(\omega_2(n))^{-1}$, we have the uniform relation*

$$\alpha'_\mu = \left(1 + O\Big(\frac{\mu^2}{n^{2/3}} + \sqrt{\frac{\log n}{\mu}}\Big)\right) \cdot 2\mu \tag{2.3}$$

apart from $O(q_2(n)n^{-1})$ restricted square partitions of n.

The method used is similar to that of Erdős, Szalay and Turán (see [2, 8]). We shall need the following lemma:

LEMMA 1. *Let $f(x) := \prod_{\nu=1}^{\infty}(1 + e^{-\nu^2 x})$, then for $x \to 0^+$ we have*

$$f(x) = \exp\Big(\frac{c_2}{\sqrt{x}} - \frac{1}{2}\log 2 + o(1)\Big). \tag{2.4}$$

PROOF. $\log f(x) = \sum_{\nu=1}^{\infty} \log(1 + e^{-\nu^2 x})$. By the Euler–Maclaurin formula (see [5], p. 524)

$$\sum_{\nu=1}^{\infty} \log(1 + e^{-\nu^2 x}) = \int_0^\infty \log(1 + e^{-\nu^2 x})d\nu - \frac{1}{2}\log 2 -$$

$$- \frac{1}{(2n)!}\int_0^\infty \Phi_{2n}(\nu)\frac{d^{2n}}{d\nu^{2n}}\{\log(1 + e^{-\nu^2 x})\}d\nu,$$

since $\frac{d^{2m-1}}{d\nu^{2m-1}}\{\log(1 + e^{-\nu^2 x})\}\Big|_{\nu=0} = 0$. Here $\Phi_{2n}(\nu)$ denotes the periodic extension of the Bernoulli polynomial on $[0, 1]$. Put $t = \nu^2 x$, $d\nu = \frac{dt}{2\sqrt{tx}}$ to get

$$\sum_{\nu=1}^{\infty} \log(1 + e^{-\nu^2 x}) = \frac{1}{2\sqrt{x}}\int_0^\infty \log(1 + e^{-t})\frac{dt}{\sqrt{t}} - \frac{1}{2}\log 2 -$$

$$- \frac{2^{2n-1}x^{n-1/2}}{(2n)!}\int_0^\infty \Phi_{2n}(t^{1/2}x^{-1/2})\Big(\sqrt{t}\frac{d}{dt}\Big)^{2n}\Big\{\log(1 + e^{-t})\Big\} \cdot t^{-1/2}dt =$$

$$= \frac{1}{2\sqrt{x}}\sum_1^\infty \frac{(-1)^{m-1}}{m}\int_0^\infty e^{-mt}t^{-1/2}dt - \frac{1}{2}\log 2 + O(x^{n-1/2}) =$$

$$= \frac{1}{2\sqrt{x}} \sum_{1}^{\infty} \frac{(-1)^{m-1}}{m^{3/2}} \int_{0}^{\infty} e^{-u} u^{-1/2} du - \frac{1}{2} \log 2 + O(x^{n-1/2}) =$$

$$= \frac{c_2}{\sqrt{x}} - \frac{1}{2} \log 2 + O(x^{n-1/2}),$$

where $c_2 = \Gamma(3/2)\zeta(3/2)(1 - 1/\sqrt{2})$.

For arbitrary nonnegative integers n, k and $\Lambda \geqq 1$, let $g(n, k, \Lambda)$ denote the number of partitions of n into distinct squares with exactly k summands less than or equal to Λ. Notice that $g(0, 0, \Lambda) = 1$; $g(0, k, \Lambda) = 0$ for $k \geqq 1$. We have the relation

$$\text{(2.5)} \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n, k, \Lambda) e^{-nx-ky} = \prod_{l \leqq \sqrt{\Lambda}} (1 + e^{-l^2 x - y}) \cdot \prod_{\nu \geqq \sqrt{\Lambda + 1}} (1 + e^{-\nu^2 x})$$

holds for every real y and positive x. In particular it holds when

$$\text{(2.6)} \quad x = x_0 = \left(\frac{c_2}{2n}\right)^{2/3}.$$

Now let us consider different values of y:

(i) For $y > 0$, $n \to \infty$, and $K \geqq 0$, we have

$$\sum_{k=0}^{K} g(n, k, \Lambda) e^{-nx_0 - Ky} \leqq \sum_{k=0}^{K} g(n, k, \Lambda) e^{-nx_0 - ky} \leqq$$

$$\leqq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n, k, \Lambda) e^{-nx_0 - ky} = \prod_{\nu=1}^{\infty} (1 + e^{-\nu^2 x_0}) \prod_{l \leqq \sqrt{\Lambda}} \frac{1 + e^{-l^2 x_0 - y}}{1 + e^{-l^2 x_0}},$$

i.e.,

$$\sum_{k=0}^{K} g(n, k, \Lambda) \leqq e^{nx_0 + Ky} \prod_{\nu=1}^{\infty} (1 + e^{-\nu^2 x_0}) \prod_{l \leqq \sqrt{\Lambda}} \left(1 - \frac{1 - e^{-y}}{1 + e^{l^2 x_0}}\right) \leqq$$

$$\leqq \exp\left\{ nx_0 + Ky + \frac{c_2}{\sqrt{x_0}} + o(1) - (1 - e^{-y}) \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + e^{l^2 x_0}} \right\} \leqq$$

$$\leqq \exp\left\{ 3\left(\frac{c_2}{2}\right)^{2/3} n^{1/3} + o(1) + y\left(K - e^{-y} \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + e^{l^2 x_0}}\right) \right\}$$

using $-(1 - e^{-y}) < -ye^{-y}$ for $y > 0$,

$$= O(n^{5/6} q_2(n)) \cdot \exp\left\{ \left(K - e^{-y} \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + e^{l^2 x_0}}\right) \right\}.$$

If $K \leqq e^{-y} \sum\limits_{l \leqq \sqrt{\Lambda}} \frac{1}{1+e^{l^2 x_0}} - \frac{3\log n}{y}$, we get $\sum\limits_{k=0}^{K} g(n,k,\Lambda) = O(q_2(n) \cdot n^{-2})$.

(ii) For $y < 0$, $n \to \infty$, and $L \geqq 0$, we have

$$\sum_{k=L}^{\infty} g(n,k,\Lambda) e^{-nx_0 - Ly} \leqq \sum_{k=L}^{\infty} g(n,k,\Lambda) e^{-nx_0 - ky} \leqq$$

$$\leqq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n,k,\Lambda) e^{-nx_0 - ky} = \prod_{\nu=1}^{\infty} (1 + e^{-\nu^2 x_0}) \prod_{l \leqq \sqrt{\Lambda}} \frac{1 + e^{-l^2 x_0 + |y|}}{1 + e^{-l^2 x_0}},$$

i.e.,

$$\sum_{k=L}^{\infty} g(n,k,\Lambda) \leqq e^{nx_0 - L|y|} \prod_{\nu=1}^{\infty} (1 + e^{-\nu^2 x_0}) \prod_{l \leqq \sqrt{\Lambda}} \left(1 + \frac{e^{|y|} - 1}{1 + e^{l^2 x_0}}\right) \leqq$$

$$\leqq \exp\left\{ nx_0 - L|y| + \frac{c_2}{\sqrt{x_0}} + o(1) + (e^{|y|} - 1) \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + e^{l^2 x_0}} \right\} \leqq$$

$$\leqq \exp\left\{ 3\left(\frac{c_2}{2}\right)^{2/3} n^{1/3} + o(1) + |y|\left(e^{|y|} \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + e^{l^2 x_0}} - L\right) \right\} =$$

$$= O(n^{5/6} q_2(n)) \cdot \exp\left\{ |y|\left(e^{|y|} \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + e^{l^2 x_0}} - L\right) \right\},$$

using $e^y - 1 < ye^y$ for $y > 0$. So if $L \geqq e^{|y|} \sum\limits_{l \leqq \sqrt{\Lambda}} \frac{1}{1+e^{l^2 x_0}} + \frac{3\log n}{|y|}$, we get

$$\sum_{k=L}^{\infty} g(n,k,\Lambda) = O(q_2(n) \cdot n^{-2}).$$

Given a partition of n into distinct squares. Π^* say, let $S(n, \Pi^*, \Lambda)$ denote the number of summands in Π^* not exceeding Λ. By the result in (i) and (ii) above, the inequalities

$$S(n, \Pi^*, \Lambda) \geqq e^{-t} \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + e^{l^2 x_0}} - \frac{3\log n}{t}, \tag{2.7}$$

$$S(n, \Pi^*, \Lambda) \leqq e^{t} \sum_{l \leqq \sqrt{\Lambda}} \frac{1}{1 + 1 + e^{l^2 x_0}} + \frac{3\log n}{t} \tag{2.8}$$

hold for arbitrary $t > 0$ with the exception of $O(q_2(n)\cdot n^{-2})$ restricted square partitions of n at most.

We suppose that $\Lambda \geqq \log^2 n$ and choose $t = t_0 = \Lambda^{-1/4}\sqrt{\log n}$. Then using (2.7) we obtain

$$S(n,\Pi^*,\Lambda) - \sum_{l\leqq\sqrt{\Lambda}} \frac{1}{1+e^{l^2x_0}} \geqq -(1-e^{-t_0}) \sum_{l\leqq\sqrt{\Lambda}} \frac{1}{1+e^{l^2x_0}} - \frac{3\log n}{t_0} \geqq$$

$$\geqq -\frac{t_0\Lambda^{1/2}}{2} - \frac{3\log n}{t_0} = -\frac{\Lambda^{1/4}\sqrt{\log n}}{2} - 3\Lambda^{1/4}\sqrt{\log n} = -\frac{7}{2}\Lambda^{1/4}\sqrt{\log n}.$$

Moreover from (2.8),

$$S(n,\Pi^*,\Lambda) - \sum_{l\leqq\sqrt{\Lambda}} \frac{1}{1+e^{l^2x_0}} \leqq (e^{t_0}-1) \sum_{l\leqq\sqrt{\Lambda}} \frac{1}{1+e^{l^2x_0}} + \frac{3\log n}{t_0} \leqq$$

$$\leqq \frac{1}{2}t_0e^{t_0}\Lambda^{1/2} + \frac{3\log n}{t_0} \leqq \frac{e}{2}\Lambda^{1/4}\sqrt{\log n} + 3\Lambda^{1/4}\sqrt{\log n}.$$

Thus

$$S(n,\Pi^*,\Lambda) = \sum_{l\leqq\sqrt{\Lambda}} \frac{1}{1+e^{l^2x_0}} + O(\Lambda^{1/4}\sqrt{\log n}) \tag{2.9}$$

holds for $\Lambda \geqq \log^2 n$ with the exception of at most $O(q_2(n)\cdot n^{-2})$ restricted square partitions of n. Hence (2.9) holds uniformly for $\log^2 n \leqq \Lambda \leqq n$ with the exception of at most $O(q_2(n)\cdot n^{-1})$ restricted square partitions of n.

If Λ is restricted by

$$\log^2 n \leqq \Lambda \leqq \frac{1}{2x_0} = \frac{n^{2/3}}{2^{1/3}c_2^{2/3}}, \tag{2.10}$$

we have

$$\frac{\sqrt{\Lambda}}{2} \geqq \sum_{l\leqq\sqrt{\Lambda}} \frac{1}{1+e^{l^2x_0}} = \sum_{l\leqq\sqrt{\Lambda}} \frac{e^{-l^2x_0}}{1+e^{-l^2x_0}} \geqq$$

$$\geqq \frac{1}{2}\sum_{l\leqq\sqrt{\Lambda}} e^{-l^2x_0} \geqq \frac{1}{2}\sum_{l\leqq\sqrt{\Lambda}} (1-l^2x_0) \geqq$$

$$\geqq \frac{[\sqrt{\Lambda}]}{2} - \frac{1}{5}\Lambda^{3/2}x_0 \geqq \frac{\sqrt{\Lambda}}{2} - \frac{\sqrt{\Lambda}}{8} = \frac{3}{8}\sqrt{\Lambda}.$$

So the uniform relation

$$S(n,\Pi^*,\Lambda) = \frac{\sqrt{\Lambda}}{2} + O(\Lambda^{3/2}x_0) + O(\sqrt{\log n}\cdot\Lambda^{1/4}) \tag{2.11}$$

holds for all but $O(q_2(n)\cdot n^{-1})$ restricted square partitions of n, under the restriction (2.10).

Let $\omega_1(n)\log n \leqq \mu \leqq n^{1/3}/\omega_2(n)$ and $n > n_0$. Since $5\mu^2$ lies within the range (2.10), by (2.11)

$$(2.12)\qquad S(n,\Pi^*,5\mu^2) = \frac{\sqrt{5}}{2}\mu + O(\mu^3 x_0) + O(\mu^{1/2}\sqrt{\log n}) =$$

$$= \frac{\sqrt{5}}{2}\mu + O\Big(\frac{\mu}{(\omega_2(n))^2}\Big) + O\Big(\frac{\mu}{\sqrt{\omega_1(n)}}\Big) = \frac{\sqrt{5}}{2}\mu + o(\mu) + o(\mu) > \mu.$$

Similarly $S(n,\Pi^*,3\mu^2) = \frac{\sqrt{3}}{2}\mu + o(\mu) < \mu$. But $S(n,\Pi^*,\alpha_\mu'^2) = \mu$, hence we have $3\mu^2 < \alpha_\mu'^2 < 5\mu^2$. Thus $\alpha_\mu'^2$ lies within the range (2.10) and $\alpha'_\mu = O(\mu)$. It follows from (2.11) that

$$\mu = S(n,\Pi^*,\alpha_\mu'^2) = \frac{\alpha'_\mu}{2} + O(\alpha_\mu'^3 x_0) + O(\alpha_\mu'^{1/2}\sqrt{\log n}) =$$

$$= \frac{\alpha'_\mu}{2} + O(\mu^3 x_0) + O(\mu^{1/2}\sqrt{\log n}),$$

which implies that

$$\alpha'_\mu = 2\mu + O(\mu^3 x_0) + O(\mu^{1/2}\sqrt{\log n}) = \bigg(1 + O\Big(\frac{\mu^2}{n^{2/3}} + \sqrt{\frac{\log n}{\mu}}\Big)\bigg)\cdot 2\mu.$$

This completes the proof of Theorem 2.

3. The small summands. In the last section, we have shown that for partitions of n into distinct squares, $\alpha'_\mu \sim 2\mu$ when $\omega_1(n)\log n \leqq \mu \leqq \leqq n^{1/3}(\omega_2(n))^{-1}$ with the exception of $O(q_2(n)n^{-1})$ restricted partitions of n. In the following, we show that a similar result holds when μ is *not too large.*

THEOREM 3. *For $2^8 \leqq k_0 \leqq \mu \leqq n^{1/12}$, we have the uniform estimation*

$$|\alpha'_\mu - 2\mu| \leqq \mu\sqrt{\frac{40\log k_0}{k_0}}$$

with the exception of $O(q_2(n)k_0^{-3/2})$ restricted square partitions of n.

Let $q_2(n;k)$ be the number of partitions of n into distinct squares $> k$. The proof of the above theorem will be similar to that of Theorem 3 in [2]. It relies on the following estimation concerning $q_2(n;k)$:

LEMMA 2. *For $1 \leqq k \leqq n^{1/5}$ and $n \to \infty$*

$$q_2(n;k) = (1+o(1))2^{-[\sqrt{k}]} \cdot q_2(n).$$

In order to prove the above lemma observe that for $\operatorname{Re} x > 0$

$$1 + \sum_{n=1}^{\infty} q_2(n;k)e^{-nx} = \prod_{\nu \geqq \sqrt{1+k}} (1+e^{-x\nu^2}).$$

Applying Cauchy's integral theorem and the saddle-point method developed by Szekeres (see [9]), we have

$$(3.1) \qquad q_2(n;k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big\{ \sum_{\nu \geqq \sqrt{k+1}} \log(1+e^{-\beta\nu^2 + i\theta\nu^2}) + \beta n - n\theta i \Big\} d\theta$$

where β is the unique positive root of

$$(3.2) \qquad \sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^2}{e^{\beta\nu^2}+1} = n.$$

Using the Euler–Maclaurin formula we find that

$$\beta = \Big(\frac{c_2}{2n}\Big)^{2/3} + O\Big(\frac{k^{3/2}}{n^{5/3}}\Big) = \Big(\frac{c_2}{2n}\Big)^{2/3} + O(n^{-41/30}), \quad \text{if} \ \ 1 \leqq k \leqq n^{1/5}.$$

Let ε be fixed with $0 < \varepsilon < 10^{-2}$. Let $\theta_0 = \beta^{5/4-\varepsilon}$. Splitting the integral in (3.1)

$$(3.3) \qquad q_2(n;k) = \frac{1}{2\pi}\Big\{ \int_{\theta_0}^{\pi} + \int_{-\theta_0}^{\theta_0} + \int_{-\pi}^{-\theta_0} \Big\} = S_1 + S_2 + S_3, \quad \text{say}.$$

We will show that the contributions of S_1 and S_3 are negligible compared to S_2.

4. The major arcs. To estimate S_2, notice that when $\theta \in (-\theta_0, \theta_0)$

$$\sum_{\nu \geqq \sqrt{k+1}} \log(1+e^{-\beta\nu^2+i\theta\nu^2}) =$$

$$= \sum_{\nu \geqq \sqrt{k+1}} \log(1+e^{-\beta\nu^2}) + i\theta \sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^2}{e^{\beta\nu^2}+1} - \frac{1}{2}\theta^2 \sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^4 e^{\beta\nu^2}}{(e^{\beta\nu^2}+1)^2} + O\Big(\frac{|\theta|^3}{\beta^{7/2}}\Big) =$$

$$= \sum_{\nu \geqq \sqrt{k+1}} \log(1+e^{-\beta\nu^2}) + ni\theta - \frac{1}{2}\theta^2 \sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^4 e^{\beta\nu^2}}{(e^{\beta\nu^2}+1)^2} + O\Big(\frac{|\theta|^3}{\beta^{7/2}}\Big).$$

But for $\theta \in (-\theta_0, \theta_0)$, $|\theta|^3 \beta^{-7/2} = O(\beta^{1/4-3\varepsilon}) = o(1)$. Hence

$$S_2 \sim \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} \exp\Big\{ \sum_{\nu \geqq \sqrt{k+1}} \log(1+e^{-\beta\nu^2}) - \frac{1}{2}\theta^2 \sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^4 e^{\beta\nu^2}}{(e^{\beta\nu^2}+1)^2} + \beta n \Big\} d\theta.$$

On the other hand we have by the Euler–Maclaurin formula

$$\sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^4 e^{\beta\nu^2}}{(e^{\beta\nu^2}+1)^2} = \sum_{\nu \geqq 1} \frac{\nu^4 e^{\nu^2\beta}}{(e^{\beta\nu^2}+1)^2} - \sum_{\nu < \sqrt{k+1}} \frac{\nu^4 e^{\nu^2\beta}}{(e^{\beta\nu^2}+1)^2} =$$

$$= \int_0^\infty \frac{t^4 e^{t^2\beta}}{(e^{\beta t^2}+1)^2} dt + R_1 - \sum_{\nu < \sqrt{k+1}} \frac{\nu^4 e^{\nu^2\beta}}{(e^{\beta\nu^2}+1)^2}$$

where

$$R_1 = \int_0^\infty \Big(t - [t] - \frac{1}{2}\Big) \cdot \Big(\frac{4t^3 e^{t^2\beta}}{(e^{\beta t^2}+1)^2} + \frac{2t^5 \beta e^{t^2\beta}}{(e^{\beta t^2}+1)^2} - \frac{4t^5 \beta e^{2\beta t^2}}{(e^{\beta t^2}+1)^3} \Big) dt.$$

Now

$$|R_1| \leqq \int_0^\infty \Big(\frac{4t^3 e^{t^2\beta}}{(e^{\beta t^2}+1)^2} + \frac{2t^5 \beta e^{t^2\beta}}{(e^{\beta t^2}+1)^2} + \frac{4t^5 \beta e^{2\beta t^2}}{(e^{\beta t^2}+1)^3} \Big) dt = O(\beta^{-2}).$$

Also,

$$\sum_{\nu < \sqrt{k+1}} \frac{\nu^4 e^{\nu^2\beta}}{(e^{\beta\nu^2}+1)^2} = O\Big(\sum_{\nu < \sqrt{k+1}} \nu^4 \Big) = O(k^{5/2}).$$

So

$$\sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^4 e^{\beta\nu^2}}{(e^{\beta\nu^2}+1)^2} = \int_0^\infty \frac{t^4 e^{t^2\beta}}{(e^{\beta t^2}+1)^2} dt + O(\beta^{-2}) + O(k^{5/2}) =$$

$$= \frac{1}{2\beta^{5/2}} \int_0^\infty \frac{y^{3/2} e^y}{(e^y+1)^2} dy + O(\beta^{-2}) = \frac{3}{4\beta^{5/2}} \int_0^\infty \frac{y^{1/2}}{e^y+1} dy + O(\beta^{-2}) = \frac{3c_2}{4\beta^{5/2}} + O(\beta^{-2}).$$

Hence for $\theta \in (-\theta_0, \theta_0)$

$$\sum_{\nu \geqq \sqrt{k+1}} \frac{\nu^4 \varepsilon^{\beta\nu^2}}{(e^{\beta\nu^2}+1)^2} \theta^2 = \frac{3c_2}{4\beta^{5/2}} \theta^2 + O(\beta^{-2} \cdot \theta^2) = \frac{3c_2}{4\beta^{5/2}} \theta^2 + o(1).$$

Thus

$$(4.1)\qquad S_2 \sim \frac{e^{\beta n}}{2\pi}\exp\Big\{\sum_{\nu\geqq\sqrt{k+1}}\log(1+e^{-\beta\nu^2})\Big\}\cdot\int_{-\theta_0}^{\theta_0}\exp\Big\{-\frac{3c_2}{8\beta^{5/2}}\theta^2\Big\}d\theta.$$

However $\theta_0^2\beta^{-5/2}=\beta^{-2\varepsilon}$, so we can replace the limits of integration in (4.1) by $-\infty$ and ∞ respectively since this will not affect the asymptotic order of S_2. Hence

$$(4.2)\qquad S_2 \sim \frac{e^{\beta n}}{2\pi}\exp\Big\{\sum_{\nu\geqq\sqrt{k+1}}\log(1+e^{-\beta\nu^2})\Big\}\cdot\int_{-\infty}^{\infty}\exp\Big\{-\frac{3c_2}{8\beta^{5/2}}\theta^2\Big\}d\theta \sim$$

$$\sim \frac{e^{\beta n}}{2\pi}\sqrt{\frac{8\pi\beta^{5/2}}{3c_2}}\cdot\exp\Big\{\sum_{\nu\geqq\sqrt{k+1}}\log(1+e^{-\beta\nu^2})\Big\} \sim$$

$$\sim \frac{c_2^{1/3}e^{\beta n}}{2^{1/3}3^{1/2}\sqrt{\pi}n^{5/6}}\cdot\exp\Big\{\sum_{\nu\geqq\sqrt{k+1}}\log(1+e^{-\beta\nu^2})\Big\}.$$

Finally we have

$$\sum_{\nu\geqq\sqrt{k+1}}\log(1+e^{-\beta\nu^2})=\sum_{\nu\geqq 1}\log(1+e^{-\beta\nu^2})-\sum_{\nu<\sqrt{k+1}}\log(1+e^{-\beta\nu^2}),$$

and

$$-\sum_{\nu<\sqrt{k+1}}\log(1+e^{-\beta\nu^2})=\sum_{\nu<\sqrt{k+1}}\log\Big(\frac{1}{2-(1-e^{-\beta\nu^2})}\Big)=$$

$$=\sum_{\nu<\sqrt{k+1}}\log\frac{1}{2}+\sum_{\nu<\sqrt{k+1}}\log\Big(\frac{1}{1-\frac{1}{2}(1-e^{-\beta\nu^2})}\Big)=$$

$$=\sum_{\nu<\sqrt{k+1}}\log\frac{1}{2}+O\Big(\sum_{\nu<\sqrt{k+1}}\frac{\frac{1}{2}(1-e^{-\beta\nu^2})}{1-\frac{1}{2}(1-e^{-\beta\nu^2})}\Big)=$$

$$=-\sum_{\nu<\sqrt{k+1}}\log 2+O\Big(\sum_{\nu<\sqrt{k+1}}(1-e^{-\beta\nu^2})\Big)=-[\sqrt{k}]\cdot\log 2+O\Big(\sum_{\nu<\sqrt{k+1}}\beta\nu^2\Big)=$$

$$=-[\sqrt{k}]\cdot\log 2+O(\beta(k^{3/2})=-[\sqrt{k}]\cdot\log 2+o(1)$$

using $-\log(1-y)<y(1-y)^{-1}$ for $|y|<1$. Thus

$$\sum_{\nu\geqq\sqrt{k+1}}\log(1+e^{-\beta\nu^2})=\sum_{\nu\geqq 1}\log(1+e^{-\beta\nu^2})-[\sqrt{k}]\cdot\log 2+o(1)=$$

$$= \frac{c_2}{\sqrt{\beta}} - \frac{1}{2}\log 2 - [\sqrt{k}] \cdot \log 2 + o(1).$$

Hence we have

$$S_2 \sim \frac{c_2^{1/3} e^{\beta n}}{2^{1/3} 3^{1/2} \sqrt{\pi} n^{5/6}} \cdot \exp\Big\{ \frac{c_2}{\sqrt{\beta}} - \frac{1}{2}\log 2 - [\sqrt{k}] \cdot \log 2 \Big\},$$

that is,

$$S_2 \sim 2^{-[\sqrt{k}]} \cdot q_2(n). \tag{4.3}$$

5. The minor arcs. Lemma 2 will follow if we can show that the integrals S_1 and S_3 are negligible compared to S_2. First let us deal with S_1. We notice that

$$S_1 = \exp\Big\{ n\beta + \sum_{\nu \geqq \sqrt{k+1}} \log(1 + e^{-\beta\nu^2}) \Big\} \times$$

$$\times \int_{\theta_0'}^{1/2} \exp\Big\{ -2n\pi i\theta + \sum_{\nu \geqq \sqrt{k+1}} \log \frac{1 + e^{-\beta\nu^2 + i2\pi\theta\nu^2}}{1 + e^{-\beta\nu^2}} \Big\} d\theta,$$

where $\theta_0' = \theta_0/(2\pi)$.

LEMMA 3. *Let θ_0 and k be as above. Then for $\theta_0' \leqq \theta \leqq 1/2$ we have*

$$|G(\theta)| := \exp\Big\{ \sum_{\nu \geqq \sqrt{k+1}} \log \Big| \frac{1 + e^{-\beta\nu^2 + i2\pi\theta\nu^2}}{1 + e^{-\beta\nu^2}} \Big| \Big\} = o(n^{-c}) \quad \textit{for every} \quad c > 0.$$

PROOF. The proof is similar to that of [11], Lemma 1. First of all we notice that

$$\log|G(\theta)| = \frac{1}{2} \sum_{\nu \geqq \sqrt{k+1}} \log\Big(1 - \frac{2(1 - \cos(2\pi\theta\nu^2))e^{\nu^2\beta}}{(e^{\beta\nu^2} + 1)^2} \Big) \leqq$$

$$\leqq - \sum_{\nu \geqq \sqrt{k+1}} \frac{(1 - \cos(2\pi\theta\nu^2))e^{\nu^2\beta}}{(e^{\beta\nu^2} + 1)^2}.$$

For those $\nu \leqq \beta^{-1/2}$, we have

$$\frac{e^{\beta\nu^2}}{(e^{\beta\nu^2} + 1)^2} \geqq \frac{e}{(e+1)^2} > \frac{1}{8}.$$

So

$$\log|G(\theta)| \leqq -\frac{1}{8} \sum_{\sqrt{k+1}\leqq\nu\leqq\beta^{-1/2}} (1-\cos(2\pi\nu^2\theta)) = \tag{5.1}$$

$$= -\frac{1}{8} \sum_{\sqrt{k+1}\leqq\nu\leqq\beta^{-1/2}} \Big(1-\frac{1}{2}(e^{2\pi i\theta\nu^2}+e^{-2\pi i\theta\nu^2})\Big) \leqq$$

$$\leqq -\frac{1}{8}(\beta^{-1/2}-\sqrt{k}-1)+\frac{1}{8}\cdot\Big|\sum_{\sqrt{k+1}\leqq\nu\leqq\beta^{-1/2}} e^{2\pi i\theta\nu^2}\Big|.$$

But

$$\Big|\sum_{\sqrt{k+1}\leqq\nu\leqq\beta^{-1/2}} e^{2\pi i\theta\nu^2}\Big| \leqq \Big|\sum_{\nu\leqq\beta^{-1/2}} e^{2\pi i\theta\nu^2}\Big| + \Big|\sum_{\nu<\sqrt{k+1}} e^{2\pi i\theta\nu^2}\Big| \leqq$$

$$\leqq \Big|\sum_{\nu\leqq\beta^{-1/2}} e^{2\pi i\theta\nu^2}\Big| + \sqrt{k+1}.$$

Moreover, applying Weyl's Inequality (see [10], Lemma 2.4 and §4 of [12]) we obtain for $\beta^{1-\varepsilon} \leqq \theta \leqq 1/2$, the inequality

$$\Big|\sum_{\nu\leqq\beta^{-1/2}} e^{2\pi i\theta\nu^2}\Big| \leqq c_0\beta^{-1/2}$$

where $0 < c_0 < 1$. Substituting these into (5.1) above we have

$$\log|G(\theta)| \leqq -c\beta^{-1/2} + c'\sqrt{k+1}.$$

However, $\beta^{-1/2} \sim cn^{1/3}$ and $\sqrt{k+1} = O(n^{1/10})$, thus $\log|G(\theta)| \leqq -c''n^{1/3}$, and this proves Lemma 3, since the case $\theta_0' \leqq \theta \leqq \beta^{1-\varepsilon}$ can be easily settled by

$$\log|G(\theta)| \leqq -\sum_{\sqrt{k+1}\leqq\nu\leqq(2\theta)^{-1/2}} \frac{(1-\cos(2\pi\theta\nu^2))e^{\nu^2\beta}}{(e^{\beta\nu^2}+1)^2}.$$

It follows that

$$S_1 = o(n^{-c})\cdot\exp\Big\{n\beta + \sum_{\nu\geqq\sqrt{k+1}} \log(1+e^{-\beta\nu^2})\Big\}$$

and in view of (4.2) we have $S_1 = o(S_2)$. The proof that $S_3 = o(S_2)$ follows similarly and this completes the proof of Lemma 2.

6. The exceptional partitions. Next we consider the set of partitions of n into distinct squares for which the square root of the μ-th smallest part, α'_μ is *not* close to 2μ. Let us suppose that

$$k_0 \leqq \mu \leqq n^{1/12}, \quad \varepsilon = \varepsilon(k_0), \quad \text{and} \quad 0 < \varepsilon < 1. \tag{6.1}$$

Our task is to get an upper bound for the number of exceptional restricted square partitions for which the inequality $|\alpha'_\mu - 2\mu| \leqq \varepsilon\mu$ does not hold. Let $E(\mu, n)$ denote the number of restricted square partitions of n with $|\alpha'_\mu - 2\mu| > \varepsilon\mu$. By (2.12) we have

$$S(n, \Pi^*, 5n^{1/6}) = \frac{\sqrt{5}}{2} n^{1/12} + o(n^{1/16}) > n^{1/12}$$

for all but $O(q_2(n) \cdot n^{-1})$ restricted square partitions of n. So for sufficiently large n, apart from $O(q_2(n) \cdot n^{-1})$ such partitions, the other have at least $[n^{1/12}]$ summands smaller than $5n^{1/6}$. Hence the inequality $\alpha'_{[n^{1/12}]} \leqq \sqrt{5}n^{1/12}$ holds for all but $O(q_2(n) \cdot n^{-1})$ restricted square partitions of n. So for μ lying within the range (6.1), we get for $E(\mu, n)$

(6.2)

$$E(\mu, n) \leqq \sum_{\mu \leqq \alpha'_\mu \leqq [(2-\varepsilon)\mu]} \; \sum_{1 \leqq \alpha'_1 < \alpha'_2 < \cdots < \alpha'_{\mu-1} < \alpha'_\mu} q_2(n - \alpha'^2_1 - \alpha'^2_2 - \cdots - \alpha'^2_\mu; \alpha'^2_\mu) +$$

$$+ \sum_{1 + [(2+\varepsilon)\mu] \leqq \alpha'_\mu \leqq [\sqrt{5}n^{1/12}]} \; \sum_{1 \leqq \alpha'_1 < \alpha'_2 < \cdots < \alpha'_{\mu-1} < \alpha'_\mu} q_2(n - \alpha'^2_2 - \alpha'^2_2 - \cdots - \alpha'^2_\mu; \alpha'^2_\mu) +$$

$$+ O(q_2(n) \cdot n^{-1}).$$

Moreover, for those α'_μ less than $[\sqrt{5}n^{1/12}]$ we have

$$(n - \alpha'^2_1 - \alpha'^2_2 - \cdots - \alpha'^2_\mu)^{1/5} \geqq n^{1/5}(1 - \mu n^{-1} \alpha'^2_\mu)^{1/5} \geqq$$

$$\geqq n^{1/5}(1 - 5n^{-3/4})^{1/5} > 5n^{1/6} > \alpha'^2_\mu.$$

Applying Lemma 2 we get

$$q_2(n - \alpha'^2_1 - \alpha'^2_2 - \cdots - \alpha'^2_{\mu-1} - \alpha'^2_\mu; \alpha'^2_\mu) =$$

$$= O(2^{-\alpha'_\mu}) \cdot q_2(n - \alpha'^2_1 - \alpha'^2_2 - \cdots - \alpha'^2_{\mu-1} - \alpha'^2_\mu) = O(2^{-\alpha'_\mu}) \cdot q_2(n).$$

Substituting into (6.2)

$$E(\mu, n) \leqq O(q_2(n)) \cdot \sum_{\alpha'_\mu = \mu}^{[(2-\varepsilon)\mu]} \binom{\alpha'_\mu - 1}{\mu - 1} 2^{-\alpha'_\mu} +$$

$$+O(q_2(n))\cdot\sum_{\alpha'_\mu=[(2+\varepsilon)\mu]+1}^{[\sqrt{5}n^{1/12}]}\binom{\alpha'_\mu-1}{\mu-1}2^{-\alpha'_\mu}+O(q_2(n)\cdot n^{-1}).$$

Following the argument in [2], p. 444, the sums of binomial coefficients in the above inequality can be estimated by means of Stirling's formula. This gives

$$E(\mu,n)=O(q_2(n))\cdot\left\{n^{-1}+\varepsilon^{-1}\exp(-\varepsilon^2\mu/8)+\varepsilon^{-1}\mu^{-1/2}\exp(-\varepsilon^2\mu/16)\right\}$$

for every $\varepsilon>0$. So

$$\sum_{k_0\leqq\mu\leqq n^{1/12}}E(\mu,n)=$$

$$=O(q_2(n))\cdot\left\{n^{-11/12}+\varepsilon^{-1}\sum_{\mu=k_0}^{\infty}\exp(-\varepsilon^2\mu/8)+\varepsilon^{-1}\sum_{\mu=k_0}^{\infty}\mu^{-1/2}\exp(-\varepsilon^2\mu/16)\right\}=$$

$$=O(q_2(n))\cdot\left\{n^{-11/12}+O(\varepsilon^{-3})\exp(-\varepsilon^2k_0/8)+O(\varepsilon^{-3}k_0^{-1/2})\exp(-\varepsilon^2k_0/16)\right\}.$$

Let $\varepsilon=\sqrt{\frac{40\log k_0}{k_0}}$. For $k_0\geqq 2^8$, we have

$$\sum_{k_0\leqq\mu\leqq n^{1/12}}E(\mu,n)=O(q_2(n)\cdot k_0^{-3/2}).$$

This shows that for $2^8\leqq k_0\leqq\mu\leqq n^{1/12}$, the inequality

$$|\alpha'_\mu-2\mu|\leqq\mu\sqrt{\frac{40\log k_0}{k_0}}$$

holds for all but $0(q_2(n)\cdot k_0^{-3/2})$ restricted square partitions of n, and we have Theorem 3.

7. Further extensions. The same argument can be used to extend the above results to partitions into distinct s-th powers. Let Π_1^* be a generic partition of n into distinct s-th powers:

$$\Pi_1^*:\quad\begin{cases}\beta_1^s+\beta_2^s+\dots+\beta_m^s=n,\quad\beta_1>\beta_2>\dots>\beta_m(\geqq 1);\\ \beta_j\text{'s integers.}\end{cases}$$

Again let $(\beta'_\mu)^s=(\beta_{m+1-\mu})^s$ be the μ-th smallest summand of Π_1^*.

THEOREM 4. *For* $\omega_1(n)$, $\omega_2(n) \to \infty$ *and* $\omega_1(n)\log n \leqq \mu \leqq \leqq n^{1/(1+s)}/\omega_2(n)$, *we have*

$$\beta'_\mu = \left(1 + O\left(\frac{\mu^s}{n^{s/(1+s)}} + \sqrt{\frac{\log n}{\mu}}\right)\right) \cdot 2\mu$$

apart from $O(q_s(n) \cdot n^{-1})$ *restricted s-th power partitions of* n.

In proving the above theorem, we make use of the following which is a natural extension of Lemma 1.

LEMMA 4. *Let* $g(x) := \prod_{\nu=1}^{\infty}(1 + e^{-\nu^s x})$, $x > 0$. *Then*

$$g(x) = \exp\left(\frac{c_s}{x^{1/s}} - \frac{1}{2}\log 2 + o(1)\right) \quad for \quad x \to 0^+.$$

Here $c_s = \int\limits_0^\infty \frac{y^{1/s}}{e^y+1}dy = (1 - 2^{-1/s})\Gamma(1 + 1/s)\zeta(1 + 1/s)$.

As to Lemma 2 and Theorem 3, we have the following extensions:

LEMMA 5. *Let* $q_s(n; k)$ *denote the number of partitions of* n *into distinct s-th powers which are greater than* k. *For* $1 \leqq k \leqq n^{1/5}$ *and* $n \to \infty$

$$q_s(n; k) = (1 + o(1))2^{-[k^{1/s}]} \cdot q_s(n).$$

THEOREM 5. *For* $2^8 \leqq k_0 \leqq \mu \leqq n^{1/6s}$, *we have the uniform estimation*

$$|\beta'_\mu - 2\mu| \leqq \mu\sqrt{\frac{40\log k_0}{k_0}}$$

with the exception of $O(q_s(n)k_0^{-3/2})$ *restricted s-th power partitions of* n.

By a more intricate argument it seems possible to obtain analogous results for partitions with summands taken from a general sequence of integers. However, since we have not carried out all the formal details, we decided to leave the general case as a conjecture.

8. Conjectures. We close this paper with two conjectures:

Let $\Lambda' = \{a_1, a_2, \dots\}$ be a strictly increasing sequence of positive integers. Let Π_2^* be a generic partition of n with summands taken from Λ':

$$\Pi_2^*: \quad \begin{cases} \lambda_1 + \lambda_2 + \dots + \lambda_m = n, & \lambda_1 > \lambda_2 > \dots > \lambda_m (\geqq 1); \\ \lambda_j \in \Lambda'. \end{cases}$$

We let $\lambda'_\mu = \lambda_{m-\mu+1}$ be the μ-th smallest part of Π_2^*. We also suppose that the sequence of integers Λ' satisfies the following two conditions (see [3], p. 54–55):

$$\text{(I)} \quad \Phi_{\Lambda'}(x) = \sum_{\substack{\lambda\in\Lambda' \\ \lambda\leqq x}} 1 = \frac{Ax^\alpha}{\log^\beta x}\left(1 + O\left(\frac{1}{\log x}\right)\right), \quad 0 < \alpha \leqq 1, \quad \beta \in \mathbf{R}.$$

(II) $\log Q(n;\Lambda') > c_1 n^{\alpha/(\alpha+1)} \log^{-\beta/(\alpha+1)} n\left(1-\log^{-1/(2\alpha+2)} n(\log\log n)^{-1}\right)$

where $Q(n;\Lambda')$ denote the number of partitions of n with distinct summands taken from Λ', and

$$c_1 = \left\{A\Gamma(\alpha+1)\zeta(\alpha+1)(1-2^{-\alpha})(\alpha+1)^{\alpha+\beta+1}\alpha^{-\alpha}\right\}^{1/(\alpha+1)}$$

where A is the constant specified in condition (I).

CONJECTURE 1. *Let $\omega_1(n)\log n \leqq \mu \leqq n^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)} n/\omega_2(n)$, where $\omega_1(n), \omega_2(n) \to \infty$ as $n \to \infty$. For sufficiently large n, the asymptotic relation*

$$\lambda'_\mu \sim 2^{1/\alpha}\alpha^{\beta/\alpha}A^{-1/\alpha}\mu^{1/\alpha}\log^{\beta/\alpha}\mu.$$

holds for almost all partitions of n into distinct parts taken from Λ'.

CONJECTURE 2. *Let a_k be the k-th term in the sequence Λ' which satisfies the two conditions stated above. Let $Q_{\Lambda'}(n;m)$ denote the number of partitions of n into distinct parts taken from Λ', such that every summand is greater than m. Then for $1 \leqq a_k \leqq n^{1/5}$ and $n \to \infty$*

$$Q_{\Lambda'}(n;a_k) = (1+o(1))2^{-k}\cdot Q(n;\Lambda'),$$

where $Q(n;\Lambda')$ denotes the number of partitions of n into distinct summands which are taken from the sequence Λ'.

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ON THE $(C, -1<\alpha<0, -1<\beta<0)$-SUMMABILITY OF SINGLE AND DOUBLE ORTHOGONAL SERIES

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In the present paper we consider necessary and sufficient conditions for the negative order Cesàro summability of single and double orthogonal series.

1. DEFINITION. Let α and β be real numbers $(\alpha, \beta > -1)$. We recall that the (C, α) and (C, β)-means of the numerical series Σa_m and Σa_{mn}, respectively, are defined as follows

$$\sigma_m^\alpha = \frac{1}{A_m^\alpha} \sum_{k=0}^{m} A_{m-k}^\alpha a_k,$$

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{A_m^\alpha A_n^\beta} \sum_{k=0}^{m} \sum_{l=0}^{n} A_{m-k}^\alpha A_{n-1}^\alpha a_{kl} \quad (m, n = 0, 1, \ldots),$$

where $A_m^\alpha = \binom{m+\alpha}{m}$ (see, e.g. [8 p. 77]).

If (Φ_m) is an orthonormal system (ONS) on $[0,1]$, then let $\sigma_m^\alpha((a_m),(\Phi_m),x)$ be the Cesàro mean of the series $\Sigma a_m \Phi_m$. Often for shortening, notations like (a_m) and (ONS) (Φ_m) will be omitted and we will use $\sigma_m^\alpha(x)$. Analogously, by the (ONS) (Φ_{mn}) on $[0,1]^2$ under $\sigma_{mn}^{\alpha\beta}((a_{mn}), (\Phi_{mn}), x, y)$ we mean the Cesàro mean of the series $\Sigma\Sigma a_{mn}\Phi_{mn}$. As in one-dimensional case, we will use $\sigma_{mn}^{\alpha\beta}(x, y)$.

2. Preliminary and main results. G. Sunouchi and S. Yano [7] have obtained a sufficient condition for the negative order Cesàro summability of single orthogonal series.

THEOREM (see [8]). *Let* (Φ_m) *be an* (ONS) *on* $[0,1]$ *and let the sequence* (a_m) *satisfy the condition*

$$\sum_{m=0}^{\infty} a_m^2 (n+1)^{-2\alpha} < \infty \quad (-1 < \alpha < 0). \tag{1}$$

Then $\sigma_m^\alpha(x)$ *converges almost everywhere (a.e.) on* $[0,1]$.

We have a two-dimensional analogue of the last result:

THEOREM 1. *Let* (Φ_{mn}) *be an* (ONS) *on* $[0,1]^2$ *and let the sequence* (a_{mn}) *satisfy the condition*

$$(2)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}^2(m+1)^{-2\alpha}(n+1)^{-2\beta}<\infty \quad (-1<\alpha,\beta<0).$$

Then $\sigma_{mn}^{\alpha\beta}$ *converges a.e. on* $[0,1]^2$ *as* $\min(m,n)\to\infty$.

We remark that condition (1) is also necessary for the Cesàro mean summability of general orthogonal series because the following theorem holds.

THEOREM 2. *Let the sequence* (a_m) *satisfy the condition*

$$(3)\qquad \sum_{m=0}^{\infty} a_m^2(m+1)^{-2\alpha}=\infty \quad (-1<\alpha<0).$$

Then there exists an (ONS) (Φ_m) *on* $[0,1]$ *for which*

$$\limsup_{m\to\infty} |\sigma_m^{\alpha}(x)|=\infty$$

a.e. on $[0,1]$.

The next result shows the sharpness of Theorem 1.

THEOREM 3. *If a sequence* (a_{mn}) *satisfies the condition*

$$(4)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}^2(m+1)^{-2\alpha}(n+1)^{-2\beta}=\infty \qquad (-1<\alpha,\beta<0),$$

then there exists an (ONS) (Φ_{mn}) *on* $[0,1]^2$, *for which*

$$\limsup_{\min(m,n)\to\infty} |\sigma_{mn}^{\alpha,\beta}(x)|=\infty$$

a.e. on $[0,1]^2$.

Finally, we note that the main results of the paper were announced in [1] and [2].

3. PROOF OF THEOREM 1. In the sequel, by C we denote positive absolute constants, not necessarily the same in each occurrence.

Let

$$(5)\qquad \hat{\sigma}_{mn}^{\alpha+1,\beta+1}(x,y)=\frac{1}{A_m^{\alpha+1}A_n^{\beta+1}}=\sum_{i=0}^{m} A_{m-i}^{\alpha}A_{n-k}^{\beta}a_{ik}\Phi_{ik}(x,y).$$

LEMMA 1. *Let* (Φ_{mn}) *be an* (ONS) *on* $[0,1]^2$ *and let the sequence* (a_{mn}) *satisfy the condition*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}^2(m+1)^{-2(\alpha+1)}(n+1)^{-2(\beta+1)} < \infty \quad (-1<\alpha,\beta<0). \tag{6}$$

Then $\hat{\sigma}_{mn}^{\alpha+1,\beta+1}(x,y)$ *tends to zero a.e. on* $[0,1]^2$, *as* $\min(m,n)\to\infty$.

PROOF. It is enough to prove that $\delta_{pq}^{\alpha+1,\beta+1}(x,y)\to 0$ a.e. on $[0,1]^2$, where

$$\delta_{pq}^{\alpha+1,\beta+1}(x,y) = \max_{\substack{2^p<m\leqq 2^{p+1}\\ 2^q<n\leqq 2^{q+1}}} |\hat{\sigma}_{mn}^{\alpha+1,\beta+1}(x,y)|.$$

By a known formula (see [5]), we obtain

$$\hat{\sigma}_{mn}^{\alpha+1,\beta+1}(x,y) = \frac{1}{A_m^{\alpha+1}A_n^{\beta+1}}\sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{(\alpha-1)/2}A_{n-k}^{(\beta-1)/2}\times$$

$$\times\sum_{\nu=0}^{i}\sum_{\mu=0}^{k} A_{i-\nu}^{(\alpha-1)/2}A_{k-\mu}^{(\beta-1)/2}a_{\nu\mu}\Phi_{\nu\mu}(x,y).$$

Hence, using Cauchy's inequality,

$$|\hat{\sigma}_{mn}^{\alpha+1,\beta+1}(x,y)|^2 \leqq \frac{1}{\{A_m^{\alpha+1}\}^2\{A_n^{\beta+1}\}^2}\sum_{i=0}^{m}\sum_{k=0}^{n}\{A_{m-i}^{(\alpha-1)/2}\}^2\{A_{n-k}^{(\beta-1)/2}\}^2\times \tag{7}$$

$$\times\sum_{i=0}^{m}\sum_{k=0}^{n}\Big\{\sum_{\nu=0}^{i}\sum_{\mu=0}^{k} A_{i-\nu}^{(\alpha-1)/2}A_{k-\mu}^{(\beta-1)/2}a_{\nu\mu}\Phi_{\nu\mu}(x,y)\Big\}^2.$$

Taking into account that

$$\sum_{i=0}^{m}\sum_{k=0}^{n}\Big\{A_{m-i}^{(\alpha-1)/2}\}^2\{A_{n-k}^{(\beta-1)/2}\}^2 \leqq C \qquad (-1<\alpha,\beta<0) \tag{8}$$

and $2^p<m\leqq 2^{p+1}$, $2^q<n\leqq 2^{q+1}$, from inequalities (7), (8) it follows that

$$\{\delta_{pq}^{\alpha+1,\beta+1}(x,y)\}^2 \leqq \frac{C}{2^{2p(\alpha+1)}2^{2q(\beta+1)}}\times$$

$$\times\sum_{i=0}^{2^{p+1}}\sum_{k=0}^{2^{q+1}}\Big\{\sum_{\nu=0}^{i}\sum_{\mu=0}^{k} A_{i-\nu}^{(\alpha-1)/2}A_{k-\mu}^{(\beta-1)/2}a_{\nu\mu}\Phi_{\nu\mu}(x,y)\Big\}^2.$$

Then, by (8),

$$\int_0^1\int_0^1 \{\delta_{pq}^{\alpha+1,\beta+1}(x,y)\}^2 dx dy \leqq$$

$$\leqq \frac{C}{2^{2p(\alpha+1)}2^{2q(\beta+1)}} \sum_{i=0}^{2^{p+1}} \sum_{k=0}^{2^{q+1}} \sum_{\nu=0}^{i} \sum_{\mu=0}^{k} \{A_{i-\nu}^{(\alpha-1)/2}\}^2 \{A_{k-\mu}^{(\beta-1)}\}^2 a_{\nu\mu}^2 \leqq$$

$$\leqq \frac{C}{2^{p(\alpha+1)}2^{2q(\beta+1)}} \sum_{\nu=0}^{2^{p+1}} \sum_{\mu=0}^{q^{q+1}} a_{\nu\mu}^2 \sum_{i=\nu}^{2^{p+1}} \sum_{k=\mu}^{2^{q+1}} \{A_{i-\nu}^{(\alpha-1)/2}\}^2 \{A_{k-\mu}^{(\beta-1)/2}\}^2 \leqq$$

$$\leqq \frac{C}{2^{2p(\alpha+1)}2^{2q(\beta+1)}} \sum_{\nu=0}^{2^{p+1}} \sum_{\mu=0}^{2^{q+1}} a_{\nu\mu}^2.$$

So, by (6),

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\int_0^1\int_0^1 \{\delta_{pq}^{\alpha+1,\beta+1}(x,y)\}^2 dx dy \leqq$$

$$\leqq C \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{2^{2p(\alpha+1)}2^{2q(\beta+1)}} \sum_{\nu=0}^{2^{p+1}} \sum_{\mu=0}^{2^{q+1}} a_{\nu\mu}^2 \leqq$$

$$\leqq C \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} a_{\nu\mu}^2 \sum_{p=\nu}^{\infty}\sum_{q=\mu}^{\infty} \frac{1}{2^{2p(\alpha+1)}2^{2q(\beta+1)}} < \infty.$$

Because the termwise integrated series is finite, B. Levi's theorem implies $\delta_{pq}^{\alpha+1,\beta+1}(x,y) \to 0$ a.e. on $[0,1]^2$. Lemma 1 is proved.

Let

$$(9)\quad R_{mn}^{\alpha+1,\beta}(x,y) = \frac{1}{A_m^{\alpha+1}A_n^{\beta}} \sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{\alpha}A_{n-k}^{\beta} a_{ik}\Phi_{ik}(x,y) \quad (\alpha,\beta > -1).$$

LEMMA 2. *Let* (Φ_{mn}) *be an* (ONS) *on* $[0,1]^2$ *and let the sequence* (a_{mn}) *satisfy the condition*

$$(10)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}^2 (m+1)^{-2(\alpha+1)} \log^2(n+2) < \infty \quad (-1 < \alpha < 0).$$

Then $R_{mn}^{\alpha+1,0}(x,y)$ *tends to zero a.e. on* $[0,1]^2$ *as* $\min(m,n) \to \infty$.

PROOF. Due to Cauchy's inequality,

$$\{R_{m,2^p}^{\alpha+1,0}(x,y)\}^2 \leqq \sum_{t=-2}^{p-1} (t+3)^2 \times$$

$$\times\Big\{\frac{1}{A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=2^t+1}^{2^{t+1}}A_{m-i}^{\alpha}a_{ik}\Phi_{ik}(x,y)\Big\}^2\sum_{t=-2}^{p-1}(t+3)^{-2},$$

where we make the following convention: for $t=-2$ and -1 by 2^t we mean -1 and 0, respectively.

Let

$$F_m^{\alpha+1}(x,y)=\sum_{t=-2}^{\infty}(t+3)^2\Big\{\frac{1}{A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=2^t+1}^{2^{t+1}}A_{m-i}^{\alpha}a_{ik}\Phi_{ik}(x,y)\Big\}^2.$$

It is obvious that

$$\{R_{m,2^p}^{\alpha+1}(x,y)\}^2\leqq CF_m^{\alpha+1}(x,y),$$

so it is enough to prove that $F_m^{\alpha+1}(x,y)\to 0$ a.e. on $[0,1]^2$ as $m\to\infty$.

Using a formula of [5] we obtain

$$F_m^{\alpha+1}(x,y)=$$

$$=\sum_{t=-2}^{\infty}(t+3)^2\Big\{\frac{1}{A_m^{\alpha+1}}\sum_{k=2^t+1}^{2^t+1}\sum_{i=0}^{m}A_{m-i}^{(\alpha-1)/2}\sum_{p=0}^{i}A_{i-p}^{(\alpha-1)/2}a_{pk}\Phi_{pk}(x,y)\Big\}^2.$$

By Cauchy's inequality and (8), when $2^q<m\leqq 2^{q+1}$, we have

$$F_m^{\alpha+1}(x,y)\leqq\sum_{t=-2}^{\infty}(t+3)^2\frac{1}{\{A_m^{\alpha+1}\}^2}\sum_{i=0}^{m}\{A_{m-i}^{(\alpha-1)/2}\}^2\cdot$$

$$\cdot\sum_{i=0}^{m}\Big\{\sum_{k=2^t+1}^{2^{t+1}}\sum_{p=0}^{i}A_{i-p}^{(\alpha-1)/2}a_{pk}\Phi_{pk}(x,y)\Big\}^2\leqq$$

$$\leqq C\sum_{t=-2}^{\infty}(t+3)^2\frac{1}{2^{2q(\alpha+1)}}\sum_{i=0}^{2^{q+1}}\Big\{\sum_{k=2^t+1}^{2^{t+1}}\sum_{p=0}^{t}A_{i-p}^{(\alpha-1)/2}a_{pk}\Phi_{pk}(x,y)\Big\}^2.$$

Now, using inequality (8) we obtain

$$\int_0^1\int_0^1\max_{2^q<m\leqq 2^{q+1}}F_m^{\alpha+1}(x,y)dxdy\leqq$$

$$\leqq C\sum_{t=-2}^{\infty}(t+3)^2\frac{1}{2^{2q(\alpha+1)}}\sum_{k=2^t+1}^{2^{t+1}}\sum_{i=0}^{2^{q+1}}\sum_{p=0}^{i}\{A_{i-p}^{(\alpha-1)/2}\}^2a_{pk}^2=$$

$$= C\sum_{t=-2}^{\infty}(t+3)^2\frac{1}{2^{2q(\alpha+1)}}\sum_{k=2^t+1}^{2^{t+1}}\sum_{p=0}^{2^{q+1}}a_{pk}^2\sum_{i=p}^{2^{q+1}}\{A_{i-p}^{(\alpha-1)/2}\}^2 \leqq$$

$$\leqq C\sum_{t=-2}^{\infty}(t+3)^2\frac{1}{2^{2q(\alpha+1)}}\sum_{k=2^t+1}^{2^{t+1}}\sum_{p=0}^{2^{q+1}}a_{pk}^2 \leqq$$

$$\leqq C\sum_{k=0}^{\infty}\sum_{i=0}^{2^{q+1}}a_{ik}^2\frac{\log^2(k+2)}{2^{2q(\alpha+1)}}.$$

So by (10)

$$\sum_{q=0}^{\infty}\int_0^1\int_0^1\max_{2^q<m\leqq 2^{q+1}}F_m^{\alpha+1}(x,y)\,dx\,dy \leqq$$

$$\leqq C\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2^{2q(\alpha+1)}}\sum_{i=0}^{2^{q+1}}a_{ik}^2\log^2(k+2) \leqq$$

$$\leqq C\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}a_{ik}^2\log^2(k+2)\sum_{2^{q+1}\geqq i}\frac{1}{2^{2q(\alpha+1)}} \leqq \infty.$$

Due to B. Levi's theorem

$$R_{m,2^p}^{\alpha+1,0}(x,y)\to 0 \quad \text{a.e. on} \quad [0,1]^2. \tag{11}$$

Further

$$\max_{2^p<n\leqq 2^{p+1}}|R_{mn}^{\alpha+1,0}(x,y)| \leqq |R_{m,2^p}^{\alpha+1,0}(x,y)| + |M_{mp}^{\alpha+1}(x,y)|, \tag{12}$$

where

$$M_{mp}^{\alpha+1}(x,y) = \max_{2^p<n\leqq 2^{p+1}}\frac{1}{A^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=2^p+1}^{n}A_{m-i}^{\alpha}a_{ik}\Phi_{ik}(x,y).$$

From the known formula (see [5]) and Cauchy's inequality it follows that

$$\{M_{mp}^{\alpha+1}(x,y)\}^2 \leqq \max_{2^p<n\leqq 2^{p+1}}\frac{1}{\{A_m^{\alpha+1}\}^2}\sum_{i=0}^{m}\{A_{m-i}^{(\alpha-1)}\}^2\times$$

$$\sum_{i=0}^{m}\Big\{\sum_{k=2^p+1}^{n}\sum_{\nu=0}^{i}A_{i-\nu}^{(\alpha-1)/2}a_{\nu k}\Phi_{\nu k}(x,y)\Big\}^2 \leqq$$

$$\leqq C \max_{2^p<n\leqq 2^{p+1}} \frac{1}{\{A_m^{\alpha+1}\}^2} \sum_{i=0}^{m} \Big\{ \sum_{k=2^p+1}^{n} \sum_{\nu=0}^{i} A_{i-\nu}^{(\alpha-1)/2} a_{\nu k} \Phi_{\nu k}(x,y) \Big\}^2.$$

Further

$$\max_{2^q<m\leqq 2^{q+1}} \{M_{mp}^{\alpha+1}(x,y)\}^2 \leqq C \max_{\substack{2^q<m\leqq 2^{q+1}\\ 2^p<n\leqq 2^{p+1}}} \frac{1}{2^{2q(\alpha+1)}} \times$$

$$\times \sum_{i=0}^{2^{q+1}} \Big\{ \sum_{k=2^p+1}^{n} \sum_{\nu=0}^{i} A_{i-\nu}^{(\alpha-1)/2} a_{\nu k} \Phi_{\nu k}(x,y) \Big\}^2.$$

Applying the Rademacher–Menshov inequality (see [4], Theorem 3) separately for each fixed i, we get

$$\int_0^1 \int_0^1 \max_{2^q<m\leqq 2^{q+1}} \{M_{mp}^{\alpha+1}(x,y)\}^2 dx dy \leqq$$

$$\leqq C \log^2 2^{p+1} \sum_{i=0}^{2^{q+1}} \frac{1}{2^{2q(\alpha+1)}} \sum_{k=2^p+1}^{2^{p+1}} \sum_{\nu=0}^{i} A_{i-\nu}^{(\alpha-1)/2}\}^2 a_{\nu k}^2 \leqq$$

$$\leqq C \log^2 2^{p+1} \sum_{k=2^p+1}^{2^{p+1}} \frac{1}{2^{2q(\alpha+1)}} \sum_{\nu=0}^{2^{q+1}} a_{\nu k}^2 \sum_{i=\nu}^{2^{q+1}} \{A_{i-\nu}^{(\alpha-1)/2}\}^2 \leqq$$

$$\leqq C \sum_{k=2^p+1}^{2^{p+1}} \frac{1}{2^{2q(\alpha+1)}} \sum_{\nu=0}^{2^{q+1}} a_{\nu k}^2 \log^2(k+2).$$

Consequently by (10)

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \int_0^1 \int_0^1 \max_{2^q<m\leqq 2^{q+1}} \{M_{mp}^{\alpha+1}(x,y)\}^2 dx dy \leqq$$

$$\leqq C \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{k=2^p+1}^{2^{p+1}} \sum_{i=0}^{2^{q+1}} \frac{1}{2^{2q(\alpha+1)}} a_{ik}^2 \log^2(k+2) \leqq$$

$$\leqq C \sum_{k=2}^{\infty} \sum_{i=0}^{\infty} a_{ik}^2 \log^2(k+2) \sum_{2^{q+1}\geqq i} \frac{1}{2^{2q(\alpha+1)}} < \infty.$$

Hence B. Levi's theorem implies that

$$M_{mp}^{\alpha+1}(x,y) \to 0 \tag{13}$$

a.e. on $[0,1]^2$.

Combining (11), (12) and (13) we find $R_{mn}^{\alpha+1}(x,y) \to 0$ a.e. on $[0,1]^2$. Lemma 2 is proved.

LEMMA 3. *Let* (Φ_{mn}) *be an* (ONS) *on* $[0,1]^2$ *and let the sequence* (a_{mn}) *satisfy the condition*

$$(14)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}^2(m+1)^{-2(\alpha+1)} < \infty \qquad (-1<\alpha<0).$$

Then the convergence of $R_{mn}^{\alpha+1,1}(x,y)$ *and* $R_{m,2^p}^{\alpha+1,0}(x,y)$ *is equivalent a.e. on* $[0,1]^2$.

PROOF. First prove the equivalence of the convergence of $R_{m,2^p}^{\alpha+1,0}(x,y)$ and $R_{m,2^p}^{\alpha+1,1}(x,y)$.

By a known transformation (see [5]) and Cauchy's inequality we get

$$\{R_{m,2^p}^{\alpha+1,0}(x,y) - R_{m,2^p}^{\alpha+1,1}(x,y)\}^p =$$

$$= \frac{1}{\{A_m^{\alpha+1}\}^2}\Big\{\sum_{i=0}^{m}\sum_{k=0}^{2^p} A_{m-i}^{(\alpha-1)/2}\sum_{q=0}^{i} A_{i-q}^{(\alpha-1)/2}\frac{k}{2^p+1}a_{qk}\Phi_{qk}(x,y)\Big\}^2 \leqq$$

$$\leqq \frac{1}{\{A_m^{\alpha+1}\}^2}\sum_{i=0}^{m}\{A_{m-i}^{(\alpha-1)/2}\}^2\sum_{i=0}^{m}\Big\{\sum_{k=0}^{2^p}\sum_{q=0}^{i} A_{i-q}^{(\alpha-1)/2}\frac{k}{2^p+1}a_{qk}\Phi_{qk}(x,y)\Big\}^2 \leqq$$

$$\leqq C\frac{1}{\{A_m^{\alpha+1}\}^2}\sum_{i=0}^{m}\Big\{\sum_{k=0}^{2^p}\sum_{q=0}^{i} A_{i-q}^{(\alpha-1)/2}\frac{k}{2^p+1}a_{qk}\Phi_{qk}(x,y)\Big\}^2.$$

Hence

$$\sum_{p=0}^{\infty}\int_0^1\int_0^1\Big\{R_{m,2^p}^{\alpha+1,0}(x,y) - R_{m,2^p}^{\alpha+1,1}(x,y)\Big\}dx\,dy \leqq$$

$$\leqq C\frac{1}{\{A_m^{\alpha+1}\}^2}\sum_{p=0}^{\infty}\sum_{i=0}^{m}\sum_{k=0}^{2^p}\sum_{q=0}^{i}\{A_{i-q}^{(\alpha-1)/2}\}^2\frac{k^2}{2^{2p}}a_{qk}^2 \leqq$$

$$\leqq C\frac{1}{\{A_m^{\alpha+1}\}^2}\sum_{p=0}^{\infty}\sum_{k=0}^{2^p}\sum_{q=0}^{m}\frac{k^2}{2^{2p}}a_{qk}^2\sum_{i=q}^{m}\{A_{i-q}^{(\alpha-1)/2}\}^2 \leqq$$

$$\leqq C\frac{1}{\{A_m^{\alpha+1}\}^2}\sum_{k=0}^{\infty}\sum_{i=0}^{m}a_{ik}^2k^2\sum_{2^p>k}^{\infty}\frac{1}{2^{2p}} \leqq C\sum_{k=0}^{\infty}\sum_{i=0}^{m}\frac{a_{ik}^2}{\{A_m^{\alpha+1}\}^2}.$$

Further

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\int_0^1\int_0^1\max_{2^q<m\leqq 2^{q+1}}\{R_{m,2^p}^{\alpha+1,0}(x,y) - R_{m,2^p}^{\alpha+1,1}(x,y) - R_{m,2^p}^{\alpha+1,1}(x,y)\}^2dx\,dy \leqq$$

$$\leqq C\sum_{q=0}^{\infty}\frac{1}{2^{2q(\alpha+1)}}\sum_{k=0}^{\infty}\sum_{i=0}^{2^{q+1}}a_{ik}^2\leqq C\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}a_{ik}^2\sum_{2^{q+1}>i}\frac{1}{2^{2q(\alpha+1)}}<\infty.$$

Then due to B. Levi's theorem $|R_{m,2^p}^{\alpha+1,1}(x,y)-R_{m,2^p}^{\alpha+1,1}(x,y)|\to 0$ a.e. on $[0,1]^2$.

Let

$$\delta_{mp}^{\alpha+1}(x,y)=\max_{2^p\leqq k<2^{p+1}}|R_{mk}^{\alpha+1,1}(x,y)-R_{m,2^p}^{\alpha+1,1}(x,y)|. \tag{15}$$

We shall prove $\delta_{mp}^{\alpha+1}(x,y)\to 0$ a.e. on $[0,1]^2$, then taking into account the above estimate, we obtain Lemma 3.

Indeed,

$$\delta_{mp}^{\alpha+1}(x,y)\leqq\sum_{s=2^p+1}^{2^{p+1}}|R_{ms}^{\alpha+1}(x,y)-R_{m,s-1}^{\alpha+1,1}(x,y)|\leqq \tag{16}$$

$$\leqq\left\{\sum_{s=2^p+1}^{2^{p+1}}|R_{ms}^{\alpha+1,1}(x,y)-R_{m,s-1}^{\alpha+1,1}(x,y)|^2\right\}^{1/2}2^{p/2}.$$

By the definition of $R_{ms}^{\alpha+1,1}(x,y)$ and a known formula (see [5]) we get

$$R_{ms}^{\alpha+1,1}(x,y)=\frac{1}{A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=0}^{s}A_{m-i}^{(\alpha-1)/2}\sum_{t=0}^{i}A_{i-t}^{(\alpha-1)/2}\Big(1-\frac{k}{s+1}\Big)a_{tk}\Phi_{tk}(x,y).$$

Then

$$R_{ms}^{\alpha+1,1}(x,y)-R_{m,s-1}^{\alpha+1,1}(x,y)=$$

$$=\frac{1}{A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=1}^{s}A_{m-i}^{(\alpha-1)/2}\sum_{t=0}^{i}A_{i-t}^{(\alpha-1)/2}\Big(\frac{1}{s}-\frac{1}{s+1}\Big)ka_{tk}\Phi_{tk}(x,y).$$

Combining the last transformation with (16) and applying a Cauchy's inequality we find

$$\{\delta_{mp}^{\alpha+1}(x,y)\}^2\leqq$$

$$\leqq\frac{2^p}{\{A_m^{\alpha+1}\}^2}\sum_{s=2^p+1}^{2^{p+1}}\Big\{\sum_{i=0}^{m}\sum_{k=1}^{s}A_{m-i}^{(\alpha-1)/2}\sum_{t=0}^{i}A_{i-t}^{(\alpha-1)/2}\Big(\frac{1}{s}-\frac{1}{s+1}\Big)ka_{tk}\Phi_{tk}(x,y)\Big\}^2\leqq$$

$$\leqq\frac{2^p}{\{A_m^{\alpha+1}\}^2}\sum_{s=2^p+1}^{2^{p+1}}\sum_{i=0}^{m}\{A_{m-i}^{(\alpha-1)/2}\}^2\sum_{i=0}^{m}\Big\{\sum_{k=1}^{s}\sum_{t=0}^{i}A_{i-t}^{(\alpha-1)/2}\Big(\frac{1}{s}-\frac{1}{s+1}\Big)k\times$$

$$\times a_{tk}\Phi_{tk}(x,y)\Big\}^2 \leqq C\frac{2^p}{\{A_m^{\alpha+1}\}^2}\sum_{s=2^p+1}^{2^{p+1}}\sum_{i=0}^{m}\Big\{\sum_{k=1}^{s}\sum_{t=0}^{i}A_{i-t}^{(\alpha-1)/2}\times\frac{1}{s^2}ka_{tk}\Phi_{tk}(x,y)\Big\}^2.$$

Consequently

$$\sum_{p=0}^{\infty}\int_0^1\int_0^1 \max_{2^q<m\leqq 2^{q+1}}\{\delta_{mp}^{\alpha+1}(x,y)\}^2dxdy \leqq$$

$$\leqq C\sum_{p=0}^{\infty}\frac{2^p}{2^{2q(\alpha+1)}}\sum_{s=2^p+1}^{2^{p+1}}\sum_{i=0}^{2^{q+1}}\sum_{k=1}^{s}\sum_{t=0}^{i}\{A_{i-t}^{(\alpha-1)/2}\}^2\frac{1}{s^4}k^2a_{tk}^2 \leqq$$

$$\leqq C\sum_{s=2}^{\infty}\frac{1}{2^{2q(\alpha+1)}}\sum_{i=0}^{2^{q+1}}\sum_{k=1}^{s}\sum_{t=0}^{i}\{A_{i-t}^{(\alpha-1)/2}\}^2\frac{1}{s^3}k^2a_{tk}^2 \leqq$$

$$\leqq C\sum_{k=1}^{\infty}\frac{1}{2^{2q(\alpha+1)}}\sum_{i=0}^{2^{q+1}}\sum_{t=0}^{i}\{A_{i-t}^{(\alpha-1)/2}\}^2k^2a_{tk}^2\sum_{s=k}^{\infty}\frac{1}{s^3} \leqq$$

$$\leqq C\sum_{k=1}^{\infty}\frac{1}{2^{2q(\alpha+1)}}\sum_{t=0}^{2^{q+1}}a_{tk}^2\sum_{i=t}^{2^{q+1}}\{A_{i-t}^{(\alpha-1)/2}\}^2 \leqq$$

$$\leqq C\sum_{k=1}^{\infty}\frac{1}{2^{2q(\alpha+1)}}\sum_{i=0}^{2^{q+1}}a_{ik}^2.$$

Thus

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\int_0^1\int_0^1 \max_{2^q<m\leqq 2^{q+1}}\{\delta_{mp}^{\alpha+1}(x,y)\}^2dxdy \leqq$$

$$\leqq C\sum_{k=1}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^{2q(\alpha+1)}}\sum_{i=0}^{2^{q+1}}a_{ik}^2 \leqq C\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}a_{ik}^2\sum_{2^q>i}\frac{1}{2^{2q(\alpha+1)}} < \infty.$$

Hence (14) and B. Levi's theorem imply (15). So the proof of Lemma 3 is complete.

The statement of the following lemma is an easy consequence of Lemmas 2 and 3.

LEMMA 4. *Let* (Φ_{mn}) *be an* (ONS) *on* $[0,1]^2$ *and let the sequence* (a_{mn}) *satisfy the condition*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}^2(m+1)^{-2(\alpha+1)}\log^2\log(n+4) < \infty. \tag{17}$$

Then $R_{mn}^{\alpha+1,1}(x,y)$ *tends to zero a.e. on* $[0,1]^2$ *as* $\min(m,n)\to 0$.

PROOF. Lemma 3 shows that it is enough to prove $R_{m,2^p}^{\alpha+1,0}(x,y)\to 0$ a.e. on $[0,1]^2$ as $\min(m,n)\to\infty$. To this end, we set

$$\hat{\Phi}_{in}(x,y)=\Big\{\sum_{k=2^{n-2}+1}^{2^{n-1}} a_{ik}\Phi_{ik}(x,y)\Big\}\Big/\Big\{\sum_{k=2^{n-2}+1}^{2^{n-1}} a_{ik}^2\Big\}^{1/2},$$

$$\hat{a}_{in}=\Big\{\sum_{k=2^{n-2}+1}^{2^{n-1}} a_{ik}^2\Big\}^{1/2}\qquad (n=0,1,\dots),$$

while keeping in mind the convention made at the beginning of the proof of Lemma 2. Obviously, (Φ_{mn}) is an (ONS). Thus

$$R_{m,2^p}^{\alpha+1,0}(x,y)=\frac{1}{A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=0}^{2^p} A_{m-i}^{\alpha}a_{ik}\Phi_{ik}(x,y)=$$

$$=\frac{1}{A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=0}^{p+1}\sum_{q=2^{k-2}+1}^{2^{k-1}} A_{m-i}^{\alpha}a_{iq}\Phi_{iq}(x,y)=$$

$$=\frac{1}{A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{k=0}^{p+1} A_{m-i}^{\alpha}\hat{a}_{ik}\hat{\Phi}_{ik}(x,y)=\hat{R}_{m,p+1}^{\alpha+1,0}(x,y).$$

By (17), $(\hat{a}_{mn})$ satisfies (10). So we can apply Lemma 2 and obtain that $\hat{R}_{mp}^{\alpha+1,0}(x,y)$ tends to zero a.e. on $[0,1]^2$, which in this case says: $R_{m,2^p}^{\alpha+1,0}(x,y)$ tends to zero a.e. on $[0,1]^2$.

LEMMA 5. *If* $\beta>-1$, $\varepsilon>0$, $\alpha>-1$, *and*

$$\frac{1}{M}\sum_{n=0}^{M}\{R_{mn}^{\alpha+1,\beta}(x,y)\}^2\to 0\quad as\quad \min(m,M)\to\infty \tag{18}$$

a.e. on $[0,1]^2$, *then*

$$R_{mn}^{\alpha+1,\beta+1/2+\varepsilon}(x,y)\to 0\quad as\quad \min(m,n)\to\infty. \tag{19}$$

PROOF. Since

$$A_n^{\beta}\leqq C(n^{\beta})\quad\text{and}\quad \sum_{k=0}^{n}\{A_{n-k}^{-1/2+\varepsilon}A_k^{\beta}\}^2\leqq C(n^{2(\beta+\varepsilon)}),$$

using a known formula (see [5]), the Cauchy inequality and condition (18) we get

$$|R_{mn}^{\alpha+1,\beta+1/2+\varepsilon}(x,y)| =$$

$$= \frac{1}{A_m^{\alpha+1}A_n^{\beta+1/2+\varepsilon}}\Big|\sum_{k=0}^{n} A_{n-k}^{-1/2+\varepsilon} A_k^{\beta}\frac{1}{A_k^{\beta}}\sum_{i=0}^{m}\sum_{p=0}^{k} A_{m-i}^{\alpha}A_{k-p}^{\beta}a_{ip}\Phi_{ip}(x,y)\Big| \leqq$$

$$\leqq \frac{1}{A_n^{\beta+1/2+\varepsilon}}\Big|\sum_{k=0}^{n}\{A_{n-k}^{-1/2+\varepsilon}A_k^{\beta}\}^2 n\frac{1}{n}\sum_{k=0}^{n}\Big\{\frac{1}{A_k^{\beta}A_m^{\alpha+1}}\sum_{i=0}^{m}\sum_{p=0}^{k}A_{m-i}^{\alpha}\times$$

$$\times A_{k-p}^{\beta}a_{ip}\Phi_{ip}(x,y)\Big\}^2\Big|^{1/2} \leqq O\Big(\frac{1}{n^{\beta+1/2+\varepsilon}}\Big)O(n^{\beta+\varepsilon})o(n^{1/2}) = o(1).$$

LEMMA 6. *If $\beta > 1/2$, $-1 < \alpha < 0$, and the condition*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}^2(m+1)^{-2(\alpha+1)} < \infty \tag{20}$$

is satisfied, then

$$T_{mN}^{\alpha+1,\beta}(x,y) = \frac{1}{N}\sum_{n=0}^{N}\{R_{mn}^{\alpha+1,\beta-1}(x,y) - R_{mn}^{\alpha+1,\beta}(x,y)\}^2 \tag{21}$$

tends to zero a.e. on $[0,1]^2$ as $\min(m,N)\to\infty$.

PROOF. If N is a positive integer, then $2^{p-1} < N \leqq 2^p$ with some non-negative integer p. (For simplicity in notation, we neglect the case $N = 0$.) Since

$$T_{mN}^{\alpha+1,\beta}(x,y) \leqq 2T_{m,2^p}^{\alpha+1,\beta}(x,y),$$

it is enough to prove that $T_{m,2^p}^{\alpha+1,\beta}(x,y)\to 0$ a.e. on $[0,1]^2$ as $\min(m,p)\to\infty$.

According to a known formula (see [5]) and an easy transformation, we obtain

$$T_{m,2^p}^{\alpha+1,\beta}(x,y) = 2\sum_{r=-2}^{p-1}\frac{1}{2^{p-r-1}}\cdot\frac{1}{2^{r+2}}\times$$

$$\times\sum_{n=2^r+1}^{2^{r+1}}\Big\{\frac{1}{A_m^{\alpha+1}\beta A_n^{\beta}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{\alpha}A_{n-k}^{\beta-1}ka_{ik}\Phi_{ik}(x,y)\Big\}^2.$$

So it is enough to prove that

$$\hat{T}_{m,r}^{\alpha+1,\beta}(x,y) = \frac{1}{2^{r+2}}\sum_{n=2^r+1}^{2^{r+1}}\Big\{\frac{1}{A_m^{\alpha+1}\beta A_n^{\beta}}\times$$

$$\times \sum_{i=0}^{m} \sum_{k=1}^{n} A_{m-i}^{\alpha} A_{n-k}^{\beta-1} k a_{ik} \Phi_{ik}(x,y) \Big\}^2 \to 0 \quad \text{as} \quad \min(m,r) \to \infty$$

a.e. on $[0,1]^2$.

Using the formula (see [5]) and the Cauchy inequality

$$\hat{T}_{m,r}^{\alpha+1,\beta}(x,y) = \frac{1}{2^{r+2}} \sum_{n=2^r+1}^{2^{r+1}} \frac{1}{\{A_m^{\alpha+1}\}^2 \beta^2 \{A_n^{\beta}\}^2} \times$$

$$\times \Big\{ \sum_{i=0}^{m} \sum_{k=1}^{n} A_{m-i}^{(\alpha-1)/2} A_{n-k}^{\beta-1} \sum_{q=0}^{i} A_{i-q}^{(\alpha-1)/2} a_{qk} k \Phi_{qk}(x,y) \Big\}^2 \leqq$$

$$\leqq \frac{1}{2^{r+2}} \sum_{n=2^r+1}^{2^{r+1}} \frac{1}{\{A_m^{\alpha+1}\}^2 \beta^2 \{A_n^{\beta}\}^2} \sum_{i=0}^{m} \{A_{m-i}^{(\alpha-1)/2}\}^2 \times$$

$$\times \sum_{i=0}^{m} \Big\{ \sum_{k=1}^{n} A_{n-k}^{\beta-1} \sum_{q=0}^{i} A_{i-q}^{(\alpha-1)/2} a_{qk} k \Phi_{qk}(x,y) \Big\}^2.$$

We can easily verify the estimate

$$\sum_{n=k}^{\infty} \frac{\{A_{n-k}^{\beta-1}\}^2}{\{A_n^{\beta}\}^2} \leqq \frac{C}{K} \qquad (\beta > 0).$$

So we have

$$\sum_{r=0}^{\infty} \int_0^1 \int_0^1 \max_{2^p < m \leqq 2^{p+1}} \hat{T}_{mr}^{\alpha+1,\beta}(x,y) dx dy \leqq$$

$$\leqq C \sum_{r=0}^{\infty} \frac{1}{2^{r+1}} \sum_{n=2^r+1}^{2^{r+1}} \frac{1}{2^{2p(\alpha+1)} \{A_n^{\beta}\}^2} \sum_{i=0}^{2^{p+1}} \sum_{k=1}^{n} \sum_{q=0}^{i} \{A_{n-k}^{\beta-1}\}^2 \times$$

$$\times \{A_{i-q}^{(\alpha-1)/2}\}^2 k^2 a_{qk}^2 \leqq C \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \frac{1}{2^{2p(\alpha+1)} \{A_n^{\beta}\}^2} \times$$

$$\times \sum_{i=0}^{2^{p+1}} \sum_{k=1}^{n} \sum_{q=0}^{i} \{A_{n-k}^{\beta-1}\}^2 \{A_{i-q}^{(\alpha-1)/2}\}^2 k a_{qk}^2 \leqq$$

$$\leqq C \sum_{k=1}^{\infty} \frac{1}{2^{2p(\alpha+1)}} \sum_{q=0}^{2^{p+1}} a_{qk}^2 k \sum_{n=k}^{\infty} \frac{\{A_{n-k}^{\beta-1}\}^2}{\{A_n^{\beta}\}^2} \sum_{i=q}^{2^{p+1}} \{A_{i-q}^{(\alpha-1)/2}\}^2 \leqq$$

$$\leqq C\sum_{k=1}^{\infty}\frac{1}{2^{2p(\alpha+1)}}\sum_{i=0}^{2^{p+1}}a_{ik}^2.$$

Hence and by (20)

$$\sum_{r=0}^{\infty}\sum_{p=1}^{\infty}\int_0^1\int_0^1\max_{2^p<m\leqq 2^{p+1}}\hat{T}_{mr}^{\alpha+1,\beta}(x,y)\,dx\,dy\leqq$$

$$\leqq C\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{1}{2^{2p(\alpha+1)}}\sum_{i=0}^{2^{p+1}}a_{ik}^2\leqq C\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}a_{ik}^2\sum_{p=i}^{\infty}\frac{1}{2^{2p(\alpha+1)}}<\infty.$$

So by B. Levi's theorem

$$\hat{T}_{mr}^{\alpha+1,\beta}(x,y)\to 0\quad\text{as}\quad\min(m,r)\to\infty$$

a.e. on $[0,1]^2$.

LEMMA 7. *If (Φ_{mn}) is an (ONS) on $[0,1]^2$ and a sequence (a_{mn}) satisfies the condition*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}^2(m+1)^{-2(\alpha+1)}\log^2\log(n+4)<\infty\quad(-1<\alpha<0),\tag{22}$$

then for arbitrary $\beta>0$

$$R_{mn}^{\alpha+1,\beta}(x,y)\to 0\quad\textit{as}\quad\min(m,n)\to\infty\tag{23}$$

a.e. on $[0,1]^2$.

PROOF. By (22), the sequence (a_{mn}) satisfies (17) and (20), thus combining Lemmas 4 and 6, for $\beta=1$ we get

$$\frac{1}{N}\sum_{n=0}^{N}\{R_{mn}^{\alpha+1,0}(x,y)\}^2\to 0\quad\text{as}\quad\min(m,n)\to\infty.$$

If we set $\beta=0$ then by Lemma 5 for arbitrary $\varepsilon>0$

$$R_{mn}^{\alpha+1,1/2+\varepsilon}(x,y)\to 0\quad\text{as}\quad\min(m,n)\to\infty$$

a.e. on $[0,1]^2$.

Using Lemma 6 (for $\beta=1/2+\varepsilon$) we get

$$\frac{1}{N}\sum_{n=0}^{N}R_{mn}^{\alpha+1,-1/2+\varepsilon}(x,y)\to 0\quad\text{as}\quad\min(m,N)\to\infty$$

a.e. on $[0,1]^2$.

By Lemma 5, we obtain

$$R_{mn}^{\alpha+1,2\varepsilon}(x,y) \to 0 \quad \text{as} \quad \min(m,n) \to \infty$$

a.e. on $[0,1]^2$.

Since ε is an arbitrary positive number, this is equivalent to Lemma 7 to be proved.

PROOF OF THEOREM 1. We will make use of the following representation:

$$\sigma_{mn}^{\alpha\beta}(x,y) - \sigma_{mn}^{\alpha+1,\beta+1}(x,y) = \tag{24}$$

$$= \frac{1}{(\alpha+1)(\beta+1)A_m^{\alpha+1}A_n^{\beta+1}} \sum_{i=1}^{m}\sum_{k=1}^{n} A_{m-i}^{\alpha}A_{n-k}^{\beta} ika_{ik}\Phi_{ik}(x,y) +$$

$$+ \frac{1}{(\alpha+1)A_m^{\alpha+1}A_n^{\beta+1}} \sum_{i=1}^{m}\sum_{k=0}^{n} A_{m-i}^{\alpha}A_{n-k}^{\beta+1} ia_{ik}\Phi_{ik}(x,y) +$$

$$+ \frac{1}{(\beta+1)A_m^{\alpha+1}A_n^{\beta+1}} \sum_{i=0}^{m}\sum_{k=1}^{n} A_{m-i}^{\alpha+1}A_{n-k}^{\beta} ka_{ik}\Phi_{ik}(x,y).$$

This follows from the identities

$$A_m^{\alpha-1} = \frac{\alpha}{\alpha+m}A_m^{\alpha} \quad \text{and} \quad A_{m-i}^{\alpha} = \frac{\alpha+m-i}{\alpha}A_{m-i}^{\alpha-1}.$$

Since (mna_{mn}) satisfies (6), and (ma_{mn}) satisfies (22), thus Lemma 1, Lemma 7 and its symmetric analogue show that the right side of (24) converges. By a theorem of Móricz (see [5]), the second term converges a.e. on $[0,1]^2$. Combining all statements we get Theorem 1.

PROOF OF THEOREM 2. Without loss of generality we assume that $a_m \neq 0$. With the notation

$$a_m^* = \min\{|a_m|, (m+1)^{\alpha}\},$$

it is clear that

$$\sum_{m=0}^{\infty} a_m^{*2}(m+1)^{-2\alpha} = \infty,$$

and there are numbers S_m, $1 > S_1 > S_2 > \ldots > S_m \to 0$ such that

$$\sum_{m=0}^{\infty} a_m^{*2}(m+1)^{-2\alpha}S_m = \infty. \tag{25}$$

Since $a_m^{*2}(m+1)^{-2\alpha}S_m \leqq 1$, we can define a stochastically independent (ONS) on $[0,1]$ for which

$$\Phi_m(x) = \begin{cases} a_m^{*-1}(m+1)^\alpha S_m^{-1/2} & \text{for } x \in A_m^1, \\ -a_m^{*-1}(m+1)^\alpha S_m^{-1/2} & \text{for } x \in A_m^2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{mes} A_m^1 = \operatorname{mes} A_m^2 = a_m^{*2}(m+1)^{-2\alpha}S_m/2$.

Then $|a_m\Phi_m(x)| \geqq S_m^{-1/2}(m+1)^\alpha$ for $x \in A_m^1 \cup A_m^2$, and by (25), the Borel–Cantelli lemma (see [3] p. 47) yields

$$\limsup_{m\to\infty} |a_m\Phi_m(x)|(m+1)^\alpha = \infty.$$

By virtue of a well-known theorem (see [6], p. 78) we get

$$\limsup_{m\to\infty} |\sigma_m^\alpha(x)| = \infty.$$

Theorem 2 is proved.

PROOF OF THEOREM 3. We begin with a few definitions and notations. By an interval $\langle a,b\rangle$ we mean either the open interval (a,b), or one of the half-closed intervals $[a,b)$ and $(a,b]$, or the closed interval $[a,b]$. By a rectangle $R = \langle a,b\rangle \times \langle c,d\rangle$ we mean a rectangle with sides parallel to the coordinate axes.

A function f defined on $[0,1]^2$ is said to be a step function if $[0,1]^2$ can be represented as the union of finitely many disjoint rectangles such that f is constant on each of these rectangles. A subset H of $[0,1]^2$ is said to be simple if H is the union of finitely many disjoint rectangles.

Given a function f defined on $[0,1]^2$ and a subrectangle $H = \langle a,b\rangle\times\langle c,d\rangle$, we set

$$f(T;x,y) = \begin{cases} f((x-a)/(b-a),(y-c)/(d-c)) & \text{for } (x,y)\in T, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, if H is a subset of $[0,1]^2$, then we denote by $H(T)$ the set into which T is carried over by the linear transformation $\hat{x} = (b-a)x + a$ and $\hat{y} = (d-c) + c$.

Now we present five lemmas.

LEMMA 8. *If* (Ψ_{mn}) *is an* (ONS) *of step functions on* $[0,1]^2$ *and* H *is a simple set of* $[0,1]^2$ *such that*

$$\operatorname{mes} H > 0, \tag{28}$$

and

$$\limsup_{\min(m,n)\to\infty} |\sigma_{mn}^{\alpha\beta}((a_{mn}),(\Psi_{mn}),x,y)| = \infty \text{ for } (x,y)\in H \ (-1<\alpha,\beta), \tag{29}$$

then it is possible to construct an (ONS) (Φ_{mn}) *on* $[0,1]^2$ *such that*

$$\limsup_{\min(m,n)\to\infty} |\sigma_{mn}^{\alpha\beta}((a_{mn}),(\Phi_{mn}),x,y)| = \infty \tag{30}$$

a.e. on $[0,1]^2$.

PROOF. The proof is modelled after the construction used in [6]. By (28), (29), using Egorov's theorem there exist an increasing sequence $\{r_p\colon r_{p+1} \geqq 2r_p;\ p = 1,2,\dots,\ r_1 = 0\}$ of integers and a sequence $\{H_p\colon p = 1,2,\dots\}$ of simple subsets of $[0,1]^2$ such that for $p = 1,2,\dots$ $\operatorname{mes} H_p > 0$, and for $(x,y) \in H_p$

$$\max_{r_p \leqq m,n < r_{p+1}} \left| \{A_m^\alpha A_n^\beta\}^{-1} \sum_{(i,k)\in Q_{mn}\setminus Q_{r_p-1,r_p-1}} A_{mi}^\alpha A_{nk}^\beta a_{ik} \Psi_{ik}(x,y) \right| \geqq$$

$$\geqq p + 4 \sum_{i=0}^{r_p-1} \sum_{k=0}^{r_p-1} |a_{ik}| M_{ik},$$

where

$$Q_{mn} = \{(i,k)\colon i = 0,1,\dots,m; k = 0,1,\dots,n\} \quad (m,n = 0,1,\dots)$$

and

$$M_{mn} = \max_{(x,y)\in[0,1]^2} |\Psi_{mn}(x,y)| \quad (m,n = 0,1,\dots).$$

Our goal is to construct an (ONS) (Φ_{mn}) of step functions on $[0,1]^2$ and a sequence $(E_p : p = 1,2,\dots)$ of simple subsets of $[0,1]^2$ such that these sets are stochastically independent, for $p = 1,2,\dots$

$$\operatorname{mes} E_p \geqq C, \tag{31}$$

for $(x,y) \in E_p$

$$\max_{r_p \leqq m,n < r_{p+1}} \{A_m^\alpha A_n^\beta\}^{-1} \left| \sum_{(i,k)\in Q_{mn}\setminus Q_{r_p-1,r_p-1}} A_{m-i}^\alpha A_{n-k}^\beta a_{ik} \Phi_{ik}(x,y) \right| \geqq \tag{32}$$

$$\geqq p + 4 \sum_{i=0}^{r_p-1} \sum_{k=0}^{r_p-1} |a_{ik}\Phi_{ik}(x,y)|,$$

and

$$\max_{(x,y)\in[0,1]^2} |\Phi_{mn}(x,y)| \leqq M_{mn} \qquad (m,n = 0,1,\dots). \tag{33}$$

We will proceed by induction on p. If $p=1$, then let

$$\Phi_{mn}(x,y)=\Psi_{mn}(x,y) \quad \text{for} \quad m,n=0,1,\dots,r_2-1 \quad \text{and} \quad E_1=H_1.$$

Conditions (31)–(33) are obviously satisfied.

Now let $p_0 \geqq 2$ be an integer and assume that the step functions $(\Phi_{mn}\colon m,n=0,1,\dots,r_{p_0}-1)$ and the simple sets $(E_p\colon p=1,2,\dots,p_0-1)$ have already been defined in such a way that these functions are orthonormal on $[0,1]^2$, these sets are stochastically independent, and relations (31)–(33) are satisfied for $p=1,2,\dots,p_0-1$. We can divide $[0,1]^2$ into a finite number of disjoint rectangles $(R_s : s=1,2,\dots,\sigma)$ such that the functions $(\Phi_{mn} : m,n=0,1,\dots,r_{p_0}-1)$ are constant on each R_s and the sets $(E_p : p=1,2,\dots,p_0-1)$ are the unions of certain R_s. Let R'_s and R''_s denote the two halves of R_s.

We set

$$\Phi_{mn}(x,y)=\sum_{s=1}^{\sigma}\{\Psi_{mn}(R'_s;x,y)-\Psi_{mn}(R''_s;x,y)\}$$

and

$$Ep_0=\bigcup_{s=1}^{\sigma}\{Hp_0(R'_s)\cup Hp_0(R''_s)\}$$

for $m,n=0,1,\dots,r_{p_0}$ and $\max(m,n)\geqq r_{p_0}$. It is easy to verify that the step functions $(\Phi_{mn} : m,n=0,1,\dots,r_{p_0})$ form an (ONS) on $[0,1]^2$, the simple sets $(E_p : p=1,2,\dots,p_0)$ are stochastically independent, and conditions (31)–(33) are satisfied.

The above induction scheme shows that the (ONS) (Φ_{mn}) and the sequence (E_p) of stochastically independent sets can be defined so that conditions (31)–(33) are satisfied for every $p=1,2,\dots$.

Putting (32) and (33) together, we can conclude for $(x,y)\in E$

$$\text{(34)} \qquad \max_{r_p\leqq m,n<r_{p+1}} |\sigma_{mn}^{\alpha\beta}((a_{mn}),(\Phi_{mn}),x,y)|\geqq p \qquad (p=1,2,\dots).$$

Setting $E=\limsup_{p\to\infty} E_p$, (31) implies $\operatorname{mes} E=1$ via the Borel–Cantelli lemma. If $(x,y)\in E$, then we have (34) for infinitely many p. Lemma 8 is proved.

Now we will prove a lemma on numerical series.

LEMMA 9. *If the $(C,-1<\alpha,\beta<0)$-means of the series*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$$

are bounded, then

$$a_{mn} = O(m^\alpha n^\beta) \quad as \quad \min(m,n) \to \infty. \tag{35}$$

PROOF. Let

$$|\sigma_{mn}^{\alpha\beta}| \leqq M. \tag{36}$$

Using a well-known formula (see [8], p. 78)

$$\frac{a_{mn}}{A_m^\alpha A_n^\beta} = \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{-\alpha-2} A_{n-k}^{-\beta-2} A_i^\alpha A_k^\beta \sigma_{ik}^{\alpha\beta} =$$

$$= \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} A_{m-i}^{-\alpha-2} A_{n-k}^{-\beta-2} A_i^\alpha A_k^\beta \sigma_{ik}^{\alpha\beta} +$$

$$+ \frac{1}{A_n^\beta} \sum_{k=0}^{n-1} A_{n-k}^{-\beta-2} A_k^\beta \sigma_{mk}^{\alpha\beta} + \frac{1}{A_m^\beta} \sum_{i=0}^{m-1} A_{m-i}^{-\alpha-2} A_i^\alpha \sigma_{in}^{\alpha\beta} + \sigma_{mn}^{\alpha\beta}.$$

Then by (36)

$$\frac{|a_{mn}|}{A_m^\alpha A_n^\beta} \leqq \frac{1}{A_m^\alpha A_n^\beta} \left| \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} A_{m-i}^{-\alpha-2} A_{n-k}^{-\beta-2} A_i^\alpha A_k^\beta \sigma_{ik}^{\alpha\beta} \right| +$$

$$+ \frac{1}{A_n^\beta} \left| \sum_{k=0}^{n-1} A_{n-k}^{-\beta-2} A_k^\beta \sigma_{mk}^{\alpha\beta} \right| + \frac{1}{A_m^\alpha} \left| \sum_{i=0}^{m-1} A_{m-i}^{-\alpha-2} A_i^\alpha \sigma_{in}^{\alpha\beta} \right| + |\sigma_{mn}^{\alpha\beta}| \leqq 4M.$$

So we get $a_{mn} = O(m^\alpha n^\beta)$. Lemma 9 is proved.

PROOF OF THEOREM 3. We will distinguish three cases:

$$\text{(a)} \quad \sum_{m=0}^{\infty} a_{m0}^2 (m+1)^{-2\alpha} = \infty, \qquad \text{(b)} \quad \sum_{n=0}^{\infty} a_{0n}^2 (n+1)^{-2\beta} = \infty, \tag{37}$$

(c) for every $r = 1, 2 \dots$,

$$\sum_{m=r}^{\infty} \sum_{n=r}^{\infty} a_{mn}^2 (m+1)^{-2\alpha} (n+1)^{-2\beta} = \infty. \tag{38}$$

By the analogy between the cases (a) and (b), we shall consider only case (a).

Case (a). Let us take an arbitrary sequence $(R_{mn} : m = 0, 1, \dots, n = 1, 2, \dots)$ of disjoint rectangles such that

$$\bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} R_{mn} \subset [0,1]^2 \setminus [0,1/2]^2.$$

Set for $m = 0, 1, \dots,$

$$\Psi_{m0}(x, y) = \begin{cases} 2\Phi_m(2x) & \text{for } (x, y) \in [0, 1/2]^2, \\ 0 & \text{otherwise}, \end{cases}$$

where (Φ_m) are the functions in Theorem 2, while for $m = 0, 1, \dots; n = 1, 2, \dots,$

$$\Psi_{mn}(x, y) = \begin{cases} (\operatorname{mes} R_{mn})^{-1/2} & \text{for } (x, y) \in R_{mn}, \\ 0 & \text{otherwise}. \end{cases}$$

It is not hard to check that (Ψ_{mn}) is an (ONS) of step functions on $[0, 1]^2$ and for a.e. (x, y) in $[0, 1/2]^2$ we have

$$\limsup_{m \to \infty} |\sigma_{mn}^{\alpha\beta}((a_{mn}), (\Psi_{mn}), x, y)| = \infty.$$

To complete the proof, we apply Lemma 8.

Case (c). Without loss of generality, we assume that $a_{mn} \neq 0$. From (38) it follows that there exists an increasing sequence $(r_j : j = 1, 2, \dots; r_1 = 0)$ of integers such that for $j = 1, 2, \dots$

$$\sum_{j=1}^{\infty} \sum_{m=r_j}^{r_{j+1}-1} \sum_{n=r_j}^{r_{j+1}-1} a_{mn}^2 (m+1)^{-2\alpha} (n+1)^{-2\beta} = \infty.$$

With the notations

$$a_{mn}^* = \min\{|a_{mn}|, (m+1)^{\alpha}(n+1)^{\beta}\},$$

$$(39) \quad N = \{(m, n) : m, n = 0, 1, \dots\}, \quad N_j = \{(m, n) : r_j \leqq m, n < r_{j+1}\},$$

it is clear that

$$\sum_{j=1}^{\infty} \sum_{(m,n) \in N_j} a_{mn}^{*2} (m+1)^{-2\alpha} (n+1)^{-2\beta} = \infty,$$

and there exists a sequence (S_{mn}), $\min\{S_{m+1'n}; S_{m'n+1}\} \geqq S_{mn}$, of positive numbers tending to zero such that

$$(40) \qquad \sum_{j=1}^{\infty} \sum_{(m,n) \in N_j} a_{mn}^{*2} (m+1)^{-2\alpha} (n+1)^{-2\beta} S_{mn} = \infty.$$

Since $a_{mn}^{*2}(m+1)^{-2\alpha}(n+1)^{-2\beta} S_{mn} \leqq 1$, we can define stochastically independent functions on $[0, 1]^2$ such that for $(m, n) \in N_j$ $(j = 1, 2, \dots)$

$$\hat{\Psi}_{mn}(x, y) = \begin{cases} a_{mn}^{*-1} (m+1)^{\alpha} (n+1)^{\beta} S_{mn}^{-1/2} & \text{for } (x, y) \in A_{mn}^1, \\ -a_{mn}^{*-1} (m+1)^{\alpha} (n+1)^{\beta} S_{mn}^{-1/2} & \text{for } (x, y) \in A_{mn}^2, \\ 0 & \text{otherwise}, \end{cases}$$

where $\operatorname{mes} A^1_{mn} = \operatorname{mes} A^2_{mn} = a^{*2}_{mn}(m+1)^{-2\alpha}(n+1)^{-2\beta}S_{mn}/2$.

Then by (40), the Borel–Cantelli lemma yields

$$\limsup_{\min(m,n)\to\infty} |a_{mn}\hat{\Psi}_{mn}(x,y)|(m+1)^{-\alpha}(n+1)^{-\beta} = \infty. \tag{41}$$

Finally, we take an arbitrary sequence $\left(R_{mn}\colon (m,n) \in N \setminus \bigcup_{j=1}^{\infty} N_j\right)$ of disjoint rectangles of $[0,1]^2 \setminus [0,1/2]^2$. Set

$$\Psi_{mn}(x,y) = \begin{cases} 2\hat{\Psi}_{mn}(2x,2y) & \text{for } (x,y) \in [0,1/2]^2, \\ 0 & \text{otherwise,} \end{cases}$$

for $(m,n) \in \bigcup_{j=1}^{\infty} N_j$, while

$$\Psi_{mn}(x,y) = \begin{cases} (\operatorname{mes} R_{mn})^{-1/2} & \text{for } (x,y) \in R_{mn}, \\ 0 & \text{otherwise,} \end{cases}$$

for $(m,n) \in N \setminus \bigcup_{j=1}^{\infty} N_j$. By (41), we have

$$\limsup_{\min(m,n)\to\infty} |a_{mn}\Psi_{mn}(x,y)|(m+1)^{-\alpha}(n+1)^{-\beta} = \infty$$

a.e. on $[0,1/2]^2$.

Then, by virtue of Lemma 9, we get

$$\limsup_{\min(m,n)\to\infty} |\sigma^{\alpha\beta}_{mn}(x,y)| = \infty$$

a.e. on $[0,1/2]^2$.

To complete the proof, we apply Lemma 8. Theorem 3 is proved.

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PRECOMPACTNESS IN THE SPACE OF PETTIS INTEGRABLE FUNCTIONS

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Let (S, Σ, μ) be a finite measure space and X a Banach space. We consider the normed space $\mathsf{P}_c(\mu, X)$ of all (μ-) Pettis integrable functions, with values into X, having an indefinite integral with compact range; the norm in $\mathsf{P}_c(\mu, X)$ is, as usual,

$$\|f\| = \sup\left\{ \int_S |x^* f(s)| d\mu : x^* \in X^*, \|x^*\| \leqq 1 \right\}.$$

The purpose of this note is to extend a characterization of precompact subsets obtained by Brooks and Dinculeanu in [1] for strongly measurable functions and by Graves and Ruess in [2] for subsets of bounded weakly measurable functions defined on perfect measure spaces. Those results were obtained as consequences of other theorems about spaces of unconditionally converging series (cf. [1]) and about spaces of compact range vector measures (cf. [2]); our approach is direct and this makes it possible to consider the most general case.

Before giving our result we need to state some terminology. If $\pi = (A_i)_{i \in I}$ is a finite partition of S, we define the conditional expectation $E(\pi, \mu)f$ of f by

$$E(\pi, \mu)f = \sum_{i \in I} [\mu(A_i)]^{-1} \left(\int_{A_i} f(s) d\mu \right) \Phi_{A_i}.$$

It is known (cf. [3]) that the family of finite partitions is directed by refinement, that $\|E(\pi, \mu)f\| \leqq \|f\|$ and that $\lim_\pi \|E(\pi, \mu)f - f\| = 0$ for all $f \in \mathsf{P}_c(\mu, X)$.

Our result is contained in the following theorem.

THEOREM 1. *Let H be a bounded subset of $\mathsf{P}_c(\mu, X)$. The following facts are equivalent:*

(a) *H is precompact,*

(b) (i) $\left\{ \int_A f(s) d\mu : f \in H \right\}$ *is relatively compact in X for all $A \in \Sigma$,*

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(ii) $\lim_{\pi} E(\pi,\mu)f = f$ *uniformly on* $f \in H$.

PROOF. (a) $\Rightarrow$ (b, i) is a direct consequence of the fact that the operator $f \to \int_A f(s)d\mu$ is linear and continuous.

(a) $\Rightarrow$ (b, ii). Observe that H is totally bounded and so, given $\varepsilon > 0$, there are $f_1, f_2, \dots, f_n \in \mathsf{P}_c(\mu, X)$ such that any $f \in H$ is at a distance less than $\varepsilon/3$ from some f_i. Hence we get

$$\|E(\pi,\mu)f - f\| \leqq \|E(\pi,\mu)f - E(\pi,\mu)f_i\| + \|E(\pi,\mu)f_i - f_i\|+$$

$$+\|f_i - f\| \leqq 2\|f_i - f\| + \|E(\pi,\mu)f_i - f_i\| \leqq (2/3)\varepsilon + \|E(\pi,\mu)f_i - f_i\|.$$

Since as already recalled $\lim_{\pi} \|E(\pi,\mu)f - f\| = 0$ for all f, there is a π' such that, if $\pi > \pi'$, then $\|E(\pi,\mu)f_i - f_i\| \leqq \varepsilon/3$ for $i = 1, 2, \dots, n$; the above inequalities conclude the proof.

(b) $\Rightarrow$ (a). Consider a sequence in H and observe that, for $n, m \in N$,

$$\|f_n - f_m\| \leqq \|f_n - E(\pi,\mu)f_n\| + \|E(\pi,\mu)f_n - E(\pi,\mu)f_m\|+$$

$$+\|E(\pi,\mu)f_m - f_m\|.$$

Using (ii) we can find a π_k, $k \in N$, such that

$$\|E(\pi_k,\mu)f - f\| \leqq 1/k \quad \text{uniformly on} \quad f \in H.$$

Moreover, by virtue of (i) we can assume (otherwise we pass to a subsequence) that $(E(\pi_k,\mu)f_n)$ is a Cauchy sequence. The inequalities considered above allow us to conclude our proof.

REMARK 1. In a sense the result above is the best possible, because if H is a set of Pettis integrable functions (it does not matter how the range of the indefinite integral is) for which Theorem 1 is true, then $f \in H$ must be in $\mathsf{P}_c(\mu, X)$ because of a result of Musial [3].

If we consider only sequences of partitions we have the following result:

THEOREM 2. *Let H be a bounded subset of $\mathsf{P}_c(\mu, X)$. The following facts are equivalent:*

(a) *H is precompact,*

(b) (i) *see Theorem* 1,

(ii) *for any sequence $(f_k) \subset H$ there is a sequence (π_n) of finite partitions, cofinal to the net (π), such that*

$$\lim_{n} \|E(\pi_n,\mu)f_k - f_k\| = 0 \quad \textit{uniformly on} \quad k \in N.$$

PROOF. That (b) implies (a) is similar to the proof of the same implication in Theorem 1. So we have just to show that (a) implies (b). Of course (b, i) is clear. Now, consider a sequence $(f_k) \subset H$; it is well known (cf. [4])

that there is a sequence (π_n) of finite partitions cofinal to the net (π) so that, for all $k \in N$, one has

$$\lim_n \|E(\pi_n, \mu) f_k - f_k\| = 0.$$

Using, as in Theorem 1, the total boundedness of (f_k) we can finish our proof.

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ON THE MAXIMAL KUROSH–AMITSUR MODEL FOR THE WEDDERBURN–ARTIN RADICAL

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1. Introduction

An associative ring A in which the sum of the nilpotent ideals is again nilpotent is said to be a ring with *Wedderburn–Artin radical* or with *nilpotent radical.* In this situation the unique maximal nilpotent ideal of A is denoted by $\mathcal{W}(A)$, and called the *Wedderburn–Artin* radical or the *nilpotent* radical of A, and also the *classical* radical (cf. Wedderburn [8], Artin [2] and Gray [3].)

A well-known example of a ring in which the nilpotent radical does *not* exist is the ring $\bigoplus_{i=1}^{\infty} \mathbf{Z}_{p^i}$, p a fixed prime. This example shows that nilpotency of rings does not define a Kurosh–Amitsur radical in the class of *all* associative rings. However, as may be easily shown, in every universal (i.e. hereditary and homomorphically closed) class of rings with nilpotent radical, nilpotency does define a Kurosh–Amitsur radical, and this is therefore also the case with the union $\mathbf{U}$ of all universal classes of rings with nilpotent radical. This subclass $\mathbf{U}$ of the class $\mathbf{T}$ of *all* rings with nilpotent radical does not coincide with $\mathbf{T}$ because $\mathbf{T}$ is not homomorphically closed — $\mathbf{T}$ (clearly) contains all free rings, but not all rings!

2. The class T of all rings with nilpotent radical

We describe here the relationship between the radical $\mathcal{W}$ of $\mathbf{T}$ on the one hand, and general radicals of the class $\mathbf{A}$ of all associative rings on the other.

PROPOSITION 1. *Let $\mathcal{R}$ be a radical of the class $\mathbf{A}$ of all associative rings. Then $\mathcal{R}(A) = \mathcal{W}(A)$ for all $A \in \mathbf{T}$ if and only if $\mathcal{R}$ is the prime radical, β.*

PROOF. ($\Leftarrow$): Any ring A in $\mathbf{T}$ has $\mathcal{W}(A) = \beta(A)$, since $\mathcal{W}(A) \subset \beta(A)$ would imply that the nonzero ring $\beta(A)/\mathcal{W}(A)$ is at the same time β-radical and semiprime.

($\Rightarrow$): The nilpotent rings are all contained in $\mathbf{T}$. Hence $\mathcal{R}(N) = N$ for every nilpotent ring N. Since β is the unique smallest radical for which all nilpotent rings are radical, $\beta \leqq \mathcal{R}$. Let A be a semiprime ring. Then $A \in \mathbf{T}$ and so $\mathcal{R}(A) = \mathcal{W}(A) = 0$. Hence $\mathcal{R} \leqq \beta$. □

The observation in part ($\Leftarrow$) of the proof, together with the fact that in an arbitrary ring A all nilpotent ideals of A are contained in $\beta(A)$, gives us:

PROPOSITION 2. *The following two conditions on a ring A are equivalent:*

(1) *A is a ring in $\mathbf{T}$.*

(2) *$\beta(A)$ is nilpotent.*

Under each of these two conditions one has that $\mathcal{W}(A) = \beta(A)$. □

3. The maximal Kurosh–Amitsur model for the Wedderburn–Artin radical

Trivial examples of Kurosh–Amitsur models for the nilpotent radical $\mathcal{W}$ are e.g. the nilpotent rings, and the simple rings. It is obvious that the union of all Kurosh–Amitsur models for $\mathcal{W}$ is a Kurosh–Amitsur model for $\mathcal{W}$. This unique maximal model happens to be simply the class $\{A \in \mathbf{T} : (I \triangleleft \triangleleft A) \Rightarrow (A/I \in \mathbf{T})\}$ — we have:

PROPOSITION 3. *The maximal universal class $\mathbf{U}$ of rings with nilpotent radical coincides with the uniquely determined largest homomorphically closed class of rings with nilpotent radical.*

PROOF. Set $\mathcal{H} := \{A \in \mathbf{T} : (I \triangleleft A) \Rightarrow (A/I \in \mathbf{T})\}$. To prove the required inclusion $\mathcal{H} \subseteqq \mathbf{U}$ it suffices to show that H is hereditary. Let $I \triangleleft A \in \mathcal{H}$ and let $J \triangleleft I$. Let $\beta(I/J) = B/J$. From $B/J^* \cong (B/J)/(J^*/J)$ we have that B/J^* is β-radical. Since the radical B/J is an ideal of A/J we have that $B \triangleleft A$, and so $B/J^* \subseteqq \beta(A/J^*)$. Since $A \in \mathcal{H}$, $B^n \subseteqq J^*$ for some positive integer n (by Proposition 2). From $J^{*3} \subseteqq J$ it follows that B/J is nilpotent, and so $I \in \mathcal{H}$ as required. □

An indication of the *difference in size* between the models $\mathbf{T}$ and $\mathbf{U}$ may be had by considering their lower radicals: $\mathcal{L}\mathbf{T}$ fills out the class $\mathbf{A}$ of all associative rings, while the following example shows that $\mathcal{L}\mathbf{U}$ is a proper radical of $\mathbf{A}$.

EXAMPLE 1. Consider the commutative polynomial ring $P = \mathbf{Z}[x_1, \dots, x_n, \dots]$ with countably infinite many indeterminates. Let S be a nonzero accessible subring of P. It follows from Andrunakievich's Lemma that S contains a nonzero ideal I of P. Let $0 \neq f(x_1, \dots, x_k) \in I$ and let J be the ideal of P generated by the set $\{(x_{k+n}f(x_1, \dots, x_k))^n : n = 1, 2, \dots\}$. For each $n \geqq 1$, $v_n = x_{k+n}f(x_1, \dots, x_k) + J$ is in $\beta(S/J)$ because $v_n^n = 0$; but $v_n^{n-1} \neq 0$. This shows that $S \notin \mathbf{U}$ and so $P \notin \mathcal{L}\mathbf{U}$. □

As to the *content* of $\mathbf{U}$, Proposition 3 ensures that the following important classes of rings be contracted within $\mathbf{U}$: the *hereditarily idempotent* rings; the *almost nilpotent* rings — which include the *nilpotent* rings; the left (and right) *artinian* rings; the left (and right) *noetherian* rings. On this *content* basis an insight into the *extent* of **U** may now be gained through:

PROPOSITION 4. *The maximal Kurosh–Amitsur model* **U** *for the nilpotent radical* $\mathcal{W}$ *has the following properties:*

(1) **U** *is closed under extensions.*

(2) **U** *is closed under finite subdirect sums.*

(3) *Let n be a positive integer. A ring A is in* **U** *if and only if the matrix ring* $M_n(A)$ *is in* **U**.

(4) *If A and B are Morita equivalent rings with unity, then A is in* **U** *if and only if B is in* **U**.

As an immediate consequence of (1) we have the useful:

COROLLARY 1. *Let A be a ring and* A^1 *its Dorroh extension. Then A is in* **U** *if and only if* A^1 *is in* **U**.

PROOF OF PROPOSITION 4. (1) Let A be a ring. Suppose that B is an ideal of A such that B and A/B are in **U**. We must show that $A \in \mathbf{U}$. Let $V \triangleleft A$ and let $\beta(A/V) = T/V$. Then $T/(V+B) \subseteqq \beta(A/(V+B))$, so there is an integer $N \geqq 1$ such that $T^N \subseteqq V + B$. Since $(T^N + V)/V \subseteqq \beta(A/V) \cap \cap (B+V)/V$, $(T^N + V)/V \subseteqq \beta((B+V)/V)$ and so, since $(B+V)/V$ is a homomorphic image of B, $(T^N+V)/V$ is nilpotent. Thus T/V is nilpotent and so $A \in \mathbf{U}$.

(2) (Cf. [4], Lemma 2.3.) Let S be a subdirect sum of two rings $S_1, S_2 \in \in \mathbf{U}$. Then there are ideals T_1 and T_2 in S such that $T_1 \cap T_2 = 0$ and $S_i \cong S/T_i$ $(i = 1,2)$. Since $T_1 \cap T_2 = 0$, $T_1 \cong (T_1 + T_2)/T_2 \triangleleft S/T_2 \cong S_2 \in \mathbf{U}$. Hence $T_1 \in \mathbf{U}$, and we know that $S/T_1 \cong S_1$ is in **U**. We conclude (using (1)) that $S \in \mathbf{U}$. The result now follows by induction.

(3) First we prove the result for rings with unity, so suppose A is a ring with unity. If $A \in \mathbf{U}$ and $J \triangleleft M_n(A)$, then $J = M_n(I)$ for some ideal I of A. Hence

$$\beta(M_n(A)/J) = \beta(M_n(A)/M_n(I)) \cong \beta(M_n(A/I)) = M_n(\beta(A/I))$$

is nilpotent, and so $M_n(A) \in \mathbf{U}$. Conversely, suppose that $M_n(A) \in \mathbf{U}$, and $K \triangleleft A$. Since $\beta(M_n(A/K)) \cong \beta(M_n(A)/M_n(K))$ is nilpotent, so too is $\beta(A/K)$, and hence $A \in \mathbf{U}$.

We now consider the case of rings A which need not have a unity. If $A \in \mathbf{U}$, then $A^1 \in \mathbf{U}$, and so by the first part of this proof $M_n(A^1) \in \mathbf{U}$. Since $M_n(A) \triangleleft M_n(A^1)$, $M_n(A) \in \mathbf{U}$. Now suppose that $M_n(A) \in \mathbf{U}$. Then $M_n(A^1) \in \mathbf{U}$ because $M_n(A^1)/M_n(A) \cong M_n(\mathbf{Z}) \in \mathbf{U}$. Hence by the first part of this proof we have that $A^1 \in \mathbf{U}$, and by Corollary 1 that $A \in \mathbf{U}$.

(4) We first prove the auxiliary result that if $A \in \mathbf{U}$ and $e^2 = e \in A$, then $eAe \in \mathbf{U}$: Let $I \triangleleft eAe$. Then $I = eJe$ where J is the ideal of A generated by I. Also $eAe/I = eAe/eJe \cong (e+J)(A/J)(e+J)$, so it is enough to prove that if a ring R has nilpotent radical and $e^2 = e \in R$, then eRe has nilpotent

radical. This, however, is true because $\beta(eRe) = e\beta(R)e$, (see e.g. [5], p. 206).

Now if A and B are Morita equivalent rings with unity there is a positive integer n and an idempotent $E \in M_n(B)$ such that $A \cong EM_n(B)E$ (cf. e.g. [1], Corollary 22.7). Our result now follows from the auxiliary result and (3). □

Proposition 4(3) can be extended to the case of structural matrix rings. We shall assume throughout that our rings have a unity — Corollary 1 can be used to extend our result to rings without unity just as in the proof of Proposition 4(3). We first summarize some of van Wyk's results from one of his recent papers [7]: Let $B = (b_{ij})$ be an $n \times n$ Boolean matrix such that $b_{ii} = 1$ for all $i = 1, \dots, n$; and for $1 \leqq i, j, k \leqq n$, if $b_{ij} = 1$ and $b_{jk} = 1$, then $b_{ik} = 1$. For any ring R the set

$$S = S(B, R) = \{(c_{ij}) \in M_n(R) : b_{ij} = 0 \Rightarrow c_{ij} = 0\}$$

is the *structural matrix ring* determined by B and R.

The antisymmetric part of S is the nilpotent ideal

$$A = \{(c_{ij}) \in S : c_{ij} \neq 0 \Rightarrow b_{ji} = 0\};$$

and A is the intersection of ideals $K_\mu : \mu = 1, \dots, \beta$ such that $S/K_\mu \cong M_k(R)$ where $k = k(\mu)$ depends on μ. Moreover, these ideals are such that if $1 \leqq \mu, \nu \leqq \beta$ and $\mu \neq \nu$, then $K_\mu + K_\nu = S$, and so it follows from the Chinese Remainder Theorem that $S/A \cong \bigoplus_{\mu=1}^{\beta} M_{k(\mu)}(R)$. We now have:

Proposition 5. *With the above notation: R is in* **U** *if and only if S is in* **U**.

Proof. We apply various parts of Proposition 4. First assume that $R \in \mathbf{U}$. From (3) and (2) we have $S/A \in \mathbf{U}$ and since A is nilpotent it follows from (1) that $S \in \mathbf{U}$.

Conversely, assume that $S \in \mathbf{U}$. Since **U** is homomorphically closed, $M_k(R) \in \mathbf{U}$ for all $k = k(\mu)$, $1 \leqq \mu \leqq \beta$, and so $R \in \mathbf{U}$ by (3). □

4. The nilpotent radical and the nil-based radicals

In the universe **U** now, the nilpotent radical $\mathcal{W}$ is a fully fledged member of the family of Kurosh–Amitsur radicals, and radical theoretic methods may be applied to it. We consider a single problem in this regard — in our view the most natural one: how does $\mathcal{W}$ compare in **U** with the nil-based radicals β (the prime radical) $<$ $\mathcal{L}$ (Levitzki's locally nilpotent radical) $<$ $\mathcal{N}$ (Köthe's nil radical)? From Proposition 1 we have that $\mathcal{W} = \beta$ on **U**, i.e. $\mathcal{W}(A) = \beta(A)$ for all $A \in \mathbf{U}$. We could as yet not decide the question *whether* $\mathcal{L} = \mathcal{W}$ *on* $\mathbf{U}$. But we know that $\mathcal{N}$ fails $\mathcal{W}$ in this respect — this is exhibited in:

EXAMPLE 2. (Cf. [6].) Let G be a finitely generated nil ring which is not nilpotent, and choose I maximal in $\{J \triangleleft G : G/J$ is not nilpotent$\}$. Set $A = G/I$. Then $\mathcal{N}(A) = A$. If $\beta(A) \neq 0$ then A has a nonzero ideal K/I such that $K^2 \subseteqq I$. Since $I \subset K$, G/K is nilpotent. Hence G/I is nilpotent. This contradiction shows that $\beta(A) = 0$; and it remains to show that $A \in \mathbf{U}$. Let $X \triangleleft A$. If $X = 0$, $\beta(A/X) = 0$ is nilpotent. If $X \neq 0$, A/X is nilpotent (by the choice of I above) so $\beta(A/X) = A/X$. Hence $A \in \mathbf{U}$. □

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ASYMPTOTIC TEST FOR INDEPENDENCE OF EXTREME VALUES

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1. Introduction

Let (X_1, Y_1), (X_2, Y_2), $\dots$ be a sequence of independent random vectors with a common distribution L with marginal R and S (in short, we write $L \in \mathcal{M}(R, S)$). Let R and S be such that with suitable norming constant a_n, b_n, c_n and d_n the random variables

$$X(n) = \frac{\max\{X_1, \dots, X_n\} - b_n}{a_n}$$

and

$$X(n) = \frac{\max\{Y_1, \dots, Y_n\} - c_n}{d_n}$$

have nondegenerate limiting distributions G and H, respectively. Assuming that $(X(n), Y(n))$ converges in distribution to some $F \in \mathcal{M}(G, H)$ we study conditions that will assure $F = GH$.

Let (u_*, v_*) be a fixed point such that $0 < G(u_*) < 1$ and $0 < H(v_*) < 1$. For $n \geqq 1$ define $u_n = a_n u_* + b_n$, $v_n = c_n v_* + d_n$,

$$\tau_n(R) = \sum_{j=1}^{n} \chi_{(X_j \leqq u_n)}, \quad \tau_n(S) = \sum_{j=1}^{n} \chi_{(Y_j \leqq v_n)},$$

and

$$\tau_n(L) = \sum_{j=1}^{n} \chi_{(X_j \leqq u_n, Y_i \leqq v_n)},$$

where χ_A denotes the indicator function of the set A. In this note we prove the following

[1] Partially supported by CNPq-Brasil.

THEOREM 1. *If*

$$|L^n(u_n, v_n) - R^n(u_n)S^n(v_n)| \to 0, \quad n \to \infty \tag{1}$$

then we have $F = GH$. That is, for all (u, v) such that $0 < G(u), H(u) < 1$ we have

$$\lim_{n\to\infty} P(X(n) \leqq u, Y(n) \leqq v) = G(u)H(v). \tag{2}$$

Theorem 1 can be viewed as an extension of Dorea–Sastrosoewignjo [2, Theorem 3.1]. It also suggests that the following test may be used for asymptotic independence:

ASYMPTOTIC TEST. *If for large n we have*

$$\frac{n\tau_n(L)}{\tau_n(R)\tau_n(S)} \geqq 1 \quad \textit{then} \quad F = GH. \tag{3}$$

Note that we have

$$\frac{\tau_n(R)}{nR(u_n)} \overset{\text{a.s.}}{\to} 1, \quad \frac{\tau_n(S)}{nS(v_n)} \overset{\text{a.s.}}{\to} 1, \quad \text{and} \quad \frac{\tau_n(L)}{nL(u_n, v_n)} \overset{\text{a.s.}}{\to} 1$$

(a.s. stands for almost sure convergence). Hence for large n we have $\frac{n\tau_n(l)}{\tau_n(R)\tau_n(S)} \approx \frac{L(u_n,v_n)}{R(u_n)S(v_n)}$. On the other hand, being F a bivariate extreme distribution we have $F \geqq GH$ so that $\liminf\left(\frac{L(u_n,v_n)}{R(u_n)S(v_n)}\right)^n \geqq 1$. Hence if $\limsup\left(\frac{L(u_n,v_n)}{R(u_n)S(v_n)}\right)^n \leqq 1$ we have (1). Since G and H are known we can select suitable (u_n, v_n) provided the norming constants a_n, b_n, c_n and d_n are known. For an account on the determination of the norming constants see Galambos [3]. If this is not the case the selection can be partially solved as follows. It is well known that G and H can only be one of the three types: $\Phi_\alpha(u) = \exp(-u^{-\alpha})$, $u > 0$; $\Psi_\alpha(u) = \exp(-(-u)^\alpha)$, $u < 0$; and $\Lambda(u) = \exp\{-\exp(-u)\}$ (where α is a positive constant). If G is of type Φ_α we have $b_n = 0$ and we can take $u_n = a_n = \beta_n = \inf\{y : R(y) \geqq 1 - \frac{1}{n}\}$ with $u_* = 1$. If G is of type Ψ_α we have $b_n = \mathcal{L} = \sup\{y : R(y) < 1\}$ and $a_n = \mathcal{L} - b_n$, so take $u_n = \beta_n$ with $u_* = -1$. If G is of type Λ we can take $u_* = 0$ and $u_n = \beta_n$. Discussions on estimation of β_n can be found in Dorea [1] and Weissmann [4].

2. Proof of results

The proof makes use of properties of dependence functions: for $T \in \mathcal{M}(A, B)$ let its associated dependence function d_T be defined by

$$d_T(A(u), B(v)) = T(u, v), \quad (u, v) \in R^2.$$

Since we assuming F a bivariate extreme distribution we have for $0 < x < 1$, $0 < y < 1$ and $s > 0$,

$$d_F^s(x^{1/s}, y^{1/s}) = d_F(x, y) \tag{4}$$

and

$$xy \leqq d_F(x, y) \leqq x \wedge y, \tag{5}$$

where $x \wedge y = \min\{x, y\}$ (see Galambos [3]). Also we can rewrite (1) as

$$|d_L^n(R(u_n), S(v_n)) - R^n(u_n)S^n(v_n)| \xrightarrow{n} 0. \tag{6}$$

Since $R^n(u_n) \to x_* = G(u_*)$ and $S^n(v_n) \to y_* = H(v_*)$ we have from (6)

$$D_L^n(R(u_n), S(v_n)) \to x_* v_* = d_F(x_*, y_*) = F(u_*, v_*).$$

It remains to show that $d_F(x, y) = xy$ for all (x, y). Or equivalently that,

$$\omega(x, y) = 1 \quad \text{for all} \quad (x, y) \tag{7}$$

where

$$\omega(x, y) = \frac{\log d_F(x, y) - \log(x \wedge y)}{\log xy - \log(x \wedge y)}.$$

Note that from (4) we have $\omega(x^s, y^s) = \omega(x, y)$ for $s > 0$. Let $s = -1/\log y$ and we have $\omega(x^s, y^s) = \omega(e^{-z}, e^{-1}) = \omega(x, y) = \omega(z)$ with $z = \log x/\log y$. For $z \geqq 1$ let $\omega_1(z) = \omega(z)$ and for $0 < z \leqq 1$ let $\omega_2(1/z) = \omega(z)$. Clearly we have $\omega_1(1) = \omega_2(1) = a$ with $0 \leqq a \leqq 1$. The remaining of the proof will be carried out in several steps.

(a) $\omega_1(z)$ and $\omega_2(z)$ are continuous on $(1.\infty)$ and right continuous at $z = 1$.

Since the proof is similar we do it for ω_2. Let $0 < y \leqq x < 1$ and $1 \leqq z = \log y/\log x$. We have $\omega(1/z) = \omega_2(z)$ and $d_F(x, y) = yx^{\omega_2(z)}$. To prove the right continuity of ω_2 let $\varepsilon > 0$ and $\delta = z\varepsilon$ then,

$$d_F(x, y) - d_F(x, y^{1+\varepsilon}) = yx^{\omega_2(z)} - y^{1+\varepsilon}x^{\omega_2(z+\delta)} \geqq 0.$$

So that $\omega_2(z) \leqq \varepsilon z + \omega_2(z + \delta)$ On the other hand $y \leqq x \leqq x^{1/(1+\varepsilon)}$ and

$$d_F(x^{1/(1+\varepsilon)}, y) - d_F(x, y) = yx^{\omega_2(z+\delta)/(1+\varepsilon)} - yx^{\omega_2(z)} \geqq 0.$$

Hence $\omega_2(z + \delta) \leqq (1 + \varepsilon)\omega_2(z)$ and the right continuity follows from

$$-\varepsilon z \leqq \omega_2(z + \delta) - \omega_2(z) \leqq \varepsilon\omega_2(z). \tag{8}$$

To see the left continuity let $y < x$ and $\varepsilon > 0$ small enough so that $y < y^{1-\varepsilon} \leqq x$. Since $d_F(x, y^{1-\varepsilon}) - d_F(x, y) \geqq 0$ and $d_F(x, y) - d_F(x, y^{1/(1-\varepsilon)}) \geqq 0$ we have for $\delta = \varepsilon z$,

$$-\varepsilon z \leqq \omega_2(z) - \omega_2(z - \delta) \leqq \varepsilon\omega_2(z). \tag{9}$$

And the left continuity for $z > 1$ follows.

(b) ω_1 and ω_2 are decreasing functions.

Again the proof is similar and we shall do it for ω_1. For $\lambda > 1$ fixed define

$$\Phi_\lambda(z) = \frac{\omega_1(\lambda z) - \omega_1(z)}{\lambda z - z}, \quad z \geqq 1.$$

We will show that $\Phi_\lambda(z)$ is increasing in z. Let $z = \log x / \log y \geqq 1$. Let $\rho \geqq 1$ then $x^\lambda \leqq x \leqq y \leqq y^{1/\rho}$ and from (4) and (5) we have $\beta - \gamma \geqq 0$ with $\beta = d_F(x, y^{1/\rho}) - d_F(x^\lambda, y^{1/\rho})$ and $\gamma = d_F(x,y) - d_F(x^\lambda, y)$. Since $d_F(x,y) = xy^{\omega_1(z)}$ we have

$$\beta = y^{[\rho z + \omega_1(\rho z)]/\rho}\left\{1 - y^{(\lambda-1)z(1+\Phi_\lambda(\rho z))}\right\}, \quad \gamma = y^{[z+\omega_1(z)]}\left\{1 - y^{(\lambda-1)z(1+\Phi_\lambda(z))}\right\},$$

and

$$\frac{1 - y^{(\lambda-1)z(1+\Phi_\lambda(\rho z))}}{1 - y^{(\lambda-1)z(1+\Phi_\lambda(z))}} \geqq y^{\omega_1(z) - \frac{\omega_1(\rho z)}{\rho}}.$$

Now let $y \uparrow 1$ maintaining z constant and we have $\Phi_\lambda(\rho z) \geqq \Phi_\lambda(z)$ for $\rho \geqq 1$. So that Φ_λ is an increasing function. To show that ω_1 is decreasing, it is enough to prove that $\Phi_\lambda(z) \leqq 0$ for all $\lambda > 1$. Assume that $\Phi_\lambda(z) = k > > 0$. Then $\omega_1(\lambda z) = \omega_1(z) + k(\lambda - 1)z$. Since $\Phi_\lambda(\lambda z) \geqq \Phi_\lambda(z) \geqq k$ we have $\omega_1(\lambda^2 z) \geqq \omega_1(z) + k(\lambda-1)z + k(\lambda-1)z\lambda$. Proceeding this way we have $\omega_1(\lambda^n z) \geqq \omega_1(z) + \sum_{i=0}^{n-1} k(\lambda-1)z\lambda^j$. Since $\lambda > 1$ we have $\lim_{n\to\infty} \omega_1(\lambda^n z) = = \infty$, a contradiction.

(c) For $i = 1, 2$ the first and second derivatives of ω_1 exist almost everywhere (a.e.) with $-1 \leqq \omega_i' \leqq 0$ and $\omega'' \geqq 0$.

Note that $\Phi_\lambda \leqq 0$ so that $\omega_1' \leqq 0$ a.e. Also (8) and (9) hold with ω_1 in place of ω_2. It follows that

$$\frac{\omega_1(z+\delta) - \omega_1(z)}{\delta} \geqq -1 \quad \text{and} \quad \frac{\omega_1(z-\delta) - \omega_1(z)}{\delta} \geqq -1$$

where $\delta = \varepsilon z$ for $\varepsilon > 0$. Hence $\omega_1' \geqq -1$ a.e.. Since ω_1' is an increasing function we have $\omega_1'' \geqq 0$ a.e. (the proof for ω_2 is similar).

(d) For $z \geqq 1$ we have

$$\omega(z) + \omega(1/z) \geqq 1 - z - a(1+z). \tag{10}$$

Let $x \leqq y$ and $z = \log x / \log y$. Since $d_F(y,y) - d_F(x,y) + d_F(x,x) \geqq 0$ we have proceeding as in step (b)

$$\frac{1 - x^{[\omega(z)+z-1-a]/z}}{1 - x^{[az - \omega(1/z)]/z}} \geqq x^{[z+\omega(1/z)-1-a]/z}.$$

And (10) follows letting $x \uparrow 1$ and maintaining z constant.

Now let $z_* = \log x_* / \log y_*$. We have $\omega(z_*) = 1$. If $z_* > 1$ then since ω_1 is decreasing and $\omega_1 \leqq 1$ we have $\omega(z) = 1$ for $1 \leqq z \leqq z_*$. Similarly if $z_* < 1$ then $\omega(z) = 1$ for $z_* \leqq z \leqq 1$. Now $\omega(1) = a = 1$ together with (10) prove (7).

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NEW ANSWERS TO PROBLEM 24 OF P. TURÁN

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1. Introduction and results

Let us consider a triangular matrix X of nodes

$$(1) \qquad -1 \leqq x_{n1} < x_{n2} < \cdots < x_{nn} \leqq 1, \qquad n = 1, 2, \ldots .$$

Sometimes omitting the superfluous notations, denote for $f \in C[-1,1]$ and for each fixed n ($\|\cdot\|$ stands for the Chebyshev norm)

$$\omega_n(x) = (x - x_1)(x - x_2)\cdots(x - x_n);$$

$$l_k(x) = \frac{\omega_n(x)}{(x - x_k)\omega_n'(x_k)}, \qquad k = 1, 2, \ldots, n;$$

$$L_n(f, x) = \sum_{k=1}^{n} f(x_k) l_k(x);$$

$$A_k(x) = \left[1 - \frac{\omega_n''(x_k)}{\omega_n'(x_k)}(x - x_k)\right] l_k^2(x) := v_k(x) l_k^2(x), \quad k = 1, 2, \ldots, n;$$

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k) A_k(x); \quad \mu_n = \left\|\sum_{k=1}^{n} |A_k(x)|\right\|; \quad \Gamma_n = \left\|\sum_{k=1}^{n} (x - x_k)^2 l_k^2(x)\right\|.$$

To draw a general conclusion from the behavior of the polynomials $H_n(f,x)$ on those of $L_n(f,x)$, P. Turán proposed his Problem 24 [1]:

Is it true that, for any matrix X satisfying

$$(2) \qquad \lim_{n\to\infty} \|H_n(f,x) - f(x)\| = 0, \quad \text{for all } f \in C[-1,1],$$

we have

$$(3) \qquad \lim_{n\to\infty} \|L_n(f,x) - f(x)\| = 0$$

for all functions f which are continuously differentiable in $[-1,1]$?

The first answer to this problem, given by P. Vértesi [2], is the following

THEOREM A. *If* $\mu_n = O(1)$, *then* (3) *holds for all* $f \in \{f : E_n(f) = o(n^{-\frac{5}{2}})\}$, *where* $E_n(f)$ *is the best uniform approximation of* $f \in C[-1,1]$ *on* $[-1,1]$ *by polynomials of degree* $\leqq n$.

Recently we have solved this problem in [3]. That is we proved the following

THEOREM B. *If* $\mu_n = O(1)$ *then* (3) *holds for all* $f \in \operatorname{Lip}\alpha$ *with* $\alpha > \alpha_1$ *and if* (2) *is true then* (3) *holds for all* $f \in \operatorname{Lip}\alpha_1$, *where*

$$\alpha_1 = \frac{\sqrt{26889} - 81}{84} \approx \frac{83}{84} \approx 0.988.$$

In [4] we improved this result and gave the following

THEOREM C. *If* $\mu_n = O(1)$ *then* (3) *holds for all* $f \in \operatorname{Lip}\alpha$ *with* $\alpha > \alpha_2$, *where*

$$a_2 = \frac{\sqrt{393} - 9}{12} \approx 0.902.$$

This shows that (2) alone can assure that $L_n(f,x)$ cannot behave "too badly."

Later in [5] we determined a class of functions for which (3) holds if

$$\mu_n = O(n^\delta), \qquad \delta \geqq 0. \tag{4}$$

That is

THEOREM D. *If* (4) *is satisfied, then* (3) *holds for all* f *with* $f^{(p)} \in \operatorname{Lip}\theta$, $0 < \theta \leqq 1$, *where*

$$p + \theta > \alpha_3$$

and

$$\alpha_3 = \begin{cases} \frac{7+15\delta+\sqrt{2209+1110\delta+225\delta^2}}{60}, & \delta \leqq \frac{2}{9} \\ \frac{8}{9} + \frac{1}{2}\delta, & \delta > \frac{2}{9}. \end{cases}$$

From this theorem we immediately obtain the following two important results for Problem 24 of P. Turán in two different directions.

THEOREM E. *If* $\mu_n = O(1)$, *then* (3) *holds for all* $f \in \operatorname{Lip}\alpha$ *with* $\alpha > \frac{9}{10}$.

THEOREM F. *If* $\mu_n = O(n^\delta)$, $\delta < \frac{2}{9}$, *then* (3) *holds for all* $f \in \operatorname{Lip} 1$.

In this paper we intend to improve Theorem D and give

THEOREM 1. *If* (4) *is satisfied, then* (3) *holds for all* f *with* $f^{(p)} \in \operatorname{Lip}\theta$, $0 < \theta \leqq 1$, *where*

$$p + \theta > \alpha^* := \begin{cases} \frac{-1+\delta+\sqrt{17+6\delta+\delta^2}}{4}, & \delta \leqq \frac{8}{5} \\ \frac{3+3\delta+\sqrt{33+30\delta+9\delta^2}}{12}, & \delta > \frac{8}{5}. \end{cases}$$

From this theorem we immediately obtain the following two new answers to Problem 24 of P. Turán in two different directions.

THEOREM 2. *If* $\mu_n = O(1)$, *then* (3) *holds for all* $f \in \operatorname{Lip}\alpha$ *with* $\alpha > \frac{1}{4}(\sqrt{17}-1) \approx 0.781$.

THEOREM 3. *If* $\mu_n = O(n^\delta)$, $\delta < \frac{1}{2}$, *then* (3) *holds for all* $f \in \operatorname{Lip} 1$.

2. Preliminaries

First we show several lemmas which are of independent interest.

LEMMA 1. *For any matrix* X *and for any subset* $N_n \subset \{1,2,\dots,n\}$, $n = 1,2,\dots$,

$$\Big\|\sum_{k\in N_n} l_{nk}^2\Big\| \leqq 100\mu_n\Delta_n(z_n)^{-\frac{2}{3}}, \qquad n=1,2,\dots$$

where

$$\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}, \qquad n=1,2,\dots$$

and $z_n \in [-1,1]$ *satisfies*

$$\sum_{k\in N_n} l_{nk}^2(z_n) = \Big\|\sum_{k\in N_n} l_{nk}^2\Big\|, \qquad n=1,2,\dots.$$

PROOF. Suppose to the contrary that there exists a number n such that

$$\Big\|\sum_{k\in N_n} l_k^2\Big\| > 100\mu_n\Delta_n(z_n)^{-\frac{2}{3}}.$$

Put

$$I_n := \Big\{k \in N_n : |x_k - z_n| \leqq \frac{1}{5}\Delta_n(z_n)^{\frac{1}{3}}, |v_k(z_n)| \leqq \frac{1}{25}\Delta_n(z_n)^{\frac{2}{3}}\Big\}.$$

Using the identity

$$\sum_{k=1}^n (x-x_k)^2 A_k(x) = 2\sum_{k=1}^n (x-x_k)^2 l_k^2(x)$$

we obtain the estimate

$$\Gamma_n = \Big\|\sum_{k=1}^n (x-x_k)^2 l_k^2(x)\Big\| = \frac{1}{2}\Big\|\sum_{k=1}^n (x-x_k)^2 A_k(x)\Big\| \leqq 2\Big\|\sum_{k=1}^n |A_k|\Big\| = 2\mu_n.$$

Then

$$\sum_{k\in N_n\setminus I_n} l_k^2(z_n) \leqq \sum_{|v_k(z_n)|>\frac{1}{25}\Delta_n(z_n)^{\frac{2}{3}}} l_k^2(z_n) + \sum_{|x_k-z_n|>\frac{1}{5}\Delta_n(z_n)^{\frac{1}{3}}} l_k^2(z_n) \leqq$$

$$\leqq 25\Delta_n(z_n)^{-\frac{2}{3}} \sum_{|v_k(z_n)|>\frac{1}{25}\Delta_n(z_n)^{\frac{2}{3}}} |v_k(z_n)|l_k^2(z_n)+$$

$$+25\Delta_n(z_n)^{-\frac{2}{3}} \sum_{|x_k-z_n|>\frac{1}{5}\Delta_n(z_n)^{\frac{1}{3}}} (x_k-z_n)^2 l_k^2(z_n) \leqq$$

$$\leqq 25\Delta_n(z_n)^{-\frac{2}{3}}(\mu_n+2\mu_n) = 75\mu_n\Delta_n(z_n)^{-\frac{2}{3}}.$$

Hence

$$\sum_{k\in I_n} l_k^2(z_n) > 25\mu_n\Delta_n(z_n)^{-\frac{2}{3}}.$$

Choose $y_n \in [-1,1]$ so that

$$\sum_{k\in I_n} l_k^2(y_n) = \Big\|\sum_{k\in I_n} l_k^2\Big\|.$$

Clearly

$$\sum_{k\in I_n} l_k^2(y_n) > 25\mu_n\Delta_n(z_n)^{-\frac{2}{3}}. \tag{5}$$

First we show three claims.

CLAIM 1.

$$|x_k - y_n| \leqq \frac{7}{10}\Delta_n(z_n)^{\frac{1}{3}}, \qquad \forall k \in I_n. \tag{6}$$

In fact, there must exist an $i \in I_n$ such that

$$|x_i - y_n| \leqq \frac{\sqrt{2}}{5}\Delta_n(z_n)^{\frac{1}{2}},$$

for otherwise

$$\sum_{k\in I_n} l_k^2(y_n) \leqq \frac{25}{2}\Delta_n(z_n)^{-\frac{2}{3}} \sum_{k\in I_n}(y_n-x_k)^2 l_k^2(y_n) \leqq 25\mu_n\Delta_n(z_n)^{-\frac{2}{3}}$$

contradicting (5). Thus

$$|x_k - y_n| \leqq |x_k - z_n| + |z_n - x_i| + |x_i - y_n| \leqq$$

$$\leqq \frac{2}{5}\Delta_n(z_n)^{\frac{1}{3}} + \frac{\sqrt{2}}{5}\Delta_n(z_n)^{\frac{1}{3}} \leqq \frac{7}{10}\Delta_n(z_n)^{\frac{1}{3}}.$$

CLAIM 2.

$$|y_n - z_n| \leqq \frac{1}{25}\Delta_n(z_n).$$

Namely otherwise, noting that $v_k(x)$ is linear and $v_k(x_k) = 1$ we have that

$$\frac{v_k(y_n) - v_k(z_n)}{y_n - z_n} = \frac{v_k(x_k) - v_k(z_n)}{x_k - z_n} = \frac{1 - v_k(z_n)}{x_k - z_n}.$$

Then for each $k \in I_n$ by (6)

$$|v_k(y_n)| = \frac{1}{|x_k - z_n|}|y_n - z_n + (x_k - y_n)v_k(z_n)| \geqq$$

$$\geqq 5\Delta_n(z_n)^{-\frac{1}{3}}\left\{\frac{1}{25}\Delta_n(z_n) - \left[\frac{7}{10}\Delta_n(z_n)^{\frac{1}{3}}\right]\left[\frac{1}{25}\Delta_n(z_n)^{\frac{2}{3}}\right]\right\} > \frac{1}{25}\Delta_n(z_n)^{\frac{2}{3}}.$$

Hence by (5)

$$\sum_{k \in I_n} |v_k(y_n)| l_k^2(y_n) > \frac{1}{25}\Delta_n(z_n)^{\frac{2}{3}} \cdot 25\mu_n \Delta_n(z_n)^{-\frac{2}{3}} = \mu_n,$$

a contradiction.

CLAIM 3. *If*

$$(7) \qquad |r_n - z_n| \leqq \frac{1}{6}\Delta_n(z_n), \qquad r_n \in [-1, 1]$$

then

$$(8) \qquad \frac{\Delta_n(z_n)}{\Delta_n(r_n)} \leqq 2.$$

Suppose without loss of generality that $z_n \geqq 0$. If $1 - z_n^2 \leqq \frac{1}{2}\Delta_n(z_n)$, which means that

$$2[n(1 - z_n^2)^{\frac{1}{2}}]^2 - [n(1 - z_n^2)^{\frac{1}{2}}] - 1 \leqq 0,$$

then we solve this inequality and get $(1 - z_n^2)^{\frac{1}{2}} \leqq \frac{1}{n}$. Thus

$$(9) \qquad \frac{\Delta_n(z_n)}{\Delta_n(r_n)} \leqq \frac{\frac{1}{n^2} + \frac{1}{n^2}}{\frac{1}{n^2}} = 2.$$

If $1 - z_n^2 > \frac{1}{2}\Delta_n(z_n)$, then $(1 - z_n^2)^{\frac{1}{2}} > \frac{1}{n}$. Hence by (7)

$$1 - r_n^2 = 1 - z_n^2 - (r_n^2 - z_n^2) \geqq 1 - z_n^2 - 2|r_n - z_n| > \frac{1}{2}\Delta_n(z_n) - \frac{1}{3}\Delta_n(z_n) =$$

$$= \frac{1}{6}\Delta_n(z_n) > \frac{(1 - z_n^2)^{\frac{1}{2}}}{6n} > \frac{1}{6n^2} > \frac{1}{9n^2},$$

i.e., $(1-r_n^2)^{\frac{1}{2}} > \frac{1}{3n}$. Thus there exists a number $\xi_n := tz_n + (1-t)r_n$ $t \in (0,1)$, such that

$$\left|\frac{\Delta_n(r_n)}{\Delta_n(z_n)} - 1\right| = \left|\frac{\Delta_n'(\xi_n)(r_n - z_n)}{\Delta_n(z_n)}\right| \leqq \frac{1}{6}|\Delta_n'(\zeta_n)| = \frac{|\xi_n|}{6n(1-\xi_n^2)^{1/2}} < \frac{1}{2},$$

because $(1-\xi_n^2)^{\frac{1}{2}} \geqq \min\{(1-z_n^2)^{\frac{1}{2}}, (1-r_n^2)^{\frac{1}{2}}\} > \frac{1}{3n}$. So $\frac{\Delta_n(z_n)}{\Delta_n(r_n)} < 2$.

To prove Lemma 1 we choose $t_n \in [-1,1]$ so that

$$(10) \qquad |t_n - z_n| = \frac{1}{25}\Delta_n(z_n).$$

So

$$|t_n - y_n| \leqq |t_n - z_n| + |z_n - y_n| \leqq \frac{2}{25}\Delta_n(z_n);$$

$$(11) \qquad |x_k - t_n| \leqq |x_k - z_n| + |z_n - t_n| \leqq \frac{1}{4}\Delta_n(z_n)^{\frac{1}{2}}, \qquad \text{for all } k \in I_n.$$

Thus for each $k \in I_n$ by (10) and (11)

$$|v_k(t_n)| = \frac{1}{|x_k - z_n|}|t_n - z_n + (x_k - t_n)v_k(z_n)| \geqq$$

$$\geqq 5\Delta_n(z_n)^{-\frac{1}{3}}\left\{\frac{1}{25}\Delta_n(z_n) - \left[\frac{1}{4}\Delta_n(z_n)^{\frac{1}{3}}\right]\left[\frac{1}{25}\Delta_n(z_n)^{\frac{2}{3}}\right]\right\} \geqq \frac{3}{20}\Delta_n(z_n)^{\frac{2}{3}}.$$

On the other hand, putting $r_n = \max\{|t_n|, |y_n|\}$, by Markov's inequality follows from (5) and (8) that

$$\sum_{k\in I_n} l_k^2(t_n) = \sum_{k\in I_n} l_k^2(y_n) + \int_{y_n}^{t_n}\Big[\sum_{k\in I_n} l_k^2(t)\Big]' dt \geqq$$

$$\geqq \Big\|\sum_{k\in I_n} l_k^2\Big\| - 4\int_{y_n}^{t_n}\Big[\Delta_n(t)^{-1}\Big\|\sum_{k\in I_n} l_k^2\Big\|\Big] dt \geqq \Big\|\sum_{k\in I_n} l_k^2\Big\|[1 - 4\Delta_n(r_n)^{-1}|t_n - y_n|]$$

$$\geqq \Big\|\sum_{k\in I_n} l_k^2\Big\|\left[1 - \frac{8}{25}\frac{\Delta_n(z_n)}{\Delta_n(r_n)}\right] > 9\mu_n\Delta_n(z_n)^{-\frac{2}{3}}.$$

Since $|v_k(t_n)| \geqq \frac{3}{20}\Delta_n(z_n)^{\frac{2}{3}}$, we get

$$\sum_{k\in I_n} |v_k(t_n)| l_k^2(t_n) > \mu_n,$$

a contradiction.

This completes the proof of Lemma 1.

By Lemma 1 and $\Delta_n^2(x) \geqq n^{-2}$ we get

COROLLARY 1. *For any matrix* X

$$\left\|\sum_{k=1}^{n} l_{nk}^2\right\| \leqq 100\mu_n n^{\frac{4}{3}}, \qquad n = 1, 2, \dots .$$

COROLLARY 2. *For any matrix* X *and for any number* $\beta \geqq 0$

$$\left\|\sum_{1-x_{nk}^2 \geqq 2n^{-2\beta}} l_{nk}^2\right\| \leqq 100\mu_n n^{\sigma}, \qquad n = 1, 2, \dots ,$$

where $\sigma := \max\left\{4\beta, \frac{2}{3}(1+\beta)\right\}$.

PROOF. Let n be fixed and let $z_n \in [-1, 1]$ satisfy

$$\sum_{k \in N_n} l_k^2(z_n) = \left\|\sum_{k \in N_n} l_k^2\right\|,$$

where $N_n = \{k : 1 - x_k^2 \geqq 2n^{-2\beta}\} \neq \emptyset$, the proof for $N_n = \emptyset$ being trivial.

If there exists an index $k \in N_n$ such that $|x_k - z_n| \leqq (50)^{-\frac{1}{2}} n^{-\frac{\sigma}{2}}$, then

$$1 - z_n^2 = 1 - [x_k + (z_n - x_k)]^2 = 1 - x_k^2 - 2x_k(z_n - x_k) - (z_n - x_k)^2 \geqq$$

$$\geqq 2n^{-2\beta} - 3|z_n - x_k| \geqq 2n^{-2\beta} - n^{-\frac{\sigma}{2}} \geqq n^{-2\beta}.$$

Hence by Lemma 1

$$\left\|\sum_{k \in N_n} l_k^2\right\| \leqq 100\mu_n (z_n)^{-\frac{2}{3}} \leqq 100\mu_n \left[\frac{(1 - z_n^2)^{\frac{1}{2}}}{n}\right]^{-\frac{2}{3}} \leqq$$

$$\leqq 100\mu_n n^{\frac{2}{3}(1+\beta)} \leqq 100\mu_n n^{\sigma}.$$

If

$$|x_k - z_n| > (50)^{-\frac{1}{2}} n^{-\frac{\sigma}{2}}, \qquad \text{for all } k \in N_n,$$

then

$$\left\|\sum_{k \in N_n} l_k^2\right\| = \sum_{k \in N_n} l_k^2(z_n) \leqq 50n^{\sigma} \sum_{k \in N_n} (z_n - x_k)^2 l_k^2(z_n) \leqq 100\mu_n n^{\sigma}.$$

This completes the proof.

Denote $M(K) :=$ the number of elements in the set K.

LEMMA 2. *Let* $z_n \in [-1,1]$. *Define* e_n *by* $z_n - n^{-\eta} \leqq e_n \leqq z_n + n^{-\eta}$. *Suppose that* $q_n := \min\{|z_n| + n^{-\eta}, |e_n|\} < 1$, $n = 1, 2, \ldots$. *If* (4) *is satisfied then*

$$M(k : |x_{nk} - z_n| \leqq n^{-\eta}, |x_{nk}| \leqq |e_n|) = O(n^{1-\eta})(1 - q_n^2)^{-\frac{1}{2}}, \quad 0 < \eta < 1$$

and

$$M(k : |x_{nk} - z_n| \leqq n^{-\eta}) = O(n^{1-\frac{\eta}{2}}), \quad 0 < \eta < 2.$$

PROOF. First we state Theorem 3.4 in [2] which says that denoting by $N(\alpha_n, \beta_n)$ the number of $\theta_{nk} = \arccos x_{nk}$ in the interval $[\alpha_n, \beta_n] \subset [0, \pi]$ one has

$$N(\alpha_n, \beta_n) = \frac{\beta_n - \alpha_n}{\pi} n + O(\ln n \cdot \ln(n\mu_n)).$$

Let $z_n \geqq 0$, say. Denote

$$a_n = \min\{e_n, z_n + n^{-\eta}\}, \quad b_n = z_n - n^{-\eta}, \quad \alpha_n = \arccos a_n, \quad \beta_n = \arccos b_n.$$

Then using

$$\beta_n - \alpha_n = \arccos b_n - \arccos a_n = \int_{b_n}^{a_n} (1 - t^2)^{-\frac{1}{2}} dt \leqq$$

$$\leqq (a_n - b_n)(1 - q_n^2)^{-\frac{1}{2}} \leqq 2n^{-\eta}(1 - q_n^2)^{-\frac{1}{2}}$$

we obtain

$$M(k : |x_{nk} - z_n| \leqq n^{-\eta}, |x_{nk}| \leqq |e_n|) = N(\alpha_n, \beta_n) =$$

$$= O(n^{1-\eta})(1 - q_n^2)^{-\frac{1}{2}} + O(\ln^2 n) = O(n^{1-\eta})(1 - q_n^2)^{-\frac{1}{2}}.$$

Meanwhile by $N(\alpha_n, \beta_n) = n(\beta_n - \alpha_n)/\pi + O(\ln^2 n)$

$$M(k : |x_{nk} - z_n| \leqq n^{-\eta}) \leqq M(k : |x_{nk} - 1| \leqq n^{-\eta}) + O(\ln^2 n) = O(n^{1-\frac{\eta}{2}}).$$

This completes the proof.

LEMMA 3. *If* (4) *is satisfied, then*

$$\lambda_n := \left\| \sum_{k=1}^{n} |l_{nk}| \right\| = O(n^{\nu}),$$

where

$$\nu > \varrho := 1 + \frac{1}{2}\delta.$$

PROOF. Let n be fixed and let $z_n \in [-1,1]$ satisfy

$$\lambda_n = \sum_{k=1}^{n} |l_k(z_n)|.$$

Put

(12) $$\beta = \frac{1}{3},$$

(13) $$\gamma_i = 2\beta - \left(\frac{1}{2}\right)^{i+1}\beta, \quad i = 0, 1, \ldots m.$$

Choose m so that

(14) $$\gamma_n \geqq 2\beta - 2(\nu - \varrho).$$

From $\Gamma_n \leqq 2\mu_n$ by the Cauchy's inequality we get

$$\left\| \sum_{k=1}^{n} |(x - x_k) l_k(x)| \right\| = O(\mu_n^{\frac{1}{2}} n^{\frac{1}{2}}).$$

Now we write

$$K_0 := \{k : |x_k - z_n| \geqq n^{-\gamma_0}\};$$

$$K_i := \{k : n^{-\gamma_{i-1}} > |x_k - z_n| \geqq n^{-\gamma_i}, 1 - x_k^2 > 2n^{-2\beta}\}, \quad i = 1, 2, \ldots, m;$$

$$K_{m+1} := \{k : n^{-\gamma_m} > |x_k - z_n|,\ 1 - x_k^2 > 2n^{-2\beta}\};$$

$$K_{m+2} := \{k : 1 - x_k^2 \leqq 2n^{-2\beta}\}.$$

Then

$$\sigma_0 := \sum_{k \in K_0} |l_k(z_n)| \leqq n^{\gamma_0} \sum_{k \in K_0} |(z_n - x_k) l_k(z_n)| = O(n^{\gamma_0 + \frac{1}{2}(1+\delta)}) = O(n^{\varrho}).$$

Write

$$d_i := \min\left\{ |z_n| + n^{-\gamma_i}, (1 - 2n^{-2\beta})^{\frac{1}{2}} \right\}, \qquad i = 0, 1, \ldots, m.$$

Then by Lemma 2 for each i, $1 \leqq i \leqq m$

$$\sigma_i := \sum_{k \in K_i} |l_k(z_n)| \leqq n^{\gamma_i} \sum_{k \in K_i} |(z_n - x_k) l_k(z_n)| \leqq$$

$$\leqq n^{\gamma_i} \left\{ \sum_{k \in K_i} 1 \right\}^{\frac{1}{2}} \left\{ \sum_{k \in K_i} (z_n - x_k)^2 l_k^2(z_n) \right\}^{\frac{1}{2}} =$$

$$= n^{\gamma_i} \left\{ O(n^{1-\gamma_{i-1}})(1-d_{i-1}^2)^{-\frac{1}{2}} \right\}^{\frac{1}{2}} O(\mu_n^{\frac{1}{2}}) = n^{\gamma_i} \{O(n^{1-\gamma_{i-1}+\beta})\}^{\frac{1}{2}} O(\mu_n^{\frac{1}{2}}) =$$

$$= O(n^{\gamma_i + \frac{1}{2}(1-\gamma_{i-1}+\beta+\delta)}) = O(n^{\varrho}).$$

Similarly (using Corollary 1 and (14))

$$\sigma_{m+1} := \sum_{k\in K_{m+1}} |l_k(z_n)| \leqq \{M\{K_{m+1}\}\}^{\frac{1}{2}} \Big\{ \sum_{k\in K_{m+1}} l_k(z_n)^2 \Big\}^{\frac{1}{2}} =$$

$$= O(n^{\frac{1}{2}(1-\gamma_m)})(1-d_m^2)^{-\frac{1}{4}} O(n^{\frac{2}{3}+\frac{1}{2}\delta}) = O(n^{\frac{7}{6}-\frac{1}{2}(\gamma_m-\beta-\delta)}) =$$

$$= O(n^{\frac{7}{6}+(\nu-\varrho-\beta)+\frac{1}{2}(\beta+\delta)}) = O(n^{\nu}).$$

$$\sigma_{m+2} := \sum_{k\in K_{m+2}} |l_k(z_n)| \leqq \{M\{K_{m+2}\}\}^{\frac{1}{2}} \Big\{ \sum_{k\in K_{m+2}} l_k(z_n)^2 \Big\}^{\frac{1}{2}} =$$

$$= O(n^{\frac{1}{2}(1-\beta)}) O(n^{\frac{2}{3}+\frac{1}{2}\delta}) = O(n^{\frac{7}{6}+\frac{1}{2}(\delta-\beta)}) = O(n^{\varrho}).$$

Thus

$$\lambda_n = \sum_{i=0}^{m+2} \sigma_i = O(n^{\nu}).$$

COROLLARY 3. *If* $\mu_n = O(1)$, *then* $\lambda_n = O(n^{\nu})$ *with arbitrary* $\nu > 1$.

3. Proof of the theorems

3.1. PROOF OF THEOREM 1. Denote $\alpha = p + \theta$. By Timan's theorem (see [6, Chap. 5, Sec. 4, Theorem 6]) there exists a polynomial P_n of degree $\leqq n-1$ such that

$$|f(x) - P_n(x)| \leqq C\Delta_n(x)^{\alpha},$$

if $f^{(p)} \in \operatorname{Lip}\theta$. For simplicity suppose without loss of generality that $C = 1$. Thus

$$|L_n(f,x)-f(x)| \leqq |L_n(f,x)-L_n(P_n,x)|+|L_n(P_n,x)-P_n(x)|+|P_n(x)-f(x)| =$$

$$= |L_n(f-P_n,x)| + |P_n(x)-f(x)| = |L_n(f-P_n,x)| + o(1)$$

and

$$|L_n(f-P_n,x)| = \Big|\sum_{k=1}^{n} [f(x_k) - P_n(x_k)] l_k(x)\Big| \leqq \sum_{k=1}^{n} \Delta_n(x_k)^{\alpha}\, l_k(x)|.$$

Let $z_n \in [-1,1]$ satisfy

$$\sum_{k=1}^{n} \Delta_n(x_k)^\alpha |l_k(z_n)| = \Big\| \sum_{k=1}^{n} \Delta_n(x_k)^\alpha |l_k(x)| \Big\|.$$

Since it is easy to check that $\alpha^* < 1 + \frac{1}{2}\delta = \varrho$, we can take α_0 so that $\alpha > \alpha_0 > \alpha^*$ and $\frac{\varrho}{\alpha_0} > 1$. Now put with the above ϱ and δ

$$\beta = \frac{\varrho}{\alpha_0} - 1, \tag{15}$$

$$\gamma_0 = \alpha_0 - \frac{1}{2} - \frac{1}{2}\delta, \tag{16}$$

$$\gamma_i = \begin{cases} (4\gamma_0 - 2\beta) - \left(\frac{3}{4}\right)^i (3\gamma_0 - 2\beta), & \gamma_{i-1} < 2\beta \\ 2\gamma_0 - \left(\frac{1}{2}\right)^i \gamma_0, & \gamma_{i-1} \geqq 2\beta \end{cases} \quad (i = 1, 2, \dots, m). \tag{17}$$

Choose m so that

$$\gamma_m > \max\{2\gamma_0 - 2(\alpha - \alpha_0), 2\beta\}. \tag{18}$$

This is possible, because we can show that

$$\gamma_0 > \beta \tag{19}$$

from which it follows that

$$\gamma_{i-1} < \gamma_i, \qquad i = 1, 2, \dots, m \tag{20}$$

and $\gamma_m \to 2\gamma_0$ as $m \to \infty$.

In order to prove (19) we note that α^* satisfies the equality

$$\alpha^* - \frac{1}{2} - \frac{1}{2}\delta = \max\left\{\frac{\varrho}{\alpha^*} - 1, \frac{\varrho}{6\alpha^*}\right\}. \tag{21}$$

In fact, if $\delta \leqq \frac{8}{5}$ then by a simple calculation

$$\alpha^* - \frac{1}{2} - \frac{1}{2}\delta = \frac{\varrho}{\alpha^*} - 1,$$

and

$$\alpha^* = \frac{-1 + \delta + \sqrt{17 + 6\delta + \delta^2}}{4} \leqq \frac{5}{6}\left(1 + \frac{1}{2}\delta\right) = \frac{5}{6}\varrho,$$

i.e., $\frac{\varrho}{\alpha^*} - 1 \geqq \frac{\varrho}{6\alpha^*}$, which means that α^* satisfies (21). Similarly, if $\delta > \frac{8}{5}$ then α^* satisfies (21), too. By (21)

$$\gamma_0 = \alpha_0 - \frac{1}{2} - \frac{1}{2}\delta > \alpha^* - \frac{1}{2} - \frac{1}{2}\delta = \max\left\{\frac{\varrho}{\alpha^*} - 1, \frac{\varrho}{6\alpha^*}\right\} >$$

$$> \max\left\{\frac{\varrho}{\alpha_0} - 1, \frac{\varrho}{6\alpha_0}\right\} = \max\left\{\beta, \frac{1}{6}(1+\beta)\right\} \geqq \beta.$$

Using (15)–(18) we write

$$S_0 := \sum_{k\in K_0} \Delta_n(x_k)^\alpha |l_k(z_n)| = O(n^{-\alpha})\sigma_0 =$$

$$= O\left(n^{\gamma_0 - \alpha + \frac{1}{2}(1+\delta)}\right) = o\left(n^{\gamma_0 - \alpha_0 + \frac{1}{2}(1+\delta)}\right) = o(1).$$

Denote

$$c_i := \max\{0, |z_n| - n^{-\gamma_i}\}, \qquad i = 0, 1, \dots, m.$$

First we prove the inequality

$$\frac{1 - |z_n| + n^{-\gamma_i}}{1 - d_i} \leqq \frac{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}} + 2n^{-\gamma_i}}{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}}}, \qquad i = 0, 1, \dots, m,$$

where $d_i := \min\{|z_n| + n^{-\gamma_i}, (1 - 2n^{-2\beta})^{\frac{1}{2}}\}$, $i = 0, 1, \dots, m$, as above.

If $|z_n| + n^{-\gamma_i} \leqq (1 - 2n^{-2\beta})^{\frac{1}{2}}$, then $d_i = |z_n| + n^{-\gamma_i}$ and hence

$$\frac{1 - |z_n| + n^{-\gamma_i}}{1 - d_i} = \frac{1 - (|z_n| + n^{-\gamma_i}) + 2n^{-\gamma_i}}{1 - (|z_n| + n^{-\gamma_i})} \leqq \frac{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}} + 2n^{-\gamma_i}}{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}}}.$$

If $|z_n| + n^{-\gamma_i} \geqq (1 - 2n^{-2\beta})^{\frac{1}{2}}$, then $d_i = (1 - 2n^{-2\beta})^{\frac{1}{2}}$ and hence

$$\frac{1 - |z_n| + n^{-\gamma_i}}{1 - d_i} = \frac{1 - (|z_n| + n^{-\gamma_i}) + 2n^{-\gamma_i}}{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}}} \leqq \frac{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}} + 2n^{-\gamma_i}}{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}}}.$$

Then we obtain

$$\frac{1 - c_i^2}{1 - d_i^2} \leqq 2\frac{1 - c_i}{1 - d_i} \leqq 2\frac{1 - |z_n| + n^{-\gamma_i}}{1 - d_i} \leqq 2\frac{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}} + 2n^{-\gamma_i}}{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}}} =$$

$$= 2 + \frac{4n^{-\gamma_i}}{1 - (1 - 2n^{-2\beta})^{\frac{1}{2}}} \leqq 2 + \frac{4n^{-\gamma_i}}{1 - (1 - n^{-2\beta})} = 2 + 4n^{2\beta - \gamma_i}.$$

Hence

$$\frac{1 - c_i^2}{1 - d_i^2} = \begin{cases} O(n^{2\beta - \gamma_i}), & \gamma_i < 2\beta \\ O(1), & \gamma_i \geqq 2\beta, \end{cases} \qquad i = 0, 1, \dots, m.$$

Thus for each i, $1 \leqq i \leqq m$, with $\gamma_{i-1} < 2\beta$

$$S_i := \sum_{k \in K_i} \Delta_n(x_k)^\alpha |l_k(z_n)| = O\Big\{ \Big[\frac{(1-c_{i-1}^2)^{\frac{1}{2}}}{n} \Big]^\alpha \Big\} \sigma_i =$$

$$= O(n^{\gamma_i - \alpha + \frac{1}{2}(1-\gamma_{i-1}+\delta)}) \frac{(1-c_{i-1}^2)^{\frac{\alpha}{2}}}{(1-d_{i-1}^2)^{\frac{1}{4}}} =$$

$$= O(n^{\gamma_i - \alpha + \frac{1}{2}(1-\gamma_{i-1}+\delta)}) \Big(\frac{1-c_{i-1}^2}{1-d_{i-1}^2} \Big)^{\frac{1}{4}} = O(n^{\gamma_i - \frac{3}{4}\gamma_{i-1} - \alpha + \frac{1}{2}(1+\beta+\delta)}) =$$

$$= o(n^{\gamma_i - \frac{3}{4}\gamma_{i-1} - \alpha_0 + \frac{1}{2}(1+\beta+\delta)}) = o(n^{\gamma_i - \frac{3}{4}\gamma_{i-1} - \gamma_0 + \frac{1}{2}\beta}) = o(1).$$

On the other hand, for each i, $1 \leqq i \leqq m$, with $\gamma_{i-1} \geqq 2\beta$

$$S_i = O(n^{\gamma_i - \alpha + \frac{1}{2}(1-\gamma_{i-1}+\delta)}) = o(n^{\gamma_i - \alpha_0 + \frac{1}{2}(1-\gamma_{i-1}+\delta)}) =$$

$$= o(n^{\gamma_i - \frac{1}{2}\gamma_{i-1} - \gamma_0}) = o(1).$$

Also we obtain by Corollary 2 that

$$S_{m+1} := \sum_{k \in K_{m+1}} \Delta_n(x_k)^\alpha |l_k(z_n)| = O\Big\{ \Big[\frac{(1-c_m^2)^{\frac{1}{2}}}{n} \Big]^\alpha \Big\} \sigma_{m+1} =$$

$$= O(n^{-\alpha})(1-c_m^2)^{\frac{\alpha}{2}} O(n^{\frac{1}{2}(1-\gamma_m)})(1-d_m^2)^{-\frac{1}{4}} O(n^{\frac{1}{2}(\sigma+\delta)}) =$$

$$= O(n^{\frac{1}{2}(1+\sigma+\delta-\gamma_m)-\alpha}) \frac{(1-c_m^2)^{\frac{\alpha}{2}}}{(1-d_m^2)^{\frac{1}{4}}} = O(n^{\frac{1}{2}(1+\sigma+\delta-\gamma_m)-\alpha}) =$$

$$= o(n^{\alpha-\alpha_0-\gamma_0+\frac{1}{2}(1+\sigma+\delta)-\alpha}) = o(n^{\frac{1}{2}\sigma - 2\gamma_0}) = o(1).$$

Finally, taking $\nu = \varrho + (\alpha - \alpha_0)$ we have

$$S_{m+2} := \sum_{k \in K_{m+2}} \Delta_n(x_k)^\alpha |l_k(z_n)| = O(n^{-\alpha(1+\beta)}) \sigma_{m+2} =$$

$$= O(n^{\nu - \alpha(1+\beta)}) = o(n^{\varrho - \alpha_0(1+\beta)}) = o(1).$$

Therefore

$$\sum_{k=1}^{n} \Delta_n(x_k)^\alpha |l_k(z_n)| = \sum_{i=0}^{m+2} S_i = o(1)$$

and (3) is true.

3.2. Proof of Theorem 2. Putting $\delta = 0$ by Theorem 1 one obtains that $\alpha^* = \frac{\sqrt{17}-1}{4}$.

3.3. Proof of Theorem 3. $\delta < \frac{1}{2}$ implies (by Theorem 1) that $\alpha^* < 1$. The rest of the proof is trivial.

4. Remark

The question if Theorems 1, 2 and 3 could be improved is still open.

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THE LOCAL BEHAVIOUR OF SOME ADDITIVE FUNCTIONS

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1. Introduction

Let f be an additive integer-valued arithmetic function; $A := f(\mathbf{N})$; $B := = f(P)$, where P is the set of primes. Throughout the paper we assume that p is a prime number, $c_1, c_2, \dots$ are positive constants. Furthermore we shall denote by $N_x(a) = N_x(a,f)$ the number of all natural numbers $n \leqq x$ for which $f(n) = a$ $(a \in A)$. We shall also use the classical notations

$$\pi(x) = \sum_{p \leqq x} 1, \qquad \omega(n) = \sum_{p|n} 1, \qquad \Omega(n) = \sum_{p^k \| n} k,$$

Γ will mean the gamma-function.

In the present paper we shall investigate the limiting behaviour of $N_x(a,f)$ for some additive functions f.

The first result in this direction was obtained by J. Hadamard [3] and C. J. de la Vallée-Poussin [2] as the law of prime numbers. Later many authors were interested in the asymptotic behaviour of $N_x(a)$ for the functions ω and Ω. Already in 1900, E. Landau [8] got an answer for fixed a. In 1953–54 L. G. Sathe [11]–[14] and A. Selberg [15] investigated $N_x(a)$ for the above mentioned functions whenever $1 \leqq a \leqq c_1 \log\log x$, where the positive constant c_1 depends only on the function investigated. They proved

$$N_x(a) \sim g(r) \frac{x(\log\log x)^{a-1}}{\log x(a-1)!}.$$

Here $r = a(\log\log x)^{-1}$, and $g(r)$ equals

$$\frac{1}{\Gamma(r+1)} \prod_p \left(1 - \frac{1}{p}\right)^r \left(1 - \frac{r}{p}\right)^{-1}$$

or

$$\frac{1}{\Gamma(r+1)} \prod_p \left(1 - \frac{1}{p}\right)^r \left(1 - \frac{r}{p-1}\right)$$

for Ω or ω, respectively.

---

[1]This work was done while the author visited the Eötvös Loránd University, Budapest.

Presently the asymptotic behaviour of $N_x(a,\Omega)$ is known for all possible values of a, i.e. for $a \leqq \log x/\log 2$ [1], [9]. (It is obvious that $N_x(a,\Omega)=0$ when $a > \log x/\log 2$.)

It was more difficult to consider the behaviour of $N_x(a,\omega)$ when a was large enough with respect to x. In spite of these difficulties an answer was given for a very wide range of a [4], [5], [6], [10].

Generalizing these results for wider sets of functions, several authors [7], [17], [18], [19] used probabilistic methods. The application of these methods limited the range of a to $\log\log x(1+o(1))$ for the functions Ω, or ω, or similar ones. Even the law of prime numbers was not included here.

In this paper we shall expand these bounds of a for some special cases of additive functions.

2. Results

Let f satisfy the following conditions:

(A) there exist non-negative constants λ_a and functions $\varepsilon_a(x)$ $(a \in B)$ such that $\lambda_0 < 1$,

$$\sum_{\substack{p \leqq x \\ f(p)=a}} 1 = \pi(x)\Big(\lambda_a + \frac{\varepsilon_a(x)}{(\log x)^\alpha}\Big),$$

and

$$\sum_{a \in B} v^a |\varepsilon_a(x)| \leqq c_2$$

with some positive constants α and v the values of which will be given later;

(B) there exist positive constants c_3, c_4, c_5, δ such that $0 < \delta < 1/3$ and

$$\sum_{p \leqq x} \frac{v^{f(p)(1+\delta)}}{p} \leqq c_3 \log\log x, \quad v^{f(p)} \leqq c_4 p^{\delta/2}, \quad \sum_{p,k \geqq 2} \frac{v^{f(p^k)}}{p^{k(1-\delta)}} \leqq c_5.$$

Let us put

$$\kappa = \kappa(z) = \sum_{a \in B} \lambda_a z^a,$$

$$G(z) = \frac{1}{\Gamma(\kappa+1)} \prod_p \Big(1 - \frac{1}{p}\Big)^\kappa \sum_{k=0}^{\infty} \frac{z^{f(p^k)}}{p^k}.$$

(In the case of $z^{f(p^k)} = 0^0$ we shall assume $0^0 = 1$.) Further let us expand the function $\kappa(\log x)^\kappa$ into powers of z and denote by $\chi(a)$ the coefficient of z^a in this expansion, i.e.

$$\kappa(\log x)^\kappa = \sum_a \chi(a) z^a.$$

Finally let $r = \chi(a-1)/\chi(a)$.

THEOREM 1. *If f is a non-negative function, $\lambda_0 + \lambda_1 = 1$ and there exists a positive constant C such that conditions* (A), (B) *are satisfied with $v = C$ and some $\alpha > 7 + 4c_3 + 2\kappa(C)$, then*

$$N_x(a) = \frac{x}{\log x} G(r)\chi(a)\Big(1 + O\Big(\frac{a}{(\log\log x)^2}\Big)\Big) \tag{1}$$

uniformly for all $x \geqq 4$ and $1 \leqq a \leqq C\lambda_1 \log\log x$. Here

$$\chi(a) = \lambda_1^a (\log x)^{\lambda_0} \frac{(\log\log x)^{a-1}}{(a-1)!}\Big(1 + \lambda_0 \frac{\log\log x}{a}\Big) \tag{2}$$

if $a \geqq 1$, and $\chi(0) = \lambda_0 (\log x)^{\lambda_0}$.

In the next theorem f may assume negative values.

THEOREM 2. *If $\lambda_0 + \lambda_1 = 1$ and there exist positive constants c and C such that the conditions* (A), (B) *are satisfied with $v = c$, $v = C$ and some $\alpha > 7 + 4c_3 + 2\kappa(C)$, then the relation* (1) *holds uniformly for all $x \geqq 4$ and $c\lambda_1 \log\log x \leqq a \leqq C\lambda_1 \log\log x$ with the same $\chi(a)$ as in* (2).

THEOREM 3. *Let f be a non-negative function, $\lambda_0 + \lambda_1 < 1$, $\lambda_1 > 0$, and let there exist a positive constant c such that condition* (A) *is satisfied with $v = c$ and some $\alpha > 7 + 4c_3 + 2\kappa(c)$. Then for every fixed $a \geqq 1$*

$$N_x(a) = \frac{x}{\log x} G(r)\chi(a)\Big(1 + O\Big(\frac{1}{\log\log x}\Big)\Big)$$

uniformly for all $x \geqq 4$.

Here we can take the same $\chi(a)$ as in (2).

Before formulating the last theorem let us assume that the greatest common divisor of the set $\{k \mid \lambda_k > 0\}$ is equal to 1, and put

$$s = s(r) = \begin{cases} \min\{k \mid k > 0,\ \lambda_k > 0\} & \text{if } r \geqq 1, \\ \min\{k \mid \text{g.c.d.}(1\operatorname{sgn}\lambda_1, \dots, k\operatorname{sgn}\lambda_k) = 1\} & \text{if } r < 1. \end{cases}$$

THEOREM 4. *Let f be a non-negative function. Assume that there exists a positive constant C such that $\kappa(C)$ is finite and conditions* (A), (B) *are satisfied with $v = C$ and some $\alpha > 7 + 4c_3 + 2\kappa(C)$. Then there exists a positive constant ε such that, uniformly for all $x \geqq 4$ and a from the range $0 < r \leqq C$, we have*

$$N_x(a) = \frac{x}{\log x} G(r)(\chi(a) +$$

$$+ O(r^{2-a-3s/2}\kappa(r)(\log x)^{\kappa(r)}(\log\log x)^{-3/2} + r^{-a-s/2}(\log x)^{\kappa(r)-\varepsilon})).$$

3. Proofs

The proofs of all these theorems are based on the following lemma of R. Skrabutėnas [16] on the mean-values of multiplicative functions.

LEMMA. *Let g be a multiplicative function. If there exist κ, $|\kappa| \leqq c_6$, $\alpha > 0$, $0 < \delta < 1/3$ such that*

$$\sum_{p \leqq x} \frac{g(p) - \kappa}{p} = c_7 + O(1/(\log x)^{\alpha}),$$

$$\sum_{p \leqq x} \frac{|g(p)|^{1+\delta}}{p} \leqq c_8 \log\log x, \qquad |g(p)| \leqq c_9 p^{\delta/2},$$

$$\sum_{p, k \geqq 2} \frac{|g(p^k)|}{p^{k(1-\delta)}} \leqq c_{10},$$

then

$$\sum_{n \leqq x} g(n) = \sum_{k=0}^{\ell} \frac{x \beta_k(\kappa)}{\Gamma(\kappa - k)(\log x)^{k+1-\kappa}} + O\Big(\frac{x}{(\log x)^{\eta}}\Big)$$

uniformly for all $x \geqq 4$. Here

(3) $\eta = \min\{\ell+1, \ell+2-\operatorname{Re}\kappa, \ell - c_8 + (\ell+1)\delta/(\delta+1)\}, \quad \ell < (\alpha-3)/2 - c_8,$

the functions $\beta_k(\kappa)$ do not depend on x and they are bounded,

$$\beta_0(\kappa) = \prod_p \Big(1 - \frac{1}{p}\Big)^{\kappa} \sum_{k=0}^{\infty} \frac{g(p^k)}{p^k}.$$

REMARK. In [16], this lemma is stated with the weaker value

$$\eta = \min\{\ell + 1, \ell + 2 - \operatorname{Re}\kappa, -c_8 + (\ell+1)\delta/(\delta+1)\}.$$

It can be seen from the proof that ℓ can be added to the last term.

Assuming $\alpha > 7 + 4c_8 + 2c_6$ and taking $\ell = 2 + [c_6 + c_8]$ ($[u]$ means the integral part of the number u) from the expression of η and the estimate of ℓ in (3) we can get easily that

$$\sum_{n \leqq x} g(n) = \frac{x \beta_0(\kappa)}{\Gamma(\kappa)(\log x)^{1-\kappa}} + O(x (\log x)^{\operatorname{Re}\kappa - 1 - c_{11}}) \tag{4}$$

with some positive constant c_{11} less than $\ell - 1 - c_6 - c_8$.

PROOF OF THEOREM 4. Let z be a complex number such that $|z| \leqq r$. Put $g = z^f$. Condition (A) allows us to represent the mean value of the function g on the primes in the form

$$\sum_{p \leqq x} g(p) = \sum_{a \in B} z^a \sum_{\substack{p \leqq x \\ f(p)=a}} 1 =$$

$$= \sum_{a \in B} z^a(\lambda_a + \varepsilon_a(x)/(\log x)^\alpha)\pi(x) = \kappa\pi(x) + O(x/(\log x)^{\alpha+1}),$$

where $\kappa = \kappa(z)$. Applying partial summation it is not difficult to obtain

$$\sum_{p \leqq x} \frac{g(p) - \kappa}{p} = d + O\Big(1/(\log x)^\alpha\Big), \tag{5}$$

where the constant d equals

$$\int_2^\infty \sum_{p \leqq u} (g(p) - \kappa) \frac{du}{u^2}.$$

Hence the conditions of the lemma are satisfied. Using the theorem of residues we obtain

$$N_x(a) = \frac{1}{2\pi i} \int_{\gamma_r} \sum_{n \leqq x} g(n) \frac{dz}{z^{a+1}},$$

where γ_r is the circle with radius r and centre at the origin. Now from (4) we have $N_x(a) = I + I_1$, where

$$I = \frac{x}{2\pi i \log x} \int_{\gamma_r} G(z)\kappa(\log x)^\kappa \frac{dz}{z^{a+1}}$$

and

$$I_1 \ll \frac{x}{r^a(\log x)^{1+c_{11}}} \int_0^\pi \exp\{\operatorname{Re} \kappa(re^{i\varphi}) \log\log x\} d\varphi. \tag{6}$$

First we shall calculate the integral I. Conditions (A), (B) guarantee the analyticity of $G(z)$ in the circle $|z| \leqq r \leqq C$. More precisely we have

$$\prod_p \Big(1 - \frac{1}{p}\Big)^\kappa \sum_{k=0}^\infty \frac{g(p^k)}{p^k} = \tag{7}$$

$$= \prod_{p \leqq p_0} \Big(1-\frac{1}{p}\Big)^{\kappa} \sum_{k=0}^{\infty} \frac{g(p^k)}{p^k} \exp\Big\{\sum_{p>p_0} \Big(\kappa \log\Big(1-\frac{1}{p}\Big) + \log\Big(\sum_{k=0}^{\infty} \frac{g(p^k)}{p^k}\Big)\Big)\Big\},$$

where p_0 is a sufficiently large but fixed prime number such that

$$\sum_{k=1}^{\infty} \frac{|g(p^k)|}{p^k} \leqq \frac{1}{2} \qquad (p > p_0).$$

From conditions (A), (B) and the relation (5) it follows that the exponent in (7) equals

$$\sum_{p>p_0} \frac{g(p)-\kappa}{p} + O(1) = O(1).$$

Hence

$$G(z) = G(r) + (z-r)G'(r) + (z-r)^2\Phi(z,r), \tag{8}$$

where

$$\Phi(z,r) = \int_0^1 (1-u)G''(r+u(z-r))du$$

is an analytic function in the above-mentioned circle, and therefore it is bounded. From (7) it follows also that there exist two positive constants c_{12}, c_{13} such that $c_{12} \leqq G(r) \leqq c_{13}$. So we can write

$$I = \frac{x}{\log x}\Big(\frac{G(r)}{2\pi i}\int_{\gamma_r} \kappa(\log x)^{\kappa}\frac{dz}{z^{a+1}} + \frac{G'(r)}{2\pi i}\int_{\gamma_r} \kappa(\log x)^{\kappa}(z-r)\frac{dz}{z^{a+1}} + \tag{9}$$

$$+\frac{1}{2\pi i}\int_{\gamma_r} \kappa(\log x)^{\kappa}(z-r)^2\Phi(z,r)\frac{dz}{z^{a+1}}\Big) = \frac{x}{\log x}(I_0+I_2+I_3).$$

Now applying again the theorem of residues we obtain

$$I_o = G(r)\chi(a)$$

and

$$I_2 = G'(r)(\chi(a-1) - r\chi(a)) = 0.$$

Further since $e^{i\varphi} - 1 = O(|\varphi|)$ we have that

$$I_3 \ll r^{2-a}\kappa(r)\int_0^{\pi} \exp\Big\{\sum_k \lambda_k r^k \cos(k\varphi)\log\log x\Big\}\varphi^2 d\varphi.$$

From the condition g.c.d. $\{k \mid \lambda_k > 0\} = 1$ it follows the existence of a positive constant c_{14} such that

$$\sum_k \lambda_k r^k (1 - \cos(k\varphi)) \geqq c_{14} r^s \varphi^2,$$

when $0 \leqq \varphi \leqq \pi$. It is not difficult to see now that

$$I_3 \ll r^{2-a}\kappa(r)(\log x)^{\kappa(r)} \int_0^\pi \exp\{-c_{14} r^s \varphi^2 \log\log x\}\varphi^2 d\varphi \ll$$

$$\ll r^{2-a-3s/2}\kappa(r)(\log x)^{\kappa(r)}(\log\log x)^{-3/2}.$$

I_1 can be estimated easily in a similar way,

$$I_1 \ll x(\log x)^{\kappa(r)-1-c_{11}} r^{-a-s/2}(\log\log x)^{-1/2}.$$

PROOF OF THEOREM 1. Note that in this case

$$\chi(a) = \lambda_1^a (\log x)^{\lambda_0} \frac{(\log\log x)^{a-1}}{(a-1)!}\left(1 + \lambda_0 \frac{\log\log x}{a}\right)$$

if $a \geqq 1$, and $\chi(0) = \lambda_0(\log x)^{\lambda_0}$. Also $\kappa(z) = \lambda_0 + \lambda_1 z$, $s = 1$, and

$$r = \frac{a}{\lambda_1 \log\log x}\left(1 - \frac{1}{\lambda_0 \log\log x + a}\right)$$

for $a > 1 - \lambda_0$.

Hence, in case $a > 1 - \lambda_0$, using Stirling's formula, we get easily that the first summand of the remainder term in Theorem 1 can be estimated in the following way:

$$r^{2-a-3/2}(\lambda_0 + \lambda_1 r)(\log x)^{\lambda_0 + \lambda_1 r}(\log\log x)^{-3/2} \ll a\chi(a)(\log\log x)^{-2}.$$

The second summand of the remainder term is evidently less than the first one.

In case $\lambda_0 = 0$ and $a = 1$ it is enough to repeat the proof of Theorem 4 using the expansion

$$G(z) = G(0) + z\Phi(z) \tag{10}$$

with

$$\Phi(z) = \int_0^1 G'(uz)du$$

instead of (8). (9) takes on the form

$$I=\frac{x}{\log x}\left(\frac{G(0)}{2\pi i}\int_{\gamma_r}(\log x)^z\frac{dz}{z}+\frac{1}{2\pi i}\int_{\gamma_r}(\log x)^z\Phi(z)dz\right)=\frac{x}{\log x}(G(0)+I_4).$$

Let us take $r=(\log\log x)^{-2}$. Then

$$I_4\ll\int_0^\pi(\log x)^r r\,d\varphi\ll(\log\log x)^{-2}. \tag{11}$$

I_1 can be estimated as in Theorem 4.

PROOF OF THEOREM 2 is even simpler than that of Theorem 1. When f takes both positive and negative values then conditions (A), (B) guarantee the analyticity of $G(z)$ in the annulus $c\leqq|z|\leqq C$ only. In this case we can use the formula (8) as well but perhaps with another function $\Phi(z,r)$. Since the range of values of r is more restricted, all previously obtained estimates hold.

PROOF OF THEOREM 3. It is not difficult to see that all the conditions of Theorem 4 are valid. Furthermore in this case we have

$$\chi(a)=\lambda_1^a(\log x)^{\lambda_0}\frac{(\log\log x)^{a-1}}{(a-1)!}\left(1+\lambda_0\frac{\log\log x}{a}\right)\left(1+O\left(\frac{1}{\log\log x}\right)\right)$$

if $a\geqq 1$, and $\chi(0)=\lambda_0(\log x)^{\lambda_0}$; also

$$r=\frac{a}{\lambda_1\log\log x}\left(1-\frac{1}{\lambda_0\log\log x+a}\right)\left(1+O\left(\frac{1}{\log\log x}\right)\right).$$

Further we can continue the proof by using the same ideas as were used in the proof of Theorem 1. Since instead of the exact value of $\chi(a)$ we use $\chi(a)$ from (2), the rate of convergence of the remainder term is a little worse than the one in Theorem 1.

4. Remarks

Here at first we shall give a result for the case $a=0$.

1. *Let f be a non-negative function, $\lambda_1>0$, and let there exist a positive constant c such that condition* (A) *is satisfied with $v=c$ and some $\alpha>7+4c_3+2\kappa(c)$. Then there exists a positive constant ε such that, uniformly for all $x\geqq 4$, we have*

$$N_x(0)=x(\log x)^{\lambda_0-1}G(0)(\lambda_0+O((\log x)^{-\varepsilon})). \tag{12}$$

PROOF. The proof is analogous to the proofs of Theorems 1–4. Using the expansion (10) for the function $G(z)$ and choosing $r=(\log x)^{-c_{11}}$, from (6) and the first part of (11) we obtain that $I_1 \ll x(\log x)^{\lambda_0-1-c_{11}}$ and $I_4 \ll r$. From these estimates we deduce easily the validity of (12).

2. Because we are not able to estimate the asymptotic behaviour of the value $\chi(a)$ in Theorem 4, the remainder term in this theorem is complicated. If we knew the asymptotic behaviour of the coefficient $\chi(a)$ in cases different from the ones in Theorems 1, 2 and 3, and maybe for a wider set of values of a, we could formulate similar theorems in those cases as well.

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THE CURVATURE OF SUBMANIFOLDS OF AN S-SPACE FORM

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0. Introduction

Many authors have studied the geometry of submanifolds of Kaehlerian and Sasakian manifolds. On the other hand, David E. Blair has initiated the study of S-manifolds, which reduce, in particular cases, to Sasakian manifolds [1].

I. Mihai [7] and Ornea [8] have studied CR-submanifolds of S-manifolds. The purpose of the present paper is to investigate some properties of invariant and anti-invariant submanifolds of an S-manifold whose invariant f-sectional curvature is constant, that is, of an S-space form. Specifically, those ones related with the curvature tensor fields and with the scalar curvature on the submanifold.

In Section 1 we review basic formulas for submanifolds in Riemannian manifolds and, in Section 2, for S-manifolds. In Sections 3 and 4 we study anti-invariant and invariant submanifolds, respectively, of an S-space form. Finally, in the last section we give some examples.

1. Preliminaries

Let N^n be a Riemannian manifold of dimension n and M^m an m-dimensional submanifold of N^n. Let g be the metric tensor field on N^n as well as the induced metric on M^m. We denote by $\overline{\nabla}$ the covariant differentiation in N^n and by ∇ the covariant differentiation in M^m determined by the induced metric. Let $T(N)$ (resp. $T(M)$) be the Lie algebra of vector fields in N^n (resp. in M^m) and $T(M)^{\perp}$ the set of all vector fields normal to M^m.

The Gauss–Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \overline{\nabla}_X V = -A_V X + D_X V, \tag{1.1}$$

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$X, Y \in T(M)$, $V \in T(M)^{\perp}$, where D is the connection in the normal bundle, σ is the second fundamental form of M^m and A_V is the Weingarten endomorphism associated with V. σ and A_V are related by $g(A_V X, Y) = g(\sigma(X,Y), V)$.

We denote by $\overline{R}$ and R the curvature tensor fields associated with $\overline{\nabla}$ and ∇, respectively. The Gauss equation is given by

$$\begin{aligned}(1.2) \qquad \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - \\ -g(\sigma(X,W),\sigma(Y,Z)), \quad X,Y,Z,W \in T(M).\end{aligned}$$

Finally, the submanifold M^m is said to be totally geodesic in N^n if its second fundamental form is identically zero and it is said to be minimal if $H \equiv 0$, where H is the mean curvature vector, defined by $H = (1/m)\,\mathrm{trace}(\sigma)$.

2. S-manifolds

Let (N^{2n+s}, g) be a $(2n+s)$-dimensional Riemannian manifold. N^{2n+s} is said to be an S-manifold if there exist on N^{2n+s} an f-structure f ([9]) of rank $2n$, s global vector fields $\xi_1, \dots, \xi_s$ (structure vector fields) and their dual 1-forms $\eta_1, \dots, \eta_s$ such that ([1])

$$(2.1) \quad \text{(i)} \qquad f\xi_\alpha = 0; \quad \eta_\alpha \circ f = 0; \quad f^2 = -I + \sum_\alpha \xi_\alpha \otimes \eta_\alpha,$$

$$g(X,Y) = g(fX, fY) + \Phi(X,Y),$$

for any $X, Y \in T(N)$, $\alpha = 1, \dots, s$, where $\Phi(X,Y) = \sum_\alpha \eta_\alpha(X)\eta_\alpha(Y)$.

(ii) The f-structure f is normal, that is

$$[f,f] + 2\sum_\alpha \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f,f]$ is the Nijenhuis torsion of f.

(iii) $\eta_1 \wedge \ldots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ and $d\eta_1 = \ldots = d\eta_s = F$, for any α, where F is the fundamental 2-form defined by $F(X,Y) = g(X, fY)$, $X, Y \in T(N)$.

In the case $s = 1$, an S-manifold is a Sasakian manifold. For $s \geqq 2$, examples of S-manifolds are given in [1], [2], [3] and [5].

For the Riemannian connection $\overline{\nabla}$ of g on an S-manifold N^{2n+s}, the following were also proved in [1]:

$$(2.2) \qquad \overline{\nabla}_X \xi_\alpha = -fX, \quad X \in T(N), \quad \alpha = 1, \dots, s,$$

$$(2.3) \qquad (\overline{\nabla}_X f)Y = \sum_\alpha [g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X], \quad X, Y \in T(N).$$

Let $\mathcal{L}$ denote the distribution determined by $-f^2$ and $\mathcal{M}$ the complementary distribution. $\mathcal{M}$ is determined by f^2+I and spanned by $\xi_1, \dots, \xi_s$. If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$ for any α and if $X \in \mathcal{M}$, then $fX = 0$.

A plane section π is called an invariant f-section if it is determined by a vector $X \in \mathcal{L}(p)$, $p \in N^{2n+s}$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature $\overline{K}(X, fX)$, denoted by $\overline{H}(X)$, is called an invariant f-sectional curvature. If N^{2n+s} is an S-manifold whose invariant f-sectional curvature is constant k, then its curvature tensor has the form ([6])

$$\overline{R}(X,Y,Z,W) = \sum_{\alpha,\beta}\{g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z)- \tag{2.4}$$
$$-g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W) + g(fY,fZ)\eta_\alpha(X)\eta_\beta(W)-$$
$$-g(fY,fW)\eta_\alpha(X)\eta_\beta(Z)\} + (1/4)(k+3s)\{g(X,W)g(fY,fZ)-$$
$$-g(X,Z)g(fY,fW) + g(fY,fW)\Phi(X,Z) - g(fY,fZ)\Phi(X,W)\}+$$
$$+(1/4)(k-s)\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W)-$$
$$-2F(X,Y)F(Z,W)\}, \quad X,Y,Z,W \in T(N)$$

Then, the S-manifold will be denoted by $N^{2n+s}(k)$ and it is said to be an S-space form.

Finally, let M^m be an m-dimensional submanifold immersed in N^{2n+s}. We denote by $\mathcal{L}_M$ the set of all vector fields in $\mathcal{L}$ which are tangent to M^m. M^m is said to be invariant if all of ξ_α ($\alpha = 1, \dots, s$) are always tangent to M^m and $fX \in T(M)$, for any $X \in T(M)$. It is easy to show that an invariant submanifold of an S-manifold is an S-manifold too and so, $m = 2p+s$. On the other hand, M^m is said to be an anti-invariant submanifold if $fX \in T(M)^\perp$, for any $X \in T(M)$.

3. Anti-invariant submanifolds of $N^{2n+s}(k)$ tangent to the structure vector fields

In this section, let M^{m+s} be an anti-invariant submanifold of $N^{2n+s}(k)$, tangent to the structure vector fields. Since $fT_x(M) \subseteqq T_x(M)^\perp$ at each point $x \in M^{m+s}$, we have the decomposition of $T_x(M)^\perp$ into the direct sum

$$T_x(M)^\perp = fT_x(M) \otimes \nu_x(M),$$

where $\nu_x(M)$ is the orthogonal complement of $fT_x(M)$ in the normal space $T_x(M)^\perp$. We note that $f\nu_x(M) \subseteqq \nu_x(M)$. For any vector field $V \in T(M)^\perp$, we write

$$fV = tV + nV, \tag{3.1}$$

where tV (resp. nV) is the tangential component (resp. normal component) of fV. Then, t is a tangent-bundle valued 1-form on the normal bundle and n is an endomorphism of the normal bundle. Moreover, if n does not vanish, it is an f-structure. We say that n is parallel if $Dn \equiv 0$, where

$$(D_X n)V = D_X nV - nD_X V, \quad X \in T(M), \ V \in T(M)^\perp.$$

From (1.1) and (2.2) it is easy to show that

$$(D_X n)V = -\sigma(X, tV) - fA_V X. \tag{3.2}$$

For later use, we prove

LEMMA 3.1. *Let M^{m+s} be an anti-invariant submanifold of an S-manifold N^{2n+s}, tangent to the structure vector fields. Then*
(i) *$A_{fX}Y = A_{fY}X$, for any $X, Y \in \mathcal{L}_M$.*
(ii) *If $Dn \equiv 0$, then $A_U = 0$, for any $U \in \nu(M)$.*

PROOF. From (1.1) and (2.3), we have

$$A_{fX}Y = -\overline{\nabla}_Y fX + D_Y fX = -\sum_\alpha g(fX, fY)\xi_\alpha - f\nabla_Y X - f\sigma(X,Y) + D_Y fX$$

if $X, Y \in \mathcal{L}_M$, so that, since σ and g are symmetric, (i) holds.
On the other hand, if $X \in T(M)$ and $U \in \nu(M)$, from (3.2), we have $0 = (\nabla_X n)U = -fA_U X$, and then, $0 = f^2 A_U X = -A_U X$. So, (ii) holds.

We choose a local field of an orthonormal frame on $N^{2n+s}(k)$

$$\{E_1, \dots, E_m, E_{m+1} = fE_1, \dots, E_{2m} = fE_m, E_{2m+1}, \dots, E_{2n},$$
$$E_{2n+1} = \xi_1, \dots, E_{2n+s} = \xi_s\}$$

such that $E_1, \dots, E_m, \xi_1, \dots, \xi_s$ are tangent to M^{m+s}. Unless otherwise stated, we use the conventions that the ranges of indices are, respectively:

$$i, j = 1, \dots, m, 2n+1, \dots, 2n+s;$$
$$y, z = 1, \dots, m;$$
$$a, b = m+1, \dots, 2n.$$

Now we denote $\sigma^a_{ij} = g(\sigma(E_i, E_j), E_a)$ and consider the symmetric $(m \times m)$-matrices $H_a = (\sigma^a_{yz})$. Form (1.1) and (2.2), we have

$$\sigma(X, \xi_\sigma) = -fX, \tag{3.3}$$

for any $X \in T(M)$, $\alpha = 1, \dots, s$. Then, if $m > 0$, M^{m+s} can not be totally geodesic. However, we can prove

PROPOSITION 3.2. *Let M^{m+s} be an anti-invariant submanifold of an S-space form $N^{2n+s}(k)$, tangent to the structure vector fields. If $Dn = 0$ and if $H_a H_b = H_b H_a$, for any a, b, then M^{m+s} is flat if and only if $k = -3s$.*

PROOF. First, since M^{m+s} is anti-invariant , from (1.1) and (2.2), we have $\nabla_X \xi_\alpha = 0$, $X \in T(M)$, $\alpha \in \{1, \dots, s\}$. Then

$$R(X, Y, \xi_\alpha, Z) = 0 = R(X, Y, Z, \xi_\alpha), \tag{3.4}$$

$X, Y, Z \in T(M)$, $\alpha \in \{1, \dots, s\}$. On the other hand, if $X, Y, Z, W \in \mathcal{L}_M$, from the Gauss equation (1.2), Lemma 3.1 and the hypothesis, we have $R(X, Y, Z, W) = \overline{R}(X, Y, Z, W)$. Then, from (2.4):

$$R(X, Y, Z, W) = (1/4)(k + 3s)(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)). \tag{3.5}$$

Now, (3.4) and (3.5) complete the proof.

If M^{m+s} is minimal, a straightforward computation, using the Gauss equation (1.2), (2.4) and (3.3), shows that

$$\rho \leqq (1/4)m(m-1)(k + 3s),$$

where ρ is the scalar curvature of M^{m+s}. On the other hand, we have

PROPOSITION 3.3. *Let M^{m+s} be an anti-invariant submanifold of an S-space form $N^{2n+s}(k)$, tangent to the structure vector fields. If $Dn = 0$ and if $H_a H_b = H_b H_a$, for any a, b, then*

$$\rho = (1/4)m(m-1)(k + 3s).$$

PROOF. By using the hypothesis, we get (3.5). Then

$$R(E_y, X, Y, E_y) = (1/4)(k + 3s)(g(X, Y) - g(X, E_y)g(Y, E_y)),$$

for any $X, Y \in \mathcal{L}_M$. So, by using (3.4), if S is the Ricci tensor field of M^{m+s}, we have

$$S(X, Y) = (1/4)(m-1)(k + 3s)g(X, Y)$$

and the result follows from (3.4) again.

4. Invariant submanifolds of $N^{2n+s}(k)$

In this section let M^{2m+s} be an invariant submanifold of an S-space form $N^{2n+s}(k)$. Then, since $g(X, fV) = -g(fX, V)$, $X \in T(M)$, $V \in T(M)^\perp$, we have $t \equiv 0$. Now, from (1.1), (2.2) and (2.3), we have

LEMMA 4.1. *Let M^{2m+s} be an invariant submanifold of an S-manifold N^{2n+s}. Then, for any $X, Y \in T(M)$, $\alpha \in \{1, \dots, s\}$*

$$\sigma(X, \xi_\alpha) = 0 \tag{4.1}$$

and

$$\sigma(X, fY) = f\sigma(X, Y) = \sigma(fX, Y). \tag{4.2}$$

PROPOSITION 4.2. *Let M^{2m+s} be an invariant submanifold of an S-space form $N^{2n+s}(k)$. If H denotes the invariant f-sectional curvature of M^{2m+s}, the $nH \leqq k$ and equality holds if and only if M^{2m+s} is totally geodesic.*

PROOF. By using the Gauss equations (1.2) and (4.2), we easily prove

$$R(X, fX, fX, X) = k - 2\|\sigma(X, X)\|^2, \tag{4.3}$$

for any $X \in \mathcal{L}_M$. Then, the first assertion is immediate from (4.3). Now if M^{2m+s} is totally geodesic, $\sigma(X, X) = 0$, for any $X \in \mathcal{L}_M$ and $H = k$. Conversely, if $H = k$, then $\sigma(X, X) = 0$, for any unit vector field $X \in \mathcal{L}_M$. Now, since σ is symmetric, the proof is complete.

We choose a local field of orthonormal frames $\{E_1, \dots, E_m, E_{m+1} = fE_1, \dots, E_{2m} = fE_m, E_{2m+1}, \dots, E_{2n}, \xi_1, \dots, \xi_s\}$ on $N^{2n+s}(k)$, such that $E_1, \dots, E_{2m}, \xi_1, \dots, \xi_s$ belong to M^{2m+s}. Unless otherwise stated, we use the conventions that the ranges of indices are:

$$i = 1, \dots, 2m;\ u, v = 1, \dots, m;$$
$$\lambda = 2m+1, \dots, 2n;\ \alpha, \beta = 1, \dots, s,$$

respectively.

PROPOSITION 4.3. *Any invariant submanifold M^{2m+s} of an S-manifold is minimal.*

PROOF. By using (4.1) and (4.2), we have

$$(2m+s)H = \sum_i \sigma(E_i, E_i) = \sum_u (\sigma(E_u, E_u) + \sigma(fE_u, fE_u)) =$$
$$= \sum_u (\sigma(E_u, E_u) + f^2\sigma(E_u, E_u)) = 0$$

and the submanifold is minimal.

Now, let S be the Ricci tensor field of M^{2m+s}. From the Gauss equation (1.2), (2.3) and Lemma 4.1, a straightforward computation gives:

$$S - (1/2)(m(k+3s) + k - s)(g - \Phi) - 2m \sum_{\alpha,\beta} \eta_\alpha \otimes \eta_\beta$$

is a negative semi-definite symmetric tensor. Moreover, if ρ is the scalar curvature of M^{2m+s}, then

$$\rho = m^2(k+3s) + m(k+s) - \|\sigma\|^2. \tag{4.4}$$

On the other hand, from Proposition 4.2, $H(X) \leqq k$, for any unit vector field $X \in \mathcal{L}_M$. Now, we can prove

THEOREM 4.4. *Let M^{2m+s} be an invariant submanifold of an S-space form $N^{2n+s}(k)$. Then M^{2m+s} is totally geodesic if and only if one of the following assertions is satisfied:*

$$\min\{k, (1/4)(k+3s)\} \leqq K(X,Y) \leqq \max\{k, (1/4)(k+3s)\}, \tag{4.5}$$

where K denotes the sectional curvature on M^{2m+s}, $X, Y \in \mathcal{L}_M$ are orthonormal vector fields and $m \geqq 2$.

$$H(X) = k, \tag{4.6}$$

where $X \in \mathcal{L}_M$ is a unit vector field.

$$S = (1/2)(m(k+3s) + k - s)(g - \Phi) + 2m \sum_{\alpha,\beta} \eta_\alpha \otimes \eta_\beta. \tag{4.7}$$

$$\rho = m^2(k+3s) + m(k+s). \tag{4.8}$$

PROOF. The restriction $m \geqq 2$ for (4.5) is natural, because if $m = 1$, then $H = K$. Now, suppose $k \geqq s$. Thus, (4.5) is

$$(1/4)(k+3s) \leqq K(X,Y) \leqq k.$$

If $g(X, fY) = 0$, by using (2.4), the Gauss equation (1.2) and (4.2), we get

$$K(X,Y) + K(X, fY) = (1/2)(k+3s) - 2\|\sigma(X,Y)\|^2.$$

Since $(1/4)(k+3s) \leqq \min\{K(X,Y), K(X,fY)\}$, we obtain $\sigma(X,Y) = 0$ and from (4.2) again, $\sigma(X, fY) = 0$. Then, using the above frame field, we have $\sigma(E_u, E_v) = 0 = \sigma(E_u, fE_v)$, for any u, v. Moreover, if we put

$$X = (1/\sqrt{2})(E_u + E_v), \quad Y = (1/\sqrt{2})(E_u - E_v),$$

then $0 = \sigma(X,Y) = (1/2)(\sigma(E_u, E_u) - \sigma(E_v, E_v))$.

Now, putting $X = (1/\sqrt{2})(E_u + fE_v)$, $Y = (1/\sqrt{2}(E_u - fE_v)$, we have $0 = (1/2)(\sigma(E_u, E_u) + \sigma(E_v, E_v))$ and so, $\sigma(E_u, E_u) = 0$, for any u. From (4.2), $\sigma(fE_u, fE_u) = 0$ too. Thus, taking account of (4.1), we get $\sigma \equiv 0$ and M^{2p+s} is totally geodesic.

Conversely, if the submanifold is totally geodesic, by using the Gauss equation (1.2) and (2.4), we obtain

$$K(X,Y) = (1/4)(k+3s) + (3/4)(k-s)g(X,fY)^2,$$

for any X and Y verifying the hypothesis. Thus, $K(X,Y) \geqq (1/4)(k+3s)$. From Cauchy–Schwarz's inequality

$$g(X,fY)^2 \leqq g(X,X)g(fY,fY) = 1$$

and so, $K(X,Y) \leqq k$.

For $k \leqq s$, the proof is analogous.

Next, it is easy to show that (4.6) is equivalent to the fact that M^{2m+s} is totally geodesic, using Proposition 4.2. Now (4.7) and (4.8) follow from the Gauss equation (1.2), (2.4) and (4.4). Consequently, the proof is complete.

Finally, we suppose that M^{2m+s} is of constant invariant f-sectional curvature c. Then $c \leqq k$ and the equality holds if and only if M^{2m+s} is totally geodesic. Now, we can prove

THEOREM 4.5. *Let $M^{2m+s}(c)$ be an invariant submanifold of an S-space form $N^{2n+s}(k)$. Then*

$$\sum_{\lambda} A_\lambda^2 = (1/2)(m+1)(c-k)f^2, \tag{4.9}$$

where we write A_λ instead of A_{E_λ},

$$\|\sigma\|^2 = m(m+1)(k-c), \tag{4.10}$$

$$\|\sigma(X,X)\|^2 = (1/2)(k-c), \tag{4.11}$$

for any unit vector field $X \in \mathcal{L}_M$, and

$$\|\sigma(X,Y)\|^2 = (1/4)(k-c), \tag{4.12}$$

for any orthonormal vector fields $X, Y \in \mathcal{L}_M$ such that $g(X,fY) = 0$.

PROOF. Since $M^{2m+s}(c)$ is an S-space form, from (2.4) we have

$$S(X,Y) = (1/2)[m(c+3s)+c-s]g(fX,fY) + 2m\sum_{\alpha,\beta}\eta_\alpha(X)\eta_\beta(Y). \tag{4.13}$$

On the other hand, from the Gauss equation (1.2)

$$S(Y,Y) = (1/2)[m(k+3s)+k-s]g(fX,fY) + \\ +2m\sum_{\alpha,\beta}\eta_\alpha(X)\eta_\beta(Y) - \sum_{\lambda} g(A_\lambda X, A_\lambda Y).$$

Since $g(A_\lambda X, A_\lambda Y) = g(A_\lambda^2 X, Y)$, we obtain (4.9). Now, from (4.13), $\rho = m^2(c+3s)+m(c+s)$ and so, by using (4.4), we have (4.10). To obtain (4.11), we only have to observe that $H(X) = c$. But from the Gauss equation (1.2), $H(x) = k - 2\|\sigma(X,X)\|^2$. Finally, from (2.4) again $K(X,Y) = (1/4)(c + 3s) = K(X, fY)$ for any orthonormal vector fields $X, Y \in \mathcal{L}_M$ such that $g(X, fY) = 0$. But

$$K(X,Y) = (1/4)(k + 3s) + g(\sigma(X,X), \sigma(Y,Y)) - \|\sigma(X,Y)\|^2$$

and

$$K(X,fY) = (1/4)(k + 3s) - g(\sigma(X,X), \sigma(Y,Y)) - \|\sigma(X,Y)\|^2.$$

The proof is complete.

It is easy to show that the $m(m + 1)$ normal vector fields $\sigma(E_i, E_j)$, $f\sigma(E_i, E_j)$, $i, j = 1, \ldots, 2m$, $i \leqq j$, are pairwise orthogonal. Using this, we can prove

THEOREM 4.6. *Let $M^{2m+s}(c)$ be an invariant submanifold of an S-space form $N^{2n+s}(k)$. If $n-m < (1/2)m(m+1)$, then $M^{2m+s}(c)$ is totally geodesic.*

PROOF. If the submanifold is not totally geodesic, then $c < k$. But if $i \neq j$, from (4.11), $\sigma(E_i, E_j) \neq 0 \neq f\sigma(E_i, E_j)$, and so $2(n - m) \geqq m(m + 1)$ which is a contradiction.

COROLLARY 4.7. *Let M^{2m+s} $(m \geqq 2)$ be an invariant submanifold of codimension 2 in an S-space form $N^{2n+s}(k)$. Then, M^{2m+s} is totally geodesic if and only if M^{2m+s} is an S-space form.*

5. Examples

(1) *Principal toroidal bundles.* Let $\pi : S^{2n+1} \to \mathbf{PC}^n$ denote the Hopf fibration, and consider

$$H^{2n+s} = \{(p_1, \ldots, p_s) \in S^{2n+1} \times \ldots \times S^{2n+1} / \pi(p_1) = \ldots = \pi(p_s)\}.$$

Using the diagonal map Δ we define a principal toroidal bundle over $\mathbf{PC}^n$ by the following commutative diagram:

$$\begin{array}{ccc} H^{2n+s} & \xrightarrow{\overline{\Delta}} & S^{2n+1} \times \ldots \times S^{2n+1} \\ \downarrow & & \downarrow{\scriptstyle \pi \times \ldots \times \pi} \\ \mathbf{PC}^n & \xrightarrow{\Delta} & \mathbf{PC}^n \times \ldots \times \mathbf{PC}^n \end{array}$$

Let ω_α be the contact form on S_α^{2n+1} ($\alpha = 1, \dots, s$) and define η_α ($\alpha = 1, \dots, s$) on H^{2n+s} by $\eta_\alpha = \overline{\Delta}^*|_{S_\alpha^{2n+1}(\omega_\alpha)}$. Then H^{2n+s} is an S-manifold (cf. [1], [2]). Moreover, the invariant f-sectional curvature of H^{2n+s} is $k = 4 - 3s$, a constant.

Now, let M^{m+s} be a submanifold of H^{2n+s}, tangent to the structure vector fields and N^m a submanifold of $\mathbf{P}\mathbf{C}^n$ such that the diagram

$$\begin{array}{ccc} M^{m+s} & \xrightarrow{\ i\ } & H^{2n+s} \\ \downarrow & & \downarrow \\ N^m & \xrightarrow{\ i'\ } & \mathbf{P}\mathbf{C}^n \end{array}$$

commutes and the immersion i is a diffeomorphism on the fibres. Then M^{m+s} is an invariant submanifold or an anti-invariant submanifold of H^{2n+s} if and only if N^m is a complex submanifold or a totally real submanifold of $\mathbf{P}\mathbf{C}^n$, respectively. Moreover, M^{m+s} is minimal if and only if N^m is minimal ([4], [7]).

Further, if N^m is a complex submanifold, M^{m+s} is totally geodesic if and only if N^m is totally geodesic. On the other hand, if N^m is a totally real submanifold, M^{m+s} is flat if and only if N^m is flat. Finally, if we denote by ρ and ρ' the scalar curvatures of M^{m+s} and N^m, respectively, we have $\rho' - sm \leqq \rho \leqq \rho'$. Moreover, M^{m+s} and N^m are anti-invariant and totally real submanifolds, respectively, if and only if $\rho = \rho'$ and they are invariant and complex submanifolds, respectively if and only if $\rho' - sm = \rho$ (cf. [4]).

(2) *Euclidean spaces.* Let E^{2n+s} be a euclidean space with cartesian coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s)$. Then an S-structure on E^{2n+s} is defined by (cf. [5])

$$\xi_\alpha = 2\partial/\partial z_\alpha \quad (\alpha = 1, \dots, s),$$

$$\eta_\alpha = (1/2)\Big(dz_\alpha - \sum_{i=1}^n y_i dx_i\Big), \quad (\alpha = 1, \dots, s),$$

$$fX = \sum_{i=1}^n Y^i \partial/\partial x_i - \sum_{i=1}^n X^i \partial/\partial y_i + \Big(\sum_{i=1}^n Y^i Y_i\Big)\Big(\sum_\alpha \partial/\partial z_\alpha\Big),$$

$$g = \sum_\alpha \eta_\alpha \otimes \eta_\alpha + (1/4)\sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i),$$

where $X = \sum_{i=1}^n (X^i \partial/\partial x_i + Y^i \partial/\partial y_i) + \sum_\alpha Z^\alpha \partial/\partial z_\alpha$.

With this structure E^{2n+s} is an S-manifold of constant invariant f-sectional curvature $k = -3s$.

We have that $E^{2m+s}(-3s)$ is a totally geodesic invariant submanifold of $E^{2m+s}(-3s)$ $(m < n)$.

On the other hand, we can consider the following natural imbedding of E^{n+s} into $E^{2n+s}(-3s)$:

$$(x_1, \dots, x_n, z_1, \dots, z_s) \mapsto (x_1, \dots, x_n, 0, \dots, 0, z_1, \dots, z_s).$$

If we denote $E_i = 2(\partial/\partial x_i + y_i \sum_\alpha \partial/\partial z_\alpha)$, $1 \leqq i \leqq n$, we get that

$$\{E_1, \dots, E_n, fE_1, \dots, fE_n, \xi_1, \dots, \xi_s\}$$

is a local field of an orthonormal frame on $E^{2n+s}(-3s)$, such that $E_1, \dots, E_n$ belong to E^{n+s}. Then it is easy to check that E^{n+s} is an anti-invariant submanifold of $E^{2n+s}(-3s)$ with f-structure n in the normal bundle parallel. Moreover, $H_{fE_i} \equiv 0$, $1 \leqq i \leqq n$. Consequently, E^{n+s} is flat.

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SOME SPECIAL PROPERTIES OF CONDITIONAL EXPECTATION

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In this paper we shall discuss some special properties of conditional expectation. In Section I we prove that the condition given in [7] is equivalent, in some sense, to the assertion of the Fatou Lemma.

The conditional expectation can be defined as an orthogonal projection. We show that the almost sure convergence of a sequence of conditional expectations of random variables $\{\mathbf{E}^{\mathcal{F}} X_n, n \geqq 1\}$ does not follow from the almost sure convergence of the sequence $\{X_n, n \geqq 1\}$, and conversely. Some suitable examples are given in Section II.

In the problem of optimal stopping one considers the value $\sup_{\tau \in T} \mathbf{E} X_\tau$. In [4] Sudderth proved that

$$\mathbf{E} \limsup X_n = \limsup_{\tau \in T} \mathbf{E}\, X_\tau$$

if $|X_n| \leqq Z$, where Z is an integrable r.v. In Section III we generalize the relation obtained by Sudderth to the conditional case. Moreover, we give an equivalent condition to this equality.

I. The conditional Fatou Lemma

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, $\{\mathcal{F}_n, n \geqq 1\}$ an increasing sequence of sub-σ-fields contained in $\mathcal{A}$ and $\{X_n, n \geqq 1\}$ a sequence of r.v. such that X_n is $\mathcal{F}_n$-measurable for every n.

The definition of conditional expectation $(E^{\mathcal{F}} X)$ with respect to the sub-σ-field $\mathcal{F} \subset \mathcal{A}$ for $X \geqq 0$ can be found in [3]. If

$$\min(E^{\mathcal{F}} X^+, E^{\mathcal{F}} X^-) < \infty \text{ a.s.}$$

where $X^+ = \max(X, 0)$, $X^- = \max(-X, 0)$, then we define the conditional expectation as follows:

$$E^{\mathcal{F}} X = E^{\mathcal{F}} X^+ - E^{\mathcal{F}} X^-.$$

In case $E^{\mathcal{F}}|X| < \infty$ a.s. we write $X \in L^1_{\mathcal{F}}$.

A random variable $\tau\colon \Omega \to \{1,2,\dots\}$ is a stopping time iff $[\tau = n] \in \mathcal{F}_n$ for every integer $n \geqq 1$. The set of all finite $(P(\tau < \infty) = 1)$ stopping times is denoted by T.

Let $\mathcal{F}$ be some sub-σ-field of $\mathcal{A}$.

DEFINITION 1. A sequence $\{X_n, n \geqq 1\}$ is uniformly $\mathcal{F}$-inegrable if for every $\mathcal{F}$-measurable r.v. $\varepsilon > 0$ a.s. there exists an $\mathcal{F}$-measurable r.v. $\lambda > 0$ a.s. such that

$$\sup E^{\mathcal{F}}|X_n| I_{[|X_n|>\lambda]} < \varepsilon \quad \text{a.s.}$$

In the sequel we need the following theorems:

THEOREM 1 [8]. *A sequence $\{X_n, n \geqq 1\}$ is uniformly $\mathcal{F}$-integrable if and only if*

1) $\sup E^{\mathcal{F}}|X_n| < \infty$ *a.s.*

and

2) *for every $\mathcal{F}$-measurable function $\varepsilon > 0$ a.s. there exists an $\mathcal{F}$-measurable function $\delta > 0$ a.s. such that*

$$E^{\mathcal{F}} I_A < \delta \quad a.s. \quad \Rightarrow \quad \sup E^{\mathcal{F}}|X_n| I_A < \varepsilon \quad a.s.$$

THEOREM 2 [7]. *If $\{X_n, n \geqq 1\}$ is a uniformly $\mathcal{F}$-integrable sequence of r.v. then*

$$\limsup E^{\mathcal{F}} X_n \leqq E^{\mathcal{F}} \limsup X_n \quad a.s.$$

If additionally $\lim X_n = X$ *a.s. then* $X \in L^1_{\mathcal{F}}$ *and* $\lim E^{\mathcal{F}} X_n = E^{\mathcal{F}} \lim X_n$ *a.s.*

We prove that the condition of conditional uniform integrability is necessary.

THEOREM 3. *If $0 \leqq X_n \to X$ a.s., $X, X_n \in L^1_{\mathcal{F}}$, $n \geqq 1$, and $\lim E^{\mathcal{F}} X_n = E^{\mathcal{F}} X$ a.s. then $\{X_n, n \geqq 1\}$ is uniformly $\mathcal{F}$-integrable.*

PROOF. It is obvious that

$$\sup E^{\mathcal{F}}|X_n| < \infty \quad \text{a.s.}$$

Put $\varepsilon > 0$ and let $A_n = [|E^{\mathcal{F}} X_k - E^{\mathcal{F}} X| < \varepsilon$, for all $k \geqq n]$, $n \geqq 1$, $A_0 = \emptyset$. It follows from the definition that A_n are $\mathcal{F}$-measurable, $A_n \subsetneqq A_{n+1}$, for $n \geqq 0$ and $\lim P(A_n) = 1$.

The sequence of r.v. $\{(X_1 - X) I_{A_n \setminus A_{n-1}}, \dots, (X_{n-1} - X) I_{A_n \setminus A_{n-1}}\}$ is uniformly $\mathcal{F}$-integrable. Then there exists a sequence of $\mathcal{F}$-measurable functions $\{\delta_n, n \geqq 1\}$ such that

$$E^{\mathcal{F}} I_A < \delta_n \quad \text{a.s.} \Rightarrow \quad E^{\mathcal{F}}|X_i - X| I_A < \varepsilon \quad \text{a.s.,} \ \ i = 1, 2, \dots, n-1.$$

By assumption $X \in L^1_{\mathcal{F}}$. Therefore, there exists an $\mathcal{F}$-measurable function $\delta > 0$ a.s. such that

$$E^{\mathcal{F}} I_A < \delta_1 \quad \text{a.s.} \quad \Rightarrow \quad E^{\mathcal{F}}|X| I_A < \varepsilon \quad \text{a.s.}$$

Let $\delta_0 = \delta_n$ on $A_n \setminus A_{n-1}$, $n \geqq 1$ and $\delta = \min(\delta_0, \delta_1)$. If $E^{\mathcal{F}} I_A < \delta$ a.s. then

$$E^{\mathcal{F}} X_n I_A = E^{\mathcal{F}}(X_n - X) I_A + E^{\mathcal{F}} X I_A \leqq \sum_{k=1}^{\infty} E^{\mathcal{F}}(X_n - X) I_{A_k \setminus A_{k-1}} I_A + \varepsilon =$$

$$= \sum_{k=1}^{n-1} E^{\mathcal{F}}(X_n - X) I_{A_k \setminus A_{k-1}} I_A + \sum_{k=n}^{\infty} E^{\mathcal{F}}(X_n - X) I_{A_k \setminus A_{k-1}} I_A + \varepsilon \leqq$$

$$\leqq \varepsilon + \sum_{k=n}^{\infty} I_{A_k \setminus A_{k-1}} E^{\mathcal{F}}(X_n - X) I_A + \varepsilon \leqq 3\varepsilon \quad \text{a.s.}$$

Then the statement of Theorem 1 completes the proof.

DEFINITION 2. An adapted sequence $\{X_n, n \geqq 1\}$ of r.v. is called a conditional amart (with respect to $\mathcal{F}$) if

(a) $X_n \in L^1_{\mathcal{F}}$ for $n \geqq 1$

and

(b) the net $L(E^{\mathcal{F}} X_\tau, X)_{\tau \in T_b}$ converges to zero for some r.v. X (where L denotes the Levy–Prokhorov metric and T_b denotes the set of bounded stopping times).

THEOREM 4 [6]. *Each $L^1_{\mathcal{F}}$-bounded conditional amart* ($\sup E^{\mathcal{F}}|X_n| < \infty$ *a.s.*) *converges almost surely.*

Now we give some generalizations of Levy's a.s. convergence theorem for martingales.

THEOREM 5. *If $X \in L^1_{\mathcal{F}}$ and $\mathcal{F} \subseteqq \mathcal{F}_1 \subseteqq \ldots \subseteqq \mathcal{F}_n \subseteqq \ldots$ and X is $\sigma\Big(\bigcup_{i=1}^{\infty} \mathcal{F}_i\Big)$-measurable then*

$$\lim_{n\to\infty} E^{\mathcal{F}_n} X = X \quad \textit{a.s. and} \quad \lim_{n\to\infty} E^{\mathcal{F}} |E^{\mathcal{F}_n} X - X| = 0 \quad \textit{a.s.}$$

$(E^{\mathcal{F}_n} X \xrightarrow{L^{\mathcal{F}}_1} X,\ n \to \infty)$.

PROOF. It is obvious that $\{E^{\mathcal{F}_n} X, \mathcal{F}_n, \mathcal{F}, n \geqq 1\}$ is a conditional amart with respect to the σ-field $\mathcal{F}$. It is easy to observe that

$$\sup E^{\mathcal{F}} |E^{\mathcal{F}_n} X| \leqq E^{\mathcal{F}} |X| < \infty \quad \text{a.s.}$$

and by Theorem 4, the sequence $\{E^{\mathcal{F}_n} X, n \geqq 1\}$ converges a.s. Moreover, the sequence $\{E^{\mathcal{F}_n} X, n \geqq 1\}$ is uniformly $\mathcal{F}$-integrable and by Theorem 4 of [8] it converges in $L^1_{\mathcal{F}}$.

Let $X_\infty = \lim_{n\to\infty} E^{\mathcal{F}_n}X$ a.s. X_∞ is $\sigma\Big(\bigcup_{i=1}^{\infty}\mathcal{F}_i\Big)$-measurable and for every $A \in \bigcup_{i=1}^{\infty}\mathcal{F}_i$ we have

$$\int_A X_\infty dP = \int_A \lim_{n\to\infty} E^{\mathcal{F}_n}X\, dP = \lim_{n\to\infty}\int_A E^{\mathcal{F}_n}X\, dP = \lim_{n\to\infty}\int_A X\, dP = \int_A X\, dP$$

which proves that $X = X_\infty$ a.s. and completes the proof of Theorem 5.

II. Examples

In this section we present some examples which show the special properties of conditional expectation.

EXAMPLE 1. Let $(\Omega, \mathcal{A}, P) = (\langle 0,1\rangle^2, \mathcal{B}\otimes\mathcal{B}, \mu)$, $\mathcal{F}_1 = \mathcal{B}\otimes\langle 0,1\rangle$ and $\mathcal{F}_2 = \langle 0,1\rangle\otimes\mathcal{B}$. A random element $X(s,t) = 1/(s^2+t^2)$, for $(s,t)\neq(0,0)$, $X(0,0)=0$ is $\mathcal{F}_1$-integrable ($X\in L^1_{\mathcal{F}_1}$) and $\mathcal{F}_2$-integrable but not integrable $X\notin L^1$.

EXAMPLE 2. Let $(\Omega, \mathcal{A}, P) = (\langle 0,1\rangle^2, \mathcal{B}\otimes\mathcal{B}, \mu)$, $\mathcal{F}_1 = \mathcal{B}\otimes\langle 0,1\rangle$ and $\mathcal{F}_2 = \langle 0,1\rangle\otimes\mathcal{B}$. Obviously $\sigma(\mathcal{F}_1\cup\mathcal{F}_2) = \mathcal{B}\otimes\mathcal{B}$. Every integer can be written in the form

$$n = (4^{L(n)+1}-1)/3 + K(n)2^{L(n)+1} + J(n)$$

where

$$L(n) = \max\{\ell \geqq 0 \colon (4^{\ell+1}-1)/3 \leqq n\},$$
$$K(n) = \max\{k \geqq 0 \colon k2^{L(n)+1} \leqq n - (4^{L(n)+1}-1)/3\},$$

and

$$J(n) = n - (4^{L(n)+1}-1)/3 - K(n)2^{L(n)+1}.$$

We define

$$X_n(s,t) = \begin{cases} 2^{L(n)+1} I_{A_n} & \text{if } K(n)=0 \text{ or } J(n)=0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$A_n = \{(s,t)\colon J(n)/(2^{L(n)+1}) \leqq s < (J(n)+1)/(2^{L(n)+1}),$$
$$K(n)/(2^{L(n)+1}) \leqq t < (K(n)+1)/(2^{L(n)+1})\}.$$

It is easy to observe that $X_n \xrightarrow{\text{a.s.}} 0$, $n\to\infty$, but

$$E^{\mathcal{F}_1}X_n \not\xrightarrow{\text{a.s.}} 0, \quad n\to\infty \quad\text{and}\quad E^{\mathcal{F}_2}X_n \not\xrightarrow{\text{a.s.}} 0, \quad n\to\infty.$$

The sequence $\{X_n, n \geqq 1\}$ is uniformly integrable but it is not uniformly $\mathcal{F}_1$-integrable and it is not uniformly $\mathcal{F}_2$-integrable.

On the other hand for $Y_n = I_{A_n}$, $n \geqq 1$, we see that $E^{\mathcal{F}_1} Y_n \xrightarrow{\text{a.s.}} 0$, $n \to \infty$, $E^{\mathcal{F}_2} Y_n \xrightarrow{\text{a.s.}} 0$, $n \to \infty$, but $Y_n \not\xrightarrow{\text{a.s.}} 0$, $n \to \infty$.

EXAMPLE 3. Set $(\Omega, \mathcal{A}, P) = (\langle 0,1\rangle, \mathcal{B}, \mu)$ and $\mathcal{F} = \sigma((1/2,1\rangle, (1/3,1/2\rangle, \dots, (1/(n+1), 1/n\rangle, \dots)$. The sequence

$$X_n(\omega) = X(\omega) = \begin{cases} 1/\omega & \text{for } \omega \neq 0 \\ 0 & \text{for } \omega = 0, \end{cases} \quad n \geqq 1$$

is uniformly $\mathcal{F}$-integrable but not uniformly integrable.

III. Some relations for randomly stopped variables

The following theorem generalizes the result given in [2] and [4].

THEOREM 6. *Let $\{X_n, n \geqq 1\}$ be a sequence of r.v. adapted to an increasing sequence of σ-fields $\{\mathcal{F}_n, n \geqq 1\}$ and let $\mathcal{F}$ be some sub-σ-field of $\mathcal{A}$. Then*

$$(1) \qquad E^{\mathcal{F}}(\limsup X_n) \leqq \operatorname*{elimsup}_{\tau \in T} E^{\mathcal{F}} X_\tau \quad a.s.$$

and

$$(2) \qquad E^{\mathcal{F}}(\liminf X_n) \geqq \operatorname*{eliminf}_{\tau \in T} E^{\mathcal{F}} X_\tau \quad a.s.$$

whenever all of the above conditional expectations are well-defined.

The definition of $\operatorname*{elimsup}_{\sigma \in T} E^{\mathcal{F}} X_\sigma = \operatorname*{essinf}_{\tau \in T}(\operatorname*{essup}_{\tau < \sigma \in T} E^{\mathcal{F}} X_\sigma)$ and $\operatorname*{eliminf}_{\tau \in T} E^{\mathcal{F}} X_\tau = \operatorname*{essup}_{\sigma \in T}(\operatorname*{essinf}_{\sigma < \tau \in T} E^{\mathcal{F}} X_\tau)$ can be found in [3], p. 121.

PROOF. It is enough to prove (1). On the set

$$A = [E^{\mathcal{F}} \limsup X_n = -\infty]$$

the inequality is obvious.

In the following we assume that $E^{\mathcal{F}} X^* > -\infty$ a.s., where $X^* := := \limsup X_n$. (If $P(A) > 0$, then it is more convenient to consider the sequence $X'_n = X_n I_{A^c}$). If $E^{\mathcal{F}} X^* < \infty$ a.s. then $X \in L^1_{\mathcal{F}}$ and by Theorem 5 the sequence $\{Y_n = E^{\mathcal{F}'_n} X, n \geq 1\}$ where $\mathcal{F}'_n = \sigma(\mathcal{F}; X_1, X_2, \dots, X_n)$, converges a.s. and in $L^1_{\mathcal{F}}$ to r.v. X^*. Choose $\varepsilon > 0$ and $\sigma \in T$. Define

$$\tau := \inf\{n \colon n \geqq \sigma \quad \text{and} \quad E^{\mathcal{F}'_n} X^* < X_n + \varepsilon\}.$$

It is obvious that $P[\tau < \infty] = 1$. Then

$$E^{\mathcal{F}} X^* = \sum_{n=1}^{\infty} E^{\mathcal{F}} X^* I_{[\tau=n]} = \sum_{n=1}^{\infty} E^{\mathcal{F}} (I_{[\tau=n]} E^{\mathcal{F}'_n} X^*) \leqq$$

$$\leqq \sum_{n=1}^{\infty} E^{\mathcal{F}} I_{[\tau=n]} (X_n + \varepsilon) = E^{\mathcal{F}} X_\tau + \varepsilon \quad \text{a.s.}$$

hence

$$E^{\mathcal{F}} X^* \leqq \operatorname*{elimsup}_{\tau \in T} E^{\mathcal{F}} X_\tau + \varepsilon \quad \text{a.s.}$$

which proves (1).

If $\{X_n, n \geqq 1\}$ are arbitrary then for every real number C we have

$$\operatorname*{elimsup}_{\tau \in T} E^{\mathcal{F}} X_\tau \geqq \operatorname*{elimsup}_{\tau \in T} E^{\mathcal{F}} \min(X_\tau, C) \geqq$$

$$\geqq E^{\mathcal{F}} \limsup \min(X_n, C) = E^{\mathcal{F}} \min(X^*, C).$$

The conclusion now follows by letting $C \to \infty$.

THEOREM 7. *Let $Z \in L^1_{\mathcal{F}}$ and $W \in L^1_{\mathcal{F}}$. If $X_n \leqq Z$ a.s. for all n, then*

$$E^{\mathcal{F}} (\limsup X_n) = \operatorname*{elimsup}_{\tau \in T} E^{\mathcal{F}} X_\tau. \tag{3}$$

If $X_n \geqq W$ a.s. for all $n \geqq 1$, then

$$E^{\mathcal{F}} (\liminf X_n) = \operatorname*{eliminf}_{\tau \in T} E^{\mathcal{F}} X_\tau. \tag{3'}$$

The proof follows from the conditional Fatou Lemma [7] and is a simple modification of the proof of Theorem 2 of [4].

COROLLARY. *Suppose $\lim X_n = X$ a.s. and there is a random variable $Z \in L^1_{\mathcal{F}}$ such that $|X_n| \leqq Z$ a.s. for all $n \geqq 1$. Then $X \in L^1_{\mathcal{F}}$ and*

$$\operatorname*{elim}_{\tau \in T} E^{\mathcal{F}} X_\tau = E^{\mathcal{F}} X. \tag{4}$$

The proof is obvious due to Theorem 6.

On the basis of the example given in [4] we see that the condition of uniform $\mathcal{F}$-integrability does not guarantee condition (4) [2].

Let T_b denote the set of all bounded stopping times (relative to $\{\mathcal{F}_n, n \geqq \geqq 1\}$).

DEFINITION 3. A sequence $\{X_n, n \geqq 1\}$ is T_b-uniform $\mathcal{F}$-integrable if for any given $\mathcal{F}$-measurable function $\varepsilon > 0$ a.s. there exists an $\mathcal{F}$-measurable function λ_0 such that

$$\operatorname*{esup}_{\tau \in T_b} E^{\mathcal{F}} |X_\tau| I_{[|X_\tau| > \lambda]} < \varepsilon \quad \text{a.s.}$$

for all $\mathcal{F}$-measurable functions $\lambda > \lambda_0$ a.s.

We can observe that if $\{X_n, n \geqq 1\}$ is T_b-uniformly $\mathcal{F}$-integrable then for any given $\mathcal{F}$-measurable function $\varepsilon > 0$ a.s. there exists an $\mathcal{F}$-measurable function λ_0 such that

(5) $$\operatorname*{esup}_{\tau \in T} E^{\mathcal{F}} |X_\tau| I_{[|X_\tau| > \lambda]} < \varepsilon \quad \text{a.s.}$$

for all $\mathcal{F}$-measurable functions $\lambda > \lambda_0$ a.s.

The sequence given in the example in [4] is uniformly integrable but it is not T_b-uniformly integrable.

THEOREM 8. *Let $X_n \in L^1_{\mathcal{F}}$, $n \geqq 1$. A sequence $\{X_n, n \geqq 1\}$ is T_b-uniformly $\mathcal{F}$-integrable iff*

(6) $$E^{\mathcal{F}} \limsup X_n = \operatorname*{elimsup}_{\tau \in T_b} E^{\mathcal{F}} X_\tau < \infty \quad \textit{a.s.}$$

and

(7) $$E^{\mathcal{F}} \liminf X_n = \operatorname*{eliminf}_{\tau \in T_b} E^{\mathcal{F}} X_\tau > -\infty \quad \textit{a.s.}$$

PROOF. Let $\{X_n, n \geqq 1\}$ be T_b-uniformly $\mathcal{F}$-integrable. It is obvious by Theorem 6 that

$$E^{\mathcal{F}} \limsup X_n \leqq \operatorname*{elimsup}_{\tau \in T} E^{\mathcal{F}} X_\tau = \operatorname*{elimsup}_{\tau \in T_b} E^{\mathcal{F}} X_\tau .$$

Moreover there exists a sequence $\{\tau_n \geqq n \text{ a.s.}, n \geqq 1\}$ such that $\tau_n \in T_b$ and

$$\operatorname*{elimsup}_{\tau \in T_b} E^{\mathcal{F}} X_\tau = \limsup E^{\mathcal{F}} X_{\tau_n} \leqq E^{\mathcal{F}} \limsup X_{\tau_n} \leqq E^{\mathcal{F}} \limsup X_n$$

on the basis of the conditional Fatou's Lemma [7].

The inequality $\operatorname{elimsup} E^{\mathcal{F}} X_\tau < \infty$ a.s. holds on the basis of Definition 3.

By analogy

$$-\infty < \operatorname*{eliminf}_{\tau \in T_b} E^{\mathcal{F}} X_\tau = E^{\mathcal{F}} \liminf X_n \quad \text{a.s.}$$

On the other hand, let $\{X_n, n \geqq 1\}$ be a sequence of nonnegative r.v. such that $X_n \in L^1_{\mathcal{F}}$, $n \geqq 1$ and

(8) $$0 = E^{\mathcal{F}} \limsup X_n = \operatorname*{elimsup}_{\tau \in T_b} E^{\mathcal{F}} X_\tau \quad \text{a.s.}$$

For every $\mathcal{F}$-measurable function $\varepsilon > 0$ a.s. we have

$$\lim_n P\{\operatorname*{esup}_{\tau \geqq n} E^{\mathcal{F}} X_\tau < \varepsilon/2\} = 1.$$

Thus, for any given $\delta > 0$, we can choose $n(\delta)$ such that

$$P(A_{n(\delta)}) = P(\operatorname*{esup}_{\tau \geqq n(\delta)} E^{\mathcal{F}} X_\tau < \varepsilon/2) > 1 - \delta$$

and $A_{n(\delta_1)} \subseteqq A_{n(\delta_2)}$, for $\delta_1 \geqq \delta_2$.

It is easy to observe that $\sup X_n < \infty$ a.s. and for every $\mathcal{F}$-measurable function $\delta > 0$ a.s., we can choose an $\mathcal{F}$-measurable function $\lambda > 0$ a.s. such that

$$E^{\mathcal{F}} I_{[\sup X_n > \lambda]} < \delta \quad \text{a.s.}$$

By uniform $\mathcal{F}$-integrability of the r.v. $X_1, X_2, \ldots, X_{n(\delta)}$ and Theorem 1 of [8] for every $\mathcal{F}$-measurable function $\varepsilon > 0$ a.s. we can choose an $\mathcal{F}$-measurable function $\lambda_{n(\delta)} > 0$ such that

$$E^{\mathcal{F}} \max(X_1, X_2, \ldots, X_n(\delta)) I_{[\sup X_n > \lambda_{n(\delta)}]} < \varepsilon/2 \quad \text{a.s.}$$

Finally, we put

$$\lambda = \begin{cases} \lambda_{n(1/k)} & \text{on } B_k = A_{n(1/k)} \setminus A_{n(1/(k-1))},\ k = 2, 3, \ldots \\ \lambda_{n(1)} & \text{on } B_1 = A_{n(1)}. \end{cases}$$

Then

$$E^{\mathcal{F}} X I_{[X_\tau > \lambda]} = \sum_{k=1}^{\infty} I_{B_k} E^{\mathcal{F}} X_\tau I_{[X_\tau > \lambda]} = \sum_{k=1}^{\infty} E^{\mathcal{F}} X_\tau I_{[X_\tau > \lambda]} I_{B_k} \leqq$$

$$\leqq \sum_{k=1}^{\infty} (E^{\mathcal{F}} X_{\tau \wedge n(1/k)} I_{[\sup X_n > \lambda]} I_{B_k} + E^{\mathcal{F}} X_{\tau \vee n(1/k)} I_{B_k} \leqq$$

$$\leqq \sum_{k=1}^{\infty} \left(E^{\mathcal{F}} \max(X_1, \ldots, X_{n(1/k)} I_{[\sup X_n > \lambda_{n(1/k)}]} I_{B_k} + (\varepsilon/2) I_{B_k} \right) \leqq$$

$$\leqq \sum_{k=1}^{\infty} \varepsilon I_{B_k} = \varepsilon \quad \text{a.s}$$

which proves that the sequence $\{X_n, n \geqq 1\}$ is T_b-uniformly $\mathcal{F}$-integrable.

If $\{X_n, n \geqq 1\}$ are arbitrary, then the conclusion follows from the inequality

$$0 \leqq |X_n| \leqq |X^*| + |X_*| + |\bar{X}_n|,$$

where $X^* = \limsup X_n$, $X_* = \liminf X_n$, $\bar{X}_n^+ = (X_n - X^*)^+ := \max(X_n - X, 0)$, $\bar{X}_n^- = (X_* - X_n)^+$ and $\bar{X}_n = \max(\bar{X}_n^+, \bar{X}_n^-)$, which completes the proof of Theorem 8.

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ISOPERIMETRIC INEQUALITIES AND AREAS OF PROJECTIONS IN $\mathbf{R}^n$

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1. Introduction

The classical isoperimetric inequality states that among all simple closed plane curves of given length L, the circle of circumference L surrounds the largest area. If A is the area enclosed by the curve C, then the isoperimetric inequality can be expressed by

$$L^2 \geqq 4\pi A \tag{1}$$

with equality if and only if C is a circle. A review of the subject including methods of proof generalizations and special forms of inequality (1) can be found in the paper by Osserman [5] and the books by Bandle [1] and Burago and Zalgaller [2].

Among the extensions of the isoperimetric inequality to higher dimensional spaces, two particular directions of generalization can be identified. In one of the curve is replaced by a closed surface and the inequality relates the surface area and volume content. The three-dimensional result was proved by Schwarz last century, and generalized to n-dimensions by Schmidt [6]. In the other approach, isoperimetric inequalities are derived for closed curves on given surfaces (see [1] and [2] again for a review of this material).

Our aim in this paper is to stimulate interest in the following isoperimetric problem. Let C be a smooth closed curve of given length L in $\mathbf{R}^n$, and let C_{jk} (or C_{kj}) be the orthogonal projection of the curve onto the Ox_jx_k plane. Obviously the curve C_{jk} will be closed although it may have corners and cusps. We can pose the following question. Over the set of all curves of given length in $\mathbf{R}^n$, what is the greatest value of

$$(\text{sum of the projected areas})/L^2,$$

and for which curve or curves is this value attained? We shall not present a complete answer to this question, but we shall prove some associated results. Areas of projected curves do not seem to have figured prominently in the literature on isoperimetric inequalities although they do arise in a paper by Schoenberg [7] who obtained an inequality relating the length of a curve in

$\mathbf{R}^n$ and the volume of its convex hull. This volume is then re-formulated in terms of areas of two-dimensional projections, but only as a sum of their products and not in the simple summation considered here. An inequality between the volumes of sets and their orthogonal projections has also been proved by Loomis and Whitney [4].

2. Projected areas

Let the closed curve C be defined in $\mathbf{R}^n$ by

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \qquad 0 \leqq t \leqq 2\pi, \quad \mathbf{x}(0) = \mathbf{x}(2\pi).$$

We shall assume that $\mathbf{x}(t)$ is continuously differentiable on $(0, 2\pi)$ with $\lim_{t\to 0-} \mathbf{x}'(t) = \lim_{t\to 2\pi+} \mathbf{x}'(t)$. We shall use Hurwitz's device [5] and parametrize C by a multiple of its arc-length s normalized on $(0, 2\pi)$. Thus we let $t = 2\pi s/L$. With this parameter it follows that

$$L^2 = 2\pi \int_0^{2\pi} \sum_{j=1}^{n} x_j'^2 dt = 2\pi \int_0^{2\pi} |\mathbf{x}'|^2 dt. \tag{2}$$

The 'areas' A_{jk} will be defined by

$$A_{jk} = \int_0^{2\pi} x_j x_k' dt,$$

from which it follows that $A_{jj} = 0$, $A_{kj} = -A_{jk}$. The number A_{jk} is a vector area whose sign will depend on the direction in which the projected curve C_{jk} is being tracked.

There are several different questions which can be posed concerning the length of L and the projected areas. One involves the absolute values of these areas which would require the greatest value of

$$\frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} |A_{jk}|/L^2$$

but this version of the isoperimetric inequality presents technical difficulties which we have not yet overcome. Instead we shall investigate the extremum principles for the sum

$$\frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_{jk} A_{jk}/L^2,$$

where ε_{jk} is a weighting factor which can be chosen from the numbers $0, 1$ and -1 although other weightings are possible. Hence partial sums of areas can be investigated with each projected curve taken in either orientation.

Consider the sum of areas defined by

$$A = \frac{1}{2} \int_0^{2\pi} \mathbf{x}^t K \mathbf{x}' dt \tag{3}$$

where $K = [\varepsilon_{jk}]$ and it is assumed that $\varepsilon_{jj} = 0$ and $\varepsilon_{kj} = -\varepsilon_{jk}$: in other words K is a skew-symmetric matrix. We shall now consider the difference

$$L^2 - 4\pi\mu A = 2\pi \int_0^{2\pi} [|\mathbf{x}'|^2 - \mu \mathbf{x}^t K \mathbf{x}'] dt. \tag{4}$$

In this expression A may be either positive or negative but the sign of A can always be reversed by replacing t by $2\pi - t$ in the integral for A. The maximum value of $|\mu|$ is required which ensures that the left-hand side of (4) is always nonnegative.

3. Wirtinger's inequality and the isoperimetric result

We shall deduce the result more or less as we obtained it, rather than deriving it from a prior knowledge of the answer. Essentially our approach was suggested by Hurwitz's proof of the plane isoperimetric inequality (see [2], p. 1183) which uses Wirtinger's inequality. We require the latter inequality in the following form.

LEMMA. *Let $\mathbf{f}(t)$ be a continuous periodic function in $\mathbf{R}^n$ of period 2π with continuous derivative $\mathbf{f}'(t)$. If*

$$\int_0^{2\pi} \mathbf{f}(t) dt = 0, \tag{5}$$

then

$$\int_0^{2\pi} |\mathbf{f}'(t)|^2 dt \geqq \int_0^{2\pi} |\mathbf{f}(t)|^2 dt \tag{6}$$

with equality if and only if

$$\mathbf{f}(t) = \mathbf{a} \cos t + \mathbf{b} \sin t \tag{7}$$

where **a** *and* **b** *are constant vectors.*

A proof can be found in [3], p. 121.

We have to arrange that the position vector $\mathbf{x}(t)$ satisfies the mean requirement (5) but this can be readily achieved by a translation of the origin. Equation (4) is now re-arranged so that the right-hand side contains terms which are identifiably nonnegative and incorporate the Wirtinger difference in the Lemma. With this in view we can re-write (4) in the form

$$L^2 - 4\pi A = (\pi/\alpha)\int_0^{2\pi}[\mu|\mathbf{x}' + \alpha K\mathbf{x}|^2 + \tag{8}$$

$$+\gamma\mu\alpha^2\{|\mathbf{x}'|^2 - |\mathbf{x}|^2\} + \mu\alpha^2\mathbf{x}^t(K^2 + \gamma I_n)\mathbf{x}]dt,$$

where $\gamma = (2\alpha - \mu)/\mu\alpha^2 > 0$, and α is any constant such that $\alpha\mu > 0$ also. By the lemma above the right-hand side of (8) will be nonnegative if the quadratic form

$$\mathbf{x}'(K^2 + \gamma I_n)\mathbf{x} \tag{9}$$

is nonnegative.

Since K is skew-symmetric, the eigenvalues of K are either imaginary in conjugate pairs or zero, and consequently those of K^2 either negative or zero. Assuming that not all ε_{jk} are zero, let the pair of eigenvalues $\pm i\lambda$ $(\lambda > 0)$ of K be such that $-\lambda^2$ is the smallest eigenvalue of K^2. All the eigenvalues in the matrix in (9) will be nonnegative if $\gamma = \lambda^2$, in which case (9) will be a nonnegative quadratic form. The parameters μ and α must be related by

$$\lambda^2\mu\alpha^2 = 2\alpha - \mu,$$

from which it follows that the maximum and minimum values of μ are $\pm 1/\lambda$ at $\alpha = \pm 1/\lambda$. Hence we have shown that

$$L^2 \geqq 4\pi|A|/\lambda, \tag{10}$$

taking account of the signs of α and μ in $\alpha\mu$, but we have not yet established that there exists a vector $\mathbf{x}$ for which equality occurs in (10). For $1/\lambda$ to be the optimum value of μ all three terms on the right-hand side of (8) must vanish along the same extremal curve.

It is necessary that equality holds in Wirtinger's inequality, and this can only occur if

$$\mathbf{x} = \mathbf{a}\cos t + \mathbf{b}\sin t. \tag{11}$$

Also $\mathbf{x}$ must satisfy

$$\mathbf{x}' + (1/\lambda)K\mathbf{x} = 0, \tag{12}$$

where we have assumed that $\alpha > 0$: a parallel argument covers the case $\alpha < 0$. Let $\mathbf{w} = \mathbf{a} + i\mathbf{b}$. Then, from (11) and (2), it follows that

$$(K - i\lambda I_n)\mathbf{w} = 0, \tag{13}$$

in other words, $\mathbf{w}$ must be an eigenvector corresponding to the eigenvalue $i\lambda$ of K.

Finally we have to show that the integral of the quadratic form (9) vanishes. Thus

$$\int_0^{2\pi} \mathbf{x}^t(K^2 + \lambda^2 I_n)\mathbf{x}dt = \pi\bar{\mathbf{w}}^t(K^2 + \lambda^2 I_n)\mathbf{w} = \pi\bar{\mathbf{w}}^t(K + i\lambda I_n)(K - i\lambda I_n)\mathbf{w} = 0,$$

by (13).

The eigenvector $\mathbf{w}$ is defined to within a constant of proportionality which is determined by the known length L of the curve. Thus from (11) and (12),

$$L^2 = 2\pi \int_0^{2\pi} |\mathbf{x}'|^2 dt = 4\pi^2|\mathbf{a}|^2,$$

since $|\mathbf{a}| = |\mathbf{b}|$ and $\mathbf{a}^t\mathbf{b} = 0$ for a skew-symmetric matrix K. The parametric equation of the extremal is therefore

$$\mathbf{x} = L(\mathbf{a}\cos t + \mathbf{b}\sin t)/2\pi|\mathbf{a}|, \tag{14}$$

for any eigenvector $\mathbf{w} = \mathbf{a} + i\mathbf{b}$ of (13).

If $\alpha < 0$, then $\mathbf{w}$ is replaced by its conjugate which results in $\mathbf{b}$ being replaced by $-\mathbf{b}$ in (14). This amounts to no more than reversing the direction in which C is described.

We have proved the following:

THEOREM. *If C is a smooth closed curve of length L in $\mathbf{R}^n$ and A is the sum of the projected areas onto all planes of coordinate pair each with an assigned index $\varepsilon_{jk} = 0, 1$ or -1, that is,*

$$A = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \varepsilon_{jk}A_{jk}, \qquad \varepsilon_{kj} = -\varepsilon_{jk},$$

then the inequality

$$L^2 - (4\pi/\lambda)|A| \geqq 0 \tag{15}$$

holds, where $i\lambda$ $(\lambda > 0)$ is an eigenvalue of $K = [\varepsilon_{jk}]$ such that λ is the largest number with this property. If $\mathbf{w}$ is the corresponding eigenvector to $i\lambda$ then equality occurs in (15) if the curve is given by

$$\mathbf{x} = L(\mathbf{a}\cos t \pm \mathbf{b}\sin t)/(2\pi|\mathbf{a}|). \tag{16}$$

4. Special cases

Before looking at some n-dimensional cases, it is perhaps helpful in visualising the projections to quote some results for $\mathbf{R}^3$. Let $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} = 1$, so that

$$K = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

The eigenvalues of K are 0, $\pm i\sqrt{3}$ and $\lambda = \sqrt{3}$. Hence inequality (15) becomes

$$L^2 \geqq (4\pi/\sqrt{3})|A|,$$

where

$$A = A_{12} + A_{13} + A_{23}.$$

We can choose $\mathbf{a} = [1\ 2\ 1]^t$, $\mathbf{b} = [-\sqrt{3}\ 0\ \sqrt{3}]^t$ so that the critical curve becomes either of the circles

$$\mathbf{x} = \frac{L}{2\pi\sqrt{6}} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cos t \pm \begin{bmatrix} -\sqrt{3} \\ 0 \\ \sqrt{3} \end{bmatrix} \sin t \right\}.$$

In fact this is the same curve but described in opposite directions. The circle is the intersection of the sphere of radius $L/2\pi$ and the plane $x_1 - x_2 + x_3 = 0$. The set of coefficients $\varepsilon_{12} = \varepsilon_{23} = 1$, $\varepsilon_{13} = -1$ leads to the critical circle equally inclined to the coordinate planes on the plane $x_1 + x_2 + x_3 = 0$. However, in both cases, the corresponding projected areas are of the same sign.

(i) *An inequality including all projected areas.* We shall consider curves in $\mathbf{R}^n$ in which all $\frac{1}{2}n(n-1)$ projected areas are included. The following alternating sign arrangement for upper triangular elements of the skew-symmetric matrix K are defined:

$$\varepsilon_{jk} = (-1)^{j+k+1} \qquad (k > j).$$

We could have taken $\varepsilon_{jk} = 1$ $(k > j)$ instead but as the case $n = 3$ suggests, this merely affects the orientations of the projections.

The set of eigenvalues $\{\omega_p\}$ $(p = 1, 2, \dots, n)$ are given by the roots of

$$D_n \equiv |K - \omega I_n| = 0.$$

Using elementary row and column operations it is easy to show that D_n satisfies the difference equation

$$D_n + 2\omega D_{n-1} - (1 - \omega^2) D_{n-2} = 0 \qquad (n \geqq 3),$$

which has the required solution

$$D_n = \frac{1}{2}(-\omega + 1)^n + \frac{1}{2}(-\omega - 1)^n.$$

Hence the eigenvalues of K are

$$\omega_p = i \cot\Big\{\Big(p - \frac{1}{2}\Big)\pi/n\Big\}, \qquad (p = 1, 2, \dots, n).$$

It follows that $\lambda = \cot(\pi/2n)$. Hence the isoperimetric inequality (15) becomes

$$L^2 \geqq 4\pi \tan(\pi/2n)|A|.$$

An eigenvector can be found by using simple row operations. We can then define

$$\mathbf{w} = \mathbf{a} + i\mathbf{b} = [r^{n-1}\ r^{n-2}\ \dots\ r\ 1]^t$$

where $r = e^{(n+1)\pi i/n}$, so that the critical curve is given by (16).

The individual projected areas are

$$\text{(17)} \qquad A_{jk} = \int_0^{2\pi} x_j x_k' dt = \frac{L^2(-1)^{j-k}}{2\pi n} \sin\{(j-k)\pi/n\}.$$

Since $\sin[(j-k)\pi/n] < 0$ for $j, k = 1, 2, \dots, n$ $(k > j)$, it follows that $\varepsilon_{jk} A_{jk} > 0$. For this extremal curve, absolute values of the projected areas are summed.

There is a difference between odd and even dimensional spaces. If $n \geqq 4$ is even, then $\frac{1}{2}n$ of the projections are circles since $r^{n-q} = -\bar{r}^q$ $(q = 1, 2, \dots, \frac{1}{2}n)$ in the elements of the eigenvector $\mathbf{w}$. The planes with circle projections have no axes in common. No projections are circles if $n \geqq 3$ is odd.

(ii) *An inequality for all coordinate planes with a common axis.* Let Ox_1 be the common axis. We can choose K to be

$$K = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

which has the eigenvalues $\pm i\sqrt{n-1}$ and zero. Thus $\lambda = \sqrt{n-1}$, and with $A = \sum_{k=2}^{n} A_{1k}$, the isoperimetric inequality becomes

$$L^2 \geqq 4\pi |A|/\sqrt{n-1}.$$

As we might expect

$$A_{1k} = L^2\ 4\pi\sqrt{n-1}$$

is independent of k for the extremal.

5. Concluding remarks

A number of general observations can be inferred from the theorem in Section 3. Most of these conclusions are fairly obvious, and we shall not enter into details of proofs. Essentially all these projection inequalities involve the dominant eigenvalue of the matrix $-K^2$.

If all ε_{jk} are zero except one, then the classical inequality can be recovered for the corresponding plane since the extremal and its projection coincide.

Suppose that for $n \geqq 4$ just two projections are allowed, which means that K has just two non-zero unit upper-triangular elements. If the planes of these projections have no axis in common, then both projections of the extremal are circles.

In principle the isoperimetric inequality for any weightings of the projections can be found. However the main question posed in the introduction concerning the sum of the projected areas still remains unresolved since our result concerns the sum of *vector* areas. In $\mathbf{R}^n$, what is the greatest value of

$$(\text{sum of the projected areas})/L^2?$$

The evidence seems strong that the answer is the one given in case (i) of Section 4.

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A RESULT CONCERNING BOUNDED SOLUTIONS OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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I. Introduction

Our paper is a contribution to the study of bounded solutions of the equation

$$x' = A(t)x + f(t) \tag{1}$$

where $A(t) : \mathbf{R} \to \mathbf{R}^{n\times n}$ is a continuous matrix, and f is continuous and bounded. We make these assumptions in order to simplify the problem, although as is shown in (2), our results can be extended to the case where A' and f are measurable.

In [1] it is proven that if the linear system

$$x' = A(t)x \tag{2}$$

has the exponential dichotomy

$$\begin{cases} |X(t)PX^{-1}(s)| \leqq K\exp\{-\alpha(t-s)\}, & t \geqq s, \\ |X(t)QX^{-1}(s)| \leqq K\exp\{-\alpha(s-t)\}, & s \geqq t \end{cases} \tag{3}$$

where $X(t)$ is a fundamental matrix of (2), K and α are constant, $K \geqq 1$, $\alpha > 0$, P is a projection ($PP = P$) and $Q = I - P$, then it follows that for B bounded, continuous, with

$$\delta = \sup_R |B(t)| \leqq \frac{\alpha}{36K^5} \tag{4}$$

the system

$$y' = [A(t) + B(t)]y \tag{5}$$

has the exponential dichotomy

$$\begin{cases} |Y(t)PY^{-1}(s)| \leqq 12K^3\exp\{-(\alpha - 6K^3\delta)(t-s)\}, & t \geqq s, \\ |Y(t)QY^{-1}(s)| \leqq 12K^3\exp\{-(\alpha - 6K^3\delta)(s-t)\}, & s \geqq t \end{cases} \tag{6}$$

and moreover

$$|Y(t)PY^{-1}(t) - X(t)PX^{-1}(t)| \leqq 144\alpha^{-1}K^{\delta}\delta. \tag{7}$$

In our paper we will generalize the estimate (7). This is the content of Theorem 3, given below.

The generalization of the estimate (7) permits us to compare the bounded solution of (1) and that of the system

$$y' = [A(t) + B(t)]y + f. \tag{8}$$

II. Previous and present results

Let us consider the system (2) with the exponential dichotomy (3). In [1] the following is proven:

THEOREM 1. *If* (3) *holds for an orthogonal projector then there exists a transformation* $T : \mathbf{R} \to \mathbf{C}^{n\times n}$ *with*

$$|T(t)| \leqq \sqrt{2}, \quad |T^{-1}(t)| \leqq K\sqrt{2} \tag{9}$$

such that the change of variable $x = T(t)z$ *transforms the system* (2) *to the system*

$$z' = C(t)z \tag{10}$$

where the continuous function $C(t)$ *is hermitian and commutes with the orthogonal projector* P. *In addition the system* (10) *admits an exponential dichotomy*

$$\begin{cases} |Z(t)PZ^{-1}(s)| \leqq 2K^3\exp\{\alpha(s-t)\}, & t \geqq s, \\ |Z(t)QZ^{-1}(s)| \leqq 2K^3\exp\{\alpha(t-s)\}, & t \geqq s. \end{cases} \tag{11}$$

Following Coppel we implement the change of variable $y = T(t)$ in (5) and obtain the system

$$u' = [C(t) + D(t)]u, \quad |D(t)| \leqq 2K\delta. \tag{12}$$

In general, $D(t)$ does not commute with the projector P. To avoid this difficulty, Coppel implements a new change of variable $u = S(t)v$, with the properties given below.

THEOREM 2. *For the system* (12) *there exists a* C^1 *function* $S : \mathbf{R} \to \mathbf{C}^{n\times n}$ *such that*

(i) $S(t) = I + H(t)$,

(ii) $|H(t)| \leqq 18\alpha^{-1}K^5\delta < \frac{1}{2}$,

(iii) $|S(t)| \leqq \frac{3}{2}$, $|S^{-1}(t) \leqq 2$,
(iv) $|S(t)PS^{-1}(t) - P| \leqq 4|H(t)|$.

The change of variables $u = S(t)v$ *transforms* (12) *to the system*

$$v' = [C(t) + E(t)]v \tag{13}$$

and the matrix E *satisfies* $E(t)P = PE(t)$ *and* $|E(t)| \leqq 3K\delta$, *moreover* (13) *admits the following:*

$$\begin{cases} |V(t)PU^{-1}(s)| \leqq 2K^2 \exp\{-(\alpha - 6K^3\delta)(t-s)\}, & t \geqq s, \\ |V(t)(I-P)^{-1}V(s)| \leqq 2K^2 \exp\{-(\alpha - 6K^3\delta)(s-t)\}, & s \geqq t. \end{cases} \tag{14}$$

We shall denote by X, Y, Z, V the fundamental matrices of the systems (2), (5), (10) and (13). Inmediately we note the relation

$$Y = T(t)S(t)V(t). \tag{15}$$

In this work we demonstrate the following two theorems.

THEOREM 3. *If* (3), (4) *and* (5) *hold, then*

$$|X(t)PX^{-1}(s) - Y(t)PY^{-1}(s)| \leqq (5K/2)^8 \delta\alpha^{-1} e^{-\alpha(t-s)/2}, \quad t \geqq s$$

and

$$|X(t)QX^{-1}(s) - Y(t)QY^{-1}(s)| \leqq (5K/2)^8 \delta\alpha^{-1} e^{-\alpha(s-t)/2}, \quad s \geqq t.$$

THEOREM 4. *If* (3), (4) *and* (5) *hold, then for the bounded solutions* x_f, y_f *of systems* (2) *and* (8) *we have the following estimate:*

$$\|x_f - y_f\| \leqq 4(5K/2)^8 \delta\alpha^{-2} \|f\|,$$

where $\|f\| = \sup_{\mathbf{R}} |f(t)|$.

III. Proof of Theorem 3

We shall employ the following

LEMMA 5. *Let* $a < 0$, $0 < \gamma < -a$, *then*

$$e^{(a+\gamma)t} - e^{at} \leqq \frac{\gamma}{|a+\gamma|},$$

for all $t \geqq 0$.

PROOF. It is easy to see that for the function

$$g(t) = e^{(a+\gamma)t} - e^{at}, \quad t \geqq 0$$

the relation $\sup_{[0,\infty)} g(t) = g(t^*)$ is valid, where $t^* = \gamma^{-1}\ln(a/(a+\gamma))$. □

PROOF OF THEOREM 3. Let

$$J := |X(t)P^{-1}(s) - Y(t)P^{-1}(s)|. \tag{16}$$

Then

$$J = |T(t)Z(t)PZ^{-1}(s)T^{-1}(s)-$$
$$-T(t)S(t)V(t)PV^{-1}(s)S^{-1}(s)T^{-1}(s)| \leqq$$
$$\leqq 2K|Z(t)PZ^{-1}(s) - S(t)V(t)PV^{-1}S^{-1}(s)| \leqq 2K(J_1 + J_2)$$

where

$$J_1 := |Z(t)PZ^{-1}(s) - V(t)PV^{-1}(s)|$$

and

$$J_2 := |V(t)PV^{-1}(s) - S(t)V(t)PV^{-1}(s)S^{-1}(s)|.$$

To estimate J_1 we observe that V, the solution of (14) can be written in the form $(PV = VP)$

$$V(t) = Z(t)Z^{-1}(s)V(t) + \int_s^t Z(t)Z^{-1}(t)D(t)V(t)\,dt. \tag{17}$$

Multiplying (17) on the right by P, and taking into account that $PV(t) = V(t)P$, $PZ = ZP$, we obtain

$$V(t)P = Z(t)PZ^{-1}(s)V(s) + \int_s^t Z(t)PZ^{-1}(\tau)E(\tau)V(\tau)P\,d\tau$$

and

$$V(t)PV^{-1}(s) = Z(t)PZ^{-1}(s) + \int_s^t Z(t)PZ^{-1}(\tau)E(\tau)V(\tau)PV^{-1}(s)\,d\tau.$$

From this for $t \geqq s$ we have

$$J_1 \leqq \int_s^t |Z(t)PZ^{-1}(\tau)||E(\tau)||V(\tau)PV^{-1}(s)|\,d\tau.$$

From (13), (16) and (17) we obtain

$$J_1 \leqq 72K^6\delta \int_s^t e^{\alpha(\tau-t)}e^{(\alpha-6K^3\delta)(s-\tau)}\,d\tau =$$

$$= 12K^3 \left[e^{(-\alpha+6K^3\delta)(t-s)} - e^{-\alpha(t-s)}\right].$$

Applying Lemma 5 we obtain

$$J_1 \leqq 72K^6\delta(\alpha - 12K^3\delta)^{-1}e^{-\alpha(t-s)/2} \leqq$$

$$\leqq 144K^6\delta\alpha^{-1}e^{-\alpha(t-s)/2} \leqq 144K^7\delta\alpha^{-1}e^{-\alpha(t-s)/2}.$$

Now we shall estimate J_2, using the identities

$$V(t)PV^{-1}(s) - S(t)V(t)PV^{-1}(s)S^{-1}(s) =$$

$$= V(t)PV^{-1}(s) - S(t)PS^{-1}(t)S(t)V(t)PV^{-1}(s)S^{-1}(s) =$$

$$= V(t)PV^{-1}(s) - PS(t)V(t)PV^{-1}(s)S^{-1}(s)+$$

$$+ \left[P - S(t)PS^{-1}(t)\right] S(t)V(t)PV^{-1}(s)S^{-1}(s).$$

Since $|P - S(t)PS^{-1}(t)| \leqq 4|H(t)|$, $|S(t)| \leqq \frac{3}{2}$, $|S^{-1}(t)| \leqq 2$, therefore

$$J_2 \leqq |V(t)PV^{-1}(s) - PS(t)V(t)PV^{-1}(s)S^{-1}(s)|+$$

$$+12|H(t)||V(t)PV^{-1}(s)|.$$

To complete the estimate of J_2 we employ the following identities:

(18) $$PS(t)V(t)PV^{-1}(s)S^{-1}(s) =$$

$$= P[I + H(t)]V(t)PV^{-1}(s)[I + H(s) + H^2(s) + \ldots] =$$

$$= V(t)PV^{-1}(s) + PH(t)V(t)PV^{-1}(s)S^{-1}(s)+$$

$$+PS(t)V(t)PV^{-1}(s)H(s)S^{-1}(s).$$

Substituting (20) in the last estimate for J_2 we obtain

(19) $$J_2 \leqq \left[|S(t)||S^{-1}(s)| + |S^{-1}(s)| + 12\right] |H(t)||V(t)PV^{-1}(s)|.$$

From Theorem 2 and (16) we have

$$J_2 \leqq 306K^5\delta\alpha^{-1}|V(t)PV^{-1}(s)| \leqq 612K^7\delta\alpha^{-1}e^{(-\alpha+6K^3\delta)(t-s)} \leqq$$

$$\leqq 612K^7\delta\alpha^{-1}e^{-\alpha(t-s)/2}.$$

Finally from (18) we obtain

$$J \leqq 2K\left[144K^7\delta\alpha^{-1} + 612K^7\delta\alpha^{-1}\right]e^{-\alpha(t-s)/2} =$$

$$= 1512K^8\delta\alpha^{-1}e^{-\alpha(t-s)/2} \leqq (5K/2)^8\delta\alpha^{-1}e^{-\alpha(t-s)/2}. \qquad \square$$

IV. Proof of Theorem 4

It is known that the unique bounded solution of (1) is given by

$$x_f(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)\,ds - \int_{t}^{\infty} X(t)QX^{-1}(s)f(s)\,ds \tag{20}$$

and the unique bounded solution of (10) is

$$y_f(t) = \int_{-\infty}^{t} Y(t)PY^{-1}(s)f(s)\,ds - \int_{t}^{\infty} Y(t)QY^{-1}(s)f(s)\,ds. \tag{21}$$

Using Theorem 3 we obtain

$$|x_f - y_f|(t) \leqq \int_{-\infty}^{t} |X(t)PX^{-1}(s) - Y(t)PY^{-1}(s)||f(s)|\,ds +$$

$$+ \int_{t}^{\infty} |X(t)QX^{-1}(s) - Y(t)QY^{-1}(s)||f(s)|\,ds \leqq$$

$$\leqq 1512K^8\delta\alpha^{-1}\left[\int_{-\infty}^{t} e^{-\alpha(t-s)/2}\,ds \int_{t}^{\infty} e^{-\alpha(s-t)/2}\,ds\right]\|f\| =$$

$$= 6048K^8\delta\alpha^{-2}\|f\| = (3K)^8\delta\alpha^{-2}\|f\|.$$

V. Application to singular perturbation problems

Let us consider the linear singular perturbed system

$$\mu x' = A(t)x \tag{22}$$

where μ is a small parameter and $A(t)$ is a bounded, uniformly continuous matrix defined on $\mathbf{R}$. We assume the following condition on the spectrum of $A(t) : |\mathrm{Re}\,\lambda(t)| \geqq \gamma > 0$, where γ is constant. Then in [3] it is proven that there exists a fundamental matrix $X(t)$ of system (24), a constant $K \geqq 1$ independent of μ, a constant $\mu_0 > 0$ and an integer k, $0 \leqq k \leqq n$, such that, for any $\mu \in (0, \mu_0)$ the following inequalities hold:

$$\begin{cases} |X(t)PX^{-1}(s)| \leqq K\exp\{-\gamma(t-s)/2\mu\}, & t \geqq s, \\ |X(t)QX^{-1}(s)| \leqq K\exp\{-\gamma(s-t)/2\mu\}, & s \geqq t. \end{cases} \tag{23}$$

Here $P = \operatorname{diag}(I_k, 0)$ and I_k is the identity matrix of dimension k and $Q = I - P$.

Let us now consider bounded solutions for the systems

$$\mu x' = A(t)x + f(t) \tag{24}$$

and

$$\mu y' = [A(t) + B(t)]y + f(t). \tag{25}$$

Then according to Theorem 3, if $\delta = \sup_{\mathbf{R}} |B(t)| \leqq \gamma/72K^5$ we have for the unique bounded solutions x_f and y_f of systems (24) and (25) the following estimate: $\|x_f - y_f\| \leqq 8(5K/2)^8\delta\gamma^{-2}$. □

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VOLUME 62

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CONTENTS

VOLUME 62

Allen, G. D., *Andrews, K. T.* and *Ward, J. D.*, Similarity and isometric equivalence of L_p-nests 25
Andrews, K. T., see *Allen, G. D.*
Blasco, J. L. and *Sanchis, M.*, On the product of two b_f-spaces 111
Bokhari, M. A., A note on certain next-to-interpolatory rational functions 49
Bregvadze, D. V., On the $(C, -1 < \alpha < 0, -1 < \beta < 0)$-summability of simple and double orthogonal series 311
Burton, A. P. and *Smith, P.*, Isoperimetric inequalities and areas of projections in $\mathbf{R}^n$ 395
Cabrerizo, J. L., *Fernández, L. M.* and *Fernández, M.*, The curvature of submanifolds of an S-space form 373
Chandra, P., A note on the degree of approximation of continuous functions 21
Cieślak, W., *Koshi, S.* and *Zając, J.*, On integral formulas for convex domains 277
de la Rosa, B. and *Stewart, P. N.*, On the maximal Kurosh–Amitsur model for the Wedderburn–Artin radical 337
Dorea, C. C. Y. and *Miasaki, E. S.*, Asymptotic test for independence of extreme values 343
Emmanuele, G., Precompactness in the space of Pettis integrable functions 333
Fabrykowski, J., A note on sums of powers of complex numbers 209
Fainleib, A. S., see *Tulyaganova, M. I.*
Fernández, L. M., see *Cabrerizo, J. L.*
Fernández, M., see *Cabrerizo, J. L.*
Garrido, M. I. and *Montalvo, F.*, On some generalizations of the Kakutani–Stone and Stone–Weierstrass theorems 199
Goldberg, V. V., On $(n+1)$-subwebs of an $(n+1)$-web and local algebras associated with them 57
Grubb, D. J., Dirichlet sets in Vilenkin groups 131
Gyires, B., On the matrix-valued exponentially convex, totally positive functions and sequences 231
Herzog, J., Weak asymptotic formulas for partitions free of small summands. II 173
Indlekofer, K.-H., The abstract prime number theorem for function fields 137
Joó, I., Note on multiplicative functions satisfying a congruence property. II 163
—, Notes to my paper "On the convergence of eigenfunction expansions in the norm of Soboleff spaces" 285
Kirchheim, B., Some further typical results on bounded Baire one functions 119

Kirchhoff, N. and *Nessel, R. J.*, Convolution processes of Fejér type and the divergence almost everywhere of a pointwise comparison 97
Koshi, S., see *Cieślak, W.*
Mathur, K. K. and *Saxena, Anjula*, Odd degree splines of higher order . 263
Mathur, K. K. and *Saxena, R. B.*, Weighted $(0,1,3)$ interpolation on the zeros of Hermite polynomials 31
Miasaki, E. S., see *Dorea, C. C. Y.*
Montalvo, F., see *Garrido, M. I.*
Naulin, R., A result concerning bounded solutions of linear systems of differential equations 403
Nessel, R. J., see *Kirchhoff, N.*
Phong, B. M., Multiplicative functions satisfying a congruence property 81
Powell, C. H. and *Wade, W. R.*, Paley sets and term by term dyadic differentiation of Walsh series 89
Prestin, J., Lagrange interpolation for functions of bounded variation ... 1
Sanchis, M., see *Blasco, J. L.*
Saxena, Anjula, see *Mathur, K. K.*
Saxena, R. B., see *Mathur, K. K.*
Shi, Y. G., New answers to Problem 24 of P. Turán 349
Smith, P., see *Burton, A. P.*
Staněk, S., On a class of five-point boundary value problems in second-order functional differential equations with parameter 253
Stepanauskas, G., The local behaviour of some additive functions 363
Stewart, P. N., see *de la Rosa, B.*
Tkačenko, M. G., On cardinal invariants of continuous images of topological groups 189
Tulyaganova, M. I. and *Fainleib, A. S.*, On the density of prime vectors in lattices 149
Varma, A. K., On Birkhoff quadrature formulas. II 15
Wade, W. R., see *Powell, C. H.*
Ward, J. D., see *Allen, G. D.*
Whelan, E. A., Direct sum decompositions of conformal rings 211
Yeung, H. F., The size of the middle summands in partitions into distinct s-th powers 293
Zahedani, H. Z., A Gelfand–Neumark theorem for commutative semifinite rank 157
Zając, J., see *Cieślak, W.*
Zięba, W., Some special properties of conditional expectation 385

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ACTA MATHEMATICA HUNGARICA / VOL. 62 No. 3–4

CONTENTS

Garrido, M. I. and *Montalvo, F.*, On some generalizations of the Kakutani–Stone and Stone–Weierstrass theorems 199
Fabrykowski, J., A note on sums of powers of complex numbers 209
Whelan, E. A., Direct sum decompositions of conformal rings 211
Gyires, B., On the matrix-valued exponentially convex, totally positive functions and sequences 231
Staněk, S., On a class of five-point boundary value problems in second-order functional differential equations with parameter 253
Mathur, K. K. and *Saxena, Anjula*, Odd degree splines of higher order 263
Cieślak, W., *Koshi, S.* and *Zając, J.*, On integral formulas for convex domains 277
Joó, I., Notes to my paper "On the convergence of eigenfunction expansions in the norm of Soboleff spaces" 285
Yeung, H. F., The size of the middle summands in partitions into distinct s-th powers 293
Bregvadze, D. V., On the $(C, -1 < \alpha < 0, -1 < \beta < 0)$-summability of simple and double orthogonal series 311
Emmanuele, G., Precompactness in the space of Pettis integrable functions 333
de la Rosa, B. and *Stewart, P. N.*, On the maximal Kurosh–Amitsur model for the Wedderburn–Artin radical 337
Dorea, C. C. Y. and *Miasaki, E. S.*, Asymptotic test for independence of extreme values 343
Shi, Y. G., New answers to Problem 24 of P. Turán 349
Stepanauskas, G., The local behaviour of some additive functions 363
Cabrerizo, J. L., *Fernández, L. M.* and *Fernández, M.*, The curvature of submanifolds of an S-space form 373
Zięba, W., Some special properties of conditional expectation 385
Burton, A. P. and *Smith, P.*, Isoperimetric inequalities and areas of projections in $\mathbf{R}^n$ 395
Naulin, R., A result concerning bounded solutions of linear systems of differential equations 403